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REDUCIBILITY OF THE QUANTUM HARMONIC OSCILLATOR IN *d*-DIMENSIONS WITH POLYNOMIAL TIME-DEPENDENT PERTURBATION

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We prove a reducibility result for a quantum harmonic oscillator in arbitrary dimension with arbitrary frequencies perturbed by a linear operator which is a polynomial of degree 2 in $(x_j, -\mathrm{i}\partial_j)$ with coefficients which depend quasiperiodically on time.

1. Introduction and statement of results

The aim of this paper is to present a reducibility result for the time-dependent Schrödinger equation

$$\mathrm{i}\dot\psi = H_\epsilon(\omega t)\psi, \quad x \in \mathbb{R}^d, \tag{1-1}$$

$$H_\epsilon(\omega t) := H_0 + \epsilon W(\omega t, x, -\mathrm{i}\nabla), \tag{1-2}$$

where

$$H_0 := -\Delta + V(x), \qquad V(x) := \sum_{j=1}^{d} \nu_j^2 x_j^2, \quad \nu_j > 0, \tag{1-3}$$

and $W(\theta, x, \xi)$ *is a real polynomial in* (x, ξ) *of degree at most* 2, with coefficients being real analytic functions of $\theta \in \mathbb{T}^n$. Here ω are parameters which are assumed to belong to the set $\mathcal{D} = (0, 2\pi)^n$.

For $\epsilon = 0$ the spectrum of (1-2) is given by

$$\sigma(H_0) = \{\lambda_k\}_{k\in\mathbb{N}^d}, \quad \lambda_k \equiv \lambda_{(k_1,\dots,k_d)} := \sum_{j=1}^{d}(2k_j+1)\nu_j, \tag{1-4}$$

with $k_j \ge 0$ integers. In particular if the frequencies ν_j are nonresonant, then the differences between couples of eigenvalues are dense on the real axis. As a consequence, in the case $\epsilon = 0$ most of the solutions of (1-1) are almost periodic with an infinite number of rationally independent frequencies.

Here we will prove that for any choice of the mechanical frequencies ν_j and for ω belonging to a set of large measure in $\mathcal{D}$, the system (1-1) is reducible: precisely there exists a time-quasiperiodic unitary transformation of $L^2(\mathbb{R}^d)$ which conjugates (1-2) to a time-independent operator. We also deduce boundedness of the Sobolev norms of the solution.

The proof exploits the fact that for polynomial Hamiltonians of degree at most 2, the correspondence between classical and quantum mechanics is exact (i.e., without error term), so that the result can be

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proven by exact quantization of the classical KAM theory, which ensures reducibility of the classical Hamiltonian system

$$h_\epsilon := h_0 + \epsilon W(\omega t, x, \xi), \quad h_0 := \sum_{j=1}^{d} \xi_j^2 + \nu_j^2 x_j^2. \tag{1-5}$$

We will use (in the Appendix) the exact correspondence between classical and quantum dynamics of quadratic Hamiltonians also to prove a complementary result. Precisely we will present a class of examples, following [Graffi and Yajima 2000], in which one generically has growth of Sobolev norms. This happens when the frequencies ω of the external forcing are resonant with some of the ν_j.

We recall that the exact correspondence between classical and quantum dynamics of quadratic Hamiltonians was already exploited in [Hagedorn et al. 1986] to prove stability/instability results for one degree of freedom time-dependent quadratic Hamiltonians.

Notwithstanding the simplicity of the proof, we think that the present result could have some interest, since this is the first example of a reducibility result for a system in which the gaps of the unperturbed spectrum are dense in $\mathbb{R}$. Furthermore it is one of the few cases in which reducibility is obtained for systems in more than one space dimension.

Indeed, most of the results on the reducibility problem for (1-1) have been obtained in the 1-dimensional case, and also the results in higher dimensions obtained up to now deal only with cases in which the spectrum of the unperturbed system has gaps whose size is bounded from below, like in the harmonic oscillator (or in the Schrödinger equation on $\mathbb{T}^d$). On the other hand we restrict here to perturbations, which although unbounded, must belong to the very special class of polynomials in x_j and $-\mathrm{i}\partial_j$. The reason is that for operators in this class, the commutator is the operator whose symbol is the Poisson bracket of the corresponding symbols, without any error term (see Remark 2.2 and Remark 2.4). In order to deal with more general perturbations one needs further ideas and techniques.

Before closing this introduction we recall some previous works on the reducibility problem for (1-1) and more generally for perturbations of the Schrödinger equation with a potential $V(x)$. As we already anticipated, most of the works deal with the 1-dimensional case. The first one is [Combescure 1987], in which the pure point nature of the Floquet operator is obtained in the case of a smoothing perturbation of the harmonic oscillator in dimension 1; see also [Kuksin 1993]. The techniques of this paper were extended in [Duclos and Šťovíček 1996; Duclos et al. 2002] in order to deal with potentials growing superquadratically (still in dimension 1) but with perturbations which were only required to be bounded.

A slightly different approach originates from the so-called KAM theory for PDEs [Kuksin 1987; Wayne 1990]. In particular the methods developed in that context in order to deal with unbounded perturbations, see [Kuksin 1997; 1998], were exploited in [Bambusi and Graffi 2001] in order to deal with the reducibility problem of (1-1) with superquadratic potential in dimension 1; see [Liu and Yuan 2010] for a further improvement. The case of bounded perturbations of the harmonic oscillator in dimension 1 was treated in [Wang 2008; Grébert and Thomann 2011].

An extension of KAM theory to NLS on $\mathbb{T}^d$ has been obtained in [Eliasson and Kuksin 2010] and its methods have been adapted to deal with the reducibility problem of quasiperiodically forced linear

Schrödinger equation in [Eliasson and Kuksin 2009]. A further reducibility result for equations in more than one space dimension is [Grébert and Paturel 2016], in which bounded perturbations of the completely resonant harmonic oscillator in $\mathbb{R}^d$ were studied. As far as we know, these are the only higher-dimensional linear systems for which reducibility is known.[1]

We remark that all these papers deal with cases where the spectrum of the unperturbed operator is formed by well-separated eigenvalues. In the higher-dimensional cases they are allowed to have high multiplicity localized in clusters. But then the perturbation must have special properties ensuring that the clusters are essentially not destroyed under the KAM iteration.

Finally we recall the works [Bambusi 2017a; 2017b], in which pseudodifferential calculus was used together with KAM theory in order to prove reducibility results for (1-1) (in dimension 1) with unbounded perturbations. The ideas of the present paper are a direct development of the ideas of [Bambusi 2017a; 2017b]. We also recall that the idea of using pseudodifferential calculus together with KAM theory in order to deal with problems involving unbounded perturbations originates from [Plotnikov and Toland 2001; Iooss et al. 2005] and has been developed in order to give a quite general theory in [Baldi et al. 2014; Berti and Montalto 2016; Montalto 2014]; see also [Feola and Procesi 2015].

In order to state our main result, we need some preparations. It is well known that (1-1) is well-posed, see for example [Maspero and Robert 2017], in the scale $\mathcal{H}^s$, $s \in \mathbb{R}$, of the weighted Sobolev spaces defined as follows. For $s \geq 0$ let

$$\mathcal{H}^s := \{\psi \in L^2(\mathbb{R}^d) : H_0^{s/2}\psi \in L^2(\mathbb{R}^d)\},$$

equipped with the natural Hilbert space norm $\|\psi\|_s := \|H_0^{s/2}\psi\|_{L^2(\mathbb{R}^d)}$. For $s < 0$, $\mathcal{H}^s$ is defined by duality. Such spaces are not dependent on ν for $\nu_j > 0$, $1 \leq j \leq d$. We also have $\mathcal{H}^s \equiv \mathrm{Dom}(-\Delta + |x|^2)^{s/2}$.

We will prove the following reducibility theorem:

Theorem 1.1. *Let ψ be a solution of* (1-1). *There exist $\epsilon_* > 0$, $C > 0$ and for all $|\epsilon| < \epsilon_*$ a closed set $\mathcal{E}_\epsilon \subset (0, 2\pi)^n$ with $\mathrm{meas}((0, 2\pi)^n \setminus \mathcal{E}_\epsilon) \leq C\epsilon^{1/9}$ and, for all $\omega \in \mathcal{E}_\epsilon$ there exists a unitary (in L^2) time-quasiperiodic map $U_\omega(\omega t)$ such that, defining φ by $U_\omega(\omega t)\varphi = \psi$, it satisfies the equation*

$$\mathrm{i}\dot{\varphi} = H_\infty \varphi, \tag{1-6}$$

with H_∞ a positive definite time-independent operator which is unitary equivalent to a diagonal operator

$$\sum_{j=1}^{d} \nu_j^\infty (x_j^2 - \partial_{x_j}^2),$$

where $\nu_j^\infty = \nu_j^\infty(\omega)$ are defined for $\omega \in \mathcal{E}_\epsilon$ and fulfill the estimates

$$|\nu_j - \nu_j^\infty| \leq C\epsilon, \quad j = 1, \dots, d.$$

Finally the following properties hold:

(i) *For all $s \geq 0$, for all $\psi \in \mathcal{H}^s$, we have $\theta \mapsto U_\omega(\theta)\psi \in C^0(\mathbb{T}^n; \mathcal{H}^s)$.*

[1] We would like to point out also [Procesi and Procesi 2012; 2015], which at present refer to the resonant *nonlinear* Schrödinger equation; it would be interesting to study if they have some consequences for reducibility theory.

(ii) *For all* $s \geq 0$, *there exists* $C_s > 0$ *such that for all* $\theta \in \mathbb{T}^n$

$$\|\mathbf{1} - U_\omega(\theta)\|_{\mathcal{L}(\mathcal{H}^{s+2};\mathcal{H}^s)} \leq C_s \epsilon. \tag{1-7}$$

(iii) *For all* $s, r \geq 0$, *the map* $\theta \mapsto U_\omega(\theta)$ *is of class* $C^r(\mathbb{T}^n; \mathcal{L}(\mathcal{H}^{s+4r+2}; \mathcal{H}^s))$.

Remark 1.2. In Theorem 1.1, if the frequencies ν_j are resonant, then the change of coordinates U_ω is close to the identity, in the sense of (1-7), but the Hamiltonian H_∞ is not necessary diagonal. However, it is always possible to diagonalize it by means of a metaplectic transformation which is not close to the identity; see Theorem 2.10 and Remark 2.11 below.

Let us denote by $\mathcal{U}_{\epsilon,\omega}(t, \tau)$ the propagator generated by (1-1) such that $\mathcal{U}_{\epsilon,\omega}(\tau, \tau) = \mathbf{1}$ for all $\tau \in \mathbb{R}$. An immediate consequence of Theorem 1.1 is that we have a Floquet decomposition:

$$\mathcal{U}_{\epsilon,\omega}(t, \tau) = U_\omega^*(\omega t)\, \mathrm{e}^{-\mathrm{i}(t-\tau)H_\infty}\, U_\omega(\omega tau). \tag{1-8}$$

Another consequence of (1-8) is that for any $s > 0$ the norm $\|\mathcal{U}_{\epsilon,\omega}(t, 0)\psi_0\|_s$ is bounded uniformly in time:

Corollary 1.3. *Let* $\omega \in \mathcal{E}_\epsilon$ *with* $|\epsilon| < \epsilon_*$. *The following is true*: *for any* $s > 0$ *one has*

$$c_s\|\psi_0\|_s \leq \|\mathcal{U}_{\epsilon,\omega}(t, 0)\psi_0\|_s \leq C_s\|\psi_0\|_s \quad \textit{for all } t \in \mathbb{R}, \textit{ for all } \psi_0 \in \mathcal{H}^s, \tag{1-9}$$

for some $c_s > 0,\ C_s > 0$.

Moreover there exists a constant c'_s *such that if the initial data* ψ_0 *is in* $\mathcal{H}^{s+2}$ *then*

$$\|\psi_0\|_s - \epsilon c'_s\|\psi_0\|_{s+2} \leq \|\mathcal{U}_{\epsilon,\omega}(t, 0)\psi_0\|_s \leq \|\psi_0\|_s + \epsilon c'_s\|\psi_0\|_{s+2} \quad \textit{for all } t \in \mathbb{R}. \tag{1-10}$$

It is interesting to compare estimate (1-9) with the corresponding estimate which can be obtained for more general perturbations $W(t, x, D)$. So denote by $\mathcal{U}(t, \tau)$ the propagator of $H_0 + W(t, x, D)$ with $\mathcal{U}(\tau, \tau) = \mathbf{1}$. Then in [Maspero and Robert 2017] it is proved that if $W(t, x, \xi)$ is a real polynomial in (x, ξ) of degree at most 2, the propagator $\mathcal{U}(t, s)$ exists, belongs to $\mathcal{L}(\mathcal{H}^s)$ for all $s \geq 0$ and fulfills

$$\|\mathcal{U}(t, 0)\psi_0\|_s \leq e^{C_s|t|}\|\psi_0\|_s \quad \text{for all } t \in \mathbb{R}$$

(the estimate is sharp!). If $W(t, x, \xi)$ is a polynomial of degree at most 1, one has

$$\|\mathcal{U}(t, 0)\psi_0\|_s \leq C_s(1 + |t|)^s\, \|\psi_0\|_s \quad \text{for all } t \in \mathbb{R}.$$

Thus estimate (1-9) improves dramatically the upper bounds proved in [Maspero and Robert 2017] when the perturbation is small and depends quasiperiodically in time with "good" frequencies.

As a final remark we recall that growth of Sobolev norms can indeed happen if the frequencies ω are not well chosen. In the Appendix, we show that the Schrödinger equation

$$\mathrm{i}\dot\psi = \left[-\tfrac{1}{2}\partial_{xx} + \tfrac{1}{2}x^2 + ax\sin\omega t\right]\psi, \quad x \in \mathbb{R}$$

(which was already studied by Graffi and Yajima [2000], who showed that the corresponding Floquet operator has continuous spectrum), exhibits growth of Sobolev norms if and only if $\omega = \pm 1$, which are clearly resonant frequencies. We also slightly generalize the example.

Another example of growth of Sobolev norms for the perturbed harmonic oscillator is given by Delort [2014]. There the perturbation is a pseudodifferential operator of order 0, periodic in time with resonant frequency $\omega = 1$.

Remark 1.4. The uniform-time estimate given in (1-9) is similar to the main result obtained in [Eliasson and Kuksin 2009] for small perturbation of the Laplace operator on the torus $\mathbb{T}^d$. Concerning perturbations of harmonic oscillators in $\mathbb{R}^d$, most known reducibility results are obtained for $d = 1$, except in [Grébert and Paturel 2016].

Remark 1.5. In [Eliasson and Kuksin 2009; Grébert and Paturel 2016] the estimate (1-10) is proved without loss of regularity; this is due to the fact that the perturbations treated in those papers are bounded operators. There are also some cases, see for example [Bambusi and Graffi 2001], in which the reducing transformation is bounded notwithstanding the fact that the perturbation is unbounded, but this is due to the fact that the unperturbed system has suitable gap properties which are not fulfilled in our case.

Remark 1.6. The $\epsilon^{1/9}$ estimate on the measure of the set of resonant frequencies is not optimal. We wrote it just for the sake of giving a simple quantitative estimate.

Remark 1.7. Denote by $\{\psi_k\}_{k\in\mathbb{N}^d}$ the set of Hermite functions, namely the eigenvectors of H_0: $H_0\psi_k = \lambda_k\psi_k$. They form an orthonormal basis of $L^2(\mathbb{R}^d)$, and writing $\psi = \sum_k c_k\psi_k$, one has

$$\|\psi\|_s^2 \simeq \sum_k (1+|k|)^{2s}|c_k|^2.$$

Denote by $\psi(t) = \sum_{k\in\mathbb{N}^d} c_k(t)\psi_k$ the solution of (1-1) written on the Hermite basis. Then (1-9) implies the following *dynamical localization* for the energy of the solution: for all $s \geq 0$, there exists $C_s \equiv C_s(\psi_0) > 0$ such that

$$\sup_{t\in\mathbb{R}} |c_k(t)| \leq C_s(1+|k|)^{-s} \quad \text{for all } k \in \mathbb{N}^d. \tag{1-11}$$

From the dynamical property (1-11) one obtains easily that every state $\psi \in L^2(\mathbb{R}^d)$ is a bounded state for the time evolution $\mathcal{U}_{\epsilon,\omega}(t,0)\psi$ under the conditions of Theorem 1.1 on (ϵ,ω). The corresponding definitions are given below.

Definition 1.8 [Enss and Veselić 1983]. A function $\psi \in L^2(\mathbb{R}^d)$ is a bounded state (or belongs to the point spectral subspace of $\{\mathcal{U}_{\epsilon,\omega}(t,0)\}_{t\in\mathbb{R}}$) if the quantum trajectory $\{\mathcal{U}_{\epsilon,\omega}(t,0)\psi : t \in \mathbb{R}\}$ is a precompact subset of $L^2(\mathbb{R}^d)$.

Corollary 1.9. *Under the conditions of Theorem 1.1 on (ϵ,ω), every state $\psi \in L^2(\mathbb{R}^d)$ is a bounded state of $\{\mathcal{U}_{\epsilon,\omega}(t,0)\}_{t\in\mathbb{R}}$.*

Proof. To prove that every state $\psi \in L^2(\mathbb{R}^d)$ is a bounded state for the time evolution $\mathcal{U}_{\epsilon,\omega}(t,0)\psi$, using that $\mathcal{H}^s$ is dense in $L^2(\mathbb{R}^d)$, it is enough to assume that $\psi \in \mathcal{H}^s$, with $s > \frac{1}{2}d$. With the notation of Remark 1.7, we write

$$\psi(t) = \psi^{(N)}(t) + R^{(N)}(t),$$

where $\psi^{(N)}(t) = \sum_{|k|\leq N} c_k(t)\psi_k$ and $R^{(N)}(t) = \sum_{|k|>N} c_k(t)\psi_k$.

Take $\delta > 0$. Applying (1-11), taking N large enough, we get that $\|R^{(N)}(t)\|_0 \le \frac{1}{2}\delta$ for all $t \in \mathbb{R}$. But $\{\psi^{(N)}(t) : t \in \mathbb{R}\}$ is a subset of a finite-dimensional linear space. So we get that $\{\mathcal{U}_{\epsilon,\omega}(t,0)\psi : \ t \in \mathbb{R}\}$ is a precompact subset of $L^2(\mathbb{R}^d)$. □

This last dynamical result is deeply connected with the spectrum of the Floquet operator. First note that Theorem 1.1 implies the following:

Corollary 1.10. *The operator U_ω induces a unitary transformation $L^2(\mathbb{T}^n) \otimes L^2(\mathbb{R}^d)$ which transforms the Floquet operator K, namely*

$$K := -\mathrm{i}\omega \cdot \frac{\partial}{\partial\theta} + H_0 + \epsilon W(\theta),$$

into

$$-\mathrm{i}\omega \cdot \frac{\partial}{\partial\theta} + H_\infty.$$

Thus one has that the spectrum of K is pure point and its eigenvalues are $\lambda_j^\infty + \omega \cdot k$.

Notice that Enss and Veselić [1983, Theorems 2.3 and 3.2] proved that the spectrum of the Floquet operator is pure point if and only if every state is a bounded state. So Corollary 1.10 gives another proof of Corollary 1.9.

2. Proof of Theorem 1.1

To start, we scale the variables x_j by defining $x_j' = \sqrt{\nu_j}x_j$ so that, defining

$$h_j(x_j, \xi_j) := \xi_j^2 + x_j^2, \quad H_j := -\partial_{x_j}^2 + x_j^2,$$

one has

$$h_0 = \sum_{j=1}^{d} \nu_j h_j, \quad H_0 = \sum_{j=1}^{d} \nu_j H_j. \tag{2-1}$$

Remark 2.1. Notice that for any positive definite quadratic Hamiltonian h on $\mathbb{R}^{2d}$ there exists a symplectic basis such that $h = \sum_{j=1}^d \nu_j h_j$, with $\nu_j > 0$ for $1 \le j \le d$; see [Hörmander 1994].

For convenience in this paper we shall consider the Weyl quantization. The Weyl quantization of a symbol f is the operator $\mathrm{Op}^w(f)$, defined as usual as

$$\mathrm{Op}^w(f)u(x) = \frac{1}{(2\pi)^d} \int_{y,\xi\in\mathbb{R}^d} e^{\mathrm{i}(x-y)\xi}\, f\left(\tfrac{1}{2}x + y, \xi\right) u(y)\, dy\, d\xi.$$

Correspondingly we will say that an operator $F = \mathrm{Op}^w(f)$ is the Weyl operator with Weyl symbol f. Notice that for polynomials f of degree at most 2 in (x, ξ), we have $\mathrm{Op}^w(f) = f(x, D) + \text{const}$, where $D = \mathrm{i}^{-1}\nabla_x$.

Most of the time we also use the notation $f^w(x, D) := \mathrm{Op}^w(f)$. In particular, in (1-2) $W(\omega t, x, -\mathrm{i}\partial_x)$ denotes the Weyl operator $W^w(\omega t, x, D)$.

Given a Hamiltonian $\chi = \chi(x, \xi)$, we will denote by ϕ_χ^t the flow of the corresponding classical Hamilton equations.

It is well known that, if f and g are symbols, then the operator $-\mathrm{i}[f^w(x,D); g^w(x,D)]$ admits a symbol denoted by $\{f; g\}_{\mathrm{M}}$ (the Moyal bracket). Two fundamental properties of quadratic polynomial symbols are given by the following well-known remarks.

Remark 2.2. If f or g is a polynomial of degree at most 2, then $\{f; g\}_{\mathrm{M}} = \{f; g\}$, where

$$\{f; g\} := \sum_{j=1}^{d} \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j} - \frac{\partial g}{\partial x_j}\frac{\partial f}{\partial \xi_j}$$

is the Poisson Bracket of f and g.

Remark 2.3. Let χ be a polynomial of degree at most 2; then it follows from the previous remark that, for any Weyl operator $f^w(x,D)$, the symbol of $e^{\mathrm{i}t\chi^w(x,D)} f^w(x,D) e^{-\mathrm{i}t\chi^w(x,D)}$ is $f \circ \phi_\chi^t$.

Remark 2.4. If f and g are not quadratic polynomials, then $\{f; g\}_{\mathrm{M}} = \{f; g\} +$ lower-order terms; similar lower-order corrections would appear in the symbol of $e^{-\mathrm{i}t\chi^w(x,D)} f^w(x,D) e^{\mathrm{i}t\chi^w(x,D)}$. That is the reason why we restrict here to the case of quadratic perturbations. In order to deal with more general perturbations one needs further ideas which will be developed elsewhere.

Next we need to know how a time-dependent transformation transforms a classical and a quantum Hamiltonian. Precisely, consider a 1-parameter family of (Hamiltonian) functions $\chi(t,x,\xi)$ (where t is thought of as an external parameter) and denote by $\phi^\tau(t,x,\xi)$ the time τ flow it generates, precisely the solution of

$$\frac{dx}{d\tau} = \frac{\partial \chi}{\partial \xi}(t,x,\xi), \qquad \frac{d\xi}{d\tau} = -\frac{\partial \chi}{\partial x}(t,x,\xi). \tag{2-2}$$

Consider the time-dependent coordinate transformation

$$(x,\xi) = \phi^1(t,x',\xi') := \phi^\tau(t,x',\xi')|_{\tau=1}. \tag{2-3}$$

Remark 2.5. Working in the extended phase space in which time and a new momentum conjugated to it are added, it is easy to see that the coordinate transformation (2-3) transforms a Hamiltonian system with Hamiltonian h into a Hamiltonian system with Hamiltonian h' given by

$$h'(t,x',\xi') = h(\phi^1(t,x',\xi')) - \int_0^1 \frac{\partial \chi}{\partial t}(t, \phi^\tau(t,x',\xi'))\, d\tau. \tag{2-4}$$

Remark 2.6. If the operator $\chi^w(t,x,D)$ is selfadjoint for any fixed t, then the transformation

$$\psi = e^{-\mathrm{i}\chi^w(t,x,D)}\psi' \tag{2-5}$$

transforms $\mathrm{i}\dot\psi = H\psi$ into $\mathrm{i}\dot\psi' = H'\psi'$ with

$$H' = e^{\mathrm{i}\chi^w(t,x,D)} H e^{-\mathrm{i}\chi^w(t,x,D)} - \int_0^1 e^{\mathrm{i}\tau\chi^w(t,x,D)} (\partial_t \chi^w(t,x,\xi))\, e^{-\mathrm{i}\tau\chi^w(t,x,D)}\, d\tau. \tag{2-6}$$

This is seen by an explicit computation. For example see Lemma 3.2 of [Bambusi 2017a].

So in view of Remark 2.3, provided that transformation (2-5) is well-defined in the quadratic case, the quantum transformed Hamiltonian (2-6) is the exact quantization of the transformed classical Hamiltonian (2-4).

To study the analytic properties of the transformation (2-5) we will use the following simplified version of Theorem 1.2 of [Maspero and Robert 2017] (to which we refer for the proof).

Theorem 2.7 [Maspero and Robert 2017]. *Let H_0 be the Hamiltonian of the harmonic oscillator. If X is an operator symmetric on $\mathcal{H}^\infty$ such that XH_0^{-1} and $[X, H_0]H_0^{-1}$ belong to $\mathcal{L}(\mathcal{H}^s)$ for any $s \geq 0$, then the Schrödinger equation*

$$\mathrm{i}\,\partial_\tau \psi = X\psi$$

is globally well-posed in $\mathcal{H}^s$ for any s, and its unitary propagator $e^{-\mathrm{i}\tau X}$ belongs to $\mathcal{L}(\mathcal{H}^s)$ for all $s \geq 0$. Furthermore one has the quantitative estimate

$$c_s\|\psi\|_s \leq \|e^{-\mathrm{i}\tau X}\psi\|_s \leq C_s\|\psi\|_s \quad \textit{for all } \tau \in [0, 1], \tag{2-7}$$

where the constants $c_s, C_s > 0$ depend only on $\|[X, H_0^s]H_0^{-s}\|_{\mathcal{L}(\mathcal{H}^0)}$.

The properties of the transformation are given by the next lemma and are closely related to the standard properties on the smoothness in time of the semigroup generated by an unbounded operator.

Lemma 2.8. *Let $\chi(\rho, x, \xi)$ be a polynomial in (x, ξ) of degree at most 2 with real coefficients depending in a C^∞-way on $\rho \in \mathbb{R}^n$. Then for all $\rho \in \mathbb{R}^n$, the operator $\chi^w(\rho, x, D)$ is selfadjoint in $L^2(\mathbb{R}^d)$. Furthermore for all $s \geq 0$, for all $\tau \in \mathbb{R}$, the following hold true*:

(i) *The map $\rho \mapsto e^{-\mathrm{i}\tau\chi^w(\rho,x,D)}$ is in $C^0(\mathbb{R}^n, \mathcal{L}(\mathcal{H}^{s+2}, \mathcal{H}^s))$.*

(ii) *For all $\psi \in \mathcal{H}^s$, the map $\rho \mapsto e^{-\mathrm{i}\tau\chi^w(\rho,x,D)}\psi$ is in $C^0(\mathbb{R}^n, \mathcal{H}^s)$.*

(iii) *For all $r \in \mathbb{N}$, the map $\rho \mapsto e^{-\mathrm{i}\tau\chi^w(\rho,x,D)}$ is in $C^r(\mathbb{R}^n, \mathcal{L}(\mathcal{H}^{s+4r+2}, \mathcal{H}^s))$.*

(iv) *If the coefficients of $\chi(\rho, x, \xi)$ are uniformly bounded in $\rho \in \mathbb{R}^n$ then for any $s > 0$ there exist $c_s > 0$, $C_s > 0$ such that we have*

$$c_s\|\psi\|_s \leq \|e^{-\mathrm{i}\tau\chi^w(\rho,x,D)}\psi\|_s \leq C_s\|\psi\|_s \quad \textit{for all } \rho \in \mathbb{R}^n, \textit{ for all } \tau \in [0, 1].$$

Proof. First we remark that in this lemma the quantity ρ plays the role of a parameter. Since $\chi(\rho, x, \xi)$ is a real-valued polynomial in (x, ξ) of degree at most 2, the operator $\chi^w(\rho, x, D)$ is selfadjoint in $L^2(\mathbb{R}^d)$, so for all $\rho \in \mathbb{R}^n$ the propagator $e^{-\mathrm{i}\tau\chi^w(\rho,x,D)}$ is unitary on $L^2(\mathbb{R}^d)$.

In order to show that $e^{-\mathrm{i}\tau\chi^w(\rho,x,D)}$ maps $\mathcal{H}^s$ to itself, for all $s > 0$, for all $\rho \in \mathbb{R}^n$, we apply Theorem 2.7. Indeed since $\chi^w(\rho, x, D)$ has a polynomial symbol, we know $\chi^w(\rho, x, D)H_0^{-1}$ and the commutator $[H_0, \chi^w(\rho, x, D)]H_0^{-1}$ belong to $\mathcal{L}(\mathcal{H}^s)$ for all $s \geq 0$. Item (iv) follows by estimate (2-7) and the fact that $\|[H_0^s, \chi^w(\rho, x, D)]H_0^{-s}\|_{\mathcal{L}(\mathcal{H}^0)}$ is bounded uniformly in ρ.

To prove item (i) we use the Duhamel formula

$$e^{-\mathrm{i}\tau B} - e^{-\mathrm{i}\tau A} = \mathrm{i}\int_0^\tau e^{-\mathrm{i}(\tau-\tau_1)A}\,(A - B)\,e^{-\mathrm{i}\tau_1 B}\,d\tau_1. \tag{2-8}$$

Then choosing $B = \chi^w(\rho + \rho', x, D)$, $A = \chi^w(\rho, x, D)$ one has that for all $0 \le \tau \le 1$

$$\|e^{-\mathrm{i}\tau\chi^w(\rho+\rho',x,D)} - e^{-\mathrm{i}\tau\chi^w(\rho,x,D)}\|_{\mathcal{L}(\mathcal{H}^{s+2},\mathcal{H}^s)} \le C\|\chi^w(\rho+\rho', x, D) - \chi^w(\rho, x, D)\|_{\mathcal{L}(\mathcal{H}^{s+2},\mathcal{H}^s)}.$$

This proves item (i). Continuity in item (ii) is deduced by (i) with a standard density argument. Finally item (iii) is proved by induction on r, again using the Duhamel formula (2-8). □

Remark 2.5, Remark 2.6 and Lemma 2.8 imply the following important proposition.

Proposition 2.9. *Let $\chi(t, x, \xi)$ be a polynomial of degree at most 2 in x and ξ with smooth time-dependent coefficients. If the transformation* (2-3) *transforms a classical system with Hamiltonian h into a Hamiltonian system with Hamiltonian h', then the transformation* (2-5) *transforms the quantum system with Hamiltonian h^w into the quantum system with Hamiltonian $(h')^w$.*

As a consequence, for quadratic Hamiltonians, the quantum KAM theorem will follow from the corresponding classical KAM theorem.

To give the needed result, consider the classical time-dependent Hamiltonian

$$h_\epsilon(\omega t, x, \xi) := \sum_{1\le j\le d} \tfrac{1}{2}\nu_j(x_j^2 + \xi_j^2) + \epsilon\, W(\omega t, x, \xi), \tag{2-9}$$

with W as in the Introduction. The following KAM theorem holds.

Theorem 2.10. *Assume $\nu_j \ge \nu_0 > 0$ for $j = 1, \dots, d$ and $\mathbb{T}^n \times \mathbb{R}^d \times \mathbb{R}^d \ni (\theta, x, \xi) \mapsto W(\theta, x, \xi) \in \mathbb{R}$ is a polynomial in (x, ξ) of degree at most 2 with coefficients which are real analytic functions of $\theta \in \mathbb{T}^n$.*

Then there exists $\epsilon_* > 0$ and $C > 0$, such that for $|\epsilon| < \epsilon_*$ the following hold true:

(i) *There exists a closed set $\mathcal{E}_\epsilon \subset (0, 2\pi)^n$ with* $\mathrm{meas}((0, 2\pi)^n \setminus \mathcal{E}_\epsilon) \le C\epsilon^{1/9}$.

(ii) *For any $\omega \in \mathcal{E}_\epsilon$, there exists an analytic map $\theta \mapsto A_\omega(\theta) \in \mathrm{sp}(2d)$ (the symplectic algebra*[2] *of dimension $2d$) and an analytic map $\theta \mapsto V_\omega(\theta) \in \mathbb{R}^{2d}$ such that the change of coordinates*

$$(x', \xi') = e^{A_\omega(\omega t)}(x, \xi) + V_\omega(\omega t) \tag{2-10}$$

conjugates the Hamiltonian equations of (2-9) *to the Hamiltonian equations of a homogeneous polynomial $h_\infty(x, \xi)$ of degree 2 which is positive definite. Finally both A_ω and V_ω are ϵ-close to zero.*

Furthermore h_∞ can be diagonalized: there exists a matrix $\mathcal{P} \in \mathrm{Sp}(2d)$ (the symplectic group of dimension $2d$) such that, setting $(y, \eta) = \mathcal{P}(x, \xi)$ we have

$$h_\infty \circ \mathcal{P}^{-1}(y, \eta) = \sum_{j=1}^{d} \nu_j^\infty(y_j^2 + \eta_j^2), \tag{2-11}$$

where $\nu_j^\infty = \nu_j^\infty(\omega)$ are defined on $\mathcal{E}_\epsilon$ and fulfill the estimates

$$|\nu_j^\infty - \nu_j| \le C\epsilon, \quad j = 1, \dots, d. \tag{2-12}$$

Remark 2.11. In general, the matrix $\mathcal{P}$ is not close to the identity. However, in the case that the frequencies ν_j are nonresonant, $\mathcal{P} = \mathbf{1}$.

[2] Recall that a real $2d \times 2d$ matrix A belongs to $\mathrm{sp}(2d)$ if and only if JA is symmetric

KAM theory in finite dimensions is nowadays standard. In particular we believe that Theorem 2.10 can be obtained combining the results of [Eliasson 1988; You 1999]. However, for the reader's convenience and the sake of being self-contained, we add in Section 3 its proof.

Theorem 1.1 follows immediately combining the results of Theorem 2.10 and Proposition 2.9.

Proof of Theorem 1.1. We see easily that the change of coordinates (2-10) has the form (2-3) with a Hamiltonian $\chi_\omega(\omega t, x, \xi)$ which is a polynomial in (x, ξ) of degree at most 2 with real, smooth and uniformly bounded coefficients in $t \in \mathbb{R}$.

Define $U_\omega(\omega t) = e^{-\mathrm{i}\chi_\omega^w(\omega t, x, D)}$. By Proposition 2.9 it conjugates the original equation (1-1) to (1-6), where $H_\infty := \mathrm{Op}^w(h_\infty)$.

Furthermore $\theta \mapsto U_\omega(\theta)$ fulfills (i)–(iv) of Lemma 2.8, from which it follows immediately that $\theta \mapsto U_\omega(\theta)$ fulfills items (i), (iii) of Theorem 1.1. Concerning item (ii), by the Taylor formula the quantity $\|\mathbf{1} - U_\omega(\theta)\|_{\mathcal{L}(\mathcal{H}^{s+2},\mathcal{H}^s)}$ is controlled by $\|\chi_\omega^w(\theta, x, D)\|_{\mathcal{L}(\mathcal{H}^{s+2},\mathcal{H}^s)}$, from which estimate (1-7) follows.

Finally using the metaplectic representation, see [Combescure and Robert 2012], and (2-11), there exists a unitary transformation in L^2, denoted by $\mathcal{R}(\mathcal{P}^{-1})$, such that

$$\mathcal{R}(\mathcal{P}^{-1})^* H_\infty \mathcal{R}(\mathcal{P}^{-1}) = \sum_{j=1}^{d} \nu_j^\infty (x_j^2 - \partial_{x_j}^2). \qquad \square$$

We prove now Corollary 1.3.

Proof of Corollary 1.3. Consider first the propagator $e^{-\mathrm{i}tH_\infty}$. We claim that

$$\sup_{t\in\mathbb{R}} \|e^{-\mathrm{i}tH_\infty}\|_{\mathcal{L}(\mathcal{H}^s)} < \infty \quad \text{for all } t \in \mathbb{R}. \tag{2-13}$$

Recall that $H_\infty = h_\infty^w(x, D)$, where $h_\infty(x, \xi)$ is a positive definite symmetric form which can be diagonalized by a symplectic matrix $\mathcal{P}$. Since h_∞ is positive definite, there exist $c_0, c_1, c_2 > 0$ such that

$$c_1 h_0(x, \xi) \le c_0 + h_\infty(x, \xi) \le c_2(1 + h_0(x, \xi)),$$

which implies $C_1 H_0 \le C_0 + H_\infty \le C_2(1 + H_0)$ as bilinear form. Thus one has the equivalence of norms

$$C_s^{-1}\|\psi\|_{\mathcal{H}^s} \le \|(H_\infty)^{s/2}\psi\|_{L^2} \le C_s\|\psi\|_{\mathcal{H}^s}.$$

Then

$$\|e^{-\mathrm{i}tH_\infty}\psi_0\|_{\mathcal{H}^s} \le C_s\|(H_\infty)^{s/2}\, e^{-\mathrm{i}tH_\infty}\psi_0\|_{L^2} = C_s\|(H_\infty)^{s/2}\,\psi_0\|_{L^2} \le C_s'\|\psi_0\|_{\mathcal{H}^s},$$

which implies (2-13).

Now let $\psi(t)$ be a solution of (1-1). By formula (1-8), $\psi(t) = U_\omega^*(\omega t)e^{-\mathrm{i}tH_\infty}U_\omega(0)\psi_0$. Then the upper bound in (1-9) follows easily from (2-13) and $\sup_t \|U_\omega(\omega t)\|_{\mathcal{L}(\mathcal{H}^s)} < \infty$, which is a consequence of Lemma 2.8. The lower bound follows by applying Lemma 2.8 (iv).

Finally estimate (1-10) follows from (1-7). $\square$

3. A classical KAM result

In this section we prove Theorem 2.10. We prefer to work in the extended phase space in which we add the angles $\theta \in \mathbb{T}^n$ as new variables and their conjugated momenta $I \in \mathbb{R}^n$. Furthermore we will use complex variables defined by

$$z_j = \frac{1}{\sqrt{2}}(\xi_j - \mathrm{i}x_j),$$

so that our phase space will be $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{C}^d$, with $\mathbb{C}^d$ considered as a real vector space. The symplectic form is $dI \wedge d\theta + \mathrm{i}\, dz \wedge d\bar{z}$ and the Hamilton equations of a Hamiltonian function $h(\theta, I, z, \bar{z})$ are

$$\dot{I} = -\frac{\partial h}{\partial \theta}, \quad \dot{\theta} = \frac{\partial h}{\partial I}, \quad \dot{z} = -\mathrm{i}\frac{\partial h}{\partial \bar{z}}.$$

In this framework h_0 takes the form $h_0 = \sum_{j=1}^d \nu_j z_j \bar{z}_j$ and W takes the form of polynomial in $(z, \bar{z})$ of degree 2, $W(\theta, x, \xi) = q(\theta, z, \bar{z})$. The Hamiltonian system associated with the time-dependent Hamiltonian h_ϵ, see (2-9), is then equivalent to the Hamiltonian system associated with the time-independent Hamiltonian $\omega \cdot I + h_\epsilon$ (written in complex variables) in the extended phase space.

General strategy. Let h be a Hamiltonian in normal form:

$$h(I, \theta, z, \bar{z}) = \omega \cdot I + \langle z, N(\omega)\bar{z}\rangle, \tag{3-1}$$

with $N \in \mathcal{M}_H$ the set of Hermitian matrices. Notice that at the beginning of the procedure N is diagonal,

$$N = N_0 = \mathrm{diag}(\nu_j,\ j = 1, \dots, d)$$

and is independent of ω. Let $q \equiv q_\omega$ be a polynomial Hamiltonian which takes real values: $q(\theta, z, \bar{z}) \in \mathbb{R}$ for $\theta \in \mathbb{T}^n$ and $z \in \mathbb{C}^d$. We write

$$q(\theta, z, \bar{z}) = \langle z, Q_{zz}(\theta)z\rangle + \langle z, Q_{z\bar{z}}(\theta)\bar{z}\rangle + \langle \bar{z}, \overline{Q}_{zz}(\theta)\bar{z}\rangle + \langle Q_z(\theta), z\rangle + \langle \overline{Q}_{\bar{z}}(\theta), \bar{z}\rangle, \tag{3-2}$$

where $Q_{zz}(\theta) \equiv Q_{zz}(\omega, \theta)$ and $Q_{z\bar{z}}(\theta) \equiv Q_{z\bar{z}}(\omega, \theta)$ are $d \times d$ complex matrices and $Q_z(\theta) \equiv Q_z(\theta, \omega)$ is a vector in $\mathbb{C}^d$. They all depend analytically on the angle

$$\theta \in \mathbb{T}^n_\sigma := \{x + iy : x \in \mathbb{T}^n,\ y \in \mathbb{R}^n,\ |y| < \sigma\}.$$

We notice that $Q_{z\bar{z}}$ is Hermitian, while Q_{zz} is symmetric. The size of such a polynomial function depending analytically on $\theta \in \mathbb{T}^n_\sigma$ and C^1 on $\omega \in \mathcal{D} = (0, 2\pi)^n$ will be controlled by the norm

$$[q]_\sigma := \sup_{\substack{|\mathrm{Im}\theta|<\sigma \\ \omega\in\mathcal{D},\ j=0,1}} \|\partial_\omega^j Q_{zz}(\omega, \theta)\| + \sup_{\substack{|\mathrm{Im}\theta|<\sigma \\ \omega\in\mathcal{D},\ j=0,1}} \|\partial_\omega^j Q_{z\bar{z}}(\omega, \theta)\| + \sup_{\substack{|\mathrm{Im}\theta|<\sigma \\ \omega\in\mathcal{D},\ j=0,1}} |\partial_\omega^j Q_z(\omega, \theta)|$$

and we denote by $\mathcal{Q}(\sigma)$ the class of Hamiltonians of the form (3-2) whose norm $[\,\cdot\,]_\sigma$ is finite.

Let us assume that $[q]_\sigma = \mathcal{O}(\epsilon)$. We search for $\chi \equiv \chi_\omega \in \mathcal{Q}(\sigma)$ with $[\chi]_\sigma = \mathcal{O}(\epsilon)$ such that its time-1 flow $\phi_\chi \equiv \phi_\chi^{t=1}$ (in the extended phase space, of course) transforms the Hamiltonian $h + q$ into

$$(h + q(\theta)) \circ \phi_\chi = h_+ + q_+(\theta), \quad \omega \in \mathcal{D}_+, \tag{3-3}$$

where $h_+ = \omega \cdot I + \langle z, N_+ \bar{z}\rangle$ is a new normal form, ϵ-close to h, the new perturbation $q_+ \in \mathcal{Q}(\sigma)$ is of size[3] $\mathcal{O}(\epsilon^{3/2})$ and $\mathcal{D}_+ \subset \mathcal{D}$ is ϵ^α-close to $\mathcal{D}$ for some $\alpha > 0$. Notice that all the functions are defined on the whole open set $\mathcal{D}$ but (3-3) holds only on $\mathcal{D}_+$, a subset of $\mathcal{D}$ from which we excised the "resonant parts".

As a consequence of the Hamiltonian structure, we have

$$(h + q(\theta)) \circ \phi_\chi = h + \{h, \chi\} + q(\theta) + \mathcal{O}(\epsilon^{3/2}), \quad \omega \in \mathcal{D}_+.$$

So to achieve the goal above we should solve the *homological equation*:

$$\{h, \chi\} = h_+ - h - q(\theta) + \mathcal{O}(\epsilon^{3/2}), \quad \omega \in \mathcal{D}_+. \tag{3-4}$$

Repeating iteratively the same procedure with h_+ instead of h, we will construct a change of variable ϕ such that

$$(h + q(\theta)) \circ \phi = \omega \cdot I + h_\infty, \quad \omega \in \mathcal{D}_\infty,$$

with $h_\infty = \langle z, N_\infty(\omega)\bar{z}\rangle$ in normal form and $\mathcal{D}_\infty$ an ϵ^α-close subset of $\mathcal{D}$. Note that we will be forced to solve the homological equation not only for the diagonal normal form N_0, but for more general normal form Hamiltonians (3-1) with N close to N_0.

Homological equation.

Proposition 3.1. *Let $\mathcal{D} = (0, 2\pi)^n$ and $\mathcal{D} \ni \omega \mapsto N(\omega) \in \mathcal{M}_H$ be a C^1 mapping that satisfies*

$$\|\partial_\omega^j (N(\omega) - N_0)\| < \frac{\min(1, \nu_0)}{\max(4, d)} \tag{3-5}$$

for $j = 0, 1$ and $\omega \in \mathcal{D}$. Let $h = \omega \cdot I + \langle z, N\bar{z}\rangle$, $q \in \mathcal{Q}(\sigma)$, $\kappa > 0$ and $K \geq 1$.

Then there exists a closed subset $\mathcal{D}' = \mathcal{D}'(\kappa, K) \subset \mathcal{D}$ satisfying

$$\operatorname{meas}(\mathcal{D} \setminus \mathcal{D}') \leq C K^n \kappa, \tag{3-6}$$

and there exist $\chi, r \in \bigcap_{0 \leq \sigma' < \sigma} \mathcal{Q}(\sigma')$ and $\mathcal{D} \ni \omega \mapsto \widetilde{N}(\omega) \in \mathcal{M}_H$ a C^1 mapping such that for all $\omega \in \mathcal{D}'$

$$\{h, \chi\} + q = \langle z, \widetilde{N}\bar{z}\rangle + r. \tag{3-7}$$

Furthermore for all $\omega \in \mathcal{D}$

$$\|\partial_\omega^j \widetilde{N}(\omega)\| \leq [q]_\sigma, \quad j = 0, 1, \tag{3-8}$$

and for all $0 \leq \sigma' < \sigma$

$$[r]_{\sigma'} \leq C \frac{e^{-1/2(\sigma - \sigma')K}}{(\sigma - \sigma')^n} [q]_\sigma, \tag{3-9}$$

$$[\chi]_{\sigma'} \leq \frac{CK}{\kappa^2 (\sigma - \sigma')^n} [q]_\sigma. \tag{3-10}$$

[3]Formally we could expect q_+ to be of size $O(\epsilon^2)$ but the small divisors and the reduction of the analyticity domain will lead to an estimate of the type $\mathcal{O}(\epsilon^{3/2})$.

Proof. Writing the Hamiltonians h, q and χ as in (3-2), the homological equation (3-7) is equivalent to the three following equations (we use that N is Hermitian, thus $\overline{N} = {}^tN$):

$$\omega \cdot \nabla_\theta X_{z\bar{z}} - \mathrm{i}[N, X_{z\bar{z}}] = \widetilde{N} - Q_{z\bar{z}} + R_{z\bar{z}}, \tag{3-11}$$

$$\omega \cdot \nabla_\theta X_{zz} - \mathrm{i}(N X_{zz} + X_{zz}\overline{N}) = -Q_{zz} + R_{zz}, \tag{3-12}$$

$$\omega \cdot \nabla_\theta X_z + \mathrm{i} N X_z = -Q_z + R_z. \tag{3-13}$$

First we solve (3-11). To simplify notation we drop the indices $z\bar{z}$. Written in Fourier variables (with respect to θ), (3-11) reads as

$$\mathrm{i}\omega \cdot k\ \widehat{X}_k - \mathrm{i}[N, \widehat{X}_k] = \delta_{k,0}\widetilde{N} - \widehat{Q}_k + \widehat{R}_k, \quad k \in \mathbb{Z}^n, \tag{3-14}$$

where $\delta_{k,j}$ denotes the Kronecker symbol.

When $k = 0$ we solve this equation by defining

$$\widehat{X}_0 = 0, \quad \widehat{R}_0 = 0 \quad \text{and} \quad \widetilde{N} = \widehat{Q}_0.$$

We notice that $\widetilde{N} \in \mathcal{M}_H$ and satisfies (3-8).

When $|k| \geq K$, (3-14) is solved by defining

$$\widehat{R}_k = \widehat{Q}_k, \quad \widehat{X}_k = 0 \quad \text{for } |k| \geq K. \tag{3-15}$$

Then we set

$$\widehat{R}_k = 0 \quad \text{for } |k| \leq K$$

in such a way that $r \in \bigcap_{0 \leq \sigma' < \sigma} \mathcal{Q}(\sigma')$ and by a standard argument r satisfies (3-9). Now it remains to solve the equations for $\widehat{X}_k,\ 0 < |k| \leq K$, which we rewrite as

$$L_k(\omega)\widehat{X}_k = i\,\widehat{Q}_k, \tag{3-16}$$

where $L_k(\omega)$ is the linear operator from $\mathcal{M}_S$, the space of symmetric matrices, into itself defined by

$$L_k(\omega) : M \mapsto (k \cdot \omega)\, M - [N(\omega), M].$$

We notice that $\mathcal{M}_S$ can be endowed with the Hermitian product: $(A, B) = \mathrm{Tr}(\bar{A}B)$ associated with the Hilbert–Schmidt norm. Since N is Hermitian, $L_k(\omega)$ is selfadjoint for this structure. As a first consequence we get

$$\|(L_k(\omega))^{-1}\| \leq \frac{1}{\min\{|\lambda|,\ \lambda \in \Sigma(L_k(\omega))\}} = \frac{1}{\min\{|k \cdot \omega - \alpha(\omega) + \beta(\omega)| : \alpha, \beta \in \Sigma(N(\omega))\}}, \tag{3-17}$$

where for any matrix A, we denote its spectrum by $\Sigma(A)$.

Let us recall an important result of perturbation theory, which is a consequence of Theorem 1.10 in [Kato 1980] (since Hermitian matrices are normal matrices):

Theorem 3.2 [Kato 1980, Theorem 1.10]. *Let $I \subset \mathbb{R}$ and $I \ni z \mapsto M(z)$ be a holomorphic curve of Hermitian matrices. Then all the eigenvalues and associated eigenvectors of $M(z)$ can be parametrized holomorphically on I.*

Let us assume for a while that N depends analytically on ω in such a way that $\omega \mapsto L_k(\omega)$ is analytic. Fix a direction $z_k \in \mathbb{R}^n$; the eigenvalue $\lambda_k(\omega) = k \cdot \omega - \alpha(\omega) + \beta(\omega)$ of $L_k(\omega)$ is C^1 in the direction[4] z_k and the associated unitary eigenvector, denoted by $v(\omega)$, is also piecewise C^1 in the direction z_k. Then, as a consequence of the hermiticity of $L_k(\omega)$ we have

$$\partial_\omega \lambda(\omega) \cdot z_k = \langle v(\omega), (\partial_\omega L_k(\omega) \cdot z_k)\, v(\omega)\rangle.$$

Therefore, if N depends analytically of ω, we deduce using (3-5) and choosing $z_k = k/|k|$

$$\left| \partial_\omega \lambda_k(\omega) \cdot \frac{k}{|k|} \right| \geq |k| - 2\|\partial_\omega N\| \geq \tfrac{1}{2} \quad \text{for } k \neq 0, \tag{3-18}$$

which extends also to the points of discontinuity of $v(\omega)$. Now given a matrix L depending on the parameter $\omega \in \mathcal{D}$, we define

$$\mathcal{D}(L, \kappa) = \{\omega \in \mathcal{D} : \|L(\omega)^{-1}\| \leq \kappa^{-1}\}$$

and we recall the following classical lemma:

Lemma 3.3. *Let $f : [0, 1] \mapsto \mathbb{R}$ be a C^1-map satisfying $|f'(x)| \geq \delta$ for all $x \in [0, 1]$ and let $\kappa > 0$. Then*

$$\operatorname{meas}\{x \in [0, 1] : |f(x)| \leq \kappa\} \leq \frac{\kappa}{\delta}.$$

Combining this lemma, (3-17) and (3-18) we deduce that, if N depends analytically of ω, then for $k \neq 0$

$$\operatorname{meas}(\mathcal{D} \setminus \mathcal{D}(L_k, \kappa)) \leq C\kappa. \tag{3-19}$$

Now it turns out that, by a density argument, this last estimate remains valid (with a larger constant C) when N is only a C^1 function of ω; the point is that (3-18) holds true uniformly for close analytic approximations of N.

In particular, defining

$$\mathcal{D}' = \bigcap_{0<|k|\leq K} \mathcal{D}(L_k, \kappa),$$

$\mathcal{D}'$ is closed and satisfies (3-6).

By construction, $\widehat{X}_k(\omega) := i L_k(\omega)^{-1} \widehat{Q}_k$ satisfies (3-16) for $0 < |k| \leq K$ and $\omega \in \mathcal{D}(L_k, \kappa)$ and

$$\|\widehat{X}_k(\omega)\| \leq \kappa^{-1} \|\widehat{Q}_k(\omega)\|, \quad \omega \in \mathcal{D}(L_k, \kappa). \tag{3-20}$$

It remains to extend $\widehat{X}_k(\,\cdot\,)$ on $\mathcal{D}$. Using again (3-5) we have for any $|k| \leq K$ and any unit vector z, $|\partial_\omega \lambda(\omega) \cdot z| \leq CK$. Therefore

$$\operatorname{dist}\big(\mathcal{D} \setminus \mathcal{D}(L_k, \kappa), \mathcal{D}\big(L_k, \tfrac{1}{2}\kappa\big)\big) \geq \frac{\kappa}{CK}$$

and we can construct (by a convolution argument) for each k, $0 < |k| \leq K$, a C^1 function g_k on $\mathcal{D}$ with

$$|g_k|_{C^0(\mathcal{D})} \leq C, \quad |g_k|_{C^1(\mathcal{D})} \leq CK\kappa^{-1} \tag{3-21}$$

[4]That is, $t \mapsto \lambda_k(\omega + t z_k)$ is a holomorphic curve on a neighborhood of 0, and we denote by $\partial_\omega \lambda(\omega) \cdot z_k$ its derivative at $t = 0$.

(the constant C is independent of k) and such that $g_k(\omega) = 1$ for $\omega \notin \mathcal{D}(L_k, \kappa)$ and $g_k(\omega) = 0$ for $\omega \in \mathcal{D}\big(L_k, \frac{1}{2}\kappa\big)$. Then $\widetilde{X}_k = g_k \widehat{X}_k$ is a C^1 extension of $\widehat{X}_k$ to $\mathcal{D}$. Similarly we define $\widetilde{Q}_k = g_k \widehat{Q}_k$ in such a way that $\widetilde{X}_k$ satisfies

$$L_k(\omega)\widetilde{X}_k(\omega) = i\,\widetilde{Q}_k(\omega), \quad 0 < |k| \le K,\ \omega \in \mathcal{D}.$$

Differentiating with respect to ω leads to

$$L_k(\omega)\,\partial_{\omega_j}\widehat{X}(k) = \mathrm{i}\,\partial_{\omega_j}\widehat{Q}(k) - k_j\widehat{X}(k) + [\partial_{\omega_j}N,\, \widehat{X}(k)], \quad 1 \le j \le n.$$

Defining $B_k(\omega) = \mathrm{i}\,\partial_{\omega_j}\widetilde{Q}_k(\omega) - k_j\widetilde{X}_k(\omega) + [\partial_{\omega_j}N(\omega),\, \widetilde{X}_k(\omega)]$ we have

$$\|\partial_{\omega_j}\widetilde{X}_k(\omega)\| \le \kappa^{-1}\|B_k(\omega)\|, \quad \omega \in \mathcal{D}.$$

Using (3-5), (3-20) and (3-21) we get for $|k| \le K$ and $\omega \in \mathcal{D}$

$$\begin{aligned}\|B_k(\omega)\| &\le \|\partial_{\omega_j}\widetilde{Q}_k(\omega)\| + K\|\widetilde{X}_k(\omega)\| + 2\|\partial_{\omega_j}N(\omega)\|\,\|\widetilde{X}_k(\omega)\| \\ &\le CK\kappa^{-1}(\|\partial_{\omega_j}\widehat{Q}(k,\omega)\| + \|\widehat{Q}(k,\omega)\|).\end{aligned}$$

Combining the last two estimates we get

$$\sup_{\omega\in\mathcal{D},\ j=0,1}\|\partial_\omega^j\widetilde{X}_k(\omega)\| \le CK\kappa^{-2}\sup_{\omega\in\mathcal{D},\ j=0,1}\|\partial_\omega^j\widehat{Q}_k(\omega)\|.$$

Thus defining

$$X_{z\bar{z}}(\omega,\theta) = \sum_{0<|k|\le K}\widetilde{X}_k(\omega)e^{ik\cdot\theta},$$

$X_{z\bar{z}}(\omega,\cdot)$ satisfies (3-11) for $\omega \in \mathcal{D}'$ and leads to (3-10) for $\chi_{z\bar{z}}(\omega,\theta,z,\bar{z}) = \langle z, X_{z\bar{z}}(\omega,\cdot)\bar{z}\rangle$.

We solve (3-13) in a similar way. We notice that in this case we face the small divisors $|\omega\cdot k - \alpha(\omega)|$, $k \in \mathbb{Z}^n$, where $\alpha \in \Sigma(N(\omega))$. In particular for $k = 0$ these quantities are $\ge \frac{1}{2}\nu_0$ since $|\alpha - \nu_j| \le \frac{1}{4}\nu_0$ for some $1 \le j \le d$ by (3-5).

Writing in Fourier variables and dropping indices zz, (3-12) reads as

$$\mathrm{i}\omega\cdot k\ \widehat{X}(k) - \mathrm{i}(N\widehat{X}(k) + \widehat{X}(k)\overline{N}) = -\widehat{Q}(k) + \widehat{R}(k). \tag{3-22}$$

So to mimic the resolution of (3-14) we have to replace the operator $L_k(\omega)$ by the operator $M_k(\omega)$, defined on $\mathcal{M}_S$ by

$$M_k(\omega)X := \omega\cdot k + NX + X\overline{N}.$$

This operator is still selfadjoint for the Hermitian product $(A, B) = \mathrm{Tr}(\bar{A}B)$ so the same strategy applies. Nevertheless we have to consider differently the case $k = 0$. In that case we use that the eigenvalues of $M_0(\omega)$ are close to eigenvalues of the operator M_0 defined by

$$M_0 : X \mapsto N_0X + X\overline{N}_0 = N_0X + XN_0,$$

with $N_0 = \mathrm{diag}(\nu_j,\ j = 1, \dots, d)$ a real and diagonal matrix. Actually in view of (3-5)

$$\|(L - L_0)M\|_{\mathrm{HS}} \le \|N - N_0\|_{\mathrm{HS}}\,\|M\|_{\mathrm{HS}} \le d\|N - N_0\|\,\|M\|_{\mathrm{HS}} \le \nu_0.$$

The eigenvalues of L_0 are $\{\nu_j + \nu_\ell : j, \ell = 1, \dots, d\}$ and they are all larger than $2\nu_0$. We conclude that all the eigenvalues of $M_0(\omega)$ satisfy $|\alpha(\omega)| \geq \nu_0$. The end of the proof follows as before. □

The KAM step. Theorem 2.10 is proved by an iterative KAM procedure. We begin with the initial Hamiltonian $h_0 + q_0$, where

$$h_0(I, \theta, z, \bar{z}) = \omega \cdot I + \langle z, N_0 \bar{z} \rangle, \tag{3-23}$$

$N_0 = \operatorname{diag}(\nu_j,\ j = 1, \dots, d)$, $\omega \in \mathcal{D} \equiv [1, 2]^n$ and the quadratic perturbation q_0 equals $\epsilon W \in \mathcal{Q}(\sigma, \mathcal{D})$ for some $\sigma > 0$. Then we construct iteratively the change of variables ϕ_m, the normal form $h_m = \omega \cdot I + \langle z, N_m \bar{z} \rangle$ and the perturbation $q_m \in \mathcal{Q}(\sigma_m, \mathcal{D}_m)$ as follows: Assume that the construction is done up to step $m \geq 0$. Then:

(i) Using Proposition 3.1 we construct χ_{m+1}, r_{m+1} and $\widetilde{N}_m$ the solution of the homological equation:

$$\{h, \chi_{m+1}\} = \langle z, \widetilde{N}_m \bar{z} \rangle - q_m(\theta) + r_{m+1}, \quad \omega \in \mathcal{D}_{m+1}, \theta \in \mathbb{T}^n_{\sigma_{m+1}}. \tag{3-24}$$

(ii) We define $h_{m+1} := \omega \cdot I + \langle z, N_{m+1} \bar{z} \rangle$ by

$$N_{m+1} = N_m + \widetilde{N}_m, \tag{3-25}$$

and

$$q_{m+1} := r_m + \int_0^1 \{(1-t)(h_{m+1} - h_m + r_{m+1}) + t q_m, \chi_{m+1}\} \circ \phi^t_{\chi_{m+1}} \,\mathrm{d}t. \tag{3-26}$$

By construction, if Q_m and N_m are Hermitian, so are R_m and S_{m+1} by the resolution of the homological equation, and also N_{m+1} and Q_{m+1}.

For any regular Hamiltonian f we have, using the Taylor expansion of $f \circ \phi^t_{\chi_{m+1}}$ between $t = 0$ and $t = 1$,

$$f \circ \phi^1_{\chi_{m+1}} = f + \{f, \chi_{m+1}\} + \int_0^1 (1-t) \{\{f, \chi_{m+1}\}, \chi_{m+1}\} \circ \phi^t_{\chi_{m+1}} \,\mathrm{d}t.$$

Therefore we get for $\omega \in \mathcal{D}_{m+1}$

$$(h_m + q_m) \circ \phi^1_{\chi_{m+1}} = h_{m+1} + q_{m+1}.$$

Iterative lemma. Following the general scheme above we have

$$(h_0 + q_0) \circ \phi^1_{\chi_1} \circ \dots \circ \phi^1_{\chi_m} = h_m + q_m,$$

where q_m is a polynomial of degree 2 and $h_m = \omega \cdot I + \langle z, N_m \bar{z} \rangle$ with N_m a Hermitian matrix. At step m the Fourier series are truncated at order K_m and the small divisors are controlled by κ_m. Now we specify the choice of all the parameters for $m \geq 0$ in terms of ϵ_m, which will control $[q_m]_{\mathcal{D}_m, \sigma_m}$.

First we define $\epsilon_0 = \epsilon$, $\sigma_0 = \sigma$, $\mathcal{D}_0 = \mathcal{D}$ and for $m \geq 1$ we choose

$$\sigma_{m-1} - \sigma_m = C_* \sigma_0 m^{-2}, \quad K_m = 2(\sigma_{m-1} - \sigma_m)^{-1} \ln \epsilon_{m-1}^{-1}, \quad \kappa_m = \epsilon_{m-1}^{1/8},$$

where $(C_*)^{-1} = 2 \sum_{j \geq 1} 1/j^2$.

Lemma 3.4. *There exists $\epsilon_* > 0$ depending on d, n such that, for $|\epsilon| \le \epsilon_*$ and*

$$\epsilon_m = \epsilon^{(3/2)^m}, \quad m \ge 0,$$

we have the following:

For all $m \ge 1$ there exist closed subsets $\mathcal{D}_m \subset \mathcal{D}_{m-1}$, $h_m = \omega \cdot I + \langle z, N_m \bar{z} \rangle$ in normal form, where $\mathcal{D}_m \ni \omega \mapsto N_m(\omega) \in \mathcal{M}_H \in C^1$, and there exist $\chi_m, q_m \in \mathcal{Q}(\mathcal{D}_m, \sigma_m)$ such that for $m \ge 1$:

(i) *The symplectomorphism*

$$\phi_m \equiv \phi_{\chi_m}(\omega) : \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2d} \to \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2d}, \quad \omega \in \mathcal{D}_m, \tag{3-27}$$

is an affine transformation in $(z, \bar{z})$, analytic in $\theta \in \mathbb{T}^n_{\sigma_m}$ and C^1 in $\omega \in \mathcal{D}_m$ of the form

$$\phi_m(I, \theta, z, \bar{z}) = (g_m(I, \theta, z, \bar{z}), \theta, \Psi_m(\theta, z, \bar{z})), \tag{3-28}$$

where, for each $\theta \in \mathbb{T}^n$, $(z, \bar{z}) \mapsto \Psi_m(\theta, z, \bar{z})$ is a symplectic change of variable on $\mathbb{C}^{2n}$.

The map ϕ_m links the Hamiltonian at step $m-1$ and the Hamiltonian at step m; *i.e.*,

$$(h_{m-1} + q_{m-1}) \circ \phi_m = h_m + q_m \quad \textit{for all } \omega \in \mathcal{D}_m.$$

(ii) *We have the estimates*

$$\operatorname{meas}(\mathcal{D}_{m-1} \setminus \mathcal{D}_m) \le \epsilon_{m-1}^{1/9}, \tag{3-29}$$

$$[\widetilde{N}_{m-1}]_{s,\beta}^{\mathcal{D}_m} \le \epsilon_{m-1}, \tag{3-30}$$

$$[q_m]_{s,\beta}^{\mathcal{D}_m,\sigma_m} \le \epsilon_m, \tag{3-31}$$

$$\|\phi_m(\omega) - \mathbf{1}\|_{\mathcal{L}(\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2d})} \le C \epsilon_{m-1}^{1/2} \quad \textit{for all } \omega \in \mathcal{D}_m. \tag{3-32}$$

Proof. At step 1, $h_0 = \omega \cdot I + \langle z, N_0 \bar{z} \rangle$ and thus hypothesis (3-5) is trivially satisfied and we can apply Proposition 3.1 to construct χ_1, N_1, r_1 and $\mathcal{D}_1$ such that for $\omega \in \mathcal{D}_1$

$$\{h_0, \chi_1\} = \langle z, (N_1 - N_0) \bar{z} \rangle - q_0 + r_1.$$

Then, using (3-6), we have

$$\operatorname{meas}(\mathcal{D} \setminus \mathcal{D}_1) \le C K_1^n \kappa_1 \le \epsilon_0^{1/9}$$

for $\epsilon = \epsilon_0$ small enough. Using (3-10) we have for ϵ_0 small enough

$$[\chi_1]_{\mathcal{D}_1, \sigma_1} \le C \frac{K_1}{\kappa_1^2 (\sigma_0 - \sigma_1)^n} \epsilon_0 \le \epsilon_0^{1/2}.$$

Similarly using (3-9), (3-8) we have

$$\|N_1 - N_0\| \le \epsilon_0 \quad \text{and} \quad [r_1]_{\mathcal{D}_1, \sigma_1} \le C \frac{\epsilon_0^{15/8}}{(\sigma_1 - \sigma_0)^n} \le \epsilon_0^{7/4}$$

for $\epsilon = \epsilon_0$ small enough. In particular we deduce $\|\phi_1 - \mathbf{1}\|_{\mathcal{L}(\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2d})} \le \epsilon_0^{1/2}$. Thus using (3-26) we get for ϵ_0 small enough

$$[q_1]_{\mathcal{D}_1, \sigma_1} \le \epsilon_0^{3/2} = \epsilon_1.$$

The form of the flow (3-28) follows since χ_1 is a Hamiltonian of the form (3-2).

Now assume that we have verified Lemma 3.4 up to step m. We want to perform the step $m+1$. We have $h_m = \omega \cdot I + \langle z, N_m \bar{z}\rangle$ and since

$$\|N_m - N_0\| \le \|N_m - N_0\| + \cdots + \|N_1 - N_0\| \le \sum_{j=0}^{m-1} \epsilon_j \le 2\epsilon_0,$$

hypothesis (3-5) is satisfied and we can apply Proposition 3.1 to construct $\mathcal{D}_{m+1}$, χ_{m+1} and q_{m+1}. Estimates (3-29)–(3-32) at step $m+1$ are proved as we have proved the corresponding estimates at step 1. □

Transition to the limit and proof of Theorem 2.10. Let $\mathcal{E}_\epsilon = \bigcap_{m\ge 0} \mathcal{D}_m$. In view of (3-29), this is a closed set satisfying

$$\operatorname{meas}(\mathcal{D} \setminus \mathcal{E}_\epsilon) \le \sum_{m \ge 0} \epsilon_m^{1/9} \le 2\epsilon_0^{1/9}.$$

Let us set $\tilde{\phi}_N = \phi_1 \circ \cdots \circ \phi_N$. Due to (3-32) it satisfies for $M \le N$ and for $\omega \in \mathcal{E}_\epsilon$

$$\|\tilde{\phi}_N - \tilde{\phi}_{\mathrm{M}}\|_{\mathcal{L}(\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2d})} \le \sum_{m=M}^{N} \epsilon_m^{1/2} \le 2\epsilon_{\mathrm{M}}^{1/2}.$$

Therefore $(\tilde{\phi}_N)_N$ is a Cauchy sequence in $\mathcal{L}(\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2d})$. Thus when $N \to \infty$, the mappings $\tilde{\phi}_N$ converge to a limit mapping $\phi_\infty \in \mathcal{L}(\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2d})$. Furthermore since the convergence is uniform on $\omega \in \mathcal{E}_\epsilon$ and $\theta \in \mathbb{T}_{\sigma/2}$, we know ϕ_∞^1 depends analytically on θ and C^1 in ω. Moreover,

$$\|\phi_\infty - \mathbf{1}\|_{\mathcal{L}(\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2d})} \le \epsilon_0^{1/2}. \tag{3-33}$$

By construction, the map $\tilde{\phi}_m$ transforms the original Hamiltonian $h_0 + q_0$ into $h_m + q_m$. When $m \to \infty$, by (3-31) we get $q_m \to 0$ and by (3-30) we get $N_m \to N$, where

$$N \equiv N(\omega) = N_0 + \sum_{k=1}^{+\infty} \widetilde{N}_k$$

is a Hermitian matrix which is C^1 with respect to $\omega \in \mathcal{E}_\epsilon$. Setting $h_\infty(z, \bar{z}) = \omega \cdot I + \langle z, N(\omega)\bar{z}\rangle$ we have proved

$$(h + q(\theta)) \circ \phi_\infty = h_\infty. \tag{3-34}$$

Furthermore for all $\omega \in \mathcal{E}_\epsilon$ we have, using (3-30),

$$\|N(\omega) - N_0\| \le \sum_{m=0}^{\infty} \epsilon_m \le 2\epsilon$$

and thus the eigenvalues of $N(\omega)$, denoted by $\nu_j^\infty(\omega)$, satisfy (2-12).

It remains to give the affine symplectomorphism ϕ_∞. At each step of the KAM procedure we have by Lemma 3.4

$$\phi_m(I, \theta, z, \bar{z}) = (g_m(I, \theta, z, \bar{z}), \theta, \Psi_m(\theta, z, \bar{z})),$$

and therefore

$$\phi_\infty(I, \theta, z, \bar{z}) = (g(I, \theta, z, \bar{z}), \theta, \Psi(\theta, z, \bar{z})),$$

where $\Psi(\theta, z, \bar{z}) = \lim_{m\to\infty} \Psi_1 \circ \Psi_2 \circ \cdots \circ \Psi_m$.

It is useful to go back to real variables (x, ξ). More precisely, write each Hamiltonian χ_m constructed in the KAM iteration in the variables (x, ξ):

$$\chi_m(\theta, x, \xi) = \frac{1}{2}\begin{bmatrix} x \\ \xi \end{bmatrix} \cdot E\, B_m(\theta)\begin{bmatrix} x \\ \xi \end{bmatrix} + U_m(\theta), \qquad E := \begin{bmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix}, \tag{3-35}$$

where $B_m(\theta)$ is a skew-symmetric matrix of dimension $2d \times 2d$ and $U_m(\theta) \in \mathbb{R}^{2d}$, and they are both of size ϵ_m. Then Ψ_m written in the real variables has the form

$$\Psi_m(\theta, x, \xi) = e^{B_m(\theta)}(x, \xi) + T_m(\theta), \quad \text{where } T_m(\theta) := \int_0^1 e^{(1-s)JB_m(\theta)}U_m(\theta)\, ds. \tag{3-36}$$

Lemma 3.5. *There exists a sequence of Hamiltonian matrices $A_l(\theta)$ and vectors $V_l(\theta) \in \mathbb{R}^{2d}$ such that*

$$\Psi_1 \circ \cdots \circ \Psi_l(x, \xi) = e^{A_l(\theta)}(x, \xi) + V_l(\theta) \quad \textit{for all } (x, \xi) \in \mathbb{R}^{2d}. \tag{3-37}$$

Furthermore, there exist an Hamiltonian matrix $A_\omega(\theta)$ and a vector $V_\omega(\theta) \in \mathbb{R}^{2d}$ such that

$$\begin{gathered} \lim_{l\to+\infty} e^{A_l(\theta)} = e^{A_\infty(\theta)}, \qquad \lim_{l\to+\infty} V_l(\theta) = V_\infty(\theta), \\ \sup_{|\mathrm{Im}\theta|\le\sigma/2} \|A_\omega(\theta)\| \le C\epsilon, \qquad \sup_{|\mathrm{Im}\theta|\le\sigma/2} |V_\omega(\theta)| \le C\epsilon, \end{gathered} \tag{3-38}$$

and for each $\theta \in \mathbb{T}^n$,

$$\Psi(\theta, x, \xi) = e^{A_\omega(\theta)}(x, \xi) + V_\omega(\theta) \quad \textit{for all } (x, \xi) \in \mathbb{R}^{2d}.$$

Proof. Recall that $\phi_j = e^{B_j} + T_j$, where T_j is a translation by the vector T_j with the estimates $\|B_j\| \le C\epsilon_j$, $\|T_j\| \le C\epsilon_j$. So we have $e^{B_j} = \mathbb{1} + S_j$ with $\|S_j\| \le C\epsilon_j$. Then the infinite product $\prod_{1\le j<+\infty} e^{B_j}$ is convergent. Moreover we have $\prod_{1\le j\le l} e^{B_j} = \mathbb{1} + M_l$ with $\|M_l\| \le C\epsilon$, so we have $\prod_{1\le j<+\infty} e^{B_j} = \mathbb{1} + M$, with $\|M\| \le C\epsilon$. This is proved by using

$$\prod_{1\le j\le l} (\mathbb{1} + S_j) = \mathbb{1} + S_l + S_{l-1}S_l + \cdots + S_1S_2\cdots S_l$$

and estimates on $\|S_j\|$.

So, M_l has a small norm and therefore $A_l := \log(\mathbb{1} + M_l)$ is well-defined. Furthermore, by construction $\mathbb{1} + M_l \in \mathrm{Sp}(2d)$ and therefore its logarithm is a Hamiltonian matrix, namely $A_l \in \mathrm{sp}(2d)$ for $1 \le l \le +\infty$.

Now we have to include the translations. By induction on l we have

$$\phi_1 \circ \cdots \circ \phi_l(x, \xi) = \mathrm{e}^{A_l}(x, \xi) + V_l,$$

with $V_{l+1} = \mathrm{e}^{A_l}T_{l+1} + V_l$ and $V_1 = T_1$. Using the previous estimates we have

$$\|V_{l+1} - V_l\| \le C\|T_{l+1}\| \le C\epsilon_l.$$

Then we get that $\lim_{l\to+\infty} V_l = V_\infty$ exists. $\square$

Appendix: An example of growth of Sobolev norms (following Graffi and Yajima)

In this appendix we are going to study the Hamiltonian

$$H := -\tfrac{1}{2}\partial_{xx} + \tfrac{1}{2}x^2 + ax\sin\omega t \tag{A-1}$$

and prove that it is reducible to the harmonic oscillator if $\omega \neq \pm 1$, while the system exhibits growth of Sobolev norms in the case $\omega = \pm 1$. Actually the result holds in a quite more general situation, but we think that the present example can give a full understanding of the situation with as few techniques as possible. We also remark that in this case it is not necessary to assume that the time-dependent part is small.

Finally we recall that (A-1) with $\omega = \pm 1$ was studied by Graffi and Yajima as an example of a Hamiltonian whose Floquet spectrum is absolutely continuous (despite the fact that the unperturbed Hamiltonian has discrete spectrum). Exploiting the results of [Enss and Veselić 1983; Bunimovich et al. 1991], one can conclude from [Graffi and Yajima 2000] that the expectation value of the energy is not bounded in this model. The novelty of the present result rests in the much more precise statement ensuring growth of Sobolev norms.

As we already pointed out, in order to get reducibility of the Hamiltonian (A-1), it is enough to study the corresponding classical Hamiltonian, in particular proving its reducibility; this is what we will do. It also turns out that the whole procedure is clearer working as much as possible at the level of the equations.

So, consider the classical Hamiltonian system

$$h := \tfrac{1}{2}(x^2 + \xi^2) + ax\sin(\omega t), \tag{A-2}$$

whose equations of motion are

$$\begin{cases} \dot{x} = \xi, \\ \dot{\xi} = -x - a\sin(\omega t) \end{cases} \iff \ddot{x} + x + a\sin(\omega t) = 0. \tag{A-3}$$

Proposition A.1. *Assume that $\omega \neq \pm 1$. Then there exists a time-periodic canonical transformation conjugating* (A-2) *to*

$$h' := \tfrac{1}{2}(x^2 + \xi^2). \tag{A-4}$$

If $\omega = \pm 1$ then the system is canonically conjugated to

$$h' := \pm\tfrac{1}{2}a\xi. \tag{A-5}$$

In both cases the transformation has the form (2-10).

Corollary A.2. *In the case $\omega = \pm 1$, for any $s > 0$ and $\psi_0 \in \mathcal{H}^s$, there exists a constant $0 < C_s = C_s(\|\psi_0\|_{\mathcal{H}^s})$ such that the solution of the Schrödinger equation with Hamiltonian* (A-1) *and initial datum ψ_0 fulfills*

$$\|\psi(t)\|_{\mathcal{H}^s} \geq C_s\langle t\rangle^s \quad \textit{for all } t \in \mathbb{R}. \tag{A-6}$$

Before proving the theorem, recall that by the general result of [Maspero and Robert 2017, Theorem 1.5], any solution of the Schrödinger equation with Hamiltonian (A-1) fulfills the a priori bound

$$\|\psi(t)\|_{\mathcal{H}^s} \leq C'_s(\|\psi_0\|_{\mathcal{H}^s} + |t|^s\|\psi_0\|_{\mathcal{H}^0}) \quad \text{for all } t \in \mathbb{R}, \tag{A-7}$$

which is therefore sharp.

Proof of Proposition A.1. We look for a translation

$$x = x' - f(t), \quad \xi = \xi' - g(t), \tag{3-8}$$

with f and g time-periodic functions to be determined in such a way to eliminate time from (A-3). Writing the equations for (x', ξ'), one gets

$$\dot{x}' = \xi' - g + \dot{f}, \quad \dot{\xi}' = -x' - a\sin(\omega t) + \dot{g} + f,$$

which reduces to the harmonic oscillator by choosing

$$\begin{cases} -a\sin(\omega t) + \dot{g} + f = 0, \\ -g + \dot{f} = 0 \end{cases} \iff \ddot{f} + f = a\sin(\omega t), \tag{3-9}$$

which has a solution of period $2\pi/\omega$ only if $\omega \neq \pm 1$. In such a case the only solution having the correct period is

$$f = \frac{a}{1-\omega^2}\sin(\omega t), \quad g = \frac{a\omega}{1-\omega^2}\cos(\omega t).$$

Then the transformation (3-8) is a canonical transformation generated as the time-1 flow of the auxiliary Hamiltonian

$$\chi := -\xi\frac{a}{1-\omega^2}\sin(\omega t) + x\frac{a\omega}{1-\omega^2}\cos(\omega t),$$

which thus conjugates the classical Hamiltonian (A-2) to the harmonic oscillator; of course the quantization of χ conjugates the quantum system to the quantum harmonic oscillator, as follows by Proposition 2.9.

We come to the resonant case, and, in order to fix ideas, we take $\omega = 1$. In such a case the flow of the harmonic oscillator is periodic of the same period as the forcing, and thus its flow can be used to reduce the system.

In a slightly more abstract way, consider a Hamiltonian system with Hamiltonian

$$H := \tfrac{1}{2}\langle z; Bz\rangle + \langle z; b(t)\rangle,$$

with $z := (x, \xi)$, B a symmetric matrix, and $b(t)$ a vector-valued time-periodic function. Then, using the formula (2-4), it is easy to see that the auxiliary time-dependent Hamiltonian

$$\chi_1 := \tfrac{1}{2}t\langle z; Bz\rangle \tag{3-10}$$

generates a time-periodic transformation which conjugates the system to

$$h' := \langle z; e^{-JBt}b(t)\rangle$$

(J being the standard symplectic matrix). An explicit computation shows that in our case

$$h' = \tfrac{1}{2}ax\sin(2t) - \tfrac{1}{2}a\xi\cos(2t) + \tfrac{1}{2}a\xi. \tag{3-11}$$

Then in order to eliminate the two time-periodic terms in (3-11) it is sufficient to use the canonical transformation generated by the Hamiltonian

$$\chi_2 := -\tfrac{1}{4}\xi a\sin(2t) - \tfrac{1}{4}xa\cos(2t), \tag{3-12}$$

which reduces to (A-5). □

Proof of Corollary A.2. To fix ideas we take $\omega = 1$. Let $\chi_1^w \equiv \frac{1}{2}t(-\partial_{xx} + x^2)$ and χ_2^w be the Weyl quantizations of the Hamiltonians (3-10) and (3-12) respectively. By the proof of Proposition A.1, the changes of coordinates

$$\psi = e^{-\mathrm{i}tH_0}\psi_1, \quad \psi_1 = e^{-\mathrm{i}\chi_2^w(t,x,D)}\varphi, \quad H_0 := \tfrac{1}{2}(-\partial_{xx} + x^2), \tag{3-13}$$

conjugate the Schrödinger equation with Hamiltonian (A-1) to the Schrödinger equation with Hamiltonian (A-2), namely the transport equation

$$\partial_t\varphi = -\tfrac{1}{2}a\partial_x\varphi.$$

The solution of this transport equation is given clearly by

$$\varphi(t,x) = \varphi_0\big(x - \tfrac{1}{2}at\big),$$

where φ_0 is the initial datum. Now a simple computation shows that

$$\liminf_{|t|\to+\infty} |t|^{-s}\|\varphi(t)\|_{\mathcal{H}^s} \geq \big(\tfrac{1}{2}|a|\big)^s\|\varphi_0\|_{\mathcal{H}^0}.$$

In particular there exists a constant $0 < C_s = C_s(\|\varphi_0\|_{\mathcal{H}^s})$ such that

$$\|\varphi(t)\|_{\mathcal{H}^s} \geq C_s\langle t\rangle^s. \tag{3-14}$$

Since the transformation (3-13) maps $\mathcal{H}^s$ to $\mathcal{H}^s$ uniformly in time (see also Lemma 2.8) estimate (3-14) holds also for the original variables. □

We remark that by a similar procedure one can also prove the following slightly more general result.

Theorem 3.3. *Consider the classical Hamiltonian system*

$$h = \sum_{j=1}^{d}\tfrac{1}{2}\nu_j(x_j^2 + \xi_j^2) + \sum_{j=1}^{d}(g_j(\omega t)x_j + f_j(\omega t)\xi_j), \tag{3-15}$$

with $f_j, g_j \in C^r(\mathbb{T}^n)$.

(1) *If there exist* $\gamma > 0$ *and* $\tau > n+1$ *such that*

$$|\omega\cdot k \pm \nu_j| \geq \frac{\gamma}{1+|k|^\tau} \quad \textit{for all } k \in \mathbb{Z}^n, \quad j = 1,\dots,d, \tag{3-16}$$

and $r > \tau + 1 + \frac{1}{2}n$, *then there exists a time-quasiperiodic canonical transformation of the form* (2-10) *conjugating the system to*[5]

$$h = \sum_{j=1}^{d}\tfrac{1}{2}\nu_j(x_j^2 + \xi_j^2).$$

(2) *If there exist* $0 \neq \bar{k} \in \mathbb{Z}^n$ *and* $\bar{j}$, *such that*

$$\omega\cdot\bar{k} - \nu_{\bar{j}} = 0, \tag{3-17}$$

[5] Actually the transformation is just a translation, so in this case one has $A \equiv 0$.

and there exist $\gamma > 0$ *and* τ *such that*

$$|\omega \cdot k \pm \nu_j| \geq \frac{\gamma}{1+|k|^\tau} \quad \textit{for all } (k, j) \neq (\bar{k}, \bar{j}) \tag{3-18}$$

and $r > \tau + 1 + \frac{1}{2}n$, *then there exists a time-quasiperiodic canonical transformation of the form* (2-10) *conjugating the system to*

$$h = \sum_{j \neq \bar{j}} \tfrac{1}{2}\nu_j (x_j^2 + \xi_j^2) + c_1 x_{\bar{j}} + c_2 \xi_{\bar{j}},$$

with $c_1, c_2 \in \mathbb{R}$.

Remark 3.4. The constants c_1, c_2 can be easily computed. If at least one of them is different from zero then the solution of the corresponding quantum system exhibits growth of Sobolev norms, as in the special model (A-1). Of course the result extends in a trivial way to the case in which more resonances are present.

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