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ON RANK-2 TODA SYSTEMS WITH ARBITRARY SINGULARITIES: LOCAL MASS AND NEW ESTIMATES

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For all rank-2 Toda systems with an arbitrary singular source, we use a unified approach to prove:

(1) The pair of local masses (σ_1, σ_2) at each blowup point has the expression

$$\sigma_i = 2(N_{i1}\mu_1 + N_{i2}\mu_2 + N_{i3}),$$

where $N_{ij} \in \mathbb{Z},\ i = 1, 2,\ j = 1, 2, 3$.

(2) At each vortex point p_t if (α_t^1, α_t^2) are integers and $\rho_i \notin 4\pi\mathbb{N}$, then all the solutions of Toda systems are uniformly bounded.

(3) If the blowup point q is a vortex point p_t and α_t^1, α_t^2 and 1 are linearly independent over Q, then

$$u^k(x) + 2\log|x - p_t| \le C.$$

The Harnack-type inequalities of 3 are important for studying the bubbling behavior near each blowup point.

1. Introduction

Let (M, g) be a Riemann surface without boundary and $\boldsymbol{K} = (k_{ij})_{n\times n}$ be the Cartan matrix of a simple Lie algebra of rank n. For example, for the Lie algebra $\mathrm{sl}(n+1)$ (the so-called A_n) we have

$$\boldsymbol{K} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{1-1}$$

In this paper we consider the solution $u = (u_1, \dots, u_n)$ of the following system defined on M:

$$\Delta_g u_i + \sum_{j=1}^{n} k_{ij}\rho_j \left(\frac{h_j e^{u_j}}{\int_M h_j e^{u_j}\, dV_g} - 1 \right) = \sum_{p_t \in S} 4\pi\alpha_t^i(\delta_{p_t} - 1), \tag{1-2}$$

where Δ_g is the Laplace–Beltrami operator $(-\Delta_g \ge 0)$, S is a finite set on M, $h_1, \dots, h_n$ are positive and smooth functions on M, $\alpha_t^i > -1$ is the strength of the Dirac mass δ_{p_t} and $\rho = (\rho_1, \dots, \rho_n)$ is a constant vector with nonnegative components. Here for simplicity we just assume that the total area of M is 1.

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Obviously, (1-2) remains the same if u_i is replaced by $u_i + c_i$ for any constant c_i. Thus we might assume that each component of $u = (u_1, \dots, u_n)$ is in

$$\mathring{H}^1(M) := \left\{ v \in L^2(M),\ \nabla v \in L^2(M) \ \text{and} \ \textstyle\int_M v\, dV_g = 0 \right\}.$$

Then (1-2) is the Euler–Lagrange equation for the following nonlinear functional $J_\rho(u)$ in $\mathring{H}^1(M)$:

$$J_\rho(u) = \frac{1}{2} \int_M \sum_{i,j=1}^n k^{ij} \nabla_g u_i \nabla_g u_j \, dV_g - \sum_{i=1}^n \rho_i \log \int_M h_i e^{u_i} \, dV_g,$$

where $(k^{ij})_{n \times n} = \boldsymbol{K}^{-1}$.

It is hard to overestimate the importance of system (1-2), as it covers a large number of equations and systems deeply rooted in geometry and physics. Even if (1-2) is reduced to a single equation with Dirac sources, it is a mean-field equation that describes metrics with conic singularities. Finding metrics with constant curvature with prescribed conic singularity is a classical problem in differential geometry and extensive references can be found in [Bartolucci and Tarantello 2002; Battaglia and Malchiodi 2014; Eremenko et al. 2014; Lin et al. 2012; 2015; Lin and Zhang 2010; 2013; 2016; Troyanov 1989; 1991; Yang 1997]. Recently profound relations among mean-field equations, the classical Lamé equation, hyperelliptic curves, modular forms and the Painlevé equation have been discovered and developed in [Chai et al. 2015; Chen et al. 2016].

The general form of (1-2) has close ties with algebraic geometry and integrable systems. Here we just briefly explain the relation between the sl($n+1$)-Toda system and the holomorphic curves in projective spaces: Let f be a holomorphic curve from a domain D of $\mathbb{R}^2$ into $\mathbb{CP}^n$. Then f can be lifted locally to $\mathbb{C}^{n+1}$ and we use $\nu(z) = [\nu_0(z), \dots, \nu_n(z)]$ to denote the lift and f_k the k-th associated curve,

$$f_k : D \to G(k, n+1) \subset \mathbb{CP}^n(\Lambda^k \mathbb{C}^{n+1}), \quad f_k(z) = [\nu(z) \wedge \nu'(z) \wedge \cdots \wedge \nu^{(k-1)}(z)],$$

where $\nu^{(j)}$ is the j-th derivative of ν with respect to z. Let

$$\Lambda_k(z) = \nu(z) \wedge \cdots \wedge \nu^{(k-1)}(z).$$

Then the well-known infinitesimal Plüker formula gives

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \|\Lambda_k(z)\|^2 = \frac{\|\Lambda_{k-1}(z)\|^2 \, \|\Lambda_{k+1}(z)\|^2}{\|\Lambda_k(z)\|^4} \quad \text{for } k = 1, 2, \dots, n, \tag{1-3}$$

where we put $\|\Lambda_0(z)\|^2 = 1$ as convention and the norm $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$ is defined by the Fubini–Study metric in $\mathbb{CP}(\Lambda^k \mathbb{C}^{n+1})$. Here we observe that (1-3) holds only for $\|\Lambda_k(z)\| > 0$, i.e., for all the unramified points $z \in M$. Now we set $\|\Lambda_{n+1}(z)\| = 1$ by normalization (analytically extended at the ramification points) and

$$U_k(z) = -\log \|\Lambda_k(z)\|^2 + k(n-k+1) \log 2, \quad 1 \le k \le n.$$

For every ramified point p we use $\{\gamma_{p,1}, \dots, \gamma_{p,n}\}$ to denote the total ramification index at p and set

$$u_i^* = \sum_{j=1}^n k_{ij} U_j, \quad \alpha_{p,i} = \sum_{j=1}^n k_{ij} \gamma_{p,j},$$

Then we have

$$\Delta u_i^* + \sum_{j=1}^{n} k_{ij} e^{u_j^*} - K_0 = 4\pi \sum_{p\in S} \alpha_{p,i}\delta_p, \quad i = 1, \dots, n, \tag{1-4}$$

where K_0 is the Gaussian curvature of the metric g.

Therefore any holomorphic curve from M to $\mathbb{CP}^n$ is associated with a solution $u^* = (u_1^*, \dots, u_n^*)$ of (1-4). Conversely, given any solution $u^* = (u_1^*, \dots, u_n^*)$ of (1-4) in $\mathbb{S}^2$, it is possible to construct a holomorphic curve of $\mathbb{S}^2$ into $\mathbb{CP}^n$ which has the given ramification index $\gamma_{p,i}$ at p if $\gamma_{p,i} \in \mathbb{N}$. One can see [Lin et al. 2012] for the details of this construction. Therefore, (1-4) is related to the following problem in more general setting: given a set of ramified points on M and its ramification indices at these points, can we find holomorphic curves into $\mathbb{CP}^n$ that satisfy the given ramification information?

Equation (1-2) is also related to many physical models from gauge field theory. For example, to describe the physics of high critical temperature superconductivity, a model related to the Chern–Simons model was proposed, which can be reduced to an $n \times n$ system with exponential nonlinearity if the gauge potential and the Higgs field are algebraically restricted. The Toda system with (1-1) is one of the limiting equations if a coupling constant tends to zero. For extensive discussions on the relationship between the Toda system and its background in Physics we refer the readers to [Bennett 1934; Ganoulis et al. 1982; Lee 1991; Mansfield 1982; Yang 2001].

In this article we are concerned with rank-2 Toda systems. There are three types of Cartan matrices of rank 2:

$$A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B_2 (= C_2) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

One of our main theorems is the following estimate:

Theorem 1.1. *Let $(k_{ij})_{2\times 2}$ be one of the matrices above, h_i be positive C^1 functions on M, $\alpha_t^i \in \mathbb{N}\cup\{0\}$, $t \in \{1, 2, \dots, N\}$ and K be a compact subset of $M \setminus S$. If $\rho_i \notin 4\pi\mathbb{N}$, then there exists a constant $C(K, \rho_1, \rho_2)$ such that for any solution $u = (u_1, u_2)$ of* (1-2)

$$|u_i(x)| \le C \quad \textit{for all } x \in K,\ i = 1, 2.$$

Our proof of Theorem 1.1 is based on the analysis of the behavior of solutions $u^k = (u_1^k, u_2^k)$ near each blowup point. A point $p \in M$ is called a blowup point if, along a sequence of points $p_k \to p$,

$$\max_{i=1,2} \{\tilde{u}_1^k(p_k), \tilde{u}_2^k(p_k)\} \to +\infty,$$

where

$$\tilde{u}_i^k(x) = u_i^k(x) + 4\pi \sum_t \alpha_t^k G(x, p_t),$$

and $G(x, y)$ is the Green's function of the Laplacian operator on M.

Suppose u^k is a sequence of solutions of (1-2). When $n = 1$, it has been proved that if u^k blows up somewhere, the mass distribution $\rho h e^{u^k} / \big(\int_M h e^{u^k}\big)$ will concentrate; that is, for a set of finite points

$p_1, p_2, \dots, p_L$ and positive numbers $m_1, \dots, m_L$

$$\frac{\rho h e^{u^k}}{\int_M h e^{u^k}} \to \sum_{i=1}^{L} m_i \delta_{p_i} \quad \text{as } k \to \infty.$$

In other words, "u_k concentrates" means $u^k(x) \to -\infty$ if x is not a blowup point. This "blowup implies concentration" was first noted by Brezis and Merle [1991] and was later proved by Li [1999], Li and Shafrir [1994] and Bartolucci and Tarantello [2002]. But for $n \geq 2$, this phenomenon might fail in general. A component u_i^k is called not concentrating if $u_i^k \not\to -\infty$ away from blowup points, or equivalently, $\tilde{u}_i^k$ converges to some smooth function w_i away from blowup points. It is natural to ask whether it is possible to have all components not concentrating. For $n = 2$, we prove it is impossible.

Theorem 1.2. *Suppose u^k is a sequence of blowup solutions of a rank-*2 *Toda system* (1-2). *Then at least one component of u^k satisfies $u_i^k(x) \to -\infty$ if x is not contained in the blowup set.*

The first example of such nonconcentration phenomenon was proved by Lin and Tarantello [2016]. The new phenomenon makes the study of systems ($n \geq 2$) much more difficult than the mean-field equation ($n = 1$). Recently, Battaglia [2015] and Lin, Yang and Zhong [Lin et al. 2017] independently proved the result of Theorem 1.2 for $n \geq 3$.

As mentioned before, our proofs of Theorems 1.1 and 1.2 are based on the asymptotic behavior of local bubbling solutions. For simplicity we set up the situation as follows:

Let $u^k = (u_1^k, u_2^k)$ be a sequence of solutions of

$$\Delta u_i^k + \sum_{j=1}^{2} k_{ij} h_j^k e^{u_j^k} = 4\pi \alpha_i \delta_0 \quad \text{in } B(0,1),\ i = 1, 2, \tag{1-5}$$

where $\alpha_i > -1$. $B(0,1)$ is the unit ball in $\mathbb{R}^2$ (we use $B(p,r)$ to denote the ball centered at p with radius r) and $(k_{ij})_{2\times 2}$ is A_2, B_2 or G_2. Throughout the paper, h_1^k, h_2^k are smooth functions satisfying $h_1^k(0) = h_2^k(0) = 1$ and

$$\frac{1}{C} \leq h_i^k \leq C, \quad \|h_i^k\|_{C^1(B(0,1))} \leq C \quad \text{in } B(0,1),\ i = 1, 2. \tag{1-6}$$

For solutions $u^k = (u_1^k, u_2^k)$ we assume

$$\begin{cases} 0 \text{ is the only blowup point of } u^k, \\ |u_i^k(x) - u_i^k(y)| \leq C \quad \text{for all } x, y \in \partial B(0,1),\ i = 1, 2, \\ \int_{B(0,1)} h_i^k e^{u_i^k} \leq C, \quad i = 1, 2. \end{cases} \tag{1-7}$$

For this sequence of blowup solutions we define the local mass by

$$\sigma_i = \lim_{r \to 0} \lim_{k \to \infty} \frac{1}{2\pi} \int_{B(0,r)} h_i^k e^{u_i^k}, \quad i = 1, 2. \tag{1-8}$$

It is known that 0 is a blowup point if and only if $(\sigma_1, \sigma_2) \neq (0, 0)$. The proof is to use ideas from [Brezis and Merle 1991] and has become standard now. We refer the readers to [Lee et al. 2017] for a complete proof. One important property of (σ_1, σ_2) is the so-called Pohozaev identity (P.I. in short)

$$k_{21}\sigma_1^2 + k_{12}k_{21}\sigma_1\sigma_2 + k_{12}\sigma_2^2 = 2k_{21}\mu_1\sigma_1 + 2k_{12}\mu_2\sigma_2, \tag{1-9}$$

where $\mu_i = 1 + \alpha_i$. Take A_2 as an example; the P.I. is

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = 2\mu_1\sigma_1 + 2\mu_2\sigma_2.$$

The proof of (1-9) was given in [Lin et al. 2015] where we initiated an algorithm to calculate all the possible (finitely many) values of local masses and (1-9) played an essential role. But the argument there seems not very efficient. In this work we add major new ingredients to our approach and improve the classification of (σ_1, σ_2) to the following sharper form:

Theorem 1.3. *Let u^k be a sequence of blowup solutions of* (1-5) *which also satisfies* (1-6) *and* (1-7). *Suppose σ_1 and σ_2 are local masses defined by* (1-8). *Then σ_i can be written as*

$$\sigma_i = 2(N_{i,1}\mu_1 + N_{i,2}\mu_2 + N_{i,3}), \quad i = 1, 2,$$

for some $N_{i,1}, N_{i,2}, N_{i,3} \in \mathbb{Z}$ $(i = 1, 2)$.

Theorem 1.3 is proved in Sections 5 and 6. In Section 5, we give an explicit procedure to calculate the local masses. Take the A_2 system as an example; we start with $\sigma_1 = 0$ and the P.I. gives $\sigma_2 = 2\mu_2$. With $\sigma_2 = 2\mu_2$, the P.I. gives $\sigma_1 = 2\mu_1 + 2\mu_2$ and so on. Let $\Gamma(\mu_1, \mu_2)$ be the set obtained by the above algorithm. Then $\Gamma(\mu_1, \mu_2)$ is equal to:

(i) $(2\mu_1, 0)$, $(2\mu_1, 2\mu_1 + 2\mu_2)$, $(2\mu_1 + 2\mu_2, 2\mu_1 + 2\mu_2)$, $(2\mu_1 + 2\mu_2, 2\mu_2)$, $(0, 2\mu_2)$ for A_2,

(ii) $(2\mu_1, 0)$, $(2\mu_1, 4\mu_1 + 2\mu_2)$, $(4\mu_1 + 2\mu_2, 4\mu_1 + 2\mu_2)$, $(4\mu_1 + 2\mu_2, 4\mu_1 + 4\mu_2)$, $(0, 2\mu_2)$, $(2\mu_1 + 2\mu_2, 2\mu_2)$, $(2\mu_1 + 2\mu_2, 4\mu_1 + 4\mu_2)$ for B_2,

(iii) $(2\mu_1, 0)$, $(2\mu_1, 6\mu_1 + 2\mu_2)$, $(6\mu_1 + 2\mu_2, 6\mu_1 + 2\mu_2)$, $(6\mu_1 + 2\mu_2, 12\mu_1 + 6\mu_2)$, $(8\mu_1 + 4\mu_2, 12\mu_1 + 6\mu_2)$, $(8\mu_1 + 4\mu_2, 12\mu_1 + 8\mu_2)$, $(0, 2\mu_2)$, $(2\mu_1 + 2\mu_2, 2\mu_2)$, $(2\mu_1 + 2\mu_2, 6\mu_1 + 6\mu_2)$, $(6\mu_1 + 4\mu_2, 6\mu_1 + 6\mu_2)$, $(6\mu_1 + 4\mu_2, 12\mu_1 + 8\mu_2)$ for G_2.

Definition 1.4. A pair of local masses $(\sigma_1, \sigma_2) \in \Gamma(\mu_1, \mu_2)$ is called special if

$$(\sigma_1, \sigma_2) = \begin{cases} (2\mu_1 + 2\mu_2, 2\mu_1 + 2\mu_2) & \text{for } A_2, \\ (4\mu_1 + 2\mu_2, 4\mu_1 + 4\mu_2) & \text{for } B_2, \\ (8\mu_1 + 4\mu_2, 12\mu_1 + 8\mu_2) & \text{for } G_2. \end{cases}$$

The analysis of local solutions in [Lin et al. 2015] describes a method to pick a family of points $\Gamma_k = \{0, x_1^k, \dots, x_N^k\}$ (if 0 is a singular point, otherwise 0 can be deleted from Γ_k) such that a tiny ball $B(x_i^k, l_j^k)$ contributes an amount of mass (which is quantized), and the following Harnack-type inequality holds:

$$u_i^k(x) + 2\log \operatorname{dist}(x, \Sigma_k) \leqslant C \quad \text{for all } x \in B(0, 1). \tag{1-10}$$

When $\alpha_1 = \alpha_2 = 0$, we can use Theorem 1.3 to calculate all the pairs of even positive integers satisfying (1-9) and the set is exactly the same as $\Gamma(1, 1)$.

It is interesting to see whether any pair of the above really consists of the local masses of some sequence of blowup solutions of (1-2). For $\boldsymbol{K} = A_2$ the existence of such a local blowup sequence has been obtained; see [Musso et al. 2016; Lin and Yan 2013].

After Σ_k is picked, the difficulty at the next step is how to calculate the mass contributed from outside $B(x_j^k, l_j^k)$ $j = 1, 2, \cdots, N$. In Section 6, we see that the mass outside this union could be very messy. However, the picture is very clean if (α_1, α_2) satisfies the following Q-condition:

$$\alpha_1, \alpha_2 \text{ and } 1 \text{ are linearly independent over } Q.$$

Theorem 1.5. *Suppose* (α_1, α_2) *satisfies the* Q*-condition. Then* $(\sigma_1, \sigma_2) \in \Gamma(\mu_1, \mu_2)$. *Furthermore, for any sequence of solutions of* (1-5) *satisfying* (1-6) *and* (1-7), *the following Harnack-type inequality holds*:

$$u_i^k(x) + 2\log|x| \leqslant C \quad \textit{for } x \in B(0,1).$$

For (1-2), let $\mu_{1,t} = \alpha_t^1 + 1$ and $\mu_{2,t} = \alpha_t^2 + 1$ at a vortex point $p_t \in S$, and define

$$\Gamma_i = \big\{2\pi(\Sigma_{t\in J}\sigma_{i,t} + 2n) \,\big|\, (\sigma_{1,t}, \sigma_{2,t}) \in \Gamma(\mu_{1,t}, \mu_{2,t}),\ J \subseteq S,\ n \in \mathbb{N}\cup\{0\}\big\}. \tag{1-11}$$

Based on Theorem 1.5, Theorem 1.1 can be extended to the following version:

Theorem 1.6. *Let* h_i *be positive* C^1 *functions on* M, *and* K *be a compact set in* M. *For every point* $p_t \in S$, *if either both* $\alpha_t^1, \alpha_t^2 \in \mathbb{N}\cup\{0\}$ *or* (α_t^1, α_t^2) *satisfies the* Q*-condition*, *then for* $\rho_i \notin \Gamma_i$ *and* $u = (u_1, u_2)$ *a solution of* (1-2), *there exists a constant* C *such that*

$$|u_i(x)| \leqslant C \quad \textit{for all } x \in K.$$

The organization of this article is as follows. In Section 2 we establish the global mass for the entire solutions of some singular Liouville equation defined in $\mathbb{R}^2$. Then in Section 3 we review some fundamental tools proved in the previous work [Lin et al. 2015]. In Section 4 we present two crucial lemmas, which play the key role in the proof of main results. In Sections 5 and 6 we discuss the local mass on each bubbling disk centered at 0 and not at 0 respectively, and then all the main results are established based on previous discussions.

2. Total mass for Liouville equation

The main purpose of this section is to prove an estimate of the total mass for the solutions of the equation

$$\begin{cases} \Delta u + e^u = \sum_{i=1}^N 4\pi\alpha_i\delta_{p_i} & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^u < \infty, \end{cases} \tag{2-1}$$

where $p_1, \ldots, p_N$ are distinct points in $\mathbb{R}^2$ and $\alpha_i > -1$ for all $1 \leqslant i \leqslant N$.

Theorem 2.1. *Suppose* u *is a solution of* (2-1) *and* $\alpha_1, \ldots, \alpha_N$ *are positive integers. Then* $\frac{1}{4\pi}\int_{\mathbb{R}^2} e^u$ *is an even integer.*

Proof. It is known that any solution u of (2-1) has, at infinity, the asymptotic behavior

$$u(z) = -2\alpha_\infty \log|z| + O(1), \quad \alpha_\infty > 1, \tag{2-2}$$

and u satisfies

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^u dx = 2\sum_{i=1}^N \alpha_i + 2\alpha_\infty. \tag{2-3}$$

We shall prove that $\alpha_\infty + \sum_{i=1}^N \alpha_i$ is an even integer. A classical Liouville theorem (see [Chou and Wan 1994]) says that u can be written as

$$u = \log \frac{8|f'(z)|^2}{(1+|f(z)|^2)^2}, \quad z \in \mathbb{R}^2, \tag{2-4}$$

for some meromorphic function f. In general, $f(z)$ is multivalued and any vertex p_i is a branch point. However if $\alpha_i \in \mathbb{N}\cup\{0\}$, then $f(z)$ is single-valued. Furthermore (2-2) implies that $f(z)$ is meromorphic at infinity. Hence for any solution u of (2-1) there is a meromorphic function f on $\mathbb{S}^2 = \mathbb{C}\cup\{\infty\}$ such that (2-4) holds. Then

$$\begin{aligned} 4\pi\Big(\sum_{j=1}^N \alpha_j + \alpha_\infty\Big) = \int_{\mathbb{R}^2} e^u &= 8\int_{\mathbb{R}^2} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2}\,dx\,dy \\ &= 8(\deg f)\int_{\mathbb{R}^2} \frac{d\tilde{x}\,d\tilde{y}}{(1+|w|^2)^2} = 8\pi(\deg f), \end{aligned}$$

where $\deg(f)$ is the degree of f as a map from $\mathbb{S}^2 = \mathbb{C}\cup\{\infty\}$ onto $\mathbb{S}^2$, and $w = f(z) = \tilde{x} + i\tilde{y}$. Thus we have

$$\sum_{j=1}^N \alpha_j + \alpha_\infty = 2\deg(f). \qquad \square$$

Theorem 2.2. *Suppose u is a solution of*

$$\begin{cases} \Delta u + e^u = 4\pi\alpha_0\delta_{p_0} + \sum_{i=1}^N 4\pi\alpha_i\delta_{p_i} & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^u < \infty, \end{cases} \tag{2-5}$$

where $p_0, p_1, \dots, p_N$ are distinct points in $\mathbb{R}^2$ and $\alpha_1, \dots, \alpha_N$ are positive integers, $\alpha_0 > -1$. Then $\frac{1}{4\pi}\int_{\mathbb{R}^2} e^u$ is equal to $2(\alpha_0+1)+2k$ for some $k\in\mathbb{Z}$ or $2k_1$ for some $k_1 \in \mathbb{N}$.

Proof. As in Theorem 2.1, there is a developing map $f(z)$ of u such that

$$u(z) = \log \frac{8|f'(z)|^2}{(1+|f(z)|^2)^2}, \quad z \in \mathbb{C}. \tag{2-6}$$

On one hand by (2-5), $u_{zz} - \frac{1}{2}u_z^2$ is a meromorphic function in $\mathbb{C}\cup\{\infty\}$ because away from the Dirac masses

$$4\big(u_{zz} - \tfrac{1}{2}u_z^2\big)_{\bar{z}} = -(e^u)_z + u_z e^u = 0.$$

By $u(z) = 2\alpha_i \log|z-p_i| + O(1)$ near p_i we have

$$u_{zz} - \tfrac{1}{2}u_z^2 = -2\Big(\sum_{j=0}^N \tfrac{1}{2}\alpha_j\big(\tfrac{1}{2}\alpha_j+1\big)(z-p_j)^{-2} + A_j(z-p_j)^{-1} + B\Big),$$

where $A_0, \dots, A_N, B \in \mathbb{C}$ are some constants. On the other hand by (2-6), a straightforward computation shows that

$$u_{zz} - \frac{1}{2}u_z^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{2-7}$$

Using the Schwarz derivative of f,

$$\{f; z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2,$$

and letting

$$I(z) = \sum_{j=0}^{N} \tfrac{1}{2}\alpha_j\left(\tfrac{1}{2}\alpha_j + 1\right)(z - p_j)^{-2} + A_j(z - p_j)^{-1} + B,$$

we can write the equation for f as

$$\{f, z\} = -2I(z). \tag{2-8}$$

A well-known classic theorem (see [Whittaker and Watson 1927]) says that for any two linearly independent solutions y_1 and y_2 of

$$y''(z) = I(z)y(z), \tag{2-9}$$

the ratio y_2/y_1 always satisfies

$$\{y_2/y_1; z\} = -2I(z).$$

By (2-8) and a basic result of the Schwarz derivative, $f(z)$ can be written as the ratio of two linearly independent solutions. This is how (2-1) is related to the complex ODE (2-9). We refer the readers to [Chai et al. 2015] for the details.

For (2-9), there is an associated monodromy representation ρ from $\pi_1(\mathbb{C} \setminus \{p_0, p_1, \dots, p_N\}; q)$ to $\mathrm{GL}(2; \mathbb{C})$, where q is a base point. Note that at any singular point p_j, the local exponents are $\frac{1}{2}\alpha_j + 1$ and $-\frac{1}{2}\alpha_j$. It is known from [Lin et al. 2012, Section 7] that e^{-u} can be locally written as

$$e^{-u} = |\nu_1|^2 + |\nu_2|^2 = \langle (\nu_1, \nu_2)^t, (\nu_1, \nu_2)^t \rangle,$$

where ν_1, ν_2 are the two fundamental solutions of (2-9). After encircling the singular point p_j once, we have $e^{-u} = \langle \rho_j(\nu_1, \nu_2)^t, \rho_j(\nu_1, \nu_2)^t \rangle$ and the value does not change. Therefore, we conclude that ρ_j is unitary and

$$\rho_j = \rho(\gamma_j) = C_j \begin{pmatrix} e^{\pi i \alpha_j} & 0 \\ 0 & e^{-\pi i \alpha_j} \end{pmatrix} C_j^{-1},$$

where $\gamma_j \in \pi_1(\mathbb{C} \setminus \{p_0, \dots, p_N\}, q)$ encircles p_j only once, $0 \le j \le N$, while the monodromy at ∞ is ρ_∞. Then we have

$$\rho_\infty \rho_N \cdots \rho_0 = I_{2\times 2}.$$

Note that $\rho_j = \pm I_{2\times 2}$ for $1 \le j \le N$. Hence

$$\rho_\infty^{-1} = D_0 \begin{pmatrix} e^{\pi i \sum_{j=0}^{N} \alpha_j} & 0 \\ 0 & e^{-\pi i \sum_{j=0}^{N} \alpha_j} \end{pmatrix} D_0^{-1}$$

for some constant invertible matrix D_0.

On the other hand, the local exponents at ∞ can be computed as follows. Let $\hat{y}(z) = y\left(\frac{1}{z}\right)$, where y is a solution of (2-9). Then we have

$$\hat{y}''(z) + \frac{2}{z}\hat{y}'(z) = \hat{I}(z)\hat{y}(z), \tag{2-10}$$

where $\hat{I}(z) = I\left(\frac{1}{z}\right)z^{-4}$. Since $I(z)$ is the Schwarz derivative of $f(z)$, by direct computation $\hat{I}(z)$ is the Schwarz derivative of $f\left(\frac{1}{z}\right)$. As before we let $\hat{u}(z) = u\left(\frac{1}{z}\right) - 4\log|z|$. Then $f\left(\frac{1}{z}\right)$ is the developing map of $\hat{u}(z)$. Since

$$\hat{u}(z) = 2(\alpha_\infty - 2)\log|z| + O(1) \quad \text{near } 0,$$

(because $u(z) = -2\alpha_\infty \log|z| + O(1)$ at infinity), we have

$$\hat{I}(z) = \tfrac{1}{2}\alpha_\infty\left(\tfrac{1}{2}\alpha_\infty - 1\right)z^{-2} + \text{higher-order terms of } z \quad \text{near } 0.$$

By (2-10) we see that the local exponents of (2-9) are $-\frac{1}{2}\alpha_\infty$ and $\frac{1}{2}\alpha_\infty - 1$. Hence $e^{\pi i\alpha_\infty}$ equals either $e^{\pi i\sum_{j=0}^{N}\alpha_j}$ or $e^{-\pi i\sum_{j=0}^{N}\alpha_j}$, which yields

$$\alpha_\infty = -\sum_{j=0}^{N}\alpha_j + 2k \quad \text{or} \quad \alpha_\infty = \sum_{j=0}^{N}\alpha_j + 2k \tag{2-11}$$

for some $k \in \mathbb{Z}$. Since

$$\frac{1}{4\pi}\int_{\mathbb{R}^2} e^u = \sum_{j=0}^{N}\alpha_j + \alpha_\infty,$$

we either have $\frac{1}{4\pi}\int_{\mathbb{R}^2} e^u = 2k$ for some $k \in \mathbb{N}$ if the first case holds or $\frac{1}{4\pi}\int_{\mathbb{R}^2} e^u = 2(\alpha_0 + 1) + 2k'$ for $k' = \sum_{i=1}^{N}\alpha_i + k - 1$ if the second case holds. □

Remark 2.3. After proving Theorems 2.1 and 2.2, we found a stronger version of both theorems in [Eremenko et al. 2014]. Because we only need the present form of both theorems, we include our proofs here to make the paper more self-contained.

3. Review of bubbling analysis from a selection process

Let $u^k = (u_1^k, u_2^k)$ be solutions of (1-5) such that (1-6) and (1-7) hold. In this section we review the process to select a set $\Sigma_k = \{0, x_1^k, \dots, x_n^k\}$ and balls $B(x_i^k, l_k)$ such that u^k has nonzero local masses in $B(x_i^k, l_k)$. This selection process was first carried out in [Lin et al. 2015]. We briefly review it below.

The set Σ_k is constructed by induction. If (1-5) has no singularity, we start with $\Sigma_k = \varnothing$. If (1-5) has a singularity, we start with $\Sigma_k = \{0\}$. By induction suppose Σ_k consists of $\{0, x_1^k, \dots, x_{m-1}^k\}$. Then we consider

$$\max_{x\in B_1}\max_{i=1,2}\left(u_i^k(x) + 2\log \operatorname{dist}(x, \Sigma_k)\right). \tag{3-1}$$

If the maximum is bounded from above by a constant independent of k, the process stops and Σ_k is exactly equal to $\{0, x_1^k, \dots, x_{m-1}^k\}$. However if the maximum tends to infinity, let q_k be where (3-1) is achieved and we set

$$d_k = \tfrac{1}{2}\operatorname{dist}(q_k, \Sigma_k)$$

and

$$S_i^k(x) = u_i^k(x) + 2\log(d_k - |x - q_k|) \quad \text{in } B(q_k, d_k),\ i = 1, 2.$$

Suppose i_0 is the component that attains

$$\max_i \max_{x \in \bar{B}(q_k, d_k)} S_i^k \tag{3-2}$$

at p_k. Then we set

$$\tilde{l}_k = \tfrac{1}{2}(d_k - |p_k - q_k|)$$

and scale u_i^k by

$$v_i^k(y) = u_i^k(p_k + e^{-\frac{1}{2}u_{i_0}^k(p_k)} y) - u_{i_0}^k(p_k) \quad \text{for } |y| \le R_k \doteqdot e^{\frac{1}{2}u_{i_0}^k(p_k)} \tilde{l}_k. \tag{3-3}$$

It can be shown that $R_k \to \infty$ and v_i^k is bounded from above over any fixed compact subset of $\mathbb{R}^2$. Thus by passing to a subsequence, v_i^k satisfies one of the following two alternatives:

(a) (v_1^k, v_2^k) converges in $C^2_{\text{loc}}(\mathbb{R}^2)$ to (v_1, v_2) which satisfies

$$\Delta v_i + \sum_{j=1}^{2} k_{ij} e^{v_j} = 0 \quad \text{in } \mathbb{R}^2,\ i = 1, 2. \tag{3-4}$$

(b) Either v_1^k converges to

$$\Delta v_1 + 2e^{v_1} = 0 \quad \text{in } \mathbb{R}^2 \tag{3-5}$$

and $v_2^k \to -\infty$ over any fixed compact subset of $\mathbb{R}^2$ or v_2^k converges to $\Delta v_2 + 2e^{v_2} = 0$ in $\mathbb{R}^2$ and $v_1^k \to -\infty$ over any fixed compact subset of $\mathbb{R}^2$.

Therefore in either case, we could choose $l_k^* \to \infty$ such that

$$v_i^k(y) + 2\log|y| \le C \quad \text{for } i = 1, 2 \text{ and } |y| \leqslant l_k^* \tag{3-6}$$

and

$$\int_{B(0, l_k^*)} h_i^k e^{v_i^k}\, dy = \int_{\mathbb{R}^2} e^{v_i(y)} + o(1).$$

By scaling back to u_i^k, we add p_k in Σ_k with

$$l_k = e^{-\frac{1}{2}u_{i_0}^k(p_k)} l_k^*.$$

We can continue in this way until the Harnack-type inequality (1-10) holds.

We summarize what the selection process has done in the following proposition (a detailed proof for a more general case can be found in [Lin et al. 2015, Proposition 2.1]):

Proposition 3A. *Let u^k be described as above. Then there exist a finite set $\Sigma_k := \{0, x_1^k, \dots, x_m^k\}$ (if 0 is not a singular point, then 0 can be deleted from Σ_k) and positive numbers $l_1^k, \dots, l_m^k \to 0$ as $k \to \infty$ such that the following hold*:

(1) *There exists $C > 0$ independent of k such that* (1-10) *holds and all the components have fast decay on $\partial B(x_j^k, l_j^k)$, $j = 1, \dots, m$. (The definition of fast decay can be found in Definition 3.1 below).*

(2) *In* $B(x_j^k, l_j^k)$ $(j = 1, \dots, m)$, *let* $R_{j,k} = e^{\frac{1}{2}u_{i_0}^k(x_j^k)} l_j^k$, $u_{i_0}^k(x_j^k) = \max_i u_i^k(x_j^k)$ *and*

$$v_i^k(y) = u_i^k(x_j^k + e^{-\frac{1}{2}u_{i_0}^k(x_j^k)} y) - u_{i_0}^k(x_j^k) \tag{3-7}$$

for $|y| \le R_{j,k}$; *then* $v^k = (v_1^k, v_2^k)$ *satisfies either* (a) *or* (b).

(3) $B(x_j^k, l_j^k) \cap B(x_i^k, l_i^k) = \varnothing$, $i \neq j$.

The inequality (1-10) is a Harnack-type inequality, because it implies the following result.

Proposition 3B. *Suppose* u^k *satisfies* (1-5)–(1-7) *and*

$$u_i^k(x) + 2\log|x - x_0| \le C \quad \textit{for } x \in B(x_0, 2r_k).$$

Then

$$|u_i^k(x_1) - u_i^k(x_2)| \le C_0 \quad \textit{for } \frac{1}{2} \le \frac{|x_1 - x_0|}{|x_2 - x_0|} \le 2 \textit{ and } x_1, x_2 \in B(x_0, r_k). \tag{3-8}$$

The proof of Proposition 3B is standard, see [Lin et al. 2015, Lemma 2.4], so we omit it here. Let $x_l^k \in \Sigma_k$ and $\tau_l^k = \frac{1}{2}\operatorname{dist}(x_l^k, \Sigma_k \setminus \{x_l^k\})$; then (3-8) implies

$$u_i^k(x) = \bar{u}_{x_l^k,i}^k(r) + O(1), \quad x \in B(x_l^k, \tau_l^k), \tag{3-9}$$

where $r = |x_l^k - x|$ and $\bar{u}_{x_l^k,i}^k$ is the average of u_i^k on $\partial B(x_l^k, r)$,

$$\bar{u}_{x_l^k,i}^k(r) = \frac{1}{2\pi r}\int_{\partial B(x_l^k, r)} u_i^k \, dS, \tag{3-10}$$

and $O(1)$ is independent of r and k.

Next we introduce the notions of slow decay and fast decay in our bubbling analysis.

Definition 3.1. We say u_i^k has fast decay on $\partial B(x_0, r_k)$ if along a subsequence

$$u_i^k(x) + 2\log|x - x_0| \le -N_k \quad \text{for all } x \in \partial B(x_0, r_k),$$

for some $N_k \to \infty$ and we say u_i^k has slow decay if there is a constant C independent of k such that

$$u_i^k(x) + 2\log|x - x_0| \ge -C \quad \text{for all } x \in \partial B(x_0, r_k).$$

Furthermore, we say u_i^k is fast-decaying in $B(x_0, s_k) \setminus B(x_0, r_k)$ if u_i^k has fast decay on $\partial B(x_0, l_k)$ for any $l_k \in [r_k, s_k]$.

The concept of fast decay is important for evaluating the Pohozaev identities. The following proposition is a direct consequence of [Lin et al. 2015, Proposition 3.1] and it says if both components are fast-decaying on the boundary, the Pohozaev identity holds for the local masses.

In the following proposition, we let $B = B(x^k, r_k)$. If $x^k \neq 0$, we assume $0 \notin B(x^k, 2r_k)$.

Proposition 3C. *Suppose both* u_1^k, u_2^k *have fast decay on* ∂B, *where* B *is given above. Then* (σ_1, σ_2) *satisfies the P.I.* (1-9), *where*

$$\sigma_i = \lim_{k \to 0} \frac{1}{2\pi}\int_B h_i^k e^{u_i^k}, \quad i = 1, 2.$$

We refer the readers to [Lin et al. 2015, Proposition 3.1] for the proof.

4. Two lemmas

In this section, we prove two crucial lemmas which play the key role in Sections 5 and 6. For Lemma 4.1, we assume:

(i) The Harnack inequality

$$u_i^k(x)+2\log|x|\le C \quad \text{for } \tfrac{1}{2}l_k\le|x|\le 2s_k,\ i=1,2,$$

holds for both components.

(ii) Both components u_i^k have fast decay on $\partial B(0,l_k)$ and $\sigma_i^k(B(0,l_k))=\sigma_i+o(1)$ for $i=1,2$, where

$$\sigma_i=\lim_{r\to 0}\lim_{k\to\infty}\frac{1}{2\pi}\int_{B(0,rs_k)}h_i^k e^{u_i^k},\quad i=1,2.$$

(iii) One of $u_i^k,\ i=1,2$, has slow decay on $\partial B(0,s_k)$.

Lemma 4.1. (a) *Assume* (i) *and* (ii). *If u_i^k has slow decay on $\partial B(0,s_k)$, then*

$$2\mu_i-\sum_{j=1}^{2}k_{ij}\sigma_j>0.$$

(b) *Assume* (i), (ii) *and* (iii). *Let u_i^k be a slow-decaying component on $\partial B(0,s_k)$. Then the other component has fast decay on $\partial B(0,s_k)$.*

Proof. (a) Suppose that u_i^k has slow decay on $\partial B(0,s_k)$. Then the scaling

$$v_j^k(y)=u_j^k(s_k y)+2\log s_k,\quad j=1,2 \text{ for } y\in B_2$$

gives

$$\Delta v_j^k(y)+\sum_{l=1}^{2}k_{jl}h_l^k(s_k y)e^{v_l^k(y)}=4\pi\alpha_j\delta_0 \quad \text{in } y\in B_2.$$

If the other component also has slow decay on $\partial B(0,s_k)$, then (v_1^k,v_2^k) converges to (v_1,v_2) which satisfies

$$\Delta v_j(y)+\sum_{l=1}^{2}k_{jl}e^{v_l}=0 \quad \text{in } B_2\setminus\{0\},\ j=1,2. \tag{4-1}$$

If the other component has fast decay on $\partial B(0,s_k)$, then $v_i^k(y)$ converges to $v_i(y)$ and $v_j(y)\to-\infty$, $j\ne i$. Furthermore, $v_i(y)$ satisfies

$$\Delta v_i(y)+2e^{v_i}=0 \quad \text{in } B_2\setminus\{0\}. \tag{4-2}$$

For any $r>0$,

$$\begin{aligned}\int_{\partial B(0,r)}\frac{\partial v_i(y)}{\partial v}\,dS&=\lim_{k\to\infty}\left(4\pi\alpha_i-\sum_{j=1}^{2}\int_{B(0,r)}k_{ij}h_j^k e^{v_j^k}\,dy\right)\\&=4\pi\alpha_i-2\pi\sum_{j=1}^{2}k_{ij}\sigma_j+o(1)\doteqdot 4\pi\beta_i+o(1),\end{aligned}$$

which implies the right-hand sides of both (4-1) and (4-2) should be replaced by $4\pi\beta_i\delta_0$. If $\beta_i \le -1$, we can use the finite energy assumption (see the bottom assumption in (1-7)) to conclude that either (4-1) or (4-2) has no solutions. Hence $\alpha_i - \frac{1}{2}\sum_{j=1}^{2} k_{ij}\sigma_j > -1$ and then (a) is proved.

(b) Since both components have fast decay on $\partial B(0, l_k)$, the pair (σ_1, σ_2) satisfies the P.I. (1-9). By a simple manipulation, the P.I. (1-9) can be written as

$$k_{21}\sigma_1(4\mu_1 - k_{12}\sigma_2 - k_{11}\sigma_1) + k_{12}\sigma_2(4\mu_2 - k_{21}\sigma_1 - k_{22}\sigma_2) = 0. \tag{4-3}$$

Note by (a),

$$4\mu_i - \sum_{l=1}^{2} k_{il}\sigma_l > 2\mu_i - \sum_{l=1}^{2} k_{il}\sigma_l \geqslant 0.$$

Hence for $j \neq i$

$$2\mu_j - \sum_{l=1}^{2} k_{jl}\sigma_l < 4\mu_j - \sum_{l=1}^{2} k_{jl}\sigma_l < 0,$$

where the last inequality is due to (4-3). By (a) again, u_j^k does not have slow decay on $\partial B(0, s_k)$. □

Our second lemma says that a fast-decaying component does not change its energy more than $o(1)$, regardless of the behavior of the other component.

Lemma 4.2. *Suppose the Harnack-type inequality holds for both components over $r \in \left[\frac{1}{2}l_k, 2s_k\right]$. If u_i^k is fast-decaying on $r \in [l_k, s_k]$, then*

$$\sigma_i^k(B(0, s_k)) = \sigma_i^k(B(0, l_k)) + o(1).$$

Proof. Obviously the conclusion holds if $s_k/l_k \leqslant C$. So we assume $s_k/l_k \to +\infty$. The Harnack-type inequality implies $u_i^k(x) = \bar{u}_i^k(r) + o(1)$ for $\frac{1}{2}l_k \leqslant |x| \leqslant 2s_k$. Thus we obtain from (1-5) that

$$\frac{d}{dr}(\bar{u}_i^k(r) + 2\log\ r) = \frac{2\mu_i - \sum_{j=1}^{2} k_{ij}\sigma_j^k(r)}{r}, \quad l_k \leqslant r \leqslant s_k,\ i = 1, 2,$$

where $\sigma_j^k(r) = \sigma_j^k(B(0, r))$ and $\sigma_j = \lim_{k\to+\infty} \sigma_j^k(l_k),\ j = 1, 2$.

Without loss of generality, we assume that $u_j^k,\ j \neq i$, is fast-decaying on $\partial B(0, l_k)$. Otherwise, we may choose $\tilde{l}_k$ such that $l_k \ll \tilde{l}_k,\ u_i^k$ remains fast-decaying for $r \in [l_k, \tilde{l}_k]$ and $\sigma_i^k(B(0, r))$ does not change more than $o(1)$, while u_j^k is fast-decaying on $\partial B(0, \tilde{l}_k)$. If $s_k/\tilde{l}_k \leqslant C$, we get the conclusion as explained above. If $s_k/\tilde{l}_k \to +\infty$, by a little abuse of notation, we may replace $\tilde{l}_k$ by l_k. Then both u_1^k, u_2^k have fast decay on $\partial B(0, l_k)$, and the P.I. holds at l_k, which implies that at least one component (say l) satisfies

$$4\mu_l - \sum_{j=1}^{2} k_{lj}\sigma_j^k(l_k) < 0.$$

Thus,

$$\frac{d}{dr}(\bar{u}_l^{(k)}(r) + 2\log\ r) \leqslant -\frac{2\mu_l + o(1)}{r} \quad \text{at } r = l_k. \tag{4-4}$$

Suppose $r_k \in [l_k, s_k]$ is the largest r such that

$$\frac{d}{dr}(\bar{u}_l^{(k)}(r) + 2\log\, r) \leqslant -\frac{\mu_l}{r} \quad \text{for } r \in [l_k, r_k]. \tag{4-5}$$

Thus, either the equality holds at $r = r_k$ or $r_k = s_k$. For simplicity, we let $\varepsilon = \mu_l$. By integrating (4-4) from l_k up to $r \leqslant r_k$, we have

$$\bar{u}_l^{(k)}(r) + 2\log\, r \leqslant \bar{u}_l^{(k)}(l_k) + 2\log(l_k) + \varepsilon\log\left(\frac{l_k}{r}\right);$$

that is for $|x| = r$,

$$e^{u_l^k(x)} \leqslant O(1)e^{\bar{u}_l^k(r)} \leqslant O(1)e^{-N_k} l_k^{\varepsilon} r^{-(2+\varepsilon)},$$

where we used $\bar{u}_l^{(k)}(l_k) + 2\log l_k \leqslant -N_k$ by the assumption of fast decay. Thus

$$\int_{l_k \leqslant |x| \leqslant r_k} e^{u_l^k(x)}\, dx \leqslant O(1)e^{-N_k} l_k^{\varepsilon} \int_{l_k}^{r_k} r^{-(1+\varepsilon)}\, dr = O(1)\frac{e^{-N_k}}{\varepsilon} \to 0$$

as $k \to +\infty$. Hence

$$\sigma_l^k(r_k) = \sigma_l^k(l_k) + o(1). \tag{4-6}$$

If both components are fast-decaying on $r \in [l_k, r_k]$, then $\lim_{k\to+\infty}(\sigma_1^k(r_k), \sigma_2^k(r_k)) = (\hat{\sigma}_1, \hat{\sigma}_2)$ also satisfies the P.I. (1-9). If $\hat{\sigma}_j > \sigma_j$, then $j \neq l$ by (4-6). We choose $r_k^* \leq r_k$ such that $\sigma_j(r_k^*) = \sigma_j^k(l_k) + \varepsilon_0$ for small ε_0, and let $\sigma_j^* = \lim_{k\to 0} \sigma_j(r_k^*)$. Then σ_j^* and σ_l satisfies the P.I. (1-9) and it yields a contradiction provided ε_0 is small. Thus, we have $\sigma_m^k(r_k) = \sigma_m^k(l_k) + o(1),\ m = 1, 2$. Then (4-4) holds at $r = r_k$, which implies $r_k = s_k$, and Lemma 4.2 is proved in this case.

If one of the components does not have fast decay on $[l_k, r_k]$, then we have $l = i$ and $u_j^k,\ j \neq i$, has slow decay on $\partial B(0, r_k^*)$ for some $r_k^* \leq r_k$. If $s_k/r_k \leq C$, then (4-6) implies the lemma. If $s_k/r_k \to +\infty$, then by the scaling of u_j^k at $r = r_k^*$, the standard argument implies that there is a sequence of $r_k^* \ll \tilde{r}_k = R_k r_k^* \ll s_k$ such that both components have fast decay on $\tilde{r}_k$ and

$$\sigma_i^k(\tilde{r}_k) = \sigma_i(r_k^*) + o(1) = \sigma_i(l_k) + o(1) \quad \text{and} \quad \sigma_j^k(\tilde{r}_k) \geq \sigma_j^k(l_k) + \varepsilon_0$$

for $j \neq i$ and $\varepsilon_0 > 0$. Therefore the assumption of Lemma 4.2 holds at $r \in [\tilde{r}_k, s_k]$. Then we repeat the argument starting from (4-4) and the lemma can be proved in a finite steps. □

Remark 4.3. Both lemmas will be used in Section 6 (and Section 5) for the case with singularity at 0 (and without singularity at 0).

5. Local mass on the bubbling disk centered at $x_l^k \neq 0$

5A. In this subsection we study the local behavior of u^k near x_l^k, where $x_l^k \neq 0$. For simplicity, we use x^k instead of x_l^k and $\bar{u}_i^k(r)$ rather than $\bar{u}_{x_l^k, i}^k(r)$. Let

$$\tau^k = \tfrac{1}{2}\,\text{dist}(x^k, \Sigma_k \setminus \{x^k\}), \quad \sigma_i^k(r) = \frac{1}{2\pi}\int_{B(x^k, r)} h_i^k e^{u_i^k}, \quad i = 1, 2.$$

By Proposition 3A, $l_k \le \tau^k$. Clearly $u^k = (u_1^k, u_2^k)$ satisfies

$$\Delta u_i^k + \sum_{j=1}^{2} k_{ij} h_j^k e^{u_j^k} = 0 \quad \text{in } B(x^k, \tau^k).$$

For a sequence s_k, we define

$$\hat{\sigma}_i(s_k) = \begin{cases} \lim_{k\to+\infty} \sigma_i^k(s_k) \text{ if } u_i^k \text{ has fast decay on } \partial B(x^k, s_k), \\ \lim_{r\to 0} \lim_{k\to+\infty} \sigma_i^k(r s_k) \text{ if } u_i^k \text{ has slow decay on } \partial B(x^k, s_k). \end{cases} \tag{5-1}$$

Recall that both components of u^k have fast decay on $\partial B(x^k, l_k)$. This is the starting point of the following proposition, which is a special case of Proposition 5.2 below.

In Proposition 5.1, (μ_1, μ_2) will be $(1, 1)$ in both lemmas of Section 4.

Proposition 5.1. *Let $u^k = (u_1^k, u_2^k)$ be the solution of* (1-5) *satisfying* (1-7) *and $\hat{\sigma}_i(s_k)$ be defined in* (5-1). *The following holds*:

(1) *At least one component u^k has fast decay on $\partial B(x^k, \tau^k)$.*

(2) $(\hat{\sigma}_1(\tau^k), \hat{\sigma}_2(\tau^k))$ *satisfies the P.I.* (1-9) *with $\mu_1 = \mu_2 = 1$.*

(3) $(\hat{\sigma}_1(\tau^k), \hat{\sigma}_2(\tau^k)) \in \Gamma(1, 1)$.

Proof. If $\tau_k / l_k \leqslant C$, (1)–(3) hold obviously for τ^k. So we assume $\tau^k / l_k \to +\infty$. First we remark that if u^k is fully bubbling in $B(x^k, l_k)$ (i.e., (1) in Proposition 3A holds), $(\hat{\sigma}_1(l_k), \hat{\sigma}_2(l_k))$ is special (see Definition 1.4) and satisfies

$$2\mu_i - \sum_{j=1}^{2} k_{ij} \hat{\sigma}_j(l_k) < 0, \quad i = 1, 2.$$

Then by Lemma 4.1, both u_i^k have fast decay on $\partial B(0, \tau^k)$ and Proposition 5.1 follows immediately.

Now we assume v_i^k defined in (3-7) and satisfies case (2) in Proposition 3A. We already know that both components have fast decay at $r = l_k$. If both components remain fast-decaying as r increases from l_k to τ^k, Lemma 4.2 implies

$$\sigma_1^k(\tau^k) = \sigma_1^k(l_k) + o(1), \quad \sigma_2^k(\tau^k) = \sigma_2^k(l_k) + o(1)$$

and we are done. So we only consider the case that at least one component changes to a slow-decaying component. For simplicity, we assume that u_1^k changes to a slow-decaying component for some $r_k \gg l_k$. By Lemma 4.2,

$$\sigma_1^k(B(x^k, r_k)) \geqslant \sigma_1^k(B(x^k, l_k)) + c_0 \quad \text{for some } c_0 > 0.$$

We might choose $s_k \leqslant r_k$ such that

$$\sigma_1^k(B(x^k, s_k)) = \sigma_1^k(B(x^k, l_k)) + \varepsilon_0,$$

and

$$\sigma_1^k(B(x^k, r)) < \sigma_1^k(B(x^k, l_k)) + \varepsilon_0 \quad \text{for all } r < s_k,$$

where $\varepsilon_0 < \frac{1}{2} c_0$ is small.

Then Lemmas 4.1 and 4.2 together imply that u_1^k has slow decay on $\partial B(x^k, s_k)$ and u_2^k has fast decay on $\partial B(x^k, s_k)$ with

$$\hat{\sigma}_1(s_k) = \sigma_1^k(l_k) + o(1) \quad \text{and} \quad \hat{\sigma}_2(s_k) = \sigma_2^k(l_k) + o(1).$$

Let $v_i^k(y) = u_i^k(x^k + s_k y) + 2\log\, s_k$. If $\tau^k/s_k \le C$, there is nothing to prove. So we assume $\tau^k/s_k \to \infty$. Then $v_1^k(y)$ converges to $v_1(y)$ and $v_2^k(y) \to -\infty$ in any compact set of $\mathbb{R}^2$ as $k \to +\infty$ and $v_1(y)$ satisfies

$$\Delta v_1 + 2e^{v_1} = -2\pi \sum_{j=1}^{2} (k_{1j}\hat{\sigma}_j(l_k))\delta(0) \quad \text{in } \mathbb{R}^2. \tag{5-2}$$

Hence there is a sequence $N_k^* \to +\infty$ as $k \to +\infty$ that satisfies

(1) $N_k^* s_k \le \tau^k$,

(2) $\int_{B(0,N_k^*)} e^{v_1}\, dy = \int_{\mathbb{R}^2} e^{v_1}\, dy + o(1)$,

(3) $v_i^k(y) + 2\log|y| \leqslant -N_k,\ i = 1, 2$, for $|y| = N_k^*$.

Scaling back to u_i^k, we obtain that $u_i^k,\ i = 1, 2$, have fast decay on $\partial B(x^k, N_k^* s_k)$.

We could use the classification theorem of [Prajapat and Tarantello 2001] to calculate the total mass of v_1, but instead we use the P.I. (1-9) to compute it. We know that both $(\hat{\sigma}_1(l_k), \hat{\sigma}_2(l_k))$ and $(\hat{\sigma}_1(N_k^* s_k), \hat{\sigma}_2(N_k^* s_k))$ satisfy the P.I. and $\hat{\sigma}_2(N_k^* s_k) = \hat{\sigma}_2(l_k)$ by Lemma 4.2. With a fixed $\sigma_2 = \hat{\sigma}_2(l_k)$, P.I. (1-9) is a quadratic polynomial in σ_1; then $\hat{\sigma}_1(l_k)$ and $\hat{\sigma}_1(N_k^* s_k)$ are two roots of the polynomial. From it, we can easily calculate $\hat{\sigma}_1(N_k^* s_k)$.

By a direct computation, we have

$$(\hat{\sigma}_1(N_k^* s_k), \hat{\sigma}_2(N_k^* s_k)) \in \Gamma(1,1) \quad \text{if } (\hat{\sigma}_1(l_k), \hat{\sigma}_2(l_k)) \in \Gamma(1,1).$$

Thus (1)–(3) hold at $r = N_k^* s_k$. By denoting $N_k^* s_k$ as l_k, we can repeat the same argument until $\tau^k/l_k \leqslant C$. Hence Proposition 5.1 is proved. □

5B. ***Local mass in a group that does not contain* 0.** In this subsection we collect some $x_i^k \in \Sigma_k$ into a group S, a subset of Σ_k satisfying the following S-conditions:

(1) $0 \notin S$ and $|S| \ge 2$.

(2) If $|S| \ge 3$ and x_i^k, x_j^k, x_l^k are three distinct elements in S, then

$$\text{dist}(x_i^k, x_j^k) \le C\, \text{dist}(x_j^k, x_l^k)$$

for some constant C independent of k.

(3) For any $x_m^k \in \Sigma_k \setminus S$, we have $\text{dist}(x_m^k, S)/\text{dist}(x_i^k, x_j^k) \to \infty$ as $k \to \infty$, where $x_i^k, x_j^k \in S$.

We write S as $S = \{x_1^k, \dots, x_m^k\}$ and let

$$l^k(S) = 2 \max_{1 \le j \le m} \text{dist}(x_1^k, x_j^k). \tag{5-3}$$

Recall $\tau_l^k = \frac{1}{2}\operatorname{dist}(x_l^k, \Sigma_k \setminus \{x_l^k\})$; by (2) and (3) above we have $l^k(S) \sim \tau_i^k$ for $1 \le i \le m$. Let

$$\tau_S^k = \tfrac{1}{2}\operatorname{dist}(x_1^k, \Sigma_k \setminus S).$$

Then by (3) above we have $\tau_S^k/\tau_i^k \to \infty$ for any $x_i^k \in S$.

By Proposition 5.1, we know that at least one of u_i^k has fast decay on $\partial B(x_1^k, \tau_1^k)$. Suppose u_1^k has fast decay on $\partial B(x_1^k, \tau_1^k)$. Then

$$u_1^k \text{ has fast decay on } \partial B(x_1^k, l^k(S)), \tag{5-4}$$

and we get

$$\begin{aligned}\sigma_1^k(B(x_1^k, l^k(S))) &= \frac{1}{2\pi}\int_{B(x_1^k, l^k(S))} h_1^k e^{u_1^k}\,dx \\ &= \frac{1}{2\pi}\int_{\bigcup_{j=1}^m B(x_j^k,\tau_j^k)} h_1^k e^{u_1^k} + \frac{1}{2\pi}\int_{B(x_1^k,l^k(S))\setminus(\bigcup_{j=1}^m B(x_j^k,\tau_j^k))} h_1^k e^{u_1^k}.\end{aligned}$$

Since u_1^k has fast decay outside of $B(x_j^k, \tau_j^k)$, we have

$$e^{u_1^k(x)} \le o(1)\max_j\{|x - x_j^k|^{-2}\} \quad \text{for } x \notin \bigcup_{j=1}^k B(x_j^k, \tau_j^k)$$

and the second integral is $o(1)$. Hence by Proposition 5.1,

$$\sigma_1^k(B(x_1^k, l^k(S))) = 2m_1 + o(1) \quad \text{for some } m_1 \in \mathbb{N}\cup\{0\}. \tag{5-5}$$

Similarly if u_2^k has fast decay on $\partial B(x_1^k, \tau_1^k)$, we have

$$\sigma_2^k(B(x_1^k, l^k(S))) = 2m_2 + o(1) \quad \text{for some } m_2 \in \mathbb{N}\cup\{0\}. \tag{5-6}$$

If u_2^k has slow decay on $\partial B(x_1^k, \tau_1^k)$, then it is easy to see that u_2^k has slow decay on $\partial B(x_j^k, \tau_j^k)$. By Proposition 5.1 we denote $n_{i,j} \in \mathbb{N}$ by

$$2n_{i,j} = \lim_{r\to 0}\lim_{k\to\infty}\sigma_i^k(B(x_j^k, r\tau_j^k)), \quad 1 \le j \le m,\ i = 1, 2.$$

Define $\hat{n}_{i,j}$ by

$$\hat{n}_{i,j} = -\sum_{l=1}^2 k_{il} n_{l,j}.$$

Then the slow decay of u_2^k on $\partial B(x_j^k, \tau_j^k)$ implies $1 + \hat{n}_{2,j} > 0$. Since $\hat{n}_{2,j} \in \mathbb{Z}$ we have $\hat{n}_{2,j} \ge 0$.

Furthermore, if we scale u^k by

$$v_i^k(y) = u_i^k(x_1^k + l^k(S)y) + 2\log l^k(S), \quad i = 1, 2,$$

the sequence v_2^k converges to $v_2(y)$ and v_1^k tends to $-\infty$ over any compact subset of $\mathbb{R}^2 \setminus \{0\}$. Then v_2 satisfies

$$\Delta v_2(y) + 2e^{v_2(y)} = 4\pi\sum_{j=1}^m \hat{n}_{2,j}\delta_{p_j} \quad \text{in } \mathbb{R}^2, \tag{5-7}$$

where $p_j = \lim_{k\to\infty}(x_j^k - x_1^k)/l^k(S)$. By Theorem 2.1

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_2} = 2N \quad \text{for some } N \in \mathbb{N}.$$

Thus using the argument in Proposition 5.1, we conclude that there is a sequence of $N_k^* \to \infty$ such that both u_i^k $(i = 1, 2)$ have fast decay on $\partial B(x_1^k, N_k^* l^k(S))$ and $\sigma_i^k(B(x_1^k, N_k^* l^k(S))) = 2m_i + o(1)$. Denote $N_k^* l^k(S)$ by l_k for simplicity; we see that (5-5) and (5-6) hold at l_k. Then by using Lemmas 4.1 and 4.2 we continue this process to obtain the following conclusion:

$$\text{At least one component of } u^k \text{ has fast decay on } \partial B(x_1^k, \tau_S^k). \tag{5-8}$$

Let $\hat{\sigma}_i^k(B(x_1^k, \tau_S^k))$ be defined as in (5-1). Then

$$\hat{\sigma}_i^k(B(x_1^k, \tau_S^k)) = 2m_i(S), \quad \text{where } m_i(S) \in \mathbb{N} \cup \{0\}, \tag{5-9}$$

and the pair $(2m_1(S), 2m_2(S))$ satisfies the P.I. (1-9).

Denote the group S by S_1. Based on this procedure, we can continue to select a new group S_2 such that the S-conditions holds except we have to modify condition (2). In (2), we consider S_1 as a single point as long as we compare the distance of distinct elements in S_2.

Set

$$\tau_{S_2}^k = \tfrac{1}{2}\operatorname{dist}(x_1^k, \Sigma_k \setminus S_2) \quad \text{for } x_1^k \in S_2.$$

Then we follow the same argument as above to obtain the same conclusion as (5-8)–(5-9).

If (1-5) does not contain a singularity, the final step is to collect all the x_i^k into the single biggest group and (5-8)–(5-9) hold. Then we get $(\sigma_1, \sigma_2) = (2m_1, 2m_2)$ (which satisfies the Pohozaev identity), where

$$\sigma_i = \lim_{r\to 0}\lim_{k\to\infty}\frac{1}{2\pi}\int_{B(0,r)} h_i^k e^{u_i^k}, \quad i = 1, 2.$$

By a direct computation, we can prove that the set of all the pairs of even integers solving (1-9) is exactly $\Gamma(1, 1)$. This proves Theorem 1.3 if (1-5) has no singularities.

If 0 is a singularity of (1-5) then Σ_k can be written as a disjoint union of $\{0\}$ and S_j $(j = 1, \dots, m)$. Here each S_j is collected by the process described above and is maximal in the following sense:

(i) $0 \notin S$, $|S| \ge 2$ and for any two distinct points x_i^k, x_j^k in S we have

$$\operatorname{dist}(x_i^k, x_j^k) \ll \tau^k(S),$$

where $\tau^k(S) = \operatorname{dist}(S, \Sigma_k \setminus S)$.

(ii) For any $0 \ne x_i^k \in \Sigma_k \setminus S$,

$$\operatorname{dist}(x_i^k, 0) \le C \operatorname{dist}(x_i^k, S)$$

for some constant C.

For S_j we define

$$\tau_{S_j}^k = \tfrac{1}{2}\operatorname{dist}(S_j, \Sigma_k \setminus S_j).$$

Then the process described above proves the main result of this section:

Proposition 5.2. *Let S_j $(j = 1, \dots, m)$ be described as above. Then* (5-8)–(5-9) *hold*, *where $B(x_1^k, \tau_S^k)$ is replaced by $B(x_i^k, \tau_{S_j}^k)$ and x_i^k is any element in S_j.*

6. Proofs of Theorems 1.2, 1.3, 1.5 and 1.6

In Proposition 5.2, we write $\Sigma_k = \{0\} \cup S_1 \cup \dots \cup S_N$. From the construction, the ratio $|x^k|/|\tilde{x}^k|$ is bounded for any $x^k, \tilde{x}^k \in S_j$. Let

$$\|S_j\| = \min_{x^k \in S_j} |x^k|$$

and arrange S_j by

$$\|S_1\| \le \|S_2\| \le \dots \le \|S_N\|.$$

Assume l is the largest number such that $\|S_l\| \le C\|S_1\|$. Then $\|S_l\| \ll \|S_{l+1}\|$.

We recall the local mass contributed by $x_j^k \in S_j$ is

$$\big(\hat{\sigma}_1(B(x_j^k, \tau_j^k)), \hat{\sigma}_2(B(x_j^k, \tau_j^k))\big) = (m_{1,j}, m_{2,j}), \quad \text{where } m_{1,j}, m_{2,j} \in 2\mathbb{N} \cup \{0\}.$$

Let

$$r_1^k = \tfrac{1}{2}\|S_1\|.$$

Then we have

$$u_i^k(x) + 2\log|x| \le C \quad \text{for } 0 < |x| \le r_1^k,\ i = 1, 2.$$

Proof of Theorem 1.3. Let

$$\tilde{u}_i^k(x) = u_i^k(x) + 2\alpha_i \log|x|, \quad i = 1, 2.$$

Then (1-5) becomes

$$\Delta\tilde{u}_i^k(x) + \sum_{j=1}^{2} k_{ij}|x|^{2\alpha_j} h_j^k(x) e^{\tilde{u}_j^k(x)} = 0, \quad |x| \le r_1^k,\ i = 1, 2.$$

Let

$$-2\log\delta_k = \max_{i \in I} \max_{x \in \bar{B}(0, r_1^k)} \frac{\tilde{u}_i^k}{1 + \alpha_i}, \tag{6-1}$$

and

$$\tilde{v}_i^k(y) = \tilde{u}_i^k(\delta_k y) + 2(1 + \alpha_i)\log\delta_k, \quad |y| \le r_1^k/\delta_k,\ i = 1, 2. \tag{6-2}$$

Then $\tilde{v}_i^k$ satisfies

$$\Delta\tilde{v}_i^k(y) + \sum_{j=1}^{2} k_{ij}|y|^{2\alpha_j} h_j^k(\delta_k y) e^{\tilde{v}_j^k(y)} = 0, \quad |y| \le r_1^k/\delta_k,\ i = 1, 2. \tag{6-3}$$

We have either

(a) $\lim_{k\to\infty} r_1^k/\delta_k = \infty$, or

(b) $r_1^k/\delta_k \le C$.

For case (a), our purpose is to prove a result similar to Proposition 5.1:

(1) *At most one component of u^k has slow decay on $\partial B(0,r_1^k)$.* As in Section 5, we define

$$\hat\sigma_{i,1}=\begin{cases}\lim_{k\to+\infty}\sigma_i^k(B(0,r_1^k)) & \text{if } u_i^k \text{ has fast decay on } \partial B(0,r_1^k),\\ \lim_{r\to 0}\lim_{k\to+\infty}\sigma_i^k(B(0,rr_1^k)) & \text{if } u_i^k \text{ has slow decay on } \partial B(0,r_1^k),\end{cases}$$

(2) $(\hat\sigma_{1,1},\hat\sigma_{2,1})$ satisfies the Pohozaev identity (1-9), and

(3) $\hat\sigma_{i,1}=2\sum_{j=1}^2 n_{i,j}\mu_j+2n_{i,3},\ n_{i,j}\in\mathbb{Z},\ i=1,2,\ j=1,2,3.$

We carry out the proof in the discussion of the following two cases.

Case 1: If both $\tilde v_i^k(y)$ converge in any compact set of $\mathbb{R}^2$, then $(\hat\sigma_{1,1},\hat\sigma_{2,1})$ can be obtained by the classification theorem in [Lin et al. 2012]:

$$(\hat\sigma_{1,1},\hat\sigma_{2,1})=\begin{cases}(2\mu_1+2\mu_2,2\mu_1+2\mu_2) & \text{for } A_2,\\ (4\mu_1+2\mu_2,4\mu_1+4\mu_2) & \text{for } B_2,\\ (8\mu_1+4\mu_2,12\mu_1+8\mu_2) & \text{for } G_2.\end{cases}$$

By Lemma 4.1, both u_i^k have fast decay on $\partial B(0,r_1^k)$. So this proves (1)–(3) in this case.

Case 2: Only one $\tilde v_i^k$ converges to $v_i(y)$ and the other tends to $-\infty$ uniformly in any compact set. Then it is easy to see that there is $l_k\ll r_1^k$ such that both u_i^k have fast decay on $\partial B(0,l_k)$ and

$$\big(\sigma_1(B(0,l_k)),\sigma_2(B(0,l_k))\big)=(2\mu_1,0)\quad\text{or}\quad\big(\sigma_1(B(0,l_k)),\sigma_2(B(0,l_k))\big)=(0,2\mu_2).$$

So this is the same situation as in the starting point for Proposition 5.1. Then the same argument of Proposition 5.1 leads to the conclusion (1)–(3).

The pair $(\hat\sigma_{1,1},\hat\sigma_{2,1})$ can be calculated by the same method in Proposition 5.1. Then $(\hat\sigma_{1,1},\hat\sigma_{2,1})\in\Gamma(\mu_1,\mu_2)$, which is given in Section 2.

To continue for $r\in[r_1^k,r_2^k]$, where $r_2^k=\frac12\|S_{l+1}\|$, we separate our discussion into two cases also.

Case 1: One component has slow decay on $\partial B(0,r_1^k)$, say u_1^k. Then we scale

$$v_i^k(y)=u_i^k(r_1^ky)+2\log r_1^k.$$

By our assumption, $v_1^k(y)$ converges to $v_1(y)$ and $v_2^k(y)\to-\infty$ in any compact set. Let $x_j^k\in S_j$ and $y_j^k=(r_1^k)^{-1}x_j^k\to p_j$ for $j\le l$. Then $v_1(y)$ satisfies

$$\Delta v_1+2e^{v_1}=4\pi\tilde\alpha_1\delta_0+4\pi\sum_{j=1}^{l}\tilde n_{1,j}\delta_{p_j}, \tag{6-4}$$

where

$$\tilde n_{1,j}=-\frac12\sum_{i=1}^2 k_{1i}m_{i,j}\quad\text{for some } m_{ij}\in\mathbb{Z}\qquad\text{and}\qquad\tilde\alpha_1=\alpha_1-\frac12\sum_{i=1}^2 k_{1i}\hat\sigma_{i,1}. \tag{6-5}$$

The finiteness of $\int_{\mathbb{R}^2}e^{v_1}$ implies

$$\tilde\alpha_1>-1\quad\text{and}\quad\tilde n_{1,j}\ge 0.$$

By Theorem 2.2, we have

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_1}\,dy = 2(\tilde{\alpha}_1+1)+2k_1, \quad \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_1}\,dy = 2k_2, \quad \text{where } k_1,k_2\in\mathbb{Z}. \tag{6-6}$$

As before, we can choose $l_k,\ r_1^k \ll l_k \ll r_2^k$, such that both u_i^k have fast decay on $\partial B(0,l_k)$. Then the new pair $(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2})$, which is defined by

$$\hat{\sigma}_{t,2} = \frac{1}{2\pi}\lim_{k\to 0}\int_{B(0,l_k)} h_t^k e^{u_t^k}, \quad t=1,2,$$

becomes

$$(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2}) = \left(\hat{\sigma}_{1,1}+\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_1}+\sum_{j=1}^{l} m_{1,j},\ \hat{\sigma}_{2,1}+\sum_{j=1}^{l} m_{2,j}\right) \tag{6-7}$$

for $m_{1j},m_{2j}\in 2\mathbb{N}\cup\{0\}$. Using (6-6), we get

$$\hat{\sigma}_{1,2} = \begin{cases} \hat{\sigma}_{1,1}+2k_2+\sum_{j=1}^{l} m_{1,j} & \text{if } \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_1}\,dy = 2k_2, \\ 2\mu_1+\hat{\sigma}_{1,1}-\sum_{i=1}^{2} k_{1i}\hat{\sigma}_{i,1}+2k_1+\sum_{j=1}^{l} m_{1,j} & \text{if } \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_1}\,dy = 2(\tilde{\alpha}_1+1)+2k_1. \end{cases} \tag{6-8}$$

We note that if $(\hat{\sigma}_{1,1},\hat{\sigma}_{2,1})\in\Gamma(\mu_1,\mu_2)$ and

$$2\mu_1+\hat{\sigma}_{1,1}-\sum_{i=1}^{2} k_{1i}\hat{\sigma}_{i,1} > 0,$$

then

$$\left(2\mu_1+\hat{\sigma}_{1,1}-\sum_{i=1}^{2} k_{1i}\hat{\sigma}_{i,1},\hat{\sigma}_{2,1}\right)\in\Gamma(\mu_1,\mu_2).$$

Let $(\sigma_1^*,\sigma_2^*) = \bigl(2\mu_1+\hat{\sigma}_{1,1}-\sum_{i=1}^{2} k_{1i}\hat{\sigma}_{i,1},\ \hat{\sigma}_{2,1}\bigr)$. We can write

$$(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2}) = (\sigma_1^*+m_1,\ \sigma_2^*+m_2), \tag{6-9}$$

with $(\sigma_1^*,\sigma_2^*)\in\Gamma(\mu_1,\mu_2)$ and $m_1,m_2\in 2\mathbb{Z}$.

Case 2: If both u_i^k have fast decay on $\partial B(0,r_1^k)$, then they have fast decay on $\partial B(0,cr_1^k)$, where we choose c bounded such that $\bigcup_{j=1}^{l} S_j \subset B\bigl(0,\frac{1}{2}cr_1^k\bigr)$. Then the new pair $(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2})$ becomes

$$(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2}) = \left(\hat{\sigma}_{1,1}+\sum_{j=1}^{l} m_{1,j},\ \hat{\sigma}_{2,1}+\sum_{j=1}^{l} m_{2,j}\right) \quad \text{for } m_{1,j},m_{2,j}\in 2\mathbb{Z}. \tag{6-10}$$

Hence, in this case we can also write

$$(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2}) = (\sigma_1^*+m_1,\ \sigma_2^*+m_2), \tag{6-11}$$

with $(\sigma_1^*,\sigma_2^*) = (\hat{\sigma}_{1,1},\hat{\sigma}_{2,1})\in\Gamma(\mu_1,\mu_2)$ and $m_1,m_2\in 2\mathbb{Z}$. Set $cr_1^k = l_k$. Then we can continue our process starting from l_k. After finitely many steps, we can prove that at most one component of u^k has slow decay on $\partial B(0,1)$ and their local masses have the expression in (3).

For case (b), i.e., $r_1^k/\delta_k \le C$, first $\tilde{v}_i^k \le 0$ implies $|y|^{2\alpha_j} h_j^k(\delta_k y)e^{\tilde{v}_j^k} \le C$ on $B(0, r_1^k/\delta_k)$. Then the fact that $\tilde{v}_i^k$ has bounded oscillation on $\partial B(0, r_1^k/\delta_k)$ further gives

$$\tilde{v}_i^k(x) = \bar{\tilde{v}}_i^k(\partial B(0, r_1^k/\delta_k)) + O(1) \quad \text{for all } x \in B(r_1^k/\delta_k),$$

where $\bar{\tilde{v}}_i^k(\partial B(0, r_1^k/\delta_k))$ stands for the average of $\tilde{v}_i^k$ on $\partial B(0, r_1^k/\delta_k)$. Direct computation shows that

$$\int_{B(0,r_1^k)} h_i^k e^{u_i^k}\, dx = \int_{B(0,r_1^k/\delta_k)} |y|^{2\alpha_i} h_i^k(\delta_k y)e^{\tilde{v}_i^k(y)}\, dy = O(1)e^{\bar{\tilde{v}}_i^k(\partial B(0,r_1^k/\delta_k))}.$$

Thus if $\bar{\tilde{v}}_i^k(\partial B(0, r_1^k/\delta_k)) \to -\infty$, we get $\int_{B(0,r_1^k)} h_i^k e^{u_i^k}\, dx = o(1)$. On the other hand, we note that $\bar{\tilde{v}}_i^k(\partial B(0, r_1^k/\delta_k)) \to -\infty$ is equivalent to u_i^k having fast decay on $\partial B(0, r_1^k)$. Consequently $\hat{\sigma}_{i,1} = 0$ if u_i^k has fast decay on $\partial B(0, r_1^k)$. So if both components have fast decay on $\partial B(0, r_1^k)$ we have $(\hat{\sigma}_{1,1}, \hat{\sigma}_{2,1}) = (0, 0)$.

If some component of u^k has slow decay, say u_2^k, according to the definition of $\hat{\sigma}_{2,1}$, we have

$$\begin{aligned}\hat{\sigma}_{2,1} &= \lim_{r\to 0}\lim_{k\to+\infty} \sigma_2^k(B(0, rr_1^k)) = \frac{1}{2\pi}\lim_{r\to 0}\lim_{k\to+\infty}\int_{B(0,rr_1^k)} h_2^k e^{u_2^k}\, dx \\ &= \frac{1}{2\pi}\lim_{r\to 0}\lim_{k\to+\infty}\int_{B(0,rr_1^k/\delta_k)} |y|^{2\alpha_2} h_2^k(\delta_k y)e^{\tilde{v}_2^k(y)}\, dy = 0,\end{aligned} \tag{6-12}$$

where we used $|y|^{2\alpha_2} h_2^k(\delta_k y)e^{\tilde{v}_2^k} \le C$ on $B(0, r_1^k/\delta_k)$. Then we still get

$$(\hat{\sigma}_{1,1}, \hat{\sigma}_{2,1}) = (0, 0).$$

Now we can continue our discussion as in case (a) and Theorem 1.3 is proved completely. □

Next, we shall prove Theorem 1.5, that is, $\Sigma_k = \{0\}$, by way of contradiction. Suppose Σ_k has points other than 0. Using the notation from the beginning of this section, we have

$$\Sigma_k = \{0\} \cup S_1 \cup \cdots \cup S_N.$$

Now suppose $r_1^k/\delta_k \to \infty$ as $k \to \infty$. Let $(\hat{\sigma}_{1,2}, \hat{\sigma}_{2,2})$ be the local masses defined by (6-7) for one of the components u_i^k having slow decay on $\partial B(0, r_1^k)$ or by (6-10) for both components having fast decay on $\partial B(0, r_1^k)$. We summarize the results in the following:

(i) $\hat{\sigma}_{i,2} = \sigma_i^* + m_i$, where $(\sigma_1^*, \sigma_2^*) \in \Gamma(\mu_1, \mu_2)$ and $m_i,\ i = 1, 2$, are even integers.

(ii) Both pairs (σ_1^*, σ_2^*) and $(\hat{\sigma}_{1,2}, \hat{\sigma}_{2,2})$ satisfy the Pohozaev identity.

Based on the description above, we now present the proof of Theorem 1.5.

Proof of Theorem 1.5. From the discussion above, we have

$$(\hat{\sigma}_{1,2}, \hat{\sigma}_{2,2}) = (\sigma_1^* + m_1,\ \sigma_2^* + m_2).$$

We note that the conclusion of Theorem 1.5 is equivalent to proving $m_i = 0,\ i = 1, 2$. In order to prove this we first observe that both $(\hat{\sigma}_{1,2}, \hat{\sigma}_{2,2})$ and (σ_1^*, σ_2^*) satisfy the P.I.

$$k_{21}\sigma_1^2 + k_{12}k_{21}\sigma_1\sigma_2 + k_{12}\sigma_2^2 = 2k_{21}\mu_1\sigma_1 + 2k_{12}\mu_2\sigma_2. \tag{6-13}$$

Thus we can write

$$k_{21}(\sigma_1^*)^2 + k_{12}k_{21}\sigma_1^*\sigma_2^* + k_{12}(\sigma_2^*)^2 = 2k_{21}\mu_1\sigma_1^* + 2k_{12}\mu_2\sigma_2^*, \tag{6-14}$$

and

$$\begin{aligned} k_{21}(\sigma_1^*+m_1)^2 + k_{12}k_{21}(\sigma_1^*+m_1)(\sigma_2^*+m_2) + k_{12}(\sigma_2^*+m_2)^2 \\ = 2k_{21}\mu_1(\sigma_1^*+m_1) + 2k_{12}\mu_2(\sigma_2^*+m_2). \end{aligned} \tag{6-15}$$

It is easy to obtain the following from (6-15) and (6-14):

$$\begin{aligned} 2k_{21}m_1\sigma_1^* + k_{12}k_{21}m_2\sigma_1^* + k_{12}k_{21}m_1\sigma_2^* + 2k_{12}m_2\sigma_2^* \\ = 2k_{21}m_1\mu_1 + 2k_{12}m_2\mu_2 - (k_{21}m_1^2 + k_{12}k_{21}m_1m_2 + k_{12}m_2^2). \end{aligned} \tag{6-16}$$

Since $(\sigma_1^*, \sigma_2^*) \in \Gamma(\mu_1, \mu_2)$, we set

$$\sigma_1^* = l_{1,1}\mu_1 + l_{1,2}\mu_2, \quad \sigma_2^* = l_{2,1}\mu_1 + l_{2,2}\mu_2.$$

Then we can rewrite (6-16) as

$$\begin{aligned} \big(2k_{21}l_{1,1}m_1 + k_{12}k_{21}l_{2,1}m_1 - 2k_{21}m_1 + 2k_{12}l_{2,1}m_2 + k_{12}k_{21}l_{1,1}m_2\big)\mu_1 \\ + \big(2k_{21}l_{1,2}m_1 + k_{12}k_{21}l_{2,2}m_1 + 2k_{12}l_{2,2}m_2 + k_{12}k_{21}l_{1,2}m_2 - 2k_{12}m_2\big)\mu_2 \\ + (k_{21}m_1^2 + k_{12}k_{21}m_1m_2 + k_{12}m_2^2) = 0. \end{aligned} \tag{6-17}$$

Since μ_1, μ_2 and 1 are linearly independent, the coefficients of μ_1 and μ_2 must vanish. Equivalently we have

$$\begin{pmatrix} 2k_{21}l_{1,1} + k_{12}k_{21}l_{2,1} - 2k_{21} & 2k_{12}l_{2,1} + k_{12}k_{21}l_{1,1} \\ 2k_{21}l_{1,2} + k_{12}k_{21}l_{2,2} & 2k_{12}l_{2,2} + k_{12}k_{21}l_{1,2} - 2k_{12} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = 0. \tag{6-18}$$

Let M_K be the coefficient matrix

$$M_K = \begin{pmatrix} 2k_{21}l_{1,1} + k_{12}k_{21}l_{2,1} - 2k_{21} & 2k_{12}l_{2,1} + k_{12}k_{21}l_{1,1} \\ 2k_{21}l_{1,2} + k_{12}k_{21}l_{2,2} & 2k_{12}l_{2,2} + k_{12}k_{21}l_{1,2} - 2k_{12} \end{pmatrix}.$$

Our goal is to show that M_k is nonsingular, which immediately implies $m_1 = m_2 = 0$ and completes the proof of Theorem 1.5. The proof of the nonsingularity of M_k is divided into the following three cases.

Case 1: $\boldsymbol{K} = A_2$. Then we can write (6-18) as

$$\begin{pmatrix} 2l_{1,1} - l_{2,1} - 2 & 2l_{2,1} - l_{1,1} \\ 2l_{1,2} - l_{2,2} & 2l_{2,2} - l_{1,2} - 2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = 0. \tag{6-19}$$

We note that

$$(l_{1,1}, l_{1,2}, l_{2,1}, l_{2,2}) \in \big\{(2,0,0,0),\ (0,0,0,2),\ (2,2,0,2),\ (2,0,2,2),\ (2,2,2,2)\big\}.$$

Then it is easy to see that M_K is nonsingular when $(l_{1,1}, l_{1,2}, l_{2,1}, l_{2,2})$ belongs the above set.

Case 2: $\boldsymbol{K} = B_2$. Then we can write (6-18) as

$$\begin{pmatrix} 2l_{1,1} - l_{2,1} - 2 & l_{2,1} - l_{1,1} \\ 2l_{1,2} - l_{2,2} & l_{2,2} - l_{1,2} - 1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = 0. \tag{6-20}$$

We note that

$$(l_{1,1}, l_{1,2}, l_{2,1}, l_{2,2}) \in \{(2,0,0,0),\ (2,0,4,2),\ (4,2,4,2),\ (0,0,0,2),\ (2,2,0,2),\ (2,2,4,4),\ (4,2,4,4)\}$$

From the above set, we can see that $4 \mid (l_{2,1} - l_{1,1})(2l_{1,2} - l_{2,2})$. As a result, if the determinant of M_K is 0, we have to make $4 \mid (2l_{1,1} - l_{2,1} - 2)$, which forces $l_{2,1} \equiv 2 \pmod 4$. However, this is impossible according to the above list. Thus M_k is nonsingular in this case.

Case 3: $\boldsymbol{K} = G_2$. Then we can write (6-18) as

$$\begin{pmatrix} 6l_{1,1}-3l_{2,1}-6 & 2l_{2,1}-3l_{1,1} \\ 6l_{1,2}-3l_{2,2} & 2l_{2,2}-3l_{1,2}-2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = 0. \tag{6-21}$$

We note that

$$(l_{1,1}, l_{1,2}, l_{2,1}, l_{2,2}) \in \{(2,0,0,0),\ (2,0,6,2),\ (6,2,6,2),\ (6,2,12,6),\ (8,4,12,6),\ (8,4,12,8),\ (0,0,0,2),\ (2,2,0,2),\ (2,2,6,6),\ (6,4,6,6),\ (6,4,12,8)\}.$$

From the above list, we have $3 \mid l_{2,1}$; then we get $9 \mid (2l_{2,1} - 3l_{1,1})(6l_{1,2} - 3l_{2,2})$. On the other hand, we see that

$$l_{1,1} \equiv 0, 2 \pmod 3 \quad \text{and} \quad l_{2,2} \equiv 0, 2 \pmod 3,$$

which implies $(6l_{1,1} - 3l_{2,1} - 6)(2l_{2,2} - 3l_{1,2} - 2)$ is not multiple of 9; therefore we have the determinant of M_K is not zero. Thus M_k is nonsingular when $\boldsymbol{K} = G_2$.

Theorem 1.5 is established. □

Finally we prove Theorems 1.2 and 1.6.

Proof of Theorems 1.2 and 1.6. Suppose there exists a sequence of blowup solutions (u_1^k, u_2^k) of (1-2) with $(\rho_1, \rho_2) = (\rho_1^k, \rho_2^k)$. First, we prove Theorem 1.2. From the previous discussion of this section, we get that at least one component (say u_1^k) of u^k has fast decay on a small ball B near each blowup point q, which means $u_1^k(x) \to -\infty$ if $x \notin S$ and x is not a blowup point. Hence Theorem 1.2 holds.

Because the mass distribution of u_1^k concentrates as $k \to +\infty$, we get that $\lim_{k\to+\infty} \rho_1^k$ is equal to the sum of the local mass σ_1 at a blowup point q, which implies $\rho_1 \in \Gamma_1$, a contradiction to the assumption. Thus, we finish the proof of Theorem 1.6. □

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