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C^1 REGULARITY OF ORTHOTROPIC p-HARMONIC FUNCTIONS IN THE PLANE

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We prove that local weak solutions of the orthotropic p-harmonic equation in $\mathbb{R}^2$ are C^1 functions.

1. Introduction

1A. ***The result.*** Let $\Omega \subset \mathbb{R}^2$ be an open set and let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local weak solution in Ω of the *orthotropic p-Laplace equation*

$$\sum_{i=1}^{2} (|u_{x_i}|^{p-2}\, u_{x_i})_{x_i} = 0. \tag{1-1}$$

This means that for every $\Omega' \Subset \Omega$ and every $\varphi \in W^{1,p}_0(\Omega')$, we have

$$\sum_{i=1}^{2} \int_{\Omega'} |u_{x_i}|^{p-2}\, u_{x_i}\, \varphi_{x_i}\, dx = 0. \tag{1-2}$$

In the recent literature, such an equation has sometimes been called the *pseudo p-Laplace equation*. We decided to adopt the terminology *orthotropic p-Laplace equation* in order to emphasize the role played by the coordinate system. Indeed, let us recall that if $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a local weak solution of the usual *p-Laplace equation*, i.e.,

$$\sum_{i=1}^{2} (|\nabla u|^{p-2}\, u_{x_i})_{x_i} = 0,$$

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then for every linear isometry $A:\mathbb{R}^2\to\mathbb{R}^2$, we know $u\circ A$ is still a local weak solution of this equation on $A^{-1}(\Omega)$. This property fails to be true for (1-1), but it still holds if A belongs to the dihedral group D_2, i.e., the group of symmetries of the square $(-1,1)\times(-1,1)$.

Equation (1-1) is the prototype of degenerate/singular elliptic equations with orthotropic structure, interest in which arose for example in the context of *optimal transport problems with congestion effects*. We refer to the introduction of [Brasco and Carlier 2014] for a detailed description of the framework and the model leading to these kinds of equations.

A function $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a local weak solution if and only if it is a local minimizer of the functional

$$\mathfrak{F}(\varphi;\Omega'):=\sum_{i=1}^{2}\frac{1}{p}\int_{\Omega'}|\varphi_{x_i}|^p\,dx,\quad \varphi\in W^{1,p}_{\mathrm{loc}}(\Omega),\ \Omega'\Subset\Omega\subset\mathbb{R}^2.$$

This easily follows from the convexity of the functional $\mathfrak{F}$. We recall that $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a local minimizer of $\mathfrak{F}$ if

$$\mathfrak{F}(u;\Omega')\le\mathfrak{F}(\varphi;\Omega')\quad\text{for every } u-\varphi\in W^{1,p}_0(\Omega'),\ \Omega'\Subset\Omega.$$

In the recent paper [Bousquet et al. 2016], we proved that for $p\ge 2$ any such local minimizer is a locally Lipschitz function; actually, the case $1<p<2$ is a mere application of [Fonseca and Fusco 1997, Theorem 2.2]. The aim of this paper is to go one step further and prove the following additional regularity.

Main Theorem. *Every local minimizer $U\in W^{1,p}_{\mathrm{loc}}(\Omega)$ of the functional $\mathfrak{F}$ is a C^1 function.*

Remark 1.1. It is easy to see that the function

$$u(x_1,x_2)=|x_1|^{\frac{p}{p-1}}-|x_2|^{\frac{p}{p-1}},\quad (x_1,x_2)\in\mathbb{R}^2,$$

is a local weak solution of (1-1). Observe that for $p>2$, we have u is not C^2, but only $C^{1,\frac{1}{p-1}}$. We conjecture this to be the sharp regularity of local weak solutions.

1B. ***Method of proof.*** The proof of the Main Theorem is greatly inspired by that of [Santambrogio and Vespri 2010, Theorem 11], which in turn exploits an idea introduced in [DiBenedetto and Vespri 1995]. However, since our equation is much more singular/degenerate than theirs, most of the estimates have to be recast and the argument needs various nontrivial adaptations. In order to neatly explain the method of proof and highlight the differences with respect to [Santambrogio and Vespri 2010], let us first recall their result.

In [Santambrogio and Vespri 2010] it is shown that in $\mathbb{R}^2$, local weak solutions of the variational equation

$$\operatorname{div}\nabla H(\nabla u)=0 \tag{1-3}$$

are such that $x\mapsto\nabla H(\nabla u(x))$ is continuous, provided that

- $\nabla H(\nabla u)\in W^{1,2}_{\mathrm{loc}}\cap L^\infty_{\mathrm{loc}}$;
- $H:\mathbb{R}^2\to[0,\infty)$ is a C^2 convex function such that there exist $M\ge 0$ and $0<\lambda\le\Lambda$ for which

$$\lambda\,|z|^{p-2}\,|\xi|^2\le\langle D^2H(z)\,\xi,\xi\rangle\le\Lambda\,|z|^{p-2}\,|\xi|^2\quad\text{for every }\xi\in\mathbb{R}^2,\ |z|\ge M. \tag{1-4}$$

The last assumption implies that (1-3) is a degenerate/singular elliptic equation, with *confined degeneracy/singularity*. Indeed, on the set where the gradient of a Lipschitz solution u satisfies $|\nabla u| \geq M$, the equation behaves as a uniformly elliptic equation. By using the terminology of [Bousquet et al. 2016], we can say that (1-3) has a *p-Laplacian-type structure at infinity*.

The proof of the continuity of $\nabla H(\nabla u)$ in [Santambrogio and Vespri 2010] relies on the following De Giorgi–type lemma: given a ball B_R of radius R, if a component $H_{x_i}(\nabla u)$ of the vector field $\nabla H(\nabla u)$ has large oscillations only on a small portion of B_R, then the global oscillation of $H_{x_i}(\nabla u)$ on the ball $B_{\frac{R}{2}}$ is reduced (in a precise quantitative sense). Such a result amounts to an L^∞ estimate for (a nonlinear function of) the gradient, which in turn relies on the Caccioppoli inequality for the linearized equation

$$\operatorname{div}(D^2 H(\nabla u)\, \nabla u_{x_i}) = 0. \tag{1-5}$$

On the contrary, if $H_{x_i}(\nabla u)$ has large oscillations on a large portion of B_R, then one exploits the fact that a function $W^{1,2} \cap L^\infty$ in the plane is such that either

(A1) its Dirichlet energy in a crown contained in B_R is large; or

(A2) the function itself is large on a circle contained in B_R.

When (A2) occurs, the structure of the linearized equation (1-5) allows us to prove a *minimum principle* for $H_{x_i}(\nabla u)$, which implies that $H_{x_i}(\nabla u)$ is large on the whole disc bounded by the above-mentioned circle. This again leads to a decay of the oscillation of $H_{x_i}(\nabla u)$ (this time because the infimum increases when shrinking the ball).

Then the continuity result of [Santambrogio and Vespri 2010] is achieved by constructing inductively a decreasing sequence of balls and using the dichotomy above at each step. The important point is that since $H_{x_i}(\nabla u)$ has finite Dirichlet energy, then possibility (A1) *can occur only finitely many times*. Hence, the oscillation of $H_{x_i}(\nabla u)$ decays to 0, as desired.

Unfortunately, our (1-1) does not have a p-Laplacian structure at infinity; i.e., (1-4) is not satisfied. Indeed, in our case we have

$$H(z) = \sum_{i=1}^{2} \frac{|z_i|^p}{p}$$

so that

$$D^2 H(z) = (p-1) \begin{bmatrix} |z_1|^{p-2} & 0 \\ 0 & |z_2|^{p-2} \end{bmatrix}, \quad z = (z_1, z_2) \in \mathbb{R}^2.$$

In particular, $D^2 H(z)$ is degenerate/singular on the union of the two axes $\{z_1{=}0\} \cup \{z_2{=}0\}$ and our equation does not fit in the framework of [Santambrogio and Vespri 2010]. Thus, even if the proof of the Main Theorem follows the guidelines illustrated above, we will have to overcome the additional difficulties linked to the more degenerate/singular structure of (1-5). In particular, in the case $p > 2$, we need a new Caccioppoli inequality, which weirdly mixes different components of the gradient (see Proposition 3.1). This is one of the main novelties of the paper.

Remark 1.2 (stream functions). For $1 < p < \infty$, let us set $p' = p/(p-1)$. When $\Omega \subset \mathbb{R}^2$ is simply connected, to every local weak solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of (1-1) one can associate a *stream function* $v \in W^{1,p'}_{\mathrm{loc}}(\Omega)$, such that

$$v_{x_1} = |u_{x_2}|^{p-2}\, u_{x_2} \quad \text{and} \quad v_{x_2} = -|u_{x_1}|^{p-2}\, u_{x_1}.$$

Existence of such a function v is a straightforward consequence of the *Poincaré Lemma*, once it is observed that (1-1) implies that the vector field

$$\big(|u_{x_1}|^{p-2}\, u_{x_1}, |u_{x_2}|^{p-2}\, u_{x_2}\big)$$

is divergence free (in the distributional sense). It is readily seen that v is a weak solution of

$$\sum_{i=1}^{2} (|v_{x_i}|^{p'-2}\, v_{x_i})_{x_i} = 0.$$

This would allow us to reduce the proof of the Main Theorem to the case $1 < p \le 2$ only. However, this kind of argument is very specific to the homogeneous equation and *already fails* in the case

$$\sum_{i=1}^{2} (|u_{x_i}|^{p-2}\, u_{x_i})_{x_i} = \lambda \in \mathbb{R},$$

which we note is covered by our method (indeed, observe that the previous equation and (1-1) have the same linearization (1-5), thus the Main Theorem still applies). More generally, we observe that our method of proof can be adapted to treat the case, as in [Santambrogio and Vespri 2010], of

$$\sum_{i=1}^{2} (|u_{x_i}|^{p-2}\, u_{x_i})_{x_i} = f$$

under suitable (not sharp) assumptions[1] on f. For these reasons, we avoided using this argument based on stream functions.

1C. ***Plan of the paper.*** First, it should be noticed that almost every section is divided in two parts, one for the degenerate case $p > 2$ and the other for the singular one $1 < p < 2$ (the case $p = 2$ corresponds to the standard Laplacian). Though the methods of proof for the two cases look very much the same, there are some important differences which lead us to think that it is better to separate the two cases.

In Section 2 we introduce the technical machinery and present some basic integrability properties of solutions and their derivatives, needed throughout the whole paper. Section 3 is devoted to some new Caccioppoli inequalities for the gradient of a local minimizer. The core of the paper is represented by Sections 4 and 5, concerning decay estimates for a nonlinear function of the gradient (case $p > 2$) or for the gradient itself (case $1 < p \le 2$). Finally, the proof of the Main Theorem is postponed to Section 6. The paper ends with Appendices A and B containing technical facts.

[1] As in the case of the ordinary p-Laplacian, see [Kuusi and Mingione 2013, Corollary 1.6], the sharp assumption should be $f \in L^{2,1}_{\mathrm{loc}}$, the latter being a Lorentz space. For $p > 2$ our proof requires $|u_{x_j}|^{\frac{p-2}{2}}\, u_{x_j} \in W^{1,2}_{\mathrm{loc}}(\Omega)$, a result which is true only when f enjoys suitable differentiability properties.

2. Preliminaries

2A. *Notation.* Given $\lambda > 0$ and a ball $B \subset \mathbb{R}^2$ of radius $R > 0$, we denote by λB the ball with the same center and radius λR.

We define for every $q > -1$ the function $g_q : \mathbb{R} \to \mathbb{R}$ as

$$g_q(t) = |t|^q\, t, \quad t \in \mathbb{R}. \tag{2-1}$$

Then g_q is a homeomorphism and $g_q^{-1} = g_{-\frac{q}{q+1}}$. Observe that

$$|t|^q\, t \le \alpha \quad \Longleftrightarrow \quad t \le |\alpha|^{-\frac{q}{q+1}}\, \alpha,$$

a fact that will be used repeatedly.

Let $U \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a given local minimizer of $\mathfrak{F}$. We fix a ball $B \Subset \Omega$. There exists $\lambda_B > 1$ such that $\lambda_B\, B \Subset \Omega$ as well. If $\{\rho_\varepsilon\}_{\varepsilon>0} \subset C_0^\infty(B_\varepsilon)$ is a smooth convolution kernel (here, B_ε refers to the ball with center 0 and radius ε), we define $U^\varepsilon := U * \rho_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$, where $\Omega_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$. By the definition of U^ε there exists $0 < \varepsilon_0 < 1$ such that for every $0 < \varepsilon < \varepsilon_0$

$$\|U^\varepsilon\|_{W^{1,p}(B)} = \|\nabla U^\varepsilon\|_{L^p(B)} + \|U^\varepsilon\|_{L^p(B)} \le \|\nabla U\|_{L^p(\lambda_B\, B)} + \|U\|_{L^p(\lambda_B\, B)}. \tag{2-2}$$

2B. *Regularization scheme, case $p > 2$.* As in [Bousquet et al. 2016], we consider the minimization problem

$$\min\left\{\sum_{i=1}^{2} \frac{1}{p} \int_B |w_{x_i}|^p\, dx + \frac{p-1}{2}\, \varepsilon \int_B |\nabla w|^2\, dx \;:\; w - U^\varepsilon \in W_0^{1,p}(B)\right\}. \tag{2-3}$$

Since the functional is strictly convex, there exists a unique solution u^ε, which is C^2 on $\overline{B}$; see, e.g., [Bousquet et al. 2016, Theorem 2.4] for the Lipschitz regularity and [Giusti 2003, Theorems 8.6 and 10.18] for the higher regularity. Moreover, u^ε satisfies the Euler–Lagrange equation

$$\sum_{i=1}^{2} \int_B (|u^\varepsilon_{x_i}|^{p-2} + (p-1)\,\varepsilon)\, u^\varepsilon_{x_i}\, \varphi_{x_i}\, dx = 0 \quad \text{for every } \varphi \in W_0^{1,p}(B).$$

We take $\varphi \in C^2$ with compact support in B. Then for $j \in \{1, 2\}$, the partial derivative φ_{x_j} is still an admissible test function. An integration by parts leads to

$$\sum_{i=1}^{2} \int_B (|u^\varepsilon_{x_i}|^{p-2} + \varepsilon)\, u^\varepsilon_{x_i x_j}\, \varphi_{x_i}\, dx = 0, \quad j = 1, 2. \tag{2-4}$$

As usual, by a density argument, the equation still holds with $\varphi \in W_0^{1,2}(B)$. We now collect some uniform estimates on u^ε.

Lemma 2.1 (uniform energy estimate). *There exists a constant $C = C(p) > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ the following estimate holds*:

$$\int_B |\nabla u^\varepsilon|^p\, dx \le C\left(\int_{\lambda_B\, B} |\nabla U|^p\, dx + \varepsilon^{\frac{p}{p-2}}\, |B|\right). \tag{2-5}$$

Moreover, the family $\{u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ converges weakly in $W^{1,p}(B)$ and strongly in $L^p(B)$ to U.

Proof. The estimate (2-5) is standard, it is sufficient to test the minimality of u^ε against U^ε, which is admissible. In particular, the family $\{u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ is uniformly bounded in $W^{1,p}(B)$. Moreover, by [Bousquet et al. 2016, Lemma 2.9] there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\varepsilon_0)$ such that u^{ε_k} converges weakly in $W^{1,p}(B)$ and strongly in $L^p(B)$ to a solution w of

$$\min\left\{\sum_{i=1}^{2} \frac{1}{p}\int_B |\varphi_{x_i}|^p\,dx \,:\, \varphi - U \in W_0^{1,p}(B)\right\}.$$

Since U is a local minimizer of $\mathfrak{F}$ and the solution of this problem is unique (by strict convexity), we get $w = U$ and full convergence of the whole family. $\square$

Lemma 2.2 (uniform regularity estimates). *For every $0<\varepsilon<\varepsilon_0$ and every $B_r \Subset B$ we have*

$$\|u^\varepsilon\|_{L^\infty(B_r)} \le C, \tag{2-6}$$

$$\|\nabla u^\varepsilon\|_{L^\infty(B_r)} \le C, \tag{2-7}$$

$$\int_{B_r} \left|\nabla\big(|u^\varepsilon_{x_j}|^{\frac{p-2}{2}}\, u^\varepsilon_{x_j}\big)\right|^2 dx \le C, \quad j=1,2, \tag{2-8}$$

for some constant $C>0$ independent of $\varepsilon>0$.

Proof. The proof of the L^∞ estimate (2-6) is standard; it can be obtained as in [Giusti 2003, Theorem 7.5]. The standing assumption required throughout [Giusti 2003, Chapter 7], namely the property (7.2) there, is satisfied in our framework since for every $z\in\mathbb{R}^2$ one has

$$\frac{1}{C}\,|z|^p \le \frac{1}{p}\sum_{i=1}^{2}|z_i|^p + \frac{p-1}{2}\,\varepsilon\,|z|^2 \le C\,(|z|^p+1)$$

for some $C = C(p) > 0$.

The Lipschitz estimate (2-7) is more delicate and is one of the main outcomes of [Bousquet et al. 2016]. Indeed, we know from Proposition 4.1 of that paper that there exists $C=C(p)>0$ such that for every $B_r \Subset B_R \Subset B$

$$\|u^\varepsilon_{x_i}\|_{L^\infty(B_r)} \le C\left(\frac{R}{R-r}\right)^8 \left[\fint_{B_R} |\nabla u^\varepsilon|^p\,dx + 1\right]^{2+\frac{1}{p}}, \quad i=1,2. \tag{2-9}$$

With the notation introduced in [Bousquet et al. 2016], this corresponds to the particular case $\delta_1=\delta_2=0$ and $f=0$ there. By combining this with (2-5), we get (2-7).

We now prove the $W^{1,2}$ estimate for the nonlinear function of ∇u^ε. We take $\eta\in C_0^\infty(B)$ a standard cut-off function such that

$$0\le\eta\le 1, \qquad \eta\equiv 1 \quad \text{on } B_r, \qquad \eta\equiv 0 \quad \text{on } \mathbb{R}^2\setminus B_R, \qquad |\nabla\eta| \le \frac{C}{R-r}.$$

Then we test (2-4) against $\varphi = u^\varepsilon_{x_j}\,\eta^2$. With standard manipulations, we get the Caccioppoli inequality

$$\sum_{i=1}^{2}\int (|u^\varepsilon_{x_i}|^{p-2}+\varepsilon)\,|u^\varepsilon_{x_i x_j}|^2\,\eta^2\,dx \le C\sum_{i=1}^{2}\int (|u^\varepsilon_{x_i}|^{p-2}+\varepsilon)\,|u^\varepsilon_{x_j}|^2\,|\eta_{x_i}|^2\,dx.$$

By dropping the term containing ε on the left and observing that

$$|u^\varepsilon_{x_i}|^{p-2}\,|u^\varepsilon_{x_i x_j}|^2 = \frac{4}{p^2}\,\Big|(|u^\varepsilon_{x_i}|^{\frac{p-2}{2}}\,u^\varepsilon_{x_i})_{x_j}\Big|^2,$$

we get

$$\sum_{i=1}^{2}\int_{B_r}\Big|(|u^\varepsilon_{x_i}|^{\frac{p-2}{2}}\,u^\varepsilon_{x_i})_{x_j}\Big|^2\,dx \le \frac{C}{(R-r)^2}\sum_{i=1}^{2}\int_{B_R}(|u^\varepsilon_{x_i}|^{p-2}+\varepsilon)\,|u^\varepsilon_{x_j}|^2\,dx, \tag{2-10}$$

where we used the properties of η. In order to conclude, it is sufficient to use again (2-5). □

From the bounds obtained in Lemma 2.2, we can deduce the following convergence result.

Proposition 2.3 (convergence). *With the notation above, for every $B_r \Subset B$ we have:*

(i) *$\{u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ converges uniformly to U on $\overline{B}_r$.*

(ii) *$\{|u^\varepsilon_{x_i}|^{\frac{p-2}{2}}\,u^\varepsilon_{x_i}\}_{0<\varepsilon<\varepsilon_0}$ converges to $|U_{x_i}|^{\frac{p-2}{2}}\,U_{x_i}$ weakly in $W^{1,2}(B_r)$ and strongly in $L^2(B_r)$. In particular, we have*

$$|U_{x_i}|^{\frac{p-2}{2}}\,U_{x_i} \in W^{1,2}(B_r).$$

(iii) *$\{\nabla u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ converges to ∇U strongly in $L^p(B_r)$.*

Proof. We already know from Lemma 2.1 that u^ε converges to U weakly in $W^{1,p}(B)$ and strongly in $L^p(B)$.

In view of (2-6) and (2-7), the Arzelà–Ascoli theorem implies that the convergence is indeed uniform on $\overline{B}_r$ for every $B_r \Subset B$.

By (2-8), there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\varepsilon_0)$ such that

$$\{|u^{\varepsilon_k}_{x_i}|^{\frac{p-2}{2}}\,u^{\varepsilon_k}_{x_i}\}_{k\in\mathbb{N}}, \quad i=1,2,$$

converges to some function $V_i \in W^{1,2}(B_r)$ weakly in $W^{1,2}(B_r)$ and strongly in $L^2(B_r)$. In particular, this is a Cauchy sequence in $L^2(B_r)$. By using the elementary inequality

$$|t-s|^p \le C\,\Big||t|^{\frac{p-2}{2}}\,t - |s|^{\frac{p-2}{2}}\,s\Big|^2, \quad t,s\in\mathbb{R},$$

where $C>0$ depends only on p, we obtain that $\{u^{\varepsilon_k}_{x_i}\}_{k\in\mathbb{N}}$ is a Cauchy sequence as well, this time in $L^p(B_r)$. This implies

$$\lim_{k\to+\infty}\|\nabla u^{\varepsilon_k}-\nabla U\|_{L^p(B_r)} = 0.$$

We now prove that $V_i = |U_{x_i}|^{\frac{p-2}{2}}\,U_{x_i}$. We use the elementary inequality

$$\Big||t|^{\frac{p-2}{2}}\,t-|s|^{\frac{p-2}{2}}\,s\Big| \le C\,\big(|t|^{\frac{p-2}{2}}+|s|^{\frac{p-2}{2}}\big)|t-s|, \quad t,s\in\mathbb{R},$$

valid for some $C=C(p)>0$. Then we obtain

$$\begin{aligned}\int_{B_r}\Big||u^{\varepsilon_k}_{x_i}|^{\frac{p-2}{2}}\,u^{\varepsilon_k}_{x_i}-|U_{x_i}|^{\frac{p-2}{2}}\,U_{x_i}\Big|^2\,dx &\le C\int_{B_r}\big(|u^{\varepsilon_k}_{x_i}|^{\frac{p-2}{2}}+|U_{x_i}|^{\frac{p-2}{2}}\big)^2\,|u^{\varepsilon_k}_{x_i}-U_{x_i}|^2\,dx\\ &\le C\bigg(\int_{B_r}\big(|u^{\varepsilon_k}_{x_i}|^{\frac{p-2}{2}}+|U_{x_i}|^{\frac{p-2}{2}}\big)^{\frac{2p}{p-2}}\,dx\bigg)^{\frac{p-2}{p}}\bigg(\int_{B_r}|u^{\varepsilon_k}_{x_i}-U_{x_i}|^p\,dx\bigg)^{\frac{2}{p}}.\end{aligned}$$

By using the strong convergence of the gradients proved above, this implies that $V_i = |U_{x_i}|^{\frac{p-2}{2}} U_{x_i}$. Since the above argument can be repeated for every subsequence of $\{u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$, it follows from the uniqueness of the limit that the convergence holds true for the whole family $\{u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$, both in (ii) and (iii). □

From the convergence results stated in the above proposition, we can obtain some regularity properties for the local minimizer U, which we state in the following theorem. These properties, which come with local scaling-invariant a priori estimates, have already been established in [Bousquet et al. 2016; Brasco and Carlier 2013; Fonseca and Fusco 1997].

Theorem 2.4 (a priori estimates, $p > 2$). *Every local minimizer $U \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of the functional $\mathfrak{F}$ is a locally Lipschitz function, such that for every $\alpha \geq \frac{p}{2}$ we have*

$$|U_{x_i}|^{\alpha-1}\, U_{x_i} \in W^{1,2}_{\mathrm{loc}}(\Omega), \quad i = 1, 2.$$

Moreover, for every $B_R \Subset \Omega$ we have

$$\|U_{x_i}\|_{L^\infty(B_{R/2})} \leq C \left(\fint_{B_R} |\nabla U|^p\, dx \right)^{\frac{1}{p}}, \qquad i = 1, 2, \tag{2-11}$$

$$\int_{B_{R/2}} \left|\nabla(|U_{x_i}|^{\alpha-1}\, U_{x_i})\right|^2 dx \leq C\, \alpha^2 \left(\fint_{B_R} |\nabla U|^p\, dx \right)^{\frac{2\alpha}{p}}, \quad i = 1, 2, \tag{2-12}$$

for some $C(p) > 0$.

Proof. Let us prove the estimates (2-11) and (2-12). By taking the limit as ε goes to 0 in (2-9) and using the convergence result of Proposition 2.3, we obtain

$$\|U_{x_i}\|_{L^\infty(B_{R/2})} \leq C \left[\fint_{B_R} |\nabla U|^p\, dx + 1 \right]^{2+\frac{1}{p}}, \quad i = 1, 2.$$

In order to obtain (2-11), it is sufficient to observe that if U is a local minimizer of $\mathfrak{F}$, then for every $\lambda > 0$ the function λU is still a local minimizer of the same functional. Thus the previous Lipschitz estimate holds true; i.e.,

$$\lambda\, \|U_{x_i}\|_{L^\infty(B_{R/2})} \leq C \left[\lambda^p \fint_{B_R} |\nabla U|^p\, dx + 1 \right]^{2+\frac{1}{p}}, \quad i = 1, 2.$$

This can be rewritten as

$$\lambda^{\frac{p}{2p+1}}\, \|U_{x_i}\|^{\frac{p}{2p+1}}_{L^\infty(B_{R/2})} - C\, \lambda^p \fint_{B_R} |\nabla U|^p\, dx \leq C, \quad i = 1, 2,$$

for a different constant $C = C(p) > 0$. If we now maximize the left-hand side with respect to $\lambda > 0$, we get (2-11) as desired.

We already know from Proposition 2.3 that $|U_{x_i}|^{\frac{p-2}{2}} U_{x_i} \in W^{1,2}_{\mathrm{loc}}(\Omega)$. By passing to the limit in (2-10) and using the convergences at our disposal from Proposition 2.3, we obtain

$$\int_{B_{R/2}} \left|\nabla(|U_{x_i}|^{\frac{p-2}{2}}\, U_{x_i})\right|^2 dx \leq \frac{C}{R^2} \int_{B_R} |\nabla U|^p\, dx,$$

which is (2-12) for $\alpha = \frac{p}{2}$. In order to prove (2-12) for a general $\alpha > \frac{p}{2}$, it is sufficient to observe that

$$|U_{x_i}|^{\alpha-1}\, U_{x_i} = \left||U_{x_i}|^{\frac{p-2}{2}}\, U_{x_i}\right|^{\frac{2}{p}\alpha-1} |U_{x_i}|^{\frac{p-2}{2}}\, U_{x_i}, \tag{2-13}$$

and the function $t \mapsto |t|^{\frac{2\alpha-p}{p}}\, t$ is C^1. By using that

$$|U_{x_i}|^{\frac{p-2}{2}}\, U_{x_i} \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega),$$

we get that $|U_{x_i}|^{\alpha-1}\, U_{x_i} \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ as well. Finally, to prove the estimate, we observe that (2-13) implies

$$\int_{B_{R/2}} \left|\nabla(|U_{x_i}|^{\alpha-1}\, U_{x_i})\right|^2 dx \le C\, \alpha^2\, \|U_{x_i}\|^{2\alpha-p}_{L^\infty(B_{R/2})} \int_{B_{R/2}} \left|\nabla(|U_{x_i}|^{\frac{p-2}{2}}\, U_{x_i})\right|^2 dx.$$

By using (2-11) and (2-12) for $\alpha = \frac{p}{2}$, we get the desired conclusion. □

We proceed with a technical result which will be needed to handle the case $p > 2$.

Lemma 2.5. *Let $p > 2$ and let $U \in W^{1,p}_{\mathrm{loc}}(\Omega)$ still denote a local minimizer of $\mathfrak{F}$. Let $\beta \in \mathbb{R}$ and set*

$$F(t) = \frac{p}{2} \int_\beta^t |s|^{\frac{p-2}{2}}\, (s-\beta)_+\, ds, \quad t \in \mathbb{R}.$$

Then $F(U_{x_j}) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and we have

$$(|U_{x_j}|^{\frac{p-2}{2}}\, U_{x_j})_{x_k}\, (U_{x_j} - \beta)_+ = (F(U_{x_j}))_{x_k} \quad \textit{a.e. in } \Omega. \tag{2-14}$$

Proof. In order to prove that $F(U_{x_j}) \in W^{1,2}_{\mathrm{loc}}(\Omega)$, we can observe that if we introduce the function

$$G(t) = F(|t|^{\frac{2-p}{p}}\, t) = \frac{p}{2} \int_\beta^{|t|^{(2-p)/p}\, t} |s|^{\frac{p-2}{2}}\, (s-\beta)_+\, ds,$$

then we have

$$F(U_{x_j}) = G(|U_{x_j}|^{\frac{p-2}{2}}\, U_{x_j}). \tag{2-15}$$

With the simple change of variable $\tau = |s|^{\frac{p-2}{2}}\, s$, the function G can be rewritten as

$$G(t) = \int_{|\beta|^{(p-2)/2}\,\beta}^t (|\tau|^{\frac{2-p}{p}}\, \tau - \beta)_+\, d\tau.$$

Hence, G is a C^1 function. By using Theorem 2.4 and (2-15), we thus get that $F(U_{x_j}) \in W^{1,2}_{\mathrm{loc}}(\Omega)$.

In order to prove (2-14), we use the approximation scheme introduced in this section. For every $\varepsilon > 0$, thanks to the smoothness of u^ε, we have

$$(|u^\varepsilon_{x_j}|^{\frac{p-2}{2}}\, u^\varepsilon_{x_j})_{x_k}\, (u^\varepsilon_{x_j} - \beta)_+ = (F(u^\varepsilon_{x_j}))_{x_k}. \tag{2-16}$$

By Proposition 2.3, we know that ∇u^ε converges to ∇U strongly in $L^p(B_r)$ and

$$|u^\varepsilon_{x_j}|^{\frac{p-2}{2}}\, u^\varepsilon_{x_j} \ \text{ weakly converges in } W^{1,2}(B_r) \text{ to } \ |U_{x_j}|^{\frac{p-2}{2}}\, U_{x_j}.$$

This implies that the left-hand side of (2-16) converges weakly in $L^1(B_r)$ to the left-hand side of (2-14).

By using the uniform bounds of Lemma 2.2, the local Lipschitz character of G and the relation (2-15), we get

$$\int_{B_r} |\nabla F(u^\varepsilon_{x_j})|^2\,dx = \int_{B_r} \big|\nabla G(|u^\varepsilon_{x_j}|^{\frac{p-2}{2}}\,u^\varepsilon_{x_j})\big|^2\,dx \le C\int_{B_r} \big|\nabla(|u^\varepsilon_{x_j}|^{\frac{p-2}{2}}\,u^\varepsilon_{x_j})\big|^2\,dx \le C,$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{B_r} \big|F(U_{x_j}) - F(u^\varepsilon_{x_j})\big|^2\,dx &= \lim_{\varepsilon\to 0}\int_{B_r} \big|G(|U_{x_j}|^{\frac{p-2}{2}}\,U_{x_j}) - G(|u^\varepsilon_{x_j}|^{\frac{p-2}{2}}\,u^\varepsilon_{x_j})\big|^2\,dx\\ &\le C\,\lim_{\varepsilon\to 0}\int_{B_r} \big||U_{x_j}|^{\frac{p-2}{2}}\,U_{x_j} - |u^\varepsilon_{x_j}|^{\frac{p-2}{2}}\,u^\varepsilon_{x_j}\big|^2\,dx = 0,\end{aligned}$$

where we used Proposition 2.3 for the last limit. We thus obtain that $F(u^\varepsilon_{x_j})$ converges weakly in $W^{1,2}(B_r)$ and strongly in $L^2(B_r)$ to $F(U_{x_j})$. We can then pass to the limit in the right-hand side of (2-16). □

We end this subsection with two results on the solutions u^ε of the problem (2-3). The first one is a standard minimum principle.

Lemma 2.6 (a minimum principle, $p > 2$). *With the notation above, let $B_r \Subset B$. We have*

$$|u^\varepsilon_{x_j}|^{\frac{p-2}{2}}\,u^\varepsilon_{x_j} \ge C \quad \text{on } \partial B_r \quad\Longleftrightarrow\quad |u^\varepsilon_{x_j}|^{\frac{p-2}{2}}\,u^\varepsilon_{x_j} \ge C \quad \text{in } B_r.$$

Proof. In the differentiated equation (2-4) we insert the test function

$$\Phi = \begin{cases}(C - |u^\varepsilon_{x_j}|^{\frac{p-2}{2}}\,u^\varepsilon_{x_j})_+ & \text{in } B_r,\\ 0 & \text{in } B\setminus B_r,\end{cases}$$

which is admissible thanks to the hypothesis. Observe that

$$|u^\varepsilon_{x_j}|^{\frac{p-2}{2}}\,u^\varepsilon_{x_j} \le C \quad\Longleftrightarrow\quad u^\varepsilon_{x_j} \le |C|^{\frac{2-p}{p}}\,C; \tag{2-17}$$

thus we obtain

$$\sum_{i=1}^{2}\int_{\{x\in B_r : u^\varepsilon_{x_j} \le |C|^{(2-p)/p}\,C\}} (|u^\varepsilon_{x_i}|^{p-2} + \varepsilon)\,|u^\varepsilon_{x_j}|^{\frac{p-2}{2}}\,|u^\varepsilon_{x_i x_j}|^2\,dx = 0.$$

Observe that the two terms are nonnegative; thus for $i = j$ we can also infer

$$\begin{aligned}0 &= \int_{\{x\in B_r : u^\varepsilon_{x_j} \le |C|^{(2-p)/p}\,C\}} |u^\varepsilon_{x_j}|^{\frac{3}{2}(p-2)}\,|u^\varepsilon_{x_j x_j}|^2\,dx\\ &= \left(\frac{4}{3p-2}\right)^2 \int_{\{x\in B_r : u^\varepsilon_{x_j} \le |C|^{(2-p)/p}\,C\}} \big|\big(|u^\varepsilon_{x_j}|^{\frac{3}{4}(p-2)}\,u^\varepsilon_{x_j}\big)_{x_j}\big|^2\,dx\\ &= \left(\frac{4}{3p-2}\right)^2 \int_{B_r} \big|\big(\min\{|u^\varepsilon_{x_j}|^{\frac{3}{4}(p-2)}\,u^\varepsilon_{x_j},\ |C|^{\frac{p-2}{2p}}\,C\}\big)_{x_j}\big|^2\,dx,\end{aligned}$$

where we used that

$$u^\varepsilon_{x_j} \le |C|^{\frac{2-p}{p}}\,C \quad\Longleftrightarrow\quad |u^\varepsilon_{x_j}|^{\frac{3}{4}(p-2)}\,u^\varepsilon_{x_j} \le |C|^{\frac{p-2}{2p}}\,C. \tag{2-18}$$

This gives

$$\left(\min\left\{|u^\varepsilon_{x_j}|^{\frac{3}{4}(p-2)}\, u^\varepsilon_{x_j},\ |C|^{\frac{p-2}{2p}}\, C\right\}\right)_{x_j} = 0 \quad \text{a.e. in } B_r,$$

so that the Sobolev function

$$\min\left\{|u^\varepsilon_{x_j}|^{\frac{3}{4}(p-2)}\, u^\varepsilon_{x_j},\ |C|^{\frac{p-2}{2p}}\, C\right\}$$

does not depend on the variable x_j in B_r. By assumption, this function is constant on ∂B_r. The last two facts imply

$$\min\left\{|u^\varepsilon_{x_j}|^{\frac{3}{4}(p-2)}\, u^\varepsilon_{x_j},\ |C|^{\frac{p-2}{2p}}\, C\right\} = |C|^{\frac{p-2}{2p}}\, C \quad \text{a.e. in } B_r,$$

which is the desired conclusion, thanks to (2-17) and (2-18). □

Finally, we will need the following result about convergence of traces.

Lemma 2.7. *Let $B_r \Subset B$. With the notation above, there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, \varepsilon_0)$ such that for almost every $s \in [0, r]$, we have*

$$\lim_{k\to+\infty} \left\| |u^{\varepsilon_k}_{x_j}|^{\frac{p-2}{2}}\, u^{\varepsilon_k}_{x_j} - |U_{x_j}|^{\frac{p-2}{2}}\, U_{x_j} \right\|_{L^\infty(\partial B_s)} = 0, \quad j = 1, 2.$$

Proof. We first observe that

$$\left\{|u^\varepsilon_{x_j}|^{\frac{p-2}{2}}\, u^\varepsilon_{x_j} - |U_{x_j}|^{\frac{p-2}{2}}\, U_{x_j}\right\}_{0<\varepsilon<\varepsilon_0}$$

weakly converges to 0 in $W^{1,2}(B_r)$, thanks to Proposition 2.3. Thus for every $0 < \tau < 1$, there exists a subsequence which strongly converges to 0 in the fractional Sobolev space $W^{\tau,2}(B_r)$. We take $\frac{1}{2} < \tau < 1$ and observe that the previous convergence implies that we can extract again a subsequence which strongly converges to 0 in $W^{\tau,2}(\partial B_s)$ for almost every $s \in [0, r]$ (see Lemma B.2). In order to conclude, it is now sufficient to use that for $\frac{1}{2} < \tau < 1$, the space $W^{\tau,2}(\partial B_s)$ is continuously embedded in $C^0(\partial B_s)$, since ∂B_s is one-dimensional; see [Adams 1975, Theorem 7.57]. □

2C. ***Regularization scheme, case* $1 < p \le 2$.** In this case, the functional in (2-3) is not smooth enough, in particular is not C^2. Thus the regularized problem is now

$$\min\left\{\sum_{i=1}^{2} \frac{1}{p} \int_B (\varepsilon + |w_{x_i}|^2)^{\frac{p}{2}} : w - U^\varepsilon \in W^{1,p}_0(B)\right\}. \tag{2-19}$$

This problem admits a unique solution u^ε, which is C^2 on $\overline{B}$; see again [Bousquet et al. 2016, Theorem 2.4] and [Giusti 2003, Theorems 8.6 and 10.18]. Moreover, the solution u^ε satisfies the corresponding Euler–Lagrange equation; i.e.,

$$\sum_{i=1}^{2} \int_B (\varepsilon + |u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\, u^\varepsilon_{x_i}\, \varphi_{x_i}\, dx = 0 \quad \text{for every } \varphi \in W^{1,p}_0(B). \tag{2-20}$$

We still have the following uniform estimate. The proof is standard routine and is left to the reader.

Lemma 2.8 (uniform energy estimate). *There exists a constant $C = C(p) > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ the following estimate holds*

$$\int_B |\nabla u^\varepsilon|^p\,dx \le C\left(\int_{\lambda_B\,B} |\nabla U|^p\,dx + \varepsilon^{\frac{p}{2}}\,|B|\right). \tag{2-21}$$

Moreover, the family $\{u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ converges weakly in $W^{1,p}(B)$ and strongly in $L^p(B)$ to U.

We will rely on the following Caccioppoli inequality to obtain certain bounds on the family $\{u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$.

Proposition 2.9 (Caccioppoli inequality for the gradient, $1 < p \le 2$). *Let $\zeta : \mathbb{R} \to \mathbb{R}$ be a C^1 monotone function; then for every $\eta \in C^2$ with compact support in B we have*

$$\begin{aligned}\sum_{i=1}^{2} \int (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,|\zeta'(u^\varepsilon_{x_j})|\,|u^\varepsilon_{x_j\,x_i}|^2\,\eta^2\,dx \\ \le C\int (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p}{2}}\,|\zeta'(u^\varepsilon_{x_j})|\,|\nabla\eta|^2\,dx + C\int (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p-1}{2}}\,|\zeta(u^\varepsilon_{x_j})|\,\big(|\nabla\eta|^2+|\eta|\,|D^2\eta|\big)\,dx\end{aligned} \tag{2-22}$$

for some $C = C(p) > 0$.

Proof. Suppose $\zeta \in C^2$; then the general result can be obtained by a standard approximation argument. To obtain (2-22), we use a trick by Fonseca and Fusco [1997] to avoid using the upper bound on the Hessian of

$$H_\varepsilon(z) := \sum_{i=1}^{2} \frac{1}{p}\,(\varepsilon + |z_i|^2)^{\frac{p}{2}}, \qquad z \in \mathbb{R}^2;$$

see also [Esposito and Mingione 1998; Fonseca et al. 2002].

We start by testing (2-20) against $\varphi = (\zeta(u^\varepsilon_{x_j})\,\eta^2)_{x_j}$. Thus we get

$$\sum_{i=1}^{2} \int_B (\varepsilon + |u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,u^\varepsilon_{x_i}\,(\zeta(u^\varepsilon_{x_j})\,\eta^2)_{x_j\,x_i}\,dx = 0.$$

By using the smoothness of u^ε and η, we have

$$\begin{aligned}(\zeta(u^\varepsilon_{x_j})\,\eta^2)_{x_j\,x_i} &= (\zeta(u^\varepsilon_{x_j})\,\eta^2)_{x_i\,x_j} = \big(\zeta'(u^\varepsilon_{x_j})\,u^\varepsilon_{x_j\,x_i}\,\eta^2 + 2\,\zeta(u^\varepsilon_{x_j})\,\eta\,\eta_{x_i}\big)_{x_j} \\ &= \big(\zeta'(u^\varepsilon_{x_j})\,u^\varepsilon_{x_j\,x_i}\,\eta^2\big)_{x_j} + 2\,\big(\zeta(u^\varepsilon_{x_j})\,\eta\,\eta_{x_i}\big)_{x_j}.\end{aligned}$$

By using an integration by parts, we thus obtain

$$-\sum_{i=1}^{2} \int_B \big((\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,u^\varepsilon_{x_i}\big)_{x_j}\,\zeta'(u^\varepsilon_{x_j})\,u^\varepsilon_{x_j\,x_i}\,\eta^2\,dx + 2\sum_{i=1}^{2} \int_B (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,u^\varepsilon_{x_i}\,(\zeta(u^\varepsilon_{x_j})\,\eta\,\eta_{x_i})_{x_j}\,dx = 0.$$

With simple manipulations, this becomes

$$\begin{aligned}&\sum_{i=1}^{2} \int_B (\varepsilon + |u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,\zeta'(u^\varepsilon_{x_j})\,|u^\varepsilon_{x_j\,x_i}|^2\,\eta^2\,dx \\ &\qquad + (p-2)\sum_{i=1}^{2} \int_B (\varepsilon + |u^\varepsilon_{x_i}|^2)^{\frac{p-4}{2}}\,|u^\varepsilon_{x_i}|^2\,\zeta'(u^\varepsilon_{x_j})\,|u^\varepsilon_{x_j\,x_i}|^2\,\eta^2\,dx \\ &= 2\sum_{i=1}^{2} \int_B (\varepsilon + |u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,u^\varepsilon_{x_i}\,\zeta'(u^\varepsilon_{x_j})\,u^\varepsilon_{x_j\,x_j}\,\eta\,\eta_{x_i}\,dx \\ &\qquad + 2\sum_{i=1}^{2} \int_B (\varepsilon + |u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,u^\varepsilon_{x_i}\,\zeta(u^\varepsilon_{x_j})\,(\eta\,\eta_{x_i})_{x_j}\,dx.\end{aligned} \tag{2-23}$$

We now observe that

$$\sum_{i=1}^{2}\int_B (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,\zeta'(u^\varepsilon_{x_j})\,|u^\varepsilon_{x_j\,x_i}|^2\,\eta^2\,dx + (p-2)\sum_{i=1}^{2}\int_B (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-4}{2}}\,|u^\varepsilon_{x_i}|^2\,\zeta'(u^\varepsilon_{x_j})\,|u^\varepsilon_{x_j\,x_i}|^2\,\eta^2\,dx$$

$$=\sum_{i=1}^{2}\int_B (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-4}{2}}\,(\varepsilon+(p-1)\,|u^\varepsilon_{x_i}|^2)\,\zeta'(u^\varepsilon_{x_j})\,|u^\varepsilon_{x_j\,x_i}|^2\,\eta^2\,dx$$

so that the left-hand side of (2-23) has a sign. Thus we obtain[2]

$$\begin{aligned}\sum_{i=1}^{2}\int_B (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-4}{2}}\,(\varepsilon+(p-1)\,|u^\varepsilon_{x_i}|^2)\,|\zeta'(u^\varepsilon_{x_j})|\,|u^\varepsilon_{x_j\,x_i}|^2\,\eta^2\,dx \\ \le 2\sum_{i=1}^{2}\int_B (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,|u^\varepsilon_{x_i}|\,|\zeta'(u^\varepsilon_{x_j})|\,|u^\varepsilon_{x_j\,x_j}|\,\eta\,|\eta_{x_i}|\,dx \\ +2\sum_{i=1}^{2}\int_B (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,|u^\varepsilon_{x_i}|\,|\zeta(u^\varepsilon_{x_j})|\,|(\eta\,\eta_{x_i})_{x_j}|\,dx.\end{aligned}\tag{2-24}$$

We now estimate the left-hand side of (2-24) from below

$$\begin{aligned}\sum_{i=1}^{2}\int_B (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-4}{2}}\,(\varepsilon+(p-1)\,|u^\varepsilon_{x_i}|^2)\,|\zeta'(u^\varepsilon_{x_j})|\,|u^\varepsilon_{x_j\,x_i}|^2\,\eta^2\,dx \\ \ge (p-1)\sum_{i=1}^{2}\int_B (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,|\zeta'(u^\varepsilon_{x_j})|\,|u^\varepsilon_{x_j\,x_i}|^2\,\eta^2\,dx \\ \ge \frac{p-1}{2}\sum_{i=1}^{2}\int_B (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,|\zeta'(u^\varepsilon_{x_j})|\,|u^\varepsilon_{x_j\,x_i}|^2\,\eta^2\,dx \\ +\frac{p-1}{2}\sum_{i=1}^{2}\int_B (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p-2}{2}}\,|\zeta'(u^\varepsilon_{x_j})|\,|u^\varepsilon_{x_j\,x_i}|^2\,\eta^2\,dx,\end{aligned}$$

where we used that $p-2<0$. We will use the last term as a *sponge term* in order to absorb the second derivatives of u^ε contained in the right-hand side.

As for the first term in the right-hand side of (2-24),

$$\begin{aligned}\int_B (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,|u^\varepsilon_{x_i}|\,|\zeta'(u^\varepsilon_{x_j})|\,|u^\varepsilon_{x_j\,x_j}|\,\eta\,|\eta_{x_i}|\,dx \\ \le \int_B (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-1}{2}}\,|\zeta'(u^\varepsilon_{x_j})|\,|u^\varepsilon_{x_j\,x_j}|\,\eta\,|\eta_{x_i}|\,dx \\ \le \int_B (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p-1}{2}}\,|\zeta'(u^\varepsilon_{x_j})|\,|u^\varepsilon_{x_j\,x_j}|\,\eta\,|\eta_{x_i}|\,dx \\ \le \frac{1}{2\tau}\int_{B_R} (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p}{2}}\,|\zeta'(u^\varepsilon_{x_j})|\,|\nabla\eta|^2\,dx+\frac{\tau}{2}\int_B (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p-2}{2}}\,|\zeta'(u^\varepsilon_{x_j})|\,|u^\varepsilon_{x_j\,x_j}|^2\,\eta^2\,dx.\end{aligned}$$

[2] Recall that by hypothesis, ζ' has constant sign.

Also, for the last term of (2-24), we simply get

$$\int_B (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,|u^\varepsilon_{x_i}|\,|\zeta(u^\varepsilon_{x_j})|\,|(\eta\,\eta_{x_i})_{x_j}|\,dx \le \int_{B_R} (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p-1}{2}}\,|\zeta(u^\varepsilon_{x_j})|\,\big(|\nabla\eta|^2+|\eta|\,|D^2\eta|\big)\,dx.$$

By using these estimates in (2-24) and taking $\tau = \frac{p-1}{2}$ in order to absorb the Hessian term on the right-hand side, we obtain

$$\sum_{i=1}^2 \int_B (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,|\zeta'(u^\varepsilon_{x_j})|\,|u^\varepsilon_{x_j\,x_i}|^2\,\eta^2\,dx$$
$$\le C\int_{B_R} (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p}{2}}\,|\zeta'(u^\varepsilon_{x_j})|\,|\nabla\eta|^2\,dx + C\int_{B_R} (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p-1}{2}}\,|\zeta(u^\varepsilon_{x_j})|\,\big(|\nabla\eta|^2+|\eta|\,|D^2\eta|\big)\,dx, \quad (2\text{-}25)$$

which is exactly (2-22). □

We now collect some bounds on the family $\{u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$.

Lemma 2.10 (uniform estimates, $1 < p \le 2$). *Let $1 < p \le 2$; then for every $B_r \Subset B$ we have*

$$\|u^\varepsilon\|_{L^\infty(B_r)} + \|\nabla u^\varepsilon\|_{L^\infty(B_r)} \le C, \tag{2-26}$$

$$\sum_{i=1}^2 \int_{B_r} (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,|u^\varepsilon_{x_i\,x_j}|^2 \le C, \quad j=1,2, \tag{2-27}$$

$$\int_{B_r} |\nabla u^\varepsilon_{x_j}|^2\,dx \le C, \quad j=1,2, \tag{2-28}$$

for some $C>0$ independent of ε.

Proof. The L^∞ estimate can be found in [Giusti 2003, Chapter 7] again, while the Lipschitz estimate follows from [Fonseca and Fusco 1997, Theorem 2.2]. More precisely, for every ball B_s such that $B_{2s} \Subset B$,

$$\sup_{B_s} (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p}{2}}\,dx \le C \fint_{B_{2s}} (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p}{2}}\,dx. \tag{2-29}$$

By covering a given ball $B_r \Subset B$ with a finite number of balls B_s such that $B_{2s} \Subset B$ and using the bound on the L^p norm of ∇u^ε, one easily gets the Lipschitz estimate in (2-26) for some constant $C>0$ which may depend on B_r but not on ε.

In order to prove (2-27), we introduce two balls $B_r \Subset B_R \Subset B$ and a standard cut-off function $\eta \in C^2$ such that

$$0\le\eta\le 1, \qquad \eta\equiv 1 \quad \text{on } B_r, \qquad \eta\equiv 0 \quad \text{on } \mathbb{R}^2\setminus B_R, \qquad |\nabla\eta| \le \frac{C}{R-r}, \qquad |D^2\eta| \le \frac{C}{(R-r)^2}.$$

By taking $\zeta(t)=t$ in (2-22), one gets

$$\sum_{i=1}^2 \int (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,|u^\varepsilon_{x_j\,x_i}|^2\,\eta^2\,dx$$
$$\le C\int (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p}{2}}\,|\nabla\eta|^2\,dx + C\int (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p-1}{2}}\,|u^\varepsilon_{x_j}|\,\big(|\nabla\eta|^2+|D^2\eta|\big)\,dx. \quad (2\text{-}30)$$

By recalling the uniform bound on the L^p norm of ∇u^ε, (2-30) gives (2-27).

We now observe that

$$\sum_{i=1}^{2}\int_B (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,|u^\varepsilon_{x_j\,x_i}|^2\,\eta^2\,dx \ge \sum_{i=1}^{2}\int_B (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p-2}{2}}\,|u^\varepsilon_{x_j\,x_i}|^2\,\eta^2\,dx$$
$$\ge \sum_{i=1}^{2}(\varepsilon+\|\nabla u^\varepsilon\|^2_{L^\infty(B_R)})^{\frac{p-2}{2}}\int_{B_r}|u^\varepsilon_{x_j\,x_i}|^2\,dx.$$

By appealing to (2-30), this yields

$$\int_{B_r}|u^\varepsilon_{x_j\,x_i}|^2\,dx \le \frac{C}{(R-r)^2}\,(\varepsilon+\|\nabla u^\varepsilon\|^2_{L^\infty(B_R)})^{\frac{2-p}{2}}\int_{B_R}(\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p}{2}}\,dx.$$

In order to conclude, it is sufficient to use (2-26) for the ball $B_R \Subset B$ and again the uniform estimate on the L^p norm of ∇u^ε. □

Proposition 2.11. *With the notation above, for every $B_r \Subset B$, we have:*

(1) $\{u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ *converges uniformly to U on $\overline{B}_r$.*

(2) $\{\nabla u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ *converges to ∇U weakly in $W^{1,2}(B_r)$ and strongly in $L^2(B_r)$. In particular, we have*

$$U_{x_i}\in W^{1,2}(B_r).$$

(3) $\big\{(\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{4}}\,u^\varepsilon_{x_i}\big\}_{0<\varepsilon<\varepsilon_0}$ *converges to $|U_{x_i}|^{\frac{p-2}{2}}\,U_{x_i}$ weakly in $W^{1,2}(B_r)$ and strongly in $L^{\frac{4}{p}}(B_r)$. In particular, we have*

$$|U_{x_i}|^{\frac{p-2}{2}}\,U_{x_i}\in W^{1,2}(B_r).$$

Proof. We already know from Lemma 2.8 that u^ε converges to U weakly in $W^{1,p}(B)$ and strongly in $L^p(B)$.

By (2-26) and the Arzelà–Ascoli theorem, the convergence of $\{u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ to U is uniform on $\overline{B}_r$ for every $B_r \Subset B$.

From estimates (2-26) and (2-28), we get that $\{u^\varepsilon_{x_i}\}_{0<\varepsilon<\varepsilon_0}$ is uniformly bounded in $W^{1,2}(B_r)$. By the Rellich–Kondrašov theorem, we can infer strong convergence in $L^2(B_r)$ to U_{x_i} for every $i=1,2$.

We now observe that

$$\big|\nabla((\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{4}}\,u^\varepsilon_{x_i})\big|^2 = \left|\frac{p-2}{2}\,(\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-6}{4}}\,|u^\varepsilon_{x_i}|^2\,\nabla u^\varepsilon_{x_i} + (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{4}}\,\nabla u^\varepsilon_{x_i}\right|^2$$
$$= (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-6}{2}}\,|\nabla u^\varepsilon_{x_i}|^2\left|\frac{p}{2}\,|u^\varepsilon_{x_i}|^2+\varepsilon\right|^2$$
$$\le (\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,|\nabla u^\varepsilon_{x_i}|^2,$$

where we used that $1<p\le 2$. By (2-27), this implies

$$\big\{(\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{4}}\,u^\varepsilon_{x_i}\big\}_{0<\varepsilon<\varepsilon_0},\quad i=1,2, \tag{2-31}$$

is bounded in $W^{1,2}(B_r)$. Again by the Rellich–Kondrašov theorem we can assume that, up to a subsequence (we do not relabel), it converges to some function $V_i\in W^{1,2}(B_r)$ weakly in $W^{1,2}(B_r)$ and strongly in $L^2(B_r)$. We now show at the same time that $V_i = |U_{x_i}|^{\frac{p-2}{2}} U_{x_i}$ and that actually we have strong convergence in $L^{\frac{4}{p}}(B_r)$. Indeed, by using the elementary inequality of Corollary A.3, we obtain

$$\begin{aligned}\int_{B_r} \Big|(\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{4}}\, u^\varepsilon_{x_i} - |U_{x_i}|^{\frac{p-2}{2}}\, U_{x_i}\Big|^{\frac{4}{p}}\,dx &\\ \le C\int_{B_r}\Big|(\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{4}}\, u^\varepsilon_{x_i} - (\varepsilon+|U_{x_i}|^2)^{\frac{p-2}{4}}\, U_{x_i}\Big|^{\frac{4}{p}}\,dx &\\ + C\int_{B_r}\Big|(\varepsilon+|U_{x_i}|^2)^{\frac{p-2}{4}}\, U_{x_i} - |U_{x_i}|^{\frac{p-2}{2}}\, U_{x_i}\Big|^{\frac{4}{p}}\,dx &\\ \le C\int_{B_r}|u^\varepsilon_{x_i}-U_{x_i}|^2\,dx + C\int_{B_r}\Big|(\varepsilon+|U_{x_i}|^2)^{\frac{p-2}{4}}\, U_{x_i} - |U_{x_i}|^{\frac{p-2}{2}}\, U_{x_i}\Big|^{\frac{4}{p}}\,dx.&\end{aligned}$$

By using the strong convergence of the gradients proved above (for the first term) and the dominated convergence theorem (for the second one), this implies $V_i = |U_{x_i}|^{\frac{p-2}{2}} U_{x_i}$ and the convergence of the full original sequence in (2-31) weakly in $W^{1,2}(B_r)$ and strongly in $L^{\frac{4}{p}}(B_r)$. □

Using the above convergence result, one can establish the following regularity properties for the local minimizer U.

Theorem 2.12 (a priori estimates, $1 < p \le 2$). *Every local minimizer $U \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of the functional $\mathfrak{F}$ is a locally Lipschitz function such that for every $\alpha \ge \frac{p}{2}$ we have*

$$|U_{x_i}|^{\alpha-1}\, U_{x_i} \in W^{1,2}_{\mathrm{loc}}(\Omega), \quad i = 1,2.$$

In particular, we have $\nabla U \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$. Moreover, for every $B_R \Subset \Omega$, we have

$$\|U_{x_j}\|_{L^\infty(B_{R/2})} \le C_1\left(\fint_{B_R}|\nabla U|^p\,dx\right)^{\frac{1}{p}}, \quad j=1,2, \tag{2-32}$$

$$\int_{B_{R/2}}\big|\nabla(|U_{x_j}|^{\alpha-1}\,U_{x_j})\big|^2\,dx \le C_2\left(\fint_{B_R}|\nabla U|^p\,dx\right)^{\frac{2\alpha}{p}}, \quad j=1,2, \tag{2-33}$$

for some $C_1 = C_1(p) > 0$ and $C_2 = C_2(p,\alpha) > 0$.

Proof. Local Lipschitz regularity and the scaling invariant estimate (2-32) follow from [Fonseca and Fusco 1997, Theorem 2.2].

We already know from Proposition 2.11 that $|U_{x_i}|^{\frac{p-2}{2}} U_{x_i} \in W^{1,2}_{\mathrm{loc}}(\Omega)$. In order to get (2-33) for $\alpha = \frac{p}{2}$, we first recall that

$$\Big|\nabla\big((\varepsilon+|u^\varepsilon_{x_j}|^2)^{\frac{p-2}{4}}u^\varepsilon_{x_j}\big)\Big|^2 \le (\varepsilon+|u^\varepsilon_{x_j}|^2)^{\frac{p-2}{2}}|\nabla u^\varepsilon_{x_j}|^2.$$

We multiply the above inequality by the cut-off function η^2 as in (2-30), associated to the balls $B_{\frac{R}{2}} \Subset B_R$. Integrating the resulting inequality, we get

$$\int_{B_{R/2}}\Big|\nabla\big((\varepsilon+|u^\varepsilon_{x_j}|^2)^{\frac{p-2}{4}}u^\varepsilon_{x_j}\big)\Big|^2\,dx \le \int_{B_R}(\varepsilon+|u^\varepsilon_{x_j}|^2)^{\frac{p-2}{2}}|\nabla u^\varepsilon_{x_j}|^2\eta^2\,dx.$$

Using (2-30), this implies

$$\int_{B_{R/2}} \left|\nabla\big((\varepsilon+|u^\varepsilon_{x_j}|^2)^{\frac{p-2}{4}} u^\varepsilon_{x_j}\big)\right|^2 dx \le \frac{C}{R^2}\int_{B_R} (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p}{2}}\, dx.$$

By taking the limit in the previous inequality and using the convergences of Proposition 2.11, we get (2-33) for $\alpha = \frac{p}{2}$.

The last part of the statement now follows as in Theorem 2.4 above (observe that this time $0 < \frac{p}{2} \le 1$). □

Remark 2.13. For later reference, we observe that for every $k, j = 1, 2$,

$$(|U_{x_j}|^{\frac{p-2}{2}} U_{x_j})_{x_k} = \frac{p}{2}\, |U_{x_j}|^{\frac{p-2}{2}} U_{x_j x_k} \quad \text{a.e. on } \{U_{x_j} \ne 0\}. \tag{2-34}$$

Since the function $t \mapsto |t|^{\frac{p-2}{2}} t$ is not C^1 for $1 < p < 2$, or locally Lipschitz, the identity (2-34) does not follow from the chain rule in a straightforward way. We start instead from the following identity, which results from the classical chain rule for smooth functions:

$$(\varepsilon+|u^\varepsilon_{x_j}|^2)^{\frac{2-p}{4}} \big((\varepsilon+|u^\varepsilon_{x_j}|^2)^{\frac{p-2}{4}} u^\varepsilon_{x_j}\big)_{x_k} = \left(\frac{\varepsilon+\frac{p}{2}|u^\varepsilon_{x_j}|^2}{\varepsilon+|u^\varepsilon_{x_j}|^2}\right) u^\varepsilon_{x_j x_k}. \tag{2-35}$$

On the left-hand side, $(\varepsilon+|u^\varepsilon_{x_j}|^2)^{\frac{2-p}{4}}$ is uniformly bounded on $B_R \Subset B$ and converges (up to a subsequence) almost everywhere to $|U_{x_j}|^{\frac{2-p}{2}}$, while

$$\big((\varepsilon+|u^\varepsilon_{x_j}|^2)^{\frac{p-2}{4}} u_{x_j}\big)_{x_k} \quad \text{weakly converges in } L^2(B_R) \text{ to } \quad (|U_{x_j}|^{\frac{p-2}{2}} U_{x_j})_{x_k}.$$

Hence, the product converges weakly in $L^2(B_R)$ to $|U_{x_j}|^{\frac{2-p}{2}} (|U_{x_j}|^{\frac{p-2}{2}} U_{x_j})_{x_k}$.

A similar argument proves that the right-hand side of (2-35) converges to $\frac{p}{2} U_{x_j x_k}$ weakly in $L^2(B_R)$. We have thus proved that for almost every $x \in B_R$,

$$|U_{x_j}|^{\frac{2-p}{2}} (|U_{x_j}|^{\frac{p-2}{2}} U_{x_j})_{x_k} = \frac{p}{2}\, U_{x_j x_k}.$$

The identity (2-34) follows at once.

As in the case $p > 2$, we end this subsection on the case $1 < p \le 2$ with two additional results on the solutions u^ε of the problem (2-19).

Lemma 2.14 (a minimum principle, $1 < p \le 2$). *Let $B_r \Subset B$. With the notation above, we have*

$$u^\varepsilon_{x_j} \ge C \quad \text{on } \partial B_r \quad \Longleftrightarrow \quad u^\varepsilon_{x_j} \ge C \quad \text{in } B_r.$$

Proof. By inserting in (2-20) a test function of the form φ_{x_j} with φ smooth with compact support in B and integrating by parts, we get

$$\sum_{i=1}^{2} \int_B \big((\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}} u^\varepsilon_{x_i}\big)_{x_j} \varphi_{x_i}\, dx = 0.$$

This is the same as

$$\sum_{i=1}^{2}\int_B(\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,u^\varepsilon_{x_i\,x_j}\,\varphi_{x_i}\,dx+(p-2)\sum_{i=1}^{2}\int_B(\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-4}{2}}\,|u^\varepsilon_{x_i}|^2\,u^\varepsilon_{x_i\,x_j}\,\varphi_{x_i}\,dx=0.$$

By the regularity of u^ε, the previous identity is still true for functions $\varphi\in W^{1,2}_0(B)$. In the previous identity, we insert the test function

$$\Phi=\begin{cases}(C-u^\varepsilon_{x_j})_+ & \text{in } B_r,\\ 0 & \text{in } B\setminus B_r,\end{cases}$$

which is admissible thanks to the hypothesis on $u^\varepsilon_{x_j}$. We obtain

$$\sum_{i=1}^{2}\int_{\{x\in B_r:u^\varepsilon_{x_j}\le C\}}(\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}}\,|u^\varepsilon_{x_i\,x_j}|^2\,dx \\ +(p-2)\sum_{i=1}^{2}\int_{\{x\in B_r:u^\varepsilon_{x_j}\le C\}}(\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-4}{2}}\,|u^\varepsilon_{x_i}|^2\,|u^\varepsilon_{x_i\,x_j}|^2\,dx=0.$$

This can be rewritten as

$$\sum_{i=1}^{2}\int_{\{x\in B_r:u^\varepsilon_{x_j}\le C\}}(\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-4}{2}}\left(\varepsilon+(p-1)\,|u^\varepsilon_{x_i}|^2\right)|u^\varepsilon_{x_j\,x_i}|^2\,dx=0,$$

which in turn implies

$$\sum_{i=1}^{2}\int_{\{x\in B_r:u^\varepsilon_{x_j}\le C\}}|u^\varepsilon_{x_j\,x_i}|^2\,dx=0;\quad \text{i.e.,}\ \int_{\{x\in B_r:u^\varepsilon_{x_j}\le C\}}|\nabla u^\varepsilon_{x_j}|^2\,dx=0.$$

From this identity, we get that the Sobolev function

$$(C-u^\varepsilon_{x_j})_+,$$

is constant in B_r and thanks to the fact that $u^\varepsilon_{x_j}\ge C$ on ∂B_r, we get

$$(C-u^\varepsilon_{x_j})_+=0\quad\text{in } B_r$$

as desired. □

Lemma 2.15. *Let $B_r\Subset B$. With the notation above, there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}\subset(0,\varepsilon_0)$ such that for almost every $s\in[0,r]$, we have*

$$\lim_{k\to+\infty}\|u^{\varepsilon_k}_{x_j}-U_{x_j}\|_{L^\infty(\partial B_s)}=0,\qquad j=1,2.$$

Proof. Observe that $\{u^\varepsilon_{x_j}-U_{x_j}\}_{0<\varepsilon<\varepsilon_0}$ weakly converges to 0 in $W^{1,2}(B_r)$, thanks to Proposition 2.11. The proof then runs similarly to that of Lemma 2.7. □

3. Caccioppoli inequalities

3A. ***The case $p > 2$.*** One of the key ingredients in the proof of the Main Theorem for $p > 2$ is the following "weird" Caccioppoli inequality for the gradient of the local minimizer U. Observe that the inequality contains quantities like the product of different components of ∇U.

Proposition 3.1. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a C^2 function such that $\Phi\,\Phi'' \ge 0$ and $\zeta : \mathbb{R} \to \mathbb{R}^+$ be a nonnegative convex function. For every $B \Subset \Omega$, every $\eta \in C_0^\infty(B)$ and every $j, k \in \{1, 2\}$,*

$$\sum_{i=1}^{2} \int \big|(|U_{x_i}|^{\frac{p-2}{2}} U_{x_i})_{x_k}\big|^2 [\Phi'(U_{x_k})]^2\, \zeta(U_{x_j})\, \eta^2\, dx \\ \le C \Big(\sum_{i=1}^{2} \int |U_{x_i}|^{p-2}\, \Phi(U_{x_k})^4\, |\eta_{x_i}|^2\, dx\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{2} \int |U_{x_i}|^{p-2}\, \zeta(U_{x_j})^2\, |\eta_{x_i}|^2\, dx\Big)^{\frac{1}{2}}. \tag{3-1}$$

Proof. By a standard approximation argument, one can assume ζ to be a smooth function. We fix $\varepsilon > 0$ and we take as above u^ε the minimizer of (2-3), subject to the boundary condition $u^\varepsilon - U^\varepsilon \in W_0^{1,p}(B)$. We divide the proof in two parts: we first show (3-1) for u^ε and then prove that we can take the limit.

<u>Caccioppoli for u^ε</u>. We consider (2-4) with k in place of j and plug in the test function

$$\varphi = \Psi(u^\varepsilon_{x_k})\, \zeta(u^\varepsilon_{x_j})\, \eta^2, \quad \text{with } \Psi(t) = \Phi(t)\, \Phi'(t),$$

where η is as in the statement. In order to simplify the notation, we write u in place of u^ε in what follows. Since

$$\varphi_{x_i} = u_{x_k x_i}\, \Psi'(u_{x_k})\, \zeta(u_{x_j})\, \eta^2 + \Psi(u_{x_k})\, (\zeta(u_{x_j}))_{x_i}\, \eta^2 + 2\, \eta\, \eta_{x_i}\, \Psi(u_{x_k})\, \zeta(u_{x_j}),$$

we obtain

$$\sum_{i=1}^{2} \int (|u_{x_i}|^{p-2} + \varepsilon)\, u^2_{x_i x_k}\, \Psi'(u_{x_k})\, \zeta(u_{x_j})\, \eta^2\, dx \\ = -\sum_{i=1}^{2} \int (|u_{x_i}|^{p-2} + \varepsilon)\, u_{x_i x_k}\, \Psi(u_{x_k})\, (\zeta(u_{x_j}))_{x_i}\, \eta^2\, dx \\ -2 \sum_{i=1}^{2} \int (|u_{x_i}|^{p-2} + \varepsilon)\, u_{x_i x_k}\, \Psi(u_{x_k})\, \zeta(u_{x_j})\, \eta\, \eta_{x_i}\, dx. \tag{3-2}$$

For the second term in the right-hand side, the Young inequality implies

$$2 \int (|u_{x_i}|^{p-2} + \varepsilon)\, u_{x_i x_k}\, \Psi(u_{x_k})\, \zeta(u_{x_j})\, \eta\, \eta_{x_i}\, dx \\ \le \frac{1}{2} \int (|u_{x_i}|^{p-2} + \varepsilon)\, u^2_{x_i x_k}\, \Phi'(u_{x_k})^2\, \zeta(u_{x_j})\, \eta^2\, dx + 2 \int (|u_{x_i}|^{p-2} + \varepsilon)\, \Phi(u_{x_k})^2\, \zeta(u_{x_j})\, \eta^2_{x_i}\, dx,$$

where we used the definition of Ψ. The first term can be absorbed into the left-hand side of (3-2), thanks to the fact that

$$\Psi' = (\Phi\, \Phi')' = \Phi'^2 + \Phi\, \Phi'' \ge \Phi'^2.$$

Hence, for the moment we have obtained

$$\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, u_{x_i x_k}^2\, \Phi'(u_{x_k})^2\, \zeta(u_{x_j})\, \eta^2\, dx$$
$$\le 2\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, |u_{x_i x_k}|\, |\Psi(u_{x_k})|\, \big|(\zeta(u_{x_j}))_{x_i}\big|\, \eta^2\, dx$$
$$+4\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, \Phi(u_{x_k})^2 \zeta(u_{x_j})\, \eta_{x_i}^2\, dx. \tag{3-3}$$

In the particular case when $\zeta\equiv 1$, we observe for later use that

$$\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, \big|(\Phi(u_{x_k}))_{x_i}\big|^2\, \eta^2\, dx = \sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, u_{x_i x_k}^2\, \Phi'(u_{x_k})^2\, \eta^2\, dx$$
$$\le 4\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, \Phi(u_{x_k})^2\, \eta_{x_i}^2\, dx. \tag{3-4}$$

We go back to (3-3). By Hölder's inequality, we can estimate the last term of the right-hand side:

$$\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, \Phi(u_{x_k})^2\, \zeta(u_{x_j})\, \eta_{x_i}^2\, dx$$
$$\le \left(\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, \Phi(u_{x_k})^4\, \eta_{x_i}^2\, dx\right)^{\frac12} \left(\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, \zeta(u_{x_j})^2\, \eta_{x_i}^2\, dx\right)^{\frac12}. \tag{3-5}$$

In a similar fashion, for the first term in the right-hand side of (3-3), we have

$$\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, |u_{x_i x_k}|\, |\Psi(u_{x_k})|\, \big|(\zeta(u_{x_j}))_{x_i}\big|\, \eta^2\, dx$$
$$\le \left(\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, u_{x_i x_k}^2\, \Psi(u_{x_k})^2 \eta^2\, dx\right)^{\frac12} \left(\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, \big|(\zeta(u_{x_j}))_{x_i}\big|^2\, \eta^2\, dx\right)^{\frac12}$$
$$= \frac12\left(\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\big|(\Phi(u_{x_k})^2)_{x_i}\big|^2 \eta^2\, dx\right)^{\frac12} \left(\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, \big|(\zeta(u_{x_j}))_{x_i}\big|^2\, \eta^2\, dx\right)^{\frac12}. \tag{3-6}$$

In the last equality, we have used the fact that

$$u_{x_i x_k}^2\, \Psi(u_{x_k})^2 = \tfrac14\, ((\Phi(u_{x_k})^2)_{x_i})^2.$$

It follows from (3-3), (3-5) and (3-6) that

$$\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon) u_{x_i x_k}^2\, \Phi'(u_{x_k})^2 \zeta(u_{x_j}) \eta^2\, dx$$
$$\le \left(\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, \big|(\Phi(u_{x_k})^2)_{x_i}\big|^2 \eta^2\, dx\right)^{\frac12} \left(\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, \big|(\zeta(u_{x_j}))_{x_i}\big|^2\, \eta^2\, dx\right)^{\frac12}$$
$$+4\left(\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, \Phi(u_{x_k})^4\, \eta_{x_i}^2\, dx\right)^{\frac12} \left(\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\, \zeta(u_{x_j})^2\, \eta_{x_i}^2\, dx\right)^{\frac12}.$$

By (3-4) with[3] Φ^2 in place of Φ, one has

$$\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\big|(\Phi(u_{x_k})^2)_{x_i}\big|^2\eta^2\,dx \le 4\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\,\Phi(u_{x_k})^4\,\eta_{x_i}^2\,dx.$$

Similarly, by using (3-4) with ζ in place of Φ and j in place of k,

$$\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\,\big|(\zeta(u_{x_j}))_{x_i}\big|^2\,\eta^2\,dx \le 4\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\,\zeta(u_{x_j})^2\,\eta_{x_i}^2\,dx.$$

Hence, we have obtained

$$\begin{aligned}\sum_{i=1}^{2}\int (|u_{x_i}|^{p-2}+\varepsilon)\,u_{x_i x_k}^2\,\Phi'(u_{x_k})^2\,\zeta(u_{x_j})\,\eta^2\,dx \qquad&\\ \le C\bigg(\int \sum_{i=1}^{2}(|u_{x_i}|^{p-2}+\varepsilon)\,\Phi(u_{x_k})^4\,\eta_{x_i}^2\,dx\bigg)^{\frac12}&\bigg(\int \sum_{i=1}^{2}(|u_{x_i}|^{p-2}+\varepsilon)\,\zeta(u_{x_j})^2\,\eta_{x_i}^2\,dx\bigg)^{\frac12}\end{aligned}$$

for some universal constant $C>0$. We now observe that

$$(|u_{x_i}|^{p-2}+\varepsilon)\,u_{x_i x_k}^2 \ge |u_{x_i}|^{p-2}u_{x_i x_k}^2 = \frac{4}{p^2}\,\big|(|u_{x_i}|^{\frac{p-2}{2}}\,u_{x_i})_{x_k}\big|^2;$$

thus, by restoring the original notation u^ε, we get

$$\begin{aligned}\sum_{i=1}^{2}\int \big|(|u^\varepsilon_{x_i}|^{\frac{p-2}{2}}\,u^\varepsilon_{x_i})_{x_k}\big|^2\,\Phi'(u^\varepsilon_{x_k})^2\,\zeta(u^\varepsilon_{x_j})\,\eta^2\,dx \qquad&\\ \le C\bigg(\sum_{i=1}^{2}\int (|u^\varepsilon_{x_i}|^{p-2}+\varepsilon)\,\Phi(u^\varepsilon_{x_k})^4\,\eta_{x_i}^2\,dx\bigg)^{\frac12}&\bigg(\sum_{i=1}^{2}\int (|u^\varepsilon_{x_i}|^{p-2}+\varepsilon)\,\zeta(u^\varepsilon_{x_j})^2\,\eta_{x_i}^2\,dx\bigg)^{\frac12}. \qquad (3\text{-}7)\end{aligned}$$

Passing to the limit $\varepsilon \to 0$. By Lemma 2.2, for every $B_r \Subset B$ the gradient ∇u^ε is uniformly bounded in $L^\infty(B_r)$. Moreover, by Proposition 2.3, up to a subsequence (we do not relabel), it converges almost everywhere to ∇U. By recalling that η has compact support in B, the dominated convergence theorem implies that the right-hand side of (3-7) converges to the corresponding quantity with U in place of u^ε and $\varepsilon = 0$.

As for the left-hand side, we use the fact that for a subsequence (still denoted by u^ε)

$$\big\|\Phi'(u^\varepsilon_{x_k})\,\sqrt{\zeta(u^\varepsilon_{x_j})}\,\eta\big\|_{L^\infty(\mathrm{spt}(\eta))} \le C, \qquad \Phi'(u^\varepsilon_{x_k})\,\sqrt{\zeta(u^\varepsilon_{x_j})}\,\eta \to \Phi'(U_{x_k})\,\sqrt{\zeta(U_{x_j})}\,\eta \quad \text{a.e.},$$

and that

$$|u^\varepsilon_{x_i}|^{\frac{p-2}{2}}\,u^\varepsilon_{x_i} \quad \text{weakly converges in } W^{1,2}(\mathrm{spt}(\eta)) \text{ to } \quad |U_{x_i}|^{\frac{p-2}{2}}\,U_{x_i},$$

still by Proposition 2.3. Hence, we can infer weak convergence in $L^2(\mathrm{spt}(\eta))$ of

$$(|u^\varepsilon_{x_i}|^{\frac{p-2}{2}}\,u^\varepsilon_{x_i})_{x_k}\,\Phi'(u^\varepsilon_{x_k})\,\sqrt{\zeta(u^\varepsilon_{x_j})}\,\eta.$$

Finally, by semicontinuity of the L^2 norm with respect to weak convergence, one gets

$$\int \big|(|U_{x_i}|^{\frac{p-2}{2}}\,U_{x_i})_{x_k}\big|^2\,\Phi'(U_{x_k})^2\,\zeta(U_{x_j})\,\eta^2\,dx \le \liminf_{\varepsilon\to 0}\int \big|(|u^\varepsilon_{x_i}|^{\frac{p-2}{2}}\,u^\varepsilon_{x_i})_{x_k}\big|^2\,\Phi'(u^\varepsilon_{x_k})^2\,\zeta(u^\varepsilon_{x_j})\,\eta^2\,dx.$$

This yields the desired estimate (3-1) for U. □

[3]Observe that Φ^2 still verifies $\Phi^2\,(\Phi^2)'' \ge 0$. Indeed, $(\Phi^2)'' = 2\,(\Phi')^2 + 2\,\Phi\,\Phi'' \ge 0$, by hypothesis.

3B. ***The case $1 < p \le 2$.*** In this case, the Caccioppoli inequality we need is more standard.

Proposition 3.2. *Let $\zeta : \mathbb{R} \to \mathbb{R}$ be a C^1 monotone function. For every $B \Subset \Omega$, every $\eta \in C_0^\infty(B)$ and every $j = 1, 2$ we have*

$$\sum_{i=1}^{2} \int_{\{U_{x_i} \neq 0\}} |U_{x_i}|^{p-2} \left|(Z(U_{x_j}))_{x_i}\right|^2 \eta^2 \, dx \\ \le C \int |\nabla U|^{p-1} \big(|\nabla U|\, |\zeta'(U_{x_j})| + |\zeta(U_{x_j})|\big) \big(|\nabla \eta|^2 + |\eta|\, |D^2 \eta|\big) \, dx, \tag{3-8}$$

where $Z : \mathbb{R} \to \mathbb{R}$ is the C^1 function defined by

$$Z(t) = \int_0^t \sqrt{|\zeta'(s)|} \, ds. \tag{3-9}$$

Proof. We fix $\varepsilon > 0$ and we take as above u^ε the minimizer of (2-19), subject to the boundary condition $u^\varepsilon - U^\varepsilon \in W_0^{1,p}(B)$. Then by Proposition 2.9, we have

$$\sum_{i=1}^{2} \int (\varepsilon + |u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}} |\zeta'(u^\varepsilon_{x_j})|\, |u^\varepsilon_{x_j\, x_i}|^2 \eta^2 \, dx \\ \le C \int (\varepsilon + |\nabla u^\varepsilon|^2)^{\frac{p}{2}} |\zeta'(u^\varepsilon_{x_j})|\, |\nabla \eta|^2 \, dx + C \int (\varepsilon + |\nabla u^\varepsilon|^2)^{\frac{p-1}{2}} |\zeta(u^\varepsilon_{x_j})| \big(|\nabla \eta|^2 + |\eta|\, |D^2 \eta|\big) \, dx$$

for some $C = C(p) > 0$. Since $p < 2$,

$$(\varepsilon + |u^\varepsilon_{x_i}|^2)^{\frac{p-2}{2}} |\zeta'(u^\varepsilon_{x_j})|\, |u^\varepsilon_{x_j\, x_i}|^2 \eta^2 \ge \Big(\big((\varepsilon + |u^\varepsilon_{x_i}|^2)^{\frac{p-2}{4}} u^\varepsilon_{x_i}\big)_{x_j} \sqrt{|\zeta'(u^\varepsilon_{x_j})|}\, \eta\Big)^2.$$

Hence,

$$\sum_{i=1}^{2} \int \Big(\big((\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{4}} u^\varepsilon_{x_i}\big)_{x_j} \sqrt{|\zeta'(u^\varepsilon_{x_j})|}\, \eta\Big)^2 \\ \le C \int (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p}{2}} |\zeta'(u^\varepsilon_{x_j})|\, |\nabla \eta|^2 \, dx + C \int (\varepsilon+|\nabla u^\varepsilon|^2)^{\frac{p-1}{2}} |\zeta(u^\varepsilon_{x_j})| \big(|\nabla \eta|^2+|\eta|\, |D^2 \eta|\big) \, dx. \tag{3-10}$$

In order to pass to the limit as ε goes to 0, we observe that by Lemma 2.10, for every $B_r \Subset B$ the gradient ∇u^ε is uniformly bounded in $L^\infty(B_r)$. Moreover, by Proposition 2.11 it converges almost everywhere to ∇U (up to a subsequence). By recalling that η has compact support in B, the dominated convergence theorem implies that the right-hand side of the above inequality converges to the corresponding quantity with U in place of u^ε and $\varepsilon = 0$.

As for the left-hand side, we observe that by Proposition 2.11

$$(\varepsilon + |u^\varepsilon_{x_i}|^2)^{\frac{p-2}{4}} u^\varepsilon_{x_i} \quad \text{weakly converges in } W^{1,2}(\mathrm{spt}(\eta)) \text{ to } \quad |U_{x_i}|^{\frac{p-2}{2}} U_{x_i},$$

and (up to a subsequence),

$$\Big\| \sqrt{|\zeta'(u^\varepsilon_{x_j})|}\, \eta \Big\|_{L^\infty(\mathrm{spt}(\eta))} \le C, \qquad \sqrt{|\zeta'(u^\varepsilon_{x_j})|}\, \eta \to \sqrt{|\zeta'(U_{x_j})|}\, \eta \quad \text{a.e.}$$

Thus as in the case $p>2$, we can infer weak convergence in $L^2(\mathrm{spt}(\eta))$ of

$$\big((\varepsilon+|u^\varepsilon_{x_i}|^2)^{\frac{p-2}{4}}u^\varepsilon_{x_i}\big)_{x_j}\sqrt{|\zeta'(u^\varepsilon_{x_j})|}\,\eta.$$

By the same semicontinuity argument as before, we get

$$\liminf_{\varepsilon\to 0}\sum_{i=1}^2\int\Big(\big((\varepsilon+|u^\varepsilon_{x_j}|^2)^{\frac{p-2}{4}}u^\varepsilon_{x_i}\big)_{x_j}\sqrt{|\zeta'(u^\varepsilon_{x_j})|}\eta\Big)^2\,dx\ge\sum_{i=1}^2\int\big|(|U_{x_i}|^{\frac{p-2}{2}}U_{x_i})_{x_j}\sqrt{|\zeta'(U_{x_j})|}\,\eta\big|^2\,dx.$$

The right-hand side is greater than or equal to

$$\sum_{i=1}^2\int_{\{U_{x_i}\neq 0\}}\big|(|U_{x_i}|^{\frac{p-2}{2}}U_{x_i})_{x_j}\big|^2\,|\zeta'(U_{x_j})|\,\eta^2\,dx=\frac{p^2}{4}\sum_{i=1}^2\int_{\{U_{x_i}\neq 0\}}\big||U_{x_i}|^{\frac{p-2}{2}}U_{x_i\,x_j}\big|^2\,|\zeta'(U_{x_j})|\,\eta^2\,dx.$$

The last equality follows from (2-34). Now, applying the standard chain rule for the C^1 function Z defined in (3-9) (remember also that $U_{x_j}\in W^{1,2}_{\mathrm{loc}}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)$) yields

$$\liminf_{\varepsilon\to 0}\sum_{i=1}^2\int\Big(\big((\varepsilon+|u^\varepsilon_{x_j}|^2)^{\frac{p-2}{4}}u^\varepsilon_{x_i}\big)_{x_j}\sqrt{|\zeta'(u^\varepsilon_{x_j})|}\,\eta\Big)^2\,dx$$
$$\ge\frac{p^2}{4}\sum_{i=1}^2\int_{\{U_{x_i}\neq 0\}}|U_{x_i}|^{p-2}\,\big|(Z(U_{x_j}))_{x_i}\big|^2\,\eta^2\,dx.$$

In view of (3-10), this completes the proof. □

4. Decay estimates for a nonlinear function of the gradient for $p>2$

We already know from Theorem 2.4 that

$$|U_{x_j}|^{\frac{p-2}{2}}\,U_{x_j}\in W^{1,2}_{\mathrm{loc}}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega).$$

This nonlinear function of the gradient of U will play a crucial role in the sequel for the case $p>2$. Thus we introduce the expedient notation

$$v_j=|U_{x_j}|^{\frac{p-2}{2}}\,U_{x_j},\quad j=1,2.$$

For every $B_R\Subset\Omega$, we will also use the following notation:

$$m_j=\inf_{B_R}v_j,\quad V_j=v_j-m_j,\quad M_j=\sup_{B_R}V_j=\operatorname*{osc}_{B_R}v_j,\quad j=1,2,\tag{4-1}$$
$$L_R=1+\|\nabla U\|_{L^\infty(B_R)}.\tag{4-2}$$

4A. ***A De Giorgi-type lemma.*** We first need the following result on the decay of the oscillation of v_j. This is the analogue of [Santambrogio and Vespri 2010, Lemma 4]. As explained in the Introduction, our operator is much more degenerate then the one considered in that paper; thus the proof has to be completely recast. We crucially rely on the Caccioppoli inequality of Proposition 3.1.

Lemma 4.1. *Let $B_R \Subset \Omega$ and $0 < \alpha < 1$. By using the notation in* (4-1) *and* (4-2), *there exists a constant $\nu = \nu(p, \alpha, L_R) > 0$ such that if*

$$\big|\{V_j > (1-\alpha)\, M_j\} \cap B_R\big| \le \nu\, M_j^{2p+4\left(1-\frac{2}{p}\right)} |B_R|,$$

then

$$0 \le V_j \le \Big(1 - \frac{\alpha}{2}\Big) M_j, \quad \text{on } B_{\frac{R}{2}}.$$

Proof. We first observe that if $M_j = 0$, then V_j identically vanishes in B_R and there is nothing to prove. Thus, we can assume that $M_j > 0$.

For $n \ge 1$, we set

$$k_n = M_j\Big(1 - \frac{\alpha}{2} - \frac{\alpha}{2^n}\Big), \quad R_n = \frac{R}{2} + \frac{R}{2^n}, \quad A_n = \{V_j > k_n\} \cap B_{R_n},$$

where the ball B_{R_n} is concentric with B_R. Let θ_n be a smooth cut-off function such that

$$0 \le \theta_n \le 1, \qquad \theta_n \equiv 1 \quad \text{on } B_{R_{n+1}}, \qquad \theta_n \equiv 0 \quad \text{on } \mathbb{R}^2 \setminus B_{R_n}, \qquad |\nabla \theta_n| \le C\, \frac{2^n}{R}.$$

Recalling the definition (2-1) of g_q, we then set for every $n \ge 1$

$$\beta_n = g^{-1}_{\frac{p-2}{2}}(m_j + k_n) = |m_j + k_n|^{\frac{2-p}{p}}\,(m_j + k_n), \tag{4-3}$$

with m_j defined in (4-1). We start from (3-1) with the choices

$$\Phi(t) = t, \quad \zeta(t) = (t - \beta_n)_+^2 \quad \text{and} \quad \eta = \theta_n.$$

Observe that

$$\zeta(U_{x_j}) = (U_{x_j} - \beta_n)_+^2 > 0 \quad \Longleftrightarrow \quad V_j > k_n,$$

and also[4]

$$0 \le \zeta(U_{x_j}) \le \big|g^{-1}_{\frac{p-2}{2}}(v_j) - g^{-1}_{\frac{p-2}{2}}(m_j + k_n)\big|^2 \le C\,|v_j - m_j - k_n|^{\frac{4}{p}} \le C M_j^{\frac{4}{p}} \quad \text{a.e. on } B_{R_n}. \tag{4-4}$$

By using (4-4) and the definition of A_n, we then obtain

$$\begin{aligned}
\sum_{i=1}^{2} \int |(v_i)_{x_k}|^2\, \zeta(U_{x_j})\, \theta_n^2
&\le C\Big(\sum_{i=1}^{2} \int |U_{x_i}|^{p-2}\, |U_{x_k}|^4\, |(\theta_n)_{x_i}|^2\, dx\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{2} \int |U_{x_i}|^{p-2}\, \zeta(U_{x_j})^2\, |(\theta_n)_{x_i}|^2\, dx\Big)^{\frac{1}{2}} \\
&\le C\, L_R^p\, M_j^{\frac{4}{p}} \Big(\int_{B_{R_n}} |\nabla \theta_n|^2\Big)^{\frac{1}{2}} \Big(\int_{A_n} |\nabla \theta_n|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

In view of the properties of θ_n, it follows that

$$\sum_{i=1}^{2} \int |(v_i)_{x_k}|^2\, \zeta(U_{x_j})\, \theta_n^2\, dx \le C\, L_R^p\, M_j^{\frac{4}{p}} \Big(\frac{2^n}{R}\Big)^2 |B_{R_n} \setminus B_{R_{n+1}}|^{\frac{1}{2}}\, |A_n|^{\frac{1}{2}} \le C\, 4^n\, L_R^p\, M_j^{\frac{4}{p}}\, \frac{|A_n|^{\frac{1}{2}}}{R}$$

[4] In the second inequality we use that $t \mapsto g^{-1}_{\frac{p-2}{2}}(t)$ is $\frac{2}{p}$-Hölder continuous.

for some $C = C(p) > 0$. Here, we have used that

$$|B_{R_n} \setminus B_{R_{n+1}}| = \pi (R_n^2 - R_{n+1}^2) = \pi\,(R_n - R_{n+1})\,(R_n + R_{n+1}) \le \frac{R}{2^{n+1}}\, 2\,\pi\, R = \pi\, \frac{R^2}{2^n}.$$

In the left-hand side, we only keep the term $i = j$ and use that by Lemma 2.5

$$(v_j)_{x_k}\, \sqrt{\zeta(U_{x_j})} = (F(U_{x_j}))_{x_k},$$

where

$$F(t) = \frac{p}{2} \int_{\beta_n}^{t} |s|^{\frac{p-2}{2}}\, \sqrt{\zeta(s)}\, ds = \frac{p}{2} \int_{\beta_n}^{t} |s|^{\frac{p-2}{2}} (s - \beta_n)_+\, ds, \quad t \in \mathbb{R}.$$

We thus obtain

$$\int \left|(F(U_{x_j}))_{x_k}\right|^2 \theta_n^2\, dx \le C\, 4^n\, L_R^{p}\, M_j^{\frac{4}{p}}\, \frac{|A_n|^{\frac12}}{R}.$$

Summing over $k = 1, 2$, this yields an estimate for the gradient of $F(U_{x_j})$, i.e.,

$$\int \left|\nabla(F(U_{x_j}))\right|^2 \theta_n^2\, dx \le C\, 4^n\, L_R^{p}\, M_j^{\frac{4}{p}}\, \frac{|A_n|^{\frac12}}{R}. \tag{4-5}$$

Since $m_j \le m_j + k_n \le m_j + M_j = \sup_{B_R} v_j$ and by the definition of L_R, we have $|m_j + k_n| \le L_R^{\frac{p}{2}}$. Hence, by the definition of β_n, see (4-3),

$$|\beta_n| \le L_R. \tag{4-6}$$

By keeping this in mind and using Lemma A.1 below,

$$0 \le F(U_{x_j}) \le C\big(|U_{x_j}|^{\frac{p-2}{2}} + |\beta_n|^{\frac{p-2}{2}}\big)\,(U_{x_j} - \beta_n)_+^2 \le C\, L_R^{\frac{p-2}{2}}\, (U_{x_j} - \beta_n)_+^2.$$

This implies that $F(U_{x_j}) = 0$ on $B_{R_n} \setminus A_n$ and also that

$$0 \le F(U_{x_j}) \le C\, L_R^{\frac{p-2}{2}}\, \zeta(U_{x_j}) \le C\, L_R^{\frac{p-2}{2}}\, M_j^{\frac{4}{p}}$$

for some $C = C(p) > 0$. In the last inequality, we have used (4-4). Hence,

$$\begin{aligned}\int |\nabla\theta_n|^2 (F(U_{x_j}))^2\, dx &\le C\, L_R^{p-2}\, M_j^{\frac{8}{p}} \int_{A_n} |\nabla\theta_n|^2\, dx \\ &\le C\, 4^n\, L_R^{p-2}\, M_j^{\frac{8}{p}}\, \frac{|A_n|}{R^2} \le C\, 4^n\, L_R^{p}\, M_j^{\frac{4}{p}}\, \frac{|A_n|^{\frac12}}{R},\end{aligned} \tag{4-7}$$

where in the last inequality we used that $|A_n|^{\frac12} \le \sqrt{\pi}\, R$ and $M_j \le 2 L_R^{\frac{p}{2}}$. By adding (4-5) and (4-7), with some simple manipulations we get

$$\int_{B_{R_n}} |\nabla(F(U_{x_j})\, \theta_n)|^2 \le C\, 4^n\, L_R^{p}\, M_j^{\frac{4}{p}}\, \frac{|A_n|^{\frac12}}{R},$$

where as usual $C = C(p) > 0$. We now rely on the following Poincaré inequality for the function $F(U_{x_j})\,\theta_n \in W_0^{1,2}(B_{R_n})$:

$$\big|\{x \in B_{R_n} : F(U_{x_j})\,\theta_n > 0\}\big| \int_{B_{R_n}} |\nabla(F(U_{x_j})\,\theta_n)|^2\,dx \geq c \int_{B_{R_n}} |F(U_{x_j})\,\theta_n|^2\,dx.$$

This inequality can be obtained as follows: for every bounded open set $\Omega \subset \mathbb{R}^2$, the Sobolev embedding $W_0^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$ implies that for every $f \in W_0^{1,2}(\Omega)$,

$$\int |f|^2\,dx \leq C\left(\int |\nabla f|\,dx\right)^2 = C\left(\int_{\{f\neq 0\}} |\nabla f|\,dx\right)^2 \leq C\,|\{x : f(x) \neq 0\}| \int |\nabla f|^2\,dx,$$

where C is a universal constant.

Since $\theta_n \equiv 1$ on $B_{R_{n+1}}$ and by construction

$$|A_n| \geq \big|\{F(U_{x_j})\,\theta_n > 0\}\big|,$$

one gets

$$\int_{B_{R_{n+1}}} |F(U_{x_j})|^2\,dx \leq C\,\frac{4^n\,L_R^p\,M_j^{\frac{4}{p}}}{R}\,|A_n|^{\frac{3}{2}}$$

for some $C = C(p) > 0$. By using that F is nondecreasing and

$$A_{n+1} = \{V_j > k_{n+1}\} \cap B_{R_{n+1}} = \{U_{x_j} > \beta_{n+1}\} \cap B_{R_{n+1}},$$

we obtain

$$\int_{B_{R_{n+1}}} |F(U_{x_j})|^2\,dx \geq \int_{A_{n+1}} |F(U_{x_j})|^2\,dx \geq |A_{n+1}|\,F(\beta_{n+1})^2.$$

This gives

$$|A_{n+1}|\,F(\beta_{n+1})^2 \leq C\,\frac{4^n\,L_R^p\,M_j^{\frac{4}{p}}}{R}|A_n|^{\frac{3}{2}}. \tag{4-8}$$

We now use the lower bound of Lemma A.1 to get

$$F(\beta_{n+1})^2 \geq c\,(\beta_{n+1} - \beta_n)^{p+2}. \tag{4-9}$$

Remember that

$$\beta_n = g_{\frac{p-2}{2}}^{-1}(m_j + k_n) \quad \text{and} \quad \beta_{n+1} = g_{\frac{p-2}{2}}^{-1}(m_j + k_{n+1}).$$

If we use again that for every $s, t \in \mathbb{R}$,

$$\big|g_{\frac{p-2}{2}}(t) - g_{\frac{p-2}{2}}(s)\big| \leq C\left(|t|^{\frac{p-2}{2}} + |s|^{\frac{p-2}{2}}\right)|t-s|,$$

then one gets

$$|k_{n+1} - k_n|^{p+2} = \big|(k_{n+1} + m_j) - (k_n + m_j)\big|^{p+2} \leq C\big(|\beta_{n+1}|^{\frac{p-2}{2}} + |\beta_n|^{\frac{p-2}{2}}\big)^{p+2}(\beta_{n+1} - \beta_n)^{p+2}.$$

By using (4-6) and (4-9) we obtain

$$|k_{n+1} - k_n|^{p+2} \leq C\,L_R^{\frac{p^2-4}{2}}\,F(\beta_{n+1})^2.$$

so that by (4-8),

$$|A_{n+1}|\,|k_{n+1}-k_n|^{p+2} \le C\,\frac{4^n\,L_R^{\frac{p^2-4+2\,p}{2}}\,M_j^{\frac{4}{p}}}{R}\,|A_n|^{\frac{3}{2}}.$$

By the definition of k_n, the previous inequality gives

$$\frac{|A_{n+1}|}{R^2} \le C\left(\frac{2^{n\,(p+4)}}{\alpha^{p+2}}\,L_R^{\frac{p^2-4+2\,p}{2}}\,M_j^{\frac{4}{p}-p-2}\right)\left(\frac{|A_n|}{R^2}\right)^{\frac{3}{2}}.$$

Since $M_j > 0$, the right-hand side is well-defined. If we now set $Y_n = |A_n|/R^2$, this finally yields

$$Y_{n+1} \le \left(C_0\,L_R^{\frac{p^2-4+2\,p}{2}}\,M_j^{\frac{4}{p}-p-2}\right)(2^{p+4})^n\,Y_n^{\frac{3}{2}} \quad \text{for every } n \in \mathbb{N}\setminus\{0\},$$

for some $C_0 = C_0(\alpha, p)$ which can be supposed to be larger than 1. If follows from Lemma B.1 that

$$\lim_{n\to+\infty} Y_n = 0, \quad \text{provided that } Y_1 \le \frac{(2^{p+4})^{-6}}{C_0^2}\,L_R^{4-p^2-2\,p}\,M_j^{2\,p+4\,(1-\frac{2}{p})}.$$

The condition on Y_1 means

$$|\{V_j > (1-\alpha)\,M_j\} \cap B_R| \le \nu\,M_j^{2\,p+4\,(1-\frac{2}{p})}\,|B_R|, \quad \text{with } \nu := \frac{(2^{p+4})^{-6}}{C_0^2\,\pi}\,L_R^{4-p^2-2p}. \tag{4-10}$$

By assuming this condition and recalling the definition of Y_n, we get

$$V_j \le \lim_{n\to+\infty} k_n = \left(1-\frac{\alpha}{2}\right)M_j \quad \text{a.e. on } B_{\frac{R}{2}}. \qquad \square$$

Remark 4.2 (quality of the constant ν). For later reference, it is useful to record that

$$\nu\,M_j^{2\,p+4\,(1-\frac{2}{p})} < \tfrac{1}{2}.$$

This follows by direct computation, using the definition of ν and observing that

$$M_j \le 2\,\|v_j\|_{L^\infty(B_R)} = 2\,\|U_{x_j}\|_{L^\infty(B_R)}^{\frac{p}{2}} \le 2\,(L_R-1)^{\frac{p}{2}}.$$

Also observe that by its definition (4-10), the constant ν is monotone nonincreasing as a function of the radius of the ball B_R (since $R \mapsto L_R$ is monotone nondecreasing and $4-p^2-2\,p<0$ for $p \ge 2$).

4B. *Alternatives.*

Lemma 4.3. *We still use the notation in* (4-1) *and* (4-2). *Let $B_R \Subset \Omega$ and let ν be the constant in Lemma 4.1 for $\alpha = \frac{1}{4}$. If we set*

$$\delta = \sqrt{\frac{\nu}{2}\,M_j^{2\,p+4\,(1-\frac{2}{p})}},$$

then exactly one of the two following alternatives occur:

$$\operatorname*{osc}_{B_{\delta R}} v_j \le \tfrac{7}{8}\,\operatorname*{osc}_{B_R} v_j, \tag{B$_1$}$$

$$\int_{B_R\setminus B_{\delta R}} |\nabla v_j|^2\,dx \ge \tfrac{1}{512\,\pi}\,\nu\,M_j^2\,M_j^{2\,p+4\,(1-\frac{2}{p})}. \tag{B$_2$}$$

Proof. We can suppose that $M_j > 0$, otherwise there is nothing to prove. We have two possibilities: either

$$\left|\left\{V_j > \tfrac{3}{4}\, M_j\right\} \cap B_R\right| < \nu\, M_j^{2p+4\left(1-\frac{2}{p}\right)}\, |B_R|,$$

or not. In the first case, by Lemma 4.1 with $\alpha = \frac{1}{4}$ we obtain

$$\operatorname*{osc}_{B_{\delta R}} v_j \le \operatorname*{osc}_{B_{R/2}} v_j \le \tfrac{7}{8} \operatorname*{osc}_{B_R} v_j,$$

which corresponds to alternative (B_1) in the statement. In the first inequality we used that $\delta < \frac{1}{2}$; see Remark 4.2.

In the second case, we appeal to Lemma B.3 with the choices

$$q = 2, \quad \varphi = V_j, \quad M = M_j \quad \text{and} \quad \gamma = \nu\, M_j^{2p+4\left(1-\frac{2}{p}\right)},$$

with δ as in the statement above. It follows that

- either
$$\int_{B_R \setminus B_{\delta R}} |\nabla V_j|^2\, dx \ge \tfrac{1}{512\pi}\, \nu\, M_j^2\, M_j^{2p+4\left(1-\frac{2}{p}\right)},$$
- or the subset of $[\delta R, R]$ given by
$$\mathcal{A} = \left\{s \in [\delta R, R] : V_j \ge \tfrac{5}{8}\, M_j,\ \mathcal{H}^1\text{-a.e. on } \partial B_s\right\}$$
has positive measure.

If the first possibility occurs, then we are done since this coincides with alternative (B_2).

In the second case, we consider u^ε the solution of the regularized problem (2-3) in a ball $B \Subset \Omega$ such that $B_R \Subset B$. Then we know from Lemma 2.7

$$\lim_{k\to+\infty} \left\| |u^{\varepsilon_k}_{x_j}|^{\frac{p-2}{2}}\, u^{\varepsilon_k}_{x_j} - v_j \right\|_{L^\infty(\partial B_s)} = 0 \quad \text{for a.e. } s \in [0, R],$$

for an infinitesimal sequence $\{\varepsilon_k\}_{n\in\mathbb{N}}$. Since $\mathcal{A}$ has positive measure, we can then choose a radius $s \in \mathcal{A}$ such that the previous convergence holds. For every $n \in \mathbb{N} \setminus \{0\}$, by taking k large enough we thus obtain

$$|u^{\varepsilon_k}_{x_j}|^{\frac{p-2}{2}}\, u^{\varepsilon_k}_{x_j} \ge \tfrac{5}{8}\, M_j + m_j - \tfrac{1}{n}, \quad \mathcal{H}^1\text{-a.e. on } \partial B_s.$$

We can now apply the minimum principle of Lemma 2.6 with $C = \frac{5}{8}\, M_j + m_j - \frac{1}{n}$ and get

$$|u^{\varepsilon_k}_{x_j}|^{\frac{p-2}{2}}\, u^{\varepsilon_k}_{x_j} \ge \tfrac{5}{8}\, M_j + m_j - \tfrac{1}{n} \quad \text{in } B_s. \tag{4-11}$$

Thanks to Proposition 2.3, we know that $\{|u^{\varepsilon_k}_{x_j}|^{\frac{p-2}{2}}\, u^{\varepsilon_k}_{x_j}\}_{k\in\mathbb{N}}$ converges strongly in $L^2(B_s)$ to v_j. It then follows from (4-11) that

$$v_j \ge \tfrac{5}{8}\, M_j + m_j - \tfrac{1}{n} \quad \text{a.e. in } B_s; \qquad \text{that is,} \quad V_j \ge \tfrac{5}{8}\, M_j - \tfrac{1}{n} \quad \text{a.e. in } B_s.$$

Hence, by the arbitrariness of n we get

$$\operatorname*{osc}_{B_{\delta R}} v_j \le \operatorname*{osc}_{B_s} v_j \le \sup_{B_R} V_j - \inf_{B_s} V_j \le \tfrac{3}{8} M_j,$$

which implies again alternative (B_1). ☐

5. Decay estimates for the gradient for $1 < p \le 2$

5A. ***A De Giorgi-type lemma.*** For every $B_R \Subset \Omega$, we introduce the alternative notation

$$m_j = \inf_{B_R} U_{x_j}, \quad V_j = U_{x_j} - m_j, \quad M_j = \sup_{B_R} V_j = \operatorname*{osc}_{B_R} U_{x_j}, \quad j = 1, 2, \tag{5-1}$$

and still use the notation (4-2) for L_R.

Lemma 5.1. *Let $B_R \Subset \Omega$ and $0 < \alpha < 1$. By using the notation in* (5-1) *and* (4-2), *there exists a constant $\nu = \nu(p, \alpha, L_R) > 0$ such that if*

$$\left|\{V_j > (1-\alpha)\, M_j\} \cap B_R\right| \le \nu\, M_j^2\, |B_R|,$$

then

$$0 \le V_j \le \left(1 - \frac{\alpha}{2}\right) M_j \quad \text{on } B_{\frac{R}{2}}.$$

Proof. We first observe that if $M_j = 0$, then V_j identically vanishes in B_R and there is nothing to prove. Thus, we can assume that $M_j > 0$.

For $n \ge 1$, we set

$$k_n = M_j\left(1 - \frac{\alpha}{2} - \frac{\alpha}{2^n}\right), \quad R_n = \frac{R}{2} + \frac{R}{2^n}, \quad A_n = \{V_j > k_n\} \cap B_{R_n},$$

where the ball B_{R_n} is concentric with B_R. Let θ_n be a cut-off function such that

$$0 \le \theta_n \le 1, \qquad \theta_n \equiv 1 \quad \text{on } B_{R_{n+1}}, \qquad \theta_n \equiv 0 \quad \text{on } \mathbb{R}^2 \setminus B_{R_n}$$
$$|\nabla\theta_n| \le C\,\frac{2^n}{R} \quad \text{and} \quad |D^2\theta_n| \le C\,\frac{4^n}{R^2}.$$

We then set for every $n \ge 1$

$$\beta_n = m_j + k_n. \tag{5-2}$$

For every $\delta > 0$, we take a C^1 nondecreasing function $\xi_\delta : \mathbb{R} \to [0 + \infty)$ such that

$$\xi_\delta(t) = 0 \quad \text{for } t \le 0, \qquad |\xi'_\delta(t)| \le C \quad \text{for } t \in \mathbb{R}, \qquad \xi'_\delta(t) = C \quad \text{for } t \ge \delta,$$

for some universal constant $C > 0$. This has to be thought of as a smooth approximation of the "positive part" function, up to the constant $C > 0$. One can take for example the function ξ_δ of the form

$$\xi_\delta(t) = \begin{cases} 0 & \text{for } t \le 0, \\ t^3/\delta^2 & \text{for } 0 < t < \delta, \\ 3\,t - 2\,\delta & \text{for } t \ge \delta. \end{cases}$$

In the setting of Proposition 3.2, we take

$$\zeta(t) = \xi_\delta(t - \beta_n) \quad \text{and} \quad \eta = \theta_n.$$

We observe that

$$\zeta(t) \le C\,(t - \beta_n)_+,$$

so that

$$\zeta(U_{x_j}) \le C\,(U_{x_j} - m_j - k_n)_+ \le C\,M_j \le 2C\,L_R. \tag{5-3}$$

By using (5-3) and the properties of ζ, one gets from (3-8)

$$\begin{aligned}
C\,\sum_{i=1}^{2} \int_{\{U_{x_j}\ge\beta_n+\delta\}\cap\{U_{x_i}\neq 0\}} |U_{x_i}|^{p-2}\,|U_{x_j\,x_i}|^2\,\theta_n^2\,dx \\
&\le C\int_{\{U_{x_j}\ge\beta_n\}} |\nabla U|^p\,\big(|\nabla\theta_n|^2+|D^2\theta_n|\big)\,dx + \int_{\{U_{x_j}\ge\beta_n\}} |\nabla U|^{p-1}\,|\zeta(U_{x_j})|\,\big(|\nabla\theta_n|^2+|D^2\theta_n|\big)\,dx \\
&\le C\,L_R^p \int_{\{U_{x_j}\ge\beta_n\}} \big(|\nabla\theta_n|^2+|D^2\theta_n|\big)\,dx.
\end{aligned}$$

Since $p<2$ and $|U_{x_i}|\le L_R$ a.e., one gets

$$\sum_{i=1}^{2} \int_{\{U_{x_j}\ge\beta_n+\delta\}} |U_{x_j\,x_i}|^2\,\theta_n^2\,dx \le C\,L_R^2 \int_{\{U_{x_j}\ge\beta_n\}} \big(|\nabla\theta_n|^2+|D^2\theta_n|\big)\,dx.$$

Here, we have also used the fact that $U_{x_j\,x_i}=0$ a.e. on the set $\{U_{x_i}=0\}$. We now take the limit as δ goes to 0 in the left-hand side. By the monotone convergence theorem, we get

$$\sum_{i=1}^{2} \int_{\{U_{x_j}\ge\beta_n\}} |U_{x_j\,x_i}|^2\,\theta_n^2\,dx \le C\,L_R^2 \int_{\{U_{x_j}\ge\beta_n\}} \big(|\nabla\theta_n|^2+|D^2\theta_n|\big)\,dx.$$

In view of the properties of θ_n, it follows that

$$\int \big|\nabla(U_{x_j}-\beta_n)_+\big|^2\,\theta_n^2\,dx \le C\,L_R^2\,4^n\,\frac{|A_n|}{R^2} \tag{5-4}$$

for some $C=C(p)>0$. Observe that

$$\int |\nabla\theta_n|^2\,(U_{x_j}-\beta_n)_+^2\,dx \le C\,L_R^2\,4^n\,\frac{|A_n|}{R^2}, \tag{5-5}$$

thanks to (5-3). By adding (5-4) and (5-5), we get

$$\int_{B_{R_n}} \big|\nabla((U_{x_j}-\beta_n)_+\,\theta_n)\big|^2\,dx \le C\,4^n\,L_R^2\,\frac{|A_n|}{R^2},$$

where as usual $C=C(p)>0$. We rely again on the Poincaré inequality and obtain

$$\big|\{x\in B_{R_n} : (U_{x_j}-\beta_n)_+\,\theta_n>0\}\big| \int_{B_{R_n}} \big|\nabla((U_{x_j}-\beta_n)_+\,\theta_n)\big|^2\,dx \ge c\int_{B_{R_n}} \big|(U_{x_j}-\beta_n)_+\,\theta_n\big|^2\,dx.$$

Since $\theta_n\equiv 1$ on $B_{R_{n+1}}$ and by construction

$$|A_n| \ge \Big|\big\{(U_{x_j}-\beta_n)_+\,\theta_n>0\big\}\Big|,$$

one gets

$$\int_{B_{R_{n+1}}} (U_{x_j}-\beta_n)_+^2\,dx \le C\,\frac{4^n\,L_R^2}{R^2}\,|A_n|^2$$

for some $C=C(p)>0$. By using that

$$A_{n+1}=\{V_j>k_{n+1}\}\cap B_{R_{n+1}}=\{U_{x_j}>\beta_{n+1}\}\cap B_{R_{n+1}},$$

we obtain

$$\int_{B_{R_{n+1}}} (U_{x_j}-\beta_n)_+^2\,dx \ge \int_{A_{n+1}} (U_{x_j}-\beta_n)_+^2\,dx \ge |A_{n+1}|\,(\beta_{n+1}-\beta_n)^2.$$

This gives

$$|A_{n+1}|\,(\beta_{n+1}-\beta_n)^2 \le C\,\frac{4^n\,L_R^2}{R^2}\,|A_n|^2. \tag{5-6}$$

By recalling the definitions of β_n and k_n, the previous inequality gives

$$\frac{|A_{n+1}|}{R^2}\le C\left(\frac{4^{2n}}{\alpha^2}\,L_R^2\,M_j^{-2}\right)\left(\frac{|A_n|}{R^2}\right)^2.$$

Since $M_j>0$, the right-hand side is well-defined. As before, we set $Y_n=|A_n|/R^2$ and obtain

$$Y_{n+1}\le (C_0\,L_R^2\,M_j^{-2})\,16^n\,Y_n^2 \quad \text{for every } n\in\mathbb{N}\setminus\{0\},$$

for some $C_0=C_0(\alpha,p)\ge 1$. Again by Lemma B.1 we get

$$\lim_{n\to+\infty} Y_n=0, \quad \text{provided that } \ Y_1\le \frac{16^{-2}}{C_0}\,L_R^{-2}\,M_j^2,$$

This means

$$|\{V_j>(1-\alpha)\,M_j\}\cap B_R|\le \nu\,M_j^2\,|B_R|, \quad \text{with } \nu:=\frac{16^{-2}}{C_0^2\,\pi}\,L_R^{-2}.$$

By assuming this condition and recalling the definition of Y_n, we get

$$V_j\le \lim_{n\to+\infty} k_n=\left(1-\frac{\alpha}{2}\right)M_j \quad \text{a.e. on } B_{\frac{R}{2}}. \qquad \square$$

Remark 5.2 (quality of the constant ν). For later reference, as in the previous case we observe that

$$\nu\,M_j^2<\tfrac{1}{2},$$

and that the constant ν is monotone nonincreasing as a function of R.

5B. *Alternatives.*

Lemma 5.3. *We still use the notation in* (5-1) *and* (4-2). *Let $B_R \Subset B_{2R}\Subset\Omega$ and let ν be the constant in Lemma 5.1 for $\alpha=\frac{1}{4}$. If we set*

$$\delta=\sqrt{\frac{\nu}{2}\,M_j^2},$$

then exactly one of the two following alternatives occur:

$$\operatorname*{osc}_{B_{\delta R}} U_{x_j} \le \tfrac{7}{8} \operatorname*{osc}_{B_R} U_{x_j}, \tag{B$_1$}$$

$$\int_{B_R\setminus B_{\delta R}} |\nabla U_{x_j}|^2\,dx \ge \tfrac{1}{512\pi}\,\nu\,M_j^4. \tag{B$_2$}$$

Proof. We can suppose that $M_j > 0$, otherwise there is nothing to prove. We have two possibilities: either

$$\big|\{V_j > \tfrac{3}{4} M_j\} \cap B_R\big| < \nu\, M_j^2\,|B_R|,$$

or not. In the first case, by Lemma 5.1 with $\alpha = \frac{1}{4}$ we obtain

$$\operatorname*{osc}_{B_{\delta R}} U_{x_j} \le \operatorname*{osc}_{B_{R/2}} U_{x_j} \le \tfrac{7}{8} \operatorname*{osc}_{B_R} U_{x_j},$$

which corresponds to alternative (B_1) in the statement. In the first inequality we used again that $\delta < \frac{1}{2}$; see Remark 5.2.

In the second case, we appeal to Lemma B.3 with the choices

$$q = 2, \quad \varphi = V_j, \quad M = M_j \quad \text{and} \quad \gamma = \nu\, M_j^2,$$

with δ as in the statement above. It follows that

- either
$$\int_{B_R\setminus B_{\delta R}} |\nabla V_j|^2\,dx \ge \tfrac{1}{512\pi}\,\nu\,M_j^4,$$
- or the set
$$\mathcal{A} = \big\{s \in [\delta R, R] : U_{x_j} - m_j \ge \tfrac{5}{8} M_j,\ \mathcal{H}^1\text{-a.e. on } \partial B_s\big\}$$
has positive measure.

Again, if the first possibility occurs, then we are done since this coincides with alternative (B_2).

In the second case, we consider u^ε the solution of the regularized problem (2-19) in a ball $B \Subset \Omega$ such that $B_R \Subset B$. Then we know from Lemma 2.15

$$\lim_{k\to+\infty} \|u^{\varepsilon_k}_{x_j} - U_{x_j}\|_{L^\infty(\partial B_s)} = 0 \quad \text{for a.e. } s \in [0, R],$$

for an infinitesimal sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$. Since $\mathcal{A}$ has positive measure, we can then choose a radius $s \in \mathcal{A}$ such that the previous convergence holds. For every $n \in \mathbb{N}\setminus\{0\}$, by taking k large enough we thus obtain

$$u^\varepsilon_{x_j} \ge \tfrac{5}{8} M_j + m_j - \tfrac{1}{n} \quad \mathcal{H}^1\text{-a.e. on } \partial B_s.$$

By proceeding as in the proof of Lemma 4.3 and using this time the minimum principle of Lemma 2.14 and Proposition 2.11, we obtain

$$U_{x_j} - m_j \ge \tfrac{5}{8} M_j - \tfrac{1}{n} \quad \text{a.e. in } B_s.$$

By arbitrariness of n, we get

$$\operatorname*{osc}_{B_{\delta R}} U_{x_j} \le \operatorname*{osc}_{B_s} U_{x_j} \le \big(\sup_{B_R} U_{x_j} - m_j\big) - \big(\inf_{B_s} U_{x_j} - m_j\big) \le \tfrac{3}{8} M_j,$$

which implies again alternative (B_1). □

6. Proof of the Main Theorem

6A. *Case p > 2.* We already observed that for every $q > -1$ the function $t \mapsto t\,|t|^q$ is a homeomorphism on $\mathbb{R}$. This implies the following.

Lemma 6.1. *Let $f : E \to \mathbb{R}$ be a measurable function such that for some $q > -1$ the function $|f|^q f$ is continuous. Then f itself is continuous.*

In view of this result, in order to prove the Main Theorem in the case $p > 2$ it is sufficient to prove that each function

$$v_j = |U_{x_j}|^{\frac{p-2}{2}}\, U_{x_j}, \quad j = 1, 2,$$

is continuous on Ω. Thus the Main Theorem for $p > 2$ is a consequence of the following.

Proposition 6.2. *Let $p > 2$, $x_0 \in \Omega$ and $R_0 > 0$ such that $B_{R_0}(x_0) \Subset \Omega$. We consider the family of balls $\{B_R(x_0)\}_{0<R\le R_0}$ centered at x_0. Then we have*

$$\lim_{R\searrow 0} \big(\operatorname*{osc}_{B_R(x_0)} v_j\big) = 0, \quad j = 1, 2.$$

Proof. For simplicity, in what follows we omit indication of the center x_0 of the balls. Since the map $R \mapsto \operatorname{osc}_{B_R} v_j$ is nondecreasing, we only need to find a decreasing sequence $\{R_n\}_{n\in\mathbb{N}}$ converging to 0 such that

$$\lim_{n\to+\infty} \big(\operatorname*{osc}_{B_{R_n}} v_j\big) = 0.$$

For simplicity we now drop the index j and write v in place of v_j. We set

$$M_0 = \operatorname*{osc}_{B_{R_0}} v \quad \text{and} \quad \delta_0 = \sqrt{\frac{\nu_0}{2}\, M_0^{2p+4\left(1-\frac{2}{p}\right)}},$$

where ν_0 is the constant of Lemma 4.1 for $R = R_0$ and $\alpha = \frac{1}{4}$. We construct by induction the sequence of triples $\{(R_n, M_n, \delta_n)\}_{n\in\mathbb{N}}$ defined by

$$M_n := \operatorname*{osc}_{B_{R_n}} v, \quad \delta_n = \sqrt{\frac{\nu_n}{2}\, M_n^{2p+4\left(1-\frac{2}{p}\right)}}, \quad R_{n+1} = \delta_n\, R_n,$$

and ν_n is the constant of Lemma 4.1 for $R = R_n$ and $\alpha = \frac{1}{4}$. Since $\delta_n < \frac{1}{2}$ for every $n \in \mathbb{N}$ (see Remark 4.2), the sequence $\{R_n\}_{n\in\mathbb{N}}$ is monotone decreasing and goes to 0. In order to conclude, we just need to prove that

$$\lim_{n\to\infty} M_n = 0. \tag{6-1}$$

Observe that we can suppose $M_n > 0$ for every $n \in \mathbb{N}$, otherwise there is nothing to prove. We set

$$I := \left\{ n \in \mathbb{N} : \int_{B_{R_n} \setminus B_{R_{n+1}}} |\nabla v|^2 \, dx \geq \frac{1}{512\,\pi}\, \nu_n \, M_n^{2p+4\left(1-\frac{2}{p}\right)} \, M_n^2 \right\},$$

and we have

$$\begin{aligned} \frac{\nu_0}{512\,\pi} \sum_{n \in I} M_n^{2p+2+4\left(1-\frac{2}{p}\right)} &\leq \frac{1}{512\,\pi} \sum_{n \in I} \nu_n \, M_n^{2p+2+4\left(1-\frac{2}{p}\right)} \\ &\leq \sum_{n \in I} \int_{B_{R_n} \setminus B_{R_{n+1}}} |\nabla v|^2 \, dx \leq \int_{B_{R_0}} |\nabla v|^2 \, dx, \end{aligned} \tag{6-2}$$

thanks to the fact that $\nu_n \geq \nu_0 > 0$ for every $n \in \mathbb{N}$ (see Remark 4.2). We now have two possibilities: either I is infinite or it is finite. If the first alternative occurs, then (6-2) and the fact that $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$ imply

$$\lim_{I \ni n \to \infty} M_n = 0.$$

This means that the monotone sequence $\{M_n\}_{n \in \mathbb{N}}$ has a subsequence which converges to 0; thus we have (6-1) and this completes the proof in that case.

Otherwise, if I is finite then there exists $\ell \in \mathbb{N}$ such that for every $n \geq \ell$ we have

$$\int_{B_{R_n} \setminus B_{R_{n+1}}} |\nabla v|^2 \, dx < \frac{1}{512\pi}\, \nu_n \, M_n^{2p+4\left(1-\frac{2}{p}\right)} \, M_n^2.$$

By Lemma 4.3, this in turn implies that

$$M_{n+1} = \operatorname*{osc}_{B_{R_{n+1}}} v \leq \tfrac{7}{8} \operatorname*{osc}_{B_{R_n}} v = \tfrac{7}{8} M_n \quad \text{for every } n \geq \ell.$$

This again implies (6-1). □

6B. ***Case*** $1 < p \leq 2$**.** The case $1 < p \leq 2$ is similar, but more direct. This time the Main Theorem follows from the result below, whose proof is exactly as above. It is sufficient to use Lemma 5.1 in place of Lemma 4.1 and Lemma 5.3 in place of Lemma 4.3. We leave the details to the reader.

Proposition 6.3. *Let* $1 < p \leq 2$, $x_0 \in \Omega$ *and* $R_0 > 0$ *such that* $B_{R_0}(x_0) \Subset \Omega$. *We consider the family of balls* $\{B_R(x_0)\}_{0<R\leq R_0}$ *centered at* x_0. *Then we have*

$$\lim_{R \searrow 0} \Big(\operatorname*{osc}_{B_R(x_0)} U_{x_j} \Big) = 0, \quad j = 1, 2.$$

Appendix A: Inequalities

In the proof of Lemma 5.1 we crucially relied on the following double-sided estimate for the function

$$F(t) = \frac{p}{2} \int_\beta^t |s|^{\frac{p-2}{2}} (s - \beta)_+ \, ds, \quad t \in \mathbb{R}.$$

Lemma A.1. *Let $\beta \in \mathbb{R}$ and $p > 2$. There exists a constant $C = C(p) > 1$ such that for every $t \in \mathbb{R}$,*

$$\frac{1}{C}\,(t-\beta)_+^{\frac{p+2}{2}} \le F(t) \le C\left(|t|^{\frac{p-2}{2}} + (\max\{0,-\beta\})^{\frac{p-2}{2}}\right)(t-\beta)_+^2. \tag{A-1}$$

Proof. Since $F(t) = 0$ when $t \le \beta$, both inequalities are true in this case. Thus let us assume that $t > \beta$. Moreover, if $\beta = 0$,

$$F(t) = \frac{p}{2}\int_0^t s^{\frac{p-2}{2}}\, s\, ds = \frac{p}{p+2}\, t^{\frac{p+2}{2}} \quad \text{for } t > 0,$$

which implies the result.

<u>Case $\beta > 0$</u>. By Hölder's inequality

$$\begin{aligned}\frac{(t-\beta)_+^p}{2^{\frac{p}{2}}} &= \left(\int_\beta^t (s-\beta)_+\, ds\right)^{\frac{p}{2}} = \left(\int_\beta^t s^{\frac{p-2}{p}}\, \frac{(s-\beta)_+}{s^{\frac{p-2}{p}}}\, ds\right)^{\frac{p}{2}} \\ &\le \left(\int_\beta^t s^{\frac{p-2}{2}}\,(s-\beta)_+\, ds\right)\left(\int_\beta^t \frac{(s-\beta)_+}{s}\, ds\right)^{\frac{p-2}{2}} \le \frac{2}{p}\, F(t)\,(t-\beta)_+^{\frac{p-2}{2}},\end{aligned}$$

where we used that $(s-\beta)_+ \le s$ and this gives the lower bound in (A-1). As for the upper bound, by the change of variables $\tau = s/\beta$ one has

$$F(t) = \beta^{\frac{p+2}{2}}\, F_+\!\left(\frac{t}{\beta}\right), \quad \text{where } F_+(X) = \frac{p}{2}\int_1^X \tau^{\frac{p-2}{2}}\,(\tau-1)\, d\tau, \quad \tau > 1.$$

Observe that

$$F_+(X) = \frac{p}{p+2}\,(X^{\frac{p+2}{2}} - 1) - (X^{\frac{p}{2}} - 1), \quad X > 1.$$

Moreover, by convexity of the function $X \mapsto X^{\frac{p}{2}}$ we have

$$-(X^{\frac{p}{2}} - 1) \le -\frac{p}{2}\,(X-1),$$

while a second-order Taylor expansion gives

$$\frac{p}{p+2}\,(X^{\frac{p+2}{2}} - 1) = \frac{p}{2}\,(X-1) + \frac{p^2}{4}\int_1^X s^{\frac{p-2}{2}}\,(X-s)\, ds \le \frac{p}{2}(X-1) + \frac{p^2}{8}\, X^{\frac{p-2}{2}}\,(X-1)^2.$$

Thus we obtain

$$F_+(X) \le \frac{p^2}{8}\, X^{\frac{p-2}{2}}\,(X-1)^2, \quad X > 1,$$

and finally for $t > \beta$

$$F(t) = \beta^{\frac{p+2}{2}}\, F_+\!\left(\frac{t}{\beta}\right) \le \frac{p^2}{8}\, t^{\frac{p-2}{2}}\,(t-\beta)^2,$$

which proves the upper bound in (A-1).

Case $\beta < 0$. This case is slightly more complicated. We introduce the function

$$F_-(X) = \frac{p}{2}\int_{-1}^{X} |s|^{\frac{p-2}{2}}\,(s+1)\,ds = \frac{p}{p+2}\,(|X|^{\frac{p+2}{2}} - 1) + (|X|^{\frac{p-2}{2}}\,X + 1), \quad X > -1.$$

It is sufficient to prove that there exists $C > 1$ such that

$$\frac{1}{C}\,(X+1)^{\frac{p+2}{2}} \le F_-(X) \le C\,(|X|^{\frac{p-2}{2}} + 1)\,(X+1)^2. \tag{A-2}$$

Indeed, $F(t) = |\beta|^{\frac{p+2}{2}}\,F_-(t/|\beta|)$ and this would give

$$\frac{1}{C}\,(t-\beta)^{\frac{p+2}{2}} \le F(t) \le C\,\big(|t|^{\frac{p-2}{2}} + |\beta|^{\frac{p-2}{2}}\big)\,(t-\beta)^2,$$

as desired.

The upper bound in (A-2) for $-1 < X < 0$ can be obtained as before, by using a second-order Taylor expansion for the first term and using that $\tau \mapsto |\tau|^{\frac{p-2}{2}}\,\tau$ is concave on $-1 < \tau < 0$. This gives

$$\begin{aligned} F_-(X) &= \frac{p}{p+2}\,(|X|^{\frac{p+2}{2}} - 1) + (|X|^{\frac{p-2}{2}}\,X + 1) \\ &\le -\frac{p}{2}\,(X+1) + \frac{p^2}{4}\int_{-1}^{X} |s|^{\frac{p-2}{2}}\,(X-s)\,ds + \frac{p}{2}\,(X+1) \\ &\le \frac{p^2}{8}\,(X+1)^2. \end{aligned}$$

Observe that the upper bound is trivial for $0 \le X \le 1$, since

$$\frac{p}{p+2}\,(|X|^{\frac{p+2}{2}} - 1) + (|X|^{\frac{p-2}{2}}\,X + 1) \le 2 \le 2\,(|X|^{\frac{p-2}{2}} + 1)\,(X+1)^2.$$

Finally, for $X > 1$ we still use a second-order Taylor expansion for the first term and the elementary inequality

$$X^{\frac{p}{2}} + 1 \le \tfrac{1}{2}\,X^{\frac{p-2}{2}}\,(X+1)^2$$

for the second one. These yield

$$F_-(X) \le \frac{p^2}{4}\int_{-1}^{X} |s|^{\frac{p-2}{2}}\,(X-s)\,ds + \frac{1}{2}\,X^{\frac{p-2}{2}}(X+1)^2 \le \left(\frac{p^2}{8} + \frac{1}{2}\right) X^{\frac{p-2}{2}}\,(X+1)^2.$$

In order to prove the lower bound, we just observe that the function

$$X \mapsto \frac{(X+1)^{\frac{p+2}{2}}}{F_-(X)}, \quad X > -1,$$

is positive continuous on $(-1, +\infty)$ and such that

$$\lim_{X\to(-1)^+} \frac{(X+1)^{\frac{p+2}{2}}}{F_-(X)} < +\infty \quad \text{and} \quad \lim_{X\to+\infty} \frac{(X+1)^{\frac{p+2}{2}}}{F_-(X)} < +\infty.$$

Thus it is bounded on $(-1, +\infty)$ and this concludes the proof of the lower bound. □

Lemma A.2. *Let $1 < q \le 2$. For every $z_0, z_1 \in \mathbb{R}^N$ we have*

$$\big||z_0|^{q-2}\, z_0 - |z_1|^{q-2}\, z_1\big| \le 2^{2-q}\, |z_0 - z_1|^{q-1}. \tag{A-3}$$

Proof. The proof is the same as that of [DiBenedetto 1993, Lemma 4.4], which proves a slightly different inequality. We first observe that if $z_1 = z_0$ there is nothing to prove; thus we can suppose $|z_1 - z_0| > 0$. Let us set

$$z_t = (1-t)z_0 + t\, z_1, \quad t \in [0,1].$$

Then we have

$$|z_1|^{q-2}\, z_1 - |z_0|^{q-2}\, z_0 = \int_0^1 \frac{d}{dt}(|z_t|^{q-2}\, z_t)\, dt = (q-1)\int_0^1 |z_t|^{q-2}\,(z_1 - z_0)\, dt,$$

which implies

$$\big||z_0|^{q-2}\, z_0 - |z_1|^{q-2}\, z_1\big| \le (q-1)\,|z_1 - z_0| \int_0^1 \big||z_0| - t\,|z_1 - z_0|\big|^{q-2}\, dt, \tag{A-4}$$

where we used that $q - 2 \le 0$. We now distinguish two cases:

$$\text{either} \quad |z_0| \ge |z_1 - z_0| \quad \text{or} \quad |z_0| < |z_1 - z_0|.$$

In the first case, we have

$$\begin{aligned}\int_0^1 \big||z_0| - t\,|z_1 - z_0|\big|^{q-2}\, dt &= \int_0^1 (|z_0| - t\,|z_1 - z_0|)^{q-2}\, dt \\ &= \frac{|z_0|^{q-1} - (|z_0| - |z_1 - z_0|)^{q-1}}{(q-1)\,|z_1 - z_0|} \le \frac{|z_1 - z_0|^{q-2}}{q-1},\end{aligned}$$

which inserted in (A-4) gives the desired conclusion. In the second case, let $0 < \kappa < 1$ be such that

$$|z_0| = \kappa\,|z_0 - z_1|,$$

then we have

$$\begin{aligned}\int_0^1 \big||z_0| - t\,|z_1 - z_0|\big|^{q-2}\, dt &= \int_0^\kappa (|z_0| - t\,|z_1 - z_0|)^{q-2}\, dt + \int_\kappa^1 (t\,|z_1 - z_0| - |z_0|)^{q-2}\, dt \\ &= \frac{|z_0|^{q-1}}{(q-1)\,|z_1 - z_0|} + \frac{(|z_1 - z_0| - |z_0|)^{q-1}}{(q-1)\,|z_1 - z_0|} \le 2^{2-q}\, \frac{|z_1 - z_0|^{q-2}}{q-1}.\end{aligned}$$

In view of (A-4), this gives the desired conclusion. □

Corollary A.3. *Let $1 < p \le 2$. For every $\varepsilon \ge 0$ and every $t, s \in \mathbb{R}$ we have*

$$\big|(\varepsilon + t^2)^{\frac{p-2}{4}}\, t - (\varepsilon + s^2)^{\frac{p-2}{4}}\, s\big| \le 2^{\frac{2-p}{2}}\, |t - s|^{\frac{p}{2}}, \quad t, s \in \mathbb{R}.$$

Proof. We use (A-3) with the choices

$$N = 2, \quad q = \tfrac{1}{2}(p+2), \quad z_0 = (t, \sqrt{\varepsilon}) \quad \text{and} \quad z_1 = (s, \sqrt{\varepsilon}).$$

This implies

$$\big|(\varepsilon + t^2)^{\frac{p-2}{4}}\,(t, \sqrt{\varepsilon}) - (\varepsilon + s^2)^{\frac{p-2}{4}}\,(s, \sqrt{\varepsilon})\big| \le 2^{\frac{2-p}{2}}\, |t - s|^{\frac{p}{2}}.$$

By further observing that

$$\left|(\varepsilon+t^2)^{\frac{p-2}{4}}\,(t,\sqrt{\varepsilon})-(\varepsilon+s^2)^{\frac{p-2}{4}}\,(s,\sqrt{\varepsilon})\right|\geq\left|(\varepsilon+t^2)^{\frac{p-2}{4}}\,t-(\varepsilon+s^2)^{\frac{p-2}{4}}\,s\right|,$$

we get the conclusion. □

Appendix B: Some general tools

In the proofs of Lemmas 4.1 and 5.1, we used the following classical result. This can be found, for example, in [Giusti 2003, Lemma 7.1].

Lemma B.1. *If $\{Y_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative numbers satisfying*

$$Y_{n+1}\leq c\,b^n\,Y_n^{1+\beta},\quad Y_1\leq c^{-\frac{1}{\beta}}b^{-\frac{\beta+1}{\beta^2}}\quad \textit{for some } c,\ \beta>0,\ b>1,$$

then $\lim_{n\to+\infty}Y_n=0$.

The next lemma is a Fubini-type result on the convergence of Sobolev functions. We denote by $\mathcal{H}^1$ the one-dimensional Hausdorff measure.

Lemma B.2. *Let $0<\tau<1$ and $1\leq p<\infty$. Let $B_R(x_0)\subset\mathbb{R}^2$ be the disc centered at x_0 with radius $R>0$ and let $\{u_n\}_{n\in\mathbb{N}}\subset W^{\tau,p}(B_R(x_0))$ be a sequence strongly converging to 0, i.e., such that*

$$\lim_{n\to\infty}\left[\int_{B_R(x_0)}|u_n|^p\,dx+\iint_{B_R(x_0)\times B_R(x_0)}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{2+\tau p}}\,dx\,dy\right]=0.$$

Then there exists a subsequence $\{u_{n_i}\}_{i\in\mathbb{N}}$ such that for almost every $r\in[0,R]$, $\{u_{n_i}\}_{i\in\mathbb{N}}$ strongly converges to 0 in $W^{\tau,p}(\partial B_r(x_0))$; i.e.,

$$\lim_{i\to\infty}\left[\int_{\partial B_r(x_0)}|u_{n_i}|^p\,d\mathcal{H}^1+\iint_{\partial B_r(x_0)\times\partial B_r(x_0)}\frac{|u_{n_i}(x)-u_{n_i}(y)|^p}{|x-y|^{1+\tau p}}\,d\mathcal{H}^1(x)\,d\mathcal{H}^1(y)\right]=0.$$

Proof. Let us consider the convergence of the double integral, since the convergence of the L^p norm is similar and simpler to prove. Without loss of generality, we can assume $x_0=0$. Then we omit indication of the center of the ball. We use polar coordinates $x=\varrho\,e^{i\,\vartheta}$. We need to show that up to a subsequence, for almost every $\varrho\in[0,R]$ we have

$$\lim_{n\to\infty}[u_n]^p_{W^{\tau,p}(\partial B_\varrho)}=\lim_{n\to\infty}\varrho^{1-\tau p}\iint_{[0,2\,\pi]\times[0,2\,\pi]}\frac{|u_n(\varrho\,e^{i\,\vartheta})-u_n(\varrho\,e^{i\,\omega})|^p}{|e^{i\,\vartheta}-e^{i\,\omega}|^{1+\tau p}}\,d\vartheta\,d\omega=0.\qquad\text{(B-1)}$$

For every $u\in W^{\tau,p}(\mathbb{R}^2)$ and $\varepsilon>0$, we introduce

$$\mathcal{W}_\varepsilon(u):=\int_\varepsilon^\infty\iint_{[0,2\,\pi]\times[0,2\,\pi]}\frac{|u(\varrho\,e^{i\,\vartheta})-u(\varrho\,e^{i\,\omega})|^p}{|e^{i\,\vartheta}-e^{i\,\omega}|^{1+\tau p}}\,d\vartheta\,d\omega\,\frac{\varrho\,d\varrho}{\varrho^{1+\tau p}}.$$

We claim that

$$\mathcal{W}_\varepsilon(u)\leq\frac{C}{\varepsilon}\,[u]^p_{W^{\tau,p}(\mathbb{R}^2)}=\frac{C}{\varepsilon}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\frac{|u(x)-u(y)|^p}{|x-y|^{2+\tau p}}\,dx\,dy\qquad\text{(B-2)}$$

for some constant $C = C(p,\tau) > 0$. Let us assume (B-2) for a moment and explain how to conclude: we can extend $\{u_n\}_{n\in\mathbb{N}}$ to a sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}} \subset W^{\tau,p}(\mathbb{R}^2)$ such that

$$\tilde{u}_n = u_n \quad \text{on } B_R \qquad \text{and} \qquad [\tilde{u}_n]^p_{W^{\tau,p}(\mathbb{R}^2)} \le C\,[u_n]^p_{W^{\tau,p}(B_R)};$$

see [Adams 1975, Lemma 7.45]. The latter and (B-2) imply that

$$\lim_{n\to\infty} \mathcal{W}_\varepsilon(\tilde{u}_n) = 0 \quad \text{for every } \varepsilon > 0.$$

By the definition of $\mathcal{W}_\varepsilon$, this means that the sequence of functions

$$f_n(\varrho) = \frac{\varrho}{\varrho^{1+\tau p}} \int_{[0,2\pi]\times[0,2\pi]} \frac{|u_n(\varrho\, e^{i\,\vartheta}) - u_n(\varrho\, e^{i\,\omega})|^p}{|e^{i\,\vartheta} - e^{i\,\omega}|^{1+\tau p}}\, d\vartheta\, d\omega$$

converges to 0 in $L^1((\varepsilon, R))$. Hence, there exists a subsequence $\{f_{n_i}\}_{i\in\mathbb{N}}$ which converges almost everywhere to 0 on (ε, R). By taking a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ converging to 0 and repeating the above argument for each ε_k, a diagonal argument leads to the existence of a subsequence, still denoted by $\{f_{n_i}\}_{i\in\mathbb{N}}$, which converges almost everywhere to 0 on $(0, R)$. Equivalently, $\{u_{n_i}\}_{i\in\mathbb{N}}$ satisfies (B-1) for almost every $\varrho \in [0, R]$.

Let us now show (B-2). The proof is similar to that of [Bethuel and Demengel 1995, Lemma A.4]. For $\varrho \ge \varepsilon$, $t \ge 0$ and $\vartheta, \omega \in [0, 2\pi]$ we have

$$\left|u(\varrho\, e^{i\,\vartheta}) - u(\varrho\, e^{i\,\omega})\right|^p \le C\left|u(\varrho\, e^{i\,\vartheta}) - u((\varrho+t)\, e^{i\,\frac{\omega+\vartheta}{2}})\right|^p + C\left|u((\varrho+t)\, e^{i\,\frac{\omega+\vartheta}{2}}) - u(\varrho\, e^{i\,\omega})\right|^p,$$

and (for $\vartheta \ne \omega$)

$$\varrho^{-\tau p-1}\,|e^{i\,\vartheta} - e^{i\,\omega}|^{-\tau p-1} = (1+\tau p)\int_0^\infty [t + \varrho\,|e^{i\,\vartheta} - e^{i\,\omega}|]^{-\tau p-2}\, dt.$$

Thus from the definition of $\mathcal{W}_\varepsilon(u)$, we obtain with simple manipulations

$$\mathcal{W}_\varepsilon(u) \le C\int_0^\infty\int_\varepsilon^\infty\int_{[0,2\pi]\times[0,2\pi]} \frac{|u(\varrho\, e^{i\,\vartheta}) - u((\varrho+t)\, e^{i\,\frac{\vartheta+\omega}{2}})|^p}{(t+\varrho\,|e^{i\,\vartheta} - e^{i\,\omega}|)^{2+\tau p}}\,\varrho\, d\vartheta\, d\omega\, d\varrho\, dt.$$

Observe that

$$\left|\varrho\, e^{i\,\vartheta} - (\varrho+t)\, e^{i\,\frac{\vartheta+\omega}{2}}\right| \le t + \varrho\left|e^{i\,\vartheta} - e^{i\,\frac{\vartheta+\omega}{2}}\right|;$$

hence,

$$\begin{aligned}\mathcal{W}_\varepsilon(u) &\le C\int_0^\infty\int_\varepsilon^\infty\int_{[0,2\pi]\times[0,2\pi]} \frac{|u(\varrho\, e^{i\,\vartheta}) - u((\varrho+t)\, e^{i\,\frac{\vartheta+\omega}{2}})|^p}{|\varrho\, e^{i\,\vartheta} - (\varrho+t)\, e^{i\,\frac{\vartheta+\omega}{2}}|^{2+\tau p}}\,\varrho\, d\vartheta\, d\omega\, d\varrho\, dt \\ &\le 2\,\frac{C}{\varepsilon}\int_{[0,\infty)\times[0,\infty)}\int_{[0,2\pi]\times[0,2\pi]} \frac{|u(\varrho\, e^{i\,\vartheta}) - u(s\, e^{i\,\psi})|^p}{|\varrho\, e^{i\,\vartheta} - s\, e^{i\,\psi}|^{2+\tau p}}\,\varrho\, s\, d\vartheta\, d\psi\, d\varrho\, ds,\end{aligned}$$

which completes the proof of (B-2). □

The following result is a general fact for bounded Sobolev functions in the plane. This is exactly the same as [Santambrogio and Vespri 2010, Lemma 5]; we reproduce the proof for the reader's convenience.

Lemma B.3. *Let $q>1$ and let $\varphi \in W^{1,q}(B_R)\cap L^\infty(B_R)$ be a function such that $0\le\varphi\le M$. Let us suppose that there exists $0<\gamma<1$ such that*

$$\left|\left\{\varphi>\tfrac{3}{4}\,M\right\}\cap B_R\right|\ge \gamma\,|B_R|.$$

If we set $\delta=\sqrt{\frac{\gamma}{2}}$, one of the following two alternatives occur:

(A_1) *either*

$$\int_{B_R\setminus B_{\delta R}} |\nabla\varphi|^q\,dx \ge \frac{R^{2-q}}{8^q\cdot 4\cdot(2\,\pi)^{q-1}}\,\gamma\,M^q;$$

(A_2) *or the subset of $[\delta R, R]$ given by*

$$\left\{s\in[\delta R,R] : \varphi\ge \tfrac{5}{8}\,M,\ \mathcal{H}^1\text{-a.e. on } \partial B_s\right\}$$

has positive measure.

Proof. We first observe that thanks to the hypothesis we have

$$\begin{aligned}\left|\left\{\varphi>\tfrac{3}{4}\,M\right\}\cap(B_R\setminus B_{\delta R})\right| &= \left|\left\{\varphi>\tfrac{3}{4}\,M\right\}\cap B_R\right| - \left|\left\{\varphi>\tfrac{3}{4}\,M\right\}\cap B_{\delta R}\right| \\ &\ge \gamma\,|B_R|-|B_{\delta R}| \\ &= (\gamma-\delta^2)\,|B_R|.\end{aligned}$$

By the definition of δ, we get

$$\left|\left\{\varphi>\tfrac{3}{4}\,M\right\}\cap(B_R\setminus B_{\delta R})\right|\ge\tfrac{1}{2}\gamma\,|B_R|.$$

We define the set

$$\mathcal{X}=\left\{s\in[\delta R,R] : \mathcal{H}^1\left(\left\{x\in\partial B_s : \varphi(x)\ge\tfrac{3}{4}\,M\right\}\right)>0\right\}.$$

Then

$$\tfrac{1}{2}\gamma\,|B_R|\le\left|\left\{\varphi>\tfrac{3}{4}\,M\right\}\cap(B_R\setminus B_{\delta R})\right|=\int_{\mathcal{X}}\int_{\partial B_s} 1_{\{\varphi>3/4\,M\}}\,d\mathcal{H}^1\,ds\le 2\,\pi\int_{\mathcal{X}} s\,ds\le 2\,\pi\,R\,|\mathcal{X}|.$$

This in turn implies that

$$|\mathcal{X}|\ge\tfrac{1}{4}\gamma\,R.$$

Let us now suppose that alternative (A_2) does not occur. This implies that

$$\mathcal{H}^1\left(\left\{x\in\partial B_s : \varphi(x)<\tfrac{5}{8}\,M\right\}\right)>0\quad\text{for a.e. } s\in[\delta R,R].$$

Thus for almost every $s\in\mathcal{X}$, we have

$$\operatorname*{osc}_{\partial B_s}\varphi\ge\tfrac{3}{4}\,M-\tfrac{5}{8}\,M=\tfrac{1}{8}M.$$

By observing that ∂B_s is one-dimensional, we obtain

$$\tfrac{1}{8}M\le\operatorname*{osc}_{\partial B_s}\varphi\le\int_{\partial B_s}|\nabla_\tau\varphi|\,d\mathcal{H}^1\le(2\,\pi\,R)^{1-\frac{1}{q}}\left(\int_{\partial B_s}|\nabla_\tau\varphi|^q\,d\mathcal{H}^1\right)^{\frac{1}{q}},$$

where ∇_τ denotes the tangential gradient (by using polar coordinates $x = \varrho\, e^{i\,\vartheta}$, this is nothing but the ϑ-derivative). By taking the power q in the previous estimate and integrating in $s \in \mathcal{X}$, we get

$$\int_{B_R \setminus B_{\delta R}} |\nabla\varphi|^q\,dx \geq \int_{\mathcal{X}} \int_{\partial B_s} |\nabla\varphi|^q\,d\mathcal{H}^1 \geq \left(\tfrac{1}{8} M\right)^q \frac{1}{(2\,\pi\,R)^{q-1}}\,|\mathcal{X}|.$$

Using the lower-bound on $|\mathcal{X}|$ yields alternative (A_1). □

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APPLICATIONS OF SMALL-SCALE QUANTUM ERGODICITY IN NODAL SETS

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The goal of this article is to draw new applications of small-scale quantum ergodicity in nodal sets of eigenfunctions. We show that if quantum ergodicity holds on balls of shrinking radius $r(\lambda) \to 0$ then one can achieve improvements on the recent upper bounds of Logunov (2016) and Logunov and Malinnikova (2016) on the size of nodal sets, according to a certain power of $r(\lambda)$. We also show that the doubling estimates and the order-of-vanishing results of Donnelly and Fefferman (1988, 1990) can be improved. Due to results of Han (2015) and Hezari and Rivière (2016), small-scale QE holds on negatively curved manifolds at logarithmically shrinking rates, and thus we get logarithmic improvements on such manifolds for the above measurements of eigenfunctions. We also get $o(1)$ improvements for manifolds with ergodic geodesic flows. Our results work for a full density subsequence of any given orthonormal basis of eigenfunctions.

1. Introduction

Let (X, g) be a smooth compact connected boundaryless Riemannian manifold of dimension n. Suppose Δ_g is the positive Laplace–Beltrami operator on (X, g) and ψ_λ is a sequence of L^2 normalized eigenfunctions of Δ_g with eigenvalues λ. It was shown in [Hezari and Rivière 2016] that if for some shrinking radius $r = r(\lambda) \to 0$ and for all geodesic balls $B_r(x)$ one has $K_1 r^n \le \|\psi_\lambda\|^2_{B_r(x)} \le K_2 r^n$, then one gets improved upper bounds[1] of the form $(r^2\lambda)^{\delta(p)}$ on the L^p norms of ψ_λ, where $\delta(p)$ is Sogge's exponent. The purpose of this article is to prove more applications of small-scale L^2 equidistribution of eigenfunctions. We will show that upper bounds on the size of nodal sets, as well as the order of vanishing of eigenfunctions, can be improved by certain powers of r. Since by [Hezari and Rivière 2016][2] such equidistribution properties hold on negatively curved manifolds[3] with $r = (\log \lambda)^{-\kappa}$ for any $\kappa \in \left(0, \frac{1}{2n}\right)$, we obtain improvements of the results of [Logunov 2016a; Logunov and Malinnikova 2016; Donnelly and Fefferman 1988; 1990a; Dong 1992]. We also get slight improvements for quantum ergodic eigenfunctions because roughly speaking they equidistribute on balls of radius $r = o(1)$.

In the following $\mathcal{H}^{n-1}(Z_{\psi_\lambda})$ means the $(n-1)$-dimensional Hausdorff measure of the nodal set of ψ_λ, denoted by Z_{ψ_λ}, and $\nu_x(\psi_\lambda)$ means the order of vanishing of ψ_λ at a point x in X.

We recall that for $n \ge 3$, a recent result of [Logunov 2016a] gives a polynomial upper bound for $\mathcal{H}^{n-1}(Z_{\psi_\lambda})$ of the form λ^α for some $\alpha > \frac{1}{2}$ depending only on n, and for $n = 2$ another recent result

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[1] It was shown by Sogge [2016] that $\|\psi_\lambda\|^2_{B_r(x)} \le K_2 r^n$ suffices.

[2] In [Han 2015], this is proved for $\kappa \in \left(0, \frac{1}{3n}\right)$.

[3] For a full density subsequence of any given orthonormal basis of eigenfunctions.

of [Logunov and Malinnikova 2016] shows upper bounds of the form $\lambda^{\frac{3}{4}-\beta}$ for some small universal $\beta \in \left(0, \frac{1}{4}\right)$. Our first result is the following refinement of the results of the above-mentioned papers and also the order-of-vanishing results of [Donnelly and Fefferman 1988; 1990a; Dong 1992].

Theorem 1.1. *Let (X, g) be a boundaryless compact Riemannian manifold of dimension n with volume measure dv_g, and ψ_λ be an eigenfunction of Δ_g of eigenvalue $\lambda > 0$. Then there exists $r_0(g) > 0$ such that if $\lambda^{-\frac{1}{2}} < r_0(g)$, and if for some $r \in [\lambda^{-\frac{1}{2}}, r_0(g)]$ and for all geodesic balls $\{B_r(x)\}_{x \in X}$ we have*

$$K_1 r^n \leq \int_{B_r(x)} |\psi_\lambda|^2 \, dv_g \leq K_2 r^n \tag{1-1}$$

for some positive constants K_1 and K_2 independent of x, then:
For $n \geq 3$,

$$\mathcal{H}^{n-1}(Z_{\psi_\lambda}) \leq c_1 r^{2\alpha-1} \lambda^{\alpha}, \tag{1-2}$$

$$\nu_x(\psi_\lambda) \leq c_2 r \sqrt{\lambda}. \tag{1-3}$$

For $n = 2$,

$$\mathcal{H}^1(Z_{\psi_\lambda}) \leq c_3 r^{\frac{1}{2}-2\beta} \lambda^{\frac{3}{4}-\beta}, \tag{1-4}$$

$$\sum_{z \in Z_{\psi_\lambda} \cap B_{r^{1/2}\lambda^{-1/4}}(x)} (\nu_z(\psi_\lambda) - 1) \leq c_4 r \sqrt{\lambda}. \tag{1-5}$$

Here, $\alpha = \alpha(n) > \frac{1}{2}$ and $\beta \in \left(0, \frac{1}{4}\right)$ are the universal exponents from [Logunov 2016a; Logunov and Malinnikova 2016], *and the constants c_1, c_2, c_3, c_4 are positive and depend only on (X, g), K_1, and K_2, and are independent of λ, r, and x. Note that the quantity on the left-hand side of* (1-5) *counts the number of singular points*

$$\mathcal{S} = \{\psi_\lambda = |\nabla \psi_\lambda| = 0\}$$

in geodesic balls of radius $r^{\frac{1}{2}} \lambda^{-\frac{1}{4}}$.

Combining this with our result [Hezari and Rivière 2016], which states that on negatively curved manifolds (1-1) holds with $r = (\log \lambda)^{-\kappa}$ for any $\kappa \in \left(0, \frac{1}{2n}\right)$, the following unconditional results on such manifolds are immediate.

Theorem 1.2. *Let (X, g) be a boundaryless compact connected smooth Riemannian manifold of dimension n, with negative sectional curvatures. Let $\{\psi_{\lambda_j}\}_{j \in \mathbb{N}}$ be any orthonormal basis of $L^2(X)$ consisting of eigenfunctions of Δ_g with eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$. Let $\epsilon > 0$ be arbitrary. Then there exists $S \subset \mathbb{N}$ of* full density[4] *such that for $j \in S$,*

$$\text{if } n \geq 3, \quad \mathcal{H}^{n-1}(Z_{\psi_{\lambda_j}}) \leq c_1 (\log \lambda_j)^{\frac{1-2\alpha}{2n}+\epsilon} \lambda_j^{\alpha},$$

$$\text{if } n = 2, \quad \mathcal{H}^1(Z_{\psi_{\lambda_j}}) \leq c_3 (\log \lambda_j)^{-\frac{1}{8}+\frac{\beta}{2}+\epsilon} \lambda_j^{\frac{3}{4}-\beta}.$$

In addition, for all dimensions

$$\nu_x(\psi_{\lambda_j}) \leq c_2 (\log \lambda_j)^{-\frac{1}{2n}+\epsilon} \sqrt{\lambda_j}.$$

[4] It means that $\lim_{N\to\infty} \frac{1}{N} \operatorname{card}(S \cap [1, N]) = 1$.

We repeat that here $\alpha = \alpha(n) > \frac{1}{2}$ and $\beta \in \big(0, \frac{1}{4}\big)$ are the universal exponents from [Logunov 2016a; Logunov and Malinnikova 2016], and c_1, c_2, c_3 depend only on (X, g) and ϵ.

We will also prove the following $o(1)$ improvements for quantum ergodic sequences of eigenfunctions. In fact equidistribution on X (instead of the phase space S^*X) suffices.

Theorem 1.3. *Let (X, g) be a boundaryless compact connected smooth Riemannian manifold of dimension n. Let $\{\psi_{\lambda_j}\}_{j\in S}$ be a sequence of eigenfunctions of Δ_g with eigenvalues $\{\lambda_j\}_{j\in S}$ such that for all $r \in \big(0, \frac{1}{2}\operatorname{inj}(g)\big)$ and all $x \in X$*

$$\int_{B_r(x)} |\psi_{\lambda_j}|^2 \to \frac{\mathrm{Vol}_g(B_r(x))}{\mathrm{Vol}_g(X)}, \quad \lambda_j \xrightarrow{j\in S} \infty. \tag{1-6}$$

Then, along this sequence, for $n \ge 3$

$$\mathcal{H}^{n-1}(Z_{\psi_{\lambda_j}}) = o(\lambda_j^{\alpha}),$$

and for $n = 2$

$$\mathcal{H}^1(Z_{\psi_{\lambda_j}}) = o(\lambda_j^{\frac{3}{4}-\beta}).$$

Also in all dimensions

$$\nu_x(\psi_{\lambda_j}) = o(\sqrt{\lambda_j}) \quad (\textit{uniformly in } x).$$

In particular the above theorem holds for manifolds with ergodic geodesic flows by the quantum ergodicity theorem of Shnirel'man [1974], Colin de Verdière [1985] and Zelditch [1987]. Hence given any orthonormal basis of eigenfunctions, on such a manifold one can pass to a full density subsequence where (1-6), whence Theorem 1.3 holds.

Remark 1.4. We point out that the equidistribution property (1-6), which is weaker than quantum ergodicity, holds for some nonergodic manifolds such as the flat torus and the rational polygons; see [Marklof and Rudnick 2012; Rivière 2013; Taylor 2015].

Main idea. The major idea in proving our upper bounds is to lower the doubling index

$$N(B_s(x)) := \log\left(\frac{\sup_{B_{2s}(x)} |\psi_\lambda|^2}{\sup_{B_s(x)} |\psi_\lambda|^2}\right)$$

under the assumption

$$K_1 r^n \le \int_{B_r(x)} |\psi_j|^2 \le K_2 r^n.$$

We recall that Donnelly and Fefferman [1988] showed that an eigenfunction ψ_λ of Δ_g with eigenvalue λ satisfies

$$N(B_s(x)) \le c\sqrt{\lambda}$$

for all $s < s_0$, where s_0 and c depend only on (X, g). We will prove in Lemma 2.1 that

$$N(B_s(x)) \le c\, r\sqrt{\lambda} \tag{1-7}$$

for all $s < 10r$, where c depends only on (X, g). We then apply this modified growth estimate to the proofs of [Logunov 2016a; Logunov and Malinnikova 2016; Donnelly and Fefferman 1988; 1990a; Dong 1992] to obtain our improvements.

Remark 1.5. It is worth mentioning that in order to prove (1-2) of Theorem 1.1, we will need the improved doubling estimates (1-7) to hold for all $0 < s < 10r$ and not just s comparable to r. This is because the doubling exponent of a ball B (or a cube Q) as defined in [Logunov 2016a], see definition (2-12), is, roughly speaking, the supremum of $N(B_s(x))$ over all balls $B_s(x)$ contained in $2B$ (or $2Q$ respectively). The main result of that paper (see Theorem 2.5) gives an upper bound on the nodal sets in terms of this maximal doubling index. For the estimates (1-4) and (1-5) we need the validity of (1-7) for $0 < s < Cr^{\frac{1}{2}}\lambda^{-\frac{1}{4}}$.

Background on the size of nodal sets. For any smooth compact connected Riemannian manifold (X, g) of dimension n, Yau's conjecture states that there exist constants $c > 0$ and $C > 0$ independent of λ such that

$$c\sqrt{\lambda} \le \mathcal{H}^{n-1}(Z_{\psi_\lambda}) \le C\sqrt{\lambda}.$$

The conjecture was proved by Donnelly and Fefferman [1988] in the real analytic case. In dimension 2 and the C^∞ case, Brüning [1978] and Yau proved the lower bound $c\sqrt{\lambda}$. Until the recent result of Logunov and Malinnikova [2016] the best upper bound in dimension 2 was $C\lambda^{\frac{3}{4}}$, which was proved independently by Donnelly and Fefferman [1990a] and Dong [1992]. The result of Logunov and Malinnikova [2016] gives $C\lambda^{\frac{3}{4}-\beta}$ for some small universal constant $\beta < \frac{1}{4}$. In dimensions $n \ge 3$ until very recently, the best lower bound was $c\lambda^{\frac{3-n}{4}}$, proved[5] by Colding and Minicozzi [2011]. However, a recent breakthrough result of Logunov [2016b] proves the lower bound $c\sqrt{\lambda}$ for all $n \ge 3$. Also another result of Logunov [2016a] shows a polynomial upper bound $C\lambda^\alpha$ for some $\alpha > \frac{1}{2}$ which depends only on n. The best upper bound before this was the exponential bound $e^{c\sqrt{\lambda}\log\lambda}$ of Hardt and Simon [1989].

Background on small-scale quantum ergodicity. First, we recall that the quantum ergodicity result of Shnirel'man [1974], Colin de Verdière [1985] and Zelditch [1987] implies in particular that if the geodesic flow of a smooth compact Riemannian manifold without boundary is ergodic then for any orthonormal basis $\{\psi_{\lambda_j}\}_{j=1}^\infty$ consisting of the eigenfunctions of Δ_g, there exists a full density subset $S \subset \mathbb{N}$ such that for any $r < \mathrm{inj}(g)$, independent of λ_j, one has

$$\|\psi_{\lambda_j}\|^2_{L^2(B_r(x))} \sim \frac{\mathrm{Vol}_g(B_r(x))}{\mathrm{Vol}_g(X)}, \quad \text{as } \lambda_j \to \infty,\ j \in S. \tag{1-8}$$

The analogous result on manifolds with piecewise smooth boundary and with ergodic billiard flows was proved by Zelditch and Zworski [1996].

The small-scale equidistribution problem asks whether (1-8) holds for r dependent on λ_j. A quantitative QE result of Luo and Sarnak [1995] shows that the Hecke eigenfunctions on the modular surface satisfy

[5]Different proofs were given later by [Hezari and Wang 2012; Hezari and Sogge 2012; Sogge and Zelditch 2012] based on the earlier work [Sogge and Zelditch 2011], and by [Steinerberger 2014] using heat equation techniques. Also logarithmic improvements of the form $\lambda^{\frac{3-n}{4}}(\log\lambda)^\alpha$ were given in [Hezari and Rivière 2016] on negatively curved manifolds and in [Blair and Sogge 2015] on nonpositively curved manifolds.

this property along a density one subsequence for $r = \lambda^{-\kappa}$ for some small $\kappa > 0$. Also, under the Generalized Riemann Hypothesis, Young [2016] proved that small-scale equidistribution holds for Hecke eigenfunctions for $r = \lambda^{-\frac{1}{4}+\epsilon}$.

This problem was studied in [Han 2015; Hezari and Rivière 2016] for the eigenfunctions of negatively curved manifolds. To be precise, it was proved that on compact negatively curved manifolds without boundary, for any $\epsilon > 0$ and any orthonormal basis $\{\psi_{\lambda_j}\}_{j=1}^{\infty}$ of $L^2(X)$ consisting of the eigenfunctions of Δ_g, there exists a subset $S \subset \mathbb{N}$ of full density such that for all $x \in X$ and $j \in S$,

$$K_1 r^n \leq \|\psi_{\lambda_j}\|^2_{L^2(B_r(x))} \leq K_2 r^n, \quad \text{with } r = (\log \lambda_j)^{-\frac{1}{2n}+\epsilon}, \tag{1-9}$$

for some positive constants K_1, K_2 which depend only on (X, g) and ϵ. The same result was proved in [Han 2015] for $r = (\log \lambda_j)^{-\frac{1}{3n}+\epsilon}$.

We also point out that although eigenfunctions on the flat torus $\mathbb{R}^n/\mathbb{Z}^n$ are not quantum ergodic, they equidistribute on the configuration space $\mathbb{R}^n/\mathbb{Z}^n$; see [Marklof and Rudnick 2012], and also [Rivière 2013; Taylor 2015] for later proofs. So one can investigate the small-scale equidistribution property for toral eigenfunctions. It was proved in [Hezari and Rivière 2017] that a commensurability of L^2 masses such as (1-9) is valid for a full density subsequence with $r = \lambda^{-\frac{1}{7n+4}}$. Lester and Rudnick [2017] improved this rate of shrinking to $r = \lambda^{-\frac{1}{2n-2}+\epsilon}$, and in fact they proved that the stronger statement (1-8) holds. They also showed that their results are almost[6] sharp. The case of interest is $n = 2$, which gives $r = \lambda^{-\frac{1}{2}+\epsilon}$. A natural conjecture is that this should be the optimal rate of shrinking on negatively curved manifolds. A recent result of [Han 2017] proves that random eigenbases on the torus enjoy small-scale QE for $r = \lambda^{-\frac{n-2}{4n}+\epsilon}$, which is better than [Lester and Rudnick 2017] for $n \geq 5$.

Some remarks.

Remark 1.6. In our proof we have used both *local and global harmonic analysis*; see [Zelditch 2008] for background. The local analysis is used in [Logunov 2016a; Logunov and Malinnikova 2016], and the global analysis is used in [Hezari and Rivière 2016] to obtain equidistribution on small balls. We emphasize that our improvements of [Logunov 2016a; Logunov and Malinnikova 2016] are robust, in the sense that any upper bounds of the form λ^{α} for $\alpha > \frac{1}{2}$ that result from a purely local analysis of eigenfunctions can be improved using our combined method.

Remark 1.7. The most important assumption of Theorem 1.1 is the lower bound $K_1 r^n \leq \int_{B_r(x)} |\psi_\lambda|^2$ and the upper bound in (1-1) can be discarded at the expense of messy estimates in Theorem 1.1. In fact using Sogge's "trivial local L^2 estimates" [2016], which assert that one always has $\int_{B_r(x)} |\psi_\lambda|^2 \leq K_2 r$, we can still prove modified doubling estimates of the form

$$\sup_{B_{2s}(x)} |\psi_\lambda|^2 \leq r^{-b} e^{cr\sqrt{\lambda}} \sup_{B_s(x)} |\psi_\lambda|^2 \quad \text{for some } b = b(n) > 0 \text{ and all } s < 10r.$$

We can use this inequality and obtain estimates similar to those in Theorem 1.1; however we have not done so for the sake of more polished estimates. Another reason that we have not discarded the assumption

[6] They showed that the equidistribution property fails for $r = \lambda^{-\frac{1}{2n-2}-\epsilon}$ for a positive density subset of some orthonormal basis.

$\int_{B_r(x)} |\psi_\lambda|^2 \le K_2 r^n$ is that all the examples (such as QE eigenfunctions) for which we know the lower bounds are satisfied, also satisfy the upper bounds in (1-1).

Remark 1.8. As we discussed in the previous section, a result of [Luo and Sarnak 1995] implies that small-scale QE holds for a full density subsequence of Hecke eigenfunctions on the modular surface for balls of radius $r = \lambda^{-\kappa}$ for some explicitly calculable $\kappa > 0$. Hence using (1-3), we get upper bounds of the form $\lambda^{\frac{1}{2}-\kappa}$ on the order of vanishing of these eigenfunctions. We could not find any arithmetic results in the literature discussing improvements on the upper bound $\sqrt{\lambda}$ of Donnelly and Fefferman. Of course a natural conjecture to impose is that for Hecke eigenfunctions $\nu_x(\psi_\lambda) \le c\lambda^\epsilon$. Although the available graphs of nodal lines of Hecke eigenfunctions with high energy do not show any singular points, i.e., places where nodal lines intersect each other, there are many almost-intersecting nodal lines.

Remark 1.9. By our discussion in the previous section on the work of [Lester and Rudnick 2017], and using (1-3), we get that for a full density subsequence of toral eigenfunctions on the 2-torus, we have $\nu_x(\psi_\lambda) \le c\lambda^\epsilon$. However, it is proved in [Bourgain and Rudnick 2011] that $\nu_x(\psi_\lambda) \le c\lambda^{\frac{c}{\log\log\lambda}}$ for all eigenfunctions on $\mathbb{T}^2$.

Remark 1.10. Theorem 1.1 is local in nature, meaning that if the eigenfunctions satisfy (1-1) for balls centered on an open set, then we get the upper bounds in this theorem on that open set. In particular we get all the upper bounds in Theorem 1.3 for eigenfunctions on ergodic billiards (and also rational polygons) as long as we stay a positive distance away from the boundary. One would expect that the results of [Logunov 2016a; Logunov and Malinnikova 2016] can be extended to the eigenfunctions of the Laplacian on manifolds with boundary (with Dirichlet or Neumann boundary conditions) using the method of [Donnelly and Fefferman 1990b].

2. Proofs of upper bounds for nodal sets and order of vanishing

The following lemma is the main ingredient of the proofs. It gives improved growth estimates for eigenfunctions under our L^2 assumption on small balls.

Lemma 2.1. *Let (X, g) be a smooth Riemannian manifold, $p \in X$ a fixed point, and $R > 0$ a fixed radius so that the geodesic ball $B_{2R}(p)$ is embedded. Then there exists $r_0(g)$ such that the following statement holds*:

Suppose $\lambda^{-\frac{1}{2}} \le r_0(g)$ and ψ_λ is a smooth function such that $\Delta_g\psi_\lambda = \lambda\psi_\lambda$ on $B_{2R}(p)$. If for some $r \in [\lambda^{-\frac{1}{2}}, r_0(g)]$ and all $x \in B_R(p)$

$$K_1 r^n \le \int_{B_r(x)} |\psi_\lambda|^2 \le K_2 r^n \tag{2-1}$$

holds for some positive constants K_1 and K_2 independent of x, then one has the refined doubling estimates

$$\text{for } \delta \in (0, 10r),\ x \in B_{\frac{R}{2}}(p), \qquad \int_{B_{2\delta}(x)} |\psi_\lambda|^2 \le e^{cr\sqrt{\lambda}} \int_{B_\delta(x)} |\psi_\lambda|^2, \tag{2-2}$$

$$\text{for } \delta \in (0, 10r),\ x \in B_{\frac{R}{2}}(p), \qquad \sup_{B_{2\delta}(x)} |\psi_\lambda|^2 \le e^{cr\sqrt{\lambda}} \sup_{B_\delta(x)} |\psi_\lambda|^2. \tag{2-3}$$

We also have

$$\text{for } \delta \in \big(0, \tfrac{1}{2}r\big),\ x \in B_{\frac{R}{2}}(p), \quad \frac{1}{\delta^n}\int_{B_\delta(x)} |\psi_\lambda|^2 \geq \left(\frac{r}{\delta}\right)^{-c\,r\sqrt{\lambda}}, \tag{2-4}$$

$$\text{for } \delta \in \big(0, \tfrac{1}{2}r\big),\ x \in B_{\frac{R}{2}}(p), \qquad \sup_{B_\delta(x)} |\psi_\lambda|^2 \geq \left(\frac{r}{\delta}\right)^{-c\,r\sqrt{\lambda}}. \tag{2-5}$$

Here c is positive and is uniform in x, r, δ, and λ, but depends on K_1, K_2, and $(B_{2R}(p), g)$.

Proof. We will give two proofs for (2-3). All other statements will follow from this as we will show. The first proof of (2-3) follows from a rescaling argument applied to the following theorem of Donnelly and Fefferman, which is a purely local result based on Carleman estimates. The second proof relies on a theorem of [Mangoubi 2013].

Theorem 2.2 [Donnelly and Fefferman 1988, Proposition 3.10(ii)]. *Let $(\widetilde{X}, \tilde{g})$ be a smooth Riemannian manifold, $p \in \widetilde{X}$ a fixed point, and $\widetilde{R} > 0$ a fixed radius such that the $\tilde{g}$-geodesic ball $\widetilde{B}_{2\widetilde{R}}(p)$ is embedded. Let $\psi_{\tilde\lambda}$ be a smooth function such that for some $\tilde\lambda \geq 1$ we have $\Delta_{\tilde g}\psi_{\tilde\lambda} = \tilde\lambda\psi_{\tilde\lambda}$ on $\widetilde{B}_{2\widetilde{R}}(p)$. Then there exists a suitably small $h_0(\tilde g) > 0$ such that for all $h \leq h_0(\tilde g)$, $\delta < \frac{1}{2}h$, and $x \in \widetilde{B}_{\frac{\widetilde{R}}{2}}(p)$,*

$$\sup_{\widetilde{B}_{2\delta}(x)} |\psi_{\tilde\lambda}|^2 \leq e^{\kappa_1\sqrt{\tilde\lambda}}\left(\frac{\sup_{\widetilde{B}_h(x)}|\psi_{\tilde\lambda}|^2}{\sup_{\widetilde{B}_{h/5}(x)\setminus\widetilde{B}_{h/10}(x)}|\psi_{\tilde\lambda}|^2}\right)^{\kappa_2}\sup_{\widetilde{B}_\delta(x)}|\psi_{\tilde\lambda}|^2. \tag{2-6}$$

The constant $h_0(\tilde g)$ is controlled by $\widetilde{R}$ and the reciprocal of the square root of $\sup_{\widetilde{B}_{2\widetilde{R}}(p)}|\mathrm{Sec}(\tilde g)|$, and the constants κ_1 and κ_2 are controlled by $\sup_{\widetilde{B}_{2\widetilde{R}}(p)}|\mathrm{Sec}(\tilde g)|$.

To prove our lemma, we define $(\widetilde{X}, \tilde g) = (X, \frac{1}{r^2}g)$, and $\widetilde{R} = \frac{1}{r}R$. Then the equation

$$\Delta_g\psi_\lambda = \lambda\psi_\lambda \quad \text{on } B_{2R}(p)$$

becomes

$$\Delta_{\tilde g}\psi_{\tilde\lambda} = \tilde\lambda\psi_{\tilde\lambda} \quad \text{on } \widetilde{B}_{2\widetilde{R}}(p),$$

with

$$\tilde\lambda = r^2\lambda \quad \text{and} \quad \psi_{\tilde\lambda} = \psi_\lambda.$$

We then note that by [Donnelly and Fefferman 1988], although not explicitly stated, we have

$$h_0(\tilde g) = C\min\Big(\tfrac{1}{2}\widetilde{R}, \big(\sup_{\widetilde{B}_{2R}(p)}|\mathrm{Sec}(\tilde g)|\big)^{-\frac{1}{2}}\Big) = \frac{C}{r}\min\Big(\tfrac{1}{2}R, \big(\sup_{B_{2R}(p)}|\mathrm{Sec}(g)|\big)^{-\frac{1}{2}}\Big)$$

for some suitably small C that is uniform in r. Hence if we set

$$r_0(g) \leq \tfrac{1}{20}C\min\Big(\tfrac{1}{2}R, \big(\sup_{B_{2R}(p)}|\mathrm{Sec}(g)|\big)^{-\frac{1}{2}}\Big)$$

then for all $r \leq r_0(g)$ we have $h_0(\tilde g) \geq 20$, and therefore we can choose $h = 20$. As a result, by (2-6)

$$\text{for } \delta \in (0, 10),\ x \in \widetilde{B}_{\frac{\widetilde{R}}{2}}(p), \qquad \sup_{\widetilde{B}_{2\delta}(x)}|\psi_{\tilde\lambda}|^2 \leq e^{\kappa_1\sqrt{\tilde\lambda}}\left(\frac{\sup_{\widetilde{B}_{20}(x)}|\psi_{\tilde\lambda}|^2}{\sup_{\widetilde{B}_4(x)\setminus\widetilde{B}_2(x)}|\psi_{\tilde\lambda}|^2}\right)^{\kappa_2}\sup_{\widetilde{B}_\delta(x)}|\psi_{\tilde\lambda}|^2.$$

Writing this inequality with respect to the metric g we get

$$\text{for } \delta \in (0, 10r),\ x \in B_{\frac{R}{2}}(p), \qquad \sup_{B_{2\delta}(x)} |\psi_\lambda|^2 \le e^{\kappa_1 r\sqrt{\lambda}} \left(\frac{\sup_{B_{20r}(x)} |\psi_\lambda|^2}{\sup_{B_{4r}(x)\setminus B_{2r}(x)} |\psi_\lambda|^2} \right)^{\kappa_2} \sup_{B_\delta(x)} |\psi_\lambda|^2. \tag{2-7}$$

Remark 2.3. We emphasize that since $|\operatorname{Sec}(\tilde{g})| = r^2 |\operatorname{Sec}(g)|$, and since r is bounded by $r_0(g)$, the constants κ_1 and κ_2 can be chosen independently from r.

We now bound the expression in parenthesis using our local L^2 assumptions (2-1). First we find y such that

$$B_r(y) \subset B_{4r}(x) \setminus B_{2r}(x).$$

Since by assumption $\int_{B_r(y)} |\psi_j|^2 \ge K_1 r^n$, we must have

$$\sup_{B_{4r}(x)\setminus B_{2r}(x)} |\psi_\lambda|^2 \ge \sup_{B_r(y)} |\psi_\lambda|^2 \ge \frac{r^n}{\operatorname{Vol}(B_r(y))} K_1.$$

By making $r_0(g)$ sufficiently smaller, we obtain that for any $r \le r_0(g)$ which satisfies (2-1), we have

$$\sup_{B_{4r}(x)\setminus B_{2r}(x)} |\psi_\lambda|^2 \ge a K_1 \tag{2-8}$$

for some constant a which is uniform in $x \in B_{\frac{R}{2}}(p)$, $r \in (0, r_0(g))$, and λ. For the numerator in the parenthesis we claim that[7]

$$\sup_{B_{20r}(x)} |\psi_\lambda|^2 \le b K_2 (r\sqrt{\lambda})^n \tag{2-9}$$

for some constant b which is uniform in $x \in B_{\frac{R}{2}}(p)$, $r \in (0, r_0(g))$ and λ. To prove (2-9) we cover $B_{20r}(x)$ using balls of radius $\frac{r}{2}$. It is therefore enough to show that

$$\sup_{B_{r/2}(y)} |\psi_\lambda|^2 \le b \lambda^{\frac{n}{2}} \sup_{z \in B_{r/2}(y)} \int_{B_r(z)} |\psi_\lambda|^2 \tag{2-10}$$

for some b that is uniform in y, r, and λ. This estimate, however, follows from standard elliptic estimates, see for example [Gilbarg and Trudinger 1998, Theorem 8.17 and Corollary 9.21], which assert that there exists $a_0 < 1$ suitably small such that for $z \in B_R(p)$ we have

$$\text{for all } s \in (0, a_0 \lambda^{-\frac{1}{2}}], \qquad \sup_{B_{s/2}(z)} |\psi_\lambda|^2 \le b_0 s^{-n} \int_{B_s(z)} |\psi_\lambda|^2 \tag{2-11}$$

for some b_0 which is uniform in λ, z, and s. Since $\lambda^{-\frac{1}{2}} \le r$, we have $B_{a_0 \lambda^{-1/2}}(z) \subset B_r(z)$ and hence to get (2-10) we just need to observe that

$$\sup_{B_{r/2}(y)} |\psi_\lambda|^2 \le \sup_{z \in B_{r/2}(y)} \sup_{B_{(a_0/2)\lambda^{-1/2}}(z)} |\psi_\lambda|^2 \le b \lambda^{\frac{n}{2}} \sup_{z \in B_{r/2}(y)} \int_{B_r(z)} |\psi_\lambda|^2,$$

[7] In fact when (X, g) is a *closed manifold* the better estimate $bK_2(r\sqrt{\lambda})^{n-1}$ holds using Sogge's local L^∞ estimates [2016], but we do not need this better estimate.

with $b = b_0 a_0^{-n}$. Now we apply (2-8) and (2-9) to (2-7) to achieve

$$\text{for } \delta \in (0, 10r),\ x \in B_{\frac{R}{2}}(p), \qquad \sup_{B_{2\delta}(x)} |\psi_\lambda|^2 \le d e^{\kappa_1 r\sqrt{\lambda}} (r\sqrt{\lambda})^{n\kappa_2} \sup_{B_\delta(x)} |\psi_\lambda|^2$$

for some uniform constant d which depends on K_1 and K_2. We note since $r\sqrt{\lambda} \ge 1$, if we choose M to be an integer larger than κ_1 and $n\kappa_2$ then

$$(r\sqrt{\lambda})^{n\kappa_2} e^{\kappa_1 r\sqrt{\lambda}} \le M!\, e^{2Mr\sqrt{\lambda}}.$$

Finally by choosing

$$c \ge \max(\log d, M \log M, 2M),$$

we get (2-3).

To prove (2-2) we use (2-3). It is enough to show that

$$\frac{\int_{B_{2\delta}(x)} |\psi_\lambda|^2}{\int_{B_\delta(x)} |\psi_\lambda|^2} \le K(\delta\sqrt{\lambda})^n \frac{\sup_{B_{2\delta}(x)} |\psi_\lambda|^2}{\sup_{B_{\delta/2}(x)} |\psi_\lambda|^2},$$

because $(\delta\sqrt{\lambda})^n \le (10r\sqrt{\lambda})^n \le e^{cr\sqrt{\lambda}}$ for some appropriate c, as we found in the above argument. The above comparison of ratios follows from the trivial estimate

$$\int_{B_{2\delta}(x)} |\psi_\lambda|^2 \le \frac{1}{a}\,(2\delta)^n \sup_{B_{2\delta}(x)} |\psi_\lambda|^2$$

applied to the numerator, and the estimate

$$\int_{B_\delta(x)} |\psi_\lambda|^2 \ge \frac{1}{b_0}\left(\min\left(\lambda^{-\frac{1}{2}}, \tfrac{1}{4}\delta\right)\right)^n \sup_{B_{\delta/2}(x)} |\psi_\lambda|^2$$

applied to the denominator. The last estimate follows from the elliptic estimate (2-11) by setting $s = \min\left(a_0\lambda^{-\frac{1}{2}}, \frac{1}{4}\delta\right)$ and writing

$$\sup_{B_{\delta/2}(x)} |\psi_\lambda|^2 \le \sup_{z\in B_{\delta/2}(x)} \sup_{B_{s/2}(z)} |\psi_\lambda|^2 \le b_0 s^{-n} \sup_{z\in B_{\delta/2}(x)} \int_{B_s(z)} |\psi_\lambda|^2 \le b_0 s^{-n} \int_{B_\delta(x)} |\psi_\lambda|^2.$$

The proofs of (2-4) and (2-5) are obtained by iterations of inequalities (2-2) and (2-3). Since they are very similar we only give the proof of (2-5). Fix $\delta \le \frac{r}{2}$ and let m be the greatest integer such that $2^{m-1}\delta \le r$. Then if we write inequalities (2-3) for $\delta, 2\delta, 4\delta, \dots, 2^{m-1}\delta$ and multiply them all we get

$$\sup_{B_\delta(x)} |\psi_\lambda|^2 \ge e^{-mcr\sqrt{\lambda}} \sup_{B_{2^m\delta}(x)} |\psi_\lambda|^2.$$

Because of the choice of m, we have $2^m\delta > r$. Hence

$$\sup_{B_\delta(x)} |\psi_\lambda|^2 \ge e^{-mcr\sqrt{\lambda}} \sup_{B_r(x)} |\psi_\lambda|^2 \ge \frac{e^{-mcr\sqrt{\lambda}}}{\mathrm{Vol}(B_r(x))} \int_{B_r(x)} |\psi_\lambda|^2 \ge aK_2 e^{-mcr\sqrt{\lambda}}.$$

Since $m \ge \log\left(\frac{r}{\delta}\right)$ and $r\sqrt{\lambda} \ge 1$, by selecting c slightly larger the lower bound (2-5) follows. $\square$

***Second proof of improved L^∞-growth estimates* (2-3).** We recall the following result of [Mangoubi 2013], which is similar to estimate (2-7).

Theorem 2.4 [Mangoubi 2013, Theorem 3.2]. *Let (X, g) be a smooth Riemannian manifold, $p \in X$, and $R > 0$ so that the geodesic ball $B_{2R}(p)$ is embedded, and set $S = \sup_{B_{2R(p)}} |\mathrm{Sec}(g)|$. Suppose ψ_λ is a smooth function such that $\Delta_g \psi_\lambda = \lambda \psi_\lambda$ on $B_{2R}(p)$ for some $\lambda \geq 0$. Then for all $\delta \leq s \leq \min\bigl(CS^{-\frac{1}{2}}, \frac{1}{6}R\bigr)$, and all $x \in B_{\frac{R}{2}}(p)$*

$$\sup_{B_{3\delta}(x)} |\psi_\lambda|^2 \leq c_0 e^{c_1 s\sqrt{\lambda}} \left(\frac{\sup_{B_{3s}(x)} |\psi_\lambda|^2}{\sup_{B_s(x)} |\psi_\lambda|^2} \right)^{1+c_2\delta^2 S} \sup_{B_{2\delta}(x)} |\psi_\lambda|^2,$$

where C, c_1 and c_2 are positive constants which depend only on R, and c_0 depends on bounds on $(g^{-1})_{ij}$, its derivatives and its ellipticity constant on the ball $B_{2R}(p)$.

Using this theorem twice, we get for $\delta \leq s \leq \min\bigl(CS^{-\frac{1}{2}}, \frac{1}{6}R\bigr)$

$$\frac{\sup_{B_{2\delta}(x)} |\psi_\lambda|^2}{\sup_{B_\delta(x)} |\psi_\lambda|^2} \leq \frac{\sup_{B_{(9/4)\delta}(x)} |\psi_\lambda|^2}{\sup_{B_{(3/2)\delta}(x)} |\psi_\lambda|^2} \frac{\sup_{B_{(3/2)\delta}(x)} |\psi_\lambda|^2}{\sup_{B_\delta(x)} |\psi_\lambda|^2} \leq c_0^2 e^{2c_1 s\sqrt{\lambda}} \left(\frac{\sup_{B_{3s}(x)} |\psi_\lambda|^2}{\sup_{B_s(x)} |\psi_\lambda|^2} \right)^{2+c_2'\delta^2 S}$$

for a new constant c_2'. Now we choose $r_0(g) \leq \frac{1}{10} \min\bigl(CS^{-\frac{1}{2}}, \frac{1}{6}R\bigr)$, we put $s = 10r$, and argue as we did following inequality (2-7).

Proof of* (1-3): *upper bound on the order of vanishing. Let us show that the upper bound (1-3) on the order of vanishing $\nu_x(\psi_\lambda)$ follows from the lower bound (2-5). Suppose ψ_λ vanishes at x to order M. Then there exists $\delta_0 > 0$ such that for all $\delta < \delta_0$

$$C_{\psi_\lambda,\delta_0} \delta^M \geq \sup_{B_\delta(x)} |\psi_\lambda|^2.$$

Therefore using (2-5), for all $0 < \delta < \min\bigl(\delta_0, \frac{1}{2}r\bigr)$

$$C_{\psi_\lambda,\delta_0} \delta^M \geq \left(\frac{\delta}{r}\right)^{cr\sqrt{\lambda}}.$$

Dividing by δ^M and letting $\delta \to 0$ we see that we must have $M \leq cr\sqrt{\lambda}$.

***Proof of* (1-2): *upper bounds on the size of nodal sets for n* ≥ 3.** The main tool is the following result.

Theorem 2.5 [Logunov 2016a, Theorem 6.1]. *Let $(\widetilde{X}, \tilde{g})$ be a smooth Riemannian manifold of dimension d, $\tilde{p} \in \widetilde{X}$, and $\widetilde{R} > 0$ so that the geodesic ball $B_{2\widetilde{R}}(\tilde{p})$ is embedded. Suppose H is a harmonic function on $B_{2\widetilde{R}}(\tilde{p})$; that is, $\Delta_{\tilde{g}} H = 0$ on $B_{2\widetilde{R}}(\tilde{p})$. Then there exists $R_0 = R_0(B_{2\widetilde{R}}(\tilde{p}), g) < \widetilde{R}$ such that for any Euclidean*[8] *cube $\widetilde{Q} \subset B_{R_0}(\tilde{p})$ one has*

$$\mathcal{H}^{d-1}\bigl(\{H = 0\} \cap \widetilde{Q}\bigr) \leq \kappa \ \mathrm{diam}(\widetilde{Q})^{d-1} N(H, \widetilde{Q})^{2\alpha}$$

[8] It means that $\widetilde{Q}$ is a cube in the chart associated to the geodesic normal coordinates at $\tilde{p}$.

for some $\alpha > \frac{1}{2}$ that is only dependent on d, and some κ that depends only on $(B_{2\widetilde{R}}(\tilde{p}), g)$. Here,

$$N(H, \widetilde{Q}) = \sup_{B^e_s(x) \subset 2\widetilde{Q}} \log\left(\frac{\sup_{B^e_{2s}(x)} |H|^2}{\sup_{B^e_s(x)} |H|^2}\right), \tag{2-12}$$

where $B^e_s(x)$ stands for the Euclidean ball of radius s centered at x in the normal chart of $B_{\widetilde{R}}(\tilde{p})$.

To prove (1-2), we use our modified growth estimates (2-2) and the above theorem. We first cover (X, g) using geodesic balls $\{B_r(x_i)\}_{x_i \in \mathcal{I}}$ such that each point in X is contained in $C(X, g)$ many of the double balls $\{B_{2r}(x_i)\}_{x_i \in \mathcal{I}}$, where $C(X, g)$ is independent of r and depends only on the injectivity radius of (X, g) and a bound on the Ricci curvature of (X, g). Such a thing is possible by the Bishop–Gromov volume comparison theorem. For a proof see, for example, [Colding and Minicozzi 2011, Lemma 2]. It is then easy to see that such a covering has at most $C_0 r^{-n}$ open balls for some uniform constant $C_0 = C_0(X, g)$. Next we estimate $\mathcal{H}^{n-1}(Z_{\psi_\lambda} \cap B_r(p))$ for each $p \in \mathcal{I}$. To do this we define

$$\widetilde{X} = X \times \mathbb{R}, \quad d = n + 1, \quad \tilde{g} = \text{product metric}.$$

We shall also use $\tilde{x} = (x, t)$. We then put

$$H(\tilde{x}) = \psi_\lambda(x) e^{t\sqrt{\lambda}}.$$

Then clearly $\Delta_{\tilde{g}} H = 0$. We now cover the compact manifold $X \times [-1, 1]$ by finitely many balls $\{\widetilde{B}_j\}_{1 \le j \le M}$ each of which satisfies the property of the ball $\widetilde{B}_{R_0}$ in Theorem 2.5. Let L_0 be the Lebesgue number of this finite cover and assume $r \le \frac{1}{2} L_0$. Also for each $p \in X$, let $Q_r(p)$ be the Euclidean cube in X of side lengths $2r$ centered at p. Then we observe that for some $1 \le j \le M$ we have

$$\widetilde{Q}_r(\tilde{p}) := Q_r(p) \times [-r, r] \subset \widetilde{B}_{2r}(\tilde{p}) \subset \widetilde{B}_{L_0}(\tilde{p}) \subset \widetilde{B}_j,$$

where $\tilde{p} = (p, 0)$. By applying Theorem 2.5 for the cube $\widetilde{Q}(\tilde{p})$ in the ball $\widetilde{B}_j$, we get that

$$\begin{aligned}
\mathcal{H}^{n-1}\big(\{\psi_\lambda = 0\} \cap B_r(p)\big) &\le \mathcal{H}^{n-1}\big(\{\psi_\lambda = 0\} \cap Q_r(p)\big) \\
&= \frac{1}{2r} \mathcal{H}^n\big(\{H = 0\} \cap \widetilde{Q}_r(\tilde{p})\big) \\
&\le \frac{\kappa}{2r} (2r)^n N(H, \widetilde{Q}_r(\tilde{p}))^{2\alpha} \\
&= \kappa' r^{n-1} N(H, \widetilde{Q}_r(\tilde{p}))^{2\alpha}.
\end{aligned}$$

Now we use our doubling estimates to show that $N(H, \widetilde{Q}_r(\tilde{p})) \le c' r \sqrt{\lambda}$ for some c' that is uniform in r, λ, and p. We emphasize that our doubling estimates involve geodesic balls, but the definition of the doubling index N in [Logunov 2016a] uses Euclidean balls $B^e_s(\tilde{x})$ in a fixed normal chart of $B_{2\widetilde{R}}(\tilde{p})$. However, by choosing R_0 sufficiently small we can make sure that

$$B_{\frac{s}{2}}(\tilde{x}) \subset B^e_s(\tilde{x}) \subset B_{\frac{3s}{2}}(\tilde{x})$$

for all $\tilde{x} \in B_{R_0}(\tilde{p})$ and all $s < R_0$. As a result of this if we assume $r < \frac{1}{20} R_0$, then using (2-3) four times we get

$$\begin{aligned} N(H, \widetilde{Q}_r(\tilde{p})) &= \sup_{B^e_s(\tilde{x}) \subset \widetilde{Q}_{2r}(\tilde{p})} \log\left(\frac{\sup_{B^e_{2s}(\tilde{x})} |H(\tilde{x})|^2}{\sup_{B^e_s(\tilde{x})} |H(\tilde{x})|^2} \right) \\ &\le \sup_{B_{s/2}(\tilde{x}) \subset \widetilde{Q}_{2r}(\tilde{p})} \log\left(\frac{\sup_{B_{3s}(\tilde{x})} |H(\tilde{x})|^2}{\sup_{B_{s/2}(\tilde{x})} |H(\tilde{x})|^2} \right) \\ &\le \sup_{B_{s/2}(x) \subset Q_{2r}(p)} \log\left(e^{5s\sqrt{\lambda}} \frac{\sup_{B_{3s}(x)} |\psi_\lambda(x)|^2}{\sup_{B_{s/4}(x)} |\psi_\lambda(x)|^2} \right) \le c' r \sqrt{\lambda}. \end{aligned}$$

Finally

$$\mathcal{H}^{n-1}(Z_{\psi_\lambda}) \le \sum_{x_i \in \mathcal{I}} \mathcal{H}^{n-1}(Z_{\psi_\lambda} \cap B_r(x_i)) \le C_0 r^{-n} \kappa' r^{n-1} (c'^2 r^2 \lambda)^\alpha \le c_1 r^{2\alpha - 1} \lambda^\alpha$$

for some c_1 that is uniform in r and λ.

Proof of* (1-4): *upper bounds on the size of nodal sets for surfaces. The main tool is the following local result.

Theorem 2.6 [Logunov and Malinnikova 2016]. *Let $(\widetilde{X}, \tilde{g})$ be a smooth Riemannian manifold of dimension $n = 2$, $p \in \widetilde{X}$ a point, and $\widetilde{R} > 0$ a radius such that the $\tilde{g}$-geodesic ball $\widetilde{B}_{2\widetilde{R}}(p)$ is embedded. Let $\psi_{\tilde{\lambda}}$ be a smooth function such that for some $\tilde{\lambda} \ge 1$ we have $\Delta_{\tilde{g}} \psi_{\tilde{\lambda}} = \tilde{\lambda} \psi_{\tilde{\lambda}}$ on $\widetilde{B}_{2\widetilde{R}}(p)$. Suppose we also know that there exists some $s_0 \le \frac{1}{10} R$ such that for all $s < s_0$ we have*

$$\frac{\sup_{\widetilde{B}_{2s}(x)} |\psi_{\tilde{\lambda}}|^2}{\sup_{\widetilde{B}_s(x)} |\psi_{\tilde{\lambda}}|^2} \le C_1 e^{c\sqrt{\tilde{\lambda}}}$$

for some constants c and C_1 that are uniform for $x \in \widetilde{B}_{\widetilde{R}}(p)$. Then

$$\mathcal{H}^1_{\tilde{g}}\big(\{\psi_{\tilde{\lambda}} = 0\} \cap \widetilde{B}_{\frac{\widetilde{R}}{2}}(p)\big) \le C_2 \tilde{\lambda}^{\frac{3}{4} - \beta}, \tag{2-13}$$

where $\beta \in \big(0, \frac{1}{4}\big)$ is a small universal constant and C_2 is controlled by c, C_1, and the $\mathcal{C}^k$ norm of $(\tilde{g}^{-1})_{ij}$ on $\widetilde{B}_{2\widetilde{R}}(p)$ for some universal k.

To prove (1-4), suppose ψ_λ is an eigenfunction of Δ_g on (X, g). We cover X by geodesic balls $\{B_{\frac{r}{2}}(x_i)\}_{x_i \in \mathcal{I}}$ of radius $\frac{1}{2} r$ in such a way that the number of them is at most $C_0 r^{-n}$. As we saw earlier, this is always possible. We then estimate the size of the nodal set of ψ_λ in each $B_{\frac{r}{2}}(x)$ using Theorem 2.6. To do this, we first define $(\widetilde{X}, \tilde{g}) = \big(X, \frac{1}{r^2} g\big)$. Under such a rescaling, a ball of radius r scales to a ball of radius 1. Hence we put $\widetilde{R} = 1$. Then the equation

$$-\Delta_g \psi_\lambda = \lambda \psi_\lambda \quad \text{on } B_{2r}(p),$$

becomes

$$-\Delta_{\tilde{g}} \psi_\lambda = \tilde{\lambda} \psi_\lambda \quad \text{on } \widetilde{B}_2(p),$$

with

$$\tilde{\lambda} = r^2\lambda \quad \text{and} \quad \psi_{\tilde{\lambda}} = \psi_\lambda.$$

We can see that the doubling condition of Theorem 2.6 is valid because for all $s \le \frac{1}{10}$, using (2-3)

$$\frac{\sup_{\widetilde{B}_{2s}(x)} |\psi_\lambda|^2}{\sup_{\widetilde{B}_s(x)} |\psi_\lambda|^2} = \frac{\sup_{B_{2sr}(x)} |\psi_\lambda|^2}{\sup_{B_{sr}(x)} |\psi_\lambda|^2} \le e^{cr\sqrt{\lambda}} = e^{c\sqrt{\tilde{\lambda}}}$$

for some c that is uniform in $\tilde{\lambda}$, s, and x, and is controlled by K_1, K_2, and the $\mathcal{C}^k$ norm of $(\tilde{g})^{ij}$ on $\widetilde{B}_2(p)$ for some universal k. Therefore, by Theorem 2.6

$$\mathcal{H}^1_g\big(\{\psi_\lambda{=}0\} \cap B_{\frac{r}{2}}(p)\big) = r^{n-1}\mathcal{H}^1_{\tilde{g}}\big(\{\psi_{\tilde{\lambda}}{=}0\} \cap \widetilde{B}_{\frac{1}{2}}(p)\big) \le C_2 r^{n-1}\tilde{\lambda}^{\frac{3}{4}-\beta}.$$

We emphasize that since $(\tilde{g})^{ij} = r^2 g^{ij}$, for small enough $r_0(g)$ and all $r < r_0(h)$, the $\mathcal{C}^k$ norm of $(\tilde{g})^{ij}$ on $\widetilde{B}_2(p)$ is bounded by the $\mathcal{C}^k$ norm of $(g)^{ij}$ on $B_{2r}(p)$. Hence C_2 is independent of r, λ, and p, and is controlled only by K_1 and K_2 and (X, g). Adding these up, we get

$$\mathcal{H}^1_g\big(\{\psi_\lambda{=}0\}\big) \le \sum_{x_i \in \mathcal{I}} \mathcal{H}^1_g\big(\{\psi_\lambda{=}0\} \cap B_{\frac{r}{2}}(x_i)\big) \le (C_0 r^{-n}) C_2 r^{n-1}\tilde{\lambda}^{\frac{3}{4}-\beta} = c_3 r^{1-2\beta}\lambda^{\frac{3}{4}-\beta}.$$

Proof of* (1-5): *number of singular points for surfaces. We shall use the results of [Dong 1992] instead of [Donnelly and Fefferman 1990a], although both methods would work. Another goal is to simplify a less detailed part of the argument of [Dong 1992]. Let us first recall some statements from that paper.

Theorem 2.7 [Dong 1992, Theorems 2.2 and 3.4]. *Let (X, g) be a smooth Riemannian manifold of dimension 2, $p \in X$, and $R > 0$ so that the geodesic ball $B_{2R}(p)$ is embedded. Suppose ψ_λ is a smooth function such that $\Delta_g \psi_\lambda = \lambda\psi_\lambda$ on $B_{2R}(p)$ for some $\lambda \ge 1$. Then for all $x \in B_{\frac{R}{2}}(p)$ and all $s < \frac{1}{8}R$*

$$\sum_{z \in Z_{\psi_\lambda} \cap B_s(x)} (\nu_z(\psi_\lambda) - 1) \le \alpha_1\sqrt{\lambda} + \alpha_2 s^2\lambda. \tag{2-14}$$

The constants α_1, α_2 are uniform in x, s, and λ, and depend only on $(B_{2R}(p), g)$.

In fact by a glance at the proof of (2-14), see [Dong 1992, Theorem 3.4, pp. 502–503], one sees that the following statement holds:

$$\sum_{z \in Z_{\psi_\lambda} \cap B_s(x)} (\nu_z(\psi_\lambda) - 1) \le \alpha_1' \log\left(\frac{\sup_{B_{4s}(x)} q_\lambda}{\sup_{B_s(x)} q_\lambda}\right) + \alpha_2 s^2\lambda, \tag{2-15}$$

where

$$q_\lambda = |\nabla\psi_\lambda|^2 + \frac{\lambda}{2}|\psi_\lambda|^2$$

and α_1' and α_2 are some uniform constants.

The estimate (2-14) follows quickly from (2-15) if one knows that

$$\text{for } s \in \big(0, \tfrac{1}{8}R\big),\ x \in B_{\frac{R}{2}}(p), \qquad \frac{\sup_{B_{4s}(x)} q_\lambda}{\sup_{B_s(x)} q_\lambda} \le \alpha_3 e^{c_2\sqrt{\lambda}}.$$

The above growth estimate is proved in [Dong 1992] using the theory of frequency functions and monotonicity formulas; see [Garofalo and Lin 1986; Han and Lin 2007; Lin 1991] for background. However the proof of the monotonicity formula associated to q_λ, see [Dong 1992, pp. 498–499], is carried out only for the Euclidean metric and the proof of the upper bound $\sqrt{\lambda}$ on the frequency function uses the methods of [Lin 1991]. Here we give a simpler proof of this growth estimate which is based on gradient estimates for solutions of elliptic equations. More precisely, we show that if doubling estimates (2-3)

$$\text{for } s \in (0, 10r),\ x \in B_{\frac{R}{2}}(p), \qquad \sup_{B_{2s}(x)} |\psi_\lambda|^2 \le e^{cr\sqrt{\lambda}} \sup_{B_s(x)} |\psi_\lambda|^2$$

hold, then

$$\text{for } s \in (\lambda^{-\frac{1}{2}}, 2r),\ x \in B_{\frac{R}{2}}(p), \qquad \frac{\sup_{B_{4s}(x)} q_\lambda}{\sup_{B_s(x)} q_\lambda} \le \alpha_3 e^{c_2 r\sqrt{\lambda}} \tag{2-16}$$

for uniform constants α_3 and c_2. For the proof we use an application of standard elliptic estimates to the gradient of eigenfunctions, as performed in [Shi and Xu 2010].

Theorem 2.8 [Shi and Xu 2010, Theorem 1]. *Let (X, g) be a smooth connected compact Riemannian manifold without boundary. Suppose ψ_λ is an eigenfunction of Δ_g with eigenvalue λ. Then*

$$\beta_1\sqrt{\lambda}\sup_X|\psi_\lambda| \le \sup_X|\nabla\psi_\lambda| \le \beta_2\sqrt{\lambda}\sup_X|\psi_\lambda|$$

for some positive constants β_1 and β_2 independent of λ.

In fact by looking at the proof of this theorem we notice that a stronger statement holds. More precisely, one can see that, see [Shi and Xu 2010, p. 23, Fact (1) and equation (6)], for all $s < \frac{1}{4}\,\mathrm{inj}(g)$

$$\beta_1\sqrt{\lambda}\sup_{B_s(x)}|\psi_\lambda| \le \sup_{B_{s+(\gamma_0/\lambda^{1/2})}(x)}|\nabla\psi_\lambda|$$

$$\sup_{B_s(x)}|\nabla\psi_\lambda| \le \beta_2\sqrt{\lambda}\sup_{B_{s+(1/\lambda^{1/2})}(x)}|\psi_\lambda|, \tag{2-17}$$

where γ_0 is a positive constant that depends only on the Riemannian manifold (X, g). In fact it is the Brüning constant that guarantees that in every ball of radius $\gamma_0/\lambda^{\frac{1}{2}}$ there is a zero of ψ_λ. However, to prove (2-16) we only need the upper bound (2-17) for the gradient.[9] Let $s \in (\lambda^{-\frac{1}{2}}, 2r)$. Then since $4s + \lambda^{-\frac{1}{2}} < 10r$, using our doubling estimate (2-3) three times, we get

$$\begin{aligned}
\sup_{B_{4s}(x)} q_\lambda &= \sup_{B_{4s}(x)} \big(|\nabla\psi_\lambda|^2 + \tfrac{1}{2}\lambda|\psi_\lambda|^2\big) \\
&\le \beta_2'\lambda \sup_{B_{4s+(1/\lambda^{1/2})}(x)} |\psi_\lambda|^2 \\
&\le \beta_2'\lambda e^{3cr\sqrt{\lambda}} \sup_{B_{s/2+(1/8\lambda^{1/2})}(x)} |\psi_\lambda|^2 \\
&\le 2\beta_2' e^{3cr\sqrt{\lambda}} \sup_{B_s(x)} q_\lambda.
\end{aligned}$$

This proves (2-16) with $\alpha_3 = 2\beta_2'$ and $c_2 = 3c$.

[9] This is proved easily by a rescaling argument and elliptic estimates such as Theorem 8.32 in [Gilbarg and Trudinger 1998].

To finish the proof of our upper bounds for the number of singular points for surfaces, we apply (2-16) to the inequality (2-15) and obtain

$$\sum_{z\in Z_{\psi_\lambda}\cap B_s(x)} (\nu_z(\psi_\lambda)-1) \le \alpha_3'' r\sqrt{\lambda}+\alpha_2 s^2\lambda.$$

We now put $s=r^{\frac{1}{2}}\lambda^{-\frac{1}{4}}$. We underline that this choice of s is in fact in the allowable range $(\lambda^{-\frac{1}{2}},2r)$ because $r\ge\lambda^{-\frac{1}{2}}$. From this, (1-5) follows immediately.

Proof of Theorem 1.3*: *upper bounds for QE eigenfunctions. This theorem follows quickly from the lemma below combined with Theorem 1.1.

Lemma 2.9. *Let $\{\psi_j\}_{j\in S}$ be a sequence of eigenfunctions of Δ_g with eigenvalues $\{\lambda_j\}_{j\in S}$ such that for all $r\in\big(0,\frac{1}{2}\operatorname{inj}(g)\big)$ and all $x\in X$*

$$\int_{B_r(x)}|\psi_j|^2\to\frac{\mathrm{Vol}_g(B_r(x))}{\mathrm{Vol}_g(X)},\quad \lambda_j\xrightarrow{j\in S}\infty. \tag{2-18}$$

Then there exists $r_0(g)$ such that for each $r\in(0,r_0(g))$ there exists Λ_r such that for $\lambda_j\ge\Lambda_r$ we have

$$K_1r^n\le\int_{B_r(x)}|\psi_j|^2\le K_2r^n$$

uniformly for all $x\in X$. Here, K_1 and K_2 are independent of r, j, and x.

We point out that this lemma is obvious when x is fixed; however to obtain uniform L^2 estimates we need to use a covering argument as follows.

Proof. First we choose $r_0(g)<\frac{1}{4}\operatorname{inj}(g)$ small enough so that for all $r<r_0(g)$

$$a_1r^n\le\mathrm{Vol}(B_{\frac{r}{2}}(x))<\mathrm{Vol}(B_{2r}(x))\le a_2r^n$$

for some positive a_1 and a_2 that are independent of r and x. Next, we cover (X,g) using geodesic balls $\{B_{\frac{r}{2}}(x_i)\}_{x_i\in\mathcal{I}}$ such that $\operatorname{card}(\mathcal{I})$ is at most C_0r^{-n}, where C_0 depends only on (X,g). The existence of such a covering was discussed in the proof of (1-2). For each $x_i\in\mathcal{I}$, by using (2-18) twice, we can find $\Lambda_{i,r}$ large enough so that for $\lambda_j\ge\Lambda_{i,r}$

$$K_1r^n\le\int_{B_{r/2}(x_i)}|\psi_j|^2\le\int_{B_{2r}(x_i)}|\psi_j|^2\le K_2r^n,$$

with $K_1=a_1/(2\,\mathrm{Vol}(X))$ and $K_2=2a_2/\mathrm{Vol}(X)$. We claim that $\Lambda_r=\max_{i\in\mathcal{I}}\{\Lambda_{i,r}\}$ would do the job for all x in X. So let x be in X and r be as above. Then $x\in B_{\frac{r}{2}}(x_i)$ for some $i\in\mathcal{I}$ and clearly one has $B_{\frac{r}{2}}(x_i)\subset B_r(x)\subset B_{2r}(x_i)$. This and the above inequalities prove the lemma. ☐

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ON RANK-2 TODA SYSTEMS WITH ARBITRARY SINGULARITIES: LOCAL MASS AND NEW ESTIMATES

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For all rank-2 Toda systems with an arbitrary singular source, we use a unified approach to prove:

(1) The pair of local masses (σ_1, σ_2) at each blowup point has the expression

$$\sigma_i = 2(N_{i1}\mu_1 + N_{i2}\mu_2 + N_{i3}),$$

where $N_{ij} \in \mathbb{Z},\ i = 1, 2,\ j = 1, 2, 3$.

(2) At each vortex point p_t if (α_t^1, α_t^2) are integers and $\rho_i \notin 4\pi\mathbb{N}$, then all the solutions of Toda systems are uniformly bounded.

(3) If the blowup point q is a vortex point p_t and α_t^1, α_t^2 and 1 are linearly independent over Q, then

$$u^k(x) + 2\log|x - p_t| \le C.$$

The Harnack-type inequalities of 3 are important for studying the bubbling behavior near each blowup point.

1. Introduction

Let (M, g) be a Riemann surface without boundary and $\boldsymbol{K} = (k_{ij})_{n\times n}$ be the Cartan matrix of a simple Lie algebra of rank n. For example, for the Lie algebra $\mathrm{sl}(n+1)$ (the so-called A_n) we have

$$\boldsymbol{K} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{1-1}$$

In this paper we consider the solution $u = (u_1, \dots, u_n)$ of the following system defined on M:

$$\Delta_g u_i + \sum_{j=1}^{n} k_{ij}\rho_j\left(\frac{h_j e^{u_j}}{\int_M h_j e^{u_j}\, dV_g} - 1\right) = \sum_{p_t\in S} 4\pi\alpha_t^i(\delta_{p_t} - 1), \tag{1-2}$$

where Δ_g is the Laplace–Beltrami operator $(-\Delta_g \ge 0)$, S is a finite set on M, $h_1, \dots, h_n$ are positive and smooth functions on M, $\alpha_t^i > -1$ is the strength of the Dirac mass δ_{p_t} and $\rho = (\rho_1, \dots, \rho_n)$ is a constant vector with nonnegative components. Here for simplicity we just assume that the total area of M is 1.

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Obviously, (1-2) remains the same if u_i is replaced by $u_i + c_i$ for any constant c_i. Thus we might assume that each component of $u = (u_1, \dots, u_n)$ is in

$$\mathring{H}^1(M) := \left\{ v \in L^2(M),\ \nabla v \in L^2(M) \ \text{ and } \ \textstyle\int_M v\, dV_g = 0 \right\}.$$

Then (1-2) is the Euler–Lagrange equation for the following nonlinear functional $J_\rho(u)$ in $\mathring{H}^1(M)$:

$$J_\rho(u) = \frac{1}{2} \int_M \sum_{i,j=1}^n k^{ij} \nabla_g u_i \nabla_g u_j \, dV_g - \sum_{i=1}^n \rho_i \log \int_M h_i e^{u_i} \, dV_g,$$

where $(k^{ij})_{n\times n} = \boldsymbol{K}^{-1}$.

It is hard to overestimate the importance of system (1-2), as it covers a large number of equations and systems deeply rooted in geometry and physics. Even if (1-2) is reduced to a single equation with Dirac sources, it is a mean-field equation that describes metrics with conic singularities. Finding metrics with constant curvature with prescribed conic singularity is a classical problem in differential geometry and extensive references can be found in [Bartolucci and Tarantello 2002; Battaglia and Malchiodi 2014; Eremenko et al. 2014; Lin et al. 2012; 2015; Lin and Zhang 2010; 2013; 2016; Troyanov 1989; 1991; Yang 1997]. Recently profound relations among mean-field equations, the classical Lamé equation, hyperelliptic curves, modular forms and the Painlevé equation have been discovered and developed in [Chai et al. 2015; Chen et al. 2016].

The general form of (1-2) has close ties with algebraic geometry and integrable systems. Here we just briefly explain the relation between the sl$(n+1)$-Toda system and the holomorphic curves in projective spaces: Let f be a holomorphic curve from a domain D of $\mathbb{R}^2$ into $\mathbb{CP}^n$. Then f can be lifted locally to $\mathbb{C}^{n+1}$ and we use $\nu(z) = [\nu_0(z), \dots, \nu_n(z)]$ to denote the lift and f_k the k-th associated curve,

$$f_k : D \to G(k, n+1) \subset \mathbb{CP}^n(\Lambda^k \mathbb{C}^{n+1}), \quad f_k(z) = [\nu(z) \wedge \nu'(z) \wedge \cdots \wedge \nu^{(k-1)}(z)],$$

where $\nu^{(j)}$ is the j-th derivative of ν with respect to z. Let

$$\Lambda_k(z) = \nu(z) \wedge \cdots \wedge \nu^{(k-1)}(z).$$

Then the well-known infinitesimal Plüker formula gives

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \|\Lambda_k(z)\|^2 = \frac{\|\Lambda_{k-1}(z)\|^2 \, \|\Lambda_{k+1}(z)\|^2}{\|\Lambda_k(z)\|^4} \quad \text{for } k = 1, 2, \dots, n, \tag{1-3}$$

where we put $\|\Lambda_0(z)\|^2 = 1$ as convention and the norm $\|\cdot\|^2 = \langle \cdot , \cdot \rangle$ is defined by the Fubini–Study metric in $\mathbb{CP}(\Lambda^k \mathbb{C}^{n+1})$. Here we observe that (1-3) holds only for $\|\Lambda_k(z)\| > 0$, i.e., for all the unramified points $z \in M$. Now we set $\|\Lambda_{n+1}(z)\| = 1$ by normalization (analytically extended at the ramification points) and

$$U_k(z) = -\log \|\Lambda_k(z)\|^2 + k(n-k+1)\log 2, \quad 1 \le k \le n.$$

For every ramified point p we use $\{\gamma_{p,1}, \dots, \gamma_{p,n}\}$ to denote the total ramification index at p and set

$$u_i^* = \sum_{j=1}^n k_{ij} U_j, \quad \alpha_{p,i} = \sum_{j=1}^n k_{ij} \gamma_{p,j},$$

Then we have

$$\Delta u_i^* + \sum_{j=1}^{n} k_{ij} e^{u_j^*} - K_0 = 4\pi \sum_{p\in S} \alpha_{p,i}\delta_p, \quad i = 1, \dots, n, \tag{1-4}$$

where K_0 is the Gaussian curvature of the metric g.

Therefore any holomorphic curve from M to $\mathbb{CP}^n$ is associated with a solution $u^* = (u_1^*, \dots, u_n^*)$ of (1-4). Conversely, given any solution $u^* = (u_1^*, \dots, u_n^*)$ of (1-4) in $\mathbb{S}^2$, it is possible to construct a holomorphic curve of $\mathbb{S}^2$ into $\mathbb{CP}^n$ which has the given ramification index $\gamma_{p,i}$ at p if $\gamma_{p,i} \in \mathbb{N}$. One can see [Lin et al. 2012] for the details of this construction. Therefore, (1-4) is related to the following problem in more general setting: given a set of ramified points on M and its ramification indices at these points, can we find holomorphic curves into $\mathbb{CP}^n$ that satisfy the given ramification information?

Equation (1-2) is also related to many physical models from gauge field theory. For example, to describe the physics of high critical temperature superconductivity, a model related to the Chern–Simons model was proposed, which can be reduced to an $n \times n$ system with exponential nonlinearity if the gauge potential and the Higgs field are algebraically restricted. The Toda system with (1-1) is one of the limiting equations if a coupling constant tends to zero. For extensive discussions on the relationship between the Toda system and its background in Physics we refer the readers to [Bennett 1934; Ganoulis et al. 1982; Lee 1991; Mansfield 1982; Yang 2001].

In this article we are concerned with rank-2 Toda systems. There are three types of Cartan matrices of rank 2:

$$A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B_2 (= C_2) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

One of our main theorems is the following estimate:

Theorem 1.1. *Let $(k_{ij})_{2\times 2}$ be one of the matrices above, h_i be positive C^1 functions on M, $\alpha_t^i \in \mathbb{N} \cup \{0\}$, $t \in \{1, 2, \dots, N\}$ and K be a compact subset of $M \setminus S$. If $\rho_i \notin 4\pi\mathbb{N}$, then there exists a constant $C(K, \rho_1, \rho_2)$ such that for any solution $u = (u_1, u_2)$ of* (1-2)

$$|u_i(x)| \le C \quad \textit{for all } x \in K,\ i = 1, 2.$$

Our proof of Theorem 1.1 is based on the analysis of the behavior of solutions $u^k = (u_1^k, u_2^k)$ near each blowup point. A point $p \in M$ is called a blowup point if, along a sequence of points $p_k \to p$,

$$\max_{i=1,2} \{\tilde{u}_1^k(p_k), \tilde{u}_2^k(p_k)\} \to +\infty,$$

where

$$\tilde{u}_i^k(x) = u_i^k(x) + 4\pi \sum_t \alpha_t^k G(x, p_t),$$

and $G(x, y)$ is the Green's function of the Laplacian operator on M.

Suppose u^k is a sequence of solutions of (1-2). When $n = 1$, it has been proved that if u^k blows up somewhere, the mass distribution $\rho h e^{u^k} / \big(\int_M h e^{u^k}\big)$ will concentrate; that is, for a set of finite points

$p_1, p_2, \dots, p_L$ and positive numbers $m_1, \dots, m_L$

$$\frac{\rho h e^{u^k}}{\int_M h e^{u^k}} \to \sum_{i=1}^{L} m_i \delta_{p_i} \quad \text{as } k \to \infty.$$

In other words, "u_k concentrates" means $u^k(x) \to -\infty$ if x is not a blowup point. This "blowup implies concentration" was first noted by Brezis and Merle [1991] and was later proved by Li [1999], Li and Shafrir [1994] and Bartolucci and Tarantello [2002]. But for $n \geq 2$, this phenomenon might fail in general. A component u_i^k is called not concentrating if $u_i^k \not\to -\infty$ away from blowup points, or equivalently, $\tilde{u}_i^k$ converges to some smooth function w_i away from blowup points. It is natural to ask whether it is possible to have all components not concentrating. For $n = 2$, we prove it is impossible.

Theorem 1.2. *Suppose u^k is a sequence of blowup solutions of a rank-*2 *Toda system* (1-2)*. Then at least one component of u^k satisfies $u_i^k(x) \to -\infty$ if x is not contained in the blowup set.*

The first example of such nonconcentration phenomenon was proved by Lin and Tarantello [2016]. The new phenomenon makes the study of systems ($n \geq 2$) much more difficult than the mean-field equation ($n = 1$). Recently, Battaglia [2015] and Lin, Yang and Zhong [Lin et al. 2017] independently proved the result of Theorem 1.2 for $n \geq 3$.

As mentioned before, our proofs of Theorems 1.1 and 1.2 are based on the asymptotic behavior of local bubbling solutions. For simplicity we set up the situation as follows:

Let $u^k = (u_1^k, u_2^k)$ be a sequence of solutions of

$$\Delta u_i^k + \sum_{j=1}^{2} k_{ij} h_j^k e^{u_j^k} = 4\pi\alpha_i \delta_0 \quad \text{in } B(0,1),\ i = 1, 2, \tag{1-5}$$

where $\alpha_i > -1$. $B(0,1)$ is the unit ball in $\mathbb{R}^2$ (we use $B(p,r)$ to denote the ball centered at p with radius r) and $(k_{ij})_{2\times 2}$ is A_2, B_2 or G_2. Throughout the paper, h_1^k, h_2^k are smooth functions satisfying $h_1^k(0) = h_2^k(0) = 1$ and

$$\frac{1}{C} \leq h_i^k \leq C, \quad \|h_i^k\|_{C^1(B(0,1))} \leq C \quad \text{in } B(0,1),\ i = 1, 2. \tag{1-6}$$

For solutions $u^k = (u_1^k, u_2^k)$ we assume

$$\begin{cases} 0 \text{ is the only blowup point of } u^k, \\ |u_i^k(x) - u_i^k(y)| \leq C \quad \text{for all } x, y \in \partial B(0,1),\ i = 1, 2, \\ \quad \int_{B(0,1)} h_i^k e^{u_i^k} \leq C, \quad i = 1, 2. \end{cases} \tag{1-7}$$

For this sequence of blowup solutions we define the local mass by

$$\sigma_i = \lim_{r \to 0} \lim_{k \to \infty} \frac{1}{2\pi} \int_{B(0,r)} h_i^k e^{u_i^k}, \quad i = 1, 2. \tag{1-8}$$

It is known that 0 is a blowup point if and only if $(\sigma_1, \sigma_2) \neq (0, 0)$. The proof is to use ideas from [Brezis and Merle 1991] and has become standard now. We refer the readers to [Lee et al. 2017] for a complete proof. One important property of (σ_1, σ_2) is the so-called Pohozaev identity (P.I. in short)

$$k_{21}\sigma_1^2 + k_{12}k_{21}\sigma_1\sigma_2 + k_{12}\sigma_2^2 = 2k_{21}\mu_1\sigma_1 + 2k_{12}\mu_2\sigma_2, \tag{1-9}$$

where $\mu_i = 1 + \alpha_i$. Take A_2 as an example; the P.I. is

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = 2\mu_1\sigma_1 + 2\mu_2\sigma_2.$$

The proof of (1-9) was given in [Lin et al. 2015] where we initiated an algorithm to calculate all the possible (finitely many) values of local masses and (1-9) played an essential role. But the argument there seems not very efficient. In this work we add major new ingredients to our approach and improve the classification of (σ_1, σ_2) to the following sharper form:

Theorem 1.3. *Let u^k be a sequence of blowup solutions of* (1-5) *which also satisfies* (1-6) *and* (1-7). *Suppose σ_1 and σ_2 are local masses defined by* (1-8). *Then σ_i can be written as*

$$\sigma_i = 2(N_{i,1}\mu_1 + N_{i,2}\mu_2 + N_{i,3}), \quad i = 1, 2,$$

for some $N_{i,1}, N_{i,2}, N_{i,3} \in \mathbb{Z}$ $(i = 1, 2)$.

Theorem 1.3 is proved in Sections 5 and 6. In Section 5, we give an explicit procedure to calculate the local masses. Take the A_2 system as an example; we start with $\sigma_1 = 0$ and the P.I. gives $\sigma_2 = 2\mu_2$. With $\sigma_2 = 2\mu_2$, the P.I. gives $\sigma_1 = 2\mu_1 + 2\mu_2$ and so on. Let $\Gamma(\mu_1, \mu_2)$ be the set obtained by the above algorithm. Then $\Gamma(\mu_1, \mu_2)$ is equal to:

(i) $(2\mu_1, 0)$, $(2\mu_1, 2\mu_1 + 2\mu_2)$, $(2\mu_1 + 2\mu_2, 2\mu_1 + 2\mu_2)$, $(2\mu_1 + 2\mu_2, 2\mu_2)$, $(0, 2\mu_2)$ for A_2,

(ii) $(2\mu_1, 0)$, $(2\mu_1, 4\mu_1 + 2\mu_2)$, $(4\mu_1 + 2\mu_2, 4\mu_1 + 2\mu_2)$, $(4\mu_1 + 2\mu_2, 4\mu_1 + 4\mu_2)$, $(0, 2\mu_2)$, $(2\mu_1 + 2\mu_2, 2\mu_2)$, $(2\mu_1 + 2\mu_2, 4\mu_1 + 4\mu_2)$ for B_2,

(iii) $(2\mu_1, 0)$, $(2\mu_1, 6\mu_1 + 2\mu_2)$, $(6\mu_1 + 2\mu_2, 6\mu_1 + 2\mu_2)$, $(6\mu_1 + 2\mu_2, 12\mu_1 + 6\mu_2)$, $(8\mu_1 + 4\mu_2, 12\mu_1 + 6\mu_2)$, $(8\mu_1 + 4\mu_2, 12\mu_1 + 8\mu_2)$, $(0, 2\mu_2)$, $(2\mu_1 + 2\mu_2, 2\mu_2)$, $(2\mu_1 + 2\mu_2, 6\mu_1 + 6\mu_2)$, $(6\mu_1 + 4\mu_2, 6\mu_1 + 6\mu_2)$, $(6\mu_1 + 4\mu_2, 12\mu_1 + 8\mu_2)$ for G_2.

Definition 1.4. A pair of local masses $(\sigma_1, \sigma_2) \in \Gamma(\mu_1, \mu_2)$ is called special if

$$(\sigma_1, \sigma_2) = \begin{cases} (2\mu_1 + 2\mu_2, 2\mu_1 + 2\mu_2) & \text{for } A_2, \\ (4\mu_1 + 2\mu_2, 4\mu_1 + 4\mu_2) & \text{for } B_2, \\ (8\mu_1 + 4\mu_2, 12\mu_1 + 8\mu_2) & \text{for } G_2. \end{cases}$$

The analysis of local solutions in [Lin et al. 2015] describes a method to pick a family of points $\Gamma_k = \{0, x_1^k, \ldots, x_N^k\}$ (if 0 is a singular point, otherwise 0 can be deleted from Γ_k) such that a tiny ball $B(x_i^k, l_j^k)$ contributes an amount of mass (which is quantized), and the following Harnack-type inequality holds:

$$u_i^k(x) + 2\log \operatorname{dist}(x, \Sigma_k) \leqslant C \quad \text{for all } x \in B(0, 1). \tag{1-10}$$

When $\alpha_1 = \alpha_2 = 0$, we can use Theorem 1.3 to calculate all the pairs of even positive integers satisfying (1-9) and the set is exactly the same as $\Gamma(1, 1)$.

It is interesting to see whether any pair of the above really consists of the local masses of some sequence of blowup solutions of (1-2). For $\boldsymbol{K} = A_2$ the existence of such a local blowup sequence has been obtained; see [Musso et al. 2016; Lin and Yan 2013].

After Σ_k is picked, the difficulty at the next step is how to calculate the mass contributed from outside $B(x_j^k, l_j^k)$ $j = 1, 2, \cdots, N$. In Section 6, we see that the mass outside this union could be very messy. However, the picture is very clean if (α_1, α_2) satisfies the following Q-condition:

$$\alpha_1, \alpha_2 \text{ and } 1 \text{ are linearly independent over } Q.$$

Theorem 1.5. *Suppose* (α_1, α_2) *satisfies the* Q*-condition. Then* $(\sigma_1, \sigma_2) \in \Gamma(\mu_1, \mu_2)$. *Furthermore, for any sequence of solutions of* (1-5) *satisfying* (1-6) *and* (1-7), *the following Harnack-type inequality holds*:

$$u_i^k(x) + 2\log|x| \leqslant C \quad \text{for } x \in B(0,1).$$

For (1-2), let $\mu_{1,t} = \alpha_t^1 + 1$ and $\mu_{2,t} = \alpha_t^2 + 1$ at a vortex point $p_t \in S$, and define

$$\Gamma_i = \big\{2\pi(\Sigma_{t\in J}\sigma_{i,t} + 2n) \,\big|\, (\sigma_{1,t}, \sigma_{2,t}) \in \Gamma(\mu_{1,t}, \mu_{2,t}),\ J \subseteq S,\ n \in \mathbb{N} \cup \{0\}\big\}. \tag{1-11}$$

Based on Theorem 1.5, Theorem 1.1 can be extended to the following version:

Theorem 1.6. *Let* h_i *be positive* C^1 *functions on* M, *and* K *be a compact set in* M. *For every point* $p_t \in S$, *if either both* $\alpha_t^1, \alpha_t^2 \in \mathbb{N} \cup \{0\}$ *or* (α_t^1, α_t^2) *satisfies the* Q*-condition, then for* $\rho_i \notin \Gamma_i$ *and* $u = (u_1, u_2)$ *a solution of* (1-2), *there exists a constant* C *such that*

$$|u_i(x)| \leqslant C \quad \text{for all } x \in K.$$

The organization of this article is as follows. In Section 2 we establish the global mass for the entire solutions of some singular Liouville equation defined in $\mathbb{R}^2$. Then in Section 3 we review some fundamental tools proved in the previous work [Lin et al. 2015]. In Section 4 we present two crucial lemmas, which play the key role in the proof of main results. In Sections 5 and 6 we discuss the local mass on each bubbling disk centered at 0 and not at 0 respectively, and then all the main results are established based on previous discussions.

2. Total mass for Liouville equation

The main purpose of this section is to prove an estimate of the total mass for the solutions of the equation

$$\begin{cases} \Delta u + e^u = \sum_{i=1}^N 4\pi\alpha_i\delta_{p_i} & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^u < \infty, \end{cases} \tag{2-1}$$

where $p_1, \dots, p_N$ are distinct points in $\mathbb{R}^2$ and $\alpha_i > -1$ for all $1 \leqslant i \leqslant N$.

Theorem 2.1. *Suppose* u *is a solution of* (2-1) *and* $\alpha_1, \dots, \alpha_N$ *are positive integers. Then* $\frac{1}{4\pi}\int_{\mathbb{R}^2} e^u$ *is an even integer.*

Proof. It is known that any solution u of (2-1) has, at infinity, the asymptotic behavior

$$u(z) = -2\alpha_\infty \log|z| + O(1), \quad \alpha_\infty > 1, \tag{2-2}$$

and u satisfies

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^u dx = 2\sum_{i=1}^{N}\alpha_i + 2\alpha_\infty. \tag{2-3}$$

We shall prove that $\alpha_\infty + \sum_{i=1}^N \alpha_i$ is an even integer. A classical Liouville theorem (see [Chou and Wan 1994]) says that u can be written as

$$u = \log\frac{8|f'(z)|^2}{(1+|f(z)|^2)^2}, \quad z \in \mathbb{R}^2, \tag{2-4}$$

for some meromorphic function f. In general, $f(z)$ is multivalued and any vertex p_i is a branch point. However if $\alpha_i \in \mathbb{N}\cup\{0\}$, then $f(z)$ is single-valued. Furthermore (2-2) implies that $f(z)$ is meromorphic at infinity. Hence for any solution u of (2-1) there is a meromorphic function f on $\mathbb{S}^2 = \mathbb{C}\cup\{\infty\}$ such that (2-4) holds. Then

$$\begin{aligned}4\pi\Big(\sum_{j=1}^{N}\alpha_j + \alpha_\infty\Big) = \int_{\mathbb{R}^2} e^u = 8\int_{\mathbb{R}^2}\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}\,dx\,dy \\ = 8(\deg f)\int_{\mathbb{R}^2}\frac{d\tilde{x}\,d\tilde{y}}{(1+|w|^2)^2} = 8\pi(\deg f),\end{aligned}$$

where $\deg(f)$ is the degree of f as a map from $\mathbb{S}^2 = \mathbb{C}\cup\{\infty\}$ onto $\mathbb{S}^2$, and $w = f(z) = \tilde{x} + i\tilde{y}$. Thus we have

$$\sum_{j=1}^{N}\alpha_j + \alpha_\infty = 2\deg(f). \qquad \square$$

Theorem 2.2. *Suppose u is a solution of*

$$\begin{cases}\Delta u + e^u = 4\pi\alpha_0\delta_{p_0} + \sum_{i=1}^N 4\pi\alpha_i\delta_{p_i} & \text{in } \mathbb{R}^2,\\ \int_{\mathbb{R}^2} e^u < \infty,\end{cases} \tag{2-5}$$

where $p_0, p_1, \dots, p_N$ are distinct points in $\mathbb{R}^2$ and $\alpha_1,\dots,\alpha_N$ are positive integers, $\alpha_0 > -1$. Then $\frac{1}{4\pi}\int_{\mathbb{R}^2} e^u$ is equal to $2(\alpha_0+1)+2k$ for some $k\in\mathbb{Z}$ or $2k_1$ for some $k_1 \in \mathbb{N}$.

Proof. As in Theorem 2.1, there is a developing map $f(z)$ of u such that

$$u(z) = \log\frac{8|f'(z)|^2}{(1+|f(z)|^2)^2}, \quad z\in\mathbb{C}. \tag{2-6}$$

On one hand by (2-5), $u_{zz} - \frac{1}{2}u_z^2$ is a meromorphic function in $\mathbb{C}\cup\{\infty\}$ because away from the Dirac masses

$$4\big(u_{zz} - \tfrac{1}{2}u_z^2\big)_{\bar{z}} = -(e^u)_z + u_z e^u = 0.$$

By $u(z) = 2\alpha_i\log|z-p_i| + O(1)$ near p_i we have

$$u_{zz} - \tfrac{1}{2}u_z^2 = -2\Big(\sum_{j=0}^{N}\tfrac{1}{2}\alpha_j\big(\tfrac{1}{2}\alpha_j + 1\big)(z-p_j)^{-2} + A_j(z-p_j)^{-1} + B\Big),$$

where $A_0, \dots, A_N, B \in \mathbb{C}$ are some constants. On the other hand by (2-6), a straightforward computation shows that

$$u_{zz} - \frac{1}{2}u_z^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{2-7}$$

Using the Schwarz derivative of f,

$$\{f; z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2,$$

and letting

$$I(z) = \sum_{j=0}^{N} \tfrac{1}{2}\alpha_j\left(\tfrac{1}{2}\alpha_j + 1\right)(z - p_j)^{-2} + A_j(z - p_j)^{-1} + B,$$

we can write the equation for f as

$$\{f, z\} = -2I(z). \tag{2-8}$$

A well-known classic theorem (see [Whittaker and Watson 1927]) says that for any two linearly independent solutions y_1 and y_2 of

$$y''(z) = I(z)y(z), \tag{2-9}$$

the ratio y_2/y_1 always satisfies

$$\{y_2/y_1; z\} = -2I(z).$$

By (2-8) and a basic result of the Schwarz derivative, $f(z)$ can be written as the ratio of two linearly independent solutions. This is how (2-1) is related to the complex ODE (2-9). We refer the readers to [Chai et al. 2015] for the details.

For (2-9), there is an associated monodromy representation ρ from $\pi_1(\mathbb{C} \setminus \{p_0, p_1, \dots, p_N\}; q)$ to $\mathrm{GL}(2; \mathbb{C})$, where q is a base point. Note that at any singular point p_j, the local exponents are $\frac{1}{2}\alpha_j + 1$ and $-\frac{1}{2}\alpha_j$. It is known from [Lin et al. 2012, Section 7] that e^{-u} can be locally written as

$$e^{-u} = |\nu_1|^2 + |\nu_2|^2 = \langle(\nu_1, \nu_2)^t, (\nu_1, \nu_2)^t\rangle,$$

where ν_1, ν_2 are the two fundamental solutions of (2-9). After encircling the singular point p_j once, we have $e^{-u} = \langle\rho_j(\nu_1, \nu_2)^t, \rho_j(\nu_1, \nu_2)^t\rangle$ and the value does not change. Therefore, we conclude that ρ_j is unitary and

$$\rho_j = \rho(\gamma_j) = C_j \begin{pmatrix} e^{\pi i \alpha_j} & 0 \\ 0 & e^{-\pi i \alpha_j} \end{pmatrix} C_j^{-1},$$

where $\gamma_j \in \pi_1(\mathbb{C} \setminus \{p_0, \dots, p_N\}, q)$ encircles p_j only once, $0 \le j \le N$, while the monodromy at ∞ is ρ_∞. Then we have

$$\rho_\infty \rho_N \cdots \rho_0 = I_{2\times 2}.$$

Note that $\rho_j = \pm I_{2\times 2}$ for $1 \le j \le N$. Hence

$$\rho_\infty^{-1} = D_0 \begin{pmatrix} e^{\pi i \sum_{j=0}^{N} \alpha_j} & 0 \\ 0 & e^{-\pi i \sum_{j=0}^{N} \alpha_j} \end{pmatrix} D_0^{-1}$$

for some constant invertible matrix D_0.

On the other hand, the local exponents at ∞ can be computed as follows. Let $\hat{y}(z) = y\left(\frac{1}{z}\right)$, where y is a solution of (2-9). Then we have

$$\hat{y}''(z) + \frac{2}{z}\hat{y}'(z) = \hat{I}(z)\hat{y}(z), \tag{2-10}$$

where $\hat{I}(z) = I\left(\frac{1}{z}\right)z^{-4}$. Since $I(z)$ is the Schwarz derivative of $f(z)$, by direct computation $\hat{I}(z)$ is the Schwarz derivative of $f\left(\frac{1}{z}\right)$. As before we let $\hat{u}(z) = u\left(\frac{1}{z}\right) - 4\log|z|$. Then $f\left(\frac{1}{z}\right)$ is the developing map of $\hat{u}(z)$. Since

$$\hat{u}(z) = 2(\alpha_\infty - 2)\log|z| + O(1) \quad \text{near } 0,$$

(because $u(z) = -2\alpha_\infty \log|z| + O(1)$ at infinity), we have

$$\hat{I}(z) = \tfrac{1}{2}\alpha_\infty\left(\tfrac{1}{2}\alpha_\infty - 1\right)z^{-2} + \text{higher-order terms of } z \quad \text{near } 0.$$

By (2-10) we see that the local exponents of (2-9) are $-\frac{1}{2}\alpha_\infty$ and $\frac{1}{2}\alpha_\infty - 1$. Hence $e^{\pi i \alpha_\infty}$ equals either $e^{\pi i \sum_{j=0}^{N}\alpha_j}$ or $e^{-\pi i \sum_{j=0}^{N}\alpha_j}$, which yields

$$\alpha_\infty = -\sum_{j=0}^{N}\alpha_j + 2k \quad \text{or} \quad \alpha_\infty = \sum_{j=0}^{N}\alpha_j + 2k \tag{2-11}$$

for some $k \in \mathbb{Z}$. Since

$$\frac{1}{4\pi}\int_{\mathbb{R}^2} e^u = \sum_{j=0}^{N}\alpha_j + \alpha_\infty,$$

we either have $\frac{1}{4\pi}\int_{\mathbb{R}^2} e^u = 2k$ for some $k \in \mathbb{N}$ if the first case holds or $\frac{1}{4\pi}\int_{\mathbb{R}^2} e^u = 2(\alpha_0 + 1) + 2k'$ for $k' = \sum_{i=1}^{N}\alpha_i + k - 1$ if the second case holds. □

Remark 2.3. After proving Theorems 2.1 and 2.2, we found a stronger version of both theorems in [Eremenko et al. 2014]. Because we only need the present form of both theorems, we include our proofs here to make the paper more self-contained.

3. Review of bubbling analysis from a selection process

Let $u^k = (u_1^k, u_2^k)$ be solutions of (1-5) such that (1-6) and (1-7) hold. In this section we review the process to select a set $\Sigma_k = \{0, x_1^k, \ldots, x_n^k\}$ and balls $B(x_i^k, l_k)$ such that u^k has nonzero local masses in $B(x_i^k, l_k)$. This selection process was first carried out in [Lin et al. 2015]. We briefly review it below.

The set Σ_k is constructed by induction. If (1-5) has no singularity, we start with $\Sigma_k = \varnothing$. If (1-5) has a singularity, we start with $\Sigma_k = \{0\}$. By induction suppose Σ_k consists of $\{0, x_1^k, \ldots, x_{m-1}^k\}$. Then we consider

$$\max_{x \in B_1}\max_{i=1,2}\left(u_i^k(x) + 2\log \operatorname{dist}(x, \Sigma_k)\right). \tag{3-1}$$

If the maximum is bounded from above by a constant independent of k, the process stops and Σ_k is exactly equal to $\{0, x_1^k, \ldots, x_{m-1}^k\}$. However if the maximum tends to infinity, let q_k be where (3-1) is achieved and we set

$$d_k = \tfrac{1}{2}\operatorname{dist}(q_k, \Sigma_k)$$

and

$$S_i^k(x) = u_i^k(x) + 2\log(d_k - |x - q_k|) \quad \text{in } B(q_k, d_k),\ i = 1, 2.$$

Suppose i_0 is the component that attains

$$\max_i \max_{x \in \bar{B}(q_k, d_k)} S_i^k \tag{3-2}$$

at p_k. Then we set

$$\tilde{l}_k = \tfrac{1}{2}(d_k - |p_k - q_k|)$$

and scale u_i^k by

$$v_i^k(y) = u_i^k(p_k + e^{-\frac{1}{2}u_{i_0}^k(p_k)}y) - u_{i_0}^k(p_k) \quad \text{for } |y| \le R_k \doteqdot e^{\frac{1}{2}u_{i_0}^k(p_k)}\tilde{l}_k. \tag{3-3}$$

It can be shown that $R_k \to \infty$ and v_i^k is bounded from above over any fixed compact subset of $\mathbb{R}^2$. Thus by passing to a subsequence, v_i^k satisfies one of the following two alternatives:

(a) (v_1^k, v_2^k) converges in $C^2_{\text{loc}}(\mathbb{R}^2)$ to (v_1, v_2) which satisfies

$$\Delta v_i + \sum_{j=1}^{2} k_{ij} e^{v_j} = 0 \quad \text{in } \mathbb{R}^2,\ i = 1, 2. \tag{3-4}$$

(b) Either v_1^k converges to

$$\Delta v_1 + 2e^{v_1} = 0 \quad \text{in } \mathbb{R}^2 \tag{3-5}$$

and $v_2^k \to -\infty$ over any fixed compact subset of $\mathbb{R}^2$ or v_2^k converges to $\Delta v_2 + 2e^{v_2} = 0$ in $\mathbb{R}^2$ and $v_1^k \to -\infty$ over any fixed compact subset of $\mathbb{R}^2$.

Therefore in either case, we could choose $l_k^* \to \infty$ such that

$$v_i^k(y) + 2\log|y| \le C \quad \text{for } i = 1, 2 \text{ and } |y| \leqslant l_k^* \tag{3-6}$$

and

$$\int_{B(0, l_k^*)} h_i^k e^{v_i^k}\, dy = \int_{\mathbb{R}^2} e^{v_i(y)} + o(1).$$

By scaling back to u_i^k, we add p_k in Σ_k with

$$l_k = e^{-\frac{1}{2}u_{i_0}^k(p_k)} l_k^*.$$

We can continue in this way until the Harnack-type inequality (1-10) holds.

We summarize what the selection process has done in the following proposition (a detailed proof for a more general case can be found in [Lin et al. 2015, Proposition 2.1]):

Proposition 3A. *Let u^k be described as above. Then there exist a finite set $\Sigma_k := \{0, x_1^k, \dots, x_m^k\}$ (if 0 is not a singular point, then 0 can be deleted from Σ_k) and positive numbers $l_1^k, \dots, l_m^k \to 0$ as $k \to \infty$ such that the following hold*:

(1) *There exists $C > 0$ independent of k such that* (1-10) *holds and all the components have fast decay on $\partial B(x_j^k, l_j^k)$, $j = 1, \dots, m$. (The definition of fast decay can be found in Definition 3.1 below).*

(2) *In* $B(x_j^k, l_j^k)$ $(j = 1, \dots, m)$, *let* $R_{j,k} = e^{\frac{1}{2}u_{i_0}^k(x_j^k)} l_j^k$, $u_{i_0}^k(x_j^k) = \max_i u_i^k(x_j^k)$ *and*

$$v_i^k(y) = u_i^k(x_j^k + e^{-\frac{1}{2}u_{i_0}^k(x_j^k)} y) - u_{i_0}^k(x_j^k) \tag{3-7}$$

for $|y| \le R_{j,k}$; *then* $v^k = (v_1^k, v_2^k)$ *satisfies either* (a) *or* (b).

(3) $B(x_j^k, l_j^k) \cap B(x_i^k, l_i^k) = \varnothing$, $i \ne j$.

The inequality (1-10) is a Harnack-type inequality, because it implies the following result.

Proposition 3B. *Suppose* u^k *satisfies* (1-5)–(1-7) *and*

$$u_i^k(x) + 2\log|x - x_0| \le C \quad \text{for } x \in B(x_0, 2r_k).$$

Then

$$|u_i^k(x_1) - u_i^k(x_2)| \le C_0 \quad \text{for } \frac{1}{2} \le \frac{|x_1 - x_0|}{|x_2 - x_0|} \le 2 \text{ and } x_1, x_2 \in B(x_0, r_k). \tag{3-8}$$

The proof of Proposition 3B is standard, see [Lin et al. 2015, Lemma 2.4], so we omit it here. Let $x_l^k \in \Sigma_k$ and $\tau_l^k = \frac{1}{2}\operatorname{dist}(x_l^k, \Sigma_k \setminus \{x_l^k\})$; then (3-8) implies

$$u_i^k(x) = \bar{u}_{x_l^k,i}^k(r) + O(1), \quad x \in B(x_l^k, \tau_l^k), \tag{3-9}$$

where $r = |x_l^k - x|$ and $\bar{u}_{x_l^k,i}^k$ is the average of u_i^k on $\partial B(x_l^k, r)$,

$$\bar{u}_{x_l^k,i}^k(r) = \frac{1}{2\pi r} \int_{\partial B(x_l^k, r)} u_i^k \, dS, \tag{3-10}$$

and $O(1)$ is independent of r and k.

Next we introduce the notions of slow decay and fast decay in our bubbling analysis.

Definition 3.1. We say u_i^k has fast decay on $\partial B(x_0, r_k)$ if along a subsequence

$$u_i^k(x) + 2\log|x - x_0| \le -N_k \quad \text{for all } x \in \partial B(x_0, r_k),$$

for some $N_k \to \infty$ and we say u_i^k has slow decay if there is a constant C independent of k such that

$$u_i^k(x) + 2\log|x - x_0| \ge -C \quad \text{for all } x \in \partial B(x_0, r_k).$$

Furthermore, we say u_i^k is fast-decaying in $B(x_0, s_k) \setminus B(x_0, r_k)$ if u_i^k has fast decay on $\partial B(x_0, l_k)$ for any $l_k \in [r_k, s_k]$.

The concept of fast decay is important for evaluating the Pohozaev identities. The following proposition is a direct consequence of [Lin et al. 2015, Proposition 3.1] and it says if both components are fast-decaying on the boundary, the Pohozaev identity holds for the local masses.

In the following proposition, we let $B = B(x^k, r_k)$. If $x^k \ne 0$, we assume $0 \notin B(x^k, 2r_k)$.

Proposition 3C. *Suppose both* u_1^k, u_2^k *have fast decay on* ∂B, *where* B *is given above. Then* (σ_1, σ_2) *satisfies the P.I.* (1-9), *where*

$$\sigma_i = \lim_{k \to 0} \frac{1}{2\pi} \int_B h_i^k e^{u_i^k}, \quad i = 1, 2.$$

We refer the readers to [Lin et al. 2015, Proposition 3.1] for the proof.

4. Two lemmas

In this section, we prove two crucial lemmas which play the key role in Sections 5 and 6. For Lemma 4.1, we assume:

(i) The Harnack inequality
$$u_i^k(x)+2\log|x|\le C \quad \text{for } \tfrac{1}{2}l_k\le |x|\le 2s_k,\ i=1,2,$$
holds for both components.

(ii) Both components u_i^k have fast decay on $\partial B(0,l_k)$ and $\sigma_i^k(B(0,l_k))=\sigma_i+o(1)$ for $i=1,2$, where
$$\sigma_i=\lim_{r\to 0}\lim_{k\to\infty}\frac{1}{2\pi}\int_{B(0,rs_k)} h_i^k e^{u_i^k}, \quad i=1,2.$$

(iii) One of $u_i^k,\ i=1,2$, has slow decay on $\partial B(0,s_k)$.

Lemma 4.1. (a) *Assume* (i) *and* (ii). *If u_i^k has slow decay on $\partial B(0,s_k)$, then*
$$2\mu_i-\sum_{j=1}^{2}k_{ij}\sigma_j>0.$$

(b) *Assume* (i), (ii) *and* (iii). *Let u_i^k be a slow-decaying component on $\partial B(0,s_k)$. Then the other component has fast decay on $\partial B(0,s_k)$.*

Proof. (a) Suppose that u_i^k has slow decay on $\partial B(0,s_k)$. Then the scaling
$$v_j^k(y)=u_j^k(s_k y)+2\log s_k, \quad j=1,2 \text{ for } y\in B_2$$
gives
$$\Delta v_j^k(y)+\sum_{l=1}^{2}k_{jl}h_l^k(s_k y)e^{v_l^k(y)}=4\pi\alpha_j\delta_0 \quad \text{in } y\in B_2.$$

If the other component also has slow decay on $\partial B(0,s_k)$, then (v_1^k,v_2^k) converges to (v_1,v_2) which satisfies
$$\Delta v_j(y)+\sum_{l=1}^{2}k_{jl}e^{v_l}=0 \quad \text{in } B_2\setminus\{0\},\ j=1,2. \tag{4-1}$$
If the other component has fast decay on $\partial B(0,s_k)$, then $v_i^k(y)$ converges to $v_i(y)$ and $v_j(y)\to-\infty$, $j\ne i$. Furthermore, $v_i(y)$ satisfies
$$\Delta v_i(y)+2e^{v_i}=0 \quad \text{in } B_2\setminus\{0\}. \tag{4-2}$$
For any $r>0$,
$$\begin{aligned}\int_{\partial B(0,r)}\frac{\partial v_i(y)}{\partial v}\,dS&=\lim_{k\to\infty}\left(4\pi\alpha_i-\sum_{j=1}^{2}\int_{B(0,r)}k_{ij}h_j^k e^{v_j^k}\,dy\right)\\&=4\pi\alpha_i-2\pi\sum_{j=1}^{2}k_{ij}\sigma_j+o(1)\doteqdot 4\pi\beta_i+o(1),\end{aligned}$$

which implies the right-hand sides of both (4-1) and (4-2) should be replaced by $4\pi\beta_i\delta_0$. If $\beta_i \le -1$, we can use the finite energy assumption (see the bottom assumption in (1-7)) to conclude that either (4-1) or (4-2) has no solutions. Hence $\alpha_i - \frac{1}{2}\sum_{j=1}^{2} k_{ij}\sigma_j > -1$ and then (a) is proved.

(b) Since both components have fast decay on $\partial B(0, l_k)$, the pair (σ_1, σ_2) satisfies the P.I. (1-9). By a simple manipulation, the P.I. (1-9) can be written as

$$k_{21}\sigma_1(4\mu_1 - k_{12}\sigma_2 - k_{11}\sigma_1) + k_{12}\sigma_2(4\mu_2 - k_{21}\sigma_1 - k_{22}\sigma_2) = 0. \tag{4-3}$$

Note by (a),

$$4\mu_i - \sum_{l=1}^{2} k_{il}\sigma_l > 2\mu_i - \sum_{l=1}^{2} k_{il}\sigma_l \geqslant 0.$$

Hence for $j \neq i$

$$2\mu_j - \sum_{l=1}^{2} k_{jl}\sigma_l < 4\mu_j - \sum_{l=1}^{2} k_{jl}\sigma_l < 0,$$

where the last inequality is due to (4-3). By (a) again, u_j^k does not have slow decay on $\partial B(0, s_k)$. $\square$

Our second lemma says that a fast-decaying component does not change its energy more than $o(1)$, regardless of the behavior of the other component.

Lemma 4.2. *Suppose the Harnack-type inequality holds for both components over $r \in \left[\frac{1}{2}l_k, 2s_k\right]$. If u_i^k is fast-decaying on $r \in [l_k, s_k]$, then*

$$\sigma_i^k(B(0, s_k)) = \sigma_i^k(B(0, l_k)) + o(1).$$

Proof. Obviously the conclusion holds if $s_k/l_k \leqslant C$. So we assume $s_k/l_k \to +\infty$. The Harnack-type inequality implies $u_i^k(x) = \bar{u}_i^k(r) + o(1)$ for $\frac{1}{2}l_k \leqslant |x| \leqslant 2s_k$. Thus we obtain from (1-5) that

$$\frac{d}{dr}(\bar{u}_i^k(r) + 2\log\, r) = \frac{2\mu_i - \sum_{j=1}^{2} k_{ij}\sigma_j^k(r)}{r}, \quad l_k \leqslant r \leqslant s_k,\ i = 1, 2,$$

where $\sigma_j^k(r) = \sigma_j^k(B(0, r))$ and $\sigma_j = \lim_{k\to+\infty} \sigma_j^k(l_k),\ j = 1, 2.$

Without loss of generality, we assume that $u_j^k,\ j \neq i$, is fast-decaying on $\partial B(0, l_k)$. Otherwise, we may choose $\tilde{l}_k$ such that $l_k \ll \tilde{l}_k,\ u_i^k$ remains fast-decaying for $r \in [l_k, \tilde{l}_k]$ and $\sigma_i^k(B(0, r))$ does not change more than $o(1)$, while u_j^k is fast-decaying on $\partial B(0, \tilde{l}_k)$. If $s_k/\tilde{l}_k \leqslant C$, we get the conclusion as explained above. If $s_k/\tilde{l}_k \to +\infty$, by a little abuse of notation, we may replace $\tilde{l}_k$ by l_k. Then both u_1^k, u_2^k have fast decay on $\partial B(0, l_k)$, and the P.I. holds at l_k, which implies that at least one component (say l) satisfies

$$4\mu_l - \sum_{j=1}^{2} k_{lj}\sigma_j^k(l_k) < 0.$$

Thus,

$$\frac{d}{dr}(\bar{u}_l^{(k)}(r) + 2\log\, r) \leqslant -\frac{2\mu_l + o(1)}{r} \quad \text{at } r = l_k. \tag{4-4}$$

Suppose $r_k \in [l_k, s_k]$ is the largest r such that

$$\frac{d}{dr}(\bar{u}_l^{(k)}(r) + 2\log\ r) \leqslant -\frac{\mu_l}{r} \quad \text{for } r \in [l_k, r_k]. \tag{4-5}$$

Thus, either the equality holds at $r = r_k$ or $r_k = s_k$. For simplicity, we let $\varepsilon = \mu_l$. By integrating (4-4) from l_k up to $r \leqslant r_k$, we have

$$\bar{u}_l^{(k)}(r) + 2\log\ r \leqslant \bar{u}_l^{(k)}(l_k) + 2\log(l_k) + \varepsilon \log\left(\frac{l_k}{r}\right);$$

that is for $|x| = r$,

$$e^{u_l^k(x)} \leqslant O(1)e^{\bar{u}_l^k(r)} \leqslant O(1)e^{-N_k} l_k^{\varepsilon} r^{-(2+\varepsilon)},$$

where we used $\bar{u}_l^{(k)}(l_k) + 2\log l_k \leqslant -N_k$ by the assumption of fast decay. Thus

$$\int_{l_k \leqslant |x| \leqslant r_k} e^{u_l^k(x)}\,dx \leqslant O(1)e^{-N_k} l_k^{\varepsilon} \int_{l_k}^{r_k} r^{-(1+\varepsilon)}\,dr = O(1)\frac{e^{-N_k}}{\varepsilon} \to 0$$

as $k \to +\infty$. Hence

$$\sigma_l^k(r_k) = \sigma_l^k(l_k) + o(1). \tag{4-6}$$

If both components are fast-decaying on $r \in [l_k, r_k]$, then $\lim_{k\to+\infty}(\sigma_1^k(r_k), \sigma_2^k(r_k)) = (\hat{\sigma}_1, \hat{\sigma}_2)$ also satisfies the P.I. (1-9). If $\hat{\sigma}_j > \sigma_j$, then $j \neq l$ by (4-6). We choose $r_k^* \leq r_k$ such that $\sigma_j(r_k^*) = \sigma_j^k(l_k) + \varepsilon_0$ for small ε_0, and let $\sigma_j^* = \lim_{k\to 0} \sigma_j(r_k^*)$. Then σ_j^* and σ_l satisfies the P.I. (1-9) and it yields a contradiction provided ε_0 is small. Thus, we have $\sigma_m^k(r_k) = \sigma_m^k(l_k) + o(1),\ m = 1, 2$. Then (4-4) holds at $r = r_k$, which implies $r_k = s_k$, and Lemma 4.2 is proved in this case.

If one of the components does not have fast decay on $[l_k, r_k]$, then we have $l = i$ and $u_j^k,\ j \neq i$, has slow decay on $\partial B(0, r_k^*)$ for some $r_k^* \leq r_k$. If $s_k/r_k \leq C$, then (4-6) implies the lemma. If $s_k/r_k \to +\infty$, then by the scaling of u_j^k at $r = r_k^*$, the standard argument implies that there is a sequence of $r_k^* \ll \tilde{r}_k = R_k r_k^* \ll s_k$ such that both components have fast decay on $\tilde{r}_k$ and

$$\sigma_i^k(\tilde{r}_k) = \sigma_i(r_k^*) + o(1) = \sigma_i(l_k) + o(1) \quad \text{and} \quad \sigma_j^k(\tilde{r}_k) \geq \sigma_j^k(l_k) + \varepsilon_0$$

for $j \neq i$ and $\varepsilon_0 > 0$. Therefore the assumption of Lemma 4.2 holds at $r \in [\tilde{r}_k, s_k]$. Then we repeat the argument starting from (4-4) and the lemma can be proved in a finite steps. □

Remark 4.3. Both lemmas will be used in Section 6 (and Section 5) for the case with singularity at 0 (and without singularity at 0).

5. Local mass on the bubbling disk centered at $x_l^k \neq 0$

5A. In this subsection we study the local behavior of u^k near x_l^k, where $x_l^k \neq 0$. For simplicity, we use x^k instead of x_l^k and $\bar{u}_i^k(r)$ rather than $\bar{u}_{x_l^k,i}^k(r)$. Let

$$\tau^k = \tfrac{1}{2}\operatorname{dist}(x^k, \Sigma_k \setminus \{x^k\}), \quad \sigma_i^k(r) = \frac{1}{2\pi}\int_{B(x^k,r)} h_i^k e^{u_i^k}, \quad i = 1, 2.$$

By Proposition 3A, $l_k \le \tau^k$. Clearly $u^k = (u_1^k, u_2^k)$ satisfies

$$\Delta u_i^k + \sum_{j=1}^{2} k_{ij} h_j^k e^{u_j^k} = 0 \quad \text{in } B(x^k, \tau^k).$$

For a sequence s_k, we define

$$\hat{\sigma}_i(s_k) = \begin{cases} \lim_{k\to+\infty} \sigma_i^k(s_k) \text{ if } u_i^k \text{ has fast decay on } \partial B(x^k, s_k), \\ \lim_{r\to 0} \lim_{k\to+\infty} \sigma_i^k(r s_k) \text{ if } u_i^k \text{ has slow decay on } \partial B(x^k, s_k). \end{cases} \tag{5-1}$$

Recall that both components of u^k have fast decay on $\partial B(x^k, l_k)$. This is the starting point of the following proposition, which is a special case of Proposition 5.2 below.

In Proposition 5.1, (μ_1, μ_2) will be $(1, 1)$ in both lemmas of Section 4.

Proposition 5.1. *Let $u^k = (u_1^k, u_2^k)$ be the solution of* (1-5) *satisfying* (1-7) *and $\hat{\sigma}_i(s_k)$ be defined in* (5-1). *The following holds*:

(1) *At least one component u^k has fast decay on $\partial B(x^k, \tau^k)$.*

(2) $(\hat{\sigma}_1(\tau^k), \hat{\sigma}_2(\tau^k))$ *satisfies the P.I.* (1-9) *with $\mu_1 = \mu_2 = 1$.*

(3) $(\hat{\sigma}_1(\tau^k), \hat{\sigma}_2(\tau^k)) \in \Gamma(1, 1)$.

Proof. If $\tau_k / l_k \leqslant C$, (1)–(3) hold obviously for τ^k. So we assume $\tau^k / l_k \to +\infty$. First we remark that if u^k is fully bubbling in $B(x^k, l_k)$ (i.e., (1) in Proposition 3A holds), $(\hat{\sigma}_1(l_k), \hat{\sigma}_2(l_k))$ is special (see Definition 1.4) and satisfies

$$2\mu_i - \sum_{j=1}^{2} k_{ij} \hat{\sigma}_j(l_k) < 0, \quad i = 1, 2.$$

Then by Lemma 4.1, both u_i^k have fast decay on $\partial B(0, \tau^k)$ and Proposition 5.1 follows immediately.

Now we assume v_i^k defined in (3-7) and satisfies case (2) in Proposition 3A. We already know that both components have fast decay at $r = l_k$. If both components remain fast-decaying as r increases from l_k to τ^k, Lemma 4.2 implies

$$\sigma_1^k(\tau^k) = \sigma_1^k(l_k) + o(1), \quad \sigma_2^k(\tau^k) = \sigma_2^k(l_k) + o(1)$$

and we are done. So we only consider the case that at least one component changes to a slow-decaying component. For simplicity, we assume that u_1^k changes to a slow-decaying component for some $r_k \gg l_k$. By Lemma 4.2,

$$\sigma_1^k(B(x^k, r_k)) \geqslant \sigma_1^k(B(x^k, l_k)) + c_0 \quad \text{for some } c_0 > 0.$$

We might choose $s_k \leqslant r_k$ such that

$$\sigma_1^k(B(x^k, s_k)) = \sigma_1^k(B(x^k, l_k)) + \varepsilon_0,$$

and

$$\sigma_1^k(B(x^k, r)) < \sigma_1^k(B(x^k, l_k)) + \varepsilon_0 \quad \text{for all } r < s_k,$$

where $\varepsilon_0 < \frac{1}{2} c_0$ is small.

Then Lemmas 4.1 and 4.2 together imply that u_1^k has slow decay on $\partial B(x^k, s_k)$ and u_2^k has fast decay on $\partial B(x^k, s_k)$ with

$$\hat{\sigma}_1(s_k) = \sigma_1^k(l_k) + o(1) \quad \text{and} \quad \hat{\sigma}_2(s_k) = \sigma_2^k(l_k) + o(1).$$

Let $v_i^k(y) = u_i^k(x^k + s_k y) + 2\log\, s_k$. If $\tau^k/s_k \le C$, there is nothing to prove. So we assume $\tau^k/s_k \to \infty$. Then $v_1^k(y)$ converges to $v_1(y)$ and $v_2^k(y) \to -\infty$ in any compact set of $\mathbb{R}^2$ as $k \to +\infty$ and $v_1(y)$ satisfies

$$\Delta v_1 + 2e^{v_1} = -2\pi \sum_{j=1}^{2} (k_{1j}\hat{\sigma}_j(l_k))\delta(0) \quad \text{in } \mathbb{R}^2. \tag{5-2}$$

Hence there is a sequence $N_k^* \to +\infty$ as $k \to +\infty$ that satisfies

(1) $N_k^* s_k \le \tau^k$,

(2) $\int_{B(0,N_k^*)} e^{v_1}\, dy = \int_{\mathbb{R}^2} e^{v_1}\, dy + o(1)$,

(3) $v_i^k(y) + 2\log|y| \leqslant -N_k,\ i = 1, 2$, for $|y| = N_k^*$.

Scaling back to u_i^k, we obtain that $u_i^k,\ i = 1, 2$, have fast decay on $\partial B(x^k, N_k^* s_k)$.

We could use the classification theorem of [Prajapat and Tarantello 2001] to calculate the total mass of v_1, but instead we use the P.I. (1-9) to compute it. We know that both $(\hat{\sigma}_1(l_k), \hat{\sigma}_2(l_k))$ and $(\hat{\sigma}_1(N_k^* s_k), \hat{\sigma}_2(N_k^* s_k))$ satisfy the P.I. and $\hat{\sigma}_2(N_k^* s_k) = \hat{\sigma}_2(l_k)$ by Lemma 4.2. With a fixed $\sigma_2 = \hat{\sigma}_2(l_k)$, P.I. (1-9) is a quadratic polynomial in σ_1; then $\hat{\sigma}_1(l_k)$ and $\hat{\sigma}_1(N_k^* s_k)$ are two roots of the polynomial. From it, we can easily calculate $\hat{\sigma}_1(N_k^* s_k)$.

By a direct computation, we have

$$(\hat{\sigma}_1(N_k^* s_k), \hat{\sigma}_2(N_k^* s_k)) \in \Gamma(1, 1) \quad \text{if } (\hat{\sigma}_1(l_k), \hat{\sigma}_2(l_k)) \in \Gamma(1, 1).$$

Thus (1)–(3) hold at $r = N_k^* s_k$. By denoting $N_k^* s_k$ as l_k, we can repeat the same argument until $\tau^k/l_k \leqslant C$. Hence Proposition 5.1 is proved. □

5B. ***Local mass in a group that does not contain* 0.** In this subsection we collect some $x_i^k \in \Sigma_k$ into a group S, a subset of Σ_k satisfying the following S-conditions:

(1) $0 \notin S$ and $|S| \ge 2$.

(2) If $|S| \ge 3$ and x_i^k, x_j^k, x_l^k are three distinct elements in S, then

$$\operatorname{dist}(x_i^k, x_j^k) \le C \operatorname{dist}(x_j^k, x_l^k)$$

for some constant C independent of k.

(3) For any $x_m^k \in \Sigma_k \setminus S$, we have $\operatorname{dist}(x_m^k, S)/\operatorname{dist}(x_i^k, x_j^k) \to \infty$ as $k \to \infty$, where $x_i^k, x_j^k \in S$.

We write S as $S = \{x_1^k, \dots, x_m^k\}$ and let

$$l^k(S) = 2 \max_{1 \le j \le m} \operatorname{dist}(x_1^k, x_j^k). \tag{5-3}$$

Recall $\tau_l^k = \frac{1}{2}\operatorname{dist}(x_l^k, \Sigma_k \setminus \{x_l^k\})$; by (2) and (3) above we have $l^k(S) \sim \tau_i^k$ for $1 \le i \le m$. Let

$$\tau_S^k = \tfrac{1}{2}\operatorname{dist}(x_1^k, \Sigma_k \setminus S).$$

Then by (3) above we have $\tau_S^k/\tau_i^k \to \infty$ for any $x_i^k \in S$.

By Proposition 5.1, we know that at least one of u_i^k has fast decay on $\partial B(x_1^k, \tau_1^k)$. Suppose u_1^k has fast decay on $\partial B(x_1^k, \tau_1^k)$. Then

$$u_1^k \text{ has fast decay on } \partial B(x_1^k, l^k(S)), \tag{5-4}$$

and we get

$$\begin{aligned}\sigma_1^k(B(x_1^k, l^k(S))) &= \frac{1}{2\pi}\int_{B(x_1^k, l^k(S))} h_1^k e^{u_1^k}\,dx \\ &= \frac{1}{2\pi}\int_{\bigcup_{j=1}^m B(x_j^k,\tau_j^k)} h_1^k e^{u_1^k} + \frac{1}{2\pi}\int_{B(x_1^k,l^k(S))\setminus(\bigcup_{j=1}^m B(x_j^k,\tau_j^k))} h_1^k e^{u_1^k}.\end{aligned}$$

Since u_1^k has fast decay outside of $B(x_j^k, \tau_j^k)$, we have

$$e^{u_1^k(x)} \le o(1)\max_j\{|x - x_j^k|^{-2}\} \quad \text{for } x \notin \bigcup_{j=1}^k B(x_j^k, \tau_j^k)$$

and the second integral is $o(1)$. Hence by Proposition 5.1,

$$\sigma_1^k(B(x_1^k, l^k(S))) = 2m_1 + o(1) \quad \text{for some } m_1 \in \mathbb{N}\cup\{0\}. \tag{5-5}$$

Similarly if u_2^k has fast decay on $\partial B(x_1^k, \tau_1^k)$, we have

$$\sigma_2^k(B(x_1^k, l^k(S))) = 2m_2 + o(1) \quad \text{for some } m_2 \in \mathbb{N}\cup\{0\}. \tag{5-6}$$

If u_2^k has slow decay on $\partial B(x_1^k, \tau_1^k)$, then it is easy to see that u_2^k has slow decay on $\partial B(x_j^k, \tau_j^k)$. By Proposition 5.1 we denote $n_{i,j} \in \mathbb{N}$ by

$$2n_{i,j} = \lim_{r\to 0}\lim_{k\to\infty}\sigma_i^k(B(x_j^k, r\tau_j^k)), \quad 1 \le j \le m,\ i = 1, 2.$$

Define $\hat{n}_{i,j}$ by

$$\hat{n}_{i,j} = -\sum_{l=1}^2 k_{il} n_{l,j}.$$

Then the slow decay of u_2^k on $\partial B(x_j^k, \tau_j^k)$ implies $1 + \hat{n}_{2,j} > 0$. Since $\hat{n}_{2,j} \in \mathbb{Z}$ we have $\hat{n}_{2,j} \ge 0$.

Furthermore, if we scale u^k by

$$v_i^k(y) = u_i^k(x_1^k + l^k(S)y) + 2\log l^k(S), \quad i = 1, 2,$$

the sequence v_2^k converges to $v_2(y)$ and v_1^k tends to $-\infty$ over any compact subset of $\mathbb{R}^2 \setminus \{0\}$. Then v_2 satisfies

$$\Delta v_2(y) + 2e^{v_2(y)} = 4\pi\sum_{j=1}^m \hat{n}_{2,j}\delta_{p_j} \quad \text{in } \mathbb{R}^2, \tag{5-7}$$

where $p_j = \lim_{k\to\infty}(x_j^k - x_1^k)/l^k(S)$. By Theorem 2.1

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_2} = 2N \quad \text{for some } N \in \mathbb{N}.$$

Thus using the argument in Proposition 5.1, we conclude that there is a sequence of $N_k^* \to \infty$ such that both u_i^k $(i = 1, 2)$ have fast decay on $\partial B(x_1^k, N_k^* l^k(S))$ and $\sigma_i^k(B(x_1^k, N_k^* l^k(S))) = 2m_i + o(1)$. Denote $N_k^* l^k(S)$ by l_k for simplicity; we see that (5-5) and (5-6) hold at l_k. Then by using Lemmas 4.1 and 4.2 we continue this process to obtain the following conclusion:

$$\text{At least one component of } u^k \text{ has fast decay on } \partial B(x_1^k, \tau_S^k). \tag{5-8}$$

Let $\hat{\sigma}_i^k(B(x_1^k, \tau_S^k))$ be defined as in (5-1). Then

$$\hat{\sigma}_i^k(B(x_1^k, \tau_S^k)) = 2m_i(S), \quad \text{where } m_i(S) \in \mathbb{N} \cup \{0\}, \tag{5-9}$$

and the pair $(2m_1(S), 2m_2(S))$ satisfies the P.I. (1-9).

Denote the group S by S_1. Based on this procedure, we can continue to select a new group S_2 such that the S-conditions holds except we have to modify condition (2). In (2), we consider S_1 as a single point as long as we compare the distance of distinct elements in S_2.

Set

$$\tau_{S_2}^k = \tfrac{1}{2}\operatorname{dist}(x_1^k, \Sigma_k \setminus S_2) \quad \text{for } x_1^k \in S_2.$$

Then we follow the same argument as above to obtain the same conclusion as (5-8)–(5-9).

If (1-5) does not contain a singularity, the final step is to collect all the x_i^k into the single biggest group and (5-8)–(5-9) hold. Then we get $(\sigma_1, \sigma_2) = (2m_1, 2m_2)$ (which satisfies the Pohozaev identity), where

$$\sigma_i = \lim_{r\to 0}\lim_{k\to\infty}\frac{1}{2\pi}\int_{B(0,r)} h_i^k e^{u_i^k}, \quad i = 1, 2.$$

By a direct computation, we can prove that the set of all the pairs of even integers solving (1-9) is exactly $\Gamma(1, 1)$. This proves Theorem 1.3 if (1-5) has no singularities.

If 0 is a singularity of (1-5) then Σ_k can be written as a disjoint union of $\{0\}$ and S_j $(j = 1, \dots, m)$. Here each S_j is collected by the process described above and is maximal in the following sense:

(i) $0 \notin S$, $|S| \ge 2$ and for any two distinct points x_i^k, x_j^k in S we have

$$\operatorname{dist}(x_i^k, x_j^k) \ll \tau^k(S),$$

where $\tau^k(S) = \operatorname{dist}(S, \Sigma_k \setminus S)$.

(ii) For any $0 \ne x_i^k \in \Sigma_k \setminus S$,

$$\operatorname{dist}(x_i^k, 0) \le C \operatorname{dist}(x_i^k, S)$$

for some constant C.

For S_j we define

$$\tau_{S_j}^k = \tfrac{1}{2}\operatorname{dist}(S_j, \Sigma_k \setminus S_j).$$

Then the process described above proves the main result of this section:

Proposition 5.2. *Let S_j ($j = 1, \dots, m$) be described as above. Then* (5-8)–(5-9) *hold, where $B(x_1^k, \tau_S^k)$ is replaced by $B(x_i^k, \tau_{S_j}^k)$ and x_i^k is any element in S_j.*

6. Proofs of Theorems 1.2, 1.3, 1.5 and 1.6

In Proposition 5.2, we write $\Sigma_k = \{0\} \cup S_1 \cup \cdots \cup S_N$. From the construction, the ratio $|x^k|/|\tilde{x}^k|$ is bounded for any $x^k, \tilde{x}^k \in S_j$. Let

$$\|S_j\| = \min_{x^k \in S_j} |x^k|$$

and arrange S_j by

$$\|S_1\| \le \|S_2\| \le \cdots \le \|S_N\|.$$

Assume l is the largest number such that $\|S_l\| \le C\|S_1\|$. Then $\|S_l\| \ll \|S_{l+1}\|$.

We recall the local mass contributed by $x_j^k \in S_j$ is

$$\big(\hat{\sigma}_1(B(x_j^k, \tau_j^k)), \hat{\sigma}_2(B(x_j^k, \tau_j^k))\big) = (m_{1,j}, m_{2,j}), \quad \text{where } m_{1,j}, m_{2,j} \in 2\mathbb{N} \cup \{0\}.$$

Let

$$r_1^k = \tfrac{1}{2}\|S_1\|.$$

Then we have

$$u_i^k(x) + 2\log|x| \le C \quad \text{for } 0 < |x| \le r_1^k,\ i = 1, 2.$$

Proof of Theorem 1.3. Let

$$\tilde{u}_i^k(x) = u_i^k(x) + 2\alpha_i \log|x|, \quad i = 1, 2.$$

Then (1-5) becomes

$$\Delta\tilde{u}_i^k(x) + \sum_{j=1}^{2} k_{ij}|x|^{2\alpha_j} h_j^k(x) e^{\tilde{u}_j^k(x)} = 0, \quad |x| \le r_1^k,\ i = 1, 2.$$

Let

$$-2\log\delta_k = \max_{i \in I} \max_{x \in \bar{B}(0, r_1^k)} \frac{\tilde{u}_i^k}{1+\alpha_i}, \tag{6-1}$$

and

$$\tilde{v}_i^k(y) = \tilde{u}_i^k(\delta_k y) + 2(1+\alpha_i)\log\delta_k, \quad |y| \le r_1^k/\delta_k,\ i = 1, 2. \tag{6-2}$$

Then $\tilde{v}_i^k$ satisfies

$$\Delta\tilde{v}_i^k(y) + \sum_{j=1}^{2} k_{ij}|y|^{2\alpha_j} h_j^k(\delta_k y) e^{\tilde{v}_j^k(y)} = 0, \quad |y| \le r_1^k/\delta_k,\ i = 1, 2. \tag{6-3}$$

We have either

(a) $\lim_{k\to\infty} r_1^k/\delta_k = \infty$, or

(b) $r_1^k/\delta_k \le C$.

For case (a), our purpose is to prove a result similar to Proposition 5.1:

(1) *At most one component of* u^k *has slow decay on* $\partial B(0, r_1^k)$. As in Section 5, we define

$$\hat{\sigma}_{i,1} = \begin{cases} \lim_{k\to+\infty} \sigma_i^k(B(0, r_1^k)) & \text{if } u_i^k \text{ has fast decay on } \partial B(0, r_1^k), \\ \lim_{r\to 0} \lim_{k\to+\infty} \sigma_i^k(B(0, r r_1^k)) & \text{if } u_i^k \text{ has slow decay on } \partial B(0, r_1^k), \end{cases}$$

(2) $(\hat{\sigma}_{1,1}, \hat{\sigma}_{2,1})$ satisfies the Pohozaev identity (1-9), and

(3) $\hat{\sigma}_{i,1} = 2\sum_{j=1}^{2} n_{i,j}\mu_j + 2n_{i,3},\ n_{i,j} \in \mathbb{Z},\ i = 1, 2,\ j = 1, 2, 3.$

We carry out the proof in the discussion of the following two cases.

Case 1: If both $\tilde{v}_i^k(y)$ converge in any compact set of $\mathbb{R}^2$, then $(\hat{\sigma}_{1,1}, \hat{\sigma}_{2,1})$ can be obtained by the classification theorem in [Lin et al. 2012]:

$$(\hat{\sigma}_{1,1}, \hat{\sigma}_{2,1}) = \begin{cases} (2\mu_1 + 2\mu_2, 2\mu_1 + 2\mu_2) & \text{for } A_2, \\ (4\mu_1 + 2\mu_2, 4\mu_1 + 4\mu_2) & \text{for } B_2, \\ (8\mu_1 + 4\mu_2, 12\mu_1 + 8\mu_2) & \text{for } G_2. \end{cases}$$

By Lemma 4.1, both u_i^k have fast decay on $\partial B(0, r_1^k)$. So this proves (1)–(3) in this case.

Case 2: Only one $\tilde{v}_i^k$ converges to $v_i(y)$ and the other tends to $-\infty$ uniformly in any compact set. Then it is easy to see that there is $l_k \ll r_1^k$ such that both u_i^k have fast decay on $\partial B(0, l_k)$ and

$$\big(\sigma_1(B(0, l_k)), \sigma_2(B(0, l_k))\big) = (2\mu_1, 0) \quad \text{or} \quad \big(\sigma_1(B(0, l_k)), \sigma_2(B(0, l_k))\big) = (0, 2\mu_2).$$

So this is the same situation as in the starting point for Proposition 5.1. Then the same argument of Proposition 5.1 leads to the conclusion (1)–(3).

The pair $(\hat{\sigma}_{1,1}, \hat{\sigma}_{2,1})$ can be calculated by the same method in Proposition 5.1. Then $(\hat{\sigma}_{1,1}, \hat{\sigma}_{2,1}) \in \Gamma(\mu_1, \mu_2)$, which is given in Section 2.

To continue for $r \in [r_1^k, r_2^k]$, where $r_2^k = \frac{1}{2}\|S_{l+1}\|$, we separate our discussion into two cases also.

Case 1: One component has slow decay on $\partial B(0, r_1^k)$, say u_1^k. Then we scale

$$v_i^k(y) = u_i^k(r_1^k y) + 2\log r_1^k.$$

By our assumption, $v_1^k(y)$ converges to $v_1(y)$ and $v_2^k(y) \to -\infty$ in any compact set. Let $x_j^k \in S_j$ and $y_j^k = (r_1^k)^{-1} x_j^k \to p_j$ for $j \le l$. Then $v_1(y)$ satisfies

$$\Delta v_1 + 2e^{v_1} = 4\pi\tilde{\alpha}_1\delta_0 + 4\pi\sum_{j=1}^{l}\tilde{n}_{1,j}\delta_{p_j}, \tag{6-4}$$

where

$$\tilde{n}_{1,j} = -\frac{1}{2}\sum_{i=1}^{2} k_{1i} m_{i,j} \quad \text{for some } m_{ij} \in \mathbb{Z} \qquad \text{and} \qquad \tilde{\alpha}_1 = \alpha_1 - \frac{1}{2}\sum_{i=1}^{2} k_{1i}\hat{\sigma}_{i,1}. \tag{6-5}$$

The finiteness of $\int_{\mathbb{R}^2} e^{v_1}$ implies

$$\tilde{\alpha}_1 > -1 \quad \text{and} \quad \tilde{n}_{1,j} \ge 0.$$

By Theorem 2.2, we have

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_1}\,dy = 2(\tilde{\alpha}_1+1)+2k_1, \quad \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_1}\,dy = 2k_2, \quad \text{where } k_1,k_2\in\mathbb{Z}. \tag{6-6}$$

As before, we can choose l_k, $r_1^k \ll l_k \ll r_2^k$, such that both u_i^k have fast decay on $\partial B(0,l_k)$. Then the new pair $(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2})$, which is defined by

$$\hat{\sigma}_{t,2} = \frac{1}{2\pi}\lim_{k\to 0}\int_{B(0,l_k)} h_t^k e^{u_t^k}, \quad t=1,2,$$

becomes

$$(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2}) = \left(\hat{\sigma}_{1,1}+\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_1}+\sum_{j=1}^{l} m_{1,j},\ \hat{\sigma}_{2,1}+\sum_{j=1}^{l} m_{2,j}\right) \tag{6-7}$$

for $m_{1j},m_{2j}\in 2\mathbb{N}\cup\{0\}$. Using (6-6), we get

$$\hat{\sigma}_{1,2} = \begin{cases} \hat{\sigma}_{1,1}+2k_2+\sum_{j=1}^{l} m_{1,j} & \text{if } \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_1}\,dy = 2k_2, \\ 2\mu_1+\hat{\sigma}_{1,1}-\sum_{i=1}^{2} k_{1i}\hat{\sigma}_{i,1}+2k_1+\sum_{j=1}^{l} m_{1,j} & \text{if } \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_1}\,dy = 2(\tilde{\alpha}_1+1)+2k_1. \end{cases} \tag{6-8}$$

We note that if $(\hat{\sigma}_{1,1},\hat{\sigma}_{2,1})\in\Gamma(\mu_1,\mu_2)$ and

$$2\mu_1+\hat{\sigma}_{1,1}-\sum_{i=1}^{2} k_{1i}\hat{\sigma}_{i,1} > 0,$$

then

$$\left(2\mu_1+\hat{\sigma}_{1,1}-\sum_{i=1}^{2} k_{1i}\hat{\sigma}_{i,1},\hat{\sigma}_{2,1}\right)\in\Gamma(\mu_1,\mu_2).$$

Let $(\sigma_1^*,\sigma_2^*) = \bigl(2\mu_1+\hat{\sigma}_{1,1}-\sum_{i=1}^{2} k_{1i}\hat{\sigma}_{i,1},\ \hat{\sigma}_{2,1}\bigr)$. We can write

$$(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2}) = (\sigma_1^*+m_1,\ \sigma_2^*+m_2), \tag{6-9}$$

with $(\sigma_1^*,\sigma_2^*)\in\Gamma(\mu_1,\mu_2)$ and $m_1,m_2\in 2\mathbb{Z}$.

Case 2: If both u_i^k have fast decay on $\partial B(0,r_1^k)$, then they have fast decay on $\partial B(0,cr_1^k)$, where we choose c bounded such that $\bigcup_{j=1}^{l} S_j \subset B\bigl(0,\frac{1}{2}cr_1^k\bigr)$. Then the new pair $(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2})$ becomes

$$(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2}) = \left(\hat{\sigma}_{1,1}+\sum_{j=1}^{l} m_{1,j},\ \hat{\sigma}_{2,1}+\sum_{j=1}^{l} m_{2,j}\right) \quad \text{for } m_{1,j},m_{2,j}\in 2\mathbb{Z}. \tag{6-10}$$

Hence, in this case we can also write

$$(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2}) = (\sigma_1^*+m_1,\ \sigma_2^*+m_2), \tag{6-11}$$

with $(\sigma_1^*,\sigma_2^*) = (\hat{\sigma}_{1,1},\hat{\sigma}_{2,1})\in\Gamma(\mu_1,\mu_2)$ and $m_1,m_2\in 2\mathbb{Z}$. Set $cr_1^k = l_k$. Then we can continue our process starting from l_k. After finitely many steps, we can prove that at most one component of u^k has slow decay on $\partial B(0,1)$ and their local masses have the expression in (3).

For case (b), i.e., $r_1^k/\delta_k \le C$, first $\tilde{v}_i^k \le 0$ implies $|y|^{2\alpha_j} h_j^k(\delta_k y)e^{\tilde{v}_j^k} \le C$ on $B(0, r_1^k/\delta_k)$. Then the fact that $\tilde{v}_i^k$ has bounded oscillation on $\partial B(0, r_1^k/\delta_k)$ further gives

$$\tilde{v}_i^k(x) = \bar{\tilde{v}}_i^k(\partial B(0, r_1^k/\delta_k)) + O(1) \quad \text{for all } x \in B(r_1^k/\delta_k),$$

where $\bar{\tilde{v}}_i^k(\partial B(0, r_1^k/\delta_k))$ stands for the average of $\tilde{v}_i^k$ on $\partial B(0, r_1^k/\delta_k)$. Direct computation shows that

$$\int_{B(0,r_1^k)} h_i^k e^{u_i^k}\, dx = \int_{B(0,r_1^k/\delta_k)} |y|^{2\alpha_i} h_i^k(\delta_k y)e^{\tilde{v}_i^k(y)}\, dy = O(1)e^{\bar{\tilde{v}}_i^k(\partial B(0,r_1^k/\delta_k))}.$$

Thus if $\bar{\tilde{v}}_i^k(\partial B(0, r_1^k/\delta_k)) \to -\infty$, we get $\int_{B(0,r_1^k)} h_i^k e^{u_i^k}\, dx = o(1)$. On the other hand, we note that $\bar{\tilde{v}}_i^k(\partial B(0, r_1^k/\delta_k)) \to -\infty$ is equivalent to u_i^k having fast decay on $\partial B(0, r_1^k)$. Consequently $\hat{\sigma}_{i,1} = 0$ if u_i^k has fast decay on $\partial B(0, r_1^k)$. So if both components have fast decay on $\partial B(0, r_1^k)$ we have $(\hat{\sigma}_{1,1}, \hat{\sigma}_{2,1}) = (0, 0)$.

If some component of u^k has slow decay, say u_2^k, according to the definition of $\hat{\sigma}_{2,1}$, we have

$$\begin{aligned}\hat{\sigma}_{2,1} &= \lim_{r\to 0}\lim_{k\to+\infty} \sigma_2^k(B(0, r r_1^k)) = \frac{1}{2\pi}\lim_{r\to 0}\lim_{k\to+\infty}\int_{B(0,rr_1^k)} h_2^k e^{u_2^k}\, dx \\ &= \frac{1}{2\pi}\lim_{r\to 0}\lim_{k\to+\infty}\int_{B(0,rr_1^k/\delta_k)} |y|^{2\alpha_2} h_2^k(\delta_k y)e^{\tilde{v}_2^k(y)}\, dy = 0,\end{aligned} \tag{6-12}$$

where we used $|y|^{2\alpha_2} h_2^k(\delta_k y)e^{\tilde{v}_2^k} \le C$ on $B(0, r_1^k/\delta_k)$. Then we still get

$$(\hat{\sigma}_{1,1}, \hat{\sigma}_{2,1}) = (0, 0).$$

Now we can continue our discussion as in case (a) and Theorem 1.3 is proved completely. □

Next, we shall prove Theorem 1.5, that is, $\Sigma_k = \{0\}$, by way of contradiction. Suppose Σ_k has points other than 0. Using the notation from the beginning of this section, we have

$$\Sigma_k = \{0\} \cup S_1 \cup \cdots \cup S_N.$$

Now suppose $r_1^k/\delta_k \to \infty$ as $k \to \infty$. Let $(\hat{\sigma}_{1,2}, \hat{\sigma}_{2,2})$ be the local masses defined by (6-7) for one of the components u_i^k having slow decay on $\partial B(0, r_1^k)$ or by (6-10) for both components having fast decay on $\partial B(0, r_1^k)$. We summarize the results in the following:

(i) $\hat{\sigma}_{i,2} = \sigma_i^* + m_i$, where $(\sigma_1^*, \sigma_2^*) \in \Gamma(\mu_1, \mu_2)$ and $m_i,\ i = 1, 2$, are even integers.

(ii) Both pairs (σ_1^*, σ_2^*) and $(\hat{\sigma}_{1,2}, \hat{\sigma}_{2,2})$ satisfy the Pohozaev identity.

Based on the description above, we now present the proof of Theorem 1.5.

Proof of Theorem 1.5. From the discussion above, we have

$$(\hat{\sigma}_{1,2}, \hat{\sigma}_{2,2}) = (\sigma_1^* + m_1,\ \sigma_2^* + m_2).$$

We note that the conclusion of Theorem 1.5 is equivalent to proving $m_i = 0,\ i = 1, 2$. In order to prove this we first observe that both $(\hat{\sigma}_{1,2}, \hat{\sigma}_{2,2})$ and (σ_1^*, σ_2^*) satisfy the P.I.

$$k_{21}\sigma_1^2 + k_{12}k_{21}\sigma_1\sigma_2 + k_{12}\sigma_2^2 = 2k_{21}\mu_1\sigma_1 + 2k_{12}\mu_2\sigma_2. \tag{6-13}$$

Thus we can write

$$k_{21}(\sigma_1^*)^2+k_{12}k_{21}\sigma_1^*\sigma_2^*+k_{12}(\sigma_2^*)^2=2k_{21}\mu_1\sigma_1^*+2k_{12}\mu_2\sigma_2^*, \tag{6-14}$$

and

$$\begin{aligned}k_{21}(\sigma_1^*+m_1)^2+k_{12}k_{21}(\sigma_1^*+m_1)(\sigma_2^*+m_2)+k_{12}(\sigma_2^*+m_2)^2&\\ =2k_{21}\mu_1(\sigma_1^*+m_1)+2k_{12}\mu_2(\sigma_2^*+m_2).&\end{aligned} \tag{6-15}$$

It is easy to obtain the following from (6-15) and (6-14):

$$\begin{aligned}2k_{21}m_1\sigma_1^*+k_{12}k_{21}m_2\sigma_1^*+k_{12}k_{21}m_1\sigma_2^*+2k_{12}m_2\sigma_2^*&\\ =2k_{21}m_1\mu_1+2k_{12}m_2\mu_2-(k_{21}m_1^2+k_{12}k_{21}m_1m_2+k_{12}m_2^2).&\end{aligned} \tag{6-16}$$

Since $(\sigma_1^*,\sigma_2^*)\in\Gamma(\mu_1,\mu_2)$, we set

$$\sigma_1^*=l_{1,1}\mu_1+l_{1,2}\mu_2,\quad \sigma_2^*=l_{2,1}\mu_1+l_{2,2}\mu_2.$$

Then we can rewrite (6-16) as

$$\begin{aligned}&\big(2k_{21}l_{1,1}m_1+k_{12}k_{21}l_{2,1}m_1-2k_{21}m_1+2k_{12}l_{2,1}m_2+k_{12}k_{21}l_{1,1}m_2\big)\mu_1\\ &\quad+\big(2k_{21}l_{1,2}m_1+k_{12}k_{21}l_{2,2}m_1+2k_{12}l_{2,2}m_2+k_{12}k_{21}l_{1,2}m_2-2k_{12}m_2\big)\mu_2\\ &\qquad+(k_{21}m_1^2+k_{12}k_{21}m_1m_2+k_{12}m_2^2)=0.\end{aligned} \tag{6-17}$$

Since μ_1,μ_2 and 1 are linearly independent, the coefficients of μ_1 and μ_2 must vanish. Equivalently we have

$$\begin{pmatrix}2k_{21}l_{1,1}+k_{12}k_{21}l_{2,1}-2k_{21} & 2k_{12}l_{2,1}+k_{12}k_{21}l_{1,1}\\ 2k_{21}l_{1,2}+k_{12}k_{21}l_{2,2} & 2k_{12}l_{2,2}+k_{12}k_{21}l_{1,2}-2k_{12}\end{pmatrix}\begin{pmatrix}m_1\\ m_2\end{pmatrix}=0. \tag{6-18}$$

Let M_K be the coefficient matrix

$$M_K=\begin{pmatrix}2k_{21}l_{1,1}+k_{12}k_{21}l_{2,1}-2k_{21} & 2k_{12}l_{2,1}+k_{12}k_{21}l_{1,1}\\ 2k_{21}l_{1,2}+k_{12}k_{21}l_{2,2} & 2k_{12}l_{2,2}+k_{12}k_{21}l_{1,2}-2k_{12}\end{pmatrix}.$$

Our goal is to show that M_k is nonsingular, which immediately implies $m_1=m_2=0$ and completes the proof of Theorem 1.5. The proof of the nonsingularity of M_k is divided into the following three cases.

Case 1: $\boldsymbol{K}=A_2$. Then we can write (6-18) as

$$\begin{pmatrix}2l_{1,1}-l_{2,1}-2 & 2l_{2,1}-l_{1,1}\\ 2l_{1,2}-l_{2,2} & 2l_{2,2}-l_{1,2}-2\end{pmatrix}\begin{pmatrix}m_1\\ m_2\end{pmatrix}=0. \tag{6-19}$$

We note that

$$(l_{1,1},l_{1,2},l_{2,1},l_{2,2})\in\big\{(2,0,0,0),\ (0,0,0,2),\ (2,2,0,2),\ (2,0,2,2),\ (2,2,2,2)\big\}.$$

Then it is easy to see that M_K is nonsingular when $(l_{1,1},l_{1,2},l_{2,1},l_{2,2})$ belongs the above set.

Case 2: $\boldsymbol{K}=B_2$. Then we can write (6-18) as

$$\begin{pmatrix}2l_{1,1}-l_{2,1}-2 & l_{2,1}-l_{1,1}\\ 2l_{1,2}-l_{2,2} & l_{2,2}-l_{1,2}-1\end{pmatrix}\begin{pmatrix}m_1\\ m_2\end{pmatrix}=0. \tag{6-20}$$

We note that

$(l_{1,1}, l_{1,2}, l_{2,1}, l_{2,2}) \in \{(2,0,0,0),\ (2,0,4,2),\ (4,2,4,2),\ (0,0,0,2),\ (2,2,0,2),\ (2,2,4,4),\ (4,2,4,4)\}$

From the above set, we can see that $4 \mid (l_{2,1} - l_{1,1})(2l_{1,2} - l_{2,2})$. As a result, if the determinant of M_K is 0, we have to make $4 \mid (2l_{1,1} - l_{2,1} - 2)$, which forces $l_{2,1} \equiv 2 \pmod 4$. However, this is impossible according to the above list. Thus M_k is nonsingular in this case.

Case 3: $\boldsymbol{K} = G_2$. Then we can write (6-18) as

$$\begin{pmatrix} 6l_{1,1} - 3l_{2,1} - 6 & 2l_{2,1} - 3l_{1,1} \\ 6l_{1,2} - 3l_{2,2} & 2l_{2,2} - 3l_{1,2} - 2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = 0. \tag{6-21}$$

We note that

$$(l_{1,1}, l_{1,2}, l_{2,1}, l_{2,2}) \in \{(2,0,0,0),\ (2,0,6,2),\ (6,2,6,2),\ (6,2,12,6),\ (8,4,12,6),\ (8,4,12,8), \\ (0,0,0,2),\ (2,2,0,2),\ (2,2,6,6),\ (6,4,6,6),\ (6,4,12,8)\}.$$

From the above list, we have $3 \mid l_{2,1}$; then we get $9 \mid (2l_{2,1} - 3l_{1,1})(6l_{1,2} - 3l_{2,2})$. On the other hand, we see that

$$l_{1,1} \equiv 0, 2 \pmod 3 \quad \text{and} \quad l_{2,2} \equiv 0, 2 \pmod 3,$$

which implies $(6l_{1,1} - 3l_{2,1} - 6)(2l_{2,2} - 3l_{1,2} - 2)$ is not multiple of 9; therefore we have the determinant of M_K is not zero. Thus M_k is nonsingular when $\boldsymbol{K} = G_2$.

Theorem 1.5 is established. □

Finally we prove Theorems 1.2 and 1.6.

Proof of Theorems 1.2 and 1.6. Suppose there exists a sequence of blowup solutions (u_1^k, u_2^k) of (1-2) with $(\rho_1, \rho_2) = (\rho_1^k, \rho_2^k)$. First, we prove Theorem 1.2. From the previous discussion of this section, we get that at least one component (say u_1^k) of u^k has fast decay on a small ball B near each blowup point q, which means $u_1^k(x) \to -\infty$ if $x \notin S$ and x is not a blowup point. Hence Theorem 1.2 holds.

Because the mass distribution of u_1^k concentrates as $k \to +\infty$, we get that $\lim_{k\to+\infty} \rho_1^k$ is equal to the sum of the local mass σ_1 at a blowup point q, which implies $\rho_1 \in \Gamma_1$, a contradiction to the assumption. Thus, we finish the proof of Theorem 1.6. □

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BEYOND THE BKM CRITERION FOR THE 2D RESISTIVE MAGNETOHYDRODYNAMIC EQUATIONS

LÉO AGÉLAS

The question of whether the two-dimensional (2D) magnetohydrodynamic (MHD) equations with only magnetic diffusion can develop a finite-time singularity from smooth initial data is a challenging open problem in fluid dynamics and mathematics. In this paper, we derive a regularity criterion less restrictive than the Beale–Kato–Majda (BKM) regularity criterion type, namely any solution $(u,b) \in C([0,T[; H^r(\mathbb{R}^2))$ with $r > 2$ remains in $H^r(\mathbb{R}^2)$ up to time T under the assumption that

$$\int_0^T \frac{\|\nabla u(t)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(t)\|_\infty)}\,dt < +\infty.$$

This regularity criterion may stand as a great improvement over the usual BKM regularity criterion, which states that if $\int_0^T \|\nabla \times u(t)\|_\infty \,dt < +\infty$ then the solution $(u,b) \in C([0,T[; H^r(\mathbb{R}^2))$ with $r > 2$ remains in $H^r(\mathbb{R}^2)$ up to time T. Furthermore, our result applies also to a class of equations arising in hydrodynamics and studied by Elgindi and Masmoudi (2014) for their L^∞ ill-posedness.

Introduction

Magnetohydrodynamic (MHD) equations describe the evolution of electrically conducting fluids in the presence of electric and magnetic fields. Examples of such fluids include plasmas, liquid metals, and salt water or electrolytes. The field of MHD was initiated by Hannes Alfvén [1942], for which he received the Nobel Prize in physics in 1970. It addresses laboratory as well as astrophysical plasmas and therefore is extensively used in very different contexts. In astrophysics, its applications range from solar wind [Marsch and Tu 1994], to the sun [Priest 1982; Priest and Forbes 2000], to the interstellar medium [Ng et al. 2003] and beyond [Zweibel and Heiles 1997]. At the same time, MHD is also relevant to large-scale motion in nuclear fusion devices such as tokamaks [Strauss 1976]. A tokamak is a toroidal device in which hydrogen isotopes in the form of a plasma reaching a temperature on the order of hundreds of millions of Kelvins is confined thanks to a very strong applied magnetic field. Tokamaks are used to study controlled fusion and are considered as one of the most promising concepts to produce fusion energy in the near future. However the main problem with this approach of confinement is that hydrodynamic instabilities arise. Numerical simulations using the MHD models are therefore of uttermost importance. Further, the proof of the existence of a smooth strong solution would allow one to guarantee a priori the convergence of some numerical approximations; see for instance [Chernyshenko et al. 2007].

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Due to their prominent roles in modeling many phenomena in astrophysics, geophysics and plasma physics, the MHD equations have been studied extensively mathematically. Furthermore, while the differences in behavior between the two-dimensional (2D) and three-dimensional (3D) hydrodynamical turbulence of neutral fluids are accepted to be important, those of the MHD system in both cases are conventionally believed to be nonsignificant [Biskamp and Schwarz 2001]. Strong statements were made by some authors that 2D simulations can be safely used to model 3D situations because the properties of the 2D and the 3D MHD turbulence are essentially the same [Biskamp 1993; Biskamp and Schwarz 2001].

Hence, the mathematical studies on the MHD equations in the two-dimensional case appear highly relevant. However up to now, the question of the spontaneous appearance of a singularity from a local classical solution of the partially viscous 2D MHD (2) or 2D inviscid MHD ((2) without the Laplacian term) remains a challenging open problem in mathematical fluid mechanics. Thus, in the absence of a well-posedness theory, the development of blow-up/nonblow-up theory is of major importance for both theoretical and practical purposes. Indeed, for a mathematical or numerical test of the actual finite-time blow-up of a given solution, it is important to have a good blow-up criterion. Thus, there have been many computational attempts to find finite-time singularities of the 2D MHD equations; see [Brachet et al. 2013; Kerr and Brandenburg 1999; Tran et al. 2013a]. Moreover, recent works on the 2D MHD equations developed regularity criteria in terms of the velocity field and dealt with the MHD equations with dissipation and magnetic diffusion given by general Fourier multiplier operators such as the fractional Laplacian operators; see [Wu 2003; 2008; 2011; Chen et al. 2010; Tran et al. 2013b; Jiu and Zhao 2014; Cao et al. 2014; Yamazaki 2014a; 2014b].

Among all the regularity criteria, one of particular interest is the Beale–Kato–Majda criterion, well-known for Euler equations, and extended in [Caflisch et al. 1997] to the inviscid MHD equations, under the assumption on both velocity field and magnetic field $\int_0^T (\|\omega(t)\|_{L^\infty} + \|j(t)\|_{L^\infty})\,dt < \infty$, where the vorticity is $\omega = \nabla \times u$ and the density is $j = \nabla \times b$. And so, the Beale–Kato–Majda criterion ensures that the solution (u, b) of the inviscid MHD equations is smooth up to time T.

Meanwhile the 2D Euler equation is globally well-posed for smooth initial data; however for the 2D inviscid MHD equations, the global well-posedness of classical solutions is still a big open problem. Despite recent developments on regularity criteria, see [Gala et al. 2017; Tran et al. 2013b; Jiu and Zhao 2014; 2015; Yamazaki 2014a; 2014b; Agélas 2016; Ye and Xu 2014; Fan et al. 2014], the global regularity issue of 2D MHD equations (2) remains a challenging open problem to date. The main reason for the unavailability of a proof of global regularity for the system of equations (2) is due to the quadratic coupling between u and b which invalidates the vorticity conservation. Indeed, the structure of the vorticity is instantaneously altered due to the effects of the magnetic fields. This fact is the source of the main difficulty connected to the global existence of classical solutions, where no strong global a priori estimates are yet known. This difficulty is revealed through the equations of the 2D inviscid MHD equations governing the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ and the current density $j = \partial_1 b_2 - \partial_2 b_1$,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = b \cdot \nabla j, \\ \partial_t j + u \cdot \nabla j = b \cdot \nabla \omega + T(\nabla u, \nabla b), \end{cases} \tag{1}$$

where,

$$T(\nabla u, \nabla b) = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) + 2\partial_2 u_2(\partial_2 b_1 + \partial_1 b_2).$$

We observe that the magnetic field contributes in the last nonlinear part of the second equation with the quadratic term $T(\nabla u, \nabla b)$.

By virtue of this difficulty, no a priori uniform bound for $\|\omega\|_{L^\infty(\mathbb{R}^2\times[0,T])}$ is known for the 2D MHD equations with only magnetic diffusion (2). Further in [Fan et al. 2014; Jiu and Zhao 2015; Agélas 2016], by considering Fourier multiplier operators magnetic diffusion slightly stronger than the Laplacian magnetic diffusion, the authors were able to obtain a uniform bound of $\|\nabla j\|_{L^1([0,T];L^\infty(\mathbb{R}^2))}$ and then from the first equation of (1) obtain a uniform bound of $\|\omega\|_{L^\infty(\mathbb{R}^2\times[0,T])}$ deriving from estimates for transport equations; see for instance Lemma 4.1 in [Kato and Ponce 1988].

However, the approach used in [Fan et al. 2014; Jiu and Zhao 2015; Agélas 2016], based on the properties of the heat equation by using singular integral representations of (2), fails in the case where we have only a Laplacian magnetic diffusion.

Then, in this paper, we consider the initial-value problem for the 2D incompressible magnetohydrodynamic equations with Laplacian magnetic diffusion,

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = -\nabla p + (b\cdot\nabla)b, \\ \partial_t b + (u\cdot\nabla)b - \Delta b = (b\cdot\nabla)u, \\ \nabla\cdot u = 0,\ \nabla\cdot b = 0, \end{cases} \tag{2}$$

with initial conditions

$$\begin{aligned} u(x,0) &= u_0(x) \quad \text{for a.e. } x\in\mathbb{R}^2, \\ b(x,0) &= b_0(x) \quad \text{for a.e. } x\in\mathbb{R}^2, \end{aligned} \tag{3}$$

which models many significant phenomena such as the magnetic reconnection in astrophysics and geomagnetic dynamo in geophysics; see [Priest and Forbes 2000]. The problem of global well-posedness of the 2D MHD equations with partial dissipation and magnetic diffusion has generated considerable interest recently [Cao and Wu 2011; Chae 2008; Jiu and Niu 2006; Lei and Zhou 2009; Zhou and Fan 2011; Jiu and Zhao 2015]. However, as of now, the problem of uniqueness and global regularity of the 2D MHD system (2) remains widely open.

Let us take a new look at the main obstruction. We start by noting that we can rewrite the first equation of (1) satisfied by ω, the vorticity of u, as

$$\partial_t\omega + (u\cdot\nabla)\omega = F - b_1\, b\cdot\nabla u_2 + b_2\, b\cdot\nabla u_1, \tag{4}$$

where

$$F = b_1(\Delta b_2 + b\cdot\nabla u_2) - b_2(\Delta b_1 + b\cdot\nabla u_1).$$

Furthermore, a uniform bound of $\|\Delta b + (b\cdot\nabla)u\|_{L^\infty(\mathbb{R}^2\times[0,T])}$ was shown recently in [Yuan and Zhao 2018] (in Section 4 we give a sketch of the proof). Moreover, we get a uniform bound of $\|b\|_{L^\infty(\mathbb{R}^2\times[0,T])}$ deriving from some estimates for the linear Stokes system, see [Giga and Sohr 1991], hence we deduce a uniform bound for $\|F\|_{L^\infty(\mathbb{R}^2\times[0,T])}$. Then, we notice that our (4) fits with the study made in [Elgindi and Masmoudi 2014] about L^∞ ill-posedness for a class of equations arising in hydrodynamics. Thus, by virtue of $\nabla u = \mathrm{R}(\omega\,\mathrm{Id})$, where R is the Riesz transform on 2×2 matrix-valued functions, see (18), we understand that the main obstruction comes from the fact that Riesz transforms do not map L^∞ into itself.

Let us specify the way in which the obstruction is characterized. We refer to Section 1 for the notation used. By using the logarithmic Sobolev inequality proved in [Kozono and Taniuchi 2000],

$$\|\nabla f\|_{L^\infty(\mathbb{R}^2)} \lesssim 1 + \|\nabla\times f\|_{L^\infty(\mathbb{R}^2)}(1+\log^+\|f\|_{W^{s,p}(\mathbb{R}^2)}) \quad \text{with } p>1,\ s>1+\frac{2}{p},$$

where $\nabla\cdot f = 0$, $\nabla\times f = -\partial_2 f_1 + \partial_1 f_2$ is the vorticity of f and $\log^+ x = \max(0,\log x)$ for any $x>0$, we infer that for all $t\in[0,T[$,

$$\|\nabla u(t)\|_\infty \lesssim_r 1 + \|\omega(t)\|_\infty(1+\log^+\|u(t)\|_{H^r}) \tag{5}$$

and also

$$\|(\nabla u,\nabla b)(t)\|_\infty \lesssim_r 1 + \|(\omega,j)(t)\|_\infty(1+\log^+\|(u,b)(t)\|_{H^r}). \tag{6}$$

Then thanks to (5) and by using estimates for transport equations, see for instance Lemma 4.1 in [Kato and Ponce 1988], from (4) we infer that for all $t\in[0,T[$

$$\|\omega(t)\|_\infty \lesssim_r \|\omega_0\|_\infty + \int_0^t (\|F(s)\|_\infty + \|b(s)\|_\infty^2)\,ds + \int_0^t \|b(s)\|_\infty^2\|\omega(s)\|_\infty(1+\log^+\|u(s)\|_{H^r})\,ds, \tag{7}$$

where $r>2$. As a consequence of the Grönwall lemma, we deduce

$$\|\omega(t)\|_\infty \le c_r\left(\|\omega_0\|_\infty + \int_0^t (\|F(s)\|_\infty + \|b(s)\|_\infty^2)\,ds\right) e^{c_r\int_0^t \|b(s)\|_\infty^2(1+\log^+\|u(s)\|_{H^r})\,ds}, \tag{8}$$

where $c_r>0$ is a real number depending only on r. Thus, the main obstruction to getting global regularity comes from the term in logarithm which appears in (8), namely $\log^+\|u(s)\|_{H^r}$. Nevertheless, thanks to (5), (7) and the estimate

$$\|(u,b)(t)\|_{H^r} \le \|(u_0,b_0)\|_{H^r} e^{\kappa_r\int_0^t \|(\nabla u,\nabla b)(\tau)\|_\infty\,d\tau} \tag{9}$$

in the Hilbert space H^r, we obtain a new estimate of $\|\nabla u(t)\|_\infty$ in Lemma 5.1, which leads to a new regularity criterion in Theorem 5.3. Our new regularity criterion states that if

$$\int_0^T \frac{\|\nabla u(t)\|_\infty^{\frac{1}{2}}}{\log(e+\|\nabla u(t)\|_\infty)}\,dt < +\infty$$

then the solution (u,b) of the 2D MHD equations (2) remains smooth up to time T. This new regularity criterion appears less restrictive than the BKM regularity criterion, which states that if $\int_0^T \|\nabla\times u(t)\|_\infty\,dt < +\infty$ then the solution (u,b) of the 2D MHD equations (2) remains smooth up to time T. Indeed, by $\nabla u = \mathrm{R}((\nabla\times u)\,\mathrm{Id})$ with R the Riesz transform on matrix-valued functions, we get

$$\|\nabla u\|_{\mathrm{BMO}(\mathbb{R}^2)} \lesssim \|\nabla\times u\|_{\mathrm{BMO}(\mathbb{R}^2)},$$

and for any $1<q<\infty$

$$\|\nabla u\|_{L^q(\mathbb{R}^2)} \lesssim \|\nabla\times u\|_{L^q(\mathbb{R}^2)}.$$

We thus expect that the blow-up rate at a time T of $\|\nabla u(t)\|_\infty$ behaves like the one of $\|\nabla\times u(t)\|_\infty\cdot(\log(e+\|\nabla\times u(t)\|_\infty))^\gamma$ for a given $\gamma\geq 0$ and due to the exponent $\frac{1}{2}$ in our regularity criterion, we can expect a great improvement over the usual BKM regularity criterion.

The paper is organized as follows:

- In Section 1, we give some notation and introduce the functional spaces.
- In Section 2, we deal with the local well-posedness of the Cauchy problem of the partially viscous magnetohydrodynamic system (2).
- In Section 3, we give two energy estimates and some estimates from the properties of heat equation by using singular integral representations of equations.
- In Section 4, we recall and give a sketch of the proof of new estimates obtained in [Yuan and Zhao 2018] related to the term $\Delta b+(b\cdot\nabla)u$.
- In Section 5, we give a new estimate for $\|\nabla u(t)\|_\infty$ in Lemma 5.1 and from this estimate, we obtain a new regularity criterion in Theorem 5.3 less restrictive than the BKM regularity criterion.

1. Some notation

For any Banach space Z, we endow the Banach space $Z\times Z$ with the norm defined for all $(f,g)\in Z\times Z$ by $\|(f,g)\|_{Z\times Z}:=\|f\|_Z+\|g\|_Z$, and for simplicity in the notation, we use $\|(f,g)\|_Z$ for $\|(f,g)\|_{Z\times Z}$. We use $X\lesssim Y$ to denote the estimate $X\leq CY$ for an absolute constant C. If we need C to depend on a parameter, we shall indicate this by subscripts; thus, for instance, $X\lesssim_s Y$ denotes the estimate $X\leq C_sY$ for some C_s depending on s.

For any $f\in L^p(\mathbb{R}^2)$, with $1\leq p\leq\infty$, we denote by $\|f\|_p$ and $\|f\|_{L^p}$, the L^p-norm of f. We denote by $\mathrm{BMO}(\mathbb{R}^2)$ the space of functions of bounded mean oscillation equipped with the norm

$$\|f\|_{\mathrm{BMO}}:=\sup_{x\in\mathbb{R}^2,\,r>0}\frac{1}{|B_{x,r}|}\int_{B_{x,r}}|f(y)-f_{B_{x,r}}|\,dy,$$

where $B_{x,r}$ is the ball of radius r centered at x, $|B_{x,r}|$ its measure and $f_{B_{x,r}}:=(1/|B_{x,r}|)\int_{B_{x,r}}f(y)\,dy$. We denote by Id the 2×2 identity matrix.

Given an absolutely integrable function $f\in L^1(\mathbb{R}^2)$, we define the Fourier transform $\hat{f}:\mathbb{R}^2\longmapsto\mathbb{C}$ by the formula,

$$\hat{f}(\xi)=\int_{\mathbb{R}^2}e^{-2\pi ix\cdot\xi}f(x)\,dx,$$

and extend it to tempered distributions. We will use also the notation $\mathcal{F}(f)$ for the Fourier transform of f. We define also the inverse Fourier transform $\check{f}:\mathbb{R}^2\longmapsto\mathbb{C}$ by the formula,

$$\check{f}(x)=\int_{\mathbb{R}^2}e^{2\pi ix\cdot\xi}f(\xi)\,d\xi.$$

For $s\in\mathbb{R}$, we define the Sobolev norm $\|f\|_{H^s(\mathbb{R}^2)}$ of a tempered distribution $f:\mathbb{R}^2\longmapsto\mathbb{R}$ by

$$\|f\|_{H^s(\mathbb{R}^2)}=\left(\int_{\mathbb{R}^2}(1+|\xi|^2)^s|\hat{f}(\xi)|^2\,d\xi\right)^{\frac{1}{2}},$$

and then we denote by $H^s(\mathbb{R}^2)$ the space of tempered distributions with finite $H^s(\mathbb{R}^2)$-norm, which matches when s is a nonnegative integer with the classical Sobolev space $H^k(\mathbb{R}^2)$, $k \in \mathbb{N}$. The Sobolev space $H^s(\mathbb{R}^2)$ can be written as $H^s(\mathbb{R}^2) = J^{-s}L^2(\mathbb{R}^2)$ where $J = (1-\Delta)^{\frac{1}{2}}$.

For $s > -1$, we also define the homogeneous Sobolev norm,

$$\|f\|_{\dot{H}^s(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi\right)^{\frac{1}{2}}, \tag{10}$$

and then we denote by $\dot{H}^s(\mathbb{R}^2)$ the space of tempered distributions with finite $\dot{H}^s(\mathbb{R}^2)$-norm. We use the Fourier transform to define the fractional Laplacian operator $(-\Delta)^\alpha$, $-1 < \alpha \le 1$, as follows:

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \hat{f}(\xi).$$

We denote by $H^s_\sigma(\mathbb{R}^2)$ the Sobolev space $H^s_\sigma(\mathbb{R}^2) := \{\psi \in H^s(\mathbb{R}^2)^2 : \operatorname{div} \psi = 0\}$.

We denote by $\mathbb{P}$ the projector onto divergence-free vector fields given by $\mathbb{P} = \mathrm{Id} - \nabla\Delta^{-1}\operatorname{div}$. The operator $\mathbb{P}$, which acts on vector-valued functions, is a projection: $\mathbb{P}$ is equal to $\mathbb{P}^2$, annihilates gradients and maps into solenoidal (divergence-free) vectors; it is a bounded operator from (vector-valued) L^q to itself for all $1 < q < \infty$ and commutes with translation. We can notice that the operator $\mathbb{P}$ can be written in the form

$$\mathbb{P} = \mathrm{Id} - \nabla\Delta^{-1}\operatorname{div}, \tag{11}$$

which yields the Helmholtz decomposition; indeed for all $v \in L^q(\mathbb{R}^2)^2$, $1 < q < \infty$,

$$\begin{aligned} v &= \mathbb{P}v + \nabla\psi, \quad \text{with } \operatorname{div} \mathbb{P}v = 0, \\ \psi &= \Delta^{-1}\operatorname{div} v. \end{aligned} \tag{12}$$

2. Local regularity of solutions of the 2D MHD equations

This section is devoted to the local well-posedness of the 2D MHD equations. By using $\mathbb{P}$, the matrix Leray operator, the first equation of (2) can be rewritten as

$$\frac{\partial u}{\partial t} + \mathbb{P}\big((u\cdot\nabla)u - (b\cdot\nabla)b\big) = 0. \tag{13}$$

For a solution (u, b) of (2), let us introduce the vorticity $\omega = \nabla \times u = -\partial_2 u_1 + \partial_1 u_2$ and the current density $j = \nabla \times b = -\partial_2 b_1 + \partial_1 b_2$. Applying $\nabla\times$ to the equations of (2), we obtain the governing equations for ω and j

$$\begin{cases} \partial_t \omega + (u\cdot\nabla)\omega = (b\cdot\nabla) j, \\ \partial_t j + (u\cdot\nabla) j - \Delta j = (b\cdot\nabla)\omega + T(\nabla u, \nabla b), \end{cases} \tag{14}$$

where,

$$T(\nabla u, \nabla b) = 2\,\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) + 2\,\partial_2 u_2(\partial_2 b_1 + \partial_1 b_2).$$

In this section we assume that the initial data satisfies $(u_0, b_0) \in H^r_\sigma(\mathbb{R}^2)$ with $r > 2$. Then, we introduce $\omega_0 = \nabla \times u_0$, the vorticity of u_0, and $j_0 = \nabla \times b$, the current density of b_0.

We assume that $(u_0, b_0) \in H^r_\sigma(\mathbb{R}^2)$ with $r > 2$, thanks to Theorem 5.1 in [Caflisch et al. 1997], valid for all integers $r \geq 3$, and by using the same arguments as in Proposition 4.3 of [Agélas 2016], valid for all real numbers $r > 2$, we get that there exists a time of existence $T > 0$ such that there exists a unique strong solution $(u, b) \in C([0, T[, H^r_\sigma(\mathbb{R}^2))$ to the 2D MHD equations (2)–(3).

Thanks to the Beale–Kato–Majda (BKM) criterion obtained in [Caflisch et al. 1997] for any integer $r \geq 3$ and extended in Proposition 4.2 of [Agélas 2016] for any real $r > 2$, we get that if $(u, b) \notin C([0, T], H^r_\sigma(\mathbb{R}^2))$, then we have

$$\int_0^T \|(\omega, j)(t)\|_{L^\infty}\, dt = +\infty. \tag{15}$$

From the first equation of (2), we can retrieve the pressure p from (u, b) with the formula

$$p = -\Delta^{-1} \operatorname{div}((u \cdot \nabla)u - (b \cdot \nabla)b). \tag{16}$$

Since $\nabla \cdot u = 0$ and $\nabla \cdot b = 0$, we get $(u \cdot \nabla)u = \nabla \cdot (u \otimes u)$ and $(b \cdot \nabla)b = \nabla \cdot (b \otimes b)$. Then by (16),

$$p = -\Delta^{-1} \operatorname{div} \nabla \cdot (u \otimes u - b \otimes b). \tag{17}$$

By introducing

$$\mathrm{R} := \Delta^{-1} \operatorname{div} \nabla \cdot \tag{18}$$

the Riesz transform on 2×2 matrix-valued functions on $\mathbb{R}^2$, we get

$$p = -\mathrm{R}(u \otimes u - b \otimes b). \tag{19}$$

Since $(u, b) \in C([0, T[, H^r(\mathbb{R}^2))$ with $r > 2$, we get $p \in C([0, T[, H^r(\mathbb{R}^2))$. Lemma X4 in [Kato and Ponce 1988] (see also [Bahouri et al. 2011, Corollary 2.86, pp. 104] for which the Besov space $B^s_{2,2}$ matches with H^s) states that $L^\infty(\mathbb{R}^2) \cap H^s(\mathbb{R}^2)$ is an algebra for any $s > 0$; i.e., for any $f \in H^s(\mathbb{R}^2)$ and $g \in H^s(\mathbb{R}^2)$, we have $\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_\infty + \|f\|_\infty \|g\|_{H^s}$. This lemma and the use of the Sobolev embedding $H^r(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, since $r > 2$, yield for all $f \in H^r(\mathbb{R}^2)$ and $g \in H^r(\mathbb{R}^2)$,

$$\|fg\|_{H^r} \lesssim \|f\|_{H^r}\|g\|_{H^r}. \tag{20}$$

Then owing to $(u, b) \in C([0, T[, H^r(\mathbb{R}^2))$, thanks to the L^2-boundedness of the Riesz transforms and (20) from (19) we infer that $p \in C([0, T[, H^r(\mathbb{R}^2))$.

Similarly to Proposition 4.1 in [Agélas 2016], we get the following local estimates in the higher Sobolev norm H^r: there exists a real $\kappa_r > 0$ depending only on r such that for all $t \in [0, T[$

$$\|(u, b)(t)\|_{H^r} \leq \|(u_0, b_0)\|_{H^r} e^{\kappa_r \int_0^t \|(\nabla u, \nabla b)(\tau)\|_\infty\, d\tau}. \tag{21}$$

3. Some estimates

In this section, we give some estimates related to the solutions of the 2D MHD equations (2).

Energy estimates. We recall some energy estimates. We state here the two following energy estimates given in [Tran et al. 2013b; Lei and Zhou 2009; Agélas 2016]: for all $t \in [0, T^*[$

$$\|u(t)\|_2^2 + \|b(t)\|_2^2 + 2\int_0^t \|\nabla b(\tau)\|_2^2\, d\tau = \|u_0\|_2^2 + \|b_0\|_2^2 \tag{22}$$

and we get also that for all $t \in [0, T^*[$

$$\|\omega(t)\|_2^2 + \|j(t)\|_2^2 + \int_0^t \|\nabla j(\tau)\|_2^2\, d\tau \leq (\|\omega_0\|_2^2 + \|j_0\|_2^2) e^{C(\|u_0\|_2^2 + \|b_0\|_2^2)}, \tag{23}$$

where $C > 0$ is an absolute constant.

Some estimates deriving from heat equation. In the lemma just below, we give the details (often omitted) of the proof of some estimates deriving from the properties of the heat kernel.

Lemma 3.1. *Let $(u_0, b_0) \in H^r(\mathbb{R}^2)$ with $r > 2$ and let $T > 0$ be such that there exists $(u, b) \in C([0, T[, H^r_\sigma(\mathbb{R}^2))$ a solution of the 2D MHD equations* (2)–(3)*. Then there exists a real $C_1 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, r and T such that*

$$\|b\|_{L^\infty(\mathbb{R}^2 \times [0,T])} \leq C_1.$$

For any real $p > 1$ and $q > 2$, we have also three real $C_2 > 0$, $C_3 > 0$ and $C_4 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, p, q, r and T such that

$$\|\nabla b\|_{L^\infty([0,T] \times L^q(\mathbb{R}^2))} \leq C_2,$$
$$\|\nabla u\|_{L^\infty([0,T] \times L^q(\mathbb{R}^2))} \leq C_3,$$
$$\|\nabla^2 b\|_{L^p([0,T] \times L^q(\mathbb{R}^2))} \leq C_4.$$

Proof. For this, we write the second equation of (2) under its integral form; then we have for all $t \in [0, T[$

$$b(t) = e^{t\Delta} b_0 + \int_0^t e^{(t-s)\Delta}((b \cdot \nabla) u(s) - (u \cdot \nabla) b(s))\, ds. \tag{24}$$

Then by using inequality (2.3) in [Kato 1984], we get

$$\begin{aligned} \left\| e^{(t-s)\Delta}((b \cdot \nabla) u(s) - (u \cdot \nabla) b(s)) \right\|_\infty &\lesssim (t-s)^{-\frac{2}{3}} \|(b \cdot \nabla) u(s) - (u \cdot \nabla) b(s)\|_{\frac{3}{2}} \\ &\lesssim (t-s)^{-\frac{2}{3}} \big(\|b(s)\|_6 \|\nabla u(s)\|_2 + \|u(s)\|_6 \|\nabla b(s)\|_2 \big). \end{aligned}$$

As a consequence, from (24) we get,

$$\|b(t)\|_\infty \lesssim \|b_0\|_\infty + \int_0^t (t-s)^{-\frac{2}{3}} \big(\|b(s)\|_6 \|\nabla u(s)\|_2 + \|u(s)\|_6 \|\nabla b(s)\|_2 \big)\, ds. \tag{25}$$

Since $\|b(s)\|_6 \lesssim \|b(s)\|_{H^1}$, $\|u(s)\|_6 \lesssim \|u(s)\|_{H^1}$ and $\|\nabla b(s)\|_2 = \|j(s)\|_2$, we have $\|\nabla u(s)\|_2 = \|\omega(s)\|_2$ due to the facts $\nabla \cdot b(s) = 0$ and $\nabla \cdot u(s) = 0$; then thanks to (22) and (23), from (25) we deduce that there exists a real $C_0 > 0$ depending only on $\|(u_0, b_0)\|_2$, $\|(\omega_0, j_0)\|_2$ such that for all $t \in [0, T[$

$$\|b(t)\|_\infty \lesssim \|b_0\|_\infty + C_0(T)^{\frac{1}{3}}. \tag{26}$$

Owing to (26) and thanks to the Sobolev embedding $H^r(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ since $r > 2$, we deduce that there exists a real $C_1 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, r and T such that for all $t \in [0, T[$

$$\|b(t)\|_\infty \leq C_1, \tag{27}$$

which concludes the first part of the proof. By virtue of (24), we get that for all $t \in [0, T[$

$$\nabla b(t) = e^{t\Delta}\nabla b_0 + \int_0^t \nabla e^{(t-s)\Delta}\big((b\cdot\nabla)u(s) - (u\cdot\nabla)b(s)\big)\, ds. \tag{28}$$

Let $2q/(q+2) < \alpha < 2$. Notice that $2q/(q+2) > 1$ since $q > 2$ and hence $\alpha > 1$. Then by using inequality (2.3′) in [Kato 1984], from (28) we deduce

$$\|\nabla b(t)\|_q \lesssim_q \|\nabla b_0\|_q + \int_0^t (t-s)^{-\left(\frac{1}{2}+\frac{1}{\alpha}-\frac{1}{q}\right)} \big\|(b\cdot\nabla)u(s) - (u\cdot\nabla)b(s)\big\|_\alpha\, ds. \tag{29}$$

Further, thanks to the Hölder inequality, we have $\|(b\cdot\nabla)u(s)\|_\alpha \le \|b(s)\|_{\frac{2\alpha}{2-\alpha}} \|\nabla u(s)\|_2$ and we also get

$$\|b(s)\|_{\frac{2\alpha}{2-\alpha}} \lesssim_\alpha \|b(s)\|_2^{\frac{2-\alpha}{\alpha}} \|\nabla b(s)\|_2^{\frac{2(\alpha-1)}{\alpha}}$$

thanks to a Gagliardo–Nirenberg inequality. Hence, we deduce for any $s \in [0, T[$

$$\begin{aligned}\|(b\cdot\nabla)u(s)\|_\alpha &\lesssim_\alpha \|b(s)\|_2^{\frac{2-\alpha}{\alpha}} \|\nabla b(s)\|_2^{\frac{2(\alpha-1)}{\alpha}} \|\nabla u(s)\|_2 \\ &\lesssim_\alpha \|b(s)\|_2^{\frac{2-\alpha}{\alpha}} \|j(s)\|_2^{\frac{2(\alpha-1)}{\alpha}} \|\omega(s)\|_2 \\ &\lesssim_\alpha \|(u,b)(s)\|_2^{\frac{2-\alpha}{\alpha}} \|(\omega,j)(s)\|_2^{\frac{3\alpha-2}{\alpha}}.\end{aligned}$$

Similarly, we get also $\|(u\cdot\nabla)b(s)\|_\alpha \lesssim_\alpha \|(u,b)(s)\|_2^{\frac{2-\alpha}{\alpha}} \|(\omega,j)(s)\|_2^{\frac{3\alpha-2}{\alpha}}$. By virtue of the two latter inequalities, it is inferred that for all $s \in [0, T[$

$$\big\|(b\cdot\nabla)u(s) - (u\cdot\nabla)b(s)\big\|_\alpha \lesssim_\alpha \|(u,b)(s)\|_2^{\frac{2-\alpha}{\alpha}} \|(\omega,j)(s)\|_2^{\frac{3\alpha-2}{\alpha}}. \tag{30}$$

Thanks to the energy estimates (22) and (23), we have $\|(u,b)(s)\|_2 \le \|(u_0,b_0)\|_2$ and $\|(\omega,j)(s)\|_2 \le \|(\omega_0,j_0)\|_2 e^{c\|(u_0,b_0)\|_2}$ with $c > 0$ an absolute constant. Then by setting

$$\eta_0 := \|(u_0,b_0)\|_2 + \|(\omega_0,j_0)\|_2 e^{c\|(u_0,b_0)\|_2},$$

from (30) we deduce that for all $s \in [0, T[$

$$\big\|(b\cdot\nabla)u(s) - (u\cdot\nabla)b(s)\big\|_\alpha \lesssim_\alpha \eta_0^2. \tag{31}$$

After plugging inequality (31) into (29), we obtain that for all $t \in [0, T[$

$$\begin{aligned}\|\nabla b(t)\|_q &\lesssim_{q,\alpha} \|\nabla b_0\|_q + \eta_0^2 \int_0^t (t-s)^{-\left(\frac{1}{2}+\frac{1}{\alpha}-\frac{1}{q}\right)}\, ds \\ &\lesssim_{q,\alpha} \|\nabla b_0\|_q + \eta_0^2 T^{\frac{q+2}{2q}-\frac{1}{\alpha}}.\end{aligned} \tag{32}$$

We choose $\alpha = \frac{1}{2}(2 + 2q/(q+2))$. Thanks to a Gagliardo–Nirenberg inequality, for any $q > 2$ we have the Sobolev embedding $H^r(\mathbb{R}^2) \hookrightarrow \dot{W}^{1,q}(\mathbb{R}^2)$ since $r > 2$; then owing to (32) we deduce that there exists a real $C_2 > 0$ depending only on $\|(u_0,b_0)\|_{H^r}$, T, r and q such that for all $t \in [0, T[$

$$\|\nabla b(t)\|_q \le C_2, \tag{33}$$

which concludes the second part of the proof.

To get an estimate of $\|\omega\|_{L^\infty([0,T];L^q)}$, we borrow some arguments used in [Jiu and Zhao 2015]. Thanks to the $L^p - L^q$ maximal regularity of the Laplacian operator, see for example [Giga and Sohr 1991], from the second equation of (2), we get that for all $t \in [0, T[$, $p > 1$ and $q > 2$

$$\begin{aligned}\int_0^t \|\nabla^2 b(s)\|_q^p &\lesssim_{p,q} \int_0^t \big\|(b\cdot\nabla)u(s) - (u\cdot\nabla)b(s)\big\|_q^p \, ds \\ &\lesssim_{p,q} \int_0^t \big(\|b(s)\|_\infty^p \, \|\omega(s)\|_q^p + \|u(s)\|_\infty^p \, \|\nabla b(s)\|_q^p\big)\, ds, \end{aligned} \tag{34}$$

where we have used the fact that $\|\nabla u(s)\|_q \lesssim_q \|\omega(s)\|_q$; see Theorem 3.1.1 in [Chemin 1998]. Then, we multiply the first equation of (14) by $\omega|\omega|^{q-2}$, integrate it over $\mathbb{R}^2$ and use the fact that $\nabla\cdot u = 0$ to obtain

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|\omega(t)\|_q^q &= \int_{\mathbb{R}^2} b(x,t)\cdot\nabla j(x,t)\omega(x,t)|\omega(x,t)|^{q-2}\, dx \\ &\le \|b(t)\|_\infty \, \|\nabla j(t)\|_q \, \|\omega(t)\|_q^{q-1},\end{aligned}$$

which yields for all $t \in [0, T[$

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_q^2 \le \|b(t)\|_\infty \, \|\nabla j(t)\|_q \, \|\omega(t)\|_q.$$

After an integration over $[0, t]$ of the inequality just above, we obtain

$$\begin{aligned}\|\omega(t)\|_q^2 &\le \|\omega_0\|_q^2 + 2\int_0^t \|b(s)\|_\infty \, \|\nabla j(s)\|_q \|\omega(s)\|_q \, ds \\ &\le \|\omega_0\|_q^2 + \int_0^t \big(\|\nabla j(s)\|_q^2 + \|b(s)\|_\infty^2 \, \|\omega(s)\|_q^2\big)\, ds\end{aligned} \tag{35}$$

Then thanks to (34), from (35) we infer that for all $t \in [0, T[$

$$\|\omega(t)\|_q^2 \lesssim \|\omega_0\|_q^2 + \int_0^t \big(\|u(s)\|_\infty^2 \, \|\nabla b(s)\|_q^2 + \|b(s)\|_\infty^2 \, \|\omega(s)\|_q^2\big)\, ds. \tag{36}$$

By using Gagliardo–Nirenberg inequalities, Young inequalities and the fact that $\|\nabla u(s)\|_q \lesssim_q \|\omega(s)\|_q$, we get

$$\|u(s)\|_\infty \lesssim_q \|u(s)\|_2 + \|\omega(s)\|_q. \tag{37}$$

By virtue of (36) and (37), we get that for all $t \in [0, T[$

$$\|\omega(t)\|_q^2 \lesssim_q \|\omega_0\|_q^2 + \int_0^t \big(\|u(s)\|_2^2 \, \|\nabla b(s)\|_q^2 + (\|\nabla b(s)\|_q^2 + \|b(s)\|_\infty^2)\|\omega(s)\|_q^2\big)\, ds. \tag{38}$$

Thanks to (22), (27) and (33), we deduce that there exists a real $C > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, T, r and q such that for all $t \in [0, T[$

$$\|\omega(t)\|_q^2 \le C + C\int_0^t \|\omega(s)\|_q^2 \, ds. \tag{39}$$

Thanks to the Grönwall inequality, we infer that for all $t \in [0, T[$

$$\|\omega(t)\|_q^2 \le Ce^{CT}.$$

By using the fact that

$$\|\nabla u(t)\|_q \lesssim_q \|\omega(t)\|_q,$$

we infer that there exists a real $C_3 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, T and q such that for all $t \in [0, T[$

$$\|\nabla u(t)\|_q \leq C_3, \tag{40}$$

which concludes the third part of the proof. By using (37) in (34) and thanks to (40), (22), (27) and (33), we complete the proof. □

4. Some new estimates

We give a sketch of the proof of Lemma 4.1 obtained in [Yuan and Zhao 2018] by exploiting the special structure of the 2D MHD equations (2).

Lemma 4.1. *Let $(u_0, b_0) \in H^r(\mathbb{R}^2)$ with $r > 2$ and let $T > 0$ be such that there exists $(u, b) \in C([0, T[, H^r_\sigma(\mathbb{R}^2))$ a solution of the 2D MHD equations (2)–(3). Then there exists a real $C > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, r and T such that*

$$\|\Delta b + (b \cdot \nabla)u\|_{L^\infty([0,T];L^\infty(\mathbb{R}^2))} \leq C, \tag{41}$$

and we have also that for any real $p \geq 2$ and $q \geq 2$,

$$\|\nabla(\Delta b + (b \cdot \nabla)u)\|_{L^p([0,T];L^q(\mathbb{R}^2))} \leq C. \tag{42}$$

Although we can deduce the proof of Lemma 4.1 from [Yuan and Zhao 2018], we prefer to give here the details of its proof, as it is at the heart of the improvements obtained in this paper. For this, we borrow some arguments used in [Yuan and Zhao 2018]. We start the proof by writing the equation satisfied by $\mathfrak{F} := \Delta b + (b \cdot \nabla)u$, that is,

$$\begin{aligned}\partial_t \mathfrak{F} - \Delta \mathfrak{F} = -(b \cdot \nabla)\mathbb{P}((u \cdot \nabla)u) + (b \cdot \nabla)\mathbb{P}((b \cdot \nabla)b) - \Delta((u \cdot \nabla)b) \\ - \nabla u\,(u \cdot \nabla)b + \nabla u\,(b \cdot \nabla)u + \nabla u\,\Delta b.\end{aligned} \tag{43}$$

This equation is obtained by applying $(b \cdot \nabla)$ and Δ respectively to the first equation of (13) and second equation of (2), multiplying the second equation of (2) by ∇u and then adding the resulting equations together. Then, by writing (43) in its integral form and using the facts that $\nabla \cdot u = 0$ and $\nabla \cdot b = 0$, we get for all $t \in [0, T[$

$$\begin{aligned}\mathfrak{F}(t) = e^{t\Delta}\mathfrak{F}(0) + \int_0^t \nabla e^{(t-s)\Delta}\big(b(s) \otimes \mathbb{P}((u \cdot \nabla)u)(s) - b(s) \otimes \mathbb{P}((b \cdot \nabla)b)(s)\big)\,ds \\ + \int_0^t \nabla e^{(t-s)\Delta}\nabla((u \cdot \nabla)b)(s)\,ds \\ + \int_0^t e^{(t-s)\Delta}\big(-\nabla u(s)\,(u(s) \cdot \nabla)b(s) + \nabla u(s)\,(b(s) \cdot \nabla)u(s) + \nabla u(s)\,\Delta b(s)\big)\,ds.\end{aligned} \tag{44}$$

Then using inequalities (2.3) and (2.3′) of [Kato 1984] stated for $1 < p \le q < +\infty$ but remaining true for $q = \infty$, we obtain for all $t \in [0, T[$

$$\begin{aligned}\|\mathfrak{F}(t)\|_\infty \le \|\mathfrak{F}(0)\|_\infty &+ \int_0^t (t-s)^{-\frac{5}{6}} \big\| \big(b(s)\otimes\mathbb{P}((u\cdot\nabla)u)(s) - b(s)\otimes\mathbb{P}((b\cdot\nabla)b)(s)\big)\big\|_3 \, ds \\ &+ \int_0^t (t-s)^{-\frac{5}{6}} \|\nabla((u\cdot\nabla)b)(s)\|_3 \, ds \\ &+ \int_0^t (t-s)^{-\frac{1}{2}} \big\| -\nabla u(s)\,(u(s)\cdot\nabla)b(s) + \nabla u(s)\,(b(s)\cdot\nabla)u(s) + \nabla u(s)\,\Delta b(s)\big\|_2 \, ds. \quad (45)\end{aligned}$$

By using the fact that $\mathbb{P}$ is a bounded operator from (vector-valued) L^q to itself for all $1 < q < \infty$ and the Hölder inequality, we get

$$\begin{aligned}&\big\|(b(s)\otimes\mathbb{P}((u\cdot\nabla)u)(s) - (b(s)\otimes\mathbb{P}((b\cdot\nabla)b)(s)\big\|_3 \\ &\qquad\qquad \lesssim \|b(s)\|_\infty\,\|u(s)\|_6\,\|\nabla u(s)\|_6 + \|b(s)\|_\infty\,\|b(s)\|_6\,\|\nabla b(s)\|_6, \\ &\|\nabla((u\cdot\nabla)b)(s)\|_3 \lesssim \|\nabla u(s)\|_6\,\|\nabla b(s)\|_6 + \|u(s)\|_6\,\|\nabla^2 b(s)\|_6, \qquad (46) \\ &\big\|-\nabla u(s)\,(u(s)\cdot\nabla)b(s) + \nabla u(s)\,(b(s)\cdot\nabla)u(s) + \nabla u(s)\,\Delta b(s)\big\|_2 \\ &\qquad\qquad \lesssim \|\nabla u(s)\|_6\,\|u(s)\|_6 \|\nabla b(s)\|_6 + \|\nabla u(s)\|_6^2\,\|b(s)\|_6 + \|\nabla u(s)\|_6\,\|\Delta b(s)\|_3.\end{aligned}$$

Furthermore, thanks to a Gagliardo–Nirenberg interpolation inequality and the fact that, since $\nabla\cdot u(s) = 0$, $\nabla\cdot b(s) = 0$, we have $\|\nabla u(s)\|_2 = \|\omega(s)\|_2$ and $\|\nabla b(s)\|_2 = \|j(s)\|_2$, we get

$$\|u(s)\|_6 \lesssim \|u(s)\|_2^{\frac{1}{3}}\,\|\omega(s)\|_2^{\frac{2}{3}}, \quad \|b(s)\|_6 \lesssim \|b(s)\|_2^{\frac{1}{3}}\,\|j(s)\|_2^{\frac{2}{3}}. \qquad (47)$$

After plugging (47) into (46) and using Lemma 3.1 with the energy inequalities (22), (23), from (45) we infer that there exists a real $C_0 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, r and T such that for all $t \in [0, T[$

$$\|\mathfrak{F}(t)\|_\infty \lesssim C_0\Big(1 + \int_0^t (t-s)^{-\frac{5}{6}}(1 + \|\nabla^2 b(s)\|_6) + (t-s)^{-\frac{1}{2}}(1 + \|\Delta b(s)\|_3)\,ds\Big). \qquad (48)$$

Thanks to the Hölder inequality used with the pairs of exponents $\left(\frac{7}{6}, 7\right)$ and $\left(\frac{3}{2}, 3\right)$, from (48) we deduce that for all $t \in [0, T[$

$$\begin{aligned}\|\mathfrak{F}(t)\|_\infty \lesssim C_0 + C_0\Big(\int_0^t (t-s)^{-\frac{35}{36}}\,ds\Big)^{\frac{6}{7}}\Big(\int_0^t (1 + \|\nabla^2 b(s)\|_6)^7\,ds\Big)^{\frac{1}{7}} \\ + C_0\Big(\int_0^t (t-s)^{-\frac{3}{4}}\,ds\Big)^{\frac{2}{3}}\Big(\int_0^t (1 + \|\Delta b(s)\|_3)^3\,ds\Big)^{\frac{1}{3}},\end{aligned}$$

which yields

$$\|\mathfrak{F}(t)\|_\infty \lesssim C_0\Big(1 + t^{\frac{1}{42}}\Big(\int_0^t (1 + \|\nabla^2 b(s)\|_6^7)\,ds\Big)^{\frac{1}{7}} + t^{\frac{1}{6}}\Big(\int_0^t (1 + \|\Delta b(s)\|_3^3)\,ds\Big)^{\frac{1}{3}}\Big). \qquad (49)$$

Then, thanks again to Lemma 3.1, from (49) one obtains that there exists a real $C_1 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, r and T such that for all $t \in [0, T[$

$$\|\mathfrak{F}(t)\|_\infty \le C_1,$$

which gives us (41), the first inequality of Lemma 4.1.

For the second inequality of Lemma 4.1, we use the $L^p - L^q$ maximal regularity of the Laplacian operator [Giga and Sohr 1991]; one has for any $1 < p < \infty$, $1 < q < \infty$ and $g = \int_0^t e^{(t-s)\Delta} f$,

$$\|\nabla^2 g\|_{L^p([0,T];L^q(\mathbb{R}^2))} \lesssim_{p,q} \|f\|_{L^p([0,T];L^q(\mathbb{R}^2))}. \tag{50}$$

Then, with the expression of $\nabla\mathfrak{F}(t)$ obtained from (44) and by using (50), inequality (2.3′) of [Kato 1984], Lemma 3.1 and the energy inequalities (22), (23), we obtain in a similar way (42), the second inequality of Lemma 4.1.

5. A new blow-up criterion

In this section, we give a new estimate for $\|\nabla u(t)\|_\infty$ in Lemma 5.1 and from this estimate, we obtain a new regularity criterion in Theorem 5.3 which is less restrictive than the BKM regularity criterion.

Lemma 5.1. *Let* $(u_0, b_0) \in H^r(\mathbb{R}^2)$ *with* $r > 2$ *and let* $T > 0$ *be such that there exists* $(u, b) \in C([0, T[, H^r_\sigma(\mathbb{R}^2))$ *a solution of the 2D MHD equations* (2)–(3). *Then there exists a real* $\gamma_0 > 0$ *depending only on* $\|(u_0, b_0)\|_{H^r}$, T *and* r *such that for all* $t \in [0, T[$

$$\|\nabla u(t)\|_\infty \le \exp\left(\gamma_0 \exp\left(\gamma_0 \int_0^t \frac{\|\nabla u(s)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(s)\|_\infty)}\, ds\right)\right). \tag{51}$$

Proof. We begin the proof with the following logarithmic Sobolev inequality, which is proved in [Kozono and Taniuchi 2000], see inequality (4.20), and stands as an improved version of that in [Beale et al. 1984]:

$$\|\nabla f\|_{L^\infty(\mathbb{R}^2)} \lesssim 1 + \|\nabla \times f\|_{L^\infty(\mathbb{R}^2)}(1 + \log^+ \|f\|_{W^{s,p}(\mathbb{R}^2)}) \quad \text{with } p > 1,\ s > 1 + \frac{2}{p}, \tag{52}$$

where $\nabla \cdot f = 0$, $\nabla \times f = -\partial_2 f_1 + \partial_1 f_2$ is the vorticity of f and $\log^+ x = \max(0, \log x)$ for any $x > 0$.

Thus, by virtue of (52), we get that for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty \le \beta_r + \beta_r \|\omega(t)\|_\infty (1 + \log^+ \|u(t)\|_{H^r}), \tag{53}$$

where $\beta_r > 0$ is a real depending only on r. Let us give an estimate of the term $1 + \log^+ \|u(t)\|_{H^r}$. Thanks to (21), we get that there exists a real $\kappa_r > 0$ depending only on r such that for all $t \in [0, T[$

$$\|(u, b)(t)\|_{H^r} \le \|(u_0, b_0)\|_{H^r} e^{\kappa_r \int_0^t \|(\nabla u, \nabla b)(\tau)\|_\infty \, d\tau}. \tag{54}$$

After taking the logarithm in the inequality (54), we observe that for all $t \in [0, T[$,

$$\log^+ \|(u, b)(t)\|_{H^r} \le \log^+ \|(u_0, b_0)\|_{H^r} + \kappa_r \int_0^t \|(\nabla u, \nabla b)(\tau)\|_\infty \, d\tau. \tag{55}$$

Thanks to Lemma 3.1 and the Sobolev embedding $W^{2,q}(\mathbb{R}^2) \hookrightarrow W^{1,\infty}(\mathbb{R}^2)$ with $q > 2$, we infer that there exists a real $\varrho_0 > 0$ depending only on r, T and $\|(u_0, b_0)\|_{H^r}$ such that

$$\int_0^T \|\nabla b(\sigma)\|_\infty \leq \varrho_0. \tag{56}$$

Then owing to (56), from (55) we infer that there exists a real $\varrho_1 \geq 1$ depending only on r, T and $\|(u_0, b_0)\|_{H^r}$ such that for all $t \in [0, T[$

$$1 + \log^+ \|(u, b)(t)\|_{H^r} \leq \varrho_1 + \kappa_r \int_0^t \|\nabla u(s)\|_\infty \, ds. \tag{57}$$

Thus, by plugging (57) into (53), we deduce that there exists a real $\varrho_2 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, T and r such that for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty \leq \beta_r + \varrho_2 \|\omega(t)\|_\infty \left(1 + \int_0^t \|\nabla u(s)\|_\infty \, ds\right). \tag{58}$$

Now, let us estimate $\|\omega(t)\|_\infty$. We observe that the first equation of (14) can be changed into

$$\partial_t \omega + u \cdot \nabla \omega = F - b_1 b \cdot \nabla u_2 + b_2 b \cdot \nabla u_1, \tag{59}$$

where $F = b_1(\Delta b_2 + b \cdot \nabla u_2) - b_2(\Delta b_1 + b \cdot \nabla u_1)$. By using estimates for transport equations, see for instance Lemma 4.1 in [Kato and Ponce 1988], we obtain that for all $t \in [0, T[$

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty + c \int_0^t \|F(s)\|_\infty \, ds + c \int_0^t \|b(s)\|_\infty^2 \, \|\nabla u(s)\|_\infty \, ds, \tag{60}$$

where $c > 0$ is a constant. Thanks to Lemmata 3.1 and 4.1, we deduce that there exist two real $\varrho_3 > 0$ and $\varrho_4 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, T and r such that for all $t \in [0, T[$

$$\begin{aligned} c \int_0^t \|F(s)\|_\infty \, ds &\leq \varrho_3, \\ \|b(t)\|_\infty^2 &\leq \varrho_4. \end{aligned} \tag{61}$$

Thus by virtue of (61), from (60) we infer that for all $t \in [0, T[$

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty + \varrho_3 + c\varrho_4 \int_0^t \|\nabla u(s)\|_\infty \, ds. \tag{62}$$

Furthermore, thanks to the Sobolev embedding $H^r(\mathbb{R}^2) \hookrightarrow W^{1,\infty}(\mathbb{R}^2)$ with $r > 2$, we get

$$\|\omega_0\|_\infty \lesssim_r \|u_0\|_{H^r}. \tag{63}$$

Hence, owing to (63), from (62) we deduce that there exists a real $\varrho_5 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, T and r such that for all $t \in [0, T[$

$$\|\omega(t)\|_\infty \leq \varrho_5 + c\varrho_4 \int_0^t \|\nabla u(s)\|_\infty \, ds. \tag{64}$$

By plugging (64) into (58), we infer that there exists a real $\varrho_6 \geq 1$ depending only on r, T and $\|(u_0, b_0)\|_{H^r}$ such that for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty \leq \varrho_6 \Big(1 + \int_0^t \|\nabla u(s)\|_\infty \, ds\Big)^2, \tag{65}$$

which yields

$$\|\nabla u(t)\|_\infty^{\frac{1}{2}} \leq \varrho_6^{\frac{1}{2}} \Big(1 + \int_0^t \|\nabla u(s)\|_\infty \, ds\Big). \tag{66}$$

We thus introduce the real function $\mathfrak{J}$ defined for all $t \in [0, T[$ by

$$\mathfrak{J}(t) := \varrho_6^{\frac{1}{2}} + \varrho_6^{\frac{1}{2}} \int_0^t \|\nabla u(s)\|_\infty \, ds. \tag{67}$$

On one hand, by virtue of (66), thanks to (67) we get that for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty^{\frac{1}{2}} \leq \mathfrak{J}(t). \tag{68}$$

On the other hand, from (67), we infer that for any $t \in [0, T[$

$$\mathfrak{J}'(t) = \varrho_6^{\frac{1}{2}} \|\nabla u(t)\|_\infty = \frac{\varrho_6^{\frac{1}{2}} \|\nabla u(t)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(t)\|_\infty^{\frac{1}{2}})} \|\nabla u(t)\|_\infty^{\frac{1}{2}} \log(e + \|\nabla u(t)\|_\infty^{\frac{1}{2}}). \tag{69}$$

Then, owing to (68), from (69), we infer that for all $t \in [0, T[$

$$\mathfrak{J}'(t) \leq \frac{\varrho_6^{\frac{1}{2}} \|\nabla u(t)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(t)\|_\infty^{\frac{1}{2}})} \mathfrak{J}(t) \log(e + \mathfrak{J}(t)). \tag{70}$$

After dividing inequality (70) by $e + \mathfrak{J}(t)$, we obtain that for all $t \in [0, T[$

$$\frac{d}{dt} \log(e + \mathfrak{J}(t)) \leq \frac{\varrho_6^{\frac{1}{2}} \|\nabla u(t)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(t)\|_\infty^{\frac{1}{2}})} \log(e + \mathfrak{J}(t)). \tag{71}$$

As a consequence of the Grönwall lemma, from (71) we get for all $t \in [0, T[$

$$\log(e + \mathfrak{J}(t)) \leq \log(e + \mathfrak{J}(0)) \exp\Big(\varrho_6^{\frac{1}{2}} \int_0^t \frac{\|\nabla u(s)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(s)\|_\infty^{\frac{1}{2}})} \, ds\Big). \tag{72}$$

From (67), we get $\mathfrak{J}(0) = \varrho_6^{\frac{1}{2}}$ and thanks to (72), we thus obtain for all $t \in [0, T[$

$$\mathfrak{J}(t) \leq \exp\Big(\log(e + \varrho_6^{\frac{1}{2}}) \exp\Big(\varrho_6^{\frac{1}{2}} \int_0^t \frac{\|\nabla u(s)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(s)\|_\infty^{\frac{1}{2}})} \, ds\Big)\Big). \tag{73}$$

Owing to (68) and (73), we obtain that for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty \le \exp\left(2\log(e+\varrho_6^{\frac{1}{2}})\exp\left(\varrho_6^{\frac{1}{2}}\int_0^t \frac{\|\nabla u(s)\|_\infty^{\frac{1}{2}}}{\log(e+\|\nabla u(s)\|_\infty^{\frac{1}{2}})}\,ds\right)\right). \tag{74}$$

Since $e+\|\nabla u(s)\|_\infty^{\frac{1}{2}} \ge (e+\|\nabla u(s)\|_\infty)^{\frac{1}{2}}$, then we get

$$\log(e+\|\nabla u(s)\|_\infty^{\frac{1}{2}}) \ge \tfrac{1}{2}\log(e+\|\nabla u(s)\|_\infty)$$

and hence from (74) we infer for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty \le \exp\left(2\log(e+\varrho_6^{\frac{1}{2}})\exp\left(2\varrho_6^{\frac{1}{2}}\int_0^t \frac{\|\nabla u(s)\|_\infty^{\frac{1}{2}}}{\log(e+\|\nabla u(s)\|_\infty)}\,ds\right)\right),$$

which concludes the proof. □

Remark 5.2. We observe that the expression of the estimate obtained in Lemma 5.1 for $\|\nabla u(t)\|_\infty$ makes a double exponential growth appear. This double exponential growth derives from taking into account in the estimate the term $\log(e+\|\nabla u(t)\|_\infty)$. We thus point out that we have also an upper bound of $\|\nabla u(t)\|_\infty$ for which we get only one single exponential growth. Indeed, from (66), thanks to the Grönwall lemma, we obtain that for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty^{\frac{1}{2}} \le \varrho_6^{\frac{1}{2}}\exp\left(\varrho_6^{\frac{1}{2}}\int_0^t \|\nabla u(s)\|_\infty^{\frac{1}{2}}\,ds\right),$$

which yields

$$\|\nabla u(t)\|_\infty \le \varrho_6\exp\left(2\varrho_6^{\frac{1}{2}}\int_0^t \|\nabla u(s)\|_\infty^{\frac{1}{2}}\,ds\right),$$

where $\varrho_6 > 0$ is a real number depending only on T, r and $\|(u_0, b_0)\|_r$.

Let us establish now, a new regularity criterion in the theorem just below.

Theorem 5.3. *Let $(u_0, b_0) \in H_\sigma^r(\mathbb{R}^2)$ with $r > 2$ and let $T > 0$ be such that there exists $(u, b) \in C([0, T[, H_\sigma^r(\mathbb{R}^2))$ a solution of the 2D MHD equations (2)–(3). If*

$$\int_0^T \frac{\|\nabla u(t)\|_\infty^{\frac{1}{2}}}{\log(e+\|\nabla u(t)\|_\infty)}\,dt < +\infty \tag{75}$$

then there cannot be blow-up of the solution u in $H^r(\mathbb{R}^2)$ at the time T, that is, $u \in C([0, T], H_\sigma^r(\mathbb{R}^2))$.

Proof. Let us assume that (75) holds. For a contradiction, we suppose that $u \notin C([0, T], H_\sigma^r(\mathbb{R}^2)$. Then we get (15). Thanks to Lemma 3.1 and the Sobolev embedding $W^{2,q}(\mathbb{R}^2) \hookrightarrow W^{1,\infty}(\mathbb{R}^2)$ with $q > 2$, we infer that $\int_0^T \|j(t)\|_\infty\,dt < +\infty$. Then from (15), we get only

$$\int_0^T \|\omega(t)\|_\infty\,dt = +\infty. \tag{76}$$

Thanks to Lemma 5.1, there exists a real $\varrho_1 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, T and r such that for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty \leq \exp\left(\varrho_1 \exp\left(\varrho_1 \int_0^t \frac{\|\nabla u(s)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(s)\|_\infty)}\, ds\right)\right). \tag{77}$$

Then from (77) and (75), we infer that $\int_0^T \|\nabla u(t)\|_\infty \, dt < +\infty$, which implies $\int_0^T \|\omega(t)\|_\infty \, dt < +\infty$. Then we obtain a contradiction with (76) and hence $u \in C([0, T], H^r_\sigma(\mathbb{R}^2)$, which concludes the proof. □

Conclusion

We obtained a new regularity criterion for the two-dimensional resistive magnetohydrodynamic (MHD) equations which is less restrictive than the BKM regularity criterion (see Theorem 5.3) by using the logarithmic Sobolev inequality. It is important to find some criteria less restrictive than the BKM regularity criterion. Indeed, due to the quadratic nonlinearity of the MHD equations, we expect that the blow-up rate of $\|\nabla u(t)\|_\infty$ at a time T be at least faster than $O(1/(T-t))$. Thus, if one investigates numerically the finite-time singularities of the solutions of such a system of equations and believes that its numerical solution computed leads to a finite-time blow-up at some time T, then one may observe a blow-up rate at the time T for $\|\nabla u(t)\|$ of the form $O(1/((T-t)^\gamma))$, $\gamma \geq 1$. Further, in all the recent numerical investigations performed to find finite-time singularities of the 2D inviscid MHD equations, the results suggest blow-up rates at a time T for $\|\nabla u(t)\|_\infty$ of the form $O(1/((T-t)^\alpha))$ with $1 \leq \alpha < 2$; see [Brachet et al. 2013; Kerr and Brandenburg 1999]. Then, for these numerical cases, with the BKM regularity criterion, one would conclude there is evidence for a finite-time singularity at some time T of the solutions of the 2D resistive MHD equations. However, with the use of our regularity criterion (see Theorem 5.3), we can confirm that in fact there is no blow-up of the solution at this time T. Then, it is dangerous to interpret the blow-up of an under-resolved computation as evidence of finite-time singularities for the 2D resistive MHD equations. Indeed, computing 2D MHD singularities numerically is an extremely challenging task. First of all, it requires huge computational resources; see [Brachet et al. 2013]. Tremendous resolutions are required to capture the nearly singular behavior of the 2D MHD equations. Secondly, one has to perform a careful convergence study.

Furthermore, we notice also that our problem fits in the class of equations considered in [Elgindi and Masmoudi 2014] in the study of L^∞ ill-posedness problem. We thus point out that by borrowing the arguments used in this paper, we can establish the same regularity criterion for another interesting open problem in mathematical fluid dynamics mentioned in [Elgindi and Masmoudi 2014] about the following type of equation in two dimensions:

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u + \nabla p &= Au, \\ \nabla \cdot u &= 0, \end{aligned} \tag{78}$$

with initial condition u_0 in a divergence-free vector field and where A is some constant matrix. Namely, as with Theorem 5.3, we get the following theorem for the system of equations (78):

Theorem 5.4. *Let $u_0 \in H^r_\sigma(\mathbb{R}^2)$ with $r > 2$ and let $T > 0$ be such that there exists $u \in C([0, T[, H^r_\sigma(\mathbb{R}^2))$ a solution of* (78). *If*

$$\int_0^T \frac{\|\nabla u(t)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(t)\|_\infty)}\, dt < +\infty$$

then there cannot be blow-up of the solution u in $H^r(\mathbb{R}^2)$ at the time T, that is, $u \in C([0, T], H^r_\sigma(\mathbb{R}^2))$.

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ON A BILINEAR STRICHARTZ ESTIMATE ON IRRATIONAL TORI

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We prove a bilinear Strichartz-type estimate for irrational tori via a decoupling-type argument, as used by Bourgain and Demeter (2015), recovering and generalizing a result of De Silva, Pavlović, Staffilani and Tzirakis (2007). As a corollary, we derive a global well-posedness result for the cubic defocusing NLS on two-dimensional irrational tori with data of infinite energy.

1. Introduction

Bourgain and Demeter [2015] proved the full range of Strichartz estimates for the Schrödinger equation on tori as a consequence of the ℓ^2 decoupling theorem. In this paper we prove in full generality the analog of the improved Strichartz estimate that first appeared in [De Silva et al. 2007] for rational tori.

1A. ***Statement of the problem and main results.*** Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus, and let $\alpha_1, \dots, \alpha_{d-1} \in \left[\frac{1}{2}, 1\right]$; we define d-dimensional torus $\mathbb{T}^d$ as $\mathbb{T}^d = \mathbb{T} \times \alpha_1 \mathbb{T} \times \cdots \times \alpha_{d-1}\mathbb{T}$. We say that the torus is irrational if at least one α_i is irrational. The torus is rational otherwise. For any $\lambda \geq 1$, we define $\mathbb{T}^d_\lambda$ as a rescaling of $\mathbb{T}^d$ by λ; i.e.,

$$\mathbb{T}^d_\lambda = \lambda \mathbb{T}^d = (\lambda\mathbb{T}) \times (\alpha_1 \lambda \mathbb{T}) \times \cdots \times (\alpha_{d-1}\lambda\mathbb{T}).$$

When $\lambda \to \infty$, one should think of $\mathbb{T}_\lambda$ as a large torus approximating $\mathbb{R}^d$. We consider the following Cauchy problem for the linear Schrödinger equation on $\mathbb{T}^d_\lambda$:

$$\begin{cases} iu_t - \Delta u = 0, & (t,x) \in \mathbb{R} \times \mathbb{T}^d_\lambda, \\ u(0,x) = u_0, & u_0 \in L^2(\mathbb{T}^d_\lambda). \end{cases} \tag{1-1}$$

Let $U_\lambda(t)u_0$ be the solution to (1-1), and let

$$\Lambda_\lambda := \frac{1}{\lambda}\left(\mathbb{Z} \times \frac{1}{\alpha_1}\mathbb{Z} \times \cdots \times \frac{1}{\alpha_{d-1}}\mathbb{Z}\right).$$

One has

$$U_\lambda(t)u_0(x) = \frac{1}{\lambda^{d/2}} \sum_{k \in \Lambda_\lambda} e^{2\pi k i x - |2\pi k|^2 i t}\hat{u}_0(k). \tag{1-2}$$

Our main theorem is the following bilinear refined Strichartz estimate.

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Theorem 1.1. *Let $\phi_1, \phi_2 \in L^2(\mathbb{T}_\lambda)$ be two initial data such that* $\operatorname{supp} \hat{\phi}_i \subset \{k : |k| \sim N_i\}$, $i = 1, 2$, *for some large $N_1 \geq N_2$, and let $\eta(t)$ be a time cut-off function*, $\operatorname{supp} \eta \subset [0, 1]$. *Then when $d = 2$,*

$$\|\eta(t)U_\lambda\phi_1 \cdot \eta(t)U_\lambda\phi_2\|_{L^2_{x,t}} \lesssim N_2^\epsilon \left(\frac{1}{\lambda} + \frac{N_2}{N_1}\right)^{1/2} \|\phi_1\|_{L^2}\|\phi_2\|_{L^2}, \tag{1-3}$$

and when $d \geq 3$

$$\|\eta(t)U_\lambda\phi_1 \cdot \eta(t)U_\lambda\phi_2\|_{L^2_{x,t}} \lesssim N_2^\epsilon \left(\frac{N_2^{d-3}}{\lambda} + \frac{N_2^{d-1}}{N_1}\right)^{1/2} \|\phi_1\|_{L^2}\|\phi_2\|_{L^2}. \tag{1-4}$$

We note that when $d = 2$, $N_1 = N_2$, and $\lambda = 1$, estimate (1-3) recovers the Strichartz inequality for the (irrational) torus after an application of Hölder's inequality, up to an N_2^ϵ-loss. When $\lambda \to \infty$, estimates (1-3) and (1-4) are consistent with the bilinear Strichartz inequality in $\mathbb{R}^{d+1}$ [Bourgain 1998]. Up to the N_2^ϵ-loss, inequality (1-3) is sharp.

Furthermore, when $\lambda \geq N_1$, the estimates fall into the so-called semiclassical regime in which the geometry of $\mathbb{T}_\lambda$ is irrelevant. We refer to [Hani 2012] for the same estimate (without N_2^ϵ-loss) on general compact manifolds. On the torus, our result improves the estimate in that paper for $\lambda \leq N_1$. Estimates (1-3) and (1-4) rely on the geometry of the torus and cannot hold on general compact manifolds.

Remark 1.2. It may also be interesting to consider trilinear estimates. In fact when one considers the quintic nonlinear Schrödinger equation, as in [Herr et al. 2011; Ionescu and Pausader 2012], trilinear estimates are fundamental. See also [Ramos 2016].

We will derive Theorem 1.1 from some bilinear decoupling-type estimates. We first introduce some basic notation.

Let P be the truncated paraboloid in $\mathbb{R}^{d+1}$,

$$P = \{(\xi, |\xi|^2) : \xi \in \mathbb{R}^d,\ |\xi| \lesssim 1\}. \tag{1-5}$$

For any function f supported on P, we define

$$Ef = \widehat{f d\sigma}, \tag{1-6}$$

where σ is the measure on P.

Note a function supported on P can be naturally understood as a function supported on the ball $B = \{\xi \in \mathbb{R}^d : |\xi| \lesssim 1\}$.

By a slight abuse of notation, for a function f supported in the ball B in $\mathbb{R}^d$, we also define

$$Ef(x,t) = \int_B e^{-2\pi i(\xi\cdot x + |\xi|^2 t)} f(\xi)\, d\xi. \tag{1-7}$$

One can see that the two definitions of Ef are essentially the same since P projects onto B.

We decompose P as a finitely overlapping union of caps θ of radius δ. Here a cap θ of radius δ is the set $\theta = \{\xi \in P : |\xi - \xi_0| \lesssim \delta\}$ for some fixed $\xi_0 \in P$. We define $Ef_\theta = \widehat{f_\theta d\sigma}$, where f_θ is f restricted to θ. We use a similar definition also when f is a function supported on the unit ball in $\mathbb{R}^d$. We have $Ef = \sum_\theta Ef_\theta$.

Now, we are ready to state our main decoupling-type estimate.

Theorem 1.3. *Given $\lambda \geq 1$, $N_1 \geq N_2 \geq 1$, let f_1 be supported on P where $|\xi| \sim 1$, and let f_2 be supported on P where $|\xi| \sim N_2/N_1$. Let $\Omega = \{(t,x) \in [0, N_1^2] \times [0, (\lambda N_1)^2]^d\}$. For a finitely overlapping covering of the ball $B = \{|\xi| \leq 1\}$ of caps $\{\theta\}$, $|\theta| = 1/(\lambda N_1)$, we have the following estimate. For any small $\epsilon > 0$, when $d = 2$,*

$$\|Ef_1 Ef_2\|_{L^2_{\mathrm{avg}}(w_\Omega)} \lesssim_\epsilon (N_2)^\epsilon \lambda^{d/2} \left(\frac{1}{\lambda} + \frac{N_2^{d-1}}{N_1} \right)^{1/2} \prod_{j=1}^{2} \left(\sum_{|\theta| = 1/(\lambda N_1)} \|Ef_{j,\theta}\|^2_{L^4_{\mathrm{avg}}(w_\Omega)} \right)^{1/2}, \tag{1-8}$$

and when $d \geq 3$,

$$\|Ef_1 Ef_2\|_{L^2_{\mathrm{avg}}(w_\Omega)} \lesssim_\epsilon (N_2)^\epsilon \lambda^{d/2} \left(\frac{N_2^{d-3}}{\lambda} + \frac{N_2^{d-1}}{N_1} \right)^{1/2} \prod_{j=1}^{2} \left(\sum_{|\theta| = 1/(\lambda N_1)} \|Ef_{j,\theta}\|^2_{L^4_{\mathrm{avg}}(w_\Omega)} \right)^{1/2}, \tag{1-9}$$

where w_Ω is a weight adapted to Ω.

The presence of the weight w in these estimates is standard. We list the basic properties of w in Section 1D, and one can refer to [Bourgain and Demeter 2017] for more details. The notation $L_{\mathrm{avg}}(w_\Omega)^2$ is explained in Section 1C.

The proof of Theorem 1.3 gives another proof of the linear decoupling theorem in [Bourgain and Demeter 2015] in dimension $d = 2$, and does not rely on multilinear Kakeya or multilinear restriction theorems in $\mathbb{R}^3$. The proof of Theorem 1.3 in dimension $d \geq 3$ relies instead on linear decoupling in $\mathbb{R}^{d+1}$ [Bourgain and Demeter 2015].

Remark 1.4. The estimates in Theorems 1.1 and 1.3 are sharp up to an N_2^ϵ. See the Appendix for examples.

Remark 1.5. The N_2^ϵ-loss in Theorem 1.1 is typical if one wants to directly use a decoupling-type argument. It may be possible to remove N_2^ϵ in the mass supercritical setting (in our case, this means $d \geq 3$), using the approach in [Killip and Vişan 2016], where the scale-invariant Strichartz estimates are studied.

Remark 1.6. Similar bilinear estimates for dimension $d \geq 3$ were also considered in [Killip and Vişan 2016] for nonrescaled tori; see Lemma 3.3 in that paper. On the other hand in this work we also consider the $d = 2$ case, which is mass critical.

1B. ***Background and motivation.*** System (1-1) and the bilinear estimates (1-3) and (1-4) naturally appear in the study of the following nonlinear Schrödinger equation on the *nonrescaled* tori:

$$\begin{cases} iu_t + \Delta u = |u|^2 u, \\ u(0) = u_0 \in H^s(\mathbb{T}^d). \end{cases} \tag{1-10}$$

Let us focus for a moment on the $d = 2$ case. The Cauchy problem is said to be locally well-posed in $H^s(\mathbb{T}^d)$ if for any initial data $u_0 \in H^s(\mathbb{T}^d)$ there exists a time $T = T(\|u_0\|_s)$ such that a unique solution to the initial value problem exists on the time interval $[0, T]$. We also require that the data-to-solution map is continuous from $H^s(\mathbb{T}^d)$ to $C_t^0 H_x^s([0, T] \times \mathbb{T}^d)$. If $T = \infty$, we say that a Cauchy problem is globally well-posed.

The initial value problem (1-10) is locally well-posed for initial data $u_0 \in H^s$, $s > 0$, via Strichartz estimates. Note that using iteration, by the energy conservation law, i.e.,

$$E(u(t)) = E(u_0) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{4}\int |u|^4,$$

all initial data in $H^1(\mathbb{T}^2)$ give rise to a global solution. Next, by the nowadays standard I-method [Colliander et al. 2002] by considering a modified version of the energy, in the rational torus case, it was proved in [De Silva et al. 2007] that (1-10) is indeed globally well-posed for initial data in H^s, $s > \frac{2}{3}$. The key estimate there was in fact (1-3) for linear solutions on *rescaled* tori, which we prove here to be available also for irrational tori.

The proof for (1-3) presented in [De Silva et al. 2007] is only for rational tori since it relies on certain types of counting lemmata that cannot directly work on irrational tori. One of the main purposes of this work in fact is to extend results on rational tori to irrational ones.

Based on the discussion we just made, as a corollary of Theorem 1.1, we have:

Corollary 1.7. *The initial value problem* (1-10) *defined on any torus* $\mathbb{T}^2$ *is globally well-posed for initial data in* $H^s(\mathbb{T}^2)$ *with* $s > \frac{2}{3}$.

Remark 1.8. Results such as Corollary 1.7 usually also give a control on the growth of Sobolev norms of the global solutions. We do not address this particular question here. We instead refer the reader to the recent work [Deng and Germain 2017].

The original Strichartz estimates needed to prove the local well-posedness of Cauchy problems such as (1-10) were first obtained in [Bourgain 1993] via number-theoretical-related counting arguments for rational tori. Recently, the striking proof of the ℓ^2 decoupling theorem [Bourgain and Demeter 2015] provided a completely different approach from which all the desired Strichartz estimates on tori, both rational and irrational, follow. This approach in particular does not depend on counting lattice points. See also [Guo et al. 2014; Deng et al. 2017]. The method of proof we implement in this present work is mostly inspired by [Bourgain and Demeter 2015] and the techniques used to prove the ℓ^2 decoupling theorem.

We quickly recall the main result in [Bourgain and Demeter 2015]. Let P be a unit parabola in $\mathbb{R}^{d+1}$, covered by finitely overlapping caps θ of radius $1/R$. Let f be a function defined on P; then one has for any $\epsilon > 0$ small,

$$\|Ef\|_{L^p(w_{B_{R^2}})} \lesssim_\epsilon R^\epsilon (R^2)^{d/4-(d+2)/(2p)} \Big(\sum_\theta \|Ef_\theta\|^2_{L^p(B_{R^2})}\Big)^{1/2}, \quad p \geq \frac{2(d+2)}{d}. \tag{1-11}$$

Note that (1-11) corresponds to Theorem 1.1 in [Bourgain and Demeter 2015], and the dimension n in the estimate (2) there corresponds to our $d+1$. Also note that the linear decoupling (1-11) not only works for those f exactly supported on P, but those f supported in an R^{-2} neighborhood of P, and in this case, cap θ would be replaced by the R^{-2} neighborhood of the original θ; see Theorem 1.1 in [Bourgain and Demeter 2015].

We remark that one key feature of this decoupling-type estimate is that one needs to work on a larger scale in physical space, i.e., the scale R^2 rather than R, in order to observe the decoupling phenomena. The proper observational scale dictated by Heisenberg's uncertainty principle is R.

Indeed, one principle, which is usually called *parallel decoupling*, indicates that if decoupling happens in a small region, then decoupling happens in a large region as well. We state a bilinear version of parallel decoupling below.

Lemma 1.9 [Bourgain and Demeter 2015; 2017]. *Let D be a domain, and $D = D_1 \cup D_2 \cup \cdots \cup D_J$, $D_i \cap D_j = \varnothing$. If for some constant $A > 0$ and for functions h_1, h_2, defined on the unit parabola, one has*

$$\|Eh_1 Eh_2\|_{L^2_{\mathrm{avg}}(w_{D_i})} \le A \prod_{j=1}^{2} \biggl(\sum_{|\theta|=1/(\lambda N_1)} \|Eh_{j,\theta}\|^2_{L^4_{\mathrm{avg}}(w_{D_i})} \biggr)^{1/2}, \quad i = 1, \dots, J, \tag{1-12}$$

then one also has

$$\|Eh_1 Eh_2\|_{L^2_{\mathrm{avg}}(w_D)} \le A \prod_{j=1}^{2} \biggl(\sum_{|\theta|=1/(\lambda N_1)} \|Eh_{j,\theta}\|^2_{L^4_{\mathrm{avg}}(w_D)} \biggr)^{1/2}. \tag{1-13}$$

The proof of this particular formulation of parallel decoupling follows by Minkowski's inequality.

As it exists, parallel decoupling is a principle rather than a concrete lemma. We state the version here solely for concreteness. It should be easy to generalize the lemma under different conditions.

1C. *Notation.* We write $A \lesssim B$ if $A \le CB$ for a constant $C > 0$, and $A \sim B$ if both $A \lesssim B$ and $B \lesssim A$. We say $A \lesssim_\epsilon B$ if the constant C depends on ϵ. Similarly for $A \sim_\epsilon B$. For a Borel set, $E \subset \mathbb{R}^d$, we denote the diameter of E by $|E|$ and the Lebesgue measure of E by $m(E)$.

We will use the usual function space L^p. We also use a (weighted) average version of L^p space; i.e.,

$$\|g\|_{L^p_{\mathrm{avg}}(A)} = \biggl(\fint_A |g|^p \biggr)^{1/p} := \biggl(\frac{1}{m(A)} \int_A |g|^p \biggr)^{1/p},$$

$$\|g\|_{L^p_{\mathrm{avg}}(w_A)} = \biggl(\frac{1}{m(A)} \int |g|^p w_A \biggr)^{1/p},$$

where w_A is a weight function described below.

For any function f, we use $\hat{f}$ to denote its Fourier transform. When we say *unit ball*, we refer to a ball of radius $r \sim 1$. We will often identify a torus as a bounded domain in Euclidean space; for example, we will view $(\mathbb{R}/\mathbb{Z})^d$ as $[0,1]^d \subset \mathbb{R}^d$. In this work, Ω is used to denote the domain $[0, N_1^2] \times [0, (\lambda N_1)^2]^d \subset \mathbb{R}^{d+1}$.

1D. *The weight w_A.* If h is a Schwartz function whose Fourier transform, $\hat{h}$, is supported in a ball of radius $1/R$, we expect h to be essentially constant on balls of radius R, and essentially

$$\|h\|_{L^p_{\mathrm{avg}}(B_R)} \sim \|h\|_{L^2_{\mathrm{avg}}(B_R)} \sim \|h\|_{L^\infty(B_R)}. \tag{1-14}$$

Expression (1-14) is not rigorous, and the introduction of the weight w_{B_R} is a standard way to overcome this technical difficulty. We refer to Lemma 4.1 in [Bourgain and Demeter 2017] for a more detailed discussion of the weight function.

For any bounded open convex set A, the weight function w_A might change from line to line and from the left-hand side of the inequality to the right-hand side, and satisfies the properties:

- $\int w_A \sim m(A)$.
- $w_A \gtrsim 1$ on A, and rapidly (polynomial-type) decays outside A.

We will usually define A to be a ball, or the product of balls in this paper.

Furthermore, let B_R be a ball centered at 0, and let μ_{B_R} be a function such that $\widehat{\mu_{B_R}}$ is about $1/(m(B_{1/R}))$ on $B_{1/R}$, and supported in $B_{2/R}$. Then μ_{B_R} is about 1 on B_R, and decays faster than any polynomial outside of B_R. Additionally, $\mu_{B_R}^2$ is positive, decays faster than any polynomial outside of B_R and is Fourier-supported in $B_{4/R}$. We take translations B' of B_R to cover the whole space, and we denote by $\mu_{B'}$ the corresponding translation of μ_{B_R} and $w_{B_R}(B') = \max_{x \in B'} w_{B_R}$. We have the useful property,

$$w_{B_R}(x) \le \sum_{B'} w_{B_R}(B') 1_{B'}(x) \lesssim \sum_{B'} w_{B_R}(B') \mu_{B'}^2(x) \lesssim w_{B_R}(x). \tag{1-15}$$

The last inequality follows from the fact that $\mu_{B'}^2$ decays faster than any polynomial outside of B'.

Lemma 1.10. *For a function f supported in $B_{1/R}$, for any $p < \infty$,*

$$\|Ef\|_{L^\infty(B_R)} \lesssim \|Ef\|_{L^p_{\mathrm{avg}}(\mu_{B_R})}.$$

We refer to the proof of Corollary 4.3 in [Bourgain and Demeter 2017] with the weight on the left-hand side being 1_{B_R} so that on the right-hand side we have a fast decay weight.

Remark 1.11. In general, Lemma 1.10 should hold for any convex set A and the dual convex body A^*.

2. Proof of Theorem 1.1 assuming Theorem 1.3

Assume Theorem 1.3, and let us prove Theorem 1.1. The argument below comes from the proof of discrete restriction and the Strichartz estimate on irrational tori assuming the ℓ^2 decoupling estimate; see Theorems 2.2 and 2.3 in [Bourgain and Demeter 2015]. The argument originally comes as observation due to Bourgain [2013]. We record it here for completeness.

Let ϕ_1, ϕ_2 be as in Theorem 1.1. We rescale ϕ_1 to be supported in the unit ball and rescale ϕ_2 to be supported in a ball of radius $\sim N_2/N_1$. Recall,

$$U_\lambda(t)\phi_j(x,t) = \frac{1}{\lambda^{d/2}} \sum_{\substack{k \in \Lambda_\lambda \\ k \sim N_1}} e^{2\pi i k \cdot x - |2\pi k|^2 t} \hat{\phi}_j(k). \tag{2-1}$$

We perform a change of variables $\xi = k/N_1$ and we let

$$h_j(\tau) = \frac{1}{\lambda^{d/2}} \sum_{\substack{\xi \in \Lambda_{\lambda N_1} \\ |\xi| \sim 1}} \hat{\phi}_j(\xi N_1) \delta_\xi(\tau), \quad j = 1, 2. \tag{2-2}$$

Note one can directly check that

$$U_\lambda(t)\phi_j(x,t) = Eh_j(-2\pi N_1 x, (2\pi)^2 N_1^2 t). \tag{2-3}$$

Without loss of generality, we suppress the constants -2π and $(2\pi)^2$.

Let $Q_0 = [0, N_1^2] \times \mathbb{T}^d_{\lambda N_1}$ and let us view $\mathbb{T}^d_{\lambda N_1}$ as a compact set in $\mathbb{R}^d$. In particular, one can construct the associated weight function w_{Q_0}. Direct computation (via change of variables) gives

$$\|U_\lambda(t)\phi_1 U_\lambda(t)\phi_2\|_{L^2([0,1]\times\mathbb{T}^d_\lambda)} \sim N_1^{-(d+2)/2} m(Q_0)^{1/2} \|Eh_1 Eh_2\|_{L^2_{\mathrm{avg}}(Q_0)} \tag{2-4}$$

and due to the periodicity of Eh_i, $i = 1, 2$, one has

$$\|Eh_1 Eh_2\|_{L^2_{\mathrm{avg}}(\Omega)} = \|Eh_1 Eh_2\|_{L^2_{\mathrm{avg}}(Q_0)}. \tag{2-5}$$

For a covering $\{\theta\}$ of caps of radius $1/(\lambda N_1)$, each cap θ contains at most one $\xi_\theta \in \Lambda_{\lambda N_1}$, corresponding to $k_\theta = N_1 \xi_\theta \in \Lambda_\lambda$. Then

$$\|Eh_{j,\theta}\|_{L^4_{\mathrm{avg}}(w_{Q_0})} \sim h_j(\xi_\theta) \sim \frac{1}{\lambda^d} \hat{\phi}_j(k_\theta),$$

and

$$\prod_{j=1}^{2} \Bigl(\sum_{|\theta|=1/(\lambda N_1)} \|Eh_j\|^2_{L^4_{\mathrm{avg}}(w_{Q_0})} \Bigr)^{1/2} \sim \lambda^{-d} \prod_{j=1}^{2} \Bigl(\frac{1}{\lambda^d} \sum_{k\in\Lambda_\lambda} |\hat{\phi}_j(k)|^2 \Bigr)^{1/2} \sim \lambda^{-d} \|\phi_1\|_{L^2} \|\phi_2\|_{L^2}.$$

For convenience of notation let

$$D_{\lambda,N_1,N_2} := \begin{cases} 1/\lambda + N_2/N_1 & \text{when } d = 2, \\ N_2^{d-3}/\lambda + N_2^{d-1}/N_1 & \text{when } d \geq 3. \end{cases} \tag{2-6}$$

Recall that $\Omega = [0, N_1]^2 \times [0, (\lambda N_1)^2]^d$; we apply Theorem 1.3 with $f_j = h_j$, and we have

$$\|Eh_1 Eh_2\|_{L^2_{\mathrm{avg}}(w_\Omega)} \lesssim_\epsilon (N_2)^\epsilon \lambda^{d/2} D^{1/2}_{\lambda,N_1,N_2} \prod_{j=1}^{2} \Bigl(\sum_{|\theta|=1/(\lambda N_1)} \|Eh_{j,\theta}\|^2_{L^4_{\mathrm{avg}}(w_\Omega)} \Bigr)^{1/2}. \tag{2-7}$$

Note that Ω can be covered by Q such that $\{Q\}$ are finitely overlapping and each Q is a translation of Q_0. Since Eh_j are periodic on x, estimate (2-7) is equivalent to

$$\|Eh_1 Eh_2\|_{L^2_{\mathrm{avg}}(w_{Q_0})} \lesssim_\epsilon (N_2)^\epsilon \lambda^{d/2} D^{1/2}_{\lambda,N_1,N_2} \prod_{j=1}^{2} \Bigl(\sum_{|\theta|=1/(\lambda N_1)} \|Eh_{j,\theta}\|^2_{L^4(w_{Q_0})} \Bigr)^{1/2}. \tag{2-8}$$

Plugging (2-8) into (2-4) gives

$$\begin{aligned} \|U_\lambda(t)\phi_1 U_\lambda(t)\phi_2\|_{L^2([0,1]\times\mathbb{T}^d_\lambda)} &\lesssim N_1^{-(d+2)/2} \cdot N_1 m(\mathbb{T}^d_{\lambda N_1})^{1/2} \lambda^{-d} \cdot (N_2)^\epsilon \lambda^{d/2} D^{1/2}_{\lambda,N_1,N_2} \|\phi_1\|_{L^2} \|\phi_2\|_{L^2} \\ &\sim (N_2)^\epsilon D^{1/2}_{\lambda,N_1,N_2} \|\phi_1\|_{L^2} \|\phi_2\|_{L^2} \end{aligned}$$

and Theorem 1.1 follows.

The rest of the paper details the proof of Theorem 1.3.

3. An overview of the proof of Theorem 1.3

First, we reduce the proof of Theorem 1.3 to the following proposition.

Proposition 3.1. *Let τ_1 be a cap of radius N_2/N_1 supported at ξ with $|\xi|\sim 1$. Let τ_2 be a cap of radius N_2/N_1 supported at ξ with $|\xi|\sim N_2/N_1$. Let f_j be a function supported in τ_j. Then for any small $\epsilon>0$, when $d=2$*

$$\|Ef_1Ef_2\|_{L^2_{\mathrm{avg}}(w_\Omega)}\lesssim_\epsilon (N_2)^\epsilon\lambda^{d/2}\left(\frac{1}{\lambda}+\frac{N_2}{N_1}\right)^{1/2}\prod_{j=1}^{2}\biggl(\sum_{\substack{|\theta|=1/(\lambda N_1)\\ \theta\subset\tau_j}}\|Ef_{j,\theta}\|^2_{L^4_{\mathrm{avg}}(w_\Omega)}\biggr)^{1/2}, \tag{3-1}$$

and when $d\geq 3$,

$$\|Ef_1Ef_2\|_{L^2_{\mathrm{avg}}(w_\Omega)}\lesssim_\epsilon (N_2)^\epsilon\lambda^{d/2}\left(\frac{N_2^{d-3}}{\lambda}+\frac{N_2^{d-1}}{N_1}\right)^{1/2}\prod_{j=1}^{2}\biggl(\sum_{\substack{|\theta|=1/(\lambda N_1)\\ \theta\subset\tau_j}}\|Ef_{j,\theta}\|^2_{L^4_{\mathrm{avg}}(w_\Omega)}\biggr)^{1/2}. \tag{3-2}$$

Now, let f_1, f_2 be as in Proposition 3.1. We define $K_0(\lambda,N_1,N_2)$ to be the best constant such that

$$\|Ef_1Ef_2\|_{L^2_{\mathrm{avg}}(w_\Omega)}\leq \lambda^{d/2}K_0(\lambda,N_1,N_2)\prod_{j=1}^{2}\biggl(\sum_{|\theta|=1/(\lambda N_1)}\|Ef_{j,\theta}\|^2_{L^4_{\mathrm{avg}}(w_\Omega)}\biggr)^{1/2}. \tag{3-3}$$

We also let $\widetilde{K}(\lambda,N_1,N_2)$ and $K(\lambda,N_1,N_2)$ be defined as the best constants such that

$$\|Ef_1Ef_2\|_{L^2_{\mathrm{avg}}(w_{[0,N_1^2]\times[0,\lambda N_1]^d})}\leq \lambda^{d/2}\widetilde{K}(\lambda,N_1,N_2)\prod_{j=1}^{2}\biggl(\sum_{|\theta|=1/(\lambda N_1)}\|Ef_{j,\theta}\|^2_{L^4_{\mathrm{avg}}(w_{[0,N_1^2]\times[0,\lambda N_1]^d})}\biggr)^{1/2}, \tag{3-4}$$

$$\|Ef_1Ef_2\|_{L^2_{\mathrm{avg}}(w_{B_{N_1^2}})}\leq \lambda^{d/2}K(\lambda,N_1,N_2)\prod_{j=1}^{2}\biggl(\sum_{|\theta|=1/(\lambda N_1)}\|Ef_{j,\theta}\|^2_{L^4_{\mathrm{avg}}(w_{B_{N_1^2}})}\biggr)^{1/2}. \tag{3-5}$$

Below we will prove that

$$\begin{aligned} K_0(\lambda,N_1,N_2)&\lesssim N_2^\epsilon\left(\frac{1}{\lambda}+\frac{N_2}{N_1}\right)^{1/2}, && d=2,\\ K_0(\lambda,N_1,N_2)&\lesssim N_2^\epsilon\left(\frac{N_2^{d-3}}{\lambda}+\frac{N_2^{d-1}}{N_1}\right)^{1/2}, && d\geq 3.\end{aligned} \tag{3-6}$$

We point out here that by parallel decoupling, Lemma 1.9, one always has

$$\begin{aligned} K_0(\lambda,N_1,N_2)&\lesssim K(\lambda,N_1,N_2),\\ K_0(\lambda,N_1,N_2)&\lesssim \widetilde{K}(\lambda,N_1,N_2).\end{aligned} \tag{3-7}$$

The proof of Proposition 3.1 or equivalently (3-6) proceeds as follows. We first show:

Lemma 3.2. *When $\lambda \geq N_1$,*

$$\widetilde{K}(\lambda, N_1, N_2) \lesssim N_2^{\epsilon} \frac{N_2^{(d-1)/2}}{N_1^{1/2}}. \tag{3-8}$$

Note that when $\lambda \geq N_1$, Proposition 3.1 follows from (3-7) and Lemma 3.2.

Then, we show:

Lemma 3.3. *When $\lambda \leq N_1$,*

$$\begin{aligned} K(\lambda, N_1, N_2) &\lesssim N_2^{\epsilon} \left(\frac{1}{\lambda} + \frac{N_2}{N_1}\right)^{1/2}, && d = 2, \\ K(\lambda, N_1, N_2) &\lesssim N_2^{\epsilon} \left(\frac{N_2^{d-3}}{\lambda} + \frac{N_2^{d-1}}{N_1}\right)^{1/2}, && d = 3. \end{aligned} \tag{3-9}$$

From (3-7), clearly Proposition 3.1 follows from Lemmas 3.2 and 3.3.

The proof of Lemma 3.3 in dimension $d = 2$ relies on induction (of scale N_2). The proof of Lemma 3.3 in dimension in $d \geq 3$ is easier and more straightforward, (in some sense, it also relies on induction, but it is enough to induct only once.)

We first show the base case:

Lemma 3.4. *When $\lambda \leq N_1$ and $N_2 \lesssim 1$, we have $K(\lambda, N_1, N_2) \lesssim 1/\lambda^{1/2}$.*

Lemma 3.4 is not as useful in dimension $d \geq 3$, we indeed have a better estimate:

Lemma 3.5. *When $d \geq 3$, $\lambda \leq N_1$ and $\lambda \leq N_1/N_2^2$, we have $K(\lambda, N_1, N_2) \lesssim (N_2^{d-3}/\lambda)^{1/2}$.*

We then show the following lemma, which ensures that we only need to induct until $\lambda \leq N_1/N_2$, when $d = 2$, and until N_1/N_2 when $d \geq 3$.

Lemma 3.6. *Let $\lambda \leq N_1$.*

Let $d = 2$. Assume we have $K(\lambda, N_1, N_2) \leq \lambda^{-1/2}$ when $\lambda < N_1/N_2$. Then

$$K(\lambda, N_1, N_2) \leq N_2^{\epsilon} \frac{N_2^{(d-1)/2}}{N_1^{1/2}} \quad \textit{when } \lambda \geq \frac{N_1}{N_2}.$$

Let $d \geq 3$. Assume we have $K(\lambda, N_1, N_2) \leq (N_2^{d-3}/\lambda)^{1/2}$ when $\lambda < N_1/N_2^2$. Then

$$K(\lambda, N_1, N_2) \leq N_2^{\epsilon} \frac{N_2^{d-1/(2)}}{N_1^{1/2}} \quad \textit{when } \lambda \geq \frac{N_1}{N_2^2}.$$

Note that when $d \geq 3$, Lemmas 3.5 and 3.6 imply Lemma 3.3. In dimension $d = 2$, we use induction (we rely on the so-called parabolic rescaling) to finish the proof of Lemma 3.3.

We end this section with an outline of the structure of the rest of the paper. We show that Proposition 3.1 implies Theorem 1.3 in Section 4. Lemmas 3.2, 3.4, and 3.6 all rely on the exploration of the so-called *transversality*, which essentially allows us to reduce the dimensionality of the problem. We first explore *transversality* in Section 5 and then we prove Lemmas 3.2, 3.4, and 3.6 in Section 6.

The details of the induction procedure (which is nontrivial) that is used to prove Lemma 3.3 in dimension $d=2$ are given in Section 5. We remark here the proof of Lemma 3.3 relies on Lemma 3.2.

Finally, we prove Lemma 3.5 at the end of Section 7, which, together with Lemma 3.6 will conclude the proof of Lemma 3.3 in dimension $d \geq 3$.

4. Proposition 3.1 implies Theorem 1.3

We first introduce one standard but important tool in the following lemma.

Lemma 4.1 [Bourgain and Demeter 2015; 2017]. *Let $\{g_\alpha\}$ be a family of functions such that* supp $\hat{g}_\alpha$ *are finitely overlapped cubes of length ρ. Let A be bounded convex open set tiled by finitely overlapped cubes Q of side length $\geq \rho^{-1}$. Then for the w_A adapted to A, the following holds*:

$$\fint_A \Big|\sum g_\alpha\Big|^2 w_A \lesssim \sum \frac{1}{m(A)} \int |g_\alpha|^2 w_A.$$

Proof. Since we can sum up the weight function over a finitely overlapping cover $\{Q\}$ of A, that is, $w_A = \sum_{Q\subset A} w_Q$, it suffices to prove the result for $A = Q$. Recall by inequality (1-15), we can cover the whole space $\mathbb{R}^n$ by translations Q' of Q:

$$\begin{aligned}
\fint_Q \Big|\sum g_\alpha\Big|^2 w_Q\,dx &\leq \frac{1}{m(Q)} \sum_{Q'} \int_{Q'} w_Q(Q') \Big|\sum g_\alpha\Big|^2 \\
&\leq \frac{1}{m(Q)} \sum_{Q'} w_Q(Q') \int \Big|\sum g_\alpha\Big|^2 \mu_{Q'}^2 \\
&= \frac{1}{m(Q)} \sum_{Q'} w_Q(Q') \int |\hat{g}_\alpha * \hat{\mu}_{Q'}|^2 \\
&\lesssim \frac{1}{m(Q)} \sum_{Q'} w_Q(Q') \sum_\alpha \int |g_\alpha|^2 \mu_{Q'}^2 \lesssim \frac{1}{m(Q)} \sum_\alpha \int |g_\alpha|^2 w_Q. \qquad \square
\end{aligned}$$

Now we can reduce Proposition 3.1 to a bilinear decoupling on two (N_2/N_1)-diameter caps.

Lemma 4.2. *Theorem 1.3 is equivalent to Proposition 3.1.*

Proof. Let f_1, f_2 be as in Theorem 1.3. Then $f_1 = \sum_{|\tau|=N_2/N_1} f_{1,\tau}$ and the $f_{1,\tau}$ are supported on finitely overlapping caps of diameter N_2/N_1.

Since $|f_2|$ is supported in a cap of diameter N_2/N_1, the supports of $\{\widehat{Ef}_{1,\tau} * \widehat{Ef_2}\}_\tau$ are in finitely overlapping cubes of length N_2/N_1. Since the scale of Ω is larger than N_1/N_2, i.e., it contains a ball of radius $> N_1/N_2$, by Lemma 4.1,

$$\fint_\Omega |Ef_1 Ef_2|^2 w_\Omega\,dx \leq \sum_{|\tau|=N_2/N_1} \Big|\fint_\Omega Ef_{1,\tau} Ef_2\Big|^2 w_\Omega\,dx.$$

Now apply Proposition 3.1 for $f_{1,\tau}$ and f_2 for each τ; Theorem 1.3 follows. $\square$

5. Transversality

Let f_1, f_2 be as in Proposition 3.1; then f_1 is supported around $(0,0,\dots,0,1,1)$ and f_2 is supported around $(0,0,\dots,0)$. The main goal of this section is to explore the transversality between $(0,0,\dots,0,1)$ and $(0,0,0,\dots,0)$, or more precisely, the transversality between the unit normal vectors of the truncated parabola at these two points. The main lemma in this section is Lemma 5.1 below, and Corollary 5.7 which essentially follows from Lemma 5.1.

We first introduce some basic notation. Let $\{e_1,\dots,e_d\}$ be the standard basis of $\mathbb{R}^d$. We will encounter caps of radius v around $(0,0,\dots,0)$ and $(0,\dots,0,1,1)$ on the parabola. Note around those two points, when v is small (which is always the case in our work), one may view those caps as their natural projection to $\mathbb{R}^{d-1}$. And their image is essentially a square/cap of radius v. We say that a (v,v^2)-plate is a d-dimensional rectangle with the short side on the e_{d-1}-direction such that its image under the orthogonal projection to R^{d-1} is a $(v\times v\times\cdots\times v\times v^2)$-rectangle.

Lemma 5.1. *Given $|\upsilon|<1$, let f_1 be a function supported on a cap of radius υ, centered at $(0,\dots,0,1,1)$ on the truncated parabola P, and let f_2 be a function supported on a cap of radius υ centered at $(0,\dots,0,0,0)$ on the paraboloid. For a covering $\{\tau_i\}$ of* supp f_i *with (υ,υ^2) plates, with the shorter side on the e_{d-1}-direction, we have the following decoupling inequality: for any $R>\upsilon^{-2}$,*

$$\int |Ef_1Ef_2|^2 w_{B_R} \lesssim \sum_{\tau_1,\tau_2}\int |Ef_{1,\tau_1}Ef_{2,\tau_2}|^2 w_{B_R}. \tag{5-1}$$

Remark 5.2. We thank J. Ramos for pointing out that Lemma 5.1 is a particular case of Proposition 2 in [Ramos 2016]. We still write a proof in this paper for clarity.

Proof. The proof is similar to the proof of the L^4 Strichartz estimate on the one-dimensional torus. From the inequality (1-15), we only need to prove that

$$\int_{B'} |Ef_1Ef_2|^2 \lesssim \sum_{\tau_1,\tau_2}\int |Ef_{1,\tau_1}Ef_{2,\tau_2}|^2\mu_{B'}^2$$

for all translations B' of B_R. Now

$$\int_{B'} |Ef_1Ef_2|^2 \le \sum_{\tau_1,\tau_2,\tau_3,\tau_4}\int_{B'} Ef_{1,\tau_1}Ef_{2,\tau_2}\overline{Ef}_{1,\tau_3}\overline{Ef}_{2,\tau_4}\mu_{B'}^2. \tag{5-2}$$

Let $\xi_i\in\tau_i,\ \xi_i=\big(\xi_{i,1},\dots,\xi_{i,d-1},\sum_{j=1}^{d-1}(\xi_i^j)^2\big)\equiv(\bar{\xi}_i,\xi_{i,d-1},|\bar{\xi}_i|^2+(\xi_i^{d-1})^2),\ i=1,2,3,4$. We have

$$\begin{aligned} |\bar{\xi}_i| &\lesssim \upsilon, \quad i=1,2,3,4,\\ |\xi_{i,d-1}-1| &\lesssim \upsilon, \quad i=1,3,\\ |\xi_{i,d-1}| &\lesssim \upsilon, \quad i=2,4. \end{aligned} \tag{5-3}$$

Essentially, for any $\tau_1,\tau_2,\tau_3,\tau_4$ such that

$$\int Ef_{1,\tau_1}Ef_{2,\tau_2}\overline{Ef}_{1,\tau_3}\overline{Ef}_{2,\tau_4}\mu_{B'}^2 \neq 0,$$

one must have for some $\xi_i \in \tau_i$,

$$\begin{aligned} \xi_1 - \xi_3 &= \xi_2 - \xi_4 + O(R^{-1}), \\ |\xi_1|^2 - |\xi_3|^2 &= |\xi|_2^2 - |\xi|_4^2 + O(R^{-1}), \end{aligned} \tag{5-4}$$

and the second formula in (5-4) implies

$$(\xi_{1,d-1} - \xi_{3,d-1})(\xi_{1,d-1} + \xi_{3,d-1}) = O(|\xi_2|^2 + |\xi_4|^2) + O(|\bar{\xi}_1|^2 + |\bar{\xi}_3|^2) + O(R^{-1}). \tag{5-5}$$

Plugging into (5-3), one has $|\xi_{1,d-1} - \xi_{3,d-1}| \lesssim \upsilon^2$, which again implies $|\xi_{2,d-1} - \xi_{4,d-1}| \lesssim \upsilon^2$.

To summarize, $\int Ef_{1,\tau_1} Ef_{2,\tau_2} \overline{Ef}_{1,\tau_3} \overline{Ef}_{2,\tau_4} \mu_{B'}^2 \neq 0$ implies the distance between τ_1 and τ_3 and the distance between τ_2 and τ_4 are both bounded by υ^2, which essentially means $\tau_i = \tau_{i+2}$, $i = 1, 2$. Applying this fact to (5-2), Lemma 5.1 follows. □

Remark 5.3. A quantitative version of estimate (5-1) can be stated as follows: assume that the support of f_1 is centered at $(0, 1/K, (1/K)^2)$ rather than $(0, 0, 1)$. From the proof we can attain the same estimate as in (5-1) by introducing an additional constant K,

$$\int |Ef_1 Ef_2|^2 w_{B_R} \lesssim K \sum_{\tau_1, \tau_2} \int |Ef_{1,\tau_1} Ef_{2,\tau_2}|^2 w_{B_R}. \tag{5-6}$$

Indeed, the proof essentially only relies on the fact that for $\xi_i \in \operatorname{supp} f_i$, $i = 1, 2$, the difference between the $d-1$ components is at least $1/K$. Similar arguments also hold for the estimate in Lemma 5.5; see Corollary 5.7 below.

Remark 5.4. We remark that for any $\alpha < \upsilon$, a function which is supported on a cap of radius α can be naturally understood as a function supported on a cap of radius υ.

Lemma 5.1 facilitates the decomposition of caps of radius υ into plates of size (υ, υ^2). We can further decompose those into caps of radius υ^2.

Lemma 5.5. *With the same notation as in Lemma 5.1, $R \geq \upsilon^{-2}$, let* $\operatorname{supp} f_i$ *be covered by finitely overlapping caps θ_i of radius υ^2, $i = 1, 2$. Then*

$$\int |Ef_1 Ef_2|^2 w_{B_R} \lesssim \upsilon^{-(d-1)} \sum_{|\theta_i| = \upsilon^2} \int |Ef_{1,\theta_1} Ef_{2,\theta_2}|^2 w_{B_R}. \tag{5-7}$$

Proof. Clearly, we need only to prove (5-7) for every ball of radius υ^{-2} contained in B_R, and then sum them together. (This is the same principle of parallel decoupling, Lemma 1.9.)

Fix a pair of (υ, υ^2) plates τ_1, τ_2:

$$\begin{aligned} \int |Ef_{1,\tau_1} Ef_{2,\tau_2}|^2 w_{B_R} &= \int \Bigg| \sum_{\substack{\theta_2 \subset \tau_2 \\ |\theta_2| = \upsilon^2}} Ef_{1,\tau_1} Ef_{2,\theta_2} \Bigg|^2 w_{B_R} \\ &\leq \upsilon^{-(d-1)} \sum_{\substack{\theta_2 \subset \tau_2 \\ |\theta_2| = \upsilon^2}} |Ef_{1,\tau_1} Ef_{2,\theta_2}|^2 w_{B_R} \lesssim \sum_{\substack{\theta_j \subset \tau_j \\ |\theta_j| = \upsilon^2}} |Ef_{1,\theta_1} Ef_{2,\theta_2}|^2 w_{B_R}. \end{aligned} \tag{5-8}$$

The last inequality follows from Lemma 4.1 and Lemma 4.2. □

Remark 5.6. Similar to Remark 5.4, for $\upsilon^2 < \alpha < \upsilon$, a cap of scale υ naturally lies in a cap of scale $\sqrt{\alpha}$. Thus if we let f_1 be a function supported on a cap of radius α centered at $(0, \dots, 0, 1, 1)$ on the paraboloid, and we let f_2 be a function supported on a cap of radius α centered at $(0, \dots, 0, 0, 0)$ on the paraboloid, then by arguing similar to the proof of Lemma 5.5, we have for $R \ge \alpha^{-1}$,

$$\int |Ef_1 Ef_2|^2 w_{B_R} \lesssim \left(\frac{\upsilon}{\alpha}\right)^{d-1} \sum_{|\theta_i|=\alpha} \int |Ef_{1,\theta_1} Ef_{2,\theta_2}|^2 w_{B_R}. \tag{5-9}$$

If we directly use the Hölder inequality for all caps in the support of f_i to estimate as in (5-8), then the interpolation in the proof of Lemma 5.5 will give us a constant υ^{-d} rather than $\upsilon^{-(d-1)}$ in (5-7), since one has υ^{-d} caps for each f_i. The bilinear transversality, i.e., the transversality between $(0, 0, \dots, 0)$ and $(0, \dots, 0, 1, 1)$, helps in reducing the dimension by 1 since in one direction we can use L^4 orthogonality, as shown in Lemma 5.1. Thus here we are able to improve the constant in (5-7) to $\upsilon^{-(d-1)}$.

Corollary 5.7. *Using the same notation as in Lemma 5.1, there exists a constant C such that for any υ, δ, and $R^{-1} \le \delta \le \upsilon$,*

$$\int |Ef_1 Ef_2|^2 w_{B_R} \lesssim \left(\frac{\upsilon}{\delta}\right)^{d-1} \left|\frac{\log \delta}{\log \upsilon}\right|^C \sum_{|\theta_i|=\delta} \int |Ef_{1,\theta_1} Ef_{2,\theta_2}|^2 w_{B_R}.$$

Proof. The proof is most clear when $\delta = \upsilon^{2^n}$ for some n. Let us first handle this case and then go to the general case. One may use induction. (This induction, however, does not rely on parabolic rescaling.) If $n = 0$, there is nothing to prove.

Assume the result holds for the case $n = k$. Let us turn to the case $n = k + 1$, where $\delta = \upsilon^{2^{k+1}}$, and so $\delta^{1/2} = \upsilon^{2^k}$; thus by the induction assumption, we have

$$\int |Ef_1 Ef_2|^2 w_{B_R} \lesssim \left(\frac{\upsilon}{\delta^{1/2}}\right)^{d-1} 2^{Ck} \sum_{|\eta_i|=\delta^{1/2}} \int |Ef_{1,\eta_1} Ef_{2,\eta_2}|^2 w_{B_R}. \tag{5-10}$$

Now note that $R \ge (\delta^{-1/2})^2$. By Lemma 5.5, we have for each pair (η_1, η_2) in (5-10) that

$$\int |Ef_{1,\eta_1} Ef_{2,\eta_2}|^2 w_{B_R} \lesssim (\delta^{1/2})^{-(d-1)} \sum_{\substack{\theta_i \subset \eta_i \\ |\theta_i|=\delta}} \int |Ef_{1,\theta_1} Ef_{2,\theta_2}|^2 w_{B_R}. \tag{5-11}$$

The case $n = k + 1$ clearly follows if one plugs (5-11) into (5-10), taking the constant C large enough.

Now we turn to the general case. We only need to work on the case $\upsilon^{2^{n+1}} < \delta < \upsilon^{2^n}$. Recall that previously, when $\delta = \upsilon^{2^n}$, we used induction as $\upsilon \to \upsilon^2 \to \upsilon^{2^2} \to \cdots \to \upsilon^{2^n} = \delta$, and in each step we used Lemma 5.5 to finish the induction $\upsilon^{2^k} \to \upsilon^{2^{k+1}}$.

In the case $\upsilon^{2^{n+1}} < \delta < \upsilon^{2^n}$ we have $\upsilon^{2^n} < \delta^{1/2}$, and we use induction as before for $\upsilon \to \upsilon^2 \to \upsilon^{2^2} \to \cdots \to \upsilon^{2^n}$, and we use (5-9) to use induction again from υ^{2^n} to δ. This ends the proof. □

6. Proofs of Lemmas 3.2, 3.4 and 3.6

We are now prepared to use transversality to prove Lemmas 3.2, 3.4, and 3.6. Recall Lemma 3.2 concerns $\widetilde{K}(\lambda, N_1, N_2)$ defined in (3-4). Furthermore, Lemmas 3.4 and 3.6 refer to $K(\lambda, N_1, N_2)$ defined in (3-5).

6A. ***Proof of Lemma 3.2.*** For convenience of notation, we let $\Omega_1 := [0, N_1^2] \times [0, \lambda N_1]^d$. Note that one can use finite overlapped balls of radius N_1^2 to cover Ω_1 since $\lambda \geq N_1$. We want to prove

$$\|Ef_1 Ef_2\|_{L^2_{\mathrm{avg}}(\omega_{\Omega_1})} \lesssim_\epsilon \lambda^{d/2} N_2^\epsilon \frac{N_2^{d-1}}{N_1} \prod_{j=1}^{2} \biggl(\sum_{|\theta|=1/(\lambda N_1)} \|Ef_{j,\theta}\|^2_{L^4_{\mathrm{avg}}(w_{\Omega_1})} \biggr)^{1/2}. \tag{6-1}$$

We first apply Corollary 5.7 with $\delta = N_1^{-2},\ \upsilon = N_2/N_1,\ R = N_1^2$. Note that $\delta \leq \upsilon$. Then we have

$$\begin{aligned}\int |Ef_1 Ef_2|^2 w_{B_{N_1^2}} &\lesssim (N_1N_2)^{d-1} \left| \frac{\log N_1}{\log N_1 - \log N_2} \right|^C \sum_{|\theta_j|=1/N_1^2} \int |Ef_{1,\theta_1} Ef_{2,\theta_2}|^2 w_{B_{N_1^2}} \\ &\lesssim (N_1N_2)^{d-1} N_2^\epsilon \sum_{|\theta_j|=1/N_1^2} \prod_{j=1}^{2} \|Ef_{j,\theta_j}\|^2_{L^4(w_{B_{N_1^2}})}. \end{aligned} \tag{6-2}$$

Remark 6.1. We avoid the case when $N_1 = N_2$, and thus $\ln N_1 - \ln N_2 = 0$, by first decomposing caps of diameter N_2/N_1 into caps of diameter $N_2/(2N_1)$ with loss of a fixed constant, then continuing with the proof as above. In all of what follows, one may assume, without loss of generality, that $N_1 \geq 2N_2$.

Via the principle of parallel decoupling, Lemma 1.9, or by summing different $B_{N_1^2}$ together, we have

$$\int |Ef_1 Ef_2|^2 w_{\Omega_1} \lesssim (N_1N_2)^{d-1} N_2^\epsilon \sum_{|\theta_j|=1/N_1^2} \prod_{j=1}^{2} \|Ef_{j,\theta_j}\|^2_{L^4(w_{\Omega_1})}. \tag{6-3}$$

Next we would like to show that

$$\|Ef_{j,\theta_j}\|^2_{L^4(w_{\Omega_1})} \leq \left(\frac{\lambda}{N_1} \right)^{d/2} \sum_{\substack{\theta_j' \subset \theta_j \\ |\theta_j'|=1/(\lambda N_1)}} \|Ef_{j,\theta_j'}\|^2_{L^4(w_{\Omega_1})}. \tag{6-4}$$

It suffices to show

$$\|Ef_{j,\theta_j}\|^2_{L^4_{\mathrm{avg}}(\Omega_1)} \leq \left(\frac{\lambda}{N_1} \right)^{d/2} \sum_{\substack{\theta_j' \subset \theta_j \\ |\theta_j'|=1/(\lambda N_1)}} \|Ef_{j,\theta_j'}\|^2_{L^4_{\mathrm{avg}}(w_{\Omega_1})}$$

and sum up as in Lemma 4.1.

Each function $Ef_{j,\theta_j'}$ is Fourier-supported in θ_j', in particular, Fourier-supported in a cylinder of radius $1/(\lambda N_1)$, height $1/N_1^2$, and Ω_1 is tiled by cylinders of radius λN_1, height N_1^2 in the t-direction. The proof of Lemma 4.1 works the same:

$$\|Ef_{j,\theta_j}\|^2_{L^2_{\mathrm{avg}}(\Omega_1)} \lesssim \sum_{\substack{\theta_j' \subset \theta_j \\ |\theta_j'|=1/(\lambda N_1)}} \|Ef_{j,\theta_j'}\|^2_{L^2_{\mathrm{avg}}(w_{B_R})} \lesssim \sum_{\substack{\theta_j' \subset \theta_j \\ |\theta_j'|=1/(\lambda N_1)}} \|Ef_{j,\theta_j'}\|^2_{L^4_{\mathrm{avg}}(w_{B_R})}.$$

For the L^∞-estimate, we apply the Cauchy–Schwarz inequality:

$$\|Ef_{j,\theta_j}\|^2_{L^\infty(\Omega_1)} \le \left(\frac{\lambda}{N_1}\right)^d \sum_{\substack{\theta_j'\subset\theta_j\\ |\theta_j'|=1/(\lambda N_1)}} \|Ef_{j,\theta_j'}\|^2_{L^\infty(\Omega_1)} \lesssim \left(\frac{\lambda}{N_1}\right)^d \sum_{\substack{\theta_j'\subset\theta_j\\ |\theta_j'|=1/(\lambda N_1)}} \|Ef_{j,\theta_j'}\|^2_{L^4_{\text{avg}}(w_{\Omega_1})}.$$

The last inequality is an application of Lemma 1.10. Note $f_{\theta_j'}$ is supported in a ball of scale $1/(\lambda N_1)$, and inside a box C of size $(1/N_1^2)\times(1/(\lambda N_1))\times\cdots\times(1/(\lambda N_1))$. We can make a affine transform of C into a cube Q^* of scale λ_{N_1}, which on the physical side would transform Ω_1 into a cube of scale λN_1. We apply Lemma 1.10 after the affine transformation and then transform back. (Note in that setting, cube is no different than a ball.)

We apply Hölder's inequality to conclude the argument.

6B. ***Proof of Lemma 3.4.*** Let $\lambda \le N_1$. We first note that we can use finitely overlapping balls $B_{\lambda N_1}$ to cover Ω and that $N_2 \lesssim 1$. Applying Corollary 5.7 with $\delta = 1/(\lambda N_1)$ and $\upsilon = N_2/N_1$ we have

$$\int |Ef_1 Ef_2|^2 w_{B_{\lambda N_1}} \lesssim (\lambda N_2)^{d-1}\left|\frac{\log\lambda+\log N_1}{\log N_1-\log N_2}\right|^C \sum_{|\theta_j|=1/(\lambda N_1)} \int |Ef_{1,\theta_1} Ef_{2,\theta_2}|^2 w_{B_{\lambda N_1}}$$
$$\lesssim (\lambda N_2)^{d-1} N_2^{\epsilon} \sum_{|\theta_j|=1/(\lambda N_1)} \prod_{j=1}^{2} \|Ef_{j,\theta_j}\|^2_{L^4(w_{B_{\lambda N_1}})}.$$

With parallel decoupling, Lemma 1.9, the desired estimate follows. (As remarked in Remark 6.1, one can assume $N_1 \ge 2N_2$.)

6C. ***Proof of Lemma 3.6.*** Let $\lambda \le N_1$.

We have the following two cases:

- Case 1: $d = 2$, $N_1 \ge \lambda \ge N_1/N_2$, and $N_2' = (N_1/\lambda)$.
- Case 2: $d \ge 3$, $N_1 \ge \lambda \ge N_1/N_2^2$, and $N_2' = (N_1/\lambda)^{1/2}$.

It is easy to check that we only need to show

$$K(\lambda, N_1, N_2) \lesssim K(\lambda, N_1, N_2')\left(\frac{N_1}{N_2'}\frac{N_2}{N_1}\right)^{(d-1)/2}. \tag{6-5}$$

We claim that

$$\|Ef_1 Ef_2\|_{L^4_{\text{avg}}(w_{B_{N_1^2}})} \lesssim \left(\frac{N_1}{N_2'}\frac{N_2}{N_1}\right)^{(d-1)/2} \prod_{j=1}^{2}\left(\sum_{|\theta|=N_2'/(N_1)} \|Ef_{j,\theta}\|^2_{L^4_{\text{avg}}(w_{B_{N_1^2}})}\right)^{1/2}. \tag{6-6}$$

Since $\lambda \le N_1$, we cover $B_{N_1^2}$ with balls of radius λN_1. Thus by parallel decoupling, to prove (6-6), we only need to show

$$\|Ef_1 Ef_2\|_{L^4_{\text{avg}}(w_{B_{\lambda N_1}})} \lesssim \left(\frac{N_1}{N_2'}\frac{N_2}{N_1}\right)^{(d-1)/2} \prod_{j=1}^{2}\left(\sum_{|\theta|=1/(\lambda N_1)} \|Ef_{j,\theta}\|^2_{L^4_{\text{avg}}(w_{B_{\lambda N_1}})}\right)^{1/2}. \tag{6-7}$$

Note that since $\lambda N_1 \geq N_1/N_2'$, estimate (6-7) follows from Corollary 5.7 by setting $\delta = N_2/N_1$, $\upsilon = N_2'/N_1$ via interpolation and local constant arguments as in Section 6A.

By the definition of $K(\lambda, N_1, N_2)$, we have that for any θ_1, θ_2 in (6-7),

$$\|Ef_{1,\theta_1} Ef_{2,\theta_2}\|_{L^4_{\mathrm{avg}}(w_{B_{\lambda N_1}})} \lesssim \lambda^{d/2} K(\lambda, N_1, N_2') \prod_{j=1}^{2} \Bigg(\sum_{\substack{|\theta_j'|=1/(\lambda N_1)\\ \theta_j \subset \theta_j}} \|Ef_{j,\theta_j'}\|^2_{L^4_{\mathrm{avg}}(w_\Omega)} \Bigg)^{1/2}. \tag{6-8}$$

Plugging (6-8) into (6-7), clearly (6-5) follows.

7. Induction procedure and proof of Lemma 3.3

To conclude the proof of Proposition 3.1, we are left with the proof of Lemma 3.3. For this lemma the proof relies on induction on N_2. The base case $N_2 \lesssim 1$ is resolved by Lemma 3.4, and by Lemma 3.6 we need only to induct until $\lambda = (N_2)^{d-1}/N_1$.

Let f_1, f_2 be as in Lemma 3.3. Applying Lemma 5.1, taking $v = N_1/N_2$ and $R = N_1^2$, we can decouple the N_2/N_1 caps into $(N_2/N_1, N_2^2/N_1^2)$ plates without any loss; i.e.,

$$\int |Ef_1 Ef_2|^2 w_{B_{N_1^2}} \lesssim \sum_{\tau_1,\tau_2} \int |Ef_{1,\tau_1} Ef_{2,\tau_2}|^2 w_{B^2_{N_1}}. \tag{7-1}$$

Here τ_i are plates as described in Lemma 5.1. We focus on the case when $d = 2$ in $\mathbb{R}^3$; the high-dimensional case will be explained in the end. When $d = 2$, the underlying plates become strips. We start with some preparation before the induction.

7A. *Preliminary preparation for the induction.*

We fix a pair of $(N_2/N_1, N_2^2/N_1^2)$ strips τ_1, τ_2 from estimate (7-1). We decompose τ_j into a union of $(N_2/(KN_1)) \times (N_2^2/N_1^2)$ strips $\{s_j\}$.

Using the notation "nonadj" short for nonadjacent, and "adj" short for adjacent, we have

$$\begin{aligned} |Ef_{\tau_j}|^2 &= \sum_{s_j} |Ef_{s_j}|^2 + \sum_{s_j, s_j' \text{adj}} |Ef_{s_j} Ef_{s_j'}| + \sum_{s_j, s_j' \text{nonadj}} |Ef_{s_j} Ef_{s_j'}| \\ &\leq 10 \sum_{s_j} |Ef_{s_j}|^2 + \sum_{s_j, s_j' \text{nonadj}} |Ef_{s_j} Ef_{s_j'}| = I_{j,1} + I_{j,2} \end{aligned} \tag{7-2}$$

and

$$\begin{aligned} \int |Ef_{\tau_1} Ef_{\tau_2}|^2 w_{B_{N_1^2}} &\leq \int \big|(Ef_{\tau_1}^2 - I_{1,1})(Ef_{\tau_2}^2 - I_{2,1})\big| + Ef_{\tau_1}^2 I_{2,1} + Ef_{\tau_2}^2 I_{1,1} + I_{1,1} I_{2,1} w_{B_{N_1^2}} \\ &\lesssim \sum_{s_j, s_j' \text{nonadj}} \int |Ef_{s_1} Ef_{s_1'} Ef_{s_2} Ef_{s_2'}| w_{B_{N_1^2}} + \sum_{s_1, s_2} \int |Ef_{s_1} Ef_{s_2}|^2 w_{B_{N_1^2}}. \end{aligned} \tag{7-3}$$

The last inequality follows from Lemmas 4.1 and 4.2.

The reason why we want to have nonadjacent parts is that we would like transversality (after rescaling) on the other direction. Formula (7-3) will the starting point of our induction.

For the second term in (7-3), we will later directly use induction (not relying on parallel rescaling) on N_2 and reduce everything to the known base case $N_2 = 1$.

For the first term, using Cauchy–Schwarz

$$\int |Ef_{s_1} Ef_{s_1'} Ef_{s_2} Ef_{s_2'}| w_{B_{N_1^2}} \le \left(\int |Ef_{s_1} Ef_{s_1'}|^2 w_{B_{N_1^2}} \right)^{1/2} \left(\int |Ef_{s_2} Ef_{s_2'}|^2 w_{B_{N_1^2}} \right)^{1/2}. \tag{7-4}$$

We point out here that in what follows we do not rely on the bilinear transversality between s_1 and s_2 (or s_1 and s_2'), which is already handled in Lemma 5.1. Instead we will rely on the bilinear transversality between s_1 and s_1' (or s_2, s_2'), since they are not adjacent. This transversality is most clear when one applies parabolic rescaling.

Let us now turn to the term $\int |Ef_{s_2} Ef_{s_2'}|^2 w_\Omega$, when s_2, s_2' are nonadjacent. The term with s_1, s_1' is handled similarly, though one may need to rotate the coordinates.

Finally we point out here that K will be chosen large later and any (fixed) power of K will not impact the final estimate. In particular, in the following estimates we will not worry about losing powers of K.

Without loss of generality, we assume

- s_2 is the strip $\{(a_1, a_2, a_1^2 + a_2^2) : |a_1| \le N_2^2/N_1^2,\ |a_2| \le N_2/KN_1\}$,
- s_2' is the strip $\{(b_1, b_2, b_1^2 + b_2^2) : |b_1| \le N_2^2/N_1^2,\ |b_2 - CN_2/KN_1| \le N_2/KN_1\},\ C \ge 10$. (Here 10 is of course just some universal constant.)

7B. *Parabolic rescaling.* The next step, parabolic scaling, is standard in decoupling-type results; we give the details here for the convenience of the reader.

Note s_2, s_2' lie on the same N_2/N_1 cap. We rescale the N_2/N_1 cap to radius 1. By a slight abuse of notation, we regard f_{s_i} as a function depending only on two variables $(\xi_{i,1}, \xi_{i,2})$. For convenience of notation, we let $h_1 = f_{s_2},\ h_2 = f_{s_2'}$. Let also $g_i(\eta_{i,1}, \eta_{i,2}) := h_i\big((N_2/N_1)\eta_{i,1}, (N_2/N_1)\eta_{i,2}\big)$.

Now

- g_1 is supported in the strip $\{(a_1, a_2, a_1^2 + a_2^2) : |a_1| \le N_2/N_1,\ |a_2| \le 1/K\}$,
- g_2 is supported in the strip $\{(b_1, b_2, b_1^2 + b_2^2) : |b_1| \le N_2/N_1,\ |b_2 - C/K| \le 1/K\},\ C \ge 10$.

Note g_1, g_2 are supported on a pair of transverse $(N_2/N_1) \times 1$ strips[1] due to the nonadjacency of s_2, s_2'. We point out here the transversality between g_1, g_2 is not as in the assumption of Lemma 5.1, but it is in the sense of Remark 5.3, which usually causes a loss of K in the estimate, but this does not matter.

The *parabolic scaling* says the following:

Claim 7.1. *Let*

$$Eg_i(y_1, y_2, y_3) = Eh_i((N_1/N_2)y_1, (N_1/N_2)y_2, (N_1^2/N_2^2)y_3),$$

let D be domain in $\mathbb{R}^3$ and let

$$\widetilde{D} := \{(y_1, y_2, y_3) : (N_1/N_2)y_1, (N_1/N_2)y_2, (N_1^2/N_2^2)y_3 \in D\}.$$

[1] Strictly speaking, we need them to support on a pair of $(N_2/N_1) \times (1/100)$ strips; we neglect this technical point here.

Then it follows from a standard change of variables technique that the following two estimates, with the same constant A, are equivalent:

$$\|Eh_1 Eh_2\|_{L^2_{\text{avg}}(w_D)} \lesssim A \prod_{j=1}^{2} \biggl(\sum_{|\theta|=1/(\lambda N_1)} \|Ef_{s_j,\theta}\|^2_{L^4_{\text{avg}}(w_D)} \biggr)^{1/2}, \tag{7-5}$$

$$\|Eg_1 Eg_2\|_{L^2_{\text{avg}}(w_{\widetilde{D}})} \lesssim A \prod_{j=1}^{2} \biggl(\sum_{|\tilde{\theta}|=1/(\lambda N_2)} \|Eg_{j,\tilde{\theta}}\|^2_{L^4_{\text{avg}}(w_{\widetilde{D}})} \biggr)^{1/2}. \tag{7-6}$$

We then concentrate on (7-6).

Take $D = B_{N_1^2}$; then $\widetilde{D} = [0, N_2^2] \times [0, N_1 N_2]^2$. (Here, without loss of generality, we regard $B_{N_1^2}$ as $[0, N_1^2]^3$.) For convenience of notation, we set $\widetilde{\Omega} = [0, N_2^2] \times [0, N_1 N_2]^2$. The parabolic rescaling gives:

Lemma 7.2. *Assume g_1, g_2 are two general functions defined on the parabola. Let g_1 be supported in a strip of size $(N_2/N_1) \times 1$ around $(0, 0, 0)$, and g_2 be supported in a strip of size $(N_2/N_1) \times 1$ around $(0, 1, 1)$. If for some constant A, one has* (*for all such g_1, g_2*)

$$\|Eg_1 Eg_2\|_{L^2_{\text{avg}}(w_{\widetilde{\Omega}})} \lesssim A \biggl(\sum_{|\tilde{\theta}|=1/(\lambda N_2)} \|Eg_{j,\tilde{\theta}}\|^2_{L^4_{\text{avg}}(w_{\widehat{\Omega}})} \biggr)^{1/2}, \tag{7-7}$$

then for the same constant A, one has

$$\begin{aligned} &\|Ef_{s_2} Ef_{s_2'}\|_{L^2_{\text{avg}}(w_{B_{N_1^2}})} \\ &\quad\lesssim K^C A \biggl(\sum_{|\theta|=1/(\lambda N_1)} \|Ef_{s_2,\theta}\|^2_{L^4_{\text{avg}}(w_{B_{N_1^2}})} \biggr)^{1/2} \biggl(\sum_{|\theta|=1/(\lambda N_1)} \|Ef_{s_2',\theta}\|^2_{L^4_{\text{avg}}(w_{B_{N_1^2}})} \biggr)^{1/2}. \end{aligned} \tag{7-8}$$

Remark 7.3. After rescaling, the relevant g_1, g_2 should be supported around $(0, 0, 0)$ and $(0, 1/K, 1/K^2)$ rather than $(0, 0, 0)$ and $(0, 0, 1)$. We state our lemma for g_1, g_2 supported around $(0, 0, 0)$ and $(0, 1, 1)$ to be consistent with the statement in Lemma 5.1. This causes a loss of K^C, but we emphasize again that any loss due to a power of K would be irrelevant in the proof.

We end this section by introducing some notation.

Let g_1, g_2 be as in Lemma 7.2. We define $A(\lambda, N_1, N_2)$ to be the best constant such that

$$\|Eg_1 Eg_2\|_{L^2_{\text{avg}}(w_{\widetilde{\Omega}})} \lesssim A(\lambda, N_1, N_2) \biggl(\sum_{|\tilde{\theta}|=1/(\lambda N_2)} \|Eg_{j,\tilde{\theta}}\|^2_{L^4_{\text{avg}}(w_{\widehat{\Omega}})} \biggr)^{1/2}. \tag{7-9}$$

Then we can restate Lemma 7.2.

Lemma 7.4. *For $j = 1, 2$, we have*

$$\begin{aligned} &\|Ef_{s_j} Ef_{s_j'}\|_{L^2_{\text{avg}}(w_{B_{N_1^2}})} \\ &\quad\lesssim K^C A(\lambda, N_1, N_2) \biggl(\sum_{|\theta|=1/(\lambda N_1)} \|Ef_{s_j,\theta}\|^2_{L^4_{\text{avg}}(w_{B_{N_1^2}})} \biggr)^{1/2} \biggl(\sum_{|\theta|=1/(\lambda N_1)} \|Ef_{s_j',\theta}\|^2_{L^4_{\text{avg}}(w_{B_{N_1^2}})} \biggr)^{1/2}. \end{aligned} \tag{7-10}$$

7C. ***The induction procedure.***

7C1. *Before induction.* Now we are ready to start the induction for the proof of Lemma 3.3. We emphasize here the induction is on N_2 (though mixed with induction on K). Note we are now in dimension $d = 2$.

We need to show that for all $1 \le N_2 \le N_1$ and $\lambda \le N_1$, one has

$$K(\lambda, N_1, N_2) \lesssim N_2^{\epsilon}\left(\frac{1}{\lambda} + \frac{N_2}{N_1}\right)^{1/2}.$$

Note the base case $N_2 = 1$ is already established in Lemma 3.4. And with Lemma 3.6, we need only to perform induction until $\lambda = N_2/N_1$.

We will work on $A(\lambda, N_1, N_2)$ defined in (7-9) to explore the transversality between nonadjacent strips. The induction process is two-fold in some sense. We will induct on N_2 to better understand $K(\lambda, N_1, N_2)$. In turn we find more information about $A(\lambda, N_1, N_2)$, which gives a better understanding of $K(\lambda, N_1, N_2)$.

This is a final summary before we start the induction. Recall, we have (7-1) and (7-3); thus we have

$$\int |Ef_1 Ef_2|^2 w_{B_{N_1^2}} \lesssim \int \sum_{s_j, s_j' \text{ nonadj}} \int |Ef_{s_1} Ef_{s_1'} Ef_{s_2} Ef_{s_2'}| w_{B_{N_1^2}} + \int_{s_1, s_2} \int |Ef_{s_1} Ef_{s_2}| w_{B_{N_1^2}}. \tag{7-11}$$

Also recall that s_1, s_1', s_2, s_2' are all $(N_2/N_1)^2 \times (N_2/KN_1)$ strips. The second term can be easily handled by direct induction, (which is not the main point of the induction procedure explained later). Indeed, if there were only the second term in (7-11), since s_1, s_2 are both contained in caps of radius (N_2/KN_1), then (7-11) already reduces the decoupling problem for f_i supported in caps of size N_2/N_1 into the decoupling problem for f_i supported in caps of size N_2/KN_1, which reduce N_2 to N_2/K.

We will focus on the first term of (7-11). The Hölder inequality gives

$$\int |Ef_{s_1} Ef_{s_1'} Ef_{s_2} Ef_{s_2'}| w_{B_{N_1^2}} \le \prod_{j=1}^{2} \left(\int |Ef_{s_j} Ef_{s_j'}|^2 w_{B_{N_1^2}}\right)^{1/2}. \tag{7-12}$$

Estimate (7-12) is the starting point of the analysis in the following subsections.

We summarize in the lemma below how (7-12) and (7-11) come together to highlight the relevance of $A(N_1, N_2, \lambda)$ in the induction procedure.

Lemma 7.5. *When $\lambda \le N_1/N_2$ and $\lambda \le N_1$, we have*

$$K(N_1, N_2, \lambda) \lesssim K^C \frac{1}{\lambda} A(N_1, N_2, \lambda) + K(N_1, N_2/K, \lambda). \tag{7-13}$$

Note that the assumption of Lemma 7.5 always holds during the induction procedure to prove Lemma 3.3.

Proof of Lemma 7.5. Applying Lemma 7.4, we have

$$\|Ef_{s_j} Ef_{s_j'}\|_{L^2_{\text{avg}}(w_{B_{N_1^2}})} \lesssim K^C A(N_1, N_2, \lambda) \left(\sum_{|\theta| = 1/(\lambda N_1)} \|Ef_{s_j,\theta}\|^2_{L^4_{\text{avg}}(w_{B_{N_1^2}})}\right)^{1/2} \left(\sum_{|\theta| = 1/(\lambda N_1)} \|Ef_{s_j',\theta}\|^2_{L^4_{\text{avg}}(w_{B_{N_1^2}})}\right)^{1/2}. \tag{7-14}$$

Plugging (7-14) into (7-12), and then plugging into (7-11), we derive

$$\|Ef_1 Ef_2\|_{l^2(w_{B_{N_1^2}})} \lesssim K^C \lambda \left(\frac{1}{\lambda}\right)^{1/2} \prod_{i=1}^{2} \left(\sum_{|\theta|=1/(\lambda N_1)} \|Ef_{i,\theta}\|^2_{L^4_{\text{avg}}(w_{B_{N_1^2}})} \right)^{1/2} + \left(\sum_{|\theta|=N_2/(\lambda K N_1)} \|Ef_{i,\theta}\|^2_{L^4_{\text{avg}}(w_{B_{N_1^2}})} \right)^{1/2}. \tag{7-15}$$

Thus we derive

$$\lambda K(N_1, N_2, \lambda) \lesssim K^C A(N_1, N_2, \lambda) + \lambda K(N_1, N_2/K, \lambda). \tag{7-16}$$

Therefore, Lemma 7.5 follows. □

Now we are ready to start with the induction procedure on N_2. We emphasize again that by Lemma 3.6 we only need to consider the case $\lambda \le N_1/N_2$.

7C2. *First induction: Case $N_2^2 \le N_1$.* It will become clear in the following proof why we choose the first splitting point at $N_1 = N_2^2$. We start with an estimate for $A(\lambda, N_1, N_2)$. We have:

Lemma 7.6. *When $N_2 \le N_1^2$, $\lambda \le N_1$, $\lambda \le N_1/N_2$,*

$$A(\lambda, N_1, N_2) \lesssim \lambda^{1/2} \equiv \lambda\lambda^{-1/2}. \tag{7-17}$$

Assuming Lemma 7.6 for the moment, let us finish the proof of Lemma 3.3 when $N_1 \ge N_2^2$. Applying Lemma 7.6 with Lemma 7.5, we derive

$$K(N_1, N_2, \lambda) \lesssim K^C \lambda \left(\frac{1}{\lambda}\right)^{1/2} + K(N_1, N_2/K, \lambda) \tag{7-18}$$

when $N_1 \ge N_2^2$ and $\lambda \le N_1/N_2$. Choosing $1 \ll K \sim N_2^{\epsilon^{10}}$, performing induction on N_2 again, and recalling that the case $N_2 \lesssim 1$ is covered by Lemma 3.4, we have Lemma 3.3 follows when $N_1 \ge N_2^2$.

Now, we turn to the proof of Lemma 7.6.

Proof of Lemma 7.6. Since $N_1 \le N_2^2$, we have $N_2/N_1 \le 1/N_2$. (It is exactly because of this that we decided our first splitting point $N_1 \le N_2^2$). Thus, the support of g_1, g_2 appearing in (7-9) is (contained in) strips of size $(1/N_2) \times 1$. Thus, in a ball of radius N_2^2, we have

$$\int |Eg_1 Eg_2| w_{B_{N_2^2}} \lesssim \sum_{\substack{|\theta_i|=1/N_2 \\ \theta_i \subset \operatorname{supp} g_i}} \int |E_{g_{1,\theta_1}} Eg_{2,\theta_2}| w_{B_{N_2^2}}. \tag{7-19}$$

The proof of (7-19) is essentially the same as the proof of Lemma 5.1 and we leave it to reader.

Note one can use balls $B_{N_2^2}$ to cover $\widetilde{\Omega} := [0, N_2^2] \times [0, N_1 N_2]^2$ (since $N_1 \ge N_2$), thus we extend (7-19) to

$$\int |Eg_1 Eg_2| w_{\widetilde{\Omega}} \lesssim \sum_{\substack{|\theta_i|=1/N_2 \\ \theta_i \subset \operatorname{supp} g_i}} \int |E_{g_{1,\theta_1}} Eg_{2,\theta_2}| w_{\widetilde{\Omega}}. \tag{7-20}$$

We claim for any fixed θ_1, θ_2, one has

$$\|Eg_{1,\theta_1} Eg_{2,\theta_2}\|_{L^2(w_{\widetilde{\Omega}})} \lesssim \lambda\lambda^{-1/2} \prod_{i=1}^{2} \Bigg(\sum_{\substack{\tilde{\theta}_i \subset \theta_i \\ \|\tilde{\theta}_i = 1/(\lambda N_2)\|}} \|Eg_{i,\tilde{\theta}_i}\|_{L^4(w_{\widetilde{\Omega}})} \Bigg)^{1/2}. \tag{7-21}$$

Plugging (7-21) into (7-20), we have

$$A(N_1, N_2, \lambda) \lesssim \lambda \left(\frac{1}{\lambda}\right)^{1/2}, \tag{7-22}$$

and then Lemma 7.6 follows.

Now we are left with the proof of (7-21). Let $N_1' = N_2,\ N_2' = N_2^2/N_1 \lesssim 1$. When $N_1' = N_2 \leq \lambda$, recall the definition of $\widetilde{K}(\lambda, N_1, N_2)$ in (3-4) and apply Lemma 3.2. Then we have

$$K(N_1', N_2', \lambda) \lesssim (N_2')^{\epsilon} \left(\frac{N_2'}{N_1'}\right)^{1/2} \lesssim \left(\frac{1}{\lambda} + \frac{N_2}{N_1}\right)^{1/2} \lesssim \lambda^{-1/2}. \tag{7-23}$$

The last inequality in (7-23) follows because we always have $\lambda \leq N_1/N_2$ in the whole induction process. Note (7-23) implies

$$\begin{aligned} &\|Eg_{1,\theta_1} Eg_{2,\theta_2}\|_{L^2(w_{[0,N_2^2]\times[0,\lambda N_2]^2})} \\ &\qquad \lesssim \lambda \widetilde{K}(N_1', N_2', \lambda) \prod_{i=1}^{2} \Bigg(\sum_{\substack{\tilde{\theta}_i \subset \theta_i \\ \|\tilde{\theta}_i = 1/(\lambda N_2)\|}} \|Eg_{i,\tilde{\theta}_i}\|_{L^4(w_{[0,N_2^2]\times[0,\lambda N_2]^2})} \Bigg)^{1/2}. \end{aligned} \tag{7-24}$$

Since $\lambda \leq N_1$, (which is also always the case during the induction process), $\widetilde{\Omega}$ can be covered by the translations of $[0, N_2^2] \times [0, \lambda N_2]$; thus (7-24) implies (7-21) by parallel decoupling, Lemma 1.9.

When $\lambda \leq N_1'$, since $N_2' \lesssim 1$, by Lemma 3.4, we have

$$K(\lambda, N_1', N_2') \lesssim \lambda^{-1/2}. \tag{7-25}$$

Thus,

$$\|Eg_{1,\theta_1} Eg_{2,\theta_2}\|_{L^2(w_{B_{N_2^2}})} \lesssim \lambda\lambda^{-1/2} \prod_{i=1}^{2} \Bigg(\sum_{\substack{\tilde{\theta}_i \subset \theta_i \\ \|\tilde{\theta}_i = 1/(\lambda N_2)\|}} \|Eg_{i,\tilde{\theta}_i}\|_{L^4(w_{B_{N_2^2}})} \Bigg)^{1/2}. \tag{7-26}$$

Since one can use $B_{N_2^2}$ and its translations to cover $\widetilde{\Omega}$, (7-26) implies (7-21) by parallel decoupling, Lemma 1.9. □

7C3. *Second induction: Case $N_2^{3/2} \leq N_1 \leq N_2^2$.*

Lemma 7.7. *When $N_2^{3/2} \leq N_1 \leq N_2^2$, $\lambda \leq N_1$ and $\lambda \leq N_1/N_2$, we have*

$$A(\lambda, N_1, N_2) \lesssim \lambda^{1/2} \equiv \lambda\lambda^{-1/2}. \tag{7-27}$$

Clearly, using Lemma 7.5 and arguing as in Section 7C2, Lemma 3.3 follows from Lemma 7.7 when $N_2^{3/2} \leq N_1 \leq N_2^2$.

Now we are left with proof of Lemma 7.7, i.e., the estimate (7-27). We will prove that estimate (7-27), in the case $N_2^{3/2} \le N_1 \le N_2^2$, follows from the fact that Lemma 3.3 holds when $N_2^2 \ge N_1$ (given Lemma 3.2).

Proof of Lemma 7.7. The proof starts similarly to the proof of Lemma 7.6; note now we have $N_2/N_1 \ge 1/N_2$. As we derived in (7-19), we have in a ball of radius N_1^2/N_2^2,

$$\int |Eg_1 Eg_2| w_{B_{(N_1/N_2)^2}} \lesssim \sum_{\substack{|\theta_i|=N_2/N_1 \\ \theta_i \subset \operatorname{supp} g_i}} \int |E_{g_{1,\theta_1}} Eg_{2,\theta_2}| w_{B_{(N_1/N_2)^2}}. \tag{7-28}$$

Note one can use $B_{(N_1/N_2)^2}$ and its translations to cover $\widetilde{\Omega}$; thus we have

$$\int |Eg_1 Eg_2| w_{\widetilde{\Omega}} \lesssim \sum_{\substack{|\theta_i|=N_2/N_1 \\ \theta_i \subset \operatorname{supp} g_i}} \int |E_{g_{1,\theta_1}} Eg_{2,\theta_2}| w_{\widetilde{\Omega}}. \tag{7-29}$$

The following procedure is essentially the same as in the first induction. Note that to prove (7-27) we only need to further show that for fixed θ_1, θ_2,

$$\|Eg_{1,\theta_1} Eg_{2,\theta_2}\|_{L^2(w_{\widetilde{\Omega}})} \lesssim \lambda\lambda^{-1/2} \prod_{i=1}^{2} \biggl(\sum_{\substack{\tilde{\theta}_i \subset \theta_i \\ |\tilde{\theta}_i|=1/(\lambda N_2)}} \|Eg_{i,\tilde{\theta}_i}\|_{L^4(w_{\widetilde{\Omega}})} \biggr)^{1/2}, \tag{7-30}$$

where now $|\theta_i| = N_2/N_1$.

Let $N_1' = N_2$ and $N_2' = N_2^2/N_1$; note we have $N_1' \ge (N_2')^2$ since $N_1 \ge N_2^{3/2}$. When $\lambda \ge N_1'$, we have by Lemma 3.2

$$\begin{aligned} \|Eg_{1,\theta_1} Eg_{2,\theta_2}\|_{L^2(w_{[0,N_2^2]\times[0,\lambda N_2]^2})} & \\ \lesssim \lambda \Bigl(\frac{N_2'}{N_1'}\Bigr)^{-1/2} & \prod_{i=1}^{2} \biggl(\sum_{\substack{\tilde{\theta}_i \subset \theta_i \\ |\tilde{\theta}_i|=1/(\lambda N_2)}} \|Eg_{i,\tilde{\theta}_i}\|_{L^4(w_{[0,N_2]\times[0,\lambda N_2]^2})} \biggr)^{1/2}. \end{aligned} \tag{7-31}$$

Since one can use $[0, N_2^2] \times [0, \lambda N_2]^2$ to cover $\widetilde{\Omega}$, (7-30) follows from (7-31) (note that $N_2'/N_1' = N_2/N_1 \le \lambda^{-1}$).

When $\lambda \le N_1'$, since one can use $B_{N_2^2}$ to cover $\widetilde{\Omega}$, to prove (7-30), we need only show

$$\|Eg_{1,\theta_1} Eg_{2,\theta_2}\|_{L^2(w_{B_{N_2^2}})} \lesssim \lambda\lambda^{-1/2} \prod_{i=1}^{2} \biggl(\sum_{\substack{\tilde{\theta}_i \subset \theta_i \\ \|\tilde{\theta}_i=1/(\lambda N_2)\|}} \|Eg_{i,\tilde{\theta}_i}\|_{L^4(w_{B_{N_2^2}})} \biggr)^{1/2}, \tag{7-32}$$

which is equivalent to $K(N_1', N_2', \lambda) \le 1/\lambda$. But recall that $N_1' \ge (N_2')^2$, thus this is exactly what we proved in first induction; i.e., Lemma 3.3 holds when $N_1 \ge N_2^2$. □

7C4. *Later inductions and the conclusion of the induction process.* Recall that the *first induction* covers the case $N_1 \ge N_2^2$ and the *second inductions* covers the case $N_2^\alpha \le N_1 \le N_2^2$, $\alpha = \frac{3}{2}$. The goal now is to use induction to cover the case $N_2^\alpha \le N_1$, all the way to $\alpha = 1$. The arguments here are similar to those for the *second induction* presented in Section 7C3. Let $N_1' = N_2$, $N_2' = N_2^2/N_1$; then $N_1' \ge (N_2')^\alpha$ is equivalent to $N_1 \ge N_2^{(2\alpha-1)/\alpha}$. Once we show that Lemma 3.3 holds when $N_2^\alpha \le N_1 \le N_2^2$, we will be able to extend Lemma 7.7 to the case when $N_2^{(2\alpha-1)/\alpha} \le N_1$, which in turn proves that Lemma 3.3 holds when $N_2^{(2\alpha-1)/\alpha} \le N_1 \le N_2^2$. The induction would not end until $\alpha = 1$. We finally point out that only an induction with finite steps is involved.

To show Lemma 3.3 for a fixed ϵ_0, we may pick an $\tilde{\epsilon} \ll \epsilon_0$, and then perform the induction for $\tilde{\epsilon}$ as above.

After we prove Lemma 3.3 for $N_1 \ge N_2^{1+\tilde{\epsilon}}$, we are left with the case $N_1 \le N_2^{1+\tilde{\epsilon}}$. We first use the Hölder inequality to shrink the size of the cap from N_2/N_1 to $N_2^{1-2\tilde{\epsilon}}/N_1$, which only gives a loss of $N_2^{C\tilde{\epsilon}} \ll N_2^{\epsilon_0}$. Then we use Lemma 3.3 in the case $N_1 \ge N_2^{1+\tilde{\epsilon}}$ again.

Thus, Lemma 3.3 holds for all the cases for our fixed ϵ_0.

7D. ***The high-dimension case.*** To handle the case $d \ge 3$, we are left with the proof of Lemma 3.5. The proof is indeed similar to previous arguments in this section and easier. The proof relies on the linear decoupling estimate in [Bourgain and Demeter 2015].

As mentioned earlier, applying Lemma 5.1, taking $v = N_2/N_1$ and $R = N_1^2$, we can decouple the N_2/N_1 caps into $(N_2/N_1, N_2^2/N_1^2)$ plates without any loss, i.e., (7-1). However, since we are in the case $\lambda \le N_1/N_2^2$, indeed $N_2^2/N_1^2 \lesssim 1/(\lambda N_1)$, we only need a weaker version of (7-1); i.e., we only want to decouple the N_2/N_1 caps into $(N_2/N_1, 1/(\lambda N_1))$ plates:

$$\int |Ef_1 Ef_2|^2 w_{B_{N_1^2}} \lesssim \sum_{\tau_1,\tau_2} \int |Ef_{1,\tau_1} Ef_{2,\tau_2}|^2 w_{B^2_{N_1}}. \tag{7-33}$$

Here τ_i are $(N_2/N_1, 1/(\lambda N_1))$ plates as described in Lemma 5.1. Note (7-33) follows from (7-1).

Now, for each τ_i fixed, we further decouple τ_i into $(1/N_1, 1/(\lambda N_1))$ plates via linear decoupling in [Bourgain and Demeter 2015], here recalled in (1-11). Note direct application of linear decoupling in dimension d gives us

$$\|Ef_{\tau_i}\|_{L^4(w_{B_{N_1^2}})} \lesssim N_2^{\epsilon} (N_2^2)^{d/4-(d+2)/8} \biggl(\sum_{v_i \subset \tau_i} \|Ef_{v_i}\|^2_{L^4(w_{B_{N_1^2}})} \biggr)^{1/2}. \tag{7-34}$$

However, we are able to use (1-11) when the dimension is $d-1$ rather than d, because our plates are so thin (of scale $1/(\lambda N_1) \le 1/N_1$), which reduces the dimension by 1. Indeed, linear decoupling (1-11) not only works for those functions which are exactly supported in parabola P but also those which are supported in an N_1^{-2} neighborhood of P. This is consistent with the uncertainty principle, since in physical space we are of scale N_1^2, and in frequency space any scale of N_1^{-2} cannot be differentiated. Since our plates are so thin, of scale $1/(\lambda N_1) \le N_1^{-2}$, one could indeed view them as N_1^{-2} neighborhoods of some $(d-1)$-dimensional parabola. To be more specific, use τ_2 as example, since τ_2 is supported at the origin. Let $\pi_t^{-1}(\tau_2)$ be the pull back image of τ_2 to the paraboloid. The Fourier inverse transform of Ef_{τ_2} is supported on $\pi_t^{-1}(\tau_2)$. One can see that if we project along the x_1-axis, the projection image of $\pi_t^{-1}(\tau_2)$ is the $(1/(\lambda N_1))^2$-neighborhood of a $(d-1)$-dimensional paraboloid (a piece of length N_2/N_1).

Now, applying $(d-1)$-dimensional linear decoupling, we improve (7-34) into

$$\|Ef_{\tau_i}\|_{L^4(w_{B_{N_1^2}})} \lesssim N_2^{\epsilon} (N_2^2)^{(d-1)/4-(d+1)/8} \biggl(\sum_{v_i \subset \tau_i} \|Ef_{v_i}\|^2_{L^4(w_{B_{N_1^2}})} \biggr)^{1/2}, \tag{7-35}$$

where v_i are $(1/N_1, 1/(\lambda N_1))$ plates.

Finally, similarly to the derivation of (6-4), we decouple v_i into caps of radius $1/(\lambda N_1)$,

$$\|Ef_{v_i}\|^4_{L^4(w_{B_{N_1^2}})} \lesssim \lambda^{(d-1)}\biggl(\sum_{\theta_i\subset v_i}\|Ef_{\theta_i}\|^2_{L^4(w_{B_{N_1^2}})}\biggr)^2. \tag{7-36}$$

We remark that each v_i can be covered by λ^{d-1} rather than λ^d caps of radius $1/(\lambda N_1)$. Plugging (7-36) into (7-35), then plugging it into (7-33), we derive

$$\|Ef_1Ef_2\|_{L^2_{\mathrm{avg}}(w_{B_{N_1^2}})} \le \lambda^{d-1/2}N_2^{(d-3)/2}\prod_{j=1}^2\biggl(\sum_{|\theta|=1/(\lambda N_1)}\|Ef_{j,\theta}\|^2_{L^4_{\mathrm{avg}}(w_{B_{N_1^2}})}\biggr)^{1/2}. \tag{7-37}$$

Thus, the desired estimate for $K(\lambda, N_1, N_2)$ follows.

Appendix: Sharpness of Theorems 1.1 and 1.3

The sharpness (up to N_2^{ϵ}) of Theorem 1.3 is provided by the following examples. One can also rescale those examples to show the sharpness of Theorem 1.1.

We take

$$Ef_1 = \sum_{\substack{\xi\in\Lambda_{\lambda N_1}\\ |\xi|\le N_2/N_1}} e^{2\pi i(\xi\cdot x+|\xi|^2t)}$$

and $f_2 = f_1(\cdot-(1,0,\dots,0))$. Then $|Ef_1|$ is about $(\lambda N_2)^d$ at $B(0, N_1/N_2)$ in $\mathbb{R}^{d+1}$. Note that it follows from the uncertainty principle that it is locally constant in any ball of size N_1/N_2 and one can easily compute $|Ef_1(0)| \sim (\lambda N_2)^d$. Also note $|Ef_1|$ has periodicity around λN_1 in all components of x (not necessarily in t). The same is true for $|Ef_2|$. Thus,

$$\|Ef_1Ef_2\|^2_{L^2(w_\Omega)} \gtrsim (\lambda N_2)^{4d}\left|B\left(0,\frac{N_1}{N_2}\right)\right|(\lambda N_1)^d \gtrsim \lambda^{5d}N_1^{2d+1}N_2^{3d-1}.$$

Each cap θ_j of radius $1/(\lambda N_1)$ contains at most one point $\xi\in\Lambda_{\lambda N_1}$. Hence $\|Ef_{j,\theta_j}\|^4_{L^4(w_\Omega)} \lesssim |\Omega| = N_1^2(\lambda N_1)^{2d}$:

$$\Pi_{j=1}^2\biggl(\sum_{|\theta_j|=1/(\lambda N_1)}\|Ef_{j,\theta_j}\|^2_{L^4(w_\Omega)}\biggr) \lesssim (\lambda N_2)^{2d}N_1^2(\lambda N_1)^{2d} \lesssim \lambda^{4d}N_1^{2d+2}N_2^{2d}.$$

This example shows that the term with N_2^{d-1}/N_1 is sharp for both $d=2$ and $d\ge 3$.

When $d=2$, we consider the example when

$$Ef_1 = \sum_{\substack{\xi\in\Lambda_{\lambda N_1}\\ \xi_1=1,\,|\xi_2|\le 1/N_1}} e^{2\pi i(\xi\cdot x+|\xi|^2t)},\quad Ef_2 = \sum_{\substack{\xi\in\Lambda_{\lambda N_1}\\ \xi_1=0\,|\xi_2|\le 1/N_1}} e^{2\pi i(\xi\cdot x+|\xi|^2t)}.$$

$|Ef_1|$ is about λ in the box of height N_1^2 (i.e., the t-direction), width N_1 (i.e., the-x_2 direction) and length $(\lambda N_1)^2$ (i.e., the x_1-direction) centered at origin. $|Ef_2|$ is the same size in the same box. Moreover,

Ef_1 and Ef_2 both have periodicity around λN_1 in x_2:

$$\|Ef_1 Ef_2\|^2_{L^2(w_\Omega)} \gtrsim \lambda^4 N_1^2 \cdot N_1 \cdot (\lambda N_1)^2 \cdot \lambda N_1 \gtrsim \lambda^7 N_1^6.$$

As calculated previously, $\|Ef_{j,\theta_j}\|^4_{L^4(w_\Omega)} = |\Omega|$, so

$$\Pi_{j=1}^2 \Bigl(\sum_{|\theta_j|=1/(\lambda N_1)} \|Ef_{j,\theta_j}\|^2_{L^4(w_\Omega)} \Bigr) \lesssim \lambda^2 \cdot |\Omega| \lesssim \lambda^6 N_1^6.$$

This example shows that when $d = 2$, the term with $1/\lambda$ is sharp.

When $d \geq 3$, we consider the example with

$$Ef_1 = \sum_{\substack{\xi \in \Lambda_{\lambda N_1}, \xi_1 = 1 \\ |(\xi_2, \dots, \xi_d)| \leq N_2/N_1}} e^{2\pi i (\xi \cdot x + |\xi|^2 t)}, \quad Ef_2 = \sum_{\substack{\xi \in \Lambda_{\lambda N_1}, \xi_1 = 0 \\ |(\xi_2, \dots, \xi_d)| \leq N_2/N_1}} e^{2\pi i (\xi \cdot x + |\xi|^2 t)}.$$

Notice that we construct the example in $d \geq 3$ differently; the support of f_j is in a thin plate of radius N_2/N_1 instead of $1/N_1$, as in two-dimensional example.

$|Ef_1|$ is about $(\lambda N_2)^{d-1}$ in a box of size $(N_1/N_2) \times \cdots \times (N_1/N_2) \times (N_1/N_2)^2 \times (\lambda N_1)^2$. $|Ef_2|$ is about $(\lambda N_2)^{d-1}$ in the same box. Both Ef_1 and Ef_2 have periodicity around λN_1 in the x_2-, …, x_d-directions:

$$\|Ef_1 Ef_2\|^2_{L^2(w_\Omega)} \gtrsim (\lambda N_1)^{4(d-1)} \Bigl(\frac{N_1}{N_2} \Bigr)^{d+1} (\lambda N_1)^2 (\lambda N_1)^{d-1} \gtrsim \lambda^{5d-3} N_1^{2d+2} N_2^{3d-5}$$

and

$$\Pi_{j=1}^2 \Bigl(\sum_{|\theta_j|=1/(\lambda N_1)} \|Ef_{j,\theta_j}\|^2_{L^4(w_\Omega)} \Bigr) \lesssim (\lambda N_2)^{2(d-1)} \cdot |\Omega| \lesssim \lambda^{4d-2} N_1^{2d+2} N_2^{2d-2}$$

This example shows that when $d \geq 3$, the term with N_2^{d-3}/λ is sharp.

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SHARP GLOBAL ESTIMATES FOR LOCAL AND NONLOCAL POROUS MEDIUM-TYPE EQUATIONS IN BOUNDED DOMAINS

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We provide a quantitative study of nonnegative solutions to nonlinear diffusion equations of porous medium-type of the form $\partial_t u + \mathcal{L}u^m = 0,\ m > 1$, where the operator $\mathcal{L}$ belongs to a general class of linear operators, and the equation is posed in a bounded domain $\Omega \subset \mathbb{R}^N$. As possible operators we include the three most common definitions of the fractional Laplacian in a bounded domain with zero Dirichlet conditions, and also a number of other nonlocal versions. In particular, $\mathcal{L}$ can be a fractional power of a uniformly elliptic operator with C^1 coefficients. Since the nonlinearity is given by u^m with $m > 1$, the equation is degenerate parabolic.

The basic well-posedness theory for this class of equations was recently developed by Bonforte and Vázquez (2015, 2016). Here we address the regularity theory: decay and positivity, boundary behavior, Harnack inequalities, interior and boundary regularity, and asymptotic behavior. All this is done in a quantitative way, based on sharp a priori estimates. Although our focus is on the fractional models, our results cover also the local case when $\mathcal{L}$ is a uniformly elliptic operator, and provide new estimates even in this setting.

A surprising aspect discovered in this paper is the possible presence of nonmatching powers for the long-time boundary behavior. More precisely, when $\mathcal{L} = (-\Delta)^s$ is a spectral power of the Dirichlet Laplacian inside a smooth domain, we can prove that

- when $2s > 1 - 1/m$, for large times all solutions behave as $\mathrm{dist}^{1/m}$ near the boundary;
- when $2s \le 1 - 1/m$, different solutions may exhibit different boundary behavior.

This unexpected phenomenon is a completely new feature of the nonlocal nonlinear structure of this model, and it is not present in the semilinear elliptic equation $\mathcal{L}u^m = u$.

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Keywords: nonlocal diffusion, nonlinear equations, bounded domains, a priori estimates, positivity, boundary behavior, regularity, Harnack inequalities.

1. Introduction

In this paper we address the question of obtaining a priori estimates, positivity, boundary behavior, Harnack inequalities, and regularity for a suitable class of weak solutions of nonlinear nonlocal diffusion equations of the form

$$\partial_t u + \mathcal{L}F(u) = 0 \quad \text{posed in } Q = (0,\infty)\times\Omega, \tag{1-1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{1,1}$ boundary, $N \geq 2$,[1] and $\mathcal{L}$ is a linear operator representing diffusion of local or nonlocal type, the prototype example being the fractional Laplacian (the class of admissible operators will be precisely described below). Although our arguments hold for a rather general class of nonlinearities $F : \mathbb{R} \to \mathbb{R}$, for the sake of simplicity we shall focus on the model case $F(u) = u^m$ with $m > 1$.

The use of nonlocal operators in diffusion equations reflects the need to model the presence of long-distance effects not included in evolution driven by the Laplace operator, and this is well documented in the literature. The physical motivation and relevance of the nonlinear diffusion models with nonlocal operators has been mentioned in many references; see for instance [Athanasopoulos and Caffarelli 2010; Bonforte and Vázquez 2014; 2015; de Pablo et al. 2011; 2012; Vázquez 2014b]. Because u usually represents a density, all data and solutions are supposed to be nonnegative. Since the problem is posed in a bounded domain, we need boundary or external conditions that we assume to be of Dirichlet type.

This kind of problem has been extensively studied when $\mathcal{L} = -\Delta$ and $F(u) = u^m$, $m > 1$, in which case the equation becomes the classical porous medium equation [Vázquez 2004; 2007; Dahlberg and Kenig 1988; Daskalopoulos and Kenig 2007]. Here, we are interested in treating nonlocal diffusion operators, in particular fractional Laplacian operators. Note that, since we are working on a bounded domain, the concept of fractional Laplacian operator admits several nonequivalent versions, the best known being the restricted fractional Laplacian (RFL), the spectral fractional Laplacian (SFL), and the censored fractional Laplacian (CFL); see Section 2A for more details. We use these names because they already appeared in some previous works [Bonforte et al. 2015; Bonforte and Vázquez 2016], but we point out that the RFL is usually known as the standard fractional Laplacian, or plainly fractional Laplacian, and the CFL is often called the regional fractional Laplacian.

The case of the SFL operator with $F(u) = u^m$, $m > 1$, was already studied by the first and third authors in [Bonforte and Vázquez 2015; 2016]. In particular, in [Bonforte and Vázquez 2016] the authors presented a rather abstract setting where they were able to treat not only the usual fractional Laplacians but also a large number of variants that will be listed below for the reader's convenience. Besides, rather general increasing nonlinearities F were allowed. The basic questions of existence and uniqueness of suitable solutions for this problem were solved in [Bonforte and Vázquez 2016] in the class of "weak dual solutions", an extension of the concept of solution introduced in [Bonforte and Vázquez 2015] that has proved to be quite flexible and efficient. A number of a priori estimates (absolute bounds and smoothing effects) were also derived in that generality.

[1] Our results work also in dimension $N = 1$ if the fractional exponent (that we shall introduce later) belongs to the range $0 < s < \frac{1}{2}$. The interval $\frac{1}{2} \leq s < 1$ requires some minor modifications that we prefer to avoid in this paper.

Since these basic facts are settled, here we focus our attention on the finer aspects of the theory, mainly sharp boundary estimates and decay estimates. Such upper and lower bounds will be formulated in terms of the first eigenfunction Φ_1 of $\mathcal{L}$, which under our assumptions will satisfy $\Phi_1 \asymp \mathrm{dist}(\cdot\,, \partial\Omega)^\gamma$ for a certain characteristic power $\gamma \in (0, 1]$ that depends on the particular operator we consider. Typical values are $\gamma = s$ (SFL), $\gamma = 1$ (RFL), and $\gamma = s - \frac{1}{2}$ for $s > \frac{1}{2}$ (CFL); see Sections 2A and 10A. As a consequence, we get various kinds of local and global Harnack-type inequalities.

It is worth mentioning that some of the boundary estimates that we obtain for the parabolic case are essentially elliptic in nature. The study of this issue for stationary problems is done in a companion paper [Bonforte et al. 2017b]. This has the advantage that many arguments are clearer, since the parabolic problem is more complicated than the elliptic one. Clarifying such differences is one of the main contributions of our present work.

Thanks to these results, in the last part of the paper we are able to prove both interior and boundary regularity, and to find the large-time asymptotic behavior of solutions.

Let us indicate here some notation of general use. The symbol ∞ will always denote $+\infty$. Given a, b, we use the notation $a \asymp b$ whenever there exist universal constants $c_0, c_1 > 0$ such that $c_0\, b \le a \le c_1 b$. We also use the symbols $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. We will always consider bounded domains Ω with boundary of class C^2. In the paper we use the short form "solution" to mean "weak dual solution", unless differently stated.

1A. ***Presentation of the results on sharp boundary behavior.*** • A basic principle in the paper is that the sharp boundary estimates depend not only on $\mathcal{L}$ but also on the behavior of the nonlinearity $F(u)$ near $u = 0$, i.e., in our case, on the exponent $m > 1$. The elliptic analysis performed in the companion paper [Bonforte et al. 2017b] combined with some standard arguments will allow us to prove that, in *all* cases, $u(t)$ approaches the separate-variable solution $\mathcal{U}(x, t) = t^{-1/(m-1)} S(x)$ in the sense that

$$\|t^{1/(m-1)} u(t, \cdot) - S\|_{\mathrm{L}^\infty(\Omega)} \xrightarrow{t\to\infty} 0, \tag{1-2}$$

where S is the solution of the elliptic problem (see Theorems 3.2 and 7.1). The behavior of the profile $S(x)$ is shown to be, when $2sm \neq \gamma(m-1)$,

$$S(x) \asymp \Phi_1(x)^{\sigma/m}, \quad \sigma := \min\left\{1, \frac{2sm}{\gamma(m-1)}\right\}. \tag{1-3}$$

Thus, the behavior strongly depends on the new parameter σ, more precisely, on whether this parameter is equal to 1 or less than 1. As we shall see later, σ encodes the interplay between the "elliptic scaling power" $2s/(m-1)$, the "eigenfunction power" γ, and the "nonlinearity power" m. When $2sm = \gamma(m-1)$ we have $\sigma = 1$, but a logarithmic correction appears:

$$S(x) \asymp \Phi_1(x)^{1/m} (1 + |\log \Phi_1(x)|)^{1/(m-1)}. \tag{1-4}$$

• This fact and the results in [Bonforte et al. 2017a] prompted us to look for estimates of the form

$$c_0(t) \frac{\Phi_1^{\sigma/m}(x_0)}{t^{1/(m-1)}} \le u(t, x_0) \le c_1 \frac{\Phi_1^{\sigma/m}(x_0)}{t^{1/(m-1)}} \quad \text{for all } t > 0,\ x_0 \in \Omega, \tag{1-5}$$

where $c_0(t)$ and c_1 are positive and independent of u, eventually with a logarithmic term appearing when $2sm = \gamma(m-1)$, as in (1-4). We will prove in this paper that the upper bound holds for the three mentioned fractional Laplacian choices, and indeed for the whole class of integrodifferential operators we will introduce below; see Theorem 4.1. Also, separate-variable solutions saturate the upper bound.

The issue of the validity of a lower bound as in (1-5) is instead much more elusive. A first indication for this is the introduction of a function $c_0(t)$ depending on t, instead of a constant. This seems to reflect the fact that the solution may take some time to reach the boundary behavior that is expected to hold uniformly for large times. Indeed, recall that in the classical PME [Aronson and Peletier 1981; Vázquez 2004; 2007], for data supported away from the boundary, some "waiting time" is needed for the support to reach the boundary.

• As proved in [Bonforte et al. 2017a], the stated lower bound holds for the RFL with $c_0(t) \sim (1\wedge t)^{m/(m-1)}$. In particular, in this nonlocal setting, infinite speed of propagation holds. Here, we show that this holds also for the CFL and a number of other operators; see Theorem 5.2. Note that for the RFL and the CFL we have $2sm > \gamma(m-1)$, in particular $\sigma = 1$, which simplifies formula (1-5).

A combination of an upper and a lower bound with matching behavior (with respect to x and t) will be called a *global Harnack principle*, and holds for all $t > 0$ for these operators; see Theorems 6.1 and 6.2.

• When $\mathcal{L}$ is the SFL, we shall see that the lower bound may fail. Of course, solutions by separation of variables satisfy the matching estimates in (1-5), eventually with an extra logarithmic term in the limit case, as in (1-4), but it came as a complete surprise to us that for the SFL the situation is not the same for "small" initial data. More precisely:

(i) We can prove that the following bounds always hold for all times:

$$c_0\left(1 \wedge \frac{t}{t_*}\right)^{m/(m-1)} \frac{\Phi_1(x_0)}{t^{1/(m-1)}} \le u(t,x_0) \le c_1 \frac{\Phi_1^{\sigma/m}(x_0)}{t^{1/(m-1)}} \tag{1-6}$$

(when $2sm = \gamma(m-1)$, a logarithmic correction $(1+|\log \Phi_1(x)|)^{1/(m-1)}$ appears in the right-hand side); see Theorem 5.1. These are nonmatching estimates.

(ii) For $2sm > \gamma(m-1)$, the sharp estimate (1-5) holds for any nonnegative nontrivial solution for large times $t \ge t_*$; see Theorem 5.3.

(iii) *Anomalous boundary behavior.* Consider now the SFL with $\sigma < 1$ (resp. $2sm = \gamma(m-1)$).[2] In this case we can find initial data for which the upper bound in (1-6) is not sharp. Depending on the initial data, there are several possible rates for the long-time behavior near the boundary. More precisely:

(a) When $u_0 \le A\Phi_1$, we have $u(t) \le F(t)\Phi_1^{1/m} \ll \Phi_1^{\sigma/m}$ (resp. $\Phi_1^{1/m} \ll \Phi_1^{1/m}(1+|\log\Phi_1|)^{1/(m-1)}$) for all times; see Theorem 5.4. In particular

$$\lim_{x\to\partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\sigma/m}} = 0 \qquad \left(\text{resp. } \lim_{x\to\partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{1/m}(1+|\log\Phi_1(x)|)^{1/(m-1)}} = 0\right) \tag{1-7}$$

for any $t > 0$.

[2] Since for the SFL $\gamma = 1$, we have $\sigma < 1$ if and only if $0 < s < s_* := (m-1)/(2m) < \frac{1}{2}$. Note that $s_* \to 0$ as we tend to the linear case $m = 1$, so this exceptional regime does not appear for linear diffusions, both fractional and standard.

(b) When $u_0 \le A\Phi_1^{1-2s/\gamma}$, we have $u(t) \le F(t)\Phi_1^{1-2s/\gamma}$ for small times; see Theorem 4.4. Notice that when $\sigma < 1$ we have always $1 - 2s/\gamma > \sigma/m$. This sets a limitation on the improvement of the lower bound, which is confirmed by another result: in Theorem 5.5 we show that lower bounds of the form $u(T,x) \ge \underline{\kappa}\,\Phi_1^{\alpha}(x)$ for data $u_0(x) \le A\Phi_1(x)$ are possible only for $\alpha \ge 1 - 2s/\gamma$.

(c) On the other hand, for "large" initial data, Theorem 6.2 shows that the desired matching estimates from above and below hold.

After discovering this strange boundary behavior, we looked for numerical confirmation. In Section 9 we will explain the numerical results obtained in [Cusimano et al. 2017]. Note that, if one looks for universal bounds independent of the initial condition, Figures 2–3 below seem to suggest that the bounds provided by (1-6) are optimal for all times and all operators.

• The current interest in more general types of nonlocal operators led us to a more general analysis where the just-explained alternative has been extended to a wide class of integrodifferential operators, subject only to a list of properties that we call (A1), (A2), (L1), (L2), (K2), (K4); a number of examples are explained in Section 2. These general classes appear also in the study of the elliptic problem [Bonforte et al. 2017b].

1B. ***Asymptotic behavior and regularity.*** Our quantitative lower and upper estimates admit formulations as local or global Harnack inequalities. They are used at the end of the paper to settle two important issues.

Sharp asymptotic behavior. Exploiting the techniques in [Bonforte et al. 2015], we can prove a sharp asymptotic behavior for our nonnegative and nontrivial solutions when the upper and lower bounds have matching powers. Such sharp results hold true for a quite general class of local and nonlocal operators. A detailed account is given in Section 7.

Regularity. By a variant of the techniques used in [Bonforte et al. 2017a], we can show interior Hölder regularity. In addition, if the kernel of the operator satisfies some suitable continuity assumptions, we show that solutions are classical in the interior and are Hölder continuous up to the boundary if the upper and lower bounds have matching powers. We refer to Section 8 for details.

2. General class of operators and their kernels

The interest of the theory developed here lies both in the sharpness of the results and in the wide range of applicability. We have just mentioned the most relevant examples appearing in the literature, and more are listed at the end of this section. Actually, our theory applies to a general class of operators with definite assumptions, and this is what we want to explain now.

Let us present the properties that have to be assumed on the class of admissible operators. Some of them already appeared in [Bonforte and Vázquez 2016]. However, to further develop our theory, more hypotheses need to be introduced. In particular, while the paper above only uses the properties of the Green function, here we shall make some assumptions also on the kernel of $\mathcal{L}$ (whenever it exists). Note that assumptions on the kernel K of $\mathcal{L}$ are needed for the positivity results, because we need to distinguish between the local and nonlocal cases. The study of the kernel K is performed in Section 10.

For convenience of reference, the list of used assumptions is (A1), (A2), (K2), (K4), (L1), (L2). The first three are assumed in all operators $\mathcal{L}$ that we use.

• *Basic assumptions on $\mathcal{L}$.* The linear operator $\mathcal{L} : \mathrm{dom}(\mathcal{L}) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and sub-Markovian; more precisely, it satisfies(A1)and(A2)below:

(A1) $\mathcal{L}$ is m-accretive on $\mathrm{L}^1(\Omega)$;

(A2) If $0 \le f \le 1$ then $0 \le \mathrm{e}^{-t\mathcal{L}} f \le 1$.

Under these assumptions, in [Bonforte and Vázquez 2016], the first and third authors proved existence, uniqueness, weighted estimates, and smoothing effects.

• *Assumptions on the kernel.* Whenever $\mathcal{L}$ is defined in terms of a kernel $K(x, y)$ via the formula

$$\mathcal{L} f(x) = \mathrm{P.V.} \int_{\mathbb{R}^N} (f(x) - f(y))\, K(x, y)\, \mathrm{d}y,$$

assumption (L1) states that there exists $\underline{\kappa}_\Omega > 0$ such that

$$\inf_{x,y\in\Omega} K(x, y) \ge \underline{\kappa}_\Omega > 0. \tag{L1}$$

We note that condition holds both for the RFL and the CFL; see Section 2A.

Whenever $\mathcal{L}$ is defined in terms of a kernel $K(x, y)$ and a zero-order term via the formula

$$\mathcal{L} f(x) = \mathrm{P.V.} \int_{\mathbb{R}^N} (f(x) - f(y))\, K(x, y)\, \mathrm{d}y + B(x) f(x),$$

assumption (L2) states that

$$K(x, y) \ge c_0 \delta^\gamma(x) \delta^\gamma(y), \quad c_0 > 0, \qquad \text{and} \qquad B(x) \ge 0, \tag{L2}$$

where, from now on, we adopt the notation $\delta(x) := \mathrm{dist}(x, \partial\Omega)$. This condition is satisfied by the SFL in a stronger form; see Section 10 and Lemma 10.1.

• *Assumptions on $\mathcal{L}^{-1}$.* In order to prove our quantitative estimates, we need to be more specific about the operator $\mathcal{L}$. Besides satisfying(A1)and (A2), we will assume that it has a left-inverse $\mathcal{L}^{-1} : \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ that can be represented by a kernel $\mathbb{G}$ (the letter "G" standing for Green function) as

$$\mathcal{L}^{-1}[f](x) = \int_\Omega \mathbb{G}(x, y) f(y)\, \mathrm{d}y,$$

where $\mathbb{G}$ satisfies the following assumption for some $s \in (0, 1]$: there exist constants $\gamma \in (0, 1]$ and $c_{0,\Omega}, c_{1,\Omega} > 0$ such that, for a.e. $x, y \in \Omega$,

$$c_{0,\Omega} \delta^\gamma(x) \delta^\gamma(y) \le \mathbb{G}(x, y) \le \frac{c_{1,\Omega}}{|x - y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x - y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y)}{|x - y|^\gamma} \wedge 1 \right). \tag{K2}$$

(Here and below we use the labels (K2) and (K4) to be consistent with the notation in [Bonforte and Vázquez 2016].) Hypothesis (K2) introduces an exponent γ which is a characteristic of the operator and will play a big role in the results. Notice that defining an inverse operator $\mathcal{L}^{-1}$ implies that we are taking into account the Dirichlet boundary conditions. See more details in Section 2 of [Bonforte and Vázquez 2016].

The lower bound in (K2) is weaker than the known bounds on the Green function for many examples under consideration; indeed, the following stronger estimate holds in many cases:

$$\mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}}\left(\frac{\delta^\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma}\wedge 1\right). \tag{K4}$$

Remarks. (i) The labels (A1), (A2), (K1), (K2), (K4) are consistent with the notation in [Bonforte and Vázquez 2016]. The label (K3) was used to mean hypothesis (K2) written in terms of Φ_1 instead of δ^γ.

(ii) In the classical local case $\mathcal{L} = -\Delta$, the Green function $\mathbb{G}$ satisfies (K4) only when $N \geq 3$, as the formulas slightly change when $N = 1, 2$. In the fractional case $s \in (0,1)$ the same problem arises when $N = 1$ and $s \in \left[\frac{1}{2}, 1\right)$. Hence, treating also these cases would require a slightly different analysis based on different but related assumptions on $\mathbb{G}$. Since our approach is very general, we expect it to work also in these remaining cases without any major difficulties. However, to simplify the presentation, from now on we assume that

$$\text{either } N \geq 2 \text{ and } s \in (0,1) \quad \text{or} \quad N = 1 \text{ and } s \in \left(0, \tfrac{1}{2}\right).$$

The role of the first eigenfunction of $\mathcal{L}$. We showed in [Bonforte et al. 2017b] that, under assumption (K1), the operator $\mathcal{L}$ is compact, has a discrete spectrum, and has a first nonnegative bounded eigenfunction Φ_1; assuming also (K2), we have

$$\Phi_1(x) \asymp \delta^\gamma(x) = \mathrm{dist}(x, \partial\Omega)^\gamma \quad \text{for all } x \in \overline{\Omega}. \tag{2-1}$$

Hence, Φ_1 encodes the parameter γ that takes care of describing the boundary behavior. We recall that we are assuming that the boundary of Ω is smooth enough, for instance $C^{1,1}$.

Remark. We note that our assumptions allow us to cover all the examples of operators described in Sections 2A and 10A.

2A. ***Main examples of operators and properties.*** When working in the whole $\mathbb{R}^N$, the fractional Laplacian admits different definitions that can be shown to be all equivalent. On the other hand, when we deal with bounded domains, there are at least three different operators in the literature, which we call the restricted (RFL), the spectral (SFL) and the censored fractional Laplacian (CFL). We will show below that these different operators exhibit quite different behaviors, so the distinction between them has to be taken into account. Let us present the statement and results for the three model cases, and we refer to Section 10A for further examples. Here, we collect the sharp results about the boundary behavior, namely the global Harnack inequalities from Theorems 6.1, 6.2, and 6.3.

The parameters γ and σ. The strong difference between the various operators $\mathcal{L}$ is reflected in the different boundary behavior of their nonnegative solutions. We will often use the exponent γ, which represents the boundary behavior of the first eigenfunction $\Phi_1 \asymp \mathrm{dist}(\cdot\,, \partial\Omega)^\gamma$; see [Bonforte et al. 2017b]. Both in the parabolic theory of this paper and the elliptic theory of [Bonforte et al. 2017b] the parameter $\sigma = \min\{1, 2sm/(\gamma(m-1))\}$ introduced in (1-3) plays a big role.

2A1. *The RFL.* We define the fractional Laplacian operator acting on a bounded domain by using the integral representation on the whole space in terms of a hypersingular kernel; namely

$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s}\ \text{P.V.}\int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z, \tag{2-2}$$

where $c_{N,s}>0$ is a normalization constant, and we "restrict" the operator to functions that are zero outside Ω. We denote such operator by $\mathcal{L}=(-\Delta_{|\Omega})^s$, and call it the *restricted fractional Laplacian*[3] (RFL). The initial and boundary conditions associated to the fractional diffusion equation (1-1) are $u(t,x)=0$ in $(0,\infty)\times\mathbb{R}^N\setminus\Omega$ and $u(0,\cdot)=u_0$. As explained in [Bonforte et al. 2015], such boundary conditions can also be understood via the Caffarelli–Silvestre extension [2007]. The sharp expression of the boundary behavior for the RFL was investigated in [Ros-Oton and Serra 2014]. We refer to [Bonforte et al. 2015] for a careful construction of the RFL in the framework of fractional Sobolev spaces, and [Blumenthal and Getoor 1960] for a probabilistic interpretation.

This operator satisfies the assumptions (A1), (A2), (L1), and also (K2) and (K4) with $\gamma=s<1$. Let us present our results in this case. Note that we have $\sigma=1$ for all $0<s<1$, and Theorem 6.1 shows the sharp boundary behavior for all times; namely for all $t>0$ and a.e. $x\in\Omega$ we have

$$\underline{\kappa}\left(1\wedge\frac{t}{t_*}\right)^{m/(m-1)}\frac{\operatorname{dist}(x,\partial\Omega)^{s/m}}{t^{1/(m-1)}} \le u(t,x) \le \bar{\kappa}\,\frac{\operatorname{dist}(x,\partial\Omega)^{s/m}}{t^{1/(m-1)}}. \tag{2-3}$$

The critical time t_* is given by a weighted L^1 norm; namely

$$t_* := \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)},$$

where $\kappa_*>0$ is a universal constant. Moreover, solutions are classical in the interior and we prove sharp Hölder continuity up to the boundary. These regularity results were first obtained in [Bonforte et al. 2017a]; we give here different proofs valid in the more general setting of this paper. See Section 8 for further details.

2A2. *The SFL.* Starting from the classical Dirichlet Laplacian Δ_Ω on the domain Ω, the so-called *spectral definition* of the fractional power of Δ_Ω may be defined via a formula in terms of the semigroup associated to the Laplacian; namely

$$(-\Delta_\Omega)^s g(x) = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{t\Delta_\Omega}g(x)-g(x)\right)\frac{\mathrm{d}t}{t^{1+s}} = \sum_{j=1}^\infty \lambda_j^s\,\hat{g}_j\,\phi_j(x), \tag{2-4}$$

where (λ_j,ϕ_j), $j=1,2,\ldots$, is the normalized spectral sequence of the standard Dirichlet Laplacian on Ω, $\hat{g}_j=\int_\Omega g(x)\phi_j(x)\,\mathrm{d}x$, and $\|\phi_j\|_{\mathrm{L}^2(\Omega)}=1$. We denote this operator by $\mathcal{L}=(-\Delta_\Omega)^s$, and call it the *spectral fractional Laplacian* (SFL) as in [Cabré and Tan 2010]. The initial and boundary conditions associated to the fractional diffusion equation (1-1) are $u(t,x)=0$ on $(0,\infty)\times\partial\Omega$ and $u(0,\cdot)=u_0$.

[3] In the literature this is often called the fractional Laplacian on domains, but this simpler name may be confusing when the spectral fractional Laplacian is also considered; see [Bonforte and Vázquez 2015]. As discussed in this paper, there are other natural versions.

Such boundary conditions can also be understood via the Caffarelli–Silvestre extension; see [Bonforte et al. 2015]. Following ideas of [Song and Vondraček 2003], we use the fact that this operator admits a kernel representation,

$$(-\Delta_\Omega)^s g(x) = c_{N,s}\ \text{P.V.}\int_\Omega [g(x)-g(z)]K(x,z)\,\mathrm{d}z + B(x)g(x), \tag{2-5}$$

where K is a singular and compactly supported kernel, which degenerates at the boundary, and $B \asymp \operatorname{dist}(\cdot\,,\partial\Omega)^{-2s}$ (see [Song and Vondraček 2003] or Lemma 10.1 for further details). This operator satisfies the assumptions (A1), (A2), (L2), and also (K2) and (K4) with $\gamma = 1$. Therefore, σ can be less than 1, depending on the values of s and m.

As we shall see, in our parabolic setting, the degeneracy of the kernel is responsible for a peculiar change of the boundary behavior of the solutions (with respect to the previous case) for small and large times. Here, the lower bounds change both for short and large times, and they strongly depend on σ and on u_0: we called this phenomenon *anomalous boundary behavior* in Section 1A. More precisely, Theorem 6.3 shows that for all $t > 0$ and all $x \in \Omega$ we have

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{m/(m-1)} \frac{\operatorname{dist}(x,\partial\Omega)}{t^{1/(m-1)}} \le u(t,x) \le \bar{\kappa}\,\frac{\operatorname{dist}(x,\partial\Omega)^{\sigma/m}}{t^{1/(m-1)}} \tag{2-6}$$

(when $2sm = \gamma(m-1)$, a logarithmic correction $(1+|\log\Phi_1(x)|)^{1/(m-1)}$ appears in the right-hand side). Such lower behavior is somehow minimal, in the sense that it holds in all cases. The basic asymptotic result (see (1-2) or Theorem 7.1) suggests that the lower bound in (2-6) could be improved by replacing $\operatorname{dist}(x,\partial\Omega)$ with $\operatorname{dist}(x,\partial\Omega)^{\sigma/m}$, at least for large times. This is shown to be true for $\sigma = 1$ (see Theorem 5.3), but it is false for $\sigma < 1$ (see Theorem 5.4), since there are "small" solutions with nonmatching boundary behavior for all times; see (1-7).

It is interesting that, in this case, one can appreciate the interplay between the "elliptic scaling power" $2s/(m-1)$ related to the invariance of the equation $\mathcal{L}S^m = S$ under the scaling $S(x) \mapsto \lambda^{-2s/(m-1)}S(\lambda x)$, the "eigenfunction power" $\gamma = 1$, and the "nonlinearity power" m, made clear through the parameter σ/m. Also in this case, thanks to the strict positivity in the interior, we can show interior space-time regularity of solutions, as well as sharp boundary Hölder regularity for large times whenever upper and lower bounds match.

2A3. *The CFL.* In the simplest case, the infinitesimal operator of the censored stochastic processes has the form

$$\mathcal{L}g(x) = \text{P.V.}\int_\Omega \frac{g(x)-g(y)}{|x-y|^{N+2s}}\,\mathrm{d}y, \quad \text{with } \tfrac12 < s < 1. \tag{2-7}$$

This operator was introduced in [Bogdan et al. 2003] (see also [Chen et al. 2010] and [Bonforte and Vázquez 2016] for further details and references).

In this case $\gamma = s - \frac12 < 2s$; hence $\sigma = 1$ for all $\frac12 < s < 1$, and Theorem 6.1 shows that for all $t > 0$ and $x \in \Omega$ we have

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{m/(m-1)} \frac{\operatorname{dist}(x,\partial\Omega)^{(s-1/2)/m}}{t^{1/(m-1)}} \le u(t,x) \le \bar{\kappa}\,\frac{\operatorname{dist}(x,\partial\Omega)^{(s-1/2)/m}}{t^{1/(m-1)}}.$$

Again, we have interior space-time regularity of solutions, as well as sharp boundary Hölder regularity for all times.

2A4. *Other examples.* There a number of examples to which our theory applies, besides the RFL, CFL and SFL, since they satisfy the list of assumptions listed in the previous section. Some are listed in the last Section 10; see more detail in [Bonforte and Vázquez 2016].

3. Reminders about weak dual solutions

We denote by $\mathrm{L}^p_{\Phi_1}(\Omega)$ the weighted L^p space $\mathrm{L}^p(\Omega,\ \Phi_1\,\mathrm{d}x)$, endowed with the norm

$$\|f\|_{\mathrm{L}^p_{\Phi_1}(\Omega)} = \left(\int_\Omega |f(x)|^p \Phi_1(x)\,\mathrm{d}x\right)^{1/p}.$$

Weak dual solutions: existence and uniqueness. We recall the definition of weak dual solutions used in [Bonforte and Vázquez 2016]. This is expressed in terms of the inverse operator $\mathcal{L}^{-1}$, and encodes the Dirichlet boundary condition. This is needed to build a theory of bounded nonnegative unique solutions to (1-1) under the assumptions of the previous section. Note that in [Bonforte and Vázquez 2016] we used the setup with the weight $\delta^\gamma = \mathrm{dist}(\,\cdot\,,\partial\Omega)^\gamma$, but the same arguments generalize immediately to the weight Φ_1; indeed under assumption (K2), these two setups are equivalent.

Definition 3.1. A function u is a *weak dual* solution to the Dirichlet problem for (1-1) in $(0,\infty)\times\Omega$ if:

- $u \in C((0,\infty) : \mathrm{L}^1_{\Phi_1}(\Omega))$ and $u^m \in \mathrm{L}^1((0,\infty) : \mathrm{L}^1_{\Phi_1}(\Omega))$.
- The identity
$$\int_0^\infty\int_\Omega \mathcal{L}^{-1}u\,\frac{\partial\psi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_0^\infty\int_\Omega u^m\psi\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{3-1}$$
holds for every test function ψ such that $\psi/\Phi_1 \in C^1_c((0,\infty) : \mathrm{L}^\infty(\Omega))$.
- A weak dual solution to the Cauchy–Dirichlet problem (CDP) is a weak dual solution to the homogeneous Dirichlet problem for (1-1) such that $u \in C([0,\infty) : \mathrm{L}^1_{\Phi_1}(\Omega))$ and $u(0,x) = u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.

This kind of solution was first introduced in [Bonforte and Vázquez 2015]; see also [Bonforte and Vázquez 2016]. Roughly speaking, we are considering the weak solution to the “dual equation” $\partial_t U = -u^m$, where $U = \mathcal{L}^{-1}u$, posed on the bounded domain Ω with homogeneous Dirichlet conditions. Such a weak solution is obtained by approximation from below as the limit of the unique mild solution provided by the semigroup theory [Bonforte and Vázquez 2016], and it was used in [Vázquez 2014a] with space domain $\mathbb{R}^N$ in the study of Barenblatt solutions. We call those solutions *minimal weak dual solutions,* and it has been proven in Theorems 4.4 and 4.5 of [Bonforte and Vázquez 2016] that such solutions exist and are unique for any nonnegative data $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. The class of weak dual solutions includes the classes of weak, mild and strong solutions, and is included in the class of very weak solutions. In this class of solutions the standard comparison result holds.

Explicit solution. When trying to understand the behavior of positive solutions with general nonnegative data, it is natural to look for solutions obtained by separation of variables. These are given by

$$\mathcal{U}_T(t,x) := (T+t)^{-1/(m-1)} S(x), \quad T \ge 0, \tag{3-2}$$

where S solves the elliptic problem

$$\begin{cases} \mathcal{L} S^m = S & \text{in } (0,+\infty)\times\Omega, \\ S = 0 & \text{on the boundary.} \end{cases} \tag{3-3}$$

The properties of S have been thoroughly studied in the companion paper [Bonforte et al. 2017b], and we summarize them here for the reader's convenience.

Theorem 3.2 (properties of asymptotic profiles). *Assume that $\mathcal{L}$ satisfies* (A1), (A2), *and* (K2). *Then there exists a unique positive solution S to the Dirichlet problem* (3-3) *with $m > 1$. Moreover, let σ be as in* (1-3), *and assume that*

- *either $\sigma = 1$ and $2sm \neq \gamma(m-1)$;*
- *or $\sigma < 1$ and* (K4) *holds.*

Then there exist positive constants c_0 and c_1 such that the following sharp absolute bounds hold true for all $x \in \Omega$:

$$c_0 \Phi_1(x)^{\sigma/m} \le S(x) \le c_1 \Phi_1(x)^{\sigma/m}. \tag{3-4}$$

When $2sm = \gamma(m-1)$ then, assuming (K4), *for all $x \in \Omega$ we have*

$$c_0 \Phi_1(x)^{1/m} (1 + |\log \Phi_1(x)|)^{1/(m-1)} \le S(x) \le c_1 \Phi_1(x)^{1/m} (1 + |\log \Phi_1(x)|)^{1/(m-1)}. \tag{3-5}$$

Remark. As observed in the proof of Theorem 7.2, by applying Theorem 6.1 to the separate-variables solution $t^{-1/(m-1)} S(x)$ we deduce that (3-4) is still true when $\sigma < 1$ if, instead of assuming (K4), we suppose that

$$K(x,y) \le c_1 |x-y|^{-(N+2s)}$$

for a.e. $x, y \in \mathbb{R}^N$ and that $\Phi_1 \in C^\gamma(\Omega)$.

When $T = 0$, the solution $\mathcal{U}_0$ in (3-2) is commonly named "friendly giant", because it takes initial data $u_0 \equiv +\infty$ (in the sense of a pointwise limit as $t \to 0$) but is bounded for all $t > 0$. This term was coined in the study of the standard porous medium equation.

In Sections 4 and 5 we will state and prove our general results concerning upper and lower bounds respectively. These sections are the crux of this paper. The combination of such upper and lower bounds will then be summarized in Section 6. Consequences of these results in terms of asymptotic behavior and regularity estimates will be studied in Sections 7 and 8 respectively.

4. Upper boundary estimates

We present a general upper bound that holds under the sole assumptions (A1), (A2), and (K2), and hence is valid for all our examples.

Theorem 4.1 (absolute boundary estimates). *Let* (A1), (A2), *and* (K2) *hold. Let $u \geq 0$ be a weak dual solution to the CDP corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$, and let σ be as in* (1-3). *Then, there exists a computable constant $k_1 > 0$, depending only on N, s, m, and Ω, such that for all $t \geq 0$ and all $x \in \Omega$*

$$u(t,x) \leq \frac{k_1}{t^{1/(m-1)}} \begin{cases} \Phi_1(x_0)^{\sigma/m} & \text{if } \gamma \neq 2sm/(m-1), \\ \Phi_1(x_0)^{1/m}(1+|\log \Phi_1(x_0)|)^{1/(m-1)} & \text{if } \gamma = 2sm/(m-1). \end{cases} \tag{4-1}$$

This absolute bound proves a strong regularization which is independent of the initial datum. It improves the absolute bound in [Bonforte and Vázquez 2016] in the sense that it exhibits a precise boundary behavior. The estimate gives the correct behavior for the solutions $\mathcal{U}_T$ in (3-2) obtained by separation of variables; see Theorem 3.2. It turns out that the estimate will be sharp for all nonnegative, nontrivial solutions in the case of the RFL and the CFL. We will also see below that the estimate is not always the correct behavior for the SFL when data are small, as explained in the Introduction (see Section 4A, and Theorem 5.4 in Section 5).

Proof of Theorem 4.1. This subsection is devoted to the proof of Theorem 4.1. The first steps are based on a few basic results of [Bonforte and Vázquez 2016] that will also be used in the rest of the paper.

Step 1: pointwise and absolute upper estimates.

Pointwise estimates. We begin by recalling the basic pointwise estimates which are crucial in the proof of all the upper and lower bounds of this paper.

Proposition 4.2 [Bonforte and Vázquez 2015; 2016]. *It holds that*

$$\int_\Omega u(t,x)\,\mathbb{G}(x,x_0)\,\mathrm{d}x \leq \int_\Omega u_0(x)\,\mathbb{G}(x,x_0)\,\mathrm{d}x \quad \textit{for all } t > 0. \tag{4-2}$$

Moreover, for every $0 < t_0 \leq t_1 \leq t$ and almost every $x_0 \in \Omega$, we have

$$\frac{t_0^{m/(m-1)}}{t_1^{m/(m-1)}}(t_1 - t_0)\,u^m(t_0,x_0) \leq \int_\Omega [u(t_0,x) - u(t_1,x)]\,\mathbb{G}(x,x_0)\,\mathrm{d}x \leq (m-1)\frac{t^{m/(m-1)}}{t_0^{1/(m-1)}}\,u^m(t,x_0). \tag{4-3}$$

Absolute upper bounds. Using the estimates above, in Theorem 5.2 of [Bonforte and Vázquez 2016] the authors proved that solutions corresponding to initial data $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$ satisfy

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \leq \frac{K_1}{t^{1/(m-1)}} \quad \text{for all } t > 0, \tag{4-4}$$

with a constant K_1 independent of u_0. For this reason, this is called "absolute bound".

Step 2: upper bounds via Green function estimates. The proof of Theorem 4.1 requires the following general statement; see [Bonforte et al. 2017b, Proposition 6.5]:

Lemma 4.3. *Let* (A1), (A2), *and* (K2) *hold, and let $v : \Omega \to \mathbb{R}$ be a nonnegative bounded function. Let σ be as in* (1-3), *and assume that, for a.e. $x_0 \in \Omega$,*

$$v(x_0)^m \leq \kappa_0 \int_\Omega v(x)\,\mathbb{G}(x,x_0)\,\mathrm{d}x. \tag{4-5}$$

Then, there exists a constant $\bar{\kappa}_\infty > 0$, *depending only on* s, γ, m, N, Ω, *such that the following bound holds true for a.e.* $x_0 \in \Omega$:

$$\int_\Omega v(x)\,\mathbb{G}(x,x_0)\,\mathrm{d}x \le \bar{\kappa}_\infty \kappa_0^{1/(m-1)} \begin{cases} \Phi_1(x_0)^\sigma & \text{if } \gamma \ne 2sm/(m-1), \\ \Phi_1(x_0)(1+|\log \Phi_1(x_0)|^{m/(m-1)}) & \text{if } \gamma = 2sm/(m-1). \end{cases} \tag{4-6}$$

<u>Step 3</u>: end of the proof of Theorem 4.1. We already know that $u(t) \in \mathrm{L}^\infty(\Omega)$ for all $t > 0$ by (4-4). Also, choosing $t_1 = 2t_0$ in (4-3) we deduce that, for $t \ge 0$ and a.e. $x_0 \in \Omega$,

$$u^m(t,x_0) \le \frac{2^{m/(m-1)}}{t} \int_\Omega u(t,x)\,\mathbb{G}(x,x_0)\,\mathrm{d}x. \tag{4-7}$$

The above inequality corresponds exactly to hypothesis (4-5) of Lemma 4.3 with the value $\kappa_0 = 2^{m/(m-1)} t^{-1}$. As a consequence, inequality (4-6) holds, and we conclude that for a.e. $x_0 \in \Omega$ and all $t > 0$

$$\int_\Omega u(t,x)\,\mathbb{G}(x,x_0)\,\mathrm{d}x \le \frac{\bar{\kappa}_\infty 2^{m/(m-1)^2}}{t^{1/(m-1)}} \begin{cases} \Phi_1(x_0)^\sigma & \text{if } \gamma \ne 2sm/(m-1), \\ \Phi_1(x_0)(1+|\log \Phi_1(x_0)|^{m/(m-1)}) & \text{if } \gamma = 2sm/(m-1). \end{cases} \tag{4-8}$$

Hence, combining this bound with (4-7), we get

$$u^m(t,x_0) \le \frac{k_1^m}{t^{m/(m-1)}} \begin{cases} \Phi_1(x_0)^\sigma & \text{if } \gamma \ne 2sm/(m-1), \\ \Phi_1(x_0)(1+|\log \Phi_1(x_0)|^{m/(m-1)}) & \text{if } \gamma = 2sm/(m-1). \end{cases}$$

This proves the upper bounds (4-1) and concludes the proof. □

4A. ***Upper bounds for small data and small times.*** As mentioned in the Introduction, the above upper bounds may not be realistic when $\sigma < 1$. We have the following estimate for small times if the initial data are sufficiently small.

Theorem 4.4. *Let* $\mathcal{L}$ *satisfy* (A1), (A2), *and* (L2). *Suppose also that* $\mathcal{L}$ *has a first eigenfunction* $\Phi_1 \asymp \mathrm{dist}(x,\partial\Omega)^\gamma$, *and assume that* $\sigma < 1$. *Finally, we assume that for all* $x, y \in \Omega$

$$K(x,y) \le \frac{c_1}{|x-y|^{N+2s}} \left(\frac{\Phi_1(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\Phi_1(y)}{|x-y|^\gamma} \wedge 1 \right) \quad \text{and} \quad B(x) \le c_1 \Phi_1(x)^{-2s/\gamma}. \tag{4-9}$$

Let $u \ge 0$ *be a weak dual solution to the CDP corresponding to* $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. *Then, for every initial data* $u_0 \le A \Phi_1^{1-2s/\gamma}$ *for some* $A > 0$, *we have*

$$u(t) \le \frac{\Phi_1^{1-2s/\gamma}}{(A^{1-m} - \widetilde{C} t)^{m-1}} \quad \text{on } [0, T_A], \quad \text{where } T_A := \frac{1}{\widetilde{C} A^{m-1}},$$

and the constant $\widetilde{C} > 0$, *that depends only on* N, s, m, λ_1, c_1, *and* Ω.

Remark. This result applies to the SFL. Notice that when $\sigma < 1$ we have always $1 - 2s/\gamma > \sigma/m$; hence in this situation small data have a smaller behavior at the boundary than the one predicted in Theorem 4.1. This is not true for "big" data, for instance for solutions obtained by separation of variables, as already said.

Proof. In view of our assumption on the initial datum, namely $u_0 \le A\Phi_1^{1-2s/\gamma}$, by comparison it is enough to prove that the function

$$\bar{u}(t,x) = F(t)\Phi_1(x)^{1-2s/\gamma}, \quad F(t) = \frac{1}{(A^{1-m} - \widetilde{C}t)^{m-1}},$$

is a supersolution (i.e., $\partial_t \bar{u} \ge -\mathcal{L}\bar{u}^m$) in $(0, T_A) \times \Omega$ provided we choose $\widetilde{C}$ sufficiently large.

To this aim, we use the following elementary inequality, whose proof is left to the interested reader: for any $\eta > 1$ and any $M > 0$ there exists $\tilde{b} = \tilde{b}(M) > 0$ such that letting $\tilde{\eta} := \eta \wedge 2$

$$a^\eta - b^\eta \le \eta\, b^{\eta-1}(a-b) + \tilde{b}|a-b|^{\tilde{\eta}} \quad \text{for all } 0 \le a, b \le M. \tag{4-10}$$

We apply inequality (4-10) to $a = \Phi_1(y)$ and $b = \Phi_1(x)$, $\eta = m(1-2s/\gamma)$, noticing that $\eta > 1$ if and only if $\sigma < 1$, and we obtain (recall that Φ_1 is bounded)

$$\begin{aligned}\bar{u}^m(t,y) - \bar{u}^m(t,x) &= F(t)^m\big(\Phi_1(y)^{m(1-2s/\gamma)} - \Phi_1(x)^{m(1-2s/\gamma)}\big) = F(t)^m(\Phi_1(y)^\eta - \Phi_1(x)^\eta)\\ &\le \eta\, F(t)^m\Phi_1(x)^{\eta-1}[\Phi_1(y) - \Phi_1(x)] + \tilde{b}\, F(t)^m|\Phi_1(y) - \Phi_1(x)|^{\tilde{\eta}}\\ &\le \eta\, F(t)^m\Phi_1(x)^{\eta-1}[\Phi_1(y) - \Phi_1(x)] + \tilde{b}\, F(t)^m c_\gamma^{\tilde{\eta}}|x-y|^{\tilde{\eta}\gamma},\end{aligned}$$

where in the last step we have used that $|\Phi_1(y) - \Phi_1(x)| \le c_\gamma|x-y|^\gamma$. Since $B \le c_1\Phi_1^{-2s/\gamma}$,

$$\int_{\mathbb{R}^N}[\Phi_1(y) - \Phi_1(x)]K(x,y)\,dy = -\mathcal{L}\Phi_1(x) + B(x)\Phi_1(x) \le -\lambda_1\Phi_1(x) + c_1\Phi_1(x)^{1-2s/\gamma}.$$

Thus, recalling that $\eta, \tilde{\eta} > 1$ and that Φ_1 is bounded, it follows that

$$\begin{aligned}-\mathcal{L}[\bar{u}^m](x) &= \int_{\mathbb{R}^N}[\bar{u}^m(t,y) - \bar{u}^m(t,x)]K(x,y)\,dy + B(x)\bar{u}^m(t,x)\\ &\le \eta\, F(t)^m\Phi_1(x)^{\eta-1}[-\lambda_1\Phi_1(x) + c_1\Phi_1(x)^{1-2s/\gamma}]\\ &\qquad + B(x)F(t)^m\Phi_1^\eta(x) + \tilde{b}\, c_\gamma^{\tilde{m}}\, F(t)^m\int_{\mathbb{R}^N}|x-y|^{\tilde{\eta}\gamma}K(x,y)\,dy\\ &\le \tilde{c}F(t)^m\bigg(\Phi_1(x)^{\eta-2s/\gamma} + \int_{\mathbb{R}^N}|x-y|^{\tilde{\eta}\gamma}K(x,y)\,dy\bigg).\end{aligned} \tag{4-11}$$

Next, we claim that, as a consequence of (4-9),

$$\int_{\mathbb{R}^N}|x-y|^{\tilde{\eta}\gamma}K(x,y)\,dy \le c_4\Phi_1(x)^{1-2s/\gamma}. \tag{4-12}$$

Postponing for the moment the proof of the above inequality, we first show how to conclude: combining (4-11) and (4-12) we have

$$-\mathcal{L}\bar{u}^m \le c_5F(t)^m\Phi_1(x)^{1-2s/\gamma} = F'(t)\Phi_1(x)^{1-2s/\gamma} = \partial_t\bar{u},$$

where we used that $F'(t) = c_5F(t)^{\tilde{m}}$ provided $\tilde{C} = c_5(m-1)$. This proves that $\bar{u}$ is a supersolution in $(0,T)\times\Omega$. Hence the proof is concluded once we prove inequality (4-12); for this, using hypothesis

(4-9) and choosing $r = \Phi_1(x)^{1/\gamma}$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} |x-y|^{\tilde\eta\gamma} K(x,y)\,\mathrm{d}y &\le c_1 \int_{B_r(x)} \frac{1}{|x-y|^{N+2s-\tilde\eta\gamma}}\,\mathrm{d}y + c_1\Phi_1(x)\int_{\Omega\setminus B_r(x)} \frac{1}{|x-y|^{N+2s+\gamma-\tilde\eta\gamma}}\,\mathrm{d}y \\ &\le c_2 r^{\tilde\eta\gamma-2s} + c_1\frac{\Phi_1(x)}{r^{2s}}\int_{\Omega\setminus B_r(x)} \frac{1}{|x-y|^{N+\gamma-\tilde\eta\gamma}}\,\mathrm{d}y \\ &= c_2 r^{\tilde\eta\gamma-2s} + c_3\frac{\Phi_1(x)}{r^{2s}} \le c_4\Phi_1(x)^{1-2s/\gamma},\end{aligned}$$

where we used that $\tilde\eta\gamma - 2s > 0$ and $\tilde\eta > 1$. □

Remark. For operators for which the previous assumptions hold with $B \equiv 0$, we can actually prove a better upper bound for “smaller data”:

Corollary 4.5. *Under the assumptions of Theorem 4.4, assume that moreover $B \equiv 0$ and $u_0 \le A\Phi_1$ for some $A > 0$. Then, we have*

$$u(t) \le \frac{\Phi_1}{(A^{1-m} - \widetilde{C}t)^{m-1}} \quad \text{on } [0, T_A], \quad \text{where } T_A := \frac{1}{\widetilde{C}A^{m-1}},$$

and the constant $\widetilde{C} > 0$ depends only on N, s, m, λ_1, c_1, and Ω.

Proof. We have to show that $\bar u(t,x) = F(t)\Phi_1(x)$ is a supersolution: we essentially repeat the proof of Theorem 4.4 with $\gamma = m$ (formally replace $1 - 2s/\gamma$ by 1), taking into account that $B \equiv 0$ and $u_0 \le A\Phi_1$. □

5. Lower bounds

This section is devoted to the proofs of all the lower bounds summarized later in the main theorems, Theorems 6.1, 6.2, and 6.3. The general situation is quite involved to describe, so we will separate several cases and we will indicate for which examples it holds for the sake of clarity.

Infinite speed of propagation: universal lower bounds. First, we are going to quantitatively establish that all nonnegative weak dual solutions of our problems are in fact positive in Ω for all $t > 0$. This result is valid for all nonlocal operators considered in this paper.

Theorem 5.1. *Let $\mathcal{L}$ satisfy* (A1), (A2), *and* (L2). *Let $u \ge 0$ be a weak dual solution to the CDP corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then there exists a constant $\underline{\kappa}_0 > 0$ such that the following inequality holds*:

$$u(t,x) \ge \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{m/(m-1)} \frac{\Phi_1(x)}{t^{1/(m-1)}} \quad \text{for all } t > 0 \text{ and a.e. } x \in \Omega. \tag{5-1}$$

Here $t_* = \kappa_* \|u_0\|^{-(m-1)}_{\mathrm{L}^1_{\Phi_1}(\Omega)}$, and the constants $\underline{\kappa}_0$ and κ_* depend only on N, s, γ, m, c_0, c_1, and Ω.

Notice that, for $t \ge t_*$, the dependence on the initial data disappears from the lower bound, as the inequality reads as

$$u(t) \ge \underline{\kappa}_0 \frac{\Phi_1}{t^{1/(m-1)}} \quad \text{for all } t \ge t_*,$$

where $\underline{\kappa}_0$ is an absolute constant. Assumption (L2) on the kernel K of $\mathcal{L}$ holds for all examples mentioned in Section 10A.

Clearly, the power in this lower bound does not match the one of the general upper bounds of Theorem 4.1; hence we cannot expect these bounds to be sharp. However, when $\sigma < 1$, for small times and small data and when $B \equiv 0$, the lower bounds (5-1) match the upper bounds of Corollary 4.5; hence they are sharp. Theorem 5.1 shows that, even in the "worst case scenario", there is a quantitative lower bound for all positive times, and shows infinite speed of propagation.

Matching lower bounds, I. Actually, in many cases the kernel of the nonlocal operator satisfies a stronger property, namely $\inf_{x,y\in\Omega} K(x,y) \ge \underline{\kappa}_\Omega > 0$ and $B \equiv 0$, in which case we can actually obtain sharp lower bounds for all times. Here we do not consider the potential logarithmic correction that may appear in the "critical case" $2sm = \gamma(m-1)$; indeed, as far as examples are concerned, the next theorem applies to the RFL and the CFL, for which $2sm > \gamma(m-1)$.

Theorem 5.2. *Let $\mathcal{L}$ satisfy* (A1), (A2), *and* (L1). *Furthermore, suppose that $\mathcal{L}$ has a first eigenfunction $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma$. Let σ be as in* (1-3) *and assume that*

- *either $\sigma = 1$;*
- *or $\sigma < 1$, $K(x,y) \le c_1|x-y|^{-(N+2s)}$ for a.e. $x, y \in \mathbb{R}^N$, and $\Phi_1 \in C^\gamma(\overline{\Omega})$.*

Let $u \ge 0$ be a weak dual solution to the CDP corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then there exists a constant $\underline{\kappa}_1 > 0$ such that the following inequality holds:

$$u(t,x) \ge \underline{\kappa}_1 \left(1 \wedge \frac{t}{t_*}\right)^{m/(m-1)} \frac{\Phi_1(x)^{\sigma/m}}{t^{1/(m-1)}} \quad \text{for all } t > 0 \text{ and a.e. } x \in \Omega, \tag{5-2}$$

where $t_* = \kappa_* \|u_0\|^{-(m-1)}_{\mathrm{L}^1_{\Phi_1}(\Omega)}$. The constants κ_* and $\underline{\kappa}_1$ depend only on N, s, γ, m, $\underline{\kappa}_\Omega$, c_1, Ω, and $\|\Phi_1\|_{C^\gamma(\Omega)}$.

Remarks. (i) As in the case of Theorem 5.1, for large times the dependence on the initial data disappears from the lower bound and we have absolute lower bounds.

(ii) The boundary behavior is sharp when $2sm \ne \gamma(m-1)$ in view of the upper bound from Theorem 4.1.

(iii) This theorem applies to the RFL and the CFL, but not to the SFL (or, more generally, spectral powers of elliptic operators); see Sections 2A and 2. In the case of the RFL, this result was obtained in Theorem 1 of [Bonforte et al. 2017a].

We have already seen the example of the separate-variables solutions (3-2) that have a very definite behavior at the boundary $\partial\Omega$. The analysis of general solutions leads to completely different situations for $\sigma = 1$ and $\sigma < 1$.

Matching lower bounds, II: The case $\sigma = 1$. When $\sigma = 1$ we can establish a quantitative lower bound near the boundary that matches the separate-variables behavior for large times (except in the case $2sm = \gamma(m-1)$ where the result is false, see Theorem 5.4 below). We do not need the assumption of nondegenerate kernel, so the SFL can be considered.

Theorem 5.3. *Let* (*A*1), (*A*2), *and* (K2) *hold, and let* $\sigma = 1$. *Let* $u \geq 0$ *be a weak dual solution to the CDP corresponding to* $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. *There exists a constant* $\underline{\kappa}_2 > 0$ *such that*

$$u(t,x) \geq \underline{\kappa}_2 \frac{\Phi_1(x)^{1/m}}{t^{1/(m-1)}} \quad \textit{for all } t \geq t_* \textit{ and a.e. } x \in \Omega. \tag{5-3}$$

Here, $t_* = \kappa_* \|u_0\|^{-(m-1)}_{\mathrm{L}^1_{\Phi_1}(\Omega)}$, *and the constants* κ_* *and* $\underline{\kappa}_2$ *depend only on* N, s, γ, m, *and* Ω.

Remarks. (i) At first sight, this theorem may seem weaker than the previous positivity result. However, this result has wider applicability since it holds under the only assumption (K2) on $\mathbb{G}$. In particular it is valid in the local case $s = 1$, where the finite speed of propagation makes it impossible to have global lower bounds for small times.

(ii) When $\mathcal{L} = -\Delta$ the result has been proven in [Aronson and Peletier 1981; Vázquez 2004] by quite different methods. On the other hand, our method is very general and immediately applies to the case when $\mathcal{L}$ is an elliptic operator with C^1 coefficients; see Section 10A.

(iii) This result fixes a small error in Theorem 7.1 of [Bonforte and Vázquez 2015], where the power σ was not present.

The anomalous lower bounds with small data. As shown in Theorem 5.1, the lower bound $u(t) \gtrsim \Phi_1$ is always valid. We now discuss the possibility of improving this bound.

Let S solve the elliptic problem (3-3). It follows by comparison whenever $u_0 \geq \epsilon_0 S$ with $\epsilon_0 > 0$ then $u(t) \geq S/(T_0 + t)^{1/(m-1)}$, where $T_0 = \epsilon_0^{1-m}$. Since $S \asymp \Phi_1^{\sigma/m}$ under (K4) (up to a possible logarithmic correction in the critical case, see Theorem 3.2), there are initial data for which the lower behavior is dictated by $\Phi_1(x)^{\sigma/m} t^{-1/(m-1)}$. More generally, as we shall see in Theorem 7.1, given any initial datum $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$ the function $v(t,x) := t^{1/(m-1)} u(t,x)$ always converges to S in $\mathrm{L}^\infty(\Omega)$ as $t \to \infty$, independently of the value of σ. Hence, one may conjecture that there should exist a waiting time $t_* > 0$ after which the lower behavior is dictated by $\Phi_1(x)^{\sigma/m} t^{-1/(m-1)}$, in analogy with what happens for the classical porous medium equation. As we shall see, this is actually *false* when $\sigma < 1$ or $2sm = \gamma(m-1)$. Since for large times $v(t,x)$ must look like $S(x)$ in uniform norm away from the boundary (by the interior regularity that we will prove later), the contrasting situation for large times could be described as a "dolphin's head" with the "snout" flatter than the "forehead". As $t \to \infty$ the forehead progressively fills the whole domain.

The next result shows that, in general, we cannot hope to prove that $u(t)$ is larger than $\Phi_1^{1/m}$. In particular, when $\sigma < 1$ or $2sm = \gamma(m-1)$, this shows that the behavior $u(t) \asymp S$ cannot hold.

Theorem 5.4. *Let* (A1), (A2), *and* (K2) *hold, and* $u \geq 0$ *be a weak dual solution to the CDP corresponding to a nonnegative initial datum* $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. *Assume that* $u_0(x) \leq C_0 \Phi_1(x)$ *a.e. in* Ω *for some* $C_0 > 0$. *Then there exists a constant* $\hat{\kappa}$, *depending only* N, s, γ, m, *and* Ω, *such that*

$$u(t,x)^m \leq C_0 \hat{\kappa} \frac{\Phi_1(x)}{t} \quad \textit{for all } t > 0 \textit{ and a.e. } x \in \Omega.$$

In particular, if $\sigma < 1$ (*resp.* $2sm = \gamma(m-1)$), *then*

$$\lim_{x\to\partial\Omega}\frac{u(t,x)}{\Phi_1(x)^{\sigma/m}} = 0 \quad \left(\textit{resp.}\ \lim_{x\to\partial\Omega}\frac{u(t,x)}{\Phi_1(x)^{1/m}(1+|\log\Phi_1(x)|)^{1/(m-1)}} = 0\right) \quad \textit{for any } t > 0.$$

The proposition above could make one wonder whether the sharp general lower bound could be given by $\Phi_1^{1/m}$, as in the case $\sigma = 1$. Recall that, under rather minimal assumptions on the kernel K associated to $\mathcal{L}$, we have a universal lower bound for $u(t)$ in terms of Φ_1 (see Theorem 5.1). Here we shall see that, under (K4), the bound $u(t) \gtrsim \Phi_1^{1/m}$ is false for $\sigma < 1$.

Theorem 5.5. *Let* (A1), (A2), *and* (K4) *hold, and let* $u \geq 0$ *be a weak dual solution to the CDP corresponding to a nonnegative initial datum* $u_0 \leq C_0\Phi_1$ *for some* $C_0 > 0$. *Assume that there exist constants* $\underline{\kappa}, T, \alpha > 0$ *such that*

$$u(T,x) \geq \underline{\kappa}\,\Phi_1^{\alpha}(x) \quad \textit{for a.e. } x \in \Omega.$$

Then $\alpha \geq 1 - 2s/\gamma$. *In particular* $\alpha > 1/m$ *if* $\sigma < 1$.

We devote the rest of this section to the proof of the above results, and to this end we collect in the first two subsections some preliminary lower bounds and results about approximate solutions.

5A. ***Lower bounds for weighted norms.*** Here we prove some useful lower bounds for weighted norms, which follow from the L^1-continuity for ordered solutions in the version proved in Proposition 8.1 of [Bonforte and Vázquez 2016].

Lemma 5.6 (backward-in-time $\mathrm{L}^1_{\Phi_1}$ lower bounds). *Let* u *be a solution to the CDP corresponding to the initial datum* $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. *For all*

$$0 \leq \tau_0 \leq t \leq \tau_0 + \frac{1}{(2\bar{K})^{1/(2s\vartheta_\gamma)}\|u(\tau_0)\|^{m-1}_{\mathrm{L}^1_{\Phi_1}(\Omega)}} \tag{5-4}$$

we have

$$\frac{1}{2}\int_\Omega u(\tau_0,x)\Phi_1(x)\,\mathrm{d}x \leq \int_\Omega u(t,x)\Phi_1(x)\,\mathrm{d}x, \tag{5-5}$$

where $\vartheta_\gamma := 1/[2s+(N+\gamma)(m-1)]$ *and* $\bar{K} > 0$ *is a computable constant.*

Proof. We recall the inequality of Proposition 8.1 of [Bonforte and Vázquez 2016], adapted to our case: for all $0 \leq \tau_0 \leq \tau, t$ we have

$$\int_\Omega u(\tau,x)\Phi_1(x)\,\mathrm{d}x \leq \int_\Omega u(t,x)\Phi_1(x)\,\mathrm{d}x + \bar{K}\|u(\tau_0)\|^{2s(m-1)\vartheta_\gamma+1}_{\mathrm{L}^1_{\Phi_1}(\Omega)}|t-\tau|^{2s\vartheta_\gamma}. \tag{5-6}$$

Choosing $\tau = \tau_0$ in the above inequality, we get

$$\left[1 - K_9\|u(\tau_0)\|^{2s(m-1)\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)}|t-\tau_0|^{2s\vartheta_\gamma}\right]\int_\Omega u(\tau_0,x)\Phi_1(x)\,\mathrm{d}x \leq \int_\Omega u(t,x)\Phi_1(x)\,\mathrm{d}x. \tag{5-7}$$

Then (5-5) follows from (5-4). □

We also need a lower bound for $\mathrm{L}^p_{\Phi_1}(\Omega)$ norms.

Lemma 5.7. *Let u be a solution to the CDP corresponding to the initial datum $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then the following lower bound holds true for any $t \in [0, t_*]$ and $p \geq 1$:*

$$c_2\left(\int_\Omega u_0(x)\Phi_1(x)\,\mathrm{d}x\right)^p \leq \int_\Omega u^p(t,x)\Phi_1(x)\,\mathrm{d}x. \tag{5-8}$$

Here $t_* = c_*\|u_0\|^{-(m-1)}_{\mathrm{L}^1_{\Phi_1}(\Omega)}$, where $c_2, c_* > 0$ are positive constants that depend only on N, s, m, p, Ω.

The proof of this lemma is an easy adaptation of the proof of Lemma 2.2 of [Bonforte et al. 2017a], so we skip it. Notice that c_* has explicit form given in [Bonforte and Vázquez 2015; 2016; Bonforte et al. 2017a], while the form of c_2 is given in the proof of Lemma 2.2 of [Bonforte et al. 2017a].

5B. ***Approximate solutions.*** To prove our lower bounds, we will need a special class of approximate solutions u_δ. We will list now the necessary details. In the case when $\mathcal{L}$ is the restricted fractional Laplacian (RFL), see Section 10A, these solutions have been used in Appendix II of [Bonforte et al. 2017a], where complete proofs can be found; the proof there holds also for the operators considered here. The interested reader can easily adapt the proofs in that paper to the current case.

Let us fix $\delta > 0$ and consider the problem

$$\begin{cases} \partial_t v_\delta = -\mathcal{L}[(v_\delta+\delta)^m - \delta^m] & \text{for any } (t,x) \in (0,\infty)\times\Omega, \\ v_\delta(t,x) = 0 & \text{for any } (t,x) \in (0,\infty)\times(\mathbb{R}^N\setminus\Omega), \\ v_\delta(0,x) = u_0(x) & \text{for any } x\in\Omega. \end{cases} \tag{5-9}$$

Next, we define

$$u_\delta := v_\delta + \delta.$$

We summarize here the basic properties of u_δ.

Approximate solutions u_δ exist, are unique, and bounded for all $(t,x) \in (0,\infty)\times\overline{\Omega}$ whenever $0 \leq u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Also, they are uniformly positive: for any $t \geq 0$,

$$u_\delta(t,x) \geq \delta > 0 \quad \text{for a.e. } x \in \Omega. \tag{5-10}$$

This implies that the equation for u_δ is never degenerate in the interior, so solutions are as smooth as the linear parabolic theory with the kernel K allows them to be (in particular, in the case of the fractional Laplacian, they are C^∞ in space and C^1 in time). Also, by a comparison principle, for all $\delta > \delta' > 0$,

$$u_\delta(t,x) \geq u_{\delta'}(t,x) \quad \text{for all } t \geq 0 \text{ and a.e. } x\in\Omega, \tag{5-11}$$

$$u_\delta(t,x) \geq u(t,x) \quad \text{for all } t \geq 0 \text{ and a.e. } x\in\Omega. \tag{5-12}$$

Furthermore, they converge in $\mathrm{L}^1_{\Phi_1}(\Omega)$ to u as $\delta \to 0$:

$$\|u_\delta(t) - u(t)\|_{\mathrm{L}^1_{\Phi_1}(\Omega)} \leq \|u_\delta(0) - u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)} = \delta\|\Phi_1\|_{\mathrm{L}^1(\Omega)}. \tag{5-13}$$

As a consequence of (5-11) and (5-13), we deduce that u_δ converges pointwise to u at almost every point: more precisely, for all $t \geq 0$,

$$u(t,x) = \lim_{\delta\to 0^+} u_\delta(t,x) \quad \text{for a.e. } x \in \Omega. \tag{5-14}$$

5C. ***Proof of Theorem 5.1.*** The proof consists in showing that

$$u(t,x) \geq \underline{u}(t,x) := k_0\, t\,\Phi_1(x)$$

for all $t \in [0, t_*]$ and a.e. $x \in \Omega$, where the parameter $k_0 > 0$ will be fixed later. Note that, once the inequality $u \geq \underline{u}$ on $[0, t_*]$ is proved, we conclude as follows: since $t \mapsto t^{1/(m-1)}\, u(t,x)$ is nondecreasing in $t > 0$ for a.e. $x \in \Omega$ [Bonforte and Vázquez 2016, (2.3)], we have

$$u(t,x) \geq \left(\frac{t_*}{t}\right)^{1/(m-1)} u(t_*, x) \geq k_0\, t_* \left(\frac{t_*}{t}\right)^{1/(m-1)} \Phi_1(x) \quad \text{for all } t \geq t_*.$$

Then, the result will follow $\underline{\kappa}_0 = k_0 t_*^{m/(m-1)}$ (note that, as we shall see below, $k_0 t_*^{m/(m-1)}$ can be chosen independently of u_0). Hence, we are left with proving that $u \geq \underline{u}$ on $[0, t_*]$.

<ins>Step 1</ins>: reduction to an approximate problem. Let us fix $\delta > 0$ and consider the approximate solutions u_δ constructed in Section 5B. We shall prove that $u_\delta \geq \underline{u}$ on $[0, t_*] \times \Omega$, so that the result will follow by the arbitrariness of δ.

<ins>Step 2</ins>: We claim that $\underline{u}(t,x) < u_\delta(t,x)$ for all $0 \leq t \leq t_*$ and $x \in \Omega$, for a suitable choice of $k_0 > 0$. Assume that the inequality $\underline{u} < u_\delta$ is false in $[0, t_*] \times \overline{\Omega}$, and let (t_c, x_c) be the first contact point between $\underline{u}$ and u_δ. Since $u_\delta = \delta > 0 = \underline{u}$ on the lateral boundary, $(t_c, x_c) \in (0, t_*] \times \Omega$. Now, since $(t_c, x_c) \in (0, t_*] \times \Omega$ is the first contact point, we necessarily have

$$u_\delta(t_c, x_c) = \underline{u}(t_c, x_c) \qquad \text{and} \qquad u_\delta(t,x) \geq \underline{u}(t,x) \quad \text{for all } t \in [0, t_c],\ x \in \overline{\Omega}. \tag{5-15}$$

Thus, as a consequence,

$$\partial_t u_\delta(t_c, x_c) \leq \partial_t \underline{u}(t_c, x_c) = k_0 \Phi_1(x_c). \tag{5-16}$$

Next, we observe the following Kato-type inequality holds: for any nonnegative function f,

$$\mathcal{L}(f^m) \leq m f^{m-1} \mathcal{L} f. \tag{5-17}$$

Indeed, by convexity, $f(x)^m - f(y)^m \leq m[f(x)]^{m-1}(f(x) - f(y))$; therefore

$$\begin{aligned}
\mathcal{L}(f^m)(x) &= \int_{\mathbb{R}^N} [f(x)^m - f(y)^m]\, K(x,y)\, \mathrm{d}y + B(x) f(x)^m \\
&\leq m[f(x)]^{m-1} \int_{\mathbb{R}^N} [f(x) - f(y)]\, K(x,y)\, \mathrm{d}y + B(x) f(x)^m \\
&= m[f(x)]^{m-1} \left(\int_{\mathbb{R}^N} [f(x) - f(y)]\, K(x,y)\, \mathrm{d}y + B(x) f(x) \right) - (m-1) B(x) f(x)^m \\
&\leq m[f(x)]^{m-1} \mathcal{L} f(x).
\end{aligned}$$

As a consequence of (5-17), since $t_c \leq t_*$ and Φ_1 is bounded,

$$\begin{aligned}
\mathcal{L}(\underline{u}^m)(t,x) \leq m \underline{u}^{m-1} \mathcal{L}(\underline{u}) &= m[k_0 t \Phi_1(x)]^{m-1}\, k_0 t \mathcal{L}(\Phi_1)(x) \\
&= m\lambda_1 [k_0 t \Phi_1(x)]^m \leq \kappa_1 (t_* k_0)^m \Phi_1(x).
\end{aligned} \tag{5-18}$$

Then, using (5-16) and (5-18), we establish an upper bound for $-\mathcal{L}(u_\delta^m - \underline{u}^m)(t_c, x_c)$ as follows:

$$-\mathcal{L}[u_\delta^m - \underline{u}^m](t_c, x_c) = \partial_t u_\delta(t_c, x_c) + \mathcal{L}(\underline{u}^m)(t_c, x_c) \le k_0\,[1 + \kappa_1 t_*^m k_0^{m-1}]\Phi_1(x_c). \tag{5-19}$$

Next, we want to prove lower bounds for $-\mathcal{L}(u_\delta^m - \psi^m)(t_c, x_c)$, and this is the point where the nonlocality of the operator enters, since we make essential use of hypothesis (L2). We recall that by (5-15) we have $u_\delta^m(t_c, x_c) = \underline{u}^m(t_c, x_c)$, so that assumption (L2) gives

$$\begin{aligned}
-\mathcal{L}[u_\delta^m - \underline{u}^m](t_c, x_c) &= -\mathcal{L}[u_\delta^m - \underline{u}^m](t_c, x_c) + B(x_c)[u_\delta^m(t_c, x_c) - \underline{u}^m(t_c, x_c)] \\
&= -\int_{\mathbb{R}^N} \big[(u_\delta^m(t_c, x_c) - u_\delta^m(t_c, y)) - (\underline{u}^m(t_c, x_c) - \underline{u}^m(t_c, y))\big] K(x_c, y)\,\mathrm{d}y \\
&= \int_\Omega [u_\delta^m(t_c, y) - \underline{u}^m(t_c, y)] K(x_c, y)\,\mathrm{d}y \\
&\ge c_0 \Phi_1(x_c) \int_\Omega [u_\delta^m(t_c, y) - \underline{u}^m(t_c, y)] \Phi_1(y)\,\mathrm{d}y,
\end{aligned}$$

from which it follows (since $\underline{u}^m = [k_0 t \Phi_1(x)]^m \le \kappa_2 (t_* k_0)^m$)

$$\begin{aligned}
-\mathcal{L}[u_\delta^m - \underline{u}^m](t_c, x_c) &\ge c_0 \Phi_1(x_c) \int_\Omega u_\delta^m(t_c, y)\Phi_1(y)\,\mathrm{d}y - c_0\Phi_1(x_c) \int_\Omega \underline{u}^m(t_c, y)\Phi_1(y)\,\mathrm{d}y. \\
&\ge c_0 \Phi_1(x_c) \int_\Omega u_\delta^m(t_c, y)\Phi_1(y)\,\mathrm{d}y - c_0\Phi_1(x_c)\kappa_3\,(t_* k_0)^m.
\end{aligned} \tag{5-20}$$

Combining the upper and lower bounds (5-19) and (5-20) we obtain

$$c_0 \Phi_1(x_c) \int_\Omega u_\delta^m(t_c, y)\Phi_1(y)\,\mathrm{d}y \le k_0\,[1 + (\kappa_1 + \kappa_3) t_*^m k_0^{m-1}]\Phi_1(x_c). \tag{5-21}$$

Hence, recalling (5-8), we get

$$c_2 \bigg(\int_\Omega u_0(x)\Phi_1(x)\,\mathrm{d}x \bigg)^m \le \int_\Omega u_\delta^m(t_c, y)\Phi_1(y)\,\mathrm{d}y \le \frac{k_0}{c_0}\big[1 + (\kappa_1 + \kappa_3) t_*^m k_0^{m-1}\big].$$

Since $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, this yields

$$c_2 \kappa_*^{m/(m-1)} t_*^{-m/(m-1)} \le \frac{k_0}{c_0}\big[1 + (\kappa_1 + \kappa_3) t_*^m k_0^{m-1}\big],$$

which gives the desired contradiction provided we choose k_0 so that $\underline{\kappa}_0 := k_0 t_*^{m/(m-1)}$ is universally small. □

5D. *Proof of Theorem 5.2.* The proof proceeds along the lines of the proof of Theorem 5.1, so we will just briefly mention the common parts.

We want to show that

$$\underline{u}(t, x) := \kappa_0\, t\, \Phi_1(x)^{\sigma/m}, \tag{5-22}$$

is a lower barrier for our problem on $[0, t_*] \times \Omega$ provided κ_0 is small enough. More precisely, as in the proof of Theorem 5.1, we aim to prove that $\underline{u} < u_\delta$ on $[0, t_*] \times \Omega$, as the lower bound for $t \ge t_*$ then follows by monotonicity.

Assume by contradiction that the inequality $\underline{u}(t,x) < u_\delta(t,x)$ is false inside $[0,t_*]\times\overline{\Omega}$. Since $\underline{u} < u_\delta$ on the parabolic boundary, letting (t_c,x_c) be the first contact point, we necessarily have $(t_c,x_c)\in(0,t_*]\times\Omega$. The desired contradiction will be obtained by combining the upper and lower bounds (which we prove below) for the quantity $-\mathcal{L}[u_\delta^m-\underline{u}^m](t_c,x_c)$, and then choosing $\kappa_0>0$ suitably small. In this direction, it is convenient in what follows to assume that

$$\kappa_0\le 1\wedge t_*^{-m/(m-1)} \quad \text{so that} \quad \kappa_0^{m-1}t_*^m\le 1. \tag{5-23}$$

Upper bound. We first establish the following upper bound: there exists a constant $\bar{A}>0$ such that

$$-\mathcal{L}[u_\delta^m-\underline{u}^m](t_c,x_c)\le \partial_t u_\delta(t_c,x_c)+\mathcal{L}\underline{u}^m(t_c,x_c)\le \bar{A}\kappa_0. \tag{5-24}$$

To prove this, we estimate $\partial_t u_\delta(t_c,x_c)$ and $\mathcal{L}\underline{u}^m(t_c,x_c)$ separately. First we notice that, since (t_δ,x_δ) is the first contact point, we have

$$u_\delta(t_\delta,x_\delta)=\underline{u}(t_\delta,x_\delta) \qquad \text{and} \qquad u_\delta(t,x)\ge \underline{u}(t,x) \quad \text{for all } t\in[0,t_\delta],\ x\in\Omega. \tag{5-25}$$

Hence, since $t_\delta\le t_*$,

$$\partial_t u_\delta(t_\delta,x_\delta)\le \partial_t\underline{u}(t_\delta,x_\delta)=\kappa_0\Phi_1(x)^{\sigma/m}\le\kappa_0\,\|\Phi_1\|_{\mathrm{L}^\infty(\Omega)}^{\sigma/m}=A_1\kappa_0, \tag{5-26}$$

where we defined $A_1:=\|\Phi_1\|_{\mathrm{L}^\infty(\Omega)}^{\sigma/m}$. Next we estimate $\mathcal{L}\underline{u}^m(t_c,x_c)$, using the Kato-type inequality (5-17); namely $\mathcal{L}[u^m]\le mu^{m-1}\mathcal{L}u$. This implies

$$\begin{aligned}\mathcal{L}[\underline{u}^m](t,x)&\le m\underline{u}^{m-1}(t,x)\,\mathcal{L}\underline{u}(t,x)=m(\kappa_0\,t)^m\Phi_1(x)^{\sigma(m-1)/m}\mathcal{L}\Phi_1^\sigma(x)\\&\le m(\kappa_0\,t_*)^m\|\Phi_1\|_{\mathrm{L}^\infty(\Omega)}^{\sigma(m-1)/m}\|\mathcal{L}\Phi_1^\sigma\|_{\mathrm{L}^\infty(\Omega)}:=A_2\kappa_0.\end{aligned} \tag{5-27}$$

Since $\kappa_0^{m-1}\,t_*^m\le 1$, see (5-23), in order to prove that A_2 is finite it is enough to bound $\|\mathcal{L}\Phi_1^\sigma\|_{\mathrm{L}^\infty(\Omega)}$. When $\sigma=1$ we simply have $\mathcal{L}\Phi_1=-\lambda_1\Phi_1$; hence $A_2\le m\lambda_1\|\Phi_1\|_{\mathrm{L}^\infty(\Omega)}^{2-1/m}$. When $\sigma<1$, we use the assumption $\Phi_1\in C^\gamma(\Omega)$ to estimate

$$|\Phi_1^\sigma(x)-\Phi_1^\sigma(y)|\le|\Phi_1(x)-\Phi_1(y)|^\sigma\le C|x-y|^{\gamma\sigma} \quad \text{for all } x,y\in\Omega. \tag{5-28}$$

Hence, since $\gamma\sigma=2sm/(m-1)>2s$ and $K(x,y)\le c_1|x-y|^{-(N+2s)}$, we see that

$$\begin{aligned}|\mathcal{L}\Phi_1^\sigma(x)|&=\left|\int_{\mathbb{R}^N}[\Phi_1^\sigma(x)-\Phi_1^\sigma(y)]K(x,y)\,\mathrm{d}y\right|\\&\le\int_\Omega|x-y|^{\gamma\sigma}K(x,y)\,\mathrm{d}y+C\|\Phi_1\|_{\mathrm{L}^\infty(\Omega)}^\sigma\int_{\mathbb{R}^N\setminus B_1}|y|^{-(N+2s)}\,\mathrm{d}y<\infty;\end{aligned}$$

hence A_2 is again finite. Combining (5-26) and (5-27), we obtain (5-24) with $\bar{A}:=A_1+A_2$.

Lower bound. We want to prove that there exists $\underline{A}>0$ such that

$$-\mathcal{L}[u_\delta^m-\underline{u}^m](t_c,x_c)\ge\frac{\underline{\kappa}_\Omega}{\|\Phi_1\|_{\mathrm{L}^\infty(\Omega)}}\int_\Omega u_\delta^m(t_c,y)\Phi_1(y)\,\mathrm{d}y-\underline{A}\kappa_0. \tag{5-29}$$

This follows by (L1) and (5-25):

$$
\begin{aligned}
-\mathcal{L}[u_\delta^m-\underline{u}^m](t_c,x_c)&=-\int_{\mathbb{R}^N}\big[(u_\delta^m(t_c,x_c)-u_\delta^m(t_c,y))-(\underline{u}^m(t_c,x_c)-\underline{u}^m(t_c,y))\big]K(x,y)\,\mathrm{d}y\\
&=\int_\Omega[u_\delta^m(t_c,y)-\underline{u}^m(t_c,y)]K(x,y)\,\mathrm{d}y\\
&\geq \underline{\kappa}_\Omega\int_\Omega[u_\delta^m(t_c,y)-\underline{u}^m(t_c,y)]\,\mathrm{d}y\\
&\geq \frac{\underline{\kappa}_\Omega}{\|\Phi_1\|_{\mathrm{L}^\infty(\Omega)}}\int_\Omega u_\delta^m(t_c,y)\Phi_1(y)\,\mathrm{d}y-\underline{A}\kappa_0,
\end{aligned}\tag{5-30}
$$

where in the last step we used that $\underline{u}^m(t_c,y)=[\kappa_0 t\Phi_1^{\sigma/m}(y)]^m\leq\kappa_2(\kappa_0t_*)^m$ and $\kappa_0^{m-1}t_*^m\leq 1$; see (5-23).

End of the proof. The contradiction can be now obtained by joining the upper and lower bounds (5-24) and (5-29). More precisely, we have proved

$$\int_\Omega u_\delta^m(t_c,y)\Phi_1(y)\,\mathrm{d}y\leq\frac{\|\Phi_1\|_{\mathrm{L}^\infty(\Omega)}}{\underline{\kappa}_\Omega}(\bar{A}+\underline{A})\kappa_0:=\bar{\kappa}\kappa_0,$$

which combined with the lower bound (5-8) yields

$$c_2\left(\int_\Omega u_0(x)\Phi_1(x)\,\mathrm{d}x\right)^m\leq\int_\Omega u_\delta^m(t_c,y)\Phi_1(y)\,\mathrm{d}y\leq\bar{\kappa}\kappa_0.$$

Setting $\kappa_0:=(1\wedge c_2/\bar{\kappa})t_*^{-m/(m-1)}$, we obtain the desired contradiction. □

5E. *Proof of Theorem 5.3.* We first recall the upper pointwise estimates (4-3): for all $0\leq t_0\leq t_1\leq t$ and a.e. $x_0\in\Omega$, we have

$$\int_\Omega u(t_0,x)\mathbb{G}(x,x_0)\,\mathrm{d}x-\int_\Omega u(t_1,x)\mathbb{G}(x,x_0)\,\mathrm{d}x\leq(m-1)\frac{t^{m/(m-1)}}{t_0^{1/(m-1)}}\,u^m(t,x_0).\tag{5-31}$$

The proof follows by estimating the two integrals on the left-hand side separately.

We begin by using the upper bounds (4-8) to get

$$\int_\Omega u(t_1,x)\mathbb{G}(x,x_0)\,\mathrm{d}x\leq\bar{\kappa}\frac{\Phi_1(x_0)}{t_1^{1/(m-1)}}\quad\text{for all }(t_1,x)\in(0,+\infty)\times\Omega.\tag{5-32}$$

Then we note that, as a consequence of (K2) and Lemma 5.6,

$$\int_\Omega u(t_0,x)\mathbb{G}(x,x_0)\,\mathrm{d}x\geq\underline{\kappa}_\Omega\Phi_1(x_0)\int_\Omega u(t_0,x)\Phi_1(x)\,\mathrm{d}x\geq\tfrac{1}{2}\underline{\kappa}_\Omega\Phi_1(x_0)\int_\Omega u_0(x)\Phi_1(x)\,\mathrm{d}x\tag{5-33}$$

provided $t_0\leq\tau_0/\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{m-1}$. Combining (5-31), (5-32), and (5-33), for all $t\geq t_1\geq t_0\geq 0$ we obtain

$$u^m(t,x_0)\geq\frac{t_0^{1/(m-1)}}{m-1}\big(\tfrac{1}{2}\underline{\kappa}_\Omega\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}-\bar{\kappa}t_1^{-1/(m-1)}\big)\frac{\Phi_1(x_0)}{t^{m/(m-1)}}.$$

Choosing

$$t_0 := \frac{\tau_0}{\|u_0\|^{m-1}_{\mathrm{L}^1_{\Phi_1}(\Omega)}} \le t_1 := t_* = \frac{\kappa_*}{\|u_0\|^{m-1}_{\mathrm{L}^1_{\Phi_1}(\Omega)}} \quad \text{with} \quad \kappa_* \ge \tau_0 \vee \left(\frac{\underline{\kappa}_\Omega}{4\bar{\kappa}}\right)^{m-1}$$

so that $\frac{1}{2}\underline{\kappa}_\Omega \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)} - \bar{\kappa} t_1^{-1/(m-1)} \ge \frac{1}{4}\underline{\kappa}_\Omega \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}$, the result follows. □

5F. *Proofs of Theorems 5.4 and 5.5.*

Proof of Theorem 5.4. Since $u_0 \le C_0 \Phi_1$ and $\mathcal{L}\Phi = \lambda_1 \Phi_1$, we have

$$\int_\Omega u_0(x)\,\mathbb{G}(x, x_0)\,\mathrm{d}x \le C_0 \int_\Omega \Phi_1(x)\,\mathbb{G}(x, x_0)\,\mathrm{d}x = C_0 \mathcal{L}^{-1}\Phi_1(x_0) = \frac{C_0}{\lambda_1}\Phi_1(x_0).$$

Since $t \mapsto \int_\Omega u(t, y)\,\mathbb{G}(x, y)\,\mathrm{d}y$ is decreasing, see (4-2), it follows that

$$\int_\Omega u(t, y)\,\mathbb{G}(x_0, y)\,\mathrm{d}y \le \frac{C_0}{\lambda_1}\Phi_1(x_0) \quad \text{for all } t \ge 0. \tag{5-34}$$

Combining this estimate with (4-7) concludes the proof. □

Proof of Theorem 5.5. Given $x_0 \in \Omega$, set $R_0 := \mathrm{dist}(x_0, \partial\Omega)$. Since $\mathbb{G}(x, x_0) \gtrsim |x - x_0|^{-(N-2s)}$ inside $B_{R_0/2}(x_0)$ by (K4), using our assumption on $u(T)$ we get

$$\int_\Omega \mathbb{G}(x, x_0) u(T, x)\,\mathrm{d}x \gtrsim \int_{B_{R_0/2}(x_0)} \frac{\Phi_1(x)^\alpha}{|x - x_0|^{N-2s}} \gtrsim \Phi_1(x_0)^\alpha R_0^{2s}.$$

Recalling that $\Phi_1(x_0) \asymp R_0^\gamma$, this yields

$$\Phi_1(x_0)^{\alpha + 2s/\gamma} \lesssim \int_\Omega \mathbb{G}(x, x_0) u(T, x)\,\mathrm{d}x.$$

Combining the above inequality with (5-34) gives

$$\Phi_1(x_0)^{\alpha+2s/\gamma} \lesssim \Phi_1(x_0) \quad \text{for all } x_0 \in \Omega,$$

which implies

$$\alpha \ge 1 - \frac{2s}{\gamma}.$$

Noticing that $1 - 2s/\gamma > 1/m$ if and only if $\sigma < 1$, this concludes the proof. □

6. Summary of the general decay and boundary results

We now present a summary of the main results, which can be summarized in various forms of upper and lower bounds, which we call the global Harnack principle, GHP for short. As already mentioned, such inequalities are important for regularity issues (see Section 8), and they play a fundamental role in formulating the sharp asymptotic behavior (see Section 7). The proof of such a GHP is obtained by combining upper and lower bounds, stated and proved in Sections 4 and 5 respectively. There are cases when the bounds do not match, for which the complicated panorama described in the Introduction holds. As explained before, as far as examples are concerned, the latter anomalous situation happens only for the SFL.

Theorem 6.1 (global Harnack principle I). *Let $\mathcal{L}$ satisfy* (A1), (A2), (K2), *and* (L1). *Furthermore, suppose that $\mathcal{L}$ has a first eigenfunction $\Phi_1 \asymp \mathrm{dist}(\cdot\,,\partial\Omega)^\gamma$. Let σ be as in* (1-3) *and assume that $2sm \neq \gamma(m-1)$ and*

- *either $\sigma = 1$*
- *or $\sigma < 1$, $K(x,y) \le c_1|x-y|^{-(N+2s)}$ for a.e. $x, y \in \mathbb{R}^N$, and $\Phi_1 \in C^\gamma(\overline{\Omega})$.*

Let $u \ge 0$ be a weak dual solution to the CDP corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then, there exist constants $\underline{\kappa}, \bar{\kappa} > 0$ such that the following inequality holds:

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{m/(m-1)} \frac{\Phi_1(x)^{\sigma/m}}{t^{1/(m-1)}} \le u(t,x) \le \bar{\kappa}\,\frac{\Phi_1(x)^{\sigma/m}}{t^{1/(m-1)}} \quad \text{for all } t>0,\ x \in \Omega. \tag{6-1}$$

The constants $\underline{\kappa}, \bar{\kappa}$ depend only on $N, s, \gamma, m, c_1, \underline{\kappa}_\Omega, \Omega$, and $\|\Phi_1\|_{C^\gamma(\Omega)}$.

Proof. We combine the upper bound (4-1) with the lower bound (5-2). The expression of t_* is explicitly given in Theorem 5.2. □

Degenerate kernels. When the kernel K vanishes on $\partial\Omega$, there are two combinations of upper/lower bounds that provide Harnack inequalities, one for small times and one for large times. As we have already seen, there is a strong difference between the cases $\sigma = 1$ and $\sigma < 1$.

Theorem 6.2 (global Harnack principle II). *Let* (A1), (A2), *and* (K2) *hold. Let $u \ge 0$ be a weak dual solution to the CDP corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Assume that*

- *either $\sigma = 1$ and $2sm \neq \gamma(m-1)$;*
- *or $\sigma < 1$, $u_0 \ge \underline{\kappa}_0 \Phi_1^{\sigma/m}$ for some $\underline{\kappa}_0 > 0$, and* (K4) *holds.*

Then there exist constants $\underline{\kappa}, \bar{\kappa} > 0$ such that the following inequality holds:

$$\underline{\kappa}\,\frac{\Phi_1(x)^{\sigma/m}}{t^{1/(m-1)}} \le u(t,x) \le \bar{\kappa}\,\frac{\Phi_1(x)^{\sigma/m}}{t^{1/(m-1)}} \quad \text{for all } t \ge t_*,\ x \in \Omega.$$

If $2sm = \gamma(m-1)$, assuming (K4) *and that $u_0 \ge \underline{\kappa}_0 \Phi_1(1+|\log\Phi_1|)^{1/(m-1)}$ for some $\underline{\kappa}_0 > 0$, then for all $t \ge t_*$ and all $x \in \Omega$*

$$\underline{\kappa}\,\frac{\Phi_1(x)^{1/m}}{t^{1/(m-1)}}(1+|\log\Phi_1(x)|)^{1/(m-1)} \le u(t,x) \le \bar{\kappa}\,\frac{\Phi_1(x)^{1/m}}{t^{1/(m-1)}}(1+|\log\Phi_1(x)|)^{1/(m-1)}.$$

The constants $\underline{\kappa}, \bar{\kappa}$ depend only on $N, s, \gamma, m, \underline{\kappa}_0, \underline{\kappa}_\Omega$, and Ω.

Proof. In the case $\sigma = 1$, we combine the upper bound (4-1) with the lower bound (5-3). The expression of t_* is explicitly given in Theorem 5.3. When $\sigma < 1$, the upper bound is still given by (4-1), while the lower bound follows by comparison with the solution $S(x)(\underline{\kappa}_0^{1-m} + t)^{-1/(m-1)}$, recalling that $S \asymp \Phi_1^{\sigma/m}$ (see Theorem 3.2). □

Remark. Local Harnack inequalities of elliptic/backward type follow as a consequence of Theorems 6.1 and 6.2, for all times and for large times respectively, see Theorem 8.2.

Note that, for small times, we cannot find matching powers for a global Harnack inequality (except for some special initial data), and such a result is actually false for $s = 1$ (in view of the finite speed of propagation). Hence, in the remaining cases, we have only the following general result.

Theorem 6.3 (nonmatching upper and lower bounds). *Let $\mathcal{L}$ satisfy* (A1), (A2), (K2), *and* (L2). *Let $u \ge 0$ be a weak dual solution to the CDP corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then, there exist constants $\underline{\kappa}, \bar{\kappa} > 0$ such that the following inequality holds when $2sm \neq \gamma(m-1)$:*

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{m/(m-1)} \frac{\Phi_1(x)}{t^{1/(m-1)}} \le u(t,x) \le \bar{\kappa}\,\frac{\Phi_1(x)^{\sigma/m}}{t^{1/(m-1)}} \quad \textit{for all } t > 0,\ x \in \Omega. \tag{6-2}$$

When $2sm = \gamma(m-1)$, a logarithmic correction $(1 + |\log \Phi_1(x)|)^{1/(m-1)}$ appears in the right-hand side.

Proof. We combine the upper bound (4-1) with the lower bound (5-1). The expression of t_* is explicitly given in Theorem 5.1. □

Remark. As already mentioned in the Introduction, in the nonmatching case, which in examples can only happen for spectral-type operators, we have the appearance of an *anomalous behavior of solutions* corresponding to “small data”: it happens for all times when $\sigma < 1$ or $2sm = \gamma(m-1)$, and it can eventually happen for short times when $\sigma = 1$.

7. Asymptotic behavior

An important application of the global Harnack inequalities of the previous section concerns the sharp asymptotic behavior of solutions. More precisely, we first show that for large times all solutions behave like the separate-variables solution $\mathcal{U}(t,x) = S(x)\,t^{-1/(m-1)}$ introduced at the end of Section 3. Then, whenever the GHP holds, we can improve this result to an estimate in relative error.

Theorem 7.1 (asymptotic behavior). *Assume that $\mathcal{L}$ satisfies* (A1), (A2), *and* (K2), *and let S be as in Theorem 3.2. Let u be any weak dual solution to the CDP. Then, unless $u \equiv 0$,*

$$\|t^{1/(m-1)}u(t,\cdot) - S\|_{\mathrm{L}^\infty(\Omega)} \xrightarrow{t\to\infty} 0. \tag{7-1}$$

Proof. The proof uses rescaling and time monotonicity arguments, and it is a simple adaptation of the proof of Theorem 2.3 of [Bonforte et al. 2015]. In those arguments, the interior $C^\alpha_x(\Omega)$ continuity is needed to improve the $\mathrm{L}^1(\Omega)$ convergence to $\mathrm{L}^\infty(\Omega)$, but the interior Hölder continuity is guaranteed by Theorem 8.1(i) below. □

We now exploit the GHP to get a stronger result.

Theorem 7.2 (sharp asymptotic behavior). *Under the assumptions of Theorem 7.1, assume that $u \not\equiv 0$. Furthermore, suppose that either the assumptions of Theorem 6.1 or of Theorem 6.2 hold. Set $\mathcal{U}(t,x) := t^{-1/(m-1)}S(x)$. Then there exists $c_0 > 0$ such that, for all $t \ge t_0 := c_0\|u_0\|^{-(m-1)}_{\mathrm{L}^1_{\Phi_1}(\Omega)}$, we have*

$$\left\|\frac{u(t,\cdot)}{\mathcal{U}(t,\cdot)} - 1\right\|_{\mathrm{L}^\infty(\Omega)} \le \frac{2}{m-1}\,\frac{t_0}{t_0+t}. \tag{7-2}$$

We remark that the constant $c_0 > 0$ only depends on N, s, γ, m, $\underline{\kappa}_0$, $\underline{\kappa}_\Omega$, and Ω.

Remark. This asymptotic result is sharp, as it can be checked by considering $u(t,x)=\mathcal{U}(t+1,x)$. For the classical case, that is $\mathcal{L}=\Delta$, we recover the classical results of [Aronson and Peletier 1981; Vázquez 2004] with a different proof.

Proof. Notice that we are in the position to use Theorem 6.1 or 6.2; namely we have

$$u(t)\asymp t^{-1/(m-1)}S=\mathcal{U}(t,\cdot)\quad \text{for all } t\ge t_*,$$

where the last equivalence follows by Theorem 3.2. Hence, we can rewrite the bounds above saying that there exist $\underline{\kappa},\bar{\kappa}>0$ such that

$$\underline{\kappa}\,\frac{S(x)}{t^{1/(m-1)}}\le u(t,x)\le\bar{\kappa}\,\frac{S(x)}{t^{1/(m-1)}}\quad \text{for all } t\ge t_* \text{ and a.e. } x\in\Omega. \tag{7-3}$$

Since $t_*=\kappa_*\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, the first inequality implies

$$\frac{S}{(t_*+t_0)^{1/(m-1)}}\le\underline{\kappa}\,\frac{S}{t_*^{1/(m-1)}}\le u(t_*)$$

for some $t_0=c_0\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}\ge t_*$. Hence, by the comparison principle,

$$\frac{S}{(t+t_0)^{1/(m-1)}}\le u(t)\quad \text{for all } t\ge t_*.$$

On the other hand, it follows by (7-3) that $u(t,x)\le\mathcal{U}_T(t,x):=S(x)(t-T)^{-1/(m-1)}$ for all $t\ge T$ provided T is large enough. If we now start to reduce T, the comparison principle combined with the upper bound (4-1) shows that u can never touch $\mathcal{U}_T$ from below in $(T,\infty)\times\Omega$. Hence we can reduce T until $T=0$, proving that $u\le\mathcal{U}_0$; for an alternative proof, see Lemma 5.4 in [Bonforte et al. 2015]. Since $t_0\ge t_*$, this shows that

$$\frac{S(x)}{(t+t_0)^{1/(m-1)}}\le u(t,x)\le\frac{S(x)}{t^{1/(m-1)}}\quad \text{for all } t\ge t_0 \text{ and a.e. } x\in\Omega.$$

Therefore

$$\left|1-\frac{u(t,x)}{\mathcal{U}(t,x)}\right|\le 1-\left(1-\frac{t_0}{t_0+t}\right)^{1/(m-1)}\le\frac{2}{m-1}\,\frac{t_0}{t_0+t}\quad \text{for all } t\ge t_0 \text{ and a.e. } x\in\Omega,$$

as desired. □

8. Regularity results

In order to obtain the regularity results, we basically require the validity of a global Harnack principle, namely Theorem 6.1, 6.2, or 6.3, depending on the situation under study. For some higher-regularity results, we will eventually need some extra assumptions on the kernels. For simplicity we assume that $\mathcal{L}$ is described by a kernel, without any lower-order term. However, it is clear that the presence of lower-order terms does not play any role in the interior regularity.

Theorem 8.1 (interior regularity). *Assume that*

$$\mathcal{L}f(x)=\mathrm{P.V.}\int_{\mathbb{R}^N}(f(x)-f(y))K(x,y)\,\mathrm{d}y+B(x)f(x),$$

with

$$K(x,y) \asymp |x-y|^{-(N+2s)} \quad \text{in } B_{2r}(x_0) \subset \Omega, \qquad K(x,y) \lesssim |x-y|^{-(N+2s)} \quad \text{in } \mathbb{R}^N \setminus B_{2r}(x_0).$$

Let u be a nonnegative bounded weak dual solution to the CDP on $(T_0, T_1) \times \Omega$, and assume that there exist $\delta, M > 0$ such that

$$\begin{aligned} 0 < \delta \le u(t,x) \quad & \text{for a.e. } (t,x) \in (T_0,T_1) \times B_{2r}(x_0), \\ 0 \le u(t,x) \le M \quad & \text{for a.e. } (t,x) \in (T_0,T_1) \times \Omega. \end{aligned}$$

(i) *Then u is Hölder continuous in the interior. More precisely, there exists $\alpha > 0$ such that, for all $0 < T_0 < T_2 < T_1$,*

$$\|u\|_{C^{\alpha/2s,\alpha}_{t,x}((T_2,T_1)\times B_r(x_0))} \le C. \tag{8-1}$$

(ii) *Assume in addition $|K(x,y) - K(x',y)| \le c|x-x'|^\beta \, |y|^{-(N+2s)}$ for some $\beta \in (0, 1 \wedge 2s)$ such that $\beta + 2s$ is not an integer. Then u is a classical solution in the interior. More precisely, for all $0 < T_0 < T_2 < T_1$,*

$$\|u\|_{C^{1+\beta/2s,2s+\beta}_{t,x}((T_2,T_1)\times B_r(x_0))} \le C. \tag{8-2}$$

The constants in the above regularity estimates depend on the solution only through the upper and lower bounds on u. These bounds can be made quantitative by means of local Harnack inequalities, of elliptic and forward type, which follow from the global ones.

Theorem 8.2 (local Harnack inequalities of elliptic/backward type). *Under the assumptions of Theorem 6.1, there exists a constant $\widehat{H} > 0$, depending only on $N, s, \gamma, m, c_1, \underline{\kappa}_\Omega, \Omega$, such that for all balls $B_R(x_0)$ such that $B_{2R}(x_0) \subset \Omega$*

$$\sup_{x \in B_R(x_0)} u(t,x) \le \frac{\widehat{H}}{(1 \wedge t/t_*)^{m/(m-1)}} \inf_{x \in B_R(x_0)} u(t,x) \quad \text{for all } t > 0. \tag{8-3}$$

Moreover, for all $t > 0$ and all $h > 0$ we have

$$\sup_{x \in B_R(x_0)} u(t,x) \le \widehat{H}\left[\left(1 + \frac{h}{t}\right)\left(1 \wedge \frac{t}{t_*}\right)^{-m}\right]^{1/(m-1)} \inf_{x \in B_R(x_0)} u(t+h,x). \tag{8-4}$$

Proof. Recalling (6-1), the bound (8-3) follows easily from the following Harnack inequality for the first eigenfunction, see for instance [Bonforte et al. 2017b]:

$$\sup_{x \in B_R(x_0)} \Phi_1(x) \le H_{N,s,\gamma,\Omega} \inf_{x \in B_R(x_0)} \Phi_1(x).$$

Since $u(t,x) \le (1 + h/t)^{1/(m-1)} u(t+h,x)$, by the time monotonicity of $t \mapsto t^{1/(m-1)} u(t,x)$, (8-4) follows. □

Remark. The same result holds for large times $t \ge t_*$ as a consequence of Theorem 6.2. Already in the local case $s = 1$, these Harnack inequalities are stronger than the known two-sided inequalities valid for solutions to the Dirichlet problem for the classical porous medium equation, see [Aronson and Caffarelli 1983; Daskalopoulos and Kenig 2007; DiBenedetto 1988; 1993; DiBenedetto et al. 2012], which are of forward type and are often stated in terms of the so-called intrinsic geometry. Note that elliptic and backward Harnack-type inequalities usually occur in the fast diffusion range $m < 1$ [Bonforte et al. 2012; Bonforte and Vázquez 2006; 2010; 2014], or for linear equations in bounded domains [Fabes et al. 1986; Safonov and Yuan 1999].

For sharp boundary regularity we need a GHP with matching powers, like Theorems 6.1 or 6.2, and when $s > \gamma/2$, we can also prove Hölder regularity up to the boundary. We leave to the interested reader to check that the presence of an extra term $B(x)u^m(t,x)$ with $0 \le B(x) \le c_1 \operatorname{dist}(x,\partial\Omega)^{-2s}$ (as in the SFL) does not affect the validity of the next result. Indeed, when considering the scaling in (8-6), the lower term scales as $\widehat{B}_r u_r^m$ with $0 \le \widehat{B}_r \le c_1$ inside the unit ball B_1.

Theorem 8.3 (Hölder continuity up to the boundary). *Under assumptions of Theorem 8.1*(ii), *assume in addition that $2s > \gamma$. Then u is Hölder continuous up to the boundary. More precisely, for all $0 < T_0 < T_2 < T_1$ there exists a constant $C > 0$ such that*

$$\|u\|_{C^{\gamma/m\vartheta,\gamma/m}_{t,x}((T_2,T_1)\times\Omega)} \le C \quad \text{with } \vartheta := 2s - \gamma\Big(1 - \frac{1}{m}\Big). \tag{8-5}$$

Remark. Since we have $u(t,x) \asymp \Phi_1(x)^{1/m} \asymp \operatorname{dist}(x,\partial\Omega)^{\gamma/m}$ (note that $2s > \gamma$ implies that $\sigma = 1$ and that $2sm \neq \gamma(m-1)$), the spacial Hölder exponent is sharp, while the Hölder exponent in time is the natural one by scaling.

8A. ***Proof of interior regularity.*** The strategy to prove Theorem 8.1 follows the lines of [Bonforte et al. 2017a] but with some modifications. The basic idea is that, because u is bounded away from zero and infinity, the equation is nondegenerate and we can use parabolic regularity for nonlocal equations to obtain the results. More precisely, interior Hölder regularity will follow by applying $C^{\alpha/2s,\alpha}_{t,x}$ estimates of [Felsinger and Kassmann 2013] for a "localized" linear problem. Once Hölder regularity is established, under a Hölder continuity assumption on the kernel we can use the Schauder estimates proved in [Dong and Zhang 2016] to conclude.

8A1. *Localization of the problem.* Up to a rescaling, we can assume $r = 2$, $T_0 = 0$, $T_1 = 1$. Also, by a standard covering argument, it is enough to prove the results with $T_2 = \frac{1}{2}$.

Take a cutoff function $\rho \in C_c^\infty(B_4)$ such that $\rho \equiv 1$ on B_3 and $\eta \in C_c^\infty(B_2)$ a cutoff function such that $\eta \equiv 1$ on B_1, and define $v = \rho u$. By construction $u = v$ on $(0,1) \times B_3$. Since $\rho \equiv 1$ on B_3, we can write the equation for v on the small cylinder $(0,1) \times B_1$ as

$$\partial_t v(t,x) = -\mathcal{L}[v^m](t,x) + g(t,x) = -L_a v(t,x) + f(t,x) + g(t,x),$$

where

$$L_a[v](t,x) := \int_{\mathbb{R}^N} (v(t,x) - v(t,y)) a(t,x,y) K(x,y)\,\mathrm{d}y,$$

$$\begin{aligned} a(t,x,y) &:= \frac{v^m(t,x) - v^m(t,y)}{v(t,x) - v(t,y)} \eta(x-y) + [1 - \eta(x-y)] \\ &= m\eta(x-y) \int_0^1 [(1-\lambda)v(t,x) + \lambda v(t,y)]^{m-1}\,\mathrm{d}\lambda + [1 - \eta(x-y)], \end{aligned}$$

$$f(t,x) := \int_{\mathbb{R}^N\setminus B_1(x)} \big(v^m(t,x) - v^m(t,y) - v(t,x) + v(t,y)\big)[1-\eta(x-y)]K(x,y)\,\mathrm{d}y,$$

$$g(t,x) := -\mathcal{L}[(1-\rho^m)u^m](t,x) = \int_{\mathbb{R}^N\setminus B_3} (1-\rho^m(y))u^m(t,y)K(x,y)\,\mathrm{d}y$$

(recall that $(1-\rho^m)u^m \equiv 0$ on $(0,1)\times B_3$).

8A2. *Hölder continuity in the interior.* Set $b := f + g$, with f and g as above. It is easy to check that, since $K(x,y) \lesssim |x-y|^{-(N+2s)}$, we have $b \in \mathrm{L}^\infty((0,1)\times B_1)$. Also, since $0<\delta\le u\le M$ inside $(0,1)\times B_1$, there exists $\Lambda>1$ such that $\Lambda^{-1}\le a(t,x,y)\le\Lambda$ for a.e. $(t,x,y)\in(0,1)\times B_1\times B_1$ with $|x-y|\le 1$. This guarantees that the linear operator L_a is uniformly elliptic, so we can apply the results in [Felsinger and Kassmann 2013] to ensure that

$$\|v\|_{C^{\alpha/2s,\alpha}_{t,x}((1/2,1)\times B_{1/2})} \le C\big(\|b\|_{L^\infty((0,1)\times B_1)} + \|v\|_{L^\infty((0,1)\times\mathbb{R}^N)}\big)$$

for some universal exponent $\alpha>0$. This proves Theorem 8.1(i).

8A3. *Classical solutions in the interior.* Now that we know that $u \in C^{\alpha/2s,\alpha}\big(\big(\tfrac12,1\big)\times B_{1/2}\big)$, we repeat the localization argument above with cutoff functions ρ and η supported inside $\big(\tfrac12,1\big)\times B_{1/2}$ to ensure that $v:=\rho u$ is Hölder continuous in $\big(\tfrac12,1\big)\times\mathbb{R}^N$. Then, to obtain higher regularity we argue as follows.

Set $\beta_1:=\min\{\alpha,\beta\}$. Thanks to the assumption on K and Theorem 8.1(i), it is easy to check that $K_a(t,x,y):=a(t,x,y)K(x,y)$ satisfies

$$|K_a(t,x,y)-K_a(t',x',y)| \le C\big(|x-x'|^{\beta_1}+|t-t'|^{\beta_1/2s}\big)|y|^{-(N+2s)}$$

inside $\big(\tfrac12,1\big)\times B_{1/2}$. Also, $f,g\in C^{\beta_1/2s,\beta_1}\big(\big(\tfrac12,1\big)\times B_{1/2}\big)$. This allows us to apply the Schauder estimates from [Dong and Zhang 2016], see also [Chang-Lara and Kriventsov 2017], to obtain that

$$\|v\|_{C^{1+\beta_1/2s,2s+\beta}_{t,x}((3/4,1)\times B_{1/4})} \le C\big(\|b\|_{C^{\beta/2s,\beta}_{t,x}((1/2,1)\times B_{1/2})} + \|v\|_{C^{\beta/2s,\beta}_{t,x}((1/2,1)\times\mathbb{R}^N)}\big).$$

In particular, $u\in C^{1+\beta_1/2s,2s+\beta_1}\big(\big(\tfrac34,1\big)\times B_{1/8}\big)$. In the case $\beta_1=\beta$ we stop here. Otherwise we set $\alpha_1:=2s+\beta$ and we repeat the argument above with $\beta_2:=\min\{\alpha_1,\beta\}$ in place of β_1. In this way, we obtain that $u\in C^{1+\beta_1/2s,2s+\beta_1}((1-2^{-4},1)\times B_{2^{-5}})$. Iterating this procedure finitely many times, we finally obtain

$$u\in C^{1+\beta/2s,2s+\beta}((1-2^{-k},1)\times B_{2^{-k-1}})$$

for some universal k. Finally, a covering argument completes the proof of Theorem 8.1(ii).

8B. ***Proof of boundary regularity.*** The proof of Theorem 8.3 follows by scaling and interior estimates. Notice that the assumption $2s>\gamma$ implies $\sigma=1$; hence $u(t)$ has matching upper and lower bounds.

Given $x_0\in\Omega$, set $r=\mathrm{dist}(x_0,\partial\Omega)/2$ and define

$$u_r(t,x):=r^{-\gamma/m}\,u(t_0+r^\vartheta t,\ x_0+rx),\quad \text{with } \vartheta:=2s-\gamma\Big(1-\frac{1}{m}\Big). \tag{8-6}$$

Note that, because $2s > \gamma$, we have $\vartheta > 0$. With this definition, we see that u_r satisfies the equation $\partial_t u_r + \mathcal{L}_r u_r^m = 0$ in $\Omega_r := (\Omega - x_0)/r$, where

$$\mathcal{L}_r f(x) = \text{P.V.}\int_{\mathbb{R}^N} (f(x) - f(y))K_r(x, y)\,\mathrm{d}y, \quad K_r(x, y) := r^{N+2s} K(x_0 + rx, x_0 + ry).$$

Note that, since $\sigma = 1$, it follows by the GHP that $u(t) \asymp \operatorname{dist}(x, \partial\Omega)^{\gamma/m}$. Hence,

$$0 < \delta \le u_r(t, x) \le M \quad \text{for all } t \in [r^{-\vartheta} T_0, r^{-\vartheta} T_1],\ x \in B_1,$$

with constants $\delta, M > 0$ that are independent of r and x_0. In addition, using again that $u(t) \asymp \operatorname{dist}(x, \partial\Omega)^{\gamma/m}$, we see that

$$u_r(t, x) \le C(1 + |x|^{\gamma/m}) \quad \text{for all } t \in [r^{-\vartheta} T_0, r^{-\vartheta} T_1] \text{ and } x \in \mathbb{R}^N.$$

Noticing that $u_r^m(t, x) \le C(1 + |x|^\gamma)$ and that $\gamma < 2s$ by assumption, we see that the tails of u_r will not create any problem. Indeed, for any $x \in B_1$,

$$\int_{\mathbb{R}^N \setminus B_2} u_r^m(t, y) K_r(x, y)^{-(N+2s)}\,\mathrm{d}y \le C \int_{\mathbb{R}^N \setminus B_2} |y|^\gamma |y|^{-(N+2s)}\,\mathrm{d}y \le \bar{C}_0,$$

where $\bar{C}_0$ is independent of r. This means that we can localize the problem using cutoff functions as done in Section 8A1, and the integrals defining the functions f and g will converge uniformly with respect to x_0 and r. Hence, we can apply Theorem 8.1(ii) to get

$$\|u_r\|_{C^{1+\beta/2s, 2s+\beta}([r^{-\vartheta} T + 1/2, r^{-\vartheta} T + 1] \times B_{1/2})} \le C \tag{8-7}$$

for all $T \in [T_0, T_1 - r^{-\vartheta}]$. Since $\gamma/m < 2s + \beta$ (because $\gamma < 2s$), it follows that

$$\|u_r\|_{L^\infty([r^{-\vartheta} T + 1/2, r^{-\vartheta} T + 1], C^{\gamma/m}(B_{1/2})} \le \|u_r\|_{C^{1+\beta/2s, 2s+\beta}([r^{-\vartheta} T + 1/2, r^{-\vartheta} T_0 + 1] \times B_{1/2})} \le C.$$

Noticing that

$$\sup_{t \in [r^{-\vartheta} T + 1/2, r^{-\vartheta} T + 1]} [u_r]_{C^{\gamma/m}(B_{1/2})} = \sup_{t \in [T + r^{\vartheta}/2, r^{-\vartheta} T + r^{-\vartheta}]} [u]_{C^{\gamma/m}(B_r(x_0))},$$

and that $T \in [T_0, T_1 - r^{-\vartheta}]$ and x_0 are arbitrary, arguing as in [Ros-Oton and Serra 2014] we deduce that, given $T_2 \in (T_0, T_1)$,

$$\sup_{t \in [T_2, T_1]} [u]_{C^{\gamma/m}(\Omega)} \le C. \tag{8-8}$$

This proves the global Hölder regularity in space. To show the regularity in time, we start again from (8-7) to get

$$\|\partial_t u_r\|_{\mathrm{L}^\infty([r^{-\vartheta} T + 1/2, r^{-\vartheta} T + 1] \times B_{1/2})} \le C.$$

By scaling, this implies

$$\|\partial_t u\|_{\mathrm{L}^\infty([T + r^{\vartheta}/2, r^{-\vartheta} T + r^{-\vartheta}] \times B_r(x_0))} \le C r^{\gamma/m - \vartheta},$$

and by the arbitrariness of T and x_0 we obtain (recall that $r = \operatorname{dist}(x_0, \partial\Omega)/2$)

$$|\partial_t u(t, x)| \le C \operatorname{dist}(x, \partial\Omega)^{\gamma/m - \vartheta} \quad \text{for all } t \in [T_2, T_1],\ x \in \Omega. \tag{8-9}$$

Note that $\gamma/m - \vartheta = \gamma - 2s < 0$ by our assumption.

Now, given $t_0, t_1 \in [T_2, T_1]$ and $x \in \Omega$, we argue as follows: if $|t_0 - t_1| \le \mathrm{dist}(x, \partial\Omega)^{\vartheta}$ then we use (8-9) to get (recall that $\gamma/m - \vartheta < 0$)

$$|u(t_1, x) - u(t_0, x)| \le C \,\mathrm{dist}(x, \partial\Omega)^{\gamma/m-\vartheta} |t_0 - t_1| \le C|t_0 - t_1|^{\gamma/m\vartheta}.$$

On the other hand, if $|t_0 - t_1| \ge \mathrm{dist}(x, \partial\Omega)^{\vartheta}$, then we use (8-8) and the fact that u vanishes on $\partial\Omega$ to obtain

$$|u(t_1, x) - u(t_0, x)| \le |u(t_1, x)| + |u(t_0, x)| \le C \,\mathrm{dist}(x, \partial\Omega)^{\gamma/m} \le C|t_0 - t_1|^{\gamma/m\vartheta}.$$

This proves that u is $(\gamma/m\vartheta)$-Hölder continuous in time, and completes the proof of Theorem 8.3. □

9. Numerical evidence

After discovering the unexpected boundary behavior, we looked for numerical confirmation. This has been given to us by the authors of [Cusimano et al. 2017], who exploited the analytical tools developed in this paper to support our results by means of accurate numerical simulations. We include here some of these simulations, courtesy of the authors. In all the figures we shall consider the spectral fractional Laplacian, so that $\gamma = 1$ (see Section 2A for more details).

We take $\Omega = (-1, 1)$, and we consider as initial datum the compactly supported function

$$u_0(x) = e^{4-1/((x-1/2)(x+1/2))} \chi_{|x|<1/2}$$

appearing in the left of Figure 1. In all the other figures, the solid line represents either $\Phi_1^{1/m}$ or Φ_1^{1-2s}, while the dotted lines represent $t^{1/(m-1)}u(t)$ for different values of t, where $u(t)$ is the solution starting from u_0. These choices are motivated by Theorems 5.3 and 5.5. Since the map $t \mapsto t^{1/(m-1)}\, u(t, x)$ is nondecreasing for all $x \in \Omega$ [Bonforte and Vázquez 2016, (2.3)], the lower dotted line corresponds to an earlier time with respect to the higher one.

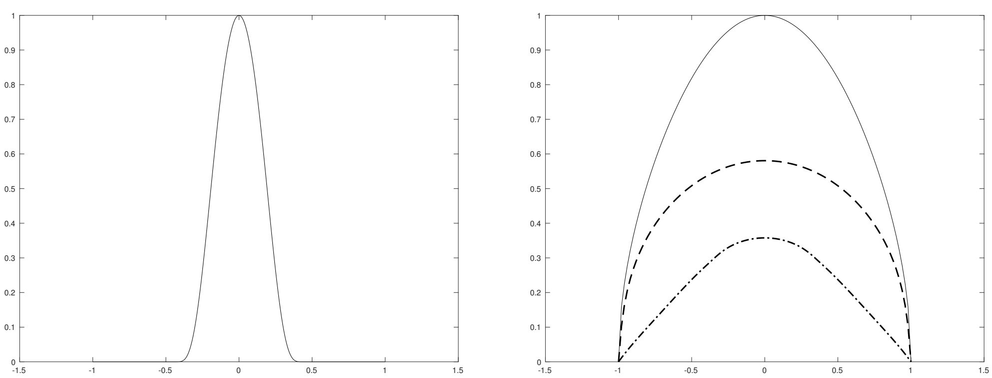

Figure 1. On the left, the initial condition u_0. On the right, the solid line represents $\Phi_1^{1/m}$, and the dotted lines represent $t^{1/(m-1)}u(t)$ at $t = 1$ and $t = 5$. The parameters are $m = 2$ and $s = \frac{1}{2}$; hence $\sigma = 1$. While $u(t)$ appears to behave as $\Phi_1 \asymp \mathrm{dist}(\cdot\,, \partial\Omega)$ for very short times, already at $t = 5$ it exhibits the matching boundary behavior predicted by Theorem 5.3.

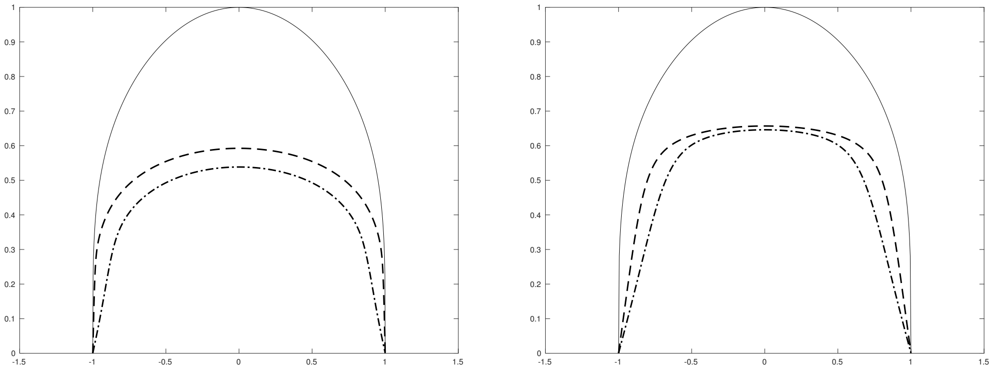

Figure 2. In both pictures, the solid line represents $\Phi_1^{1/m}$. On the left, the dotted lines represent $t^{1/(m-1)}u(t)$ at $t = 30$ and $t = 150$, with parameters $m = 4$ and $s = \frac{3}{4}$; hence $\sigma = 1$. In this case $u(t)$ appears to behave as $\Phi_1 \asymp \mathrm{dist}(\,\cdot\,, \partial\Omega)$ for quite some time, and only around $t = 150$ it exhibits the matching boundary behavior predicted by Theorem 5.3. On the right, the dotted lines represent $t^{1/(m-1)}u(t)$ at $t = 150$ and $t = 600$ with parameters $m = 4$ and $s = \frac{1}{5}$; hence $\sigma = \frac{8}{15} < 1$. In this case $u(t)$ seems to exhibit a linear boundary behavior even after long time (this linear boundary behavior is a universal lower bound for all times by Theorem 5.1). The second picture may lead one to conjecture that, in the case $\sigma < 1$ and $u_0 \lesssim \Phi_1$, the behavior $u(t) \asymp \Phi_1$ holds for all times. However, as shown in Figure 3, there are cases when $u(t) \gg \Phi_1^{1-2s}$ for large times.

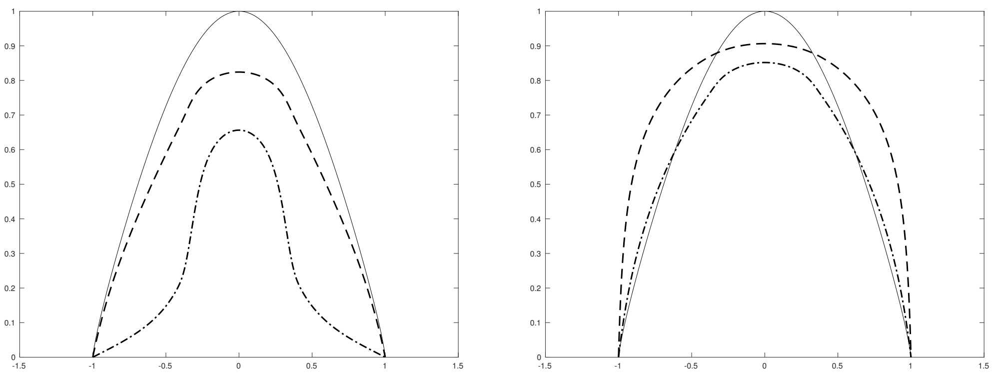

Figure 3. In both pictures we use the parameters $m = 2$ and $s = \frac{1}{10}$; hence $\sigma = \frac{2}{5} < 1$, and the solid line represents Φ_1^{1-2s}. On the left, the dotted lines represent $t^{1/(m-1)}u(t)$ at $t = 4$ and $t = 25$, on the right we see $t = 40$ and $t = 150$. Note that $u(t) \asymp \Phi_1$ for short times. Then, after some time, $u(t)$ starts looking more like Φ_1^{1-2s}, and for large times ($t = 150$) it becomes much larger than Φ_1^{1-2s}.

Comparing Figures 2 and 3, it seems that when $\sigma < 1$ there is no hope of finding a universal behavior of solutions for large times. In particular, the bound provided by (1-6) seems to be optimal.

10. Complements, extensions and further examples

Elliptic versus parabolic. The exceptional boundary behaviors we have found for some operators and data came as a surprise to us, since the solution to the corresponding "elliptic setting" $\mathcal{L}S^m = S$ satisfies $S \asymp \Phi_1^{\sigma/m}$ (with a logarithmic correction when $2sm \neq \gamma(m-1)$); hence separate-variable solutions always satisfy (1-5) (see (3-2) and Theorem 3.2).

About the kernel of operators of the spectral type. In this section we study the properties of the kernel of $\mathcal{L}$. While in some situations $\mathcal{L}$ may not have a kernel (for instance, in the local case), in other situations that may not be so obvious from its definition. In the next lemma it is shown in particular that the SFL, defined by (2-4), admits a representation of the form (2-5). We state hereby the precise result, mentioned in [Abatangelo 2015] and proven in [Song and Vondraček 2003] for the SFL.

Lemma 10.1 (spectral kernels). *Let $s \in (0,1)$, and let $\mathcal{L}$ be the s-th spectral power of a linear elliptic second-order operator $\mathcal{A}$, and let $\Phi_1 \asymp \mathrm{dist}(\cdot\,,\partial\Omega)^\gamma$ be the first positive eigenfunction of $\mathcal{A}$. Let $H(t,x,y)$ be the heat kernel of $\mathcal{A}$, and assume that it satisfies the following bounds: there exist constants $c_0, c_1, c_2 > 0$ such that for all $0 < t \le 1$*

$$c_0\left(\frac{\Phi_1(x)}{t^{\gamma/2}}\wedge 1\right)\left(\frac{\Phi_1(y)}{t^{\gamma/2}}\wedge 1\right)\frac{\mathrm{e}^{-c_1|x-y|^2/t}}{t^{N/2}} \le H(t,x,y) \le c_0^{-1}\left(\frac{\Phi_1(x)}{t^{\gamma/2}}\wedge 1\right)\left(\frac{\Phi_1(y)}{t^{\gamma/2}}\wedge 1\right)\frac{\mathrm{e}^{-|x-y|^2/(c_1 t)}}{t^{N/2}} \tag{10-1}$$

and

$$0 \le H(t,x,y) \le c_2\Phi_1(x)\Phi_1(y) \quad \textit{for all } t \ge 1. \tag{10-2}$$

Then the operator $\mathcal{L}$ can be expressed in the form

$$\mathcal{L}f(x) = \mathrm{P.V.}\int_{\mathbb{R}^N}(f(x)-f(y))\,K(x,y)\,\mathrm{d}y + B(x)u(x) \tag{10-3}$$

with a kernel $K(x,y)$ supported in $\overline{\Omega}\times\overline{\Omega}$ satisfying

$$K(x,y) \asymp \frac{1}{|x-y|^{N+2s}}\left(\frac{\Phi_1(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\Phi_1(y)}{|x-y|^\gamma}\wedge 1\right) \quad \textit{and} \quad B(x) \asymp \Phi_1(x)^{-2s/\gamma}. \tag{10-4}$$

The proof of this lemma follows the ideas of [Song and Vondraček 2003]; indeed assumptions of Lemma 10.1 allow us to adapt the proof of that paper to our case with minor changes.

Method and generality. Our work is part of a current effort aimed at extending the theory of evolution equations of parabolic type to a wide class of nonlocal operators, in particular operators with general kernels that have been studied by various authors; see for instance [del Teso et al. 2017; de Pablo et al. 2016; Serra 2015]. Our approach is different from many others: indeed, even if the equation is nonlinear, we concentrate on the properties of the inverse operator $\mathcal{L}^{-1}$ (more precisely, on its kernel given by the Green function $\mathbb{G}$), rather than on the operator $\mathcal{L}$ itself. Once this setting is well-established and good linear estimates for the Green function are available, the calculations and estimates are very general. Hence, the method is applicable to a very large class of equations, both for elliptic and parabolic problems, as well as to more general nonlinearities than $F(u) = u^m$; see also related comments in the works [Bonforte and Vázquez 2015; 2016; Bonforte et al. 2015].

Finite and infinite propagation. In all cases considered in the paper for $s < 1$ we prove that the solution becomes strictly positive inside the domain at all positive times. This is called *infinite speed of propagation*, a property that does not hold in the limit $s = 1$ for any $m > 1$ [Vázquez 2007] (in that case, finite speed of propagation holds and a free boundary appears). Previous results on this infinite speed of propagation can be found in [Bonforte et al. 2017a; de Pablo et al. 2012]. We recall that infinite speed of propagation is typical of the evolution with nonlocal operators representing long-range interactions, but it is not true for the standard porous medium equation; hence a trade-off takes place when both effects are combined. All our models fall on the side of infinite propagation, but we recall that finite propagation holds for a related nonlocal model called "nonlinear porous medium flow with fractional potential pressure"; see [Caffarelli and Vázquez 2011].

The local case. Since $2sm > \gamma(m-1)$ when $s = 1$ (independently of $m > 1$), our results give a sharp behavior in the local case after a "waiting time". Although this is well-known for the classical porous medium equation, our results apply also to the case of the uniformly elliptic operator in divergence form with C^1 coefficients, and yield new results in this setting. Actually one can check that, even when the coefficients are merely measurable, many of our results are still true and they provided universal upper and lower estimates. At least to our knowledge, such general results are completely new.

10A. ***Further examples of operators.*** Here we briefly exhibit a number of examples to which our theory applies, besides the RFL, CFL and SFL already discussed in Section 2. These include a wide class of local and nonlocal operators. We just sketch the essential points, referring to [Bonforte and Vázquez 2016] for a more detailed exposition.

Censored fractional Laplacian (CFL) and operators with more general kernels. As already mentioned in Section 2A, assumptions (A1), (A2), and (K2) are satisfied with $\gamma = s - \frac{1}{2}$. Moreover, it follows by [Bogdan et al. 2003; Chen et al. 2010] that we can also consider operators of the form:

$$\mathcal{L} f(x) = \mathrm{P.V.} \int_\Omega (f(x) - f(y)) \frac{a(x, y)}{|x-y|^{N+2s}} \, \mathrm{d}y, \quad \text{with } \tfrac{1}{2} < s < 1,$$

where $a(x, y)$ is a symmetric function of class C^1 bounded between two positive constants. The Green function $\mathbb{G}(x, y)$ of $\mathcal{L}$ satisfies the stronger assumption (K4); see Corollary 1.2 of [Chen et al. 2010].

Fractional operators with more general kernels. Consider integral operators of the form

$$\mathcal{L} f(x) = \mathrm{P.V.} \int_{\mathbb{R}^N} (f(x) - f(y)) \frac{a(x, y)}{|x-y|^{N+2s}} \, \mathrm{d}y,$$

where a is a measurable symmetric function, bounded between two positive constants, and satisfying

$$|a(x, y) - a(x, x)| \chi_{|x-y|<1} \le c|x-y|^\sigma, \quad \text{with } 0 < s < \sigma \le 1,$$

for some $c > 0$ (actually, one can allow even more general kernels; see [Bonforte and Vázquez 2016; Kim and Kim 2014]). Then, for all $s \in (0, 1]$, the Green function $\mathbb{G}(x, y)$ of $\mathcal{L}$ satisfies (K4) with $\gamma = s$; see Corollary 1.4 of [Kim and Kim 2014].

Spectral powers of uniformly elliptic operators. Consider a linear operator $\mathcal{A}$ in divergence form,

$$\mathcal{A} = -\sum_{i,j=1}^{N} \partial_i (a_{ij} \partial_j),$$

with uniformly elliptic C^1 coefficients. The uniform ellipticity allows one to build a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with discrete spectrum (λ_k, ϕ_k). Using the spectral theorem, we can construct the spectral power of such operator as

$$\mathcal{L} f(x) := \mathcal{A}^s f(x) := \sum_{k=1}^{\infty} \lambda_k^s \hat{f}_k \phi_k(x), \quad \text{where } \hat{f}_k = \int_\Omega f(x) \phi_k(x) \, \mathrm{d}x$$

(we refer to the books [Davies 1990; 1995] for further details), and the Green function satisfies (K2) with $\gamma = 1$; see [Davies 1995, Chapter 4.6]. Then, the first eigenfunction Φ_1 is comparable to $\mathrm{dist}(\cdot, \partial\Omega)$. Also, Lemma 10.1 applies, see for instance [Davies 1995], and allow us to get sharp upper and lower estimates for the kernel K of $\mathcal{L}$, as in (10-4).

Other examples. As explained in Section 3 of [Bonforte and Vázquez 2016], our theory may also be applied to: (i) sums of two fractional operators; (ii) the sum of the Laplacian and a nonlocal operator kernel; (iii) Schrödinger equations for nonsymmetric diffusions; (iv) gradient perturbation of restricted fractional Laplacians. Finally, it is worth mentioning that our arguments readily extend to operators on manifolds for which the required bounds hold.

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BLOW-UP OF A CRITICAL SOBOLEV NORM FOR ENERGY-SUBCRITICAL AND ENERGY-SUPERCRITICAL WAVE EQUATIONS

THOMAS DUYCKAERTS AND JIANWEI YANG

We consider a wave equation in three space dimensions, with a power-like nonlinearity which is either focusing or defocusing. The exponent is greater than 3 (conformally supercritical) and not equal to 5 (not energy-critical). We prove that for any radial solution which does not scatter to a linear solution, an adapted scale-invariant Sobolev norm goes to infinity at the maximal time of existence. The proof uses a conserved generalized energy for the radial linear wave equation, new Strichartz estimates adapted to this generalized energy, and a bound from below of the generalized energy of any nonzero solution outside wave cones. It relies heavily on the fact that the equation does not have any nontrivial stationary solution. Our work yields a qualitative improvement on previous results on energy-subcritical and energy-supercritical wave equations, with a unified proof.

1. Introduction

1A. ***Motivation and background.*** Consider the semilinear wave equation in 1+3 dimensions

$$(\partial_t^2 - \Delta)u = \iota |u|^{2m} u, \tag{1-1}$$

with initial data

$$u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \tag{1-2}$$

where $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$. The parameters $m > 1$ and $\iota \in \{\pm 1\}$ are fixed. The equation is *focusing* when $\iota = 1$ and *defocusing* when $\iota = -1$. It has the following scaling invariance: if $u(t,x)$ is a solution of (1-1) and $\lambda > 0$, then $\lambda^{\frac{1}{m}} u(\lambda t, \lambda x)$ is also a solution. It is well-posed in the scale-invariant Sobolev space $\dot{\mathcal{H}}^{s_c} := \dot{H}^{s_c}(\mathbb{R}^3) \times \dot{H}^{s_c-1}(\mathbb{R}^3)$, where $s_c = \frac{3}{2} - \frac{1}{m}$ is the critical Sobolev exponent. Equation (1-1) is *energy-subcritical* if $s_c < 1$ (equivalently $m < 2$), *energy-critical* if $s_c = 1$ ($m = 2$) and *energy-supercritical* if $s_c > 1$ ($m > 2$).

The dynamics of (1-1) depend in a crucial way on the value of m and the sign of ι.

The energy-critical case $m = 2$ is particular. The conserved energy

$$E(\vec{u}(t)) = \frac{1}{2}\int |\nabla u(t,x)|^2\,dx + \frac{1}{2}\int (\partial_t u(t,x))^2\,dx - \frac{\iota}{2m+2}\int |u(t,x)|^{2m+2}\,dx$$

is well-defined in $\dot{\mathcal{H}}^{s_c} = \dot{\mathcal{H}}^1 = \dot{H}^1 \times L^2$. When the nonlinearity is defocusing, the conservation of the energy implies that all solutions are bounded in $\dot{\mathcal{H}}^1$. It was proved in the 90s that all solutions are global

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and scatter to a linear solution in the energy space, i.e., that there exists a solution u_{L} of the linear wave equation

$$(\partial_t^2-\Delta)u_{\mathrm{L}}=0,\ (t,x)\in\mathbb{R}\times\mathbb{R}^3, \tag{1-3}$$

with initial data in $\dot{\mathcal{H}}^1$, such that

$$\lim_{t\to+\infty}\|\vec{u}(t)-\vec{u}_{\mathrm{L}}(t)\|_{\dot{\mathcal{H}}^1}=0; \tag{1-4}$$

see [Grillakis 1990; 1992; Ginibre et al. 1992; Shatah and Struwe 1993; 1994; Kapitanski 1994; Ginibre and Velo 1995; Nakanishi 1999; Bahouri and Shatah 1998]. In the focusing case, there exist solutions that do not scatter. Indeed, there exist solutions of (1-1) that blow up in finite time with a *type I* behavior; i.e., there are solutions u such that

$$\lim_{t\to T_+(u)}\|\vec{u}(t)\|_{\dot{\mathcal{H}}^1}=+\infty,$$

where $T_+(u)$ is the maximal time of existence of u. Furthermore, the equation also admits stationary solutions and more generally traveling waves. It was proved in [Duyckaerts et al. 2013] that any radial solution that does not scatter and is not a type I blow-up solution decouples asymptotically as a sum of rescaled stationary solutions and a dispersive term. This includes global nonscattering solutions (see [Krieger and Schlag 2007; Donninger and Krieger 2013], and also [Martel and Merle 2016; Jendrej 2016] in higher space dimensions, for examples of such solutions) and solutions that blow up in finite time but remain bounded in the energy space, called *type II blow-up* solutions (see, e.g., [Krieger et al. 2009; Krieger and Schlag 2014a] and, in higher dimensions [Hillairet and Raphaël 2012; Jendrej 2017]).

The case $m\neq 2$ is quite different. It is known that stationary solutions do not exist in the critical Sobolev space, even for focusing nonlinearity, see, e.g., [Joseph and Lundgren 1973; Farina 2007, Theorem 2], and it is conjectured that any solution that does not satisfy

$$\lim_{t\to T_+(u)}\|\vec{u}(t)\|_{\dot{\mathcal{H}}^{s_c}}=+\infty \tag{1-5}$$

is global and scatters to a linear solution for positive times. A slightly weaker version of this result was proved in many works; namely, if the solution does not scatter, then

$$\limsup_{t\to T_+(u)}\|\vec{u}(t)\|_{\dot{\mathcal{H}}^{s_c}}=+\infty. \tag{1-6}$$

See [Kenig and Merle 2011; Duyckaerts et al. 2014] for the radial case, $m>2$, [Shen 2013; Rodriguez 2017] for the radial case, $1<m<2$, [Killip and Visan 2011] for the defocusing nonradial case, $m>2$, [Dodson and Lawrie 2015] for the radial case, $m=1$, and also [Killip et al. 2014] for the nonradial defocusing case, $1\leq m<2$, where (1-6) is proved for finite time blow-up solutions with initial data in the energy space.

Note that none of the preceding works excludes the existence of a nonscattering solution of (1-1) such that

$$\limsup_{t\to T_+(u)}\|\vec{u}(t)\|_{\dot{\mathcal{H}}^{s_c}}=+\infty \quad\text{and}\quad \liminf_{t\to T_+(u)}\|\vec{u}(t)\|_{\dot{\mathcal{H}}^{s_c}}<\infty.$$

In [Duyckaerts and Roy 2015], this type of solution was ruled out in the case $m > 2$: for any radial nonscattering solution of the equation, the critical Sobolev norm goes to infinity as $t \to T_+(u)$.

It is interesting to compare the theorems cited above with analogous ones for other equations, and in particular for the nonlinear Schrödinger equation

$$i\partial_t v - \Delta v = \iota|v|^{2m} v. \tag{1-7}$$

For the defocusing equation ($\iota = -1$), the fact that the bound of a critical norm implies scattering is known in the cubic case in three space dimensions [Kenig and Merle 2010] and in energy-supercritical cases in large space dimensions [Killip and Visan 2010]. Merle and Raphaël [2008] considered the focusing equation (1-7) with $\iota = 1$ and an L^2 supercritical (i.e., pseudoconformally supercritical), energy subcritical nonlinearity, that is, $\frac{2}{3} < m < 2$ when the number of space dimensions is three. This condition is the analogue of the condition $1 < m < 2$ (conformally supercritical and energy subcritical power) for the wave equation. They proved that if u is radial with initial data in the intersection of $\dot{H}^1$ and the critical Sobolev space, and if $T_+(v)$ is finite, then

$$\|v(t)\|_{L^{3m}} \geq \frac{1}{C}|\log(T_+(v) - t)|^{\alpha}$$

for some constant $\alpha > 0$. Note that in this case there exists a global, bounded, nonscattering solution. The space L^{3m} is scale-invariant and strictly larger than the critical Sobolev space. Analogous results are known for Navier–Stokes equations; see [Iskauriaza et al. 2003; Kenig and Koch 2011; Seregin 2012; Gallagher et al. 2013; 2016]. For example, it is proved in [Seregin 2012] that the scale-invariant L^3 norm of a solution blowing-up in finite time goes to infinity at the blow-up time.

Going back to (1-1) with $m \neq 2$, many questions remain open:

- Is it true that all nonscattering solutions of (1-1) satisfy (1-5) in the nonradial case, or if $1 < m < 2$?
- Can one lower the regularity of the scale-invariant norm used in (1-5), as in the case of nonlinear Schrödinger and Navier–Stokes equations?
- Is it possible to give an explicit lower-bound of the critical norm, in the spirit of [Merle and Raphaël 2008]?

In this article, we give a partial answer to the first two questions in the radial case. This is based on a new well-posedness theory for (1-1), in a scale-invariant weighted Sobolev space $\mathcal{L}^m$ which is not Hilbertian, but is related to a conserved quantity of the linear wave equation and is compatible with the finite speed of propagation.

1B. ***Strichartz estimates and local well-posedness.*** Consider the following norm for radial functions (u_0, u_1) on $\mathbb{R}^3$:

$$\|(u_0, u_1)\|_{\mathcal{L}^m} = \left(\int_0^{+\infty} (|r\partial_r u_0|^m + |ru_1|^m)\, dr\right)^{\frac{1}{m}},$$

and define the space $\mathcal{L}^m$ as the closure of radial, smooth, compactly supported functions for this norm. Note that $\mathcal{L}^2$ is exactly[1] $\dot{\mathcal{H}}^1_{\text{rad}}$. The $\mathcal{L}^m$ norm was introduced in [Duyckaerts and Roy 2015], in the case $m > 2$, as a scale-invariant substitute to the energy norm $\dot{H}^1 \times L^2$ norm. Let us mention that $\dot{\mathcal{H}}^{s_c}_{\text{rad}} \subset \mathcal{L}^m$ if $m > 2$, and $\mathcal{L}^m \subset \dot{\mathcal{H}}^{s_c}_{\text{rad}}$ if $1 < m < 2$ (see Proposition 2.2 below). It was observed in [Duyckaerts and Roy 2015] that the $\mathcal{L}^m$ norm is almost conserved for solutions of the linear wave equation: we will indeed introduce in Section 2 a *conserved* quantity (the generalized energy) which is equivalent to this norm. We first prove Strichartz estimates for the linear wave equation. If I is a real interval, we denote by $S(I)$ the space defined by the norm

$$\|f\|_{S(I)} = \left(\int_I \left(\int_0^{+\infty} |f(t,r)|^{(2m+1)m} r^m \, dr\right)^{\frac{1}{m}} dt\right)^{\frac{1}{2m+1}}.$$

Theorem 1. *Let v be a solution of the linear wave equation*

$$\partial_t^2 v - \Delta v = 0, \quad (v, \partial_t v)_{\restriction t=0} = (v_0, v_1) \in \mathcal{L}^m.$$

Then $v \in S(\mathbb{R})$ and

$$\|v\|_{S(\mathbb{R})} \le C \|(v_0, v_1)\|_{\mathcal{L}^m}.$$

Note that Theorem 1 generalizes, in the radial case, the $L^5 L^{10}$ Strichartz/Sobolev estimate for finite-energy solutions of the linear wave equation to the case $m \neq 2$. Let us mention that we prove more general Strichartz estimates, including estimates for the nonhomogeneous wave equation (see Section 2B for the details). As a consequence, we obtain local well-posedness in $\mathcal{L}^m$ for (1-1):

Theorem 2. *For $m > 1$,* (1-1) *is locally well-posed in $\mathcal{L}^m$. For any initial data (u_0, u_1) in $\mathcal{L}^m$, there exists a unique solution u of* (1-1), (1-2) *defined on a maximal interval of existence $I_{\max}(u) = (T_-(u), T_+(u))$ such that $\vec{u} \in C^0(I_{\max}(u), \mathcal{L}^m)$ and for all compact intervals $J \Subset I_{\max}(u)$, we have $u \in S(J)$. Furthermore,*

$$T_+(u) < \infty \quad \Longrightarrow \quad \|u\|_{S([0,T_+(u)))} = +\infty.$$

We obtain Theorem 1 and the other generalized Strichartz estimates of Section 2B by interpolating between the known generalized Strichartz estimates of [Ginibre and Velo 1995], see also [Lindblad and Sogge 1995], in correspondence to the case $m = 2$, and Strichartz-type estimates obtained by a new method, based on the continuity of the Hardy–Littlewood maximal function from L^1 to L^1_w (see Section 2B).

We also construct a profile decomposition for sequences of functions that are bounded in $\mathcal{L}^m$, which is adapted to (1-1), in the spirit of the one of [Bahouri and Gérard 1999] which corresponds to the case $m = 2$. This construction is based on a refined Sobolev embedding due to Chamorro [2011]. The fact that $\mathcal{L}^m$ is not a Hilbert space yields a new technical difficulty, namely that the usual Pythagorean expansion of the norm does not seem to be valid and must be replaced by a weaker statement, closer to Bessel's inequality than to the Pythagorean theorem. We refer to [Solimini 1995; Jaffard 1999] for other non-Hilbertian profile decompositions where this type of inequality also appears.

The definition of the space $\mathcal{L}^m$ does not involve any fractional derivatives and is technically easier to handle than the space $\dot{\mathcal{H}}^{s_c}$ with $m \neq 2$, where the latter are all defined by norms that are not compatible

[1] Throughout the article, the index rad denotes the subspace of radial elements of a given space of distributions on $\mathbb{R}^3$.

with finite speed of propagation. We hope that the Strichartz estimates and profile decomposition proved in this article will find applications for nonlinear wave equations apart from (1-1).

1C. ***Blow-up of the critical Sobolev norm for the nonlinear equation.*** Our second result is that the dichotomy proved in [Duyckaerts and Roy 2015] remains valid in $\mathcal{L}^m$, as long as $m \neq 2$:

Theorem 3. *Assume $m > 1$ and $m \neq 2$. Let u be a radial solution of* (1-1), (1-2), *with $(u_0, u_1) \in \mathcal{L}^m$ and maximal positive time of existence T_+. Then one of the following holds*:

(1) $\lim_{t \to T_+(u)} \|\vec{u}(t)\|_{\mathcal{L}^m} = +\infty$.

(2) $T_+(u) = +\infty$ *and u scatters forward in time to a linear solution*; *i.e.*, *there exists a solution u_{L} of* (1-3), *with initial data $\mathcal{L}^m$, such that*

$$\lim_{t \to +\infty} \|\vec{u}(t) - \vec{u}_{\mathrm{L}}(t)\|_{\mathcal{L}^m} = 0.$$

In the energy-supercritical case $m > 2$, Theorem 3 improves the result of [Duyckaerts and Roy 2015] since $\dot{\mathcal{H}}^{s_c}$ is continuously embedded into $\mathcal{L}^m$. In the case $1 < m < 2$, we know $\mathcal{L}^m$ is continuously embedded into $\dot{\mathcal{H}}^{s_c}$ and Theorem 3 is not strictly stronger than the result of [Shen 2013]. However, Theorem 3 is also new, since it says that as least some scale-invariant norm of u must go to infinity as t goes to $T_+(u)$. It is very natural to conjecture that the $\dot{\mathcal{H}}^{s_c}$ norm of the solution also goes to infinity, but this is still an open question.

Once the Strichartz estimates, well-posed theory and profile decomposition in $\mathcal{L}^m$ are known, the proof of Theorem 3 (sketched in Sections 4, 5 and 6) is very close to the proof of the corresponding result in [Duyckaerts and Roy 2015], with some simplifications due to the use of the space $\mathcal{L}^m$ instead of $\dot{\mathcal{H}}^{s_c}$ throughout the proof. As in [loc. cit.], we use the *channels of energy method* initiated in [Duyckaerts et al. 2011], and the main ingredient of the proof is an exterior energy estimate for radial solutions of the linear wave equation for the $\mathcal{L}^m$-energy, which generalizes the exterior energy estimate used in [Duyckaerts et al. 2011; 2013; 2014].

According to Theorem 3, there are three potential types of dynamics for (1-1): scattering, finite time blow-up solutions such that the critical norm goes to infinity at the blow-up time, and global solutions such that the critical norm goes to infinity as t goes to infinity. Only two of these dynamics are known to exist: scattering (for both focusing and defocusing nonlinearities) and finite time blow-up (for focusing nonlinearity only). Indeed, in the focusing case, it is possible to construct blow-up solutions with smooth, compactly supported initial data using finite speed of propagation and the ordinary differential equation $y'' = |y|^{2m} y$. Another type of blow-up solution was constructed by C. Collot [2014] for some energy-supercritical nonlinearity in large space dimension: in this case the scale-invariant Sobolev norms blow up logarithmically.

It is natural to conjecture that all solutions in $\mathcal{L}^m$ are global in the defocusing case. For $m < 2$, this follows from conservation of the energy if the data is assumed to be in $\dot{\mathcal{H}}^1$, and only the case of low-regularity solution is open. For supercritical nonlinearity $m > 2$, it is a very delicate issue even for smooth initial data, as the recent construction by T. Tao [2016] of a finite time blow-up solution for a defocusing *system*[2] of energy supercritical wave equation suggests.

The existence of global solutions blowing-up at infinity with initial data in $\mathcal{L}^m$ (or $\dot{\mathcal{H}}^{s_c}$) is also completely open. We refer to [Krieger and Schlag 2014b; Luk et al. 2016, Appendix A] for two different constructions of global, smooth, nonscattering solutions in the case $m = 3$. The initial data of these solutions do not belong either to the critical Sobolev spaces $\dot{\mathcal{H}}^{\frac{7}{6}}$ or to the $\mathcal{L}^3$ space, but are, however, in all spaces $\dot{\mathcal{H}}^s$, $s > \frac{7}{6}$. These constructions and Theorem 3 seem to suggest that any global solution with initial data decaying sufficiently at infinity actually scatters, but we do not know of any rigorous result in this direction.

Let us finally mention [Beceanu and Soffer 2017] on (1-1) with supercritical nonlinearity $m > 2$, where global existence is proved for a class of outgoing initial data.

The outline of the paper is as follows: in Section 2, we prove the Strichartz estimate for the linear wave equation and deduce the Cauchy theory for (1-1). In Section 3, we construct the profile decomposition. In Section 4, we prove the exterior energy property for nonzero solutions of (1-1), which is the core of the proof of Theorem 3. In Section 5, we introduce the radiation term (i.e., the dispersive part) of a solution which is bounded in the critical space for a sequence of times. In Section 6, we conclude the proof.

Notation. If a and b are two positive quantities we write $a \lesssim b$ when there exists a constant $C > 0$ such that $a \le Cb$, where the constant will be clear from the context. When the constant depends on some other quantity M, we emphasize the dependence by writing $a \lesssim_M b$. We will write $a \approx b$ when we have both $a \lesssim b$ and $b \lesssim a$. We will write $a \ll b$ or $a \gg b$ if there exists a sufficiently large constant $C > 0$ such that $Ca \le b$ or $a \ge Cb$ respectively. We use $\mathcal{S}(\mathbb{R}^d)$ to denote the Schwartz class of functions on the Euclidean space $\mathbb{R}^d$.

If f is a radial function depending on t and $r := |x|$, let

$$\vec{f} := (f, \partial_t f) \quad \text{and} \quad [f]_{\pm}(t, r) = (\partial_r \pm \partial_t)(rf).$$

Given $s \ge 0$ and n a positive integer, we define

$$\dot{\mathcal{H}}^s(\mathbb{R}^n) := \dot{H}^s(\mathbb{R}^n) \times \dot{H}^{s-1}(\mathbb{R}^n),$$

where $\dot{H}^s$ denotes the standard homogeneous Sobolev space. We let $L^p_t(I, L^q_x)$ be the space of measurable functions f on $I \times \mathbb{R}^3$ such that

$$\|f\|_{L^p_t(I, L^q_x)} = \left(\int_I \left(\int_{\mathbb{R}^3} |f(t,x)|^q \, dx \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}} < \infty.$$

Unless specified, the functional spaces (L^p, $\dot{H}^s$, etc...) are spaces of functions or distributions on $\mathbb{R}^3$ with the Lebesgue measure. On a measurable space $(\Omega, d\mu)$ where μ is positive, the weak L^q quasinorm of a function f is defined as

$$\|f\|_{L^q_w} := \sup_{\lambda > 0} \lambda \left(\mu\{x \in \Omega : |f(x)| > \lambda\} \right)^{\frac{1}{q}}.$$

[2]The unknown u is $\mathbb{R}^{40}$-valued.

We shall also use the weighted Lebesgue norm $L^q(\mathbb{R}^n, \omega)$, defined as

$$\|f\|_{L^q(\mathbb{R}^n,\omega)} := \left(\int_{\mathbb{R}^n} |f(x)|^q \omega(x)\, dx\right)^{\frac{1}{q}}$$

for some measurable function $\omega(x)$ as a weight. For $q > 1$, we use $q' = \frac{q}{q-1}$ to mean its Lebesgue conjugate.

We denote by $\mathcal{T}_R$ the operator

$$f \mapsto \mathcal{T}_R(f) := \begin{cases} f(R), & |x| \le R, \\ f(|x|), & |x| \ge R. \end{cases}$$

Let $S_{\mathrm{L}}(t)$ denote the linear propagator; i.e.,

$$S_{\mathrm{L}}(t)(w_0, w_1) := \cos(tD) w_0 + \frac{\sin(tD)}{D} w_1, \quad D = \sqrt{-\triangle}.$$

If u is a function of t and r, we will denote by $F(\partial_{r,t} u)$ the sum $F(\partial_r u) + F(\partial_t u)$; for example, $|\partial_{t,r} u|^m := |\partial_t u|^m + |\partial_r u|^m$.

2. Strichartz estimates and local well-posedness

2A. *Preliminaries.* For $m > 1$, we denote by $\dot{\mathcal{W}}^{1,m}$ the closure of $C^\infty_{0,\mathrm{rad}}$ for the norm $\|\cdot\|_{\dot{\mathcal{W}}^{1,m}}$ defined by

$$\|\varphi\|_{\dot{\mathcal{W}}^{1,m}} := \left(\int_0^{+\infty} |\partial_r \varphi(r)|^m r^m\, dr\right)^{\frac{1}{m}}.$$

Proposition 2.1. *We have $f \in \dot{\mathcal{W}}^{1,m}$ if and only if $f(r) \in C^0_{\mathrm{rad}}((0, +\infty))$ satisfies the conditions*

$$\int_0^{+\infty} |r \partial_r f(r)|^m\, dr < +\infty, \tag{2-1}$$

$$\lim_{r\to 0} r^{\frac{1}{m}} f(r) = \lim_{r\to\infty} r^{\frac{1}{m}} f(r) = 0. \tag{2-2}$$

The proof is given in the Appendix.

We denote by $\mathcal{L}^m$ the closure of $(C^\infty_{0,\mathrm{rad}})^2$ for the norm $\|\cdot\|_{\mathcal{L}^m}$,

$$\|(u_0, u_1)\|_{\mathcal{L}^m} := \|u_0\|_{\dot{\mathcal{W}}^{1,m}} + \left(\int_0^{+\infty} |u_1(r)|^m r^m\, dr\right)^{\frac{1}{m}}.$$

Then:

Proposition 2.2. (1) *If $m > 2$ and $(u_0, u_1) \in \dot{\mathcal{H}}^{s_c}$, then $(u_0, u_1) \in \mathcal{L}^m$ and*

$$\|(u_0, u_1)\|_{\mathcal{L}^m} \lesssim \|(u_0, u_1)\|_{\dot{\mathcal{H}}^{s_c}}.$$

(2) *If $1 < m < 2$ and $(u_0, u_1) \in \mathcal{L}^m$, then $(u_0, u_1) \in \dot{\mathcal{H}}^{s_c}$ and*

$$\|(u_0, u_1)\|_{\dot{\mathcal{H}}^{s_c}} \lesssim \|(u_0, u_1)\|_{\mathcal{L}^m}.$$

(3) *If* $u_0 \in \dot{\mathcal{W}}^{1,m}$, *then* $u_0 \in L^{3m}(\mathbb{R}^3)$ *and*

$$\|u_0\|_{L^{3m}} \lesssim \|u_0\|_{\dot{\mathcal{W}}^{1,m}}.$$

(4) *If* $u_0 \in \dot{\mathcal{W}}^{1,m}$, *and* $R > 0$, *then*

$$R|u_0(R)|^m + \int_R^{+\infty} |\partial_r(ru_0)|^m\,dr \approx \int_R^{+\infty} |\partial_r u_0|^m r^m\,dr,$$

where the implicit constant does not depend on R.

Proof. For the proofs of properties (1), (3), (4), see [Kenig and Merle 2011, Lemma 3.2; Duyckaerts and Roy 2015, Lemmas 3.2 and 3.3]. We prove (2) by duality from (1). Assume $m \in (1,2)$ and let m' be the Lebesgue dual exponent of m. Let $(u_0,u_1) \in \mathcal{L}^m$ and $\varphi, \psi \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^3)$. Note that

$$\int_0^\infty r^2 \partial_r u_0 \partial_r \varphi\,dr = \int_0^\infty \partial_r(ru_0)\partial_r(r\varphi)\,dr.$$

By Hölder's inequality and (1),

$$\left|\int_0^\infty r^2\partial_r u_0 \partial_r\varphi\,dr\right| + \left|\int_0^\infty r^2 u_1\psi\,dr\right| \le \|(u_0,u_1)\|_{\mathcal{L}^m}\|(\varphi,\psi)\|_{\mathcal{L}^{m'}} \le \|(u_0,u_1)\|_{\mathcal{L}^m}\|(\varphi,\psi)\|_{\dot{\mathcal{H}}^{1/2+1/m}}.$$

This yields the announced result. □

Let $v(t,x)$ be a solution to the Cauchy problem

$$(\partial_t^2 - \Delta)v(t,x) = 0, \quad (v,\partial_t v)|_{t=0} = (v_0,v_1), \quad t \in \mathbb{R},\ x \in \mathbb{R}^3, \tag{2-3}$$

where the initial data is in $\mathcal{L}^m$. Define $r = |x|$ and set

$$F(\sigma) = \frac{1}{2}\sigma\, v_0(|\sigma|) + \frac{1}{2}\int_0^{|\sigma|} r\, v_1(r)\,dr. \tag{2-4}$$

An explicit computation, using

$$(\partial_t^2 - \partial_r^2)(rv) = 0 \tag{2-5}$$

yields

$$v(t,r) = \frac{1}{r}\big(F(t+r) - F(t-r)\big). \tag{2-6}$$

We have

$$[v]_+(t,r) = (\partial_r + \partial_t)(rv) = 2\dot{F}(t+r), \quad [v]_-(t,r) = (\partial_r - \partial_t)(rv) = 2\dot{F}(t-r). \tag{2-7}$$

If $(v_0,v_1) \in \mathcal{L}^m$, we define

$$E_m(v_0,v_1) = \int_0^{+\infty} \big(|\partial_r(rv_0) + rv_1|^m + |\partial_r(rv_0) - rv_1|^m\big)\,dr,$$

so that

$$E_m(\vec{v}(t)) = \int_0^{+\infty} \big|[v]_+(t,r)\big|^m\,dr + \int_0^{+\infty} \big|[v]_-(t,r)\big|^m\,dr.$$

Proposition 2.3. *Assume* $1 < m < +\infty$. *Let* $(v_0, v_1) \in \mathcal{L}^m$ *and* $v(t,r)$ *be given by* (2-3).

(1) Equivalence of energy and $\mathcal{L}^m$ norm.
$$\|(v_0,v_1)\|_{\mathcal{L}^m}^m \approx \int_0^{+\infty} |\partial_r(rv_0)|^m\,dr + \int_0^{+\infty} |rv_1|^m\,dr \approx E_m(v_0,v_1).$$

(2) Energy conservation. *$E_m(\vec{v})$ is independent of time. We call E_m the $\mathcal{L}^m$-modified energy for* (1-3).

(3) Exterior energy bound. *If $R > 0$, the following holds for all $t \ge 0$ or for all $t \le 0$*:
$$\int_R^{+\infty} |\partial_r(rv_0)|^m + |rv_1|^m\,dr \lesssim \int_{R+|t|}^{+\infty} |\partial_r(rv)|^m + |\partial_t(rv)|^m\,dr.$$

Property (2) follows from direct computation, and the formula (2-5). Let us mention that the notation E_m has a slightly different meaning in [Duyckaerts and Roy 2015].

Remark 2.4. Note that
$$E_2(v(t)) = \int_{\mathbb{R}^3} |\nabla v(t,x)|^2\,dx + \int_{\mathbb{R}^3} |\partial_t v(t,x)|^2\,dx, \tag{2-8}$$
which coincides (up to a constant) with the standard energy functional for (2-3). Moreover, from (2-6) we know for any $m \in (1,+\infty)$, there exists $C_m > 0$ such that
$$C_m^{-1}\|\vec{v}(0)\|_{\mathcal{L}^m} \le \|\vec{v}(t)\|_{\mathcal{L}^m} \le C_m\|\vec{v}(0)\|_{\mathcal{L}^m} \quad \text{for all } t. \tag{2-9}$$
Thus $\|\vec{v}(t)\|_{\mathcal{L}^m}$ enjoys the pseudoconservation law, namely (2-9), and extends the classical energy to the general case $m > 1$.

From the conservation of the energy, we deduce the following energy estimate for the equation with a right-hand side.

Corollary 2.5. *Consider the problem*
$$(\partial_t^2 - \Delta)u(t,x) = f(t,x), \quad (u,\partial_t u)|_{t=0} = (u_0,u_1), \quad t \in \mathbb{R},\ x \in \mathbb{R}^3, \tag{2-10}$$
with $(u_0,u_1) \in \mathcal{L}^m$ for a fixed $m > 1$, and f radial. Then we have the following inequality as long as the right-hand side is finite:
$$\sup_{t\in\mathbb{R}}\left(\int_0^\infty \big[|\partial_r(ru)|^m(t)+|\partial_t(ru)|^m(t)\big]\,dr\right)^{\frac{1}{m}} \le C\left(\|(u_0,u_1)\|_{\mathcal{L}^m}+\int_{-\infty}^{+\infty}\left(\int_0^\infty |rf(t,r)|^m\,dr\right)^{\frac{1}{m}} dt\right) \tag{2-11}$$

Proof. Write $u(t,r) = u_{\mathrm{L}}(t,r) + u_{\mathcal{N}}(t,r)$ with
$$u_{\mathrm{L}}(t,r) = S_{\mathrm{L}}(t)(u_0,\,u_1), \quad u_{\mathcal{N}}(t) = \int_0^t \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} f(s)\,ds.$$
The bound for $\|\vec{u}_{\mathrm{L}}\|_{\mathcal{L}^m}$ follows from (2-9) and the conservation of the $\mathcal{L}^m$ modified energy. Moreover,
$$\|\vec{u}_{\mathcal{N}}(t,r)\|_{\mathcal{L}^m} \le \int_0^t \left\|\left(\frac{\sin\big((t-s)\sqrt{-\Delta}\big)}{\sqrt{-\Delta}} f(s), \cos\big((t-s)\sqrt{-\Delta}\big) f(s)\right)\right\|_{\mathcal{L}^m} ds,$$
and the estimate on $u_{\mathcal{N}}$ follows again from (2-9) and the conservation of the $\mathcal{L}^m$-modified energy. □

2B. ***Strichartz estimates in weighted Sobolev spaces.*** Let Ω be a measurable subset of $\mathbb{R}_t \times (0,+\infty)$ of the form $\Omega = \bigcup_{t\in\mathbb{R}}\{t\}\times J_t$, where for all t, we have J_t is a measurable subset of $(0,+\infty)$. If f is a measurable function on Ω, we let

$$\|f\|_{S(\Omega)} = \left(\int_{\mathbb{R}}\left(\int_{J_t} |f(t,r)|^{(2m+1)m} r^m \, dr\right)^{\frac{1}{m}} dt\right)^{\frac{1}{2m+1}}.$$

If $\Omega = I\times(0,+\infty)$, where I is a time interval, we will set $S(\Omega)=S(I)$ to lighten notation:

$$\|f\|_{S(I)} = \left(\int_{I}\left(\int_0^{+\infty} |f(t,r)|^{(2m+1)m} r^m \, dr\right)^{\frac{1}{m}} dt\right)^{\frac{1}{2m+1}}.$$

In this subsection we prove the following Strichartz estimate:

Proposition 2.6. *Let $m>1$ and assume $v(t,x)$ is the solution of the Cauchy problem* (2-3) *with radial initial data $(v_0,v_1)\in\mathcal{L}^m$. Then there exists a constant C such that*

$$\|v\|_{S(\mathbb{R})} \le C\|\vec{v}(0)\|_{\mathcal{L}^m}. \tag{2-12}$$

We also have its analogue for the inhomogeneous part:

Proposition 2.7. *Let $m>1$ and $u(t,r)$ be the solution of* (2-10) *with $\vec{u}(0)=(0,0)$. Assume*

$$\|f\|_{L^1_t L^m_x(r^m\,dr)} := \int_{-\infty}^{+\infty}\left(\int_0^{+\infty}|f(t,r)|^m r^m\,dr\right)^{\frac{1}{m}} dt < \infty.$$

Then we have

$$\|u\|_{S(\mathbb{R})} \le C\|f\|_{L^1_t L^m_x(r^m dr)}. \tag{2-13}$$

We start by proving auxiliary symmetric Strichartz-type estimates in Section 2B1, using the weak continuity in L^1 of the Hardy–Littlewood maximal function. In Section 2B2 we will interpolate these estimates with standard Strichartz inequalities to obtain the key estimates (2-12) and (2-13).

2B1. *A family of symmetric Strichartz estimates.* With the explicit expression (2-6), we are ready to deduce a crucial estimate for the linear wave equation (2-3) with $\vec{v}(0)\in\mathcal{L}^m$.

Proposition 2.8. *Let $v(t,x)=S_{\mathrm{L}}(t)(v_0,v_1)$ be a radial solution of* (2-3). *Then for any $m\in(1,+\infty)$ and $\alpha\in(1,+\infty)$, there is a constant C such that the following* a priori *estimate is valid*:

$$\left(\int_{\mathbb{R}}\int_0^{+\infty}|v(t,r)|^{\alpha m} r^{\alpha-2}\,dr\,dt\right)^{\frac{1}{\alpha m}} \le C\|\vec{v}(0)\|_{\mathcal{L}^m}. \tag{2-14}$$

Proof. We assume $v_1\equiv 0$ first. Then from (2-4) and the fundamental theorem of calculus,

$$v(t,r) = \frac{1}{2r}\int_{t-r}^{t+r}\partial_s(s\,v_0(|s|))\,ds, \quad r=|x|. \tag{2-15}$$

Let us consider the operator

$$\mathcal{T}: G(s)\mapsto \frac{1}{2r}\int_{t-r}^{t+r} G(s)\,ds. \tag{2-16}$$

First, it is clear that

$$\sup_{(t,r)\in\mathbb{R}\times\mathbb{R}_+} |\mathcal{T}G(t,r)| \le \|G\|_{L^\infty(\mathbb{R};\,ds)}. \tag{2-17}$$

Next, we demonstrate the weak-type estimate

$$\|\mathcal{T}G\|_{L^\alpha_w(\mathbb{R}\times\mathbb{R}_+;\,r^{\alpha-2}drdt)} \le C\|G\|_{L^1(\mathbb{R};\,ds)}, \tag{2-18}$$

or equivalently, there is $C>0$ such that for any $\lambda>0$ we have

$$\iint_{\mathcal{E}_\lambda} r^{\alpha-2}drdt \le C\left(\frac{\|G\|_{L^1}}{\lambda}\right)^{\alpha}, \tag{2-19}$$

where $\mathcal{E}_\lambda=\{(t,r)\in\mathbb{R}\times\mathbb{R}_+ : |\mathcal{T}G(t,r)|>\lambda\}$.

Given this, we have, interpolating between (2-17) and (2-18),

$$\left(\int_{\mathbb{R}}\int_0^{+\infty} |\mathcal{T}G(t,r)|^{\alpha m} r^{\alpha-2}\,dr\,dt\right)^{\frac{1}{\alpha}} \le C\int_{\mathbb{R}} |G(s)|^m\,ds; \tag{2-20}$$

see Theorem 5.3.2 in [Bergh and Löfström 1976]. The estimate (2-14) with $v_1\equiv 0$ now follows by using (2-20) with

$$G(s)=\partial_s(s\,v_0(|s|)).$$

To show (2-19), one observes that on $\mathcal{E}_\lambda$,

$$0<r<\frac{\|G\|_{L^1}}{\lambda} \quad\text{and}\quad (\mathcal{M}G)(t)>\lambda,$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal function. Therefore, we can bound the left-hand side of (2-19) as follows:

$$\int_0^{\frac{1}{2\lambda}\|G\|_{L^1}} r^{\alpha-2}\,dr \int_{\{t\in\mathbb{R}|(\mathcal{M}G)(t)>\lambda\}} dt \le C\left(\frac{\|G\|_{L^1}}{\lambda}\right)^{\alpha}, \tag{2-21}$$

where we have used the weak estimate $\mathcal{M}:\, L^1(\mathbb{R})\to L^1_w(\mathbb{R})$.

The case $v_0\equiv 0$ follows from the same argument. Indeed, in this case we have

$$v(t,r)=\frac{1}{2r}\int_{t-r}^{t+r} s v_1(|s|)\,ds. \tag{2-22}$$

Letting $G(s)=sv_1(|s|)$ and applying (2-20) we are done. □

Let $u(t,x)$ be a solution to the nonhomogeneous Cauchy problem (2-10), where $f(t,x)$ is radial in the space variable and locally integrable. If we set

$$g(t,\rho)=\rho f(t,|\rho|), \tag{2-23}$$

then we have

$$u(t,r)=\frac{1}{2r}\int_0^t\int_{\tau-r}^{\tau+r} g(t-\tau,\,\sigma)\,d\sigma\,d\tau. \tag{2-24}$$

After a change of variables, we obtain

$$u(t,r)=\frac{1}{2r}\int_{t-r}^{t+r} G(t,\rho)\,d\rho, \tag{2-25}$$

with

$$G(t,\rho)=\int_0^t g(s,\rho-s)\,ds.$$

A proof very close to the one of Proposition 2.8 yields symmetric Strichartz estimates for the nonhomogeneous equation:

Proposition 2.9. *Let $u(t,x)$ be a radial solution of the problem* (2-10) *with initial data $\vec{u}(0)=(0,0)$. Then for any $m\in(1,+\infty)$ and $\alpha\in(1,+\infty)$ there is a constant C such that we have*

$$\left(\int_{\mathbb{R}}\int_0^{+\infty}|u(t,r)|^{\alpha m}r^{\alpha-2}\,dr\,dt\right)^{\frac{1}{\alpha m}}\le C\int_{\mathbb{R}}\left(\int_0^{+\infty}|rf(t,r)|^m\,dr\right)^{\frac{1}{m}}dt. \tag{2-26}$$

Proof. In view of (2-25), we have

$$|u(t,r)|\le \mathcal{T}\widetilde{G}(t,r),$$

where $\mathcal{T}$ is defined as in (2-16) and

$$\widetilde{G}(\rho)=\int_{-\infty}^{+\infty}|g(s,\rho-s)|\,ds,$$

with g given by (2-23). Noting that $m>1$, we obtain (2-26) by using (2-20) and Minkowski's inequality. □

Remark 2.10. Notice that from (2-15) and (2-22), one may deduce the following end-point Strichartz estimate for linear wave equations in three dimensions with radial initial data

$$\|S_{\mathrm{L}}(t)(v_0,v_1)\|_{L^2(\mathbb{R}_t,\,L^\infty(\mathbb{R}^3_x))}\le C\left(\|v_0\|_{\dot{H}^1(\mathbb{R}^3)}+\|v_1\|_{L^2(\mathbb{R}^3)}\right), \tag{2-27}$$

where $(v_0,v_1)\in\dot{H}^1_{\mathrm{rad}}(\mathbb{R}^3)\times L^2_{\mathrm{rad}}(\mathbb{R}^3)$. In fact, we may assume without loss of generality that (v_0,v_1) belongs to the Schwartz class. Then (2-27) follows from (2-15) and (2-22) by using the L^2-boundedness of the Hardy–Littlewood maximal function and integration by parts.

2B2. *Proof of the key Strichartz inequality.* We prove here Propositions 2.6 and 2.7. Let us first recall the following classical Strichartz estimates for wave equations; see [Ginibre and Velo 1995].

Theorem 2.11. *Consider $v(t,x)$, the solution of the linear Cauchy problem*

$$\begin{cases}(\partial_t^2-\Delta)v=h(t,x), & (x,t)\in\mathbb{R}^3\times\mathbb{R},\\ v|_{t=0}=v_0\in\dot{H}^1(\mathbb{R}^3), & \\ \partial_t v|_{t=0}=v_1\in L^2(\mathbb{R}^3), & \end{cases} \tag{2-28}$$

so that

$$v(t)=S_{\mathrm{L}}(t)(v_0,v_1)+\int_0^t\frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}}h(s)\,ds.$$

Let $2 \le q, \sigma \le \infty$ and let the following conditions be satisfied:

$$\frac{1}{q}+\frac{1}{\sigma}\le\frac{1}{2},\quad (q,\sigma)\neq(2,\infty),\quad \frac{1}{q}+\frac{3}{\sigma}=\frac{1}{2}.$$

Then there exists $C>0$ such that v satisfies the estimate

$$\|v\|_{L^q(\mathbb{R},\,L^\sigma(\mathbb{R}^3))}\le C\big(\|v_0\|_{\dot H^1(\mathbb{R}^3)}+\|v_1\|_{L^2(\mathbb{R}^3)}+\|h\|_{L^1(\mathbb{R}\,;\,L^2(\mathbb{R}^3))}\big). \tag{2-29}$$

We are now ready to prove Proposition 2.6

Proof. Since (2-12) is classical when $m=2$, it suffices to consider below the cases for $m>2$ and $1<m<2$ separately.

If $m>2$, we define $m^*=2m$ and take $\alpha=\frac{4}{3}(2m+1)$. Then we have from (2-14)

$$\left(\int_{-\infty}^{+\infty}\int_0^{+\infty}|v(t,r)\,r^{\gamma_1}|^{\alpha m^*}r^{\gamma_2}\,dr\,dt\right)^{\frac{1}{\alpha m^*}}\le C\|\vec v(0)\|_{\mathcal{L}^{m*}}, \tag{2-30}$$

where

$$\gamma_1=\frac{5m-2}{5m(2m+1)},\quad \gamma_2=\frac{2}{5},$$

so that $\gamma_1\alpha m^*+\gamma_2=\alpha-2$. Let

$$q=\frac{8m(2m+1)}{8m^2-11m+6},\quad \sigma=\frac{8m(2m+1)}{5m-2}.$$

Then (2-29) yields

$$\left(\int_{-\infty}^{+\infty}\left(\int_0^{+\infty}|v(t,r)\,r^{\gamma_1}|^{\sigma}r^{\gamma_2}\,dr\right)^{\frac{q}{\sigma}}dt\right)^{\frac{1}{q}}\le C\|\vec v(0)\|_{\mathcal{L}^2}. \tag{2-31}$$

In view of

$$\frac{1}{m}=\frac{\theta}{2}+\frac{1-\theta}{m^*},\quad \frac{1}{2m+1}=\frac{\theta}{q}+\frac{1-\theta}{\alpha m^*},\quad \frac{1}{m(2m+1)}=\frac{\theta}{\sigma}+\frac{1-\theta}{\alpha m^*},\quad \theta=\frac{1}{m-1},$$

and the fact that $\gamma_1 m(2m+1)+\gamma_2=m$, we obtain (2-12) by interpolating (2-30) and (2-31); see Theorem 5.1.2 in [Bergh and Löfström 1976].

If $1<m<2$, we set

$$m^*=\frac{m+1}{2},\quad \alpha=\frac{8(2m+1)}{3m+5},\quad \theta=\frac{2(m-1)}{m(3-m)},$$
$$q=\frac{8(2m+1)}{10-m},\quad \sigma=\frac{8(2m+1)}{3m-2},$$
$$\gamma_1=\frac{3m-2}{6m^2+11m+4}=\frac{3m-2}{(2m+1)(3m+4)},\quad \gamma_2=\frac{6m}{3m+4}.$$

One can verify that (2-30) and (2-31) along with the interpolation relations as in the first case remain valid. □

Using the same argument as above and (2-26), we obtain Proposition 2.7.

We conclude this subsection with some additional Strichartz-type estimates that will be useful in the construction of the profile decomposition in Section 3 and follow from Proposition 2.8 and (2-27).

Proposition 2.12. *Assume $m > 2$ and $v(t,x)$ is the solution of the Cauchy problem* (2-3) *with radial initial data $(v_0, v_1) \in \mathcal{L}^m$. Let*

$$a = \frac{2m(m-1)(m+2)}{m^2+3m-2}, \quad b = \frac{2m(m-1)(m+2)}{m-2}.$$

Then there exists a constant C such that

$$\left(\int_{-\infty}^{+\infty}\left(\int_0^{+\infty} |v(t,r)|^b r^m \, dr\right)^{\frac{a}{b}} dt\right)^{\frac{1}{a}} \le C\|\vec{v}(0)\|_{\mathcal{L}^m}. \tag{2-32}$$

Proof. Indeed, from (2-14), we have

$$\left(\int_{-\infty}^{+\infty}\int_0^{+\infty} |v(t,r)|^{2m(m+2)} r^m \, dr\, dt\right)^{\frac{1}{2m(m+2)}} \le C\|\vec{v}(0)\|_{\mathcal{L}^{2m}}. \tag{2-33}$$

Interpolating (2-33) with (2-27), we are done. □

The choice of (a,b) above is not suitable in the case $m < 2$, where we will use the following estimates:

Proposition 2.13. *Assume $1 < m < 2$ and $v(t,x)$ is the solution of the Cauchy problem* (2-3) *with radial initial data $(v_0, v_1) \in \mathcal{L}^m$. Let*

$$a = \frac{m(m+2)(3-m)}{m^2-m+2}, \quad b = \frac{m(m+2)(3-m)}{2(2-m)}.$$

Then there exists a constant C such that

$$\left(\int_{-\infty}^{+\infty}\left(\int_0^{+\infty} |v(t,r)|^b r^m \, dr\right)^{\frac{a}{b}} dt\right)^{\frac{1}{a}} \le C\|\vec{v}(0)\|_{\mathcal{L}^m}. \tag{2-34}$$

Proof. Let $m^* = \frac{m+1}{2}$. From (2-14), we have

$$\left(\int_{-\infty}^{+\infty}\int_0^{+\infty} |v(t,r)|^{m^*(m+2)} r^m \, dr\, dt\right)^{\frac{1}{(m+2)m^*}} \le C\|\vec{v}(0)\|_{\mathcal{L}^{m^*}}. \tag{2-35}$$

Interpolating (2-35) with (2-27), we are done. □

Remark 2.14. In both propositions, we have $m < a < 2m+1 < \frac{b}{m} < \infty$.

Remark 2.15. The interpolations we used in the above two propositions are based on the complex method. In fact, we used Theorems 5.1.1 and 5.1.2 in [Bergh and Löfström 1976].

Remark 2.16. Notice that when $m = 2$, we have $(a,b) = (2,\infty)$ coincides with the end-point Strichartz estimate (2-27).

2C. ***Local well-posedness.*** Consider here the Cauchy problem for the nonlinear wave equations (1-1), (1-2), with $(u_0, u_1) \in \mathcal{L}^m$, $m > 1$. In this subsection, we prove the following small-data well-posedness statement, which implies Theorem 2:

Proposition 2.17. *There exists $\delta_0 > 0$ such that if $0 \in I \subset \mathbb{R}$ is an interval and*

$$\|S_{\mathrm{L}}(t)(u_0, u_1)\|_{S(I)} = \delta \le \delta_0, \tag{2-36}$$

then there exists a unique solution $u \in S(I)$ to the Cauchy problem (1-1), (1-2) *for $t \in I$ such that $\vec{u} \in C^0(I, \mathcal{L}^m)$. Moreover,*

$$\|u\|_{S(I)} \le 2\delta \tag{2-37}$$

and we have

$$\sup_{t\in I} \|\vec{u}(t)\|_{\mathcal{L}^m} \le C_m\big(\|(u_0, u_1)\|_{\mathcal{L}^m} + \delta^{2m+1}\big). \tag{2-38}$$

Remark 2.18. From the assumption on the initial data and the Strichartz-type inequality (2-12), we see that for each $(u_0, u_1) \in \mathcal{L}^m$ and $\delta > 0$, there is an interval $I = I(u_0, u_1, \delta)$ such that (2-36) holds. Using this observation and standard arguments, it is easy to construct from Proposition 2.17 a maximal solution of (1-1), (1-2) that satisfies the conclusion of Theorem 2.

Proof. Let C_0 be the constant in the estimates (2-12) and (2-13) . Consider

$$\mathfrak{X} = \{v \text{ on } \mathbb{R}\times\mathbb{R}^3 \mid v(t,x) = v(t,|x|),\ \|v\|_{S(I)} \le 2\delta\},$$

where

$$0 < \delta < \min\big(C_0^{-\frac{1}{p-1}} 2^{-\frac{p}{p-1}}, 2^{-\frac{p+2}{p-1}} (pC_0)^{-\frac{1}{p-1}}\big), \quad p = 2m+1.$$

Define

$$\Phi_{(u_0,u_1)}(v) = S_{\mathrm{L}}(t)(u_0, u_1) + \iota \int_0^t \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} |v|^{2m} v(s)\, ds. \tag{2-39}$$

If $v, w \in \mathfrak{X}$, we have from (2-13)

$$\|\Phi_{(u_0,u_1)}(v)\|_{S(I)} \le \delta + C_0(2\delta)^p \le 2\delta,$$

and by the Hölder inequality

$$\begin{aligned}\big\|\Phi_{(u_0,u_1)}(v) - \Phi_{(u_0,v_0)}(w)\big\|_{S(I)} &\le 2pC_0\big(\|v\|_{S(I)}^{p-1} + \|w\|_{S(I)}^{p-1}\big)\|v-w\|_{S(I)} \\ &\le 4\,p\,C_0(2\delta)^{p-1}\|v-w\|_{S(I)} \\ &\le \tfrac{1}{2}\|v-w\|_{S(I)}\end{aligned}$$

for all $v, w \in \mathfrak{X}$. Thus, there exists a unique fixed point $u \in \mathfrak{X}$ such that

$$u = \Phi_{u_0,u_1}(u).$$

Note that (2-37) follows from the construction and (2-38) follows from the energy estimates and (2-37). □

2D. ***Exterior long-time perturbation theory.*** We conclude this section by a long-time perturbation theory result for (1-1) with initial data in $\mathcal{L}^m$. Taking into account the finite speed of propagation, we will give a statement that works as well when the estimates are restricted to the exterior $\{r > A + |t|\}$ of a wave cone. This generalization will be very useful when using the channels of energy arguments.

Lemma 2.19. *Let $M > 0$. There exist $\varepsilon_M > 0$, $C_M > 0$ with the following properties. Let $T \in (0, +\infty]$, $u, \tilde{u} \in S((0,T))$ such that $\vec{u}, \vec{\tilde{u}} \in C^0([0,T), \mathcal{L}^m)$. Assume that u is a solution of* (1-1), (1-2) *on* $[0,T)$ *and that*[3]

$$\begin{cases} \partial_t^2 \tilde{u} - \Delta \tilde{u} = \iota \mathbb{1}_{\{r \geq (A+|t|)_+\}} |\tilde{u}|^{2m} \tilde{u} + e, \\ \tilde{u}_{\restriction t=0} = (\tilde{u}_0, \tilde{u}_1), \end{cases} \tag{2-40}$$

where $e \in L^1_t L^m_x(r^m\, dr)$, $A \in \mathbb{R} \cup \{-\infty\}$. Let

$$R_{\mathrm{L}}(t) = S_{\mathrm{L}}(t)((u_0, u_1) - (\tilde{u}_0, \tilde{u}_1)).$$

Assume

$$\|\tilde{u}\|_{S(\{t \in (0,T),\, r \geq (A+|t|)_+\})} \leq M, \tag{2-41}$$

$$\int_0^T \left(\int_{(A+|t|)_+}^{+\infty} |r\, e|^m\, dr \right)^{\frac{1}{m}} dt + \|R_{\mathrm{L}}\|_{S(\{t \in [0,T),\, r \geq (A+|t|)_+\})} = \varepsilon \leq \varepsilon_M. \tag{2-42}$$

Then $u(t) = \tilde{u}(t) + R_{\mathrm{L}}(t) + \epsilon(t)$ with

$$\|\epsilon\|_{S(\{t[0,T),\, r \geq (A+|t|)_+\})} + \sup_{t \in [0,T)} \int_{(A+|t|)_+} |r\, \partial_{t,r} \epsilon|^m\, dr \leq C_M \varepsilon.$$

In the lemma, we have set $(A + |t|)_+ = \max(0, A + |t|)$. By convention, if $A = -\infty$, this quantity equals 0 for all t. Note that the case $A = -\infty$ corresponds to the usual long-time perturbation theory statement;[4] see, e.g., [Tao and Visan 2005].

Sketch of the proof. We let, for $t \in [0,T)$,

$$\mathfrak{E}(t) = \left(\int_{(A+|t|)_+}^{+\infty} |\epsilon(t,r)|^{(2m+1)m} r^m\, dr \right)^{\frac{1}{(2m+1)m}},$$

$$\widetilde{\mathfrak{U}}(t) = \left(\int_{(A+|t|)_+}^{+\infty} |\tilde{u}(t,r)|^{(2m+1)m} r^m\, dr \right)^{\frac{1}{(2m+1)m}},$$

$$\mathfrak{R}(t) = \left(\int_{(A+|t|)_+}^{+\infty} |R_{\mathrm{L}}(t,r)|^{(2m+1)m} r^m\, dr \right)^{\frac{1}{(2m+1)m}}.$$

By the assumptions (2-41), (2-42),

$$\|\widetilde{\mathfrak{U}}\|_{L^{2m+1}(0,T)} \leq M, \quad \|\mathfrak{R}\|_{L^{2m+1}(0,T)} \leq \varepsilon.$$

[3] in the sense that $\tilde{u}$ satisfies the usual integral equation

[4] Traditionally the "linear part" of the solution $R_{\mathrm{L}}(t)$ is incorporated into $\tilde{u}$. For convenience we preferred to distinguish between these two components.

Since

$$(\partial_t^2-\Delta)\epsilon=\iota(|u|^{2m}u-|\tilde{u}|^{2m}\tilde{u})+e,$$

we obtain by (2-11), Strichartz estimates and finite speed of propagation that for all $\theta\in[0,T)$,

$$\sup_{t\in[0,\theta]}\left[\left(\int_{(A+|t|)_+}^{+\infty}|r\partial_{t,r}\epsilon|^m\,dr\right)^{\frac{1}{m}}+\|\vec{\epsilon}(t)\|_{\mathcal{L}^m}+\|\mathfrak{E}(t)\|_{L^{2m+1}}\right]$$
$$\le C\int_0^\theta\left(\int_{(A+|t|)_+}^{+\infty}\left(\left||\tilde{u}|^{2m}\tilde{u}-|u|^{2m}u\right|^m+|e|^m\right)r^m\,dr\right)^{\frac{1}{m}}dt. \tag{2-43}$$

We have

$$\int_0^\theta\left(\int_{(A+|t|)_+}^{+\infty}|e|^m r^m\,dr\right)^{\frac{1}{m}}dt\le\varepsilon$$

and, using Hölder's inequality

$$\begin{aligned}\int_0^\theta&\left(\int_{(A+|t|)_+}^{+\infty}\left||\tilde{u}|^{2m}\tilde{u}-|u|^{2m}u\right|^m r^m\,dr\right)^{\frac{1}{m}}dt\\ &\lesssim\int_0^\theta(\mathfrak{E}(t)+\mathfrak{R}(t))(\widetilde{\mathfrak{U}}(t)^{2m}+\mathfrak{R}(t)^{2m}+\mathfrak{E}(t)^{2m})\,dt\\ &\le C\left(\|\mathfrak{E}\|^{2m+1}_{L^{2m+1}(0,\theta)}+\|\mathfrak{R}\|^{2m+1}_{L^{2m+1}(0,\theta)}+\int_0^\theta\mathfrak{R}(t)\widetilde{\mathfrak{U}}(t)^{2m}\,dt+\int_0^\theta\mathfrak{E}(t)\widetilde{\mathfrak{U}}(t)^{2m}\,dt\right)\\ &\le C\left(\|\mathfrak{E}\|^{2m+1}_{L^{2m+1}(0,\theta)}+\varepsilon^{2m+1}+M^{2m}\varepsilon+\int_0^\theta\mathfrak{E}(t)\widetilde{\mathfrak{U}}(t)^{2m}\,dt\right).\end{aligned}$$

Collecting the above, we obtain, for all $\theta\in[0,T)$,

$$\|\mathfrak{E}\|_{L^{2m+1}(0,\theta)}\le C\left(\varepsilon+\varepsilon^{2m+1}+M^{2m}\varepsilon+\|\mathfrak{E}\|^{2m+1}_{L^{2m+1}(0,\theta)}+\int_0^\theta\mathfrak{E}(t)\widetilde{\mathfrak{U}}(t)^{2m}\,dt\right).$$

This is a Grönwall-type inequality classical in this context. Using, e.g., Lemma 8.1 in [Fang et al. 2011], we deduce that for all $\theta\in[0,T)$,

$$\|\mathfrak{E}\|_{L^{2m+1}(0,\theta)}\le C\left(\varepsilon+\varepsilon^{2m+1}+M^{2m}\varepsilon+\|\mathfrak{E}\|^{2m+1}_{L^{2m+1}(0,\theta)}\right)\Phi(CM^{2m}),$$

where $\Phi(s)=2\Gamma(3+2s)$, and Γ is the usual Gamma function. Using a standard bootstrap argument, we deduce, assuming that $\varepsilon\le\varepsilon_M$ for some small ε_M,

$$\|\mathfrak{E}\|_{L^{2m+1}(0,\theta)}\le C_M\varepsilon,$$

and going back to (2-43) and the computations that follow this inequality, we obtain also the desired bound on the $\mathcal{L}^m$ norm of ϵ. □

3. Profile decomposition

3A. ***Linear profile decomposition.*** The main result of this section is the following:

Theorem 3.1. *Let $(u_{\mathrm{L},n})_n$ be a sequence of radial solutions of* (1-3) *such that $(\vec{u}_{\mathrm{L},n}(0))_n$ is bounded in $\mathcal{L}^m$. Then there exists a subsequence of $(u_{\mathrm{L},n})_n$ (still denoted by $(u_{\mathrm{L},n})_n$) and, for all $j \geq 1$, a solution U_{L}^j of* (1-3) *with initial data (U_0^j, U_1^j) in $\mathcal{L}^m$ and sequences $(\lambda_{j,n})_n \in (0,\infty)^{\mathbb{N}}$, $(t_{j,n})_n \in \mathbb{R}^{\mathbb{N}}$ such that the following properties hold*:

- Pseudo-orthogonality. *For all $j, k \geq 1$, one has*
$$j \neq k \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{\lambda_{j,n}}{\lambda_{k,n}} + \frac{\lambda_{k,n}}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{k,n}|}{\lambda_{j,n}} = +\infty. \tag{3-1}$$
- Weak convergence. *For all $j \geq 1$,*
$$\left(\lambda_{j,n}^{\frac{1}{m}} u_{\mathrm{L},n}(t_{j,n}, \lambda_{j,n}\,\cdot), \lambda_{j,n}^{\frac{1}{m}+1} \partial_t u_{\mathrm{L},n}(t_{j,n}, \lambda_{j,n}\,\cdot)\right) \xrightarrow[n\to\infty]{} (U_0^j, U_1^j), \tag{3-2}$$
weakly in $\mathcal{L}^m$.
- Bessel-type inequality. *For all $J \geq 1$,*
$$\lim_{n\to\infty} E_m(u_{0,n}, u_{1,n}) - \sum_{j=1}^{J} E_m(\vec{U}_{\mathrm{L}}^j(0)) \geq 0. \tag{3-3}$$
- Vanishing in the dispersive norm.
$$\lim_{J\to\infty} \lim_{n\to\infty} \|w_n^J\|_{S(\mathbb{R})} = 0, \tag{3-4}$$

In the above, we have taken
$$w_n^J(t,x) = u_{\mathrm{L},n}(t,x) - \sum_{j=1}^{J} U_{\mathrm{L},n}^j(t,x), \tag{3-5}$$
$$U_{\mathrm{L},n}^j(t,x) = \frac{1}{\lambda_{j,n}^{\frac{1}{m}}} U_{\mathrm{L}}^j\left(\frac{t - t_{j,n}}{\lambda_{j,n}}, \frac{x}{\lambda_{j,n}}\right). \tag{3-6}$$

Theorem 3.1 generalizes (in the radial setting) the profile decomposition of [Bahouri and Gérard 1999] to sequences that are bounded in $\mathcal{L}^m$ instead of the classical energy space. The only difference between the two decompositions is the fact that the Pythagorean expansion proved in that paper is replaced by the weaker property (3-3). One cannot hope, in this context, to have an exact Pythagorean expansion; see the example on p. 387 of [Jaffard 1999].

The proof of Theorem 3.1 is based on the following two propositions, which we will prove in Sections 3B and 3C respectively.

Proposition 3.2. *Let $(u_{\mathrm{L},n})_n$ be a sequence of radial solutions to the linear wave equation and set $(u_{0,n}, u_{1,n}) = \vec{u}_{\mathrm{L},n}(0)$. Assume for $m \in (1, +\infty)$, the sequence $(\vec{u}_{\mathrm{L},n}(0))_n$ is bounded in $\mathcal{L}^m$ and that for* *all sequences* $(\lambda_n)_n \in (0,\infty)^{\mathbb{N}}$ *and* $(t_n)_n \in \mathbb{R}^{\mathbb{N}}$,

$$\left(\frac{1}{\lambda_n^{\frac{1}{m}}} u_{\mathrm{L},n}\left(\frac{-t_n}{\lambda_n}, \frac{\cdot}{\lambda_n}\right), \frac{1}{\lambda_n^{1+\frac{1}{m}}} \partial_t u_{\mathrm{L},n}\left(\frac{-t_n}{\lambda_n}, \frac{\cdot}{\lambda_n}\right)\right)_n \tag{3-7}$$

converges weakly to 0 *in* $\mathcal{L}^m$ *as* $n \to +\infty$. *Then*

$$\lim_{n\to+\infty} \|u_{\mathrm{L},n}\|_{S(\mathbb{R})} = 0. \tag{3-8}$$

Proposition 3.3. *Let* $J \geq 1$ *and* $(U_{\mathrm{L}}^j)_{j=1,\dots,J}$ *be solutions of the linear wave equations with initial data in* $\mathcal{L}^m$. *For all* $j = 1, \dots, J$, *we let* $(\lambda_{j,n})_n \in (0,\infty)^{\mathbb{N}}$ *and* $(t_{j,n})_n \in \mathbb{R}^{\mathbb{N}}$ *be sequences of parameters that satisfy the pseudo-orthogonality property* (3-1). *Let* $(u_{\mathrm{L},n})$ *be a sequence of solutions of the linear wave equation with initial data in* $\mathcal{L}^m$. *Let* w_n^J *be defined by* (3-5), (3-6) *and assume that for all* $j \in \{1, \dots, J\}$,

$$\left(\lambda_{j,n}^{\frac{1}{m}} w_n^J(t_{j,n}, \lambda_{j,n}\,\cdot), \lambda_{j,n}^{\frac{1}{m}+1} \partial_t w_n^J(t_{j,n}, \lambda_{j,n}\,\cdot)\right) \xrightarrow[n\to\infty]{} 0 \quad \textit{weakly in } \mathcal{L}^m. \tag{3-9}$$

Then the Bessel-type inequality (3-3) *holds.*

Proof of the theorem. The proof of Theorem 3.1, assuming Proposition 3.2 and 3.3, is quite standard, at least in the Hilbertian setting. We give it for the sake of completeness. We mainly need to check that it is harmless that we have only a Bessel-type inequality (3-3) in the $\mathcal{L}^m$ setting, which is not Hilbertian, instead of a more precise Pythagorean expansion.

We construct the profiles U_{L}^j and the parameters $\lambda_{j,n}$, $t_{j,n}$ by induction.

Let $J \geq 1$ and assume that for $1 \leq j \leq J-1$, we have constructed profiles U_{L}^j such that (3-1) and (3-2) hold after extraction of a subsequence in n (if $J = 1$ we do not assume anything and set $w_n^0 = u_{\mathrm{L},n}$). Note that this implies (3-3) by Proposition 3.3. Let $\mathcal{A}_J$ be the set of $(U_0, U_1) \in \mathcal{L}^m$ such that there exist sequences $(\lambda_n)_n$, $(t_n)_n$ of parameters such that, after extraction of a subsequence,

$$\left(\lambda_n^{\frac{1}{m}} w_n^{J-1}(t_n, \lambda_n\,\cdot), \lambda_n^{\frac{1}{m}+1} \partial_t w_n^{J-1}(t_n, \lambda_n\,\cdot)\right) \xrightarrow[n\to\infty]{} (U_0, U_1)$$

weakly in $\mathcal{L}^m$, where w_n^{J-1} is defined by (3-5). We distinguish two cases.

Case 1: $\mathcal{A}_J = \{(0,0)\}$. In this case we stop the process and let $U_{\mathrm{L}}^j = 0$ for all $j \geq J$.

Case 2: There exists a nonzero element in $\mathcal{A}_J$. In this case, we choose $(U_0^J, U_1^J) \in \mathcal{A}_J$ such that

$$E_m(U_0^J, U_1^J) \geq \tfrac{1}{2} \sup_{(U_0,U_1)\in\mathcal{A}_J} E_m(U_0, U_1), \tag{3-10}$$

and we choose sequences $(\lambda_{J,n})_n$ and $(t_{J,n})_n$ such that, (after extraction of subsequences in n),

$$\left(\lambda_{J,n}^{\frac{1}{m}} w_n^{J-1}(t_{J,n}, \lambda_{J,n}\,\cdot), \lambda_{J,n}^{\frac{1}{m}+1} \partial_t w_n^{J-1}(t_{J,n}, \lambda_{J,n}\,\cdot)\right) \xrightarrow[n\to\infty]{} (U_0^J, U_1^J) \tag{3-11}$$

weakly in $\mathcal{L}^m$. Note that (3-2) holds for $j = J$ thanks to (3-11). Furthermore, (3-1) for $j \in \{1, \dots, J-1\}$, $k = J$ follows from (3-2) (for $j \in \{1, \dots, J-1\}$), (3-11) and the fact that $(U_0^J, U_1^J) \neq (0,0)$. Finally, as already observed, (3-3) is a consequence of (3-1), (3-2) and Proposition 3.3.

If there exists a $J \geq 1$ such that Case 1 above holds, then we are done: indeed, in this case, w_n^J does not depend on J for large n, and (3-4) is an immediate consequence of the definition of $\mathcal{A}_J$ and Proposition 3.2.

Next assume that Case 2 holds for all $J \geq 1$. Using a diagonal extraction argument, we obtain, for all $j \geq 1$, profiles U_{L}^j, and sequences of parameters $(\lambda_n^j)_n$ and $(t_n^j)_n$ such that (3-1), (3-2) and (3-3) hold for all j, k, J. It remains to prove (3-4). In view of Proposition 3.2, it is sufficient to prove

$$\lim_{J\to\infty} \sup_{(A_0,A_1)\in\mathcal{A}_J} \|(A_0, A_1)\|_{\mathcal{L}^m} = 0.$$

This follows from (3-10), the equivalence between $E_m^{\frac{1}{m}}$ and the $\mathcal{L}^m$ norm, and the fact that, by (3-3),

$$\lim_{J\to\infty} E_m(U_0^J, U_1^J) = 0. \qquad \square$$

3B. ***Convergence to* 0 *of the Strichartz norm.*** First of all, let us introduce the notation $\dot{B}^s_{\infty,\infty}(\mathbb{R}^d)$ for the homogeneous Besov space on $\mathbb{R}^d$, which is defined as follows. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be a radial function, supported in $\left\{\xi \in \mathbb{R}^d : \frac{1}{2} \leq |\xi| \leq 2\right\}$ and such that

$$\sum_{j\in\mathbb{Z}} \psi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^3 \setminus \{0\}.$$

We denote by $\dot{\Delta}_j$ the Littlewood–Paley projector

$$\dot{\Delta}_j f(x) = \big(\psi(2^{-j}\cdot)\hat{f}(\cdot)\big)^\vee(x), \quad j \in \mathbb{Z},$$

where

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx$$

is the Fourier transform on $\mathbb{R}^d$ and we use

$$g^\vee(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} g(\xi)e^{ix\cdot\xi}\,d\xi$$

to denote the inverse Fourier transform. For a tempered distribution f on $\mathbb{R}^d$, we set

$$\|f\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^d)} := \sup_{j\in\mathbb{Z}} 2^{js} \big\|\dot{\Delta}_j f\big\|_{L^\infty(\mathbb{R}^d)}.$$

If $\|f\|_{\dot{B}^s_{\infty,\infty}} < +\infty$, we say f belongs to $\dot{B}^s_{\infty,\infty}$.

We have the following refined Sobolev inequality in weighted norms.

Lemma 3.4. *Let $\omega(x) \in A_p$ with $1 < p < +\infty$; i.e.,*

$$\sup_B \left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} < +\infty, \tag{3-12}$$

where the supremum is taken over all balls B in $\mathbb{R}^d$. If $\nabla f \in L^p(\mathbb{R}^d, \omega(x)dx)$ and $f \in \dot{B}^{-\beta}_{\infty,\infty}(\mathbb{R}^d)$, then

$$\|f\|_{L^q(\mathbb{R}^d,\omega)} \le C\|\nabla f\|^{\theta}_{L^p(\mathbb{R}^d,\omega)}\|f\|^{1-\theta}_{\dot{B}^{-\beta}_{\infty,\infty}(\mathbb{R}^d)}, \tag{3-13}$$

where $1 < p < q < +\infty$, $\theta = \frac{p}{q}$, $\beta = \frac{\theta}{1-\theta}$.

The refined Sobolev inequality (3-13) in weighted norms was proved in [Chamorro 2011], where the author considered more general situations with the underlying domain $\mathbb{R}^d$ replaced by stratified Lie groups. The above lemma follows immediately since the Euclidean space $\mathbb{R}^d$ with its natural group structure is an example of a stratified Lie group. Notice that $1 \in A_p$, and one recovers the classical result on the refined Sobolev inequalities established first in [Gerard et al. 1997].

With Lemma 3.4 at hand, we are ready to prove the Proposition 3.2.

Proof. Since $((u_{0,n}, u_{1,n}))_n$ is bounded in $\mathcal{L}^m$, there exists $A > 0$ such that

$$\int_0^{+\infty} |r\partial_r u_{0,n}(r)|^m\,dr + \int_0^{+\infty} |ru_{1,n}(r)|^m\,dr \le A < +\infty$$

for all n.

Assuming (3-8) fails, we have for some constant c_0 having the property that $0 < c_0 \le C\,A^{\frac{1}{m}}$, that

$$\limsup_{n\to\infty} \|u_{\mathrm{L},n}\|_{L^{2m+1}_t(\mathbb{R},\,L^{m(2m+1)}_x(\mathbb{R}^3,r^{m-2}))} = c_0, \tag{3-14}$$

where C is the constant in (2-12), (2-32) and (2-34). From (2-32), (2-34) and Hölder's inequality, we know that up to a subsequence, there exists some $\theta \in (0,\,1)$ such that

$$\lim_{n\to\infty} \|u_{\mathrm{L},n}\|_{L^{\infty}_t(\mathbb{R},\,L^{m(m+1)}_x(\mathbb{R}^3,r^{m-2}))} \ge \left(\frac{c_0}{(CA^{\frac{1}{m}})^{\theta}}\right)^{\frac{1}{1-\theta}}. \tag{3-15}$$

For $m > 1$, we denote by $[m]$ the greatest integer less than or equal to m and by $\{m\} := m - [m]$ the fractional part of m. Notice that $\{m\} \in [0,1)$ and $\{m\} = 0$ if and only if $m \in \mathbb{N}$.

Let $d = [m]+1$ and $\omega(x) = |x|^{\gamma}$ with $\gamma = \{m\}$, $x \in \mathbb{R}^d$. It is easy to see that $\omega \in A_m$, see for example [Grafakos 2014], and we have the following refined Sobolev inequality in view of Lemma 3.4:

$$\|f\|_{L^{m(m+1)}(\mathbb{R}^d,|x|^{\gamma})} \le C_0\|\nabla f\|^{\frac{1}{m+1}}_{L^m(\mathbb{R}^d,|x|^{\gamma})}\|f\|^{\frac{m}{m+1}}_{\dot{B}^{-1/m}_{\infty,\infty}(\mathbb{R}^d)}. \tag{3-16}$$

If we apply (3-16) to functions $u_{\mathrm{L},n}(t,|x|)$ with respect to the spatial variable $x \in \mathbb{R}^{[m]+1}$, we obtain by transferring the formula into polar coordinates

$$\int_0^{+\infty} |u_{\mathrm{L},n}(t,r)|^{m(m+1)} r^m\,dr \le C_0^{m(m+1)} \int_0^{+\infty} |r\partial_r u_{\mathrm{L},n}(t,r)|^m\,dr \\ \times \sup_{j\in\mathbb{Z}}\ \sup_{x\in\mathbb{R}^{[m]+1}} \left(2^{-\frac{j}{m}} \int_{\mathbb{R}^{[m]+1}} \psi^{\vee}(y)\,u_{\mathrm{L},n}(t,|x-2^{-j}y|)\,dy\right)^{m^2}. \tag{3-17}$$

In view of the conservation of the $\mathcal{L}^m$-energy, and the fact that the norms $\|\cdot\|_{\mathcal{L}^m}$ and $(E_m)^{\frac{1}{m}}$ are equivalent, there exists some $N > 0$ such that if $n \geq N$

$$\sup_{t\in\mathbb{R}} \sup_{j\in\mathbb{Z}} \sup_{x\in\mathbb{R}^{[m]+1}} \left| \int_{\mathbb{R}^{[m]+1}} 2^{-\frac{j}{m}} u_{\mathrm{L},n}(t, |x - 2^{-j} y|)\, \psi^{\vee}(y)\, dy \right| \geq \delta_0, \tag{3-18}$$

where

$$\delta_0 = \tfrac{1}{2} c_0^{\frac{m+1}{(1-\theta)m}} (C^{\frac{\theta}{1-\theta}} C_0)^{-\frac{m+1}{m}} C_m^{-\frac{1}{m}} A^{-\frac{m+1}{m^2}\left(\frac{\theta}{1-\theta}+\frac{1}{m+1}\right)} > 0,$$

and C_m is the constant in (2-9).

As a result of (3-18), we have a family of $(t_n^0)_n$ in $\mathbb{R}^{\mathbb{N}}$, a sequence of $(j_n)_n \in \mathbb{Z}^{\mathbb{N}}$ and $(x_n)_n$ in $(\mathbb{R}^{[m]+1})^{\mathbb{N}}$such that

$$\left| \int_{\mathbb{R}^{[m]+1}} 2^{-\frac{j_n}{m}} u_{\mathrm{L},n}(t_n^0, |x_n - 2^{-j_n} y|) \psi^{\vee}(y)\, dy \right| \geq \frac{\delta_0}{2}, \quad n \geq N.$$

Setting $\varphi(\cdot) = \psi^{\vee}(\cdot)$, $\lambda_n = 2^{j_n}$, $t_n = -t_n^0 \lambda_n$, and $y_n = \lambda_n x_n$, we will obtain a contradiction by letting $n \to \infty$ provided, up to some subsequences,

$$\int_{\mathbb{R}^{[m]+1}} \frac{1}{\lambda_n^{\frac{1}{m}}} u_{\mathrm{L},n}\left(\frac{-t_n}{\lambda_n}, \frac{|y - y_n|}{\lambda_n} \right) \varphi(y)\, dy \to 0, \quad n \to +\infty. \tag{3-19}$$

To prove this, we divide the argument into two cases.

Case 1: $\limsup_{n\to\infty} |y_n| = +\infty$. Up to a subsequence, we may assume

$$0 < |y_1| \ll |y_2| \ll \cdots \ll |y_n| \ll |y_{n+1}| \cdots \to +\infty, \quad n \to +\infty. \tag{3-20}$$

Define

$$V_n(y) = \frac{1}{\lambda_n^{\frac{1}{m}}} u_{\mathrm{L},n}\left(-\frac{t_n}{\lambda_n}, \frac{|y|}{\lambda_n} \right).$$

Note that V_n is a radial function on $\mathbb{R}^{[m]+1}$. Then from the radial Sobolev embedding (see (4) in Proposition 2.2), we have

$$|V_n(y)| \leq \frac{1}{|y|^{\frac{1}{m}}} \left(\int_0^{+\infty} \left| r \partial_r u_{\mathrm{L},n}\left(-\frac{t_n}{\lambda_n}, r \right) \right|^m dr \right)^{\frac{1}{m}} \leq C_m \left(\frac{A}{|y|} \right)^{\frac{1}{m}} \tag{3-21}$$

for all n. As a consequence, (3-19) is bounded by

$$c_n := \int_{\mathbb{R}^{[m]+1}} |y - y_n|^{-\frac{1}{m}} |\varphi(y)|\, dy, \tag{3-22}$$

and it suffices to show

$$\lim_{n\to+\infty} c_n = 0. \tag{3-23}$$

We write

$$c_n = \int_{|y-y_n|\leq 1} |y - y_n|^{-\frac{1}{m}} |\varphi(y)|\, dy + \int_{|y-y_n|\geq 1} |y - y_n|^{-\frac{1}{m}} |\varphi(y)|\, dy.$$

The first term is bounded by

$$\Big(\sup_{|y-y_n|\le 1}|\varphi(y)|\Big)\int_{|z|\le 1}|z|^{-\frac{1}{m}} \underset{n\to\infty}{\longrightarrow} 0,$$

while the second one goes to zero by dominated convergence. Hence (3-23).

Case 2: There exists $c>0$ such that $|y_n|\le c<+\infty$ for all n. We have, up to some subsequences, $y_n\to y_*$ as $n\to\infty$, where $y_*\in\mathbb{R}^{[m]+1}$ such that $|y_*|\le c$. Setting $\tau_n\varphi(\cdot)=\varphi(\cdot+y_n)$ and $\tau_*\varphi(\cdot)=\varphi(\cdot+y_*)$, we have

$$\tau_n\varphi\to\tau_*\varphi,\quad n\to+\infty,\quad \text{in } \mathcal{S}(\mathbb{R}^{[m]+1}). \tag{3-24}$$

From the condition that (3-7) converges weakly to zero in $\mathcal{L}^m$, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^{[m]+1}} V_n(x)\,\tau_*\varphi(x)\,dx=0.$$

In fact, considered as a function on $\mathbb{R}^3$, we have, by (3) in Proposition 2.2,

$$V_n \underset{n\to\infty}{\rightharpoonup} 0 \quad \text{weakly in } L^{3m}(\mathbb{R}^3).$$

Furthermore,

$$\begin{aligned}\int_{\mathbb{R}^{[m]+1}} V_n(x)\,\tau_*\varphi(x)\,dx &= \int_0^{+\infty}\int_{S^{[m]}}\tau_*\varphi(r\omega)\,d\sigma(\omega)\,V_n(r)r^{[m]}\,dr\\ &=\int_0^{+\infty}\underbrace{\Big(\int_{S^{[m]}}\tau_*\varphi(r\omega)\,d\sigma(\omega)\,r^{[m]-2}\Big)}_{:=\Psi(r)}V_n(r)r^2\,dr \underset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

since $\Psi(r)$ can be considered as a radial function in $L^{(3m)'}(\mathbb{R}^3)$ for $1<m<+\infty$. On the other hand, we have by the fundamental theorem of calculus and integration by parts

$$\int_{\mathbb{R}^{[m]+1}} V_n(|y|)\,(\tau_n\varphi(y)-\tau_*\varphi(y))\,dy=\int_0^1\int_{\mathbb{R}^{[m]+1}}\big\langle\nabla V_n(y),(y_*-y_n)\varphi(y+s(y_n-y_*)+y_*)\big\rangle\,dy\,ds.$$

After using Hölder's inequality and the energy estimate, we see the term on the right-hand side is bounded by

$$C_mA^{\frac{1}{m}}|y_n-y_*|\int_0^1\Big(\int_{\mathbb{R}^{[m]+1}}\big|\varphi(y+s(y_n-y_*)+y_*)\big|^{\frac{m}{m-1}}|y|^{-\frac{m-[m]}{m-1}}\,dy\Big)^{\frac{m-1}{m}}\,ds.$$

Notice that $\varphi\in\mathcal{S}(\mathbb{R}^{[m]+1})$, $|y_*|\le c$ and $|y|^{-\frac{m-[m]}{m-1}}$ is integrable near the origin of $\mathbb{R}^{[m]+1}$ when $m>1$. We have

$$\lim_{n\to\infty}\int_{\mathbb{R}^{[m]+1}} V_n(y)\,\big(\tau_n\varphi(y)-\tau_*\varphi(y)\big)\,dy=0. \qquad \square$$

3C. ***Bessel-type inequality.*** In this subsection we prove Proposition 3.3.

We let $\{u_{\mathrm{L},n}\}_{n\in\mathbb{N}}$ and, for $1\le j\le J$, let U_{L}^j and $(\lambda_{j,n},t_{j,n})_n$ be as in Proposition 3.3, and define $U_{\mathrm{L},n}^j$ by (3-6) and w_n^J by (3-5).

First of all, we have the explicit formula for $[U_{\mathrm{L}}^j]_\pm(t,r)$

$$[U_{\mathrm{L}}^j]_+(t,r) = 2\dot{F}^j(t+r), \quad [U_{\mathrm{L}}^j]_-(t,r) = 2\dot{F}^j(t-r), \quad j \geq 1, \tag{3-25}$$

with

$$F^j(\sigma) = \frac{1}{2}\sigma U_0^j(|\sigma|) + \frac{1}{2}\int_0^{|\sigma|} \varrho\, U_1^j(\varrho)\, d\varrho.$$

In view of (2-7), one easily verifies that

$$[U_{\mathrm{L},n}^j]_\pm(t,r) = \frac{1}{\lambda_{j,n}^{\frac{1}{m}}}[U_{\mathrm{L}}^j]_\pm\left(\frac{t-t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}}\right).$$

Up to subsequences, we may assume, after translating in time and rescaling U_{L}^j if necessary,

$$j \geq 1, \quad \lim_{n\to\infty} -\frac{t_{j,n}}{\lambda_{j,n}} = \pm\infty \qquad \text{or} \qquad \text{for all } n, \quad t_{j,n} = 0. \tag{3-26}$$

Step 1: decoupling of linear profiles. In this step, we prove

$$\lim_{n\to+\infty} E_m\left(\sum_{j=1}^J \vec{U}_{\mathrm{L},n}^j(0)\right) = \sum_{j=1}^J E_m(\vec{U}_{\mathrm{L}}^j(0)). \tag{3-27}$$

Recall that for any solution u of the linear wave equation, we have

$$E_m(\vec{u}(0)) = E_m(\vec{u}(t)) = \sum_{\pm}\int_0^{+\infty} \big|[u]_\pm(t,r)\big|^m\, dr,$$

where $[u]_\pm$ is defined in (2-7). Hence (for constants $C>0$ that depend on J and m, but not on n)

$$\begin{aligned}\left|E_m\left(\sum_{j=1}^J U_{\mathrm{L},n}^j(0)\right) - \sum_{j=1}^J E_m(U_{\mathrm{L}}^j(0))\right| &= \left|E_m\left(\sum_{j=1}^J U_{\mathrm{L},n}^j(0)\right) - \sum_{j=1}^J E_m(U_{\mathrm{L},n}^j(0))\right| \\ &\leq C\sum_{\substack{j\neq k\\ \pm}}\int_0^{+\infty} \big|[U_{\mathrm{L},n}^j]_\pm(0,r)\big|^{m-1}\big|[U_{\mathrm{L},n}^k]_\pm(0,r)\big|\, dr \\ &\leq C\sum_{\substack{j\neq k\\ \pm}} \underbrace{\int_0^{+\infty} \left|\frac{1}{\lambda_{j,n}^{\frac{1}{m}}}\dot{F}^j\left(\frac{-t_{j,n}\pm r}{\lambda_{j,n}}\right)\right|^{m-1}\left|\frac{1}{\lambda_{k,n}^{\frac{1}{m}}}\dot{F}^k\left(\frac{-t_{k,n}\pm r}{\lambda_{k,n}}\right)\right| dr.}_{I_{j,k,n}^\pm}\end{aligned}$$

We are thus reduced to proving that each of the terms $I_{j,k,n}^\pm$ $(j\neq k)$ goes to 0 as n goes to infinity. By density we may assume

$$U_0^j,\, U_1^j,\, U_0^k,\, U_1^k \in C_0^\infty,$$

and thus $\dot{F}^j, \dot{F}^k \in C_0^\infty$. We will only consider $I_{j,k,n}^+$, whereas the proof for $I_{j,k,n}^-$ is the same. Extracting subsequences and arguing by contradiction, we can distinguish without loss of generality between the following three cases.

Case 1: We assume $\lim_{n\to\infty} \frac{\lambda_{k,n}}{\lambda_{j,n}} = 0$. By the change of variable $s = \frac{-t_{k,n}+r}{\lambda_{k,n}}$, we obtain

$$I^{+}_{j,k,n} = \int_{-\frac{t_{k,n}}{\lambda_{k,n}}}^{+\infty} \left(\frac{\lambda_{k,n}}{\lambda_{j,n}}\right)^{1-\frac{1}{m}} \left|\dot{F}^{j}\left(\frac{\lambda_{k,n}s + t_{k,n} - t_{j,n}}{\lambda_{j,n}}\right)\right|^{m-1} |\dot{F}^{k}(s)|\, ds \lesssim \left(\frac{\lambda_{k,n}}{\lambda_{j,n}}\right)^{1-\frac{1}{m}}, \tag{3-28}$$

where we have used that $\dot{F}^{j}$ and $\dot{F}^{k}$ are bounded and compactly supported. Since $\frac{\lambda_{k,n}}{\lambda_{j,n}}$ goes to 0 as n goes to infinity, we are done.

Case 2: We assume $\lim_{n\to\infty} \frac{\lambda_{j,n}}{\lambda_{k,n}} = 0$. We argue similarly by using the change of variable $s = \frac{-t_{j,n}+r}{\lambda_{j,n}}$.

Case 3: We assume that the sequence $\left(\frac{\lambda_{j,n}}{\lambda_{k,n}} + \frac{\lambda_{k,n}}{\lambda_{j,n}}\right)_n$ is bounded. We use as in Case 1 the change of variable $s = \frac{-t_{k,n}+r}{\lambda_{k,n}}$. By the pseudo-orthogonality condition (3-1) we see that

$$\lim_{n\to\infty} \frac{|t_{j,n} - t_{k,n}|}{\lambda_{j,n}} = +\infty,$$

and thus, as a consequence of the first line of (3-28), $I^{+}_{j,k,n}$ is 0 for large n, which concludes Step 1.

Step 2: end of the proof. For $1 < m < +\infty$, we introduce the notation

$$\begin{aligned}
\Phi^{j}_{n,0}(r) &= \frac{1}{2r} \sum_{\pm} \int_0^r \left|[U^{j}_{\mathrm{L},n}]_{\pm}(0,s)\right|^{m-2} [U^{j}_{\mathrm{L},n}]_{\pm}(0,s)\, ds,\\
\Phi^{j}_{n,1}(r) &= \frac{1}{2r} \sum_{\pm} \pm \left|[U^{j}_{\mathrm{L},n}]_{\pm}(0,r)\right|^{m-2} [U^{j}_{\mathrm{L},n}]_{\pm}(0,r),
\end{aligned}$$

and let $\Phi^{j}_{n,\mathrm{L}}(t)$ be the solution of the linear wave equations with initial data $(\Phi^{j}_{n,0}, \Phi^{j}_{n,1}) \in \mathcal{L}^{m'}$, where $m' = \frac{m}{m-1}$. Then we have

$$[\Phi^{j}_{n,\mathrm{L}}]_{\pm}(0,r) = \left|[U^{j}_{\mathrm{L},n}]_{\pm}(0,r)\right|^{m-2} [U^{j}_{\mathrm{L},n}]_{\pm}(0,r),$$

and note that

$$E_m(\vec{U}^{j}_{\mathrm{L}}(0)) = E_m(\vec{U}^{j}_{\mathrm{L},n}(0)) = \int_0^{+\infty} \sum_{\pm} [\Phi^{j}_{n,\mathrm{L}}]_{\pm}(0) [U^{j}_{\mathrm{L},n}]_{\pm}(0)\, dr. \tag{3-29}$$

From the weak convergence condition satisfied by the remainder term w^{J}_{n}, we have by time translation and changing variables

$$\begin{aligned}
&\int_0^{+\infty} \left([\Phi^{j}_{n,\mathrm{L}}]_{+}(0,r)[w^{J}_{n}]_{+}(0,r) + [\Phi^{j}_{n,\mathrm{L}}]_{-}(0,r)[w^{J}_{n}]_{-}(0,r)\right) dr\\
&\quad = \int_0^{+\infty} \left|[U^{j}_{\mathrm{L}}]_{+}(0,r)\right|^{m-2} [U^{j}_{\mathrm{L}}]_{+}(0,r)\, \lambda^{\frac{1}{m}}_{j,n} [w^{J}_{n}]_{+}(t_{j,n}, \lambda_{j,n} r)\, dr\\
&\qquad + \int_0^{+\infty} \left|[U^{j}_{\mathrm{L}}]_{-}(0,r)\right|^{m-2} [U^{j}_{\mathrm{L}}]_{-}(0,r)\, \lambda^{\frac{1}{m}}_{j,n} [w^{J}_{n}]_{-}(t_{j,n}, \lambda_{j,n} r)\, dr,
\end{aligned}$$

which goes to zero as $n \to +\infty$ for $1 \le j \le J$. Furthermore,

$$\int_0^{+\infty} \left|[\Phi^{j}_{n,\mathrm{L}}]_{\pm}(0,r)[U^{k}_{\mathrm{L},n}]_{\pm}(0,r)\right| dr = \int_0^{+\infty} \left|[U^{j}_{\mathrm{L},n}]_{\pm}(0,r)\right|^{m-1} \left|[U^{k}_{\mathrm{L},n}]_{\pm}(0,r)\right| dr,$$

and, by Step 1, this goes to 0 as n goes to infinity if $j \neq k$. Hence from (3-29), we have

$$\sum_{j=1}^{J} E_m(\vec{U}_{\mathrm{L}}^j(0)) = \lim_{n\to+\infty}\left[\int_0^{+\infty}[u_{\mathrm{L},n}]_+(0,r)\left(\sum_{j=1}^{J}[\Phi_{n,\mathrm{L}}^j]_+(0,r)\right)dr + \int_0^{+\infty}[u_{\mathrm{L},n}]_-(0,r)\left(\sum_{j=1}^{J}[\Phi_{n,\mathrm{L}}^j]_-(0,r)\right)dr\right],$$

which is bounded after using Hölder's inequality by

$$\left[\lim_{n\to+\infty} E_{m'}\left(\sum_{j=1}^{J}\vec{\Phi}_{n,\mathrm{L}}^j(0,r)\right)\right]^{\frac{1}{m}'}\left[\limsup_{n\to+\infty} E_m(\vec{u}_{\mathrm{L},n}(0))\right]^{\frac{1}{m}}.$$

Furthermore, by the decoupling property proved in Step 1 we obtain

$$\lim_{n\to+\infty} E_{m'}\left(\sum_{j=1}^{J}\vec{\Phi}_{n,\mathrm{L}}^j(0,r)\right) = \sum_{j=1}^{J} E_{m'}(\vec{\Phi}_{n,\mathrm{L}}^j(0)) = \sum_{j=1}^{J} E_m(\vec{U}_{\mathrm{L}}^j(0))$$

and this concludes the result.

3D. ***Approximation by sum of profiles.*** We next write a lemma approximating a nonlinear solution by a sum of profiles outside a wave cone. This type of approximation is only available in space-time slabs where the S norm of all the profiles remain finite. To satisfy this assumption, we will work outside a sufficiently large wave cone.

Let $\{(u_{0,n}, u_{1,n})\}_n$ be a sequence of functions in $\mathcal{L}^m$ that has a profile decomposition with profiles (U_0^j, U_1^j) and parameters $(\lambda_{j,n}, t_{j,n})_n$, $j \geq 1$. Extracting subsequences and time-translating the profiles, we can assume that for all $j \geq 1$ one of the following holds:

$$\lim_{n\to\infty} -\frac{t_{j,n}}{\lambda_{j,n}} \in \{\pm\infty\} \quad \text{or} \tag{3-30}$$

$$\text{for all } n, \quad t_{j,n} = 0. \tag{3-31}$$

We will denote by $\mathcal{J}_\infty$ the set of indices j such that (3-30) holds and by $\mathcal{J}_0$ the set of indices such that (3-31) holds. We assume:

(1) There exist $j_0 \geq 1$, $A > 0$ and a global solution U^{j_0} of

$$\begin{cases} \partial_t^2 U^{j_0} - \Delta U^{j_0} = \iota|U^{j_0}|^{2m}U^{j_0}\mathbb{1}_{\{r\geq|t|+A\}}, \\ \vec{U}^{j_0}(0,r) = \vec{U}_{\mathrm{L}}^{j_0}(0,r), \quad r \geq A, \end{cases}$$

such that $\vec{U}^{j_0}(0) \in \mathcal{L}^m$ and $\|U^{j_0}\|_{S(\{r\geq|t|+A\})} < \infty$.

(2) If $j \in \mathcal{J}_0 \setminus \{j_0\}$, then the solution of (1-1) with initial $\vec{U}_{\mathrm{L}}^j(0)$ scatters in both time directions or

$$\lim_{n\to\infty}\frac{\lambda_{j,n}}{\lambda_{j_0,n}} = 0.$$

For $j \geq 1$, we define U^j as follows:

- U^{j_0} is defined as in point (1) above.
- If $j \in \mathcal{J}_0$ and $\lim_{n\to\infty}\frac{\lambda_{j,n}}{\lambda_{j_0,n}} = 0$, then U^j is the solution of (1-1) with initial data $\vec{U}_{\mathrm{L}}^j(0)$.

- If $j \in \mathcal{J}_0$ and $\lim_{n\to\infty} \frac{\lambda_{j,n}}{\lambda_{j_{0,n}}} = \infty$, then $U^j = 0$.
- If $j \in \mathcal{J}_\infty$, then $U^j = U^j_{\mathrm{L}}$.

We let U^j_n be the corresponding modulated profiles:

$$U^j_n(t,x) = \frac{1}{\lambda_{j,n}^{\frac{1}{m}}} U^j\left(\frac{t-t_{j,n}}{\lambda_{j,n}}, \frac{x}{\lambda_{j,n}}\right).$$

Lemma 3.5. *Assume that points* (1) *and* (2) *above hold, let u_n be the solution of* (1-1) *with initial data* $(u_{0,n}, u_{1,n})$, *and I_n be its maximal interval of existence. Then*

$$u_n(t,x) = \sum_{j=1}^{J} U^j_n(t,x) + w^J_n(t,x) + \varepsilon^J_n(t,x),$$

where

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left(\|\varepsilon^J_n\|_{S(\{t\in I_n,\, r\geq A\lambda_{j_{0,n}}+|t|\})} + \sup_{t\in I_n} \int_{A\lambda_{j_{0},n}+|t|}^{+\infty} |r\partial_{t,r}\vec{\varepsilon}^J_n(t,r)|^m \, dr \right) = 0.$$

Proof. This follows from Lemma 2.19 with

$$\tilde{u}_n = \sum_{j\in\mathcal{J}_0} U^j_n.$$

We omit the details of the proof that are by now standard; see, e.g., the proof of the main theorem in [Bahouri and Gérard 1999]. ☐

3E. *Exterior energy of a sum of profiles.*

Proposition 3.6. *Let $\{(u_{0,n}, u_{1,n})\}_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{L}^m$ that has a profile decomposition with profiles $\{U^j_{\mathrm{L}}\}_{j\geq 1}$ and parameters $\{(t_{j,n}, \lambda_{j,n})_n\}_{j\geq 1}$. Let $\{(\theta_n, \rho_n, \sigma_n)\}_{n\in\mathbb{N}}$ be a sequence such that $0 \leq \rho_n < \sigma_n \leq \infty$, $\theta_n \in \mathbb{R}$. Let $k \geq 1$. Then, extracting a subsequence if necessary*

$$o_n(1) + \int_{\rho_n}^{\sigma_n} |r\partial_{r,t} u_{\mathrm{L},n}(\theta_n, r)|^m \, dr \geq \int_{\rho_n}^{\sigma_n} |r\partial_{r,t} U^k_{\mathrm{L},n}(\theta_n, r)|^m \, dr, \tag{3-32}$$

where $\lim_n o_n(1) = 0$, $u_{\mathrm{L},n}$ is the solution of the linear wave equation with initial data $(u_{0,n}, u_{1,n})$ and $U^k_{\mathrm{L},n}$ is defined in (3-6).

See [Duyckaerts and Roy 2015, Proposition 3.12] for the proof.

4. Exterior energy for solutions of the nonlinear equation

4A. ***Preliminaries on singular stationary solutions.*** We recall from [Duyckaerts et al. 2014; Duyckaerts and Roy 2015; Shen 2013] the following result on existence of stationary solutions for (1-1).

Proposition 4.1. *Let $\ell \in \mathbb{R}\setminus\{0\}$. Assume $m > 1$, $m \neq 2$. There exists $R_\ell \geq 0$ and a maximal radial C^2 solution Z_ℓ of*

$$\Delta Z_\ell + \iota|Z_\ell|^{2m} Z_\ell = 0 \quad \text{on } \mathbb{R}^3 \cap \{|x| > R_\ell\} \tag{4-1}$$

such that

$$|rZ_\ell(r) - \ell| + |r^2 Z'_\ell(r) + \ell| \lesssim \frac{1}{r^{2m-2}}, \quad r \gg 1. \tag{4-2}$$

Furthermore,

- *if* $\iota = +1$ (*focusing nonlinearity*), *then* $R_\ell = 0$ *and* $Z_\ell \notin L^{3m}(\mathbb{R}^3)$,
- *if* $\iota = -1$ (*defocusing nonlinearity*), *then* $R_\ell > 0$ *and*

$$\lim_{r \to R_\ell} |Z_\ell(r)| = +\infty. \tag{4-3}$$

Remark 4.2. We will construct Z_1 and let

$$Z_\ell = \frac{\pm 1}{|\ell|^{\frac{1}{m-1}}} Z_1\left(\frac{r}{|\ell|^{\frac{m}{m-1}}}\right)$$

(where $\pm$ is the sign of ℓ), which will satisfy the conclusion of Proposition 4.1 for all $\ell \in \mathbb{R} \setminus \{0\}$. In particular,

$$R_\ell = R_1 |\ell|^{\frac{m}{m-1}}.$$

Let us mention that the uniqueness of Z_ℓ can be proved by elementary arguments. However, it will follow from Proposition 4.3 and we will not prove it here.

Proof. The proof is essentially contained in [Duyckaerts et al. 2014; Shen 2013] (focusing case for $m > 2$ and $m \in (1, 2)$ respectively) and [Duyckaerts and Roy 2015] (defocusing case for $m > 2$). We give a sketch for the sake of completeness.

We assume $\ell = 1$ (see Remark 4.2).

Existence for large r. Letting $g = rZ_1$, we see that the equation on Z_1 is equivalent to

$$g''(r) = -\frac{\iota}{r^{2m}} |g(r)|^{2m} g(r). \tag{4-4}$$

It is sufficient to find a fixed point for the operator A defined by

$$A(g) = 1 - \int_r^\infty \int_s^\infty \frac{\iota}{\sigma^{2m}} |g(\sigma)|^{2m} g(\sigma)\, d\sigma\, ds$$

in the ball

$$B = \{g \in C^0([r_0, +), \mathbb{R}) : d(g, 1) \le M\},$$

where r_0 and M are two large parameters and

$$d(g, h) := \sup_{r \ge r_0} (r^{2m-2} |g(r) - h(r)|).$$

Noting that (B, d) is a complete metric space, it is easy to prove that A is a contraction on B assuming $M \gg 1$ and $r_0 \gg 1$ (depending on M), and thus that A has a fixed point g_1. The fact that $Z_1 := \frac{1}{r} g_1$ satisfies the estimates (4-2) follows easily. Let $R_1 \ge 0$ such that $(R_1, +\infty)$ is the maximal interval of existence of g_1 as a solution of the ordinary differential equation.

Focusing case. We next assume $\iota = 1$ and prove that $R_1 = 0$ and $Z_\ell \notin L^{3m}$. Let

$$G(r) = \frac{1}{2} g'(r)^2 + \frac{1}{(2m+2) r^{2m}} |g(r)|^{2m+2}.$$

By (4-4), if $r \in (R_1, +\infty)$,

$$G'(r) = -\frac{m}{(m+1)r^{2m+1}} |g(r)|^{2m+2}.$$

Hence

$$|G'(r)| \le \frac{C}{r}\, G(r).$$

This proves that G is bounded on $(R_1, +\infty)$ if $R_1 > 0$, a contradiction with the standard ODE blow-up criterion. Thus $R_1 = 0$.

The fact that $Z_1 \notin L^{3m}(\mathbb{R}^3)$ is nontrivial but classical. Assume by contradiction that $Z_1 \in L^{3m}$. Then one can prove, see [Duyckaerts et al. 2014], that Z_1 is a solution in the distributional sense on $\mathbb{R}^3$ of

$$-\Delta Z_1 = |Z_1|^{2m} Z_1.$$

Noting that $|Z_1|^{2m} \in L^{\frac{3}{2}}$, one can use [Trudinger 1968] to prove that $Z_1 \in L^\infty$, and thus, by elliptic regularity, that Z_1 is C^2 on $\mathbb{R}^3$. To deduce a contradiction, we introduce, as in [Shen 2013], the function $v(r) = r^{\frac{1}{m}} Z_1$. It is easy to check, using (4-2), for the limits at infinity and the fact that Z_1 is C^2 for the limit at 0, that

$$\lim_{r\to 0^+} v(r) = \lim_{r\to 0^+} r v'(r) = \lim_{r\to+\infty} v(r) = \lim_{r\to+\infty} r v'(r) = 0.$$

Furthermore,

$$v'' + \frac{2}{r}\left(1 - \frac{1}{m}\right) v' + \frac{1}{r^2}\left(\frac{1}{m^2} - \frac{1}{m}\right) v + \frac{1}{r^2} |v|^{2m} v = 0.$$

Integrating the identity

$$\frac{d}{dr}\left(r^2 \frac{|v'(r)|^2}{2} - \frac{m-1}{2m^2} v^2(r) + \frac{|v(r)|^{2m+2}}{2m+2} \right) = \frac{2-m}{m} r |v'(r)|^2 \tag{4-5}$$

between 0 and $+\infty$, one sees that v must be a constant, a contradiction with the construction of Z_1. Note that we have used in this last step that the constant $\frac{2-m}{m}$ in the right-hand side of the identity (4-5) is nonzero, i.e., $m \neq 2$.

Defocusing case. Assume $\iota = -1$. We prove that $R_1 > 0$ by contradiction. Assume $R_1 = 0$ and let

$$h(s) := Z_\ell\left(\frac{1}{s}\right).$$

Then

$$h''(s) = \frac{1}{s^4} |h(s)|^{2m} h(s)$$

and by (4-2),

$$\lim_{s\to 0^+} \frac{h(s)}{s} = \lim_{s\to 0^+} h'(s) = 1.$$

By a classical ODE argument, see [Duyckaerts and Roy 2015] for the details, one can prove that h blows up in finite time, a contradiction. This proves that $R_1 > 0$. The condition (4-3) follows from the standard ODE blow-up criterion. $\square$

4B. ***Statement.*** One of the main ingredients of the proof of Theorem 3 is a bound from below of the exterior $\mathcal{L}^m$-energy for nonzero, $\mathcal{L}^m$ solutions of (1-1). It is similar to [Duyckaerts et al. 2013, Propostions 2.1 and 2.2] and [Duyckaerts and Roy 2015, Propositions 4.1 and 4.2]. The statements in these articles are divided between two cases, whether the support of $(u_0, u_1) - (Z_\ell, 0)$ is compact for all $\ell \neq 0$ or not. We give below a unified statement.

If $(u_0, u_1) \in \mathcal{L}^m$ and $A > 0$ we will denote by $\mathcal{T}_A(u_0, u_1)$ the element of $\mathcal{L}^m$ defined by

$$\mathcal{T}_A(u_0, u_1)(r) = (u_0, u_1)(r) \quad \text{if } r > A, \tag{4-6}$$

$$\mathcal{T}_A(u_0, u_1)(r) = (u_0(A), 0) \quad \text{if } r \leq A. \tag{4-7}$$

We note that

$$\|\mathcal{T}_A(u_0, u_1)\|^m_{\mathcal{L}^m} = \int_A^{+\infty} \left(|\partial_r u_0(r)ht|^m + |u_1(r)|^m\right) r^m\, dr. \tag{4-8}$$

We denote by ess supp the essential support of a function defined on a domain D of $\mathbb{R}^3$:

$$\operatorname{ess\,supp}(f) = D \setminus \bigcup \{\Omega \subset D \mid \Omega \text{ is open and } f = 0 \text{ a.e. in } \Omega\}.$$

Recall from Proposition 4.1 the definition of Z_1 and R_1.

Proposition 4.3. *Let u be a radial solution of* (1-1) *with* $(u_0, u_1) \in \mathcal{L}^m$. *Assume that* (u_0, u_1) *is not identically* 0. *Then there exist* $A > 0$, $\eta > 0$ *such that, if* $(\tilde{u}_0, \tilde{u}_1) = \mathcal{T}_A(u_0, u_1)$, *and $\tilde{u}$ is the solution of*

$$\partial_t^2 \tilde{u} - \Delta \tilde{u} = \iota |\tilde{u}|^{2m} \tilde{u} \mathbb{1}_{\{r \geq A + |t|\}} \tag{4-9}$$

with initial data $(\tilde{u}_0, \tilde{u}_1)$, *then $\tilde{u}$ is global, scatters in $\mathcal{L}^m$ and the following holds for all $t \geq 0$ or for all $t \leq 0$*:

$$\int_{A+|t|}^{+\infty} |\partial_r \tilde{u}(r)|^m r^m\, dr + \int_{A+|t|}^{+\infty} |\partial_t \tilde{u}(r)|^m r^m\, dr \geq \eta. \tag{4-10}$$

The proof of Proposition 4.3 is very close to the proofs of the analogous propositions in [Duyckaerts et al. 2014; Duyckaerts and Roy 2015]. We give a sketch of proof for the sake of completeness.

4C. ***Sketch of proof of Proposition 4.3.*** We argue by contradiction, assuming that for all $A > 0$ the solution $\tilde{u}$ of (4-9) with initial data $\mathcal{T}_A(u_0, u_1)$ is not a scattering solution, or is scattering and satisfies

$$\liminf_{t \to \pm\infty} \int_{A+|t|} |\partial_{t,r} \tilde{u}(t, r)|^m r^m\, dr = 0. \tag{4-11}$$

We let

$$v(r) = r u(r), \quad v_0(r) = r u_0(r), \quad v_1(r) = r u_1(r).$$

Step 1: In this step we prove that there exists $\varepsilon_0 > 0$ such that, if $A > 0$ is such that

$$\int_A^{+\infty} (|\partial_r u_0|^m + |u_1|^m) r^m\, dr = \varepsilon \leq \varepsilon_0, \tag{4-12}$$

then

$$\int_A^{+\infty} |\partial_r v_0|^m + |v_1|^m \, dr \le \frac{C}{A^{(2m+1)(m-1)}} |v_0(A)|^{m(2m+1)}, \tag{4-13}$$

$$\text{for all } B \in [A, 2A], \quad |v_0(B) - v_0(A)| \le C A^{2-2m} |v_0(A)|^{2m+1} \le C\varepsilon^2 |v_0(A)|. \tag{4-14}$$

We first assume (4-13) and prove (4-14). By the Hölder inequality and (4-13) we have

$$|v_0(B) - v_0(A)| \le \int_A^{2A} |\partial_r v_0(r)| \, dr \le A^{\frac{m-1}{m}} \left(\int_A^{2A} |\partial_r v_0|^m \, dr \right)^{\frac{1}{m}} \le C A^{2-2m} |v_0(A)|^{2m+1}. \tag{4-15}$$

Furthermore, by (4-12) and (4) in Proposition 2.2,

$$\frac{1}{A^{m-1}} |v_0(A)|^m = A |u_0(A)|^m \lesssim \varepsilon,$$

which yields

$$|v_0(A)|^{2m} \lesssim \varepsilon^2 A^{2m-2}.$$

Combining with (4-15), we obtain the second inequality of (4-14).

We next prove (4-13). Let

$$(\tilde{u}_0, \tilde{u}_1) = \mathcal{T}_A(u_0, u_1).$$

Let $\tilde{u}$ and $\tilde{u}_{\mathrm{L}}$ be the solutions of the nonlinear wave equation (1-1) and the linear wave equation (1-3), respectively, with initial data $(\tilde{u}_0, \tilde{u}_1)$. By the small data theory, $\tilde{u}$ is global and

$$\sup_{t \in \mathbb{R}} \|\vec{\tilde{u}}(t) - \vec{\tilde{u}}_{\mathrm{L}}(t)\|_{\mathcal{L}^m} \le C \varepsilon^{2m+1}. \tag{4-16}$$

Using the exterior energy property (3) in Proposition 2.3, we have that the following holds for all $t \ge 0$ or for all $t \le 0$:

$$\int_A^{+\infty} (|\partial_r (v_0)|^m + |v_1|^m) \, dr \le C \int_{A+|t|}^{+\infty} |\partial_{r,t}(r\tilde{u}_{\mathrm{L}})(t,r)|^m \, dr \le C \int_{A+|t|}^{+\infty} |\partial_{r,t} \tilde{u}_{\mathrm{L}}(t,r)|^m r^m \, dr.$$

Using (4-16), we obtain that the following holds for all $t \ge 0$ or for all $t \le 0$:

$$\int_A^{+\infty} (|\partial_r (v_0)|^m + |v_1|^m) \, dr \le C \left(\int_{A+|t|}^{+\infty} |\partial_{r,t} \tilde{u}(t,r)|^m \, dr + \varepsilon^{(2m+1)m} \right). \tag{4-17}$$

Using (4-11) and the definition (4-12) of ε, and letting $t \to +\infty$ or $t \to -\infty$, we obtain

$$\frac{1}{C} \int_A^{+\infty} (|\partial_r v_0|^m + |v_1|^m) \, dr \le \left(\int_A^{+\infty} (|\partial_r u_0|^m + |u_1|^m) r^m \, dr \right)^{2m+1}.$$

By (4) in Proposition 2.2, and since $A|u_0(A)|^m = \frac{1}{A^{m-1}} |v_0(A)|^m$,

$$\int_A^{+\infty} (|\partial_r v_0|^m + |v_1|^m) \, dr \le C \left(\int_A^{+\infty} (|\partial_r v_0|^m + |v_1|^m) \, dr + \frac{1}{A^{m-1}} |v_0(A)|^m \right)^{2m+1}.$$

Since $\int_A^{+\infty} (|\partial_r v_0|^m + |v_1|^m) \, dr$ is small, we deduce (4-13).

Step 2: We prove that there exists $\ell \in \mathbb{R} \setminus 0$ such that

$$\lim_{r\to\infty} v_0(r) = \ell, \tag{4-18}$$

and that there exists a constant $M > 0$ (depending on u) such that

$$|v_0(r) - \ell| \le \frac{M}{r^{2m-2}} \tag{4-19}$$

for large r.

Let $\varepsilon > 0$ and fix A_0 such that

$$\int_{A_0}^{+\infty} (|\partial_r u_0|^m + |u_1|^m) r^m \, dr = \varepsilon \le \varepsilon_0, \tag{4-20}$$

where ε_0 is given by Step 1. By (4-14),

$$\text{for all } k \ge 0, \quad |v_0(2^{k+1}A_0)| \le (1 + C\varepsilon^2)(|v_0(2^k A_0)|).$$

Hence, by a straightforward induction,

$$\text{for all } k \ge 0, \quad |v_0(2^{k+1}A_0)| \le (1 + C\varepsilon^2)^k |v_0(A_0)|.$$

Using (4-14) again, we deduce

$$\left|v_0(2^{k+1}A_0) - v_0(2^k A_0)\right| \le C(2^k A_0)^{2-2m} (1 + C\varepsilon^2)^{k(2m+1)} |v_0(A_0)|^{2m+1}. \tag{4-21}$$

Choosing ε small enough (so that $2^{2-2m}(1 + C\varepsilon^2)^{2m+1} < 1$), we see that

$$\sum_{k\ge 1} \left|v_0(2^{k+1}A_0) - v_0(2^k A_0)\right| < \infty,$$

and thus that $v_0(2^k A_0)$ has a limit ℓ as $k \to +\infty$. Using (4-14) again, we deduce

$$\lim_{r\to\infty} |v_0(r)| = \ell.$$

Summing (4-21) over all $k \ge 0$, we deduce, using that v_0 is bounded, that there exists a constant $M > 0$, such that $|v_0(A_0) - \ell| \le M A_0^{2-2m}$ for A_0 large enough. This yields (4-19).

It remains to prove that $\ell \ne 0$. We argue by contradiction. By (4-19), if $\ell = 0$, then

$$|v_0(r)| \le \frac{M}{r^{2m-2}}.$$

On the other hand, using (4-14) and an easy induction argument, we obtain that for all $\varepsilon > 0$, for all A_0 satisfying (4-20),

$$|v_0(2^k A_0)| \ge (1 - C\varepsilon^2)^k |v_0(A_0)|.$$

Combining with the previous bound, we obtain

$$(1 - C\varepsilon^2)^k |v_0(A_0)| \le \frac{M}{(2^k A_0)^{2m-2}},$$

a contradiction if ε is chosen small enough unless $v_0(A_0)=0$. Using (4-13), we see that this would imply $v_0(r)=0$ and $v_1(r)=0$ for almost all $r\geq A_0$. Since this is true for any A_0 such that (4-20) holds, an obvious bootstrap argument proves that $(v_0,v_1)=(0,0)$ almost everywhere, contradicting our assumption.

Step 3: Recall from Proposition 4.1 the definition of R_ℓ. Let, for $r>R_\ell$,

$$(g_0,g_1)(r):=(u_0(r)-Z_\ell(r),u_1(r)),\quad (h_0,h_1)(r)=r(g_0(r),g_1(r)).$$

If $\varepsilon>0$, we fix $A_\varepsilon>R_\ell$ such that

$$\int_{A_\varepsilon}^{+\infty}|\partial_r Z_\ell|^m r^m\,dr+\|Z_\ell\|^m_{S(\{r\geq A_\varepsilon+|t|\})}\leq\frac{\varepsilon^m}{C},\tag{4-22}$$

In this step, we prove that for all $\varepsilon>0$, if $A>A_\varepsilon$ satisfies

$$\int_A^{+\infty}(|\partial_r g_0|^m+|g_1|^m)r^m\,dr<\frac{\varepsilon^m}{C}\tag{4-23}$$

then

$$\int_A^{+\infty}|\partial_r h_0|^m+|h_1|^m\,dr\leq\frac{\varepsilon}{A^{m-1}}|h_0(A)|^m.\tag{4-24}$$

Fix $A>A_\varepsilon$, let $(\tilde u_0,\tilde u_1)=\mathcal{T}_A(u_0,u_1)$, and let $\tilde u$ be the solution of the nonlinear wave equation (1-1) with initial data $(\tilde u_0,\tilde u_1)$ at $t=0$. Note that by (4-23) and small data theory, $\tilde u$ is global and scatters in both time directions. Note also that by our assumption, $\tilde u$ satisfies (4-11).

Define $\tilde g$ as the solution to the equation

$$\begin{cases}\partial_t^2\tilde g-\Delta\tilde g=\mathbb{1}_{\{r\geq A+|t|\}}(|\tilde u|^{2m}\tilde u-|Z_\ell|^{2m}Z_\ell),\\ \vec{\tilde g}_{\restriction t=0}=\mathcal{T}_A(g_0,g_1),\end{cases}\tag{4-25}$$

and $\tilde g_{\mathrm{L}}$ the solution of the free wave equation with the same initial data. Notice that $(\partial_t^2-\Delta)(\tilde u-Z_\ell)=(\partial_t^2-\Delta)\tilde g$ for $r>A+|t|$ and $\vec{\tilde g}(0,r)=(\tilde u_0-Z_\ell,\tilde u_1)(r)$ for $r>A$. Thus, by finite speed of propagation, $\tilde g=\tilde u-Z_\ell$ for $r>A+|t|$, and we can rewrite the first equation in (4-25):

$$\partial_t^2\tilde g-\Delta\tilde g=\mathbb{1}_{\{r\geq A+|t|\}}\big(|Z_\ell+\tilde g|^{2m}(Z_\ell+\tilde g)-|Z_\ell|^{2m}Z_\ell\big).\tag{4-26}$$

Using (4-26), Strichartz estimates and the Hölder inequality, we see that for all time intervals I containing 0

$$\|\tilde g-\tilde g_{\mathrm{L}}\|_{S(I)}+\sup_{t\in I_{\max}(u)}\|\vec{\tilde g}(t)-\vec{\tilde g}_{\mathrm{L}}(t)\|_{\mathcal{L}^m}\leq C\big(\|Z_\ell\|^{2m}_{S(\{r\geq A+|t|\})}\|\tilde g\|_{S(I)}+\|\tilde g\|^{2m+1}_{S(I)}\big).$$

By (4-23), (4-22) and a straightforward bootstrap argument, we deduce that for all intervals I with $0\in I$,

$$\|\tilde g\|_{S(I)}\leq C\|\tilde g_{\mathrm{L}}\|_{S(I)}\leq C\|\mathcal{T}_A(g_0,g_1)\|_{\mathcal{L}^m}\leq C\varepsilon,$$

and

$$\sup_{t\in\mathbb{R}}\|\vec{\tilde g}(t)-\vec{\tilde g}_{\mathrm{L}}(t)\|_{\mathcal{L}^m}\leq C\varepsilon^{2m}\|\mathcal{T}_A(g_0,g_1)\|_{\mathcal{L}^m}.\tag{4-27}$$

By the exterior energy property (3) in Proposition 2.3, the following holds for all $t \geq 0$ or for all $t \leq 0$:

$$\int_A^{+\infty} (|h_0|^m + |h_1|^m)\, dr \leq C \int_{A+|t|}^{+\infty} |\partial_{t,r} \tilde{g}_{\mathrm{L}}|^m r^m\, dr$$
$$\leq C\biggl((\varepsilon^{2m} \|\mathcal{T}_A(g_0, g_1)\|_{\mathcal{L}^m})^m + \int_{A+|t|}^{+\infty} |\partial_{t,r} \tilde{g}|^m r^m\, dr \biggr),$$

where in the last line we used (4-27).

Letting $t \to \pm\infty$ and using (4-11), we deduce

$$\int_A^{+\infty} |\partial_r h_0|^m + |h_1|^m\, dr \leq C \varepsilon^{2m^2} \int_A^{+\infty} (|\partial_r g_0|^m + |g_1|^m) r^m\, dr.$$

The desired estimate (4-24) follows, taking ε small and using (4) in Proposition 2.2.

Step 4: Fix a small $\varepsilon > 0$ and let A_ε be as in Step 3, i.e., such that (4-22) holds. In this step, we prove that $r \leq A_\varepsilon$ on $\operatorname{ess\,supp}(u_0 - Z_\ell, u_1)$.

Indeed, if not, we obtain from (4-24) that there exists $A > A_\varepsilon$ such that $h_0(A) \neq 0$. Using a similar argument to that in Step 1, we deduce from (4-24) that for all $A \geq A_\varepsilon$ such that (4-23) holds,

$$\text{for all } B \in [A, 2A], \quad |h_0(A) - h_0(B)| \leq C\varepsilon |h_0(A)|. \tag{4-28}$$

If $\operatorname{ess\,supp}(u_0 - Z_\ell, u_1)$ is not bounded, we deduce by (4-24) that $h_0(A) \neq 0$ for all large $A > 0$. If $\varepsilon > 0$ is small enough, we deduce using (4-28) that

$$\lim_{r \to +\infty} r^\alpha h_0(r) = +\infty,$$

where $\alpha \in (0, 2m - 2)$ is fixed. Since

$$v_0(r) - \ell = h_0(r) - \ell + r Z_\ell,$$

this contradicts (4-19) in Step 2 and the asymptotic estimate (4-2) of Z_ℓ.

We have proved that $\operatorname{ess\,supp}(u_0 - Z_\ell, u_1)$ is bounded. Using (4-24), (4-28) and a straightforward bootstrap argument, we deduce that $r \leq A_\varepsilon$ on the support of $\operatorname{ess\,supp}(u_0 - Z_\ell, u_1)$.

Step 5: Fix a small $\varepsilon > 0$. We have proved in Step 4 that $(u_0, u_1)(r) = (Z_\ell(r), 0)$ for almost every $r \geq A_\varepsilon$, where A_ε depends only on ℓ. We will prove $(u_0, u_1)(r) = (Z_\ell(r), 0)$ for $r > R_\ell$, a contradiction with Proposition 4.1 since $(u_0, u_1) \in \mathcal{L}^m$.

We argue by contradiction, assuming that there exists $B > R_\ell$ such that $B \in \operatorname{ess\,supp}(u_0 - Z_\ell, u_1)$. Using a similar argument to that in Step 3, but on small time intervals (see, e.g., the proof of Proposition 2.2(a), §2.2.1 in [Duyckaerts et al. 2013]), we prove that the following holds for all $t \geq 0$ or for all $t \leq 0$:

$$B + |t| \in \operatorname{ess\,supp}\bigl((u(t) - Z_\ell, \partial_t u(t))\bigr). \tag{4-29}$$

Choose t_0 such that

$$B + |t_0| > A_\varepsilon \quad \text{on } \operatorname{ess\,supp}\bigl((u(t_0) - Z_\ell, \partial_t u(t_0))\bigr). \tag{4-30}$$

It is easy to see that u satisfies the following: for all $A > |t_0|$ the solution $\tilde{u}$ of

$$\partial_t^2 \tilde{u} - \Delta \tilde{u} = \iota |\tilde{u}|^{2m} \tilde{u} \mathbb{1}_{\{r \geq A + |t - t_0|\}}$$

with initial data $\mathcal{T}_A(\vec{u}(t_0))$ at $t = t_0$ is not a scattering solution, or is scattering and satisfies

$$\liminf_{t \to \pm\infty} \int_{A+|t-t_0|}^{+\infty} |\partial_{t,r} \tilde{u}(t,r)|^m r^m \, dr = 0.$$

We can then go through Steps 1–4 above, but with initial data at $t = t_0$, and restricting to $r > |t_0|$. Note that by finite speed of propagation, the limit ℓ obtained in Step 2 for $t = 0$ and for $t = t_0$ is the same; i.e.,

$$\lim_{r \to +\infty} r u(t_0, r) = \lim_{r \to +\infty} r u(0, r).$$

By the conclusion of Step 4, we obtain that $r < \max(A_\varepsilon, t_0)$ on $\operatorname{ess\,supp}(\vec{u}(t_0) - Z_\ell)$, contradicting (4-30). □

5. Dispersive term

This section concerns the existence of a "dispersive" component for a solution u of (1-1) that remains bounded in $\mathcal{L}^m$ along a sequence of times. This component is the strong limit of $\vec{u}(t)$, in $\mathcal{L}^m$, outside the origin in the finite time blow-up case (see Section 5A), and a solution of the linear wave equation in the global case (see Section 5B).

5A. *Regular part in the finite time blow-up case.*

Proposition 5.1. *Let u be a radial solution of* (1-1), (1-2). *Assume*

$$T_+(u) < \infty, \qquad \liminf_{t \to T_+(u)} \|\vec{u}(t)\|_{\mathcal{L}^m} < \infty.$$

Then there exists a solution v of (1-1), *defined in a neighborhood of $t = T_+$, such that for all t in $I_{\max}(u) \cap I_{\max}(v)$,*

$$\text{for all } r > T_+ - t, \quad \vec{u}(t,r) = \vec{v}(t,r).$$

We omit the proof; see Section 6.3 in [Duyckaerts and Roy 2015] for a very close proof.

5B. *Extraction of the radiation term in the global case.* We prove here:

Proposition 5.2. *Let u be a radial solution of* (1-1), (1-2). *Assume*

$$T_+(u) = +\infty, \qquad \liminf_{t \to +\infty} \|\vec{u}(t)\|_{\mathcal{L}^m} < \infty.$$

Then there exists a solution v_L of the free wave equation (1-3) *such that for all $A \in \mathbb{R}$,*

$$\lim_{t \to +\infty} \int_{|x| \geq A+|t|} \big(|\partial_t (u - v_L)|^m + |\partial_r (u - v_L)|^m \big) r^m \, dr = 0. \tag{5-1}$$

The proof relies on the following lemma, which is a consequence of finite speed of propagation, Strichartz estimates and the small data theory. We omit the proof, which is an easy adaptation of the proofs of Claims 2.3 and 2.4 in [Duyckaerts et al. 2016] where the usual energy is replaced by the $\mathcal{L}^m$-energy:

Lemma 5.3. *There exists $\varepsilon_1 > 0$ with the following property. Let u be a solution of* (1-1), (1-2) *such that $T_+(u) = +\infty$. Let $T \geq 0$ and $A \geq 0$. Assume $\|S_{\mathrm{L}}(\cdot - T)\vec{u}(T)\|_{S(\{|x|\geq A+t,\, t\geq T\})} = \varepsilon' < \varepsilon_1$. Then $\|u\|_{S(\{|x|\geq A+t,\, t\geq T\})} \leq 2\varepsilon'$, and there exists a solution v_{L} of the linear wave equation such that* (5-1) *holds.*

Proof of Proposition 5.2. See also Section 3.3 in [Duyckaerts et al. 2013].

Step 1: Let $t_n \to +\infty$ such that the sequence $(\vec{u}(t_n))_n$ is bounded in $\mathcal{L}^m$. In this step we prove that there exists $\delta > 0$ such that for large n,

$$\|S_{\mathrm{L}}(\cdot)\vec{u}(t_n)\|_{S(\{|x|\geq(1-\delta)t_n+t,\, t\geq 0\})} < \varepsilon_1, \tag{5-2}$$

where ε_1 is given by Lemma 5.3. We argue by contradiction, assuming (after extraction of subsequences) that there exists a sequence $\delta_n \to 0$ such that

$$\|S_{\mathrm{L}}(\cdot)\vec{u}(t_n)\|_{S(\{|x|\geq(1-\delta_n)t_n+t,\, t\geq 0\})} \geq \varepsilon_1. \tag{5-3}$$

Extracting subsequences again, we can assume that the sequence $(\vec{u}(t_n))_n$ has a profile decomposition with profiles U_{L}^j and parameters $(\lambda_{j,n}, t_{j,n})_n$. Let J be a large integer such that

$$\left\| S_{\mathrm{L}}(\cdot)\left(\vec{u}(t_n) - \sum_{j=1}^{J} \vec{U}_{\mathrm{L},n}^j(0)\right)\right\|_{S(\mathbb{R})} \leq \frac{\varepsilon_1}{2}.$$

A contradiction will follow if we prove (possibly extracting subsequences in n) that for all $j \in \{1, \dots, J\}$,

$$\lim_{n\to\infty} \|S_{\mathrm{L}}(\cdot)\vec{U}_{\mathrm{L},n}^j(0)\|_{S(\{|x|\geq(1-\delta_n)t_n+t,\, t\geq 0\})} = 0. \tag{5-4}$$

We have

$$\|S_{\mathrm{L}}(\cdot)\vec{U}_{\mathrm{L},n}^j(0)\|_{S(\{r\geq(1-\delta_n)t_n+t,\, t\geq 0\})} = \|U_{\mathrm{L}}^j\|_{S(A_{j,n})},$$

where

$$A_{j,n} := \left\{(t,r) \in \mathbb{R}\times(0,\infty) : t \geq -\frac{t_{j,n}}{\lambda_{j,n}} \text{ and } r \geq \frac{(1-\delta_n)t_n}{\lambda_{j,n}} + \left|t + \frac{t_{j,n}}{\lambda_{j,n}}\right|\right\}.$$

As a consequence, we see that we can extract subsequences so that the characteristic function of $A_{j,n}$ goes to 0 pointwise unless $\frac{t_{j,n}}{\lambda_{j,n}}$ and $\frac{t_n}{\lambda_{j,n}}$ are bounded. Time translating the profile U_{L}^j and extracting again, we can assume

$$\lim_{n\to\infty} \frac{t_n}{\lambda_{j,n}} = \tau_0 \in [0,\infty) \quad \text{for all } n,\ t_{j,n} = 0.$$

By finite speed of propagation and the small data theory,

$$\lim_{A\to+\infty} \limsup_{n\to+\infty} \int_{|x|\geq t_n+A} |r\partial_{r,t}u(t_n)|^m \, dr = 0. \tag{5-5}$$

By Proposition 3.6, for all $A \in \mathbb{R}$, we have that for large n,

$$\begin{aligned}\int_{t_n+A}^{+\infty} |r(\partial_{r,t} u(t_n))|^m \, dr &\geq \frac{1}{2} \int_{t_n+A}^{+\infty} |r(\partial_{r,t} U^j_{\mathrm{L,n}}(0))|^m \, dr \\ &= \frac{1}{2} \int_{\frac{t_n+A}{\lambda_{j,n}}}^{+\infty} |r(\partial_{r,t} U^j_{\mathrm{L}}(0))|^m \, dr \underset{n\to\infty}{\longrightarrow} \frac{1}{2} \int_{\tau_0}^{+\infty} |r(\partial_{r,t} U^j_{\mathrm{L}}(0))|^m \, dr.\end{aligned}$$

Combining with (5-5), we see that if U^j_{L} is not identically 0, then τ_0 is strictly positive, and we can rescale the profile U^j_{L} to assume $\tau_0 = 1$, and $\lambda_{j,n} = t_n$. Using (5-5) we see that ess supp $\vec{U}^j_{\mathrm{L}}(0)$ is included in the unit ball of $\mathbb{R}^3$, which implies

$$\|U^j_{\mathrm{L}}\|_{S(A_{j,n})} = \|U^j_{\mathrm{L}}\|_{S(\{t\geq 0,\, r\geq (1-\delta_n)+t\})} \underset{n\to\infty}{\longrightarrow} 0,$$

concluding the proof of (5-4) in this case. Step 1 is complete.

Step 2: By Step 1 and Lemma 5.3, for all $A \in \mathbb{R}$, there exists a solution v^A_{L} of the free wave equation such that

$$\lim_{t\to+\infty} \int_{|x|\geq A+|t|} \big(|\partial_t (u - v^A_{\mathrm{L}})|^m + |\partial_r (u - v^A_{\mathrm{L}})|^m\big) r^m \, dr = 0. \tag{5-6}$$

We consider the sequence $t_n \to +\infty$ of Step 1 and assume, extracting a subsequence if necessary, that $\vec{u}(t_n)$ has a profile decomposition $(U^j_{\mathrm{L}}, (\lambda_{j,n}, t_{j,n})_n)_{j\geq 1}$. Reordering the profiles and rescaling and time-translating U^1_{L} if necessary, we can assume, without loss of generality, that $t_{1,n} = t_n$ and $\lambda_{1,n} = 1$ for all n. In other words, $\vec{U}^1_{\mathrm{L}}(0)$ is the weak limit, as n goes to infinity, of $\vec{S}_{\mathrm{L}}(-t_n)\vec{u}(t_n)$. Note that U^1_{L} might be identically 0.

Fix $A \in \mathbb{R}$. Then

$$\vec{u}(t_n) - \vec{v}^A_{\mathrm{L}}(t_n) = \vec{U}^1_{\mathrm{L}}(t_n) - \vec{v}^A_{\mathrm{L}}(t_n) + \sum_{j=2}^{J} \vec{U}^j_{\mathrm{L},n}(0) + \vec{w}^J_n(t_n);$$

i.e., $\vec{u}(t_n) - \vec{v}^A_{\mathrm{L}}(t_n)$ has a profile decomposition $(\widetilde{U}^j_{\mathrm{L}}, (\lambda_{j,n}, t_{j,n})_n)_{j\geq 1}$, with $\widetilde{U}^j_{\mathrm{L}} = U^j_{\mathrm{L}}$ if $j \geq 2$, and $\widetilde{U}^1_{\mathrm{L}} = U^1_{\mathrm{L}} - v^A_{\mathrm{L}}$. By Proposition 3.6,

$$\limsup_{n\to\infty} \int_{r\geq A+t_n} |r\partial_{r,t}(u - v^A_{\mathrm{L}})(t_n)|^m \, dr \geq \limsup_{n\to\infty} \int_{r\geq t_n+A} |r\partial_{r,t}(U^1_{\mathrm{L}} - v^A_{\mathrm{L}})(t_n)|^m,$$

and thus, by (5-6)

$$\lim_{n\to\infty} \int_{r\geq t_n+A} |r\partial_{r,t}(U^1_{\mathrm{L}} - v^A_{\mathrm{L}})(t_n)|^m = 0.$$

Using (5-6) again, we obtain

$$\lim_{n\to\infty} \int_{r\geq t_n+A} |r\partial_{r,t}(U^1_{\mathrm{L}} - u)(t_n)|^m = 0.$$

This is valid for all $A \in \mathbb{R}$. A simple argument using finite speed of propagation and small data theory yields

$$\lim_{t\to\infty} \int_{r\geq t+A} |r\,\partial_{r,t}(U_{\mathrm{L}}^1 - u)(t)|^m = 0,$$

concluding the proof of the proposition with $v_{\mathrm{L}} = U_{\mathrm{L}}^1$. □

6. Scattering/blow-up dichotomy

In this section we prove Theorem 3. Let u be a solution of (1-1). Consider the property:

$$\liminf_{t\to T_+(u)} \|\vec{u}(t)\|_{\mathcal{L}^m} < \infty. \tag{6-1}$$

We must prove:

(1) If (6-1) holds then $T_+(u) = +\infty$.

(2) If $T_+(u) = +\infty$ and (6-1) holds, then u scatters to a linear solution in $\mathcal{L}^m$.

The proofs of (1) and (2) are very similar, and are simplified versions of the corresponding proofs in [Duyckaerts and Roy 2015]. We will only sketch the proof of (2) and explain the necessary modification to obtain (1).

6A. ***Proof of scattering.*** Let u be a global solution and let $t_n \to +\infty$ such that $\vec{u}(t_n)$ is bounded. Let v_{L} be the linear component of u, given by Proposition 5.2. Extracting subsequences, we can assume that $(\vec{u}(t_n) - \vec{v}_{\mathrm{L}}(t_n))_n$ has a profile decomposition with profiles U_{L}^j and parameters $(\lambda_{j,n}, t_{j,n})_n$. As before, we denote by $U_{\mathrm{L},n}^j$ the modulated profiles; see (3-6). Extracting subsequences and translating the profiles in time if necessary, one of the following three cases holds.

Case 1: Assume

$$\forall j \geq 1, \quad U_{\mathrm{L}}^j \equiv 0 \quad \text{or} \quad \lim_{n\to\infty} \frac{-t_{j,n}}{\lambda_{j,n}} = -\infty. \tag{6-2}$$

Let $T \gg 1$ such that $\|v_{\mathrm{L}}\|_{S((T,+\infty))} < \frac{\delta_0}{2}$, where δ_0 is given by the small data theory (see Proposition 2.17). By (6-2), for all j,

$$\lim_{n\to\infty} \|U_{\mathrm{L},n}^j\|_{S((T-t_n,0))} = \lim_{n\to\infty} \|U_{\mathrm{L}}^j\|_{S\left(\left(\frac{T-t_n-t_{j,n}}{\lambda_{j,n}}, \frac{-t_{j,n}}{\lambda_{j,n}}\right)\right)} = 0.$$

Thus for large n,

$$\|S_{\mathrm{L}}(\cdot)\vec{u}(t_n)\|_{S((T-t_n,0))} < \delta_0.$$

By Proposition 2.17, for large n,

$$\|u\|_{S((T,t_n))} = \|u(t_n + \cdot)\|_{S((T-t_n,0))} < 2\delta_0.$$

Letting $n \to \infty$, we deduce $\|u\|_{S((T,+\infty))} < 2\delta_0$, and thus u scatters.

Case 2: We assume

$$\forall j \geq 1, \quad U_{\mathrm{L}}^{j} \equiv 0 \quad \text{or} \quad \lim_{n\to\infty} \frac{-t_{j,n}}{\lambda_{j,n}} \in \{\pm\infty\}. \tag{6-3}$$

and

$$\exists j_0 \geq 1, \quad U_{\mathrm{L}}^{j_0} \not\equiv 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{-t_{j_0,n}}{\lambda_{j_0,n}} = +\infty. \tag{6-4}$$

We will use a channel of energy argument based on the following observation, which is a direct consequence of the explicit form of the solution; see (2-4), (2-6):

Claim 6.1. *Let u_{L} be a nonzero solution of the linear wave equation* (1-3) *with initial data in $\mathcal{L}^m$. Then there exists $A \in \mathbb{R}$ such that*

$$\liminf_{t\to+\infty} \int_{A+t}^{+\infty} r^m |\partial_{r,t} u_{\mathrm{L}}|^m \, dr > 0.$$

If $j \geq 1$, we have

$$\|U_{\mathrm{L},n}^{j}\|_{S(\{t\geq 0,\, r\geq t\})} = \|U_{\mathrm{L}}^{j}\|_{S\left(\left\{t\geq -\frac{t_{j,n}}{\lambda_{j,n}},\, r\geq t+\frac{t_{j,n}}{\lambda_{j,n}}\right\}\right)}.$$

Noting that under the assumptions of Case 2,

$$\text{for all } j \geq 1, \quad \mathbb{1}_{\left\{t\geq -\frac{t_{j,n}}{\lambda_{j,n}},\, r\geq t+\frac{t_{j,n}}{\lambda_{j,n}}\right\}} \xrightarrow[n\to\infty]{} 0$$

pointwise, otherwise $U_{\mathrm{L}}^{j} \equiv 0$. We obtain

$$\text{for all } j \geq 1, \quad \lim_{n\to\infty} \|U_{\mathrm{L},n}^{j}\|_{S(\{r\geq t\geq 0\})} = 0$$

and thus

$$\lim_{n\to\infty} \|S_{\mathrm{L}}(t)\vec{u}(t_n)\|_{S(\{r\geq t\geq 0\})} = 0.$$

By the small data theory (see Proposition 2.17) and finite speed of propagation

$$\lim_{n\to\infty} \left(\|u(t_n+\cdot)\|_{S(\{r\geq t\geq 0\})} + \sup_{t\geq 0} \int_t^{+\infty} \left|\partial_{t,r}(u(t_n+t) - S_{\mathrm{L}}(t)\vec{u}(t_n))\right|^m r^m \, dr \right) = 0. \tag{6-5}$$

Let j_0 be as in (6-4). By Claim 6.1, there exists $A \in \mathbb{R}$ such that

$$\liminf_{t\to+\infty} \int_{\lambda_{j_0,n}A - t_{j_0,n}+t}^{\infty} |r\partial_{t,r} U_{\mathrm{L},n}^{j_0}|^m \, dr > 0.$$

For large n, we have $\lambda_{j_0,n}A - t_{j_0,n} \geq 0$. By Proposition 3.6, we deduce from (6-5) that for large n,

$$\liminf_{t\to+\infty} \int_{\lambda_{j_0,n}A - t_{j_0,n}+t-t_n}^{\infty} r^m |\partial_{t,r}(u(t,r) - v_{\mathrm{L}}(t,r))|^m \, dr > 0,$$

contradicting the definition of v_{L}.

Case 3: In this last case we assume

$$\exists j \geq 1,\ \forall n, \quad t_{j,n} = 0 \quad \text{and} \quad U_{\mathrm{L}}^{j} \not\equiv 0. \tag{6-6}$$

This is the core of the proof, where we use Proposition 4.3, and thus the fact that (1-1) has no nonzero stationary solution in $\mathcal{L}^m$.

We will use Section 3D to approximate u, outside appropriate wave cones, by a sum of profiles. As in Section 3D, we let $\mathcal{J}_0$ be the set of indices j such that $t_{j,n}=0$ for all n and $\mathcal{J}_\infty$ the set of j such that $\frac{t_{j,n}}{\lambda_{j,n}}$ goes to $+\infty$ or $-\infty$. Extracting subsequences and translating the profiles in time if necessary, we can assume $\mathbb{N}\setminus\{0\}=\mathcal{J}_0\cup\mathcal{J}_\infty$. Let $\delta_1>0$ be a small number, smaller than the number given by the small data theory, and such that there exists $j\in\mathcal{J}_0$ with $\|\vec{U}_{\mathrm{L}}^j(0)\|_{\mathcal{L}^m}>\delta_1$. We let $j_0\in\mathcal{J}_0$ such that $\|\vec{U}_{\mathrm{L}}^{j_0}(0)\|_{\mathcal{L}^m}>\delta_1$, and

$$\big(j\in\mathcal{J}_0 \quad\text{and}\quad \|\vec{U}_{\mathrm{L}}^j(0)\|_{\mathcal{L}^m}>\delta_1\big) \quad\Longrightarrow\quad \lim_{n\to\infty}\frac{\lambda_{j_0,n}}{\lambda_{j,n}}=+\infty. \tag{6-7}$$

We note that by Proposition 3.3, there exists a finite number of $j\in\mathcal{J}_0$ with $\|\vec{U}_{\mathrm{L}}^j(0)\|_{\mathcal{L}^m}>\delta_1$, so that, in view of the pseudo-orthogonality property (3-1), j_0 is well-defined. By Proposition 4.3, there exist A, $\eta>0$, $U^{j_0}\in S(\mathbb{R})$ such that $\vec{U}^{j_0}\in C^0(\mathbb{R},\mathcal{L}^m)$,

$$(\partial_t^2-\Delta)U^{j_0}=\iota|U^{j_0}|^{2m}U^{j_0}\mathbb{1}_{\{r\geq A+|t|\}},\quad \vec{U}^{j_0}(0)=\mathcal{T}_A(\vec{U}_{\mathrm{L}}^{j_0}(0)), \tag{6-8}$$

and the following holds for all $t\geq 0$ or for all $t\leq 0$:

$$\int_{|t|+A}^{+\infty}|r\partial_{t,r}U^{j_0}|^m\,dr\geq\eta. \tag{6-9}$$

Note that $(U_{\mathrm{L}}^j,\lambda_{j,n},t_{j,n})_{j\geq 0}$ with $U_{\mathrm{L}}^0=v_{\mathrm{L}}$ and $\lambda_{0,n}=1, t_{0,n}=t_n$ is a profile decomposition of $\vec{u}(t_n)$. According to Lemma 3.5,

$$u(t+t_n)=v_{\mathrm{L}}(t+t_n)+\sum_{j=1}^J U_n^j(t)+w_n^J(t)+\varepsilon_n^J(t),\quad t\in[-t_n,+\infty), \tag{6-10}$$

where the modulated profiles U_n^j for $j\neq j_0$ are defined in Section 3D and

$$\limsup_{n\to\infty}\bigg(\|\varepsilon_n^J\|_{S(\{t\in[-t_n,+\infty),\,r>A\lambda_{j_0,n}+|t|\})}+\sup_{t\geq -t_n}\int_{|t|+A\lambda_{j_0,n}}^{+\infty}|r\partial_{t,r}\varepsilon_n^J(t,r)|^m\,dr\bigg)$$

goes to 0 as J goes to infinity. It can be deduced from Proposition 3.6 that for all sequences $(\theta_n)_n$ in $[-t_n,+\infty)$,

$$o_n(1)+\int_{A\lambda_{j_0,n}+|\theta_n|}^{+\infty}\big|r\partial_{r,t}(u-v_{\mathrm{L}})(t_n+\theta_n,r)\big|^m\,dr\geq\int_{A\lambda_{j_0,n}+|\theta_n|}^{+\infty}|r\partial_{r,t}U_n^{j_0}(\theta_n,r)|^m\,dr. \tag{6-11}$$

Indeed, this can be proved by noticing that (6-10) (and its time derivative) at $t=\theta_n$ can be considered as a profile decomposition of the sequence $((\vec{u}-\vec{v}_{\mathrm{L}})(\theta_n+t_n))_n$ and using Proposition 3.6 and finite speed of propagation. We refer to the proof of (3.18) in [Duyckaerts and Roy 2015] for a detailed proof in a very similar setting.

If (6-9) holds for $t\geq 0$, then by (6-11), for large n,

$$\limsup_{t\to+\infty}\int_{t+A\lambda_{j_0,n}}^{+\infty}|r\partial_{t,r}(u(t_n,r)-v_{\mathrm{L}}(t_n,r))|^m\,dr\geq\frac{\eta}{2},$$

contradicting the definition of v_{L}.

If (6-9) holds for $t \leq 0$, we use (6-11) at $\theta_n = -t_n$ together with (6-9) and obtain that for large n

$$\int_{t_n + A\lambda_{j_0,n}}^{+\infty} |r\partial_{t,r}(u(0,r) - v_{\mathrm{L}}(0,r))|^m \, dr \geq \frac{\eta}{2},$$

a contradiction since $\vec{u}(0) \in \mathcal{L}^m$. □

6B. ***Proof of global existence.*** We argue by contradiction, assuming that (6-1) holds and that $T_+ = T_+(u)$ is finite. Let v be the regular part of u at $t = T_+$, defined by Proposition 5.1. Recall that v is a solution of (1-1) defined in a neighborhood of $T_+(u)$ and such that

$$\text{for all } t \in I_{\max}(u) \cap I_{\max}(v), \text{ for all } r > T_+ - t, \quad \vec{u}(t,r) = \vec{v}(t,r). \tag{6-12}$$

As in Section 6A, we consider a sequence $t_n \to T_+$ such that $\vec{u}(t_n)$ is bounded in $\mathcal{L}^m$, and we assume (extracting subsequences if necessary) that $(\vec{u}(t_n) - \vec{v}(t_n))_n$ has a profile decomposition with profiles U_{L}^j and parameters $(\lambda_{j,n}, t_{j,n})_n$. We distinguish again between three cases.

Case 1: We assume (6-2). By the same proof as in Case 1 of Section 6A, we obtain

$$\lim_{n\to\infty} \left\| S_{\mathrm{L}}(\cdot)(\vec{u}(t_n) - \vec{v}(t_n)) \right\|_{S((-\infty,0))} = 0.$$

By Lemma 2.19, if $T < T_+(u)$ is in the domain of definition of v, close to $T_+(u)$,

$$\lim_{n\to\infty} \|\vec{u}(t_n)\|_{S((T,t_n))} < \infty,$$

which contradicts the blow-up criterion

$$\|\vec{u}(t_n)\|_{S((T,T_+(u)))} = +\infty,$$

Case 2: We assume (6-3) and (6-4). Fix $j_0 \geq 1$ such that (6-4) holds. Using Claim 6.1 and an argument very similar to the one of Case 2 of Section 6A, we obtain that for large n,

$$\liminf_{t\to T_+(u)} \int_{r \geq A\lambda_{j_0,n} - t_{j_0,n} + t - t_n} |\partial_{t,r}(u-v)(t,r)|^m \, r^m \, dr > 0,$$

where $A \in \mathbb{R}$ is given by Claim 6.1, contradicting (6-12) (since for large n, we have $A\lambda_{j_0,n} - t_{j_0,n} \geq 0$).

Case 3: We assume (6-6). We define $\mathcal{J}_0$, $\mathcal{J}_\infty$ as in Case 3 of Section 6A and choose $j_0 \in \mathcal{J}_0$ such that (6-7) holds. Using Proposition 4.3, we obtain A, $\eta > 0$, and a solution $U^{j_0} \in S(\mathbb{R})$ of (6-8), such that (6-9) holds for all $t \geq 0$ or for all $t \leq 0$. We distinguish two cases.

If (6-9) holds for all $t \geq 0$, then we prove using Lemma 2.19 and Proposition 3.6 that for large n,

$$\liminf_{t\to T_+(u)} \int_{r \geq A\lambda_{j_0,n} + t - t_n} |\partial_{t,r}(u-v)(t,r)|^m \, r^m \, dr > 0,$$

a contradiction with (6-12).

If (6-9) holds for all $t \le 0$, we let $T \in [0, T_+(u))$ such that T is in the domain of definition of v. Using Lemma 2.19 and Proposition 3.6, we deduce that for large n

$$\int_{r \ge A\lambda_{j_0,n}+t_n-T} |\partial_{t,r}(u-v)(T,r)|^m\, r^m\, dr \ge \frac{\eta}{2},$$

a contradiction for large n, since $\partial_{t,r}(u-v)(T,r)$ is supported in $|x| \le T_+ - T$. This concludes the sketch of proof.

Appendix: Proof of Proposition 2.1

The "only if" part. First of all, we have a sequence of smooth radial functions $(f_n)_n$ with compact supports such that

$$\int_0^{+\infty} |\partial_r(f-f_n)(r)|^m r^m\, dr \to 0, \quad n \to \infty. \tag{A-1}$$

As a consequence, we clearly have (2-1). Notice that for $0 < r < r' < +\infty$, we have

$$|f(r') - f(r)| \le \frac{C_m}{r^{\frac{1}{m}}} \left(\int_r^{r'} |s\partial_s f(s)|^m\, ds \right)^{\frac{1}{m}},$$

and this yields that $f(r)$ is continuous.

To see (2-2), we first prove

$$|f(r)| \le \frac{1}{r^{\frac{1}{m}}} \left(\int_r^{+\infty} |s\partial_s f(s)|^m\, ds \right)^{\frac{1}{m}},$$

and

$$|rf(r)| \le r^{\frac{m-1}{m}} \left(\int_0^{r} |\partial_s (sf(s))|^m\, ds \right)^{\frac{1}{m}}.$$

Indeed, if $f \in C_0^\infty((0, +\infty))$, then the preceding inequality follows from the fundamental theorem of calculus and the Hölder inequality. The case of a general function f can be deduced from (A-1). The desired estimate (2-2) is an immediate consequence of these two inequalities.

The"if" part. Given a radial function $f(x)$ on $\mathbb{R}^3$, satisfying the conditions (2-1), (2-2), we are to construct a sequence of smooth radial functions $f_n(x)$ compactly supported in $\mathbb{R}^3$ such that (A-1) holds.

To achieve this, we take a smooth radial function $\varphi(x)$ on $\mathbb{R}^3$ such that $\varphi(x) = 1$ for $|x| \le 1$ and $\varphi(x) = 0$ if $|x| \ge 2$. Let $(\varepsilon_n)_n$ be a sequence of positive numbers, tending to zero as $n \to \infty$. Define

$$f_n(x) = \varphi(\varepsilon_n x)\left(1 - \varphi\left(\frac{x}{\varepsilon_n}\right)\right)(f * \zeta_{\varepsilon_n})(x), \tag{A-2}$$

where $\zeta_\varepsilon(\varrho)$ is the usual approximate delta function supported in $-\frac{\varepsilon}{2} < \varrho < 0$ and $f * \zeta_\varepsilon$ denotes the radial convolution as in [Strauss 1977], namely

$$f * \zeta_\varepsilon(x) = \int_{-\frac{\varepsilon}{2}}^{0} \zeta_\varepsilon(\varrho)\, f(|x| - \varrho)\, d\varrho.$$

Then it is clear that $f_n(x)$ is smooth, radial and supported in $\{x \in \mathbb{R}^3 \mid \varepsilon_n \le |x| \le \frac{2}{\varepsilon_n}\}$. We have

$$\partial_r(f(r) - f_n(r)) = -\varepsilon_n(\partial_r\varphi)(\varepsilon_n r)\left(1-\varphi\left(\frac{r}{\varepsilon_n}\right)\right)f(r) \tag{A-3}$$

$$+\varphi(\varepsilon_n r)\frac{1}{\varepsilon_n}(\partial_r\varphi)\left(\frac{r}{\varepsilon_n}\right)f(r) \tag{A-4}$$

$$+\left[1-\varphi(\varepsilon_n r)\left(1-\varphi\left(\frac{r}{\varepsilon_n}\right)\right)\right]\partial_r f(r) \tag{A-5}$$

$$+\partial_r\left[\varphi(\varepsilon_n r)\left(1-\varphi\left(\frac{r}{\varepsilon_n}\right)\right)\right]\int_{-\frac{\varepsilon_n}{2}}^{0}(f(r)-f(r-\varrho))\,\zeta_{\varepsilon_n}(\varrho)\,d\varrho \tag{A-6}$$

$$+\varphi(\varepsilon_n r)\left(1-\varphi\left(\frac{r}{\varepsilon_n}\right)\right)\int_{-\frac{\varepsilon_n}{2}}^{0}(\partial_r f(r)-\partial_r f(r-\varrho))\,\zeta_{\varepsilon_n}(\varrho)\,d\varrho. \tag{A-7}$$

In view of (2-1), one easily sees that multiplying by r on both sides of the above identity, raising them to the power m and integrating over $(0,+\infty)$, we have the contributions of (A-6), (A-7) go to zero as $n\to\infty$. In fact, this is immediate for (A-7) in view of the boundedness of φ and the fact that ζ_ε is an approximation of the identity. For (A-6), we need to estimate two terms produced correspondingly by the cases when ∂_r hits on $\varphi(\varepsilon_n r)$ and $\varphi\left(\frac{r}{\varepsilon_n}\right)$. In the first case, we use the fundamental theorem of calculus to write

$$f(r)-f(r-\varrho)=\int_0^1 \varrho\,\partial_r f(r-\theta\varrho)\,d\theta.$$

Applying Minkowski's inequality, we are led to estimating

$$\varepsilon_n\int_{-\frac{\varepsilon_n}{2}}^{0}\int_0^1\left(\int_{\frac{1}{2\varepsilon_n}}^{\frac{4}{\varepsilon_n}}|r\partial_r f(r)|^m\,dr\right)^{\frac{1}{m}}|\varrho\,\zeta_{\varepsilon_n}(\varrho)|\,d\theta\,d\varrho,$$

which is clearly tending to zero as $n\to\infty$. A similar argument applies to the second case. In fact, applying the same trick will lead us to estimating

$$\int_{-\frac{\varepsilon_n}{2}}^{0}\int_0^1\left(\int_{\frac{\varepsilon_n}{2}}^{2\varepsilon_n}|r\partial_r f(r)|^m\,dr\right)^{\frac{1}{m}}\frac{|\varrho|}{\varepsilon_n}\,\zeta_{\varepsilon_n}(\varrho)\,d\theta\,d\varrho,$$

which tends to zero as $n\to\infty$.

Next, by invoking (2-2), one sees that the contribution from (A-3) is bounded by

$$\Big(\sup_{\frac{1}{\varepsilon_n}\le r\le\frac{2}{\varepsilon_n}} r^{\frac{1}{m}}\big|f(r)|\Big)^m\cdot\int_{\frac{1}{\varepsilon_n}}^{\frac{2}{\varepsilon_n}}|\varphi'(\varepsilon_n r)|^m\varepsilon_n^m r^{m-1}\,dr\to 0,\quad n\to\infty.$$

Similar argument applies to (A-4) thanks to (2-2). Finally, the contribution of (A-5) is easily seen to be bounded by

$$\int_0^{2\varepsilon_n}|r\partial_r f(r)|^m\,dr+\int_{\frac{1}{\varepsilon_n}}^{+\infty}|r\partial_r f(r)|^m dr\longrightarrow 0,\quad n\to\infty. \qquad \square$$

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GLOBAL WEAK SOLUTIONS FOR GENERALIZED SQG IN BOUNDED DOMAINS

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We prove the existence of global L^2 weak solutions for a family of generalized inviscid surface quasigeostrophic (SQG) equations in bounded domains of $\mathbb{R}^2$. In these equations, the active scalar is transported by a velocity field which is determined by the scalar through a more singular nonlocal operator compared to the SQG equation. The result is obtained by establishing appropriate commutator representations for the weak formulation together with good bounds for them in bounded domains.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with smooth boundary. Define

$$\Lambda = \sqrt{-\Delta}, \tag{1-1}$$

where $-\Delta$ is the Laplacian operator in Ω with homogeneous Dirichlet boundary condition.

We consider the following family of active scalars

$$\partial_t \theta + u \cdot \nabla \theta = 0, \tag{1-2}$$

where $\theta = \theta(x,t)$, $u = u(x,t)$ with $(x,t) \in \Omega \times [0,\infty)$ and with the velocity u given by

$$u = \nabla^{\perp} \psi, \tag{1-3}$$

$$\psi = \Lambda^{-\alpha}\theta, \quad \alpha \in [0,2]. \tag{1-4}$$

Here, fractional powers of the Laplacian $-\Delta$ are based on eigenfunction expansions (see the first subsection of Section 2 below for definitions and notations) and ψ is called the stream function. By (1-3) the velocity u is automatically divergence-free. The case $\alpha = 2$ corresponds to the two-dimensional Euler equation in the vorticity formulation. When $\alpha = 1$, (1-2) is the surface quasigeostrophic (SQG) equation of geophysical significance [Held et al. 1995], which also serves as a two-dimensional model of the three-dimensional Euler equations in view of many striking physical and mathematical analogies between them [Constantin et al. 1994]. The global regularity issue is known for the two-dimensional Euler equations but remains open for any $\alpha < 2$. Growth of solutions when $\alpha = 1, 2$ and $\Omega = \mathbb{R}^2, \mathbb{T}^2$ was studied in [Córdoba and Fefferman 2002]; nonexistence of simple hyperbolic blow-up when $\alpha = 1$ and $\Omega = \mathbb{R}^2$ was confirmed in [Córdoba 1998]. We refer to [Chae et al. 2011] for a regularity criterion when $\alpha \in [1,2]$ and $\Omega = \mathbb{R}^2$. On the other hand, finite time blow-up for patch solutions of (1-2) in the half plane with

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small $\alpha < 2$ was recently shown in [Kiselev et al. 2016]. The velocity u becomes more singular when α decreases, and in particular, u is not in $L^2(\Omega)$ if θ is in $L^2(\Omega)$ and $\alpha < 1$. Equations (1-2) with $\alpha \in (0, 1)$ were introduced in [Chae et al. 2012] to understand solutions to the SQG-type equations with even more singular velocity fields. More precisely, that paper established the existence of global L^2 weak solutions on the torus $\mathbb{T}^2$, together with local existence and uniqueness of strong solutions in $\mathbb{R}^2$. The borderline case $\alpha = 0$ is surprisingly easy due to the cancellation of the nonlinear term: (1-2) reduces to the simple equation $\partial_t \theta = 0$, and thus $\theta(\cdot\,, t) = \theta(\cdot\,, 0)$ for all $t > 0$. On the other hand, if $\alpha < 0$ then the stream function $\psi = \Lambda^{-\alpha}\theta$ is not well-defined when $\theta \in L^2(\Omega)$, noticing that there is no dissipation in the equation.

In this paper, we are interested in the issue of global weak solutions for (1-2) with $\alpha \in (0, 1)$ in arbitrary (smooth) bounded domains of $\mathbb{R}^2$. Let us recall that the existence of global weak solutions for SQG ($\alpha = 1$) were first proved in [Resnick 1995] in the periodic case. This highlights a difference between the nonlinearities of the SQG equation and the three-dimensional Euler equations: SQG has weak continuity in L^2, while the Euler equations do not. The weak continuity of SQG is due to a remarkable commutator structure which was subsequently revisited in [Chae et al. 2011] and used in the proof of absence of anomalous dissipation in [Constantin et al. 2014]. In [Constantin and Nguyen 2016], this structure was adapted to arbitrary bounded domains to take into account the lack of translation invariance of the fractional Laplacian in domains: a new commutator between the fractional Laplacian and differentiation appears. In addition to that, with the more singular constitutive laws (1-4), in order to establish the weak continuity of the nonlinearity $u \cdot \nabla\theta$ we will need to find appropriate commutator representations for which good bounds can be derived. Let us emphasize that many known commutator estimates for fractional Laplacians in the whole space (or on tori) are too expensive for bounded domains due to possible singularity near the boundary or the lack of powerful tools of Fourier analysis. For further results on the fractional Laplacian and SQG in bounded domains, we refer to [Cabré and Tan 2010; Caffarelli and Silvestre 2007; Constantin and Ignatova 2016; 2017].

Our main result is:

Theorem 1.1. *Let $\alpha \in (0, 1)$ and $\theta_0 \in L^2(\Omega)$. There is a weak solution of* (1-2), $\theta \in L^\infty([0, \infty); L^2(\Omega))$ *with initial data θ_0. That is, for any $T \geq 0$ and $\phi \in C_0^\infty(\Omega \times (0, T))$, θ satisfies*

$$\int_0^T \int_\Omega \theta(x, t)\, \partial_t \phi(x, t)\, dx\, dt + \int_0^T \mathcal{N}(\psi, \phi)\, dt = 0 \tag{1-5}$$

and the initial data

$$\theta(\cdot\,, 0) = \theta_0(\cdot) \quad \textit{in } H^{-\varepsilon}(\Omega) \textit{ for all } \varepsilon > 0 \tag{1-6}$$

is attained. Here,

$$\mathcal{N}(\psi, \phi) = \frac{1}{2} \int_\Omega [\Lambda^\alpha, \nabla^\perp]\psi \cdot \nabla\phi\psi\, dx - \frac{1}{2} \int_\Omega \Lambda^{-1+\alpha}\nabla^\perp\psi \cdot \Lambda^{1-\alpha}[\Lambda^\alpha, \nabla\phi]\psi\, dx. \tag{1-7}$$

Moreover, θ *obeys the energy inequality*

$$\|\theta(\cdot\,, t)\|_{L^2(\Omega)}^2 \leq \|\theta_0\|_{L^2(\Omega)}^2 \quad \textit{a.e. } t \geq 0. \tag{1-8}$$

Additionally, the stream function ψ *is in* $C([0,\infty); D(\Lambda^{\alpha-\varepsilon}))$ *for any* $\varepsilon > 0$ *and its* $D(\Lambda^{\frac{\alpha}{2}})$ *norm is conserved*,

$$\|\psi(\cdot,t)\|_{D(\Lambda^{\alpha/2})} = \|\psi(\cdot,0)\|_{D(\Lambda^{\alpha/2})} \quad \text{for all } t > 0. \tag{1-9}$$

In Theorem 1.1 and what follows,

$$[A,B] := AB - BA$$

denotes the commutator of two operators A and B.

When $\alpha = 0$, we have $u = R^{\perp}\theta$, where $R = (\partial_{x_1}\Lambda^{-1}, \partial_{x_2}\Lambda^{-1})$ denotes the Riesz transforms. As $R : L^2(\Omega) \to L^2(\Omega)$ is continuous, we have $u\theta \in L^1(\Omega)$ if $\theta \in L^2(\Omega)$. In that case, θ is a weak solution of (1-2) if

$$\int_0^T\int_\Omega \theta(x,t)\,\partial_t\phi(x,t)\,dx\,dt + \int_0^T\int_\Omega u(x,t)\,\theta(x,t)\cdot\nabla\phi(x,t)\,dx\,dt = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega\times(0,T)).$$

The global existence of such solutions was proved in [Constantin and Nguyen 2016]. However, when $\alpha < 1$, we have u is less regular than θ and the second integral in the preceding formulation is not well-defined. Nevertheless, taking into account the nonlinearity structure to explore extra cancellations, this integral has the commutator representation (1-7), which makes sense provided only $\theta \in L^2(\Omega)$, as will be proved in Lemma 3.4 below using the heat kernel approach. Let us note that the two objects are equal if $\psi \in H_0^1(\Omega)$, or equivalently, $\theta \in D(\Lambda^{1-\alpha})$. This representation is good enough to well define the nonlinearity but another representation, see (3-5), will be needed for the compactness argument. The point is that these two representations are equivalent provided only $\theta \in L^2(\Omega)$ (see Lemma 3.3 below). Unlike the proof in [Constantin and Nguyen 2016], which uses only Galerkin approximations, Theorem 1.1 will be proved by a two-tier approximation procedure: Galerkin approximations for each vanishing viscosity approximation. This is because the nonlinearity $u\theta$ is not well-defined in $L^1(\Omega)$ (see Remark 3.6 below).

The paper is organized as follows. In Section 2, we present the functional setup of fractional Laplacian in domains and necessary commutator estimates, which can be of independent interest. The proof of Theorem 1.1 is presented in Section 3. Finally, the proofs of the commutator estimates announced in Section 2 are given Appendices A and B.

2. Preliminaries

Fractional Laplacian. Let Ω be an open bounded set of $\mathbb{R}^d$, $d \ge 2$, with smooth boundary. The Laplacian $-\Delta$ is defined on $D(-\Delta) = H^2(\Omega)\cap H_0^1(\Omega)$. Let $\{w_j\}_{j=1}^\infty$ be an orthonormal basis of $L^2(\Omega)$ comprised of L^2-normalized eigenfunctions w_j of $-\Delta$; i.e.,

$$-\Delta w_j = \lambda_j w_j, \quad w_j|_{\partial\Omega} = 0, \quad \int_\Omega w_j^2\,dx = 1,$$

with $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \to \infty$.

The fractional Laplacian is defined using eigenfunction expansions,

$$\Lambda^s f \equiv (-\Delta)^{\frac{s}{2}} f := \sum_{j=1}^{\infty} \lambda_j^{\frac{s}{2}} f_j, w_j \qquad \text{with } f = \sum_{j=1}^{\infty} f_j w_j, \quad f_j = \int_\Omega f w_j \, dx,$$

for $s \geq 0$ and

$$f \in D(\Lambda^s) := \{f \in L^2(\Omega) : \Lambda^s f \in L^2(\Omega)\}.$$

The norm of $f = \sum_{j=1}^{\infty} f_j w_j$ in $D(\Lambda^s)$, $s \geq 0$, is defined by

$$\|f\|_{D(\Lambda^s)} := \|\Lambda^s f\|_{L^2(\Omega)} = \left(\sum_{j=1}^{\infty} \lambda_j^s f_j^2\right)^{\frac{1}{2}}.$$

It is also well-known that $D(\Lambda)$ and $H_0^1(\Omega)$ are isometric. In the language of interpolation theory, see [Lions and Magenes 1972, Chapter 1],

$$D(\Lambda^s) = [L^2(\Omega), D(-\Delta)]_{\frac{s}{2}} \quad \text{for all } s \in [0, 2].$$

As mentioned above,

$$H_0^1(\Omega) = D(\Lambda) = [L^2(\Omega), D(-\Delta)]_{\frac{1}{2}};$$

hence

$$D(\Lambda^s) = [L^2(\Omega), H_0^1(\Omega)]_s \quad \text{for all } s \in [0, 1]. \tag{2-1}$$

Consequently, we can identify $D(\Lambda^s)$ with usual Sobolev spaces:

$$D(\Lambda^s) = \begin{cases} H_0^s(\Omega) & \text{if } s \in \left(\frac{1}{2}, 1\right], \\ H_{00}^{\frac{1}{2}}(\Omega) := \{u \in H^{\frac{1}{2}}(\Omega) : u/\sqrt{d(x)} \in L^2(\Omega)\} & \text{if } s = \frac{1}{2}, \\ H^s(\Omega) & \text{if } s \in \left[0, \frac{1}{2}\right). \end{cases} \tag{2-2}$$

see [Lions and Magenes 1972, Chapter 1]. Here and below $d(x)$ is the distance to the boundary of the domain:

$$d(x) = d(x, \partial\Omega). \tag{2-3}$$

Next, for $s > 0$ we define

$$\Lambda^{-s} f = \sum_{j=1}^{\infty} \lambda_j^{-\frac{s}{2}} f_j w_j$$

if $f = \sum_{j=1}^{\infty} f_j w_j \in D(\Lambda^{-s})$ with

$$D(\Lambda^{-s}) := \left\{\sum_{j=1}^{\infty} f_j w_j \in \mathscr{D}'(\Omega) : f_j \in \mathbb{R}, \ \sum_{j=1}^{\infty} \lambda_j^{-\frac{s}{2}} f_j w_j \in L^2(\Omega)\right\};$$

moreover,

$$\|f\|_{D(\Lambda^{-s})} := \|\Lambda^{-s} f\|_{L^2(\Omega)} = \left(\sum_{j=1}^{\infty} \lambda_j^{-s} f_j^2\right)^{\frac{1}{2}}.$$

It is easy to check that $D(\Lambda^{-s})$ is the dual of $D(\Lambda^s)$ with respect to the pivot space $L^2(\Omega)$.

We have the following relation between $D(\Lambda^s)$ and $H^s(\Omega)$ when $s \geq 0$.

Lemma 2.1. *The continuous embedding*

$$D(\Lambda^s) \subset H^s(\Omega) \tag{2-4}$$

holds for any $s \geq 0$.

Proof. By interpolation, it suffices to prove (2-4) for $s \in \{0, 1, 2, \dots\}$. The case $s = 0$ is obvious and the case $s = 1$ follows from (2-2). Assume by induction (2-4) for $s \leq m$ with $m \geq 1$. Let $\theta \in D(\Lambda^{m+1})$. Then $f := -\Delta\theta \in D(\Lambda^{m-1})$ and thus $f \in H^{m-1}(\Omega)$ by the induction hypothesis. On the other hand, θ vanishes on the boundary $\partial\Omega$ in the trace sense because $\theta \in D(\Lambda^1) = H_0^1(\Omega)$. Elliptic regularity then implies that $\theta \in H^{m+1}(\Omega)$ and

$$\|\theta\|_{H^{m+1}} \leq C\|f\|_{H^{m-1}} \leq C\|\Delta\theta\|_{m-1,D} = C\|\theta\|_{m+1,D},$$

which is (2-4) for $s = m + 1$. □

Lemma 2.2. *The operator*

$$\Lambda^\mu \nabla : D(\Lambda^\gamma) \to D(\Lambda^{\gamma-1-\mu}) \tag{2-5}$$

is continuous for any $\gamma \in [0, 1]$ *and* $\mu \leq \gamma - 1$.

Proof. We first note that the gradient operator ∇ is continuous from $H_0^1(\Omega)$ to $L^2(\Omega)$ and from $L^2(\Omega)$ to $H^{-1}(\Omega)$; hence by interpolation,

$$\nabla : [L^2, H_0^1]_\gamma \to [H^{-1}, L^2]_\gamma$$

for any $\gamma \in [0, 1]$. From the interpolation (2-1) we deduce that

$$\begin{aligned} &[L^2, H_0^1]_\gamma = D(\Lambda^\gamma), \\ &[H^{-1}, L^2]_\gamma = ([H^1, L^2]_\gamma)^* = ([L^2, H^1]_{1-\gamma})^* = D(\Lambda^{1-\gamma})^* = D(\Lambda^{\gamma-1}). \end{aligned}$$

Thus, for any $\gamma \in [0, 1]$,

$$\nabla : D(\Lambda^\gamma) \to D(\Lambda^{\gamma-1}),$$

and hence

$$\Lambda^\mu \nabla : D(\Lambda^\gamma) \to D(\Lambda^{\gamma-1-\mu})$$

provided $\mu \leq \gamma - 1$. □

Remark 2.3. The above fractional Laplacian is the *spectral* one. In $\mathbb{R}^d$ the well-known integral representation

$$(-\Delta_{\mathbb{R}^d})^s f(x) = c_{d,s}\ \text{P.V.}\int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+2s}}\, dy, \quad s > 0,$$

holds; here P.V. stands for the principal value integral. For any domain $\Omega \subset \mathbb{R}^d$, the *restricted fractional Laplacian* $(-\Delta|_\Omega)^s$ is defined by

$$(-\Delta|_\Omega)^s f = ((-\Delta_{\mathbb{R}^d})^s \tilde{f})|_\Omega$$

for $f:\Omega\to\mathbb{R}$ and $\tilde{f}$ the zero-extension of f outside Ω. It was proved in [Bonforte et al. 2015] (see Section 3.1 there) that $(-\Delta|_\Omega)^s$ is an isomorphism from $D(\Lambda^s)$ onto its dual $D(\Lambda^s)^*$ with respect to the bilinear form

$$B(f,g)=\int_\Omega \mathcal{L}^{\frac12} f\mathcal{L}^{\frac12} g,\quad \mathcal{L}=(-\Delta|_\Omega)^s.$$

Hence for any scalar $\theta\in D(\Lambda^{\frac{\alpha}{2}})^*\supset L^2(\Omega)$ the stream function ψ can be defined alternatively by

$$\psi=((-\Delta|_\Omega)^{\frac{\alpha}{2}})^{-1}\theta. \tag{2-6}$$

Note that the resulting ψ is different from the one defined in (1-4). It would be interesting to see if the results in this paper still hold with this definition. We also refer to [Ros-Oton and Serra 2014] for the Hölder regularity of the ψ given by (2-6).

Commutator estimates. Due to the lack of translation invariance, the fractional Laplacian does not commute with differentiation. The following theorem provides a bound for the commutator.

Theorem 2.4 [Constantin and Nguyen 2016, Theorem 2.2]. *Let $p,\ q\in[1,\infty]$, $s\in(0,2)$ and a satisfy*

$$a(\cdot)d(\cdot)^{-s-1-\frac{d}{p}}\in L^q(\Omega).$$

Then the operator $a[\Lambda^s,\nabla]$ can be uniquely extended from $C_0^\infty(\Omega)$ to $L^p(\Omega)$ such that there exists a positive constant $C=C(d,s,p,\Omega)$ such that

$$\|a[\Lambda^s,\nabla]f\|_{L^q(\Omega)}\le C\|a(\cdot)d(\cdot)^{-s-1-\frac{d}{p}}\|_{L^q(\Omega)}\|f\|_{L^p(\Omega)} \tag{2-7}$$

holds for all $f\in L^p(\Omega)$.

The bound (2-7) is remarkable in that the commutator between an operator of order $s>0$ and an operator of order 1, which happens to vanish when $\Omega=\mathbb{R}^d$, is of order 0. The price is a singularity of the form $d(x)^{-s-1-\frac{d}{p}}$, which counts the order of Λ^s and ∇.

Remark 2.5. Let us explain how Theorem 2.4 follows from [Constantin and Nguyen 2016]. In that paper, using the heat kernel representation of the fractional Laplacian together with a cancellation of the heat kernel of $\mathbb{R}^d$, we proved the pointwise estimate for $f\in C_0^\infty(\Omega)$,

$$|[\Lambda^s,\nabla]f(x)|\le C(d,s,p,\Omega)d(x)^{-s-1-\frac{d}{p}}\|f\|_{L^p(\Omega)}.$$

The estimate (2-7) then follows by extension by continuity.

The next commutator estimate for negative powers of Laplacian is needed to handle the situation of more singular velocity.

Theorem 2.6. *Let $s\in(0,d)$ and $a\in W^{1,\infty}(\Omega)$. Let $p,r\in(1,\infty)$ satisfy*

$$\frac{1}{p}+\frac{d-s}{d}=1+\frac{1}{r}.$$

Then the operator $[\Lambda^{-s}, a]$ *can be uniquely extended from* $C_0^\infty(\Omega)$ *to* $L^p(\Omega)$ *with values in* $W_0^{1,r}(\Omega)$ *such that there exists* $C = C(s, d, p, r, \Omega) > 0$ *such that*

$$\|[\Lambda^{-s}, a] f\|_{W_0^{1,r}(\Omega)} \le C \|a\|_{W^{1,\infty}(\Omega)} \|f\|_{L^p(\Omega)}$$

for all $f \in L^p(\Omega)$.

In particular, for any $p \in (1, \infty)$, $s \in \left(0, \frac{d}{p}\right)$, *there exists* $C = C(s, d, p, \Omega) > 0$ *such that*

$$\|[\Lambda^{-s}, a] f\|_{W_0^{1,p}(\Omega)} \le C \|a\|_{W^{1,\infty}(\Omega)} \|f\|_{L^p(\Omega)} \tag{2-8}$$

for all $f \in L^p(\Omega)$.

With the same method of proof, we obtain:

Theorem 2.7. *Let* $s \in (0, 1)$ *and* $a \in C^\gamma(\Omega)$ *with* $\gamma \in (0, 1]$ *and* $s < \gamma$. *Let* $p, r \in (1, \infty)$ *satisfy*

$$\frac{1}{p} + \frac{d+s-\gamma}{d} = 1 + \frac{1}{r}.$$

Then the operator $[\Lambda^s, a]$ *can be uniquely extended from* $C_0^\infty(\Omega)$ *to* $L^p(\Omega)$ *with values in* $L^r(\Omega)$ *such that there exists* $C = C(s, \gamma, p, r, d, \Omega) > 0$ *such that*

$$\|[\Lambda^s, a] f\|_{L^r(\Omega)} \le C \|a\|_{C^\gamma(\Omega)} \|f\|_{L^p(\Omega)} \tag{2-9}$$

for all $f \in L^p(\Omega)$.

In particular, for any $p \in (1, \infty)$, *if*

$$s \in \left(\max\left\{\gamma - \frac{d}{p}, 0\right\}, \max\left\{\gamma - \frac{d}{p} + d, \gamma\right\}\right)$$

then there exists $C = C(s, \gamma, p, d, \Omega) > 0$ *such that*

$$\|[\Lambda^s, a] f\|_{L^p(\Omega)} \le C \|a\|_{C^\gamma(\Omega)} \|f\|_{L^p(\Omega)}. \tag{2-10}$$

Remark 2.8. In view of the identity

$$\Lambda^{-s}[\Lambda^s, a] f = [a, \Lambda^{-s}] \Lambda^s f,$$

it follows from (2-8) that

$$\|[\Lambda^s, a] f\|_{D(\Lambda^{1-s})} \le C \|a\|_{W^{1,\infty}(\Omega)} \|f\|_{D(\Lambda^s)}, \quad s \in \left(0, \tfrac{d}{2}\right). \tag{2-11}$$

This exhibits a gain of $1-s$ derivatives of $[\Lambda^s, a]$ when acting on $D(\Lambda^s)$. On the other hand, the estimate (2-10) shows a gain of s derivatives when acting on L^2. Both (2-8) and (2-10) make use of the fact that Ω is bounded.

The proofs of Theorems 2.6, 2.7 are given in Appendices A and B.

3. Proof of Theorem 1.1

Commutator representations. First, we adapt the well-known commutator representation of the nonlinearity in SQG [Resnick 1995], see also [Chae et al. 2012; Constantin et al. 2001; Constantin and Nguyen 2016], to take into account the lack of translation invariance of the fractional Laplacian and the more singular constitutive law (1-4):

Lemma 3.1. *Let $\psi \in H_0^1(\Omega)$, $u = \nabla^\perp \psi$, and $\theta = \Lambda^\alpha \psi$. Let $\phi \in C_0^\infty(\Omega)$ be a test function. Then*

$$\int_\Omega \theta u \cdot \nabla\phi \, dx = \frac{1}{2}\int_\Omega [\Lambda^\alpha, \nabla^\perp]\psi \cdot \nabla\phi\psi \, dx - \frac{1}{2}\int_\Omega \nabla^\perp\psi \cdot [\Lambda^\alpha, \nabla\phi]\psi \, dx \tag{3-1}$$

holds.

Proof. We have

$$\int_\Omega \theta u \cdot \nabla\phi \, dx = \int_\Omega \Lambda^\alpha \psi \nabla^\perp\psi \cdot \nabla\phi \, dx = -\int_\Omega \psi \nabla^\perp \Lambda^\alpha \psi \cdot \nabla\phi \, dx,$$

where we integrated by parts and used the fact that $\nabla^\perp \cdot \nabla\phi = 0$. The first and middle terms are well defined because $\theta u = \theta\nabla^\perp\psi \in L^1(\Omega)$, noticing that $\psi \in H_0^1(\Omega)$ and $\theta = \Lambda^\alpha\psi \in D(\Lambda^{1-\alpha}) \subset L^2(\Omega)$. The last term is defined because $\nabla\phi \cdot \nabla^\perp\Lambda^\alpha\psi \in H^{-1}(\Omega)$ and $\psi \in H_0^1(\Omega)$. Commuting $\nabla^\perp$ with Λ^α and then with $\nabla\phi$ leads to

$$\begin{aligned}
\int_\Omega \theta u \cdot \nabla\phi \, dx &= -\int_\Omega \psi[\nabla^\perp, \Lambda^\alpha]\psi \cdot \nabla\phi \, dx - \int_\Omega \psi\Lambda^\alpha\nabla^\perp\psi \cdot \nabla\phi \, dx \\
&= -\int_\Omega \psi[\nabla^\perp, \Lambda^\alpha]\psi \cdot \nabla\phi \, dx - \int_\Omega \nabla^\perp\psi \cdot \Lambda^\alpha(\psi\nabla\phi) \, dx \\
&= -\int_\Omega [\nabla^\perp, \Lambda^\alpha]\psi \cdot \nabla\phi\psi \, dx - \int_\Omega \nabla^\perp\psi \cdot [\Lambda^\alpha, \nabla\phi]\psi \, dx - \int_\Omega \nabla^\perp\psi \cdot \nabla\phi\Lambda^\alpha\psi \, dx \\
&= -\int_\Omega [\nabla^\perp, \Lambda^\alpha]\psi \cdot \nabla\phi\psi \, dx - \int_\Omega \nabla^\perp\psi \cdot [\Lambda^\alpha, \nabla\phi]\psi \, dx - \int_\Omega \theta u \cdot \nabla\phi \, dx.
\end{aligned}$$

The above calculations are justified by means of Theorems 2.4 and 2.7. Noticing that the last term on the right-hand side is exactly the negative of the left-hand side, we proved (3-1). □

Remark 3.2. The representation (3-1) was derived in [Constantin and Nguyen 2016] for the SQG equation ($\alpha = 1$). When $\Omega = \mathbb{R}^2$ or $\mathbb{T}^2$, (3-1) reduces to

$$\int_\Omega \theta u \cdot \nabla\phi \, dx = -\frac{1}{2}\int_\Omega \nabla^\perp\psi \cdot [\Lambda^\alpha, \nabla\phi]\psi \, dx.$$

Integrating by parts yields

$$-\frac{1}{2}\int_\Omega \nabla^\perp\psi \cdot [\Lambda^\alpha, \nabla\phi]\psi \, dx = \frac{1}{2}\int_\Omega \psi\nabla^\perp \cdot [\Lambda^\alpha, \nabla\phi]\psi \, dx = \frac{1}{2}\int_\Omega \psi[\Lambda^\alpha\nabla^\perp, \nabla\phi]\psi \, dx,$$

where we used in the second equality the fact that $\nabla^\perp \cdot \nabla\phi = 0$. This representation was invoked in [Chae et al. 2012] to prove the existence of global L^2 weak solutions of (1-2) in the periodic setting. More precisely, the authors proved the commutator estimate

$$\|[\Lambda^s\nabla, g]h\|_{L^2(\mathbb{T}^2)} \le C\|h\|_{L^2(\mathbb{T}^2)}\|g\|_{H^{s+2+\varepsilon}(\mathbb{T}^2)} + C\|\Lambda^s h\|_{L^2(\mathbb{T}^2)}\|g\|_{H^{2+\varepsilon}(\mathbb{T}^2)}$$

for any $s, \varepsilon > 0$. In arbitrary bounded domains, we were not able to establish such a commutator estimate.

We observe that by virtue of Theorem 2.4, the first integral on the right-hand side of (3-1) is well-defined provided only $\psi \in L^2(\Omega)$; moreover,

$$\left|\int_\Omega [\Lambda^\alpha, \nabla^\perp]\psi \cdot \nabla\phi\psi\,dx\right| \le C\|\nabla\phi d(\cdot)^{-\alpha-2}\|_{L^2(\Omega)}\|\psi\|^2_{L^2(\Omega)},$$

where by applying the Hardy inequality three times, together with the fact that $\alpha \in (0,1)$, we get

$$\|\nabla\phi d(\cdot)^{-\alpha-2}\|_{L^2(\Omega)} \le C\|\nabla\phi d(\cdot)^{-3}\|_{L^2} \le C\|\nabla^4\phi\|_{L^2(\Omega)} \le C\|\phi\|_{H^4(\Omega)}.$$

Consequently,

$$\left|\int_\Omega [\Lambda^\alpha, \nabla^\perp]\psi \cdot \nabla\phi\psi\,dx\right| \le C\|\phi\|_{H^4(\Omega)}\|\psi\|^2_{L^2(\Omega)}. \tag{3-2}$$

Regarding the second integral, we prove:

Lemma 3.3. *Assume $\psi \in D(\Lambda^\alpha)$. Then*

$$\mathcal{N}_2(\psi,\phi) := \int_\Omega \Lambda^{-1+\alpha}\nabla^\perp\psi \cdot \Lambda^{1-\alpha}[\Lambda^\alpha, \nabla\phi]\psi\,dx \tag{3-3}$$

satisfies

$$|\mathcal{N}_2(\psi,\phi)| \le C\|\nabla\phi\|_{W^{1,\infty}}\|\psi\|^2_{D(\Lambda^\alpha)}. \tag{3-4}$$

For any $\delta \in (0, \min(\alpha, 1-\alpha))$ we have

$$\mathcal{N}_2(\psi,\phi) = \int_\Omega \Lambda^{-1+\alpha-\delta}\nabla^\perp\psi\cdot\Lambda[\nabla\phi, \Lambda^{-\alpha+\delta}]\Lambda^\alpha\psi\,dx + \int_\Omega \Lambda^{-1+\alpha}\nabla^\perp\psi\cdot\Lambda[\nabla\phi, \Lambda^{-\delta}]\Lambda^\delta\psi\,dx. \tag{3-5}$$

Moreover,

$$|\mathcal{N}_2(\psi,\phi)| \le C\|\nabla\phi\|_{W^{1,\infty}}\|\psi\|_{D(\Lambda^{\alpha-\delta})}\|\psi\|_{D(\Lambda^\alpha)} + C\|\nabla\phi\|_{W^{1,\infty}}\|\psi\|_{D(\Lambda^\alpha)}\|\psi\|_{D(\Lambda^\delta)}. \tag{3-6}$$

Proof. (1) By Lemma 2.2,

$$\|\Lambda^{-1+\alpha}\nabla^\perp\psi\|_{L^2} \le \|\psi\|_{D(\Lambda^\alpha)}.$$

On the other hand, a direct calculation gives

$$\Lambda^{-\alpha}[\Lambda^\alpha, \nabla\phi]\psi = [\nabla\phi, \Lambda^{-\alpha}]\Lambda^\alpha\psi,$$

which, by virtue of Theorem 2.6, belongs to $D(\Lambda)$ and satisfies

$$\|\Lambda[\nabla\phi, \Lambda^{-\alpha}]\Lambda^\alpha\psi\|_{L^2} \le C\|\nabla\phi\|_{W^{1,\infty}}\|\Lambda^\alpha\psi\|_{L^2} = C\|\nabla\phi\|_{W^{1,\infty}}\|\psi\|_{D(\Lambda^\alpha)}.$$

Therefore, the integral defining $\mathcal{N}_2(\psi,\phi)$ in (3-3) makes sense and obeys the bound (3-4).

(2) Let $\delta \in [0, \min(\alpha, 1-\alpha))$. According to (3-3),

$$\begin{aligned}\mathcal{N}_2(\psi,\phi) &= \langle \Lambda^{-1+\alpha}\nabla^\perp\psi, \Lambda^{1-\alpha}[\Lambda^\alpha, \nabla\phi]\psi\rangle_{L^2,L^2}\\ &= \langle \Lambda^{-1+\alpha-\delta}\nabla^\perp\psi, \Lambda^{1-\alpha+\delta}[\Lambda^\alpha, \nabla\phi]\psi\rangle_{D(\Lambda^\delta),D(\Lambda^{-\delta})}.\end{aligned}$$

Now we write

$$\begin{aligned}\Lambda^{1-\alpha+\delta}[\Lambda^\alpha,\nabla\phi]\psi &= \Lambda\Lambda^{-\alpha+\delta}[\Lambda^\alpha,\nabla\phi]\psi\\ &= \Lambda\big\{\Lambda^\delta(\nabla\phi\psi) - \Lambda^{-\alpha+\delta}(\nabla\phi\Lambda^\alpha\psi)\big\}\\ &= \Lambda\big\{[\Lambda^\delta,\nabla\phi]\psi + \nabla\phi\Lambda^\delta\psi - \Lambda^{-\alpha+\delta}(\nabla\phi\Lambda^\alpha\psi)\big\}\\ &= \Lambda\big\{[\Lambda^\delta,\nabla\phi]\psi + \nabla\phi\Lambda^{-\alpha+\delta}\Lambda^\alpha\psi - \Lambda^{-\alpha+\delta}(\nabla\phi\Lambda^\alpha\psi)\big\}\\ &= \Lambda[\Lambda^\delta,\nabla\phi]\psi + \Lambda[\nabla\phi,\Lambda^{-\alpha+\delta}]\Lambda^\alpha\psi,\end{aligned}$$

where, according to (2-11),

$$[\Lambda^\delta,\nabla\phi]\psi \in D(\Lambda^{1-\delta}),$$

so

$$\Lambda[\Lambda^\delta,\nabla\phi]\psi \in D(\Lambda^{-\delta});$$

on the other hand, according to Theorem 2.6,

$$\Lambda[\nabla\phi,\Lambda^{-\alpha+\delta}]\Lambda^\alpha\psi \in L^2(\Omega) \subset D(\Lambda^{-\delta}).$$

Thus, we can write

$$\begin{aligned}I &= \langle \Lambda^{-1+\alpha-\delta}\nabla^\perp\psi, \Lambda[\nabla\phi,\Lambda^{-\alpha+\delta}]\Lambda^\alpha\psi\rangle_{D(\Lambda^\delta),D(\Lambda^{-\delta})} + \langle \Lambda^{-1+\alpha-\delta}\nabla^\perp\psi, \Lambda[\Lambda^\delta,\nabla\phi]\psi\rangle_{D(\Lambda^\delta),D(\Lambda^{-\delta})}\\ &= \int_\Omega \Lambda^{-1+\alpha-\delta}\nabla^\perp\psi\cdot\Lambda[\nabla\phi,\Lambda^{-\alpha+\delta}]\Lambda^\alpha\psi\,dx + \int_\Omega \Lambda^{-1+\alpha}\nabla^\perp\psi\cdot\Lambda^{1-\delta}[\Lambda^\delta,\nabla\phi]\psi\,dx\\ &= \int_\Omega \Lambda^{-1+\alpha-\delta}\nabla^\perp\psi\cdot\Lambda[\nabla\phi,\Lambda^{-\alpha+\delta}]\Lambda^\alpha\psi\,dx + \int_\Omega \Lambda^{-1+\alpha}\nabla^\perp\psi\cdot\Lambda[\nabla\phi,\Lambda^{-\delta}]\Lambda^\delta\psi\,dx.\end{aligned}$$

As in (1), an application of Theorems 2.4, 2.6, and (2-5), with $(\gamma = \alpha-\delta, \mu = -1-\alpha-\delta)$ and $(\gamma = \alpha, \mu = -1+\alpha)$, leads to the bound (3-6). □

Let us define

$$\begin{aligned}\mathcal{N}_1(\psi,\phi) &= \int_\Omega [\Lambda^\alpha,\nabla^\perp]\psi\cdot\nabla\phi\psi\,dx,\\ \mathcal{N}(\psi,\phi) &= \tfrac{1}{2}\mathcal{N}_1(\psi,\phi) - \tfrac{1}{2}\mathcal{N}_2(\psi,\phi).\end{aligned} \tag{3-7}$$

Putting together the above considerations, we have proved:

Lemma 3.4. *If* $\psi \in H_0^1(\Omega)$ *then*

$$\int_\Omega u\theta\cdot\nabla\phi = \mathcal{N}(\psi,\phi).$$

If $\theta \in L^2(\Omega)$ *then*

$$|\mathcal{N}(\psi,\phi)| \le C\|\phi\|_{H^4}\|\psi\|_{L^2}^2 + C\|\nabla\phi\|_{W^{1,\infty}}\|\psi\|_{D(\Lambda^\alpha)}^2$$

and for any $\delta \in (0, \min(\alpha, 1-\alpha))$,

$$|\mathcal{N}(\psi,\phi)| \le C\|\phi\|_{H^4}\|\psi\|_{L^2}^2 + C\|\nabla\phi\|_{W^{1,\infty}}\|\psi\|_{D(\Lambda^{\alpha-\delta})}\|\psi\|_{D(\Lambda^{\alpha})} + C\|\nabla\phi\|_{W^{1,\infty}}\|\psi\|_{D(\Lambda^{\alpha})}\|\psi\|_{D(\Lambda^{\delta})}.$$

Viscosity approximations. Let us fix $\theta_0 \in L^2(\Omega)$ and a positive time T. For each fixed $\varepsilon > 0$ we consider the viscosity approximation of (1-2)

$$\begin{cases} \partial_t \theta^\varepsilon + u^\varepsilon \cdot \nabla\theta^\varepsilon - \varepsilon\Delta\theta^\varepsilon = 0, & t > 0, \\ \theta^\varepsilon = \theta_0, & t = 0, \end{cases} \tag{3-8}$$

with $u^\varepsilon = \nabla^\perp \psi^\varepsilon,\ \psi^\varepsilon = \Lambda^{-\alpha}\theta^\varepsilon$.

Equation (3-8) can be solved using the Galerkin approximation method as follows. Denote by $\mathbb{P}_m$ the projection in $L^2(\Omega)$ onto the linear span $L^2_m(\Omega)$ of eigenfunctions $\{w_1, \dots, w_m\}$; i.e.,

$$\mathbb{P}_m f = \sum_{j=1}^{m} f_j w_j \quad \text{for } f = \sum_{j=1}^{\infty} f_j w_j.$$

We recall the following lemma which shows that $\mathbb{P}_m\phi$ are good approximations of ϕ in any Sobolev space for $\phi \in C_0^\infty(\Omega)$.

Lemma 3.5 [Constantin and Nguyen 2016, Lemma 3.1]. *Let* $\phi \in C_0^\infty(\Omega)$. *For all* $k \in \mathbb{N}$ *we have*

$$\lim_{m\to\infty} \|(\mathbb{I} - \mathbb{P}_m)\phi\|_{H^k(\Omega)} = 0. \tag{3-9}$$

The m-th Galerkin approximation of (3-8) is the following ODE system in the finite-dimensional space $\mathbb{P}_m L^2(\Omega) = L^2_m$:

$$\begin{cases} \dot\theta^\varepsilon_m + \mathbb{P}_m(u^\varepsilon_m \cdot \nabla\theta^\varepsilon_m) - \varepsilon\Delta\theta^\varepsilon_m = 0, & t > 0, \\ \theta^\varepsilon_m = P_m\theta_0, & t = 0, \end{cases} \tag{3-10}$$

with $\theta_m(x,t) = \sum_{j=1}^m \theta_j^{(m)}(t) w_j(x)$ and $u_m = \nabla^\perp\Lambda^{-\alpha}\theta_m$ automatically satisfying $\operatorname{div} u_m = 0$. Note that in general $u_m \notin L^2_m$. The existence of solutions of (3-10) at fixed m follows from the fact that this is an ODE:

$$\frac{d\theta_l^{(m)}}{dt} + \sum_{j,k=1}^{m} \gamma_{jkl}^{(m)} \theta_j^{(m)} \theta_k^{(m)} + \varepsilon\lambda_l\theta_l^{(m)} = 0,$$

with

$$\gamma_{jkl}^{(m)} = \lambda_j^{-\frac{\alpha}{2}} \int_\Omega (\nabla^\perp w_j \cdot \nabla w_k) w_l \, dx.$$

Since $\mathbb{P}_m$ is self-adjoint in L^2, u_m is divergence-free and w_j vanishes at the boundary $\partial\Omega$, integration by parts with θ_m gives

$$\int_\Omega \theta_m \mathbb{P}_m(u_m \cdot \nabla\theta_m)\, dx = \int_\Omega \theta_m u_m \cdot \nabla\theta_m \, dx = 0$$

and

$$-\int_\Omega \Delta\theta^\varepsilon_m \theta^\varepsilon_m \, dx = \int_\Omega |\nabla\theta^\varepsilon_m|^2 \, dx.$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\|\theta_m(\cdot,t)\|^2_{L^2(\Omega)}+\varepsilon\|\nabla\theta^\varepsilon_m\|^2_{L^2(\Omega)}=0,$$

and thus for $t\in[0,T]$,

$$\tfrac{1}{2}\|\theta^\varepsilon_m(\cdot,t)\|^2_{L^2(\Omega)}+\varepsilon\int_0^t\|\nabla\theta^\varepsilon_m(\cdot,s)\|^2_{L^2(\Omega)}\,ds=\tfrac{1}{2}\|\theta^\varepsilon_m(\cdot,0)\|^2_{L^2(\Omega)}\le\tfrac{1}{2}\|\theta_0\|^2_{L^2(\Omega)}. \tag{3-11}$$

This can be seen directly on the ODE because $\gamma^{(m)}_{jkl}$ is antisymmetric in k,l. Therefore, the smooth solution θ^ε_m of (3-10) exists globally and obeys the L^2 bound (3-11). The sequence $(\theta^\varepsilon_m)_m$ is thus uniformly in m bounded in $L^\infty([0,T];L^2(\Omega))\cap L^2([0,T];H^1_0(\Omega))$. Consequently, for any $p\in[1,\infty)$ and any $q\in[1,2/(1-\alpha)]$, we have

$$\theta^\varepsilon_m\in L^2([0,T];H^1_0(\Omega))\subset L^2([0,T];L^p(\Omega)),$$
$$u^\varepsilon_m=\nabla^\perp\Lambda^{-\alpha}\theta_m\in L^2([0,T];H^\alpha(\Omega))\subset L^2([0,T];L^q(\Omega)),$$

with bounds uniform with respect to m, where we have used Lemma 2.1 to have

$$\Lambda^{-\alpha}\theta_m\in L^2([0,T];D(\Lambda^{1+\alpha}))\subset L^2([0,T];H^{1+\alpha}(\Omega)).$$

In particular,

$$\begin{aligned}\|u^\varepsilon_m\cdot\nabla\theta^\varepsilon_m\|_{L^1([0,T];H^{-1}(\Omega))}&=\|\operatorname{div}(u^\varepsilon_m\cdot\theta^\varepsilon_m)\|_{L^1([0,T];H^{-1}(\Omega))}\\&\le C\|\theta^\varepsilon_m\|^2_{L^2([0,T];H^1(\Omega))}\le\frac{C}{\varepsilon}\|\theta_0\|^2_{H^1(\Omega)},\end{aligned} \tag{3-12}$$

where (3-11) was invoked in the last inequality. Therefore, using (3-10) we obtain that $(\partial_t\theta^\varepsilon_m)_m$ is uniformly in m bounded in $L^1([0,T];H^{-1}(\Omega))$. Then according to the Aubin–Lions lemma [Lions 1969], there exist a θ^ε,

$$\theta^\varepsilon\in L^\infty([0,T];L^2(\Omega))\cap L^2([0,T];H^1_0(\Omega)), \tag{3-13}$$

and a subsequence of $(\theta^\varepsilon_m)_m$ such that

$$\theta^\varepsilon_m\to\theta^\varepsilon\quad\text{strongly in } L^p([0,T];H^{-\mu}(\Omega))\cap L^2([0,T];H^{1-\mu}_0(\Omega)) \tag{3-14}$$

for any $p<\infty$ and $\mu\in(0,1)$.

Integrating by parts the first equation of (3-10) against any test function $\phi\in C^\infty_0(\Omega\times(0,T))$ gives

$$\int_0^T\int_\Omega\theta^\varepsilon_m\,\partial_t\phi\,dx\,dt+\int_0^T\int_\Omega\theta^\varepsilon_m u^\varepsilon_m\cdot\nabla\mathbb{P}_m\phi(x,t)\,dx\,dt+\varepsilon\int_0^T\int_\Omega\theta^\varepsilon_m\Delta\phi\,dx\,dt=0. \tag{3-15}$$

In the limit $m\to\infty$, the first term and the third term converge respectively to

$$\int_0^T\int_\Omega\theta^\varepsilon\,\partial_t\phi\,dx\,dt,\quad \varepsilon\int_0^T\int_\Omega\theta^\varepsilon\Delta\phi\,dx\,dt.$$

It remains to study the nonlinear term:

$$\begin{aligned}N &:= \int_0^T\int_\Omega \theta_m^\varepsilon u_m^\varepsilon\cdot\nabla\mathbb{P}_m\phi\,dx\,dt-\int_0^T\int_\Omega \theta^\varepsilon u^\varepsilon\cdot\nabla\phi\,dx\,dt\\ &= \int_0^T\int_\Omega \theta_m^\varepsilon u_m^\varepsilon\cdot\nabla(\mathbb{P}_m\phi-\phi)\,dx\,dt+\int_0^T\int_\Omega(\theta_m^\varepsilon-\theta^\varepsilon)u_m^\varepsilon\cdot\nabla\phi\,dx\,dt+\int_0^T\int_\Omega\theta^\varepsilon(u_m^\varepsilon-u^\varepsilon)\cdot\nabla\phi\,dx\,dt\\ &=: N_1+N_2+N_3.\end{aligned}$$

Lemma 3.5 ensures that $\lim_{m\to\infty} N_1 = 0$. On the other hand, the strong convergence (3-14) with sufficiently small μ implies $\lim_{m\to\infty} N_2 = \lim_{m\to\infty} N_3 = 0$. Thus, we have proved that θ^ε satisfies

$$\int_0^T\int_\Omega \theta^\varepsilon\,\partial_t\phi\,dx\,dt+\int_0^T\int_\Omega\theta^\varepsilon u^\varepsilon\cdot\nabla\phi\,dx\,dt+\varepsilon\int_0^T\int_\Omega\theta^\varepsilon\Delta\phi\,dx\,dt=0$$

for all $\phi\in C_0^\infty(\Omega\times(0,T))$. Here, θ^ε has the regularity (3-13), and in view of (3-11),

$$\|\theta^\varepsilon\|^2_{L^\infty([0,T];L^2(\Omega))}+2\varepsilon\|\theta^\varepsilon\|^2_{L^2([0,T];H_0^1(\Omega))}\le\|\theta_0\|^2_{L^2(\Omega)}. \tag{3-16}$$

Since $\psi^\varepsilon(\cdot,t)\in D(\Lambda^{1+\alpha})\subset H_0^1(\Omega)$ for a.e. $t>0$, using Lemma 3.4 for the representation of the nonlinearity, we obtain for all $\phi\in C_0^\infty(\Omega\times(0,T))$,

$$\int_0^T\int_\Omega\theta^\varepsilon\,\partial_t\phi\,dx\,dt+\int_0^T\mathcal{N}(\psi^\varepsilon,\phi)\,dt+\varepsilon\int_0^T\int_\Omega\theta^\varepsilon\Delta\phi\,dx\,dt=0. \tag{3-17}$$

Moreover, integrating by parts (3-10) with ψ_m^ε leads to

$$\frac{1}{2}\frac{d}{dt}\|\psi_m^\varepsilon(\cdot,t)\|^2_{D(\Lambda^{\alpha/2})}+\varepsilon\|\psi_m^\varepsilon(\cdot,t)\|^2_{D(\Lambda^{1+\alpha/2})}=0,$$

where we used the fact that the nonlinear term vanishes:

$$\int_\Omega\psi_m^\varepsilon\mathbb{P}_m(u_m^\varepsilon\cdot\nabla\theta_m^\varepsilon)\,dx=\int_\Omega\psi_m^\varepsilon\operatorname{div}(\nabla^\perp\psi_m^\varepsilon\theta_m)\,dx=-\int_\Omega\nabla\psi_m^\varepsilon\cdot\nabla^\perp\psi_m^\varepsilon\theta_m\,dx=0.$$

Consequently, integrating in time and letting $m\to\infty$ results in

$$\|\psi^\varepsilon(\cdot,t)\|^2_{D(\Lambda^{\alpha/2})}+2\varepsilon\int_0^t\|\psi^\varepsilon(\cdot,s)\|^2_{D(\Lambda^{1+\alpha/2})}\,ds=\|\psi^\varepsilon(\cdot,0)\|^2_{D(\Lambda^{\alpha/2})}\quad\text{for all } t>0. \tag{3-18}$$

Vanishing viscosity. In order to extract a convergent subsequence of θ^ε we need, in addition to (3-16), a uniform bound for $\partial_t\theta^\varepsilon$ in a lower norm. Let us note that the bound (3-12) is not uniform in ε. By (3-13), $\theta^\varepsilon(\cdot,t)\in D(\Lambda)$ for a.e. $t>0$, which implies $\psi^\varepsilon(\cdot,t)=\Lambda^{-\alpha}\theta^\varepsilon(\cdot,t)\in D(\Lambda^{1+\alpha})\subset D(\Lambda)$ for a.e. $t>0$. Lemma 3.4 then gives

$$\left|\int_\Omega\theta^\varepsilon u^\varepsilon\cdot\nabla\phi\,dx\right|\le C\|\phi\|_{H^4(\Omega)}\|\psi^\varepsilon\|^2_{D(\Lambda^\alpha)}\le C\|\phi\|_{H^4(\Omega)}\|\theta_0\|^2_{L^2(\Omega)},$$

and hence, in view of (3-17),

$$\left|\int_0^T\int_\Omega\theta^\varepsilon\,\partial_t\phi\,dx\,dt\right|\le C\|\phi\|_{L^1([0,T];H^4(\Omega))}\big(\|\theta_0\|_{L^2(\Omega)}+\|\theta_0\|^2_{L^2(\Omega)}\big)$$

for all $\phi \in C_0^\infty(\Omega \times (0,T))$. Consequently,

$$\|\partial_t \theta^\varepsilon\|_{L^\infty([0,T];H^{-4}(\Omega))} \le C\big(\|\theta_0\|_{L^2(\Omega)} + \|\theta_0\|^2_{L^2(\Omega)}\big). \tag{3-19}$$

In view of the uniform bounds (3-16) and (3-19), the Aubin–Lions lemma ensures the existence of a θ,

$$\theta \in L^\infty([0,T]; L^2(\Omega)) \cap C([0,T]; H^{-\nu}(\Omega)) \quad \text{for all } \nu > 0,$$

and a subsequence θ^ε such that

$$\theta^\varepsilon \rightharpoonup \theta \quad \text{weakly in } L^2([0,T]; L^2(\Omega)), \tag{3-20}$$

$$\theta^\varepsilon \to \theta \quad \text{strongly in } C([0,T]; H^{-\nu}(\Omega)) \quad \text{for all } \nu > 0. \tag{3-21}$$

Consequently, with $\psi := \Lambda^{-\alpha}\theta$,

$$\psi \in L^\infty([0,T]; D(\Lambda^\alpha)) \cap C([0,T]; D(\Lambda^{\alpha-\nu}) \quad \text{for all } \nu > 0,$$

we have

$$\psi^\varepsilon \rightharpoonup \psi \quad \text{weakly in } L^2([0,T]; D(\Lambda^\alpha)), \tag{3-22}$$

$$\psi^\varepsilon \to \psi \quad \text{strongly in } C([0,T]; D(\Lambda^{\alpha-\nu})) \quad \text{for all } \nu > 0. \tag{3-23}$$

Let $\phi \in C_0^\infty(\Omega \times (0,T))$ a be fixed test function, we send ε to 0 in the weak formulation (3-17). The first term converges to $\int_0^T\int_\Omega \theta\, \partial_t\phi\, dx\, dt$ and the last term converges to 0. Regarding the nonlinear term, we shall prove that

$$R^\varepsilon := \int_0^T \mathcal{N}(\psi^\varepsilon, \phi) - \mathcal{N}(\psi, \phi)\, dt$$

converges to 0. In view of (3-1), (3-5), we have $2R^\varepsilon = \sum_{j=1}^6 I_j^\varepsilon$ with

$$\begin{aligned}
I_1^\varepsilon &= \int_\Omega [\Lambda^\alpha, \nabla^\perp](\psi_\varepsilon - \psi)\cdot\nabla\phi\,\psi_\varepsilon\, dx,\\
I_2^\varepsilon &= \int_\Omega [\Lambda^\alpha, \nabla^\perp]\psi\cdot\nabla\phi(\psi_\varepsilon - \psi)\, dx,\\
I_3^\varepsilon &= -\int_0^T\int_\Omega \Lambda^{-1+\alpha-\delta}\nabla^\perp(\psi^\varepsilon - \psi)\cdot\Lambda[\nabla\phi, \Lambda^{-\alpha+\delta}]\Lambda^\alpha\psi^\varepsilon\, dx\, dt,\\
I_4^\varepsilon &= -\int_0^T\int_\Omega \Lambda^{-1+\alpha-\delta}\nabla^\perp\psi\cdot\Lambda[\nabla\phi, \Lambda^{-\alpha+\delta}]\Lambda^\alpha(\psi_\varepsilon - \psi)\, dx\, dt,\\
I_5^\varepsilon &= -\int_\Omega \Lambda^{-1+\alpha}\nabla^\perp(\psi_\varepsilon - \psi)\cdot\Lambda[\nabla\phi, \Lambda^{-\delta}]\Lambda^\delta\psi\, dx,\\
I_6^\varepsilon &= -\int_\Omega \Lambda^{-1+\alpha}\nabla^\perp\psi_\varepsilon\cdot\Lambda[\nabla\phi, \Lambda^{-\delta}]\Lambda^\delta(\psi_\varepsilon - \psi)\, dx,
\end{aligned}$$

where $\delta \in (0, \min(\alpha, 1-\alpha))$.

By virtue of Theorem 2.4 and the fact that $\phi \in C_0^\infty(\Omega)$,

$$|I_1^\varepsilon| \le C(\phi)\|\psi_\varepsilon - \psi\|_{L^2(\Omega)}\|\psi_\varepsilon\|_{L^2(\Omega)}, \quad |I_2^\varepsilon| \le C(\phi)\|\psi_\varepsilon - \psi\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)}.$$

Hence $\lim_{\varepsilon\to 0} I_1^\varepsilon = \lim_{\varepsilon\to 0} I_2^\varepsilon = 0$ in view of the convergence (3-23) with $\nu < \alpha$.

As for (3-6),

$$|I_3^\varepsilon| \le C\|\nabla\phi\|_{L^1([0,T];W^{1,\infty})}\|\psi^\varepsilon-\psi\|_{L^\infty([0,T];D(\Lambda^{\alpha-\delta}))}\|\psi^\varepsilon\|_{L^\infty([0,T];D(\Lambda^\alpha))},$$

which combined with (3-23) leads to $\lim_{\varepsilon\to 0} I_3^\varepsilon = 0$. Because $\Lambda[\nabla\phi, \Lambda^{-\alpha+\delta}]\Lambda^\alpha$ is norm continuous from $L^2([0,T]; D(\Lambda^\alpha))$ to $L^2([0,T]; L^2(\Omega))$ (according to Theorem 2.6), it is weak-weak continuous, and thus $\lim_{\varepsilon\to 0} I_4^\varepsilon = 0$ noticing that by (2-5),

$$\Lambda^{-1+\alpha-\delta}\nabla^\perp\psi \in L^\infty([0,T]; D(\Lambda^\delta)) \subset L^2([0,T]; L^2(\Omega)).$$

Similarly, $\lim_{\varepsilon\to 0} I_5^\varepsilon = 0$ since $\Lambda^{-1+\alpha}\nabla^\perp(\psi_\varepsilon-\psi) \rightharpoonup 0$ in $L^2([0,T]; D(\Lambda^\alpha))$ by (3-22), and since $\Lambda[\nabla\phi, \Lambda^{-\delta}]\Lambda^\delta\psi \in L^2([0,T]; L^2(\Omega))$ by Theorem 2.6. Finally, by (2-5) and Theorem 2.6,

$$|I_6^\varepsilon| \le \|\Lambda^{-1+\alpha}\nabla^\perp\psi_\varepsilon\|_{L^2(\Omega)}\|\Lambda[\nabla\phi, \Lambda^{-\delta}]\Lambda^\delta(\psi_\varepsilon-\psi)\|_{L^2(\Omega)} \le \|\psi_\varepsilon\|_{D(\Lambda^\alpha)}\|\psi_\varepsilon-\psi\|_{D(\Lambda^\delta)} \to 0,$$

noticing that $\delta < \alpha$. We conclude that

$$\int_0^T\int_\Omega \theta\,\partial_t\phi\,dx\,dt + \int_0^T \mathcal{N}(\psi,\phi)\,dt = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega\times(0,T)).$$

Moreover, because of the strong convergence (3-21) the initial data is attained:

$$\theta(\,\cdot\,,0) = \lim_{\varepsilon\to 0}\theta^\varepsilon(\,\cdot\,,0) = \lim_{\varepsilon\to 0}\theta_0(\,\cdot\,) = \theta_0(\,\cdot\,) \quad \text{in } H^{-\nu}(\Omega) \text{ for all } \nu > 0.$$

Let us now show the conservation (1-9). In view of (3-16) and the fact that $\theta^\varepsilon = \Lambda^\alpha\psi^\varepsilon$ we have

$$\|\Lambda^\alpha\psi^\varepsilon\|^2_{L^\infty([0,T];L^2(\Omega))} + 2\varepsilon\|\Lambda^{1+\alpha}\psi^\varepsilon\|^2_{L^2([0,T];L^2(\Omega))} \le \|\theta_0\|^2_{L^2(\Omega)}.$$

By interpolation,

$$\|\Lambda^{1+\frac{\alpha}{2}}\psi^\varepsilon\|_{L^2(\Omega)} \le C\|\Lambda^{1+\alpha}\psi^\varepsilon\|^a_{L^2(\Omega)}\|\Lambda^\alpha\psi^\varepsilon\|^{1-a}_{L^2(\Omega)}, \quad a = 1-\tfrac{\alpha}{2}.$$

Hölder's inequality then yields

$$\begin{aligned}\|\Lambda^{1+\frac{\alpha}{2}}\psi^\varepsilon\|^2_{L^2([0,T];L^2(\Omega))} &\le C\|\Lambda^\alpha\psi^\varepsilon\|^{2(1-a)}_{L^\infty([0,T];L^2(\Omega))}\|\Lambda^{1+\alpha}\psi^\varepsilon\|^{2a}_{L^2([0,T];L^2(\Omega))}T^{\frac{\alpha}{2}}\\ &\le CT^{\frac{\alpha}{2}}\|\theta_0\|^2_{L^2(\Omega)}\varepsilon^{-1+\frac{\alpha}{2}} \quad \text{for all } T > 0.\end{aligned}$$

In particular,

$$\lim_{\varepsilon\to 0}\varepsilon\|\Lambda^{1+\frac{\alpha}{2}}\psi^\varepsilon\|^2_{L^2([0,T];L^2(\Omega))} = 0 \quad \text{for all } T > 0.$$

Letting $\varepsilon\to 0$ in (3-18) we obtain (1-9).

Finally, the energy inequality (1-8) follows from (3-16) and lower semicontinuity.

Remark 3.6. If we implement directly the Galerkin approximations for (1-2) then in view of (3-1), we need to bound

$$\left|\int_\Omega [\Lambda^\alpha, \nabla^\perp]\psi_m\cdot\nabla\mathbb{P}_m\phi\,\psi_m\,dx\right|.$$

However, the commutator $[\Lambda^\alpha, \nabla^\perp]$ then cannot be bounded by means of Theorem 2.4 because $\nabla\mathbb{P}_m\phi$ does not vanish on the boundary even though ϕ has compact support. In [Constantin and Nguyen 2016], we overcame this by first using Lemma 3.5 and the fact that $u_m\theta_m$ is uniformly bounded in $L^1(\Omega)$ to approximate $\int_\Omega u_m\theta_m\nabla\mathbb{P}_m\phi$ by $\int_\Omega u_m\theta_m\nabla\phi$. When $\alpha<1$, this argument breaks down since $u_m\theta_m$ is not anymore uniformly bounded in $L^1(\Omega)$. This explains why we proceeded in the proof of Theorem 1.1 using vanishing viscosity approximations.

Appendix A: Proof of Theorem 2.6

In view of the identity

$$D^{-r}=c_r\int_0^\infty t^{-1+r}e^{-tD}\,dt$$

with $D,r>0$ we have the representation of negative powers of Laplacian via heat kernel:

$$\Lambda^{-s}f(x)=c_s\int_0^\infty t^{-1+\frac{s}{2}}e^{t\Delta}f(x)\,dt,\quad s>0. \tag{A-1}$$

Let $H(x,y,t)$ denote the heat kernel of Ω; i.e.,

$$e^{t\Delta}f(x)=\int_\Omega H(x,y,t)f(y)\,dy\quad\text{for all } x\in\Omega.$$

We have from [Li and Yau 1986] the following bounds on H and its gradient:

$$H(x,y,t)\le Ct^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{Kt}}, \tag{A-2}$$

$$|\nabla_x H(x,y,t)|\le Ct^{-\frac{1}{2}-\frac{d}{2}}e^{-\frac{|x-y|^2}{Kt}} \tag{A-3}$$

for all $(x,y)\in\Omega\times\Omega$ and $t>0$.

We will also use the elementary estimate

$$\int_0^\infty t^{-1-\frac{m}{2}}e^{-\frac{p^2}{Kt}}\,dt\le C_{K,m}p^{-m},\quad m,p,K>0. \tag{A-4}$$

Let $f\in C_0^\infty(\Omega)$. Using (A-1) we have

$$\begin{aligned}[\Lambda^{-s},a]f(x)&=c_s\int_0^\infty t^{-1+\frac{s}{2}}\int_\Omega H(x,y,t)a(y)f(y)\,dt-c_s\,a(x)\int_0^\infty t^{-1+\frac{s}{2}}\int_\Omega H(x,y,t)f(y)\,dt\\&=c_s\int_0^\infty t^{-1+\frac{s}{2}}\int_\Omega H(x,y,t)[a(y)-a(x)]f(y)\,dt.\end{aligned} \tag{A-5}$$

In view of (A-2), (A-4), and the assumption that $s<d$, we deduce that

$$\begin{aligned}|[\Lambda^{-s},a]f(x)|&\le C\|a\|_{L^\infty}\int_\Omega\int_0^\infty t^{-1+\frac{s}{2}-\frac{d}{2}}e^{-\frac{|x-y|^2}{Kt}}\,dt\,|f(y)|\,dy\\&\le C\|a\|_{L^\infty}\int_\Omega\frac{|f(y)|}{|x-y|^{d-s}}\,dy.\end{aligned} \tag{A-6}$$

Let us recall the Hardy–Littlewood–Sobolev inequality. Let $\alpha \in (0,d)$ and $(p,r) \in (1,\infty)$ satisfy

$$\frac{1}{p} + \frac{\alpha}{d} = 1 + \frac{1}{r}. \tag{A-7}$$

A constant C then exists such that

$$\|f * |\cdot|^{-\alpha}\|_{L^r(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}. \tag{A-8}$$

Applying (A-8) with $\alpha = d - s$ leads to

$$\|[\Lambda^{-s}, a]f\|_{L^r(\Omega)} \le C\|a\|_{L^\infty}\|f\|_{L^p(\Omega)}. \tag{A-9}$$

Let γ_0 denote the trace operator for Ω. It is readily seen that $\gamma_0(\Lambda^{-s}f) = 0$ because $\Lambda^{-s}f \in D(\Lambda^m)$ for all $m \ge 0$; hence $\gamma_0(a\Lambda^{-s}f) = \gamma_0(a)\gamma_0(\Lambda^{-s}f) = 0$. In addition, $af \in H_0^1(\Omega) = D(\Lambda)$; hence $\Lambda^{-s}(af) \in D(\Lambda^{1+s}) \subset H_0^1(\Omega)$ and $\gamma_0(\Lambda^{-s}(af)) = 0$. We deduce that

$$\gamma_0([\Lambda^{-s}, a]f) = 0. \tag{A-10}$$

Next, for gradient bound we differentiate (A-5) and obtain

$$\begin{aligned}\nabla[\Lambda^{-s}, a]f(x) = c_s\int_0^\infty t^{-1+\frac{s}{2}}\int_\Omega \nabla_x H(x,y,t)[a(y)-a(x)]f(y)\,dt \\ - c_s\int_0^\infty t^{-1+\frac{s}{2}}\int_\Omega H(x,y,t)\nabla a(x) f(y)\,dt \\ =: I + II.\end{aligned}$$

The term II can be treated as above and we have

$$\|II\|_{L^r(\Omega)} \le C\|\nabla a\|_{L^\infty}\|f\|_{L^p(\Omega)}. \tag{A-11}$$

For I, we use the gradient estimate (A-3) for the heat kernel and the fact that

$$|a(x) - a(y)| \le \|\nabla a\|_{L^\infty}|x-y|$$

to arrive at

$$\begin{aligned}|I(x)| &\le C\|\nabla a\|_{L^\infty}\int_\Omega\int_0^\infty t^{-1+\frac{s}{2}-\frac{1}{2}-\frac{d}{2}} e^{-\frac{|x-y|^2}{Kt}}\,dt|x-y||f(y)|\,dy \\ &\le C\|\nabla a\|_{L^\infty}\int_\Omega \frac{|f(y)|}{|x-y|^{d-s}}\,dy.\end{aligned}$$

Appealing to (A-8) as before gives

$$\|I\|_{L^r(\Omega)} \le C\|\nabla a\|_{L^\infty}\|f\|_{L^p(\Omega)},$$

which, combined with (A-9), (A-11), (A-10), leads to

$$\|[\Lambda^{-s}, a]f\|_{W_0^{1,r}(\Omega)} \le C\|a\|_{W^{1,\infty}(\Omega)}\|f\|_{L^p(\Omega)}, \tag{A-12}$$

where p, r satisfy (A-8) with $\alpha = d - s$. Using the density of $C_0^\infty(\Omega)$ in $L^p(\Omega)$ for $p \in (1,\infty)$, and extension by continuity, we conclude that the estimate (A-12) holds for any $f \in L^p(\Omega)$.

Now, for any $p \in (0, \infty)$, if $s < \frac{d}{p}$ then $r \in (1, \infty)$ given by

$$\frac{1}{r} = \frac{1}{p} - \frac{s}{d}$$

satisfies (A-8). Because $r > p$ and Ω is bounded, the continuous embedding $W_0^{1,r}(\Omega) \subset W_0^{1,p}(\Omega)$ yields

$$\|[\Lambda^{-s}, a]f\|_{W_0^{1,p}(\Omega)} \le C\|a\|_{W^{1,\infty}(\Omega)}\|f\|_{L^p(\Omega)}. \tag{A-13}$$

Appendix B: Proof of Theorem 2.7

In view of the identity

$$\lambda^{\frac{s}{2}} = c_s \int_0^\infty t^{-1-\frac{s}{2}}(1 - e^{-t\lambda})\, dt$$

with $0 < s < 2$ and

$$1 = c_s \int_0^\infty t^{-1-\frac{s}{2}}(1 - e^{-t})\, dt,$$

we have the representation of the fractional Laplacian via heat kernel

$$\Lambda^s f(x) = c_s \int_0^\infty t^{-1-\frac{s}{2}}(1 - e^{t\Delta}) f(x)\, dt, \quad 0 < s < 2. \tag{B-1}$$

Appealing to this representation, we have for $f \in C_0^\infty(\Omega)$

$$[\Lambda^s, a]f(x) = c_s \int_0^\infty t^{-1-\frac{s}{2}} \int_\Omega H(x, y, t)\, dt\, [a(x) - a(y)] f(y)\, dy.$$

In view of (A-2), the fact that

$$|a(x) - a(y)| \le \|a\|_{C^\gamma}|x - y|^\gamma,$$

and (A-4), we deduce that

$$\begin{aligned} |[\Lambda^s, a]f(x)| &\le c_s\|a\|_{C^\gamma} \int_\Omega \int_0^\infty t^{-1-\frac{s}{2}-\frac{d}{2}} e^{-\frac{|x-y|^2}{Kt}}\, dt\, |x - y|^\gamma |f(y)|\, dy \\ &\le c_s\|a\|_{C^\gamma} \int_\Omega \frac{|f(y)|}{|x - y|^{d+s-\gamma}}\, dy. \end{aligned}$$

Then as in the proof of Theorem 2.6, if $s < \gamma$ (note that $d + s - \gamma > 0$), an application of the Hardy–Littlewood–Sobolev inequality leads to the bound (2-9). Finally, (2-10) follows from (2-9) and the fact that Ω is bounded.

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SCALE-FREE UNIQUE CONTINUATION PRINCIPLE FOR SPECTRAL PROJECTORS, EIGENVALUE-LIFTING AND WEGNER ESTIMATES FOR RANDOM SCHRÖDINGER OPERATORS

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We prove a scale-free, quantitative unique continuation principle for functions in the range of the spectral projector $\chi_{(-\infty,E]}(H_L)$ of a Schrödinger operator H_L on a cube of side $L \in \mathbb{N}$, with bounded potential. Previously, such estimates were known only for individual eigenfunctions and for spectral projectors $\chi_{(E-\gamma,E]}(H_L)$ with small γ. Such estimates are also called, depending on the context, uncertainty principles, observability estimates, or spectral inequalities. Our main application of such an estimate is to find lower bounds for the lifting of eigenvalues under semidefinite positive perturbations, which in turn can be applied to derive a Wegner estimate for random Schrödinger operators with nonlinear parameter-dependence. Another application is an estimate of the control cost for the heat equation in a multiscale domain in terms of geometric model parameters. Let us emphasize that previous uncertainty principles for individual eigenfunctions or spectral projectors onto small intervals were not sufficient to study such applications.

1. Introduction

We prove a *quantitative unique continuation* inequality, announced in [Nakić et al. 2015b], for functions in the range of the spectral projector $\chi_{(-\infty,E]}(H_L)$ of a Schrödinger operator H_L on a cube of side $L \in \mathbb{N}$. Depending on the area of mathematics and the context, estimates of this type have various names: quantitative unique continuation principles (UCP), uncertainty principles, spectral inequalities, observability or sampling estimates, or bounds on the vanishing order. For our applications it is crucial (i) to exhibit explicitly the dependence of the quantitative unique continuation inequality on the model parameters, and (ii) to allow energy intervals $(-\infty, E]$ of arbitrary length, that is, for arbitrary E. If the observability or sampling set respects in a certain way the underlying lattice structure, our estimate is independent of L; for this reason we call it *scale-free*. This property is crucial for applications where one studies spectral properties of the Schrödinger operator H_L in the thermodynamic limit $L \nearrow \infty$.

A key motivation to study scale-free quantitative unique continuation estimates comes from the theory of random Schrödinger operators, in particular with *nonlinear dependence* on the random variables. The class of operators considered here includes the random breather model as studied in [Combes et al. 1996; 2001; Täufer and Veselić 2015; 2016]. Models with nonlinear randomness constitute a step towards a better understanding of the universality of Anderson localization.

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We establish eigenvalue-lifting estimates, Wegner bounds, and the continuity of the integrated density of states. (We defer precise definitions to Section 2.) In fact, there are a number of previous papers which have derived a scale-free UCP and eigenvalue-lifting estimates under special assumptions.

Naturally, the first situation to be considered was the case where the Schrödinger operator is the pure Laplacian $H = -\Delta$, i.e., the background potential V vanishes identically. For instance, [Kirsch 1996] derives a UCP which is valid for energies in an interval at zero, i.e., the bottom of the spectrum, if one has a periodic arrangement of sampling sets. The proof uses detailed information about hitting probabilities of Brownian motion paths, and is related to Harnack inequalities. A very elementary approach to eigenvalue-lifting estimates is provided by the spatial averaging trick, used in [Bourgain and Kenig 2005; Germinet et al. 2007] in periodic situations, and extended to nonperiodic situations in [Germinet 2008]. It is applicable to energies near zero. A different approach for eigenvalue-lifting was derived in [Boutet de Monvel et al. 2006]. In [Boutet de Monvel et al. 2011] it was shown how one can conclude an uncertainty principle at low energies based on an eigenvalue-lifting estimate. Related results have been derived for energies near spectral edges in [Kirsch et al. 1998; Combes et al. 2001] using resolvent comparison. In one space dimension, eigenvalue-lifting results and Wegner estimates have been proven in [Veselić 1996; Kirsch and Veselić 2002]. There a periodic arrangement of the sampling set is assumed. The proof carries over to the case of nonperiodic arrangements verbatim, which has been spelled out and used in the context of quantum graphs in [Helm and Veselić 2007]. In the case that both the deterministic background potential and the sampling set are periodic, an uncertainty principle and a Wegner estimate, which are valid for arbitrary bounded energy regions, have been proven in [Combes et al. 2003; 2007]. These papers make use of Floquet theory; hence they are a priori restricted to periodic background potentials as well as periodic sampling sets. An alternative proof for the result in [Combes et al. 2007], with more explicit control of constants, has been worked out in [Germinet and Klein 2013]. The case where the background potential is periodic but the impurities need not be periodically arranged has been considered in [Boutet de Monvel et al. 2006; Germinet 2008] for low energies. Our main theorem *unifies and generalizes all the results* mentioned so far and makes the dependence on the model parameters quantitative. Indeed, our scale-free unique continuation principle answers positively a question asked in [Rojas-Molina and Veselić 2013]. A partial answer was given already in [Klein 2013]. While [Rojas-Molina and Veselić 2013] concerns the case of a single eigenfunctions, [Klein 2013] uses a very nice perturbation argument to treat linear combinations of eigenfunctions corresponding to eigenvalues which lie in an interval whose size is smaller than an explicitly determined number. For a broader discussion we refer to the summer school notes [Täufer et al. 2016].

A second application of our scale-free UCP is in the control theory of the heat equation. Here one asks whether one can drive a given initial state to a desired state with a control function living in a specified subset, and what the minimal L^2-norm of the control function (called control cost) is. Recently, the search for optimal placement of the control set and the dependence of the control cost on geometric features of this set has received much attention; see, e.g., [Privat et al. 2015b; 2015a]. Our scale-free UCP gives an explicit estimate of the control cost with respect to the model parameters in multiscale domains. While this is of interest in itself, *our main motivation* to include the application to control theory in our paper is to bring to attention the relation between methods and ideas from this field and the theory of random Schrödinger operators. This relation has not been explored before and it seems that it can be effectively used in other problems of random operators.

Other authors have applied our main result, as announced in [Nakić et al. 2015b], to prove decorrelation estimates for eigenvalues of random Schrödinger operators [Shirley 2015] and lower bounds on averaged spectral shift functions [Dietlein et al. 2017]. We will generalize the methods of the present paper to certain unbounded domains in $\mathbb{R}^d$ in a forthcoming paper, while two of us have extended the results to certain infinite-dimensional spectral subspaces in [Täufer and Tautenhahn 2017].

Our proof of the scale-free unique continuation estimate uses two Carleman and nested interpolation bounds to obtain propagation of smallness estimates, an idea used before, e.g., in [Lebeau and Robbiano 1995; Jerison and Lebeau 1999]. Roughly speaking, one of the Carleman estimates establishes propagation of smallness from a set of codimension one to a small ball, and the other one from a small ball to a larger ball. To obtain explicit estimates we need explicit weight functions. The first Carleman estimate includes a boundary term and uses a parabolic weight function as proposed in [Jerison and Lebeau 1999]. The second Carleman estimate is similar to the ones in [Escauriaza and Vessella 2003; Bourgain and Kenig 2005]. However, neither of the two is quite sufficient for our purposes, so we use a variant developed in [Nakić et al. 2015a]. A similar result was established recently in [Davey 2014]. Moreover, at first sight it seems that one can get our result simply by summing up doubling estimates (which are a standard consequence of Carleman estimates). However, the prefactor in the doubling estimate depends on the ambient space, in particular its diameter. In our case we consider a family of domains Λ_L, $L \in \mathbb{N}$, and the diameter grows unboundedly in L; hence the constant in the doubling estimate becomes worse and worse. Thus, to eliminate the L-dependence we have to use techniques developed in the context of random Schrödinger operators to accommodate for the multiscale structure of the underlying domain and sampling set.

In the next section we state our main results. Section 3 is devoted to the proof of the scale-free unique continuation principle, Section 4 to proofs concerning random Schrödinger operators, and Section 5 to the observability estimate of the control equation, while certain technical aspects are deferred to the Appendix.

2. Results

Scale-free unique continuation and eigenvalue lifting. Let $d \in \mathbb{N}$. For $L > 0$ we denote by $\Lambda_L = (-L/2, L/2)^d \subset \mathbb{R}^d$ the cube with side length L, and by Δ_L the Laplace operator on $L^2(\Lambda_L)$ with Dirichlet, Neumann or periodic boundary conditions. Moreover, for a measurable and bounded $V\colon \mathbb{R}^d \to \mathbb{R}$ we denote by $V_L : \Lambda_L \to \mathbb{R}$ its restriction to Λ_L given by $V_L(x) = V(x)$ for $x \in \Lambda_L$, and by

$$H_L = -\Delta_L + V_L \quad \text{on } L^2(\Lambda_L)$$

the corresponding Schrödinger operator. Note that H_L has purely discrete spectrum. For $x \in \mathbb{R}^d$ and $r > 0$ we denote by $B(x, r)$ the ball with center x and radius r with respect to Euclidean norm. If the ball is centered at zero we write $B(r) = B(0, r)$.

Definition 2.1. Let $G > 0$ and $\delta > 0$. We say that a sequence $z_j \in \mathbb{R}^d$, $j \in (G\mathbb{Z})^d$, is (G, δ)-*equidistributed*, if

$$\text{for all } j \in (G\mathbb{Z})^d \text{ we have } B(z_j, \delta) \subset \Lambda_G + j.$$

Corresponding to a (G, δ)-equidistributed sequence we define for $L \in G\mathbb{N}$ the set

$$W_\delta(L) = \bigcup_{j \in (G\mathbb{Z})^d} B(z_j, \delta) \cap \Lambda_L.$$

Theorem 2.2. *There is $N = N(d)$ such that for all $\delta \in \left(0, \frac{1}{2}\right)$, all $(1, \delta)$-equidistributed sequences, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $L \in \mathbb{N}$, all $E \geq 0$ and all $\phi \in \operatorname{Ran}(\chi_{(-\infty,E]}(H_L))$ we have*

$$\|\phi\|^2_{L^2(W_\delta(L))} \geq C_{\mathrm{sfuc}} \|\phi\|^2_{L^2(\Lambda_L)}, \tag{1}$$

where

$$C_{\mathrm{sfuc}} = C_{\mathrm{sfuc}}(d, \delta, E, \|V\|_\infty) := \delta^{N(1+\|V\|_\infty^{2/3}+\sqrt{E})}.$$

The result can be formulated in terms of spectral projectors. This is the convenient form to use in the context of random Schrödinger operators.

Corollary 2.3. *Under the same assumptions as in the above theorem, we have in the sense of quadratic forms*

$$\chi_{(-\infty,E]}(H_L)\ \chi_{W_\delta(L)}\ \chi_{(-\infty,E]}(H_L) \geq \delta^{N(1+\|V\|_\infty^{2/3}+\sqrt{E})} \chi_{(-\infty,E]}(H_L). \tag{2}$$

Here $\chi_{W_\delta(L)}$ denotes the multiplication operator with a characteristic function, and $\chi_{(-\infty,E]}(H_L)$ denotes a spectral projector.

The crucial point here is that we allow energy intervals $(-\infty, E]$ of arbitrary length. It is not possible to achieve this result with the methods of [Rojas-Molina and Veselić 2013; Klein 2013]. For $t, L > 0$ and a measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$ we define the Schrödinger operator $H_{t,L} = -t\Delta_L + V_L$ on $L^2(\Lambda_L)$. By scaling we obtain the following corollary.

Corollary 2.4. *Let $N = N(d)$ be the constant from Theorem 2.2. Then, for all $G, t > 0$, all $\delta \in (0, G/2)$, all (G, δ)-equidistributed sequences, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $L \in G\mathbb{N}$, all $E \geq 0$ and all $\phi \in \operatorname{Ran}(\chi_{(-\infty,E]}(H_{t,L}))$ we have*

$$\|\phi\|^2_{L^2(W_\delta(L))} \geq C^{G,t}_{\mathrm{sfuc}} \|\phi\|^2_{L^2(\Lambda_L)},$$

where

$$C^{G,t}_{\mathrm{sfuc}} = C^{G,t}_{\mathrm{sfuc}}(d, \delta, E, \|V\|_\infty) := \left(\frac{\delta}{G}\right)^{N(1+G^{4/3}\|V\|_\infty^{2/3}/t^{2/3}+G\sqrt{E/t})}.$$

Note that the set $W_\delta(L)$ depends on G and the choice of the (G, δ)-equidistributed sequence. In particular, there is a constant $M = M(d, G, t) \geq 1$ such that

$$C^{G,t}_{\mathrm{sfuc}} \geq \delta^{M(1+\|V\|_\infty^{2/3}+\sqrt{|E|})}. \tag{3}$$

We also emphasize that Theorem 2.2 and Corollary 2.4 also hold for $E < 0$, since

$$\operatorname{Ran}(\chi_{(-\infty,E]}(H)) \subset \operatorname{Ran}(\chi_{(-\infty,0]}(H))$$

for any self-adjoint operator H.

Remark 2.5 (previous results). If $L = G$ the result is closely related to doubling estimates and bounds on the vanishing order; see [Lebeau and Robbiano 1995; Kukavica 1998; Jerison and Lebeau 1999; Bakri 2013]. These results, however, do not study the dependence of the bound on geometric data, e.g., the diameter of the domain or manifold. In the context of random Schrödinger operators results like (1) have been proven before under additional assumptions and using other methods: for $V \equiv 0$ and energies close to the minimum of the spectrum in [Kirsch 1996; Bourgain and Kenig 2005], near spectral edges of periodic Schrödinger operators in [Kirsch et al. 1998], and for periodic geometries $W_\delta(L)$ and potentials in [Combes et al. 2003]. More recently and using similar methods to ours, bounds like (1) have been established for individual eigenfunctions in [Rojas-Molina and Veselić 2013]. This has then been extended in [Klein 2013] to linear combinations of eigenfunctions corresponding to eigenvalues which are close to each other. For more references and a broader discussion of the history see, e.g., [Rojas-Molina and Veselić 2013; Klein 2013; Täufer et al. 2016].

As an application to spectral theory we have the following corollary. A proof is given at the end of Section 3.

Corollary 2.6. *Let $E, \alpha, G > 0$, $\delta \in (0, G/2)$, $L \in G\mathbb{N}$ and $A, B : \Lambda_L \to \mathbb{R}$ be measurable, bounded potentials and assume that*

$$B \geq \alpha \chi_{W_\delta(L)}$$

for a (G, δ)-equidistributed sequence. Denote the eigenvalues of a self-adjoint operator H with discrete spectrum by $\lambda_i(H)$, enumerated increasingly and counting multiplicities. Then for all $i \in \mathbb{N}$ with $\lambda_i(-\Delta + A + B) \leq E$, we have

$$\lambda_i(-\Delta_L + A + B) \geq \lambda_i(-\Delta_L + A) + \alpha C_{\mathrm{sfuc}}^{G,1}(d, \delta, E, \|A + B\|_\infty).$$

Remark 2.7 (generalizations). In [Täufer and Tautenhahn 2017] it has been proven that Corollary 2.4 holds also if $\chi_{(-\infty,E]}(H_L)$ is replaced by $\exp(-tH_L)$ for sufficiently large $t > 0$. An adaptation of our methods allows us to treat Schrödinger operators H on the whole of $\mathbb{R}^d$ instead on cubes. This will be discussed in our forthcoming paper. An important consequence of this result is a lifting estimate for boundaries of the essential spectrum, quite analogous to Corollary 2.6. Finally, let us remark that an analog of Theorem 2.2 for the case $V \equiv 0$ where the equidistributed set needs only to be measurable (and not open) has been established in [Egidi and Veselić 2016] using different methods.

Application to random breather Schrödinger operators. An important application of our result is in the spectral theory of random Schrödinger operators. The above scale-free unique continuation estimate is the key for proving the Wegner estimate formulated below, which is a bound on the expected number of eigenvalues in a short energy interval of a finite box restriction of our random Hamiltonian. Together with a so-called initial scale estimate, Wegner estimates facilitate a proof of Anderson localization via multiscale analysis. For more background on multiscale analysis and localization and on Wegner estimates consult, e.g., the monographs [Stollmann 2001] and [Veselić 2008], respectively.

The main point is that the potentials we are dealing with here exhibit a *nonlinear dependence* on the random parameters ω_j. Due to this challenge, it is not clear how to apply previously established versions of (1), as discussed in Remark 2.5, to such models. We emphasize that our scale-free unique continuation principle and Wegner estimate are valid for all bounded energy intervals, not only near the bottom of the spectrum.

Let us introduce a simple, but paradigmatic example of the models we are considering. (The general case will be studied in the next paragraph.)

Let $\mathcal{D}$ be a countable set to be specified later. For $0 \le \omega_- < \omega_+ < 1$ we define the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with

$$\Omega = \bigtimes_{j \in \mathcal{D}} \mathbb{R}, \quad \mathcal{A} = \bigotimes_{j \in \mathcal{D}} \mathcal{B}(\mathbb{R}) \quad \text{and} \quad \mathbb{P} = \bigotimes_{j \in \mathcal{D}} \mu,$$

where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra and μ is a probability measure with $\operatorname{supp} \mu \subset [\omega_-, \omega_+]$ and a bounded density ν_μ. Hence, the projections $\omega \mapsto \omega_k$ give rise to a sequence of independent and identically distributed random variables ω_j, $j \in \mathcal{D}$. We denote by $\mathbb{E}$ the expectation with respect to the measure $\mathbb{P}$. The standard random breather model is defined as

$$H_\omega = -\Delta + V_\omega(x) \quad \text{with } V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \chi_{B_{\omega_j}}(x - j), \tag{4}$$

and the restriction of H_ω to the box Λ_L is denoted by $H_{\omega,L}$. Here obviously $\mathcal{D} = \mathbb{Z}^d$. Denote by $\chi_{[E-\varepsilon,E+\varepsilon]}(H_{\omega,L})$ the spectral projector of $H_{\omega,L}$. We formulate now a version of our general Theorem 2.10 applied to the standard random breather model.

Theorem 2.8 (Wegner estimate for the standard random breather model). *Assume that* $[\omega_-, \omega_+] \subset \left[0, \frac{1}{4}\right]$, *fix* $E_0 \in \mathbb{R}$, *and set* $\varepsilon_{\max} = \frac{1}{4} \cdot 8^{-N(2+|E_0+1|^{1/2})}$, *where N is the constant from Theorem 2.2. Then there is* $C = C(d, E_0) \in (0, \infty)$ *such that for all* $\varepsilon \in (0, \varepsilon_{\max}]$ *and* $E \ge 0$ *with* $[E-\varepsilon, E+\varepsilon] \subset (-\infty, E_0]$, *we have*

$$\mathbb{E}\left[\operatorname{Tr}[\chi_{[E-\varepsilon,E+\varepsilon]}(H_{\omega,L})]\right] \le C \|\nu\|_\infty \varepsilon^{[N(2+|E_0+1|^{1/2})]^{-1}} |\ln \varepsilon|^d L^d.$$

Theorem 2.8 implies local Hölder continuity of the integrated density of states (IDS) and is sufficient for the multiscale analysis proof of spectral localization; see the next paragraph.

Remark 2.9 (previous results on the random breather model). The paper [Combes et al. 1996] introduced random breather potentials, while a Wegner estimate was proven in [Combes et al. 2001], however, excluding any bounded and any continuous single site potential; see the Appendix. Lifshitz tails for random breather Schrödinger operators were proven in [Kirsch and Veselić 2010]. All of the papers mentioned so far approached the breather model using techniques which have been developed for the alloy-type model. Consequently, at some stage the nonlinear dependence on the random variables was linearized, giving rise to certain differentiability conditions. As a result, characteristic functions of cubes or balls, which would be the most basic example one can think of, were excluded as single-site potentials. Only [Veselić 2007] considers a simple nondifferentiable example, namely the standard random breather potential in one dimension, and proves a Lifshitz tail estimate. This will be extended to multidimensional models in the forthcoming paper [Schumacher and Veselić 2017].

More general nonlinear models and localization. We formulate now a Wegner estimate for a general class of models, which includes the standard random breather potential, considered in the last paragraph as a special case. We state also an initial scale estimate which implies localization.

Here, in the general setting, we assume that $\mathcal{D} \subset \mathbb{R}^d$ is a Delone set; i.e., there are $0 < G_1 < G_2$ such that for any $x \in \mathbb{R}^d$ we have $\sharp\{\mathcal{D} \cap (\Lambda_{G_1} + x)\} \le 1$ and $\sharp\{\mathcal{D} \cap (\Lambda_{G_2} + x)\} \ge 1$. Here, $\sharp\{\,\cdot\,\}$ stands for the cardinality. In other words, Delone sets are relatively dense and uniformly discrete subsets of $\mathbb{R}^d$. For more background about Delone sets, see, for example, the contributions in [Kellendonk et al. 2015]. The reader unacquainted with the concept of a Delone set can always think of $\mathcal{D} = \mathbb{Z}^d$.

Furthermore, let $\{u_t : t \in [0,1]\} \subset L_0^\infty(\mathbb{R}^d)$ be functions such that there are $G_u \in \mathbb{N}$, $u_{\max} \ge 0$, $\alpha_1, \beta_1 > 0$ and $\alpha_2, \beta_2 \ge 0$ with

$$\begin{aligned} \forall t \in [0,1], \quad & \operatorname{supp} u_t \subset \Lambda_{G_u}, \\ \forall t \in [0,1], \quad & \|u_t\|_\infty \le u_{\max}, \\ \forall t \in [\omega_-, \omega_+],\ \delta \le 1 - \omega_+,\ \exists x_0 \in \Lambda_{G_u}, \quad & u_{t+\delta} - u_t \ge \alpha_1 \delta^{\alpha_2} \chi_{B(x_0, \beta_1 \delta^{\beta_2})}. \end{aligned} \tag{5}$$

We define the family of Schrödinger operators H_ω, $\omega \in \Omega$, on $L^2(\mathbb{R}^d)$ given by

$$H_\omega := -\Delta + V_\omega, \quad \text{where } V_\omega(x) = \sum_{j \in \mathcal{D}} u_{\omega_j}(x - j).$$

Note that for all $\omega \in [0,1]^{\mathcal{D}}$ we have

$$\|V_\omega\|_\infty \le K_u := u_{\max} \lceil G_u / G_1 \rceil^d;$$

see Lemma 4.1. Assumption (5) includes many prominent models of random Schrödinger operators — linear and nonlinear. We give some examples.

Standard random breather model: Let μ be the uniform distribution on $\left[0, \frac{1}{4}\right]$ and let $u_t(x) = \chi_{B(0,t)}$, $j \in \mathbb{Z}^d$. Then $V_\omega = \sum_{j \in \mathbb{Z}^d} \chi_{B(j,\omega_j)}$ is the characteristic function of a disjoint union of balls with random radii. This model was introduced in the previous subsection.

General random breather models: Let $0 \le u \in L_0^\infty(\mathbb{R}^d)$ and define $u_t(x) := u(x/t)$ for $t > 0$ and $u_0 :\equiv 0$ and assume that the family $\{u_t : t \in [0,1]\}$ satisfies (5). Natural examples are discussed in the Appendix. They include the characteristic function of bounded convex sets, the hat-potential $(1 - |x|)\chi_{\{|x|<1\}}$ or the bump function $\exp(1/(|x|^2 - 1))\chi_{\{|x|<1\}}$. Then $V_\omega(x) = \sum_{j \in \mathbb{Z}^d} u_{\omega_j}(x - j)$ is a sum of random dilations of a single-site potential u at each lattice site $j \in \mathbb{Z}^d$.

Alloy-type model: Let $0 \le u \in L_0^\infty(\mathbb{R}^d)$, $u \ge \alpha > 0$, on some open set and let $u_t(x) := tu(x)$. Then $V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \omega_j u(x - j)$ is a sum of copies of u at all lattice sites $j \in \mathbb{Z}^d$, multiplied with ω_j.

Delone-alloy-type model: Let $\mathcal{D} \subset \mathbb{R}^d$ be a Delone set, $0 \le u \in L_0^\infty(\mathbb{R}^d)$, $u \ge \alpha > 0$, on some nonempty open set and let $u_t(x) := tu(x)$. Then $V_\omega(x) = \sum_{j \in \mathcal{D}} \omega_j u(x - j)$ is a sum of copies of u at all Delone points $j \in \mathcal{D}$, multiplied with ω_j. See [Germinet et al. 2015] for background on such models.

For $L > 0$ we denote by $H_{\omega,L}$ the restriction of H_ω to $L^2(\Lambda_L)$ with Dirichlet boundary conditions. Following the methods developed in [Hundertmark et al. 2006], we obtain a Wegner estimate under our general assumption (5).

Theorem 2.10 (Wegner estimate). *For all $E_0 \in \mathbb{R}$ there are constants $C, \kappa, \varepsilon_{\max} > 0$, depending only on d, E_0, K_u, G_u, G_2, α_1, α_2, β_1, β_2, ω_+ and $\|\nu_\mu\|_\infty$, such that for all $L \in (G_2 + G_u)\mathbb{N}$, all $E \in \mathbb{R}$ and $\varepsilon \leq \varepsilon_{\max}$ with $[E - \varepsilon, E + \varepsilon] \subset (-\infty, E_0]$ we have*

$$\mathbb{E}\big[\mathrm{Tr}[\chi_{[E-\varepsilon,E+\varepsilon]}(H_{\omega,L})]\big] \leq C\varepsilon^{1/\kappa}|\ln \varepsilon|^d L^d. \tag{6}$$

Theorem 2.11 (initial scale estimate). *Let κ be as in Theorem 2.10 for $E_0 = d\pi^2 + K_u$. Assume that there are $t_0, C > 0$ such that*

$$0 \in \operatorname{supp} \mu \qquad \text{and} \qquad \text{for all } t \in [0, t_0], \quad \mu([0, t]) \leq C t^{d\kappa}.$$

Then there is $L_0 = L_0(t_0, \delta_{\max}, \kappa, G_u, G_1) \geq 1$ such that for all $L \in (G_2 + G_u)\mathbb{N},\ L \geq L_0$, we have

$$\mathbb{P}\bigg(\bigg\{\omega \in \Omega : \lambda_1(H_{\omega,L}) - \lambda_1(H_{0,L}) \geq \frac{1}{L^{3/2}}\bigg\}\bigg) \geq 1 - \frac{C}{L^{d/2}},$$

where $H_{0,L}$ is obtained from $H_{\omega,L}$ by setting ω_j to zero for all $j \in \mathcal{D}$.

Remark 2.12 (discussion on initial scale estimate). Theorem 2.11 may serve as an initial scale estimate for a proof of localization via multiscale analysis. More precisely, by using the Combes–Thomas estimate, an initial scale estimate in some neighborhood of $a := \inf \sigma(H_0)$ follows. Note that the exponents $\frac{3}{2}$ and $\frac{d}{2}$ in Theorem 2.11 can be modified to some extent by adapting the proof and the assumption on the measure μ. Localization in a neighborhood I_a of a follows via multiscale analysis, e.g., à la [Stollmann 2001]. The question of whether $\sigma(H_\omega) \cap I_a \neq \varnothing$ for almost all $\omega \in \Omega$ has to be settled. This is, however, satisfied for all examples mentioned above. In the special case of the standard random breather model one can get rid of the assumption on μ by proving and using the Lifshitz tail behavior of the integrated density of states; see [Veselić 2007] for the one-dimensional case, and the forthcoming paper of Schumacher and Veselić for the multidimensional one.

Application to control theory. We consider the controlled heat equation with heat generation term $(-V)$

$$\begin{cases} \partial_t u - \Delta u + V u = f\chi_\omega, & u \in L^2([0, T] \times \Omega), \\ u = 0 & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0, & u_0 \in L^2(\Omega), \end{cases} \tag{7}$$

where ω is an open subset of the connected $\Omega \subset \mathbb{R}^d$, $T > 0$ and $V \in L^\infty(\Omega)$. In (7) u is the state and f is the control function which acts on the system through the control set ω.

Definition 2.13. For initial data $u_0 \in L^2(\Omega)$ and time $T > 0$, the set of reachable states $R(T, u_0)$ is

$$R(T, u_0) = \big\{u(T, \cdot) : \text{there exists } f \in L^2([0, T] \times \omega) \text{ such that } u \text{ is solution of (7) with RHS}\big\}.$$

The system (7) is called null controllable at time T if $0 \in R(T; u_0)$ for all $u_0 \in L^2(\Omega)$. The controllability cost $\mathcal{C}(T, u_0)$ at time T for the initial state u_0 is

$$\mathcal{C}(T, u_0) = \inf\big\{\|f\|_{L^2([0,T]\times\omega)} : u \text{ is solution of (7) and } u(T, \cdot) = 0\big\}.$$

Since the system is linear, null controllability implies that the range of the semigroup generated by the heat equation is reachable too. It is well known that null controllability holds for any time $T > 0$, connected Ω and any nonempty and open set $\omega \subset \Omega$ on which the control acts; see [Fursikov and Imanuvilov 1996].

It is also known, see for instance [Tucsnak and Weiss 2009, Theorem 11.2.1], that null controllability of the system (7) at time T is equivalent to final state observability on the set ω at time T of the system

$$\begin{cases} \partial_t u - \Delta u + V u = 0, & u \in L^2([0,T] \times \Omega), \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u_0, & u_0 \in L^2(\Omega). \end{cases} \tag{8}$$

Definition 2.14. The system (8) is called final state observable on the set ω at time T if there exists $\kappa_T = \kappa_T(\Omega, \omega, V)$ such that for every initial state $u_0 \in L^2(\Omega)$ the solution $u \in L^2([0,T] \times \Omega)$ of (8) satisfies

$$\|u(T,\cdot)\|_\Omega^2 \le \kappa_T \|u\|^2_{L^2([0,T]\times\omega)}. \tag{9}$$

Moreover, the controllability cost $\mathcal{C}(T, u_0)$ of (7) coincides with the infimum over all observability costs $\sqrt{\kappa_T}$ in (9) times $\|u_0\|_\Omega$; see, for example, the proof of [Tucsnak and Weiss 2009, Theorem 11.2.1].

The problem of obtaining explicit bounds on $\mathcal{C}(T, u_0)$ received much consideration in the literature, see, for example, [Güichal 1985; Fernández-Cara and Zuazua 2000; Phung 2004; Tenenbaum and Tucsnak 2007; Miller 2006; 2004; 2010; Ervedoza and Zuazua 2011; Lissy 2012], especially the case of small time, i.e., when T goes to zero. The dependencies of the controllability cost on T and $\|V\|_\infty$ are today well understood; see, for example, [Zuazua 2007]. However, the dependence on the geometry of the control set is less clear: in the known estimates the geometry enters only in terms of the distance to the boundary or in terms of the geometrical optics condition. To find an optimal control set is a very difficult problem; see for instance the recent articles [Privat et al. 2015a; 2015b].

We are interested in the situation $\Omega = \Lambda_L \subset \mathbb{R}^d$ and $\omega = W_\delta(L)$ for a (G,δ)-equidistributed sequence with $L \in G\mathbb{N}$, $G > 0$ and $\delta < G/2$. In this specific setting we will give an estimate on the controllability cost. The novelty of our result is that the observability cost is independent of the scale L and the specific choice of the (G,δ)-equidistributed sequence. Moreover, the dependencies on $\|V\|_\infty$ and on the size of the control set via δ are known explicitly. As far as we are aware, this is the first time that such a scale-free estimate is obtained.

By the equivalence between null-controllability and final state observability, it is sufficient to construct an estimate of the form (9). In order to find such an estimate, we will combine Corollary 2.4 with results from [Miller 2010] to obtain the following theorem.

Theorem 2.15. *For every $G > 0$, $\delta \in (0, G/2)$ and $K_V \ge 0$ there is $T' = T'(G, \delta, K_V) > 0$ such that for all $T \in (0, T']$, all (G,δ)-equidistributed sequences, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}^d$ with $\|V\|_\infty \le K_V$ and all $L \in G\mathbb{N}$, the system*

$$\begin{cases} \partial_t u - \Delta_L u + V_L u = 0, & u \in L^2([0,T] \times \Lambda_L), \\ u = 0 & \textit{on } (0,T) \times \partial\Lambda_L, \\ u(0,\cdot) = u_0, & u_0 \in L^2(\Lambda_L) \end{cases}$$

is final state observable on the set $W_\delta(L)$ with cost κ_T satisfying

$$\kappa_T \le 4a_0 b_0 \mathrm{e}^{2c_*/T},$$

where $a_0 = (\delta/G)^{-N(1+G^{4/3}\|V\|_\infty^{2/3})}$, $b_0 = \mathrm{e}^{2\|V\|_\infty}$, $c_* \le \ln(G/\delta)^2(NG + 4/\ln 2)^2$ *and* $N = N(d)$ *is the constant from Theorem 2.2.*

Remark 2.16. (i) The same result holds also in the case of the controlled heat equation with periodic or Neumann boundary conditions with obvious modifications.

(ii) Null controllability of the heat equation implies a stronger type of controllability, so-called approximate controllability. Following [Fernández-Cara and Zuazua 2000], one can find an estimate for the cost of approximate controllability from the proof of Theorem 2.15. We will not pursue it in this paper.

3. Proof of the scale-free unique continuation principle

Carleman inequalities. We denote by $\mathbb{R}^{d+1}_+ := \{x \in \mathbb{R}^{d+1} : x_{d+1} \ge 0\}$ the $(d+1)$-dimensional half-space and by $B_r^+ := \{x \in \mathbb{R}^{d+1}_+ : |x| < r\}$ the $(d+1)$-dimensional half-ball. For $x \in \mathbb{R}^{d+1}$ we denote by x' the projection on the first d coordinates; i.e., for $x = (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1}$ we use the notation $x' = (x_1, \dots, x_d) \in \mathbb{R}^d$. By $|x|$ and $|x'|$ we denote the Euclidean norms and by Δ the Laplacian on $\mathbb{R}^{d+1}$. For functions $f \in C^\infty(\mathbb{R}^{d+1}_+)$ we use the notation $f_0 = f|_{x_{d+1}=0}$.

In the appendix of [Lebeau and Robbiano 1995], the authors state a Carleman estimate for complex-valued functions with support in B_r^+ by using a real-valued weight function $\psi \in C^\infty(\mathbb{R}^{d+1})$ satisfying the two conditions

$$\text{for all } x \in B_r^+ \text{ we have } (\partial_{d+1}\psi)(x) \ne 0, \tag{10}$$

and for all $\xi \in \mathbb{R}^{d+1}$ and $x \in B_r^+$ there holds

$$\left.\begin{aligned} 2\langle \xi, \nabla\psi\rangle = 0,\\ |\xi|^2 = |\nabla\psi|^2 \end{aligned}\right\} \quad\Longrightarrow\quad \sum_{j,k=1}^{d+1} (\partial_{jk}\psi)\big(\xi_j\xi_k + (\partial_j\psi)(\partial_k\psi)\big) > 0. \tag{11}$$

As proposed in [Jerison and Lebeau 1999] we choose $r < 2 - \sqrt{2}$ and the special weight function $\psi : \mathbb{R}^{d+1} \to \mathbb{R}$,

$$\psi(x) = -x_{d+1} + \tfrac{1}{2}x_{d+1}^2 - \tfrac{1}{4}|x'|^2. \tag{12}$$

Note that $\psi(x) \le 0$ for all $x \in B_2^+$. This function ψ indeed satisfies the assumptions (10) and (11). Condition (10) is trivial for $r < 1$. In order to show the implication (11) we show

$$|\xi|^2 = |\nabla\psi|^2 \quad\Longrightarrow\quad \sum_{j,k=1}^{d+1} \partial_{jk}\psi(\xi_j\xi_k + \partial_j\psi\,\partial_k\psi) > 0. \tag{13}$$

We use the hypothesis of (13) and calculate

$$\sum_{j,k=1}^{d+1} \partial_{jk}\psi(\xi_j\xi_k + \partial_j\psi\,\partial_k\psi) = -\tfrac{1}{2}\sum_{i=1}^{d}\xi_i^2 + \xi_{d+1}^2 - \tfrac{1}{8}|x'|^2 + (x_{d+1}-1)^2 = \tfrac{3}{2}\xi_{d+1}^2 - \tfrac{1}{4}|x'|^2 + \tfrac{1}{2}(x_{d+1}-1)^2.$$

Since $|x'|^2 \le r^2$ and $(x_{d+1}-1)^2 \ge (1-r)^2$, assumption (13) is satisfied if $r < 2-\sqrt{2}$. Now let

$$C^\infty_{\mathrm{c},0}(B_r^+) = \big\{g : \mathbb{R}^{d+1}_+ \to \mathbb{C} : g \equiv 0 \text{ on } \{x_{d+1}=0\},\ \exists \phi \in C^\infty(\mathbb{R}^{d+1}) \text{ with } \operatorname{supp}\phi \subset \{x \in \mathbb{R}^{d+1} : |x| < r\} \text{ and } g \equiv \phi \text{ on } \mathbb{R}^{d+1}_+\big\}.$$

Hence, as a corollary of Proposition 1 in the appendix of [Lebeau and Robbiano 1995] we have:

Proposition 3.1. *Let $\psi \in C^\infty(\mathbb{R}^{d+1};\mathbb{R})$ be as in* (12) *and $\rho \in (0, 2-\sqrt{2})$. Then there are constants $\beta_0, C_1 \ge 1$ such that for all $\beta \ge \beta_0$, and all $g \in C^\infty_{\mathrm{c},0}(B_\rho^+)$ we have*

$$\int_{\mathbb{R}^{d+1}} \mathrm{e}^{2\beta\psi}(\beta|\nabla g|^2 + \beta^3|g|^2) \le C_1\bigg(\int_{\mathbb{R}^{d+1}} \mathrm{e}^{2\beta\psi}|\Delta g|^2 + \beta\int_{\mathbb{R}^d} \mathrm{e}^{2\beta\psi_0}|(\partial_{d+1}g)_0|^2\bigg).$$

We will need another Carleman estimate with a weight function whose level sets can be explicitly controlled.

Proposition 3.2 [Nakić et al. 2015a]. *Let $\rho > 0$ and $w : \mathbb{R}^d \to \mathbb{R}$,*

$$w(x) = \frac{|x|}{\rho}\int_0^{|x|/\rho} \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t.$$

In particular,

$$\text{for all } x \in B(\rho), \quad \frac{|x|}{\rho\mathrm{e}} \le w(x) \le \frac{|x|}{\rho}.$$

Then there are constants $\alpha_0, C_2 \ge 1$ depending only on the dimension such that for all $\alpha \ge \alpha_0$, and all $u \in W^{2,2}(\mathbb{R}^d)$ with support in $B(\rho)\setminus\{0\}$ we have

$$\int_{\mathbb{R}^d}\big(\alpha\rho^2 w^{1-2\alpha}|\nabla u|^2 + \alpha^3 w^{-1-2\alpha}|u|^2\big)\,\mathrm{d}x \le C_2\rho^4\int_{\mathbb{R}^d} w^{2-2\alpha}|\Delta u|^2\,\mathrm{d}x.$$

This variant of the Carleman estimate is essentially given in [Escauriaza and Vessella 2003], albeit that paper concerns parabolic operators. For elliptic operators, in [Bourgain and Kenig 2005] a weaker statement than Proposition 3.2, without the gradient term on the left-hand side, was spelled out and proven explicitly. A version of Proposition 3.2 for divergence-type elliptic operators is stated in [Kenig et al. 2011]. While this covers more general operators than we are interested in here, it lacks a quantitative statement about the admissible functions u. An explicit proof of Proposition 3.2, i.e., for the pure Laplacian, was first given in [Klein and Tsang 2016]. See also [Nakić et al. 2015a] for the case of divergence-type elliptic operators. The paper [Davey 2014] also contains a Carleman estimate which is less explicit than Proposition 3.2, but would still be sufficient for the purpose of the proof of Theorem 2.2.

Extension to larger boxes. For each measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$ and each $L \in \mathbb{N}$ we denote the eigenvalues of the corresponding operator H_L by E_k, $k \in \mathbb{N}$, enumerated in increasing order and counting multiplicities, and fix a corresponding sequence ϕ_k, $k \in \mathbb{N}$, of normalized eigenfunctions. Note that we suppress the dependence of E_k and ϕ_k on V and L.

Given V and L we define an extension of the potential V_L and the eigenfunctions ϕ_k to the set Λ_{RL} for some $R \in \mathbb{N}_{\mathrm{odd}} = \{1, 3, 5, \dots\}$ to be chosen later on. The extension will depend on the type of boundary conditions we are considering for the Laplace operator.

Extension for periodic boundary conditions: We extend the potential V_L as well as the function ϕ_k, defined on the box Λ_L, periodically to $\widetilde{V}, \tilde{\psi} : \mathbb{R}^d \to \mathbb{R}$ and then restrict them to Λ_{RL}. By the very definition of the operator domain of Δ_{Λ_L} with periodic boundary conditions the extension $\tilde{\psi}$ is locally in the Sobolev space $W^{2,2}(\mathbb{R}^d)$.

Extension for Dirichlet and Neumann boundary conditions: The potential V_L will be extended by symmetric reflections with respect to the hypersurfaces forming the boundaries of Λ_L. In the first step we extend $V_L : \Lambda_L \to \mathbb{R}$ to the set

$$\{x \in \Lambda_{3L} : x_i \in (-L/2, L/2),\ i \in \{2, \dots, d\}\}$$

by

$$V_L(x) = \begin{cases} V_L(x) & \text{if } x \in \Lambda_L, \\ 0 & \text{if } x_1 \in \{-L/2, L/2\}, \\ V_L(L - x_1, x_2, \dots, x_d) & \text{if } x_1 > L/2, \\ V_L(-L - x_1, x_2, \dots, x_d) & \text{if } x_1 < -L/2. \end{cases}$$

Now we iteratively extend V_L in the remaining $d-1$ directions using the same procedure and obtain a function $V_L : \Lambda_{3L} \to \mathbb{R}$. Iterating this procedure we obtain a function $V_L : \Lambda_{RL} \to \mathbb{R}$. The extensions of the eigenfunctions will depend on the boundary conditions. In the case of Dirichlet boundary conditions, we extend an eigenfunction similarly to the potential by antisymmetric reflections, while in the case of Neumann boundary conditions, we extend by symmetric reflections.

The extensions of the functions and V_L and $\phi_k,\ k \in \mathbb{N}$, to the set Λ_{RL} will again be denoted by V_L and $\phi_k,\ k \in \mathbb{N}$. The reader should be reminded that (the extended) $V_L : \Lambda_{RL} \to \mathbb{R}$ does in general not coincide with $V_{RL} : \Lambda_{RL} \to \mathbb{R}$. Note that for all three boundary conditions, $V_L : \Lambda_{RL} \to \mathbb{R}$ takes values in $[-\|V\|_\infty, \|V\|_\infty]$, the extended ϕ_k are elements of $W^{2,2}(\Lambda_{RL})$ with corresponding boundary conditions and they satisfy $\Delta\phi_k = (V_L - E_k)\phi_k$ on Λ_{RL}. Furthermore, the orthogonality relations remain valid.

Ghost dimension. For a measurable and bounded $V : \mathbb{R}^d \to \mathbb{R},\ L \in \mathbb{N},\ E \geq 0$ and $\phi \in \operatorname{Ran}(\chi_{(-\infty,E]}(H_L))$ we have

$$\phi = \sum_{\substack{k \in \mathbb{N} \\ E_k \leq E}} \alpha_k \phi_k, \quad \text{with} \quad \alpha_k = \langle \phi_k, \phi \rangle.$$

Since the ϕ_k extend to Λ_{RL} as explained in the previous subsection, the function ϕ also extends to Λ_{RL}. We set $\omega_k := \sqrt{|E_k|}$ and define the function $F : \Lambda_{RL} \times \mathbb{R} \to \mathbb{C}$ by

$$F(x, x_{d+1}) = \sum_{\substack{k \in \mathbb{N} \\ E_k \leq E}} \alpha_k \phi_k(x)\, \mathrm{s}_k(x_{d+1}),$$

where $s_k : \mathbb{R} \to \mathbb{R}$ is given by

$$\mathrm{s}_k(t) = \begin{cases} \sinh(\omega_k t)/\omega_k, & E_k > 0, \\ t, & E_k = 0, \\ \sin(\omega_k t)/\omega_k, & E_k < 0. \end{cases}$$

Note that we suppress the dependence of ϕ and ϕ_k on V, L, E. Furthermore, the sums are finite since H_L is lower semibounded with purely discrete spectrum. The function F satisfies

$$\Delta F = \sum_{i=1}^{d+1} \partial_i^2 F = V_L F \quad \text{on } \Lambda_{RL} \times \mathbb{R}$$

and

$$\partial_{d+1} F(x,0) = \sum_{\substack{k\in\mathbb{N} \\ E_k \leq E}} \alpha_k \phi_k(x) \quad \text{for } x \in \Lambda_{RL}.$$

In particular, for all $x \in \Lambda_L$ we have $\partial_{d+1}F(x,0) = \phi$. This way we recover the original function we are interested in.

Let us also fix the geometry. For $\delta \in \left(0, \frac{1}{2}\right)$ we choose

$$\begin{aligned} \psi_1 &= -\tfrac{1}{16}\delta^2, & \psi_2 &= -\tfrac{1}{8}\delta^2, & \psi_3 &= -\tfrac{1}{4}\delta^2, \\ r_1 &= \tfrac{1}{2} - \tfrac{1}{8}\sqrt{16-\delta^2}, & r_2 &= 1, & r_3 &= 6\mathrm{e}\sqrt{d}, \\ R_1 &= 1 - \tfrac{1}{4}\sqrt{16-\delta^2}, & R_2 &= 3\sqrt{d}, & R_3 &= 9\mathrm{e}\sqrt{d}, \end{aligned}$$

and define for $i \in \{1,2,3\}$ the sets

$$\begin{aligned} S_i &:= \left\{x \in \mathbb{R}^{d+1} : \psi(x) > \psi_i,\, x_{d+1} \in [0,1]\right\} \subset \mathbb{R}^{d+1}_+, \\ V_i &:= B(R_i) \setminus \overline{B(r_i)} \subset \mathbb{R}^{d+1}. \end{aligned}$$

Let $R \in \mathbb{N}$ be the least power of 3 larger than $2R_3 + 2$. For $i \in \{1,2,3\}$ and $x \in \mathbb{R}^d$ we denote by $S_i(x) = S_i + (x,0)$ and $V_i(x) = V_i + (x,0)$ the translates of the sets $S_i \subset \mathbb{R}^{d+1}$ and $V_i \subset \mathbb{R}^{d+1}$. Moreover, for $L \in \mathbb{N}$ and a $(1,\delta)$-equidistributed sequence $z_j \in \mathbb{R}^d$, $j \in \mathbb{Z}^d$, we define $Q_L = \mathbb{Z}^d \cap \Lambda_L$, $U_i(L) = \bigcup_{j\in Q_L} S_i(z_j)$, $X_1 = \Lambda_L \times [-1,1]$ and $\widetilde{X}_{R_3} = \Lambda_{L+2R_3} \times [-R_3, R_3]$. Note that $W_\delta(L)$ is a disjoint union. In the following lemma we collect some consequences of our geometric setting. We will first restrict our attention to the case $L \in \mathbb{N}_{\mathrm{odd}}$, and consider the case of even integers thereafter.

Lemma 3.3. (i) *For all $\delta \in \left(0, \frac{1}{2}\right)$ we have $S_1 \subset S_2 \subset S_3 \subset B_\delta^+ \subset \mathbb{R}^{d+1}_+$.*

(ii) *For all $L \in \mathbb{N}_{\mathrm{odd}}$ with $L \geq 5$, all $\delta \in \left(0, \frac{1}{2}\right)$ and all $(1,\delta)$-equidistributed sequences z_j we have $\bigcup_{j\in Q_L} V_2(z_j) \supset X_1$.*

(iii) *There is a constant K_d, depending only on d, such that for all $L \in \mathbb{N}_{\mathrm{odd}}$, all $\delta \in \left(0, \frac{1}{2}\right)$, all $(1,\delta)$-equidistributed sequences z_j, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $E \geq 0$ and all $\phi \in \operatorname{Ran}(\chi_{(-\infty,E]}(H_L))$ we have*

$$\sum_{j\in Q_L} \|F\|^2_{H^1(V_3(z_j))} \leq K_d \|F\|^2_{H^1\left(\bigcup_{j\in Q_L} V_3(z_j)\right)}.$$

(iv) *For all $L \in \mathbb{N}_{\mathrm{odd}}$, $\delta \in \left(0, \frac{1}{2}\right)$ and all $(1,\delta)$-equidistributed sequences z_j we have $\bigcup_{j\in Q_L} V_3(z_j) \subset X_{R_3}$.*

We note that part (ii) of Lemma 3.3 will be applied with L replaced by $5L$.

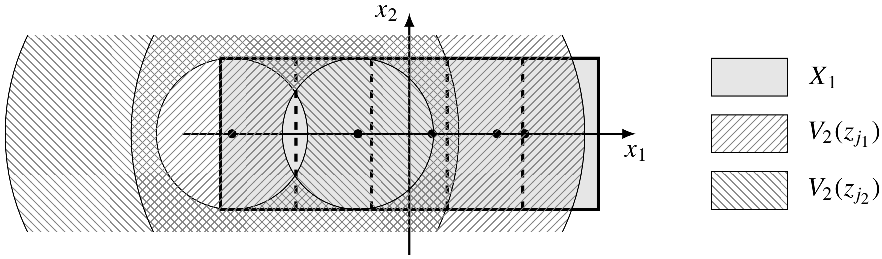

Figure 1. Illustration for (ii) in the case $d = 1$, $L = 5$ and for some configuration z_j, $j \in Q_L$. The set $\left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1, 1]$ is covered by $V_2(z_{j_1})$ and $V_2(z_{j_2})$.

Proof. Parts (i) and (iv) are obvious.

To show (ii), we first prove that $\left[-\frac{1}{2}, \frac{1}{2}\right]^d \times [-1, 1]$ can be covered by the sets $V_2(z_j)$. Let us take $j_1 = (-1, 0, \dots, 0)$, $j_2 = (-2, 0, \dots, 0)$, $j_1, j_2 \in Q_L$.

Then

$$\left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^d \times [-1, 1] \subset V_2(z_{j_1}) \cup V_2(z_{j_2}); \tag{14}$$

see Figure 1. Indeed, let $x = (x_1, \dots, x_{d+1})$ be an arbitrary point from $\left[-\frac{1}{2}, \frac{1}{2}\right]^d \times [-1, 1]$. Then (14) is not satisfied only if $|(z_{j_1}, 0) - x|^2 < 1$ and $|(z_{j_2}, 0) - x|^2 > R_2^2$. Since $z_{j_1} \in \left(-\frac{3}{2} + \delta, -\frac{1}{2} - \delta\right) \times \left(-\frac{1}{2} + \delta, \frac{1}{2} - \delta\right)^{d-1}$ and $z_{j_2} \in \left(-\frac{5}{2} + \delta, -\frac{3}{2} - \delta\right) \times \left(-\frac{1}{2} + \delta, \frac{1}{2} - \delta\right)^{d-1}$, it follows that

$$\left(-\tfrac{1}{2} - \delta - x_1\right)^2 + x_{d+1}^2 < 1 \quad \text{and} \quad \left(-\tfrac{5}{2} + \delta - x_1\right)^2 + (d-1)(1-\delta)^2 + x_{d+1}^2 > 9d.$$

Plugging the first relation into the second, we obtain

$$9d < (d-1)(1-\delta)^2 + 2(1-\delta)(3 + 2x_1) + 1 \le (d-1)(1-\delta)^2 + 8(1-\delta) + 1.$$

But this relation is satisfied only for $d < 1$. Since $L \ge 5$ the same argument applies to cover every elementary cell $\left(\left[-\frac{1}{2}, \frac{1}{2}\right] + i\right) \times [-1, 1]$, $i \in Q_L$, by two neighboring sets $V_2(z_j)$.

Now we turn to the proof of (iii). Since $R \ge 2R_3 + 2$, the function F is defined on $V_3(z_j)$ for all $j \in Q_L$. For all $x \in \bigcup_{j \in Q_L} V_3(z_j)$, the number of indices $j \in Q_L$ such that $V_3(z_j) \ni x$ is bounded from above by $(2R_3 + 2)^d$. Hence,

$$\text{for all } x \in \widetilde{X}_{R_3}, \quad \sum_{j \in Q_L} \chi_{V_3(z_j)}(x) \le (2R_3 + 2)^d \chi_{\bigcup_{j \in Q_L} V_3(z_j)}(x),$$

and thus

$$\begin{aligned} \sum_{j \in Q_L} \|F\|^2_{H^1(V_3(z_j))} &= \int_{\widetilde{X}_{R_3}} \Bigg(\sum_{j \in Q_L} \chi_{V_3(z_j)}(x) \Bigg) (|F(x)|^2 + |\nabla F(x)|^2) \, \mathrm{d}x \\ &\le (2R_3 + 2)^d \|F\|^2_{H^1(\bigcup_{j \in Q_L} V_3(z_j))}. \end{aligned}$$

Hence we can take $K_d = (2R_3 + 2)^d$. □

Interpolation inequalities.

Proposition 3.4. *For all $\delta \in \left(0, \frac{1}{2}\right)$, all $(1,\delta)$-equidistributed sequences z_j, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $L \in \mathbb{N}_{\mathrm{odd}}$, all $E \geq 0$ and all $\phi \in \operatorname{Ran}(\chi_{(-\infty,E]}(H_L))$:*

(a) *There is $\beta_1 = \beta_1(d, \|V\|_\infty) \geq 1$ such that for all $\beta \geq \beta_1$ we have*

$$\|F\|^2_{H^1(U_1(L))} \leq \widetilde{D}_1(\beta)\|F\|^2_{H^1(U_3(L))} + \widehat{D}_1(\beta)\|(\partial_{d+1}F)_0\|^2_{L^2(W_\delta(L))},$$

where β_1 is given in (16), *and $\widetilde{D}_1(\beta)$ and $\widehat{D}_1(\beta)$ are given in* (17).

(b) *We have*

$$\|F\|_{H^1(U_1(L))} \leq D_1\|(\partial_{d+1}F)_0\|^{1/2}_{L^2(W_\delta(L))}\|F\|^{1/2}_{H^1(U_3(L))},$$

where D_1 is given in (21).

Proof. First we recall that $\Delta F = V_L F$, $\partial_{d+1}F(x',0) = \phi(x')$ and $B^+_\delta \supset S_3$. Now we choose a cutoff function $\chi \in C^\infty(\mathbb{R}^{d+1}; [0,1])$ with $\operatorname{supp}\chi \subset \bar{S}_3$, $\chi(x) = 1$ if $x \in S_2$ and

$$\max\{\|\Delta\chi\|_\infty, \||\nabla\chi|\|_\infty\} \leq \frac{\widetilde{\Theta}_1}{\delta^4} =: \Theta_1,$$

where $\widetilde{\Theta}_1 = \widetilde{\Theta}_1(d)$ depends only on the dimension. This is due to the fact that the distance of S_2 and $\mathbb{R}^{d+1}_+ \setminus S_3$ is bounded from below by $\delta^2/16$. Let φ be a nonnegative function in $C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ with the properties that $\|\varphi\|_1 = 1$ and $\operatorname{supp}\varphi \subset B(1)$. For $\varepsilon > 0$ we define $\varphi_\varepsilon : \mathbb{R}^d \to \mathbb{R}^+_0$ by $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$. The function φ_ε belongs to $C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ and satisfies $\operatorname{supp}\varphi_\varepsilon \subset (\varepsilon)$. Now we continuously extend the eigenfunctions $\phi_k : \Lambda_{RL} \to \mathbb{R}$ to the set $\mathbb{R}^d$ by zero and define for $\varepsilon > 0$ the function $F_\varepsilon : \mathbb{R}^d \times \mathbb{R}$ by

$$F_\varepsilon(x, x_{d+1}) = \sum_{\substack{k\in\mathbb{N}\\ E_k \leq E}} \alpha_k(\varphi_\varepsilon * \phi_k)(x)\, \mathrm{s}_k(x_{d+1}).$$

By construction, the function $g = \chi F_\varepsilon$ is an element of $C^\infty_{\mathrm{c},0}(B^+_\delta)$. Hence, we can apply Proposition 3.1 with $g = \chi F_\varepsilon$ and $\rho = \frac{1}{2}$ and obtain for all $\beta \geq \beta_0 \geq 1$

$$\int_{S_3} \mathrm{e}^{2\beta\psi}\left(\beta|\nabla(\chi F_\varepsilon)|^2 + \beta^3|\chi F_\varepsilon|^2\right) \leq C_1\int_{S_3} \mathrm{e}^{2\beta\psi}|\Delta(\chi F_\varepsilon)|^2 + \beta C_1\int_{B(\delta)} \mathrm{e}^{2\beta\psi_0}|(\partial_{d+1}(\chi F_\varepsilon))_0|^2. \quad (15)$$

Note that β_0 and C_1 only depend on the dimension. By [Ziemer 1989, Theorem 1.6.1(iii)] we have $\varphi_\varepsilon * \phi_k \to \phi_k$, $\nabla(\varphi_\varepsilon * \phi_k) \to \nabla\phi_k$ and $\Delta(\varphi_\varepsilon * \phi_k) \to \Delta\phi_k$ in $L^2(S_3)$ as ε tends to zero. Consequently, the same holds for F_ε, ∇F_ε and ΔF_ε and thus we obtain (15) with F_ε replaced by F. For the first summand on the right-hand side we have the upper bound

$$\begin{aligned}\int_{S_3} \mathrm{e}^{2\beta\psi}|\Delta(\chi F)|^2 &\leq 3\int_{S_3} \mathrm{e}^{2\beta\psi}\left(4|\nabla\chi|^2|\nabla F|^2 + |\Delta\chi|^2|F|^2 + |\Delta F|^2|\chi|^2\right)\\ &\leq 3\mathrm{e}^{2\beta\psi_2}\int_{S_3\setminus S_2}\left(4\Theta_1^2|\nabla F|^2 + \Theta_1^2|F|^2\right) + \int_{S_3} 3\mathrm{e}^{2\beta\psi}|V_L F\chi|^2\\ &\leq 12\Theta_1^2\mathrm{e}^{2\beta\psi_2}\|F\|^2_{H^1(S_3)} + 3\|V\|^2_\infty\int_{S_3} \mathrm{e}^{2\beta\psi}|\chi F|^2.\end{aligned}$$

The second summand is bounded from above by $\beta C_1 \int_{B(\delta)} |(\partial_{d+1} F)_0|^2$, since $F = 0$ and $\psi \leq 0$ on $\{x_{d+1} = 0\}$. Hence,

$$\beta \int_{S_3} \mathrm{e}^{2\beta\psi} |\nabla(\chi F)|^2 + (\beta^3 - 3\|V\|_\infty^2 C_1) \int_{S_3} \mathrm{e}^{2\beta\psi} |\chi F|^2 \leq 12 C_1 \Theta_1^2 \mathrm{e}^{2\beta\psi_2} \|F\|^2_{H^1(S_3)} + C_1 \beta \|(\partial_{d+1} F)_0\|^2_{L^2(B(\delta))}.$$

Additionally to $\beta \geq \beta_0$ we choose $\beta \geq (6\|V\|_\infty^2 C_1)^{1/3} =: \tilde{\beta}_0$. This ensures that for all

$$\beta \geq \beta_1 := \max\{\beta_0, \tilde{\beta}_0\} \tag{16}$$

we have

$$\frac{1}{2} \int_{S_3} \mathrm{e}^{2\beta\psi} \bigl(\beta |\nabla(\chi F)|^2 + \beta^3 |\chi F|^2\bigr) \leq 12 C_1 \Theta_1^2 \mathrm{e}^{2\beta\psi_2} \|F\|^2_{H^1(S_3)} + C_1 \beta \|(\partial_{d+1} F)_0\|^2_{L^2(B(\delta))}.$$

Since $\beta \geq 1$, $S_3 \supset S_1$, $\chi = 1$ and $\mathrm{e}^{2\beta\psi} \geq \mathrm{e}^{2\beta\psi_1}$ on S_1, we obtain

$$\mathrm{e}^{2\beta\psi_1} \|F\|^2_{H^1(S_1)} \leq 24 C_1 \Theta_1^2 \mathrm{e}^{2\beta\psi_2} \|F\|^2_{H^1(S_3)} + 2C_1 \|(\partial_{d+1} F)_0\|^2_{L^2(B(\delta))}.$$

We apply this inequality for translates $S_i(z_j)$ and obtain by summing over $j \in Q_L = \mathbb{Z}^d \cap \Lambda_L$

$$\mathrm{e}^{2\beta\psi_1} \sum_{j \in Q_L} \|F\|^2_{H^1(S_1(z_j))} \leq 24 C_1 \Theta_1^2 \mathrm{e}^{2\beta\psi_2} \sum_{j \in Q_L} \|F\|^2_{H^1(S_3(z_j))} + 2C_1 \sum_{j \in Q_L} \|(\partial_{d+1} F)_0\|^2_{L^2(B(z_j,\delta))}.$$

Recall that $U_i(L) = \bigcup_{j \in Q_L} S_i(z_j)$ and $W_\delta(L) = \bigcup_{j \in Q_L} B(z_j, \delta)$. Hence, for all $\beta \geq \beta_1$ we have

$$\|F\|^2_{H^1(U_1(L))} \leq \widetilde{D}_1 \|F\|^2_{H^1(U_3(L))} + \widehat{D}_1 \|(\partial_{d+1} F)_0\|^2_{L^2(W_\delta(L))},$$

where

$$\widetilde{D}_1(\beta) = 24 C_1 \Theta_1^2 \mathrm{e}^{2\beta(\psi_2 - \psi_1)} \quad \text{and} \quad \widehat{D}_1(\beta) = 2C_1 \mathrm{e}^{-2\beta\psi_1}. \tag{17}$$

We choose β such that

$$\mathrm{e}^{\beta} = \left[\frac{1}{12\Theta_1^2} \frac{\|(\partial_{d+1} F)_0\|^2_{L^2(W_\delta(L)}}{\|F\|^2_{H^1(U_3(L))}}\right]^{1/(2\psi_2)}. \tag{18}$$

Now we distinguish two cases. If $\beta \geq \beta_1$ we obtain by using $\psi_1 = 2\psi_2$

$$\|F\|^2_{H^1(U_1(L))} \leq 8\sqrt{3} C_1 \Theta_1 \|F\|_{H^1(U_3(L))} \|(\partial_{d+1} F)_0\|_{L^2(W_\delta(L))}. \tag{19}$$

If $\beta < \beta_1$ we use Lemma 5.2 of [Le Rousseau and Lebeau 2012]. In particular, one concludes from (18) that

$$\|F\|^2_{H^1(U_3(L))} < \frac{1}{12\Theta_1^2} \mathrm{e}^{-2\beta_1\psi_2} \|(\partial_{d+1} F)_0\|^2_{L^2(W_\delta(L))}.$$

This gives us in the case $\beta < \beta_1$

$$\|F\|^2_{H^1(U_1(L))} \leq \|F\|^2_{H^1(U_3(L))} < \frac{\mathrm{e}^{-\beta_1\psi_2}}{\sqrt{12}\Theta_1} \|F\|_{H^1(U_3(L))} \|(\partial_{d+1} F)_0\|_{L^2(W_\delta(L))}. \tag{20}$$

If we set

$$D_1^2 = \max\left\{8\sqrt{3} C_1 \Theta_1, \frac{\mathrm{e}^{-\beta_1\psi_2}}{\Theta_1\sqrt{12}}\right\}, \tag{21}$$

we conclude the statement of the proposition from inequalities(19) and (20). □

Now we deduce from the second Carleman estimate, Proposition 3.2, another interpolation inequality.

Proposition 3.5. *For all $\delta \in \left(0, \frac{1}{2}\right)$, all $(1, \delta)$-equidistributed sequences z_j, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $L \in \mathbb{N}_{\mathrm{odd}}$, all $E \geq 0$ and all $\phi \in \operatorname{Ran}(\chi_{(-\infty,E]}(H_L))$:*

(a) *There is $\alpha_1 = \alpha_1(d, \|V\|_\infty) \geq 1$ such that for all $\alpha \geq \alpha_1$ we have*

$$\|F\|^2_{H^1(X_1)} \leq \widetilde{D}_2(\alpha)\|F\|^2_{H^1(U_1(L))} + \widehat{D}_2(\alpha)\|F\|_{H^1(\widetilde{X}_{R_3})},$$

where α_1 is given in (23), *and $\widetilde{D}_2(\alpha)$ and $\widehat{D}_2(\alpha)$ are given in* (27).

(b) *We have*

$$\|F\|_{H^1(X_1)} \leq D_2\|F\|^{\gamma}_{H^1(U_1(L))}\|F\|^{1-\gamma}_{H^1(\widetilde{X}_{R_3})},$$

where γ and D_2 are given in (32) *and* (33).

Proof. We choose a cutoff function $\chi \in C_c^\infty(\mathbb{R}^{d+1}; [0, 1])$ with $\operatorname{supp}\chi \subset B(R_3) \setminus \overline{B(r_1)}$, $\chi(x) = 1$ if $x \in B(r_3) \setminus \overline{B(R_1)}$,

$$\max\{\|\Delta\chi\|_{\infty,V_1}, \||\nabla\chi|\|_{\infty,V_1}\} \leq \frac{\widetilde{\Theta}_2}{\delta^4} =: \Theta_2,$$
$$\max\{\|\Delta\chi\|_{\infty,V_3}, \||\nabla\chi|\|_{\infty,V_3}\} \leq \Theta_3,$$

where $\widetilde{\Theta}_2$ depends only on the dimension and Θ_3 is an absolute constant. We set $u = \chi F$. We apply Proposition 3.2 with $\rho = R_3$ to the function u and obtain for all $\alpha \geq \alpha_0 \geq 1$

$$\int_{B(R_3)} \left(\alpha R_3^2 w^{1-2\alpha}|\nabla u|^2 + \alpha^3 w^{-1-2\alpha}|u|^2\right) \mathrm{d}x \leq C_2 R_3^4 \int_{B(R_3)} w^{2-2\alpha}|\Delta u|^2 \,\mathrm{d}x.$$

Since $w \leq 1$ on $B(R_3)$ we can replace the exponent of the weight function w at all three places by $2-2\alpha$; i.e.,

$$\int_{B(R_3)} \left(\alpha R_3^2 w^{2-2\alpha}|\nabla u|^2 + \alpha^3 w^{2-2\alpha}|u|^2\right) \mathrm{d}x \leq C_2 R_3^4 \int_{B(R_3)} w^{2-2\alpha}|\Delta u|^2 \,\mathrm{d}x =: I. \tag{22}$$

For the right-hand side we use

$$\Delta u = 2(\nabla\chi)(\nabla F) + (\Delta\chi)F + (\Delta F)\chi,$$

and $\Delta F = V_L F$, and obtain

$$I \leq 3C_2 R_3^4 \int_{B(R_3)} w^{2-2\alpha}\left(4|(\nabla\chi)(\nabla F)|^2 + |(\Delta\chi)F|^2 + \|V\|^2_\infty|\chi F|^2\right) \mathrm{d}x =: I_1 + I_2 + I_3.$$

If we choose α sufficiently large, i.e.,

$$\alpha \geq (6C_2 R_3^4\|V\|^2_\infty)^{1/3} =: \tilde{\alpha}_0,$$

we can subsume the term I_3 into the left-hand side of (22). We obtain for all

$$\alpha \geq \alpha_1 := \max\{\alpha_0, \tilde{\alpha}_0\} \tag{23}$$

the estimate

$$\int_{B(R_3)} \left(\alpha R_3^2 w^{2-2\alpha}|\nabla u|^2 + \tfrac{1}{2}\alpha^3 w^{2-2\alpha}|u|^2\right) \mathrm{d}x \leq I_1 + I_2.$$

For the "new" left-hand side we have the lower bound

$$I_1 + I_2 \geq \int_{B(R_3)} \bigl(\alpha R_3^2 w^{2-2\alpha} |\nabla u|^2 + \tfrac{1}{2}\alpha^3 w^{2-2\alpha}|u|^2\bigr)\,\mathrm{d}x \geq \frac{1}{2}\left(\frac{R_3}{R_2}\right)^{2\alpha-2} \|F\|^2_{H^1(V_2)}.$$

For I_1 and I_2 we have the estimates

$$I_1 \leq 3C_2R_3^4\biggl[4\Theta_2^2\left(\frac{\mathrm{e}R_3}{r_1}\right)^{2\alpha-2}\int_{V_1}|\nabla F|^2 + 4\Theta_3^2\left(\frac{\mathrm{e}R_3}{r_3}\right)^{2\alpha-2}\int_{V_3}|\nabla F|^2\biggr],$$

$$I_2 \leq 3C_2R_3^4\biggl[\Theta_2^2\left(\frac{\mathrm{e}R_3}{r_1}\right)^{2\alpha-2}\int_{V_1}|F|^2 + \Theta_3^2\left(\frac{\mathrm{e}R_3}{r_3}\right)^{2\alpha-2}\int_{V_3}|F|^2\biggr].$$

Putting everything together, the Carleman estimate from Proposition 3.2 implies for $\alpha \geq \alpha_1$

$$\|F\|^2_{H^1(V_2)} \leq 24C_2R_3^4\biggl[\Theta_2^2\left(\frac{\mathrm{e}R_2}{r_1}\right)^{2\alpha-2}\|F\|^2_{H^1(V_1)} + \Theta_3^2\left(\frac{\mathrm{e}R_2}{r_3}\right)^{2\alpha-2}\|F\|^2_{H^1(V_3)}\biggr]. \tag{24}$$

By translation, (24) is still true if we replace V_1, V_2 and V_3 by its translates $V_1(z_j)$, $V_2(z_j)$ and $V_3(z_j)$ for all $j \in Q_L$. Hence,

$$\sum_{j\in Q_L}\|F\|^2_{H^1(V_2(z_j))} \leq 24C_2R_3^4\biggl[\Theta_2^2\left(\frac{\mathrm{e}R_2}{r_1}\right)^{2\alpha-2}\sum_{j\in Q_L}\|F\|^2_{H^1(V_1(z_j))} + \Theta_3^2\left(\frac{\mathrm{e}R_2}{r_3}\right)^{2\alpha-2}\sum_{j\in Q_L}\|F\|^2_{H^1(V_3(z_j))}\biggr]. \tag{25}$$

For all $L \in \mathbb{N}_{\mathrm{odd}}$ Lemma 3.3 tells us that $\bigcup_{k\in Q_5}\bigcup_{j\in Q_L} V_2(z_j+kL) \supset X_1 = \Lambda_L \times [-1, 1]$ and the left-hand side is bounded from below by

$$\sum_{j\in Q_L}\|F\|^2_{H^1(V_2(z_j))} = \frac{1}{5^d}\sum_{k\in Q_5}\sum_{j\in Q_L}\|F\|^2_{H^1(V_2(z_j+kL))} \geq \frac{1}{5^d}\|F\|^2_{H^1(X_1)}.$$

Since $V_1(z_j)\cap\mathbb{R}^{d+1}_+ \subset S_1(z_j)$, $S_1(z_i)\cap S_1(z_j) = \varnothing$ for $i \neq j$, and since F is antisymmetric with respect to its last coordinate, we have

$$\sum_{j\in Q_L}\|F\|^2_{H^1(V_1(z_j))} \leq 2\sum_{j\in Q_L}\|F\|^2_{H^1(S_1(z_j))} = 2\|F\|^2_{H^1(U_1(L))}.$$

For the second summand on the right-hand side of (25), we find by Lemma 3.3(iii) that there exists a constant K_d such that

$$\sum_{j\in Q_L}\|F\|^2_{H^1(V_3(z_j))} \leq K_d\|F\|^2_{H^1(\bigcup_{j\in Q_L} V_3(z_j))}.$$

Moreover, since $\bigcup_{j\in Q_L} V_3(z_j) \subset \widetilde{X}_{R_3} = \Lambda_{L+R_3}\times[-R_3, R_3]$, we have

$$\sum_{j\in Q_L}\|F\|^2_{H^1(V_3(z_j))} \leq K_d\|F\|_{H^1(\widetilde{X}_{R_3})}.$$

Putting everything together we obtain for all $\alpha \geq \alpha_1$

$$\frac{1}{5^d}\|F\|^2_{H^1(X_1)} \leq \widetilde{D}_2(\alpha)\|F\|^2_{H^1(U_1(L))} + \widehat{D}_2(\alpha)\|F\|_{H^1(\widetilde{X}_{R_3})}, \tag{26}$$

where

$$\widetilde{D}_2(\alpha) = 48C_2R_3^4\Theta_2^2\left(\frac{\mathrm{e}R_2}{r_1}\right)^{2\alpha-2} \quad \text{and} \quad \widehat{D}_2(\alpha) = 24C_2R_3^4\Theta_3^2K_d\left(\frac{\mathrm{e}R_2}{r_3}\right)^{2\alpha-2}. \tag{27}$$

If we let $c_1 = 48C_2\Theta_2^2R_3^4r_1^2/(\mathrm{e}R_2)^2,\ c_2 = 24C_2\Theta_3^2K_dR_3^4r_3^2/(\mathrm{e}R_2)^2$,

$$p^+ = 2\ln\left(\frac{\mathrm{e}R_2}{r_1}\right) > 0 \quad \text{and} \quad p^- = 2\ln\left(\frac{\mathrm{e}R_2}{r_3}\right) < 0,$$

then (26) reads as

$$\frac{1}{5^d}\|F\|^2_{H^1(X_1)} \le c_1\mathrm{e}^{p^+\alpha}\|F\|^2_{H^1(U_1(L))} + c_2\mathrm{e}^{p^-\alpha}\|F\|^2_{H^1(\widetilde{X}_{R_3})}. \tag{28}$$

We choose α such that

$$\mathrm{e}^\alpha = \left(\frac{c_2}{c_1}\frac{\|F\|^2_{H^1(\widetilde{X}_{R_3})}}{\|F\|^2_{H^1(U_1(L))}}\right)^{1/(p^+-p^-)}. \tag{29}$$

If $\alpha \ge \alpha_1$ we obtain from (28) that

$$\frac{1}{5^d}\|F\|^2_{H^1(X_1)} \le 2c_1^\gamma c_2^{1-\gamma}\|F\|^{2\gamma}_{H^1(U_1(L))}\|F\|^{2-2\gamma}_{H^1(\widetilde{X}_{R_3})}, \quad \text{where} \quad \gamma = \frac{-p^-}{p^+-p^-}. \tag{30}$$

If $\alpha < \alpha_1$, we proceed as in the last part of the proof of Proposition 3.4; i.e., we conclude from (29) that

$$\|F\|^2_{H^1(\widetilde{X}_{R_3})} < \frac{c_1}{c_2}\mathrm{e}^{\alpha_1(p^+-p^-)}\|F\|^2_{H^1(U_1(L))},$$

and thus

$$\|F\|^2_{H^1(X_1)} \le \|F\|^{2(p^+-p^-)/(p^+-p^-)}_{H^1(\widetilde{X}_{R_3})} < \|F\|^{2(1-\gamma)}_{H^1(\widetilde{X}_{R_3})}\left(\frac{c_1}{c_2}\mathrm{e}^{\alpha_1(p^+-p^-)}\right)^\gamma\|F\|^{2\gamma}_{H^1(U_1(L))}. \tag{31}$$

We calculate

$$\gamma = \frac{\ln 2}{\ln(r_3/r_1)}, \tag{32}$$

set

$$D_2^2 = \max\left\{5^d\,192\cdot 9^4C_2\Theta_3^2K_d\mathrm{e}^4d^2\left(\frac{2\Theta_2^2r_1^2}{\Theta_3^2K_dr_3^2}\right)^\gamma, \left(\frac{2\Theta_2^2}{\Theta_3^2K_d}\left(\frac{r_3}{r_1}\right)^{2(\alpha_1-1)}\right)^\gamma\right\} \tag{33}$$

and conclude the statement of the proposition from (30) and (31). □

Proofs of Theorem 2.2 and Corollary 2.6.

Proposition 3.6. *For all $T > 0$, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $L \in \mathbb{N}_{\mathrm{odd}}$, all $E \ge 0$ and all $\phi \in \mathrm{Ran}(\chi_{(-\infty,E]}(H_L))$ we have*

$$\frac{T}{2}\sum_{\substack{k\in\mathbb{N}\\E_k\le E}}|\alpha_k|^2 \le \frac{\|F\|^2_{H^1(\Lambda_{RL}\times[-T,T])}}{R^d} \le 2T(1+(1+\|V\|_\infty)T^2)\sum_{\substack{k\in\mathbb{N}\\E_k\le E}}\beta_k(T)|\alpha_k|^2,$$

where

$$\beta_k(T) = \begin{cases}1 & \text{if } E_k \le 0,\\ \mathrm{e}^{2T\sqrt{E_k}} & \text{if } E_k > 0.\end{cases}$$

Proof. For the function $F : \Lambda_{RL} \times \mathbb{R} \to \mathbb{C}$ we have for $T > 0$

$$\|F\|^2_{H^1(\Lambda_{RL}\times[-T,T])} = \int_{-T}^{T}\int_{\Lambda_{RL}} \big(|\partial_{d+1}F|^2 + |\nabla' F|^2 + |F|^2\big)\,\mathrm{d}x.$$

Note that $\|\phi_k\|_{L^2(\Lambda_{RL})} = R^d$. By Green's theorem we have

$$\int_{\Lambda_{RL}} |\nabla' F|^2\,\mathrm{d}x' = \int_{\Lambda_{RL}} \Big(-\sum_{i=1}^{d} \partial_i^2 F\Big)\bar{F}\,\mathrm{d}x' = -\int_{\Lambda_{RL}} V|F|^2\,\mathrm{d}x' + \int_{\Lambda_{RL}} (\partial_{d+1}^2 F)\bar{F}\,\mathrm{d}x'$$

for all $x_{d+1} \in \mathbb{R}$. First we estimate

$$\begin{aligned}\|F\|^2_{H^1(\Lambda_{RL}\times[-T,T])} &= \int_{-T}^{T}\int_{\Lambda_{RL}} \big(|\partial_{d+1}F|^2 - V|F|^2 + (\partial_{d+1}^2 F)\bar{F} + |F|^2\big)\,\mathrm{d}x\\ &\le \int_{-T}^{T}\int_{\Lambda_{RL}} \big(|\partial_{d+1}F|^2 + (\partial_{d+1}^2 F)\bar{F} + (1+\|V\|_\infty)|F|^2\big)\,\mathrm{d}x = 2R^d \sum_{\substack{k\in\mathbb{N}\\ E_k\le E}} |\alpha_k|^2 I_k,\end{aligned}$$

where

$$\begin{aligned}I_k &:= \int_0^T \big((1+\|V\|_\infty)\,\mathrm{s}_k(x_{d+1})^2 + \mathrm{s}_k'(x_{d+1})^2 + \mathrm{s}_k''(x_{d+1})\,\mathrm{s}_k(x_{d+1})\big)\,\mathrm{d}x_{d+1}\\ &= (1+\|V\|_\infty)\int_0^T \mathrm{s}_k(x_{d+1})^2\,\mathrm{d}x_{d+1} + \mathrm{s}_k'(T)\,\mathrm{s}_k(T).\end{aligned}$$

If $E_k \le 0$, we estimate using $s_k(t) \le t$ and $s_k'(t)s_k(t) \le t$ for $t > 0$

$$I_k \le \tfrac{1}{3}(1+\|V\|_\infty)T^3 + T \le ((1+\|V\|_\infty)T^3 + T)\beta_k(T).$$

For $E_k > 0$ we use $\sinh(\omega_k t)/\omega_k \le t\cosh(\omega_k t)$ for $t > 0$ and $\cosh(\omega_k T)^2 \le \mathrm{e}^{2\omega_k T}$ to obtain

$$\begin{aligned}I_k &= (1+\|V\|_\infty)\int_0^T \frac{\sinh^2(\omega_k x_{d+1})}{\omega_k^2}\,\mathrm{d}x_{d+1} + \sinh(\omega_k T)\cosh(\omega_k T)/\omega_k\\ &\le ((1+\|V\|_\infty)T^3\cosh^2(\omega_k T) + T\cosh^2(\omega_k T)) \le ((1+\|V\|_\infty)T^3 + T)\beta_k(T).\end{aligned}$$

This shows the upper bound. For the lower bound we drop the gradient term and obtain

$$\|F\|^2_{H^1(\Lambda_{RL}\times[-T,T])} \ge \int_{-T}^{T}\int_{\Lambda_{RL}} (|\partial_{d+1}F|^2 + |F|^2)\,\mathrm{d}x = 2\cdot R^d \sum_{\substack{k\in\mathbb{N}\\ E_k\le E}} |\alpha_k|^2 \tilde{I}_k,$$

where

$$\tilde{I}_k := \int_0^T [\mathrm{s}_k(x_{d+1})^2 + \mathrm{s}_k'(x_{d+1})^2]\,\mathrm{d}x_{d+1}.$$

If $E_k = 0$, the lower bound $\tilde{I}_k \ge T$ follows immediately. Else, we have $\mathrm{s}_k(t)^2 \ge \sin^2(\omega_k t)/\omega_k$ and $\mathrm{s}_k'(t)^2 \ge \cos(\omega_k t)$, whence

$$\tilde{I}_k \ge \int_0^T \frac{\sin^2(\omega_k x_{d+1})}{\omega_k^2} + \cos^2(\omega_k x_{d+1})\,\mathrm{d}x_{d+1} \ge \int_0^T \cos^2(\omega_k x_{d+1})\,\mathrm{d}x_{d+1} = \frac{T}{2} + \frac{\sin(2\omega_k T)}{4\omega_k}.$$

Now, if $2\omega_k T < \pi$, the sinus term is positive and we drop it to find $\tilde{I}_k \geq T/2$. If $2\omega_k T \geq \pi$, we have $\sin(2\omega_k T) \geq -1$ and estimate

$$\tilde{I}_k \geq \frac{T}{2} - \frac{1}{4\omega_k} = \frac{T}{2} - \frac{\pi}{4\pi\omega_k} \geq \frac{T}{2} - \frac{T}{2\pi} \geq \frac{T}{4}. \qquad \square$$

Proof of Theorem 2.2. First we consider the case $L \in \mathbb{N}_{\text{odd}}$. We note that Proposition 3.6 remains true if we replace Λ_{RL} by Λ_L and R^d by 1; i.e., for all $T > 0$ and $L \in \mathbb{N}_{\text{odd}}$ we have

$$\frac{T}{2} \sum_{\substack{k\in\mathbb{N}\\ E_k\leq E}} |\alpha_k|^2 \leq \|F\|^2_{H^1(\Lambda_L\times[-T,T])} \leq 2T(1+(1+\|V\|_\infty)T^2) \sum_{\substack{k\in\mathbb{N}\\ E_k\leq E}} \beta_k(T)|\alpha_k|^2. \tag{34}$$

We have $\widetilde{X}_{R_3} \subset \Lambda_{RL} \times [-R_3, R_3]$. By (34) and Proposition 3.6 we have

$$\frac{\|F\|^2_{H^1(\widetilde{X}_{R_3})}}{\|F\|^2_{H^1(X_1)}} \leq \frac{\|F\|^2_{H^1(\Lambda_{RL}\times[-R_3,R_3])}}{\|F\|^2_{H^1(X_1)}} \leq \widetilde{D}_3^2 D_4^2$$

with

$$\widetilde{D}_3^2 = \frac{\sum_{E_k\leq E}\theta_k|\alpha_k|^2}{\sum_{E_k\leq E}|\alpha_k|^2} \quad \text{and} \quad D_4^2 = 4\cdot R^d R_3(1+(1+\|V\|_\infty)R_3^2),$$

where $\theta_k = \beta_k(R_3)$. We use Propositions 3.4 and 3.5 and obtain

$$\|F\|_{H^1(\widetilde{X}_{R_3})} \leq \widetilde{D}_3 D_4 \|F\|_{H^1(X_1)} \leq D_1^\gamma D_2 \widetilde{D}_3 D_4 \|F\|^{1-\gamma}_{H^1(\widetilde{X}_{R_3})} \|(\partial_{d+1}F)_0\|^{\gamma/2}_{L^2(W_\delta(L))} \|F\|^{\gamma/2}_{H^1(U_3(L))}.$$

Since $U_3(L) \subset \widetilde{X}_{R_3}$ we have

$$\|F\|_{H^1(\widetilde{X}_{R_3})} \leq D_1^2 D_2^{2/\gamma} \widetilde{D}_3^{2/\gamma} D_4^{2/\gamma} \|(\partial_{d+1}F)_0\|_{L^2(W_\delta(L))}.$$

By (34), the square of the left-hand side is bounded from below by

$$\|F\|^2_{H^1(\widetilde{X}_{R_3})} \geq \|F\|^2_{H^1(\Lambda_L\times[-R_3,R_3])} \geq \tfrac{1}{2}R_3 \sum_{\substack{k\in\mathbb{N}\\ E_k\leq E}} |\alpha_k|^2.$$

Putting everything together we obtain by using $(\partial_{d+1}F)_0 = \phi$

$$\tfrac{1}{2}R_3 \sum_{\substack{k\in\mathbb{N}\\ E_k\leq E}} |\alpha_k|^2 \leq D_1^4 (D_2 \widetilde{D}_3 D_4)^{4/\gamma} \|\phi\|^2_{L^2(W_\delta(L))}.$$

In order to end the proof we will give an upper bound on $\widetilde{D}_3$ which is independent of α_k, $k \in \mathbb{N}$. For this purpose, we recall that $\theta_k = \beta_k(R_3)$. Since $\theta_k \leq \mathrm{e}^{2R_3\sqrt{E}}$ for all $k \in \mathbb{N}$ with $E_k \leq E$, we have

$$\widetilde{D}_3^4 \leq D_3^4 := \mathrm{e}^{4R_3\sqrt{E}}.$$

Hence, using $\sum_{E_k\leq E}|\alpha_k|^2 = \|\phi\|^2_{L^2(\Lambda_L)}$, we obtain for all $L \in \mathbb{N}_{\text{odd}}$ the estimate

$$\widetilde{C}_{\text{sfuc}} \|\phi\|^2_{L^2(\Lambda_L)} \leq \|\phi\|^2_{L^2(W_\delta(L))},$$

where $\widetilde{C}_{\mathrm{sfuc}} = \widetilde{C}_{\mathrm{sfuc}}(d, \delta, E, \|V\|_\infty) = D_1^{-4}(D_2 D_3 D_4)^{-4/\gamma}$. From the definitions of D_i, $i \in \{1, 2, 3, 4\}$, and γ one calculates that

$$\widetilde{C}_{\mathrm{sfuc}} \geq \delta^{\widetilde{N}\left(1+\|V\|_\infty^{2/3}+\sqrt{E}\right)}$$

with some constant $\widetilde{N} = \widetilde{N}(d)$. Now we treat the case of $L \in \mathbb{N}_{\mathrm{even}} = \{2, 4, 6, \dots\}$. By a scaling argument as in Corollary 2.2 of [Rojas-Molina and Veselić 2013], we immediately obtain that for all $G > 0$, $\delta \in (0, G/2)$, $L/G \in \mathbb{N}_{\mathrm{odd}}$ and all (G, δ)-equidistributed sequences q_j we have

$$\|\phi\|^2_{L^2(W^q_\delta(L))} \geq \widetilde{C}^G_{\mathrm{sfuc}} \|\phi\|^2_{L^2(\Lambda_L)} \tag{35}$$

and $\widetilde{C}^G_{\mathrm{sfuc}}(d, \delta, E, \|V\|_\infty) = \widetilde{C}_{\mathrm{sfuc}}(d, \delta/G, EG^2, \|V\|_\infty G^2)$. Here $W^q_\delta(L)$ denotes the set $W_\delta(L)$ corresponding to the sequence q_j. Now we define

$$G = \begin{cases} L/(L/2-1) & \text{if } L \in 4\mathbb{N}, \\ 2 & \text{otherwise}, \end{cases}$$

which satisfies $G \in [2, 4]$ and $L/G \in \mathbb{N}_{\mathrm{odd}}$. Since $G \geq 2$, every elementary cell $\Lambda_G + j$, $j \in (G\mathbb{Z})^d$, contains at least one elementary cell $\Lambda_1 + j$, $j \in \mathbb{Z}^d$. Hence we can choose a (G, δ)-equidistributed subsequence q_j of z_j. We apply (35) to this subsequence and obtain

$$\|\phi\|^2_{L^2(W_\delta(L))} \geq \|\phi\|^2_{L^2(W^q_\delta(L))} \geq \widetilde{C}^G_{\mathrm{sfuc}} \|\phi\|_{L^2(\Lambda_L)}.$$

Note that $W_\delta(L)$ corresponds to the sequence z_j. Putting everything together we obtain the statement of the theorem with

$$\min\bigl\{\widetilde{C}_{\mathrm{sfuc}}, \inf_{G \in [2,4]} \widetilde{C}^G_{\mathrm{sfuc}}\bigr\} \geq \delta^{N\left(1+\|V\|_\infty^{2/3}+\sqrt{E}\right)} =: C_{\mathrm{sfuc}}$$

and some constant $N = N(d)$. For the last inequality we use that $\left(\frac{1}{4}\right)^{\widetilde{N}} \geq \delta^{2\widetilde{N}}$. □

Proof of Corollary 2.6. We denote the normalized eigenfunctions of $-\Delta_L + A + B$ corresponding to the eigenvalues $\lambda_i(-\Delta_L + A + B)$ by ϕ_i. Then we have

$$\begin{aligned} \lambda_i(-\Delta_L + A + B) &= \langle \phi_i, (-\Delta_L + A + B)\phi_i \rangle \\ &= \max_{\substack{\phi \in \mathrm{Span}\{\phi_1, \dots, \phi_i\} \\ \|\phi\|=1}} \langle \phi, (-\Delta_L + A)\phi \rangle + \langle \phi, B\phi \rangle \\ &\geq \max_{\substack{\phi \in \mathrm{Span}\{\phi_1, \dots, \phi_i\} \\ \|\phi\|=1}} \langle \phi, (-\Delta_L + A)\phi \rangle + \alpha \langle \phi, \chi_{W_\delta(L)}\phi \rangle. \end{aligned}$$

By Corollary 2.4, we conclude that for all $\phi \in \mathrm{Span}\{\phi_1, \dots, \phi_i\}$, $\|\phi\| = 1$, we have

$$\langle \phi, \chi_{W_\delta(L)}\phi \rangle \geq C^{G,1}_{\mathrm{sfuc}}(d, \delta, E, \|A + B\|_\infty)$$

and furthermore, by the variational characterization of eigenvalues, we find

$$\max_{\substack{\phi \in \mathrm{Span}\{\phi_1, \dots, \phi_i\} \\ \|\phi\|=1}} \langle \phi, (-\Delta_L + A)\phi \rangle \geq \inf_{\dim \mathcal{D} = i} \max_{\substack{\phi \in \mathcal{D} \\ \|\phi\|=1}} \langle \phi, (-\Delta_L + A)\phi \rangle = \lambda_i(-\Delta_L + A).$$

Thus, we obtain the statement of the corollary. □

4. Proof of Wegner and initial scale estimates

Recall that $0 < G_1 < G_2$ are the numbers from the Delone property such that $\sharp\{\mathcal{D} \cap (\Lambda_{G_1} + x)\} \le 1$, $\sharp\{\mathcal{D} \cap (\Lambda_{G_2} + x)\} \ge 1$ for any $x \in \mathbb{R}^d$, and that for all $t \in [0, 1]$ we have $\operatorname{supp} u_t \subset \Lambda_{G_u}$. Let $\delta_{\max} := 1 - \omega_+$ and $K_u := u_{\max} \lceil G_u/G_1 \rceil^d$. For $\omega \in [\omega_-, \omega_+]^{\mathcal{D}}$ and $\delta \le \delta_{\max}$, we use the notation $V_{\omega+\delta}$ for the potential V_ω, where every ω_j, $j \in \mathcal{D}$, has been replaced by $\omega_j + \delta$. The following lemma is a consequence of the properties of a Delone set, in particular $\sharp \Lambda_L \cap \mathcal{D} \le \lceil L/G_1 \rceil^d$, and our assumption (5).

Lemma 4.1. (i) *For all $\omega \in [\omega_-, \omega_+]^{\mathcal{D}}$, all $0 < \delta \le \delta_{\max}$ and all $L \in (G_2 + G_u)\mathbb{N}$, the difference $V_{\omega+\delta} - V_\omega$ is on Λ_L bounded from below by $\alpha_1 \delta^{\alpha_2}$ times the characteristic function of $W_{\beta_1 \delta^{\beta_2}}(L)$ which corresponds to a $(G_2 + G_u, \beta_1 \delta^{\beta_2})$-equidistributed sequence.*

(ii) *For all $\omega \in [0, 1]^{\mathcal{D}}$ we have $\|V_\omega\|_\infty \le K_u$.*

(iii) *For all $L \in (G_2 + G_u)\mathbb{N}$, we have*

$$\sharp\big\{ j \in \mathcal{D} : \exists t \in [0, 1],\ \operatorname{supp} u_t(\cdot - j) \cap \Lambda_L \ne \varnothing \big\} \le \lceil (L + G_u)/G_1 \rceil^d \le (2L/G_1)^d.$$

Proof of Theorem 2.10. Note that for all $E_0 \in \mathbb{R}$, $\lambda_i(H_{\omega,L}) \le E_0$ implies, by Lemma 4.1(ii), that $\lambda_i(H_{\omega+\delta,L}) \le E_0 + \|V_{\omega+\delta} - V_\omega\| \le E_0 + 2K_u$. Now we apply Corollary 2.6 with $A = V_\omega$ and $B = V_{\omega+\delta} - V_\omega$ (both restricted to Λ_L). Together with Lemma 4.1(i), we obtain for all $E_0 \in \mathbb{R}$, all $L \in G_u\mathbb{N}$, all $\omega \in [\omega_-, \omega_+]^{\mathcal{D}}$, all $\delta \le \delta_{\max}$ and all $i \in \mathbb{N}$ with $\lambda_i(H_{\omega,L}) \le E_0$ the inequality

$$\lambda_i(H_{\omega+\delta,L}) \ge \lambda_i(H_{\omega,L}) + \alpha_1 \delta^{\alpha_2} C_{\text{sfuc}}^{G_2+G_u,1}(d, \beta_1 \delta^{\beta_2}, E_0 + 2K_u, K_u).$$

In particular, there is $\kappa = \kappa(d, \omega_+, \alpha_1, \alpha_2, \beta_1, \beta_2, G_2, G_u, K_u, E_0) > 0$ such that

$$\lambda_i(H_{\omega+\delta,L}) \ge \lambda_i(H_{\omega,L}) + \delta^\kappa. \tag{36}$$

Now let $\varepsilon > 0$, satisfying $\varepsilon \le \varepsilon_{\max} := \delta_{\max}^\kappa/4$. We choose $\delta := (4\varepsilon)^{1/\kappa}$, whence

$$\lambda_i(H_{\omega+\delta,L}) \ge \lambda_i(H_{\omega,L}) + 4\varepsilon. \tag{37}$$

Let $\rho \in C^\infty(\mathbb{R}, [-1, 0])$ be smooth, nondecreasing such that $\rho = -1$ on $(-\infty; -\varepsilon]$ and $\rho = 0$ on $[\varepsilon; \infty)$. We can assume $\|\rho'\|_\infty \le 1/\varepsilon$. It holds that

$$\chi_{[E-\varepsilon; E+\varepsilon]}(x) \le \rho(x - E + 2\varepsilon) - \rho(x - E - 2\varepsilon) = \rho(x - E - 2\varepsilon + 4\varepsilon) - \rho(x - E - 2\varepsilon)$$

for all $x \in \mathbb{R}$ and together with (37) this implies

$$\begin{aligned} \mathbb{E}\big[\operatorname{Tr}[\chi_{[E-\varepsilon; E+\varepsilon]}(H_{\omega,L})]\big] &\le \mathbb{E}\big[\operatorname{Tr}[\rho(H_{\omega,L} - E - 2\varepsilon + 4\varepsilon) - \rho(H_{\omega,L} - E - 2\varepsilon)]\big] \\ &\le \mathbb{E}\big[\operatorname{Tr}[\rho(H_{\omega+\delta,L} - E - 2\varepsilon) - \rho(H_{\omega,L} - E - 2\varepsilon)]\big]. \end{aligned} \tag{38}$$

Now let $\tilde{\Lambda}_L := \{ j \in \mathcal{D} : \exists t \in [0, 1],\ \operatorname{supp} u_t(\cdot - j) \cap \Lambda_L \ne \varnothing\}$ be the set of lattice sites which can influence the potential within Λ_L. Note that $\sharp\tilde{\Lambda}_L \le (2L/G_1)^d$. We enumerate the points in $\tilde{\Lambda}_L$ by $k : \{1, \dots, \sharp\tilde{\Lambda}_L\} \to \mathcal{D}$, $n \mapsto k(n)$. The upper bound in (38) will be expanded in a telescopic sum by changing the $|\tilde{\Lambda}_L|$ indices from ω_j to $\omega_j + \delta$ successively. In order to do that some notation is needed.

Given $\omega \in [\omega_-, \omega_+]^{\mathcal{D}}$, $n \in \{1, \dots, |\tilde{\Lambda}_L|\}$, $\delta \in [0, \delta_{\max}]$ and $t \in [\omega_-, \omega_+]$, we define $\tilde{\omega}^{(n,\delta)}(t) \in [\omega_-, 1]^{\mathcal{D}}$ inductively via

$$(\tilde{\omega}^{(1,\delta)}(t))_j := \begin{cases} t & \text{if } j = k(1), \\ \omega_j & \text{else} \end{cases} \quad \text{and} \quad (\tilde{\omega}^{(n,\delta)}(t))_j := \begin{cases} t & \text{if } j = k(n), \\ (\tilde{\omega}^{(n-1,\delta)}(\omega_j + \delta))_j & \text{else.} \end{cases}$$

The function $\tilde{\omega}^{(n,\delta)} : [\omega_-, 1] \to [\omega_-, 1]^{\mathcal{D}}$ is the rank-1 perturbation of ω in the $k(n)$-th coordinate with the additional requirement that all sites $k(1), \dots, k(n-1)$ have already been blown up by δ. We define

$$\Theta_n(t) := \operatorname{Tr}[\rho(H_{\tilde{\omega}^{(n,\delta)}(t),L} - E - 2\varepsilon)] \quad \text{for } n = 1, \dots, |\tilde{\Lambda}_L|.$$

Note that

$$\begin{aligned} \Theta_1(\omega_{k(1)}) &= \operatorname{Tr}[\rho(H_{\omega,L} - E - 2\varepsilon)], \\ \Theta_n(\omega_{k(n)}) &= \Theta_{n-1}(\omega_{k(n-1)} + \delta) \quad \text{for } n = 2, \dots, |\tilde{\Lambda}_L| \quad \text{and} \\ \Theta_{|\tilde{\Lambda}_L|}(\omega_{k(|\tilde{\Lambda}_L|)} + \delta) &= \operatorname{Tr}[\rho(H_{\omega+\delta,L} - E - 2\varepsilon)]. \end{aligned}$$

Hence the upper bound in (38) is

$$\begin{aligned} \mathbb{E}\big[\operatorname{Tr}[\rho(H_{\omega+\delta,L} - E - 2\varepsilon)] - \operatorname{Tr}[\rho(H_{\omega,L} - E - 2\varepsilon]\big] &= \mathbb{E}\big[\Theta_{|\tilde{\Lambda}_L|}(\omega_{k(|\tilde{\Lambda}_L|)} + \delta) - \Theta_1(\omega_{k(1)})\big] \\ &= \sum_{n=1}^{|\tilde{\Lambda}_L|} \mathbb{E}\big[\Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)})\big]. \end{aligned}$$

Due to the product structure of the probability space, we can apply Fubini's theorem to each summand and obtain

$$\mathbb{E}\big[\Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)})\big] = \mathbb{E}\bigg[\int_{\omega_-}^{\omega_+} \Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)}) \,\mathrm{d}\mu(\omega_{k(n)})\bigg].$$

Note that $\Theta_n : [\omega_-, 1] \to \mathbb{R}$ is monotone and bounded. We will use the following lemma.

Lemma 4.2. *Let $-\infty < \omega_- < \omega_+ \leq +\infty$. Assume that μ is a probability distribution with bounded density ν_μ and support in the interval $[\omega_-, \omega_+]$ and let Θ be a nondecreasing, bounded function. Then for all $\delta > 0$*

$$\int_{\mathbb{R}} [\Theta(\lambda + \delta) - \Theta(\lambda)] \,\mathrm{d}\mu(\lambda) \leq \|\nu_\mu\|_\infty \cdot \delta[\Theta(\omega_+ + \delta) - \Theta(\omega_-)].$$

Proof of Lemma 4.2. We calculate

$$\begin{aligned} &\int_{\mathbb{R}} [\Theta(\lambda + \delta) - \Theta(\lambda)] \,\mathrm{d}\mu(\lambda) \\ &\qquad \leq \|\nu_\mu\|_\infty \int_{\omega_-}^{\omega_+} [\Theta(\lambda + \delta) - \Theta(\lambda)] \,\mathrm{d}\lambda = \|\nu_\mu\|_\infty \bigg[\int_{\omega_- + \delta}^{\omega_+ + \delta} \Theta(\lambda) \,\mathrm{d}\lambda - \int_{\omega_-}^{\omega_+} \Theta(\lambda) \,\mathrm{d}\lambda\bigg] \\ &\qquad = \|\nu_\mu\|_\infty \bigg[\int_{\omega_+}^{\omega_+ + \delta} \Theta(\lambda) \,\mathrm{d}\lambda - \int_{\omega_-}^{\omega_- + \delta} \Theta(\lambda) \,\mathrm{d}\lambda\bigg] \leq \|\nu_\mu\|_\infty \cdot \delta[\Theta(\omega_+ + \delta) - \Theta(\omega_-)]. \quad \square \end{aligned}$$

Thus, we find for all $n = 1, \dots, |\tilde{\Lambda}_L|$

$$\int_{\omega_-}^{\omega_+} \big[\Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)})\,\mathrm{d}\mu(\omega_{k(n)})\big] \le \|\nu_\mu\|_\infty \cdot \delta[\Theta_n(\omega_+ + \delta) - \Theta_n(\omega_-)].$$

We will also need the following result; see, e.g., Theorem 2 in [Hundertmark et al. 2006].

Proposition 4.3. *Let $H_0 := -\Delta + A$ be a Schrödinger operator with a bounded potential $A \ge 0$, and let $H_1 := H_0 + B$ for some bounded $B \ge 0$ with compact support. Denote the corresponding Dirichlet restrictions to Λ by H_0^Λ and H_1^Λ, respectively. There are constants K_1, K_2 depending only on d and monotonously on* diam supp B *such that for any smooth, bounded function $g : \mathbb{R} \to \mathbb{R}$ with compact support in $(-\infty, E_0]$ and the property that $g(H_1^\Lambda) - g(H_0^\Lambda)$ is trace class we have*

$$\operatorname{Tr}[g(H_1^\Lambda) - g(H_0^\Lambda)] \le K_1 \mathrm{e}^{E_0} + K_2 (\ln(1 + \|g'\|_\infty)^d)\|g'\|_1.$$

Proposition 4.3 implies:

Lemma 4.4. *Let $0 < \varepsilon \le \varepsilon_{\max}$. Then $\Theta_n(\omega_+ + \delta) - \Theta_n(\omega_-) \le (K_1 \mathrm{e}^{E_0} + 2^d K_2)|\ln \varepsilon|^d$, where K_1, K_2 are as in Proposition 4.3 and thus only depend on d and on G_u.*

Proof of Lemma 4.4. Let $g(\,\cdot\,) := \rho(\,\cdot\, - E - 2\varepsilon))$. By our choice of ρ, we know g has support in $(-\infty, E_0]$, $\|g'\|_\infty \le 1/\varepsilon$ and $\|g'\|_1 = 1$. We define the operators

$$H_0^\Lambda := H(\tilde{\omega}^{(n,\delta)}(\omega_-), L) \quad \text{and} \quad H_1^\Lambda := H(\tilde{\omega}^{(n,\delta)}(\omega_+ + \delta), L).$$

They are lower semibounded operators with purely discrete spectrum and since g has support in $(-\infty, E_0]$, the difference $g(H_1^\Lambda) - g(H_0^\Lambda)$ is trace class. By the previous proposition

$$\Theta_n(\omega_+ + \delta) - \Theta_n(\omega_-) = \operatorname{Tr}[g(H_1^\Lambda) - g(H_0^\Lambda)] \le K_1 \mathrm{e}^{E_0} + K_2(\ln(1 + 1/\varepsilon))^d.$$

To conclude, note that $\varepsilon \le \varepsilon_{\max} < \frac{1}{2}$ and thus $\ln(1 + 1/\varepsilon) \le 2|\ln \varepsilon|$ and $1 \le |\ln \varepsilon| \le |\ln \varepsilon|^d$. ☐

Putting everything together and recalling $\delta = (4\varepsilon)^{1/\kappa}$ we find

$$\begin{aligned}\mathbb{E}\big[\operatorname{Tr}[\chi_{[E-\varepsilon, E+\varepsilon]}(H_{\omega,L})]\big] &\le (K_1 \mathrm{e}^{E_0} + 2^d K_2)\|\nu_\mu\|_\infty \cdot \delta |\ln \varepsilon|^d |\tilde{\Lambda}_L| \\ &\le (K_1 \mathrm{e}^{E_0} + 2^d K_2)\|\nu_\mu\|_\infty \cdot (4\varepsilon)^{1/\kappa} |\ln \varepsilon|^d (2/G_1)^d L^d.\end{aligned}$$ ☐

Proof of Theorem 2.11. We follow the ideas developed in [Barbaroux et al. 1997; Kirsch et al. 1998]. Let $t \le \delta_{\max}$, $V_{t,L}$ be the restriction of V_ω to Λ_L obtained by setting all random variables to t, and $H_{t,L} = -\Delta_{\Lambda_L} + V_{L,t}$ on $L^2(\Lambda_L)$ with Dirichlet boundary conditions. Note that $H_{0,L} = -\Delta_{\Lambda_L} + V_{0,L}$ and that the first eigenvalue of $H_{t,L}$ is bounded from above by $d(\pi/L)^2 + K_u$. Inequality (36) with $E_0 = d\pi^2 + K_u$, $\omega_k = 0$, $k \in \mathcal{D}$, and $\delta = t$ yields that there is $\kappa = \kappa(d, \delta_{\max}, \alpha_1, \alpha_2, \beta_1, \beta_2, G_2, G_u, K_u)$ such that for all $t \le \delta_{\max}$

$$\lambda_1(H_{t,L}) \ge \lambda_1(H_{0,L}) + t^\kappa.$$

We choose $t = L^{-7/(4\kappa)}$ and L sufficiently large such that $t < \min\{\delta_{\max}, t_0\}$. Then,

$$\lambda_1(H_{t,L}) - \lambda_1(H_{0,L}) \ge L^{-7/4}.$$

Let $\Omega_0 := \{\omega \in \Omega : \lambda_1(H_{\omega,L}) \geq \lambda_1(H_{t,L})\}$. Since the potential values in Λ_L only depend on ω_k, $k \in \Lambda_{L+G_u} \cap \mathcal{D}$, we calculate using $\sharp \Lambda_{L+G_u} \cap \mathcal{D} \leq \lceil (L+G_u)/G_1 \rceil^d$ and our assumption on the measure μ that

$$\mathbb{P}(\Omega_0) \geq 1 - \mathbb{P}(\exists \gamma \in \Lambda_{L+G_u} \cap \mathcal{D},\ \omega_\gamma \leq t) \geq 1 - \left\lceil \frac{L+G_u}{G_1} \right\rceil^d \mu([0,t]) \geq 1 - \left\lceil \frac{L+G_u}{G_1} \right\rceil^d \frac{C}{L^{7d/4}}.$$

Since $\lceil (L+G_u)/G_1 \rceil^d \leq L^{5d/4}$ for L sufficiently large, we obtain the statement of the theorem. □

5. Proof of the observability estimate

We want to apply [Miller 2010, Theorem 2.2] where we choose $A = \Delta_L - V_L$ on $L^2(\Lambda_L)$ with Dirichlet boundary conditions, $C = \chi_{W_\delta(L)}$ and $C_0 = \mathrm{Id}$. Note that A is self-adjoint with spectrum contained in $(-\infty, \|V\|_\infty]$. For $\lambda > 0$ we define the increasing sequence of spectral subspaces $\mathcal{E}_\lambda := \operatorname{Ran} \chi_{[-\lambda,\infty)}(\Delta_L - V_L)$.

We need to check [Miller 2010, (5),(6),(7)]. By spectral calculus, we have for all $\lambda > 0$

$$\|\mathrm{e}^{(\Delta_L - V_L)t} u\|_{\Lambda_L} \leq \mathrm{e}^{-\lambda t} \|u\|_{\Lambda_L}, \quad u \in \mathcal{E}_\lambda^\perp = \operatorname{Ran} \chi_{(-\infty,-\lambda)}(\Delta_L - V_L), \quad t > 0.$$

Furthermore, Corollary 2.4 implies for all $\lambda > 0$ and $u \in \mathcal{E}_\lambda$

$$\|u\|_{\Lambda_L}^2 \leq a_0 \mathrm{e}^{-N \ln(\delta/G) G \sqrt{\lambda}} \|u\|_{W_\delta(L)}^2.$$

For $T \leq 1$ we have $\mathrm{e}^{2T\|V\|_\infty}/T \leq \mathrm{e}^{2\|V\|_\infty} \mathrm{e}^{2/T}$, whence

$$\|\mathrm{e}^{T(\Delta - V)} u\|_{\Lambda_L}^2 \leq \frac{\mathrm{e}^{2T\|V\|_\infty}}{T} \int_0^T \|\mathrm{e}^{t(\Delta - V)} u\|_{\Lambda_L}^2 \,\mathrm{d}t \leq \mathrm{e}^{2\|V\|_\infty} \mathrm{e}^{2/T} \int_0^T \|\mathrm{e}^{t(\Delta - V)} u\|_{\Lambda_L}^2 \,\mathrm{d}t.$$

Thus we found [Miller 2010, (5),(6),(7)] with $m_0 = 1,\ m = 0,\ \alpha = \nu = \frac{1}{2},\ a_0$ and b_0 as in the theorem, $a = -(N/2)\ln(\delta/G) G > 0,\ b = 1$ and $\beta = 1$. By [Miller 2010, Theorem 2.2 and Corollary 1(i)], there exists $T' > 0$ such that for all $T \leq T'$

$$\kappa_T \leq 4 a_0 b_0 \mathrm{e}^{2c_*/T}, \quad \text{where } c_* = 4(\sqrt{a+2} - \sqrt{a})^{-4}.$$

From the proof in [Miller 2010], it can be inferred that T' only depends on $m_0, \alpha, \beta, a, b, a_0, b_0$ and on our choice of $T \leq 1$. Thus, in our case, T' only depends on G, δ and $\|V\|_\infty$.

Using $\sqrt{a+2} - \sqrt{a} = \int_a^{a+2} (2\sqrt{x})^{-1} \,\mathrm{d}x \geq (a+2)^{-1/2}$ and the fact that from $\delta \leq G/2$, it follows that $2 \leq 2a/a_{\min}$, where $a_{\min} := (N/2)\ln(2) G$, and we obtain

$$c_* \leq 4(a+2)^2 \leq 4a^2(1 + 2/a_{\min})^2 = \ln(G/\delta)^2 (NG + 4/\ln 2)^2.$$

Appendix: On single-site potentials for the breather model

Our assumptions. In this section we discuss our conditions on the single-site potential in the random breather model. Recall that the ω_j were supported in $[\omega_-, \omega_+] \subset [0, 1)$, whence we consider $t \in [\omega_-, \omega_+]$ and $\delta \in [0, 1 - \omega_+]$.

Definition A.1. We say that a family $\{u_t\}_{t\in[0,1]}$ of measurable functions $u_t : \mathbb{R}^d \to \mathbb{R}$ satisfies:

- condition (A) if the u_t are uniformly bounded, have uniform compact support and if there are $\alpha_1, \beta_1 > 0$ and $\alpha_2, \beta_2 \geq 0$ such that for all $t \in [\omega_-, \omega_+]$, $\delta \leq 1-\omega_+$ there is $x_0 = x_0(t, \delta) \in \mathbb{R}^d$ with
$$u_{t+\delta} - u_t \geq \alpha_1 \delta^{\alpha_2} \chi_{B(x_0, \beta_1 \delta^{\beta_2})}, \tag{39}$$
- condition (B) if u_t is the dilation of a function u by t, defined as $u_t(x) := u(x/t)$ for $t > 0$ and $u_0 \equiv 0$, where u is the characteristic function of a bounded, convex, open set K with $0 \in \overline{K}$,
- condition (C) if u_t is the dilation of a measurable function u which is positive, radially symmetric, compactly supported, bounded with monotonously decreasing radial part $r_u : [0, \infty) \to [0, \infty)$ and such there is a point $\tilde{x} > 0$ where r_u is differentiable, $r_u'(\tilde{x}) < 0$ and $r_u(\tilde{x}) > 0$,
- condition (D) if u_t is the dilation of a measurable function u which is positive, radially symmetric, radially decreasing, compactly supported, bounded and which has a discontinuity away from 0,
- condition (E) if u_{1-t} is the dilation of a measurable function which is nonpositive, radially symmetric, radially increasing, compactly supported, bounded, and such there is a point $\tilde{x} > 0$ where the radial part r_u is differentiable, $r_u'(\tilde{x}) > 0$ and $r_u(\tilde{x}) < 0$.

Remark A.2. Condition (A) is the abstract assumption we used in the proof of the Wegner estimate for the random breather model. Conditions (B) to (E) are relatively easy to verify for specific examples of single-site potentials. In particular, (C) holds for many natural choices of single-site potentials such as the smooth function $\chi_{|x|<1} \exp(1/(|x|^2 - 1))$ or the hat-potential $\chi_{|x|<1}(1 - |x|)$. Furthermore, we note that if we have families $\{u_t\}_{t\in[0,1]}$ and $\{v_t\}_{t\in[0,1]}$ where u_t satisfies (A) and $v_{t+\delta} - v_t \geq 0$ for all $t \in [\omega_-, \omega_+]$ and $\delta \in (0, 1-\omega_+]$, then the family $\{u_t + v_t\}_{t\in[0,1]}$ also satisfies (A).

Lemma A.3. *We have that each of the assumptions* (*B*) *to* (*E*) *implies* (*A*).

Proof. Assume (B). We will show (A) with $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_2 = 1$ and $\beta_1 = c$, and hence it is enough to show the existence of a $c\delta$-ball in $K_{t+\delta} \backslash K_t$.

For $K \subset \mathbb{R}^d$ and $t > 0$ we define $K_t := \{x \in \mathbb{R}^d : x/t \in K\}$ and $K_0 := \varnothing$. Without loss of generality let $x := (1, 0, \dots, 0)$ be a point in $\overline{K}$ which maximizes $|x|$ over $\overline{K}$. For $\lambda \in \mathbb{R}$ define the half-space $H_\lambda := \{x \in \mathbb{R}^d : x_1 \leq \lambda\}$, where x_1 stands for the first coordinate of x. By scaling, the existence of a $c\delta$-ball in $K_{t+\delta} \backslash K_t$ is equivalent to the existence of a $c\delta/(t+\delta)$-ball in $K \backslash K_{t/(t+\delta)}$. By maximality of $(1, 0, \dots, 0)$, we have $K \subset H_1$ and hence $K_{t/(t+\delta)} \subset H_{t/(t+\delta)}$. Thus, it is sufficient to find a $c\delta/(t+\delta)$-ball in $K \backslash H_{t/(t+\delta)}$. By convexity of K, the set $\left\{z \in K : z_1 = \frac{1}{2}\right\}$ is nonempty and since K is open, we find $z_0 \in K$ with $z_1 = \frac{1}{2}$ and $0 < c < \frac{1}{2}$ such that $B(z_0, c) \subset K$. We define for $\lambda \in [0, 1)$ the set $X(\lambda) \subset \mathbb{R}^d$ as $X(\lambda) := B(z_0 + \lambda((1, 0, \dots, 0) - z_0), c \cdot (1 - \lambda))$. By convexity and the fact that $(1, 0, \dots, 0) \in \overline{K}$, we have $X(\lambda) \subset K$. In fact, let $\{x_n\}_{n\in\mathbb{N}} \subset K$ be a sequence with $x_n \to (1, 0, \dots, 0)$. We define open sets $X_n(\lambda)$ by replacing $(1, 0, \dots, 0)$ by x_n in the definition of $X(\lambda)$. By convexity of K, every X_n is a subset of K, whence $\bigcup_{n\in\mathbb{N}} X_n(\lambda) \subset K$. Furthermore we have $X(\lambda) \subset \bigcup_{n\in\mathbb{N}} X_n(\lambda)$. Thus $X(\lambda) \subset K$. We now choose $\lambda := t/(t+\delta)$. Then $X(\lambda) \cap H_\lambda = \varnothing$. Noting that $c(1-\lambda) = c\delta/(t+\delta)$, we see that $X(\lambda)$ is the desired $c\delta/(t+\delta)$-ball.

Now we assume (C). Let $r_u'(\tilde{x}) = -C_1$. Then there is $\tilde{\varepsilon} > 0$ such that

$$r_u(\tilde{x}+\varepsilon) - r_u(\tilde{x}) \in \Big[-2\varepsilon C_1, -\frac{\varepsilon}{2}C_1\Big] \quad \text{for all } |\varepsilon| < \tilde{\varepsilon}. \tag{40}$$

It is sufficient to prove the following: there are $C_2, C_3 > 0$ such that for every $0 \le t \le \omega_+$ and every $0 < \delta \le 1-\omega_+$ there is $\hat{x} = \hat{x}(t,\delta)$ such that

$$r_u\left(\frac{\hat{x}+C_2\delta}{t+\delta}\right) - r_u\left(\frac{\hat{x}}{t}\right) \ge C_3\delta. \tag{41}$$

Indeed, by monotonicity of r_u, (41) implies that for every $x \in [\hat{x}, \hat{x}+C_2\delta]$ we have

$$r_u\left(\frac{x}{t+\delta}\right) - r_u\left(\frac{x}{t}\right) \ge r_u\left(\frac{\hat{x}+C_2\delta}{t+\delta}\right) - r_u\left(\frac{\hat{x}}{t}\right) \ge C_3\delta,$$

whence (A) holds with $x_0 := (\hat{x}+C_2\delta/2)e_1,\ \alpha_1 = C_3,\ \beta_1 = C_2/2,\ \alpha_2 = \beta_2 = 1$.

In order to see (41), let $\hat{x} = (t+\delta)\tilde{x}$. We choose $\kappa \in \big(0, \frac{1}{4}\big)$ and assume that $\tilde{x} - 4\kappa\tilde{\varepsilon} > 0$ (this is no restriction since (40) also holds for smaller $\tilde{\varepsilon}$). Furthermore, we define $C_2 := \kappa\tilde{\varepsilon}$. Now we distinguish two cases. If $\tilde{x}\delta/t \le \tilde{\varepsilon}$, then (40) implies

$$\begin{aligned} r_u\left(\frac{\hat{x}+C_2\delta}{t+\delta}\right) - r_u\left(\frac{\hat{x}}{t}\right) &= r_u\left(\tilde{x}+\kappa\frac{\tilde{\varepsilon}\delta}{t+\delta}\right) - r_u(\tilde{x}) + r_u(\tilde{x}) - r_u\left(\tilde{x}+\tilde{x}\frac{\delta}{t}\right) \\ &\ge -2\kappa C_1\frac{\tilde{\varepsilon}\delta}{t+\delta} + C_1\frac{\tilde{x}\delta}{2t} \ge \delta\frac{C_1}{2}\frac{\tilde{x}-4\kappa\tilde{\varepsilon}}{t+\delta}. \end{aligned}$$

If $\tilde{x}\delta/t > \tilde{\varepsilon}$, we use $r_u(\tilde{x}) - r_u(\tilde{x}+\tilde{x}\delta/t) \ge r_u(\tilde{x}) - r_u(\tilde{x}+\tilde{\varepsilon})$ and (40) to obtain

$$r_u\left(\frac{\hat{x}+C_2\delta}{t+\delta}\right) - r_u\left(\frac{\hat{x}}{t}\right) \ge -2\kappa C_1\frac{\tilde{\varepsilon}\delta}{t+\delta} + C_1\frac{\tilde{\varepsilon}}{2} = \frac{C_1\tilde{\varepsilon}}{2}\left(1-\frac{4\kappa\delta}{t+\delta}\right) \ge \frac{C_1\tilde{\varepsilon}}{2}(1-4\kappa).$$

Hence

$$r_u\left(\frac{\hat{x}+C_2\delta}{t+\delta}\right) - r_u\left(\frac{\hat{x}}{t}\right) \ge C_3\delta, \qquad \text{where } C_3 := \min\left\{\frac{C_1(\tilde{x}-4\kappa\tilde{\varepsilon})}{2}, \frac{C_1\tilde{\varepsilon}(1-4\kappa)}{2(1-\omega_+)}\right\} > 0.$$

The fact that (D) implies (A) is a consequence of (B). In fact, a function u as in (D) can be decomposed as $u = v + w$, where v is (a multiple of) a characteristic function of a ball, centered at the origin, and w is positive, radially symmetric and decreasing. Indeed, let x_0 be the point of discontinuity with the smallest norm. Then we can take $v = (u(x_0-) - u(x_0+))\chi_{B(0,|x_0|)}$, where χ_A denotes the characteristic function of the set A.

The function v satisfies (A) by (B) (since balls are convex) and we have $w_{t+\delta} - w_t \ge 0$. By Remark A.2, the family $\{u_t\}_{t\in[0,1]} = \{v_t + w_t\}_{t\in[0,1]}$ also satisfies (A). The case (E) is an adaptation of (C). □

Earlier assumptions. For certain types of random breather potentials, Wegner estimates have been given before; see [Combes et al. 1996; 2001]. As we will show below, none of these results covers the *standard breather model*. The methods of [Combes et al. 1996; 2001] seem to be motivated by reducing, thanks to linearization, the random breather model to a model of alloy type and then applying methods designed for the latter one. They are not focused to take advantage of the inherent, albeit nonlinear, monotonicity of the random breather model. The following assumptions on the single site potential are considered in [Combes et al. 1996] and [Combes et al. 2001], respectively.

Definition A.4. We say that a measurable function $u : \mathbb{R}^d \to [0, \infty)$ satisfies:

- condition (F) if u is compactly supported, in $C^2(\mathbb{R}^d)$, nonzero in a neighborhood of the origin and for some $c_0 > 0$ we have the inequalities

$$-x \cdot \nabla u \geq 0 \quad \text{for all } x \in \mathbb{R}^d \qquad \text{and} \qquad \left| \frac{(x, \mathrm{Hess}[u]x)}{x \cdot \nabla u} \right| \leq c_0 < \infty \quad \text{for all } x \in \mathbb{R}^d \backslash \{0\}, \tag{42}$$

- condition (G) if $u \not\equiv 0$ is compactly supported, in $C^1(B_1 \backslash \{0\})$, and there is $\varepsilon_0 > 0$ such that

$$-x \cdot \nabla u - \varepsilon_0 u \geq 0 \quad \text{for all } x \in \mathbb{R}^d \backslash \{0\}. \tag{43}$$

We have the following lemma.

Lemma A.5. *We have that*

- (F) *never holds*,
- (G) *implies that u has a singularity at the origin.*

Proof. We first show the statements in dimension one. Assume (F) and let $x_0 := \min \operatorname{supp} u$. Note that $x_0 < 0$. By the first inequality in (42) we have that $u' \geq 0$ for $x \in (x_0, 0)$. The second inequality in (42) implies

$$|u''(x)| \leq \frac{c_0 u'(x)}{|x|} \leq \frac{2c_0 u'(x)}{|x_0|} \quad \text{for all } x \in (x_0, x_0/2),$$

whence we have

$$u'(x) = \int_{x_0}^{x} u''(y)\, \mathrm{d}y \leq \int_{x_0}^{x} |u''(y)|\, \mathrm{d}y \leq \frac{2c_0}{|x_0|} \int_{x_0}^{x} u'(y)\, \mathrm{d}y,$$

and iteratively

$$\begin{aligned} u'(x) &\leq \frac{(2c_0)^n}{|x_0|^n} \int_{x_0}^{x} \int_{x_0}^{x^{(1)}} \cdots \int_{x_0}^{x^{(n-1)}} u'(x^{(n)})\, \mathrm{d}x^{(n)} \cdots \mathrm{d}x^{(1)} \\ &\leq \|u'\|_\infty \cdot \frac{(2c_0)^n}{|x_0|^n} \int_{x_0}^{x} \int_{x_0}^{x^{(1)}} \cdots \int_{x_0}^{x^{(n-1)}} \mathrm{d}x^{(n)} \cdots \mathrm{d}x^{(1)} \\ &= \|u'\|_\infty \cdot \left(\frac{2c_0(x - x_0)}{|x_0|} \right)^n (n!)^{-1} \to 0 \quad \text{as } n \to \infty \end{aligned}$$

for all $x \in (x_0, x_0/2)$. We found $u' \equiv 0$ on $(x_0, x_0/2)$, which is a contradiction.

Now we assume (G). The function u cannot have its supremum at a point of differentiability for else it would have to be zero at its maximum, which would imply $u \equiv 0$. Condition (43) implies that u is increasing on the negative half-axis and decreasing on the positive half-axis. We conclude that the supremum has to be the limit at the only possible nondifferentiable point $x = 0$ and we will show that this limit is ∞. By monotonicity of u and the assumption $u \not\equiv 0$, there is $\delta_0 > 0$ such that

$$u(x) \geq u(\delta_0) > 0 \quad \text{on } (0, \delta_0) \qquad \text{or} \qquad u(x) \geq u(-\delta_0) > 0 \quad \text{on } (-\delta_0, 0).$$

Without loss of generality, we assume $u(x) \geq u(\delta_0) > 0$ on $(0, \delta_0)$. Furthermore, from (43) it follows that

$$-u'(x) \geq \varepsilon_0 \frac{u(x)}{x} \quad \text{for } x > 0.$$

Using this inequality we estimate for $0 < x < \delta_0$:

$$\begin{aligned} u(x) \geq u(x) - u(\delta_0) &= -\int_x^{\delta_0} u'(s)\,\mathrm{d}s \geq \varepsilon_0 \int_x^{\delta_0} \frac{u(s)}{s}\,\mathrm{d}s \\ &\geq \varepsilon_0 u(\delta_0) \int_x^{\delta_0} s^{-1}\mathrm{d}s = \varepsilon_0 u(\delta_0)[\ln(\delta_0) - \ln(x)] \to \infty \text{ as } x \to 0. \end{aligned}$$

Now we show the claim in higher dimensions. If the single site potential $U : \mathbb{R}^d \to [0, \infty)$ does not vanish identically there is a point y such that $U(y) > 0$. Assume without loss of generality that y lies on the x_1-axis and define $u : \mathbb{R} \to [0, \infty)$ by $u(x_1) = U(x_1, 0, \dots, 0)$. Note that if U satisfies the assumption (F) or (G), respectively, then u satisfies (F) or (G) as well and the one-dimensional argument can be applied to u. Hence, the statement of the lemma also holds for U. □

In the light of the comments made at the beginning of this section, the occurrence of a singularity is not surprising since in the case of a single-site potential with a polynomial singularity, $u(x) = |x|^{-\alpha}$, we have

$$u(x/\omega_j) = |x/\omega_j|^{-\alpha} = \omega_j^\alpha |x|^{-\alpha} = \omega_j^\alpha u(x),$$

and thus the random breathing would correspond to a multiplication, which would allow to reduce the breather model to the well-understood alloy-type model $V_\omega(x) = \sum_j \omega_j u(x - j)$.

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