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BEYOND THE BKM CRITERION FOR THE 2D RESISTIVE MAGNETOHYDRODYNAMIC EQUATIONS

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The question of whether the two-dimensional (2D) magnetohydrodynamic (MHD) equations with only magnetic diffusion can develop a finite-time singularity from smooth initial data is a challenging open problem in fluid dynamics and mathematics. In this paper, we derive a regularity criterion less restrictive than the Beale–Kato–Majda (BKM) regularity criterion type, namely any solution $(u, b) \in C([0, T[; H^r(\mathbb{R}^2))$ with $r > 2$ remains in $H^r(\mathbb{R}^2)$ up to time T under the assumption that

$$\int_0^T \frac{\|\nabla u(t)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(t)\|_\infty)}\, dt < +\infty.$$

This regularity criterion may stand as a great improvement over the usual BKM regularity criterion, which states that if $\int_0^T \|\nabla \times u(t)\|_\infty\, dt < +\infty$ then the solution $(u, b) \in C([0, T[; H^r(\mathbb{R}^2))$ with $r > 2$ remains in $H^r(\mathbb{R}^2)$ up to time T. Furthermore, our result applies also to a class of equations arising in hydrodynamics and studied by Elgindi and Masmoudi (2014) for their L^∞ ill-posedness.

Introduction

Magnetohydrodynamic (MHD) equations describe the evolution of electrically conducting fluids in the presence of electric and magnetic fields. Examples of such fluids include plasmas, liquid metals, and salt water or electrolytes. The field of MHD was initiated by Hannes Alfvén [1942], for which he received the Nobel Prize in physics in 1970. It addresses laboratory as well as astrophysical plasmas and therefore is extensively used in very different contexts. In astrophysics, its applications range from solar wind [Marsch and Tu 1994], to the sun [Priest 1982; Priest and Forbes 2000], to the interstellar medium [Ng et al. 2003] and beyond [Zweibel and Heiles 1997]. At the same time, MHD is also relevant to large-scale motion in nuclear fusion devices such as tokamaks [Strauss 1976]. A tokamak is a toroidal device in which hydrogen isotopes in the form of a plasma reaching a temperature on the order of hundreds of millions of Kelvins is confined thanks to a very strong applied magnetic field. Tokamaks are used to study controlled fusion and are considered as one of the most promising concepts to produce fusion energy in the near future. However the main problem with this approach of confinement is that hydrodynamic instabilities arise. Numerical simulations using the MHD models are therefore of uttermost importance. Further, the proof of the existence of a smooth strong solution would allow one to guarantee a priori the convergence of some numerical approximations; see for instance [Chernyshenko et al. 2007].

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Due to their prominent roles in modeling many phenomena in astrophysics, geophysics and plasma physics, the MHD equations have been studied extensively mathematically. Furthermore, while the differences in behavior between the two-dimensional (2D) and three-dimensional (3D) hydrodynamical turbulence of neutral fluids are accepted to be important, those of the MHD system in both cases are conventionally believed to be nonsignificant [Biskamp and Schwarz 2001]. Strong statements were made by some authors that 2D simulations can be safely used to model 3D situations because the properties of the 2D and the 3D MHD turbulence are essentially the same [Biskamp 1993; Biskamp and Schwarz 2001].

Hence, the mathematical studies on the MHD equations in the two-dimensional case appear highly relevant. However up to now, the question of the spontaneous appearance of a singularity from a local classical solution of the partially viscous 2D MHD (2) or 2D inviscid MHD ((2) without the Laplacian term) remains a challenging open problem in mathematical fluid mechanics. Thus, in the absence of a well-posedness theory, the development of blow-up/nonblow-up theory is of major importance for both theoretical and practical purposes. Indeed, for a mathematical or numerical test of the actual finite-time blow-up of a given solution, it is important to have a good blow-up criterion. Thus, there have been many computational attempts to find finite-time singularities of the 2D MHD equations; see [Brachet et al. 2013; Kerr and Brandenburg 1999; Tran et al. 2013a]. Moreover, recent works on the 2D MHD equations developed regularity criteria in terms of the velocity field and dealt with the MHD equations with dissipation and magnetic diffusion given by general Fourier multiplier operators such as the fractional Laplacian operators; see [Wu 2003; 2008; 2011; Chen et al. 2010; Tran et al. 2013b; Jiu and Zhao 2014; Cao et al. 2014; Yamazaki 2014a; 2014b].

Among all the regularity criteria, one of particular interest is the Beale–Kato–Majda criterion, well-known for Euler equations, and extended in [Caflisch et al. 1997] to the inviscid MHD equations, under the assumption on both velocity field and magnetic field $\int_0^T (\|\omega(t)\|_{L^\infty} + \|j(t)\|_{L^\infty})\,dt < \infty$, where the vorticity is $\omega = \nabla \times u$ and the density is $j = \nabla \times b$. And so, the Beale–Kato–Majda criterion ensures that the solution (u, b) of the inviscid MHD equations is smooth up to time T.

Meanwhile the 2D Euler equation is globally well-posed for smooth initial data; however for the 2D inviscid MHD equations, the global well-posedness of classical solutions is still a big open problem. Despite recent developments on regularity criteria, see [Gala et al. 2017; Tran et al. 2013b; Jiu and Zhao 2014; 2015; Yamazaki 2014a; 2014b; Agélas 2016; Ye and Xu 2014; Fan et al. 2014], the global regularity issue of 2D MHD equations (2) remains a challenging open problem to date. The main reason for the unavailability of a proof of global regularity for the system of equations (2) is due to the quadratic coupling between u and b which invalidates the vorticity conservation. Indeed, the structure of the vorticity is instantaneously altered due to the effects of the magnetic fields. This fact is the source of the main difficulty connected to the global existence of classical solutions, where no strong global a priori estimates are yet known. This difficulty is revealed through the equations of the 2D inviscid MHD equations governing the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ and the current density $j = \partial_1 b_2 - \partial_2 b_1$,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = b \cdot \nabla j, \\ \partial_t j + u \cdot \nabla j = b \cdot \nabla \omega + T(\nabla u, \nabla b), \end{cases} \tag{1}$$

where,

$$T(\nabla u, \nabla b) = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) + 2\partial_2 u_2(\partial_2 b_1 + \partial_1 b_2).$$

We observe that the magnetic field contributes in the last nonlinear part of the second equation with the quadratic term $T(\nabla u, \nabla b)$.

By virtue of this difficulty, no a priori uniform bound for $\|\omega\|_{L^\infty(\mathbb{R}^2\times[0,T])}$ is known for the 2D MHD equations with only magnetic diffusion (2). Further in [Fan et al. 2014; Jiu and Zhao 2015; Agélas 2016], by considering Fourier multiplier operators magnetic diffusion slightly stronger than the Laplacian magnetic diffusion, the authors were able to obtain a uniform bound of $\|\nabla j\|_{L^1([0,T];L^\infty(\mathbb{R}^2))}$ and then from the first equation of (1) obtain a uniform bound of $\|\omega\|_{L^\infty(\mathbb{R}^2\times[0,T])}$ deriving from estimates for transport equations; see for instance Lemma 4.1 in [Kato and Ponce 1988].

However, the approach used in [Fan et al. 2014; Jiu and Zhao 2015; Agélas 2016], based on the properties of the heat equation by using singular integral representations of (2), fails in the case where we have only a Laplacian magnetic diffusion.

Then, in this paper, we consider the initial-value problem for the 2D incompressible magnetohydrodynamic equations with Laplacian magnetic diffusion,

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = -\nabla p + (b\cdot\nabla)b, \\ \partial_t b + (u\cdot\nabla)b - \Delta b = (b\cdot\nabla)u, \\ \nabla\cdot u = 0,\ \nabla\cdot b = 0, \end{cases} \tag{2}$$

with initial conditions

$$\begin{aligned} u(x,0) &= u_0(x) \quad \text{for a.e. } x\in\mathbb{R}^2, \\ b(x,0) &= b_0(x) \quad \text{for a.e. } x\in\mathbb{R}^2, \end{aligned} \tag{3}$$

which models many significant phenomena such as the magnetic reconnection in astrophysics and geomagnetic dynamo in geophysics; see [Priest and Forbes 2000]. The problem of global well-posedness of the 2D MHD equations with partial dissipation and magnetic diffusion has generated considerable interest recently [Cao and Wu 2011; Chae 2008; Jiu and Niu 2006; Lei and Zhou 2009; Zhou and Fan 2011; Jiu and Zhao 2015]. However, as of now, the problem of uniqueness and global regularity of the 2D MHD system (2) remains widely open.

Let us take a new look at the main obstruction. We start by noting that we can rewrite the first equation of (1) satisfied by ω, the vorticity of u, as

$$\partial_t\omega + (u\cdot\nabla)\omega = F - b_1\, b\cdot\nabla u_2 + b_2\, b\cdot\nabla u_1, \tag{4}$$

where

$$F = b_1(\Delta b_2 + b\cdot\nabla u_2) - b_2(\Delta b_1 + b\cdot\nabla u_1).$$

Furthermore, a uniform bound of $\|\Delta b + (b\cdot\nabla)u\|_{L^\infty(\mathbb{R}^2\times[0,T])}$ was shown recently in [Yuan and Zhao 2018] (in Section 4 we give a sketch of the proof). Moreover, we get a uniform bound of $\|b\|_{L^\infty(\mathbb{R}^2\times[0,T])}$ deriving from some estimates for the linear Stokes system, see [Giga and Sohr 1991], hence we deduce a uniform bound for $\|F\|_{L^\infty(\mathbb{R}^2\times[0,T])}$. Then, we notice that our (4) fits with the study made in [Elgindi and Masmoudi 2014] about L^∞ ill-posedness for a class of equations arising in hydrodynamics. Thus, by virtue of $\nabla u = \mathrm{R}(\omega\,\mathrm{Id})$, where R is the Riesz transform on 2×2 matrix-valued functions, see (18), we understand that the main obstruction comes from the fact that Riesz transforms do not map L^∞ into itself.

Let us specify the way in which the obstruction is characterized. We refer to Section 1 for the notation used. By using the logarithmic Sobolev inequality proved in [Kozono and Taniuchi 2000],

$$\|\nabla f\|_{L^\infty(\mathbb{R}^2)} \lesssim 1 + \|\nabla\times f\|_{L^\infty(\mathbb{R}^2)}(1+\log^+\|f\|_{W^{s,p}(\mathbb{R}^2)}) \quad \text{with } p>1,\ s>1+\frac{2}{p},$$

where $\nabla\cdot f = 0$, $\nabla\times f = -\partial_2 f_1 + \partial_1 f_2$ is the vorticity of f and $\log^+ x = \max(0,\log x)$ for any $x>0$, we infer that for all $t\in[0,T[$,

$$\|\nabla u(t)\|_\infty \lesssim_r 1 + \|\omega(t)\|_\infty(1+\log^+\|u(t)\|_{H^r}) \tag{5}$$

and also

$$\|(\nabla u,\nabla b)(t)\|_\infty \lesssim_r 1 + \|(\omega,j)(t)\|_\infty(1+\log^+\|(u,b)(t)\|_{H^r}). \tag{6}$$

Then thanks to (5) and by using estimates for transport equations, see for instance Lemma 4.1 in [Kato and Ponce 1988], from (4) we infer that for all $t\in[0,T[$

$$\|\omega(t)\|_\infty \lesssim_r \|\omega_0\|_\infty + \int_0^t (\|F(s)\|_\infty + \|b(s)\|_\infty^2)\,ds + \int_0^t \|b(s)\|_\infty^2\|\omega(s)\|_\infty(1+\log^+\|u(s)\|_{H^r})\,ds, \tag{7}$$

where $r>2$. As a consequence of the Grönwall lemma, we deduce

$$\|\omega(t)\|_\infty \le c_r\left(\|\omega_0\|_\infty + \int_0^t (\|F(s)\|_\infty + \|b(s)\|_\infty^2)\,ds\right) e^{c_r \int_0^t \|b(s)\|_\infty^2(1+\log^+\|u(s)\|_{H^r})\,ds}, \tag{8}$$

where $c_r>0$ is a real number depending only on r. Thus, the main obstruction to getting global regularity comes from the term in logarithm which appears in (8), namely $\log^+\|u(s)\|_{H^r}$. Nevertheless, thanks to (5), (7) and the estimate

$$\|(u,b)(t)\|_{H^r} \le \|(u_0,b_0)\|_{H^r} e^{\kappa_r \int_0^t \|(\nabla u,\nabla b)(\tau)\|_\infty\,d\tau} \tag{9}$$

in the Hilbert space H^r, we obtain a new estimate of $\|\nabla u(t)\|_\infty$ in Lemma 5.1, which leads to a new regularity criterion in Theorem 5.3. Our new regularity criterion states that if

$$\int_0^T \frac{\|\nabla u(t)\|_\infty^{\frac12}}{\log(e+\|\nabla u(t)\|_\infty)}\,dt < +\infty$$

then the solution (u,b) of the 2D MHD equations (2) remains smooth up to time T. This new regularity criterion appears less restrictive than the BKM regularity criterion, which states that if $\int_0^T \|\nabla\times u(t)\|_\infty\,dt < +\infty$ then the solution (u,b) of the 2D MHD equations (2) remains smooth up to time T. Indeed, by $\nabla u = \mathrm{R}((\nabla\times u)\,\mathrm{Id})$ with R the Riesz transform on matrix-valued functions, we get

$$\|\nabla u\|_{\mathrm{BMO}(\mathbb{R}^2)} \lesssim \|\nabla\times u\|_{\mathrm{BMO}(\mathbb{R}^2)},$$

and for any $1<q<\infty$

$$\|\nabla u\|_{L^q(\mathbb{R}^2)} \lesssim \|\nabla\times u\|_{L^q(\mathbb{R}^2)}.$$

We thus expect that the blow-up rate at a time T of $\|\nabla u(t)\|_\infty$ behaves like the one of $\|\nabla \times u(t)\|_\infty \cdot (\log(e + \|\nabla \times u(t)\|_\infty))^\gamma$ for a given $\gamma \geq 0$ and due to the exponent $\frac{1}{2}$ in our regularity criterion, we can expect a great improvement over the usual BKM regularity criterion.

The paper is organized as follows:

- In Section 1, we give some notation and introduce the functional spaces.
- In Section 2, we deal with the local well-posedness of the Cauchy problem of the partially viscous magnetohydrodynamic system (2).
- In Section 3, we give two energy estimates and some estimates from the properties of heat equation by using singular integral representations of equations.
- In Section 4, we recall and give a sketch of the proof of new estimates obtained in [Yuan and Zhao 2018] related to the term $\Delta b + (b \cdot \nabla) u$.
- In Section 5, we give a new estimate for $\|\nabla u(t)\|_\infty$ in Lemma 5.1 and from this estimate, we obtain a new regularity criterion in Theorem 5.3 less restrictive than the BKM regularity criterion.

1. Some notation

For any Banach space Z, we endow the Banach space $Z \times Z$ with the norm defined for all $(f, g) \in Z \times Z$ by $\|(f, g)\|_{Z \times Z} := \|f\|_Z + \|g\|_Z$, and for simplicity in the notation, we use $\|(f, g)\|_Z$ for $\|(f, g)\|_{Z \times Z}$. We use $X \lesssim Y$ to denote the estimate $X \leq CY$ for an absolute constant C. If we need C to depend on a parameter, we shall indicate this by subscripts; thus, for instance, $X \lesssim_s Y$ denotes the estimate $X \leq C_s Y$ for some C_s depending on s.

For any $f \in L^p(\mathbb{R}^2)$, with $1 \leq p \leq \infty$, we denote by $\|f\|_p$ and $\|f\|_{L^p}$, the L^p-norm of f. We denote by $\mathrm{BMO}(\mathbb{R}^2)$ the space of functions of bounded mean oscillation equipped with the norm

$$\|f\|_{\mathrm{BMO}} := \sup_{x \in \mathbb{R}^2,\, r>0} \frac{1}{|B_{x,r}|} \int_{B_{x,r}} |f(y) - f_{B_{x,r}}|\, dy,$$

where $B_{x,r}$ is the ball of radius r centered at x, $|B_{x,r}|$ its measure and $f_{B_{x,r}} := (1/|B_{x,r}|) \int_{B_{x,r}} f(y)\, dy$. We denote by Id the 2×2 identity matrix.

Given an absolutely integrable function $f \in L^1(\mathbb{R}^2)$, we define the Fourier transform $\hat{f} : \mathbb{R}^2 \longmapsto \mathbb{C}$ by the formula,

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} e^{-2\pi i x \cdot \xi} f(x)\, dx,$$

and extend it to tempered distributions. We will use also the notation $\mathcal{F}(f)$ for the Fourier transform of f. We define also the inverse Fourier transform $\check{f} : \mathbb{R}^2 \longmapsto \mathbb{C}$ by the formula,

$$\check{f}(x) = \int_{\mathbb{R}^2} e^{2\pi i x \cdot \xi} f(\xi)\, d\xi.$$

For $s \in \mathbb{R}$, we define the Sobolev norm $\|f\|_{H^s(\mathbb{R}^2)}$ of a tempered distribution $f : \mathbb{R}^2 \longmapsto \mathbb{R}$ by

$$\|f\|_{H^s(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2\, d\xi \right)^{\frac{1}{2}},$$

and then we denote by $H^s(\mathbb{R}^2)$ the space of tempered distributions with finite $H^s(\mathbb{R}^2)$-norm, which matches when s is a nonnegative integer with the classical Sobolev space $H^k(\mathbb{R}^2),\ k \in \mathbb{N}$. The Sobolev space $H^s(\mathbb{R}^2)$ can be written as $H^s(\mathbb{R}^2) = J^{-s}L^2(\mathbb{R}^2)$ where $J = (1-\Delta)^{\frac{1}{2}}$.

For $s > -1$, we also define the homogeneous Sobolev norm,

$$\|f\|_{\dot{H}^s(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |\xi|^{2s}|\hat{f}(\xi)|^2\, d\xi\right)^{\frac{1}{2}}, \tag{10}$$

and then we denote by $\dot{H}^s(\mathbb{R}^2)$ the space of tempered distributions with finite $\dot{H}^s(\mathbb{R}^2)$-norm. We use the Fourier transform to define the fractional Laplacian operator $(-\Delta)^\alpha,\ -1 < \alpha \le 1$, as follows:

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha}\hat{f}(\xi).$$

We denote by $H^s_\sigma(\mathbb{R}^2)$ the Sobolev space $H^s_\sigma(\mathbb{R}^2) := \{\psi \in H^s(\mathbb{R}^2)^2 : \operatorname{div}\psi = 0\}$.

We denote by $\mathbb{P}$ the projector onto divergence-free vector fields given by $\mathbb{P} = \mathrm{Id} - \nabla\Delta^{-1}\operatorname{div}$. The operator $\mathbb{P}$, which acts on vector-valued functions, is a projection: $\mathbb{P}$ is equal to $\mathbb{P}^2$, annihilates gradients and maps into solenoidal (divergence-free) vectors; it is a bounded operator from (vector-valued) L^q to itself for all $1 < q < \infty$ and commutes with translation. We can notice that the operator $\mathbb{P}$ can be written in the form

$$\mathbb{P} = \mathrm{Id} - \nabla\Delta^{-1}\operatorname{div}, \tag{11}$$

which yields the Helmholtz decomposition; indeed for all $v \in L^q(\mathbb{R}^2)^2$, $1 < q < \infty$,

$$\begin{aligned} v &= \mathbb{P}v + \nabla\psi, \quad \text{with } \operatorname{div}\mathbb{P}v = 0,\\ \psi &= \Delta^{-1}\operatorname{div} v. \end{aligned} \tag{12}$$

2. Local regularity of solutions of the 2D MHD equations

This section is devoted to the local well-posedness of the 2D MHD equations. By using $\mathbb{P}$, the matrix Leray operator, the first equation of (2) can be rewritten as

$$\frac{\partial u}{\partial t} + \mathbb{P}\big((u\cdot\nabla)u - (b\cdot\nabla)b\big) = 0. \tag{13}$$

For a solution (u, b) of (2), let us introduce the vorticity $\omega = \nabla\times u = -\partial_2 u_1 + \partial_1 u_2$ and the current density $j = \nabla\times b = -\partial_2 b_1 + \partial_1 b_2$. Applying $\nabla\times$ to the equations of (2), we obtain the governing equations for ω and j

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega = (b\cdot\nabla)j, \\ \partial_t j + (u\cdot\nabla)j - \Delta j = (b\cdot\nabla)\omega + T(\nabla u, \nabla b), \end{cases} \tag{14}$$

where,

$$T(\nabla u, \nabla b) = 2\,\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) + 2\,\partial_2 u_2(\partial_2 b_1 + \partial_1 b_2).$$

In this section we assume that the initial data satisfies $(u_0, b_0) \in H^r_\sigma(\mathbb{R}^2)$ with $r > 2$. Then, we introduce $\omega_0 = \nabla\times u_0$, the vorticity of u_0, and $j_0 = \nabla\times b$, the current density of b_0.

We assume that $(u_0, b_0) \in H^r_\sigma(\mathbb{R}^2)$ with $r > 2$, thanks to Theorem 5.1 in [Caflisch et al. 1997], valid for all integers $r \geq 3$, and by using the same arguments as in Proposition 4.3 of [Agélas 2016], valid for all real numbers $r > 2$, we get that there exists a time of existence $T > 0$ such that there exists a unique strong solution $(u, b) \in C([0, T[, H^r_\sigma(\mathbb{R}^2))$ to the 2D MHD equations (2)–(3).

Thanks to the Beale–Kato–Majda (BKM) criterion obtained in [Caflisch et al. 1997] for any integer $r \geq 3$ and extended in Proposition 4.2 of [Agélas 2016] for any real $r > 2$, we get that if $(u, b) \notin C([0, T], H^r_\sigma(\mathbb{R}^2))$, then we have

$$\int_0^T \|(\omega, j)(t)\|_{L^\infty}\, dt = +\infty. \tag{15}$$

From the first equation of (2), we can retrieve the pressure p from (u, b) with the formula

$$p = -\Delta^{-1} \operatorname{div}((u \cdot \nabla)u - (b \cdot \nabla)b). \tag{16}$$

Since $\nabla \cdot u = 0$ and $\nabla \cdot b = 0$, we get $(u \cdot \nabla)u = \nabla \cdot (u \otimes u)$ and $(b \cdot \nabla)b = \nabla \cdot (b \otimes b)$. Then by (16),

$$p = -\Delta^{-1} \operatorname{div} \nabla \cdot (u \otimes u - b \otimes b). \tag{17}$$

By introducing

$$\mathrm{R} := \Delta^{-1} \operatorname{div} \nabla \cdot \tag{18}$$

the Riesz transform on 2×2 matrix-valued functions on $\mathbb{R}^2$, we get

$$p = -\mathrm{R}(u \otimes u - b \otimes b). \tag{19}$$

Since $(u, b) \in C([0, T[, H^r(\mathbb{R}^2))$ with $r > 2$, we get $p \in C([0, T[, H^r(\mathbb{R}^2))$. Lemma X4 in [Kato and Ponce 1988] (see also [Bahouri et al. 2011, Corollary 2.86, pp. 104] for which the Besov space $B^s_{2,2}$ matches with H^s) states that $L^\infty(\mathbb{R}^2) \cap H^s(\mathbb{R}^2)$ is an algebra for any $s > 0$; i.e., for any $f \in H^s(\mathbb{R}^2)$ and $g \in H^s(\mathbb{R}^2)$, we have $\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_\infty + \|f\|_\infty\|g\|_{H^s}$. This lemma and the use of the Sobolev embedding $H^r(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, since $r > 2$, yield for all $f \in H^r(\mathbb{R}^2)$ and $g \in H^r(\mathbb{R}^2)$,

$$\|fg\|_{H^r} \lesssim \|f\|_{H^r}\|g\|_{H^r}. \tag{20}$$

Then owing to $(u, b) \in C([0, T[, H^r(\mathbb{R}^2))$, thanks to the L^2-boundedness of the Riesz transforms and (20) from (19) we infer that $p \in C([0, T[, H^r(\mathbb{R}^2))$.

Similarly to Proposition 4.1 in [Agélas 2016], we get the following local estimates in the higher Sobolev norm H^r: there exists a real $\kappa_r > 0$ depending only on r such that for all $t \in [0, T[$

$$\|(u, b)(t)\|_{H^r} \leq \|(u_0, b_0)\|_{H^r} e^{\kappa_r \int_0^t \|(\nabla u, \nabla b)(\tau)\|_\infty\, d\tau}. \tag{21}$$

3. Some estimates

In this section, we give some estimates related to the solutions of the 2D MHD equations (2).

Energy estimates. We recall some energy estimates. We state here the two following energy estimates given in [Tran et al. 2013b; Lei and Zhou 2009; Agélas 2016]: for all $t \in [0, T^*[$

$$\|u(t)\|_2^2 + \|b(t)\|_2^2 + 2\int_0^t \|\nabla b(\tau)\|_2^2\, d\tau = \|u_0\|_2^2 + \|b_0\|_2^2 \tag{22}$$

and we get also that for all $t \in [0, T^*[$

$$\|\omega(t)\|_2^2 + \|j(t)\|_2^2 + \int_0^t \|\nabla j(\tau)\|_2^2 \, d\tau \le (\|\omega_0\|_2^2 + \|j_0\|_2^2) e^{C(\|u_0\|_2^2 + \|b_0\|_2^2)}, \tag{23}$$

where $C > 0$ is an absolute constant.

Some estimates deriving from heat equation. In the lemma just below, we give the details (often omitted) of the proof of some estimates deriving from the properties of the heat kernel.

Lemma 3.1. *Let $(u_0, b_0) \in H^r(\mathbb{R}^2)$ with $r > 2$ and let $T > 0$ be such that there exists $(u, b) \in C([0, T[, H^r_\sigma(\mathbb{R}^2))$ a solution of the 2D MHD equations* (2)–(3). *Then there exists a real $C_1 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, r and T such that*

$$\|b\|_{L^\infty(\mathbb{R}^2 \times [0,T])} \le C_1.$$

For any real $p > 1$ and $q > 2$, we have also three real $C_2 > 0$, $C_3 > 0$ and $C_4 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, p, q, r and T such that

$$\begin{aligned} \|\nabla b\|_{L^\infty([0,T] \times L^q(\mathbb{R}^2))} &\le C_2, \\ \|\nabla u\|_{L^\infty([0,T] \times L^q(\mathbb{R}^2))} &\le C_3, \\ \|\nabla^2 b\|_{L^p([0,T] \times L^q(\mathbb{R}^2))} &\le C_4. \end{aligned}$$

Proof. For this, we write the second equation of (2) under its integral form; then we have for all $t \in [0, T[$

$$b(t) = e^{t\Delta} b_0 + \int_0^t e^{(t-s)\Delta} ((b \cdot \nabla) u(s) - (u \cdot \nabla) b(s)) \, ds. \tag{24}$$

Then by using inequality (2.3) in [Kato 1984], we get

$$\begin{aligned} \left\| e^{(t-s)\Delta} ((b \cdot \nabla) u(s) - (u \cdot \nabla) b(s)) \right\|_\infty &\lesssim (t-s)^{-\frac{2}{3}} \|(b \cdot \nabla) u(s) - (u \cdot \nabla) b(s)\|_{\frac{3}{2}} \\ &\lesssim (t-s)^{-\frac{2}{3}} \big(\|b(s)\|_6 \, \|\nabla u(s)\|_2 + \|u(s)\|_6 \, \|\nabla b(s)\|_2 \big). \end{aligned}$$

As a consequence, from (24) we get,

$$\|b(t)\|_\infty \lesssim \|b_0\|_\infty + \int_0^t (t-s)^{-\frac{2}{3}} \big(\|b(s)\|_6 \, \|\nabla u(s)\|_2 + \|u(s)\|_6 \, \|\nabla b(s)\|_2 \big) \, ds. \tag{25}$$

Since $\|b(s)\|_6 \lesssim \|b(s)\|_{H^1}$, $\|u(s)\|_6 \lesssim \|u(s)\|_{H^1}$ and $\|\nabla b(s)\|_2 = \|j(s)\|_2$, we have $\|\nabla u(s)\|_2 = \|\omega(s)\|_2$ due to the facts $\nabla \cdot b(s) = 0$ and $\nabla \cdot u(s) = 0$; then thanks to (22) and (23), from (25) we deduce that there exists a real $C_0 > 0$ depending only on $\|(u_0, b_0)\|_2$, $\|(\omega_0, j_0)\|_2$ such that for all $t \in [0, T[$

$$\|b(t)\|_\infty \lesssim \|b_0\|_\infty + C_0 (T)^{\frac{1}{3}}. \tag{26}$$

Owing to (26) and thanks to the Sobolev embedding $H^r(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ since $r > 2$, we deduce that there exists a real $C_1 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, r and T such that for all $t \in [0, T[$

$$\|b(t)\|_\infty \le C_1, \tag{27}$$

which concludes the first part of the proof. By virtue of (24), we get that for all $t \in [0, T[$

$$\nabla b(t) = e^{t\Delta}\nabla b_0 + \int_0^t \nabla e^{(t-s)\Delta}\big((b\cdot\nabla)u(s) - (u\cdot\nabla)b(s)\big)\,ds. \tag{28}$$

Let $2q/(q+2) < \alpha < 2$. Notice that $2q/(q+2) > 1$ since $q > 2$ and hence $\alpha > 1$. Then by using inequality (2.3′) in [Kato 1984], from (28) we deduce

$$\|\nabla b(t)\|_q \lesssim_q \|\nabla b_0\|_q + \int_0^t (t-s)^{-\left(\frac{1}{2}+\frac{1}{\alpha}-\frac{1}{q}\right)} \big\|(b\cdot\nabla)u(s) - (u\cdot\nabla)b(s)\big\|_\alpha\,ds. \tag{29}$$

Further, thanks to the Hölder inequality, we have $\|(b\cdot\nabla)u(s)\|_\alpha \le \|b(s)\|_{\frac{2\alpha}{2-\alpha}}\|\nabla u(s)\|_2$ and we also get

$$\|b(s)\|_{\frac{2\alpha}{2-\alpha}} \lesssim_\alpha \|b(s)\|_2^{\frac{2-\alpha}{\alpha}}\,\|\nabla b(s)\|_2^{\frac{2(\alpha-1)}{\alpha}}$$

thanks to a Gagliardo–Nirenberg inequality. Hence, we deduce for any $s \in [0, T[$

$$\begin{aligned}\|(b\cdot\nabla)u(s)\|_\alpha &\lesssim_\alpha \|b(s)\|_2^{\frac{2-\alpha}{\alpha}}\,\|\nabla b(s)\|_2^{\frac{2(\alpha-1)}{\alpha}}\,\|\nabla u(s)\|_2\\ &\lesssim_\alpha \|b(s)\|_2^{\frac{2-\alpha}{\alpha}}\,\|j(s)\|_2^{\frac{2(\alpha-1)}{\alpha}}\,\|\omega(s)\|_2\\ &\lesssim_\alpha \|(u,b)(s)\|_2^{\frac{2-\alpha}{\alpha}}\,\|(\omega,j)(s)\|_2^{\frac{3\alpha-2}{\alpha}}.\end{aligned}$$

Similarly, we get also $\|(u\cdot\nabla)b(s)\|_\alpha \lesssim_\alpha \|(u,b)(s)\|_2^{\frac{2-\alpha}{\alpha}}\,\|(\omega,j)(s)\|_2^{\frac{3\alpha-2}{\alpha}}$. By virtue of the two latter inequalities, it is inferred that for all $s \in [0, T[$

$$\big\|(b\cdot\nabla)u(s) - (u\cdot\nabla)b(s)\big\|_\alpha \lesssim_\alpha \|(u,b)(s)\|_2^{\frac{2-\alpha}{\alpha}}\,\|(\omega,j)(s)\|_2^{\frac{3\alpha-2}{\alpha}}. \tag{30}$$

Thanks to the energy estimates (22) and (23), we have $\|(u,b)(s)\|_2 \le \|(u_0,b_0)\|_2$ and $\|(\omega,j)(s)\|_2 \le \|(\omega_0,j_0)\|_2 e^{c\|(u_0,b_0)\|_2}$ with $c > 0$ an absolute constant. Then by setting

$$\eta_0 := \|(u_0,b_0)\|_2 + \|(\omega_0,j_0)\|_2 e^{c\|(u_0,b_0)\|_2},$$

from (30) we deduce that for all $s \in [0, T[$

$$\big\|(b\cdot\nabla)u(s) - (u\cdot\nabla)b(s)\big\|_\alpha \lesssim_\alpha \eta_0^2. \tag{31}$$

After plugging inequality (31) into (29), we obtain that for all $t \in [0, T[$

$$\begin{aligned}\|\nabla b(t)\|_q &\lesssim_{q,\alpha} \|\nabla b_0\|_q + \eta_0^2 \int_0^t (t-s)^{-\left(\frac{1}{2}+\frac{1}{\alpha}-\frac{1}{q}\right)}\,ds\\ &\lesssim_{q,\alpha} \|\nabla b_0\|_q + \eta_0^2 T^{\frac{q+2}{2q}-\frac{1}{\alpha}}.\end{aligned} \tag{32}$$

We choose $\alpha = \frac{1}{2}(2 + 2q/(q+2))$. Thanks to a Gagliardo–Nirenberg inequality, for any $q > 2$ we have the Sobolev embedding $H^r(\mathbb{R}^2) \hookrightarrow \dot W^{1,q}(\mathbb{R}^2)$ since $r > 2$; then owing to (32) we deduce that there exists a real $C_2 > 0$ depending only on $\|(u_0,b_0)\|_{H^r}$, T, r and q such that for all $t \in [0, T[$

$$\|\nabla b(t)\|_q \le C_2, \tag{33}$$

which concludes the second part of the proof.

To get an estimate of $\|\omega\|_{L^\infty([0,T];L^q)}$, we borrow some arguments used in [Jiu and Zhao 2015]. Thanks to the $L^p - L^q$ maximal regularity of the Laplacian operator, see for example [Giga and Sohr 1991], from the second equation of (2), we get that for all $t \in [0, T[$, $p > 1$ and $q > 2$

$$\begin{aligned}\int_0^t \|\nabla^2 b(s)\|_q^p &\lesssim_{p,q} \int_0^t \big\|(b\cdot\nabla)u(s) - (u\cdot\nabla)b(s)\big\|_q^p \, ds \\ &\lesssim_{p,q} \int_0^t \big(\|b(s)\|_\infty^p \, \|\omega(s)\|_q^p + \|u(s)\|_\infty^p \, \|\nabla b(s)\|_q^p\big) \, ds, \end{aligned} \tag{34}$$

where we have used the fact that $\|\nabla u(s)\|_q \lesssim_q \|\omega(s)\|_q$; see Theorem 3.1.1 in [Chemin 1998]. Then, we multiply the first equation of (14) by $\omega|\omega|^{q-2}$, integrate it over $\mathbb{R}^2$ and use the fact that $\nabla\cdot u = 0$ to obtain

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|\omega(t)\|_q^q &= \int_{\mathbb{R}^2} b(x,t)\cdot\nabla j(x,t)\omega(x,t)|\omega(x,t)|^{q-2} \, dx \\ &\le \|b(t)\|_\infty \, \|\nabla j(t)\|_q \, \|\omega(t)\|_q^{q-1},\end{aligned}$$

which yields for all $t \in [0, T[$

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_q^2 \le \|b(t)\|_\infty \, \|\nabla j(t)\|_q \, \|\omega(t)\|_q.$$

After an integration over $[0, t]$ of the inequality just above, we obtain

$$\begin{aligned}\|\omega(t)\|_q^2 &\le \|\omega_0\|_q^2 + 2\int_0^t \|b(s)\|_\infty \, \|\nabla j(s)\|_q \|\omega(s)\|_q \, ds \\ &\le \|\omega_0\|_q^2 + \int_0^t \big(\|\nabla j(s)\|_q^2 + \|b(s)\|_\infty^2 \, \|\omega(s)\|_q^2\big) \, ds\end{aligned} \tag{35}$$

Then thanks to (34), from (35) we infer that for all $t \in [0, T[$

$$\|\omega(t)\|_q^2 \lesssim \|\omega_0\|_q^2 + \int_0^t \big(\|u(s)\|_\infty^2 \, \|\nabla b(s)\|_q^2 + \|b(s)\|_\infty^2 \, \|\omega(s)\|_q^2\big) \, ds. \tag{36}$$

By using Gagliardo–Nirenberg inequalities, Young inequalities and the fact that $\|\nabla u(s)\|_q \lesssim_q \|\omega(s)\|_q$, we get

$$\|u(s)\|_\infty \lesssim_q \|u(s)\|_2 + \|\omega(s)\|_q. \tag{37}$$

By virtue of (36) and (37), we get that for all $t \in [0, T[$

$$\|\omega(t)\|_q^2 \lesssim_q \|\omega_0\|_q^2 + \int_0^t \big(\|u(s)\|_2^2 \, \|\nabla b(s)\|_q^2 + (\|\nabla b(s)\|_q^2 + \|b(s)\|_\infty^2)\|\omega(s)\|_q^2\big) \, ds. \tag{38}$$

Thanks to (22), (27) and (33), we deduce that there exists a real $C > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, T, r and q such that for all $t \in [0, T[$

$$\|\omega(t)\|_q^2 \le C + C\int_0^t \|\omega(s)\|_q^2 \, ds. \tag{39}$$

Thanks to the Grönwall inequality, we infer that for all $t \in [0, T[$

$$\|\omega(t)\|_q^2 \le Ce^{CT}.$$

By using the fact that

$$\|\nabla u(t)\|_q \lesssim_q \|\omega(t)\|_q,$$

we infer that there exists a real $C_3 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, T and q such that for all $t \in [0, T[$

$$\|\nabla u(t)\|_q \leq C_3, \tag{40}$$

which concludes the third part of the proof. By using (37) in (34) and thanks to (40), (22), (27) and (33), we complete the proof. □

4. Some new estimates

We give a sketch of the proof of Lemma 4.1 obtained in [Yuan and Zhao 2018] by exploiting the special structure of the 2D MHD equations (2).

Lemma 4.1. *Let $(u_0, b_0) \in H^r(\mathbb{R}^2)$ with $r > 2$ and let $T > 0$ be such that there exists $(u, b) \in C([0, T[, H^r_\sigma(\mathbb{R}^2))$ a solution of the 2D MHD equations (2)–(3). Then there exists a real $C > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, r and T such that*

$$\|\Delta b + (b\cdot\nabla)u\|_{L^\infty([0,T];L^\infty(\mathbb{R}^2))} \leq C, \tag{41}$$

and we have also that for any real $p \geq 2$ and $q \geq 2$,

$$\|\nabla(\Delta b + (b\cdot\nabla)u)\|_{L^p([0,T];L^q(\mathbb{R}^2))} \leq C. \tag{42}$$

Although we can deduce the proof of Lemma 4.1 from [Yuan and Zhao 2018], we prefer to give here the details of its proof, as it is at the heart of the improvements obtained in this paper. For this, we borrow some arguments used in [Yuan and Zhao 2018]. We start the proof by writing the equation satisfied by $\mathfrak{F} := \Delta b + (b\cdot\nabla)u$, that is,

$$\begin{aligned}\partial_t\mathfrak{F} - \Delta\mathfrak{F} = -(b\cdot\nabla)\mathbb{P}((u\cdot\nabla)u) + (b\cdot\nabla)\mathbb{P}((b\cdot\nabla)b) - \Delta((u\cdot\nabla)b)\\ - \nabla u\,(u\cdot\nabla)b + \nabla u\,(b\cdot\nabla)u + \nabla u\,\Delta b.\end{aligned} \tag{43}$$

This equation is obtained by applying $(b\cdot\nabla)$ and Δ respectively to the first equation of (13) and second equation of (2), multiplying the second equation of (2) by ∇u and then adding the resulting equations together. Then, by writing (43) in its integral form and using the facts that $\nabla\cdot u = 0$ and $\nabla\cdot b = 0$, we get for all $t \in [0, T[$

$$\begin{aligned}\mathfrak{F}(t) = e^{t\Delta}\mathfrak{F}(0) + \int_0^t \nabla e^{(t-s)\Delta}\big(b(s)\otimes\mathbb{P}((u\cdot\nabla)u)(s) - b(s)\otimes\mathbb{P}((b\cdot\nabla)b)(s)\big)\,ds\\ + \int_0^t \nabla e^{(t-s)\Delta}\nabla((u\cdot\nabla)b)(s)\,ds\\ + \int_0^t e^{(t-s)\Delta}\big(-\nabla u(s)\,(u(s)\cdot\nabla)b(s) + \nabla u(s)\,(b(s)\cdot\nabla)u(s) + \nabla u(s)\,\Delta b(s)\big)\,ds.\end{aligned} \tag{44}$$

Then using inequalities (2.3) and (2.3′) of [Kato 1984] stated for $1 < p \le q < +\infty$ but remaining true for $q = \infty$, we obtain for all $t \in [0, T[$

$$\begin{aligned}\|\mathfrak{F}(t)\|_\infty \le \|\mathfrak{F}(0)\|_\infty + \int_0^t (t-s)^{-\frac{5}{6}} \left\| \big(b(s) \otimes \mathbb{P}((u\cdot\nabla)u)(s) - b(s) \otimes \mathbb{P}((b\cdot\nabla)b)(s)\big) \right\|_3 ds \\ + \int_0^t (t-s)^{-\frac{5}{6}} \|\nabla((u\cdot\nabla)b)(s)\|_3 \, ds \\ + \int_0^t (t-s)^{-\frac{1}{2}} \left\| -\nabla u(s)\,(u(s)\cdot\nabla)b(s) + \nabla u(s)\,(b(s)\cdot\nabla)u(s) + \nabla u(s)\,\Delta b(s) \right\|_2 ds. \qquad (45)\end{aligned}$$

By using the fact that $\mathbb{P}$ is a bounded operator from (vector-valued) L^q to itself for all $1 < q < \infty$ and the Hölder inequality, we get

$$\begin{aligned}&\left\| (b(s) \otimes \mathbb{P}((u\cdot\nabla)u)(s) - (b(s) \otimes \mathbb{P}((b\cdot\nabla)b)(s) \right\|_3 \\ &\qquad \lesssim \|b(s)\|_\infty \, \|u(s)\|_6 \, \|\nabla u(s)\|_6 + \|b(s)\|_\infty \, \|b(s)\|_6 \, \|\nabla b(s)\|_6, \\ &\|\nabla((u\cdot\nabla)b)(s)\|_3 \lesssim \|\nabla u(s)\|_6 \, \|\nabla b(s)\|_6 + \|u(s)\|_6 \, \|\nabla^2 b(s)\|_6, \qquad (46) \\ &\left\| -\nabla u(s)\,(u(s)\cdot\nabla)b(s) + \nabla u(s)\,(b(s)\cdot\nabla)u(s) + \nabla u(s)\,\Delta b(s) \right\|_2 \\ &\qquad \lesssim \|\nabla u(s)\|_6 \, \|u(s)\|_6 \|\nabla b(s)\|_6 + \|\nabla u(s)\|_6^2 \, \|b(s)\|_6 + \|\nabla u(s)\|_6 \, \|\Delta b(s)\|_3.\end{aligned}$$

Furthermore, thanks to a Gagliardo–Nirenberg interpolation inequality and the fact that, since $\nabla\cdot u(s) = 0$, $\nabla \cdot b(s) = 0$, we have $\|\nabla u(s)\|_2 = \|\omega(s)\|_2$ and $\|\nabla b(s)\|_2 = \|j(s)\|_2$, we get

$$\|u(s)\|_6 \lesssim \|u(s)\|_2^{\frac{1}{3}} \, \|\omega(s)\|_2^{\frac{2}{3}}, \quad \|b(s)\|_6 \lesssim \|b(s)\|_2^{\frac{1}{3}} \, \|j(s)\|_2^{\frac{2}{3}}. \qquad (47)$$

After plugging (47) into (46) and using Lemma 3.1 with the energy inequalities (22), (23), from (45) we infer that there exists a real $C_0 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, r and T such that for all $t \in [0, T[$

$$\|\mathfrak{F}(t)\|_\infty \lesssim C_0\Big(1 + \int_0^t (t-s)^{-\frac{5}{6}} (1 + \|\nabla^2 b(s)\|_6) + (t-s)^{-\frac{1}{2}} (1 + \|\Delta b(s)\|_3) \, ds\Big). \qquad (48)$$

Thanks to the Hölder inequality used with the pairs of exponents $\left(\frac{7}{6}, 7\right)$ and $\left(\frac{3}{2}, 3\right)$, from (48) we deduce that for all $t \in [0, T[$

$$\begin{aligned}\|\mathfrak{F}(t)\|_\infty \lesssim C_0 + C_0 \Big(\int_0^t (t-s)^{-\frac{35}{36}} \, ds\Big)^{\frac{6}{7}} \Big(\int_0^t (1 + \|\nabla^2 b(s)\|_6)^7 \, ds\Big)^{\frac{1}{7}} \\ + C_0 \Big(\int_0^t (t-s)^{-\frac{3}{4}} \, ds\Big)^{\frac{2}{3}} \Big(\int_0^t (1 + \|\Delta b(s)\|_3)^3 \, ds\Big)^{\frac{1}{3}},\end{aligned}$$

which yields

$$\|\mathfrak{F}(t)\|_\infty \lesssim C_0\Big(1 + t^{\frac{1}{42}} \Big(\int_0^t (1 + \|\nabla^2 b(s)\|_6^7) \, ds\Big)^{\frac{1}{7}} + t^{\frac{1}{6}} \Big(\int_0^t (1 + \|\Delta b(s)\|_3^3) \, ds\Big)^{\frac{1}{3}}\Big). \qquad (49)$$

Then, thanks again to Lemma 3.1, from (49) one obtains that there exists a real $C_1 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, r and T such that for all $t \in [0, T[$

$$\|\mathfrak{F}(t)\|_\infty \le C_1,$$

which gives us (41), the first inequality of Lemma 4.1.

For the second inequality of Lemma 4.1, we use the $L^p - L^q$ maximal regularity of the Laplacian operator [Giga and Sohr 1991]; one has for any $1 < p < \infty$, $1 < q < \infty$ and $g = \int_0^t e^{(t-s)\Delta} f$,

$$\|\nabla^2 g\|_{L^p([0,T];L^q(\mathbb{R}^2))} \lesssim_{p,q} \|f\|_{L^p([0,T];L^q(\mathbb{R}^2))}. \tag{50}$$

Then, with the expression of $\nabla\mathfrak{F}(t)$ obtained from (44) and by using (50), inequality (2.3′) of [Kato 1984], Lemma 3.1 and the energy inequalities (22), (23), we obtain in a similar way (42), the second inequality of Lemma 4.1.

5. A new blow-up criterion

In this section, we give a new estimate for $\|\nabla u(t)\|_\infty$ in Lemma 5.1 and from this estimate, we obtain a new regularity criterion in Theorem 5.3 which is less restrictive than the BKM regularity criterion.

Lemma 5.1. *Let $(u_0, b_0) \in H^r(\mathbb{R}^2)$ with $r > 2$ and let $T > 0$ be such that there exists $(u, b) \in C([0, T[, H^r_\sigma(\mathbb{R}^2))$ a solution of the 2D MHD equations* (2)–(3). *Then there exists a real $\gamma_0 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, T and r such that for all $t \in [0, T[$*

$$\|\nabla u(t)\|_\infty \le \exp\biggl(\gamma_0 \exp\biggl(\gamma_0 \int_0^t \frac{\|\nabla u(s)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(s)\|_\infty)}\, ds\biggr)\biggr). \tag{51}$$

Proof. We begin the proof with the following logarithmic Sobolev inequality, which is proved in [Kozono and Taniuchi 2000], see inequality (4.20), and stands as an improved version of that in [Beale et al. 1984]:

$$\|\nabla f\|_{L^\infty(\mathbb{R}^2)} \lesssim 1 + \|\nabla \times f\|_{L^\infty(\mathbb{R}^2)}(1 + \log^+ \|f\|_{W^{s,p}(\mathbb{R}^2)}) \quad \text{with } p > 1,\ s > 1 + \frac{2}{p}, \tag{52}$$

where $\nabla \cdot f = 0$, $\nabla \times f = -\partial_2 f_1 + \partial_1 f_2$ is the vorticity of f and $\log^+ x = \max(0, \log x)$ for any $x > 0$.

Thus, by virtue of (52), we get that for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty \le \beta_r + \beta_r \|\omega(t)\|_\infty (1 + \log^+ \|u(t)\|_{H^r}), \tag{53}$$

where $\beta_r > 0$ is a real depending only on r. Let us give an estimate of the term $1 + \log^+ \|u(t)\|_{H^r}$. Thanks to (21), we get that there exists a real $\kappa_r > 0$ depending only on r such that for all $t \in [0, T[$

$$\|(u, b)(t)\|_{H^r} \le \|(u_0, b_0)\|_{H^r} e^{\kappa_r \int_0^t \|(\nabla u, \nabla b)(\tau)\|_\infty \, d\tau}. \tag{54}$$

After taking the logarithm in the inequality (54), we observe that for all $t \in [0, T[$,

$$\log^+ \|(u, b)(t)\|_{H^r} \le \log^+ \|(u_0, b_0)\|_{H^r} + \kappa_r \int_0^t \|(\nabla u, \nabla b)(\tau)\|_\infty \, d\tau. \tag{55}$$

Thanks to Lemma 3.1 and the Sobolev embedding $W^{2,q}(\mathbb{R}^2) \hookrightarrow W^{1,\infty}(\mathbb{R}^2)$ with $q > 2$, we infer that there exists a real $\varrho_0 > 0$ depending only on r, T and $\|(u_0, b_0)\|_{H^r}$ such that

$$\int_0^T \|\nabla b(\sigma)\|_\infty \leq \varrho_0. \tag{56}$$

Then owing to (56), from (55) we infer that there exists a real $\varrho_1 \geq 1$ depending only on r, T and $\|(u_0, b_0)\|_{H^r}$ such that for all $t \in [0, T[$

$$1 + \log^+ \|(u, b)(t)\|_{H^r} \leq \varrho_1 + \kappa_r \int_0^t \|\nabla u(s)\|_\infty \, ds. \tag{57}$$

Thus, by plugging (57) into (53), we deduce that there exists a real $\varrho_2 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, T and r such that for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty \leq \beta_r + \varrho_2 \|\omega(t)\|_\infty \left(1 + \int_0^t \|\nabla u(s)\|_\infty \, ds\right). \tag{58}$$

Now, let us estimate $\|\omega(t)\|_\infty$. We observe that the first equation of (14) can be changed into

$$\partial_t \omega + u \cdot \nabla \omega = F - b_1 b \cdot \nabla u_2 + b_2 b \cdot \nabla u_1, \tag{59}$$

where $F = b_1(\Delta b_2 + b \cdot \nabla u_2) - b_2(\Delta b_1 + b \cdot \nabla u_1)$. By using estimates for transport equations, see for instance Lemma 4.1 in [Kato and Ponce 1988], we obtain that for all $t \in [0, T[$

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty + c \int_0^t \|F(s)\|_\infty \, ds + c \int_0^t \|b(s)\|_\infty^2 \|\nabla u(s)\|_\infty \, ds, \tag{60}$$

where $c > 0$ is a constant. Thanks to Lemmata 3.1 and 4.1, we deduce that there exist two real $\varrho_3 > 0$ and $\varrho_4 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, T and r such that for all $t \in [0, T[$

$$\begin{aligned} c \int_0^t \|F(s)\|_\infty \, ds &\leq \varrho_3, \\ \|b(t)\|_\infty^2 &\leq \varrho_4. \end{aligned} \tag{61}$$

Thus by virtue of (61), from (60) we infer that for all $t \in [0, T[$

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty + \varrho_3 + c\varrho_4 \int_0^t \|\nabla u(s)\|_\infty \, ds. \tag{62}$$

Furthermore, thanks to the Sobolev embedding $H^r(\mathbb{R}^2) \hookrightarrow W^{1,\infty}(\mathbb{R}^2)$ with $r > 2$, we get

$$\|\omega_0\|_\infty \lesssim_r \|u_0\|_{H^r}. \tag{63}$$

Hence, owing to (63), from (62) we deduce that there exists a real $\varrho_5 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, T and r such that for all $t \in [0, T[$

$$\|\omega(t)\|_\infty \leq \varrho_5 + c\varrho_4 \int_0^t \|\nabla u(s)\|_\infty \, ds. \tag{64}$$

By plugging (64) into (58), we infer that there exists a real $\varrho_6 \geq 1$ depending only on r, T and $\|(u_0, b_0)\|_{H^r}$ such that for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty \leq \varrho_6 \left(1 + \int_0^t \|\nabla u(s)\|_\infty \, ds\right)^2, \tag{65}$$

which yields

$$\|\nabla u(t)\|_\infty^{\frac{1}{2}} \leq \varrho_6^{\frac{1}{2}} \left(1 + \int_0^t \|\nabla u(s)\|_\infty \, ds\right). \tag{66}$$

We thus introduce the real function $\mathfrak{J}$ defined for all $t \in [0, T[$ by

$$\mathfrak{J}(t) := \varrho_6^{\frac{1}{2}} + \varrho_6^{\frac{1}{2}} \int_0^t \|\nabla u(s)\|_\infty \, ds. \tag{67}$$

On one hand, by virtue of (66), thanks to (67) we get that for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty^{\frac{1}{2}} \leq \mathfrak{J}(t). \tag{68}$$

On the other hand, from (67), we infer that for any $t \in [0, T[$

$$\mathfrak{J}'(t) = \varrho_6^{\frac{1}{2}} \|\nabla u(t)\|_\infty = \frac{\varrho_6^{\frac{1}{2}} \|\nabla u(t)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(t)\|_\infty^{\frac{1}{2}})} \|\nabla u(t)\|_\infty^{\frac{1}{2}} \log(e + \|\nabla u(t)\|_\infty^{\frac{1}{2}}). \tag{69}$$

Then, owing to (68), from (69), we infer that for all $t \in [0, T[$

$$\mathfrak{J}'(t) \leq \frac{\varrho_6^{\frac{1}{2}} \|\nabla u(t)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(t)\|_\infty^{\frac{1}{2}})} \mathfrak{J}(t) \log(e + \mathfrak{J}(t)). \tag{70}$$

After dividing inequality (70) by $e + \mathfrak{J}(t)$, we obtain that for all $t \in [0, T[$

$$\frac{d}{dt} \log(e + \mathfrak{J}(t)) \leq \frac{\varrho_6^{\frac{1}{2}} \|\nabla u(t)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(t)\|_\infty^{\frac{1}{2}})} \log(e + \mathfrak{J}(t)). \tag{71}$$

As a consequence of the Grönwall lemma, from (71) we get for all $t \in [0, T[$

$$\log(e + \mathfrak{J}(t)) \leq \log(e + \mathfrak{J}(0)) \exp\left(\varrho_6^{\frac{1}{2}} \int_0^t \frac{\|\nabla u(s)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(s)\|_\infty^{\frac{1}{2}})} \, ds\right). \tag{72}$$

From (67), we get $\mathfrak{J}(0) = \varrho_6^{\frac{1}{2}}$ and thanks to (72), we thus obtain for all $t \in [0, T[$

$$\mathfrak{J}(t) \leq \exp\left(\log(e + \varrho_6^{\frac{1}{2}}) \exp\left(\varrho_6^{\frac{1}{2}} \int_0^t \frac{\|\nabla u(s)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(s)\|_\infty^{\frac{1}{2}})} \, ds\right)\right). \tag{73}$$

Owing to (68) and (73), we obtain that for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty \le \exp\left(2\log(e + \varrho_6^{\frac{1}{2}}) \exp\left(\varrho_6^{\frac{1}{2}} \int_0^t \frac{\|\nabla u(s)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(s)\|_\infty^{\frac{1}{2}})}\, ds\right)\right). \tag{74}$$

Since $e + \|\nabla u(s)\|_\infty^{\frac{1}{2}} \ge (e + \|\nabla u(s)\|_\infty)^{\frac{1}{2}}$, then we get

$$\log(e + \|\nabla u(s)\|_\infty^{\frac{1}{2}}) \ge \tfrac{1}{2}\log(e + \|\nabla u(s)\|_\infty)$$

and hence from (74) we infer for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty \le \exp\left(2\log(e + \varrho_6^{\frac{1}{2}}) \exp\left(2\varrho_6^{\frac{1}{2}} \int_0^t \frac{\|\nabla u(s)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(s)\|_\infty)}\, ds\right)\right),$$

which concludes the proof. □

Remark 5.2. We observe that the expression of the estimate obtained in Lemma 5.1 for $\|\nabla u(t)\|_\infty$ makes a double exponential growth appear. This double exponential growth derives from taking into account in the estimate the term $\log(e + \|\nabla u(t)\|_\infty)$. We thus point out that we have also an upper bound of $\|\nabla u(t)\|_\infty$ for which we get only one single exponential growth. Indeed, from (66), thanks to the Grönwall lemma, we obtain that for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty^{\frac{1}{2}} \le \varrho_6^{\frac{1}{2}} \exp\left(\varrho_6^{\frac{1}{2}} \int_0^t \|\nabla u(s)\|_\infty^{\frac{1}{2}}\, ds\right),$$

which yields

$$\|\nabla u(t)\|_\infty \le \varrho_6 \exp\left(2\varrho_6^{\frac{1}{2}} \int_0^t \|\nabla u(s)\|_\infty^{\frac{1}{2}}\, ds\right),$$

where $\varrho_6 > 0$ is a real number depending only on T, r and $\|(u_0, b_0)\|_r$.

Let us establish now, a new regularity criterion in the theorem just below.

Theorem 5.3. *Let* $(u_0, b_0) \in H^r_\sigma(\mathbb{R}^2)$ *with* $r > 2$ *and let* $T > 0$ *be such that there exists* $(u, b) \in C([0, T[, H^r_\sigma(\mathbb{R}^2))$ *a solution of the* 2D *MHD equations* (2)–(3). *If*

$$\int_0^T \frac{\|\nabla u(t)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(t)\|_\infty)}\, dt < +\infty \tag{75}$$

then there cannot be blow-up of the solution u *in* $H^r(\mathbb{R}^2)$ *at the time* T, *that is*, $u \in C([0, T], H^r_\sigma(\mathbb{R}^2))$.

Proof. Let us assume that (75) holds. For a contradiction, we suppose that $u \notin C([0, T], H^r_\sigma(\mathbb{R}^2)$. Then we get (15). Thanks to Lemma 3.1 and the Sobolev embedding $W^{2,q}(\mathbb{R}^2) \hookrightarrow W^{1,\infty}(\mathbb{R}^2)$ with $q > 2$, we infer that $\int_0^T \|j(t)\|_\infty\, dt < +\infty$. Then from (15), we get only

$$\int_0^T \|\omega(t)\|_\infty\, dt = +\infty. \tag{76}$$

Thanks to Lemma 5.1, there exists a real $\varrho_1 > 0$ depending only on $\|(u_0, b_0)\|_{H^r}$, T and r such that for all $t \in [0, T[$

$$\|\nabla u(t)\|_\infty \le \exp\biggl(\varrho_1 \exp\biggl(\varrho_1 \int_0^t \frac{\|\nabla u(s)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(s)\|_\infty)}\, ds\biggr)\biggr). \tag{77}$$

Then from (77) and (75), we infer that $\int_0^T \|\nabla u(t)\|_\infty\, dt < +\infty$, which implies $\int_0^T \|\omega(t)\|_\infty\, dt < +\infty$. Then we obtain a contradiction with (76) and hence $u \in C([0, T], H^r_\sigma(\mathbb{R}^2)$, which concludes the proof. □

Conclusion

We obtained a new regularity criterion for the two-dimensional resistive magnetohydrodynamic (MHD) equations which is less restrictive than the BKM regularity criterion (see Theorem 5.3) by using the logarithmic Sobolev inequality. It is important to find some criteria less restrictive than the BKM regularity criterion. Indeed, due to the quadratic nonlinearity of the MHD equations, we expect that the blow-up rate of $\|\nabla u(t)\|_\infty$ at a time T be at least faster than $O(1/(T-t))$. Thus, if one investigates numerically the finite-time singularities of the solutions of such a system of equations and believes that its numerical solution computed leads to a finite-time blow-up at some time T, then one may observe a blow-up rate at the time T for $\|\nabla u(t)\|$ of the form $O(1/((T-t)^\gamma))$, $\gamma \ge 1$. Further, in all the recent numerical investigations performed to find finite-time singularities of the 2D inviscid MHD equations, the results suggest blow-up rates at a time T for $\|\nabla u(t)\|_\infty$ of the form $O(1/((T-t)^\alpha))$ with $1 \le \alpha < 2$; see [Brachet et al. 2013; Kerr and Brandenburg 1999]. Then, for these numerical cases, with the BKM regularity criterion, one would conclude there is evidence for a finite-time singularity at some time T of the solutions of the 2D resistive MHD equations. However, with the use of our regularity criterion (see Theorem 5.3), we can confirm that in fact there is no blow-up of the solution at this time T. Then, it is dangerous to interpret the blow-up of an under-resolved computation as evidence of finite-time singularities for the 2D resistive MHD equations. Indeed, computing 2D MHD singularities numerically is an extremely challenging task. First of all, it requires huge computational resources; see [Brachet et al. 2013]. Tremendous resolutions are required to capture the nearly singular behavior of the 2D MHD equations. Secondly, one has to perform a careful convergence study.

Furthermore, we notice also that our problem fits in the class of equations considered in [Elgindi and Masmoudi 2014] in the study of L^∞ ill-posedness problem. We thus point out that by borrowing the arguments used in this paper, we can establish the same regularity criterion for another interesting open problem in mathematical fluid dynamics mentioned in [Elgindi and Masmoudi 2014] about the following type of equation in two dimensions:

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u + \nabla p &= Au, \\ \nabla \cdot u &= 0, \end{aligned} \tag{78}$$

with initial condition u_0 in a divergence-free vector field and where A is some constant matrix. Namely, as with Theorem 5.3, we get the following theorem for the system of equations (78):

Theorem 5.4. *Let $u_0 \in H^r_\sigma(\mathbb{R}^2)$ with $r > 2$ and let $T > 0$ be such that there exists $u \in C([0, T[, H^r_\sigma(\mathbb{R}^2))$ a solution of* (78). *If*

$$\int_0^T \frac{\|\nabla u(t)\|_\infty^{\frac{1}{2}}}{\log(e + \|\nabla u(t)\|_\infty)}\, dt < +\infty$$

then there cannot be blow-up of the solution u in $H^r(\mathbb{R}^2)$ at the time T, that is, $u \in C([0, T], H^r_\sigma(\mathbb{R}^2))$.

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