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THE ENDPOINT PERTURBED BRASCAMP–LIEB INEQUALITIES WITH EXAMPLES

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We prove the folklore endpoint multilinear k_j-plane conjecture originating in a paper of Bennett, Carbery and Tao where the almost sharp multilinear Kakeya estimate was proved. Along the way we prove a more general result, namely the endpoint multilinear k_j-variety theorem. Finally, we generalize our results to the endpoint perturbed Brascamp–Lieb inequalities using techniques in earlier sections.

1. Introduction

The endpoint multilinear k_j-plane theorem. The multilinear k_j-plane conjecture was implicitly proved by Bennett, Carbery and Tao [2006], except for the endpoint case. In the first part of this paper we formulate and prove the endpoint case. In fact we will prove the endpoint multilinear k_j-variety theorem, which is more general.

The proof uses the polynomial method. We will set up the polynomial like Guth [2010] did in his proof of the endpoint multilinear Kakeya conjecture. Then we make some crucial new observations and development of the theory, enabling us to estimate "the quantitative interaction of the polynomial with itself" in terms of its visibility. As a result, we are able to deal with the codimension difficulty and complete the proof.

The multilinear k_j-plane estimate is a natural generalization of the famous multilinear Kakeya estimate. Albeit weaker than linear Kakeya, the multilinear Kakeya theorem and the methods it inspired recently had remarkable applications to classical harmonic analysis problems as well [Bourgain and Guth 2011; Bourgain 2013a; 2013b; Guth 2016b; 2016c; Bourgain and Demeter 2015]. See the beginning of [Guth 2015] for a good introduction.

The nonendpoint case of the multilinear Kakeya conjecture was proved by Bennett, Carbery and Tao [Bennett et al. 2006] and later Guth [2010] proved the endpoint case, which we state below.

Theorem 1.1. *For $1 \le j \le d$, let $\{T_{j,a} : 1 \le a \le A(j)\}$ be a family of unit cylinders in $\mathbb{R}^d$. We set $v_{j,a}$ to be the direction of the core line of the cylinder $T_{j,a}$. Assume the core lines of cylinders from different families are "quantitatively transversal"; i.e., for any $1 \le a_j \le A(j)$, we have $v_{1,a_1} \wedge v_{2,a_2} \wedge \cdots \wedge v_{d,a_d} \ge \theta > 0$, where θ is fixed. Then we have*

$$\int_{\mathbb{R}^d} \biggl(\prod_{j=1}^{d} \sum_{a=1}^{A(j)} \chi_{T_{j,a}} \biggr)^{1/(d-1)} \lesssim_d \theta^{-1/(d-1)} \prod_{j=1}^{d} A(j)^{1/(d-1)}. \tag{1-1}$$

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Keywords: Brascamp–Lieb inequality, polynomial method.

Guth's approach to proving Theorem 1.1 is very different from the approach of Bennett, Carbery and Tao. He was able to take a polynomial that approximates the intersection of tubes sufficiently well, along the way employing some nice tools and lemmas from algebraic topology and integral geometry.

In the Kakeya setting we have cylinders which are neighborhoods of lines. A natural analogue is to replace lines with higher-dimensional affine subspaces and this will exactly be our multilinear k_j-plane setting. In Remark 5.4 of [Bennett et al. 2006], the authors note that their techniques can be also used to obtain nonendpoint cases of multilinear k-plane transform estimates considered in [Oberlin and Stein 1982]. There is also a k-plane version of the Kakeya problem [Bourgain 1991] that could be relevant here.

They did not state the result precisely and we will state what we can get from their proof below. If we go check the proof, similar techniques in [Guth 2015] can also give us the result. Here we allow subspaces of different dimensions and hence call the theorem a "multilinear k_j-plane theorem".

Before stating the theorem we introduce our terminology to describe a "higher-dimensional" analogue of cylinders.

Definition 1.2. In a space of dimension d, for any $1 \le b < d$ define a *b-slab* to be the Cartesian product of a b-dimensional ball B_1 and a $(d-b)$-dimensional ball B_2 (the spaces spanned by both balls are required to be orthogonal). The radius of B_1 will be called the *size* of our b-slab and the radius of B_2 will be called the *radius* of it. The Cartesian product of B_1 and the center of B_2 is called the *core* of this b-slab.

By the above definition, a 1-slab is a cylinder. Its length is the size in our language. Our definitions of radius and core are consistent with familiar definitions for cylinders. As explained above, we call our theorem a k_j-plane theorem because when the size is large, a k-slab looks flat and is like a "fattened" k-plane.

Theorem 1.3 (multilinear k_j-plane theorem with R^ε loss [Bennett et al. 2006]). *Assume R is a large positive number. Assume $K_1, K_2, \ldots, K_n \subsetneqq \{1, 2, \ldots, d\}$ are disjoint and $K_1 \cup \cdots \cup K_n = \{1, 2, \ldots, d\}$. Let $k_j = |K_j|$.*

For $1 \le j \le n$, let $\{T_{j,a} : 1 \le a \le A(j)\}$ be a family of k_j-slabs of size $\le R$ and radius 1. Assume that for any $1 \le a_j \le A(j)$, the core of T_{j,a_j} is on a k_j-plane that forms an angle $< \delta$ against the k_j-plane spanned by all $\boldsymbol{e}_i$, $i \in K_j$.

Then when $\delta > 0$ is sufficiently small depending on d, we have

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^{n} \sum_{a=1}^{A(j)} \chi_{T_{j,a}} \right)^{1/(n-1)} \lesssim_{\varepsilon,d} R^\varepsilon \prod_{j=1}^{n} A(j)^{1/(n-1)}. \tag{1-2}$$

When $n = d$ and $K_j = \{j\}$, this theorem is the multilinear Kakeya theorem with R^ε loss, which is the main theorem of [Bennett et al. 2006]. In [Guth 2015], a simpler proof of this special case is also given, and it can be generalized easily to prove the whole Theorem 1.3.

We can obtain various k_j-plane theorems by taking different n and K_j in Theorem 1.3. As we saw in Theorem 1.1, Guth [2010] was able to remove the R^ε in the multilinear Kakeya case. So in general we would also expect the removal of R^ε. Conceptually, this will allow us to have slabs with "size ∞" (that are actually 1-neighborhoods of k_j planes) in the theorem. It turns out to be true and will be proved in this paper.

Theorem 1.4 (multilinear k_j-plane theorem). *Take the same assumptions as Theorem 1.3, but with no restriction on the size of slabs. We have*

$$\int_{\mathbb{R}^d}\Bigl(\prod_{j=1}^{n}\sum_{a=1}^{A(j)}\chi_{T_{j,a}}\Bigr)^{1/(n-1)} \lesssim_d \prod_{j=1}^{n} A(j)^{1/(n-1)}. \tag{1-3}$$

Theorem 1.4 has an affine-invariant version, just like the multilinear Kakeya case, which was first pointed out in [Bourgain and Guth 2011]. We will actually prove this version (Theorem 1.5 below). Theorem 1.4 is a direct corollary of it.

In order to state the theorem, we introduce some notation. For any $q \le d$ vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_q$, we define $|\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \cdots \wedge \boldsymbol{v}_q|$ to be the volume of the parallelepiped generated by $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_q$. Moreover, for any m (affine) subspaces $V_1, V_2, \dots, V_m$ with a total dimension d, we can define $|V_1 \wedge V_2 \wedge \cdots \wedge V_m|$ to be $|\boldsymbol{v}_{1,1} \wedge \cdots \wedge \boldsymbol{v}_{1,d_1} \wedge \boldsymbol{v}_{2,1} \wedge \cdots \wedge \boldsymbol{v}_{2,d_2} \wedge \cdots \wedge \boldsymbol{v}_{m,1} \cdots \wedge \boldsymbol{v}_{m,d_m}|$, where $\{\boldsymbol{v}_{j,i} : 1 \le i \le d_j\}$ form an orthonormal basis of the linear subspace parallel to V_j.

Theorem 1.5 (affine invariant multilinear k_j-plane theorem). *Assume the positive integers* $1 \le k_1, \dots, k_n \le d-1$ *satisfy* $\sum_{j=1}^{n} k_j = d$. *For* $1 \le j \le n$, *let* $\{T_{j,a} : 1 \le a \le A(j)\}$ *be a family of* k_j*-slabs of radius* 1. *Assume the core* k_j*-plane of* $T_{j,a}$ *is parallel to the linear subspace* $H_{j,a}$. *Then for any real numbers* ρ_{j,a_j} *we have*

$$\int_{\mathbb{R}^d}\Bigl(\sum_{a_1=1}^{A(1)}\cdots\sum_{a_n=1}^{A(n)}\prod_{j=1}^{n}\rho_{j,a_j}\chi_{T_{j,a_j}}(x)\cdot H_{1,a_1}\wedge\cdots\wedge H_{n,a_n}\Bigr)^{1/(n-1)}\mathrm{d}x \lesssim_d \prod_{j=1}^{n}\Bigl(\sum_{a_j=1}^{A(j)}|\rho_{j,a_j}|\Bigr)^{1/(n-1)}. \tag{1-4}$$

Remark 1.6. We refer the reader to [Bennett and Bez 2010] for an explanation of why the exponents are as they appear in Theorem 1.5. Also we note that in that paper the authors already observed the affine-invariant Finner inequality, which is an "unperturbed" version of Theorem 1.5.

Our Theorem 1.5 has some application in the multilinear restriction theorem too. For each $1 \le j \le n$ assume $\Sigma_j : U_j \to \mathbb{R}^d$ is a smooth parametrization of a subset of a smooth submanifold Ω_j whose closure is compact. Also assume $\sum_{j=1}^{n} \dim \Omega_j = d$. Here we assume U_j is a neighborhood of the origin 0. We can associate the *extension operator* to Σ_j as follows:

$$E_j g_j(\xi) = \int_{U_j} e^{2\pi i \xi\cdot\Sigma_j(x)} g_j(x)\,\mathrm{d}x. \tag{1-5}$$

Assume $T_{\Sigma_1(0)}\Omega_1 \wedge \cdots \wedge T_{\Sigma_n(0)}\Omega_n \neq 0$. Then just like the classical multilinear restriction case discussed in [Bennett et al. 2006], we can form the endpoint multilinear restriction conjecture:

Conjecture 1.7 (endpoint multilinear k_j-restriction conjecture). *Assume we have* Σ_j *as above such that* $T_{\Sigma_1(0)}\Omega_1 \wedge \cdots \wedge T_{\Sigma_n(0)}\Omega_n \neq 0$. *Then when the* U_j *are sufficiently small, we have*

$$\int_{\mathbb{R}^d}\prod_{j=1}^{n}|E_j g_j|^{2/(n-1)} \lesssim_d \prod_{j=1}^{n}\|g_j\|_{L^2(U_j)}^{2/(n-1)}. \tag{1-6}$$

The methods in [Bennett et al. 2006] yield the following local variant of the conjectured (1-6) with R^{ε}-loss:

$$\int_{B(0,R)} \prod_{j=1}^{n} |E_j g_j|^{2/(n-1)} \lesssim_{d,\varepsilon} R^{\varepsilon} \prod_{j=1}^{n} \|g_j\|_{L^2(U_j)}^{2/(n-1)}. \tag{1-7}$$

We can use Theorem 1.4 to slightly improve (1-7). Using exactly the same proof techniques as in the proof of Theorem 4.2 in [Bennett 2014], from Theorem 1.4 we deduce that there exists a $\kappa = \kappa(d) > 0$ such that

$$\int_{B(0,R)} \prod_{j=1}^{n} |E_j g_j|^{2/(n-1)} \lesssim_{d} (\log R)^{\kappa} \prod_{j=1}^{n} \|g_j\|_{L^2(U_j)}^{2/(n-1)}. \tag{1-8}$$

The endpoint perturbed Brascamp–Lieb inequalities. Everything in the previous section in its unperturbed version, including the Loomis–Whitney inequality and the multilinear k_j-plane theorem, is a special case of the Brascamp–Lieb inequalities. In this paper we also generalize the Brascamp–Lieb inequalities in the same way we do with the multilinear k_j-plane theorem, with some new combinatorial ideas. We state our endpoint perturbed Brascamp–Lieb inequalities in this section.

We first briefly review the Brascamp–Lieb inequalities. We will mostly follow the notational convention in [Bennett et al. 2008; 2010], which are two important references in the literature. Assume that in $\mathbb{R}^d$ we have n linear surjections $B_j : \mathbb{R}^d \to E_j$. Then for certain positive numbers p_j, $1 \le j \le n$, the following Brascamp–Lieb inequality holds for any measurable function f_j on $E_j (1 \le j \le n)$ with some $C > 0$:

$$\int_{\mathbb{R}^d} \prod_{j=1}^{n} (f_j \circ B_j)^{p_j} \le C \prod_{j=1}^{n} \biggl(\int_{E_j} f_j \biggr)^{p_j}. \tag{1-9}$$

If this is the case, we call the minimum possible constant C such that (1-9) holds the *Brascamp–Lieb constant* $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p})$. Here we use $\boldsymbol{B}$ to denote the data $(B_1, \dots, B_n)$ and $\boldsymbol{p}$ to denote the data $(p_1, \dots, p_n)$. The pair $(\boldsymbol{B}, \boldsymbol{p})$ is called the corresponding *Brascamp–Lieb datum.* If (1-9) fails for any finite C, we define $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) = +\infty$. Note that no a priori assumptions are made on the relationship between d and n here.

Lieb [Lieb 1990] showed:

Theorem 1.8. $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) = \mathrm{BL}_g(\boldsymbol{B}, \boldsymbol{p})$, *where*

$$\mathrm{BL}_g(\boldsymbol{B}, \boldsymbol{p}) = \sup \biggl(\frac{\prod_{j=1}^{n} (\det_{E_j} A_j)^{p_j}}{\det\bigl(\sum_{j=1}^{n} p_j B_j^* A_j B_j\bigr)} \biggr)^{1/2} \tag{1-10}$$

with the supremum is taken over all $A_j : E_j \to E_j$ such that A_j is a positive definite linear transform.

An alternative way to state Theorem 1.8 is that the Brascamp–Lieb constant is what one would obtain by restricting attention to the special case in which each f_j is a certain Gaussian function.

Subsequently, Bennett, Carbery, Christ and Tao [Bennett et al. 2008; 2010] determined a necessary and sufficient condition for $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) = \mathrm{BL}_g(\boldsymbol{B}, \boldsymbol{p}) < +\infty$. They proved that $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) = \mathrm{BL}_g(\boldsymbol{B}, \boldsymbol{p}) < +\infty$ is equivalent to the following two conditions:

(1) Scaling condition:

$$\sum_j p_j \dim E_j = d. \tag{1-11}$$

(2) Dimension condition: for any linear subspace $V \subseteq \mathbb{R}^d$,

$$\dim V \le \sum_j p_j \dim(B_j V). \tag{1-12}$$

So we know when we can have the actual Brascamp–Lieb inequality (1-9) thanks to their work.

Inequality (1-9) has an equivalent version that is easier to understand intuitively. We state it in the following proposition and refer the readers to [Bennett 2012] for this observation.

Proposition 1.9 (combinatorial Brascamp–Lieb). *Assume we have a Brascamp–Lieb datum $(\boldsymbol{B}, \boldsymbol{p})$ in $\mathbb{R}^d$. Assume $k_j = \dim\ker B_j$ and we have n families of slabs. Assume the j-th family $\mathbb{T}_j$ consists of only k_j-slabs of radius* 1 *whose cores are all parallel to* $\ker B_j$. *Also assume each $|\mathbb{T}_j|$ is finite. Then* $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) < +\infty$ *if and only if we always have*

$$\int_{\mathbb{R}^d} \prod_{j=1}^n \Bigl(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j}\Bigr)^{p_j} \lesssim \prod_{j=1}^n |\mathbb{T}_j|^{p_j}. \tag{1-13}$$

In light of the last subsection, a perturbed version of this proposition should be true. This can indeed be proved; recently, Bennett, Bez, Flock and Lee [Bennett et al. 2015, Theorem 1.2] proved the following (nonendpoint) theorem via generalizations of Guth's method [2015].

Theorem 1.10 (perturbed Brascamp–Lieb with R^ε-loss [Bennett et al. 2015]). *Assume we have a Brascamp–Lieb datum $(\boldsymbol{B}, \boldsymbol{p})$ in $\mathbb{R}^d$ with* $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) < +\infty$. *Let $k_j = \dim\ker B_j$. Assume we have n families of slabs and the j-th family $\mathbb{T}_j$ consists of only k_j-slabs of radius* 1 *and size $\le R$. Assume each $|\mathbb{T}_j|$ is finite. Also assume that each slab in the j-th family has its core k_j-plane within a δ-neighborhood of* $\ker B_j$ *on the corresponding Grassmannian* (*with a given standard metric*). *Then when δ is sufficiently small depending on $(\boldsymbol{B}, \boldsymbol{p})$ we have*

$$\int_{\mathbb{R}^d} \prod_{j=1}^n \Bigl(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j}\Bigr)^{p_j} \lesssim_{d,\boldsymbol{p},\mathrm{BL}(\boldsymbol{B},\boldsymbol{p}),\varepsilon} R^\varepsilon \prod_{j=1}^n |\mathbb{T}_j|^{p_j}. \tag{1-14}$$

They conjectured that R^ε can be removed here (see inequalities (7) and (8) of [Bennett et al. 2015]) and we prove their conjecture in the last section of this paper.

Theorem 1.11 (endpoint perturbed Brascamp–Lieb theorem). *Assume we have a Brascamp–Lieb datum $(\boldsymbol{B}, \boldsymbol{p})$ in $\mathbb{R}^d$ with* $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) < +\infty$. *Let $k_j = \dim\ker B_j$. Assume we have n families of slabs and the j-th family $\mathbb{T}_j$ consists of only k_j-slabs of radius* 1. *Assume each $|\mathbb{T}_j|$ is finite. Also assume that each slab in the j-th family has its core k_j-plane within a δ-neighborhood of* $\ker B_j$ *on the corresponding Grassmannian* (*with a given standard metric*). *Then when δ is sufficiently small depending on $(\boldsymbol{B}, \boldsymbol{p})$ we have*

$$\int_{\mathbb{R}^d} \prod_{j=1}^n \Bigl(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j}\Bigr)^{p_j} \lesssim_{d,\boldsymbol{p},\mathrm{BL}(\boldsymbol{B},\boldsymbol{p})} \prod_{j=1}^n |\mathbb{T}_j|^{p_j}. \tag{1-15}$$

Remark 1.12. Theorem 1.11 formally implies the stability of Brascamp–Lieb constants, which was a result of Bennett, Bez, Flock, and Lee [Bennett et al. 2015]. However, it is worth noticing (see the proof later in Section 8) that the main result in [Bennett et al. 2015] is an input rather than an output in our proof of Theorem 1.11. In particular, we do not have a new proof of the main result in [Bennett et al. 2015] in this paper.

Our proof of Theorem 1.11 will follow the same scheme as the proof of Theorem 1.4. Some new difficulties present themselves and we deal with them in due course.

Like what we had in the end of last subsection, our perturbed Brascamp–Lieb theorem has some impact on the endpoint Brascamp–Lieb-type restriction conjecture formulated in [Bennett et al. 2015]. To introduce it, we use the same setup that we did in Conjecture 1.7, but this time we don't assume that $\sum_j k_j = d$ or that $T_{\Sigma_1(0)}\Omega_1 \wedge \cdots \wedge T_{\Sigma_n(0)}\Omega_n \neq 0$. Instead, we assume that there exists $\boldsymbol{p} = (p_1, \dots, p_n)$, $p_j > 0$, such that $\mathrm{BL}(\boldsymbol{B}(\Sigma), \boldsymbol{p}) < \infty$, where $\boldsymbol{B}(\Sigma) = (T_{\Sigma_1(0)}\Omega_1, \dots, T_{\Sigma_n(0)}\Omega_n)$ (here we abuse the notation a bit and for each component we really mean the linear subspace of $\mathbb{R}^d$ parallel to it).

Conjecture 1.13 (endpoint Brascamp–Lieb-type restriction conjecture). *With the above setup, when the U_j are sufficiently small, we have*

$$\int_{\mathbb{R}^d} \prod_{j=1}^{n} |E_j g_j|^{2p_j} \lesssim_{d,\boldsymbol{p},\mathrm{BL}(\boldsymbol{B}(\Sigma),\boldsymbol{p})} \prod_{j=1}^{n} \|g_j\|_{L^2(U_j)}^{2p_j}. \tag{1-16}$$

In [Bennett et al. 2015] a local variant of (1-16) with R^ε-loss is proved:

$$\int_{B(0,R)} \prod_{j=1}^{n} |E_j g_j|^{2p_j} \lesssim_{d,\boldsymbol{p},\mathrm{BL}(\boldsymbol{B}(\Sigma),\boldsymbol{p}),\varepsilon} R^\varepsilon \prod_{j=1}^{n} \|g_j\|_{L^2(U_j)}^{2p_j}. \tag{1-17}$$

By Theorem 1.11 and again the same method as in the proof of Theorem 4.2 in [Bennett 2014], we can slightly improve (1-17): there is a $\kappa = \kappa(\mathrm{BL}(\boldsymbol{B}(\Sigma), \boldsymbol{p})) > 0$ such that

$$\int_{B(0,R)} \prod_{j=1}^{n} |E_j g_j|^{2p_j} \lesssim_{d,\boldsymbol{p}} (\log R)^\kappa \prod_{j=1}^{n} \|g_j\|_{L^2(U_j)}^{2p_j}. \tag{1-18}$$

Idea of the proofs. When looking to remove the factor R^ε in Theorems 1.3 and 1.10, the methods in [Bennett et al. 2006] or [Guth 2015] do not feel very appealing. Instead we will follow the path led by Guth [2010] and try to come up with a version of the so-called polynomial method.

However, there is a major difficulty to generalizing Guth's argument: note that the zero set of one polynomial has codimension 1. In the setting of [Guth 2010], because a line has dimension 1, a line will intersect the above zero set at discrete points. And the number of such points is controlled by the degree of the polynomial. Hence we can do some counting to obtain estimates. In particular, Guth's proof relies heavily on the following *cylinder estimate*.

Lemma 1.14 (cylinder estimate). *Let T be a cylinder of radius* 1 *and P be a polynomial of degree D. Let $\boldsymbol{v}$ be a unit vector parallel to the core line of T. If we define $Z(P)$ to be the zero set of P, then the directed volume* (*see Definition 2.1*) *satisfies*

$$V_{Z(P)\cap T}(\boldsymbol{v}) \lesssim_d D. \tag{1-19}$$

In the k_j-plane setting, the zero set of a single polynomial no longer interacts well with a k_j-plane: because the latter generally has a smaller codimension, it won't intersect the former at discrete points in general. Due to this issue we cannot do counting and seem to lose our main weapon (Lemma 1.14).

In this paper, we deal with this difficulty and obtain our Theorem 1.4. The main idea is the following: for a k-plane, instead of finding one single polynomial, we would like to take zero sets of k polynomials to interact with it. Because the codimensions of the k-plane and zero sets of the k polynomials add up to d, they will intersect at points and it is possible to do counting to estimate the intersection again.

Along this line, we are taking more than one polynomial to approximate an arbitrary set of N cubes. We would like the zero sets of all the polynomials to be "transverse"; with this requirement we can choose at most d such polynomials. Like the original polynomial method, we would like to know how low the degrees of our polynomials can be. Guth [2010] showed that we can always choose the first polynomial to be of degree $\lesssim_d N^{1/d}$. But for the second polynomial this degree bound may already be no longer valid. Think about N unit squares lining up on a line in the plane $\mathbb{R}^2$. Any polynomial with degree significantly less than N would have most of its zero set "almost parallel" to the line, see [Guth 2016a], and hence two such polynomials cannot interact transversely at most of the squares. However, in this example it is possible to find two transverse polynomials with degree product N. One can also look at examples of cube grids, or more generally transverse intersections of hypersurfaces, and similar phenomena happen there. Based on the above discussion, we are willing to ask the following question in the spirit of the polynomial method.

Question 1.15. *Given any N disjoint unit cubes in $\mathbb{R}^d$ and $A_\nu > 1$ for each given cube Q_ν, do there always exist d polynomials $P_1, P_2, \dots, P_d$ such that $\prod_{i=1}^d \deg P_i$ is roughly $\sum_\nu A_\nu$, and the zero sets of all P_i have "quantitative interaction" $\gtrsim_d A_\nu$ at each of the above cubes?*

We notice that it looks like a "continuous version" of the inverse Bézout's theorem; see for example [Tao 2012]. The analogue is very difficult in algebraic geometry, see [Tao 2012] for part of the reason, and is conceivably very hard in its current continuous version too. We believe it can be formulated as an explicit question with an affirmative answer though. One can make this question rigorous by specifying the meaning of "quantitative interaction"; see the discussion below and (6-9) for a result of this flavor.

Luckily enough, we find the full power of this hard version is not needed this time. Instead, it will be equally useful to have a positive answer to the following "softer" question.

Question 1.16. *Given any N disjoint unit cubes in $\mathbb{R}^d$ and $A_\nu > 1$ for each given cube Q_ν, do there always exist d polynomials $P_1, P_2, \dots, P_d$ and positive numbers $\alpha_\nu > 1$ such that $\prod_{i=1}^d \deg P_i$ is roughly $\sum_\nu \alpha_\nu A_\nu$, and the zero sets of all P_i have quantitative interaction $\gtrsim_d \alpha_\nu A_\nu$ at each of the above cubes?*

This question is weaker than Question 1.15 because there we have the additional requirement that $a_\nu = 1$. In other words, we allow polynomials of higher degree here but "with the right multiplicity". In general, higher-degree polynomials, even with the right multiplicity, do not necessarily work as well as ones with lowest possible degree; see for example some estimates in [Guth 2016a]. But in this application it makes no difference, as we are in a situation similar to what we have in [Guth 2010].

Surprisingly, it turns out that after some further refinement of the question, we find that we can take $P_1, \dots, P_d$ all to be the same P and that we can obtain P by the refined polynomial method of Guth involving visibility. Once this is clear we are able to prove our theorem with a great amount of help from (multi-)linear algebra and geometry.

To be more specific, we find that we can take a single nonzero polynomial (that is complicated enough to look like the product of several transverse polynomials) such that the following holds: If we define $Z(P)$ to be the zero set of P, then for each relevant Q_ν, d copies of $Z(P) \cap Q_\nu$ interact in a sufficiently transverse manner. Since the d copies of $Z(P) \cap Q_\nu$ interact in a very transverse way, and the copies are all the same, for any j and any Q_ν we deduce that k_j copies of $Z(P) \cap Q_\nu$ interact sufficiently transversely with the part of the j-th family of slabs inside Q_ν. But for any j, the j-th family has a limited capacity of transverse interaction with k_j copies of $Z(P)$ by Bézout's theorem. This gives us an estimate that leads to Theorem 1.4.

As we saw above, we end up taking one single polynomial d times. Nevertheless, we choose to keep the entire thought process on "d transverse polynomials" here because after all, it is how we eventually come up with the solution and the reader might find our thought process useful elsewhere. Also, Question 1.15, which remains open, is still fundamental, as it's a general one concerning the polynomial approximation of any N cubes. For example, it implies the existence of the polynomial in the polynomial method. Its discrete analogue is also open; see [Tao 2012]. But progress in various subcases has been made.

In the multilinear k_j-plane setting, our method actually proves a stronger theorem (multilinear k_j-variety theorem, Theorem 6.1) which largely generalizes Theorem 1.4. We will state its exact form after a bit more preparation. Here let us briefly describe it.

Let's take a new viewpoint. Knowing that a point belongs to a slab of radius 1 is equivalent to knowing the existence of another point on the core of the slab that lies in its 1-neighborhood. Also note that the union of all cores (k_j-planes) of the j-th family of slabs can be viewed as an algebraic variety of degree $A(j)$ and dimension k_j. This variety is a smoothly embedded k_j-manifold except some zero-volume subset. Our Theorem 1.4 is basically saying that the n families of k_j-planes have limited capacity of "transversally interaction". We will prove that this is the general case for any n algebraic varieties with total dimension d in Theorem 6.1.

This multilinear k_j-variety theorem immediately has interesting special cases. For instance, we have a theorem about collections of sphere shells in the flavor of Theorem 1.4.

The proof of Theorem 1.11 is with almost the same machine, but we have some new difficulties: When we use this machine, we want to know how well each k_j-plane interacts with our polynomial. However, the information on the Brascamp–Lieb constant seems to be very hard to use when we try to look at things "locally", as we do in the proof of Theorem 1.4. We address this issue in Section 7 and Section 8 by proving a weaker "integral version" of our previous pointwise estimate. Albeit weaker, it already leads to a proof of Theorem 1.11.

Like the situation of Theorem 1.4, Theorem 1.11 has a generalization to algebraic varieties (Theorem 8.1) and we prove the latter to automatically imply the former. Again the current form is quite strong and interesting in its own right.

Outline of the paper. In Sections 2 and 3 we review Guth's polynomial method [2010] and develop all we need in this subject. Section 4 consists of linear algebra preliminaries and Section 5 consists of integral geometry preliminaries. We prove Theorem 1.4 and Theorem 1.5 in Section 6 and Theorem 1.11 in Section 8 after some preparation (Section 7). We will prove them by generalizing to versions about algebraic varieties.

2. Polynomial with high visibility

In this section, we review the refined polynomial method by Guth [2010]. We review the definition and properties of visibility and state Guth's theorem that we can find a polynomial with reasonable degree and large visibility in many cubes. Along the way we define a relevant notion, namely the *fading zone*, for future convenience.

Definition 2.1. In $\mathbb{R}^d$, for any compact smooth hypersurface Z (possibly with boundary) and any vector $\boldsymbol{v}$, define the *directed volume*

$$V_Z(\boldsymbol{v}) = \int_Z |\boldsymbol{v}\cdot\boldsymbol{n}|\,\mathrm{dVol}_Z, \tag{2-1}$$

where $\boldsymbol{n}$ is the normal vector at the corresponding point of Z.

If $\boldsymbol{v}$ is a unit vector, there is a formula for $V_Z(\boldsymbol{v})$ that is geometrically more meaningful. Let $\pi_{\boldsymbol{v}}$ be the orthogonal projection of $\mathbb{R}^d$ onto the subspace $\boldsymbol{v}^\perp$. Then for almost $y\in\boldsymbol{v}^\perp$, we have $|Z\cap\pi_{\boldsymbol{v}}^{-1}(y)|$ is finite and, see [Guth 2010],

$$V_Z(\boldsymbol{v}) = \int_{\boldsymbol{v}^\perp} |Z\cap\pi_{\boldsymbol{v}}^{-1}(y)|\,\mathrm{d}y. \tag{2-2}$$

Definition 2.2. The *fading zone* $F(Z)$ is defined to be the set $\{\boldsymbol{v} : |\boldsymbol{v}|\le 1,\ V_Z(\boldsymbol{v})\le 1\}$. It is a nonempty convex compact subset of the unit ball; see [Guth 2010]. The visibility $\mathrm{Vis}[Z] = 1/|F(Z)|$.

First we explain the heuristic meaning of the two concepts. Imagine that it is midnight and we are looking at a glittering object with exactly the same shape as Z from a fixed distance. To describe the situation mathematically, we can find a vector $\boldsymbol{v}$ such that its direction is the direction of the object and its length is the brightness of the object. Then we can intuitively think that Z fades away when $\boldsymbol{v}$ enters the fading zone. And naturally the less visible the object is, the larger we want the fading zone to be. Hence we can define the visibility to be the inverse of the volume of the fading zone. See the beginning of Section 6 in [Guth 2010] for how to intuitively understand visibility and a few simple examples.

It is good to keep in mind that in this paper we will mostly deal with hypersurfaces Z with $V_Z(\boldsymbol{v})\gtrsim_d 1$ for any unit vector $\boldsymbol{v}$. For hypersurfaces that don't satisfy this we will typically fix it by taking its union with several hyperplanes parallel to coordinate hyperplanes.

Clearly as long as Z has finite volume, $F(Z)$ has a nonempty interior.

We are interested in polynomials and want to use the notions above to study them. Recall that the space of degree D algebraic hypersurfaces in $\mathbb{R}^d$ is parametrized by $\mathbb{RP}^K$ for $K=\binom{D+d}{d}-1$ in the following way: any such hypersurface corresponds to a polynomial P up to a scalar. By viewing P also as the $\binom{D+d}{d}$-tuple of its coefficients we find this parametrization [Guth 2010]. We want to think of the directed volume and the visibility as functions over $\mathbb{RP}^K$. However, as Guth [2010] pointed out, they are bad functions that may even be discontinuous.

Following [Guth 2010], we get around this difficulty by looking at the mollified versions of them. If we take the standard metric on $\mathbb{RP}^K$, we will mollify those functions over small balls around some $P \in \mathbb{RP}^K$. In the rest of this paper, we take ε to be a very small positive number depending on all the constants, and in application on the set of cubes and visibility conditions. This kind of assumption is often dangerous but as we can eventually see, here it does no harm at all (mainly because all the algebraic hypersurfaces of degree D satisfy the same intersection estimate (5-5) uniformly), just like the case of [Guth 2010]. There instead of the intersection estimate, we have the cylinder estimate (1-19) as a special case counterpart.

For any $P \in \mathbb{RP}^K$, let $B(P, \varepsilon)$ be the ε-neighborhood of P. Let $Z(P)$ denote the zero set of P. Note that for any P, the set of singular points on $Z(P)$ has zero $(d-1)$-dimensional Hausdorff measure. And the rest of $Z(P)$ is a smooth embedded hypersurface by the implicit function theorem.

Definition 2.3. For any bounded open set U and any vector $\boldsymbol{v}$, define the *mollified directed volume*

$$\overline{V}_{Z(P)\cap U}(\boldsymbol{v}) = \frac{1}{|B(P,\varepsilon)|}\int_{B(P,\varepsilon)} V_{Z(P')\cap U}(\boldsymbol{v})\,\mathrm{d}P'. \tag{2-3}$$

Define the *mollified fading zone* and *mollified visibility* based on the mollified directional volumes:

$$\bar{F}(Z(P)\cap U) = \{\boldsymbol{v} : |\boldsymbol{v}| \le 1 : \overline{V}_{Z(P)\cap U}(\boldsymbol{v}) \le 1\}, \tag{2-4}$$

$$\overline{\mathrm{Vis}}[Z(P)\cap U] = \frac{1}{|\bar{F}(Z(P)\cap U)|}. \tag{2-5}$$

Like we had before for $F(Z)$, $\bar{F}(Z(P)\cap U)$ is a convex compact subset of the unit ball with a nonempty interior. By John's ellipsoid theorem [1948], for any convex set Γ with interior, there is an ellipsoid $\mathrm{Ell}(\Gamma)$ such that $\mathrm{Ell}(\Gamma) \subseteq \Gamma \subseteq C_d\,\mathrm{Ell}(\Gamma)$ and $|\,\mathrm{Ell}(\Gamma)| \sim_d |\Gamma|$. It is easy to see that if the convex set is symmetric about the origin (which will be the case for all convex sets considered in this paper), then we may require the ellipsoid to be symmetric about the origin too. We assume so henceforth in the paper. We call any such $\mathrm{Ell}(\Gamma)$ an *elliptical approximation* of Γ.

$\overline{V}_{Z(P)\cap U}(\boldsymbol{v})$ and $\overline{\mathrm{Vis}}[Z(P)\cap U]$ are continuous with respect to $P \in \mathbb{RP}^M$ [Guth 2010]. In the same paper, Guth also proved the following key lemma.

Lemma 2.4 (large visibility on many cubes [Guth 2010]). *For any finite set of cubes $Q_1, \dots, Q_N$ and nonnegative integers $M(Q_i)$, $1 \le i \le N$, there exists a polynomial P of degree $\le D$ (but viewed as a degree-D polynomial when we mollify) such that $\overline{\mathrm{Vis}}(Z(P)\cap Q_k) \ge M(Q_k)$ and $D \lesssim_d \left(\sum_{i=1}^N M(Q_k)\right)^{1/d}$.*

3. Wedge-product estimate based on visibility

As we are actually dealing with the mollification version of everything, it is convenient to have a generalized definition of visibility on any space of finite measure. The generalized setup here will also be cleaner and more flexible in the inductive arguments which are needed.

Assume we have a measure space (X, μ) with $\mu(X) < \infty$ and a vector-valued measurable function $f : X \to \mathbb{R}^d$. For any vector $\boldsymbol{v} \in \mathbb{R}^d$ define the *total absolute inner product* of $\boldsymbol{v}$ and f as

$$V_{X,f}(\boldsymbol{v}) = \int_X |\boldsymbol{v} \cdot f(x)| \, \mathrm{d}\mu(x) \tag{3-1}$$

(the directed volume of the last section being the example we have in mind).

Define the fading zone $F(X, f) = \{\boldsymbol{v} \le 1 : V_{X,f}(\boldsymbol{v}) \le 1\}$ and visibility $\mathrm{Vis}[X, f] = 1/|F(X, f)|$. As we had in the end of the last section, we have an elliptical approximation $\mathrm{Ell}(F(X, f))$ such that $\mathrm{Ell}(F(X, f)) \subseteq F(X, f) \subseteq C_d \,\mathrm{Ell}(F(X, f))$.

Next we obtain a lower bound of a wedge product integral in terms of visibility.

Theorem 3.1 (wedge product estimate). *Assume that for any unit vector $\boldsymbol{v}$ we have $V_{X,f}(\boldsymbol{v}) \ge 1$. Then*

$$\int \cdots \int_{X^d} \left| \bigwedge_{i=1}^{d} f(x_i) \right| \mathrm{d}\mu(x_1)\, \mathrm{d}\mu(x_2) \cdots \mathrm{d}\mu(x_d) \gtrsim_d \mathrm{Vis}[X, f]. \tag{3-2}$$

Proof. We do induction on the dimension d to prove the theorem. First observe that if $\mathrm{Ell}(F(X, f))$ is an elliptical approximation of $F(X, f)$, then for any linear subspace W of $\mathbb{R}^d$, we have $\mathrm{Ell}(F(X, f)) \cap W$ (an ellipsoid) is also an elliptical approximation of $F(X, f) \cap W$ by definition (this may seem problematic as the C_d will vary, but for the conclusion only finitely many intermediate dimensions are involved in the whole induction process and we can set C_d of them to all be the same).

For $d = 1$, by definition we easily see

$$\mathrm{Vis}[X, f] = \frac{1}{2} \int_X |f(x)| \, \mathrm{d}\mu(x) \tag{3-3}$$

and the conclusion holds. Note that even in the argument here we are using the hypothesis to ensure we have (3-3).

Assume the conclusion holds for $d < d_0$ and $d_0 > 1$. Now we deal with the case $d = d_0$. Assume $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d_0}$ are parallel to the semiprincipal axes of any elliptical approximation $\mathrm{Ell}(F(X, f))$, respectively, and that they form an orthonormal basis (we can arbitrarily choose a set of orthogonal semiprincipal axes if there is ambiguity defining the semiprincipal axes). Among them we assume $\boldsymbol{v}_1$ is parallel to a semiminor axis (i.e., a shortest semiprincipal axis) that has length t_1. Taking $\boldsymbol{v} = \lambda \boldsymbol{v}_1$, where $\lambda \sim_{d_0} t_1$ in (3-1), we deduce

$$\int_X |f(x)| \, \mathrm{d}\mu(x) \ge \frac{1}{t_1}. \tag{3-4}$$

Next for any unit vector $\boldsymbol{v} \in \mathbb{R}^{d_0}$, we prove

$$\int \cdots \int_{X^{d_0 - 1}} \left| f(x_1) \wedge \cdots \wedge f(x_{d_0 - 1}) \wedge \boldsymbol{v} \right| \mathrm{d}\mu(x_1)\, \mathrm{d}\mu(x_2) \cdots \mathrm{d}\mu(x_{d_0 - 1}) \gtrsim_{d_0} t_1 \cdot \mathrm{Vis}[X, f]. \tag{3-5}$$

Let $\pi_{\boldsymbol{v}^\perp}$ be the orthogonal projection from $\mathbb{R}^{d_0}$ to its subspace $\boldsymbol{v}^\perp$. Define $f_{\boldsymbol{v}^\perp} = \pi_{\boldsymbol{v}^\perp} \circ f$. If we identify $\mathbb{R}^{d_0 - 1}$ with $\boldsymbol{v}^\perp$, then $f_{\boldsymbol{v}^\perp}$ is another $(d_0 - 1)$-dimensional vector-valued function on X. By definition, we know $V_{X,f}(\boldsymbol{w}) = V_{X, f_{\boldsymbol{v}^\perp}}(\boldsymbol{w})$ for any $\boldsymbol{w} \in \boldsymbol{v}^\perp$. Hence $F(X, f_{\boldsymbol{v}^\perp}) = F(X, f) \cap \boldsymbol{v}^\perp$. By the previous discussion, we know we can choose $\mathrm{Ell}(F(X, f_{\boldsymbol{v}^\perp}))$ to be $\mathrm{Ell}(F(X, f)) \cap \boldsymbol{v}^\perp$. But among all the (d_0-1)-dimensional sections of $\mathrm{Ell}(F(X, f))$ passing through the origin, the section cut by $\boldsymbol{v}_1^\perp$ has the largest volume (see also Lemma 7.4), which is $\sim_{d_0} |\mathrm{Ell}(F(X, f))|/t_1 = 1/(t_1 \cdot \mathrm{Vis}[X, f])$. Hence

$$\mathrm{Vis}[X, f_{\boldsymbol{v}^\perp}] = \frac{1}{|F(X, f_{\boldsymbol{v}^\perp})|} \sim_{d_0} \frac{1}{|\mathrm{Ell}(F(X, f_{\boldsymbol{v}^\perp}))|} \gtrsim_{d_0} t_1 \cdot \mathrm{Vis}[X, f].$$

By induction hypothesis we have

$$\begin{aligned}\int \cdots \int_{X^{d_0-1}} & \big| f(x_1) \wedge \cdots \wedge f(x_{d_0-1}) \wedge \boldsymbol{v} \big| \, \mathrm{d}\mu(x_1)\, \mathrm{d}\mu(x_2) \cdots \mathrm{d}\mu(x_{d_0-1}) \\ &= \int \cdots \int_{X^{d_0-1}} \big| f_{\boldsymbol{v}^\perp}(x_1) \wedge \cdots \wedge f_{\boldsymbol{v}^\perp}(x_{d_0-1}) \big| \, \mathrm{d}\mu(x_1)\, \mathrm{d}\mu(x_2) \cdots \mathrm{d}\mu(x_{d_0-1}) \\ &\gtrsim_{d_0} \mathrm{Vis}[X, f_{\boldsymbol{v}^\perp}] \gtrsim_{d_0} t_1 \cdot \mathrm{Vis}[X, f]. \end{aligned} \tag{3-6}$$

This is (3-5).

Combining (3-4) and (3-5), we have

$$\begin{aligned}\int \cdots \int_{X^d} & \Big| \bigwedge_{i=1}^{d} f(x_i) \Big| \, \mathrm{d}\mu(x_1)\, \mathrm{d}\mu(x_2) \cdots \mathrm{d}\mu(x_d) \\ &= \int_X |f(x)| \bigg(\int \cdots \int_{X^{d_0-1}} \Big| f(x_1) \wedge \cdots \wedge f(x_{d_0-1}) \wedge \frac{f(x)}{|f(x)|} \Big| \, \mathrm{d}\mu(x_1)\, \mathrm{d}\mu(x_2) \cdots \mathrm{d}\mu(x_{d_0-1}) \bigg) \mathrm{d}\mu(x) \\ &\gtrsim_{d_0} t_1 \cdot \mathrm{Vis}[X, f] \cdot \int_X |f(x)| \, \mathrm{d}\mu(x) \gtrsim_{d_0} \mathrm{Vis}[X, f], \end{aligned} \tag{3-7}$$

which concludes the induction. □

4. Linear algebra preliminaries

Our proof relies heavily on linear algebra. In this section we do the linear algebraic part and prove two useful lemmas.

Lemma 4.1. *Assume $V_1, \dots, V_n \subseteq \mathbb{R}^d$ and $k_j = \dim V_j$ satisfies $\sum_{j=1}^n k_j = d$. Then for any vectors $\boldsymbol{w}_1, \dots, \boldsymbol{w}_d \in \mathbb{R}^d$, we have*

$$\max \prod_{j=1}^{n} \big| (V_j)^\perp \wedge \boldsymbol{w}_{i_{j,1}} \wedge \cdots \wedge \boldsymbol{w}_{i_{j,k_j}} \big| \gtrsim_d |V_1 \wedge \cdots \wedge V_n| \cdot \Big| \bigwedge_{i=1}^{d} \boldsymbol{w}_i \Big|, \tag{4-1}$$

where the maximum is taken over $1 \le i_{j,h} \le d$ for $1 \le j \le n,\ 1 \le h \le k_j$, where each $1 \le i \le d$ is chosen exactly once among all $i_{j,h}$.

Proof. Assume that $\{\boldsymbol{v}_{j,h}\}_{1 \le h \le k_j}$ is an orthonormal basis of V_j. Then by definition we have

$$\Big| (V_1 \wedge \cdots \wedge V_n) \cdot \Big(\bigwedge_{i=1}^{d} \boldsymbol{w}_i \Big) \Big| = |(\boldsymbol{v}_{j,h} \cdot \boldsymbol{w}_i)|. \tag{4-2}$$

By the generalized Laplace cofactor expansion, the determinant on the right-hand side of (4-2) is a sum of terms in the form

$$\pm \det(\boldsymbol{v}_{1,h}\cdot\tilde{\boldsymbol{w}}_{1,h})\det(\boldsymbol{v}_{2,h}\cdot\tilde{\boldsymbol{w}}_{2,h})\cdots\det(\boldsymbol{v}_{n,h}\cdot\tilde{\boldsymbol{w}}_{n,h}) \tag{4-3}$$

where $\tilde{\boldsymbol{w}}_{1,1},\tilde{\boldsymbol{w}}_{1,2},\dots,\tilde{\boldsymbol{w}}_{1,k_1},\tilde{\boldsymbol{w}}_{2,1},\dots,\tilde{\boldsymbol{w}}_{2,k_2},\dots,\tilde{\boldsymbol{w}}_{n,1},\dots,\tilde{\boldsymbol{w}}_{n,k_n}$ is a rearrangement of $\boldsymbol{w}_1,\dots,\boldsymbol{w}_d$. Hence for some such rearrangement we have

$$|V_1\wedge\cdots\wedge V_n|\cdot\left|\bigwedge_{i=1}^{d}\boldsymbol{w}_i\right|\lesssim_d\left|\det(\boldsymbol{v}_{1,h}\cdot\tilde{\boldsymbol{w}}_{1,h})\det(\boldsymbol{v}_{2,h}\cdot\tilde{\boldsymbol{w}}_{2,h})\cdots\det(\boldsymbol{v}_{n,h}\cdot\tilde{\boldsymbol{w}}_{n,h})\right|. \tag{4-4}$$

By the properties of the Hodge $*$-operator, we then have

$$|\det(\boldsymbol{v}_{j,h}\cdot\tilde{\boldsymbol{w}}_{j,h})|=\left|(*\boldsymbol{v}_{j,1}\wedge\cdots\wedge\boldsymbol{v}_{j,k_j})\wedge\tilde{\boldsymbol{w}}_{j,1}\wedge\cdots\wedge\tilde{\boldsymbol{w}}_{j,k_j}\right|=|V_j^{\perp}\wedge\tilde{\boldsymbol{w}}_{j,1}\wedge\cdots\wedge\tilde{\boldsymbol{w}}_{j,k_j}|, \tag{4-5}$$

which concludes the proof. □

The rest of this section is dedicated to the computation of a determinant that will be useful in the next section.

Lemma 4.2. *Assume that $0\le c_j\le d$ are integers, $1\le j\le m$, satisfying $\sum_{j=1}^m c_j=d$. For any j, assume $\boldsymbol{v}_{j,1},\boldsymbol{v}_{j,2},\dots,\boldsymbol{v}_{j,d}$ is an orthonormal basis of $\mathbb{R}^d$ (written as column vectors). Then we have*

$$\begin{aligned}&\left|\det\begin{pmatrix}\boldsymbol{v}_{1,c_1+1}&\cdots&\boldsymbol{v}_{1,d}&\boldsymbol{v}_{2,c_2+1}&\cdots&\boldsymbol{v}_{2,d}&0&\cdots&0&\cdots&0&\cdots&0\\ \boldsymbol{v}_{1,c_1+1}&\cdots&\boldsymbol{v}_{1,d}&0&\cdots&0&\boldsymbol{v}_{3,c_3+1}&\cdots&\boldsymbol{v}_{3,d}&\cdots&0&\cdots&0\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\ \boldsymbol{v}_{1,c_1+1}&\cdots&\boldsymbol{v}_{1,d}&0&\cdots&0&0&\cdots&0&\cdots&\boldsymbol{v}_{m,c_m+1}&\cdots&\boldsymbol{v}_{m,d}\end{pmatrix}\right|\\ &\qquad\qquad=\left|\det\bigl(\boldsymbol{v}_{1,1}\cdots\boldsymbol{v}_{1,c_1}\boldsymbol{v}_{2,1}\cdots\boldsymbol{v}_{2,c_2}\cdots\boldsymbol{v}_{m,1}\cdots\boldsymbol{v}_{m,c_m}\bigr)\right|.\end{aligned} \tag{4-6}$$

Proof. For $2\le j\le m$, let $A_j=(\boldsymbol{v}_{1,1}\cdots\boldsymbol{v}_{1,c_1}\mathbf{0}\cdots\mathbf{0}\cdots\boldsymbol{v}_{j,1}\cdots\boldsymbol{v}_{j,c_j}\cdots\mathbf{0}\cdots\mathbf{0})$. The rule here is that its first c_1 columns are $\boldsymbol{v}_{1,1},\dots,\boldsymbol{v}_{1,c_1}$ and its $\bigl(\sum_{j'<j}c_{j'}+1\bigr)$-th to $\bigl(\sum_{j'<j}c_{j'}\bigr)$-th columns are $\boldsymbol{v}_{j,1},\dots,\boldsymbol{v}_{j,c_j}$, while its other columns are zero vectors. The left-hand side of (4-6) is equal to

$$\left|\det\begin{pmatrix}I&0&0&0&0&0&0&0&0&0&0&0&0&0\\ A_2&\boldsymbol{v}_{1,c_1+1}&\cdots&\boldsymbol{v}_{1,d}&\boldsymbol{v}_{2,c_2+1}&\cdots&\boldsymbol{v}_{2,d}&0&\cdots&0&\cdots&0&\cdots&0\\ A_3&\boldsymbol{v}_{1,c_1+1}&\cdots&\boldsymbol{v}_{1,d}&0&\cdots&0&\boldsymbol{v}_{3,c_3+1}&\cdots&\boldsymbol{v}_{3,d}&\cdots&0&\cdots&0\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\ A_m&\boldsymbol{v}_{1,c_1+1}&\cdots&\boldsymbol{v}_{1,d}&0&\cdots&0&0&\cdots&0&\cdots&\boldsymbol{v}_{m,c_m+1}&\cdots&\boldsymbol{v}_{m,d}\end{pmatrix}\right|.$$

We exchange the columns to make it look better. For simplicity let $V_j=(\boldsymbol{v}_{j,1}\cdots\boldsymbol{v}_{j,d})$. This is an orthogonal matrix. We also define a matrix $B_j=(b_j(k,l))$, $1\le k\le d$, such that $b_j(k,l)=1$ if $l\le c_j$ and $k=l+\sum_{j'<j}c_{j'}$, and $b_j(k,l)=0$ otherwise. Then after rearranging the columns of the matrix above we

find the determinant (in absolute value) is equal to

$$\left|\det\begin{pmatrix} B_1 & B_2 & B_3 & \cdots & B_m \\ V_1 & V_2 & 0 & \cdots & 0 \\ V_1 & 0 & V_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ V_1 & 0 & 0 & \cdots & V_m \end{pmatrix}\right|.$$

We can multiply the j-th column by $V_j^{-1} = V_j^t$ on the right, then subtract all the j-th columns, $j > 1$, from the first column. This preserves the determinant. Note the definition of B_j, if we define $\Delta = (\boldsymbol{v}_{1,1} \cdots \boldsymbol{v}_{1,c_1} - \boldsymbol{v}_{2,1} \cdots - \boldsymbol{v}_{2,c_2} \cdots - \boldsymbol{v}_{m,1} \cdots - \boldsymbol{v}_{m,c_m})$, then the determinant is

$$\left|\det\begin{pmatrix} \Delta^t & B_2V_2^t & B_3V_3^t & \cdots & B_mV_m^t \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I \end{pmatrix}\right|.$$

Equation (4-6) then follows directly. □

5. Integral geometry preliminaries

In this section we prepare some integral geometry tools for our proof of Theorem 1.4. First we generalize (2-2) to the following lemma.

Lemma 5.1. *Assume in $\mathbb{R}^d$ we have m smooth compact submanifolds $Z_1, Z_2, \dots, Z_m$ (possibly with boundary) with codimensions $c_1, \dots, c_m$ respectively. If $\sum_{j=1}^m c_j = d$ then for any measurable subset $U \subseteq \mathbb{R}^{d(m-1)} = (\mathbb{R}^d)^{m-1}$, we have*

$$\int_{Z_1}\int_{Z_2}\cdots\int_{Z_m} \chi_U(\overrightarrow{p_1p_2}, \dots, \overrightarrow{p_1p_m})\big|(T_{p_1}Z_1)^\perp \wedge \cdots \wedge (T_{p_m}Z_m)^\perp\big|\,\mathrm{dVol}_1 \cdots \mathrm{dVol}_m$$
$$= \int_{\boldsymbol{v}_2,\dots,\boldsymbol{v}_m \in \mathbb{R}^d,\ (\boldsymbol{v}_2,\dots,\boldsymbol{v}_m)\in U} \big|(Z_1) \cap (Z_2 + \boldsymbol{v}_2) \cap \cdots \cap (Z_{m-1} + \boldsymbol{v}_{m-1}) \cap (Z_m + \boldsymbol{v}_m)\big|\,\mathrm{d}\boldsymbol{v}_2 \cdots \mathrm{d}\boldsymbol{v}_m, \quad (5\text{-}1)$$

where $p_j \in Z_j$, $T_{p_j}Z_j$ is the tangent space of Z_j at p_j, dVol_j is the volume element on the j-th submanifold, and $Z_j + \boldsymbol{v}_j = \{p_j + \boldsymbol{v}_j : p_j \in Z_j\}$ is the translation of Z_j along the vector $\boldsymbol{v}_j$. The $|\cdot|$ on the right-hand side defines cardinality.

This lemma has a lot of information so we pause a bit and go through several examples to understand it better.

When $d = 2$, if Z_1 and Z_2 are two nonparallel line segments and U is the whole $\mathbb{R}^2$, the integrand on the right-hand side of (5-1) is the characteristic function of a parallelogram generalized by Z_1 and Z_2. Hence the right-hand side is the area of the parallelogram, which is easily seen to be equal to the left-hand side. When $d = 3$, if Z_1 is a line segment, Z_2 is a parallelogram in a plane and U is the whole $\mathbb{R}^3$, the situation is totally analogous.

When $d=3$, Z_1 is a whole line and Z_2 is a smooth surface of finite area, and we can take U to be the point set between two planes orthogonal to Z_1 with distance 1. It is a simple exercise to show that (5-1) then becomes (2-2). Hence it is indeed a generalization of the latter.

Finally let's look at a more complicated example. Again take $d=3$ and $U=\mathbb{R}^6$. Take a parallelepiped $\Omega = ABCD - A_1B_1C_1D_1$. Take three parallelograms $Z_1 = ABCD,\ Z_2 = ABB_1A_1,\ Z_3 = ADD_1A_1$. Define $\boldsymbol{u} = \overrightarrow{AB},\ \boldsymbol{v} = \overrightarrow{AD},\ \boldsymbol{w} = \overrightarrow{AA_1}$. Again the integrand on the right-hand side is a characteristic function. We find it is plainly equal to $\mathrm{Vol}(\Omega)^2$. Now the left-hand side is equal to

$$\begin{aligned}|\boldsymbol{u}\times\boldsymbol{v}|\cdot|\boldsymbol{v}\times\boldsymbol{w}|\cdot|\boldsymbol{w}\times\boldsymbol{u}|\cdot\left|\frac{\boldsymbol{u}\times\boldsymbol{v}}{|\boldsymbol{u}\times\boldsymbol{v}|}\wedge\frac{\boldsymbol{v}\times\boldsymbol{w}}{|\boldsymbol{v}\times\boldsymbol{w}|}\wedge\frac{\boldsymbol{w}\times\boldsymbol{u}}{|\boldsymbol{w}\cdot\boldsymbol{u}|}\right| &= |(\boldsymbol{u}\times\boldsymbol{v})\wedge(\boldsymbol{v}\times\boldsymbol{w})\wedge(\boldsymbol{w}\times\boldsymbol{u})|\\ &= |((\boldsymbol{u}\times\boldsymbol{v})\times(\boldsymbol{v}\times\boldsymbol{w}))\cdot(\boldsymbol{w}\times\boldsymbol{u})|\\ &= |(\boldsymbol{v}\cdot(\boldsymbol{u}\times\boldsymbol{w}))\boldsymbol{v}\cdot(\boldsymbol{w}\times\boldsymbol{u})| = \mathrm{Vol}(\Omega)^2. \quad (5\text{-}2)\end{aligned}$$

Proof of Lemma 5.1. Without loss of generality we can assume U is open and bounded. By the multilinear feature of both sides of (5-1), we only need to consider this problem locally. Hence we can assume each Z_j is smoothly parametrized by a domain in $\mathbb{R}^{d-c_j}$. In other words we may assume $Z_j : x_i = f_{j,i}(y_{j,1},\dots,y_{j,d-c_j})$ and that the $(d-c_j)$ vectors $\boldsymbol{w}_{j,l} = (\partial f_{j,i}/\partial y_{j,l})_{1\le i\le d}$ have a nonzero wedge product at any point $p_j \in Z_j$. They span the tangent space $T_{p_j}Z_j$ and will be written as column vectors below.

Look at the cartesian product $Z = Z_1\times Z_2\times\cdots\times Z_m \subseteq (\mathbb{R}^d)^m \cong \mathbb{R}^{dm}$. This is a smooth submanifold of dimension $\sum_{j=1}^m (d-c_j) = d(m-1)$. Use $x_{j,i},\ 1\le i\le d$, to denote the standard Euclidean coordinates in the j-th copy of $\mathbb{R}^d$ and let $v_{j,i} = x_{j,i} - x_{1,i},\ j>1$. For simplicity let $\boldsymbol{x}_j = (x_{j,i})_{1\le i\le d}$ and $\boldsymbol{v}_j = (v_{j,i})_{1\le i\le d}$. Notice that the right-hand side of (5-1) is equal to

$$\int_Z \chi_U((\boldsymbol{v}_j)_{2\le j\le m})\,|\,\mathrm{d}\boldsymbol{v}_2\,\mathrm{d}\boldsymbol{v}_3\cdots\,\mathrm{d}\boldsymbol{v}_m|.$$

Define the density form $\theta = |\mathrm{d}\boldsymbol{v}_2\,\mathrm{d}\boldsymbol{v}_3\ \cdots\ \mathrm{d}\boldsymbol{v}_m| = |\mathrm{d}v_{2,1}\wedge\mathrm{d}v_{2,2}\wedge\cdots\wedge\mathrm{d}v_{2,d}\wedge\cdots\wedge\mathrm{d}v_{m,1}\wedge\cdots\wedge\mathrm{d}v_{m,d}|$. On the manifold Z it is a multiple of the volume density element

$$|\mathrm{d}V| = \prod_{j=1}^m \left|\bigwedge_{l=1}^{d-c_j}\boldsymbol{w}_{j,l}\right|\left|\bigwedge_{1\le j\le m,1\le l\le d-c_j}\mathrm{d}y_{j,l}\right|.$$

Next we find $\theta/|\mathrm{d}V|$.

We have

$$\frac{\theta}{|\mathrm{d}V|} = \frac{1}{\prod_{j=1}^m\left|\bigwedge_{l=1}^{d-c_j}\boldsymbol{w}_{j,l}\right|}\left|\left(\frac{\partial v_{j,i}}{\partial y_{j,l}}\right)\right|.$$

And by change of variable we have

$$\left|\left(\frac{\partial v_{j,i}}{\partial y_{j,l}}\right)\right| = \left|\det\begin{pmatrix} -\boldsymbol{w}_{1,1} & \cdots & -\boldsymbol{w}_{1,d-c_1} & \boldsymbol{w}_{2,1} & \cdots & \boldsymbol{w}_{2,d-c_2} & \cdots & 0 & \cdots & 0 & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ -\boldsymbol{w}_{1,1} & \cdots & -\boldsymbol{w}_{1,d-c_1} & 0 & \cdots & 0 & \cdots & \boldsymbol{w}_{m,1} & \cdots & \boldsymbol{v}_{m,d-c_m} & \cdots & \cdots & \cdots \end{pmatrix}\right|. \quad (5\text{-}3)$$

This looks very much like the left-hand side of (4-6). Indeed, the extra negative signs do not change the determinant and can be ignored. The only essential difference here is that for each j, our $\{w_{j,l}\}_{1\le l\le d-c_j}$ is not a set of orthonormal vectors. If we do a change of variable to make them orthonormal we will extract a factor of $\left|\bigwedge_{l=1}^{d-c_j} \boldsymbol{w}_{j,l}\right|$ from right-hand side of (5-3) for each j. We then apply Lemma 4.2 and get

$$\frac{\theta}{|\mathrm{d}V|} = \left|\bigwedge_{j=1}^{m} (T_{p_j} Z_j)^\perp\right|. \tag{5-4}$$

Hence the right-hand side of (5-1) is equal to $\int_Z \chi_U((\boldsymbol{v}_j)_{2\le j\le m})\left|\bigwedge_{j=1}^{m}(T_{p_j}Z_j)^\perp\right| \mathrm{d}V$. Note that $\mathrm{d}V$ is the product of all dVol_j. This can easily be recognized as the left-hand side. □

In application, we want to look at the case where each V_j is the zero set of an algebraic variety of codimension c_j. Such a V_j may contain singular points, but they form a subset of measure 0 when we take the $(d-c_j)$-dimensional Hausdorff measure. Hence almost all points on V_j are smooth points and we can apply our Lemma 5.1 to obtain the following theorem.

Theorem 5.2 (intersection estimate). *Assume in $\mathbb{R}^d$ we have m algebraic subvarieties $Z_1, Z_2, \dots, Z_m$ with codimensions $c_1, \dots, c_m$ and degrees $s_1, \dots, s_m$ respectively. If $\sum_{j=1}^m c_j = d$ then for any measurable subset $U \subseteq \mathbb{R}^{d(m-1)} = (\mathbb{R}^d)^{m-1}$, we have*

$$\int_{Z_1}\int_{Z_2}\cdots\int_{Z_m} \chi_U(\overrightarrow{p_1p_2}, \dots, \overrightarrow{p_1p_m})\left|(T_{p_1}Z_1)^\perp\wedge\cdots\wedge(T_{p_m}Z_m)^\perp\right| \mathrm{dVol}_1\cdots\mathrm{dVol}_m \le \mathrm{Vol}(U)\prod_{j=1}^m s_j, \tag{5-5}$$

where dVol_j is the $(d-c_j)$-dimensional volume element on the j-th subvariety. Almost all $p_j \in Z_j$ are smooth points and we define $T_{p_j}Z_j$ to be the tangent space of Z_j at p_j.

Proof. Inequality (5-5) follows directly from Lemma 5.1 and Bézout's theorem. □

Theorem 5.2 generalizes the cylinder estimate in [Guth 2010], which was recorded as Lemma 1.14 in our current paper.

6. Proofs of Theorems 1.4 and 1.5

In this section, we prove Theorem 1.5 and deduce Theorem 1.4 as a corollary. As briefly described in the Introduction, we actually prove a generalized theorem about any n varieties.

Basically, our multilinear k_j-variety theorem says that for n algebraic subvarieties of $\mathbb{R}^d$ with their codimension adding up to d, their tubular neighborhoods will provide us with an inequality similar to Theorem 1.5 if we take their "amount of interaction" into account. In particular, if we take each subvariety to be a union of k_j-planes we obtain Theorems 1.5 and 1.4 (see the end of this section).

Theorem 6.1 (multilinear k_j-variety theorem). *Assume $1 \le k_j \le d-1$, $1 \le j \le n$, satisfy $\sum_{j=1}^n k_j = d$. Assume that for $1 \le j \le n$, $H_j \subseteq \mathbb{R}^d$ is part of a k_j-dimensional algebraic subvariety of degree $A(j)$. Let $\mathrm{d}\sigma_j$ denote the k_j-dimensional (Hausdorff) volume measure of H_j. Then with respect to this measure, almost all $y_j \in H_j$ are smooth points. For a smooth point $y_j \in H_j$, let $T_{y_j}H_j$ denote the tangent space of*

H_j at y_j. Then

$$\int_{\mathbb{R}^d}\left(\int_{H_1\times H_2\times\cdots\times H_n}\chi_{\{\operatorname{dist}(y_j,x)\le 1\}}\left|\bigwedge_{j=1}^{n}T_{y_j}H_j\right|\mathrm{d}\sigma_1(y_1)\cdots\mathrm{d}\sigma_n(y_n)\right)^{1/(n-1)}\mathrm{d}x\lesssim_d\prod_{j=1}^{n}A(j)^{1/(n-1)}. \tag{6-1}$$

We give an outline of the proof before we actually do it. For the convenience of the statement, we wrote Theorem 6.1 in an integral form. However, because of the truncation $\chi_{\{\operatorname{dist}(y_j,x)\le 1\}}$ it is really of discrete flavor. In other words, around any unit cube, we only take into account the part of the varieties near this cube. By Lemma 2.4, we can find a polynomial with large visibility around each relevant cube. In the lemma, it is possible to assign different weights to different cubes in the above movement. We assign the weights according to the cubes' "popularity" among H_j, as done in [Guth 2010] for the multilinear Kakeya theorem.

We will see it does not matter if we multiply all the weights by the same large positive number simultaneously. As long as the weights are large enough, we can add hyperplanes to the polynomial which do not essentially increase its degree and make its zero set satisfy the assumption of Theorem 3.1 at each relevant cube. Then we can invoke Theorem 3.1 for the resulting zero set $Z(P)$ at all relevant cubes to show that d copies of $Z(P)$ have enough interaction there. Now around each relevant cube we are ready to assign some copies of $Z(P)$ to each variety H_j and use Lemma 4.1 to show that those "have enough interaction". On the other hand, we can use Theorem 5.2 to bound the amount of interaction from above. Hence we obtain a nontrivial inequality. All quantities there work out as they supposed to and we obtain our theorem.

Proof of Theorem 6.1. We only need to prove the case where each H_j is compact and take a limiting argument to complete the proof. Fix a large constant N in terms of d; for example, $N=100e^d$ should be more than sufficient.

Consider the standard lattice of unit cubes in $\mathbb{R}^d$. For each cube Q_ν in the lattice, let O_ν be its center. Let

$$G(Q_\nu)=\int_{H_1\times H_2\times\cdots\times H_n}\prod_{j=1}^{n}\chi_{\{\operatorname{dist}(y_j,O_\nu)\le N\}}\left|\bigwedge_{j=1}^{n}T_{y_j}H_j\right|\mathrm{d}\sigma_1(y_1)\cdots\mathrm{d}\sigma_n(y_n). \tag{6-2}$$

Obviously

$$G(Q_\nu)\ge\int_{H_1\times H_2\times\cdots\times H_n}\prod_{j=1}^{n}\chi_{\{\operatorname{dist}(y_j,x)\le 1\}}\left|\bigwedge_{j=1}^{n}T_{y_j}H_j\right|\mathrm{d}\sigma_1(y_1)\cdots\mathrm{d}\sigma_n(y_n) \tag{6-3}$$

for any $x\in Q_\nu$. Hence it suffices to prove that under assumptions of Theorem 6.1, we have

$$\sum_\nu G(Q_\nu)^{1/(n-1)}\lesssim_d\prod_{j=1}^{m}A(j)^{1/(n-1)}. \tag{6-4}$$

We only have finitely many relevant cubes Q_ν such that $G(Q_\nu)\ne 0$. Hence we can choose a huge cube of side length S containing all of the relevant cubes. By Guth's lemma, Lemma 2.4, we can find a polynomial P of degree $\lesssim_d S$ such that for each cube Q_ν,

$$\overline{\mathrm{Vis}}[Z(P)\cap Q_\nu] \ge S^d G(Q_\nu)^{1/(n-1)}\left(\sum_\nu G(Q_\nu)^{1/(n-1)}\right)^{-1}. \tag{6-5}$$

Adding $\lesssim_d S$ hyperplanes to P (in other words multiplying P by linear equations of those hyperplanes) if necessary, we may assume that for all Q_ν where $G(Q_\nu) > 0$ we have $\overline{V}_{Z(P)\cap Q_\nu}(\boldsymbol{v}) \ge |\boldsymbol{v}|$. Hence we are in a good position to use the wedge product estimate (3-2).

Before we move on let us remark on a technical issue. If we do have to add hyperplanes at this point, we need to modify our Definition 2.3 a little bit: Assume all the hyperplanes we added form a zero set of a polynomial P_0. We call the original polynomial in Guth's lemma P_{old} and our P is actually $P_{\mathrm{old}}P_0$. Then when we are talking about the mollified directed volume, mollified visibility, etc. around P, we want to look at all $P'P_0$, where $P' \in B(P_{\mathrm{old}}, \varepsilon)$ instead of all $P' \in B(P, \varepsilon)$. For example, the definition (2-3) should now be modified to

$$\overline{V}_{Z(P)\cap U}(\boldsymbol{v}) = \frac{1}{|B(P_{\mathrm{old}}, \varepsilon)|}\int_{B(P_{\mathrm{old}},\varepsilon)} V_{Z(P'P_0)\cap U}(\boldsymbol{v})\,\mathrm{d}P'. \tag{6-6}$$

We also note that an alternative strategy to "adding hyperplanes" is given in [Carbery and Valdimarsson 2013] (see Lemmas 3 and 6 and the argument on page 1654 there). It is a very detailed and clear account.

For the rest of the section for simplicity of the notation, we deal with the case where no hyperplanes are added. For the general case the proof is identical except for proper correction of notation.

For any $y_1 \in H_1 \cap B(O_\nu, N), \dots, y_n \in H_n \cap B(O_\nu, N)$, $P_1, \dots, P_d \in B(P, \varepsilon)$(see Section 2), $p_1 \in Z(P_1)\cap B(O_\nu, N), \dots, p_d \in Z(P_d)\cap B(O_\nu, N)$, by Lemma 4.1, we can find some $i_{j,h}$ for $1 \le j \le n$, $1 \le h \le k_j$ such that

$$\prod_{j=1}^n \left|(T_{y_j}H_j)^\perp \wedge (T_{p_{i_{j,1}}}Z(P_{i_{j,1}}))^\perp \wedge \cdots \wedge (T_{p_{i_{j,k_j}}}Z(P_{i_{j,k_j}}))^\perp\right| \gtrsim_d \left|\bigwedge_{j=1}^n T_{y_j}H_j\right| \cdot \left|\bigwedge_{i=1}^d (T_{p_i}Z(P_i))^\perp\right| \tag{6-7}$$

and all $i_{j,h}$ are distinct and form exactly the set $\{1, 2, \dots, d\}$.

Integrating over $(H_1 \cap B(O_\nu, N)) \times \cdots \times (H_n \cap B(O_\nu, N))$, we obtain

$$\begin{aligned} G(Q_\nu)\cdot&\left|\bigwedge_{i=1}^d (T_{p_i}Z(P_i))^\perp\right| \\ &\lesssim_d \sum_{(i_{j,h})}\int_{H_1\cap B(O_\nu,N)}\cdots\int_{H_n\cap B(O_\nu,N)}\prod_{j=1}^n\left|(T_{y_j}H_j)^\perp\wedge(T_{p_{i_{j,1}}}Z(P_{i_{j,1}}))^\perp\wedge\cdots\wedge(T_{p_{i_{j,k_j}}}Z(P_{i_{j,k_j}}))^\perp\right| \\ &\qquad\qquad \mathrm{d}\sigma_1(y_1)\cdots\mathrm{d}\sigma_n(y_n). \end{aligned} \tag{6-8}$$

Here we sum over all possible choices of $\{i_{j,h} : 1 \le j \le n,\ 1 \le h \le k_j\}$ such that all $i_{j,h}$ are distinct and form exactly the set $\{1, 2, \dots, d\}$.

Integrate (6-8) over $P_1, \dots, P_d \in B(P, \varepsilon)$ and $p_i \in Z(P_i) \cap B(O_\nu, N)$ (we abuse the notation a bit and write $\mathrm{d}p = \mathrm{d}\sigma(p)$ where $\mathrm{d}\sigma$ is the $(d-1)$-dimensional Hausdorff volume measure on $Z(P)$). Taking Definition 2.3 into account, we use wedge product estimate (Theorem 3.1) and (6-5), (6-8) and deduce

$$
\begin{aligned}
\sum_{(i_{j,h})} \frac{1}{|B(P,\varepsilon)|^d} \int\!\cdots\!\int_{B(P,\varepsilon)^d} \int_{H_1\cap B(O_\nu,N)} \cdots \int_{H_n\cap B(O_\nu,N)} \int_{Z(P_1)\cap B(O_\nu,N)} \cdots \int_{Z(P_d)\cap B(O_\nu,N)} \\
\prod_{j=1}^n \big|(T_{y_j}H_j)^\perp \wedge (T_{p_{i_{j,1}}}Z(P_{i_{j,1}}))^\perp \wedge \cdots \wedge (T_{p_{i_{j,k_j}}}Z(P_{i_{j,k_j}}))^\perp\big| \\
\mathrm{d}p_1 \cdots \mathrm{d}p_d \,\mathrm{d}\sigma_1(y_1) \cdots \mathrm{d}\sigma_n(y_n)\,\mathrm{d}P_1 \cdots \mathrm{d}P_d \\
\gtrsim_d G(Q_\nu)\cdot \overline{\mathrm{Vis}}[Z(P)\cap Q_\nu] \\
\gtrsim_d S^d G(Q_\nu)^{n/(n-1)} \Big(\sum_\nu G(Q_\nu)^{1/(n-1)}\Big)^{-1}. \qquad \text{(6-9)}
\end{aligned}
$$

Rewrite (6-9) as

$$
\begin{aligned}
\sum_{(i_{j,h})} \prod_{j=1}^n \Big(\frac{1}{|B(P,\varepsilon)|^{k_j}} \int\!\cdots\!\int_{B(P,\varepsilon)^{k_j}} \frac{1}{S^{k_j}\cdot A(j)} \int_{H_j\cap B(O_\nu,N)} \int_{Z(P_{i_{j,1}})\cap B(O_\nu,N)} \cdots \int_{Z(P_{i_{j,k_j}})\cap B(O_\nu,N)} \\
\big|(T_{y_j}H_j)^\perp \wedge (T_{p_{i_{j,1}}}Z(P_{i_{j,1}}))^\perp \wedge \cdots \wedge (T_{p_{i_{j,k_j}}}Z(P_{i_{j,k_j}}))^\perp\big| \\
\mathrm{d}p_{i_{j,1}} \cdots \mathrm{d}p_{i_{j,k_j}} \,\mathrm{d}\sigma_j(y_j)\,\mathrm{d}P_{i_{j,1}} \cdots \mathrm{d}P_{i_{j,k_j}} \Big) \\
\gtrsim_d \frac{1}{\prod_{j=1}^n A(j)} G(Q_\nu)^{n/(n-1)} \Big(\sum_\nu G(Q_\nu)^{1/(n-1)}\Big)^{-1}. \qquad \text{(6-10)}
\end{aligned}
$$

By the arithmetic-geometric mean inequality we deduce

$$
\begin{aligned}
\frac{1}{\big(\prod_{j=1}^n A(j)\big)^{1/n}} G(Q_\nu)^{1/(n-1)} \Big(\sum_\nu G(Q_\nu)^{1/(n-1)}\Big)^{-1/n} \\
\lesssim_d \sum_{(i_{j,h})} \sum_{j=1}^n \frac{1}{|B(P,\varepsilon)|^{k_j}} \int\!\cdots\!\int_{B(P,\varepsilon)^{k_j}} \frac{1}{S^{k_j}\cdot A(j)} \int_{H_j\cap B(O_\nu,N)} \int_{Z(P_{i_{j,1}})\cap B(O_\nu,N)} \cdots \int_{Z(P_{i_{j,k_j}})\cap B(O_\nu,N)} \\
\big|(T_{y_j}H_j)^\perp \wedge (T_{p_{i_{j,1}}}Z(P_{i_{j,1}}))^\perp \wedge \cdots \wedge (T_{p_{i_{j,k_j}}}Z(P_{i_{j,k_j}}))^\perp\big| \\
\mathrm{d}p_{i_{j,1}} \cdots \mathrm{d}p_{i_{j,k_j}} \,\mathrm{d}\sigma_j(y_j)\,\mathrm{d}P_{i_{j,1}} \cdots \mathrm{d}P_{i_{j,k_j}} \qquad \text{(6-11)}
\end{aligned}
$$

Sum (6-11) over ν, and then invoke the intersection estimate Theorem 5.2 with $U = \{(u_i)_{1\le i\le k_j+1} : u_i \in \mathbb{R}^d, \operatorname{dist}(u_i, u_{i'}) < N^2\}$ (it suffices to choose U large enough). Note that $\deg P_j = S$ and $\deg H_j = A(j)$, we have

$$
\begin{aligned}
\frac{1}{\big(\prod_{j=1}^n A(j)\big)^{1/n}} \Big(\sum_\nu G(Q_\nu)^{1/(n-1)}\Big) \Big(\sum_\nu G(Q_\nu)^{1/(n-1)}\Big)^{-1/n} \\
\lesssim_d \sum_{(i_{j,h})} \sum_{j=1}^n \frac{1}{|B(P,\varepsilon)|^{k_j}} \int\!\cdots\!\int_{B(P,\varepsilon)^{k_j}} \frac{1}{S^{k_j}\cdot A(j)} \int_{H_j} \int_{Z(P_{i_{j,1}})} \cdots \int_{Z(P_{i_{j,k_j}})} \\
\chi_U(y_j, p_{i_{j,1}}, \ldots, p_{i_{j,k_j}})\cdot \big|(T_{y_j}H_j)^\perp \wedge (T_{p_{i_{j,1}}}Z(P_{i_{j,1}}))^\perp \wedge \cdots \wedge (T_{p_{i_{j,k_j}}}Z(P_{i_{j,k_j}}))^\perp\big| \\
\mathrm{d}p_{i_{j,1}} \cdots \mathrm{d}p_{i_{j,k_j}} \,\mathrm{d}\sigma_j(y_j)\,\mathrm{d}P_{i_{j,1}} \cdots \mathrm{d}P_{i_{j,k_j}} \\
\lesssim_d \sum_{(i_{j,h})} \sum_{j=1}^n \frac{1}{|B(P,\varepsilon)|^{k_j}} \int\!\cdots\!\int_{B(P,\varepsilon)^{k_j}} 1\,\mathrm{d}P_{i_{j,1}} \cdots \mathrm{d}P_{i_{j,k_j}} \lesssim_d 1 \qquad \text{(6-12)}
\end{aligned}
$$

and (6-4) holds. ☐

Theorems 1.5 and 1.4 follow easily from Theorem 6.1. To prove Theorem 1.5 it suffices to prove the case where all ρ_{j,a_j} are rational numbers. Then without loss of generality we may assume further that they are integers. By considering multiple copies of the U_{j,a_j}, we can further assume they are all 1. Then one just takes the j-th variety to be the union of the cores of the j-th family of slabs and apply Theorem 6.1 (after a scaling). Theorem 1.4 is a direct corollary of Theorem 1.5.

7. An analogue of Lemma 4.1

In the rest of this paper, we prove Theorem 1.11. In this section we prove a lemma (Lemma 7.5) analogous to Lemma 4.1, which will be used in the proof the same way as Lemma 4.1 was used in the proof of Theorem 1.4. This lemma is weaker in appearance than Lemma 4.1, but it turns out that it serves our purpose.

Definition 7.1. In $\mathbb{R}^d$, given a convex body Γ centered at the origin, define its *dual body* Γ^* to be $\{\boldsymbol{v} \in \mathbb{R}^d : |(\boldsymbol{v}, \boldsymbol{x})| \le 1 \text{ for all } x \in \Gamma\}$, where $(\,\cdot\,,\,\cdot\,)$ is the Euclidean inner product on $\mathbb{R}^d$.

It is trivial by definition that if two convex bodies Γ_1 and Γ_2 satisfy $\Gamma_1 \subseteq \Gamma_2$ then $\Gamma_1^* \supseteq \Gamma_2^*$.

By John's ellipsoid theorem, we need to mainly consider ellipsoids as examples of convex bodies. Next we develop several lemmas concerning ellipsoids. From now on, when we talk about an ellipsoid in Euclidean space, we always assume the ellipsoid has the same dimension as the background space.

Lemma 7.2. *If the Γ in Definition 7.1 is a (closed) ellipsoid centered at O (the origin), then Γ^* is also an ellipsoid centered at O. We call Γ^* the* dual ellipsoid *of Γ. Choose a set of principal axes of Γ (the wording is because the choices might not be unique); then they are also a set of principal axes of Γ^*. Moreover, the lengths of the corresponding principal axes of Γ and Γ^*, when divided by 2, are reciprocal to each other. Hence $(\Gamma^*)^* = \Gamma$ and* $\mathrm{Vol}(\Gamma) \cdot \mathrm{Vol}(\Gamma^*) = C_d > 0$ *is a constant depending only on d.*

Proof. Trivially the dual body of the unit ball is again the unit ball. Assume Γ_0 has a dual body Γ_0^*. Then for any positive definite linear transform A, we have by definition

$$\begin{aligned}(A\Gamma_0)^* &= \{\boldsymbol{v} \in \mathbb{R}^d : |(\boldsymbol{v}, A\boldsymbol{x})| \le 1 \text{ for all } x \in \Gamma_0\} \\ &= \{\boldsymbol{v} \in \mathbb{R}^d : |(A^*\boldsymbol{v}, \boldsymbol{x})| \le 1 \text{ for all } x \in \Gamma_0\} \\ &= (A^*)^{-1}\Gamma_0^* = A^{-1}\Gamma_0^*. \end{aligned} \tag{7-1}$$

Now we can use a positive definite linear transformation A to transform the closed unit ball to our Γ; by the computation above, Γ^* is A^{-1} acting on the unit ball, so it is an ellipsoid. Also the principal axes of Γ correspond to an orthonormal basis that diagonalizes A. This basis also diagonalizes A^{-1}. Hence the principal axes of Γ are also principal axes of Γ^*. The rest of the lemma is obvious. □

Lemma 7.3. *Suppose we have a subspace $V \subseteq \mathbb{R}^d$ and $\Gamma \in \mathbb{R}^d$ is an ellipsoid centered at O. Let $\pi_V(\,\cdot\,)$ be the orthogonal projection onto V. Then $\pi_V(\Gamma^*)$ and $\Gamma \cap V$ are dual to each other (in V with respect to the induced inner product). Note these two are both ellipsoids.*

Proof. If V has dimension 1, then the lemma is true by definition of the dual body (note by Lemma 7.2, the two ellipsoids are dual to each other).

For general V, by the last paragraph for any $V' \subseteq V$ of dimension 1 we have $\pi_{V'}(\pi_V(\Gamma^*))$ and $(\Gamma \cap V) \cap V'$ are dual to each other. But given the ellipsoid $\Gamma_V = \Gamma \cap V \subseteq V$, apparently there is only one possible set $Y \subseteq V$ such that for any $V' \subseteq V$ of dimension 1, $\pi_{V'}(\Gamma_V)$ and $Y \cap V'$ are dual to each other (since $Y \cap V'$ is determined by Γ_V via this property). Now by last paragraph again, the dual of Γ_V in V is this unique Y. Hence $\pi_V(\Gamma^*)$ has to be this dual. □

Lemma 7.4. *For any subspace $V \subseteq \mathbb{R}^d$ of dimension d', we define π_V to be the orthogonal projection onto V as usual. Then for any (closed) ellipsoid $\Gamma \subseteq \mathbb{R}^d$, we have*

$$|\pi_V(\Gamma)|\,|\Gamma \cap V^\perp| = C_{d,d'}|\Gamma|, \tag{7-2}$$

where $C_{d,d'} > 0$ only depends on d and d'. Here we use the corresponding standard Lebesgue measures on V, $V^\perp$ and $\mathbb{R}^d$, respectively.

Proof. It is well known that in $\mathbb{R}^d$, an ellipsoid defined by $\{\boldsymbol{x} : (\boldsymbol{x}, A\boldsymbol{x}) \le 1\}$ has volume $C_d/(\det A)^{1/2}$, where A is a positive definite linear transform and $(\cdot\,,\cdot)$ is the Euclidean inner product. Assume $\Gamma = \{\boldsymbol{x} : \|T\boldsymbol{x}\|^2 \le 1\}$, where T is a nondegenerate linear transform. Since we can multiply T by any orthogonal transform on the left, we may assume $TV^\perp = V^\perp$. Then by last paragraph,

$$|\Gamma| = \frac{C_d}{|\det T|}, \tag{7-3}$$

$$|\Gamma \cap V^\perp| = \frac{C_{d,d'}}{|\det T|_{V^\perp}|}. \tag{7-4}$$

Meanwhile, $\boldsymbol{x} \in V$ belongs to $\pi_V(\Gamma)$ if and only if $\inf_{\boldsymbol{v}\in V^\perp} \|T(\boldsymbol{x}+\boldsymbol{v})\| \le 1$. By the method of least squares, $\inf_{\boldsymbol{v}\in V^\perp} \|T(\boldsymbol{x}+\boldsymbol{v})\| = \|\pi_{(TV^\perp)^\perp}(T\boldsymbol{x})\| = \|\pi_V(T\boldsymbol{x})\|$. Hence

$$|\pi_V(\Gamma)| = \frac{C_{d'}}{|\det(\pi_V \circ T|_V)|}. \tag{7-5}$$

Now notice $\pi V^\perp = V^\perp$. Hence when written in matrix form it is easy to verify $\det T|_{V^\perp} \cdot \det(\pi_V \circ T|_V) = \det T$. This together with (7-3)–(7-5) implies (7-2). □

Now we are ready to develop an analogue of Lemma 4.1. We recall that in Section 3 we defined the total absolute inner product $V_{X,f}(\boldsymbol{v})$, the fading zone $F(X, f)$, visibility $\mathrm{Vis}[X, f]$, and chose an elliptical approximation $\mathrm{Ell}(F(X, f))$ for any measurable vector-valued function $f : X \to \mathbb{R}^d$.

Lemma 7.5. *Fix positive integers d and $1 \le k_1, \dots, k_n < d$. Let $\mathbb{R}^d$ be the standard Euclidean space.*

Assume a Brascamp–Lieb datum $(\boldsymbol{B}, \boldsymbol{p})$ such that all B_j are orthogonal projections from $\mathbb{R}^d$ to a subspace and $\dim \ker B_j = k_j$. *Assume $E_j = B_j(\mathbb{R}^d) = (\ker B_j)^\perp$. Assume we have the scaling condition $\sum_{j=1}^n p_j \dim E_j = d$.*

For any measurable vector valued function $f : X \to \mathbb{R}^d$ on some measure space satisfying $V_{X,f}(\boldsymbol{v}) \ge 1$ for all $\boldsymbol{v} \in \mathbb{R}^d$, we have

$$\prod_{j=1}^n \biggl(\int_{X^{k_j}} |E_j \wedge f(x_1) \wedge \cdots \wedge f(x_{k_j})|\,\mathrm{d}x_1 \cdots \mathrm{d}x_{k_j}\biggr)^{p_j} \gtrsim_{d,\boldsymbol{p}} (\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}))^{-1} (\mathrm{Vis}[X, f])^{\sum_{j=1}^n p_j - 1}. \tag{7-6}$$

Proof. Similar to the proof of Theorem 3.1, we define $\pi_{\ker B_j}$ to be the orthogonal projection onto $\ker B_j$ as before and $f_{\ker B_j} = \pi_{\ker B_j} \circ f$. Then

$$\int_{X^{k_j}} \left| E_j \wedge f(x_1) \wedge \cdots \wedge f(x_{k_j}) \right| \mathrm{d}x_1 \cdots \mathrm{d}x_{k_j} = \int_{X^{k_j}} \left| f_{\ker B_j}(x_1) \wedge \cdots \wedge f_{\ker B_j}(x_{k_j}) \right| \mathrm{d}x_1 \cdots \mathrm{d}x_{k_j}. \tag{7-7}$$

Similar to the proof of Theorem 3.1, we know $F(X, f_{\ker B_j}) = F(X, f) \cap \ker B_j$. Hence we can take $\mathrm{Ell}(F(X, f_{\ker B_j}))$ to be $\mathrm{Ell}(F(X, f)) \cap \ker B_j$. By (7-7), Theorem 3.1, Lemma 7.2 and Lemma 7.3,

$$\begin{aligned}\int_{X^{k_j}} \left| E_j \wedge f(x_1) \wedge \cdots \wedge f(x_{k_j}) \right| \mathrm{d}x_1 \cdots \mathrm{d}x_{k_j} &= \int_{X^{k_j}} \left| f_{\ker B_j}(x_1) \wedge \cdots \wedge f_{\ker B_j}(x_{k_j}) \right| \mathrm{d}x_1 \cdots \mathrm{d}x_{k_j} \\ &\gtrsim_d \frac{1}{|\mathrm{Ell}(F(X, f)) \cap \ker B_j|} \\ &\gtrsim_d |(\mathrm{Ell}(F(X, f)) \cap \ker B_j)^*| \\ &= |\pi_{\ker B_j}(\mathrm{Ell}(F(X, f))^*)|. \end{aligned} \tag{7-8}$$

Hence it suffices to prove

$$\prod_{j=1}^{n} \left| \pi_{\ker B_j}(\mathrm{Ell}(F(X, f))^*) \right|^{p_j} \gtrsim_{d,\boldsymbol{p}} (\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}))^{-1} (\mathrm{Vis}[X, f])^{\sum_{j=1}^{n} p_j - 1}. \tag{7-9}$$

At this point we invoke the definition of $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p})$. For any ellipsoid Γ, we choose $f_j = \chi_{\pi_{E_j}(\Gamma^*)}$ in (1-9). Then by definition $\prod_{j=1}^{n} (f_j \circ B_j)^{p_j} \geq \chi_{\Gamma^*}$. Hence

$$|\Gamma^*| \leq \int_{\mathbb{R}^d} \prod_{j=1}^{n} (f_j \circ B_j)^{p_j} \leq \mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) \prod_{j=1}^{n} \left(\int_{E_j} f_j \right)^{p_j} = \mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) \prod_{j=1}^{n} |\pi_{E_j}(\Gamma^*)|^{p_j}. \tag{7-10}$$

In other words,

$$\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) \cdot |\Gamma| \cdot \prod_{j=1}^{n} |\pi_{E_j}(\Gamma^*)|^{p_j} \gtrsim_d 1. \tag{7-11}$$

By Lemmas 7.2, 7.3 and 7.4, we have

$$|\pi_{E_j}(\Gamma^*)| \sim_{k_j} \frac{1}{|\Gamma \cap E_j|} \sim_{k_j, d} \frac{|\pi_{\ker B_j}(\Gamma)|}{|\Gamma|}. \tag{7-12}$$

Hence

$$\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) \cdot |\Gamma| \cdot \prod_{j=1}^{n} \left(\frac{|\pi_{\ker B_j}(\Gamma)|}{|\Gamma|} \right)^{p_j} \gtrsim_{d,\boldsymbol{p}} 1. \tag{7-13}$$

Take $\Gamma = \mathrm{Ell}(F(X, f))^*$. By Lemma 7.2 again, we have $|\Gamma| = |\mathrm{Ell}(F(X, f))^*| \sim_d 1/|\mathrm{Ell}(F(X, f))| = \mathrm{Vis}[X, f]$. This fact and (7-13) imply (7-9), which in turn implies (7-6). □

8. Proof of Theorem 1.11

We are ready to prove Theorem 1.11. Just like the proof of Theorem 1.4, we prove a stronger theorem concerning algebraic varieties. This theorem can also be considered as an analogue of Theorem 6.1.

Theorem 8.1 (variety version of Brascamp–Lieb). *Assume we have positive integers $k_1, \dots, k_n \le d$ and rational numbers $p_1, \dots, p_n > 0$. Choose a common denominator τ of all p_j and assume $p_j = \tau_j/\tau$, with $\tau_j \in \mathbb{Z}^+$ satisfying the scaling condition $\sum_j p_j(d-k_j) = d$.*

Assume that for $1 \le j \le n$, $H_j \subseteq \mathbb{R}^d$ is part of a k_j-dimensional algebraic subvariety of degree $A(j)$. Let $\mathrm{d}\sigma_j$ denote the k_j-dimensional (Hausdorff) volume measure of H_j. Then under this measure, almost all $y_j \in H_j$ are smooth points. For a smooth point $y_j \in H_j$, let $T_{y_j}H_j$ denote the tangent space of H_j at y_j.

For $\sum_j \tau_j$ smooth points $\boldsymbol{y} = (y_{1,1}, \dots, y_{1,\tau_1}, y_{2,1}, \dots, y_{2,\tau_2}, \dots, y_{n,\tau_n})$, $y_{j,l} \in H_j$, there exists a unique Brascamp–Lieb datum $(\boldsymbol{B}(\boldsymbol{y}), \boldsymbol{p}(\boldsymbol{y}))$ with $\sum_j \tau_j$ projections B_j all being orthogonal projections within $\mathbb{R}^d$ as the following: Define $(\boldsymbol{B}(\boldsymbol{y}), \boldsymbol{p}(\boldsymbol{y})) = (B_{1,1}, \dots, B_{1,\tau_1}, B_{2,1}, \dots, B_{2,\tau_2}, \dots, B_{n,\tau_n}, 1/\tau, \dots, 1/\tau)$ such that $\ker B_{j,l} = T_{y_{j,l}}H_j$ *and all components of $\boldsymbol{p}$ are $1/\tau$. Then*

$$\int_{\mathbb{R}^d}\left(\int_{H_1^{\tau_1}\times\cdots\times H_n^{\tau_n}}\prod_{j=1}^{n}\prod_{k=1}^{\tau_j}\chi_{\{\operatorname{dist}(y_{j,k},x)\le 1\}}\,\mathrm{BL}(\boldsymbol{B}(\boldsymbol{y}),\boldsymbol{p}(\boldsymbol{y}))^{-\tau}\,\mathrm{d}\sigma_1(y_{1,1})\cdots\mathrm{d}\sigma_1(y_{1,\tau_1})\cdots\mathrm{d}\sigma_n(y_{n,\tau_n})\right)^{1/\tau}\mathrm{d}x \lesssim_{d,\tau_1,\dots,\tau_n,\tau}\prod_{j=1}^{n}A(j)^{p_j}. \tag{8-1}$$

Let us explain the motivation of Theorem 8.1 before proving it. If we want to naturally generalize Theorem 6.1 to the Brascamp–Lieb setting, first of all we have to come up with a reasonable integral like the left-hand side of (6-1) to put on the left-hand side. However the fact that in (6-1) all $p_j = 1/(n-1)$ no longer holds in our situation. In fact, the p_j might even all be irrational numbers. A natural way would be approximating (p_j) by rational tuples. This works (see below) but eventually we need all the p_j to be the same to get a quantity analogous to left-hand of (6-1).

Another remark before we move on. It's good to keep in mind that we may assume $\tau_1 = \cdots = \tau_n = 1$ in this theorem without loss of generality. This is trivial to see. But we keep the theorem in its current form here so it is more straightforward to apply.

Proof that Theorem 8.1 implies Theorem 1.11. Note that the conditions (1-11) and (1-12) only have rational coefficients. Hence it is possible to choose $(n+1)$ different rational $\boldsymbol{p}'$ close enough to $\boldsymbol{p}$ such that the conditions (1-11) and (1-12) are satisfied (that is, $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}') < +\infty$), and that $\boldsymbol{p}$ lies in the convex hull of those $\boldsymbol{p}'$. By interpolation we only need to prove the case when $\boldsymbol{p}$ is a rational vector.

Next in order to apply the result of Theorem 8.1 to prove Theorem 1.11, we claim that if a Brascamp–Lieb datum $(\boldsymbol{B}, \boldsymbol{p})$ is such that $p_j = \tau_j/\tau$, where τ all τ_j are positive integers, then $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) = \mathrm{BL}(\boldsymbol{B}', \boldsymbol{p}')$, where $\boldsymbol{B}' = (B_1, \dots, B_1, \dots, B_n, \dots, B_n)$ contains τ_j copies of B_j, and $\boldsymbol{p}' = (1/\tau, \dots, 1/\tau)$. In fact, looking at the definition (1-9) of $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p})$, we have

$$\mathrm{BL}(\boldsymbol{B}', \boldsymbol{p}') = \sup_{\{f_{j,l}\}}\frac{\int_{\mathbb{R}^d}\prod_{j=1}^{n}\prod_{l=1}^{\tau_j}(f_{j,l}\circ B_j)^{1/\tau}}{\prod_{j=1}^{n}\prod_{l=1}^{\tau_j}\left(\int_{H_j} f_{j,l}\right)^{1/\tau}}. \tag{8-2}$$

Since we can always take $f_{j,l} = f_j$ for all l, we deduce $\mathrm{BL}(\boldsymbol{B}', \boldsymbol{p}') \ge \mathrm{BL}(\boldsymbol{B}, \boldsymbol{p})$. On the other hand, in the definition of $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p})$ we can take $f_j = f_{j,l_j}$ for every possible tuple $(l_1, \dots, l_n)$ satisfying $1 \le l_j \le \tau_j$

to deduce

$$\int_{\mathbb{R}^d}\prod_{j=1}^{n}(f_{j,l_j}\circ B_j)^{\tau_j/\tau}\le \mathrm{BL}(\boldsymbol{B},\boldsymbol{p})\prod_{j=1}^{n}\left(\int_{H_j} f_{j,l_j}\right)^{\tau_j/\tau}. \tag{8-3}$$

Then we let (l_j) run through all possible tuples and invoke Hölder to conclude that

$$\int_{\mathbb{R}^d}\prod_{j=1}^{n}\prod_{l=1}^{\tau_j}(f_{j,l}\circ B_j)^{1/\tau}\le \mathrm{BL}(\boldsymbol{B},\boldsymbol{p})\prod_{j=1}^{n}\prod_{l=1}^{\tau_j}\left(\int_{H_j} f_{j,l}\right)^{1/\tau}. \tag{8-4}$$

Hence $\mathrm{BL}(\boldsymbol{B}',\boldsymbol{p}')\le \mathrm{BL}(\boldsymbol{B},\boldsymbol{p})$. Therefore $\mathrm{BL}(\boldsymbol{B}',\boldsymbol{p}')=\mathrm{BL}(\boldsymbol{B},\boldsymbol{p})$.

By Theorem 1.1 in [Bennett et al. 2015], BL is a locally bounded function. It is then not hard to derive Theorem 1.11 from Theorem 8.1 when $\boldsymbol{p}'$ is a fixed rational number. □

Proof of Theorem 8.1. It's plain that we may assume $\tau_1=\cdots=\tau_n=1$. For short we write $B_j=B_{j,1}$ and $y_j=y_{j,1}$.

The proof will be almost identical to that of Theorem 6.1. In the current proof, we omit some details for familiar manipulations in that proof to reduce redundancy and refer the reader to it.

Take the N and set up the unit cube lattice in $\mathbb{R}^d$ as in the proof of Theorem 6.1. Again let O_ν be the center of any cube Q_ν in the lattice. This time we define

$$G(Q_\nu)=\int_{H_1\times\cdots\times H_n}\prod_{j=1}^{n}\chi_{\mathrm{dist}(y_j,O_\nu)\le N}\,\mathrm{BL}(\boldsymbol{B}(\boldsymbol{y}),\boldsymbol{p}(\boldsymbol{y}))^{-\tau}\,\mathrm{d}\sigma_1(y_1)\cdots\mathrm{d}\sigma_n(y_n). \tag{8-5}$$

Similar to the proof of Theorem 6.1, it suffices to show

$$\sum_\nu G(Q_\nu)^{1/\tau}\lesssim_{d,n}\prod_{j=1}^{n}A(j)^{1/\tau}. \tag{8-6}$$

Again we may assume for the moment that each H_j is compact and use a limiting argument. Then we can again choose a large cube of side length S that contains all the relevant cubes. Finally we can find a polynomial P of degree $\lesssim_d S$ such that for each Q_ν,

$$\overline{\mathrm{Vis}}[Z(P)\cap Q_\nu]\ge S^d G(Q_\nu)^{1/\tau}\left(\sum_\nu G(Q_\nu)^{1/\tau}\right)^{-1}. \tag{8-7}$$

As before we have to make the technical comment that after adding some hyperplanes and changing the definition of $\overline{\mathrm{Vis}}$ accordingly, we may assume for all Q_ν with $G(Q_\nu)>0$ we have

$$\overline{V}_{Z(P)\cap Q_\nu}(\boldsymbol{v})\ge|\boldsymbol{v}|$$

(so that we are allowed to apply (7-6)). We only deal with the case where no hyperplanes are added so that the notation is simpler.

Similar to what we did in the proof of Theorem 6.1, we choose $B_j=T_{y_j}H_j$, all $p_j=1/\tau$ and integrate (7-6) over $y_j\in H_j\cap B(O_\nu,N)$. Then we choose the measure space X in (7-6) to be

$$\{p\in Z(P')\cap B(O_\nu,N):P'\in B(P,\varepsilon)\}$$

(the measure is just the surface measure on each $Z(P')$ joint with the standard measure on $B(P,\varepsilon)$, which is $\mathrm{d}p\,\mathrm{d}P'$, where $P' \in B(P,\varepsilon)$ and $p \in Z(P') \cap B(O_\nu, N)$) and deduce

$$\begin{aligned}
&\frac{1}{|B(P,\varepsilon)|^{(n-\tau)d}} \int \cdots \int_{B(P,\varepsilon)^{(n-\tau)d}} \int_{H_1\cap B(O_\nu,N)} \cdots \int_{H_n \cap B(O_\nu,N)} \int_{Z(P_1)\cap B(O_\nu,N)} \cdots \int_{Z(P_{(n-\tau)d})\cap B(O_\nu,N)} \\
&\qquad \prod_{j=1}^{n} \big|(T_{y_j}H_j)^\perp \wedge (T_{p_{k_1+\cdots+k_{j-1}+1}} Z(P_{k_1+\cdots+k_{j-1}+1}))^\perp \wedge \cdots \wedge (T_{p_{k_1+\cdots+k_j}} Z(P_{k_1+\cdots+k_j}))^\perp\big| \\
&\qquad\qquad \mathrm{d}p_1 \cdots \mathrm{d}p_{(n-\tau)d}\, \mathrm{d}\sigma_1(y_1) \cdots \mathrm{d}\sigma_n(y_n)\, \mathrm{d}P_1 \cdots \mathrm{d}P_{(n-\tau)d} \\
&\quad \gtrsim_{d,n} G(Q_\nu) \cdot \overline{\mathrm{Vis}}[Z(P)\cap Q_\nu]^{n-\tau} \\
&\quad \gtrsim_{d,n} S^{(n-\tau)d} G(Q_\nu)^{n/\tau} \Big(\sum_\nu G(Q_\nu)^{1/\tau}\Big)^{-(n-\tau)}.
\end{aligned} \tag{8-8}$$

As before we rewrite it as

$$\begin{aligned}
&\prod_{j=1}^{n} \bigg(\frac{1}{|B(P,\varepsilon)|^{k_j}} \int \cdots \int_{B(P,\varepsilon)^{k_j}} \frac{1}{S^{k_j}\cdot A(j)} \int_{H_j\cap B(O_\nu,N)} \int_{Z(P_1)\cap B(O_\nu,N)} \cdots \int_{Z(P_{k_j})\cap B(O_\nu,N)} \\
&\qquad \big|(T_{y_j}H_j)^\perp \wedge (T_{p_1}Z(P_1))^\perp \wedge \cdots \wedge (T_{p_{k_j}} Z(P_{k_j}))^\perp\big|\, \mathrm{d}p_1 \cdots \mathrm{d}p_{k_j}\, \mathrm{d}\sigma_j(y_j)\, \mathrm{d}P_1 \cdots \mathrm{d}P_{k_j} \bigg) \\
&\quad \gtrsim_{d,n} \frac{1}{\prod_{j=1}^n A(j)} G(Q_\nu)^{n/\tau} \Big(\sum_\nu G(Q_\nu)^{1/\tau}\Big)^{-(n-\tau)}.
\end{aligned} \tag{8-9}$$

Here note that since $\sum_{j=1}^n (d-k_j) = \tau d$ by assumption, we have $\sum_{j=1}^n k_j = (n-\tau)d$. We have used this fact in the above inequality chain (8-9).

By the arithmetic-geometric mean inequality we have

$$\begin{aligned}
&\sum_{j=1}^{n} \bigg(\frac{1}{|B(P,\varepsilon)|^{k_j}} \int \cdots \int_{B(P,\varepsilon)^{k_j}} \frac{1}{S^{k_j}\cdot A(j)} \int_{H_j\cap B(O_\nu,N)} \int_{Z(P_1)\cap B(O_\nu,N)} \cdots \int_{Z(P_{k_j})\cap B(O_\nu,N)} \\
&\qquad \big|(T_{y_j}H_j)^\perp \wedge (T_{p_1}Z(P_1))^\perp \wedge \cdots \wedge (T_{p_{k_j}} Z(P_{k_j}))^\perp\big|\, \mathrm{d}p_1 \cdots \mathrm{d}p_{k_j}\, \mathrm{d}\sigma_j(y_j)\, \mathrm{d}P_1 \cdots \mathrm{d}P_{k_j} \bigg) \\
&\quad \gtrsim_{d,n} \frac{1}{\big(\prod_{j=1}^n A(j)\big)^{1/n}} G(Q_\nu)^{1/\tau} \Big(\sum_\nu G(Q_\nu)^{1/\tau}\Big)^{-(n-\tau)/n}.
\end{aligned} \tag{8-10}$$

Like we did in the proof of Theorem 6.1, summing over ν and applying the intersection estimate Theorem 5.2 with $U = \{(u_i)_{1\le i\le k_j+1} : u_i \in \mathbb{R}^d, \operatorname{dist}(u_i, u_i') < N^2\}$, we deduce

$$\frac{1}{\big(\prod_{j=1}^n A(j)\big)^{1/n}} \Big(\sum_\nu G(Q_\nu)^{1/\tau}\Big) \Big(\sum_\nu G(Q_\nu)^{1/\tau}\Big)^{-(n-\tau)/n} \lesssim_{d,n} 1, \tag{8-11}$$

which implies (8-6) and concludes the proof. □

Remark 8.2. For the perturbed Brascamp–Lieb theorem itself, Theorem 1.11, it is conceivable that one can directly work with the framework of arguments in [Carbery and Valdimarsson 2013], without applying a rational approximation argument as we did in this section. Nevertheless, we still decided to keep the current approach as we feel that Theorem 8.1 here may be of independent interest, and that rationality seems indispensable for us to state the theorem (and prove it).

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SQUARE FUNCTION ESTIMATES FOR DISCRETE RADON TRANSFORMS

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We show $\ell^p(\mathbb{Z}^d)$-boundedness, for $p \in (1, \infty)$, of discrete singular integrals of Radon type with the aid of appropriate square function estimates, which can be thought of as a discrete counterpart of Littlewood–Paley theory. It is a very robust approach which allows us to proceed as in the continuous case.

1. Introduction

Assume that $K \in \mathcal{C}^1(\mathbb{R}^k \setminus \{0\})$ is a Calderón–Zygmund kernel satisfying the differential inequality

$$|y|^k |K(y)| + |y|^{k+1} |\nabla K(y)| \le 1 \tag{1-1}$$

for all $y \in \mathbb{R}^k$ with $|y| \ge 1$ and the cancellation condition

$$\sup_{\lambda \ge 1} \left| \int_{1 \le |y| \le \lambda} K(y)\, \mathrm{d}y \right| \le 1. \tag{1-2}$$

Let $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{d_0}) : \mathbb{Z}^k \to \mathbb{Z}^{d_0}$ be a polynomial mapping, where each component $\mathcal{P}_j : \mathbb{Z}^k \to \mathbb{Z}$ is a polynomial of k variables with integer coefficients and $\mathcal{P}_j(0) = 0$. In the present article, as in [Ionescu and Wainger 2006], we are interested in the discrete singular Radon transform $T^{\mathcal{P}}$ defined by

$$T^{\mathcal{P}} f(x) = \sum_{y \in \mathbb{Z}^k \setminus \{0\}} f(x - \mathcal{P}(y)) K(y) \tag{1-3}$$

for a finitely supported function $f : \mathbb{Z}^{d_0} \to \mathbb{R}$. We prove the following theorem.

Theorem A. *For every $p \in (1, \infty)$ there is $C_p > 0$ such that for all $f \in \ell^p(\mathbb{Z}^{d_0})$ we have*

$$\|T^{\mathcal{P}} f\|_{\ell^p} \le C_p \|f\|_{\ell^p}. \tag{1-4}$$

Moreover, the constant C_p is independent of the coefficients of the polynomial mapping $\mathcal{P}$.

Theorem A was proven by Ionescu and Wainger [2006]. The operator $T^{\mathcal{P}}$ is a discrete analogue of the continuous Radon transform $R^{\mathcal{P}}$ defined by

$$R^{\mathcal{P}} f(x) = \text{p.v.} \int_{\mathbb{R}^k} f(x - \mathcal{P}(y)) K(y)\, \mathrm{d}y. \tag{1-5}$$

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Nowadays the operators $R^{\mathcal{P}}$ and their $L^p(\mathbb{R}^{d_0})$-boundedness properties for $p \in (1, \infty)$ are very well understood. We refer to [Stein 1993] for a detailed exposition, and see also [Christ et al. 1999] for more general cases. The key ingredient in proving $L^p(\mathbb{R}^{d_0})$ bounds for $R^{\mathcal{P}}$ is Littlewood–Paley theory. More precisely, we begin with $L^2(\mathbb{R}^{d_0})$ theory which, based on some oscillatory integral estimates for dyadic pieces of the multiplier corresponding to $R^{\mathcal{P}}$, provides bounds with acceptable decays. Then appealing to Littlewood–Paley theory and interpolation it is possible to obtain general $L^p(\mathbb{R}^{d_0})$ bounds for all $p \in (1, \infty)$. Now, one would like to follow the same scheme in the discrete case. However, the situation for $T^{\mathcal{P}}$ is much more complicated due to arithmetic nature of this operator. Although $\ell^2(\mathbb{Z}^{d_0})$ theory is based on estimates for some oscillatory integrals, or rather exponential sums associated with dyadic pieces of the multiplier corresponding to $T^{\mathcal{P}}$ as was shown in [Ionescu and Wainger 2006], $\ell^p(\mathbb{Z}^{d_0})$ theory does not fall under the Littlewood–Paley paradigm as it does in the continuous case.

The main aim of this paper is to give a new proof of Theorem A using square function techniques. We construct a suitable square function which allows us to proceed as in the continuous case to obtain $\ell^p(\mathbb{Z}^{d_0})$ theory for the operator (1-3). Our square function gives a new insight for these sort of problems, see especially [Mirek et al. 2015; 2017], and can be thought as a discrete counterpart of Littlewood–Paley theory.

There is also an interesting open question concerning the estimates of $T^{\mathcal{P}}$ at the endpoint for $p = 1$. This is unknown even in the continuous case. For instance, if we consider a Radon transform $R^{\mathcal{P}}$ along the parabola $\mathcal{P}(y) = (y, y^2)$ in $\mathbb{R}^2$, i.e.,

$$R^{\mathcal{P}} f(x_1, x_2) = \text{p.v.} \int_{\mathbb{R}} f(x_1 - y, x_2 - y^2) \, \frac{\mathrm{d}y}{y},$$

then the question about weak-type (1, 1)-estimates for $R^{\mathcal{P}}$ is one of the major unsolved problems in harmonic analysis. The best known result to date belongs to Seeger, Tao and Wright [Seeger et al. 2004].

In view of the recent negative results of [Buczolich and Mauldin 2010] and [LaVictoire 2011], at the endpoint for $p = 1$, for Bourgain's maximal functions corresponding to the discrete averaging operators along n^2 or n^k with $k \geq 2$, we expect that similar phenomena may occur for discrete singular Radon transforms.

Outline of the strategy of our proof. Recall from [Stein 1993, Chapter 6, §4.5, Chapter 13, §5.3] that given a kernel K satisfying (1-1) and (1-2) there are functions $(K_n : n \in \mathbb{Z})$ and a constant $C > 0$ such that for $x \neq 0$,

$$K(x) = \sum_{n \in \mathbb{Z}}^{\infty} K_n(x), \tag{1-6}$$

where for each $n \in \mathbb{Z}$, the kernel K_n is supported inside $2^{n-2} \leq |x| \leq 2^n$, satisfies

$$|x|^k |K_n(x)| + |x|^{k+1} |\nabla K_n(x)| \leq C \tag{1-7}$$

for all $x \in \mathbb{R}^k$ such that $|x| \geq 1$, and has integral 0. Thus in view of (1-7), instead of (1-4), it suffices to show that for every $p \in (1, \infty)$ there is a constant $C_p > 0$ such that

$$\Big\| \sum_{n \geq 0} T_n^{\mathcal{P}} f \Big\|_{\ell^p} \leq C_p \|f\|_{\ell^p} \tag{1-8}$$

for all $f \in \ell^p(\mathbb{Z}^d)$, where

$$T_n^{\mathcal{P}} f(x) = \sum_{y \in \mathbb{Z}^k} f(x - \mathcal{P}(y)) K_n(y) \tag{1-9}$$

for each $n \in \mathbb{Z}$. The summation in (1-8) can be taken over nonnegative integers, since $\sum_{n<0} T_n^{\mathcal{P}} f \equiv 0$.

As we mentioned before, the proof of inequality (1-8) will strongly follow the scheme of the proof of the corresponding inequality from the continuous setup. Now we describe the key points of our approach. To avoid some technicalities assume that $\mathcal{P}(x) = (x^d, \ldots, x)$ is a moment curve for some $d = d_0 \geq 2$ and $k = 1$. Let m_n be the Fourier multiplier associated with the operator $T_n^{\mathcal{P}}$; i.e., $T_n^{\mathcal{P}} f = \mathcal{F}^{-1}(m_n \hat{f})$. As in [Mirek et al. 2015; 2017], we introduce a family of appropriate projections $(\Xi_n(\xi) : n \geq 0)$ which will localize the asymptotic behaviour of $m_n(\xi)$. Namely, let η be a smooth bump function with a small support, fix $l \in \mathbb{N}$ and define for each integer $n \geq 0$ the projection

$$\Xi_n(\xi) = \sum_{a/q \in \mathscr{U}_{n^l}} \eta(\mathcal{E}_n^{-1}(\xi - a/q)), \tag{1-10}$$

where $\mathcal{E}_n$ is a diagonal $d \times d$ matrix with positive entries $(\varepsilon_j : 1 \leq j \leq d)$ such that $\varepsilon_j \leq e^{-n^{1/5}}$ and

$$\mathscr{U}_{n^l} = \big\{a/q \in \mathbb{T}^d \cap \mathbb{Q}^d : a = (a_1, \ldots, a_d) \in \mathbb{N}_q^d \text{ and } \gcd(a_1, \ldots, a_d, q) = 1 \text{ and } q \in P_{n^l}\big\}$$

for some family P_{n^l} such that $\mathbb{N}_{n^l} \subseteq P_{n^l} \subseteq \mathbb{N}_{e^{n^{1/10}}}$. All details are described in Section 2. Exploiting the ideas of [Ionescu and Wainger 2006], we prove that for every $p \in (1, \infty)$ there is a constant $C_{l,p} > 0$ such that

$$\|\mathcal{F}^{-1}(\Xi_n \hat{f})\|_{\ell^p} \leq C_{l,p} \log(n+2) \|f\|_{\ell^p}. \tag{1-11}$$

Inequality (1-11) will be essential in our proof. Observe that (1-10) allows us to dominate (1-8) as

$$\Big\| \sum_{n \geq 0} T_n^{\mathcal{P}} f \Big\|_{\ell^p} \leq \Big\| \sum_{n \geq 0} \mathcal{F}^{-1}(m_n \Xi_n \hat{f}) \Big\|_{\ell^p} + \Big\| \sum_{n \geq 0} \mathcal{F}^{-1}(m_n(1 - \Xi_n) \hat{f}) \Big\|_{\ell^p}, \tag{1-12}$$

and we can employ the ideas from the circle method of Hardy and Littlewood, which are implicit in the behaviour of the projections Ξ_n and $1 - \Xi_n$. Namely, the second norm on the right-hand side of (1-12) is bounded, since the multiplier $m_n(1 - \Xi_n)$ is highly oscillatory. Thus appealing to (1-11) and a variant of Weyl's inequality with logarithmic decay, which has been proven in [Mirek et al. 2015], see Theorem 3.1, we can conclude that there is a constant $C_p > 0$ such that for each $n \geq 0$ we have

$$\big\| \mathcal{F}^{-1}(m_n(1 - \Xi_n) \hat{f}) \big\|_{\ell^p} \leq C_p (n+1)^{-2} \|f\|_{\ell^p}.$$

Now the whole difficulty lies in proving

$$\Big\| \sum_{n \geq 0} \mathcal{F}^{-1}(m_n \Xi_n \hat{f}) \Big\|_{\ell^p} \leq C_p \|f\|_{\ell^p}. \tag{1-13}$$

For this purpose we construct new multipliers of the form

$$\Delta_{n,s}^j(\xi) = \sum_{a/q \in \mathscr{U}_{(s+1)^l} \setminus \mathscr{U}_{s^l}} \big(\eta(\mathcal{E}_{n+j}^{-1}(\xi - a/q)) - \eta(\mathcal{E}_{n+j+1}^{-1}(\xi - a/q))\big) \eta(\mathcal{E}_s^{-1}(\xi - a/q)) \tag{1-14}$$

such that

$$\Xi_n(\xi) \simeq \sum_{j\in\mathbb{Z}} \sum_{s\geq 0} \Delta^j_{n,s}(\xi).$$

Moreover, we will be able to show in Theorem 3.3, using Theorem 2.2, that for each $p \in (1, \infty)$ there is a constant $C_p > 0$ such that

$$\left\| \left(\sum_{n\in\mathbb{Z}} |\mathcal{F}^{-1}(\Delta^j_{n,s}\hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p \log(s+2)\|f\|_{\ell^p} \tag{1-15}$$

for any $s \geq 0$, uniformly in $j \in \mathbb{Z}$. Estimate (1-15) can be thought of as a discrete counterpart of the Littlewood–Paley inequality; see Theorem 3.3. This is the key ingredient in our proof, which combined with the robust $\ell^2(\mathbb{Z}^d)$ estimate

$$\left\| \left(\sum_{n\in\mathbb{Z}} |\mathcal{F}^{-1}(m_n\Delta^j_{n,s}\hat{f})|^2 \right)^{1/2} \right\|_{\ell^2} \leq C2^{-\varepsilon|j|}(s+1)^{-\delta l}\|f\|_{\ell^2}, \tag{1-16}$$

allows us to deduce (1-13). The last bound follows, since for each $a/q \in \mathscr{U}_{(s+1)^l} \setminus \mathscr{U}_{s^l}$ we have

$$m_n(\xi) = G(a/q)\Phi_n(\xi - a/q) + \mathcal{O}(2^{-n/2}),$$

where $G(a/q)$ is the Gaussian sum and Φ_n is a continuous counterpart of m_n; precise definitions can be found at the beginning of Section 3. This observation leads to (1-16), because $|G(a/q)| \leq Cq^{-\delta}$ and $q \geq s^l$ if $a/q \in \mathscr{U}_{(s+1)^l} \setminus \mathscr{U}_{s^l}$. The decay in $|j|$ in (1-16) follows from the assumption on the support of $\Delta^j_{n,s}$ and the behaviour of $\Phi_n(\xi - a/q)$; see Section 3 for more details.

The ideas of exploiting projection (1-10) were initiated in [Mirek et al. 2015] in the context of $\ell^p(\mathbb{Z}^{d_0})$-boundedness of maximal functions corresponding to the averaging Radon operators

$$M^{\mathcal{P}}_N f(x) = N^{-k} \sum_{y\in\mathbb{N}^k_N} f(x - \mathcal{P}(y)), \tag{1-17}$$

where $\mathbb{N}^k_N = \{1, 2, \dots, N\}^k$, and the truncated singular Radon transforms

$$T^{\mathcal{P}}_N f(x) = \sum_{y\in\mathbb{B}_N\setminus\{0\}} f(x - \mathcal{P}(y))K(y), \tag{1-18}$$

where $\mathbb{B}_N = \{x \in \mathbb{Z}^k : |x| \leq N\}$. These ideas, on the one hand, resulted in a new proof for Bourgain's maximal operators [Bourgain 1988a; 1988b; 1989]. On the other hand, they turned out to be flexible enough to attack $\ell^p(\mathbb{Z}^{d_0})$-boundedness of maximal functions for operators with signs like in (1-18). In fact, in [Mirek et al. 2015] we provided some vector-valued estimates for the maximal functions associated with (1-17) and (1-18). These estimates found applications in variational estimates for (1-17) and (1-18), which were the subject of [Mirek et al. 2017]. Our approach falls within the scope of a general scheme which was recently developed in [Mirek et al. 2015; 2017] and resulted in some unification in the theory of discrete analogues in harmonic analysis. The novelty of this paper is that it provides a counterpart of the Littlewood–Paley square function, which is useful in the problems with arithmetic flavour. Furthermore, this square function theory is also an invaluable ingredient in the estimates of variational seminorm in [Mirek et al. 2017].

The paper is organized as follows. In Section 2 we prove Theorem 2.2, which is essential in our approach and guarantees (1-11). Ionescu and Wainger [2006] proved this result with $(\log N)^D$ loss in norm, where $D > 0$ is a large power. In [Mirek et al. 2015] we provided a slightly different proof and showed that $\log N$ is possible. Moreover $\log N$ loss is sharp for the method which we used. Since Theorem 2.2 is a deep theorem, which uses the most sophisticated tools developed to date in the field of discrete analogues, we have decided, for the sake of completeness, to provide necessary details. In Section 3 we prove Theorem A. To understand more quickly the proof of Theorem A, the reader may begin by looking at Section 3 first. These sections can be read independently, assuming the results from Section 2.

Basic reductions. We set

$$\deg \mathcal{P} = \max\{\deg \mathcal{P}_j : 1 \le j \le d_0\}$$

and define the set

$$\Gamma = \{\gamma \in \mathbb{Z}^k \setminus \{0\} : 0 < |\gamma| \le \deg \mathcal{P}\}$$

with the lexicographic order. Let d be the cardinality of Γ. Then we can identify $\mathbb{R}^d$ with the space of all vectors whose coordinates are labelled by multi-indices $\gamma \in \Gamma$. Next we introduce the canonical polynomial mapping

$$\mathcal{Q} = (\mathcal{Q}_\gamma : \gamma \in \Gamma) : \mathbb{Z}^k \to \mathbb{Z}^d,$$

where $\mathcal{Q}_\gamma(x) = x^\gamma$ and $x^\gamma = x_1^{\gamma_1} \cdot \dots \cdot x_k^{\gamma_k}$. The canonical polynomial mapping $\mathcal{Q}$ determines anisotropic dilations. Namely, let A be a diagonal $d \times d$ matrix such that

$$(Av)_\gamma = |\gamma| v_\gamma$$

for any $v \in \mathbb{R}^d$ and $\gamma \in \Gamma$, where $|\gamma| = \gamma_1 + \dots + \gamma_k$. Then for every $t > 0$ we set

$$t^A = \exp(A \log t);$$

i.e., $t^A x = (t^{|\gamma|} x_\gamma : \gamma \in \Gamma)$ for $x \in \mathbb{R}^d$, and we see that $\mathcal{Q}(tx) = t^A \mathcal{Q}(x)$.

Observe also that each $\mathcal{P}_j$ can be expressed as

$$\mathcal{P}_j(x) = \sum_{\gamma \in \Gamma} c_j^\gamma x^\gamma$$

for some $c_j^\gamma \in \mathbb{R}$. Moreover, the coefficients $(c_j^\gamma : \gamma \in \Gamma, j \in \{1, \dots, d_0\})$ define a linear transformation $L : \mathbb{R}^d \to \mathbb{R}^{d_0}$ such that $L\mathcal{Q} = \mathcal{P}$. Indeed, it is enough to set

$$(Lv)_j = \sum_{\gamma \in \Gamma} c_j^\gamma v_\gamma$$

for each $j \in \{1, \dots, d_0\}$ and $v \in \mathbb{R}^d$. Now instead of proving Theorem A we show the following.

Theorem B. *For every $p \in (1, \infty)$ there is $C_p > 0$ such that for all $f \in \ell^p(\mathbb{Z}^d)$ we have*

$$\|T^{\mathcal{Q}} f\|_{\ell^p} \le C_p \|f\|_{\ell^p}. \tag{1-19}$$

In view of [Stein 1993, Section 11] we can perform a lifting procedure, which allows us to replace the underlying polynomial mapping $\mathcal{P}$ from (1-4) by the canonical polynomial mapping $\mathcal{Q}$. Moreover, it shows that (1-19) implies (1-4) with the same constant C_p; see also [Ionescu and Wainger 2006] for more details. Therefore, the matters are reduced to proving (1-19) for the canonical polynomial mapping. The advantage of working with the canonical polynomial mapping $\mathcal{Q}$ is that it has all coefficients equal to 1, and the uniform bound in this case is immediate. From now on for simplicity of notation we will write $T = T^{\mathcal{Q}}$.

Notation. Throughout the whole article $C > 0$ will stand for a positive constant (possibly large constant) whose value may change from occurrence to occurrence. If there is an absolute constant $C > 0$ such that $A \le CB$ ($A \ge CB$) then we will write $A \lesssim B$ ($A \gtrsim B$). Moreover, we will write $A \simeq B$ if $A \lesssim B$ and $A \gtrsim B$ hold simultaneously, and we will write $A \lesssim_\delta B$ ($A \gtrsim_\delta B$) to indicate that the constant $C > 0$ depends on some $\delta > 0$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and for $N \in \mathbb{N}$ we set

$$\mathbb{N}_N = \{1, 2, \dots, N\}.$$

For a vector $x \in \mathbb{R}^d$ we will use the norms

$$|x|_\infty = \max\{|x_j| : 1 \le j \le d\} \quad \text{and} \quad |x| = \Big(\sum_{j=1}^{d} |x_j|^2\Big)^{1/2}.$$

If γ is a multi-index from $\mathbb{N}_0^k$ then $|\gamma| = \gamma_1 + \cdots + \gamma_k$. Although, we use $|\cdot|$ for the length of a multi-index $\gamma \in \mathbb{N}_0^k$ and the Euclidean norm of $x \in \mathbb{R}^d$, their meaning will be always clear from the context and it will cause no confusions in the sequel.

2. Ionescu–Wainger-type multipliers

For a function $f \in L^1(\mathbb{R}^d)$ let $\mathcal{F}$ denote the Fourier transform on $\mathbb{R}^d$ defined as

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}^d} f(x) e^{2\pi i \xi \cdot x}\, \mathrm{d}x.$$

If $f \in \ell^1(\mathbb{Z}^d)$ we set

$$\hat{f}(\xi) = \sum_{x \in \mathbb{Z}^d} f(x) e^{2\pi i \xi \cdot x}.$$

To simplify the notation, we denote by $\mathcal{F}^{-1}$ the inverse Fourier transform on $\mathbb{R}^d$ and the inverse Fourier transform on the torus $\mathbb{T}^d \equiv [0, 1)^d$ (Fourier coefficients). The meaning of $\mathcal{F}^{-1}$ will be always clear from the context. Let $\eta : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that $0 \le \eta(x) \le 1$ and

$$\eta(x) = \begin{cases} 1 & \text{for } |x| \le 1/(16d), \\ 0 & \text{for } |x| \ge 1/(8d). \end{cases}$$

Remark 2.1. We will additionally assume that η is a convolution of two nonnegative smooth functions ϕ and ψ with compact supports contained in $(-1/(8d), 1/(8d))^d$.

This section is intended to prove Theorem 2.2, which is inspired by the ideas of [Ionescu and Wainger 2006]. Let $\rho > 0$ and for every $N \in \mathbb{N}$ define

$$N_0 = \lfloor N^{\rho/2} \rfloor + 1 \quad \text{and} \quad Q_0 = (N_0!)^D,$$

where $D = D_\rho = \lfloor 2/\rho \rfloor + 1$. Let $\mathbb{P}_N = \mathbb{P} \cap (N_0, N]$, where $\mathbb{P}$ is the set of all prime numbers. For any $V \subseteq \mathbb{P}_N$ we define

$$\Pi(V) = \bigcup_{k \in \mathbb{N}_D} \Pi_k(V),$$

where for any $k \in \mathbb{N}_D$

$$\Pi_k(V) = \{p_1^{\gamma_1} \cdot \dots \cdot p_k^{\gamma_k} : \gamma_l \in \mathbb{N}_D \text{ and } p_l \in V \text{ are distinct for all } 1 \le l \le k\}.$$

In other words $\Pi(V)$ is the set of all products of prime factors from V of length at most D, at powers between 1 and D. Now we introduce the sets

$$P_N = \{q = Q \cdot w : Q \mid Q_0 \text{ and } w \in \Pi(\mathbb{P}_N) \cup \{1\}\}.$$

It is not difficult to see that every integer $q \in \mathbb{N}_N$ can be uniquely written as $q = Q \cdot w$, where $Q \mid Q_0$ and $w \in \Pi(\mathbb{P}_N) \cup \{1\}$. Moreover, for sufficiently large $N \in \mathbb{N}$ we have

$$q = Q \cdot w \le Q_0 \cdot w \le (N_0!)^D N^{D^2} \le e^{N^\rho};$$

thus we have $\mathbb{N}_N \subseteq P_N \subseteq \mathbb{N}_{e^{N^\rho}}$. Furthermore, if $N_1 \le N_2$ then $P_{N_1} \subseteq P_{N_2}$.

For a subset $S \subseteq \mathbb{N}$ we define

$$\mathcal{R}(S) = \{a/q \in \mathbb{T}^d \cap \mathbb{Q}^d : a \in A_q \text{ and } q \in S\},$$

where for each $q \in \mathbb{N}$

$$A_q = \{a \in \mathbb{N}_q^d : \gcd(q, (a_\gamma : \gamma \in \Gamma)) = 1\}.$$

Finally, for each $N \in \mathbb{N}$ we will consider the sets

$$\mathscr{U}_N = \mathcal{R}(P_N). \tag{2-1}$$

It is easy to see, if $N_1 \le N_2$ then $\mathscr{U}_{N_1} \subseteq \mathscr{U}_{N_2}$.

We will assume that Θ is a multiplier on $\mathbb{R}^d$ and for every $p \in (1, \infty)$ there is a constant $\boldsymbol{A}_p > 0$ such that for every $f \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ we have

$$\|\mathcal{F}^{-1}(\Theta \mathcal{F} f)\|_{L^p} \le \boldsymbol{A}_p \|f\|_{L^p}. \tag{2-2}$$

For each $N \in \mathbb{N}$ we define the new periodic multiplier

$$\Delta_N(\xi) = \sum_{a/q \in \mathscr{U}_N} \Theta(\xi - a/q) \eta_N(\xi - a/q), \tag{2-3}$$

where $\eta_N(\xi) = \eta(\mathcal{E}_N^{-1}\xi)$ and $\mathcal{E}_N$ is a diagonal $d \times d$ matrix with positive entries $(\varepsilon_\gamma : \gamma \in \Gamma)$ such that $\varepsilon_\gamma \le e^{-N^{2\rho}}$. The main result is the following.

Theorem 2.2. *Let Θ be a multiplier on $\mathbb{R}^d$ obeying* (2-2). *Then for every $\rho > 0$ and $p \in (1, \infty)$ there is a constant $C_{\rho,p} > 0$ such that for any $N \in \mathbb{N}$ and $f \in \ell^p(\mathbb{Z}^d)$ we have*

$$\|\mathcal{F}^{-1}(\Delta_N \hat{f})\|_{\ell^p} \le C_{\rho,p}(\boldsymbol{A}_p + 1) \log N \|f\|_{\ell^p}. \tag{2-4}$$

The main constructing blocks have been gathered in the next three subsections. Theorem 2.2 is a consequence of Theorem 2.6 and Proposition 2.7 proved below. To prove Theorem 2.2 we find some $C_\rho > 0$ and disjoint sets $\mathscr{U}_N^i \subseteq \mathscr{U}_N$ such that

$$\mathscr{U}_N = \bigcup_{1 \le i \le C_\rho \log N} \mathscr{U}_N^i$$

and we show that Δ_N with the summation restricted to $\mathscr{U}_N^i$ is bounded on $\ell^p(\mathbb{Z}^d)$ for every $p \in (1, \infty)$. In order to construct $\mathscr{U}_N^i$, we need a suitable partition of integers from the set $\Pi(\mathbb{P}_N) \cup \{1\}$; see also [Ionescu and Wainger 2006].

Fundamental combinatorial lemma. We begin with the following definition.

Definition 2.3. A subset $\Lambda \subseteq \Pi(V)$ has property $\mathcal{O}$ if there is $k \in \mathbb{N}_D$ and there are sets $S_1, S_2, \dots, S_k$ with the following properties:

(i) For each $1 \le j \le k$ there is $\beta_j \in \mathbb{N}$ such that $S_j = \{q_{j,1}, \dots, q_{j,\beta_j}\}$.

(ii) For every $q_{j,s} \in S_j$ there are $p_{j,s} \in V$ and $\gamma_j \in \mathbb{N}_D$ such that $q_{j,s} = p_{j,s}^{\gamma_j}$.

(iii) For every $w \in \Lambda$ there are unique numbers $q_{1,s_1} \in S_1, \dots, q_{k,s_k} \in S_k$ such that $w = q_{1,s_1} \cdot \dots \cdot q_{k,s_k}$.

(iv) If $(j, s) \ne (j', s')$ then $(q_{j,s}, q_{j',s'}) = 1$.

Now three comments are in order.

- The set $\Lambda = \{1\}$ has property $\mathcal{O}$ corresponding to $k = 0$.
- If Λ has property $\mathcal{O}$, then each subset $\Lambda' \subseteq \Lambda$ has property $\mathcal{O}$ as well.
- If a set Λ has property $\mathcal{O}$ then each element of Λ has the same number of prime factors $k \le D$.

The main result is the following.

Lemma 2.4. *For every $\rho > 0$ there exists a constant $C_\rho > 0$ such that for every $N \in \mathbb{N}$ the set $\mathscr{U}_N$ can be written as a disjoint union of at most $C_\rho \log N$ sets $\mathscr{U}_N^i = \mathcal{R}(P_N^i)$, where*

$$P_N^i = \{q = Q \cdot w : Q \mid Q_0 \text{ and } w \in \Lambda_i(\mathbb{P}_N)\} \tag{2-5}$$

and $\Lambda_i(\mathbb{P}_N) \subseteq \Pi(\mathbb{P}_N) \cup \{1\}$ has property $\mathcal{O}$ for each integer $1 \le i \le C_\rho \log N$.

Proof. We have to prove that for every $V \subseteq \mathbb{P}_N$ the set $\Pi(V)$ can be written as a disjoint union of at most $C_k \log N$ sets with property $\mathcal{O}$. Fix $k \in \mathbb{N}_D$, let $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{N}_D^k$ be a multi-index and observe that

$$\Pi_k(V) = \bigcup_{\gamma \in \mathbb{N}_D^k} \Pi_k^\gamma(V),$$

where

$$\Pi_k^\gamma(V) = \{p_1^{\gamma_1} \cdot \dots \cdot p_k^{\gamma_k} : p_l \in V \text{ are distinct for all } 1 \le l \le k\}.$$

Since there are D^k possible choices of exponents $\gamma_1, \dots, \gamma_k \in \mathbb{N}_D$ when $k \in \mathbb{N}_D$, it only suffices to prove that every $\Pi_k^\gamma(V)$ can be partitioned into a union (not necessarily disjoint) of at most $C_k \log N$ sets with property $\mathcal{O}$.

We claim that for each $k \in \mathbb{N}$ there is a constant $C_k > 0$ and a family

$$\pi = \{\pi_i(V) : 1 \le i \le C_k \log |V|\} \tag{2-6}$$

of partitions of V such that

(i) for every $1 \le i \le C_k \log |V|$, each $\pi_i(V) = \{V_1^i, \dots, V_k^i\}$ consists of pairwise disjoint subsets of V and $V = V_1^i \cup \dots \cup V_k^i$;

(ii) for every $E \subseteq V$ with at least k elements, there exists $\pi_i(V) = \{V_1^i, \dots, V_k^i\} \in \pi$ such that $E \cap V_j^i \neq \varnothing$ for every $1 \le j \le k$.

Assume for a moment we have constructed a family π as in (2-6). Then one sees that for a fixed $\gamma \in \mathbb{N}_D^k$ we have

$$\Pi_k^\gamma(V) = \bigcup_{1 \le i \le C_k \log |V|} \Pi_{k,i}^\gamma(V), \tag{2-7}$$

where

$$\Pi_{k,i}^\gamma(V) = \{p_1^{\gamma_1} \cdot \dots \cdot p_k^{\gamma_k} : p_j \in V_j^i \text{ and } V_j^i \in \pi_i(V) \text{ for each } 1 \le j \le k\}.$$

Indeed, the sum on the right-hand side of (2-7) is contained in $\Pi_k^\gamma(V)$ since each $\Pi_{k,i}^\gamma(V)$ is. For the opposite inclusion take $p_1^{\gamma_1} \cdot \dots \cdot p_k^{\gamma_k} \in \Pi_k^\gamma(V)$ and let $E = \{p_1, \dots, p_k\}$; then property (ii) for the family (2-6) ensures that there is $\pi_i(V) = \{V_1^i, \dots, V_k^i\} \in \pi$ such that $E \cap V_j^i \neq \varnothing$ for every $1 \le j \le k$. Therefore, $p_1^{\gamma_1} \cdot \dots \cdot p_k^{\gamma_k} \in \Pi_{k,i}^\gamma(V)$. Furthermore, we see that for each $1 \le i \le C_k \log N$, the sets $\Pi_{k,i}^\gamma(V)$ have property $\mathcal{O}$.

The proof will be completed if we construct the family π as in (2-6) for the set V. We assume, for simplicity, that $V = \mathbb{N}_N$ but the result is true for all $V \subseteq \mathbb{N}_N$ containing at least k elements. Now it will be more comfortable to work with surjective mappings $f : \mathbb{N}_N \mapsto \mathbb{N}_k$ rather than with partitions of $\mathbb{N}_N$ into k nonempty subsets. It will cause no changes to us, since every surjection $f : \mathbb{N}_N \mapsto \mathbb{N}_k$ determines a partition $\{f^{-1}[\{m\}] : 1 \le m \le k\}$ of $\mathbb{N}_N$ into k nonempty subsets.

For the proof we employ a probabilistic argument. Indeed, let $f : \mathbb{N}_N \mapsto \mathbb{N}_k$ be a random surjective mapping. Assume that for every $n \in \mathbb{N}_N$ and $m \in \mathbb{N}_k$ we have $\mathbb{P}(\{f(n) = m\}) = 1/k$ independently of all other $n \in \mathbb{N}_N$. For every $E \subseteq \mathbb{N}_N$ with k elements we have $\mathbb{P}(\{|f[E]| = k\}) = k!/k^k$. It suffices to show that for some $r \simeq_k \log N$ and $f_1, \dots, f_r$ random surjections we have

$$\mathbb{P}\big(\{\forall_{E \subseteq \mathbb{N}_N}\ |E| = k\ \exists_{1 \le l \le r}\ |f_l[E]| = k\}\big) > 0.$$

In other words, for each $E \subseteq \mathbb{N}_N$ with cardinality k it is always possible to find, with a positive probability, among at most $C_k \log N$ random surjections at least one $f : \mathbb{N}_N \mapsto \mathbb{N}_k$ such that $|f[E]| = k$. Then the set $\{f^{-1}[\{m\}] : 1 \le m \le k\}$ is a partition of $\mathbb{N}_N$ and $E \cap f^{-1}[\{m\}] \neq \varnothing$ for every $1 \le m \le k$.

The task now is the determine the exact value of $r \simeq_k \log N$. Take now $1 \le r \le N$ independent random surjections $f_1, \dots, f_r$ and observe that

$$\mathbb{P}\big(\big\{\exists_{E\subseteq\mathbb{N}_N}\ |E|=k\ \forall_{1\le l\le r}\ |f_l[E]|<k\big\}\big) \le \sum_{E\subseteq\mathbb{N}_N:|E|=k} \mathbb{P}\big(\{\forall_{1\le l\le r}\ |f_l[E]|<k\}\big) = \sum_{E\subseteq\mathbb{N}_N:|E|=k} \Big(1-\frac{k!}{k^k}\Big)^r$$
$$= \binom{N}{k}\Big(1-\frac{k!}{k^k}\Big)^r \le \Big(\frac{eN}{k}\Big)^k e^{-rk!/k^k} = e^{k\log(eN/k)-rk!/k^k}.$$

Therefore

$$\mathbb{P}\big(\{\exists_{E\subseteq\mathbb{N}_N}\ |E|=k\ \forall_{1\le l\le r}\ |f_l[E]|<k\}\big) < 1$$

if and only if

$$r > \frac{k^{k+1}}{k!}\log\Big(\frac{eN}{k}\Big).$$

Thus taking

$$r = \Big\lceil \frac{k^{k+1}}{k!}\log\Big(\frac{eN}{k}\Big)\Big\rceil + 1 \simeq C_k \log N,$$

we see that it does the job. This completes the proof of Lemma 2.4. □

Further reductions and square function estimates. Now we can write

$$\Delta_N = \sum_{1\le i\le C_\rho \log N} \Delta_N^i,$$

where for each $1 \le i \le C_\rho \log N$

$$\Delta_N^i(\xi) = \sum_{a/q\in\mathscr{U}_N^i} \Theta(\xi - a/q)\eta_N(\xi - a/q) \tag{2-8}$$

with $\mathscr{U}_N^i$ as in Lemma 2.4. The proof of Theorem 2.2 will be completed if we show that for every $p \in (1,\infty)$ and $\rho > 0$, there is a constant $C > 0$ such that for any $N \in \mathbb{N}$ and $1 \le i \le C_\rho \log N$ we have

$$\|\mathcal{F}^{-1}(\Delta_N^i \hat{f})\|_{\ell^p} \le C(\boldsymbol{A}_p + 1)\|f\|_{\ell^p} \tag{2-9}$$

for every $f \in \ell^p(\mathbb{Z}^d)$.

Let

$$\Lambda \subseteq \Pi(\mathbb{P}_N) \cup \{1\} \tag{2-10}$$

be a set with property $\mathcal{O}$; see Definition 2.3. Define

$$\mathscr{U}_N^\Lambda = \mathcal{R}\big(\{q = Q\cdot w : Q \mid Q_0 \text{ and } w \in \Lambda\}\big)$$

and $\mathscr{W}_N = \mathcal{R}(\Lambda)$, and we introduce

$$\Delta_N^\Lambda(\xi) = \sum_{a/q\in\mathscr{U}_N^\Lambda} \Theta(\xi - a/q)\eta_N(\xi - a/q). \tag{2-11}$$

We show that for every $p \in (1, \infty)$ and $\rho > 0$, there is a constant $C > 0$ such that for any $N \geq 8^{\max\{p,p'\}/\rho}$ and for any set Λ as in (2-10) and for every $f \in \ell^p(\mathbb{Z}^d)$ we have

$$\|\mathcal{F}^{-1}(\Delta_N^\Lambda \hat{f})\|_{\ell^p} \leq C(\boldsymbol{A}_p + 1)\|f\|_{\ell^p}. \tag{2-12}$$

For $N \leq 8^{\max\{p,p'\}/\rho}$ the bound in (2-12) is obvious, since we allow the constant $C > 0$ to depend on p and ρ. Moreover, by the duality and interpolation, it suffices to prove (2-12) for $p = 2r$, where $r \in \mathbb{N}$. If $\Lambda = \Lambda_i(\mathbb{P}_N)$, as in Lemma 2.4, for some $1 \leq i \leq C_\rho \log N$, then we see that $\mathscr{U}_N^\Lambda = \mathscr{U}_N^i$ and $\Delta_N^\Lambda = \Delta_N^i$, and consequently (2-12) implies (2-9) as desired.

The function $\Theta(\xi)\eta_N(\xi)$ is regarded as a periodic function on $\mathbb{T}^d$; thus

$$\Delta_N^\Lambda(\xi) = \sum_{a/q \in \mathscr{U}_N^\Lambda} \Theta(\xi - a/q)\eta_N(\xi - a/q) = \sum_{b \in \mathbb{N}_{Q_0}^d} \sum_{a/w \in \mathscr{W}_N} \Theta(\xi - b/Q_0 - a/w)\eta_N(\xi - b/Q_0 - a/w),$$

where we have used the fact that if $(q_1, q_2) = 1$ then for every $a \in \mathbb{Z}^d$, there are unique $a_1, a_2 \in \mathbb{Z}^d$, such that $a_1/q_1, a_2/q_2 \in [0, 1)^d$ and

$$\frac{a}{q_1 q_2} = \frac{a_1}{q_1} + \frac{a_2}{q_2} \pmod{\mathbb{Z}^d}. \tag{2-13}$$

Since Λ has property $\mathcal{O}$, according to Definition 2.3 there is an integer $1 \leq k \leq 2/\rho + 1$ and there are sets $S_1, \ldots, S_k$ such that for any $j \in \mathbb{N}_k$ we have $S_j = \{q_{j,1}, \ldots, q_{j,\beta_j}\}$ for some $\beta_j \in \mathbb{N}$.

Now for each $j \in \mathbb{N}_k$ we introduce

$$\mathcal{U}_{\{j\}} = \{a_{j,s}/q_{j,s} \in \mathbb{T}^d \cap \mathbb{Q}^d : s \in \mathbb{N}_{\beta_j} \text{ and } a_{j,s} \in A_{q_{j,s}}\}$$

and for any $M = \{j_1, \ldots, j_m\} \subseteq \mathbb{N}_k$ let

$$\mathcal{U}_M = \{u_{j_1} + \cdots + u_{j_m} \in \mathbb{T}^d \cap \mathbb{Q}^d : u_{j_l} \in \mathcal{U}_{\{j_l\}} \text{ for any } l \in \mathbb{N}_m\}.$$

For any sequence $\sigma = (s_{j_1}, \ldots, s_{j_m}) \in \mathbb{N}_{\beta_{j_1}} \times \cdots \times \mathbb{N}_{\beta_{j_m}}$ determined by the set M, let us define

$$\mathcal{V}_M^\sigma = \{a_{j_1,s_{j_1}}/q_{j_1,s_{j_1}} + \cdots + a_{j_m,s_{j_m}}/q_{j_m,s_{j_m}} \in \mathbb{T}^d \cap \mathbb{Q}^d : a_{j_l,s_{j_l}} \in A_{q_{j_l,s_{j_l}}} \text{ for any } l \in \mathbb{N}_m\}.$$

Note that $\mathcal{V}_M^\sigma$ is a subset of $\mathcal{U}_M$ with fixed denominators $q_{j_1,s_{j_1}}, \ldots, q_{j_m,s_{j_m}}$. If $M = \varnothing$ then we have $\mathcal{U}_M = \mathcal{V}_M = \{0\}$. Let

$$\chi(\xi) = \mathbb{1}_\Lambda(\xi) \quad \text{and} \quad \Omega_N(\xi) = \Theta(\xi)\eta_N(\xi).$$

Then again by (2-13) we obtain

$$\begin{aligned} \Delta_N^\Lambda(\xi) &= \sum_{a/w \in \mathscr{W}_N} \sum_{b \in \mathbb{N}_{Q_0}^d} \Theta(\xi - b/Q_0 - a/w)\eta_N(\xi - b/Q_0 - a/w) \\ &= \sum_{s_1 \in \mathbb{N}_{\beta_1}} \sum_{a_{1,s_1} \in A_{q_{1,s_1}}} \cdots \sum_{s_k \in \mathbb{N}_{\beta_k}} \sum_{a_{k,s_k} \in A_{q_{k,s_k}}} m_{a_{1,s_1}/q_{1,s_1} + \cdots + a_{k,s_k}/q_{k,s_k}}(\xi) = \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} m_u(\xi), \end{aligned} \tag{2-14}$$

where

$$m_u(\xi) = m_{a_{1,s_1}/q_{1,s_1}+\cdots+a_{k,s_k}/q_{k,s_k}}(\xi) = \chi(q_{1,s_1}\cdot\cdots\cdot q_{k,s_k}) \sum_{b\in\mathbb{N}^d_{Q_0}} \Omega_N\Bigl(\xi - b/Q_0 - \sum_{j=1}^{k} a_{j,s_j}/q_{j,s_j}\Bigr) \tag{2-15}$$

for $u = a_{1,s_1}/q_{1,s_1} + \cdots + a_{k,s_k}/q_{k,s_k}$.

From now on we will write, for every $u \in \mathcal{U}_{\mathbb{N}_k}$,

$$f_u(x) = \mathcal{F}^{-1}(m_u\hat{f})(x) \tag{2-16}$$

with $f \in \ell^{2r}(\mathbb{Z}^d)$ and $r \in \mathbb{N}$. Therefore,

$$\mathcal{F}^{-1}(\Delta_N^{\Lambda}\hat{f})(x) = \sum_{u\in\mathcal{U}_{\mathbb{N}_k}} f_u(x) \tag{2-17}$$

and the proof of inequality (2-12) will follow from the theorem below.

Theorem 2.5. *Suppose that $\rho > 0$ and $r \in \mathbb{N}$ are given. Then there is a constant $C_{\rho,r} > 0$ such that for any $N > 8^{2r/\rho}$ and for any set Λ as in* (2-10) *and for every $f \in \ell^{2r}(\mathbb{Z}^d)$ we have*

$$\Bigl\| \sum_{u\in\mathcal{U}_{\mathbb{N}_k}} f_u \Bigr\|_{\ell^{2r}} \le C_{\rho,r}\|f\|_{\ell^{2r}}. \tag{2-18}$$

Moreover, the integer $k \in \mathbb{N}_D$, the set $\mathcal{U}_{\mathbb{N}_k}$ and consequently the sets $S_1, \dots, S_k$ are determined by the set Λ as it was described above.

The estimate (2-18) will follow from Theorem 2.6 and Proposition 2.7 formulated below. Let us introduce a suitable square function which will be useful in bounding (2-18). For any $M \subseteq \mathbb{N}_k$ and $L = \{j_1, \dots, j_l\} \subseteq M$ and any sequence $\sigma = (s_{j_1}, \dots, s_{j_l}) \in \mathbb{N}_{\beta_{j_1}} \times \cdots \times \mathbb{N}_{\beta_{j_l}}$ determined by the set L, let us define the square function $\mathcal{S}^{\sigma}_{L,M}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})$ associated with the sequence $(f_u : u \in \mathcal{U}_{\mathbb{N}_k})$ of complex-valued functions as in (2-16) by setting

$$\mathcal{S}^{\sigma}_{L,M}(f_u(x) : u \in \mathcal{U}_{\mathbb{N}_k}) = \Bigl(\sum_{w\in\mathcal{U}_{M^c}} \Bigl| \sum_{u\in\mathcal{U}_{M\setminus L}} \sum_{v\in\mathcal{V}^{\sigma}_L} f_{w+u+v}(x) \Bigr|^2 \Bigr)^{1/2}, \tag{2-19}$$

where $M^c = \mathbb{N}_k \setminus M$. For some $s_{j_i} \in \{s_{j_1}, \dots, s_{j_l}\}$ we will write

$$\|\mathcal{S}^{\sigma}_{L,M}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^2_{s_{j_i}}} = \Bigl(\sum_{s_{j_i}\in\mathbb{N}_{\beta_{j_i}}} \bigl| \mathcal{S}^{(s_{j_1},\dots,s_{j_l})}_{L,M}(f_u(x) : u \in \mathcal{U}_{\mathbb{N}_k}) \bigr|^2 \Bigr)^{1/2}, \tag{2-20}$$

which defines some function which depends on $x \in \mathbb{Z}^d$ and on each $s_{j_n} \in \{s_{j_1}, \dots, s_{j_l}\} \setminus \{s_{j_i}\}$.

For the proof of (2-18) we have to exploit the fact that the Fourier transform of f_u is defined as a sum of disjointly supported smooth cut-off functions. Then appropriate subsums of $\sum_{u\in\mathcal{U}_{\mathbb{N}_k}} f_u$ should be strongly orthogonal to each other.

Theorem 2.5 will be proved as a consequence of Theorem 2.6 and Proposition 2.7 below.

Theorem 2.6. *Suppose that $\rho > 0$ and $r \in \mathbb{N}$ are given. Then there is a constant $C_{\rho,r} > 0$ such that for any $N > 8^{2r/\rho}$ and for any set Λ as in (2-10) and for every $f \in \ell^{2r}(\mathbb{Z}^d)$ we have*

$$\Big\| \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} f_u \Big\|_{\ell^{2r}}^{2r} \le C_{\rho,r} \sum_{\substack{M \subseteq \mathbb{N}_k \\ M = \{j_1, \dots, j_m\}}} \sum_{\sigma \in \mathbb{N}_{\beta_{j_1}} \times \cdots \times \mathbb{N}_{\beta_{j_m}}} \| \mathcal{S}^{\sigma}_{M,M}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^{2r}}^{2r}. \tag{2-21}$$

Moreover, the integer $k \in \mathbb{N}_D$, the set $\mathcal{U}_{\mathbb{N}_k}$ and consequently the sets $S_1, \dots, S_k$ are determined by the set Λ as it was described above the formulation of Theorem 2.5.

Proof. Under the assumptions of Theorem 2.5, there is a constant $C_r > 0$ such that for any $M \subseteq \mathbb{N}_k$ and $L = \{j_1, \dots, j_l\} \subseteq M$ and $j_n \in M \setminus L$ and for any $\sigma = (s_{j_1}, \dots, s_{j_l}) \in \mathbb{N}_{\beta_{j_1}} \times \cdots \times \mathbb{N}_{\beta_{j_l}}$ determined by the set L we have

$$\| \mathcal{S}^{\sigma}_{L,M}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^{2r}} \le C_r \big\| \| \mathcal{S}^{\sigma \oplus s_{j_n}}_{L \cup \{j_n\}, M}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^2_{s_{j_n}}} \big\|_{\ell^{2r}}, \tag{2-22}$$

where $\sigma \oplus s_{j_n} = (s_{j_1}, \dots, s_{j_l}, s_{j_n}) \in \mathbb{N}_{\beta_{j_1}} \times \cdots \times \mathbb{N}_{\beta_{j_l}} \times \mathbb{N}_{\beta_{j_n}}$ is the sequence determined by the set $L \cup \{s_{j_n}\}$. Moreover, the right-hand side in (2-22) can be controlled in the following way:

$$\begin{aligned} &\big\| \| \mathcal{S}^{\sigma \oplus s_{j_n}}_{L \cup \{j_n\}, M}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^2_{s_{j_n}}} \big\|_{\ell^{2r}}^{2r} \\ &\qquad \le C_r \sum_{s_{j_n} \in \mathbb{N}_{\beta_{j_n}}} \| \mathcal{S}^{\sigma \oplus s_{j_n}}_{L \cup \{j_n\}, M}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^{2r}}^{2r} + C_r \| \mathcal{S}^{\sigma}_{L, M \setminus \{j_n\}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^{2r}}^{2r}. \end{aligned} \tag{2-23}$$

The proofs of (2-22) and (2-23) can be found in [Mirek et al. 2015]. Therefore, (2-22) combined with (2-23) yields

$$\| \mathcal{S}^{\sigma}_{L,M}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^{2r}}^{2r} \le C_r \sum_{s_{j_n} \in \mathbb{N}_{\beta_{j_n}}} \| \mathcal{S}^{\sigma \oplus s_{j_n}}_{L \cup \{j_n\}, M}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^{2r}}^{2r} + C_r \| \mathcal{S}^{\sigma}_{L, M \setminus \{j_n\}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^{2r}}^{2r}. \tag{2-24}$$

Applying (2-24) recursively we obtain

$$\begin{aligned} \Big\| \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} f_u \Big\|_{\ell^{2r}}^{2r} &= \| \mathcal{S}_{\varnothing, \mathbb{N}_k}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^{2r}}^{2r} \\ &\lesssim_r \sum_{s_k \in \mathbb{N}_{\beta_{j_k}}} \| \mathcal{S}^{(s_k)}_{\{k\}, \mathbb{N}_k}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^{2r}}^{2r} + \| \mathcal{S}_{\varnothing, \mathbb{N}_{k-1}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^{2r}}^{2r} \\ &\lesssim_r \sum_{s_{k-1} \in \mathbb{N}_{\beta_{j_{k-1}}}} \sum_{s_k \in \mathbb{N}_{\beta_{j_k}}} \| \mathcal{S}^{(s_{k-1}, s_k)}_{\{k-1,k\}, \mathbb{N}_k}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^{2r}}^{2r} + \sum_{s_k \in \mathbb{N}_{\beta_{j_k}}} \| \mathcal{S}^{(s_k)}_{\{k\}, \mathbb{N}_k \setminus \{k-1\}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^{2r}}^{2r} \\ &\quad + \sum_{s_{k-1} \in \mathbb{N}_{\beta_{j_{k-1}}}} \| \mathcal{S}^{(s_{k-1})}_{\{k-1\}, \mathbb{N}_{k-1}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^{2r}}^{2r} + \| \mathcal{S}_{\varnothing, \mathbb{N}_{k-2}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \|_{\ell^{2r}}^{2r} \\ &\lesssim_r \cdots \lesssim_{\rho,r} \sum_{\substack{M \subseteq \mathbb{N}_k \\ M = \{j_1, \dots, j_m\}}} \sum_{\sigma \in \mathbb{N}_{\beta_{j_1}} \times \cdots \times \mathbb{N}_{\beta_{j_m}}} \sum_{x \in \mathbb{Z}^d} \bigg(\sum_{w \in \mathcal{U}_{M^c}} \Big| \sum_{v \in \mathcal{V}^{\sigma}_M} f_{w+v}(x) \Big|^2 \bigg)^r. \end{aligned} \tag{2-25}$$

The proof of (2-21) is completed. □

Concluding remarks and the proof of Theorem 2.5. Now Theorem 2.6 reduces the proof of inequality (2-18) to showing the estimate

$$\sum_{\substack{M\subseteq\mathbb{N}_k\\ M=\{j_1,\dots,j_m\}}}\ \sum_{\sigma\in\mathbb{N}_{\beta_{j_1}}\times\cdots\times\mathbb{N}_{\beta_{j_m}}} \|\mathcal{S}^{\sigma}_{M,M}(f_u:u\in\mathcal{U}_{\mathbb{N}_k})\|^{2r}_{\ell^{2r}}\lesssim_r \|f\|^{2r}_{\ell^{2r}} \tag{2-26}$$

for any $f\in\ell^{2r}(\mathbb{Z}^d)$ which is a characteristic function of a finite set in $\mathbb{Z}^d$. Firstly, we prove the following.

Proposition 2.7. *Under the assumptions of Theorem 2.5, there exists a constant $C_{\rho,r}>0$ such that for any $M=\{j_1,\dots,j_m\}\subseteq\mathbb{N}_k$, any $\sigma=(s_{j_1},\dots,s_{j_m})\in\mathbb{N}_{\beta_{j_1}}\times\cdots\times\mathbb{N}_{\beta_{j_m}}$ determined by the set M and $f\in\ell^{2r}(\mathbb{Z}^d)$ we have*

$$\|\mathcal{S}^{\sigma}_{M,M}(f_u:u\in\mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}\le C_{\rho,r}\,\boldsymbol{A}_r\bigg\|\mathcal{S}^{\sigma}_{M,M}\bigg(\mathcal{F}^{-1}\bigg(\sum_{b\in\mathbb{N}_{Q_0}}\eta_N(\xi-b/Q_0-u)\hat{f}(\xi)\bigg):u\in\mathcal{U}_{\mathbb{N}_k}\bigg)\bigg\|_{\ell^{2r}}. \tag{2-27}$$

Proof. We assume, without of loss of generality, that $N\in\mathbb{N}$ is large. Let $B_h=q_{j_1,s_{j_1}}\cdot\dots\cdot q_{j_m,s_{j_m}}\cdot Q_0\le e^{N^\rho}$ and observe that according to the notation from (2-16) and (2-14), we have

$$\begin{aligned}
&\|\mathcal{S}^{\sigma}_{M,M}(f_u:u\in\mathcal{U}_{\mathbb{N}_k})\|^{2r}_{\ell^{2r}}\\
&\qquad=\sum_{x\in\mathbb{Z}^d}\bigg(\sum_{w\in\mathcal{U}_{M^c}}\bigg|\sum_{v\in\mathcal{V}^{\sigma}_M}f_{w+v}(x)\bigg|^2\bigg)^r\\
&\qquad\le\sum_{x\in\mathbb{Z}^d}\bigg(\sum_{w\in\mathcal{U}_{M^c}}\bigg|\mathcal{F}^{-1}\bigg(\sum_{v\in\mathcal{V}^{\sigma}_M}\sum_{b\in\mathbb{N}_{Q_0}}\Theta(\xi-b/Q_0-v-w)\,\eta_N(\xi-b/Q_0-v-w)\hat{f}(\xi)\bigg)(x)\bigg|^2\bigg)^r\\
&\qquad=\sum_{n\in\mathbb{N}^d_{B_h}}\sum_{x\in\mathbb{Z}^d}\bigg(\sum_{w\in\mathcal{U}_{M^c}}|\mathcal{F}^{-1}(\Theta\eta_N G(\xi;n,w))(B_hx+n)|^2\bigg)^r,
\end{aligned}\tag{2-28}$$

where

$$G(\xi;n,w)=\sum_{v\in\mathcal{V}^{\sigma}_M}\sum_{b\in\mathbb{N}^d_{Q_0}}\hat{f}(\xi+b/Q_0+v+w)e^{-2\pi i(b/Q_0+v)\cdot n}. \tag{2-29}$$

We know that for each $0<p<\infty$ there is a constant $C_p>0$ such that for any $d\in\mathbb{N}$ and $\lambda_1,\dots,\lambda_d\in\mathbb{C}^d$ we have

$$\bigg(\int_{\mathbb{C}^d}|\lambda_1z_1+\dots+\lambda_dz_d|^pe^{-\pi|z|^2}\,\mathrm{d}z\bigg)^{1/p}=C_p(|\lambda_1|^2+\dots+|\lambda_d|^2)^{1/2}. \tag{2-30}$$

By Proposition 4.5 from [Mirek et al. 2015], with the sequence of multipliers $\Theta_N=\Theta$ for all $N\in\mathbb{N}$ and Θ as in (2-2), we have

$$\big\|\mathcal{F}^{-1}(\Theta\eta_NG(\xi;n,w))(B_hx+n)\big\|_{\ell^{2r}(x)}\le C_{\rho,r}\boldsymbol{A}_{2r}\big\|\mathcal{F}^{-1}(\eta_NG(\xi;n,w))(B_hx+n)\big\|_{\ell^{2r}(x)} \tag{2-31}$$

since $\inf_{\gamma\in\Gamma}\varepsilon_\gamma^{-1}\ge e^{N^{2\rho}}\ge 2e^{(d+1)N^\rho}\ge B_h$ for sufficiently large $N\in\mathbb{N}$.

Therefore, combining (2-31) with (2-30), we obtain

$$\begin{aligned}
\sum_{n\in\mathbb{N}^d_{B_h}}\sum_{x\in\mathbb{Z}^d}\Big(\sum_{w\in\mathcal{U}_{M^c}}\big|\mathcal{F}^{-1}(\Theta\eta_N G(\xi;n,w))(B_hx+n)\big|^2\Big)^r
&= C_{2r}^{2r}\int_{\mathbb{C}^d}\sum_{n\in\mathbb{N}^d_{B_h}}\sum_{x\in\mathbb{Z}^d}\Big|\mathcal{F}^{-1}\Big(\Theta\eta_N\Big(\sum_{w\in\mathcal{U}_{M^c}} z_w G(\xi;n,w)\Big)\Big)(B_hx+n)\Big|^{2r}e^{-\pi|z|^2}\,\mathrm{d}z\\
&\lesssim_r \int_{\mathbb{C}^d}\sum_{n\in\mathbb{N}^d_{B_h}}\sum_{x\in\mathbb{Z}^d}\Big|\mathcal{F}^{-1}\Big(\sum_{w\in\mathcal{U}_{M^c}} z_w\eta_N G(\xi;n,w)\Big)(B_hx+n)\Big|^{2r}e^{-\pi|z|^2}\,\mathrm{d}z\\
&\lesssim_r \sum_{n\in\mathbb{N}^d_{B_h}}\sum_{x\in\mathbb{Z}^d}\Big(\sum_{w\in\mathcal{U}_{M^c}}\big|\mathcal{F}^{-1}(\eta_N G(\xi;n,w))(B_hx+n)\big|^2\Big)^r\\
&\lesssim_r \sum_{x\in\mathbb{Z}^d}\Big(\sum_{w\in\mathcal{U}_{M^c}}\Big|\mathcal{F}^{-1}\Big(\sum_{v\in\mathcal{V}^\sigma_M}\sum_{b\in\mathbb{N}_{Q_0}}\eta_N(\xi-b/Q_0-v-w)\hat{f}(\xi)\Big)(x)\Big|^2\Big)^r.
\end{aligned}\tag{2-32}$$

This completes the proof of Proposition 2.7. □

Now we are able to finish the proof of Theorem 2.5.

Proof of Theorem 2.5. It remains to show that there exists a constant $C_{\rho,r}>0$ such that for any $M=\{j_1,\dots,j_m\}\subseteq\mathbb{N}_k$ any $\sigma=(s_{j_1},\dots,s_{j_m})\in\mathbb{N}_{\beta_{j_1}}\times\cdots\times\mathbb{N}_{\beta_{j_m}}$ determined by the set M and $f\in\ell^{2r}(\mathbb{Z}^d)$ we have

$$\sum_{\sigma\in\mathbb{N}_{\beta_{j_1}}\times\cdots\times\mathbb{N}_{\beta_{j_m}}}\Big\|\mathcal{S}^\sigma_{M,M}\Big(\mathcal{F}^{-1}\Big(\sum_{b\in\mathbb{N}_{Q_0}}\eta_N(\xi-b/Q_0-u)\hat{f}(\xi)\Big):u\in\mathcal{U}_{\mathbb{N}_k}\Big)\Big\|_{\ell^{2r}}^{2r}\le C_{\rho,r}^{2r}\|f\|_{\ell^{2r}}.\tag{2-33}$$

Since there are 2^k possible choices of sets $M\subseteq\mathbb{N}_k$ and $k\in\mathbb{N}_D$, (2-26) will follow and the proof of Theorem 2.5 will be completed. If $r=1$ then Plancherel's theorem does the job since the functions $\eta_N(\xi-b/Q_0-v-w)$ are disjointly supported for all $b/Q_0\in\mathbb{N}_{Q_0}$, $w\in\mathcal{U}_{M^c}$, $v\in\mathcal{V}^\sigma_M$ and $\sigma=(s_{j_1},\dots,s_{j_m})\in\mathbb{N}_{\beta_{j_1}}\times\cdots\times\mathbb{N}_{\beta_{j_m}}$. For general $r\ge 2$, since $\|f\|_{\ell^{2r}}^{2r}=\|f\|_{\ell^2}^2$ because we have assumed that f is a characteristic function of a finite set in $\mathbb{Z}^d$, it suffices to prove for any $x\in\mathbb{Z}^d$ that

$$\sum_{w\in\mathcal{U}_{M^c}}\Big|\mathcal{F}^{-1}\Big(\sum_{v\in\mathcal{V}^\sigma_M}\sum_{b\in\mathbb{N}_{Q_0}}\eta_N(\xi-b/Q_0-v-w)\hat{f}(\xi)\Big)(x)\Big|^2\le C_{\rho,r}.\tag{2-34}$$

In fact, since $\|f\|_{\ell^\infty}=1$, it is enough to show

$$\Big\|\mathcal{F}^{-1}\Big(\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)\sum_{v\in\mathcal{V}^\sigma_M}\sum_{b\in\mathbb{N}_{Q_0}}\eta_N(\xi-b/Q_0-v-w)\Big)\Big\|_{\ell^1}\le C_{\rho,r}\tag{2-35}$$

for any sequence of complex numbers $(\alpha(w):w\in\mathcal{U}_{M^c})$ such that

$$\sum_{w\in\mathcal{U}_{M^c}}|\alpha(w)|^2=1.\tag{2-36}$$

Computing the Fourier transform we obtain

$$\mathcal{F}^{-1}\bigg(\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)\sum_{v\in\mathcal{V}_M^\sigma}\sum_{b\in\mathbb{N}_{Q_0}}\eta_N(\xi-b/Q_0-v-w)\bigg)(x)$$
$$=\bigg(\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)e^{-2\pi i x\cdot w}\bigg)\cdot\det(\mathcal{E}_N)\mathcal{F}^{-1}\eta(\mathcal{E}_N x)\cdot\bigg(\sum_{v\in\mathcal{V}_M^\sigma}\sum_{b\in\mathbb{N}_{Q_0}}e^{-2\pi i x\cdot(b/Q_0+v)}\bigg). \tag{2-37}$$

The function

$$\sum_{v\in\mathcal{V}_M^\sigma}\sum_{b\in\mathbb{N}_{Q_0}}e^{-2\pi i x\cdot(b/Q_0+v)} \tag{2-38}$$

can be written as a sum of 2^m functions

$$\sum_{b\in\mathbb{N}_Q}e^{-2\pi i x\cdot(b/Q)}=\begin{cases}Q^d & \text{if } x\equiv 0 \pmod{Q},\\ 0 & \text{otherwise,}\end{cases} \tag{2-39}$$

where possible values of Q are products of Q_0 and $p_{j_i,s_{j_i}}^{\gamma_i}$ or $p_{j_i,s_{j_i}}^{\gamma_i-1}$ for $i\in\mathbb{N}_m$. Therefore, the proof of (2-35) will be completed if we show that

$$\bigg\|\bigg(\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)e^{-2\pi i Qx\cdot w}\bigg)\cdot Q^d\det(\mathcal{E}_N)\mathcal{F}^{-1}\eta(Q\mathcal{E}_N x)\bigg\|_{\ell^1(x)}\le C_{\rho,r} \tag{2-40}$$

for any integer $Q\le e^{N^\rho}$ such that $(Q,q_{j,s})=1$, for all $j\in M^c$ and $s\in\mathbb{N}_{\beta_j}$.

Recall that, according to Remark 2.1, in our case $\eta=\phi*\psi$ for some smooth functions ϕ, ψ supported in $(-1/(8d),1/(8d))^d$. Therefore, by the Cauchy–Schwarz inequality we only need to prove that

$$Q^{d/2}\det(\mathcal{E}_N)^{1/2}\|\mathcal{F}^{-1}\phi(Q\mathcal{E}_N x)\|_{\ell^2(x)}\le C_{\rho,r} \tag{2-41}$$

and

$$Q^{d/2}\det(\mathcal{E}_N)^{1/2}\bigg\|\bigg(\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)e^{-2\pi i Qx\cdot w}\bigg)\cdot\mathcal{F}^{-1}\psi(Q\mathcal{E}_N x)\bigg\|_{\ell^2(x)}\le C_{\rho,r}. \tag{2-42}$$

Since $(Q,q_{j,s})=1$, for all $j\in M^c$ and $s\in\mathbb{N}_{\beta_j}$, we know $Qw\notin\mathbb{Z}^d$ for any $w\in\mathcal{U}_{M^c}$ and its denominator is bounded by N^D. We can assume, without of loss of generality, that $Qw\in[0,1)^d$ by the periodicity of $x\mapsto e^{-2\pi i x\cdot Qw}$. Inequality (2-41) easily follows from Plancherel's theorem. In order to prove (2-42), observe that by the change of variables one has

$$\bigg(\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)e^{-2\pi i x\cdot Qw}\bigg)\cdot\mathcal{F}^{-1}\psi(Q\mathcal{E}_N x)=Q^{-d}\det(\mathcal{E}_N)^{-1}\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)\mathcal{F}^{-1}\big(\psi(Q^{-1}\mathcal{E}_N^{-1}(\cdot-Qw))\big)(x).$$

Therefore, Plancherel's theorem and the last identity yield

$$Q^d\det(\mathcal{E}_N)\bigg\|\bigg(\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)e^{-2\pi i Qx\cdot w}\bigg)\cdot\mathcal{F}^{-1}\psi(Q\mathcal{E}_N x)\bigg\|_{\ell^2(x)}^2$$
$$=\sum_{w\in\mathcal{U}_{M^c}}|\alpha(w)|^2\int_{\mathbb{R}^d}\big|\psi(\xi-\mathcal{E}_N^{-1}w)\big|^2\,d\xi+\sum_{\substack{w_1,w_2\in\mathcal{U}_{M^c}\\ w_1\neq w_2}}\alpha(w_1)\overline{\alpha(w_2)}\int_{\mathbb{R}^d}\psi(\xi)\psi(\xi-\mathcal{E}_N^{-1}(w_1-w_2))\,d\xi. \tag{2-43}$$

The first sum on the right-hand side of (2-43) is bounded in view of (2-36). The second one vanishes since the function ψ is supported in $(-1/(8d), 1/(8d))^d$ and $|\mathcal{E}_N^{-1}(w_1 - w_2)|_\infty \geq e^{N^{2\rho}} N^{-2D} > 1$ for sufficiently large N. The proof of Theorem 2.5 is completed. $\square$

3. Proof of Theorem B

To prove inequality (1-19) in Theorem B, in view of the decomposition of the kernel K into dyadic pieces as in (1-6), it suffices to show that for every $p \in (1, \infty)$ there is a constant $C_p > 0$ such that for all $f \in \ell^p(\mathbb{Z}^d)$ we have

$$\Big\| \sum_{n \geq 0} T_n f \Big\|_{\ell^p} \leq C_p \|f\|_{\ell^p}, \tag{3-1}$$

where

$$T_n f(x) = \sum_{y \in \mathbb{Z}^k} f(x - \mathcal{Q}(y)) K_n(y) \tag{3-2}$$

with the kernel K_n as in (1-6) for each $n \in \mathbb{Z}$.

Exponential sums and $\ell^2(\mathbb{Z}^d)$ approximations. Recall that for $q \in \mathbb{N}$

$$A_q = \big\{ a \in \mathbb{N}_q^d : \gcd(q, (a_\gamma : \gamma \in \Gamma)) = 1 \big\}.$$

Now for $q \in \mathbb{N}$ and $a \in A_q$ we define the Gaussian sums

$$G(a/q) = q^{-k} \sum_{y \in \mathbb{N}_q^k} e^{2\pi i (a/q) \cdot \mathcal{Q}(y)}.$$

Let us observe that there exists $\delta > 0$ such that

$$|G(a/q)| \lesssim q^{-\delta}. \tag{3-3}$$

This follows from the multidimensional variant of Weyl's inequality; see [Stein and Wainger 1999, Proposition 3].

Let P be a polynomial in $\mathbb{R}^k$ of degree $d \in \mathbb{N}$ such that

$$P(x) = \sum_{\gamma \in \Gamma} \xi_\gamma x^\gamma.$$

Given $N \geq 1$, let Ω_N be a convex set in $\mathbb{R}^k$ such that

$$\Omega_N \subseteq \{ x \in \mathbb{R}^k : |x - x_0| \leq cN \}$$

for some $x_0 \in \mathbb{R}^k$ and $c > 0$. We define the Weyl sums

$$S_N = \sum_{n \in \Omega_N \cap \mathbb{Z}^k} e^{2\pi i P(n)} \varphi(n), \tag{3-4}$$

where $\varphi : \mathbb{R}^k \mapsto \mathbb{C}$ is a continuously differentiable function which for some $C > 0$ satisfies

$$|\varphi(x)| \leq C \quad \text{and} \quad |\nabla \varphi(x)| \leq C(1 + |x|)^{-1}. \tag{3-5}$$

In [Mirek et al. 2015] we proved Theorem 3.1, which is a refinement of the estimates for the multidimensional Weyl sums S_N, where the limitations $N^{\varepsilon} \le q \le N^{k-\varepsilon}$ from [Stein and Wainger 1999, Proposition 3] are replaced by the weaker restrictions $(\log N)^{\beta} \le q \le N^{k}(\log N)^{-\beta}$ for appropriate β. Namely:

Theorem 3.1. *Assume that there is a multi-index γ_0 such that $0 < |\gamma_0| \le d$ and*

$$\left|\xi_{\gamma_0} - \frac{a}{q}\right| \le \frac{1}{q^2}$$

for some integers a, q such that $0 \le a \le q$ and $(a, q) = 1$. Then for any $\alpha > 0$ there is $\beta_\alpha > 0$ so that, for any $\beta \ge \beta_\alpha$, if

$$(\log N)^{\beta} \le q \le N^{|\gamma_0|}(\log N)^{-\beta} \tag{3-6}$$

then there is a constant $C > 0$ such that

$$|S_N| \le C N^{k}(\log N)^{-\alpha}. \tag{3-7}$$

The implied constant C is independent of N.

Let $(m_n : n \ge 0)$ be a sequence of multipliers on $\mathbb{T}^d$, corresponding to the operators (3-2). Then for any finitely supported function $f : \mathbb{Z}^d \mapsto \mathbb{C}$ we see that

$$T_n f(x) = \mathcal{F}^{-1}(m_n \hat{f})(x),$$

where

$$m_n(\xi) = \sum_{y \in \mathbb{Z}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_n(y).$$

For $n \ge 0$ we set

$$\Phi_n(\xi) = \int_{\mathbb{R}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_n(y)\, \mathrm{d}y.$$

Using multidimensional version of van der Corput's lemma, see [Stein and Wainger 2001, Proposition 2.1], we obtain

$$|\Phi_n(\xi)| \lesssim \min\{1, |2^{nA}\xi|_\infty^{-1/d}\}. \tag{3-8}$$

Moreover, if $n \ge 1$ we have

$$|\Phi_n(\xi)| = \left|\Phi_n(\xi) - \int_{\mathbb{R}^k} K_n(y)\, \mathrm{d}y\right| \lesssim \min\{1, |2^{nA}\xi|_\infty\}. \tag{3-9}$$

The next proposition shows relations between m_n and Φ_n.

Proposition 3.2. *There is a constant $C > 0$ such that for every $n \in \mathbb{N}$ and for every $\xi \in \left[\frac{1}{2}, \frac{1}{2}\right)^d$ satisfying*

$$\left|\xi_\gamma - \frac{a_\gamma}{q}\right| \le L_1^{-|\gamma|} L_2$$

for all $\gamma \in \Gamma$, where $1 \le q \le L_3 \le 2^{n/2}$, $a \in A_q$, $L_1 \ge 2^n$ and $L_2 \ge 1$ we have

$$\left|m_n(\xi) - G(a/q)\Phi_n(\xi - a/q)\right| \le C\left(L_3 2^{-n} + L_2 L_3 2^{-n} \sum_{\gamma \in \Gamma} (2^n/L_1)^{|\gamma|}\right) \le C L_2 L_3 2^{-n}. \tag{3-10}$$

Proof. Let $\theta = \xi - a/q$. For any $r \in \mathbb{N}_q^k$, if $y \equiv r \pmod q$ then for each $\gamma \in \Gamma$

$$\xi_\gamma y^\gamma \equiv \theta_\gamma y^\gamma + (a_\gamma/q) r^\gamma \pmod 1;$$

thus

$$\xi \cdot \mathcal{Q}(y) \equiv \theta \cdot \mathcal{Q}(y) + (a/q) \cdot \mathcal{Q}(r) \pmod 1.$$

Therefore,

$$\sum_{y \in \mathbb{Z}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_n(y) = \sum_{r \in \mathbb{N}_q^k} e^{2\pi i (a/q) \cdot \mathcal{Q}(r)} \sum_{y \in \mathbb{Z}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy+r)} K_n(qy+r).$$

If $2^{n-2} \le |qy+r|, |qy| \le 2^n$ then by the mean value theorem we obtain

$$\left|\theta \cdot \mathcal{Q}(qy+r) - \theta \cdot \mathcal{Q}(qy)\right| \lesssim |r| \sum_{\gamma \in \Gamma} |\theta_\gamma| \cdot 2^{n(|\gamma|-1)} \lesssim q \sum_{\gamma \in \Gamma} L_1^{-|\gamma|} L_2 2^{n(|\gamma|-1)} \lesssim L_2 L_3 2^{-n} \sum_{\gamma \in \Gamma} (2^n/L_1)^{|\gamma|}$$

and

$$\left|K_n(qy+r) - K_n(qy)\right| \lesssim 2^{-n(k+1)} L_3.$$

Thus

$$\sum_{y \in \mathbb{Z}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_n(y) = G(a/q) \cdot q^k \sum_{y \in \mathbb{Z}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K_n(qy) + \mathcal{O}\bigg(L_3 2^{-n} + L_2 L_3 2^{-n} \sum_{\gamma \in \Gamma} (2^n/L_1)^{|\gamma|}\bigg).$$

Now one can replace the sum on the right-hand side by the integral. Indeed, again by the mean value theorem we obtain

$$\begin{aligned}
&\left|\sum_{y \in \mathbb{Z}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K_n(qy) - \int_{\mathbb{R}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qt)} K_n(qt)\,\mathrm{d}t\right| \\
&\qquad = \left|\sum_{y \in \mathbb{Z}^k} \int_{[0,1)^k} \big(e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K_n(qy) - e^{2\pi i \theta \cdot \mathcal{Q}(q(y+t))} K_n(q(y+t))\,\mathrm{d}t\big)\right| \\
&\qquad = \mathcal{O}\bigg(q^{-k} L_3 2^{-n} + q^{-k} L_2 L_3 2^{-n} \sum_{\gamma \in \Gamma} (2^n/L_1)^{|\gamma|}\bigg). \qquad \square
\end{aligned}$$

Discrete Littlewood–Paley theory. Fix $j, n \in \mathbb{Z}$ and $N \in \mathbb{N}$ and let $\mathcal{E}_N$ be a diagonal $d \times d$ matrix with positive entries $(\varepsilon_\gamma : \gamma \in \Gamma)$ such that $\varepsilon_\gamma \le e^{-N^{2\rho}}$ with $\rho > 0$ as in Section 2. Let us consider the multipliers

$$\Omega_N^{j,n}(\xi) = \sum_{a/q \in \mathscr{U}_N} \Phi_{j,n}(\xi - a/q)\,\eta_N(\xi - a/q) \tag{3-11}$$

with $\eta_N(\xi) = \eta(\mathcal{E}_N^{-1}\xi)$ and $\Phi_{j,n}(\xi) = \Phi(2^{nA + jI}\xi)$, where Φ is a Schwartz function such that $\Phi(0) = 0$.

If $\mathscr{U}_N = \{0\}$ then $\Omega_N^{j,n}(\xi)$ can be treated as a standard Littlewood–Paley projector. Now we formulate an abstract theorem which can be thought of as a discrete variant of Littlewood–Paley theory. Its proof will be based on Theorem 2.2. Here we obtain a square function estimate which will be used in the proof of inequality (3-1).

Theorem 3.3. *For every $p \in (1, \infty)$ there is a constant $C_p > 0$ such that for all $-\infty \le M_1 \le M_2 \le \infty$, $j \in \mathbb{Z}$ and $N \in \mathbb{N}$ and every $f \in \ell^p(\mathbb{Z}^d)$ we have*

$$\bigg\| \bigg(\sum_{M_1 \le n \le M_2} |\mathcal{F}^{-1}(\Omega_N^{j,n} \hat{f})|^2 \bigg)^{1/2} \bigg\|_{\ell^p} \le C_p \log N \|f\|_{\ell^p}. \tag{3-12}$$

Proof. By Khintchine's inequality, (3-12) is equivalent to

$$\bigg(\int_0^1 \bigg\| \sum_{M_1 \le n \le M_2} \varepsilon_n(t) \mathcal{F}^{-1}(\Omega_N^{j,n} \hat{f}) \bigg\|_{\ell^p}^p \,\mathrm{d}t \bigg)^{1/p} \lesssim \log N \|f\|_{\ell^p}. \tag{3-13}$$

Observe that the multiplier from (3-13) can be rewritten as

$$\sum_{M_1 \le n \le M_2} \varepsilon_n(t) \Omega_N^{j,n}(\xi) = \sum_{a/q \in \mathscr{U}_N} \sum_{M_1 \le n \le M_2} \mathfrak{m}_n(\xi - a/q)\, \eta_N(\xi - a/q)$$

with the functions

$$\mathfrak{m}_n(\xi) = \varepsilon_n(t)\, \Phi(2^{nA + jI} \xi).$$

We observe that

$$|\mathfrak{m}_n(\xi)| \lesssim \min\big\{ |2^{nA+jI}\xi|_\infty, |2^{nA+jI}\xi|_\infty^{-1} \big\}.$$

The first bound follows from the mean value theorem, since

$$|\Phi(2^{nA+jI}\xi)| = \big|\Phi(2^{nA+jI}\xi) - \Phi(0)\big| \lesssim |2^{nA+jI}\xi| \sup_{\xi \in \mathbb{R}^d} |\nabla \Phi(\xi)| \lesssim |2^{nA+jI}\xi|_\infty.$$

The second bound follows since Φ is a Schwartz function. Moreover, for every $p \in (1, \infty)$ there is $C_p > 0$ such that

$$\big\| \sup_{n \in \mathbb{Z}} |\mathcal{F}^{-1}(\mathfrak{m}_n \mathcal{F} f)| \big\|_{L^p} \le C_p \|f\|_{L^p}$$

for every $f \in L^p(\mathbb{R}^d)$. Therefore, by [Stein 1993], the multiplier

$$\sum_{M_1 \le n \le M_2} \mathfrak{m}_n(\xi)$$

corresponds to a continuous singular integral; thus it defines a bounded operator on $L^p(\mathbb{R}^d)$ for all $p \in (1, \infty)$ with the bound independent of $j \in \mathbb{Z}$ and $-\infty \le M_1 \le M_2 \le \infty$. Hence, Theorem 2.2 applies and the multiplier

$$\sum_{M_1 \le n \le M_2} \varepsilon_n(t) \Omega_N^{j,n}(\xi)$$

defines a bounded operator on $\ell^p(\mathbb{Z}^d)$ with the $\log N$ loss, and (3-13) is established. □

Remark 3.4. If the function Φ is a real-valued function then we have

$$\bigg\| \sum_{M_1 \le n \le M_2} \mathcal{F}^{-1}(\Omega_N^{j,n} \hat{f}_n) \bigg\|_{\ell^p} \le C_p \log N \bigg\| \bigg(\sum_{M_1 \le n \le M_2} |f_n|^2 \bigg)^{1/2} \bigg\|_{\ell^p}. \tag{3-14}$$

This is the dual version of inequality (3-12) for any sequence of functions $(f_n : M_1 \le n \le M_2)$ such that

$$\Big\| \Big(\sum_{M_1 \le n \le M_2} |f_n|^2 \Big)^{1/2} \Big\|_{\ell^p} < \infty.$$

We have gathered all necessary ingredients to prove inequality (3-1).

Proof of inequality (3-1). Let $\chi > 0$ and $l \in \mathbb{N}$ be the numbers whose precise values will be adjusted later. As in [Mirek et al. 2015], we will consider for every $n \in \mathbb{N}_0$ the multipliers

$$\Xi_n(\xi) = \sum_{a/q \in \mathscr{U}_{n^l}} \eta(2^{n(A-\chi I)}(\xi - a/q))^2 \tag{3-15}$$

with $\mathscr{U}_N$ as defined in Section 2. Theorem 2.2 yields, for every $p \in (1, \infty)$, that

$$\|\mathcal{F}^{-1}(\Xi_n \hat{f})\|_{\ell^p} \lesssim \log(n+2) \|f\|_{\ell^p}. \tag{3-16}$$

The implicit constant in (3-16) depends on $\rho > 0$ from Theorem 2.2. From now on we will assume that $l \in \mathbb{N}$ and $\rho > 0$ are related by the equation

$$10\rho l = 1. \tag{3-17}$$

Assume that $f : \mathbb{Z}^d \mapsto \mathbb{C}$ has finite support and $f \ge 0$. Observe that

$$\Big\| \sum_{n \ge 0} T_n f \Big\|_{\ell^p} \le \Big\| \sum_{n \ge 0} \mathcal{F}^{-1}(m_n \Xi_n \hat{f}) \Big\|_{\ell^p} + \Big\| \sum_{n \ge 0} \mathcal{F}^{-1}(m_n (1 - \Xi_n) \hat{f}) \Big\|_{\ell^p}. \tag{3-18}$$

Without of loss of generality we may assume that $p \ge 2$; the case $1 < p \le 2$ follows by the duality then.

***The estimate of the second norm in* (3-18).** It suffices to show that

$$\big\| \mathcal{F}^{-1}(m_n(1 - \Xi_n)\hat{f}) \big\|_{\ell^p} \lesssim (n+1)^{-2} \|f\|_{\ell^p}. \tag{3-19}$$

For this purpose we define for every $x \in \mathbb{Z}^d$ the Radon averages

$$M_N f(x) = N^{-k} \sum_{y \in \mathbb{N}_N^k} f(x - \mathcal{Q}(y)).$$

From [Mirek et al. 2015] follows that for every $p \in (1, \infty)$ there is a constant $C_p > 0$ such that for every $f \in \ell^p(\mathbb{Z}^d)$ we have

$$\big\| \sup_{N \in \mathbb{N}} |M_N f| \big\|_{\ell^p} \le C_p \|f\|_{\ell^p}. \tag{3-20}$$

Then for every $1 < p < \infty$, by (3-16) and (3-20) we obtain

$$\big\| \mathcal{F}^{-1}(m_n(1 - \Xi_n)\hat{f}) \big\|_{\ell^p} \le \| \sup_{N \in \mathbb{N}} M_N f \|_{\ell^p} + \big\| \sup_{N \in \mathbb{N}} M_N(|\mathcal{F}^{-1}(\Xi_n \hat{f})|) \big\|_{\ell^p} \lesssim \log(n+2) \|f\|_{\ell^p} \tag{3-21}$$

since we have a pointwise bound

$$|\mathcal{F}^{-1}(m_n \hat{f})(x)| = |T_n f(x)| \lesssim M_{2^n} f(x). \tag{3-22}$$

We show that it is possible to improve estimate (3-21) for $p = 2$. Indeed, by Theorem 3.1 we will show that for big enough $\alpha > 0$, which will be specified later, and for all $n \in \mathbb{N}_0$ we have

$$\left|m_n(\xi)(1 - \Xi_n(\xi))\right| \lesssim (n+1)^{-\alpha}. \tag{3-23}$$

By Dirichlet's principle, we have for every $\gamma \in \Gamma$

$$|\xi_\gamma - a_\gamma/q_\gamma| \le q_\gamma^{-1} n^\beta 2^{-n|\gamma|},$$

where $1 \le q_\gamma \le n^{-\beta} 2^{n|\gamma|}$. In order to apply Theorem 3.1 we must show that there exists some $\gamma \in \Gamma$ such that $n^\beta \le q_\gamma \le n^{-\beta} 2^{n|\gamma|}$. Suppose for a contradiction that for every $\gamma \in \Gamma$ we have $1 \le q_\gamma < n^\beta$; then for some $q \le \operatorname{lcm}(q_\gamma : \gamma \in \Gamma) \le n^{\beta d}$ we have

$$|\xi_\gamma - a'_\gamma/q| \le n^\beta 2^{-n|\gamma|},$$

where $\gcd(q, \gcd(a'_\gamma : \gamma \in \Gamma)) = 1$. Hence, taking $a' = (a'_\gamma : \gamma \in \Gamma)$ we have $a'/q \in \mathscr{U}_{n^l}$ provided that $\beta d < l$. On the other hand, if $1 - \Xi_n(\xi) \neq 0$ then for every $a'/q \in \mathscr{U}_{n^l}$ there exists $\gamma \in \Gamma$ such that

$$|\xi_\gamma - a'_\gamma/q| > (16d)^{-1} 2^{-n(|\gamma| - \chi)}.$$

Therefore

$$2^{\chi n} < 16 d n^\beta$$

but this is impossible when $n \in \mathbb{N}$ is large. Hence, there is $\gamma \in \Gamma$ such that $n^\beta \le q_\gamma \le n^{-\beta} 2^{n|\gamma|}$. Thus by Theorem 3.1,

$$|m_n(\xi)| \lesssim (n+1)^{-\alpha}$$

provided that $1 - \Xi_n(\xi) \neq 0$. This yields (3-23) and we obtain

$$\left\|\mathcal{F}^{-1}(m_n(1 - \Xi_n)\hat{f})\right\|_{\ell^2} \lesssim (1+n)^{-\alpha} \log(n+2) \|f\|_{\ell^2}. \tag{3-24}$$

Interpolating (3-24) with (3-21) we obtain

$$\left\|\mathcal{F}^{-1}(m_n(1 - \Xi_n)\hat{f})\right\|_{\ell^p} \lesssim (1+n)^{-c_p \alpha} \log(n+2) \|f\|_{\ell^p}. \tag{3-25}$$

for some $c_p > 0$. Choosing $\alpha > 0$ and $l \in \mathbb{N}$ appropriately large, one obtains (3-19).

***The estimate of the first norm in* (3-18).** Note that for any $\xi \in \mathbb{T}^d$ such that

$$|\xi_\gamma - a_\gamma/q| \le 2^{-n(|\gamma| - \chi)}$$

for every $\gamma \in \Gamma$ with $1 \le q \le e^{n^{1/10}}$, we have

$$m_n(\xi) = G(a/q)\Phi_n(\xi - a/q) + q^{-\delta} E_{2^n}(\xi), \tag{3-26}$$

where

$$|E_{2^n}(\xi)| \lesssim 2^{-n/2}. \tag{3-27}$$

Proposition 3.2, with $L_1 = 2^n$, $L_2 = 2^{\chi n}$ and $L_3 = e^{n^{1/10}}$, establishes (3-26) and (3-27), since for sufficiently large $n \in \mathbb{N}$ we have

$$q^\delta |E_{2^n}(\xi)| \lesssim q^\delta L_2 L_3 2^{-n} \lesssim (e^{-n((1-\chi)\log 2 - 2n^{-9/10})}) \lesssim 2^{-n/2}$$

provided $\chi > 0$ is sufficiently small. Now for every $j, n \in \mathbb{N}_0$ we introduce the multiplier

$$\Xi_n^j(\xi) = \sum_{a/q \in \mathscr{U}_{n^l}} \eta(2^{nA+jI}(\xi - a/q))^2 \tag{3-28}$$

and we note that

$$\begin{aligned}
&\Big\| \sum_{n \ge 0} \mathcal{F}^{-1}(m_n \Xi_n \hat{f}) \Big\|_{\ell^p} \\
&\quad \le \Big\| \sum_{n \ge 0} \mathcal{F}^{-1}\Big(\sum_{-\lfloor \chi n \rfloor \le j < n} m_n (\Xi_n^j - \Xi_n^{j+1}) \hat{f} \Big) \Big\|_{\ell^p} + \Big\| \sum_{n \ge 0} \mathcal{F}^{-1}\big(m_n (\Xi_n^{-\chi n} - \Xi_n^{-\lfloor \chi n \rfloor}) \hat{f} + m_n \Xi_n^n \hat{f} \big) \Big\|_{\ell^p} \\
&\quad = I_p^1 + I_p^2.
\end{aligned} \tag{3-29}$$

We will estimate I_p^1 and I_p^2 separately. For this purpose observe that by (3-26) and (3-27), for every $a/q \in \mathscr{U}_{n^l}$ we have

$$\begin{aligned}
|m_n(\xi)| &\lesssim q^{-\delta} |\Phi_n(\xi - a/q)| + q^{-\delta} |E_{2^n}(\xi)| \\
&\lesssim q^{-\delta} \big(\min\{1, |2^{nA}(\xi - a/q)|_\infty, |2^{nA}(\xi - a/q)|_\infty^{-1/d}\} + 2^{-n/2} \big),
\end{aligned} \tag{3-30}$$

where the last inequality follows from (3-8) and (3-9). Therefore by (3-30) we get

$$\big| m_n(\xi) \big(\eta(2^{nA - \chi n I}(\xi - a/q))^2 - \eta(2^{nA - \lfloor \chi n \rfloor I}(\xi - a/q)) \big)^2 \big| \lesssim q^{-\delta} (2^{-\chi n/d} + 2^{-n/2}) \tag{3-31}$$

since $\eta(2^{nA - \chi n I}(\xi - a/q)) \ge \eta(2^{nA - \lfloor \chi n \rfloor I}(\xi - a/q)$. Moreover, for any integer $-\chi n \le j < n$ we get

$$\big| m_n(\xi) \big(\eta(2^{nA + jI}(\xi - a/q))^2 - \eta(2^{nA + (j+1)I}(\xi - a/q))^2 \big) \big| \lesssim q^{-\delta} (2^{-|j|/d} + 2^{-n/2}). \tag{3-32}$$

Bounding I_p^2. It will suffice to show, for some $\varepsilon = \varepsilon_p > 0$, that

$$\big\| \mathcal{F}^{-1}\big(m_n (\Xi_n^{-\chi n} - \Xi_n^{-\lfloor \chi n \rfloor}) \hat{f} + m_n \Xi_n^n \hat{f} \big) \big\|_{\ell^p} \lesssim 2^{-\varepsilon n} \| f \|_{\ell^p}. \tag{3-33}$$

Observe that for any $1 < p < \infty$, by (3-22), (3-20) and (3-16) we have

$$\big\| \mathcal{F}^{-1}(m_n \Xi_n^n \hat{f}) \big\|_{\ell^p} \le \big\| \sup_{N \in \mathbb{N}} M_N(|\mathcal{F}^{-1}(\Xi_n^n \hat{f})|) \big\|_{\ell^p} \lesssim \big\| \mathcal{F}^{-1}(\Xi_n^n \hat{f}) \big\|_{\ell^p} \lesssim \log(n+2) \| f \|_{\ell^p} \tag{3-34}$$

and in a similar way we obtain

$$\big\| \mathcal{F}^{-1}\big(m_n (\Xi_n^{-\chi n} - \Xi_n^{-\lfloor \chi n \rfloor}) \hat{f} \big) \big\|_{\ell^p} \lesssim \log(n+2) \| f \|_{\ell^p}. \tag{3-35}$$

For $p = 2$, by Plancherel's theorem and (3-30) we obtain

$$\big\| \mathcal{F}^{-1}(m_n \Xi_n^n \hat{f}) \big\|_{\ell^2} = \bigg(\int_{\mathbb{T}^d} \sum_{a/q \in \mathscr{U}_{n^l}} |m_n(\xi)|^2 \eta(2^{nA+nI}(\xi - a/q))^4 |\hat{f}(\xi)|^2 \, \mathrm{d}\xi \bigg)^{1/2} \lesssim 2^{-n/(2d)} \| f \|_{\ell^2}. \tag{3-36}$$

By (3-31) we obtain

$$\begin{aligned}
&\big\| \mathcal{F}^{-1}\big(m_n (\Xi_n^{-\chi n} - \Xi_n^{-\lfloor \chi n \rfloor}) \hat{f} \big) \big\|_{\ell^2} \\
&\qquad = \bigg(\int_{\mathbb{T}^d} \sum_{a/q \in \mathscr{U}_{n^l}} |m_n(\xi)|^2 \big(\eta(2^{nA - \chi n I}(\xi - a/q))^2 - \eta(2^{nA - \lfloor \chi n \rfloor I}(\xi - a/q))^2 \big)^2 |\hat{f}(\xi)|^2 \, \mathrm{d}\xi \bigg)^{1/2} \\
&\qquad \lesssim 2^{-\chi n/(2d)} \| f \|_{\ell^2}.
\end{aligned} \tag{3-37}$$

Therefore, by interpolation (3-34) with (3-36) and (3-35) with (3-37) we obtain for every $p \in (1, \infty)$ that

$$\big\|\mathcal{F}^{-1}\big(m_n(\Xi_n^{-\chi n} - \Xi_n^{-\lfloor \chi n \rfloor})\hat{f} + m_n \Xi_n^n \hat{f}\big)\big\|_{\ell^p} \lesssim 2^{-\varepsilon n}\|f\|_{\ell^p},$$

which in turn implies (3-33) and $I_p^2 \lesssim \|f\|_{\ell^p}$.

Bounding I_p^1. Define for any $0 \le s < n$ the new multiplier

$$\Delta_{n,s}^j(\xi) = \sum_{a/q \in \mathscr{U}_{(s+1)^l} \setminus \mathscr{U}_{s^l}} \big(\eta(2^{nA+jI}(\xi - a/q))^2 - \eta(2^{nA+(j+1)I}(\xi - a/q))^2\big)\eta(2^{s(A-\chi I)}(\xi - a/q))^2$$

and we observe that by the definition (3-28) we have

$$\Xi_n^j(\xi) - \Xi_n^{j+1}(\xi) = \sum_{0 \le s < n} \Delta_{n,s}^j(\xi).$$

Moreover,

$$\eta(2^{nA+jI}\xi)^2 - \eta(2^{nA+(j+1)I}\xi)^2 = \big(\eta(2^{nA+jI}\xi)^2 - \eta(2^{nA+(j+1)I}\xi)^2\big)\cdot\big(\eta(2^{nA+(j-1)I}\xi) - \eta(2^{nA+(j+2)I}\xi)\big).$$

Thus we see

$$\Delta_{n,s}^j(\xi) = \Delta_{n,s}^{j,1}(\xi)\cdot\Delta_{n,s}^{j,2}(\xi),$$

where

$$\Delta_{n,s}^{j,1}(\xi) = \sum_{a/q \in \mathscr{U}_{(s+1)^l} \setminus \mathscr{U}_{s^l}} \big(\eta(2^{nA+(j-1)I}(\xi - a/q)) - \eta(2^{nA+(j+2)I}(\xi - a/q))\big)\eta(2^{s(A-\chi I)}(\xi - a/q))$$

and

$$\Delta_{n,s}^{j,2}(\xi) = \sum_{a/q \in \mathscr{U}_{(s+1)^l} \setminus \mathscr{U}_{s^l}} \big(\eta(2^{nA+jI}(\xi - a/q))^2 - \eta(2^{nA+(j+1)I}(\xi - a/q))^2\big)\eta(2^{s(A-\chi I)}(\xi - a/q)).$$

Moreover, $\Delta_{n,s}^{j,1}$ and $\Delta_{n,s}^{j,2}$ are the multipliers which satisfy the assumptions of Theorem 3.3. Therefore,

$$\begin{aligned} I_p^1 &= \Big\|\sum_{n \ge 0}\mathcal{F}^{-1}\Big(\sum_{-\chi n \le j < n}\sum_{0 \le s < n}\Delta_{n,s}^{j,1} m_n \Delta_{n,s}^{j,2}\hat{f}\Big)\Big\|_{\ell^p} \\ &\le \sum_{s \ge 0}\sum_{j \in \mathbb{Z}}\Big\|\sum_{n \ge \max\{j, -j/\chi, s\}}\mathcal{F}^{-1}(\Delta_{n,s}^{j,1} m_n \Delta_{n,s}^{j,2}\hat{f})\Big\|_{\ell^p} \\ &\lesssim \sum_{s \ge 0}\sum_{j \in \mathbb{Z}}\log s\,\Big\|\Big(\sum_{n \ge \max\{j, -j/\chi, s\}}\big|\mathcal{F}^{-1}(m_n\Delta_{n,s}^{j,2}\hat{f})\big|^2\Big)^{1/2}\Big\|_{\ell^p}. \end{aligned} \tag{3-38}$$

In the last step we used (3-14). The task now is to show that for some $\varepsilon = \varepsilon_p > 0$

$$\Big\|\Big(\sum_{n \ge \max\{j, -j/\chi, s\}}\big|\mathcal{F}^{-1}(m_n\Delta_{n,s}^{j,2}\hat{f})\big|^2\Big)^{1/2}\Big\|_{\ell^p} \lesssim s^{-2}2^{-\varepsilon j}\|f\|_{\ell^p}. \tag{3-39}$$

This in turn will imply $I_p^1 \lesssim \|f\|_{\ell^p}$ and the proof will be completed. We have assumed that $p \ge 2$; then for every $g \in \ell^r(\mathbb{Z}^d)$ such that $g \ge 0$ with $r = (p/2)' > 1$ we have by (3-22), the Cauchy–Schwarz inequality and (3-20) that

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}}\big|\mathcal{F}^{-1}(m_n\Delta_{n,s}^{j,2}\hat{f})(x)\big|^2 g(x) &\lesssim \sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}} M_{2^n}\big(\big|\mathcal{F}^{-1}(\Delta_{n,s}^{j,2}\hat{f})\big|\big)(x)^2 g(x)\\ &\le \sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}} M_{2^n}\big(\big|\mathcal{F}^{-1}(\Delta_{n,s}^{j,2}\hat{f})\big|^2\big)(x) g(x)\\ &= \sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}}\big|\mathcal{F}^{-1}(\Delta_{n,s}^{j,2}\hat{f})(x)\big|^2 M_{2^n}^* g(x)\\ &\lesssim \Big\|\Big(\sum_{n\in\mathbb{Z}}\big|\mathcal{F}^{-1}(\Delta_{n,s}^{j,2}\hat{f})\big|^2\Big)^{1/2}\Big\|_{\ell^p}^2 \big\|\sup_{N\in\mathbb{N}} M_N^* g\big\|_{\ell^r}\\ &\lesssim \Big\|\Big(\sum_{n\in\mathbb{Z}}\big|\mathcal{F}^{-1}(\Delta_{n,s}^{j,2}\hat{f})\big|^2\Big)^{1/2}\Big\|_{\ell^p}^2 \|g\|_{\ell^r}.\end{aligned}\tag{3-40}$$

Therefore, by Theorem 3.3 we have

$$\Big\|\Big(\sum_{n\in\mathbb{Z}}\big|\mathcal{F}^{-1}(m_n\Delta_{n,s}^{j,2}\hat{f})\big|^2\Big)^{1/2}\Big\|_{\ell^p} \lesssim \Big\|\Big(\sum_{n\in\mathbb{Z}}\big|\mathcal{F}^{-1}(\Delta_{n,s}^{j,2}\hat{f})\big|^2\Big)^{1/2}\Big\|_{\ell^p} \lesssim \log s\|f\|_{\ell^p}.\tag{3-41}$$

We refine the estimate in (3-41) for $p=2$. Indeed, define

$$\begin{aligned}\varrho_{n,j}(\xi) &= \big(\eta(2^{nA+jI}\xi)^2-\eta(2^{nA+(j+1)I}\xi)^2\big)\eta(2^{s(A-\chi I)}\xi),\\ \Psi_n(\xi) &= \min\big\{|2^{nA}\xi|_\infty, |2^{nA}\xi|_\infty^{-1/d}, 1\big\}.\end{aligned}$$

By Plancherel's theorem we have

$$\begin{aligned}&\Big\|\Big(\sum_{n\ge\max\{j,-j/\chi,s\}}\big|\mathcal{F}^{-1}(m_n\Delta_{n,s}^{j,2}\hat{f})\big|^2\Big)^{1/2}\Big\|_{\ell^2}\\ &\qquad= \Big(\int_{\mathbb{T}^d}\sum_{n\ge\max\{j,-j/\chi,s\}}\sum_{a/q\in\mathscr{U}_{(s+1)^l}\setminus\mathscr{U}_{s^l}}|m_n(\xi)|^2\varrho_{n,j}(\xi-a/q)^2|\hat{f}(\xi)|^2\,\mathrm{d}\xi\Big)^{1/2}\\ &\qquad\lesssim (s+1)^{-\delta l}2^{-|j|/(2d)}\|f\|_{\ell^2}.\end{aligned}\tag{3-42}$$

The last estimate is implied by (3-30). Namely, by (3-30) we may write

$$\begin{aligned}&\sum_{n\ge\max\{j,-j/\chi,s\}}\sum_{a/q\in\mathscr{U}_{(s+1)^l}\setminus\mathscr{U}_{s^l}}|m_n(\xi)|^2\varrho_{n,j}(\xi-a/q)^2\\ &\quad\lesssim \sum_{n\ge\max\{j,-j/\chi,s\}}\sum_{a/q\in\mathscr{U}_{(s+1)^l}\setminus\mathscr{U}_{s^l}} q^{-2\delta}(\Psi_n(\xi-a/q)+2^{-n/2})(2^{-|j|/d}+2^{-n/2})\eta(2^{s(A-\chi I)}(\xi-a/q))^2\\ &\quad\lesssim (s+1)^{-2\delta l}2^{-|j|/(2d)}.\end{aligned}\tag{3-43}$$

The last line follows, since we have used the lower bound for $q\ge s^l$ if $a/q\in\mathscr{U}_{(s+1)^l}\setminus\mathscr{U}_{s^l}$. Moreover,

$$\sum_{n\ge 0}(\Psi_n(\xi-a/q)+2^{-n/2})\lesssim 1 \quad\text{and}\quad \sum_{a/q\in\mathscr{U}_{(s+1)^l}\setminus\mathscr{U}_{s^l}}\eta(2^{s(A-\chi I)}(\xi-a/q))\lesssim 1$$

by the disjointness of the supports of the $\eta(2^{s(A-\chi I)}(\xi - a/q))$ whenever $a/q \in \mathscr{U}_{(s+1)^l} \setminus \mathscr{U}_{s^l}$. Since $l \in \mathbb{N}$ can be as large as we wish, interpolating (3-42) with (3-41) we obtain (3-39) and the proof of (3-1) and consequently Theorem A is completed. $\square$

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ON THE KATO PROBLEM AND EXTENSIONS FOR DEGENERATE ELLIPTIC OPERATORS

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We study the Kato problem for divergence form operators whose ellipticity may be degenerate. The study of the Kato conjecture for degenerate elliptic equations was begun by Cruz-Uribe and Rios (2008, 2012, 2015). In these papers the authors proved that given an operator $L_w = -w^{-1}\operatorname{div}(A\nabla)$, where w is in the Muckenhoupt class A_2 and A is a w-degenerate elliptic measure (that is, $A = wB$ with $B(x)$ an $n \times n$ bounded, complex-valued, uniformly elliptic matrix), then L_w satisfies the weighted estimate $\|\sqrt{L_w} f\|_{L^2(w)} \approx \|\nabla f\|_{L^2(w)}$. In the present paper we solve the L^2-Kato problem for a family of degenerate elliptic operators. We prove that under some additional conditions on the weight w, the following unweighted L^2-Kato estimates hold:

$$\|L_w^{1/2} f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)}.$$

This extends the celebrated solution to the Kato conjecture by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian, allowing the differential operator to have some degree of degeneracy in its ellipticity. For example, we consider the family of operators $L_\gamma = -|x|^\gamma \operatorname{div}(|x|^{-\gamma} B(x)\nabla)$, where B is any bounded, complex-valued, uniformly elliptic matrix. We prove that there exists $\varepsilon > 0$, depending only on dimension and the ellipticity constants, such that

$$\|L_\gamma^{1/2} f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)}, \quad -\varepsilon < \gamma < \frac{2n}{n+2}.$$

The case $\gamma = 0$ corresponds to the case of uniformly elliptic matrices. Hence, our result gives a range of γ's for which the classical Kato square root proved in Auscher et al. (2002) is an interior point.

Our main results are obtained as a consequence of a rich Calderón–Zygmund theory developed for certain operators naturally associated with L_w. These results, which are of independent interest, establish estimates on $L^p(w)$, and also on $L^p(v\,dw)$ with $v \in A_\infty(w)$, for the associated semigroup, its gradient, the functional calculus, the Riesz transform, and vertical square functions. As an application, we solve some unweighted L^2-Dirichlet, regularity and Neumann boundary value problems for degenerate elliptic operators.

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1. Introduction

We study the degenerate elliptic operators $L_w = -w^{-1} \operatorname{div} A\nabla$, where w is in the Muckenhoupt class A_2 and $A(x)$ is an $n \times n$ complex-valued matrix that satisfies the degenerate ellipticity condition

$$\lambda w(x)|\xi|^2 \leq \operatorname{Re}\langle A(x)\xi, \xi\rangle, \quad |\langle A(x)\xi, \eta\rangle| \leq \Lambda w(x)|\xi||\eta|, \quad \xi, \eta \in \mathbb{C}^n, \text{ a.e. } x \in \mathbb{R}^n.$$

Equivalently, $A(x) = w(x)B(x)$, where B is an $n \times n$ complex-valued matrix that satisfies the uniform ellipticity conditions

$$\lambda|\xi|^2 \leq \operatorname{Re}\langle B(x)\xi, \xi\rangle, \quad |\langle B(x)\xi, \eta\rangle| \leq \Lambda|\xi||\eta|, \quad \xi, \eta \in \mathbb{C}^n, \text{ a.e. } x \in \mathbb{R}^n.$$

Such operators were first studied (with A a real symmetric matrix) by Fabes, Kenig and Serapioni [Fabes et al. 1982]. When A is complex-valued and uniformly elliptic (i.e., $w \equiv 1$), a landmark result was the proof by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian [Auscher et al. 2002] of the Kato conjecture, which states that for all $f \in H^1$,

$$\|L^{1/2} f\|_2 \approx \|\nabla f\|_2.$$

The proof of this long-standing conjecture led naturally to the study of the operators associated with L: the semigroup e^{-tL}, its gradient $\sqrt{t}\nabla e^{-tL}$, the Riesz transform $\nabla L^{-1/2}$, the H^∞ functional calculus and square functions; for details and complete references, see [Auscher 2007]. These estimates are interesting in themselves; moreover, it is well known that L^p estimates for these operators yield regularity results for boundary value problems for L; for details, see the introduction to [Auscher and Tchamitchian 1998].

In [Cruz-Uribe and Rios 2015] (see also [Cruz-Uribe and Rios 2008; 2012; Auscher et al. 2015]), the first and third authors solved the Kato problem for degenerate elliptic operators: they showed that if $w \in A_2$ and A satisfies the degenerate ellipticity conditions, then for all $f \in H^1(w)$,

$$\|L_w^{1/2} f\|_{L^2(w)} \approx \|\nabla f\|_{L^2(w)}. \tag{1.1}$$

In this paper we consider the problem of determining those A_2 weights such that the classical Kato problem can be solved for L_w, that is, finding weights such that L_w satisfies the unweighted estimate

$$\|L_w^{1/2} f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)}$$

for f in a class of nice functions (a posteriori, by standard density arguments, the estimate can be extended to all $f \in H^1(\mathbb{R}^n)$). We solve this problem in two steps. The first is to prove weighted L^p estimates for some operators associated with L_w (the semigroup, its gradient, the Riesz transform, the functional calculus, and square functions). These results, which are of interest in their own right, are analogous to those obtained in the uniformly elliptic case. However, a significant technical obstruction is that given a weight $w \in A_2$, while it is the case that there exists $\varepsilon > 0$ such that $w \in A_{2-\varepsilon}$, it is easy to construct examples to show that ε may be arbitrarily small. Therefore, our bounds in the range $1 < p < 2$ need to take this into account.

The second step is to find conditions on the weight w so that these operators satisfy *unweighted* L^2 estimates. Both steps are carried out simultaneously, and the proofs are intertwined. Our approach is to apply the theory of off-diagonal estimates on balls developed by Auscher and the second author [Auscher and Martell 2006; 2007a; 2007b; 2008]. We will in fact prove weighted estimates on $L^p(v\,dw)$, where v satisfies Muckenhoupt and reverse Hölder conditions with respect to the measure $dw = w\,dx$; $L^p(w)$ estimates are then obtained by taking $v = 1$, and unweighted estimates by taking $v = w^{-1}$.

The unweighted L^2 estimates are delicate, since they require a careful estimate of the constants that appear. Nevertheless, we are able to give useful sufficient conditions; e.g., $w \in A_1 \cap \mathrm{RH}_{n/2+1}$. (For definitions of these classes, see Section 2 below.) For example, we have the following result that is a special case of one of our main results (cf. Theorem 11.11).

Theorem 1.2. *Let $L_w = -w^{-1} \operatorname{div} A\nabla$ be a degenerate elliptic operator as above. If $w \in A_1 \cap \mathrm{RH}_{n/2+1}$, then the Kato problem can be solved for L_w: for every $f \in H^1(\mathbb{R}^n)$,*

$$\|L_w^{1/2} f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)}.$$

The implicit constants depend only on the dimension, the ellipticity constants, and the A_1 and $\mathrm{RH}_{n/2+1}$ constants of w.

Furthermore, if we define $L_\gamma = -|x|^\gamma \operatorname{div}(|x|^{-\gamma} B(x)\nabla)$, where B is an $n \times n$ complex-valued matrix that satisfies the uniform ellipticity condition, then there exists $0 < \varepsilon < \frac{1}{2}$ small enough (depending only on the dimension and the ratio Λ/λ) such that

$$\|L_\gamma^{1/2} f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)}, \quad -\varepsilon < \gamma < \frac{2n}{n+2}.$$

Remark 1.3. In Theorem 1.2 the operator $L_w^{1/2}$ is a priori only defined on $H^1(w)$; however, this means that it is defined on $C_0^\infty(\mathbb{R}^n)$ and so by a standard density argument we can extend our results to all $f \in H^1(\mathbb{R}^n)$. Hereafter we will make this extension without further comment.

We emphasize that in Theorem 1.2, when $\gamma = 0$ we are back at the uniformly elliptic case, which is the celebrated solution to the Kato square root problem by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian [Auscher et al. 2002]. Here we are able to find a range of γ's for which the same estimates hold and the classical Kato square root problem (i.e., $\gamma = 0$) is an interior point in that range.

These unweighted L^2 estimates have important applications to boundary value problems for degenerate elliptic operators. Consider, for example, the following Dirichlet problem on $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times [0,\infty)$:

$$\begin{cases} \partial_t^2 u - L_w u = 0 & \text{on } \mathbb{R}^{n+1}_+, \\ u = f & \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases}$$

If $f \in L^2(\mathbb{R}^n)$, then $u(x,t) = e^{-tL_w^{1/2}} f(x)$ is a solution, and if L_w has a bounded H^∞ functional calculus on L^2, then $\sup_{t>0} \|u(\cdot\,,t)\|_2 \lesssim \|f\|_2$. Similar results hold for the corresponding Neumann and regularity problems.

Our proofs are unavoidably technical, and the results for each operator considered build upon what was proved previously for other operators. We have organized the material as follows. In Section 2 we gather some essential definitions and results about weights, degenerate elliptic operators, and off-diagonal estimates. Central to all of our subsequent work are Theorems 2.35 and 2.39, which were proved in [Auscher and Martell 2006].

In Sections 3, 4, and 5 we prove estimates for the semigroup e^{-tL_w}, $t > 0$, the H^∞ functional calculus (i.e., operators $\varphi(L_w)$ where $\varphi \in \mathcal{H}^\infty$), the vertical square function associated to the semigroup,

$$g_{L_w} f(x) = \left(\int_0^\infty \left|(tL_w)^{1/2} e^{-tL_w} f(x)\right|^2 \frac{dt}{t} \right)^{1/2},$$

and its discrete analog. Here and in subsequent sections we prove both $L^p(w)$ estimates and weighted $L^p(v\,dw)$ estimates. In many cases these results are proved simultaneously, with the unweighted results (i.e., in $L^p(w)$) following from the weighted results (i.e., in $L^p(v\,dw)$) by taking $v = 1$.

In Section 6 we prove the so-called reverse inequality, $\|L_w^{1/2}\|_{L^p(w)} \lesssim \|\nabla f\|_{L^p(w)}$, that generalizes the $L^2(w)$ estimate in (1.1). We note that while the equivalence in (1.1) follows at once from the reverse inequality for $p = 2$ by duality, the two inequalities behave differently when $p \neq 2$.

In Sections 7 and 8 we prove estimates for the gradient of the semigroup, $\sqrt{t}\nabla e^{-tL_w}$. The proof that there exists $q_+ > 2$ such that this operator satisfies $L^p(w)$ estimates for $2 < p < q_+$ is quite involved as it requires preliminary estimates for the Riesz transform and the Hodge projection. We note that, as opposed to the nondegenerate case, here we cannot use "global" embeddings, nor can we rescale. Also we cannot expect to obtain that the gradient of the semigroup maps globally $L^2(w)$ into $L^p(w)$ for $p \neq 2$. All these difficulties arise naturally from the lack of isotropy of the natural underlying measure $w(x)\,dx$ and make the typical arguments used in the uniformly elliptic case (see [Auscher 2007, Chapter 4]) unusable. We also note that in some sense our result is the best possible: even in the nondegenerate case it is known [Auscher 2007] that given any $p > 2$ there exists a matrix A and operator L such that gradient of the semigroup is not bounded on L^p.

In Section 9 we prove $L^p(w)$ estimates for the Riesz transform $\nabla L^{-1/2}$, and in Section 10 we prove $L^p(w)$ estimates for the square function associated to the gradient of the semigroup,

$$G_{L_w} f(x) = \left(\int_0^\infty |t^{1/2} \nabla e^{-tL_w} f(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

In Section 11 we prove unweighted L^2 inequalities for the operators we have considered in previous sections. These are a consequence of the weighted estimates and are obtained by taking $v = w^{-1}$. The main problem is determining conditions on w for these to hold. We essentially have two different kinds of estimates, one for operators that do not involve the gradient, and one for those that do. The latter are more delicate as they involve careful bounds for the parameter q_+ from Section 8 in terms of the weight w. We also show that we get unweighted L^p estimates for p very close to 2.

Finally, in Section 12 we describe in more detail the application of our results to L^2 boundary value problems for degenerate elliptic operators. The results in this section are the culmination of our work, as they depend on all the estimates derived in previous sections.

As we were completing this project, we learned that related results had been obtained independently by other authors. Le [2015] studied (among other things) the $L^p(w)$ theory for some of the operators considered here and proved estimates for values of p in the range $(2-\varepsilon, 2+\varepsilon)$. His proofs differ from ours in a number of details. Hofmann, Le and Morris [Hofmann et al. 2015] established some Carleson measure estimates and considered the Dirichlet problem for degenerate elliptic operators. Also, very recently we learned that Yang and Zhang [2017] proved Kato-type estimates in $L^p(w)$ for p in the range $(p_0, 2]$. Finally, we note that the paper [Chen et al. 2016] complements our work here as it considers the conical square functions associated to the operator L_w.

2. Preliminaries

Throughout, n will denote the dimension of the underlying space $\mathbb{R}^n$ and we will always assume $n \geq 2$. If we write $A \lesssim B$ we mean that there exists a constant C such that $A \leq CB$. We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. The constant C in these estimates may depend on the dimension n and other (fixed) parameters that should be clear from the context. All constants, explicit or implicit, may change at each appearance.

Given a ball B, let $r(B)$ denote the radius of B. Let λB denote the concentric ball with radius $r(\lambda B) = \lambda r(B)$.

Weights. By a weight w we mean a nonnegative, locally integrable function. For brevity, we will often write dw for $w\,dx$. We will use the following notation for averages: given a set E such that $0 < w(E) < \infty$,

$$\fint_E f\,dw = \frac{1}{w(E)} \int_E f\,dw,$$

or, if $0 < |E| < \infty$,

$$\fint_E f\,dx = \frac{1}{|E|} \int_E f\,dx.$$

We state some definitions and basic properties of Muckenhoupt weights. For further details, see [Duoandikoetxea 2001; García-Cuerva and Rubio de Francia 1985]. We say that $w \in A_p$, $1 < p < \infty$, if

$$[w]_{A_p} = \sup_Q \fint_Q w(x)\,dx \left(\fint_Q w(x)^{1-p'}\,dx \right)^{p-1} < \infty.$$

When $p = 1$, we say that $w \in A_1$ if

$$[w]_{A_1} = \sup_Q \fint_Q w(x)\,dx \operatorname*{ess\,sup}_{x\in Q} w(x)^{-1} < \infty.$$

We say that $w \in \mathrm{RH}_s$, $1 < s < \infty$, if

$$[w]_{\mathrm{RH}_s} = \sup_Q \left(\fint_Q w(x)\,dx\right)^{-1} \left(\fint_Q w(x)^s\,dx\right)^{1/s} < \infty,$$

and we say that $w \in \mathrm{RH}_\infty$ if

$$[w]_{\mathrm{RH}_\infty} = \sup_Q \left(\fint_Q w(x)\,dx\right)^{-1} \operatorname*{ess\,sup}_{x\in Q} w(x) < \infty.$$

Let

$$A_\infty = \bigcup_{1\le p<\infty} A_p = \bigcup_{1<s\le\infty} \mathrm{RH}_s.$$

Weights in the A_p and RH_s classes have a self-improving property: if $w \in A_p$, there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$, and similarly if $w \in \mathrm{RH}_s$, then $w \in \mathrm{RH}_{s+\delta}$ for some $\delta > 0$. Hereafter, given $w \in A_p$, let

$$r_w = \inf\{p : w \in A_p\}, \quad s_w = \sup\{q : w \in \mathrm{RH}_q\}.$$

An important property of A_p weights is that they are doubling: given $w \in A_p$, for all $\tau \ge 1$ and any ball B,

$$w(\tau B) \le [w]_{A_p} \tau^{pn} w(B).$$

In particular, hereafter let $D \le pn$ be the doubling order of w, that is, the smallest exponent such that this inequality holds.

As a consequence of this doubling property, we have that with the ordinary Euclidean distance $|\cdot|$, $(\mathbb{R}^n, dw, |\cdot|)$ is a space of homogeneous type. In this setting we can define the new weight classes $A_p(w)$ and $\mathrm{RH}_s(w)$ by replacing Lebesgue measure in the definitions above with dw; e.g., $v \in A_p(w)$ if

$$[v]_{A_p(w)} = \sup_Q \fint_Q v(x)\,dw \left(\fint_Q v(x)^{1-p'}\,dw\right)^{p-1} < \infty.$$

It follows at once from these definitions that there is a "duality" relationship between the weighted and unweighted A_p and RH_s conditions: $v = w^{-1} \in A_p(w)$ if and only if $w \in \mathrm{RH}_{p'}$ and $v = w^{-1} \in \mathrm{RH}_s(w)$ if and only if $w \in A_{s'}$.

Weighted Poincaré–Sobolev inequalities were proved in [Fabes et al. 1982].

Theorem 2.1. *Given $w \in A_p$, $p \ge 1$, let $p_w^* = pnr_w/(nr_w - p)$ if $p < nr_w$, and $p_w^* = \infty$ otherwise. Then for every $p \le q < p_w^*$, ball B and $f \in C_0^\infty(B)$,*

$$\left(\fint_B |f(x)|^q\,dw(x)\right)^{1/q} \le C r(B) \left(\fint_B |\nabla f(x)|^p\,dw\right)^{1/p}. \tag{2.2}$$

Moreover, if $f \in C^\infty(B)$, then

$$\left(\fint_B |f(x) - f_{B,w}|^q \, dw(x)\right)^{1/q} \le C r(B) \left(\fint_B |\nabla f(x)|^p \, dw\right)^{1/p}, \tag{2.3}$$

where $f_{B,w} = \fint_B f \, dw$.

Remark 2.4. In the special case when $w \in A_1$ and $1 < p < n$ we can also take $q = p_w^* = p^*$, i.e., the regular Sobolev exponent. See [Pérez 1999, Theorem 2.5.2].

Remark 2.5. If we let $q = np/(n-1) < p_w^*$, then we can get a sharp estimate for the constant C in (2.2) and (2.3): it is of the form $C(p,n)[w]_{A_p}^{\kappa}$, where $\kappa = (np-1)/(np(p-1))$. This follows from the sharp weighted estimates for the fractional integral operator due to Alberico, Cianchi and Sbordone [Alberico et al. 2009] and the standard pointwise estimates used to prove Poincaré–Sobolev inequalities; see [Fabes et al. 1982] for details.

Remark 2.6. By a standard density argument, once we know that (2.3) holds for smooth functions in B we can easily extend that estimate to any function $f \in L^q(w)$ with $\nabla f \in L^p(w)$. Details are left to the reader.

Degenerate elliptic operators. Given $w \in A_2$ and constants $0 < \lambda \le \Lambda < \infty$, let $\mathcal{E}_n(w, \lambda, \Lambda)$ denote the class of $n \times n$ matrices $A = (A_{ij}(x))_{i,j=1}^n$ of complex-valued, measurable functions satisfying the degenerate ellipticity condition

$$\lambda w(x)|\xi|^2 \le \operatorname{Re}\langle A\xi, \xi\rangle, \quad |\langle \mathcal{A}\xi, \eta\rangle| \le \Lambda w(x)|\xi||\eta|, \quad \xi, \eta \in \mathbb{C}^n. \tag{2.7}$$

Given $A \in \mathcal{E}_n(w, \lambda, \Lambda)$, we define the degenerate elliptic operator in divergence form

$$L_w = -w^{-1} \operatorname{div} A \nabla.$$

These operators were developed in [Cruz-Uribe and Rios 2008] and we refer the reader there for complete details. Here we sketch the key ideas.

Given a weight $w \in A_2$, the space $H^1(w)$ is the weighted Sobolev space that is the completion of C_c^∞ with respect to the norm

$$\|f\|_{H^1(w)} = \left(\int_{\mathbb{R}^n} \left(|f(x)|^2 + |\nabla f(x)|^2\right) dw\right)^{1/2}.$$

Note that the space defined above would usually be denoted by $H_0^1(w)$. The space $H^1(w)$ is defined as the set of distributions for which both f and $|\nabla f|$ belong to $L^2(w)$. However, since the underlying domain is $\mathbb{R}^n$, this definition implies that the "boundary" values vanish in the $L^2(w)$-sense, and both definitions agree [Miller 1982].

Given a matrix $A \in \mathcal{E}_n(w, \lambda, \Lambda)$, define $\mathfrak{a}(f, g)$ to be the sesquilinear form

$$\mathfrak{a}(f, g) = \int_{\mathbb{R}^n} A(x) \nabla f(x) \cdot \overline{\nabla g(x)} \, dx. \tag{2.8}$$

Since $w \in A_2$ and A satisfies (2.7), $\mathfrak{a}$ is a closed, maximally accretive, continuous sesquilinear form. Therefore, there exists an operator L_w whose domain $\mathcal{D}(L_w) \subset H^1(w)$ is dense in $L^2(w)$ and such that for every $f \in \mathcal{D}(L_w)$ and every $g \in H^1(w)$,

$$\mathfrak{a}(f, g) = \langle L_w f, g\rangle_w = \int_{\mathbb{R}^n} L_w f(x)\,\overline{g(x)}\,dw. \tag{2.9}$$

We note that the operator L_w is one-to-one. Indeed, if $u, v \in \mathcal{D}(L_w)$ are such that $L_w u = L_w v$, then for all $g \in H^1(w)$

$$0 = \int_{\mathbb{R}^n} A(x)\nabla(u(x) - v(x)) \cdot \overline{\nabla g(x)}\,dx.$$

Taking $g = u - v$ implies $\nabla u(x) = \nabla v(x)$ and so $u = v$.

The properties of the sesquilinear form guarantee that on $L^2(w)$ there exists a bounded, strongly continuous semigroup e^{-tL_w}. Further, it has a holomorphic extension. Let

$$\Sigma_\omega = \{z \in \mathbb{C} : z \neq 0, |\arg(z)| < \omega\}$$

and define $\vartheta, \vartheta^* \in \left[0, \frac{\pi}{2}\right)$ by

$$\vartheta = \sup\{|\arg\langle Lf, f\rangle_w| : f \in \mathcal{D}(L_w)\}, \quad \vartheta^* = \arctan\sqrt{\frac{\Lambda^2}{\lambda^2} - 1}.$$

Then there exists a complex semigroup e^{-zL_w} on $\Sigma_{\pi/2-\vartheta}$ of bounded operators on $L^2(w)$. By the weighted ellipticity condition (2.7), we have $0 \le \vartheta \le \vartheta^* < \frac{\pi}{2}$.

Holomorphic functional calculus. Our operator L_w is "an operator of type ω" with $\omega = \vartheta$, as defined in [McIntosh 1986]. Indeed, the ellipticity conditions imply that L_w is closed and densely defined, its spectrum is contained in Σ_ϑ, and its resolvent satisfies standard decay estimates [Cruz-Uribe and Rios 2008]. Therefore, we can define an $L^2(w)$ functional calculus as in [McIntosh 1986].

Given $\mu \in (\vartheta, \pi)$, let $\mathcal{H}^\infty(\Sigma_\mu)$ be the collection of bounded holomorphic functions on Σ_μ. To define $\varphi(L_w)$ for $\varphi \in \mathcal{H}^\infty(\Sigma_\mu)$ we first consider a smaller class: we say that $\varphi \in \mathcal{H}_0^\infty(\Sigma_\mu)$ if for some c, $s > 0$ it satisfies

$$|\varphi(z)| \le c|z|^s(1 + |z|)^{-2s}, \quad z \in \Sigma_\mu.$$

We then have an integral representation of $\varphi(L_w)$. Let Γ_θ be the boundary of Σ_θ with positive orientation, and let $\vartheta < \theta < \nu < \min\left(\mu, \frac{\pi}{2}\right)$; then

$$\varphi(L_w) = \int_{\Gamma_{\pi/2-\theta}} e^{-zL_w}\eta(z)\,dz, \tag{2.10}$$

where

$$\eta(z) = \frac{1}{2\pi i}\int_{\gamma_\nu(z)} e^{\zeta z}\varphi(\zeta)\,d\zeta \tag{2.11}$$

and $\gamma_\nu(z) = \mathbb{R}^+ e^{i\,\mathrm{sign}(\mathrm{Im}(z))\nu}$. Note that

$$|\eta(z)| \lesssim \min\{1, |z|^{-s-1}\}, \quad z \in \Gamma_{\pi/2-\theta},$$

so the representation (2.10) converges in $L^2(w)$, and we have the bound

$$\|\varphi(L_w)f\|_{L^2(w)} \le C\|\varphi\|_\infty \|f\|_{L^2(w)}, \quad f \in \mathcal{H}_0^\infty(\Sigma_\mu). \tag{2.12}$$

Now, since L_w is a one-to-one operator of type ω, it has dense range [Cowling et al. 1996, Theorem 2.3], and so the results in [McIntosh 1986] (see also [Cowling et al. 1996, Corollary 2.2]) imply that L_w has an H^∞ functional calculus and (2.12) extends to all of $\mathcal{H}^\infty(\Sigma_\mu)$. Moreover, in [McIntosh 1986, Section 8] the equivalence between the existence of this H^∞ functional calculus and square function estimates for L_w and L_w^* is established:

$$\left(\int_0^\infty \|\varphi(tL_w)\|_{L^2(w)}^2 \frac{dt}{t}\right)^{1/2} \le C\|\varphi\|_\infty \|f\|_{L^2(w)}, \quad \varphi \in \mathcal{H}_0^\infty(\Sigma_\mu), \tag{2.13}$$

with similar estimates for L_w^*.

The operators $\varphi(L_w)$ also have the following properties:

- If φ and ψ are bounded holomorphic functions, then we have the operator identity $\varphi(L)\psi(L) = (\varphi\psi)(L)$.
- Given any sequence $\{\varphi_k\}$ of bounded holomorphic functions converging uniformly on compact subsets of Σ_μ to φ, we have that $\varphi_k(L_w)$ converges to $\varphi(L_w)$ in the strong operator topology (of operators on $L^2(w)$).

Remark 2.14. The H^∞ functional calculus can be extended to more general holomorphic functions, such as powers, for which the operators $\varphi(L_w)$ can be defined as unbounded operators; see [Haase 2006; McIntosh 1986].

Gaffney-type estimates. The semigroup and its gradient satisfy Gaffney-type estimates on $L^2(w)$. Below, we will see that these are a particular case of what we will call full off-diagonal estimates; see Definition 2.33.

Theorem 2.15. *Given $w \in A_2$ and $A \in \mathcal{E}_n(w, \lambda, \Lambda)$, for any closed sets E and F, for $f \in L^2(w)$ and for all $z \in \Sigma_\nu$, where $0 < \nu < \frac{\pi}{2} - \vartheta$,*

(1) $\|e^{-zL_w}(f\chi_E)\chi_F\|_{L^2(w)} \le Ce^{-cd(E,F)^2/|z|}\|f\chi_E\|_{L^2(w)}$,

(2) $\|\sqrt{z}\nabla e^{-zL_w}(f\chi_E)\chi_F\|_{L^2(w)} \le Ce^{-cd(E,F)^2/|z|}\|f\chi_E\|_{L^2(w)}$,

(3) $\|zL_w e^{-zL_w}(f\chi_E)\chi_F\|_{L^2(w)} \le Ce^{-cd(E,F)^2/|z|}\|f\chi_E\|_{L^2(w)}$.

Proof. The semigroup estimate (1) was proved in [Cruz-Uribe and Rios 2008, Theorem 1.6] for real z, but the same proof can be readily modified to prove the analytic version. Alternatively, estimates (1) and (2) follow from the resolvent bounds

$$\|(1+z^2L_w)^{-1}(f\chi_E)\chi_F\|_{L^2(w)} \le Ce^{-cd(E,F)/|z|}\|f\chi_E\|_{L^2(w)}, \tag{2.16}$$

$$\|z\nabla(1+z^2L_w)^{-1}(f\chi_E)\chi_F\|_{L^2(w)} \le Ce^{-cd(E,F)/|z|}\|f\chi_E\|_{L^2(w)}, \tag{2.17}$$

obtained in [Cruz-Uribe and Rios 2015, Lemma 2.10] for $z \in \Sigma_{\pi/2+\nu}$, together with the integral representation of the semigroup

$$e^{-zL_w} f = \frac{1}{2\pi} \int_\Gamma e^{z\zeta} (\zeta + L_w)^{-1} f \, d\zeta,$$

where Γ is the boundary of Σ_θ with positive orientation and $\frac{\pi}{2} < \theta < \frac{\pi}{2} + \nu - \arg(z)$.

Finally, from (2.16) and (2.17) we obtain the estimate

$$\|z^2 L_w (1 + z^2 L_w)^{-1} (f \chi_E) \chi_F \|_{L^2(w)} \le C e^{-cd(E,F)/|z|} \|f \chi_E\|_{L^2(w)},$$

and then by the same kind of argument we get (3). □

The Kato estimate. The starting point for all of our estimates is the $L^2(w)$ Kato estimates for the square root operator $L_w^{1/2}$ proved in [Cruz-Uribe and Rios 2015] (see also [Auscher et al. 2015] for a different proof). This operator is the unique, maximal accretive operator such that $L_w^{1/2} L_w^{1/2} = L_w$. It has the integral representation

$$L_w^{1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \sqrt{t} L_w e^{-tL_w} \frac{dt}{t}.$$

(For further details, see [Auscher and Tchamitchian 1998; McIntosh 1986].)

Theorem 2.18 [Cruz-Uribe and Rios 2015, Theorem 1.1]. *Given $w \in A_2$ and $A \in \mathcal{E}_n(w, \lambda, \Lambda)$, the domain of L_w is $H^1(w)$ and there exist constants c and C, depending on n, Λ/λ and $[w]_{A_2}$, such that for all $f \in H^1(w)$,*

$$c\|\nabla f\|_{L^2(w)} \le \|L_w^{1/2} f\|_{L^2(w)} \le C\|\nabla f\|_{L^2(w)}. \tag{2.19}$$

The Riesz transform associated to L_w is the operator $\nabla L_w^{-1/2}$. Formally, by (2.19) we have that the Riesz transform is a bounded operator on $L^2(w, \mathbb{C}^n)$. To legitimize this, we define

$$\nabla L_w^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \sqrt{t} \nabla e^{-tL_w} \frac{dt}{t}. \tag{2.20}$$

However, it is not immediate that this integral converges at 0 or ∞. To rectify this, for $\varepsilon > 0$ define

$$S_\varepsilon = S_\varepsilon(L_w) = \frac{1}{\sqrt{\pi}} \int_\varepsilon^{1/\varepsilon} \sqrt{t} e^{-tL_w} \frac{dt}{t}. \tag{2.21}$$

Since $S_\varepsilon(z)$ is a uniformly bounded holomorphic function on the right half-plane for all $0 < \varepsilon < 1$, by the $L^2(w)$ functional calculus described above, $S_\varepsilon(L_w)$ is uniformly bounded on $L^2(w)$ for that range of ε. Further, for $f \in L_c^\infty$, we have $S_\varepsilon f \in \mathcal{D}(L_w) \subset \mathcal{D}(L_w^{1/2})$, and so by inequality (2.19) and the functional calculus,

$$\|\nabla S_\varepsilon f\|_{L^2(w)} \lesssim \|L^{1/2} S_\varepsilon f\|_{L^2(w)} = \|\varphi_\varepsilon(L_w) f\|_{L^2(w)}, \tag{2.22}$$

where

$$\varphi_\varepsilon(z) = \frac{1}{\sqrt{\pi}} \int_\varepsilon^{1/\varepsilon} \sqrt{t} \sqrt{z} e^{-tz} \frac{dt}{t}.$$

The sequence $\{\varphi_\varepsilon\}$ is uniformly bounded and converges uniformly to 1 on compact subsets of the sector Σ_μ, $0 < \mu < \frac{\pi}{2}$. Therefore, $L^{1/2}S_\varepsilon f \to f$ strongly in $L^2(w)$. If we combine this fact with (2.22) we see that $\{\nabla S_\varepsilon f\}$ is Cauchy and so it converges in $L^2(w)$. We therefore define

$$\nabla L^{-1/2} f = \lim_{\varepsilon \to 0} \nabla S_\varepsilon f,$$

where the limit is in $L^2(w)$.

Given this definition, hereafter, when we are proving $L^2(w)$ estimates for the Riesz transform, we should actually prove estimates for ∇S_ε that are independent of ε. These arguments will remain implicit unless there are details we need to emphasize.

Off-diagonal estimates. Off-diagonal estimates as we define them were introduced in [Auscher and Martell 2007b] and we will refer repeatedly to this paper for further information and results. Throughout this section we will assume that given a weight w, we have $w \in A_2$.

Given a ball B, for $j \geq 2$ we define the annuli $C_j(B) = 2^{j+1}B \setminus 2^j B$. We let $C_1(B) = 4B$. By a slight abuse of notation, we will define

$$⨍_{C_j(B)} h\, dw = \frac{1}{w(2^{j+1}B)} \int_{C_j(B)} h\, dw.$$

If $w \in A_2$ (as it will be hereafter), then $w(2^{j+1}B) \approx w(C_j(B))$, so this definition is equivalent to the one given above up to a constant. Finally, for $s > 0$ we set $\Upsilon(s) = \max\{s, s^{-1}\}$.

Definition 2.23. Given $1 \leq p \leq q \leq \infty$, a family $\{T_t\}_{t>0}$ of sublinear operators satisfies $L^p(w) - L^q(w)$ off-diagonal estimates on balls, denoted by

$$T_t \in \mathcal{O}(L^p(w) \to L^q(w)),$$

if there exist constants $\theta_1, \theta_2 > 0$ and $c > 0$ such that for every $t > 0$ and for any ball B, setting $r = r(B)$,

$$\left(⨍_B |T_t(\chi_B f)|^q\, dw\right)^{1/q} \lesssim \Upsilon\left(\frac{r}{\sqrt{t}}\right)^{\theta_2} \left(⨍_B |f|^p\, dw\right)^{1/p}, \tag{2.24}$$

and for all $j \geq 2$,

$$\left(⨍_B |T_t(\chi_{C_j(B)} f)|^q\, dw\right)^{1/q} \lesssim 2^{j\theta_1} \Upsilon\left(\frac{2^j r}{\sqrt{t}}\right)^{\theta_2} e^{-c4^j r^2/t} \left(⨍_{C_j(B)} |f|^p\, dw\right)^{1/p}, \tag{2.25}$$

$$\left(⨍_{C_j(B)} |T_t(\chi_B f)|^q\, dw\right)^{1/q} \lesssim 2^{j\theta_1} \Upsilon\left(\frac{2^j r}{\sqrt{t}}\right)^{\theta_2} e^{-c4^j r^2/t} \left(⨍_B |f|^p\, dw\right)^{1/p}. \tag{2.26}$$

If the family of sublinear operators $\{T_z\}_{z \in \Sigma_\mu}$ is defined on a complex sector Σ_μ, we say that it satisfies $L^p(w) - L^q(w)$ off-diagonal estimates on balls in Σ_μ if (2.24)–(2.26) hold for $z \in \Sigma_\mu$ with t replaced by $|z|$ in the right-hand terms. We denote this by $T_z \in \mathcal{O}(L^p(w) \to L^q(w), \Sigma_\mu)$.

We give some basic properties of off-diagonal estimates on balls as a series of lemmas taken from [Auscher and Martell 2007b, Section 2.2]. The first follows immediately by real interpolation, the second by Hölder's inequality, and the third by duality.

Lemma 2.27. *Given* $1 \le p_i \le q_i \le \infty$, $i = 1, 2$, *if* $T_t \in \mathcal{O}(L^{p_1}(w) \to L^{q_1}(w))$ *and* $T_t : L^{p_2}(w) \to L^{q_2}(w)$ *is uniformly bounded*, *then* $T_t \in \mathcal{O}(L^{p_\theta}(w) \to L^{q_\theta}(w))$, $0 < \theta < 1$, *where*

$$\frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Lemma 2.28. *If* $1 \le p \le p_1 \le q_1 \le q \le \infty$, *then*

$$\mathcal{O}(L^p(w) \to L^q(w)) \subset \mathcal{O}(L^{p_1}(w) \to L^{q_1}(w)).$$

Lemma 2.29. *If for some* $1 \le p \le q \le \infty$, *we have* $T_t \in \mathcal{O}(L^p(w) \to L^q(w))$, *and the operators* T_t *are linear*, *then* $T_t^* \in \mathcal{O}(L^{q'}(w) \to L^{p'}(w))$. (*Here* T_t^* *is the dual operator for the inner product* $\int_{\mathbb{R}^n} fg\, dw$.)

Lemma 2.30 [Auscher and Martell 2007b, Theorem 2.3]. (1) *If* $T_t \in \mathcal{O}(L^p(w) \to L^p(w))$, $1 \le p \le \infty$, *then* $T_t : L^p(w) \to L^p(w)$ *is uniformly bounded.*

(2) *If* $1 \le p \le q \le r \le \infty$, $T_t \in \mathcal{O}(L^q(w) \to L^r(w))$, *and* $S_t \in \mathcal{O}(L^p(w) \to L^q(w))$, *then* $T_t \circ S_t \in \mathcal{O}(L^p(w) \to L^r(w))$.

Remark 2.31. If $p < q$, then $T_t \in \mathcal{O}(L^p(w) \to L^q(w))$ does not guarantee that T_t is bounded from $L^p(w)$ to $L^q(w)$.

Remark 2.32. Since complex sectors Σ_μ, $0 \le \mu < \pi$, are closed under addition, the proof of Lemma 2.30 extends to give off-diagonal estimates on complex sectors $\mathcal{O}(L^p(w) \to L^q(w), \Sigma_\mu)$.

Definition 2.33. Given $1 \le p \le q \le \infty$, a family of operators $\{T_t\}$ satisfies full off-diagonal estimates from $L^p(w)$ to $L^q(w)$, denoted by

$$T_t \in \mathcal{F}(L^p(w) \to L^q(w)),$$

if there exist constants $C, c, \theta > 0$ such that given any closed sets E, F,

$$\|T_t(f\chi_E)\chi_F\|_{L^q(w)} \le C t^{-\theta} e^{-cd^2(E,F)/t} \|f\chi_E\|_{L^p(w)}.$$

The connection between full off-diagonal estimates and off-diagonal estimates on balls is given in the following lemma from [Auscher and Martell 2007b, Section 3.1].

Lemma 2.34. *Given* $1 \le p \le q \le \infty$:

(1) *if* $T_t \in \mathcal{F}(L^p(w) \to L^q(w))$, *then* $T_t : L^p(w) \to L^q(w)$ *is uniformly bounded*;

(2) $T_t \in \mathcal{F}(L^p(w) \to L^p(w))$ *if and only if* $T_t \in \mathcal{O}(L^p(w) \to L^p(w))$.

The importance of off-diagonal estimates is that they will let us prove weighted norm inequalities for the operators we are interested in. To do so we will make repeated use of two results first proved in [Auscher and Martell 2007a]; however, we will use special cases of these results as given in [Auscher and Martell 2006, Theorems 2.2 and 2.4].

Theorem 2.35. *Given $w \in A_2$ and $1 \le p_0 < q_0 \le \infty$, let T be a sublinear operator acting on $L^{p_0}(w)$, $\{\mathcal{A}_r\}_{r>0}$ a family of operators acting from a subspace $\mathcal{D}$ of $L^{p_0}(w)$ into $L^{p_0}(w)$, and S an operator from $\mathcal{D}$ into the space of measurable functions on $\mathbb{R}^n$. Suppose that for every $f \in \mathcal{D}$ and ball B with radius r,*

$$\left(\fint_B |T(I-\mathcal{A}_r)f|^{p_0}\,dw\right)^{1/p_0} \le \sum_{j\ge 1} g(j)\left(\fint_{2^{j+1}B} |Sf|^{p_0}\,dw\right)^{1/p_0}, \tag{2.36}$$

$$\left(\fint_B |T\mathcal{A}_r f|^{q_0}\,dw\right)^{1/q_0} \le \sum_{j\ge 1} g(j)\left(\fint_{2^{j+1}B} |Tf|^{p_0}\,dw\right)^{1/p_0}, \tag{2.37}$$

where $\sum g(j) < \infty$. Then for every p, $p_0 < p < q_0$, and weights

$$v \in A_{p/p_0}(w) \cap \mathrm{RH}_{(q_0/p)'}(w),$$

there is a constant C such that for all $f \in \mathcal{D}$,

$$\|Tf\|_{L^p(v\,dw)} \le C\|Sf\|_{L^p(v\,dw)}.$$

Remark 2.38. In Theorem 2.35 and Theorem 2.39 below, the case $q_0 = \infty$ is understood in the sense that the $L^{q_0}(w)$-average is replaced by the essential supremum. Also in Theorem 2.35, if $q_0 = \infty$, then the condition on v becomes $v \in A_{p/p_0}$.

Theorem 2.39. *Given $w \in A_2$ with doubling order D, and $1 \le p_0 < q_0 \le \infty$, let $T : L^{q_0}(w) \to L^{q_0}(w)$ be a sublinear operator, and $\{\mathcal{A}_r\}_{r>0}$ a family of linear operators acting from L^∞_c into $L^{q_0}(w)$. Suppose that for every ball B with radius r, $f \in L^\infty_c$ with $\operatorname{supp}(f) \subset B$ and $j \ge 2$,*

$$\left(\fint_{C_j(B)} |T(I-\mathcal{A}_r)f|^{p_0}\,dw\right)^{1/p_0} \le g(j)\left(\fint_B |f|^{p_0}\,dw\right)^{1/p_0}. \tag{2.40}$$

Suppose further that for every $j \ge 1$,

$$\left(\fint_{C_j(B)} |\mathcal{A}_r f|^{q_0}\,dw\right)^{1/q_0} \le g(j)\left(\fint_B |f|^{p_0}\,dw\right)^{1/p_0}, \tag{2.41}$$

where $\sum g(j)2^{Dj} < \infty$. Then for all p, $p_0 < p < q_0$, there exists a constant C such that for all $f \in L^\infty_c$,

$$\|Tf\|_{L^p(w)} \le C\|f\|_{L^p(w)}.$$

3. Off-diagonal estimates for the semigroup e^{-tL_w}

In this section we consider off-diagonal estimates for the semigroup associated to L_w. Throughout this and subsequent sections, let $w \in A_2$ and $A \in \mathcal{E}_n(w, \Lambda, \lambda)$ be fixed. Our goal is to characterize the set of pairs (p, q), $p \le q$, such that these operators are in $\mathcal{O}(L^p(w) \to L^q(w))$. By Theorem 2.15 we have

$$e^{-tL_w} \in \mathcal{F}(L^2(w) \to L^2(w)) \subset \mathcal{O}(L^2(w) \to L^2(w)).$$

We will show that in the (p, q)-plane this set contains a right triangle; see Figure 1.

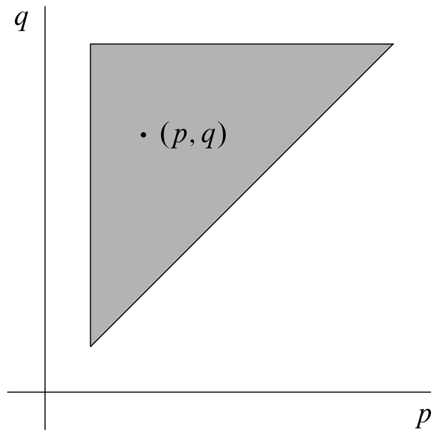

Figure 1. (p,q) such that $e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^q(w))$

Let $\widetilde{\mathcal{J}}(L_w) \subset [1,\infty]$ be the set of all exponents p such that $e^{-tL_w} : L^p(w) \to L^p(w)$ is uniformly bounded for all $t > 0$. By Theorem 2.15 and Lemma 2.34, $2 \in \widetilde{\mathcal{J}}(L_w)$, and if it contains more than one point, then by interpolation $\widetilde{\mathcal{J}}(L_w)$ is an interval. The set of pairs (p,q) such that $e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^p(w))$ is completely characterized by the next result.

Proposition 3.1. *There exists an interval $\mathcal{J}(L_w) \subset [1,\infty]$ such that $p, q \in \mathcal{J}(L_w)$ if and only if $e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^q(w))$. Furthermore, $\mathcal{J}(L_w)$ has the following properties*:

(1) $\mathcal{J}(L_w) \subset \widetilde{\mathcal{J}}(L_w)$.

(2) $\operatorname{Int} \mathcal{J}(L_w) = \operatorname{Int} \widetilde{\mathcal{J}}(L_w)$.

(3) *If $p_-(L_w)$ and $p_+(L_w)$ are respectively the left and right endpoints of $\mathcal{J}(L_w)$, then $p_-(L_w) \le (2^*_w)'$ and $p_+(L_w) \ge 2^*_w$, where 2^*_w is as in Theorem 2.1. In particular, $2 \in \operatorname{Int}(\mathcal{J}(L_w))$.*

Remark 3.2. The smaller the value of r_w, the better our bounds on the size of the set $\mathcal{J}(L_w)$. In the limiting case when $w \in A_1$, we have $p_-(L_w) \le 2n/(n+2)$ and $p_+(L_w) \ge 2n/(n-2)$. These values should be compared to the estimates in [Auscher 2007, Corollary 4.6] for the nondegenerate case that corresponds to the case $w = 1$.

We get two corollaries to Proposition 3.1. The first gives us weighted off-diagonal estimates.

Corollary 3.3. *Let $p_-(L_w) < p \le q < p_+(L_w)$. If $v \in A_{p/p_-(L_w)}(w) \cap \mathrm{RH}_{(p_+(L_w)/q)'}(w)$, then $e^{-tL_w} \in \mathcal{O}(L^p(v\,dw) \to L^q(v\,dw))$.*

Proof. By Proposition 3.1, if $p_-(L_w) < p \le q < p_+(L_w)$, then $e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^q(w))$. Therefore, by [Auscher and Martell 2007b, Proposition 2.6], if $v \in A_{p/p_-(L_w)}(w) \cap \mathrm{RH}_{(p_+(L_w)/q)'}(w)$, then we have $e^{-tL_w} \in \mathcal{O}(L^p(v\,dw) \to L^q(v\,dw))$. $\square$

As our second corollary we get off-diagonal estimates for the holomorphic extension of the semigroup.

Corollary 3.4. *For any ν, $0<\nu<\frac{\pi}{2}-\vartheta$, and for any $p\le q$ such that $e^{-tL_w}\in\mathcal{O}(L^p(w)\to L^q(w))$, then for all $m\in\mathbb{N}\cup\{0\}$, $(zL_w)^m e^{-zL_w}\in\mathcal{O}(L^p(w)\to L^q(w),\Sigma_\nu)$.*

Proof. This follows from [Auscher and Martell 2007b, Theorem 4.3] and the fact that, by Theorem 2.15, $e^{-zL_w}\in\mathcal{F}(L^2(w)\to L^2(w))$ for these values of z. □

Proof of Proposition 3.1. Fix $2<q<2^*_w$. (If $w\in A_1$ we let $q=2^*_w=2^*$.) We will show that $e^{-tL_w}\in\mathcal{O}(L^2(w)\to L^q(w))$. Given this, then we also have $e^{-tL_w}\in\mathcal{O}(L^{q'}(w)\to L^2(w))$. For if L^*_w is the adjoint of L_w (with respect to $L^2(w)$), then $L^*_w=-w^{-1}\operatorname{div}(A^*\nabla f)$ and the same estimates hold for L^*_w. Hence, $e^{-tL^*_w}\in\mathcal{O}(L^2(w)\to L^q(w))$, and so by Lemma 2.29, $e^{-tL_w}\in\mathcal{O}(L^{q'}(w)\to L^2(w))$. Since e^{-tL_w} is a semigroup, by Lemma 2.30 we have $e^{-tL_w}\in\mathcal{O}(L^{q'}(w)\to L^q(w))$. Therefore, by [Auscher and Martell 2007b, Proposition 4.1], we have that there exists an interval $\mathcal{J}(L_w)$ and properties (1) and (2) hold. Moreover, we have $[q',q]\subset\mathcal{J}(L_w)$, so if we let $q\to 2^*_w$, then we immediately get property (3).

It therefore remains to prove that $e^{-tL_w}\in\mathcal{O}(L^2(w)\to L^q(w))$. We first show (2.24). Fix B and for brevity write $r=r(B)$ and $C_j=C_j(B)$. By our choice of q, the Poincaré inequality (2.3) holds. Moreover, as we noted above, e^{-tL_w}, $\sqrt{t}\,\nabla e^{-tL_w}\in\mathcal{O}(L^2(w)\to L^2(w))$; we may assume that the same exponents θ_1, θ_2 hold for both operators. We thus get that

$$\begin{aligned}
&\Big(\fint_B |e^{-tL_w}(\chi_B f)|^q\,dw\Big)^{1/q}\\
&\qquad\le |(e^{-tL_w}(\chi_B f))_{B,w}| + \Big(\fint_B \big|e^{-tL_w}(\chi_B f)(x)-(e^{-tL_w}(\chi_B f))_{B,w}\big|^q\,dw(x)\Big)^{1/q}\\
&\qquad\lesssim \Big(\fint_B |e^{-tL_w}(\chi_B f)|^2\,dw\Big)^{1/2} + r\Big(\fint_B |\nabla e^{-tL_w}(\chi_B f)|^2\,dw\Big)^{1/2}\\
&\qquad\lesssim \Big(1+\frac{r}{\sqrt{t}}\Big)\Upsilon\Big(\frac{r}{\sqrt{t}}\Big)^{\theta_2}\Big(\fint_B |f|^2\,dw\Big)^{1/2}\\
&\qquad\lesssim \Upsilon\Big(\frac{r}{\sqrt{t}}\Big)^{1+\theta_2}\Big(\fint_B |f|^2\,dw\Big)^{1/2}.
\end{aligned}$$

The proof that (2.25) holds is gotten by nearly the same argument:

$$\begin{aligned}
&\Big(\fint_B |e^{-tL_w}(\chi_{C_j} f)|^q\,dw\Big)^{1/q}\\
&\qquad\le |(e^{-tL_w}(\chi_{C_j} f))_{B,w}| + \Big(\fint_B \big|e^{-tL_w}(\chi_{C_j} f)(x)-(e^{-tL_w}(\chi_{C_j} f))_{B,w}\big|^q\,dw(x)\Big)^{1/q}\\
&\qquad\lesssim \Big(\fint_B |e^{-tL_w}(\chi_{C_j} f)|^2\,dw\Big)^{1/2} + r\Big(\fint_B |\nabla e^{-tL_w}(\chi_{C_j} f)|^2\,dw\Big)^{1/2}\\
&\qquad\lesssim 2^{j\theta_1}\Big(1+\frac{r}{\sqrt{t}}\Big)\Upsilon\Big(\frac{2^j r}{\sqrt{t}}\Big)^{\theta_2} e^{-c4^j r^2/t}\Big(\fint_{C_j} |f|^2\,dw\Big)^{1/2}\\
&\qquad\lesssim 2^{j\theta_1}\,\Upsilon\Big(\frac{2^j r}{\sqrt{t}}\Big)^{1+\theta_2} e^{-c4^j r^2/t}\Big(\fint_{C_j} |f|^2\,dw\Big)^{1/2}.
\end{aligned}$$

Finally, to prove that (2.26) holds we use a covering argument. Fix $j \geq 2$; then we can cover the annulus C_j by a collection of balls $\{B_k\}_{k=1}^N$, $r(B_k) = 2^{j-2}r$, with centers $x_{B_k} \in C_j$. The number of balls required, N, depends only on the dimension. For any such ball, since dw is a doubling measure, we have

$$\begin{aligned}
&\left(\fint_{B_k} |e^{-tL_w}(\chi_B f)|^q \, dw\right)^{1/q} \\
&\qquad \leq |(e^{-tL_w}(\chi_B f))_{B_k,w}| + \left(\fint_{B_k} \left|e^{-tL_w}(\chi_B f)(x) - (e^{-tL_w}(\chi_B f))_{B_k,w}\right|^q \, dw(x)\right)^{1/q} \\
&\qquad \lesssim \left(\fint_{B_k} |e^{-tL_w}(\chi_B f)|^2 \, dw\right)^{1/2} + r(B_k)\left(\fint_{B_k} |\nabla e^{-tL_w}(\chi_B f)|^2 \, dw\right)^{1/2} \\
&\qquad \lesssim \left(\fint_{2^{j+2}B\setminus 2^{j-1}B} |e^{-tL_w}(\chi_B f)|^2 \, dw\right)^{1/2} + 2^j r\left(\fint_{2^{j+2}B\setminus 2^{j-1}B} |\nabla e^{-tL_w}(\chi_B f)|^2 \, dw\right)^{1/2}.
\end{aligned}$$

If $j \geq 3$, then $2^{j+2}B \setminus 2^{j-1}B = C_{j+1} \cup C_j \cup C_{j-1}$; then to estimate the last two terms we use the fact that e^{-tL_w}, $\sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^2(w))$ and apply (2.26) with $p = q = 2$ in each annulus C_i, $j-1 \leq i \leq j+1$. (These annuli have comparable measure since dw is a doubling measure, so we can divide the average up into three averages). If $j = 2$, then $2^4 B \setminus 2B = C_3 \cup C_2 \cup (4B \setminus 2B)$. On C_3 and C_2 we argue as before using (2.26). On $4B \setminus B$ we apply [Auscher and Martell 2007b, Lemma 6.1]. (We note that in the notation there, $\widehat{C}_1(B) = 4B \setminus 2B$.)

If we combine all of these estimates, we get that for every $j \geq 2$,

$$\begin{aligned}
\left(\fint_{B_k} |e^{-tL_w}(\chi_B f)|^q \, dw\right)^{1/q} &\lesssim 2^{j\theta_1}\left(1 + \frac{2^j r}{\sqrt{t}}\right)\Upsilon\left(\frac{2^j r}{\sqrt{t}}\right)^{\theta_2}\left(\fint_B |f|^2 \, dw\right)^{1/2} \\
&\lesssim 2^{j\theta_1}\,\Upsilon\left(\frac{2^j r}{\sqrt{t}}\right)^{1+\theta_2} e^{-c4^j r^2/t}\left(\fint_B |f|^2 \, dw\right)^{1/2}.
\end{aligned}$$

Since $C_j \subset \bigcup_k B_k$, we can sum in k to get

$$\begin{aligned}
\left(\fint_{C_j(B)} |e^{-tL_w}(\chi_B f)|^q \, dw\right)^{1/q} &\lesssim \sum_{k=1}^N \left(\fint_{B_k} |e^{-tL_w}(\chi_B f)|^q \, dw\right)^{1/q} \\
&\lesssim 2^{j\theta_1}\,\Upsilon\left(\frac{2^j r}{\sqrt{t}}\right)^{1+\theta_2} e^{-c4^j r^2/t}\left(\fint_B |f|^2 \, dw\right)^{1/2}. \qquad \square
\end{aligned}$$

This completes the proof that $e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^q(w))$.

4. The functional calculus

In this section we show that the operator L_w has an $L^p(w)$ holomorphic functional calculus. As we discussed in Section 2 above, we know already that if φ is a bounded holomorphic function on Σ_μ, $\mu \in (\vartheta, \pi)$, then $\varphi(L_w)$ is a bounded operator on $L^2(w)$. Recall that for any $\mu \in (\vartheta, \pi)$, we say that $\varphi \in \mathcal{H}_0^\infty(\Sigma_\mu)$ if for some $c, s > 0$,

$$|\varphi(z)| \leq c|z|^s(1 + |z|)^{-2s}, \quad z \in \Sigma_\mu. \tag{4.1}$$

We say that L_w has a bounded holomorphic functional calculus on $L^p(w)$ if for any such φ,

$$\|\varphi(L_w)f\|_{L^p(w)} \le C\|\varphi\|_\infty \|f\|_{L^p(w)}, \quad f \in L^p(w) \cap L^2(w), \tag{4.2}$$

where C depends only on p, w, ϑ and μ (but not on the decay of φ). By a standard density argument, (4.2) implies that $\varphi(L_w)$ extends to a bounded operator on all of $L^p(w)$. Furthermore, we then have this inequality holds if φ is any bounded holomorphic function. For the details of this extension, see [Haase 2006; McIntosh 1986].

Proposition 4.3. *Let $p_-(L_w) < p < p_+(L_w)$ and $\mu \in (\vartheta, \pi)$. Then for any $\varphi \in \mathcal{H}_0^\infty(\Sigma_\mu)$,*

$$\|\varphi(L_w)f\|_{L^p(w)} \le C\|\varphi\|_\infty \|f\|_{L^p(w)}, \tag{4.4}$$

with C independent of φ and f. Hence, L_w has a bounded holomorphic functional calculus on $L^p(w)$. Moreover, if $v \in A_{p/p_-(L_w)}(w) \cap \mathrm{RH}_{(p_+(L_w)/p)'}(w)$ then L_w also has a bounded holomorphic functional calculus on $L^p(v\,dw)$:

$$\|\varphi(L_w)f\|_{L^p(v\,dw)} \le C\|\varphi\|_\infty \|f\|_{L^p(v\,dw)}, \tag{4.5}$$

with C independent of φ and f.

Proof. For brevity, let $p_- = p_-(L_w)$ and $p_+ = p_+(L_w)$. By density it will suffice to assume that $f \in L_c^\infty$. Fix $\varphi \in \mathcal{H}_0^\infty(\Sigma_\mu)$; by linearity we may assume that $\|\varphi\|_\infty = 1$.

We divide the proof into two steps. We first obtain (4.4) for $p_- < p < 2$ by applying Theorem 2.39 and following the ideas in [Auscher 2007]. To do so, we will pick $q_0 = 2$ and $p_0 > p_-$ arbitrarily close to p_-. In the second step, using some ideas from [Auscher and Martell 2006], we will use Theorem 2.35 to get (4.5); in particular this yields (4.4) for every $2 < p < p_+$ by taking $v \equiv 1$. To apply Theorem 2.35 we will choose $p_0 > p_-$ arbitrarily close to p_- and $q_0 < p_+$ arbitrarily close to p_+. We will also use the fact that $\varphi(L_w)$ is bounded on $L^{p_0}(w)$; this follows from the first step choosing $p_- < p_0 < 2$.

To apply Theorem 2.39, fix $p_- < p_0 < p < 2$ and let $q_0 = 2$, $T = \varphi(L_w)$, and

$$\mathcal{A}_r f(x) = (I - (I - e^{-r^2 L_w})^m) f(x), \tag{4.6}$$

where m is a positive integer that will be chosen below. We first show that inequality (2.41) holds. By Proposition 3.1 we have $e^{-tL_w} \in \mathcal{O}(L^{p_0}(w) \to L^2(w))$. Using

$$\mathcal{A}_r = \sum_{k=1}^m \binom{m}{k} (-1)^{k+1} e^{-kr^2 L_w}, \tag{4.7}$$

and that for each fixed m and $1 \le k \le m$

$$\Upsilon\Big(\frac{r}{\sqrt{k}t}\Big) \le \sqrt{m}\,\Upsilon\Big(\frac{r}{t}\Big) \quad \text{and} \quad \exp\Big(-\frac{c}{k}\frac{4^j r^2}{t^2}\Big) \le \exp\Big(-\frac{c}{m}\frac{4^j r^2}{t^2}\Big),$$

Proposition 3.1 implies

$$\mathcal{A}_r \in \mathcal{O}(L^p(w) \to L^q(w)) \quad \text{for all } p_-(L_w) < p \le q < p_+(L_w). \tag{4.8}$$

In particular, we have $\mathcal{A}_r \in \mathcal{O}(L^{p_0}(w) \to L^2(w))$. Thus, given any ball B with radius r, if $\operatorname{supp}(f) \subset B$, then for all $j \geq 1$,

$$\left(\fint_{C_j(B)} |\mathcal{A}_r f|^2 \, dw\right)^{1/2} \lesssim 2^{j\theta_1} \Upsilon(2^j)^{\theta_2} e^{-c4^j} \left(\fint_B |f|^{p_0} \, dw\right)^{1/p_0}. \tag{4.9}$$

This establishes (2.41) with $g(j) = C\, 2^{j(\theta_1+\theta_2)} e^{-c4^j}$, for in this case we have

$$\sum_{j\geq 1} 2^{j(\theta_1+\theta_2+D)} e^{-c4^j} < \infty,$$

where D is the doubling constant of w.

We next prove that (2.40) holds. Since $\varphi(z)(1-e^{-r^2 z})^m \in \mathcal{H}_0^\infty(\Sigma_{\{\min\{\mu,\pi/2\}\}})$, by the functional calculus representation (2.10) we have

$$\varphi(L_w)(I - \mathcal{A}_r) f = \int_\Gamma e^{-zL_w} f \eta(z) \, dz,$$

where $\Gamma = \partial \Sigma_{\pi/2-\theta}$, with $0 < \vartheta < \theta < \nu < \min\{\mu, \frac{\pi}{2}\}$, and we choose θ so that the hypotheses of Corollary 3.4 are satisfied for $z \in \Gamma$. Moreover, we have the estimate

$$|\eta(z)| \lesssim \frac{r^{2m}}{|z|^{m+1}};$$

see [Auscher 2007, Section 5.1] for details.

We can now argue as follows: given a ball B with radius r, for each $j \geq 2$, by Minkowski's inequality and Corollary 3.4 (since $p_0 \in \operatorname{Int} \mathcal{J}(L_w)$),

$$\begin{aligned}
&\left(\fint_{C_j(B)} |\varphi(L_w)(I - \mathcal{A}_{r(B)}) f|^{p_0} \, dw\right)^{1/p_0} \\
&\qquad = \left(\fint_{C_j(B)} \left|\int_\Gamma e^{-zL_w} f \eta(z) \, dz\right|^{p_0} dw\right)^{1/p_0} \\
&\qquad \lesssim \int_\Gamma \left(\fint_{C_j(B)} |e^{-zL_w} f|^{p_0} \, dw\right)^{1/p_0} \frac{r^{2m}}{|z|^{m+1}} \, |dz| \\
&\qquad \lesssim \left(\fint_B |f|^{p_0} \, dw\right)^{1/p_0} \int_\Gamma \frac{r^{2m}}{|z|^{m+1}} 2^{j\theta_1} \Upsilon\left(\frac{2^j r}{\sqrt{|z|}}\right)^{\theta_2} e^{-c(r^2/|z|)4^j} \, |dz| \\
&\qquad = \left(\fint_B |f|^{p_0} \, dw\right)^{1/p_0} 2^{j(\theta_1-2m)} \int_0^\infty \sigma^{2m} \Upsilon(\sigma)^{\theta_2} e^{-c\sigma^2} \frac{d\sigma}{\sigma} \\
&\qquad \lesssim 2^{j(\theta_1-2m)} \left(\fint_B |f|^{p_0} \, dw\right)^{1/p_0};
\end{aligned} \tag{4.10}$$

the final inequality holds (i.e., the integral in σ converges) provided $2m > \theta_2$. Moreover, if we choose $2m > \theta_1 + D$, we have that (2.40) holds with $g(j) = C\, 2^{(j-1)(\theta_1-2m)}$ and

$$\sum_{j\geq 2} g(j) 2^{jD} \lesssim \sum_{j\geq 2} 2^{j(\theta_1+D-2m)} < \infty.$$

We have shown that inequalities (2.40) and (2.41) hold, and so by Theorem 2.39 inequality (4.4) holds for all p such that $p_- < p \le 2$.

We will now apply Theorem 2.35 to show that (4.5) holds for $p_- < p < p_+$. (Inequality (4.4) then follows for $2 < p < p_+$ if we take $v \equiv 1$.) Fix p, $p_- < p < p_+$, and $v \in A_{p/p_-}(w) \cap \mathrm{RH}_{(p_+/p)'}(w)$. By the openness properties of the A_q and RH_s classes there exist p_0, q_0 such that

$$p_- < p_0 < \min\{p, 2\} \le p < q_0 < p_+, \quad v \in A_{p/p_0}(w) \cap \mathrm{RH}_{(q_0/p)'}(w).$$

Let $T = \varphi(L_w)$, $\mathcal{A}_r = I - (I - e^{-r^2 L_w})^m$, $S = I$, and fix the above values of p_0 and q_0. By the previous argument we have that $\varphi(L_w)$ is bounded on $L^{p_0}(w)$.

We first show that (2.36) holds. Fix a ball B and decompose f as

$$f = \sum_{j\ge 1} f\chi_{C_j(B)} := \sum_{j \ge 1} f_j. \tag{4.11}$$

Then, by the same functional calculus argument as given above, we have that for each j,

$$\begin{aligned}
&\biggl(\fint_B \bigl|\varphi(L_w)(I-\mathcal{A}_r) f_j\bigr|^{p_0}\,dw\biggr)^{1/p_0}\\
&\qquad = \biggl(\fint_B \biggl|\int_\Gamma e^{-zLw} f_j \eta(z)\,dz\biggr|^{p_0} dw\biggr)^{1/p_0}\\
&\qquad \lesssim \int_\Gamma \biggl(\fint_B |e^{-zLw} f_j|^{p_0}\,dw\biggr)^{1/p_0} \frac{r^{2m}}{|z|^{m+1}}|dz|\\
&\qquad \lesssim \biggl(\fint_{C_j(B)} |f|^{p_0}\,dw\biggr)^{1/p_0} 2^{j(\theta_1-2m)} \int_\Gamma \biggl(\frac{2^j r}{\sqrt{|z|}}\biggr)^{2m} \Upsilon\biggl(\frac{2^j r}{\sqrt{|z|}}\biggr)^{\theta_2} e^{-c4^j r^2/|z|}\,\frac{|dz|}{|z|}\\
&\qquad \lesssim 2^{j(\theta_1-2m)} \biggl(\fint_{C_j(B)} |f|^{p_0}\,dw\biggr)^{1/p_0};
\end{aligned}$$

the last inequality holds provided $2m > \theta_2$. Hence, since $2^{j+1}B \supset C_j$, by Minkowski's inequality we have (since the sum $\sum f_j$ is finite for $f \in L^\infty_c$)

$$\begin{aligned}
\biggl(\fint_B |\varphi(L_w)(I-\mathcal{A}_r) f|^{p_0}\,dw\biggr)^{1/p_0} &\le \sum_{j\ge1} \biggl(\fint_B \bigl|\varphi(L_w)(I-\mathcal{A}_r) f_j\bigr|^{p_0}\,dw\biggr)^{1/p_0}\\
&\lesssim \sum_{j\ge1} 2^{j(\theta_1-2m)} \biggl(\fint_{2^{j+1}B} |f|^{p_0}\,dw\biggr)^{1/p_0}.
\end{aligned}$$

This establishes (2.36) with $g(j) = C\,2^{j(\theta_1-2m)}$. If we take $2m > \max\{\theta_1, \theta_2\}$, then $\sum g(j) < \infty$.

We now show that (2.37) holds. Fix a ball B and $j \ge 1$. Since $\mathcal{A}_r \in \mathcal{O}(L^{p_0}(w) \to L^{q_0}(w))$ (see (4.8)),

$$\biggl(\fint_B \bigl|\mathcal{A}_r\bigl(\chi_{C_j(B)}\varphi(L_w) f\bigr)\bigr|^{q_0}\,dw\biggr)^{1/q_0} \lesssim 2^{j\theta_1}\Upsilon(2^j)^{\theta_2} e^{-c4^j} \biggl(\fint_{C_j(B)} |\varphi(L_w) f|^{p_0}\,d\mu\biggr)^{1/p_0}.$$

Therefore, since $\varphi(L_w)$ and $\mathcal{A}_r$ commute, by Minkowski's inequality we obtain

$$\left(\fint_B |\varphi(L_w)\mathcal{A}_r f|^{q_0}\,dw\right)^{1/q_0} \lesssim \sum_{j\geq 1} 2^{j(\theta_1+\theta_2)} e^{-c4^j} \left(\fint_{C_j(B)} |\varphi(L_w) f|^{p_0}\,d\mu\right)^{1/p_0}.$$

This establishes (2.37) with $g(j) = C\,2^{j(\theta_1+\theta_2)}e^{-c4^j}$; again, $\sum g(j) < \infty$. Therefore, our proof is complete. □

5. Square function estimates for the semigroup

In this section we prove $L^p(w)$ norm inequalities for the vertical square function associated to the semigroup e^{-tL_w}:

$$g_{L_w} f(x) = \left(\int_0^\infty |(tL_w)^{1/2} e^{-tL_w} f(x)|^2 \,\frac{dt}{t}\right)^{1/2}.$$

Proposition 5.1. *Let $p_-(L_w) < p < p_+(L_w)$. Then*

$$\|g_{L_w} f\|_{L^p(w)} \approx \|f\|_{L^p(w)}. \tag{5.2}$$

Conversely if for some p the equivalence (5.2) holds, then $p \in \widetilde{\mathcal{J}}(L_w)$ — i.e., the interior of the interval on which (5.2) holds is $(p_-(L_w), p_+(L_w))$.

Moreover, if $v \in A_{p/p_-(L_w)}(w) \cap \mathrm{RH}_{(p_+(L_w)/p)'}(w)$, then

$$\|g_{L_w} f\|_{L^p(v\,dw)} \approx \|f\|_{L^p(v\,dw)}. \tag{5.3}$$

We note that the upper bounds in the previous result could be obtained by combining Proposition 4.3 with the operator theory methods developed in [Cowling et al. 1996]. To reach a wider audience we present a self-contained harmonic analysis proof. We will use an auxiliary Hilbert space related to square functions, following the approach in [Auscher and Martell 2006]. Let $\mathbb{H}$ denote the Hilbert space $L^2\big((0,\infty), \frac{dt}{t}\big)$ with norm

$$|||h||| = \left(\int_0^\infty |h(t)|^2\,\frac{dt}{t}\right)^{1/2}.$$

In particular, we have

$$g_{L_w} f(x) = |||\varphi(L,\cdot) f(x)|||,$$

where $\varphi(z,t) = (tz)^{1/2}e^{-tz}$. Furthermore, we define $L^p_{\mathbb{H}}(w)$ to be the space of $\mathbb{H}$-valued functions with the norm

$$\|h\|_{L^p_{\mathbb{H}}(w)} = \left(\int_{\mathbb{R}^n} |||h(x,\cdot)|||^p\,dw(x)\right)^{1/p}.$$

The following lemma lets us extend scalar-valued inequalities to $\mathbb{H}$-valued inequalities. For a proof, see [Auscher and Martell 2006, Lemma 7.4].

Lemma 5.4. *Given a Borel measure μ on $\mathbb{R}^n$, let $\mathcal{D}$ be a subspace of $\mathcal{M}$, the space of measurable functions in $\mathbb{R}^n$, and let S, T be linear operators from $\mathcal{D}$ into $\mathcal{M}$. Fix $1 \le p \le q < \infty$ and suppose there exists $C_0 > 0$ such that for all $f \in \mathcal{D}$,*

$$\|Tf\|_{L^q(\mu)} \le C_0 \sum_{j\ge 1} \alpha_j \|Sf\|_{L^p(F_j,\mu)},$$

where the F_j are measurable subsets of $\mathbb{R}^n$ and $\alpha_j \ge 0$. Then there is an $\mathbb{H}$-valued inequality with the same constant: for all $f : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ such that for almost all $t > 0$, $f(\,\cdot\,, t) \in \mathcal{D}$,

$$\|Tf\|_{L^q_{\mathbb{H}}(\mu)} \le C_0 \sum_{j\ge 1} \alpha_j \|Sf\|_{L^p_{\mathbb{H}}(F_j,\mu)}.$$

The extension of a linear operator T on $\mathbb{C}$-valued functions to $\mathbb{H}$-valued functions is defined for $x \in \mathbb{R}^n$ and $t > 0$ by $(Th)(x,t) = T(h(\,\cdot\,,t))(x)$; that is, t can be considered as a parameter and T acts only on the variable in $\mathbb{R}^n$.

Proof of Proposition 5.1. We shall first prove the upper bound inequalities. We first claim that the upper bound inequality in (5.2) holds for $p = 2$. Indeed, since $\varphi(z) = z^{1/2}e^{-z} \in \mathcal{H}^\infty_0(\Sigma_\mu)$, it follows from (2.13) that we have the bound

$$\|g_{L_w} f\|_{L^2(w)} \lesssim \|f\|_{L^2(w)}.$$

For brevity, let $p_- = p_-(L_w)$ and $p_+ = p_+(L_w)$. As in previous proofs, we divide our proof into two steps. We will first prove the upper bound in (5.2) for $p_- < p < 2$ by applying Theorem 2.39. Fix $p_- < p < q_0 = 2$, and let $\mathcal{A}_r = I - (I - e^{-r^2 L_w})^m$, where m will be chosen below. Notice that, by (4.8), $\mathcal{A}_r$ is bounded on $L^{q_0}(w)$ for each m. Fix $f \in L^\infty_c$; the result for general $f \in L^p(w)$ then follows by a density argument.

We have $(tL_w)^{1/2}e^{-tL_w}(I - \mathcal{A}_r)f = \varphi(L_w, t)f$, where

$$\varphi(z,t) = (tz)^{1/2}e^{-tz}(1 - e^{-r^2 z})^m.$$

Moreover, since $\varphi(\,\cdot\,,t) \in \mathcal{H}^\infty_0(\Sigma_{\{\min\{\mu,\pi/2\}\}})$, by the functional calculus representation (2.10) we have

$$(tL_w)^{1/2}e^{-tL_w}(I - \mathcal{A}_r)f = \int_\Gamma \eta(z,t)e^{-zL_w} f\, dz,$$

where $\Gamma = \partial\Sigma_{\pi/2-\theta}$, with $0 < \vartheta < \theta < \nu < \min\{\mu, \frac{\pi}{2}\}$, and we choose θ so that the hypotheses of Corollary 3.4 are satisfied for $z \in \Gamma$. Moreover, we have the estimate [Auscher 2007; Auscher and Martell 2006]

$$|\eta(z,t)| \lesssim \frac{t^{1/2} r^{2m}}{(|z| + t)^{m+3/2}}, \quad z \in \Gamma.$$

Therefore,

$$|||\eta(z,\cdot)||| = \left(\int_0^\infty |\eta(z,t)|^2 \frac{dt}{t}\right)^{1/2} \lesssim \frac{r^{2m}}{|z|^{m+1}}. \tag{5.5}$$

Now let $f \in L^\infty_c$ with $\operatorname{supp}(f) \subset B$. For $j \geq 2$, we have

$$\begin{aligned}
\left(\fint_{C_j(B)} |g_{L_w}(I-\mathcal{A}_r)f|^p\,dw\right)^{1/p} &= \left(\fint_{C_j(B)} \left|\left(\int_0^\infty \left|\int_{\Gamma_{\pi/2-\theta}} \eta(z,t)e^{-zL_w} f\,dz\right|^2 \frac{dt}{t}\right)^{1/2}\right|^p dw\right)^{1/p}\\
&\leq \left(\fint_{C_j(B)} \left|\int_{\Gamma_{\pi/2-\theta}} |e^{-zL_w} f|\,|||\eta(z,\cdot)|||\,d|z|\right|^p dw\right)^{1/p}\\
&\lesssim \int_{\Gamma_{\pi/2-\theta}} \left(\fint_{C_j(B)} |e^{-zL_w} f|^p\,dw\right)^{1/p} \frac{r^{2m}}{|z|^{m+1}}\,d|z|\\
&\lesssim 2^{j\theta_1} \left(\fint_B |f|^p\,dw\right)^{1/p} \int_{\Gamma_{\pi/2-\theta}} \Upsilon\left(\frac{2^j r}{\sqrt{|z|}}\right)^{\theta_2} e^{-c4^j r^2/|z|} \frac{r^{2m}}{|z|^m} \frac{d|z|}{|z|}\\
&\lesssim 2^{j\theta_1} 4^{-mj} \left(\fint_B |f|^p\,dw\right)^{1/p}; \qquad (5.6)
\end{aligned}$$

in the second inequality we applied (5.5) and the off-diagonal estimates for e^{-zL_w} from Corollary 3.4, and the last inequality holds provided $2m > \theta_2$. Thus, if we take $m > \theta_1 + D$, where D is the doubling order of w, the operator g_{L_w} satisfies (2.40) in Theorem 2.39 with $g(j) = C\,2^{j(\theta_1-2m)}$. Since we already established (2.41) in (4.9) with $g(j) = C\,2^{j(\theta_1+\theta_2)}4^{-mj}$, the hypotheses of Theorem 2.39 are satisfied if $m > \theta_1 + \theta_2 + D$. Therefore, for each $p_- < p < 2$ there exists a constant C such that

$$\|g_{L_w} f\|_{L^p(w)} \leq C\|f\|_{L^p(w)}. \qquad (5.7)$$

In the second part of the proof we will show that if $p_- < p < p_+$ and $v \in A_{p/p_-}(w) \cap \mathrm{RH}_{(p_+/p)'}(w)$, then the upper bound inequality in (5.3) holds. If we take $v \equiv 1$, then we immediately get (5.2). To do so, first note that if we fix p and v, then by the openness properties of weights there exist p_0, q_0 such that

$$p_- < p_0 < \min\{p,2\} \leq \max\{p,2\} < q_0 < p_+$$

and $v \in A_{p_0/p_-}(w) \cap \mathrm{RH}_{(q_0/p)'}(w)$.

We will apply Theorem 2.35 with $T = g_{L_w}$, $S = I$ and $\mathcal{D} = L^{p_0}(w)$ (again, note that by (4.8), $\mathcal{A}_r$ is bounded on $L^{p_0}(w)$). We first prove that inequality (2.36) holds. For each $j \geq 1$, let $f_j = f\chi_{C_j(B)}$; then we can argue exactly as we did in the proof of (5.6), exchanging the roles of B and $C_j(B)$, to get

$$\left(\fint_B |g_{L_w}(I-\mathcal{A}_r)f_j|^p\,dw\right)^{1/p} \lesssim 2^{j\theta_1}4^{-mj}\left(\fint_{2^{j+1}B} |f|^p\,dw\right)^{1/p}.$$

Inequality (2.36) follows if we sum over all j and take $g(j) = 2^{j\theta_1}4^{-mj}$.

We will now show that inequality (2.37) holds. To do so, we need to prove a vector-valued version of a key inequality. By Proposition 3.1, given a ball B with radius r, we have for all $j \geq 1$, g with $\operatorname{supp}(g) \subset C_j(B)$, and $1 \leq k \leq m$,

$$\left(\fint_B |e^{-kr^2L_w} g|^{q_0}\,dw\right)^{1/q_0} \leq C_0\,2^{j(\theta_1+\theta_2)}e^{-\alpha 4^j}\left(\fint_{C_j(B)} |g|^{p_0}\,dw\right)^{1/p_0}. \qquad (5.8)$$

We now apply Lemma 5.4 with $S = I$ and $T : L^{p_0}(w) \to L^{q_0}(w)$ given by

$$Tg = (C_0 2^{j(\theta_1+\theta_2)} e^{-\alpha 4^j})^{-1} \frac{w(2^{j+1}B)^{1/p_0}}{w(B)^{1/q_0}} \chi_B e^{-kr^2 L_w}(g\chi_{C_j(B)}).$$

This yields the $\mathbb{H}$-valued extension of (5.8): for all $g \in L^{p_0}_{\mathbb{H}}(w)$ with $\operatorname{supp}(g(\,\cdot\,,t)) \subset C_j(B)$, $t > 0$, we have

$$\left(\fint_B |||e^{-kr^2 L_w} g(x,\cdot)|||^{q_0}\,dw\right)^{1/q_0} \le C_0 2^{j(\theta_1+\theta_2)} e^{-\alpha 4^j} \left(\fint_{C_j(B)} |||g(x,\cdot)|||^{p_0}\,dw\right)^{1/p_0}. \tag{5.9}$$

Given an arbitrary $g \in L^{p_0}_{\mathbb{H}}(w)$, decompose it as

$$g(x,t) = \sum_{j\ge 1} g(x,t)\chi_{C_j(B)}(x) = \sum_{j\ge 1} g_j(x,t).$$

Then inequality (5.9) yields

$$\begin{aligned}\left(\fint_B |||e^{-kr^2 L_w} g(x,\cdot)|||^{q_0}\,dw\right)^{1/q_0} &\le \sum_{j\ge 1}\left(\fint_B |||e^{-kr^2 L_w} g_j(x,\cdot)|||^{q_0}\,dw\right)^{1/q_0} \\ &\lesssim \sum_{j\ge 1} 2^{j(\theta_1+\theta_2)} e^{-\alpha 4^j} \left(\fint_{2^{j+1}B} |||g(x,\cdot)|||^{p_0}\,dw\right)^{1/p_0}. \end{aligned} \tag{5.10}$$

Define $g(x,t) = (tL_w)^{1/2} e^{-tL_w} f(x)$. Then $g_{L_w} f(x) = |||g(x,\cdot)|||$; by our choice of p_0 and the first step of the proof we have $g \in L^{p_0}_{\mathbb{H}}(w)$. Moreover, since for each $t > 0$ we know that $(tL_w)^{1/2} e^{-tL_w}$ and $e^{-kr^2 L_w}$ commute,

$$g_{L_w}(e^{-kr^2 L_w} f)(x) = |||e^{-kr^2 L_w} g(x,\cdot)|||.$$

We can now use (4.7) and (5.10) to get

$$\begin{aligned}\left(\fint_B |g_{L_w}\mathcal{A}_r f|^{q_0}\,dw\right)^{1/q_0} &\lesssim \sum_{k=1}^{m}\left(\fint_B |||e^{-kr^2 L_w} g(x,\cdot)|||^{q_0}\,dw\right)^{1/q_0} \\ &\lesssim \sum_{j\ge 1} 2^{j(\theta_1+\theta_2)} e^{-\alpha 4^j} \left(\fint_{2^{j+1}B} |g_{L_w} f|^{p}\,dw\right)^{1/p_0}. \end{aligned}$$

This proves (2.37) with $g(j) = C\,2^{j(\theta_1+\theta_2)} e^{-c4^j}$. Therefore, by Theorem 2.35 we get

$$\|g_{L_w} f\|_{L^p(v\,dw)} \lesssim \|f\|_{L^p(v\,dw)}.$$

It remains to show the reverse inequalities. We will prove the lower bound in (5.3); then the lower bound in (5.2) holds if we take $v \equiv 1$. Fix $p_- < p < p_+$ and $v \in A_{p/p_-(L_w)}(w) \cap \mathrm{RH}_{(p_+(L_w)/p)'}(w)$. By the duality properties of weights [Auscher and Martell 2007a, Lemma 4.4] and since $p_\pm(L_w)' = p_\mp(L_w^*)$, where L_w^* is the adjoint (on $L^2(w)$) of L_w,

$$v^{1-p'} \in A_{p'/p_-(L^*)}(w) \cap \mathrm{RH}_{(p_+(L^*)/p')'}(w). \tag{5.11}$$

We now proceed as in the proof of [Auscher and Martell 2006, Theorem 7.3]. Given $F \in L^p_{\mathbb{H}}(v\,dw) \cap L^2_{\mathbb{H}}(w)$ and $x \in \mathbb{R}^n$, we set

$$T_{L_w} F(x) = \int_0^\infty (t\,L_w)^{1/2}\, e^{-tL_w} F(x,t)\,\frac{dt}{t}. \tag{5.12}$$

Recall that $(t\,L_w)^{1/2}\, e^{-tL_w} F(x,t) = (t\,L_w)^{1/2}\, e^{-tL_w}(F(\,\cdot\,,t))(x)$. Hence, T_{L_w} maps $\mathbb{H}$-valued functions to $\mathbb{C}$-valued functions. For $h \in L^{p'}(v^{1-p'}\,dw) \cap L^2(w)$ with $\|h\|_{L^{p'}(v^{1-p'}\,dw)} = 1$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n} T_{L_w} F\bar{h}\,dw\right| &= \left|\int_{\mathbb{R}^n}\int_0^\infty F(x,t)\overline{(t\,L_w^*)^{1/2}\, e^{-tL_w^*} h(x)}\,\frac{dt}{t}\,dw(x)\right| \\ &\le \int_{\mathbb{R}^n} |||F(x,\cdot)|||\, g_{L_w^*} h(x)\,dw(x) \\ &\lesssim \|F\|_{L^p_{\mathbb{H}}(v\,dw)} \|g_{L_w^*} h\|_{L^{p'}(v^{1-p'}\,dw)} \lesssim \|F\|_{L^p_{\mathbb{H}}(v\,dw)},\end{aligned}$$

where the last estimate uses the fact that $g_{L_w^*}$ is bounded on $L^{p'}(v^{1-p'}\,dw)$. This follows from the upper bound in (5.3) (with L_w^* in place of L_w), which we proved above, and (5.11). Taking the supremum over all such functions h and using a standard density argument we have obtained that T_{L_w} is bounded from $L^p_{\mathbb{H}}(v\,dw)$ to $L^p(v\,dw)$.

Next, given $f \in L^p(v\,dw) \cap L^2(dw)$, if we define $F(x,t) = (tL_w)^{1/2} e^{-tL_w} f(x)$, then $F \in L^p_{\mathbb{H}}(v\,dw) \cap L^2_{\mathbb{H}}(w)$ since $\|F\|_{L^p_{\mathbb{H}}(v\,dw)} = \|g_{L_w} f\|_{L^p(v\,dw)}$ and analogously for $L^2(w)$. Also, by the $L^2(w)$ functional calculus we have

$$f(x) = 2\int_0^\infty (tL_w)^{1/2} e^{-tL_w} F(x,t)\,\frac{dt}{t} = 2T_{L_w} F(x). \tag{5.13}$$

Therefore,

$$\|f\|_{L^p(v\,dw)} = 2\|T_{L_w} F\|_{L^p(v\,dw)} \lesssim \|F\|_{L^p_{\mathbb{H}}(v\,dw)} = \|g_{L_w} f\|_{L^p(v\,dw)},$$

and this completes the proof of (5.3).

To finish the proof of Proposition 5.1 we need to show that the equivalence of norms in (5.2) implies that the semigroup is uniformly bounded. However, this follows immediately from the definition of g_{L_w} and the semigroup property: for any $s > 0$,

$$g_{L_w}(e^{-sL_w} f)(x) = \left(\int_0^\infty |L_w^{1/2} e^{-(s+t)L_w} f(x)|^2\,dt\right)^{1/2} \le g_{L_w} f(x). \qquad \square$$

We conclude this section by proving a version of Proposition 5.1 for the “adjoint” of a discrete square function. We will need this estimate in the proof of Proposition 6.1 below.

Proposition 5.14. *Define the holomorphic function ψ on the sector $\Sigma_{\pi/2}$ by*

$$\psi(z) = \frac{1}{\sqrt{\pi}}\int_1^\infty z e^{-tz}\,\frac{dt}{\sqrt{t}}. \tag{5.15}$$

If $p_-(L_w) < p < p_+(L_w)$, then for any sequence of functions $\{\beta_k\}_{k\in\mathbb{Z}}$,

$$\Big\| \sum_{k\in\mathbb{Z}} \psi(4^k L_w)\, \beta_k \Big\|_{L^p(w)} \lesssim \Big\| \Big(\sum_{k\in\mathbb{Z}} |\beta_k|^2 \Big)^{1/2} \Big\|_{L^p(w)}. \tag{5.16}$$

Proof. By duality and since $p_\pm(L_w)' = p_\mp(L_w^*)$, it will suffice to show that for every $p_-(L_w^*) < p < p_+(L_w^*)$,

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |\bar{\psi}(4^k L_w^*) h|^2 \Big)^{1/2} \Big\|_{L^p(w)} \lesssim \|h\|_{L^p(w)}. \tag{5.17}$$

The function ψ satisfies $|\psi(z)| \le C|z|^{1/2} e^{-c|z|}$ uniformly on subsectors Σ_μ, $0 \le \mu < \frac{\pi}{2}$. Thus the operator on the left-hand side of (5.17) is a discrete analog of the square function $g_{L_w^*}$, changing continuous times t to discrete times 4^k and $z^{1/2}e^{-z}$ to $\bar{\psi}(z)$. Since $\bar{\psi}(z)$ has the same quantitative properties as $z^{1/2}e^{-z}$ (decay at 0 and at infinity), we can repeat the previous argument and obtain the desired estimates as in the proof of Proposition 5.1. □

Remark 5.18. In Proposition 5.14 we can also get $L^p(v\,dw)$ estimates, but in the proof of Proposition 6.1 below we will only need the unweighted estimates. Further details and the precise statements are left to the interested reader.

6. Reverse inequalities

In this section we will prove $L^p(w)$ estimates of the form $\|L_w^{1/2} f\|_{L^p(w)} \le C\|\nabla f\|_{L^p(w)}$, which generalize the $L^2(w)$ Kato estimates in Theorem 2.18. These are referred to as reverse inequalities since if we replace f by $L_w^{-1/2} f$, then formally we get a reverse-type inequality for the Riesz transform: $\|f\|_{L^p(w)} \le C\|\nabla L_w^{-1/2} f\|_{L^p(w)}$.

Since these estimates involve the gradient, in proving them we will rely (implicitly and explicitly) on the weighted Poincaré inequality (2.3). This will require an additional assumption on p when $p < 2$. To state it simply, define

$$(p_-(L_w))_{w,*} = \frac{n r_w\, p_-(L_w)}{n r_w + p_-(L_w)} < p_-(L_w).$$

Proposition 6.1. *Let* $\max\{r_w, (p_-(L_w))_{w,*}\} < p < p_+(L_w)$. *Then for all* $f \in \mathcal{S}$,

$$\|L_w^{1/2} f\|_{L^p(w)} \le C\|\nabla f\|_{L^p(w)}, \tag{6.2}$$

with C independent of f. Furthermore, if

$$\max\{r_w, p_-(L_w)\} < p < p_+(L_w) \quad \textit{and} \quad v \in A_{p/\max\{r_w, p_-(L_w)\}}(w) \cap \mathrm{RH}_{(p_+(L_w)/p)'}(w),$$

then for all $f \in \mathcal{S}$,

$$\|L_w^{1/2} f\|_{L^p(v\,dw)} \le C\|\nabla f\|_{L^p(v\,dw)}. \tag{6.3}$$

Remark 6.4. The quantity $\max\{r_w, (p_-(L_w))_{w,*}\}$ can be equal to either term. For instance, it equals r_w if $p_-(L_w) \le n' r_w$. From Proposition 3.1 we know that $p_-(L_w) < (2_w^*)' = 2n r_w/(n r_w + 2)$, but this only implies the previous inequality for some values of n and r_w.

Proof. As before, let $p_- = p_-(L_w)$ and $p_+ = p_+(L_w)$. Fix p, $\max\{r_w, (p_-)_{w,*}\} < p < 2$, and $f \in \mathcal{S}$. We will first show

$$\|L_w^{1/2} f\|_{L^{p,\infty}(w)} \lesssim \|\nabla f\|_{L^p(w)}. \tag{6.5}$$

First note that since $p > r_w$, we have $w \in A_p$. Therefore, given $\alpha > 0$ we can form the Calderón–Zygmund decomposition given in [Auscher and Martell 2006, Lemma 6.6]. There exist a collection of balls $\{B_i\}_i$, smooth functions $\{b_i\}_i$ and a function $g \in L^1_{\mathrm{loc}}(w)$ such that

$$f = g + \sum_i b_i \tag{6.6}$$

and the following properties hold:

$$|\nabla g(x)| \le C\alpha \quad \text{for } w\text{-a.e. } x, \tag{6.7}$$

$$\operatorname{supp}(b_i) \subset B_i \quad \text{and} \quad \int_{B_i} |\nabla b_i|^p \, dw \le C\alpha^p w(B_i), \tag{6.8}$$

$$\sum_i w(B_i) \le \frac{C}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw, \tag{6.9}$$

$$\sum_i \chi_{B_i} \le N, \tag{6.10}$$

$$\left(\fint_{B_i} |b_i|^q \, dw \right)^{1/q} \lesssim C\alpha r(B_i) \quad \text{for } 1 \le q \le p_w^*, \tag{6.11}$$

where C and N depend only on n, p, q and the doubling constant of w.

To prove (6.5) we will prove the corresponding weak-type estimates with f replaced by g and b_i. For g, we use the $L^2(w)$ Kato estimate (2.19), (6.7), and the fact that $p < 2$ to get

$$\begin{aligned}
w\left(\left\{ |L_w^{1/2} g| > \frac{\alpha}{3} \right\}\right) &\lesssim \frac{1}{\alpha^2} \int_{\mathbb{R}^n} |L_w^{1/2} g|^2 \, dw \\
&\lesssim \frac{1}{\alpha^2} \int_{\mathbb{R}^n} |\nabla g|^2 \, dw \\
&\lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla g|^p \, dw \\
&\lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw + \frac{1}{\alpha^p} \int_{\mathbb{R}^n} \left| \sum_i \nabla b_i \right|^p dw \\
&\lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw,
\end{aligned}$$

where the last estimate follows from (6.10), (6.8), and (6.9).

To prove a weak-type estimate for $L_w^{1/2}\big(\sum_i b_i\big)$, let $r_i = 2^k$ if $2^k \le r(B_i) < 2^{k+1}$. Then for all i, $r_i \sim r(B_i)$. Write

$$L_w^{1/2} = \frac{1}{\sqrt{\pi}} \int_0^{r_i^2} L_w e^{-tL_w} \frac{dt}{\sqrt{t}} + \frac{1}{\sqrt{\pi}} \int_{r_i^2}^{\infty} L_w e^{-tL_w} \frac{dt}{\sqrt{t}} = T_i + U_i;$$

then we have

$$
\begin{aligned}
w\Big(\Big\{\Big|\sum_i L_w^{1/2} b_i\Big| > \frac{2\alpha}{3}\Big\}\Big)& \\
&\le w\Big(\bigcup_i 4B_i\Big) + w\Big(\Big\{\Big|\sum_i U_i b_i\Big| > \frac{\alpha}{3}\Big\}\Big) + w\Big(\Big(\mathbb{R}^n \setminus \bigcup_i 4B_i\Big) \bigcap \Big\{\Big|\sum_i T_i b_i\Big| > \frac{\alpha}{3}\Big\}\Big) \\
&\lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw + I_1 + I_2,
\end{aligned}
$$

where the last inequality follows from (6.9).

We first estimate I_2. Since $p > (p_-)_{w,*}$ we have $p_w^* > ((p_-)_{w,*})_w^* = p_-$, and we can choose $q \in \mathcal{J}(L_w)$ such that (6.11) is satisfied. By Corollary 3.4, $tL_w e^{-tL_w} \in \mathcal{O}(L^q(w) \to L^q(w))$, and so

$$
\begin{aligned}
I_2 &\lesssim \frac{1}{\alpha} \sum_i \sum_{j\ge 2} \int_{C_j(B_i)} |T_i b_i| \, dw \\
&\lesssim \frac{1}{\alpha} \sum_i \sum_{j\ge 2} w(2^j B_i) \int_0^{r_i^2} \fint_{C_j(B_i)} |tL_w e^{-tL_w} b_i| \, dw \, \frac{dt}{t^{3/2}} \\
&\lesssim \frac{1}{\alpha} \sum_i \sum_{j\ge 2} 2^{jD}\, w(B_i) \int_0^{r_i^2} 2^{j\theta_1} \Upsilon\Big(2^j r_i/\sqrt{t}\Big)^{\theta_2} e^{-c\,4^j r_i^2/t} \, \frac{dt}{t^{3/2}} \Big(\fint_{B_i} |b_i|^q \, dw\Big)^{1/q} \\
&\lesssim \sum_i \sum_{j\ge 2} 2^{jD} e^{-c4^j}\, w(B_i) \\
&\lesssim \sum_i w(B_i) \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw,
\end{aligned}
$$

where we have used (6.11) and (6.9), and D is the doubling order of dw.

We will now estimate I_1. For q as above, by Proposition 4.3 we have an $L^q(w)$ functional calculus for L_w. Therefore, we can write U_i as $r_i^{-1}\psi(r_i^2 L_w)$ with ψ defined by (5.15). Let $\beta_k = \sum_{i\, r_i = 2^k} b_i/r_i$; then,

$$
\sum_i U_i\, b_i = \sum_{k\in\mathbb{Z}} \psi(4^k\, L_w)\Big(\sum_{i\, r_i = 2^k} \frac{b_i}{r_i}\Big) = \sum_{k\in\mathbb{Z}} \psi(4^k\, L_w)\beta_k.
$$

Therefore, by Proposition 5.14, (6.10), (6.11), the fact that $r_i \sim r(B_i)$ and (6.9), we have

$$
\begin{aligned}
I_1 &\lesssim \frac{1}{\alpha^q} \Big\|\sum_i U_i b_i\Big\|_{L^q(w)}^q \lesssim \frac{1}{\alpha^q} \Big\|\Big(\sum_{k\in\mathbb{Z}} |\beta_k|^2\Big)^{1/2}\Big\|_{L^q(w)}^q \\
&\lesssim \frac{1}{\alpha^q} \int_{\mathbb{R}^n} \sum_i \frac{|b_i|^q}{r_i^q} \, dw \lesssim \sum_i w(B_i) \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw.
\end{aligned}
$$

If we combine all of the estimates we have obtained, we get (6.5) as desired.

To prove (6.2) from the weak-type estimate (6.5) we will use an interpolation argument from [Auscher and Martell 2006]. Fix p and r such that $\max\{r_w, (p_-)_{w,*}\} < r < p < 2$. Then by (6.5) and (2.19) we have that for every $f \in \mathcal{S}$,

$$\|L_w^{1/2} f\|_{L^{r,\infty}(w)} \lesssim \|\nabla f\|_{L^r(w)}, \quad \|L_w^{1/2} f\|_{L^2(w)} \lesssim \|\nabla f\|_{L^2(w)}. \tag{6.12}$$

Formally, to apply Marcinkiewicz interpolation, we let $g = \nabla f$ to get a weak (r, r) and strong $(2, 2)$ inequality; this would immediately yield a strong (p, p) inequality. To formalize this we must justify this substitution.

For every $q > r_w$, by [Auscher and Martell 2006, Lemma 6.7] we have that

$$\mathcal{E} = \{(-\Delta)^{1/2} f : f \in \mathcal{S},\ \operatorname{supp} \hat{f} \subset \mathbb{R}^n \setminus \{0\}\}$$

is dense in $L^q(w)$, where $\hat{f}$ denotes the Fourier transform of f. Moreover, since $r > r_w$, we have $w \in A_r$ and the Riesz transforms, $R_j = \partial_j(-\Delta)^{-1/2}$, are bounded on $L^r(w)$ [García-Cuerva and Rubio de Francia 1985]. It follows from this and the identity $-I = R_1^2 + \cdots + R_n^2$ that for $g \in L^r(w)$,

$$\|g\|_{L^r(w)} \sim \|\nabla(-\Delta)^{-1/2} g\|_{L^r(w)}.$$

Thus, for $g \in \mathcal{E}$, we know $L_w^{1/2}(-\Delta)^{-1/2} g = L_w^{1/2} f$ if $f = (-\Delta)^{-1/2} g$ and $\|\nabla f\|_{L^r(w)} \sim \|g\|_{L^r(w)}$ for $r > r_w$. Thus (6.12) becomes weighted weak (r, r) and strong $(2, 2)$ inequalities for $T = L_w^{1/2}(-\Delta)^{-1/2}$, and this operator is defined a priori on $\mathcal{E}$. Since $\mathcal{E}$ is dense in each $L^q(w)$, we can extend T by density in both cases and their restrictions to the space of simple functions agree. Hence, we can apply Marcinkiewicz interpolation and conclude, again by density, that (6.2) holds for all p with $r < p < 2$. Since r is arbitrary, we get (6.2) in the range $\max\{r_w, (p_-)_{w,*}\} < p < 2$.

For the second step of the proof we will prove (6.3) using Theorem 2.35. Inequality (6.2) for its full range of exponents then follows by letting $v = 1$. Define $\tilde{p}_- = \max\{r_w, p_-\} < 2$, and fix $\tilde{p}_- < p < p_+$ and $v \in A_{p/\tilde{p}_-}(w) \cap \mathrm{RH}_{(p_+/p)'}(w)$. By the openness properties of A_q and RH_s weights, there exist p_0, q_0 such that

$$\tilde{p}_- < p_0 < \min\{p, 2\} \le p < q_0 < p_+, \quad v \in A_{p/p_0}(w) \cap \mathrm{RH}_{(q_0/p)'}(w).$$

To apply Theorem 2.35, let $T = L_w^{1/2}$, $S = \nabla$, and $\mathcal{A}_r = I - (I - e^{-r^2 L_w})^m$, where the value of m will be fixed below. We will first show that (2.37) holds. By (4.8) we have $\mathcal{A}_r \in \mathcal{O}(L^{p_0}(w) \to L^{q_0}(w))$ since $p_0, q_0 \in \mathcal{J}(L_w)$. Let $h = L_w^{1/2} f$ and decompose h as we decomposed f in (4.11). Then, since $L_w^{1/2}$ and $\mathcal{A}_r$ commute, it follows that

$$\begin{aligned}\left(\fint_B |L_w^{1/2} \mathcal{A}_r f|^{q_0}\, dw\right)^{1/q_0} &\lesssim \sum_{j \ge 1} \left(\fint_B |\mathcal{A}_r h_j|^{q_0}\, dw\right)^{1/q_0} \\ &\lesssim \sum_{j \ge 1} 2^{j\theta_1} \Upsilon(2^j)^{\theta_2} e^{-c4^j} \left(\fint_{C_j} |h|^{p_0}\, dw\right)^{1/p_0} \\ &\le \sum_{j \ge 1} 2^{j(\theta_1+\theta_2)} e^{-c4^j} \left(\fint_{2^{j+1}B} |L_w^{1/2} f|^{p_0}\, dw\right)^{1/p_0}.\end{aligned}$$

This gives us (2.37) with $g(j) = C\, 2^{j(\theta_1+\theta_2)} e^{-c4^j}$; clearly, $\sum g(j) < \infty$.

We now prove that (2.36) holds. Fix $f \in \mathcal{S}$ and let $\varphi(z) = z^{1/2}(1-e^{-r^2 z})^m$ so that

$$\varphi(L_w) f = L_w^{1/2}(I - e^{-r^2 L_w})^m f.$$

By the conservation property [Cruz-Uribe and Rios 2015; Auscher 2007, Section 2.5],

$$\varphi(L_w)\, f = \varphi(L_w)\,(f - f_{4B,w}) = \sum_{j\geq 1} \varphi(L_w)\, h_j, \tag{6.13}$$

where $h_j = (f - f_{4B,w})\,\phi_j$, $\phi_j = \chi_{C_j(B)}$ for $j \geq 3$, ϕ_1 is a smooth function with support in $4B$, $0 \leq \phi_1 \leq 1$, $\phi_1 = 1$ in $2B$ and $\|\nabla\phi_1\|_\infty \leq C/r$, and ϕ_2 is chosen so that $\sum_{j\geq 1}\phi_j = 1$.

We estimate each term in the right-hand side of (6.13) separately. When $j = 1$, since $p_- < p_0 < p_+$, by the bounded holomorphic functional calculus on $L^{p_0}(w)$ (Proposition 4.3) and the fact that $\varphi(L_w)\, h_1 = (I - e^{-r^2 L_w})^m\, L_w^{1/2} h_1$, we have

$$\|\varphi(L_w)\, h_1\|_{L^{p_0}(w)} \lesssim \|L_w^{1/2} h_1\|_{L^{p_0}(w)}$$

uniformly in r. By the above argument we have that (6.2) holds for $p = p_0$ since $\tilde{p}_- < p_0 < 2$. Further, since $f \in \mathcal{S}$, we have $h_1 \in \mathcal{S}$ by our choice of ϕ_1. This, together with the $L^{p_0}(w)$-Poincaré inequality (2.3) (since $p_0 > r_w$, $w \in A_{p_0}$) and the definition of h_1 yield

$$\begin{aligned}\|L_w^{1/2} h_1\|_{L^{p_0}(w)} &\lesssim \|\nabla h_1\|_{L^{p_0}(w)}\\ &\lesssim \|(\nabla f)\chi_{4B}\|_{L^{p_0}(w)} + r^{-1}\,\|(f - f_{4B,w})\chi_{4B}\|_{L^{p_0}(w)} \lesssim \|(\nabla f)\chi_{4B}\|_{L^{p_0}(w)}.\end{aligned}$$

Therefore,

$$\left(\fint_B |\varphi(L_w)\, h_1|^{p_0}\, dw\right)^{1/p_0} \lesssim \left(\fint_{4B} |\nabla f|^{p_0}\, dw\right)^{1/p_0}.$$

When $j \geq 3$, the functions η associated with φ by (2.11) satisfy

$$|\eta(z)| \lesssim \frac{r^{2m}}{|z|^{m+3/2}}, \quad z \in \Gamma_{\pi/2-\theta}.$$

Since $p_0 \in \mathcal{J}(L_w)$, by Corollary 3.4, $e^{-zLw} \in \mathcal{O}(L^{p_0}(w) \to L^{p_0}(w), \Sigma_\mu)$. This, together with the representation (2.10), gives us that

$$\begin{aligned}\left(\fint_B |\varphi(L_w) h_j|^{p_0}\, dw\right)^{1/p_0} &\leq \int_{\Gamma_{\pi/2-\theta}} \left(\fint_B |e^{-zL} h_j|^{p_0}\, dw\right)^{1/p_0} |\eta(z)|\,|dz|\\ &\lesssim 2^{j\theta_1} \int_{\Gamma_{\pi/2-\theta}} \Upsilon\left(\frac{2^j r}{\sqrt{|z|}}\right)^{\theta_2} e^{-\alpha 4^j r^2/|z|} \frac{r^{2m}}{|z|^{m+3/2}}\, |dz| \left(\fint_{C_j(B)} |h_j|^{p_0}\, dw\right)^{1/p_0}\\ &\lesssim 2^{j(\theta_1 - 2m - 1)} r^{-1} \left(\fint_{2^{j+1}B} |f - f_{4B,w}|^{p_0}\, dw\right)^{1/p_0}\\ &\lesssim 2^{j(\theta_1 - 2m - 1)} \sum_{l=1}^{j} 2^l \left(\fint_{2^{l+1}B} |\nabla f|^{p_0}\, dx\right)^{1/p_0},\end{aligned}$$

provided $2m+1>\theta_2$. The last estimate follows from the $L^{p_0}(w)$-Poincaré inequality (2.3) (here we again use that $p_0>r_w$ and so $w\in A_{p_0}$):

$$\begin{aligned}\Big(\fint_{2^{j+1}B}|f-f_{4B,w}|^{p_0}\,dw\Big)^{1/p_0} &\le \Big(\fint_{2^{j+1}B}|f-f_{2^{j+1}B,w}|^{p_0}\,dw\Big)^{1/p_0}+\sum_{l=2}^{j}|f_{2^lB,w}-f_{2^{l+1}B,w}|\\ &\lesssim \sum_{l=1}^{j}\Big(\fint_{2^{l+1}B}|f-f_{2^{l+1}B}|^{p_0}\,dx\Big)^{1/p_0}\\ &\lesssim r\sum_{l=1}^{j}2^l\Big(\fint_{2^{l+1}B}|\nabla f|^{p_0}\,dx\Big)^{1/p_0}. \end{aligned}\tag{6.14}$$

When $j=2$ we can argue similarly, using the fact that

$$|h_2|\le|f-f_{4B,w}|\,\chi_{8B\setminus 2B}\le|f-f_{2B,w}|\,\chi_{8B\setminus 2B}+|f_{4B,w}-f_{2B,w}|\,\chi_{8B\setminus 2B}.$$

If we combine these estimates, then by (6.13) and Minkowski's inequality we get

$$\Big(\fint_B|\varphi(L_w)h|^{p_0}\,dw\Big)^{1/p_0}\lesssim\sum_{j\ge1}\Big(\fint_B|\varphi(L_w)h_j|^{p_0}\,dw\Big)^{1/p_0}\le\sum_{j\ge1}g(j)\Big(\fint_B|\nabla f|^{p_0}\,dw\Big)^{1/p_0}$$

with $g(j)=C_m\,2^{j(\theta_1-2m)}$ provided $2m+1>\theta_2$. If we further assume that $2m>\theta_1$, then $\sum_j g(j)<\infty$. This proves that (2.36) holds. Therefore, by Theorem 2.35 we get (6.3) as desired. □

7. The gradient of the semigroup $\sqrt{t}\nabla e^{-tL_w}$

Let $\widetilde{\mathcal{K}}(L_w)\subset[1,\infty]$ be the set of all exponents p such that $\sqrt{t}\nabla e^{-tL_w}:L^p(w)\to L^p(w)$ is uniformly bounded for all $t>0$. By Theorem 2.15 and Lemma 2.34, $2\in\widetilde{\mathcal{K}}(L_w)$ and if it contains more than one point, then by interpolation $\widetilde{\mathcal{K}}(L_w)$ is an interval. In this section we give a partial description of the set of (p,q) such that $\sqrt{t}\nabla e^{-tL_w}\in\mathcal{O}(L^p(w)\to L^q(w))$.

Proposition 7.1. *There exists an interval $\mathcal{K}(L_w)$ such that if $p,\,q\in\mathcal{K}(L_w)$, $p\le q$, then $\sqrt{t}\,\nabla e^{-tL_w}\in\mathcal{O}(L^p(w)\to L^q(w))$. Moreover, $\mathcal{K}(L_w)$ has the following properties*:

(1) $\mathcal{K}(L_w)\subset\widetilde{\mathcal{K}}(L_w)$.

(2) *If $q_-(L_w)$ and $q_+(L_w)$ are the left and right endpoints of $\mathcal{K}(L_w)$, then $q_-(L_w)=p_-(L_w)$, $2\le q_+(L_w)\le(q_+(L_w))^*_w\le p_+(L_w)$. In particular, $2\in\mathcal{K}(L_w)$ and $\mathcal{K}(L_w)\subset\mathcal{J}(L_w)$.*

(3) *If $q\ge2$ and $p<q$, and if $\sqrt{t}\,\nabla e^{-tL_w}\in\mathcal{O}(L^p(w)\to L^q(w))$, then $p,\,q\in\mathcal{K}(L_w)$.*

(4) $\sup\widetilde{\mathcal{K}}(L_w)=q_+(L_w)$.

Remark 7.2. Unlike in the unweighted case [Auscher and Martell 2007b], we are unable to give a complete characterization of $\mathcal{K}(L_w)$. More precisely, if we have an off-diagonal estimate and $p<q<2$, then we cannot prove that $p,\,q\in\mathcal{K}(L_w)$.

Remark 7.3. In Section 8 below we will show that $q_+(L_w)>2$; in particular, this gives that $2\in\operatorname{Int}\mathcal{K}(L_w)$.

As an immediate consequence of Proposition 7.1 we get weighted inequalities for the gradient of the semigroup. The proof is identical to the proofs of Corollaries 3.3 and 3.4.

Corollary 7.4. *Let $q_-(L_w) < p \le q < q_+(L_w)$. If $v \in A_{p/q_-(L_w)}(w) \cap \mathrm{RH}_{(q_+(L_w)/q)'}(w)$, then $\sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^p(v\,dw) \to L^q(v\,dw))$ and $\sqrt{z}\,\nabla e^{-zL_w} \in \mathcal{O}(L^p(v\,dw) \to L^q(v\,dw), \Sigma_\nu)$ for all ν, $0 < \nu < \frac{\pi}{2} - \vartheta$.*

The proof of Proposition 7.1 requires two lemmas.

Lemma 7.5. *Given $w \in A_\infty$ and a family of sublinear operators $\{T_t\}_{t>0}$ such that $T_t \in \mathcal{O}(L^p(w) \to L^q(w))$, with $1 \le p < q \le \infty$, there exist $\alpha, \beta > 0$ such that for any ball B with radius r and for any $t > 0$,*

$$\left(\fint_B |T_t(\chi_B f)|^q \, dw\right)^{1/q} \lesssim \max\left\{\left(\frac{r}{\sqrt{t}}\right)^\alpha, \left(\frac{r}{\sqrt{t}}\right)^\beta\right\} \left(\fint_B |f|^p \, dw\right)^{1/p}. \tag{7.6}$$

Proof. This result is implicit in [Auscher and Martell 2007b, Proof of Proposition 2.4, p. 306]; here we reprove it with a small improvement in the constant. There it was shown that in Definition 2.23 it is sufficient to consider the case where $r \approx \sqrt{t}$. But in this case we get that $\Upsilon(r/\sqrt{t}) \approx 1$ and for all $j \ge 2$, $\Upsilon(2^j r/\sqrt{t}) \approx 2^j$. The argument in [loc. cit., p. 306] shows that if we assume that (2.24)–(2.26) hold when $r \approx \sqrt{t}$, then (2.24) holds in general with constant $\max\{1, (r/\sqrt{t})^\alpha\}$ for some $\alpha > 0$ depending on p, q and w. In this maximum the 1 occurs when $r \le \sqrt{t}$; therefore, to prove (7.6) we need to show that if $r \le \sqrt{t}$, then we can replace 1 by the better constant $(r/\sqrt{t})^\beta$ for some $\beta > 0$.

Fix $r \le \sqrt{t}$. If $B = B(x, r)$, then $B \subset B_t = B(x, \sqrt{t})$. As in [loc. cit., p. 306] we apply (2.24) to T_t and B_t; this yields

$$\begin{aligned}
\left(\fint_B |T_t(\chi_B f)|^q \, dw\right)^{1/q} &\le \left(\frac{w(B_t)}{w(B)}\right)^{1/q} \left(\fint_{B_t} |T_t(\chi_B f)|^q \, dw\right)^{1/q} \\
&\lesssim \left(\frac{w(B_t)}{w(B)}\right)^{1/q} \left(\fint_{B_t} |\chi_B f|^p \, dw\right)^{1/p} \\
&\le \left(\frac{w(B)}{w(B_t)}\right)^{1/p-1/q} \left(\fint_B |f|^p \, dw\right)^{1/p}.
\end{aligned}$$

Since $w \in A_\infty$, we have that for some $\theta > 0$,

$$\frac{w(B)}{w(B_t)} \lesssim \left(\frac{|B|}{|B_t|}\right)^\theta = \left(\frac{r}{\sqrt{t}}\right)^{\theta n}.$$

Since $p < q$ we have

$$\left(\fint_B |T_t(\chi_B f)|^q \, dw\right)^{1/q} \lesssim \left(\frac{r}{\sqrt{t}}\right)^{(1/p-1/q)\theta n} \left(\fint_B |f|^p \, dw\right)^{1/p}.$$

Therefore, if we combine this with the argument from [loc. cit., p. 306] described above, we get that (7.6) holds with $\beta = (1/p - 1/q)\,\theta n$. □

The second lemma gives the close connection between off-diagonal estimates for e^{-tL_w} and $\sqrt{t}\,\nabla e^{-tL_w}$ for $p < 2$.

Lemma 7.7. *Given $1 \le p < 2$ the following are equivalent*:

(1) $e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^2(w))$.

(2) $\sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^2(w))$.

(3) $tL_w\, e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^2(w))$.

Proof. We follow the proof of [Auscher and Martell 2007b, Lemma 5.3]. To prove that (1) implies (2), note that by Theorem 2.15, $\sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^2(w))$. If we compose this with (1), by Lemma 2.30 and the semigroup property, we get (2).

To prove that (2) implies (3), define $S_t \vec{f} = \sqrt{t} e^{-tL_w}(w^{-1}\operatorname{div}(A\vec{f}))$. By duality, we have

$$\begin{aligned}\langle S_t \vec{f}, g\rangle_{L^2(w)} &= \langle w^{-1}\operatorname{div}(A\vec{f})), \sqrt{t} e^{-tL_w^*} g\rangle_{L^2(w)} = \langle \operatorname{div}(A\vec{f})), \sqrt{t} e^{-tL_w^*} g\rangle_{L^2} \\ &= -\langle \vec{f}, A^* \sqrt{t}\,\nabla e^{-tL_w^*} g\rangle_{L^2} = \langle \vec{f}, w^{-1} A^* \sqrt{t}\,\nabla e^{-tL_w^*} g\rangle_{L^2(w)}.\end{aligned}$$

The matrix $w^{-1}A^*$ is uniformly elliptic, and so multiplication by it is bounded on $L^2(w)$. Furthermore, $\sqrt{t}\,\nabla e^{-tL_w^*} \in \mathcal{O}(L^2(w) \to L^2(w))$. Therefore, it follows that $S_t \in \mathcal{O}(L^2(w) \to L^2(w))$. If we combine this with (2), we get that $-tL_w\, e^{-2tL_w} = S_t \circ \sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^2(w))$. This proves (3).

Finally we show that (3) implies (1). We first prove (2.24). Fix B and f, g such that $\big(\fint_B |f|^p\, dw\big)^{1/p} = \big(\fint_B |g|^2\, dw\big)^{1/2} = 1$, and assume also that $f \in L^2(B, dw)$. Define

$$h(t) = \fint_B e^{-tL_w}(\chi_B f)(x) g(x)\, dw(x).$$

By duality it will suffice to show that $|h(t)| \lesssim \Upsilon(r/\sqrt{t})^\theta$. (Note that our assumption implies that $th'(t)$ satisfies such a bound.) First, we claim that

$$\lim_{t\to\infty} h(t) = 0.$$

To see this we use the fact (discussed in Section 2) that L_w has a bounded holomorphic functional calculus on $L^2(w)$. Given this, since $z \mapsto e^{-tz}$ converges to 0 uniformly on compact subsets of $\operatorname{Re} z > 0$, we get the desired limit.

Hence, we can write $h(t) = -\int_t^\infty h'(s)\, ds$. Notice that $|th'(t)| \lesssim \Upsilon(r/\sqrt{t})^{\theta_2}$ but this does not give a convergent integral. However, if we apply Lemma 7.5 to $tL_w\, e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^2(w))$, we get that $|th'(t)| \lesssim \widetilde{\Upsilon}(r/\sqrt{t})$ with $\widetilde{\Upsilon}(s) = \max\{s^\alpha, s^\beta\}$. It follows from this estimate that

$$|h(t)| \le \int_t^\infty |h'(s)|\, ds \lesssim \int_t^\infty \widetilde{\Upsilon}\Big(\frac{r}{\sqrt{s}}\Big)\frac{ds}{s} \approx \int_0^{r/\sqrt{t}} \widetilde{\Upsilon}(s)\frac{ds}{s} \lesssim \widetilde{\Upsilon}\Big(\frac{r}{\sqrt{t}}\Big) \lesssim \Upsilon\Big(\frac{r}{\sqrt{t}}\Big)^{\alpha+\beta}.$$

To prove (2.25) we argue as before, but with $\big(\fint_{C_j(B)} |f|^p\, dw\big)^{1/p} = \big(\fint_B |g|^2\, dw\big)^{1/2} = 1$ and

$$h(t) = \fint_B e^{-tL_w}(\chi_{C_j(B)} f)(x) g(x)\, dw(x).$$

Since $d(B, C_j(B)) > 0$, by Theorem 2.15 and Hölder's inequality, $h(t) \to 0$ as $t \to 0$. Therefore, $h(t) = \int_0^t h'(s)\,ds$. Since $tL_w\, e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^2(w))$, we have

$$h(t) \le \int_0^t |h'(s)|\,ds \lesssim 2^{j\theta_1} \int_0^t \Upsilon\Big(\frac{2^j r}{\sqrt{s}}\Big)^{\theta_2} e^{-c4^j r^2/s}\,\frac{ds}{s}$$
$$\approx 2^{j\theta_1} \int_{2^j r/\sqrt{t}}^{\infty} \Upsilon(s)^{\theta_2} e^{-c s^2}\,\frac{ds}{s} \lesssim 2^{j\theta_1}\Upsilon\Big(\frac{2^j r}{\sqrt{t}}\Big)^{\theta_2} e^{-c4^j r^2/t}.$$

This is (2.25).

Finally, the proof of (2.26) is essentially the same and we omit the details. This completes the proof that (3) implies (1). □

Proof of Proposition 7.1. Define the sets $\mathcal{K}_-(L_w)$ and $\mathcal{K}_+(L_w)$ to be

$$\mathcal{K}_-(L_w) = \{p \in [1,2] : \sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^2(w))\},$$
$$\mathcal{K}_+(L_w) = \{p \in [2,\infty] : \sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^p(w))\},$$

and let $\mathcal{K}(L_w) = \mathcal{K}_-(L_w) \cup \mathcal{K}_+(L_w)$. The set is nonempty, since $2 \in \mathcal{K}(L_w)$. By Lemma 2.28 it is an interval. Now fix $p, q \in \mathcal{K}(L_w)$ with $p < q$. If $p < q \le 2$ or $2 \le p < q$, then by Lemma 2.28, $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{O}(L^p(w) \to L^q(w))$ since $p, q \in \mathcal{K}_-(L_w)$ or $p, q \in \mathcal{K}_+(L_w)$. If $p \le 2 < q$, then $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{O}(L^2(w) \to L^q(w))$ and by Lemma 7.7, $e^{-tL} \in \mathcal{O}(L^p(w) \to L^2(w))$. Hence, by Lemma 2.30 and the semigroup property, $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{O}(L^p(w) \to L^q(w))$. Thus, in every case we get the desired off-diagonal estimate.

We now prove (1)–(4). By Lemma 2.30, off-diagonal estimates on balls imply uniform boundedness, and so $\mathcal{K}(L_w) \subset \widetilde{\mathcal{K}}(L_w)$. This proves (1).

To prove (2), we first note that if $p < 2$, then by Lemma 7.7, $p \in \mathcal{J}(L_w)$ if and only if $p \in \mathcal{K}_-(L_w)$. Thus $\mathcal{J}(L_w) \cap [1,2] = \mathcal{K}_-(L_w)$ and so $q_-(L_w) = p_-(L_w)$. To show that $(q_+(L_w))^*_w \le p_+(L_w)$, first note that if $q_+(L_w) = 2$, then by Proposition 3.1 we have $(q_+(L_w))^*_w = 2^*_w \le p_+(L_w)$. If $q_+(L_w) > 2$, then we proceed as in the proof of this proposition. Let $2 < p < q_+(L_w)$ and $p < q < p^*_w$. Then by (2.3), and the facts that $e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^2(w))$ and $\sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^p(w))$, we get

$$\Big(\fint_B |e^{-tL_w}(\chi_B f)|^q\,dw\Big)^{1/q} \lesssim \Big(\fint_B |e^{-tL_w}(\chi_B f)|^2\,dw\Big)^{1/2} + r\Big(\fint_B |\nabla e^{-tL_w}(\chi_B f)|^p\,dw\Big)^{1/p}$$
$$\lesssim \Upsilon\Big(\frac{r}{\sqrt{t}}\Big)^{1+\theta_2}\Big(\fint_B |f|^2\,dw\Big)^{1/2}.$$

This gives us inequality (2.24). The other two inequalities in Definition 2.23 can be proved in exactly the same way. Thus $e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^q(w))$, which implies $q \le p_+(L_w)$. Letting $p \nearrow q_+(L_w)$ and $q \nearrow p^*_w$, we conclude that $(q_+(L_w))^*_w \le p_+(L_w)$.

The last estimate implies in particular that $q_+(L_w) \le p_+(L_w)$. If $q_+(L_w) < \infty$, we clearly have $q_+(L_w) < p_+(L_w)$ and so $\mathcal{K}_+(L_w) \subset \mathcal{J}(L_w)$. Otherwise, $p_+(L) = \infty$ and again we have $\mathcal{K}_+(L_w) \subset \mathcal{J}(L_w)$. This completes the proof of (2).

To prove (3), suppose first that $2 \le p < q$ and $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{O}(L^p(w) \to L^q(w))$. We will show that $p, q \in \mathcal{K}(L_w)$. Since we also have $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{O}(L^2(w) \to L^2(w))$, by interpolation (Lemma 2.27), $\sqrt{t}\,\nabla e^{-tL} \in \mathcal{O}(L^{p_\theta}(w) \to L^{q_\theta}(w))$, where $1/p_\theta = (1-\theta)/p + \theta/2$, $1/q_\theta = (1-\theta)/q + \theta/2$ and $\theta \in (0,1)$. If $p \notin \mathcal{K}_+(L_w)$, then $q > \sup \mathcal{K}_+(L_w)$. We can choose θ such that $p_\theta < \sup \mathcal{K}_+(L_w) < q_\theta$. Since $\mathcal{K}_+(L_w) \subset \mathcal{J}(L_w)$, we have $p_\theta \in \mathcal{J}(L_w)$; i.e., $e^{-tL} \in \mathcal{O}(L^2(w) \to L^{p_\theta}(w))$. By composition and the semigroup property, $\sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^{q_\theta}(w))$; hence, $q_\theta \in \mathcal{K}_+(L_w)$, a contradiction. Therefore, $p \in \mathcal{K}_+(L_w)$. As we have $\sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^q(w))$ by assumption and $e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^p(w))$ since $p \in \mathcal{J}(L_w)$, by composition and the semigroup property, $\sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^q(w))$. Hence, $q \in \mathcal{K}_+(L_w)$.

The case $p < 2 \le q$ is straightforward. Since $\sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^q(w))$, by Lemma 2.28 we have $\sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^q(w))$ and $\sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^2(w))$. Hence, $p \in \mathcal{K}_-(L_w)$ and $q \in \mathcal{K}_+(L_w)$.

Finally, we prove (4). Suppose to the contrary that $\sup \widetilde{\mathcal{K}}(L_w) > q_+(L_w)$. Then there exist p, q such that $q_+(L_w) < p < q < \sup \widetilde{\mathcal{K}}(L_w)$. Fix r such that $p_-(L_w) = q_-(L_w) < r < 2$. Then we have that $\sqrt{t}\,\nabla e^{-tL_w}$ is uniformly bounded on $L^q(w)$ and in $\mathcal{O}(L^r(w) \to L^2(w))$. By Lemma 2.27 we can interpolate between these to get that $\sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^s(w) \to L^p(w))$ for some $s < p$. But then by the above converse, we have $p \in \mathcal{K}(L_w)$, which is a contradiction. □

8. An upper bound for $\mathcal{K}(L_w)$

In this section we will prove that $q_+(L_w) > 2$; that is, the set $\mathcal{K}(L_w)$ contains 2 in its interior. In general, all we can say is that $q_+(L_w) > 2$; as noted in [Auscher 2007, Section 4.5], even in the unweighted case this is the best possible bound, since given any $\varepsilon > 0$ it is possible to find an operator L such that $q_+(L) < 2 + \varepsilon$. In Section 11 below we will give some estimates for $q_+(L_w)$ in terms of $[w]_{A_2}$.

We have broken the proof that $q_+(L_w) > 2$ into a series of discrete steps where we borrow some ideas from [Auscher and Coulhon 2005]. We first prove a reverse Hölder inequality and use Gehring's inequality to get a higher-integrability estimate. We then prove that the Hodge projection is bounded on $L^q(w)$ for a range of $q > 2$ and use this to prove the Riesz transform is also bounded for exponents greater than 2. (In Section 9 we give a more complete discussion of the Riesz transform.) From this we deduce that $q_+(L_w) > 2$.

A reverse Hölder inequality. Fix a ball B_0 and let $u \in H^1(w)$ be any weak solution of $L_w u = 0$ in $4B_0$. Then for any ball B such that $3B \subset 4B_0$, we can again prove via a standard argument a Caccioppoli inequality:

$$\left(\fint_B |\nabla u|^2 \, dw \right)^{1/2} \le \frac{C_1}{r} \left(\fint_{2B} |u - u_{2B,w}|^2 \, dw \right)^{1/2},$$

where $C_1 = C(n, \Lambda/\lambda)[w]_{A_2}^{1/2} \ge 1$. Fix q such that

$$\max\left\{ \frac{2(n-1)}{n}, r_w, \frac{2nr_w}{2+nr_w} \right\} < q < 2; \tag{8.1}$$

such a q exists since $r_w < 2$. Our choice of q guarantees that $2 < q_w^*$ and also that $2 < nq/(q-1)$. Then, by the weighted Poincaré inequality, Theorem 2.1,

$$\frac{1}{r}\left(\fint_{2B} |u - u_{2B,w}|^2\, dw\right)^{1/2} \le C_2\left(\fint_{2B} |\nabla u|^q\, dw\right)^{1/q}, \tag{8.2}$$

where $C_2 = C(n)[w]_{A_2}^{\kappa} \ge 1$ and $\kappa = (nq-1)/(nq(q-1))$. (By our choice of q we can get this sharp estimate; see Remark 2.5. Since $q < 2$ we could write $[w]_{A_q}$, but we use that $[w]_{A_q} \le [w]_{A_2}$.) If we combine these inequalities, we get a reverse Hölder inequality:

$$\left(\fint_{B} |\nabla u|^2\, dw\right)^{1/2} \le C_1 C_2\left(\fint_{2B} |\nabla u|^q\, dw\right)^{1/q}.$$

We now apply Gehring's lemma in the setting of spaces of homogeneous type [Björn and Björn 2011, Theorem 3.22] to get that there exists $p_0 > 2$ such that for every such B,

$$\left(\fint_{B} |\nabla u|^{p_0}\, dw\right)^{1/p_0} \le C_0\left(\fint_{2B} |\nabla u|^2\, dw\right)^{1/2}. \tag{8.3}$$

Moreover, we can take the values $C_0 = 8C_1^2 C_2^2 [w]_{A_2}^{31}$ and

$$p_0 = 2 + \frac{2-q}{2^{4/q+1} C_1^2 C_2^2 [w]_{A_2}^{6/q+17}}. \tag{8.4}$$

In Section 11 below we will need these precise values. Here, it suffices to note that in inequality (8.3) we have $p_0 > 2$.

The Hodge projection. Define the Hodge projection operator by

$$T = \nabla L_w^{-1/2} (\nabla (L_w^*)^{-1/2})^*,$$

where the adjoint operators are defined with respect to the inner product in $L^2(w)$. As we noted in Section 2, the Riesz transform is bounded on $L^2(w)$; hence, the Hodge projection is also bounded. By duality, $(\nabla(L_w^*)^{-1/2})^* \vec{f} = -L_w^{-1/2}(w^{-1}\operatorname{div}(w\vec{f}))$, and so

$$T\vec{f} = -\nabla L_w^{-1/2} L_w^{-1/2}(w^{-1}\operatorname{div}(w\vec{f}) = -\nabla L_w^{-1}(w^{-1}\operatorname{div}(w\vec{f})).$$

Now fix $\vec{f} \in L^2(w, \mathbb{C}^n) \cap L^{p_0}(w, \mathbb{C}^n)$ such that $\operatorname{supp}(\vec{f}) \subset \mathbb{R}^n \setminus 4B_0$. Let $u \in H^1(w)$ be a solution to the equation

$$L_w u = w^{-1}\operatorname{div}(w\vec{f});$$

by a standard Lax–Milgram argument because A satisfies (2.7) [Fabes et al. 1982, Theorem 2.2], we know u exists. Then

$$T\vec{f} = -\nabla L_w^{-1} L_w u = -\nabla u,$$

where equality is in the sense of distributions. In particular, since $f = 0$ on $4B_0$, we know $L_w u = 0$ on $4B_0$. Therefore, we can apply (8.3) to u: on any ball B such that $3B \subset 4B_0$,

$$\left(\fint_B |T\vec{f}|^{p_0}\,dw\right)^{1/p_0} = \left(\fint_B |\nabla u|^{p_0}\,dw\right)^{1/p_0} \le C_0\left(\fint_{2B} |\nabla u|^2\,dw\right)^{1/2} = \left(\fint_{2B} |T\vec{f}|^2\,dw\right)^{1/2}.$$

As a consequence of this inequality, by [Auscher and Martell 2007a, Theorem 3.14] (see also Section 5 of the same paper), for all q, $2 \le q < p_0$, we have $T : L^q(w, \mathbb{C}^n) \to L^q(w, \mathbb{C}^n)$.

Boundedness of the Riesz transform. To show that the Riesz transform $\nabla L_w^{-1/2}$ is bounded, fix q such that

$$\max\{p_-(L_w^*), r_w, p_0'\} = \max\left\{p_-(L_w^*), r_w, p_0', \frac{n r_w\, p_-(L_w^*)}{n r_w + p_-(L_w^*)}\right\} < q' < 2.$$

(The reason for including $p_-(L_w^*)$ will be made clear below.) By the above argument we have that T^* is bounded on $L^{q'}(w)$, where $T^*\vec{f} = -\nabla(L_w^*)^{-1}(w^{-1}\operatorname{div}(w\vec{f}))$. Furthermore, by Proposition 6.1, we have

$$\|(L_w^*)^{1/2} f\|_{L^{q'}(w)} \le C\|\nabla f\|_{L^{q'}(w)}.$$

Therefore,

$$\begin{aligned}
\|(\nabla L_w^{-1/2})^*\vec{f}\|_{L^{q'}(w)} &= \|(L_w^*)^{-1/2}(w^{-1}\operatorname{div}(w\vec{f}))\|_{L^{q'}(w)} \\
&= \|(L_w^*)^{1/2}(L_w^*)^{-1}(w^{-1}\operatorname{div}(w\vec{f}))\|_{L^{q'}(w)} \\
&\lesssim \|\nabla(L_w^*)^{-1}(w^{-1}\operatorname{div}(w\vec{f}))\|_{L^{q'}(w)} \\
&= \|T^*\vec{f}\|_{L^{q'}(w)} \lesssim \|\vec{f}\|_{L^{q'}(w)}.
\end{aligned}$$

Hence, by duality we have $\nabla L_w^{-1/2} : L^q(w) \to L^q(w)$ for all q such that

$$2 < q < \min\{p_+(L_w), r_w', p_0\} = q_w;$$

here we have used the fact that by duality, $p_-(L_w^*)' = p_+(L_w)$.

Boundedness of the gradient of the semigroup. Finally, we show that if $2 < q < q_w$, then $\sqrt{t}\nabla e^{-tL_w} : L^q(w) \to L^q(w)$. The desired estimate for $q_+(L_w)$ follows from this: by Proposition 7.1, part (4),

$$q_+(L_w) = \sup \widetilde{\mathcal{K}}(L_w) \ge q_w > 2.$$

Fix such a q; then by the above estimate for the Riesz transform,

$$\begin{aligned}
\|\sqrt{t}\nabla e^{-tL_w} f\|_{L^q(w)} &= \|\nabla L_w^{-1/2}(tL_w)^{1/2} e^{-tL_w} f\|_{L^q(w)} \\
&\lesssim \|(tL_w)^{1/2} e^{-tL_w} f\|_{L^q(w)} = \|\varphi_t(L_w) f\|_{L^q(w)},
\end{aligned}$$

where $\varphi_t(z) = (tz)^{1/2} e^{-tz}$. For all $t > 0$, this is a uniformly bounded holomorphic function in the right half-plane. Therefore, since $2 < q < p_+(L_w)$, by Proposition 4.3 we have

$$\|\sqrt{t}\nabla e^{-tL_w} f\|_{L^q(w)} \lesssim \|\varphi_t\|_\infty \|f\|_{L^q(w)} \lesssim \|f\|_{L^q(w)}$$

and the bound is independent of t. This completes the proof that $q_+(L_w) > 2$.

9. Riesz transform estimates

In this section we prove $L^p(w)$ norm inequalities for the Riesz transform $\nabla L_w^{-1/2}$. We have already proved such inequalities for a small range of values $q > 2$ in Section 8. Here we prove the following result.

Proposition 9.1. *Let $q_-(L_w) < p < q_+(L_w)$. Then there exists a constant C such that*

$$\|\nabla L_w^{-1/2} f\|_{L^p(w)} \le C\|f\|_{L^p(w)}. \tag{9.2}$$

Furthermore, if $v \in A_{p/q_-(L_w)}(w) \cap \mathrm{RH}_{(q_+(L_w)/p)'}(w)$, then

$$\|\nabla L_w^{-1/2} f\|_{L^p(v\,dw)} \le C\|f\|_{L^p(v\,dw)}. \tag{9.3}$$

To prove Proposition 9.1 we would like to follow the same outline as the proof of Proposition 4.3. The first step (i.e., proving (9.2) holds when $q_-(L_w) < p < 2$) does work with the appropriate changes. However, the second step (i.e., the proof that (9.3) holds) runs into difficulties since $\nabla L_w^{-1/2}$ and the auxiliary operators $\mathcal{A}_r$ do not commute. One approach to overcoming this obstacle would be to adapt the proof in [Auscher and Martell 2006]; see also [Auscher 2007]. In this case we would need to use an $L^{p_0}(w)$-Poincaré inequality, which may not hold unless we assume $w \in A_{p_0}$. This would yield estimates in the range $\max\{r_w, q_-(L_w)\} < p < q_+(L_w)$, analogous to those in Proposition 6.1.

There is, however, an alternative approach. In [Auscher and Martell 2008] the authors considered Riesz transforms associated with the Laplace–Beltrami operator of a complete, noncompact Riemannian manifold. Their proof avoids Poincaré inequalities for p close to 1 as these may not hold. Instead, they use a duality argument based on ideas in [Bernicot and Zhao 2008]; this requires that they first prove that the Riesz transform is bounded for $p > 2$ in the appropriate range of values. This reverses the order used in the proof of Proposition 4.3.

Proof of Proposition 9.1. For brevity, let $q_- = q_-(L_w)$ and $q_+ = q_+(L_w)$. To implement the approach sketched above, we divide the proof in two steps. First we will prove that (9.2) holds when $2 < p < q_+$. We do so using Theorem 2.35 and some ideas from [Auscher 2007; Auscher and Martell 2006]. We note that since the Riesz transform and $\mathcal{A}_r$ do not commute, we will use an $L^2(w)$-Poincaré inequality. This holds since $w \in A_2$; the problem with using the Poincaré inequality only occurs with exponents less than 2. The second step is to prove that (9.3) holds by adapting the proof in [Auscher and Martell 2008]. Here we will use duality and a result from [Auscher and Martell 2007a] that is based on good-λ inequalities. Inequality (9.2) then holds when $q_- < p < 2$ by taking $v \equiv 1$.

To apply Theorem 2.35, fix $2 < p < q_+$ and let $T = \nabla L_w^{-1/2}$, $S = I$ and $\mathcal{D} = L_c^\infty$. Let $p_0 = 2$ and fix q_0 such that $2 < p < q_0 < q_+$. As before we take $\mathcal{A}_r = I - (I - e^{-r^2 L_w})^m$, where m will be chosen below. We first show that (2.36) holds. Let $f \in L_c^\infty$ and decompose it as in (4.11); then we have

$$\left(\fint_B \left|\nabla L_w^{-1/2}(I - e^{-r^2 L_w})^m f\right|^2 dw\right)^{1/2} \le \sum_{j\ge 1}\left(\fint_B \left|\nabla L_w^{-1/2}(I - e^{-r^2 L_w})^m f_j\right|^2 dw\right)^{1/2}.$$

To estimate the first term, note that $\nabla L_w^{-1/2}$ and $e^{-r^2 L_w}$ are bounded on $L^2(w)$ by Theorems 2.15 and 2.18. Hence,

$$\Big(\fint_B \big|\nabla L_w^{-1/2}(I-e^{-r^2L_w})^m f_1\big|^2\,dw\Big)^{1/2} \lesssim \Big(\fint_{4B} |f|^2\,dw\Big)^{1/2}. \tag{9.4}$$

Fix $j \ge 2$; to get the desired L^2 estimates we will use the L^2 bounds for the gradient of the square function. If $h \in L^2(w)$, by (2.20)

$$\nabla L_w^{-1/2}(I-e^{-r^2L_w})^m h = \frac{1}{\sqrt{\pi}}\int_0^\infty \sqrt{t}\,\nabla\varphi(L_w,t)h\,\frac{dt}{t}, \tag{9.5}$$

where $\varphi(z,t) = e^{-tz}\,(1-e^{-r^2z})^m \in \mathcal{H}_0^\infty(\Sigma_\mu)$. We can therefore use the integral representation (2.10) for $\varphi(\,\cdot\,,t)$. The function $\eta(\,\cdot\,,t)$ in this representation satisfies

$$|\eta(z,t)| \lesssim \frac{r^{2m}}{(|z|+t)^{m+1}}, \quad z\in\Gamma,\ t>0.$$

By Theorem 2.15, $\sqrt{z}\,\nabla e^{-zLw} \in \mathcal{O}(L^2(w)\to L^2(w))$; hence,

$$\begin{aligned}
\Big(\fint_B\Big|\int_\Gamma \eta(z)\,\sqrt{t}\,\nabla e^{-zLw} f_j\,dz\Big|^2 dw\Big)^{1/2}
&\le \int_\Gamma\Big(\fint_B |\sqrt{z}\,\nabla e^{-zLw} f_j|^2\,dw\Big)^{1/2}\frac{\sqrt{t}}{\sqrt{|z|}}|\eta(z)|\,|dz| \\
&\lesssim 2^{j\theta_1}\int_\Gamma \Upsilon\Big(\frac{2^j r}{\sqrt{|z|}}\Big)^{\theta_2} e^{-\alpha\,4^j r^2/|z|}\frac{\sqrt{t}}{\sqrt{|z|}}\,|\eta(z)|\,|dz|\Big(\fint_{C_j(B)}|f|^2\,dw\Big)^{1/2} \\
&\lesssim 2^{j\theta_1}\int_0^\infty \Upsilon\Big(\frac{2^j r}{\sqrt{s}}\Big)^{\theta_2} e^{-\alpha 4^j r^2/s}\frac{\sqrt{t}}{\sqrt{s}}\,\frac{r^{2m}}{(s+t)^{m+1}}\,ds\Big(\fint_{C_j(B)}|f|^2\,dw\Big)^{1/2}.
\end{aligned} \tag{9.6}$$

When $2m > \theta_2$,

$$\int_0^\infty\int_0^\infty \Upsilon\Big(\frac{2^j r}{\sqrt{s}}\Big)^{\theta_2} e^{-\alpha\,4^j r^2/s}\frac{\sqrt{t}}{\sqrt{s}}\,\frac{r^{2m}}{(s+t)^{m+1}}\,ds\frac{dt}{t} = C\,4^{-jm}. \tag{9.7}$$

If we insert this into the representation (2.10) we get

$$\begin{aligned}
\Big(\fint_B\big|\nabla e^{-tL_w}(I-e^{-r^2L_w})^m f_j\big|^2\,dw\Big)^{1/2} &\lesssim \int_0^\infty\Big(\fint_B|\sqrt{t}\,\nabla\varphi(L_w,t)f_j|^2\,dw\Big)^{1/2}\frac{dt}{t} \\
&\lesssim 2^{j(\theta_1-2m)}\Big(\fint_{C_j(B)}|f|^2\,dw\Big)^{1/2}.
\end{aligned} \tag{9.8}$$

If we now combine (9.4) and (9.8) we get (2.36) with $g(j) = C_m\,2^{j(\theta_1-2m)}$; if we also fix $2m > \theta_1$, we get that $\sum g(j) < \infty$.

We now show that (2.37) holds. As we remarked above, the Riesz transform does not commute with $\mathcal{A}_r$. To overcome this obstacle, we will prove an off-diagonal estimate for the gradient of the semigroup (using the $L^2(w)$-Poincaré inequality), and then use an approximation argument to get the desired estimate for the Riesz transform.

More precisely, we claim that for every $f \in H^1(w)$ and $1 \le k \le m$,

$$\left(\fint_B |\nabla e^{-k r^2 L_w} f|^{q_0}\, dw\right)^{1/q_0} \le \sum_{j\ge 1} g(j) \left(\fint_{2^{j+1}B} |\nabla f|^2\, dw\right)^{1/2}, \tag{9.9}$$

where $g(j) = C_m 2^j \sum_{l\ge j} 2^{l\theta} e^{-\alpha 4^l}$. Assume for the moment that (9.9) holds. Then for every $\varepsilon > 0$ we can apply this estimate to $S_\varepsilon f$, defined by (2.21), since $S_\varepsilon f \in H^1(w)$. Moreover, we have that $\mathcal{A}_r$ and S_ε commute, and so if we expand $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$ and apply (9.9), we get

$$\left(\fint_B |\nabla S_\varepsilon \mathcal{A}_r f|^{q_0}\, dw\right)^{1/q_0} \le C_m \sum_{j\ge 1} g(j) \left(\fint_{2^{j+1}B} |\nabla S_\varepsilon f|^2\, dw\right)^{1/2}.$$

If we let ε go to 0, we obtain (2.37). (The justification of this uses the observations made in Section 2 after (2.21) and is left to the reader.) Moreover, we have $\sum_{j\ge 1} g(j) < \infty$, and so by Theorem 2.35 with $v \equiv 1$, which trivially satisfies $v \in A_{p/2}(w) \cap \mathrm{RH}_{(q_0/p)'}(w)$, we have that (9.2) holds for $f \in L^\infty_c$ and for every $2 < p < q_+$.

To complete this step we need to prove (9.9). Fix $1 \le k \le m$ and $f \in H^1(w)$. Let $h = f - f_{4B,w}$, where $f_{4B,w} = \fint_{4B} f\, dw$. Then by the conservation property (see [Cruz-Uribe and Rios 2015], or the proof in [Auscher 2007, Section 2.5]), $e^{-tL_w} 1 = 1$ for all $t > 0$, and so

$$\nabla e^{-k r^2 L_w} f = \nabla e^{-k r^2 L_w}(f - f_{4B,w}) = \nabla e^{-k r^2 L_w} h = \sum_{j\ge 1} \nabla e^{-k r^2 L_w} h_j,$$

where $h_j = h\, \chi_{C_j(B)}$. Hence,

$$\left(\fint_B |\nabla e^{-k r^2 L_w} f|^{q_0}\, dw\right)^{1/q_0} \le \sum_{j\ge 1}\left(\fint_B |\nabla e^{-k r^2 L_w} h_j|^{q_0}\, dw\right)^{1/q_0}.$$

Since $2 < q_0 < q_+$, by Proposition 7.1, $\sqrt{t}\,\nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^{q_0}(w))$. If we apply this and the $L^2(w)$-Poincaré inequality (see Remark 2.6 with $p = q = 2$), then for each $j \ge 1$ we get

$$\begin{aligned}
&\left(\fint_B |\nabla e^{-k r^2 L_w} h_j|^{q_0}\, dw\right)^{1/q_0} \\
&\qquad \lesssim \frac{2^{j(\theta_1+\theta_2)} e^{-\alpha 4^j}}{r} \left(\fint_{C_j(B)} |h_j|^2\, dw\right)^{1/2} \\
&\qquad \le \frac{2^{j(\theta_1+\theta_2)} e^{-\alpha 4^j}}{r} \left(\fint_{2^{j+1}B} |f - f_{4B,w}|^2\, dw\right)^{1/2} \\
&\qquad \le \frac{2^{j(\theta_1+\theta_2)} e^{-\alpha 4^j}}{r} \left(\left(\fint_{2^{j+1}B} |f - f_{2^{j+1}B,w}|^2\, dw\right)^{1/2} + \sum_{l=2}^{j} |f_{2^l B,w} - f_{2^{l+1}B,w}|\right) \\
&\qquad \lesssim \frac{2^{j,(\theta_1+\theta_2)} e^{-\alpha 4^j}}{r} \sum_{l=1}^{j}\left(\fint_{2^{l+1}B} |f - f_{2^{l+1}B,w}|^2\, dw\right)^{1/2} \\
&\qquad \lesssim 2^{j(\theta_1+\theta_2)} e^{-\alpha 4^j} \sum_{l=1}^{j} 2^l \left(\fint_{2^{l+1}B} |\nabla f|^2\, dw\right)^{1/2}.
\end{aligned}$$

If we combine these two estimates and exchange the order of summation we get (9.9) with $\theta = \theta_1 + \theta_2$. This completes the proof that (9.2) holds when $2 < p < q_+$.

For the second step of our proof we show that (9.3) holds for all p, $q_- < p < q_+$, and $v \in A_{p/q_-}(w) \cap \mathrm{RH}_{(q_+/p)'}(w)$. Fix such a p and v; then by the openness properties of A_q and RH_s weights, there exist p_0, q_0 such that

$$q_- < p_0 < \min\{p, 2\} \le \max\{p, 2\} < q_0 < q_+ \quad \text{and} \quad v \in A_{p/p_0}(w) \cap \mathrm{RH}_{(q_0/p)'}(w).$$

By the duality properties of weights [Auscher and Martell 2007a, Lemma 4.4],

$$u = v^{1-p'} \in A_{p'/q_0'}(w) \cap \mathrm{RH}_{(p_0'/p')'}(w).$$

Let $T = \nabla L_w^{-1/2}$; then T is bounded from $L^p(\mathbb{R}^n, v\,dw)$ to $L^p(\mathbb{R}^n; \mathbb{C}^n, v\,dw)$ if and only if T^* is bounded from $L^{p'}(\mathbb{R}^n; \mathbb{C}^n, u\,dw)$ to $L^{p'}(\mathbb{R}^n; u\,dw)$. (Note that T takes scalar-valued functions to vector-valued functions and T^* does the opposite.)

Therefore, it will suffice to prove the boundedness of T^*. We will do so using a particular case of [Auscher and Martell 2007a, Theorem 3.1]. This result is stated there in the Euclidean setting but it extends to spaces of homogeneous type. Here we give the weighted version we need; see [loc. cit., Section 5].

Theorem 9.10. *Fix $1 < q < \infty$, $a \ge 1$ and $u \in \mathrm{RH}_{s'}(w)$, $1 < s < \infty$. Then there exists $C > 1$ with the following property: suppose $F \in L^1(w)$ and G are nonnegative measurable functions such that for any ball B there exist nonnegative functions G_B and H_B with $F(x) \le G_B(x) + H_B(x)$ for a.e. $x \in B$ and, for all $x \in B$,*

$$\left(\fint_B H_B^q\,dw\right)^{1/q} \le a\,M_w F(x), \quad \fint_B G_B\,dw \le G(x), \tag{9.11}$$

where M_w is the Hardy–Littlewood maximal function with respect to dw. Then for $1 < t < q/s$,

$$\|M_w F\|_{L^t(u\,dw)} \le C\,\|G\|_{L^t(u\,dw)}. \tag{9.12}$$

To apply Theorem 9.10, fix $\vec{f} \in L_c^\infty(\mathbb{R}^n; \mathbb{C}^n)$, and let $h = T^*\vec{f}$ and $F = |h|^{q_0'}$. Then $F \in L^1(w)$; by the argument above, since $2 < q_0 < q_+$, we have that T is bounded from $L^{q_0}(\mathbb{R}^n, w)$ to $L^{q_0}(\mathbb{R}^n; \mathbb{C}^n, w)$, thus, T^* is bounded from $L^{q_0'}(\mathbb{R}^n; \mathbb{C}^n, w)$ to $L^{q_0'}(\mathbb{R}^n, w)$.

Now let $\mathcal{A}_r = I - (I - e^{-r^2 L_w})^m$, where $m > 0$ will be fixed below. Given a ball B with radius r, we define

$$F \le 2^{q_0'-1}\,|(I - \mathcal{A}_r)^* h|^{q_0'} + 2^{q_0'-1}\,|\mathcal{A}_r^* h|^{q_0'} \equiv G_B + H_B,$$

where, as before, the adjoint is with respect to $L^2(w)$. To complete the proof, suppose for the moment that we could prove (9.11) with $q = p_0'/q_0'$ and $G = M_w(|\vec{f}|^{q_0'})$. Since $u \in \mathrm{RH}_{(p_0'/p')'}(w)$, by the openness property of reverse Hölder weights, $u \in \mathrm{RH}_{s'}(w)$ for some $s < p_0'/p'$. Then if we let $t = p'/q_0' = (p_0'/q_0')/(p_0'/p') < q/s$, we have $u \in A_t(w)$, and so M_w is bounded on $L^t(u\,dw)$. Therefore, by (9.12),

$$\|T^*\vec{f}\|_{L^{p'}(u\,dw)}^{q_0'} \le \|M_w F\|_{L^t(u\,dw)} \le C\,\|G\|_{L^t(u\,dw)} = C\,\|M_w(|\vec{f}|^{q_0'})\|_{L^t(u\,dw)} \lesssim \|\vec{f}\|_{L^{p'}(u\,dw)}^{q_0'}.$$

To complete the proof we need to show that (9.11) holds. We first estimate H_B. By duality there exists $g \in L^{p_0}(B, dw/w(B))$ with norm 1 such that for all $x \in B$,

$$\begin{aligned}
\left(\fint_B H_B^q \, dw\right)^{1/(q q_0')} &\lesssim w(B)^{-1} \int_{\mathbb{R}^n} |h||\mathcal{A}_r g| \, dw \\
&\lesssim \sum_{j=1}^{\infty} 2^{jD} \left(\fint_{C_j(B)} |h|^{q_0'} \, dw\right)^{1/q_0'} \left(\fint_{C_j(B)} |\mathcal{A}_r g|^{q_0} \, dw\right)^{1/q_0} \\
&\lesssim M_w F(x)^{1/q_0'} \sum_{j=1}^{\infty} 2^{j(D+\theta_1+\theta_2)} e^{-\alpha\, 4^j} \left(\fint_B |g|^{p_0} \, dw\right)^{1/p_0} \lesssim M_w F(x)^{1/q_0'},
\end{aligned}$$

where in the second-to-last inequality we used the fact that by our choice of p_0, q_0, we have $e^{-tL_w} \in \mathcal{O}(L^{p_0}(w) \to L^{q_0}(w))$, and so $\mathcal{A}_r$ is as well.

We now estimate G_B. Again by duality there exists $g \in L^{q_0}(B, dw/w(B))$ with norm 1 such that for all $x \in B$,

$$\begin{aligned}
\left(\fint_B G_B \, dw\right)^{1/q_0'} &\lesssim w(B)^{-1} \int_{\mathbb{R}^n} |\vec{f}||T(I - \mathcal{A}_r)g| \, dw \\
&\lesssim \sum_{j=1}^{\infty} 2^{jD} \left(\fint_{C_j(B)} |\vec{f}|^{q_0'} \, dw\right)^{1/q_0'} \left(\fint_{C_j(B)} |T(I - \mathcal{A}_r)g|^{q_0} \, dw\right)^{1/q_0} \\
&\leq M_w(|\vec{f}|^{q_0'})(x)^{1/q_0'} \sum_{j=1}^{\infty} 2^{jD} \left(\fint_{C_j(B)} |T(I - \mathcal{A}_r)g|^{q_0} \, d\mu\right)^{1/q_0}.
\end{aligned} \tag{9.13}$$

To estimate each term in the sum, we argue as in the first half of the proof. When $j = 1$, we know that $\nabla L_w^{-1/2}$ and $e^{-r^2 L_w}$ are bounded on $L^{q_0}(w)$ by the first part of the proof and Theorem 2.15. Hence,

$$\left(\fint_{4B} |\nabla L_w^{-1/2}(I - e^{-r^2 L_w})^m g|^{q_0} \, dw\right)^{1/q_0} \lesssim \left(\fint_B |g|^{q_0} \, dw\right)^{1/q_0} = 1. \tag{9.14}$$

For $j \geq 2$ we use the integral representation (9.5). If we estimate as in (9.6), with the roles of B and $C_j(B)$ switched and using the fact that $\sqrt{z}\, \nabla e^{-zLw} \in \mathcal{O}(L^{q_0}(w) \to L^{q_0}(w))$ since $2 < q_0 < q_+$, we see that

$$\begin{aligned}
&\left(\fint_{C_j(B)} \left| \int_\Gamma \eta(z) \sqrt{t}\, \nabla e^{-zLw} g \, dz \right|^{q_0} dw\right)^{1/q_0} \\
&\qquad \leq \int_\Gamma \left(\fint_{C_j(B)} |\sqrt{z} \nabla e^{-zLw} g|^{q_0} \, dw\right)^{1/q_0} \frac{\sqrt{t}}{\sqrt{|z|}} |\eta(z)||dz| \\
&\qquad \lesssim 2^{j\theta_1} \int_\Gamma \Upsilon\left(\frac{2^j r}{\sqrt{|z|}}\right)^{\theta_2} e^{-\alpha 4^j r^2/|z|} \frac{\sqrt{t}}{\sqrt{|z|}} |\eta(z)||dz| \left(\fint_B |g|^{q_0} \, dw\right)^{1/2} \\
&\qquad \lesssim 2^{j\theta_1} \int_0^\infty \Upsilon\left(\frac{2^j r}{\sqrt{s}}\right)^{\theta_2} e^{-\alpha 4^j r^2/s} \frac{\sqrt{t}}{\sqrt{s}} \frac{r^{2m}}{(s+t)^{m+1}} \, ds.
\end{aligned}$$

If we take $2m > \theta_2$, we can combine this with (9.7). We can then insert this estimate into the representation (2.10) to get that for every $j \geq 2$,

$$\left(\fint_{C_j(B)} |\nabla e^{-tL_w}(I - e^{-r^2 L_w})^m g|^{q_0}\, dw\right)^{1/q_0} \lesssim \int_0^\infty \left(\fint_{C_j(B)} \left|\sqrt{t}\,\nabla\varphi(L_w,t)g\right|^{q_0} dw\right)^{1/q_0} \frac{dt}{t} \lesssim 2^{j(\theta_1 - 2m)}. \tag{9.15}$$

Taken together, (9.13)–(9.15) yield

$$\left(\fint_B G_B\, dw\right)^{1/q_0'} \lesssim M_w(|\vec{f}|^{q_0'})(x)^{1/q_0'} \sum_{j=1}^\infty 2^{j(D+\theta_1-2m)} \lesssim M_w(|\vec{f}|^{q_0'})(x)^{1/q_0'} = G(x)^{1/q_0'},$$

provided we take m large enough so that $D + \theta_1 - 2m < 0$. This completes the estimate of H_B and G_B and so completes our proof. □

10. Square function estimates for the gradient of the semigroup

In this section we prove $L^p(w)$ estimates for the vertical square function

$$G_{L_w} f(x) = \left(\int_0^\infty |t^{1/2}\nabla e^{-tL_w} f(x)|^2 \frac{dt}{t}\right)^{1/2}.$$

Proposition 10.1. *Let $q_-(L_w) < p < q_+(L_w)$. Then*

$$\|G_{L_w} f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}. \tag{10.2}$$

Furthermore, if $v \in A_{p/q_-(L_w)}(w) \cap \mathrm{RH}_{(q_+(L_w)/p)'}(w)$, then

$$\|G_{L_w} f\|_{L^p(v\,dw)} \lesssim \|f\|_{L^p(v\,dw)}. \tag{10.3}$$

We can also prove a reverse inequality for G_{L_w}. To do so we need to introduce an auxiliary operator. Define the weighted Laplacian by $\Delta_w = -w^{-1} \operatorname{div} w\nabla$; i.e., Δ_w is the operator L_w if we take the matrix A to be wI, where I is the identity matrix.

Proposition 10.4. *Let $q_+(\Delta_w)' < p < \infty$. Then*

$$\|f\|_{L^p(w)} \lesssim \|G_{L_w} f\|_{L^p(w)}. \tag{10.5}$$

Furthermore, if $v \in A_{p/q_+(\Delta_w)'}(w)$, then

$$\|f\|_{L^p(v\,dw)} \lesssim \|G_{L_w} f\|_{L^p(v\,dw)}. \tag{10.6}$$

Proof of Proposition 10.1. The proof could be done in a way similar to those for the square function g_{L_w} in Section 5. However, we will give a shorter proof that uses the Riesz transform estimates from Section 9.

Let $q_- = q_-(L_w)$ and $q_+ = q_+(L_w)$. Fix p,

$$q_- = p_-(L_w) < p < q_+ \leq p_+(L_w),$$

and $v \in A_{p/q_-}(w) \cap \mathrm{RH}_{(q_+/p)'}(w)$. Then by Proposition 9.1, the Riesz transform is bounded on $L^p(v\,dw)$, and so by Lemma 5.4 it has a bounded extension to $L^p_{\mathbb{H}}(v\,dw)$; i.e., if $g(x,t) \in L^p_{\mathbb{H}}(v\,dw)$, then $\|\nabla L_w^{-1/2} g\|_{L^p_{\mathbb{H}}(v\,dw)} \lesssim \|g\|_{L^p_{\mathbb{H}}(v\,w)}$, where the extension of $\nabla L_w^{-1/2}$ to $\mathbb{H}$-valued functions is defined for $x \in \mathbb{R}^n$ and $t > 0$ by $(\nabla L_w^{-1/2} g)(x,t) = \nabla L_w^{-1/2}(g(\,\cdot\,,t))(x)$.

Define $g_f(x,t) = (tL_w)^{1/2} e^{-tL_w} f(x)$ and $G_f(x,t) = t^{1/2}\nabla e^{-tL_w} f(x)$; then we clearly have $\|g_{L_w} f\|_{L^p(v\,dw)} = \|g_f\|_{L^p_{\mathbb{H}}(v\,dw)}$ and $\|G_{L_w} f\|_{L^p(v\,dw)} = \|G_f\|_{L^p_{\mathbb{H}}(v\,dw)}$. Furthermore, $G_f(x,t) = \nabla L_w^{-1/2}(g_f(\,\cdot\,,t))(x) = (\nabla L_w^{-1/2} g_f)(x,t)$. Hence,

$$
\begin{aligned}
\|G_{L_w} f\|_{L^p(v\,dw)} = \|G_f\|_{L^p_{\mathbb{H}}(v\,dw)} &= \|\nabla L_w^{-1/2} g_f\|_{L^p_{\mathbb{H}}(v\,dw)} \\
&\lesssim \|g_f\|_{L^p_{\mathbb{H}}(v\,dw)} = \|g_{L_w} f\|_{L^p(v\,dw)} \lesssim \|f\|_{L^p(v\,dw)}.
\end{aligned}
$$

To prove the last inequality, we used Proposition 5.1; we also used the fact that $q_- = p_-(L_w) < p < q_+ \le p_+(L_w)$ and $v \in A_{p/q_-}(w) \cap \mathrm{RH}_{(q_+/p)'}(w)$, which together imply $v \in A_{p/p_-(L_w)}(w) \cap \mathrm{RH}_{(p_+(L_w)/p)'}(w)$. This proves (10.3). To prove inequality (10.2), we take $v \equiv 1$. □

To prove Proposition 10.4 we need the following identity relating G_{L_w} and Δ_w. It is a straightforward extension of a similar unweighted result given in [Auscher 2007, Section 7.1]. For completeness we include the proof.

Lemma 10.7. *If $f, g \in L^\infty_c(w)$ then*

$$
\left| \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dw \right| \le (\Lambda + 1) \int_{\mathbb{R}^n} G_{L_w} f(x) \overline{G_{\Delta_w} g(x)}\,dw.
$$

Proof. By the definition and properties of the operators L_w and Δ_w we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dw &= \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} e^{-\varepsilon L_w} f(x) \overline{e^{-\varepsilon \Delta_w} g(x)}\,dw - \lim_{R \uparrow \infty} \int_{\mathbb{R}^n} e^{-RL_w} f(x) \overline{e^{-R\Delta_w} g(x)}\,dw \\
&= -\int_0^\infty \frac{d}{dt} \int_{\mathbb{R}^n} e^{-tL_w} f(x) \overline{e^{-t\Delta_w} g(x)}\,dw\,dt \\
&= \int_0^\infty \int_{\mathbb{R}^n} \big(L_w e^{-tL_w} f(x) \overline{e^{-t\Delta_w} g(x)} + e^{-tL_w} f(x) \overline{\Delta_w e^{-t\Delta_w} g(x)} \big)\,dw\,dt \\
&= \int_0^\infty \int_{\mathbb{R}^n} (A(x)w(x)^{-1} + I) \big(\nabla e^{-tL_w} f(x)\ \overline{\nabla e^{-t\Delta_w} g(x)} \big)\,dw\,dt.
\end{aligned}
$$

Since $\|Aw^{-1}\|_\infty \le \Lambda$, if we apply Hölder's inequality in the t variable we get the desired result. □

Proof of Proposition 10.4. As a consequence of the Gaussian estimate for weighted operators with real symmetric coefficients that were proved in [Cruz-Uribe and Rios 2008], we have that $\Delta_w \in \mathcal{O}(L^1(w) \to L^\infty(w))$. In particular, $q_-(\Delta_w) = p_-(L_{\Delta_w}) = 1$. Further, by the results in Section 8 we have $q_+(\Delta_w) > 2$.

Therefore, by Proposition 10.1, if $1 < p' < q_+(\Delta_w)$, and

$$
u \in A_{p'}(w) \cap \mathrm{RH}_{(q_+(\Delta_w)/p')'}(w), \tag{10.8}
$$

then

$$
\|G_{\Delta_w} f\|_{L^{p'}(u\,dw)} \lesssim \|f\|_{L^{p'}(u\,dw)}. \tag{10.9}
$$

We want to apply inequality (10.9) with $u = v^{1-p'}$. By [Auscher and Martell 2007a, Lemma 4.4], the condition (10.8) is equivalent to $v \in A_{p/q_+(w)'}(w)$.

Now fix $f, g \in L^\infty_c$, and a weight $v \in A_{p/q_+(w)'}(w)$. Then by Lemma 10.7, for $q_+(\Delta_w)' < p < \infty$,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)g(x)\,dw\right| &\le (\Lambda+1)\int_{\mathbb{R}^n} G_{L_w}f(x)G_{\Delta_w}g(x)\,dw\\ &= (\Lambda+1)\int_{\mathbb{R}^n} G_{L_w}f(x)G_{\Delta_w}g(x)v^{1/p}\,v^{-1/p}\,dw\\ &\le (\Lambda+1)\|G_{L_w}f\|_{L^p(v\,dw)}\|G_{\Delta_w}g\|_{L^{p'}(v^{1-p'}\,dw)}\\ &\lesssim \|G_{L_w}f\|_{L^p(v\,dw)}\|g\|_{L^{p'}(v^{1-p'}\,dw)};\end{aligned}$$

the last inequality follows from (10.9). If we take $g = \operatorname{sign}(f)|f|^{p-1}v$, we get

$$\|f\|^p_{L^p(v\,dw)} \lesssim \|G_{L_w}f\|_{L^p(v\,w)}\big\||f|^{p-1}v\big\|_{L^{p'}(v^{1-p'}\,dw)} = \|G_{L_w}f\|_{L^p(v\,dw)}\|f\|^{p/p'}_{L^p(v\,dw)}.$$

This immediately gives us the desired inequality. □

11. Unweighted L^2 Kato estimates

In this section we prove unweighted L^2 estimates for the operators we have considered in the previous sections. These will all be consequences of the weighted $L^p(v\,dw)$ estimates we have already proved: it will only be necessary to find further conditions on $w \in A_2$ so that the weight $v = w^{-1}$ satisfies the requisite conditions.

We are particularly interested in power weights and we recall some well-known facts about them. Define $w_\alpha(x) = |x|^\alpha$, $\alpha > -n$; this restriction guarantees that w_α is locally integrable. We can exactly determine the Muckenhoupt A_p and reverse Hölder RH_s classes of these weights in terms of α: if $-n < \alpha \le 0$, then $w \in A_1$; for $1 < p < \infty$, we have $w \in A_p$ if $-n < \alpha < n(p-1)$. Furthermore, if $0 \le \alpha < \infty$, then $w \in \mathrm{RH}_\infty$; for $1 < q < \infty$, we have $w \in \mathrm{RH}_q$ if $-n/q < \alpha < \infty$. Hence, we easily see that

$$r_{w_\alpha} = \max\{1, 1+\alpha/n\}, \quad s_{w_\alpha} = (\max\{1, (1+\alpha/n)^{-1}\})'. \tag{11.1}$$

We first consider the semigroup e^{-tL_w}, the functional calculus, and the square function g_{L_w}, since these estimates will depend on $p_-(L_w)$ and $p_+(L_w)$ and we have good estimates for these quantities.

Theorem 11.2. *Given a weight $w \in A_2$, suppose $1 \le r_w < 1+\frac{2}{n}$ and $s_w > \frac{n}{2}r_w+1$. Then $e^{-tL_w} : L^2 \to L^2$ is uniformly bounded for all $t > 0$. Similarly, $\varphi(L_w) : L^2 \to L^2$, where φ is any bounded holomorphic function on Σ_μ, $\mu \in (\vartheta, \pi)$, and $g_{L_w} : L^2 \to L^2$.*

In particular, these L^2 estimates hold if we assume that $w \in A_1 \cap \mathrm{RH}_{1+n/2}$, or more generally if $w \in A_r \cap \mathrm{RH}_{(n/2)r+1}$ for $1 < r \le 1+\frac{2}{n}$, or if we take the power weights

$$w_\alpha(x) = |x|^\alpha, \quad -\frac{2n}{n+2} < \alpha < 2.$$

Proof. Let $p = q = 2$, $p_0 = (2^*_w)'$, $q_0 = 2^*_w$, and let $v = w^{-1}$. Then by Proposition 3.1, Corollary 3.3 and the nesting properties of weights, $e^{-tL_w} \in \mathcal{O}(L^2 \to L^2)$ if $w^{-1} \in A_{2/p_0}(w) \cap \mathrm{RH}_{(q_0/2)'}(w)$; in particular, by Lemma 2.30, $e^{-tL_w} : L^2 \to L^2$ is uniformly bounded. However, this weight condition is equivalent to

$$w \in \mathrm{RH}_{(2/p_0)'} \cap A_{q_0/2}.$$

A straightforward computation shows that

$$\frac{q_0}{2} = \frac{nr_w}{nr_w - 2}, \quad \left(\frac{2}{p_0}\right)' = \frac{n}{2} r_w + 1.$$

Since $r_w < 1 + \frac{2}{n}$, we have $r_w < nr_w/(nr_w - 2)$, so we automatically have $w \in A_{q_0/2}$. Therefore, the desired bounds hold if we have $s_w > \frac{n}{2} r_w + 1$. If $w \in A_r \cap \mathrm{RH}_{(n/2)r+1}$ with $1 \le r \le 1 + \frac{2}{n}$, then $r_w \le r$ and $s_w > \frac{n}{2} r + 1 \ge \frac{n}{2} r_w + 1$. The desired conclusion for power weights follows at once from (11.1).

The same argument holds for $\varphi(L_w)$ and g_{L_w}, using Proposition 4.3 or Proposition 5.1, respectively. □

It is straightforward to construct weights more general than power weights that satisfy the conditions on r_w and s_w in the above theorems. For instance, $w \in A_{1+2/n} \cap \mathrm{RH}_{2+n/2}$ (which corresponds to the choice $r = 1 + \frac{2}{n}$) if and only if there exist $u_1, u_2 \in A_1$ such that

$$w = u_1^{2/(n+4)} u_2^{-2/n}.$$

This follows from the Jones factorization theorem and the properties of A_1 weights; see [Cruz-Uribe and Neugebauer 1995].

Remark 11.3. We can modify the proof of Theorem 11.2 to get unweighted L^p estimates for values of p close to 2. We leave the details to the interested reader.

For the reverse inequalities we must take into account the slightly stronger hypotheses in Proposition 6.1; otherwise, the proof of the following result follows exactly as in the proof of Theorem 11.2.

Theorem 11.4. *Given a weight $w \in A_2$, suppose that*

$$1 \le r_w < 1 + \frac{2}{n} \quad \textit{and} \quad s_w > \max\left\{\left(\frac{2}{r_w}\right)', \frac{n}{2} r_w + 1\right\}.$$

Then

$$\|L_w^{1/2} f\|_{L^2} \le C \|\nabla f\|_{L^2}, \quad f \in \mathcal{S}. \tag{11.5}$$

In particular, this is the case if we either assume that $w \in A_1 \cap \mathrm{RH}_{1+n/2}$, or more generally that $w \in A_r \cap \mathrm{RH}_{\max\{(2/r)', (n/2)r+1\}}$, with $1 < r \le 1 + \frac{2}{n}$, or for power weights if we take

$$w_\alpha(x) = |x|^\alpha, \quad -\frac{2n}{n+2} = -\min\left\{\frac{n}{2}, \frac{2n}{n+2}\right\} < \alpha < 2.$$

Remark 11.6. Note that $\max\left\{\left(\frac{2}{r}\right)', \frac{n}{2} r + 1\right\} = \frac{n}{2} r + 1$ provided $r \le 2 - \frac{2}{n}$ and this always holds if $n \ge 4$ as $1 + \frac{2}{n} \le 2 - \frac{2}{n}$. In this case, the conditions in the second part of Theorem 11.4 simplify to the same conditions as in Theorem 11.2.

Remark 11.7. We note that in Theorems 11.2 and 11.4 we can replace $1 \le r_w < 1 + \frac{2}{n}$ with the possibly weaker condition $1 \le r_w < p_+(L_w)/2$. The proof only requires us to take $q_0 = p_+(L_w)$.

For the gradient of the semigroup $\sqrt{t}\nabla e^{-tL_w}$, the Riesz transform $\nabla L_w^{-1/2}$, and the square function G_{L_w}, our estimates depend on $q_+(L_w)$.

Theorem 11.8. *Given a weight $w \in A_2$, suppose $1 \le r_w < q_+(L_w)/2$ and $s_w > \frac{n}{2} r_w + 1$. Then $\sqrt{t}\nabla e^{-tL_w} : L^2 \to L^2$ is uniformly bounded for all $t > 0$. Similarly, we have $\nabla L_w^{-1/2} : L^2 \to L^2$ and $G_{L_w} : L^2 \to L^2$.*

In particular, this is the case if we assume that $w \in A_1 \cap \mathrm{RH}_{n/2+1}$. Furthermore, these L^2 estimates hold if the following is true: given $\Theta \ge 1$ there exists $\varepsilon_0 = \varepsilon_0(\Theta, n, \Lambda/\lambda)$, $0 < \varepsilon_0 \le \frac{1}{2n}$, such that $w \in A_{1+\varepsilon} \cap \mathrm{RH}_{(n/2)(1+\varepsilon)+1}$, $0 \le \varepsilon < \varepsilon_0$, and $[w]_{A_2} \le \Theta$.

For power weights, there exists $\varepsilon_1 = \varepsilon_1(n, \Lambda/\lambda)$, $0 < \varepsilon_1 \le \frac{1}{2}$, such that these estimate holds for

$$w_\alpha(x) = |x|^\alpha, \quad -\frac{2n}{n+2} < \alpha < \varepsilon_1.$$

Proof. We will prove this result for $\sqrt{t}\nabla e^{-tL_w}$ using Proposition 7.1. The proof for $\nabla L_w^{-1/2}$ or G_{L_w} is exactly the same, using Proposition 9.1 or Proposition 10.1.

By Proposition 7.1, $\sqrt{t}\nabla e^{-tL_w} : L^2 \to L^2$ if $w^{-1} = v \in A_{2/q_-(L_w)}(w) \cap \mathrm{RH}_{q_+(L_w)/2)'}(w)$, which is equivalent to

$$w \in \mathrm{RH}_{(2/q_-(L_w))'} \cap A_{q_+(L_w)/2}.$$

Therefore, we need $r_w < q_+(L_w)/2$. Furthermore, since we have $q_-(L_w) = p_-(L_w) \le (2_w^*)'$, we can take

$$s_w > \left(\frac{2}{(2_w^*)'}\right)' = \frac{n}{2} r_w + 1.$$

To get the particular examples stated in the theorem, note first that if we let $r_w = 1$, then it clearly suffices to assume $w \in A_1 \cap \mathrm{RH}_{n/2+1}$, since we showed in Section 8 that $q_+(L_w) > 2$ for every $w \in A_2$.

We now prove the condition for weights $w \in A_{1+\varepsilon}$. In this case it is more difficult to satisfy the condition $r_w < q_+(L_w)/2$ since the right-hand side can be very close to 1, depending on w. Assume then that $w \in A_{1+\varepsilon} \cap \mathrm{RH}_{(n/2)\,(1+\varepsilon)+1}$, with $0 \le \varepsilon < \varepsilon_0 \le \frac{1}{2n}$, $[w]_{A_2} \le \Theta$, and with $\varepsilon_0 > 0$ to be fixed below. Then we have

$$s_w > \frac{n}{2}\,(1+\varepsilon) + 1 \ge \frac{n}{2}\, r_w + 1.$$

Therefore, in order to apply the first half of the theorem we need to show that we can choose ε_0 sufficiently small so that $r_w < q_+(L_w)/2$. To do so we will use the notation and computations from Section 8. There we showed that $q_+(L_w) \ge q_w$, and so it will suffice to show that

$$2r_w < q_w = \min\{r_w', p_+(L_w), p_0\}. \tag{11.9}$$

We will compare r_w to each term in the minimum in turn.

The first two terms are straightforward. First, we have $r_w < 1 + \varepsilon < 1 + \frac{1}{2n} < \frac{3}{2}$ and so $2r_w < r_w'$. Second, $r_w < 1 + \frac{1}{2n} < 1 + \frac{2}{n}$, and it follows at once from this that $2r_w < 2_w^*$. By Proposition 3.1, $2_w^* \le p_+(L_w)$ and so $2r_w < p_+(L_w)$.

Finally, we estimate p_0, the exponent from the higher-integrability condition (8.3). We will use the formula (8.4). First, we need to fix the exponent q from the Poincaré inequality (8.2). Let $q = 2 - 1/n$; this value satisfies (8.1) since $r_w < 1 + \frac{1}{2n} < 1 + \frac{1}{n}$. With this choice of q (that only depends on n), we have

$$p_0 = 2 + \frac{2-q}{2^{4/q+1} C_1^2 C_2^2 [w]_{A_2}^{6/q+17}} = 2 + \frac{1}{nC(n, \Lambda/\lambda)\, [w]_{A_2}^{\theta_n}},$$

where $C(n, \Lambda/\lambda) \geq 1$ depends only on n and the ratio Λ/λ of the ellipticity constants of the matrix A used to define L_w, and where $\theta_n \geq 1$ depends only on n. Then, since we also assumed that $[w]_{A_2} \leq \Theta$, we get that

$$p_0 = 2 + \frac{1}{nC(n, \Lambda/\lambda)\, [w]_{A_2}^{\theta_n}} \geq 2 + \frac{1}{nC(n, \Lambda/\lambda)\, \Theta^{\theta_n}} = 2 + 2\varepsilon_0,$$

and $\varepsilon_0 = (2nC(n, \Lambda/\lambda)\, \Theta^{\theta_n})^{-1}$ is such that $0 < \varepsilon_0 \leq \frac{1}{2n}$. Thus $2r_w < 2(1+\varepsilon) < 2(1+\varepsilon_0) \leq p_0$ and so $2r_w < p_0$. This completes the proof that (11.9) is satisfied, and so the L^2 estimates hold for weights that satisfy $w \in A_{1+\varepsilon} \cap \mathrm{RH}_{(n/2)(1+\varepsilon)+1}$.

Finally, we consider power weights. First, it is easy to see that

$$w_\alpha(x) = |x|^\alpha, \quad \frac{-2n}{n+2} < \alpha \leq 0$$

yields the desired estimates, since in this case $r_w = 1$ and $s_w > \frac{n}{2} + 1 = \frac{n}{2} r_w + 1$.

Now consider the case $\alpha > 0$. If we assume that $\alpha < \frac{1}{2}$, then $w \in A_{1+1/(2n)} \cap \mathrm{RH}_\infty$. Moreover, it is straightforward to show that for all such α, there exists Θ, depending only on n, such that $[w_\alpha]_{A_2} \leq \Theta$. Now apply the above argument to find $\varepsilon_0 \in \left(0, \frac{1}{2n}\right]$; this value will only depend on n and the ratio Λ/λ. If we let $\varepsilon_1 = n\varepsilon_0$ and assume that $0 < \alpha < \varepsilon_1$, then $\alpha < \frac{1}{2}$ and $w_\alpha \in A_{1+\varepsilon}$ for some $\varepsilon < \varepsilon_0$ as desired. □

To find examples of weights other than power weights to which Theorem 11.8 apply, we argue as before. If $u_1 \in A_1$, then

$$w = u_1^{2/(n+2)} \in A_1 \cap \mathrm{RH}_{n/2+1}.$$

To get weights that are not in A_1, take $u \in A_2$ and let $w = u^\theta$. If θ is sufficiently small (depending on n, the ratio Λ/λ and $[u]_{A_2}$), we can show that w satisfies the final conditions given in Theorem 11.8. Details are left to the interested reader.

Remark 11.10. To get the unweighted lower estimate

$$\|f\|_{L^2} \leq C \|G_{L_w} f\|_{L^2},$$

we note that by (10.6) we need $w^{-1} \in A_{2/q_+(\Delta_w)'}(w)$, or equivalently, $w \in \mathrm{RH}_{(2/q_+(\Delta_w)')'}$. Hence, it suffices to assume

$$s_w > 1 + \frac{q_+(\Delta_w)}{q_+(\Delta_w) - 2}.$$

Arguing as above we can construct weights that satisfy this condition; details are left to the interested reader.

If we combine Theorems 11.4, 11.8, and Remark 11.7 we solve the Kato square root problem for degenerate elliptic operators.

Theorem 11.11. *Let $L_w = -w^{-1} \operatorname{div} A\nabla$ be a degenerate elliptic operator with $w \in A_2$. If*

$$1 \le r_w < \frac{q_+(L_w)}{2} \quad \text{and} \quad s_w > \max\left\{\left(\frac{2}{r_w}\right)', \frac{n}{2} r_w + 1\right\},$$

then the Kato problem can be solved for L_w; that is, for every $f \in H^1(\mathbb{R}^n)$,

$$\|L_w^{1/2} f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)}, \tag{11.12}$$

where the implicit constants depend only on the dimension, the ellipticity constants λ, Λ, and w.

In particular, (11.12) holds if $w \in A_1 \cap \mathrm{RH}_{n/2+1}$. Further, (11.12) holds if the following is true: given $\Theta \ge 1$ there exists $\varepsilon_0 = \varepsilon_0(\Theta, n, \Lambda/\lambda)$, $0 < \varepsilon_0 \le \frac{1}{2n}$, such that $w \in A_{1+\varepsilon} \cap \mathrm{RH}_{\max\{(2/(1+\varepsilon))', (n/2)(1+\varepsilon)+1\}}$, $0 \le \varepsilon < \varepsilon_0$, and $[w]_{A_2} \le \Theta$.

For power weights, there exists $\varepsilon_1 = \varepsilon_1(n, \Lambda/\lambda)$, $0 < \varepsilon_1 \le \frac{1}{2}$, such that inequality (11.12) holds (with w_α in place of w) if

$$w_\alpha(x) = |x|^\alpha, \quad -\frac{2n}{n+2} < \alpha < \varepsilon_1.$$

We can restate the final part of Theorem 11.11 as follows: consider the family of operators $L_\gamma = -|x|^\gamma \operatorname{div}(|x|^{-\gamma} B(x)\nabla)$, where B is an $n \times n$ complex-valued matrix that satisfies the uniform ellipticity condition

$$\lambda|\xi|^2 \le \operatorname{Re}\langle B(x)\xi, \xi\rangle, \quad |\langle B(x)\xi, \eta\rangle| \le \Lambda|\xi||\eta|, \quad \xi, \eta \in \mathbb{C}^n, \text{ a.e. } x \in \mathbb{R}^n.$$

Then,

$$\|L_\gamma^{1/2} f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)}, \quad -\varepsilon_1 < \gamma < \frac{2n}{n+2}. \tag{11.13}$$

When $\gamma = 0$ we get the classical Kato square root problem solved by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian [Auscher et al. 2002]. Inequality (11.13) shows that we can find an open interval containing 0 such that if γ is in this interval, the same estimate holds.

12. Applications to L^2 boundary value problems

In this section we apply the results from the previous section to some L^2 boundary value problems involving the degenerate elliptic operator L_w. We follow the ideas in [Auscher and Tchamitchian 1998] and consider semigroup solutions: for the Dirichlet or regularity problems we let $u(x,t) = e^{-tL_w^{1/2}} f(x)$; for the Neumann problem we let $u(x,t) = -L_w^{-1/2} e^{-tL_w^{1/2}} f(x)$. In each case, for $t > 0$ fixed, $L_w u(\cdot, t)$ makes sense in a weak sense since $u(\cdot, t)$ is in the domain of L_w. Further, derivatives in t are well-defined because of the semigroup properties. Finally, note that by the strong continuity of the semigroup and the off-diagonal estimates, in the context of the following results we have $e^{-tL_w^{1/2}} f \to f$ as $t \to 0^+$ in L^2; see [Auscher and Martell 2007b, Section 4.2]. Further details are left to the interested reader.

We first consider the Dirichlet problem on $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times [0,\infty)$:

$$\begin{cases} \partial_t^2 u - L_w u = 0 & \text{on } \mathbb{R}^n, \\ u|_{\partial\mathbb{R}^{n+1}_+} = f & \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases} \tag{12.1}$$

Theorem 12.2. *Given a weight $w \in A_2$, suppose $1 \le r_w < 1 + \frac{2}{n}$ and $s_w > \frac{n}{2} r_w + 1$. Then for any $f \in L^2(\mathbb{R}^n)$, we have that $u(x,t) = e^{-tL_w^{1/2}} f(x)$ is a solution of* (12.1) *with convergence to the boundary data as $t \to 0^+$ in the L^2-sense. Furthermore, we have*

$$\sup_{t>0} \|u(\cdot,t)\|_{L^2} \le C\|f\|_{L^2}. \tag{12.3}$$

In particular, this is the case if we assume that $w \in A_1 \cap \mathrm{RH}_{1+n/2}$, or $w \in A_r \cap \mathrm{RH}_{n/2\,r+1}$ with $1 < r \le 1 + \frac{2}{n}$, or if we take the power weights

$$w_\alpha(x) = |x|^\alpha, \quad -\frac{2n}{n+2} < \alpha < 2.$$

Proof. Formally, it is clear that u is a solution to (12.1), and this formalism can be justified by appealing to the theory of maximal accretive operators; see [Kato 1966]. Alternatively, the weighted estimates for the functional calculus in Proposition 4.3 show that both $(\partial^2/\partial t^2)u(\cdot,t)$ and $L_w u(\cdot,t)$ belong to L^2 for each $t > 0$ and that they are equal in the L^2-sense. To see that inequality (12.3) holds, it suffices to let $\varphi_t(z) = e^{-t\sqrt{z}}$. Then φ_t is a bounded holomorphic function on Σ_μ, and so by Theorem 11.2 we get the desired bound. □

Remark 12.4. Note that as observed in Remark 11.7, in the previous result we can replace $1 \le r_w < 1 + \frac{2}{n}$ with the possibly weaker condition $1 \le r_w < p_+(L_w)/2$. Also, by Proposition 4.3 we also have that for u as in Theorem 12.2 and all $k \ge 1$,

$$\sup_{t>0} \left\| t^k \frac{\partial^k}{\partial t^k} u(\cdot,t) \right\|_{L^2} = \sup_{t>0} \left\| (t^k L_w^{1/2})^k e^{-tL_w^{1/2}} f(\cdot) \right\|_{L^2} \le C\|f\|_{L^2}. \tag{12.5}$$

For the regularity problem we have the following.

Theorem 12.6. *Given a weight $w \in A_2$, suppose*

$$1 \le r_w < \frac{q_+(L_w)}{2} \quad \text{and} \quad s_w > \max\left\{ \left(\frac{2}{r_w}\right)', \frac{n}{2} r_w + 1 \right\}.$$

Then for any $f \in H^1(\mathbb{R}^n)$, we have $u(x,t) = e^{-tL_w^{1/2}} f(x)$ is a solution of (12.1) *with convergence to the boundary data as $t \to 0^+$ in the L^2-sense. Furthermore, we have*

$$\sup_{t>0} \|\nabla_{x,t} u(\cdot,t)\|_{L^2} \le C\|\nabla f\|_{L^2}. \tag{12.7}$$

In particular, (12.7) *holds if we assume that $w \in A_1 \cap \mathrm{RH}_{1+n/2}$. Furthermore, it holds if the following is true: given $\Theta \ge 1$ there exists $\varepsilon_0 = \varepsilon_0(\Theta, n, \Lambda/\lambda)$, $0 < \varepsilon_0 \le \frac{1}{2n}$, such that $w \in A_{1+\varepsilon} \cap \mathrm{RH}_{\max\{(2/(1+\varepsilon))', (n/2)(1+\varepsilon)+1\}}$, $0 \le \varepsilon < \varepsilon_0$, and $[w]_{A_2} \le \Theta$.*

For power weights, there exists $\varepsilon_1 = \varepsilon_1(n, \Lambda/\lambda)$, $0 < \varepsilon_1 \le \frac{1}{2}$, *such that* (12.7) *holds if*

$$w_\alpha(x) = |x|^\alpha, \quad -\frac{n}{2} < \alpha < \varepsilon_1.$$

Proof. Arguing as before, it suffices to prove that (12.7) holds. For any $t > 0$ we have, by Theorem 11.11,

$$\begin{aligned}\|\nabla_{x,t} u(\cdot\,,t)\|_{L^2} &\le \|\nabla L_w^{-1/2} L_w^{1/2} e^{-tL_w^{1/2}} f\|_{L^2} + \|L_w^{1/2} e^{-tL_w^{1/2}} f\|_{L^2} \\ &\lesssim \|L_w^{1/2} e^{-tL_w^{1/2}} f\|_{L^2} = \|e^{-tL_w^{1/2}} L_w^{1/2} f\|_{L^2} \lesssim \|L_w^{1/2} f\|_{L^2} \lesssim \|\nabla f\|_{L^2}. \qquad \square\end{aligned}$$

Note that under the hypothesis of Theorem 12.6, and as observed in Remark 12.4, we have that $u(\cdot\,,t) = e^{-tL_w^{1/2}} f$ satisfies (12.3) and (12.5). Additionally, from the functional calculus estimates on L^2 it follows that

$$\sup_{t>0} \|t\nabla_{x,t} u(\cdot\,,t)\|_{L^2} \lesssim \|tL_w^{1/2} e^{-tL_w^{1/2}} f\|_{L^2} \lesssim \|f\|_{L^2}. \tag{12.8}$$

Finally, we consider the Neumann problem

$$\begin{cases} \partial_t^2 u - L_w u = 0 & \text{on } \mathbb{R}^n, \\ \partial_t u|_{\partial\mathbb{R}^{n+1}_+} = f & \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases} \tag{12.9}$$

Theorem 12.10. *Given a weight* $w \in A_2$, *suppose* $1 \le r_w < q_+(L_w)/2$ *and* $s_w > \frac{n}{2} r_w + 1$. *Then for any* $f \in L^2(\mathbb{R}^n)$, *we have* $u(x,t) = -L_w^{-1/2} e^{-tL_w^{1/2}} f(x)$ *is a solution of* (12.9) *with convergence of* $\partial_t u(\cdot\,,t) \to f$ *as* $t \to 0^+$ *in the* L^2*-sense. Furthermore, we have*

$$\sup_{t>0} \|\nabla_{x,t} u(\cdot\,,t)\|_{L^2} \le C\|f\|_{L^2}. \tag{12.11}$$

In particular, (12.11) *holds if we assume that* $w \in A_1 \cap \mathrm{RH}_{1+n/2}$. *Furthermore, it holds if the following is true: given* $\Theta \ge 1$ *there exists* $\varepsilon_0 = \varepsilon_0(\Theta, n, \Lambda/\lambda)$, $0 < \varepsilon_0 \le \frac{1}{2n}$, *such that* $w \in A_{1+\varepsilon} \cap \mathrm{RH}_{(n/2)(1+\varepsilon)+1}$, $0 \le \varepsilon < \varepsilon_0$, *and* $[w]_{A_2} \le \Theta$.

For power weights, there exists $\varepsilon_1 = \varepsilon_1(n, \Lambda/\lambda)$, $0 < \varepsilon_1 \le \frac{1}{2}$, *such that* (12.11) *holds if*

$$w_\alpha(x) = |x|^\alpha, \quad -\frac{2n}{n+2} < \alpha < \varepsilon_1.$$

Proof. Again, u is clearly a formal solution of (12.9); see [Kato 1966]. The proof that (12.11) holds is similar to the proof of (12.7):

$$\|\nabla_{x,t} u(\cdot\,,t)\|_{L^2} \le \|\nabla L_w^{-1/2} e^{-tL_w^{1/2}} f\|_{L^2} + \|e^{-tL_w^{1/2}} f\|_{L^2} \lesssim \|e^{-tL_w^{1/2}} f\|_{L^2} \lesssim \|f\|_{L^2},$$

where we have used Theorem 11.8 (for the Riesz transform) and Theorem 11.2 (for the functional calculus with $\varphi(z) = e^{-t\sqrt{z}}$). $\square$

Remark 12.12. As we noted in Remark 11.3, we can also get unweighted L^p bounds for these operators for values of p close to 2. As a consequence we can also get estimates for L^p boundary value problems for the same values of p. Details are left to the reader.

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SMALL DATA GLOBAL REGULARITY FOR HALF-WAVE MAPS

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We formulate the half-wave maps problem with target S^2 and prove global regularity in sufficiently high spatial dimensions for a class of small critical data in Besov spaces.

1. The problem

Let $u:\mathbb{R}^{n+1}\to S^2\hookrightarrow\mathbb{R}^3$ be smooth, and assume that it converges to some $p\in S^2$ at spatial infinity. Further, assume that on each fixed time slice $\nabla_{t,x}u\in L^p(\mathbb{R}^n)$ for some $p\in(1,\infty)$. Denote by $\times$ the standard vectorial product in three dimensions. We call this a *half-wave map*, provided it satisfies the relation

$$\partial_t u = u\times(-\triangle)^{\frac{1}{2}}u. \tag{1-1}$$

Here we define the operator $(-\triangle)^{\frac{1}{2}}$ via

$$(-\triangle)^{\frac{1}{2}}u = -\sum_{j=1}^{n}(-\triangle)^{-\frac{1}{2}}\partial_j(\partial_j u),$$

a specification necessary on account of the fact that u does not vanish at infinity, but instead approaches some $p\in S^2$, while ∇u does vanish at infinity. In fact, the expression $(-\triangle)^{\frac{1}{2}}u$ under our current definition is then well-defined since $\nabla_{t,x}u(t,\cdot)\in L^p(\mathbb{R}^n)$ for some $p\in(1,\infty)$, for all t.

We note that the model (1-1) appears formally related to the much-studied Schrödinger maps problem, which can be written in the form

$$\partial_t u = u\times\triangle u,$$

and moreover, we shall see shortly that (1-1) also appears closely related to the classical wave maps problem with target S^2. We also note that we have a formally conserved quantity

$$E(t) := \int_{\mathbb{R}^n}|(-\triangle)^{\frac{1}{4}}u|^2\,dx, \tag{1-2}$$

where we let $(-\triangle)^{\frac{1}{4}}u=-\sum_{j=1}^{n}(-\triangle)^{-\frac{3}{4}}\partial_j(\partial_j u)$. Such kinds of quantities have been considered in the works of Da Lio and Rivière in the study of fractional harmonic maps; see for instance [Da Lio and Rivière 2011a; 2011b; Da Lio 2013]. We also note that on account of the results on fractional harmonic maps previously mentioned, this model moreover displays a very rich class of static solutions; see also [Millot and Sire 2015].

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On the other hand, (1-1) scales just like wave maps, which means that in all dimensions $n \geq 2$ the problem (1-1) is formally supercritical.

We formulated the model (1-1) as a nonlocal wave analogue to Schrödinger maps in 2014, but have since learned from E. Lenzmann[1] that it already exists in the physics literature. We learned from Lenzmann that the half-wave map equation arises as the continuum version of the so-called integrable spin Calogero–Moser systems, which in turn comes from the completely integrable quantum spin systems called Haldane–Shastry systems.[2] Recent work by Schikorra and Lenzmann [2017] completely classifies the travelling wave solutions for this model in the critical case $n = 1$.

In the present note, our goal is to approach the issue of global solutions corresponding to small data, attempting to parallel the developments in [Tataru 1998; Tao 2001a]. We will see that (1-1) can be reformulated as a nonlinear wave-type equation of the schematic form

$$\Box u = F(u)\nabla_{t,x}u \cdot \nabla_{t,x}u, \tag{1-3}$$

although this is an oversimplification as the true underlying wave equation displays nonlocal expressions. It has been known now for a while, see [Sterbenz 2004], that (1-3) admits global solutions corresponding to initial data of small critical, i.e., scaling invariant, Besov $\dot{B}_2^{\frac{n}{2},1}$ norms, provided one restricts oneself to spatial dimensions $n \geq 6$, and that passing to lower dimensions appears to require some sort of null-structure. Here, we show that (1-1) does have enough of an intrinsic null-structure to allow for the following.

Theorem 1.1. *Let $n \geq 5$. Let*

$$u[0] = (u(0,\cdot), u_t(0,\cdot)) = (u_0, u_1) : \mathbb{R}^n \to S^2 \times TS^2$$

be a smooth data pair with $u_1 = u_0 \times (-\triangle)^{\frac{1}{2}} u_0$, and such that u_0 is constant outside of a compact subset of $\mathbb{R}^n$ (this condition in particular ensures that $(-\triangle)^{\frac{1}{2}} u_0$ is well-defined). Also, assume the smallness condition

$$\|u[0]\|_{\dot{B}_2^{n/2,1} \times \dot{B}_2^{n/2-1,1}} < \epsilon,$$

where $\epsilon \ll 1$ is sufficiently small. Then problem (1-1) *admits a global smooth solution.*

To prove this theorem, we shall have to reformulate (1-1) as a wave equation, which we do next.

Remark 1.2. We note that the restriction $n \geq 5$ comes from the fact that we use the $L_t^2 L_x^4$-Strichartz estimate, which is not available in spatial dimension $n = 4$. However, it is quite likely that this can be circumvented, and that the structures exhibited in this paper suffice to push the result to $n = 4$. However, both the issue of passing to the critical space $\dot{H}^{\frac{n}{2}}$, as well as going to lower spatial dimensions $n \leq 3$, appear nontrivial, as there are novel trilinear terms which no longer seem to have the same strong null-structure as the leading term coming from the wave maps equation.

[1] The name of half-wave map was suggested by Lenzmann.

[2] Lenzmann provided us with the references [Haldane 1988; Shastry 1988; Hikami and Wadati 1993; Blom and Langmann 1998] and we refer to his work for an account on the passage from the physics to the mathematical model.

2. Passage to a wave equation

Departing from (1-1), we compute

$$\begin{aligned}\partial_t^2 u &= \partial_t u \times (-\triangle)^{\frac{1}{2}} u + u \times (-\triangle)^{\frac{1}{2}}\, \partial_t u \\ &= (u \times (-\triangle)^{\frac{1}{2}} u) \times (-\triangle)^{\frac{1}{2}} u + u \times (-\triangle)^{\frac{1}{2}} (u \times (-\triangle)^{\frac{1}{2}} u).\end{aligned}$$

Then using the basic formula $a \times (b \times c) = b(a \cdot c) - c(a \cdot b),\ a, b, c \in \mathbb{R}^3$, we rewrite the first term on the right as

$$(u \times (-\triangle)^{\frac{1}{2}} u) \times (-\triangle)^{\frac{1}{2}} u = -u((-\triangle)^{\frac{1}{2}} u \cdot (-\triangle)^{\frac{1}{2}} u) + (-\triangle)^{\frac{1}{2}} u (u \cdot (-\triangle)^{\frac{1}{2}} u).$$

For the second term on the right above, introducing a commutator term, we write it in the form

$$\begin{aligned}u \times (-\triangle)^{\frac{1}{2}} (u \times (-\triangle)^{\frac{1}{2}} u) &= u \times (-\triangle)^{\frac{1}{2}} (u \times (-\triangle)^{\frac{1}{2}} u) - u \times (u \times (-\triangle) u) + u \times (u \times (-\triangle) u) \\ &= u \times (-\triangle)^{\frac{1}{2}} (u \times (-\triangle)^{\frac{1}{2}} u) - u \times (u \times (-\triangle) u) + u(u \cdot (-\triangle) u) + \triangle u.\end{aligned}$$

Using the fact that $u \cdot u = 1$, whence

$$u \cdot \triangle u + \nabla u \cdot \nabla u = 0,$$

we arrive at the equation

$$\begin{aligned}(\partial_t^2 - \triangle) u = -u((-\triangle)^{\frac{1}{2}} u \cdot (-\triangle)^{\frac{1}{2}} u) + (-\triangle)^{\frac{1}{2}} u (u \cdot (-\triangle)^{\frac{1}{2}} u) + u \times (-\triangle)^{\frac{1}{2}} (u \times (-\triangle)^{\frac{1}{2}} u) \\ - u \times (u \times (-\triangle) u) + u(\nabla u \cdot \nabla u).\end{aligned}$$

Carefully note that ∇u here only involves the spatial derivatives. In order to make this appear closer to the wave maps equation and introduce better null-structure, we have to also make the time derivatives visible on the right-hand side, for which the first line on the right-hand side is pivotal. In fact, we get

$$\big(-u((-\triangle)^{\frac{1}{2}} u \cdot (-\triangle)^{\frac{1}{2}} u) + (-\triangle)^{\frac{1}{2}} u (u \cdot (-\triangle)^{\frac{1}{2}} u)\big) \cdot u = -\big|u \times (-\triangle)^{\frac{1}{2}} u\big|^2 = -|\partial_t u|^2,$$

and so the equation becomes

$$\begin{aligned}(\partial_t^2 - \triangle) u = u(\nabla u \cdot \nabla u - \partial_t u \cdot \partial_t u) + \Pi_{u_\perp}((-\triangle)^{\frac{1}{2}} u)(u \cdot (-\triangle)^{\frac{1}{2}} u) \\ + u \times (-\triangle)^{\frac{1}{2}} (u \times (-\triangle)^{\frac{1}{2}} u) - u \times (u \times (-\triangle) u),\end{aligned} \tag{2-1}$$

where $\Pi_{u_\perp}$ denotes projection onto the orthogonal complement of u. Thus we see that formally the nonlinearity involves the precise wave maps source term, as well as two error terms, which formally behave like

$$u \nabla u \cdot \nabla u.$$

3. Technical preliminaries

Our main tools shall be the classical Strichartz estimates, combined with some $X^{s,b}$-space technology. Specifically, we let P_k, $k \in \mathbb{Z}$, be standard Littlewood–Paley multipliers on $\mathbb{R}^n$ (acting on the spatial variables), and furthermore, we denote by $Q_j,\ j \in \mathbb{Z}$, multipliers which localise a space-time function $F(t, x)$

to dyadic distance $\sim 2^j$ from the light cone $|\tau| = |\xi|$ on the Fourier side. Specifically, letting $\widetilde{F}(\tau,\xi)$ denote the space time Fourier transform of F, while $\hat{f}(\xi)$ denotes the Fourier transform with respect to the spatial variables, and letting $\chi \in C_0^\infty(\mathbb{R}_+)$ be a smooth cutoff satisfying

$$\sum_{k\in\mathbb{Z}} \chi\Big(\frac{x}{2^k}\Big) = 1 \quad \text{for all } x \in \mathbb{R}_+,$$

we set

$$\widehat{P_k f}(\xi) = \chi\Big(\frac{|\xi|}{2^k}\Big)\hat{f}(\xi), \quad \widetilde{Q_j F} = \chi\Big(\frac{\big||\tau|-|\xi|\big|}{2^j}\Big)\widetilde{F}(\tau,\xi).$$

Using these ingredients one can then define the norms

$$\|u\|_{\dot{X}^{n/2,1/2,\infty}} := \sup_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\|\nabla_x^{\frac{n}{2}} Q_j u\|_{L^2_{t,x}}, \quad \|F\|_{\dot{X}^{n/2-1,-1/2,1}} := \sum_{j\in\mathbb{Z}} 2^{-\frac{j}{2}}\|\nabla_x^{\frac{n}{2}-1} Q_j F\|_{L^2_{t,x}}.$$

In addition to these, we rely on the classical Strichartz norms, which are the mixed-type Lebesgue norms $\|\cdot\|_{L_t^pL_x^q}$, $\frac{1}{p}+\frac{n-1}{2q} \le \frac{n-1}{4}$, $p \ge 2$, where we shall always restrict to $n \ge 5$. Call such pairs (p,q) admissible.

We shall freely use the fact that Fourier localisers of the form P_kQ_j act in bounded fashion on spaces of the form $L_t^pL_x^2$, $1 \le p \le \infty$; see e.g., [Tao 2001b]. We can now define a norm controlling our solutions as follows:

$$\|u\|_S := \sum_{k\in\mathbb{Z}} \sup_{(p,q)\text{ admissible}} 2^{(\frac{1}{p}+\frac{n}{q}-1)k}\|\nabla_{t,x}P_ku\|_{L_t^pL_x^q} + \|\nabla_{t,x}P_ku\|_{\dot{X}^{n/2-1,1/2,\infty}} =: \sum_{k\in\mathbb{Z}}\|P_ku\|_{S_k}. \tag{3-1}$$

We also introduce

$$\|F\|_N := \sum_{k\in\mathbb{Z}}\|P_kF\|_{L_t^1\dot{H}^{n/2-1}+\dot{X}^{n/2-1,-1/2,1}}, \tag{3-2}$$

as well as the norms

$$\|u\|_{\dot{B}_2^{r,1}} := \sum_{k\in\mathbb{Z}}\|P_ku\|_{\dot{H}^r}.$$

Then the following inequality is by now completely standard; see e.g., [Krieger 2008; Tao 2001a; Tataru 1998]:

Proposition 3.1.
$$\|u\|_S \lesssim \|u[0]\|_{\dot{B}_2^{n/2,1}\times\dot{B}_2^{n/2-1,1}} + \|\Box u\|_N. \tag{3-3}$$

Sketch of proof. The fact that

$$\sum_{k\in\mathbb{Z}} \sup_{(p,q)\text{ admissible}} 2^{(\frac{1}{p}+\frac{n}{q}-1)k}\|\nabla_{t,x}P_ku\|_{L_t^pL_x^q} \lesssim \sum_{k\in\mathbb{Z}}\big[\|P_ku[0]\|_{\dot{H}^{n/2}\times\dot{H}^{n/2-1}} + \|\Box P_ku\|_{L_t^1\dot{H}^{n/2-1}}\big]$$

is a direct consequence of the Strichartz estimates; see, e.g., [Shatah and Struwe 1998]. The fact that

$$\|\nabla_{t,x}P_ku\|_{\dot{X}^{n/2-1,1/2,\infty}} \lesssim \|\Box P_ku\|_{L_t^1\dot{H}^{n/2-1}}$$

follows by localising the modulation and applying Holder's inequality: setting $k=0$, as we may by scaling invariance,

$$2^{\frac{j}{2}}\|\nabla_{t,x}P_0Q_ju\|_{L^2_{t,x}}\sim 2^{\frac{j}{2}}(1+2^j)\big\|\chi(|\xi|)\widehat{Q_ju}(t,\xi)\big\|_{L^2_tL^2_\xi}$$
$$\lesssim 2^j(1+2^j)\big\|\chi(|\xi|)\widehat{Q_ju}(t,\xi)\big\|_{L^2_\xi L^1_t}\lesssim\|\Box P_0Q_ju\|_{L^1_tL^2_x}.$$

Furthermore, the fact that

$$\sup_{(p,q)\text{ admissible}}2^{(\frac{1}{p}+\frac{n}{q}-1)k}\|\nabla_{t,x}P_kQ_ju\|_{L^p_tL^q_x}\lesssim\|\Box P_kQ_ju\|_{\dot{X}^{n/2-1,-1/2,1}}\tag{3-4}$$

is a consequence of the fact that the function P_kQ_ju may be represented as a weighted average of free waves, in conjunction with the standard Strichartz estimates: putting $k=0$ as we may, write

$$\begin{aligned}\widetilde{P_0Q_ju}(\tau,\xi)&=\chi(|\xi|)\chi\bigg(\frac{\big||\tau|-|\xi|\big|}{2^j}\bigg)\tilde{u}(\tau,\xi)\\&=\chi(|\xi|)\sum_{\pm,\pm}\int\chi\Big(\pm\frac{a}{2^j}\Big)\tilde{u}(|\xi|+a,\xi)\delta(\pm\tau-|\xi|-a)\,da\\&=\chi(|\xi|)\sum_{\pm,\pm}\int_{a\sim\pm2^j}\widetilde{e^{\pm ita}u_a^{\pm,\pm}}(\tau,\xi)\,da.\end{aligned}$$

Here each $u_a^{\pm,\pm}$ is a free wave and we have

$$\sum_{\pm,\pm}\int_{a\sim\pm2^j}\|u_a^{\pm,\pm}\|_{L^\infty_tL^2_x}\,da\lesssim 2^{\frac{j}{2}}\sum_{\pm,\pm}\bigg(\int_{a\sim\pm2^j}\|u_a^{\pm,\pm}\|^2_{L^\infty_tL^2_x}\,da\bigg)^{\frac{1}{2}}\lesssim 2^{\frac{j}{2}}\|P_0Q_ju\|_{L^2_{t,x}},$$

where we have used Plancherel's theorem in the last step. This gives the case $j\le k$ in (3-4) as a direct consequence of the Strichartz estimates, while the case $j\ge k$ follows directly from Bernstein's inequality.

The fact that

$$\|\nabla_{t,x}P_ku\|_{\dot{X}^{n/2-1,1/2,\infty}}\lesssim\|\Box P_ku\|_{\dot{X}^{n/2-1,-1/2,1}}$$

is immediate. This concludes our sketch of the proof of the proposition. □

In order to deal with the nonlocal expressions such as $(-\triangle)^{\frac{1}{2}}(u\times(-\triangle)^{\frac{1}{2}}u)$, the following simple lemma shall be useful:

Lemma 3.2. *Consider the bilinear expression* (*where $\chi_{k_j}(\cdot)$ smoothly localises to the annulus $|\xi|\sim 2^{k_j}$*)

$$F(u,v)(x):=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}m(\xi,\eta)\,e^{ix\cdot(\xi+\eta)}\chi_{k_1}(\xi)\,\hat{u}(\xi)\,\chi_{k_2}(\eta)\,\hat{v}(\eta)\,d\xi\,d\eta$$

where the multiplier $m(\xi,\eta)$ is C^∞ with respect to the coordinates on the support of $\chi_{k_1}(\xi)\cdot\chi_{k_2}(\eta)$, and satisfies the pointwise bounds

$$|m(\xi,\eta)|\le\gamma\lesssim 1,\qquad\big|(2^{k_1}\nabla_\xi)^i(2^{k_2}\nabla_\eta)^jm(\xi,\eta)\big|\lesssim_{i,j}1\quad\textit{for all } i,j.$$

Then if $\|\cdot\|_Z$, $\|\cdot\|_Y$, $\|\cdot\|_X$ *are translation invariant norms with the property that*

$$\|u\cdot v\|_Z \le \|u\|_X\cdot\|v\|_Y,$$

then it follows that

$$\|F(u,v)\|_Z \lesssim \gamma^{(1-)}\|P_{k_1}u\|_X\,\|P_{k_2}v\|_Y,$$

where the implied constant only depends on the size of finitely many derivatives of m.

Proof. This follows by Fourier expansion of the multiplier $m(\xi,\eta)$: write

$$m(\xi,\eta)\,\chi_{k_1}(\xi)\,\chi_{k_2}(\eta)=\sum_{m\in\mathbb{Z}^n}\sum_{p\in\mathbb{Z}^n}a_{mp}\,e^{i(2^{-k_1}m\cdot\xi+2^{-k_2}p\cdot\eta)},$$

where we have

$$\begin{aligned}|a_{m_1m_2}| &\le (2^{-k_1}|m|+2^{-k_2}|p|)^{-Mn}\big\|\nabla_{\xi,\eta}^{Mn}\big[m(\xi,\eta)\chi_{k_1}(\xi)\chi_{k_2}\big]\big\|_{L^\infty_{\xi,\eta}}\\ &\lesssim_{M,n}[|m|+|p|]^{-Mn},\end{aligned}$$

while we also get the trivial bound $|a_{m_1m_2}|\le\gamma$. It follows that

$$\begin{aligned}F(u,v)(x)=&\sum_{\substack{m,p\in\mathbb{Z}^n\\ |m|+|p|<\gamma^{-1/(nM)}}}a_{mp}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{u}(\xi)\,\hat{v}(\eta)\,e^{i([2^{-k_1}m+x]\cdot\xi+[2^{-k_2}p+x]\cdot\eta)}\,d\xi\,d\eta\\ &+\sum_{\substack{m,p\in\mathbb{Z}^n\\ |m|+|p|\ge\gamma^{-1/(nM)}}}a_{mp}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{u}(\xi)\,\hat{v}(\eta)\,e^{i([2^{-k_1}m+x]\cdot\xi+[2^{-k_2}p+x]\cdot\eta)}\,d\xi\,d\eta\end{aligned}$$

and so

$$\|F(u,v)\|_Z\lesssim\|u\|_X\,\|v\|_Y\Bigg[\sum_{\substack{m,p\in\mathbb{Z}^n\\ |m|+|p|<\gamma^{-1/(nM)}}}\gamma+\sum_{\substack{m,p\in\mathbb{Z}^n\\ |m|+|p|\ge\gamma^{-1/(nM)}}}(|m|+|p|)^{-Mn}\Bigg]\lesssim\|u\|_X\,\|v\|_Y\gamma^{1-\frac{1}{M}}.$$

Here the constant M may be chosen arbitrarily large (with implied constant depending on M). □

4. Multilinear estimates

Here we gather the multilinear estimates which allow us to obtain a solution for (2-1) by means of a suitable iteration scheme:

Proposition 4.1. *Assume that* u *takes values in* S^2 *and converges to* $p\in S^2$ *at spatial infinity. Then using the norms* $\|\cdot\|_S$, $\|\cdot\|_N$ *introduced in the previous section, we have the bounds*

$$\big\|P_k\big[u(\nabla u\cdot\nabla u-\partial_t u\cdot\partial_t u)\big]\big\|_N\lesssim(1+\|u\|_S)\|u\|_S\bigg(\sum_{k_1\in\mathbb{Z}}2^{-\sigma|k-k_1|}\|P_{k_1}u\|_{S_{k_1}}\bigg).\tag{4-1}$$

Furthermore, if $\tilde{u}$ *maps into a small neighbourhood of* S^2 *and* $\|\tilde{u}\|_S\lesssim 1$, *we have the similar bound*

$$\big\|P_k\big(\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac12}u)(u\cdot(-\triangle)^{\frac12}u)\big)\big\|_N\lesssim\prod_{v=u,\tilde{u}}(1+\|v\|_S)\|u\|_S\bigg(\sum_{k_1\in\mathbb{Z}}2^{-\sigma|k-k_1|}\|P_{k_1}u\|_{S_{k_1}}\bigg),\tag{4-2}$$

as well as

$$\big\| P_k\big(\Pi_{\tilde u_\perp}\big[u\times(-\triangle)^{\frac12}(u\times(-\triangle)^{\frac12}u)-u\times(u\times(-\triangle)u)\big]\big)\big\|_N \lesssim \prod_{v=u,\tilde u}(1+\|v\|_S)\|u\|_S\bigg(\sum_{k_1\in\mathbb{Z}}2^{-\sigma|k-k_1|}\|P_{k_1}u\|_{S_{k_1}}\bigg). \tag{4-3}$$

We also have corresponding difference estimates: assuming that $u^{(j)},\ j=1,2$, *map into* S^2, *while* $\tilde u^{(j)}$ *map into a small neighbourhood of* S^2, *then using the notation*

$$\triangle_{1,2}F^{(j)}=F^{(1)}-F^{(2)}$$

we have

$$\begin{aligned}\big\|\triangle_{1,2}P_k\big[u^{(j)}(\nabla u^{(j)}\cdot\nabla u^{(j)}-\partial_t u^{(j)}\cdot\partial_t u^{(j)})\big]\big\|_N &\\ \lesssim \big(1+\max_j\|u^{(j)}\|_S\big)\big(\max_j\|u^{(j)}\|_S\big)&\bigg(\sum_{k_1\in\mathbb{Z}}2^{-\sigma|k-k_1|}\|P_{k_1}u^{(1)}-P_k u^{(2)}\|_{S_{k_1}}\bigg)\\ +\big(1+\max_j\|u^{(j)}\|_S\big)\big(\|u^{(1)}-u^{(2)}\|_S\big)&\bigg(\max_j\sum_{k_1\in\mathbb{Z}}2^{-\sigma|k-k_1|}\|P_{k_1}u^{(j)}\|_{S_{k_1}}\bigg),\end{aligned}\tag{4-4}$$

and similarly

$$\begin{aligned}\big\|P_k\triangle_{1,2}\big(\Pi_{\tilde u^{(j)}_\perp}((-\triangle)^{\frac12}u^{(j)})(u^{(j)}\cdot(-\triangle)^{\frac12}u^{(j)})\big)\big\|_N &\\ \lesssim \max_j\prod_{v=u^{(j)},\tilde u^{(j)}}(1+\|v\|_S)\|u^{(j)}\|_S&\bigg(\sum_{k_1\in\mathbb{Z}}2^{-\sigma|k-k_1|}\|P_{k_1}u^{(1)}-P_{k_2}u^{(2)}\|_{S_{k_1}}\bigg)\\ +\max_j\prod_{v=u^{(j)},\tilde u^{(j)}}(1+\|v\|_S)\|u^{(1)}-u^{(2)}\|_S&\bigg(\max_j\sum_{k_1\in\mathbb{Z}}2^{-\sigma|k-k_1|}\|P_{k_1}u^{(j)}\|_{S_{k_1}}\bigg)\\ +\max_j(1+\|u^{(j)}\|_S)\|\tilde u^{(1)}-\tilde u^{(2)}\|_S&\bigg(\max_j\sum_{k_1\in\mathbb{Z}}2^{-\sigma|k-k_1|}\|P_{k_1}u^{(j)}\|_{S_{k_1}}\bigg).\end{aligned}\tag{4-5}$$

The analogous difference estimate for (4-3) *is similar. In fact, in all these estimates the choice* $\sigma=1$ *works.*

Proof. We shall only deal in detail with the case $n=5$, since the case $n\geq 6$ is simpler due to the better decay with respect to large frequencies. Also, we note that then the desired estimates follow for a slightly different functional framework from [Sterbenz 2004]. We observe that the proof of (4-1) is really quite standard and follows for example from [Tataru 1998]. For completeness's sake, we include a simple version here.

Before giving the proof, we note that the $X^{s,b}$-type components of our spaces are only used to prove (4-1), and not (4-2), (4-3). The key fact behind the proof of (4-1) is the identity

$$2[u_t\cdot v_t-\nabla_x u\cdot\nabla_x v]=\Box(uv)-\Box u v-u\Box v;$$

see for example [Krieger 2008] for further discussion and earlier references.

On the other hand, the estimates (4-2), (4-3) only use Strichartz norms; there the idea is to move a derivative from a high- to a low-frequency factor, using algebraic relations such as

$$a\times(b\times c)=b(a\cdot c)-c(a\cdot b),\ a,b,c\in\mathbb{R}^3.$$

Proof of (4-1) To achieve it, we localise the second and third factors to frequencies $\sim 2^{k_1}, 2^{k_2}$, respectively, and we shall restrict the output logarithmic frequency k to size 0. This is possible on account of the scaling invariance of the estimate. We shall obtain exponential gains in terms of these frequencies in certain cases, and summation over all allowed frequencies will result in the desired bound (4-1).

(1) *High-high interactions* $\max\{k_1,k_2\}>10$. This is schematically written as

$$P_0[u\,\nabla_{t,x}u_{k_1}\nabla_{t,x}u_{k_2}].$$

Then if $k_1=k_2+O(1)$, we estimate this by

$$\big\|P_0[u\,\nabla_{t,x}u_{k_1}\nabla_{t,x}u_{k_2}]\big\|_{L_t^1L_x^2}\lesssim\|\nabla_{t,x}u_{k_1}\|_{L_t^2L_x^4}\,\|\nabla_{t,x}u_{k_2}\|_{L_t^2L_x^4}\lesssim 2^{-\frac{3}{2}k_1}\prod_{j=1,2}\|u_{k_j}\|_{S_{k_j}}.$$

If $k_2>k_1+10$, say then we estimate it by

$$\begin{aligned}\big\|P_0[u\,\nabla_{t,x}u_{k_1}\nabla_{t,x}u_{k_2}]\big\|_{L_t^1L_x^2}&=\big\|P_0[P_{k_2+O(1)}u\,\nabla_{t,x}u_{k_1}\nabla_{t,x}u_{k_2}]\big\|_{L_t^1L_x^2}\\&\lesssim\|P_{k_2+O(1)}u\|_{L_t^2L_x^4}\,\|\nabla_{t,x}u_{k_2}\|_{L_t^2L_x^4}\,\|\nabla_{t,x}u_{k_1}\|_{L_t^\infty L_x^2+L_{t,x}^\infty}\\&\lesssim 2^{-\frac{5}{2}k_2}\|u_{k_2}\|_{S_{k_2}}\,\|u_{k_2+O(1)}\|_{S_{k_2+O(1)}}\,\|u_{k_1}\|_{S_{k_1}}.\end{aligned}$$

The case $k_1>k_2+10$ is of course the same. Summation over the suitable ranges of k_1, k_2 implies (4-1) in this case with $\sigma=\frac{3}{2}$.

(2) *High-low interactions* $\max\{k_1,k_2\}<-10$. Here one places $\nabla_{t,x}u_{k_j}$, $j=1,2$, into $L_t^2L_x^\infty$ and $u=P_{O(1)}u$ into $L_t^\infty L_x^2$.

(3) *Low-high interactions* $\max\{k_1,k_2\}\in[-10,10]$. This is the most delicate case. We may assume that $k_1<k_2-10$, since else we argue as in (1). Thus $k_2\in[-10,10]$. Note that then

$$\begin{aligned}\big\|P_0[u_{\geq k_1-10}\,\nabla_{t,x}u_{k_1}\nabla_{t,x}u_{k_2}]\big\|_{L_t^1L_x^2}&\lesssim\|u_{\geq k_1-10}\|_{L_t^2L_x^\infty}\,\|\nabla_{t,x}u_{k_1}\|_{L_t^2L_x^\infty}\,\|\nabla_{t,x}u_{k_2}\|_{L_t^\infty L_x^2}\\&\lesssim\|u\|_S\,\|u_{k_2}\|_{S_{k_2}}\,\|u_{k_1}\|_{S_{k_1}},\end{aligned}$$

which can be summed over $k_1<k_2-10$. One similarly estimates

$$P_0[Q_{\geq k_1-10}\,u_{<k_1-10}\nabla_{t,x}u_{k_1}\nabla_{t,x}u_{k_2}].$$

We have now reduced to estimating

$$P_0[Q_{<k_1-10}\,u_{<k_1-10}\,\partial_\alpha u_{k_1}\,\partial^\alpha u_{k_2}].$$

Here note that

$$\begin{aligned}\big\|P_0\big[Q_{<k_1-10}\,u_{<k_1-10}\,\partial_\alpha u_{k_1}\,\partial^\alpha Q_{>k_1-10}\,u_{k_2}\big]\big\|_{L_t^1L_x^2}&\lesssim\|\partial_\alpha u_{k_1}\|_{L_t^2L_x^\infty}\,\|\partial^\alpha Q_{>k_1-10}\,u_{k_2}\|_{L_{t,x}^2}\\&\lesssim\|u_{k_1}\|_{S_{k_1}}\,\|u_{k_2}\|_{S_{k_2}},\end{aligned}$$

again summable over $k_1 < k_2 - 10$. Also, we get

$$\begin{aligned}
&\big\| P_0\big[Q_{<k_1-10}\, u_{<k_1-10}\, \partial_\alpha Q_{\geq k_1+10} u_{k_1}\, \partial^\alpha Q_{<k_1-10}\, u_{k_2}\big]\big\|_{X^{\frac{n}{2}-1,-\frac{1}{2},1}} \\
&\qquad \lesssim \sum_{j\geq k_1+10} 2^{-\frac{j}{2}} \| Q_j \nabla_{t,x}\, , u_{k_1}\|_{L_t^2 L_x^\infty} \|\partial^\alpha Q_{<k_1-10}\, u_{k_2}\|_{L_t^\infty L_x^2} \lesssim \|u_{k_1}\|_{S_{k_1}} \|u_{k_2}\|_{S_{k_2}},
\end{aligned}$$

and hence it is summable over $k_1 < k_2 - 10$. Finally, we expand the expression out using its null-structure:

$$\begin{aligned}
&2P_0[Q_{<k_1-10}\, u_{<k_1-10}\, \partial_\alpha Q_{<k_1+10}\, u_{k_1}\, \partial^\alpha Q_{<k_1-10}\, u_{k_2}] \\
&\qquad = P_0\big[Q_{<k_1-10}\, u_{<k_1-10}\, \Box(Q_{<k_1+10}\, u_{k_1}\, Q_{<k_1-10}\, u_{k_2})\big] \\
&\qquad\quad - P_0\big[Q_{<k_1-10}\, u_{<k_1-10}\, \Box Q_{<k_1+10}\, u_{k_1}\, Q_{<k_1-10}\, u_{k_2}\big] \\
&\qquad\quad - P_0\big[Q_{<k_1-10}\, u_{<k_1-10}\, Q_{<k_1+10}\, u_{k_1}\, \Box Q_{<k_1-10}\, u_{k_2}\big].
\end{aligned}$$

Then we bound each of these:

$$\begin{aligned}
&P_0\big[Q_{<k_1-10}\, u_{<k_1-10}\, \Box(Q_{<k_1+10}\, u_{k_1}\; Q_{<k_1-10}\, u_{k_2})\big] \\
&\qquad = \Box P_0\big[Q_{<k_1-10}\, u_{<k_1-10}(Q_{<k_1+10}\, u_{k_1}\; Q_{<k_1-10}\, u_{k_2})\big] \\
&\qquad\quad - P_0\big[\nabla_{t,x} Q_{<k_1-10}\, u_{<k_1-10}\, \nabla_{t,x}(Q_{<k_1+10}\, u_{k_1}\, Q_{<k_1-10}\, u_{k_2})\big] \\
&\qquad\quad - P_0\big[\nabla_{t,x}^2 Q_{<k_1-10}\, u_{<k_1-10}(Q_{<k_1+10}\, u_{k_1}\; Q_{<k_1-10}\, u_{k_2})\big].
\end{aligned}$$

The last two terms on the right can be easily placed into $L_t^1 L_x^2$ using the $L_t^2 L_x^\infty$ norm for the low-frequency factors, while for the first term on the right, we get

$$\begin{aligned}
&\big\|\Box P_0\big[Q_{<k_1-10}\, u_{<k_1-10}(Q_{<k_1+10}\, u_{k_1}\; Q_{<k_1-10}\, u_{k_2})\big]\big\|_{X^{\frac{n}{2}-1,-\frac{1}{2},1}} \\
&\qquad \lesssim \sum_{j<k_1+20} 2^{\frac{j}{2}} \big\| P_0 Q_j \big[Q_{<k_1-10}\, u_{<k_1-10}(Q_{<k_1+10}\, u_{k_1}\; Q_{<k_1-10}\, u_{k_2})\big]\big\|_{L_{t,x}^2} \\
&\qquad \lesssim 2^{\frac{k_1}{2}} \|Q_{<k_1+10}\, u_{k_1}\|_{L_t^2 L_x^\infty} \|Q_{<k_1-10}\, u_{k_2}\|_{L_t^\infty L_x^2} \\
&\qquad \lesssim \|u_{k_1}\|_{S_{k_1}} \|u_{k_2}\|_{S_{k_2}}.
\end{aligned}$$

Further, we get

$$\begin{aligned}
&\big\|P_0\big[Q_{<k_1-10}\, u_{<k_1-10}\, \Box Q_{<k_1+10}\, u_{k_1}\; Q_{<k_1-10}\, u_{k_2}\big]\big\|_{L_t^1 L_x^2} \\
&\qquad \lesssim \|\Box Q_{<k_1+10}\, u_{k_1}\|_{L_t^2 L_x^4} \|Q_{<k_1-10}\, u_{k_2}\|_{L_t^2 L_x^4} \\
&\qquad \lesssim 2^{\frac{k_1}{4}} \|u_{k_1}\|_{S_{k_1}} \|u_{k_2}\|_{S_{k_2}}.
\end{aligned}$$

To close things, we also get

$$\begin{aligned}
&\big\|P_0\big[Q_{<k_1-10}\, u_{<k_1-10}\; Q_{<k_1+10}\, u_{k_1}\, \Box Q_{<k_1-10}\, u_{k_2}\big]\big\|_{L_t^1 L_x^2} \\
&\qquad \lesssim \|Q_{<k_1+10}\, u_{k_1}\|_{L_t^2 L_x^\infty} \|\Box Q_{<k_1-10}\, u_{k_2}\|_{L_{t,x}^2} \\
&\qquad \lesssim \|u_{k_1}\|_{S_{k_1}} \|u_{k_2}\|_{S_{k_2}}
\end{aligned}$$

and the desired bound follows again by summing over $k_1 < k_2 - 10$.

Proof of (4-2) Here we shall be able to get by only using Strichartz-type norms, by taking advantage of the condition $u\cdot u=1$. Using the standard Littlewood–Paley trichotomy, we have

$$0=u\cdot u-p\cdot p=\sum_{|k_1-k_2|\le 10}u_{k_1}u_{k_2}+2\sum_{k_1}u_{k_1}\cdot u_{<k_1-10}. \tag{4-6}$$

This implies

$$0=\sum_{|k_1-k_2|<10}(-\triangle)^{\frac12}(u_{k_1}u_{k_2})+2\sum_{k_1}(-\triangle)^{\frac12}(u_{k_1}\cdot u_{<k_1-10}). \tag{4-7}$$

Here the first term is better, since the outer derivative falls on the low-frequency output. We shall use this to replace the second term on the right by the first. Write

$$\begin{aligned}\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u)(u\cdot(-\triangle)^{\frac12}u)=&\sum_{|k_1-k_2|\le 10}\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u)(u_{k_1}\cdot(-\triangle)^{\frac12}u_{k_2})\\&+\sum_{k_1}\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u)(u_{k_1}\cdot(-\triangle)^{\frac12}u_{<k_1-10})\\&+\sum_{k_2}\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u)(u_{<k_2-10}\cdot(-\triangle)^{\frac12}u_{k_2}).\end{aligned} \tag{4-8}$$

Then for the first term on the right we infer

$$\begin{aligned}&\Big\|P_0\Big[\sum_{|k_1-k_2|\le 10}\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u)(u_{k_1}\cdot(-\triangle)^{\frac12}u_{k_2})\Big]\Big\|_{L_t^1L_x^2}\\&\quad\lesssim\sum_{\substack{|k_1-k_2|\le 10\\k_1<-20}}\big\|P_{[-20,20]}(\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u))\big\|_{L_t^\infty L_x^2}\|u_{k_1}\|_{L_t^2L_x^\infty}\|(-\triangle)^{\frac12}u_{k_2}\|_{L_t^2L_x^\infty}\\&\qquad+\sum_{\substack{|k_1-k_2|\le 10\\k_1\ge-20}}\|\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u)\|_{L_t^\infty L_x^2+L_{t,x}^\infty}\|u_{k_1}\|_{L_t^2L_x^4}\|(-\triangle)^{\frac12}u_{k_2}\|_{L_t^2L_x^4}.\end{aligned}$$

Then using a further elementary frequency decomposition it is easy to see (see the Appendix) that

$$\begin{aligned}\big\|P_{[-20,20]}(\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u))\big\|_{L_t^\infty L_x^2}&\lesssim\sum_{k_3\in\mathbb{Z}}2^{-|k_3|}\|P_{k_3}u\|_{S_{k_3}}(1+\|\tilde u\|_S),\\\|(\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u))\|_{L_t^\infty L_x^2+L_{t,x}^\infty}&\lesssim\sum_{k_3\in\mathbb{Z}}2^{-|k_3|}\|P_{k_3}u\|_{S_{k_3}}(1+\|\tilde u\|_S),\end{aligned}$$

and so we obtain that

$$\begin{aligned}&\sum_{\substack{|k_1-k_2|\le 10\\k_1<-20}}\big\|P_{[-20,20]}(\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u))\big\|_{L_t^\infty L_x^2}\|u_{k_1}\|_{L_t^2L_x^\infty}\|(-\triangle)^{\frac12}u_{k_2}\|_{L_t^2L_x^\infty}\\&\quad\lesssim\sum_{\substack{|k_1-k_2|\le 10\\k_1<-20}}2^{\frac{k_2-k_1}{2}}\prod_{j=1,2}\|P_{k_j}u\|_{S_{k_j}}\Big(\sum_{k_3\in\mathbb{Z}}2^{-|k_3|}\|P_{k_3}u\|_{S_{k_3}}(1+\|\tilde u\|_S)\Big)\\&\quad\lesssim\Big(\sum_{k_3\in\mathbb{Z}}2^{-|k_3|}\|P_{k_3}u\|_{S_{k_3}}\Big)\|u\|_S^2(1+\|\tilde u\|_S),\end{aligned}$$

as well as

$$\begin{aligned}\sum_{\substack{|k_1-k_2|\le 10\\ k_1\ge -20}} &\big\|\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u)\big\|_{L_t^\infty L_x^2+L_{t,x}^\infty}\,\|u_{k_1}\|_{L_t^2L_x^4}\,\|(-\triangle)^{\frac12}u_{k_2}\|_{L_t^2L_x^4}\\ &\lesssim\bigg(\sum_{\substack{|k_1-k_2|\le 10\\ k_1\ge -20}}2^{-\frac52 k_1}\,\|u_{k_1}\|_{S_{k_1}}\,\|u_{k_2}\|_{S_{k_2}}\bigg)\sum_{k_3\in\mathbb{Z}}2^{-|k_3|}\|P_{k_3}u\|_{S_{k_3}}(1+\|\tilde u\|_S)\\ &\lesssim\bigg(\sum_{k_3\in\mathbb{Z}}2^{-|k_3|}\|P_{k_3}u\|_{S_{k_3}}\bigg)\|u\|_S^2(1+\|\tilde u\|_S).\end{aligned}$$

This concludes the required bound for the first term on the right-hand side of (4-8).

Now we pass to the second term. We write it as a sum of three terms:

$$\begin{aligned}\sum_{k_1}\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u)(u_{k_1}\cdot(-\triangle)^{\frac12}u_{<k_1-10})&=\sum_{k_1\ge 5}\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u)(u_{k_1}\cdot(-\triangle)^{\frac12}u_{<k_1-10})\\ &\quad+\sum_{k_1\in[-5,5]}\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u)(u_{k_1}\cdot(-\triangle)^{\frac12}u_{<k_1-10})\\ &\quad+\sum_{k_1<-5}\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u)(u_{k_1}\cdot(-\triangle)^{\frac12}u_{<k_1-10}).\end{aligned}$$

Then we get

$$\begin{aligned}\bigg\|P_0\bigg(\sum_{k_1\ge 5}&\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u)(u_{k_1}\cdot(-\triangle)^{\frac12}u_{<k_1-10})\bigg)\bigg\|_{L_t^1L_x^2}\\ &\lesssim\sum_{k_1\ge 5}\big\|P_{[k_1-5,k_1+5]}(\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u))\big\|_{L_t^\infty L_x^2}\,\|u_{k_1}\|_{L_t^2L_x^\infty}\,\|(-\triangle)^{\frac12}u_{<k_1-10}\|_{L_t^2L_x^\infty}\\ &\lesssim\sum_{k_1\ge 5,\,k_2<k_1-10}\ \sum_{k_3}2^{-\frac32|k_3|}\,\|P_{k_3}u\|_{S_{k_3}}(1+\|\tilde u\|_S)\,\|u_{k_1}\|_{S_{k_1}}\,\|u_{k_2}\|_{S_{k_2}}\\ &\lesssim\bigg(\sum_{k_3}2^{-\frac32|k_3|}\|P_{k_3}u\|_{S_{k_3}}\bigg)(1+\|\tilde u\|_S)\,\|u\|_S^2.\end{aligned}$$

Similarly, for the term of intermediate k_1, we have

$$\begin{aligned}\bigg\|P_0\bigg[\sum_{k_1\in[-5,5]}&\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u)(u_{k_1}\cdot(-\triangle)^{\frac12}u_{<k_1-10})\bigg]\bigg\|_{L_t^1L_x^2}\\ &\lesssim\sum_{k_1\in[-5,5]}\big\|P_{<10}\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u)\big\|_{L_t^2L_x^\infty}\,\|u_{k_1}\|_{S_{k_1}}\,\|(-\triangle)^{\frac12}u_{<k_1-10}\|_{L_t^2L_x^\infty},\end{aligned}$$

and one closes by observing (see the Appendix for the first bound) that

$$\big\|P_{<10}\Pi_{\tilde u_\perp}((-\triangle)^{\frac12}u)\big\|_{L_t^2L_x^\infty}\lesssim(\|\tilde u\|_S+1)\,\|u\|_S,\qquad\|(-\triangle)^{\frac12}u_{<k_1-10}\|_{L_t^2L_x^\infty}\lesssim\|u\|_S.$$

Finally, for the range of low $k_1<-5$, we place both u_{k_1} and $(-\triangle)^{\frac12}u_{<k_1-10}$ into $L_t^2L_x^\infty$ and observe that

$$\|u_{k_1}\|_{L_t^2L_x^\infty}\|(-\triangle)^{\frac12}u_{<k_1-10}\|_{L_t^2L_x^\infty}\lesssim\|u_{k_1}\|_{S_{k_1}}\|u\|_S.$$

Then we close by using that

$$P_0\Big(\sum_{k_1<-5}\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u)(u_{k_1}\cdot(-\triangle)^{\frac{1}{2}}u_{<k_1-10})\Big) = P_0\Big(\sum_{k_1<-5}P_{[-2,2]}(\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u))(u_{k_1}\cdot(-\triangle)^{\frac{1}{2}}u_{<k_1-10})\Big),$$

as well as

$$\big\|P_{[-2,2]}(\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u))\big\|_{L_t^\infty L_x^2}\lesssim(1+\|\tilde{u}\|_S)\Big(\sum_{k_3}2^{-\frac{3}{2}|k_3|}\|P_{k_3}u\|_{S_{k_3}}\Big).$$

This concludes the required bound for the second term on the right in (4-8).

Finally, the third term in (4-8) is the most delicate, as the derivative $(-\triangle)^{\frac{1}{2}}$ lands on the higher-frequency term u_{k_2}. To deal with it, we note, using Lemma 3.2, that the difference

$$\sum_{k_2}\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u)(u_{<k_2-10}\cdot(-\triangle)^{\frac{1}{2}}u_{k_2})-\sum_{k_2}\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u)(-\triangle)^{\frac{1}{2}}(u_{<k_2-10}\cdot u_{k_2})$$

can be estimated like the second term on the right in (4-8), and hence it suffices to bound

$$\sum_{k_2}\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u)(-\triangle)^{\frac{1}{2}}(u_{<k_2-10}\cdot u_{k_2})=-\sum_{|k_3-k_4|<10}\tfrac{1}{2}\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u)(-\triangle)^{\frac{1}{2}}(u_{k_3}\cdot u_{k_4}),$$

where we have used (4-6). This term is again straightforward to estimate: we have

$$\Big\|P_0\Big[\sum_{\substack{|k_3-k_4|<10\\ k_3<-20}}\tfrac{1}{2}\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u)(-\triangle)^{\frac{1}{2}}(u_{k_3}\cdot u_{k_4})\Big]\Big\|_{L_t^1L_x^2} \lesssim\sum_{\substack{|k_3-k_4|<10\\ k_3<-20}}\big\|P_{[-10,10]}[\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u)]\big\|_{L_t^\infty L_x^2}\|(-\triangle)^{\frac{1}{2}}(u_{k_3}\cdot u_{k_4})\|_{L_t^1L_x^\infty},$$

and we close for the case $k_3<-20$ by observing that

$$\sum_{\substack{|k_3-k_4|<10\\ k_3<-20}}\|(-\triangle)^{\frac{1}{2}}(u_{k_3}\cdot u_{k_4})\|_{L_t^1L_x^\infty}\lesssim\sum_{\substack{|k_3-k_4|<10\\ k_3<-20}}2^{k_3}\|u_{k_3}\|_{L_t^2L_x^\infty}\|u_{k_4}\|_{L_t^2L_x^\infty}\lesssim\|u\|_S^2,$$

as well as

$$\big\|P_{[-10,10]}[\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u)]\big\|_{L_t^\infty L_x^2}\lesssim(1+\|\tilde{u}\|_S)\sum_{k_3}2^{-\frac{3}{2}|k_3|}\|P_{k_3}u\|_{S_{k_3}}.$$

On the other hand, if $k_3>-20$, we place both $u_{k_{3,4}}$ into $L_t^2L_x^4$. We omit the simple details. This finally concludes the bound of estimate (4-2).

Proof of (4-3) We commence by observing that we may in fact get rid of the outer operator $\Pi_{\tilde{u}^\perp}$, since one easily checks that

$$\|P_0[\Pi_{\tilde{u}^\perp}F]\|_{L_t^1L_x^2}\lesssim(1+\|\tilde{u}\|_S)\sum_{k_1}2^{-|k_1|}\|P_{k_1}F\|_{L_t^1\dot{H}^{n/2-1}}.$$

Then assuming that we have proved the bound

$$\|P_{k_1}F\|_{L_t^1\dot{H}^{n/2-1}} \lesssim \sum_{k_2} 2^{-\sigma|k_1-k_2|}\|P_{k_2}u\|_{S_{k_2}}$$

for some $\sigma > 1$, we then infer the bound

$$\|P_0[\Pi_{\tilde{u}^\perp}F]\|_{L_t^1L_x^2} \lesssim (1+\|\tilde{u}\|_S)\sum_{k_2} 2^{-|k_2|}\|P_{k_2}u\|_{S_{k_2}}.$$

Next, localising the last two factors to dyadic frequencies, and the output to frequency ~ 1 as we may, we arrive at the expression

$$P_0\big[u\times(-\triangle)^{\frac{1}{2}}(u_{k_1}\times(-\triangle)^{\frac{1}{2}}u_{k_2}) - u\times(u_{k_1}\times(-\triangle)u_{k_2})\big].$$

Then we first dispose of the easy cases:

Both frequencies large: $\max\{k_1,k_2\} > 10$. If $k_1 = k_2 + O(1)$, we simply place both high-frequency factors into $L_t^2L_x^4$, resulting in

$$\begin{aligned}\big\|P_0\big[u\times(-\triangle)^{\frac{1}{2}}(u_{k_1}\times(-\triangle)^{\frac{1}{2}}u_{k_2}) - u\times(u_{k_1}\times(-\triangle)u_{k_2})\big]\big\|_{L_t^1L_x^2} &\lesssim 2^{2k_1}\|P_{k_1}u\|_{L_t^2L_x^4}\,\|u_{k_2}\|_{L_t^2L_x^4}\\ &\lesssim 2^{2k_1-(1+\frac{5}{2})k_1}\prod_{j=1,2}\|u_{k_j}\|_{S_{k_j}},\end{aligned}$$

whence we have

$$\begin{aligned}&\Big\|\sum_{k_1=k_2+O(1)>10} P_0\big[u\times(-\triangle)^{\frac{1}{2}}(u_{k_1}\times(-\triangle)^{\frac{1}{2}}u_{k_2}) - u\times(u_{k_1}\times(-\triangle)u_{k_2})\big]\Big\|_{L_t^1L_x^2}\\ &\qquad\lesssim \sum_{k_1=k_2+O(1)>10} 2^{-\frac{3}{2}|k_1|}\prod_{j=1,2}\|u_{k_j}\|_{S_{k_j}} \lesssim \Big(\sum_{k_1} 2^{-\frac{3}{2}|k_1|}\|P_{k_1}u\|_{S_{k_1}}\Big)\|u\|_S.\end{aligned}$$

On the other hand, if $k_2 \gg k_1$, we use

$$\begin{aligned}&P_0\big[u\times(-\triangle)^{\frac{1}{2}}(u_{k_1}\times(-\triangle)^{\frac{1}{2}}u_{k_2}) - u\times(u_{k_1}\times(-\triangle)u_{k_2})\big]\\ &\qquad= P_0\big[P_{k_2+O(1)}u\times(-\triangle)^{\frac{1}{2}}(u_{k_1}\times(-\triangle)^{\frac{1}{2}}u_{k_2}) - P_{k_2+O(1)}u\times(u_{k_1}\times(-\triangle)u_{k_2})\big].\end{aligned}$$

Then place the first and third factors into $L_t^2L_x^4$ and the middle factor into $L_t^\infty L_x^2 + L_{t,x}^\infty$. The case $k_2 \ll k_1$ is similar.

Both frequencies small: $\max\{k_1,k_2\} < -10$. Here we observe that Lemma 3.2 allows us to place one derivative $(-\triangle)^{\frac{1}{2}}$ onto the factor u_{k_1}, even if $k_1 < k_2 - 10$. Thus we reduce to bounding the schematic expression

$$P_0[P_{[-5,5]}u\nabla_x u_{k_1}\nabla_x u_{k_2}],$$

which is straightforward since we can place the second and third factors into $L_t^2L_x^\infty$. We omit the simple details.

One frequency intermediate, the other small: $\max\{k_1,k_2\} \in [-10,10]$. This case is a bit more difficult, and we shall exploit the geometric structure of the expression. We split this further into two cases:

(i) $k_1 \in [-10, 10], k_2 < 10$. Here the difference structure inherent in the term is not helpful. In fact, we can immediately estimate

$$\begin{aligned}\big\| P_0[u \times (u_{k_1} \times (-\triangle) u_{k_2})]\big\|_{L_t^1 L_x^2} &\lesssim \|u_{k_1}\|_{L_t^2 L_x^4} \, \|(-\triangle) u_{k_2}\|_{L_t^2 L_x^4} \\ &\lesssim 2^{\frac{k_2}{4}} \|u_{k_2}\|_{S_{k_2}} \, \|u_{k_1}\|_{S_{k_1}},\end{aligned}$$

and here of course we can sum over $k_2 < 10$ to infer the desired bound. Next, using Lemma 3.2 allows us to replace the term

$$P_0\big[u \times (-\triangle)^{\frac{1}{2}} (u_{k_1} \times (-\triangle)^{\frac{1}{2}} u_{k_2})\big]$$

by

$$P_0\big[u \times ((-\triangle)^{\frac{1}{2}} u_{k_1} \times (-\triangle)^{\frac{1}{2}} u_{k_2})\big]$$

up to a term which is estimated like $P_0[u \times (u_{k_1} \times (-\triangle) u_{k_2})]$. Before exploiting the algebraic structure of the term above, we reduce the first factor u to frequency $< 2^{k_2 - 10}$, which we can on account of

$$\begin{aligned}&\big\| P_0\big[u_{\geq k_2-10} \times ((-\triangle)^{\frac{1}{2}} u_{k_1} \times (-\triangle)^{\frac{1}{2}} u_{k_2})\big]\big\|_{L_t^1 L_x^2} \\ &\qquad \lesssim \|u_{\geq k_2-10}\|_{L_t^2 L_x^\infty} \, \|(-\triangle)^{\frac{1}{2}} u_{k_1}\|_{L_t^\infty L_x^2} \, \|(-\triangle)^{\frac{1}{2}} u_{k_2}\|_{L_t^2 L_x^\infty} \\ &\qquad \lesssim \|u_{k_1}\|_{S_{k_1}} \, \|u_{k_2}\|_{S_{k_2}} \, \|u\|_S.\end{aligned}$$

Summing over $k_2 < 10$ and recalling that $k_1 \in [-10, 10]$ leads to the desired bound.

Consider now the expression

$$P_0\big[u_{<k_2-10} \times ((-\triangle)^{\frac{1}{2}} u_{k_1} \times (-\triangle)^{\frac{1}{2}} u_{k_2})\big].$$

Write this as

$$\begin{aligned}&P_0\big[u_{<k_2-10} \times ((-\triangle)^{\frac{1}{2}} u_{k_1} \times (-\triangle)^{\frac{1}{2}} u_{k_2})\big] \\ &\qquad = P_0\big[(-\triangle)^{\frac{1}{2}} u_{k_1} (u_{<k_2-10} \cdot (-\triangle)^{\frac{1}{2}} u_{k_2}) - (-\triangle)^{\frac{1}{2}} u_{k_2} (u_{<k_2-10} \cdot (-\triangle)^{\frac{1}{2}} u_{k_1})\big].\end{aligned}$$

In order to estimate this, we use a frequency-localised version of (4-6). Specifically, we have

$$0 = 2u_k \cdot u_{<k-10} + \sum_{k_1 = k_2 + O(1)} P_k(u_{k_1} \cdot u_{k_2}) + 2^{-k} L(u_k, \nabla_x u_{<k-10}), \tag{4-9}$$

where L is a bilinear operator of the form used in Lemma 3.2 with a bounded kernel $m(\xi, \eta)$. We conclude the schematic relation

$$(-\triangle)^{\frac{1}{2}} u_k \cdot u_{<k-10} = -\tfrac{1}{2} (-\triangle)^{\frac{1}{2}} \sum_{k_1 = k_2 + O(1)} P_k(u_{k_1} \cdot u_{k_2}) + L(u_k, \nabla_x u_{<k-10}).$$

It follows that we can write

$$\begin{aligned}&P_0\big[(-\triangle)^{\frac{1}{2}} u_{k_1} (u_{<k_2-10} \cdot (-\triangle)^{\frac{1}{2}} u_{k_2})\big] \\ &\qquad = -\tfrac{1}{2} P_0\Big[(-\triangle)^{\frac{1}{2}} u_{k_1} \sum_{k_3 = k_4 + O(1)} (-\triangle)^{\frac{1}{2}} P_{k_2}(u_{k_3} \cdot u_{k_4})\Big] + P_0\big[(-\triangle)^{\frac{1}{2}} u_{k_1} L(u_{k_2}, \nabla_x u_{<k_2-10})\big],\end{aligned}$$

and here we have (keeping in mind that $k_1 \in [-10, 10]$)

$$\begin{aligned}\Big\| P_0\Big[(-\triangle)^{\frac{1}{2}} u_{k_1} \sum_{k_3=k_4+O(1)} (-\triangle)^{\frac{1}{2}} P_{k_2}(u_{k_3} \cdot u_{k_4})\Big]\Big\|_{L_t^1 L_x^2} \\ \lesssim 2^{k_2} \sum_{k_3=k_4+O(1)\geq k_2} \|(-\triangle)^{\frac{1}{2}} u_{k_1}\|_{L_t^\infty L_x^2} \|u_{k_3}\|_{L_t^2 L_x^\infty} \|u_{k_4}\|_{L_t^2 L_x^\infty} \\ \lesssim \|(-\triangle)^{\frac{1}{2}} u_{k_1}\|_{L_t^\infty L_x^2} \sum_{k_3=k_4+O(1)\geq k_2} 2^{k_2-k_3} \|u_{k_3}\|_{S_{k_3}} \|u_{k_4}\|_{S_{k_4}},\end{aligned}$$

and here we can sum over $k_2 < 10$ to arrive at an upper bound of $\lesssim \|u_{k_1}\|_{S_{k_1}} \|u\|_S^2$, as desired. We also have the simple bound

$$\begin{aligned}\big\| P_0\big[(-\triangle)^{\frac{1}{2}} u_{k_1} L(u_{k_2}, \nabla_x u_{<k_2-10})\big]\big\|_{L_t^1 L_x^2} &\lesssim \|(-\triangle)^{\frac{1}{2}} u_{k_1}\|_{L_t^\infty L_x^2} \|u_{k_2}\|_{L_t^2 L_x^\infty} \|\nabla_x u_{<k_2-10}\|_{L_t^2 L_x^\infty} \\ &\lesssim \|(-\triangle)^{\frac{1}{2}} u_{k_1}\|_{L_t^\infty L_x^2} \|u_{k_2}\|_{S_{k_2}} \|u\|_S,\end{aligned}$$

and summing over $k_2 < 10$, we arrive again at the bound

$$\lesssim \|u_{k_1}\|_{S_{k_1}} \|u\|_S^2.$$

This concludes the case (i).

(ii) $k_2 \in [-10, 10], k_1 < 10$. Proceeding in analogy to case (i), we immediately reduce to the expression

$$P_0\big[u_{<k_1-10} \times (-\triangle)^{\frac{1}{2}}(u_{k_1} \times (-\triangle)^{\frac{1}{2}} u_{k_2}) - u_{<k_1-10} \times (u_{k_1} \times (-\triangle) u_{k_2})\big].$$

Here we first note that on account of Lemma 3.2 we have

$$\begin{aligned}\big\| P_0\big[u_{<k_1-10} \times (-\triangle)^{\frac{1}{2}}(u_{k_1} \times (-\triangle)^{\frac{1}{2}} u_{k_2}) - (-\triangle)^{\frac{1}{2}}\big(u_{<k_1-10} \times (u_{k_1} \times (-\triangle)^{\frac{1}{2}} u_{k_2})\big)\big]\big\|_{L_t^1 L_x^2} \\ \lesssim \|(-\triangle)^{\frac{1}{2}} u_{<k_1-10}\|_{L_t^2 L_x^\infty} \|u_{k_1}\|_{L_t^2 L_x^\infty} \|(-\triangle)^{\frac{1}{2}} u_{k_2}\|_{L_t^\infty L_x^2} \\ \lesssim \|u\|_S \|u_{k_1}\|_{S_{k_1}} \|u_{k_2}\|_{S_{k_2}}.\end{aligned}$$

Then summation over $k_1 < 10$ gives the required bound.

Next, we expand out

$$\begin{aligned}&P_0\big[(-\triangle)^{\frac{1}{2}}\big(u_{<k_1-10} \times (u_{k_1} \times (-\triangle)^{\frac{1}{2}} u_{k_2})\big) - u_{<k_1-10} \times (u_{k_1} \times (-\triangle) u_{k_2})\big] \\ &\quad = P_0(-\triangle)^{\frac{1}{2}}\big(u_{k_1}(u_{<k_1-10} \cdot (-\triangle)^{\frac{1}{2}} u_{k_2}) - (-\triangle)^{\frac{1}{2}} u_{k_2}(u_{<k_1-10} \cdot u_{k_1})\big) \\ &\qquad - P_0\big(u_{k_1}(u_{<k_1-10} \cdot (-\triangle) u_{k_2}) - (-\triangle) u_{k_2}(u_{<k_1-10} \cdot u_{k_1})\big).\end{aligned} \tag{4-10}$$

Then pairing up these last four terms suitably, we have

$$\begin{aligned}&P_0(-\triangle)^{\frac{1}{2}}\big(u_{k_1}(u_{<k_1-10} \cdot (-\triangle)^{\frac{1}{2}} u_{k_2})\big) - P_0\big(u_{k_1}(u_{<k_1-10} \cdot (-\triangle) u_{k_2})\big) \\ &\quad = P_0(-\triangle)^{\frac{1}{2}}\big(u_{k_1}(-\triangle)^{\frac{1}{2}}(u_{<k_1-10} \cdot u_{k_2})\big) - P_0\big(u_{k_1}(-\triangle)(u_{<k_1-10} \cdot u_{k_2})\big) + u_{k_1} L(\nabla_x u_{<k_1-10}, u_{k_2}) \\ &\quad = L\big((-\triangle)^{\frac{1}{2}} u_{k_1}, (-\triangle)^{\frac{1}{2}}(u_{<k_1-10} \cdot u_{k_2})\big) + u_{k_1} L(\nabla_x u_{<k_1-10}, u_{k_2}).\end{aligned}$$

The last term is straightforward since

$$\big\|u_{k_1} L(\nabla_x u_{<k_1-10}, u_{k_2})\big\|_{L_t^1 L_x^2} \lesssim \|u_{k_1}\|_{L_t^2 L_x^\infty} \, \|\nabla_x u_{<k_1-10}\|_{L_t^2 L_x^\infty} \, \|u_{k_2}\|_{L_t^\infty L_x^2} \\ \lesssim \|u\|_S \, \|u_{k_1}\|_{S_{k_1}} \, \|u_{k_2}\|_{S_{k_2}},$$

and we can sum over $k_1 < 10$. Further, we see that

$$L\big((-\triangle)^{\frac{1}{2}} u_{k_1}, (-\triangle)^{\frac{1}{2}}(u_{<k_1-10} \cdot u_{k_2})\big) = L\big((-\triangle)^{\frac{1}{2}} u_{k_1}, (-\triangle)^{\frac{1}{2}}(u_{<k_2-10} \cdot u_{k_2})\big) + \text{error},$$

where the term error here is estimated exactly like the previous term. But then taking advantage of (4-9), we find

$$L\big((-\triangle)^{\frac{1}{2}} u_{k_1}, (-\triangle)^{\frac{1}{2}}(u_{<k_2-10} \cdot u_{k_2})\big) \\ = -\frac{1}{2} \sum_{k_3=k_4+O(1)>k_2} L\big((-\triangle)^{\frac{1}{2}} u_{k_1}, (-\triangle)^{\frac{1}{2}} P_{k_2}(u_{k_3} \cdot u_{k_4})\big) \\ + 2^{-k_2} L\big((-\triangle)^{\frac{1}{2}} u_{k_1}, (-\triangle)^{\frac{1}{2}} L(\nabla_x u_{<k_2-10}, u_{k_2})\big).$$

Then we have

$$\bigg\| -\frac{1}{2} \sum_{k_3=k_4+O(1)>k_2} L\big((-\triangle)^{\frac{1}{2}} u_{k_1}, (-\triangle)^{\frac{1}{2}} P_{k_2}(u_{k_3} \cdot u_{k_4})\big) \bigg\|_{L_t^1 L_x^2} \\ \lesssim \sum_{k_3=k_4+O(1)>k_2} 2^{k_2} \|(-\triangle)^{\frac{1}{2}} u_{k_1}\|_{L_t^2 L_x^\infty} \, \|u_{k_3}\|_{L_t^2 L_x^\infty} \, \|u_{k_4}\|_{L_t^\infty L_x^2}.$$

The preceding sum can be further bounded by

$$\lesssim \sum_{k_3=k_4+O(1)>k_2} 2^{k_2} 2^{\frac{k_1-k_3}{2}} 2^{-\frac{5}{2}k_4} \|u_{k_1}\|_{S_{k_1}} \, \|u_{k_3}\|_{S_{k_3}} \, \|u_{k_4}\|_{S_{k_4}} \\ \lesssim \bigg(\sum_{k_1} 2^{-|k_4-k_2|} \, \|u_{k_4}\|_{S_{k_4}} \, \|u\|_S \bigg) \|u_{k_1}\|_{S_{k_1}}.$$

This can be summed over $k_1 < 10$ to yield the desired kind of bound.

Finally, we have the simpler bound

$$\big\|2^{-k_2} L\big((-\triangle)^{\frac{1}{2}} u_{k_1}, (-\triangle)^{\frac{1}{2}} L(\nabla_x u_{<k_2-10}, u_{k_2})\big)\big\|_{L_t^1 L_x^2} \\ \lesssim \|(-\triangle)^{\frac{1}{2}} u_{k_1}\|_{L_t^2 L_x^\infty} \, \|\nabla_x u_{<k_2-10}\|_{L_t^2 L_x^\infty} \, \|u_{k_2}\|_{L_t^\infty L_x^2},$$

which after summation over $k_1 < 10$ is again bounded by $\lesssim \|u\|_S^2 \, \|u_{k_2}\|_{S_{k_2}}$.

Returning to (4-10), it remains to bound the difference

$$P_0(-\triangle)^{\frac{1}{2}}\big[(-\triangle)^{\frac{1}{2}} u_{k_2}(u_{<k_1-10} \cdot u_{k_1})\big] - P_0\big[(-\triangle) u_{k_2}(u_{<k_1-10} \cdot u_{k_1})\big] \\ = -L\bigg((-\triangle)^{\frac{1}{2}} u_{k_2}, \sum_{k_3=k_4+O(1)>k_1} (-\triangle)^{\frac{1}{2}} P_{k_1}(u_{k_3} \cdot u_{k_4})\bigg) \\ + L\big((-\triangle)^{\frac{1}{2}} u_{k_2}, (-\triangle)^{\frac{1}{2}} 2^{-k_1} L(\nabla_x u_{<k_1-10}, u_{k_1})\big).$$

Then the first term is bounded by

$$\begin{aligned}\Big\|\tfrac{1}{2}L\Big((-\triangle)^{\frac{1}{2}}u_{k_2},\sum_{k_3=k_4+O(1)>k_1}(-\triangle)^{\frac{1}{2}}P_{k_1}(u_{k_3}\cdot u_{k_4})\Big)\Big\|_{L_t^1L_x^2}\\ \lesssim 2^{k_1}\sum_{k_3=k_4+O(1)>k_1}\big\|(-\triangle)^{\frac{1}{2}}u_{k_2}\big\|_{L_t^\infty L_x^2}\,\|u_{k_3}\|_{L_t^2L_x^\infty}\,\|u_{k_4}\|_{L_t^2L_x^\infty}\\ \lesssim \|u_{k_2}\|_{S_{k_2}}\sum_{k_3=k_4+O(1)>k_1}2^{k_1-k_3}\,\|u_{k_3}\|_{S_{k_3}}\|u_{k_4}\|_{S_{k_4}}.\end{aligned}$$

This expression can be summed over k_1 to give the desired bound. Similarly, we get

$$\begin{aligned}\big\|L\big((-\triangle)^{\frac{1}{2}}u_{k_2},(-\triangle)^{\frac{1}{2}}2^{-k_1}L(\nabla_x u_{<k_1-10},u_{k_1})\big)\big\|_{L_t^1L_x^2}\\ \lesssim \|(-\triangle)^{\frac{1}{2}}u_{k_2}\|_{L_t^\infty L_x^2}\,\|\nabla_x u_{<k_1-10}\|_{L_t^2L_x^\infty}\,\|u_{k_1}\|_{L_t^2L_x^\infty}\\ \lesssim \|u_{k_2}\|_{S_{k_2}}\,\|u_{k_1}\|_{S_{k_1}}\,\|u\|_S,\end{aligned}$$

and summation over $k_1 < 10$ yields the desired bound. This concludes case (ii), and thereby of (4-3).

The estimates (4-4), (4-5) are proved similarly, after passing to the differences. One only needs to make sure to reformulate the terms as in the preceding using (4-6), (4-9), before passing to the differences. □

5. The iteration scheme

Here we solve (2-1). Specifically, we prove the following.

Theorem 5.1. *Let $n \ge 5$. Let $u[0] = (u, u_t) : \mathbb{R}^n \to S^2 \times TS^2$ be a smooth data pair with $u \cdot u_t = 0$ pointwise, and such that u is constant outside of a compact subset of $\mathbb{R}^n$. Also, assume the smallness condition*

$$\|u[0]\|_{\dot{B}_2^{n/2,1}\times\dot{B}_2^{n/2-1,1}} < \epsilon,$$

where $\epsilon \ll 1$ is sufficiently small. Then problem (2-1) *admits a global smooth solution with these data.*

Proof. We do this by means of a suitable iteration scheme: first, let $u^{(0)} = p$, where $p \in S^2$ is the limit of the initial data $u|_{t=0}$ at spatial infinity. Then let $u^{(1)}$ be the wave map into S^2 with the given data (which is possible since $u_t(0,\cdot)\cdot u(0,\cdot) = 0$ from our assumption), thus solving

$$(\partial_t^2 - \triangle)u^{(1)} = u^{(1)}(\nabla u^{(1)}\cdot\nabla u^{(1)} - \partial_t u^{(1)}\cdot\partial_t u^{(1)}).$$

It is given by $u^{(1)} = p + \sum_{k\in\mathbb{Z}} u_k^{(1)}$, and its existence follows via simple iteration from (4-1) and the corresponding difference estimate. Then we define the higher iterates $u^{(j)}$, $j \ge 2$, via the following iterative scheme:

$$\begin{aligned}(\partial_t^2 - \triangle)u^{(j)}\\ = u^{(j)}(\nabla u^{(j)}\cdot\nabla u^{(j)} - \partial_t u^{(j)}\cdot\partial_t u^{(j)})\\ + \Pi_{u_\perp^{(j)}}((-\triangle)^{\frac{1}{2}}u^{(j-1)})(u^{(j-1)}\cdot(-\triangle)^{\frac{1}{2}}u^{(j-1)})\\ + \Pi_{u_\perp^{(j)}}\big[u^{(j-1)}\times(-\triangle)^{\frac{1}{2}}(u^{(j-1)}\times(-\triangle)^{\frac{1}{2}}u^{(j-1)}) - u^{(j-1)}\times(u^{(j-1)}\times(-\triangle)u^{(j-1)})\big].\end{aligned} \tag{5-1}$$

This equation defines $u^{(j)}$ implicitly, and so to actually compute it, we have to run a subiteration

$$\begin{aligned}(\partial_t^2-\triangle)&u^{(j,i)}\\ &=u^{(j,i-1)}(\nabla u^{(j,i-1)}\cdot\nabla u^{(j,i-1)}-\partial_t u^{(j,i-1)}\cdot\partial_t u^{(j,i-1)})\\ &\quad+\Pi_{u_\perp^{(j,i-1)}}((-\triangle)^{\frac12}u^{(j-1)})(u^{(j-1)}\cdot(-\triangle)^{\frac12}u^{(j-1)})\\ &\quad\quad+\Pi_{u_\perp^{(j,i-1)}}\big[u^{(j-1)}\times(-\triangle)^{\frac12}(u^{(j-1)}\times(-\triangle)^{\frac12}u^{(j-1)})-u^{(j-1)}\times(u^{(j-1)}\times(-\triangle)u^{(j-1)})\big]\end{aligned}\tag{5-2}$$

for $i\geq 1$, while $u^{(j,0)}$ is the free wave evolution of the data $u[0]$. Then we again have $u^{(j,i)}=p+\sum_k u_k^{(j,i)}$, and in particular each $u^{(j,i)}$ is close to S^2 with respect to the L^∞ norm, while convergence with respect to $\|\cdot\|_S$ follows from Proposition 4.1. We also get higher regularity of each $u^{(j,i)}$ and $u^{(j)}$ by differentiating the equation.

Our choice of iterative scheme (5-1) implies

$$\Box(u^{(j)}\cdot u^{(j)}-1)=(u^{(j)}\cdot u^{(j)}-1)(\nabla u^{(j)}\cdot\nabla u^{(j)}-\partial_t u^{(j)}\cdot\partial_t u^{(j)}),$$

as well as $(u^{(j)}\cdot u^{(j)}-1)[0]=(0,0)$, which inductively gives that $u^{(j)}$ maps into S^2 for all j. Finally, convergence of the $u^{(j)}$ with respect to $\|\cdot\|_S$ follows again via Proposition 4.1. Differentiating (5-1) then also gives higher regularity of the limit function u. The latter is then easily seen to solve (2-1). For later purposes, we also note that Proposition 4.1 in conjunction with the assumptions that $(u-p)|_{t=0}\in C_0^\infty$ and $u_t|_{t=0}=u\times(-\triangle)^{\frac12}|_{t=0}$ imply that we have improved control over low frequencies: $u(t,\cdot)\in\dot H^{\frac n2-\frac12}, u_t(t,\cdot)\in\dot H^{\frac n2-\frac32}$ for all t. $\square$

6. Proof of Theorem 1.1

It remains to show that the solution $u(t,x)$ obtained in Theorem 5.1 actually solves (1-1). For this introduce the quantity

$$X:=u_t-u\times(-\triangle)^{\frac12}u,$$

as well as the energy type functional

$$\widetilde{E}(t):=\frac12\int_{\mathbb{R}^n}\big|(-\triangle)^{\frac n4-\frac34}X(t,\cdot)\big|^2\,dx.$$

Note that we have $\nabla_{t,x}u\in\dot H^{\frac n2-\frac32}$ as observed previously, and hence $\widetilde{E}(t)$ is well defined and also continuously differentiable (on account of the higher-regularity properties of u). Retracing the steps that led to the final wave equation (2-1), we deduce

$$\partial_t X=-X\times(-\triangle)^{\frac12}u-u\times(-\triangle)^{\frac12}X-u(X\cdot(u\times(-\triangle)^{\frac12}u+u_t)),$$

and so we deduce

$$\begin{aligned}\frac{d}{dt}\widetilde{E}(t)=&-\int_{\mathbb{R}^n}(-\triangle)^{\frac n4-\frac34}\big(X\times(-\triangle)^{\frac12}u+u\times(-\triangle)^{\frac12}X\big)\cdot(-\triangle)^{\frac n4-\frac34}X\,dx\\ &-\int_{\mathbb{R}^n}(-\triangle)^{\frac n4-\frac34}\big(u(X\cdot(u\times(-\triangle)^{\frac12}u+u_t))\big)\cdot(-\triangle)^{\frac n4-\frac34}X\,dx.\end{aligned}$$

Then we note that[3]

$$
\begin{aligned}
&\big\|(-\triangle)^{\frac{n}{4}-\frac{3}{4}}\big(X\times(-\triangle)^{\frac{1}{2}}u+u\times(-\triangle)^{\frac{1}{2}}X\big)-u\times(-\triangle)^{\frac{n}{4}-\frac{1}{4}}X\big\|_{L_x^2}\\
&\qquad\lesssim\|(-\triangle)^{\frac{n}{4}-\frac{3}{4}}X\|_{L_x^2}\,\|(-\triangle)^{\frac{1}{2}}u\|_{L_x^\infty}+\|X\|_{L_x^{2n/3}}\,\|(-\triangle)^{\frac{n}{4}-\frac{1}{4}}u\|_{L_x^{2n/(n-3)}}\\
&\qquad\qquad+\|(-\triangle)^{\frac{n}{4}-\frac{3}{4}}u\|_{L_x^{2n/(n-5)}}\,\|(-\triangle)^{\frac{1}{2}}X\|_{L_x^{2n/5}}\lesssim_u\|(-\triangle)^{\frac{n}{4}-\frac{3}{4}}X\|_{L_x^2}^2
\end{aligned}
$$

on account of Sobolev's embedding and higher regularity of u, and further, we observe that

$$
\begin{aligned}
&\int_{\mathbb{R}^n}(u\times(-\triangle)^{\frac{n}{4}-\frac{1}{4}}X)\cdot(-\triangle)^{\frac{n}{4}-\frac{3}{4}}X\,dx\\
&\qquad=\int_{\mathbb{R}^n}(-\triangle)^{\frac{1}{4}}(u\times(-\triangle)^{\frac{n}{4}-\frac{2}{4}}X)\cdot(-\triangle)^{\frac{n}{4}-\frac{3}{4}}X\,dx+O\big(\|(-\triangle)^{\frac{n}{4}-\frac{3}{4}}X\|_{L_x^2}^2\,\|\nabla_x u\|_{L_x^\infty}\big)\\
&\qquad=O\big(\|(-\triangle)^{\frac{n}{4}-\frac{3}{4}}X\|_{L_x^2}^2\,\|\nabla_x u\|_{L_x^\infty}\big).
\end{aligned}
$$

Similarly, we infer

$$
\left|\int_{\mathbb{R}^n}(-\triangle)^{\frac{n}{4}-\frac{3}{4}}\big(u(X\cdot(u\times(-\triangle)^{\frac{1}{2}}u+u_t))\big)\cdot(-\triangle)^{\frac{n}{4}-\frac{3}{4}}X\,dx\right|\lesssim_u\|(-\triangle)^{\frac{n}{4}-\frac{3}{4}}X\|_{L_x^2}^2.
$$

But then the preceding implies that

$$
\frac{d}{dt}\widetilde{E}(t)\le C(u)\widetilde{E}(t)
$$

and furthermore $\widetilde{E}(0)=0$, which implies $\widetilde{E}(t)=0$ throughout. It follows that $X=0$ identically, which completes the proof of Theorem 1.1.

Appendix

Here we prove some bounds related to the projection operator $\Pi_{\tilde{u}_\perp}$ used in the proof of Proposition 4.1.

Lemma A.1. *Assume that $\tilde{u}:\mathbb{R}^{5+1}\to S^2$ maps into a small neighbourhood of S^2 with $\|\tilde{u}\|_S\lesssim 1$. Then for any $a\in\mathbb{Z}$ we have the bounds*

$$
\big\|P_{[-a,a]}(\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u))\big\|_{L_t^\infty L_x^2}\lesssim_a\sum_{k_3\in\mathbb{Z}}2^{-|k_3|}\|P_{k_3}u\|_{S_{k_3}}(1+\|\tilde{u}\|_S),\tag{A-1}
$$

$$
\big\|(\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u))\big\|_{L_t^\infty L_x^2+L_{t,x}^\infty}\lesssim\sum_{k_3\in\mathbb{Z}}2^{-|k_3|}\|P_{k_3}u\|_{S_{k_3}}(1+\|\tilde{u}\|_S),\tag{A-2}
$$

$$
\big\|P_{<a}\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u)\big\|_{L_t^2L_x^\infty}\lesssim_a(\|\tilde{u}\|_S+1)\|u\|_S.\tag{A-3}
$$

Proof of (A-1). Note that we can write

$$
\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u)=(-\triangle)^{\frac{1}{2}}u-F(\tilde{u})\cdot(-\triangle)^{\frac{1}{2}}u
$$

[3]Here $2n/(n-5)$ gets replaced by ∞ if $n=5$.

for a suitable C^∞ function $F:\mathbb{R}^3\to\mathbb{R}^3$, which in addition to all its derivatives is bounded. Then since

$$\big\|P_{[-a,a]}(-\triangle)^{\frac{1}{2}}u\big\|_{L_t^\infty L_x^2}\lesssim_a\sum_{k_3\in\mathbb{Z}}2^{-|k_3|}\|P_{k_3}u\|_{S_{k_3}},$$

it suffices to consider $\big\|P_{[-a,a]}\big[F(\tilde u)\cdot(-\triangle)^{\frac{1}{2}}u\big]\big\|_{L_t^\infty L_x^2}$. To deal with this expression, observe first that

$$\begin{aligned}\|P_lF(\tilde u)\|_{L_t^\infty L_x^2}&\lesssim 2^{-l}\big\|P_l\big[P_{<l-20}(\nabla\tilde u)F'(\tilde u)\big]\big\|_{L_t^\infty L_x^2}+2^{-\frac{5}{2}l}\|\tilde u\|_{\dot H^{n/2}}\\&\lesssim 2^{-2l}\big\|P_l\big[P_{<l-20}(\nabla\tilde u)P_{<l-20}(\nabla\tilde u)F''(\tilde u)\big]\big\|_{L_t^\infty L_x^2}+2^{-\frac{5}{2}l}\|\tilde u\|_{\dot H^{n/2}}\\&\lesssim 2^{-3l}\big\|P_l\big[P_{<l-20}(\nabla\tilde u)P_{<l-20}(\nabla\tilde u)P_{<l-20}(\nabla\tilde u)F'''(\tilde u)\big]\big\|_{L_t^\infty L_x^2}+2^{-\frac{5}{2}l}\|\tilde u\|_{\dot H^{n/2}}\end{aligned}$$

and we can estimate the last term by

$$\begin{aligned}&2^{-3l}\big\|P_l\big[P_{<l-20}(\nabla\tilde u)P_{<l-20}(\nabla\tilde u)P_{<l-20}(\nabla\tilde u)F'''(\tilde u)\big]\big\|_{L_t^\infty L_x^2}\\&\qquad\lesssim 2^{-3l}\|P_{<l-20}(\nabla\tilde u)\|^3_{L_t^\infty L_x^6}\lesssim 2^{-\frac{5}{2}l}\|\tilde u\|_S^3\lesssim 2^{-\frac{5}{2}l}\|\tilde u\|_S,\end{aligned}$$

whence in summary $\|P_lF(\tilde u)\|_{L_t^\infty L_x^2}\lesssim 2^{-\frac{5}{2}l}\|\tilde u\|_S$. To conclude, we estimate

$$\begin{aligned}\big\|P_{[-a,a]}[F(\tilde u)\cdot(-\triangle)^{\frac{1}{2}}u]\big\|_{L_t^\infty L_x^2}\le{}&\big\|P_{[-a,a]}\big[P_{<-a-10}[F(\tilde u)]\cdot(-\triangle)^{\frac{1}{2}}u\big]\big\|_{L_t^\infty L_x^2}\\&+\big\|P_{[-a,a]}\big[P_{[-a-10,a+10]}[F(\tilde u)]\cdot(-\triangle)^{\frac{1}{2}}u\big]\big\|_{L_t^\infty L_x^2}\\&+\big\|P_{[-a,a]}\big[P_{>a+10}[F(\tilde u)]\cdot(-\triangle)^{\frac{1}{2}}u\big]\big\|_{L_t^\infty L_x^2},\end{aligned}$$

and we have

$$\begin{aligned}\big\|P_{[-a,a]}\big[P_{<-a-10}[F(\tilde u)]\cdot(-\triangle)^{\frac{1}{2}}u\big]\big\|_{L_t^\infty L_x^2}&\lesssim\big\|P_{[-a-20,a+20]}(-\triangle)^{\frac{1}{2}}u\big\|_{L_t^\infty L_x^2}\\&\lesssim_a\sum_{k_3\in\mathbb{Z}}2^{-|k_3|}\|P_{k_3}u\|_{S_{k_3}},\end{aligned}$$

$$\begin{aligned}\big\|P_{[-a,a]}\big[P_{[-a-10,a+10]}[F(\tilde u)]\cdot(-\triangle)^{\frac{1}{2}}u\big]\big\|_{L_t^\infty L_x^2}&\lesssim\big\|P_{[-a-10,a+10]}[F(\tilde u)]\big\|_{L_t^\infty L_x^2}\big\|P_{<a+10}(-\triangle)^{\frac{1}{2}}u\big\|_{L^\infty_{t,x}}\\&\lesssim_a\|\tilde u\|_S\sum_{k_3\in\mathbb{Z}}2^{-|k_3|}\|P_{k_3}u\|_{S_{k_3}},\end{aligned}$$

where we have used the preceding bound for $P_lF(\tilde u)$ to control $\big\|P_{[-a-10,a+10]}[F(\tilde u)]\big\|_{L_t^\infty L_x^2}$. Finally, we get

$$\begin{aligned}\big\|P_{[-a,a]}\big[P_{>a+10}[F(\tilde u)]\cdot(-\triangle)^{\frac{1}{2}}u\big]\big\|_{L_t^\infty L_x^2}&\le\sum_{k_1=k_2+O(1)>a+10}\big\|P_{[-a,a]}\big[P_{k_1}[F(\tilde u)]\cdot P_{k_2}(-\triangle)^{\frac{1}{2}}u\big]\big\|_{L_t^\infty L_x^2}\\&\lesssim_a\sum_{k_1=k_2+O(1)>a+10}\|P_{k_1}[F(\tilde u)]\|_{L_t^\infty L_x^2}\|P_{k_2}(-\triangle)^{\frac{1}{2}}u\|_{L_t^\infty L_x^2}\\&\lesssim_a\|\tilde u\|_S\sum_{k_3\in\mathbb{Z}}2^{-|k_3|}\|P_{k_3}u\|_{S_{k_3}},\end{aligned}$$

where we have used Bernstein's and Holder's inequalities as well as the preceding bound for $P_lF(\tilde u)$. $\square$

Proof of (A-2). This is similar to the preceding bound; one places

$$P_{<0}[(\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u))]$$

into $L^\infty_{t,x}$ and

$$P_{\geq 0}[(\Pi_{\tilde{u}_\perp}((-\triangle)^{\frac{1}{2}}u))]$$

into $L^\infty_t L^2_x$. □

Proof of (A-3). We use the preceding bounds, and reduce to bounding $\|P_{<a}(F(\tilde{u})(-\triangle)^{\frac{1}{2}}u)\|_{L^2_t L^\infty_x}$. Then

$$\begin{aligned}\|P_{<a}(F(\tilde{u})(-\triangle)^{\frac{1}{2}}u)\|_{L^2_t L^\infty_x} &\\ &\lesssim \|P_{<a}(F(\tilde{u})P_{<a+10}(-\triangle)^{\frac{1}{2}}u)\|_{L^2_t L^\infty_x} + \|P_{<a}(F(\tilde{u})P_{\geq a+10}(-\triangle)^{\frac{1}{2}}u)\|_{L^2_t L^\infty_x}\end{aligned}$$

and we can bound

$$\begin{aligned}\|P_{<a}(F(\tilde{u})P_{<a+10}(-\triangle)^{\frac{1}{2}}u)\|_{L^2_t L^\infty_x} &\lesssim \sum_{k<a+10} \|P_k(-\triangle)^{\frac{1}{2}}u\|_{L^2_t L^\infty_x}\\ &\lesssim \sum_{k<a+10} 2^{\frac{k}{2}}\|P_k u\|_{S_k} \lesssim_a \|u\|_S,\end{aligned}$$

as well as

$$\begin{aligned}\|P_{<a}(F(\tilde{u})P_{\geq a+10}(-\triangle)^{\frac{1}{2}}u)\|_{L^2_t L^\infty_x} &\lesssim \sum_{k_1=k_2+O(1)\geq a+10} \|P_{<a}(P_{k_2}F(\tilde{u})P_{k_1}(-\triangle)^{\frac{1}{2}}u)\|_{L^2_t L^\infty_x}\\ &\lesssim 2^{\frac{5}{2}a} \sum_{k_1=k_2+O(1)\geq a+10} \|P_{k_2}F(\tilde{u})\|_{L^\infty_t L^2_x} \|P_{k_1}(-\triangle)^{\frac{1}{2}}u\|_{L^2_t L^\infty_x}\\ &\lesssim 2^{\frac{5}{2}a} \sum_{k_1=k_2+O(1)\geq a+10} 2^{-\frac{5}{2}k_2}\|\tilde{u}\|_S \cdot 2^{\frac{k_1}{2}}\|P_{k_1}u\|_{S_{k_1}}\\ &\lesssim_a \|\tilde{u}\|_S \|u\|_S.\end{aligned}$$

□

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THE SEMIGROUP GENERATED BY THE DIRICHLET LAPLACIAN OF FRACTIONAL ORDER

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In the whole space $\mathbb{R}^d$, linear estimates for heat semigroup in Besov spaces are well established, which are estimates of L^p–L^q type, with maximal regularity, etc. This paper is concerned with such estimates for the semigroup generated by the Dirichlet Laplacian of fractional order in terms of the Besov spaces on an arbitrary open set of $\mathbb{R}^d$.

1. Introduction

Let Ω be an arbitrary open set of $\mathbb{R}^d$ with $d \geq 1$. We consider the Dirichlet Laplacian A on $L^2(\Omega)$,

$$A = -\Delta = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2},$$

with the domain

$$\mathcal{D}(A) := \{ f \in H_0^1(\Omega) : \Delta f \in L^2(\Omega) \}.$$

We consider the fractional Laplacian and the semigroup

$$A^{\frac{\alpha}{2}} = \int_{-\infty}^{\infty} \lambda^{\frac{\alpha}{2}} \, dE_A(\lambda), \quad e^{-tA^{\alpha/2}} = \int_{-\infty}^{\infty} e^{-t\lambda^{\alpha/2}} \, dE_A(\lambda), \quad t \geq 0.$$

Here, $\alpha > 0$ and $\{E_A(\lambda)\}_{\lambda \in \mathbb{R}}$ denotes the spectral resolution of identity, which is determined uniquely for the self-adjoint operator A by the spectral theorem. The motivation of the study of the fractional Laplacian comes from the study of fluid mechanics, stochastic processes, finance etc.; see for instance [Applebaum 2009; Bertoin 1996; Chen et al. 2010; Vlahos et al. 2008]. We also refer to [Di Nezza et al. 2012; Vázquez 2012; 2014], where one can find some results on fundamental properties of fractional Sobolev spaces and applications to partial differential equations.

In the paper [Iwabuchi et al. 2016a], based on spectral theory for the Dirichlet Laplacian A on $L^2(\Omega)$, a kind of L^p theory was established and the Besov spaces on an open set Ω were introduced, where regularity of functions is measured by A. The purpose of this paper is to develop linear estimates for the semigroup generated by the Dirichlet Laplacian of fractional order in the homogeneous Besov spaces $\dot{B}^s_{p,q}(A)$, namely, the estimate of L^p–L^q type, smoothing effects, continuity in time of the semigroup, equivalent norms with the semigroup and maximal regularity estimates. Such estimates with the heat semigroup in the case when $\Omega = \mathbb{R}^d$ are well established; see [Bahouri et al. 2011; Chemin 2004; Danchin 2005; 2007; Danchin and Mucha 2009; Hieber and Prüss 1997; Kozono et al. 2003; Lemarié-Rieusset

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2002; Ogawa and Shimizu 2010; 2016; Peetre 1976]. In this paper we consider open sets of $\mathbb{R}^d$ and the semigroup generated by the fractional Laplacian with the Dirichlet boundary condition.

Let us recall the definitions of spaces of test functions and tempered distributions and the Besov spaces associated with the Dirichlet Laplacian; see [Iwabuchi et al. 2016a]. We take $\phi_0(\cdot) \in C_0^\infty(\mathbb{R})$ to be a nonnegative function on $\mathbb{R}$ such that

$$\operatorname{supp}\phi_0 \subset \{\lambda \in \mathbb{R} : 2^{-1} \le \lambda \le 2\}, \quad \sum_{j\in\mathbb{Z}} \phi_0(2^{-j}\lambda) = 1 \quad \text{for } \lambda > 0, \tag{1-1}$$

and $\{\phi_j\}_{j\in\mathbb{Z}}$ is defined by letting

$$\phi_j(\lambda) := \phi_0(2^{-j}\lambda) \quad \text{for } \lambda \in \mathbb{R}. \tag{1-2}$$

Definition. (i) (linear topological spaces $\mathcal{X}_0(\Omega)$ and $\mathcal{X}_0'(\Omega)$) $\mathcal{X}_0(\Omega)$ is defined by letting

$$\mathcal{X}_0(\Omega) := \bigl\{ f \in L^1(\Omega) \cap \mathcal{D}(A) : A^M f \in L^1(\Omega) \cap \mathcal{D}(A) \text{ for all } M \in \mathbb{N} \bigr\},$$

equipped with the family of seminorms $\{p_{0,M}(\cdot)\}_{M=1}^\infty$ given by

$$p_{0,M}(f) := \|f\|_{L^1(\Omega)} + \sup_{j\in\mathbb{N}} 2^{Mj} \|\phi_j(\sqrt{A}) f\|_{L^1(\Omega)}.$$

(ii) (linear topological spaces $\mathcal{Z}_0(\Omega)$ and $\mathcal{Z}_0'(\Omega)$) $\mathcal{Z}_0(\Omega)$ is defined by letting

$$\mathcal{Z}_0(\Omega) := \Bigl\{ f \in \mathcal{X}_0(\Omega) : \sup_{j\le 0} 2^{M|j|} \|\phi_j(\sqrt{A}) f\|_{L^1(\Omega)} < \infty \text{ for all } M \in \mathbb{N} \Bigr\},$$

equipped with the family of seminorms $\{q_{0,M}(\cdot)\}_{M=1}^\infty$ given by

$$q_{0,M}(f) := \|f\|_{L^1(\Omega)} + \sup_{j\in\mathbb{Z}} 2^{M|j|} \|\phi_j(\sqrt{A}) f\|_{L^1(\Omega)}.$$

Definition. For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, $\dot{B}^s_{p,q}(A)$ is defined by letting

$$\dot{B}^s_{p,q}(A) := \{ f \in \mathcal{Z}_0'(\Omega) : \|f\|_{\dot{B}^s_{p,q}(A)} < \infty \},$$

where

$$\|f\|_{\dot{B}^s_{p,q}(A)} := \bigl\| \{ 2^{sj} \|\phi_j(\sqrt{A}) f\|_{L^p(\Omega)} \}_{j\in\mathbb{Z}} \bigr\|_{\ell^q(\mathbb{Z})}.$$

Let us mention the basic properties of $\mathcal{X}_0(\Omega)$, $\mathcal{Z}_0(\Omega)$, their duals, and $\dot{B}^s_{p,q}(A)$ and explain the operators $\phi_j(\sqrt{A})$ and the Laplacian of fractional order.

Proposition [Iwabuchi et al. 2016a]. *Let $s, \alpha \in \mathbb{R}$ and $1 \le p, q, r \le \infty$. Then the following hold*:

(i) *$\mathcal{X}_0(\Omega)$ and $\mathcal{Z}_0(\Omega)$ are Fréchet spaces and enjoy $\mathcal{X}_0(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathcal{X}_0'(\Omega)$ and $\mathcal{Z}_0(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathcal{Z}_0'(\Omega)$.*

(ii) *$\dot{B}^s_{p,q}(A)$ is a Banach space and enjoys $\mathcal{Z}_0(\Omega) \hookrightarrow \dot{B}^s_{p,q}(A) \hookrightarrow \mathcal{Z}_0'(\Omega)$.*

(iii) *If $p, q < \infty$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$, the dual space of $\dot{B}^s_{p,q}(A)$ is $\dot{B}^{-s}_{p',q'}(A)$.*

(iv) *If $r \le p$, then $\dot{B}^{s+d(\frac{1}{r}-\frac{1}{p})}_{r,q}(A)$ is embedded to $\dot{B}^s_{p,q}(A)$.*

(v) *For any $f \in \dot{B}^{s+\alpha}_{p,q}(A)$, we have $A^{\frac{\alpha}{2}} f \in \dot{B}^s_{p,q}(A)$.*

It should be noted that $\phi_j(\sqrt{A})$ and A are defined in $L^2(\Omega)$ initially and by the argument in [Iwabuchi et al. 2016a] they can be realized as operators in $\mathcal{Z}_0'(\Omega)$ and Besov spaces. In the proof, the uniform boundedness in $L^p(\Omega)$ of $\phi_j(\sqrt{A})$ with respect to $j \in \mathbb{Z}$ is essential. Uniformity in $L^2(\Omega)$ is proved easily by the spectral theorem, while that in $L^1(\Omega)$ is not trivial. For any open set $\Omega \subset \mathbb{R}^d$, $L^1(\Omega)$ boundedness is known in some papers; see Proposition 6.1 in [Thinh Duong et al. 2002] and also Theorem 1.1 in [Iwabuchi et al. 2017]. Let us explain the strategy of its proof as in [Iwabuchi et al. 2017] (see also the comment below Lemma 2.2). The uniform boundedness in $L^1(\Omega)$ is proved via estimates in amalgam spaces $\ell^1(L^2)_\theta$, where the side length of each cube is scaled by $\theta^{\frac{1}{2}}$, $\theta = 2^{-2j}$ (see Section 2), together with the Gaussian upper bounds of the kernel of e^{-tA}. That scaling fits for the scaled operator $\phi_j(\sqrt{A}) = \phi_0(2^{-j}\sqrt{A})$, and we can handle the norm in $\ell^1(L^2)_\theta$ through the estimates in $L^2(\Omega)$, since its norm is defined locally with $L^2(\Omega)$. The Gaussian upper bounds of the kernel of e^{-tA} are necessary in order to estimate the $L^1(\Omega)$ norm via $\ell^1(L^2)_\theta$. Once the $L^1(\Omega)$ estimate is proved, the $L^p(\Omega)$ case is assured by interpolation and a duality argument.

As for the Laplacian of fractional order, it was shown in the proof of Proposition 3.2 in [Iwabuchi et al. 2016a] that $A^{\frac{\alpha}{2}}$ is a continuous operator from $\mathcal{Z}_0'(\Omega)$ to itself, which is proved as follows: Show the continuity of $A^{\frac{\alpha}{2}}$ in $\mathcal{Z}_0(\Omega)$ first with the boundedness of spectral multipliers

$$\|A^{\frac{\alpha}{2}}\phi_j(\sqrt{A})\|_{L^1(\Omega)\to L^1(\Omega)} \le C2^{\alpha j}$$

for all $j \in \mathbb{Z}$ and consider their dual operator together with the approximation of the identity

$$f = \sum_{j\in\mathbb{Z}} \phi_j(\sqrt{A})f \quad \text{in } \mathcal{Z}_0'(\Omega) \text{ for any } f \in \mathcal{Z}_0'(\Omega).$$

Hence, we define $A^{\frac{\alpha}{2}}$ by

$$A^{\frac{\alpha}{2}} f = \sum_{j\in\mathbb{Z}} (A^{\frac{\alpha}{2}}\phi_j(\sqrt{A}))f \quad \text{in } \mathcal{Z}_0'(\Omega) \text{ for any } f \in \mathcal{Z}_0'(\Omega).$$

Noting that $e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})$ with $t \ge 0$ is also bounded in $L^1(\Omega)$ (see Lemma 2.1 and (3-1) below), we also define $e^{-tA^{\alpha/2}}$ by

$$e^{-tA^{\alpha/2}} f = \sum_{j\in\mathbb{Z}} (e^{-tA^{\alpha/2}}\phi_j(\sqrt{A}))f \quad \text{in } \mathcal{Z}_0'(\Omega) \text{ for any } f \in \mathcal{Z}_0'(\Omega).$$

We state four theorems on the semigroup generated by $A^{\frac{\alpha}{2}}$: the estimates of L^p–L^q type and smoothing effects, continuity in time, equivalent norms with semigroup and maximal regularity estimates, referring to the results in the case when $\Omega = \mathbb{R}^d$ and $\alpha = 2$.

We start by considering estimates of L^p–L^q type and smoothing effects. When $\Omega = \mathbb{R}^n$, it is well known that

$$\|e^{t\Delta}f\|_{L^q(\mathbb{R}^d)} \le Ct^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}, \quad \|\nabla e^{t\Delta}f\|_{L^p(\mathbb{R}^d)} \le Ct^{-\frac{1}{2}}\|f\|_{L^p(\mathbb{R}^d)},$$

where $1 \le p, q \le \infty$ and $f \in L^p(\mathbb{R}^d)$. Hence one can show that

$$\|e^{t\Delta}f\|_{\dot{B}^{s_2}_{p_2,q}(A)} \le Ct^{-\frac{d}{2}(\frac{1}{p_1}-\frac{1}{p_2})-\frac{s_2-s_1}{2}}\|f\|_{\dot{B}^{s_1}_{p_1,q}(A)},$$

where $s_2 \ge s_1$, $1 \le p_1 \le p_2 \le \infty$ and $1 \le q \le \infty$. The following gives the linear estimates for the semigroup generated by $A^{\frac{\alpha}{2}}$ on an open set.

Theorem 1.1. *Let $\alpha > 0$, $t \ge 0$, $s, s_1, s_2 \in \mathbb{R}$ and $1 \le p, p_1, p_2, q, q_1, q_2 \le \infty$:*

(i) *$e^{-tA^{\alpha/2}}$ is a bounded linear operator in $\dot{B}^s_{p,q}(A)$; i.e., there exists a constant $C > 0$ such that for any $f \in \dot{B}^s_{p,q}(A)$*

$$e^{-tA^{\alpha/2}} f \in \dot{B}^s_{p,q}(A) \quad \text{and} \quad \|e^{-tA^{\alpha/2}} f\|_{\dot{B}^s_{p,q}(A)} \le C \|f\|_{\dot{B}^s_{p,q}(A)}. \tag{1-3}$$

(ii) *If $s_2 \ge s_1$, $p_1 \le p_2$ and*

$$d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) + s_2 - s_1 > 0,$$

then there exists a constant $C > 0$ such that

$$\|e^{-tA^{\alpha/2}} f\|_{\dot{B}^{s_2}_{p_2,q_2}(A)} \le C t^{-\frac{d}{\alpha}(\frac{1}{p_1}-\frac{1}{p_2})-\frac{s_2-s_1}{\alpha}} \|f\|_{\dot{B}^{s_1}_{p_1,q_1}(A)} \tag{1-4}$$

for any $f \in \dot{B}^{s_1}_{p_1,q_1}(A)$.

Remark. In the estimate (1-4), the regularity on indices q_1 and q_2 is gained without loss of the singularity at $t = 0$. This estimate is known in the case when $\Omega = \mathbb{R}^n$ and $\alpha = 2$; see [Kozono et al. 2003].

As for the continuity in time of the heat semigroup $e^{t\Delta}$ when $\Omega = \mathbb{R}^d$, it is well known that for $1 \le p < \infty$

$$\lim_{t \to 0} \|e^{t\Delta} f - f\|_{L^p(\mathbb{R}^d)} = 0 \quad \text{for any } f \in L^p(\mathbb{R}^d).$$

In the case when $p = \infty$, the above strong convergence does not hold in general, while it holds in the dual weak sense. The following theorem is concerned with such continuity in the Besov spaces on an open set.

Theorem 1.2. *Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$:*

(i) *Assume that $q < \infty$ and $f \in \dot{B}^s_{p,q}(A)$. Then*

$$\lim_{t \to 0} \|e^{-tA^{\alpha/2}} f - f\|_{\dot{B}^s_{p,q}(A)} = 0.$$

(ii) *Assume that $1 < p \le \infty$, $q = \infty$ and $f \in \dot{B}^s_{p,\infty}(A)$. Then $e^{-tA^{\alpha/2}} f$ converges to f in the dual weak sense as $t \to 0$; namely,*

$$\lim_{t \to 0} \sum_{j \in \mathbb{Z}} \int_\Omega \{\phi_j(\sqrt{A})(e^{-tA^{\alpha/2}} f - f)\} \overline{\Phi_j(\sqrt{A}) g}\, dx = 0$$

for any $g \in \dot{B}^{-s}_{p',1}(A)$.

Remark. Related to Theorem 1.2(ii), it should be noted that the predual of $\dot{B}^s_{p,q}(A)$ is $\dot{B}^{-s}_{p',q'}(A)$ for $1 < p, q \le \infty$, where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. In fact, we can regard $f \in \dot{B}^s_{p,q}(A)$ as an element of the dual of $\dot{B}^{-s}_{p',q'}(A)$ by

$$\langle f, g \rangle = \sum_{j \in \mathbb{Z}} \int_\Omega \{\phi_j(\sqrt{A}) f\} \overline{\Phi_j(\sqrt{A}) g}\, dx$$

for any $g \in \dot{B}^{-s}_{p',q'}(A)$, see [Iwabuchi et al. 2016a], where $\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1}$.

As for the characterization of the norm by using the semigroup when $\Omega = \mathbb{R}^d$, it is known that

$$\|f\|_{\dot{B}^s_{p,q}(A)} \simeq \left\{ \int_0^\infty \left(t^{-\frac{s}{2}} \|e^{t\Delta} f\|_{L^p(\mathbb{R}^d)}\right)^q \frac{dt}{t} \right\}^{\frac{1}{q}},$$

where $s < 0$; see, e.g., [Lemarié-Rieusset 2002]. We consider the equivalent norm of Besov spaces on an open set by using the semigroup generated by $A^{\frac{\alpha}{2}}$.

Theorem 1.3. *Let $\alpha > 0$, $s, s_0 \in \mathbb{R}$, $s_0 > \frac{s}{\alpha}$ and $1 \le p, q \le \infty$. Then there exists a constant $C > 0$ such that*

$$C^{-1} \|f\|_{\dot{B}^s_{p,q}(A)} \le \left\{ \int_0^\infty \left(t^{-\frac{s}{\alpha}} \|(tA^{\frac{\alpha}{2}})^{s_0} e^{-tA^{\alpha/2}} f\|_X\right)^q \frac{dt}{t} \right\}^{\frac{1}{q}} \le C \|f\|_{\dot{B}^s_{p,q}(A)} \tag{1-5}$$

for any $f \in \dot{B}^s_{p,q}(A)$, where $X = L^p(\Omega)$, $\dot{B}^0_{p,r}(A)$ with $1 \le r \le \infty$.

Since the equivalence (1-5) is closely related to the real interpolation in the Besov spaces, we mention that the interpolation is also available; see, e.g., [Bergh and Löfström 1976; Triebel 1983] and also Proposition A.1 in the Appendix.

The last result is concerned with the maximal regularity estimates. When $\Omega = \mathbb{R}^d$, the Cauchy problem which we should consider is

$$\begin{cases} \partial_t u - \Delta u = f, & t > 0,\ x \in \mathbb{R}^d, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

For $1 < p, q < \infty$, the solution u of the above problem satisfies

$$\|\partial_t u\|_{L^q(0,\infty;L^p(\mathbb{R}^d))} + \|\Delta u\|_{L^q(0,\infty;L^p(\mathbb{R}^d))} \le C \|u_0\|_{\dot{B}^{2-2/q}_{p,q}(A)} + C \|f\|_{L^q(0,\infty;L^p(\mathbb{R}^d))},$$

provided that $u_0 \in \dot{B}^{2-\frac{2}{q}}_{p,q}(A)$ and $f \in L^q(0, \infty; L^p(\mathbb{R}^d))$; see [Hieber and Prüss 1997; Lemarié-Rieusset 2002]. We note that maximal regularity such as the above is well-studied in the general framework on Banach spaces with unconditional martingale differences (UMD); see [Amann 1995; Da Prato and Grisvard 1975; Denk et al. 2003; Dore and Venni 1987; Ladyzhenskaya and Ural'tseva 1968; Weis 2001]. We also note that the cases when $p, q = 1, \infty$ require a different treatment from UMD since the spaces are not reflexive. In terms of Besov spaces, one can consider $\dot{B}^0_{p,q}(A)$ for all indices p, q with $1 \le p, q \le \infty$; see [Danchin 2005; 2007; Danchin and Mucha 2009; Hieber and Prüss 1997; Ogawa and Shimizu 2010; 2016]. Our result on the maximal regularity estimates on open sets is formulated in the following way.

Theorem 1.4. *Let $s \in \mathbb{R}$, $\alpha > 0$ and $1 \le p, q \le \infty$. Assume that $u_0 \in \dot{B}^{s+\alpha-\frac{\alpha}{q}}_{p,q}(A)$, $f \in L^q(0, \infty; \dot{B}^s_{p,q}(A))$. Let u be given by*

$$u(t) = e^{-tA^{\alpha/2}} u_0 + \int_0^t e^{-(t-\tau)A^{\alpha/2}} f(\tau)\, d\tau.$$

Then there exists a constant $C > 0$ independent of u_0 and f such that

$$\|\partial_t u\|_{L^q(0,\infty;\dot{B}^s_{p,q}(A))} + \|A^{\frac{\alpha}{2}} u\|_{L^q(0,\infty;\dot{B}^s_{p,q}(A))} \le C \|u_0\|_{\dot{B}^{s+\alpha-\alpha/q}_{p,q}(A)} + C \|f\|_{L^q(0,\infty;\dot{B}^s_{p,q}(A))}. \tag{1-6}$$

The proofs of our theorems are based on the boundedness of the spectral multiplier of the operator $e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})$:

$$\big\|e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})\big\|_{L^p(\Omega)\to L^p(\Omega)} \le C\,\big\|e^{-t2^{\alpha j}(\cdot)^\alpha}\phi_0(\sqrt{\cdot}\,)\big\|_{H^s(\mathbb{R})} \quad \text{for all } j\in\mathbb{Z},$$

where $s>\frac{d}{2}+\frac{1}{2}$ (see Lemma 2.1 below). The above inequality implies

$$\big\|e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})\big\|_{L^p(\Omega)\to L^p(\Omega)} \le Ce^{-C^{-1}t2^{\alpha j}},$$

and this estimate allows us to show our theorems in a method analogous to those in the case when $\Omega=\mathbb{R}^d$. In this paper, we give proofs of all theorems by estimating directly so that the paper is self-contained. Here, we note that our proofs can be applicable to the estimates for e^{-tA} in the inhomogeneous Besov spaces and hence similar theorems are able to be obtained. On the other hand, for the semigroup generated by the fractional Laplacian, since there appear to be problems around low frequencies, we show only the result for the heat semigroup in Section 7 (see Theorem 7.2 below). It should be also noted that our argument can be applied not only to the Dirichlet Laplacian but also to more general operators A such that the Gaussian upper bounds for e^{-tA} hold.

This paper is organized as follows. In Section 2, we prepare a lemma to prove our results. Sections 3–6 are devoted to proofs of theorems. In Section 7, we state the result for the inhomogeneous Besov spaces. In the Appendix, we show the characterization of Besov spaces by real interpolation.

Before closing this section, we introduce some notation. We denote by $\|\cdot\|_{L^p}$ the $L^p(\Omega)$ norm and by $\|\cdot\|_{\dot{B}^s_{p,q}}$ the $\dot{B}^s_{p,q}(A)$ norm. We use the notation $\|\cdot\|_{H^s(\mathbb{R})}$ to represent the $H^s(\mathbb{R})$ norm for functions, e.g., $\phi_j(\lambda)$, $e^{-t\lambda^{\alpha/2}}$, whose variables are spectral parameters. We denote by $\mathcal{S}(\mathbb{R})$ the Schwartz class.

2. Preliminaries

In this section we introduce the following lemma on the boundedness of the scaled spectral multiplier.

Lemma 2.1. *Let $N>\frac{d}{2}$, $1\le p\le\infty$, $\delta>0$ and $a,b>0$. Then there exists a constant $C>0$ such that for any $\phi\in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\phi\subset[a,b]$, $G\in C^\infty((0,\infty))\cap C(\mathbb{R})$ and $f\in L^p(\Omega)$ we have*

$$\big\|G(\sqrt{A})\phi(2^{-j}\sqrt{A})f\big\|_{L^p} \le C\,\big\|G(2^j\sqrt{\cdot}\,)\phi(\sqrt{\cdot}\,)\big\|_{H^{N+1/2+\delta}(\mathbb{R})}\|f\|_{L^p} \tag{2-1}$$

for all $j\in\mathbb{Z}$.

Remark. As is seen from the proof below, the constant C on the right-hand side of (2-1) depends on the interval $[a,b]$ containing the support of ϕ.

To prove Lemma 2.1, we introduce a set $\mathscr{A}_N$ of some bounded operators on $L^2(\Omega)$ and scaled amalgam spaces $\ell^1(L^2)_\theta$ for $\theta>0$ to prepare a lemma. Hereafter, for $k\in\mathbb{Z}^d$, $C_\theta(k)$ denotes a cube with the center $\theta^{\frac{1}{2}}k$ and side length $\theta^{\frac{1}{2}}$, namely,

$$C_\theta(k) := \big\{x\in\Omega : |x_j-\theta^{\frac{1}{2}}k_j|\le 2^{-1}\theta^{\frac{1}{2}} \text{ for } j=1,2,\dots,d\big\},$$

and $\chi_{C_\theta(k)}$ is a characteristic function whose support is $C_\theta(k)$.

Definition. For $N \in \mathbb{N}$, we denote by $\mathscr{A}_N$ the set of all bounded operators T on $L^2(\Omega)$ such that

$$\|T\|_{\mathscr{A}_N} := \sup_{k\in\mathbb{Z}^d} \big\| |\cdot - \theta^{\frac{1}{2}} k|^N T \chi_{C_\theta(k)} \big\|_{L^2\to L^2} < \infty.$$

Definition. The space $\ell^1(L^2)_\theta$ is defined by letting

$$\ell^1(L^2)_\theta := \{ f \in L^2_{\mathrm{loc}}(\overline{\Omega}) : \|f\|_{\ell^1(L^2)_\theta} < \infty \},$$

where

$$\|f\|_{\ell^1(L^2)_\theta} := \sum_{k\in\mathbb{Z}^d} \|f\|_{L^2(C_\theta(k))}.$$

Lemma 2.2 [Iwabuchi et al. 2017; Iwabuchi et al. 2016b]. (i) *Let $N \in \mathbb{N}$ and $N > \frac{d}{2}$. Then there exists a constant $C > 0$ such that*

$$\|T\|_{\ell^1(L^2)_\theta \to \ell^1(L^2)_\theta} \le C \Big(\|T\|_{L^2\to L^2} + \theta^{-\frac{d}{4}} \|T\|_{\mathscr{A}_N}^{\frac{d}{2N}} \|T\|_{L^2\to L^2}^{1-\frac{d}{2N}} \Big) \tag{2-2}$$

for any $T \in \mathscr{A}_N$ and $\theta > 0$.

(ii) *Let $N \in \mathbb{N}$. Then there exists a constant $C > 0$ such that*

$$\|\psi((M+\theta A)^{-1})\|_{\mathscr{A}_N} \le C \theta^{\frac{N}{2}} \int_{-\infty}^{\infty} (1+|\xi|^2)^{\frac{N}{2}} |\hat{\psi}(\xi)| \, d\xi \tag{2-3}$$

for any $\psi \in \mathcal{S}(\mathbb{R})$ and $\theta > 0$.

(iii) *Let $M > 0$ and $\beta > \frac{d}{4}$. Then there exists a constant $C > 0$ such that*

$$\|(M+\theta A)^{-\beta}\|_{L^1\to \ell^1(L^2)_\theta} \le C\theta^{-\frac{d}{2}} \tag{2-4}$$

for any $\theta > 0$.

Remark. Lemma 2.2 is useful to prove the L^1 boundedness of spectral multipliers and let us briefly remind how to prove Lemma 2.2 as in [Iwabuchi et al. 2017; Iwabuchi et al. 2016b]. The original idea is by Jensen and Nakamura [1994; 1995], who studied the Schrödinger operators on $\mathbb{R}^d$. In the first inequality (2-2), we start with the decomposition $T = \sum_{m,k\in\mathbb{Z}^d} \chi_{C_\theta(m)} T \chi_{C_\theta(k)}$, and it suffices to show that for each $k \in \mathbb{Z}$ a sum of operator norms $\sum_{m\in\mathbb{Z}} \|\chi_{C_\theta(m)} T \chi_{C_\theta(k)}\|_{L^2\to L^2}$ is bounded by the right-hand side of (2-2). The first term $\|T\|_{L^2\to L^2}$ is obtained just by applying $L^2(\Omega)$ boundedness to the $L^2(C_\theta(m))$ norm with $m = k$. The second term is obtained by decomposing the sum into two cases when $0 < |m-k| \le \omega$ and $|m-k| > \omega$ for $\omega > 0$, applying the $L^2(\Omega)$ boundedness to the case $|m-k| \le \omega$ and the Schwarz inequality to the case $|m-k| > \omega$ for sequences $|m-k|^{-N}$, $|m-k|^N \|\chi_{C_\theta(m)} T \chi_{C_\theta(k)}\|_{L^2}$, and minimizing by taking suitable ω. As for the second one (2-3), we utilize the formula

$$\psi((M+\theta A)^{-1}) = (2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-it(M+\theta A)^{-1}} \hat{\psi}(t) \, dt.$$

To estimate $\|e^{-it(M+\theta A)^{-1}}\|_{A_N}$, we consider the commutator of $(x - \theta^{1/2} k)$ and $e^{-it(M+\theta A)^{-1}}$, which is rewritten with θ, $(M+\theta A)^{-1}$, $\nabla (M+\theta A)^{-1}$ and is able to be handled by the use of $L^2(\Omega)$

boundedness, which proves (2-3). As for the last one (2-4), thanks to the formula

$$(M+\theta A)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-Mt} e^{-t\theta A}\, dt$$

and the Young inequality, we get

$$\|(M+\theta A)^{-\beta} f\|_{\ell^1(L^2)_\theta} \le \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-Mt} \left(\int_\Omega \|e^{-t\theta A}(\,\cdot\,, y)\|_{\ell^1(L^2)_\theta} |f(y)|\, dy \right) dt,$$

where $\Gamma(\beta)$ is the Gamma function. By the Gaussian upper bounds of $e^{-t\theta A}$, we have

$$\|e^{-t\theta A}(\,\cdot\,, y)\|_{\ell^1(L^2)_\theta} \le C\theta^{-\frac{d}{4}}(1+t^{-\frac{d}{4}}).$$

These estimates yield the inequality (2-4), since the integrability with respect to $t \in (0,\infty)$ is assured by $\beta > \frac{4}{d}$.

Proof of Lemma 2.1. Instead of the inequality (2-1), by replacing $2^{-j}\sqrt{A}$ and $\sqrt{A}$ with $2^{-2j}A$ and A, respectively, it is sufficient to show that

$$\|G(A)\phi(2^{-2j}A)f\|_{L^p} \le C\|G(2^{2j}\,\cdot\,)\phi(\,\cdot\,)\|_{H^{N+1/2+\delta}(\mathbb{R})}\|f\|_{L^p}, \tag{2-5}$$

where $\operatorname{supp}\phi \subset [a^2, b^2]$.

First we consider the case when $p=1$. By decomposing Ω into cubes $C_\theta(k)$ and the Hölder inequality, we get

$$\|G(A)f\phi(2^{-2j}A)\|_{L^1} \le C\theta^{\frac{d}{2}}\|G(A)\phi(2^{-2j}A)f\|_{\ell^1(L^2)_\theta}. \tag{2-6}$$

For fixed real numbers $M>0$ and $\beta > \frac{d}{2}$, let ψ be such that

$$\psi(\mu) := G(2^{2j}(\mu^{-1}-M))\phi(\mu^{-1}-M)\mu^{-\beta}. \tag{2-7}$$

It is easy to check that

$$\psi \in C_0^\infty((0,\infty)) \quad \text{and} \quad \operatorname{supp}\psi \subset \left[\frac{1}{M+b}, \frac{1}{M+a}\right],$$

and

$$G(\lambda)\phi(2^{-2j}\lambda) = G(2^{2j}\cdot 2^{-2j}\lambda)\phi(2^{-2j}\lambda)\mu^{-\beta}\cdot\mu^{\beta} = \psi(\mu)\mu^{\beta},$$

where λ and μ are real numbers with

$$2^{-2j}\lambda = \mu^{-1} - M.$$

The above equality yields that

$$G(A)\phi(2^{-2j}A) = \psi((M+2^{-2j}A)^{-1})(M+2^{-2j}A)^{-\beta}. \tag{2-8}$$

Then it follows from (2-6), (2-8) and the estimate (2-4) in Lemma 2.2 that

$$\begin{aligned}
&\|G(A)\phi(2^{-2j}A)f\|_{L^1} \\
&\qquad \le C\theta^{\frac{d}{2}} \big\|\psi((M+2^{-2j}A)^{-1})(M+2^{-2j}A)^{-\beta} f\big\|_{\ell^1(L^2)_\theta} \\
&\qquad \le C\theta^{\frac{d}{2}} \|\psi((M+2^{-2j}A)^{-1})\|_{\ell^1(L^1)_\theta\to\ell^1(L^2)_\theta} \|(M+2^{-2j}A)^{-\beta}\|_{L^1\to\ell^1(L^2)_\theta}\|f\|_{L^1} \\
&\qquad \le C\|\psi((M+2^{-2j}A)^{-1})\|_{\ell^1(L^1)_\theta\to\ell^1(L^2)_\theta}\|f\|_{L^1}.
\end{aligned} \tag{2-9}$$

By comparing the estimates (2-5) and (2-9), all we have to do is to show that

$$\|\psi((M+2^{-2j}A)^{-1})\|_{\ell^1(L^2)_\theta\to\ell^1(L^2)_\theta} \le C\|G(2^j\,\cdot)\phi(\cdot)\|_{H^{N+1/2+\delta}(\mathbb{R})}. \tag{2-10}$$

To apply the estimate (2-2), we consider the operator norms $\|\cdot\|_{L^2\to L^2}$ and $\|\cdot\|_{\mathscr{A}_N}$ of $\psi((M+2^{-2j}A)^{-1})$. On the operator norm $\|\cdot\|_{L^2\to L^2}$, we have from $N>\frac{d}{2}$ and the embedding $H^{N+\frac{1}{2}+\delta}(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$ that

$$\|\psi((M+2^{-2j}A)^{-1})\|_{L^2\to L^2} \le \|\psi\|_{L^\infty(\mathbb{R})} \le \|\psi\|_{H^{N+1/2+\delta}(\mathbb{R})}$$

for any $\delta>0$. As for $\|\psi((M+2^{-2j}A)^{-1})\|_{\mathscr{A}_N}$, by applying the estimate (2-3) and the Hölder inequality, for any $\delta>0$ there exists $C>0$ such that

$$\begin{aligned}\|\psi((M+\theta A)^{-1})\|_{\mathscr{A}_N} &\le C\theta^{\frac{N}{2}}\int_{-\infty}^{\infty}(1+|\xi|^2)^{\frac{N}{2}}|\hat{\psi}(\xi)|\,d\xi\\ &\le C\theta^{\frac{N}{2}}\|(1+|\xi|^2)^{-\frac{1}{2}-\delta}\|_{L^2(\mathbb{R})}\|(1+|\xi|^2)^{\frac{N}{2}+\frac{1}{2}+\delta}\hat{\psi}\|_{L^2(\mathbb{R})}\\ &\le C\theta^{\frac{N}{2}}\|\psi\|_{H^{N+1/2+\delta}(\mathbb{R})}.\end{aligned}$$

Then we deduce from the above two estimates and (2-2) that

$$\begin{aligned}&\|\psi((M+2^{-2j}A)^{-1})\|_{\ell^1(L^2)_\theta\to\ell^1(L^2)_\theta}\\ &\qquad\le C\big\{\|\psi\|_{H^{N+1/2+\delta}(\mathbb{R})}+\theta^{-\frac{d}{4}}(\theta^{\frac{N}{2}}\|\psi\|_{H^{N+1/2+\delta}(\mathbb{R})})^{\frac{d}{2N}}(\|\psi\|_{H^{N+1/2+\delta}(\mathbb{R})})^{1-\frac{d}{2N}}\big\}\\ &\qquad\le C\|\psi\|_{H^{N+1/2+\delta}(\mathbb{R})}.\end{aligned}$$

Since ψ is defined by (2-7) and the support is bounded and away from the origin, we see from the change of variables by $\mu=(\lambda+M)^{-1}$ that

$$\|\psi(\cdot)\|_{H^{N+1/2+\delta}(\mathbb{R})} \le C\|G(2^{2j}\,\cdot)\phi(\cdot)\|_{H^{N+1/2+\delta}(\mathbb{R})}.$$

Hence the estimate (2-10) is obtained by the above two estimates, and the estimate (2-5) in the case when $p=1$ is proved.

We next consider the case when $p=\infty$. Since the dual space of $L^1(\Omega)$ is $L^\infty(\Omega)$ and $C_0^\infty(\Omega)$ is dense in $L^1(\Omega)$, the following holds:

$$\|G(A)\phi(2^{-j}A)f\|_{L^\infty} = \sup_{g\in C_0^\infty,\,\|g\|_{L^1}=1}\left|\int_\Omega\big(G(A)\phi(2^{-j}A)f\big)\bar{g}\,dx\right|.$$

On the right-hand side of the above equality, we have from the duality argument for the operator $G(A)\phi(2^{-j}A)$, the Hölder inequality and the estimate (2-5) with $p=1$ that

$$\begin{aligned}\left|\int_\Omega\big(G(A)\phi(2^{-j}A)f\big)\,\bar{g}\,dx\right| &= \big|{}_{\mathcal{X}_0'}\langle G(A)\phi(2^{-j}A)f,g\rangle_{\mathcal{X}_0}\big| = \big|{}_{\mathcal{X}_0'}\langle f,G(A)\phi(2^{-j}A)g\rangle_{\mathcal{X}_0}\big|\\ &= \left|\int_\Omega f\,\overline{G(A)\phi(2^{-j}A)g}\,dx\right| \le \|f\|_{L^\infty}\|G(A)\phi(2^{-j}A)g\|_{L^1}\\ &\le \|f\|_{L^\infty}\|G(2^{2j}\,\cdot)\phi(\cdot)\|_{H^{N+1/2+\delta}(\mathbb{R})}\|g\|_{L^1},\end{aligned}$$

where $g\in C_0^\infty$. This proves (2-5) in the case when $p=\infty$.

As for the case when $1<p<\infty$, the Riesz–Thorin theorem allows us to obtain the estimate (2-5). $\square$

3. Proof of Theorem 1.1

Proof of (1-3). Put $\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1}$. By applying the estimate (2-1) in Lemma 2.1 with

$$G = G_t(\lambda) = e^{-t\lambda^\alpha},$$

we have

$$\begin{aligned}\big\|\phi_j(\sqrt{A})e^{-tA^{\alpha/2}} f\big\|_{L^p} &= \big\|\big(G_t(\sqrt{A})\Phi_j(\sqrt{A})\big)(\phi_j(\sqrt{A})f)\big\|_{L^p} \\ &\le C\big\|G_t(2^j\sqrt{\cdot}\,)\Phi_0(\sqrt{\cdot}\,)\big\|_{H^{N+1/2+\delta}(\mathbb{R})}\big\|\phi_j(\sqrt{A})f\big\|_{L^p},\end{aligned}$$

where $N > \frac{d}{2}$ and $\delta > 0$. Here it is easy to check that there exists $C > 0$ such that

$$\big\|G_t(2^j\sqrt{\cdot}\,)\Phi_0(\sqrt{\cdot}\,)\big\|_{H^{N+1/2+\delta}(\mathbb{R})} \le Ce^{-C^{-1}t2^{\alpha j}} \quad \text{for any } j \in \mathbb{Z},$$

and hence,

$$\big\|\phi_j(\sqrt{A})e^{-tA^{\alpha/2}} f\big\|_{L^p} \le Ce^{-C^{-1}t2^{\alpha j}}\big\|\phi_j(\sqrt{A})f\big\|_{L^p} \quad \text{for any } j \in \mathbb{Z}. \tag{3-1}$$

By multiplying 2^{sj} and taking the $\ell^q(\mathbb{Z})$ norm in the above inequality, we obtain the assertion (1-3). □

Proof of (1-4). By the inequalities

$$\|e^{-tA^{\alpha/2}} f\|_{\dot{B}^{s_2}_{p_2,q_2}} \le \|e^{-tA^{\alpha/2}} f\|_{\dot{B}^{s_2}_{p_2,1}}, \quad \|f\|_{\dot{B}^{s_1}_{p_1,\infty}} \le \|f\|_{\dot{B}^{s_1}_{p_1,q_1}},$$

which are assured from the embedding relations in the Besov spaces, and taking $s_1 = 0$ for the sake of simplicity, it is sufficient to show

$$\|e^{-tA^{\alpha/2}} f\|_{\dot{B}^{s_2}_{p_2,1}} \le Ct^{-\frac{d}{\alpha}(\frac{1}{p_1}-\frac{1}{p_2})-\frac{s_2}{\alpha}}\|f\|_{\dot{B}^0_{p_1,\infty}}, \tag{3-2}$$

where

$$s_2 \ge 0, \quad p_1 \le p_2 \quad \text{and} \quad d\Big(\frac{1}{p_1}-\frac{1}{p_2}\Big) + s_2 > 0.$$

It follows from the embedding $\dot{B}^{s_2+d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_1,1} \hookrightarrow \dot{B}^{s_2}_{p_2,1}$ and the estimate (3-1) that

$$\|e^{-tA^{\alpha/2}} f\|_{\dot{B}^{s_2}_{p_2,1}} \le C\|e^{-tA^{\alpha/2}} f\|_{\dot{B}^{s_2+d(1/p_1-1/p_2)}_{p_1,1}} \le C\sum_{j\in\mathbb{Z}} 2^{s_2 j+d(\frac{1}{p_1}-\frac{1}{p_2})j}e^{-ct2^{\alpha j}}\big\|\phi_j(\sqrt{A})f\big\|_{L^{p_1}}.$$

Since $s_2 + d\big(\frac{1}{p_1}-\frac{1}{p_2}\big) > 0$, we get

$$\begin{aligned}&\sum_{j\in\mathbb{Z}} 2^{s_2 j+d(\frac{1}{p_1}-\frac{1}{p_2})j}e^{-ct2^{\alpha j}}\big\|\phi_j(\sqrt{A})f\big\|_{L^{p_1}} \\ &\qquad = t^{-\frac{s_2}{\alpha}-\frac{d}{\alpha}(\frac{1}{p_1}-\frac{1}{p_2})}\sum_{j\in\mathbb{Z}}\big\{(t2^{\alpha j})^{\frac{s_2}{\alpha}+\frac{d}{\alpha}(\frac{1}{p_1}-\frac{1}{p_2})}e^{-ct2^{\alpha j}}\big\}\big\|\phi_j(\sqrt{A})f\big\|_{L^{p_1}} \\ &\qquad \le Ct^{-\frac{s_2}{\alpha}-\frac{d}{\alpha}(\frac{1}{p_1}-\frac{1}{p_2})}\|f\|_{\dot{B}^0_{p_1,\infty}},\end{aligned}$$

which proves (3-2). □

4. Proof of Theorem 1.2

Proof of (i). Let $f \in \dot{B}^s_{p,q}(A)$. We take f_N such that

$$f_N := \sum_{|j|\le N} \phi_j(\sqrt{A}) f \quad \text{for } N \in \mathbb{N}.$$

Since $q < \infty$, for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$\| f_N - f \|_{\dot{B}^s_{p,q}} < \varepsilon \quad \text{for any } N \ge N_0.$$

The above inequality and boundedness (1-3) in Theorem 1.1 imply

$$\begin{aligned}
\| e^{-tA^{\alpha/2}} f - f \|_{\dot{B}^s_{p,q}} &\le \| e^{-tA^{\alpha/2}} f_N - f_N \|_{\dot{B}^s_{p,q}} + \| e^{-tA^{\alpha/2}} (f_N - f) \|_{\dot{B}^s_{p,q}} + \| f_N - f \|_{\dot{B}^s_{p,q}} \\
&\le \| e^{-tA^{\alpha/2}} f_N - f_N \|_{\dot{B}^s_{p,q}} + C \| f_N - f \|_{\dot{B}^s_{p,q}} \\
&\le \| e^{-tA^{\alpha/2}} f_N - f_N \|_{\dot{B}^s_{p,q}} + C\varepsilon
\end{aligned}$$

for any $t > 0$ provided that $N \ge N_0$. Then all we have to do is to show that

$$\lim_{t\to 0} \| e^{-tA^{\alpha/2}} f_N - f_N \|_{\dot{B}^s_{p,q}} = 0. \tag{4-1}$$

We prove (4-1). Noting that the spectrum of f_N is restricted and

$$\| e^{-tA^{\alpha/2}} f_N - f_N \|_{\dot{B}^s_{p,q}} = \left\{ \sum_{j=-N-1}^{N+1} \left(2^{sj} \left\| \phi_j(\sqrt{A})(e^{-tA^{\alpha/2}} - 1) f_N \right\|_{L^p} \right)^q \right\}^{\frac{1}{q}},$$

we may consider the convergence of $\left\| \phi_j(\sqrt{A})(e^{-tA^{\alpha/2}} - 1) f_N \right\|_{L^p}$ for each j. For each $j = 0, \pm 1, \pm 2, \dots, \pm(N+1)$, it follows from (2-1) in Lemma 2.1 with

$$G = G_t(\lambda) = e^{-t\lambda^\alpha} - 1$$

that

$$\begin{aligned}
\left\| \phi_j(\sqrt{A})(e^{-tA^{\alpha/2}} - 1) f_N \right\|_{L^p} &= \left\| (G_t(\sqrt{A}) \Phi_j(\sqrt{A}))(\phi_j(\sqrt{A}) f_N) \right\|_{L^p} \\
&\le C \left\| G_t(2^j \sqrt{\cdot}\,) \Phi_0(\sqrt{\cdot}\,) \right\|_{H^{N+d/2+\delta}} \left\| \phi_j(\sqrt{A}) f_N \right\|_{L^p},
\end{aligned}$$

where $\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1}$. Here it is readily checked that

$$\lim_{t\to 0} \left\| G_t(2^j \sqrt{\cdot}\,) \Phi_0(\sqrt{\cdot}\,) \right\|_{H^{N+d/2+\delta}} = 0 \quad \text{for each } j,$$

and hence, (4-1) is obtained. □

Proof of (ii). Put $\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1}$. By considering the dual operator of $e^{-tA^{\alpha/2}} - 1$, we have

$$\sum_{j\in\mathbb{Z}} \int_\Omega \{ \phi_j(\sqrt{A})(e^{-tA^{\alpha/2}} - 1) f \} \overline{\Phi_j(\sqrt{A}) g}\, dx = \sum_{j\in\mathbb{Z}} \int_\Omega \{ \phi_j(\sqrt{A}) f \} \overline{\Phi_j(\sqrt{A})(e^{-tA^{\alpha/2}} - 1) g}\, dx. \tag{4-2}$$

It follows from the Hölder inequality that

$$\sum_{j\in\mathbb{Z}}\int_\Omega \big|\{\phi_j(\sqrt{A})f\}\overline{\Phi_j(\sqrt{A})(e^{-tA^{\alpha/2}}-1)g}\big|\,dx$$
$$\le \sum_{j\in\mathbb{Z}} 2^{sj}\,\|\phi_j(\sqrt{A})f\|_{L^p}\cdot 2^{-sj}\,\|\Phi_j(\sqrt{A})(e^{-tA^{\alpha/2}}-1)g\|_{L^{p'}}$$
$$\le C\|f\|_{\dot B^s_{p,\infty}}\|(e^{-tA^{\alpha/2}}-1)g\|_{\dot B^{-s}_{p',1}}, \tag{4-3}$$

which assures the absolute convergence of the series in (4-2) by the boundedness of $e^{-tA^{\alpha/2}}$ in $\dot B^{-s}_{p',1}$ from (1-3) in Theorem 1.1. The above estimate and the assertion (i) of Theorem 1.2 imply

$$\bigg|\sum_{j\in\mathbb{Z}}\int_\Omega \{\phi_j(\sqrt{A})(e^{-tA^{\alpha/2}}-1)f\}\overline{\Phi_j(\sqrt{A})g}\,dx\bigg| \le C\|f\|_{\dot B^s_{p,\infty}}\|(e^{-tA^{\alpha/2}}-1)g\|_{\dot B^{-s}_{p',1}}\to 0 \quad \text{as } t\to 0. \ \square$$

5. Proof of Theorem 1.3

To prove Theorem 1.3 we will need the following lemma.

Lemma 5.1. *Let $\alpha>0$, $s_0\in\mathbb{R}$ and $1\le p\le\infty$. Then there exists $C>0$ such that*

$$\begin{aligned}C^{-1}(t2^{\alpha j})^{s_0}e^{-Ct2^{\alpha j}}\,\|\phi_j(\sqrt{A})f\|_{L^p} &\le \|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})f\|_{L^p}\\ &\le C(t2^{\alpha j})^{s_0}e^{-C^{-1}t2^{\alpha j}}\,\|\phi_j(\sqrt{A})f\|_{L^p}\end{aligned} \tag{5-1}$$

for any $t>0$, $j\in\mathbb{Z}$ and $f\in L^p(\Omega)$.

Proof. Put $\Phi_j:=\phi_{j-1}+\phi_j+\phi_{j+1}$. We start by proving the second inequality of the estimate (5-1). By applying the estimate (2-1) in Lemma 2.1 with

$$G=G_t(\lambda)=(t\lambda^\alpha)^{s_0}e^{-t\lambda^\alpha},$$

we have

$$\begin{aligned}\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})f\|_{L^p} &= \big\|\big(G_t(\sqrt{A})\Phi_j(\sqrt{A})\big)(\phi_j(\sqrt{A})f)\big\|_{L^p}\\ &\le C\,\|G_t(2^j\sqrt{\cdot}\,)\Phi_0(\sqrt{\cdot}\,)\|_{H^{N+1/2+\delta}(\mathbb{R})}\|\phi_j(\sqrt{A})f\|_{L^p},\end{aligned} \tag{5-2}$$

where $N>\frac{d}{2}$ and $\delta>0$. Here it is easy to check that there exists $C>0$ such that

$$\|G_t(2^j\sqrt{\cdot}\,)\Phi_0(\sqrt{\cdot}\,)\|_{H^{N+1/2+\delta}(\mathbb{R})}\le C(t2^{\alpha j})^{s_0}e^{-C^{-1}t2^{\alpha j}} \quad \text{for any } j\in\mathbb{Z}, \tag{5-3}$$

and hence,

$$\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})f\|_{L^p}\le C(t2^{\alpha j})^{s_0}e^{-C^{-1}t2^{\alpha j}}\,\|\phi_j(\sqrt{A})f\|_{L^p} \quad \text{for any } j\in\mathbb{Z}.$$

This proves the second inequality of (5-1).

We turn to the first inequality of (5-1). Since $\phi_j(\sqrt{A})f$ is written as

$$\begin{aligned}\phi_j(\sqrt{A})f &= \big((tA^{\frac{\alpha}{2}})^{-s_0}e^{tA^{\alpha/2}}\Phi_j(\sqrt{A})\big)\big((tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})f\big)\\ &=: \big((tA^{\frac{\alpha}{2}})^{-s_0}e^{tA^{\alpha/2}}\Phi_j(\sqrt{A})\big)F,\end{aligned}$$

all we have to do is to show that

$$\big\|(tA^{\frac{\alpha}{2}})^{-s_0}e^{tA^{\alpha/2}}\Phi_j(\sqrt{A})F\big\|_{L^p} \le C(t2^{\alpha j})^{-s_0}e^{Ct\lambda^\alpha}\|F\|_{L^p}. \tag{5-4}$$

Applying (2-1) in Lemma 2.1 with

$$G = \widetilde{G}_t(\lambda) = (t\lambda^\alpha)^{-s_0}e^{t\lambda^\alpha}$$

to the left-hand side of (5-4), we have from an similar argument to (5-2) and (5-3) that

$$\begin{aligned}\big\|(tA^{\frac{\alpha}{2}})^{-s_0}e^{tA^{\alpha/2}}\Phi_j(\sqrt{A})F\big\|_{L^p} &\le C\big\|\widetilde{G}_t(2^j\sqrt{\cdot}\,)\Phi_0(\sqrt{\cdot}\,)\big\|_{H^{N+1/2+\varepsilon}(\mathbb{R})}\|F\|_{L^p}\\ &\le C(t2^{\alpha j})^{-s_0}e^{Ct\lambda^\alpha}\|F\|_{L^p}.\end{aligned}$$

This proves (5-4) and the first inequality of (5-1) is obtained. □

In what follows, we show the inequality (1-5) for $f \in \dot{B}^s_{p,q}(A)$ to prove Theorem 1.3. We note that the proof below concerns the case when $q < \infty$ only, since the case when $q = \infty$ is also shown analogously with some modification.

Proof of the first inequality of (1-5). By the embedding $L^p(\Omega)$, $\dot{B}^0_{p,r}(A) \hookrightarrow \dot{B}^0_{p,\infty}(A)$, it is sufficient to show that

$$C^{-1}\|f\|_{\dot{B}^s_{p,q}} \le \bigg\{\int_0^\infty \big(t^{-\frac{s}{\alpha}}\big\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}f\big\|_{\dot{B}^0_{p,\infty}}\big)^q\,\frac{dt}{t}\bigg\}^{\frac{1}{q}}. \tag{5-5}$$

We have from the definition of norm $\|\cdot\|_{\dot{B}^0_{p,\infty}}$ and the first inequality of estimate (5-1) in Lemma 5.1 that

$$\begin{aligned}&\bigg\{\int_0^\infty \big(t^{-\frac{s}{\alpha}}\big\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}f\big\|_{\dot{B}^0_{p,\infty}}\big)^q\,\frac{dt}{t}\bigg\}^{\frac{1}{q}}\\ &\qquad\ge C^{-1}\bigg\{\int_0^\infty \big(t^{-\frac{s}{\alpha}}\sup_{j\in\mathbb{Z}}(t2^{\alpha j})^{s_0}e^{-Ct2^{\alpha j}}\big\|\phi_j(\sqrt{A})f\big\|_{L^p}\big)^q\,\frac{dt}{t}\bigg\}^{\frac{1}{q}}.\end{aligned}$$

Decomposing $(0,\infty)$ in the last line by

$$(0,\infty) = \bigcup_{k\in\mathbb{Z}}[2^{-\alpha(k+1)}, 2^{-\alpha k}], \tag{5-6}$$

we get

$$\begin{aligned}&\bigg\{\int_0^\infty \big(t^{-\frac{s}{\alpha}}\big\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}f\big\|_{\dot{B}^0_{p,\infty}}\big)^q\,\frac{dt}{t}\bigg\}^{\frac{1}{q}}\\ &\qquad\ge C^{-1}\bigg\{\sum_{k\in\mathbb{Z}}\int_{2^{-\alpha(k+1)}}^{2^{-\alpha k}} \big(t^{-\frac{s}{\alpha}}\sup_{j\in\mathbb{Z}}(t2^{\alpha j})^{s_0}e^{-Ct2^{\alpha j}}\big\|\phi_j(\sqrt{A})f\big\|_{L^p}\big)^q\,\frac{dt}{t}\bigg\}^{\frac{1}{q}}\\ &\qquad\ge C^{-1}\bigg\{\sum_{k\in\mathbb{Z}}\big(2^{sk}\sup_{j\in\mathbb{Z}}(2^{\alpha(j-k)})^{s_0}e^{-C2^{\alpha(j-k)}}\big\|\phi_j(\sqrt{A})f\big\|_{L^p}\big)^q\bigg\}^{\frac{1}{q}}.\end{aligned} \tag{5-7}$$

Here it follows from the Hölder inequality that

$$\sup_{j\in\mathbb{Z}}(2^{\alpha(j-k)})^{s_0}e^{-C2^{\alpha(j-k)}}\|\phi_j(\sqrt{A})f\|_{L^p} \\ \geq C^{-1}\biggl\{\sum_{j\in\mathbb{Z}}\biggl(\frac{1}{1+\alpha^2|j-k|^2}\cdot(2^{\alpha(j-k)})^{s_0}e^{-C2^{\alpha(j-k)}}\|\phi_j(\sqrt{A})f\|_{L^p}\biggr)^q\biggr\}^{\frac{1}{q}}.$$

Then we deduce from (5-7) and the above inequality that

$$\begin{aligned}\biggl\{\int_0^\infty&\bigl(t^{-\frac{s}{\alpha}}\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}f\|_{\dot B^0_{p,\infty}}\bigr)^q\,\frac{dt}{t}\biggr\}^{\frac{1}{q}}\\ &\geq C^{-1}\biggl\{\sum_{k\in\mathbb{Z}}(2^{sk})^q\sum_{j\in\mathbb{Z}}\biggl(\frac{1}{1+\alpha^2|j-k|^2}\cdot(2^{\alpha(j-k)})^{s_0}e^{-C2^{\alpha(j-k)}}\|\phi_j(\sqrt{A})f\|_{L^p}\biggr)^q\biggr\}^{\frac{1}{q}}\\ &=C^{-1}\biggl\{\sum_{j\in\mathbb{Z}}(2^{sj}\|\phi_j(\sqrt{A})f\|_{L^p})^q\sum_{k\in\mathbb{Z}}\biggl(\frac{2^{-s(j-k)}}{1+\alpha^2|j-k|^2}\cdot(2^{\alpha(j-k)})^{s_0}e^{-C2^{\alpha(j-k)}}\biggr)^q\biggr\}^{\frac{1}{q}}\\ &=C^{-1}\|f\|_{\dot B^s_{p,q}}\biggl\{\sum_{k\in\mathbb{Z}}\biggl(\frac{2^{(s_0\alpha-s)k}}{1+\alpha^2|k|^2}\cdot e^{-C2^{\alpha k}}\biggr)^q\biggr\}^{\frac{1}{q}}.\end{aligned}$$

Since $s_0>\frac{s}{\alpha}$ and the summation appearing in the last line converges, we obtain (5-5). ☐

Proof of the second inequality of (1-5). By the embedding $\dot B^0_{p,1}(A)\hookrightarrow L^p(\Omega)$, $\dot B^0_{p,q}(A)$, it is sufficient to show that

$$\biggl\{\int_0^\infty\bigl(t^{-\frac{s}{\alpha}}\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}f\|_{\dot B^0_{p,1}}\bigr)^q\,\frac{dt}{t}\biggr\}^{\frac{1}{q}}\leq C\|f\|_{\dot B^s_{p,q}(A)}. \tag{5-8}$$

Analogously to the proof of (5-5), we apply the second inequality of (5-1) in Lemma 5.1 instead of the first one and the decomposition (5-6) to get

$$\biggl\{\int_0^\infty\bigl(t^{-\frac{s}{\alpha}}\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}f\|_{\dot B^0_{p,\infty}}\bigr)^q\,\frac{dt}{t}\biggr\}^{\frac{1}{q}} \\ \leq\biggl\{\sum_{k\in\mathbb{Z}}\biggl(2^{sk}\sum_{j\in\mathbb{Z}}(2^{\alpha(j-k)})^{s_0}e^{-C^{-1}2^{\alpha(j-k)}}\|\phi_j(\sqrt{A})f\|_{L^p}\biggr)^q\biggr\}^{\frac{1}{q}}.$$

Here the Hölder inequality yields that

$$\sum_{j\in\mathbb{Z}}(2^{\alpha(j-k)})^{s_0}e^{-C^{-1}2^{\alpha(j-k)}}\|\phi_j(\sqrt{A})f\|_{L^p} \\ \leq C\biggl\{\sum_{j\in\mathbb{Z}}\bigl((1+\alpha^2|j-k|^2)(2^{\alpha(j-k)})^{s_0}e^{-C^{-1}2^{\alpha(j-k)}}\|\phi_j(\sqrt{A})f\|_{L^p}\bigr)^q\biggr\}^{\frac{1}{q}}.$$

Then we have from the above two estimates that

$$\begin{aligned}
&\left\{\int_0^\infty \left(t^{-\frac{s}{\alpha}}\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}f\|_{\dot B^0_{p,\infty}}\right)^q \frac{dt}{t}\right\}^{\frac1q}\\
&\qquad \le C\left\{\sum_{k\in\mathbb{Z}}(2^{sk})^q\sum_{j\in\mathbb{Z}}\left((1+\alpha^2|j-k|^2)(2^{\alpha(j-k)})^{s_0}e^{-C^{-1}2^{\alpha(j-k)}}\|\phi_j(\sqrt{A})f\|_{L^p}\right)^q\right\}^{\frac1q}\\
&\qquad = C\left\{\sum_{j\in\mathbb{Z}}\left(2^{sj}\|\phi_j(\sqrt{A})f\|_{L^p}\right)^q\sum_{k\in\mathbb{Z}}\left(2^{-s(j-k)}(1+\alpha^2|j-k|^2)(2^{\alpha(j-k)})^{s_0}e^{-C^{-1}2^{\alpha(j-k)}}\right)^q\right\}^{\frac1q}\\
&\qquad = C\|f\|_{\dot B^s_{p,q}}\left\{\sum_{k\in\mathbb{Z}}\left((1+\alpha^2|k|^2)2^{(s_0\alpha-s)k}e^{-C^{-1}2^{\alpha k}}\right)^q\right\}^{\frac1q}.
\end{aligned}$$

Since $s_0 > \frac{s}{\alpha}$ and the summation appearing in the last line converges, we obtain (5-8). □

6. Proof of Theorem 1.4

Proof of (1-6). It is sufficient to prove the case when $s = 0$ thanks to the lifting property in the proposition on page 684. We also consider the case when $q < \infty$ only, since the case when $q = \infty$ is also shown analogously. First we prove that

$$\|A^{\frac{\alpha}{2}}u\|_{L^q(0,\infty;\dot B^0_{p,q})} \le C\|u_0\|_{\dot B^{\alpha-\alpha/q}_{p,q}} + C\|f\|_{L^q(0,\infty;\dot B^0_{p,q})}. \tag{6-1}$$

By the definition of u and the triangle inequality, we get

$$\|A^{\frac{\alpha}{2}}u\|_{L^q(0,\infty;\dot B^0_{p,q})} \le \|A^{\frac{\alpha}{2}}e^{-tA^{\alpha/2}}u_0\|_{L^q(0,\infty;\dot B^0_{p,q})} + \left\|A^{\frac{\alpha}{2}}\int_0^t e^{-(t-\tau)A^{\alpha/2}}f(\tau)\,d\tau\right\|_{L^q(0,\infty;\dot B^0_{p,q})}. \tag{6-2}$$

On the first term of the right-hand side in the above inequality, it follows from the estimate (1-5) for $s_0 = 1,\ s = \alpha - \frac{\alpha}{q}$ that

$$\|A^{\frac{\alpha}{2}}e^{-tA^{\alpha/2}}u_0\|_{L^q(0,\infty;\dot B^0_{p,q})} \le C\|u_0\|_{L^q(0,\infty;\dot B^{\alpha-\alpha/q}_{p,q})}. \tag{6-3}$$

As for the second one, we start by proving that

$$\left\|\phi_j(\sqrt{A})A^{\frac{\alpha}{2}}\int_0^t e^{-(t-\tau)A^{\alpha/2}}f(\tau)\,d\tau\right\|_{L^p} \le C2^{\frac{\alpha}{q}j}\left\{\int_0^t \left(e^{-C^{-1}(t-\tau)2^{\alpha j}}\|\phi_j(\sqrt{A})f\|_{L^p}\right)^q d\tau\right\}^{\frac1q}. \tag{6-4}$$

The above estimate (6-4) is verified by applying the estimate (5-1) in Lemma 5.1 and the Hölder inequality; in fact, we get

$$\left\|\phi_j(\sqrt{A})A^{\frac{\alpha}{2}}\int_0^t e^{-(t-\tau)A^{\alpha/2}}f(\tau)\,d\tau\right\|_{L^p} \le C2^{\alpha j}\int_0^t e^{-C^{-1}(t-\tau)2^{\alpha j}}\|\phi_j(\sqrt{A})f(\tau)\|_{L^p}\,d\tau$$

$$
\begin{aligned}
&\le C2^{\alpha j}\|e^{-(2C)^{-1}(t-\tau)2^{\alpha j}}\|_{L^{q/(q-1)}(\{0\le\tau\le t\})}\left\{\int_0^t \bigl(e^{-(2C)^{-1}(t-\tau)2^{\alpha j}}\,\|\phi_j(\sqrt{A})f(\tau)\|_{L^p}\bigr)^q\,d\tau\right\}^{\frac{1}{q}}\\
&\le C2^{\frac{\alpha}{q}j}\left\{\int_0^t \bigl(e^{-(2C)^{-1}(t-\tau)2^{\alpha j}}\,\|\phi_j(\sqrt{A})f(\tau)\|_{L^p}\bigr)^q\,d\tau\right\}^{\frac{1}{q}}.
\end{aligned}
$$

By the estimate (6-4), we have

$$
\begin{aligned}
&\left\|A^{\frac{\alpha}{2}}\int_0^t e^{-(t-\tau)A^{\alpha/2}}f(\tau)\,d\tau\right\|_{L^q(0,\infty;\dot{B}^0_{p,q})}\\
&\qquad\le C\left[\int_0^\infty\sum_{j\in\mathbb{Z}}\left\{2^{\frac{\alpha}{q}j}\left(\int_0^t \bigl(e^{-(2C)^{-1}(t-\tau)2^{\alpha j}}\,\|\phi_j(\sqrt{A})f(\tau)\|_{L^p}\bigr)^q\,d\tau\right)^{\frac{1}{q}}\right\}^q dt\right]^{\frac{1}{q}}\\
&\qquad= C\left[\int_0^\infty\sum_{j\in\mathbb{Z}}\|\phi_j(\sqrt{A})f(\tau)\|_{L^p}^q\left(2^{\alpha j}\int_\tau^\infty e^{-q(2C)^{-1}(t-\tau)2^{\alpha j}}\,dt\right)d\tau\right]^{\frac{1}{q}}\\
&\qquad= C\|f\|_{L^q(0,\infty;\dot{B}^0_{p,q}(A))}.
\end{aligned}
\tag{6-5}
$$

Then the estimates (6-2), (6-3) and (6-5) imply the inequality (6-1). The estimate for $\partial_t u$, i.e., the inequality

$$
\|\partial_t u\|_{L^q(0,\infty;\dot{B}^0_{p,q})}\le C\|u_0\|_{\dot{B}^{\alpha-\alpha/q}_{p,q}}+C\|f\|_{L^q(0,\infty;\dot{B}^0_{p,q})},
$$

is verified by the estimate (6-1) and the equality

$$
\partial_t u=-A^{\frac{\alpha}{2}}u+f.
$$

Hence we obtain the estimate (1-6) and the proof is complete. □

7. Results for the inhomogeneous Besov spaces

We should mention that similar theorems also hold for the heat semigroup in the inhomogeneous Besov spaces $B^s_{p,q}(A)$. We also note that the semigroup generated by the fractional Laplacian cannot be treated analogously by the direct application of boundedness of the scaled spectral multiplier in Lemma 2.1 (see the comment below Theorem 7.2).

First we recall the definition of $B^s_{p,q}(A)$. Let ψ be as in $C_0^\infty((-\infty,\infty))$ such that

$$
\psi(\lambda^2)+\sum_{j\in\mathbb{N}}\phi_j(\lambda)=1\quad\text{for any }\lambda\ge 0.
$$

The inhomogeneous Besov space $B^s_{p,q}(A)$ is defined as follows; see [Iwabuchi et al. 2016a].

Definition. For $s\in\mathbb{R}$ and $1\le p,q\le\infty$, $B^s_{p,q}(A)$ is defined by letting

$$
B^s_{p,q}(A):=\{f\in\mathcal{X}_0'(\Omega):\ \|f\|_{B^s_{p,q}(A)}<\infty\},
$$

where

$$
\|f\|_{B^s_{p,q}(A)}:=\|\psi(A)f\|_{L^p}+\left\|\{2^{sj}\,\|\phi_j(\sqrt{A})f\|_{L^p}\}_{j\in\mathbb{N}}\right\|_{\ell^q(\mathbb{N})}.
$$

The high-frequency part is able to be treated in the same way as the proof for the homogeneous case by using Lemma 2.1. As for the low-frequency part, we employ the pointwise estimate of the kernel of e^{-tA}

$$0 \le e^{-tA}(x,y) \le (4\pi t)^{-\frac{d}{2}} \exp\left(\frac{|x-y|^2}{4t}\right),$$

which assures the boundedness of e^{-tA} in $L^p(\Omega)$ and also $B^s_{p,q}(A)$ as well as the case when $\Omega = \mathbb{R}^d$. In order to treat continuity in time of e^{-tA}, we need the following obtained by a proof similar to that of Lemma 2.1.

Lemma 7.1. *Let $N > \frac{d}{2}$, $1 \le p \le \infty$, $\delta > 0$, $\psi \in C_0^\infty((-\infty,\infty))$ and $G \in H^{N+\frac{1}{2}+\delta}(\mathbb{R})$. Then there exists a positive constant C such that for any $f \in L^p(\Omega)$*

$$\|G(A)\psi(A)f\|_{L^p} \le C\|G(\cdot)\psi(\cdot)\|_{H^{N+1/2+\delta}(\mathbb{R})}\|f\|_{L^p}. \tag{7-1}$$

We take G such that

$$G(\lambda) := e^{-t\lambda} - 1 \quad \text{for any } \lambda \in \mathbb{R}$$

to apply the above lemma. For the above G it is easy to check that

$$\|G(\cdot)\psi(\cdot)\|_{H^{N+1/2+\delta}(\mathbb{R})} \to 0 \quad \text{as } t \to 0.$$

Hence for any $f \in B^s_{p,q}(A)$, it follows from (7-1) that

$$\lim_{t\to 0}\|\psi(A)(e^{-tA}f - f)\|_{L^p} = 0.$$

According to the boundedness and the continuity of e^{-tA}, we obtain the following result for the inhomogeneous Besov spaces.

Theorem 7.2. *Let $s \in \mathbb{R}$, $1 \le p, p_1, p_2, q \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let Ψ and Ψ_j with $j \in \mathbb{N}$ be such that*

$$\begin{aligned}
\Psi(A) &:= \psi(A) + \phi_1(\sqrt{A}),\\
\Phi_1(\sqrt{A}) &:= \psi(A) + \phi_1(\sqrt{A}) + \phi_2(\sqrt{A}),\\
\Phi_j(\sqrt{A}) &:= \phi_{j-1}(\sqrt{A}) + \phi_j(\sqrt{A}) + \phi_{j+1}(\sqrt{A}) \quad \textit{for } j \ge 2:
\end{aligned}$$

(i) *There exists a constant $C > 0$ such that*

$$\|e^{-tA}f\|_{B^s_{p,q}(A)} \le C\|f\|_{B^s_{p,q}(A)}$$

for any $f \in B^s_{p,q}(A)$. If $p_1 \le p_2$, then there exists a constant $C > 0$ such that

$$\|e^{-tA}f\|_{B^s_{p_2,q}(A)} \le Ct^{-\frac{d}{2}(\frac{1}{p_1}-\frac{1}{p_2})}\|f\|_{B^s_{p_1,q}(A)}$$

for any $f \in B^s_{p_1,q}(A)$.

(ii) *If $q < \infty$ and $f \in B^s_{p,q}(A)$, then*

$$\lim_{t\to 0}\|e^{-tA}f - f\|_{B^s_{p,q}(A)} = 0.$$

If $q=\infty,\ 1<p\le\infty$ *and* $f\in B^s_{p,\infty}(A)$, *then* $e^{-tA}f$ *converges to* f *in the dual weak sense as* $t\to 0$; *namely*,

$$\lim_{t\to 0}\biggl[\int_\Omega \{\psi(A)(e^{-tA}f-f)\}\,\overline{\Psi(A)g}\,dx+\sum_{j\in\mathbb{N}}\int_\Omega \{\phi_j(\sqrt{A})(e^{-tA}f-f)\}\,\overline{\Phi_j(\sqrt{A})g}\,dx\biggr]=0$$

for any $g\in \dot{B}^{-s}_{p',1}(A)$.

(iii) *Let* $T>0,\ s,s_0\in\mathbb{R}$ *and* $s_0>\frac{s}{2}$. *Then*

$$\|f\|_{B^s_{p,q}(A)}\simeq \|\psi(TA)f\|_{L^p}+\biggl\{\int_0^T\bigl(t^{-\frac{s}{2}}\|(tA)^{s_0}e^{-tA}f\|_X\bigr)^q\,\frac{dt}{t}\biggr\}^{\frac{1}{q}}$$

for any $f\in B^s_{p,q}(A)$, *where* $X=L^p(\Omega),\ B^0_{p,r}(A)$ *with* $1\le r\le\infty$.

(iv) *Let* $T>0,\ u_0\in B^{s+2-2/q}_{p,q}(A)$ *and* $f\in L^q(0,T;B^s_{p,q}(A))$. *Assume that* u *satisfies*

$$u(t)=e^{-tA}u_0+\int_0^t e^{-(t-\tau)A}f(\tau)\,d\tau.$$

Then there exists a constant $C=C(T)>0$ *independent of* u_0 *and* f *such that*

$$\|\partial_t u\|_{L^q(0,T;B^s_{p,q}(A))}+\|A^{\frac{\alpha}{2}}u\|_{L^q(0,T;B^s_{p,q}(A))}\le C\|u_0\|_{B^{s+2-2/q}_{p,q}(A)}+C\|f\|_{L^q(0,T;B^s_{p,q}(A))}.$$

Remark. Let us mention what is obtained by the abstract theory for sectorial operators by Da Prato and Grisvard [1975]; see also [Haase 2006; Lunardi 1995]. Let $X=B^0_{p,q}(A)$. We can consider A as a sectorial operator with the domain $D(A^\alpha)=B^2_{p,q}(A)$. Let $0<T<\infty,\ 1<q<\infty,\ 1\le p,r\le\infty$, $\theta\in(0,1)$ and $\alpha>0$. Then for any $f\in L^q(0,T;(X,D(A^\alpha))_{\theta,r})$ the equation

$$\begin{cases}\dfrac{du}{dt}+Au=f, & 0<t<T,\\ u(0)=0\end{cases}$$

admits a unique solution u satisfying

$$\Bigl\|\frac{du}{dt}\Bigr\|_{L^q(0,T;(X,D(A^\alpha))_{\theta,r})}+\|Au\|_{L^q(0,T;(X,D(A^\alpha))_{\theta,r})}\le C\|f\|_{L^q(0,T;(X,D(A^\alpha))_{\theta,r})},$$

where C depends on T. Here we note that $(X,D(A^\alpha))_{\theta,r}=B^{2\alpha\theta}_{p,r}(A)$ and $2\alpha\theta$ is possibly an arbitrary positive number since $\alpha>0$ and $\theta\in(0,1)$.

Let us give a few remarks on the semigroup generated by $A^{\frac{\alpha}{2}}$. If we consider applying Lemma 7.1 directly, it is impossible to obtain the boundedness of $e^{-tA^{\alpha/2}}$ for general α. In fact, taking

$$G=G_t(\lambda)=e^{-t|\lambda|^{\alpha/2}},$$

and applying (7-1), we see that the $H^{N+\frac{1}{2}+\delta}(\mathbb{R})$ norm of the above $G=G_t(\lambda)$ is not finite for small $\lambda>0$ because of less regularity around $\lambda=0$. On the other hand, if α is even or sufficiently large, the $H^{N+\frac{1}{2}+\delta}(\mathbb{R})$ norm of $e^{-t|\lambda|^{\alpha/2}}$ is finite and we can get some results. However this argument does not reach the optimal estimate, and hence, we do not treat it in this paper and will treat it in a future work.

Appendix: Real interpolation

We now give a remark that real interpolation can be considered in the Besov spaces $\dot{B}^s_{p,q}(A)$ and $B^s_{p,q}(A)$ on open sets as well as the whole space case. We recall the definition of real interpolation spaces $(X_0, X_1)_{\theta,q}$ for Banach spaces X_0 and X_1; see, e.g., [Bergh and Löfström 1976; Peetre 1968; Triebel 1983].

Definition. Let $0 < \theta < 1$ and $1 \le q \le \infty$. $(X_0, X_1)_{\theta,q}$ is defined by letting

$$(X_0, X_1)_{\theta,q} := \left\{ a \in X_0 + X_1 : \|a\|_{(X_0,X_1)_{\theta,q}} := \left\{ \int_0^\infty (t^{-\theta} K(t,a))^q \,\frac{dt}{t} \right\}^{\frac{1}{q}} < \infty \right\},$$

where $K(t,a)$ is Peetre's K-function

$$K(t,a) := \inf\bigl\{ \|a_0\|_{X_0} + t\|a_1\|_{X_1} : a = a_0 + a_1,\ a_0 \in X_0,\ a_1 \in X_1 \bigr\}.$$

As well as in the case when $\Omega = \mathbb{R}^d$, we obtain the following.

Proposition A.1. *Let $0 < \theta < 1$, $s, s_0, s_1 \in \mathbb{R}$ and $1 \le p, q, q_0, q_1 \le \infty$. Assume that $s_0 \ne s_1$ and $s = (1-\theta)s_0 + \theta s_1$. Then*

$$(\dot{B}^{s_0}_{p,q_0}(A),\ \dot{B}^{s_1}_{p,q_1}(A))_{\theta,q} = \dot{B}^s_{p,q}(A),$$
$$(B^{s_0}_{p,q_0}(A),\ B^{s_1}_{p,q_1}(A))_{\theta,q} = B^s_{p,q}(A).$$

We omit the proof of the above proposition since one can show it analogously to the whole space case; see, e.g., [Triebel 1983].

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KLEIN'S PARADOX AND THE RELATIVISTIC δ-SHELL INTERACTION IN $\mathbb{R}^3$

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Under certain hypotheses of smallness on the regular potential V, we prove that the Dirac operator in $\mathbb{R}^3$, coupled with a suitable rescaling of V, converges in the strong resolvent sense to the Hamiltonian coupled with a δ-shell potential supported on Σ, a bounded C^2 surface. Nevertheless, the coupling constant depends nonlinearly on the potential V; Klein's paradox comes into play.

1. Introduction

Klein's paradox is a counterintuitive relativistic phenomenon related to scattering theory for high-barrier (or equivalently low-well) potentials for the Dirac equation. When an electron is approaching a barrier, its wave function can be split in two parts: the reflected one and the transmitted one. In a nonrelativistic situation, it is well known that the transmitted wave-function decays exponentially depending on the high of the potential; see [Thaller 2005]. In the case of the Dirac equation it has been observed, in [Klein 1929] for the first time, that the transmitted wave-function depends weakly on the power of the barrier, and it becomes almost transparent for very high barriers. This means that outside the barrier the wave-function behaves like an electronic solution and inside the barrier it behaves like a positronic one, violating the principle of the conservation of the charge. This incongruence comes from the fact that, in the Dirac equation, the behavior of electrons and positrons is described by different components of the same spinor wave-function; see [Katsnelson et al. 2006]. Roughly speaking, this contradiction derives from the fact that even if a very high barrier is reflective for electrons, it is attractive for the positrons.

From a mathematical perspective, the problem appears when approximating the Dirac operator coupled with a δ-shell potential by the corresponding operator using local potentials with shrinking support. The idea of coupling Hamiltonians with singular potentials supported on subsets of lower dimension with respect to the ambient space (commonly called *singular perturbations*) is quite classic in quantum mechanics. One important example is the model of a particle in a 1-dimensional lattice that analyses the evolution of an electron on a straight line perturbed by a potential caused by ions in the periodic structure of the crystal that create an electromagnetic field. Kronig and Penney [1931] idealized this system: in their model the electron is free to move in regions of the whole space separated by some periodical barriers which are zero everywhere except at a single point, where they take infinite value. In modern language, this corresponds to a δ-point potential. For the Schrödinger operator, this problem is described in [Albeverio et al. 1988] for finite and infinite δ-point interactions and in [Exner 2008] for

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singular potentials supported on hypersurfaces. The reader may look at [Dittrich et al. 1989; Behrndt and Holzmann 2016; Arrizabalaga et al. 2014; 2015; Ourmières-Bonafos and Vega 2016] for the case of the Dirac operator, and to [Posilicano 2008] for a much more general scenario.

Nevertheless, one has to keep in mind that, even if this kind of model is more easily mathematically understood, since the analysis can be reduced to an algebraic problem, it is an ideal model that cannot be physically reproduced. This is the reason why it is interesting to approximate these kinds of operators by more regular ones. For instance, in one dimension, if $V \in C_c^\infty(\mathbb{R})$ then

$$V_\epsilon(t) := \frac{1}{\epsilon} V\Big(\frac{t}{\epsilon}\Big) \to \Big(\int V\Big)\delta_0 \quad \text{when } \epsilon \to 0$$

in the sense of distributions, where δ_0 denotes the Dirac measure at the origin. In [Albeverio et al. 1988] it is proved that $\Delta + V_\epsilon \to \Delta + \big(\int V\big)\delta_0$ in the norm resolvent sense when $\epsilon \to 0$, and in [Behrndt et al. 2017] this result is generalized to higher dimensions for singular perturbations on general smooth hypersurfaces.

These kinds of results do not hold for the Dirac operator. In fact, in [Šeba 1989] it is proved that, in the 1-dimensional case, the convergence holds in the norm resolvent sense but the coupling constant does depend nonlinearly on the potential V, unlike in the case of Schrödinger operators. This nonlinear phenomenon, which may also occur in higher dimensions, is a consequence of the fact that, in a sense, the free Dirac operator is critical with respect to the set where the δ-shell interaction is performed, unlike the Laplacian (the Dirac/Laplace operator is a first/second-order differential operator, respectively, and the set where the interaction is performed has codimension 1 with respect to the ambient space). The present paper is devoted to the study of the 3-dimensional case, where we investigate if it is possible to obtain the same results as in one dimension. For δ-shell interactions on bounded smooth hypersurfaces, we get the same nonlinear phenomenon on the coupling constant but we are only able to show convergence in the strong resolvent sense.

Given $m \geq 0$, the free Dirac operator in $\mathbb{R}^3$ is defined by

$$H := -i\alpha\cdot\nabla + m\beta,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$,

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad \text{for } j = 1,2,3, \qquad \beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \qquad \mathbb{I}_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1-1}$$

is the family of *Pauli matrices*. It is well known that H is self-adjoint on the Sobolev space $H^1(\mathbb{R}^3)^4 =: D(H)$; see [Thaller 1992, Theorem 1.1]. Throughout this article we assume that $m > 0$.

In the sequel $\Omega \subset \mathbb{R}^3$ denotes a bounded C^2 domain and $\Sigma := \partial\Omega$ denotes its boundary. By a C^2 domain we mean the following: for each point $Q \in \Sigma$ there exist a ball $B \subset \mathbb{R}^3$ centered at Q, a C^2 function $\psi : \mathbb{R}^2 \to \mathbb{R}$ and a coordinate system $\{(x, x_3) : x \in \mathbb{R}^2,\ x_3 \in \mathbb{R}\}$ such that, with respect to this coordinate system, $Q = (0, 0)$ and

$$B \cap \Omega = B \cap \{(x, x_3) : x_3 > \psi(x)\},$$
$$B \cap \Sigma = B \cap \{(x, x_3) : x_3 = \psi(x)\}.$$

By compactness, one can find a finite covering of Σ made of such coordinate systems; thus the Lipschitz constant of those ψ can be taken to be uniformly bounded on Σ.

Set $\Omega_\epsilon := \{x \in \mathbb{R}^3 : d(x, \Sigma) < \epsilon\}$ for $\epsilon > 0$. Following [Behrndt et al. 2017, Appendix B], there exists $\eta > 0$ small enough depending on Σ such that for every $0 < \epsilon \leq \eta$ one can parametrize Ω_ϵ as

$$\Omega_\epsilon = \{x_\Sigma + t\nu(x_\Sigma) : x_\Sigma \in \Sigma,\ t \in (-\epsilon, \epsilon)\}, \tag{1-2}$$

where $\nu(x_\Sigma)$ denotes the outward (with respect to Ω) unit normal vector field on Σ evaluated at x_Σ. This parametrization is a bijective correspondence between Ω_ϵ and $\Sigma \times (-\epsilon, \epsilon)$; it can be understood as *tangential* and *normal coordinates*. For $t \in [-\eta, \eta]$, we set

$$\Sigma_t := \{x_\Sigma + t\nu(x_\Sigma) : x_\Sigma \in \Sigma\}. \tag{1-3}$$

In particular, $\Sigma_t = \partial\Omega_t \setminus \Omega$ if $t > 0$, $\Sigma_t = \partial\Omega_{|t|} \cap \Omega$ if $t < 0$ and $\Sigma_0 = \Sigma$. Let σ_t denote the surface measure on Σ_t and, for simplicity of notation, we set $\sigma := \sigma_0$, the surface measure on Σ.

Given $V \in L^\infty(\mathbb{R})$ with $\operatorname{supp} V \subset [-\eta, \eta]$ and $0 < \epsilon \leq \eta$ define

$$V_\epsilon(t) := \frac{\eta}{\epsilon} V\left(\frac{\eta t}{\epsilon}\right)$$

and, for $x \in \mathbb{R}^3$,

$$\boldsymbol{V}_\epsilon(x) := \begin{cases} V_\epsilon(t) & \text{if } x \in \Omega_\epsilon, \text{ where } x = x_\Sigma + t\nu(x_\Sigma) \text{ for a unique } (x_\Sigma, t) \in \Sigma \times (-\epsilon, \epsilon), \\ 0 & \text{if } x \notin \Omega_\epsilon. \end{cases} \tag{1-4}$$

Finally, set

$$\begin{aligned} \boldsymbol{u}_\epsilon &:= |\boldsymbol{V}_\epsilon|^{1/2}, & \boldsymbol{v}_\epsilon &:= \operatorname{sign}(\boldsymbol{V}_\epsilon)|\boldsymbol{V}_\epsilon|^{1/2}, \\ u(t) &:= |\eta V(\eta t)|^{1/2}, & v(t) &:= \operatorname{sign}(V(\eta t))u(t). \end{aligned} \tag{1-5}$$

Note that $\boldsymbol{u}_\epsilon, \boldsymbol{v}_\epsilon \in L^\infty(\mathbb{R}^3)$ are supported in $\overline{\Omega}_\epsilon$ and $u, v \in L^\infty(\mathbb{R})$ are supported in $[-1, 1]$.

Definition 1.1. Given $\eta,\ \delta > 0$, we say that $V \in L^\infty(\mathbb{R})$ is (δ, η)-small if

$$\operatorname{supp} V \subset [-\eta, \eta] \quad \text{and} \quad \|V\|_{L^\infty(\mathbb{R})} \leq \frac{\delta}{\eta}.$$

Observe that if V is (δ, η)-small then $\|V\|_{L^1(\mathbb{R})} \leq 2\delta$; this is the reason why we call it a "small" potential.

In this article we study the asymptotic behavior, in a strong resolvent sense, of the couplings of the free Dirac operator with electrostatic and Lorentz scalar short-range potentials of the forms

$$H + \boldsymbol{V}_\epsilon \quad \text{and} \quad H + \beta\boldsymbol{V}_\epsilon, \tag{1-6}$$

respectively, where $\boldsymbol{V}_\epsilon$ is given by (1-4) for some (δ, η)-small V with δ and η small enough only depending on Σ. By [Thaller 1992, Theorem 4.2], both couplings in (1-6) are self-adjoint operators on $H^1(\mathbb{R}^3)^4$.

Given $\eta > 0$ small enough so that (1-2) holds, and given u and v as in (1-5) for some $V \in L^\infty(\mathbb{R})$ with $\operatorname{supp} V \subset [-\eta, \eta]$, set

$$\mathcal{K}_V f(t) := \tfrac{1}{2} i \int_{\mathbb{R}} u(t) \operatorname{sign}(t-s) v(s) f(s)\, ds \quad \text{for } f \in L^1_{\text{loc}}(\mathbb{R}). \tag{1-7}$$

The main result in this article reads as follows.

Theorem 1.2. *There exist η_0, $\delta > 0$ small enough only depending on Σ such that, for any $0 < \eta \leq \eta_0$ and (δ, η)-small V,*

$$H + \boldsymbol{V}_\epsilon \to H + \lambda_e \delta_\Sigma \qquad \textit{in the strong resolvent sense when } \epsilon \to 0, \tag{1-8}$$

$$H + \beta \boldsymbol{V}_\epsilon \to H + \lambda_s \beta\, \delta_\Sigma \quad \textit{in the strong resolvent sense when } \epsilon \to 0, \tag{1-9}$$

where

$$\lambda_e := \int_{\mathbb{R}} v(t)\, ((1 - \mathcal{K}_V^2)^{-1} u)(t)\, dt \in \mathbb{R}, \tag{1-10}$$

$$\lambda_s := \int_{\mathbb{R}} v(t)\, ((1 + \mathcal{K}_V^2)^{-1} u)(t)\, dt \in \mathbb{R}, \tag{1-11}$$

and $H + \lambda_e \delta_\Sigma$ and $H + \lambda_s \beta\, \delta_\Sigma$ are the electrostatic and Lorentz scalar shell interactions given by (2-9) *and* (2-11), *respectively.*

To define λ_e in (1-10) and λ_s in (1-11), the invertibility of $1 \pm \mathcal{K}_V^2$ is required. However, since $\mathcal{K}_V$ is a Hilbert–Schmidt operator, we know that $\|\mathcal{K}_V\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})}$ is controlled by the norm of its kernel in $L^2(\mathbb{R} \times \mathbb{R})$, which is exactly $\|u\|_{L^2(\mathbb{R})} \|v\|_{L^2(\mathbb{R})} = \|V\|_{L^1(\mathbb{R})} \leq 2\delta < 1$, assuming that $\delta < \frac{1}{2}$ and that V is (δ, η)-small with $\eta \leq \eta_0$. We must stress that the way to construct λ_e and λ_s is the same as in the 1-dimensional case; see [Šeba 1989, Theorem 1].

From Theorem 1.2 we deduce that if $a \in \sigma(H + \lambda_e \delta_\Sigma)$, where $\sigma(\,\cdot\,)$ denotes the spectrum, then there exists a sequence $\{a_\epsilon\}$ such that $a_\epsilon \in \sigma(H + \boldsymbol{V}_\epsilon)$ and $a_\epsilon \to a$ when $\epsilon \to 0$. The kind of instruments we used to prove Theorem 1.2 suggest to us that the norm resolvent convergence may not hold in general; thus we cannot ensure that the vice-versa spectral implication also holds. Nevertheless, if Σ is a sphere, one has more information than in the general scenario; see [Mas and Pizzichillo 2017]. The Lorentz scalar case is analogous.

The nonlinear behavior of the limiting coupling constant with respect to the approximating potentials mentioned in the first paragraphs of the Introduction is depicted by (1-10) and (1-11); the reader may compare this to the analogous result [Behrndt et al. 2017, Theorem 1.1] in the nonrelativistic scenario. However, unlike in that result, in Theorem 1.2 we demand a smallness assumption on the potential, the (δ, η)-smallness from Definition 1.1. We use this assumption in Corollary 3.3 below, where the strong convergence of some inverse operators $(1 + B_\epsilon(a))^{-1}$ when $\epsilon \to 0$ is shown. The proof of Theorem 1.2 follows the strategy of [Behrndt et al. 2017, Theorem 1.1], but dealing with the Dirac operator instead of the Laplacian makes a big difference at this point. In the nonrelativistic scenario, the fundamental solution of $-\Delta + a^2$ in $\mathbb{R}^3$ for $a > 0$ has exponential decay at infinity and behaves like $1/|x|$ near the origin, which is locally integrable in $\mathbb{R}^2$ and thus its integral tends to zero as we integrate on shrinking balls in

$\mathbb{R}^2$ centered at the origin. These facts are used in [Behrndt et al. 2017] to show that their corresponding $(1+B_\epsilon(a))^{-1}$ can be uniformly bounded in ϵ just by taking a big enough. In our situation, the fundamental solution of $H-a$ in $\mathbb{R}^3$ can still be taken with exponential decay at infinity for $a \in \mathbb{C}\setminus\mathbb{R}$, but it is not locally absolutely integrable in $\mathbb{R}^2$. Actually, its most singular part behaves like $x/|x|^3$ near the origin, and thus it yields a singular integral operator in $\mathbb{R}^2$. This means that the contribution near the origin cannot be disregarded as in [Behrndt et al. 2017] just by shrinking the domain of integration and taking $a \in \mathbb{C}\setminus\mathbb{R}$ big enough; something else is required. We impose smallness on V to obtain smallness on $B_\epsilon(a)$ and ensure the uniform invertibility of $1+B_\epsilon(a)$ with respect to ϵ; this is the only point where the (δ,η)-smallness is used.

Let $\eta_0,\ \delta>0$ be as in Theorem 1.2. Take $0<\eta\le\eta_0$ and $V=\frac{1}{2}\tau\chi_{(-\eta,\eta)}$ for some $\tau\in\mathbb{R}$ such that $0<|\tau|\eta\le 2\delta$. Then, arguing as in [Šeba 1989, Remark 1], one gets that

$$\int_{\mathbb{R}} v\,(1-\mathcal{K}_V^2)^{-1}u=\sum_{n=0}^{\infty}\int_{\mathbb{R}} v\,\mathcal{K}_V^{2n}u=2\tan\big(\tfrac{1}{2}\tau\eta\big).$$

Since V is (δ,η)-small, using (1-10) and (1-8) we obtain that

$$H+\boldsymbol{V}_\epsilon\to H+2\tan\big(\tfrac{1}{2}\tau\eta\big)\delta_\Sigma\quad\text{in the strong resolvent sense when }\epsilon\to 0,$$

analogously to [Šeba 1989, Remark 1]. Similarly, one can check that $\int v\,(1+\mathcal{K}_V^2)^{-1}u=2\tanh\big(\frac{1}{2}\tau\eta\big)$. Then, (1-11) and (1-9) yield

$$H+\beta\,\boldsymbol{V}_\epsilon\to H+2\tanh\big(\tfrac{1}{2}\tau\eta\big)\beta\delta_\Sigma\quad\text{in the strong resolvent sense when }\epsilon\to 0.$$

Regarding the structure of the paper, Section 2 is devoted to the preliminaries, which refer to basic rudiments with a geometric measure-theory flavor and spectral properties of the short-range and shell interactions appearing in Theorem 1.2. In Section 3 we present the first main step to proving Theorem 1.2, a decomposition of the resolvent of the approximating interaction into three concrete operators. This type of decomposition, which is made through a scaling operator, already appears in [Behrndt et al. 2017; Šeba 1989]. Section 3 also contains some auxiliary results concerning these three operators, whose proofs are carried out later on, and the proof of Theorem 1.2; see Section 3A. Sections 4, 5, 6 and 7 are devoted to proving all those auxiliary results presented in Section 3.

2. Preliminaries

As usual, in the sequel the letter "C" (or "c") stands for some constant which may change its value at different occurrences. We will also make use of constants with subscripts, both to highlight the dependence on some other parameters and to stress that they retain their value from one equation to another. The precise meaning of the subscripts will be clear from the context in each situation.

2A. ***Geometric and measure-theoretic considerations.*** In this section we recall some geometric and measure-theoretic properties of Σ and the domains presented in (1-2). At the end, we provide some growth estimates of the measures associated to the layers introduced in (1-3).

The following definition and propositions correspond to Definition 2.2 and Propositions 2.4 and 2.6 in [Behrndt et al. 2017], respectively. The reader should look at that paper for the details.

Definition 2.1 (Weingarten map). Let Σ be parametrized by the family $\{\varphi_i, U_i, V_i\}_{i\in I}$; that is, I is a finite set, $U_i \subset \mathbb{R}^2$, $V_i \subset \mathbb{R}^3$, $\Sigma \subset \bigcup_{i\in I} V_i$ and $\varphi_i(U_i) = V_i \cap \Sigma$ for all $i \in I$. For

$$x = \varphi_i(u) \in \Sigma \cap V_i$$

with $u \in U_i$, $i \in I$, one defines the Weingarten map $W(x) : T_x \to T_x$, where T_x denotes the tangent space of Σ on x, as the linear operator acting on the basis vector $\{\partial_j \varphi_i(u)\}_{j=1,2}$ of T_x as

$$W(x)\,\partial_j\varphi_i(u) := -\partial_j \nu(\varphi_i(u)).$$

Proposition 2.2. *The Weingarten map $W(x)$ is symmetric with respect to the inner product induced by the first fundamental form and its eigenvalues are uniformly bounded for all $x \in \Sigma$.*

Given $0 < \epsilon \le \eta$ and Ω_ϵ as in (1-2), let $i_\epsilon : \Sigma \times (-\epsilon, \epsilon) \to \Omega_\epsilon$ be the bijection defined by

$$i_\epsilon(x_\Sigma, t) := x_\Sigma + t\nu(x_\Sigma).$$

For future purposes, we also introduce the projection $P_\Sigma : \Omega_\epsilon \to \Sigma$ given by

$$P_\Sigma(x_\Sigma + t\nu(x_\Sigma)) := x_\Sigma. \tag{2-1}$$

For $1 \le p < +\infty$, let $L^p(\Omega_\epsilon)$ and $L^p(\Sigma \times (-1, 1))$ be the Banach spaces endowed with the norms

$$\|f\|^p_{L^p(\Omega_\epsilon)} := \int_{\Omega_\epsilon} |f|^p\, d\mathcal{L}, \quad \|f\|^p_{L^p(\Sigma\times(-1,1))} := \int_{-1}^{1}\int_\Sigma |f|^p\, d\sigma\, dt, \tag{2-2}$$

respectively, where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^3$. The Banach spaces corresponding to the endpoint case $p = +\infty$ are defined, as usual, in terms of essential suprema with respect to the measures associated to Ω_ϵ and $\Sigma \times (-1, 1)$ in (2-2), respectively.

Proposition 2.3. *If $\eta > 0$ is small enough, there exist $0 < c_1, c_2 < +\infty$ such that*

$$c_1\|f\|_{L^1(\Omega_\epsilon)} \le \|f \circ i_\epsilon\|_{L^1(\Sigma\times(-\epsilon,\epsilon))} \le c_2\|f\|_{L^1(\Omega_\epsilon)} \quad \textit{for all } f \in L^1(\Omega_\epsilon),\ 0 < \epsilon \le \eta.$$

Moreover, if W denotes the Weingarten map associated to Σ from Definition 2.1,

$$\int_{\Omega_\epsilon} f(x)\,dx = \int_{-\epsilon}^{\epsilon}\int_\Sigma f(x_\Sigma + t\nu(x_\Sigma)) \det(1 - tW(x_\Sigma))\, d\sigma(x_\Sigma)\, dt \quad \textit{for all } f \in L^1(\Omega_\epsilon). \tag{2-3}$$

The eigenvalues of the Weingarten map $W(x)$ are the principal curvatures of Σ on $x \in \Sigma$, and they are independent of the parametrization of Σ. Therefore, the term $\det(1-tW(x_\Sigma))$ in (2-3) is also independent of the parametrization of Σ.

Remark 2.4. Let $h : \Omega_\epsilon \to (-\epsilon, \epsilon)$ be defined by $h(x_\Sigma + t\nu(x_\Sigma)) := t$. Then $|\nabla h| = 1$ in Ω_ϵ, so the coarea formula, see for example [Ambrosio et al. 2000, Remark 2.94], gives

$$\int_{\Omega_\epsilon} f(x)\,dx = \int_{-\epsilon}^{\epsilon}\int_{\Sigma_t} f(x)\, d\sigma_t(x)\, dt \quad \text{for all } f \in L^1(\Omega_\epsilon).$$

In view of (2-3), one deduces that

$$\int_{\Sigma_t} f\,d\sigma_t = \int_{\Sigma} f(x_\Sigma + t\nu(x_\Sigma))\det(1 - tW(x_\Sigma))\,d\sigma(x_\Sigma) \tag{2-4}$$

for all $t \in (-\epsilon, \epsilon)$ and all $f \in L^1(\Sigma_t)$.

In the following lemma we give uniform growth estimates on the measures σ_t for $t \in [-\eta, \eta]$ that exhibit their 2-dimensional nature. These estimates will be used many times in the sequel, mostly for the case of σ.

Lemma 2.5. *If $\eta > 0$ is small enough, there exist $c_1, c_2 > 0$ such that*

$$\sigma_t(B_r(x)) \le c_1 r^2 \quad \text{for all } x \in \mathbb{R}^3,\ r > 0,\ t \in [-\eta, \eta], \tag{2-5}$$

$$\sigma_t(B_r(x)) \ge c_2 r^2 \quad \text{for all } x \in \Sigma_t,\ 0 < r < 2\operatorname{diam}(\Omega_\eta),\ t \in [-\eta, \eta], \tag{2-6}$$

where $B_r(x)$ is the ball of radius r centered at x.

Proof. We first prove (2-5). Let $r_0 > 0$ be a constant small enough, to be fixed later on. If $r \ge r_0$, then

$$\sigma_t(B_r(x)) \le \max_{t\in[-\eta,\eta]} \sigma_t(\mathbb{R}^3) \le C = \frac{C}{r_0^2} r_0^2 \le C_0 r^2,$$

where $C_0 := C/r_0^2 > 0$ only depends on r_0 and η. Therefore, we can assume that $r < r_0$. Let us see that we can also suppose that $x \in \Sigma_t$. In fact, if η and r_0 are small enough and $0 < r < r_0$, given $x \in \mathbb{R}^3$ one can always find $\tilde{x} \in \Sigma_t$ such that $\sigma_t(B_r(x)) \le 2\sigma_t(B_r(\tilde{x}))$ (if $x \in \Omega_\eta$ just take $\tilde{x} = P_\Sigma x + t\nu(P_\Sigma x)$). Then if (2-5) holds for $\tilde{x}$, one gets $\sigma_t(B_r(x)) \le 2\sigma_t(B_r(\tilde{x})) \le Cr^2$, as desired.

Thus, it is enough to prove (2-5) for $x \in \Sigma_t$ and $r < r_0$. If r_0 and η are small enough, covering Σ_t by local chards we can find an open and bounded set $V_{t,r} \subset \mathbb{R}^2$ and a C^1 diffeomorphism $\varphi_t : \mathbb{R}^2 \to \varphi_t(\mathbb{R}^2) \subset \mathbb{R}^3$ such that $\varphi_t(V_{t,r}) = \Sigma_t \cap B_r(x)$. By means of a rotation if necessary, we can further assume that φ_t is of the form $\varphi_t(y') = (y', T_t(y'))$, i.e., φ_t is the graph of a C^1 function $T_t : \mathbb{R}^2 \to \mathbb{R}$, and that $\max_{t\in[-\eta,\eta]} \|\nabla T_t\|_\infty \le C$ (this follows from the regularity of Σ). Then, if $x' \in V_{t,r}$ is such that $\varphi_t(x') = x$, for any $y' \in V_{t,r}$ we get

$$r^2 \ge |\varphi_t(y') - \varphi_t(x')|^2 \ge |y' - x'|^2,$$

which means that $V_{t,r} \subset \{y' \in \mathbb{R}^2 : |x' - y'| < r\} =: B' \subset \mathbb{R}^2$. Denoting by $\mathcal{H}^2$ the 2-dimensional Hausdorff measure, from [Mattila 1995, Theorem 7.5] we get

$$\sigma_t(B_r(x)) = \mathcal{H}^2(\varphi_t(V_{t,r})) \le \mathcal{H}^2(\varphi_t(B')) \le \|\nabla\varphi_t\|_\infty^2 \mathcal{H}^2(B') \le Cr^2$$

for all $t \in [-\eta, \eta]$, so (2-5) is finally proved.

Let us now deal with (2-6). Given $r_0 > 0$, by the regularity and boundedness of Σ it is clear that $\inf_{t\in[-\eta,\eta],\, x\in\Sigma_t} \sigma_t(B_{r_0}(x)) \ge C > 0$. As before, for any $r_0 \le r < 2\operatorname{diam}(\Omega_\eta)$ we easily see that

$$\sigma_t(B_r(x)) \ge \sigma_t(B_{r_0}(x)) \ge C = \frac{C}{4\operatorname{diam}(\Omega_\eta)^2} 4\operatorname{diam}(\Omega_\eta)^2 \ge C_1 r^2,$$

where $C_1 := C/(4\,\mathrm{diam}(\Omega_\eta)^2) > 0$ only depends on r_0 and η. Hence (2-6) is proved for all $r_0 \leq r < 2\,\mathrm{diam}(\Omega_\eta)$.

The case $0 < r < r_0$ is treated, as before, using the local parametrization of Σ_t around x by the graph of a function. Taking η and r_0 small enough, we may assume the existence of $V_{t,r}$ and φ_t as above, so let us set $\varphi_t(x') = x$ for some $x' \in V_{t,r}$. The fact that φ_t is of the form $\varphi_t(y') = (y', T_t(y'))$ and that $\varphi_t(V_{t,r}) = \Sigma_t \cap B_r(x)$ implies $B'' := \{y' \in \mathbb{R}^2 : |x' - y'| < C_2 r\} \subset V_{t,r}$ for some $C_2 > 0$ small enough only depending on $\max_{t\in[-\eta,\eta]} \|\nabla T_t\|_\infty$, which is finite by assumption. Then, we easily see that

$$\sigma_t(B_r(x)) = \sigma_t(\varphi_t(V_{t,r})) \geq \sigma_t(\varphi_t(B'')) = \int_{B''} \sqrt{1 + |\nabla T_t(y')|^2}\, dy' \geq \int_{B''} dy' = Cr^2,$$

where $C > 0$ only depends on C_2. □

2B. ***Shell interactions for Dirac operators.*** In this section we briefly recall some useful instruments regarding the δ-shell interactions studied in [Arrizabalaga et al. 2014; 2015]. The reader should look at [Arrizabalaga et al. 2015, Sections 2 and 5] for the details.

Let $a \in \mathbb{C}$. A fundamental solution of $H - a$ is given by

$$\phi^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}\Big(a + m\beta + (1 + \sqrt{m^2 - a^2}|x|)\, i\alpha \cdot \frac{x}{|x|^2}\Big) \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\},$$

where $\sqrt{m^2 - a^2}$ is chosen with positive real part whenever $a \in (\mathbb{C} \setminus \mathbb{R}) \cup ((-m, m) \times \{0\})$. To guarantee the exponential decay of ϕ^a at ∞, from now on we assume that $a \in (\mathbb{C} \setminus \mathbb{R}) \cup ((-m, m) \times \{0\})$. Given $G \in L^2(\mathbb{R}^3)^4$ and $g \in L^2(\sigma)^4$ we define

$$\Phi^a(G, g)(x) := \int_{\mathbb{R}^3} \phi^a(x - y)\, G(y)\, dy + \int_\Sigma \phi^a(x - y) g(y)\, d\sigma(y) \quad \text{for } x \in \mathbb{R}^3 \setminus \Sigma. \tag{2-7}$$

Then, $\Phi^a : L^2(\mathbb{R}^3)^4 \times L^2(\sigma)^4 \to L^2(\mathbb{R}^3)^4$ is linear and bounded and $\Phi^a(G, 0) \in H^1(\mathbb{R}^3)^4$. We also set

$$\Phi^a_\sigma G := \mathrm{tr}_\sigma(\Phi^a(G, 0)) \in L^2(\sigma)^4,$$

where tr_σ is the trace operator on Σ. Finally, given $x \in \Sigma$ we define

$$C^a_\sigma g(x) := \lim_{\epsilon \searrow 0} \int_{\Sigma\cap\{|x-y|>\epsilon\}} \phi^a(x - y) g(y)\, d\sigma(y) \quad \text{and} \quad C^a_\pm g(x) := \lim_{\Omega_\pm \ni y \xrightarrow{nt} x} \Phi^a(0, g)(y),$$

where $\Omega_\pm \ni y \xrightarrow{nt} x$ means that y tends to x nontangentially from the interior/exterior of Ω, respectively; i.e., $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}$. The operators C^a_σ and $C^a_\pm$ are linear and bounded in $L^2(\sigma)^4$. Moreover, the following Plemelj–Sokhotski jump formulae hold:

$$C^a_\pm = \mp\tfrac{1}{2} i(\alpha \cdot \nu) + C^a_\sigma. \tag{2-8}$$

Let $\lambda_e \in \mathbb{R}$. Using Φ^a, we define the electrostatic δ-shell interaction appearing in Theorem 1.2 as

$$\begin{aligned} D(H + \lambda_e\delta_\Sigma) &:= \big\{\Phi^0(G, g) : G \in L^2(\mathbb{R}^3)^4,\ g \in L^2(\sigma)^4,\ \lambda_e \Phi^0_\sigma G = -(1 + \lambda_e C^0_\sigma) g\big\}, \\ (H + \lambda_e\delta_\Sigma)\varphi &:= H\varphi + \tfrac{1}{2}\lambda_e(\varphi_+ + \varphi_-)\sigma \quad \text{for } \varphi \in D(H + \lambda_e\delta_\Sigma), \end{aligned} \tag{2-9}$$

where $H\varphi$ in the right-hand side of the second statement in (2-9) is understood in the sense of distributions and $\varphi_\pm$ denotes the boundary traces of φ when one approaches Σ from $\Omega_\pm$. In particular, one has $(H+\lambda_e\delta_\Sigma)\varphi = G \in L^2(\mathbb{R}^3)^4$ for all $\varphi = \Phi^0(G, g) \in D(H+\lambda_e\delta_\Sigma)$. We should mention that one recovers the free Dirac operator in $H^1(\mathbb{R}^3)^4$ when $\lambda_e = 0$.

From [Arrizabalaga et al. 2015, Section 3.1] we know that $H+\lambda_e\delta_\Sigma$ is self-adjoint for all $\lambda_e \neq \pm 2$. Additionally, if $\lambda_e \neq 0$, given $a \in (-m, m)$ and $\varphi = \Phi^0(G, g) \in D(H+\lambda_e\delta_\Sigma)$,

$$(H+\lambda_e\delta_\Sigma - a)\varphi = 0 \quad \text{if and only if} \quad \Big(\frac{1}{\lambda_e} + C_\sigma^a\Big) g = 0. \tag{2-10}$$

This corresponds to the Birman–Schwinger principle in the electrostatic δ-shell interaction setting. Since the case $\lambda_e = 0$ corresponds to the free Dirac operator, it can be excluded from this consideration because it is well known that the free Dirac operator doesn't have pure point spectrum. Moreover, the relation (2-10) can be easily extended to the case of $a \in (\mathbb{C}\setminus\mathbb{R}) \cup ((-m, m)\times\{0\})$ (one still has exponential decay of a fundamental solution of $H-a$).

In the same vein, given $\lambda_s \in \mathbb{R}$, we define the Lorentz scalar δ-shell interaction as

$$\begin{aligned} &D(H+\lambda_s\beta\,\delta_\Sigma) := \big\{\Phi^0(G, g) : G \in L^2(\mathbb{R}^3)^4,\ g \in L^2(\sigma)^4,\ \lambda_s\Phi_\sigma^0 G = -(\beta + \lambda_s C_\sigma^0) g\big\}, \\ &(H+\lambda_s\beta\,\delta_\Sigma)\varphi := H\varphi + \tfrac{1}{2}\lambda_s\beta(\varphi_+ + \varphi_-)\sigma \quad \text{for } \varphi \in D(H+\lambda_s\beta\,\delta_\Sigma). \end{aligned} \tag{2-11}$$

From [Arrizabalaga et al. 2015, Section 5.1] we know that $H+\lambda_s\beta\,\delta_\Sigma$ is self-adjoint for all $\lambda_s \in \mathbb{R}$. Additionally, given $\lambda_s \neq 0$, $a \in (\mathbb{C}\setminus\mathbb{R}) \cup ((-m, m)\times\{0\})$ and $\varphi = \Phi^0(G, g) \in D(H+\lambda_s\beta\,\delta_\Sigma)$, arguing as in (2-10) one gets

$$(H+\lambda_s\beta\,\delta_\Sigma - a)\varphi = 0 \quad \text{if and only if} \quad \Big(\frac{\beta}{\lambda_s} + C_\sigma^a\Big) g = 0. \tag{2-12}$$

The following lemma describes the resolvent operator of the δ-shell interactions presented in (2-9) and (2-11).

Lemma 2.6. *Given $\lambda_e,\ \lambda_s \in \mathbb{R}$ with $\lambda_e \neq \pm 2$, $a \in \mathbb{C}\setminus\mathbb{R}$ and $F \in L^2(\mathbb{R}^3)^4$, the following identities hold*:

$$(H+\lambda_e\delta_\Sigma - a)^{-1}F = (H-a)^{-1}F - \lambda_e\Phi^a(0, (1+\lambda_e C_\sigma^a)^{-1}\Phi_\sigma^a F), \tag{2-13}$$

$$(H+\lambda_s\beta\,\delta_\Sigma - a)^{-1}F = (H-a)^{-1}F - \lambda_s\Phi^a(0, (\beta+\lambda_s C_\sigma^a)^{-1}\Phi_\sigma^a F). \tag{2-14}$$

Proof. We will only show (2-13); the proof of (2-14) is analogous. Since $H+\lambda_e\delta_\Sigma$ is self-adjoint for $\lambda_e \neq \pm 2$, we know $(H+\lambda_e\delta_\Sigma - a)^{-1}$ is well-defined and bounded in $L^2(\mathbb{R}^3)^4$. For $\lambda_e = 0$ there is nothing to prove, so we assume $\lambda_e \neq 0$.

Let $\varphi = \Phi^0(G, g) \in D(H+\lambda_e\delta_\Sigma)$ as in (2-9) and $F = (H+\lambda_e\delta_\Sigma - a)\varphi \in L^2(\mathbb{R}^3)^4$. Then,

$$F = (H+\lambda_e\delta_\Sigma - a)\Phi^0(G, g) = G - a\Phi^0(G, g). \tag{2-15}$$

If we apply H on both sides of (2-15) and we use that $H\Phi^0(G, g) = G + g\sigma$ in the sense of distributions, we get $HF = HG - a(G + g\sigma)$; that is, $(H-a)G = (H-a)F + aF + ag\sigma$. Convolving with ϕ^a the left- and right-hand sides of this last equation, we obtain $G = F + a\Phi^a(F, 0) + a\Phi^a(0, g)$; thus

$G-F=a\Phi^a(F,g)$. This, combined with (2-15), yields

$$\Phi^0(G,g)=\Phi^a(F,g). \tag{2-16}$$

Therefore, taking nontangential boundary values on Σ from inside/outside of Ω in (2-16) we obtain

$$\Phi^0_\sigma G+C^0_\pm g=\Phi^a_\sigma F+C^a_\pm g.$$

Since $\Phi^0(G,g)\in D(H+\lambda_e\delta_\Sigma)$, thanks to (2-9) and (2-8) we conclude that

$$\Phi^a_\sigma F=-\Big(\frac{1}{\lambda_e}+C^a_\sigma\Big)g. \tag{2-17}$$

Since $a\in\mathbb{C}\setminus\mathbb{R}$ and $H+\lambda_e\delta_\Sigma$ is self-adjoint for $\lambda_e\neq\pm 2$, by (2-10) we see that $\operatorname{Kernel}(1/\lambda_e+C^a_\sigma)=\{0\}$. Moreover, using the ideas of the proof of [Arrizabalaga et al. 2014, Lemma 3.7] and that $\lambda_e\neq\pm 2$, one can show that $1/\lambda_e+C^a_\sigma$ has closed range. Finally, since we are taking the square root so that

$$\overline{\sqrt{m^2-a^2}}=\sqrt{m^2-\bar a^2},$$

following Lemma 3.1 of the same paper we see that $\overline{(\phi^a)^t}(x)=\phi^{\bar a}(-x)$. Here, $(\phi^a)^t$ denotes the transpose matrix of ϕ^a. Thus we conclude that $(\operatorname{Range}(1/\lambda_e+C^a_\sigma))^\perp=\operatorname{Kernel}(1/\lambda_e+C^{\bar a}_\sigma)=\{0\}$, and so $1/\lambda_e+C^a_\sigma$ is invertible. Then, by (2-17), we obtain

$$g=-\Big(\frac{1}{\lambda_e}+C^a_\sigma\Big)^{-1}\Phi^a_\sigma F. \tag{2-18}$$

Thanks to (2-16) and (2-18), we finally get

$$\begin{aligned}(H+\lambda_e\delta_\Sigma-a)^{-1}F=\varphi=\Phi^0(G,g)=\Phi^a(F,g)&=\Phi^a\Big(F,-\Big(\frac{1}{\lambda_e}+C^a_\sigma\Big)^{-1}\Phi^a_\sigma F\Big)\\ &=\Phi^a(F,0)-\lambda_e\Phi^a(0,(1+\lambda_e C^a_\sigma)^{-1}\Phi^a_\sigma F),\end{aligned}$$

and the lemma follows because $\Phi^a(\,\cdot\,,0)=(H-a)^{-1}$ as a bounded operator in $L^2(\mathbb{R}^3)^4$. ☐

2C. ***Coupling the free Dirac operator with short-range potentials as in* (1-6).** Given $\boldsymbol{V}_\epsilon$ as in (1-4), set

$$H^e_\epsilon:=H+\boldsymbol{V}_\epsilon \quad\text{and}\quad H^s_\epsilon:=H+\beta\boldsymbol{V}_\epsilon.$$

Recall that these operators are self-adjoint on $H^1(\mathbb{R}^3)^4$. In the following, we give the resolvent formulae for H^e_ϵ and H^s_ϵ.

Throughout this section we make an abuse of notation. Remember that, given $G\in L^2(\mathbb{R}^3)^4$ and $g\in L^2(\sigma)^4$, in (2-7) we already defined $\Phi^a(G,g)$. However, now we make the identification $\Phi^a(\,\cdot\,)\equiv\Phi^a(\,\cdot\,,0)$; that is, in this section we identify Φ^a with an operator acting on $L^2(\mathbb{R}^3)^4$ by always assuming that the second entrance in Φ^a vanishes. Additionally, in this section we use the symbol $\sigma(\,\cdot\,)$ to denote the spectrum of an operator, the reader should not confuse it with the symbol σ for the surface measure on Σ.

Proposition 2.7. *Let $\boldsymbol{u}_\epsilon$ and $\boldsymbol{v}_\epsilon$ be as in* (1-5). *Then*:

(i) *$a\in\rho(H^e_\epsilon)$ if and only if $-1\in\rho(\boldsymbol{u}_\epsilon\Phi^a\boldsymbol{v}_\epsilon)$, where $\rho(\,\cdot\,)$ denotes the resolvent set.*

(ii) *$a \in \sigma_{pp}(H_\epsilon^e)$ if and only if $-1 \in \sigma_{pp}(\boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)$, where $\sigma_{pp}(\cdot)$ denotes the pure point spectrum. Moreover, the multiplicity of a as an eigenvalue of H_ϵ^e coincides with the multiplicity of -1 as eigenvalue of $\boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon$.*

Furthermore, the following resolvent formula holds:

$$(H_\epsilon^e - a)^{-1} = \Phi^a - \Phi^a \boldsymbol{v}_\epsilon (1 + \boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)^{-1} \boldsymbol{u}_\epsilon \Phi^a. \tag{2-19}$$

Proof. To prove (i) and (ii) it is enough to verify that the assumptions of [Konno and Kuroda 1966, Lemma 1] are satisfied. That is, we just need to show that $a \in \sigma_{pp}(H_\epsilon^e)$ if and only if $-1 \in \sigma_{pp}(\boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)$ and that there exists $a \in \rho(H_\epsilon^e)$ such that $-1 \in \rho(\boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)$.

Assume that $a \in \sigma_{pp}(H_\epsilon^e)$. Then $(H + \boldsymbol{V}_\epsilon - a)F = 0$ for some $F \in L^2(\mathbb{R}^3)^4$ with $F \not\equiv 0$, so $(H-a)F = -\boldsymbol{V}_\epsilon F$. Using that $\sigma(H) = \sigma_{\text{ess}}(H)$, where $\sigma_{\text{ess}}(\cdot)$ denotes the essential spectrum, it is not hard to show that indeed $\boldsymbol{V}_\epsilon F \not\equiv 0$. Since $\boldsymbol{V}_\epsilon = \boldsymbol{v}_\epsilon \boldsymbol{u}_\epsilon$, by setting $G = \boldsymbol{u}_\epsilon F \in L^2(\mathbb{R}^3)^4$ we get that $G \not\equiv 0$ and

$$(H-a)F = -\boldsymbol{v}_\epsilon G. \tag{2-20}$$

From [Thaller 1992, Theorem 4.7] we know that $\sigma_{\text{ess}}(H + \boldsymbol{V}_\epsilon) = \sigma_{\text{ess}}(H) = \sigma(H)$. Since $\sigma(H_\epsilon^e)$ is the disjoint union of the pure point spectrum and the essential spectrum, we have $\sigma_{pp}(H_\epsilon^e) \subset \rho(H)$, which means that $(H-a)^{-1} = \Phi^a$ is a bounded operator on $L^2(\mathbb{R}^3)^4$. By (2-20), $F = -\Phi^a \boldsymbol{v}_\epsilon G$. If we multiply both sides of this last equation by $\boldsymbol{u}_\epsilon$ we obtain $G = \boldsymbol{u}_\epsilon F = -\boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon G$, so $-1 \in \sigma_{pp}(\boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)$ as desired.

On the contrary, assume now that there exists a nontrivial $G \in L^2(\mathbb{R}^3)^4$ such that $\boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon G = -G$. If we take $F = \Phi^a \boldsymbol{v}_\epsilon G \in L^2(\mathbb{R}^3)$, we easily see that $F \not\equiv 0$ and $\boldsymbol{V}_\epsilon F = -(H-a)F$, which means that a is an eigenvalue of H_ϵ^e.

To conclude the first part of the proof, it remains to show that there exists $a \in \rho(H_\epsilon^e)$ such that $-1 \in \rho(\boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)$. By [Thaller 1992, Theorem 4.23] we know that $\sigma_{pp}(H_\epsilon^e)$ is a finite sequence contained in $(-m, m)$, so we can chose $a \in (-m, m) \cap \rho(H_\epsilon^e)$. Moreover, by [Šeba 1988, Lemma 2], $\boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon$ is a compact operator. Then, by Fredholm's alternative, either $-1 \in \sigma_{pp}(\boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)$ or $-1 \in \rho(\boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)$. But we can discard the first option, otherwise $a \in \sigma_{pp}(H_\epsilon^e)$, in contradiction with $a \in \rho(H_\epsilon^e)$.

Let us now prove (2-19). Writing $\boldsymbol{V}_\epsilon = \boldsymbol{v}_\epsilon \boldsymbol{u}_\epsilon$ and using that $(H-a)^{-1} = \Phi^a$, we have

$$\begin{aligned}
&(H_\epsilon^e - a)(\Phi^a - \Phi^a \boldsymbol{v}_\epsilon (1 + \boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)^{-1} \boldsymbol{u}_\epsilon \Phi^a) \\
&\qquad = 1 - \boldsymbol{v}_\epsilon (1 + \boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)^{-1} \boldsymbol{u}_\epsilon \Phi^a + \boldsymbol{v}_\epsilon \boldsymbol{u}_\epsilon \Phi^a - \boldsymbol{v}_\epsilon (-1 + 1 + \boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)(1 + \boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)^{-1} \boldsymbol{u}_\epsilon \Phi^a \\
&\qquad = 1 - \boldsymbol{v}_\epsilon (1 + \boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)^{-1} \boldsymbol{u}_\epsilon \Phi^a + \boldsymbol{v}_\epsilon \boldsymbol{u}_\epsilon \Phi^a + \boldsymbol{v}_\epsilon (1 + \boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)^{-1} \boldsymbol{u}_\epsilon \Phi^a - \boldsymbol{v}_\epsilon \boldsymbol{u}_\epsilon \Phi^a = 1,
\end{aligned}$$

as desired. □

The following result can be proved in the same way; we leave the details for the reader.

Proposition 2.8. *Let $\boldsymbol{u}_\epsilon$ and $\boldsymbol{v}_\epsilon$ be as in* (1-5). *Then*:

(i) *$a \in \rho(H_\epsilon^s)$ if and only if $-1 \in \rho(\beta \boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)$.*

(ii) *$a \in \sigma_{pp}(H_\epsilon^s)$ if and only if $-1 \in \sigma_{pp}(\beta \boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)$. Moreover, the multiplicity of a as an eigenvalue of H_ϵ^s coincides with the multiplicity of -1 as eigenvalue of $\beta \boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon$.*

Furthermore, the following resolvent formula holds:

$$(H_\epsilon^s - a)^{-1} = \Phi^a - \Phi^a \boldsymbol{v}_\epsilon (\beta + \boldsymbol{u}_\epsilon \Phi^a \boldsymbol{v}_\epsilon)^{-1} \boldsymbol{u}_\epsilon \Phi^a. \tag{2-21}$$

3. The main decomposition and the proof of Theorem 1.2

Following the ideas in [Šeba 1989; Behrndt et al. 2017], the first key step to proving Theorem 1.2 is to decompose $(H^e_\epsilon - a)^{-1}$ and $(H^s_\epsilon - a)^{-1}$, using a scaling operator, in terms of the operators $A_\epsilon(a)$, $B_\epsilon(a)$ and $C_\epsilon(a)$ introduced below (see Lemma 3.1).

Let $\eta_0 > 0$ be some constant small enough to be fixed later on. In particular, we take η_0 so that (1-2) holds for all $0 < \epsilon \le \eta_0$. Given $0 < \epsilon \le \eta_0$, define

$$\begin{aligned} &\mathcal{I}_\epsilon : L^2(\Sigma \times (-\epsilon,\epsilon))^4 \to L^2(\Omega_\epsilon)^4 && \text{by } (\mathcal{I}_\epsilon f)(x_\Sigma + t\nu(x_\Sigma)) := f(x_\Sigma, t),\\ &\mathcal{S}_\epsilon : L^2(\Sigma \times (-1,1))^4 \to L^2(\Sigma \times (-\epsilon,\epsilon))^4 && \text{by } (\mathcal{S}_\epsilon g)(x_\Sigma, t) := \frac{1}{\sqrt{\epsilon}}\, g\Big(x_\Sigma, \frac{t}{\epsilon}\Big). \end{aligned}$$

Thanks to the regularity of Σ, $\mathcal{I}_\epsilon$ is well-defined, bounded and invertible for all $0 < \epsilon \le \eta_0$ if η_0 is small enough. Note also that $\mathcal{S}_\epsilon$ is a unitary and invertible operator.

Let $0 < \eta \le \eta_0$, $V \in L^\infty(\mathbb{R})$ with $\operatorname{supp} V \subset [-\eta, \eta]$ and $u, v \in L^\infty(\mathbb{R})$ be the functions with support in $[-1,1]$ introduced in (1-5); that is,

$$u(t) := |\eta V(\eta t)|^{1/2} \quad \text{and} \quad v(t) := \operatorname{sign}(V(\eta t))u(t). \tag{3-1}$$

Using the notation related to (2-3), for $0 < \epsilon \le \eta_0$ we consider the integral operators

$$\begin{aligned} &A_\epsilon(a) : L^2(\Sigma \times (-1,1))^4 \to L^2(\mathbb{R}^3)^4,\\ &B_\epsilon(a) : L^2(\Sigma \times (-1,1))^4 \to L^2(\Sigma \times (-1,1))^4,\\ &C_\epsilon(a) : L^2(\mathbb{R}^3)^4 \to L^2(\Sigma \times (-1,1))^4 \end{aligned} \tag{3-2}$$

defined by

$$\begin{aligned} (A_\epsilon(a)g)(x) &:= \int_{-1}^1 \int_\Sigma \phi^a(x - y_\Sigma - \epsilon s \nu(y_\Sigma)) v(s) \det(1 - \epsilon s W(y_\Sigma)) g(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds,\\ (B_\epsilon(a)g)(x_\Sigma, t) &:= u(t) \int_{-1}^1 \int_\Sigma \phi^a(x_\Sigma + \epsilon t \nu(x_\Sigma) - y_\Sigma - \epsilon s \nu(y_\Sigma)) v(s)\\ &\qquad\qquad \times \det(1 - \epsilon s W(y_\Sigma)) g(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds,\\ (C_\epsilon(a)g)(x_\Sigma, t) &:= u(t) \int_{\mathbb{R}^3} \phi^a(x_\Sigma + \epsilon t \nu(x_\Sigma) - y) g(y)\, dy. \end{aligned} \tag{3-3}$$

Recall that, given $F \in L^2(\mathbb{R}^3)^4$ and $f \in L^2(\sigma)^4$, in (2-7) we defined $\Phi^a(F, f)$. However, in Section 2C we made the identification $\Phi^a(\cdot) \equiv \Phi^a(\cdot, 0)$, which enabled us to write $(H - a)^{-1} = \Phi^a$. Here, and in the sequel, we recover the initial definition for Φ^a given in (2-7) and we assume that $a \in \mathbb{C} \setminus \mathbb{R}$; now we must write $(H - a)^{-1} = \Phi^a(\cdot, 0)$, which is a bounded operator in $L^2(\mathbb{R}^3)^4$.

Proceeding as in the proof of [Behrndt et al. 2017, Lemma 3.2], one can show the following result.

Lemma 3.1. *The following operator identities hold for all* $0 < \epsilon \le \eta$:

$$\begin{aligned} A_\epsilon(a) &= \Phi^a(\cdot, 0) \boldsymbol{v}_\epsilon\, \mathcal{I}_\epsilon\, \mathcal{S}_\epsilon,\\ B_\epsilon(a) &= \mathcal{S}_\epsilon^{-1} \mathcal{I}_\epsilon^{-1} \boldsymbol{u}_\epsilon\, \Phi^a(\cdot, 0) \boldsymbol{v}_\epsilon\, \mathcal{I}_\epsilon\, \mathcal{S}_\epsilon,\\ C_\epsilon(a) &= \mathcal{S}_\epsilon^{-1} \mathcal{I}_\epsilon^{-1} \boldsymbol{u}_\epsilon\, \Phi^a(\cdot, 0). \end{aligned} \tag{3-4}$$

Moreover, *the following resolvent formulae hold*:

$$(H_\epsilon^e - a)^{-1} = (H-a)^{-1} + A_\epsilon(a)(1+B_\epsilon(a))^{-1}C_\epsilon(a), \tag{3-5}$$

$$(H_\epsilon^s - a)^{-1} = (H-a)^{-1} + A_\epsilon(a)(\beta+B_\epsilon(a))^{-1}C_\epsilon(a). \tag{3-6}$$

In (3-4), $A_\epsilon(a) = \Phi^a(\,\cdot\,,0)\boldsymbol{v}_\epsilon\,\mathcal{I}_\epsilon\,\mathcal{S}_\epsilon$ means that $A_\epsilon(a)g = \Phi^a(\boldsymbol{v}_\epsilon\,\mathcal{I}_\epsilon\,\mathcal{S}_\epsilon\,g, 0)$ for all $g \in L^2(\Sigma\times(-1,1))^4$, and similarly for $B_\epsilon(a)$ and $C_\epsilon(a)$. Since both $\mathcal{I}_\epsilon$ and $\mathcal{S}_\epsilon$ are an isometry, $V \in L^\infty(\mathbb{R})$ is supported in $[-\eta,\eta]$ and $\Phi^a(\,\cdot\,,0)$ is bounded by assumption, from (3-4) we deduce that $A_\epsilon(a)$, $B_\epsilon(a)$ and $C_\epsilon(a)$ are well-defined and bounded, so (3-2) is fully justified. Once (3-4) is proved, the resolvent formulae (3-5) and (3-6) follow from (2-19) and (2-21), respectively. We stress that, in (2-19) and (2-21), there is the abuse of notation in the definition of Φ^a, commented on before.

Lemma 3.1 connects $(H_\epsilon^e - a)^{-1}$ and $(H_\epsilon^s - a)^{-1}$ to $A_\epsilon(a)$, $B_\epsilon(a)$ and $C_\epsilon(a)$. When $\epsilon \to 0$, the limit of the former ones is also connected to the limit of the latter ones. We now introduce those limit operators for $A_\epsilon(a)$, $B_\epsilon(a)$ and $C_\epsilon(a)$ when $\epsilon \to 0$. Let

$$\begin{aligned} A_0(a) &: L^2(\Sigma\times(-1,1))^4 \to L^2(\mathbb{R}^3)^4,\\ B_0(a) &: L^2(\Sigma\times(-1,1))^4 \to L^2(\Sigma\times(-1,1))^4,\\ B' &: L^2(\Sigma\times(-1,1))^4 \to L^2(\Sigma\times(-1,1))^4,\\ C_0(a) &: L^2(\mathbb{R}^3)^4 \to L^2(\Sigma\times(-1,1))^4 \end{aligned} \tag{3-7}$$

be the operators given by

$$\begin{aligned} (A_0(a)g)(x) &:= \int_{-1}^1\int_\Sigma \phi^a(x-y_\Sigma)v(s)g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\ (B_0(a)g)(x_\Sigma,t) &:= \lim_{\epsilon\to0} u(t)\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|>\epsilon}\phi^a(x_\Sigma-y_\Sigma)v(s)g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\ (B'g)(x_\Sigma,t) &:= (\alpha\cdot\nu(x_\Sigma))\,\tfrac{1}{2}i\,u(t)\int_{-1}^1 \operatorname{sign}(t-s)v(s)g(x_\Sigma,s)\,ds,\\ (C_0(a)g)(x_\Sigma,t) &:= u(t)\int_{\mathbb{R}^3}\phi^a(x_\Sigma-y)g(y)\,dy. \end{aligned} \tag{3-8}$$

The next theorem corresponds to the core of this article. Its proof is quite technical and is carried out in Sections 4, 5 and 6. We also postpone the proof of (3-7) to those sections, where each operator is studied in detail. Anyway, the boundedness of B' is trivial.

Theorem 3.2. *The following convergences of operators hold in the strong sense*:

$$A_\epsilon(a) \to A_0(a) \qquad \textit{when } \epsilon\to0, \tag{3-9}$$

$$B_\epsilon(a) \to B_0(a)+B' \quad \textit{when } \epsilon\to0, \tag{3-10}$$

$$C_\epsilon(a) \to C_0(a) \qquad \textit{when } \epsilon\to0. \tag{3-11}$$

The proof of the following corollary is also postponed to Section 7. It combines Theorem 3.2, (3-5) and (3-6), but it requires some fine estimates developed in Sections 4, 5 and 6.

Corollary 3.3. *There exist η_0, $\delta > 0$ small enough only depending on Σ such that, for any $a \in \mathbb{C}\setminus\mathbb{R}$ with $|a| \le 1$, $0 < \eta \le \eta_0$ and (δ, η)-small V (see* Definition 1.1)*, the following convergences of operators hold in the strong sense*:

$$(H + \boldsymbol{V}_\epsilon - a)^{-1} \to (H - a)^{-1} + A_0(a)(1 + B_0(a) + B')^{-1} C_0(a) \quad \textit{when } \epsilon \to 0,$$
$$(H + \beta \boldsymbol{V}_\epsilon - a)^{-1} \to (H - a)^{-1} + A_0(a)(\beta + B_0(a) + B')^{-1} C_0(a) \quad \textit{when } \epsilon \to 0.$$

In particular, $(1 + B_0(a) + B')^{-1}$ and $(\beta + B_0(a) + B')^{-1}$ are well-defined bounded operators in $L^2(\Sigma \times (-1, 1))^4$.

3A. ***Proof of Theorem 1.2.*** Thanks to [Reed and Simon 1980, Theorem VIII.19], to prove the theorem it is enough to show that, for some $a \in \mathbb{C}\setminus\mathbb{R}$, the following convergences of operators hold in the strong sense:

$$(H + \boldsymbol{V}_\epsilon - a)^{-1} \to (H + \lambda_e \delta_\Sigma - a)^{-1} \quad \text{when } \epsilon \to 0, \tag{3-12}$$
$$(H + \beta \boldsymbol{V}_\epsilon - a)^{-1} \to (H + \lambda_s \beta \delta_\Sigma - a)^{-1} \quad \text{when } \epsilon \to 0. \tag{3-13}$$

Thus, from now on, we fix $a \in \mathbb{C}\setminus\mathbb{R}$ with $|a| \le 1$.

We introduce the operators

$$\widehat{V} : L^2(\Sigma \times (-1, 1))^4 \to L^2(\Sigma)^4 \quad \text{and} \quad \widehat{U} : L^2(\Sigma)^4 \to L^2(\Sigma \times (-1, 1))^4$$

given by

$$\widehat{V} f(x_\Sigma) := \int_{-1}^{1} v(s)\, f(x_\Sigma, s)\, ds \quad \text{and} \quad \widehat{U} f(x_\Sigma, t) := u(t)\, f(x_\Sigma).$$

Observe that, by Fubini's theorem,

$$A_0(a) = \Phi^a(0, \cdot)\widehat{V}, \quad B_0(a) = \widehat{U} C^a_\sigma \widehat{V}, \quad C_0(a) = \widehat{U}\Phi^a_\sigma. \tag{3-14}$$

Hence, from Corollary 3.3 and (3-14) we deduce that, in the strong sense,

$$(H + \boldsymbol{V}_\epsilon - a)^{-1} \to (H - a)^{-1} + \Phi^a(0, \cdot)\widehat{V}(1 + \widehat{U} C^a_\sigma \widehat{V} + B')^{-1}\widehat{U}\Phi^a_\sigma \quad \text{when } \epsilon \to 0, \tag{3-15}$$
$$(H + \beta \boldsymbol{V}_\epsilon - a)^{-1} \to (H - a)^{-1} + \Phi^a(0, \cdot)\widehat{V}(\beta + \widehat{U} C^a_\sigma \widehat{V} + B')^{-1}\widehat{U}\Phi^a_\sigma \quad \text{when } \epsilon \to 0. \tag{3-16}$$

For convenience of notation, set

$$\widetilde{\mathcal{K}} g(x_\Sigma, t) := \mathcal{K}_V(g(x_\Sigma, \cdot))(t) \quad \text{for } g \in L^2(\Sigma \times (-1, 1)),$$

where $\mathcal{K}_V$ is as in (1-7). Then, we get

$$1 + B' = \mathbb{I}_4 + (\alpha \cdot \nu)\widetilde{\mathcal{K}}\mathbb{I}_4 = \begin{pmatrix} \mathbb{I}_2 & (\sigma \cdot \nu)\widetilde{\mathcal{K}}\mathbb{I}_2 \\ (\sigma \cdot \nu)\widetilde{\mathcal{K}}\mathbb{I}_2 & \mathbb{I}_2 \end{pmatrix}.$$

Here, $\sigma := (\sigma_1, \sigma_2, \sigma_3)$, see (1-1), $\mathbb{I}_4$ denotes the 4×4 identity matrix and $\widetilde{\mathcal{K}}\mathbb{I}_4$ denotes the diagonal 4×4 operator matrix whose nontrivial entries are $\widetilde{\mathcal{K}}$, and analogously for $\widetilde{\mathcal{K}}\mathbb{I}_2$. Since the operators that compose the matrix $1+B'$ commute, if we set $\mathcal{K} := \widetilde{\mathcal{K}}\mathbb{I}_4$, we get

$$(1+B')^{-1} = (1-\widetilde{\mathcal{K}}^2)^{-1} \otimes \begin{pmatrix} \mathbb{I}_2 & -(\sigma\cdot\nu)\widetilde{\mathcal{K}}\mathbb{I}_2 \\ -(\sigma\cdot\nu)\widetilde{\mathcal{K}}\mathbb{I}_2 & \mathbb{I}_2 \end{pmatrix} = (1-\mathcal{K}^2)^{-1} - (\alpha\cdot\nu)(1-\mathcal{K}^2)^{-1}\mathcal{K}. \tag{3-17}$$

With this at hand, we can compute

$$\begin{aligned}(1+\widehat{U}C_\sigma^a\widehat{V}+B')^{-1} &= \big(1+(1+B')^{-1}\widehat{U}C_\sigma^a\widehat{V}\big)^{-1}(1+B')^{-1} \\ &= \big(1+(1-\mathcal{K}^2)^{-1}\widehat{U}C_\sigma^a\widehat{V} - (\alpha\cdot\nu)(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C_\sigma^a\widehat{V}\big)^{-1} \\ &\qquad\qquad \circ\big((1-\mathcal{K}^2)^{-1} - (\alpha\cdot\nu)(1-\mathcal{K}^2)^{-1}\mathcal{K}\big).\end{aligned} \tag{3-18}$$

Notice that

$$\begin{aligned}&\widehat{V}\big(1+(1-\mathcal{K}^2)^{-1}\widehat{U}C_\sigma^a\widehat{V} - (\alpha\cdot\nu)(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C_\sigma^a\widehat{V}\big) \\ &\qquad = \big(1+\widehat{V}(1-\mathcal{K}^2)^{-1}\widehat{U}C_\sigma^a - (\alpha\cdot\nu)\widehat{V}(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C_\sigma^a\big)\widehat{V},\end{aligned}$$

which obviously yields

$$\begin{aligned}&\widehat{V}\big(1+(1-\mathcal{K}^2)^{-1}\widehat{U}C_\sigma^a\widehat{V} - (\alpha\cdot\nu)(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C_\sigma^a\widehat{V}\big)^{-1} \\ &\qquad = \big(1+\widehat{V}(1-\mathcal{K}^2)^{-1}\widehat{U}C_\sigma^a - (\alpha\cdot\nu)\widehat{V}(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C_\sigma^a\big)^{-1}\widehat{V}.\end{aligned} \tag{3-19}$$

Additionally, by the definition of $\mathcal{K}_V$ in (1-7), we see that

$$\begin{aligned}\widehat{V}(1-\mathcal{K}^2)^{-1}\widehat{U} &= \bigg(\int_{\mathbb{R}} v\,(1-\mathcal{K}_V^2)^{-1}u\bigg)\mathbb{I}_4 = \lambda_e\mathbb{I}_4, \\ \widehat{V}(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U} &= \bigg(\int_{\mathbb{R}} v\,(1-\mathcal{K}_V^2)^{-1}\mathcal{K}_V u\bigg)\mathbb{I}_4 = 0.\end{aligned} \tag{3-20}$$

Indeed, from (1-10) in Theorem 1.2, $\lambda_e = \int_{\mathbb{R}} v\,(1-\mathcal{K}_V^2)^{-1}u$. Let us focus on $\int_{\mathbb{R}} v\,(1-\mathcal{K}_V^2)^{-1}\mathcal{K}_V u$. Note that, for any $n\geq 0$,

$$\int_{\mathbb{R}} v\mathcal{K}_V^{2n+1}u = \big(-\tfrac{1}{2}i\big)^{2n+1}\int_{(-\eta,\eta)^{2n+2}} V(t_0)V(t_1)\cdots V(t_{2n+1})\,\mathrm{sign}(t_0-t_1)\cdots\mathrm{sign}(t_{2n}-t_{2n+1})\,dt_0\,dt_1\cdots dt_{2n+1}.$$

Set $s_j := t_{2n+1-j}$ for $j\in\{0,\dots,2n+1\}$. Then,

$$\mathrm{sign}(t_0-t_1)\cdots\mathrm{sign}(t_{2n}-t_{2n+1}) = (-1)^{2n+1}\,\mathrm{sign}(s_0-s_1)\cdots\mathrm{sign}(s_{2n}-s_{2n+1});$$

thus, by Fubini's theorem, $\int_{\mathbb{R}} v\mathcal{K}_V^{2n+1}u = 0$. This implies $\int_{\mathbb{R}} v\,(1-\mathcal{K}_V^2)^{-1}\mathcal{K}_V u = 0$ by a Neumann series argument, and therefore $\widehat{V}(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U} = 0$.

Hence, combining (3-19) and (3-20) we have

$$\widehat{V}\big(1+(1-\mathcal{K}^2)^{-1}\widehat{U}C_\sigma^a\widehat{V} - (\alpha\cdot\nu)(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C_\sigma^a\widehat{V}\big)^{-1} = (1+\lambda_e C_\sigma^a)^{-1}\widehat{V}. \tag{3-21}$$

Then, from (3-18), (3-21) and (3-20), we finally get

$$\Phi^a(0,\cdot)\widehat{V}(1+\widehat{U}C_\sigma^a\widehat{V}+B')^{-1}\widehat{U}\Phi_\sigma^a = \Phi^a(0,\cdot)(1+\lambda_e C_\sigma^a)^{-1}\lambda_e\Phi_\sigma^a.$$

This last identity combined with (3-15) and (2-13) yields (3-12).

The proof of (3-13) follows the same lines. Similarly to (3-17),

$$(\beta+B')^{-1}=(1+\mathcal{K}^2)^{-1}\beta-(\alpha\cdot\nu)(1+\mathcal{K}^2)^{-1}.$$

One can then make the computations analogous to (3-18)–(3-21). Since

$$\lambda_s=\int_{\mathbb{R}} v\,(1+\mathcal{K}_V^2)^{-1}u,$$

we now get

$$\Phi^a(0,\cdot\,)\widehat{V}(\beta+\widehat{U}C_\sigma^a\widehat{V}+B')^{-1}\widehat{U}\Phi_\sigma^a=\Phi^a(0,\cdot\,)(\beta+\lambda_s C_\sigma^a)^{-1}\lambda_s\Phi_\sigma^a.$$

From this, (3-16) and (2-14) we obtain (3-13). This finishes the proof of Theorem 1.2, except for the boundedness stated in (3-7), the proof of Corollary 3.3 in Section 7, and Theorem 3.2, whose proof is broken up as follows: (3-9) in Section 6, (3-10) in Section 5 and (3-11) in Section 4.

4. Proof of (3-11): $C_\epsilon(a)\to C_0(a)$ in the strong sense when $\epsilon\to 0$

Recall from (3-3) and (3-8) that $C_\epsilon(a)$ with $0<\epsilon\le\eta_0$ and $C_0(a)$ are defined by

$$(C_\epsilon(a)g)(x_\Sigma,t)=u(t)\int_{\mathbb{R}^3}\phi^a(x_\Sigma+\epsilon t\nu(x_\Sigma)-y)g(y)\,dy,$$
$$(C_0(a)g)(x_\Sigma,t)=u(t)\int_{\mathbb{R}^3}\phi^a(x_\Sigma-y)g(y)\,dy.$$

Let us first show that $C_\epsilon(a)$ is bounded from $L^2(\mathbb{R}^3)^4$ to $L^2(\Sigma\times(-1,1))^4$ with a norm uniformly bounded on $0\le\epsilon\le\eta_0$. For this purpose, we write

$$(C_\epsilon(a)g)(x_\Sigma,t)=u(t)(\phi^a*g)(x_\Sigma+\epsilon t\nu(x_\Sigma)),\tag{4-1}$$

where ϕ^a*g denotes the convolution of the matrix-valued function ϕ^a with the vector-valued function $g\in L^2(\mathbb{R}^3)^4$. Since we are assuming that $a\in\mathbb{C}\setminus\mathbb{R}$ and, in the definition of ϕ^a, we are taking $\sqrt{m^2-a^2}$ with positive real part, the same arguments as the ones in the proof of [Arrizabalaga et al. 2014, Lemma 2.8] (essentially Plancherel's theorem) show that

$$\|\phi^a*g\|_{H^1(\mathbb{R}^3)^4}\le C\|g\|_{L^2(\mathbb{R}^3)^4}\quad\text{for all } g\in L^2(\mathbb{R}^3)^4,$$

where $C>0$ only depends on a. Additionally, thanks to the C^2 regularity of Σ, if η_0 is small enough it is not hard to show that the Sobolev trace inequality from $H^1(\mathbb{R}^3)^4$ to $L^2(\Sigma_{\epsilon t})^4$ holds for all $0\le\epsilon\le\eta_0$ and $t\in[-1,1]$ with a constant only depending on η_0 (and Σ, of course). Combining these two facts, we obtain that

$$\|\phi^a*g\|_{L^2(\Sigma_{\epsilon t})^4}\le C\|g\|_{L^2(\mathbb{R}^3)^4}\quad\text{for all } g\in L^2(\mathbb{R}^3)^4,\ 0\le\epsilon\le\eta_0 \text{ and } t\in[-1,1].\tag{4-2}$$

By Proposition 2.2, if η_0 is small enough there exists $C>0$ such that

$$C^{-1}\le\det(1-\epsilon tW(P_\Sigma x))\le C\quad\text{for all } 0<\epsilon\le\eta_0,\ t\in(-1,1) \text{ and } x\in\Sigma_{\epsilon t}.\tag{4-3}$$

Therefore, an application of (4-1), (2-4), (4-3) and (4-2) finally yields

$$\begin{aligned}\|C_\epsilon(a)g\|^2_{L^2(\Sigma\times(-1,1))^4} &= \int_{-1}^{1}\int_\Sigma \big|u(t)(\phi^a * g)(x_\Sigma+\epsilon t\nu(x_\Sigma))\big|^2\,d\sigma(x_\Sigma)\,dt\\ &\le \|u\|^2_{L^\infty(\mathbb{R})}\int_{-1}^{1}\int_{\Sigma_{\epsilon t}}\big|\det(1-\epsilon t W(P_\Sigma x))^{-1/2}(\phi^a * g)(x)\big|^2\,d\sigma_{\epsilon t}(x)\,dt\\ &\le C\|u\|^2_{L^\infty(\mathbb{R})}\int_{-1}^{1}\|\phi^a * g\|^2_{L^2(\Sigma_{\epsilon t})^4}\,dt \le C\|u\|^2_{L^\infty(\mathbb{R})}\|g\|^2_{L^2(\mathbb{R}^3)^4}.\end{aligned}$$

That is, if η_0 is small enough there exists $C_1>0$ only depending on η_0 and a such that

$$\|C_\epsilon(a)\|_{L^2(\mathbb{R}^3)^4\to L^2(\Sigma\times(-1,1))^4} \le C_1\|u\|_{L^\infty(\mathbb{R})} \quad \text{for all } 0\le\epsilon\le\eta_0. \tag{4-4}$$

In particular, the boundedness stated in (3-7) holds for $C_0(a)$.

In order to prove the strong convergence of $C_\epsilon(a)$ to $C_0(a)$ when $\epsilon\to 0$, fix $g\in L^2(\mathbb{R}^3)^4$. We must show that, given $\delta>0$, there exists $\epsilon_0>0$ such that

$$\|C_\epsilon(a)g - C_0(a)g\|_{L^2(\Sigma\times(-1,1))^4} \le \delta \quad \text{for all } 0\le\epsilon\le\epsilon_0. \tag{4-5}$$

For every $0<d\le\eta_0$, using (4-4) we can estimate

$$\begin{aligned}&\|C_\epsilon(a)g - C_0(a)g\|_{L^2(\Sigma\times(-1,1))^4}\\ &\le \|C_\epsilon(a)(\chi_{\Omega_d}g)\|_{L^2(\Sigma\times(-1,1))^4} + \|C_0(a)(\chi_{\Omega_d}g)\|_{L^2(\Sigma\times(-1,1))^4} + \|(C_\epsilon(a)-C_0(a))(\chi_{\mathbb{R}^3\setminus\Omega_d}g)\|_{L^2(\Sigma\times(-1,1))^4}\\ &\le 2C_1\|u\|_{L^\infty(\mathbb{R})}\|\chi_{\Omega_d}g\|_{L^2(\mathbb{R}^3)^4} + \|(C_\epsilon(a)-C_0(a))(\chi_{\mathbb{R}^3\setminus\Omega_d}g)\|_{L^2(\Sigma\times(-1,1))^4}.\end{aligned} \tag{4-6}$$

On one hand, since $g\in L^2(\mathbb{R}^3)^4$ and $\mathcal{L}(\Sigma)=0$ ($\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^3$), we can take $d>0$ small enough so that

$$\|\chi_{\Omega_d}g\|_{L^2(\mathbb{R}^3)^4} \le \frac{\delta}{4C_1\|u\|_{L^\infty(\mathbb{R})}}. \tag{4-7}$$

On the other hand, note that

$$|(x_\Sigma+\epsilon t\nu(x_\Sigma)) - x_\Sigma| = \epsilon|t||\nu(x_\Sigma)| \le \epsilon \le \tfrac12 d = \tfrac12\,\operatorname{dist}(\Sigma,\mathbb{R}^3\setminus\Omega_d) \le \tfrac12\,|x_\Sigma - y| \tag{4-8}$$

for all $0\le\epsilon\le\frac12 d,\ t\in(-1,1),\ x_\Sigma\in\Sigma$ and $y\in\mathbb{R}^3\setminus\Omega_d$.

As we said before, we are assuming that $a\in\mathbb{C}\setminus\mathbb{R}$ and, in the definition of ϕ^a, we are taking $\sqrt{m^2-a^2}$ with positive real part, so the components of $\phi^a(x)$ decay exponentially as $|x|\to\infty$. In particular, there exist $C, r>0$ only depending on a such that

$$\begin{aligned}|\partial\phi^a(x)| &\le Ce^{-r|x|} \quad \text{for all } |x|\ge 1,\\ |\partial\phi^a(x)| &\le C|x|^{-3} \quad \text{for all } 0<|x|<1,\end{aligned} \tag{4-9}$$

where by the left-hand side in (4-9) we mean the absolute value of any derivative of any component of the matrix $\phi^a(x)$. Therefore, using the mean value theorem, (4-9) and (4-8), we see that there exists $C_{a,d}>0$

only depending on a and d such that

$$\left|\phi^a(x_\Sigma+\epsilon t\nu(x_\Sigma)-y)-\phi^a(x_\Sigma-y)\right|\leq C_{a,d}\,\frac{\epsilon}{|x_\Sigma-y|^3}$$

for all $0\leq\epsilon\leq\frac{1}{2}d$, $t\in(-1,1)$, $x_\Sigma\in\Sigma$ and $y\in\mathbb{R}^3\setminus\Omega_d$. Hence, we can easily estimate

$$\begin{aligned}
&\left|(C_\epsilon(a)-C_0(a))(\chi_{\mathbb{R}^3\setminus\Omega_d}g)(x_\Sigma,t)\right|\\
&\qquad\leq\|u\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}^3\setminus\Omega_d}\left|\phi^a(x_\Sigma+\epsilon t\nu(x_\Sigma)-y)-\phi^a(x_\Sigma-y)\right||g(y)|\,dy\\
&\qquad\leq C_{a,d}\|u\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}^3\setminus\Omega_d}\frac{\epsilon|g(y)|}{|x_\Sigma-y|^3}\,dy\\
&\qquad\leq C_{a,d}\,\epsilon\|u\|_{L^\infty(\mathbb{R})}\left(\int_{\mathbb{R}^3\setminus B_d(x_\Sigma)}\frac{dy}{|x_\Sigma-y|^6}\right)^{1/2}\|g\|_{L^2(\mathbb{R}^3)^4}\leq C'_{a,d}\,\epsilon\|u\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\mathbb{R}^3)^4},
\end{aligned}$$

where $C'_{a,d}>0$ only depends on a and d. Then,

$$\|(C_\epsilon(a)-C_0(a))(\chi_{\mathbb{R}^3\setminus\Omega_d}g)\|_{L^2(\Sigma\times(-1,1))^4}\leq C'_{a,d}\,\epsilon\|u\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\mathbb{R}^3)^4}\tag{4-10}$$

for a possibly bigger constant $C'_{a,d}>0$.

With these ingredients, the proof of (4-5) is straightforward. Given $\delta>0$, take $d>0$ small enough so that (4-7) holds. For this fixed d, take

$$\epsilon_0=\min\left\{\frac{\delta}{2C'_{a,d}\|u\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\mathbb{R}^3)^4}},\frac{d}{2}\right\}.$$

Then, (4-5) follows from (4-6), (4-7) and (4-10). In conclusion, we have shown that

$$\lim_{\epsilon\to0}\|(C_\epsilon(a)-C_0(a))g\|_{L^2(\Sigma\times(-1,1))^4}=0\quad\text{for all } g\in L^2(\mathbb{R}^3)^4,\tag{4-11}$$

which is (3-11).

5. Proof of (3-10): $B_\epsilon(a)\to B_0(a)+B'$ in the strong sense when $\epsilon\to0$

Recall from (3-3) and (3-8) that $B_\epsilon(a)$ with $0<\epsilon\leq\eta_0$, and $B_0(a)$ and B' are defined by

$$\begin{aligned}
(B_\epsilon(a)g)(x_\Sigma,t)&=u(t)\int_{-1}^1\int_\Sigma\phi^a(x_\Sigma+\epsilon t\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))v(s)\\
&\qquad\qquad\times\det(1-\epsilon sW(y_\Sigma))g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\
(B_0(a)g)(x_\Sigma,t)&=\lim_{\epsilon\to0}u(t)\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|>\epsilon}\phi^a(x_\Sigma-y_\Sigma)v(s)g(y_\Sigma,s)\,ds\,d\sigma(y_\Sigma),\\
(B'g)(x_\Sigma,t)&=(\alpha\cdot\nu(x_\Sigma))\,\tfrac{1}{2}i\,u(t)\int_{-1}^1\operatorname{sign}(t-s)v(s)g(x_\Sigma,s)\,ds.
\end{aligned}$$

We already know that $B_\epsilon(a)$ and B' are bounded in $L^2(\Sigma\times(-1,1))^4$. Let us postpone to Section 5B the proof of the boundedness of $B_0(a)$ stated in (3-7). The first step to proving (3-10) is to decompose ϕ^a as in [Arrizabalaga et al. 2015, Lemma 3.2]; that is,

$$\begin{aligned}\phi^a(x) &= \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}\left(a+m\beta+\sqrt{m^2-a^2}\,i\alpha\cdot\frac{x}{|x|}\right)+\frac{e^{-\sqrt{m^2-a^2}|x|}-1}{4\pi}\,i\alpha\cdot\frac{x}{|x|^3}+\frac{i}{4\pi}\,\alpha\cdot\frac{x}{|x|^3}\\ &=: \omega_1^a(x)+\omega_2^a(x)+\omega_3(x).\end{aligned}\tag{5-1}$$

Then we can write

$$\begin{aligned}B_\epsilon(a) &= B_{\epsilon,\omega_1^a}+B_{\epsilon,\omega_2^a}+B_{\epsilon,\omega_3},\\ B_0(a) &= B_{0,\omega_1^a}+B_{0,\omega_2^a}+B_{0,\omega_3},\end{aligned}\tag{5-2}$$

where B_{ϵ,ω_1^a}, B_{ϵ,ω_2^a} and B_{ϵ,ω_3} are defined as $B_\epsilon(a)$ but replacing ϕ^a by ω_1^a, ω_2^a and ω_3, respectively, and analogously for the case of $B_0(a)$.

For $j=1,2$, we see that $|\omega_j^a(x)|=O(|x|^{-1})$ and $|\partial\omega_j^a(x)|=O(|x|^{-2})$ for $|x|\to 0$, with the understanding that $|\omega_j^a(x)|$ means the absolute value of any component of the matrix $\omega_j^a(x)$ and $|\partial\omega_j^a(x)|$ means the absolute value of any first-order derivative of any component of $\omega_j^a(x)$. Therefore, the integrals defining B_{ϵ,ω_j^a} and B_{0,ω_j^a} are of fractional type for $j=1,2$ (recall Lemma 2.5) and they are taken over bounded sets, so the strong convergence follows by standard methods. However, one can also follow the arguments in the proof of [Behrndt et al. 2017, Lemma 3.4] to show, for $j=1,2$, the convergence of B_{ϵ,ω_j^a} to B_{0,ω_j^a} in the norm sense when $\epsilon\to 0$; that is,

$$\lim_{\epsilon\to 0}\|B_{\epsilon,\omega_j^a}-B_{0,\omega_j^a}\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4}=0\quad\text{for } j=1,2.\tag{5-3}$$

A comment is in order. Since the integrals involved in (5-3) are taken over $\Sigma\times(-1,1)$, which is bounded, the exponential decay at infinity from [Behrndt et al. 2017, Proposition A.1] is not necessary in the setting of (3-10); hence the local estimates of $|\omega_j^a(x)|$ and $|\partial\omega_j^a(x)|$ near the origin are enough to adapt the proof of Lemma 3.4 of the same paper to get (5-3).

Thanks to (5-2) and (5-3), to prove (3-10) we only need to show that $B_{\epsilon,\omega_3}\to B_{0,\omega_3}+B'$ in the strong sense when $\epsilon\to 0$. This will be done in two main steps. First, we will show that

$$\lim_{\epsilon\to 0}B_{\epsilon,\omega_3}g(x_\Sigma,t)=B_{0,\omega_3}g(x_\Sigma,t)+B'g(x_\Sigma,t)\quad\text{for almost all } (x_\Sigma,t)\in\Sigma\times(-1,1)\tag{5-4}$$

and all $g\in L^\infty(\Sigma\times(-1,1))^4$ such that $\sup_{|t|<1}|g(x_\Sigma,t)-g(y_\Sigma,t)|\le C|x_\Sigma-y_\Sigma|$ for all $x_\Sigma,\ y_\Sigma\in\Sigma$ and some $C>0$ which may depend on g. This is done in Section 5A. Then, for a general $g\in L^2(\Sigma\times(-1,1))^4$, we will estimate $|B_{\epsilon,\omega_3}g(x_\Sigma,t)|$ in terms of some bounded maximal operators that will allow us to prove the pointwise limit (5-4) for almost every $(x_\Sigma,t)\in\Sigma\times(-1,1)$ and the desired strong convergence of B_{ϵ,ω_3} to $B_{0,\omega_3}+B'$; see Section 5B.

5A. ***The pointwise limit of $B_{\epsilon,\omega_3}g(x_\Sigma,t)$ when $\epsilon\to 0$ for g in a dense subspace of $L^2(\Sigma\times(-1,1))^4$.*** Observe that the function u in front of the definitions of B_{ϵ,ω_3}, B_{0,ω_3} and B' does not affect the validity of the limit in (5-4), so we can assume without loss of generality that $u\equiv 1$ in $(-1,1)$.

We are going to prove (5-4) by showing the pointwise limit component by component; that is, we are going to work in $L^\infty(\Sigma\times(-1,1))$ instead of $L^\infty(\Sigma\times(-1,1))^4$. In order to do so, we need to introduce some definitions. Set

$$k(x) := \frac{x}{4\pi|x|^3} \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\}. \tag{5-5}$$

Given $t \in (-1, 1)$ and $0 < \epsilon \leq \eta_0$ with η_0 small enough and $f \in L^\infty(\Sigma \times (-1, 1))$ such that

$$\sup_{|t|<1} |f(x_\Sigma, t) - f(y_\Sigma, t)| \leq C|x_\Sigma - y_\Sigma|$$

for all $x_\Sigma,\ y_\Sigma \in \Sigma$ and some $C > 0$, we define

$$T_t^\epsilon f(x_\Sigma) := \int_{-1}^1 \int_\Sigma k(x_\Sigma + \epsilon t \nu(x_\Sigma) - y_\Sigma - \epsilon s \nu(y_\Sigma)) f(y_\Sigma, s) \det(1 - \epsilon s W(y_\Sigma))\, d\sigma(y_\Sigma)\, ds.$$

By (2-4),

$$T_t^\epsilon f(x_\Sigma) = \int_{-1}^1 \int_{\Sigma_{\epsilon s}} k(x_{\epsilon t} - y_{\epsilon s}) f(P_\Sigma y_{\epsilon s}, s)\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds, \tag{5-6}$$

where $x_{\epsilon t} := x_\Sigma + \epsilon t \nu(x_\Sigma)$, $y_{\epsilon s} := y_\Sigma + \epsilon s \nu(y_\Sigma)$ and P_Σ is given by (2-1). We also set

$$T_t f(x_\Sigma) := \lim_{\delta \to 0} \int_{-1}^1 \int_{|x_\Sigma - y_\Sigma| > \delta} k(x_\Sigma - y_\Sigma) f(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds + \tfrac{1}{2}\nu(x_\Sigma) \int_{-1}^1 \operatorname{sign}(t - s) f(x_\Sigma, s)\, ds.$$

We are going to prove that

$$\lim_{\epsilon \to 0} T_t^\epsilon f(x_\Sigma) = T_t f(x_\Sigma) \tag{5-7}$$

for almost all $(x_\Sigma, t) \in \Sigma \times (-1, 1)$. Once this is proved, it is not hard to get (5-4). Indeed, note that $k = (k_1, k_2, k_3)$ with $k_j(x) := x_j/(4\pi|x|^3)$ being the scalar components of the vector kernel $k(x)$. Thus, we can write

$$T_t^\epsilon f(x_\Sigma) = \big((T_t^\epsilon f(x_\Sigma))_1, (T_t^\epsilon f(x_\Sigma))_2, (T_t^\epsilon f(x_\Sigma))_3\big),$$

where each $(T_t^\epsilon f(x_\Sigma))_j$ is defined as in (5-6) but replacing k by k_j. Then, (5-7) holds if and only if $(T_t^\epsilon f(x_\Sigma))_j \to (T_t f(x_\Sigma))_j$ when $\epsilon \to 0$ for $j = 1, 2, 3$. From these limits, if we let $f(y_\Sigma, s)$ in the definitions of $T_t^\epsilon f$ and $T_t f$ be the different components of $v(s) g(y_\Sigma, s)$, we easily deduce (5-4). Thus, we are reduced to proving (5-7).

The proof of (5-7) follows the strategy of the proof of [Hofmann et al. 2010, Proposition 3.30]. Set

$$E(x) := -\frac{1}{4\pi|x|} \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\},$$

the fundamental solution of the Laplace operator in $\mathbb{R}^3$. Note that $\nabla E = k = (k_1, k_2, k_3)$. In particular, if we set $\nu = (\nu_1, \nu_2, \nu_3)$ and $x = (x_1, x_2, x_3)$, for $x \in \mathbb{R}^3$ and $y \in \Sigma$ with $x \neq y$ we have the decomposition

$$\begin{aligned} k_j(x - y) &= \partial_{x_j} E(x - y) = |\nu(y)|^2\, \partial_{x_j} E(x - y) \\ &= \sum_n \nu_n(y)^2 \partial_{x_j} E(x - y) + \sum_n \nu_j(y) \nu_n(y) \partial_{x_n} E(x - y) - \sum_n \nu_j(y) \nu_n(y) \partial_{x_n} E(x - y) \\ &= \nu_j(y) \sum_n \partial_{x_n} E(x - y) \nu_n(y) + \sum_n \big(\nu_n(y) \partial_{x_j} E(x - y) - \nu_j(y) \partial_{x_n} E(x - y)\big) \nu_n(y) \\ &= \nu_j(y) \nabla_{\nu(y)} E(x - y) + \sum_n \nabla_{\nu(y)}^{j,n} E(x - y) \nu_n(y), \end{aligned} \tag{5-8}$$

where we have taken

$$\begin{aligned}\nabla_{\nu(y)}E(x-y) &:= \sum_n \nu_n(y)\partial_{x_n}E(x-y) = \nabla_x E(x-y)\cdot\nu(y),\\ \nabla^{j,n}_{\nu(y)}E(x-y) &:= \nu_n(y)\partial_{x_j}E(x-y) - \nu_j(y)\partial_{x_n}E(x-y).\end{aligned} \tag{5-9}$$

For $j,\ n\in\{1,2,3\}$ we define

$$\begin{aligned}T^\epsilon_\nu f(x_\Sigma,t) &:= \int_{-1}^1\int_{\Sigma_{\epsilon s}} \nabla_{\nu_{\epsilon s}(y_{\epsilon s})}E(x_{\epsilon t}-y_{\epsilon s})f(P_\Sigma y_{\epsilon s},s)\,d\sigma_{\epsilon s}(y_{\epsilon s})\,ds,\\ T^\epsilon_{j,n} f(x_\Sigma,t) &:= \int_{-1}^1\int_{\Sigma_{\epsilon s}} \nabla^{j,n}_{\nu_{\epsilon s}(y_{\epsilon s})}E(x_{\epsilon t}-y_{\epsilon s})f(P_\Sigma y_{\epsilon s},s)\,d\sigma_{\epsilon s}(y_{\epsilon s})\,ds,\end{aligned} \tag{5-10}$$

where $\nu_{\epsilon s}(y_{\epsilon s}) := \nu(y_\Sigma)$ is a normal vector field to $\Sigma_{\epsilon s}$. Additionally, the terms $\nabla_{\nu_{\epsilon s}(y_{\epsilon s})}E(x_{\epsilon t}-y_{\epsilon s})$ and $\nabla^{j,n}_{\nu_{\epsilon s}(y_{\epsilon s})}E(x_{\epsilon t}-y_{\epsilon s})$ in (5-10) are defined as in (5-9) with the obvious replacements.

Given $f\in L^\infty(\Sigma\times(-1,1))$ such that $\sup_{|t|<1}|f(x_\Sigma,t)-f(y_\Sigma,t)|\le C|x_\Sigma-y_\Sigma|$ for all $x_\Sigma,\ y_\Sigma\in\Sigma$ and some $C>0$, by (5-8) we see that

$$(T^\epsilon_t f(x_\Sigma))_j = T^\epsilon_\nu h_j(x_\Sigma,t) + \sum_n T^\epsilon_{j,n}h_n(x_\Sigma,t), \tag{5-11}$$

where $h_n(P_\Sigma y_{\epsilon s},s) := (\nu_{\epsilon s}(y_{\epsilon s}))_n\, f(P_\Sigma y_{\epsilon s},s)$ for $n=1,2,3$. We are going to prove that

$$\begin{aligned}\lim_{\epsilon\to0} T^\epsilon_\nu h_j(x_\Sigma,t) = \lim_{\delta\to0}\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|>\delta} &\nabla_{\nu(y_\Sigma)}E(x_\Sigma-y_\Sigma)h_j(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds\\ &+\frac12\int_{-1}^1 \operatorname{sign}(t-s)h_j(x_\Sigma,s)\,ds,\end{aligned} \tag{5-12}$$

$$\lim_{\epsilon\to0} T^\epsilon_{j,n}h_n(x_\Sigma,t) = \lim_{\delta\to0}\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|>\delta} \nabla^{j,n}_{\nu(y_\Sigma)}E(x_\Sigma-y_\Sigma)h_n(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds \tag{5-13}$$

for $n=1,2,3$. Then, combining (5-11), (5-12) and (5-13), we obtain (5-7). Therefore, it is enough to show (5-12) and (5-13).

We first deal with (5-12). Remember that $\nabla E = k$ so, given $\delta>0$, from (5-9) and (5-10) we can split $T^\epsilon_\nu h_j(x_\Sigma,t)$ as

$$\begin{aligned}T^\epsilon_\nu h_j(x_\Sigma,t) &= \int_{-1}^1\int_{|x_{\epsilon s}-y_{\epsilon s}|>\delta} k(x_{\epsilon t}-y_{\epsilon s})\cdot\nu_{\epsilon s}(y_{\epsilon s})\,h_j(P_\Sigma y_{\epsilon s},s)\,d\sigma_{\epsilon s}(y_{\epsilon s})\,ds\\ &\quad+\int_{-1}^1\int_{|x_{\epsilon s}-y_{\epsilon s}|\le\delta} k(x_{\epsilon t}-y_{\epsilon s})\cdot\nu_{\epsilon s}(y_{\epsilon s})(h_j(P_\Sigma y_{\epsilon s},s)-h_j(P_\Sigma x_{\epsilon s},s))\,d\sigma_{\epsilon s}(y_{\epsilon s})\,ds\\ &\quad+\int_{-1}^1 h_j(P_\Sigma x_{\epsilon s},s)\int_{|x_{\epsilon s}-y_{\epsilon s}|\le\delta} k(x_{\epsilon t}-y_{\epsilon s})\cdot\nu_{\epsilon s}(y_{\epsilon s})\,d\sigma_{\epsilon s}(y_{\epsilon s})\,ds\\ &=: \mathscr{A}_{\epsilon,\delta}+\mathscr{B}_{\epsilon,\delta}+\mathscr{C}_{\epsilon,\delta},\end{aligned}$$

and we easily see that

$$\lim_{\epsilon\to0} T^\epsilon_\nu h_j(x_\Sigma,t) = \lim_{\delta\to0}\lim_{\epsilon\to0}(\mathscr{A}_{\epsilon,\delta}+\mathscr{B}_{\epsilon,\delta}+\mathscr{C}_{\epsilon,\delta}). \tag{5-14}$$

We study the three terms on the right-hand side of (5-14) separately.

For the case of $\mathscr{A}_{\epsilon,\delta}$, note that $k \in C^\infty(\mathbb{R}^3 \setminus B_\delta(0))^3$ and it has polynomial decay at ∞, so

$$|k(x)| + |\partial k(x)| \leq C < +\infty \quad \text{for all } x \in \mathbb{R}^3 \setminus B_\delta(0),$$

where $C > 0$ only depends on δ, and ∂k denotes any first-order derivative of any component of k. Moreover, h_j is bounded on $\Sigma \times (-1, 1)$ and Σ is bounded and of class C^2. Therefore, for a fixed $\delta > 0$, the uniform boundedness of the integrand combined with the regularity of k and Σ and the dominated convergence theorem yields

$$\lim_{\epsilon\to 0} \mathscr{A}_{\epsilon,\delta} = \int_{-1}^{1} \int_{|x_\Sigma - y_\Sigma| > \delta} k(x_\Sigma - y_\Sigma) \cdot \nu(y_\Sigma)\, h_j(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds. \tag{5-15}$$

Then, if we let $\delta \to 0$, from (5-15) we get the first term on the right-hand side of (5-12).

Recall that the function h_j appearing in $\mathscr{B}_{\epsilon,\delta}$ is constructed from the one in (5-4) using υ, see below (5-7), and $\nu_{\epsilon s}$, see below (5-11). Hence $h_j \in L^\infty(\Sigma \times (-1, 1))$ and $\sup_{|t|<1} |h_j(x_\Sigma, t) - h_j(y_\Sigma, t)| \leq C|x_\Sigma - y_\Sigma|$ for all $x_\Sigma,\ y_\Sigma \in \Sigma$ and some $C > 0$. Thus, if η_0 and δ are small enough, by the mean value theorem there exists $C > 0$ such that

$$\bigl|k(x_{\epsilon t} - y_{\epsilon s}) \cdot \nu_{\epsilon s}(y_{\epsilon s})(h_j(P_\Sigma y_{\epsilon s}, s) - h_j(P_\Sigma x_{\epsilon s}, s))\bigr| \leq C \frac{|P_\Sigma y_{\epsilon s} - P_\Sigma x_{\epsilon s}|}{|x_{\epsilon t} - y_{\epsilon s}|^2} \leq \frac{C}{|y_{\epsilon s} - x_{\epsilon s}|} \tag{5-16}$$

for all $0 \leq \epsilon \leq \eta_0$ and $|x_{\epsilon s} - y_{\epsilon s}| \leq \delta$. In the last inequality in (5-16) we used that P_Σ is Lipschitz on Ω_{η_0} and that $|x_{\epsilon s} - y_{\epsilon s}| \leq C|x_{\epsilon t} - y_{\epsilon s}|$ if $|x_{\epsilon s} - y_{\epsilon s}| \leq \delta$ and δ is small enough (due to the regularity of Σ). From the local integrability of the right-hand side of (5-16) with respect to $\sigma_{\epsilon s}$ (see Lemma 2.5) and standard arguments, we easily deduce the existence of $C_\delta > 0$ such that $\sup_{0\leq\epsilon\leq\eta_0} |\mathscr{B}_{\epsilon,\delta}| \leq C_\delta$ and $C_\delta \to 0$ when $\delta \to 0$; see [Behrndt et al. 2017, (A.7)] for a similar argument. Then, we have

$$\bigl|\lim_{\delta\to 0}\lim_{\epsilon\to 0} \mathscr{B}_{\epsilon,\delta}\bigr| \leq \lim_{\delta\to 0} \sup_{0\leq\epsilon\leq\eta_0} |\mathscr{B}_{\epsilon,\delta}| \leq \lim_{\delta\to 0} C_\delta = 0. \tag{5-17}$$

Let us finally focus on $\mathscr{C}_{\epsilon,\delta}$. Since $k = \nabla E$, from (5-9) we get

$$\int_{|x_{\epsilon s} - y_{\epsilon s}| \leq \delta} k(x_{\epsilon t} - y_{\epsilon s}) \cdot \nu_{\epsilon s}(y_{\epsilon s})\, d\sigma_{\epsilon s}(y_{\epsilon s}) = \int_{|x_{\epsilon s} - y_{\epsilon s}| \leq \delta} \nabla_{\nu_{\epsilon s}(y_{\epsilon s})} E(x_{\epsilon t} - y_{\epsilon s})\, d\sigma_{\epsilon s}(y_{\epsilon s}).$$

Consider the set

$$D^\epsilon_\delta(t, s) := \begin{cases} B_\delta(x_{\epsilon s}) \setminus \overline{\Omega(\epsilon, s)} & \text{if } t \leq s, \\ B_\delta(x_{\epsilon s}) \cap \Omega(\epsilon, s) & \text{if } t > s, \end{cases}$$

where $\Omega(\epsilon, s)$ denotes the bounded connected component of $\mathbb{R}^3 \setminus \Sigma_{\epsilon s}$ that contains Ω if $s \geq 0$ and that is included in Ω if $s < 0$.

Set $E_x(y) := E(x - y)$ for $x,\ y \in \mathbb{R}^3$ with $x \neq y$. Then $\Delta E_{x_{\epsilon t}} = 0$ in $D^\epsilon_\delta(t, s)$ and $\nabla E_{x_{\epsilon t}}(y) = -\nabla E(x_{\epsilon t} - y)$. If $\nu_{\partial D^\epsilon_\delta(t,s)}$ denotes the normal vector field on $\partial D^\epsilon_\delta(t, s)$ pointing outside $D^\epsilon_\delta(t, s)$, by the

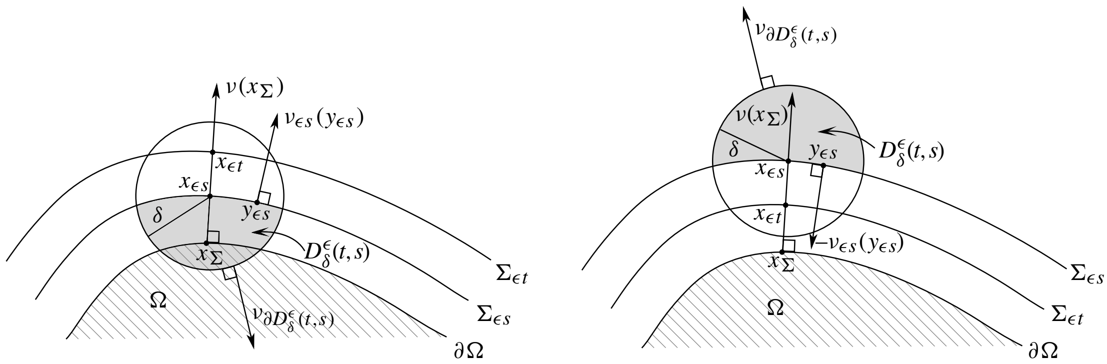

Figure 1. The set $D^\epsilon_\delta(t,s)$ in the case $t > s > 0$ (left) and $s > t > 0$ (right).

divergence theorem,

$$\begin{aligned}0 = \int_{D^\epsilon_\delta(t,s)} \Delta E_{x_{\epsilon t}}(y)\,dy &= -\int_{\partial D^\epsilon_\delta(t,s)} \nabla E(x_{\epsilon t}-y)\cdot \nu_{\partial D^\epsilon_\delta(t,s)}(y)\,d\mathcal{H}^2(y)\\ &= -\operatorname{sign}(t-s)\int_{|x_{\epsilon s}-y_{\epsilon s}|\le\delta} \nabla_{\nu_{\epsilon s}(y_{\epsilon s})} E(x_{\epsilon t}-y_{\epsilon s})\,d\sigma_{\epsilon s}(y_{\epsilon s})\\ &\qquad -\int_{\{y\in\mathbb{R}^3:|x_{\epsilon s}-y|=\delta\}\cap A^\epsilon_{t,s}} \nabla E(x_{\epsilon t}-y)\cdot\frac{y-x_{\epsilon s}}{|y-x_{\epsilon s}|}\,d\mathcal{H}^2(y), \end{aligned}\tag{5-18}$$

where

$$A^\epsilon_{t,s} := \mathbb{R}^3\setminus\overline{\Omega(\epsilon,s)}\quad \text{if } t\le s \qquad\text{and}\qquad A^\epsilon_{t,s} := \Omega(\epsilon,s)\quad\text{if } t>s.$$

Remember also that $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure. Since $\nabla E = k$, from (5-18) and (5-9) we deduce that

$$\int_{|x_{\epsilon s}-y_{\epsilon s}|\le\delta} k(x_{\epsilon t}-y_{\epsilon s})\cdot\nu_{\epsilon s}(y_{\epsilon s})\,d\sigma_{\epsilon s}(y_{\epsilon s}) = \operatorname{sign}(t-s)\int_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}} k(x_{\epsilon t}-y)\cdot\frac{x_{\epsilon s}-y}{|x_{\epsilon s}-y|}\,d\mathcal{H}^2(y).\tag{5-19}$$

Note that $x_{\epsilon t}\notin D^\epsilon_\delta(t,s)$ by construction; see Figure 1. Moreover, by the regularity of Σ, given $\delta>0$ small enough we can find $\epsilon_0>0$ so that $|x_{\epsilon t}-y|\ge\frac{1}{2}\delta$ for all $0<\epsilon\le\epsilon_0$, $s,t\in[-1,1]$ and $y\in\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}$. In particular,

$$|k(x_{\epsilon t}-y)|\le C<+\infty\quad\text{for all } y\in\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s},\tag{5-20}$$

where C only depends on δ and ϵ_0. Then,

$$\begin{aligned}\chi_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}}(y)\,k(x_{\epsilon t}-y)\cdot\frac{x_{\epsilon s}-y}{|x_{\epsilon s}-y|}\,d\mathcal{H}^2(y) &= \chi_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}}(y)\,\frac{x_{\epsilon t}-y}{4\pi|x_{\epsilon t}-y|^3}\cdot\frac{x_{\epsilon s}-y}{|x_{\epsilon s}-y|}\,d\mathcal{H}^2(y)\\ &\to \frac{\chi_{\partial B_\delta(x_\Sigma)\cap D(t,s)}(y)}{4\pi|x_\Sigma-y|^2}\,d\mathcal{H}^2(y)\quad\text{when }\epsilon\to 0,\end{aligned}\tag{5-21}$$

where

$$D(t,s) := \mathbb{R}^3\setminus\overline{\Omega}\quad\text{if } t\le s\qquad\text{and}\qquad D(t,s) := \Omega\quad\text{if } t>s.$$

The limit in (5-21) refers to weak-$*$ convergence of finite Borel measures in $\mathbb{R}^3$ (acting on the variable y). Using (5-21), the uniform estimate (5-20), the boundedness of h_j and the dominated convergence theorem, we see that

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{-1}^{1}\operatorname{sign}(t-s)h_j(x_\Sigma,s)\int_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}} k(x_{\epsilon t}-y)\cdot\frac{x_{\epsilon s}-y}{|x_{\epsilon s}-y|}\,d\mathcal{H}^2(y)\,ds&\\ =\int_{-1}^{1}\operatorname{sign}(t-s)h_j(x_\Sigma,s)\int_{\partial B_\delta(x_\Sigma)\cap D(t,s)}\frac{1}{4\pi|x_\Sigma-y|^2}\,d\mathcal{H}^2(y)\,ds&\\ =\int_{-1}^{1}\operatorname{sign}(t-s)h_j(x_\Sigma,s)\frac{\mathcal{H}^2(\partial B_\delta(x_\Sigma)\cap D(t,s))}{\mathcal{H}^2(\partial B_\delta(x_\Sigma))}\,ds.&\end{aligned}$$

Then, using the regularity of Σ and the dominated convergence theorem once again, we get

$$\begin{aligned}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{-1}^{1}\operatorname{sign}(t-s)h_j(x_\Sigma,s)\int_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}} k(x_{\epsilon t}-y)\cdot\frac{x_{\epsilon s}-y}{|x_{\epsilon s}-y|}\,d\mathcal{H}^2(y)\,ds&\\ =\frac{1}{2}\int_{-1}^{1}\operatorname{sign}(t-s)h_j(x_\Sigma,s)\,ds.&\end{aligned}\tag{5-22}$$

By (5-19), (5-22) and the definition of $\mathscr{C}_{\epsilon,\delta}$ before (5-14), we get

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\mathscr{C}_{\epsilon,\delta}=\frac{1}{2}\int_{-1}^{1}\operatorname{sign}(t-s)h_j(x_\Sigma,s)\,ds.\tag{5-23}$$

The proof of (5-12) is a straightforward combination of (5-14), (5-15), (5-17) and (5-23).

To prove (5-13) we use the same approach as in (5-12); that is, we split $T^\epsilon_{j,n}h_n(x_\Sigma,t)$ as

$$T^\epsilon_{j,n}h_n(x_\Sigma,t)=:\mathscr{A}_{\epsilon,\delta}+\mathscr{B}_{\epsilon,\delta}+\mathscr{C}_{\epsilon,\delta},$$

like above (5-14). The first two terms can be treated analogously and one gets the desired result; the details are left for the reader. To estimate $\mathscr{C}_{\epsilon,\delta}$ we use the notation introduced before. Recall that $E_{x_{\epsilon t}}$ is smooth in $\overline{D^\epsilon_\delta(t,s)}$ (assuming $t\neq s$) and $k(x_{\epsilon t}-y)=\nabla E(x_{\epsilon t}-y)=-\nabla E_{x_{\epsilon t}}(y)$. So, by the divergence theorem, see also (5-9),

$$\begin{aligned}\int_{\partial D^\epsilon_\delta(t,s)}&\nabla^{j,n}_{\nu_{\partial D^\epsilon_\delta(t,s)}(y)}E(x_{\epsilon t}-y)\,d\mathcal{H}^2(y)\\ &=\int_{\partial D^\epsilon_\delta(t,s)}\big((\nu_{\partial D^\epsilon_\delta(t,s)}(y))_n\partial_{x_j}E(x_{\epsilon t}-y)-(\nu_{\partial D^\epsilon_\delta(t,s)}(y))_j\partial_{x_n}E(x_{\epsilon t}-y)\big)\,d\mathcal{H}^2(y)\\ &=\int_{D^\epsilon_\delta(t,s)}(\partial_{y_j}\partial_{y_n}E_{x_{\epsilon t}}-\partial_{y_n}\partial_{y_j}E_{x_{\epsilon t}})(y)\,dy=0.\end{aligned}\tag{5-24}$$

Since $\partial D^\epsilon_\delta(t,s)=(B_\delta(x_{\epsilon s})\cap\Sigma_{\epsilon s})\cup(\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s})$, from (5-24) we have

$$\left|\int_{|x_{\epsilon s}-y_{\epsilon s}|\le\delta}\nabla^{j,n}_{\nu_{\epsilon s}(y_{\epsilon s})}E(x_{\epsilon st}-y_{\epsilon s})\,d\sigma_{\epsilon s}(y_{\epsilon s})\right|=\left|\int_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}}\nabla^{j,n}_{\nu_{\partial D^\epsilon_\delta(t,s)}(y)}E(x_{\epsilon t}-y)\,d\mathcal{H}^2(y)\right|.$$

Observe that

$$\begin{aligned}\chi_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}}(y)\,\nabla^{j,n}_{\nu_{\partial D^\epsilon_\delta(t,s)}(y)}E(x_{\epsilon t}-y)\,d\mathcal{H}^2(y)\\ &= \chi_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}}(y)\big((\nu_{\partial D^\epsilon_\delta(t,s)}(y))_j\,\partial_{y_n}E_{x_{\epsilon t}}(y)-(\nu_{\partial D^\epsilon_\delta(t,s)}(y))_n\,\partial_{y_j}E_{x_{\epsilon t}}(y)\big)\,d\mathcal{H}^2(y)\\ &\to \chi_{\partial B_\delta(x_\Sigma)\cap D(t,s)}(y)\bigg(\frac{(y-x_\Sigma)_j}{|y-x_\Sigma|}\partial_{y_n}E_{x_\Sigma}(y)-\frac{(y-x_\Sigma)_n}{|y-x_\Sigma|}\partial_{y_j}E_{x_\Sigma}(y)\bigg)\,d\mathcal{H}^2(y)=0 \qquad (5\text{-}25)\end{aligned}$$

when $\epsilon\to 0$. The limit measure in (5-25) vanishes because its density function corresponds to a tangential derivative of E_{x_Σ} on $\partial B_\delta(x_\Sigma)$, which is a constant function on $\partial B_\delta(x_\Sigma)$. Therefore, arguing as in the proof of (5-12) but replacing (5-21) by (5-25), we have that, now,

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\mathscr{C}_{\epsilon,\delta}=0.$$

This yields (5-13) and concludes the proof of (5-4).

5B. ***A pointwise estimate of $|B_{\epsilon,\omega_3}g(x_\Sigma,t)|$ by maximal operators.*** We begin this section by setting

$$k(x):=\frac{x_j}{4\pi|x|^3}\quad\text{for } j=1,2,3,\ \ x=(x_1,x_2,x_3)\in\mathbb{R}^3\setminus\{0\}. \qquad (5\text{-}26)$$

In (5-5) we already introduced a kernel k which, in fact, corresponds to the vectorial version of the ones introduced in (5-26). So, by an abuse of notation, throughout this section we mean by $k(x)$ any of the components of the kernel given in (5-5).

Note that $k(-x)=-k(x)$ for all $x\in\mathbb{R}^3\setminus\{0\}$ and, besides, there exists $C>0$ such that

$$\begin{aligned}|k(x-y)|&\le\frac{C}{|x-y|^2}&&\text{for all } x,y\in\mathbb{R}^3 \text{ such that } |x-y|>0,\\ |k(z-y)-k(x-y)|&\le C\frac{|z-x|}{|x-y|^3}&&\text{for all } x,y,z\in\mathbb{R}^3 \text{ with } 0<|z-x|\le\tfrac12|x-y|.\end{aligned} \qquad (5\text{-}27)$$

As in Section 5A, we are going to work componentwise. More precisely, in order to deal with the different components of $B_{\epsilon,\omega_3}g(x_\Sigma,t)$ for $g\in L^2(\Sigma\times(-1,1))^4$, we are going to study the following scalar version. Given $0<\epsilon\le\eta_0$, $g\in L^2(\Sigma\times(-1,1))$ and $(x_\Sigma,t)\in\Sigma\times(-1,1)$, define

$$\begin{aligned}&\widetilde{B}_\epsilon g(x_\Sigma,t)\\ &\quad:=u(t)\int_{-1}^1\int_\Sigma k(x_\Sigma+\epsilon t\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))v(s)\det(1-\epsilon s W(y_\Sigma))g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds, \qquad (5\text{-}28)\end{aligned}$$

where u and v are as in (3-1) for some $0<\eta\le\eta_0$. It is clear that pointwise estimates of $|\widetilde{B}_\epsilon g(x_\Sigma,t)|$ for a given $g\in L^2(\Sigma\times(-1,1))$ directly transfer to pointwise estimates of $|B_{\epsilon,\omega_3}h(x_\Sigma,t)|$ for a given $h\in L^2(\Sigma\times(-1,1))^4$, so we are reduced to estimating $|\widetilde{B}_\epsilon g(x_\Sigma,t)|$ for $g\in L^2(\Sigma\times(-1,1))$.

A key ingredient to finding those suitable pointwise estimates is to relate $\widetilde{B}_\epsilon$ to the Hardy–Littlewood maximal operator and some maximal singular integral operators from Calderón–Zygmund theory. The Hardy–Littlewood maximal operator is given by

$$M_*f(x_\Sigma):=\sup_{\delta>0}\frac{1}{\sigma(B_\delta(x_\Sigma))}\int_{B_\delta(x_\Sigma)}|f|\,d\sigma,\quad M_*:L^2(\Sigma)\to L^2(\Sigma)\text{ bounded}; \qquad (5\text{-}29)$$

see [Mattila 1995, Theorem 2.19] for a proof of the boundedness. The above-mentioned maximal singular integral operators are

$$T_* f(x_\Sigma) := \sup_{\delta>0}\left|\int_{|x_\Sigma-y_\Sigma|>\delta} k(x_\Sigma-y_\Sigma) f(y_\Sigma)\,d\sigma(y_\Sigma)\right|, \quad T_*: L^2(\Sigma)\to L^2(\Sigma) \text{ bounded;} \tag{5-30}$$

see [David 1988, Proposition 4 bis] for a proof of the boundedness. We also introduce some integral versions of these maximal operators to connect them to the space $L^2(\Sigma\times(-1,1))$. Set

$$\begin{aligned}\widetilde{M}_* g(x_\Sigma) &:= \left(\int_{-1}^{1} M_*(g(\,\cdot\,,s))(x_\Sigma)^2\,ds\right)^{1/2}, & \widetilde{M}_*&: L^2(\Sigma\times(-1,1))\to L^2(\Sigma) \text{ bounded,}\\ \widetilde{T}_* g(x_\Sigma) &:= \int_{-1}^{1} T_*(g(\,\cdot\,,s))(x_\Sigma)\,ds, & \widetilde{T}_*&: L^2(\Sigma\times(-1,1))\to L^2(\Sigma) \text{ bounded.}\end{aligned} \tag{5-31}$$

Indeed, by Fubini's theorem and (5-29),

$$\begin{aligned}\|\widetilde{M}_* g\|^2_{L^2(\Sigma)} &= \int_\Sigma\int_{-1}^{1} M_*(g(\,\cdot\,,s))(x_\Sigma)^2\,ds\,d\sigma(x_\Sigma) = \int_{-1}^{1}\|M_*(g(\,\cdot\,,s))\|^2_{L^2(\Sigma)}\,ds\\ &\le C\int_{-1}^{1}\|g(\,\cdot\,,s)\|^2_{L^2(\Sigma)}\,ds = C\|g\|^2_{L^2(\Sigma\times(-1,1))}.\end{aligned}$$

By the Cauchy–Schwarz inequality, Fubini's theorem and (5-30), we also see that $\widetilde{T}_*$ is bounded, so (5-31) is fully justified.

Let us focus for a moment on the boundedness of $B_0(a)$ stated in (3-7). The fact that, for $g\in L^2(\Sigma\times(-1,1))^4$, the limit in the definition of $(B_0(a)g)(x_\Sigma,t)$ exists for almost every $(x_\Sigma,t)\in\Sigma\times(-1,1)$ is a consequence of the decomposition (see (5-1))

$$\phi^a = \omega_1^a + \omega_2^a + \omega_3,$$

the integrals of fractional type on bounded sets in the case of ω_1^a and ω_2^a and, for ω_3, that

$$\lim_{\epsilon\to 0}\int_{|x_\Sigma-y_\Sigma|>\epsilon} k(x_\Sigma-y_\Sigma) f(y_\Sigma)\,d\sigma(y_\Sigma) \quad \text{exists for } \sigma\text{-almost every } x_\Sigma\in\Sigma \tag{5-32}$$

if $f\in L^2(\Sigma)$ (see [Mattila 1995, Theorem 20.27] for a proof) and that

$$\int_{-1}^{1} v(s) g(\,\cdot\,,s)\,ds \in L^2(\Sigma)^4.$$

Of course, (5-32) directly applies to B_{0,ω_3} (see (5-2) for the definition). From the boundedness of $\widetilde{T}_*$ and working component by component, we easily see that B_{0,ω_3} is bounded in $L^2(\Sigma\times(-1,1))^4$. By the comments regarding B_{0,ω_1^a} and B_{0,ω_2^a} from the paragraph which contains (5-3), we also get that $B_0(a)$ is bounded in $L^2(\Sigma\times(-1,1))^4$, which gives (3-7) in this case.

With the maximal operators at hand, we proceed to pointwise estimate $|\widetilde{B}_\epsilon g(x_\Sigma,t)|$ for $g\in L^2(\Sigma\times(-1,1))$. Set

$$g_\epsilon(y_\Sigma,s) := v(s)\det(1-\epsilon s W(y_\Sigma)) g(y_\Sigma,s). \tag{5-33}$$

Then, since the eigenvalues of W are uniformly bounded by Proposition 2.2, there exists $C>0$ only depending on η_0 such that

$$|g_\epsilon(y_\Sigma,s)| \leq C\|v\|_{L^\infty(\mathbb{R})}|g(y_\Sigma,s)| \quad \text{for all } 0<\epsilon\leq\eta_0,\ (y_\Sigma,s)\in\Sigma\times(-1,1). \tag{5-34}$$

Additionally, the regularity and boundedness of Σ imply the existence of $L>0$ such that

$$|\nu(x_\Sigma)-\nu(y_\Sigma)| \leq L|x_\Sigma-y_\Sigma| \quad \text{for all } x_\Sigma, y_\Sigma\in\Sigma. \tag{5-35}$$

We make the following splitting of $\widetilde{B}_\epsilon g(x_\Sigma,t)$ (see (5-28) for the definition):

$$\begin{aligned}
\widetilde{B}_\epsilon g(x_\Sigma,t) = {}& u(t)\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|\leq 4\epsilon|t-s|} k(x_\Sigma+\epsilon t\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds\\
&+u(t)\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|>4\epsilon|t-s|}\big(k(x_\Sigma+\epsilon t\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))\\
&\qquad\qquad -k(x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))\big)g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds\\
&+u(t)\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|>4\epsilon|t-s|}\big(k(x_\Sigma+\epsilon s(\nu(x_\Sigma)-\nu(y_\Sigma))-y_\Sigma)-k(x_\Sigma-y_\Sigma)\big)\\
&\qquad\qquad\times g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds\\
&+u(t)\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|>4\epsilon|t-s|} k(x_\Sigma-y_\Sigma)g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds\\
=:{}& \widetilde{B}_{\epsilon,1}g(x_\Sigma,t)+\widetilde{B}_{\epsilon,2}g(x_\Sigma,t)+\widetilde{B}_{\epsilon,3}g(x_\Sigma,t)+\widetilde{B}_{\epsilon,4}g(x_\Sigma,t).
\end{aligned} \tag{5-36}$$

We are going to estimate the four terms on the right-hand side of (5-36) separately.

Concerning $\widetilde{B}_{\epsilon,1}g(x_\Sigma,t)$, note that

$$\epsilon|t-s| = \operatorname{dist}(x_\Sigma+\epsilon t\nu(x_\Sigma),\Sigma_{\epsilon s}) \leq \big|x_\Sigma+\epsilon t\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))\big|$$

for all $(y_\Sigma,s)\in\Sigma\times(-1,1)$; thus $|k(x_\Sigma+\epsilon t\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))|\leq 1/(\epsilon^2|t-s|^2)$ by (5-27), and then

$$\begin{aligned}
|\widetilde{B}_{\epsilon,1}g(x_\Sigma,t)| &\leq \|u\|_{L^\infty(\mathbb{R})}\int_{-1}^1\frac{1}{\epsilon^2|t-s|^2}\int_{|x_\Sigma-y_\Sigma|\leq 4\epsilon|t-s|}|g_\epsilon(y_\Sigma,s)|\,d\sigma(y_\Sigma)\,ds\\
&\leq C\|u\|_{L^\infty(\mathbb{R})}\int_{-1}^1 M_*(g_\epsilon(\,\cdot\,,s))(x_\Sigma)\,ds \leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma),
\end{aligned} \tag{5-37}$$

where we used the Cauchy–Schwarz inequality and (5-34) in the last inequality above.

For the case of $\widetilde{B}_{\epsilon,2}g(x_\Sigma,t)$, we split the integral over Σ on dyadic annuli as follows. Set

$$N := \left[\left|\log_2\left(\frac{\operatorname{diam}(\Omega_{\eta_0})}{\epsilon|t-s|}\right)\right|\right]+1 \tag{5-38}$$

for $t\neq s$, where $[\,\cdot\,]$ denotes the integer part. Then, $2^N\epsilon|t-s|>\operatorname{diam}(\Omega_{\eta_0})$ and

$$|\widetilde{B}_{\epsilon,2}g(x_\Sigma,t)| \leq \|u\|_{L^\infty(\mathbb{R})}\int_{-1}^1\sum_{n=2}^N\int_{2^{n+1}\epsilon|t-s|\geq|x_\Sigma-y_\Sigma|>2^n\epsilon|t-s|}\cdots\,d\sigma(y_\Sigma)\,ds, \tag{5-39}$$

where "$\cdots$" means $\big|k(x_\Sigma+\epsilon t\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))-k(x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))\big|\,|g_\epsilon(y_\Sigma,s)|$. By (5-35),

$$\begin{aligned}(1-\eta_0 L)|x_\Sigma-y_\Sigma| &\le |x_\Sigma-y_\Sigma|-\eta_0|\nu(x_\Sigma)-\nu(y_\Sigma)|\\ &\le \big|x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma)\big|\\ &\le |x_\Sigma-y_\Sigma|+\eta_0|\nu(x_\Sigma)-\nu(y_\Sigma)|\le(1+\eta_0 L)|x_\Sigma-y_\Sigma|,\end{aligned}$$

thus if we take $\eta_0\le 1/(2L)$ we get

$$\tfrac{1}{2}|x_\Sigma-y_\Sigma|\le\big|x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma)\big|\le 2|x_\Sigma-y_\Sigma|. \tag{5-40}$$

Additionally, for $2^{n+1}\epsilon|t-s|\ge|x_\Sigma-y_\Sigma|>2^n\epsilon|t-s|$, using (5-40) we see that

$$\begin{aligned}\big|x_\Sigma+\epsilon t\nu(x_\Sigma)-(x_\Sigma+\epsilon s\nu(x_\Sigma))\big| &= \epsilon|t-s|<2^{-n}|x_\Sigma-y_\Sigma|\\ &\le 2^{-n+1}\big|x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma)\big|\\ &\le \tfrac{1}{2}\big|x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma)\big|\end{aligned} \tag{5-41}$$

for all $n=2,\dots,N$. Therefore, combining (5-41), (5-27) and (5-40) we finally get

$$\begin{aligned}&\big|k(x_\Sigma+\epsilon t\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))-k(x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))\big|\\ &\qquad\le C\frac{|x_\Sigma+\epsilon t\nu(x_\Sigma)-(x_\Sigma+\epsilon s\nu(x_\Sigma))|}{|x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma)|^3}\le\frac{C\epsilon|t-s|}{|x_\Sigma-y_\Sigma|^3}<\frac{C}{2^{3n}\epsilon^2|t-s|^2}\end{aligned}$$

for all $s,t\in(-1,1)$, $0<\epsilon\le\eta_0$, $n=2,\dots,N$ and $2^{n+1}\epsilon|t-s|\ge|x_\Sigma-y_\Sigma|>2^n\epsilon|t-s|$. Plugging this estimate into (5-39) we obtain

$$\begin{aligned}|\widetilde{B}_{\epsilon,2}g(x_\Sigma,t)| &\le C\|u\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\sum_{n=2}^{N}\int_{2^{n+1}\epsilon|t-s|\ge|x_\Sigma-y_\Sigma|>2^n\epsilon|t-s|}\frac{|g_\epsilon(y_\Sigma,s)|}{2^{3n}\epsilon^2|t-s|^2}\,d\sigma(y_\Sigma)\,ds\\ &\le C\|u\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\sum_{n=2}^{N}\frac{1}{2^n}\int_{|x_\Sigma-y_\Sigma|\le 2^{n+1}\epsilon|t-s|}\frac{|g_\epsilon(y_\Sigma,s)|}{(2^{n+1}\epsilon|t-s|)^2}\,d\sigma(y_\Sigma)\,ds\\ &\le C\|u\|_{L^\infty(\mathbb{R})}\sum_{n=2}^{\infty}\frac{1}{2^n}\int_{-1}^{1}M_*(g_\epsilon(\,\cdot\,,s))(x_\Sigma)\,ds\le C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma),\end{aligned} \tag{5-42}$$

where we used the Cauchy–Schwarz inequality and (5-34) in the last inequality above.

Let us deal now with $\widetilde{B}_{\epsilon,3}g(x_\Sigma,t)$. Since $0<\epsilon\le\eta_0$ and $s\in(-1,1)$, if we take $\eta_0\le 1/(2L)$ as before, from (5-35) we see that

$$\big|(x_\Sigma+\epsilon s(\nu(x_\Sigma)-\nu(y_\Sigma)))-x_\Sigma\big|=\epsilon|s|\,|\nu(x_\Sigma)-\nu(y_\Sigma)|\le\tfrac{1}{2}|x_\Sigma-y_\Sigma|,$$

and then, by (5-27),

$$\big|k(x_\Sigma+\epsilon s(\nu(x_\Sigma)-\nu(y_\Sigma))-y_\Sigma)-k(x_\Sigma-y_\Sigma)\big|\le C\frac{\epsilon|s|\,|\nu(x_\Sigma)-\nu(y_\Sigma)|}{|x_\Sigma-y_\Sigma|^3}\le\frac{C\epsilon}{|x_\Sigma-y_\Sigma|^2}. \tag{5-43}$$

Splitting the integral which defines $\widetilde{B}_{\epsilon,3}g(x_\Sigma,t)$ into dyadic annuli as in (5-39), and using (5-43), (5-34) and (5-38), we get

$$\begin{aligned}|\widetilde{B}_{\epsilon,3}g(x_\Sigma,t)| &\leq C\|u\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\sum_{n=2}^{N}\epsilon\int_{2^{n+1}\epsilon|t-s|\geq|x_\Sigma-y_\Sigma|>2^n\epsilon|t-s|}\frac{|g_\epsilon(y_\Sigma,s)|}{|x_\Sigma-y_\Sigma|^2}\,d\sigma(y_\Sigma)\,ds\\ &\leq C\|u\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\epsilon\sum_{n=2}^{N}M_*(g_\epsilon(\,\cdot\,,s))(x_\Sigma)\,ds\\ &\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\epsilon\left|\log_2\left(\frac{\operatorname{diam}(\Omega_{\eta_0})}{\epsilon|t-s|}\right)\right|M_*(g(\,\cdot\,,s))(x_\Sigma)\,ds.\end{aligned}\tag{5-44}$$

Note that

$$\epsilon\left|\log_2\left(\frac{\operatorname{diam}(\Omega_{\eta_0})}{\epsilon|t-s|}\right)\right|\leq\epsilon\big(C+|\log_2\epsilon|+\big|\log_2|t-s|\big|\big)\leq C\big(1+\big|\log_2|t-s|\big|\big)$$

for all $0<\epsilon\leq\eta_0$, where $C>0$ only depends on η_0. Hence, from (5-44) and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}|\widetilde{B}_{\epsilon,3}g(x_\Sigma,t)| &\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\big(1+\big|\log_2|t-s|\big|\big)M_*(g(\,\cdot\,,s))(x_\Sigma)\,ds\\ &\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\left(\int_{-1}^{1}\big(1+\big|\log_2|t-s|\big|\big)^2\,ds\right)^{1/2}\widetilde{M}_*g(x_\Sigma)\\ &\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma),\end{aligned}\tag{5-45}$$

where we also used that $t\in(-1,1)$, so $\int_{-1}^{1}\big(1+\big|\log_2|t-s|\big|\big)^2\,ds\leq C\big(1+\int_0^2|\log_2 r|^2\,dr\big)<+\infty$, in the last inequality above.

The term $|\widetilde{B}_{\epsilon,4}g(x_\Sigma,t)|$ can be estimated using the maximal operator $\widetilde{T}_*$ as follows. Let $\lambda_1(y_\Sigma)$ and $\lambda_2(y_\Sigma)$ denote the eigenvalues of the Weingarten map $W(y_\Sigma)$. By definition,

$$\begin{aligned}g_\epsilon(y_\Sigma,s)&=v(s)\det(1-\epsilon sW(y_\Sigma))g(y_\Sigma,s)\\ &=v(s)\big(1+\epsilon^2s^2\lambda_1(y_\Sigma)\lambda_2(y_\Sigma)-\epsilon s\lambda_1(y_\Sigma)-\epsilon s\lambda_2(y_\Sigma)\big)g(y_\Sigma,s).\end{aligned}$$

Therefore, the triangle inequality yields

$$\begin{aligned}|\widetilde{B}_{\epsilon,4}g(x_\Sigma,t)| &\leq\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\big(T_*(g(\,\cdot\,,s))(x_\Sigma)+\eta_0^2T_*(\lambda_1\lambda_2g(\,\cdot\,,s))(x_\Sigma)\\ &\qquad\qquad+\eta_0T_*(\lambda_1g(\,\cdot\,,s))(x_\Sigma)+\eta_0T_*(\lambda_2g(\,\cdot\,,s))(x_\Sigma)\big)\,ds\\ &\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\big(\widetilde{T}_*g(x_\Sigma)+\widetilde{T}_*(\lambda_1\lambda_2g)(x_\Sigma)+\widetilde{T}_*(\lambda_1g)(x_\Sigma)+\widetilde{T}_*(\lambda_2g)(x_\Sigma)\big).\end{aligned}\tag{5-46}$$

Combining (5-36), (5-37), (5-42), (5-45) and (5-46) and taking the supremum on ϵ we finally get that

$$\begin{aligned}\sup_{0<\epsilon\leq\eta_0}|\widetilde{B}_\epsilon g(x_\Sigma,t)| &\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\big(\widetilde{M}_*g(x_\Sigma)+\widetilde{T}_*g(x_\Sigma)\\ &\qquad+\widetilde{T}_*(\lambda_1\lambda_2g)(x_\Sigma)+\widetilde{T}_*(\lambda_1g)(x_\Sigma)+\widetilde{T}_*(\lambda_2g)(x_\Sigma)\big),\end{aligned}\tag{5-47}$$

where $C > 0$ only depends on η_0. Define

$$\widetilde{B}_* g(x_\Sigma, t) := \sup_{0<\epsilon\leq\eta_0} |\widetilde{B}_\epsilon g(x_\Sigma, t)| \quad \text{for } (x_\Sigma, t) \in \Sigma \times (-1, 1).$$

Then, from (5-47), the boundedness of $\widetilde{M}_*$ and $\widetilde{T}_*$ from $L^2(\Sigma \times (-1, 1))$ to $L^2(\Sigma)$, see (5-31), and the fact that $\|\lambda_1\|_{L^\infty(\Sigma)}$ and $\|\lambda_2\|_{L^\infty(\Sigma)}$ are finite by Proposition 2.2, we easily conclude that there exists $C > 0$ only depending on η_0 such that

$$\|\widetilde{B}_* g\|_{L^2(\Sigma\times(-1,1))} \leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))}. \tag{5-48}$$

5C. $B_{\epsilon,\omega_3} \to B_{0,\omega_3} + B'$ *in the strong sense when* $\epsilon \to 0$ *and conclusion of the proof of* (3-10). To begin this section, we present a standard result in harmonic analysis about the existence of a limit almost everywhere for a sequence of operators acting on a fixed function and its convergence in the strong sense. General statements can be found in [Duoandikoetxea 2001, Theorem 2.2 and the remark below it] and [Torchinsky 1986, Proposition 6.2], for example. For the sake of completeness, here we present a concrete version with its proof.

Lemma 5.1. *Let $b \in \mathbb{N}$ and (X, μ_X) and (Y, μ_Y) be two Borel measure spaces. Let $\{W_\epsilon\}_{0<\epsilon\leq\eta_0}$ be a family of bounded linear operators from $L^2(\mu_X)^b$ to $L^2(\mu_Y)^b$ such that, if*

$$W_* g(y) := \sup_{0<\epsilon\leq\eta_0} |W_\epsilon g(y)| \quad \text{for } g \in L^2(\mu_X)^b \text{ and } y \in Y,$$

then

$$W_* : L^2(\mu_X)^b \to L^2(\mu_Y)$$

is a bounded sublinear operator. Suppose that for any $g \in S$, where $S \subset L^2(\mu_X)^b$ is a dense subspace, $\lim_{\epsilon\to 0} W_\epsilon g(y)$ exists for μ_Y-a.e. $y \in Y$. Then, for any $g \in L^2(\mu_X)^b$, we know $\lim_{\epsilon\to 0} W_\epsilon g(y)$ exists for μ_Y-a.e. $y \in Y$ and

$$\lim_{\epsilon\to 0}\Big\| W_\epsilon g - \lim_{\delta\to 0} W_\delta g\Big\|_{L^2(\mu_Y)^b} = 0. \tag{5-49}$$

In particular, $\lim_{\epsilon\to 0} W_\epsilon$ defines a bounded operator from $L^2(\mu_X)^b$ to $L^2(\mu_Y)^b$.

Proof. We start by proving that, for any $g \in L^2(\mu_X)^b$, $\lim_{\epsilon\to 0} W_\epsilon g(y)$ exists for μ_Y-a.e. $y \in Y$. Take $g_k \in S$ such that $\|g_k - g\|_{L^2(\mu_X)^b} \to 0$ for $k \to \infty$, and fix $\lambda > 0$. Since $\lim_{\epsilon\to 0} W_\epsilon g_k(y)$ exists for μ_Y-a.e. $y \in Y$, the Chebyshev inequality yields

$$\begin{aligned}
&\mu_Y\Big(\Big\{y \in Y : \Big|\limsup_{\epsilon\to 0} W_\epsilon g(y) - \liminf_{\epsilon\to 0} W_\epsilon g(y)\Big| > \lambda\Big\}\Big)\\
&\qquad\leq \mu_Y\Big(\Big\{y \in Y : \Big|\limsup_{\epsilon\to 0} W_\epsilon (g - g_k)(y)\Big| + \Big|\liminf_{\epsilon\to 0} W_\epsilon (g_k - g)(y)\Big| > \lambda\Big\}\Big)\\
&\qquad\leq \mu_Y(\{y \in Y : 2W_*(g - g_k)(y) > \lambda\})\\
&\qquad\leq \frac{4}{\lambda^2}\, \|W_*(g - g_k)\|^2_{L^2(\mu_Y)}\\
&\qquad\leq \frac{C}{\lambda^2}\, \|g - g_k\|^2_{L^2(\mu_X)^b}.
\end{aligned}$$

Letting $k \to \infty$ we deduce that

$$\mu_Y\Big(\Big\{y \in Y : \big|\limsup_{\epsilon\to 0} W_\epsilon g(y) - \liminf_{\epsilon\to 0} W_\epsilon g(y)\big| > \lambda\Big\}\Big) = 0.$$

Since this holds for all $\lambda > 0$, we finally get that $\lim_{\epsilon\to 0} W_\epsilon g(y)$ exists μ_Y-a.e.

Note that $|W_\epsilon g(y) - \lim_{\delta\to 0} W_\delta g(y)| \le 2W_* g(y)$ and $W_* g \in L^2(\mu_Y)$. Thus, (5-49) follows by the dominated convergence theorem. The last statement in the lemma is also a consequence of the boundedness of W_*. □

Thanks to Lemma 5.1 and the results in Sections 5A and 5B, we are ready to conclude the proof of (3-10). As we said before (5-4), to obtain (3-10) we only need to show that $B_{\epsilon,\omega_3} \to B_{0,\omega_3} + B'$ in the strong sense when $\epsilon \to 0$. From (5-4), we know that

$$\lim_{\epsilon\to 0} B_{\epsilon,\omega_3} g(x_\Sigma, t) = B_{0,\omega_3} g(x_\Sigma, t) + B' g(x_\Sigma, t) \quad \text{for almost all } (x_\Sigma, t) \in \Sigma \times (-1, 1)$$

and all $g \in L^\infty(\Sigma \times (-1, 1))^4$ such that $\sup_{|t|<1} |g(x_\Sigma, t) - g(y_\Sigma, t)| \le C_g |x_\Sigma - y_\Sigma|$ for all $x_\Sigma,\ y_\Sigma \in \Sigma$ and some $C_g > 0$ (it may depend on g). Note also that this set of functions g is dense in $L^2(\Sigma \times (-1, 1))^4$. Additionally, thanks to (5-48) we see that, if $\eta_0 > 0$ is small enough and we set

$$B_{*,\omega_3} g(x_\Sigma, t) := \sup_{0<\epsilon\le\eta_0} |B_{\epsilon,\omega_3} g(x_\Sigma, t)| \quad \text{for } (x_\Sigma, t) \in \Sigma \times (-1, 1),$$

then there exists $C > 0$ only depending on η_0 such that

$$\|B_{*,\omega_3} g\|_{L^2(\Sigma\times(-1,1))} \le C \|u\|_{L^\infty(\mathbb{R})} \|v\|_{L^\infty(\mathbb{R})} \|g\|_{L^2(\Sigma\times(-1,1))^4}. \tag{5-50}$$

Thus, from Lemma 5.1 we get that, for any $g \in L^2(\Sigma \times (-1, 1))^4$, the pointwise limit $\lim_{\epsilon\to 0} B_{\epsilon,\omega_3} g(x_\Sigma, t)$ exists for almost every $(x_\Sigma, t) \in \Sigma \times (-1, 1)$. Recall also that $B_{0,\omega_3} + B'$ is bounded in $L^2(\Sigma \times (-1, 1))^4$ (see the comment before (5-33) for B_{0,ω_3}, the case of B' is trivial), so one can easily adapt the proof of Lemma 5.1 to also show that, for any $g \in L^2(\Sigma \times (-1, 1))^4$,

$$\lim_{\epsilon\to 0} B_{\epsilon,\omega_3} g(x_\Sigma, t) = B_{0,\omega_3} g(x_\Sigma, t) + B' g(x_\Sigma, t) \quad \text{for almost all } (x_\Sigma, t) \in \Sigma \times (-1, 1).$$

Finally, (5-49) in Lemma 5.1 yields

$$\lim_{\epsilon\to 0} \|(B_{\epsilon,\omega_3} - B_{0,\omega_3} - B')g\|_{L^2(\Sigma\times(-1,1))^4} = 0 \quad \text{for all } g \in L^2(\Sigma \times (-1, 1))^4,$$

which is the required strong convergence of B_{ϵ,ω_3} to $B_{0,\omega_3} + B'$. This finishes the proof of (3-10).

6. Proof of (3-9): $A_\epsilon(a) \to A_0(a)$ in the strong sense when $\epsilon \to 0$

Recall from (3-3) and (3-8) that $A_\epsilon(a)$ with $0 < \epsilon \le \eta_0$ and $A_0(a)$ are defined by

$$(A_\epsilon(a)g)(x) = \int_{-1}^{1} \int_\Sigma \phi^a(x - y_\Sigma - \epsilon s \nu(y_\Sigma)) v(s) \det(1 - \epsilon s W(y_\Sigma)) g(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds,$$

$$(A_0(a)g)(x) = \int_{-1}^{1} \int_\Sigma \phi^a(x - y_\Sigma) v(s) g(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds.$$

We already know that $A_\epsilon(a)$ is bounded from $L^2(\Sigma\times(-1,1))^4$ to $L^2(\mathbb{R}^3)^4$. To show the boundedness of $A_0(a)$ (and conclude the proof of (3-7)) just note that, by Fubini's theorem, for every $x\in\mathbb{R}^3\setminus\Sigma$ we have

$$(A_0(a)g)(x)=\int_\Sigma \phi^a(x-y_\Sigma)\biggl(\int_{-1}^1 v(s)g(y_\Sigma,s)\,ds\biggr)\,d\sigma(y_\Sigma),$$

and $\int_{-1}^1 v(s)g(\,\cdot\,,s)\,ds\in L^2(\Sigma)^4$ if $g\in L^2(\Sigma\times(-1,1))^4$. Since $a\in\mathbb{C}\setminus\mathbb{R}$, [Arrizabalaga et al. 2014, Lemma 2.1] shows that $A_0(a)$ is bounded from $L^2(\Sigma\times(-1,1))^4$ to $L^2(\mathbb{R}^3)^4$.

We begin the proof of (3-9) by splitting

$$A_\epsilon(a)g=\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}A_\epsilon(a)g+\chi_{\Omega_{\eta_0}}A_\epsilon(a)g. \tag{6-1}$$

Let us treat first the case of $\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}A_\epsilon(a)$. As we said before, since $a\in\mathbb{C}\setminus\mathbb{R}$, the components of $\phi^a(x)$ decay exponentially when $|x|\to\infty$. In particular, there exist $C,r>0$ only depending on a and η_0 such that

$$|\phi^a(x)|,\,|\partial\phi^a(x)|\le Ce^{-r|x|}\quad\text{for all } |x|\ge\tfrac12\eta_0, \tag{6-2}$$

where the left-hand side of (6-2) means the absolute value of any component of the matrix $\phi^a(x)$ and of any first-order derivative of it, respectively.

Note that $\eta_0=\operatorname{dist}(\mathbb{R}^3\setminus\Omega_{\eta_0},\Sigma)$. Hence, if $x\in\mathbb{R}^3\setminus\Omega_{\eta_0}$, $y_\Sigma\in\Sigma$, $0\le\epsilon\le\frac12\eta_0$ and $s\in(-1,1)$ then, for any $0\le q\le 1$,

$$\begin{aligned}\bigl|q(x-y_\Sigma-\epsilon s\nu(y_\Sigma))+(1-q)(x-y_\Sigma)\bigr|&=|x-y_\Sigma-q\epsilon s\nu(y_\Sigma)|\\&\ge|x-y_\Sigma|-q\epsilon|s|\ge|x-y_\Sigma|-\tfrac12\eta_0\ge\tfrac12|x-y_\Sigma|\ge\tfrac12\eta_0.\end{aligned} \tag{6-3}$$

Thus (6-2) applies to $[x,y_\Sigma]_q:=q(x-y_\Sigma-\epsilon s\nu(y_\Sigma))+(1-q)(x-y_\Sigma)$, and a combination of the mean value theorem and (6-3) gives

$$\bigl|\phi^a(x-y_\Sigma-\epsilon s\nu(y_\Sigma))-\phi^a(x-y_\Sigma)\bigr|\le\epsilon\max_{0\le q\le1}|\partial\phi^a([x,y_\Sigma]_q)|\le C\epsilon e^{-(r/2)|x-y_\Sigma|}. \tag{6-4}$$

Set $\tilde g_\epsilon(y_\Sigma,s):=\det(1-\epsilon sW(y_\Sigma))g(y_\Sigma,s)$. On one hand, from (6-4), Proposition 2.2 and the Cauchy–Schwarz inequality, we get that

$$\begin{aligned}&\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(x)\bigl|(A_\epsilon(a)g)(x)-(A_0(a)g_\epsilon)(x)\bigr|\\&\qquad\le C\|v\|_{L^\infty(\mathbb{R})}\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(x)\int_{-1}^1\int_\Sigma\epsilon e^{-(r/2)|x-y_\Sigma|}|\tilde g_\epsilon(y_\Sigma,s)|\,d\sigma(y_\Sigma)\,ds\\&\qquad\le C\epsilon\|v\|_{L^\infty(\mathbb{R})}\|\tilde g_\epsilon\|_{L^2(\Sigma\times(-1,1))^4}\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(x)\biggl(\int_\Sigma e^{-r|x-y_\Sigma|}\,d\sigma(y_\Sigma)\biggr)^{1/2}\\&\qquad\le C\epsilon\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))^4}\xi(x),\end{aligned}$$

where

$$\xi(x):=\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(x)\biggl(\int_\Sigma e^{-r|x-y_\Sigma|}\,d\sigma(y_\Sigma)\biggr)^{1/2}.$$

Since $\xi \in L^2(\mathbb{R}^3)$ because $\sigma(\Sigma) < +\infty$, we deduce that

$$\|\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(A_\epsilon(a)g - A_0(a)\tilde{g}_\epsilon)\|_{L^2(\mathbb{R}^3)^4} \leq C\epsilon\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))^4}. \tag{6-5}$$

On the other hand, by Proposition 2.2 we have

$$|\tilde{g}_\epsilon(y_\Sigma, s) - g(y_\Sigma, s)| = \big|\det(1-\epsilon s W(y_\Sigma)) - 1\big|\,|g(y_\Sigma, s)| \leq C\epsilon|g(y_\Sigma, s)|.$$

This, together with the fact that $A_0(a)$ is bounded from $L^2(\Sigma\times(-1,1))^4$ to $L^2(\mathbb{R}^3)^4$, see above (6-1), implies

$$\begin{aligned}\|\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}A_0(a)(\tilde{g}_\epsilon - g)\|_{L^2(\mathbb{R}^3)^4} &\leq C\|v\|_{L^\infty(\mathbb{R})}\|\tilde{g}_\epsilon - g\|_{L^2(\Sigma\times(-1,1))^4}\\ &\leq C\epsilon\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))^4}.\end{aligned} \tag{6-6}$$

Using the triangle inequality, (6-5) and (6-6), we finally get that

$$\|\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(A_\epsilon(a) - A_0(a))g\|_{L^2(\mathbb{R}^3)^4} \leq C\epsilon\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))^4} \tag{6-7}$$

for all $0 \leq \epsilon \leq \frac{1}{2}\eta_0$, where $C > 0$ only depends on a and η_0. In particular, this implies

$$\lim_{\epsilon\to 0}\|\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(A_\epsilon(a) - A_0(a))\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\mathbb{R}^3)^4} = 0. \tag{6-8}$$

Let us deal now with $\chi_{\Omega_{\eta_0}}A_\epsilon(a)$. Consider the decomposition of ϕ^a given by (5-1). Then, as in (5-2), we write

$$\begin{aligned}A_\epsilon(a) &= A_{\epsilon,\omega_1^a} + A_{\epsilon,\omega_2^a} + A_{\epsilon,\omega_3},\\ A_0(a) &= A_{0,\omega_1^a} + A_{0,\omega_2^a} + A_{0,\omega_3},\end{aligned} \tag{6-9}$$

where A_{ϵ,ω_1^a}, A_{ϵ,ω_2^a} and A_{ϵ,ω_3} are defined as $A_\epsilon(a)$ but replacing ϕ^a by ω_1^a, ω_2^a and ω_3, respectively, and analogously for the case of $A_0(a)$. For $j = 1, 2$, the arguments used to show (5-3) in the case of B_{ϵ,ω_j^a} also apply to $\chi_{\Omega_{\eta_0}}A_{\epsilon,\omega_j^a}$; thus we now get

$$\lim_{\epsilon\to 0}\|\chi_{\Omega_{\eta_0}}(A_{\epsilon,\omega_j^a} - A_{0,\omega_j^a})\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\mathbb{R}^3)^4} = 0 \quad \text{for } j = 1, 2. \tag{6-10}$$

It only remains to show the strong convergence of $\chi_{\Omega_{\eta_0}}A_{\epsilon,\omega_3}$. This case is treated similarly to what we did in Sections 5A, 5B and 5C, as follows.

6A. *The pointwise limit of $A_{\epsilon,\omega_3}g(x)$ when $\epsilon\to 0$ for $g \in L^2(\Sigma\times(-1,1))^4$.* This case is much easier than the one in Section 5A. For a fixed $x \in \mathbb{R}^3\setminus\Sigma$, we can always find $\delta_x, C_x > 0$ small enough such that

$$|x - y_\Sigma - \epsilon s\nu(y_\Sigma)| \geq C_x \quad \text{for all } y_\Sigma \in \Sigma,\ s \in (-1,1) \text{ and } 0 \leq \epsilon \leq \delta_x.$$

In particular, for a fixed $x \in \mathbb{R}^3\setminus\Sigma$, we have $|\omega_3(x - y_\Sigma - \epsilon s\nu(y_\Sigma))| \leq C$ uniformly on $y_\Sigma \in \Sigma$, $s \in (-1,1)$ and $0 \leq \epsilon \leq \delta_x$, where $C > 0$ depends on x. By Proposition 2.2 and the dominated convergence theorem, given $g \in L^2(\Sigma\times(-1,1))^4$, we have

$$\lim_{\epsilon\to 0}A_{\epsilon,\omega_3}g(x) = A_{0,\omega_3}g(x) \quad \text{for } \mathcal{L}\text{-a.e. } x \in \mathbb{R}^3, \tag{6-11}$$

where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^3$.

6B. ***A pointwise estimate of $\chi_{\Omega_{\eta_0}}(x)|A_{\epsilon,\omega_3}g(x)|$ by maximal operators.*** Given $0 \le \epsilon \le \frac{1}{4}\eta_0$, we divide the study of $\chi_{\Omega_{\eta_0}}(x)A_{\epsilon,\omega_3}g(x)$ into two different cases, i.e., $x \in \Omega_{\eta_0} \setminus \Omega_{4\epsilon}$ and $x \in \Omega_{4\epsilon}$. As we did in Section 5B, we are going to work componentwise; that is, we consider $\mathbb{C}$-valued functions instead of $\mathbb{C}^4$-valued functions. With this in mind, for $g \in L^2(\Sigma \times (-1,1))$ we set

$$\widetilde{A}_\epsilon g(x) := \int_{-1}^{1}\int_\Sigma k(x - y_\Sigma - \epsilon s\nu(y_\Sigma))v(s)\det(1-\epsilon sW(y_\Sigma))g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,$$

where k is given by (5-26).

In what follows, we can always assume that $x \in \mathbb{R}^3 \setminus \Sigma$ because $\mathcal{L}(\Sigma) = 0$. In case that $x \in \Omega_{4\epsilon}$, we can write $x = x_\Sigma + \epsilon t\nu(x_\Sigma)$ for some $t \in (-4,4)$, and then $\widetilde{A}_\epsilon g(x)$ coincides with $\widetilde{B}_\epsilon g(x_\Sigma,t)$, see (5-28), except for the term $u(t)$. Therefore, one can carry out all the arguments involved in the estimate of $\widetilde{B}_\epsilon g(x_\Sigma,t)$, that is, from (5-28) to (5-48), with minor modifications to get the following result: Define

$$\widetilde{A}_* g(x_\Sigma,t) := \sup_{0<\epsilon\le\eta_0/4} |\widetilde{A}_\epsilon g(x_\Sigma + \epsilon t\nu(x_\Sigma))| \quad \text{for } (x_\Sigma,t) \in \Sigma \times (-4,4). \tag{6-12}$$

Then, if η_0 is small enough, there exists $C > 0$ only depending on η_0 such that

$$\Big\|\sup_{|t|<4} \widetilde{A}_* g(\,\cdot\,,t)\Big\|_{L^2(\Sigma)} \le C\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))} \quad \text{for all } g \in L^2(\Sigma\times(-1,1)). \tag{6-13}$$

For the proof of (6-13), a remark is in order. The fact that in the present situation $t \in (-4,4)$ instead of $t \in (-1,1)$, as in the definition of $\widetilde{B}_\epsilon g(x_\Sigma,t)$ in (5-28), only affects the arguments used to get (5-47) at the comment just below (5-45). Now one should use that

$$\int_0^5 |\log_2 r|^2\,dr < +\infty$$

to prove the estimate analogous to (5-45) and to derive the counterpart of (5-47); that is,

$$\widetilde{A}_* g(x_\Sigma,t) \le C\|v\|_{L^\infty(\mathbb{R})}\big(\widetilde{M}_* g(x_\Sigma) + \widetilde{T}_* g(x_\Sigma) + \widetilde{T}_*(\lambda_1\lambda_2 g)(x_\Sigma) + \widetilde{T}_*(\lambda_1 g)(x_\Sigma) + \widetilde{T}_*(\lambda_2 g)(x_\Sigma)\big)$$

for all $(x_\Sigma,t) \in \Sigma \times (-4,4)$, where λ_1 and λ_2 are the eigenvalues of the Weingarten map. Combining this estimate, whose right-hand side is independent of $t \in (-4,4)$, the boundedness of $\widetilde{M}_*$ and $\widetilde{T}_*$ from $L^2(\Sigma \times (-1,1))$ to $L^2(\Sigma)$, see (5-31), and Proposition 2.2, we get (6-13).

Finally, thanks to (6-12), (2-3), Proposition 2.2 and (6-13), for η_0 small enough we conclude

$$\begin{aligned}\Big\|\sup_{0\le\epsilon\le\eta_0/4} \chi_{\Omega_{4\epsilon}}|\widetilde{A}_\epsilon g|\Big\|_{L^2(\mathbb{R}^3)} &\le \Big\|\sup_{|t|<4} \widetilde{A}_* g(P_\Sigma\cdot,t)\Big\|_{L^2(\Omega_{\eta_0})}\\ &\le C\Big\|\sup_{|t|<4} \widetilde{A}_* g(\,\cdot\,,t)\Big\|_{L^2(\Sigma)}\\ &\le C\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))}.\end{aligned} \tag{6-14}$$

We now focus on $\chi_{\Omega_{\eta_0}\setminus\Omega_{4\epsilon}}\widetilde{A}_\epsilon$ for $0 \le \epsilon \le \frac{1}{4}\eta_0$. Similarly to what we did in (5-36), we set

$$g_\epsilon(y_\Sigma,s) := v(s)\det(1-\epsilon sW(y_\Sigma))g(y_\Sigma,s),$$

see (5-33), and we split $\widetilde{A}_\epsilon g(x)$ into $\widetilde{A}_\epsilon g(x)=\widetilde{A}_{\epsilon,1}g(x)+\widetilde{A}_{\epsilon,2}g(x)+\widetilde{A}_{\epsilon,3}g(x)+\widetilde{A}_{\epsilon,4}g(x)$, where

$$\begin{aligned}
\widetilde{A}_{\epsilon,1}g(x)&:=\int_{-1}^{1}\int_{\Sigma}\big(k(x-y_\Sigma-\epsilon s\nu(y_\Sigma))-k(x-y_\Sigma)\big)g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\
\widetilde{A}_{\epsilon,2}g(x)&:=\int_{-1}^{1}\int_{|x_\Sigma-y_\Sigma|\le 4\,\mathrm{dist}(x,\Sigma)}k(x-y_\Sigma)g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\
\widetilde{A}_{\epsilon,3}g(x)&:=\int_{-1}^{1}\int_{|x_\Sigma-y_\Sigma|> 4\,\mathrm{dist}(x,\Sigma)}(k(x-y_\Sigma)-k(x_\Sigma-y_\Sigma))g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\
\widetilde{A}_{\epsilon,4}g(x)&:=\int_{-1}^{1}\int_{|x_\Sigma-y_\Sigma|> 4\,\mathrm{dist}(x,\Sigma)}k(x_\Sigma-y_\Sigma)g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds.
\end{aligned}$$

From now on we assume $x\in\Omega_{\eta_0}\setminus\Omega_{4\epsilon}$ and, as always, $y_\Sigma\in\Sigma$. Note that

$$|(y_\Sigma-\epsilon s\nu(y_\Sigma))-y_\Sigma|\le\epsilon\le\tfrac14\,\mathrm{dist}(x,\Sigma)\le\tfrac14\,|x-y_\Sigma|,$$

so (5-27) gives $|k(x-y_\Sigma-\epsilon s\nu(y_\Sigma))-k(x-y_\Sigma)|\le C\epsilon|x-y_\Sigma|^{-3}$. Furthermore, we have $|x-y_\Sigma|\ge C|x_\Sigma-y_\Sigma|$ for all $y_\Sigma\in\Sigma$ and some $C>0$ only depending on η_0. We can split the integral on Σ, which defines $\widetilde{A}_{\epsilon,1}g(x)$ in dyadic annuli as we did in (5-39), see also (5-42), to obtain

$$\begin{aligned}
|\widetilde{A}_{\epsilon,1}g(x)|&\le C\int_{-1}^{1}\int_{|x_\Sigma-y_\Sigma|<\mathrm{dist}(x,\Sigma)}\frac{\epsilon|g_\epsilon(y_\Sigma,s)|}{\mathrm{dist}(x,\Sigma)^3}\,d\sigma(y_\Sigma)\,ds\\
&\qquad\qquad+C\int_{-1}^{1}\sum_{n=0}^{\infty}\int_{2^n\,\mathrm{dist}(x,\Sigma)<|x_\Sigma-y_\Sigma|\le 2^{n+1}\,\mathrm{dist}(x,\Sigma)}\frac{\epsilon|g_\epsilon(y_\Sigma,s)|}{|x-y_\Sigma|^3}\,d\sigma(y_\Sigma)\,ds\\
&\le C\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma)+C\int_{-1}^{1}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_{|x_\Sigma-y_\Sigma|\le 2^{n+1}\,\mathrm{dist}(x,\Sigma)}\frac{|g_\epsilon(y_\Sigma,s)|}{(2^n\,\mathrm{dist}(x,\Sigma))^2}\,d\sigma(y_\Sigma)\,ds\\
&\le C\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma)+C\sum_{n=0}^{\infty}\frac{1}{2^n}\int_{-1}^{1}M_*(g_\epsilon(\,\cdot\,,s))(x_\Sigma)\,ds\le C\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma).
\end{aligned}\tag{6-15}$$

Using that $|k(x-y_\Sigma)|\le C|x-y_\Sigma|^{-2}\le C\,\mathrm{dist}(x,\Sigma)^{-2}$ by (5-27), it is easy to show that

$$|\widetilde{A}_{\epsilon,2}g(x)|\le C\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma).\tag{6-16}$$

Since $\mathrm{dist}(x,\Sigma)=|x-x_\Sigma|$, the same arguments as in (6-15) yield

$$|\widetilde{A}_{\epsilon,3}g(x)|\le C\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma).\tag{6-17}$$

Finally, the same arguments as in (5-46) show that

$$|\widetilde{A}_{\epsilon,4}g(x)|\le C\|v\|_{L^\infty(\mathbb{R})}\big(\widetilde{T}_*g(x_\Sigma)+\widetilde{T}_*(\lambda_1\lambda_2g)(x_\Sigma)+\widetilde{T}_*(\lambda_1g)(x_\Sigma)+\widetilde{T}_*(\lambda_2g)(x_\Sigma)\big).\tag{6-18}$$

Therefore, thanks to (6-15)–(6-18) we conclude that

$$\begin{aligned}
\sup_{0\le\epsilon\le\eta_0/4}\ &\chi_{\Omega_{\eta_0}\setminus\Omega_{4\epsilon}}(x)|\widetilde{A}_\epsilon g(x)|\\
&\le C\|v\|_{L^\infty(\mathbb{R})}\big(\widetilde{M}_*g(x_\Sigma)+\widetilde{T}_*g(x_\Sigma)+\widetilde{T}_*(\lambda_1\lambda_2g)(x_\Sigma)+\widetilde{T}_*(\lambda_1g)(x_\Sigma)+\widetilde{T}_*(\lambda_2g)(x_\Sigma)\big),
\end{aligned}$$

and then, similarly to what we did in (6-14), a combination of (5-31) and Proposition 2.2 gives

$$\Big\| \sup_{0\leq\epsilon\leq\eta_0/4} \chi_{\Omega_{\eta_0}\setminus\Omega_{4\epsilon}} |\widetilde{A}_\epsilon g| \Big\|_{L^2(\mathbb{R}^3)} \leq C\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))}. \tag{6-19}$$

Finally, combining (6-14) and (6-19) we get that, if $\eta_0 > 0$ is small enough, then

$$\Big\| \sup_{0\leq\epsilon\leq\eta_0/4} \chi_{\Omega_{\eta_0}} |\widetilde{A}_\epsilon g| \Big\|_{L^2(\mathbb{R}^3)} \leq C\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))}, \tag{6-20}$$

where $C > 0$ only depends on η_0.

6C. $A_{\epsilon,\omega_3} \to A_{0,\omega_3}$ *in the strong sense when* $\epsilon \to 0$ *and conclusion of the proof of* (3-9). It only remains to put all the pieces together. Despite that the proof follows more or less the same lines as the one in Section 5C, this case is easier. Namely, now we don't need to appeal to Lemma 5.1 because the dominated convergence theorem suffices (the developments in Section 6A hold for all $g \in L^2(\Sigma\times(-1,1))^4$, not only for a dense subspace like in Section 5A).

Working component by component and using (6-20) we see that, if we set

$$A_{*,\omega_3}g(x) := \sup_{0\leq\epsilon\leq\eta_0/4} |A_{\epsilon,\omega_3}g(x)| \quad \text{for } x \in \mathbb{R}^3\setminus\Sigma,$$

then there exists $C > 0$ only depending on $\eta_0 > 0$ (being η_0 small enough) such that

$$\|\chi_{\Omega_{\eta_0}} A_{*,\omega_3}g\|_{L^2(\mathbb{R}^3)^4} \leq C\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))^4}. \tag{6-21}$$

Moreover, given $g \in L^2(\Sigma\times(-1,1))^4$, in (6-11) we showed that $\lim_{\epsilon\to 0} A_{\epsilon,\omega_3}g(x) = A_{0,\omega_3}g(x)$ for $\mathcal{L}$-a.e. $x \in \mathbb{R}^3$. Thus (6-21) and the dominated convergence theorem show that

$$\lim_{\epsilon\to 0} \|\chi_{\Omega_{\eta_0}}(A_{\epsilon,\omega_3} - A_{0,\omega_3})g\|_{L^2(\mathbb{R}^3)^4} = 0. \tag{6-22}$$

Then, combining (6-1), (6-9), (6-8), (6-10) and (6-22), we conclude that

$$\begin{aligned}
\lim_{\epsilon\to 0}\|(A_\epsilon(a) - A_0(a))g\|^2_{L^2(\mathbb{R}^3)^4} \leq \lim_{\epsilon\to 0}\big(&\|\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(A_\epsilon(a) - A_0(a))g\|^2_{L^2(\mathbb{R}^3)^4}\\
&+ \|\chi_{\Omega_{\eta_0}}(A_{\epsilon,\omega_1^a} - A_{0,\omega_1^a})g\|^2_{L^2(\mathbb{R}^3)^4}\\
&+ \|\chi_{\Omega_{\eta_0}}(A_{\epsilon,\omega_2^a} - A_{0,\omega_2^a})g\|^2_{L^2(\mathbb{R}^3)^4}\\
&+ \|\chi_{\Omega_{\eta_0}}(A_{\epsilon,\omega_3} - A_{0,\omega_3})g\|^2_{L^2(\mathbb{R}^3)^4}\big) = 0
\end{aligned}$$

for all $g \in L^2(\Sigma\times(-1,1))^4$. This is precisely (3-9).

7. Proof of Corollary 3.3

We first prove an auxiliary result.

Lemma 7.1. *Let $a \in \mathbb{C}\setminus\mathbb{R}$ and $\eta_0 > 0$ be such that* (1-2) *holds for all* $0 < \epsilon \leq \eta_0$. *If η_0 is small enough, then for any $0 < \eta \leq \eta_0$ and $V \in L^\infty(\mathbb{R})$ with* $\operatorname{supp} V \subset [-\eta,\eta]$ *we have*

$$\begin{aligned}
&\|A_\epsilon(a)\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\mathbb{R}^3)^4},\\
&\|B_\epsilon(a)\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4},\\
&\|C_\epsilon(a)\|_{L^2(\mathbb{R}^3)^4\to L^2(\Sigma\times(-1,1))^4}
\end{aligned}$$

are uniformly bounded for all $0 \le \epsilon \le \eta_0$, *with bounds that only depend on* a, η_0 *and* V. *Furthermore*, *if* η_0 *is small enough there exists* $\delta > 0$ *only depending on* η_0 *such that*

$$\|B_\epsilon(a)\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} \le \tfrac{1}{3} \tag{7-1}$$

for all $|a| \le 1,\ 0 \le \epsilon \le \eta_0,\ 0 < \eta \le \eta_0$ *and all* (δ, η)*-small* V.

Proof. The first statement in the lemma comes as a byproduct of the developments carried out in Sections 4, 5 and 6; see (4-4) for the case of $C_\epsilon(a)$, (5-50) and the paragraph which contains (5-3) for $B_\epsilon(a)$, and (6-7), (6-10) and (6-21) for $A_\epsilon(a)$. We should stress that these developments are valid for any $V \in L^\infty(\mathbb{R})$ with $\operatorname{supp} V \subset [-\eta, \eta]$, where $0 < \eta \le \eta_0$; hence the (δ, η)-small assumption on V in Theorem 1.2 is only required to prove the explicit bound in the second part of the lemma, which will yield the strong convergence of $(1 + B_\epsilon(a))^{-1}$ and $(\beta + B_\epsilon(a))^{-1}$ to $(1 + B_0(a) + B')^{-1}$ and $(\beta + B_0(a) + B')^{-1}$, respectively, in Corollary 3.3.

Recall the decomposition

$$B_\epsilon(a) = B_{\epsilon,\omega_1^a} + B_{\epsilon,\omega_2^a} + B_{\epsilon,\omega_3} \tag{7-2}$$

given by (5-2). Thanks to (5-50), there exists $C_0 > 0$ only depending on η_0 such that

$$\|B_{\epsilon,\omega_3}\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} \le C_0\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})} \quad \text{for all } 0 < \epsilon \le \eta_0. \tag{7-3}$$

The comments in the paragraph which contains (5-3) and an inspection of the proof of [Behrndt et al. 2017, Lemma 3.4] show that there also exists $C_1 > 0$ only depending on η_0 such that, for any $|a| \le 1$ and $j = 1, 2$,

$$\|B_{\epsilon,\omega_j^a}\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} \le C_1\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})} \quad \text{for all } 0 < \epsilon \le \eta_0. \tag{7-4}$$

Note that the kernel defining B_{ϵ,ω_2^a} is given by

$$\omega_2^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|} - 1}{4\pi}\, i\alpha \cdot \frac{x}{|x|^3},$$

so

$$|\omega_2^a(x)| = O\biggl(\frac{\sqrt{|m^2-a^2|}}{|x|}\biggr) \quad \text{for } |x| \to 0.$$

Therefore, the kernel is of fractional type with respect to σ, but the estimate blows up as $|a| \to \infty$. This is the reason why we restrict ourselves to $|a| \le 1$ in (7-4), where we have a uniform bound with respect to a. However, for proving Theorem 1.2, one fixed $a \in \mathbb{C} \setminus \mathbb{R}$ suffices, say $a = i$; see (3-12) and (3-13).

From (7-2), (7-3) and (7-4), we derive that

$$\|B_\epsilon(a)\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} \le (C_0 + 2C_1)\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})} \quad \text{for all } 0 < \epsilon \le \eta_0. \tag{7-5}$$

If V is (δ, η)-small (see Definition 1.1) then $\|V\|_{L^\infty(\mathbb{R})} \le \delta/\eta$, so (1-5) yields

$$\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})} = \eta\|V\|_{L^\infty(\mathbb{R})} \le \delta.$$

Taking $\delta > 0$ small enough so that $(C_0 + 2C_1)\delta \le \frac{1}{3}$, from (7-5) we finally get (7-1) for all $0 < \epsilon \le \eta_0$. The case of $B_0(a)$ follows similarly, just recall the paragraph previous to (5-33) taking into account that the dependence of the norm of $B_0(a)$ with respect to $\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}$ is the same as in the case of $0 < \epsilon \le \eta_0$. □

7A. ***Proof of Corollary 3.3.*** We are going to prove the corollary for $(H + \boldsymbol{V}_\epsilon - a)^{-1}$; the case of $(H + \beta \boldsymbol{V}_\epsilon - a)^{-1}$ follows by the same arguments. Let $\eta_0,\ \delta > 0$ be as in Lemma 7.1 and take $a \in \mathbb{C} \setminus \mathbb{R}$ with $|a| \le 1$. It is trivial to show that

$$\|B'\|_{L^2(\Sigma\times(-1,1))^4 \to L^2(\Sigma\times(-1,1))^4} \le C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}$$

for some $C > 0$ only depending on Σ. Using (1-5), we can take a smaller $\delta > 0$ so that, for any (δ, η)-small V with $0 < \eta \le \eta_0$,

$$\|B'\|_{L^2(\Sigma\times(-1,1))^4 \to L^2(\Sigma\times(-1,1))^4} \le C\delta \le \tfrac{1}{3}.$$

Then, from this and (7-1) in Lemma 7.1 (with $\epsilon = 0$) we deduce that

$$\begin{aligned}\|(1 + B_0(a) + B')g\|_{L^2(\Sigma\times(-1,1))^4} &\ge \|g\|_{L^2(\Sigma\times(-1,1))^4} - \|(B_0(a) + B')g\|_{L^2(\Sigma\times(-1,1))^4}\\ &\ge \tfrac{1}{3}\|g\|_{L^2(\Sigma\times(-1,1))^4}\end{aligned}$$

for all $g \in L^2(\Sigma \times (-1, 1))^4$. Therefore, $1 + B_0(a) + B'$ is invertible and

$$\|(1 + B_0(a) + B')^{-1}\|_{L^2(\Sigma\times(-1,1))^4 \to L^2(\Sigma\times(-1,1))^4} \le 3.$$

This justifies the last comment in the corollary. Similar considerations also apply to $1 + B_\epsilon(a)$, so in this case we deduce that

$$\|(1 + B_\epsilon(a))^{-1}\|_{L^2(\Sigma\times(-1,1))^4 \to L^2(\Sigma\times(-1,1))^4} \le \tfrac{3}{2} \tag{7-6}$$

for all $0 < \epsilon \le \eta_0$. Note also that

$$(1 + B_\epsilon(a))^{-1} - (1 + B_0(a) + B')^{-1} = (1 + B_\epsilon(a))^{-1}(B_0(a) + B' - B_\epsilon(a))(1 + B_0(a) + B')^{-1}. \tag{7-7}$$

Given $g \in L^2(\Sigma \times (-1, 1))^4$, set $f = (1 + B_0(a) + B')^{-1}g \in L^2(\Sigma \times (-1, 1))^4$. Then, by (7-7) and (7-6), we see that

$$\begin{aligned}&\big\|\big((1 + B_\epsilon(a))^{-1} - (1 + B_0(a) + B')^{-1}\big)g\big\|_{L^2(\Sigma\times(-1,1))^4}\\ &\qquad = \big\|(1 + B_\epsilon(a))^{-1}(B_0(a) + B' - B_\epsilon(a))f\big\|_{L^2(\Sigma\times(-1,1))^4}\\ &\qquad \le \tfrac{3}{2}\big\|(B_0(a) + B' - B_\epsilon(a))f\big\|_{L^2(\Sigma\times(-1,1))^4}.\end{aligned} \tag{7-8}$$

By (3-10) in Theorem 3.2, the right-hand side of (7-8) converges to zero when $\epsilon \to 0$. Therefore, we deduce that $(1 + B_\epsilon(a))^{-1}$ converges strongly to $(1 + B_0(a) + B')^{-1}$ when $\epsilon \to 0$. Since the composition of strongly convergent operators is strongly convergent, using (3-5) and Theorem 3.2, we finally obtain the desired strong convergence

$$(H + \boldsymbol{V}_\epsilon - a)^{-1} \to (H - a)^{-1} + A_0(a)(1 + B_0(a) + B')^{-1}C_0(a) \quad \text{when } \epsilon \to 0.$$

Corollary 3.3 is finally proved.

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DIMENSION-FREE L^p ESTIMATES FOR VECTORS OF RIESZ TRANSFORMS ASSOCIATED WITH ORTHOGONAL EXPANSIONS

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An explicit Bellman function is used to prove a bilinear embedding theorem for operators associated with general multidimensional orthogonal expansions on product spaces. This is then applied to obtain L^p boundedness, $1 < p < \infty$, of appropriate vectorial Riesz transforms, in particular in the case of Jacobi polynomials. Our estimates for the L^p norms of these Riesz transforms are both dimension-free and linear in $\max(p, p/(p-1))$. The approach we present allows us to avoid the use of both differential forms and general spectral multipliers.

1. Introduction

The classical Riesz transforms on $\mathbb{R}^d$ are the operators

$$R_i f(x) = \partial_{x_i}(-\Delta_{\mathbb{R}^d})^{-1/2} f(x), \quad i = 1, \dots, d.$$

E. M. Stein [1983] proved that the vector of Riesz transforms

$$\boldsymbol{R} f = (R_1 f, \dots, R_d f)$$

has L^p bounds which are independent of the dimension. More precisely

$$\|\boldsymbol{R} f\|_{L^p(\mathbb{R}^d)} \leqslant C_p \|f\|_{L^p(\mathbb{R}^d)}, \quad 1 < p < \infty, \tag{1-1}$$

where C_p is independent of the dimension d. Note that (1-1) is formally the same as the a priori bound

$$\big\| |\nabla f| \big\|_{L^p(\mathbb{R}^d)} \leqslant C_p \|(-\Delta)^{1/2} f\|_{L^p(\mathbb{R}^d)}.$$

Later it was realized that, for $1 < p < 2$, one may take $C_p \leqslant C(p-1)^{-1}$ in (1-1); see [Bañuelos 1986; Duoandikoetxea and Rubio de Francia 1985]. It is worth mentioning that the best constant in (1-1) remains unknown when $d \geq 2$; the best results to date are given in [Bañuelos and Wang 1995] (see also [Dragičević and Volberg 2006] for an analytic proof) and [Iwaniec and Martin 1996].

The main goal of this paper is to generalize (1-1) to product settings different from $\mathbb{R}^d = \mathbb{R} \times \cdots \times \mathbb{R}$ with the product Lebesgue measure. Our starting point is the observation that the classical Riesz transform can be written as $R_i = \delta_i \big(\sum_{i=1}^d L_i\big)^{-1/2}$, where $\delta_i = \partial_{x_i}$, and $L_i = \delta_i^* \delta_i$. The generalized Riesz transforms we pursue are of the same form,

$$R_i = \delta_i L^{-1/2}, \quad i = 1, \dots, d, \tag{1-2}$$

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with δ_i being an operator on $L^2(X_i, \mu_i)$,

$$L_i = \delta_i^* \delta_i + a_i \quad \text{and} \quad L = \sum_{i=1}^{d} L_i.$$

Here a_i is a nonnegative constant. The adjoint δ_i^* is taken with respect to the inner product on $L^2(X_i, \mu_i)$, where μ_i is a nonnegative Borel measure on X_i such that $d\mu_i(x_i) = w_i(x_i)\,dx_i$ for some positive and smooth function w_i on X_i. To be precise, if 0 is an L^2 eigenvalue of L, then the definition of R_i needs to be slightly modified; this is properly explained in the next section. Throughout the paper we assume that each X_i, $i = 1, \dots, d$, is an open interval in $\mathbb{R}$, an open half-line in $\mathbb{R}$, or the real line; we also set $X = X_1 \times \cdots \times X_d$ and $\mu = \mu_1 \otimes \cdots \otimes \mu_d$. We consider δ_i given by

$$\delta_i f(x) = p_i(x_i)\,\partial_{x_i} + q_i(x_i), \qquad x_i \in X_i,$$

for some real-valued functions $p_i \in C^\infty(X_i)$ and $q_i \in C^\infty(X_i)$. We remark that a significant difference between the classical Riesz transforms and the general Riesz transforms (1-2) lies in the fact that the operators δ_i and δ_i^* do not need to commute.

There are two assumptions which are critical to our results. Firstly, a computation, see [Nowak and Stempak 2006, p. 683], shows that the commutator $[\delta_i, \delta_i^*]$ is a function which we call v_i. We assume that v_i is nonnegative; see (A1). Secondly, it is not hard to see that $L = \sum_{i=1}^d L_i$ may be written as $L = \widetilde{L} + r$, where $\widetilde{L}$ is a purely differential operator (without a zero-order potential term) and r is the potential term. We impose that $\sum_{i=1}^d q_i^2$ is controlled pointwise from above by a constant times r, namely $\sum_{i=1}^d q_i^2 \le K \cdot r$ for some $K \geqslant 0$; see (A2). In several cases we will consider, we can take $K = 1$ or $K = 0$. In particular if $q_1 = \cdots = q_d = 0$ then the bound (A2) holds with $K = 0$. When 0 is not an L^2 eigenvalue of L, our main result can be summarized as follows.

Main result (informal). *Set $p^* = \max(p, p/(p-1))$. Then the vectorial Riesz transform $\boldsymbol{R} f = (R_1 f, \dots, R_d f)$ with R_i given by* (1-2) *satisfies the bounds*

$$\|\boldsymbol{R} f\|_{L^p(X,\mu)} \leqslant 24(1+\sqrt{K})(p^*-1)\|f\|_{L^p(X,\mu)}, \qquad 1 < p < \infty.$$

In other words, introducing $\delta f = (\delta_1 f, \dots, \delta_d f)$, we have

$$\big\| |\delta f| \big\|_{L^p(X,\mu)} \leqslant 24(1+\sqrt{K})(p^*-1)\|L^{1/2} f\|_{L^p(X,\mu)}, \qquad 1 < p < \infty.$$

The rigorous statement of our main result is contained in Theorem 1. In order to prove it we need some extra technical assumptions. For the sake of clarity of the presentation we decided to concentrate on the case of orthogonal expansions, when each of the operators $L_i = \delta_i^* \delta_i + a_i$ has a decomposition in terms of an orthonormal basis. Our precise setting is described in detail in Section 2. We follow the approach of [Nowak and Stempak 2006]; in fact the present paper may be thought of as an L^p counterpart for a large part of the L^2 results from that paper. Adding the technical assumptions (T1), (T2), and (T3) to the crucial assumptions (A1) and (A2), we state our main result, Theorem 1, in Section 3. In all the cases we will consider, the projection Π appearing in Theorem 1 is the identity operator or has its L^p norm bounded by 2 for all $1 \le p \le \infty$. Moreover, we have $\Pi = I$ if and only if 0 is not an L^2 eigenvalue of L.

From Theorem 1 we obtain several new dimension-free bounds on L^p, $1 < p < \infty$, for vectors of Riesz transforms connected with classical multidimensional orthogonal expansions. For more details we refer to the examples in Section 5. For instance, in Section 5.3 we obtain the dimension-free boundedness for the vector of Riesz transforms in the case of Jacobi polynomial expansions. This answers a question left open in [Nowak and Sjögren 2008]. Moreover, the approach we present gives a unified way to treat dimension-free estimates for vectors of Riesz transforms. In most of the previous cases, separate papers were written for each of the classical orthogonal expansions. More unified approaches were recently presented by Forzani, Sasso, and Scotto in [Forzani et al. 2015] and by the author in [Wróbel 2014]. However, these papers treat only dimension-free estimates for scalar Riesz transforms and not for the vector of Riesz transforms.

Let us remark that Theorem 1 formally cannot be applied to some cases where the crucial assumptions on v_i and r continue to hold. This is true when L has a purely continuous spectrum, for instance for the classical Riesz transforms on $\mathbb{R}^d$ (when $v_i = 0$ and $r = 0$). However, it is not difficult to modify the proof of Theorem 1 so that it remains valid for the classical Riesz transforms. We believe that a similar procedure can be applied to other cases outside the scope of Theorem 1, as long as the crucial assumptions (A1) and (A2) are satisfied.

We deduce Theorem 1 from a bilinear embedding theorem (see Theorem 3) together with a bilinear formula (see Proposition 2). The main tool that is used to prove Theorem 3 is the Bellman function technique. This method was introduced to harmonic analysis by Nazarov, Treil, and Volberg [Nazarov et al. 1999]. Before that paper, Bellman functions appeared implicitly in [Burkholder 1984; 1988; 1991]. The proof of Theorem 3 is presented in Section 4 and is based on subtle properties of a particular Bellman function. This approach was devised by Dragičević and Volberg [2006; 2011; 2012]. Carbonaro and Dragičević [2013; 2016a; 2016b; 2017] developed the method further. The approach from [Carbonaro and Dragičević 2013] was recently adapted by Mauceri and Spinelli [2015] to the case of the Laguerre operator. Our paper generalizes simultaneously [Dragičević and Volberg 2012] (as we admit a nonnegative potential r) and [Dragičević and Volberg 2006; Mauceri and Spinelli 2015] (as we consider general p_i in $\delta_i = p_i\, \partial_{x_i} + q_i$).

In some applications of the Bellman function method, the authors needed to prove dimension-free bounds on L^p for certain spectral multipliers related to the considered operators; see [Dragičević and Volberg 2006; 2012] for such a situation. In other papers mentioned in the previous paragraph they needed to consider operators acting on differential forms; see [Carbonaro and Dragičević 2013; Mauceri and Spinelli 2015]. One of the merits of our approach is that we avoid using both general spectral multipliers and differential forms. This is achieved by means of the bilinear formula from Proposition 2. This formula relates the Riesz transform R_i with an integral where only δ_i and two kinds of semigroups (one for L and one for $L + v_i$) are present; see (3-1).

For the sake of simplicity we use a Bellman function with real entries in Section 4. Thus our main results, Theorems 1 and 3, apply to real-valued functions. Of course they can be easily extended to complex-valued functions with the constants being twice as large. One may improve the estimates further by using a Bellman function with complex arguments, as was done in [Dragičević and Volberg 2006; 2011; 2012].

Notation. We finish this section by introducing the general notation used in the paper. By $\mathbb{N}$ we denote the set of nonnegative integers. For $N \in \mathbb{N}$ and Y being an open subset of $\mathbb{R}^N$, the symbol $C^n(Y)$, $n \in \mathbb{N}$, denotes the space of real-valued functions which have continuous partial derivatives in Y up to the order n. In particular $C^0(Y) = C(Y)$ denotes the space of continuous functions on Y equipped with the supremum norm. By $C^\infty(Y)$ we mean the space of infinitely differentiable functions on Y. Whenever we say that ν is a measure on Y we mean that ν is a Borel measure on Y. The symbols ∇f and $\operatorname{Hess} f$ stand for the gradient and the Hessian of a function $f\colon \mathbb{R}^N \to \mathbb{R}$. For $a, b \in \mathbb{R}^N$, we denote by $\langle a, b\rangle$ the inner product on $\mathbb{R}^N$ and set $|a|^2 = \langle a, a\rangle$. The actual N should be clear from the context (in fact we always have $N \in \{1, d, d+1\}$). For $p \in (1, \infty)$ we set

$$p^* = \max\left(p, \frac{p}{p-1}\right).$$

2. Preliminaries

All the functions we consider are real-valued. Our notations will closely follow that of [Nowak and Stempak 2006].

For $i = 1, \dots, d$, let X_i be the real line $\mathbb{R}$, an open half-line in $\mathbb{R}$ or an open interval in $\mathbb{R}$ of the form

$$X_i = (\sigma_i, \Sigma_i), \quad \text{where } -\infty \leqslant \sigma_i < \Sigma_i \leqslant \infty.$$

Consider the measure spaces $(X_i, \mathcal{B}_i, \mu_i)$, where $\mathcal{B}_i$ denotes the σ-algebra of Borel subsets of X_i and μ_i is a Borel measure on X_i. We impose that $d\mu_i(x_i) = w_i(x_i)\, dx_i$, where w_i is a positive C^∞ function on X_i. Note that in [Nowak and Stempak 2006] the authors assumed that $X_1 = \cdots = X_d$; this is, however, not needed in our paper. Throughout the article we let

$$X = X_1 \times \cdots \times X_d, \quad \mu = \mu_1 \otimes \cdots \otimes \mu_d,$$

and abbreviate

$$L^p := L^p(X, \mu), \quad \|\cdot\|_p = \|\cdot\|_{L^p}, \quad \text{and} \quad \|\cdot\|_{p\to p} = \|\cdot\|_{L^p \to L^p}.$$

This notation is also used for vector-valued functions. Namely, if $\boldsymbol{g} = (g_1, \dots, g_N)\colon X \to \mathbb{R}^N$ for some $N \in \mathbb{N}$, then

$$\|\boldsymbol{g}\|_p = \left(\int_X |\boldsymbol{g}(x)|^p \, d\mu(x)\right)^{1/p}, \quad \text{with } |\boldsymbol{g}(x)| = \left(\sum_{i=1}^N |g_i(x)|^2\right)^{1/2}.$$

We shall also write $\langle f, g\rangle_{L^2}$ for $\langle f, g\rangle_{L^2(X,\mu)}$.

Let δ_i, $i = 1, \dots, d$, be the operators acting on $C_c^\infty(X_i)$ functions via

$$\delta_i = p_i\, \partial_{x_i} + q_i.$$

Here p_i and q_i are real-valued functions on X_i, with $p_i, q_i \in C^\infty(X_i)$. We assume that $p_i(x_i) \neq 0$ for $x_i \in X_i$. We shall also denote by p and q the exponents of L^p and L^q spaces. This will not lead to any confusion as the functions p_i and q_i will always appear with the index $i = 1, \dots, d$.

Let δ_i^* be the formal adjoint of δ_i with respect to the inner product on $L^2(X_i,\mu_i)$; i.e.,

$$\delta_i^* f = -\frac{1}{w_i}\,\partial_{x_i}(p_i w_i f) + q_i f, \qquad f \in C_c^\infty(X_i).$$

A simple calculation, see [Nowak and Stempak 2006, p. 683], shows that the commutator

$$[\delta_i,\delta_i^*] = \delta_i\delta_i^* - \delta_i^*\delta_i = p_i\left(2q_i' - \left(p_i\frac{w_i'}{w_i}\right)' - p_i''\right) =: v_i \tag{2-1}$$

is a locally integrable function (0-order operator). Most of the assumptions made in this section are of a technical nature. The first of the two assumptions that are crucial to our results is

$$\textit{the functions } v_i,\ i=1,\dots,d,\ \textit{are nonnegative}. \tag{A1}$$

The property (A1) has been (explicitly or implicitly) instrumental for establishing the main results in [Harboure et al. 2004; Mauceri and Spinelli 2015; Nowak and Sjögren 2008; Stempak and Wróbel 2013]. It is also explicitly stated by Forzani, Sasso, and Scotto as Assumption H1(c) in [Forzani et al. 2015].

For a scalar $a_i \ge 0$ we let L_i and L be given on $C_c^\infty(X)$ by

$$L_i := \delta_i^*\delta_i + a_i, \qquad L = \sum_{i=1}^d L_i.$$

Here each L_i can be considered to act either on $C_c^\infty(X_i)$ or on $C_c^\infty(X)$; thus the definition of L makes sense. Note that both L_i and L are symmetric on $C_c^\infty(X)$ with respect to the inner product on L^2. We assume that for each $i=1,\dots,d$, there is an orthonormal basis $\{\varphi^i_{k_i}\}_{k_i\in\mathbb{N}}$ which consists of $L^2(X_i,\mu_i)$ eigenvectors of L_i that correspond to nonnegative eigenvalues $\{\lambda^i_{k_i}\}_{k_i\in\mathbb{N}}$; i.e.,

$$L_i\varphi^i_{k_i} = \lambda^i_{k_i}\varphi^i_{k_i}.$$

Then, it must be that $\lambda_{k_i} \ge a_i$ for $k_i \in \mathbb{N}$ and $i=1,\dots,d$. We require that the sequence $\{\lambda^i_{k_i}\}_{k_i\in\mathbb{N}}$ is strictly increasing and that $\lim_{k_i\to\infty}\lambda^i_{k_i} = \infty$. Note that our assumptions on p_i, q_i, and w_i imply that L_i is hypoelliptic. Therefore we have $\varphi^i_{k_i} \in C^\infty(X_i)$. Setting, for $k=(k_1,\dots,k_d)\in\mathbb{N}^d$,

$$\varphi_k = \varphi^1_{k_1}\otimes\cdots\otimes\varphi^d_{k_d}, \tag{2-2}$$

we obtain an orthonormal basis of eigenvectors on L^2 for the operator $L = L_1+\cdots+L_d$. The eigenvalue corresponding to φ_k is

$$\lambda_k := \lambda^1_{k_1}+\cdots+\lambda^d_{k_d},$$

so that $L\varphi_k = \lambda_k\varphi_k$. We consider the self-adjoint extension of L (still denoted by the same symbol) given by

$$Lf = \sum_{k\in\mathbb{N}^d}\lambda_k\,\langle f,\varphi_k\rangle_{L^2}\varphi_k$$

on the domain

$$\operatorname{Dom}(L) = \left\{ f\in L^2\colon \sum_{k\in\mathbb{N}^d}|\lambda_k|^2|\langle f,\varphi_k\rangle_{L^2}|^2 < \infty \right\}.$$

We assume that the eigenfunctions $\varphi^i_{k_i}$, $i = 1, \ldots, d$, are such that

$$\langle \delta_i \varphi^i_{k_i}, \delta_i \varphi^i_{m_i} \rangle_{L^2(X_i,\mu_i)} = \langle \delta_i^* \delta_i \, \varphi^i_{k_i}, \varphi^i_{m_i} \rangle_{L^2(X_i,\mu_i)} \tag{T1}$$

for $i = 1, \ldots, d$, and $k_i, m_i \in \mathbb{N}$; see [Nowak and Stempak 2006, (2.8)]. The condition (T1) implies that the functions

$$\delta_i \varphi_k = \varphi^1_{k_1} \otimes \cdots \otimes \delta_i \varphi^i_{k_i} \otimes \cdots \otimes \varphi^d_{k_d} \tag{2-3}$$

are pairwise orthogonal on L^2 and

$$\langle \delta_i \varphi_k, \delta_i \varphi_k \rangle_{L^2} = \lambda^i_{k_i} - a_i;$$

see [Nowak and Stempak 2006, Lemmas 5 and 6]. Moreover, since $\varphi_k \in C^\infty(X)$, we also see that $\delta_i \varphi_k \in C^\infty(X)$.

We remark that our assumptions differ slightly from those in [Nowak and Stempak 2006]. Namely, we assume that the coefficients p_i, q_i, and the weight w_i are C^∞ functions, whereas they considered p_i, q_i, w_i that possessed only a finite order of smoothness. The smoothness of these functions is in fact needed to easily conclude that L_i is hypoelliptic and that $\varphi_k \in C^\infty(X)$, which is an issue that was overlooked† in [Nowak and Stempak 2006].

We also impose a boundary condition on the functions $\varphi^i_{k_i}$ and $\delta_i \varphi^i_{k_i}$. Namely, we require that for each $i = 1, \ldots, d$, if $z_i \in \{\sigma_i, \Sigma_i\}$, then

$$\begin{aligned} &\lim_{x_i \to z_i} \big[(1 + |\varphi^i_{k_i}|^{s_1} + |\delta_i \varphi^i_{k_i}|^{s_2})(p_i^2 \, w_i \, \partial_{x_i} \varphi^i_{k_i})\big](x_i) = 0, \\ &\lim_{x_i \to z_i} \big[(1 + |\varphi^i_{k_i}|^{s_1} + |\delta_i \varphi^i_{k_i}|^{s_2})(p_i^2 \, w_i \, \partial_{x_i} \delta_i \varphi^i_{k_i})\big](x_i) = 0 \end{aligned} \tag{T2}$$

for all $k_i \in \mathbb{N}$ and $s_1, s_2 > 0$. Condition (T2) is close to Assumption H1(a) from [Forzani et al. 2015]. Observe that the term $|\varphi^i_{k_i}|^{s_1} + |\delta_i \varphi^i_{k_i}|^{s_2}$ in (T2) is significant only when the functions $\varphi^i_{k_i}$ and $\delta_i \varphi^i_{k_i}$ are unbounded on X_i.

Let

$$A = a_1 + \cdots + a_d, \qquad \Lambda_0 = \lambda^1_0 + \cdots + \lambda^d_0.$$

Then Λ_0 is the smallest eigenvalue of L. We set

$$\mathbb{N}^d_\Lambda = \begin{cases} \mathbb{N}^d, & \Lambda_0 > 0, \\ \mathbb{N}^d \setminus \{(0, \ldots, 0)\}, & \Lambda_0 = 0 \end{cases}$$

and define

$$\Pi f = \sum_{k \in \mathbb{N}^d_\Lambda} \langle f, \varphi_k \rangle_{L^2} \varphi_k.$$

Then in the case $\Lambda_0 > 0$ we have $\Pi = I$, while in the case $\Lambda_0 = 0$ the operator Π is the projection onto the orthogonal complement of the vector $\varphi_{(0,\ldots,0)}$. The Riesz transforms studied in this paper are formally

†The hypoellipticity of L_i is not necessary for the theory from [Nowak and Stempak 2006] to work (Nowak, personal communication, 2017). When not having this property, one has to add instead some extra assumptions (much weaker than smoothness) on the regularity of the eigenfunctions φ_k.

of the form

$$R_i := \delta_i L^{-1/2}\Pi,$$

while the rigorous definition of R_i is

$$R_i f = \sum_{k\in\mathbb{N}^d_\Lambda} \lambda_k^{-1/2}\langle f, \varphi_k\rangle_{L^2}\,\delta_i\varphi_k.$$

In many of the considered cases, $\Pi \equiv I$ so that $R_i = \delta_i L^{-1/2}$.

It was proved in [Nowak and Stempak 2006, Proposition 1] that the vector of Riesz transforms

$$\boldsymbol{R} f = (R_1 f, \dots, R_d f)$$

satisfies

$$\|\boldsymbol{R} f\|_{2\to 2} \leqslant \|f\|_2.$$

The main goal of this paper is to prove similar estimates for p in place of 2. We aim for these estimates to be dimension-free and linear in p^*. More precisely, we shall prove that for $1 < p < \infty$ it holds that

$$\|\boldsymbol{R} f\|_{p\to p} \leqslant C(p^*-1)\|f\|_p.$$

Here C is a constant that is independent of both p and the dimension d.

To state and prove our main results we need several auxiliary objects. Firstly, we let

$$\mathfrak{d}_i = p_i\,\partial_{x_i}. \tag{2-4}$$

That is, $\mathfrak{d}_i$ is the "differential" part of δ_i. In many (though not all) of our applications we will have $q_i \equiv 0$ and thus $\delta_i \equiv \mathfrak{d}_i$. The formal adjoint of $\mathfrak{d}_i$ on $L^2(X_i, \mu_i)$ is

$$\mathfrak{d}_i^* f = -\frac{1}{w_i}\,\partial_{x_i}(p_i w_i f), \qquad f \in C_c^\infty(X_i). \tag{2-5}$$

A computation shows that $L_i = \mathfrak{d}_i^*\mathfrak{d}_i + r_i$, with

$$r_i = a_i + \left(q_i^2 - p_i q_i' - p_i' q_i - p_i q_i \frac{w_i'}{w_i}\right). \tag{2-6}$$

We shall also need

$$\widetilde{L} := \sum_{i=1}^d \mathfrak{d}_i^*\mathfrak{d}_i = L - r, \qquad \text{where } r := \sum_{i=1}^d r_i.$$

Then $\widetilde{L}$ is the potential-free component of L and the potential r is a locally integrable function on X. We assume that

$$\textit{there is a constant } K \geqslant 0 \textit{ such that } \sum_{i=1}^d q_i^2(x_i) \le K\cdot r(x) \tag{A2}$$

for all $x \in X$. This is our second (and last) crucial assumption. In many of our examples we shall have $q_1 = \dots = q_d = 0$ and thus $r = A$ and (A2) holding with $K = 0$.

Next we define

$$M_i := \sum_{j\neq i} \delta_j^*\delta_j + \delta_i\delta_i^* = L + [\delta_i, \delta_i^*] = L + v_i,$$

see [Nowak and Stempak 2006, (5.1)], and set

$$c_k^i = \|\delta_i \varphi_k\|_2^{-1}$$

if $\delta_i\varphi_k \neq 0$ and $c_k^i = 0$ in the other case. Then $\{c_k^i \delta_i \varphi_k\}_{k\in\mathbb{N}^d}$ (excluding those $c_k^i\delta_i\varphi_k$ which vanish) is an orthonormal system of eigenvectors of M_i such that $M_i(c_k^i\delta_i\varphi_k)$ equals $\lambda_k c_k^i \delta_i\varphi_k$.

We define

$$\mathcal{D} = \operatorname{lin}\{\varphi_k : k \in \mathbb{N}^d\}, \quad \mathcal{D}_i = \delta_i[\mathcal{D}] = \operatorname{lin}\{\delta_i\varphi_k : k \in \mathbb{N}^d\},$$

and make the technical assumption that

$$\text{both } \mathcal{D} \text{ and } \mathcal{D}_i,\ i = 1, \dots, d, \text{ are dense subspaces of } L^p,\ 1 \leq p < \infty. \tag{T3}$$

In most of our applications the condition (T3) will follow from [Forzani et al. 2015, Lemma 7.5], which is itself a consequence of [Berg and Christensen 1981, Theorem 5].

Lemma 1 [Forzani et al. 2015, Lemma 7.5]. *Assume that ν is a measure on X such that, for some $\varepsilon > 0$, we have*

$$\int_X \exp\biggl(\varepsilon \sum_{i=1}^d |y_i|\biggr)\, d\nu(y) < \infty.$$

Then, for each $1 \leqslant p < \infty$, multivariable polynomials on X are dense in $L^p(X, \nu)$.

In what follows we consider the self-adjoint extension of M_i given by

$$M_i f = \sum_{k\in\mathbb{N}^d} \lambda_k \langle f, c_k^i\, \delta_i\varphi_k\rangle_{L^2} c_k^i\, \delta_i\varphi_k,$$

on the domain

$$\operatorname{Dom}(M_i) = \biggl\{ f \in L^2 \colon \sum_{k\in\mathbb{N}^d} |\lambda_k|^2 |\langle f, c_k^i \delta_i\varphi_k\rangle_{L^2}|^2 < \infty \biggr\}.$$

Keeping the symbol M_i for this self-adjoint extension is a slight abuse of notation, which however will not lead to any confusion. Finally, we shall need the semigroups

$$P_t := e^{-tL^{1/2}} \quad \text{and} \quad Q_t^i := e^{-tM_i^{1/2}}.$$

These are formally defined on L^2 as

$$P_t f = \sum_{k\in\mathbb{N}^d} e^{-t\lambda_k^{1/2}} \langle f, \varphi_k\rangle_{L^2}\varphi_k, \quad Q_t^i f = \sum_{k\in\mathbb{N}^d} e^{-t\lambda_k^{1/2}} \langle f, c_k^i\, \delta_i\varphi_k\rangle_{L^2} c_k^i\, \delta_i\varphi_k.$$

Note that for $t > 0$ we have $P_t[\mathcal{D}] \subseteq \mathcal{D}$ and $Q_t^i[\mathcal{D}_i] \subseteq \mathcal{D}_i,\ i = 1, \dots, d$.

3. General results for Riesz transforms

Recall that we are in the setting of the previous section. In particular the assumptions (A1), (A2), and the technical assumptions (T1), (T2), (T3), are in force. The following is the main result of our paper.

Theorem 1. *For each* $1 < p < \infty$ *we have*

$$\|\boldsymbol{R} f\|_p \leqslant 24(1+\sqrt{K})(p^*-1)\|\Pi f\|_{L^p}, \quad f \in L^p.$$

Remark. In all the examples we consider in Section 5 the projection Π satisfies $\|\Pi\|_{p\to p} \leqslant 2,\ 1 \leqslant p \leqslant \infty$. In fact in many of the examples Π equals the identity operator.

In order to prove Theorem 1 we need two ingredients. The first of these ingredients is a bilinear formula that relates the Riesz transform with an integral in which both P_t and Q_t^i are present.

Proposition 2. *Let* $i = 1, \ldots, d$. *Then the formula*

$$\langle R_i f, g\rangle_{L^2} = -4\int_0^\infty \langle \delta_i P_t \Pi f, \partial_t Q_t^i g\rangle_{L^2}\, t\, dt \tag{3-1}$$

holds for $f \in \mathcal{D}$ *and* $g \in \mathcal{D}_i$.

Before proving the proposition let us make two remarks.

Remark 1. Formulas similar to (3-1) were proved before, though, depending on the context, they may have involved spectral multipliers of the operator L. However, treating these spectral multipliers appropriately was achieved with variable success. A way of avoiding multipliers was first devised in [Carbonaro and Dragičević 2013] for Riesz transforms on manifolds. In such a setting, the above formula is a special case of the identity (3) there. The approach in [Carbonaro and Dragičević 2013] was adapted in [Mauceri and Spinelli 2015] to the case of Hodge–Laguerre operators. In the case of Laguerre polynomial expansions (see Section 5.2) the formula (3-1) is a special case of [Mauceri and Spinelli 2015, (5.1)]. We note that both in [Carbonaro and Dragičević 2013] and [Mauceri and Spinelli 2015] the authors needed to consider the Riesz transform as well as the formula (3-1) for differential forms; this is not needed in our approach.

Remark 2. Note that if the operators δ_i and δ_i^* commute, then $Q_t^i = P_t$ and the formula (3-1) can be formally obtained via the spectral theorem. The problem is that often these operators do not commute. A way to overcome this noncommutativity problem was devised by Nowak and Stempak [2013]. They introduced a symmetrization T_i of δ_i that does commute with its adjoint; in fact $T_i^* = -T_i$. This symmetrization is defined on $L^2(\widetilde{X})$, where

$$\widetilde{X} = (X_1 \cup (-X_1)) \times \cdots \times (X_d \cup (-X_d)).$$

Set $T = -\sum_{i=1}^d T_i^2$ and let $S_t = e^{-tT^{1/2}}$. The formula (3-1) for T_i is then formally

$$\langle T_i T^{-1/2} f, g\rangle_{L^2(\widetilde{X})} = -4\int_0^\infty \langle T_i S_t f, \partial_t S_t g\rangle_{L^2(\widetilde{X})}\, t\, dt. \tag{3-2}$$

This leads to a proof of (3-1) different from the one presented in our paper. Namely, a computation shows that applying (3-2) to functions $f : \widetilde{X} \to \mathbb{R}$ and $g : \widetilde{X} \to \mathbb{R}$, which are both even in all the variables, we arrive at (3-1).

Proof of Proposition 2. We start with proving (3-1) for $f=\varphi_k$ and $g=\delta_i\varphi_n$, with some $k,n\in\mathbb{N}^d$. If $k=0$ and $\Lambda_0=0$ then both sides of (3-1) vanish. Thus we can assume that $\lambda_k>0$. A computation shows that

$$\langle \delta_i L^{-1/2} f, g\rangle_{L^2}=\lambda_k^{-1/2}\langle \delta_i f, g\rangle_{L^2}$$

and

$$\begin{aligned}-4\int_0^\infty \langle \delta_i P_t f, \partial_t Q_t^i g\rangle_{L^2}\, t\,dt &= -4\int_0^\infty \big\langle e^{-t\lambda_k^{1/2}}\delta_i f, -\lambda_n^{1/2}e^{-t\lambda_n^{1/2}} g\big\rangle_{L^2}\, t\,dt\\ &= 4\lambda_n^{1/2}\int_0^\infty e^{-t(\lambda_k^{1/2}+\lambda_n^{1/2})}\, t\,dt\cdot \langle \delta_i f, g\rangle_{L^2} = \frac{4\lambda_n^{1/2}}{(\lambda_k^{1/2}+\lambda_n^{1/2})^2}\cdot\langle \delta_i f, g\rangle_{L^2};\end{aligned}$$

hence

$$\langle \delta_i L^{-1/2} f, g\rangle + 4\int_0^\infty \langle \delta_i P_t f, \partial_t Q_t^i g\rangle_{L^2}\, t\,dt = \left(\lambda_k^{-1/2}-\frac{4\lambda_n^{1/2}}{(\lambda_k^{1/2}+\lambda_n^{1/2})^2}\right)\cdot\langle \delta_i f, g\rangle_{L^2}. \tag{3-3}$$

Now $\delta_i f$ is also an L^2 eigenvector for M_i corresponding to the eigenvalue λ_k. Consequently, since eigenspaces for M_i corresponding to different eigenvalues are orthogonal, $\langle \delta_i f, g\rangle$ is nonzero only if $\lambda_n=\lambda_k$. Coming back to (3-3) we obtain (3-1) for $f=\varphi_k$ and $g=\delta_i\varphi_n$.

Finally, by linearity (3-1) holds also for $f\in\mathcal{D}$ and $g\in\mathcal{D}_i$. □

The second ingredient we need to prove Theorem 1 is a bilinear embedding, as was the case in [Carbonaro and Dragičević 2013; Dragičević and Volberg 2006; 2012; Mauceri and Spinelli 2014; 2015]. For $N\in\mathbb{N}$ (the cases interesting to us being $N=1$ and $N=d$) we take $F=(f_1,\dots,f_N):X\times(0,\infty)\to\mathbb{R}^N$ and set

$$|F|_*^2 := r|F|^2+|\partial_t F|^2+\sum_{i=1}^d |\mathfrak{d}_i F|^2. \tag{3-4}$$

The absolute values $|\cdot|$ in (3-4) denote the Euclidean norms on $\mathbb{R}^N$ of the vectors $F(x,t)$, $\partial_t F(x,t)=(\partial_t f_1(x,t),\dots,\partial_t f_N(x,t))$, and $\mathfrak{d}_i F(x,t)=(\mathfrak{d}_i f_1(x,t),\dots,\mathfrak{d}_i f_N(x,t))$, where $(x,t)\in X\times(0,\infty)$. Below we only state our bilinear embedding. The proof of it is presented in the next section.

Theorem 3. *Let $f:X\to\mathbb{R}$ and $\boldsymbol{g}=(g_1,\dots,g_d):X^d\to\mathbb{R}^d$ and assume that $f\in\mathcal{D}$ and $g_i\in\mathcal{D}_i$ for $i=1,\dots,d$. Define*

$$F(x,t)=P_t\,\Pi f(x)\quad\textit{and}\quad G(x,t)=Q_t\boldsymbol{g}=(Q_t^1 g_1,\dots,Q_t^d g_d).$$

Then

$$\int_0^\infty\int_X |F(x,t)|_*|G(x,t)|_*\,d\mu(x)\,t\,dt \leqslant 6(p^*-1)\|\Pi f\|_p\|\boldsymbol{g}\|_q. \tag{3-5}$$

Remark. The theorem can be slightly generalized, at least at a formal level. Namely in Theorem 3, we do not need that $v_i=[\delta_i,\delta_i^*]$. It is enough to have any $v_i\geq 0$ and take $Q_t=e^{-tM_i}$ with $M_i=L+v_i$.

Our main theorem is an immediate corollary of Proposition 2 and Theorem 3.

Proof of Theorem 1. It is enough to prove that for each $f \in L^p$ and $g_i \in L^q$, $i = 1, \dots, d$, the absolute value of $\sum_{i=1}^d \langle R_i f, g_i \rangle$ does not exceed

$$24(1+\sqrt{K})(p^*-1)\|\Pi f\|_p \left\| \left(\sum_{i=1}^d |g_i|^2 \right)^{1/2} \right\|_q .$$

A density argument based on the assumption (T3) allows us to take $f \in \mathcal{D}$ and $g_i \in \mathcal{D}_i$, $i = 1, \dots, d$. From Proposition 2 we have

$$-\tfrac{1}{4}\langle R_i f, g_i \rangle_{L^2} = \int_0^\infty \langle \eth_i P_t \Pi f, \partial_t Q_t^i g_i \rangle_{L^2}\, t\, dt + \int_0^\infty \langle q_i P_t \Pi f, \partial_t Q_t^i g_i \rangle_{L^2}\, t\, dt$$

and thus, assumption (A2) gives

$$\begin{aligned} \left| \sum_{i=1}^d \langle R_i f, g_i \rangle_{L^2} \right| &\leqslant 4 \int_0^\infty \int_X \left(\left(\sum_{i=1}^d |\eth_i P_t \Pi f(x)|^2 \right)^{1/2} + \sqrt{K}\sqrt{r(x)}\,|P_t \Pi f(x)| \right) |G(x,t)|_*\, d\mu(x)\, t\, dt \\ &\leq 4(1+\sqrt{K}) \int_0^\infty \int_X |F(x,t)|_* |G(x,t)|_*\, d\mu(x)\, t\, dt. \end{aligned}$$

Now, Theorem 3 completes the proof. □

4. Bilinear embedding theorem

This section is devoted to the proof of our embedding theorem, Theorem 3. We shall follow closely the reasoning from [Carbonaro and Dragičević 2013; Mauceri and Spinelli 2015].

4.1. ***The Bellman function.*** Before proceeding to the proof of Theorem 3 we need to introduce its most important ingredient: the Bellman function.

Choose $p \geqslant 2$. Let $q = p/(p-1)$,

$$\gamma = \gamma(p) = \tfrac{1}{8} q(q-1),$$

and define $\beta_p \colon [0,\infty)^2 \to [0,\infty]$ by

$$\beta_p(s_1, s_2) = s_1^p + s_2^q + \gamma \begin{cases} s_1^2 s_2^{2-q}, & s_1^p \leqslant s_2^q, \\ (2/p)\, s_1^p + \big(2/q - 1\big) s_2^q, & s_1^p \geqslant s_2^q. \end{cases}$$

For $m = (m_1, m_2) \in \mathbb{N}^2$, the Nazarov–Treil Bellman function corresponding to p, m is the function

$$B = B_{p,m} \colon \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to [0, \infty)$$

given, for any $\zeta \in \mathbb{R}^{m_1}$ and $\eta \in \mathbb{R}^{m_2}$, by

$$B_{p,m}(\zeta, \eta) = \tfrac{1}{2} \beta_p(|\zeta|, |\eta|).$$

The function B originated in an article by F. Nazarov and S. Treil [1996]. It was employed (and simplified) in [Carbonaro and Dragičević 2013; 2017; Dragičević and Volberg 2006; 2011; 2012]. Note that B is

$C^1(\mathbb{R}^{m_1+m_2})$ and is C^2 everywhere except on the set

$$\{(\zeta,\eta)\in\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}\colon \eta=0 \text{ or } |\zeta|^p=|\eta|^q\}.$$

To remedy the nonsmoothness of B we consider the regularization

$$B_{\kappa,p,m}=B_\kappa:=B*_{\mathbb{R}^{m_1+m_2}}\psi_\kappa,$$

where

$$\psi(x)=c_m e^{-1/(1-|x|^2)}\chi_{B^{m_1+m_2}}(x)\quad\text{and}\quad \psi_\kappa(x)=\frac{1}{\kappa^{m_1+m_2}}\psi(x/\kappa),$$

with c_m such that $\int_{\mathbb{R}^{m_1+m_2}}\psi_\kappa(x)\,dx=1$. Here $\chi_{B^{m_1+m_2}}$ stands for the characteristic function of the (m_1+m_2)-dimensional Euclidean ball centered at the origin and of radius 1. Since both B and ψ_κ are biradial also B_κ is biradial. Hence, there is $\beta_\kappa=\beta_{\kappa,p}$ acting from $[0,\infty)^2$ to $\mathbb{R}$ such that

$$B_\kappa(\zeta,\eta)=\tfrac12\beta_\kappa(|\zeta|,|\eta|),\quad \zeta\in\mathbb{R}^{m_1},\ \eta\in\mathbb{R}^{m_2}.$$

We shall need some properties of β_κ and B_κ that were essentially proved in [Carbonaro and Dragičević 2013; Dragičević and Volberg 2012; Mauceri and Spinelli 2014; 2015].

Proposition 4. *Let $\kappa\in(0,1)$. Then, for $s_i>0$, $i=1,2$, we have*

(i) $0\leqslant\beta_\kappa(s_1,s_2)\leqslant(1+\gamma(p))((s_1+\kappa)^p+(s_2+\kappa)^q)$,

(ii) $0\leqslant\partial_{s_1}\beta_\kappa(s)\leqslant C_p\max((s_1+\kappa)^{p-1},s_2+\kappa)$ *and* $0\leqslant\partial_{s_2}\beta_\kappa(s)\leqslant C_p(s_2+\kappa)^{q-1}$, *with C_p a positive constant.*

The function B_κ belongs to $C^\infty(\mathbb{R}^{m_1+m_2})$, and for any $\xi=(\zeta,\eta)\in\mathbb{R}^{m_1+m_2}$ there exists a positive $\tau_\kappa=\tau_\kappa(|\zeta|,|\eta|)$ such that for $\omega=(\omega_1,\omega_2)\in\mathbb{R}^{m_1+m_2}$ we have

(iii) $\langle\operatorname{Hess}(B_\kappa)(\xi)\omega,\omega\rangle\geqslant\frac12\gamma(p)(\tau_\kappa|\omega_1|^2+\tau_\kappa^{-1}|\omega_2|^2)$.

Moreover, there is a continuous function $E_\kappa\colon\mathbb{R}^{m_1+m_2}\to\mathbb{R}$ for which

(iv) $\langle(\nabla B_\kappa)(\xi),\xi\rangle\geqslant\frac12\gamma(p)(\tau_\kappa|\zeta|^2+\tau_\kappa^{-1}|\eta|^2)-\kappa E_\kappa(\xi)$,

(v) $|E_\kappa(\xi)|\le C_{m,p}(|\zeta|^{p-1}+|\eta|+|\eta|^{q-1}+\kappa^{q-1})$.

Proof (*sketch*). Let $\tau=\tau(|\zeta|,|\eta|)$ be the function from [Carbonaro and Dragičević 2013, Theorem 3] and define $\tau_\kappa=\tau*_{\mathbb{R}^{d+1}}\psi_\kappa$. With exactly this τ_κ, items (i), (ii), and (iii) were proved in [Mauceri and Spinelli 2014, Proposition 6.3].

Let

$$E_\kappa(\xi)=-\int_{\mathbb{R}^{m_1+m_2}}\langle\nabla B(\xi-\kappa s),s\rangle\,\psi_\kappa(s)\,ds,\quad \xi\in\mathbb{R}^{m_1+m_2};$$

see [Dragičević and Volberg 2012, (2.10)]. Item (iv) (with these τ_κ and E_κ) follows from [Dragičević and Volberg 2012, Theorem 4(iii′)], together with the observation from [Carbonaro and Dragičević 2013; Dragičević and Volberg 2012] that

$$(\tau*\psi_\kappa)(\xi)(\tau^{-1}*\psi_\kappa)(\xi)\ge\left(\int_{\mathbb{R}^{d+1}}(\tau(y)\psi_\kappa(x-y))^{1/2}(\tau^{-1}(y)\psi_\kappa(x-y))^{1/2}\,dy\right)^{1/2}=1.$$

Item (v) is proved in [Dragičević and Volberg 2012, p. 207]. Note that, our Bellman function B_κ coincides with $-\frac{1}{2}Q_\kappa$ from that paper (when Q_κ is restricted to real arguments).

We remark that in [Dragičević and Volberg 2012, Theorem 4 (iii′)] a stronger statement is proved with an additional negative term $-B_\kappa(\xi)$ on the left-hand side of (iv). □

4.2. ***Proof of Theorem 3.*** Define $u : X \times (0,\infty) \to \mathbb{R} \times \mathbb{R}^d$ by

$$u = u(x,t) = (P_t\,\Pi f(x),\, Q_t\boldsymbol{g}(x)) = \big(P_t\,\Pi f(x),\, Q_t^1 g_1(x), \dots, Q_t^d g_d(x)\big).$$

Assume first that $p \geq 2$ and set

$$b_\kappa = B_\kappa \circ u : X \times (0,\infty) \to [0,\infty).$$

Here $B_\kappa = B_{\kappa,d,p}$ is the Bellman function from Proposition 4 with $m_1 = 1$ and $m_2 = d$. For each $i = 1,\dots,d$, we fix a sequence $\{\sigma_i^n\}_{n\in\mathbb{N}}$ which converges to σ_i, and a sequence $\{\Sigma_i^n\}_{n\in\mathbb{N}}$ which converges to Σ_i. We also impose that $\sigma_i < \sigma_i^n < \Sigma_i^n < \Sigma_i$ for $i = 1,\dots,d,\ n \in \mathbb{N}$. Defining

$$X_i^n = [\sigma_i^n, \Sigma_i^n] \quad \text{and} \quad X_n = X_1^n \times \cdots \times X_d^n,$$

where $n \in \mathbb{N}$, we see that $\{X_n\}_{n\in\mathbb{N}}$ is an increasing family of compact subsets of X such that $X = \bigcup_n X_n$. We shall estimate the integral

$$I(n,\varepsilon) := \int_0^\infty \int_{X_n} (\partial_t^2 - \widetilde{L})(b_{\kappa(n)})(x,t)\, d\mu(x)\, te^{-\varepsilon t}\, dt \tag{4-1}$$

from below and above and then, first let $n \to \infty$ and then $\varepsilon \to 0^+$. Here $\kappa(n)$ is a small quantity depending on n which will be determined in the proof. Since X_n is compact, $f \in \mathcal{D}$ and $g_i \in \mathcal{D}_i,\ i = 1,\dots,d$, the integral (4-1) is in fact absolutely convergent. In what follows we will often briefly write κ instead of $\kappa(n)$.

***The lower estimate of* (4-1) *for* $p \geq 2$.** The key result here is Proposition 5 below. Its proof hinges on the assumption (A1).

Proposition 5. *For $x \in X$ and $t > 0$ it holds that*

$$((\partial_t^2 - \widetilde{L})b_\kappa)(x,t) \geqslant \gamma\,|F(x,t)|_*\,|G(x,t)|_* - \kappa\, r(x) E_\kappa(u(x,t)). \tag{4-2}$$

Proof. Set $\mathfrak{d}_0 := \partial_t$. To justify (4-2) we shall need the pointwise equality

$$(\partial_t^2 - \widetilde{L})b_\kappa = r\,\langle \nabla B_\kappa(u), u\rangle + \sum_{i=1}^d v_i \cdot (\partial_{\eta_i} B_\kappa(u) \cdot Q_t^i g_i) + \sum_{i=0}^d \big\langle \mathrm{Hess}(B_\kappa)(\mathfrak{d}_i u), \mathfrak{d}_i u\big\rangle. \tag{4-3}$$

We first we focus on proving (4-3).

From the chain rule we have $\mathfrak{d}_i b_\kappa = p_i \langle \nabla B_\kappa(u), \partial_{x_i} u\rangle$. Moreover, a computation shows that, for $i = 1,\dots,d$,

$$\mathfrak{d}_i^* = -p_i\partial_{x_i} - p_i\frac{w_i'}{w_i} - p_i' \quad \text{and} \quad \mathfrak{d}_i^*\mathfrak{d}_i = -p_i^2\partial_{x_i}^2 - \left(p_i\frac{w_i'}{w_i} + 2p_i'\right)p_i\partial_{x_i}.$$

Consequently, applying once again the chain rule we obtain for $i=0,\dots,d$,

$$\begin{aligned}\mathfrak{d}_i^*\mathfrak{d}_i b_\kappa &= -p_i\partial_{x_i}(p_i\langle\nabla B_\kappa(u),\partial_{x_i}u\rangle)-\left(p_i\frac{w_i'}{w_i}+p_i'\right)p_i\langle\nabla B_\kappa(u),\partial_{x_i}u\rangle\\ &= -p_i^2\partial_{x_i}(\langle\nabla B_\kappa(u),\partial_{x_i}u\rangle)-p_ip_i'\langle\nabla B_\kappa(u),\partial_{x_i}u\rangle-\left(p_i\frac{w_i'}{w_i}+p_i'\right)p_i\langle\nabla B_\kappa(u),\partial_{x_i}u\rangle\\ &= \left\langle\nabla B_\kappa(u),-p_i^2\partial_{x_i}^2u-\left(p_i\frac{w_i'}{w_i}+2p_i'\right)p_i\partial_{x_i}u\right\rangle-p_i^2\big\langle\mathrm{Hess}(B_\kappa)(\partial_{x_i}u),\partial_{x_i}u\big\rangle\\ &= \big\langle\nabla B_\kappa(u),\mathfrak{d}_i^*\mathfrak{d}_iu\big\rangle-\big\langle\mathrm{Hess}(B_\kappa)(\mathfrak{d}_iu),\mathfrak{d}_iu\big\rangle.\end{aligned}$$

Now, summing the above formula in $i=0,\dots,d$, we obtain

$$(\mathfrak{d}_0^2-\widetilde{L})b_\kappa=\big\langle\nabla B_\kappa(u),(\mathfrak{d}_0^2-\widetilde{L})u\big\rangle+\sum_{i=0}^d\big\langle\mathrm{Hess}(B_\kappa)(u)(\mathfrak{d}_iu),\mathfrak{d}_iu\big\rangle. \tag{4-4}$$

The formula (4-4) implies (4-3). Indeed we have

$$(\partial_t^2-L)u=\big((\partial_t^2-L)P_tf,(\partial_t^2-L)Q_t\boldsymbol{g}\big),$$

where

$$(\partial_t^2-L)P_tf=0$$

and

$$(\partial_t^2-L)Q_t\boldsymbol{g}=\big((\partial_t^2-L)Q_t^1g_1,\dots,(\partial_t^2-L)Q_t^dg_d\big).$$

Moreover,

$$(\partial_t^2-L)Q_t^ig_i=(\partial_t^2-M_i)Q_t^ig_i+v_i\cdot Q_t^ig_i=v_i\cdot Q_t^ig_i,$$

and using (4-4) the equation (4-3) follows.

Having demonstrated (4-3) we pass to the proof of (4-2). Proposition 4(ii) implies $(\partial_{\eta_i}B_\kappa(u)\cdot Q_t^ig_i)\geq 0$. Thus (4-3) together with the assumption (A1) produce

$$(\partial_t^2-\widetilde{L})b_\kappa\geqslant r\,\langle\nabla B_\kappa(u),u\rangle+\sum_{i=0}^d\big\langle\mathrm{Hess}(B_\kappa)(\mathfrak{d}_iu),\mathfrak{d}_iu\big\rangle. \tag{4-5}$$

Finally, (4-2) is a consequence of (4-5), items (iii) and (iv) from Proposition 4, and the inequality between the arithmetic and geometric mean. □

Coming back to the proof of the lower estimate in (4-1) we now take $\{\kappa(n)\}_{n\in\mathbb{N}}$ such that $|\kappa(n)|\leqslant 1$, $\lim_n\kappa(n)=0$ and

$$|\kappa(n)|^{1/2}\int_{X_n}|r(x)E_{\kappa(n)}(u(x,t))|\,d\mu(x)\leqslant 1. \tag{4-6}$$

To see that such a sequence exists we use Proposition 4(v) and the fact that $P_tf\in\mathcal{D}$ and $Q_t^ig_i\in\mathcal{D}_i$ (hence also $P_tf\in C^\infty(X)$ and $Q_t^ig_i\in C^\infty(X)$). Next, (4-2), together with (4-6), leads to

$$\liminf_{n\to\infty}I(n,\varepsilon)\geq\gamma\int_0^\infty\int_X|F(x,t)|_*\,|G(x,t)|_*\,d\mu(x)\,te^{-\varepsilon t}\,dt,$$

and, consequently, by the monotone convergence theorem

$$\liminf_{\varepsilon\to 0^+}\liminf_{n\to\infty} I(n,\varepsilon) \geq \gamma(p)\int_0^\infty\int_X |F(x,t)|_*|G(x,t)|_*\,d\mu(x)\,t\,dt. \tag{4-7}$$

This is our lower estimate of (4-1).

***The upper estimate of* (4-1) *for* $p \geq 2$.** The main ingredients here are the technical assumptions (T2) and (T3). We split the integral in (4-1) as

$$\begin{aligned} I(n,\varepsilon) &= I_1(n,\varepsilon) - I_2(n,\varepsilon) \\ &:= \int_0^\infty\int_{X_n} \partial_t^2(b_{\kappa(n)})(x,t)\,d\mu(x)\,te^{-\varepsilon t}\,dt - \int_0^\infty\int_{X_n} \widetilde{L}(b_{\kappa(n)})(x,t)\,d\mu(x)\,te^{-\varepsilon t}\,dt. \end{aligned}$$

First we prove that

$$\lim_{n\to\infty} I_2(n,\varepsilon) = 0. \tag{4-8}$$

To see this we recall that $\widetilde{L} = \sum_{i=1}^d \mathfrak{d}_i^*\mathfrak{d}_i$ with $\mathfrak{d}_i$ given by (2-4) and $\mathfrak{d}_i^*$ being the formal adjoint of $\mathfrak{d}_i$ on L^2. Then,

$$I_2(n,\varepsilon) = \sum_{i=1}^d I_2^i(n,\varepsilon) := \sum_{i=1}^d \int_0^\infty\int_{X_n} (\mathfrak{d}_i^*\mathfrak{d}_i)(b_{\kappa(n)})(x,t)\,d\mu(x)\,te^{-\varepsilon t}\,dt,$$

and it is enough to prove that each of the integrals $I_2^i(n,\varepsilon)$ goes to 0 as $n\to\infty$. As the reasoning is symmetric in $i = 1,\dots,d$, we present it only for $I_2^1(n,\varepsilon)$. Define

$$X^{(1)} = X_2\times\cdots\times X_d,\quad x^{(1)} = (x_2,\dots,x_d),\quad\text{and}\quad \mu^{(1)} = \mu_2\otimes\cdots\otimes\mu_d.$$

Formula (2-5) together with integration by parts in the x_1-variable produces

$$I_2^1(n,\varepsilon) = \int_0^\infty\int_{X^{(1)}} \big((p_1^2 w_1\,\partial_{x_1}b_\kappa)(\Sigma_1^n, x^{(1)}) - (p_1^2 w_1\,\partial_{x_1}b_\kappa)(\sigma_1^n, x^{(1)})\big)\,d\mu^{(1)}(x^1)\,te^{-\varepsilon t}\,dt.$$

Call z_1^n either of the quantities σ_1^n or Σ_1^n. Then the chain rule gives

$$\begin{aligned} (p_1^2 w_1\,\partial_{x_1}b_\kappa)(z_1^n, x^{(1)}) &= p_1^2(z_1^n)w_1(z_1^n)\partial_{x_1}P_t f(z_1^n,x^{(1)})\,\partial_\zeta B_\kappa\big(P_t f(z_1^n,x^{(1)}), Q_t\boldsymbol{g}(z_1^n,x^{(1)})\big) \\ &\quad + p_1^2(z_1^n)w_1(z_1^n)\big\langle\partial_{x_1}Q_t\boldsymbol{g}(z_1^n,x^{(1)}), \nabla_\eta B_\kappa\big(P_t f(z_1^n,x^{(1)}), Q_t\boldsymbol{g}(z_1^n,x^{(1)})\big)\big\rangle. \end{aligned} \tag{4-9}$$

Since $f\in\mathcal{D}$ and $g_i\in\mathcal{D}_i$ we have that $P_t f\in\mathcal{D}$ and $Q_t\boldsymbol{g}\in\mathcal{D}_1\otimes\cdots\otimes\mathcal{D}_d$. Recall that φ_k is defined by (2-2), while $\delta_i\varphi_k$, $i = 1,\dots,d$, are given by (2-3). Now, Proposition 4(ii) implies

$$|\nabla_{\zeta,\eta}B_\kappa(\zeta,\eta)| \leqslant C_{p,q}(|\zeta|^{p-1} + |\eta|^{q-1} + |\eta| + \kappa^{q-1}). \tag{4-10}$$

Therefore, since $|\kappa(n)|\leqslant 1$, a calculation based on (4-9) together with the assumptions (T2), (T3), and Hölder's inequality produces $\lim_n I_2^1(n,\varepsilon) = 0$.

Now we focus on $I_1(n,\varepsilon)$. Since $f\in\mathcal{D},\ g_i\in\mathcal{D}_i,\ i=1,\dots,d,\ B_\kappa\in C^\infty(\mathbb{R}^{d+1})$ and we integrate over $x\in X_n$, the double integral is absolutely convergent. Thus Fubini's theorem gives

$$I_1(n,\varepsilon)=\int_{X_n}\int_0^\infty \partial_t^2(b_{\kappa(n)})(x,t)\,te^{-\varepsilon t}\,dt\,d\mu(x).$$

Integrating by parts in the inner integral twice we obtain

$$\begin{aligned}I_1(n,\varepsilon)&=-\int_{X_n}\int_0^\infty \partial_t(b_{\kappa(n)})(x,t)(1-\varepsilon t)e^{-\varepsilon t}\,dt\,d\mu(x)\\&=\int_{X_n}b_{\kappa(n)}(x,0)\,d\mu(x)+\varepsilon^2\int_{X_n}\int_0^\infty b_{\kappa(n)}(x,t)\,te^{-\varepsilon t}\,dt\,d\mu(x)-2\varepsilon\int_{X_n}\int_0^\infty b_{\kappa(n)}(x,t)e^{-\varepsilon t}\,dt\,d\mu(x)\\&\leqslant\int_{X_n}b_{\kappa(n)}(x,0)\,d\mu(x)+\varepsilon^2\int_{X_n}\int_0^\infty b_{\kappa(n)}(x,t)\,te^{-\varepsilon t}\,dt\,d\mu(x)\\&:=I_1^1(n)+I_1^2(n,\varepsilon).\end{aligned}$$

In the first two equalities above we neglected the boundary terms by using the chain rule together with (4-10).

First we treat $I_1^2(n,\varepsilon)$. Proposition 4(i) gives

$$I_1^2(n,\varepsilon)\leqslant\varepsilon^2C_p\int_{X_n}\int_0^\infty\bigl(|P_t\Pi f(x)|^p+|Q_t\boldsymbol{g}(x)|^q+\max(\kappa(n)^p,\kappa(n)^q)\bigr)\,te^{-\varepsilon t}\,dt\,d\mu(x).$$

Take $\kappa(n)$ which satisfies (4-6) and

$$\max(\kappa(n)^{p-1/2},\kappa(n)^{q-1/2})\,\mu(X_n)\leqslant 1. \tag{4-11}$$

Then, since $f\in\mathcal{D}$ and $g_i\in\mathcal{D}_i,\ i=1,\dots,d$, we have

$$\limsup_{n\to\infty}I_1^2(n,\varepsilon)\leqslant\varepsilon^2C_p\int_X\int_0^\infty|P_t\Pi f(x)|^p+|Q_t\boldsymbol{g}(x)|^q\,t\,dt\,d\mu(x)\leqslant C(p,f,g)\,\varepsilon^2,$$

and, consequently,

$$\limsup_{\varepsilon\to0^+}\ \limsup_{n\to\infty}I_1^2(n,\varepsilon)=0. \tag{4-12}$$

Coming back to $I_1^1(n)$ we use Proposition 4(i) to estimate

$$I_1^1(n)\leqslant\tfrac12(1+\gamma)\int_{X_n}\bigl(|\Pi f(x)|+\kappa(n)\bigr)^p\,d\mu(x)+\tfrac12(1+\gamma)\int_{X_n}\bigl(|\boldsymbol{g}(x)|+\kappa(n)\bigr)^q\,d\mu(x).$$

Now for each $\varepsilon>0$ we split the first integral onto $\int_{|\kappa(n)|\leqslant\varepsilon|\Pi f(x)|}$ and $\int_{|\kappa(n)|>\varepsilon|\Pi f(x)|}$ and the second integral onto $\int_{|\kappa(n)|\leqslant\varepsilon|\boldsymbol{g}(x)|}$ and $\int_{|\kappa(n)|>\varepsilon|\boldsymbol{g}(x)|}$. Then we obtain

$$\begin{aligned}I_1^1(n)\leqslant\tfrac12(1+\gamma)\bigl((1+\varepsilon)^p\|\Pi f\|_p^p+(1+\varepsilon)^q\|\boldsymbol{g}\|_q^q\bigr)&\\+\tfrac12(1+\gamma)\bigl((1+\varepsilon^{-1})^p\kappa(n)^p\mu(X_n)+(1+\varepsilon^{-1})^q\kappa(n)^q\mu(X_n)\bigr).&\end{aligned}$$

Since $\kappa(n)$ satisfies (4-6) and (4-11) we arrive at

$$\limsup_{\varepsilon\to 0^+}\limsup_{n\to\infty} I_1^1(n) \leqslant \tfrac{1}{2}(1+\gamma)\big(\|\Pi f\|_p^p + \|\boldsymbol{g}\|_q^q\big).$$

Recalling (4-8) and (4-12) we thus proved

$$\limsup_{\varepsilon\to 0^+}\limsup_{n\to\infty} I(n,\varepsilon) \leqslant \tfrac{1}{2}(1+\gamma(p))\big(\|\Pi f\|_p^p + \|\boldsymbol{g}\|_q^q\big), \tag{4-13}$$

which is the upper estimate of (4-1) we need.

Completion of the proof of the bilinear embedding. Consider first $p \geq 2$. Combining the lower estimate (4-7) and the upper estimate (4-13) we obtain

$$\int_0^\infty\int_X |F(x,t)|_* |G(x,t)|_* \, d\mu(x)\, t\, dt \leqslant \frac{1+\gamma(p)}{2\gamma(p)}\big(\|\Pi f\|_p^p + \|\boldsymbol{g}\|_q^q\big). \tag{4-14}$$

Finally, a polarization argument finishes the proof. More precisely, for $s > 0$ we replace f with sf and $\boldsymbol{g}$ with $s^{-1}\boldsymbol{g}$ on both sides of (4-14). Then, the left-hand side is unchanged, while minimizing the right-hand side over $s > 0$ we obtain

$$\int_0^\infty\int_X |F(x,t)|_* |G(x,t)|_* \, d\mu(x)\, t\, dt \leqslant \frac{1+\gamma(p)}{2\gamma(p)}\left(\left(\frac{p}{q}\right)^{1/p} + \left(\frac{q}{p}\right)^{1/q}\right)\|\Pi f\|_p \|\boldsymbol{g}\|_q. \tag{4-15}$$

Using the above inequality, a calculation leads to (3-5). We sketch the argument below.

Note that for $p \geq 2$ we have $p^* = p$ and recall that $\gamma(p) = \frac{1}{8}q(q-1)$. Thus, for $1 < q \leq 2$ we obtain

$$\begin{aligned}\frac{1+\gamma(p)}{2\gamma(p)}\left(\left(\frac{p}{q}\right)^{1/p} + \left(\frac{q}{p}\right)^{1/q}\right) &= \tfrac{1}{2}(8+q(q-1))(q-1)^{1/q-1}(p-1)\\ &\leq (q+3)(q-1)^{1/q-1}(p^*-1).\end{aligned} \tag{4-16}$$

Setting $s = q-1$ we need to maximize the function $H(s) := (s+4)s^{-s/(s+1)}$ for $s \in (0,1]$. Let

$$h(s) = \log(s+4) - \frac{s\log s}{s+1},$$

so that $H(s) = e^{h(s)}$. Then we have

$$h'(s) = \frac{1}{s+4} - \frac{\log s}{(s+1)^2} - \frac{1}{s+1} \quad \text{and} \quad h''(s) = -\frac{1}{(s+4)^2} + \frac{2\log s}{(s+1)^3} + \frac{s-1}{s(s+1)^2};$$

consequently, $h''(s) < 0$ for $s \in (0,1)$. Observe that $h'\big(\frac{7}{20}\big) > 0$ and $h'\big(\frac{2}{5}\big) < 0$. Therefore h' has a unique zero inside the interval $\big(\frac{7}{20}, \frac{2}{5}\big)$ and h attains a global maximum there. Obviously, the same is true for $H = e^h$. Now it is easy to see that

$$\max_{7/20\leq s\leq 2/5} H(s) < \tfrac{22}{5}\cdot\big(\tfrac{7}{20}\big)^{-2/7} < 6,$$

and thus also $\sup_{0<s\leq 1} H(s) < 6$. Hence, coming back to (4-16) we obtain

$$\frac{1+\gamma(p)}{2\gamma(p)}\left(\left(\frac{p}{q}\right)^{1/p} + \left(\frac{q}{p}\right)^{1/q}\right) \leq 6\,(p^*-1).$$

In view of (4-15) this implies (3-5) and completes the proof of Theorem 3 for $p \geq 2$.

The proof of Theorem 3 for $p \le 2$ proceeds analogously once we switch p with q and $P_t f$ with $Q_t \boldsymbol{g}$ in the definition of b_κ. Namely, we consider $\widetilde{b}_\kappa(x,t) = \widetilde{B}_\kappa(Q_t \boldsymbol{g}, P_t f)$, where $\widetilde{B}_\kappa(\zeta,\eta) = B_{\kappa,q,(d,1)}(\zeta,\eta)$, $\zeta \in \mathbb{R}^d$, $\eta \in \mathbb{R}$. Here $B_{\kappa,q,(d,1)}$ is the function from Proposition 4 with $m_1 = d$ and $m_2 = 1$. Then we repeat the argument used for $p \ge 2$. The function $\widetilde{B}_\kappa$ satisfies items (iii)–(v) of Proposition 4 with p replaced by q. Therefore both the lower estimate (4-7) and the upper estimate (4-13) hold with $\gamma(p)$ replaced by $\gamma(q)$.

5. Examples

Throughout this section we apply Theorem 1 to the examples of orthogonal systems considered in [Nowak and Stempak 2006, Section 7]. This is possible for all of these systems except for the Fourier–Bessel expansions [Nowak and Stempak 2006, Section 7.8]. In this case the condition (T2) fails. Despite this failure we think that it might be possible to treat also the Fourier–Bessel expansions by the methods of the present paper. It might be also interesting to try to apply the methods of our paper to the Riesz transforms considered in [Nowak and Sjögren 2012] (in the case of Jacobi trigonometric polynomial expansions).

In all of the examples we present, for more details the reader is kindly referred to [Nowak and Stempak 2006, Sections 7.1–7.7]. The formulas for v_i and $r = \sum_{i=1}^{d} r_i$ in the examples below follow directly from (2-1) and (2-6). Recall that

$$\mu = \mu_1 \otimes \cdots \otimes \mu_d, \quad X = X_1 \times \cdots \times X_d, \quad L^p = L^p(X,\mu), \quad \|\cdot\|_p = \|\cdot\|_{L^p},$$

and

$$p^* = \max\left(p, \frac{p}{p-1}\right).$$

5.1. ***Ornstein–Uhlenbeck operator: Hermite polynomial expansions.*** Here we consider

$$p_i = 1, \quad q_i = 0, \quad a_i = 0, \quad w_i(x_i) = \pi^{-1/2} e^{-x_i^2}, \quad d\mu_i(x_i) = w_i(x_i)\,dx_i$$

on $X_i = \mathbb{R}$. Then

$$\delta_i = \mathfrak{d}_i = \partial_{x_i}, \quad \delta_i^* = -\partial_{x_i} + 2x_i, \quad v_i = [\delta_i, \delta_i^*] = 2, \quad r = 0, \tag{5-1}$$

and

$$L = \sum_{i=1}^{d} L_i = -\Delta + 2\langle x, \nabla \rangle$$

is the Ornstein–Uhlenbeck operator on $X = \mathbb{R}^d$. The operator L is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$ with the self-adjoint extension given by

$$Lf = \sum_{k \in \mathbb{N}^d} |k| \langle f, \widetilde{H}_k \rangle_{L^2} \widetilde{H}_k.$$

In the formula above $|k| = k_1 + \cdots + k_d$, the symbol L^2 stands for $L^2 = L^2(\mathbb{R}^d, \mu)$, while $\{\widetilde{H}_k\}_{k \in \mathbb{N}^d}$ is the system of L^2 normalized Hermite polynomials; see [Nowak and Stempak 2006, Section 7.1; Lebedev 1972, p. 60]. In this section we take

$$\varphi_k = \widetilde{H}_k, \quad k \in \mathbb{N}^d.$$

Note that μ is a probability measure in this setting. The projection Π becomes

$$\Pi f = \sum_{k\in\mathbb{N}^d,\, k\neq 0} \langle f, \widetilde{H}_k\rangle_{L^2} \widetilde{H}_k, \quad f \in L^2.$$

Then

$$(I-\Pi) f = \langle f, \widetilde{H}_0\rangle_{L^2} \widetilde{H}_0,$$

and, since $\widetilde{H}_0 = 1$, the operator $I-\Pi$ is the projection onto the constants given by

$$(I-\Pi) f(x) = \int_X f(y)\, d\mu(y), \quad x \in X.$$

Hence, by Holder's inequality $\|(I-\Pi)f\|_p \leqslant \|f\|_p$, and, consequently,

$$\|\Pi f\|_p \leqslant 2\|f\|_p, \quad 1 \leqslant p \leqslant \infty. \tag{5-2}$$

Next

$$\delta_i \widetilde{H}_k = \sqrt{2k_j}\, \widetilde{H}_{k-e_j}, \tag{5-3}$$

where, by convention $\widetilde{H}_{k-e_j} = 0$ if $k_j = 0$. This convention is also used for the examples presented in the next sections. The Riesz transform is defined by

$$R_i f = \sum_{k\in\mathbb{N}^d,\, k\neq 0} \left(\frac{k_j}{|k|}\right)^{1/2} \langle f, \widetilde{H}_k\rangle_{L^2} \widetilde{H}_{k-e_i}, \quad f \in L^2.$$

Dimension-free estimates for the vector $\boldsymbol{R} f = (R_1 f, \dots, R_d f)$ were proved by Meyer [1984]; see also [Gundy 1986; Gutiérrez 1994; Pisier 1988] for different proofs. Later Dragičević and Volberg [2006, Corollary 0.4] found a proof which uses the Bellman function method. The best result in terms of the size of the constants is due to Arcozzi [1998, Corollary 2.4] who proved that $\|\boldsymbol{R} f\|_p \leqslant 2(p^*-1)\|f\|_p$, $1 < p < \infty$. An application of Theorem 1 produces similar, though weaker, bounds.

Theorem 6. *Fix $1 < p < \infty$. Then, for $f \in L^p$ such that $\int_X f(y)\, d\mu(y) = 0$, we have*

$$\|\boldsymbol{R} f\|_p \leqslant 24(p^*-1)\|f\|_p. \tag{5-4}$$

Remark. Using (5-2) we may extend the bound (5-4) to all $f \in L^p$ with 24 being replaced by 48.

Proof. We apply Theorem 1. In order to do so we need to check that its assumptions are satisfied.

By (5-1) we see that (A1) and (A2) (with $K = 0$) hold. Condition (T1) is proved by an easy calculation based on integration by parts. The assumption (T2) is also straightforward. Finally, (T3) follows from Lemma 1 and (5-3).

Now, if $\int_X f(y)\, d\mu(y) = 0$ then $\Pi f = f$. Thus, an application of Theorem 1 completes the proof. □

5.2. ***Laguerre operator: Laguerre polynomial expansions.*** Here, for a parameter $\alpha \in (-1,\infty)^d$, we consider

$$p_i = \sqrt{x_i}, \quad q_i = 0, \quad a_i = 0, \quad w_i(x_i) = \frac{1}{\Gamma(\alpha_i+1)} x_i^{\alpha_i} e^{-x_i} dx_i, \quad d\mu_i(x_i) = w_i(x_i)\, dx_i$$

on $X_i = (0, \infty)$. Then $\delta_i = \mathfrak{d}_i = \sqrt{x_i}\,\partial_{x_i}$, and thus

$$\delta_i^* = -\sqrt{x_i}\,\partial_{x_i} - \frac{\alpha_i + \frac{1}{2}}{\sqrt{x_i}} + \sqrt{x_i}, \quad v_i = [\delta_i, \delta_i^*] = \frac{\alpha_i + \frac{1}{2} + x_i}{2x_i}, \quad r = 0. \tag{5-5}$$

In this case

$$L = \sum_{i=1}^d L_i = \sum_{i=1}^d -x_i\,\partial_{x_i}^2 - (\alpha_i + 1 - x_i)\,\partial_{x_i}$$

is the Laguerre operator on $X = (0, \infty)^d$. It is symmetric on $C_c^\infty((0,\infty)^d)$ and has a self-adjoint extension

$$Lf = \sum_{k\in\mathbb{N}^d} |k| \langle f, \widetilde{L}_k^\alpha\rangle_{L^2} \widetilde{L}_k^\alpha.$$

Here $L^2 = L^2((0,\infty)^d, \mu)$, while $\{\widetilde{L}_k^\alpha\}_{k\in\mathbb{N}^d}$ is the system of L^2 normalized Laguerre polynomials; see [Nowak and Stempak 2006, Section 7.2; Lebedev 1972, p. 76]. These Laguerre polynomials are our functions φ_k in this section; namely

$$\varphi_k = \widetilde{L}_k^\alpha, \quad k \in \mathbb{N}^d.$$

Next we have

$$\delta_i \widetilde{L}_k^\alpha = \sqrt{k_j}\sqrt{x_i}\,\widetilde{L}_{k-e_i}^{\alpha+e_i}, \tag{5-6}$$

while the projection Π becomes

$$\Pi f = \sum_{k\in\mathbb{N}^d,\,k\neq 0} \langle f, \widetilde{L}_k^\alpha\rangle_{L^2} \widetilde{L}_k^\alpha, \quad f \in L^2.$$

A repetition of the argument from the previous section shows that $\Pi f = f$ if and only if $\int_X f(y)\,d\mu(y) = 0$ and

$$\|\Pi f\|_p \leqslant 2\|f\|_p, \quad 1 \leqslant p \leqslant \infty. \tag{5-7}$$

The Riesz transform is then given by

$$R_i f = \sum_{k\in\mathbb{N}^d, k\neq 0} \left(\frac{k_j}{|k|}\right)^{1/2} \langle f, \widetilde{L}_k^\alpha\rangle_{L^2} \sqrt{x_i}\,\widetilde{L}_{k-e_i}^{\alpha+e_i}, \quad f \in L^2.$$

Dimension-free bounds for single Riesz transforms R_i were first studied by Gutiérrez, Incognito and Torrea [Gutiérrez et al. 2001] for half-integer multi-indices, and generalized[†] by Nowak [2004] to multi-indices $\alpha \in \left[-\frac{1}{2}, \infty\right)^d$. Moreover in [Graczyk et al. 2005], Graczyk, Loeb, López, Nowak, and Urbina proved dimension-free estimates on L^p for the vector of Riesz-Laguerre transforms and half-integer multi-indices α. Recently, the author [Wróbel 2014, Theorem 4.1(b)] obtained dimension-free bounds on L^p for scalar Riesz transforms and general parameters $\alpha \in (-1, \infty)^d$, while Mauceri and Spinelli [2015, Theorem 5.2] proved a dimension-free bound for the vectorial Riesz transforms $\boldsymbol{R} f = (R_1 f, \ldots, R_d f)$, and $\alpha \in \left[-\frac{1}{2}, \infty\right)^d$. All the bounds mentioned in this paragraph are also independent of the parameter α

[†] In [Nowak 2004, Theorem 13] the author also states an estimate on L^p for the vector of Riesz-Laguerre transforms that is dimension-free for certain values of α. Unfortunately this result is not properly proved there (Nowak, personal communication, 2017). This is due to a problem in the proof of the vectorial g-function bound from [Nowak 2004, Theorem 7(b)].

(appropriately restricted). Moreover, the estimate from [Mauceri and Spinelli 2015, Theorem 5.2] is also linear in p^*.

By using Theorem 1 we obtain a result which coincides with [Mauceri and Spinelli 2015, Theorem 5.2] in the case of Riesz transforms acting on functions.

Theorem 7. *Fix $\alpha \in \left[-\frac{1}{2}, \infty\right)^d$ and $1 < p < \infty$. Then, for $f \in L^p$ which satisfy $\int_X f(y)\, d\mu(y) = 0$, we have*

$$\|\boldsymbol{R} f\|_p \leqslant 24(p^* - 1)\|f\|_p.$$

Remark. By (5-7) we have the same bound for general $f \in L^p$ with the constant being twice as large.

Proof. We are going to apply Theorem 1, so we need to verify its assumptions.

By (5-5) we see that if $\alpha \in \left[-\frac{1}{2}, \infty\right)^d$, then (A1) and (A2) (with $K = 0$) are satisfied. Moreover, the assumptions (T1) and (T2) follow from a direct calculation. Next, for such α the condition (T3) can be deduced from Lemma 1 together with (5-6).

Now, if $\int_X f(y)\, d\mu(y) = 0$ then $\Pi f = f$. Therefore, using Theorem 1 we complete the proof of Theorem 7. □

5.3. *Jacobi operator: Jacobi polynomial expansions.*

In this section for parameters $\alpha, \beta \in (-1, \infty)^d$ we consider

$$p_i = \sqrt{1 - x_i^2}, \quad q_i = 0, \quad a_i = 0,$$
$$w_i(x_i) = \frac{1}{C(\alpha_i, \beta_i)} (1 - x_i)^{\alpha_i} (1 + x_i)^{\beta_i}\, dx_i, \quad d\mu_i(x_i) = w_i(x_i)\, dx_i, \quad X_i = (-1, 1),$$

where $C(\alpha_i, \beta_i)$ is such that $\mu_i(X_i) = 1$. Then $\delta_i = \mathfrak{d}_i = \sqrt{1 - x_i^2}\, \partial_{x_i}$, and

$$\begin{aligned} \delta_i^* &= -\sqrt{1 - x_i^2}\, \partial_{x_i} + \left(\alpha_i + \tfrac{1}{2}\right)\sqrt{\frac{1 + x_i}{1 - x_i}} - \left(\beta_i + \tfrac{1}{2}\right)\sqrt{\frac{1 - x_i}{1 + x_i}}, \\ v_i &= [\delta_i, \delta_i^*] = \frac{\alpha_i + \frac{1}{2}}{1 - x_i} + \frac{\beta_i + \frac{1}{2}}{1 + x_i}, \quad r = 0. \end{aligned} \tag{5-8}$$

Here

$$L = \sum_{i=1}^d L_i = \sum_{i=1}^d -(1 - x_i^2)\partial_{x_i}^2 - \big(\beta_i - \alpha_i - (\alpha_i + \beta_i + 2)x_i\big)\, \partial_{x_i}$$

is the Jacobi operator on $X = (-1, 1)^d$. Let $L^2 = L^2((-1, 1)^d, \mu)$, and denote by $\{\widetilde{P}_k^{\alpha,\beta}\}_{k \in \mathbb{N}^d}$ the system of L^2 normalized Jacobi polynomials; see [Nowak and Stempak 2006, Section 7.1; Szegő 1975, Chapter 4]. These Jacobi polynomials are our functions φ_k in this section; namely

$$\varphi_k = \widetilde{P}_k^{\alpha,\beta}, \quad k \in \mathbb{N}.$$

The Jacobi operator is symmetric on $C_c^\infty((-1, 1)^d)$ and has a self-adjoint extension

$$Lf = \sum_{k \in \mathbb{N}^d} \lambda_k \langle f, \widetilde{P}_k^{\alpha,\beta} \rangle_{L^2} \widetilde{P}_k^{\alpha,\beta},$$

where $\lambda_k=\sum_{i=1}^d \lambda^i_{k_i}$ with $\lambda^i_{k_i}=k_i(k_i+\alpha_i+\beta_i+1)$, $i=1,\dots,d$. Similarly to the previous two sections the projection Π is

$$\Pi f=\sum_{k\in\mathbb{N}^d,\,k\neq 0}\langle f,\widetilde{P}^{\alpha,\beta}_k\rangle_{L^2}\widetilde{P}^{\alpha,\beta}_k,\qquad f\in L^2.$$

Moreover, $\Pi f=f$ precisely when $\int_X f(y)\,d\mu(y)=0$ and we have

$$\|\Pi f\|_p\leqslant 2\|f\|_p,\qquad 1\leqslant p\leqslant\infty. \tag{5-9}$$

The action of δ_i on Jacobi polynomials is given by

$$\delta_i\widetilde{P}^{\alpha,\beta}_k=\sqrt{k_i(k_i+\alpha_i+\beta_i+1)}\sqrt{1-x_i^2}\,\widetilde{P}^{\alpha+e_i,\beta+e_i}_{k-e_i}, \tag{5-10}$$

and the Riesz transform becomes

$$R_i f=\sum_{k\in\mathbb{N}^d,\,k\neq 0}\left(\frac{\lambda^i_{k_i}}{\lambda_k}\right)^{1/2}\langle f,\widetilde{P}^{\alpha,\beta}_k\rangle_{L^2}\sqrt{1-x_i^2}\,\widetilde{P}^{\alpha+e_i,\beta+e_i}_{k-e_i},\qquad f\in L^2.$$

Dimension- and parameter-free estimates for single Riesz transforms R_i are due to Nowak and Sjögren [2008], who proved them for $\alpha,\beta\in\left[-\frac12,\infty\right)^d$.

An application of Theorem 1 generalizes [Nowak and Sjögren 2008, Theorem 5.1] to the vectorial Riesz transforms $\boldsymbol{R}f=(R_1f,\dots,R_df)$. This result is new according to our knowledge. Moreover, we obtain an explicit estimate which is linear in p^*.

Theorem 8. *Fix $\alpha,\beta\in\left[-\frac12,\infty\right)^d$ and $1<p<\infty$. Then, for $f\in L^p$ which satisfy $\int_X f(y)\,d\mu(y)=0$, we have*

$$\|\boldsymbol{R}f\|_p\leqslant 24\,(p^*-1)\|f\|_p,\qquad f\in L^p. \tag{5-11}$$

Remark. As in the previous two sections (5-11) holds for all $f\in L^p$ with $48\,(p^*-1)$ in place of $24\,(p^*-1)$. This follows from (5-9).

Proof. We are going to apply Theorem 1, so we need to verify its assumptions for parameters $\alpha,\beta\in\left[-\frac12,\infty\right)^d$.

By (5-8) we see that if $\alpha,\beta\in\left[-\frac12,\infty\right)^d$, then (A1) and (A2) (with $K=0$) are satisfied. Similarly, using (5-10) one can see that, for such α and β, the conditions (T1) and (T2) also hold. The assumption (T3) follows from Lemma 1 together with (5-10).

Now, since $\int_X f(y)\,d\mu(y)=0$ implies $\Pi f=f$, an application of Theorem 1 completes the proof of Theorem 8. □

5.4. ***Harmonic oscillator: Hermite function expansions.*** Here we take

$$p_i=1,\quad q_i=x_i,\quad a_i=1,\quad w_i(x_i)=1,\quad d\mu_i(x_i)=dx_i,\quad X_i=\mathbb{R},$$

so that

$$\delta_i=\partial_{x_i}+x_i,\quad \mathfrak{d}_i=\partial_{x_i},\quad \delta_i^*=-\partial_{x_i}+x_i,\quad v_i=[\delta_i,\delta_i^*]=2,\quad r(x)=|x|^2, \tag{5-12}$$

and L is the harmonic oscillator

$$L = \sum_{i=1}^{d} L_i = -\Delta + |x|^2.$$

It is well known that L is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$ with the self-adjoint extension given by

$$Lf = \sum_{k\in\mathbb{N}^d} (2|k|+d)\langle f, h_k\rangle_{L^2} h_k.$$

Here $L^2 = L^2(\mathbb{R}^d, dx)$, while $\{h_k\}_{k\in\mathbb{N}^d}$ is the system of L^2 normalized Hermite functions; see [Nowak and Stempak 2006, Section 7.4]. The functions h_k are our φ_k in this section. They are of the form $h_k = h_{k_1} \otimes \cdots \otimes h_{k_d}$, where

$$h_{k_i}(x_i) = \widetilde{H}_{k_i}(x_i) e^{-x_i^2/2}, \quad x_i \in \mathbb{R}, \tag{5-13}$$

with $\widetilde{H}_{k_i}$ being the Hermite polynomial from Section 5.1. Note that as 0 is not an L^2 eigenvalue of L, the projection Π equals the identity operator.

Next

$$\delta_i h_k = \sqrt{2k_j}\, h_{k-e_j}, \tag{5-14}$$

and thus the Riesz transform is

$$R_i f = \sum_{k\in\mathbb{N}^d,\, k\neq 0} \left(\frac{2k_j}{2|k|+d}\right)^{1/2} \langle f, h_k\rangle_{L^2} h_{k-e_i}, \quad f \in L^2.$$

Here dimension-free bounds for the vector of Riesz transforms can be deduced, by means of transference, from the paper of Coulhon, Müller, and Zienkiewicz [Coulhon et al. 1996]; see also [Harboure et al. 2004; Lust-Piquard 2006] for different proofs. Moreover, a dimension-free bound for the vector of Riesz transforms which is additionally linear in p^* was proved in [Dragičević and Volberg 2012, Proposition 4].

Using Theorem 1 we are able to obtain a more explicit estimate for the vector $\boldsymbol{R} f$ than in [Dragičević and Volberg 2012]. However, contrary to that paper, our method says nothing about the vector of "adjoint" transforms $\boldsymbol{R}^* f = (\delta_1^* L^{-1/2} f, \ldots, \delta_d^* L^{-1/2} f)$.

Theorem 9. *For $1 < p < \infty$ we have*

$$\|\boldsymbol{R} f\|_p \leqslant 48(p^*-1)\|f\|_p, \quad f \in L^p.$$

Proof. We apply Theorem 1. In order to do so we need to check that its assumptions are satisfied.

The equation (5-12) gives (A1) and (A2) with $K=1$. Condition (T1) is straightforward. The assumption (T2) holds since, by (5-13), Hermite functions h_{k_i} vanish rapidly at $\pm\infty$. Finally, (T3) follows from (5-14) and the (well-known) density of Hermite functions in L^p, $1 \leqslant p < \infty$.

Thus, an application of Theorem 1 is justified and the proof of Theorem 9 is completed. □

5.5. ***Laguerre operator: Laguerre function expansions of Hermite type.*** For a parameter $\alpha \in (-1,\infty)^d$ we consider

$$p_i = 1, \quad q_i = x_i - \frac{\alpha_i + \frac{1}{2}}{x_i}, \quad a_i = 1, \quad w_i(x_i) = 1, \quad d\mu_i(x_i) = dx_i, \quad X_i = (0,\infty),$$

so that

$$\begin{gathered}\delta_i=\partial_{x_i}+x_i-\frac{\alpha_i+\frac{1}{2}}{x_i},\quad \mathfrak{d}_i=\partial_{x_i},\quad \delta_i^*=-\partial_{x_i}+x_i-\frac{\alpha_i+\frac{1}{2}}{x_i},\\ v_i=[\delta_i,\delta_i^*]=2,\quad r(x)=|x|^2+\sum_{i=1}^d\frac{\alpha_i^2-\frac{1}{4}}{x_i^2}.\end{gathered}\tag{5-15}$$

Here L is the Laguerre operator

$$L=\sum_{i=1}^d L_i=-\Delta+|x|^2+\sum_{i=1}^d\frac{\alpha_i^2-\frac{1}{4}}{x_i^2}.$$

Then L is symmetric on $C_c^\infty(\mathbb{R}^d)$ and has a self-adjoint extension given by

$$Lf=\sum_{k\in\mathbb{N}^d}(4|k|+2d+2|\alpha|)\langle f,\varphi_k^\alpha\rangle_{L^2}\varphi_k^\alpha.$$

In the above formula we set $|k|=k_1+\cdots+k_d$ and $|\alpha|=\alpha_1+\cdots+\alpha_d$; note that $|\alpha|$ may be negative. By L^2 we mean $L^2((0,\infty)^d,dx)$, while $\{\varphi_k^\alpha\}_{k\in\mathbb{N}^d}$ stands for the system of L^2 normalized Laguerre functions of Hermite type; see [Nowak and Stempak 2006, Section 7.5]. The functions φ_k^α are the tensor products $\varphi_k^\alpha=\varphi_{k_1}^{\alpha_1}\otimes\cdots\otimes\varphi_{k_d}^{\alpha_d}$ with

$$\varphi_{k_i}^{\alpha_i}(x_i)=\sqrt{2}\,\widetilde{L}_{k_i}^{\alpha_i}(x_i^2)\,x_i^{\alpha_i+1/2}e^{-x_i^2/2},\quad x_i>0,\tag{5-16}$$

and $\widetilde{L}_{k_i}^{\alpha_i}$ the Laguerre polynomials from Section 5.2. In this section we take

$$\varphi_k=\varphi_k^\alpha.$$

As 0 is not an L^2 eigenvalue of L, the projection Π equals the identity operator.

Next

$$\delta_i\varphi_k^\alpha=-2\sqrt{k_j}\,\varphi_{k-e_j}^{\alpha+e_j},\tag{5-17}$$

and thus the Riesz transform is

$$R_if=-\sum_{k\in\mathbb{N}^d,\,k\neq 0}\left(\frac{4k_i}{4|k|+2|\alpha|+2d}\right)^{1/2}\langle f,\varphi_k^\alpha\rangle_{L^2}\varphi_{k-e_j}^{\alpha+e_j},\quad f\in L^2.$$

Dimension-free bounds for single Riesz transforms R_i were obtained by Stempak and the author [Stempak and Wróbel 2013, Theorem 5.1] for a certain restricted range of the parameter α.

In this section, for $\alpha\in\left(\frac{1}{2},\infty\right)^d$ we define

$$C(\alpha)=\max_{i=1,\dots,d}\frac{\alpha_i+\frac{1}{2}}{\alpha_i-\frac{1}{2}}.$$

By using Theorem 1 we obtain the following strengthening of [Stempak and Wróbel 2013, Theorem 5.1] in the case $\alpha\in\left(\frac{1}{2},\infty\right)^d$.

Theorem 10. *Let $\alpha \in \left(\frac{1}{2}, \infty\right)^d$. Then, for $1 < p < \infty$, we have*

$$\|\boldsymbol{R} f\|_p \leqslant 24(1+\sqrt{C(\alpha)})(p^*-1)\|f\|_p, \quad f \in L^p.$$

Proof. We apply Theorem 1. In order to do so we need to check that its assumptions are satisfied.

The formula (5-15) gives (A1) and (A2) for $\alpha \in \left(\frac{1}{2}, \infty\right)^d$ with $K = C(\alpha)$. Conditions (T1) and (T2) follow from (5-16) and (5-17). Finally, (T3) follows from [Nowak 2003, Lemma 5.2] and (5-17).

Thus, an application of Theorem 1 is justified and the proof of Theorem 10 is completed. □

5.6. ***Laguerre operator: Laguerre function expansions of convolution type.*** For a parameter $\alpha \in (-1, \infty)^d$ we consider

$$p_i = 1, \quad q_i = x_i, \quad a_i = 2\alpha_i + 2,$$
$$w_i(x_i) = x_i^{2\alpha_i+1}, \quad d\mu_i(x_i) = w_i(x_i)\,dx_i, \quad X_i = (0, \infty),$$

so that

$$\begin{gathered} \delta_i = \partial_{x_i} + x_i, \quad \mathfrak{d}_i = \partial_{x_i}, \quad \delta_i^* = -\partial_{x_i} + x_i - \frac{2\alpha_i+1}{x_i}, \\ v_i = [\delta_i, \delta_i^*] = 2 + \frac{2\alpha+1}{x_i^2}, \quad r(x) = |x|^2. \end{gathered} \tag{5-18}$$

Here L is the Laguerre operator

$$L = \sum_{i=1}^d L_i = -\Delta + |x|^2 - \sum_{i=1}^d \frac{2\alpha_i+1}{x_i}\,\partial_{x_i}.$$

Then L is symmetric on $C_c^\infty((0,\infty)^d)$ and has a self-adjoint extension given by

$$Lf = \sum_{k\in\mathbb{N}^d} (4|k| + 2d + 2|\alpha|)\langle f, \ell_k^\alpha\rangle_{L^2} \ell_k^\alpha.$$

Here $L^2 = L^2((0,\infty)^d, w(x)dx)$, while $\{\ell_k^\alpha\}_{k\in\mathbb{N}^d}$ is the system of L^2 normalized Laguerre functions of convolution type; see [Nowak and Stempak 2006, Section 7.6]. The functions ℓ_k^α are of the form $\ell_k^\alpha = \ell_{k_1}^{\alpha_1} \otimes \cdots \otimes \ell_{k_d}^{\alpha_d}$ with

$$\ell_{k_i}^{\alpha_i}(x_i) = \sqrt{2}\,\widetilde{L}_{k_i}^{\alpha_i}(x_i^2)\,e^{-x_i^2/2}, \quad x_i > 0, \tag{5-19}$$

and $\widetilde{L}_{k_i}^{\alpha_i}$ the Laguerre polynomials from Section 5.2. In this section we take

$$\varphi_k = \ell_k^\alpha.$$

Also here, as 0 is not an L^2 eigenvalue of L, the projection Π equals the identity operator.

Next

$$\delta_i \ell_k^\alpha = -2\sqrt{k_i}\,x_i\,\ell_{k-e_i}^{\alpha+e_i}, \tag{5-20}$$

and thus the Riesz transform is

$$R_i f = -\sum_{k\in\mathbb{N}^d,\, k\neq 0} \left(\frac{4k_i}{4|k| + 2|\alpha| + 2d}\right)^{1/2} \langle f, \ell_k^\alpha\rangle_{L^2} \ell_{k-e_i}^{\alpha+e_i}, \quad f \in L^2.$$

The boundedness of these Riesz transforms on L^p was proved by Nowak and Stempak [2007, Theorem 3.4]. Later Nowak and Szarek [2012, Theorem 4.1] enlarged the range of admitted parameters α. In both of these papers Calderón–Zygmund theory was used; thus the L^p bounds depended on the dimension d. Applying Theorem 1 we obtain a dimension-free bound for the vectorial Riesz transform $\boldsymbol{R}f = (R_1 f, \dots, R_d f)$.

Theorem 11. *Let $\alpha \in \left[-\frac{1}{2}, \infty\right)^d$. Then, for $1 < p < \infty$, we have*

$$\|\boldsymbol{R}f\|_p \leqslant 48(p^* - 1)\|f\|_p, \quad f \in L^p. \tag{5-21}$$

Proof. A continuity argument based on (5-19) and (5-20) shows that it suffices to prove (5-21) for $\alpha \in \left(-\frac{1}{2}, \infty\right)^d$. We are going to apply Theorem 1. In order to do so we need to check that its assumptions are satisfied.

The formula (5-18) gives (A1) and (A2) with $K = 1$. Conditions (T1) and (T2) follow from (5-19) and (5-20). It remains to prove (T3). For the space $\mathcal{D}$ this condition follows from [Nowak 2003, Lemma 4.3]. In the case of $\mathcal{D}_i$, $i = 1, \dots, d$, the assumption (T3) can be deduced from (T3) for $\mathcal{D}$ together with (5-20).

Thus, an application of Theorem 1 is justified and the proof of Theorem 11 is completed. □

5.7. *Jacobi operator: Jacobi function expansions.* For parameters $\alpha, \beta \in (-1, \infty)^d$ we consider

$$p_i = 1, \quad q_i = -\tfrac{1}{4}(2\alpha_i + 1)\cot\left(\tfrac{1}{2}x_i\right) + \tfrac{1}{4}(2\beta_i + 1)\tan\left(\tfrac{1}{2}x_i\right), \quad a_i = \tfrac{1}{4}(\alpha_i + \beta_1 + 1)^2,$$
$$w_i(x_i) = 1, \quad d\mu_i(x_i) = dx_i, \quad X_i = (0, \pi),$$

so that

$$\delta_i = \partial_{x_i} - \tfrac{1}{4}(2\alpha_i+1)\cot\left(\tfrac{1}{2}x_i\right) + \tfrac{1}{4}(2\beta_i+1)\tan\left(\tfrac{1}{2}x_i\right), \quad \mathfrak{d}_i = \partial_{x_i},$$
$$\delta_i^* = -\partial_{x_i} - \tfrac{1}{4}(2\alpha_i+1)\cot\left(\tfrac{1}{2}x_i\right) + \tfrac{1}{4}(2\beta_i+1)\tan\left(\tfrac{1}{2}x_i\right), \quad v_i = [\delta_i, \delta_i^*] = \frac{2\alpha_i+1}{8\cos^2\left(\frac{1}{2}x_i\right)} + \frac{2\beta_i+1}{8\sin^2\left(\frac{1}{2}x_i\right)}, \tag{5-22}$$
$$r(x) = \sum_{i=1}^d \tfrac{1}{16}(2\alpha_i+1)^2\cot^2\left(\tfrac{1}{2}x_i\right) + \tfrac{1}{16}(2\beta_i+1)^2\tan^2\left(\tfrac{1}{2}x_i\right) + \tfrac{1}{16}\left((\alpha_i+\beta_i+1)^2 - (2\alpha_i+1)(2\beta_i+1)\right).$$

Here L is the Jacobi operator

$$L = \sum_{i=1}^d L_i = -\Delta + \sum_{i=1}^d \left(\frac{4\alpha_i^2 - 1}{16\sin^2\left(\frac{1}{2}x_i\right)} + \frac{4\beta_i^2 - 1}{16\cos^2\left(\frac{1}{2}x_i\right)}\right).$$

Then L is symmetric on $C_c^\infty((0, \pi)^d)$ and has a self-adjoint extension given by

$$Lf = \sum_{k \in \mathbb{N}^d} \lambda_k \langle f, \phi_k^{\alpha,\beta}\rangle_{L^2} \phi_k^{\alpha,\beta}.$$

Here $\lambda_k = \sum_{i=1}^d \lambda_{k_i}^i$ with $\lambda_{k_i}^i = \left(k_i + \frac{1}{2}(\alpha_i + \beta_i + 1)\right)^2$, $L^2 = L^2((0, \pi)^d, dx)$, while $\{\phi_k^{\alpha,\beta}\}_{k \in \mathbb{N}^d}$ is the system of L^2 normalized Jacobi functions; see [Nowak and Stempak 2006, Section 7.7]. These Jacobi functions have the tensor product form $\phi_k^{\alpha,\beta} = \phi_{k_1}^{\alpha_1,\beta_1} \otimes \dots \otimes \phi_{k_d}^{\alpha_d,\beta_d}$ with

$$\phi_{k_i}^{\alpha_i,\beta_i}(x_i) = 2^{(\alpha_i+\beta_i+1)/2} \widetilde{P}_{k_i}^{\alpha_i,\beta_i}(\cos x_i)\left(\sin\left(\tfrac{1}{2}x_i\right)\right)^{\alpha_i+1/2}\left(\cos\left(\tfrac{1}{2}x_i\right)\right)^{\beta_i+1/2} \tag{5-23}$$

for $x_i \in (0, \pi)$, and $\widetilde{P}_{k_i}^{\alpha_i,\beta_i}$ being the Jacobi polynomials from Section 5.3. In this section we take

$$\varphi_k = \phi_k^{\alpha,\beta}.$$

In the case when $\alpha, \beta \in \left[\frac{1}{2}, \infty\right)^d$, the L^2 kernel of L is trivial, and thus the projection Π equals the identity operator.

Next

$$\delta_i \phi_k^{\alpha,\beta} = -\sqrt{k_i(k_i+\alpha_i+\beta_i+1)}\,\phi_{k-e_i}^{\alpha+e_i,\beta+e_i}, \tag{5-24}$$

and thus the Riesz transform is

$$R_i f = -\sum_{k\in\mathbb{N}^d,\, k_i\neq 0} \left(\frac{k_i(k_i+\alpha_i+\beta_i+1)}{\lambda_k}\right)^{1/2} \langle f, \phi_k^{\alpha,\beta}\rangle_{L^2} \phi_{k-e_i}^{\alpha+e_i,\beta+e_i}, \qquad f \in L^2.$$

In the case $d = 1$ the L^p boundedness of these Riesz transforms was proved by Stempak [2007]. Using Theorem 1 we obtain the following multidimensional bounds.

Theorem 12. *Let $\alpha, \beta \in \left[\frac{1}{2}, \infty\right)^d$. Then, for $1 < p < \infty$, we have*

$$\|\boldsymbol{R} f\|_p \leqslant 48\,(p^*-1)\|f\|_p, \qquad f \in L^p.$$

Proof. A continuity argument based on (5-23) and (5-24) allows us to focus on $\alpha, \beta \in \left(\frac{1}{2}, \infty\right)^d$. We are going to apply Theorem 1 for such parameters α and β. In order to do so we need to check that its assumptions are satisfied.

The formula (5-22) gives (A1) and (A2) (with $K = 1$). Conditions (T1) and (T2) follow from (5-23) and (5-24), while (T3) can be deduced from the density of polynomials in $C((-1, 1))$ together with (5-23) and (5-24).

Thus, an application of Theorem 1 is permitted and the proof of Theorem 12 is completed. □

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REDUCIBILITY OF THE QUANTUM HARMONIC OSCILLATOR IN d-DIMENSIONS WITH POLYNOMIAL TIME-DEPENDENT PERTURBATION

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We prove a reducibility result for a quantum harmonic oscillator in arbitrary dimension with arbitrary frequencies perturbed by a linear operator which is a polynomial of degree 2 in $(x_j, -\mathrm{i}\partial_j)$ with coefficients which depend quasiperiodically on time.

1. Introduction and statement of results

The aim of this paper is to present a reducibility result for the time-dependent Schrödinger equation

$$\mathrm{i}\dot{\psi} = H_\epsilon(\omega t)\psi, \quad x \in \mathbb{R}^d, \tag{1-1}$$

$$H_\epsilon(\omega t) := H_0 + \epsilon W(\omega t, x, -\mathrm{i}\nabla), \tag{1-2}$$

where

$$H_0 := -\Delta + V(x), \qquad V(x) := \sum_{j=1}^{d} \nu_j^2 x_j^2, \quad \nu_j > 0, \tag{1-3}$$

and $W(\theta, x, \xi)$ *is a real polynomial in* (x, ξ) *of degree at most* 2, with coefficients being real analytic functions of $\theta \in \mathbb{T}^n$. Here ω are parameters which are assumed to belong to the set $\mathcal{D} = (0, 2\pi)^n$.

For $\epsilon = 0$ the spectrum of (1-2) is given by

$$\sigma(H_0) = \{\lambda_k\}_{k\in\mathbb{N}^d}, \quad \lambda_k \equiv \lambda_{(k_1,\dots,k_d)} := \sum_{j=1}^{d}(2k_j + 1)\nu_j, \tag{1-4}$$

with $k_j \geq 0$ integers. In particular if the frequencies ν_j are nonresonant, then the differences between couples of eigenvalues are dense on the real axis. As a consequence, in the case $\epsilon = 0$ most of the solutions of (1-1) are almost periodic with an infinite number of rationally independent frequencies.

Here we will prove that for any choice of the mechanical frequencies ν_j and for ω belonging to a set of large measure in $\mathcal{D}$, the system (1-1) is reducible: precisely there exists a time-quasiperiodic unitary transformation of $L^2(\mathbb{R}^d)$ which conjugates (1-2) to a time-independent operator. We also deduce boundedness of the Sobolev norms of the solution.

The proof exploits the fact that for polynomial Hamiltonians of degree at most 2, the correspondence between classical and quantum mechanics is exact (i.e., without error term), so that the result can be

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proven by exact quantization of the classical KAM theory, which ensures reducibility of the classical Hamiltonian system

$$h_\epsilon := h_0 + \epsilon W(\omega t, x, \xi), \quad h_0 := \sum_{j=1}^{d} \xi_j^2 + \nu_j^2 x_j^2. \tag{1-5}$$

We will use (in the Appendix) the exact correspondence between classical and quantum dynamics of quadratic Hamiltonians also to prove a complementary result. Precisely we will present a class of examples, following [Graffi and Yajima 2000], in which one generically has growth of Sobolev norms. This happens when the frequencies ω of the external forcing are resonant with some of the ν_j.

We recall that the exact correspondence between classical and quantum dynamics of quadratic Hamiltonians was already exploited in [Hagedorn et al. 1986] to prove stability/instability results for one degree of freedom time-dependent quadratic Hamiltonians.

Notwithstanding the simplicity of the proof, we think that the present result could have some interest, since this is the first example of a reducibility result for a system in which the gaps of the unperturbed spectrum are dense in $\mathbb{R}$. Furthermore it is one of the few cases in which reducibility is obtained for systems in more than one space dimension.

Indeed, most of the results on the reducibility problem for (1-1) have been obtained in the 1-dimensional case, and also the results in higher dimensions obtained up to now deal only with cases in which the spectrum of the unperturbed system has gaps whose size is bounded from below, like in the harmonic oscillator (or in the Schrödinger equation on $\mathbb{T}^d$). On the other hand we restrict here to perturbations, which although unbounded, must belong to the very special class of polynomials in x_j and $-\mathrm{i}\partial_j$. The reason is that for operators in this class, the commutator is the operator whose symbol is the Poisson bracket of the corresponding symbols, without any error term (see Remark 2.2 and Remark 2.4). In order to deal with more general perturbations one needs further ideas and techniques.

Before closing this introduction we recall some previous works on the reducibility problem for (1-1) and more generally for perturbations of the Schrödinger equation with a potential $V(x)$. As we already anticipated, most of the works deal with the 1-dimensional case. The first one is [Combescure 1987], in which the pure point nature of the Floquet operator is obtained in the case of a smoothing perturbation of the harmonic oscillator in dimension 1; see also [Kuksin 1993]. The techniques of this paper were extended in [Duclos and Šťovíček 1996; Duclos et al. 2002] in order to deal with potentials growing superquadratically (still in dimension 1) but with perturbations which were only required to be bounded.

A slightly different approach originates from the so-called KAM theory for PDEs [Kuksin 1987; Wayne 1990]. In particular the methods developed in that context in order to deal with unbounded perturbations, see [Kuksin 1997; 1998], were exploited in [Bambusi and Graffi 2001] in order to deal with the reducibility problem of (1-1) with superquadratic potential in dimension 1; see [Liu and Yuan 2010] for a further improvement. The case of bounded perturbations of the harmonic oscillator in dimension 1 was treated in [Wang 2008; Grébert and Thomann 2011].

An extension of KAM theory to NLS on $\mathbb{T}^d$ has been obtained in [Eliasson and Kuksin 2010] and its methods have been adapted to deal with the reducibility problem of quasiperiodically forced linear

Schrödinger equation in [Eliasson and Kuksin 2009]. A further reducibility result for equations in more than one space dimension is [Grébert and Paturel 2016], in which bounded perturbations of the completely resonant harmonic oscillator in $\mathbb{R}^d$ were studied. As far as we know, these are the only higher-dimensional linear systems for which reducibility is known.[1]

We remark that all these papers deal with cases where the spectrum of the unperturbed operator is formed by well-separated eigenvalues. In the higher-dimensional cases they are allowed to have high multiplicity localized in clusters. But then the perturbation must have special properties ensuring that the clusters are essentially not destroyed under the KAM iteration.

Finally we recall the works [Bambusi 2017a; 2017b], in which pseudodifferential calculus was used together with KAM theory in order to prove reducibility results for (1-1) (in dimension 1) with unbounded perturbations. The ideas of the present paper are a direct development of the ideas of [Bambusi 2017a; 2017b]. We also recall that the idea of using pseudodifferential calculus together with KAM theory in order to deal with problems involving unbounded perturbations originates from [Plotnikov and Toland 2001; Iooss et al. 2005] and has been developed in order to give a quite general theory in [Baldi et al. 2014; Berti and Montalto 2016; Montalto 2014]; see also [Feola and Procesi 2015].

In order to state our main result, we need some preparations. It is well known that (1-1) is well-posed, see for example [Maspero and Robert 2017], in the scale $\mathcal{H}^s$, $s \in \mathbb{R}$, of the weighted Sobolev spaces defined as follows. For $s \geq 0$ let

$$\mathcal{H}^s := \{\psi \in L^2(\mathbb{R}^d) : H_0^{s/2}\psi \in L^2(\mathbb{R}^d)\},$$

equipped with the natural Hilbert space norm $\|\psi\|_s := \|H_0^{s/2}\psi\|_{L^2(\mathbb{R}^d)}$. For $s < 0$, $\mathcal{H}^s$ is defined by duality. Such spaces are not dependent on ν for $\nu_j > 0$, $1 \leq j \leq d$. We also have $\mathcal{H}^s \equiv \mathrm{Dom}(-\Delta + |x|^2)^{s/2}$.

We will prove the following reducibility theorem:

Theorem 1.1. *Let ψ be a solution of* (1-1). *There exist $\epsilon_* > 0$, $C > 0$ and for all $|\epsilon| < \epsilon_*$ a closed set $\mathcal{E}_\epsilon \subset (0, 2\pi)^n$ with* $\mathrm{meas}((0, 2\pi)^n \setminus \mathcal{E}_\epsilon) \leq C\epsilon^{1/9}$ *and, for all $\omega \in \mathcal{E}_\epsilon$ there exists a unitary* (*in L^2*) *time-quasiperiodic map $U_\omega(\omega t)$ such that, defining φ by $U_\omega(\omega t)\varphi = \psi$, it satisfies the equation*

$$\mathrm{i}\dot{\varphi} = H_\infty \varphi, \tag{1-6}$$

with H_∞ a positive definite time-independent operator which is unitary equivalent to a diagonal operator

$$\sum_{j=1}^{d} \nu_j^\infty (x_j^2 - \partial_{x_j}^2),$$

where $\nu_j^\infty = \nu_j^\infty(\omega)$ are defined for $\omega \in \mathcal{E}_\epsilon$ and fulfill the estimates

$$|\nu_j - \nu_j^\infty| \leq C\epsilon, \quad j = 1, \ldots, d.$$

Finally the following properties hold:

(i) *For all $s \geq 0$, for all $\psi \in \mathcal{H}^s$, we have $\theta \mapsto U_\omega(\theta)\psi \in C^0(\mathbb{T}^n; \mathcal{H}^s)$.*

[1] We would like to point out also [Procesi and Procesi 2012; 2015], which at present refer to the resonant *nonlinear* Schrödinger equation; it would be interesting to study if they have some consequences for reducibility theory.

(ii) *For all $s \geq 0$, there exists $C_s > 0$ such that for all $\theta \in \mathbb{T}^n$*

$$\|\mathbf{1} - U_\omega(\theta)\|_{\mathcal{L}(\mathcal{H}^{s+2};\mathcal{H}^s)} \leq C_s \epsilon. \tag{1-7}$$

(iii) *For all $s, r \geq 0$, the map $\theta \mapsto U_\omega(\theta)$ is of class $C^r(\mathbb{T}^n; \mathcal{L}(\mathcal{H}^{s+4r+2}; \mathcal{H}^s))$.*

Remark 1.2. In Theorem 1.1, if the frequencies ν_j are resonant, then the change of coordinates U_ω is close to the identity, in the sense of (1-7), but the Hamiltonian H_∞ is not necessary diagonal. However, it is always possible to diagonalize it by means of a metaplectic transformation which is not close to the identity; see Theorem 2.10 and Remark 2.11 below.

Let us denote by $\mathcal{U}_{\epsilon,\omega}(t, \tau)$ the propagator generated by (1-1) such that $\mathcal{U}_{\epsilon,\omega}(\tau, \tau) = \mathbf{1}$ for all $\tau \in \mathbb{R}$. An immediate consequence of Theorem 1.1 is that we have a Floquet decomposition:

$$\mathcal{U}_{\epsilon,\omega}(t, \tau) = U_\omega^*(\omega t)\, \mathrm{e}^{-\mathrm{i}(t-\tau)H_\infty}\, U_\omega(\omega tau). \tag{1-8}$$

Another consequence of (1-8) is that for any $s > 0$ the norm $\|\mathcal{U}_{\epsilon,\omega}(t, 0)\psi_0\|_s$ is bounded uniformly in time:

Corollary 1.3. *Let $\omega \in \mathcal{E}_\epsilon$ with $|\epsilon| < \epsilon_*$. The following is true: for any $s > 0$ one has*

$$c_s\|\psi_0\|_s \leq \|\mathcal{U}_{\epsilon,\omega}(t, 0)\psi_0\|_s \leq C_s\|\psi_0\|_s \quad \textit{for all } t \in \mathbb{R}, \textit{ for all } \psi_0 \in \mathcal{H}^s, \tag{1-9}$$

for some $c_s > 0$, $C_s > 0$.

Moreover there exists a constant c'_s such that if the initial data ψ_0 is in $\mathcal{H}^{s+2}$ then

$$\|\psi_0\|_s - \epsilon c'_s\|\psi_0\|_{s+2} \leq \|\mathcal{U}_{\epsilon,\omega}(t, 0)\psi_0\|_s \leq \|\psi_0\|_s + \epsilon c'_s\|\psi_0\|_{s+2} \quad \textit{for all } t \in \mathbb{R}. \tag{1-10}$$

It is interesting to compare estimate (1-9) with the corresponding estimate which can be obtained for more general perturbations $W(t, x, D)$. So denote by $\mathcal{U}(t, \tau)$ the propagator of $H_0 + W(t, x, D)$ with $\mathcal{U}(\tau, \tau) = \mathbf{1}$. Then in [Maspero and Robert 2017] it is proved that if $W(t, x, \xi)$ is a real polynomial in (x, ξ) of degree at most 2, the propagator $\mathcal{U}(t, s)$ exists, belongs to $\mathcal{L}(\mathcal{H}^s)$ for all $s \geq 0$ and fulfills

$$\|\mathcal{U}(t, 0)\psi_0\|_s \leq e^{C_s|t|}\|\psi_0\|_s \quad \text{for all } t \in \mathbb{R}$$

(the estimate is sharp!). If $W(t, x, \xi)$ is a polynomial of degree at most 1, one has

$$\|\mathcal{U}(t, 0)\psi_0\|_s \leq C_s(1 + |t|)^s\, \|\psi_0\|_s \quad \text{for all } t \in \mathbb{R}.$$

Thus estimate (1-9) improves dramatically the upper bounds proved in [Maspero and Robert 2017] when the perturbation is small and depends quasiperiodically in time with "good" frequencies.

As a final remark we recall that growth of Sobolev norms can indeed happen if the frequencies ω are not well chosen. In the Appendix, we show that the Schrödinger equation

$$\mathrm{i}\dot{\psi} = \left[-\tfrac{1}{2}\partial_{xx} + \tfrac{1}{2}x^2 + ax\sin\omega t\right]\psi, \quad x \in \mathbb{R}$$

(which was already studied by Graffi and Yajima [2000], who showed that the corresponding Floquet operator has continuous spectrum), exhibits growth of Sobolev norms if and only if $\omega = \pm 1$, which are clearly resonant frequencies. We also slightly generalize the example.

Another example of growth of Sobolev norms for the perturbed harmonic oscillator is given by Delort [2014]. There the perturbation is a pseudodifferential operator of order 0, periodic in time with resonant frequency $\omega = 1$.

Remark 1.4. The uniform-time estimate given in (1-9) is similar to the main result obtained in [Eliasson and Kuksin 2009] for small perturbation of the Laplace operator on the torus $\mathbb{T}^d$. Concerning perturbations of harmonic oscillators in $\mathbb{R}^d$, most known reducibility results are obtained for $d = 1$, except in [Grébert and Paturel 2016].

Remark 1.5. In [Eliasson and Kuksin 2009; Grébert and Paturel 2016] the estimate (1-10) is proved without loss of regularity; this is due to the fact that the perturbations treated in those papers are bounded operators. There are also some cases, see for example [Bambusi and Graffi 2001], in which the reducing transformation is bounded notwithstanding the fact that the perturbation is unbounded, but this is due to the fact that the unperturbed system has suitable gap properties which are not fulfilled in our case.

Remark 1.6. The $\epsilon^{1/9}$ estimate on the measure of the set of resonant frequencies is not optimal. We wrote it just for the sake of giving a simple quantitative estimate.

Remark 1.7. Denote by $\{\psi_k\}_{k\in\mathbb{N}^d}$ the set of Hermite functions, namely the eigenvectors of H_0: $H_0\psi_k = \lambda_k\psi_k$. They form an orthonormal basis of $L^2(\mathbb{R}^d)$, and writing $\psi = \sum_k c_k\psi_k$, one has

$$\|\psi\|_s^2 \simeq \sum_k (1+|k|)^{2s}|c_k|^2.$$

Denote by $\psi(t) = \sum_{k\in\mathbb{N}^d} c_k(t)\psi_k$ the solution of (1-1) written on the Hermite basis. Then (1-9) implies the following *dynamical localization* for the energy of the solution: for all $s \geq 0$, there exists $C_s \equiv C_s(\psi_0) > 0$ such that

$$\sup_{t\in\mathbb{R}} |c_k(t)| \leq C_s(1+|k|)^{-s} \quad \text{for all } k \in \mathbb{N}^d. \tag{1-11}$$

From the dynamical property (1-11) one obtains easily that every state $\psi \in L^2(\mathbb{R}^d)$ is a bounded state for the time evolution $\mathcal{U}_{\epsilon,\omega}(t,0)\psi$ under the conditions of Theorem 1.1 on (ϵ,ω). The corresponding definitions are given below.

Definition 1.8 [Enss and Veselić 1983]. A function $\psi \in L^2(\mathbb{R}^d)$ is a bounded state (or belongs to the point spectral subspace of $\{\mathcal{U}_{\epsilon,\omega}(t,0)\}_{t\in\mathbb{R}}$) if the quantum trajectory $\{\mathcal{U}_{\epsilon,\omega}(t,0)\psi : t \in \mathbb{R}\}$ is a precompact subset of $L^2(\mathbb{R}^d)$.

Corollary 1.9. *Under the conditions of Theorem 1.1 on (ϵ,ω), every state $\psi \in L^2(\mathbb{R}^d)$ is a bounded state of $\{\mathcal{U}_{\epsilon,\omega}(t,0)\}_{t\in\mathbb{R}}$.*

Proof. To prove that every state $\psi \in L^2(\mathbb{R}^d)$ is a bounded state for the time evolution $\mathcal{U}_{\epsilon,\omega}(t,0)\psi$, using that $\mathcal{H}^s$ is dense in $L^2(\mathbb{R}^d)$, it is enough to assume that $\psi \in \mathcal{H}^s$, with $s > \frac{1}{2}d$. With the notation of Remark 1.7, we write

$$\psi(t) = \psi^{(N)}(t) + R^{(N)}(t),$$

where $\psi^{(N)}(t) = \sum_{|k|\leq N} c_k(t)\psi_k$ and $R^{(N)}(t) = \sum_{|k|>N} c_k(t)\psi_k$.

Take $\delta > 0$. Applying (1-11), taking N large enough, we get that $\|R^{(N)}(t)\|_0 \le \frac{1}{2}\delta$ for all $t \in \mathbb{R}$. But $\{\psi^{(N)}(t) : t \in \mathbb{R}\}$ is a subset of a finite-dimensional linear space. So we get that $\{\mathcal{U}_{\epsilon,\omega}(t,0)\psi : \ t \in \mathbb{R}\}$ is a precompact subset of $L^2(\mathbb{R}^d)$. □

This last dynamical result is deeply connected with the spectrum of the Floquet operator. First note that Theorem 1.1 implies the following:

Corollary 1.10. *The operator U_ω induces a unitary transformation $L^2(\mathbb{T}^n) \otimes L^2(\mathbb{R}^d)$ which transforms the Floquet operator K, namely*

$$K := -\mathrm{i}\omega \cdot \frac{\partial}{\partial\theta} + H_0 + \epsilon W(\theta),$$

into

$$-\mathrm{i}\omega \cdot \frac{\partial}{\partial\theta} + H_\infty.$$

Thus one has that the spectrum of K is pure point and its eigenvalues are $\lambda_j^\infty + \omega \cdot k$.

Notice that Enss and Veselić [1983, Theorems 2.3 and 3.2] proved that the spectrum of the Floquet operator is pure point if and only if every state is a bounded state. So Corollary 1.10 gives another proof of Corollary 1.9.

2. Proof of Theorem 1.1

To start, we scale the variables x_j by defining $x_j' = \sqrt{\nu_j}x_j$ so that, defining

$$h_j(x_j, \xi_j) := \xi_j^2 + x_j^2, \qquad H_j := -\partial_{x_j}^2 + x_j^2,$$

one has

$$h_0 = \sum_{j=1}^{d} \nu_j h_j, \qquad H_0 = \sum_{j=1}^{d} \nu_j H_j. \tag{2-1}$$

Remark 2.1. Notice that for any positive definite quadratic Hamiltonian h on $\mathbb{R}^{2d}$ there exists a symplectic basis such that $h = \sum_{j=1}^{d} \nu_j h_j$, with $\nu_j > 0$ for $1 \le j \le d$; see [Hörmander 1994].

For convenience in this paper we shall consider the Weyl quantization. The Weyl quantization of a symbol f is the operator $\mathrm{Op}^w(f)$, defined as usual as

$$\mathrm{Op}^w(f)u(x) = \frac{1}{(2\pi)^d} \int_{y,\xi\in\mathbb{R}^d} e^{\mathrm{i}(x-y)\xi}\, f\big(\tfrac{1}{2}x + y, \xi\big)\, u(y)\, dy\, d\xi.$$

Correspondingly we will say that an operator $F = \mathrm{Op}^w(f)$ is the Weyl operator with Weyl symbol f. Notice that for polynomials f of degree at most 2 in (x, ξ), we have $\mathrm{Op}^w(f) = f(x, D) + \mathrm{const}$, where $D = \mathrm{i}^{-1}\nabla_x$.

Most of the time we also use the notation $f^w(x, D) := \mathrm{Op}^w(f)$. In particular, in (1-2) $W(\omega t, x, -\mathrm{i}\partial_x)$ denotes the Weyl operator $W^w(\omega t, x, D)$.

Given a Hamiltonian $\chi = \chi(x, \xi)$, we will denote by ϕ_χ^t the flow of the corresponding classical Hamilton equations.

It is well known that, if f and g are symbols, then the operator $-\mathrm{i}[f^w(x,D); g^w(x,D)]$ admits a symbol denoted by $\{f;g\}_{\mathrm{M}}$ (the Moyal bracket). Two fundamental properties of quadratic polynomial symbols are given by the following well-known remarks.

Remark 2.2. If f or g is a polynomial of degree at most 2, then $\{f;g\}_{\mathrm{M}} = \{f;g\}$, where

$$\{f;g\} := \sum_{j=1}^{d} \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j} - \frac{\partial g}{\partial x_j}\frac{\partial f}{\partial \xi_j}$$

is the Poisson Bracket of f and g.

Remark 2.3. Let χ be a polynomial of degree at most 2; then it follows from the previous remark that, for any Weyl operator $f^w(x,D)$, the symbol of $e^{\mathrm{i}t\chi^w(x,D)} f^w(x,D) e^{-\mathrm{i}t\chi^w(x,D)}$ is $f \circ \phi^t_\chi$.

Remark 2.4. If f and g are not quadratic polynomials, then $\{f;g\}_{\mathrm{M}} = \{f;g\} +$ lower-order terms; similar lower-order corrections would appear in the symbol of $e^{-\mathrm{i}t\chi^w(x,D)} f^w(x,D) e^{\mathrm{i}t\chi^w(x,D)}$. That is the reason why we restrict here to the case of quadratic perturbations. In order to deal with more general perturbations one needs further ideas which will be developed elsewhere.

Next we need to know how a time-dependent transformation transforms a classical and a quantum Hamiltonian. Precisely, consider a 1-parameter family of (Hamiltonian) functions $\chi(t,x,\xi)$ (where t is thought of as an external parameter) and denote by $\phi^\tau(t,x,\xi)$ the time τ flow it generates, precisely the solution of

$$\frac{dx}{d\tau} = \frac{\partial \chi}{\partial \xi}(t,x,\xi), \qquad \frac{d\xi}{d\tau} = -\frac{\partial \chi}{\partial x}(t,x,\xi). \tag{2-2}$$

Consider the time-dependent coordinate transformation

$$(x,\xi) = \phi^1(t,x',\xi') := \phi^\tau(t,x',\xi')|_{\tau=1}. \tag{2-3}$$

Remark 2.5. Working in the extended phase space in which time and a new momentum conjugated to it are added, it is easy to see that the coordinate transformation (2-3) transforms a Hamiltonian system with Hamiltonian h into a Hamiltonian system with Hamiltonian h' given by

$$h'(t,x',\xi') = h(\phi^1(t,x',\xi')) - \int_0^1 \frac{\partial \chi}{\partial t}(t, \phi^\tau(t,x',\xi'))\, d\tau. \tag{2-4}$$

Remark 2.6. If the operator $\chi^w(t,x,D)$ is selfadjoint for any fixed t, then the transformation

$$\psi = e^{-\mathrm{i}\chi^w(t,x,D)}\psi' \tag{2-5}$$

transforms $\mathrm{i}\dot\psi = H\psi$ into $\mathrm{i}\dot\psi' = H'\psi'$ with

$$H' = e^{\mathrm{i}\chi^w(t,x,D)} H e^{-\mathrm{i}\chi^w(t,x,D)} - \int_0^1 e^{\mathrm{i}\tau\chi^w(t,x,D)} (\partial_t \chi^w(t,x,\xi))\, e^{-\mathrm{i}\tau\chi^w(t,x,D)}\, d\tau. \tag{2-6}$$

This is seen by an explicit computation. For example see Lemma 3.2 of [Bambusi 2017a].

So in view of Remark 2.3, provided that transformation (2-5) is well-defined in the quadratic case, the quantum transformed Hamiltonian (2-6) is the exact quantization of the transformed classical Hamiltonian (2-4).

To study the analytic properties of the transformation (2-5) we will use the following simplified version of Theorem 1.2 of [Maspero and Robert 2017] (to which we refer for the proof).

Theorem 2.7 [Maspero and Robert 2017]. *Let H_0 be the Hamiltonian of the harmonic oscillator. If X is an operator symmetric on $\mathcal{H}^\infty$ such that XH_0^{-1} and $[X, H_0]H_0^{-1}$ belong to $\mathcal{L}(\mathcal{H}^s)$ for any $s \geq 0$, then the Schrödinger equation*

$$\mathrm{i}\,\partial_\tau \psi = X\psi$$

is globally well-posed in $\mathcal{H}^s$ for any s, and its unitary propagator $e^{-\mathrm{i}\tau X}$ belongs to $\mathcal{L}(\mathcal{H}^s)$ for all $s \geq 0$. Furthermore one has the quantitative estimate

$$c_s\|\psi\|_s \leq \|e^{-\mathrm{i}\tau X}\psi\|_s \leq C_s\|\psi\|_s \quad \textit{for all } \tau \in [0, 1], \tag{2-7}$$

where the constants $c_s, C_s > 0$ depend only on $\|[X, H_0^s]H_0^{-s}\|_{\mathcal{L}(\mathcal{H}^0)}$.

The properties of the transformation are given by the next lemma and are closely related to the standard properties on the smoothness in time of the semigroup generated by an unbounded operator.

Lemma 2.8. *Let $\chi(\rho, x, \xi)$ be a polynomial in (x, ξ) of degree at most 2 with real coefficients depending in a C^∞-way on $\rho \in \mathbb{R}^n$. Then for all $\rho \in \mathbb{R}^n$, the operator $\chi^w(\rho, x, D)$ is selfadjoint in $L^2(\mathbb{R}^d)$. Furthermore for all $s \geq 0$, for all $\tau \in \mathbb{R}$, the following hold true:*

(i) *The map $\rho \mapsto e^{-\mathrm{i}\tau\chi^w(\rho,x,D)}$ is in $C^0(\mathbb{R}^n, \mathcal{L}(\mathcal{H}^{s+2}, \mathcal{H}^s))$.*

(ii) *For all $\psi \in \mathcal{H}^s$, the map $\rho \mapsto e^{-\mathrm{i}\tau\chi^w(\rho,x,D)}\psi$ is in $C^0(\mathbb{R}^n, \mathcal{H}^s)$.*

(iii) *For all $r \in \mathbb{N}$, the map $\rho \mapsto e^{-\mathrm{i}\tau\chi^w(\rho,x,D)}$ is in $C^r(\mathbb{R}^n, \mathcal{L}(\mathcal{H}^{s+4r+2}, \mathcal{H}^s))$.*

(iv) *If the coefficients of $\chi(\rho, x, \xi)$ are uniformly bounded in $\rho \in \mathbb{R}^n$ then for any $s > 0$ there exist $c_s > 0$, $C_s > 0$ such that we have*

$$c_s\|\psi\|_s \leq \|e^{-\mathrm{i}\tau\chi^w(\rho,x,D)}\psi\|_s \leq C_s\|\psi\|_s \quad \textit{for all } \rho \in \mathbb{R}^n,\ \textit{for all } \tau \in [0, 1].$$

Proof. First we remark that in this lemma the quantity ρ plays the role of a parameter. Since $\chi(\rho, x, \xi)$ is a real-valued polynomial in (x, ξ) of degree at most 2, the operator $\chi^w(\rho, x, D)$ is selfadjoint in $L^2(\mathbb{R}^d)$, so for all $\rho \in \mathbb{R}^n$ the propagator $e^{-\mathrm{i}\tau\chi^w(\rho,x,D)}$ is unitary on $L^2(\mathbb{R}^d)$.

In order to show that $e^{-\mathrm{i}\tau\chi^w(\rho,x,D)}$ maps $\mathcal{H}^s$ to itself, for all $s > 0$, for all $\rho \in \mathbb{R}^n$, we apply Theorem 2.7. Indeed since $\chi^w(\rho, x, D)$ has a polynomial symbol, we know $\chi^w(\rho, x, D)H_0^{-1}$ and the commutator $[H_0, \chi^w(\rho, x, D)]H_0^{-1}$ belong to $\mathcal{L}(\mathcal{H}^s)$ for all $s \geq 0$. Item (iv) follows by estimate (2-7) and the fact that $\|[H_0^s, \chi^w(\rho, x, D)]H_0^{-s}\|_{\mathcal{L}(\mathcal{H}^0)}$ is bounded uniformly in ρ.

To prove item (i) we use the Duhamel formula

$$e^{-\mathrm{i}\tau B} - e^{-\mathrm{i}\tau A} = \mathrm{i}\int_0^\tau e^{-\mathrm{i}(\tau-\tau_1)A}\,(A - B)\,e^{-\mathrm{i}\tau_1 B}\,d\tau_1. \tag{2-8}$$

Then choosing $B = \chi^w(\rho+\rho', x, D)$, $A = \chi^w(\rho, x, D)$ one has that for all $0 \le \tau \le 1$

$$\|e^{-\mathrm{i}\tau\chi^w(\rho+\rho',x,D)} - e^{-\mathrm{i}\tau\chi^w(\rho,x,D)}\|_{\mathcal{L}(\mathcal{H}^{s+2},\mathcal{H}^s)} \le C\|\chi^w(\rho+\rho', x, D) - \chi^w(\rho, x, D)\|_{\mathcal{L}(\mathcal{H}^{s+2},\mathcal{H}^s)}.$$

This proves item (i). Continuity in item (ii) is deduced by (i) with a standard density argument. Finally item (iii) is proved by induction on r, again using the Duhamel formula (2-8). □

Remark 2.5, Remark 2.6 and Lemma 2.8 imply the following important proposition.

Proposition 2.9. *Let $\chi(t, x, \xi)$ be a polynomial of degree at most 2 in x and ξ with smooth time-dependent coefficients. If the transformation (2-3) transforms a classical system with Hamiltonian h into a Hamiltonian system with Hamiltonian h', then the transformation (2-5) transforms the quantum system with Hamiltonian h^w into the quantum system with Hamiltonian $(h')^w$.*

As a consequence, for quadratic Hamiltonians, the quantum KAM theorem will follow from the corresponding classical KAM theorem.

To give the needed result, consider the classical time-dependent Hamiltonian

$$h_\epsilon(\omega t, x, \xi) := \sum_{1\le j\le d} \tfrac{1}{2}\nu_j(x_j^2+\xi_j^2) + \epsilon\, W(\omega t, x, \xi), \tag{2-9}$$

with W as in the Introduction. The following KAM theorem holds.

Theorem 2.10. *Assume $\nu_j \ge \nu_0 > 0$ for $j = 1, \dots, d$ and $\mathbb{T}^n \times \mathbb{R}^d \times \mathbb{R}^d \ni (\theta, x, \xi) \mapsto W(\theta, x, \xi) \in \mathbb{R}$ is a polynomial in (x, ξ) of degree at most 2 with coefficients which are real analytic functions of $\theta \in \mathbb{T}^n$.*

Then there exists $\epsilon_* > 0$ and $C > 0$, such that for $|\epsilon| < \epsilon_*$ the following hold true:

(i) *There exists a closed set $\mathcal{E}_\epsilon \subset (0, 2\pi)^n$ with $\operatorname{meas}((0, 2\pi)^n \setminus \mathcal{E}_\epsilon) \le C\epsilon^{1/9}$.*

(ii) *For any $\omega \in \mathcal{E}_\epsilon$, there exists an analytic map $\theta \mapsto A_\omega(\theta) \in \mathrm{sp}(2d)$ (the symplectic algebra[2] of dimension $2d$) and an analytic map $\theta \mapsto V_\omega(\theta) \in \mathbb{R}^{2d}$ such that the change of coordinates*

$$(x', \xi') = e^{A_\omega(\omega t)}(x, \xi) + V_\omega(\omega t) \tag{2-10}$$

conjugates the Hamiltonian equations of (2-9) to the Hamiltonian equations of a homogeneous polynomial $h_\infty(x, \xi)$ of degree 2 which is positive definite. Finally both A_ω and V_ω are ϵ-close to zero.

Furthermore h_∞ can be diagonalized: there exists a matrix $\mathcal{P} \in \mathrm{Sp}(2d)$ (the symplectic group of dimension $2d$) such that, setting $(y, \eta) = \mathcal{P}(x, \xi)$ we have

$$h_\infty \circ \mathcal{P}^{-1}(y, \eta) = \sum_{j=1}^{d} \nu_j^\infty(y_j^2 + \eta_j^2), \tag{2-11}$$

where $\nu_j^\infty = \nu_j^\infty(\omega)$ are defined on $\mathcal{E}_\epsilon$ and fulfill the estimates

$$|\nu_j^\infty - \nu_j| \le C\epsilon, \quad j = 1, \dots, d. \tag{2-12}$$

Remark 2.11. In general, the matrix $\mathcal{P}$ is not close to the identity. However, in the case that the frequencies ν_j are nonresonant, $\mathcal{P} = \mathbf{1}$.

[2]Recall that a real $2d \times 2d$ matrix A belongs to $\mathrm{sp}(2d)$ if and only if JA is symmetric

KAM theory in finite dimensions is nowadays standard. In particular we believe that Theorem 2.10 can be obtained combining the results of [Eliasson 1988; You 1999]. However, for the reader's convenience and the sake of being self-contained, we add in Section 3 its proof.

Theorem 1.1 follows immediately combining the results of Theorem 2.10 and Proposition 2.9.

Proof of Theorem 1.1. We see easily that the change of coordinates (2-10) has the form (2-3) with a Hamiltonian $\chi_\omega(\omega t, x, \xi)$ which is a polynomial in (x, ξ) of degree at most 2 with real, smooth and uniformly bounded coefficients in $t \in \mathbb{R}$.

Define $U_\omega(\omega t) = e^{-\mathrm{i}\chi_\omega^w(\omega t, x, D)}$. By Proposition 2.9 it conjugates the original equation (1-1) to (1-6), where $H_\infty := \mathrm{Op}^w(h_\infty)$.

Furthermore $\theta \mapsto U_\omega(\theta)$ fulfills (i)–(iv) of Lemma 2.8, from which it follows immediately that $\theta \mapsto U_\omega(\theta)$ fulfills items (i), (iii) of Theorem 1.1. Concerning item (ii), by the Taylor formula the quantity $\|\mathbf{1} - U_\omega(\theta)\|_{\mathcal{L}(\mathcal{H}^{s+2}, \mathcal{H}^s)}$ is controlled by $\|\chi_\omega^w(\theta, x, D)\|_{\mathcal{L}(\mathcal{H}^{s+2}, \mathcal{H}^s)}$, from which estimate (1-7) follows.

Finally using the metaplectic representation, see [Combescure and Robert 2012], and (2-11), there exists a unitary transformation in L^2, denoted by $\mathcal{R}(\mathcal{P}^{-1})$, such that

$$\mathcal{R}(\mathcal{P}^{-1})^* H_\infty \mathcal{R}(\mathcal{P}^{-1}) = \sum_{j=1}^{d} \nu_j^\infty (x_j^2 - \partial_{x_j}^2). \qquad \square$$

We prove now Corollary 1.3.

Proof of Corollary 1.3. Consider first the propagator $e^{-\mathrm{i}tH_\infty}$. We claim that

$$\sup_{t \in \mathbb{R}} \|e^{-\mathrm{i}tH_\infty}\|_{\mathcal{L}(\mathcal{H}^s)} < \infty \quad \text{for all } t \in \mathbb{R}. \tag{2-13}$$

Recall that $H_\infty = h_\infty^w(x, D)$, where $h_\infty(x, \xi)$ is a positive definite symmetric form which can be diagonalized by a symplectic matrix $\mathcal{P}$. Since h_∞ is positive definite, there exist $c_0, c_1, c_2 > 0$ such that

$$c_1 h_0(x, \xi) \le c_0 + h_\infty(x, \xi) \le c_2(1 + h_0(x, \xi)),$$

which implies $C_1 H_0 \le C_0 + H_\infty \le C_2(1 + H_0)$ as bilinear form. Thus one has the equivalence of norms

$$C_s^{-1} \|\psi\|_{\mathcal{H}^s} \le \|(H_\infty)^{s/2} \psi\|_{L^2} \le C_s \|\psi\|_{\mathcal{H}^s}.$$

Then

$$\|e^{-\mathrm{i}tH_\infty} \psi_0\|_{\mathcal{H}^s} \le C_s \|(H_\infty)^{s/2}\, e^{-\mathrm{i}tH_\infty} \psi_0\|_{L^2} = C_s \|(H_\infty)^{s/2}\, \psi_0\|_{L^2} \le C_s' \|\psi_0\|_{\mathcal{H}^s},$$

which implies (2-13).

Now let $\psi(t)$ be a solution of (1-1). By formula (1-8), $\psi(t) = U_\omega^*(\omega t) e^{-\mathrm{i}tH_\infty} U_\omega(0)\psi_0$. Then the upper bound in (1-9) follows easily from (2-13) and $\sup_t \|U_\omega(\omega t)\|_{\mathcal{L}(\mathcal{H}^s)} < \infty$, which is a consequence of Lemma 2.8. The lower bound follows by applying Lemma 2.8 (iv).

Finally estimate (1-10) follows from (1-7). $\square$

3. A classical KAM result

In this section we prove Theorem 2.10. We prefer to work in the extended phase space in which we add the angles $\theta \in \mathbb{T}^n$ as new variables and their conjugated momenta $I \in \mathbb{R}^n$. Furthermore we will use complex variables defined by

$$z_j = \frac{1}{\sqrt{2}}(\xi_j - \mathrm{i}x_j),$$

so that our phase space will be $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{C}^d$, with $\mathbb{C}^d$ considered as a real vector space. The symplectic form is $dI \wedge d\theta + \mathrm{i}\, dz \wedge d\bar{z}$ and the Hamilton equations of a Hamiltonian function $h(\theta, I, z, \bar{z})$ are

$$\dot{I} = -\frac{\partial h}{\partial \theta}, \quad \dot{\theta} = \frac{\partial h}{\partial I}, \quad \dot{z} = -\mathrm{i}\frac{\partial h}{\partial \bar{z}}.$$

In this framework h_0 takes the form $h_0 = \sum_{j=1}^d \nu_j z_j \bar{z}_j$ and W takes the form of polynomial in $(z, \bar{z})$ of degree 2, $W(\theta, x, \xi) = q(\theta, z, \bar{z})$. The Hamiltonian system associated with the time-dependent Hamiltonian h_ϵ, see (2-9), is then equivalent to the Hamiltonian system associated with the time-independent Hamiltonian $\omega \cdot I + h_\epsilon$ (written in complex variables) in the extended phase space.

General strategy. Let h be a Hamiltonian in normal form:

$$h(I, \theta, z, \bar{z}) = \omega \cdot I + \langle z, N(\omega)\bar{z}\rangle, \tag{3-1}$$

with $N \in \mathcal{M}_H$ the set of Hermitian matrices. Notice that at the beginning of the procedure N is diagonal,

$$N = N_0 = \mathrm{diag}(\nu_j,\ j = 1, \dots, d)$$

and is independent of ω. Let $q \equiv q_\omega$ be a polynomial Hamiltonian which takes real values: $q(\theta, z, \bar{z}) \in \mathbb{R}$ for $\theta \in \mathbb{T}^n$ and $z \in \mathbb{C}^d$. We write

$$q(\theta, z, \bar{z}) = \langle z, Q_{zz}(\theta)z\rangle + \langle z, Q_{z\bar{z}}(\theta)\bar{z}\rangle + \langle \bar{z}, \overline{Q}_{zz}(\theta)\bar{z}\rangle + \langle Q_z(\theta), z\rangle + \langle \overline{Q}_{\bar{z}}(\theta), \bar{z}\rangle, \tag{3-2}$$

where $Q_{zz}(\theta) \equiv Q_{zz}(\omega, \theta)$ and $Q_{z\bar{z}}(\theta) \equiv Q_{z\bar{z}}(\omega, \theta)$ are $d \times d$ complex matrices and $Q_z(\theta) \equiv Q_z(\theta, \omega)$ is a vector in $\mathbb{C}^d$. They all depend analytically on the angle

$$\theta \in \mathbb{T}^n_\sigma := \{x + iy : x \in \mathbb{T}^n,\ y \in \mathbb{R}^n,\ |y| < \sigma\}.$$

We notice that $Q_{z\bar{z}}$ is Hermitian, while Q_{zz} is symmetric. The size of such a polynomial function depending analytically on $\theta \in \mathbb{T}^n_\sigma$ and C^1 on $\omega \in \mathcal{D} = (0, 2\pi)^n$ will be controlled by the norm

$$[q]_\sigma := \sup_{\substack{|\mathrm{Im}\theta|<\sigma \\ \omega\in\mathcal{D},\ j=0,1}} \|\partial_\omega^j Q_{zz}(\omega, \theta)\| + \sup_{\substack{|\mathrm{Im}\theta|<\sigma \\ \omega\in\mathcal{D},\ j=0,1}} \|\partial_\omega^j Q_{z\bar{z}}(\omega, \theta)\| + \sup_{\substack{|\mathrm{Im}\theta|<\sigma \\ \omega\in\mathcal{D},\ j=0,1}} |\partial_\omega^j Q_z(\omega, \theta)|$$

and we denote by $\mathcal{Q}(\sigma)$ the class of Hamiltonians of the form (3-2) whose norm $[\,\cdot\,]_\sigma$ is finite.

Let us assume that $[q]_\sigma = \mathcal{O}(\epsilon)$. We search for $\chi \equiv \chi_\omega \in \mathcal{Q}(\sigma)$ with $[\chi]_\sigma = \mathcal{O}(\epsilon)$ such that its time-1 flow $\phi_\chi \equiv \phi_\chi^{t=1}$ (in the extended phase space, of course) transforms the Hamiltonian $h + q$ into

$$(h + q(\theta)) \circ \phi_\chi = h_+ + q_+(\theta), \quad \omega \in \mathcal{D}_+, \tag{3-3}$$

where $h_+ = \omega\cdot I + \langle z, N_+\bar z\rangle$ is a new normal form, ϵ-close to h, the new perturbation $q_+ \in \mathcal{Q}(\sigma)$ is of size[3] $\mathcal{O}(\epsilon^{3/2})$ and $\mathcal{D}_+ \subset \mathcal{D}$ is ϵ^α-close to $\mathcal{D}$ for some $\alpha > 0$. Notice that all the functions are defined on the whole open set $\mathcal{D}$ but (3-3) holds only on $\mathcal{D}_+$, a subset of $\mathcal{D}$ from which we excised the "resonant parts".

As a consequence of the Hamiltonian structure, we have

$$(h + q(\theta)) \circ \phi_\chi = h + \{h, \chi\} + q(\theta) + \mathcal{O}(\epsilon^{3/2}), \quad \omega \in \mathcal{D}_+.$$

So to achieve the goal above we should solve the *homological equation*:

$$\{h, \chi\} = h_+ - h - q(\theta) + \mathcal{O}(\epsilon^{3/2}), \quad \omega \in \mathcal{D}_+. \tag{3-4}$$

Repeating iteratively the same procedure with h_+ instead of h, we will construct a change of variable ϕ such that

$$(h + q(\theta)) \circ \phi = \omega \cdot I + h_\infty, \quad \omega \in \mathcal{D}_\infty,$$

with $h_\infty = \langle z, N_\infty(\omega)\bar z\rangle$ in normal form and $\mathcal{D}_\infty$ an ϵ^α-close subset of $\mathcal{D}$. Note that we will be forced to solve the homological equation not only for the diagonal normal form N_0, but for more general normal form Hamiltonians (3-1) with N close to N_0.

Homological equation.

Proposition 3.1. *Let $\mathcal{D} = (0, 2\pi)^n$ and $\mathcal{D} \ni \omega \mapsto N(\omega) \in \mathcal{M}_H$ be a C^1 mapping that satisfies*

$$\|\partial_\omega^j (N(\omega) - N_0)\| < \frac{\min(1, \nu_0)}{\max(4, d)} \tag{3-5}$$

for $j = 0, 1$ and $\omega \in \mathcal{D}$. Let $h = \omega \cdot I + \langle z, N\bar z\rangle$, $q \in \mathcal{Q}(\sigma)$, $\kappa > 0$ and $K \geq 1$.

Then there exists a closed subset $\mathcal{D}' = \mathcal{D}'(\kappa, K) \subset \mathcal{D}$ satisfying

$$\operatorname{meas}(\mathcal{D} \setminus \mathcal{D}') \leq C K^n \kappa, \tag{3-6}$$

and there exist $\chi, r \in \bigcap_{0 \leq \sigma' < \sigma} \mathcal{Q}(\sigma')$ and $\mathcal{D} \ni \omega \mapsto \widetilde N(\omega) \in \mathcal{M}_H$ a C^1 mapping such that for all $\omega \in \mathcal{D}'$

$$\{h, \chi\} + q = \langle z, \widetilde N \bar z\rangle + r. \tag{3-7}$$

Furthermore for all $\omega \in \mathcal{D}$

$$\|\partial_\omega^j \widetilde N(\omega)\| \leq [q]_\sigma, \quad j = 0, 1, \tag{3-8}$$

and for all $0 \leq \sigma' < \sigma$

$$[r]_{\sigma'} \leq C \frac{e^{-1/2(\sigma-\sigma')K}}{(\sigma - \sigma')^n} [q]_\sigma, \tag{3-9}$$

$$[\chi]_{\sigma'} \leq \frac{CK}{\kappa^2 (\sigma - \sigma')^n} [q]_\sigma. \tag{3-10}$$

[3]Formally we could expect q_+ to be of size $O(\epsilon^2)$ but the small divisors and the reduction of the analyticity domain will lead to an estimate of the type $\mathcal{O}(\epsilon^{3/2})$.

Proof. Writing the Hamiltonians h, q and χ as in (3-2), the homological equation (3-7) is equivalent to the three following equations (we use that N is Hermitian, thus $\overline{N} = {}^tN$):

$$\omega \cdot \nabla_\theta X_{z\bar{z}} - \mathrm{i}[N, X_{z\bar{z}}] = \widetilde{N} - Q_{z\bar{z}} + R_{z\bar{z}}, \tag{3-11}$$

$$\omega \cdot \nabla_\theta X_{zz} - \mathrm{i}(N X_{zz} + X_{zz}\overline{N}) = -Q_{zz} + R_{zz}, \tag{3-12}$$

$$\omega \cdot \nabla_\theta X_z + \mathrm{i} N X_z = -Q_z + R_z. \tag{3-13}$$

First we solve (3-11). To simplify notation we drop the indices $z\bar{z}$. Written in Fourier variables (with respect to θ), (3-11) reads as

$$\mathrm{i}\omega \cdot k\ \widehat{X}_k - \mathrm{i}[N, \widehat{X}_k] = \delta_{k,0}\widetilde{N} - \widehat{Q}_k + \widehat{R}_k, \quad k \in \mathbb{Z}^n, \tag{3-14}$$

where $\delta_{k,j}$ denotes the Kronecker symbol.

When $k = 0$ we solve this equation by defining

$$\widehat{X}_0 = 0, \quad \widehat{R}_0 = 0 \quad \text{and} \quad \widetilde{N} = \widehat{Q}_0.$$

We notice that $\widetilde{N} \in \mathcal{M}_H$ and satisfies (3-8).

When $|k| \geq K$, (3-14) is solved by defining

$$\widehat{R}_k = \widehat{Q}_k, \quad \widehat{X}_k = 0 \quad \text{for } |k| \geq K. \tag{3-15}$$

Then we set

$$\widehat{R}_k = 0 \quad \text{for } |k| \leq K$$

in such a way that $r \in \bigcap_{0 \leq \sigma' < \sigma} \mathcal{Q}(\sigma')$ and by a standard argument r satisfies (3-9). Now it remains to solve the equations for $\widehat{X}_k,\ 0 < |k| \leq K$, which we rewrite as

$$L_k(\omega)\widehat{X}_k = i\,\widehat{Q}_k, \tag{3-16}$$

where $L_k(\omega)$ is the linear operator from $\mathcal{M}_S$, the space of symmetric matrices, into itself defined by

$$L_k(\omega) : M \mapsto (k \cdot \omega)\, M - [N(\omega), M].$$

We notice that $\mathcal{M}_S$ can be endowed with the Hermitian product: $(A, B) = \mathrm{Tr}(\bar{A}B)$ associated with the Hilbert–Schmidt norm. Since N is Hermitian, $L_k(\omega)$ is selfadjoint for this structure. As a first consequence we get

$$\|(L_k(\omega))^{-1}\| \leq \frac{1}{\min\{|\lambda|,\ \lambda \in \Sigma(L_k(\omega))\}} = \frac{1}{\min\{|k \cdot \omega - \alpha(\omega) + \beta(\omega)| : \alpha, \beta \in \Sigma(N(\omega))\}}, \tag{3-17}$$

where for any matrix A, we denote its spectrum by $\Sigma(A)$.

Let us recall an important result of perturbation theory, which is a consequence of Theorem 1.10 in [Kato 1980] (since Hermitian matrices are normal matrices):

Theorem 3.2 [Kato 1980, Theorem 1.10]. *Let $I \subset \mathbb{R}$ and $I \ni z \mapsto M(z)$ be a holomorphic curve of Hermitian matrices. Then all the eigenvalues and associated eigenvectors of $M(z)$ can be parametrized holomorphically on I.*

Let us assume for a while that N depends analytically on ω in such a way that $\omega \mapsto L_k(\omega)$ is analytic. Fix a direction $z_k \in \mathbb{R}^n$; the eigenvalue $\lambda_k(\omega) = k \cdot \omega - \alpha(\omega) + \beta(\omega)$ of $L_k(\omega)$ is C^1 in the direction[4] z_k and the associated unitary eigenvector, denoted by $v(\omega)$, is also piecewise C^1 in the direction z_k. Then, as a consequence of the hermiticity of $L_k(\omega)$ we have

$$\partial_\omega \lambda(\omega) \cdot z_k = \langle v(\omega), (\partial_\omega L_k(\omega) \cdot z_k)\, v(\omega)\rangle.$$

Therefore, if N depends analytically of ω, we deduce using (3-5) and choosing $z_k = k/|k|$

$$\left| \partial_\omega \lambda_k(\omega) \cdot \frac{k}{|k|} \right| \geq |k| - 2\|\partial_\omega N\| \geq \tfrac{1}{2} \quad \text{for } k \neq 0, \tag{3-18}$$

which extends also to the points of discontinuity of $v(\omega)$. Now given a matrix L depending on the parameter $\omega \in \mathcal{D}$, we define

$$\mathcal{D}(L, \kappa) = \{\omega \in \mathcal{D} : \|L(\omega)^{-1}\| \leq \kappa^{-1}\}$$

and we recall the following classical lemma:

Lemma 3.3. *Let $f : [0, 1] \mapsto \mathbb{R}$ be a C^1-map satisfying $|f'(x)| \geq \delta$ for all $x \in [0, 1]$ and let $\kappa > 0$. Then*

$$\operatorname{meas}\{x \in [0, 1] : |f(x)| \leq \kappa\} \leq \frac{\kappa}{\delta}.$$

Combining this lemma, (3-17) and (3-18) we deduce that, if N depends analytically of ω, then for $k \neq 0$

$$\operatorname{meas}(\mathcal{D} \setminus \mathcal{D}(L_k, \kappa)) \leq C\kappa. \tag{3-19}$$

Now it turns out that, by a density argument, this last estimate remains valid (with a larger constant C) when N is only a C^1 function of ω; the point is that (3-18) holds true uniformly for close analytic approximations of N.

In particular, defining

$$\mathcal{D}' = \bigcap_{0<|k|\leq K} \mathcal{D}(L_k, \kappa),$$

$\mathcal{D}'$ is closed and satisfies (3-6).

By construction, $\widehat{X}_k(\omega) := i L_k(\omega)^{-1} \widehat{Q}_k$ satisfies (3-16) for $0 < |k| \leq K$ and $\omega \in \mathcal{D}(L_k, \kappa)$ and

$$\|\widehat{X}_k(\omega)\| \leq \kappa^{-1} \|\widehat{Q}_k(\omega)\|, \quad \omega \in \mathcal{D}(L_k, \kappa). \tag{3-20}$$

It remains to extend $\widehat{X}_k(\cdot)$ on $\mathcal{D}$. Using again (3-5) we have for any $|k| \leq K$ and any unit vector z, $|\partial_\omega \lambda(\omega) \cdot z| \leq CK$. Therefore

$$\operatorname{dist}\bigl(\mathcal{D} \setminus \mathcal{D}(L_k, \kappa), \mathcal{D}\bigl(L_k, \tfrac{1}{2}\kappa\bigr)\bigr) \geq \frac{\kappa}{CK}$$

and we can construct (by a convolution argument) for each k, $0 < |k| \leq K$, a C^1 function g_k on $\mathcal{D}$ with

$$|g_k|_{C^0(\mathcal{D})} \leq C, \quad |g_k|_{C^1(\mathcal{D})} \leq CK\kappa^{-1} \tag{3-21}$$

[4]That is, $t \mapsto \lambda_k(\omega + t z_k)$ is a holomorphic curve on a neighborhood of 0, and we denote by $\partial_\omega \lambda(\omega) \cdot z_k$ its derivative at $t = 0$.

(the constant C is independent of k) and such that $g_k(\omega) = 1$ for $\omega \notin \mathcal{D}(L_k, \kappa)$ and $g_k(\omega) = 0$ for $\omega \in \mathcal{D}\big(L_k, \frac{1}{2}\kappa\big)$. Then $\widetilde{X}_k = g_k \widehat{X}_k$ is a C^1 extension of $\widehat{X}_k$ to $\mathcal{D}$. Similarly we define $\widetilde{Q}_k = g_k \widehat{Q}_k$ in such a way that $\widetilde{X}_k$ satisfies

$$L_k(\omega)\widetilde{X}_k(\omega) = i\,\widetilde{Q}_k(\omega), \quad 0 < |k| \le K,\ \omega \in \mathcal{D}.$$

Differentiating with respect to ω leads to

$$L_k(\omega)\,\partial_{\omega_j}\widehat{X}(k) = \mathrm{i}\,\partial_{\omega_j}\widehat{Q}(k) - k_j\widehat{X}(k) + [\partial_{\omega_j}N, \widehat{X}(k)], \quad 1 \le j \le n.$$

Defining $B_k(\omega) = \mathrm{i}\,\partial_{\omega_j}\widetilde{Q}_k(\omega) - k_j\widetilde{X}_k(\omega) + [\partial_{\omega_j}N(\omega), \widetilde{X}_k(\omega)]$ we have

$$\|\partial_{\omega_j}\widetilde{X}_k(\omega)\| \le \kappa^{-1}\|B_k(\omega)\|, \quad \omega \in \mathcal{D}.$$

Using (3-5), (3-20) and (3-21) we get for $|k| \le K$ and $\omega \in \mathcal{D}$

$$\begin{aligned}\|B_k(\omega)\| &\le \|\partial_{\omega_j}\widetilde{Q}_k(\omega)\| + K\|\widetilde{X}_k(\omega)\| + 2\|\partial_{\omega_j}N(\omega)\|\,\|\widetilde{X}_k(\omega)\|\\ &\le CK\kappa^{-1}(\|\partial_{\omega_j}\widehat{Q}(k,\omega)\| + \|\widehat{Q}(k,\omega)\|).\end{aligned}$$

Combining the last two estimates we get

$$\sup_{\omega\in\mathcal{D},\, j=0,1}\|\partial_\omega^j\widetilde{X}_k(\omega)\| \le CK\kappa^{-2}\sup_{\omega\in\mathcal{D},\, j=0,1}\|\partial_\omega^j\widehat{Q}_k(\omega)\|.$$

Thus defining

$$X_{z\bar{z}}(\omega,\theta) = \sum_{0<|k|\le K}\widetilde{X}_k(\omega)e^{ik\cdot\theta},$$

$X_{z\bar{z}}(\omega,\cdot)$ satisfies (3-11) for $\omega \in \mathcal{D}'$ and leads to (3-10) for $\chi_{z\bar{z}}(\omega,\theta,z,\bar{z}) = \langle z, X_{z\bar{z}}(\omega,\cdot)\bar{z}\rangle$.

We solve (3-13) in a similar way. We notice that in this case we face the small divisors $|\omega\cdot k - \alpha(\omega)|$, $k \in \mathbb{Z}^n$, where $\alpha \in \Sigma(N(\omega))$. In particular for $k = 0$ these quantities are $\ge \frac{1}{2}\nu_0$ since $|\alpha - \nu_j| \le \frac{1}{4}\nu_0$ for some $1 \le j \le d$ by (3-5).

Writing in Fourier variables and dropping indices zz, (3-12) reads as

$$\mathrm{i}\omega\cdot k\ \widehat{X}(k) - \mathrm{i}(N\widehat{X}(k) + \widehat{X}(k)\overline{N}) = -\widehat{Q}(k) + \widehat{R}(k). \tag{3-22}$$

So to mimic the resolution of (3-14) we have to replace the operator $L_k(\omega)$ by the operator $M_k(\omega)$, defined on $\mathcal{M}_S$ by

$$M_k(\omega)X := \omega\cdot k + NX + X\overline{N}.$$

This operator is still selfadjoint for the Hermitian product $(A, B) = \mathrm{Tr}(\bar{A}B)$ so the same strategy applies. Nevertheless we have to consider differently the case $k = 0$. In that case we use that the eigenvalues of $M_0(\omega)$ are close to eigenvalues of the operator M_0 defined by

$$M_0 : X \mapsto N_0X + X\overline{N}_0 = N_0X + XN_0,$$

with $N_0 = \mathrm{diag}(\nu_j,\ j = 1,\dots,d)$ a real and diagonal matrix. Actually in view of (3-5)

$$\|(L - L_0)M\|_{\mathrm{HS}} \le \|N - N_0\|_{\mathrm{HS}}\,\|M\|_{\mathrm{HS}} \le d\|N - N_0\|\,\|M\|_{\mathrm{HS}} \le \nu_0.$$

The eigenvalues of L_0 are $\{\nu_j+\nu_\ell : j,\ell=1,\dots,d\}$ and they are all larger than $2\nu_0$. We conclude that all the eigenvalues of $M_0(\omega)$ satisfy $|\alpha(\omega)|\geq \nu_0$. The end of the proof follows as before. □

The KAM step. Theorem 2.10 is proved by an iterative KAM procedure. We begin with the initial Hamiltonian h_0+q_0, where

$$h_0(I,\theta,z,\bar z)=\omega\cdot I+\langle z, N_0\bar z\rangle, \tag{3-23}$$

$N_0=\operatorname{diag}(\nu_j,\ j=1,\dots,d)$, $\omega\in\mathcal{D}\equiv[1,2]^n$ and the quadratic perturbation q_0 equals $\epsilon W\in\mathcal{Q}(\sigma,\mathcal{D})$ for some $\sigma>0$. Then we construct iteratively the change of variables ϕ_m, the normal form $h_m=\omega\cdot I+\langle z, N_m\bar z\rangle$ and the perturbation $q_m\in\mathcal{Q}(\sigma_m,\mathcal{D}_m)$ as follows: Assume that the construction is done up to step $m\geq 0$. Then:

(i) Using Proposition 3.1 we construct χ_{m+1}, r_{m+1} and $\widetilde{N}_m$ the solution of the homological equation:

$$\{h,\chi_{m+1}\}=\langle z,\widetilde{N}_m\bar z\rangle-q_m(\theta)+r_{m+1},\quad \omega\in\mathcal{D}_{m+1},\theta\in\mathbb{T}^n_{\sigma_{m+1}}. \tag{3-24}$$

(ii) We define $h_{m+1}:=\omega\cdot I+\langle z, N_{m+1}\bar z\rangle$ by

$$N_{m+1}=N_m+\widetilde{N}_m, \tag{3-25}$$

and

$$q_{m+1}:=r_m+\int_0^1\{(1-t)(h_{m+1}-h_m+r_{m+1})+tq_m,\chi_{m+1}\}\circ\phi^t_{\chi_{m+1}}\,\mathrm{d}t. \tag{3-26}$$

By construction, if Q_m and N_m are Hermitian, so are R_m and S_{m+1} by the resolution of the homological equation, and also N_{m+1} and Q_{m+1}.

For any regular Hamiltonian f we have, using the Taylor expansion of $f\circ\phi^t_{\chi_{m+1}}$ between $t=0$ and $t=1$,

$$f\circ\phi^1_{\chi_{m+1}}=f+\{f,\chi_{m+1}\}+\int_0^1(1-t)\{\{f,\chi_{m+1}\},\chi_{m+1}\}\circ\phi^t_{\chi_{m+1}}\,\mathrm{d}t.$$

Therefore we get for $\omega\in\mathcal{D}_{m+1}$

$$(h_m+q_m)\circ\phi^1_{\chi_{m+1}}=h_{m+1}+q_{m+1}.$$

Iterative lemma. Following the general scheme above we have

$$(h_0+q_0)\circ\phi^1_{\chi_1}\circ\cdots\circ\phi^1_{\chi_m}=h_m+q_m,$$

where q_m is a polynomial of degree 2 and $h_m=\omega\cdot I+\langle z,N_m\bar z\rangle$ with N_m a Hermitian matrix. At step m the Fourier series are truncated at order K_m and the small divisors are controlled by κ_m. Now we specify the choice of all the parameters for $m\geq 0$ in terms of ϵ_m, which will control $[q_m]_{\mathcal{D}_m,\sigma_m}$.

First we define $\epsilon_0=\epsilon$, $\sigma_0=\sigma$, $\mathcal{D}_0=\mathcal{D}$ and for $m\geq 1$ we choose

$$\sigma_{m-1}-\sigma_m=C_*\sigma_0 m^{-2},\quad K_m=2(\sigma_{m-1}-\sigma_m)^{-1}\ln\epsilon_{m-1}^{-1},\quad \kappa_m=\epsilon_{m-1}^{1/8},$$

where $(C_*)^{-1}=2\sum_{j\geq 1}1/j^2$.

Lemma 3.4. *There exists $\epsilon_* > 0$ depending on d, n such that, for $|\epsilon| \le \epsilon_*$ and*

$$\epsilon_m = \epsilon^{(3/2)^m}, \quad m \ge 0,$$

we have the following:

For all $m \ge 1$ there exist closed subsets $\mathcal{D}_m \subset \mathcal{D}_{m-1}$, $h_m = \omega \cdot I + \langle z, N_m \bar{z}\rangle$ in normal form, where $\mathcal{D}_m \ni \omega \mapsto N_m(\omega) \in \mathcal{M}_H \in C^1$, and there exist $\chi_m, q_m \in \mathcal{Q}(\mathcal{D}_m, \sigma_m)$ such that for $m \ge 1$:

(i) *The symplectomorphism*

$$\phi_m \equiv \phi_{\chi_m}(\omega) : \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2d} \to \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2d}, \quad \omega \in \mathcal{D}_m, \tag{3-27}$$

is an affine transformation in $(z, \bar{z})$, analytic in $\theta \in \mathbb{T}^n_{\sigma_m}$ and C^1 in $\omega \in \mathcal{D}_m$ of the form

$$\phi_m(I, \theta, z, \bar{z}) = (g_m(I, \theta, z, \bar{z}), \theta, \Psi_m(\theta, z, \bar{z})), \tag{3-28}$$

where, for each $\theta \in \mathbb{T}^n$, $(z, \bar{z}) \mapsto \Psi_m(\theta, z, \bar{z})$ is a symplectic change of variable on $\mathbb{C}^{2n}$.

The map ϕ_m links the Hamiltonian at step $m-1$ and the Hamiltonian at step m; *i.e.*,

$$(h_{m-1} + q_{m-1}) \circ \phi_m = h_m + q_m \quad \textit{for all } \omega \in \mathcal{D}_m.$$

(ii) *We have the estimates*

$$\operatorname{meas}(\mathcal{D}_{m-1} \setminus \mathcal{D}_m) \le \epsilon_{m-1}^{1/9}, \tag{3-29}$$

$$[\widetilde{N}_{m-1}]_{s,\beta}^{\mathcal{D}_m} \le \epsilon_{m-1}, \tag{3-30}$$

$$[q_m]_{s,\beta}^{\mathcal{D}_m,\sigma_m} \le \epsilon_m, \tag{3-31}$$

$$\|\phi_m(\omega) - \mathbf{1}\|_{\mathcal{L}(\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2d})} \le C\epsilon_{m-1}^{1/2} \quad \textit{for all } \omega \in \mathcal{D}_m. \tag{3-32}$$

Proof. At step 1, $h_0 = \omega \cdot I + \langle z, N_0 \bar{z}\rangle$ and thus hypothesis (3-5) is trivially satisfied and we can apply Proposition 3.1 to construct χ_1, N_1, r_1 and $\mathcal{D}_1$ such that for $\omega \in \mathcal{D}_1$

$$\{h_0, \chi_1\} = \langle z, (N_1 - N_0)\bar{z}\rangle - q_0 + r_1.$$

Then, using (3-6), we have

$$\operatorname{meas}(\mathcal{D} \setminus \mathcal{D}_1) \le C K_1^n \kappa_1 \le \epsilon_0^{1/9}$$

for $\epsilon = \epsilon_0$ small enough. Using (3-10) we have for ϵ_0 small enough

$$[\chi_1]_{\mathcal{D}_1,\sigma_1} \le C \frac{K_1}{\kappa_1^2 (\sigma_0 - \sigma_1)^n} \epsilon_0 \le \epsilon_0^{1/2}.$$

Similarly using (3-9), (3-8) we have

$$\|N_1 - N_0\| \le \epsilon_0 \quad \text{and} \quad [r_1]_{\mathcal{D}_1,\sigma_1} \le C \frac{\epsilon_0^{15/8}}{(\sigma_1 - \sigma_0)^n} \le \epsilon_0^{7/4}$$

for $\epsilon = \epsilon_0$ small enough. In particular we deduce $\|\phi_1 - \mathbf{1}\|_{\mathcal{L}(\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2d})} \le \epsilon_0^{1/2}$. Thus using (3-26) we get for ϵ_0 small enough

$$[q_1]_{\mathcal{D}_1,\sigma_1} \le \epsilon_0^{3/2} = \epsilon_1.$$

The form of the flow (3-28) follows since χ_1 is a Hamiltonian of the form (3-2).

Now assume that we have verified Lemma 3.4 up to step m. We want to perform the step $m+1$. We have $h_m = \omega \cdot I + \langle z, N_m \bar{z}\rangle$ and since

$$\|N_m - N_0\| \le \|N_m - N_0\| + \cdots + \|N_1 - N_0\| \le \sum_{j=0}^{m-1} \epsilon_j \le 2\epsilon_0,$$

hypothesis (3-5) is satisfied and we can apply Proposition 3.1 to construct $\mathcal{D}_{m+1}$, χ_{m+1} and q_{m+1}. Estimates (3-29)–(3-32) at step $m+1$ are proved as we have proved the corresponding estimates at step 1. □

Transition to the limit and proof of Theorem 2.10. Let $\mathcal{E}_\epsilon = \bigcap_{m\ge 0} \mathcal{D}_m$. In view of (3-29), this is a closed set satisfying

$$\operatorname{meas}(\mathcal{D} \setminus \mathcal{E}_\epsilon) \le \sum_{m\ge 0} \epsilon_m^{1/9} \le 2\epsilon_0^{1/9}.$$

Let us set $\tilde{\phi}_N = \phi_1 \circ \cdots \circ \phi_N$. Due to (3-32) it satisfies for $M \le N$ and for $\omega \in \mathcal{E}_\epsilon$

$$\|\tilde{\phi}_N - \tilde{\phi}_{\mathrm{M}}\|_{\mathcal{L}(\mathbb{R}^n\times\mathbb{T}^n\times\mathbb{C}^{2d})} \le \sum_{m=M}^{N} \epsilon_m^{1/2} \le 2\epsilon_{\mathrm{M}}^{1/2}.$$

Therefore $(\tilde{\phi}_N)_N$ is a Cauchy sequence in $\mathcal{L}(\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2d})$. Thus when $N \to \infty$, the mappings $\tilde{\phi}_N$ converge to a limit mapping $\phi_\infty \in \mathcal{L}(\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2d})$. Furthermore since the convergence is uniform on $\omega \in \mathcal{E}_\epsilon$ and $\theta \in \mathbb{T}_{\sigma/2}$, we know ϕ^1_∞ depends analytically on θ and C^1 in ω. Moreover,

$$\|\phi_\infty - \mathbf{1}\|_{\mathcal{L}(\mathbb{R}^n\times\mathbb{T}^n\times\mathbb{C}^{2d})} \le \epsilon_0^{1/2}. \tag{3-33}$$

By construction, the map $\tilde{\phi}_m$ transforms the original Hamiltonian $h_0 + q_0$ into $h_m + q_m$. When $m \to \infty$, by (3-31) we get $q_m \to 0$ and by (3-30) we get $N_m \to N$, where

$$N \equiv N(\omega) = N_0 + \sum_{k=1}^{+\infty} \widetilde{N}_k$$

is a Hermitian matrix which is C^1 with respect to $\omega \in \mathcal{E}_\epsilon$. Setting $h_\infty(z, \bar{z}) = \omega \cdot I + \langle z, N(\omega)\bar{z}\rangle$ we have proved

$$(h + q(\theta)) \circ \phi_\infty = h_\infty. \tag{3-34}$$

Furthermore for all $\omega \in \mathcal{E}_\epsilon$ we have, using (3-30),

$$\|N(\omega) - N_0\| \le \sum_{m=0}^{\infty} \epsilon_m \le 2\epsilon$$

and thus the eigenvalues of $N(\omega)$, denoted by $\nu_j^\infty(\omega)$, satisfy (2-12).

It remains to give the affine symplectomorphism ϕ_∞. At each step of the KAM procedure we have by Lemma 3.4

$$\phi_m(I, \theta, z, \bar{z}) = (g_m(I, \theta, z, \bar{z}), \theta, \Psi_m(\theta, z, \bar{z})),$$

and therefore

$$\phi_\infty(I, \theta, z, \bar{z}) = (g(I, \theta, z, \bar{z}), \theta, \Psi(\theta, z, \bar{z})),$$

where $\Psi(\theta, z, \bar{z}) = \lim_{m\to\infty} \Psi_1 \circ \Psi_2 \circ \cdots \circ \Psi_m$.

It is useful to go back to real variables (x, ξ). More precisely, write each Hamiltonian χ_m constructed in the KAM iteration in the variables (x, ξ):

$$\chi_m(\theta, x, \xi) = \frac{1}{2}\begin{bmatrix} x \\ \xi \end{bmatrix} \cdot E\, B_m(\theta) \begin{bmatrix} x \\ \xi \end{bmatrix} + U_m(\theta), \qquad E := \begin{bmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix}, \tag{3-35}$$

where $B_m(\theta)$ is a skew-symmetric matrix of dimension $2d \times 2d$ and $U_m(\theta) \in \mathbb{R}^{2d}$, and they are both of size ϵ_m. Then Ψ_m written in the real variables has the form

$$\Psi_m(\theta, x, \xi) = e^{B_m(\theta)}(x, \xi) + T_m(\theta), \quad \text{where } T_m(\theta) := \int_0^1 e^{(1-s)JB_m(\theta)} U_m(\theta)\, ds. \tag{3-36}$$

Lemma 3.5. *There exists a sequence of Hamiltonian matrices $A_l(\theta)$ and vectors $V_l(\theta) \in \mathbb{R}^{2d}$ such that*

$$\Psi_1 \circ \cdots \circ \Psi_l(x, \xi) = e^{A_l(\theta)}(x, \xi) + V_l(\theta) \quad \textit{for all } (x, \xi) \in \mathbb{R}^{2d}. \tag{3-37}$$

Furthermore, there exist an Hamiltonian matrix $A_\omega(\theta)$ and a vector $V_\omega(\theta) \in \mathbb{R}^{2d}$ such that

$$\begin{aligned} &\lim_{l\to+\infty} e^{A_l(\theta)} = e^{A_\infty(\theta)}, && \lim_{l\to+\infty} V_l(\theta) = V_\infty(\theta), \\ &\sup_{|\mathrm{Im}\theta|\le\sigma/2} \|A_\omega(\theta)\| \le C\epsilon, && \sup_{|\mathrm{Im}\theta|\le\sigma/2} |V_\omega(\theta)| \le C\epsilon, \end{aligned} \tag{3-38}$$

and for each $\theta \in \mathbb{T}^n$,

$$\Psi(\theta, x, \xi) = e^{A_\omega(\theta)}(x, \xi) + V_\omega(\theta) \quad \textit{for all } (x, \xi) \in \mathbb{R}^{2d}.$$

Proof. Recall that $\phi_j = e^{B_j} + T_j$, where T_j is a translation by the vector T_j with the estimates $\|B_j\| \le C\epsilon_j$, $\|T_j\| \le C\epsilon_j$. So we have $e^{B_j} = \mathbb{I} + S_j$ with $\|S_j\| \le C\epsilon_j$. Then the infinite product $\prod_{1\le j<+\infty} e^{B_j}$ is convergent. Moreover we have $\prod_{1\le j\le l} e^{B_j} = \mathbb{I} + M_l$ with $\|M_l\| \le C\epsilon$, so we have $\prod_{1\le j<+\infty} e^{B_j} = \mathbb{I} + M$, with $\|M\| \le C\epsilon$. This is proved by using

$$\prod_{1\le j\le l} (\mathbb{I} + S_j) = \mathbb{I} + S_l + S_{l-1}S_l + \cdots + S_1 S_2 \cdots S_l$$

and estimates on $\|S_j\|$.

So, M_l has a small norm and therefore $A_l := \log(\mathbb{I} + M_l)$ is well-defined. Furthermore, by construction $\mathbb{I} + M_l \in \mathrm{Sp}(2d)$ and therefore its logarithm is a Hamiltonian matrix, namely $A_l \in \mathrm{sp}(2d)$ for $1 \le l \le +\infty$.

Now we have to include the translations. By induction on l we have

$$\phi_1 \circ \cdots \circ \phi_l(x, \xi) = \mathrm{e}^{A_l}(x, \xi) + V_l,$$

with $V_{l+1} = \mathrm{e}^{A_l} T_{l+1} + V_l$ and $V_1 = T_1$. Using the previous estimates we have

$$\|V_{l+1} - V_l\| \le C\|T_{l+1}\| \le C\epsilon_l.$$

Then we get that $\lim_{l\to+\infty} V_l = V_\infty$ exists. $\square$

Appendix: An example of growth of Sobolev norms (following Graffi and Yajima)

In this appendix we are going to study the Hamiltonian

$$H := -\tfrac{1}{2}\partial_{xx} + \tfrac{1}{2}x^2 + ax\sin\omega t \tag{A-1}$$

and prove that it is reducible to the harmonic oscillator if $\omega \neq \pm 1$, while the system exhibits growth of Sobolev norms in the case $\omega = \pm 1$. Actually the result holds in a quite more general situation, but we think that the present example can give a full understanding of the situation with as few techniques as possible. We also remark that in this case it is not necessary to assume that the time-dependent part is small.

Finally we recall that (A-1) with $\omega = \pm 1$ was studied by Graffi and Yajima as an example of a Hamiltonian whose Floquet spectrum is absolutely continuous (despite the fact that the unperturbed Hamiltonian has discrete spectrum). Exploiting the results of [Enss and Veselić 1983; Bunimovich et al. 1991], one can conclude from [Graffi and Yajima 2000] that the expectation value of the energy is not bounded in this model. The novelty of the present result rests in the much more precise statement ensuring growth of Sobolev norms.

As we already pointed out, in order to get reducibility of the Hamiltonian (A-1), it is enough to study the corresponding classical Hamiltonian, in particular proving its reducibility; this is what we will do. It also turns out that the whole procedure is clearer working as much as possible at the level of the equations.

So, consider the classical Hamiltonian system

$$h := \tfrac{1}{2}(x^2 + \xi^2) + ax\sin(\omega t), \tag{A-2}$$

whose equations of motion are

$$\begin{cases} \dot{x} = \xi, \\ \dot{\xi} = -x - a\sin(\omega t) \end{cases} \iff \ddot{x} + x + a\sin(\omega t) = 0. \tag{A-3}$$

Proposition A.1. *Assume that $\omega \neq \pm 1$. Then there exists a time-periodic canonical transformation conjugating* (A-2) *to*

$$h' := \tfrac{1}{2}(x^2 + \xi^2). \tag{A-4}$$

If $\omega = \pm 1$ then the system is canonically conjugated to

$$h' := \pm\tfrac{1}{2}a\xi. \tag{A-5}$$

In both cases the transformation has the form (2-10).

Corollary A.2. *In the case $\omega = \pm 1$, for any $s > 0$ and $\psi_0 \in \mathcal{H}^s$, there exists a constant $0 < C_s = C_s(\|\psi_0\|_{\mathcal{H}^s})$ such that the solution of the Schrödinger equation with Hamiltonian* (A-1) *and initial datum ψ_0 fulfills*

$$\|\psi(t)\|_{\mathcal{H}^s} \geq C_s\langle t\rangle^s \quad \textit{for all } t \in \mathbb{R}. \tag{A-6}$$

Before proving the theorem, recall that by the general result of [Maspero and Robert 2017, Theorem 1.5], any solution of the Schrödinger equation with Hamiltonian (A-1) fulfills the a priori bound

$$\|\psi(t)\|_{\mathcal{H}^s} \leq C'_s(\|\psi_0\|_{\mathcal{H}^s} + |t|^s\|\psi_0\|_{\mathcal{H}^0}) \quad \text{for all } t \in \mathbb{R}, \tag{A-7}$$

which is therefore sharp.

Proof of Proposition A.1. We look for a translation

$$x = x' - f(t), \quad \xi = \xi' - g(t), \tag{3-8}$$

with f and g time-periodic functions to be determined in such a way to eliminate time from (A-3). Writing the equations for (x', ξ'), one gets

$$\dot{x}' = \xi' - g + \dot{f}, \quad \dot{\xi}' = -x' - a\sin(\omega t) + \dot{g} + f,$$

which reduces to the harmonic oscillator by choosing

$$\begin{cases} -a\sin(\omega t) + \dot{g} + f = 0, \\ -g + \dot{f} = 0 \end{cases} \iff \ddot{f} + f = a\sin(\omega t), \tag{3-9}$$

which has a solution of period $2\pi/\omega$ only if $\omega \neq \pm 1$. In such a case the only solution having the correct period is

$$f = \frac{a}{1-\omega^2}\sin(\omega t), \quad g = \frac{a\omega}{1-\omega^2}\cos(\omega t).$$

Then the transformation (3-8) is a canonical transformation generated as the time-1 flow of the auxiliary Hamiltonian

$$\chi := -\xi\frac{a}{1-\omega^2}\sin(\omega t) + x\frac{a\omega}{1-\omega^2}\cos(\omega t),$$

which thus conjugates the classical Hamiltonian (A-2) to the harmonic oscillator; of course the quantization of χ conjugates the quantum system to the quantum harmonic oscillator, as follows by Proposition 2.9.

We come to the resonant case, and, in order to fix ideas, we take $\omega = 1$. In such a case the flow of the harmonic oscillator is periodic of the same period as the forcing, and thus its flow can be used to reduce the system.

In a slightly more abstract way, consider a Hamiltonian system with Hamiltonian

$$H := \tfrac{1}{2}\langle z; Bz\rangle + \langle z; b(t)\rangle,$$

with $z := (x, \xi)$, B a symmetric matrix, and $b(t)$ a vector-valued time-periodic function. Then, using the formula (2-4), it is easy to see that the auxiliary time-dependent Hamiltonian

$$\chi_1 := \tfrac{1}{2}t\langle z; Bz\rangle \tag{3-10}$$

generates a time-periodic transformation which conjugates the system to

$$h' := \langle z; e^{-JBt}b(t)\rangle$$

(J being the standard symplectic matrix). An explicit computation shows that in our case

$$h' = \tfrac{1}{2}ax\sin(2t) - \tfrac{1}{2}a\xi\cos(2t) + \tfrac{1}{2}a\xi. \tag{3-11}$$

Then in order to eliminate the two time-periodic terms in (3-11) it is sufficient to use the canonical transformation generated by the Hamiltonian

$$\chi_2 := -\tfrac{1}{4}\xi a\sin(2t) - \tfrac{1}{4}xa\cos(2t), \tag{3-12}$$

which reduces to (A-5). $\square$

Proof of Corollary A.2. To fix ideas we take $\omega = 1$. Let $\chi_1^w \equiv \frac{1}{2}t(-\partial_{xx} + x^2)$ and χ_2^w be the Weyl quantizations of the Hamiltonians (3-10) and (3-12) respectively. By the proof of Proposition A.1, the changes of coordinates

$$\psi = e^{-\mathrm{i}tH_0}\psi_1, \quad \psi_1 = e^{-\mathrm{i}\chi_2^w(t,x,D)}\varphi, \quad H_0 := \tfrac{1}{2}(-\partial_{xx} + x^2), \tag{3-13}$$

conjugate the Schrödinger equation with Hamiltonian (A-1) to the Schrödinger equation with Hamiltonian (A-2), namely the transport equation

$$\partial_t \varphi = -\tfrac{1}{2}a\partial_x\varphi.$$

The solution of this transport equation is given clearly by

$$\varphi(t,x) = \varphi_0\big(x - \tfrac{1}{2}at\big),$$

where φ_0 is the initial datum. Now a simple computation shows that

$$\liminf_{|t|\to+\infty} |t|^{-s}\|\varphi(t)\|_{\mathcal{H}^s} \geq \big(\tfrac{1}{2}|a|\big)^s \|\varphi_0\|_{\mathcal{H}^0}.$$

In particular there exists a constant $0 < C_s = C_s(\|\varphi_0\|_{\mathcal{H}^s})$ such that

$$\|\varphi(t)\|_{\mathcal{H}^s} \geq C_s\langle t\rangle^s. \tag{3-14}$$

Since the transformation (3-13) maps $\mathcal{H}^s$ to $\mathcal{H}^s$ uniformly in time (see also Lemma 2.8) estimate (3-14) holds also for the original variables. □

We remark that by a similar procedure one can also prove the following slightly more general result.

Theorem 3.3. *Consider the classical Hamiltonian system*

$$h = \sum_{j=1}^{d} \tfrac{1}{2}\nu_j(x_j^2 + \xi_j^2) + \sum_{j=1}^{d}(g_j(\omega t)x_j + f_j(\omega t)\xi_j), \tag{3-15}$$

with $f_j, g_j \in C^r(\mathbb{T}^n)$.

(1) *If there exist* $\gamma > 0$ *and* $\tau > n+1$ *such that*

$$|\omega\cdot k \pm \nu_j| \geq \frac{\gamma}{1+|k|^\tau} \quad \textit{for all } k \in \mathbb{Z}^n, \quad j = 1,\dots,d, \tag{3-16}$$

and $r > \tau + 1 + \frac{1}{2}n$, *then there exists a time-quasiperiodic canonical transformation of the form* (2-10) *conjugating the system to*[5]

$$h = \sum_{j=1}^{d} \tfrac{1}{2}\nu_j(x_j^2 + \xi_j^2).$$

(2) *If there exist* $0 \neq \bar{k} \in \mathbb{Z}^n$ *and* $\bar{j}$, *such that*

$$\omega\cdot\bar{k} - \nu_{\bar{j}} = 0, \tag{3-17}$$

[5] Actually the transformation is just a translation, so in this case one has $A \equiv 0$.

and there exist $\gamma > 0$ and τ such that

$$|\omega \cdot k \pm \nu_j| \geq \frac{\gamma}{1+|k|^\tau} \quad \textit{for all } (k, j) \neq (\bar{k}, \bar{j}) \tag{3-18}$$

and $r > \tau + 1 + \frac{1}{2}n$, then there exists a time-quasiperiodic canonical transformation of the form (2-10) *conjugating the system to*

$$h = \sum_{j \neq \bar{j}} \tfrac{1}{2}\nu_j(x_j^2 + \xi_j^2) + c_1 x_{\bar{j}} + c_2 \xi_{\bar{j}},$$

with $c_1, c_2 \in \mathbb{R}$.

Remark 3.4. The constants c_1, c_2 can be easily computed. If at least one of them is different from zero then the solution of the corresponding quantum system exhibits growth of Sobolev norms, as in the special model (A-1). Of course the result extends in a trivial way to the case in which more resonances are present.

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EIGENFUNCTION SCARRING AND IMPROVEMENTS IN L^∞ BOUNDS

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We study the relationship between L^∞ growth of eigenfunctions and their L^2 concentration as measured by defect measures. In particular, we show that scarring in the sense of concentration of defect measure on certain submanifolds is incompatible with maximal L^∞ growth. In addition, we show that a defect measure which is too diffuse, such as the Liouville measure, is also incompatible with maximal eigenfunction growth.

1. Introduction

Let (M, g) be a C^∞ compact manifold of dimension n without boundary. Consider the eigenfunctions

$$(-\Delta_g - \lambda_j^2)u_{\lambda_j} = 0, \quad \|u_{\lambda_j}\|_{L^2} = 1 \tag{1-1}$$

as $\lambda_j \to \infty$. It is well known [Avakumović 1956; Levitan 1952; Hörmander 1968], see also [Zworski 2012, Chapter 7], that solutions to (1-1) satisfy

$$\|u_{\lambda_j}\|_{L^\infty(M)} \le C\lambda_j^{(n-1)/2} \tag{1-2}$$

and that this bound is saturated, e.g., on the sphere. It is natural to consider the situations which produce sharp examples for (1-2). In many cases, one expects polynomial improvements to (1-2), but rigorous results along these lines are few and far between [Iwaniec and Sarnak 1995]. In the case of negatively curved manifolds, log improvements can be obtained [Bérard 1977]. However, at present, under general dynamical assumptions, known results involve o-improvements to (1-2) [Toth and Zelditch 2002; Sogge et al. 2011; Sogge and Zelditch 2002; 2003; 2016a; 2016b]. These papers all study the connections between the growth of L^∞ norms of eigenfunctions and the global geometry of the manifold (M, g). In this note, we examine the relationship between L^∞ growth and L^2 concentration of eigenfunctions. We measure L^2 concentration using the concept of a *defect measure* — a sequence $\{u_{\lambda_j}\}$ has defect measure μ if for any $a \in S^0_{\text{hom}}(T^*M \setminus \{0\})$,

$$\langle a(x, D)u_{\lambda_j}, u_{\lambda_j}\rangle \to \int_{S^*M} a(x, \xi)\, d\mu. \tag{1-3}$$

By an elementary compactness/diagonalization argument, it follows that any sequence of eigenfunctions u_{λ_j} solving (1-1) possesses a further subsequence that has a defect measure in the sense of (1-3) [Zworski 2012, Chapter 5; Gérard 1991]. Moreover, a standard commutator argument shows that if $\{u_{\lambda_j}\}$ is any

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sequence of L^2-normalized Laplace eigenfunctions, the associated defect measure μ is invariant under the geodesic flow; that is, if $G_t : S^*M \to S^*M$ is the geodesic flow, i.e., the Hamiltonian flow of $p = \frac{1}{2}|\xi|_g^2$, $(G_t)_*\mu = \mu$ for all $t \in \mathbb{R}$.

Definition 1.1. We say that an eigenfunction subsequence is *strongly scarring* provided that for any defect measure μ associated to the sequence, $\operatorname{supp} \mu$ is a finite union of periodic geodesics.

Theorem 1. *Let* $\{u_{\lambda_j}\}$ *be a strongly scarring sequence of solutions to* (1-1). *Then*

$$\|u_{\lambda_j}\|_{L^\infty} = o(\lambda_j^{(n-1)/2}).$$

We also have improved L^∞ bounds when eigenfunctions are *quantum ergodic*, that is, their defect measure is the Liouville measure on S^*M, μ_L; see, e.g., [Shnirelman 1974; Colin de Verdière 1985; Zelditch 1987] for the standard quantum ergodicity theorem.

Theorem 2. *Let* $\{u_{\lambda_j}\}$ *be a quantum ergodic sequence of solutions to* (1-1). *Then*

$$\|u_{\lambda_j}\|_{L^\infty} = o(\lambda_j^{(n-1)/2}).$$

Theorems 1 and 2 are corollaries of our next theorem, where we relax the assumptions on μ and make the following definitions. Define the *time-T flow-out* by

$$\Lambda_{x,T} := \bigcup_{t=-T}^{T} G_t(S_x^*M).$$

Definition 1.2. Let $\mathcal{H}^n$ be the n-dimensional Hausdorff measure on S^*M induced by the Sasaki metric on T^*M; see for example [Blair 2010, Chapter 9] for a treatment of the Sasaki metric. We say that the subsequence u_{λ_j}, $j = 1, 2, \dots$, is *admissible at x* if for any defect measure μ associated to the sequence there exists $T > 0$ such that

$$\mathcal{H}^n(\operatorname{supp} \mu|_{\Lambda_{x,T}}) = 0. \tag{1-4}$$

We say that the subsequence is *admissible* if it is admissible at x for every $x \in M$.

We note that in (1-4), $\mu|_{\Lambda_{x,T}}$ denotes the defect measure restricted to the flow-out $\Lambda_{x,T}$; for any A that is μ-measurable,

$$\mu|_{\Lambda_{x,T}}(A) := \mu(A \cap \Lambda_{x,T}).$$

Theorem 3. *Let* $\{u_{\lambda_j}\}$ *be a sequence of L^2-normalized Laplace eigenfunctions that is admissible in the sense of* (1-4). *Then*

$$\|u_{\lambda_j}\|_{L^\infty} = o(\lambda_j^{(n-1)/2}).$$

Remark 1.3. We choose to use the Sasaki metric to define $\mathcal{H}^n$ for concreteness, but this is not important and we could replace the Sasaki metric by any other metric on S^*M.

Theorem 3 can be interpreted as saying that eigenfunctions which strongly scar are too concentrated to have maximal L^∞ growth, while diffuse eigenfunctions are too spread out to have maximal growth. However, the reason the admissibility assumption is satisfied differs in these cases. In the diffuse case (see Theorem 2), one has $\mu|_{\Lambda_{x,T}} = 0$, so that the admissibility assumption is trivially verified. In the case where the eigenfunctions strongly scar (see Theorem 1), $\mu|_{\Lambda_{x,T}} \neq 0$ but the Hausdorff dimension of $\operatorname{supp} \mu|_{\Lambda_{x,T}}$ is $< n$; so again, (1-4) is satisfied. The zonal harmonics on the sphere S^2, which saturate the L^∞ bound (1-2), lie precisely between being diffuse and strongly scarring (see Section 4).

Observe that the condition μ is diffuse is much more general than $\mu = \mu_L$. Jakobson and Zelditch [1999] show that any invariant measure on S^*S^n where S^n is the round sphere can be obtained as a defect measure for a sequence of eigenfunctions and in particular many non-Liouville but diffuse measures occur.

Remark 1.4. We note that the results here hold for any $o(\lambda)$ quasimode of $(-\Delta_g - \lambda^2)$ that is compactly microlocalized in frequency; see [Galkowski 2017].

Relation with previous results. Theorem 2 is related to [Sogge et al. 2011, Theorem 3], where the $o(h^{(1-n)/2})$ sup bound is proved for all Laplace eigenfunctions on a C^ω surface with ergodic geodesic flow. However, in Theorem 2, we make no analyticity or dynamical assumptions on (M, g) whatsoever, only an assumption on the particular defect measure associated with the eigenfunction sequence. Recently, Hezari [2016] and Sogge [2016] gave independent proofs of Theorem 2.

One consequence of the work of Sogge is the relation between L^p norms for eigenfunctions and the push forward of defect measures to the base manifold M. In particular, he showed [Sogge 2016, (3.3)] that

$$\|u_\lambda\|_{L^\infty(M)} \le C\lambda^{(n-1)/2} \sup_{x\in M} \delta^{-1/2} \|u_\lambda\|_{L^2(B_\delta(x))} \tag{1-5}$$

when $\lambda^{-1} \le \delta \le \operatorname{inj}(M, g)$ and $\lambda \ge 1$. We note that when u_λ are quantum ergodic, $\|u_\lambda\|_{L^2(B_\delta(x))} \approx \delta^{n/2}$ and so the $o(\lambda^{(n-1)/2})$-bound in Theorem 2 follows from (1-5) as well; see also Corollary 1.2 in [Sogge 2016].

However, neither the scarring result in Theorem 1 nor the more general bound in Theorem 3 follow from (1-5). To compare and contrast with (1-5), we observe that (1-5) implies for any $\delta > 0$ independent of λ,

$$\limsup_{\lambda\to\infty} \lambda^{(1-n)/2} \|u_\lambda\|_{L^\infty(M)} \le C \sup_{x\in M} \delta^{-1/2} \left(\mu(S^* B_\delta(x))\right)^{1/2}.$$

Our main estimate in (3-12) says that for any $x(\lambda)$ with $d(x(\lambda), x) = o(1)$,

$$\limsup_{\lambda\to\infty} \lambda^{(1-n)/2} |u_\lambda(x(\lambda))| \le C'_\delta \big(\mathcal{H}^n(\operatorname{supp} \mu|_{A_x(\delta/2,3\delta)})\big)^{1/2}, \tag{1-6}$$

where for $\delta_2 > \delta_1$ we have $A_x(\delta_1, \delta_2) = \Lambda_{x,\delta_2} \setminus \Lambda_{x,\delta_1}$. This microlocalized bound allows us to deal with the more general scarring-type cases as well. In particular, the key differences are that we have replaced $S^* B_\delta(x)$ by $A_x(\delta/2, 2\delta) \subset \Lambda_x$ and the defect measure by Hausdorff n-measure. We note however that unlike (1-5), $\delta > 0$ can be arbitrarily small but is fixed independent of λ in (1-6).

Sogge and Zelditch [2002] proved that any manifold on which (1-2) is sharp must have a self-focal point. That is, a point x such that $|\mathcal{L}_x| > 0$, where

$$\mathcal{L}_x := \{\xi \in S^*_x M : \text{ there exists } T \text{ such that } \exp_x T\xi = x\}$$

and $|\cdot|$ denotes the normalized surface measure on the sphere. Subsequently, in [Sogge et al. 2011] the authors showed that one can replace $\mathcal{L}_x$ by the set of recurrent directions $\mathcal{R}_x \subset \mathcal{L}_x$ and the assumption $|\mathcal{R}_x| > 0$ for some $x \in M$ is necessary to saturate the maximal bound in (1-2). Here,

$$\mathcal{R}_x := \left\{ \xi \in S_x^*M : \xi \in \Big(\bigcap_{T>0} \overline{\bigcup_{t \geq T} G_t(x,\xi) \cap S_x^*M} \Big) \bigcap \Big(\bigcap_{T>0} \overline{\bigcup_{t \leq -T} G_t(x,\xi) \cap S_x^*M} \Big) \right\}.$$

The example of the triaxial ellipsoid with x equal to an umbilic point shows that latter assumption is weaker than the former. Indeed, in such a case $|\mathcal{L}_x| = 1$, whereas $|\mathcal{R}_x| = 0$. Most recently, in [Sogge and Zelditch 2016a; 2016b], it was proved that for real-analytic surfaces, the maximal L^∞ bound can only be achieved if there exists a periodic point $x \in M$ for the geodesic flow, i.e., a point (x,ξ) such that all geodesics starting at $(x,\xi) \in S^*M$ close up smoothly after some finite time $T > 0$.

Together with our analysis, the results of [Sogge et al. 2011] imply that any sequence of eigenfunctions, $\{u_\lambda\}$ having maximal L^∞ growth near x and defect measure μ must have $\mu(\Lambda_{x,T}) > 0$ for all $T > 0$ and $|\mathcal{R}_x| > 0$. In particular, the results of that paper show that u_λ can only have maximal L^∞ growth near a point with a positive measure set of recurrent points and Theorem 3 shows that a point with maximal L^∞ growth must have $\mu(\Lambda_{x,T}) > 0$. As far as the authors are aware, the results in [Sogge et al. 2011; Sogge and Zelditch 2016a; 2016b] do not give additional information about μ.

On the other hand, under an additional regularity assumption on the measure μ, Theorem 3 can be used to show that when u_λ has maximal growth near x, the measure $\mu|_{\Lambda_{x,T}}$ is not mutually singular with respect to $\mathcal{H}^n$. Since the measure for a zonal harmonic is a smooth multiple of $\mathcal{H}^n$ (see Section 4), this implies that the measure μ resembles the defect measure of a zonal harmonic. In [Galkowski 2017], the first author removed the necessity for any additional regularity assumption and gave a full characterization of defect measures for eigenfunctions with maximal L^∞ growth, in particular proving that if u_λ has maximal growth near x and defect measure μ, then $\mu|_{\Lambda_{x,T}}$ is not mutually singular with respect to $\mathcal{H}^n$. Finally, we note that unlike [Sogge et al. 2011; Sogge and Zelditch 2002; 2016a; 2016b], the analysis here is entirely local.

2. A local version of Theorem 3

In the following, we will freely use semiclassical pseudodifferential calculus where the semiclassical parameter is h with $h^{-1} = \lambda \in \operatorname{Spec}\sqrt{-\Delta_g}$. We write $r(x,y) : M \times M \to \mathbb{R}$ for the Riemannian distance from x to y and write $B(x,\delta)$ for the geodesic ball of radius δ around x. We start with a local result:

Theorem 4. *Let $\{u_h\}$ be sequence of Laplace eigenfunctions that is admissible at x. Then for any $\delta(h) = o(1)$,*

$$\|u_h\|_{L^\infty(B(x,\delta(h)))} = o(h^{(1-n)/2}).$$

Theorem 3 is an easy consequence of Theorem 4.

Proof that Theorem 4 implies Theorem 3. Suppose that u is admissible and

$$\limsup_{h\to 0} h^{(n-1)/2}\|u_h\|_{L^\infty} \neq 0.$$

Then, there exist $c > 0$, $h_k \to 0$, x_{h_k} so that

$$|u_{h_k}(x_{h_k})| \geq c h_k^{-(n-1)/2}.$$

Since M is compact, by taking a subsequence, we may assume $x_{h_k} \to x$. But then $r(x, x_{h_k}) = o(1)$ and since u is admissible at x, Theorem 4 implies

$$\limsup_{k\to\infty} h_k^{(n-1)/2} |u_{h_k}(x_{h_k})| = 0. \qquad \square$$

3. Proof of Theorem 4

In view of the above, it suffices to prove the local result: Theorem 4.

Proof. Fix $T > 3\delta > 0$ and let $\rho \in \mathcal{S}(\mathbb{R})$ with $\rho(0) = 1$ and $\operatorname{supp} \hat{\rho} \subset (\delta, 2\delta)$. Let

$$S^*M(\varepsilon) := \big\{(x,\xi) : \big||\xi|_x - 1\big| \leq \varepsilon\big\}$$

and $\chi(x,\xi) \in C_0^\infty(T^*M)$ be a cutoff near the cosphere S^*M with $\chi(x,\xi) = 1$ for $(x,\xi) \in S^*M(\varepsilon)$ and $\chi(x,\xi) = 0$ when $(x,\xi) \in T^*M \setminus S^*M(2\varepsilon)$. Let $\chi(x, hD) \in \mathrm{Op}_h(C_0^\infty(T^*M))$ be the corresponding h-pseudodifferential cutoff. Also, in the following, we will use the notation

$$\Gamma_x := \operatorname{supp} \mu|_{\Lambda_{x,T}}$$

to denote the support of the restricted defect measure corresponding to the eigenfunction sequence $\{u_{h_j}\}$ in Theorem 3.

Then, we have

$$u_h = \rho\Big(\frac{1}{2h}[-h^2\Delta - 1]\Big) u_h = \int_{\mathbb{R}} \hat{\rho}(t) e^{i(t/2)[-h^2\Delta-1]/h} \chi(y, hD_y) u_h \, dt + O_\varepsilon(h^\infty). \tag{3-1}$$

Microlocalization to the flow-out Λ_x. Set

$$V(t,x,y,h) := \big(\hat{\rho}(t) e^{i(t/2)[-h^2\Delta-1]/h} \chi(y, hD_y)\big)(t,x,y).$$

Then, by Egorov's theorem [Zworski 2012, Theorem 11.1]

$$\mathrm{WF}_h'(V(t,\cdot,\cdot,h)) \subset \big\{(x,\xi,y,\eta) : (x,\xi) = G_t(y,\eta),\ \big||\xi|_x - 1\big| \leq 2\varepsilon\big\}; \tag{3-2}$$

see, e.g., [Dyatlov and Zworski 2017, Definition E.37] for a definition of WF_h'.

Let $b_{x,\varepsilon}(x, hD) \in \mathrm{Op}_h(C_0^\infty(T^*M))$ be a family of h-pseudodifferential cutoffs with principal symbols

$$b_{x,\varepsilon} \in C_0^\infty\big(\big\{(y,\eta) : (y,\eta) = G_t(x_0,\xi) \text{ for some } (x_0,\xi) \in S^*_{x_0}M(3\varepsilon) \text{ with } r(x,x_0) < 2\varepsilon,\ \delta/2 < t < 3\delta\big\}\big),$$

with

$$b_{x,\varepsilon} \equiv 1 \quad \text{on } \big\{(y,\eta) : (y,\eta) = G_t(x_0,\xi) \text{ for some } (x_0,\xi) \in S^*_{x_0}M(2\varepsilon) \text{ with } r(x,x_0) < \varepsilon,\ \delta < t < 2\delta\big\}.$$

By the definition of WF_h' together with (3-1) and (3-2), it follows that for $r(x(h), x) = o(1)$,

$$u_h(x(h)) = \int_M \overline{V}(x(h), y, h)\, b_{x,\varepsilon}(y, hD_y)\, u_h(y)\, dy + O_\varepsilon(h^\infty), \tag{3-3}$$

where,

$$\overline{V}(x(h), y, h) := \int_{\mathbb{R}} \hat{\rho}(t)\big(e^{i(t/2)[-h^2\Delta-1]/h}\chi(y, hD_y)\big)(t, x(h), y)\, dt.$$

By a standard stationary phase argument,

$$\overline{V}(x, y, h) = h^{(1-n)/2} e^{-ir(x,y)/h} a(x, y, h)\, \hat{\rho}(r(x, y)) + O_\varepsilon(h^\infty), \tag{3-4}$$

where $a(x, y, h) \in S^0(1)$.

To see this, observe that by [Zworski 2012, Theorem 10.4]

$$\overline{V}(x, y, h) = (2\pi h)^{-n} \int e^{i\varphi(t,x,y,\eta)/h}\, \alpha(t, x, y, \eta, h)\, \hat{\rho}(t)\, d\eta\, dt + O(h^\infty),$$

where $b \in C_c^\infty$ and φ solves

$$\partial_t\varphi = \tfrac{1}{2}(|\partial_x\varphi|^2_{g(x)} - 1), \quad \varphi(0, x, y, \eta) = \langle x - y, \eta\rangle. \tag{3-5}$$

In particular, for all (t, x, y, η), we have $\exp(tH_{|\xi|^2_g/2})(\partial_\eta\varphi + y, \eta) = (x, \partial_x\varphi)$. The phase function

$$\varphi(t, x, y, \eta) = \langle \exp_y^{-1}(x), \eta\rangle + \tfrac{1}{2}t(|\eta|^2_y - 1)$$

satisfies (3-5).

We next perform stationary phase in (t, η). First, observe that the phase is stationary at

$$\exp(tH_{|\xi|^2_g/2})(y, \eta) = (x, \partial_x\varphi), \quad |\partial_x\varphi|_{g(x)} = 1.$$

In particular, $t = r(x, y)$ and the geodesic through (y, η) passes through x. Since $\operatorname{supp}\hat{\rho} \subset (\delta, 2\delta)$, by performing nonstationary phase, we may assume $t \in (\delta, 2\delta)$ and hence $\delta < r(x, y) < 2\delta$. Then, we observe that $\partial^2_{(t,\eta)}\varphi$ is nondegenerate for $t \in (\delta, 2\delta)$. The solutions (t_c, η_c) of the critical point equations $\partial_t\varphi = 0$ and $\partial_\eta\varphi = 0$ are given by

$$t_c = |\exp_y^{-1}(x)| = r(x, y), \quad \eta_c = -\frac{\exp_y^{-1}(x)}{r(x, y)}.$$

Consequently, (3-4) follows from an application of stationary phase; see also [Sogge 1993, Lemma 5.1.3; Burq et al. 2007, Theorem 4].

Then, in view of (3-4) and (3-3),

$$\begin{gathered} u_h(x(h)) = v_h(x(h)) + O_\varepsilon(h^\infty), \\ v_h(x(h)) = h^{(1-n)/2}\int_{\delta/2<r(x,y)<2\delta} e^{-ir(x(h),y)/h} a(x(h), y, h)\, \hat{\rho}(r(x(h), y))\, b_{x,\varepsilon}(y, hD_y)\, u_h(y)\, dy. \end{gathered} \tag{3-6}$$

Now, note that for any $\psi \in C_0^\infty(M)$,

$$v_h(x(h)) = I_1(x(h), h) + I_2(x(h), h), \tag{3-7}$$

where

$$I_1 := (2\pi h)^{(1-n)/2}\int_{\delta/2<r(x,y)<2\delta} e^{-ir(x(h),y)/h} a(x(h), y, h)\, \hat{\rho}(r(x(h), y))\, \psi(y)\, (b_{x,\varepsilon}(y, hD_y)u_h)\, dy,$$

$$I_2 := (2\pi h)^{(1-n)/2}\int_{\delta/2<r(x,y)<2\delta} e^{-ir(x(h),y)/h} a(x(h), y, h)\, \hat{\rho}(r(x(h), y))\, (1-\psi(y))\, (b_{x,\varepsilon}(y, hD_y)u_h)\, dy.$$

Therefore, by Cauchy–Schwarz applied to I_1 and I_2,

$$|h^{(n-1)/2}v_h(x(h))| \le C_\delta\big(\|\psi\|_{L^2}\|b_{x,\varepsilon}(y,hD_y)u_h(y)\|_{L^2} + \big\|(1-\psi(y))[b_{x,\varepsilon}(y,hD_y)u_h]\big\|_{L^2}\big).$$

Hence letting $h \to 0$ then $\varepsilon \to 0$, and using that

$$\|b_{x,\varepsilon}(y,hD_y)u_h(y)\|_{L^2} \le (\sup|b_{x,\varepsilon}| + o_\varepsilon(1))\|u_h\|_{L^2},$$

see for example [Zworski 2012, Theorem 5.1], we have

$$\limsup_{h\to 0} h^{(n-1)/2}|u_h(x(h))| \le C_\delta\left(\|\psi\|_{L^2} + \left(\int_{\Lambda_{x,3\delta}\setminus\Lambda_{x,\delta/2}} (1-\psi(y))^2\,d\mu\right)^{\frac{1}{2}}\right). \tag{3-8}$$

Further microlocalization along $\boldsymbol{\operatorname{supp}\mu|_{\Lambda_x}}$**.** Let $\mathcal{H}^n$ be the n-dimensional Hausdorff measure on the flow-out Λ_x. By assumption, $\mathcal{H}^n(\operatorname{supp}\mu|_{\Lambda_x}) = 0$. In view of the microlocalization above, we are only interested in the annular subset

$$A_x(\delta/2, 3\delta) := \Lambda_{x,3\delta} \setminus \Lambda_{x,\delta/2}.$$

Since $\mathcal{H}^n$ is Radon, for any $\varepsilon_1 > 0$, there exist n-dimensional balls $B(r_j) \subset A_x(\delta/4, 4\delta)$, $j = 1, 2, \dots,$ with radii $r_j > 0$, $j = 1, 2, \dots$, such that

$$\operatorname{supp}\mu|_{A_x(\delta/2,3\delta)} \subset \bigcup_{j=1}^{\infty} B(r_j), \quad \mathcal{H}^n\Big(\bigcup_{j=1}^{\infty} B(r_j)\Big) < \mathcal{H}^n(\operatorname{supp}\mu|_{A_x(\delta/2,3\delta)}) + \varepsilon_1.$$

Note that for $\delta > 0$ small enough, the canonical projection $\pi : T^*M \to M$ restricts to a diffeomorphism

$$\pi : A_x\big(\tfrac{1}{4}\delta, 4\delta\big) \to \{y \in M : \delta/4 < r(x,y) < 4\delta\}.$$

Consider the closed set

$$K = \pi(\operatorname{supp}\mu|_{A_x(\delta/2,3\delta)}) \subset M$$

with open covering

$$G := \pi\Big(\bigcup_{j=1}^{\infty} B(r_j)\Big) \quad \text{satisfying } \mathcal{H}^n(G) = \mathcal{H}^n(K) + O(\varepsilon_1). \tag{3-9}$$

By the C^∞ Urysohn lemma, there exists $\chi_{\Gamma_x} \in C_0^\infty(M; [0,1])$ with

$$\chi_{\Gamma_x}|_K = 1, \quad \operatorname{supp}\chi_{\Gamma_x} \subset G. \tag{3-10}$$

(Note that χ_{Γ_x} depends on ε_1, but we suppress this dependence to simplify notation.) We now apply (3-8) with $\psi = \chi_{\Gamma_x}$. First, observe that by (3-9) and (3-10)

$$\|\chi_{\Gamma_x}\|_{L^2} \le (\mathcal{H}^n(G))^{1/2} \le (\mathcal{H}^n(K))^{1/2} + O(\varepsilon_1^{1/2}). \tag{3-11}$$

Next, by construction, for all $\varepsilon_1 > 0$,

$$(1-\chi_{\Gamma_x})(y) = 0 \quad \text{for all } y \in \pi(\operatorname{supp}\mu|_{\Lambda_{x,4\delta}\setminus\Lambda_{x,\delta/4}})$$

and hence

$$\int_{\Lambda_{x,3\delta}\setminus\Lambda_{x,\delta/2}} (1-\chi_{\Gamma_x})^2\,d\mu = 0.$$

Using this together with (3-11) in (3-8) and sending $\varepsilon_1 \to 0$ gives

$$\limsup_{h\to 0} h^{(n-1)/2}|u_h(x(h))| \le C_\delta\big(\mathcal{H}^n(\pi(\operatorname{supp}\mu|_{A_x(\delta/2,3\delta)}))\big)^{1/2} \le C'_\delta\big(\mathcal{H}^n(\operatorname{supp}\mu|_{A_x(\delta/2,3\delta)})\big)^{1/2}, \tag{3-12}$$

where the last inequality follows from the fact that $\pi|_{A(\delta/2,3\delta)}$ is a diffeomorphism. Finally, since u_h is admissible at x,

$$\mathcal{H}^n(\operatorname{supp}\mu|_{A_x(\delta/2,3\delta)}) = 0. \qquad \square$$

Remark 3.1. For $r(x(h),x) = o(1)$, the estimate

$$\limsup_{h\to 0} h^{(n-1)/2}|u_h(x(h))| \le C'_\delta\big(\mathcal{H}^n(\operatorname{supp}\mu|_{A_x(\delta/2,3\delta)})\big)^{1/2}$$

in (3-12) holds for any sequence of eigenfunctions with defect measure μ. It gives a quantitative estimate relating the behaviour of the defect measure to L^∞ norms of eigenfunctions. This estimate can also be obtained as a consequence of [Galkowski 2017, Theorem 2] by replacing the absolutely continuous part of μ with $1_{\operatorname{supp}\mu|_{\Lambda_x}}\,d\mathcal{H}^n$.

4. The example of zonal harmonics

Let (S^2, g_{can}) be the round sphere and (r,θ) be polar variables centred at the north pole $p=(0,0,1)\in\mathbb{R}^3$. The geodesic flow is a completely integrable system with Hamiltonian

$$H = |\xi|_g^2 = \xi_r^2 + (\sin r)^{-2}\xi_\theta^2, \quad r\in(0,\pi), \tag{4-1}$$

and Claurault integral $p=\xi_\theta$ satisfying $\{H,p\}=0$. The associated moment mapping is $\mathcal{P}=(H,p): T^*S^2 \to \mathbb{R}^2$ and the connected components of the level sets are, by the Liouville–Arnold theorem, Lagrangian tori Λ_c indexed by the values of the moment map $(1,c)\in\mathcal{P}(T^*S^2)$.

The associated quantum integrable system is given by the Laplacian Δ_g and the rotation operator hD_θ. The corresponding L^2-normalized joint eigenfunctions are the standard spherical harmonics Y_m^k with

$$-\Delta_g Y_m^k = k(k+1)Y_m^k, \quad hD_\theta Y_m^k = mY_m^k.$$

These eigenfunctions can be separated into various sequences (i.e., *ladders*) associated with different values ($\in\mathcal{P}(T^*S^2)$; specifically, the correspondence is given by $c=\lim_{m\to\infty} m/k$). The eigenfunctions with maximal L^∞ blow-up are the sequence of *zonal* harmonics given by

$$u_h(r,\theta) = Y_0^k(r,\theta) = \frac{\sqrt{2k+1}}{2\pi}\int_0^{2\pi}(\cos r + i\sin r\cos\tau)^k\,d\tau, \quad h=k^{-1},\ k=1,2,3\ldots. \tag{4-2}$$

It is obvious from (4-2) that

$$|Y_0^k(p)| \approx k^{1/2}$$

and thus attains the maximal sup growth at p (similarly, at the south pole). At the classical level, the zonals $u_h = Y_0^k$ concentrate microlocally on the Lagrangian tori $\Lambda_0 = \mathcal{P}^{-1}(1,0)$. From the formula (4-1) it is clear that away from the poles (where (r,θ) are honest coordinates),

$$\Lambda_0 \setminus \{\pm p\} = \{(r,\theta,\xi_r=\pm 1,\xi_\theta=0) : r \in (0,\pi)\} \cong S^2 \setminus \{\pm p\}. \tag{4-3}$$

The choice of $\xi_r = \pm 1$ determines the Lagrangian torus (there are two of them) and also, either torus clearly covers the entire sphere. At the poles themselves, the projection $\pi_{\Lambda_0} : \Lambda_0 \to S^2$ has a blowdown singularity with

$$\pi_{\Lambda_0}^{-1}(\pm p) = S^*_{\pm}(S^2) \cong S^1. \tag{4-4}$$

To see this, consider the behaviour at p (with a similar computation at $-p$). Rewriting the integral in involution in Euclidean coordinates $(x,y,z) \in \mathbb{R}^3$, one has $H = (x\xi_y - y\xi_x)^2 + (x\xi_z - z\xi_x)^2 + (y\xi_z - z\xi_y)^2$ and $\xi_\theta = x\xi_y - y\xi_x$. Setting $H = 1$, $x\xi_y - y\xi_x = 0$ and $(x,y,z) = (0,0,1)$ gives

$$\pi_{\Lambda_0}^{-1}(p) \cong \{(\xi_x,\xi_y) \in \mathbb{R}^2 : \xi_x^2 + \xi_y^2 = 1\}.$$

It is then clear from (4-3) and (4-4) that $\pi_{\Lambda_0} : \Lambda_0 \to S^2$ is surjective and a diffeomorphism away from the poles (modulo choice of Lagrangian cover) and the fibres above the poles are $S^*_{\pm}(S^2) \cong S^1$. We also note that the Lagrangian $\Lambda_0 = \Lambda_{p,2\pi}$ is the 2π-flow-out Lagrangian of $S^*_p(S^2)$ and the cylinder $A_p(\delta/2, 3\delta)$ is just a local slice of this Lagrangian.

The defect measure μ associated with the zonals is

$$d\mu = |d\theta_1\, d\theta_2|,$$

where $(\theta_1,\theta_2; I_1, I_2) \in \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}^2$ are symplectic action-angle variables defined in a neighbourhood of the Lagrangian torus Λ_0 [Toth and Zelditch 2003]. One can choose one of the angle variables $\theta_1 \in S^*_p(S^2)$ to parametrize the circle fibre above p (a homology generator of the torus). Then, by the Liouville–Arnold theorem, the geodesic flow on the torus $\Lambda_0 = \{I_1 = c_1, I_2 = c_2\}$ is affine with

$$\theta_j(t) = \theta_j(0) + \alpha_j t, \qquad \alpha_j = \frac{\partial H}{\partial I_j} \neq 0.$$

It is then clear that

$$\mu(\Lambda_{p,\delta}) = \int_0^{2\pi} d\theta_1 \cdot \int_{|t|<\delta} \alpha_2\, dt \approx \delta \neq 0$$

and $\operatorname{supp} \mu|_{\Lambda_p} = \Lambda_p$. Therefore, this case violates the assumption in Theorem 3 and that is of course consistent with the maximal L^∞ growth of zonal harmonics.

The analysis above extends in a straightforward fashion to the case of a more general sphere of rotation [Toth and Zelditch 2003].

5. Eigenfunctions of Schrödinger operators

Consider a Schrödinger operator $P(h) = -h^2\Delta_g + V$ with $V \in C^\infty(M;\mathbb{R})$ on a compact, closed Riemannian manifold (M,g) and let u_h be an L^2-normalized eigenfunction with

$$P(h)u_h = E(h)u_h, \qquad E(h) = E + o(1), \quad E > \min V, \qquad \|u_h\|_{L^2} = 1. \tag{5-1}$$

Any sequence u_h of solutions to (5-1) has a subsequence u_{h_k} with a defect measure μ in the sense that for $a \in C_0^\infty(T^*M)$

$$\langle a(x, hD)u_h, u_h\rangle \to \int_{T^*M} a\, d\mu.$$

Such a measure μ is supported on $\{p = 0\}$ and is invariant under the bicharacteristic flow $G_t := \exp(tH_p)$.

In analogy with the homogeneous case, we define for $x \in M$ the *time-T flow-out* by

$$\Lambda_{x,T,V} := \bigcup_{t=-T}^{T} G_t(\Sigma_x),$$

where

$$\Sigma_x = \{\xi \in T_x^*M : |\xi|_g^2 + V(x) = E\}.$$

Definition 5.1. Let $\mathcal{H}^n$ be n-dimensional Hausdorff measure on $\{|\xi|_g^2 + V(x) = E\}$ induced by the Sasaki metric on T^*M. We say that the sequence u_h of solutions to (5-1) is *admissible at x* if for any defect measure μ associated to the sequence, there exists $T > 0$ so that

$$\mathcal{H}^n(\operatorname{supp} \mu|_{\Lambda_{x,T,V}}) = 0. \tag{5-2}$$

With these definitions we have the analog of Theorem 3:

Theorem 5. *Let $B \subset V^{-1}(E)$ be a closed ball in the classically allowable region and μ be a defect measure associated with the eigenfunction sequence u_h. Then, if the eigenfunction sequence is admissible for all $x \in B$ in the sense of* (5-2),

$$\sup_{x\in B} |u_h(x)| = o(h^{(1-n)/2}).$$

Proof. In analogy with the homogeneous case [Christianson et al. 2015, Lemma 5.1], we have

$$\rho(h^{-1}[P(h) - E]))(x, y) = h^{(1-n)/2} a(x, y, h)e^{-iA(x,y)/h} + R(x, y, h),$$

where $A(x, y) \in [(2C_0)^{-1}\varepsilon, 2C_0\varepsilon]$ for some $C_0 > 1$ and is the action function defined to be the integral of the Lagrangian $L(x, \xi) = |\xi|_g^2 - V(x)$ along the bicharacteristic in $\{p = E\}$ starting at (y, η) and ending at (x, ξ). For (x, y) in a small neighbourhood of the diagonal, there is a unique such η satisfying this condition. The remainder $R(x, y, h)$ is equal to $O(h^\infty)$ pointwise and with all derivatives. The proof then follows using the same argument as in the homogeneous case. □

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