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We prove the existence of global L^2 weak solutions for a family of generalized inviscid surface quasi-geostrophic (SQG) equations in bounded domains of $\mathbb{R}^2$. In these equations, the active scalar is transported by a velocity field which is determined by the scalar through a more singular nonlocal operator compared to the SQG equation. The result is obtained by establishing appropriate commutator representations for the weak formulation together with good bounds for them in bounded domains.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with smooth boundary. Define

$$\Lambda = \sqrt{-\Delta}, \tag{1-1}$$

where $-\Delta$ is the Laplacian operator in Ω with homogeneous Dirichlet boundary condition.

We consider the following family of active scalars

$$\partial_t \theta + u \cdot \nabla \theta = 0, \tag{1-2}$$

where $\theta = \theta(x, t),\ u = u(x, t)$ with $(x, t) \in \Omega \times [0, \infty)$ and with the velocity u given by

$$u = \nabla^{\perp} \psi, \tag{1-3}$$

$$\psi = \Lambda^{-\alpha} \theta, \quad \alpha \in [0, 2]. \tag{1-4}$$

Here, fractional powers of the Laplacian $-\Delta$ are based on eigenfunction expansions (see the first subsection of Section 2 below for definitions and notations) and ψ is called the stream function. By (1-3) the velocity u is automatically divergence-free. The case $\alpha = 2$ corresponds to the two-dimensional Euler equation in the vorticity formulation. When $\alpha = 1$, (1-2) is the surface quasigeostrophic (SQG) equation of geophysical significance [Held et al. 1995], which also serves as a two-dimensional model of the three-dimensional Euler equations in view of many striking physical and mathematical analogies between them [Constantin et al. 1994]. The global regularity issue is known for the two-dimensional Euler equations but remains open for any $\alpha < 2$. Growth of solutions when $\alpha = 1, 2$ and $\Omega = \mathbb{R}^2, \mathbb{T}^2$ was studied in [Córdoba and Fefferman 2002]; nonexistence of simple hyperbolic blow-up when $\alpha = 1$ and $\Omega = \mathbb{R}^2$ was confirmed in [Córdoba 1998]. We refer to [Chae et al. 2011] for a regularity criterion when $\alpha \in [1, 2]$ and $\Omega = \mathbb{R}^2$. On the other hand, finite time blow-up for patch solutions of (1-2) in the half plane with

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small $\alpha < 2$ was recently shown in [Kiselev et al. 2016]. The velocity u becomes more singular when α decreases, and in particular, u is not in $L^2(\Omega)$ if θ is in $L^2(\Omega)$ and $\alpha < 1$. Equations (1-2) with $\alpha \in (0, 1)$ were introduced in [Chae et al. 2012] to understand solutions to the SQG-type equations with even more singular velocity fields. More precisely, that paper established the existence of global L^2 weak solutions on the torus $\mathbb{T}^2$, together with local existence and uniqueness of strong solutions in $\mathbb{R}^2$. The borderline case $\alpha = 0$ is surprisingly easy due to the cancellation of the nonlinear term: (1-2) reduces to the simple equation $\partial_t \theta = 0$, and thus $\theta(\cdot, t) = \theta(\cdot, 0)$ for all $t > 0$. On the other hand, if $\alpha < 0$ then the stream function $\psi = \Lambda^{-\alpha}\theta$ is not well-defined when $\theta \in L^2(\Omega)$, noticing that there is no dissipation in the equation.

In this paper, we are interested in the issue of global weak solutions for (1-2) with $\alpha \in (0, 1)$ in arbitrary (smooth) bounded domains of $\mathbb{R}^2$. Let us recall that the existence of global weak solutions for SQG ($\alpha = 1$) were first proved in [Resnick 1995] in the periodic case. This highlights a difference between the nonlinearities of the SQG equation and the three-dimensional Euler equations: SQG has weak continuity in L^2, while the Euler equations do not. The weak continuity of SQG is due to a remarkable commutator structure which was subsequently revisited in [Chae et al. 2011] and used in the proof of absence of anomalous dissipation in [Constantin et al. 2014]. In [Constantin and Nguyen 2016], this structure was adapted to arbitrary bounded domains to take into account the lack of translation invariance of the fractional Laplacian in domains: a new commutator between the fractional Laplacian and differentiation appears. In addition to that, with the more singular constitutive laws (1-4), in order to establish the weak continuity of the nonlinearity $u \cdot \nabla\theta$ we will need to find appropriate commutator representations for which good bounds can be derived. Let us emphasize that many known commutator estimates for fractional Laplacians in the whole space (or on tori) are too expensive for bounded domains due to possible singularity near the boundary or the lack of powerful tools of Fourier analysis. For further results on the fractional Laplacian and SQG in bounded domains, we refer to [Cabré and Tan 2010; Caffarelli and Silvestre 2007; Constantin and Ignatova 2016; 2017].

Our main result is:

Theorem 1.1. *Let $\alpha \in (0, 1)$ and $\theta_0 \in L^2(\Omega)$. There is a weak solution of* (1-2), $\theta \in L^\infty([0, \infty); L^2(\Omega))$ *with initial data θ_0. That is, for any $T \geq 0$ and $\phi \in C_0^\infty(\Omega \times (0, T))$, θ satisfies*

$$\int_0^T \int_\Omega \theta(x, t)\, \partial_t \phi(x, t)\, dx\, dt + \int_0^T \mathcal{N}(\psi, \phi)\, dt = 0 \tag{1-5}$$

and the initial data

$$\theta(\cdot, 0) = \theta_0(\cdot) \quad \textit{in } H^{-\varepsilon}(\Omega) \textit{ for all } \varepsilon > 0 \tag{1-6}$$

is attained. Here,

$$\mathcal{N}(\psi, \phi) = \frac{1}{2} \int_\Omega [\Lambda^\alpha, \nabla^\perp]\psi \cdot \nabla\phi\psi\, dx - \frac{1}{2} \int_\Omega \Lambda^{-1+\alpha} \nabla^\perp \psi \cdot \Lambda^{1-\alpha}[\Lambda^\alpha, \nabla\phi]\psi\, dx. \tag{1-7}$$

Moreover, θ obeys the energy inequality

$$\|\theta(\cdot, t)\|^2_{L^2(\Omega)} \leq \|\theta_0\|^2_{L^2(\Omega)} \quad \textit{a.e. } t \geq 0. \tag{1-8}$$

Additionally, the stream function ψ *is in* $C([0,\infty); D(\Lambda^{\alpha-\varepsilon}))$ *for any* $\varepsilon > 0$ *and its* $D(\Lambda^{\frac{\alpha}{2}})$ *norm is conserved*,

$$\|\psi(\cdot,t)\|_{D(\Lambda^{\alpha/2})} = \|\psi(\cdot,0)\|_{D(\Lambda^{\alpha/2})} \quad \textit{for all } t > 0. \tag{1-9}$$

In Theorem 1.1 and what follows,

$$[A,B] := AB - BA$$

denotes the commutator of two operators A and B.

When $\alpha = 0$, we have $u = R^{\perp}\theta$, where $R = (\partial_{x_1}\Lambda^{-1}, \partial_{x_2}\Lambda^{-1})$ denotes the Riesz transforms. As $R : L^2(\Omega) \to L^2(\Omega)$ is continuous, we have $u\theta \in L^1(\Omega)$ if $\theta \in L^2(\Omega)$. In that case, θ is a weak solution of (1-2) if

$$\int_0^T\int_\Omega \theta(x,t)\,\partial_t\phi(x,t)\,dx\,dt + \int_0^T\int_\Omega u(x,t)\,\theta(x,t)\cdot\nabla\phi(x,t)\,dx\,dt = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega\times(0,T)).$$

The global existence of such solutions was proved in [Constantin and Nguyen 2016]. However, when $\alpha < 1$, we have u is less regular than θ and the second integral in the preceding formulation is not well-defined. Nevertheless, taking into account the nonlinearity structure to explore extra cancellations, this integral has the commutator representation (1-7), which makes sense provided only $\theta \in L^2(\Omega)$, as will be proved in Lemma 3.4 below using the heat kernel approach. Let us note that the two objects are equal if $\psi \in H_0^1(\Omega)$, or equivalently, $\theta \in D(\Lambda^{1-\alpha})$. This representation is good enough to well define the nonlinearity but another representation, see (3-5), will be needed for the compactness argument. The point is that these two representations are equivalent provided only $\theta \in L^2(\Omega)$ (see Lemma 3.3 below). Unlike the proof in [Constantin and Nguyen 2016], which uses only Galerkin approximations, Theorem 1.1 will be proved by a two-tier approximation procedure: Galerkin approximations for each vanishing viscosity approximation. This is because the nonlinearity $u\theta$ is not well-defined in $L^1(\Omega)$ (see Remark 3.6 below).

The paper is organized as follows. In Section 2, we present the functional setup of fractional Laplacian in domains and necessary commutator estimates, which can be of independent interest. The proof of Theorem 1.1 is presented in Section 3. Finally, the proofs of the commutator estimates announced in Section 2 are given Appendices A and B.

2. Preliminaries

Fractional Laplacian. Let Ω be an open bounded set of $\mathbb{R}^d$, $d \ge 2$, with smooth boundary. The Laplacian $-\Delta$ is defined on $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. Let $\{w_j\}_{j=1}^\infty$ be an orthonormal basis of $L^2(\Omega)$ comprised of L^2-normalized eigenfunctions w_j of $-\Delta$; i.e.,

$$-\Delta w_j = \lambda_j w_j, \quad w_j|_{\partial\Omega} = 0, \quad \int_\Omega w_j^2\,dx = 1,$$

with $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \to \infty$.

The fractional Laplacian is defined using eigenfunction expansions,

$$\Lambda^s f \equiv (-\Delta)^{\frac{s}{2}} f := \sum_{j=1}^{\infty} \lambda_j^{\frac{s}{2}} f_j, w_j \qquad \text{with } f = \sum_{j=1}^{\infty} f_j w_j, \quad f_j = \int_\Omega f w_j \, dx,$$

for $s \geq 0$ and

$$f \in D(\Lambda^s) := \{ f \in L^2(\Omega) : \Lambda^s f \in L^2(\Omega) \}.$$

The norm of $f = \sum_{j=1}^{\infty} f_j w_j$ in $D(\Lambda^s),\ s \geq 0$, is defined by

$$\|f\|_{D(\Lambda^s)} := \|\Lambda^s f\|_{L^2(\Omega)} = \left(\sum_{j=1}^{\infty} \lambda_j^s f_j^2 \right)^{\frac{1}{2}}.$$

It is also well-known that $D(\Lambda)$ and $H_0^1(\Omega)$ are isometric. In the language of interpolation theory, see [Lions and Magenes 1972, Chapter 1],

$$D(\Lambda^s) = [L^2(\Omega), D(-\Delta)]_{\frac{s}{2}} \quad \text{for all } s \in [0, 2].$$

As mentioned above,

$$H_0^1(\Omega) = D(\Lambda) = [L^2(\Omega), D(-\Delta)]_{\frac{1}{2}};$$

hence

$$D(\Lambda^s) = [L^2(\Omega), H_0^1(\Omega)]_s \quad \text{for all } s \in [0, 1]. \tag{2-1}$$

Consequently, we can identify $D(\Lambda^s)$ with usual Sobolev spaces:

$$D(\Lambda^s) = \begin{cases} H_0^s(\Omega) & \text{if } s \in \left(\frac{1}{2}, 1\right], \\ H_{00}^{\frac{1}{2}}(\Omega) := \{ u \in H^{\frac{1}{2}}(\Omega) : u/\sqrt{d(x)} \in L^2(\Omega) \} & \text{if } s = \frac{1}{2}, \\ H^s(\Omega) & \text{if } s \in \left[0, \frac{1}{2}\right). \end{cases} \tag{2-2}$$

see [Lions and Magenes 1972, Chapter 1]. Here and below $d(x)$ is the distance to the boundary of the domain:

$$d(x) = d(x, \partial\Omega). \tag{2-3}$$

Next, for $s > 0$ we define

$$\Lambda^{-s} f = \sum_{j=1}^{\infty} \lambda_j^{-\frac{s}{2}} f_j w_j$$

if $f = \sum_{j=1}^{\infty} f_j w_j \in D(\Lambda^{-s})$ with

$$D(\Lambda^{-s}) := \left\{ \sum_{j=1}^{\infty} f_j w_j \in \mathscr{D}'(\Omega) : f_j \in \mathbb{R},\ \sum_{j=1}^{\infty} \lambda_j^{-\frac{s}{2}} f_j w_j \in L^2(\Omega) \right\};$$

moreover,

$$\|f\|_{D(\Lambda^{-s})} := \|\Lambda^{-s} f\|_{L^2(\Omega)} = \left(\sum_{j=1}^{\infty} \lambda_j^{-s} f_j^2 \right)^{\frac{1}{2}}.$$

It is easy to check that $D(\Lambda^{-s})$ is the dual of $D(\Lambda^s)$ with respect to the pivot space $L^2(\Omega)$.

We have the following relation between $D(\Lambda^s)$ and $H^s(\Omega)$ when $s \geq 0$.

Lemma 2.1. *The continuous embedding*

$$D(\Lambda^s) \subset H^s(\Omega) \tag{2-4}$$

holds for any $s \geq 0$.

Proof. By interpolation, it suffices to prove (2-4) for $s \in \{0, 1, 2, \dots\}$. The case $s = 0$ is obvious and the case $s = 1$ follows from (2-2). Assume by induction (2-4) for $s \leq m$ with $m \geq 1$. Let $\theta \in D(\Lambda^{m+1})$. Then $f := -\Delta\theta \in D(\Lambda^{m-1})$ and thus $f \in H^{m-1}(\Omega)$ by the induction hypothesis. On the other hand, θ vanishes on the boundary $\partial\Omega$ in the trace sense because $\theta \in D(\Lambda^1) = H_0^1(\Omega)$. Elliptic regularity then implies that $\theta \in H^{m+1}(\Omega)$ and

$$\|\theta\|_{H^{m+1}} \leq C\|f\|_{H^{m-1}} \leq C\|\Delta\theta\|_{m-1,D} = C\|\theta\|_{m+1,D},$$

which is (2-4) for $s = m + 1$. $\square$

Lemma 2.2. *The operator*

$$\Lambda^\mu \nabla : D(\Lambda^\gamma) \to D(\Lambda^{\gamma-1-\mu}) \tag{2-5}$$

is continuous for any $\gamma \in [0, 1]$ *and* $\mu \leq \gamma - 1$.

Proof. We first note that the gradient operator ∇ is continuous from $H_0^1(\Omega)$ to $L^2(\Omega)$ and from $L^2(\Omega)$ to $H^{-1}(\Omega)$; hence by interpolation,

$$\nabla : [L^2, H_0^1]_\gamma \to [H^{-1}, L^2]_\gamma$$

for any $\gamma \in [0, 1]$. From the interpolation (2-1) we deduce that

$$\begin{aligned}
&[L^2, H_0^1]_\gamma = D(\Lambda^\gamma),\\
&[H^{-1}, L^2]_\gamma = ([H^1, L^2]_\gamma)^* = ([L^2, H^1]_{1-\gamma})^* = D(\Lambda^{1-\gamma})^* = D(\Lambda^{\gamma-1}).
\end{aligned}$$

Thus, for any $\gamma \in [0, 1]$,

$$\nabla : D(\Lambda^\gamma) \to D(\Lambda^{\gamma-1}),$$

and hence

$$\Lambda^\mu \nabla : D(\Lambda^\gamma) \to D(\Lambda^{\gamma-1-\mu})$$

provided $\mu \leq \gamma - 1$. $\square$

Remark 2.3. The above fractional Laplacian is the *spectral* one. In $\mathbb{R}^d$ the well-known integral representation

$$(-\Delta_{\mathbb{R}^d})^s f(x) = c_{d,s}\ \text{P.V.}\int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+2s}}\, dy, \quad s > 0,$$

holds; here P.V. stands for the principal value integral. For any domain $\Omega \subset \mathbb{R}^d$, the *restricted fractional Laplacian* $(-\Delta|_\Omega)^s$ is defined by

$$(-\Delta|_\Omega)^s f = ((-\Delta_{\mathbb{R}^d})^s \tilde{f})|_\Omega$$

for $f:\Omega\to\mathbb{R}$ and $\tilde f$ the zero-extension of f outside Ω. It was proved in [Bonforte et al. 2015] (see Section 3.1 there) that $(-\Delta|_\Omega)^s$ is an isomorphism from $D(\Lambda^s)$ onto its dual $D(\Lambda^s)^*$ with respect to the bilinear form

$$B(f,g)=\int_\Omega \mathcal{L}^{\frac12} f\mathcal{L}^{\frac12} g,\quad \mathcal{L}=(-\Delta|_\Omega)^s.$$

Hence for any scalar $\theta\in D(\Lambda^{\frac{\alpha}{2}})^*\supset L^2(\Omega)$ the stream function ψ can be defined alternatively by

$$\psi=((-\Delta|_\Omega)^{\frac{\alpha}{2}})^{-1}\theta. \tag{2-6}$$

Note that the resulting ψ is different from the one defined in (1-4). It would be interesting to see if the results in this paper still hold with this definition. We also refer to [Ros-Oton and Serra 2014] for the Hölder regularity of the ψ given by (2-6).

Commutator estimates. Due to the lack of translation invariance, the fractional Laplacian does not commute with differentiation. The following theorem provides a bound for the commutator.

Theorem 2.4 [Constantin and Nguyen 2016, Theorem 2.2]. *Let $p,\ q\in[1,\infty]$, $s\in(0,2)$ and a satisfy*

$$a(\cdot)d(\cdot)^{-s-1-\frac{d}{p}}\in L^q(\Omega).$$

Then the operator $a[\Lambda^s,\nabla]$ can be uniquely extended from $C_0^\infty(\Omega)$ to $L^p(\Omega)$ such that there exists a positive constant $C=C(d,s,p,\Omega)$ such that

$$\|a[\Lambda^s,\nabla]f\|_{L^q(\Omega)}\le C\|a(\cdot)d(\cdot)^{-s-1-\frac{d}{p}}\|_{L^q(\Omega)}\|f\|_{L^p(\Omega)} \tag{2-7}$$

holds for all $f\in L^p(\Omega)$.

The bound (2-7) is remarkable in that the commutator between an operator of order $s>0$ and an operator of order 1, which happens to vanish when $\Omega=\mathbb{R}^d$, is of order 0. The price is a singularity of the form $d(x)^{-s-1-\frac{d}{p}}$, which counts the order of Λ^s and ∇.

Remark 2.5. Let us explain how Theorem 2.4 follows from [Constantin and Nguyen 2016]. In that paper, using the heat kernel representation of the fractional Laplacian together with a cancellation of the heat kernel of $\mathbb{R}^d$, we proved the pointwise estimate for $f\in C_0^\infty(\Omega)$,

$$|[\Lambda^s,\nabla]f(x)|\le C(d,s,p,\Omega)d(x)^{-s-1-\frac{d}{p}}\|f\|_{L^p(\Omega)}.$$

The estimate (2-7) then follows by extension by continuity.

The next commutator estimate for negative powers of Laplacian is needed to handle the situation of more singular velocity.

Theorem 2.6. *Let $s\in(0,d)$ and $a\in W^{1,\infty}(\Omega)$. Let $p,r\in(1,\infty)$ satisfy*

$$\frac{1}{p}+\frac{d-s}{d}=1+\frac{1}{r}.$$

Then the operator $[\Lambda^{-s}, a]$ *can be uniquely extended from* $C_0^\infty(\Omega)$ *to* $L^p(\Omega)$ *with values in* $W_0^{1,r}(\Omega)$ *such that there exists* $C = C(s, d, p, r, \Omega) > 0$ *such that*

$$\|[\Lambda^{-s}, a]f\|_{W_0^{1,r}(\Omega)} \le C\|a\|_{W^{1,\infty}(\Omega)}\|f\|_{L^p(\Omega)}$$

for all $f \in L^p(\Omega)$.

In particular, for any $p \in (1,\infty)$, $s \in \left(0, \frac{d}{p}\right)$, *there exists* $C = C(s, d, p, \Omega) > 0$ *such that*

$$\|[\Lambda^{-s}, a]f\|_{W_0^{1,p}(\Omega)} \le C\|a\|_{W^{1,\infty}(\Omega)}\|f\|_{L^p(\Omega)} \tag{2-8}$$

for all $f \in L^p(\Omega)$.

With the same method of proof, we obtain:

Theorem 2.7. *Let* $s \in (0,1)$ *and* $a \in C^\gamma(\Omega)$ *with* $\gamma \in (0,1]$ *and* $s < \gamma$. *Let* $p, r \in (1,\infty)$ *satisfy*

$$\frac{1}{p} + \frac{d+s-\gamma}{d} = 1 + \frac{1}{r}.$$

Then the operator $[\Lambda^s, a]$ *can be uniquely extended from* $C_0^\infty(\Omega)$ *to* $L^p(\Omega)$ *with values in* $L^r(\Omega)$ *such that there exists* $C = C(s, \gamma, p, r, d, \Omega) > 0$ *such that*

$$\|[\Lambda^s, a]f\|_{L^r(\Omega)} \le C\|a\|_{C^\gamma(\Omega)}\|f\|_{L^p(\Omega)} \tag{2-9}$$

for all $f \in L^p(\Omega)$.

In particular, for any $p \in (1,\infty)$, *if*

$$s \in \left(\max\left\{\gamma - \frac{d}{p}, 0\right\}, \max\left\{\gamma - \frac{d}{p} + d, \gamma\right\}\right)$$

then there exists $C = C(s, \gamma, p, d, \Omega) > 0$ *such that*

$$\|[\Lambda^s, a]f\|_{L^p(\Omega)} \le C\|a\|_{C^\gamma(\Omega)}\|f\|_{L^p(\Omega)}. \tag{2-10}$$

Remark 2.8. In view of the identity

$$\Lambda^{-s}[\Lambda^s, a]f = [a, \Lambda^{-s}]\Lambda^s f,$$

it follows from (2-8) that

$$\|[\Lambda^s, a]f\|_{D(\Lambda^{1-s})} \le C\|a\|_{W^{1,\infty}(\Omega)}\|f\|_{D(\Lambda^s)}, \quad s \in \left(0, \tfrac{d}{2}\right). \tag{2-11}$$

This exhibits a gain of $1-s$ derivatives of $[\Lambda^s, a]$ when acting on $D(\Lambda^s)$. On the other hand, the estimate (2-10) shows a gain of s derivatives when acting on L^2. Both (2-8) and (2-10) make use of the fact that Ω is bounded.

The proofs of Theorems 2.6, 2.7 are given in Appendices A and B.

3. Proof of Theorem 1.1

Commutator representations. First, we adapt the well-known commutator representation of the nonlinearity in SQG [Resnick 1995], see also [Chae et al. 2012; Constantin et al. 2001; Constantin and Nguyen 2016], to take into account the lack of translation invariance of the fractional Laplacian and the more singular constitutive law (1-4):

Lemma 3.1. *Let $\psi \in H_0^1(\Omega)$, $u = \nabla^\perp \psi$, and $\theta = \Lambda^\alpha \psi$. Let $\phi \in C_0^\infty(\Omega)$ be a test function. Then*

$$\int_\Omega \theta u \cdot \nabla\phi \, dx = \frac{1}{2}\int_\Omega [\Lambda^\alpha, \nabla^\perp]\psi \cdot \nabla\phi\psi \, dx - \frac{1}{2}\int_\Omega \nabla^\perp\psi \cdot [\Lambda^\alpha, \nabla\phi]\psi \, dx \tag{3-1}$$

holds.

Proof. We have

$$\int_\Omega \theta u \cdot \nabla\phi \, dx = \int_\Omega \Lambda^\alpha\psi\nabla^\perp\psi \cdot \nabla\phi \, dx = -\int_\Omega \psi\nabla^\perp\Lambda^\alpha\psi \cdot \nabla\phi \, dx,$$

where we integrated by parts and used the fact that $\nabla^\perp \cdot \nabla\phi = 0$. The first and middle terms are well defined because $\theta u = \theta\nabla^\perp\psi \in L^1(\Omega)$, noticing that $\psi \in H_0^1(\Omega)$ and $\theta = \Lambda^\alpha\psi \in D(\Lambda^{1-\alpha}) \subset L^2(\Omega)$. The last term is defined because $\nabla\phi \cdot \nabla^\perp\Lambda^\alpha\psi \in H^{-1}(\Omega)$ and $\psi \in H_0^1(\Omega)$. Commuting $\nabla^\perp$ with Λ^α and then with $\nabla\phi$ leads to

$$\begin{aligned}
\int_\Omega \theta u \cdot \nabla\phi \, dx &= -\int_\Omega \psi[\nabla^\perp, \Lambda^\alpha]\psi \cdot \nabla\phi \, dx - \int_\Omega \psi\Lambda^\alpha\nabla^\perp\psi \cdot \nabla\phi \, dx \\
&= -\int_\Omega \psi[\nabla^\perp, \Lambda^\alpha]\psi \cdot \nabla\phi \, dx - \int_\Omega \nabla^\perp\psi \cdot \Lambda^\alpha(\psi\nabla\phi) \, dx \\
&= -\int_\Omega [\nabla^\perp, \Lambda^\alpha]\psi \cdot \nabla\phi\psi \, dx - \int_\Omega \nabla^\perp\psi \cdot [\Lambda^\alpha, \nabla\phi]\psi \, dx - \int_\Omega \nabla^\perp\psi \cdot \nabla\phi\Lambda^\alpha\psi \, dx \\
&= -\int_\Omega [\nabla^\perp, \Lambda^\alpha]\psi \cdot \nabla\phi\psi \, dx - \int_\Omega \nabla^\perp\psi \cdot [\Lambda^\alpha, \nabla\phi]\psi \, dx - \int_\Omega \theta u \cdot \nabla\phi \, dx.
\end{aligned}$$

The above calculations are justified by means of Theorems 2.4 and 2.7. Noticing that the last term on the right-hand side is exactly the negative of the left-hand side, we proved (3-1). ☐

Remark 3.2. The representation (3-1) was derived in [Constantin and Nguyen 2016] for the SQG equation ($\alpha = 1$). When $\Omega = \mathbb{R}^2$ or $\mathbb{T}^2$, (3-1) reduces to

$$\int_\Omega \theta u \cdot \nabla\phi \, dx = -\frac{1}{2}\int_\Omega \nabla^\perp\psi \cdot [\Lambda^\alpha, \nabla\phi]\psi \, dx.$$

Integrating by parts yields

$$-\frac{1}{2}\int_\Omega \nabla^\perp\psi \cdot [\Lambda^\alpha, \nabla\phi]\psi \, dx = \frac{1}{2}\int_\Omega \psi\nabla^\perp \cdot [\Lambda^\alpha, \nabla\phi]\psi \, dx = \frac{1}{2}\int_\Omega \psi[\Lambda^\alpha\nabla^\perp, \nabla\phi]\psi \, dx,$$

where we used in the second equality the fact that $\nabla^\perp \cdot \nabla\phi = 0$. This representation was invoked in [Chae et al. 2012] to prove the existence of global L^2 weak solutions of (1-2) in the periodic setting. More precisely, the authors proved the commutator estimate

$$\|[\Lambda^s\nabla, g]h\|_{L^2(\mathbb{T}^2)} \le C\|h\|_{L^2(\mathbb{T}^2)}\|g\|_{H^{s+2+\varepsilon}(\mathbb{T}^2)} + C\|\Lambda^s h\|_{L^2(\mathbb{T}^2)}\|g\|_{H^{2+\varepsilon}(\mathbb{T}^2)}$$

for any $s, \varepsilon > 0$. In arbitrary bounded domains, we were not able to establish such a commutator estimate.

We observe that by virtue of Theorem 2.4, the first integral on the right-hand side of (3-1) is well-defined provided only $\psi \in L^2(\Omega)$; moreover,

$$\left|\int_\Omega [\Lambda^\alpha, \nabla^\perp]\psi \cdot \nabla\phi\psi\, dx\right| \le C\|\nabla\phi d(\cdot)^{-\alpha-2}\|_{L^2(\Omega)}\|\psi\|^2_{L^2(\Omega)},$$

where by applying the Hardy inequality three times, together with the fact that $\alpha \in (0,1)$, we get

$$\|\nabla\phi d(\cdot)^{-\alpha-2}\|_{L^2(\Omega)} \le C\|\nabla\phi d(\cdot)^{-3}\|_{L^2} \le C\|\nabla^4\phi\|_{L^2(\Omega)} \le C\|\phi\|_{H^4(\Omega)}.$$

Consequently,

$$\left|\int_\Omega [\Lambda^\alpha, \nabla^\perp]\psi \cdot \nabla\phi\psi\, dx\right| \le C\|\phi\|_{H^4(\Omega)}\|\psi\|^2_{L^2(\Omega)}. \tag{3-2}$$

Regarding the second integral, we prove:

Lemma 3.3. *Assume $\psi \in D(\Lambda^\alpha)$. Then*

$$\mathcal{N}_2(\psi,\phi) := \int_\Omega \Lambda^{-1+\alpha}\nabla^\perp\psi \cdot \Lambda^{1-\alpha}[\Lambda^\alpha, \nabla\phi]\psi\, dx \tag{3-3}$$

satisfies

$$|\mathcal{N}_2(\psi,\phi)| \le C\|\nabla\phi\|_{W^{1,\infty}}\|\psi\|^2_{D(\Lambda^\alpha)}. \tag{3-4}$$

For any $\delta \in (0, \min(\alpha, 1-\alpha))$ we have

$$\mathcal{N}_2(\psi,\phi) = \int_\Omega \Lambda^{-1+\alpha-\delta}\nabla^\perp\psi\cdot\Lambda[\nabla\phi, \Lambda^{-\alpha+\delta}]\Lambda^\alpha\psi\, dx + \int_\Omega \Lambda^{-1+\alpha}\nabla^\perp\psi\cdot\Lambda[\nabla\phi, \Lambda^{-\delta}]\Lambda^\delta\psi\, dx. \tag{3-5}$$

Moreover,

$$|\mathcal{N}_2(\psi,\phi)| \le C\|\nabla\phi\|_{W^{1,\infty}}\|\psi\|_{D(\Lambda^{\alpha-\delta})}\|\psi\|_{D(\Lambda^\alpha)} + C\|\nabla\phi\|_{W^{1,\infty}}\|\psi\|_{D(\Lambda^\alpha)}\|\psi\|_{D(\Lambda^\delta)}. \tag{3-6}$$

Proof. (1) By Lemma 2.2,

$$\|\Lambda^{-1+\alpha}\nabla^\perp\psi\|_{L^2} \le \|\psi\|_{D(\Lambda^\alpha)}.$$

On the other hand, a direct calculation gives

$$\Lambda^{-\alpha}[\Lambda^\alpha, \nabla\phi]\psi = [\nabla\phi, \Lambda^{-\alpha}]\Lambda^\alpha\psi,$$

which, by virtue of Theorem 2.6, belongs to $D(\Lambda)$ and satisfies

$$\|\Lambda[\nabla\phi, \Lambda^{-\alpha}]\Lambda^\alpha\psi\|_{L^2} \le C\|\nabla\phi\|_{W^{1,\infty}}\|\Lambda^\alpha\psi\|_{L^2} = C\|\nabla\phi\|_{W^{1,\infty}}\|\psi\|_{D(\Lambda^\alpha)}.$$

Therefore, the integral defining $\mathcal{N}_2(\psi,\phi)$ in (3-3) makes sense and obeys the bound (3-4).

(2) Let $\delta \in [0, \min(\alpha, 1-\alpha))$. According to (3-3),

$$\begin{aligned}\mathcal{N}_2(\psi,\phi) &= \langle \Lambda^{-1+\alpha}\nabla^\perp\psi, \Lambda^{1-\alpha}[\Lambda^\alpha,\nabla\phi]\psi\rangle_{L^2,L^2}\\ &= \langle \Lambda^{-1+\alpha-\delta}\nabla^\perp\psi, \Lambda^{1-\alpha+\delta}[\Lambda^\alpha,\nabla\phi]\psi\rangle_{D(\Lambda^\delta),D(\Lambda^{-\delta})}.\end{aligned}$$

Now we write

$$\begin{aligned}\Lambda^{1-\alpha+\delta}[\Lambda^\alpha,\nabla\phi]\psi &= \Lambda\Lambda^{-\alpha+\delta}[\Lambda^\alpha,\nabla\phi]\psi\\ &= \Lambda\{\Lambda^\delta(\nabla\phi\psi) - \Lambda^{-\alpha+\delta}(\nabla\phi\Lambda^\alpha\psi)\}\\ &= \Lambda\{[\Lambda^\delta,\nabla\phi]\psi + \nabla\phi\Lambda^\delta\psi - \Lambda^{-\alpha+\delta}(\nabla\phi\Lambda^\alpha\psi)\}\\ &= \Lambda\{[\Lambda^\delta,\nabla\phi]\psi + \nabla\phi\Lambda^{-\alpha+\delta}\Lambda^\alpha\psi - \Lambda^{-\alpha+\delta}(\nabla\phi\Lambda^\alpha\psi)\}\\ &= \Lambda[\Lambda^\delta,\nabla\phi]\psi + \Lambda[\nabla\phi,\Lambda^{-\alpha+\delta}]\Lambda^\alpha\psi,\end{aligned}$$

where, according to (2-11),

$$[\Lambda^\delta,\nabla\phi]\psi \in D(\Lambda^{1-\delta}),$$

so

$$\Lambda[\Lambda^\delta,\nabla\phi]\psi \in D(\Lambda^{-\delta});$$

on the other hand, according to Theorem 2.6,

$$\Lambda[\nabla\phi,\Lambda^{-\alpha+\delta}]\Lambda^\alpha\psi \in L^2(\Omega) \subset D(\Lambda^{-\delta}).$$

Thus, we can write

$$\begin{aligned}I &= \langle \Lambda^{-1+\alpha-\delta}\nabla^\perp\psi, \Lambda[\nabla\phi,\Lambda^{-\alpha+\delta}]\Lambda^\alpha\psi\rangle_{D(\Lambda^\delta),D(\Lambda^{-\delta})} + \langle \Lambda^{-1+\alpha-\delta}\nabla^\perp\psi, \Lambda[\Lambda^\delta,\nabla\phi]\psi\rangle_{D(\Lambda^\delta),D(\Lambda^{-\delta})}\\ &= \int_\Omega \Lambda^{-1+\alpha-\delta}\nabla^\perp\psi\cdot\Lambda[\nabla\phi,\Lambda^{-\alpha+\delta}]\Lambda^\alpha\psi\,dx + \int_\Omega \Lambda^{-1+\alpha}\nabla^\perp\psi\cdot\Lambda^{1-\delta}[\Lambda^\delta,\nabla\phi]\psi\,dx\\ &= \int_\Omega \Lambda^{-1+\alpha-\delta}\nabla^\perp\psi\cdot\Lambda[\nabla\phi,\Lambda^{-\alpha+\delta}]\Lambda^\alpha\psi\,dx + \int_\Omega \Lambda^{-1+\alpha}\nabla^\perp\psi\cdot\Lambda[\nabla\phi,\Lambda^{-\delta}]\Lambda^\delta\psi\,dx.\end{aligned}$$

As in (1), an application of Theorems 2.4, 2.6, and (2-5), with $(\gamma = \alpha-\delta, \mu = -1-\alpha-\delta)$ and $(\gamma = \alpha, \mu = -1+\alpha)$, leads to the bound (3-6). □

Let us define

$$\begin{aligned}\mathcal{N}_1(\psi,\phi) &= \int_\Omega [\Lambda^\alpha,\nabla^\perp]\psi\cdot\nabla\phi\psi\,dx,\\ \mathcal{N}(\psi,\phi) &= \tfrac{1}{2}\mathcal{N}_1(\psi,\phi) - \tfrac{1}{2}\mathcal{N}_2(\psi,\phi).\end{aligned} \tag{3-7}$$

Putting together the above considerations, we have proved:

Lemma 3.4. *If* $\psi \in H_0^1(\Omega)$ *then*

$$\int_\Omega u\theta\cdot\nabla\phi = \mathcal{N}(\psi,\phi).$$

If $\theta \in L^2(\Omega)$ *then*

$$|\mathcal{N}(\psi,\phi)| \le C\|\phi\|_{H^4}\|\psi\|_{L^2}^2 + C\|\nabla\phi\|_{W^{1,\infty}}\|\psi\|_{D(\Lambda^\alpha)}^2$$

and for any $\delta \in (0, \min(\alpha, 1-\alpha))$,

$$|\mathcal{N}(\psi,\phi)| \le C\|\phi\|_{H^4}\|\psi\|_{L^2}^2 + C\|\nabla\phi\|_{W^{1,\infty}}\|\psi\|_{D(\Lambda^{\alpha-\delta})}\|\psi\|_{D(\Lambda^\alpha)} + C\|\nabla\phi\|_{W^{1,\infty}}\|\psi\|_{D(\Lambda^\alpha)}\|\psi\|_{D(\Lambda^\delta)}.$$

Viscosity approximations. Let us fix $\theta_0 \in L^2(\Omega)$ and a positive time T. For each fixed $\varepsilon > 0$ we consider the viscosity approximation of (1-2)

$$\begin{cases} \partial_t\theta^\varepsilon + u^\varepsilon\cdot\nabla\theta^\varepsilon - \varepsilon\Delta\theta^\varepsilon = 0, & t>0, \\ \theta^\varepsilon = \theta_0, & t=0, \end{cases} \tag{3-8}$$

with $u^\varepsilon = \nabla^\perp\psi^\varepsilon$, $\psi^\varepsilon = \Lambda^{-\alpha}\theta^\varepsilon$.

Equation (3-8) can be solved using the Galerkin approximation method as follows. Denote by $\mathbb{P}_m$ the projection in $L^2(\Omega)$ onto the linear span $L^2_m(\Omega)$ of eigenfunctions $\{w_1,\dots,w_m\}$; i.e.,

$$\mathbb{P}_m f = \sum_{j=1}^m f_j w_j \quad \text{for } f = \sum_{j=1}^\infty f_j w_j.$$

We recall the following lemma which shows that $\mathbb{P}_m\phi$ are good approximations of ϕ in any Sobolev space for $\phi \in C_0^\infty(\Omega)$.

Lemma 3.5 [Constantin and Nguyen 2016, Lemma 3.1]. *Let* $\phi \in C_0^\infty(\Omega)$. *For all* $k \in \mathbb{N}$ *we have*

$$\lim_{m\to\infty}\|(\mathbb{I}-\mathbb{P}_m)\phi\|_{H^k(\Omega)} = 0. \tag{3-9}$$

The m-th Galerkin approximation of (3-8) is the following ODE system in the finite-dimensional space $\mathbb{P}_m L^2(\Omega) = L^2_m$:

$$\begin{cases} \dot\theta^\varepsilon_m + \mathbb{P}_m(u^\varepsilon_m\cdot\nabla\theta^\varepsilon_m) - \varepsilon\Delta\theta^\varepsilon_m = 0, & t>0, \\ \theta^\varepsilon_m = P_m\theta_0, & t=0, \end{cases} \tag{3-10}$$

with $\theta_m(x,t) = \sum_{j=1}^m \theta_j^{(m)}(t)w_j(x)$ and $u_m = \nabla^\perp\Lambda^{-\alpha}\theta_m$ automatically satisfying $\operatorname{div} u_m = 0$. Note that in general $u_m \notin L^2_m$. The existence of solutions of (3-10) at fixed m follows from the fact that this is an ODE:

$$\frac{d\theta_l^{(m)}}{dt} + \sum_{j,k=1}^m \gamma_{jkl}^{(m)}\theta_j^{(m)}\theta_k^{(m)} + \varepsilon\lambda_l\theta_l^{(m)} = 0,$$

with

$$\gamma_{jkl}^{(m)} = \lambda_j^{-\frac{\alpha}{2}}\int_\Omega(\nabla^\perp w_j\cdot\nabla w_k)w_l\,dx.$$

Since $\mathbb{P}_m$ is self-adjoint in L^2, u_m is divergence-free and w_j vanishes at the boundary $\partial\Omega$, integration by parts with θ_m gives

$$\int_\Omega \theta_m\mathbb{P}_m(u_m\cdot\nabla\theta_m)\,dx = \int_\Omega \theta_m u_m\cdot\nabla\theta_m\,dx = 0$$

and

$$-\int_\Omega \Delta\theta^\varepsilon_m\theta^\varepsilon_m\,dx = \int_\Omega|\nabla\theta^\varepsilon_m|^2\,dx.$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\|\theta_m(\cdot,t)\|^2_{L^2(\Omega)}+\varepsilon\|\nabla\theta^\varepsilon_m\|^2_{L^2(\Omega)}=0,$$

and thus for $t\in[0,T]$,

$$\tfrac{1}{2}\|\theta^\varepsilon_m(\cdot,t)\|^2_{L^2(\Omega)}+\varepsilon\int_0^t\|\nabla\theta^\varepsilon_m(\cdot,s)\|^2_{L^2(\Omega)}\,ds=\tfrac{1}{2}\|\theta^\varepsilon_m(\cdot,0)\|^2_{L^2(\Omega)}\le\tfrac{1}{2}\|\theta_0\|^2_{L^2(\Omega)}. \tag{3-11}$$

This can be seen directly on the ODE because $\gamma^{(m)}_{jkl}$ is antisymmetric in k,l. Therefore, the smooth solution θ^ε_m of (3-10) exists globally and obeys the L^2 bound (3-11). The sequence $(\theta^\varepsilon_m)_m$ is thus uniformly in m bounded in $L^\infty([0,T];L^2(\Omega))\cap L^2([0,T];H^1_0(\Omega))$. Consequently, for any $p\in[1,\infty)$ and any $q\in[1,2/(1-\alpha)]$, we have

$$\theta^\varepsilon_m\in L^2([0,T];H^1_0(\Omega))\subset L^2([0,T];L^p(\Omega)),$$
$$u^\varepsilon_m=\nabla^\perp\Lambda^{-\alpha}\theta_m\in L^2([0,T];H^\alpha(\Omega))\subset L^2([0,T];L^q(\Omega)),$$

with bounds uniform with respect to m, where we have used Lemma 2.1 to have

$$\Lambda^{-\alpha}\theta_m\in L^2([0,T];D(\Lambda^{1+\alpha}))\subset L^2([0,T];H^{1+\alpha}(\Omega)).$$

In particular,

$$\begin{aligned}\|u^\varepsilon_m\cdot\nabla\theta^\varepsilon_m\|_{L^1([0,T];H^{-1}(\Omega))}&=\|\operatorname{div}(u^\varepsilon_m\cdot\theta^\varepsilon_m)\|_{L^1([0,T];H^{-1}(\Omega))}\\&\le C\|\theta^\varepsilon_m\|^2_{L^2([0,T];H^1(\Omega))}\le\frac{C}{\varepsilon}\|\theta_0\|^2_{H^1(\Omega)},\end{aligned} \tag{3-12}$$

where (3-11) was invoked in the last inequality. Therefore, using (3-10) we obtain that $(\partial_t\theta^\varepsilon_m)_m$ is uniformly in m bounded in $L^1([0,T];H^{-1}(\Omega))$. Then according to the Aubin–Lions lemma [Lions 1969], there exist a θ^ε,

$$\theta^\varepsilon\in L^\infty([0,T];L^2(\Omega))\cap L^2([0,T];H^1_0(\Omega)), \tag{3-13}$$

and a subsequence of $(\theta^\varepsilon_m)_m$ such that

$$\theta^\varepsilon_m\to\theta^\varepsilon\quad\text{strongly in } L^p([0,T];H^{-\mu}(\Omega))\cap L^2([0,T];H^{1-\mu}_0(\Omega)) \tag{3-14}$$

for any $p<\infty$ and $\mu\in(0,1)$.

Integrating by parts the first equation of (3-10) against any test function $\phi\in C^\infty_0(\Omega\times(0,T))$ gives

$$\int_0^T\int_\Omega\theta^\varepsilon_m\,\partial_t\phi\,dx\,dt+\int_0^T\int_\Omega\theta^\varepsilon_m u^\varepsilon_m\cdot\nabla\mathbb{P}_m\phi(x,t)\,dx\,dt+\varepsilon\int_0^T\int_\Omega\theta^\varepsilon_m\Delta\phi\,dx\,dt=0. \tag{3-15}$$

In the limit $m\to\infty$, the first term and the third term converge respectively to

$$\int_0^T\int_\Omega\theta^\varepsilon\,\partial_t\phi\,dx\,dt,\quad \varepsilon\int_0^T\int_\Omega\theta^\varepsilon\Delta\phi\,dx\,dt.$$

It remains to study the nonlinear term:

$$\begin{aligned} N &:= \int_0^T\int_\Omega \theta_m^\varepsilon u_m^\varepsilon\cdot\nabla\mathbb{P}_m\phi\,dx\,dt - \int_0^T\int_\Omega \theta^\varepsilon u^\varepsilon\cdot\nabla\phi\,dx\,dt \\ &= \int_0^T\int_\Omega \theta_m^\varepsilon u_m^\varepsilon\cdot\nabla(\mathbb{P}_m\phi-\phi)\,dx\,dt + \int_0^T\int_\Omega (\theta_m^\varepsilon-\theta^\varepsilon)u_m^\varepsilon\cdot\nabla\phi\,dx\,dt + \int_0^T\int_\Omega \theta^\varepsilon(u_m^\varepsilon-u^\varepsilon)\cdot\nabla\phi\,dx\,dt \\ &=: N_1+N_2+N_3. \end{aligned}$$

Lemma 3.5 ensures that $\lim_{m\to\infty} N_1 = 0$. On the other hand, the strong convergence (3-14) with sufficiently small μ implies $\lim_{m\to\infty} N_2 = \lim_{m\to\infty} N_3 = 0$. Thus, we have proved that θ^ε satisfies

$$\int_0^T\int_\Omega \theta^\varepsilon\,\partial_t\phi\,dx\,dt + \int_0^T\int_\Omega \theta^\varepsilon u^\varepsilon\cdot\nabla\phi\,dx\,dt + \varepsilon\int_0^T\int_\Omega \theta^\varepsilon\Delta\phi\,dx\,dt = 0$$

for all $\phi\in C_0^\infty(\Omega\times(0,T))$. Here, θ^ε has the regularity (3-13), and in view of (3-11),

$$\|\theta^\varepsilon\|^2_{L^\infty([0,T];L^2(\Omega))} + 2\varepsilon\|\theta^\varepsilon\|^2_{L^2([0,T];H_0^1(\Omega))} \le \|\theta_0\|^2_{L^2(\Omega)}. \tag{3-16}$$

Since $\psi^\varepsilon(\cdot,t)\in D(\Lambda^{1+\alpha})\subset H_0^1(\Omega)$ for a.e. $t>0$, using Lemma 3.4 for the representation of the nonlinearity, we obtain for all $\phi\in C_0^\infty(\Omega\times(0,T))$,

$$\int_0^T\int_\Omega \theta^\varepsilon\,\partial_t\phi\,dx\,dt + \int_0^T \mathcal{N}(\psi^\varepsilon,\phi)\,dt + \varepsilon\int_0^T\int_\Omega \theta^\varepsilon\Delta\phi\,dx\,dt = 0. \tag{3-17}$$

Moreover, integrating by parts (3-10) with ψ_m^ε leads to

$$\frac{1}{2}\frac{d}{dt}\|\psi_m^\varepsilon(\cdot,t)\|^2_{D(\Lambda^{\alpha/2})} + \varepsilon\|\psi_m^\varepsilon(\cdot,t)\|^2_{D(\Lambda^{1+\alpha/2})} = 0,$$

where we used the fact that the nonlinear term vanishes:

$$\int_\Omega \psi_m^\varepsilon\mathbb{P}_m(u_m^\varepsilon\cdot\nabla\theta_m^\varepsilon)\,dx = \int_\Omega \psi_m^\varepsilon\,\mathrm{div}(\nabla^\perp\psi_m^\varepsilon\theta_m)\,dx = -\int_\Omega \nabla\psi_m^\varepsilon\cdot\nabla^\perp\psi_m^\varepsilon\theta_m\,dx = 0.$$

Consequently, integrating in time and letting $m\to\infty$ results in

$$\|\psi^\varepsilon(\cdot,t)\|^2_{D(\Lambda^{\alpha/2})} + 2\varepsilon\int_0^t \|\psi^\varepsilon(\cdot,s)\|^2_{D(\Lambda^{1+\alpha/2})}\,ds = \|\psi^\varepsilon(\cdot,0)\|^2_{D(\Lambda^{\alpha/2})} \quad \text{for all } t>0. \tag{3-18}$$

Vanishing viscosity. In order to extract a convergent subsequence of θ^ε we need, in addition to (3-16), a uniform bound for $\partial_t\theta^\varepsilon$ in a lower norm. Let us note that the bound (3-12) is not uniform in ε. By (3-13), $\theta^\varepsilon(\cdot,t)\in D(\Lambda)$ for a.e. $t>0$, which implies $\psi^\varepsilon(\cdot,t)=\Lambda^{-\alpha}\theta^\varepsilon(\cdot,t)\in D(\Lambda^{1+\alpha})\subset D(\Lambda)$ for a.e. $t>0$. Lemma 3.4 then gives

$$\left|\int_\Omega \theta^\varepsilon u^\varepsilon\cdot\nabla\phi\,dx\right| \le C\|\phi\|_{H^4(\Omega)}\|\psi^\varepsilon\|^2_{D(\Lambda^\alpha)} \le C\|\phi\|_{H^4(\Omega)}\|\theta_0\|^2_{L^2(\Omega)},$$

and hence, in view of (3-17),

$$\left|\int_0^T\int_\Omega \theta^\varepsilon\,\partial_t\phi\,dx\,dt\right| \le C\|\phi\|_{L^1([0,T];H^4(\Omega))}\big(\|\theta_0\|_{L^2(\Omega)} + \|\theta_0\|^2_{L^2(\Omega)}\big)$$

for all $\phi \in C_0^\infty(\Omega \times (0,T))$. Consequently,

$$\|\partial_t \theta^\varepsilon\|_{L^\infty([0,T];H^{-4}(\Omega))} \le C\big(\|\theta_0\|_{L^2(\Omega)} + \|\theta_0\|_{L^2(\Omega)}^2\big). \tag{3-19}$$

In view of the uniform bounds (3-16) and (3-19), the Aubin–Lions lemma ensures the existence of a θ,

$$\theta \in L^\infty([0,T];L^2(\Omega)) \cap C([0,T];H^{-\nu}(\Omega)) \quad \text{for all } \nu > 0,$$

and a subsequence θ^ε such that

$$\theta^\varepsilon \rightharpoonup \theta \quad \text{weakly in } L^2([0,T];L^2(\Omega)), \tag{3-20}$$

$$\theta^\varepsilon \to \theta \quad \text{strongly in } C([0,T];H^{-\nu}(\Omega)) \quad \text{for all } \nu > 0. \tag{3-21}$$

Consequently, with $\psi := \Lambda^{-\alpha}\theta$,

$$\psi \in L^\infty([0,T];D(\Lambda^\alpha)) \cap C([0,T];D(\Lambda^{\alpha-\nu}) \quad \text{for all } \nu > 0,$$

we have

$$\psi^\varepsilon \rightharpoonup \psi \quad \text{weakly in } L^2([0,T];D(\Lambda^\alpha)), \tag{3-22}$$

$$\psi^\varepsilon \to \psi \quad \text{strongly in } C([0,T];D(\Lambda^{\alpha-\nu})) \quad \text{for all } \nu > 0. \tag{3-23}$$

Let $\phi \in C_0^\infty(\Omega \times (0,T))$ a be fixed test function, we send ε to 0 in the weak formulation (3-17). The first term converges to $\int_0^T\int_\Omega \theta\, \partial_t\phi \, dx\, dt$ and the last term converges to 0. Regarding the nonlinear term, we shall prove that

$$R^\varepsilon := \int_0^T \mathcal{N}(\psi^\varepsilon,\phi) - \mathcal{N}(\psi,\phi)\, dt$$

converges to 0. In view of (3-1), (3-5), we have $2R^\varepsilon = \sum_{j=1}^6 I_j^\varepsilon$ with

$$\begin{aligned}
I_1^\varepsilon &= \int_\Omega [\Lambda^\alpha, \nabla^\perp](\psi_\varepsilon - \psi)\cdot \nabla\phi\psi_\varepsilon\, dx,\\
I_2^\varepsilon &= \int_\Omega [\Lambda^\alpha, \nabla^\perp]\psi\cdot \nabla\phi(\psi_\varepsilon - \psi)\, dx,\\
I_3^\varepsilon &= -\int_0^T\int_\Omega \Lambda^{-1+\alpha-\delta}\nabla^\perp(\psi^\varepsilon-\psi)\cdot\Lambda[\nabla\phi, \Lambda^{-\alpha+\delta}]\Lambda^\alpha\psi^\varepsilon\, dx\, dt,\\
I_4^\varepsilon &= -\int_0^T\int_\Omega \Lambda^{-1+\alpha-\delta}\nabla^\perp\psi\cdot\Lambda[\nabla\phi, \Lambda^{-\alpha+\delta}]\Lambda^\alpha(\psi_\varepsilon-\psi)\, dx\, dt,\\
I_5^\varepsilon &= -\int_\Omega \Lambda^{-1+\alpha}\nabla^\perp(\psi_\varepsilon-\psi)\cdot\Lambda[\nabla\phi, \Lambda^{-\delta}]\Lambda^\delta\psi\, dx,\\
I_6^\varepsilon &= -\int_\Omega \Lambda^{-1+\alpha}\nabla^\perp\psi_\varepsilon\cdot\Lambda[\nabla\phi, \Lambda^{-\delta}]\Lambda^\delta(\psi_\varepsilon-\psi)\, dx,
\end{aligned}$$

where $\delta \in (0, \min(\alpha, 1-\alpha))$.

By virtue of Theorem 2.4 and the fact that $\phi \in C_0^\infty(\Omega)$,

$$|I_1^\varepsilon| \le C(\phi)\|\psi_\varepsilon - \psi\|_{L^2(\Omega)}\|\psi_\varepsilon\|_{L^2(\Omega)}, \quad |I_2^\varepsilon| \le C(\phi)\|\psi_\varepsilon - \psi\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)}.$$

Hence $\lim_{\varepsilon\to 0} I_1^\varepsilon = \lim_{\varepsilon\to 0} I_2^\varepsilon = 0$ in view of the convergence (3-23) with $\nu < \alpha$.

As for (3-6),

$$|I_3^\varepsilon| \le C\|\nabla\phi\|_{L^1([0,T];W^{1,\infty})}\|\psi^\varepsilon-\psi\|_{L^\infty([0,T];D(\Lambda^{\alpha-\delta}))}\|\psi^\varepsilon\|_{L^\infty([0,T];D(\Lambda^\alpha))},$$

which combined with (3-23) leads to $\lim_{\varepsilon\to 0} I_3^\varepsilon = 0$. Because $\Lambda[\nabla\phi, \Lambda^{-\alpha+\delta}]\Lambda^\alpha$ is norm continuous from $L^2([0,T]; D(\Lambda^\alpha))$ to $L^2([0,T]; L^2(\Omega))$ (according to Theorem 2.6), it is weak-weak continuous, and thus $\lim_{\varepsilon\to 0} I_4^\varepsilon = 0$ noticing that by (2-5),

$$\Lambda^{-1+\alpha-\delta}\nabla^\perp\psi \in L^\infty([0,T]; D(\Lambda^\delta)) \subset L^2([0,T]; L^2(\Omega)).$$

Similarly, $\lim_{\varepsilon\to 0} I_5^\varepsilon = 0$ since $\Lambda^{-1+\alpha}\nabla^\perp(\psi_\varepsilon-\psi) \rightharpoonup 0$ in $L^2([0,T]; D(\Lambda^\alpha))$ by (3-22), and since $\Lambda[\nabla\phi, \Lambda^{-\delta}]\Lambda^\delta\psi \in L^2([0,T]; L^2(\Omega))$ by Theorem 2.6. Finally, by (2-5) and Theorem 2.6,

$$|I_6^\varepsilon| \le \|\Lambda^{-1+\alpha}\nabla^\perp\psi_\varepsilon\|_{L^2(\Omega)}\|\Lambda[\nabla\phi, \Lambda^{-\delta}]\Lambda^\delta(\psi_\varepsilon-\psi)\|_{L^2(\Omega)} \le \|\psi_\varepsilon\|_{D(\Lambda^\alpha)}\|\psi_\varepsilon-\psi\|_{D(\Lambda^\delta)} \to 0,$$

noticing that $\delta<\alpha$. We conclude that

$$\int_0^T\int_\Omega \theta\,\partial_t\phi\,dx\,dt + \int_0^T \mathcal{N}(\psi,\phi)\,dt = 0 \quad \text{for all } \phi\in C_0^\infty(\Omega\times(0,T)).$$

Moreover, because of the strong convergence (3-21) the initial data is attained:

$$\theta(\cdot,0) = \lim_{\varepsilon\to 0}\theta^\varepsilon(\cdot,0) = \lim_{\varepsilon\to 0}\theta_0(\cdot) = \theta_0(\cdot) \quad \text{in } H^{-\nu}(\Omega) \text{ for all } \nu>0.$$

Let us now show the conservation (1-9). In view of (3-16) and the fact that $\theta^\varepsilon = \Lambda^\alpha\psi^\varepsilon$ we have

$$\|\Lambda^\alpha\psi^\varepsilon\|^2_{L^\infty([0,T];L^2(\Omega))} + 2\varepsilon\|\Lambda^{1+\alpha}\psi^\varepsilon\|^2_{L^2([0,T];L^2(\Omega))} \le \|\theta_0\|^2_{L^2(\Omega)}.$$

By interpolation,

$$\|\Lambda^{1+\frac{\alpha}{2}}\psi^\varepsilon\|_{L^2(\Omega)} \le C\|\Lambda^{1+\alpha}\psi^\varepsilon\|^a_{L^2(\Omega)}\|\Lambda^\alpha\psi^\varepsilon\|^{1-a}_{L^2(\Omega)}, \quad a = 1-\tfrac{\alpha}{2}.$$

Hölder's inequality then yields

$$\begin{aligned}\|\Lambda^{1+\frac{\alpha}{2}}\psi^\varepsilon\|^2_{L^2([0,T];L^2(\Omega))} &\le C\|\Lambda^\alpha\psi^\varepsilon\|^{2(1-a)}_{L^\infty([0,T];L^2(\Omega))}\|\Lambda^{1+\alpha}\psi^\varepsilon\|^{2a}_{L^2([0,T];L^2(\Omega))}T^{\frac{\alpha}{2}}\\ &\le CT^{\frac{\alpha}{2}}\|\theta_0\|^2_{L^2(\Omega)}\varepsilon^{-1+\frac{\alpha}{2}} \quad \text{for all } T>0.\end{aligned}$$

In particular,

$$\lim_{\varepsilon\to 0}\varepsilon\|\Lambda^{1+\frac{\alpha}{2}}\psi^\varepsilon\|^2_{L^2([0,T];L^2(\Omega))} = 0 \quad \text{for all } T>0.$$

Letting $\varepsilon\to 0$ in (3-18) we obtain (1-9).

Finally, the energy inequality (1-8) follows from (3-16) and lower semicontinuity.

Remark 3.6. If we implement directly the Galerkin approximations for (1-2) then in view of (3-1), we need to bound

$$\left|\int_\Omega [\Lambda^\alpha, \nabla^\perp]\psi_m\cdot\nabla\mathbb{P}_m\phi\,\psi_m\,dx\right|.$$

However, the commutator $[\Lambda^\alpha, \nabla^\perp]$ then cannot be bounded by means of Theorem 2.4 because $\nabla\mathbb{P}_m\phi$ does not vanish on the boundary even though ϕ has compact support. In [Constantin and Nguyen 2016], we overcame this by first using Lemma 3.5 and the fact that $u_m\theta_m$ is uniformly bounded in $L^1(\Omega)$ to approximate $\int_\Omega u_m\theta_m\nabla\mathbb{P}_m\phi$ by $\int_\Omega u_m\theta_m\nabla\phi$. When $\alpha<1$, this argument breaks down since $u_m\theta_m$ is not anymore uniformly bounded in $L^1(\Omega)$. This explains why we proceeded in the proof of Theorem 1.1 using vanishing viscosity approximations.

Appendix A: Proof of Theorem 2.6

In view of the identity

$$D^{-r}=c_r\int_0^\infty t^{-1+r}e^{-tD}\,dt$$

with $D, r>0$ we have the representation of negative powers of Laplacian via heat kernel:

$$\Lambda^{-s}f(x)=c_s\int_0^\infty t^{-1+\frac{s}{2}}e^{t\Delta}f(x)\,dt,\quad s>0. \tag{A-1}$$

Let $H(x,y,t)$ denote the heat kernel of Ω; i.e.,

$$e^{t\Delta}f(x)=\int_\Omega H(x,y,t)f(y)\,dy\quad\text{for all } x\in\Omega.$$

We have from [Li and Yau 1986] the following bounds on H and its gradient:

$$H(x,y,t)\le Ct^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{Kt}}, \tag{A-2}$$

$$|\nabla_x H(x,y,t)|\le Ct^{-\frac{1}{2}-\frac{d}{2}}e^{-\frac{|x-y|^2}{Kt}} \tag{A-3}$$

for all $(x,y)\in\Omega\times\Omega$ and $t>0$.

We will also use the elementary estimate

$$\int_0^\infty t^{-1-\frac{m}{2}}e^{-\frac{p^2}{Kt}}\,dt\le C_{K,m}p^{-m},\quad m,p,K>0. \tag{A-4}$$

Let $f\in C_0^\infty(\Omega)$. Using (A-1) we have

$$\begin{aligned}[\Lambda^{-s},a]f(x)&=c_s\int_0^\infty t^{-1+\frac{s}{2}}\int_\Omega H(x,y,t)a(y)f(y)\,dt-c_s\,a(x)\int_0^\infty t^{-1+\frac{s}{2}}\int_\Omega H(x,y,t)f(y)\,dt\\&=c_s\int_0^\infty t^{-1+\frac{s}{2}}\int_\Omega H(x,y,t)[a(y)-a(x)]f(y)\,dt.\end{aligned} \tag{A-5}$$

In view of (A-2), (A-4), and the assumption that $s<d$, we deduce that

$$\begin{aligned}|[\Lambda^{-s},a]f(x)|&\le C\|a\|_{L^\infty}\int_\Omega\int_0^\infty t^{-1+\frac{s}{2}-\frac{d}{2}}e^{-\frac{|x-y|^2}{Kt}}\,dt|f(y)|\,dy\\&\le C\|a\|_{L^\infty}\int_\Omega\frac{|f(y)|}{|x-y|^{d-s}}\,dy.\end{aligned} \tag{A-6}$$

Let us recall the Hardy–Littlewood–Sobolev inequality. Let $\alpha \in (0, d)$ and $(p, r) \in (1, \infty)$ satisfy

$$\frac{1}{p} + \frac{\alpha}{d} = 1 + \frac{1}{r}. \tag{A-7}$$

A constant C then exists such that

$$\|f * |\cdot|^{-\alpha}\|_{L^r(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}. \tag{A-8}$$

Applying (A-8) with $\alpha = d - s$ leads to

$$\|[\Lambda^{-s}, a]f\|_{L^r(\Omega)} \le C\|a\|_{L^\infty}\|f\|_{L^p(\Omega)}. \tag{A-9}$$

Let γ_0 denote the trace operator for Ω. It is readily seen that $\gamma_0(\Lambda^{-s}f) = 0$ because $\Lambda^{-s}f \in D(\Lambda^m)$ for all $m \ge 0$; hence $\gamma_0(a\Lambda^{-s}f) = \gamma_0(a)\gamma_0(\Lambda^{-s}f) = 0$. In addition, $af \in H_0^1(\Omega) = D(\Lambda)$; hence $\Lambda^{-s}(af) \in D(\Lambda^{1+s}) \subset H_0^1(\Omega)$ and $\gamma_0(\Lambda^{-s}(af)) = 0$. We deduce that

$$\gamma_0([\Lambda^{-s}, a]f) = 0. \tag{A-10}$$

Next, for gradient bound we differentiate (A-5) and obtain

$$\begin{aligned}\nabla[\Lambda^{-s}, a]f(x) &= c_s\int_0^\infty t^{-1+\frac{s}{2}}\int_\Omega \nabla_x H(x, y, t)[a(y) - a(x)]f(y)\,dt\\ &\qquad - c_s\int_0^\infty t^{-1+\frac{s}{2}}\int_\Omega H(x, y, t)\nabla a(x) f(y)\,dt\\ &=: I + II.\end{aligned}$$

The term II can be treated as above and we have

$$\|II\|_{L^r(\Omega)} \le C\|\nabla a\|_{L^\infty}\|f\|_{L^p(\Omega)}. \tag{A-11}$$

For I, we use the gradient estimate (A-3) for the heat kernel and the fact that

$$|a(x) - a(y)| \le \|\nabla a\|_{L^\infty}|x - y|$$

to arrive at

$$\begin{aligned}|I(x)| &\le C\|\nabla a\|_{L^\infty}\int_\Omega\int_0^\infty t^{-1+\frac{s}{2}-\frac{1}{2}-\frac{d}{2}} e^{-\frac{|x-y|^2}{Kt}}\,dt|x - y||f(y)|\,dy\\ &\le C\|\nabla a\|_{L^\infty}\int_\Omega \frac{|f(y)|}{|x - y|^{d-s}}\,dy.\end{aligned}$$

Appealing to (A-8) as before gives

$$\|I\|_{L^r(\Omega)} \le C\|\nabla a\|_{L^\infty}\|f\|_{L^p(\Omega)},$$

which, combined with (A-9), (A-11), (A-10), leads to

$$\|[\Lambda^{-s}, a]f\|_{W_0^{1,r}(\Omega)} \le C\|a\|_{W^{1,\infty}(\Omega)}\|f\|_{L^p(\Omega)}, \tag{A-12}$$

where p, r satisfy (A-8) with $\alpha = d - s$. Using the density of $C_0^\infty(\Omega)$ in $L^p(\Omega)$ for $p \in (1, \infty)$, and extension by continuity, we conclude that the estimate (A-12) holds for any $f \in L^p(\Omega)$.

Now, for any $p \in (0, \infty)$, if $s < \frac{d}{p}$ then $r \in (1, \infty)$ given by

$$\frac{1}{r} = \frac{1}{p} - \frac{s}{d}$$

satisfies (A-8). Because $r > p$ and Ω is bounded, the continuous embedding $W_0^{1,r}(\Omega) \subset W_0^{1,p}(\Omega)$ yields

$$\|[\Lambda^{-s}, a]f\|_{W_0^{1,p}(\Omega)} \le C\|a\|_{W^{1,\infty}(\Omega)}\|f\|_{L^p(\Omega)}. \tag{A-13}$$

Appendix B: Proof of Theorem 2.7

In view of the identity

$$\lambda^{\frac{s}{2}} = c_s \int_0^\infty t^{-1-\frac{s}{2}}(1-e^{-t\lambda})\,dt$$

with $0 < s < 2$ and

$$1 = c_s \int_0^\infty t^{-1-\frac{s}{2}}(1-e^{-t})\,dt,$$

we have the representation of the fractional Laplacian via heat kernel

$$\Lambda^s f(x) = c_s \int_0^\infty t^{-1-\frac{s}{2}}(1-e^{t\Delta})f(x)\,dt, \quad 0 < s < 2. \tag{B-1}$$

Appealing to this representation, we have for $f \in C_0^\infty(\Omega)$

$$[\Lambda^s, a]f(x) = c_s \int_0^\infty t^{-1-\frac{s}{2}} \int_\Omega H(x, y, t)\,dt\,[a(x) - a(y)]f(y)\,dy.$$

In view of (A-2), the fact that

$$|a(x) - a(y)| \le \|a\|_{C^\gamma}|x-y|^\gamma,$$

and (A-4), we deduce that

$$\begin{aligned}
|[\Lambda^s, a]f(x)| &\le c_s\|a\|_{C^\gamma} \int_\Omega \int_0^\infty t^{-1-\frac{s}{2}-\frac{d}{2}} e^{-\frac{|x-y|^2}{Kt}}\,dt\,|x-y|^\gamma |f(y)|\,dy \\
&\le c_s\|a\|_{C^\gamma} \int_\Omega \frac{|f(y)|}{|x-y|^{d+s-\gamma}}\,dy.
\end{aligned}$$

Then as in the proof of Theorem 2.6, if $s < \gamma$ (note that $d + s - \gamma > 0$), an application of the Hardy–Littlewood–Sobolev inequality leads to the bound (2-9). Finally, (2-10) follows from (2-9) and the fact that Ω is bounded.

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