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# ON RANK-2 TODA SYSTEMS WITH ARBITRARY SINGULARITIES: LOCAL MASS AND NEW ESTIMATES

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For all rank-2 Toda systems with an arbitrary singular source, we use a unified approach to prove:

(1) The pair of local masses $(\sigma_1, \sigma_2)$ at each blowup point has the expression

$$\sigma_i = 2(N_{i1}\mu_1 + N_{i2}\mu_2 + N_{i3}),$$

where $N_{ij} \in \mathbb{Z}$, $i = 1, 2$, $j = 1, 2, 3$.

(2) At each vortex point $p_t$ if $(\alpha_t^1, \alpha_t^2)$ are integers and $\rho_i \notin 4\pi\mathbb{N}$, then all the solutions of Toda systems are uniformly bounded.

(3) If the blowup point $q$ is a vortex point $p_t$ and $\alpha_t^1, \alpha_t^2$ and 1 are linearly independent over $Q$, then

$$u^k(x) + 2\log|x - p_t| \le C.$$

The Harnack-type inequalities of 3 are important for studying the bubbling behavior near each blowup point.

## 1. Introduction

Let $(M, g)$ be a Riemann surface without boundary and $\boldsymbol{K} = (k_{ij})_{n\times n}$ be the Cartan matrix of a simple Lie algebra of rank $n$. For example, for the Lie algebra $\mathrm{sl}(n+1)$ (the so-called $A_n$) we have

$$\boldsymbol{K} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{1-1}$$

In this paper we consider the solution $u = (u_1, \dots, u_n)$ of the following system defined on $M$:

$$\Delta_g u_i + \sum_{j=1}^{n} k_{ij}\rho_j \left( \frac{h_j e^{u_j}}{\int_M h_j e^{u_j}\, dV_g} - 1 \right) = \sum_{p_t \in S} 4\pi\alpha_t^i (\delta_{p_t} - 1), \tag{1-2}$$

where $\Delta_g$ is the Laplace–Beltrami operator ($-\Delta_g \ge 0$), $S$ is a finite set on $M$, $h_1, \dots, h_n$ are positive and smooth functions on $M$, $\alpha_t^i > -1$ is the strength of the Dirac mass $\delta_{p_t}$ and $\rho = (\rho_1, \dots, \rho_n)$ is a constant vector with nonnegative components. Here for simplicity we just assume that the total area of $M$ is 1.

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Obviously, (1-2) remains the same if $u_i$ is replaced by $u_i + c_i$ for any constant $c_i$. Thus we might assume that each component of $u = (u_1, \dots, u_n)$ is in

$$\mathring{H}^1(M) := \left\{ v \in L^2(M),\ \nabla v \in L^2(M) \ \text{ and } \ \textstyle\int_M v\, dV_g = 0 \right\}.$$

Then (1-2) is the Euler–Lagrange equation for the following nonlinear functional $J_\rho(u)$ in $\mathring{H}^1(M)$:

$$J_\rho(u) = \frac{1}{2} \int_M \sum_{i,j=1}^{n} k^{ij} \nabla_g u_i \nabla_g u_j \, dV_g - \sum_{i=1}^{n} \rho_i \log \int_M h_i e^{u_i} \, dV_g,$$

where $(k^{ij})_{n\times n} = \boldsymbol{K}^{-1}$.

It is hard to overestimate the importance of system (1-2), as it covers a large number of equations and systems deeply rooted in geometry and physics. Even if (1-2) is reduced to a single equation with Dirac sources, it is a mean-field equation that describes metrics with conic singularities. Finding metrics with constant curvature with prescribed conic singularity is a classical problem in differential geometry and extensive references can be found in [Bartolucci and Tarantello 2002; Battaglia and Malchiodi 2014; Eremenko et al. 2014; Lin et al. 2012; 2015; Lin and Zhang 2010; 2013; 2016; Troyanov 1989; 1991; Yang 1997]. Recently profound relations among mean-field equations, the classical Lamé equation, hyperelliptic curves, modular forms and the Painlevé equation have been discovered and developed in [Chai et al. 2015; Chen et al. 2016].

The general form of (1-2) has close ties with algebraic geometry and integrable systems. Here we just briefly explain the relation between the sl$(n+1)$-Toda system and the holomorphic curves in projective spaces: Let $f$ be a holomorphic curve from a domain $D$ of $\mathbb{R}^2$ into $\mathbb{CP}^n$. Then $f$ can be lifted locally to $\mathbb{C}^{n+1}$ and we use $\nu(z) = [\nu_0(z), \dots, \nu_n(z)]$ to denote the lift and $f_k$ the $k$-th associated curve,

$$f_k : D \to G(k, n+1) \subset \mathbb{CP}^n(\Lambda^k \mathbb{C}^{n+1}), \qquad f_k(z) = [\nu(z) \wedge \nu'(z) \wedge \cdots \wedge \nu^{(k-1)}(z)],$$

where $\nu^{(j)}$ is the $j$-th derivative of $\nu$ with respect to $z$. Let

$$\Lambda_k(z) = \nu(z) \wedge \cdots \wedge \nu^{(k-1)}(z).$$

Then the well-known infinitesimal Plüker formula gives

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \|\Lambda_k(z)\|^2 = \frac{\|\Lambda_{k-1}(z)\|^2 \, \|\Lambda_{k+1}(z)\|^2}{\|\Lambda_k(z)\|^4} \quad \text{for } k = 1, 2, \dots, n, \tag{1-3}$$

where we put $\|\Lambda_0(z)\|^2 = 1$ as convention and the norm $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$ is defined by the Fubini–Study metric in $\mathbb{CP}(\Lambda^k \mathbb{C}^{n+1})$. Here we observe that (1-3) holds only for $\|\Lambda_k(z)\| > 0$, i.e., for all the unramified points $z \in M$. Now we set $\|\Lambda_{n+1}(z)\| = 1$ by normalization (analytically extended at the ramification points) and

$$U_k(z) = -\log \|\Lambda_k(z)\|^2 + k(n-k+1) \log 2, \quad 1 \le k \le n.$$

For every ramified point $p$ we use $\{\gamma_{p,1}, \dots, \gamma_{p,n}\}$ to denote the total ramification index at $p$ and set

$$u_i^* = \sum_{j=1}^{n} k_{ij} U_j, \quad \alpha_{p,i} = \sum_{j=1}^{n} k_{ij} \gamma_{p,j},$$

Then we have

$$\Delta u_i^* + \sum_{j=1}^{n} k_{ij} e^{u_j^*} - K_0 = 4\pi \sum_{p \in S} \alpha_{p,i} \delta_p, \quad i = 1, \dots, n, \tag{1-4}$$

where $K_0$ is the Gaussian curvature of the metric $g$.

Therefore any holomorphic curve from $M$ to $\mathbb{CP}^n$ is associated with a solution $u^* = (u_1^*, \dots, u_n^*)$ of (1-4). Conversely, given any solution $u^* = (u_1^*, \dots, u_n^*)$ of (1-4) in $\mathbb{S}^2$, it is possible to construct a holomorphic curve of $\mathbb{S}^2$ into $\mathbb{CP}^n$ which has the given ramification index $\gamma_{p,i}$ at $p$ if $\gamma_{p,i} \in \mathbb{N}$. One can see [Lin et al. 2012] for the details of this construction. Therefore, (1-4) is related to the following problem in more general setting: given a set of ramified points on $M$ and its ramification indices at these points, can we find holomorphic curves into $\mathbb{CP}^n$ that satisfy the given ramification information?

Equation (1-2) is also related to many physical models from gauge field theory. For example, to describe the physics of high critical temperature superconductivity, a model related to the Chern–Simons model was proposed, which can be reduced to an $n \times n$ system with exponential nonlinearity if the gauge potential and the Higgs field are algebraically restricted. The Toda system with (1-1) is one of the limiting equations if a coupling constant tends to zero. For extensive discussions on the relationship between the Toda system and its background in Physics we refer the readers to [Bennett 1934; Ganoulis et al. 1982; Lee 1991; Mansfield 1982; Yang 2001].

In this article we are concerned with rank-2 Toda systems. There are three types of Cartan matrices of rank 2:

$$A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B_2 (= C_2) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

One of our main theorems is the following estimate:

**Theorem 1.1.** *Let $(k_{ij})_{2\times 2}$ be one of the matrices above, $h_i$ be positive $C^1$ functions on $M$, $\alpha_t^i \in \mathbb{N} \cup \{0\}$, $t \in \{1, 2, \dots, N\}$ and $K$ be a compact subset of $M \setminus S$. If $\rho_i \notin 4\pi\mathbb{N}$, then there exists a constant $C(K, \rho_1, \rho_2)$ such that for any solution $u = (u_1, u_2)$ of (1-2)*

$$|u_i(x)| \le C \quad \text{for all } x \in K, \ i = 1, 2.$$

Our proof of Theorem 1.1 is based on the analysis of the behavior of solutions $u^k = (u_1^k, u_2^k)$ near each blowup point. A point $p \in M$ is called a blowup point if, along a sequence of points $p_k \to p$,

$$\max_{i=1,2} \{\tilde{u}_1^k(p_k), \tilde{u}_2^k(p_k)\} \to +\infty,$$

where

$$\tilde{u}_i^k(x) = u_i^k(x) + 4\pi \sum_t \alpha_t^k G(x, p_t),$$

and $G(x, y)$ is the Green's function of the Laplacian operator on $M$.

Suppose $u^k$ is a sequence of solutions of (1-2). When $n = 1$, it has been proved that if $u^k$ blows up somewhere, the mass distribution $\rho h e^{u^k} / \left(\int_M h e^{u^k}\right)$ will concentrate; that is, for a set of finite points

$p_1, p_2, \dots, p_L$ and positive numbers $m_1, \dots, m_L$

$$\frac{\rho h e^{u^k}}{\int_M h e^{u^k}} \to \sum_{i=1}^{L} m_i \delta_{p_i} \quad \text{as } k \to \infty.$$

In other words, "$u_k$ concentrates" means $u^k(x) \to -\infty$ if $x$ is not a blowup point. This "blowup implies concentration" was first noted by Brezis and Merle [1991] and was later proved by Li [1999], Li and Shafrir [1994] and Bartolucci and Tarantello [2002]. But for $n \geq 2$, this phenomenon might fail in general. A component $u_i^k$ is called not concentrating if $u_i^k \not\to -\infty$ away from blowup points, or equivalently, $\tilde{u}_i^k$ converges to some smooth function $w_i$ away from blowup points. It is natural to ask whether it is possible to have all components not concentrating. For $n = 2$, we prove it is impossible.

**Theorem 1.2.** *Suppose $u^k$ is a sequence of blowup solutions of a rank-2 Toda system (1-2). Then at least one component of $u^k$ satisfies $u_i^k(x) \to -\infty$ if $x$ is not contained in the blowup set.*

The first example of such nonconcentration phenomenon was proved by Lin and Tarantello [2016]. The new phenomenon makes the study of systems ($n \geq 2$) much more difficult than the mean-field equation ($n = 1$). Recently, Battaglia [2015] and Lin, Yang and Zhong [Lin et al. 2017] independently proved the result of Theorem 1.2 for $n \geq 3$.

As mentioned before, our proofs of Theorems 1.1 and 1.2 are based on the asymptotic behavior of local bubbling solutions. For simplicity we set up the situation as follows:

Let $u^k = (u_1^k, u_2^k)$ be a sequence of solutions of

$$\Delta u_i^k + \sum_{j=1}^{2} k_{ij} h_j^k e^{u_j^k} = 4\pi\alpha_i \delta_0 \quad \text{in } B(0,1),\ i = 1, 2, \tag{1-5}$$

where $\alpha_i > -1$. $B(0,1)$ is the unit ball in $\mathbb{R}^2$ (we use $B(p,r)$ to denote the ball centered at $p$ with radius $r$) and $(k_{ij})_{2\times 2}$ is $A_2$, $B_2$ or $G_2$. Throughout the paper, $h_1^k, h_2^k$ are smooth functions satisfying $h_1^k(0) = h_2^k(0) = 1$ and

$$\frac{1}{C} \leq h_i^k \leq C, \quad \|h_i^k\|_{C^1(B(0,1))} \leq C \quad \text{in } B(0,1),\ i = 1, 2. \tag{1-6}$$

For solutions $u^k = (u_1^k, u_2^k)$ we assume

$$\begin{cases} 0 \text{ is the only blowup point of } u^k, \\ |u_i^k(x) - u_i^k(y)| \leq C \quad \text{for all } x, y \in \partial B(0,1),\ i = 1, 2, \\ \int_{B(0,1)} h_i^k e^{u_i^k} \leq C, \quad i = 1, 2. \end{cases} \tag{1-7}$$

For this sequence of blowup solutions we define the local mass by

$$\sigma_i = \lim_{r \to 0} \lim_{k \to \infty} \frac{1}{2\pi} \int_{B(0,r)} h_i^k e^{u_i^k}, \quad i = 1, 2. \tag{1-8}$$

It is known that 0 is a blowup point if and only if $(\sigma_1, \sigma_2) \neq (0, 0)$. The proof is to use ideas from [Brezis and Merle 1991] and has become standard now. We refer the readers to [Lee et al. 2017] for a complete proof. One important property of $(\sigma_1, \sigma_2)$ is the so-called Pohozaev identity (P.I. in short)

$$k_{21}\sigma_1^2 + k_{12}k_{21}\sigma_1\sigma_2 + k_{12}\sigma_2^2 = 2k_{21}\mu_1\sigma_1 + 2k_{12}\mu_2\sigma_2, \tag{1-9}$$

where $\mu_i = 1 + \alpha_i$. Take $A_2$ as an example; the P.I. is

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = 2\mu_1\sigma_1 + 2\mu_2\sigma_2.$$

The proof of (1-9) was given in [Lin et al. 2015] where we initiated an algorithm to calculate all the possible (finitely many) values of local masses and (1-9) played an essential role. But the argument there seems not very efficient. In this work we add major new ingredients to our approach and improve the classification of $(\sigma_1, \sigma_2)$ to the following sharper form:

**Theorem 1.3.** *Let $u^k$ be a sequence of blowup solutions of (1-5) which also satisfies (1-6) and (1-7). Suppose $\sigma_1$ and $\sigma_2$ are local masses defined by (1-8). Then $\sigma_i$ can be written as*

$$\sigma_i = 2(N_{i,1}\mu_1 + N_{i,2}\mu_2 + N_{i,3}), \quad i = 1, 2,$$

*for some $N_{i,1}, N_{i,2}, N_{i,3} \in \mathbb{Z}$ $(i = 1, 2)$.*

Theorem 1.3 is proved in Sections 5 and 6. In Section 5, we give an explicit procedure to calculate the local masses. Take the $A_2$ system as an example; we start with $\sigma_1 = 0$ and the P.I. gives $\sigma_2 = 2\mu_2$. With $\sigma_2 = 2\mu_2$, the P.I. gives $\sigma_1 = 2\mu_1 + 2\mu_2$ and so on. Let $\Gamma(\mu_1, \mu_2)$ be the set obtained by the above algorithm. Then $\Gamma(\mu_1, \mu_2)$ is equal to:

(i) $(2\mu_1, 0)$, $(2\mu_1, 2\mu_1 + 2\mu_2)$, $(2\mu_1 + 2\mu_2, 2\mu_1 + 2\mu_2)$, $(2\mu_1 + 2\mu_2, 2\mu_2)$, $(0, 2\mu_2)$ for $A_2$,

(ii) $(2\mu_1, 0)$, $(2\mu_1, 4\mu_1 + 2\mu_2)$, $(4\mu_1 + 2\mu_2, 4\mu_1 + 2\mu_2)$, $(4\mu_1 + 2\mu_2, 4\mu_1 + 4\mu_2)$, $(0, 2\mu_2)$, $(2\mu_1 + 2\mu_2, 2\mu_2)$, $(2\mu_1 + 2\mu_2, 4\mu_1 + 4\mu_2)$ for $B_2$,

(iii) $(2\mu_1, 0)$, $(2\mu_1, 6\mu_1 + 2\mu_2)$, $(6\mu_1 + 2\mu_2, 6\mu_1 + 2\mu_2)$, $(6\mu_1 + 2\mu_2, 12\mu_1 + 6\mu_2)$, $(8\mu_1 + 4\mu_2, 12\mu_1 + 6\mu_2)$, $(8\mu_1 + 4\mu_2, 12\mu_1 + 8\mu_2)$, $(0, 2\mu_2)$, $(2\mu_1 + 2\mu_2, 2\mu_2)$, $(2\mu_1 + 2\mu_2, 6\mu_1 + 6\mu_2)$, $(6\mu_1 + 4\mu_2, 6\mu_1 + 6\mu_2)$, $(6\mu_1 + 4\mu_2, 12\mu_1 + 8\mu_2)$ for $G_2$.

**Definition 1.4.** A pair of local masses $(\sigma_1, \sigma_2) \in \Gamma(\mu_1, \mu_2)$ is called special if

$$(\sigma_1, \sigma_2) = \begin{cases} (2\mu_1 + 2\mu_2, 2\mu_1 + 2\mu_2) & \text{for } A_2, \\ (4\mu_1 + 2\mu_2, 4\mu_1 + 4\mu_2) & \text{for } B_2, \\ (8\mu_1 + 4\mu_2, 12\mu_1 + 8\mu_2) & \text{for } G_2. \end{cases}$$

The analysis of local solutions in [Lin et al. 2015] describes a method to pick a family of points $\Gamma_k = \{0, x_1^k, \ldots, x_N^k\}$ (if 0 is a singular point, otherwise 0 can be deleted from $\Gamma_k$) such that a tiny ball $B(x_i^k, l_j^k)$ contributes an amount of mass (which is quantized), and the following Harnack-type inequality holds:

$$u_i^k(x) + 2\log \operatorname{dist}(x, \Sigma_k) \leqslant C \quad \text{for all } x \in B(0, 1). \tag{1-10}$$

When $\alpha_1 = \alpha_2 = 0$, we can use Theorem 1.3 to calculate all the pairs of even positive integers satisfying (1-9) and the set is exactly the same as $\Gamma(1, 1)$.

It is interesting to see whether any pair of the above really consists of the local masses of some sequence of blowup solutions of (1-2). For $\boldsymbol{K} = A_2$ the existence of such a local blowup sequence has been obtained; see [Musso et al. 2016; Lin and Yan 2013].

After $\Sigma_k$ is picked, the difficulty at the next step is how to calculate the mass contributed from outside $B(x_j^k, l_j^k)$ $j = 1, 2, \cdots, N$. In Section 6, we see that the mass outside this union could be very messy. However, the picture is very clean if $(\alpha_1, \alpha_2)$ satisfies the following $Q$-condition:

$$\alpha_1, \alpha_2 \text{ and } 1 \text{ are linearly independent over } Q.$$

**Theorem 1.5.** *Suppose $(\alpha_1, \alpha_2)$ satisfies the $Q$-condition. Then $(\sigma_1, \sigma_2) \in \Gamma(\mu_1, \mu_2)$. Furthermore, for any sequence of solutions of (1-5) satisfying (1-6) and (1-7), the following Harnack-type inequality holds:*

$$u_i^k(x) + 2\log|x| \leqslant C \quad \textit{for } x \in B(0,1).$$

For (1-2), let $\mu_{1,t} = \alpha_t^1 + 1$ and $\mu_{2,t} = \alpha_t^2 + 1$ at a vortex point $p_t \in S$, and define

$$\Gamma_i = \left\{2\pi(\Sigma_{t\in J}\sigma_{i,t} + 2n) \,\middle|\, (\sigma_{1,t}, \sigma_{2,t}) \in \Gamma(\mu_{1,t}, \mu_{2,t}),\ J \subseteq S,\ n \in \mathbb{N} \cup \{0\}\right\}. \tag{1-11}$$

Based on Theorem 1.5, Theorem 1.1 can be extended to the following version:

**Theorem 1.6.** *Let $h_i$ be positive $C^1$ functions on $M$, and $K$ be a compact set in $M$. For every point $p_t \in S$, if either both $\alpha_t^1, \alpha_t^2 \in \mathbb{N} \cup \{0\}$ or $(\alpha_t^1, \alpha_t^2)$ satisfies the $Q$-condition, then for $\rho_i \notin \Gamma_i$ and $u = (u_1, u_2)$ a solution of (1-2), there exists a constant $C$ such that*

$$|u_i(x)| \leqslant C \quad \textit{for all } x \in K.$$

The organization of this article is as follows. In Section 2 we establish the global mass for the entire solutions of some singular Liouville equation defined in $\mathbb{R}^2$. Then in Section 3 we review some fundamental tools proved in the previous work [Lin et al. 2015]. In Section 4 we present two crucial lemmas, which play the key role in the proof of main results. In Sections 5 and 6 we discuss the local mass on each bubbling disk centered at 0 and not at 0 respectively, and then all the main results are established based on previous discussions.

## 2. Total mass for Liouville equation

The main purpose of this section is to prove an estimate of the total mass for the solutions of the equation

$$\begin{cases} \Delta u + e^u = \sum_{i=1}^N 4\pi\alpha_i\delta_{p_i} & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^u < \infty, \end{cases} \tag{2-1}$$

where $p_1, \ldots, p_N$ are distinct points in $\mathbb{R}^2$ and $\alpha_i > -1$ for all $1 \leqslant i \leqslant N$.

**Theorem 2.1.** *Suppose $u$ is a solution of (2-1) and $\alpha_1, \ldots, \alpha_N$ are positive integers. Then $\frac{1}{4\pi}\int_{\mathbb{R}^2} e^u$ is an even integer.*

*Proof.* It is known that any solution $u$ of (2-1) has, at infinity, the asymptotic behavior

$$u(z) = -2\alpha_\infty \log|z| + O(1), \quad \alpha_\infty > 1, \tag{2-2}$$

and $u$ satisfies

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^u\,dx = 2\sum_{i=1}^N \alpha_i + 2\alpha_\infty. \tag{2-3}$$

We shall prove that $\alpha_\infty + \sum_{i=1}^N \alpha_i$ is an even integer. A classical Liouville theorem (see [Chou and Wan 1994]) says that $u$ can be written as

$$u = \log\frac{8|f'(z)|^2}{(1+|f(z)|^2)^2}, \quad z \in \mathbb{R}^2, \tag{2-4}$$

for some meromorphic function $f$. In general, $f(z)$ is multivalued and any vertex $p_i$ is a branch point. However if $\alpha_i \in \mathbb{N}\cup\{0\}$, then $f(z)$ is single-valued. Furthermore (2-2) implies that $f(z)$ is meromorphic at infinity. Hence for any solution $u$ of (2-1) there is a meromorphic function $f$ on $\mathbb{S}^2 = \mathbb{C}\cup\{\infty\}$ such that (2-4) holds. Then

$$\begin{aligned}4\pi\Bigl(\sum_{j=1}^N \alpha_j + \alpha_\infty\Bigr) = \int_{\mathbb{R}^2} e^u &= 8\int_{\mathbb{R}^2}\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}\,dx\,dy \\ &= 8(\deg f)\int_{\mathbb{R}^2}\frac{d\tilde{x}\,d\tilde{y}}{(1+|w|^2)^2} = 8\pi(\deg f),\end{aligned}$$

where $\deg(f)$ is the degree of $f$ as a map from $\mathbb{S}^2 = \mathbb{C}\cup\{\infty\}$ onto $\mathbb{S}^2$, and $w = f(z) = \tilde{x} + i\tilde{y}$. Thus we have

$$\sum_{j=1}^N \alpha_j + \alpha_\infty = 2\deg(f). \qquad \square$$

**Theorem 2.2.** *Suppose u is a solution of*

$$\begin{cases}\Delta u + e^u = 4\pi\alpha_0\delta_{p_0} + \sum_{i=1}^N 4\pi\alpha_i\delta_{p_i} & \text{in } \mathbb{R}^2,\\ \int_{\mathbb{R}^2} e^u < \infty,\end{cases} \tag{2-5}$$

*where $p_0, p_1, \dots, p_N$ are distinct points in $\mathbb{R}^2$ and $\alpha_1, \dots, \alpha_N$ are positive integers, $\alpha_0 > -1$. Then $\frac{1}{4\pi}\int_{\mathbb{R}^2} e^u$ is equal to $2(\alpha_0+1)+2k$ for some $k \in \mathbb{Z}$ or $2k_1$ for some $k_1 \in \mathbb{N}$.*

*Proof.* As in Theorem 2.1, there is a developing map $f(z)$ of $u$ such that

$$u(z) = \log\frac{8|f'(z)|^2}{(1+|f(z)|^2)^2}, \quad z \in \mathbb{C}. \tag{2-6}$$

On one hand by (2-5), $u_{zz} - \frac{1}{2}u_z^2$ is a meromorphic function in $\mathbb{C}\cup\{\infty\}$ because away from the Dirac masses

$$4\bigl(u_{zz} - \tfrac{1}{2}u_z^2\bigr)_{\bar{z}} = -(e^u)_z + u_z e^u = 0.$$

By $u(z) = 2\alpha_i \log|z-p_i| + O(1)$ near $p_i$ we have

$$u_{zz} - \tfrac{1}{2}u_z^2 = -2\Bigl(\sum_{j=0}^N \tfrac{1}{2}\alpha_j\bigl(\tfrac{1}{2}\alpha_j+1\bigr)(z-p_j)^{-2} + A_j(z-p_j)^{-1} + B\Bigr),$$

where $A_0, \dots, A_N, B \in \mathbb{C}$ are some constants. On the other hand by (2-6), a straightforward computation shows that

$$u_{zz} - \frac{1}{2}u_z^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{2-7}$$

Using the Schwarz derivative of $f$,

$$\{f; z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2,$$

and letting

$$I(z) = \sum_{j=0}^{N} \tfrac{1}{2}\alpha_j\big(\tfrac{1}{2}\alpha_j + 1\big)(z - p_j)^{-2} + A_j(z - p_j)^{-1} + B,$$

we can write the equation for $f$ as

$$\{f, z\} = -2I(z). \tag{2-8}$$

A well-known classic theorem (see [Whittaker and Watson 1927]) says that for any two linearly independent solutions $y_1$ and $y_2$ of

$$y''(z) = I(z)y(z), \tag{2-9}$$

the ratio $y_2/y_1$ always satisfies

$$\{y_2/y_1; z\} = -2I(z).$$

By (2-8) and a basic result of the Schwarz derivative, $f(z)$ can be written as the ratio of two linearly independent solutions. This is how (2-1) is related to the complex ODE (2-9). We refer the readers to [Chai et al. 2015] for the details.

For (2-9), there is an associated monodromy representation $\rho$ from $\pi_1(\mathbb{C} \setminus \{p_0, p_1, \dots, p_N\}; q)$ to $\mathrm{GL}(2; \mathbb{C})$, where $q$ is a base point. Note that at any singular point $p_j$, the local exponents are $\frac{1}{2}\alpha_j + 1$ and $-\frac{1}{2}\alpha_j$. It is known from [Lin et al. 2012, Section 7] that $e^{-u}$ can be locally written as

$$e^{-u} = |\nu_1|^2 + |\nu_2|^2 = \langle (\nu_1, \nu_2)^t, (\nu_1, \nu_2)^t \rangle,$$

where $\nu_1, \nu_2$ are the two fundamental solutions of (2-9). After encircling the singular point $p_j$ once, we have $e^{-u} = \langle \rho_j(\nu_1, \nu_2)^t, \rho_j(\nu_1, \nu_2)^t \rangle$ and the value does not change. Therefore, we conclude that $\rho_j$ is unitary and

$$\rho_j = \rho(\gamma_j) = C_j \begin{pmatrix} e^{\pi i \alpha_j} & 0 \\ 0 & e^{-\pi i \alpha_j} \end{pmatrix} C_j^{-1},$$

where $\gamma_j \in \pi_1(\mathbb{C} \setminus \{p_0, \dots, p_N\}, q)$ encircles $p_j$ only once, $0 \le j \le N$, while the monodromy at $\infty$ is $\rho_\infty$. Then we have

$$\rho_\infty \rho_N \cdots \rho_0 = I_{2\times 2}.$$

Note that $\rho_j = \pm I_{2\times 2}$ for $1 \le j \le N$. Hence

$$\rho_\infty^{-1} = D_0 \begin{pmatrix} e^{\pi i \sum_{j=0}^{N} \alpha_j} & 0 \\ 0 & e^{-\pi i \sum_{j=0}^{N} \alpha_j} \end{pmatrix} D_0^{-1}$$

for some constant invertible matrix $D_0$.

On the other hand, the local exponents at $\infty$ can be computed as follows. Let $\hat{y}(z)=y\big(\frac{1}{z}\big)$, where $y$ is a solution of (2-9). Then we have

$$\hat{y}''(z)+\frac{2}{z}\hat{y}'(z)=\hat{I}(z)\hat{y}(z), \tag{2-10}$$

where $\hat{I}(z)=I\big(\frac{1}{z}\big)z^{-4}$. Since $I(z)$ is the Schwarz derivative of $f(z)$, by direct computation $\hat{I}(z)$ is the Schwarz derivative of $f\big(\frac{1}{z}\big)$. As before we let $\hat{u}(z)=u\big(\frac{1}{z}\big)-4\log|z|$. Then $f\big(\frac{1}{z}\big)$ is the developing map of $\hat{u}(z)$. Since

$$\hat{u}(z)=2(\alpha_\infty-2)\log|z|+O(1)\quad\text{near }0,$$

(because $u(z)=-2\alpha_\infty\log|z|+O(1)$ at infinity), we have

$$\hat{I}(z)=\tfrac{1}{2}\alpha_\infty\big(\tfrac{1}{2}\alpha_\infty-1\big)z^{-2}+\text{higher-order terms of } z\quad\text{near }0.$$

By (2-10) we see that the local exponents of (2-9) are $-\frac{1}{2}\alpha_\infty$ and $\frac{1}{2}\alpha_\infty-1$. Hence $e^{\pi i\alpha_\infty}$ equals either $e^{\pi i\sum_{j=0}^{N}\alpha_j}$ or $e^{-\pi i\sum_{j=0}^{N}\alpha_j}$, which yields

$$\alpha_\infty=-\sum_{j=0}^{N}\alpha_j+2k\quad\text{or}\quad\alpha_\infty=\sum_{j=0}^{N}\alpha_j+2k \tag{2-11}$$

for some $k\in\mathbb{Z}$. Since

$$\frac{1}{4\pi}\int_{\mathbb{R}^2}e^u=\sum_{j=0}^{N}\alpha_j+\alpha_\infty,$$

we either have $\frac{1}{4\pi}\int_{\mathbb{R}^2}e^u=2k$ for some $k\in\mathbb{N}$ if the first case holds or $\frac{1}{4\pi}\int_{\mathbb{R}^2}e^u=2(\alpha_0+1)+2k'$ for $k'=\sum_{i=1}^{N}\alpha_i+k-1$ if the second case holds. □

**Remark 2.3.** After proving Theorems 2.1 and 2.2, we found a stronger version of both theorems in [Eremenko et al. 2014]. Because we only need the present form of both theorems, we include our proofs here to make the paper more self-contained.

## 3. Review of bubbling analysis from a selection process

Let $u^k=(u_1^k,u_2^k)$ be solutions of (1-5) such that (1-6) and (1-7) hold. In this section we review the process to select a set $\Sigma_k=\{0,x_1^k,\dots,x_n^k\}$ and balls $B(x_i^k,l_k)$ such that $u^k$ has nonzero local masses in $B(x_i^k,l_k)$. This selection process was first carried out in [Lin et al. 2015]. We briefly review it below.

The set $\Sigma_k$ is constructed by induction. If (1-5) has no singularity, we start with $\Sigma_k=\varnothing$. If (1-5) has a singularity, we start with $\Sigma_k=\{0\}$. By induction suppose $\Sigma_k$ consists of $\{0,x_1^k,\dots,x_{m-1}^k\}$. Then we consider

$$\max_{x\in B_1}\max_{i=1,2}\big(u_i^k(x)+2\log\operatorname{dist}(x,\Sigma_k)\big). \tag{3-1}$$

If the maximum is bounded from above by a constant independent of $k$, the process stops and $\Sigma_k$ is exactly equal to $\{0,x_1^k,\dots,x_{m-1}^k\}$. However if the maximum tends to infinity, let $q_k$ be where (3-1) is achieved and we set

$$d_k=\tfrac{1}{2}\operatorname{dist}(q_k,\Sigma_k)$$

and

$$S_i^k(x) = u_i^k(x) + 2\log(d_k - |x - q_k|) \quad \text{in } B(q_k, d_k),\ i = 1, 2.$$

Suppose $i_0$ is the component that attains

$$\max_i \max_{x \in \bar{B}(q_k, d_k)} S_i^k \tag{3-2}$$

at $p_k$. Then we set

$$\tilde{l}_k = \tfrac{1}{2}(d_k - |p_k - q_k|)$$

and scale $u_i^k$ by

$$v_i^k(y) = u_i^k(p_k + e^{-\frac{1}{2}u_{i_0}^k(p_k)} y) - u_{i_0}^k(p_k) \quad \text{for } |y| \le R_k \doteqdot e^{\frac{1}{2}u_{i_0}^k(p_k)} \tilde{l}_k. \tag{3-3}$$

It can be shown that $R_k \to \infty$ and $v_i^k$ is bounded from above over any fixed compact subset of $\mathbb{R}^2$. Thus by passing to a subsequence, $v_i^k$ satisfies one of the following two alternatives:

(a) $(v_1^k, v_2^k)$ converges in $C^2_{\text{loc}}(\mathbb{R}^2)$ to $(v_1, v_2)$ which satisfies

$$\Delta v_i + \sum_{j=1}^{2} k_{ij} e^{v_j} = 0 \quad \text{in } \mathbb{R}^2,\ i = 1, 2. \tag{3-4}$$

(b) Either $v_1^k$ converges to

$$\Delta v_1 + 2e^{v_1} = 0 \quad \text{in } \mathbb{R}^2 \tag{3-5}$$

and $v_2^k \to -\infty$ over any fixed compact subset of $\mathbb{R}^2$ or $v_2^k$ converges to $\Delta v_2 + 2e^{v_2} = 0$ in $\mathbb{R}^2$ and $v_1^k \to -\infty$ over any fixed compact subset of $\mathbb{R}^2$.

Therefore in either case, we could choose $l_k^* \to \infty$ such that

$$v_i^k(y) + 2\log|y| \le C \quad \text{for } i = 1, 2 \text{ and } |y| \leqslant l_k^* \tag{3-6}$$

and

$$\int_{B(0, l_k^*)} h_i^k e^{v_i^k}\, dy = \int_{\mathbb{R}^2} e^{v_i(y)} + o(1).$$

By scaling back to $u_i^k$, we add $p_k$ in $\Sigma_k$ with

$$l_k = e^{-\frac{1}{2}u_{i_0}^k(p_k)} l_k^*.$$

We can continue in this way until the Harnack-type inequality (1-10) holds.

We summarize what the selection process has done in the following proposition (a detailed proof for a more general case can be found in [Lin et al. 2015, Proposition 2.1]):

**Proposition 3A.** *Let $u^k$ be described as above. Then there exist a finite set $\Sigma_k := \{0, x_1^k, \dots, x_m^k\}$ (if 0 is not a singular point, then 0 can be deleted from $\Sigma_k$) and positive numbers $l_1^k, \dots, l_m^k \to 0$ as $k \to \infty$ such that the following hold*:

(1) *There exists $C > 0$ independent of $k$ such that (1-10) holds and all the components have fast decay on $\partial B(x_j^k, l_j^k)$, $j = 1, \dots, m$. (The definition of fast decay can be found in Definition 3.1 below).*

(2) *In* $B(x_j^k, l_j^k)$ $(j = 1, \dots, m)$, *let* $R_{j,k} = e^{\frac{1}{2}u_{i_0}^k(x_j^k)} l_j^k$, $u_{i_0}^k(x_j^k) = \max_i u_i^k(x_j^k)$ *and*

$$v_i^k(y) = u_i^k(x_j^k + e^{-\frac{1}{2}u_{i_0}^k(x_j^k)} y) - u_{i_0}^k(x_j^k) \tag{3-7}$$

*for* $|y| \le R_{j,k}$; *then* $v^k = (v_1^k, v_2^k)$ *satisfies either* (a) *or* (b).

(3) $B(x_j^k, l_j^k) \cap B(x_i^k, l_i^k) = \varnothing$, $i \ne j$.

The inequality (1-10) is a Harnack-type inequality, because it implies the following result.

**Proposition 3B.** *Suppose* $u^k$ *satisfies* (1-5)–(1-7) *and*

$$u_i^k(x) + 2\log|x - x_0| \le C \quad \text{for } x \in B(x_0, 2r_k).$$

*Then*

$$|u_i^k(x_1) - u_i^k(x_2)| \le C_0 \quad \text{for } \frac{1}{2} \le \frac{|x_1 - x_0|}{|x_2 - x_0|} \le 2 \text{ and } x_1, x_2 \in B(x_0, r_k). \tag{3-8}$$

The proof of Proposition 3B is standard, see [Lin et al. 2015, Lemma 2.4], so we omit it here. Let $x_l^k \in \Sigma_k$ and $\tau_l^k = \frac{1}{2}\operatorname{dist}(x_l^k, \Sigma_k \setminus \{x_l^k\})$; then (3-8) implies

$$u_i^k(x) = \bar{u}_{x_l^k,i}^k(r) + O(1), \quad x \in B(x_l^k, \tau_l^k), \tag{3-9}$$

where $r = |x_l^k - x|$ and $\bar{u}_{x_l^k,i}^k$ is the average of $u_i^k$ on $\partial B(x_l^k, r)$,

$$\bar{u}_{x_l^k,i}^k(r) = \frac{1}{2\pi r}\int_{\partial B(x_l^k,r)} u_i^k \, dS, \tag{3-10}$$

and $O(1)$ is independent of $r$ and $k$.

Next we introduce the notions of slow decay and fast decay in our bubbling analysis.

**Definition 3.1.** We say $u_i^k$ has fast decay on $\partial B(x_0, r_k)$ if along a subsequence

$$u_i^k(x) + 2\log|x - x_0| \le -N_k \quad \text{for all } x \in \partial B(x_0, r_k),$$

for some $N_k \to \infty$ and we say $u_i^k$ has slow decay if there is a constant $C$ independent of $k$ such that

$$u_i^k(x) + 2\log|x - x_0| \ge -C \quad \text{for all } x \in \partial B(x_0, r_k).$$

Furthermore, we say $u_i^k$ is fast-decaying in $B(x_0, s_k) \setminus B(x_0, r_k)$ if $u_i^k$ has fast decay on $\partial B(x_0, l_k)$ for any $l_k \in [r_k, s_k]$.

The concept of fast decay is important for evaluating the Pohozaev identities. The following proposition is a direct consequence of [Lin et al. 2015, Proposition 3.1] and it says if both components are fast-decaying on the boundary, the Pohozaev identity holds for the local masses.

In the following proposition, we let $B = B(x^k, r_k)$. If $x^k \ne 0$, we assume $0 \notin B(x^k, 2r_k)$.

**Proposition 3C.** *Suppose both* $u_1^k, u_2^k$ *have fast decay on* $\partial B$, *where* $B$ *is given above. Then* $(\sigma_1, \sigma_2)$ *satisfies the P.I.* (1-9), *where*

$$\sigma_i = \lim_{k \to 0} \frac{1}{2\pi}\int_B h_i^k e^{u_i^k}, \quad i = 1, 2.$$

We refer the readers to [Lin et al. 2015, Proposition 3.1] for the proof.

## 4. Two lemmas

In this section, we prove two crucial lemmas which play the key role in Sections 5 and 6. For Lemma 4.1, we assume:

(i) The Harnack inequality

$$u_i^k(x) + 2\log|x| \le C \quad \text{for } \tfrac{1}{2}l_k \le |x| \le 2s_k,\ i = 1, 2,$$

holds for both components.

(ii) Both components $u_i^k$ have fast decay on $\partial B(0, l_k)$ and $\sigma_i^k(B(0, l_k)) = \sigma_i + o(1)$ for $i = 1, 2$, where

$$\sigma_i = \lim_{r\to 0}\lim_{k\to\infty}\frac{1}{2\pi}\int_{B(0,rs_k)} h_i^k e^{u_i^k}, \quad i = 1, 2.$$

(iii) One of $u_i^k$, $i = 1, 2$, has slow decay on $\partial B(0, s_k)$.

**Lemma 4.1.** (a) *Assume (i) and (ii). If $u_i^k$ has slow decay on $\partial B(0, s_k)$, then*

$$2\mu_i - \sum_{j=1}^{2} k_{ij}\sigma_j > 0.$$

(b) *Assume (i), (ii) and (iii). Let $u_i^k$ be a slow-decaying component on $\partial B(0, s_k)$. Then the other component has fast decay on $\partial B(0, s_k)$.*

*Proof.* (a) Suppose that $u_i^k$ has slow decay on $\partial B(0, s_k)$. Then the scaling

$$v_j^k(y) = u_j^k(s_k y) + 2\log s_k, \quad j = 1, 2 \text{ for } y \in B_2$$

gives

$$\Delta v_j^k(y) + \sum_{l=1}^{2} k_{jl} h_l^k(s_k y) e^{v_l^k(y)} = 4\pi\alpha_j\delta_0 \quad \text{in } y \in B_2.$$

If the other component also has slow decay on $\partial B(0, s_k)$, then $(v_1^k, v_2^k)$ converges to $(v_1, v_2)$ which satisfies

$$\Delta v_j(y) + \sum_{l=1}^{2} k_{jl} e^{v_l} = 0 \quad \text{in } B_2\backslash\{0\},\ j = 1, 2. \tag{4-1}$$

If the other component has fast decay on $\partial B(0, s_k)$, then $v_i^k(y)$ converges to $v_i(y)$ and $v_j(y) \to -\infty$, $j \ne i$. Furthermore, $v_i(y)$ satisfies

$$\Delta v_i(y) + 2e^{v_i} = 0 \quad \text{in } B_2\backslash\{0\}. \tag{4-2}$$

For any $r > 0$,

$$\int_{\partial B(0,r)} \frac{\partial v_i(y)}{\partial \nu}\, dS = \lim_{k\to\infty}\left(4\pi\alpha_i - \sum_{j=1}^{2}\int_{B(0,r)} k_{ij} h_j^k e^{v_j^k}\, dy\right)$$

$$= 4\pi\alpha_i - 2\pi\sum_{j=1}^{2} k_{ij}\sigma_j + o(1) \doteqdot 4\pi\beta_i + o(1),$$

which implies the right-hand sides of both (4-1) and (4-2) should be replaced by $4\pi\beta_i\delta_0$. If $\beta_i \le -1$, we can use the finite energy assumption (see the bottom assumption in (1-7)) to conclude that either (4-1) or (4-2) has no solutions. Hence $\alpha_i - \frac{1}{2}\sum_{j=1}^{2} k_{ij}\sigma_j > -1$ and then (a) is proved.

(b) Since both components have fast decay on $\partial B(0, l_k)$, the pair $(\sigma_1, \sigma_2)$ satisfies the P.I. (1-9). By a simple manipulation, the P.I. (1-9) can be written as

$$k_{21}\sigma_1(4\mu_1 - k_{12}\sigma_2 - k_{11}\sigma_1) + k_{12}\sigma_2(4\mu_2 - k_{21}\sigma_1 - k_{22}\sigma_2) = 0. \tag{4-3}$$

Note by (a),

$$4\mu_i - \sum_{l=1}^{2} k_{il}\sigma_l > 2\mu_i - \sum_{l=1}^{2} k_{il}\sigma_l \geqslant 0.$$

Hence for $j \neq i$

$$2\mu_j - \sum_{l=1}^{2} k_{jl}\sigma_l < 4\mu_j - \sum_{l=1}^{2} k_{jl}\sigma_l < 0,$$

where the last inequality is due to (4-3). By (a) again, $u_j^k$ does not have slow decay on $\partial B(0, s_k)$. □

Our second lemma says that a fast-decaying component does not change its energy more than $o(1)$, regardless of the behavior of the other component.

**Lemma 4.2.** *Suppose the Harnack-type inequality holds for both components over $r \in \left[\frac{1}{2}l_k, 2s_k\right]$. If $u_i^k$ is fast-decaying on $r \in [l_k, s_k]$, then*

$$\sigma_i^k(B(0, s_k)) = \sigma_i^k(B(0, l_k)) + o(1).$$

*Proof.* Obviously the conclusion holds if $s_k/l_k \leqslant C$. So we assume $s_k/l_k \to +\infty$. The Harnack-type inequality implies $u_i^k(x) = \bar{u}_i^k(r) + o(1)$ for $\frac{1}{2}l_k \leqslant |x| \leqslant 2s_k$. Thus we obtain from (1-5) that

$$\frac{d}{dr}(\bar{u}_i^k(r) + 2\log r) = \frac{2\mu_i - \sum_{j=1}^{2} k_{ij}\sigma_j^k(r)}{r}, \quad l_k \leqslant r \leqslant s_k,\ i = 1, 2,$$

where $\sigma_j^k(r) = \sigma_j^k(B(0, r))$ and $\sigma_j = \lim_{k\to+\infty}\sigma_j^k(l_k)$, $j = 1, 2$.

Without loss of generality, we assume that $u_j^k$, $j \neq i$, is fast-decaying on $\partial B(0, l_k)$. Otherwise, we may choose $\tilde{l}_k$ such that $l_k \ll \tilde{l}_k$, $u_i^k$ remains fast-decaying for $r \in [l_k, \tilde{l}_k]$ and $\sigma_i^k(B(0, r))$ does not change more than $o(1)$, while $u_j^k$ is fast-decaying on $\partial B(0, \tilde{l}_k)$. If $s_k/\tilde{l}_k \leqslant C$, we get the conclusion as explained above. If $s_k/\tilde{l}_k \to +\infty$, by a little abuse of notation, we may replace $\tilde{l}_k$ by $l_k$. Then both $u_1^k, u_2^k$ have fast decay on $\partial B(0, l_k)$, and the P.I. holds at $l_k$, which implies that at least one component (say $l$) satisfies

$$4\mu_l - \sum_{j=1}^{2} k_{lj}\sigma_j^k(l_k) < 0.$$

Thus,

$$\frac{d}{dr}(\bar{u}_l^{(k)}(r) + 2\log r) \leqslant -\frac{2\mu_l + o(1)}{r} \quad \text{at } r = l_k. \tag{4-4}$$

Suppose $r_k \in [l_k, s_k]$ is the largest $r$ such that

$$\frac{d}{dr}(\bar{u}_l^{(k)}(r) + 2\log\ r) \leqslant -\frac{\mu_l}{r} \quad \text{for } r \in [l_k, r_k]. \tag{4-5}$$

Thus, either the equality holds at $r = r_k$ or $r_k = s_k$. For simplicity, we let $\varepsilon = \mu_l$. By integrating (4-4) from $l_k$ up to $r \leqslant r_k$, we have

$$\bar{u}_l^{(k)}(r) + 2\log\ r \leqslant \bar{u}_l^{(k)}(l_k) + 2\log(l_k) + \varepsilon \log\left(\frac{l_k}{r}\right);$$

that is for $|x| = r$,

$$e^{u_l^k(x)} \leqslant O(1)e^{\bar{u}_l^k(r)} \leqslant O(1)e^{-N_k} l_k^{\varepsilon} r^{-(2+\varepsilon)},$$

where we used $\bar{u}_l^{(k)}(l_k) + 2\log l_k \leqslant -N_k$ by the assumption of fast decay. Thus

$$\int_{l_k \leqslant |x| \leqslant r_k} e^{u_l^k(x)}\, dx \leqslant O(1)e^{-N_k} l_k^{\varepsilon} \int_{l_k}^{r_k} r^{-(1+\varepsilon)}\, dr = O(1)\frac{e^{-N_k}}{\varepsilon} \to 0$$

as $k \to +\infty$. Hence

$$\sigma_l^k(r_k) = \sigma_l^k(l_k) + o(1). \tag{4-6}$$

If both components are fast-decaying on $r \in [l_k, r_k]$, then $\lim_{k\to+\infty}(\sigma_1^k(r_k), \sigma_2^k(r_k)) = (\hat{\sigma}_1, \hat{\sigma}_2)$ also satisfies the P.I. (1-9). If $\hat{\sigma}_j > \sigma_j$, then $j \neq l$ by (4-6). We choose $r_k^* \leq r_k$ such that $\sigma_j(r_k^*) = \sigma_j^k(l_k) + \varepsilon_0$ for small $\varepsilon_0$, and let $\sigma_j^* = \lim_{k\to 0} \sigma_j(r_k^*)$. Then $\sigma_j^*$ and $\sigma_l$ satisfies the P.I. (1-9) and it yields a contradiction provided $\varepsilon_0$ is small. Thus, we have $\sigma_m^k(r_k) = \sigma_m^k(l_k) + o(1),\ m = 1, 2$. Then (4-4) holds at $r = r_k$, which implies $r_k = s_k$, and Lemma 4.2 is proved in this case.

If one of the components does not have fast decay on $[l_k, r_k]$, then we have $l = i$ and $u_j^k,\ j \neq i$, has slow decay on $\partial B(0, r_k^*)$ for some $r_k^* \leq r_k$. If $s_k/r_k \leq C$, then (4-6) implies the lemma. If $s_k/r_k \to +\infty$, then by the scaling of $u_j^k$ at $r = r_k^*$, the standard argument implies that there is a sequence of $r_k^* \ll \tilde{r}_k = R_k r_k^* \ll s_k$ such that both components have fast decay on $\tilde{r}_k$ and

$$\sigma_i^k(\tilde{r}_k) = \sigma_i(r_k^*) + o(1) = \sigma_i(l_k) + o(1) \quad \text{and} \quad \sigma_j^k(\tilde{r}_k) \geq \sigma_j^k(l_k) + \varepsilon_0$$

for $j \neq i$ and $\varepsilon_0 > 0$. Therefore the assumption of Lemma 4.2 holds at $r \in [\tilde{r}_k, s_k]$. Then we repeat the argument starting from (4-4) and the lemma can be proved in a finite steps. □

**Remark 4.3.** Both lemmas will be used in Section 6 (and Section 5) for the case with singularity at 0 (and without singularity at 0).

## 5. Local mass on the bubbling disk centered at $x_l^k \neq 0$

**5A.** In this subsection we study the local behavior of $u^k$ near $x_l^k$, where $x_l^k \neq 0$. For simplicity, we use $x^k$ instead of $x_l^k$ and $\bar{u}_i^k(r)$ rather than $\bar{u}_{x_l^k, i}^k(r)$. Let

$$\tau^k = \tfrac{1}{2}\operatorname{dist}(x^k, \Sigma_k \setminus \{x^k\}), \quad \sigma_i^k(r) = \frac{1}{2\pi}\int_{B(x^k, r)} h_i^k e^{u_i^k}, \quad i = 1, 2.$$

By Proposition 3A, $l_k \le \tau^k$. Clearly $u^k = (u_1^k, u_2^k)$ satisfies

$$\Delta u_i^k + \sum_{j=1}^{2} k_{ij} h_j^k e^{u_j^k} = 0 \quad \text{in } B(x^k, \tau^k).$$

For a sequence $s_k$, we define

$$\hat{\sigma}_i(s_k) = \begin{cases} \lim_{k\to+\infty} \sigma_i^k(s_k) \text{ if } u_i^k \text{ has fast decay on } \partial B(x^k, s_k), \\ \lim_{r\to 0} \lim_{k\to+\infty} \sigma_i^k(r s_k) \text{ if } u_i^k \text{ has slow decay on } \partial B(x^k, s_k). \end{cases} \tag{5-1}$$

Recall that both components of $u^k$ have fast decay on $\partial B(x^k, l_k)$. This is the starting point of the following proposition, which is a special case of Proposition 5.2 below.

In Proposition 5.1, $(\mu_1, \mu_2)$ will be $(1, 1)$ in both lemmas of Section 4.

**Proposition 5.1.** *Let $u^k = (u_1^k, u_2^k)$ be the solution of (1-5) satisfying (1-7) and $\hat{\sigma}_i(s_k)$ be defined in (5-1). The following holds*:

(1) *At least one component $u^k$ has fast decay on $\partial B(x^k, \tau^k)$.*

(2) *$(\hat{\sigma}_1(\tau^k), \hat{\sigma}_2(\tau^k))$ satisfies the P.I. (1-9) with $\mu_1 = \mu_2 = 1$.*

(3) *$(\hat{\sigma}_1(\tau^k), \hat{\sigma}_2(\tau^k)) \in \Gamma(1, 1)$.*

*Proof.* If $\tau_k / l_k \leqslant C$, (1)–(3) hold obviously for $\tau^k$. So we assume $\tau^k / l_k \to +\infty$. First we remark that if $u^k$ is fully bubbling in $B(x^k, l_k)$ (i.e., (1) in Proposition 3A holds), $(\hat{\sigma}_1(l_k), \hat{\sigma}_2(l_k))$ is special (see Definition 1.4) and satisfies

$$2\mu_i - \sum_{j=1}^{2} k_{ij} \hat{\sigma}_j(l_k) < 0, \quad i = 1, 2.$$

Then by Lemma 4.1, both $u_i^k$ have fast decay on $\partial B(0, \tau^k)$ and Proposition 5.1 follows immediately.

Now we assume $v_i^k$ defined in (3-7) and satisfies case (2) in Proposition 3A. We already know that both components have fast decay at $r = l_k$. If both components remain fast-decaying as $r$ increases from $l_k$ to $\tau^k$, Lemma 4.2 implies

$$\sigma_1^k(\tau^k) = \sigma_1^k(l_k) + o(1), \qquad \sigma_2^k(\tau^k) = \sigma_2^k(l_k) + o(1)$$

and we are done. So we only consider the case that at least one component changes to a slow-decaying component. For simplicity, we assume that $u_1^k$ changes to a slow-decaying component for some $r_k \gg l_k$. By Lemma 4.2,

$$\sigma_1^k(B(x^k, r_k)) \geqslant \sigma_1^k(B(x^k, l_k)) + c_0 \quad \text{for some } c_0 > 0.$$

We might choose $s_k \leqslant r_k$ such that

$$\sigma_1^k(B(x^k, s_k)) = \sigma_1^k(B(x^k, l_k)) + \varepsilon_0,$$

and

$$\sigma_1^k(B(x^k, r)) < \sigma_1^k(B(x^k, l_k)) + \varepsilon_0 \quad \text{for all } r < s_k,$$

where $\varepsilon_0 < \frac{1}{2} c_0$ is small.

Then Lemmas 4.1 and 4.2 together imply that $u_1^k$ has slow decay on $\partial B(x^k, s_k)$ and $u_2^k$ has fast decay on $\partial B(x^k, s_k)$ with

$$\hat{\sigma}_1(s_k) = \sigma_1^k(l_k) + o(1) \quad \text{and} \quad \hat{\sigma}_2(s_k) = \sigma_2^k(l_k) + o(1).$$

Let $v_i^k(y) = u_i^k(x^k + s_k y) + 2\log\, s_k$. If $\tau^k/s_k \le C$, there is nothing to prove. So we assume $\tau^k/s_k \to \infty$. Then $v_1^k(y)$ converges to $v_1(y)$ and $v_2^k(y) \to -\infty$ in any compact set of $\mathbb{R}^2$ as $k \to +\infty$ and $v_1(y)$ satisfies

$$\Delta v_1 + 2e^{v_1} = -2\pi \sum_{j=1}^{2} (k_{1j}\hat{\sigma}_j(l_k))\delta(0) \quad \text{in } \mathbb{R}^2. \tag{5-2}$$

Hence there is a sequence $N_k^* \to +\infty$ as $k \to +\infty$ that satisfies

(1) $N_k^* s_k \le \tau^k$,

(2) $\int_{B(0,N_k^*)} e^{v_1}\, dy = \int_{\mathbb{R}^2} e^{v_1}\, dy + o(1)$,

(3) $v_i^k(y) + 2\log|y| \leqslant -N_k$, $i = 1, 2$, for $|y| = N_k^*$.

Scaling back to $u_i^k$, we obtain that $u_i^k$, $i = 1, 2$, have fast decay on $\partial B(x^k, N_k^* s_k)$.

We could use the classification theorem of [Prajapat and Tarantello 2001] to calculate the total mass of $v_1$, but instead we use the P.I. (1-9) to compute it. We know that both $(\hat{\sigma}_1(l_k), \hat{\sigma}_2(l_k))$ and $(\hat{\sigma}_1(N_k^* s_k), \hat{\sigma}_2(N_k^* s_k))$ satisfy the P.I. and $\hat{\sigma}_2(N_k^* s_k) = \hat{\sigma}_2(l_k)$ by Lemma 4.2. With a fixed $\sigma_2 = \hat{\sigma}_2(l_k)$, P.I. (1-9) is a quadratic polynomial in $\sigma_1$; then $\hat{\sigma}_1(l_k)$ and $\hat{\sigma}_1(N_k^* s_k)$ are two roots of the polynomial. From it, we can easily calculate $\hat{\sigma}_1(N_k^* s_k)$.

By a direct computation, we have

$$(\hat{\sigma}_1(N_k^* s_k), \hat{\sigma}_2(N_k^* s_k)) \in \Gamma(1,1) \quad \text{if } (\hat{\sigma}_1(l_k), \hat{\sigma}_2(l_k)) \in \Gamma(1,1).$$

Thus (1)–(3) hold at $r = N_k^* s_k$. By denoting $N_k^* s_k$ as $l_k$, we can repeat the same argument until $\tau^k/l_k \leqslant C$. Hence Proposition 5.1 is proved. □

**5B.** ***Local mass in a group that does not contain 0.*** In this subsection we collect some $x_i^k \in \Sigma_k$ into a group $S$, a subset of $\Sigma_k$ satisfying the following $S$-conditions:

(1) $0 \notin S$ and $|S| \ge 2$.

(2) If $|S| \ge 3$ and $x_i^k, x_j^k, x_l^k$ are three distinct elements in $S$, then

$$\operatorname{dist}(x_i^k, x_j^k) \le C \operatorname{dist}(x_j^k, x_l^k)$$

for some constant $C$ independent of $k$.

(3) For any $x_m^k \in \Sigma_k \setminus S$, we have $\operatorname{dist}(x_m^k, S)/\operatorname{dist}(x_i^k, x_j^k) \to \infty$ as $k \to \infty$, where $x_i^k, x_j^k \in S$.

We write $S$ as $S = \{x_1^k, \dots, x_m^k\}$ and let

$$l^k(S) = 2 \max_{1 \le j \le m} \operatorname{dist}(x_1^k, x_j^k). \tag{5-3}$$

Recall $\tau_l^k = \frac{1}{2}\operatorname{dist}(x_l^k, \Sigma_k \setminus \{x_l^k\})$; by (2) and (3) above we have $l^k(S) \sim \tau_i^k$ for $1 \le i \le m$. Let

$$\tau_S^k = \tfrac{1}{2}\operatorname{dist}(x_1^k, \Sigma_k \setminus S).$$

Then by (3) above we have $\tau_S^k/\tau_i^k \to \infty$ for any $x_i^k \in S$.

By Proposition 5.1, we know that at least one of $u_i^k$ has fast decay on $\partial B(x_1^k, \tau_1^k)$. Suppose $u_1^k$ has fast decay on $\partial B(x_1^k, \tau_1^k)$. Then

$$u_1^k \text{ has fast decay on } \partial B(x_1^k, l^k(S)), \tag{5-4}$$

and we get

$$\begin{aligned}\sigma_1^k(B(x_1^k, l^k(S))) &= \frac{1}{2\pi}\int_{B(x_1^k, l^k(S))} h_1^k e^{u_1^k}\,dx \\ &= \frac{1}{2\pi}\int_{\bigcup_{j=1}^m B(x_j^k,\tau_j^k)} h_1^k e^{u_1^k} + \frac{1}{2\pi}\int_{B(x_1^k,l^k(S))\setminus(\bigcup_{j=1}^m B(x_j^k,\tau_j^k))} h_1^k e^{u_1^k}.\end{aligned}$$

Since $u_1^k$ has fast decay outside of $B(x_j^k, \tau_j^k)$, we have

$$e^{u_1^k(x)} \le o(1)\max_j\{|x - x_j^k|^{-2}\} \quad \text{for } x \notin \bigcup_{j=1}^k B(x_j^k, \tau_j^k)$$

and the second integral is $o(1)$. Hence by Proposition 5.1,

$$\sigma_1^k(B(x_1^k, l^k(S))) = 2m_1 + o(1) \quad \text{for some } m_1 \in \mathbb{N} \cup \{0\}. \tag{5-5}$$

Similarly if $u_2^k$ has fast decay on $\partial B(x_1^k, \tau_1^k)$, we have

$$\sigma_2^k(B(x_1^k, l^k(S))) = 2m_2 + o(1) \quad \text{for some } m_2 \in \mathbb{N} \cup \{0\}. \tag{5-6}$$

If $u_2^k$ has slow decay on $\partial B(x_1^k, \tau_1^k)$, then it is easy to see that $u_2^k$ has slow decay on $\partial B(x_j^k, \tau_j^k)$. By Proposition 5.1 we denote $n_{i,j} \in \mathbb{N}$ by

$$2n_{i,j} = \lim_{r\to 0}\lim_{k\to\infty}\sigma_i^k(B(x_j^k, r\tau_j^k)), \quad 1 \le j \le m,\ i = 1, 2.$$

Define $\hat{n}_{i,j}$ by

$$\hat{n}_{i,j} = -\sum_{l=1}^{2} k_{il} n_{l,j}.$$

Then the slow decay of $u_2^k$ on $\partial B(x_j^k, \tau_j^k)$ implies $1 + \hat{n}_{2,j} > 0$. Since $\hat{n}_{2,j} \in \mathbb{Z}$ we have $\hat{n}_{2,j} \ge 0$.

Furthermore, if we scale $u^k$ by

$$v_i^k(y) = u_i^k(x_1^k + l^k(S)y) + 2\log l^k(S), \quad i = 1, 2,$$

the sequence $v_2^k$ converges to $v_2(y)$ and $v_1^k$ tends to $-\infty$ over any compact subset of $\mathbb{R}^2 \setminus \{0\}$. Then $v_2$ satisfies

$$\Delta v_2(y) + 2e^{v_2(y)} = 4\pi\sum_{j=1}^m \hat{n}_{2,j}\delta_{p_j} \quad \text{in } \mathbb{R}^2, \tag{5-7}$$

where $p_j = \lim_{k\to\infty}(x_j^k - x_1^k)/l^k(S)$. By Theorem 2.1

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_2} = 2N \quad \text{for some } N \in \mathbb{N}.$$

Thus using the argument in Proposition 5.1, we conclude that there is a sequence of $N_k^* \to \infty$ such that both $u_i^k$ $(i = 1, 2)$ have fast decay on $\partial B(x_1^k, N_k^* l^k(S))$ and $\sigma_i^k(B(x_1^k, N_k^* l^k(S))) = 2m_i + o(1)$. Denote $N_k^* l^k(S)$ by $l_k$ for simplicity; we see that (5-5) and (5-6) hold at $l_k$. Then by using Lemmas 4.1 and 4.2 we continue this process to obtain the following conclusion:

$$\text{At least one component of } u^k \text{ has fast decay on } \partial B(x_1^k, \tau_S^k). \tag{5-8}$$

Let $\hat{\sigma}_i^k(B(x_1^k, \tau_S^k))$ be defined as in (5-1). Then

$$\hat{\sigma}_i^k(B(x_1^k, \tau_S^k)) = 2m_i(S), \quad \text{where } m_i(S) \in \mathbb{N} \cup \{0\}, \tag{5-9}$$

and the pair $(2m_1(S), 2m_2(S))$ satisfies the P.I. (1-9).

Denote the group $S$ by $S_1$. Based on this procedure, we can continue to select a new group $S_2$ such that the $S$-conditions holds except we have to modify condition (2). In (2), we consider $S_1$ as a single point as long as we compare the distance of distinct elements in $S_2$.

Set

$$\tau_{S_2}^k = \tfrac{1}{2}\,\mathrm{dist}(x_1^k, \Sigma_k \setminus S_2) \quad \text{for } x_1^k \in S_2.$$

Then we follow the same argument as above to obtain the same conclusion as (5-8)–(5-9).

If (1-5) does not contain a singularity, the final step is to collect all the $x_i^k$ into the single biggest group and (5-8)–(5-9) hold. Then we get $(\sigma_1, \sigma_2) = (2m_1, 2m_2)$ (which satisfies the Pohozaev identity), where

$$\sigma_i = \lim_{r\to 0}\lim_{k\to\infty}\frac{1}{2\pi}\int_{B(0,r)} h_i^k e^{u_i^k}, \quad i = 1, 2.$$

By a direct computation, we can prove that the set of all the pairs of even integers solving (1-9) is exactly $\Gamma(1,1)$. This proves Theorem 1.3 if (1-5) has no singularities.

If 0 is a singularity of (1-5) then $\Sigma_k$ can be written as a disjoint union of $\{0\}$ and $S_j$ $(j = 1, \dots, m)$. Here each $S_j$ is collected by the process described above and is maximal in the following sense:

(i) $0 \notin S$, $|S| \geq 2$ and for any two distinct points $x_i^k, x_j^k$ in $S$ we have

$$\mathrm{dist}(x_i^k, x_j^k) \ll \tau^k(S),$$

where $\tau^k(S) = \mathrm{dist}(S, \Sigma_k \setminus S)$.

(ii) For any $0 \neq x_i^k \in \Sigma_k \setminus S$,

$$\mathrm{dist}(x_i^k, 0) \leq C\,\mathrm{dist}(x_i^k, S)$$

for some constant $C$.

For $S_j$ we define

$$\tau_{S_j}^k = \tfrac{1}{2}\,\mathrm{dist}(S_j, \Sigma_k \setminus S_j).$$

Then the process described above proves the main result of this section:

**Proposition 5.2.** *Let $S_j$ $(j = 1, \dots, m)$ be described as above. Then (5-8)–(5-9) hold, where $B(x_1^k, \tau_S^k)$ is replaced by $B(x_i^k, \tau_{S_j}^k)$ and $x_i^k$ is any element in $S_j$.*

## 6. Proofs of Theorems 1.2, 1.3, 1.5 and 1.6

In Proposition 5.2, we write $\Sigma_k = \{0\} \cup S_1 \cup \dots \cup S_N$. From the construction, the ratio $|x^k|/|\tilde{x}^k|$ is bounded for any $x^k, \tilde{x}^k \in S_j$. Let

$$\|S_j\| = \min_{x^k \in S_j} |x^k|$$

and arrange $S_j$ by

$$\|S_1\| \le \|S_2\| \le \dots \le \|S_N\|.$$

Assume $l$ is the largest number such that $\|S_l\| \le C\|S_1\|$. Then $\|S_l\| \ll \|S_{l+1}\|$.

We recall the local mass contributed by $x_j^k \in S_j$ is

$$\big(\hat{\sigma}_1(B(x_j^k, \tau_j^k)), \hat{\sigma}_2(B(x_j^k, \tau_j^k))\big) = (m_{1,j}, m_{2,j}), \quad \text{where } m_{1,j}, m_{2,j} \in 2\mathbb{N} \cup \{0\}.$$

Let

$$r_1^k = \tfrac{1}{2}\|S_1\|.$$

Then we have

$$u_i^k(x) + 2\log|x| \le C \quad \text{for } 0 < |x| \le r_1^k,\ i = 1, 2.$$

*Proof of Theorem 1.3.* Let

$$\tilde{u}_i^k(x) = u_i^k(x) + 2\alpha_i \log|x|, \quad i = 1, 2.$$

Then (1-5) becomes

$$\Delta \tilde{u}_i^k(x) + \sum_{j=1}^{2} k_{ij} |x|^{2\alpha_j} h_j^k(x) e^{\tilde{u}_j^k(x)} = 0, \quad |x| \le r_1^k,\ i = 1, 2.$$

Let

$$-2\log\delta_k = \max_{i \in I} \max_{x \in \bar{B}(0, r_1^k)} \frac{\tilde{u}_i^k}{1 + \alpha_i}, \tag{6-1}$$

and

$$\tilde{v}_i^k(y) = \tilde{u}_i^k(\delta_k y) + 2(1 + \alpha_i)\log\delta_k, \quad |y| \le r_1^k/\delta_k,\ i = 1, 2. \tag{6-2}$$

Then $\tilde{v}_i^k$ satisfies

$$\Delta \tilde{v}_i^k(y) + \sum_{j=1}^{2} k_{ij} |y|^{2\alpha_j} h_j^k(\delta_k y) e^{\tilde{v}_j^k(y)} = 0, \quad |y| \le r_1^k/\delta_k,\ i = 1, 2. \tag{6-3}$$

We have either

(a) $\lim_{k\to\infty} r_1^k/\delta_k = \infty$, or

(b) $r_1^k/\delta_k \le C$.

For case (a), our purpose is to prove a result similar to Proposition 5.1:

(1) *At most one component of $u^k$ has slow decay on $\partial B(0,r_1^k)$.* As in Section 5, we define

$$\hat{\sigma}_{i,1}=\begin{cases}\lim_{k\to+\infty}\sigma_i^k(B(0,r_1^k)) & \text{if } u_i^k \text{ has fast decay on } \partial B(0,r_1^k),\\ \lim_{r\to 0}\lim_{k\to+\infty}\sigma_i^k(B(0,rr_1^k)) & \text{if } u_i^k \text{ has slow decay on } \partial B(0,r_1^k),\end{cases}$$

(2) $(\hat{\sigma}_{1,1},\hat{\sigma}_{2,1})$ satisfies the Pohozaev identity (1-9), and

(3) $\hat{\sigma}_{i,1}=2\sum_{j=1}^2 n_{i,j}\mu_j+2n_{i,3},\ n_{i,j}\in\mathbb{Z},\ i=1,2,\ j=1,2,3.$

We carry out the proof in the discussion of the following two cases.

Case 1: If both $\tilde{v}_i^k(y)$ converge in any compact set of $\mathbb{R}^2$, then $(\hat{\sigma}_{1,1},\hat{\sigma}_{2,1})$ can be obtained by the classification theorem in [Lin et al. 2012]:

$$(\hat{\sigma}_{1,1},\hat{\sigma}_{2,1})=\begin{cases}(2\mu_1+2\mu_2,2\mu_1+2\mu_2) & \text{for } A_2,\\ (4\mu_1+2\mu_2,4\mu_1+4\mu_2) & \text{for } B_2,\\ (8\mu_1+4\mu_2,12\mu_1+8\mu_2) & \text{for } G_2.\end{cases}$$

By Lemma 4.1, both $u_i^k$ have fast decay on $\partial B(0,r_1^k)$. So this proves (1)–(3) in this case.

Case 2: Only one $\tilde{v}_i^k$ converges to $v_i(y)$ and the other tends to $-\infty$ uniformly in any compact set. Then it is easy to see that there is $l_k\ll r_1^k$ such that both $u_i^k$ have fast decay on $\partial B(0,l_k)$ and

$$\big(\sigma_1(B(0,l_k)),\sigma_2(B(0,l_k))\big)=(2\mu_1,0)\quad\text{or}\quad\big(\sigma_1(B(0,l_k)),\sigma_2(B(0,l_k))\big)=(0,2\mu_2).$$

So this is the same situation as in the starting point for Proposition 5.1. Then the same argument of Proposition 5.1 leads to the conclusion (1)–(3).

The pair $(\hat{\sigma}_{1,1},\hat{\sigma}_{2,1})$ can be calculated by the same method in Proposition 5.1. Then $(\hat{\sigma}_{1,1},\hat{\sigma}_{2,1})\in\Gamma(\mu_1,\mu_2)$, which is given in Section 2.

To continue for $r\in[r_1^k,r_2^k]$, where $r_2^k=\frac{1}{2}\|S_{l+1}\|$, we separate our discussion into two cases also.

Case 1: One component has slow decay on $\partial B(0,r_1^k)$, say $u_1^k$. Then we scale

$$v_i^k(y)=u_i^k(r_1^k y)+2\log r_1^k.$$

By our assumption, $v_1^k(y)$ converges to $v_1(y)$ and $v_2^k(y)\to-\infty$ in any compact set. Let $x_j^k\in S_j$ and $y_j^k=(r_1^k)^{-1}x_j^k\to p_j$ for $j\le l$. Then $v_1(y)$ satisfies

$$\Delta v_1+2e^{v_1}=4\pi\tilde{\alpha}_1\delta_0+4\pi\sum_{j=1}^{l}\tilde{n}_{1,j}\delta_{p_j}, \tag{6-4}$$

where

$$\tilde{n}_{1,j}=-\frac{1}{2}\sum_{i=1}^{2}k_{1i}m_{i,j}\quad\text{for some } m_{ij}\in\mathbb{Z}\qquad\text{and}\qquad\tilde{\alpha}_1=\alpha_1-\frac{1}{2}\sum_{i=1}^{2}k_{1i}\hat{\sigma}_{i,1}. \tag{6-5}$$

The finiteness of $\int_{\mathbb{R}^2}e^{v_1}$ implies

$$\tilde{\alpha}_1>-1\quad\text{and}\quad\tilde{n}_{1,j}\ge 0.$$

By Theorem 2.2, we have

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_1}\,dy = 2(\tilde{\alpha}_1+1)+2k_1, \quad \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_1}\,dy = 2k_2, \quad \text{where } k_1,k_2\in\mathbb{Z}. \tag{6-6}$$

As before, we can choose $l_k$, $r_1^k \ll l_k \ll r_2^k$, such that both $u_i^k$ have fast decay on $\partial B(0,l_k)$. Then the new pair $(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2})$, which is defined by

$$\hat{\sigma}_{t,2} = \frac{1}{2\pi}\lim_{k\to 0}\int_{B(0,l_k)} h_t^k e^{u_t^k}, \quad t=1,2,$$

becomes

$$(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2}) = \left(\hat{\sigma}_{1,1}+\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_1}+\sum_{j=1}^{l} m_{1,j},\ \hat{\sigma}_{2,1}+\sum_{j=1}^{l} m_{2,j}\right) \tag{6-7}$$

for $m_{1j},m_{2j}\in 2\mathbb{N}\cup\{0\}$. Using (6-6), we get

$$\hat{\sigma}_{1,2} = \begin{cases} \hat{\sigma}_{1,1}+2k_2+\sum_{j=1}^{l} m_{1,j} & \text{if } \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_1}\,dy = 2k_2, \\ 2\mu_1+\hat{\sigma}_{1,1}-\sum_{i=1}^{2} k_{1i}\hat{\sigma}_{i,1}+2k_1+\sum_{j=1}^{l} m_{1,j} & \text{if } \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_1}\,dy = 2(\tilde{\alpha}_1+1)+2k_1. \end{cases} \tag{6-8}$$

We note that if $(\hat{\sigma}_{1,1},\hat{\sigma}_{2,1})\in\Gamma(\mu_1,\mu_2)$ and

$$2\mu_1+\hat{\sigma}_{1,1}-\sum_{i=1}^{2} k_{1i}\hat{\sigma}_{i,1}>0,$$

then

$$\left(2\mu_1+\hat{\sigma}_{1,1}-\sum_{i=1}^{2} k_{1i}\hat{\sigma}_{i,1},\hat{\sigma}_{2,1}\right)\in\Gamma(\mu_1,\mu_2).$$

Let $(\sigma_1^*,\sigma_2^*) = \big(2\mu_1+\hat{\sigma}_{1,1}-\sum_{i=1}^{2} k_{1i}\hat{\sigma}_{i,1},\ \hat{\sigma}_{2,1}\big)$. We can write

$$(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2}) = (\sigma_1^*+m_1,\ \sigma_2^*+m_2), \tag{6-9}$$

with $(\sigma_1^*,\sigma_2^*)\in\Gamma(\mu_1,\mu_2)$ and $m_1,m_2\in 2\mathbb{Z}$.

Case 2: If both $u_i^k$ have fast decay on $\partial B(0,r_1^k)$, then they have fast decay on $\partial B(0,cr_1^k)$, where we choose $c$ bounded such that $\bigcup_{j=1}^{l} S_j \subset B\big(0,\frac{1}{2}cr_1^k\big)$. Then the new pair $(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2})$ becomes

$$(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2}) = \left(\hat{\sigma}_{1,1}+\sum_{j=1}^{l} m_{1,j},\ \hat{\sigma}_{2,1}+\sum_{j=1}^{l} m_{2,j}\right) \quad \text{for } m_{1,j},m_{2,j}\in 2\mathbb{Z}. \tag{6-10}$$

Hence, in this case we can also write

$$(\hat{\sigma}_{1,2},\hat{\sigma}_{2,2}) = (\sigma_1^*+m_1,\ \sigma_2^*+m_2), \tag{6-11}$$

with $(\sigma_1^*,\sigma_2^*) = (\hat{\sigma}_{1,1},\hat{\sigma}_{2,1})\in\Gamma(\mu_1,\mu_2)$ and $m_1,m_2\in 2\mathbb{Z}$. Set $cr_1^k = l_k$. Then we can continue our process starting from $l_k$. After finitely many steps, we can prove that at most one component of $u^k$ has slow decay on $\partial B(0,1)$ and their local masses have the expression in (3).

For case (b), i.e., $r_1^k/\delta_k \le C$, first $\tilde{v}_i^k \le 0$ implies $|y|^{2\alpha_j} h_j^k(\delta_k y) e^{\tilde{v}_j^k} \le C$ on $B(0, r_1^k/\delta_k)$. Then the fact that $\tilde{v}_i^k$ has bounded oscillation on $\partial B(0, r_1^k/\delta_k)$ further gives

$$\tilde{v}_i^k(x) = \bar{\tilde{v}}_i^k(\partial B(0, r_1^k/\delta_k)) + O(1) \quad \text{for all } x \in B(r_1^k/\delta_k),$$

where $\bar{\tilde{v}}_i^k(\partial B(0, r_1^k/\delta_k))$ stands for the average of $\tilde{v}_i^k$ on $\partial B(0, r_1^k/\delta_k)$. Direct computation shows that

$$\int_{B(0,r_1^k)} h_i^k e^{u_i^k}\, dx = \int_{B(0,r_1^k/\delta_k)} |y|^{2\alpha_i} h_i^k(\delta_k y) e^{\tilde{v}_i^k(y)}\, dy = O(1) e^{\bar{\tilde{v}}_i^k(\partial B(0,r_1^k/\delta_k))}.$$

Thus if $\bar{\tilde{v}}_i^k(\partial B(0, r_1^k/\delta_k)) \to -\infty$, we get $\int_{B(0,r_1^k)} h_i^k e^{u_i^k}\, dx = o(1)$. On the other hand, we note that $\bar{\tilde{v}}_i^k(\partial B(0, r_1^k/\delta_k)) \to -\infty$ is equivalent to $u_i^k$ having fast decay on $\partial B(0, r_1^k)$. Consequently $\hat{\sigma}_{i,1} = 0$ if $u_i^k$ has fast decay on $\partial B(0, r_1^k)$. So if both components have fast decay on $\partial B(0, r_1^k)$ we have $(\hat{\sigma}_{1,1}, \hat{\sigma}_{2,1}) = (0, 0)$.

If some component of $u^k$ has slow decay, say $u_2^k$, according to the definition of $\hat{\sigma}_{2,1}$, we have

$$\begin{aligned}\hat{\sigma}_{2,1} &= \lim_{r\to 0} \lim_{k\to+\infty} \sigma_2^k(B(0, r r_1^k)) = \frac{1}{2\pi} \lim_{r\to 0} \lim_{k\to+\infty} \int_{B(0,r r_1^k)} h_2^k e^{u_2^k}\, dx \\ &= \frac{1}{2\pi} \lim_{r\to 0} \lim_{k\to+\infty} \int_{B(0,r r_1^k/\delta_k)} |y|^{2\alpha_2} h_2^k(\delta_k y) e^{\tilde{v}_2^k(y)}\, dy = 0,\end{aligned} \tag{6-12}$$

where we used $|y|^{2\alpha_2} h_2^k(\delta_k y) e^{\tilde{v}_2^k} \le C$ on $B(0, r_1^k/\delta_k)$. Then we still get

$$(\hat{\sigma}_{1,1}, \hat{\sigma}_{2,1}) = (0, 0).$$

Now we can continue our discussion as in case (a) and Theorem 1.3 is proved completely. □

Next, we shall prove Theorem 1.5, that is, $\Sigma_k = \{0\}$, by way of contradiction. Suppose $\Sigma_k$ has points other than 0. Using the notation from the beginning of this section, we have

$$\Sigma_k = \{0\} \cup S_1 \cup \cdots \cup S_N.$$

Now suppose $r_1^k/\delta_k \to \infty$ as $k \to \infty$. Let $(\hat{\sigma}_{1,2}, \hat{\sigma}_{2,2})$ be the local masses defined by (6-7) for one of the components $u_i^k$ having slow decay on $\partial B(0, r_1^k)$ or by (6-10) for both components having fast decay on $\partial B(0, r_1^k)$. We summarize the results in the following:

(i) $\hat{\sigma}_{i,2} = \sigma_i^* + m_i$, where $(\sigma_1^*, \sigma_2^*) \in \Gamma(\mu_1, \mu_2)$ and $m_i$, $i = 1, 2$, are even integers.

(ii) Both pairs $(\sigma_1^*, \sigma_2^*)$ and $(\hat{\sigma}_{1,2}, \hat{\sigma}_{2,2})$ satisfy the Pohozaev identity.

Based on the description above, we now present the proof of Theorem 1.5.

*Proof of Theorem 1.5.* From the discussion above, we have

$$(\hat{\sigma}_{1,2}, \hat{\sigma}_{2,2}) = (\sigma_1^* + m_1,\ \sigma_2^* + m_2).$$

We note that the conclusion of Theorem 1.5 is equivalent to proving $m_i = 0$, $i = 1, 2$. In order to prove this we first observe that both $(\hat{\sigma}_{1,2}, \hat{\sigma}_{2,2})$ and $(\sigma_1^*, \sigma_2^*)$ satisfy the P.I.

$$k_{21}\sigma_1^2 + k_{12}k_{21}\sigma_1\sigma_2 + k_{12}\sigma_2^2 = 2k_{21}\mu_1\sigma_1 + 2k_{12}\mu_2\sigma_2. \tag{6-13}$$

Thus we can write

$$k_{21}(\sigma_1^*)^2+k_{12}k_{21}\sigma_1^*\sigma_2^*+k_{12}(\sigma_2^*)^2=2k_{21}\mu_1\sigma_1^*+2k_{12}\mu_2\sigma_2^*, \tag{6-14}$$

and

$$\begin{aligned}k_{21}(\sigma_1^*+m_1)^2+k_{12}k_{21}(\sigma_1^*+m_1)(\sigma_2^*+m_2)+k_{12}(\sigma_2^*+m_2)^2&\\ =2k_{21}\mu_1(\sigma_1^*+m_1)+2k_{12}\mu_2(\sigma_2^*+m_2).&\end{aligned} \tag{6-15}$$

It is easy to obtain the following from (6-15) and (6-14):

$$\begin{aligned}2k_{21}m_1\sigma_1^*+k_{12}k_{21}m_2\sigma_1^*+k_{12}k_{21}m_1\sigma_2^*+2k_{12}m_2\sigma_2^*&\\ =2k_{21}m_1\mu_1+2k_{12}m_2\mu_2-(k_{21}m_1^2+k_{12}k_{21}m_1m_2+k_{12}m_2^2).&\end{aligned} \tag{6-16}$$

Since $(\sigma_1^*,\sigma_2^*)\in\Gamma(\mu_1,\mu_2)$, we set

$$\sigma_1^*=l_{1,1}\mu_1+l_{1,2}\mu_2,\quad \sigma_2^*=l_{2,1}\mu_1+l_{2,2}\mu_2.$$

Then we can rewrite (6-16) as

$$\begin{aligned}&\big(2k_{21}l_{1,1}m_1+k_{12}k_{21}l_{2,1}m_1-2k_{21}m_1+2k_{12}l_{2,1}m_2+k_{12}k_{21}l_{1,1}m_2\big)\mu_1\\ &\quad+\big(2k_{21}l_{1,2}m_1+k_{12}k_{21}l_{2,2}m_1+2k_{12}l_{2,2}m_2+k_{12}k_{21}l_{1,2}m_2-2k_{12}m_2\big)\mu_2\\ &\qquad+(k_{21}m_1^2+k_{12}k_{21}m_1m_2+k_{12}m_2^2)=0.\end{aligned} \tag{6-17}$$

Since $\mu_1,\mu_2$ and 1 are linearly independent, the coefficients of $\mu_1$ and $\mu_2$ must vanish. Equivalently we have

$$\begin{pmatrix}2k_{21}l_{1,1}+k_{12}k_{21}l_{2,1}-2k_{21} & 2k_{12}l_{2,1}+k_{12}k_{21}l_{1,1}\\ 2k_{21}l_{1,2}+k_{12}k_{21}l_{2,2} & 2k_{12}l_{2,2}+k_{12}k_{21}l_{1,2}-2k_{12}\end{pmatrix}\begin{pmatrix}m_1\\ m_2\end{pmatrix}=0. \tag{6-18}$$

Let $M_K$ be the coefficient matrix

$$M_K=\begin{pmatrix}2k_{21}l_{1,1}+k_{12}k_{21}l_{2,1}-2k_{21} & 2k_{12}l_{2,1}+k_{12}k_{21}l_{1,1}\\ 2k_{21}l_{1,2}+k_{12}k_{21}l_{2,2} & 2k_{12}l_{2,2}+k_{12}k_{21}l_{1,2}-2k_{12}\end{pmatrix}.$$

Our goal is to show that $M_k$ is nonsingular, which immediately implies $m_1=m_2=0$ and completes the proof of Theorem 1.5. The proof of the nonsingularity of $M_k$ is divided into the following three cases.

<u>Case 1</u>: $\boldsymbol{K}=A_2$. Then we can write (6-18) as

$$\begin{pmatrix}2l_{1,1}-l_{2,1}-2 & 2l_{2,1}-l_{1,1}\\ 2l_{1,2}-l_{2,2} & 2l_{2,2}-l_{1,2}-2\end{pmatrix}\begin{pmatrix}m_1\\ m_2\end{pmatrix}=0. \tag{6-19}$$

We note that

$$(l_{1,1},l_{1,2},l_{2,1},l_{2,2})\in\big\{(2,0,0,0),\ (0,0,0,2),\ (2,2,0,2),\ (2,0,2,2),\ (2,2,2,2)\big\}.$$

Then it is easy to see that $M_K$ is nonsingular when $(l_{1,1},l_{1,2},l_{2,1},l_{2,2})$ belongs the above set.

<u>Case 2</u>: $\boldsymbol{K}=B_2$. Then we can write (6-18) as

$$\begin{pmatrix}2l_{1,1}-l_{2,1}-2 & l_{2,1}-l_{1,1}\\ 2l_{1,2}-l_{2,2} & l_{2,2}-l_{1,2}-1\end{pmatrix}\begin{pmatrix}m_1\\ m_2\end{pmatrix}=0. \tag{6-20}$$

We note that

$$(l_{1,1}, l_{1,2}, l_{2,1}, l_{2,2}) \in \{(2,0,0,0),\ (2,0,4,2),\ (4,2,4,2),\ (0,0,0,2),\ (2,2,0,2),\ (2,2,4,4),\ (4,2,4,4)\}$$

From the above set, we can see that $4 \mid (l_{2,1} - l_{1,1})(2l_{1,2} - l_{2,2})$. As a result, if the determinant of $M_K$ is 0, we have to make $4 \mid (2l_{1,1} - l_{2,1} - 2)$, which forces $l_{2,1} \equiv 2 \pmod 4$. However, this is impossible according to the above list. Thus $M_k$ is nonsingular in this case.

Case 3: $\boldsymbol{K} = G_2$. Then we can write (6-18) as

$$\begin{pmatrix} 6l_{1,1} - 3l_{2,1} - 6 & 2l_{2,1} - 3l_{1,1} \\ 6l_{1,2} - 3l_{2,2} & 2l_{2,2} - 3l_{1,2} - 2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = 0. \tag{6-21}$$

We note that

$$(l_{1,1}, l_{1,2}, l_{2,1}, l_{2,2}) \in \{(2,0,0,0),\ (2,0,6,2),\ (6,2,6,2),\ (6,2,12,6),\ (8,4,12,6),\ (8,4,12,8),\ (0,0,0,2),\ (2,2,0,2),\ (2,2,6,6),\ (6,4,6,6),\ (6,4,12,8)\}.$$

From the above list, we have $3 \mid l_{2,1}$; then we get $9 \mid (2l_{2,1} - 3l_{1,1})(6l_{1,2} - 3l_{2,2})$. On the other hand, we see that

$$l_{1,1} \equiv 0, 2 \pmod 3 \quad \text{and} \quad l_{2,2} \equiv 0, 2 \pmod 3,$$

which implies $(6l_{1,1} - 3l_{2,1} - 6)(2l_{2,2} - 3l_{1,2} - 2)$ is not multiple of 9; therefore we have the determinant of $M_K$ is not zero. Thus $M_k$ is nonsingular when $\boldsymbol{K} = G_2$.

Theorem 1.5 is established. □

Finally we prove Theorems 1.2 and 1.6.

*Proof of Theorems 1.2 and 1.6.* Suppose there exists a sequence of blowup solutions $(u_1^k, u_2^k)$ of (1-2) with $(\rho_1, \rho_2) = (\rho_1^k, \rho_2^k)$. First, we prove Theorem 1.2. From the previous discussion of this section, we get that at least one component (say $u_1^k$) of $u^k$ has fast decay on a small ball $B$ near each blowup point $q$, which means $u_1^k(x) \to -\infty$ if $x \notin S$ and $x$ is not a blowup point. Hence Theorem 1.2 holds.

Because the mass distribution of $u_1^k$ concentrates as $k \to +\infty$, we get that $\lim_{k \to +\infty} \rho_1^k$ is equal to the sum of the local mass $\sigma_1$ at a blowup point $q$, which implies $\rho_1 \in \Gamma_1$, a contradiction to the assumption. Thus, we finish the proof of Theorem 1.6. □

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