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We prove the folklore endpoint multilinear k_j-plane conjecture originating in a paper of Bennett, Carbery and Tao where the almost sharp multilinear Kakeya estimate was proved. Along the way we prove a more general result, namely the endpoint multilinear k_j-variety theorem. Finally, we generalize our results to the endpoint perturbed Brascamp–Lieb inequalities using techniques in earlier sections.

1. Introduction

The endpoint multilinear k_j-plane theorem. The multilinear k_j-plane conjecture was implicitly proved by Bennett, Carbery and Tao [2006], except for the endpoint case. In the first part of this paper we formulate and prove the endpoint case. In fact we will prove the endpoint multilinear k_j-variety theorem, which is more general.

The proof uses the polynomial method. We will set up the polynomial like Guth [2010] did in his proof of the endpoint multilinear Kakeya conjecture. Then we make some crucial new observations and development of the theory, enabling us to estimate "the quantitative interaction of the polynomial with itself" in terms of its visibility. As a result, we are able to deal with the codimension difficulty and complete the proof.

The multilinear k_j-plane estimate is a natural generalization of the famous multilinear Kakeya estimate. Albeit weaker than linear Kakeya, the multilinear Kakeya theorem and the methods it inspired recently had remarkable applications to classical harmonic analysis problems as well [Bourgain and Guth 2011; Bourgain 2013a; 2013b; Guth 2016b; 2016c; Bourgain and Demeter 2015]. See the beginning of [Guth 2015] for a good introduction.

The nonendpoint case of the multilinear Kakeya conjecture was proved by Bennett, Carbery and Tao [Bennett et al. 2006] and later Guth [2010] proved the endpoint case, which we state below.

Theorem 1.1. *For $1 \le j \le d$, let $\{T_{j,a} : 1 \le a \le A(j)\}$ be a family of unit cylinders in $\mathbb{R}^d$. We set $v_{j,a}$ to be the direction of the core line of the cylinder $T_{j,a}$. Assume the core lines of cylinders from different families are "quantitatively transversal"; i.e., for any $1 \le a_j \le A(j)$, we have $v_{1,a_1} \wedge v_{2,a_2} \wedge \cdots \wedge v_{d,a_d} \ge \theta > 0$, where θ is fixed. Then we have*

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^{d} \sum_{a=1}^{A(j)} \chi_{T_{j,a}} \right)^{1/(d-1)} \lesssim_d \theta^{-1/(d-1)} \prod_{j=1}^{d} A(j)^{1/(d-1)}. \tag{1-1}$$

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Keywords: Brascamp–Lieb inequality, polynomial method.

Guth's approach to proving Theorem 1.1 is very different from the approach of Bennett, Carbery and Tao. He was able to take a polynomial that approximates the intersection of tubes sufficiently well, along the way employing some nice tools and lemmas from algebraic topology and integral geometry.

In the Kakeya setting we have cylinders which are neighborhoods of lines. A natural analogue is to replace lines with higher-dimensional affine subspaces and this will exactly be our multilinear k_j-plane setting. In Remark 5.4 of [Bennett et al. 2006], the authors note that their techniques can be also used to obtain nonendpoint cases of multilinear k-plane transform estimates considered in [Oberlin and Stein 1982]. There is also a k-plane version of the Kakeya problem [Bourgain 1991] that could be relevant here.

They did not state the result precisely and we will state what we can get from their proof below. If we go check the proof, similar techniques in [Guth 2015] can also give us the result. Here we allow subspaces of different dimensions and hence call the theorem a "multilinear k_j-plane theorem".

Before stating the theorem we introduce our terminology to describe a "higher-dimensional" analogue of cylinders.

Definition 1.2. In a space of dimension d, for any $1 \le b < d$ define a *b-slab* to be the Cartesian product of a b-dimensional ball B_1 and a $(d-b)$-dimensional ball B_2 (the spaces spanned by both balls are required to be orthogonal). The radius of B_1 will be called the *size* of our b-slab and the radius of B_2 will be called the *radius* of it. The Cartesian product of B_1 and the center of B_2 is called the *core* of this b-slab.

By the above definition, a 1-slab is a cylinder. Its length is the size in our language. Our definitions of radius and core are consistent with familiar definitions for cylinders. As explained above, we call our theorem a k_j-plane theorem because when the size is large, a k-slab looks flat and is like a "fattened" k-plane.

Theorem 1.3 (multilinear k_j-plane theorem with R^ε loss [Bennett et al. 2006]). *Assume R is a large positive number. Assume $K_1, K_2, \dots, K_n \subsetneq \{1, 2, \dots, d\}$ are disjoint and $K_1 \cup \cdots \cup K_n = \{1, 2, \dots, d\}$. Let $k_j = |K_j|$.*

For $1 \le j \le n$, let $\{T_{j,a} : 1 \le a \le A(j)\}$ be a family of k_j-slabs of size $\le R$ and radius 1. Assume that for any $1 \le a_j \le A(j)$, the core of T_{j,a_j} is on a k_j-plane that forms an angle $< \delta$ against the k_j-plane spanned by all $\boldsymbol{e}_i$, $i \in K_j$.

Then when $\delta > 0$ is sufficiently small depending on d, we have

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^{n} \sum_{a=1}^{A(j)} \chi_{T_{j,a}} \right)^{1/(n-1)} \lesssim_{\varepsilon,d} R^\varepsilon \prod_{j=1}^{n} A(j)^{1/(n-1)}. \tag{1-2}$$

When $n = d$ and $K_j = \{j\}$, this theorem is the multilinear Kakeya theorem with R^ε loss, which is the main theorem of [Bennett et al. 2006]. In [Guth 2015], a simpler proof of this special case is also given, and it can be generalized easily to prove the whole Theorem 1.3.

We can obtain various k_j-plane theorems by taking different n and K_j in Theorem 1.3. As we saw in Theorem 1.1, Guth [2010] was able to remove the R^ε in the multilinear Kakeya case. So in general we would also expect the removal of R^ε. Conceptually, this will allow us to have slabs with "size ∞" (that are actually 1-neighborhoods of k_j planes) in the theorem. It turns out to be true and will be proved in this paper.

Theorem 1.4 (multilinear k_j-plane theorem). *Take the same assumptions as Theorem 1.3, but with no restriction on the size of slabs. We have*

$$\int_{\mathbb{R}^d} \biggl(\prod_{j=1}^{n} \sum_{a=1}^{A(j)} \chi_{T_{j,a}}\biggr)^{1/(n-1)} \lesssim_d \prod_{j=1}^{n} A(j)^{1/(n-1)}. \tag{1-3}$$

Theorem 1.4 has an affine-invariant version, just like the multilinear Kakeya case, which was first pointed out in [Bourgain and Guth 2011]. We will actually prove this version (Theorem 1.5 below). Theorem 1.4 is a direct corollary of it.

In order to state the theorem, we introduce some notation. For any $q \le d$ vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_q$, we define $|\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \cdots \wedge \boldsymbol{v}_q|$ to be the volume of the parallelepiped generated by $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_q$. Moreover, for any m (affine) subspaces $V_1, V_2, \dots, V_m$ with a total dimension d, we can define $|V_1 \wedge V_2 \wedge \cdots \wedge V_m|$ to be $|\boldsymbol{v}_{1,1} \wedge \cdots \wedge \boldsymbol{v}_{1,d_1} \wedge \boldsymbol{v}_{2,1} \wedge \cdots \wedge \boldsymbol{v}_{2,d_2} \wedge \cdots \wedge \boldsymbol{v}_{m,1} \cdots \wedge \boldsymbol{v}_{m,d_m}|$, where $\{\boldsymbol{v}_{j,i} : 1 \le i \le d_j\}$ form an orthonormal basis of the linear subspace parallel to V_j.

Theorem 1.5 (affine invariant multilinear k_j-plane theorem). *Assume the positive integers $1 \le k_1, \dots, k_n \le d-1$ satisfy $\sum_{j=1}^{n} k_j = d$. For $1 \le j \le n$, let $\{T_{j,a} : 1 \le a \le A(j)\}$ be a family of k_j-slabs of radius* 1. *Assume the core k_j-plane of $T_{j,a}$ is parallel to the linear subspace $H_{j,a}$. Then for any real numbers ρ_{j,a_j} we have*

$$\int_{\mathbb{R}^d} \biggl(\sum_{a_1=1}^{A(1)} \cdots \sum_{a_n=1}^{A(n)} \prod_{j=1}^{n} \rho_{j,a_j} \chi_{T_{j,a_j}}(x) \cdot H_{1,a_1} \wedge \cdots \wedge H_{n,a_n}\biggr)^{1/(n-1)} \mathrm{d}x \lesssim_d \prod_{j=1}^{n} \biggl(\sum_{a_j=1}^{A(j)} |\rho_{j,a_j}|\biggr)^{1/(n-1)}. \tag{1-4}$$

Remark 1.6. We refer the reader to [Bennett and Bez 2010] for an explanation of why the exponents are as they appear in Theorem 1.5. Also we note that in that paper the authors already observed the affine-invariant Finner inequality, which is an "unperturbed" version of Theorem 1.5.

Our Theorem 1.5 has some application in the multilinear restriction theorem too. For each $1 \le j \le n$ assume $\Sigma_j : U_j \to \mathbb{R}^d$ is a smooth parametrization of a subset of a smooth submanifold Ω_j whose closure is compact. Also assume $\sum_{j=1}^{n} \dim \Omega_j = d$. Here we assume U_j is a neighborhood of the origin 0. We can associate the *extension operator* to Σ_j as follows:

$$E_j g_j(\xi) = \int_{U_j} e^{2\pi i \xi \cdot \Sigma_j(x)} g_j(x)\, \mathrm{d}x. \tag{1-5}$$

Assume $T_{\Sigma_1(0)}\Omega_1 \wedge \cdots \wedge T_{\Sigma_n(0)}\Omega_n \neq 0$. Then just like the classical multilinear restriction case discussed in [Bennett et al. 2006], we can form the endpoint multilinear restriction conjecture:

Conjecture 1.7 (endpoint multilinear k_j-restriction conjecture). *Assume we have Σ_j as above such that $T_{\Sigma_1(0)}\Omega_1 \wedge \cdots \wedge T_{\Sigma_n(0)}\Omega_n \neq 0$. Then when the U_j are sufficiently small, we have*

$$\int_{\mathbb{R}^d} \prod_{j=1}^{n} |E_j g_j|^{2/(n-1)} \lesssim_d \prod_{j=1}^{n} \|g_j\|_{L^2(U_j)}^{2/(n-1)}. \tag{1-6}$$

The methods in [Bennett et al. 2006] yield the following local variant of the conjectured (1-6) with R^ε-loss:

$$\int_{B(0,R)} \prod_{j=1}^{n} |E_j g_j|^{2/(n-1)} \lesssim_{d,\varepsilon} R^\varepsilon \prod_{j=1}^{n} \|g_j\|_{L^2(U_j)}^{2/(n-1)}. \tag{1-7}$$

We can use Theorem 1.4 to slightly improve (1-7). Using exactly the same proof techniques as in the proof of Theorem 4.2 in [Bennett 2014], from Theorem 1.4 we deduce that there exists a $\kappa = \kappa(d) > 0$ such that

$$\int_{B(0,R)} \prod_{j=1}^{n} |E_j g_j|^{2/(n-1)} \lesssim_{d} (\log R)^\kappa \prod_{j=1}^{n} \|g_j\|_{L^2(U_j)}^{2/(n-1)}. \tag{1-8}$$

The endpoint perturbed Brascamp–Lieb inequalities. Everything in the previous section in its unperturbed version, including the Loomis–Whitney inequality and the multilinear k_j-plane theorem, is a special case of the Brascamp–Lieb inequalities. In this paper we also generalize the Brascamp–Lieb inequalities in the same way we do with the multilinear k_j-plane theorem, with some new combinatorial ideas. We state our endpoint perturbed Brascamp–Lieb inequalities in this section.

We first briefly review the Brascamp–Lieb inequalities. We will mostly follow the notational convention in [Bennett et al. 2008; 2010], which are two important references in the literature. Assume that in $\mathbb{R}^d$ we have n linear surjections $B_j : \mathbb{R}^d \to E_j$. Then for certain positive numbers p_j, $1 \le j \le n$, the following Brascamp–Lieb inequality holds for any measurable function f_j on $E_j (1 \le j \le n)$ with some $C > 0$:

$$\int_{\mathbb{R}^d} \prod_{j=1}^{n} (f_j \circ B_j)^{p_j} \le C \prod_{j=1}^{n} \biggl(\int_{E_j} f_j\biggr)^{p_j}. \tag{1-9}$$

If this is the case, we call the minimum possible constant C such that (1-9) holds the *Brascamp–Lieb constant* $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p})$. Here we use $\boldsymbol{B}$ to denote the data $(B_1, \dots, B_n)$ and $\boldsymbol{p}$ to denote the data $(p_1, \dots, p_n)$. The pair $(\boldsymbol{B}, \boldsymbol{p})$ is called the corresponding *Brascamp–Lieb datum.* If (1-9) fails for any finite C, we define $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) = +\infty$. Note that no a priori assumptions are made on the relationship between d and n here.

Lieb [Lieb 1990] showed:

Theorem 1.8. $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) = \mathrm{BL}_g(\boldsymbol{B}, \boldsymbol{p})$, *where*

$$\mathrm{BL}_g(\boldsymbol{B}, \boldsymbol{p}) = \sup\biggl(\frac{\prod_{j=1}^{n} (\det_{E_j} A_j)^{p_j}}{\det\bigl(\sum_{j=1}^{n} p_j B_j^* A_j B_j\bigr)}\biggr)^{1/2} \tag{1-10}$$

with the supremum is taken over all $A_j : E_j \to E_j$ such that A_j is a positive definite linear transform.

An alternative way to state Theorem 1.8 is that the Brascamp–Lieb constant is what one would obtain by restricting attention to the special case in which each f_j is a certain Gaussian function.

Subsequently, Bennett, Carbery, Christ and Tao [Bennett et al. 2008; 2010] determined a necessary and sufficient condition for $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) = \mathrm{BL}_g(\boldsymbol{B}, \boldsymbol{p}) < +\infty$. They proved that $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) = \mathrm{BL}_g(\boldsymbol{B}, \boldsymbol{p}) < +\infty$ is equivalent to the following two conditions:

(1) Scaling condition:

$$\sum_j p_j \dim E_j = d. \tag{1-11}$$

(2) Dimension condition: for any linear subspace $V \subseteq \mathbb{R}^d$,

$$\dim V \leq \sum_j p_j \dim(B_j V). \tag{1-12}$$

So we know when we can have the actual Brascamp–Lieb inequality (1-9) thanks to their work.

Inequality (1-9) has an equivalent version that is easier to understand intuitively. We state it in the following proposition and refer the readers to [Bennett 2012] for this observation.

Proposition 1.9 (combinatorial Brascamp–Lieb). *Assume we have a Brascamp–Lieb datum $(\boldsymbol{B}, \boldsymbol{p})$ in $\mathbb{R}^d$. Assume $k_j = \dim\ker B_j$ and we have n families of slabs. Assume the j-th family $\mathbb{T}_j$ consists of only k_j-slabs of radius* 1 *whose cores are all parallel to* $\ker B_j$. *Also assume each $|\mathbb{T}_j|$ is finite. Then* $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) < +\infty$ *if and only if we always have*

$$\int_{\mathbb{R}^d} \prod_{j=1}^{n} \Bigl(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j}\Bigr)^{p_j} \lesssim \prod_{j=1}^{n} |\mathbb{T}_j|^{p_j}. \tag{1-13}$$

In light of the last subsection, a perturbed version of this proposition should be true. This can indeed be proved; recently, Bennett, Bez, Flock and Lee [Bennett et al. 2015, Theorem 1.2] proved the following (nonendpoint) theorem via generalizations of Guth's method [2015].

Theorem 1.10 (perturbed Brascamp–Lieb with R^ε-loss [Bennett et al. 2015]). *Assume we have a Brascamp–Lieb datum $(\boldsymbol{B}, \boldsymbol{p})$ in $\mathbb{R}^d$ with* $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) < +\infty$. *Let $k_j = \dim\ker B_j$. Assume we have n families of slabs and the j-th family $\mathbb{T}_j$ consists of only k_j-slabs of radius* 1 *and size $\leq R$. Assume each $|\mathbb{T}_j|$ is finite. Also assume that each slab in the j-th family has its core k_j-plane within a δ-neighborhood of* $\ker B_j$ *on the corresponding Grassmannian* (*with a given standard metric*). *Then when δ is sufficiently small depending on $(\boldsymbol{B}, \boldsymbol{p})$ we have*

$$\int_{\mathbb{R}^d} \prod_{j=1}^{n} \Bigl(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j}\Bigr)^{p_j} \lesssim_{d, \boldsymbol{p}, \mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}), \varepsilon} R^\varepsilon \prod_{j=1}^{n} |\mathbb{T}_j|^{p_j}. \tag{1-14}$$

They conjectured that R^ε can be removed here (see inequalities (7) and (8) of [Bennett et al. 2015]) and we prove their conjecture in the last section of this paper.

Theorem 1.11 (endpoint perturbed Brascamp–Lieb theorem). *Assume we have a Brascamp–Lieb datum $(\boldsymbol{B}, \boldsymbol{p})$ in $\mathbb{R}^d$ with* $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) < +\infty$. *Let $k_j = \dim\ker B_j$. Assume we have n families of slabs and the j-th family $\mathbb{T}_j$ consists of only k_j-slabs of radius* 1. *Assume each $|\mathbb{T}_j|$ is finite. Also assume that each slab in the j-th family has its core k_j-plane within a δ-neighborhood of* $\ker B_j$ *on the corresponding Grassmannian* (*with a given standard metric*). *Then when δ is sufficiently small depending on $(\boldsymbol{B}, \boldsymbol{p})$ we have*

$$\int_{\mathbb{R}^d} \prod_{j=1}^{n} \Bigl(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j}\Bigr)^{p_j} \lesssim_{d, \boldsymbol{p}, \mathrm{BL}(\boldsymbol{B}, \boldsymbol{p})} \prod_{j=1}^{n} |\mathbb{T}_j|^{p_j}. \tag{1-15}$$

Remark 1.12. Theorem 1.11 formally implies the stability of Brascamp–Lieb constants, which was a result of Bennett, Bez, Flock, and Lee [Bennett et al. 2015]. However, it is worth noticing (see the proof later in Section 8) that the main result in [Bennett et al. 2015] is an input rather than an output in our proof of Theorem 1.11. In particular, we do not have a new proof of the main result in [Bennett et al. 2015] in this paper.

Our proof of Theorem 1.11 will follow the same scheme as the proof of Theorem 1.4. Some new difficulties present themselves and we deal with them in due course.

Like what we had in the end of last subsection, our perturbed Brascamp–Lieb theorem has some impact on the endpoint Brascamp–Lieb-type restriction conjecture formulated in [Bennett et al. 2015]. To introduce it, we use the same setup that we did in Conjecture 1.7, but this time we don't assume that $\sum_j k_j = d$ or that $T_{\Sigma_1(0)}\Omega_1 \wedge \cdots \wedge T_{\Sigma_n(0)}\Omega_n \neq 0$. Instead, we assume that there exists $\boldsymbol{p} = (p_1, \ldots, p_n)$, $p_j > 0$, such that $\mathrm{BL}(\boldsymbol{B}(\Sigma), \boldsymbol{p}) < \infty$, where $\boldsymbol{B}(\Sigma) = (T_{\Sigma_1(0)}\Omega_1, \ldots, T_{\Sigma_n(0)}\Omega_n)$ (here we abuse the notation a bit and for each component we really mean the linear subspace of $\mathbb{R}^d$ parallel to it).

Conjecture 1.13 (endpoint Brascamp–Lieb-type restriction conjecture). *With the above setup, when the U_j are sufficiently small, we have*

$$\int_{\mathbb{R}^d} \prod_{j=1}^{n} |E_j g_j|^{2p_j} \lesssim_{d,\boldsymbol{p},\mathrm{BL}(\boldsymbol{B}(\Sigma),\boldsymbol{p})} \prod_{j=1}^{n} \|g_j\|_{L^2(U_j)}^{2p_j}. \tag{1-16}$$

In [Bennett et al. 2015] a local variant of (1-16) with R^ε-loss is proved:

$$\int_{B(0,R)} \prod_{j=1}^{n} |E_j g_j|^{2p_j} \lesssim_{d,\boldsymbol{p},\mathrm{BL}(\boldsymbol{B}(\Sigma),\boldsymbol{p}),\varepsilon} R^\varepsilon \prod_{j=1}^{n} \|g_j\|_{L^2(U_j)}^{2p_j}. \tag{1-17}$$

By Theorem 1.11 and again the same method as in the proof of Theorem 4.2 in [Bennett 2014], we can slightly improve (1-17): there is a $\kappa = \kappa(\mathrm{BL}(\boldsymbol{B}(\Sigma), \boldsymbol{p})) > 0$ such that

$$\int_{B(0,R)} \prod_{j=1}^{n} |E_j g_j|^{2p_j} \lesssim_{d,\boldsymbol{p}} (\log R)^\kappa \prod_{j=1}^{n} \|g_j\|_{L^2(U_j)}^{2p_j}. \tag{1-18}$$

Idea of the proofs. When looking to remove the factor R^ε in Theorems 1.3 and 1.10, the methods in [Bennett et al. 2006] or [Guth 2015] do not feel very appealing. Instead we will follow the path led by Guth [2010] and try to come up with a version of the so-called polynomial method.

However, there is a major difficulty to generalizing Guth's argument: note that the zero set of one polynomial has codimension 1. In the setting of [Guth 2010], because a line has dimension 1, a line will intersect the above zero set at discrete points. And the number of such points is controlled by the degree of the polynomial. Hence we can do some counting to obtain estimates. In particular, Guth's proof relies heavily on the following *cylinder estimate*.

Lemma 1.14 (cylinder estimate). *Let T be a cylinder of radius* 1 *and P be a polynomial of degree D. Let $\boldsymbol{v}$ be a unit vector parallel to the core line of T. If we define $Z(P)$ to be the zero set of P, then the directed volume* (*see Definition 2.1*) *satisfies*

$$V_{Z(P)\cap T}(\boldsymbol{v}) \lesssim_d D. \tag{1-19}$$

In the k_j-plane setting, the zero set of a single polynomial no longer interacts well with a k_j-plane: because the latter generally has a smaller codimension, it won't intersect the former at discrete points in general. Due to this issue we cannot do counting and seem to lose our main weapon (Lemma 1.14).

In this paper, we deal with this difficulty and obtain our Theorem 1.4. The main idea is the following: for a k-plane, instead of finding one single polynomial, we would like to take zero sets of k polynomials to interact with it. Because the codimensions of the k-plane and zero sets of the k polynomials add up to d, they will intersect at points and it is possible to do counting to estimate the intersection again.

Along this line, we are taking more than one polynomial to approximate an arbitrary set of N cubes. We would like the zero sets of all the polynomials to be "transverse"; with this requirement we can choose at most d such polynomials. Like the original polynomial method, we would like to know how low the degrees of our polynomials can be. Guth [2010] showed that we can always choose the first polynomial to be of degree $\lesssim_d N^{1/d}$. But for the second polynomial this degree bound may already be no longer valid. Think about N unit squares lining up on a line in the plane $\mathbb{R}^2$. Any polynomial with degree significantly less than N would have most of its zero set "almost parallel" to the line, see [Guth 2016a], and hence two such polynomials cannot interact transversely at most of the squares. However, in this example it is possible to find two transverse polynomials with degree product N. One can also look at examples of cube grids, or more generally transverse intersections of hypersurfaces, and similar phenomena happen there. Based on the above discussion, we are willing to ask the following question in the spirit of the polynomial method.

Question 1.15. *Given any N disjoint unit cubes in $\mathbb{R}^d$ and $A_\nu > 1$ for each given cube Q_ν, do there always exist d polynomials $P_1, P_2, \dots, P_d$ such that $\prod_{i=1}^d \deg P_i$ is roughly $\sum_\nu A_\nu$, and the zero sets of all P_i have "quantitative interaction" $\gtrsim_d A_\nu$ at each of the above cubes?*

We notice that it looks like a "continuous version" of the inverse Bézout's theorem; see for example [Tao 2012]. The analogue is very difficult in algebraic geometry, see [Tao 2012] for part of the reason, and is conceivably very hard in its current continuous version too. We believe it can be formulated as an explicit question with an affirmative answer though. One can make this question rigorous by specifying the meaning of "quantitative interaction"; see the discussion below and (6-9) for a result of this flavor.

Luckily enough, we find the full power of this hard version is not needed this time. Instead, it will be equally useful to have a positive answer to the following "softer" question.

Question 1.16. *Given any N disjoint unit cubes in $\mathbb{R}^d$ and $A_\nu > 1$ for each given cube Q_ν, do there always exist d polynomials $P_1, P_2, \dots, P_d$ and positive numbers $\alpha_\nu > 1$ such that $\prod_{i=1}^d \deg P_i$ is roughly $\sum_\nu \alpha_\nu A_\nu$, and the zero sets of all P_i have quantitative interaction $\gtrsim_d \alpha_\nu A_\nu$ at each of the above cubes?*

This question is weaker than Question 1.15 because there we have the additional requirement that $a_\nu = 1$. In other words, we allow polynomials of higher degree here but "with the right multiplicity". In general, higher-degree polynomials, even with the right multiplicity, do not necessarily work as well as ones with lowest possible degree; see for example some estimates in [Guth 2016a]. But in this application it makes no difference, as we are in a situation similar to what we have in [Guth 2010].

Surprisingly, it turns out that after some further refinement of the question, we find that we can take $P_1, \dots, P_d$ all to be the same P and that we can obtain P by the refined polynomial method of Guth involving visibility. Once this is clear we are able to prove our theorem with a great amount of help from (multi-)linear algebra and geometry.

To be more specific, we find that we can take a single nonzero polynomial (that is complicated enough to look like the product of several transverse polynomials) such that the following holds: If we define $Z(P)$ to be the zero set of P, then for each relevant Q_ν, d copies of $Z(P) \cap Q_\nu$ interact in a sufficiently transverse manner. Since the d copies of $Z(P) \cap Q_\nu$ interact in a very transverse way, and the copies are all the same, for any j and any Q_ν we deduce that k_j copies of $Z(P) \cap Q_\nu$ interact sufficiently transversely with the part of the j-th family of slabs inside Q_ν. But for any j, the j-th family has a limited capacity of transverse interaction with k_j copies of $Z(P)$ by Bézout's theorem. This gives us an estimate that leads to Theorem 1.4.

As we saw above, we end up taking one single polynomial d times. Nevertheless, we choose to keep the entire thought process on "d transverse polynomials" here because after all, it is how we eventually come up with the solution and the reader might find our thought process useful elsewhere. Also, Question 1.15, which remains open, is still fundamental, as it's a general one concerning the polynomial approximation of any N cubes. For example, it implies the existence of the polynomial in the polynomial method. Its discrete analogue is also open; see [Tao 2012]. But progress in various subcases has been made.

In the multilinear k_j-plane setting, our method actually proves a stronger theorem (multilinear k_j-variety theorem, Theorem 6.1) which largely generalizes Theorem 1.4. We will state its exact form after a bit more preparation. Here let us briefly describe it.

Let's take a new viewpoint. Knowing that a point belongs to a slab of radius 1 is equivalent to knowing the existence of another point on the core of the slab that lies in its 1-neighborhood. Also note that the union of all cores (k_j-planes) of the j-th family of slabs can be viewed as an algebraic variety of degree $A(j)$ and dimension k_j. This variety is a smoothly embedded k_j-manifold except some zero-volume subset. Our Theorem 1.4 is basically saying that the n families of k_j-planes have limited capacity of "transversally interaction". We will prove that this is the general case for any n algebraic varieties with total dimension d in Theorem 6.1.

This multilinear k_j-variety theorem immediately has interesting special cases. For instance, we have a theorem about collections of sphere shells in the flavor of Theorem 1.4.

The proof of Theorem 1.11 is with almost the same machine, but we have some new difficulties: When we use this machine, we want to know how well each k_j-plane interacts with our polynomial. However, the information on the Brascamp–Lieb constant seems to be very hard to use when we try to look at things "locally", as we do in the proof of Theorem 1.4. We address this issue in Section 7 and Section 8 by proving a weaker "integral version" of our previous pointwise estimate. Albeit weaker, it already leads to a proof of Theorem 1.11.

Like the situation of Theorem 1.4, Theorem 1.11 has a generalization to algebraic varieties (Theorem 8.1) and we prove the latter to automatically imply the former. Again the current form is quite strong and interesting in its own right.

Outline of the paper. In Sections 2 and 3 we review Guth's polynomial method [2010] and develop all we need in this subject. Section 4 consists of linear algebra preliminaries and Section 5 consists of integral geometry preliminaries. We prove Theorem 1.4 and Theorem 1.5 in Section 6 and Theorem 1.11 in Section 8 after some preparation (Section 7). We will prove them by generalizing to versions about algebraic varieties.

2. Polynomial with high visibility

In this section, we review the refined polynomial method by Guth [2010]. We review the definition and properties of visibility and state Guth's theorem that we can find a polynomial with reasonable degree and large visibility in many cubes. Along the way we define a relevant notion, namely the *fading zone*, for future convenience.

Definition 2.1. In $\mathbb{R}^d$, for any compact smooth hypersurface Z (possibly with boundary) and any vector $\boldsymbol{v}$, define the *directed volume*

$$V_Z(\boldsymbol{v}) = \int_Z |\boldsymbol{v}\cdot\boldsymbol{n}|\,\mathrm{dVol}_Z, \tag{2-1}$$

where $\boldsymbol{n}$ is the normal vector at the corresponding point of Z.

If $\boldsymbol{v}$ is a unit vector, there is a formula for $V_Z(\boldsymbol{v})$ that is geometrically more meaningful. Let $\pi_{\boldsymbol{v}}$ be the orthogonal projection of $\mathbb{R}^d$ onto the subspace $\boldsymbol{v}^\perp$. Then for almost $y \in \boldsymbol{v}^\perp$, we have $|Z \cap \pi_{\boldsymbol{v}}^{-1}(y)|$ is finite and, see [Guth 2010],

$$V_Z(\boldsymbol{v}) = \int_{\boldsymbol{v}^\perp} |Z \cap \pi_{\boldsymbol{v}}^{-1}(y)|\,\mathrm{d}y. \tag{2-2}$$

Definition 2.2. The *fading zone* $F(Z)$ is defined to be the set $\{\boldsymbol{v} : |\boldsymbol{v}| \le 1,\ V_Z(\boldsymbol{v}) \le 1\}$. It is a nonempty convex compact subset of the unit ball; see [Guth 2010]. The visibility $\mathrm{Vis}[Z] = 1/|F(Z)|$.

First we explain the heuristic meaning of the two concepts. Imagine that it is midnight and we are looking at a glittering object with exactly the same shape as Z from a fixed distance. To describe the situation mathematically, we can find a vector $\boldsymbol{v}$ such that its direction is the direction of the object and its length is the brightness of the object. Then we can intuitively think that Z fades away when $\boldsymbol{v}$ enters the fading zone. And naturally the less visible the object is, the larger we want the fading zone to be. Hence we can define the visibility to be the inverse of the volume of the fading zone. See the beginning of Section 6 in [Guth 2010] for how to intuitively understand visibility and a few simple examples.

It is good to keep in mind that in this paper we will mostly deal with hypersurfaces Z with $V_Z(\boldsymbol{v}) \gtrsim_d 1$ for any unit vector $\boldsymbol{v}$. For hypersurfaces that don't satisfy this we will typically fix it by taking its union with several hyperplanes parallel to coordinate hyperplanes.

Clearly as long as Z has finite volume, $F(Z)$ has a nonempty interior.

We are interested in polynomials and want to use the notions above to study them. Recall that the space of degree D algebraic hypersurfaces in $\mathbb{R}^d$ is parametrized by $\mathbb{RP}^K$ for $K = \binom{D+d}{d} - 1$ in the following way: any such hypersurface corresponds to a polynomial P up to a scalar. By viewing P also as the $\binom{D+d}{d}$-tuple of its coefficients we find this parametrization [Guth 2010]. We want to think of the directed volume and the visibility as functions over $\mathbb{RP}^K$. However, as Guth [2010] pointed out, they are bad functions that may even be discontinuous.

Following [Guth 2010], we get around this difficulty by looking at the mollified versions of them. If we take the standard metric on $\mathbb{RP}^K$, we will mollify those functions over small balls around some $P \in \mathbb{RP}^K$. In the rest of this paper, we take ε to be a very small positive number depending on all the constants, and in application on the set of cubes and visibility conditions. This kind of assumption is often dangerous but as we can eventually see, here it does no harm at all (mainly because all the algebraic hypersurfaces of degree D satisfy the same intersection estimate (5-5) uniformly), just like the case of [Guth 2010]. There instead of the intersection estimate, we have the cylinder estimate (1-19) as a special case counterpart.

For any $P \in \mathbb{RP}^K$, let $B(P, \varepsilon)$ be the ε-neighborhood of P. Let $Z(P)$ denote the zero set of P. Note that for any P, the set of singular points on $Z(P)$ has zero $(d-1)$-dimensional Hausdorff measure. And the rest of $Z(P)$ is a smooth embedded hypersurface by the implicit function theorem.

Definition 2.3. For any bounded open set U and any vector $\boldsymbol{v}$, define the *mollified directed volume*

$$\bar{V}_{Z(P)\cap U}(\boldsymbol{v}) = \frac{1}{|B(P,\varepsilon)|}\int_{B(P,\varepsilon)} V_{Z(P')\cap U}(\boldsymbol{v})\,\mathrm{d}P'. \tag{2-3}$$

Define the *mollified fading zone* and *mollified visibility* based on the mollified directional volumes:

$$\bar{F}(Z(P)\cap U) = \{\boldsymbol{v} : |\boldsymbol{v}| \le 1 : \bar{V}_{Z(P)\cap U}(\boldsymbol{v}) \le 1\}, \tag{2-4}$$

$$\overline{\mathrm{Vis}}[Z(P)\cap U] = \frac{1}{|\bar{F}(Z(P)\cap U)|}. \tag{2-5}$$

Like we had before for $F(Z)$, $\bar{F}(Z(P)\cap U)$ is a convex compact subset of the unit ball with a nonempty interior. By John's ellipsoid theorem [1948], for any convex set Γ with interior, there is an ellipsoid $\mathrm{Ell}(\Gamma)$ such that $\mathrm{Ell}(\Gamma) \subseteq \Gamma \subseteq C_d\,\mathrm{Ell}(\Gamma)$ and $|\,\mathrm{Ell}(\Gamma)| \sim_d |\Gamma|$. It is easy to see that if the convex set is symmetric about the origin (which will be the case for all convex sets considered in this paper), then we may require the ellipsoid to be symmetric about the origin too. We assume so henceforth in the paper. We call any such $\mathrm{Ell}(\Gamma)$ an *elliptical approximation* of Γ.

$\bar{V}_{Z(P)\cap U}(\boldsymbol{v})$ and $\overline{\mathrm{Vis}}[Z(P)\cap U]$ are continuous with respect to $P \in \mathbb{RP}^M$ [Guth 2010]. In the same paper, Guth also proved the following key lemma.

Lemma 2.4 (large visibility on many cubes [Guth 2010]). *For any finite set of cubes $Q_1, \dots, Q_N$ and nonnegative integers $M(Q_i)$, $1 \le i \le N$, there exists a polynomial P of degree $\le D$ (but viewed as a degree-D polynomial when we mollify) such that $\overline{\mathrm{Vis}}(Z(P)\cap Q_k) \ge M(Q_k)$ and $D \lesssim_d \left(\sum_{i=1}^N M(Q_k)\right)^{1/d}$.*

3. Wedge-product estimate based on visibility

As we are actually dealing with the mollification version of everything, it is convenient to have a generalized definition of visibility on any space of finite measure. The generalized setup here will also be cleaner and more flexible in the inductive arguments which are needed.

Assume we have a measure space (X, μ) with $\mu(X) < \infty$ and a vector-valued measurable function $f : X \to \mathbb{R}^d$. For any vector $\boldsymbol{v} \in \mathbb{R}^d$ define the *total absolute inner product* of $\boldsymbol{v}$ and f as

$$V_{X,f}(\boldsymbol{v}) = \int_X |\boldsymbol{v} \cdot f(x)| \,\mathrm{d}\mu(x) \tag{3-1}$$

(the directed volume of the last section being the example we have in mind).

Define the fading zone $F(X, f) = \{\boldsymbol{v} \leq 1 : V_{X,f}(\boldsymbol{v}) \leq 1\}$ and visibility $\mathrm{Vis}[X, f] = 1/|F(X, f)|$. As we had in the end of the last section, we have an elliptical approximation $\mathrm{Ell}(F(X, f))$ such that $\mathrm{Ell}(F(X, f)) \subseteq F(X, f) \subseteq C_d \,\mathrm{Ell}(F(X, f))$.

Next we obtain a lower bound of a wedge product integral in terms of visibility.

Theorem 3.1 (wedge product estimate). *Assume that for any unit vector $\boldsymbol{v}$ we have $V_{X,f}(\boldsymbol{v}) \geq 1$. Then*

$$\int \cdots \int_{X^d} \left| \bigwedge_{i=1}^{d} f(x_i) \right| \mathrm{d}\mu(x_1)\,\mathrm{d}\mu(x_2) \cdots \mathrm{d}\mu(x_d) \gtrsim_d \mathrm{Vis}[X, f]. \tag{3-2}$$

Proof. We do induction on the dimension d to prove the theorem. First observe that if $\mathrm{Ell}(F(X, f))$ is an elliptical approximation of $F(X, f)$, then for any linear subspace W of $\mathbb{R}^d$, we have $\mathrm{Ell}(F(X, f)) \cap W$ (an ellipsoid) is also an elliptical approximation of $F(X, f) \cap W$ by definition (this may seem problematic as the C_d will vary, but for the conclusion only finitely many intermediate dimensions are involved in the whole induction process and we can set C_d of them to all be the same).

For $d = 1$, by definition we easily see

$$\mathrm{Vis}[X, f] = \frac{1}{2} \int_X |f(x)| \,\mathrm{d}\mu(x) \tag{3-3}$$

and the conclusion holds. Note that even in the argument here we are using the hypothesis to ensure we have (3-3).

Assume the conclusion holds for $d < d_0$ and $d_0 > 1$. Now we deal with the case $d = d_0$. Assume $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d_0}$ are parallel to the semiprincipal axes of any elliptical approximation $\mathrm{Ell}(F(X, f))$, respectively, and that they form an orthonormal basis (we can arbitrarily choose a set of orthogonal semiprincipal axes if there is ambiguity defining the semiprincipal axes). Among them we assume $\boldsymbol{v}_1$ is parallel to a semiminor axis (i.e., a shortest semiprincipal axis) that has length t_1. Taking $\boldsymbol{v} = \lambda \boldsymbol{v}_1$, where $\lambda \sim_{d_0} t_1$ in (3-1), we deduce

$$\int_X |f(x)| \,\mathrm{d}\mu(x) \geq \frac{1}{t_1}. \tag{3-4}$$

Next for any unit vector $\boldsymbol{v} \in \mathbb{R}^{d_0}$, we prove

$$\int \cdots \int_{X^{d_0 - 1}} \left| f(x_1) \wedge \cdots \wedge f(x_{d_0-1}) \wedge \boldsymbol{v} \right| \mathrm{d}\mu(x_1)\,\mathrm{d}\mu(x_2) \cdots \mathrm{d}\mu(x_{d_0-1}) \gtrsim_{d_0} t_1 \cdot \mathrm{Vis}[X, f]. \tag{3-5}$$

Let $\pi_{\boldsymbol{v}^\perp}$ be the orthogonal projection from $\mathbb{R}^{d_0}$ to its subspace $\boldsymbol{v}^\perp$. Define $f_{\boldsymbol{v}^\perp} = \pi_{\boldsymbol{v}^\perp} \circ f$. If we identify $\mathbb{R}^{d_0-1}$ with $\boldsymbol{v}^\perp$, then $f_{\boldsymbol{v}^\perp}$ is another $(d_0{-}1)$-dimensional vector-valued function on X. By definition, we know $V_{X,f}(\boldsymbol{w}) = V_{X,f_{\boldsymbol{v}^\perp}}(\boldsymbol{w})$ for any $\boldsymbol{w} \in \boldsymbol{v}^\perp$. Hence $F(X, f_{\boldsymbol{v}^\perp}) = F(X, f) \cap \boldsymbol{v}^\perp$. By the previous discussion, we know we can choose $\mathrm{Ell}(F(X, f_{\boldsymbol{v}^\perp}))$ to be $\mathrm{Ell}(F(X, f)) \cap \boldsymbol{v}^\perp$. But among all the (d_0-1)-dimensional sections of $\mathrm{Ell}(F(X, f))$ passing through the origin, the section cut by $\boldsymbol{v}_1^\perp$ has the largest volume (see also Lemma 7.4), which is $\sim_{d_0} |\mathrm{Ell}(F(X, f))|/t_1 = 1/(t_1 \cdot \mathrm{Vis}[X, f])$. Hence

$$\mathrm{Vis}[X, f_{\boldsymbol{v}^\perp}] = \frac{1}{|F(X, f_{\boldsymbol{v}^\perp})|} \sim_{d_0} \frac{1}{|\mathrm{Ell}(F(X, f_{\boldsymbol{v}^\perp}))|} \gtrsim_{d_0} t_1 \cdot \mathrm{Vis}[X, f].$$

By induction hypothesis we have

$$\begin{aligned}\int \cdots \int_{X^{d_0-1}} \big| f(x_1) \wedge \cdots \wedge f(x_{d_0-1}) \wedge \boldsymbol{v} \big| \, \mathrm{d}\mu(x_1)\, \mathrm{d}\mu(x_2) \cdots \mathrm{d}\mu(x_{d_0-1}) \\ = \int \cdots \int_{X^{d_0-1}} \big| f_{\boldsymbol{v}^\perp}(x_1) \wedge \cdots \wedge f_{\boldsymbol{v}^\perp}(x_{d_0-1}) \big| \, \mathrm{d}\mu(x_1)\, \mathrm{d}\mu(x_2) \cdots \mathrm{d}\mu(x_{d_0-1}) \\ \gtrsim_{d_0} \mathrm{Vis}[X, f_{\boldsymbol{v}^\perp}] \gtrsim_{d_0} t_1 \cdot \mathrm{Vis}[X, f]. \end{aligned} \tag{3-6}$$

This is (3-5).

Combining (3-4) and (3-5), we have

$$\begin{aligned}\int \cdots \int_{X^d} \left| \bigwedge_{i=1}^{d} f(x_i) \right| \mathrm{d}\mu(x_1)\, \mathrm{d}\mu(x_2) \cdots \mathrm{d}\mu(x_d) \\ = \int_X |f(x)| \left(\int \cdots \int_{X^{d_0-1}} \left| f(x_1) \wedge \cdots \wedge f(x_{d_0-1}) \wedge \frac{f(x)}{|f(x)|} \right| \mathrm{d}\mu(x_1)\, \mathrm{d}\mu(x_2) \cdots \mathrm{d}\mu(x_{d_0-1}) \right) \mathrm{d}\mu(x) \\ \gtrsim_{d_0} t_1 \cdot \mathrm{Vis}[X, f] \cdot \int_X |f(x)| \, \mathrm{d}\mu(x) \gtrsim_{d_0} \mathrm{Vis}[X, f], \end{aligned} \tag{3-7}$$

which concludes the induction. ☐

4. Linear algebra preliminaries

Our proof relies heavily on linear algebra. In this section we do the linear algebraic part and prove two useful lemmas.

Lemma 4.1. *Assume $V_1, \dots, V_n \subseteq \mathbb{R}^d$ and $k_j = \dim V_j$ satisfies $\sum_{j=1}^n k_j = d$. Then for any vectors $\boldsymbol{w}_1, \dots, \boldsymbol{w}_d \in \mathbb{R}^d$, we have*

$$\max \prod_{j=1}^{n} \big| (V_j)^\perp \wedge \boldsymbol{w}_{i_{j,1}} \wedge \cdots \wedge \boldsymbol{w}_{i_{j,k_j}} \big| \gtrsim_d |V_1 \wedge \cdots \wedge V_n| \cdot \left| \bigwedge_{i=1}^{d} \boldsymbol{w}_i \right|, \tag{4-1}$$

where the maximum is taken over $1 \le i_{j,h} \le d$ for $1 \le j \le n$, $1 \le h \le k_j$, where each $1 \le i \le d$ is chosen exactly once among all $i_{j,h}$.

Proof. Assume that $\{\boldsymbol{v}_{j,h}\}_{1 \le h \le k_j}$ is an orthonormal basis of V_j. Then by definition we have

$$\left| (V_1 \wedge \cdots \wedge V_n) \cdot \left(\bigwedge_{i=1}^{d} \boldsymbol{w}_i \right) \right| = |(\boldsymbol{v}_{j,h} \cdot \boldsymbol{w}_i)|. \tag{4-2}$$

By the generalized Laplace cofactor expansion, the determinant on the right-hand side of (4-2) is a sum of terms in the form

$$\pm \det(\boldsymbol{v}_{1,h} \cdot \tilde{\boldsymbol{w}}_{1,h}) \det(\boldsymbol{v}_{2,h} \cdot \tilde{\boldsymbol{w}}_{2,h}) \cdots \det(\boldsymbol{v}_{n,h} \cdot \tilde{\boldsymbol{w}}_{n,h}) \tag{4-3}$$

where $\tilde{\boldsymbol{w}}_{1,1}, \tilde{\boldsymbol{w}}_{1,2}, \dots, \tilde{\boldsymbol{w}}_{1,k_1}, \tilde{\boldsymbol{w}}_{2,1}, \dots, \tilde{\boldsymbol{w}}_{2,k_2}, \dots, \tilde{\boldsymbol{w}}_{n,1}, \dots, \tilde{\boldsymbol{w}}_{n,k_n}$ is a rearrangement of $\boldsymbol{w}_1, \dots, \boldsymbol{w}_d$. Hence for some such rearrangement we have

$$|V_1 \wedge \cdots \wedge V_n| \cdot \left| \bigwedge_{i=1}^{d} \boldsymbol{w}_i \right| \lesssim_d \left| \det(\boldsymbol{v}_{1,h} \cdot \tilde{\boldsymbol{w}}_{1,h}) \det(\boldsymbol{v}_{2,h} \cdot \tilde{\boldsymbol{w}}_{2,h}) \cdots \det(\boldsymbol{v}_{n,h} \cdot \tilde{\boldsymbol{w}}_{n,h}) \right|. \tag{4-4}$$

By the properties of the Hodge $*$-operator, we then have

$$|\det(\boldsymbol{v}_{j,h} \cdot \tilde{\boldsymbol{w}}_{j,h})| = \left|(*\boldsymbol{v}_{j,1} \wedge \cdots \wedge \boldsymbol{v}_{j,k_j}) \wedge \tilde{\boldsymbol{w}}_{j,1} \wedge \cdots \wedge \tilde{\boldsymbol{w}}_{j,k_j}\right| = |V_j^{\perp} \wedge \tilde{\boldsymbol{w}}_{j,1} \wedge \cdots \wedge \tilde{\boldsymbol{w}}_{j,k_j}|, \tag{4-5}$$

which concludes the proof. □

The rest of this section is dedicated to the computation of a determinant that will be useful in the next section.

Lemma 4.2. *Assume that* $0 \le c_j \le d$ *are integers*, $1 \le j \le m$, *satisfying* $\sum_{j=1}^{m} c_j = d$. *For any* j, *assume* $\boldsymbol{v}_{j,1}, \boldsymbol{v}_{j,2}, \dots, \boldsymbol{v}_{j,d}$ *is an orthonormal basis of* $\mathbb{R}^d$ (*written as column vectors*). *Then we have*

$$\left|\det\begin{pmatrix} \boldsymbol{v}_{1,c_1+1} & \cdots & \boldsymbol{v}_{1,d} & \boldsymbol{v}_{2,c_2+1} & \cdots & \boldsymbol{v}_{2,d} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \boldsymbol{v}_{1,c_1+1} & \cdots & \boldsymbol{v}_{1,d} & 0 & \cdots & 0 & \boldsymbol{v}_{3,c_3+1} & \cdots & \boldsymbol{v}_{3,d} & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{v}_{1,c_1+1} & \cdots & \boldsymbol{v}_{1,d} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \boldsymbol{v}_{m,c_m+1} & \cdots & \boldsymbol{v}_{m,d} \end{pmatrix}\right|$$
$$= \left|\det\big(\boldsymbol{v}_{1,1} \cdots \boldsymbol{v}_{1,c_1} \boldsymbol{v}_{2,1} \cdots \boldsymbol{v}_{2,c_2} \cdots \boldsymbol{v}_{m,1} \cdots \boldsymbol{v}_{m,c_m}\big)\right|. \tag{4-6}$$

Proof. For $2 \le j \le m$, let $A_j = (\boldsymbol{v}_{1,1} \cdots \boldsymbol{v}_{1,c_1} \boldsymbol{0} \cdots \boldsymbol{0} \cdots \boldsymbol{v}_{j,1} \cdots \boldsymbol{v}_{j,c_j} \cdots \boldsymbol{0} \cdots \boldsymbol{0})$. The rule here is that its first c_1 columns are $\boldsymbol{v}_{1,1}, \dots, \boldsymbol{v}_{1,c_1}$ and its $\big(\sum_{j'<j} c_{j'} + 1\big)$-th to $\big(\sum_{j'<j} c_{j'}\big)$-th columns are $\boldsymbol{v}_{j,1}, \dots, \boldsymbol{v}_{j,c_j}$, while its other columns are zero vectors. The left-hand side of (4-6) is equal to

$$\left|\det\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_2 & \boldsymbol{v}_{1,c_1+1} & \cdots & \boldsymbol{v}_{1,d} & \boldsymbol{v}_{2,c_2+1} & \cdots & \boldsymbol{v}_{2,d} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ A_3 & \boldsymbol{v}_{1,c_1+1} & \cdots & \boldsymbol{v}_{1,d} & 0 & \cdots & 0 & \boldsymbol{v}_{3,c_3+1} & \cdots & \boldsymbol{v}_{3,d} & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_m & \boldsymbol{v}_{1,c_1+1} & \cdots & \boldsymbol{v}_{1,d} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \boldsymbol{v}_{m,c_m+1} & \cdots & \boldsymbol{v}_{m,d} \end{pmatrix}\right|.$$

We exchange the columns to make it look better. For simplicity let $V_j = (\boldsymbol{v}_{j,1} \cdots \boldsymbol{v}_{j,d})$. This is an orthogonal matrix. We also define a matrix $B_j = (b_j(k,l))$, $1 \le k \le d$, such that $b_j(k,l) = 1$ if $l \le c_j$ and $k = l + \sum_{j'<j} c_{j'}$, and $b_j(k,l) = 0$ otherwise. Then after rearranging the columns of the matrix above we

find the determinant (in absolute value) is equal to

$$\left|\det\begin{pmatrix} B_1 & B_2 & B_3 & \cdots & B_m \\ V_1 & V_2 & 0 & \cdots & 0 \\ V_1 & 0 & V_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ V_1 & 0 & 0 & \cdots & V_m \end{pmatrix}\right|.$$

We can multiply the j-th column by $V_j^{-1} = V_j^t$ on the right, then subtract all the j-th columns, $j > 1$, from the first column. This preserves the determinant. Note the definition of B_j, if we define $\Delta = (\boldsymbol{v}_{1,1} \cdots \boldsymbol{v}_{1,c_1} - \boldsymbol{v}_{2,1} \cdots - \boldsymbol{v}_{2,c_2} \cdots - \boldsymbol{v}_{m,1} \cdots - \boldsymbol{v}_{m,c_m})$, then the determinant is

$$\left|\det\begin{pmatrix} \Delta^t & B_2V_2^t & B_3V_3^t & \cdots & B_mV_m^t \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I \end{pmatrix}\right|.$$

Equation (4-6) then follows directly. □

5. Integral geometry preliminaries

In this section we prepare some integral geometry tools for our proof of Theorem 1.4. First we generalize (2-2) to the following lemma.

Lemma 5.1. *Assume in $\mathbb{R}^d$ we have m smooth compact submanifolds $Z_1, Z_2, \ldots, Z_m$ (possibly with boundary) with codimensions $c_1, \ldots, c_m$ respectively. If $\sum_{j=1}^m c_j = d$ then for any measurable subset $U \subseteq \mathbb{R}^{d(m-1)} = (\mathbb{R}^d)^{m-1}$, we have*

$$\begin{aligned}\int_{Z_1}\int_{Z_2}\cdots\int_{Z_m} \chi_U(\overrightarrow{p_1p_2}, \ldots, \overrightarrow{p_1p_m})\bigl|(T_{p_1}Z_1)^\perp \wedge \cdots \wedge (T_{p_m}Z_m)^\perp\bigr|\,\mathrm{dVol}_1 \cdots \mathrm{dVol}_m \\ = \int_{\boldsymbol{v}_2,\ldots,\boldsymbol{v}_m \in \mathbb{R}^d,\,(\boldsymbol{v}_2,\ldots,\boldsymbol{v}_m)\in U} \bigl|(Z_1) \cap (Z_2 + \boldsymbol{v}_2) \cap \cdots \cap (Z_{m-1} + \boldsymbol{v}_{m-1}) \cap (Z_m + \boldsymbol{v}_m)\bigr|\,\mathrm{d}\boldsymbol{v}_2 \cdots \mathrm{d}\boldsymbol{v}_m, \quad (5\text{-}1)\end{aligned}$$

where $p_j \in Z_j$, $T_{p_j}Z_j$ is the tangent space of Z_j at p_j, dVol_j is the volume element on the j-th submanifold, and $Z_j + \boldsymbol{v}_j = \{p_j + \boldsymbol{v}_j : p_j \in Z_j\}$ is the translation of Z_j along the vector $\boldsymbol{v}_j$. The $|\cdot|$ on the right-hand side defines cardinality.

This lemma has a lot of information so we pause a bit and go through several examples to understand it better.

When $d = 2$, if Z_1 and Z_2 are two nonparallel line segments and U is the whole $\mathbb{R}^2$, the integrand on the right-hand side of (5-1) is the characteristic function of a parallelogram generalized by Z_1 and Z_2. Hence the right-hand side is the area of the parallelogram, which is easily seen to be equal to the left-hand side. When $d = 3$, if Z_1 is a line segment, Z_2 is a parallelogram in a plane and U is the whole $\mathbb{R}^3$, the situation is totally analogous.

When $d = 3$, Z_1 is a whole line and Z_2 is a smooth surface of finite area, and we can take U to be the point set between two planes orthogonal to Z_1 with distance 1. It is a simple exercise to show that (5-1) then becomes (2-2). Hence it is indeed a generalization of the latter.

Finally let's look at a more complicated example. Again take $d = 3$ and $U = \mathbb{R}^6$. Take a parallelepiped $\Omega = ABCD - A_1B_1C_1D_1$. Take three parallelograms $Z_1 = ABCD,\ Z_2 = ABB_1A_1,\ Z_3 = ADD_1A_1$. Define $\boldsymbol{u} = \overrightarrow{AB},\ \boldsymbol{v} = \overrightarrow{AD},\ \boldsymbol{w} = \overrightarrow{AA_1}$. Again the integrand on the right-hand side is a characteristic function. We find it is plainly equal to $\mathrm{Vol}(\Omega)^2$. Now the left-hand side is equal to

$$\begin{aligned}|\boldsymbol{u}\times\boldsymbol{v}|\cdot|\boldsymbol{v}\times\boldsymbol{w}|\cdot|\boldsymbol{w}\times\boldsymbol{u}|\cdot\left|\frac{\boldsymbol{u}\times\boldsymbol{v}}{|\boldsymbol{u}\times\boldsymbol{v}|}\wedge\frac{\boldsymbol{v}\times\boldsymbol{w}}{|\boldsymbol{v}\times\boldsymbol{w}|}\wedge\frac{\boldsymbol{w}\times\boldsymbol{u}}{|\boldsymbol{w}\cdot\boldsymbol{u}|}\right| &= |(\boldsymbol{u}\times\boldsymbol{v})\wedge(\boldsymbol{v}\times\boldsymbol{w})\wedge(\boldsymbol{w}\times\boldsymbol{u})|\\ &= |((\boldsymbol{u}\times\boldsymbol{v})\times(\boldsymbol{v}\times\boldsymbol{w}))\cdot(\boldsymbol{w}\times\boldsymbol{u})|\\ &= |(\boldsymbol{v}\cdot(\boldsymbol{u}\times\boldsymbol{w}))\boldsymbol{v}\cdot(\boldsymbol{w}\times\boldsymbol{u})| = \mathrm{Vol}(\Omega)^2. \quad (5\text{-}2)\end{aligned}$$

Proof of Lemma 5.1. Without loss of generality we can assume U is open and bounded. By the multilinear feature of both sides of (5-1), we only need to consider this problem locally. Hence we can assume each Z_j is smoothly parametrized by a domain in $\mathbb{R}^{d-c_j}$. In other words we may assume $Z_j : x_i = f_{j,i}(y_{j,1}, \dots, y_{j,d-c_j})$ and that the $(d - c_j)$ vectors $\boldsymbol{w}_{j,l} = (\partial f_{j,i}/\partial y_{j,l})_{1\le i\le d}$ have a nonzero wedge product at any point $p_j \in Z_j$. They span the tangent space $T_{p_j}Z_j$ and will be written as column vectors below.

Look at the cartesian product $Z = Z_1 \times Z_2 \times\cdots\times Z_m \subseteq (\mathbb{R}^d)^m \cong \mathbb{R}^{dm}$. This is a smooth submanifold of dimension $\sum_{j=1}^m (d - c_j) = d(m-1)$. Use $x_{j,i},\ 1 \le i \le d$, to denote the standard Euclidean coordinates in the j-th copy of $\mathbb{R}^d$ and let $v_{j,i} = x_{j,i} - x_{1,i},\ j > 1$. For simplicity let $\boldsymbol{x}_j = (x_{j,i})_{1\le i\le d}$ and $\boldsymbol{v}_j = (v_{j,i})_{1\le i\le d}$. Notice that the right-hand side of (5-1) is equal to

$$\int_Z \chi_U((\boldsymbol{v}_j)_{2\le j\le m})|\,\mathrm{d}\boldsymbol{v}_2\,\mathrm{d}\boldsymbol{v}_3\cdots\,\mathrm{d}\boldsymbol{v}_m|.$$

Define the density form $\theta = |\mathrm{d}\boldsymbol{v}_2\,\mathrm{d}\boldsymbol{v}_3\,\cdots\,\mathrm{d}\boldsymbol{v}_m| = |\mathrm{d}v_{2,1}\wedge\mathrm{d}v_{2,2}\wedge\cdots\wedge\mathrm{d}v_{2,d}\wedge\cdots\wedge\mathrm{d}v_{m,1}\wedge\cdots\wedge\mathrm{d}v_{m,d}|$. On the manifold Z it is a multiple of the volume density element

$$|\mathrm{d}V| = \prod_{j=1}^m \left|\bigwedge_{l=1}^{d-c_j}\boldsymbol{w}_{j,l}\right|\left|\bigwedge_{1\le j\le m,1\le l\le d-c_j}\mathrm{d}y_{j,l}\right|.$$

Next we find $\theta/|\mathrm{d}V|$.

We have

$$\frac{\theta}{|\mathrm{d}V|} = \frac{1}{\prod_{j=1}^m\left|\bigwedge_{l=1}^{d-c_j}\boldsymbol{w}_{j,l}\right|}\left|\left(\frac{\partial v_{j,i}}{\partial y_{j,l}}\right)\right|.$$

And by change of variable we have

$$\left|\left(\frac{\partial v_{j,i}}{\partial y_{j,l}}\right)\right| = \left|\det\begin{pmatrix} -\boldsymbol{w}_{1,1} & \cdots & -\boldsymbol{w}_{1,d-c_1} & \boldsymbol{w}_{2,1} & \cdots & \boldsymbol{w}_{2,d-c_2} & \cdots & 0 & \cdots & 0 & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ -\boldsymbol{w}_{1,1} & \cdots & -\boldsymbol{w}_{1,d-c_1} & 0 & \cdots & 0 & \cdots & \boldsymbol{w}_{m,1} & \cdots & \boldsymbol{v}_{m,d-c_m} & \cdots & \cdots & \cdots \end{pmatrix}\right|. \quad (5\text{-}3)$$

This looks very much like the left-hand side of (4-6). Indeed, the extra negative signs do not change the determinant and can be ignored. The only essential difference here is that for each j, our $\{w_{j,l}\}_{1\le l\le d-c_j}$ is not a set of orthonormal vectors. If we do a change of variable to make them orthonormal we will extract a factor of $\bigl|\bigwedge_{l=1}^{d-c_j} \boldsymbol{w}_{j,l}\bigr|$ from right-hand side of (5-3) for each j. We then apply Lemma 4.2 and get

$$\frac{\theta}{|\mathrm{d}V|} = \left|\bigwedge_{j=1}^{m} (T_{p_j} Z_j)^{\perp}\right|. \tag{5-4}$$

Hence the right-hand side of (5-1) is equal to $\int_Z \chi_U((\boldsymbol{v}_j)_{2\le j\le m})\bigl|\bigwedge_{j=1}^{m}(T_{p_j}Z_j)^{\perp}\bigr|\,\mathrm{d}V$. Note that $\mathrm{d}V$ is the product of all dVol_j. This can easily be recognized as the left-hand side. □

In application, we want to look at the case where each V_j is the zero set of an algebraic variety of codimension c_j. Such a V_j may contain singular points, but they form a subset of measure 0 when we take the $(d-c_j)$-dimensional Hausdorff measure. Hence almost all points on V_j are smooth points and we can apply our Lemma 5.1 to obtain the following theorem.

Theorem 5.2 (intersection estimate). *Assume in $\mathbb{R}^d$ we have m algebraic subvarieties $Z_1, Z_2, \dots, Z_m$ with codimensions $c_1, \dots, c_m$ and degrees $s_1, \dots, s_m$ respectively. If $\sum_{j=1}^m c_j = d$ then for any measurable subset $U \subseteq \mathbb{R}^{d(m-1)} = (\mathbb{R}^d)^{m-1}$, we have*

$$\int_{Z_1}\int_{Z_2}\cdots\int_{Z_m} \chi_U(\overrightarrow{p_1p_2}, \dots, \overrightarrow{p_1p_m})\bigl|(T_{p_1}Z_1)^{\perp}\wedge\cdots\wedge(T_{p_m}Z_m)^{\perp}\bigr|\,\mathrm{dVol}_1\cdots\,\mathrm{dVol}_m \le \mathrm{Vol}(U)\prod_{j=1}^{m} s_j, \tag{5-5}$$

where dVol_j is the $(d-c_j)$-dimensional volume element on the j-th subvariety. Almost all $p_j \in Z_j$ are smooth points and we define $T_{p_j}Z_j$ to be the tangent space of Z_j at p_j.

Proof. Inequality (5-5) follows directly from Lemma 5.1 and Bézout's theorem. □

Theorem 5.2 generalizes the cylinder estimate in [Guth 2010], which was recorded as Lemma 1.14 in our current paper.

6. Proofs of Theorems 1.4 and 1.5

In this section, we prove Theorem 1.5 and deduce Theorem 1.4 as a corollary. As briefly described in the Introduction, we actually prove a generalized theorem about any n varieties.

Basically, our multilinear k_j-variety theorem says that for n algebraic subvarieties of $\mathbb{R}^d$ with their codimension adding up to d, their tubular neighborhoods will provide us with an inequality similar to Theorem 1.5 if we take their "amount of interaction" into account. In particular, if we take each subvariety to be a union of k_j-planes we obtain Theorems 1.5 and 1.4 (see the end of this section).

Theorem 6.1 (multilinear k_j-variety theorem). *Assume $1 \le k_j \le d-1$, $1 \le j \le n$, satisfy $\sum_{j=1}^n k_j = d$. Assume that for $1 \le j \le n$, $H_j \subseteq \mathbb{R}^d$ is part of a k_j-dimensional algebraic subvariety of degree $A(j)$. Let $\mathrm{d}\sigma_j$ denote the k_j-dimensional (Hausdorff) volume measure of H_j. Then with respect to this measure, almost all $y_j \in H_j$ are smooth points. For a smooth point $y_j \in H_j$, let $T_{y_j}H_j$ denote the tangent space of*

H_j at y_j. Then

$$\int_{\mathbb{R}^d}\left(\int_{H_1\times H_2\times\cdots\times H_n}\chi_{\{\mathrm{dist}(y_j,x)\le 1\}}\left|\bigwedge_{j=1}^{n}T_{y_j}H_j\right|\mathrm{d}\sigma_1(y_1)\cdots\,\mathrm{d}\sigma_n(y_n)\right)^{1/(n-1)}\mathrm{d}x\lesssim_d\prod_{j=1}^{n}A(j)^{1/(n-1)}. \tag{6-1}$$

We give an outline of the proof before we actually do it. For the convenience of the statement, we wrote Theorem 6.1 in an integral form. However, because of the truncation $\chi_{\{\mathrm{dist}(y_j,x)\le 1\}}$ it is really of discrete flavor. In other words, around any unit cube, we only take into account the part of the varieties near this cube. By Lemma 2.4, we can find a polynomial with large visibility around each relevant cube. In the lemma, it is possible to assign different weights to different cubes in the above movement. We assign the weights according to the cubes' "popularity" among H_j, as done in [Guth 2010] for the multilinear Kakeya theorem.

We will see it does not matter if we multiply all the weights by the same large positive number simultaneously. As long as the weights are large enough, we can add hyperplanes to the polynomial which do not essentially increase its degree and make its zero set satisfy the assumption of Theorem 3.1 at each relevant cube. Then we can invoke Theorem 3.1 for the resulting zero set $Z(P)$ at all relevant cubes to show that d copies of $Z(P)$ have enough interaction there. Now around each relevant cube we are ready to assign some copies of $Z(P)$ to each variety H_j and use Lemma 4.1 to show that those "have enough interaction". On the other hand, we can use Theorem 5.2 to bound the amount of interaction from above. Hence we obtain a nontrivial inequality. All quantities there work out as they supposed to and we obtain our theorem.

Proof of Theorem 6.1. We only need to prove the case where each H_j is compact and take a limiting argument to complete the proof. Fix a large constant N in terms of d; for example, $N=100e^d$ should be more than sufficient.

Consider the standard lattice of unit cubes in $\mathbb{R}^d$. For each cube Q_ν in the lattice, let O_ν be its center. Let

$$G(Q_\nu)=\int_{H_1\times H_2\times\cdots\times H_n}\prod_{j=1}^{n}\chi_{\{\mathrm{dist}(y_j,O_\nu)\le N\}}\left|\bigwedge_{j=1}^{n}T_{y_j}H_j\right|\mathrm{d}\sigma_1(y_1)\cdots\,\mathrm{d}\sigma_n(y_n). \tag{6-2}$$

Obviously

$$G(Q_\nu)\ge\int_{H_1\times H_2\times\cdots\times H_n}\prod_{j=1}^{n}\chi_{\{\mathrm{dist}(y_j,x)\le 1\}}\left|\bigwedge_{j=1}^{n}T_{y_j}H_j\right|\mathrm{d}\sigma_1(y_1)\cdots\,\mathrm{d}\sigma_n(y_n) \tag{6-3}$$

for any $x\in Q_\nu$. Hence it suffices to prove that under assumptions of Theorem 6.1, we have

$$\sum_\nu G(Q_\nu)^{1/(n-1)}\lesssim_d\prod_{j=1}^{m}A(j)^{1/(n-1)}. \tag{6-4}$$

We only have finitely many relevant cubes Q_ν such that $G(Q_\nu)\ne 0$. Hence we can choose a huge cube of side length S containing all of the relevant cubes. By Guth's lemma, Lemma 2.4, we can find a polynomial P of degree $\lesssim_d S$ such that for each cube Q_ν,

$$\overline{\mathrm{Vis}}[Z(P)\cap Q_\nu] \geq S^d G(Q_\nu)^{1/(n-1)}\biggl(\sum_\nu G(Q_\nu)^{1/(n-1)}\biggr)^{-1}. \tag{6-5}$$

Adding $\lesssim_d S$ hyperplanes to P (in other words multiplying P by linear equations of those hyperplanes) if necessary, we may assume that for all Q_ν where $G(Q_\nu)>0$ we have $\overline{V}_{Z(P)\cap Q_\nu}(\boldsymbol{v})\geq|\boldsymbol{v}|$. Hence we are in a good position to use the wedge product estimate (3-2).

Before we move on let us remark on a technical issue. If we do have to add hyperplanes at this point, we need to modify our Definition 2.3 a little bit: Assume all the hyperplanes we added form a zero set of a polynomial P_0. We call the original polynomial in Guth's lemma P_{old} and our P is actually $P_{\mathrm{old}}P_0$. Then when we are talking about the mollified directed volume, mollified visibility, etc. around P, we want to look at all $P'P_0$, where $P'\in B(P_{\mathrm{old}},\varepsilon)$ instead of all $P'\in B(P,\varepsilon)$. For example, the definition (2-3) should now be modified to

$$\overline{V}_{Z(P)\cap U}(\boldsymbol{v}) = \frac{1}{|B(P_{\mathrm{old}},\varepsilon)|}\int_{B(P_{\mathrm{old}},\varepsilon)} V_{Z(P'P_0)\cap U}(\boldsymbol{v})\,\mathrm{d}P'. \tag{6-6}$$

We also note that an alternative strategy to "adding hyperplanes" is given in [Carbery and Valdimarsson 2013] (see Lemmas 3 and 6 and the argument on page 1654 there). It is a very detailed and clear account.

For the rest of the section for simplicity of the notation, we deal with the case where no hyperplanes are added. For the general case the proof is identical except for proper correction of notation.

For any $y_1\in H_1\cap B(O_\nu,N),\dots,y_n\in H_n\cap B(O_\nu,N)$, $P_1,\dots,P_d\in B(P,\varepsilon)$(see Section 2), $p_1\in Z(P_1)\cap B(O_\nu,N),\dots,p_d\in Z(P_d)\cap B(O_\nu,N)$, by Lemma 4.1, we can find some $i_{j,h}$ for $1\leq j\leq n$, $1\leq h\leq k_j$ such that

$$\prod_{j=1}^n\bigl|(T_{y_j}H_j)^\perp\wedge(T_{p_{i_{j,1}}}Z(P_{i_{j,1}}))^\perp\wedge\cdots\wedge(T_{p_{i_{j,k_j}}}Z(P_{i_{j,k_j}}))^\perp\bigr| \gtrsim_d \biggl|\bigwedge_{j=1}^n T_{y_j}H_j\biggr|\cdot\biggl|\bigwedge_{i=1}^d(T_{p_i}Z(P_i))^\perp\biggr| \tag{6-7}$$

and all $i_{j,h}$ are distinct and form exactly the set $\{1,2,\dots,d\}$.

Integrating over $(H_1\cap B(O_\nu,N))\times\cdots\times(H_n\cap B(O_\nu,N))$, we obtain

$$\begin{aligned} G(Q_\nu)\cdot&\biggl|\bigwedge_{i=1}^d(T_{p_i}Z(P_i))^\perp\biggr| \\ &\lesssim_d \sum_{(i_{j,h})}\int_{H_1\cap B(O_\nu,N)}\cdots\int_{H_n\cap B(O_\nu,N)}\prod_{j=1}^n\bigl|(T_{y_j}H_j)^\perp\wedge(T_{p_{i_{j,1}}}Z(P_{i_{j,1}}))^\perp\wedge\cdots\wedge(T_{p_{i_{j,k_j}}}Z(P_{i_{j,k_j}}))^\perp\bigr| \\ &\qquad\qquad\mathrm{d}\sigma_1(y_1)\cdots\mathrm{d}\sigma_n(y_n). \end{aligned} \tag{6-8}$$

Here we sum over all possible choices of $\{i_{j,h}:1\leq j\leq n,\ 1\leq h\leq k_j\}$ such that all $i_{j,h}$ are distinct and form exactly the set $\{1,2,\dots,d\}$.

Integrate (6-8) over $P_1,\dots,P_d\in B(P,\varepsilon)$ and $p_i\in Z(P_i)\cap B(O_\nu,N)$ (we abuse the notation a bit and write $\mathrm{d}p=\mathrm{d}\sigma(p)$ where $\mathrm{d}\sigma$ is the $(d-1)$-dimensional Hausdorff volume measure on $Z(P)$). Taking Definition 2.3 into account, we use wedge product estimate (Theorem 3.1) and (6-5), (6-8) and deduce

$$\sum_{(i_{j,h})} \frac{1}{|B(P,\varepsilon)|^d} \int \cdots \int_{B(P,\varepsilon)^d} \int_{H_1 \cap B(O_\nu,N)} \cdots \int_{H_n \cap B(O_\nu,N)} \int_{Z(P_1)\cap B(O_\nu,N)} \cdots \int_{Z(P_d)\cap B(O_\nu,N)} \prod_{j=1}^{n} \left|(T_{y_j} H_j)^\perp \wedge (T_{p_{i_{j,1}}} Z(P_{i_{j,1}}))^\perp \wedge \cdots \wedge (T_{p_{i_{j,k_j}}} Z(P_{i_{j,k_j}}))^\perp\right| \,\mathrm{d}p_1 \cdots \mathrm{d}p_d \,\mathrm{d}\sigma_1(y_1) \cdots \mathrm{d}\sigma_n(y_n) \,\mathrm{d}P_1 \cdots \mathrm{d}P_d$$

$$\gtrsim_d G(Q_\nu) \cdot \overline{\mathrm{Vis}}[Z(P) \cap Q_\nu]$$

$$\gtrsim_d S^d G(Q_\nu)^{n/(n-1)} \Big(\sum_\nu G(Q_\nu)^{1/(n-1)}\Big)^{-1}. \tag{6-9}$$

Rewrite (6-9) as

$$\sum_{(i_{j,h})} \prod_{j=1}^{n} \Big(\frac{1}{|B(P,\varepsilon)|^{k_j}} \int \cdots \int_{B(P,\varepsilon)^{k_j}} \frac{1}{S^{k_j} \cdot A(j)} \int_{H_j \cap B(O_\nu,N)} \int_{Z(P_{i_{j,1}})\cap B(O_\nu,N)} \cdots \int_{Z(P_{i_{j,k_j}})\cap B(O_\nu,N)} \left|(T_{y_j} H_j)^\perp \wedge (T_{p_{i_{j,1}}} Z(P_{i_{j,1}}))^\perp \wedge \cdots \wedge (T_{p_{i_{j,k_j}}} Z(P_{i_{j,k_j}}))^\perp\right| \,\mathrm{d}p_{i_{j,1}} \cdots \mathrm{d}p_{i_{j,k_j}} \,\mathrm{d}\sigma_j(y_j) \,\mathrm{d}P_{i_{j,1}} \cdots \mathrm{d}P_{i_{j,k_j}} \Big)$$

$$\gtrsim_d \frac{1}{\prod_{j=1}^n A(j)} G(Q_\nu)^{n/(n-1)} \Big(\sum_\nu G(Q_\nu)^{1/(n-1)}\Big)^{-1}. \tag{6-10}$$

By the arithmetic-geometric mean inequality we deduce

$$\frac{1}{\big(\prod_{j=1}^n A(j)\big)^{1/n}} G(Q_\nu)^{1/(n-1)} \Big(\sum_\nu G(Q_\nu)^{1/(n-1)}\Big)^{-1/n}$$

$$\lesssim_d \sum_{(i_{j,h})} \sum_{j=1}^{n} \frac{1}{|B(P,\varepsilon)|^{k_j}} \int \cdots \int_{B(P,\varepsilon)^{k_j}} \frac{1}{S^{k_j} \cdot A(j)} \int_{H_j \cap B(O_\nu,N)} \int_{Z(P_{i_{j,1}})\cap B(O_\nu,N)} \cdots \int_{Z(P_{i_{j,k_j}})\cap B(O_\nu,N)} \left|(T_{y_j} H_j)^\perp \wedge (T_{p_{i_{j,1}}} Z(P_{i_{j,1}}))^\perp \wedge \cdots \wedge (T_{p_{i_{j,k_j}}} Z(P_{i_{j,k_j}}))^\perp\right| \,\mathrm{d}p_{i_{j,1}} \cdots \mathrm{d}p_{i_{j,k_j}} \,\mathrm{d}\sigma_j(y_j) \,\mathrm{d}P_{i_{j,1}} \cdots \mathrm{d}P_{i_{j,k_j}} \tag{6-11}$$

Sum (6-11) over ν, and then invoke the intersection estimate Theorem 5.2 with $U = \{(u_i)_{1 \le i \le k_j+1} : u_i \in \mathbb{R}^d, \operatorname{dist}(u_i, u_{i'}) < N^2\}$ (it suffices to choose U large enough). Note that $\deg P_j = S$ and $\deg H_j = A(j)$, we have

$$\frac{1}{\big(\prod_{j=1}^n A(j)\big)^{1/n}} \Big(\sum_\nu G(Q_\nu)^{1/(n-1)}\Big) \Big(\sum_\nu G(Q_\nu)^{1/(n-1)}\Big)^{-1/n}$$

$$\lesssim_d \sum_{(i_{j,h})} \sum_{j=1}^{n} \frac{1}{|B(P,\varepsilon)|^{k_j}} \int \cdots \int_{B(P,\varepsilon)^{k_j}} \frac{1}{S^{k_j} \cdot A(j)} \int_{H_j} \int_{Z(P_{i_{j,1}})} \cdots \int_{Z(P_{i_{j,k_j}})} \chi_U(y_j, p_{i_{j,1}}, \ldots, p_{i_{j,k_j}}) \cdot \left|(T_{y_j} H_j)^\perp \wedge (T_{p_{i_{j,1}}} Z(P_{i_{j,1}}))^\perp \wedge \cdots \wedge (T_{p_{i_{j,k_j}}} Z(P_{i_{j,k_j}}))^\perp\right| \,\mathrm{d}p_{i_{j,1}} \cdots \mathrm{d}p_{i_{j,k_j}} \,\mathrm{d}\sigma_j(y_j) \,\mathrm{d}P_{i_{j,1}} \cdots \mathrm{d}P_{i_{j,k_j}}$$

$$\lesssim_d \sum_{(i_{j,h})} \sum_{j=1}^{n} \frac{1}{|B(P,\varepsilon)|^{k_j}} \int \cdots \int_{B(P,\varepsilon)^{k_j}} 1 \,\mathrm{d}P_{i_{j,1}} \cdots \mathrm{d}P_{i_{j,k_j}} \lesssim_d 1 \tag{6-12}$$

and (6-4) holds. □

Theorems 1.5 and 1.4 follow easily from Theorem 6.1. To prove Theorem 1.5 it suffices to prove the case where all ρ_{j,a_j} are rational numbers. Then without loss of generality we may assume further that they are integers. By considering multiple copies of the U_{j,a_j}, we can further assume they are all 1. Then one just takes the j-th variety to be the union of the cores of the j-th family of slabs and apply Theorem 6.1 (after a scaling). Theorem 1.4 is a direct corollary of Theorem 1.5.

7. An analogue of Lemma 4.1

In the rest of this paper, we prove Theorem 1.11. In this section we prove a lemma (Lemma 7.5) analogous to Lemma 4.1, which will be used in the proof the same way as Lemma 4.1 was used in the proof of Theorem 1.4. This lemma is weaker in appearance than Lemma 4.1, but it turns out that it serves our purpose.

Definition 7.1. In $\mathbb{R}^d$, given a convex body Γ centered at the origin, define its *dual body* Γ^* to be $\{\boldsymbol{v} \in \mathbb{R}^d : |(\boldsymbol{v}, \boldsymbol{x})| \leq 1 \text{ for all } x \in \Gamma\}$, where $(\cdot\,,\cdot)$ is the Euclidean inner product on $\mathbb{R}^d$.

It is trivial by definition that if two convex bodies Γ_1 and Γ_2 satisfy $\Gamma_1 \subseteq \Gamma_2$ then $\Gamma_1^* \supseteq \Gamma_2^*$.

By John's ellipsoid theorem, we need to mainly consider ellipsoids as examples of convex bodies. Next we develop several lemmas concerning ellipsoids. From now on, when we talk about an ellipsoid in Euclidean space, we always assume the ellipsoid has the same dimension as the background space.

Lemma 7.2. *If the Γ in Definition 7.1 is a (closed) ellipsoid centered at O (the origin), then Γ^* is also an ellipsoid centered at O. We call Γ^* the* dual ellipsoid *of Γ. Choose a set of principal axes of Γ (the wording is because the choices might not be unique); then they are also a set of principal axes of Γ^*. Moreover, the lengths of the corresponding principal axes of Γ and Γ^*, when divided by 2, are reciprocal to each other. Hence $(\Gamma^*)^* = \Gamma$ and $\mathrm{Vol}(\Gamma) \cdot \mathrm{Vol}(\Gamma^*) = C_d > 0$ is a constant depending only on d.*

Proof. Trivially the dual body of the unit ball is again the unit ball. Assume Γ_0 has a dual body Γ_0^*. Then for any positive definite linear transform A, we have by definition

$$\begin{aligned}(A\Gamma_0)^* &= \{\boldsymbol{v} \in \mathbb{R}^d : |(\boldsymbol{v}, A\boldsymbol{x})| \leq 1 \text{ for all } x \in \Gamma_0\} \\ &= \{\boldsymbol{v} \in \mathbb{R}^d : |(A^*\boldsymbol{v}, \boldsymbol{x})| \leq 1 \text{ for all } x \in \Gamma_0\} \\ &= (A^*)^{-1}\Gamma_0^* = A^{-1}\Gamma_0^*. \end{aligned} \tag{7-1}$$

Now we can use a positive definite linear transformation A to transform the closed unit ball to our Γ; by the computation above, Γ^* is A^{-1} acting on the unit ball, so it is an ellipsoid. Also the principal axes of Γ correspond to an orthonormal basis that diagonalizes A. This basis also diagonalizes A^{-1}. Hence the principal axes of Γ are also principal axes of Γ^*. The rest of the lemma is obvious. □

Lemma 7.3. *Suppose we have a subspace $V \subseteq \mathbb{R}^d$ and $\Gamma \in \mathbb{R}^d$ is an ellipsoid centered at O. Let $\pi_V(\cdot)$ be the orthogonal projection onto V. Then $\pi_V(\Gamma^*)$ and $\Gamma \cap V$ are dual to each other (in V with respect to the induced inner product). Note these two are both ellipsoids.*

Proof. If V has dimension 1, then the lemma is true by definition of the dual body (note by Lemma 7.2, the two ellipsoids are dual to each other).

For general V, by the last paragraph for any $V' \subseteq V$ of dimension 1 we have $\pi_{V'}(\pi_V(\Gamma^*))$ and $(\Gamma \cap V) \cap V'$ are dual to each other. But given the ellipsoid $\Gamma_V = \Gamma \cap V \subseteq V$, apparently there is only one possible set $Y \subseteq V$ such that for any $V' \subseteq V$ of dimension 1, $\pi_{V'}(\Gamma_V)$ and $Y \cap V'$ are dual to each other (since $Y \cap V'$ is determined by Γ_V via this property). Now by last paragraph again, the dual of Γ_V in V is this unique Y. Hence $\pi_V(\Gamma^*)$ has to be this dual. $\square$

Lemma 7.4. *For any subspace $V \subseteq \mathbb{R}^d$ of dimension d', we define π_V to be the orthogonal projection onto V as usual. Then for any* (*closed*) *ellipsoid $\Gamma \subseteq \mathbb{R}^d$, we have*

$$|\pi_V(\Gamma)|\,|\Gamma \cap V^{\perp}| = C_{d,d'}|\Gamma|, \tag{7-2}$$

where $C_{d,d'} > 0$ only depends on d and d'. Here we use the corresponding standard Lebesgue measures on V, $V^{\perp}$ and $\mathbb{R}^d$, respectively.

Proof. It is well known that in $\mathbb{R}^d$, an ellipsoid defined by $\{\boldsymbol{x} : (\boldsymbol{x}, A\boldsymbol{x}) \le 1\}$ has volume $C_d/(\det A)^{1/2}$, where A is a positive definite linear transform and $(\cdot\,,\cdot)$ is the Euclidean inner product. Assume $\Gamma = \{\boldsymbol{x} : \|T\boldsymbol{x}\|^2 \le 1\}$, where T is a nondegenerate linear transform. Since we can multiply T by any orthogonal transform on the left, we may assume $TV^{\perp} = V^{\perp}$. Then by last paragraph,

$$|\Gamma| = \frac{C_d}{|\det T|}, \tag{7-3}$$

$$|\Gamma \cap V^{\perp}| = \frac{C_{d,d'}}{|\det T|_{V^{\perp}}|}. \tag{7-4}$$

Meanwhile, $\boldsymbol{x} \in V$ belongs to $\pi_V(\Gamma)$ if and only if $\inf_{\boldsymbol{v}\in V^{\perp}} \|T(\boldsymbol{x}+\boldsymbol{v})\| \le 1$. By the method of least squares, $\inf_{\boldsymbol{v}\in V^{\perp}} \|T(\boldsymbol{x}+\boldsymbol{v})\| = \|\pi_{(TV^{\perp})^{\perp}}(T\boldsymbol{x})\| = \|\pi_V(T\boldsymbol{x})\|$. Hence

$$|\pi_V(\Gamma)| = \frac{C_{d'}}{|\det(\pi_V \circ T|_V)|}. \tag{7-5}$$

Now notice $\pi V^{\perp} = V^{\perp}$. Hence when written in matrix form it is easy to verify $\det T|_{V^{\perp}} \cdot \det(\pi_V \circ T|_V) = \det T$. This together with (7-3)–(7-5) implies (7-2). $\square$

Now we are ready to develop an analogue of Lemma 4.1. We recall that in Section 3 we defined the total absolute inner product $V_{X,f}(\boldsymbol{v})$, the fading zone $F(X, f)$, visibility $\mathrm{Vis}[X, f]$, and chose an elliptical approximation $\mathrm{Ell}(F(X, f))$ for any measurable vector-valued function $f : X \to \mathbb{R}^d$.

Lemma 7.5. *Fix positive integers d and $1 \le k_1, \dots, k_n < d$. Let $\mathbb{R}^d$ be the standard Euclidean space.*

Assume a Brascamp–Lieb datum $(\boldsymbol{B}, \boldsymbol{p})$ such that all B_j are orthogonal projections from $\mathbb{R}^d$ to a subspace and $\dim\ker B_j = k_j$. *Assume $E_j = B_j(\mathbb{R}^d) = (\ker B_j)^{\perp}$. Assume we have the scaling condition $\sum_{j=1}^n p_j \dim E_j = d$.*

For any measurable vector valued function $f : X \to \mathbb{R}^d$ on some measure space satisfying $V_{X,f}(\boldsymbol{v}) \ge 1$ for all $\boldsymbol{v} \in \mathbb{R}^d$, we have

$$\prod_{j=1}^n \left(\int_{X^{k_j}} |E_j \wedge f(x_1) \wedge \cdots \wedge f(x_{k_j})| \,\mathrm{d}x_1 \cdots \mathrm{d}x_{k_j} \right)^{p_j} \gtrsim_{d,\boldsymbol{p}} (\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}))^{-1} (\mathrm{Vis}[X, f])^{\sum_{j=1}^n p_j - 1}. \tag{7-6}$$

Proof. Similar to the proof of Theorem 3.1, we define $\pi_{\ker B_j}$ to be the orthogonal projection onto $\ker B_j$ as before and $f_{\ker B_j} = \pi_{\ker B_j} \circ f$. Then

$$\int_{X^{k_j}} \left| E_j \wedge f(x_1) \wedge \cdots \wedge f(x_{k_j}) \right| \mathrm{d}x_1 \cdots \mathrm{d}x_{k_j} = \int_{X^{k_j}} \left| f_{\ker B_j}(x_1) \wedge \cdots \wedge f_{\ker B_j}(x_{k_j}) \right| \mathrm{d}x_1 \cdots \mathrm{d}x_{k_j}. \tag{7-7}$$

Similar to the proof of Theorem 3.1, we know $F(X, f_{\ker B_j}) = F(X, f) \cap \ker B_j$. Hence we can take $\mathrm{Ell}(F(X, f_{\ker B_j}))$ to be $\mathrm{Ell}(F(X, f)) \cap \ker B_j$. By (7-7), Theorem 3.1, Lemma 7.2 and Lemma 7.3,

$$\begin{aligned} \int_{X^{k_j}} \left| E_j \wedge f(x_1) \wedge \cdots \wedge f(x_{k_j}) \right| \mathrm{d}x_1 \cdots \mathrm{d}x_{k_j} &= \int_{X^{k_j}} \left| f_{\ker B_j}(x_1) \wedge \cdots \wedge f_{\ker B_j}(x_{k_j}) \right| \mathrm{d}x_1 \cdots \mathrm{d}x_{k_j} \\ &\gtrsim_d \frac{1}{|\mathrm{Ell}(F(X, f)) \cap \ker B_j|} \\ &\gtrsim_d |(\mathrm{Ell}(F(X, f)) \cap \ker B_j)^*| \\ &= |\pi_{\ker B_j}(\mathrm{Ell}(F(X, f))^*)|. \end{aligned} \tag{7-8}$$

Hence it suffices to prove

$$\prod_{j=1}^{n} \left| \pi_{\ker B_j}(\mathrm{Ell}(F(X, f))^*) \right|^{p_j} \gtrsim_{d,\boldsymbol{p}} (\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}))^{-1} (\mathrm{Vis}[X, f])^{\sum_{j=1}^{n} p_j - 1}. \tag{7-9}$$

At this point we invoke the definition of $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p})$. For any ellipsoid Γ, we choose $f_j = \chi_{\pi_{E_j}(\Gamma^*)}$ in (1-9). Then by definition $\prod_{j=1}^{n} (f_j \circ B_j)^{p_j} \geq \chi_{\Gamma^*}$. Hence

$$|\Gamma^*| \leq \int_{\mathbb{R}^d} \prod_{j=1}^{n} (f_j \circ B_j)^{p_j} \leq \mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) \prod_{j=1}^{n} \left(\int_{E_j} f_j \right)^{p_j} = \mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) \prod_{j=1}^{n} |\pi_{E_j}(\Gamma^*)|^{p_j}. \tag{7-10}$$

In other words,

$$\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) \cdot |\Gamma| \cdot \prod_{j=1}^{n} |\pi_{E_j}(\Gamma^*)|^{p_j} \gtrsim_d 1. \tag{7-11}$$

By Lemmas 7.2, 7.3 and 7.4, we have

$$|\pi_{E_j}(\Gamma^*)| \sim_{k_j} \frac{1}{|\Gamma \cap E_j|} \sim_{k_j, d} \frac{|\pi_{\ker B_j}(\Gamma)|}{|\Gamma|}. \tag{7-12}$$

Hence

$$\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) \cdot |\Gamma| \cdot \prod_{j=1}^{n} \left(\frac{|\pi_{\ker B_j}(\Gamma)|}{|\Gamma|} \right)^{p_j} \gtrsim_{d,\boldsymbol{p}} 1. \tag{7-13}$$

Take $\Gamma = \mathrm{Ell}(F(X, f))^*$. By Lemma 7.2 again, we have $|\Gamma| = |\mathrm{Ell}(F(X, f))^*| \sim_d 1/|\mathrm{Ell}(F(X, f))| = \mathrm{Vis}[X, f]$. This fact and (7-13) imply (7-9), which in turn implies (7-6). $\square$

8. Proof of Theorem 1.11

We are ready to prove Theorem 1.11. Just like the proof of Theorem 1.4, we prove a stronger theorem concerning algebraic varieties. This theorem can also be considered as an analogue of Theorem 6.1.

Theorem 8.1 (variety version of Brascamp–Lieb). *Assume we have positive integers $k_1, \dots, k_n \le d$ and rational numbers $p_1, \dots, p_n > 0$. Choose a common denominator τ of all p_j and assume $p_j = \tau_j/\tau$, with $\tau_j \in \mathbb{Z}^+$ satisfying the scaling condition $\sum_j p_j(d-k_j) = d$.*

Assume that for $1 \le j \le n$, $H_j \subseteq \mathbb{R}^d$ is part of a k_j-dimensional algebraic subvariety of degree $A(j)$. Let $\mathrm{d}\sigma_j$ denote the k_j-dimensional (Hausdorff) volume measure of H_j. Then under this measure, almost all $y_j \in H_j$ are smooth points. For a smooth point $y_j \in H_j$, let $T_{y_j}H_j$ denote the tangent space of H_j at y_j.

For $\sum_j \tau_j$ smooth points $\boldsymbol{y} = (y_{1,1}, \dots, y_{1,\tau_1}, y_{2,1}, \dots, y_{2,\tau_2}, \dots, y_{n,\tau_n})$, $y_{j,l} \in H_j$, there exists a unique Brascamp–Lieb datum $(\boldsymbol{B}(\boldsymbol{y}), \boldsymbol{p}(\boldsymbol{y}))$ with $\sum_j \tau_j$ projections B_j all being orthogonal projections within $\mathbb{R}^d$ as the following: Define $(\boldsymbol{B}(\boldsymbol{y}), \boldsymbol{p}(\boldsymbol{y})) = (B_{1,1}, \dots, B_{1,\tau_1}, B_{2,1}, \dots, B_{2,\tau_2}, \dots, B_{n,\tau_n}, 1/\tau, \dots, 1/\tau)$ such that $\ker B_{j,l} = T_{y_{j,l}}H_j$ *and all components of $\boldsymbol{p}$ are $1/\tau$. Then*

$$\int_{\mathbb{R}^d}\left(\int_{H_1^{\tau_1}\times\cdots\times H_n^{\tau_n}} \prod_{j=1}^{n}\prod_{k=1}^{\tau_j} \chi_{\{\mathrm{dist}(y_{j,k},x)\le 1\}}\, \mathrm{BL}(\boldsymbol{B}(\boldsymbol{y}), \boldsymbol{p}(\boldsymbol{y}))^{-\tau}\, \mathrm{d}\sigma_1(y_{1,1})\cdots \mathrm{d}\sigma_1(y_{1,\tau_1})\cdots \mathrm{d}\sigma_n(y_{n,\tau_n})\right)^{1/\tau} \mathrm{d}x \lesssim_{d,\tau_1,\dots,\tau_n,\tau} \prod_{j=1}^{n} A(j)^{p_j}. \tag{8-1}$$

Let us explain the motivation of Theorem 8.1 before proving it. If we want to naturally generalize Theorem 6.1 to the Brascamp–Lieb setting, first of all we have to come up with a reasonable integral like the left-hand side of (6-1) to put on the left-hand side. However the fact that in (6-1) all $p_j = 1/(n-1)$ no longer holds in our situation. In fact, the p_j might even all be irrational numbers. A natural way would be approximating (p_j) by rational tuples. This works (see below) but eventually we need all the p_j to be the same to get a quantity analogous to left-hand of (6-1).

Another remark before we move on. It's good to keep in mind that we may assume $\tau_1 = \cdots = \tau_n = 1$ in this theorem without loss of generality. This is trivial to see. But we keep the theorem in its current form here so it is more straightforward to apply.

Proof that Theorem 8.1 implies Theorem 1.11. Note that the conditions (1-11) and (1-12) only have rational coefficients. Hence it is possible to choose $(n+1)$ different rational $\boldsymbol{p}'$ close enough to $\boldsymbol{p}$ such that the conditions (1-11) and (1-12) are satisfied (that is, $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}') < +\infty$), and that $\boldsymbol{p}$ lies in the convex hull of those $\boldsymbol{p}'$. By interpolation we only need to prove the case when $\boldsymbol{p}$ is a rational vector.

Next in order to apply the result of Theorem 8.1 to prove Theorem 1.11, we claim that if a Brascamp–Lieb datum $(\boldsymbol{B}, \boldsymbol{p})$ is such that $p_j = \tau_j/\tau$, where τ all τ_j are positive integers, then $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) = \mathrm{BL}(\boldsymbol{B}', \boldsymbol{p}')$, where $\boldsymbol{B}' = (B_1, \dots, B_1, \dots, B_n, \dots, B_n)$ contains τ_j copies of B_j, and $\boldsymbol{p}' = (1/\tau, \dots, 1/\tau)$. In fact, looking at the definition (1-9) of $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p})$, we have

$$\mathrm{BL}(\boldsymbol{B}', \boldsymbol{p}') = \sup_{\{f_{j,l}\}} \frac{\int_{\mathbb{R}^d} \prod_{j=1}^{n}\prod_{l=1}^{\tau_j} (f_{j,l}\circ B_j)^{1/\tau}}{\prod_{j=1}^{n}\prod_{l=1}^{\tau_j} \left(\int_{H_j} f_{j,l}\right)^{1/\tau}}. \tag{8-2}$$

Since we can always take $f_{j,l} = f_j$ for all l, we deduce $\mathrm{BL}(\boldsymbol{B}', \boldsymbol{p}') \ge \mathrm{BL}(\boldsymbol{B}, \boldsymbol{p})$. On the other hand, in the definition of $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p})$ we can take $f_j = f_{j,l_j}$ for every possible tuple $(l_1, \dots, l_n)$ satisfying $1 \le l_j \le \tau_j$

to deduce

$$\int_{\mathbb{R}^d} \prod_{j=1}^{n} (f_{j,l_j} \circ B_j)^{\tau_j/\tau} \le \mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) \prod_{j=1}^{n} \biggl(\int_{H_j} f_{j,l_j} \biggr)^{\tau_j/\tau}. \tag{8-3}$$

Then we let (l_j) run through all possible tuples and invoke Hölder to conclude that

$$\int_{\mathbb{R}^d} \prod_{j=1}^{n} \prod_{l=1}^{\tau_j} (f_{j,l} \circ B_j)^{1/\tau} \le \mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) \prod_{j=1}^{n} \prod_{l=1}^{\tau_j} \biggl(\int_{H_j} f_{j,l} \biggr)^{1/\tau}. \tag{8-4}$$

Hence $\mathrm{BL}(\boldsymbol{B}', \boldsymbol{p}') \le \mathrm{BL}(\boldsymbol{B}, \boldsymbol{p})$. Therefore $\mathrm{BL}(\boldsymbol{B}', \boldsymbol{p}') = \mathrm{BL}(\boldsymbol{B}, \boldsymbol{p})$.

By Theorem 1.1 in [Bennett et al. 2015], BL is a locally bounded function. It is then not hard to derive Theorem 1.11 from Theorem 8.1 when $\boldsymbol{p}'$ is a fixed rational number. □

Proof of Theorem 8.1. It's plain that we may assume $\tau_1 = \cdots = \tau_n = 1$. For short we write $B_j = B_{j,1}$ and $y_j = y_{j,1}$.

The proof will be almost identical to that of Theorem 6.1. In the current proof, we omit some details for familiar manipulations in that proof to reduce redundancy and refer the reader to it.

Take the N and set up the unit cube lattice in $\mathbb{R}^d$ as in the proof of Theorem 6.1. Again let O_ν be the center of any cube Q_ν in the lattice. This time we define

$$G(Q_\nu) = \int_{H_1 \times \cdots \times H_n} \prod_{j=1}^{n} \chi_{\mathrm{dist}(y_j, O_\nu) \le N} \, \mathrm{BL}(\boldsymbol{B}(\boldsymbol{y}), \boldsymbol{p}(\boldsymbol{y}))^{-\tau} \, \mathrm{d}\sigma_1(y_1) \cdots \mathrm{d}\sigma_n(y_n). \tag{8-5}$$

Similar to the proof of Theorem 6.1, it suffices to show

$$\sum_{\nu} G(Q_\nu)^{1/\tau} \lesssim_{d,n} \prod_{j=1}^{n} A(j)^{1/\tau}. \tag{8-6}$$

Again we may assume for the moment that each H_j is compact and use a limiting argument. Then we can again choose a large cube of side length S that contains all the relevant cubes. Finally we can find a polynomial P of degree $\lesssim_d S$ such that for each Q_ν,

$$\overline{\mathrm{Vis}}[Z(P) \cap Q_\nu] \ge S^d G(Q_\nu)^{1/\tau} \biggl(\sum_{\nu} G(Q_\nu)^{1/\tau} \biggr)^{-1}. \tag{8-7}$$

As before we have to make the technical comment that after adding some hyperplanes and changing the definition of $\overline{\mathrm{Vis}}$ accordingly, we may assume for all Q_ν with $G(Q_\nu) > 0$ we have

$$\overline{V}_{Z(P) \cap Q_\nu}(\boldsymbol{v}) \ge |\boldsymbol{v}|$$

(so that we are allowed to apply (7-6)). We only deal with the case where no hyperplanes are added so that the notation is simpler.

Similar to what we did in the proof of Theorem 6.1, we choose $B_j = T_{y_j} H_j$, all $p_j = 1/\tau$ and integrate (7-6) over $y_j \in H_j \cap B(O_\nu, N)$. Then we choose the measure space X in (7-6) to be

$$\{p \in Z(P') \cap B(O_\nu, N) : P' \in B(P, \varepsilon)\}$$

(the measure is just the surface measure on each $Z(P')$ joint with the standard measure on $B(P,\varepsilon)$, which is $\mathrm{d}p\,\mathrm{d}P'$, where $P' \in B(P,\varepsilon)$ and $p \in Z(P') \cap B(O_\nu, N)$) and deduce

$$\begin{aligned}
&\frac{1}{|B(P,\varepsilon)|^{(n-\tau)d}} \int \cdots \int_{B(P,\varepsilon)^{(n-\tau)d}} \int_{H_1 \cap B(O_\nu,N)} \cdots \int_{H_n \cap B(O_\nu,N)} \int_{Z(P_1)\cap B(O_\nu,N)} \cdots \int_{Z(P_{(n-\tau)d})\cap B(O_\nu,N)} \\
&\qquad \prod_{j=1}^{n} \bigl|(T_{y_j} H_j)^\perp \wedge (T_{p_{k_1+\cdots+k_{j-1}+1}} Z(P_{k_1+\cdots+k_{j-1}+1}))^\perp \wedge \cdots \wedge (T_{p_{k_1+\cdots+k_j}} Z(P_{k_1+\cdots+k_j}))^\perp\bigr| \\
&\qquad\qquad \mathrm{d}p_1 \cdots \mathrm{d}p_{(n-\tau)d}\, \mathrm{d}\sigma_1(y_1) \cdots \mathrm{d}\sigma_n(y_n)\, \mathrm{d}P_1 \cdots \mathrm{d}P_{(n-\tau)d} \\
&\gtrsim_{d,n} G(Q_\nu) \cdot \overline{\mathrm{Vis}}[Z(P) \cap Q_\nu]^{n-\tau} \\
&\gtrsim_{d,n} S^{(n-\tau)d} G(Q_\nu)^{n/\tau} \Bigl(\sum_\nu G(Q_\nu)^{1/\tau}\Bigr)^{-(n-\tau)}.
\end{aligned} \tag{8-8}$$

As before we rewrite it as

$$\begin{aligned}
&\prod_{j=1}^{n} \Biggl(\frac{1}{|B(P,\varepsilon)|^{k_j}} \int \cdots \int_{B(P,\varepsilon)^{k_j}} \frac{1}{S^{k_j} \cdot A(j)} \int_{H_j \cap B(O_\nu,N)} \int_{Z(P_1)\cap B(O_\nu,N)} \cdots \int_{Z(P_{k_j})\cap B(O_\nu,N)} \\
&\qquad \bigl|(T_{y_j} H_j)^\perp \wedge (T_{p_1} Z(P_1))^\perp \wedge \cdots \wedge (T_{p_{k_j}} Z(P_{k_j}))^\perp\bigr| \,\mathrm{d}p_1 \cdots \mathrm{d}p_{k_j}\, \mathrm{d}\sigma_j(y_j)\, \mathrm{d}P_1 \cdots \mathrm{d}P_{k_j} \Biggr) \\
&\gtrsim_{d,n} \frac{1}{\prod_{j=1}^n A(j)} G(Q_\nu)^{n/\tau} \Bigl(\sum_\nu G(Q_\nu)^{1/\tau}\Bigr)^{-(n-\tau)}.
\end{aligned} \tag{8-9}$$

Here note that since $\sum_{j=1}^n (d-k_j) = \tau d$ by assumption, we have $\sum_{j=1}^n k_j = (n-\tau)d$. We have used this fact in the above inequality chain (8-9).

By the arithmetic-geometric mean inequality we have

$$\begin{aligned}
&\sum_{j=1}^{n} \Biggl(\frac{1}{|B(P,\varepsilon)|^{k_j}} \int \cdots \int_{B(P,\varepsilon)^{k_j}} \frac{1}{S^{k_j} \cdot A(j)} \int_{H_j \cap B(O_\nu,N)} \int_{Z(P_1)\cap B(O_\nu,N)} \cdots \int_{Z(P_{k_j})\cap B(O_\nu,N)} \\
&\qquad \bigl|(T_{y_j} H_j)^\perp \wedge (T_{p_1} Z(P_1))^\perp \wedge \cdots \wedge (T_{p_{k_j}} Z(P_{k_j}))^\perp\bigr| \,\mathrm{d}p_1 \cdots \mathrm{d}p_{k_j}\, \mathrm{d}\sigma_j(y_j)\, \mathrm{d}P_1 \cdots \mathrm{d}P_{k_j} \Biggr) \\
&\gtrsim_{d,n} \frac{1}{\bigl(\prod_{j=1}^n A(j)\bigr)^{1/n}} G(Q_\nu)^{1/\tau} \Bigl(\sum_\nu G(Q_\nu)^{1/\tau}\Bigr)^{-(n-\tau)/n}.
\end{aligned} \tag{8-10}$$

Like we did in the proof of Theorem 6.1, summing over ν and applying the intersection estimate Theorem 5.2 with $U = \{(u_i)_{1\le i\le k_j+1} : u_i \in \mathbb{R}^d, \operatorname{dist}(u_i, u_i') < N^2\}$, we deduce

$$\frac{1}{\bigl(\prod_{j=1}^n A(j)\bigr)^{1/n}} \Bigl(\sum_\nu G(Q_\nu)^{1/\tau}\Bigr) \Bigl(\sum_\nu G(Q_\nu)^{1/\tau}\Bigr)^{-(n-\tau)/n} \lesssim_{d,n} 1, \tag{8-11}$$

which implies (8-6) and concludes the proof. □

Remark 8.2. For the perturbed Brascamp–Lieb theorem itself, Theorem 1.11, it is conceivable that one can directly work with the framework of arguments in [Carbery and Valdimarsson 2013], without applying a rational approximation argument as we did in this section. Nevertheless, we still decided to keep the current approach as we feel that Theorem 8.1 here may be of independent interest, and that rationality seems indispensable for us to state the theorem (and prove it).

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