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LARGE SETS AVOIDING PATTERNS

ROBERT FRASER AND MALABIKA PRAMANIK

We construct subsets of Euclidean space of large Hausdorff dimension and full Minkowski dimension that do not contain nontrivial patterns described by the zero sets of functions. The results are of two types. Given a countable collection of v -variate vector-valued functions $f_q : (\mathbb{R}^n)^v \rightarrow \mathbb{R}^m$ satisfying a mild regularity condition, we obtain a subset of \mathbb{R}^n of Hausdorff dimension $m/(v-1)$ that avoids the zeros of f_q for every q . We also find a set that simultaneously avoids the zero sets of a family of uncountably many functions sharing the same linearization. In contrast with previous work, our construction allows for nonpolynomial functions, as well as uncountably many patterns. In addition, it highlights the dimensional dependence of the avoiding set on v , the number of input variables.

1. Introduction

Identification of geometric and algebraic patterns in large sets has been a focal point of interest in modern analysis, geometric measure theory and additive combinatorics. A fundamental and representative result in the discrete setting that has been foundational in the development of a rich theory is Szemerédi's theorem [1975], which states that every subset of the integers with positive asymptotic density contains an arbitrarily long arithmetic progression. There is now an abundance of similar results in the continuum setting, all of which guarantee existence of configurations under appropriate assumptions on size, often stated in terms of Lebesgue measure, Hausdorff dimension or Banach density. While this body of work has contributed significantly to our understanding of such phenomena, a complete picture concerning existence or avoidance of patterns in sets is yet to emerge. In this paper, we will be concerned with the “avoidance” aspect of the problem. Namely, given a function $f : \mathbb{R}^{nv} \rightarrow \mathbb{R}^m$ satisfying certain conditions, how large a set $E \subset \mathbb{R}^n$ can one construct that carries no nontrivial solution of the equation $f(x_1, \dots, x_v) = 0$? In other words, we aim to find as large a set E as possible such that $f(x_1, \dots, x_v)$ is nonzero for any choice of distinct points $x_1, \dots, x_v \in E$.

In the discrete regime, results of this type can be traced back to Salem and Spencer [1942] and Behrend [1946], who identified large subsets of the integers avoiding progressions. The Euclidean formulation of this problem appears to be of relatively recent vintage. Keleti [1999] constructed a subset E of the real numbers of full Hausdorff dimension avoiding all nontrivial “one-dimensional rectangles”. More precisely, this means that there exist no solutions of the equation $x_2 - x_1 - x_4 + x_3 = 0$ with $x_1 < x_2 \leq x_3 < x_4$, $x_i \in E$, $1 \leq i \leq 4$. In particular, such a set contains no nontrivial arithmetic progression, as can be seen by setting $x_2 = x_3$. A counterpoint to [Keleti 1999] is a result of Łaba and the second author [Łaba and

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Pramanik 2009], who established existence of three-term progressions in special “random-like” subsets of \mathbb{R} that support measures satisfying an appropriate ball condition and a Fourier decay estimate. Higher-dimensional variants of this theme may be found in [Chan et al. 2016; Henriot et al. 2016]. On the other hand, large Hausdorff dimensionality, while failing to ensure specific patterns, is sometimes sufficient to ensure existence or even abundance of certain configuration classes; see for instance [Greenleaf and Iosevich 2012; Greenleaf et al. 2015; 2017; Bennett et al. 2016]. Harangi, Keleti, Kiss, Maga, Máthé, Mattila, and Strenner [Harangi et al. 2013] showed that sets of sufficiently large Hausdorff dimension contain points that generate specific angles.

Nonexistence of patterns such as the one proved by Keleti [1999] is the primary focus of this article. A main contribution of [Keleti 1999] is best described as a Cantor-type construction with memory, where selection of basic intervals at each stage is contingent on certain selections made at a much earlier step of the construction, so as to prevent certain algebraic relations from taking place. This idea has been instrumental in a large body of subsequent work involving nonexistence of configurations. For example, Keleti [2008] used this to show that for any countable set A , it is possible to construct a full-dimensional subset E of \mathbb{R} such that

$$x_2 - x_1 + a(x_3 - x_2) = 0$$

has no solutions for any $a \in A$, where x_1, x_2 and x_3 are distinct points in E . Maga [2010] exploited this idea to demonstrate a full-dimensional subset $E \subset \mathbb{R}^n$ not containing the vertices of any parallelogram. He also constructed a full-dimensional planar set that misses all similar copies of a given triangle. Other results in this direction of considerable generality, extending their predecessors in [Keleti 1999; 2008; Maga 2010], are due to Máthé [2012]. Given any countable collection of polynomials $p_j : \mathbb{R}^{m_j} \rightarrow \mathbb{R}$ of degree at most d with rational coefficients, the main result of [Máthé 2012] ensures the existence of a subset $E \subseteq \mathbb{R}^n$ of Hausdorff dimension n/d such that $p_j(x_1, \dots, x_{m_j})$ is nonzero for any choice of distinct points $x_1, \dots, x_{m_j} \in E$. The same conclusion continues to hold if the polynomials p_j are replaced by $p_j(\Phi_{j,1}(x_1), \dots, \Phi_{j,m_j}(x_{m_j}))$, where $\Phi_{j,k}$ are C^1 -diffeomorphisms of \mathbb{R}^n . Interestingly, the Hausdorff dimension bound in [Máthé 2012], while depending on the ambient dimension n and the maximum degree d of the polynomials, is independent of the number of input vectors m_j in p_j , which may continue to grow without bound.

This paper uses similar ideas to present two results in a somewhat different direction. The first complements Máthé’s result mentioned above. It applies to a countable family of functions $f : \mathbb{R}^{n_v} \rightarrow \mathbb{R}^m$ with a fixed v that are not necessarily polynomials with rational coefficients. Further, in contrast with [Máthé 2012], the Hausdorff dimension of the obtained set depends on the number of vector variables v . The second result is of a perturbative flavour, and gives a set of positive Hausdorff dimension that simultaneously avoids zeros of *all* functions with a common linearization and bounded higher-order terms. To the best of our knowledge, such uniform avoidance results are new. Some points of tenuous similarity may be found in [Harangi et al. 2013], where the authors construct sets that avoid angles within a specific range, but the ideas, methods and goals are very different.

1A. Main results. Our first result is most general in dimension one, where we need very mild restrictions on the functions whose zeros we want to avoid. The higher-dimensional, vector-valued version of this result applies with some additional restrictions. We state these two separately.

Theorem 1.1. For any $\eta > 0$ and integer $v \geq 3$, let $f_q : \mathbb{R}^v \rightarrow \mathbb{R}$ be a countable family of functions in v variables with the following properties:

- (a) There exists $r_q < \infty$ such that $f_q \in C^{r_q}([0, \eta]^v)$.
- (b) For each q , some partial derivative of f_q of order $r_q \geq 1$ does not vanish at any point of $[0, \eta]^v$.

Then there exists a set $E \subseteq [0, \eta]$ of Hausdorff dimension at least $1/(v-1)$ and Minkowski dimension 1 such that $f_q(x_1, \dots, x_v)$ is not equal to zero for any v -tuple of distinct points $x_1, \dots, x_v \in E$ and any function f_q .

Theorem 1.2. Fix $\eta > 0$ and positive integers m, n, v such that $v \geq 3$, and $m \leq n(v-1)$. Let $f_q : \mathbb{R}^{nv} \rightarrow \mathbb{R}^m$ be a countable family of C^2 functions with the following property: for every q on $[0, \eta]^{nv}$, the derivative $Df_q(x_1, \dots, x_v)$ has full rank at every point (x_1, \dots, x_v) in the zero set of f_q such that $x_r \neq x_s$ for all $r \neq s$.

Then there exists a set $E \subseteq [0, \eta]^n$ of Hausdorff dimension at least $m/(v-1)$ and Minkowski dimension n such that $f_q(x_1, \dots, x_v)$ is not equal to zero for any v -tuple of distinct points $x_1, \dots, x_v \in E^n$ and any function f_q .

Remarks. (a) If one seeks to avoid zeros of a single function f , then Theorem 1.1 is nontrivial only when the components of $\nabla f(x)$ sum to zero at every point x in the zero set of f . If this is not the case, then there is necessarily some interval I such that $f(x_1, \dots, x_v)$ is nonzero for points x_i in the interval I .

(b) The points $x_1, \dots, x_v \in E$ that ensure $f(x_1, \dots, x_v) \neq 0$ in Theorems 1.1 and 1.2 are taken to be distinct. This assumption, while needed for the proof, is often nonrestrictive for the purpose of applications. In fact, one can typically augment the family $\{f_q\}$ by $\{g_q\}$, where the function g_q equals f_q with certain input variables coincident, and apply the theorems above to the augmented family provided the nonvanishing derivative assumptions are met. For instance, Keleti's function $f(x_1, x_2, x_3, x_4) = (x_2 - x_1) - (x_4 - x_3) = -x_1 + x_2 + x_3 - x_4$ identifies "one-dimensional rectangles" in general, and three-term arithmetic progressions only if $x_2 = x_3$. In order to obtain a set that avoids both using our setup, we would need to apply our Theorem 1.1 to the collection $\{f, g\}$, where $g(x_1, x_2, x_3, x_4) = f(x_1, x_2, x_2, x_4) = -x_1 + 2x_2 - x_4$.

(c) Theorem 1.2 is sharp in certain instances, for example when $m = n(v-1)$. On the other hand, Theorem 1.1 need not be sharp for specific choices of f_q , as Keleti's example shows. Our result would only ensure a set of Hausdorff dimension $\frac{1}{3}$ for this example. Given the similarity in our respective methods of proof, the contrast in the results requires a word of explanation. In [Keleti 1999], one had explicit knowledge of the function f (which was linear), and hence of the structure of its zero set. This arithmetic structure was exploited heavily in the construction. Our assumptions on $\{f_q\}$ are too weak to offer explicit information concerning algebraic dependencies in the zero set, and hence our proof is based on a "worst-case analysis", which is true generically, but results in worse bounds. However, our method of proof is robust enough to accommodate special structures in zero sets, and yields better dimensional bounds in those settings. We substantiate this comment with more precise details at the appropriate juncture of the proof; see the remark on page 1092.

(d) Our conjecture is that the dimensional lower bound of $1/(v - 1)$ in Theorem 1.1 is sharp for certain generic functions, but we are currently unaware of any result in the literature that addresses the optimality of this bound in the setup that we describe. Partial evidence in support of this conjecture is provided in [Körner 2009], where the author constructs a set of Fourier dimension $1/(v - 1)$ avoiding all v -variate rational linear relations. We hope to return to this issue in the future.

(e) Even though our results do not recover those of [Keleti 2008; Maga 2010; Máthé 2012] in all instances where these results are applicable, the Hausdorff dimension provided in Theorems 1.1 and 1.2 offers new bounds in settings where previously none were available, for instance where the functions are nonpolynomials with mild regularity. It also improves the bound given in [Máthé 2012] for polynomials with rational coefficients in the regime where the degree d is much larger than the number of variables v . On the other hand, for polynomials of low degree the result in [Máthé 2012] improves ours, obtaining the best bound when $d = 1$.

(f) Finding the optimal dimension of a zero-avoiding set for a *specific and explicitly stated function* remains an interesting open question. For the quadratic polynomial $f(x_1, x_2, x_3) = (x_3 - x_1) - (x_2 - x_1)^2$, the zero-avoiding set is guaranteed to be of Hausdorff dimension at least $\frac{1}{2}$, both according to [Máthé 2012] and Theorem 1.1. It is not known whether this bound is optimal.

Our second result is about a set on which no function f with a given linearization and controlled higher-order term is zero.

Theorem 1.3. *Given any constant $K > 0$ and a vector $\alpha \in \mathbb{R}^v$ such that*

$$\alpha \cdot u \neq 0 \quad \text{for every } u \in \{0, 1\}^v \text{ with } u \neq 0, u \neq (1, 1, \dots, 1), \tag{1-1}$$

and such that

$$\sum_{j=1}^v \alpha_j = 0, \tag{1-2}$$

there exists a positive constant $c(\alpha)$ and a set $E = E(K, \alpha) \subseteq [0, 1]$ of Hausdorff dimension $c(\alpha) > 0$ with the following property.

The set E does not contain any nontrivial solution of the equation

$$f(x_1, \dots, x_v) = 0, \quad (x_1, \dots, x_v) \text{ not all identical,}$$

for any C^2 function f of the form

$$f(x_1, \dots, x_v) = \sum_{j=1}^v \alpha_j x_j + G(x_1, \dots, x_v), \tag{1-3}$$

where

$$|G(x)| \leq K \sum_{j=2}^v (x_j - x_1)^2. \tag{1-4}$$

Remarks. (a) The condition (1-1) implies that α does not lie in any coordinate hyperplane.

(b) The proof of Theorem 1.3 can be used to obtain a corresponding result with *finitely* many linearizations. There is a loss in the Hausdorff dimension as more linear functions are added to the family, so the proof fails for families of functions with countably many linearizations.

(c) It is interesting to note that the dimensional constant $c(\alpha)$ does not depend on K . Of course the set E does, and is uniform for all functions f obeying (1-3) and (1-4) with a fixed value of K .

1B. Layout. Section 2 is devoted to geometric applications of Theorems 1.1, 1.2 and 1.3. Optimality of these results (or lack thereof) in various settings is discussed, and comparison with earlier work is presented. Section 3 is a collection of geometric algorithms needed for the proofs of Theorems 1.1 and 1.2. The proofs themselves are executed in Sections 4 and 5.

2. Examples

2A. Subsets of curves avoiding isosceles triangles. This subsection is given over to the following question: suppose we are given a small segment of a simple C^2 curve $\Gamma \subset \mathbb{R}^n$ with nonvanishing curvature bounded above by K , parameterized by a C^2 function $\gamma : [0, \eta] \rightarrow \mathbb{R}^n$ with nonvanishing derivative. How large can the Hausdorff dimension of a subset $E \subseteq [0, \eta]$ be if there do not exist three points $x_1, x_2, x_3 \in E$ such that $\{\gamma(x_1), \gamma(x_2), \gamma(x_3)\} \subseteq \Gamma$ are the vertices of an isosceles triangle?

The existence of an isosceles triangle with vertices on Γ will be determined using one of the functions

$$f_1(t_1, t_2, t_3) = |\gamma(t_1) - \gamma(t_2)|^2 - |\gamma(t_2) - \gamma(t_3)|^2, \tag{2-1}$$

$$f_2(t_1, t_2, t_3) = d(\gamma(t_1), \gamma(t_2)) - d(\gamma(t_2), \gamma(t_3)). \tag{2-2}$$

Here d is the “signed distance” along the curve Γ defined by

$$d(\gamma(t_1), \gamma(t_2)) = \begin{cases} |\gamma(t_1) - \gamma(t_2)| & \text{if } t_1 \geq t_2, \\ -|\gamma(t_1) - \gamma(t_2)| & \text{if } t_1 < t_2. \end{cases} \tag{2-3}$$

For reasons to be explained shortly, we will want to avoid the zero set of f_1 or f_2 . In order to apply Theorem 1.1, we need to verify that these functions are differentiable. This is evident for f_1 . In Lemma A.1 of the Appendix, we have shown that the signed distance d is differentiable, which provides the same conclusion for f_2 .

Let f be either the function f_1 or f_2 given in (2-1) or (2-2). In either case, we have that if $f(t_1, t_2, t_3) = 0$, then $\gamma(t_1), \gamma(t_2), \gamma(t_3)$ form the vertices of an isosceles triangle or points in an arithmetic progression. Conversely, let x, y, z be distinct points of Γ that form an isosceles triangle, with $|x - y| = |y - z|$. Then there exist $t_1 < t_2 < t_3$ such that some permutation of $\gamma(t_1), \gamma(t_2), \gamma(t_3)$ will be the points x, y, z . It is not difficult to see that if η is sufficiently small depending on $|\gamma'(0)|$ and the curvature K , then y can be neither $\gamma(t_1)$ nor $\gamma(t_3)$. We include a proof of this in Lemma A.2 in the Appendix. Therefore $y = \gamma(t_2)$, in which case $f(t_1, t_2, t_3) = 0$.

2A1. A set avoiding isosceles triangles along a single curve. We will first discuss the problem of avoiding isosceles triangles along a single curve Γ . For this variant of the problem, γ may be any parameterization of Γ satisfying the conditions laid out above.

Let us first consider the case where Γ is parameterized by a polynomial function γ of degree d with rational coefficients; i.e., $\gamma(t) = (p_1(t), p_2(t), \dots, p_n(t))$. Let us observe that the result in [Máthé 2012]

does not apply to the nonpolynomial function $f_2(t_1, t_2, t_3)$, but does apply to

$$f_1(t_1, t_2, t_3) = [(p_1(t_1) - p_1(t_2))^2 + \cdots + (p_n(t_1) - p_n(t_2))^2] - [(p_1(t_2) - p_1(t_3))^2 + \cdots + (p_n(t_2) - p_n(t_3))^2],$$

which is a polynomial of degree at most $2d$. Applying [Máthé 2012] then gives a subset of Γ of Hausdorff dimension $1/(2d)$ that does not contain the vertices of any isosceles triangle.

If Γ is a general (not necessarily polynomial) C^2 curve with parameterization $\gamma(t)$, and $f(t_1, t_2, t_3)$ is either f_1 or f_2 described above, then Theorem 1.1 demonstrates the existence of a subset E of $[0, 1]$ of Hausdorff dimension $\frac{1}{2}$ such that $f(t_1, t_2, t_3) \neq 0$ for any choice of $t_1, t_2, t_3 \in E$. Under γ , this lifts to a subset of Γ of Hausdorff dimension $\frac{1}{2}$ that does not contain the vertices of an isosceles triangle. Even for the case of functions with a rational polynomial parameterization, this set has a larger Hausdorff dimension than the one provided by [Máthé 2012].

Incidentally, it is instructive to compare the above with the case where the curve γ is a line, even though the curvature for the latter is zero. Here we will view three-term arithmetic progressions as degenerate isosceles triangles. Set $\gamma(t) = at + b$ for some $a, b \in \mathbb{R}^n$, $a \neq 0$. Then the function $f(t_1, t_2, t_3) = t_1 + t_3 - 2t_2$ is equal to zero precisely when $\gamma(t_1)$, $\gamma(t_2)$ and $\gamma(t_3)$ lie in arithmetic progression. Keleti's result [1999], as well as [Máthé 2012], applied to this f show that there is a subset of Γ of Hausdorff dimension 1 that does not contain any arithmetic progressions. Theorem 1.1 on the other hand provides a set with Hausdorff dimension $\frac{1}{2}$, which is suboptimal.

2A2. *A set avoiding isosceles triangles along all curves with bounded curvature.* We will also ask a question related to the one above, this time considering only C^2 curves given by arclength parameterization. How large a set $E \subset [0, 1]$ can we construct such that $\gamma(E)$ does not contain any isosceles triangle for any $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ with $|\gamma'(t)| \equiv 1$ and with curvature at most K ?

For any such curve γ , the function f_2 defined in (2-2) will be differentiable everywhere, with $\partial f_2 / \partial t_1 = \partial f_2 / \partial t_3 \equiv 1$ and $\partial f_2 / \partial t_2 \equiv -2$, as we have verified in Lemma A.1(b). Thus the function f_2 will satisfy the conditions of Theorem 1.3. One therefore obtains a subset $E \subset [0, 1]$ of positive Hausdorff dimension such that $f_2(t_1, t_2, t_3) \neq 0$ whenever $t_1, t_2, t_3 \in E$ are distinct, no matter which γ we choose in this class. Thus the points parameterized by E manage to avoid isosceles triangles on all curves Γ with a fixed bounded curvature.

How large a Hausdorff dimension can we get? A careful scrutiny of Lemma 5.1, Proposition 5.2 and Theorem 1.3 shows that one can ensure sets of Hausdorff dimension at least $\log 2 / \log 3$. For more details, we refer the reader to the proofs of these results in Section 5A and the remarks following them.

2A3. *Discussion on optimality.* Clearly Theorem 1.2 is optimal when $m = n(v - 1)$. On the other hand, we can use Theorem 1.3 together with the example above to give a polynomial with rational coefficients for which neither [Máthé 2012] nor Theorem 1.1 gives the optimal bounds. Consider a polynomial of the form

$$p(t_1, t_2, t_3) = t_1 - 2t_2 + t_3 + q(t_1, t_2, t_3),$$

where $q(t_1, t_2, t_3)$ is a nontrivial homogeneous quadratic polynomial in $(t_2 - t_1)$ and $(t_3 - t_1)$ with rational coefficients. We are of course interested in finding a set E (as large as possible) such that $p(t_1, t_2, t_3) \neq 0$

for any choice of distinct points $t_1, t_2, t_3 \in E$. Both [Máthé 2012] and Theorem 1.1 provide such a set E , with dimension at least $\frac{1}{2}$ in both cases. Theorem 1.3 provides such a set E as well. Note that p has the same linearization as the functions f described in the previous section above. Hence, as described at the end of Section 2A2, the set E obtained via Theorem 1.3 is a set of dimension at least $\log 2 / \log 3 > \frac{1}{2}$, proving the claimed suboptimality statement.

In fact, we can use this framework to construct other examples. Notice that it is possible to ask for sets E that avoid triangles that are not necessarily isosceles, for instance triangles where the side-length ratio is a prescribed constant κ . The results in [Keleti 1999; Máthé 2012] and Theorem 1.1 all apply to give a set with the same Hausdorff dimension $\frac{1}{2}$ as above not containing t_1, t_2, t_3 such that $|\gamma(t_2) - \gamma(t_1)| = \kappa |\gamma(t_3) - \gamma(t_1)|$. However, the Hausdorff dimension bound in Theorem 1.3 becomes worse as κ moves farther away from 1. Still, for κ close to 1, Theorem 1.3 outperforms Theorem 1.1, giving rise to a family of polynomials whose zeros can be avoided by a set of unusually large Hausdorff dimension.

2B. A subset of a curve not containing certain kinds of trapezoids. The following is a geometric example of Theorem 1.2. Call a trapezoid $ABCD$ with AD parallel to BC “special” if the side lengths obey the restriction $|BC|^2 = |AB||CD|$. Given a curve $\Gamma \subset \mathbb{R}^2$ parameterized by a smooth function $\gamma : [0, \eta] \rightarrow \mathbb{R}^2$, we aim to find a subset E of $[0, \eta]$ with the following property: for any choice of $t_1 < t_2 < t_3 < t_4$ in E , the trapezoid $ABCD$ with

$$A = \gamma(t_1), \quad B = \gamma(t_2), \quad C = \gamma(t_3), \quad D = \gamma(t_4)$$

is not special. For simplicity and ease of exposition, we may assume that the components of γ' are strictly positive on $[0, \eta]$ and that the curvature is also of constant sign, say Γ is strictly convex.

Notice that the special trapezoid assumption places two essentially independent conditions on $\gamma(t_1), \gamma(t_2), \gamma(t_3)$, and $\gamma(t_4)$. One is that two sides need to be parallel, and the other is the condition on the side lengths. Accordingly, we define two functions f_1 and f_2 as follows:

$$f_1(t_1, t_2, t_3, t_4) = \det[(\gamma(t_4) - \gamma(t_1))^t, (\gamma(t_3) - \gamma(t_2))^t], \tag{2-4}$$

$$f_2(t_1, t_2, t_3, t_4) = d(\gamma(t_4), \gamma(t_3)) d(\gamma(t_2), \gamma(t_1)) - d(\gamma(t_3), \gamma(t_2))^2. \tag{2-5}$$

Here in (2-4), a^t and b^t represent the transpose of the planar row vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ respectively, while

$$\det[a^t, b^t] = \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1$$

denotes the (signed) length of the cross-product $a \times b$. Alternatively, $\det[a^t, b^t]$ may be interpreted as the signed area of the parallelogram whose sides are the vectors a and b . The determinant vanishes if either a or b is zero, or if the two vectors are parallel.

Returning to (2-4) and (2-5), f_1 is zero if and only if AD is parallel to BC , while f_2 is zero if and only if $|BC|^2 = |AB||CD|$. We therefore seek to avoid the zeros of the smooth vector-valued function $f = (f_1, f_2)$. We verify in Lemma A.3 of the Appendix that the derivative Df is of full rank on the zero

set of f . Applying Theorem 1.2 with $n = 1$, $m = 2$, $v = 4$, we obtain a set E of Hausdorff dimension $\frac{2}{3}$ such that the points on Γ indexed by E avoids special trapezoids as explained above. Thus, there is a subset of Γ of Hausdorff dimension $\frac{2}{3}$ that does not contain any special trapezoids.

3. Avoidance of zeros on a single scale

The proofs of Theorems 1.1 and 1.2 are based on an iterative construction whose primary building block relies on an algorithm: given a set $T \subseteq \mathbb{R}^{nv}$ contained in the domain of a suitably nonsingular function $f : \mathbb{R}^{nv} \rightarrow \mathbb{R}^m$, one identifies a subset $S \subseteq T$ that stays away from the zero set of f . This zero-avoiding subset S , which is a union of cubes in \mathbb{R}^{nv} (and as such of positive Lebesgue measure and full Hausdorff dimension), does not immediately yield the set we seek because it is typically not the v -fold Cartesian product of a set in \mathbb{R}^n with itself, and hence does not meet the specifications of the theorems. However, the algorithm can be used iteratively on many different scales and for many functions in the construction of the set E whose existence has been asserted in the theorems. Our objective in this section is to describe this algorithm. The versions that we need for Theorems 1.1 and 1.2 are very similar in principle, although the exact statements differ somewhat. These appear in Propositions 3.1 and 3.4 below respectively.

3A. Building block in dimension one. Let f be a real-valued C^1 function of v variables and nonvanishing gradient defined in a neighbourhood of the origin containing $[0, 1]^v$. Suppose that we are given an index $i_0 \in \{1, 2, \dots, v\}$, an integer $M \geq 1$, a small constant $c_0 > 0$ and compact subsets $T_1, \dots, T_v \subseteq [0, 1]$ with the following properties:

$$\begin{aligned} \text{Each } T_i \text{ is a union of closed intervals of length } M^{-1} \text{ with disjoint interiors.} \\ \text{Let us denote by } \mathcal{J}_M(T_i) \text{ this collection of intervals.} \end{aligned} \tag{3-1}$$

$$\text{int}(T_i) \cap \text{int}(T_{i'}) = \emptyset \text{ if } i \neq i'. \tag{3-2}$$

$$\left| \frac{\partial f}{\partial x_{i_0}}(x) \right| \geq c_0 \text{ and } |\nabla f(x)| \leq c_0^{-1} \text{ for all } x \in T_1 \times \dots \times T_v. \tag{3-3}$$

Proposition 3.1. *Given f, M, i_0, c_0 and $\mathbb{T} = (T_1, \dots, T_v)$ obeying (3-1) and (3-3) above, there exist a small rational constant $c_1 > 0$ and an integer N_0 (depending on all these quantities), for which the following conclusions hold.*

There is a sequence of arbitrarily large integers $N \geq N_0$ with $N/M, c_1 N \in \mathbb{N}$ such that for each N in this sequence, one can find compact subsets $S_i \subseteq T_i$ for all $1 \leq i \leq v$ such that:

- (a) *There are no solutions of $f(x) = 0$ with $x \in S_1 \times \dots \times S_v$.*
- (b) *For each $J \in \mathcal{J}_M(T_i)$, let us decompose J into closed intervals of length N^{-1} with disjoint interiors and call the resulting collection of intervals $\mathcal{I}_N(J, i)$. Then for each $i \neq i_0$ and each $I \in \mathcal{I}_N(J, i)$, the set $S_i \cap I$ is an interval of length $c_1 N^{1-v}$.*
- (c) *For every $J \in \mathcal{J}_M(T_{i_0})$, there exists $\mathcal{I}'_N(J, i_0) \subseteq \mathcal{I}_N(J, i_0)$ with*

$$\#(\mathcal{I}'_N(J, i_0)) \geq \left(1 - \frac{1}{M}\right) \#(\mathcal{I}_N(J, i_0)) \tag{3-4}$$

such that for each $I \in \mathcal{I}'_N(J, i_0)$,

$$|\mathcal{S}_{i_0} \cap I| \geq \frac{c_1}{N}. \tag{3-5}$$

Unlike part (b), $\mathcal{S}_{i_0} \cap I$ need not be an interval; however, it can be written as a union of intervals of length $c_1 N^{1-v}$ with disjoint interiors.

Proof. Without loss of generality, we may set $i_0 = v$. For $i \neq v$, we define

$$\mathcal{S}_i = \bigcup \{ [a_i, a_i + c_1 N^{1-v}] : [a_i, b_i] = I \in \mathcal{I}_N(J, i) \text{ for some } J \in \mathcal{J}_M(T_i) \},$$

where the small positive constant c_1 and the integer N will be specified shortly. In other words, \mathcal{S}_i consists of the leftmost $c_1 N^{1-v}$ -subintervals of all the $1/N$ -intervals that constitute T_i . It is clear that the conclusion (b) holds for this choice of \mathcal{S}_i .

We now proceed to define the subcollection $\mathcal{I}'_N(J, v)$ and the set \mathcal{S}_v that obey the requirements in (c). Consider the collection

$$\mathbb{A}_N := \prod_{i=1}^{v-1} \{ a_i : [a_i, b_i] = I \in \mathcal{I}_N(J, i) \text{ for some } J \in \mathcal{J}_M(T_i) \}$$

consisting of $(v-1)$ -tuples of the form $a' = (a_1, \dots, a_{v-1})$, where each a_i is a left endpoint of an interval in $\mathcal{I}_N(J, i)$ for some $J \in \mathcal{J}_M(T_i)$. For each i , the number of possible choices for $1/N$ -intervals $I \subseteq [0, 1]$ and hence for a_i is at most N . Thus

$$\#(\mathbb{A}_N) \leq N^{v-1}. \tag{3-6}$$

We will prove in Lemma 3.2 below that for every fixed $a' \in \mathbb{A}_N$,

$$\#\{x_v : f(a', x_v) = 0\} \leq M. \tag{3-7}$$

Assuming this for the moment, define

$$\mathbb{B} := \{x_v : \exists a' \in \mathbb{A}_N \text{ such that } f(a', x_v) = 0\}.$$

In light of (3-6) and (3-7), we find that

$$\#(\mathbb{B}) \leq MN^{v-1}. \tag{3-8}$$

The subcollection $\mathcal{I}'_N(J, v) \subseteq \mathcal{I}_N(J, v)$ specified in part (c) is chosen as follows: we declare

$$I \in \mathcal{I}'_N(J, v) \quad \text{if} \quad \#(\mathbb{B} \cap I) \leq M^3 N^{v-2}.$$

In view of (3-8) and the pigeonhole principle, it follows that

$$\#(\mathcal{I}_N(J, v) \setminus \mathcal{I}'_N(J, v)) \leq \frac{MN^{v-1}}{M^3 N^{v-2}} = \frac{N}{M^2}. \tag{3-9}$$

The fact that $\#(\mathcal{I}_N(J, v)) = N/M$ then implies (3-4).

We now decompose each $I \in \mathcal{I}'_N(J, v)$ into consecutive subintervals of length $C_0 c_1 / N^{v-1}$ with disjoint interiors, and denote the successive intervals by $\tilde{I}_\ell(I)$:

$$I = \bigcup \{ \tilde{I}_\ell(I) : 1 \leq \ell \leq N^{v-2} / (C_0 c_1) \}.$$

Here C_0 is a constant integer depending on f, M and T_1, \dots, T_v , as has been specified in Lemma 3.3 below. The integer N is chosen large enough so that $N^{v-2}/(C_0c_1)$ is an integer. All intervals $\tilde{I}_\ell(I)$ that intersect \mathbb{B} , together with their adjacent neighbours, are then discarded. This still leaves open the possibility that the subintervals $\tilde{I}_\ell(I)$ at the edges of I , namely $\ell = 1$ and $\ell = N^{v-2}/(C_0c_1)$, are proximate to a part of \mathbb{B} lying in an adjacent interval I' , so we remove these edge subintervals as well. The remaining subset of T_v is defined to be S_v . More specifically,

$$S_v = \bigcup \left\{ \tilde{I}_\ell(I) : \tilde{I}_k(I) \cap \mathbb{B} = \emptyset \text{ for } \begin{array}{l} I \in \mathcal{I}'_N(J, v), J \in \mathcal{J}_M(T_v), \\ |k - \ell| \leq 1, 1 < \ell < N^{v-2}/(C_0c_1) \end{array} \right\}.$$

Clearly S_v can be viewed a union of intervals of length c_1/N^{v-1} . The definition of $\mathcal{I}'_N(J, v)$ implies that the total length of the discarded subintervals in each $I \in \mathcal{I}'_N(J, v)$ is at most $3C_0c_1M^3N^{v-2}/N^{v-1} = 3M^3C_0c_1/N$. The claim (3-5) now follows by choosing $c_1 > 0$ small enough so as to satisfy $3M^3C_0c_1 < (1 - c_1)$.

Finally, Lemma 3.3 below shows that given $x' = (x_1, \dots, x_{v-1}) \in S_1 \times S_2 \times \dots \times S_{v-1}$, any x_v obeying $f(x', x_v) = 0$ should necessarily lie within a C_0c_1/N^{v-1} neighbourhood of \mathbb{B} . Since the set $S_v \subseteq T_v$ was created so as to avoid these neighbourhoods, conclusion (b) follows. \square

Remark. We take this opportunity to point out the distinction of our selection algorithm as compared to, say, [Keleti 1999; Máthé 2012]. The length c_1N^{1-v} of the intervals $S_i \cap I$ (for $i \neq i_0$) is the main contributing factor to the dimensional lower bound of Theorem 1.1. These intervals can be chosen slightly differently and also possibly longer if additional information is available about the zero set of f , as indicated in part (c) of the remark on page 1085.

For example, suppose that $f : \mathbb{R}^v \rightarrow \mathbb{R}$ is a linear function, say

$$f(x_1, \dots, x_v) = \sum_{i=1}^v \alpha_i x_i \tag{3-10}$$

with nonzero integer coefficients as in [Keleti 1999], and that $i_0 = v$. Without loss of generality suppose also that in the notation of Proposition 3.1 each T_i is a finite union of intervals J of the form $\mathbb{Z}/M + [0, 1/M]$. Then for $i < v$, a possible choice of S_i could be as follows: for each $I = [k/N, (k + 1)/N] \in \mathcal{I}_N(J, i)$ with k any integer, we set

$$S_i \cap I := \left[\frac{k}{N}, \frac{k + c_1}{N} \right]$$

for some small positive constant c_1 to be chosen shortly. If $x_i \in S_i$ for $i < v$, then any x_v with $f(x_1, x_2, \dots, x_v) = 0$ has to be of the form

$$x_v = -\frac{1}{\alpha_v} \sum_{i=1}^{v-1} \alpha_i x_i, \quad \text{so that } \text{dist}\left(x_v, \frac{\mathbb{Z}}{|\alpha_v|N}\right) < \frac{1}{4|\alpha_v|N}, \quad \text{where } c_1 \sum_{i=1}^{v-1} |\alpha_i| < \frac{1}{4}.$$

Let us then choose S_v as follows: for any $I = |\alpha_v|^{-1}[k/N, (k + 1)/N] \subseteq T_v$,

$$S_v \cap I := \frac{1}{|\alpha_v|} \left[\frac{k}{N} + \frac{1 - c_1}{2N}, \frac{k}{N} + \frac{1 + c_1}{2N} \right].$$

This implies

$$\text{dist}\left(S_v \cap I, \frac{\mathbb{Z}}{|\alpha_v|N}\right) > \frac{1}{4|\alpha_v|N},$$

provided $c_1 > 0$ is chosen small enough. Thus the construction above ensures that $S_1 \times S_2 \cdots \times S_v$ contains no zeros of f . Further, the size of this Cartesian product is significantly larger than the one obtained in Proposition 3.1. Tracking these new choices of S_i through the rest of the proof yields a set E of full Hausdorff dimension that avoids all zeros of (3-10), which is the result of [Keleti 1999].

Lemma 3.2. *For f and \mathbb{A}_N as in Proposition 3.1, the inequality (3-7) holds for every fixed $a' \in \mathbb{A}_N$.*

Proof. Given $a' \in \mathbb{A}_N$, we claim that for every $J \in \mathcal{J}_M(T_v)$, there exists at most one $x_v \in J$ such that $f(a', x_v) = 0$. Since the number of $J \in \mathcal{J}_M(T_v)$ is at most M , the desired conclusion would follow once the claim is established.

To prove the claim, let us assume if possible that there exist $x_v, y_v \in J$, $x_v \neq y_v$, such that $f(a', x_v) = f(a', y_v) = 0$. By Rolle's theorem, this ensures the existence of some point $z_v \in J$ where $\partial f / \partial x_v(a', x_v) = 0$. But this contradicts the hypothesis (3-3) that the partial derivative $\partial f / \partial x_v$ is nonzero on $T_1 \times \cdots \times T_v$. \square

Lemma 3.3. *Let f, M and T_1, \dots, T_v be as in Proposition 3.1. Then there exists a constant C_0 depending on these quantities, and in particular on c_0 , such that for the choice of S_1, S_2, \dots, S_{v-1} as specified in the proof of the proposition,*

$$\text{dist}(x_v, \mathbb{B}) \leq \frac{C_0 c_1}{N^{v-1}}$$

for any x_v obeying $f(x) = 0$, with $x' = (x_1, \dots, x_{v-1}) \in S_1 \times \cdots \times S_{v-1}$.

Proof. Let $\mathbb{J} = J_1 \times \cdots \times J_v = \mathbb{J}' \times J_v \in \prod_{i=1}^v \mathcal{J}_M(T_i)$ be a v -dimensional cube of side length $1/M$ such that the zero set of f intersects \mathbb{J} . The nonvanishing derivative condition (3-3) then implies, in view of the implicit function theorem, that there exists a $(v-1)$ -variate C^1 function $g_{\mathbb{J}}$ defined on \mathbb{J}' and a constant $C_0 > 0$ depending on c_0, M, T_1, \dots, T_v such that

$$f(x) = 0, x \in \mathbb{J}, \quad \text{implies} \quad x_v = g_{\mathbb{J}}(x'), x' \in \mathbb{J}', \tag{3-11}$$

$$|\nabla g_{\mathbb{J}}| \leq \frac{C_0}{\sqrt{v}} \quad \text{on } \mathbb{J}'. \tag{3-12}$$

Given $x = (x', x_v) \in S_1 \times \cdots \times S_v$ such that $f(x) = 0$, let \mathbb{J}_x denote the v -dimensional $1/M$ -cube \mathbb{J} in which x lies, and let $\mathbb{J}'_x = I_1 \times \cdots \times I_{v-1} = \prod_{i=1}^{v-1} [a_i, b_i] \in \prod_{i=1}^{v-1} \mathcal{I}_N(J_i, i)$ be the $(v-1)$ -dimensional subcube of \mathbb{J}'_x of side length $1/N$ containing x' . Then

$$x_v = g_{\mathbb{J}}(x'), \quad a' = (a_1, \dots, a_{v-1}) \in \mathbb{A}_N, \quad g_{\mathbb{J}}(a') \in \mathbb{B}, \quad \text{and} \quad |x' - a'| \leq \frac{c_1 \sqrt{v}}{N^{v-1}}.$$

Further, (3-12) implies

$$\text{dist}(x_v, \mathbb{B}) \leq |g_{\mathbb{J}}(a') - g_{\mathbb{J}}(x')| \leq \|\nabla g_{\mathbb{J}}\|_{\infty} |x' - a'| \leq \frac{C_0}{\sqrt{v}} \times \frac{c_1 \sqrt{v}}{N^{v-1}} = \frac{C_0 c_1}{N^{v-1}},$$

which is the conclusion of the lemma. \square

3B. Building block in higher dimensions. Given positive integers $m, n \geq 1$ and $v \geq 3$ with $m \leq n(v - 1)$, let $f : \mathbb{R}^{nv} \rightarrow \mathbb{R}^m$ be a C^2 function whose zero set has nontrivial intersection with $[0, 1]^{nv}$. Suppose that $M \geq M_0$ is a large integer, $c_0 > 0$ is a small constant and $T_1, \dots, T_v \subseteq [0, 1]^n$ are sets with the following properties:

$$\begin{aligned} \text{Each } T_i \text{ is expressible as a union of closed axis-parallel cubes of side length } M^{-1} \\ \text{with disjoint interiors, the collection of which will be called } \mathcal{J}_M(T_i). \end{aligned} \tag{3-13}$$

As before, $\text{int}(T_i) \cap \text{int}(T_{i'}) = \emptyset$ if $i \neq i'$.

$$\begin{aligned} \text{On } \{x \in T_1 \times \dots \times T_v : f(x) = 0\} \text{ the matrix } Df \text{ is of full rank, with} \\ \text{the singular values of } Df \text{ bounded above and below by } c_0^{-1} \text{ and } c_0 \text{ respectively.} \end{aligned} \tag{3-14}$$

$$\text{On } [0, 1]^{nv}, \text{ the matrix norm of the Hessian } D^2 f \text{ is bounded above by } c_0^{-1}. \tag{3-15}$$

Proposition 3.4. *Given f, M and c_0 as above, there exists a rational constant $c_1 > 0$ and an integer N_0 depending on these quantities for which the following conclusions hold. For $N \geq N_0$, set $\ell = c_1 N^{n(1-v)/m}$. If N is such that $N/M, 1/(\ell N) \in \mathbb{Z}$, then one can find compact subsets $S_i \subseteq T_i$ for all $1 \leq i \leq v$ such that:*

- (a) *There are no solutions to $f(x) = 0$ with $x \in S_1 \times \dots \times S_v$.*
- (b) *For each $1 \leq i \leq v$ and $J \in \mathcal{J}_M(T_i)$, let us decompose J into closed axis-parallel cubes of length N^{-1} with disjoint interiors and call the resulting collection of cubes $\mathcal{I}_N(J, i)$. There exists $\mathcal{I}'_N(J, i) \subseteq \mathcal{I}_N(J, i)$ such that*

$$S_i \subseteq \bigcup \{I : J \in \mathcal{J}_M(T_i), I \in \mathcal{I}'_N(J, i)\}.$$

More precisely, for each $I \in \mathcal{I}'_N(J, i)$, the set $S_i \cap I$ is a single axis-parallel cube of side length $\ell = c_1 N^{n(1-v)/m}$, provided $i \neq v$. For $i = v$ and $I \in \mathcal{I}'_N(J, v)$, the set $S_v \cap I$ is not necessarily a single cube of side length ℓ , but a union of such cubes, with the property that

$$|S_v \cap I| \geq \left(1 - \frac{1}{M}\right) \frac{1}{N^n}. \tag{3-16}$$

- (c) *The subcollections $\mathcal{I}'_N(J, i)$ of cubes are large subsets of the ambient collection $\mathcal{I}_N(J, i)$, in the sense that for all $1 \leq i \leq v, J \in \mathcal{J}_M(T_i)$,*

$$\#(\mathcal{I}'_N(J, i)) \geq \left(1 - \frac{1}{M}\right) \#(\mathcal{I}_N(J, i)). \tag{3-17}$$

Remarks. (a) The proof will show that the constant c_1 in Proposition 3.4 may be chosen as a small constant multiple of M^{-R} , where $R = [(n + 1)v + 1]/m$. For the purposes of application, M is negligible compared to N , and hence the specific power of M that appears in the expression for ℓ is not critical to the proof. The power of N , which is $-(n/m)(v - 1)$, is of utmost importance and the principal reason that the Hausdorff dimension of the set $E \subseteq \mathbb{R}^n$ in Theorem 1.2 is equal to $m/(v - 1)$.

(b) The restriction $m \leq n(v - 1)$ justifies on one hand the dimensional constraint on the set E which lies in \mathbb{R}^n . On a technical note, it is also necessary for the assumption $\ell \ll N^{-1}$ that permeates the proof. If $m < n(v - 1)$, the chosen value of $\ell = c_0 M^{-R} N^{-n(v-1)/m}$ will be less than $1/N$ if N is sufficiently large. If $m = n(v - 1)$, the chosen value of ℓ will be less than $1/N$ provided that M is sufficiently large.

(c) The special treatment of the variable x_v in the proposition is for convenience only. The result holds for x_v replaced by x_{i_0} for any $1 \leq i_0 \leq v$.

Proof. Let $Z_f = \{x = (x_1, \dots, x_v) \in ([0, 1]^n)^v : f(x) = 0\}$ be the zero set of the function f , which we wish to avoid. The assumptions (3-14) and (3-15) ensure that $Z_f \cap (T_1 \times \dots \times T_v)$ is an $(nv-m)$ -dimensional submanifold of $[0, 1]^{nv}$; see for example [Sharpe 1997, Theorem 2.13]. Further, the coarea formula gives that Z_f is coverable by at most $C\epsilon^{m-nv}$ many cubes of side length ϵ , for all sufficiently small ϵ . Here C is a large constant depending only on c_0 and independent of ϵ . The proof consists of projecting Z_f successively onto the coordinates x_1, x_2, \dots and selecting the sets S_i so as to avoid the projected zero sets. The main ingredient of this argument is described in Lemma 3.5. We ask the reader to view the statement of the lemma first. Assuming the lemma, the remainder of the proof proceeds as follows.

Fix a parameter $\ell \ll 1/N$ soon to be specified. Recalling that $\mathcal{I}_{\alpha^{-1}}(J, i)$ denotes the collection of axis-parallel subcubes of side length α that constitute a partition of $J \in \mathcal{J}_M(T_i)$, let us define the collection of “bad boxes” \mathbb{B}_1 as

$$\mathbb{B}_1 = \left\{ Q \in \prod_{i=1}^v \mathcal{I}_{\ell^{-1}}(J_i, i) : Q \cap Z_f \neq \emptyset \text{ for some } J_i \in \mathcal{J}_M(T_i) \right\}. \tag{3-18}$$

In other words, a box of side length ℓ in $T_1 \times \dots \times T_v$ is considered bad if it contains a point in the zero set of the function f . The discussion in the preceding paragraph shows that

$$\#(\mathbb{B}_1) \leq C\ell^{m-nv}, \tag{3-19}$$

where C is a constant that depends only on the function f and the value c_0 .

The construction of S_1, \dots, S_v now proceeds as follows. At the first step, we project the boxes in \mathbb{B}_1 onto their (x_2, \dots, x_v) -coordinates (each n -dimensional), and use Lemma 3.5 below with $r = v$, $T = T_1$, $T' = T_2 \times \dots \times T_v$ and $\mathbb{B} = \mathbb{B}_1$ to arrive at a set $S_1 \subseteq T_1$ and a family of $n(v-1)$ -dimensional boxes $\mathbb{B}' = \mathbb{B}_2$ obeying the conclusions of that lemma. Clearly the set S_1 obeys the requirements of part (b) of the proposition. Lemma 3.5 also ensures that

$$\#(\mathbb{B}_2) \leq M^{n+1} N^n \ell^n \#(\mathbb{B}_1) \leq CM^{n+1} N^n \ell^{m-n(v-1)},$$

and that $f(x) \neq 0$ for any $x = (x_1, x')$ such that $x_1 \in S_1$ and any $x' \in T_2 \times \dots \times T_v$ that is not contained in the cubes constituting \mathbb{B}_2 .

We now inductively follow a procedure similar to the above. At the end of step j , we will have selected sets $S_1 \subseteq T_1, \dots, S_j \subseteq T_j$ and will be left with a family \mathbb{B}_{j+1} of $n(v-j)$ -dimensional cubes of side length ℓ , such that

$$\#(\mathbb{B}_{j+1}) \leq CM^{(n+1)j} N^{jn} \ell^{m-n(v-j)} \tag{3-20}$$

and

$$f(x'', x') \neq 0 \text{ for } x'' = (x_1, \dots, x_j) \in \prod_{i=1}^j S_i, x' \in \prod_{i=j+1}^v T_i, x' \text{ not contained in any of the cubes in } \mathbb{B}_{j+1}. \tag{3-21}$$

We can then apply Lemma 3.5 with

$$T = T_{j+1}, \quad T' = T_{j+2} \times \dots \times T_v, \quad \mathbb{B} = \mathbb{B}_{j+1},$$

to arrive at a set $S_{j+1} \subseteq T_{j+1}$ meeting the requirement of part (b) of the proposition. The lemma also gives a family $\mathbb{B}' = \mathbb{B}_{j+2}$ of $n(v-j-1)$ -dimensional cubes of side length ℓ , whose cardinality obeys the inequality (3-20) with j replaced by $j + 1$, allowing us to carry the induction forward.

We continue this construction for $v - 1$ steps, obtaining sets S_1, \dots, S_{v-1} and a collection \mathbb{B}_v consisting of at most $CM^{(n+1)(v-1)}N^{n(v-1)}\ell^{m-n}$ cubes of side length ℓ and dimension n contained in T_v . The set S_v is then defined according to the prescription of Lemma 3.6, the conclusion of which verifies part (a) of the proposition for S_1, \dots, S_v . □

3B1. Projections of bad boxes. It remains to justify the projection mechanism used repeatedly in Proposition 3.4. We set this up below.

Fix $2 \leq r \leq v$, and consider sets $T \subseteq [0, 1]^n$ and $T' \subseteq [0, 1]^{n(r-1)}$ expressible as unions of closed axis-parallel cubes of side length M^{-1} and disjoint interiors. As before, we denote by $\mathcal{J}_M(T)$ and $\mathcal{J}_M(T')$ the respective collections of these cubes. Given any $J \in \mathcal{J}_M(T)$, we decompose J into axis-parallel subcubes of side length N^{-1} ; the corresponding collection is termed $\mathcal{I}_N(J)$. We will also need to fix a subset $B \subseteq T \times T'$, which we view as a union of a collection \mathbb{B} of cubes of side length ℓ . Here M, N and ℓ are as specified in Proposition 3.4. Since ℓ/N is taken to be an integer, we may assume that each cube in \mathbb{B} is contained in exactly one cube in $\mathcal{I}_N(J)$.

Lemma 3.5. *Given T, T', B as above, there exist sets $S \subseteq T, B' \subseteq T'$ and a collection of boxes $\mathbb{B}' \subseteq \mathbb{B}$ with the following properties:*

- (a) *The set S is a union of closed axis-parallel cubes with side length ℓ and disjoint interiors. More precisely, for every $J \in \mathcal{J}_M(T)$, there exists $\mathcal{I}'_N(J) \subseteq \mathcal{I}_N(J)$ such that*

$$\#(\mathcal{I}'_N(J)) \geq (1 - M^{-1})\#(\mathcal{I}_N(J)),$$

and $S \cap I$ is a single ℓ -cube for each $I \in \mathcal{I}'_N(J)$. For $I \in \mathcal{I}_N(J) \setminus \mathcal{I}'_N(J)$, the interior of the set $S \cap I$ is empty.

- (b) *The set B' is the union of the ℓ -cubes in \mathbb{B}' .*
- (c) $\#(\mathbb{B}') \leq M^{n+1}N^n\ell^n\#(\mathbb{B})$.
- (d) $(S \times T') \cap B \subseteq S \times B'$.

Proof. Fix $J \in \mathcal{J}_M(T)$. For $I \in \mathcal{I}_N(J)$, define a “slab”

$$W_N[I] := \bigcup \{Q = I \times I' \subseteq T \times T' : Q \text{ is a cube of side length } N^{-1}\}.$$

Thus a slab is the union of all of the axis-parallel boxes in $T \times T'$ of side length $1/N$ whose projection onto the x_1 -coordinate is the cube I . Similarly, given an n -dimensional cube I of side length ℓ , we define a “wafer” $W_{\ell^{-1}}[I]$ to be the union of all cubes of side length ℓ that project onto I in the x_1 -space. Let us observe that a slab is an essentially disjoint union of exactly $N^{-n}\ell^{-n}$ wafers, and that the total number of wafers supported by J is $M^{-n}\ell^{-n}$. A wafer in turn is a union of ℓ -cubes.

Let us agree to call a wafer $W_{\ell^{-1}}[I]$ “good” if it contains at most $M^{n+1}\ell^n\#(\mathbb{B})$ boxes of \mathbb{B} . The pigeonhole principle dictates that the proportion of bad wafers is $\leq 1/M$. We will call a slab $W_N[I]$

“good” if it contains at least one good wafer. Again pigeonholing implies that no more than a $1/M$ -fraction of the slabs can be bad. Let us define $\mathcal{I}'_N(J)$ as the collection all cubes $I \in \mathcal{I}_N(J)$ such that $W_N[I]$ is good. For each cube $I \in \mathcal{I}'_N(J)$, we select one cube $I_0 = I_0(I) \subset I$ of side length ℓ such that $W_{\ell^{-1}}[I_0]$ is a good wafer. The set S is now defined to be the union of all selected ℓ -cubes $I_0(I)$, with $I \in \mathcal{I}'_N(J)$ and $J \in \mathcal{J}_M(T)$. Clearly, S satisfies part (a) of the lemma.

Let B' be the union of the collection \mathbb{B}' of all ℓ -cubes $Q' \subseteq T'$ such that $Q \times Q' \in \mathbb{B}$ for some ℓ -cube $Q \subseteq S$. Then (b) and (d) hold by definition. The selection algorithm for S gives that for a given cube $Q \subseteq S$, the number of Q' such that $Q \times Q' \in \mathbb{B}$ is $\leq M^{n+1} \ell^n \#(\mathbb{B})$. On the other hand, each $Q \subseteq S$ comes from a distinct slab. Hence the total number of possible choices for $Q \subseteq S$ is no more than the total number of slabs, which is bounded above by N^n . Combining all of this we get (c) as desired. \square

A version of the lemma above is needed for the extreme case $r = 1$. We needed this in the final step of the iterative process described in Proposition 3.4, specifically in the construction of S_v .

Lemma 3.6. *Fix parameters $\ell \ll N^{-1} \ll M^{-1}$. Let $T \subseteq [0, 1]^n$ be a union of closed axis-parallel cubes with side length M^{-1} and disjoint interiors. Let $B \subseteq T$ be a union of similar cubes with side length ℓ . Decompose T into similar axis-parallel cubes of side length N^{-1} , denoting the corresponding collection by \mathbb{T} . The collection of ℓ -cubes composing B is termed \mathbb{B} . Suppose that*

$$\#(\mathbb{B}) \leq CM^{(n+1)(v-1)} N^{n(v-1)} \ell^{m-n},$$

with

$$\ell \leq C^{-1/m} M^{-(1/m)((n+1)v+1)} N^{-n(v-1)/m}.$$

Then there exist $S \subseteq T$ and $\mathbb{T}^* \subseteq \mathbb{T}$ such that

- (a) $S \cap B = \emptyset$,
- (b) $\#(\mathbb{T}^*) \geq (1 - 1/M)\#(\mathbb{T})$,
- (c) S is a union of a large number of ℓ -cubes coming from \mathbb{T}^* . More precisely, $|S \cap I| \geq (1 - M^{-1})N^{-n}$ for each $I \in \mathbb{T}^*$.

Proof. Decomposing each cube $I \in \mathbb{T}$ into subcubes of side length ℓ , we declare I to be good if it contains $\leq M^{n+1} N^{-n} \#(\mathbb{B})$ subcubes that are in \mathbb{B} . As in the proof of Lemma 3.5, the pigeonhole principle ensures that the fraction of bad cubes in \mathbb{T} is at most M^{-1} . Define \mathbb{T}^* to be the collection of good cubes in \mathbb{T} , and S to be the union of all subcubes of side length ℓ that are contained in the cubes of \mathbb{T}^* but which are disjoint from B . The relation between ℓ , M and N implies that for every $I \in \mathbb{T}^*$,

$$|I \cap B| \leq M^{n+1} N^{-n} \#(\mathbb{B}) \ell^n \leq CM^{(n+1)v} N^{n(v-2)} \ell^m \leq M^{-1} N^{-n},$$

which justifies the size conclusion for S . \square

4. Proof of Theorems 1.1 and 1.2

We present the construction of the set E in Theorem 1.1 in complete detail. The construction for Theorem 1.2 is similar. The small variations needed for this have been discussed in Section 4C.

4A. A sequence of differential operators. We will need to define a sequence of privileged derivatives in order to prove Theorem 1.1. For η and r_q as in the statement of Theorem 1.1, let α_q be a v -dimensional multi-index with $|\alpha_q| = r_q$ such that $\partial^{\alpha_q} f_q / \partial x^{\alpha_q}$ is nonvanishing everywhere on $[0, \eta]$. Here $\partial^\beta / \partial x^\beta$ denotes, following standard convention, the differential operator $\partial^{\beta_1 + \dots + \beta_v} / \partial x_1^{\beta_1} \dots \partial x_v^{\beta_v}$ of order $|\beta| = \beta_1 + \dots + \beta_v$, if $\beta = (\beta_1, \dots, \beta_v)$. We now define for each q a finite sequence of privileged differential operators of diminishing order

$$\mathcal{D}_q^k = \frac{\partial^{\alpha_{qk}}}{\partial x^{\alpha_{qk}}}, \quad 0 \leq k \leq r_q. \tag{4-1}$$

Here $\alpha_{q,r_q} = \alpha_q$, and $\alpha_{q,k-1}$ is obtained by reducing the largest entry of α_{qk} by 1 and leaving the others unchanged. If there are multiple entries of α_{qk} with the largest value, we pick any one. Clearly $|\alpha_{qk}| = k$.

4B. Construction of E . The construction is of Cantor type with a certain memory-retaining feature inspired by the constructions of Keleti [1999; 2008]. This distinctive feature is the existence of an accompanying queue that is, on one hand, generated by the construction and on the other, contributes to it. More precisely, the j -th iteration of the construction is predicated on the j -th member of the queue; at the same time the j -th step also adds a large number of new members to the queue, which become significant at a later stage.

Step 0: At the initializing step, we set for $k = 1, \dots, v$,

$$I_k[0] = \left[(k-1)\frac{\eta}{v}, \frac{k\eta}{v} \right], \quad \mathcal{E}_0 = \{I_1[0], \dots, I_v[0]\}, \quad M_0 = \frac{v}{\eta}.$$

Letting Σ_0 denote the collection of injective mappings from $\{1, \dots, v-1\}$ into $\{1, \dots, v\}$, we define an ordered queue

$$\mathcal{Q}_0 = \{(1, m, \mathbb{I}_\sigma[0]) : 0 \leq m \leq r_1 - 1, \sigma \in \Sigma_0\},$$

where

$$\mathbb{I}_\sigma[0] = (I_{\sigma(1)}[0], \dots, I_{\sigma(v-1)}[0]).$$

The ordering in \mathcal{Q}_0 is as follows: Viewing Σ_0 as a collection of $(v-1)$ -tuples with values from $\{1, \dots, v\}$, we first endow Σ_0 with the lexicographic ordering, writing $\Sigma_0 = \{\sigma_1 < \sigma_2 < \dots\}$. Then $(1, m, \mathbb{I}_{\sigma_r}[0])$ precedes $(1, m', \mathbb{I}_{\sigma_{r'}}[0])$ in the list \mathcal{Q}_0 if one of the following scenarios holds: (a) $r < r'$, no matter what m, m' might be, or (b) $r = r'$ and $m > m'$.

Step 1: Consider the first member of \mathcal{Q}_0 , which is $(1, r_1 - 1, \mathbb{I}_{\sigma_1}[0])$. Recalling the definition (4-1), we proceed to verify the hypotheses of Proposition 3.1, with

$$f = \mathcal{D}_1^{r_1-1} f_1, \quad (T_i : i \neq v) = \mathbb{I}_{\sigma_1}[0], \quad M = M_0.$$

Here $i_0 = i_0(1)$ is the unique index in $\{1, 2, \dots, v\}$ such that $\partial f / \partial x_{i_0} = \mathcal{D}_1^{r_1} f_1$, which is nonzero on $[0, \eta]$. The set T_v will be the complement in $[0, \eta]$ of $\bigcup_i \{T_i : i \neq v\}$. The conclusion of Proposition 3.1 therefore holds for some small constant $d_0 = c_1(M_0, \mathbb{T}) > 0$ and for arbitrarily large integers N_1 obeying the divisibility criteria of the proposition. We choose such an integer N_1 large enough so that $N_1 > e^{M_0/d_0}$.

Proposition 3.1 then ensures the existence of subsets $S_j \subset T_j$ for $1 \leq j \leq v$, each of which is a union of intervals of length $\ell_1 = d_0/N_1^{v-1}$ with

$$\mathcal{D}_1^{r_1-1} f_1(x) \neq 0 \quad \text{for } x = (x_1, \dots, x_v) \in S_1 \times \dots \times S_v.$$

These constitute the basic intervals for the first stage.

Let $\mathcal{E}_1 = \{I_1[1], I_2[1], \dots, I_{L_1}[1]\}$ be an enumeration of the first-stage basic intervals, and Σ_1 the collection of injective mappings from $\{1, \dots, v-1\}$ to $\{1, \dots, L_1\}$. We view an element of Σ_1 as an ordered $(v-1)$ -tuple of distinct indices from $\{1, \dots, L_1\}$. As before, Σ_1 is arranged lexicographically. Set

$$\mathcal{Q}'_1 = \{(q, k, \mathbb{l}_\sigma[1]) : 1 \leq q \leq 2, 0 \leq k \leq r_q - 1, \sigma \in \Sigma_1\},$$

with $\mathbb{l}_\sigma[1] = (I_{\sigma(1)}[1], \dots, I_{\sigma(v-1)}[1])$. The list \mathcal{Q}'_1 is assigned the following ordering. An element of the form $(q, k, \mathbb{l}_\sigma[1])$ will precede $(q', k', \mathbb{l}_{\sigma'}[1])$ if one of the following conditions holds:

- (a) $\sigma < \sigma'$ (irrespective of the relative values of q, q', k, k'), or
- (b) $\sigma = \sigma', q < q'$ (irrespective of the relative values of k, k'), or
- (c) $\sigma = \sigma', q = q'$ and $k > k'$.

The list \mathcal{Q}'_1 is appended to \mathcal{Q}_0 to arrive at the updated queue \mathcal{Q}_1 at the end of the first step.

The general step: In general, at the end of step j , we have the following quantities:

- The j -th iterate of the construction E_j , which is the union of the j -th level basic intervals of length $\ell_j = d_{j-1}/N_j^{v-1}$. Here d_j is a sequence of small constants obtained as c_1 from repeated applications of Proposition 3.1 and depending on the collection of functions $\{f_q : q \leq j\}$. In particular, d_j only depends on parameters involved in the first j steps of the construction. The sequence N_j is chosen to be rapidly increasing. For instance, choosing

$$N_{j+1} > \exp \left[\prod_{k=1}^j \left(\frac{N_k}{d_k} \right)^R \right] \quad \text{for all } j \geq 1 \tag{4-2}$$

and some fixed large constant R would suffice.

- The collection of the j -th level basic intervals that constitute E_j , which we denote by

$$\mathcal{E}_j = \{I_1[j], I_2[j], \dots, I_{L_j}[j]\}.$$

- The updated queue $\mathcal{Q}_j = \mathcal{Q}_{j-1} \cup \mathcal{Q}'_j$, with

$$\mathcal{Q}'_j = \{(q, k, \mathbb{l}_\sigma[j]) : 1 \leq q \leq j+1, 0 \leq k \leq r_q - 1, \sigma \in \Sigma_j\}.$$

Here Σ_j is the collection of all injective maps from $\{1, \dots, v-1\}$ to $\{1, \dots, L_j\}$, which is viewed as the collection of all $(v-1)$ -dimensional vectors with distinct entries taking values in $\{1, \dots, L_j\}$ and endowed with the lexicographical order. The new list \mathcal{Q}'_j is ordered in the same way as described in Step 1 and appended to \mathcal{Q}_{j-1} . Notice that the number of members in the list \mathcal{Q}_j is much larger than j .

We also know that $\mathcal{D}_q^k f_q(x)$ is nonzero for certain choices of k , q , and x with $k \leq r_q - 1$. Specifically, given any tuple of the form (q, k, \mathbb{I}) that appears among the first j members of the list \mathcal{Q}_j , the construction yields that

$$|\mathcal{D}_q^k f_q(x)| > 0 \quad \text{if } x_i \in E_j \cap I_i \text{ for } i \neq v, \quad x_v \in E_j \setminus (I_1 \cup \cdots \cup I_{v-1}). \quad (4-3)$$

Here the $(v-1)$ -tuple of intervals \mathbb{I} has been labelled as $\mathbb{I} = (I_i : i \neq v)$. We will continue to use this notation for the remainder of this subsection.

At step $j+1$, we refer to the $(j+1)$ -th entry of the queue \mathcal{Q}_j , which we denote by (q_0, k_0, \mathbb{I}) . Set i_0 to be the distinguished index such that

$$\frac{\partial}{\partial x_{i_0}} [\mathcal{D}_{q_0}^{k_0} f_{q_0}] = \mathcal{D}_{q_0}^{k_0+1} f_{q_0}.$$

Two cases can occur, depending on whether k_0 is maximal for the given q_0 or not. If it is, that means $k_0 = r_{q_0} - 1$ for some $1 \leq q_0 \leq j+1$. We want to apply Proposition 3.1 with $M^{-1} = \ell_j$, the index i_0 as described in the paragraph above, and

$$f = \mathcal{D}_{q_0}^{r_{q_0}-1} f_{q_0}, \quad T_i = \begin{cases} E_j \cap I_i & \text{if } i \neq v, \\ E_j \setminus \bigcup_{i \neq v} T_i & \text{if } i = v. \end{cases} \quad (4-4)$$

In this case, the nonvanishing derivative condition required for the application of Proposition 3.1 is ensured by the hypothesis of Theorem 1.1.

The other possibility is when $k_0 < r_{q_0} - 1$. Given the specified ordering on \mathcal{Q}_j , we conclude that $(q_0, k_0 + 1, \mathbb{I})$ must be the j -th member of \mathcal{Q}_j , and hence, by the induction hypothesis, (4-3) holds with $q = q_0$ and $k = k_0 + 1$. We can now apply Proposition 3.1 with $f = \mathcal{D}_{q_0}^{k_0} f_{q_0}$, $M^{-1} = \ell_j$, and the same choices of i_0 and T_1, \dots, T_v as in (4-4) above.

In either case, we obtain a collection \mathcal{E}_{j+1} of $(j+1)$ -th level basic cubes of length $\ell_{j+1} = d_j / N_{j+1}^{v-1}$, the union of which is E_{j+1} , and for which (4-3) holds with $q = q_0$, $k = k_0$ and j replaced by $j+1$. This completes the induction.

The set E is now defined as $E = \bigcap_{j=1}^{\infty} E_j$. We will establish shortly that E meets the requirements of Theorem 1.1.

4C. Modifications to the construction of E for Theorem 1.2. The main distinction for Theorem 1.2 is that we only need to consider the first derivative Df_q of f_q , so there is no need for the higher-order differential operators \mathcal{D}_q^k , and hence no need for distinguished indices i_0 . What this means is that the elements of the queue \mathcal{Q}'_j are of the form $(q, \mathbb{I}_\sigma[j])$, where q ranges from 1 to j and \mathbb{I}_σ is a tuple of cubes instead of intervals, and one needs to appeal to Proposition 3.4 instead of Proposition 3.1. The number of subcubes of $[0, \eta]^{nv}$ at the initializing step needs to be chosen large enough, so that their side lengths do not exceed M_0^{-1} , as specified in the hypotheses of Proposition 3.4. This is simply to ensure that Proposition 3.4 is applicable. The small parameters d_j and large parameters N_j are still assumed to obey a relation of the form (4-2), with the constant R possibly depending on v, n, m . The side length ℓ_j of a j -th level basic cube is now

$$\ell_j = d_{j-1} N_j^{-n(v-1)/m}. \quad (4-5)$$

From this point onward, no distinction will be made between Theorem 1.1 and the $m = 1, n = 1$ case of Theorem 1.2. The computation of the Hausdorff and Minkowski dimensions of the set E in these two cases proceeds in exactly the same manner.

4D. Nonexistence of solutions. Fix any $q \geq 1$, and a tuple $x = (x_1, \dots, x_v)$ of distinct points in E . Since $\ell_j \rightarrow 0$, the minimum separation between the points x_1, \dots, x_v exceeds ℓ_j for some j . In other words, there exists a step $j \geq q$ in the construction of E where these points lie in distinct basic intervals (in the case of Theorem 1.1) or cubes (in the case of Theorem 1.2) of that step. Suppose that $\mathbb{I}^* = (I_1^*, \dots, I_{v-1}^*)$ is the tuple of j -th stage basic intervals such that $x_i \in I_i^*$ for $1 \leq i \leq v - 1$ and $x_v \in E_j \setminus (I_1^* \cup \dots \cup I_{v-1}^*)$. Then the tuple $(q, 0, \mathbb{I}^*)$ (or (q, \mathbb{I}^*) in the case of Theorem 1.2) belongs to the list \mathcal{Q}_j . Suppose that it is the j_0 -th member of \mathcal{Q}_j , $j_0 \gg j$. This tuple then plays a decisive role at the j_0 -th step of the construction, at the end of which we obtain (either from Proposition 3.1 or 3.4) that f_q does not vanish on $\prod_{i=1}^v E_{j_0} \cap I_i^*$. Since x lies in this set, we are done.

4E. Hausdorff dimension of E . Frostman’s lemma dictates that the Hausdorff dimension of a Borel set E is the supremum value of $\alpha > 0$ for which one can find a probability measure supported on E with $\sup_{x,r} \mu(B(x; r))/r^\alpha < \infty$, where $B(x; r)$ denotes a ball centred at x of radius r . Keeping in mind that any ball is coverable by a fixed number of cubes, we aim to construct a probability measure μ on E with the property that for every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\mu(I) \leq C_\epsilon l(I)^{m/(v-1)-\epsilon} \quad \text{for all cubes } I. \tag{4-6}$$

Here $l(I)$ denotes the side length of I .

Let us recall that \mathcal{E}_j denotes the collection of all basic cubes with side length ℓ_j at step j of the construction. Decomposing each cube in \mathcal{E}_j into equal subcubes of length $1/N_{j+1}$, we denote by \mathcal{F}_{j+1} the resulting collection of subcubes that contain a cube from \mathcal{E}_{j+1} . Let F_{j+1} be the union of the cubes in \mathcal{F}_{j+1} . We define a sequence of measures ν_{j+1} and μ_j supported respectively on F_{j+1} and E_j as follows. The measure μ_0 is the uniform measure on $[0, \eta]^n$. Given μ_j , the measure ν_{j+1} will be supported on F_{j+1} and will be defined by evenly splitting the measure μ_j of each cube in \mathcal{E}_j among its children in \mathcal{F}_{j+1} . Given ν_j , the measure μ_j will be supported on E_j and will be defined by evenly splitting the measure ν_j of each cube in \mathcal{F}_j among its children in \mathcal{E}_j . It follows from the mass distribution principle that the measures μ_j have a weak limit μ . We claim that μ obeys the desired requirement (4-6).

The proof of the claim rests on the following proposition, which describes the mass distribution on the basic cubes of the construction.

Proposition 4.1. *Let $K \in \mathcal{E}_j, J \in \mathcal{F}_{j+1}$ with $J \subset K$. Then:*

- (a) $\mu(K)/|K| \leq \mu(J)/|J| \leq 2\mu(K)/|K|$.
- (b) $\mu(J) \leq M_j |J|$, where $M_j = \prod_{k=1}^j 2(\ell_k N_k)^{-n}$.

Proof. We first prove part (a). Each $K \in \mathcal{E}_j$ decomposes into $(\ell_j N_{j+1})^n$ subcubes of side length $1/N_{j+1}$. Propositions 3.1 and 3.4 assert that at least a $(1-1/M)$ -fraction of these subcubes contain a cube from \mathcal{E}_{j+1} and hence lie in \mathcal{F}_{j+1} . The number of descendants $J \in \mathcal{F}_{j+1}$ of a given cube $K \in \mathcal{E}_j$ is therefore

at most $(\ell_j N_{j+1})^n = |K|/|J|$ and at least $\frac{1}{2}(\ell_j N_{j+1})^n = |K|/(2|J|)$. Since $\mu(K)$ is evenly distributed among such J , part (a) follows.

We prove part (b) by applying part (a) iteratively. Suppose that \bar{J} is the cube in \mathcal{F}_j that contains K . Then

$$\frac{\mu(J)}{|J|} \leq 2 \frac{\mu(K)}{|K|} \leq 2 \frac{\mu(\bar{J})}{|K|} = \frac{2|\bar{J}|}{|K|} \frac{\mu(\bar{J})}{|\bar{J}|} = \frac{2}{(\ell_j N_j)^n} \frac{\mu(\bar{J})}{|\bar{J}|}. \quad \square$$

We are now ready to apply Proposition 4.1 to prove (4-6). Suppose that I is a cube with side length between ℓ_{j+1} and ℓ_j . There are two possibilities: either $1/N_{j+1} \leq l(I) \leq \ell_j$ or $\ell_{j+1} \leq l(I) < 1/N_{j+1}$.

In the first case I can be covered by at most $C|I|N_{j+1}^n$ cubes of side length $1/N_{j+1}$, all of which could be in \mathcal{F}_{j+1} . If J is a generic member of \mathcal{F}_{j+1} , we obtain from Proposition 4.1 that

$$\begin{aligned} \mu(I) &\leq C|I|N_{j+1}^n \mu(J) \leq C|I|N_{j+1}^n M_j |J| \leq C M_j |I| \\ &\leq C \frac{2M_{j-1}}{(\ell_j N_j)^n} |I| \leq C M_{j-1} d_{j-1}^{-m/(v-1)} \ell_j^{m/(v-1)-n} |I| \leq C_\epsilon \ell_j^{m/(v-1)-n-\epsilon} |I| \leq C_\epsilon l(I)^{m/(v-1)-\epsilon}. \end{aligned}$$

Here the penultimate inequality follows from the relation (4-5) and the rapid growth condition (4-2).

Let us turn to the complementary case, when $\ell_{j+1} \leq l(I) \leq N_{j+1}^{-1}$. If $\mu(I) > 0$, the cube I intersects at least one cube J in \mathcal{F}_{j+1} in which case it is contained in the union of at most $3^n - 1$ cubes of the same dimension adjacent to it. Proposition 4.1 then yields

$$\mu(I) \leq C_n \mu(J) \leq C_n M_j |J| = C_n M_j N_{j+1}^{-n} = C_n M_j d_j^{-m/(v-1)} \ell_{j+1}^{m/(v-1)} \leq C_\epsilon \ell_{j+1}^{m/(v-1)-\epsilon} \leq C_\epsilon l(I)^{m/(v-1)-\epsilon},$$

applying (4-2) as before at the penultimate stage. This establishes the claim (4-6).

4F. Minkowski dimension of E . In order to establish the full Minkowski dimension of E , we show that for any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$\mathcal{N}_\ell(E) \geq c_\epsilon \ell^{-n+\epsilon} \quad \text{for any } 0 < \ell \ll 1. \quad (4-7)$$

Here $\mathcal{N}_\ell(A)$ denotes the ℓ -covering number of a Borel set A , i.e., the smallest number of closed cubes of side length ℓ required to cover A . Given $0 < \ell \ll 1$, we first fix the index j such that $\ell_{j+1} \leq \ell < \ell_j$. As before, we study two cases.

4F1. Case 1. If $\ell \in [\ell_{j+1}, 1/N_{j+1})$, we select $I \in \mathcal{E}_j$ of side length ℓ_j to be one of the ‘‘special cubes’’ for step $j + 1$; i.e.,

$$\begin{aligned} I &\subseteq T_{i_0(j+1)} && \text{for Theorem 1.1,} \\ I &\subseteq T_v[j + 1] && \text{for Theorem 1.2.} \end{aligned} \quad (4-8)$$

Here $i_0(j + 1) \in \{1, \dots, v\}$ denotes the preferred index at step $j + 1$ of the construction, based on which Proposition 3.1 is applied. On the other hand, $T_v[j + 1]$ denotes the choice of T_v at the $(j+1)$ -th step for the purpose of applying Proposition 3.4. In either case, $I \in \mathcal{E}_j$ can be partitioned into $(\ell_j N_{j+1})^n$ subcubes of side length $1/N_{j+1}$. It follows from (3-4) and (3-17) in Propositions 3.1 and 3.4 that at least half of these subcubes lie in \mathcal{F}_{j+1} . Further, the conclusions (3-5) and (3-16) of the propositions say that for

each $J \in \mathcal{F}_{j+1}$, $J \subset I$,

$$|J \cap E_{j+1}| \geq \begin{cases} d_j N_{j+1}^{-1} & \text{for Theorem 1.1,} \\ \frac{1}{2} N_{j+1}^{-n} & \text{for Theorem 1.2,} \end{cases}$$

so that combining the two

$$|J \cap E_{j+1}| \geq d_j N_{j+1}^{-n} \quad \text{for any } n. \tag{4-9}$$

Let \mathcal{Q}_ℓ be a collection of cubes of side length ℓ that cover $I \cap E$, with $\#(\mathcal{Q}_\ell) = \mathcal{N}_\ell(I \cap E)$. Given any $Q \in \mathcal{Q}_\ell$, let Q^* denote the axis-parallel cube with the same centre as Q , but side length $4\ell\sqrt{n}$. Our main claim is that

$$I \cap E_{j+1} = \bigcup \{J \cap E_{j+1} : J \subseteq I, J \in \mathcal{F}_{j+1}\} \tag{4-10}$$

$$\subseteq \bigcup \{Q^* : Q \in \mathcal{Q}_\ell\}, \tag{4-11}$$

so that

$$|I \cap E_{j+1}| \leq \#(\mathcal{Q}_\ell)(4\sqrt{n}\ell)^n. \tag{4-12}$$

Assuming the claim for now, the proof of (4-7) proceeds as follows,

$$\begin{aligned} \mathcal{N}_\ell(E) &\geq \mathcal{N}_\ell(I \cap E) = \#(\mathcal{Q}_\ell) \geq c_n \frac{|I \cap E_{j+1}|}{\ell^n} \geq c_n \sum_J \left\{ \frac{|J \cap E_{j+1}|}{\ell^n} : J \subseteq I, J \in \mathcal{F}_{j+1} \right\} \\ &\geq \frac{c_n}{\ell^n} \times \frac{(\ell_j N_{j+1})^n}{2} \times d_j N_{j+1}^{-n} = c_n \frac{d_j \ell_j^n}{2\ell^n} \geq c_{\epsilon,n} \ell^{-n+\epsilon}. \end{aligned}$$

Let us pause for a moment to explain the steps above. The second inequality in the sequence follows from (4-12) with $c_n = (4\sqrt{n})^{-n}$. The third inequality uses (4-10) and the disjointness of the cubes J ; the fourth follows from (4-9) and the counting argument for $\#(\mathcal{F}_{j+1})$ preceding it. The final inequality is a consequence of the rapid growth condition (4-2) and the assumption $\ell \leq 1/N_{j+1}$. Together they imply that for any $\epsilon > 0$, there is a constant $c_\epsilon > 0$ such that $d_j \ell_j^n \geq c_\epsilon N_{j+1}^{-\epsilon} \geq c_\epsilon \ell^\epsilon$.

Proof of the claim. It remains to verify (4-10)–(4-12). The equality in (4-10) is part of the definitions of \mathcal{E}_j and \mathcal{F}_{j+1} . The estimate (4-12) is an easy consequence of (4-11). To establish (4-11), pick any $x \in I \cap E_{j+1}$. Since E_{j+1} is by definition a union of the cubes in \mathcal{E}_{j+1} , there must exist a basic interval $I' \in \mathcal{E}_{j+1}$ containing x . The set $I' \cap E$ is nonempty by construction, so we pick an element y in this set. Then $|x - y| \leq \text{diam}(I') = \sqrt{n}\ell_{j+1}$. Since $y \in I \cap E$, there must be a cube $Q_y \in \mathcal{Q}_\ell$ containing y ; let $c(Q_y)$ denote the centre of Q_y . The assumption $\ell \geq \ell_{j+1}$ gives that

$$|x - c(Q_y)| \leq |x - y| + |y - c(Q_y)| \leq \sqrt{n}\ell_{j+1} + \frac{1}{2}\ell\sqrt{n} \leq 2\ell\sqrt{n}.$$

This means that $x \in Q_y^*$, as desired. □

4F2. Case 2. In the second case, where $\ell \in [1/N_{j+1}, \ell_j]$, the analysis is similar, with minor variations in numerology. Since ℓ is larger, we need to start from a coarser scale. Pick a cube $I \in \mathcal{E}_{j-1}$ that is “special” for the j -th step, in the sense that (4-8) holds with j replaced by $j - 1$. As before, we decompose I into cubes $J \in \mathcal{F}_j$; the number of such cubes J is at least $\frac{1}{2}(\ell_{j-1} N_j)^n$. Let \mathcal{Q}_ℓ again denote a covering of

$I \cap E$ by ℓ -cubes, with $\#(\mathcal{Q}_\ell) = \mathcal{N}_\ell(I \cap E)$. Set F_j to be the union of the intervals in \mathcal{F}_j . This time, we will need the following analogues of (4-10)–(4-12), to be proven shortly:

$$I \cap F_{j+1} = \bigcup \{J \cap F_{j+1} : J \subseteq I, J \in \mathcal{F}_j\} \subseteq \bigcup \{Q^* : Q \in \mathcal{Q}_\ell\}, \tag{4-13}$$

so that

$$|I \cap F_{j+1}| \leq \#(\mathcal{Q}_\ell)(4\sqrt{n}\ell)^n. \tag{4-14}$$

Further,

$$|J \cap F_{j+1}| \geq \frac{1}{2}d_{j-1}N_j^{-n} \quad \text{for each } J \in \mathcal{F}_j, J \subseteq I. \tag{4-15}$$

Assuming these, an argument analogous to the previous case leads to

$$\begin{aligned} \mathcal{N}_\ell(E) &\geq \mathcal{N}_\ell(E \cap I) = \#(\mathcal{Q}_\ell) \geq c_n \ell^{-n} |I \cap F_{j+1}| \\ &\geq c_n \ell^{-n} \sum \{|J \cap F_{j+1}| : J \subseteq I, J \in \mathcal{F}_j\} \\ &\geq c_n \ell^{-n} \frac{1}{2}(\ell_{j-1}N_j)^n \times \frac{1}{2}d_{j-1}N_j^{-n} = c_n \frac{d_{j-1}\ell_{j-1}^n}{4\ell^n} \geq c_{\epsilon,n} \ell^{-n+\epsilon}. \end{aligned}$$

The second inequality in the sequence above uses (4-14), and the fourth uses (4-15). The last step uses the assumption $\ell < \ell_j$, which implies in view of (4-2) that

$$\ell^\epsilon < \ell_j^\epsilon < c_\epsilon^{-1}d_{j-1}\ell_{j-1}^n \quad \text{for every } \epsilon > 0.$$

Proof of the claim. Returning to the claims surrounding $I \cap F_{j+1}$, we briefly comment on (4-13) and (4-15), whose proofs constitute the only points of departure from the previous case. Let us start with (4-13). For any $x \in I \cap F_{j+1}$, we focus on a cube $J' \in \mathcal{F}_{j+1}$ such that $x \in J'$. Choosing $y \in J' \cap E$ and $Q_y \in \mathcal{Q}_\ell$ containing y , we see that $|x - y| \leq \text{diam}(J') = \sqrt{n}/N_{j+1}$. Keeping in mind that $\ell \geq 1/N_{j+1}$, one obtains

$$|x - c(Q_y)| \leq |x - y| + |y - c(Q_y)| \leq \frac{\sqrt{n}}{N_{j+1}} + \ell\sqrt{n} \leq 2\ell\sqrt{n},$$

where $c(Q_y)$ denotes the centre of Q_y , as before. This in turn implies (4-13).

To prove (4-15), let us fix $J \in \mathcal{F}_j$, $J \subseteq I$, and observe that $J \cap E_j$ is a union of basic ℓ_j -cubes. The special choice of $I \in \mathcal{E}_{j-1}$ dictates that (4-9) holds with j replaced by $j - 1$; i.e., $|J \cap E_j| \geq d_{j-1}N_j^{-n}$. Thus, the number of basic ℓ_j -cubes in $J \cap E_j$ at the j -th level is at least $d_{j-1}N_j^{-n}/\ell_j^n$. At step $j + 1$, each j -th level basic cube contributes at least $\frac{1}{2}(\ell_j N_{j+1})^n$ subcubes of side length $1/N_{j+1}$ to \mathcal{F}_{j+1} , according to Propositions 3.1 and 3.4. Combining all of this yields,

$$|J \cap F_{j+1}| \geq \frac{d_{j-1}N_j^{-n}}{\ell_j^n} \times \frac{1}{2}(\ell_j N_{j+1})^n \times N_{j+1}^{-n} = \frac{1}{2}d_{j-1}N_j^{-n},$$

as claimed. □

5. Zero sets of functions with a common linearization

We now turn our attention to the proof of Theorem 1.3. Not surprisingly in view of the other results in this paper, it is also predicated on an iterative algorithm, which has been encapsulated in Proposition 5.2 below. The following lemma provides a preparatory step.

Let $\alpha \in \mathbb{R}^v$ be as in the statement of Theorem 1.3, and let \mathfrak{C} be a nonempty proper subset of the index set $\{1, 2, \dots, v\}$. Let $\delta > 0$. Consider disjoint intervals $[a_1, b_1]$ and $[a_2, b_2]$ of length λ , with $a_1 < b_1 \leq a_2 < b_2$. We define two quantities ϵ_{left} and ϵ_{right} depending on \mathfrak{C} , a_1, b_1, a_2, b_2 and δ as follows:

$$\epsilon_{\text{left}} := \sup \left\{ \epsilon : \left| \sum_{j=1}^v \alpha_j z_j \right| \geq \delta \lambda \text{ for } \begin{cases} z_j \in [a_1, a_1 + \epsilon \lambda] \text{ for all } j \notin \mathfrak{C}, \\ z_j \in [a_2, a_2 + \epsilon \lambda] \text{ for all } j \in \mathfrak{C} \end{cases} \right\}, \tag{5-1}$$

$$\epsilon_{\text{right}} := \sup \left\{ \epsilon : \left| \sum_{j=1}^v \alpha_j z_j \right| \geq \delta \lambda \text{ for } \begin{cases} z_j \in [a_1, a_1 + \epsilon \lambda] \text{ for all } j \notin \mathfrak{C}, \\ z_j \in [b_2 - \epsilon \lambda, b_2] \text{ for all } j \in \mathfrak{C} \end{cases} \right\}. \tag{5-2}$$

Lemma 5.1. *Given any $\alpha \in \mathbb{R}^v$ as in Theorem 1.3, there exists $\delta_0 > 0$ depending only on α such that for any $\lambda > 0$ and any choice of intervals $\mathfrak{I}_1 = [a_1, b_1]$ and $\mathfrak{I}_2 = [a_2, b_2]$ of equal length λ with $a_1 < b_1 \leq a_2 < b_2$, the following property holds. For any $\delta < \delta_0$, there exists $\epsilon_0 = \epsilon_0(\mathfrak{C}, \delta)$ (not depending on a_1, a_2, b_1, b_2 , or λ) such that $\max(\epsilon_{\text{left}}, \epsilon_{\text{right}}) \geq \epsilon_0$.*

In particular, there exist subintervals $\widehat{\mathfrak{I}}_1 \subseteq \mathfrak{I}_1$ and $\widehat{\mathfrak{I}}_2 \subseteq \mathfrak{I}_2$ with $|\widehat{\mathfrak{I}}_1| = |\widehat{\mathfrak{I}}_2| = \epsilon_0 \lambda$ and $\text{dist}(\widehat{\mathfrak{I}}_1, \widehat{\mathfrak{I}}_2) \geq (1 - \epsilon_0)\lambda$ such that

$$|\alpha \cdot x| \geq \delta \lambda \text{ for all } x \in \mathbb{R}^v \text{ such that } \begin{cases} x_j \in \widehat{\mathfrak{I}}_1 & \text{for } j \notin \mathfrak{C}, \\ x_j \in \widehat{\mathfrak{I}}_2 & \text{for } j \in \mathfrak{C}. \end{cases}$$

Proof. Set $g(y) = \sum_j \alpha_j y_j$, and consider $g(z^*)$, where $z^* = (z_1^*, \dots, z_v^*)$ is defined to be the v -dimensional vector with $z_j^* = a_1$ if $j \notin \mathfrak{C}$ and $z_j^* = a_2$ if $j \in \mathfrak{C}$. Setting $C^* = \sum_j |\alpha_j|$, we note that

$$|g(z) - g(z^*)| \leq C^* \epsilon \lambda \text{ whenever } |z_j - z_j^*| \leq \epsilon \lambda, \quad 1 \leq j \leq v. \tag{5-3}$$

If $|g(z^*)| > (\delta + \epsilon_0 C^*)\lambda$, then (5-3) implies that $|g(z)| \geq \delta \lambda$ for any z as in (5-1). Therefore $\epsilon_{\text{left}} \geq \epsilon_0$, and the conclusion of the lemma holds with $\widehat{\mathfrak{I}}_1 = [a_1, a_1 + \epsilon_0 \lambda]$, $\widehat{\mathfrak{I}}_2 = [a_2, a_2 + \epsilon_0 \lambda]$. Otherwise, let $\hat{z} = (\hat{z}_1, \dots, \hat{z}_v)$ be the v -dimensional vector with $\hat{z}_j = a_1$ if $j \notin \mathfrak{C}$ and $\hat{z}_j = b_2$ if $j \in \mathfrak{C}$. Then $g(\hat{z}) = g(z^*) + \alpha \cdot (\hat{z} - z^*) = g(z^*) \pm (b_2 - a_2)C_0 = g(z^*) \pm \lambda C_0$, where $C_0 = |\sum_{j \in \mathfrak{C}} \alpha_j| > 0$. Thus, for z as in (5-2), we obtain the estimate

$$\begin{aligned} |g(z)| &\geq |g(\hat{z})| - |\alpha \cdot (z - \hat{z})| \geq |C_0 \lambda \pm g(z^*)| - C^* \epsilon_0 \lambda \\ &\geq C_0 \lambda - (\delta + C^* \epsilon_0) \lambda - C^* \epsilon_0 \lambda \geq C_0 \lambda - (\delta + 2\epsilon_0 C^*) \lambda, \end{aligned}$$

which is greater than or equal to $\delta \lambda$ provided that $\delta < \frac{1}{2} C_0 =: \delta_0$ and $\epsilon_0 \leq (C_0 - 2\delta)/(2C^*)$. One has $\epsilon_{\text{right}} \geq \epsilon_0$ for this choice of ϵ_0 , with the conclusion of the lemma verified for $\widehat{\mathfrak{I}}_1 = [a_1, a_1 + \epsilon_0 \lambda]$, $\widehat{\mathfrak{I}}_2 = [b_2 - \epsilon_0 \lambda, b_2]$. □

Remarks. (a) Let us consider the example $\alpha = (1, -2, 1)$, which corresponds to a linear function g that picks out three-term arithmetic progressions. Choose \mathfrak{C} to be $\{3\}$. For $x_1, x_2 \in [a_1, a_1 + \epsilon \lambda]$ and $x_3 \in [a_2, a_2 + \epsilon \lambda]$, it is easy to see that

$$x_1 - 2x_2 + x_3 \geq a_1 + a_2 - 2(a_1 + \epsilon \lambda) = a_2 - a_1 - 2\epsilon \lambda \geq (1 - 2\epsilon)\lambda.$$

We can thus take $\epsilon_{\text{left}} = \frac{1}{2}(1 - \delta)$. On the other hand, if $x_1, x_2 \in [a_1, a_1 + \epsilon\lambda]$ and $x_3 \in [b_2 - \epsilon\lambda, b_2]$, then

$$x_1 - 2x_2 + x_3 \geq a_1 + b_2 - \epsilon\lambda - 2(a_1 + \epsilon\lambda) = b_2 - a_1 - 3\epsilon\lambda \geq (2 - 3\epsilon)\lambda.$$

Thus $\epsilon_{\text{right}} = \frac{1}{3}(2 - \delta)$. The point is that, in the above lemma, it is possible in certain instances for both ϵ_{left} and ϵ_{right} to be bounded from below. The lemma guarantees that at least one of them will be.

(b) It is important to be aware that the above proof does not necessarily give the best possible ϵ_0 for a given δ because the signs of the components of α are not taken into account. When dealing with a specific α , it is often possible to improve the bound on ϵ_0 given above.

Proposition 5.2. *Given any $\alpha \in \mathbb{R}^v$ obeying the hypotheses of Theorem 1.3, there exist fixed small constants $0 < \epsilon < \frac{3}{4}$ and $\delta(\epsilon) > 0$ depending on α with the following property.*

Let I be any interval say of length ℓ , and let I_1 and I_2 denote the two halves of I . Then one can find subintervals I'_1 and I'_2 of I_1 and I_2 of length $\epsilon\ell$ such that

$$|\alpha \cdot x| \geq \delta\ell \quad \text{for every sufficiently small } \delta \leq \delta(\epsilon),$$

and for any choice of $x_1, x_2, \dots, x_v \in I'_1 \cup I'_2$, not all of which are in I'_i for a single $i = 1, 2$. The subintervals I'_1 and I'_2 are separated by at least $\frac{1}{4}\ell$.

Proof. Let $\{\mathfrak{C}_1, \mathfrak{C}_2, \dots, \mathfrak{C}_R\}$ be an enumeration of all nonempty, proper subsets of $\{1, 2, \dots, v\}$. Given any $x = (x_1, \dots, x_v)$ such that $x_j \in I$ for all j but not all the x_j lie in a single I_1 or I_2 , there exists $1 \leq m \leq R$ such that $j \in \mathfrak{C}_m$ if and only if $x_j \in I_2$. Set

$$C_m := \left| \sum_{j \in \mathfrak{C}_m} \alpha_j \right| \quad \text{and} \quad \delta_0 = \frac{1}{2} \min(C_1, \dots, C_R),$$

so that Lemma 5.1 can be applied for any $\delta < \delta_0$ and any $\mathfrak{C} = \mathfrak{C}_m$, $1 \leq m \leq R$.

Starting with I_1 and I_2 , we apply Lemma 5.1 with $\mathfrak{C} = \mathfrak{C}_1$, $\mathfrak{J}_1 = I_1$, $\mathfrak{J}_2 = I_2$ and $\lambda = \frac{1}{2}\ell$. For a small but fixed $\delta_1 > 0$ with $2\delta_1 \leq \delta_0$, this gives a constant $\epsilon_1 = \epsilon_0(\mathfrak{C}_1, 2\delta_1) > 0$ and two subintervals $I_1^{(1)} \subseteq I_1$ and $I_2^{(1)} \subseteq I_2$ of length $\frac{1}{2}\epsilon_1\ell$ obeying the conclusions of the lemma. Without loss of generality, we can assume that $\epsilon_1 \leq \frac{1}{2}$, so that

$$\text{dist}(I_1^{(1)}, I_2^{(1)}) \geq (1 - \epsilon_1)\frac{1}{2}\ell \geq \frac{1}{4}\ell. \tag{5-4}$$

For $2 \leq k \leq R$, we continue to apply Lemma 5.1 recursively, with the same value δ_1 , and

$$\mathfrak{C} = \mathfrak{C}_k, \quad \mathfrak{J}_1 = I_1^{(k-1)}, \quad \mathfrak{J}_2 = I_2^{(k-1)}, \quad \lambda = \frac{1}{2}\epsilon_1 \cdots \epsilon_{k-1}\ell.$$

At the end of the k -th step, this yields a constant $\epsilon_k = \epsilon_0(\mathfrak{C}_k, 2\delta)$ and subintervals $I_1^{(k)} \subseteq I_1^{(k-1)} \subseteq I_1$, $I_2^{(k)} \subseteq I_2^{(k-1)} \subseteq I_2$ each of length $\frac{1}{2}\epsilon_1 \cdots \epsilon_k\ell$ such that for any $m \leq k$,

$$|\alpha \cdot x| \geq \delta_1\epsilon_1 \cdots \epsilon_{k-1}\ell \quad \text{for all } x \text{ such that } \begin{cases} x_j \in I_1^{(k)} & \text{for } j \notin \mathfrak{C}_m, \\ x_j \in I_2^{(k)} & \text{for } j \in \mathfrak{C}_m. \end{cases}$$

The conclusion of the proposition then holds for

$$I'_1 = I_1^{(R)}, \quad I'_2 = I_2^{(R)}, \quad \epsilon = \frac{1}{2} \prod_{k=1}^R \epsilon_k \quad \text{and} \quad \delta(\epsilon) = \delta_1\epsilon_1 \cdots \epsilon_{R-1}.$$

The separation condition is an easy consequence of the one in Lemma 5.1. Specifically, since $I'_i \subseteq I_i^{(1)}$ for $i = 1, 2$, the relation (5-4) yields

$$\text{dist}(I'_1, I'_2) \geq \text{dist}(I_1^{(1)}, I_2^{(1)}) \geq \frac{1}{4}\ell. \quad \square$$

Remarks. (a) Tracking the parameters from Lemma 5.1, we find that the constant ϵ claimed in Proposition 5.2 obeys the estimate

$$\epsilon \geq \frac{1}{2} \prod_{m=1}^R \frac{(C_m - 2\delta_1)}{(2C^*)}, \tag{5-5}$$

where recall

$$C^* = \sum_{j=1}^v |\alpha_j|$$

and C_m and δ_1 are as in the proof of the proposition.

(b) In view of the remarks made at the end of Lemma 5.1, it is not surprising that the bound on ϵ in the preceding inequality is not always optimal. Returning to the example $\alpha = (1, -2, 1)$, we leave the reader to verify that given any small $\delta > 0$ and $I = [a, a + \ell]$, the choice $I'_1 = [a, a + \frac{1}{3}(1 - \delta)\ell]$ and $I'_2 = [a + \frac{1}{3}(2 + \delta)\ell, a + \ell]$ meets the requirements of the proposition. Thus for this α , the best choice of ϵ is at least $\frac{1}{3}(1 - \delta)$, which is much better than the one provided by the proof.

5A. Proof of Theorem 1.3. Take ϵ and $\delta = \delta(\epsilon)$ to be the positive α -dependent constants given by Proposition 5.2. Recall that $g(x_1, \dots, x_v) = \sum_{j=1}^v \alpha_j x_j$.

Start with $E_0 = [0, \eta]$ where $0 < \eta \ll 1$ is chosen sufficiently small so as to ensure $2Kv\eta < \delta$. Applying Proposition 5.2 with $I = E_0$, we arrive at subintervals $I'_1 = J_1 \subseteq [0, \frac{1}{2}\eta]$ and $I'_2 = J_2 \subseteq [\frac{1}{2}\eta, \eta]$ of length $\ell_1 = \epsilon\eta$ that obey its conclusions. Let $E_1 = J_1 \cup J_2$ with $|J_1| = |J_2| = \ell_1$. In general, if E_j is a disjoint union of 2^j basic intervals of length $\ell_j = \epsilon^j\eta$, then at step $j + 1$, we apply Proposition 5.2 to each such interval to find two subintervals of length $\ell_{j+1} = \epsilon\ell_j = \epsilon^{j+1}\eta$ and separated by a length of at least $\frac{1}{4}\ell_j$, which form the basic intervals of E_{j+1} .

Defining $E = \bigcap_{j=1}^\infty E_j$, we now show that $f(x_1, \dots, x_v) \neq 0$ if x_1, \dots, x_v are not all identical and f is of the form (1-3). For any such choice of x_1, \dots, x_v , there exists a largest index j such that x_1, x_2, \dots, x_v all lie in a basic interval I at step j . This means that if I'_1 and I'_2 are the two subintervals of I generated by Proposition 5.2, then x_1, \dots, x_v lie in $I'_1 \cup I'_2$, but not all of them lie in a single I'_i . If I is of length ℓ_j , it follows from Proposition 5.2 that $|g(x)| \geq \delta\ell_j$. But $|f(x) - g(x)| \leq Kv\ell_j^2$ according to (1-4), so this implies $|f(x)| \geq \frac{1}{2}\delta\ell_j$ for $\ell_j < \eta$.

We recall that the $(j+1)$ -th step of the construction generates exactly two children from each parent, and these are separated by at least $\frac{1}{4}\ell_j$. It now follows from standard results, see for instance [Falconer 2003, Example 4.6, page 64], that the Hausdorff dimension of E is bounded from below by

$$\lim_{j \rightarrow \infty} \frac{\log(2^j)}{-\log(2\ell_j/4)} = \lim_{j \rightarrow \infty} \frac{\log(2^j)}{-\log(\epsilon^j\eta/2)} = \frac{\log 2}{-\log \epsilon}.$$

This establishes the existence of the set claimed by the theorem, with $c(\alpha) = \log 2 / \log(1/\epsilon)$, where ϵ is at least as large as the bound given in (5-5). □

Remark. We return to the example $\alpha = (1, -2, 1)$ that we have been following across this section to show that the avoiding set in this instance can be chosen to have Hausdorff dimension $\log 2/\log 3$. We have referred to this fact in certain examples occurring in Sections 2A2 and 2A3.

Choose a slowly decreasing sequence $\delta_j = 1/(j + C)$ for some fixed large constant C . We have seen, in part (b) of the remark on page 1107, that $\epsilon(\delta_j) = \epsilon_j$ can be chosen as $\frac{1}{3}(1 - \delta_j)$. Let us now use the same Cantor construction as in the proof given above, but using the parameter δ_j at step j instead of a fixed δ . The following consequences are immediate:

$$\ell_j = \epsilon_1 \cdots \epsilon_j \eta \quad \text{so that } \ell_j \leq \frac{C\eta 3^{-j}}{j+C},$$

$$|g(x)| \geq \delta_j \ell_j \quad \text{and} \quad |f(x) - g(x)| \leq K v \ell_j^2 \quad \text{so that } |f(x)| \geq (\delta_j - K v \ell_j) \ell_j \geq \left(\frac{1}{j+C} - \frac{K v \eta C}{j+C} 3^{-j} \right) \ell_j > 0,$$

where $x = (x_1, \dots, x_v)$ is as in the second paragraph of Section 5A. This proves the nonexistence of nontrivial zeros of f . Further, the Hausdorff dimension is bounded from below by

$$\lim_{j \rightarrow \infty} \frac{\log(2^j)}{-\log(2\ell_j/4)} = \lim_{j \rightarrow \infty} \frac{\log(2^j)}{-\log(3^{-j} \eta \prod_{k=1}^j (1 - \delta_k)/2)} = \frac{\log 2}{\log 3},$$

establishing the claim.

Appendix

We collect here the proofs of a few technical facts mentioned in Section 2.

Lemma A.1. *Given a C^2 parameterization $\gamma : [0, \eta] \rightarrow \mathbb{R}^n$ of a curve Γ , let us recall the definition of the signed distance function d from (2-3). Set $F(t_1, t_2) = d(\gamma(t_1), \gamma(t_2))$. Then:*

- (a) F is differentiable on $[0, \eta]^2$.
- (b) If γ is the arclength parameterization, i.e., $|\gamma'(t)| \equiv 1$, then

$$\frac{\partial F}{\partial t_1}(t, t) = 1, \quad \frac{\partial F}{\partial t_2}(t, t) = -1.$$

Proof. Since differentiability is obvious for $t_1 \neq t_2$, it suffices to verify it when $t_1 = t_2 = t$. We consider two cases. If $h \geq k$, then

$$\begin{aligned} F(t+h, t+k) &= d(\gamma(t+h), \gamma(t+k)) = |\gamma(t+h) - \gamma(t+k)| \\ &= |\gamma'(t)| |h - k| + O(h^2 + k^2) \\ &= |\gamma'(t)|(h - k) + O(h^2 + k^2). \end{aligned}$$

On the other hand if $h < k$, we have

$$\begin{aligned} d(\gamma(t+h), \gamma(t+k)) &= -|\gamma(t+h) - \gamma(t+k)| \\ &= -|\gamma'(t)| |h - k| + O(h^2 + k^2) \\ &= |\gamma'(t)|(h - k) + O(h^2 + k^2). \end{aligned}$$

This establishes the first part of the lemma, with

$$\frac{\partial F}{\partial t_1}(t, t) = |\gamma'(t)|, \quad \frac{\partial F}{\partial t_2}(t, t) = -|\gamma'(t)|.$$

The second part is now obvious. □

Lemma A.2. *Let $\gamma : [0, \eta] \rightarrow \mathbb{R}^n$ be an injective parameterization of a C^2 curve with*

$$\gamma'(0) \neq 0 \quad \text{and} \quad \sup\{\|\gamma''(t)\| : t \in [0, \eta]\} \leq K.$$

If η is sufficiently small depending on $|\gamma'(0)|$ and K , then there are no isosceles triangles $\gamma(t_1), \gamma(t_2), \gamma(t_3)$ with $0 \leq t_1 < t_2 < t_3 \leq \eta$ whose sides of equal length meet at $\gamma(t_1)$ or at $\gamma(t_3)$.

Proof. Since d has already been shown to be differentiable in the previous lemma, we compute

$$d(\gamma(t_3), \gamma(t_1)) - d(\gamma(t_2), \gamma(t_1)) = \int_{t_2}^{t_3} \frac{\partial}{\partial t} d(\gamma(t), \gamma(t_1)) = \int_{t_2}^{t_3} \gamma'(t) \cdot \frac{\gamma(t) - \gamma(t_1)}{|\gamma(t) - \gamma(t_1)|}. \tag{A-1}$$

For $t, t_1 \in [0, \eta]$ with $t > t_1$, we obtain

$$\begin{aligned} \frac{\gamma(t) - \gamma(t_1)}{|\gamma(t) - \gamma(t_1)|} &= \frac{[\gamma'(t_1)(t - t_1) + O(K(t - t_1)^2)]}{|[\gamma'(t_1)(t - t_1) + O(K(t - t_1)^2)]|} \\ &= \frac{\gamma'(t_1) + O(K\eta)}{|\gamma'(t_1) + O(K\eta)|} = \frac{\gamma'(0) + O(K\eta)}{|\gamma'(0) + O(K\eta)|} \\ &= \frac{\gamma'(0)}{|\gamma'(0)|} \left[1 + O\left(\frac{K\eta}{|\gamma'(0)|}\right) \right]. \end{aligned}$$

Using this, the integrand in (A-1) may be estimated as follows:

$$\gamma'(t) \cdot \frac{\gamma(t) - \gamma(t_1)}{|\gamma(t) - \gamma(t_1)|} = [\gamma'(0) + O(K\eta)] \cdot \frac{\gamma'(0)}{|\gamma'(0)|} \left[1 + O\left(\frac{K\eta}{|\gamma'(0)|}\right) \right] \geq \frac{1}{2} |\gamma'(0)| \neq 0,$$

provided $K\eta$ is small relative to $|\gamma'(0)|$. This shows that

$$d(\gamma(t_3), \gamma(t_1)) - d(\gamma(t_2), \gamma(t_1)) \geq \frac{1}{2} |\gamma'(0)| (t_3 - t_2) \neq 0,$$

proving that $\gamma(t_1)$ cannot be the vertex at the intersection of two equal sides in an isosceles triangle. A similar argument works for $\gamma(t_3)$. □

Lemma A.3. *Given a curve Γ as described in Section 2B, let us recall the function $f = (f_1, f_2)$ given by (2-4) and (2-5). Then $Df(t)$ is of full rank at every point $t = (t_1, t_2, t_3, t_4)$ with distinct entries and $f(t) = 0$.*

Proof. To prove that Df has rank 2 on the zero set of f , it suffices to show that the 2×2 submatrix with entries $\partial f_i / \partial t_j$ with $i = 1, 2$ and $j = 1, 4$ is nonsingular. We will do this by proving that $\partial f_1 / \partial t_j$ are nonzero and of the same sign for $j = 1, 4$, whereas for $\partial f_2 / \partial t_j$ the signs are reversed.

We begin by computing $\partial f_1 / \partial t_j$ on the zero set of f_1 , where

$$\frac{\gamma_2(t_3) - \gamma_2(t_2)}{\gamma_1(t_3) - \gamma_1(t_2)} = \frac{\gamma_2(t_4) - \gamma_2(t_1)}{\gamma_1(t_4) - \gamma_1(t_1)}. \tag{A-2}$$

Feeding this into the formula for the derivatives, we find that

$$\begin{aligned}\frac{\partial f_1}{\partial t_1} &= -\gamma'_1(t_1)(\gamma_2(t_3) - \gamma_2(t_2)) + \gamma'_2(t_1)(\gamma_1(t_3) - \gamma_1(t_2)) = \gamma'_1(t_1)(\gamma_1(t_3) - \gamma_1(t_2))F_1, \\ \frac{\partial f_1}{\partial t_4} &= \gamma'_1(t_4)(\gamma_2(t_3) - \gamma_2(t_2)) - \gamma'_2(t_4)(\gamma_1(t_3) - \gamma_1(t_2)) = \gamma'_1(t_4)(\gamma_1(t_3) - \gamma_1(t_2))F_4,\end{aligned}$$

where

$$F_1 = -\frac{\gamma_2(t_4) - \gamma_2(t_1)}{\gamma_1(t_4) - \gamma_1(t_1)} + \frac{\gamma'_2(t_1)}{\gamma'_1(t_1)} \quad \text{and} \quad F_4 = \frac{\gamma_2(t_4) - \gamma_2(t_1)}{\gamma_1(t_4) - \gamma_1(t_1)} - \frac{\gamma'_2(t_4)}{\gamma'_1(t_4)}.$$

Since γ'_1 is assumed to be of fixed positive sign on $[0, \eta]$, we have

$$\text{sign}\left(\frac{\partial f_1}{\partial t_1} \cdot \frac{\partial f_1}{\partial t_4}\right) = \text{sign}(F_1 F_4).$$

But $\gamma'_2(t_j)/\gamma'_1(t_j)$ is the slope of the tangent to the curve Γ at the point t_j , whereas $(\gamma_2(t_4) - \gamma_2(t_1))/(\gamma_1(t_4) - \gamma_1(t_1))$ is the slope of the chord joining t_1 and t_4 . Since we have assumed that Γ is strictly convex, this yields that F_1 and F_4 are of the same sign, which is the desired conclusion.

We turn to $\partial f_2/\partial t_j$ for $j = 1, 4$. Let us observe that f_2 is nonzero if $t_4 - t_3$ and $t_2 - t_1$ have opposite signs. In what follows, we will therefore restrict to the case where $(t_4 - t_3)(t_2 - t_1) > 0$. We find that

$$\frac{\partial}{\partial t_4} d(\gamma(t_4), \gamma(t_3)) = \gamma'(t_4) \cdot \frac{\gamma(t_4) - \gamma(t_3)}{|\gamma(t_4) - \gamma(t_3)|},$$

so

$$\frac{\partial f_2}{\partial t_4} = \gamma'(t_4) \cdot \frac{\gamma(t_4) - \gamma(t_3)}{|\gamma(t_4) - \gamma(t_3)|} d(\gamma(t_2), \gamma(t_1)).$$

Similarly

$$\frac{\partial f_2}{\partial t_1} = -\gamma'(t_1) \cdot \frac{\gamma(t_2) - \gamma(t_1)}{|\gamma(t_2) - \gamma(t_1)|} d(\gamma(t_4), \gamma(t_3)).$$

In the regime where $(t_4 - t_3)(t_2 - t_1) > 0$, these two quantities are of opposite signs, completing the proof. \square

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References

- [Behrend 1946] F. A. Behrend, “On sets of integers which contain no three terms in arithmetical progression”, *Proc. Nat. Acad. Sci. U.S.A.* **32** (1946), 331–332. MR Zbl
- [Bennett et al. 2016] M. Bennett, A. Iosevich, and K. Taylor, “Finite chains inside thin subsets of \mathbb{R}^d ”, *Anal. PDE* **9**:3 (2016), 597–614. MR Zbl

- [Chan et al. 2016] V. Chan, I. Łaba, and M. Pramanik, “Finite configurations in sparse sets”, *J. Anal. Math.* **128** (2016), 289–335. MR Zbl
- [Falconer 2003] K. Falconer, *Fractal geometry: mathematical foundations and applications*, 2nd ed., John Wiley & Sons, Hoboken, NJ, 2003. MR Zbl
- [Greenleaf and Iosevich 2012] A. Greenleaf and A. Iosevich, “On triangles determined by subsets of the Euclidean plane, the associated bilinear operators and applications to discrete geometry”, *Anal. PDE* **5**:2 (2012), 397–409. MR Zbl
- [Greenleaf et al. 2015] A. Greenleaf, A. Iosevich, B. Liu, and E. Palsson, “A group-theoretic viewpoint on Erdős–Falconer problems and the Mattila integral”, *Rev. Mat. Iberoam.* **31**:3 (2015), 799–810. MR Zbl
- [Greenleaf et al. 2017] A. Greenleaf, A. Iosevich, and M. Pramanik, “On necklaces inside thin subsets of \mathbb{R}^d ”, *Math. Res. Lett.* **24**:2 (2017), 347–362. MR
- [Harangi et al. 2013] V. Harangi, T. Keleti, G. Kiss, P. Maga, A. Máthé, P. Mattila, and B. Strenner, “How large dimension guarantees a given angle?”, *Monatsh. Math.* **171**:2 (2013), 169–187. MR
- [Henriot et al. 2016] K. Henriot, I. Łaba, and M. Pramanik, “On polynomial configurations in fractal sets”, *Anal. PDE* **9**:5 (2016), 1153–1184. MR Zbl
- [Keleti 1999] T. Keleti, “A 1-dimensional subset of the reals that intersects each of its translates in at most a single point”, *Real Anal. Exchange* **24**:2 (1999), 843–844. MR Zbl
- [Keleti 2008] T. Keleti, “Construction of one-dimensional subsets of the reals not containing similar copies of given patterns”, *Anal. PDE* **1**:1 (2008), 29–33. MR Zbl
- [Körner 2009] T. W. Körner, “Fourier transforms of measures and algebraic relations on their supports”, *Ann. Inst. Fourier (Grenoble)* **59**:4 (2009), 1291–1319. MR Zbl
- [Łaba and Pramanik 2009] I. Łaba and M. Pramanik, “Arithmetic progressions in sets of fractional dimension”, *Geom. Funct. Anal.* **19**:2 (2009), 429–456. MR Zbl
- [Maga 2010] P. Maga, “Full dimensional sets without given patterns”, *Real Anal. Exchange* **36**:1 (2010), 79–90. MR Zbl
- [Máthé 2012] A. Máthé, “Sets of large dimension not containing polynomial configurations”, preprint, 2012. arXiv
- [Salem and Spencer 1942] R. Salem and D. C. Spencer, “On sets of integers which contain no three terms in arithmetical progression”, *Proc. Nat. Acad. Sci. U. S. A.* **28** (1942), 561–563. MR Zbl
- [Sharpe 1997] R. W. Sharpe, *Differential geometry*, Graduate Texts in Mathematics **166**, Springer, 1997. MR Zbl
- [Szemerédi 1975] E. Szemerédi, “On sets of integers containing no k elements in arithmetic progression”, *Acta Arith.* **27** (1975), 199–245. MR Zbl

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ROBERT FRASER: rgf@math.ubc.ca
 University of British Columbia, Vancouver, BC, Canada

MALABIKA PRAMANIK: malabika@math.ubc.ca
 University of British Columbia, Vancouver, BC, Canada

ON MINIMIZERS OF AN ISOPERIMETRIC PROBLEM WITH LONG-RANGE INTERACTIONS UNDER A CONVEXITY CONSTRAINT

MICHAEL GOLDMAN, MATTEO NOVAGA AND BERARDO RUFFINI

We study a variational problem modeling the behavior at equilibrium of charged liquid drops under a convexity constraint. After proving the well-posedness of the model, we show $C^{1,1}$ -regularity of minimizers for the Coulombic interaction in dimension two. As a by-product we obtain that balls are the unique minimizers for small charge. Eventually, we study the asymptotic behavior of minimizers, as the charge goes to infinity.

1. Introduction

We are interested in the existence and regularity of minimizers of the problem

$$\min\{\mathcal{F}_{Q,\alpha}(E) : E \subset \mathbb{R}^N \text{ convex body, } |E| = V\}, \quad (1-1)$$

where, for $E \subset \mathbb{R}^N$, $V, Q > 0$ and $\alpha \in [0, N)$, we have set

$$\mathcal{F}_{Q,\alpha}(E) := P(E) + Q^2 \mathcal{I}_\alpha(E). \quad (1-2)$$

Here $P(E) := \mathcal{H}^{N-1}(\partial E)$ stands for the perimeter of E and, letting $\mathcal{P}(E)$ be the set of probability measures supported on the closure of E , we set for $\alpha \in (0, N)$,

$$\mathcal{I}_\alpha(E) := \inf_{\mu \in \mathcal{P}(E)} \int_{E \times E} \frac{d\mu(x) d\mu(y)}{|x - y|^\alpha}, \quad (1-3)$$

and for $\alpha = 0$,

$$\mathcal{I}_0(E) := \inf_{\mu \in \mathcal{P}(E)} \int_{E \times E} \log\left(\frac{1}{|x - y|}\right) d\mu(x) d\mu(y). \quad (1-4)$$

Notice that, up to rescaling, we can assume, as we shall do for the rest of the paper, that $V = 1$.

Starting from the seminal work [Strutt (Lord Rayleigh) 1882] (in the Coulombic case $N = 3$, $\alpha = 1$), the functional (1-2) has been extensively studied in the physical literature to model the shape of charged liquid drops; see [Goldman et al. 2015]. In particular, it is known that the ball is a *linearly* stable critical point for (1-1) if the charge Q is not too large; see for instance [Fontelos and Friedman 2004]. However, quite surprisingly, the authors showed in [Goldman et al. 2015] that, without the convexity constraint, (1-2) never admits minimizers under a volume constraint for any $Q > 0$ and $\alpha < N - 1$. In particular, this implies that in this model a charged drop is always *nonlinearly* unstable. This result is in sharp contrast with experiments, see for instance [Zeleny 1917; Taylor 1964], where there is evidence of stability of the

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ball for small charges. This suggests that the energy $\mathcal{F}_{Q,\alpha}(E)$ does not include all the physically relevant contributions.

As shown in [Goldman et al. 2015], a possible way to gain well-posedness of the problem is requiring some extra regularity of the admissible sets. In this paper, we consider an alternative type of constraint, namely the convexity of admissible sets. This assumption seems reasonable as long as the minimizers remain strictly convex, that is, for small enough charges. Let us point out that in [Muratov and Novaga 2016], still another regularizing mechanism is proposed. There, well-posedness is obtained by adding an entropic term which prevents charges from concentrating too much on the boundary of E . We point out that it has been recently shown in [Muratov et al. 2016] that in the borderline case $\alpha = 1$, $N = 2$ such a regularization is not needed for the model to be well-posed. For a more exhaustive discussion about the physical motivations and the literature on related problems we refer to the papers [Muratov and Novaga 2016; Goldman et al. 2015].

Using the compactness properties of convex sets, our first result is the existence of minimizers for every charge $Q > 0$.

Theorem 1.1. *For every $\alpha \in [0, N)$ and every Q , (1-1) admits a minimizer.*

We then study the regularity of minimizers. As often in variational problems with convexity constraints, regularity (or singularity) of minimizers is hard to deal with in dimensions larger than two; see [Lamboley et al. 2012, 2016]. We thus restrict ourselves to $N = 2$. Since our analysis strongly uses the regularity of *equilibrium measures*, i.e., the minimizer of (1-3), we are further reduced to studying the case $\alpha = N - 2$ (that is, $\alpha = 0$ in this case). The second main result of the paper is then:

Theorem 1.2. *Let $N = 2$ and $\alpha = 0$. Then for every $Q > 0$, the minimizers of (1-1) are of class $C^{1,1}$.*

Since we are able to prove uniform $C^{1,1}$ estimates as Q goes to zero, building upon our previous stability results established in [Goldman et al. 2015], we get:

Corollary 1.3. *If $N = 2$ and $\alpha = 0$, for Q small enough, the only minimizers of (1-1) are balls.*

The proof of Theorem 1.2 is based on the natural idea of comparing the minimizers with a competitor made by “cutting out the angles”. However, the nonlocal nature of the problem makes the estimates nontrivial. As already mentioned, a crucial point is an estimate on the integrability of the equilibrium measures. This is obtained by drawing a connection with harmonic measures (see Section 3). Let us point out¹ that, up to proving the regularity of the shape functional \mathcal{I}_0 and computing its shape derivative, one could have obtained a proof of Theorem 1.2 by applying the abstract regularity result of [Lamboley et al. 2012]. Nevertheless, since our proof has a nice geometrical flavor and since regularity of \mathcal{I}_0 is not known in dimension two (see for instance [Jerison 1996; Crasta et al. 2005; Novaga and Ruffini 2015] for the proof in higher dimensions), we decided to keep it.

We remark that, differently from the two-dimensional case, when $N = 3$ we expect the onset of singularities at a critical value $Q_c > 0$, with the shape of a spherical cone with a prescribed angle. Such singularities are also observed in experiments and are usually called *Taylor cones*; see [Taylor 1964;

¹This was suggested to us by J. Lamboley.

Zeleny 1917]. At the moment we are not able to show the presence of such singularities in our model, and this will be the subject of future research.

Eventually, in Section 6, we study the behavior of the optimal sets when the charge goes to infinity. Even though this regime is less significant from the point of view of the applications, we believe that it is still mathematically interesting. Building on Γ -convergence results, we prove:

Theorem 1.4. *Let $\alpha \in [0, 1)$ and $N \geq 2$. Then, every minimizer E_Q of (1-1) satisfies (up to a rigid motion)*

$$Q^{-\frac{2N(N-1)}{1+(N-1)\alpha}} E_Q \rightarrow [0, L_{N,\alpha}] \times \{0\}^{N-1},$$

where the convergence is in the Hausdorff topology and where

$$L_{N,\alpha} := \left(\frac{\alpha(N-1)\mathcal{I}_\alpha([0, 1])}{N^{\frac{N-2}{N-1}}\omega_{N-1}^{\frac{1}{N-1}}} \right)^{\frac{(N-1)}{1+\alpha(N-1)}} \quad \text{for } \alpha \in (0, 1) \quad \text{and} \quad L_{N,0} := \frac{(N-1)^{N-1}}{\omega_{N-1} N^{N-2}},$$

ω_N being the volume of the unit ball in \mathbb{R}^N . For $\alpha = 1$ and $N = 2, 3$, we have

$$Q^{-\frac{2(N-1)}{N}} (\log Q)^{-1+\frac{1}{N}} E_Q \rightarrow [0, L_{N,1}] \times \{0\}^{N-1},$$

where

$$L_{N,1} := \left(\frac{4(N-1)}{N^{\frac{N-2}{N-1}}\omega_{N-1}^{\frac{1}{N-1}}} \right)^{\frac{N-1}{N}}.$$

An obvious consequence of this result is that the ball cannot be a minimizer for Q large enough. For a careful analysis of the loss of linear stability of the ball we refer to [Fontelos and Friedman 2004].

2. Existence of minimizers

We now show that the minimum in (1-1) is achieved. We begin with a simple lemma linking estimates on the energy with estimates on the size of the convex body.

Lemma 2.1. *Let $N \geq 2$, and $\lambda_1, \dots, \lambda_N > 0$. Letting $E := \prod_{i=1}^N [0, \lambda_i]$, $V := |E|$ and $\Phi := V^{-\frac{N-2}{N-1}} P(E)$, it holds that²*

$$\max_i \lambda_i \lesssim \Phi^{N-1} \quad \text{and} \quad \min_i \lambda_i \sim V^{\frac{1}{N-1}} \Phi^{-1}, \tag{2-1}$$

where the involved constants depend only on the dimension. Moreover, letting i_{\max} be such that $\lambda_{i_{\max}} = \max_i \lambda_i$, it holds for $\alpha > 0$ that

$$\lambda_{i_{\max}} \gtrsim \mathcal{I}_\alpha(E)^{-\frac{1}{\alpha}} \quad \text{and} \quad \lambda_i \lesssim \mathcal{I}_\alpha(E)^{\frac{1}{\alpha}} \Phi^{N-2} V^{\frac{1}{N-1}} \quad \text{for } i \neq i_{\max}, \tag{2-2}$$

and for $\alpha = 0$,

$$\lambda_{i_{\max}} \gtrsim \exp(-\mathcal{I}_0(E)) \quad \text{and} \quad \lambda_i \lesssim \exp(\mathcal{I}_0(E)) \Phi^{N-2} V^{\frac{1}{N-1}} \quad \text{for } i \neq i_{\max}, \tag{2-3}$$

where the constants implicitly appearing in (2-2) and (2-3) depend only on N and α .

²Here and in the rest of the paper, we write $f \lesssim g$ if there exists $C > 0$ such that $f \leq Cg$. If $f \lesssim g$ and $g \lesssim f$, we will simply write $f \sim g$.

Proof. Without loss of generality, we can assume that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$. Then, since $V = \prod_{i=1}^N \lambda_i$ and $P(E) \lesssim \prod_{i=1}^{N-1} \lambda_i$, taking the ratio of these two quantities, we obtain $\lambda_N \gtrsim VP(E)^{-1} = V^{\frac{1}{N-1}} \Phi^{-1}$. Now, since the λ_i are decreasing (in particular $\lambda_i \geq \lambda_N$ for all i), this implies

$$\Phi \gtrsim V^{-\frac{N-2}{N-1}} \prod_{i=1}^{N-1} \lambda_i = V^{-\frac{N-2}{N-1}} \lambda_1 \prod_{i=2}^{N-1} \lambda_i \gtrsim V^{-\frac{N-2}{N-1}} \lambda_1 V^{\frac{N-2}{N-1}} \Phi^{-(N-2)},$$

yielding (2-1).

Assume now that $\alpha > 0$. Then, from $\text{diam}(E) \sim \lambda_1$, we get $\mathcal{I}_\alpha(E) \gtrsim \lambda_1^{-\alpha}$. If $N = 2$, together with $\lambda_1 \lambda_2 = V$, this implies (2-2). If $N \geq 3$, we infer as above that

$$\Phi \gtrsim V^{-\frac{N-2}{N-1}} \lambda_1 \lambda_2 \prod_{i=3}^{N-1} \lambda_i \gtrsim V^{-\frac{N-2}{N-1}} \mathcal{I}_\alpha(E)^{-\frac{1}{\alpha}} \lambda_2 V^{\frac{N-3}{N-1}} \Phi^{-(N-3)} \gtrsim V^{-\frac{1}{N-1}} \Phi^{-(N-3)} \mathcal{I}_\alpha(E)^{-\frac{1}{\alpha}} \lambda_2.$$

This gives (2-2). The case $\alpha = 0$ follows analogously, using the fact that $\mathcal{I}_0(E) \geq C - \log \lambda_1$. □

The next result follows directly from John’s lemma [1948].

Lemma 2.2. *There exists a dimensional constant $C_N > 0$ such that for every convex body $E \subset \mathbb{R}^N$, up to a rotation and a translation, there exists $\mathcal{R} := \prod_{i=1}^N [0, \lambda_i]$ such that*

$$\mathcal{R} \subseteq E \subseteq C_N \mathcal{R}.$$

As a consequence $\text{diam}(E) \sim \text{diam}(\mathcal{R})$, $|E| \sim |\mathcal{R}|$, $P(E) \sim P(\mathcal{R})$ and $\mathcal{I}_\alpha(E) \sim \mathcal{I}_\alpha(\mathcal{R})$ for $\alpha > 0$ (and $\exp(-\mathcal{I}_0(E)) \sim \exp(-\mathcal{I}_0(\mathcal{R}))$).

With these two preliminary results at hand, we can prove existence of minimizers for (1-1).

Theorem 2.3. *For every $Q > 0$ and $\alpha \in [0, N)$, (1-1) has a minimizer.*

Proof. Let E_n be a minimizing sequence and let us prove that $\text{diam}(E_n)$ is uniformly bounded. Let \mathcal{R}_n be the parallelepipeds given by Lemma 2.2. Since $\text{diam}(E_n) \sim \text{diam}(\mathcal{R}_n)$, it is enough to estimate $\text{diam}(\mathcal{R}_n)$ from above. Let us begin with the case $\alpha > 0$. In this case, since $\mathcal{I}_\alpha(\mathcal{R}_n) \geq 0$, by (2-1), applied with $V = 1$, we get

$$\text{diam}(\mathcal{R}_n) \lesssim P(\mathcal{R}_n)^{N-1} \lesssim \mathcal{F}_{Q,\alpha}(E_n)^{N-1}.$$

In the case $\alpha = 0$, from (2-1) and (2-3) applied to $V = 1$, we get

$$P(\mathcal{R}_n) \gtrsim \exp\left(-\frac{\mathcal{I}_0(\mathcal{R}_n)}{N-1}\right)$$

so that

$$\mathcal{F}_{Q,0}(\mathcal{R}_n) \gtrsim \exp\left(-\frac{\mathcal{I}_0(\mathcal{R}_n)}{N-1}\right) + Q^2 \mathcal{I}_0(\mathcal{R}_n).$$

From this we obtain that $|\mathcal{I}_0(\mathcal{R}_n)|$ is bounded and thus also $P(\mathcal{R}_n)$ is bounded, whence, arguing as above, we obtain a uniform bound on $\text{diam}(\mathcal{R}_n)$.

Since the E_n are convex sets, up to a translation, we can extract a subsequence which converges in the Hausdorff (and L^1) topology to some convex body E of volume 1. Since the perimeter functional is lower

semicontinuous with respect to the L^1 convergence, and the Riesz potential \mathcal{I}_α is lower semicontinuous with respect to the Hausdorff convergence, see [Landkof 1972; Saff and Totik 1997; Goldman et al. 2015, Proposition 2.2], we get that E is a minimizer of (1-1). \square

3. Regularity of the planar charge distribution for the logarithmic potential

We now focus on the case $N = 2$ and $\alpha = 0$. Relying on classical results on harmonic measures, we show that for every convex set E , the corresponding optimal measure μ for $\mathcal{I}_0(E)$ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner \partial E$ with L^p estimates. Upon making that connection between μ and harmonic measures, this fact is fairly classical. However, since we could not find a proper reference, we recall (and slightly adapt) a few useful results. Let us point out that most definitions and results of this section extend to the case $N \geq 3$ and $\alpha = N - 2$, and to more general classes of sets. In particular, for bounded Lipschitz sets, the fact that harmonic measures are absolutely continuous with respect to the surface measure with L^p densities for $p > 2$ was established in [Dahlberg 1977], and extended later to more general domains; see for instance [Kenig and Toro 1997; 1999; Jerison and Kenig 1982]. The interest for harmonic measures stems from the fact that they bear a lot of geometric information; see in particular [Alt and Caffarelli 1981; Kenig and Toro 1999]. The main result of this section is the following.

Theorem 3.1. *Let E_n be a sequence of compact convex bodies converging to a convex body E and let μ_n be the associated equilibrium measures. Then, $\mu_n = f_n \mathcal{H}^1 \llcorner \partial E_n$ and there exists $p > 2$ and $M > 0$ (depending only on E) such that $f_n \in L^p(\partial E_n)$ with*

$$\|f_n\|_{L^p(\partial E_n)} \leq M.$$

Moreover, if E is smooth, then p can be taken arbitrarily large.

Remark 3.2. By applying the previous result with $E_n = E$, we get that the equilibrium measure of a convex set is always in some $L^p(\partial E)$ with $p > 2$. We stress also that the exponent p and the bound on the L^p norm of its equilibrium measure depend indeed on the set: for instance, a sequence of convex sets with smooth boundaries converging to a square cannot have equilibrium measures with densities uniformly bounded in L^p for $p > 4$.

We will define here $\Omega := E^c$. Let us recall the definition of harmonic measures; see [Garnett and Marshall 2005; Kenig and Toro 1999].

Definition 3.3. Let Ω be a Lipschitz open set (bounded or unbounded) such that $\mathbb{R}^2 \setminus \partial\Omega$ has two connected components, and let $X \in \Omega$. We denote by G_Ω^X the Green function of Ω with pole at X , i.e., the unique distributional solution of

$$-\Delta G_\Omega^X = \delta_X \quad \text{in } \Omega \quad \text{and} \quad G_\Omega^X = 0 \quad \text{on } \partial\Omega,$$

and by ω_Ω^X the harmonic measure of Ω with pole at X , that is, the unique (positive) measure such that for every $f \in C^0(\partial\Omega)$, the solution u of

$$-\Delta u = 0 \quad \text{in } \Omega \quad \text{and} \quad u = f \quad \text{on } \partial\Omega$$

satisfies

$$u(X) = \int_{\partial\Omega} f(y) d\omega_\Omega^X(y).$$

If Ω is unbounded with $\partial\Omega$ bounded and $0 \in \bar{\Omega}^c$, we call ω_Ω^∞ the *harmonic measure of Ω with pole at infinity*, that is, the unique probability measure on $\partial\Omega$ satisfying

$$\int_{\partial\Omega} \phi d\omega^\infty = \int_\Omega u \Delta \phi \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^2),$$

where u is the solution of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \lim_{|z| \rightarrow +\infty} \{u(z) - \frac{1}{2\pi} \log |z|\} & \text{exists and is finite.} \end{cases} \tag{3-1}$$

When it is clear from the context, we omit the dependence of G^X , ω^X or ω^∞ on the domain Ω .

Remark 3.4. For smooth domains, it is not hard to check that $\omega^X = \partial_\nu G^X \mathcal{H}^1 \llcorner \partial\Omega$, and that $\omega^\infty = \partial_\nu u \mathcal{H}^1 \llcorner \partial\Omega$, where ν is the inward unit normal to Ω . Moreover, for Ω unbounded, if h^∞ is the harmonic function in Ω with $h^\infty(z) = -\frac{1}{2\pi} \log |z|$ on $\partial\Omega$, then the function u from (3-1) can also be defined by $u(z) = \frac{1}{2\pi} \log |z| + h^\infty(z)$.

We may now make the connection between harmonic measures and equilibrium measures. For E a Lipschitz bounded open set containing 0, let μ be the optimal measure for $\mathcal{I}_0(E)$ and let

$$v(x) := \int_{\partial E} -\log(|x - y|) d\mu(y).$$

Since

$$-\Delta v = 2\pi\mu \quad \text{in } \mathbb{R}^2, \quad v < \mathcal{I}_0(E) \quad \text{in } E^c \quad \text{and} \quad v = \mathcal{I}_0(E) \quad \text{on } \partial E,$$

if we let $u := (2\pi)^{-1}(\mathcal{I}_0(E) - v)$, we see that it satisfies (3-1) for $\Omega = E^c$. Therefore, $\mu = \omega_{E^c}^\infty$ (recall that $\mu(\partial E) = 1$). For Lipschitz sets Ω , it is well known that ω^∞ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner \partial\Omega$ with density in $L^p(\partial\Omega)$ for some $p > 1$; see [Garnett and Marshall 2005, Theorem 4.2]. However, we will need a stronger result, namely that it is in $L^p(\partial\Omega)$ for some $p > 2$, with estimates on the L^p norm depending only on the geometry of Ω .

Given a convex body E and a point $x \in \partial E$, we call the *angle* of ∂E at x the angle spanned by the tangent cone $\bigcup_{\lambda > 0} \lambda(E - x)$.

We now state a crucial lemma which relates in a quantitative way the regularity of E with the integrability properties of the corresponding harmonic measure. This result is a slight adaptation of [Warschawski and Schober 1966, Theorem 2].

Lemma 3.5. *Let E be a convex body containing the origin in its interior, let $\bar{\zeta} \in (0, \pi]$ be the minimal angle of ∂E , and let $p_c := \pi/(\pi - \bar{\zeta}) + 1$ if $\bar{\zeta} < \pi$ and $p_c := +\infty$ if $\bar{\zeta} = \pi$. Let also E_n be a sequence of convex bodies converging to E in the Hausdorff topology. Then, for every $1 \leq p < p_c$, there exists*

$C(p, \partial E)$ such that for n large enough (depending on p), every conformal map $\psi_n : E_n^c \rightarrow B_1$ with $\psi_n(\infty) = 0$ satisfies

$$\int_{\partial E_n} |\psi_n'|^p \leq C(p, \partial E), \tag{3-2}$$

where we indicate by $|\psi_n'|$ the absolute value of the derivative of ψ_n (seen as a complex function). In particular, for n large enough, $\psi_n' \in L^p(\partial E_n)$ for some $p > 2$.

Proof. The scheme of the proof follows that of [Warschawski and Schober 1966, Theorem 2, Equation (9)]; thus we limit ourselves to pointing out the main differences. We begin by noticing that although Theorem 2 of that paper is written for bounded sets, up to composing with the map $z \rightarrow z^{-1}$, this does not create any difficulties.

We first introduce some notation from [Warschawski and Schober 1966]. Given a convex body E we let $\partial E = \{\gamma(s) : s \in [0, L]\}$ be an arc-length parametrization of ∂E . Notice that, for every s , the left and right derivatives $\gamma'_\pm(s)$ exist and the angle $v(s)$ between $\gamma'(s)$ and a fixed direction, say e_1 , is a function of bounded variation. Up to changing the orientation of ∂E , we can assume that v is increasing. We then let

$$\bar{\eta} := \max_s [v(s^+) - v(s^-)] \geq 0,$$

where $v(s^\pm)$ are the left and right limits at s of v . Notice that $\bar{\zeta} = \pi - \bar{\eta}$ is the minimal angle of ∂E .

Letting $\varphi_n := \psi_n^{-1}$, we want to prove that there exists $C(p, \partial E)$ such that

$$\int_{\partial B_1} |\varphi_n'|^{-p} \leq C(p, \partial E)$$

for n large enough and for $p < \pi/\bar{\eta}$. By a change of variables, this yields (3-2). Let $p < p' < \pi/\bar{\eta}$, and let as in [Warschawski and Schober 1966],

$$h := \frac{1}{2\pi}(p\bar{\eta} + \pi) \quad \text{and} \quad h' := \frac{1}{2\pi}(p'\bar{\eta} + \pi),$$

so that

$$\frac{\pi h}{p} > \frac{\pi h'}{p'} > \bar{\eta}.$$

Let now v^n (respectively v) be the angle functions corresponding to the sets E_n (respectively E). As in [Warschawski and Schober 1966], there exists $\delta > 0$ such that for $s - s' \leq \delta$,

$$v(s) - v(s') \leq \frac{\pi h'}{p'}.$$

By the convexity of E_n and by the convergence of E_n to E , for n large enough and for $s - s' \leq \delta$ we get that

$$v^n(s) - v^n(s') \leq \frac{\pi h}{p}.$$

Let $L_n := \mathcal{H}^1(\partial E_n)$ and let us extend v^n to \mathbb{R} by letting for $s \geq 0$,

$$v^n(s) := v^n\left(L_n \left\lfloor \frac{s}{L_n} \right\rfloor\right) + v^n\left(s - L_n \left\lfloor \frac{s}{L_n} \right\rfloor\right),$$

and similarly for $s \leq 0$, so that v^n is an increasing function with $(v^n)'$ periodic of period L_n . Let now $k_n := \lceil L_n/\delta \rceil \in \mathbb{N}$ and $\delta_n := L/k_n$. By the convergence of E_n to E , we have k_n and δ_n are uniformly bounded from above and below. For $t \in [0, \delta_n]$, and $0 \leq j \leq k_n$, let $s_j^t := t + j\delta_n$. Since

$$\begin{aligned} \int_0^{\delta_n} \sum_{j=0}^{k_n-1} \int_{s_j^t}^{s_{j+1}^t} \frac{v^n(s) - v^n(s_j^t)}{s - s_j^t} ds dt &= \sum_{j=0}^{k_n-1} \int_0^{\delta_n} \int_0^{\delta_n} \frac{v^n(s+t+j\delta_n) - v^n(t+j\delta_n)}{s} dt ds \\ &= \int_0^{\delta_n} \frac{1}{s} \sum_{j=0}^{k_n-1} \int_0^{\delta_n} v^n(s+t+j\delta_n) - v^n(t+j\delta_n) dt ds \\ &= \int_0^{\delta_n} \frac{1}{s} \left(\int_{L_n}^{L_n+s} v^n(t) dt - \int_0^s v^n(t) dt \right) ds \\ &\leq 2\delta_n \sup_{[0, 2L_n]} |v^n| \lesssim \delta_n \|v\|_\infty, \end{aligned}$$

we can find $\bar{t} \in (0, \delta_n)$ such that

$$\sum_{j=0}^{k_n-1} \int_{s_j^{\bar{t}}}^{s_{j+1}^{\bar{t}}} \frac{v^n(s) - v^n(s_j^{\bar{t}})}{s - s_j^{\bar{t}}} ds \lesssim \|v\|_\infty.$$

For notational simplicity, let us simply define $s_j := s_j^{\bar{t}}$. Arguing as above, we can further assume that

$$\sum_{j=0}^{k_n-1} \int_{s_j}^{s_{j+1}} \frac{v^n(s_{j+1}) - v^n(s)}{s_{j+1} - s} ds \lesssim \|v\|_\infty.$$

The proof then follows almost exactly as in [Warschawski and Schober 1966, Theorem 2], by replacing the pointwise quantity

$$G_j^n := \sup_{s_j < s < s_{j+1}} \frac{v^n(s) - v^n(s_j)}{s - s_j}$$

by the integral ones. There is just one additional change in the proof: letting

$$0 \leq \lambda_j^n := v^n(s_{j+1}) - v^n(s_j) \leq \frac{\pi h}{p},$$

we see that in the estimates of [Warschawski and Schober 1966, Theorem 2], the quantity $\max_{\lambda_j^n \neq 0} 1/\lambda_j^n$ appears and could be unbounded in n . Let $\gamma_n(s)$ be the arc-length parametrization of ∂E_n and let $\theta_n(s)$ be such that $\gamma_n(s) = \varphi_n(e^{i\theta_n(s)})$. For $0 < r < 1$ and $j \in [0, k_n - 1]$, if $\lambda_j^n \neq 0$, we have

$$\frac{1}{\lambda_j^n} \int_{s_j}^{s_{j+1}} dv^n(s) \int_{\theta_n(s_j)}^{\theta_n(s_{j+1})} \frac{dt}{|e^{i\theta_n(s)} - r e^{it}|^h} \lesssim \frac{1}{1-h}.$$

Using this estimate, the proof concludes exactly as in [Warschawski and Schober 1966, Theorem 2]. \square

We can now prove Theorem 3.1.

Proof of Theorem 3.1. Without loss of generality we can assume that the sets E_n and E contain the origin in their interior. As observed above, we then have $\mu_n = \omega_{E_n}^\infty$. Let ψ_n be a conformal mapping from E_n^c

to B_1 with $\psi_n(\infty) = 0$. We have

$$\mu_n = \omega_{E_n^c}^\infty = (\psi_n^{-1})\# \omega_{B_1}^0 = (\psi_n^{-1})\# \frac{\mathcal{H}^1 \llcorner \partial B_1}{2\pi} = \frac{|\psi_n'|}{2\pi} \mathcal{H}^1 \llcorner \partial E_n.$$

Then, Lemma 3.5 gives the desired estimate. □

We will also need a similar estimate for $C^{1,\beta}$ sets.

Lemma 3.6. *Let E be a convex set with boundary of class $C^{1,\beta}$. Then, the optimal charge distribution μ is of class $C^{0,\beta}$ and in particular it is in $L^\infty(\partial E)$. Moreover, $\|\mu\|_{C^{0,\beta}}$ depends only on the $C^{1,\beta}$ norm of ∂E .*

Proof. Up to translation we can assume that $0 \in E$ with $\text{dist}(0, \partial E) \geq c$ (with c depending only on the $C^{1,\beta}$ character of ∂E). By [Pommerenke 1992, Theorem 3.6], there exists a conformal mapping ψ of class $C^{1,\beta}$ which maps E^c into B_1 with $\psi(\infty) = 0$ and $\|\psi\|_{C^{1,\beta}(E^c)}$ controlled by the $C^{1,\beta}$ character of ∂E . Since, as before, $\mu = (\psi^{-1})\# \omega_{B_1}^0$, the claim follows by Lemma 3.5. □

4. $C^{1,1}$ -regularity of minimizers for $N = 2$ and $\alpha = 0$

We now show that any minimizer of (1-1) has boundary of class $C^{1,1}$. We begin by showing that we can drop the volume constraint by adding a volume penalization to the functional. This penalization is commonly used in isoperimetric-type problems; see for instance [Esposito and Fusco 2011; Goldman and Novaga 2012]. Let Λ be a positive number and define the functional

$$\mathcal{G}_\Lambda(E) := P(E) + Q^2 \mathcal{I}_0(E) + \Lambda ||E| - 1|.$$

Lemma 4.1. *For every $Q_0 > 0$, there exists $\bar{\Lambda} > 0$ such that, if $\Lambda > \bar{\Lambda}$ and $Q \leq Q_0$, the minimizers of*

$$\min_{E \subseteq \mathbb{R}^2, E \text{ convex}} \mathcal{G}_\Lambda(E) \tag{4-1}$$

are also minimizers of (1-1) and vice versa. Furthermore, the diameter of the minimizers of (4-1) is uniformly bounded by a constant depending only on Q_0 .

Proof. Let us fix $Q_0 > 0$ and let $Q < Q_0$. Let B be a ball with $|B| = 1$. Then for any $E \subset \mathbb{R}^2$ such that $\mathcal{G}_\Lambda(E) \leq \mathcal{G}_\Lambda(B)$ we have

$$\text{diam}(E) - Q^2 \log(\text{diam}(E)) \leq \mathcal{G}_\Lambda(E) \leq \mathcal{G}_\Lambda(B) = \mathcal{F}_{Q,0}(B) \lesssim 1,$$

where the constant involved depends only on Q_0 . For such sets, $\text{diam}(E)$ is bounded by a constant R depending only on Q_0 , and thus $I_0(E) \geq I_0(B_R)$. This implies that every minimizing sequence is uniformly bounded so that, up to passing to a subsequence, it converges in Hausdorff distance to a minimizer of \mathcal{G}_Λ whose diameter is bounded by R . Moreover, for

$$\Lambda > \bar{\Lambda} := P(B) + Q_0^2 (\mathcal{I}_0(B) + |\mathcal{I}_0(B_R)|)$$

we have that $|E| > 0$. Indeed, for $|E| = 0$ the inequality $\mathcal{G}_\Lambda(E) \leq \mathcal{G}_\Lambda(B)$ implies $\Lambda \leq \bar{\Lambda}$.

Notice that the minimum in (4-1) is always less than or equal to the minimum in (1-1). We are thus left to prove the opposite inequality. Assume that E is not a minimizer for $\mathcal{F}_{Q,0}$. In this case we get

$$\sigma := ||E| - 1| > 0.$$

From the uniform bound on the diameter of E we deduce that $\Lambda\sigma$ is itself also bounded by a constant (again depending only on Q_0). From now on we assume that $|E| < 1$, or equivalently, $|E| = 1 - \sigma$, since the other case is analogous. Let us define

$$F := \frac{1}{(1 - \sigma)^{\frac{1}{2}}} E,$$

so that $|F| = 1$. Then, by the minimality of E , the homogeneity of the perimeter and recalling that

$$\mathcal{I}_0(\lambda E) = \mathcal{I}_0(E) - \log(\lambda),$$

a Taylor expansion gives

$$\begin{aligned} \Lambda\sigma &= \mathcal{G}_\Lambda(E) - \mathcal{F}_{Q,0}(E) \leq \mathcal{G}_\Lambda(F) - \mathcal{F}_{Q,0}(E) \\ &= P(E)(1 - \sigma)^{-\frac{1}{2}} + Q^2\mathcal{I}_0(E) + \frac{1}{2}\log(1 - \sigma) - \mathcal{F}_{Q,0}(E) \\ &\leq P(E)((1 - \sigma)^{-\frac{1}{2}} - 1) \leq \frac{1}{2}P(E)\sigma, \end{aligned}$$

so that $\Lambda \leq \frac{1}{2}P(E) \lesssim 1$. Therefore, if Λ is large enough, we must have $\sigma = 0$ or equivalently that E is also a minimizer of $\mathcal{F}_{Q,0}$. □

Let now E be a minimizer of (4-1). In order to prove the regularity of E , we shall construct a competitor in the following way: Since E is a convex body, there exists ε_0 such that for $\varepsilon \leq \varepsilon_0$, and every $x_0 \in \partial E$, we have $\partial E \cap \partial B_\varepsilon(x_0) = \{x_1^\varepsilon, x_2^\varepsilon\}$ (in particular $|x_0 - x_i^\varepsilon| = \varepsilon$). Let us fix x_0 . For $\varepsilon \leq \varepsilon_0$, let $x_1^\varepsilon, x_2^\varepsilon$ be given as above and let L_ε be the line joining x_1^ε to x_2^ε . Denote by H_ε^+ the half-space with boundary L_ε containing x_0 (and H_ε^- be its complementary). We then define our competitor as

$$E_\varepsilon := E \cap H_\varepsilon^-.$$

Let us fix some further notation (see Figure 1):

- We denote by $\Pi : \partial E \cap H_\varepsilon^+ \rightarrow L_\varepsilon$ the projection of the cap of ∂E inside H_ε^+ , on L_ε . We shall extend Π to the whole ∂E as the identity, outside $\partial E \cap H_\varepsilon^+$.
- If $f \mathcal{H}^1 \llcorner \partial E$ is the optimal measure for $\mathcal{I}_0(E)$, we let $f_\varepsilon := \Pi_{\#} f$ (which is defined on ∂E_ε) so that $\mu_\varepsilon := f_\varepsilon \mathcal{H}^1 \llcorner \partial E_\varepsilon$ is a competitor for $\mathcal{I}_0(E_\varepsilon)$.
- For $x, y \in \partial E$, we denote by $\gamma_\varepsilon(x, y)$ the acute angle between the line $L_{x,y}$ joining x to y and L_ε (if $L_{x,y}$ is parallel to L_ε , we set $\gamma_\varepsilon(x, y) = 0$).
- If $y = x_0$, then we define $\gamma_\varepsilon(x) := \gamma_\varepsilon(x, x_0)$.
- We let $\gamma_\varepsilon := \gamma_\varepsilon(x_1^\varepsilon) = \gamma_\varepsilon(x_2^\varepsilon)$.
- We let $\partial B_{3\varepsilon}(x_0) \cap \partial E = \{x_1^{3\varepsilon}, x_2^{3\varepsilon}\}$. As before, we define $H_{3\varepsilon}^+$ as the half-space bounded by $L_{x_1^{3\varepsilon}, x_2^{3\varepsilon}}$ containing x_0 and $H_{3\varepsilon}^-$ as its complement. Let then $\Sigma_\varepsilon := \partial E \cap H_\varepsilon^+$, $\Sigma_{3\varepsilon} := \partial E \cap H_{3\varepsilon}^+$ and $\Gamma_\varepsilon := \partial E \cap H_{3\varepsilon}^-$.

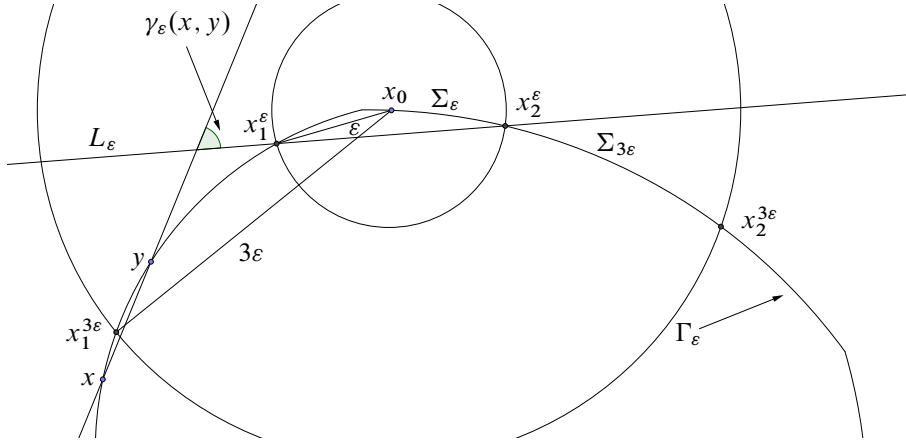


Figure 1

- We let $\Delta V := |E| - |E_\varepsilon|$, $\Delta P := P(E) - P(E_\varepsilon)$ and $\Delta \mathcal{I}_0 := \mathcal{I}_0(E_\varepsilon) - \mathcal{I}_0(E)$.

We point out some simple remarks:

- Thanks to Theorem 3.1 we have that the optimal measure f satisfies $f \in L^p(\partial E)$ for some $p = p(E) > 2$.
- If E is a convex body then γ_ε is bounded away from $\frac{\pi}{2}$ and $|x_1^{3\varepsilon} - x_1^\varepsilon| \sim |x_2^{3\varepsilon} - x_2^\varepsilon| \sim \varepsilon$.
- The quantities ΔV , ΔP and $\Delta \mathcal{I}_0$ are nonnegative by definition.
- All the constants involved up to now depend only on the Lipschitz character of ∂E . In particular, if E_n is a sequence of convex bodies converging to a convex body E , then these constants depend only on the geometry of E .

Before stating the main result of this section, we prove two regularity lemmas.

Lemma 4.2. *Let $0 < \beta \leq 1$ and $C, \varepsilon_0 > 0$ be given. Then, every convex body E such that for every $x_0 \in \partial E$ and every $\varepsilon \leq \varepsilon_0$,*

$$\Delta V \leq C\varepsilon^{2+\beta}, \tag{4-2}$$

is $C^{1,\beta}$ with $C^{1,\beta}$ -norm depending only on the Lipschitz character of ∂E , ε_0 and C .

Proof. Let $x_0 \in \partial E$ be fixed. Since E is convex, there exist $R > 0$ and a convex function $u : I \rightarrow \mathbb{R}$ such that $\partial E \cap B_R(x_0) = \{(t, u(t)) : t \in I\}$ for some interval $I \subset \mathbb{R}$. Furthermore, $\|u'\|_{L^\infty} \lesssim 1$. Let $\bar{x} \in \partial E \cap B_R(x_0)$. Without loss of generality, we can assume that $\bar{x} = 0 = (0, u(0))$. By the convexity of u , up to adding a linear function, we can further assume that $u \geq 0$ in I . Thanks to the Lipschitz bound on u , for $x = (t, u(t)) \in \partial E \cap B_R(x_0)$, we have

$$|x| = (t^2 + |u(t)|^2)^{\frac{1}{2}} \sim t. \tag{4-3}$$

Let now $\varepsilon > 0$. For $\delta > 0$, let $-1 \ll t_1^\delta < 0 < t_2^\delta \ll 1$ such that $x_i^\delta = (t_i^\delta, u(t_i^\delta))$ for $i = 1, 2$ (see the notation above). By (4-3), there exists $\lambda > 0$, depending only on the Lipschitz character of u , such that

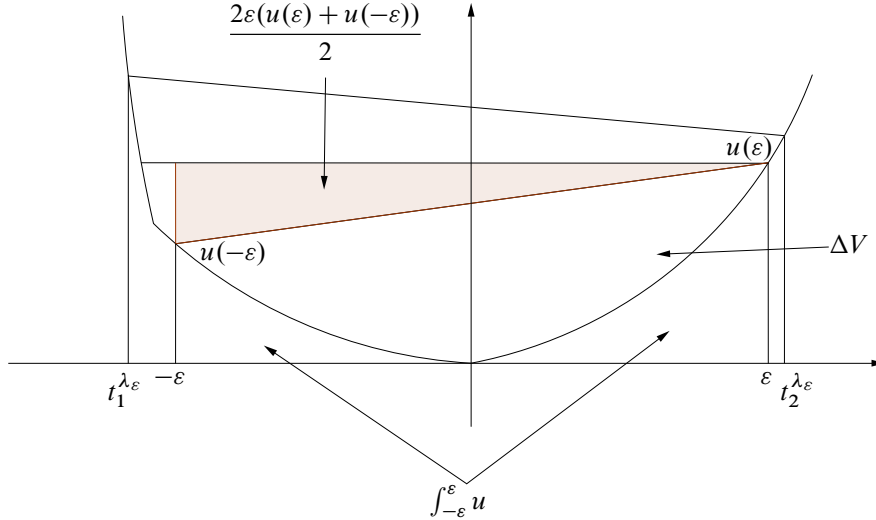


Figure 2

$|t_i^{\lambda_\varepsilon}| \geq \varepsilon$. Without loss of generality, we can now assume that $u(-\varepsilon) \leq u(\varepsilon)$. In particular, considering the ΔV associated to λ_ε , we have that (see Figure 2)

$$\begin{aligned} \Delta V &\geq 2\varepsilon u(\varepsilon) - \frac{2\varepsilon(u(\varepsilon) - u(-\varepsilon))}{2} - \int_{-\varepsilon}^{\varepsilon} u(t) dt \\ &= \varepsilon(u(\varepsilon) + u(-\varepsilon)) - \int_{-\varepsilon}^{\varepsilon} u(t) dt. \end{aligned}$$

Since u is decreasing in $[-\varepsilon, 0]$ and increasing in $[0, \varepsilon]$, this means that both

$$\varepsilon u(\varepsilon) - \int_0^\varepsilon u \lesssim \varepsilon^{2+\beta} \quad \text{and} \quad \varepsilon u(-\varepsilon) - \int_{-\varepsilon}^0 u \lesssim \varepsilon^{2+\beta} \tag{4-4}$$

hold. Let us prove that this implies that for $|t|$ small enough

$$u(t) \lesssim |t|^{1+\beta}. \tag{4-5}$$

We can assume without loss of generality that $t > 0$. By (4-4) and the monotonicity of u ,

$$tu(t) \leq Ct^{2+\beta} + \int_0^{\frac{t}{2}} u + \int_{\frac{t}{2}}^t u \leq Ct^{2+\beta} + \frac{1}{2}t(u(\frac{1}{2}t) + u(t)),$$

from which we obtain

$$u(t) - u(\frac{1}{2}t) \lesssim t^{1+\beta}.$$

Applying this for $k \geq 0$ to $t_k = 2^{-k}t$ and summing over k we obtain

$$u(t) \lesssim \sum_{k=0}^{\infty} (2^{-k}t)^{1+\beta} \lesssim t^{1+\beta},$$

that is, (4-5).

In other words, we have proven that u is differentiable in zero with $u'(0) = 0$ and that for $|t|$ small enough,

$$|u(t) - u(0) - u'(0)t| \lesssim |t|^{1+\beta}.$$

Since the point zero was arbitrarily chosen, this yields that u is differentiable everywhere and that for $t, s \in I$ with $|t - s|$ small enough,

$$|u(t) - u(s) - u'(s)(t - s)| \lesssim |t - s|^{\beta+1},$$

which is equivalent to the $C^{1,\beta}$ regularity of ∂E .³ □

Lemma 4.3. *Suppose that the minimizer E for (4-1) has boundary of class $C^{1,\beta}$ for some $0 < \beta < 1$. Then, there exists $R > 0$ (depending only on the $C^{1,\beta}$ character of ∂E) such that for every $x_0 \in \partial E$, $x \in \Sigma_\varepsilon$ and $y \in B_R(x_0)$,*

$$\gamma_\varepsilon(x, y) \lesssim \varepsilon^\beta + |x - y|^\beta. \tag{4-6}$$

Proof. Without loss of generality, we can assume that $x_0 = 0$. As in the proof of Lemma 4.2, since E is convex and of class $C^{1,\beta}$ in the ball $B_R(0)$ for a small enough R , we know that ∂E is a graph over its tangent of a $C^{1,\beta}$ function u . Up to a rotation, we can further assume that this tangent is horizontal so that for some interval $I \subset \mathbb{R}$, we have $\partial E \cap B_R(0) = \{(t, u(t)) : t \in I\}$. In particular, if $x = (t, u(t)) \in \partial E \cap B_R(0)$, $|u(t)| \lesssim |t|^{1+\beta}$ and $|u'(t)| \lesssim |t|^\beta$.

For $x = (t, u(t)) \in \Sigma_\varepsilon$ and $y = (s, u(s)) \in B_R(0)$, let $\tilde{\gamma}_\varepsilon(x, y)$ be the angle between $L_{x,y}$ and the horizontal line; i.e., $\tan(\tilde{\gamma}_\varepsilon(x, y)) = |u(t) - u(s)| / |t - s|$. Let us begin by estimating $\tilde{\gamma}_\varepsilon$. First, if $|x - y| \lesssim \varepsilon$ (which thanks to (4-3) amounts to $|t - s| \lesssim \varepsilon$ and thus $|t| + |s| \lesssim \varepsilon$ since $x \in \Sigma_\varepsilon$),

$$\tilde{\gamma}_\varepsilon(x, y) \sim \frac{|u(t) - u(s)|}{|t - s|} \leq \sup_{r \in [s,t]} |u'(r)| \lesssim \varepsilon^\beta.$$

Otherwise, if $|x - y| \gg \varepsilon$, since $|x| \lesssim \varepsilon$, we have $|x - y| \sim |y| \sim |s|$ and thus

$$\tilde{\gamma}_\varepsilon(x, y) \lesssim \frac{|u(t)| + |u(s)|}{|t - s|} \lesssim \frac{\varepsilon^{1+\beta} + |s|^{1+\beta}}{|s|} \lesssim |s|^\beta \lesssim |x - y|^\beta.$$

Putting these estimates together, we find

$$\tilde{\gamma}_\varepsilon(x, y) \lesssim \varepsilon^\beta + |x - y|^\beta. \tag{4-7}$$

Let ξ_ε be the angle between L_ε and the horizontal line (see Figure 3). Since $\gamma_\varepsilon(x, y) = \tilde{\gamma}_\varepsilon \pm \xi_\varepsilon$, (4-6) holds provided that we can show

$$\xi_\varepsilon \lesssim \varepsilon^\beta. \tag{4-8}$$

Let $t_1^\varepsilon, t_2^\varepsilon \sim \varepsilon$ be such that $x_1^\varepsilon = (-t_1^\varepsilon, u(-t_1^\varepsilon))$ and $x_2^\varepsilon = (t_2^\varepsilon, u(t_2^\varepsilon))$. We see that ξ_ε is maximal if $u(-t_1^\varepsilon) = 0$, and then $t_1^\varepsilon = \varepsilon$. In that case, $\tan \xi_\varepsilon = u(t_2^\varepsilon) / (\varepsilon + t_2^\varepsilon)$.

³Indeed, for $|s - t| \leq \varepsilon_1$, we have $|u'(t) - u'(s)| \leq |t - s|^{-1} (|u(t) - u(s) - u'(s)(t - s)| + |u(s) - u(t) - u'(t)(s - t)|) \lesssim |t - s|^\beta$.

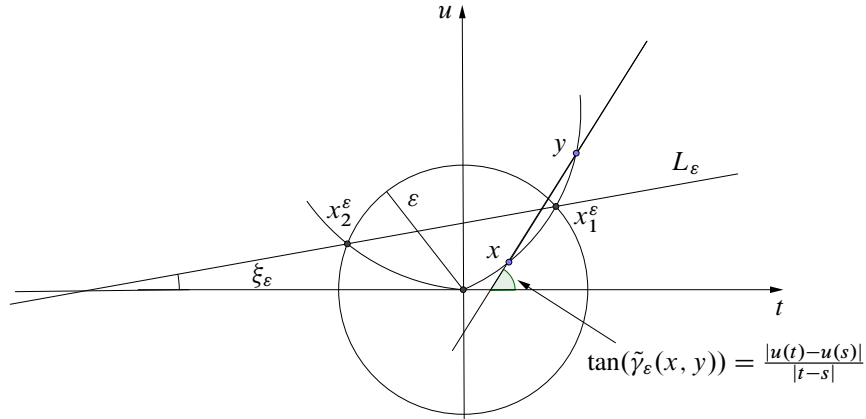


Figure 3

Since $u(t_2^\varepsilon) \lesssim \varepsilon^{1+\beta}$, and $t_\varepsilon^2 \lesssim \varepsilon$, we obtain

$$\xi_\varepsilon \sim \tan \xi_\varepsilon \lesssim \frac{\varepsilon^{1+\beta}}{\varepsilon} = \varepsilon^\beta,$$

proving (4-8). This concludes the proof of (4-6). □

We pass now to the main result of this section.

Theorem 4.4. *Every minimizer of (4-1) is $C^{1,1}$. Moreover, for every Q_0 and every $Q \leq Q_0$, the $C^{1,1}$ character of ∂E depends only on Q_0 , the Lipschitz character of ∂E and $\|f\|_{L^p(\partial E)}$.*

Proof. Let E be a minimizer of (4-1), $x_0 \in \partial E$ be fixed and let $\varepsilon \leq \varepsilon_0$. With the above notation in force, we begin by observing that using E_ε as a competitor, by the minimality of E for (4-1), we have

$$Q^2 \Delta \mathcal{I}_0 \geq \Delta P - \Lambda \Delta V. \tag{4-9}$$

We are thus going to estimate $\Delta \mathcal{I}_0$, ΔP and ΔV in terms of ε and γ_ε . This will give us a quantitative decay estimate for γ_ε . This in turn, in light of (4-10) below and Lemma 4.2, will provide the desired regularity of E .

Step 1 (volume estimate): In this first step, we prove that

$$\Delta V \sim \varepsilon^2 \gamma_\varepsilon. \tag{4-10}$$

By construction, we have

$$\Delta V = |E| - |E_\varepsilon| = |E \cap H_\varepsilon^+|.$$

By convexity, we first have that the triangle with vertices $x_0, x_1^\varepsilon, x_2^\varepsilon$ is contained inside $E \cap H_\varepsilon^+$. By convexity again, letting \bar{x}_1^ε be the point of $\partial B_\varepsilon(x_0)$ diametrically opposed to x_1^ε (and similarly for \bar{x}_2^ε), we get that $E \cap H_\varepsilon^+$ is contained in the union of the triangles of vertices $x_1^\varepsilon, x_2^\varepsilon, \bar{x}_1^\varepsilon$ and $x_1^\varepsilon, x_2^\varepsilon, \bar{x}_2^\varepsilon$ (see Figure 4).

Therefore, we obtain

$$\Delta V \sim \varepsilon^2 \cos \gamma_\varepsilon \sin \gamma_\varepsilon \sim \varepsilon^2 \gamma_\varepsilon.$$

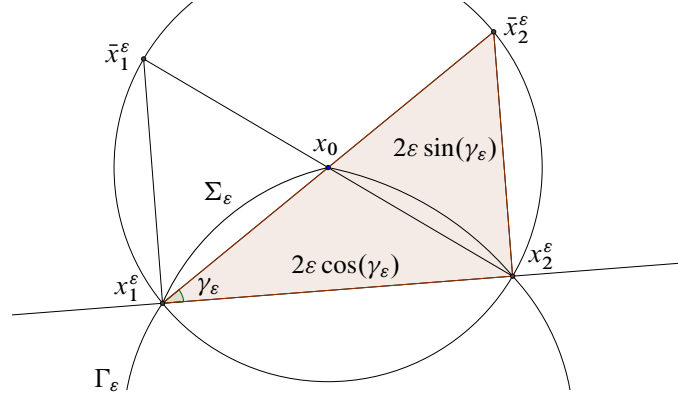


Figure 4. ΔV is contained in the union of the triangles of vertices $x_1^\varepsilon, x_2^\varepsilon, \bar{x}_1^\varepsilon$ and $x_1^\varepsilon, x_2^\varepsilon, \bar{x}_2^\varepsilon$.

Step 2 (perimeter estimate): Since the triangle with vertices $x_0, x_1^\varepsilon, x_2^\varepsilon$ is contained inside $E \cap H_\varepsilon^+$, it holds that

$$\Delta P = P(E) - P(E_\varepsilon) \geq 2\varepsilon(1 - \cos \gamma_\varepsilon) \gtrsim \varepsilon \gamma_\varepsilon^2. \tag{4-11}$$

Step 3 (nonlocal energy estimate): We now estimate $\Delta \mathcal{I}_0$. Since μ_ε is a competitor for $\mathcal{I}_0(E_\varepsilon)$, recalling that Π is the identity outside Σ_ε , we have

$$\begin{aligned} \Delta \mathcal{I}_0 &= \mathcal{I}_0(E_\varepsilon) - \mathcal{I}_0(E) \\ &\leq \int_{\partial E_\varepsilon \times \partial E_\varepsilon} f_\varepsilon(x) f_\varepsilon(y) \log\left(\frac{1}{|x-y|}\right) - \int_{\partial E \times \partial E} f(x) f(y) \log\left(\frac{1}{|x-y|}\right) \\ &= \int_{\partial E \times \partial E} f(x) f(y) \log\left(\frac{1}{|\Pi(x) - \Pi(y)|}\right) - \int_{\partial E \times \partial E} f(x) f(y) \log\left(\frac{1}{|x-y|}\right) \\ &= \int_{\partial E \times \partial E} f(x) f(y) \log\left(\frac{|x-y|}{|\Pi(x) - \Pi(y)|}\right). \end{aligned}$$

Since for $x, y \in \Sigma_\varepsilon^c$, we have $|\Pi(x) - \Pi(y)| = |x - y|$, we get

$$\begin{aligned} \Delta \mathcal{I}_0 &\leq \int_{\Sigma_{3\varepsilon} \times \Sigma_{3\varepsilon}} f(x) f(y) \log\left(\frac{|x-y|}{|\Pi(x) - \Pi(y)|}\right) + 2 \int_{\Sigma_\varepsilon} \int_{\Gamma_\varepsilon} f(x) f(y) \log\left(\frac{|x-y|}{|\Pi(x) - \Pi(y)|}\right) \\ &=: I_1 + 2I_2. \end{aligned}$$

We first estimate I_1 :

$$\begin{aligned} I_1 &= \int_{\Sigma_{3\varepsilon} \times \Sigma_{3\varepsilon}} f(x) f(y) \log\left(1 + \frac{|x-y| - |\Pi(x) - \Pi(y)|}{|\Pi(x) - \Pi(y)|}\right) \\ &\leq \int_{\Sigma_{3\varepsilon} \times \Sigma_{3\varepsilon}} f(x) f(y) \frac{|x-y| - |\Pi(x) - \Pi(y)|}{|\Pi(x) - \Pi(y)|}. \end{aligned}$$

Since for any $x, y \in \Sigma_{3\varepsilon}$ we have (with equality if $x, y \in \Sigma_\varepsilon$),

$$\cos(\gamma_\varepsilon(x, y))|x-y| \leq |\Pi(x) - \Pi(y)|,$$

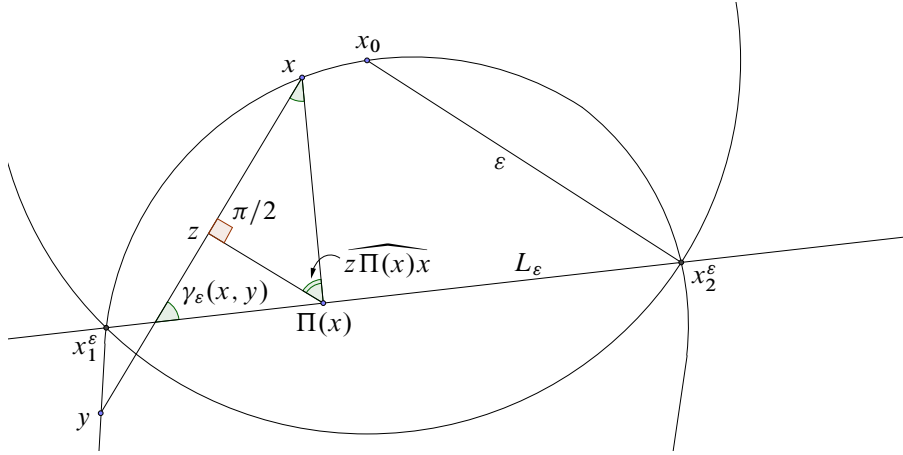


Figure 5. The angle $\widehat{z\Pi(x)x}$ equals $\gamma_\epsilon(x, y)$.

we get

$$I_1 \leq \int_{\Sigma_{3\epsilon} \times \Sigma_{3\epsilon}} f(x)f(y) \left(\frac{1}{\cos(\gamma_\epsilon(x, y))} - 1 \right) \lesssim \int_{\Sigma_{3\epsilon} \times \Sigma_{3\epsilon}} \gamma_\epsilon^2(x, y) f(x)f(y). \tag{4-12}$$

Using then Hölder’s inequality (recall that $f \in L^p(\partial E)$ for some $p > 2$) to get

$$\int_{\Sigma_{3\epsilon}} f \leq \left(\int_{\Sigma_{3\epsilon}} f^p \right)^{\frac{1}{p}} \mathcal{H}^1(\Sigma_{3\epsilon})^{\frac{p-1}{p}} \lesssim \epsilon^{\frac{p-1}{p}}, \tag{4-13}$$

and $\gamma_\epsilon(x, y) \lesssim 1$, we obtain

$$I_1 \lesssim \epsilon^{2\frac{p-1}{p}}. \tag{4-14}$$

We can now estimate I_2 :

$$\begin{aligned} I_2 &= \int_{\Sigma_\epsilon} \int_{\Gamma_\epsilon} f(x)f(y) \log \left(1 + \left(\frac{|x-y| - |\Pi(x)-y|}{|\Pi(x)-y|} \right) \right) \\ &\leq \int_{\Sigma_\epsilon} \int_{\Gamma_\epsilon} f(x)f(y) \left(\frac{|x-y| - |\Pi(x)-y|}{|\Pi(x)-y|} \right). \end{aligned}$$

Denote by z the projection of $\Pi(x)$ on the line containing x and y . Then, since the projection is a 1-Lipschitz function, it holds that $|z-y| \leq |\Pi(x)-y|$. Thus,

$$|x-y| - |y-\Pi(x)| = |x-z| + |z-y| - |y-\Pi(x)| \leq |x-z|.$$

Arguing as in Step 1, we get $|x-\Pi(x)| \leq |\bar{x}_2^\epsilon - x_2^\epsilon| \lesssim \epsilon \gamma_\epsilon$. Furthermore, the angle $\widehat{z\Pi(x)x}$ equals $\gamma_\epsilon(x, y)$ (see Figure 5), so that

$$|x-y| - |y-\Pi(x)| \leq |x-z| = |x-\Pi(x)| \sin(\gamma_\epsilon(x, y)) \lesssim \epsilon \gamma_\epsilon \gamma_\epsilon(x, y).$$

On the other hand, since $|y - x| \geq 2\varepsilon$ (indeed $|x - x_0| \leq \varepsilon$ and $|y - x_0| \geq 3\varepsilon$), we have

$$|y - \Pi(x)| \geq |y - x| - |x - \Pi(x)| \gtrsim |y - x| - \varepsilon \gtrsim |y - x|.$$

Therefore,

$$I_2 \lesssim \varepsilon \gamma_\varepsilon \int_{\Sigma_\varepsilon} \int_{\Gamma_\varepsilon} \frac{f(x)f(y)\gamma_\varepsilon(x,y)}{|y-x|}. \tag{4-15}$$

There exists $M > 0$ which depends only on the Lipschitz character of ∂E such that for $x \in \Sigma_\varepsilon$ and $y \in \Gamma_\varepsilon \cap B_M(x_0)$,

$$|y - x| \geq \min_{i=1,2} |y - x_i^\varepsilon|.$$

Let $\Gamma_\varepsilon^N := \Gamma_\varepsilon \cap B_M(x_0)$ and $\Gamma_\varepsilon^F := \Gamma_\varepsilon \cap B_M^c(x_0)$. We then have

$$\begin{aligned} I_2 &\lesssim \varepsilon \gamma_\varepsilon \left(\int_{\Sigma_\varepsilon \times \Gamma_\varepsilon^N} \frac{f(x)f(y)\gamma_\varepsilon(x,y)}{\min_i |y - x_i^\varepsilon|} + \int_{\Sigma_\varepsilon \times \Gamma_\varepsilon^F} f(x)f(y)\gamma_\varepsilon(x,y) \right) \\ &=: I_2^N + I_2^F. \end{aligned}$$

We begin by estimating I_2^F . Since $\gamma_\varepsilon(x,y) \lesssim 1$, using Hölder’s inequality we find

$$\begin{aligned} I_2^F &\lesssim \varepsilon \gamma_\varepsilon \left(\int_{\Gamma_\varepsilon} f \right) \left(\int_{\Sigma_\varepsilon} f \right) \\ &\leq \varepsilon \gamma_\varepsilon \|f\|_{L^p \mathcal{H}^1(\Gamma_\varepsilon)}^{1-\frac{1}{p}} \|f\|_{L^p \mathcal{H}^1(\Sigma_\varepsilon)}^{1-\frac{1}{p}} \\ &\lesssim \varepsilon \gamma_\varepsilon \mathcal{H}^1(\Sigma_\varepsilon)^{1-\frac{1}{p}} \\ &\lesssim \varepsilon^{2-\frac{1}{p}} \gamma_\varepsilon. \end{aligned} \tag{4-16}$$

We can now estimate I_2^N . Recall that

$$I_2^N := \varepsilon \gamma_\varepsilon \int_{\Sigma_\varepsilon \times \Gamma_\varepsilon^N} \frac{f(x)f(y)\gamma_\varepsilon(x,y)}{\min_i |y - x_i^\varepsilon|}. \tag{4-17}$$

As before, we use $\gamma_\varepsilon(x,y) \lesssim 1$ together with Hölder’s inequality applied twice to get

$$\int_{\Sigma_\varepsilon \times \Gamma_\varepsilon^N} \frac{f(x)f(y)\gamma_\varepsilon(x,y)}{\min_i |y - x_i^\varepsilon|} \lesssim \varepsilon^{1-\frac{1}{p}} \left(\int_{\Gamma_\varepsilon^N} \frac{1}{\min_i |y - x_i^\varepsilon|^{\frac{p}{p-1}}} \right)^{\frac{p-1}{p}}.$$

Since E is convex, its boundary can be locally parametrized by Lipschitz functions so that, if M is small enough (depending only on the Lipschitz regularity of ∂E), then for $y \in \Gamma_\varepsilon^N$, we have

$$\min_i \ell(y, \tilde{x}_i^\varepsilon) \sim \min_i |y - \tilde{x}_i^\varepsilon|$$

(where $\ell(x,y)$ denotes the geodesic distance on ∂E). From this we get

$$\int_{\Gamma_\varepsilon^N} \frac{1}{\min_i |y - x_i^\varepsilon|^{\frac{p}{p-1}}} \lesssim \varepsilon^{-\frac{1}{p-1}}.$$

From this we conclude that

$$I_2^N \lesssim \gamma_\varepsilon \varepsilon^{2-\frac{2}{p}}. \tag{4-18}$$

Step 4 (C^{1,β} regularity): We now prove that E has boundary of class $C^{1,\beta}$. To this aim, we can assume that $\Delta V \ll \Delta P$. Indeed, if $\Delta V \gtrsim \Delta P$, thanks to (4-10) and (4-11), we would get $\gamma_\varepsilon \lesssim \varepsilon$ and thus $\Delta V \lesssim \varepsilon^3$, which by Lemma 4.2 would already ensure the $C^{1,1}$ regularity of ∂E . Using (4-9), (4-11), (4-14), (4-16) and (4-18), we get

$$Q^2(\varepsilon^{1-\frac{2}{p}} + \gamma_\varepsilon(\varepsilon^{1-\frac{1}{p}} + \varepsilon^{1-\frac{2}{p}})) \gtrsim \gamma_\varepsilon^2. \tag{4-19}$$

Now since $\varepsilon^{1-\frac{1}{p}} \lesssim \varepsilon^{1-\frac{2}{p}}$, this reduces further to

$$Q^2(\varepsilon^{1-\frac{2}{p}} + \gamma_\varepsilon \varepsilon^{1-\frac{2}{p}}) \gtrsim \gamma_\varepsilon^2. \tag{4-20}$$

We can now distinguish two cases. Either $Q^2 \varepsilon^{2(\frac{1}{2}-\frac{1}{p})} \gtrsim \gamma_\varepsilon^2$ and then $\gamma_\varepsilon \lesssim Q \varepsilon^{(\frac{1}{2}-\frac{1}{p})}$ or $Q^2 \gamma_\varepsilon \varepsilon^{1-\frac{2}{p}} \gtrsim \gamma_\varepsilon^2$ and then $\gamma_\varepsilon \lesssim Q^2 \varepsilon^{1-\frac{2}{p}}$. Thus in both cases, since $p > 2$, we find $\gamma_\varepsilon \lesssim Q \varepsilon^\beta$ for some $\beta > 0$ and we can conclude, by means of (4-10) and Lemma 4.2, that ∂E is $C^{1,\beta}$.

Step 5 (C^{1,1} regularity): Thanks to Lemma 3.6, we get that $f \in L^\infty$ with $\|f\|_{L^\infty}$ depending only on the Lipschitz character of ∂E and on $\|f\|_{L^p}$. Using this new information, we can improve (4-14), (4-16) and (4-18) to

$$I_1 \lesssim \varepsilon^2, \quad I_2^F \lesssim \gamma_\varepsilon \varepsilon^2, \quad \text{and} \quad I_2^N \lesssim \gamma_\varepsilon \varepsilon^2 |\log \varepsilon|. \tag{4-21}$$

Arguing as in Step 4, we find $\gamma_\varepsilon \lesssim Q \varepsilon^{\frac{1}{2}}$ and thus ∂E is of class $C^{1,\frac{1}{2}}$. In order to get higher regularity, we need to get a better estimate on $\gamma_\varepsilon(x, y)$.

Going back to (4-12) and using (4-6) with $\beta = \frac{1}{2}$, we find the improved estimate

$$I_1 \lesssim \varepsilon^3. \tag{4-22}$$

If we also use (4-6) in (4-17), we obtain

$$\begin{aligned} I_2^N &\lesssim \varepsilon \gamma_\varepsilon \int_{\Sigma_\varepsilon \times \Gamma_\varepsilon^N} \frac{\varepsilon^{\frac{1}{2}} + |x - y|^{\frac{1}{2}}}{\min_i |y - \tilde{x}_i^\varepsilon|} \\ &\lesssim \varepsilon \gamma_\varepsilon \int_{\Sigma_\varepsilon \times \Gamma_\varepsilon^N} \frac{\varepsilon^{\frac{1}{2}} + \min_i \{|x - \tilde{x}_i^\varepsilon|^{\frac{1}{2}} + |y - \tilde{x}_i^\varepsilon|^{\frac{1}{2}}\}}{\min_i |y - \tilde{x}_i^\varepsilon|} \\ &\lesssim \varepsilon \gamma_\varepsilon \int_{\Sigma_\varepsilon \times \Gamma_\varepsilon^N} \frac{\varepsilon^{\frac{1}{2}} + \min_i |y - \tilde{x}_i^\varepsilon|^{\frac{1}{2}}}{\min_i |y - \tilde{x}_i^\varepsilon|} \\ &\lesssim \varepsilon^2 \gamma_\varepsilon \int_{\Gamma_\varepsilon^N} \frac{\varepsilon^{\frac{1}{2}}}{\min_i |y - \tilde{x}_i^\varepsilon|} + \frac{1}{\min_i |y - \tilde{x}_i^\varepsilon|^{\frac{1}{2}}} \\ &\lesssim \varepsilon^2 \gamma_\varepsilon (\varepsilon^{\frac{1}{2}} |\log \varepsilon| + 1) \lesssim \varepsilon^2 \gamma_\varepsilon. \end{aligned}$$

As in the beginning of Step 4, we can assume that $\Delta V \ll \Delta P$, so that by (4-9) and (4-11) we have $Q^2 \Delta \mathcal{I}_0 \gtrsim \Delta P \gtrsim \varepsilon \gamma_\varepsilon^2$. By the previous estimate for I_2^N , (4-22) and the second inequality in (4-21) we have

eventually get

$$Q^2 \varepsilon^2 \gamma_\varepsilon \sim Q^2 (\varepsilon^3 + \varepsilon^2 \gamma_\varepsilon) \gtrsim \varepsilon \gamma_\varepsilon^2,$$

which leads to $\gamma_\varepsilon \lesssim Q^2 \varepsilon$. By using again Lemma 4.2, the proof is concluded. \square

5. Minimality of the ball for $N = 2$ and Q small

We now use the regularity result obtained in Section 4 to prove that for small charges, the only minimizers of $\mathcal{F}_{Q,0}$ in dimension two are balls.

Theorem 5.1. *Let $N = 2$ and $\alpha = 0$. There exists $Q_0 > 0$ such that for $Q < Q_0$, up to translations, the only minimizer of (1-1) is the ball.*

Proof. Let E_Q be a minimizer of $\mathcal{F}_{Q,0}$ and let B be a ball of measure 1. By the minimality of E_Q , we have

$$P(E_Q) - P(B) \leq Q^2 (\mathcal{I}_0(B) - \mathcal{I}_0(E_Q)) \leq Q^2 (\mathcal{I}_0(B) + |\mathcal{I}_0(E_Q)|). \tag{5-1}$$

By Lemma 4.1 the diameter of E_Q is uniformly bounded and so is $|\mathcal{I}_0(E_Q)|$. Using the quantitative isoperimetric inequality, see [Fusco et al. 2008], we infer

$$|E_Q \Delta B|^2 \lesssim P(E_Q) - P(B) \leq Q^2 (\mathcal{I}_0(B) + |\mathcal{I}_0(E_Q)|).$$

This implies that E_Q converges to B in L^1 as $Q \rightarrow 0$. From the convexity of E_Q , this implies the convergence also in the Hausdorff metric. Since the sets E_Q are all uniformly bounded and of fixed volume, they are uniformly Lipschitz. Theorem 4.4 then implies that ∂E_Q are $C^{1,1}$ -regular sets with $C^{1,1}$ norm uniformly bounded. Therefore, thanks to the Arzelà–Ascoli theorem, we can write

$$\partial E_Q = \{(1 + \varphi_Q(x))x : x \in \partial B\},$$

with $\|\varphi_Q\|_{C^{1,\beta}}$ converging to 0 as $Q \rightarrow 0$ for every $\beta < 1$. From Lemma 3.6 we infer that the optimal measures μ_Q for E_Q are uniformly $C^{0,\beta}$ and in particular are uniformly bounded. Using now [Goldman et al. 2015, Proposition 6.3], we get that for small enough Q ,

$$\|\mu_Q\|_{L^\infty}^2 (P(E_Q) - P(B)) \gtrsim \mathcal{I}_0(B) - \mathcal{I}_0(E_Q).$$

Putting this into (5-1), we then obtain

$$P(E_Q) - P(B) \lesssim Q^2 (P(E_Q) - P(B)),$$

from which we deduce that for Q small enough, $P(E_Q) = P(B)$. Since, up to translations, the ball is the unique solution of the isoperimetric problem, this implies $E_Q = B$. \square

6. Asymptotic behavior as $Q \rightarrow +\infty$

We now characterize the limit shape of (suitably rescaled) minimizers of $\mathcal{F}_{Q,\alpha}$, with $\alpha \in [0, 1]$, as the charge Q tends to $+\infty$. For this, we fix a sequence $Q_n \rightarrow +\infty$.

The case $\alpha \in [0, 1)$. For $n \in \mathbb{N}$, we let $V_n := Q_n^{-\frac{2N(N-1)}{1+(N-1)\alpha}}$ (so that $V_n \rightarrow 0$ as $n \rightarrow +\infty$) and

$$\begin{aligned} \mathcal{A}_{n,\alpha} &:= \{E \subset \mathbb{R}^N \text{ convex body, } |E| = V_n\}, \\ \widehat{\mathcal{F}}_{n,\alpha}(E) &:= V_n^{-\frac{N-2}{N-1}} P(E) + \mathcal{I}_\alpha(E) \quad \text{for } E \in \mathcal{A}_{n,\alpha}. \end{aligned}$$

It is straightforward to check that if E is a minimizer of (1-1), then the rescaled set

$$\widehat{E} := Q_n^{-\frac{2(N-1)}{1+(N-1)\alpha}} E$$

is a minimizer of $\widehat{\mathcal{F}}_{n,\alpha}$ in the class $\mathcal{A}_{n,\alpha}$.

We begin with a compactness result for a sequence of sets of equibounded energy.

Proposition 6.1. *Let $\alpha \in [0, 1)$ and let $E_n \in \mathcal{A}_{n,\alpha}$ be such that*

$$\sup_n \widehat{\mathcal{F}}_{n,\alpha}(E_n) < +\infty.$$

Then, up to extracting a subsequence and up to rigid motions, the sets E_n converge in the Hausdorff topology to the segment $[0, L] \times \{0\}^{N-1}$ for some $L \in (0, +\infty)$.

Proof. The bound on $\mathcal{I}_\alpha(E_n)$ directly implies with (2-2) (or (2-3) in the case $\alpha = 0$) that the diameter of E_n is uniformly bounded from below.

Let us show that the diameter of E_n is also uniformly bounded from above. Arguing as in Theorem 2.3, let $\mathcal{R}_n = \prod_{i=1}^N [0, \lambda_i^n]$ be the parallelepipeds given by Lemma 2.2, and assume without loss of generality that $\lambda_1^n \geq \lambda_2^n \geq \dots \geq \lambda_N^n$. In the case $\alpha > 0$, (2-1) directly gives the bound, while for $\alpha = 0$, we get using (2-1) and (2-3), that $|\mathcal{I}_0(\mathcal{R}_n)|$ is uniformly bounded, from which the bound on the diameter follows, using once again (2-1). Moreover, from (2-2) and (2-3), we obtain that $\lambda_i^n \sim V_n^{\frac{1}{N-1}}$ (where the constants depend on $\widehat{\mathcal{F}}_{n,\alpha}(E_n)$) for $i = 2, \dots, N$. The convex bodies E_n are therefore compact in the Hausdorff topology and any limit set is a nontrivial segment of length $L \in (0, +\infty)$. □

In the proof of the Γ -convergence result we will use the following result.

Lemma 6.2. *Let $0 < \gamma < \beta$ with $\beta \geq 1$, $V > 0$ and $L > 0$, then*

$$\min \left\{ \int_0^L f^\gamma : \int_0^L f^\beta = V, f \text{ concave and } f \geq 0 \right\} = \frac{(\beta + 1)^{\frac{\gamma}{\beta}}}{\gamma + 1} L^{1-\frac{\gamma}{\beta}} V^{\frac{\gamma}{\beta}}. \tag{6-1}$$

Proof. For $L, V > 0$, let

$$M(L, V) := \min \left\{ \int_0^L f^\gamma : \int_0^L f^\beta = V, f \text{ concave and } f \geq 0 \right\}.$$

Let us now prove (6-1). By scaling, we can assume that $L = V = 1$. Thanks to the concavity and positivity constraints, existence of a minimizer for (6-1) follows. Let f be such a minimizer. Let us prove that we can assume that f is nonincreasing. Notice first that by definition, it holds that

$$M(1, 1) = \int_0^1 f^\gamma.$$

Up to a rearrangement, we can assume that f is symmetric around the point $\frac{1}{2}$, so that f is nonincreasing in $[\frac{1}{2}, 1]$ and

$$\int_{\frac{1}{2}}^1 f^\gamma = \frac{1}{2}M(1, 1) = M(\frac{1}{2}, \frac{1}{2}).$$

Finally letting $\hat{f}(x) := f(\frac{1}{2}(x + \frac{1}{2}))$ for $x \in [0, 1]$, we have that \hat{f} is nonincreasing, admissible for (6-1) and

$$\int_0^1 \hat{f}^\gamma = 2 \int_{\frac{1}{2}}^1 f^\gamma = M(1, 1),$$

so that \hat{f} is also a minimizer for (6-1).

Assume now that f is not affine in $(0, 1)$. Then there is $\bar{x} > 0$ such that for all $0 < x \leq \bar{x}$

$$f(x) > f(0) - (f(0) - f(1))x.$$

Let $\tilde{f} := \lambda - (\lambda - f(1))x$ with $\lambda > f(0)$ chosen so that

$$\int_0^1 f^{\beta-1} \tilde{f} = \int_0^1 f^\beta. \tag{6-2}$$

Now, let $g := \tilde{f} - f$. Since $f + g = \tilde{f}$ is concave, for every $0 \leq t \leq 1$, we have $f + tg$ is a concave function. For $\delta \in \mathbb{R}$, let $f_{t,\delta} := f + t(g + \delta(1 - x))$. Let finally δ_t be such that

$$\int_0^1 f_{t,\delta_t}^\beta = \int_0^1 f^\beta.$$

Thanks to (6-2) and since $\beta \geq 1$, we have $|\delta_t| = O(t)$. Since f_{t,δ_t} is concave, by the minimality of f we get

$$\int_0^1 f_{t,\delta_t}^\gamma - \int_0^1 f^\gamma \geq 0.$$

Dividing by t and taking the limit as t goes to zero, we obtain

$$\int_0^1 f^{\gamma-1} g \geq 0.$$

Let $z \in (0, 1)$ be the unique point such that $\tilde{f}(z) = f(z)$ (so that $\tilde{f}(x) > f(x)$ for $x < z$ and $\tilde{f}(x) < f(x)$ for $x > z$). We then have

$$\begin{aligned} 0 &\leq \int_0^1 f^{\beta-1} \frac{\tilde{f} - f}{f^{\beta-\gamma}} \\ &= \int_0^z f^{\beta-1} \frac{\tilde{f} - f}{f^{\beta-\gamma}} + \int_z^1 f^{\beta-1} \frac{\tilde{f} - f}{f^{\beta-\gamma}} \\ &< \frac{1}{f^{\beta-\gamma}(z)} \left(\int_0^z f^{\beta-1} (\tilde{f} - f) + \int_z^1 f^{\beta-1} (\tilde{f} - f) \right) \\ &= \frac{1}{f^{\beta-\gamma}(z)} \int_0^1 f^{\beta-1} (\tilde{f} - f), \end{aligned}$$

which contradicts (6-2).

We are left to study the case when f is linear. Assume that $f(1) > 0$ and let

$$\delta := \frac{\int_0^1 f^{\beta-1}}{\int_0^1 x f^{\beta-1}} > 1,$$

so that in particular, $\int_0^1 f^{\beta-1}(1 - \delta x) = 0$. Up to adjusting the volume as in the previous case, for $t > 0$ small enough, $f + t(1 - \delta x)$ is admissible. From this, arguing as above, we find that

$$\int_0^1 f^{\gamma-1}(1 - \delta x) \geq 0.$$

By splitting the integral around the point $\bar{z} = \delta^{-1} \in (0, 1)$ and proceeding as above, we get again a contradiction. As a consequence, we obtain that $f(x) = \lambda(1 - x)$, with $\lambda = (\beta + 1)^{\frac{1}{\beta}}$ so that the volume constraint is satisfied. This concludes the proof of (6-1). \square

We now prove the following Γ -convergence result.

Theorem 6.3. *For $\alpha \in [0, 1)$, the functionals $\widehat{\mathcal{F}}_{n,\alpha}$ Γ -converge in the Hausdorff topology, as $n \rightarrow +\infty$, to the functional*

$$\widehat{\mathcal{F}}_{\alpha}(E) := \begin{cases} C_N L^{\frac{1}{N-1}} + \mathcal{I}_{\alpha}([0, 1])/L^{\alpha} & \text{if } E \simeq [0, L] \times \{0\}^{N-1} \text{ and } \alpha > 0, \\ C_N L^{\frac{1}{N-1}} + \mathcal{I}_0([0, 1]) - \log L & \text{if } E \simeq [0, L] \times \{0\}^{N-1} \text{ and } \alpha = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $E \simeq F$ means that $E = F$ up to a rigid motion, and $C_N := \omega_{\frac{N-1}{N-1}} N^{\frac{N-2}{N-1}}$ with ω_N the volume of the ball of radius 1 in \mathbb{R}^N (so that for $N = 2$ we have $C_2 = 2$).

Proof. By Proposition 6.1 we know that the Γ -limit is $+\infty$ on the sets which are not segments.

Let us first prove the Γ -limsup inequality. Given $L \in (0, +\infty)$, we are going to construct E_n symmetric with respect to the hyperplane $\{0\} \times \mathbb{R}^{N-1}$. For $t \in [0, \frac{L}{2}]$, we let

$$r(t) := \left(\frac{N V_n}{\omega_{N-1} L} \right)^{\frac{1}{N-1}} \left(1 - \frac{2t}{L} \right)$$

and then

$$E_n \cap (\mathbb{R}^+ \times \mathbb{R}^{N-1}) := \left\{ (t, B_{r(t)}^{N-1}) : t \in \left[0, \frac{L}{2} \right] \right\},$$

where $B_{r(t)}^{N-1}$ is the ball of radius $r(t)$ in \mathbb{R}^{N-1} . With this definition, $|E_n| = V_n$, so that $E_n \in \mathcal{A}_{n,\alpha}$. We then compute

$$\begin{aligned} P(E_n) &= 2 \int_0^{\frac{L}{2}} \mathcal{H}^{N-2}(\mathbb{S}^{N-2}) r(t)^{N-2} \sqrt{1 + |r'|^2} \\ &= 2(N-1) \omega_{N-1} \left(\frac{N V_n}{\omega_{N-1} L} \right)^{\frac{N-2}{N-1}} \int_0^{\frac{L}{2}} \left(1 - \frac{2t}{L} \right)^{N-2} \left(1 + \frac{c_N}{L^2} \left(\frac{V_n}{L} \right)^{\frac{2}{N-1}} \right)^{\frac{1}{2}} \\ &= C_N V_n^{\frac{N-2}{N-1}} L^{\frac{1}{N-1}} + o(V_n^{\frac{N-2}{N-1}}). \end{aligned}$$

Letting μ_α be the optimal measure for $\mathcal{I}_\alpha\left(\left[-\frac{L}{2}, \frac{L}{2}\right]\right)$, we then have

$$\widehat{\mathcal{F}}_{n,\alpha}(E_n) \leq C_N L^{\frac{1}{N-1}} + \mathcal{I}_\alpha([0, L]) + o(1),$$

which gives the Γ -limsup inequality.

We now turn to the Γ -liminf inequality. Let $E_n \in \mathcal{A}_{n,\alpha}$ be such that $E_n \rightarrow [0, L] \times \{0\}^{N-1}$ in the Hausdorff topology. Since \mathcal{I}_α is continuous under Hausdorff convergence, it is enough to prove that

$$\liminf_{n \rightarrow +\infty} V_n^{-\frac{N-2}{N-1}} P(E_n) \geq C_N L^{\frac{1}{N-1}}. \tag{6-3}$$

Let $L_n := \text{diam}(E_n)$. By Hausdorff convergence, we have that $L_n \rightarrow L$. Moreover, up to a rotation and a translation, we can assume that $[0, L_n] \times \{0\}^{N-1} \subset E_n$. For $N = 2$, we directly obtain $P(E_n) \geq 2L_n$, which gives (6-3). We thus assume from now on that $N \geq 3$. Let \tilde{E}_n be the set obtained from E_n after a Schwarz symmetrization around the axis $\mathbb{R} \times \{0\}^{N-1}$. By Brunn’s principle [1887], \tilde{E}_n is still a convex set with $P(E_n) \geq P(\tilde{E}_n)$ and $|E_n| = |\tilde{E}_n|$. We thus have

$$\tilde{E}_n = \bigcup_{t \in [0, L_n]} \{t\} \times B_{r(t)}^{N-1}$$

for an appropriate function $r(t)$, and, by Fubini’s theorem,

$$\int_0^{L_n} r(t)^{N-1} dt = \frac{V_n}{\omega_{N-1}}.$$

By the coarea formula [Ambrosio et al. 2000, Theorem 2.93], we then get

$$P(\tilde{E}_n) \geq \mathcal{H}^{N-2}(\mathbb{S}^{N-2}) \int_0^{L_n} r(t)^{N-2} \sqrt{1 + |r'(t)|^2} dt \geq \mathcal{H}^{N-2}(\mathbb{S}^{N-2}) \int_0^{L_n} r(t)^{N-2} dt.$$

Applying then Lemma 6.2 with $\gamma = N - 2$ and $\beta = N - 1$, we obtain (6-3). □

Remark 6.4. For $\alpha \in [0, 1)$ and $N \geq 2$, it is easy to optimize \widehat{F}_α in L and obtain the values $L_{N,\alpha}$ given in Theorem 1.4.

From Proposition 6.1, Theorem 6.3 and the uniqueness of the minimizers for \widehat{F}_α , we directly obtain the following asymptotic result for minimizers of (1-1).

Corollary 6.5. *Let $\alpha \in [0, 1)$ and $N \geq 2$. Then, up to rescalings and rigid motions, every sequence E_n of minimizers of (1-1) converges in the Hausdorff topology to $[0, L_{N,\alpha}] \times \{0\}^{N-1}$.*

The case $N = 2, 3$ and $\alpha = 1$. In the case $\alpha \geq 1$, the energy \mathcal{I}_α is infinite on segments and thus a Γ -limit of the same type as the one obtained in Theorem 6.3 cannot be expected. Nevertheless in the Coulombic case $N = 3, \alpha = 1$ we can use a dual formulation of the nonlocal part of the energy to obtain the Γ -limit. As a by-product, we can also treat the case $N = 2, \alpha = 1$.

For $N = 2, 3$ and $n \in \mathbb{N}$, we let

$$\begin{aligned} \mathcal{A}_{n,1} &:= \{E \subset \mathbb{R}^3 \text{ convex body, } |E| = Q_n^{-2(N-1)} (\log Q_n)^{-(N-1)}\}, \\ \widehat{\mathcal{F}}_{n,1}(E) &:= Q_n^{2(N-2)} (\log Q_n)^{N-2} P(E) + \frac{\mathcal{I}_1(E)}{\log Q_n} \quad \text{for } E \in \mathcal{A}_{n,1}. \end{aligned}$$

As before, if E is a minimizer of (1-1), then the rescaled set

$$\widehat{E} := Q_n^{-\frac{2(N-1)}{N}} (\log Q_n)^{-\frac{(N-1)}{N}} E$$

is a minimizer of $\widehat{\mathcal{F}}_{n,1}$ in $\mathcal{A}_{n,1}$.

Let $C_\varepsilon := [0, 1] \times B_\varepsilon \subset \mathbb{R}^3$ be a narrow cylinder of radius $\varepsilon > 0$ (where B_ε denotes a two-dimensional ball of radius ε). We begin by proving the following estimate on $\mathcal{I}_1(C_\varepsilon)$:

Proposition 6.6. *It holds that*

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{I}_1(C_\varepsilon)}{|\log \varepsilon|} = 2. \tag{6-4}$$

As a consequence, for every $L > 0$,

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{I}_1([0, L] \times B_\varepsilon)}{|\log \varepsilon|} = \frac{2}{L}. \tag{6-5}$$

Proof. The equality in (6-4) is well known; see for instance [Maxwell 1877]. We include here a proof for the reader’s convenience.

To show that

$$\lim_{\varepsilon \rightarrow 0} |\log \varepsilon|^{-1} \mathcal{I}_1(C_\varepsilon) \leq 2,$$

we use $\mu_\varepsilon := (1/(\pi\varepsilon^2))\chi_{C_\varepsilon}$ as a test measure in the definition of $\mathcal{I}_1(C_\varepsilon)$. Then, noting that for every $y \in C_\varepsilon$,

$$\int_{C_\varepsilon+y} \frac{dz}{|z|} \leq \int_{[-\frac{1}{2}, \frac{1}{2}] \times B_\varepsilon} \frac{dz}{|z|},$$

we obtain

$$\begin{aligned} \mathcal{I}_1(C_\varepsilon) &\leq \frac{1}{\pi^2 \varepsilon^4} \int_{C_\varepsilon \times C_\varepsilon} \frac{dx dy}{|x - y|} = \frac{1}{\pi^2 \varepsilon^4} \int_{C_\varepsilon} \left(\int_{C_\varepsilon+y} \frac{dz}{|z|} \right) dy \\ &\leq \frac{1}{\pi \varepsilon^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{B_\varepsilon} \frac{1}{(z_1^2 + |(z_2, z_3)|^2)^{\frac{1}{2}}} = \frac{4}{\varepsilon^2} \int_0^{\frac{1}{2}} \int_0^\varepsilon \frac{r}{(z_1^2 + r^2)^{\frac{1}{2}}} \\ &= \frac{4}{\varepsilon^2} \int_0^{\frac{1}{2}} \sqrt{z_1^2 + \varepsilon^2} - z_1 \\ &= \frac{4}{\varepsilon^2} \left(\frac{1}{8} \sqrt{1 + 4\varepsilon^2} - \frac{1}{8} + \frac{\varepsilon^2}{2} \log \left(\frac{1}{2\varepsilon} + \sqrt{1 + \frac{1}{4\varepsilon^2}} \right) \right) \\ &= 2|\log \varepsilon| + o(|\log \varepsilon|). \end{aligned}$$

In order to show the opposite inequality, we recall the following definition of capacity of a set E :

$$\text{Cap}(E) := \min \left\{ \int_{\mathbb{R}^3} |\nabla \phi|^2 : \chi_E \leq \phi, \phi \in H_0^1(\mathbb{R}^3) \right\}.$$

Then, if E is compact, we have [Landkof 1972; Goldman et al. 2015]

$$\mathcal{I}_1(E) = \frac{4\pi}{\text{Cap}(E)}.$$

Thus (6-4) will be proved once we show that

$$\text{Cap}(C_\varepsilon)|\log \varepsilon| \leq 2\pi + o(1). \tag{6-6}$$

For this, let $\lambda > 0$ and $\mu > 0$ to be fixed later and let

$$f_\lambda(x') := \begin{cases} 1 & \text{for } |x'| \leq \varepsilon, \\ 1 - \log(|x'|/\varepsilon)/\log(\lambda/\varepsilon) & \text{for } \varepsilon \leq |x'| \leq \lambda, \\ 0 & \text{for } |x'| \geq \lambda \end{cases}$$

and

$$\rho_\mu(z) := \begin{cases} 0 & \text{for } z \leq -\mu, \\ (z + \mu)/\mu & \text{for } -\mu \leq z \leq 0, \\ 1 & \text{for } 0 \leq z \leq 1, \\ 1 - (z - 1)/\mu & \text{for } 1 \leq z \leq 1 + \mu, \\ 0 & \text{for } z \geq 1 + \mu. \end{cases}$$

We finally let $\phi(x', z) := f_\lambda(x')\rho_\mu(z)$. Since $\rho_\mu, f_\lambda \leq 1$ and $|\rho'_\mu| \leq \mu^{-1}$, by the definition of $\text{Cap}(C_\varepsilon)$, we have

$$\begin{aligned} \text{Cap}(C_\varepsilon) &\leq \int_0^1 \frac{2\pi}{\log(\lambda/\varepsilon)^2} \int_\varepsilon^\lambda \frac{1}{r} + C \left(\frac{\mu}{\log(\lambda/\varepsilon)} + \frac{\lambda^2}{\mu} \right) \\ &\leq \frac{2\pi}{\log(\lambda/\varepsilon)} + C \left(\frac{\mu}{\log(\lambda/\varepsilon)} + \frac{\lambda^2}{\mu} \right). \end{aligned}$$

We now choose $\lambda := |\log \varepsilon|^{-1} \gg \varepsilon$ and $\mu := |\log \lambda|^{-1} = (\log|\log \varepsilon|)^{-1}$ so that $\log(\lambda/\varepsilon) = |\log \varepsilon| + \log|\log \varepsilon|$, $\mu \rightarrow 0$ and $\mu \gg \lambda$; thus

$$\frac{\mu}{\log(\lambda/\varepsilon)} + \frac{\lambda^2}{\mu} = o(|\log \varepsilon|^{-1})$$

and we find (6-6).

The equality in (6-5) then follows by scaling. □

As a simple corollary we get the two-dimensional result

Corollary 6.7.
$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{I}_1([0, 1] \times [0, \varepsilon])}{|\log \varepsilon|} = 2. \tag{6-7}$$

Proof. The upper bound is obtained as above by testing with $\mu_\varepsilon := \varepsilon^{-1} \chi_{[0,1] \times [0,\varepsilon]}$. By identifying $[0, 1] \times [0, \varepsilon]$ with $[0, 1] \times [0, \varepsilon] \times \{0\} \subset C_\varepsilon$ we get that $\mathcal{I}_1([0, 1] \times [0, \varepsilon]) \geq \mathcal{I}_1(C_\varepsilon)$. This gives, together with (6-4), the corresponding lower bound. □

We can now prove a compactness result analogous to Proposition 6.1.

Proposition 6.8. *Let $E_n \in \mathcal{A}_{n,1}$ be such that $\sup_n \widehat{\mathcal{F}}_{n,1}(E_n) < +\infty$. Then, up to extracting a subsequence and up to rigid motions, the sets E_n converge in the Hausdorff topology to a segment $[0, L] \times \{0\}^N$ for some $L \in (0, +\infty)$.*

Proof. We argue as in the proof of Proposition 6.1. Since the case $N = 2$ is easier, we focus on $N = 3$. Let $\mathcal{R}_n = \prod_{i=1}^3 [0, \lambda_{i,n}]$ be given by Lemma 2.2 and let us assume without loss of generality that $i \mapsto \lambda_{i,n}$ is decreasing. Then (2-1) applied with $V = Q_n^{-4}(\log Q_n)^{-2}$, directly yields an upper bound on $\lambda_{1,n}$ (and thus on $\text{diam}(E_n)$).

We now show that the diameter of E_n is also uniformly bounded from below. Unfortunately, (2-2) does not give the right bound and we need to refine it using (6-4). As in Proposition 6.1, the energy bound $\mathcal{I}_1(E_n) \lesssim \log Q_n$, directly implies that

$$\lambda_{1,n} \gtrsim \frac{1}{\log Q_n},$$

from which, using (2-1) and $\prod_{i=1}^3 \lambda_{i,n} \sim Q_n^{-4}(\log Q_n)^{-2}$, we get

$$\lambda_{2,n} \lesssim Q_n^{-2}.$$

In particular, it follows that

$$\frac{\lambda_{2,n}}{\lambda_{1,n}} \lesssim \frac{\log Q_n}{Q_n^2}.$$

By Proposition 6.6, letting $\varepsilon_n := Q_n^{-2} \log Q_n$ we get

$$\begin{aligned} \lambda_{1,n} \log Q_n &\gtrsim \lambda_{1,n} \mathcal{I}_1(E_n) \sim \lambda_{1,n} \mathcal{I}_1(\mathcal{R}_n) \\ &= \mathcal{I}_1\left(\prod_{i=1}^3 \left[0, \frac{\lambda_{i,n}}{\lambda_{1,n}}\right]\right) \gtrsim \mathcal{I}_1(C_{\varepsilon_n}) \\ &\sim |\log \varepsilon_n| \sim \log Q_n, \end{aligned}$$

which implies

$$\lambda_{1,n} \gtrsim 1,$$

and gives a lower bound on the diameter of E_n .

Arguing as in the proof of (2-2), we then get

$$\lambda_{3,n} \leq \lambda_{2,n} \lesssim Q_n^{-2}(\log Q_n)^{-1}. \tag{6-8}$$

It follows that the sets E_n are compact in the Hausdorff topology, and any limit set is a segment of length $L \in (0, +\infty)$. □

Arguing as in Theorem 6.3, we obtain the following result.

Theorem 6.9. *The functionals $\widehat{\mathcal{F}}_{n,1}$ Γ -converge in the Hausdorff topology to the functional*

$$\widehat{\mathcal{F}}_1(E) := \begin{cases} C_N L^{\frac{1}{N-1}} + \frac{4}{L} & \text{if } E \simeq [0, L] \times \{0\}^{N-1}, \\ +\infty & \text{otherwise,} \end{cases}$$

where C_N is defined as in Theorem 6.3.

Proof. Since the case $N = 2$ is easier, we focus on $N = 3$. The compactness and lower bound for the perimeter are obtained exactly as in Theorem 6.3. For the upper bound, for $L > 0$ and $n \in \mathbb{N}$, we define E_n as in the proof of Theorem 6.3 by first letting $V_n := Q_n^{-4}(\log Q_n)^{-2}$ (recall that $N = 3$) and then

for $t \in [0, \frac{L}{2}]$,

$$r(t) := \left(\frac{3V_n}{\pi L}\right)^{\frac{1}{2}} \left(1 - \frac{2t}{L}\right)$$

and

$$E_n \cap (\mathbb{R}^+ \times \mathbb{R}^2) := \bigcup_{t \in [0, \frac{L}{2}]} \{t\} \times B_{r(t)}^2,$$

where $B_{r(t)}^2$ is the ball of radius $r(t)$ in \mathbb{R}^2 .

As in the proof of Theorem 6.3, we have

$$\lim_{n \rightarrow +\infty} Q_n^2 \log Q_n P(E_n) = C_3 L^{\frac{1}{2}}.$$

Let μ_n be the optimal measure for $\mathcal{I}_1(E_n)$, and let

$$\varepsilon_n := \left(\frac{3V_n}{\pi L}\right)^{\frac{1}{2}}.$$

For $L > \delta > 0$, we have $[-\frac{L-\delta}{2}, \frac{L-\delta}{2}] \times B_{\varepsilon_n}^2 \subset E_n$ so that by (6-5),

$$\mathcal{I}_1(E_n) \leq \mathcal{I}_1\left(\left[-\frac{L-\delta}{2}, \frac{L-\delta}{2}\right] \times B_{\varepsilon_n}^2\right) = \frac{|\log V_n|}{(L-\delta)} + o(|\log V_n|).$$

Recalling that $|\log V_n| = 4|\log Q_n| + o(|\log Q_n|)$, we then get

$$\overline{\lim}_{n \rightarrow +\infty} \frac{\mathcal{I}_1(E_n)}{\log(Q_n)} \leq \frac{4}{L-\delta}.$$

Letting $\delta \rightarrow 0^+$, we obtain the upper bound.

We are left to prove the lower bound for the nonlocal part of the energy. Let E_n be a sequence of convex sets such that $E_n \rightarrow [0, L] \times \{0\}^2$ and such that $|E_n| = Q_n^{-4}(\log Q_n)^{-2}$. We can assume that $\sup_n \widehat{\mathcal{F}}_{n,1}(E_n) < +\infty$, since otherwise there is nothing to prove. Let $\delta > 0$. Up to a rotation and a translation, we can assume that $[0, L-\delta] \times \{0\}^2 \subset E_n \subset [0, L+\delta] \times \mathbb{R}^2$ for n large enough. Let now $x^1 = (x_1^1, x_2^1, x_3^1)$ be such that

$$|(x_2^1, x_3^1)| = \max_{x \in E_n} |(x_2, x_3)|.$$

Up to a rotation of axis $\mathbb{R} \times \{0\}^2$, we can assume that $x^1 = (a, \ell_1^n, 0)$ for some $\ell_1^n \geq 0$. Let finally x^2 be such that

$$|x^2 \cdot e_3| = \max_{x \in E_n} |x \cdot e_3|$$

so that $x^2 = (b, c, \ell_2^n)$ with $\ell_2^n \leq \ell_1^n$. Since by definition $E_n \subset [0, L+\delta] \times [-\ell_1^n, \ell_1^n] \times [-\ell_2^n, \ell_2^n]$, we have $Q_n^{-4}(\log Q_n)^{-2} = |E_n| \lesssim \ell_1^n \ell_2^n (L+\delta)$. On the other hand, by convexity, the tetrahedron T with vertices $0, x_1, x_2$ and $(L-\delta, 0, 0)$ is contained in E_n . We thus have $|E_n| \geq |T|$. Since

$$|T| = \frac{1}{8} |\det(x^1, x^2, (L-\delta, 0, 0))| = \frac{1}{8} (L-\delta) \ell_1^n \ell_2^n,$$

we also have $Q_n^{-4}(\log Q_n)^{-2} \gtrsim \ell_1^n \ell_2^n (L - \delta)$. Arguing as in the proof of (2-2), we get from the energy bound, $(L - \delta)\ell_1^n \lesssim Q_n^{-2}(\log Q_n)^{-1}$, and thus

$$\ell_1^n \ell_2^n \gtrsim \frac{1}{(L - \delta) Q_n^4 (\log Q_n)^2}.$$

From this we get $\ell_1^n \sim \ell_2^n \sim Q_n^{-2}(\log Q_n)^{-1}$, where the constants involved might depend on L . We therefore have $E_n \subset [0, L + \delta] \times B_{C Q_n^{-2}(\log Q_n)^{-1}}$ for C large enough. From this we infer that

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \frac{\mathcal{I}_1(E_n)}{\log Q_n} &\geq \liminf_{n \rightarrow +\infty} \frac{\mathcal{I}_1([0, L + \delta] \times B_{C Q_n^{-2}(\log Q_n)^{-1}})}{\log Q_n} \\ &\geq 2 \liminf_{n \rightarrow +\infty} \frac{\mathcal{I}_1([0, L + \delta] \times B_{C Q_n^{-2}(\log Q_n)^{-1}})}{\log(C Q_n^{-2}(\log Q_n)^{-1})} \geq 4(L + \delta)^{-1}, \end{aligned}$$

where the last inequality follows from (6-5). Letting $\delta \rightarrow 0$, we conclude the proof. \square

Remark 6.10. As before, optimizing $\hat{\mathcal{F}}_1$ with respect to L , one easily obtains the values of $L_{N,1}$ given in Theorem 1.4.

Remark 6.11. By analogy with results obtained in the setting of minimal Riesz energy point configurations [Hardin and Saff 2005; Martínez-Finkelshtein et al. 2004], we believe that for every $N \geq 2$, $\alpha > 1$ and $L > 0$, (6-5) can be generalized to

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{I}_\alpha([0, L] \times [0, \varepsilon]^{N-1})}{\varepsilon^{1-\alpha}} = \frac{C_\alpha}{L^\alpha} \quad (6-9)$$

for some constant C_α depending only on α . This result would permit one to extend Theorem 6.9 beyond $\alpha = 1$. Let us point out that showing that the right-hand side of (6-9) is bigger than the left-hand side can be easily obtained by plugging in the uniform measure as a test measure. However, we are not able to prove the reverse inequality.

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References

- [Alt and Caffarelli 1981] H. W. Alt and L. A. Caffarelli, “Existence and regularity for a minimum problem with free boundary”, *J. Reine Angew. Math.* **1981**:325 (1981), 105–144. MR Zbl
- [Ambrosio et al. 2000] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Clarendon, New York, 2000. MR Zbl
- [Brunn 1887] H. Brunn, *Über Ovale und Eiflächen*, dissertation, Ludwig Maximilian University of Munich, 1887. JFM
- [Crasta et al. 2005] G. Crasta, I. Fragalà, and F. Gazzola, “On a long-standing conjecture by Pólya–Szegő and related topics”, *Z. Angew. Math. Phys.* **56**:5 (2005), 763–782. MR Zbl
- [Dahlberg 1977] B. E. J. Dahlberg, “Estimates of harmonic measure”, *Arch. Rational Mech. Anal.* **65**:3 (1977), 275–288. MR Zbl

- [Esposito and Fusco 2011] L. Esposito and N. Fusco, “A remark on a free interface problem with volume constraint”, *J. Convex Anal.* **18**:2 (2011), 417–426. MR Zbl
- [Fontelos and Friedman 2004] M. A. Fontelos and A. Friedman, “Symmetry-breaking bifurcations of charged drops”, *Arch. Ration. Mech. Anal.* **172**:2 (2004), 267–294. MR Zbl
- [Fusco et al. 2008] N. Fusco, F. Maggi, and A. Pratelli, “The sharp quantitative isoperimetric inequality”, *Ann. of Math. (2)* **168**:3 (2008), 941–980. MR Zbl
- [Garnett and Marshall 2005] J. B. Garnett and D. E. Marshall, *Harmonic measure*, New Mathematical Monographs **2**, Cambridge University Press, 2005. MR Zbl
- [Goldman and Novaga 2012] M. Goldman and M. Novaga, “Volume-constrained minimizers for the prescribed curvature problem in periodic media”, *Calc. Var. Partial Differential Equations* **44**:3-4 (2012), 297–318. MR Zbl
- [Goldman et al. 2015] M. Goldman, M. Novaga, and B. Ruffini, “Existence and stability for a non-local isoperimetric model of charged liquid drops”, *Arch. Ration. Mech. Anal.* **217**:1 (2015), 1–36. MR Zbl
- [Hardin and Saff 2005] D. P. Hardin and E. B. Saff, “Minimal Riesz energy point configurations for rectifiable d -dimensional manifolds”, *Adv. Math.* **193**:1 (2005), 174–204. MR Zbl
- [Jerison 1996] D. Jerison, “A Minkowski problem for electrostatic capacity”, *Acta Math.* **176**:1 (1996), 1–47. MR Zbl
- [Jerison and Kenig 1982] D. S. Jerison and C. E. Kenig, “Boundary behavior of harmonic functions in nontangentially accessible domains”, *Adv. in Math.* **46**:1 (1982), 80–147. MR Zbl
- [John 1948] F. John, “Extremum problems with inequalities as subsidiary conditions”, pp. 187–204 in *Studies and essays presented to R. Courant on his 60th birthday, January 8, 1948*, Interscience, New York, 1948. MR Zbl
- [Kenig and Toro 1997] C. E. Kenig and T. Toro, “Harmonic measure on locally flat domains”, *Duke Math. J.* **87**:3 (1997), 509–551. MR Zbl
- [Kenig and Toro 1999] C. E. Kenig and T. Toro, “Free boundary regularity for harmonic measures and Poisson kernels”, *Ann. of Math. (2)* **150**:2 (1999), 369–454. MR Zbl
- [Lamboley et al. 2012] J. Lamboley, A. Novruzzi, and M. Pierre, “Regularity and singularities of optimal convex shapes in the plane”, *Arch. Ration. Mech. Anal.* **205**:1 (2012), 311–343. MR Zbl
- [Lamboley et al. 2016] J. Lamboley, A. Novruzzi, and M. Pierre, “Estimates of first and second order shape derivatives in nonsmooth multidimensional domains and applications”, *J. Funct. Anal.* **270**:7 (2016), 2616–2652. MR Zbl
- [Landkof 1972] N. S. Landkof, *Foundations of modern potential theory*, Die Grundlehren der Mathematischen Wissenschaften **180**, Springer, 1972. MR Zbl
- [Martínez-Finkelshtein et al. 2004] A. Martínez-Finkelshtein, V. Maymeskul, E. A. Rakhmanov, and E. B. Saff, “Asymptotics for minimal discrete Riesz energy on curves in \mathbb{R}^d ”, *Canad. J. Math.* **56**:3 (2004), 529–552. MR Zbl
- [Maxwell 1877] C. Maxwell, “On the electrical capacity of a long narrow cylinder, and of a disk of sensible thickness”, *Proc. Lond. Math. Soc.* **9** (1877), 94–99. MR Zbl
- [Muratov and Novaga 2016] C. B. Muratov and M. Novaga, “On well-posedness of variational models of charged drops”, *Proc. A.* **472**:2187 (2016), art. id. 20150808. MR Zbl
- [Muratov et al. 2016] C. B. Muratov, M. Novaga, and B. Ruffini, “On equilibrium shapes of charged flat drops”, preprint, 2016. To appear in *Comm. Pure Appl. Math.* arXiv
- [Novaga and Ruffini 2015] M. Novaga and B. Ruffini, “Brunn–Minkowski inequality for the 1-Riesz capacity and level set convexity for the $1/2$ -Laplacian”, *J. Convex Anal.* **22**:4 (2015), 1125–1134. MR
- [Pommerenke 1992] C. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften **299**, Springer, 1992. MR Zbl
- [Saff and Totik 1997] E. B. Saff and V. Totik, *Logarithmic potentials with external fields*, Grundlehren der Mathematischen Wissenschaften **316**, Springer, 1997. MR Zbl
- [Strutt (Lord Rayleigh) 1882] J. W. Strutt (Lord Rayleigh), “On the equilibrium of liquid conducting masses charged with electricity”, *Phil. Mag.* **14** (1882), 184–186.
- [Taylor 1964] G. Taylor, “Disintegration of water drops in an electric field”, *Proc. Roy. Soc. Lond. A* **280**:1382 (1964), 383–397. Zbl

[Warschawski and Schober 1966] S. E. Warschawski and G. E. Schober, “On conformal mapping of certain classes of Jordan domains”, *Arch. Rational Mech. Anal.* **22** (1966), 201–209. MR Zbl

[Zeleny 1917] J. Zeleny, “Instability of electricified liquid surfaces”, *Phys. Rev.* **10**:1 (1917), 1–6.

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MICHAEL GOLDMAN: goldman@math.univ-paris-diderot.fr

Université Paris-Diderot, Sorbonne Paris-Cité, Sorbonne Université, CNRS, Laboratoire Jacques-Louis Lions, Paris, France

MATTEO NOVAGA: matteo.novaga@unipi.it

Dipartimento di Matematica, Università di Pisa, Pisa, Italy

BERARDO RUFFINI: berardo.ruffini@umontpellier.fr

Institut Montpellierain Alexander Grothendieck, University of Montpellier, CNRS, Montpellier, France

NONAUTONOMOUS MAXIMAL L^p -REGULARITY UNDER FRACTIONAL SOBOLEV REGULARITY IN TIME

STEPHAN FACKLER

We prove nonautonomous maximal L^p -regularity results on UMD spaces, replacing the common Hölder assumption by a weaker fractional Sobolev regularity in time. This generalizes recent Hilbert space results by Dier and Zacher. In particular, on $L^q(\Omega)$ we obtain maximal L^p -regularity for $p \geq 2$ and elliptic operators in divergence form with uniform VMO-modulus in space and $W^{\alpha,p}$ -regularity for $\alpha > \frac{1}{2}$ in time.

1. Introduction

In this work we improve some known results on maximal L^p -regularity of nonautonomous abstract Cauchy problems with time-dependent domains of the form

$$\begin{cases} \dot{u}(t) + A(t)u(t) = f(t), \\ u(0) = u_0. \end{cases} \quad (\text{NACP})$$

In particular, we obtain new stronger results if the operators $A(t)$ are elliptic operators in divergence form. Let us right away start with the definition.

Definition 1.1. For a family $(A(t))_{t \in [0, T]}$ of closed linear operators on some Banach space X the problem (NACP) has *maximal L^p -regularity* if for all $f \in L^p([0, T]; X)$ and all initial values u_0 in the real interpolation space $(D(A(0)), X)_{1/p, p}$ there exists a unique solution $u \in L^p([0, T]; X)$ satisfying $u(t) \in D(A(t))$ for almost all $t \in [0, T]$ as well as $\dot{u}, A(\cdot)u(\cdot) \in L^p([0, T]; X)$ and if there exists $C > 0$ such that one has the maximal regularity a priori estimate

$$\|u\|_{W^{1,p}([0, T]; X)} + \|A(\cdot)u(\cdot)\|_{L^p([0, T]; X)} \leq C(\|f\|_{L^p([0, T]; X)} + \|u_0\|_{(D(A(0)), X)_{1/p, p}}).$$

Observe that $W^{1,p}([0, T]; X) \hookrightarrow C([0, T]; X)$ and therefore the initial condition makes sense. Maximal regularity results have profound applications to nonlinear parabolic problems, as we will exemplify in Section 8.

We now give a summary of the previously known results on maximal L^p -regularity. The autonomous case $A(t) = A$ is well understood. Here, maximal L^p -regularity holds for one $p \in (1, \infty)$ if and only if it holds for all $p \in (1, \infty)$. Further, maximal L^p -regularity for $u_0 = 0$ implies maximal L^p -regularity for

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all $u_0 \in D(A, X)_{1/p, p}$. On Hilbert spaces an operator A has maximal L^p -regularity if and only if $-A$ generates an analytic semigroup. In non-Hilbert spaces, not every generator of an analytic semigroup has maximal regularity; see [Kalton and Lancien 2000; Fackler 2014]. Here, an additional \mathcal{R} -boundedness assumption is needed. We refer to Section 3, [Denk et al. 2003] and [Kunstmann and Weis 2004] for details.

Let us come to the nonautonomous case. Here the best understood setting is that of nonautonomous forms on Hilbert spaces. For this let V, H be two complex Hilbert spaces with a dense embedding $V \hookrightarrow H$. A mapping $a : [0, T] \times V \times V \rightarrow \mathbb{C}$ is called a *coercive, bounded sesquilinear form* if $a(t, \cdot, \cdot)$ is sesquilinear for all $t \in [0, T]$ and if there exist $\alpha, M > 0$ such that for all $u, v \in V$

$$\begin{aligned} \operatorname{Re} a(t, u, u) &\geq \alpha \|u\|_V^2, \\ |a(t, u, v)| &\leq M \|u\|_V \|v\|_V. \end{aligned} \tag{1-1}$$

This induces operators $\mathcal{A}(t) : V \rightarrow V'$. We denote their parts in H by $A(t)$. It has been shown in [Haak and Ouhabaz 2015] that the operators $(A(t))_{t \in [0, T]}$ satisfy maximal L^p -regularity for all $p \in (1, \infty)$ if $t \mapsto A(t)$ is α -Hölder continuous for some $\alpha > \frac{1}{2}$. For $\alpha > \frac{1}{2}$ and maximal L^2 -regularity this has been improved to the fractional Sobolev regularity $\mathcal{A} \in \dot{W}^{\alpha, 2}([0, T]; \mathcal{B}(V, V'))$ [Dier and Zacher 2017]. If one considers elliptic divergence form operators

$$L(t) = -\operatorname{div}(A(t)\nabla \cdot)$$

for coefficients $A(t) = (a_{ij}(t))$ realized by the form method (see Section 7), this translates into the regularity of the mappings $t \mapsto a_{ij}(t, \cdot) \in L^\infty$, i.e., $a_{ij} \in \dot{W}^{\alpha, 2}([0, T]; L^\infty)$ for some $\alpha > \frac{1}{2}$. The less regularity one needs here, the more applicable the results are to nonlinear problems in the form of a priori estimates. In the special case of elliptic operators in divergence form, some more refined results are available; see [Auscher and Egert 2016; Fackler 2017b]. However, all results have in common that one needs some differentiability in time of order at least $\frac{1}{2}$. This is no coincidence. Recent counterexamples to Lions' problem by the author [Fackler 2017a] show that maximal L^p -regularity can fail if $\mathcal{A} \in C^{1/2}([0, T]; \mathcal{B}(V, V'))$. For more details see the recent survey on maximal L^2 -regularity of nonautonomous forms [Arendt et al. 2017]. Dealing with nonlinear problems, one needs some form of Sobolev embedding to carry out the usual iteration procedure. In higher dimensional cases maximal regularity on $X = L^2(\Omega)$ is too weak for the embeddings to hold. Therefore one is interested in maximal regularity on $X = L^q(\Omega)$ for q big enough.

Nonautonomous maximal L^p -regularity on Banach spaces is far more involved. The classical works for time-dependent domains are [Hieber and Monniaux 2000a; 2000b]. Although the general method used there is applicable on Banach spaces, maximal L^p -regularity was first only obtained on Hilbert spaces in a nonform setting [Hieber and Monniaux 2000a] and in [Hieber and Monniaux 2000b] extrapolated to $X = L^q(\Omega)$ for smooth bounded domains Ω and elliptic operators assuming $a_{ij} \in C^\alpha([0, T]; C^1(\bar{\Omega}))$ for some $\alpha > \frac{1}{2}$. A true generalization of this approach to Banach spaces was obtained in [Portal and Štrkalj 2006] using the emerging concept of \mathcal{R} -boundedness. Already the results in [Hieber and Monniaux 2000b] indicate a fundamental new issue in the non-Hilbert space setting. Whereas on L^2 the coefficients only need to be measurable in space, on L^q all known results require some regularity in space. Recently,

the author lowered the needed regularity in space and showed maximal L^p -regularity on $L^q(\Omega)$ for elliptic operators in divergence form if the coefficients have a uniform VMO-modulus [Fackler 2015].

The aim of this work is to generalize the results in both [Dier and Zacher 2017] and [Fackler 2015]. We show maximal L^p -regularity on UMD Banach spaces assuming fractional Sobolev regularity as in [Dier and Zacher 2017]. To give a flavor of the proved results let us formulate a particular consequence of our general result for elliptic operators in divergence form.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 -domain, $T > 0$ and $a_{ij} \in L^\infty([0, T] \times \Omega)$ for $i, j = 1, \dots, n$. Assume further that there exists $\delta > 0$ such that for almost all $(t, x) \in [0, T] \times \Omega$ and all $\xi \in \mathbb{C}^n$ the ellipticity estimate*

$$\operatorname{Re} \sum_{i,j=1}^n a_{ij}(t, x) \xi_i \bar{\xi}_j \geq \delta |\xi|^2$$

holds and that for $t \in [0, T]$ the functions $x \mapsto a_{ij}(t, x)$ lie in $\operatorname{VMO}(\Omega)$ with uniform VMO-modulus. Then for all $q \in (1, \infty)$ the nonautonomous problem (NACP) associated to the operators $(-\operatorname{div} A \nabla \cdot)_{t \in [0, T]}$ has maximal L^p -regularity

- (a) *for $p \in (1, 2]$ if $a_{ij} \in \dot{W}^{1/2+\varepsilon, 2}([0, T]; L^\infty(\Omega))$ for some $\varepsilon > 0$,*
- (b) *for $p \in [2, \infty)$ if $a_{ij} \in \dot{W}^{1/2+\varepsilon, p}([0, T]; L^\infty(\Omega))$ for some $\varepsilon > 0$.*

Here, the divergence form operators on $L^q(\Omega)$ are compatible with the operator on $L^2(\Omega)$ obtained via the form method (for a precise definition see Section 7). Note that in comparison to [Hieber and Monniaux 2000b], the regularity in space is lowered from $C^1(\bar{\Omega})$ to $\operatorname{VMO}(\Omega)$ and the time regularity $C^{1/2+\varepsilon}$ is replaced by $\dot{W}^{1/2+\varepsilon, p}$ in the case $p \geq 2$. This is the lower time regularity we aim for and leads to more refined results in nonlinear PDE, as we illustrate in Section 8. The general result makes use of some more technical definitions and we postpone its formulation to Section 3.

The obtained results are even new in the Hilbert space case as [Dier and Zacher 2017] fully relies on Hilbert space methods and therefore only deals with the case $p = 2$. Our result is the first improvement of the time regularity on non-Hilbert spaces since the classical work [Acquistapace and Terreni 1987]. Since we establish maximal L^p -regularity for elliptic operators on $L^q(\Omega)$ for $q > 2$, we obtain existence results for *strong* solutions of quasilinear parabolic equations in divergence form. Such results cannot be obtained with maximal regularity results on Hilbert spaces. We further show that our results are optimal in the sense that in general we cannot relax the regularity to some $\alpha \leq \frac{1}{2}$.

Note that, in contrast, elliptic operators in nondivergence form have time-independent domains and one can therefore obtain maximal L^p -regularity only assuming the time dependence to be measurable; see for example [Gallarati and Veraar 2017b; Dong and Kim 2016] for recent results. However, note that in correspondence with our results, one still needs a variant of VMO-regularity in space.

This work is structured as follows. The first sections introduce the necessary mathematical background. The main result and the strategy of proof is then presented in Section 3. The proof of the main result is given in Section 6. As a consequence, we obtain in Theorem 7.4 the stated result for elliptic operators. Section 8 uses this result to establish strong solutions of quasilinear elliptic equations. We discuss the optimality of our results in Section 9.

2. Extrapolation spaces and the fundamental identity

Using ideas established in [Acquistapace and Terreni 1987] and their previous works, we show that maximal L^p -regularity solutions of (NACP) satisfy a certain integral equation. It turns out that this equation is better approachable with analytic tools. We recall some basic definitions first and introduce the fundamental concept of extrapolation spaces. For $\varphi \in (0, \pi)$ we denote by $\Sigma_\varphi := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \varphi\}$ the sector of angle φ . If λ does not lie in the spectrum $\sigma(A) \subset \mathbb{C}$ of A , we write $R(\lambda, A) = (\lambda - A)^{-1}$ for its resolvent.

Definition 2.1. A linear operator $A : D(A) \rightarrow X$ on a Banach space X is *sectorial of angle φ* if the spectrum $\sigma(A)$ of A is contained in $\overline{\Sigma}_\varphi$ for some $\varphi \in (0, \frac{\pi}{2})$ and if

$$\sup_{\lambda \notin \overline{\Sigma}_\varphi} (|\lambda| + 1) \|R(\lambda, A)\| < \infty.$$

A family of linear operators $A_i : D(A_i) \rightarrow X$ for $i \in I$ is *uniformly sectorial* if $\sigma(A_i) \subset \overline{\Sigma}_\varphi$ for some $\varphi \in (0, \frac{\pi}{2})$ and all $i \in I$ and if there exists $C > 0$ with

$$\sup_{\lambda \notin \overline{\Sigma}_\varphi} (|\lambda| + 1) \|R(\lambda, A_i)\| \leq C \quad \text{for all } i \in I.$$

Recall that a closed operator A is sectorial if and only if $-A$ generates an exponentially stable analytic semigroup [Engel and Nagel 2000, Chapter II, Section 4 and Chapter V, Section I]. In particular, A is invertible.

In the following we need interpolation and extrapolation spaces associated to a sectorial operator A on some Banach space X , a fully developed theory carefully presented in [Amann 1995]. We only discuss spaces associated to the complex interpolation method $[\cdot, \cdot]_\theta$ [Bergh and Löfström 1976, Chapter 4]. The results to be obtained hold for several other, but not all, scales of interpolation and extrapolation spaces. As a unified treatment would lead to a more abstract presentation, we focus on this important setting.

We define $X_{1,A} = D(A)$ endowed with the norm $x \mapsto \|Ax\|$ and $X_{-1,A}$ as the completion of X with respect to the norm $x \mapsto \|A^{-1}x\|$. For $\theta \in (0, 1)$ we further let $X_{\theta,A} = [X, X_{1,A}]_\theta$ and $X_{-\theta,A} = [X, X_{-1,A}]_\theta$. The operator $A : X_{1,A} \rightarrow X$ and its extension $A_{-1} : X \rightarrow X_{-1,A}$ are isometries. By interpolation, for $\theta \in (0, 1)$ the part $A_{-\theta}$ of A_{-1} in $X_{-\theta,A}$ is an isometry $A_{-\theta} : X_{1-\theta,A} \rightarrow X_{-\theta,A}$. The operator A_{-1} is sectorial on $X_{-1,A}$ with $\rho(A_{-1}) = \rho(A)$ and satisfies the same sectorial estimates as A . By interpolation, the same holds for the operators $A_{-\theta}$ on $X_{-\theta,A}$. Considering duality, if X is reflexive, one has $(X_{\theta,A})' \simeq X'_{-\theta,A}$ and $(A_\theta)' = A'_{-\theta}$ with respect to the pairing induced by $\langle \cdot, \cdot \rangle_{X, X'}$. Extrapolation spaces allow us to define a weaker notion of solution for (NACP).

Proposition 2.2. *Let $(A(t))_{t \in [0, T]}$ for $T > 0$ be uniformly sectorial operators on some Banach space X . If u is a maximal L^p -regularity solution of (NACP) for the initial value $u_0 = 0$ in the sense of Definition 1.1, then for every fixed $t \in [0, T]$ one has in $X_{-1, A(t)}$*

$$\begin{aligned} u(t) &= \int_0^t e^{-(t-s)A_{-1}(t)} (A_{-1}(t) - A(s))u(s) ds + \int_0^t e^{-(t-s)A(t)} f(s) ds \\ &=: \int_0^t K_1(t, s)u(s) ds + \int_0^t K_2(t, s)f(s) ds =: (S_1 u)(t) + (S_2 f)(t). \end{aligned} \quad (2-1)$$

Proof. Fix $t \in (0, T)$. Consider $v : [0, t] \rightarrow X$ given by $v(s) = e^{-(t-s)A(t)}u(s)$. Then v is differentiable in X almost everywhere and for almost every $s \in (0, t)$ we have

$$\begin{aligned} \dot{v}(s) &= A(t)e^{-(t-s)A(t)}u(s) + e^{-(t-s)A(t)}\dot{u}(s) \\ &= e^{-(t-s)A_{-1}(t)}(A_{-1}(t) - A(s))u(s) + e^{-(t-s)A(t)}f(s). \end{aligned}$$

Notice that $(A_{-1}(t) - A(s))u(s) \in X_{-1,A(t)}$ for almost every $s \in (0, T)$. The fundamental theorem of calculus gives

$$v(t) = v(0) + \int_0^t \dot{v}(s) ds.$$

Inserting the explicit terms for v and \dot{v} and using $u(0) = 0$ yields (2-1). □

3. Formulation of the main result and strategy of proof

The crucial assumption we make is that on a certain extrapolation space the operators get independent of t . For concrete differential operators endowed with some boundary condition this is often satisfied. For this we refer to [Triebel 1978, Section 4.3] for operators with smooth coefficients and to the results originating from the positive solution of the Kato square root problem in [Auscher et al. 2002] for operators with rough coefficients (see also Section 7).

Definition 3.1. For $\theta \in [0, 1]$ a family $(A(t))_{t \in [0, T]}$ of sectorial operators on some Banach space X is called θ -stable if there exists a Banach space $X_{\theta,A}$ and $K \geq 0$ such that for all $t \in [0, T]$ the spaces $X_{\theta,A(t)}$ and $X_{\theta,A}$ agree as vector spaces and

$$K^{-1}\|x\|_{\theta,A} \leq \|x\|_{\theta,A(t)} \leq K\|x\|_{\theta,A} \quad \text{for all } x \in X_{\theta,A} \tag{3-1}$$

and if the same also holds for some space $X_{\theta-1,A}$ and all spaces $X_{\theta-1,A(t)}$.

Note that $(A(t))_{t \in [0, T]}$ is 1-stable if and only if the domains $D(A(t))$ agree for all $t \in [0, T]$ and their norms are uniformly equivalent. Further, as already mentioned in the Introduction, even for the autonomous case $A(t) = A$, maximal L^p -regularity may fail on non-Hilbert spaces. This has to do with particular features of harmonic analysis on Banach spaces. In particular, it is by now well-understood that the classical multiplier results only hold in the vector-valued setting if one makes additional assumptions both on the Banach space and the multiplier. We now introduce the necessary background.

Definition 3.2. A Banach space X is called a *UMD space* if for one, or by Hörmander’s condition all $p \in (1, \infty)$, the vector-valued Hilbert transform

$$(Hf)(x) = \lim_{\varepsilon \downarrow 0} \int_{|t| \geq \varepsilon} \frac{f(x-t)}{t} dt$$

initially defined on $C_c^\infty(\mathbb{R}^n; X)$ extends to a bounded operator $L^p(\mathbb{R}; X) \rightarrow L^p(\mathbb{R}; X)$.

In different words, on UMD spaces one of the most basic Fourier multipliers $m(\xi) = \mathbb{1}_{\mathbb{R}_{>0}}(\xi)$ is bounded. Only on those spaces a reasonable multiplier theory can be developed. For our purposes it is sufficient to know that Hilbert and L^p -spaces for $p \in (1, \infty)$ are UMD spaces and that all UMD spaces

are reflexive. For detailed information on UMD spaces we refer to [Rubio de Francia 1986; Burkholder 2001], whereas more on \mathcal{R} -boundedness, to be defined now, can be found in [Denk et al. 2003; Kunstmann and Weis 2004].

Definition 3.3. Let X and Y be Banach spaces. A subset $\mathcal{T} \subseteq \mathcal{B}(X, Y)$ is called \mathcal{R} -bounded if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$, $T_1, \dots, T_n \in \mathcal{T}$, $x_1, \dots, x_n \in X$ and all independent identically distributed random variables $\varepsilon_1, \dots, \varepsilon_n$ on some probability space $(\Omega, \Sigma, \mathbb{P})$ with $\mathbb{P}(\varepsilon_k = \pm 1) = \frac{1}{2}$ one has

$$\mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k T_k x_k \right\|_Y \leq C \mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|_X.$$

The smallest constant $C \geq 0$ for which this holds is denoted by $\mathcal{R}(\mathcal{T})$. Further, we define $\text{Rad } X$ as the closure in $L^1(\Omega, \Sigma, \mathbb{P}; X)$ of finite sums of the form $\sum_{k=1}^n \varepsilon_k x_k$.

Note that the definition of \mathcal{R} -boundedness depends only on the distribution of the random variables and is therefore independent of the probability space. The same holds for the definition of $\text{Rad } X$ up to canonical isomorphisms. We write $\mathcal{R}^{X \rightarrow Y}$ to indicate between which spaces the mapping is considered if it is not clear from the context. Every \mathcal{R} -bounded set is bounded in $\mathcal{B}(X, Y)$. If both $X = Y$ are Hilbert spaces, then the converse holds as well. Further, Kahane’s contraction principle states that $\{z \text{ Id} : |z| \leq 1\}$ has \mathcal{R} -bound at most 2 on every Banach space. By a celebrated theorem of Weis [2001], on a UMD space the autonomous problem $A(t) = A$ has maximal L^p -regularity for one and then for all $p \in (1, \infty)$ if and only if A is \mathcal{R} -sectorial, the \mathcal{R} -boundedness analogue of sectorial operators, up to shifts.

Definition 3.4. A linear operator $A : D(A) \rightarrow X$ on a Banach space X is called \mathcal{R} -sectorial of angle φ if $\sigma(A)$ of A is contained in $\bar{\Sigma}_\varphi$ for some $\varphi \in (0, \frac{\pi}{2})$ and if

$$\mathcal{R}\{(|\lambda| + 1)R(\lambda, A) : \lambda \notin \bar{\Sigma}_\varphi\} < \infty.$$

A family of linear operators $A_i : D(A_i) \rightarrow X$ for $i \in I$ is uniformly \mathcal{R} -sectorial if $\sigma(A_i) \subset \bar{\Sigma}_\varphi$ for some $\varphi \in (0, \frac{\pi}{2})$ and all $i \in I$ and if there exists $C > 0$ with

$$\mathcal{R}\{(|\lambda| + 1)R(\lambda, A_i) : \lambda \notin \bar{\Sigma}_\varphi\} \leq C \quad \text{for all } i \in I.$$

The main point in our maximal L^p -regularity result is that it only assumes the operators to lie in a fractional Sobolev space.

Definition 3.5. Let X be a Banach space, $p \in (1, \infty)$ and $\alpha \in (0, 1)$. A Bochner-measurable function $f : [0, T] \rightarrow X$ lies in the homogeneous fractional Sobolev space $\dot{W}^{\alpha,p}([0, T]; X)$ provided

$$\|f\|_{\dot{W}^{\alpha,p}([0,T];X)} = \left(\int_0^T \int_0^T \frac{\|f(t) - f(s)\|_X^p}{|t - s|^{1+\alpha p}} ds dt \right)^{1/p} < \infty.$$

The inhomogeneous Sobolev space $W^{\alpha,p}([0, T]; X)$ is the space of all $f \in L^p([0, T]; X)$ such that $\|f\|_{\dot{W}^{\alpha,p}([0,T];X)} < \infty$.

We remark that there exist equivalent definitions of fractional Sobolev spaces based on Littlewood–Paley decompositions [Amann 2000, Section 3, (3.5)]. The usual embedding results for Sobolev spaces into Hölder spaces hold: for $\alpha \in (0, 1)$ and $p \in (1, \infty)$ with $\alpha > \frac{1}{p}$ one has $W^{\alpha,p}([0, T]; X) \hookrightarrow C^{\alpha-1/p}([0, T]; X)$ [Simon 1990, Corollary 26]. We are now ready to present our general maximal L^p -regularity result that in particular implies Theorem 1.2 presented in the Introduction.

Theorem 3.6. *For $T > 0$ and $\theta \in (0, 1]$ let $(A(t))_{t \in [0, T]}$ be a θ -stable family of uniformly \mathcal{R} -sectorial operators on some UMD space X with fractional regularity $A_{-1} \in \dot{W}^{\alpha,q}([0, T]; \mathcal{B}(X_{\theta,A}, X_{\theta-1,A}))$. Then the nonautonomous problem (NACP) has maximal L^p -regularity*

- (a) for $p \in (1, \frac{1}{1-\theta})$, $q = \frac{1}{1-\theta}$ and $\alpha > 1 - \theta$,
- (b) for $p \in [\frac{1}{1-\theta}, \infty)$, $q = p$ and $\alpha > 1 - \theta$.

Let us compare the above conditions with the Acquistapace–Terreni condition [1987] used in [Hieber and Monniaux 2000b; Portal and Štrkalj 2006]. Apart from some uniform \mathcal{R} -boundedness assumptions they require that there exist constants $0 \leq \gamma < \beta \leq 1$ such that for all $t, s \in [0, T]$ and all $\lambda \notin \bar{\Sigma}_\varphi$ for some $\varphi \in (0, \frac{\pi}{2})$ one has the estimate

$$\|A(t)R(\lambda, A(t))(A(t)^{-1} - A(s)^{-1})\|_{\mathcal{B}(X)} \lesssim \frac{|t-s|^\beta}{1+|\lambda|^{1-\gamma}}.$$

In principle, no regularity assumptions on the domain like θ -stability are made. However, in concrete examples some stability is usually necessary and one chooses $\gamma = 1 - \theta$ to verify the estimate; see for example [Fackler 2015]. Then one requires $\beta > 1 - \theta$ and one arrives at the usual Hölder regularity assumptions. However, for example for elliptic operators with irregular coefficients substantial effort is needed to verify the above inequality from the assumed Hölder regularity on the coefficients. Exactly this is done in [Fackler 2015], where as intermediate steps reformulations of the problem that are close to—but more general than—our setting are used.

The improvement of C^α - to $\dot{W}^{\alpha,p}$ -regularity has direct consequences to applications of maximal regularity to nonlinear PDE. As one can see in Theorem 8.1 and Remark 8.2 our result gives existence results under more relaxed regularity assumptions.

Strategy of proof. In Section 4, we first show existence and uniqueness of less regular integrated solutions than is needed for maximal L^p -regularity. This can be done only assuming some continuity on the operators $A(t)$ on the extrapolation spaces. Afterwards in Section 5, we show that we can bootstrap the regularity of these solutions if the operators are α -Hölder continuous for some *arbitrarily small* exponent $\alpha > 0$. With respect to this we note that our assumptions on the fractional Sobolev space are in a such way that the fractional Sobolev space embeds into the space of α -Hölder continuous functions for *some* $\alpha > 0$. After that we show in Section 6 that this higher regularity of the solutions implies maximal L^p -regularity.

4. Existence and uniqueness of integrated solutions

In this section we show that under certain assumptions a unique solution of (2-1) exists. We next show by interpolation that, given an \mathcal{R} -sectorial operator, one obtains corresponding \mathcal{R} -boundedness estimates on

the induced extrapolation spaces. The following result is not new [Haak et al. 2006, Lemma 6.9]; we give a proof for the sake of completeness. For its proof we use the fact that for an interpolation couple (X, Y) of UMD spaces we have by [Kaip and Saal 2012, Proposition 3.14]

$$[\text{Rad}(X), \text{Rad}(Y)]_\theta = \text{Rad}([X, Y]_\theta). \tag{4-1}$$

Here one uses the facts that $[L^1(\Omega, \Sigma, \mathbb{P}; X), L^1(\Omega, \Sigma, \mathbb{P}; Y)]_\theta = L^1(\Omega, \Sigma, \mathbb{P}; [X, Y]_\theta)$ and that the $\text{Rad}(X)$ -spaces are complemented in the vector-valued $L^1(\Omega, \Sigma, \mathbb{P}; X)$ -spaces if X is UMD.

Lemma 4.1. *Let $A : D(A) \rightarrow X$ be an \mathcal{R} -sectorial operator on a UMD space X . Then for all $\theta_2, \theta_1 \in [-1, 1]$ with $\theta_2 > \theta_1$ and $\theta_2 - \theta_1 \leq 1$ one has with φ as in Definition 3.4 and with constants independent of A*

$$\mathcal{R}^{X_{\theta_1, A} \rightarrow X_{\theta_2, A}} \{(1 + |\lambda|)^{1 - (\theta_2 - \theta_1)} R(\lambda, A) : \lambda \notin \bar{\Sigma}_\varphi\} \lesssim \mathcal{R}^{X \rightarrow X} \{(|\lambda| + 1)R(\lambda, A) : \lambda \notin \bar{\Sigma}_\varphi\}.$$

Proof. The assertion holds for $\theta_1 = \theta_2 \in \{-1, 1\}$. By complex interpolation and (4-1) this extends to $\theta_1 = \theta_2 \in [-1, 1]$. Since $AR(\lambda, A) = \lambda R(\lambda, A) - \text{Id}$, one has for all $\theta_1 \in [-1, 0]$

$$\mathcal{R}^{X_{\theta_1, A} \rightarrow X_{\theta_1 + 1, A}} \{R(\lambda, A) : \lambda \notin \bar{\Sigma}_\varphi\} < \infty.$$

For the case of general θ_2 with $\theta_2 - \theta_1 \leq 1$ consider for given $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \notin -\bar{\Sigma}_\varphi$ and $x_1, \dots, x_n \in X$ the mapping $S = \{z \in \mathbb{C} : \text{Re } z \in [0, 1]\} \rightarrow \text{Rad}(X_{\theta_1, A}) + \text{Rad}(X_{\theta_1 + 1, A})$ given by

$$\mathcal{T}_z : \sum_{k=1}^n \varepsilon_k x_k \mapsto \sum_{k=1}^n \varepsilon_k (1 + \lambda_k)^z R(\lambda_k, -A) x_k.$$

The mapping $z \mapsto \mathcal{T}_z$ is continuous on S and analytic in the interior of S and it follows from Kahane’s contraction principle that the norms of \mathcal{T}_{it} and \mathcal{T}_{1+it} as operators in $\mathcal{B}(\text{Rad}(X_{\theta_1, A}), \text{Rad}(X_{\theta_1 + 1, A}))$ and $\mathcal{B}(\text{Rad}(X_{\theta_1, A}), \text{Rad}(X_{\theta_1, A}))$ are bounded by $e^{|t|\varphi}$ up to a uniform constant. Hence, it follows from the generalized Stein interpolation theorem [Voigt 1992] and (4-1) that for $\alpha \in (0, 1)$

$$\mathcal{T}_\alpha : \text{Rad}(X_{\theta_1, A}) \rightarrow \text{Rad}(X_{\theta_1 + \alpha, A}),$$

which gives the statement by unwinding the definitions of \mathcal{R} -boundedness. □

Remark 4.2. Curiously, the above result fails for the negative Laplacian and the real interpolation method [Haak et al. 2006, Example 6.13]. Hence, this is one step where one cannot work with arbitrary extrapolation spaces.

We establish the existence of a unique solution of (2-1) assuming Hölder regularity of arbitrarily low order.

Definition 4.3. A function $f : [0, T] \rightarrow X$ with values in some Banach space X is α -Hölder continuous for $\alpha \in (0, 1]$ if $\|f(t) - f(s)\| \leq C|t - s|^\alpha$ for some $C \geq 0$ and all $t, s \in [0, T]$. We denote by $C^\alpha([0, T]; X)$ the space of all such functions.

We are now ready to prove the existence of integrated solutions.

Proposition 4.4. For $T > 0$ and $\theta \in (0, 1]$ let $(A(t))_{t \in [0, T]}$ be a θ -stable family of uniformly \mathcal{R} -sectorial operators on some UMD space X . Suppose there exist $\alpha \in (0, 1]$ with $A_{-1} \in C^\alpha([0, T]; \mathcal{B}(X_{\theta, A}, X_{\theta-1, A}))$. Then for all $p \in (1, \infty)$ and $f \in L^p([0, T]; X)$ there exists a unique solution u of the integral equation (2-1) in $L^p([0, T]; X_{\theta, A})$. Further, one has $u \in W^{1,p}([0, T]; X_{\theta-1, A}) \cap L^p([0, T]; X_{\theta, A})$,

$$\begin{cases} \dot{u}(t) + A_{\theta-1}(t)u(t) = f(t), \\ u(0) = 0, \end{cases} \tag{WNACP}$$

and $\|u\|_{L^p([0, T]; X_{\theta, A})}$ only depends on $\|f\|_{L^p([0, T]; X_{\theta-1, A})}$, T , α , θ , K in (3-1) and the constants in the Hölder and \mathcal{R} -sectorial estimates.

Proof. First note that by the uniform sectorial estimates and the properties of extrapolation spaces we have the uniform estimate

$$\|e^{-(t-s)A_{-1}(t)}\|_{\mathcal{B}(X_{\theta-1, A}, X_{\theta, A})} \lesssim |t-s|^{-1}.$$

Using this together with the assumed Hölder regularity on $A_{-1}(\cdot)$ we get

$$\|K_1(t, s)\|_{\mathcal{B}(X_{\theta, A}, X_{\theta, A})} \lesssim |t-s|^{\alpha-1}. \tag{4-2}$$

By Young’s inequality for convolutions we then have the norm estimate

$$\|S_1 u\|_{L^p([0, T]; X_{\theta, A})} \leq \int_0^T s^{\alpha-1} ds \|u\|_{L^p([0, T]; X_{\theta, A})} = \alpha^{-1} T^\alpha \|u\|_{L^p([0, T]; X_{\theta, A})}.$$

Let us show the uniqueness of solutions of (2-1) in $L^p([0, T]; X_{\theta, A})$. Since the equation is linear, it suffices to consider a solution with $u = S_1 u$. Now, for sufficiently small T_0 we have $\|S_1\| < 1$. Hence, $\text{Id} - S_1$ is invertible and consequently $u|_{[0, T_0]} = 0$. Using this information we see that (2-1) for $t > T_0$ reduces to

$$u(t) = \int_{T_0}^t e^{-(t-s)A_{-1}(t)} (A_{-1}(t) - A_{-1}(s)) u(s) ds.$$

By the same argument as before we see that the operator defined by the right-hand side is bounded and invertible on $L^p([T_0, 2T_0]; X_{\theta, A})$. Hence, $u|_{[T_0, 2T_0]} = 0$. Iterating this argument finitely many times gives $u = 0$.

Since $t \mapsto A_{\theta-1} \in \mathcal{B}(X_{\theta, A}, X_{\theta-1, A})$ is a fortiori continuous, it follows from perturbation arguments and Lemma 4.1 that (WNACP) has nonautonomous maximal L^p -regularity for all $p \in (1, \infty)$; see [Prüss and Schnaubelt 2001, Theorem 2.5; Arendt et al. 2007, Theorem 2.7]. This means there exists a unique $w \in W^{1,p}([0, T]; X_{\theta-1, A}) \cap L^p([0, T]; X_{\theta, A})$ satisfying (WNACP) and the corresponding maximal L^p -regularity estimate. Using the same argument as in Proposition 2.2, we see that w satisfies (2-1). By the uniqueness shown in the first part, we have $w = u$. □

5. Bootstrapping regularity

Again, assuming Hölder regularity of arbitrarily small order, we improve the regularity of the obtained integrated solutions with the help of the following bootstrapping result.

Proposition 5.1. For $T > 0$ and $\theta \in (0, 1]$ let $(A(t))_{t \in [0, T]}$ be a θ -stable family of uniformly sectorial operators on some Banach space X satisfying $A_{-1} \in C^\alpha([0, T]; \mathcal{B}(X_{\theta, A}, X_{\theta-1, A}))$ for some $\alpha \in (0, 1]$. If either

- (a) $p \in (\frac{1}{1-\theta}, \infty)$ and $q \in (1, \infty]$, or
- (b) $p = \frac{1}{1-\theta}$ and $q \in (1, \infty)$, or
- (c) $p \in (1, \frac{1}{1-\theta})$ and $q \in (1, \frac{p}{1-p(1-\theta)}]$,

then there exists $C_{pq} > 0$ depending only on T, K in (3-1) and the constants of the sectorial and Hölder estimates such that for all solutions $u \in L^p([0, T]; X_{\theta, A})$ of (2-1) for some right-hand side $f \in L^p([0, T]; X)$ one has

$$\|u\|_{L^q([0, T]; X_{\theta, A})} \leq C_{pq} (\|u\|_{L^p([0, T]; X_{\theta, A})} + \|f\|_{L^p([0, T]; X)}).$$

Proof. By Young’s inequality for convolutions and the kernel estimate (4-2) we have for $q, p, r \in (1, \infty)$ with $\frac{1}{r} + \frac{1}{p} = 1 + \frac{1}{q}$ the estimate

$$\begin{aligned} \left(\int_0^T \|(S_1 u)(t)\|_{X_{\theta, A}}^q dt \right)^{1/q} &\leq \left(\int_0^T \left(\int_0^t (t-s)^{\alpha-1} \|u(s)\|_{X_{\theta, A}} ds \right)^q dt \right)^{1/q} \\ &\lesssim \|s \mapsto s^{\alpha-1}\|_{L^{r, \infty}} \left(\int_0^T \|u(s)\|_{X_{\theta, A}}^p ds \right)^{1/p}. \end{aligned}$$

The weak L^r norm is finite for $r \in (1, \frac{1}{1-\alpha}]$. Hence, S_1 is a bounded operator $L^p([0, T]; X_{\theta, A}) \rightarrow L^q([0, T]; X_{\theta, A})$ for all $p \in (1, \frac{1}{\alpha})$ and $q \in [1, \frac{p}{1-p\alpha}]$. If $p > \frac{1}{\alpha}$, then

$$\begin{aligned} \|(S_1 u)(t)\|_{X_{\theta, A}} &\leq \left(\int_0^t \|K_1(t, s)\|^{p'} ds \right)^{1/p'} \left(\int_0^t \|u(s)\|_{X_{\theta, A}}^p ds \right)^{1/p} \\ &\leq \left(\int_0^t |t-s|^{p'(\alpha-1)} ds \right)^{1/p'} \|u\|_{L^p([0, T]; X_{\theta, A})}. \end{aligned}$$

Hence, $S_1 : L^p([0, T]; X_{\theta, A}) \rightarrow L^\infty([0, T]; X_{\theta, A})$ is bounded for $p > \frac{1}{\alpha}$.

Interpolating the analytic estimate

$$\|e^{-(t-s)A(t)}\|_{\mathcal{B}(X, D(A(t)))} \lesssim |t-s|^{-1}$$

with the boundedness of the semigroups $\|e^{-(t-s)A(t)}\|_{\mathcal{B}(X)} \lesssim 1$, one sees that the kernel of S_2 satisfies

$$\|K_2(t, s)\|_{\mathcal{B}(X, X_{\theta, A(t)})} = \|e^{-(t-s)A(t)}\|_{\mathcal{B}(X, X_{\theta, A(t)})} \lesssim |t-s|^{-\theta}. \tag{5-1}$$

Using Young’s inequality together with the kernel estimate (5-1) and θ -stability, we obtain for $p, q, r \in (1, \infty)$ with $\frac{1}{r} + \frac{1}{p} = 1 + \frac{1}{q}$ the estimate

$$\begin{aligned} \left(\int_0^T \|(S_2 f)(t)\|_{X_{\theta, A}}^q dt \right)^{1/q} &\lesssim \left(\int_0^T \left(\int_0^t (t-s)^{-\theta} \|f(s)\|_X ds \right)^q dt \right)^{1/q} \\ &\lesssim \|s \mapsto s^{-\theta}\|_{L^{r, \infty}} \left(\int_0^T \|f(s)\|_X^p ds \right)^{1/p}. \end{aligned}$$

This time the $L^{r,\infty}$ -norm is finite for $r \in (1, \theta^{-1}]$. Hence, $S_2 : L^p([0, T]; X) \rightarrow L^q([0, T]; X_{\theta,A})$ is bounded for all $p < \frac{1}{1-\theta}$ and $q \in [1, \frac{p}{1-p(1-\theta)}]$. Further, one has $S_2 : L^p([0, T]; X) \rightarrow L^\infty([0, T]; X_{\theta,A})$ for $p > \frac{1}{1-\theta}$. For the stated result, we iterate the obtained regularity improvement finitely often to bootstrap the regularity of u . □

6. Maximal regularity results under fractional Sobolev regularity

In this section we come to the heart of the proof. To the solution obtained in Proposition 4.4 we apply $A_{-1}(t)$ to both sides of (2-1). This gives $A_{-1}(t)u(t) = A_{-1}(t)(S_1u)(t) + A_{-1}(t)(S_2f)(t)$. We show that both summands lie in $L^p([0, T]; X)$. The second summand requires some preliminary work. We rely on the following Hölder continuity of the \mathcal{R} -boundedness constant.

Lemma 6.1. *For $\theta \in (0, 1]$ let $(A(t))_{t \in \mathbb{R}}$ be a θ -stable family of uniformly \mathcal{R} -sectorial operators on some UMD space X . Suppose there exists $\alpha \in (0, 1]$ with $A_{-1} \in C^\alpha([0, T]; \mathcal{B}(X_{\theta,A}, X_{\theta-1,A}))$. Then for all $k \in \mathbb{N}_0$ there exists a constant $C_k > 0$ depending only on K in (3-1) and the constants in the Hölder and \mathcal{R} -sectorial estimate of Definition 3.4 such that for all $t, h \in \mathbb{R}$*

$$\mathcal{R}^{X \rightarrow X} \left\{ (1 + |\xi|)^k \left(\frac{\partial}{\partial \xi} \right)^k [i\xi(R(i\xi, A(t+h)) - R(i\xi, A(t)))] : \xi \in \mathbb{R} \right\} \leq C_k |h|^\alpha.$$

Proof. We first establish the case $k = 0$. For all $t, h \in \mathbb{R}$ the resolvent identity gives

$$R(i\xi, A(t+h)) - R(i\xi, A(t)) = R(i\xi, A_{-1}(t+h))[A_{-1}(t) - A_{-1}(t+h)]R(i\xi, A(t)).$$

By the assumed Hölder regularity on A_{-1} and Lemma 4.1 we get for all $t, h \in \mathbb{R}$

$$\begin{aligned} &\mathcal{R}^{X \rightarrow X} \{i\xi(R(i\xi, A(t+h)) - R(i\xi, A(t)))\} \\ &\lesssim \mathcal{R}^{X_{\theta-1,A} \rightarrow X} \{(1 + |\xi|)^\theta R(i\xi, A_{-1}(t+h))\} \|A_{-1}(t+h) - A_{-1}(t)\|_{\mathcal{B}(X_{\theta,A}, X_{\theta-1,A})} \\ &\qquad \qquad \qquad \times \mathcal{R}^{X \rightarrow X_{\theta,A}} \{(1 + |\xi|)^{1-\theta} R(i\xi, A(t))\} \\ &\lesssim |h|^\alpha. \end{aligned}$$

For the case $k \geq 1$ notice that the map $S : z \mapsto R(z, A(t+h)) - R(z, A(t)) \in \mathcal{B}(X)$ is analytic on the complement of some shifted sector $\Sigma_\varphi + \varepsilon$ and that the above estimate holds there by the same argument. It follows from the Cauchy integral representation of derivatives [Kunstmann and Weis 2004, Example 2.16] that for $S(z) = z(R(z, A(t+h)) - R(z, A(t)))$

$$\mathcal{R} \left\{ (1 + |z|)^k \left(\frac{d}{dz} \right)^k S(z) : z \notin \Sigma_\varphi \right\} \lesssim \mathcal{R} \left\{ S \left(i\xi + \frac{\varepsilon}{2} \right) : \xi \in \mathbb{R} \right\} \lesssim |h|^\alpha. \quad \square$$

Proposition 6.2. *For $T > 0$ and $\theta \in (0, 1]$ let $(A(t))_{t \in \mathbb{R}}$ be a θ -stable family of uniformly \mathcal{R} -sectorial operators on some UMD space X . Suppose there exists $\alpha \in (0, 1]$ with $A_{-1} \in C^\alpha([0, T]; \mathcal{B}(X_{\theta,A}, X_{\theta-1,A}))$. Then $A(\cdot)S_2 : L^p([0, T]; X) \rightarrow L^p([0, T]; X)$ is bounded for all $p \in (1, \infty)$ and its norm only depends on p, K in (3-1) and the constants in the Hölder and \mathcal{R} -sectorial estimates.*

Proof. It is shown in [Hieber and Monniaux 2000b, p. 1053; Fackler 2015, Section 2.4.1] that the boundedness of $A(\cdot)S_2$ follows from the boundedness of the pseudodifferential operator

$$(\widehat{S}f)(t) = \int_{-\infty}^{\infty} a(t, \xi) \widehat{f}(\xi) e^{2\pi i t \xi} d\xi$$

for the operator-valued symbol $a : \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{B}(X)$ given by

$$a(t, \xi) = \begin{cases} i\xi R(i\xi, A(0)), & t < 0, \\ i\xi R(i\xi, A(t)), & t \in [0, T], \\ i\xi R(i\xi, A(T)), & t > T. \end{cases}$$

Such operators are well-studied and understood. Applying [Hytönen and Portal 2008, Theorem 17] and [Hytönen and Portal 2008, Remark 20] (the dependence on the constants is not explicitly stated) in the one-dimensional and one-parameter case, we see that $S : L^p([0, T]; X) \rightarrow L^p([0, T]; X)$ is bounded for all $p \in (1, \infty)$ provided

$$\mathcal{R} \left\{ (1 + |\xi|)^k \left(\frac{\partial}{\partial \xi} \right)^k [a(t + h, \xi) - a(t, \xi)] : \xi \in \mathbb{R} \right\} \lesssim |h|^\alpha$$

holds for some $\alpha \in (0, 1]$ and all $k = 0, 1, 2$. This is the \mathcal{R} -analogue of the condition considered by Yamazaki [1986] and therefore called an \mathcal{R} -Yamazaki symbol. The fact that a is indeed an \mathcal{R} -Yamazaki symbol has been verified in Lemma 6.1. □

The next proposition shows that in many cases it is sufficient to show maximal L^p -regularity for initial value zero. This is well known in the autonomous case. The arguments have been used before; see for example [Dier and Zacher 2017, Theorem 6.2].

Proposition 6.3. *Let X be a Banach space, $p \in (1, \infty)$, $T > 0$ and $(A(t))_{t \in [0, T]}$ a family of uniformly sectorial operators:*

(a) *Suppose that the nonautonomous operator $(B(t))_{t \in [0, T+1]}$,*

$$B(t) = \begin{cases} A(0) & \text{for } t \in [0, 1], \\ A(t - 1) & \text{for } t \in [1, T + 1], \end{cases}$$

has maximal L^p -regularity for $u_0 = 0$. Then $(A(t))_{t \in [0, T]}$ has maximal L^p -regularity for all initial values $u_0 \in (D(A(0)), X)_{1/p, p}$. Further, the maximal regularity estimate only additionally depends on a constant controlled by the sectorial estimate for $A(0)$.

(b) *Suppose additionally that for all $t_0 \in (0, T]$ the nonautonomous problem associated to $(C_{t_0}(t))_{t \in [0, t_0+2]}$, where*

$$C_{t_0}(t) = \begin{cases} A(0) & \text{for } t \in [0, 1], \\ A(t - 1) & \text{for } t \in [1, 1 + t_0], \\ A(t_0) & \text{for } t \in [1 + t_0, 2 + t_0], \end{cases}$$

has maximal L^p -regularity for $u_0 = 0$. Then the unique solution of (NACP) for $(A(t))_{t \in [0, T]}$ satisfies $u(t) \in (D(A(t)), X)_{1/p, p}$ for all $t \in [0, T]$ and $u_0 \in (D(A(0)), X)_{1/p, p}$.

Proof. We start with the first part. By the characterization of real interpolation spaces via the trace method [Lunardi 1995, Proposition 1.2.10] and a cut-off argument, there is some $C > 0$ such that for all $u_0 \in (D(A(0)), X)_{1/p,p}$ there exists $v \in W^{1,p}([0, 1]; X) \cap L^p([0, 1]; D(A(0)))$ with $v(0) = 0$, $v(1) = u_0$ and

$$\|A(0)v\|_{L^p([0,1];X)} + \|\dot{v}\|_{L^p([0,1];X)} \leq C \|u_0\|_{(D(A(0)),X)_{1/p,p}}.$$

For given $f \in L^p([0, T]; X)$ we define $g \in L^p([0, T + 1]; X)$ as

$$g(t) = \begin{cases} \dot{v}(t) + A(0)v(t) & \text{for } t \in [0, 1), \\ f(t - 1) & \text{for } t \in [1, T + 1]. \end{cases}$$

By assumption $(B(t))_{t \in [0, T+1]}$ has maximal L^p -regularity for $u_0 = 0$. We denote by w the unique solution of (NACP) for $(B(t))_{t \in [0, T+1]}$ with right-hand side g . By the uniqueness of mild solutions in the autonomous case we have $w = v$ on $[0, 1]$. In particular, we have $w(1) = v(1) = u_0$. As a consequence we see that $u(t) = w(t + 1)$ solves (NACP) for $u(0) = w(1) = u_0$. Further,

$$\begin{aligned} \|u\|_{W^{1,p}([0,T];X)} + \|A(\cdot)u(\cdot)\|_{L^p([0,T];X)} &\lesssim \|g\|_{L^p([0,T+1];X)} \\ &\lesssim \|f\|_{L^p([0,T];X)} + \|u_0\|_{(D(A(0)),X)_{1/p,p}}. \end{aligned}$$

For the uniqueness observe that a second solution \tilde{u} of (NACP) with right-hand side f and $u(0) = u_0$ yields a solution $z = (\dot{v} + A(0)v)\mathbb{1}_{[0,1)} + \tilde{u}(\cdot - 1)\mathbb{1}_{[1,t_0+1]}$ of (NACP) for $(B(t))_{t \in [0, T+1]}$ that agrees with $u(\cdot - 1)$ on $[1, T + 1]$ by the uniqueness of solutions.

For the second part and fixed $t_0 \in (0, T]$ let z be the solution of (NACP) for $(C_{t_0}(t))_{t \in [0, t_0+2]}$ and the right-hand side $\tilde{g} = g\mathbb{1}_{[0, t_0+1]}$. Then z agrees with the solution w of the first part on $[0, t_0 + 1]$ and solves the autonomous problem $\dot{z}(s) + A(t_0)z(s) = 0$ on $[t_0 + 1, t_0 + 2]$. Since functions in $W^{1,p}([t_0 + 1, t_0 + 2]; X) \cap L^p([t_0 + 1, t_0 + 2]; D(A(t_0)))$ take values in the corresponding trace spaces [Amann 1995, Theorem III.4.10.2], we have $u(t_0) \in (D(A(t_0)), X)_{1/p,p}$. \square

We are now ready to prove our general maximal regularity result.

Theorem 6.4. *For $T > 0$ and $\theta \in (0, 1]$ let $(A(t))_{t \in [0, T]}$ be a θ -stable family of uniformly \mathcal{R} -sectorial operators on some UMD space X with fractional regularity $A_{-1} \in \dot{W}^{\alpha,q}([0, T]; \mathcal{B}(X_{\theta,A}, X_{\theta-1,A}))$. Then the nonautonomous problem (NACP) has maximal L^p -regularity*

- (a) for $p \in (1, \frac{1}{1-\theta})$, $q = \frac{1}{1-\theta}$ and $\alpha > 1 - \theta$,
- (b) for $p \in [\frac{1}{1-\theta}, \infty)$, $q = p$ and $\alpha > 1 - \theta$.

In this case the unique maximal L^p -regularity solution u of (NACP) satisfies $u(t) \in (D(A(t)), X)_{1/p,p}$ for all $t \in [0, T]$ and there exists a constant $C_p > 0$ with

$$\|u\|_{W^{1,p}([0,T];X)} + \|A(\cdot)u(\cdot)\|_{L^p([0,T];X)} \leq C(\|f\|_{L^p([0,T];X)} + \|u_0\|_{(D(A(0)),X)_{1/p,p}}),$$

which only depends on T, α, θ, K in (3-1), $\|A_{-1}\|_{\dot{W}^{\alpha,q}([0,T];\mathcal{B}(X_{\theta,A},X_{\theta-1,A}))}$ and the constants in the \mathcal{R} -sectorial estimates.

Proof. First note that under the made regularity assumptions, we have $A_{-1} \in C^\gamma([0, T]; \mathcal{B}(X_{\theta,A}, X_{\theta-1,A}))$ for some $\gamma > 0$. Further, let $u \in W^{1,p}([0, T]; X_{\theta-1,A}) \cap L^p([0, T]; X_{\theta,A})$ be the unique solution of (2-1) given by Proposition 4.4. We show that u has the higher regularity $A_{-1}(t)u(t) \in L^p([0, T]; X)$. For this we use the decomposition of $A_{-1}(t)u(t)$ given by (2-1).

Let us start with the integrability of $A_{-1}(t)(S_1u)(t)$. We will omit subindices in the following estimates. For $g \in L^{p'}([0, T]; X')$ we have, where $A'(t)$ is the adjoint,

$$\begin{aligned} & \int_0^T \int_0^t \langle g(t), A(t)e^{-(t-s)A(t)}(A(t) - A(s))u(s) \rangle_{X',X} ds dt \\ &= \int_0^T \int_0^t \langle A'(t)e^{-(t-s)A'(t)}g(t), (A(t) - A(s))u(s) \rangle_{X'_{1-\theta,A'(t)}, X_{\theta-1,A(t)}} ds dt. \end{aligned} \tag{6-1}$$

We now distinguish between the cases $p \in (\frac{1}{1-\theta}, \infty)$, $p = \frac{1}{1-\theta}$ and $p \in (1, \frac{1}{1-\theta})$. In the first case we know from Proposition 5.1 that $u \in L^\infty([0, T]; X_{\theta,A})$. Hence, up to constants (6-1) is dominated by

$$\begin{aligned} & \left(\int_0^T \int_0^T \frac{\|A(t) - A(s)\|_{\mathcal{B}(X_{\theta-1,A})}^p ds dt}{|t-s|^{1+p\alpha}} \right)^{1/p} \\ & \quad \times \left(\int_0^T \int_0^t \|A'(t)e^{-(t-s)A'(t)}g(t)\|_{X'_{1-\theta,A'(t)}}^{p'} |t-s|^{p'(1/p+\alpha)} ds dt \right)^{1/p'} \\ & \lesssim \|A\|_{\dot{W}^{\alpha,p}} \|u\|_{L^\infty([0,T]; X_{\theta,A})} \left(\int_0^T \int_0^t (t-s)^{p'(1/p+\alpha+\theta-2)} ds \|g(t)\|_{X'}^{p'} dt \right)^{1/p'}. \end{aligned}$$

The inner integral is finite because of the assumption $\alpha > 1 - \theta$. Since $g \in L^{p'}([0, T]; X')$ is arbitrary, we get $A_{-1}(\cdot)S_1u \in L^p([0, T]; X)$. The case $p = \frac{1}{1-\theta}$ follows similarly, using $u \in L^{q'}([0, T]; X_{\theta,A})$ for some big q' and the fact that the condition $\alpha > 1 - \theta$ leaves a little room. Let us come to the case $p \in (1, \frac{1}{1-\theta})$. Here Proposition 5.1 shows that $u \in L^{p/(1-p(1-\theta))}([0, T]; X_{\theta,A})$. Hence, using Hölder's inequality, for $\beta > 0$ the expression in (6-1) is dominated by

$$\begin{aligned} & \left(\int_0^T \int_0^T \frac{\|A(t) - A(s)\|_{\mathcal{B}(X_{\theta,A}, X_{\theta-1,A})}^{1/(1-\theta)}}{|t-s|^{1+\alpha(1-\theta)^{-1}}} ds dt \right)^{1-\theta} \left(\int_0^T \int_0^t (t-s)^{p'(\alpha+\beta-1)} ds \|g(t)\|_{X'}^{p'} dt \right)^{1/p'} \\ & \quad \times \left(\int_0^T \int_s^T (t-s)^{-\beta p/(1-p(1-\theta))} dt \|u(s)\|_{X_{\theta,A}}^{p/(1-p(1-\theta))} ds \right)^{1/p-(1-\theta)}. \end{aligned}$$

The last integral is finite for $\beta < \theta - \frac{1}{p'}$. Since $\alpha > 1 - \theta$, we can find $\beta \in (0, \theta - \frac{1}{p'})$ for which the second integral is finite as well.

Further, $A_{-1}(\cdot)(S_2f)(\cdot)$ lies in $L^p([0, T]; X)$ by Proposition 6.2. This shows that the solution satisfies $u(t) \in D(A(t))$ for almost all $t \in [0, T]$ and $A(\cdot)u(\cdot) \in L^p([0, T]; X)$. Since u solves (WNACP), it follows that $\dot{u} \in L^p([0, T]; X)$. This shows maximal L^p -regularity in the case $u_0 = 0$. It remains to verify the maximal regularity estimate. By the estimates obtained in the first part of the proof we have for

some case-dependent $q' \in (p, \infty]$

$$\begin{aligned}
\|A(\cdot)u(\cdot)\|_{L^p([0,T];X)} &= \|A_{-1}(\cdot)u(\cdot)\|_{L^p([0,T];X)} \\
&\leq \|A_{-1}(\cdot)(S_1u)(\cdot)\|_{L^p([0,T];X)} + \|A_{-1}(\cdot)(S_2f)(\cdot)\|_{L^p([0,T];X)} \\
&\lesssim \|A\|_{\dot{W}^{\alpha,q}} \|u\|_{L^{q'}([0,T];X_{\theta,A})} + \|f\|_{L^p([0,T];X)} \\
&\lesssim C_{pq'} \|A\|_{\dot{W}^{\alpha,q}} (\|u\|_{L^p([0,T];X_{\theta,A})} + \|f\|_{L^p([0,T];X)}) + \|f\|_{L^p([0,T];X)} \\
&\lesssim C_{pq'} \|A\|_{\dot{W}^{\alpha,q}} \|f\|_{L^p([0,T];X)} + \|f\|_{L^p([0,T];X)}.
\end{aligned}$$

Here we have used the estimates obtained in the first part of the proof, Proposition 5.1 and Proposition 4.4 in the third, fourth and fifth lines respectively. Since u solves (NACP) and the operators $(A(t))_{t \in [0,T]}$ are uniformly sectorial, this implies the maximal regularity estimate for $u_0 = 0$.

The case of general initial values $u_0 \in (D(A(0)), X)_{1/p,p}$ follows from Proposition 6.3. Here we use the fact that for $q > \alpha^{-1}$ functions in $\dot{W}^{\alpha,q}$ can be extended with the same regularity by their values at the endpoints [Dier and Zacher 2017, Proposition 7.8]. \square

Remark 6.5. Compared to the result in [Portal and Štrkalj 2006] we need a weaker \mathcal{R} -boundedness result. Further, the time regularity is lowered to some fractional Sobolev space at the cost of more regularity on the domain spaces. In order to obtain maximal L^p -regularity for all $p \in [(1-\theta)^{-1}, \infty)$ our result requires $A_{-1} \in \bigcap_{p \in [(1-\theta)^{-1}, \infty)} \bigcup_{\varepsilon > 0} \dot{W}^{1-\theta+\varepsilon,p}([0, T]; \mathcal{B}(X_{\theta,A}, X_{\theta-1,A}))$. This is slightly less restrictive than the α -Hölder continuity for some $\alpha > 1 - \theta$ assumed usually.

For nonautonomous problems given by sesquilinear forms on Hilbert spaces one obtains by the same line of thought the following improvement of [Dier and Zacher 2017], where only the case $p = 2$ was treated. Let us shortly recall how the form setting is related to the general setting considered by us. Given, as in (1-1), a coercive, bounded nonautonomous sesquilinear form on some Hilbert space V one gets operators $\mathcal{A}(t) : V \rightarrow V'$ with $\mathcal{A}(t)u = a(t, u, \cdot)$. Given a second Hilbert space with dense embedding $V \hookrightarrow H$ and the associated triple $V \hookrightarrow H \hookrightarrow V'$ one considers their restrictions $A(t)$ on H , i.e., $D(A(t)) = \{u \in V : \mathcal{A}(t)u \in H\}$. One then obtains an associated problem (NACP) for $(A(t))_{t \in [0,T]}$ on H . The spaces V and V' can be seen as replacements of $X_{1/2,A}$ and $X_{-1/2,A}$. Hence, $(A(t))$ is $\frac{1}{2}$ -stable in some sense.

Corollary 6.6. *Let V, H be Hilbert spaces with dense embedding $V \hookrightarrow H$ and let $a : [0, T] \times V \times V \rightarrow \mathbb{C}$ be a coercive, bounded nonautonomous sesquilinear form as in (1-1). Then the associated problem (NACP) on H has maximal L^p -regularity*

- (a) for $p \in (1, 2]$ provided $\mathcal{A} \in \dot{W}^{1/2+\varepsilon,2}([0, T]; \mathcal{B}(V, V'))$ for some $\varepsilon > 0$,
- (b) for $p \in [2, \infty)$ provided $\mathcal{A} \in \dot{W}^{1/2+\varepsilon,p}([0, T]; \mathcal{B}(V, V'))$ for some $\varepsilon > 0$.

The constants in the maximal L^p -regularity estimate only depend on T, ε , the constants α, M in (1-1) and the fractional Sobolev norm of \mathcal{A} .

Proof. Repeat the previous proof for $X = H$ and replace $X_{1/2,A}$ and $X_{-1/2,A}$ with V and V' . \square

Note that V and V' only agree with the complex interpolation spaces $X_{1/2,A(t)}$ and $X_{-1/2,A(t)}$ if the operators $A(t)$ satisfy the so-called Kato square root property; see [Auscher 2002] for a short introduction to this topic. However, this is not necessary to carry out the argument. In the UMD setting the case $\theta = \frac{1}{2}$ is also of particular interest. We obtain the following corollary relevant for concrete applications (which holds for other values of θ as well).

Corollary 6.7. *Let $T > 0$ and $(A(t))_{t \in [0,T]}$ be uniformly sectorial on a UMD space X such that for some $\omega \in (0, \frac{\pi}{2})$ and $M > 0$ the imaginary powers satisfy*

$$\|A(t)^{is}\| \leq M e^{\omega|s|}$$

uniformly for all $t \in [0, T]$ and $s \in \mathbb{R}$. Further, suppose that there exist Banach spaces $X_{1/2}$ and $X_{-1/2}$ for which for all $t \in [0, T]$ the spaces $D(A(t)^{1/2})$ and $D(A(t)^{-1/2})$ agree with $X_{1/2}$ and $X_{-1/2}$ as vector spaces and the respective norms are uniformly equivalent for some constant $K > 0$. Then the nonautonomous Cauchy problem (NACP) for $(A(t))_{t \in [0,T]}$ has maximal L^p -regularity

- (a) *for $p \in (1, 2]$ if $A_{-1} \in \dot{W}^{1/2+\varepsilon,2}([0, T]; \mathcal{B}(X_{1/2}, X_{-1/2}))$ for some $\varepsilon > 0$,*
- (b) *for $p \in [2, \infty)$ if $A_{-1} \in \dot{W}^{1/2+\varepsilon,p}([0, T]; \mathcal{B}(X_{1/2}, X_{-1/2}))$ for some $\varepsilon > 0$.*

The constants in the maximal L^p -regularity estimates only depend on p, T, ε, K in (3-1), M, ω , the fractional Sobolev norm of A_{-1} and the constants in the sectorial estimates.

Proof. Since the operators $A(t)$ have uniformly bounded imaginary powers, it follows from [Denk et al. 2003, Theorem 4.5] that for $\varphi \in (\omega, \pi)$

$$\sup_{t \in [0,T]} \mathcal{R}\{\lambda R(\lambda, A(t)) : \lambda \notin \bar{\Sigma}_\varphi\} < \infty.$$

Since uniformly bounded analytic families are uniformly \mathcal{R} -bounded on compact subsets of a common domain [Weis 2001, Proposition 2.6], the operators $(A(t))_{t \in [0,T]}$ are uniformly \mathcal{R} -sectorial. Further, the fractional domains spaces $D(A(t)^{1/2})$ and $D(A(t)^{-1/2})$ are uniformly equivalent to $X_{1/2,A(t)}$ and $X_{-1/2,A(t)}$ [Fackler 2015, Proposition 2.5]. As a consequence of the assumptions, the family $(A(t))_{t \in [0,T]}$ is $\frac{1}{2}$ -stable. This means that we can apply Theorem 6.4. □

Remark 6.8. Corollary 6.7 holds under the slightly weaker assumption that the operators $(A(t))_{t \in [0,T]}$ are uniformly \mathcal{R} -sectorial. For this one uses the scale $X_{\theta,A} = D(A^\theta)$ for $|\theta| \in (0, 1)$ and repeats the proof of Theorem 6.4. The main difference is that one has to use [Haak et al. 2006, Lemma 6.9(1)] instead of Lemma 4.1.

7. Nonautonomous maximal regularity for elliptic operators

We now illustrate the consequences of our results for nonautonomous problems governed by elliptic operators in divergence form. We concentrate on pure second-order operators with VMO-coefficients subject to Dirichlet boundary conditions, as the used results are already involved and spread over the

literature in this special case. On a bounded domain $\Omega \subset \mathbb{R}^n$ we consider bounded measurable coefficients $A = (a_{ij}) : \Omega \rightarrow \mathbb{C}^{n \times n}$ and the bounded sesquilinear form

$$a : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \rightarrow \mathbb{C}, \quad (u, v) \mapsto \int_{\Omega} A \nabla u \overline{\nabla v}.$$

Further, we assume that (a_{ij}) satisfies for some $\delta > 0$ and all $\xi \in \mathbb{C}^n$ the estimate

$$\operatorname{Re} \sum_{i,j=1}^n a_{ij}(x) \xi_i \bar{\xi}_j \geq \delta |\xi|^2.$$

Then the operator L_2 on $L^2(\Omega)$ associated to a is sectorial. Further, one has for $u \in D(L_2) \subset W_0^{1,2}(\Omega)$ the identity $L_2 u = -\operatorname{div}(A \nabla u)$ in the sense of distributions. One can show that if Ω has C^1 -boundary and if the coefficients lie in VMO, then L_2 induces for all $q \in (1, \infty)$ compatible sectorial operators L_q on $L^q(\Omega)$ (see the proof of Theorem 7.2). These operators are realizations of $-\operatorname{div}(A \nabla \cdot)$ on $L^q(\Omega)$. For further details on the form method we refer to [Ouhabaz 2005].

Definition 7.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. A bounded measurable function $f : \Omega \rightarrow \mathbb{C}$ is of vanishing mean oscillation if one has $\inf_{r>0} \eta_f(r) = 0$ for the modulus

$$\eta_f(r) := \sup_{B:d(B) \leq r} \left(\frac{1}{|B \cap \Omega|} \int_{B \cap \Omega} |f(x) - f_{B \cap \Omega}|^2 dx \right)^{1/2},$$

where $f_{B \cap \Omega}$ denotes the mean of f over $B \cap \Omega$ and the supremum is taken over all balls $B \subset \mathbb{R}^n$ centered in Ω whose diameter $d(B)$ does not exceed r .

We need the following variant of the Kato square root property on $L^q(\Omega)$.

Theorem 7.2. Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded C^1 -domain, $q \in (1, \infty)$ and $A = (a_{ij})_{1 \leq i, j \leq n} \in L^\infty(\Omega; \mathbb{C}^{n \times n})$ be complex-valued coefficients with

$$\operatorname{Re} \sum_{i,j=1}^n a_{ij}(x) \xi_i \bar{\xi}_j \geq \delta |\xi|^2 \quad \text{for all } \xi \in \mathbb{C}^n,$$

for some $\delta > 0$ and almost every $x \in \Omega$. Let L_q be the realization of $-\operatorname{div}(A \nabla \cdot)$ on $L^q(\Omega)$ subject to Dirichlet boundary conditions. If $a_{ij} \in \operatorname{VMO}(\Omega)$ for all $i, j = 1, \dots, n$, then there exists $\lambda_0 \geq 0$ such that the following holds:

- (a) $L_q + \lambda$ is a sectorial operator on $L^q(\Omega)$ for all $\lambda \geq \lambda_0$ and

$$\|f\|_q + \|\nabla f\|_q \simeq \|(L_q + \lambda)^{1/2} f\|_q \quad \text{for all } f \in W_0^{1,q}(\Omega).$$

- (b) The operator L_q extends to an isomorphism $W_0^{1,q}(\Omega) \xrightarrow{\simeq} W^{-1,q}(\Omega)$.

The constant λ_0 only depends on $\Omega, q, \eta_{a_{ij}}, \delta$ and $\|A\|_\infty$. With an additional dependence on λ , the same holds for the constant in the equivalence in (a), the isomorphism in (b) and the sectorial estimates of the operators $L_q + \lambda$.

Proof. Under the made assumptions, the operator L_2 satisfies local Gaussian estimates [Auscher and Tchamitchian 2001a, Theorem 7]. Although not explicitly stated, the coefficients in the estimate only depend on the claimed constants. This has several consequences. First, for λ sufficiently large the operator

$L_2 + \lambda$ satisfies global Gaussian estimates [Auscher and Tchamitchian 1998, Section 1.4.5, Theorem 18] and extends to a sectorial operator $L_q + \lambda$ on $L^q(\Omega)$. Secondly, it essentially follows from [Auscher and Tchamitchian 2001b, Theorem 4] that $\|(L_q + \lambda)^{1/2}\|_q \lesssim \|f\|_q + \|\nabla f\|_q$. Here are two additional points to consider. First, the theorem is only stated in the case $\lambda = 0$. The case $\lambda \neq 0$ can be obtained by including terms of lower order in the argument or by arguing as in [Auscher and Tchamitchian 1998, p. 135]. The second point is the not explicitly stated dependence on the constants. However, taking a close look at the proof in [Auscher and Tchamitchian 2001b] one sees that most auxiliary results give the explicit dependence on the constants (in [Auscher and Tchamitchian 2001b, p. 162] such a dependence is explicitly stated in a special case). One crucial point needed here is the dependence in the case $p = 2$, which is well known. This can be found in [Axelsson et al. 2006, Theorem 1] for a broad class of Lipschitz domains and a combination of [Egert et al. 2014, Theorem 4.2; 2016, Theorems 3.1 and 3.3 and Section 6] yields the dependence for general bounded Lipschitz domains and therefore a fortiori for C^1 -domains.

Now, as in [Auscher and Tchamitchian 1998, p. 135], the converse inequality follows if $(L_q + \lambda)^{-1}$ extends to a bounded operator from $W^{-1,q}(\Omega) = (W_0^{1,q'}(\Omega))'$ into $W_0^{1,q}(\Omega)$. Notice that

$$\|u\|_{W^{-1,q}(\Omega)} = \inf \left\{ \|g\|_q + \sum_{k=1}^n \|F_k\|_q : g, F_k \in L^q(\Omega) \text{ with } u = g + \operatorname{div} F \right\}.$$

It is shown in [Dong and Kim 2010, Theorem 4] that for $\lambda \geq 0$ there exists $C \geq 0$ such that for all $F_k, g \in L^q(\Omega)$ there is a unique $u \in W_0^{1,q}(\Omega)$ with $-\operatorname{div}(A\nabla u) + \lambda u = g + \operatorname{div} F$ and

$$\|u\|_{W^{1,q}(\Omega)} \leq C \left(\|g\|_q + \sum_{k=1}^n \|F_k\|_q \right).$$

Here, our required dependence on the constants can be found in the lemmata in [Dong and Kim 2010, Section 7]. Note that the above estimate is exactly the boundedness of $(\lambda + L_q)^{-1} : W^{-1,q}(\Omega) \rightarrow W_0^{1,q}(\Omega)$, which is a uniform isomorphism by the uniqueness of $u \in W_0^{1,q}(\Omega)$. \square

Remark 7.3. The estimate $\|L^{1/2} f\|_q \lesssim \|\nabla f\|_q$ is known under more general assumptions on the coefficients and the domain [Auscher and Tchamitchian 2001b, Theorem 4]. The same holds for the boundedness of $(L_q + \lambda)^{-1} : W^{-1,q}(\Omega) \rightarrow W_0^{1,q}(\Omega)$ for which originating from [Krylov 2007] many results have been obtained in the last years. For a complete list of references we refer to the introduction of [Dong and Kim 2016] and for a proof of similar results within the framework of maximal regularity to [Gallarati and Veraar 2017a; 2017b].

Theorem 7.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 -domain, $T > 0$ and $a_{ij} \in L^\infty([0, T] \times \Omega)$ for $i, j = 1, \dots, n$. Assume further that the following properties are satisfied:*

- (1) *There exists $\delta > 0$ such that for almost all $(t, x) \in [0, T] \times \Omega$ and all $\xi \in \mathbb{C}^n$*

$$\operatorname{Re} \sum_{i,j=1}^n a_{ij}(t, x) \xi_i \bar{\xi}_j \geq \delta |\xi|^2.$$

- (2) *The functions $x \mapsto a_{ij}(t, x)$ lie in $\operatorname{VMO}(\Omega)$ and there is $\eta : [0, 1] \rightarrow [0, \infty]$ with $\lim_{r \downarrow 0} \eta(r) = 0$ and $\eta_{a_{ij}(t, \cdot)} \leq \eta$ for all $t \in [0, T]$ and $i, j = 1, \dots, n$.*

For $q \in (1, \infty)$ let $L_q(t) = -\operatorname{div}(A(t)\nabla \cdot)$ be realizations on $L^q(\Omega)$. Then for all $q \in (1, \infty)$ the nonautonomous problem (NACP) associated to $(L_q(t))_{t \in [0, T]}$ has maximal L^p -regularity

- (a) for $p \in (1, 2]$ if $a_{ij} \in \dot{W}^{1/2+\varepsilon, 2}([0, T]; L^\infty(\Omega))$ for some $\varepsilon > 0$,
- (b) for $p \in [2, \infty)$ if $a_{ij} \in \dot{W}^{1/2+\varepsilon, p}([0, T]; L^\infty(\Omega))$ for some $\varepsilon > 0$.

The maximal L^p -regularity estimate depends only on $p, q, T, \Omega, \delta, \eta, \varepsilon, \|a_{ij}\|_\infty$ and the homogeneous Sobolev norm in (a) or (b).

Proof. Thanks to the Gaussian estimates discussed in the proof of Theorem 7.2, for sufficiently large λ the operators $L_q(t) + \lambda$ have uniformly bounded imaginary powers with $\|(L_q(t) + \lambda)^{is}\| \leq M e^{\omega|s|}$ for some $M > 0$ and $\omega \in (0, \frac{\pi}{2})$. This follows from the general result [Duong and Robinson 1996, Theorem 4.3] (which even gives a bounded H^∞ -calculus), which does not state the dependence on the constants explicitly. Further, it follows from Theorem 7.2 that $D((L_q(t) + \lambda)^{1/2}) \simeq W_0^{1,q}(\Omega)$ holds uniformly in $t \in [0, T]$. Moreover, the operator $L_q(t) + \lambda$ extends to an isomorphism $W_0^{1,q}(\Omega) \xrightarrow{\sim} W^{-1,q}(\Omega)$ which is uniform in $t \in [0, T]$. Consequently, for $u \in L^q(\Omega)$ one has

$$\begin{aligned} \|u\|_{D((L_q(t)+\lambda)^{-1/2})} &= \|(L_q(t) + \lambda)^{-1/2}u\|_{L^q(\Omega)} \\ &= \|(L_q(t) + \lambda)^{1/2}(L_q(t) + \lambda)^{-1}u\|_{L^q(\Omega)} \simeq \|(L_q(t) + \lambda)^{-1}u\|_{W_0^{1,q}(\Omega)} \\ &\simeq \|u\|_{W^{-1,q}(\Omega)}. \end{aligned}$$

Therefore $X_{1/2} = W^{1,q}(\Omega)$ and $X_{-1/2} = W^{-1,q}(\Omega)$ in Corollary 6.7.

It remains to check the time regularity. For $u \in W_0^{1,2}(\Omega) \cap W_0^{1,q}(\Omega)$ and $v \in W_0^{1,2}(\Omega) \cap W_0^{1,q'}(\Omega)$ one has

$$|(L_q(t)u - L_q(s)u, v)| = \left| \int_{\Omega} (A(t) - A(s))\nabla u \nabla v \right| \leq \|A(t) - A(s)\|_\infty \|\nabla u\|_q \|\nabla v\|_{q'}.$$

By density this extends to all $u \in W_0^{1,q}(\Omega)$ and all $v \in W_0^{1,q'}(\Omega)$. Hence, it follows that $L_q(\cdot) + \lambda \in \dot{W}^{\alpha,r}([0, T]; \mathcal{B}(W_0^{1,q}(\Omega), W^{-1,q}(\Omega)))$ with α and r as in the assumptions. Now, Corollary 6.7 applies and yields maximal L^p -regularity for $(L_q(t) + \lambda)_{t \in [0, T]}$ and λ big enough. By a rescaling argument this is equivalent to the maximal L^p -regularity of $(L_q(t))_{t \in [0, T]}$. \square

8. Applications to quasilinear parabolic problems

We now use Theorem 7.4 to solve quasilinear parabolic equations. It may be a little bit confusing that in the result below Hölder assumptions on the coefficients are made. The point for concrete applications is not that we can replace Hölder regularity by fractional Sobolev regularity, but that the fractional Sobolev regularity in Theorem 7.4 allows us to loosen the assumed Hölder regularity. We will comment on this point later.

Theorem 8.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 -domain and $T > 0$. For coefficients $A = (a_{ij}) : \mathbb{C} \rightarrow \mathbb{C}^{n \times n}$, $p \in [2, \infty)$, $q \in (1, \infty)$, an inhomogeneous part $f \in L^p([0, T]; L^q(\Omega))$ and an initial value $u_0 \in L^q(\Omega)$*

satisfying the condition $u_0 \in (D(\operatorname{div} A(u_0)\nabla \cdot), L^q(\Omega))_{1/p,p}$ consider the problem

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) - \operatorname{div}(A(u(t, x))\nabla u(t, x)) = f(t, x), \\ u(t, x) = 0 \quad \text{on } [0, T] \times \partial\Omega, \\ u(0, x) = u_0(x) \quad \text{on } \Omega. \end{cases} \tag{QLP}$$

Suppose that the following assumptions are satisfied:

- (1) The coefficients a_{ij} are β -Hölder continuous for some $\beta > \frac{1}{2}$.
- (2) For all $M > 0$ there exist $\delta(M) > 0$ such that for all $|u| \leq M$

$$\operatorname{Re} \sum_{i,j=1}^n a_{ij}(u) \xi_i \bar{\xi}_j \geq \delta(M) |\xi|^2 \quad \text{for all } \xi \in \mathbb{C}^n.$$

If $q > n$ and $\beta > \frac{q}{2(q-n)}$, then there exists $C \geq 0$ such that for

$$\|f\|_{L^p([0,T];L^q(\Omega))} + \|u_0\|_{(D(\operatorname{div} A(u_0)\nabla \cdot), L^q(\Omega))_{1/p,p}} \leq C$$

the quasilinear problem (QLP) has a solution

$$u \in W^{1,p}([0, T]; L^q(\Omega)) \cap \operatorname{BUC}([0, T] \times \bar{\Omega})$$

with $u(t) \in D(\operatorname{div} A(u(t, \cdot))\nabla \cdot)$ for almost every $t \in [0, T]$ and $\operatorname{div} A(u)\nabla u \in L^p([0, T]; L^q(\Omega))$. A fortiori, $u \in C^{\alpha-1/p}([0, T]; C^{1-\alpha-n/q}(\bar{\Omega}))$ for $\alpha \in (\frac{1}{p}, 1 - \frac{n}{q})$.

Proof. Choose $\alpha \in (\frac{1}{2\beta}, 1 - \frac{n}{q})$, which is possible by our assumptions. Now, choose $\delta > 0$ with $\alpha - \delta > \frac{1}{2\beta}$ and $\alpha + \delta < 1 - \frac{n}{q}$. Further, let

$$\mathcal{M} = \{v \in W^{\alpha-\delta,p}([0, T]; W_0^{1-\alpha-\delta,q}(\Omega)) : v(0) = u_0\}$$

and \mathcal{M}_R for $R > 0$ be the ball $B(0, R)$ in \mathcal{M}_R . For $v \in \mathcal{M}_R$ consider the problem

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) - \operatorname{div}(A(v(t, x))\nabla u(t, x)) = f(t, x), \\ u(t, x) = 0 \quad \text{on } [0, T] \times \partial\Omega, \\ u(0, x) = u_0(x) \quad \text{on } \Omega. \end{cases} \tag{LP}$$

Since $\alpha + \delta < 1 - \frac{n}{q}$ and $\alpha - \delta > \frac{1}{2\beta} \geq \frac{1}{2} \geq \frac{1}{p}$, we have $v \in W^{\alpha-\delta,p}([0, T]; \operatorname{BUC}(\bar{\Omega}))$ and \mathcal{M} is compactly embedded in $\operatorname{BUC}([0, T] \times \bar{\Omega})$. By the Arzelà–Ascoli theorem, the functions in \mathcal{M}_R are uniformly equicontinuous on $[0, T] \times \Omega$. As a consequence (2) of Theorem 7.4 is satisfied and one can find uniform ellipticity constants for $A \circ v$ with $v \in \mathcal{M}_R$. For $\varepsilon > 0$ with $r := (\alpha - \delta - \varepsilon)\beta > \frac{1}{2}$ we have

$$\begin{aligned} \|a_{ij} \circ v\|_{\dot{W}^{r,p}([0,T];L^\infty(\Omega))}^p &= \int_0^T \int_0^T \frac{\|a_{ij}(v(t, \cdot)) - a_{ij}(v(s, \cdot))\|_\infty^p}{|t-s|^{1+pr}} ds dt \\ &\lesssim \int_0^T \int_0^T \frac{\|v(t, \cdot) - v(s, \cdot)\|_\infty^{\beta p}}{|t-s|^{1+pr}} ds dt = \|v\|_{\dot{W}^{r\beta-1,\beta p}([0,T];L^\infty(\Omega))}^{\beta p} \\ &= \|v\|_{\dot{W}^{\alpha-\delta-\varepsilon,\beta p}([0,T];L^\infty(\Omega))}^{\beta p} \lesssim \|v\|_{\dot{W}^{\alpha-\delta,p}([0,T];L^\infty(\Omega))}^{\beta p}. \end{aligned} \tag{8-1}$$

This means that the coefficients $A \circ v$ satisfy the assumptions of Theorem 7.4. Hence, (LP) has a unique solution u and there is $C_R \geq 0$ independent of $v \in \mathcal{M}_R$ with

$$\|u\|_{W^{1,p}([0,T];L^q(\Omega))} + \|\operatorname{div} A(v)\nabla u\|_{L^p([0,T];L^q(\Omega))} \leq C_R(\|f\|_{L^p([0,T];L^q(\Omega))} + \|u_0\|),$$

where the norm of u_0 is taken in $(D(\operatorname{div} A(u_0)\nabla \cdot), L^q(\Omega))_{1/p,p}$. Further, by the real interpolation formula for vector-valued Besov spaces [Amann 2000, Corollary 4.3] one has for $\theta \in (\frac{1}{2}, 1 - \frac{1}{q})$ and sufficiently small $\varepsilon > 0$, uniformly in $v \in \mathcal{M}_R$, the embeddings

$$\begin{aligned} u \in W^{1,p}([0, T]; L^q(\Omega)) \cap L^p([0, T]; W_0^{1,q}(\Omega)) &\hookrightarrow (L^p([0, T]; W_0^{1,q}(\Omega)), W^{1,p}([0, T]; L^q(\Omega)))_{\theta,p} \\ &\hookrightarrow W^{\theta-\varepsilon,p}([0, T]; (W_0^{1,q}(\Omega), L^q(\Omega))_{\theta,p}) \\ &= W^{\theta-\varepsilon,p}([0, T]; B_{0,p}^{1-\theta,q}(\Omega)) \\ &\hookrightarrow W^{\theta-\varepsilon,p}([0, T]; W_0^{1-\theta-\varepsilon,q}(\Omega)). \end{aligned} \tag{8-2}$$

All estimates hold uniformly for $v \in \mathcal{M}_R$. The embedding (8-2) implies that for sufficiently small

$$\|f\|_{L^p([0,T];L^q(\Omega))} + \|u_0\|_{(D(\operatorname{div} A(u_0)\nabla \cdot), L^q(\Omega))_{1/p,p}}$$

we obtain a well-defined map

$$S_R : \mathcal{M}_R \rightarrow \mathcal{M}_R, \quad v \mapsto u, \quad \text{where } u \text{ is the solution of (LP).}$$

It follows from (8-2) and the compact embedding results for vector-valued Sobolev spaces [Amann 2000, Theorem 5.1] that $S_R\mathcal{M}_R$ is a precompact subset of \mathcal{M}_R . We next show that S_R is continuous. For this let $v_n \rightarrow v$ in \mathcal{M}_R and let $u_n = S_R v_n$. After passing to a subsequence we may assume that $v_n \rightarrow v$ in $\text{BUC}([0, T] \times \bar{\Omega})$ and that u_n converges weakly to some u in

$$W^{1,p}([0, T]; L^q(\Omega)) \cap L^p([0, T]; W_0^{1,q}(\Omega)).$$

Now, let $g \in L^{p'}([0, T]; W_0^{1,q'}(\Omega))$. Note that $A^T(v_n)\nabla g \rightarrow A^T(v)\nabla g$ in $L^{q'}(\Omega)$ by the dominated convergence theorem. Since u_n solves (LP) we have

$$\begin{aligned} \int_0^T \langle f(t), g(t) \rangle dt &= \int_0^T \langle \dot{u}_n(t), g(t) \rangle dt + \int_0^T \langle A(v_n(t))\nabla u_n(t), \nabla g(t) \rangle dt \\ &= \int_0^T \langle \dot{u}_n(t), g(t) \rangle dt + \int_0^T \langle \nabla u_n(t), A^T(v_n(t))\nabla g(t) \rangle dt. \end{aligned}$$

Taking limits on both sides of the equation, we get

$$\int_0^T \langle f(t), g(t) \rangle dt = \int_0^T \langle \dot{u}(t), g(t) \rangle dt + \int_0^T \langle A(v(t))\nabla u(t), \nabla g(t) \rangle dt.$$

Since g is arbitrary and $u_0 = u_n(0) \rightarrow u(0)$, this implies that u solves (LP) on $W^{-1,q}(\Omega)$, i.e., is the unique integrated solution of (LP) given by Proposition 4.4. Hence, $S_R v = u$. Since the same argument

works for arbitrary subsequences, we have shown that S_R is continuous. Now, by Schauder’s fixed point theorem there is some $u \in \mathcal{M}_R$ with $S_R u = u$. Using Theorem 7.4 for $v = u$ we see that

$$\begin{aligned} \|u\|_{W^{1,p}([0,T];L^q(\Omega))} + \|\operatorname{div} A(u)\nabla u\|_{L^p([0,T];L^q(\Omega))} \\ \leq C(\|f\|_{L^p([0,T];L^q(\Omega))} + \|u_0\|_{(D(\operatorname{div} A(u_0)\nabla \cdot),L^q(\Omega))_{1/p,p}}). \quad \square \end{aligned}$$

Remark 8.2. We illustrate the benefits of Theorem 7.4 for quasilinear equations with the help of Theorem 8.1. First, maximal regularity results for nonautonomous problems governed by elliptic operators before [Fackler 2015] assumed C^1 -regularity in space. In particular, such results cannot deal with Hölder continuous coefficients a_{ij} as in Theorem 8.1 because the composition $a_{ij} \circ v$ in (8-1) would fail to have the necessary C^1 -smoothness.

Further, in (8-1) one needs from a conceptual point of view that the composition operator $v \mapsto a_{ij} \circ v$ maps into the Sobolev space $\dot{W}^{\alpha,p}([0, T]; L^\infty(\Omega))$ for some $\alpha > \frac{1}{2}$ in order to apply Theorem 7.4. Although v lies in some fractional Sobolev space and one only requires the image to lie in a different fractional Sobolev space, the only useful sufficient condition the author is aware of is to assume that the coefficients a_{ij} are Hölder continuous. Nevertheless, the less restrictive fractional Sobolev assumption in Theorem 7.4 is useful as it allows us to relax the assumed regularity. To illustrate this point explicitly, let us calculate the necessary regularity if one needs to check that $a_{ij} \circ v$ is in $C^\alpha([0, T]; L^\infty(\Omega))$ for some $\alpha > \frac{1}{2}$. Using the same notation as before one has

$$\|a_{ij}(v(t, \cdot)) - a_{ij}(v(s, \cdot))\|_\infty \lesssim \|v(t, \cdot) - v(s, \cdot)\|_\infty^\beta.$$

Now, ignoring the technical aspect of having an additional $\delta > 0$ of room, for functions $v \in W^{\alpha,p}([0, T]; W^{1-\alpha,q}(\Omega))$ we have for $\alpha \in (\frac{1}{p}, 1 - \frac{n}{q})$ the embedding

$$W^{\alpha,p}([0, T]; W^{1-\alpha,q}(\Omega)) \hookrightarrow C^{\alpha-1/p}([0, T]; \operatorname{BUC}(\bar{\Omega})).$$

Consequently, we have

$$\|a_{ij}(v(t, \cdot)) - a_{ij}(v(s, \cdot))\|_\infty \lesssim |t - s|^{\beta(\alpha-1/p)}.$$

Since $\alpha < 1 - \frac{n}{q}$, for maximal regularity with Hölder assumptions one needs

$$\beta \cdot \left(1 - \frac{n}{q} - \frac{1}{p}\right) > \frac{1}{2} \iff \beta > \frac{q}{2(q-n-q/p)}.$$

In particular, this is a stronger condition than $\beta > \frac{q}{2(q-n)}$, as used in Theorem 8.1. This improvement comes from the fact that the p -integrability is for free in the fractional Sobolev result, whereas in the Hölder case one has to sacrifice some differentiation order for the Sobolev embeddings.

Remark 8.3. We can only deduce the existence of solutions for small data in Theorem 7.4 because the constant in the maximal regularity estimate depends on the VMO-modulus of the coefficients and their fractional Sobolev norm. If one has estimates on solutions of (QLP) independent of these regularity data, the Leray–Schauder principle would yield solutions for arbitrary f and u_0 .

Further note that the reasoning of Theorem 8.1 works for a far more general class of problems. For example, the coefficients $A(u)$ may depend in a nonlocal way on u , e.g., on the history of the solution as in [Amann 2005; 2006].

9. Optimality of the results

In this section we show that the maximal regularity results obtained in Theorem 6.4 are optimal or close to optimal. In fact, even in the form setting considered in Corollary 6.6, maximal regularity may fail if one relaxes the assumed regularity, i.e., for maximal L^p -regularity $\mathcal{A} \in \dot{W}^{\alpha,p}([0, T]; \mathcal{B}(V, V'))$ for some $\alpha > \frac{1}{2}$. It was shown in [Fackler 2017a, Theorem 5.1] that there is a symmetric nonautonomous form with $\mathcal{A} \in C^{1/2}([0, T]; \mathcal{B}(V, V'))$ and $f \in L^\infty([0, T]; V)$ for which the unique solution given by Proposition 4.4 satisfies $\dot{u}(t) \notin H$ for almost all $t \in [0, T]$, although $u \in L^\infty([0, T]; V)$ holds as one aims for in the bootstrapping result given in Proposition 5.1. As a consequence, maximal L^p -regularity fails for all $p \in [1, \infty]$. Note that $C^{1/2}([0, T]; \mathcal{B}(V, V')) \hookrightarrow \dot{W}^{\alpha,q}([0, T]; \mathcal{B}(V, V'))$ for all $\alpha \in (0, \frac{1}{2})$ and all $q \in [1, \infty]$. Hence, Theorem 6.4 fails for $\alpha < \frac{1}{2}$ in all possible variants.

This leaves open the critical case $\alpha = \frac{1}{2}$. Note that for $q \in (1, 2)$ the space $\dot{W}^{1/2,q}([0, T]; \mathcal{B}(V, V'))$ contains piecewise constant forms. Hence, as observed by Dier [2014, Section 5.2], the failure of the Kato square root property for general forms implies that maximal L^2 -regularity may not hold for $q < 2$. Example 7.2 in [Fackler 2017b] shows that for $p > 2$ maximal L^p -regularity on $L^2(\Omega)$ for $\mathcal{A} \in \dot{W}^{1/2,q}([0, T]; \mathcal{L}(V, V'))$ with $q \in (1, 2)$ does not even hold for elliptic operators. Note that for $p \in (1, 2)$ these arguments based on the incompatibility of trace spaces break down.

Refining the arguments in [Fackler 2017a], we show that for symmetric forms maximal L^p -regularity may fail for all $p \in [1, \infty]$ under the regularity $\mathcal{A} \in \dot{W}^{1/2,q}([0, T]; \mathcal{B}(V, V'))$ for some $q > 2$.

Example 9.1. We take $H = L^2([0, \frac{1}{2}])$ and $V = L^2([0, \frac{1}{2}], w)$ with $w(x) = (x|\log x|)^{-3/2}$. Further, we consider $u(t, x) = c(x)(\sin(t\varphi(x)) + d)$ for $\varphi(x) = w(x)$, $c(x) = x \cdot |\log x|$ and some sufficiently large $d > 0$. Note that for all $t \in [0, T]$

$$\|\dot{u}(t)\|_H^2 \simeq \int_0^{1/2} |c(x)\varphi(x)|^2 dx = \int_0^{1/2} x^{-1} \frac{1}{|\log x|} dx = \infty.$$

Hence, $\dot{u}(t) \notin H$ for all $t \in [0, T]$. Following the ideas and arguments in [Fackler 2017a] we now show that u is indeed an integrated solution of a nonautonomous problem associated to some coercive, bounded symmetric sesquilinear form $a : [0, T] \times V \times V \rightarrow \mathbb{C}$ and inhomogeneous part $f(t) = u(t) \in L^\infty([0, T]; V)$. For this one defines the form $a(t, \cdot, \cdot)$ on the set $\langle u(t) \rangle \times V$ as

$$a(t, c \cdot u(t), v) = c [(f(t)|v)_H - \langle \dot{u}(t), v \rangle_{V',V}] \tag{9-1}$$

and then extends the form to $V \times V$ by the same procedure as in [Fackler 2017a, Section 4]. Following Section 5 of that paper, one checks the regularity of the extended forms. By the explicit formula for the extension, one sees that it suffices to control the regularity of (duality) products of the functions $u : [0, T] \rightarrow V$, $w^{-1}u : [0, T] \rightarrow V$ and $\dot{u} : [0, T] \rightarrow V'$. Since $\dot{W}^{\alpha,p} \cap L^\infty$ is an algebra under pointwise multiplication, the regularity question boils down to the regularity of these individual functions. Further,

one sees that for our concrete choice of u , the relevant fractional norms are dominated by that of $u : [0, T] \rightarrow V$. Hence, one only has to check the regularity of $u : [0, T] \rightarrow V$, which we do now.

We show explicitly that $u \in \dot{W}^{1/2,q}([0, T]; V)$ for all $q \in (2, \infty)$. Note that on the one hand

$$|\sin(t\varphi(x)) - \sin(s\varphi(x))|^2 \leq |t - s|^2 \varphi^2(x) = |t - s|^2 x^{-3} |\log x|^{-3}. \tag{9-2}$$

On the other hand the left-hand side can clearly be estimated by 4. Now, let $\psi(x) = 2x^{3/2} |\log x|^{3/2}$. Then (9-2) gives the sharper estimate if and only if $|t - s| \leq \psi(x)$ or equivalently $x \geq \psi^{-1}(|t - s|)$. Splitting the fractional norm, we obtain

$$\begin{aligned} & \left(\int_0^T \int_0^T \frac{\|u(t) - u(s)\|_V^q}{|t - s|^{1+q/2}} dt ds \right)^{1/q} \\ &= \left(\int_0^T \int_{-t}^{T-t} \frac{\|u(t) - u(t+r)\|_V^q}{|r|^{1+q/2}} dr dt \right)^{1/q} \\ &\lesssim \left(\int_0^T \int_{-t}^{T-t} \left(\int_0^{\psi^{-1}(|r|)} x^{1/2} |\log x|^{1/2} dx \right)^{q/2} \frac{dr}{|r|^{1+q/2}} dt \right)^{1/q} \\ &\quad + \left(\int_0^T \int_{-t}^{T-t} \left(\int_{\psi^{-1}(|r|)}^{1/2} x^{-5/2} |\log x|^{-5/2} dx \right)^{q/2} \frac{dr}{|r|^{1-q/2}} dt \right)^{1/q}. \tag{9-3} \end{aligned}$$

Now, for the innermost integral of the first term we have for $F(x) = x^{3/2} |\log x|^{1/2}$

$$\begin{aligned} \int_0^{\psi^{-1}(|r|)} x^{1/2} |\log x|^{1/2} dx &\lesssim \int_0^{\psi^{-1}(|r|)} F'(x) dx = F(\psi^{-1}(|r|)) \\ &\lesssim \psi(\psi^{-1}(|r|)) |\log \psi^{-1}(|r|)|^{-1} = |r| |\log \psi^{-1}(|r|)|^{-1} \lesssim |r| |\log r|^{-1}. \end{aligned}$$

Analogously, for the second term we have for $F(x) = -x^{-3/2} |\log x|^{-5/2}$

$$\begin{aligned} \int_{\psi^{-1}(|r|)}^{1/2} x^{-5/2} |\log x|^{-5/2} dx &\lesssim \int_{\psi^{-1}(|r|)}^{1/2} F'(x) dx \leq -F(\psi^{-1}(|r|)) \\ &\lesssim \frac{1}{\psi(\psi^{-1}(|r|))} |\log \psi^{-1}(|r|)|^{-1} \lesssim |r|^{-1} |\log r|^{-1}. \end{aligned}$$

Hence, (9-3) is dominated up to a constant by the finite expression

$$\left(\int_0^T |\log r|^{-q/2} \frac{dr}{|r|} \right)^{1/q}$$

for $q > 2$.

Hence, for maximal L^2 -regularity of forms the only case left open is that of $\dot{W}^{1/2,2}([0, T]; \mathcal{B}(V, V'))$ regularity, which we are not able to answer at the moment. Note that there is also a positive result assuming some half differentiability. Namely, it was shown by Auscher and Egert [2016] that for elliptic operators one has maximal L^2 -regularity if the coefficients a_{ij} satisfy $\partial^{1/2} a_{ij} \in \text{BMO}$. This in particular implies $a_{ij} \in \dot{H}^{1/2,q}$ for all $q \in (1, \infty)$, which in turn implies $a_{ij} \in \dot{W}^{1/2,q}$ for all $q \geq 2$, which in general is not sufficient for maximal L^p -regularity by the above example. In the other direction, the inclusion

$\dot{W}^{1/2,q} \hookrightarrow \dot{H}^{1/2,q}$ does only hold for $q \in (1, 2]$. Hence, for $q \in (1, 2)$ the space $\dot{H}^{1/2,q}$ contains step functions. Note that in the critical case one has $\dot{H}^{1/2,2} = \dot{W}^{1/2,2}$; i.e., the Besov and the Bessel scale give rise to the same problem.

References

- [Acquistapace and Terreni 1987] P. Acquistapace and B. Terreni, “A unified approach to abstract linear nonautonomous parabolic equations”, *Rend. Sem. Mat. Univ. Padova* **78** (1987), 47–107. MR Zbl
- [Amann 1995] H. Amann, *Linear and quasilinear parabolic problems, I: Abstract linear theory*, Monographs in Mathematics **89**, Birkhäuser, Boston, 1995. MR Zbl
- [Amann 2000] H. Amann, “Compact embeddings of vector-valued Sobolev and Besov spaces”, *Glas. Mat. Ser. III* **35**:1 (2000), 161–177. MR Zbl
- [Amann 2005] H. Amann, “Quasilinear parabolic problems via maximal regularity”, *Adv. Differential Equations* **10**:10 (2005), 1081–1110. MR Zbl
- [Amann 2006] H. Amann, “Quasilinear parabolic functional evolution equations”, pp. 19–44 in *Recent advances on elliptic and parabolic issues* (Zürich, Switzerland, 2004), edited by M. Chipot and H. Ninomiya, World Scientific, Nackensack, NJ, 2006. Zbl
- [Arendt et al. 2007] W. Arendt, R. Chill, S. Fornaro, and C. Poupaud, “ L^p -maximal regularity for non-autonomous evolution equations”, *J. Differential Equations* **237**:1 (2007), 1–26. MR Zbl
- [Arendt et al. 2017] W. Arendt, D. Dier, and S. Fackler, “J. L. Lions’ problem on maximal regularity”, *Arch. Math. (Basel)* **109**:1 (2017), 59–72. MR Zbl
- [Auscher 2002] P. Auscher, “Lectures on the Kato square root problem”, pp. 1–18 in *Surveys in analysis and operator theory* (Canberra, 2001), edited by A. Hassell, Proc. Centre Math. Appl. Austral. Nat. Univ. **40**, Austral. Nat. Univ., Canberra, 2002. MR Zbl
- [Auscher and Egert 2016] P. Auscher and M. Egert, “On non-autonomous maximal regularity for elliptic operators in divergence form”, *Arch. Math. (Basel)* **107**:3 (2016), 271–284. MR Zbl
- [Auscher and Tchamitchian 1998] P. Auscher and P. Tchamitchian, *Square root problem for divergence operators and related topics*, Astérisque **249**, Société Mathématique de France, Paris, 1998. MR Zbl
- [Auscher and Tchamitchian 2001a] P. Auscher and P. Tchamitchian, “Gaussian estimates for second order elliptic divergence operators on Lipschitz and C^1 domains”, pp. 15–32 in *Evolution equations and their applications in physical and life sciences* (Bad Herrenalb, 1998), edited by G. Lumer and L. Weis, Lecture Notes in Pure and Appl. Math. **215**, Dekker, New York, 2001. MR Zbl
- [Auscher and Tchamitchian 2001b] P. Auscher and P. Tchamitchian, “Square roots of elliptic second order divergence operators on strongly Lipschitz domains: L^p theory”, *Math. Ann.* **320**:3 (2001), 577–623. MR Zbl
- [Auscher et al. 2002] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, and P. Tchamitchian, “The solution of the Kato square root problem for second order elliptic operators on \mathbb{R}^n ”, *Ann. of Math. (2)* **156**:2 (2002), 633–654. MR Zbl
- [Axelsson et al. 2006] A. Axelsson, S. Keith, and A. McIntosh, “The Kato square root problem for mixed boundary value problems”, *J. London Math. Soc. (2)* **74**:1 (2006), 113–130. MR Zbl
- [Bergh and Löfström 1976] J. Bergh and J. Löfström, *Interpolation spaces: an introduction*, Grundlehren der Mathematischen Wissenschaften **223**, Springer, 1976. MR Zbl
- [Burkholder 2001] D. L. Burkholder, “Martingales and singular integrals in Banach spaces”, pp. 233–269 in *Handbook of the geometry of Banach spaces, I*, edited by W. B. Johnson and J. Lindenstrauss, North-Holland, Amsterdam, 2001. MR Zbl
- [Denk et al. 2003] R. Denk, M. Hieber, and J. Prüss, *\mathcal{R} -boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc. **788**, Amer. Math. Soc., Providence, RI, 2003. MR Zbl
- [Dier 2014] D. Dier, *Non-autonomous Cauchy problems governed by forms: maximal regularity and invariance*, Ph.D. thesis, Universität Ulm, 2014, available at <https://oparu.uni-ulm.de/xmlui/handle/123456789/3297>.

- [Dier and Zacher 2017] D. Dier and R. Zacher, “Non-autonomous maximal regularity in Hilbert spaces”, *J. Evol. Equ.* **17**:3 (2017), 883–907. MR
- [Dong and Kim 2010] H. Dong and D. Kim, “Elliptic equations in divergence form with partially BMO coefficients”, *Arch. Ration. Mech. Anal.* **196**:1 (2010), 25–70. MR Zbl
- [Dong and Kim 2016] H. Dong and D. Kim, “On L_p -estimates for elliptic and parabolic equations with A_p weights”, preprint, 2016. arXiv
- [Duong and Robinson 1996] X. T. Duong and D. W. Robinson, “Semigroup kernels, Poisson bounds, and holomorphic functional calculus”, *J. Funct. Anal.* **142**:1 (1996), 89–128. MR Zbl
- [Egert et al. 2014] M. Egert, R. Haller-Dintelmann, and P. Tolksdorf, “The Kato square root problem for mixed boundary conditions”, *J. Funct. Anal.* **267**:5 (2014), 1419–1461. MR Zbl
- [Egert et al. 2016] M. Egert, R. Haller-Dintelmann, and P. Tolksdorf, “The Kato square root problem follows from an extrapolation property of the Laplacian”, *Publ. Mat.* **60**:2 (2016), 451–483. MR Zbl
- [Engel and Nagel 2000] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics **194**, Springer, 2000. MR Zbl
- [Fackler 2014] S. Fackler, “The Kalton–Lancien theorem revisited: maximal regularity does not extrapolate”, *J. Funct. Anal.* **266**:1 (2014), 121–138. MR Zbl
- [Fackler 2015] S. Fackler, “Non-autonomous maximal L^p -regularity for rough divergence form elliptic operators”, preprint, 2015. arXiv
- [Fackler 2017a] S. Fackler, “J.-L. Lions’ problem concerning maximal regularity of equations governed by non-autonomous forms”, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **34**:3 (2017), 699–709. MR Zbl
- [Fackler 2017b] S. Fackler, “Non-autonomous maximal regularity for forms given by elliptic operators of bounded variation”, *J. Differential Equations* **263**:6 (2017), 3533–3549. MR Zbl
- [Rubio de Francia 1986] J. L. Rubio de Francia, “Martingale and integral transforms of Banach space valued functions”, pp. 195–222 in *Probability and Banach spaces (Zaragoza, 1985)*, edited by J. Bastero and M. San Miguel, Lecture Notes in Math. **1221**, Springer, 1986. MR Zbl
- [Gallarati and Veraar 2017a] C. Gallarati and M. Veraar, “Evolution families and maximal regularity for systems of parabolic equations”, *Adv. Differential Equations* **22**:3-4 (2017), 169–190. MR Zbl
- [Gallarati and Veraar 2017b] C. Gallarati and M. Veraar, “Maximal regularity for non-autonomous equations with measurable dependence on time”, *Potential Anal.* **46**:3 (2017), 527–567. MR Zbl
- [Haak and Ouhabaz 2015] B. H. Haak and E. M. Ouhabaz, “Maximal regularity for non-autonomous evolution equations”, *Math. Ann.* **363**:3-4 (2015), 1117–1145. MR Zbl
- [Haak et al. 2006] B. H. Haak, M. Haase, and P. C. Kunstmann, “Perturbation, interpolation, and maximal regularity”, *Adv. Differential Equations* **11**:2 (2006), 201–240. MR Zbl
- [Hieber and Monniaux 2000a] M. Hieber and S. Monniaux, “Heat-kernels and maximal $L^p - L^q$ -estimates: the non-autonomous case”, *J. Fourier Anal. Appl.* **6**:5 (2000), 467–481. MR Zbl
- [Hieber and Monniaux 2000b] M. Hieber and S. Monniaux, “Pseudo-differential operators and maximal regularity results for non-autonomous parabolic equations”, *Proc. Amer. Math. Soc.* **128**:4 (2000), 1047–1053. MR Zbl
- [Hytönen and Portal 2008] T. Hytönen and P. Portal, “Vector-valued multiparameter singular integrals and pseudodifferential operators”, *Adv. Math.* **217**:2 (2008), 519–536. MR Zbl
- [Kaip and Saal 2012] M. Kaip and J. Saal, “The permanence of \mathcal{R} -boundedness and property (α) under interpolation and applications to parabolic systems”, *J. Math. Sci. Univ. Tokyo* **19**:3 (2012), 359–407. MR Zbl
- [Kalton and Lancien 2000] N. J. Kalton and G. Lancien, “A solution to the problem of L^p -maximal regularity”, *Math. Z.* **235**:3 (2000), 559–568. MR Zbl
- [Krylov 2007] N. V. Krylov, “Parabolic and elliptic equations with VMO coefficients”, *Comm. Partial Differential Equations* **32**:1-3 (2007), 453–475. MR Zbl

- [Kunstmann and Weis 2004] P. C. Kunstmann and L. Weis, “Maximal L_p -regularity for parabolic equations, Fourier multiplier theorems and H^∞ -functional calculus”, pp. 65–311 in *Functional analytic methods for evolution equations*, edited by R. N. M. Iannelli and S. Piazzera, Lecture Notes in Math. **1855**, Springer, 2004. MR Zbl
- [Lunardi 1995] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Progress in Nonlinear Differential Equations and their Applications **16**, Birkhäuser, Basel, 1995. MR Zbl
- [Ouhabaz 2005] E. M. Ouhabaz, *Analysis of heat equations on domains*, London Mathematical Society Monographs Series **31**, Princeton University Press, 2005. MR Zbl
- [Portal and Štrkalj 2006] P. Portal and Ž. Štrkalj, “Pseudodifferential operators on Bochner spaces and an application”, *Math. Z.* **253**:4 (2006), 805–819. MR Zbl
- [Prüss and Schnaubelt 2001] J. Prüss and R. Schnaubelt, “Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time”, *J. Math. Anal. Appl.* **256**:2 (2001), 405–430. MR Zbl
- [Simon 1990] J. Simon, “Sobolev, Besov and Nikolskii fractional spaces: imbeddings and comparisons for vector valued spaces on an interval”, *Ann. Mat. Pura Appl.* (4) **157** (1990), 117–148. MR Zbl
- [Triebel 1978] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland Mathematical Library **18**, North-Holland, Amsterdam, 1978. MR Zbl
- [Voigt 1992] J. Voigt, “Abstract Stein interpolation”, *Math. Nachr.* **157** (1992), 197–199. MR Zbl
- [Weis 2001] L. Weis, “Operator-valued Fourier multiplier theorems and maximal L_p -regularity”, *Math. Ann.* **319**:4 (2001), 735–758. MR Zbl
- [Yamazaki 1986] M. Yamazaki, “The L^p -boundedness of pseudodifferential operators with estimates of parabolic type and product type”, *J. Math. Soc. Japan* **38**:2 (1986), 199–225. MR Zbl

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STEPHAN FACKLER: stephan.fackler@uni-ulm.de
Institute of Applied Analysis, Ulm University, Ulm, Germany



TRANSFERENCE OF BILINEAR RESTRICTION ESTIMATES TO QUADRATIC VARIATION NORMS AND THE DIRAC–KLEIN–GORDON SYSTEM

TIMOTHY CANDY AND SEBASTIAN HERR

Firstly, bilinear Fourier restriction estimates — which are well known for free waves — are extended to adapted spaces of functions of bounded quadratic variation, under quantitative assumptions on the phase functions. This has applications to nonlinear dispersive equations, in particular in the presence of resonances. Secondly, critical global well-posedness and scattering results for massive Dirac–Klein–Gordon systems in dimension three are obtained, in resonant as well as in nonresonant regimes. The results apply to small initial data in scale-invariant Sobolev spaces exhibiting a small amount of angular regularity.

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1. Introduction

The Fourier restriction conjecture was shaped in the 1970s by work of Stein, among others, and has generated significant advances in the field of harmonic analysis and dispersive partial differential equations since then; see, e.g., [Stein 1993; Tao 2004] for a survey and references.

As an example, let $n \geq 2$ and C be a compact subset of the cone, say $C = \{(|\xi|, \xi) : \frac{1}{2} \leq |\xi| \leq 2\} \subset \mathbb{R}^{n+1}$, and g be a Schwartz function on \mathbb{R}^{n+1} . Equivalently to the Fourier restriction operator $\mathcal{R} : g \mapsto \hat{g}|_C$, consider its adjoint, the Fourier extension operator

$$\mathcal{E}f(t, x) = \int_{\mathbb{R}^n} e^{-i(t,x) \cdot (|\xi|, \xi)} f(\xi) d\xi$$

for smooth f with $\text{supp}(f)$ contained in the unit annulus. The function $\mathcal{E}f$ can be viewed as the inverse Fourier transform of a surface-measure supported on the cone C , and defines a function on \mathbb{R}^{n+1} which

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solves the wave equation. The Fourier restriction conjecture for the cone is equivalent to establishing the corresponding Fourier extension estimate

$$\|\mathcal{E}f\|_{L^p_{t,x}(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^q(\mathbb{R}^n)}$$

within the optimal range of p, q . In the special case $q = 2$ this holds if and only if $p \geq (2n + 2)/(n - 1)$, and in the literature on dispersive equations this is stated as

$$\|e^{-it|\nabla|}f\|_{L^p_{t,x}(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^2_x}$$

and called a Strichartz estimate [1977] for the wave equation; see also [Keel and Tao 1998].

In the course of proving Fourier extension estimates for the cone, it became apparent that a key role was played by bilinear estimates. Indeed, a major breakthrough was achieved by Wolff [2001], when he proved that for every $p > (n + 3)/(n + 1)$, $n \geq 2$, we have

$$\|e^{-it|\nabla|}fe^{-it|\nabla|}g\|_{L^p_{t,x}(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^2_x}\|g\|_{L^2_x},$$

provided the supports of \hat{f} and \hat{g} are angularly separated and contained in the unit annulus. As a result Wolff was able to prove the linear restriction conjecture for C in dimension $n = 3$. It is important to note that, in the presence of angular separation, a larger set of p can be covered in the bilinear estimate than would follow from a simple application of Hölder’s inequality together with the linear estimates.

In parallel to these developments, bilinear estimates proved useful in the context of nonlinear dispersive equations; see, e.g., [Klainerman and Machedon 1993; Bourgain 1998; Foschi and Klainerman 2000]. The perturbative approach to dispersive equations is based on constructing adapted function spaces in which nonlinear terms can be effectively estimated. Bilinear estimates for solutions to the homogeneous equation, which go beyond simple almost orthogonality considerations, give precise control over dynamic interactions of products of linear solutions. However, to apply these homogeneous estimates to the nonlinear problem necessitates the transfer of such genuinely bilinear estimates to adapted function spaces.

Such a *transference principle* was implemented first in $X^{s,b}$ spaces; see [Ginibre et al. 1997, Lemma 2.3] and [Klainerman and Selberg 2002, Proposition 3.7]. Let us briefly illustrate it by looking at the following example. Suppose that $u, v \in L^{\infty}_t L^2_x$ are superpositions of modulated solutions of the homogeneous equation, i.e.,

$$u(t) = \int_{\mathbb{R}} e^{it\lambda} e^{it|\nabla|} F_{\lambda} d\lambda, \quad v(t) = \int_{\mathbb{R}} e^{it\lambda'} e^{it|\nabla|} G_{\lambda'} d\lambda',$$

which is true for $u, v \in X^{0,b}$ if $b > \frac{1}{2}$. Suppose in addition, that the spatial Fourier supports of u, v are angularly separated. Then, for any $p > (n + 3)/(n + 1)$, Wolff’s estimate transfers to

$$\|uv\|_{L^p_{t,x}(\mathbb{R}^{n+1})} \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \|e^{it|\nabla|}F_{\lambda}e^{it|\nabla|}G_{\lambda'}\|_{L^p_{t,x}(\mathbb{R}^{n+1})} d\lambda d\lambda' \lesssim \left(\int_{\mathbb{R}} \|F_{\lambda}\|_{L^2_x} d\lambda\right) \left(\int_{\mathbb{R}} \|G_{\lambda'}\|_{L^2_x} d\lambda'\right),$$

which is equivalent to the bilinear estimate holding for functions in $X^{0,b}$. Another strategy involves certain atomic function spaces introduced in [Koch and Tataru 2005]. Suppose that

$$u(t) = \sum_{J \in \mathcal{I}} \mathbb{1}_J(t) e^{it|\nabla|} f_J, \quad v(t) = \sum_{J' \in \mathcal{I}'} \mathbb{1}_{J'}(t) e^{it|\nabla|} g_{J'}$$

for finite partitions $\mathcal{I}, \mathcal{I}'$ of \mathbb{R} and $f_J, g_{J'} \in L^2_x$. Then, under the above angular separation assumption, Wolff’s bound implies

$$\|uv\|_{L^p_{t,x}(\mathbb{R}^{n+1})} \leq \left(\sum_{J \in \mathcal{I}} \sum_{J' \in \mathcal{I}'} \|e^{it|\nabla|} f_J e^{it|\nabla|} g_{J'}\|_{L^p_{t,x}(\mathbb{R}^{n+1})} \right)^{\frac{1}{p}} \lesssim \left(\sum_{J \in \mathcal{I}} \|f_J\|_{L^2_x}^p \right)^{\frac{1}{p}} \left(\sum_{J' \in \mathcal{I}'} \|g_{J'}\|_{L^2_x}^p \right)^{\frac{1}{p}}.$$

As a consequence, we deduce that Wolff’s bilinear estimate holds for angularly separated functions in the atomic space U^p ; see Definition 3.4 below. This is one instance of the transference principle in U^p , which has been formalised in [Hadac et al. 2009, Proposition 2.19].

For many applications, the above superposition requirements are too strong, partly due to the duality theory for the spaces $X^{0,b}$ for $b > \frac{1}{2}$ and U^p for $p \leq 2$. Nevertheless, variations of the above strategies have been successfully employed in numerous applications to nonlinear global-in-time problems in the case $p \geq 2$. In the case $p < 2$, the only result we are aware of is [Sterbenz and Tataru 2010, Lemma 5.7 and its proof], where this approach is used in conjunction with an interpolation argument to give a partial result only; see Remark 6.2 for further details.

It turned out that one of the most powerful function spaces in the context of adapted function spaces is the space of functions of bounded quadratic variation V^2 , which is slightly bigger than U^2 . Our first main result of this paper is the corresponding transference principle in V^2 for a quite general class of surfaces in Theorem 1.1 below.

We start with some definitions. Define $\mathcal{Z} = \{(t_j)_{j \in \mathbb{Z}} : t_j \in \mathbb{R} \text{ and } t_j < t_{j+1}\}$ to be the set of increasing sequences of real numbers and $1 \leq p < \infty$. Given a function $\rho : \mathbb{R} \rightarrow L^2_x$, we define the p -variation of ρ to be

$$|\rho|_{V^p} = \sup_{(t_j) \in \mathcal{Z}} \left(\sum_{j \in \mathbb{Z}} \|\rho(t_j) - \rho(t_{j-1})\|_{L^2_x}^p \right)^{\frac{1}{p}}.$$

The Banach space V^p is then defined to be all right continuous functions $\rho : \mathbb{R} \rightarrow L^2_x$ such that

$$\|\rho\|_{V^p} = \|\rho\|_{L^\infty_t L^2_x} + |\rho|_{V^p} < \infty.$$

Given a phase $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ we let V^p_Φ denote the space of all functions u such that $e^{-it\Phi(-i\nabla)}u \in V^p$ equipped with the obvious norm $\|u\|_{V^p_\Phi} = \|e^{-it\Phi(-i\nabla)}u\|_{V^p}$. In other words, the space V^p_Φ contains all functions $u \in L^\infty_t L^2_x$ such that the pull-back along the linear flow has bounded p -variation; in particular we have

$$\|e^{it\Phi(-i\nabla)}f\|_{V^p_\Phi} = \|f\|_{L^2_x}.$$

Before stating Theorem 1.1, we need to introduce the assumptions that we impose on our phases, which are motivated by [Lee and Vargas 2010; Bejenaru 2017]. Examples will be discussed in Section 2. Let $\Phi_j : \mathbb{R}^n \rightarrow \mathbb{R}$ and Λ_j be a convex subset of $\{\frac{1}{16} \leq |\xi| \leq 16\}$. Given $\mathfrak{h} = (a, h) \in \mathbb{R}^{1+n}$ and $\{j, k\} = \{1, 2\}$ we define the hypersurfaces

$$\Sigma_j(\mathfrak{h}) = \{\xi \in \Lambda_j \cap (\Lambda_k + h) : \Phi_j(\xi) = \Phi_k(\xi - h) + a\}.$$

With this notation, we are ready to state the main assumption; cf. [Bejenaru 2017; Lee and Vargas 2010].

Assumption 1 (transversality/curvature/regularity). *There exist $\mathbf{D}_1, \mathbf{D}_2 > 0$ and $N \in \mathbb{N}$ such that for $\Phi_1, \Phi_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ the following hold true:*

(i) *For every $\{j, k\} = \{1, 2\}$, $\mathfrak{h} \in \mathbb{R}^{1+n}$, $\xi, \xi' \in \Sigma_j(\mathfrak{h})$, and $\eta \in \Lambda_k$ we have the estimate*

$$|(\nabla\Phi_j(\xi) - \nabla\Phi_j(\xi')) \wedge (\nabla\Phi_j(\xi) - \nabla\Phi_k(\eta))| \geq \mathbf{D}_1 |\xi - \xi'|.$$

(ii) *We have $\Phi_j \in C^N(\Lambda_j)$ with the derivative bound*

$$\sup_{1 \leq |\kappa| \leq N} \|\partial^\kappa \Phi_j\|_{L^\infty(\Lambda_j)} \leq \mathbf{D}_2.$$

The condition (i) in Assumption 1 is somewhat difficult to interpret, but one immediate consequence is the bound

$$|\nabla\Phi_j(\xi) - \nabla\Phi_j(\xi')| \geq \frac{\mathbf{D}_1 |\xi - \xi'|}{\|\nabla\Phi_1\|_{L^\infty} + \|\nabla\Phi_2\|_{L^\infty}}, \tag{1-1}$$

which holds for every $\xi, \xi' \in \Sigma_j(\mathfrak{h})$. To some extent, this is a *curvature* condition, as it shows that the normal direction varies on $\Sigma_j(\mathfrak{h})$. Another consequence of (i) is that for every $\xi \in \Lambda_1$, $\eta \in \Lambda_2$ we have the *transversality* bound

$$|\nabla\Phi_1(\xi) - \nabla\Phi_2(\eta)| \geq \frac{\mathbf{D}_1}{\min\{\|\nabla^2\Phi_1\|_{L^\infty}, \|\nabla^2\Phi_2\|_{L^\infty}\}}. \tag{1-2}$$

This follows by simply observing that for every $\xi \in \Lambda_1$ there is $\mathfrak{h} \in \mathbb{R}^{1+n}$ such that $\xi \in \Sigma_1(\mathfrak{h})$. Our first main result can now be stated as follows.

Theorem 1.1. *Let $n \geq 2$, $p > (n + 3)/(n + 1)$, and $\mathbf{D}_1, \mathbf{D}_2, \mathbf{R}_0 > 0$. For $j = 1, 2$, let $\Lambda_j, \Lambda_j^* \subset \{\frac{1}{16} \leq |\xi| \leq 16\}$ with Λ_j convex and $\Lambda_j^* + 1/\mathbf{R}_0 \subset \Lambda_j$. There exists $N \in \mathbb{N}$ and a constant $C > 0$ such that, for any phases Φ_1 and Φ_2 satisfying Assumption 1, and any $u \in V_{\Phi_1}^2$, $v \in V_{\Phi_2}^2$ with $\text{supp } \hat{u}(t) \subset \Lambda_1^*$, $\text{supp } \hat{v}(t) \subset \Lambda_2^*$, we have*

$$\|uv\|_{L_{t,x}^p(\mathbb{R}^{1+n})} \leq C \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2}.$$

Note that the constants N and C depend on the parameters $p > (n + 3)/(n + 1)$, $n \geq 2$, and $\mathbf{D}_1, \mathbf{D}_2, \mathbf{R}_0 > 0$, but are otherwise independent of the phase Φ_j , the sets Λ_j, Λ_j^* , and the functions u and v . Moreover, as the conditions in Assumption 1 are invariant under translations, the condition that $\Lambda_j \subset \{\frac{1}{16} \leq |\xi| \leq 16\}$ can be replaced with the condition that the sets Λ_j are simply contained in balls of radius 16. In other words, the *location* of the sets Λ_j plays no role. We refer the reader to Corollary 6.1 for a generalisation of Theorem 1.1 to mixed norms. Further, we refer to Corollary 6.4 for a generalisation to more general frequency scales in the case of hyperboloids, which is also shown to be sharp.

Let us summarise the developments for solutions to the homogeneous equation, i.e.,

$$u = e^{it\Phi_1(-i\nabla)} f, \quad v = e^{it\Phi_2(-i\nabla)} g.$$

First estimates of this type for nontrivial $p < 2$ are due to Bourgain [1991; 1995] in the case of the cone, i.e., $\Phi_1(\xi) = \Phi_2(\xi) = |\xi|$. Subsequently, these have been improved by Tao, Vargas and Vega [Tao et al. 1998], Moyua, Vargas and Vega [Moyua et al. 1999], Tao and Vargas [2000a], before finally Tao

[2001] proved the endpoint case $p = (n + 3)/(n + 1)$; see also Remark 5.1. Actually, we observe that the vector-valued inequality in [Tao 2001] is strong enough to deduce the estimate in U^2 in the case of the wave equation; see Remark 5.2. Related estimates for null-forms have been proved by Tao and Vargas [2000b], Klainerman, Rodnianski and Tao [Klainerman et al. 2002], Lee and Vargas [2008], and Lee, Rogers and Vargas [Lee et al. 2008]. In the case of the paraboloid, i.e., $\Phi_1(\xi) = \Phi_2(\xi) = |\xi|^2$, the result for homogeneous solutions is due to Tao [2003], with generalisations by Lee [2006a; 2006b], Lee and Vargas [2010], and Bejenaru [2017] under more general curvature and transversality conditions, as well as by Buschenhenke, Müller and Vargas [2017] for surfaces of finite type. For our approach, the most important references are [Tao 2003] concerning notation and general line of proof and [Lee and Vargas 2010; Bejenaru 2017], concerning the assumptions on the phases and its consequences. Throughout the paper, we shall point out similarities and differences in more detail.

We would like to highlight the fact that we explicitly track the dependence of the constants on the phases in Theorem 1.1 based on the global, quantitative Assumption 1; in particular we avoid abstract localisation arguments. This is helpful for applications to dispersive equations, as we will see below. The main novelty of this result, however, lies in the fact that it holds for $V_{\Phi_j}^2$ -functions in the range $p \leq 2$.

Now, we turn to the application of Theorem 1.1 to nonlinear dispersive equations with a quadratic nonlinearity which exhibit resonances. Roughly speaking, by a resonance we mean the scenario that a product of two solutions to the homogeneous equation creates another solution of the homogeneous equation; see Section 8 for details. This leads to the lack of oscillations in the Duhamel integral and hence to strong nonlinear effects. In many instances, one finds that the Fourier supports intersect transversally in the resonant sets. As an example, we mention the local well-posedness theory for the Zakharov system [Bejenaru et al. 2009; Bejenaru and Herr 2011], where this is exploited in terms of a nonlinear Loomis–Whitney inequality [Bennett et al. 2005; Bejenaru et al. 2010; Bennett and Bez 2010; Koch and Steinerberger 2015]. This is a special case of the multilinear restriction theory [Bennett et al. 2006; Bennett and Bez 2010]. Here, we will exploit transversality in resonant sets via Theorem 1.1 and prove global-in-time estimates which go beyond the range of linear Strichartz estimates.

With this approach, we address the Dirac–Klein–Gordon system

$$\begin{aligned} -i\gamma^\mu \partial_\mu \psi + M\psi &= \phi\psi, \\ \square\phi + m^2\phi &= \psi^\dagger \gamma^0 \psi. \end{aligned} \tag{1-3}$$

Here, $\psi : \mathbb{R}^{1+3} \rightarrow \mathbb{C}^4$ is a spinor field, $\psi^\dagger = \bar{\psi}^t$, $\phi : \mathbb{R}^{1+3} \rightarrow \mathbb{R}$ is a scalar field, $\square := \partial_t^2 - \Delta_x$ is the d’Alembertian operator, and $M, m \geq 0$. We use the summation convention with respect to $\mu = 0, \dots, 4$, and the Dirac matrices $\gamma^\mu \in \mathbb{C}^{4 \times 4}$ are given by

$$\gamma^0 = \text{diag}(1, 1, -1, -1), \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix},$$

with the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We are interested in the system (1-3) with the initial condition

$$\psi(0) = \psi_0 : \mathbb{R}^3 \rightarrow \mathbb{C}^4 \quad \text{and} \quad (\phi(0), \partial_t \phi(0)) = (\phi_0, \phi_1) : \mathbb{R}^3 \rightarrow \mathbb{R} \times \mathbb{R}. \quad (1-4)$$

In the massless case, (1-3) can be rescaled and the scale-invariant Sobolev space for (ψ_0, ϕ_0, ϕ_1) is

$$L^2(\mathbb{R}^3; \mathbb{C}^4) \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^3; \mathbb{R}) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3; \mathbb{R}).$$

Let $\langle \Omega \rangle^\sigma = (1 - \Delta_{\mathbb{S}^2})^{\frac{\sigma}{2}}$ denote σ angular derivatives; see Section 7B for precise definitions. Our second main result is the following.

Theorem 1.2. *Suppose that either $2M \geq m > 0$ and $\sigma > 0$, or that $m > 2M > 0$ and $\sigma > \frac{7}{30}$. Then, for initial data satisfying*

$$\|\langle \Omega \rangle^\sigma \psi_0\|_{L^2(\mathbb{R}^3)} + \|\langle \Omega \rangle^\sigma \phi_0\|_{H^{1/2}(\mathbb{R}^3)} + \|\langle \Omega \rangle^\sigma \phi_1\|_{H^{-1/2}(\mathbb{R}^3)} \ll 1,$$

the system (1-3)–(1-4) is globally well-posed and solutions (ψ, ϕ) scatter to free solutions as $t \rightarrow \pm\infty$.

As the proof relies on contraction arguments in adapted function spaces, the notion of global well-posedness in Theorem 1.2 includes persistence of regularity and the local Lipschitz continuity of the flow map and it provides a certain uniqueness class. Note that the angular regularity does not affect the scaling of the spaces. In summary, Theorem 1.2 establishes global well-posedness and scattering in the critical Sobolev space for small initial data with a bit of angular regularity.

In the case $2M > m > 0$, which we call *nonresonant regime* due to Lemma 8.7, this theorem improves Wang’s result [2015] by both relaxing the angular regularity hypothesis and replacing Besov spaces by Sobolev spaces. We also mention the previous subcritical result [Bejenaru and Herr 2017] without additional angular regularity, where the possibility of a Besov endpoint result with an $\epsilon > 0$ of angular regularity was discussed in Remark 4.2. In the case $m > 2M > 0$, which we call the *resonant regime* due to Lemma 8.7, this appears to be the first global well-posedness and scattering result in critical spaces for (1-3). A similar comment applies to the case $2M = m > 0$, which we call the *weakly resonant regime*. It is the resonant regime where we employ Theorem 1.1; see also Remark 7.6. Concerning further comments on the number of angular derivatives required in the resonant case, we refer to Remark 8.4.

We shall only mention a few selected results on this well-studied system (1-3). We refer the reader to [D’Ancona et al. 2007] for previous local results and to [Chadam and Glassey 1974; Bachelot 1988; Bejenaru and Herr 2017; Wang 2015] for previous global results on this system. Concerning its relevance in physics we refer the reader to [Bjorken and Drell 1964].

The organisation of the paper is as follows: In Section 2, we discuss a sufficient condition on the phases, verify Assumption 1 in the case of the Schrödinger, the wave, and the Klein–Gordon equations, and derive important consequences, in particular the dispersive inequality, and a bilinear estimate for homogeneous solutions in $L^2_{t,x}$. In Section 3, we study wave packets, atomic spaces and tubes. In Section 4, we state and prove a crucial localised version of Theorem 1.1. The proof proceeds by performing an induction-on-scales argument, and reducing the problem to obtaining a crucial L^2 -bound which in turn follows from a combinatorial estimate. Section 5 is devoted to the globalisation lemma, which removes the localisation assumption used in Section 4, and hence concludes the proof of Theorem 1.1. In Section 6, we generalise

Theorem 1.1 to mixed norms and, in the case of hyperboloids, give an extension to general scales and discuss counterexamples. In Section 7 we prepare the analysis of the Dirac–Klein–Gordon system and prove Theorem 1.2 under the hypothesis that certain bilinear estimates hold true. In Section 8 we discuss some auxiliary estimates and finally provide proofs of the bilinear estimates used in Section 7.

2. On Assumption 1: examples and consequences

We now discuss examples, and consider in detail a number of key consequences of Assumption 1. All of this is known to experts, at least in the specific cases we are interested in. The main objective is to verify that Assumption 1 allows for a unified treatment which allows us to track the dependence of constants on the phases.

2A. A sufficient condition. Let $\text{diam}(\Lambda_j) = \sup_{\xi, \xi' \in \Lambda_j} |\xi - \xi'|$. The condition (i) in Assumption 1 is somewhat difficult to check (essentially since we insist on a *global* condition rather than just a local condition using the Hessian of Φ_j). In practise it is easier to check the following marginally stronger conditions.

Lemma 2.1. *Assume that the following three conditions hold:*

(i) For all $\xi \in \Lambda_1$ and $\eta \in \Lambda_2$

$$|\nabla\Phi_1(\xi) - \nabla\Phi_2(\eta)| \geq A_1. \tag{2-1}$$

(ii) For $j = 1, 2$, and every $\mathfrak{h} \in \mathbb{R}^{1+n}$ and $\xi, \xi' \in \Sigma_j(\mathfrak{h})$

$$\left| (\nabla\Phi_j(\xi) - \nabla\Phi_j(\xi')) \cdot \frac{\xi - \xi'}{|\xi - \xi'|} \right| \geq A_2 |\xi - \xi'|. \tag{2-2}$$

(iii) The sets Λ_1 and Λ_2 satisfy

$$\text{diam}(\Lambda_1) + \text{diam}(\Lambda_2) \leq \frac{A_1 A_2}{2(\|\nabla^2\Phi_1\|_{L^\infty(\Lambda_1)} + \|\nabla^2\Phi_2\|_{L^\infty(\Lambda_2)})^2}. \tag{2-3}$$

Then, condition (i) in Assumption 1 holds with $D_1 = \frac{1}{2}A_1 A_2$.

Proof. The first step is to observe that for vectors $x, y \in \mathbb{R}^n$, and $\omega \in \mathbb{S}^{n-1}$ we have

$$|x \wedge y| \geq |y| |x \cdot \omega| - |x| |y \cdot \omega|. \tag{2-4}$$

Indeed, this follows from

$$|x \wedge y|^2 = |x|^2 |y|^2 - (x \cdot y)^2 = |y|^2 \left| x - \frac{x \cdot y}{|y|^2} y \right|^2,$$

which implies

$$|x \wedge y| = |y| \left| x - \frac{x \cdot y}{|y|^2} y \right| \geq |y| \left| x \cdot \omega - \frac{x \cdot y}{|y|^2} y \cdot \omega \right| \geq |y| \left(|x \cdot \omega| - \frac{|x|}{|y|} |y \cdot \omega| \right).$$

In particular, if we let $x = \nabla\Phi_j(\xi) - \nabla\Phi_j(\xi')$, $y = \nabla\Phi_j(\xi) - \nabla\Phi_k(\eta)$, and $\omega = (\xi - \xi')/|\xi - \xi'|$, then since $|x| \leq \|\nabla^2\Phi_j\|_{L^\infty(\Lambda_j)} |\xi - \xi'|$ (using the convexity of Λ_j), the lower bound (i) in Assumption 1

would follow from (2-2), (2-4), and the transversality condition (2-1), provided that

$$\left| (\nabla\Phi_j(\xi) - \nabla\Phi_k(\eta)) \cdot \frac{\xi - \xi'}{|\xi - \xi'|} \right| \leq \frac{A_1 A_2}{2\|\nabla^2\Phi_j\|_{L^\infty(\Lambda_j)}}. \quad (2-5)$$

The proof of (2-5) requires the condition $\xi, \xi' \in \Sigma_j(h)$ together with the assumption (2-3) on the size of the sets Λ_j . Let

$$\sigma_j(x, z) = \Phi_j(x) - \Phi_j(z) - \nabla\Phi_j(z) \cdot (x - z).$$

A computation gives

$$\begin{aligned} \nabla\Phi_j(z) \cdot (x - y) &= (\Phi_j(x) - \sigma_j(x, z) - \Phi_j(z) - \nabla\Phi_j(z) \cdot z) - (\Phi_j(y) - \sigma_j(y, z) - \Phi_j(z) - \nabla\Phi_j(z) \cdot z) \\ &= \Phi_j(x) - \Phi_j(y) + \sigma_j(y, z) - \sigma_j(x, z), \end{aligned}$$

and hence, using the assumption $\xi, \xi' \in \Sigma_j(h)$, we see that

$$\begin{aligned} (\nabla\Phi_j(\xi) - \nabla\Phi_k(\eta)) \cdot (\xi - \xi') &= \Phi_j(\xi) - \Phi_j(\xi') + \sigma_j(\xi', \xi) - (\Phi_j(\xi - h) - \Phi_k(\xi' - h) + \sigma_k(\xi' - h, \eta) - \sigma_k(\xi - h, \eta)) \\ &= \sigma_j(\xi', \xi) + \sigma_k(\xi - h, \eta) - \sigma_k(\xi' - h, \eta). \end{aligned}$$

If we now observe that

$$\sigma_j(x, z) - \sigma_j(y, z) = \int_0^1 [\nabla\Phi_j(y + t(x - y)) - \nabla\Phi_j(z)] \cdot (x - y) dt \leq \|\nabla^2\Phi_j\|_{L^\infty(\Lambda_j)} \text{diam}(\Lambda_j) |x - y|$$

we then deduce the bound

$$\left| (\nabla\Phi_j(\xi) - \nabla\Phi_k(\eta)) \cdot \frac{\xi - \xi'}{|\xi - \xi'|} \right| \leq \text{diam}(\Lambda_1) \|\nabla^2\Phi_1\|_{L^\infty(\Lambda_1)} + \text{diam}(\Lambda_2) \|\nabla^2\Phi_2\|_{L^\infty(\Lambda_2)}.$$

Consequently (2-5) follows from (2-3). \square

2B. The Schrödinger, the wave and the Klein–Gordon equations. We now consider some examples of phases satisfying Assumption 1. It is enough to check the conditions in Lemma 2.1. In particular, by making the sets Λ_j slightly smaller if necessary, it suffices to ensure that the transversality condition (2-1) and curvature condition (2-2) hold.

Firstly, consider the Schrödinger case

$$\Phi_j(\xi) = \frac{1}{2}|\xi|^2.$$

Then the condition (2-1) in Lemma 2.1 becomes

$$|\nabla\Phi_1(\xi) - \nabla\Phi_2(\eta)| = |\xi - \eta|;$$

thus we simply require that the sets Λ_j have some separation. Assuming that the diameters of the sets Λ_j are sufficiently small, we just need to ensure that (2-2) holds. However (2-2) is just

$$\left| (\nabla\Phi_j(\xi) - \nabla\Phi_j(\xi')) \cdot \frac{\xi - \xi'}{|\xi - \xi'|} \right| = |\xi - \xi'|$$

and so (2-2) clearly holds (with constant $A_2 = 1$).

Secondly, consider the case

$$\Phi_j(\xi) = \langle \xi \rangle_{m_j} = (m_j^2 + |\xi|^2)^{\frac{1}{2}},$$

where the mass satisfies $m_j \geq 0$. To simplify notation, we assume that for $\xi \in \Lambda_j$ there is a constant $A > 0$ such that

$$\frac{1}{A} \leq \langle \xi \rangle_{m_j} \leq A.$$

To check the transversality condition (2-1) we note that

$$\begin{aligned} |\nabla \Phi_1(\xi) - \nabla \Phi_2(\eta)|^2 &= \left| \frac{\xi}{\langle \xi \rangle_{m_1}} - \frac{\eta}{\langle \eta \rangle_{m_2}} \right|^2 \\ &= \left(\frac{|\xi|}{\langle \xi \rangle_{m_1}} - \frac{|\eta|}{\langle \eta \rangle_{m_2}} \right)^2 + \frac{2|\xi||\eta|}{\langle \xi \rangle_{m_1} \langle \eta \rangle_{m_2}} \left(1 - \frac{\xi \cdot \eta}{|\xi||\eta|} \right) \\ &= \left(\frac{(m_2|\xi| + m_1|\eta|)(m_2|\xi| - m_1|\eta|)}{\langle \xi \rangle_{m_1} \langle \eta \rangle_{m_2} (|\xi| \langle \eta \rangle_{m_2} + |\eta| \langle \xi \rangle_{m_1})} \right)^2 + \frac{2|\xi||\eta|}{\langle \xi \rangle_{m_1} \langle \eta \rangle_{m_2}} \left(1 - \frac{\xi \cdot \eta}{|\xi||\eta|} \right) \end{aligned} \tag{2-6}$$

(in particular, we *always* have transversality if $|\xi| \approx |\eta| \approx 1$ and $m_1 \ll m_2$).

On the other hand, to check the condition (2-2), we use the following elementary bound.

Lemma 2.2. *Let $\ell \geq 2$ and $(a, h) \in \mathbb{R}^{1+\ell}$. If $x, y \in \{z \in \mathbb{R}^\ell : |z| = |z - h| + a\}$ we have the inequality*

$$\left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2 \geq |x - y|^2 \left| \frac{x}{|x|} - \frac{x - h}{|x - h|} \right|^4 \frac{|x - h|^2}{16|x||y||x - h|^2 + 4(|x - h| + |x|)^2|y|^2}.$$

Proof. The condition $x \in \{z \in \mathbb{R}^\ell : |z| = |z - h| + a\}$ implies $|x - h|^2 = (|x| - a)^2$ and hence

$$\frac{x}{|x|} \cdot h = \frac{|h|^2 - a^2}{2|x|} + a.$$

Therefore

$$\left| \frac{x}{|x|} - \frac{y}{|y|} \right| \geq \frac{|h|^2 - a^2}{2|h|} \left| \frac{1}{|x|} - \frac{1}{|y|} \right| = \frac{|x - h|}{2|h||y|} \left| \frac{x}{|x|} - \frac{x - h}{|x - h|} \right|^2 ||x| - |y||,$$

where we used the identities $h = x - (x - h)$ and $a = |x| - |x - h|$. The lemma now follows by noting that

$$|x - y|^2 = |x||y| \left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2 + ||x| - |y||^2. \quad \square$$

We now show that (2-2) holds. A computation gives

$$\begin{aligned} |(\nabla \Phi_j(\xi) - \nabla \Phi_j(\xi')) \cdot (\xi - \xi')| &= \left| \frac{|\xi|^2}{\langle \xi \rangle_{m_j}} + \frac{|\xi'|^2}{\langle \xi' \rangle_{m_j}} - \frac{\xi \cdot \xi'}{\langle \xi \rangle_{m_j}} - \frac{\xi \cdot \xi'}{\langle \xi' \rangle_{m_j}} \right| \\ &= \left| \langle \xi \rangle_{m_j} + \langle \xi' \rangle_{m_j} - \frac{\xi \cdot \xi' + m_j^2}{\langle \xi \rangle_{m_j}} - \frac{\xi \cdot \xi' + m_j^2}{\langle \xi' \rangle_{m_j}} \right| \\ &= \frac{\langle \xi \rangle_{m_j} + \langle \xi' \rangle_{m_j}}{2} \left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2, \end{aligned}$$

where we let $x = (m_j, \xi)$ and $y = (m_j, \xi')$. If we now note that the surface $\Phi_j(\xi) = \Phi_k(\xi - h) + a$ can be written as $|x| = |y - h'| + a$ with $h' = (m_k - m_j, h)$, then an application of Lemma 2.2 gives

$$|(\nabla\Phi_j(\xi) - \nabla\Phi_j(\xi')) \cdot (\xi - \xi')| \geq \frac{A_1^4}{32A^6} |\xi - \xi'|^2.$$

Therefore, by Lemma 2.1, we see that (i) in Assumption 1 holds with $D_1 = A_1^5/(64A^6)$. Note that the above argument also applies in the case of the wave equation $m_1 = m_2 = 0$.

2C. The dispersive inequality. To simplify the statements to follow, we fix constants $R_0 \geq 1$, $D_1, D_2 > 0$ and $N > n + 1$, and assume that we have phases Φ_1, Φ_2 satisfying Assumption 1 and sets Λ_j, Λ_j^* with Λ_j convex and $\Lambda_j^* + 1/R_0 \subset \Lambda_j \subset \{1/16 \leq |\xi| \leq 16\}$.

As a consequence of the curvature-type bound (1-1) relative to the $(n-1)$ -dimensional surface $\Sigma_j(\mathfrak{h})$, we expect that we should have the dispersive inequality

$$\|e^{it\Phi_j(-i\nabla)} f\|_{L_x^\infty} \lesssim t^{-\frac{n-1}{2}} \|f\|_{L_x^1} \tag{2-7}$$

for $f \in L^1$ with $\text{supp } \hat{f} \subset \Lambda_j$. To prove this decay in practise, the standard approach would involve a stationary phase argument. However, as we only have curvature information on the surfaces $\Sigma_j(\mathfrak{h})$, and these surfaces are somewhat involved to work with, the standard approach via stationary phase arguments, keeping track of the constants, seems difficult to implement. Consequently, we instead present a different argument, using an approach via wave packets. Roughly speaking, fixing some large time $t \approx R$, the idea is to cover Λ_j with balls of size $R^{-1/2}$ and decompose $e^{it\Phi_j(-i\nabla)} f$ as

$$e^{it\Phi_j(-i\nabla)} f = \sum_{\xi_0 \in R^{-1/2}\mathbb{Z}^n \cap \text{supp } \hat{f}} K_{\xi_0} * f$$

for some smooth kernels $K_{\xi_0}(t, x)$ with $\|K_{\xi_0}(t)\|_{L_x^\infty} \leq R^{-n/2}$. Then since $\Sigma_j(\mathfrak{h})$ is a hypersurface, by restricting to points close to $\Sigma_j(\mathfrak{h})$ we should have

$$\begin{aligned} \|e^{it\Phi_j(-i\nabla)} f\|_{L_x^\infty} &\leq \|f\|_{L_x^1} \left\| \sum_{\xi_0 \in R^{-1/2}\mathbb{Z}^n \cap \text{supp } \hat{f}} K_{\xi_0}(t, x) \right\|_{L_x^\infty} \\ &\lesssim \|f\|_{L_x^1} R^{\frac{1}{2}} \sup_{\mathfrak{h}} \left\| \sum_{\xi_0 \in R^{-1/2}\mathbb{Z}^n \cap (\Sigma_j(\mathfrak{h}) + R^{-1/2})} K_{\xi_0}(t, x) \right\|_{L_x^\infty}. \end{aligned}$$

The condition (i) in Assumption 1 then shows that, for times $t \approx R$, the spatial supports of the kernels $K_{\xi_0}(t, x)$ are essentially disjoint, and hence

$$\left\| \sum_{\xi_0 \in R^{-1/2}\mathbb{Z}^n \cap (\Sigma_j(\mathfrak{h}) + R^{-1/2})} K_{\xi_0}(t, x) \right\|_{L_x^\infty} \approx \sup_{\xi_0 \in R^{-1/2}\mathbb{Z}^n \cap (\Sigma_j(\mathfrak{h}) + R^{-1/2})} \|K_{\xi_0}(t)\|_{L_x^\infty} \lesssim R^{-n/2} \approx t^{-n/2},$$

which would then give the desired dispersive estimate (2-7).

In the remainder of this subsection, we fill in the details of the argument sketched above. We first require a technical lemma involving the surfaces $\Sigma_j(\mathfrak{h})$.

Lemma 2.3. *Let $\{j, k\} = \{1, 2\}$, $\mathfrak{h} = (a, h) \in \mathbb{R}^{1+n}$, and $r \geq 2(\mathbf{D}_2/\mathbf{D}_1)\mathbf{R}_0$. Assume $\xi_0 \in (\Lambda_j^* + 1/(2\mathbf{R}_0)) \cap (\Lambda_k^* + h + 1/(2\mathbf{R}_0))$ and*

$$|\Phi_j(\xi_0) - \Phi_k(\xi_0 - h) - a| \leq \frac{1}{r}.$$

Then $|\xi_0 - \Sigma_j(\mathfrak{h})| \leq \mathbf{D}_2/(\mathbf{D}_1 r)$.

Proof. Define $F(\xi) = \Phi_1(\xi) - \Phi_2(\xi - h) - a$; by replacing F with $-F$ if necessary, we may assume that $F(\xi_0) \geq 0$. We need to show there exists $|\xi - \xi_0| \leq \mathbf{D}_2/(\mathbf{D}_1 r)$ such that $F(\xi) = 0$. To this end, let $\xi(s)$ be the solution to

$$\partial_s \xi(s) = -\frac{\nabla F(\xi(s))}{|\nabla F(\xi(s))|}, \quad \xi(0) = \xi_0.$$

Note that, for times $s \in [0, \mathbf{D}_2/(r\mathbf{D}_1)]$, we have $|\xi(s) - \xi_0| \leq s$. On the other hand, since $|F(\xi_0)| \leq 1/r$ by assumption, the transversality property (1-2) implies

$$F(\xi(s)) = F(\xi_0) - \int_0^s |\nabla F(\xi(s'))| ds' \leq \frac{1}{r} - s \frac{\mathbf{D}_1}{\mathbf{D}_2}.$$

Consequently $F(\xi(s))$ must be zero for some $s \in [0, \mathbf{D}_2/(r\mathbf{D}_1)]$ and hence the result follows. \square

We now come to the proof of the dispersive inequality.

Lemma 2.4 (dispersion). *Let $j = 1, 2$. For any $f \in L_x^1$ with $\text{supp } \hat{f} \subset \Lambda_j^* + 1/(2\mathbf{R}_0)$ and any $t \geq 1$ we have*

$$\|e^{it\Phi_j(-i\nabla)} f\|_{L_x^\infty} \lesssim t^{-\frac{n-1}{2}} \|f\|_{L_x^1},$$

where the implied constant depends only $\mathbf{R}_0, \mathbf{D}_1, \mathbf{D}_2$, and $n \geq 2$.

Proof. It is enough to consider the case $j = 1$ and $R \leq t \leq 2R$ with $R \geq (10\mathbf{R}_0)^2$. Since $\Lambda_2^* + 1/(2\mathbf{R}_0)$ contains a ball of size $(2\mathbf{R}_0)^{-1}$, we can find a finite set $H \subset \mathbb{R}^n$ such that $\#H \lesssim \mathbf{R}_0^n$ and $\Lambda_1 = \bigcup_{h \in H} \Lambda_1 \cap (\Lambda_2^* + 1/(2\mathbf{R}_0)h)$. In particular, by decomposing $\text{supp } \hat{f}$ into $\mathcal{O}(\mathbf{R}_0^n)$ sets, it is enough to consider the case $\text{supp } \hat{f} \subset (\Lambda_1^* + 1/(2\mathbf{R}_0)) \cap (\Lambda_2^* + 1/(2\mathbf{R}_0) + h)$. Let $\rho \in C_0^\infty(|\xi| \leq 1)$ such that

$$\sum_{k \in \mathbb{Z}^n} \rho(\xi - k) = 1.$$

The support assumption on \hat{f} , together with the fact that $R \geq (10\mathbf{R}_0)^2$, implies

$$(e^{it\Phi_1(-i\nabla)} f)(x) = \sum_{\xi_0 \in R^{-1/2}\mathbb{Z}^n \cap (\text{supp } \hat{f} + \frac{1}{10\mathbf{R}_0})} K_{\xi_0}(t) * f(x),$$

where $K_{\xi_0}(t, x) = \int_{\mathbb{R}^n} \rho(R^{\frac{1}{2}}(\xi - \xi_0)) e^{it\Phi_1(\xi)} e^{ix \cdot \xi} d\xi$. Since $R \leq t \leq 2R$, our goal is to show that

$$\left\| \sum_{\xi_0 \in R^{-1/2}\mathbb{Z}^n \cap (\text{supp } \hat{f} + \frac{1}{10\mathbf{R}_0})} |K_{\xi_0}(t, x)| \right\|_{L_x^\infty} \lesssim R^{-\frac{n-1}{2}}.$$

We would like to write this sum in a way which involves the hypersurfaces $\Sigma_1(h)$. Fix $0 < \delta \ll \mathbf{D}_1/(\mathbf{D}_1 + \mathbf{D}_2)$ and let $\delta^* = (\mathbf{D}_1/\mathbf{D}_2)\delta$. Given $\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z} \cap (\text{supp } \hat{f} + 1/(10\mathbf{R}_0))$, we can find $a \in \delta^* R^{-\frac{1}{2}}\mathbb{Z}$ with $|a| \leq 2\mathbf{D}_2$ such that

$$|\Phi_1(\xi_0) - \Phi_2(\xi_0 - h) - a| \leq \delta^* R^{-\frac{1}{2}}.$$

Therefore, an application of Lemma 2.3 with $r = R^{\frac{1}{2}}/\delta^*$ implies $\xi_0 \in \Sigma_1(a, h) + \delta R^{-\frac{1}{2}}$; hence we have

$$\begin{aligned} \sum_{\xi_0 \in R^{-1/2}\mathbb{Z}^n \cap (\text{supp } \hat{f} + \frac{1}{10\mathbf{R}_0})} |K_{\xi_0}(t, x)| &\leq \sum_{\substack{a \in \delta^* R^{-1/2}\mathbb{Z} \\ |a| \leq 2\mathbf{D}_2}} \sum_{\xi_0 \in R^{-1/2}\mathbb{Z}^n \cap (\Sigma_1(a, h) + \delta R^{-1/2})} |K_{\xi_0}(t, x)| \\ &\lesssim R^{\frac{1}{2}} \sup_h \sum_{\xi_0 \in R^{-1/2}\mathbb{Z}^n \cap (\Sigma_1(h) + \delta R^{-1/2})} |K_{\xi_0}(t, x)|. \end{aligned}$$

We now exploit the localisation of the kernel, together with the partial curvature condition (1-1). Write

$$K_{\xi_0}(t, x) = R^{-\frac{n}{2}} \int_{\mathbb{R}^n} \rho(\xi) e^{it[\Phi_1(R^{-1/2}\xi + \xi_0) - R^{-1/2}\nabla\Phi_1(\xi_0) \cdot \xi]} e^{iR^{-1/2}(x + t\nabla\Phi_1(\xi_0)) \cdot \xi} d\xi.$$

Integrating by parts $n + 1$ times gives

$$|K_{\xi_0}(t, x)| \lesssim R^{-\frac{n}{2}} (1 + R^{-\frac{1}{2}}|x + t\nabla\Phi_1(\xi_0)|)^{-n-1}. \tag{2-8}$$

Let $\xi'_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n \cap (\Sigma_1(a, h) + R^{-\frac{1}{2}})$ denote the minimum of $|x + t\nabla\Phi_1(\xi_0)|$. We claim that for every $\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n \cap (\Sigma_1(a, h) + R^{-\frac{1}{2}})$ we have

$$|x + t\nabla\Phi_1(\xi_0)| \geq \frac{1}{4}\mathbf{D}_1 R |\xi_0 - \xi'_0|. \tag{2-9}$$

Assuming this holds for the moment, we would then obtain

$$\begin{aligned} \sum_{\xi_0 \in R^{-1/2}\mathbb{Z}^n \cap (\text{supp } \hat{f} + \frac{1}{10\mathbf{R}_0})} |K_{\xi_0}(t, x)| &\lesssim R^{\frac{1}{2}} \sup_h \sum_{\xi_0 \in R^{-1/2}\mathbb{Z}^n \cap (\Sigma_1(h) + R^{-1/2})} |K_{\xi_0}(t, x)| \\ &\lesssim R^{-\frac{n-1}{2}} \sum_{\xi_0 \in R^{-1/2}\mathbb{Z}^n} (1 + R^{\frac{1}{2}}|\xi_0 - \xi'_0|)^{-n-1} \lesssim R^{-\frac{n-1}{2}} \end{aligned}$$

as required. Thus it only remains to verify (2-9). This is immediate if $R\mathbf{D}_1|\xi_0 - \xi'_0| \leq 2|x + t\nabla\Phi_1(\xi'_0)|$. Thus we may assume that $R\mathbf{D}_1|\xi_0 - \xi'_0| \geq 2|x + t\nabla\Phi_1(\xi'_0)|$. Note that this implies $|\xi - \xi_0| \geq R^{-\frac{1}{2}}$. By construction, there exists $\xi, \xi' \in \Sigma_1(h)$ such that $|\xi - \xi_0| \leq \delta R^{-\frac{1}{2}}$, $|\xi' - \xi'_0| \leq \delta R^{-\frac{1}{2}}$. Therefore, applying the lower bound (1-1), we deduce that

$$\begin{aligned} |x + t\nabla\Phi_1(\xi_0)| &\geq t|\nabla\Phi(\xi) - \nabla\Phi(\xi')| - |x + t\nabla\Phi_1(\xi'_0)| - t|\nabla\Phi_1(\xi_0) - \nabla\Phi_1(\xi)| - t|\nabla\Phi_1(\xi'_0) - \nabla\Phi_1(\xi')| \\ &\geq R\mathbf{D}_1|\xi - \xi'| - |x + t\nabla\Phi_1(\xi'_0)| - 4\mathbf{D}_2\delta R^{\frac{1}{2}} \\ &\geq \frac{1}{2}R\mathbf{D}_1|\xi_0 - \xi'_0| - 4(\mathbf{D}_1 + \mathbf{D}_2)\delta R^{\frac{1}{2}} \geq \frac{1}{4}R\mathbf{D}_1|\xi_0 - \xi'_0|, \end{aligned}$$

provided that we choose $\delta \ll \mathbf{D}_1/(\mathbf{D}_1 + \mathbf{D}_2)$. Hence we obtain (2-9) and thus result follows. \square

Remark 2.5. By the standard TT^* -argument, this implies the linear Strichartz-type estimates for wave admissible pairs. We omit the details and refer to [Keel and Tao 1998].

2D. Classical bilinear estimate in $L^2_{t,x}$. The main use of the transversality property (1-2) contained in Assumption 1 is to deduce the following well-known bilinear estimate, which dates back at least to [Bourgain 1998, Lemma 111] in the case of the Schrödinger equation and $n = 2$.

Lemma 2.6. *Let $0 < r < 1$ and $f, g \in L^2_x$. Assume that the supports of \hat{f} and \hat{g} are contained in balls of radius r intersected with Λ_1 and Λ_2 respectively, and for all $\xi \in \Lambda_1$ and $\eta \in \Lambda_2$*

$$|\nabla\Phi_1(\xi) - \nabla\Phi_2(\eta)| \geq C_0. \tag{2-10}$$

Then,

$$\|e^{it\Phi_1(-i\nabla)} f e^{it\Phi_2(-i\nabla)} g\|_{L^2_{t,x}(\mathbb{R}^{1+n})} \lesssim \left(\frac{r^{n-1}}{C_0}\right)^{\frac{1}{2}} \|f\|_{L^2_x} \|g\|_{L^2_x}.$$

Proof. For $m = 1, \dots, n$ let

$$\Omega_m = \left\{ (\xi, \eta) \in \Lambda_1 \times \Lambda_2 : |\partial_m \Phi_1(\xi) - \partial_m \Phi_2(\eta)| \geq \frac{C_0}{2n} \right\}.$$

Condition (2-10) and the support assumptions on \hat{f} and \hat{g} imply that we have the decomposition

$$\widehat{(e^{it\Phi_1(-i\nabla)} f e^{it\Phi_2(-i\nabla)} g)}(\xi) = \sum_{m=1}^n \int_{\mathbb{R}^n} \hat{f}(\xi - \eta) \hat{g}(\eta) \mathbb{1}_{\Omega_m}(\xi - \eta, \eta) e^{it(\Phi_1(\xi - \eta) + \Phi_2(\eta))} d\eta.$$

Consider the $m = 1$ term and write $\eta = (\eta_1, \eta') \in \mathbb{R} \times \mathbb{R}^{n-1}$. The change of variables $(\eta_1, \eta') \mapsto (\tau, \eta')$, where $\tau = \Phi_1(\xi - \eta) + \Phi_2(\eta)$, gives

$$\begin{aligned} \int_{\mathbb{R}^n} \hat{f}(\xi - \eta) \hat{g}(\eta) \mathbb{1}_{\Omega_1}(\xi - \eta, \eta) e^{it(\Phi_1(\xi - \eta) + \Phi_2(\eta))} d\eta \\ = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \frac{\hat{f}(\xi - \eta^*) \hat{g}(\eta^*)}{\partial_1 \Phi_1(\xi - \eta^*) - \partial_1 \Phi_2(\eta^*)} \mathbb{1}_{\Omega_1}(\xi - \eta^*, \eta^*) d\eta' e^{it\tau} d\tau, \end{aligned}$$

where $\eta^* = (\eta_1[\tau, \xi, \eta'], \eta')$. Thus an application of Plancherel, followed by Hölder in η' , shows that

$$\begin{aligned} & \left\| \int_{\mathbb{R}^n} \hat{f}(\xi - \eta) \hat{g}(\eta) \mathbb{1}_{\Omega_m}(\xi - \eta, \eta) e^{it(\Phi_1(\xi - \eta) + \Phi_2(\eta))} d\eta \right\|_{L^2_{t,\xi}} \\ &= \left\| \int_{\mathbb{R}^{n-1}} \frac{\hat{f}(\xi - \eta^*) \hat{g}(\eta^*)}{\partial_1 \Phi_1(\xi - \eta^*) - \partial_1 \Phi_2(\eta^*)} \mathbb{1}_{\Omega_1}(\xi - \eta^*, \eta^*) d\eta' \right\|_{L^2_{\tau,\xi}} \\ &\leq (2r)^{\frac{n-1}{2}} \frac{2n}{C_0^{\frac{1}{2}}} \left\| \frac{\hat{f}(\xi - \eta^*) \hat{g}(\eta^*)}{|\partial_1 \Phi_1(\xi - \eta^*) - \partial_1 \Phi_2(\eta^*)|^{\frac{1}{2}}} \right\|_{L^2_{\tau,\xi,\eta'}} = 2n \left(\frac{(2r)^{n-1}}{C_0} \right)^{\frac{1}{2}} \|f\|_{L^2_x} \|g\|_{L^2_x}, \end{aligned}$$

where the last equality follows by undoing the change of variables. Since the terms with $1 < m \leq n$ are identical, the lemma follows. □

2E. Geometric consequences. The last step in the proof of Theorem 1.1 requires a combinatorial Kakeya-type bound. This bound relies on the fact that certain tubes intersect transversally, and is the main reason for introducing the condition (i) in Assumption 1. The following is motivated by [Lee and Vargas 2010; Bejenaru 2017]; see also Section 9 of [Tao 2003].

Let $\mathfrak{h} \in \mathbb{R}^{1+n}$ and define the conic hypersurface

$$\mathcal{C}_j(\mathfrak{h}) = \{(r, -r\nabla\Phi_j(\xi)) : r \in \mathbb{R}, \xi \in \Sigma_j(\mathfrak{h})\}.$$

A computation shows that the tangent plane to $\mathcal{C}_j(\mathfrak{h})$ is spanned by the vectors

$$(1, -\nabla\Phi_j(\xi)) \quad \text{and} \quad H\Phi_j(\xi)v \quad \text{for } v \in T_\xi\Sigma_j(\mathfrak{h}),$$

where $H\Phi_j(\xi)$ denotes the Hessian of Φ_j at ξ . On the other hand, as we will see in the proof Lemma 2.7 below, the condition (i) in Assumption 1 implies

$$|(1, -\nabla\Phi_j(\xi)) \wedge (1, -\nabla\Phi_k(\eta)) \wedge (0, \nabla\Phi_j(\xi) - \nabla\Phi_j(\xi'))| \gtrsim |\xi - \xi'|$$

for every $\xi, \xi' \in \Sigma_j(\mathfrak{h})$. Hence, letting $\xi' \rightarrow \xi$ in $\Sigma_j(\mathfrak{h})$, we can interpret (i) in Assumption 1 as saying that, for every $v \in T_\xi\Sigma_j(\mathfrak{h})$, we have

$$|(1, -\nabla\Phi_j(\xi)) \wedge (1, -\nabla\Phi_k(\eta)) \wedge (0, H\Phi_j(\xi)v)| \gtrsim |v|.$$

In particular, the vector $(1, -\nabla\Phi_k(\eta))$ must be transversal to the surface $\mathcal{C}_j(\mathfrak{h})$ for every $\eta \in \Lambda_k$. A more quantitative version of this statement — and the one we make use of in practice — is given by the following.

Lemma 2.7. *Let $\mathfrak{h} \in \mathbb{R}^{1+n}$ and $\{j, k\} = \{1, 2\}$. For every $\eta \in \Lambda_j$ and $p, q \in \mathcal{C}_k(\mathfrak{h})$ we have*

$$|(p - q) \wedge (1, -\nabla\Phi_j(\eta))| \geq \frac{\mathbf{D}_1 |p - q|}{(1 + \|\nabla\Phi_k\|_{L^\infty(\Lambda_k)}) \|\nabla^2\Phi_k\|_{L^\infty(\Lambda_k)}}.$$

Proof. Let $w, w', w'' \in \mathbb{R}^n$. The identity

$$|x \wedge y \wedge z| = \inf_{v \in \text{span}\{x, y\}} \frac{|v \wedge z|}{|v|} |x \wedge y|$$

implies

$$\begin{aligned} |(1, w'') \wedge (1, w) \wedge (0, w - w')| &= |(1, w'') \wedge (0, w - w'') \wedge (0, w - w')| \\ &= \inf_{v \in W} \frac{|v \wedge (1, w'')|}{|v|} |(0, w - w'') \wedge (0, w - w')| \\ &\geq |(w - w'') \wedge (w - w')|, \end{aligned}$$

where $W = \text{span}\{(0, w - w''), (0, w - w')\}$. Consequently, applying the wedge product identity once more, we deduce that for every $v \in \text{span}\{(1, w), (0, w - w')\}$

$$|v \wedge (1, w'')| \geq \frac{|(w - w'') \wedge (w - w')|}{(1 + |w|)|w - w'|} |v|. \tag{2-11}$$

Fix $\eta \in \Lambda_j$ and $p, q \in \mathcal{C}_k(\mathfrak{h})$. By definition, this implies that we have $\xi, \xi' \in \Sigma_j(\mathfrak{h})$ and $r, r' > 0$ such that $p = (r, -r\nabla\Phi_k(\xi))$ and $q = (r', -r'\nabla\Phi_k(\xi'))$. Clearly, due to the convexity of Λ_k we have $|\nabla\Phi_k(\xi) - \nabla\Phi_k(\xi')| \leq \|\nabla^2\Phi_k\|_{L^\infty(\Lambda_k)}|\xi - \xi'|$. If we now let $w = -\nabla\Phi_k(\xi)$, $w' = -\nabla\Phi_k(\xi')$, and $w'' = -\nabla\Phi_j(\eta)$ in (2-11), then we deduce from (i) in Assumption 1 that

$$|v \wedge (1, -\nabla\Phi_j(\eta))| \geq \frac{\mathbf{D}_1|v|}{(1 + \|\nabla\Phi_k\|_{L^\infty(\Lambda_k)})\|\nabla^2\Phi_k\|_{L^\infty(\Lambda_k)}}$$

for every $v \in \text{span}\{(1, -\nabla\Phi_k(\xi)), (0, \nabla\Phi_k(\xi) - \nabla\Phi_k(\xi'))\}$. Taking $v = p - q$ and observing that we can write

$$(p - q) = (r - r')(1, -\nabla\Phi_k(\xi)) + r'(0, \nabla\Phi_k(\xi) - \nabla\Phi_k(\xi')),$$

the required bound now follows. □

3. Wave packets, atomic spaces, and tubes

We now discuss the wave packet decomposition. To some extent, we follow the arguments in [Tao 2003], but use a slightly different notation by using projections labelled by phase-space points as in [Lee and Vargas 2010]. Again, this helps us to carefully track constants. In addition, we consider certain atomic decompositions. Concerning the phases Φ_j , it turns out that the only property we require in the construction of wave packets below, is (ii) in Assumption 1. Consequently, throughout this section, we fix constants $\mathbf{R}_0 \geq 1$, $\mathbf{D}_2 > 0$ and $N > n + 1$, and assume that for $j = 1, 2$ we have sets Λ_j, Λ_j^* with Λ_j convex and $\Lambda_j^* + 1/\mathbf{R}_0 \subset \Lambda_j \subset \{\frac{1}{16} \leq |\xi| \leq 16\}$, and phases $\Phi_j : \Lambda_j \rightarrow \mathbb{R}$ such that

$$\sup_{1 \leq |\kappa| \leq N} \|\partial^\kappa \Phi_j\|_{L^\infty(\Lambda_j)} \leq \mathbf{D}_2.$$

3A. Wave packets. Let $R \geq 1$ and define the cylinder

$$Q_R = \{(t, x) \in \mathbb{R}^{1+n} : \frac{1}{2}R < t < R, |x| < R\},$$

and $\mathcal{X} = R^{\frac{1}{2}}\mathbb{Z}^n \times R^{-\frac{1}{2}}\mathbb{Z}^n$. Define

$$\mathcal{X}_j = \{(x_0, \xi_0) \in \mathcal{X} : \xi_0 \in \Lambda_j^* + 3R^{-\frac{1}{2}}\}$$

to be the set of phase points which are within $3R^{-\frac{1}{2}}$ of Λ_j^* . Note that provided $R \geq (3\mathbf{R}_0)^2$, if $\gamma = (x_0, \xi_0) \in \mathcal{X}_j$, then $\xi_0 \in \Lambda_j$. Given a point $\gamma = (x_0, \xi_0) \in \mathcal{X}$ in phase-space, we let $x(\gamma) = x_0$ and $\xi(\gamma) = \xi_0$ denote the projections onto the first and second components respectively. Fix $\eta, \rho \in \mathcal{S}(\mathbb{R}^n)$ such that $\text{supp } \hat{\eta} \subset \{|\xi| \leq 1\}$, $\text{supp } \rho \subset \{|\xi| \leq 1\}$, and for all $x, \xi \in \mathbb{R}^n$

$$\sum_{k \in \mathbb{Z}^n} \eta(x - k) = \sum_{k \in \mathbb{Z}^n} \rho(\xi - k) = 1.$$

Given $\gamma \in \mathcal{X}$ and $f \in L^2_x(\mathbb{R}^n)$, define the phase-space localisation operator

$$(L_\gamma f)(x) = \eta\left(\frac{x - x(\gamma)}{R^{\frac{1}{2}}}\right) \left[\rho\left(\frac{-i\nabla - \xi(\gamma)}{R^{-\frac{1}{2}}}\right) f \right](x).$$

Note that by definition we have

$$f = \sum_{\gamma \in \mathcal{X}} L_\gamma f, \quad \text{supp } \widehat{L_\gamma f} \subset \{\xi \in \mathbb{R}^n : |\xi - \xi(\gamma)| \leq 2R^{-\frac{1}{2}}\}.$$

Moreover, letting

$$w_\gamma(x) = \left(1 + \frac{|x - x(\gamma)|}{R^{\frac{1}{2}}}\right)^{N-1+\frac{n+1}{2}},$$

for any $\Gamma \subset \mathcal{X}$ we have the orthogonality bounds

$$\left\| \sum_{\gamma \in \Gamma} L_\gamma f \right\|_{L_x^2} \lesssim \left(\sum_{\gamma \in \Gamma} \|w_\gamma(x)L_\gamma f(x)\|_{L_x^2}^2 \right)^{\frac{1}{2}} \lesssim \|f\|_{L_x^2}. \tag{3-1}$$

To simplify notation, we define the slightly larger phase-space localisation operators $L_\gamma^\# = \omega_\gamma(x)L_\gamma$. It is worth noting that $L_\gamma^\# f$ no longer has compact Fourier support; this does not pose any problems in the arguments to follow, as the only properties that we require are the trivial bound $\|L_\gamma f\|_{L_x^2} \leq \|L_\gamma^\# f\|_{L_x^2}$ and the orthogonality bound in (3-1).

To define wave packets, we conjugate the phase-space localisation operator L_γ with the flow $e^{it\Phi_j(-i\nabla)}$.

Definition 3.1 (wave packets). Let $j = 1, 2$, $R \geq (3R_0)^2$, and $u \in L_t^\infty L_x^2(\mathbb{R}^{1+n})$. Given a point $\gamma_j \in \mathcal{X}_j$, we define

$$(\mathcal{P}_{\gamma_j} u)(t) = e^{it\Phi_j(-i\nabla)} L_{\gamma_j} (e^{-it\Phi_j(-i\nabla)} u(t)).$$

Similarly, we define

$$(\mathcal{P}_{\gamma_j}^\# u)(t) = e^{it\Phi_j(-i\nabla)} L_{\gamma_j}^\# (e^{-it\Phi_j(-i\nabla)} u(t)).$$

We also require the associated tubes T_γ .

Definition 3.2 (tubes). Let $j = 1, 2$ and $\gamma_j \in \mathcal{X}_j$. Then we define the tube $T_{\gamma_j} \subset \mathbb{R}^{1+n}$ as

$$T_{\gamma_j} = \{(t, x) \in \mathbb{R}^{1+n} : \frac{1}{2}R \leq t \leq R, |x - x(\gamma) + t\nabla\Phi_j(\xi(\gamma))| \leq R^{\frac{1}{2}}\}.$$

The most important properties of the wave packets $\mathcal{P}_{\gamma_j} u$ are summarised in the following.

Proposition 3.3 (properties of wave packets). Let $j = 1, 2$. For any $R \geq (3R_0)^2$, $f \in L_x^2$ with $\text{supp } \hat{f} \subset \Lambda_j^*$, and $u = e^{it\Phi_j(-i\nabla)} f$, we have $u = \sum_{\gamma_j \in \mathcal{X}_j} \mathcal{P}_{\gamma_j} u$, $\text{supp } \widehat{\mathcal{P}_{\gamma_j} u} \subset \{\xi - \xi(\gamma) : |\xi - \xi(\gamma)| \leq 2R^{-\frac{1}{2}}\}$, and given any $\Gamma_j \subset \mathcal{X}_j$ we have the orthogonality bound

$$\left\| \sum_{\gamma_j \in \Gamma_j} \mathcal{P}_{\gamma_j} u \right\|_{L_t^\infty L_x^2} \lesssim \left(\sum_{\gamma_j \in \Gamma_j} \|L_{\gamma_j}^\# f\|_{L_x^2}^2 \right)^{\frac{1}{2}} \lesssim \|f\|_{L_x^2}. \tag{3-2}$$

Moreover, the wave packets $\mathcal{P}_{\gamma_j} u$ are concentrated on the tubes T_{γ_j} in the sense that for every $r \geq R^{\frac{1}{2}}$, and any ball $B \subset \mathbb{R}^{1+n}$, we have the bound

$$\left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \text{dist}(T_{\gamma_j}, B) > r}} \mathcal{P}_{\gamma_j} u \right\|_{L_{t,x}^\infty(B \cap Q_R)} \lesssim \left(\frac{r}{R^{\frac{1}{2}}}\right)^{\frac{n+3}{2}-N} \left(\sum_{\gamma_j \in \Gamma_j} \|L_{\gamma_j}^\# f\|_{L_x^2}^2 \right)^{\frac{1}{2}}. \tag{3-3}$$

Here, the implied constants depend only on R_0, D_2, N and $n \geq 2$.

Proof. This result is somewhat standard; see for instance [Tao 2003, Lemma 4.1] and [Lee 2006a, Lemma 2.2] for related estimates. We only prove the localisation property (3-3), as the remaining properties follow directly from the definition of \mathcal{P}_{γ_j} , together with the analogous properties of the phase-space localisation operator L_{γ_j} . Let $\gamma_j = (x_0, \xi_0)$ and write

$$\begin{aligned} \mathcal{P}_{\gamma_j} u(t, x) &= \int_{\mathbb{R}^n} \widehat{(L_{\gamma_j} f)}(\xi) e^{it\Phi_j(\xi)} e^{ix \cdot \xi} d\xi \\ &= \int_{\mathbb{R}^n} K_{\xi_0}(t, x - y) (L_{\gamma_j} f)(y) dy, \end{aligned}$$

where, as in the proof of Lemma 2.4, the kernel is given by $K_{\xi_0}(t, x) = \int_{\mathbb{R}^n} \rho(R^{\frac{1}{2}}(\xi - \xi_0)) e^{it\Phi_j(\xi)} e^{ix \cdot \xi} d\xi$. Note that, as in (2-8), integrating by parts $N - 1$ times, and using the fact that $|t| \leq R$, $R \gg 1$, we deduce that

$$K_{\xi_0}(t, x) \lesssim R^{-\frac{n}{2}} \left(1 + \frac{|x + t\nabla\Phi_j(\xi_0)|}{R^{\frac{1}{2}}} \right)^{1-N}.$$

Plugging this bound into the identity for $\mathcal{P}_{\gamma_j} u(t, x)$, we deduce that

$$\begin{aligned} |\mathcal{P}_{\gamma_j} u(t, x)| &\lesssim R^{-\frac{n}{2}} \left(1 + \frac{|x - x_0 + t\nabla\Phi_j(\xi_0)|}{R^{\frac{1}{2}}} \right)^{1-N} \int_{\mathbb{R}^n} \left(1 + \frac{|y - x_0|}{R^{\frac{1}{2}}} \right)^{N-1} |L_{\gamma_j} f(y)| dy \\ &\lesssim R^{-\frac{n}{4}} \left(1 + \frac{|x - x_0 + t\nabla\Phi_j(\xi_0)|}{R^{\frac{1}{2}}} \right)^{1-N} \|L_{\gamma_j}^{\#} f\|_{L_x^2}. \end{aligned}$$

Since there are $\mathcal{O}(R^{\frac{n}{2}})$ choices of ξ_0 , and

$$\begin{aligned} |x - x_0 + t\nabla\Phi_j(\xi_0)| &= |(t, x) - (t, x_0 - t\nabla\Phi_j(\xi_0))| \\ &\geq \text{dist}((t, x), T_{\gamma_j}), \end{aligned}$$

an application of Hölder’s inequality gives for any $(t, x) \in B$

$$\begin{aligned} &\sum_{\substack{\gamma_j \in \Gamma_j \\ \text{dist}(T_{\gamma_j}, B) \geq r}} |\mathcal{P}_{\gamma_j} u(t, x)| \\ &\lesssim R^{-\frac{n}{4}} \left(\sum_{\substack{\gamma_j \in \mathcal{X}_j \\ \text{dist}(T_{\gamma_j}, B) \geq r}} \left(1 + \frac{|x - x_0 + t\nabla\Phi_j(\xi_0)|}{R^{\frac{1}{2}}} \right)^{2-2N} \right)^{\frac{1}{2}} \left(\sum_{\gamma_j \in \Gamma_j} \|L_{\gamma_j}^{\#} f\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{r}{R^{\frac{1}{2}}} \right)^{\frac{n+3}{2}-N} \sup_{\xi_0} \left(\sum_{x_0 \in R^{1/2}\mathbb{Z}^n} \left(1 + \frac{|x - x_0 + t\nabla\Phi_j(\xi_0)|}{R^{\frac{1}{2}}} \right)^{-n-1} \right)^{\frac{1}{2}} \left(\sum_{\gamma_j \in \Gamma_j} \|L_{\gamma_j}^{\#} f\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{r}{R^{\frac{1}{2}}} \right)^{\frac{n+3}{2}-N} \left(\sum_{\gamma_j \in \Gamma_j} \|L_{\gamma_j}^{\#} f\|_{L_x^2}^2 \right)^{\frac{1}{2}} \end{aligned}$$

as required. □

3B. Atomic spaces and wave packets. Closely related to the V^p spaces, are the slightly smaller U^p spaces; see [Koch and Tataru 2005; Hadac et al. 2009; Koch et al. 2014].

Definition 3.4. Let $1 \leq p < \infty$. A function $\rho : \mathbb{R} \rightarrow L^2_x$ is called a U^p atom if there exists a decomposition $\rho = \sum_{J \in \mathcal{I}} \mathbb{1}_J(t) f_J$ subordinate to a finite partition

$$\mathcal{I} = \{(-\infty, t_1), [t_2, t_3), \dots, [t_N, \infty)\}$$

of \mathbb{R} such that

$$\|f_J\|_{\ell^p_t L^2_x} := \left(\sum_{J \in \mathcal{I}} \|f_J\|_{L^2_x}^p \right)^{\frac{1}{p}} \leq 1.$$

The atomic Banach space U^p is then defined as

$$U^p = \left\{ \sum_j c_j \rho_j : (c_j) \in \ell^1(\mathbb{N}), \rho_j \text{ a } U^p \text{ atom} \right\}$$

with the induced norm

$$\|\rho\|_{U^p} = \inf_{\substack{\rho = \sum_k c_k \phi_k \\ \phi_k \text{ a } U^p \text{ atom}}} \sum_k |c_k|.$$

The space U^p_{Φ} is the set of all $u : \mathbb{R} \rightarrow L^2_x$ such that $e^{-it\Phi(-i\nabla)}u \in U^p$ with the obvious norm.

Let $u = \sum_J \mathbb{1}_J(t) e^{it\Phi_j(-i\nabla)} f_J$ be a $U^2_{\Phi_j}$ atom. Since $\mathbb{1}_J(t)$ commutes with spatial Fourier multipliers, we have

$$\begin{aligned} \mathcal{P}_{\gamma_j} u &= \sum_J \mathbb{1}_J(t) e^{it\Phi_j(-i\nabla)} L_{\gamma_j} f_J, \\ \mathcal{P}^{\#}_{\gamma_j} u &= \sum_J \mathbb{1}_J(t) e^{it\Phi_j(-i\nabla)} L^{\#}_{\gamma_j} f_J. \end{aligned}$$

Proposition 3.3 gives the following.

Corollary 3.5 (wave packets for $U^2_{\Phi_j}$ atoms). *Let $j = 1, 2$. For any $R \geq (3\mathbf{R}_0)^2$ and $U^2_{\Phi_j}$ atom $u = \sum_J \mathbb{1}_J(t) e^{it\Phi_j(-i\nabla)} f_J$ with $\text{supp } \hat{u} \subset \Lambda_j^*$, we have $u = \sum_{\gamma_j \in \mathcal{X}_j} \mathcal{P}_{\gamma_j} u$, $\text{supp } \widehat{\mathcal{P}_{\gamma_j} u} \subset \{|\xi - \xi(\gamma)| \leq 2R^{-\frac{1}{2}}\}$, and given any $\Gamma_j \subset \mathcal{X}_j$ we have the orthogonality bound*

$$\left\| \sum_{\gamma_j \in \Gamma_j} \mathcal{P}_{\gamma_j} u \right\|_{L^\infty_t L^2_x} \lesssim \left(\sum_{\gamma_j \in \Gamma_j} \|L^{\#}_{\gamma_j} f_J\|_{\ell^2_t L^2_x}^2 \right)^{\frac{1}{2}} \lesssim \|f_J\|_{\ell^2_t L^2_x}. \tag{3-4}$$

Moreover, the wave packets $\mathcal{P}_{\gamma_j} u$ are concentrated on the tubes T_{γ_j} in the sense that for every $r \geq R^{\frac{1}{2}}$, and any ball $B \subset \mathbb{R}^{1+n}$, we have the bound

$$\left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \text{dist}(T_{\gamma_j}, B) > r}} \mathcal{P}_{\gamma_j} u \right\|_{L^\infty_{t,x}(B \cap \mathcal{Q}_R)} \lesssim \left(\frac{r}{R^{\frac{1}{2}}} \right)^{\frac{n+3}{2} - N} \left(\sum_{\gamma_j \in \Gamma_j} \|L^{\#}_{\gamma_j} f_J\|_{\ell^2_t L^2_x}^2 \right)^{\frac{1}{2}}. \tag{3-5}$$

Here, the implied constants depend only on $\mathbf{R}_0, \mathbf{D}_2, N$ and $n \geq 2$.

3C. Sets and relations of tubes. We repeat the definitions and notation used by Tao [2003], but as above we adopt the point of view that the basic objects are the phase-space elements $\gamma \in \mathcal{X}_j$, rather than the associated tubes T_{γ_j} .

For $\delta > 0$, let \mathcal{B} be a collection of (space-time) balls of radius $R^{1-\delta}$ which form a finitely overlapping cover of Q_R . Similarly let \mathbf{q} denote a collection of finitely overlapping cubes q of radius $R^{\frac{1}{2}}$ which cover the cylinder Q_R . Let $R^\delta q$ denote a cube of radius $R^{\delta+\frac{1}{2}}$ with the same centre as q . Given a collection $\Gamma_j \subset \mathcal{X}_j$, and a cube $q \in \mathbf{q}$, we define

$$\Gamma_j(q) = \{\gamma_j \in \Gamma_j : T_{\gamma_j} \cap R^\delta q \neq \emptyset\},$$

so $\Gamma_j(q)$ is the subcollection of our phase-space decomposition such that the associated tube T_{γ_j} intersects a slight enlargement of the cube $q \in \mathbf{q}$. In the remainder of this subsection, the implied constants may depend on $n \geq 2$ only. Given $1 \leq \mu_1, \mu_2 \lesssim R^{100n}$, define

$$\mathbf{q}(\mu_1, \mu_2) = \{q \in \mathbf{q} : \mu_j \leq \#\Gamma_j(q) < 2\mu_j, j = 1, 2\}.$$

Thus, roughly, $\mathbf{q}(\mu_1, \mu_2)$ restricts to those elements of \mathbf{q} which are intersected by μ_j tubes T_{γ_j} , $\gamma_j \in \Gamma_j$. Given $\gamma_j \in \Gamma_j$, we let

$$\lambda(\gamma_j, \mu_1, \mu_2) = \#\{q \in \mathbf{q}(\mu_1, \mu_2) : T_{\gamma_j} \cap R^\delta q \neq \emptyset\}$$

and for every $1 \leq \lambda_j \lesssim R^{100n}$ we define

$$\Gamma_j[\lambda_j, \mu_1, \mu_2] = \{\gamma_j \in \Gamma_j : \lambda_j \leq \lambda(\gamma_j, \mu_1, \mu_2) < 2\lambda_j\}.$$

So $\Gamma_j[\lambda_j, \mu_1, \mu_2]$ essentially restricts to $\gamma_j \in \Gamma_j$ such that the associated tubes T_{γ_j} intersect λ_j cubes in $\mathbf{q}(\mu_1, \mu_2)$. Clearly

$$\bigcup_{1 \leq \lambda_j, \mu_1, \mu_2 \lesssim R^{100n}} \Gamma_j[\lambda_j, \mu_1, \mu_2] = \Gamma_j.$$

The following relation \sim between balls in \mathcal{B} and $\gamma_j \in \Gamma_j$ plays a key role in the arguments to follow.

Definition 3.6. Given $\gamma_j \in \Gamma_j[\lambda_j, \mu_1, \mu_2]$, we let $B(\gamma_j, \lambda_j, \mu_1, \mu_2) \in \mathcal{B}$ denote a ball which maximises

$$\#\{q \in \mathbf{q}(\mu_1, \mu_2) : T_{\gamma_j} \cap R^\delta q \neq \emptyset, q \cap B(\gamma_j, \lambda_j, \mu_1, \mu_2) \neq \emptyset\}.$$

If $B \in \mathcal{B}$, and $\gamma_j \in \Gamma_j[\lambda_j, \mu_1, \mu_2]$, we then define $\gamma_j \sim_{\lambda_j, \mu_1, \mu_2} B$ if $B \subset 10B(\gamma_j, \lambda_j, \mu_1, \mu_2)$. To extend this definition to general points $\gamma_j \in \Gamma_j$, we simply say that $\gamma_j \sim B$ if there exists some $1 \leq \lambda_j, \mu_1, \mu_2 \lesssim R^{100n}$ such that $\gamma_j \sim_{\lambda_j, \mu_1, \mu_2} B$.

Remark 3.7. This definition has the following important consequences:

(i) Let $\gamma_j \in \Gamma_j$ and consider the set $\{B \in \mathcal{B} : \gamma_j \sim B\}$. Since there are at most $\mathcal{O}(R^\epsilon)$ dyadic $1 \leq \lambda_j, \mu_1, \mu_2 \leq R^{100n}$ such that $\gamma_j \in \Gamma_j[\lambda_j, \mu_1, \mu_2]$, and only $\mathcal{O}(1)$ balls B such that $\gamma_j \sim_{\lambda_j, \mu_1, \mu_2} B$, we have

$$\#\{B \in \mathcal{B} : \gamma_j \sim B\} \leq \sum_{\substack{1 \leq \lambda_j, \mu_1, \mu_2 \leq R^{100n} \\ \gamma_j \in \Gamma_j[\lambda_j, \mu_1, \mu_2]}} \#\{B \in \mathcal{B} : \gamma_j \sim_{\lambda_j, \mu_1, \mu_2} B\} \lesssim \sum_{1 \leq \lambda_j, \mu_1, \mu_2 \leq R^{100n}} 1 \lesssim R^\epsilon.$$

(ii) Fix $1 \leq \lambda_1, \mu_1, \mu_2 \lesssim R^{100n}$ and let $\gamma_j \in \Gamma_j[\lambda_j, \mu_1, \mu_2]$. By definition, we have

$$\begin{aligned} \lambda_j &\leq \#\{q \in \mathbf{q}(\mu_1, \mu_2) : T_{\gamma_j} \cap R^\delta q \neq \emptyset\} \\ &\leq \sum_{B \in \mathcal{B}} \#\{q \in \mathbf{q}(\mu_1, \mu_2) : T_{\gamma_j} \cap R^\delta q \neq \emptyset, q \cap B \neq \emptyset\} \\ &\leq \#\mathcal{B} \#\{q \in \mathbf{q}(\mu_1, \mu_2) : T_{\gamma_j} \cap R^\delta q \neq \emptyset, q \cap B(\gamma_j, \lambda_1, \mu_1, \mu_2) \neq \emptyset\}, \end{aligned}$$

where we used the maximal property of the ball $B(\gamma_j, \lambda_j, \mu_1, \mu_2)$. Therefore, as $\#\mathcal{B} \lesssim R^{(n+1)\delta}$, we deduce the lower bound

$$\#\{q \in \mathbf{q}(\mu_1, \mu_2) : T_{\gamma_j} \cap R^\delta q \neq \emptyset, q \cap B(\gamma_j, \lambda_j, \mu_1, \mu_2) \neq \emptyset\} \gtrsim R^{-(n+1)\delta} \lambda_j.$$

4. A local bilinear restriction estimate

The main step in the proof of Theorem 1.1 is proving the following spatially localised version in U_Φ^2 .

Theorem 4.1. *Let $n \geq 2$ and $\alpha > 0$. Let $R_0 \geq 1$ and $D_1, D_2 > 0$. For $j = 1, 2$, let $\Lambda_j, \Lambda_j^* \subset \{\frac{1}{16} \leq |\xi| \leq 16\}$ with Λ_j convex and $\Lambda_j^* + 1/R_0 \subset \Lambda_j$. There exists $N \in \mathbb{N}$ and a constant $C > 0$ such that, for any phases Φ_1 and Φ_2 satisfying Assumption 1, any $u \in U_{\Phi_1}^2, v \in U_{\Phi_2}^2$ with $\text{supp } \hat{u}(t) \subset \Lambda_1^*, \text{supp } \hat{v}(t) \subset \Lambda_2^*$, and any $R \geq 1$, we have*

$$\|uv\|_{L_{t,x}^{(n+3)/(n+1)}(Q_R)} \leq CR^{2\alpha} \|u\|_{U_{\Phi_1}^2} \|v\|_{U_{\Phi_2}^2}.$$

In the remainder of this section we give the proof of Theorem 4.1. The proof is broken up into three key steps. The first step is to use an induction-on-scales argument to reduce to proving an $L_{t,x}^2$ bound. We then use the localisation properties of the wave packet decomposition to show that the $L_{t,x}^2$ bound follows from a combinatorial Keakeya-type bound. The final step is prove the combinatorial estimate using a ‘bush’ argument.

4A. Induction on scales. Let $\alpha > 0$ and fix $R_0 \geq 1, D_1, D_2 > 0$. Fix $N = ((\alpha + 1)/\alpha)(100n)^2$. For $j = 1, 2$, let $\Lambda_j, \Lambda_j^* \subset \{\frac{1}{16} \leq |\xi| \leq 16\}$ with Λ_j convex and $\Lambda_j^* + 1/R_0 \subset \Lambda_j$. It is enough to show that there exists a constant $C > 0$ such that, for any phases Φ_1 and Φ_2 satisfying Assumption 1, any $R \geq (3R_0)^2$, and any $U_{\Phi_j}^2$ atoms $u = \sum_J \mathbb{1}_J(t) e^{it\Phi_1(-i\nabla)} f_J, v = \sum_{J'} \mathbb{1}_{J'}(t) e^{it\Phi_2(-i\nabla)} g_{J'}$, with $\text{supp } \hat{f} \subset \Lambda_1^*, \text{supp } \hat{g}_{J'} \subset \Lambda_2^*$, we have

$$\|uv\|_{L_{t,x}^{(n+3)/(n+1)}(Q_R)} \leq CR^{2\alpha}. \tag{4-1}$$

To simplify the notation to follow, we now work under the assumption that any implicit constants may now depend on $\alpha, n \geq 2$, and the constants R_0, D_1, D_2 , but will be independent of R and the particular choice of phases Φ_j satisfying Assumption 1.

The proof of (4-1) proceeds along the same lines as Tao’s argument for the paraboloid [2003]. Namely, we use an induction-on-scales argument to deduce the estimate at scale R by applying a weaker estimate at a smaller scale $R^{1-\delta}$. We start by observing that it suffices to show that, for every $\Gamma_j \subset \mathcal{X}_j$ such that

$\#\Gamma_j \leq R^{10n}$, and any $\beta \geq \alpha$, we have

$$\left\| \sum_{\gamma_j \in \Gamma_j} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{(n+3)/(n+1)}(Q_R)} \lesssim R^\beta (\#\Gamma_1 \#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^\# f_J\|_{\ell_J^2 L_x^2} \|L_{\gamma_2}^\# g_{J'}\|_{\ell_J^2 L_x^2}. \quad (4-2)$$

To deduce (4-1) from (4-2), we let

$$\mathcal{X}_1(v_1) = \{\gamma_1 \in \mathcal{X}_1 : v_1 \leq \|L_{\gamma_1}^\# f_J\|_{\ell_J^2 L_x^2} \leq 2v_1, T_{\gamma_1} \cap 2Q_R \neq \emptyset\}$$

and $\mathcal{X}_2(v_2)$ similarly, where $v_j \in 2^{\mathbb{Z}}$. An application of Corollary 3.5 gives the decomposition $u = \sum_{\gamma_j \in \mathcal{X}_j} \mathcal{P}_{\gamma_j} u$, as well as the bounds

$$\left\| \sum_{\substack{\gamma_1 \in \mathcal{X}_1 \\ T_{\gamma_1} \cap 2Q_R = \emptyset}} \mathcal{P}_{\gamma_1} u \right\|_{L_{t,x}^\infty(Q_R)} \lesssim R^{-99n}$$

and

$$\left(\sum_{\gamma_j \in \mathcal{X}_j} \|\mathcal{P}_{\gamma_1} u\|_{L_t^\infty L_x^2}^2 \right)^{\frac{1}{2}} \lesssim \left(\sum_{\gamma_j \in \mathcal{X}_j} \|L_{\gamma_1}^\# f_J\|_{\ell_J^2 L_x^2}^2 \right)^{\frac{1}{2}} \lesssim 1.$$

The analogous bounds hold for v . Moreover $\#\{\gamma_j \in \mathcal{X}_j : T_{\gamma_j} \cap 2Q_R \neq \emptyset\} \lesssim R^{n+1}$. Collecting these properties together, we deduce that $\mathcal{X}_1(v_1) = \emptyset$ for $v_1 \gg 1$ and

$$\left\| u - \sum_{R^{-100n} \leq v_1 \lesssim 1} \sum_{\gamma_1 \in \mathcal{X}_1(v_1)} \mathcal{P}_{\gamma_1} u \right\|_{L_{t,x}^\infty(Q_R)} \lesssim R^{-90n}.$$

A similar argument shows that

$$\left\| v - \sum_{R^{-100n} \leq v_2 \lesssim 1} \sum_{\gamma_2 \in \mathcal{X}_2(v_2)} \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^\infty(Q_R)} \lesssim R^{-90n}.$$

Therefore, applying the bound (4-2) with $\Gamma_j = \mathcal{X}_j(v_j)$ and $\beta = \alpha$, we obtain

$$\begin{aligned} & \|uv\|_{L_{t,x}^{(n+3)/(n+1)}(Q_R)} \\ & \leq \left\| uv - \sum_{R^{-100n} \leq v_j \lesssim 1} \sum_{\gamma_j \in \mathcal{X}_j(v_j)} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{(n+3)/(n+1)}(Q_R)} \\ & \quad + \sum_{R^{-100n} \leq v_j \lesssim 1} \left\| \sum_{\gamma_j \in \mathcal{X}_j(v_j)} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{(n+3)/(n+1)}(Q_R)} \\ & \lesssim 1 + \log(R) R^\alpha \sup_{v_j} (\#\mathcal{X}_1(v_1) \#\mathcal{X}_2(v_2))^{\frac{1}{2}} \sup_{\gamma_j \in \mathcal{X}_j(v_j)} \|L_{\gamma_1}^\# f_J\|_{\ell_J^2 L_x^2} \|L_{\gamma_2}^\# g_{J'}\|_{\ell_J^2 L_x^2} \lesssim R^{2\alpha}, \end{aligned}$$

where the last line follows from the orthogonality properties of the phase-space localisation operators (3-1). Hence (4-1) follows.

The proof of (4-2) proceeds via an induction-on-scales argument. The first step is to note that we already have (4-2) provided we take $\beta > 0$ sufficiently large. Indeed, a crude argument by Hölder and

Bernstein inequalities implies the bound with $\beta = (n + 1)/(n + 3)$ (which could be improved by using linear Strichartz estimates as indicated in Remark 2.5). Suppose we could show that, if (4-2) holds for some $\beta > \alpha$, then for every $\epsilon > 0$ we have

$$\begin{aligned} \left\| \sum_{\gamma_j \in \Gamma_j} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{(n+3)/(n+1)}(Q_R)} & \lesssim R^{2\epsilon} (R^{(1-\delta)\beta} + R^{D\delta}) (\#\Gamma_1 \#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^\# f_J\|_{\ell_J^2 L_x^2} \|L_{\gamma_2}^\# g_{J'}\|_{\ell_J^2 L_x^2}, \quad (4-3) \end{aligned}$$

where $\delta = \alpha/(D + \alpha)$ and $D \geq 0$ is some constant which depends only on the dimension n . Then, since $D\delta < \alpha$, by taking $\epsilon > 0$ sufficiently small, we deduce that we must have (4-2) for some $\beta' < \beta$. Iterating this argument then gives (4-2) for $\beta = \alpha$. Consequently, our aim is to prove (4-3), under the assumption that we already have (4-2) for some $\beta > \alpha$.

We now fix $\Gamma_j \subset \mathcal{X}_j$ such that $\#\Gamma_j \leq R^{10n}$, and $\beta > \alpha$. Let \mathcal{B} denote a collection of balls B of radius $R^{1-\delta}$ which form a finitely overlapping cover of Q_R . Let \sim denote the relation between points $\gamma_j \in \Gamma_j$ and balls $B \in \mathcal{B}$ given by Definition 3.6. It is important to note that the relation \sim depends only on the fixed sets Γ_j , and *not* on u and v . We have the decomposition

$$\begin{aligned} \left\| \sum_{\gamma_j \in \Gamma_j} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{(n+3)/(n+1)}(Q_R)} & \leq \sum_{B \in \mathcal{B}} \left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_j \sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{(n+3)/(n+1)}(B)} + \sum_{B \in \mathcal{B}} \left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_1 \not\sim B \text{ or } \gamma_2 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{(n+3)/(n+1)}(B)}. \end{aligned}$$

For the first term, which contains the tubes which are concentrated on B , we apply the induction assumption at scale $R^{1-\delta}$ to deduce that

$$\begin{aligned} \sum_{B \in \mathcal{B}} \left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_j \sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{(n+3)/(n+1)}(B)} & \lesssim R^{(1-\delta)\beta} \sum_{B \in \mathcal{B}} (\#\{\gamma_1 \in \Gamma_1 : \gamma_1 \sim B\} \#\{\gamma_2 \in \Gamma_2 : \gamma_2 \sim B\})^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^\# f_J\|_{\ell_J^2 L_x^2} \|L_{\gamma_2}^\# g_{J'}\|_{\ell_J^2 L_x^2} \\ & \lesssim R^\epsilon R^{(1-\delta)\beta} (\#\Gamma_1 \#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^\# f_J\|_{\ell_J^2 L_x^2} \|L_{\gamma_2}^\# g_{J'}\|_{\ell_J^2 L_x^2}, \end{aligned}$$

where the last line follows from (i) in Remark 3.7. For the second term, as we can now safely lose factors of R^δ ; we may ignore the sum over the balls B (as there are only $\mathcal{O}(R^{\delta(n+1)})$ balls). Thus, after replacing D with $D - n - 1$, we need to prove the bound

$$\begin{aligned} \left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_1 \not\sim B \text{ or } \gamma_2 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{(n+3)/(n+1)}(B)} & \lesssim R^{\epsilon + D\delta} (\#\Gamma_1 \#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^\# f_J\|_{\ell_J^2 L_x^2} \|L_{\gamma_2}^\# g_{J'}\|_{\ell_J^2 L_x^2}. \quad (4-4) \end{aligned}$$

To this end, an application of Hölder together with the orthogonality property of the tube decomposition gives

$$\begin{aligned} \left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_1 \not\sim B \text{ or } \gamma_2 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L^1_{t,x}(B)} &\lesssim R \left(\sum_{\gamma_1 \in \Gamma_1} \|L_{\gamma_1}^\# f_J\|_{\ell^2 J L^2_x} \right)^{\frac{1}{2}} \left(\sum_{\gamma_2 \in \Gamma_2} \|L_{\gamma_2}^\# g_{J'}\|_{\ell^2 J' L^2_x} \right)^{\frac{1}{2}} \\ &\lesssim R (\#\Gamma_1 \#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^\# f_J\|_{\ell^2 J L^2_x} \|L_{\gamma_2}^\# g_{J'}\|_{\ell^2 J' L^2_x}. \end{aligned}$$

In particular, the convexity of the L^p norms implies (4-4) follows from the $L^2_{t,x}$ bound

$$\left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_1 \not\sim B \text{ or } \gamma_2 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L^2_{t,x}(B)} \lesssim R^{\epsilon + D\delta - \frac{n-1}{4}} (\#\Gamma_1 \#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^\# f_J\|_{\ell^2 J L^2_x} \|L_{\gamma_2}^\# g_{J'}\|_{\ell^2 J' L^2_x}. \tag{4-5}$$

Thus we have reduced the problem of obtaining the $L^{\frac{n+3}{n+1}}_{t,x}$ estimate (4-3) to proving the $L^2_{t,x}$ bound (4-5).

Remark 4.2. The fact that the above reduction can be done in U^2_Φ , is the key reason why we can extend the homogeneous bilinear Fourier restriction estimates to U^2_Φ .

Our goal in the following two subsections is to prove the bound (4-5), and thus complete the proof of Theorem 4.1. As in the previous subsections, we essentially follow the argument of Tao [2003], but apply the results of Section 2 in place of analogous results for the paraboloid. The general strategy is to first use the transversality via Lemma 2.6 to reduce to counting intersections of tubes. The number of tubes is then controlled by using (i) in Assumption 1 via Lemma 2.7 together with a ‘bush’ argument. The notation for various cubes and tubes introduced in Section 3C is used heavily in what follows.

4B. The L^2 bound: initial reductions and transversality. Recall that the ball $B \in \mathcal{B}$ is now fixed. Write

$$\sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_1 \not\sim B \text{ or } \gamma_2 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v = \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_1 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v + \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_1 \sim B \text{ and } \gamma_2 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v.$$

We only prove the bound for the first term, as an identical argument can handle the second term (just replace Γ_1 with $\{\gamma_1 \in \Gamma_1 : \gamma_1 \sim B\}$ and reverse the roles of u and v). The first step is to make a number of reductions exploiting the spatial localisation properties of the wave packets, together with a dyadic pigeon-hole argument to fix various quantities. To this end, decompose into cubes $q \in \mathbf{q}$:

$$\left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_1 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L^2_{t,x}(B)} \leq \left(\sum_{\substack{q \in \mathbf{q} \\ q \subset 2B}} \left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_1 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L^2_{t,x}(q)} \right)^{\frac{1}{2}}.$$

Note that the concentration property of the wave packet decomposition implies

$$\left\| \sum_{\substack{\gamma_1 \in \Gamma_1 \\ T_{\gamma_1} \cap R^\delta q = \emptyset}} \mathcal{P}_{\gamma_1} u \right\|_{L^\infty_{t,x}(q)} \lesssim R^{-\delta(N - \frac{n+3}{2})} (\#\Gamma_1)^{\frac{1}{2}} \sup_{\gamma_1 \in \Gamma_1} \|L_{\gamma_1}^\# f_J\|_{\ell^2 J L^2_x}.$$

A similar bound holds for v . By our choice of N , we have $\delta(N - (n + 3)/2) \geq 100n$. Therefore, as $\#\Gamma_j \lesssim R^{10n}$ and $\#\mathbf{q} \lesssim R^{2n}$, it suffices to prove

$$\left(\sum_{\substack{\mathbf{q} \in \mathbf{q} \\ \mathbf{q} \subset 2B}} \left\| \sum_{\substack{\gamma_j \in \Gamma_j(\mathbf{q}) \\ \gamma_1 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L^2_{t,x}(\mathbf{q})}^2 \right)^{\frac{1}{2}} \lesssim R^{\epsilon + D\delta - \frac{n-1}{4}} (\#\Gamma_1)^{\frac{1}{2}} (\#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L^{\#}_{\gamma_1} f_J\|_{\ell^2_J L^2_x} \|L^{\#}_{\gamma_2} g_{J'}\|_{\ell^2_{J'} L^2_x}. \tag{4-6}$$

Let $\Gamma_1^{\not\sim B}(\mathbf{q}) = \{\gamma_1 \in \Gamma_1(\mathbf{q}) : \gamma_1 \not\sim B\}$ and decompose into

$$\left(\sum_{\substack{\mathbf{q} \in \mathbf{q} \\ \mathbf{q} \subset 2B}} \left\| \sum_{\substack{\gamma_j \in \Gamma_j(\mathbf{q}) \\ \gamma_1 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L^2_{t,x}(\mathbf{q})}^2 \right)^{\frac{1}{2}} \leq \sum_{\substack{1 \leq \lambda_1 \\ \mu_1, \mu_2 \lesssim R^{100n}}} \left(\sum_{\substack{\mathbf{q} \in \mathbf{q}(\mu_1, \mu_2) \\ \mathbf{q} \subset 2B}} \left\| \sum_{\substack{\gamma_1 \in \Gamma_1^{\not\sim B}(\mathbf{q}) \cap \Gamma_1[\lambda_1, \mu_1, \mu_2] \\ \gamma_2 \in \Gamma_2(\mathbf{q})}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L^2_{t,x}(\mathbf{q})}^2 \right)^{\frac{1}{2}}.$$

Clearly, as we can freely lose R^ϵ , (4-6) follows from proving the estimate for fixed λ_1, μ_1, μ_2 ,

$$\left(\sum_{\substack{\mathbf{q} \in \mathbf{q}(\mu_1, \mu_2) \\ \mathbf{q} \subset 2B}} \left\| \sum_{\substack{\gamma_1 \in \Gamma_1^{\not\sim B}(\mathbf{q}) \cap \Gamma_1[\lambda_1, \mu_1, \mu_2] \\ \gamma_2 \in \Gamma_2(\mathbf{q})}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L^2_{t,x}(\mathbf{q})}^2 \right)^{\frac{1}{2}} \lesssim R^{\epsilon + D\delta - \frac{n-1}{4}} (\#\Gamma_1)^{\frac{1}{2}} (\#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L^{\#}_{\gamma_1} f_J\|_{\ell^2_J L^2_x} \|L^{\#}_{\gamma_2} g_{J'}\|_{\ell^2_{J'} L^2_x}. \tag{4-7}$$

To make the notation slightly less cumbersome, we introduce the shorthand

$$\Gamma_1^*(\mathbf{q}) = \Gamma_1^{\not\sim B}(\mathbf{q}) \cap \Gamma_1[\lambda_1, \mu_1, \mu_2].$$

Given $\mathbf{q} \in \mathbf{q}$ and $\mathfrak{h} \in \mathbb{R}^{1+n}$, we define the set

$$\Gamma_1^{**}(\mathbf{q}, \mathfrak{h}) = \Gamma_1^{**}[\lambda_1, \mu_1, \mu_2](\mathbf{q}, \mathfrak{h}) = \{\gamma_1 \in \Gamma_1^*(\mathbf{q}) : \xi(\gamma_1) \in \Sigma_1(\mathfrak{h}) + \mathcal{O}(R^{-\frac{1}{2}})\}.$$

Thus $\Gamma_1^{**}(\mathbf{q}, \mathfrak{h})$ consists of all $\gamma_1 \in \Gamma_1^*(\mathbf{q})$ such that $\xi(\gamma_1)$ lies within $CR^{-\frac{1}{2}}$ of the surface $\Sigma_1(\mathfrak{h})$. If we expand the square of the $L^2_{t,x}$ in (4-7) we get

$$\left\| \sum_{\substack{\gamma_1 \in \Gamma_1^*(\mathbf{q}) \\ \gamma_2 \in \Gamma_2(\mathbf{q})}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L^2_{t,x}(\mathbf{q})}^2 \leq \sum_{\substack{\gamma_1 \in \Gamma_1^*(\mathbf{q}) \\ \gamma'_2 \in \Gamma_2(\mathbf{q})}} \sum_{\gamma'_1 \in \Gamma_1^*(\mathbf{q})} \sum_{\gamma_2 \in \Gamma_2(\mathbf{q})} |\langle \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v, \mathcal{P}_{\gamma'_1} u \mathcal{P}_{\gamma'_2} v \rangle_{L^2_{t,x}}|.$$

We now exploit the Fourier localisation properties of the wave packets to deduce that the inner product vanishes unless

$$\begin{aligned} \xi(\gamma_1) + \xi(\gamma_2) &= \xi(\gamma'_1) + \xi(\gamma'_2) + \mathcal{O}(R^{-\frac{1}{2}}), \\ \Phi_1(\xi(\gamma_1)) + \Phi_2(\xi(\gamma_2)) &= \Phi_1(\xi(\gamma'_1)) + \Phi_2(\xi(\gamma'_2)) + \mathcal{O}(R^{-\frac{1}{2}}). \end{aligned} \tag{4-8}$$

In particular, if we take $\mathfrak{h}_{\gamma_1, \gamma'_2} = (\Phi_1(\xi(\gamma_1)) - \Phi_2(\xi(\gamma'_2)), \xi(\gamma_1) - \xi(\gamma'_2))$, then an application of Lemma 2.3 implies

$$\begin{aligned} & \left\| \sum_{\substack{\gamma_1 \in \Gamma_1^*(q) \\ \gamma_2 \in \Gamma_2(q)}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^2}^2 \\ & \leq \sum_{\substack{\gamma_1 \in \Gamma_1^*(q) \\ \gamma'_2 \in \Gamma_2(q)}} \sum_{\gamma_1 \in \Gamma_1^{**}(q, \mathfrak{h}_{\gamma_1, \gamma'_2})} \sum_{\substack{\gamma_2 \in \Gamma_2(q) \\ \xi(\gamma_2) = \xi(\gamma'_1) + \xi(\gamma'_2) - \xi(\gamma_1) + \mathcal{O}(R^{-1/2})}} \left| \langle \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v, \mathcal{P}_{\gamma'_1} u \mathcal{P}_{\gamma'_2} v \rangle_{L_{t,x}^2} \right|. \end{aligned}$$

On the other hand, an application of Lemma 2.6 easily gives the U_{Φ}^2 bound

$$\| \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \|_{L_{t,x}^2} \lesssim R^{-\frac{n-1}{4}} \| L_{\gamma_1}^{\#} f_J \|_{\ell_J^2 L_x^2} \| L_{\gamma_2}^{\#} g_{J'} \|_{\ell_{J'}^2 L_x^2}.$$

If we now note that, for fixed γ_1, γ'_2 , and γ'_1 and any $q \in \mathfrak{q}$, we have

$$\#\{ \gamma_2 \in \Gamma_2 : T_{\gamma_2} \cap R^{\delta} q \neq \emptyset, \xi(\gamma_2) = \xi(\gamma'_1) + \xi(\gamma'_2) - \xi(\gamma_1) + \mathcal{O}(R^{-\frac{1}{2}}) \} \lesssim R^{n\delta}$$

then an application of Cauchy–Schwarz gives

$$\begin{aligned} & \left\| \sum_{\substack{\gamma_1 \in \Gamma_1^*(q) \\ \gamma_2 \in \Gamma_2(q)}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^2}^2 \\ & \lesssim R^{D\delta - \frac{n-1}{2}} \#\Gamma_1^*(q) \#\Gamma_2(q) \sup_{\mathfrak{h}} \#\Gamma_1^{**}(q, \mathfrak{h}) \sup_{\gamma_j \in \Gamma_j} \| L_{\gamma_1}^{\#} f_J \|_{\ell_J^2 L_x^2}^2 \| L_{\gamma_2}^{\#} g_{J'} \|_{\ell_{J'}^2 L_x^2}^2. \end{aligned}$$

Consequently the bound (4-7) follows from the combinatorial estimate

$$\sum_{\substack{q \in \mathfrak{q}(\mu_1, \mu_2) \\ q \subset 2B}} \#\Gamma_1^*(q) \#\Gamma_2(q) \sup_{\mathfrak{h} \in \mathbb{R}^{1+n}} \#\Gamma_1^{**}(q, \mathfrak{h}) \lesssim R^{D\delta} \#\Gamma_1 \#\Gamma_2. \tag{4-9}$$

We now simplify this bound slightly by exploiting the dyadic localisations we performed earlier. More precisely, by definition, for every $q \in \mathfrak{q}(\mu_1, \mu_2)$, we have $\#\Gamma_2(q) \leq 2\mu_2$. On the other hand, by exchanging the order of summation, recalling the shorthand $\Gamma_1^*(q) = \Gamma_1^{\mathcal{J}^B}(q) \cap \Gamma_1[\lambda_1, \mu_1, \mu_2]$, and using the definition of the set $\Gamma_1[\lambda_1, \mu_1, \mu_2]$, we deduce that

$$\begin{aligned} \sum_{\substack{q \in \mathfrak{q}(\mu_1, \mu_2) \\ q \subset 2B}} \#\Gamma_1^*(q) & \leq \sum_{q \in \mathfrak{q}(\mu_1, \mu_2)} \#(\Gamma_1(q) \cap \Gamma[\lambda_1, \mu_1, \mu_2]) \\ & = \sum_{\gamma_1 \in \Gamma[\lambda_1, \mu_1, \mu_2]} \#\{q \in \mathfrak{q}(\mu_1, \mu_2) : T_{\gamma_1} \cap R^{\delta} q \neq \emptyset\} \leq 2\lambda_1 \#\Gamma_1 \end{aligned}$$

Therefore, we have reduced the bound (4-9) to proving the combinatorial Kakeya-type estimate

$$\sup_{\substack{\mathfrak{h} \in \mathbb{R}^{1+n} \\ q \in \mathfrak{q}(\mu_1, \mu_2), q \subset 2B}} \Gamma_1^{**}[\lambda_1, \mu_1, \mu_2](q, \mathfrak{h}) \lesssim R^{D\delta} \frac{\#\Gamma_2}{\lambda_1 \mu_2}. \tag{4-10}$$

The proof of this bound is the focus of the next subsection.

4C. The L^2 bound: the combinatorial estimate. We have reduced the proof of Theorem 4.1 to obtaining the combinatorial bound (4-10), which is essentially well known to experts as it does not see the difference between homogeneous solutions and $V_{\mathbb{F}_j}^2$ -functions. For completeness, we include the proof here. We follow the “bush” argument used in [Tao 2003], making some minor adjustments only to relate it to Assumption 1. Recall that we have fixed a ball $B \in \mathcal{B}$. Fix any $h \in \mathbb{R}^{1+n}$ and $q_0 \in \mathbf{q}(\mu_1, \mu_2)$ with $q_0 \subset 2B$. Our goal is to prove

$$\#\Gamma_1^{**}(q_0, h) \lesssim R^{D\delta} \frac{\#\Gamma_2}{\lambda_1 \mu_2}.$$

The first step is to exploit the fact that γ_1 is not concentrated on B . Recall from Section 3C that for $\gamma_1 \in \Gamma_1$ we have defined the ball $B(\gamma_1, \lambda_1, \mu_1, \mu_2) \in \mathcal{B}$ to be (a) maximiser for the quantity

$$\#\{q \in \mathbf{q}(\mu_1, \mu_2) : T_{\gamma_j} \cap R^\delta q \neq \emptyset, q \cap B(\gamma_j, \lambda_j, \mu_1, \mu_2) \neq \emptyset\}.$$

Let $\gamma_1 \in \Gamma_1^{**}(q_0, h)$. By construction this implies $\gamma_1 \in \Gamma_1^{\not\subset B}(q_0)$, and hence by the definition of the relation \sim , we have $B \not\subset 10B(\gamma_1, \lambda_1, \mu_1, \mu_2)$. Since $q_0 \subset 2B$ and the balls in \mathcal{B} have radius $R^{1-\delta}$, we must have $\text{dist}(q_0, B(\gamma_1, \lambda_1, \mu_1, \mu_2)) \gtrsim R^{1-\delta}$. In particular, by (ii) in Remark 3.7, we have for every $\gamma_1 \in \Gamma_1^{**}(q_0, h)$

$$\begin{aligned} \#\{q \in \mathbf{q}(\mu_1, \mu_2) : T_{\gamma_1} \cap R^\delta q \neq \emptyset, \text{dist}(q, q_0) \gtrsim R^{1-\delta}\} \\ \gtrsim \#\{q \in \mathbf{q}(\mu_1, \mu_2) : T_{\gamma_1} \cap R^\delta q \neq \emptyset, q \cap B(\gamma_1, \lambda_1, \mu_1, \mu_2) \neq \emptyset\} \gtrsim R^{-D\delta} \lambda_1. \end{aligned}$$

On the other hand, since for $q \in \mathbf{q}(\mu_1, \mu_2)$ we have $\#\Gamma_2(q) \geq \mu_2$, we deduce that

$$\#\{(q, \gamma_2) \in \mathbf{q}(\mu_1, \mu_2) \times \Gamma_2 : T_{\gamma_1} \cap R^\delta q \neq \emptyset, T_{\gamma_2} \cap R^\delta q \neq \emptyset, \text{dist}(q, q_0) \gtrsim R^{1-\delta}\} \gtrsim R^{-D\delta} \lambda_1 \mu_2.$$

Summing up over $\gamma_1 \in \Gamma_1^{**}(q_0, h)$ and then changing the order of summation gives

$$\begin{aligned} \lambda_1 \mu_2 \#\Gamma_1^{**}(q_0, h) \\ \lesssim R^{D\delta} \sum_{\gamma_1 \in \Gamma_1^{**}(q_0, h)} \#\{(q, \gamma_2) \in \mathbf{q}(\mu_1, \mu_2) \times \Gamma_2 : T_{\gamma_1} \cap R^\delta q \neq \emptyset, T_{\gamma_2} \cap R^\delta q \neq \emptyset, \text{dist}(q, q_0) \gtrsim R^{1-\delta}\} \\ = R^{D\delta} \sum_{\gamma_2 \in \Gamma_2} \#\{(q, \gamma_1) \in \mathbf{q}(\mu_1, \mu_2) \times \Gamma_1^{**}(q_0, h) : T_{\gamma_1} \cap R^\delta q \neq \emptyset, T_{\gamma_2} \cap R^\delta q \neq \emptyset, \text{dist}(q, q_0) \gtrsim R^{1-\delta}\}. \end{aligned}$$

Therefore the required bound (4-10) follows from the lemma below; see [Tao 2003, Lemma 8.1].

Lemma 4.3. *Let $q_0 \in \mathbf{q}$, $h \in \mathbb{R}^{1+n}$, and $\gamma_2 \in \Gamma_2$. Then*

$$\#\{(q, \gamma_1) \in \mathbf{q}(\mu_1, \mu_2) \times \Gamma_1^{**}(q_0, h) : T_{\gamma_1} \cap R^\delta q \neq \emptyset, T_{\gamma_2} \cap R^\delta q \neq \emptyset, \text{dist}(q, q_0) \gtrsim R^{1-\delta}\} \lesssim R^{D\delta}.$$

Proof. Define the bush (or “fan”) at q_0 by

$$\text{Bush}(q_0) = \bigcup_{\gamma_1 \in \Gamma_1^{**}(q_0, h)} T_{\gamma_1}.$$

Thus $\text{Bush}(q_0) \subset \mathbb{R}^{1+n}$ is the union of all tubes T_{γ_1} (associated to phase-space elements $\gamma_1 \in \Gamma_1^{**}(q_0, \mathfrak{h})$) passing through a neighbourhood of the cube q_0 . Our goal is then to bound the sum

$$\sum_{\substack{q \in \mathbf{q}(\mu_1, \mu_2) \\ q \subset \text{Bush}(q_0) \cap T_{\gamma_2} + \mathcal{O}(R^{1/2+\delta}) \\ \text{dist}(q, q_0) \gtrsim R^{1-\delta}}} \#\{\gamma_1 \in \Gamma_1^{**}(q_0, \mathfrak{h}) : T_{\gamma_1} \cap R^\delta q \neq \emptyset\}. \tag{4-11}$$

We first count the number of possible cubes in the outer summation. The idea is to first show that

$$\text{Bush}(q_0) \subset (t_0, x_0) + \mathcal{C}_1(\mathfrak{h}) + \mathcal{O}(R^{\frac{1}{2}+D\delta}), \tag{4-12}$$

where (t_0, x_0) denotes the centre of the cube q_0 , and the conic hypersurface $\mathcal{C}_1(\mathfrak{h})$ is given by

$$\mathcal{C}_1(\mathfrak{h}) = \{(r, -r \nabla \Phi_1(\xi)) : r \in \mathbb{R}, \xi \in \Sigma_1(\mathfrak{h})\}.$$

If we had (4-12), an application of Lemma 2.7 would then show that $\text{Bush}(q_0) \cap T_{\gamma_2}$ is contained in a ball of radius $R^{\frac{1}{2}+D\delta}$, and hence the outer summation in (4-11) only contains $\mathcal{O}(R^{D\delta})$ terms. To show the inclusion (4-12), suppose that $(t, x) \in \text{Bush}(q_0)$. Then $(t, x) \in T_{\gamma_1}$ for some $\gamma_1 \in \Gamma_1^{**}(q_0, \mathfrak{h})$. By construction, we have $\xi(\gamma) = \xi^* + \mathcal{O}(R^{-\frac{1}{2}})$ for some $\xi^* \in \Sigma_1(\mathfrak{h})$. On the other hand, since $T_{\gamma_1} \cap R^\delta q_0 \neq \emptyset$, we have

$$x - x_0 + (t - t_0) \nabla \Phi_1(\xi(\gamma_1)) = [x - x(\gamma) + t \nabla \Phi_1(\xi(\gamma_1))] - [x_0 - x(\gamma) + t_0 \nabla \Phi_1(\xi(\gamma_1))] = \mathcal{O}(R^{\frac{1}{2}+\delta}).$$

Therefore, since $|t - t_0| \lesssim R$, we can write

$$\begin{aligned} &(t, x) - (t_0, x_0) \\ &= (t - t_0, -(t - t_0) \nabla \Phi_1(\xi^*)) + (0, x - x_0 + (t - t_0) \nabla \Phi_1(\xi(\gamma_1))) + (0, (t - t_0)[\nabla \Phi_1(\xi^*) - \nabla \Phi_1(\gamma(\xi))]) \\ &= (t - t_0, -(t - t_0) \nabla \Phi_1(\xi^*)) + \mathcal{O}(R^{\frac{1}{2}+\delta}) \end{aligned}$$

and hence we have (4-12). Consequently, the outer sum in (4-11) is only over $\mathcal{O}(R^{C\delta})$ cubes.

Fix $q \in \mathbf{q}(\mu_1, \mu_2)$ with $\text{dist}(q, q_0) \gtrsim R^{1-\delta}$. As the outer sum in (4-11) only adds $\mathcal{O}(R^{D\delta})$, the required bound now follows from

$$\#\{\gamma_1 \in \Gamma_1 : \xi(\gamma_1) \in \Sigma_1(\mathfrak{h}) + \mathcal{O}(R^{-\frac{1}{2}}), T_{\gamma_1} \cap R^\delta q \neq \emptyset, T_{\gamma_1} \cap R^\delta q_0 \neq \emptyset\} \lesssim R^\delta. \tag{4-13}$$

The point is that since the cubes q and q_0 are at a distance $R^{1-\delta}$ apart, the condition that T_{γ_1} must intersect *both* cubes, essentially fixes the tube T_{γ_1} . Since $\xi(\gamma_1) \in \Sigma_1(\mathfrak{h}) + \mathcal{O}(R^{-\frac{1}{2}})$, the bound (1-1) implies that fixing the tube T_{γ_1} also more or less fixes the phase-space element γ_1 (note that without the bound (1-1), the set in (4-13) could potentially contain far more than $\mathcal{O}(R^\delta)$ points). In more detail, let

$$\gamma_1, \gamma'_1 \in \{\gamma_1 \in \Gamma_1 : \xi(\gamma_1) \in \Sigma_1(\mathfrak{h}) + \mathcal{O}(R^{-\frac{1}{2}}), T_{\gamma_1} \cap R^\delta q \neq \emptyset, T_{\gamma_1} \cap R^\delta q_0 \neq \emptyset\}.$$

In light of (1-1), the estimate (4-13) would follow from the bounds

$$|x(\gamma_1) - x(\gamma'_1)| \lesssim R^{\frac{1}{2}+\delta}, \quad |v(\gamma_1) - v(\gamma'_1)| \lesssim R^{-\frac{1}{2}+\delta}, \tag{4-14}$$

where for ease of notation we define the *velocity* as $v(\gamma_1) = \Phi_1(\xi(\gamma_1))$. We now exploit the condition that the tubes T_{γ_1} and $T_{\gamma'_1}$ intersect the cubes q and q_0 . Let (t_q, x_q) denote the centre of the cube q and (t_0, x_0) the centre of q_0 . Since $|v(\gamma_1)| \leq D_2$ and

$$x_0 - x_q + (t_0 - t_q)v(\gamma_1) = (x_0 - x(\gamma_1) + t_0v(\gamma_1)) - (x_q - x(\gamma_1) + t_qv(\gamma_1)) = \mathcal{O}(R^{\frac{1}{2} + D\delta}),$$

the separation of the cubes q and q_0 implies $R^{1 - C\delta} \lesssim |t_0 - t_q| \lesssim R$. A computation shows that

$$(t_0 - t_q)(v(\gamma_1) - v(\gamma'_1)) = \mathcal{O}(R^{\frac{1}{2} + D\delta}), \quad x(\gamma_1) - x(\gamma'_1) = t_0(v(\gamma'_1) - v(\gamma_1)) + \mathcal{O}(R^{\frac{1}{2} + D\delta}),$$

and hence the bound on $|t_0 - t_q|$ gives (4-14). □

5. The globalisation lemma

We now complete the proof of Theorem 1.1 by showing that it follows from the localised bound in Theorem 4.1. The proof of Theorem 1.1 proceeds by using a strategy sketched in Section 8 of [Klainerman et al. 2002], together with an interpolation argument to replace $U_{\Phi_j}^2$ with $V_{\Phi_j}^2$.

Proof of Theorem 1.1. The first step is to show that by exploiting the (approximate) finite speed of propagation of frequency-localised waves, the bilinear estimate on Q_R implies the same estimate holds on $I_R \times \mathbb{R}^n$ with $I_R = [0, R]$. The second step is to remove the remaining temporal localisation and R^α -factor by using duality, together with the dispersive decay in Lemma 2.4. Finally we use a simple interpolation argument to replace $U_{\Phi_j}^2$ with the larger $V_{\Phi_j}^2$ space.

Step 1: from Q_R to $I_R \times \mathbb{R}^n$. Let $R \geq (10R_0)^2$, $u \in U_{\Phi_j}^2$ and $v \in U_{\Phi_j}^2$, with $\text{supp } \hat{u} \subset \Lambda_1^*$ and $\text{supp } \hat{v} \subset \Lambda_2^*$. Assuming Theorem 4.1, our goal is to prove that for every $\alpha > 0$ we have

$$\|uv\|_{L_{t,x}^{(n+3)/(n+1)}(I_R \times \mathbb{R}^n)} \lesssim R^\alpha \|u\|_{U_{\Phi_j}^2} \|v\|_{U_{\Phi_j}^2}. \tag{5-1}$$

It is enough to consider the case where u and v are atoms; thus we have the decomposition

$$u = \sum_J \mathbb{1}_J(t) e^{it\Phi_1(-i\nabla)} f_J, \quad v = \sum_{J'} \mathbb{1}_{J'}(t) e^{it\Phi_2(-i\nabla)} g_{J'},$$

with

$$\sum_J \|f_J\|_{L^2}^2 + \sum_{J'} \|g_{J'}\|_{L^2}^2 \leq 1,$$

and we may assume that $\text{supp } \hat{f}_J \subset \Lambda_1^*$ and $\text{supp } \hat{g}_{J'} \subset \Lambda_2^*$ (using sharp Fourier cutoffs). By translation invariance, the bound (5-1) then follows from

$$\|uv\|_{L_{t,x}^{(n+3)/(n+1)}(Q_R)} \lesssim R^\alpha \left(\sum_J \|(1 + R^{-1}|x|)^{-(n+1)} f_J\|_{L_x^2}^2 \right)^{\frac{1}{2}} \left(\sum_{J'} \|(1 + R^{-1}|x|)^{-(n+1)} g_{J'}\|_{L_x^2}^2 \right)^{\frac{1}{2}} \tag{5-2}$$

since we can then sum up over the centres of balls (or cubes) of radius R which cover \mathbb{R}^n . The inequality (5-2) is a reflection of the fact that, as u and v are localised to frequencies of size ≈ 1 , we expect that the waves $e^{it\Phi_j(-i\nabla)} f_J$ should travel with velocity 1. In particular, u and v on Q_R should only depend on the data in $\{|x| \lesssim R\}$. It turns out that this is true, modulo a rapidly decreasing tail.

Let $\rho \in \mathcal{S}$ with $\text{supp } \hat{\rho} \subset \{|\xi| \leq 1\}$ and $\rho \gtrsim 1$ on $|x| \leq 1$. To prove (5-2), we start by noting that since the left-hand integral is only over \mathcal{Q}_R , we may replace uv with $\rho(R^{-1}x)u(t, x)\rho(R^{-1}x)v(y)$. We can write

$$\begin{aligned} \rho\left(\frac{x}{R}\right)(e^{it\Phi_j(-i\nabla)} f)(x) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} R^n \hat{\rho}(R(\xi - \eta)) e^{it\Phi_j(\eta)} \hat{f}(\eta) d\eta e^{i\xi \cdot x} d\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} R^n \hat{\rho}(R(\xi - \eta)) \hat{f}(\eta) F(t, R(\xi - \eta), \eta) d\eta e^{i\xi \cdot x} e^{i\Phi_j(\xi)} d\xi, \end{aligned} \tag{5-3}$$

where $F(t, \xi, \eta) = \chi(\xi, \eta) e^{it(\Phi_j(\frac{\xi}{R} + \eta) - \Phi_j(\eta))}$ and $\chi \in C_0^\infty(\{|\xi| \leq 2\} \times (\Lambda_j^* + 1/R_0))$ with $\chi = 1$ on $\{|\xi| \leq 2\} \times \Lambda_j^*$. The oscillating component of F is essentially constant for $|t| \leq R$. To exploit this, we expand F using a Fourier series to get

$$F(t, \xi, \eta) = \sum_{k \in \mathbb{Z}^{2n}} c_k(t) e^{ik \cdot (\xi, \eta)}, \quad c_k(t) = \int_{\mathbb{R}^{2n}} F(t, \xi, \eta) e^{ik \cdot (\xi, \eta)} d\xi d\eta,$$

and by (ii) in Assumption 1, the coefficients satisfy $|c_k(t)| \lesssim_{\mathbf{R}_0, \mathbf{D}_2} (1 + |k_1|)^{-2(n+1)} (1 + |k_2|)^{-2(n+1)}$ with $k = (k_1, k_2)$. Applying this expansion to $\rho(R^{-1}x)u$ and $\rho(R^{-1}x)v$ we obtain the decompositions

$$\begin{aligned} \rho(R^{-1}x)u &= \sum_J \sum_k c_k(t) \mathbb{1}_J(t) e^{it\Phi_1(-i\nabla)} f_{k,J}, \\ \rho(R^{-1}x)v &= \sum_{J'} \sum_k c'_k(t) \mathbb{1}_{J'}(t) e^{it\Phi_2(-i\nabla)} g_{k,J'}, \end{aligned} \tag{5-4}$$

where the coefficients c_k, c'_k are independent of J and J' , and the functions $f_{k,J}$ and $g_{k,J'}$ are given by

$$f_{k,J}(x) = \rho\left(\frac{x}{R} + k_1\right) f_J(x + k_2), \quad g_{k,J'}(x) = \rho\left(\frac{x}{R} + k_1\right) g_{J'}(x + k_2),$$

with $k = (k_1, k_2)$. Note that $\text{supp } \hat{f}_{k,J} \subset \Lambda_1^* + 1/(2\mathbf{R}_0)$ since $R \geq (10\mathbf{R}_0)^2$, thus the $f_{k,J}$ satisfy the support conditions in Theorem 4.1 (with Λ_j^* replaced with $\Lambda_j^* + 1/\mathbf{R}_0$, and \mathbf{R}_0 replaced with $2\mathbf{R}_0$). A similar comment applies to the $g_{k',J'}$. Therefore, plugging the decomposition (5-4) into the left-hand side of (5-2), we deduce via an application of Theorem 4.1 that

$$\begin{aligned} &\|uv\|_{L_{t,x}^{(n+3)/(n+1)}(\mathcal{Q}_R)} \\ &\lesssim \sum_{k, k' \in \mathbb{Z}^n \times \mathbb{Z}^n} (1 + |k|)^{-2(n+1)} (1 + |k'|)^{-2(n+1)} \\ &\quad \times \left\| \sum_{J, J'} \mathbb{1}_J(t) e^{it\Phi_1(-i\nabla)} f_{k,J} \mathbb{1}_{J'}(t) e^{it\Phi_2(-i\nabla)} g_{k',J'} \right\|_{L_{t,x}^{(n+3)/(n+1)}(\mathcal{Q}_R)} \\ &\lesssim R^\alpha \sum_{k, k'} (1 + |k|)^{-2(n+1)} (1 + |k'|)^{-2(n+1)} \\ &\quad \times \left(\sum_J \|(1 + R^{-1}|x - k_1 + Rk_2|)^{-(n+1)} f_J\|_{L_x^2}^2 \right)^{\frac{1}{2}} \left(\sum_{J'} \|(1 + R^{-1}|x - k'_1 + Rk'_2|)^{-(n+1)} g_{J'}\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim R^\alpha \left(\sum_J \|(1 + R^{-1}|x|)^{-(n+1)} f_J\|_{L_x^2}^2 \right)^{\frac{1}{2}} \left(\sum_{J'} \|(1 + R^{-1}|x|)^{-(n+1)} g_{J'}\|_{L_x^2}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Thus we obtain (5-2) and hence (5-1).

Step 2: from $I_R \times \mathbb{R}^n$ to \mathbb{R}^{1+n} . Let $u \in U_{\Phi_1}^2$ and $v \in U_{\Phi_2}^2$, with $\text{supp } \hat{u} \subset \Lambda_1^*$ and $\text{supp } \hat{v} \subset \Lambda_2^*$. Our goal is to show that for every $p > (n + 3)/(n + 1)$

$$\|uv\|_{L_{t,x}^p} \lesssim \|u\|_{U_{\Phi_1}^2} \|v\|_{U_{\Phi_2}^2}. \tag{5-5}$$

In fact the argument below gives the marginally stronger (though essentially equivalent) bound

$$\|uv\|_{L_t^p L_x^{(n+3)/(n+1)}} \lesssim \|u\|_{U_{\Phi_1}^2} \|v\|_{U_{\Phi_2}^2}. \tag{5-6}$$

To deduce (5-5) from (5-6), note that the dispersive estimate in Lemma 2.4, together with the abstract Strichartz estimates of [Keel and Tao 1998, Theorem 1.2], implies there exists $1 < a < b < \infty$ such that $\|uv\|_{L_t^a L_x^b} \lesssim 1$. On the other hand, the Fourier support assumptions imply that we have the trivial bound $\|uv\|_{L_t^\infty L_x^p(\mathbb{R}^{1+n})} \lesssim 1$ for every $p \geq 1$. Thus interpolation gives (5-5) from (5-6).

We now turn to the proof of (5-6). As in Step 1, we may assume that u and v are atoms with the decomposition

$$u = \sum_J \mathbb{1}_J(t) e^{it\Phi_1(-i\nabla)} f_J, \quad v = \sum_{J'} \mathbb{1}_{J'}(t) e^{it\Phi_2(-i\nabla)} g_{J'},$$

with $\text{supp } \hat{f}_J \subset \Lambda_1^*$, $\text{supp } \hat{g}_{J'} \subset \Lambda_2^*$, and

$$\sum_J \|f_J\|_{L^2}^2 + \sum_{J'} \|g_{J'}\|_{L^2}^2 \leq 1.$$

By real interpolation it is enough to show that for every $q > (n + 3)/(n + 1)$ we have

$$\|uv\|_{L_t^{q,\infty} L_x^{(n+3)/(n+1)}} \lesssim 1,$$

where $L_t^{q,\infty}$ is the Lorentz norm. Applying duality, this follows from the estimate

$$\int_\Omega \|uv\|_{L_x^{(n+3)/(n+1)}} dt \lesssim |\Omega|^{\frac{1}{q'}} \tag{5-7}$$

for every measurable $\Omega \subset \mathbb{R}$. Define the Fourier localised solution operator

$$\mathcal{U}_j(t)[h] = e^{it\Phi_j(-i\nabla)} P_{\Lambda_j^*} h,$$

where we let

$$\widehat{P_{\Lambda_j^*} h}(\xi) = \rho_{\Lambda_j^*}(\xi) \hat{h}(\xi)$$

with $\rho \in C_0^\infty(\Lambda_j^* + 1/(10R_0))$ and $\rho = 1$ on Λ_j^* . If we interpolate Lemma 2.4 with the trivial $L_t^\infty L_x^2$ bound and apply duality, we deduce that for every $1 \leq a \leq 2$

$$\int_{\substack{(t,t') \in \Omega \times \Omega \\ |t-t'| \gtrsim R}} \langle \mathcal{U}_j^*(t)[G(t)], \mathcal{U}_j^*(t')[G(t')] \rangle_{L_x^2} dt dt' \lesssim |\Omega|^2 R^{-\frac{n-1}{2}(\frac{2}{a}-1)} \|G\|_{L_t^\infty L_x^a}^2, \tag{5-8}$$

where \mathcal{U}_j^* denotes the L_x^2 adjoint of \mathcal{U}_j . The dispersive bound (5-8) together with the bilinear estimate (5-1) are the key inequalities required in the proof of (5-7).

We now begin the proof of (5-7). If $|\Omega| \lesssim 1$, then (5-7) follows by putting $uv \in L_t^\infty L_x^{\frac{n+3}{n+1}}$ and using the Sobolev embedding. Thus we may assume that $|\Omega| \gg 1$. Let us set $J'_\Omega := \Omega \cap J'$. An application

of duality gives

$$\begin{aligned} \int_{\Omega} \|uv\|_{L_x^{(n+3)/(n+1)}} dt &\leq \sup_{\|F\|_{L_t^\infty L_x^{(n+3)/2}} \leq 1} \left| \int_{\Omega} \langle F, uv \rangle_{L_x^2} dt \right| \\ &= \sup_{\|F\|_{L_t^\infty L_x^{(n+3)/2}} \leq 1} \left| \sum_{J'} \int_{J'_\Omega} \langle F, u \mathcal{U}_2(t)[g_{J'}] \rangle_{L_x^2} dt \right| \\ &\lesssim \sup_{\|F\|_{L_t^\infty L_x^{(n+3)/2}} \leq 1} \left(\sum_{J'} \left\| \int_{J'_\Omega} \mathcal{U}_2^*(t)[F\bar{u}] dt \right\|_{L_x^2}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

If we expand the square of the L_x^2 norm, we have via (5-8)

$$\begin{aligned} \sum_{J'} \left\| \int_{J'_\Omega} \mathcal{U}_2^*(t)[F\bar{u}] dt \right\|_{L_x^2}^2 &= \sum_{J'} \int_{t,t' \in J'_\Omega} \langle \mathcal{U}_2^*(t)[F\bar{u}], \mathcal{U}_2^*(t')[F\bar{u}] \rangle_{L_x^2} dt dt' \\ &= \sum_{J'} \int_{\substack{t,t' \in J'_\Omega \\ |t-t'| \gtrsim R}} \langle \mathcal{U}_2^*(t)[F\bar{u}], \mathcal{U}_2^*(t')[F\bar{u}] \rangle_{L_x^2} dt dt' \\ &\quad + \sum_{J'} \sum_{|I-I'| \leq R} \int_{J'_\Omega \cap I} \int_{J'_\Omega \cap I'} \langle \mathcal{U}_2^*(t)[F\bar{u}], \mathcal{U}_2^*(t')[F\bar{u}] \rangle_{L_x^2} dt dt' \\ &\lesssim |\Omega|^2 R^{-\frac{n-1}{2}(a-1)} \|F\bar{u}\|_{L_t^\infty L_x^a}^2 + \sum_{J',I} \left\| \int_{J'_\Omega \cap I} \mathcal{U}_2^*(t)[F\bar{u}] dt \right\|_{L_x^2}^2 \\ &\lesssim |\Omega|^2 R^{-\frac{2(n-1)}{n+3}} \|F\|_{L_t^\infty L_x^{(n+3)/2}}^2 \|u\|_{L_t^\infty L_x^2}^2 + \sum_{J',I} \left\| \int_{J'_\Omega \cap I} \mathcal{U}_2^*(t)[F\bar{u}] dt \right\|_{L_x^2}^2, \end{aligned}$$

where

$$\frac{1}{a} = \frac{2}{n+3} + \frac{1}{2}.$$

Here we always take I (and I') to be a decomposition of \mathbb{R} into intervals of size R . We now essentially repeat the previous argument, but expand u instead of v to obtain

$$\begin{aligned} \sum_{J',I} \left\| \int_{J'_\Omega \cap I} \mathcal{U}_2^*(t)[F\bar{u}] dt \right\|_{L_x^2}^2 &\leq \sup_{\sum_{J',I} \|g_{J',I}\|_{L_x^2}^2 \leq 1} \left| \sum_{J',I} \int_{J'_\Omega \cap I} \langle F, \bar{u} \mathcal{U}_2(t)g_{J',I} \rangle_{L_x^2} dt \right|^2 \\ &\lesssim \sup_{\sum_{J',I} \|g_{J',I}\|_{L_x^2}^2 \leq 1} \left| \sum_{J,I} \int_{J_\Omega \cap I} \langle \mathcal{U}_1^*(t)[F\bar{v}_I], f_J \rangle_{L_x^2} dt \right|^2 \\ &\lesssim \sup_{\sum_{J',I} \|g_{J',I}\|_{L_x^2}^2 \leq 1} \sum_J \left\| \sum_I \int_{J_\Omega \cap I} \mathcal{U}_1^*(t)[F\bar{v}_I] dt \right\|_{L_x^2}^2, \end{aligned}$$

where we take

$$v_I = \sum_{J'} \mathbb{1}_{J'}(t) \mathcal{U}_2(t)g_{J',I}.$$

Again expanding out the L_x^2 norm, and applying (5-8), we have

$$\begin{aligned} & \sum_J \left\| \sum_I \int_{J_\Omega \cap I} \mathcal{U}_1^*(t)[F\bar{v}_I] dt \right\|_{L_x^2}^2 \\ &= \sum_J \sum_{|I-I'| \gg R} \int_{J_\Omega \cap I} \int_{J_\Omega \cap I'} \langle \mathcal{U}_1^*(t)[F\bar{v}_I], \mathcal{U}_1(t')[F\bar{v}_{I'}] \rangle_{L_x^2} dt dt' \\ & \quad + \sum_J \sum_{|I-I'| \lesssim R} \int_{J_\Omega \cap I} \int_{J_\Omega \cap I'} \langle \mathcal{U}_1^*(t)[F\bar{v}_I], \mathcal{U}_1^*(t')[F\bar{v}_{I'}] \rangle_{L_x^2} dt dt' \\ & \lesssim |\Omega|^2 R^{-\frac{2(n-1)}{n+3}} \|F\|_{L_t^\infty L_x^{(n+3)/2}}^2 \sup_I \|v_I\|_{L_t^\infty L_x^2}^2 + \sum_{J,I} \left\| \int_{J_\Omega \cap I} \mathcal{U}_1(t)[Fv_I] dt \right\|_{L_x^2}^2. \end{aligned}$$

Collecting the above chain of estimates together, and using the fact that

$$\|v_I\|_{L_t^\infty L_x^2}^2 \leq \sum_{I,J'} \|g_{J',I}\|_{L_x^2}^2 \leq 1$$

together with another application of duality, we see that

$$\begin{aligned} \int_\Omega \|uv\|_{L_x^{(n+3)/(n+1)}} dt &\lesssim |\Omega| R^{-\frac{n-1}{n+3}} + \sup_{\substack{\|F\|_{L_t^\infty L_x^{(n+3)/2}} \leq 1 \\ \sum_{I,J'} \|g_{I,J'}\|_{L_x^2}^2 \leq 1}} \left(\sum_{J,I} \left\| \int_{J_\Omega \cap I} \mathcal{U}_1(t)[F\bar{v}_I] dt \right\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\ &\leq |\Omega| R^{-\frac{n-1}{n+3}} + \sup_{\substack{\sum_{I,J'} \|g_{I,J'}\|_{L_x^2}^2 \leq 1 \\ \sum_{I,J} \|f_{I,J}\|_{L_x^2}^2 \leq 1}} \sum_I \int_{\Omega \cap I} \|u_I v_I\|_{L_x^{(n+3)/(n+2)}} dt, \end{aligned}$$

where we define $u_I = \sum_{I,J} \mathbb{1}_J(t) \mathcal{U}_1(t)[f_{I,J}]$. Observe that

$$\sum_I \|u_I\|_{U_{\Phi_1}^2}^2 \leq \sum_{I,J} \|f_{I,J}\|_{L_x^2}^2 \leq 1,$$

and that u_I satisfies the support properties in Theorem 4.1 (with Λ_j^* replaced by $\Lambda_j^* + 1/(10\mathbf{R}_0)$, and \mathbf{R}_0 replaced by $2\mathbf{R}_0$). A similar comment applies to v_I . Consequently, an application of (5-1) gives for any $\alpha > 0$

$$\begin{aligned} \sum_I \int_{\Omega \cap I} \|u_I v_I\|_{L_x^{(n+3)/(n+1)}} dt &\leq |\Omega|^{\frac{2}{n+3}} \sum_I \|u_I v_I\|_{L_{t,x}^{(n+3)/(n+1)}(I \times \mathbb{R}^n)} \\ &\lesssim |\Omega|^{\frac{2}{n+3}} R^\alpha \left(\sum_{I,J} \|f_{I,J}\|_{L_x^2}^2 \right)^{\frac{1}{2}} \left(\sum_{I,J'} \|g_{I,J'}\|_{L_x^2}^2 \right)^{\frac{1}{2}} \leq |\Omega|^{\frac{2}{n+3}} R^\alpha \end{aligned}$$

and therefore

$$\int_\Omega \|uv\|_{L_x^{(n+3)/(n+1)}} dt \lesssim |\Omega| R^{-\frac{n-1}{n+3}} + |\Omega|^{\frac{2}{n+3}} R^\alpha.$$

To complete the proof, we choose $R = |\Omega|^C$ with $C > 0$ sufficiently large so that $|\Omega| R^{-\frac{n-1}{n+3}} \leq |\Omega|^{\frac{1}{q'}}$. On the other hand, since $q > (n+3)/(n+1)$, we can take

$$\alpha = \frac{1}{2C} \left(\frac{n+1}{n+3} - \frac{1}{q} \right),$$

which implies

$$|\Omega|^{\frac{2}{n+3}} R^\alpha = |\Omega|^{\frac{2}{n+3} + \alpha C} \leq |\Omega|^{\frac{1}{q'}}.$$

Therefore we obtain (5-7) as required.

Step 3: from U_{Φ}^2 to V_{Φ}^2 . Let $p > (n+3)/(n+1)$, $u \in V_{\Phi_1}^2$ and $v \in V_{\Phi_2}^2$, with $\text{supp } \hat{u} \subset \Lambda_1^*$ and $\text{supp } \hat{v} \subset \Lambda_2^*$. An application of [Koch and Tataru 2005, Lemma 6.4]; see also [Hadac et al. 2009, Propositions 2.5 and 2.20], gives a decomposition $u = \sum_{k \in \mathbb{N}} u_k$ and $v = \sum_{k \in \mathbb{N}} v_k$ such that u_k, v_k retain the correct Fourier support properties (we can just use sharp Fourier cutoffs here) and for any $r \geq 2$ we have the bounds

$$\|u_k\|_{U_{\Phi_1}^r} \lesssim 2^{k(\frac{2}{r}-1)} \|u\|_{V_{\Phi_1}^2}, \quad \|v_k\|_{U_{\Phi_2}^r} \lesssim 2^{k(\frac{2}{r}-1)} \|v\|_{V_{\Phi_2}^2}.$$

Let $(n+3)/(n+1) < q < p$, and take $\theta = q/p < 1$. Then an application of (5-5) (with $p = q$), together with the convexity of L^p norms, gives

$$\begin{aligned} \|uv\|_{L_{t,x}^p} &\leq \sum_{k,k'} \|u_k v_{k'}\|_{L_{t,x}^p} \leq \sum_{k,k'} \|u_k v_{k'}\|_{L_{t,x}^q}^\theta \|u_k v_{k'}\|_{L_{t,x}^\infty}^{1-\theta} \\ &\leq \sum_{k',k} (\|u_k\|_{U_{\Phi_1}^2} \|v_{k'}\|_{U_{\Phi_2}^2})^\theta (\|u_k\|_{U_{\Phi_1}^\infty} \|v_{k'}\|_{U_{\Phi_2}^\infty})^{1-\theta} \\ &\lesssim \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2} \sum_{k,k'} 2^{-k(1-\theta)} 2^{-k'(1-\theta)} \lesssim \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2}, \end{aligned}$$

where we used the Sobolev embedding and the fact that the Fourier support of u, v is contained in the unit ball to control the $L_{t,x}^\infty$ norm. Thus Theorem 1.1 follows. \square

Remark 5.1. The argument in Step 3 above, using (5-6), also implies the slightly stronger estimate

$$\|uv\|_{L_t^p L_x^{(n+3)/(n+1)}(\mathbb{R}^{1+n})} \leq C \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2}.$$

This is well known in the case of homogeneous solutions; see, e.g., [Tao 2003]. However, the estimate in the endpoint $p = q = (n+3)/(n+1)$ remains open. For homogeneous solutions it is known only in the case of the cone [Tao 2001].

Remark 5.2. In fact, since Tao’s endpoint result [2001, Theorem 1.1] holds for Hilbert-space-valued waves, we observe that one can deduce the U^2 -estimate for the cone directly. This follows by noting that, given U^2 -atoms $u = \sum_{I \in \mathcal{I}} \mathbb{1}_I u_I$ and $v = \sum_{J \in \mathcal{J}} \mathbb{1}_J v_J$, we have

$$|uv| \leq \left(\sum_{I \in \mathcal{I}} |u_I|^2 \right)^{\frac{1}{2}} \left(\sum_{J \in \mathcal{J}} |v_J|^2 \right)^{\frac{1}{2}} = |U||V|.$$

with ℓ^2 -valued waves U and V .

6. Mixed norms and generalisations to small scales

We now give some consequences of the bilinear estimate in Theorem 1.1. Namely, we state an extension to mixed $L_t^q L_x^r$ spaces, and, in the case of the hyperboloid, we give a small-scale version of Theorem 1.1. The small-scale estimate will play a key role in our application to the Dirac–Klein–Gordon system.

6A. Mixed norms. Let Φ_1 and Φ_2 be phases satisfying Assumption 1. A standard TT^* argument, see for instance [Keel and Tao 1998], together with Lemma 2.4 implies that, provided

$$\frac{1}{q} + \frac{n-1}{2r} \leq \frac{n-1}{4}$$

and $q > 2$, we have the Strichartz-type bound

$$\|e^{it\Phi_j(-i\nabla)} f\|_{L_t^q L_x^r(\mathbb{R}^{1+n})} \lesssim \|f\|_{L_x^2}. \tag{6-1}$$

As in Step 3 of the proof of the globalisation lemma, by decomposing V^2 into U^a atoms, see [Koch and Tataru 2005, Lemma 6.4] or [Hadac et al. 2009, Propositions 2.5 and 2.20], we see that

$$\|uv\|_{L_t^a L_x^b} \lesssim \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2}$$

for any

$$\frac{1}{a} + \frac{n-1}{2b} \leq \frac{n-1}{2}.$$

Interpolating with Theorem 1.1 then gives the following mixed norm version.

Corollary 6.1. *Let $n \geq 2$ and assume that $a > 1$,*

$$\frac{1}{a} + \frac{n+1}{2b} < \frac{n+1}{2},$$

and

$$\frac{1}{a} + \frac{n-1}{4b} < \begin{cases} \frac{n+1}{4}, & n \geq 3, \\ \frac{1}{2} + \frac{5}{12b}, & n = 2. \end{cases} \tag{6-2}$$

Let Φ_1, Φ_2 , and u, v be as in the statement of Theorem 1.1. Then

$$\|uv\|_{L_t^a L_x^b} \lesssim \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2}.$$

Remark 6.2. Let $p > (n + 3)/(n + 1)$. It is possible to deduce a weaker version of Theorem 1.1 and Corollary 6.1 directly from the homogeneous estimate

$$\|e^{it\Phi_1(-i\nabla)} f e^{it\Phi_2(-i\nabla)} g\|_{L_{t,x}^p(\mathbb{R}^{1+n})} \lesssim \|f\|_{L_x^2} \|g\|_{L_x^2}, \tag{6-3}$$

where the phases satisfy the conditions in Assumption 1, and $f, g \in L^2$ have the required support conditions. We sketch the argument as follows. By interpolating (6-3) with the trivial $L_t^\infty L_x^2$ bound, we deduce that for every $a > 2$ we have

$$\|e^{it\Phi_1(-i\nabla)} f e^{it\Phi_2(-i\nabla)} g\|_{L_t^a L_x^{(n+1)/n}} \lesssim \|f\|_{L_x^2} \|g\|_{L_x^2}.$$

By decomposing V^2 functions into U^a atoms [Koch and Tataru 2005; Hadac et al. 2009; Koch et al. 2014] and using the convexity of the L^p spaces, we see that for $a > 2$

$$\|uv\|_{L_t^a L_x^{(n+1)/n}} \lesssim \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2}.$$

Consequently, as in the proof of Corollary 6.1, by interpolating with the standard Strichartz estimates, we obtain

$$\|uv\|_{L_t^a L_x^b} \lesssim \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2}$$

provided that $a > 1$,

$$\frac{1}{a} + \frac{n+1}{2b} < \frac{n+1}{2},$$

and

$$\frac{1}{a} < \begin{cases} \frac{n-1}{n+3} \left(\frac{n}{2} - \frac{n+1}{2b} \right) + \frac{1}{2}, & n \geq 3, \\ \frac{1}{2}, & n = 2. \end{cases} \tag{6-4}$$

In particular, the homogeneous bounds contained in [Lee and Vargas 2010; Bejenaru 2017] imply a weaker version of our main result, with (6-2) in Corollary 6.1 replaced with (6-4). Note that condition (6-4) is much more restrictive than (6-2). This is most apparent in the low-dimensional cases; for instance if $n = 2$ then Corollary 6.4 allows $a < 2$, while (6-4) only allows the somewhat trivial (from a V^2 perspective) $a > 2$. To summarise, our main result, Theorem 1.1 not only clarifies the dependence of the constant on the global properties of the phases Φ_1 and Φ_2 , but also presents a significant strengthening of the allowed exponents for the V^2 estimate.

We observe that the above argument, namely deducing a V^2 bound directly from the homogeneous estimate, has been used in [Sterbenz and Tataru 2010, Lemma 5.7 and its proof] in the case of the cone.

Remark 6.3. In the special case of the hyperboloid, $\Phi_j = \langle \xi \rangle_{m_j}$, or the paraboloid, $\Phi_j = |\xi|^2$, the Strichartz bound (6-1) holds in the larger region

$$\frac{1}{q} + \frac{n}{2r} \leq \frac{n}{4}.$$

This can be used to improve the range of exponents in Corollary 6.1; in particular (6-2) can be replaced with

$$\frac{1}{a} + \frac{n}{3b} < \frac{n+1}{3}.$$

However, it is important to note that, in the case of the hyperboloid, some care has to be taken as the constant will now depend on the masses m_j .

6B. Small scale bilinear restriction estimates. In the case of hyperboloids we now generalise Theorem 1.1, similarly to [Lee and Vargas 2008] in the case of the cone. Given $0 < \alpha \lesssim 1$, we define \mathcal{C}_α to be a collection of finitely overlapping caps of radius α on the sphere \mathbb{S}^{n-1} . If $\kappa \in \mathcal{C}_\alpha$, we define $\omega(\kappa)$ to be the centre of the cap κ .

We consider the case $\Phi_j(\xi) = -\pm_j \langle \xi \rangle$ and define the corresponding $V_{\pm, m}^2$ space as $V_{\pm, m}^2 = V_{\Phi_j}^2$; thus

$$\|u\|_{V_{\pm, m}^2} = \|e^{\pm it(\nabla)_m} u(t)\|_{V^2}. \tag{6-5}$$

We define the corresponding $U_{\pm, m}^2$ space similarly. Rescaling Theorem 1.1 then gives the following optimal result.

Corollary 6.4. *Let $p > (n + 3)/(n + 1)$, $0 \leq m_1, m_2 \leq 1$:*

(i) *For any $\lambda \gtrsim m_1 + m_2$, $(m_1 + m_2)/\lambda \lesssim \alpha \lesssim 1$, $\kappa, \kappa' \in \mathcal{C}_\alpha$ with $\theta(\pm_1\kappa, \pm_2\kappa') \approx \alpha$, and*

$$\text{supp } \hat{u} \subset \left\{ |\xi| \approx \lambda, \frac{\xi}{|\xi|} \in \kappa \right\}, \quad \text{supp } \hat{v} \subset \left\{ |\xi| \approx \lambda, \frac{\xi}{|\xi|} \in \kappa' \right\},$$

we have the bilinear estimate

$$\|uv\|_{L^p_{t,x}} \lesssim \alpha^{n-1-\frac{n+1}{p}} \lambda^{n-\frac{n+1}{p}} \|u\|_{V^2_{\pm_1, m_1}} \|v\|_{V^2_{\pm_2, m_2}}.$$

(ii) *For any $\lambda \gtrsim m_1 + m_2$, $0 < \alpha \ll (m_1 + m_2)/\lambda$, $\kappa, \kappa' \in \mathcal{C}_\alpha$, $c_1 \approx c_2 \approx \lambda$ with*

$$\theta(\pm_1\kappa, \pm_2\kappa') \lesssim \alpha, \quad |m_1c_1 - m_2c_2| \approx \alpha\lambda^2,$$

and

$$\text{supp } \hat{u} \subset \left\{ \left| |\xi \cdot \omega(\kappa)| - c_1 \right| \ll \alpha\lambda^2, \frac{\xi}{|\xi|} \in \kappa \right\}, \quad \text{supp } \hat{v} \subset \left\{ \left| |\xi \cdot \omega(\kappa')| - c_2 \right| \ll \alpha\lambda^2, \frac{\xi}{|\xi|} \in \kappa' \right\},$$

we have the bilinear estimate

$$\|uv\|_{L^p_{t,x}} \lesssim \alpha^{n-\frac{n+2}{p}} \lambda^{n+1-\frac{n+2}{p}} \|u\|_{V^2_{\pm_1, m_1}} \|v\|_{V^2_{\pm_2, m_2}}.$$

Proof. Fix $\pm_1 = +$ and $\pm_2 = \pm$, the remaining cases follow from a reflection. We start with (i). If $\alpha \approx 1$, then the estimate follows from rescaling in x together with an application of Theorem 1.1. Thus we may assume that $0 < \alpha \ll 1$, and after a rotation, that κ is centred at e_1 and κ' is centred at $\pm(1-\alpha^2)^{\frac{1}{2}}e_1 + \alpha e_2$. Similarly to [Lee and Vargas 2008], we define the rescaled functions

$$u_{\lambda, \alpha}(t, x) = u\left(\frac{t}{\alpha^2\lambda}, \frac{x_1}{\lambda} + \frac{t}{\alpha^2\lambda}, \frac{x'}{\alpha\lambda}\right), \quad v_{\lambda, \alpha}(t, x) = v\left(\frac{t}{\alpha^2\lambda}, \frac{x_1}{\lambda} + \frac{t}{\alpha^2\lambda}, \frac{x'}{\alpha\lambda}\right)$$

(where we write $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$) and the phases

$$\Phi_1(\xi) = \frac{-1}{\alpha^2\lambda}((m_1^2 + \lambda^2\xi_1^2 + \alpha^2\lambda^2|\xi'|^2)^{\frac{1}{2}} - \lambda\xi_1), \quad \Phi_2(\xi) = \frac{\mp 1}{\alpha^2\lambda}((m_2^2 + \lambda^2\xi_1^2 + \alpha^2\lambda^2|\xi'|^2)^{\frac{1}{2}} \mp \lambda\xi_1),$$

with associated sets

$$\Lambda_1 = \{\xi_1 \approx 1, |\xi'| \ll 1\}, \quad \Lambda_2 = \{\xi_1 \approx \pm 1, \xi_2 \approx 1, |\xi''| \ll 1\}$$

where we write $\xi = (\xi_1, \xi_2, \xi'') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$. A computation gives $\text{supp } \hat{u}_{\lambda, \alpha} \subset \Lambda_1$ and

$$[e^{-it\Phi_1(-i\nabla)}u_{\lambda, \alpha}(t)](x) = \left[e^{i\frac{t}{\alpha^2\lambda}\langle \nabla \rangle_{m_1}} u\left(\frac{t}{\alpha^2\lambda}\right) \right] \left(\frac{x_1}{\lambda}, \frac{x'}{\alpha\lambda} \right).$$

Similarly we can check that $\text{supp } \hat{v}_{\lambda, \alpha} \subset \Lambda_2$ and

$$[e^{-it\Phi_2(-i\nabla)}v_{\lambda, \alpha}(t)](x) = \left[e^{\pm i\frac{t}{\alpha^2\lambda}\langle \nabla \rangle_{m_2}} v\left(\frac{t}{\alpha^2\lambda}\right) \right] \left(\frac{x_1}{\lambda}, \frac{x'}{\alpha\lambda} \right).$$

Therefore, after rescaling together with an application of Theorem 1.1, it is enough to check that the phases Φ_j satisfy Assumption 1 on the sets Λ_j . To this end, we start by noting that we can write

$$\nabla\Phi_1(\xi) = \frac{1}{(\lambda^{-2}m_1^2 + \xi_1^2 + \alpha^2|\xi'|^2)^{\frac{1}{2}}} \left(\frac{-(m_1/(\alpha\lambda))^2 - |\xi'|^2}{(\lambda^{-2}m_1^2 + \xi_1^2 + \alpha^2|\xi'|^2)^{\frac{1}{2}} + \xi_1}, \xi' \right),$$

which shows that (ii) in Assumption 1 holds with D_2 depending only on N and n . A similar argument shows that Φ_2 satisfies (ii) in Assumption 1. On the other hand, to check condition (i) in Assumption 1, we invoke Lemma 2.1. First, we observe that for any $\xi \in \Lambda_1$, $\eta \in \Lambda_2$, we have

$$\begin{aligned} |\nabla\Phi_1(\xi) - \nabla\Phi_2(\eta)| &\geq |\partial_2\Phi_1(\xi) - \partial_2\Phi_2(\eta)| \\ &= \left| \frac{\xi_2}{(\lambda^{-2}m_1^2 + \xi_1^2 + \alpha^2|\xi'|^2)^{\frac{1}{2}}} \mp \frac{\eta_2}{(\lambda^{-2}m_2^2 + \eta_1^2 + \alpha^2|\eta'|^2)^{\frac{1}{2}}} \right| \gtrsim 1, \end{aligned}$$

and hence we can take $A_1 \approx 1$. It remains to check (2-2) in Lemma 2.1. We make use of the following elementary inequality; if $(h^*, a^*) \in \mathbb{R}^{n+1} \times \mathbb{R}^1$ and $x, y \in \{z \in \mathbb{R}^{n+1} : |z| = |z - h^*| + a^*\}$, then

$$\left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2 \geq \frac{1}{4|x||y|} \left(\frac{|x \wedge y|^2}{|x||y|} + \frac{|(x - h^*) \wedge (y - h^*)|^2}{|x - h^*||y - h^*|} \right). \tag{6-6}$$

To prove (6-6), we start by observing that since $x, y \in \{z \in \mathbb{R}^{n+1} : |z| = |z - h^*| + a^*\}$, we have

$$\begin{aligned} \left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2 &= \frac{1}{|x||y|} (|x - y|^2 - ||x| - |y||^2) \\ &= \frac{1}{|x||y|} (|(x - h^*) - (y - h^*)|^2 - ||x - h^*| - |y - h^*||^2) \\ &= \frac{|x - h^*||y - h^*|}{|x||y|} \left| \frac{x - h^*}{|x - h^*|} - \frac{y - h^*}{|y - h^*|} \right|^2. \end{aligned}$$

The inequality (6-6) now follows from the identity $|\omega - \omega^*|^2 \geq \frac{1}{2}|\omega \wedge \omega^*|^2$ for $\omega, \omega^* \in \mathbb{S}^{n+1}$. We now return to checking (2-2) in Lemma 2.1; we only check the case $j = 1$ as the remaining case is identical. Let $\xi, \eta \in \Sigma_1(a, h)$ for some $(a, h) \in \mathbb{R}^{1+n}$ such that $\xi - h, \eta - h \in \Lambda_2$. A computation gives

$$\begin{aligned} &|(\nabla\Phi_j(\xi) - \nabla\Phi_j(\eta)) \cdot (\xi - \eta)| \\ &= \alpha^{-2} \left| \left(\frac{(\xi_1, \alpha^2\xi')}{|(\lambda^{-1}m_1, \xi_1, \alpha^2\xi')|} - \frac{(\eta_1, \alpha^2\eta')}{|(\lambda^{-1}m_1, \eta_1, \alpha^2\eta')|} \right) \cdot (\xi - \eta) \right| \\ &= \alpha^{-2} \frac{|(\lambda^{-1}m_1, \xi_1, \alpha^2\xi')| + |(\lambda^{-1}m_1, \eta_1, \alpha^2\eta')|}{2} \left| \frac{(\lambda^{-1}m_1, \xi_1, \alpha^2\xi')}{|(\lambda^{-1}m_1, \xi_1, \alpha^2\xi')|} - \frac{(\lambda^{-1}m_1, \eta_1, \alpha^2\eta')}{|(\lambda^{-1}m_1, \eta_1, \alpha^2\eta')|} \right|^2 \\ &\approx \alpha^{-2} \left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2, \end{aligned} \tag{6-7}$$

where we take $x = (\lambda^{-1}m_1, \xi_1, \alpha^2\xi')$ and $y = (\lambda^{-1}m_1, \eta_1, \alpha^2\eta')$. Note that the condition $\xi \in \Sigma_1(a, h)$ becomes $|x| = |x - h^*| + a^*$ with $h^* = (\lambda^{-1}m_2 - \lambda^{-1}m_1, h_1, \alpha h')$ and $a^* = \alpha^2 a$. In particular, since

$|x| \approx |y| \approx |x - h^*| \approx |y - h^*| \approx 1$, an application of (6-6) gives

$$\left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2 \gtrsim |x \wedge y|^2 + |(x - h^*) \wedge (y - h^*)|^2. \quad (6-8)$$

The required bound (2-2) with $A_2 \approx 1$ now follows in the region $|\xi_1 - \eta_1| \lesssim |\xi' - \eta'|$ by noting that

$$|x \wedge y| \geq \alpha |\xi_1 \eta' - \eta_1 \xi'| \geq \alpha (|\xi' - \eta'| |\xi_1| - |\xi'| |\xi_1 - \eta_1|) \approx \alpha |\xi' - \eta'| \approx \alpha |\xi - \eta|$$

and applying the inequalities (6-7) and (6-8). On the other hand, if $|\xi_1 - \eta_1| \gg |\xi' - \eta'|$, then as $\xi - h, \eta - h \in \Lambda_2$, we have

$$\begin{aligned} |(x - h^*) \wedge (y - h^*)| &\geq \alpha |(\xi_1 - h_1)(\eta_2 - h_2) - (\eta_1 - h_1)(\xi_2 - h_2)| \\ &\geq \alpha (|\xi_1 - \eta_1| |\eta_2 - h_2| - |\xi_2 - \eta_2| |\eta_1 - h_1|) \approx \alpha |\xi_1 - \eta_1| \approx \alpha |\xi - \eta|, \end{aligned}$$

which again gives (2-2) with $A_2 \approx 1$. Thus the phases Φ_j satisfy Assumption 1 with $D_1 \approx D_2 \approx 1$ and therefore part (i) follows.

We now turn to the proof of part (ii). The argument is similar to (i), but we need a further rescaling to exploit the radial separation condition. As before, after rotating, we may assume that $\omega(\kappa_1) = e_1$. Define the rescaled functions

$$u_{\lambda, \alpha}^\#(t, x) = u\left(\frac{t}{\alpha^2 \lambda}, \frac{x_1}{\alpha \lambda^2} + \frac{t c_1}{\alpha^2 \lambda \langle c_1 \rangle_{m_1}}, \frac{x'}{\alpha \lambda}\right), \quad v_{\lambda, \alpha}^\#(t, x) = v\left(\frac{t}{\alpha^2 \lambda}, \frac{x_1}{\alpha \lambda^2} + \frac{t c_1}{\alpha^2 \lambda \langle c_1 \rangle_{m_1}}, \frac{x'}{\alpha \lambda}\right)$$

(where, as previously, we write $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$) and the phases

$$\begin{aligned} \Phi_1(\xi) &= \frac{-1}{\alpha^2 \lambda} \left((m_1^2 + (\alpha \lambda^2 \xi_1)^2 + \alpha^2 \lambda^2 |\xi'|^2)^{\frac{1}{2}} - \frac{\alpha \lambda^2 c_1}{\langle c_1 \rangle_{m_1}} \xi_1 \right), \\ \Phi_2(\xi) &= \frac{\mp 1}{\alpha^2 \lambda} \left((m_2^2 + (\alpha \lambda^2 \xi_1)^2 + \alpha^2 \lambda^2 |\xi'|^2)^{\frac{1}{2}} \mp \frac{\alpha \lambda^2 c_1}{\langle c_1 \rangle_{m_1}} \xi_1 \right) \end{aligned}$$

with associated sets

$$\Lambda_1 = \left\{ \left| \xi_1 - \frac{1}{\alpha \lambda^2} c_1 \right| \ll 1, |\xi'| \ll 1 \right\}, \quad \Lambda_2 = \left\{ \left| \xi_1 \mp \frac{1}{\alpha \lambda^2} c_2 \right| \ll 1, |\xi'| \lesssim 1 \right\}.$$

As previously, a computation shows that $\text{supp } \hat{u}_{\lambda, \alpha}^\# \subset \Lambda_1$, $\text{supp } \hat{v}_{\lambda, \alpha}^\# \subset \Lambda_2$ and we have the identities

$$\begin{aligned} [e^{-it\Phi_1(-i\nabla)} u_{\lambda, \alpha}^\#(t)](x) &= \left[e^{it\langle \nabla \rangle_{m_1}} u\left(\frac{t}{\alpha^2 \lambda}\right) \right] \left(\frac{x_1}{\alpha \lambda^2}, \frac{x'}{\alpha \lambda} \right), \\ [e^{-it\Phi_2(-i\nabla)} v_{\lambda, \alpha}^\#(t)](x) &= \left[e^{\pm it\langle \nabla \rangle_{m_2}} v\left(\frac{t}{\alpha^2 \lambda}\right) \right] \left(\frac{x_1}{\alpha \lambda^2}, \frac{x'}{\alpha \lambda} \right). \end{aligned}$$

Thus, as in the proof of (i), after rescaling and an application of Theorem 1.1, it is enough to check that the phases Φ_j satisfy Assumption 1 on the sets Λ_j . To this end, note that we can write

$$\partial_1 \Phi_1 = \frac{m_1^2 / (\alpha \lambda^3) ((\alpha \lambda^2 \xi_1)^2 - c_1^2) - (c_1 / \lambda)^2 \alpha \lambda |\xi'|^2}{f(\alpha \lambda \xi_1, \alpha \xi')}$$

for some smooth function f with $f \approx 1$ on Λ_1 . Since $\partial_{\xi_1}^M [(\alpha\lambda^2\xi_1)^2 - c_1^2] \lesssim \alpha\lambda^3$ for all $M \geq 0$ and $\xi_1 \in \Lambda$, we see that Φ_1 satisfies (ii) in Assumption 1 with constant depending only on n and N . A similar argument, using the fact that

$$\frac{\lambda}{\alpha} \left| \frac{c_1}{\langle c_1 \rangle_{m_1}} - \frac{c_2}{\langle c_2 \rangle_{m_2}} \right| \approx 1,$$

shows that Φ_2 also satisfies (ii) in Assumption 1. On the other hand, to check (i) in Assumption 1, we use Lemma 2.1. Concerning the transversality condition (2-1), we observe that for $\xi \in \Lambda_1$, $\eta \in \Lambda_2$, we have $|\xi_1| \approx |\eta_1| \approx 1/(\alpha\lambda)$ and

$$|\xi_1^2 m_2^2 - \eta_1^2 m_1^2| \approx \frac{m_1 + m_2}{\alpha\lambda}, \quad \alpha^2 |\xi_1^2 |\eta'|^2 - \eta_1^2 |\xi'|^2| \lesssim \lambda^{-2} \ll \lambda^{-2} \frac{m_1 + m_2}{\alpha\lambda}.$$

Therefore

$$\begin{aligned} |\nabla\Phi_1(\xi) - \nabla\Phi_2(\eta)| &= \left| \frac{(\lambda^2\xi_1, \xi')}{(\lambda^{-2}m_1^2 + \alpha^2\lambda^2\xi_1^2 + \alpha^2|\xi'|^2)^{\frac{1}{2}}} \mp \frac{(\lambda^2\eta_1, \eta')}{(\lambda^{-2}m_2^2 + \alpha^2\lambda^2\eta_1^2 + \alpha^2|\eta'|^2)^{\frac{1}{2}}} \right| \\ &\gtrsim \lambda^3 \alpha |\xi_1^2 (\lambda^{-2}m_2^2 + \alpha^2\lambda^2\eta_1^2 + \alpha^2|\eta'|^2) - \eta_1^2 (\lambda^{-2}m_1^2 + \alpha^2\lambda^2\xi_1^2 + \alpha^2|\xi'|^2)| \\ &\approx m_1 + m_2 \gtrsim 1, \end{aligned}$$

so that (2-1) holds with $A_1 \approx 1$. We now check the curvature condition (2-2) for $j = 1$. Let $\xi, \eta \in \Sigma_1(a, h)$. Repeating the computation (6-7) we deduce that

$$|(\nabla\Phi_1(\xi) - \nabla\Phi_1(\eta)) \cdot (\xi - \eta)| \approx \alpha^{-2} \left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2 \gtrsim \alpha^{-2} (|x \wedge y|^2 + |(x - h^*) \wedge (y - h^*)|^2),$$

where $x = (\lambda^{-1}m_1, \alpha\lambda\xi_1, \alpha\xi')$, $y = (\lambda^{-1}m_1, \alpha\lambda\eta_1, \alpha\eta')$, $h^* = (\lambda^{-1}m_2 - \lambda^{-1}m_1, \alpha\lambda h_1, \alpha h')$, and we used the fact that $x, y, x - h^*, y - h^*$ all have length 1. It thus remains to show that

$$|x \wedge y| + |(x - h^*) \wedge (y - h^*)| \gtrsim \alpha |\xi - \xi'|$$

since then (2-2) holds with $A_2 \approx 1$. If $|\xi_1 - \eta_1| \lesssim |\xi' - \eta'|$ we simply observe as previously that

$$|x \wedge y| \geq \alpha |\alpha\lambda\xi_1\eta' - \alpha\lambda\eta_1\xi'| \geq \alpha (|\xi' - \eta'| \alpha\lambda |\xi_1| - |\xi'| \alpha\lambda |\xi_1 - \eta_1|) \approx \alpha |\xi' - \eta'| \approx \alpha |\xi - \eta|$$

On the other hand, if $|\xi_1 - \eta_1| \gtrsim |\xi' - \eta'|$, then as $\xi - h, \eta - h \in \Lambda_2$, we have

$$|x \wedge y| + |(x - h^*) \wedge (y - h^*)| \geq \alpha m_1 |\xi_1 - \eta_1| + \alpha m_2 |(\xi_1 - h_1) - (\eta_1 - h_2)| \gtrsim \alpha |\xi - \eta|.$$

An identical argument shows that Φ_2 also satisfies the curvature condition. Thus the phases Φ_j satisfy Assumption 1 with $D_1 \approx D_2 \approx 1$ and therefore part (ii) follows. \square

The α and λ dependence in Corollary 6.4 is sharp. At least for (ii), this can be seen with the following example. Let

$$\Omega_j = \{|\xi_1 - c_j| \ll \alpha\lambda^2, |\xi'| \ll \alpha\lambda\},$$

with $|c_1 - c_2| \lesssim \alpha\lambda^2$, $c_1 \approx c_2 \approx \lambda$, and $\alpha \ll \lambda^{-1}$. Define $\hat{f}(\xi) = \mathbb{1}_{\Omega_1}(\xi)$, $\hat{g}(\xi) = \mathbb{1}_{\Omega_2}(\xi)$ and

$$u = e^{it\langle \nabla \rangle} f, \quad v = e^{it\langle \nabla \rangle} g.$$

Then

$$\|u\|_{V_{\langle \nabla \rangle}^2} = \|f\|_{L_x^2} = |\Omega_1|^{\frac{1}{2}}$$

and similarly $\|v\|_{V_{\langle \nabla \rangle}^2} = |\Omega_2|^{\frac{1}{2}}$. On the other hand we have

$$(uv)(t, x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{u}(t, \xi) \hat{v}(t, \eta) e^{ix \cdot (\xi + \eta)} d\xi d\eta = \int_{\Omega_1} \int_{\Omega_2} e^{it((\xi) + (\eta))} e^{ix \cdot (\xi + \eta)} d\xi d\eta.$$

The idea is to try and find a set $A \subset \mathbb{R}^{1+n}$ such that the phase is essentially constant for $(t, x) \in A$. We start by noting that for $\xi \in \Omega_1$ we have

$$\langle \xi \rangle - \frac{1 + c_1 \xi_1}{\langle c_1 \rangle} \approx \lambda^{-3} |(1 + |\xi|^2)(1 + c_1^2) - (1 + c_1 \xi_1)^2| = \lambda^{-3} |(\xi_1 - c_1)^2 + (1 + c_1^2)|\xi'|^2| \approx \alpha^2 \lambda,$$

and hence

$$\left| \langle \xi \rangle - \langle c_1 \rangle^{-1} - \frac{c_1}{\langle c_1 \rangle} \xi_1 \right| \lesssim \alpha^2 \lambda.$$

Similarly, since

$$\left| \frac{c_1}{\langle c_1 \rangle} - \frac{c_2}{\langle c_2 \rangle} \right| \approx \lambda^{-2} |c_1 \langle c_2 \rangle - c_2 \langle c_1 \rangle| \approx \lambda^{-3} |c_1 - c_2| \approx \frac{\alpha}{\lambda},$$

we deduce that for $\eta \in \Omega_2$

$$\left| \langle \eta \rangle - \langle c_2 \rangle^{-1} - \left(\frac{c_2}{\langle c_2 \rangle} - \frac{c_1}{\langle c_1 \rangle} \right) c_2 - \frac{c_1}{\langle c_1 \rangle} \eta_1 \right| \leq \left| \langle \eta \rangle - \langle c_2 \rangle^{-1} - \frac{c_2}{\langle c_2 \rangle} \eta_1 \right| + \left| \frac{c_1}{\langle c_1 \rangle} - \frac{c_2}{\langle c_2 \rangle} \right| |\eta_1 - c_2| \lesssim \alpha^2 \lambda.$$

In particular, for $|t| \ll (\alpha^2 \lambda)^{-1}$, $|x_1 + (c_1 / \langle c_1 \rangle)t| \ll (\alpha \lambda^2)^{-1}$, and $|x'| \ll (\alpha \lambda)^{-1}$, the phase is essentially constant and hence

$$\begin{aligned} |(uv)(t, x)| &= \left| \int_{\Omega_1} \int_{\Omega_2} e^{it((\xi) - \langle c_1 \rangle^{-1} - \frac{c_1}{\langle c_1 \rangle} \xi_1)} e^{it((\eta) - \langle c_2 \rangle^{-1} - (\frac{c_2}{\langle c_2 \rangle} - \frac{c_1}{\langle c_1 \rangle})c_2 - \frac{c_1}{\langle c_1 \rangle} \eta_1)} \right. \\ &\quad \left. \times e^{i(x_1 + t \frac{c_1}{\langle c_1 \rangle})(\xi_1 + \eta_1 - c_1 - c_2) + x' \cdot (\xi' + \eta')} d\xi d\eta \right| \\ &\gtrsim |\Omega_1| |\Omega_2|, \end{aligned}$$

which then implies

$$\|uv\|_{L_{t,x}^p} \gtrsim (\alpha^{n+2} \lambda^{n+2})^{-\frac{1}{p}} \times |\Omega_1| |\Omega_2|.$$

Therefore, if the estimate

$$\|uv\|_{L_{t,x}^p} \leq C(\alpha, \lambda) \|u\|_{V_{\langle \nabla \rangle}^2} \|v\|_{V_{\langle \nabla \rangle}^2}$$

holds, then we must have

$$(\alpha \lambda)^{-\frac{n+2}{p}} |\Omega_1| |\Omega_2| \lesssim C |\Omega_1|^{\frac{1}{2}} |\Omega_2|^{\frac{1}{2}}.$$

Since $|\Omega_1| \approx |\Omega_2| \approx \alpha^n \lambda^{n+1}$, after rearranging, this becomes $C \gtrsim \alpha^{n - \frac{n+2}{p}} \lambda^{n+1 - \frac{n+2}{p}}$, which matches the bound obtained in Corollary 6.4.

7. The Dirac–Klein–Gordon system

We now set up notation and reduce the DKG system to the first-order system (7-3). We then give the proof of Theorem 1.2, up to the crucial nonlinear estimates, which are postponed to Section 8. In the remainder of this article, as we now only consider the DKG system, the dimension is fixed to $n = 3$.

7A. Notation and setup. Fix a smooth function $\rho \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \rho \subset \{\frac{1}{2} < t < 2\}$ and

$$\sum_{\lambda \in 2^{\mathbb{Z}}} \rho\left(\frac{t}{\lambda}\right) = 1,$$

and let $\rho_1 = \sum_{\lambda \leq 1} \rho(t/\lambda)$ with $\rho_1(0) = 1$. Similarly, we let Q_μ be a finitely overlapping collection of cubes of diameter $\mu/1000$ covering \mathbb{R}^3 , and fix $(\rho_q)_{q \in Q_\mu}$ to be a corresponding subordinate partition of unity. We now define the standard dyadic Fourier cutoffs, for $\lambda \in 2^{\mathbb{N}}$, $\lambda > 1$, $q \in Q$, $d \in 2^{\mathbb{Z}}$

$$P_\lambda = \rho\left(\frac{|\cdot - i\nabla|}{\lambda}\right), \quad P_1 = \rho_1(|\cdot - i\nabla|), \quad P_q = \rho_q(|\cdot - i\nabla|), \quad C_d^{\pm, m} = \rho\left(\frac{-i\partial_t \pm \langle -i\nabla \rangle_m}{d}\right).$$

We also let $C_{\leq d}^{\pm, m} = \sum_{d' \leq d} C_{d'}^{\pm, m}$, and any related multipliers such as $C_{\geq d}^{\pm, m}$ are defined analogously. To simplify notation somewhat, we make the convention that

$$C_d = C_d^{+, 1}, \quad C_d^\pm = \Pi_\pm C_d^{\pm, M},$$

where M will denote the mass of the spinor in (1-3) and Π_\pm is as defined below. Given $\alpha \leq 1$, we let $(\rho_\kappa)_{\kappa \in C_\alpha}$ be a smooth partition of unity subordinate to the conic sectors $\{\xi \neq 0, \xi/|\xi| \in \kappa\}$, and define the angular Fourier localisation multipliers as

$$R_\kappa = \rho_\kappa(-i\nabla).$$

We use the well-known fact that for any $1 \leq p, q \leq \infty$ the modulation cutoff multipliers are uniformly disposable in $L_t^q L_x^r$ for certain scales; namely we have the bounds

$$\|C_d^{\pm, m} P_\lambda R_\kappa u\|_{L_t^q L_x^r} + \|C_{\leq d}^{\pm, m} P_\lambda R_\kappa u\|_{L_t^q L_x^r} \lesssim \|P_\lambda R_\kappa u\|_{L_t^q L_x^r}, \tag{7-1}$$

provided that $\kappa \in C_\alpha$ and $d \gtrsim \alpha^2 \lambda$ and $\alpha \gtrsim \lambda^{-1}$; see, e.g., [Bejenaru and Herr 2015, Lemma 4.1]. Similarly, by writing

$$C_d^{\pm, m} = e^{\mp i t \langle \nabla \rangle_m} \rho\left(\frac{-i\partial_t}{d}\right) e^{\pm i t \langle \nabla \rangle_m},$$

and using the fact that convolution with $L_t^1(\mathbb{R})$ functions is bounded on V^2 , we deduce that for every $d \in 2^{\mathbb{Z}}$

$$\|C_{\leq d}^{\pm, m} u\|_{V_{\pm, m}^2} \lesssim \|u\|_{V_{\pm, m}^2}. \tag{7-2}$$

To deal with solutions to the Dirac equation, we follow the, by now, standard approach used in [D’Ancona et al. 2007; Bejenaru and Herr 2017] and define the projections

$$\Pi_\pm(\xi) = \frac{1}{2} \left(I \pm \frac{1}{\langle \xi \rangle_M} (\xi_j \gamma^0 \gamma^j + M \gamma^0) \right)$$

and the associated Fourier multiplier $\widehat{(\Pi_{\pm} f)}(\xi) = \Pi_{\pm}(\xi) \widehat{f}(\xi)$. A computation shows that $\Pi_+ \Pi_- = \Pi_- \Pi_+ = 0$ and $\Pi_{\pm}^2 = \Pi_{\pm}$. Moreover, given any spinor ψ we have

$$\psi = \Pi_+ \psi + \Pi_- \psi, \quad (-i \gamma^\mu \partial_\mu + M) \Pi_{\pm} \psi = \gamma^0 (-i \partial_t \pm \langle -i \nabla \rangle_M) \psi.$$

As in [Bejenaru and Herr 2017], we can now reduce the original system (1-3) to a first-order system as follows. Suppose we have a solution (ψ_{\pm}, ϕ_+) to

$$\begin{aligned} (-i \partial_t \pm \langle \nabla \rangle_M) \psi_{\pm} &= \Pi_{\pm} (\mathfrak{R}(\phi_+) \gamma^0 \psi) \\ (-i \partial_t + \langle \nabla \rangle_m) \phi_+ &= \langle \nabla \rangle_m^{-1} (\psi^\dagger \gamma^0 \psi) \\ \psi_{\pm}(0) &= f_{\pm}, \\ \phi_+(0) &= g_+, \end{aligned} \tag{7-3}$$

where $\psi = \Pi_+ \psi_+ + \Pi_- \psi_-$ and the data (f_{\pm}, g_+) satisfies $\Pi_{\pm} f_{\pm} = f_{\pm}$. If we let $\phi = \mathfrak{R}(\phi_+)$, then since $\psi^\dagger \gamma^0 \psi$ is real-valued, we deduce that

$$\begin{aligned} 2(\phi + i \langle \nabla \rangle_m^{-1} \partial_t \phi) &= \phi_+ + i \langle \nabla \rangle_m^{-1} \partial_t \phi_+ + \overline{(\phi_+ - i \langle \nabla \rangle_m^{-1} \partial_t \phi_+)} \\ &= 2\phi_+ - \langle \nabla \rangle_m^{-2} (\psi^\dagger \gamma^0 \psi) + \langle \nabla \rangle_m^{-2} \overline{(\psi^\dagger \gamma^0 \psi)} = 2\phi_+. \end{aligned}$$

Consequently, if we take $g_+ = \phi(0) + i \langle \nabla \rangle_m^{-1} \partial_t \phi(0)$, a simple computation shows that (ψ, ϕ) is a solution to the original DKG system (1-3). Note that, after rescaling, it suffices to consider the case $m = 1$. Therefore, to prove Theorem 1.2, it is enough to construct global solutions to the reduced system (7-3) with $m = 1$.

7B. Analysis on the sphere. We require some basic facts on analysis on the sphere \mathbb{S}^2 , which can be found in, for instance, [Stein and Weiss 1971; Strichartz 1972; Sterbenz 2005]. Let Y_ℓ denote the set of homogeneous harmonic polynomials of degree ℓ , and let $y_{\ell,n}$, $n = 0, \dots, 2\ell$, be an orthonormal basis for Y_ℓ with respect to the inner product

$$\langle y_{\ell,n}, y_{\ell',n'} \rangle_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} [y_{\ell,n}(\omega)]^\dagger y_{\ell',n'}(\omega) d\mathbb{S}(\omega).$$

Given $f \in L^2(\mathbb{R}^3)$, we have the orthogonal (in $L^2(\mathbb{R}^3)$) decomposition

$$f(x) = \sum_{\ell} \sum_{n=0}^{2\ell} \langle f(|x|\omega), y_{\ell,n}(\omega) \rangle_{L^2_{\omega}(\mathbb{S}^2)} y_{\ell,n} \left(\frac{x}{|x|} \right).$$

For $N > 1$, we define the spherical Littlewood–Paley projections

$$\begin{aligned} (H_N f)(x) &= \sum_{\ell \in \mathbb{N}} \sum_{n=0}^{2\ell} \rho \left(\frac{\ell}{N} \right) \langle f(|x|\cdot), y_{\ell,n} \rangle_{L^2(\mathbb{S}^2)} y_{\ell,n} \left(\frac{x}{|x|} \right), \\ (H_1 f)(x) &= \sum_{\ell \in \mathbb{N}} \sum_{n=0}^{2\ell} \rho_{\leq 1}(\ell) \langle f(|x|\cdot), y_{\ell,n} \rangle_{L^2(\mathbb{S}^2)} y_{\ell,n} \left(\frac{x}{|x|} \right). \end{aligned}$$

Fractional powers of the angular derivatives $\langle \Omega \rangle$ are then defined as

$$\langle \Omega \rangle^\sigma f = \sum_{N \in 2^{\mathbb{N}}} N^\sigma H_N f. \tag{7-4}$$

If we let $\Omega_{ij} = x_i \partial_j - x_j \partial_i$ denote the standard infinitesimal generators of the rotations on \mathbb{R}^3 , then a computation gives

$$\|\Omega_{ij} H_N f\|_{L_x^2(\mathbb{R}^3)} \approx N \|H_N f\|_{L_x^2(\mathbb{R}^3)}.$$

In addition, if $\Delta_{\mathbb{S}^2}$ denotes the Laplacian on the sphere of radius $|x|$, then $\Delta_{\mathbb{S}^2} = \sum_{j < k} \Omega_{ij}^2$. These facts are not explicitly required in the following, and we shall only make use of the spectral definition (7-4). More important for our purposes, are the basic properties of the multipliers H_N .

Lemma 7.1. *Let $N \geq 1$. Then H_N is uniformly (in N) bounded on $L^p(\mathbb{R}^3)$, and H_N commutes with all radial Fourier multipliers. Moreover, if $N' \geq 1$, then either $N \sim N'$ or*

$$H_N \Pi_{\pm} H_{N'} = 0.$$

Proof. The first claim follows from [Strichartz 1972]. To prove the second claim, let T be a radial Fourier multiplier with $\widehat{Tf}(\xi) = \sigma(|\xi|) \hat{f}(\xi)$. It is enough to show that, if $f(x) = a(|x|) y_\ell(x/|x|)$ for some $y_\ell \in Y_\ell$, then $Tf = b(|x|) y_\ell(x/|x|)$ for some $b(|x|)$ depending on a and σ . But this follows directly from [Stein and Weiss 1971, page 158]. To prove the final claim, suppose that $N \gg N'$ or $N \ll N'$. Our goal is to show that $H_N \Pi_{\pm} H_{N'} = 0$. Since H_N commutes with radial Fourier multipliers, it is enough to show that $H_N(\partial_j f) = 0$ in the case $f(x) = a(|x|) y_{\ell'}(x/|x|)$ with $y_{\ell'} \in Y_{\ell'}$ and $\frac{1}{2}N' \leq \ell' \leq 2N'$. Since $\partial_j = (x_j/|x|)\partial_r + \sum_k (x_k/|x|^2)\Omega_{jk}$, where $\partial_r = (x/|x|) \cdot \nabla$, and $\partial_r(y_{\ell'}(x/|x|)) = 0$, we can reduce further to just showing that $H_N(x_k \Omega_{jk} y_{\ell'}) = 0$, which corresponds to checking that

$$\langle y_\ell, x_k \Omega_{kj} y_{\ell'} \rangle_{L^2(\mathbb{S}^2)} = 0 \tag{7-5}$$

for every $\frac{1}{2}N \leq \ell \leq 2N$. Since $x_k \Omega_{kj} y_{\ell'}$ is a polynomial of order $\ell' + 1$, by the orthogonality of the polynomials y_ℓ , (7-5) clearly holds if $\ell > \ell' + 1$. On the other hand, after an application of integration by parts, we obtain

$$\langle y_\ell, x_k \Omega_{kj} y_{\ell'} \rangle_{L^2(\mathbb{S}^2)} = \langle \Omega_{kj}(x_k y_\ell), y_{\ell'} \rangle_{L^2(\mathbb{S}^2)}$$

since $\Omega_{kj}(x_k y_\ell)$ is a polynomial of order $\ell + 1$; we see that again (7-5) holds if $\ell' > \ell + 1$. □

An application of Lemma 7.1 shows that H_N commutes with the P_λ and C_d multipliers since we may write $C_d^{\pm, m} = e^{\mp i t \langle \nabla \rangle m} \rho(-i \partial_t / d) e^{\pm i t \langle \nabla \rangle m}$. On the other hand, it is important to note that H_N does not commute with the cube and cap localisation operators R_k and P_q .

7C. Norms and the energy inequality. Fix $0 < \sigma \ll 1$,

$$\frac{1}{2} < \frac{1}{a} < \frac{1}{2} + \frac{\sigma}{1000},$$

and $b = 3/a - 1$, and define

$$\|u\|_{Y_{\lambda, N}^{\pm, m}} = \lambda^{\frac{1}{a} - b} \sup_{d \in 2^{\mathbb{Z}}} d^b \|C_d^{\pm, m} P_\lambda H_N u\|_{L_t^a L_x^2}$$

and

$$\|u\|_{F_{\lambda,N}^{\pm,m}} = \|P_\lambda H_N u\|_{V_{\pm,m}^2} + \|u\|_{Y_{\lambda,N}^{\pm,m}}.$$

We also let

$$\|u\|_{F_{\pm,m}^{s,\sigma}} = \left(\sum_{\lambda \geq 1} \sum_{N \geq 1} \lambda^{2s} N^{2\sigma} \|u\|_{F_{\lambda,N}^{\pm,m}}^2 \right)^{\frac{1}{2}}$$

and define the Banach space

$$F_{\pm,m}^{s,\sigma} = \{u \in C(\mathbb{R}, \langle \Omega \rangle^{-\sigma} H^s) : \|u\|_{F_{\pm,m}^{s,\sigma}} < \infty\}.$$

For the remainder of this section, let $\sigma_M = \sigma$ if $M \geq \frac{1}{2}$ and $\sigma_M = \frac{7}{30} + \sigma$ if $0 < M < \frac{1}{2}$. Thus σ_M corresponds to amount of angular regularity in the statement of Theorem 1.2. We will construct a solution $(\psi_\pm, \phi_\pm) \in F_{\pm,M}^{0,\sigma_M} \times F_{\pm,1}^{\frac{1}{2},\sigma_M}$ to the reduced system (7-3). Thus we work in a frequency-localised V^2 space, with the additional component $Y_{\lambda,N}^{\pm,m}$ needed to control the solution in the high modulation region; for the latter see [Bejenaru and Herr 2015, Section 4].

There are three basic properties of $V_{\pm,m}^2$ which we exploit in the following. The first is a simple bound in the high-modulation region; see [Hadac et al. 2009, Corollary 2.18] for a proof.

Lemma 7.2. *Let $m \geq 0$ and $2 \leq q \leq \infty$. For any $d \in 2^{\mathbb{Z}}$ we have*

$$\|C_d^{\pm,m} u\|_{L_t^q L_x^2} \lesssim d^{-\frac{1}{q}} \|u\|_{V_{\pm,m}^2}.$$

The second key property is a standard energy inequality, which reduces the problem of estimating a Duhamel integral in $F_{\lambda,N}^{\pm,M}$ to controlling a trilinear integral.

Lemma 7.3. *Let $F \in L_t^\infty L_x^2$, and suppose that*

$$\sup_{\|P_\lambda H_N v\|_{V_{\pm,m}^2} \lesssim 1} \left| \int_{\mathbb{R}} \langle P_\lambda H_N v(t), F(t) \rangle_{L_x^2} dt \right| < \infty.$$

If $u \in C(\mathbb{R}, L_x^2)$ satisfies $-i \partial_t u \pm \langle \nabla \rangle_m u = F$, then $P_\lambda H_N u \in V_{\pm,m}^2$ and we have the bound

$$\|P_\lambda H_N u\|_{V_{\pm,m}^2} \lesssim \|P_\lambda H_N u(0)\|_{L^2} + \sup_{\|P_\lambda H_N v\|_{V_{\pm,m}^2} \lesssim 1} \int_{\mathbb{R}} \langle P_\lambda H_N v(t), F(t) \rangle_{L_x^2} dt. \tag{7-6}$$

Proof. See [Koch and Steinerberger 2015] or [Hadac et al. 2009, Proposition 2.10] for details on the duality. It is also possible to prove this directly as follows. Clearly it is enough to consider the case $u(0) = 0$, thus $u(t) = \int_0^t e^{\mp i(t-s)\langle \nabla \rangle_m} F(s) ds$. Let $K > 0$ and $(t_k) \in \mathcal{Z}$. A computation gives the identity

$$\left(\sum_{|k| < K} \|e^{\pm i t_k \langle \nabla \rangle_m} P_\lambda H_N u(t_k) - e^{\pm i t_{k-1} \langle \nabla \rangle_m} P_\lambda H_N u(t_{k-1})\|_{L_x^2}^2 \right)^{\frac{1}{2}} = \int_{\mathbb{R}} \langle P_\lambda H_N v(s), F(s) \rangle_{L_x^2} ds$$

with

$$v(s) = A^{-1} \sum_{|k| < K} \mathbb{1}_{[t_{k-1}, t_k)}(s) (e^{\mp i(s-t_k)\langle \nabla \rangle_m} u(t_k) - e^{\mp i(s-t_{k-1})\langle \nabla \rangle_m} u(t_{k-1}))$$

and

$$A = \left(\sum_{|k| < K} \|e^{\pm i t_k \langle \nabla \rangle_m} P_\lambda H_N u(t_k) - e^{\pm i t_{k-1} \langle \nabla \rangle_m} P_\lambda H_N u(t_{k-1})\|_{L_x^2}^2 \right)^{\frac{1}{2}}.$$

It is easy to check that

$$\|P_\lambda H_N v\|_{V_{\pm, m}^2} \lesssim 1.$$

Thus, by taking the sup over the above inequality, and then letting $K \rightarrow \infty$ we deduce the bound (7-6). Since u is also continuous, we obtain $u \in V_{\pm, m}^2$ as required. \square

Note that the norm on v can in fact be taken to be the stronger $U_{\pm, m}^2$ norm, but we do not require this improvement here.

The final result we require on the $V_{\pm, m}^2$ spaces, concerns the question of scattering.

Lemma 7.4. *Let $u \in V_{\pm, m}^2$. Then there exists $f \in L_x^2$ such that $\|u(t) - e^{\mp i t \langle \nabla \rangle} f\|_{L_x^2} \rightarrow 0$ as $t \rightarrow \infty$.*

Clearly, this result can be extended to elements of the space $F_{\pm, m}^{s, \sigma_M}$. In other words, if we construct a solution in $F_{\pm, m}^{s, \sigma_M}$, then we immediately deduce the solution must scatter to a linear solution as $t \rightarrow \pm\infty$.

7D. Proof of Theorem 1.2. We now come to the proof of Theorem 1.2. In light of Lemma 7.4, it is enough to construct a solution $(\psi_\pm, \phi_\pm) \in F_{\pm, M}^{0, \sigma_M} \times F_{\pm, 1}^{\frac{1}{2}, \sigma_M}$ to the reduced system (7-3). Note that we may always assume that $\psi_\pm = \Pi_\pm \psi_\pm$, provided that this is satisfied at $t = 0$. Define the Duhamel integral

$$\mathcal{I}_m^\pm[F] = \int_0^t e^{\mp i(t-s)\langle \nabla \rangle_m} F(s) ds.$$

Note that $\mathcal{I}_m^\pm[F]$ solves the equation

$$(-i \partial_t \pm \langle \nabla \rangle_m) \mathcal{I}_m^\pm[F] = F$$

with vanishing data at $t = 0$. Moreover, we can check that for every $1 < p < \infty$ we have

$$\|C_d^{\pm, m} \mathcal{I}_m^\pm[F]\|_{L_t^p L_x^2} \lesssim d^{-1} \|C_d^{\pm, m} F\|_{L_t^p L_x^2}. \tag{7-7}$$

If we had the bounds

$$\begin{aligned} \|\Pi_{\pm 1} \mathcal{I}_M^{\pm 1}[\phi \gamma^0 \Pi_{\pm 2} \varphi]\|_{F_{\pm 1, M}^{0, \sigma_M}} &\lesssim \|\phi\|_{F_{+1}^{1/2, \sigma_M}} \|\varphi\|_{F_{M, \pm 2}^{0, \sigma_M}}, \\ \|\langle \nabla \rangle^{-1} \mathcal{I}_1^+[(\Pi_{\pm 1} \psi)^\dagger \gamma^0 \Pi_{\pm 2} \varphi]\|_{F_{+1}^{1/2, \sigma_M}} &\lesssim \|\psi\|_{F_{M, \pm 1}^{0, \sigma_M}} \|\varphi\|_{F_{M, \pm 2}^{0, \sigma_M}}, \end{aligned} \tag{7-8}$$

then a standard fixed-point argument in $F_{\pm, M}^{0, \sigma_M} \times F_{+1}^{\frac{1}{2}, \sigma_M}$ would give the required solution to (7-3), provided of course that the data (f_\pm, g_+) satisfied

$$\|\langle \Omega \rangle^{\sigma_M} f_\pm\|_{L^2} + \|\langle \Omega \rangle^{\sigma_M} g_+\|_{H^{1/2}} \ll 1.$$

Let

$$\phi_{\mu, N} = P_\mu H_N \phi, \quad \psi_{\lambda_1, N_1} = P_{\lambda_1} H_{N_1}, \quad \varphi_{\lambda_2, N_2} = P_{\lambda_2} H_{N_2} \varphi.$$

We have the following frequency-localised estimates.

Theorem 7.5. Fix $M > 0$. Then there exists $\epsilon > 0$ such that

$$\begin{aligned} & \left\| \Pi_{\pm_1} \mathcal{I}_M^{\pm_1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm_2} \varphi_{\lambda_2, N_2}] \right\|_{F_{\lambda_1, N_1}^{\pm_1, M}} \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^{\sigma_M} \left(\frac{\min\{\mu, \lambda_1, \lambda_2\}}{\max\{\mu, \lambda_1, \lambda_2\}} \right)^\epsilon \|\phi\|_{F_{\mu, N}^{\pm_1, 1}} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm_2, M}} \end{aligned} \quad (7-9)$$

and

$$\begin{aligned} & \left\| \mathcal{I}_1^+ [(\Pi_{\pm_1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm_2} \varphi_{\lambda_2, N_2}] \right\|_{F_{\mu, N}^{\pm_1, 1}} \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N_1, N_2\})^{\sigma_M} \left(\frac{\min\{\mu, \lambda_1, \lambda_2\}}{\max\{\mu, \lambda_1, \lambda_2\}} \right)^\epsilon \|\psi\|_{F_{\lambda_1, N_1}^{\pm_1, M}} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm_2, M}}. \end{aligned} \quad (7-10)$$

Remark 7.6. The proof of Theorem 7.5 in the resonant regime $0 < M < \frac{1}{2}$ relies on the small-scale V^2 estimates in Corollary 6.4. However, it is possible to prove a weaker version of Theorem 7.5, with σ_M replaced with some larger σ , provided only that a *robust* version of the *homogeneous* bilinear restriction estimate (6-3) holds. More precisely, by following the proof of Corollary 6.4, and then interpolating with the Klein–Gordon Strichartz estimates as in Remarks 6.2 and 6.3, it is possible to show that (6-3) implies the V^2 bound

$$\|uv\|_{L_t^a L_x^b(\mathbb{R}^{1+3})} \lesssim \lambda^{1+\frac{1}{a}-\frac{1}{b}} \|u\|_{V_{\pm_1, m_1}^2} \|v\|_{V_{\pm_2, m_2}^2}$$

in the range

$$\frac{1}{a} + \frac{2}{b} < 2, \quad \frac{1}{a} + \frac{6}{5b} < \frac{7}{5},$$

where u and v have Fourier support in 1-separated angular wedges of size $1 \times 1 \times \lambda$ at distance λ from the origin. The case $a = 2-$ and $b = \frac{4}{3}+$ can be used together with the $L_t^{2+} L_x^{4-}$ angular Strichartz bound from [Cho and Lee 2013, Theorem 1.1] instead of the argument used in the high-high case in the proof of Theorem 8.8 below. However, the estimate obtained is weaker than the one in Theorem 7.5. Moreover, it still requires a robust version of the homogeneous bilinear estimate (6-3) for which we can track the dependence of the constant on the phases Φ_j due to the lack of homogeneity of the Klein–Gordon phase. Irrespective of fact the Theorem 1.1 applies to V^2 -functions, a key advantage of our formulation of Theorem 1.1, in comparison to [Bejenaru 2017; Lee and Vargas 2010], is that it allows us to read off the above-mentioned dependence.

The standard Littlewood–Paley trichotomy implies that the left-hand sides of (7-9) and (7-10) are zero unless

$$\max\{\mu, \lambda_1, \lambda_2\} \approx \text{med}\{\mu, \lambda_1, \lambda_2\} \gtrsim \min\{\mu, \lambda_1, \lambda_2\} \quad (7-11)$$

and

$$\max\{N, N_1, N_2\} \approx \text{med}\{N, N_1, N_2\} \gtrsim \min\{N, N_1, N_2\}$$

It is now easy to check that the bilinear estimates (7-8), follow from Theorem 7.5. Consequently, we have reduced the proof of Theorem 1.2 to proving the frequency-localised bilinear estimates in Theorem 7.5. As the proof of Theorem 7.5 requires a number of preliminary results, we postpone the proof until Section 8D.

8. Linear and multilinear estimates

In this section our goal is give the proof of Theorem 7.5. To this end, we first provide some linear estimates and adapt them to our functional setup, prove an auxiliary trilinear estimate in V^2 , and eventually give the proof of the crucial Theorem 7.5 in Section 8D.

8A. Auxiliary estimates. As is well known, see for example [D’Ancona et al. 2007], the system (7-3) exhibits null structure. To exploit the null structure of the product $\psi^\dagger \gamma^0 \psi$, we start by noting that for any $x, y \in \mathbb{R}^3$, we have the identity

$$\begin{aligned} & [\Pi_{\pm_1} f]^\dagger \gamma^0 \Pi_{\pm_2} g \\ &= [(\Pi_{\pm_1} - \Pi_{\pm_1}(x))f]^\dagger \gamma^0 \Pi_{\pm_2} g + [\Pi_{\pm_1}(x)f]^\dagger \gamma^0 (\Pi_{\pm_2} - \Pi_{\pm_2}(y))g + f^\dagger \Pi_{\pm_1}(x) \gamma^0 \Pi_{\pm_2}(y)g. \end{aligned}$$

This is then exploited by using the null-form-type bound

$$|\Pi_{\pm_1}(x) \gamma^0 \Pi_{\pm_2}(y)| \lesssim \theta(\pm_1 x, \pm_2 y) + \frac{|\pm_1 |x| \pm_2 |y||}{\langle x \rangle \langle y \rangle}, \tag{8-1}$$

which follows from (2-6) by observing that

$$\begin{aligned} \Pi_{\pm_1}(x) \gamma^0 \Pi_{\pm_2}(y) &= \Pi_{\pm_1}(x) (\Pi_{\pm_1}(x) \gamma^0 - \gamma^0 \Pi_{\mp_2}(y)) \Pi_{\pm_2}(y) \\ &= \Pi_{\pm_1}(x) \left(\left(\frac{\pm_2 \eta_j}{\langle \eta \rangle_M} - \frac{\pm_1 \xi_j}{\langle \xi \rangle_M} \right) \gamma^j + \left(\frac{\pm_1 M}{\langle \xi \rangle_M} + \frac{\pm_2 M}{\langle \eta \rangle_M} \right) I \right) \Pi_{\pm_2}(y), \end{aligned}$$

together with the following lemma; see [Bejenaru 2017, Lemma 3.3] for a similar statement to part (i).

Lemma 8.1. *Let $1 < r < \infty$:*

(i) *If $\lambda \geq 1$, $\alpha \gtrsim \lambda^{-1}$, $\kappa \in C_\alpha$, then*

$$\|(\Pi_{\pm_1} - \Pi_{\pm_1}(\lambda \omega(\kappa))) R_\kappa P_\lambda f\|_{L_x^r} \lesssim \alpha \|R_\kappa P_\lambda u\|_{L_x^r}.$$

(ii) *If $\lambda \geq 1$, $0 < \alpha \lesssim \lambda^{-1}$, $\kappa \in C_\alpha$, $q \in Q_{\lambda^2 \alpha}$ with centre ξ_0 , then*

$$\|(\Pi_{\pm_1} - \Pi_{\pm_1}(\xi_0)) R_\kappa P_q P_\lambda f\|_{L_x^r} \lesssim \alpha \|R_\kappa P_q P_\lambda u\|_{L_x^r}.$$

Proof. Concerning part (i), see [Bejenaru 2017, Proof of Lemma 3.3]. Concerning part (ii), we may assume $|\xi_0| \approx \lambda$ and, due to boundedness, we may replace the symbol of $R_\kappa P_q P_\lambda$ by a smooth cutoff χ_E to the parallelepiped E with centre ξ_0 of side lengths $\alpha \mu^2 \times \alpha \mu \times \alpha \mu$ with long side pointing in the direction ξ_0 . After rotating ξ_0 to $\xi_0 = |\xi_0|(1, 0, 0)$, the operator has the symbol

$$m(\xi) = \left(\pm B^j \left(\frac{\xi_j}{\langle \xi \rangle_M} - \frac{\xi_{0,j}}{\langle \xi_0 \rangle_M} \right) \pm \frac{1}{2} \gamma^0 \left(\frac{1}{\langle \xi \rangle_M} - \frac{1}{\langle \xi_0 \rangle_M} \right) \right) \chi_E(\xi)$$

for certain $B^1, B^2, B^3 \in \mathbb{C}^{4 \times 4}$. It suffices to prove the kernel bound

$$|(\mathcal{F}_x^{-1} m)(x)| \lesssim \alpha^4 \lambda^4 (1 + \alpha \lambda^2 |x_1| + \alpha \lambda |x'|)^{-4}, \quad x = (x_1, x'), \tag{8-2}$$

as it implies $\|\mathcal{F}_x^{-1}m\|_{L^1(\mathbb{R}^3)} \lesssim \alpha$. In the support of χ_E we obtain, from (2-6) and a simple computation,

$$|m(\xi)| \lesssim \lambda^{-3}||\xi| - |\xi_0|| + \theta(\xi, \xi_0) + \lambda^{-2}||\xi| - |\xi_0|| \lesssim \alpha.$$

From the localisation of χ_E , where $|\partial_{\xi_1}^\ell (\xi_j / \langle \xi \rangle_M)| \lesssim \lambda^{-\ell-1}$, and the Leibniz rule, we conclude for $\ell > 0$

$$|\partial_{\xi_1}^\ell m(\xi)| \lesssim \alpha(\alpha\lambda^2)^{-\ell} + \sum_{0 < \ell_1 \leq \ell} \lambda^{-\ell_1-1}(\alpha\lambda^2)^{\ell_1-\ell} \lesssim \alpha(\alpha\lambda^2)^{-\ell}.$$

Integration by parts now implies (8-2) if $\alpha\lambda^2|x_1| \geq \alpha\lambda|x'|$. For $k = 2, 3$, we have $|\partial_{\xi_k}^\ell (\xi_j / \langle \xi \rangle_M)| \lesssim \lambda^{-\ell}$ within the support of χ_E ; hence we conclude for $\ell > 0$

$$|\partial_{\xi_k}^\ell m(\xi)| \lesssim \alpha(\alpha\lambda)^{-\ell} + \sum_{0 < \ell_1 \leq \ell} \lambda^{-\ell_1}(\alpha\lambda)^{\ell_1-\ell} \lesssim \alpha(\alpha\lambda)^{-\ell}.$$

Integration by parts now implies (8-2) in the region where $\alpha\lambda^2|x_1| \leq \alpha\lambda|x_k|$. □

The proof of Theorem 7.5 requires a number of standard linear estimates for homogeneous solutions to the Klein–Gordon equation. We start by recalling the Strichartz estimates for the wave and Klein–Gordon equations.

Lemma 8.2 (wave Strichartz). *Let $m \geq 0$ and $2 < q \leq \infty$. If $0 < \mu \leq \lambda$, $N \geq 1$, and $1/r = 1/2 - 1/q$ then for every $q \in Q_\mu$ we have*

$$\|e^{\mp it\langle \nabla \rangle^m} P_q P_\lambda f\|_{L_t^q L_x^r} \lesssim \mu^{\frac{1}{2}-\frac{1}{r}} \lambda^{\frac{1}{2}-\frac{1}{r}} \|P_q P_\lambda f\|_{L_x^2}.$$

Moreover, by spending additional angular regularity we have

$$\|e^{\mp it\langle \nabla \rangle^m} P_\lambda H_N f\|_{L_t^q L_x^4} \lesssim \lambda^{\frac{3}{4}-\frac{1}{q}} N \|P_\lambda H_N f\|_{L_x^2}.$$

Proof. The proof of the first estimate can be found in [Bejenaru and Herr 2017, Lemma 3.1]. The second follows by simple modification of the argument in the appendix to [Sterbenz 2005]. More precisely, after interpolating with the $L_t^\infty L_x^2$ estimate, we need to show that

$$\|e^{\mp it\langle \nabla \rangle^m} H_N P_\lambda f\|_{L_t^2 L_x^r} \lesssim N \lambda^{3(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}} \|H_N \lambda f\|_{L_x^2}.$$

After rescaling, and following the argument on [Sterbenz 2005, pp. 226–227], it is enough to prove that for every $\epsilon > 0$ we have the space-time Morawetz-type bound

$$\|(1 + |x|)^{-\frac{1}{2}-\epsilon} \nabla u\|_{L_{t,x}^2} \lesssim \|(\partial_t u(0), \nabla u(0))\|_{L_x^2} \tag{8-3}$$

for functions u with $\square u + mu = 0$, and the constant in (8-3) is independent of m . However the proof of (8-3) follows the same argument as the wave case in [Sterbenz 2005]; the only change is to replace the wave-energy-momentum tensor with the Klein–Gordon version

$$Q_{\alpha\beta} = \frac{1}{2}(\partial_\alpha \phi \overline{\partial_\beta \phi} + \partial_\beta \phi \overline{\partial_\alpha \phi} - g_{\alpha\beta}(\partial^\gamma \phi \overline{\partial_\gamma \phi} + m^2|\phi|^2)).$$

We omit the details. □

The amount of angular regularity required for the $L_t^{2+} L_x^4$ Strichartz estimate to hold, is much less than that stated in Lemma 8.2. In fact, in [Sterbenz 2005], it is shown that the same estimate holds with $N^{\frac{1}{2}+}$. However, as the sharp number of angular derivatives is not required in the arguments we use in the present paper, we have elected to simply state the result with a whole angular derivative. On the other hand, the number of angular derivatives required in the following Klein–Gordon regime, plays a crucial role.

Lemma 8.3 (Klein–Gordon Strichartz). *Let $m > 0$ and $\frac{3}{10} < 1/r < \frac{5}{14}$. Then for every $\epsilon > 0$ we have*

$$\|e^{\mp it \langle \nabla \rangle^m} P_\lambda H_N f\|_{L_{t,x}^r} \lesssim \lambda^{2-\frac{5}{r}} N^{7(\frac{1}{r}-\frac{3}{10})+\epsilon} \|P_\lambda H_N f\|_{L_x^2}.$$

Proof. This is a special case of [Cho and Lee 2013, Theorem 1.1]. □

Remark 8.4. Without angular regularity, the optimal $L_{t,x}^r$ Strichartz estimate for the Klein–Gordon equation is $r = \frac{10}{3}$; see for instance [Machihara et al. 2003]. However, in the resonant region, we are forced to take r slightly below 3; thus the additional angular regularity is essential to obtain the additional integrability in time. In other words, the angular regularity is used not just to obtain the scale-invariant endpoint, but also plays a crucial role in controlling the resonant interaction. Note that the number of angular derivatives required in Lemma 8.3 is not expected to be optimal, and any improvement in this direction has an impact on Theorem 1.2.

We have seen that the addition of angular regularity improves the range of available Strichartz estimates. An alternative way to exploit additional angular regularity is given by the following angular-concentration-type bound.

Lemma 8.5 [Sterbenz 2005, Lemma 5.2]. *Let $2 \leq p < \infty$, and $0 \leq s < 2/p$. If $\lambda, N \geq 1$, $\alpha \gtrsim \lambda^{-1}$, and $\kappa \in C_\alpha$ we have*

$$\|R_\kappa P_\lambda H_N f\|_{L_x^p(\mathbb{R}^3)} \lesssim \alpha^s N^s \|P_\lambda H_N f\|_{L_x^p(\mathbb{R}^3)}.$$

Finally, we need to estimate various square sums of norms. As we work in V^2 , this causes a slight loss in certain estimates. However, as we have some angular derivatives to work with, this loss can always be absorbed elsewhere.

Lemma 8.6. *Let $(P_j)_{j \in \mathcal{J}}$ and $(\mathcal{M}_j)_{j \in \mathcal{J}}$ be a collection of spatial Fourier multipliers. Suppose that the symbols of P_j have finite overlap, and*

$$\|\mathcal{M}_j P_j f\|_{L_x^2} \lesssim \delta \|P_j f\|_{L_x^2}$$

for some $\delta > 0$.

(i) *Let $q > 2$, $r \geq 2$. Suppose that there exists $A > 0$ such that for every j we have the bound*

$$\|e^{\mp it \langle \nabla \rangle^m} P_j f\|_{L_t^q L_x^r} \leq A \|P_j f\|_{L_x^2}.$$

Then for every $\epsilon > 0$ we have

$$\left(\sum_{j \in \mathcal{J}} \|\mathcal{M}_j P_j v\|_{L_t^q L_x^r}^2 \right)^{\frac{1}{2}} \lesssim \delta (\#\mathcal{J})^\epsilon A \|v\|_{V_{\pm,m}^2}.$$

(ii) Fix $p_0 > 1$. Suppose that there exists $A > 0$ such that $\|P_j f\|_{L_x^\infty} \lesssim A \|f\|_{L_x^2}$. Moreover, suppose that for every $p > p_0$ there exists $B_p > 0$, and for any $j \in \mathcal{J}$ there exists $\mathcal{K}_j \subset \mathcal{J}$ with $\#\mathcal{K}_j \lesssim 1$ such that for every $k \in \mathcal{K}_j$

$$\|P_j u P_k v\|_{L_{t,x}^p} \lesssim B_p \|P_j u\|_{U_{\pm 1, m_1}^2} \|P_k v\|_{U_{\pm 2, m_2}^2}.$$

Then for every $q > p_0$ and $p_0/q < \theta < 1$ we have

$$\sum_{j \in \mathcal{J}, k \in \mathcal{K}_j} \|P_j u \mathcal{M}_k P_k v\|_{L_{t,x}^q} \lesssim \delta (\#\mathcal{J})^{1-\theta} A^{1-\theta} B_{\theta q}^\theta \|u\|_{V_{\pm 1, m_1}^2} \|v\|_{V_{\pm 2, m_2}^2}.$$

Proof. We start with the proof of (i). Let $2 \leq p \leq q$ and suppose that $\phi = \sum_{I \in \mathcal{I}} \mathbb{1}_I(t) e^{\mp i t \langle \nabla \rangle^m} f_I$ is a U^p atom; thus $\sum_I \|f_I\|_{L_x^2}^p \leq 1$. The assumed linear estimate, together with the finite overlap of the Fourier multipliers P_j implies

$$\begin{aligned} \left(\sum_{j \in \mathcal{J}} \|\mathcal{M}_j P_j \phi\|_{L_t^q L_x^r}^p \right)^{\frac{1}{p}} &\leq \left(\sum_{I \in \mathcal{I}} \sum_{j \in \mathcal{J}} \|e^{\mp i t \langle \nabla \rangle^m} \mathcal{M}_j P_j f_I\|_{L_t^q L_x^r}^p \right)^{\frac{1}{p}} \\ &\leq A \left(\sum_{j \in \mathcal{J}} \sum_{I \in \mathcal{I}} \|\mathcal{M}_j P_j f_I\|_{L_x^2}^p \right)^{\frac{1}{p}} \leq \delta A \left(\sum_{I \in \mathcal{I}} \left(\sum_{j \in \mathcal{J}} \|P_j f_I\|_{L_x^2}^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \lesssim \delta A. \end{aligned}$$

Consequently the atomic definition of $U_{\pm, m}^p$ then implies that for any $2 \leq p \leq q$

$$\left(\sum_{j \in \mathcal{J}} \|\mathcal{M}_j P_j u\|_{L_t^q L_x^r}^p \right)^{\frac{1}{p}} \lesssim A \delta \|u\|_{U_{\pm, m}^p}. \tag{8-4}$$

Let $v \in V_{\pm, m}^2$. There exists a decomposition $v = \sum_{\ell \in \mathbb{N}} v_\ell$ such that for every $p \geq 2$ we have

$$\|v_\ell\|_{U_{\pm, m}^p} \lesssim 2^{\ell(\frac{2}{p}-1)} \|v\|_{V_{\pm, m}^2};$$

see, e.g., [Koch and Tataru 2005, Lemma 6.4] or [Hadac et al. 2009, Propositions 2.5 and 2.20]. An application of Hölder’s inequality, together with (8-4) gives for any $2 < p \leq q$

$$\begin{aligned} \left(\sum_{j \in \mathcal{J}} \|\mathcal{M}_j P_j v\|_{L_t^q L_x^r}^2 \right)^{\frac{1}{2}} &\lesssim (\#\mathcal{J})^{\frac{1}{2}-\frac{1}{p}} \sum_{\ell \in \mathbb{N}} \left(\sum_{j \in \mathcal{J}} \|\mathcal{M}_j P_j v_\ell\|_{L_t^q L_x^r}^p \right)^{\frac{1}{p}} \\ &\lesssim \delta A (\#\mathcal{J})^{\frac{1}{2}-\frac{1}{p}} \sum_{\ell \in \mathbb{N}} \|v_\ell\|_{U_{\pm, m}^p} \\ &\lesssim \delta A (\#\mathcal{J})^{\frac{1}{2}-\frac{1}{p}} \|v\|_{V_{\pm, m}^2} \sum_{\ell \in \mathbb{N}} 2^{\ell(\frac{2}{p}-1)} \\ &\lesssim \delta A (\#\mathcal{J})^{\frac{1}{2}-\frac{1}{p}} \|v\|_{V_{\pm, m}^2}. \end{aligned}$$

Thus (i) follows by taking p sufficiently close to 2.

We now turn to the proof of (ii). As in the proof of (i), we have the decompositions $u = \sum_{\ell \in \mathbb{N}} u_\ell$ and $v = \sum_{\ell \in \mathbb{N}} v_\ell$ with $\|u_\ell\|_{U_{\pm 1, m_1}^r} \lesssim 2^{\ell(\frac{2}{r}-1)}$ and $\|v_\ell\|_{U_{\pm 2, m_2}^r} \lesssim 2^{\ell(\frac{2}{r}-1)}$ for every $r \geq 2$. Let $q > p_0$ and $p_0/q < \theta < 1$. Then the convexity of the L^q norms together with Hölder’s inequality, our assumed bilinear estimate, and the U^2 summation argument used in (i) implies

$$\begin{aligned} & \sum_{j \in \mathcal{J}, k \in \mathcal{K}_j} \|P_j u \mathcal{M}_k P_k v\|_{L_{t,x}^q} \\ & \lesssim (\#\mathcal{J})^{1-\theta} \sum_{\ell, \ell' \in \mathbb{N}} \left(\sum_{j \in \mathcal{J}, k \in \mathcal{K}_j} \|P_j u_\ell \mathcal{M}_k P_k v_{\ell'}\|_{L_{t,x}^{\theta q}} \right)^\theta \left(\sup_{j, k \in \mathcal{J}} \|P_j u_\ell \mathcal{M}_k P_k v_{\ell'}\|_{L_{t,x}^\infty} \right)^{1-\theta} \\ & \lesssim \delta (\#\mathcal{J})^{1-\theta} A^{1-\theta} B_{\theta q}^\theta \sum_{\ell, \ell' \in \mathbb{N}} (\|u_\ell\|_{U_{\pm 1, m_1}^2} \|v_{\ell'}\|_{U_{\pm 2, m_2}^2})^\theta (\|u_\ell\|_{U_{\pm 1, m_1}^\infty} \|v_{\ell'}\|_{U_{\pm 2, m_2}^\infty})^{1-\theta} \\ & \lesssim \delta (\#\mathcal{J})^{1-\theta} A^{1-\theta} B_{\theta q}^\theta \|u\|_{V_{\pm 1, m_1}^2} \|v\|_{V_{\pm 2, m_2}^2} \sum_{\ell, \ell' \in \mathbb{N}} 2^{-\ell(1-\theta)} 2^{-\ell'(1-\theta)} \\ & \lesssim \delta (\#\mathcal{J})^{1-\theta} A^{1-\theta} B_{\theta q}^\theta \|u\|_{V_{\pm 1, m_1}^2} \|v\|_{V_{\pm 2, m_2}^2}. \end{aligned}$$

Therefore (ii) follows. □

Clearly the previous lemma allows us to extend Corollary 6.4, and the linear estimates discussed above, to frequency-localised functions in $V_{\pm, m}^2$. For instance, for any $1 \leq \mu \lesssim \lambda$, $\alpha \gtrsim \lambda^{-1}$, and $\epsilon > 0$, $q > 2$, we have by Lemma 8.2

$$\left(\sum_{q \in Q_\mu} \sum_{\kappa \in C_\alpha} \|R_\kappa P_q u_{\lambda, N}\|_{L_{t,x}^4}^2 \right)^{\frac{1}{2}} \lesssim \alpha^{-\epsilon} \left(\frac{\mu}{\lambda} \right)^{\frac{1}{4}-\epsilon} \lambda^{\frac{1}{2}} \|u_{\lambda, N}\|_{V_{\pm, m}^2}, \tag{8-5}$$

$$\left(\sum_{\kappa \in C_\alpha} \|R_\kappa u_{\lambda, N}\|_{L_t^q L_x^4}^2 \right)^{\frac{1}{2}} \lesssim \alpha^{-\epsilon} \lambda^{\frac{3}{4}-\frac{1}{q}} N \|u_{\lambda, N}\|_{V_{\pm, m}^2}, \tag{8-6}$$

where we use the shorthand $u_{\lambda, N} = P_\lambda P_N u$. Similarly, an application of Corollary 6.4, Lemma 8.1, and (ii) in Lemma 8.6 gives for every $q > \frac{3}{2}$ and $\epsilon > 0$

$$\begin{aligned} & \left(\sum_{\kappa, \kappa'' \in C_{\mu^{-1}}} \sum_{\substack{q, q'' \in Q_\mu \\ |q-q''| \approx \mu \text{ or } |\kappa-\kappa''| \approx \mu^{-1}}} \|R_{\kappa''} P_{q''} \phi_{\mu, N} [(\Pi_+ - \Pi_+(\mu\omega(\kappa))) R_\kappa P_q \psi_{\mu, N_1}]^\dagger\|_{L_{t,x}^q(\mathbb{R}^{1+3})}^2 \right)^{\frac{1}{2}} \\ & \lesssim \mu^\epsilon \|\phi_{\mu, N}\|_{V_{+, 1}^2} \|\psi_{\mu, N_1}\|_{V_{+, M}^2}, \end{aligned} \tag{8-7}$$

where $\omega(\kappa)$ denotes the centre of the cap $\kappa \in C_{\mu^{-1}}$. This bilinear bound plays a key role in controlling the solution to the DKG system in the resonant region.

8B. General resonance identity. After an application of Lemma 7.3, proving the bilinear estimates in Theorem 7.5 for the V^2 component of the norm, reduces to estimating trilinear expressions of the form

$$\int_{\mathbb{R}^{1+3}} \phi \psi^\dagger \gamma^0 \varphi \, dx \, dt. \tag{8-8}$$

Suppose ϕ , ψ , and φ have small modulation; thus $\text{supp } \tilde{\phi} \subset \{|\tau + \langle \xi \rangle| \leq d\}$, $\text{supp } \tilde{\psi} \subset \{|\tau \pm_1 \langle \xi \rangle_M| \leq d\}$, and $\text{supp } \tilde{\varphi} \subset \{|\tau \pm_2 \langle \xi \rangle_M| \leq d\}$ for some $d \in 2^{\mathbb{Z}}$. If $\xi \in \text{supp } \hat{\psi}$ and $\eta \in \text{supp } \hat{\phi}$, then it is easy to check that the integral (8-8) vanishes unless

$$|\langle \xi - \eta \rangle \mp_1 \langle \xi \rangle_M \pm_2 \langle \eta \rangle_M| \lesssim d.$$

To exploit this, we define the modulation function

$$\mathfrak{M}_{\pm_1, \pm_2}(\xi, \eta) = |\langle \xi - \eta \rangle \mp_1 \langle \xi \rangle_M \pm_2 \langle \eta \rangle_M|.$$

Clearly we have the symmetry properties $\mathfrak{M}_{+,+}(\xi, \eta) = \mathfrak{M}_{-,-}(\eta, \xi)$ and $\mathfrak{M}_{\pm, \mp}(\xi, \eta) = \mathfrak{M}_{\pm, \mp}(\eta, \xi)$. The proof of our global existence results requires a careful analysis of the zero sets of $\mathfrak{M}_{\pm_1, \pm_2}$; the key tool is the following.

Lemma 8.7. *Let $M > 0$:*

(i) *(Nonresonant interactions). We have*

$$\begin{aligned} \mathfrak{M}_{-,+}(\xi, \eta) &\gtrsim \langle \xi \rangle + \langle \eta \rangle, & \mathfrak{M}_{\pm, \pm}(\xi, \eta) &\gtrsim \frac{1}{\langle \xi - \eta \rangle} \left(\frac{(|\xi| - |\eta|)^2}{\langle \xi \rangle \langle \eta \rangle} + |\xi| |\eta| \theta^2(\xi, \eta) + 1 \right), \\ \mathfrak{M}_{-,-}(\xi, \eta) &\gtrsim \frac{|\xi - \eta| |\xi|}{\langle \xi \rangle + \langle \eta \rangle} \theta^2(\xi - \eta, -\xi), & \mathfrak{M}_{+,+}(\xi, \eta) &\gtrsim \frac{|\xi - \eta| |\eta|}{\langle \xi \rangle + \langle \eta \rangle} \theta^2(\xi - \eta, \eta). \end{aligned}$$

(ii) *(Resonant interactions). We have*

$$\begin{aligned} \mathfrak{M}_{+,-}(\xi, \eta) &\approx \frac{1}{\langle \xi \rangle + \langle \eta \rangle} \left| M^2 \frac{(|\xi| - |\eta|)^2}{\langle \xi \rangle_M \langle \eta \rangle_M + |\xi| |\eta| + M^2} + |\xi| |\eta| + \xi \cdot \eta + \frac{4M^2 - 1}{2} \right|, \\ \mathfrak{M}_{+,-}(\xi, \eta) &\gtrsim \frac{1}{\langle \eta \rangle} \left| \frac{(|\xi| - M|\xi - \eta|)^2}{\langle \xi \rangle_M \langle \xi - \eta \rangle + |\xi| |\xi - \eta| + M} + |\xi| |\xi - \eta| - \xi \cdot (\xi - \eta) + \frac{2M - 1}{2} \right|. \end{aligned}$$

Proof. We begin by noting that, if we let $m_1, m_2, m_3 \geq 0$, then for any $x, y \in \mathbb{R}^n$ we have the identity

$$\begin{aligned} &| \langle x - y \rangle_{m_3}^2 - (\langle x \rangle_{m_1} \pm \langle y \rangle_{m_2})^2 | \\ &= | \mp 2 \langle x \rangle_{m_1} \langle y \rangle_{m_2} - 2x \cdot y + (m_3^2 - m_1^2 - m_2^2) | \\ &= | 2(\langle x \rangle_{m_1} \langle y \rangle_{m_2} - (|x||y| + m_1 m_2)) + 2(|x||y| \pm x \cdot y) \pm ((m_1 \pm m_2)^2 - m_3^2) | \\ &= 2 \left| \frac{(m_1 |y| - m_2 |x|)^2}{\langle x \rangle_{m_1} \langle y \rangle_{m_2} + |x||y| + m_1 m_2} + |x||y| \pm x \cdot y \pm \frac{(m_1 \pm m_2)^2 - m_3^2}{2} \right|. \end{aligned} \tag{8-9}$$

We now turn to (i). The bound for $\mathfrak{M}_{-,+}$ is clear. On the other hand, by taking $x = \xi$, $y = \eta$, $m_1 = m_2 = M$, $m_3 = 1$ in (8-9), we have

$$\begin{aligned} \mathfrak{M}_{\pm, \pm}(\xi, \eta) &\geq | \langle \xi - \eta \rangle - |\langle \xi \rangle_M - \langle \eta \rangle_M| | \approx \frac{1}{\langle \xi - \eta \rangle} | \langle \xi - \eta \rangle^2 - (\langle \xi \rangle_M - \langle \eta \rangle_M)^2 | \\ &\approx \frac{1}{\langle \xi - \eta \rangle} \left(\frac{(|\xi| - |\eta|)^2}{\langle \xi \rangle \langle \eta \rangle} + |\xi| |\eta| \theta^2(\xi, \eta) + 1 \right). \end{aligned}$$

Similarly, taking $x = \xi - \eta$ and $y = \xi$, gives

$$\begin{aligned} \mathfrak{M}_{-,-}(\xi, \eta) &= \frac{|\langle \eta \rangle_M^2 - (\langle \xi - \eta \rangle + \langle \xi \rangle_M)^2|}{\langle \eta \rangle_M + \langle \xi - \eta \rangle + \langle \xi \rangle_M} \\ &\gtrsim \frac{|\xi - \eta| |\xi|}{\langle \xi \rangle + \langle \eta \rangle} \theta^2(\xi - \eta, -\xi). \end{aligned}$$

Using the symmetry $\mathfrak{M}_{-,-}(\xi, \eta) = \mathfrak{M}_{+,+}(\eta, \xi)$ gives the remaining bound in (i). To prove (ii), we note that another application of (8-9) gives

$$\begin{aligned} \mathfrak{M}_{+,-}(\xi, \eta) &\approx \frac{1}{\langle \xi \rangle + \langle \eta \rangle} |\langle \xi - \eta \rangle^2 - (\langle \xi \rangle_M + \langle \eta \rangle_M)^2| \\ &\approx \frac{1}{\langle \xi \rangle + \langle \eta \rangle} \left| M^2 \frac{(|\xi| - |\eta|)^2}{\langle \xi \rangle_M \langle \eta \rangle_M + |\xi| |\eta| + M^2} + |\xi| |\eta| + \xi \cdot \eta + \frac{4M^2 - 1}{2} \right|, \end{aligned}$$

from which the first inequality in (ii) follows. The second inequality in (ii) follows from a similar application of (8-9). \square

8C. The trilinear estimates. Suppose we would like to bound an expression of the form $P_\lambda H_N \mathcal{I}_m^\pm[F]$ in $V_{\pm, m}^2$. An application of the energy inequality, Lemma 7.3, implies we have

$$\|P_\lambda H_N \mathcal{I}_m^\pm[F]\|_{V_{\pm, m}^2} \lesssim \sup_{\|P_\lambda H_N u\|_{V_{\pm, m}^2} \lesssim 1} \left| \int_{\mathbb{R}^{1+3}} (P_\lambda H_N u)^\dagger F \, dx \, dt \right|.$$

Thus to bound the V^2 component of $\|\mathcal{I}_m^\pm[F]\|_{F_{\lambda, N}^{\pm, m}}$, it is enough to control $\int_{\mathbb{R}^{1+3}} (P_\lambda H_N u)^\dagger F \, dx \, dt$. Consequently, to estimate the V^2 component of the norms in Theorem 7.5, the key step is to prove the following trilinear estimate. To simplify notation somewhat, we define $\mathbf{B}_\epsilon = (\min\{\mu, \lambda_1, \lambda_2\} / \max\{\mu, \lambda_1, \lambda_2\})^\epsilon$ if $M \geq \frac{1}{2}$, and if $0 < M < \frac{1}{2}$ we let

$$\mathbf{B}_\epsilon = \begin{cases} (\min\{\mu, \lambda_1, \lambda_2\} / \max\{\mu, \lambda_1, \lambda_2\})^\epsilon, & \mu \ll \max\{\lambda_1, \lambda_2\} \text{ or } \mu \gg \min\{\lambda_1, \lambda_2\}, \\ 1 + \mu^{-\frac{1}{6} + \sigma} (\min\{N, N_1, N_2\})^{\frac{7}{30}}, & \mu \approx \lambda_1 \approx \lambda_2. \end{cases}$$

Theorem 8.8. *Let $M > 0$. For every $\sigma/100 < \delta \ll 1$ we have*

$$\begin{aligned} &\left| \int_{\mathbb{R}^{3+1}} \phi_{\mu, N} (\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2} \, dx \, dt \right| \\ &\lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^\delta \mathbf{B}_{\min\{\frac{\delta}{8}, \frac{1}{2a} - \frac{1}{4}\}} \|\phi\|_{F_{\mu, N}^{+, 1}} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm 2, M}} \end{aligned} \quad (8-10)$$

and

$$\begin{aligned} &\left| \int_{\mathbb{R}^{3+1}} \phi_{\mu, N} (\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2} \, dx \, dt \right| \\ &\lesssim \mu^{\frac{1}{2}} (\min\{N_1, N_2\})^\delta \mathbf{B}_{\min\{\frac{\delta}{8}, \frac{1}{2a} - \frac{1}{4}\}} \|\phi_{\mu, N}\|_{V_{\pm 1}^2} \|\psi\|_{F_{\lambda_1, N_1}^{\pm 1, M}} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm 2, M}}. \end{aligned} \quad (8-11)$$

In the region $\lambda_2 \gg \lambda_1$ we have the slightly stronger bound

$$\begin{aligned} & \left| \int_{\mathbb{R}^{3+1}} \phi_{\mu,N} (\Pi_{\pm_1} \psi_{\lambda_1,N_1})^\dagger \gamma^0 \Pi_{\pm_2} \varphi_{\lambda_2,N_2} dx dt \right| \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^\delta \left(\frac{\lambda_1}{\lambda_2} \right)^{\frac{\delta}{8}} \|\phi_{\mu,N}\|_{V_{+,1}^2} \|\psi_{\lambda_1,N_1}\|_{V_{\pm_1,M}^2} \|\varphi_{\lambda_2,N_2}\|_{V_{\pm_2,M}^2}. \end{aligned} \quad (8-12)$$

Similarly, when $\mu \ll \lambda_1$, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^{3+1}} \phi_{\mu,N} (\Pi_{\pm_1} \psi_{\lambda_1,N_1})^\dagger \gamma^0 \Pi_{\pm_2} \varphi_{\lambda_2,N_2} dx dt \right| \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N_1, N_2\})^\delta \left(\frac{\mu}{\lambda_1} \right)^{\frac{\delta}{8}} \|\phi_{\mu,N}\|_{V_{+,1}^2} \|\psi_{\lambda_1,N_1}\|_{V_{\pm_1,M}^2} \|\varphi_{\lambda_2,N_2}\|_{V_{\pm_2,M}^2}. \end{aligned} \quad (8-13)$$

Proof. We begin by decomposing the modulation (or distance to the relevant characteristic surface) as

$$\begin{aligned} & \phi_{\mu,N} (\Pi_{\pm_1} \psi_{\lambda_1,N_1})^\dagger \gamma^0 \Pi_{\pm_2} \varphi_{\lambda_2,N_2} \\ & = \sum_{d \in 2^{\mathbb{Z}}} C_d \phi_{\mu,N} (C_{\leq d}^{\pm_1} \psi_{\lambda_1,N_1})^\dagger \gamma^0 C_{\leq d}^{\pm_2} \varphi_{\lambda_2,N_2} + C_{<d} \phi_{\mu,N} (C_d^{\pm_1} \psi_{\lambda_1,N_1})^\dagger \gamma^0 C_{\leq d}^{\pm_2} \varphi_{\lambda_2,N_2} \\ & \quad + C_{<d} \phi_{\mu,N} (C_{<d}^{\pm_1} \psi_{\lambda_1,N_1})^\dagger \gamma^0 C_d^{\pm_2} \varphi_{\lambda_2,N_2} \\ & = \sum_{d \in 2^{\mathbb{Z}}} A_0 + A_1 + A_2. \end{aligned}$$

Keeping in mind (7-11), we now divide the proof into cases depending on the relative sizes of the frequency and the modulation. Namely, we consider separately the low-modulation cases

$$\lambda_1 \approx \lambda_2 \gg \mu \text{ and } d \lesssim \mu, \quad \mu \gg \min\{\lambda_1, \lambda_2\} \text{ and } d \lesssim \min\{\lambda_1, \lambda_2\}, \quad \lambda_1 \approx \lambda_2 \approx \mu \text{ and } d \lesssim \mu,$$

and the high-modulation cases

$$\lambda_1 \approx \lambda_2 \gtrsim \mu \text{ and } d \gg \mu, \quad \mu \gg \min\{\lambda_1, \lambda_2\} \text{ and } d \gg \min\{\lambda_1, \lambda_2\}.$$

Clearly, this covers all possible frequency combinations. The first case in the low-modulation regime, where the two spinors are high-frequency, is the easiest, as this case interacts very favourably with the null structure. The second case, when $\mu \gg \min\{\lambda_1, \lambda_2\}$, is more difficult, and is the main obstruction to the scale-invariant Sobolev result. The final case, when $\mu \approx \lambda_1 \approx \lambda_2$, is the only resonant interaction, and this is where the bilinear estimates in Corollary 6.4 play a crucial role. In the remaining high-modulation cases $d \gg \min\{\mu, \lambda_1, \lambda_2\}$, the null structure of the system no longer plays any role, and we need to exploit the $Y_{\lambda,N}^{\pm,m}$ norms to gain the off-diagonal decay term.

High-low, I: $\mu \ll \lambda_1 \approx \lambda_2$ and $d \lesssim \mu$. Our goal is to show that

$$\begin{aligned} & \sum_{d \ll \lambda_1} \left| \int_{\mathbb{R}^{1+3}} A_0 dx dt \right| + \left| \int_{\mathbb{R}^{1+3}} A_1 dx dt \right| + \left| \int_{\mathbb{R}^{1+3}} A_2 dx dt \right| \\ & \lesssim \mu^{\frac{1}{2}} N_{\min}^\delta \left(\frac{\mu}{\lambda_1} \right)^{\frac{1}{4}} \|\phi_{\mu,N}\|_{V_{+,1}^2} \|\psi_{\lambda_1,N_1}\|_{V_{\pm_1,M}^2} \|\varphi_{\lambda_2,N_2}\|_{V_{\pm_2,M}^2}, \end{aligned} \quad (8-14)$$

where we let $N_{\min} = \min\{N, N_1, N_2\}$. Clearly this gives the bounds (8-10), (8-11), and (8-13).

We now prove the bound (8-14). An application of Lemma 8.7 implies that we must have $\pm_1 = \pm_2$ and moreover, that the sum over the modulation is restricted to the region $\mu^{-1} \lesssim d \lesssim \mu$ (in particular this case is nonresonant). To estimate the first term, A_0 , we note that after another application of Lemma 8.7, we have the almost orthogonal decomposition

$$A_0 = \sum_{\substack{\kappa, \kappa' \in C_\alpha \\ |\kappa - \kappa'| \lesssim \alpha}} \sum_{\substack{q, q' \in Q_\mu \\ |q - q'| \lesssim \mu}} C_d \phi_{\mu, N} (C_{\leq d}^{\pm_1} R_\kappa P_q \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_{\leq d}^{\pm_2} R_{\kappa'} P_{q'} \varphi_{\lambda_2, N_2},$$

where $\alpha = (d\mu)^{\frac{1}{2}} \lambda_1^{-1}$. Then, using the null-structure by writing

$$C_{\leq d}^{\pm_1} R_\kappa P_{\lambda_1} = C_{\leq d}^{\pm_1, M} (\Pi_{\pm_1} - \Pi_{\pm_1}(\lambda_1 \omega)) R_\kappa P_{\lambda_1} + C_{\leq d}^{\pm_1, M} \Pi_{\pm_1}(\lambda_1 \omega_\kappa) R_\kappa P_{\lambda_1}$$

(here ω_κ denotes the centre of the cap κ) and applying Lemma 8.1, together with the uniform disposability of $C_{\leq d}^{\pm_1, M}$ from (7-1), we obtain for every $\epsilon > 0$

$$\begin{aligned} \left| \int A_0 dx dt \right| &\lesssim \sum_{\substack{\kappa, \kappa' \in C_\alpha \\ |\kappa - \kappa'| \lesssim \alpha}} \sum_{\substack{q, q' \in Q_\mu \\ |q - q'| \lesssim \mu}} \alpha \|C_d \phi_{\mu, N}\|_{L_{t,x}^2} \|R_\kappa P_q \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|R_{\kappa'} P_{q'} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ &\lesssim \mu^{\frac{1}{2}} \alpha^{-\epsilon} \left(\frac{\mu}{\lambda_1}\right)^{\frac{1}{2} - \epsilon} \|\phi_{\mu, N}\|_{V_{\pm_1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm_1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm_2, M}^2}, \end{aligned} \tag{8-15}$$

where we used Lemma 7.2 to control the $L_{t,x}^2$ norm of the high-modulation term, and the bound (8-5). On the other hand, we have the decomposition

$$A_0 = \sum_{\substack{\kappa, \kappa', \kappa'' \in C_\beta \\ |\kappa - \kappa'|, |\kappa'' \pm_2 \kappa'| \lesssim \beta}} C_d R_{\kappa''} \phi_{\mu, N} (C_{\leq d}^{\pm_1} R_\kappa \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_{\leq d}^{\pm_2} R_{\kappa'} \varphi_{\lambda_2, N_2},$$

where $\beta = d^{\frac{1}{2}} \mu^{-\frac{1}{2}}$, again by almost orthogonality and Lemma 8.7. As above, we obtain for every $\epsilon > 0$

$$\begin{aligned} \left| \int A_0 dx dt \right| &\lesssim \sum_{\substack{\kappa, \kappa', \kappa'' \in C_\beta \\ |\kappa - \kappa'|, |\kappa'' \pm_2 \kappa'| \lesssim \beta}} \beta \|C_d R_{\kappa''} \phi_{\mu, N}\|_{L_{t,x}^2} \|R_\kappa \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|R_{\kappa'} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ &\lesssim \beta^{1-\epsilon} d^{-\frac{1}{2}} \lambda (\beta N_{\min})^{\frac{1}{4}} \|\phi_{\mu, N}\|_{V_{\pm_1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm_1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm_2, M}^2}, \end{aligned} \tag{8-16}$$

where we used the angular concentration Lemma 8.5 on the lowest angular-frequency term. Combining (8-15) and (8-16), by taking $\epsilon > 0$ sufficiently small, we obtain for every $0 < \delta \ll 1$

$$\begin{aligned} \sum_{\mu^{-1} \lesssim d \lesssim \mu} \left| \int A_0 dx dt \right| &\lesssim \sum_{\mu^{-1} \lesssim d \lesssim \mu} \left(\frac{d}{\mu}\right)^{\frac{\delta}{4}} N_{\min}^\delta \left(\frac{\mu}{\lambda}\right)^{\frac{1}{4}} \mu^{\frac{1}{2}} \|P_\mu H_N \phi\|_{V_{\pm_1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm_1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm_2, M}^2} \\ &\lesssim N_{\min}^\delta \left(\frac{\mu}{\lambda}\right)^{\frac{1}{4}} \mu^{\frac{1}{2}} \|\phi_{\mu, N}\|_{V_{\pm_1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm_1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm_2, M}^2}, \end{aligned}$$

which gives (8-14) for the A_0 term. Next, we deal with the A_1 term. The argument is similar to the above, but the initial decomposition is slightly different as we no longer require the cube decomposition. Instead, we need to decompose the ϕ term into caps to ensure that the $C_{<d}$ multiplier is disposable. In more detail, the resonance bound in Lemma 8.7 gives

$$A_1 = \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_\alpha \\ |\kappa - \kappa'| \lesssim \alpha}} \sum_{\substack{\kappa'' \in \mathcal{C}_\beta \\ |\kappa'' \pm 2\kappa'| \lesssim \beta}} C_{<d} R_{\kappa''} \phi_{\mu, N} (C_d^{\pm 1} R_\kappa \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_{\leq d}^{\pm 2} R_{\kappa'} \varphi_{\lambda_2, N_2},$$

where $\alpha = (d\mu/\lambda_1^2)^{\frac{1}{2}}$ and $\beta = (d/\mu)^{\frac{1}{2}}$. By exploiting the null structure as previously, we then obtain for every $\epsilon > 0$

$$\begin{aligned} \left| \int A_1 dx dt \right| &\lesssim \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_\alpha \\ |\kappa - \kappa'| \lesssim \alpha}} \sum_{\substack{\kappa'' \in \mathcal{C}_\beta \\ |\kappa'' \pm 2\kappa'| \lesssim \beta}} \alpha \|R_{\kappa''} \phi_{\mu, N}\|_{L^4_{t,x}} \|C_d^{\pm 1} R_\kappa \psi_{\lambda_1, N_1}\|_{L^2_{t,x}} \|R_{\kappa'} \varphi_{\lambda_2, N_2}\|_{L^4_{t,x}} \\ &\lesssim \alpha^{1-\epsilon} \mu^{\frac{1}{2}} d^{-\frac{1}{2}} \lambda_2^{\frac{1}{2}} \|\phi_{\mu, N}\|_{V^2_{+,1}} \|\psi_{\lambda_1, N_1}\|_{V^2_{\pm 1, M}} \|\varphi_{\lambda_2, N_2}\|_{V^2_{\pm 2, M}}, \end{aligned} \tag{8-17}$$

where we used Lemma 7.2 to control the $L^2_{t,x}$ norm of the high-modulation term, and again used (8-5). To gain a power of d , we again exploit the angular concentration estimate by exploiting a similar argument to (8-16) to deduce that

$$\begin{aligned} \left| \int A_1 dx dt \right| &\lesssim \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa - \kappa'|, |\kappa'' \pm 2\kappa'| \lesssim \beta}} \beta \|R_{\kappa''} \phi_{\mu, N}\|_{L^4_{t,x}} \|C_d^{\pm 1} R_\kappa \psi_{\lambda_1, N_1}\|_{L^2_{t,x}} \|R_{\kappa'} \varphi_{\lambda_2, N_2}\|_{L^4_{t,x}} \\ &\lesssim \beta^{1-\epsilon} d^{-\frac{1}{2}} \lambda_2^{\frac{1}{2}} \mu^{\frac{1}{2}} (\beta N_{\min})^{\frac{1}{4}} \|\phi_{\mu, N}\|_{V^2_{+,1}} \|\psi_{\lambda_1, N_1}\|_{V^2_{\pm 1, M}} \|\varphi_{\lambda_2, N_2}\|_{V^2_{\pm 2, M}}. \end{aligned} \tag{8-18}$$

Combining (8-17) and (8-18) as in the A_0 case, and summing up over $\mu^{-1} \lesssim d \lesssim \mu$ with ϵ sufficiently small, we obtain (8-14). The remaining term A_2 can be handled in an identical manner to the A_1 . Thus the bound (8-14) follows.

High-low, II: $\mu \gg \min\{\lambda_1, \lambda_2\}$ and $d \lesssim \min\{\lambda_1, \lambda_2\}$. Let $\{j, k\} = \{1, 2\}$ and $\lambda_j \geq \lambda_k$. Our goal is to prove that

$$\begin{aligned} \sum_{d \lesssim \lambda_k} \left| \int_{\mathbb{R}^{1+3}} A_0 dx dt \right| + \left| \int_{\mathbb{R}^{1+3}} A_j dx dt \right| \\ \lesssim \mu^{\frac{1}{2}} N_{\min}^\delta \left(\frac{\lambda_k}{\mu} \right)^{\frac{1}{8}} \|\phi_{\mu, N}\|_{V^2_{+,1}} \|\psi_{\lambda_1, N_1}\|_{V^2_{\pm 1, M}} \|\varphi_{\lambda_2, N_2}\|_{V^2_{\pm 2, M}}. \end{aligned} \tag{8-19}$$

On the other hand, for the A_k term, we have the weaker bounds

$$\sum_{d \lesssim \lambda_k} \left| \int_{\mathbb{R}^{1+3}} A_k dx dt \right| \lesssim \mu^{\frac{1}{2}} \left(\frac{\lambda_k}{\mu} \right)^{\frac{\delta}{8}} (\min\{N, N_j\})^\delta \|\phi_{\mu, N}\|_{V^2_{+,1}} \|\psi_{\lambda_1, N_1}\|_{V^2_{\pm 1, M}} \|\varphi_{\lambda_2, N_2}\|_{V^2_{\pm 2, M}} \tag{8-20}$$

and

$$\sum_{d \lesssim \lambda_k} \left| \int_{\mathbb{R}^{1+3}} A_k dx dt \right| \lesssim \mu^{\frac{1}{2}} \left(\frac{\lambda_k}{\mu} \right)^{\frac{1}{2a} - \frac{1}{4}} N_k^\delta \|\phi_{\mu, N}\|_{V_{+,1}^2} \begin{cases} \|\psi\|_{F_{\lambda_1, N_1}^{\pm 1, M}} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}, & k = 1, \\ \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, 1}^2} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm 2, M}}, & k = 2, \end{cases} \quad (8-21)$$

where

$$\frac{1}{2} < \frac{1}{a} < \frac{1}{2} + \frac{\sigma}{1000}$$

is as in the definition of the $Y_{\lambda, N}^{\pm, m}$ norm. Clearly (8-19), (8-20), and (8-21) give the estimates claimed in Theorem 8.8. Note that we have a weaker bound when the low-frequency term has modulation away from the hyperboloid, and for this interaction, we are forced to exploit the $Y_{\lambda, N}^{\pm, m}$ norms.

We begin the proof of (8-19), (8-20), and (8-21) by observing that since the estimate is essentially symmetric in ψ and φ , it is enough to consider the case $\mu \approx \lambda_1 \gg \lambda_2$; in other words, we only consider the case $j = 1$ and $k = 2$. As in the previous case, Lemma 8.7 implies that we only have a nonzero contribution if $\pm_1 = +$ and $\lambda_2^{-1} \lesssim d \lesssim \lambda_2$. To control the A_0 term, we decompose into caps of radius $\beta = (d/\lambda_2)^{\frac{1}{2}}$ and cubes of diameter λ_2 . Lemma 8.7 implies that we have the almost orthogonality identity

$$A_0 = \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_\beta \\ |\kappa \mp 2\kappa'| \lesssim \beta}} \sum_{\substack{q, q' \in \mathcal{Q}_{\lambda_2} \\ |q - q'| \lesssim \lambda_2}} P_{q'} C_d \phi_{\mu, N} (P_q R_\kappa C_{\leq d}^+ \psi_{\lambda_1, N_1})^\dagger \gamma^0 R_{\kappa'} C_{\leq d}^{\pm 2} \varphi_{\lambda_2, N_2}.$$

Thus exploiting the null structure as previously, disposing of the $C_d^{\pm, m}$ multipliers using (7-1), and applying the $L_{t,x}^4$ Strichartz estimate, we obtain for every $\epsilon > 0$

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_0 dx dt \right| &\lesssim \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_\beta \\ |\kappa \mp 2\kappa'| \lesssim \beta}} \sum_{\substack{q, q' \in \mathcal{Q}_{\lambda_2} \\ |q - q'| \lesssim \lambda_2}} \beta \|P_{q'} C_d \phi_{\mu, N}\|_{L_{t,x}^2} \|P_q R_\kappa \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|R_{\kappa'} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ &\lesssim \beta^{-\epsilon} \mu^{\frac{1}{2}} \left(\frac{\lambda_2}{\mu} \right)^{\frac{1}{4} - \epsilon} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{+,M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \quad (8-22)$$

On the other hand, by decomposing into

$$A_0 = \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa \mp 2\kappa'|, |\kappa'' \pm 2\kappa'| \lesssim \beta}} R_{\kappa''} C_d \phi_{\mu, N} (R_\kappa C_{\leq d}^+ \psi_{\lambda_1, N_1})^\dagger \gamma^0 R_{\kappa'} C_{\leq d}^{\pm 2} \varphi_{\lambda_2, N_2}$$

and using the angular concentration bound Lemma 8.5 on the smallest angular-frequency term, a similar argument gives

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_0 dx dt \right| &\lesssim \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa \mp 2\kappa'|, |\kappa'' \pm 2\kappa'| \lesssim \beta}} \beta \|C_d R_{\kappa''} \phi_{\mu, N}\|_{L_{t,x}^2} \|R_\kappa \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|R_{\kappa'} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ &\lesssim \mu^{\frac{1}{2}} \beta^{\frac{1}{4} - \epsilon} N_{\min}^{\frac{1}{4}} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{+,M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \quad (8-23)$$

As in the previous case, combining (8-22) and (8-23) with ϵ sufficiently small gives (8-19) for the A_0 term. The A_1 term can be estimated by an identical argument (since the high-modulation term is again at frequency μ). To control the A_2 component, we start by again applying Lemma 8.7 and decomposing into

$$A_2 = \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_\beta \\ |\kappa \mp 2\kappa'| \lesssim \beta}} \sum_{\substack{q, q' \in \mathcal{Q}_{\lambda_2} \\ |q - q'| \lesssim \lambda_2}} P_{q'} C_{<d} \phi_{\mu, N} (P_q R_\kappa C_{\leq d}^+ \psi_{\lambda_1, N_1})^\dagger \gamma^0 R_{\kappa'} C_d^{\pm 2} \varphi_{\lambda_2, N_2},$$

where as usual $\beta = (d/\lambda_2)^{\frac{1}{2}}$. Applying the, by now, standard null-form bound, (7-1), and the $L_{t,x}^4$ Strichartz estimate, we conclude that for every $\epsilon > 0$

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_2 dx dt \right| &\lesssim \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_\beta \\ |\kappa \mp 2\kappa'| \lesssim \beta}} \sum_{\substack{q, q' \in \mathcal{Q}_{\lambda_2} \\ |q - q'| \lesssim \lambda_2}} \beta \|P_{q'} \phi_{\mu, N}\|_{L_{t,x}^4} \|P_q R_\kappa \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|R_{\kappa'} C_d^{\pm 2} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^2} \\ &\lesssim \mu^{\frac{1}{2}} \beta^{-\epsilon} \left(\frac{\mu}{\lambda_2}\right)^\epsilon \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{+,M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \tag{8-24}$$

Note that we get no high frequency gain here (in fact we have a slight loss due to the sum over cubes). On the other hand, by decomposing all three terms into caps of size β , using null structure, the $L_t^q L_x^4$ Strichartz estimate in Lemma 8.2, and Bernstein's inequality followed by Lemma 7.2 for φ_{λ_2, N_2} , we obtain for any $2 < q < 2 + \frac{2}{3}$

$$\begin{aligned} &\left| \int_{\mathbb{R}^{1+3}} A_2 dx dt \right| \\ &\lesssim \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa \mp 2\kappa'|, |\kappa'' \pm 2\kappa'| \lesssim \beta}} \beta \|R_{\kappa''} \phi_{\mu, N}\|_{L^{q/(q-2)}_t L^{2q/(4-q)}_x} \|R_\kappa \psi_{\lambda_1, N_1}\|_{L_t^q L_x^4} \|R_{\kappa'} C_d^{\pm 2} \varphi_{\lambda_2, N_2}\|_{L_t^q L_x^{4q/(5q-8)}} \\ &\lesssim \mu^{\frac{1}{2}} \left(\frac{d}{\lambda_2}\right)^{\frac{1}{q} - \frac{1}{4} - \epsilon} \left(\frac{\lambda_2}{\mu}\right)^{\frac{5}{q} - \frac{9}{4}} N_1 \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{+,M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2} \end{aligned} \tag{8-25}$$

(schematically, we are putting the product into $L_t^{\infty-} L_x^{2+} \times L_t^{2+} L_x^4 \times L_t^{2+} L_x^{4-}$). Switching the roles of $\phi_{\mu, N}$ and ψ_{λ_1, N_1} , and combining (8-24) and (8-25) with q close to 2, and $\epsilon > 0$ sufficiently small, we obtain (8-20).

It remains to prove (8-21); thus we need to consider the case where φ also has the smallest angular frequency. We begin by again using Lemma 8.7 to get the decomposition

$$A_2 = \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa \mp 2\kappa'|, |\kappa'' \pm 2\kappa'| \lesssim \beta}} \sum_{\substack{q, q'' \in \mathcal{Q}_{\lambda_2} \\ |q - q''| \lesssim \lambda_2}} R_{\kappa''} P_{q''} C_{<d} \phi_{\mu, N} (R_\kappa P_q C_{\leq d}^+ \psi_{\lambda_1, N_1})^\dagger \gamma^0 R_{\kappa'} C_d^{\pm 2} \varphi_{\lambda_2, N_2},$$

where $\beta = (d/\lambda_2)^{\frac{1}{2}}$. An application of Bernstein's inequality, Lemma 7.2, and the angular concentration lemma for φ , together with the null-form bound, and Lemma 8.2, implies that for any $\epsilon > 0$ sufficiently

small

$$\begin{aligned}
 & \left| \int_{\mathbb{R}^{1+3}} A_2 dx dt \right| \\
 & \lesssim \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa \mp 2\kappa'|, |\kappa'' \pm 2\kappa'| \lesssim \beta}} \sum_{\substack{q, q'' \in \mathcal{Q}_{\lambda_2} \\ |q - q''| \lesssim \lambda_2}} \beta \|R_{\kappa''} P_{q''} \phi_{\mu, N}\|_{L_t^{2a/(a-1)} L_x^{2a}} \\
 & \quad \times \|R_\kappa P_q \psi_{\lambda_1, N_1}\|_{L_t^{2a/(a-1)} L_x^{2a}} \|R_{\kappa'} C_d^{\pm 2} \varphi_{\lambda_2, N_2}\|_{L_t^a L_x^{a/(a-1)}} \\
 & \lesssim \beta^{1-\epsilon} \left(\frac{\mu}{\lambda_2}\right)^\epsilon (\mu \lambda_2)^{1-\frac{1}{a}} (\beta^2 \lambda_2^3)^{\frac{1}{a}-\frac{1}{2}} (\beta N_2)^\delta \|P_\mu H_N \phi\|_{V_{\pm,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm,1,M}^2} \|C_d^{\pm 2} \varphi_{\lambda_2, N_2}\|_{L_t^a L_x^2} \\
 & \lesssim \mu^{\frac{1}{2}} N_2^\delta \left(\frac{\lambda_2}{\mu}\right)^{\frac{1}{2a}-\frac{1}{4}} \left(\frac{d}{\lambda_2}\right)^{\frac{1}{2}(\frac{\delta}{2}-b+\frac{1}{a})} \|\phi_{\mu, N}\|_{V_{\pm,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm,1,M}^2} \|\varphi\|_{Y_{\lambda_2, N_2}^{\pm 2, M}},
 \end{aligned}$$

which gives (8-21) since

$$\frac{1}{2} < \frac{1}{a} < \frac{1}{2} + \frac{\sigma}{1000} \quad \text{and} \quad b - \frac{1}{a} = \frac{2}{a} - 1 < \frac{\sigma}{500} < \frac{\delta}{5}.$$

High-high: $\mu \approx \lambda_1 \approx \lambda_2$ and $d \lesssim \mu$. Our goal is to prove that if $M \geq \frac{1}{2}$, then for any $\delta > 0$ we have the bound

$$\begin{aligned}
 & \sum_{d \lesssim \mu} \left| \int_{\mathbb{R}^{1+3}} A_0 dx dt \right| + \left| \int_{\mathbb{R}^{1+3}} A_1 dx dt \right| + \left| \int_{\mathbb{R}^{1+3}} A_2 dx dt \right| \\
 & \lesssim \mu^{\frac{1}{2}} N_{\min}^\delta \|\phi_{\mu, N}\|_{V_{\pm,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm,1,M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}, \quad (8-26)
 \end{aligned}$$

while if $0 < M < \frac{1}{2}$, for every $s, \delta > 0$, we have

$$\begin{aligned}
 & \left| \int_{\mathbb{R}^{1+3}} \sum_{d \lesssim \mu} A_0 + A_1 + A_2 dx dt \right| \\
 & \lesssim \mu^{\frac{1}{2}} N_{\min}^\delta (1 + \mu^{-\frac{1}{6}+s} N_{\min}^{\frac{7}{30}}) \|\phi_{\mu, N}\|_{V_{\pm,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm,1,M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \quad (8-27)
 \end{aligned}$$

The key difference from the previous cases, is that if $0 < M \leq \frac{1}{2}$, we no longer have the nonresonant bound $d \gtrsim \mu^{-1}$, and thus we also have to estimate the *resonant* interactions $d \ll \mu^{-1}$. This is particularly challenging in light of the fact that in the *strongly resonant regime*, $0 < M < \frac{1}{2}$, there is no gain from the null structure when $d \ll \mu^{-1}$. However, we do have *transversality* in the region $d \ll \mu^{-1}$, and consequently, we can apply the key bilinear restriction estimate in Corollary 6.4. On the other hand, in the *weakly resonant regime*, $M = \frac{1}{2}$, somewhat surprisingly and in stark contrast to the cases $M \neq \frac{1}{2}$, the null structure gives cancellation for *all* modulation scales.

We start by considering the nonresonant region $\mu^{-1} \lesssim d \lesssim \mu$. By decomposing into caps of radius $\beta = (d/\mu)^{\frac{1}{2}}$, an application of Lemma 8.7 gives the identity

$$A_0 = \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\pm_1 \kappa \mp 2\kappa'|, |\kappa'' \pm 2\kappa'| \lesssim \beta}} R_{\kappa''} C_d \phi_{\mu, N} (R_\kappa C_{\leq d}^{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 R_{\kappa'} C_{\leq d}^{\pm 2} \varphi_{\lambda_2, N_2}.$$

Thus by applying the $L^4_{t,x}$ Strichartz bound, exploiting the null structure as previously (here we need the assumption $d \gtrsim \mu^{-1}$), and using the angular concentration bound in Lemma 8.5 on N_{\min} , we obtain for every $\epsilon > 0$

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_0 \, dx \, dt \right| &\lesssim \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\pm_1 \kappa \mp_2 \kappa', |\kappa'' \pm_2 \kappa'| \lesssim \beta}} \beta \|R_{\kappa''} C_d \phi_{\mu, N}\|_{L^2_{t,x}} \|R_\kappa \psi_{\lambda_1, N_1}\|_{L^4_{t,x}} \|R_{\kappa'} \varphi_{\lambda_2, N_2}\|_{L^4_{t,x}} \\ &\lesssim \beta^{1-\epsilon} d^{-\frac{1}{2}} \mu (\beta N_{\min})^\delta \|\phi_{\mu, N}\|_{V^2_{+,1}} \|\psi_{\lambda_1, N_1}\|_{V^2_{\pm_1, M}} \|\varphi_{\lambda_2, N_2}\|_{V^2_{\pm_2, M}}. \end{aligned}$$

Taking $\delta > 0$ and $\epsilon > 0$ sufficiently small, and summing up over the modulation $\mu^{-1} \lesssim d \lesssim \mu$ then gives (8-26) and (8-27) for A_0 in the region $\mu^{-1} \lesssim d \lesssim \mu$. A similar argument bounds the A_1 and A_2 terms in (8-26) and (8-27), provided the sum over modulation is restricted to $\mu^{-1} \lesssim d \lesssim \mu$.

We now consider the case $d \ll \mu^{-1}$. Note that if $M > \frac{1}{2}$, then using Lemma 8.7, we see that $A_0 = A_1 = A_2 = 0$ and thus (8-26) is immediate. On the other hand, if we are in the weakly resonant regime $M = \frac{1}{2}$, then another application of Lemma 8.7 implies that $\pm_1 = +$ and $\pm_2 = -$, and we have the decomposition

$$A_0 = \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa + \kappa', |\kappa'' - \kappa| \lesssim \beta}} \sum_{\substack{q, q' \in \mathcal{Q}_{\mu^2 \beta} \\ |q + q'| \lesssim \mu^2 \beta}} R_{\kappa''} C_d \phi_{\mu, N} (R_\kappa P_q C_{\leq d}^+ \psi_{\lambda_1, N_1})^\dagger \gamma^0 R_{\kappa'} P_{q'} C_{\leq d}^- \varphi_{\lambda_2, N_2},$$

where $\beta = (d/\mu)^{\frac{1}{2}}$. Therefore, using the null-form-type bound (8-1), together with (ii) in Lemma 8.1 to exploit the null structure, the orthogonality estimate in Lemma 8.6, and an application of Lemma 8.2 gives for every $\epsilon > 0$

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_0 \, dx \, dt \right| &\lesssim \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa - \kappa', |\kappa'' - \kappa| \lesssim \beta}} \sum_{\substack{q, q' \in \mathcal{Q}_{\mu^2 \beta} \\ |q + q'| \lesssim \mu^2 \beta}} \beta \|R_{\kappa''} C_d \phi_{\mu, N}\|_{L^2_{t,x}} \|R_\kappa P_q \psi_{\lambda_1, N_1}\|_{L^4_{t,x}} \|R_{\kappa'} P_{q'} \varphi_{\lambda_2, N_2}\|_{L^4_{t,x}} \\ &\lesssim \beta \times d^{-\frac{1}{2}} \times \mu \times \beta^{-\epsilon} (\mu \beta)^{-\epsilon} \times (\beta N_{\min})^\delta \|\phi_{\mu, N}\|_{V^2_{+,1}} \|\psi_{\lambda_1, N_1}\|_{V^2_{+, M}} \|\varphi\|_{V^2_{-, M}}, \end{aligned}$$

where we used the angular concentration bound in Lemma 8.5 on the term with smallest angular frequency. Choosing $\epsilon > 0$ sufficiently small, and summing up over $0 < d \ll \mu^{-1}$ then gives (8-26) for the A_0 term. An identical argument bounds the A_1 and A_2 terms.

It remains to prove (8-27) when $0 < d \ll \mu^{-1}$. Another application of Lemma 8.7, implies that we must have $\pm_1 = +$ and $\pm_2 = -$, as well as the key orthogonality identity

$$\begin{aligned} &\sum_{d \ll \mu^{-1}} A_0 + A_1 + A_2 \\ &= C_{\ll \mu^{-1}} \phi_{\mu, N} (C_{\ll \mu^{-1}}^+ \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_{\ll \mu^{-1}}^- \varphi_{\lambda_2, N_2} \\ &= \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_{\mu^{-1}} \\ |\kappa + \kappa'| \lesssim \mu^{-1}}} \sum_{\substack{q, q'' \in \mathcal{Q}_\mu \\ |q - q''| \approx \mu \text{ or } |\kappa - \kappa''| \approx \mu^{-1}}} R_{\kappa''} P_{q''} C_{\ll \mu^{-1}} \phi_{\mu, N} (R_\kappa P_q C_{\ll \mu^{-1}}^+ \psi_{\lambda_1, N_1})^\dagger \gamma^0 R_{\kappa'} C_{\ll \mu^{-1}}^- \varphi_{\lambda_2, N_2}. \end{aligned}$$

Note that the summation is restricted to terms for which $R_{\kappa''} P_{q''} C_{\ll \mu^{-1}} \phi_{\mu, N}$ and $R_{\kappa} P_q C_{\ll \mu^{-1}}^+ \psi_{\lambda_1, N_1}$ have either angular orthogonality or radial orthogonality. In either case, we may apply Corollary 6.4, via the bound (8-7), the null structure bound in Lemma 8.1, and the Klein–Gordon angular Strichartz estimate in Lemma 8.3, to deduce that for every $\frac{3}{2} < q < \frac{14}{9}$ and $\epsilon > 0$ we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^{1+3}} \sum_{d \ll \mu^{-1}} A_0 + A_1 + A_2 \, dx \, dt \right| \\ & \lesssim \mu^{-1} \sum_{\kappa, \kappa'' \in \mathcal{C}_{\mu^{-1}}} \sum_{\substack{q, q' \in \mathcal{Q}_{\mu} \\ |q - q''| \approx \mu \text{ or } |\kappa - \kappa''| \approx \mu^{-1}}} \|R_{\kappa''} P_{q''} C_{\ll \mu^{-1}} \phi_{\mu, N} (R_{\kappa} P_q C_{\ll \mu^{-1}}^+ \psi_{\lambda_1, N_1})^\dagger\|_{L_{t,x}^q} \|\varphi_{\lambda_2, N_2}\|_{L_{t,x}^{q'}} \\ & \lesssim \mu^{\frac{5}{q} - 3 + \epsilon} N_2^{7(\frac{7}{10} - \frac{1}{q}) + \epsilon} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{+,M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{-,M}^2}, \end{aligned}$$

where for ease of reading we suppressed the $\Pi_{\pm}(\omega_{\kappa})$ matrices used to extract the null-form gain of μ^{-1} . Choosing q sufficiently close to $\frac{3}{2}$, and $\epsilon > 0$ sufficiently small, then gives (8-27) in the case $N_2 = N_{\min}$. To deal with remaining cases, we just reverse the roles of ϕ , ψ , and φ , again apply Lemma 8.7 to deduce the required transversality, and always use the angular Strichartz estimate from Lemma 8.3 on the term with smallest angular frequency. This completes the proof of (8-27).

High modulation, I: $\mu \lesssim \lambda_1 \approx \lambda_2$ and $d \gg \mu$. In this region, our goal is to prove that

$$\sum_{d \gg \mu} \left| \int_{\mathbb{R}^{1+3}} A_1 \, dx \, dt \right| + \left| \int_{\mathbb{R}^{1+3}} A_2 \, dx \, dt \right| \lesssim \mu^{\frac{1}{2}} \left(\frac{\mu}{\lambda_1} \right)^{\frac{1}{8}} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2} \tag{8-28}$$

and for every $\delta > 0$, the weaker bounds

$$\sum_{d \gg \mu} \left| \int_{\mathbb{R}^{1+3}} A_0 \, dx \, dt \right| \lesssim \mu^{\frac{1}{2}} \left(\frac{\mu}{\lambda_1} \right)^{\frac{\delta}{8}} (\min\{N_1, N_2\})^{\delta} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}, \tag{8-29}$$

$$\sum_{d \gg \mu} \left| \int_{\mathbb{R}^{1+3}} A_0 \, dx \, dt \right| \lesssim \mu^{\frac{1}{2}} \left(\frac{\mu}{\lambda_1} \right)^{\frac{1}{a} - \frac{1}{2}} \|\phi\|_{Y_{\mu, N}^{+,1}} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}, \tag{8-30}$$

where a is as in the definition of the $Y_{\lambda, N}^{\pm, m}$ norm. We start with proving the estimates (8-29) and (8-30), under the additional restriction of the sums to the range $d \gtrsim \lambda_1$.

To bound the A_0 component, decomposing ψ and φ into cubes of size μ , together with an application of the $L_{t,x}^4$ Strichartz estimate gives for all $\epsilon > 0$

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_0 \, dx \, dt \right| & \lesssim \sum_{\substack{q, q' \in \mathcal{Q}_{\mu} \\ |q - q'| \lesssim \mu}} \|C_d \phi_{\mu, N}\|_{L_{t,x}^2} \|P_q \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|P_{q'} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ & \lesssim \mu^{\frac{1}{2}} \left(\frac{\lambda_1}{\mu} \right)^{\epsilon} \left(\frac{\lambda_1}{d} \right)^{\frac{1}{2}} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \tag{8-31}$$

As in the proof of (8-25), if we instead apply the $L_t^q L_x^4$ bound, together with Bernstein’s inequality for ϕ , we obtain for any $2 < q < 2 + \frac{2}{11}$

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_0 dx dt \right| &\lesssim \|C_d \phi_{\mu,N}\|_{L_t^q L_x^{4q/(5q-8)}} \|\psi_{\lambda_1,N_1}\|_{L_t^q L_x^4} \|\varphi_{\lambda_2,N_2}\|_{L_t^{q/(q-2)} L_x^{2q/(4-q)}} \\ &\lesssim \mu^{\frac{1}{2}} \left(\frac{\lambda_1}{d}\right)^{\frac{1}{q}} \left(\frac{\mu}{\lambda_1}\right)^{\frac{6}{q}-\frac{11}{4}} N_1 \|\phi_{\mu,N}\|_{V_{+,1}^2} \|\psi_{\lambda_1,N_1}\|_{V_{\pm_1,M}^2} \|\varphi_{\lambda_2,N_2}\|_{V_{\pm_2,M}^2} \end{aligned} \quad (8-32)$$

(schematically, we are putting the product into $L_t^{2+} L_x^{4-} \times L_t^{2+} L_x^4 \times L_t^{\infty-} L_x^{2+}$). Switching the roles of ψ_{λ_1,N_1} and φ_{λ_2,N_2} , and combining (8-31) and (8-32) with q sufficiently close to 2 and $\epsilon > 0$ sufficiently small, followed by summing up over $d \gtrsim \lambda_1$, we obtain (8-29). On the other hand, to obtain (8-30), we again use Lemma 8.2 to deduce that

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_0 dx dt \right| &\lesssim \sum_{\substack{q,q' \in \mathcal{Q}_\mu \\ |q-q'| \lesssim \mu}} \|C_d \phi_{\mu,N}\|_{L_t^q L_x^{a/(a-1)}} \|P_q \psi_{\lambda_1,N_1}\|_{L_t^{2a/(a-1)} L_x^{2a}} \|P_{q'} \varphi_{\lambda_2,N_2}\|_{L_t^{2a/(a-1)} L_x^{2a}} \\ &\lesssim \mu^{\frac{1}{2}} \left(\frac{\lambda_1}{d}\right)^b \left(\frac{\mu}{\lambda}\right)^{b+\frac{1}{a}-1-\epsilon} \|\phi\|_{Y_{\mu,N}^{+,1}} \|\psi_{\lambda_1,N_1}\|_{V_{\pm_1,M}^2} \|\varphi_{\lambda_2,N_2}\|_{V_{\pm_2,M}^2}, \end{aligned}$$

which then gives (8-30) if we choose ϵ sufficiently small as

$$\frac{1}{a} > \frac{1}{2} \quad \text{and} \quad b + \frac{1}{a} - 1 = 4\left(\frac{1}{a} - \frac{1}{2}\right)$$

(here a, b are as in the definition of the $Y_\lambda^{\pm,m}$ norm).

We now turn to the estimates for A_1 and A_2 . By symmetry, it is enough to consider the A_1 term. After decomposing into cubes of size μ and applying the $L_{t,x}^4$ Strichartz estimate, we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_1 dt dx \right| &\lesssim \sum_{\substack{q,q' \in \mathcal{Q}_\mu \\ |q-q'| \lesssim \mu}} \|\phi_{\mu,N}\|_{L_{t,x}^4} \|C_d^{\pm 1} P_q \psi_{\lambda_1,N_1}\|_{L_{t,x}^2} \|P_{q'} \varphi_{\lambda_2,N_2}\|_{L_{t,x}^4} \\ &\lesssim \mu^{\frac{1}{2}} \left(\frac{\mu}{\lambda_1}\right)^{\frac{1}{4}-\epsilon} \left(\frac{\lambda_1}{d}\right)^{\frac{1}{2}} \|\phi_{\mu,N}\|_{V_{+,1}^2} \|\psi_{\lambda_1,N_1}\|_{V_{\pm_1,M}^2} \|\varphi_{\lambda_2,N_2}\|_{V_{\pm_2,M}^2}. \end{aligned}$$

Summing up over $d \gtrsim \lambda_1$ and choosing ϵ sufficiently small then gives (8-28).

Next, we consider the parts of the sums where $\mu \ll d \ll \lambda_1$. Since $\mathfrak{M}_{\pm_1, \pm_2} \lesssim d \ll \lambda_1$, we must have $\pm_1 = \pm_2$; hence $\mathfrak{M}_{\pm_1, \pm_2} \lesssim \mu$.

For A_0 this implies the decomposition

$$\begin{aligned} \int_{\mathbb{R}^{1+3}} A_0 dx dt &= \int_{\mathbb{R}^{1+3}} C_d \phi_{\mu,N} (C_{\approx d}^{\pm 1} \psi_{\lambda_1,N_1})^\dagger \gamma^0 C_{\leq d}^{\pm 2} \varphi_{\lambda_2,N_2} dx dt \\ &\quad + \int_{\mathbb{R}^{1+3}} C_d \phi_{\mu,N} (C_{\ll d}^{\pm 1} \psi_{\lambda_1,N_1})^\dagger \gamma^0 C_{\approx d}^{\pm 2} \varphi_{\lambda_2,N_2} dx dt. \end{aligned} \quad (8-33)$$

Concerning the first term, using null-structure,

$$\begin{aligned} & \left| \int_{\mathbb{R}^{1+3}} C_d \phi_{\mu,N} (C_{\approx d}^{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_{\leq d}^{\pm 2} \varphi_{\lambda_2, N_2} dx dt \right| \\ & \lesssim \frac{\mu}{\lambda_1} \sum_{\substack{q, q' \in \mathcal{Q}_\mu \\ |q - q'| \lesssim \mu}} \|C_d \phi_{\mu,N}\|_{L_t^2 L_x^\infty} \|P_q C_{\approx d}^{\pm 1} \psi_{\lambda_1, N_1}\|_{L_{t,x}^2} \|P_{q'} C_{\leq d}^{\pm 2} \varphi_{\lambda_2, N_2}\|_{L_t^\infty L_x^2} \\ & \lesssim \left(\frac{d}{\mu}\right)^{-1} \left(\frac{\mu}{\lambda_1}\right)^{1-\epsilon} \mu^{\frac{1}{2}} \|\phi_{\mu,N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}, \end{aligned}$$

which can be summed up with respect to $\mu \ll d \ll \lambda_1$. The second term in (8-33) can be treated along the same lines.

Similarly, for A_1 we have the decomposition

$$\begin{aligned} \int_{\mathbb{R}^{1+3}} A_1 dx dt &= \int_{\mathbb{R}^{1+3}} C_{\approx d} \phi_{\mu,N} (C_d^{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_{\leq d}^{\pm 2} \varphi_{\lambda_2, N_2} dx dt \\ &+ \int_{\mathbb{R}^{1+3}} C_{\ll d} \phi_{\mu,N} (C_d^{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_{\approx d}^{\pm 2} \varphi_{\lambda_2, N_2} dx dt. \end{aligned} \quad (8-34)$$

The first term can be estimated the same way as the first contribution to A_0 . For the second term, we use the decomposition

$$\begin{aligned} & \left| \int_{\mathbb{R}^{1+3}} C_{\ll d} \phi_{\mu,N} (C_d^{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_{\approx d}^{\pm 2} \varphi_{\lambda_2, N_2} dx dt \right| \\ & \lesssim \frac{\mu}{\lambda_1} \sum_{\substack{q, q' \in \mathcal{Q}_\mu \\ |q - q'| \lesssim \mu}} \|\phi_{\mu,N}\|_{L_{t,x}^\infty} \|P_q C_d^{\pm 1} \psi_{\lambda_1, N_1}\|_{L_{t,x}^2} \|P_{q'} C_{\approx d}^{\pm 2} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^2} \\ & \lesssim \left(\frac{d}{\mu}\right)^{-1} \left(\frac{\mu}{\lambda_1}\right)^{1-\epsilon} \mu^{\frac{1}{2}} \|\phi_{\mu,N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}, \end{aligned}$$

which, as above, can be summed up with respect to $\mu \ll d \ll \lambda_1$.

A similar argument treats the A_2 term.

High modulation, II: $\mu \gg \min\{\lambda_1, \lambda_2\}$ and $d \gg \min\{\lambda_1, \lambda_2\}$. Our goal is to prove the bound

$$\begin{aligned} & \sum_{d \gg \min\{\lambda_1, \lambda_2\}} \left| \int_{\mathbb{R}^{1+3}} A_0 dx dt \right| + \left| \int_{\mathbb{R}^{1+3}} A_1 dx dt \right| + \left| \int_{\mathbb{R}^{1+3}} A_2 dx dt \right| \\ & \lesssim \mu^{\frac{1}{2}} \left(\frac{\min\{\lambda_1, \lambda_2\}}{\mu}\right)^{\frac{1}{4}} \|\phi_{\mu,N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \quad (8-35)$$

As the estimate is essentially symmetric in λ_1 and λ_2 , we may assume that $\lambda_1 \geq \lambda_2$.

First, we consider the contribution to the sum where $d \gtrsim \mu$. The bound for A_0 follows by decomposing into cubes of size λ_2 and applying the standard $L_{t,x}^4$ Strichartz estimate to obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}^{3+1}} A_0 dt dx \right| \lesssim \sum_{\substack{q, q'' \in \mathcal{Q}_{\lambda_2} \\ |q - q''| \lesssim \lambda_2}} \|C_d P_{q''} \phi_{\mu,N}\|_{L_{t,x}^2} \|P_q \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|\varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ & \lesssim \mu^{\frac{1}{2}} \left(\frac{\lambda_2}{\mu}\right)^{\frac{3}{4}-\epsilon} \left(\frac{\mu}{d}\right)^{\frac{1}{2}} \|\phi_{\mu,N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}, \end{aligned}$$

which easily gives (8-35) in the range $d \gtrsim \mu$ for the A_0 term, provided we choose ϵ sufficiently small. The proof for the A_1 term is identical (as we do not exploit any null structure here). On the other hand, to estimate the A_2 term, we again decompose into cubes of size λ_2 and apply the $L^4_{t,x}$ Strichartz estimate to deduce that

$$\begin{aligned} \left| \int_{\mathbb{R}^{3+1}} A_2 dt dx \right| &\lesssim \sum_{\substack{q, q'' \in Q_{\lambda_2} \\ |q - q''| \lesssim \lambda_2}} \|P_{q''} \phi_{\mu, N}\|_{L^4_{t,x}} \|P_q \psi_{\lambda_1, N_1}\|_{L^4_{t,x}} \|C_d^{\pm 2} \varphi_{\lambda_2, N_2}\|_{L^2_{t,x}} \\ &\lesssim \mu^{\frac{1}{2}} \left(\frac{\lambda_2}{\mu}\right)^{\frac{1}{2} - \epsilon} \left(\frac{\mu}{d}\right)^{\frac{1}{2}} \|\phi_{\mu, N}\|_{V^2_{+,1}} \|\psi_{\lambda_1, N_1}\|_{V^2_{\pm 1, M}} \|\varphi_{\lambda_2, N_2}\|_{V^2_{\pm 2, M}}. \end{aligned}$$

Therefore (8-35) follows in the range $d \gtrsim \mu$.

Second, we consider the contribution to the sum where $\lambda_2 \ll d \ll \mu$. Concerning A_0 , as in the first high modulation case, we have the decomposition (8-33). To bound the first term in (8-33), we have

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} C_d \phi_{\mu, N} (C_{\approx d}^{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_{\leq d}^{\pm 2} \varphi_{\lambda_2, N_2} dx dt \right| &\lesssim \|C_d \phi_{\mu, N}\|_{L^2_{t,x}} \|C_{\approx d}^{\pm 1} \psi_{\lambda_1, N_1}\|_{L^2_{t,x}} \|C_{\leq d}^{\pm 2} \varphi_{\lambda_2, N_2}\|_{L^\infty_{t,x}} \\ &\lesssim \left(\frac{d}{\lambda_2}\right)^{-1} \left(\frac{\lambda_2}{\mu}\right)^{\frac{1}{2}} \mu^{\frac{1}{2}} \|\phi_{\mu, N}\|_{V^2_{+,1}} \|\psi_{\lambda_1, N_1}\|_{V^2_{\pm 1, M}} \|\varphi_{\lambda_2, N_2}\|_{V^2_{\pm 2, M}}. \end{aligned}$$

To bound the second term in (8-33), we have

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} C_d \phi_{\mu, N} (C_{\ll d}^{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_{\approx d}^{\pm 2} \varphi_{\lambda_2, N_2} dx dt \right| &\lesssim \|C_d \phi_{\mu, N}\|_{L^2_{t,x}} \|C_{\ll d}^{\pm 1} \psi_{\lambda_1, N_1}\|_{L^\infty_t L^2_x} \|C_{\approx d}^{\pm 2} \varphi_{\lambda_2, N_2}\|_{L^2_t L^\infty_x} \\ &\lesssim \left(\frac{d}{\lambda_2}\right)^{-1} \left(\frac{\lambda_2}{\mu}\right)^{\frac{1}{2}} \mu^{\frac{1}{2}} \|\phi_{\mu, N}\|_{V^2_{+,1}} \|\psi_{\lambda_1, N_1}\|_{V^2_{\pm 1, M}} \|\varphi_{\lambda_2, N_2}\|_{V^2_{\pm 2, M}}. \end{aligned}$$

Concerning A_1 , as in the first high-modulation case, we have the decomposition (8-34), and we can repeat the argument above for the A_0 terms.

Concerning A_2 , we have the decomposition

$$\begin{aligned} \int_{\mathbb{R}^{1+3}} A_2 dx dt &= \int_{\mathbb{R}^{1+3}} C_{\approx d} \phi_{\mu, N} (C_{\leq d}^{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_d^{\pm 2} \varphi_{\lambda_2, N_2} dx dt \\ &\quad + \int_{\mathbb{R}^{1+3}} C_{\ll d} \phi_{\mu, N} (C_{\approx d}^{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_d^{\pm 2} \varphi_{\lambda_2, N_2} dx dt. \end{aligned}$$

The first term can be treated in the same manner as the second contribution to A_0 . For the second term we have

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} C_{\ll d} \phi_{\mu, N} (C_{\approx d}^{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_d^{\pm 2} \varphi_{\lambda_2, N_2} dx dt \right| &\lesssim \|\phi_{\mu, N}\|_{L^\infty_t L^2_x} \|C_{\approx d}^{\pm 1} \psi_{\lambda_1, N_1}\|_{L^2_{t,x}} \|C_d^{\pm 2} \varphi_{\lambda_2, N_2}\|_{L^2_t L^\infty_x} \\ &\lesssim \left(\frac{d}{\lambda_2}\right)^{-1} \left(\frac{\lambda_2}{\mu}\right)^{\frac{1}{2}} \mu^{\frac{1}{2}} \|\phi_{\mu, N}\|_{V^2_{+,1}} \|\psi_{\lambda_1, N_1}\|_{V^2_{\pm 1, M}} \|\varphi_{\lambda_2, N_2}\|_{V^2_{\pm 2, M}}. \quad \square \end{aligned}$$

8D. Proof of Theorem 7.5. We begin with the proof of (7-9). An application of the energy inequality in Lemma 7.3 gives

$$\begin{aligned} & \|P_{\lambda_1} H_{N_1} \Pi_{\pm_1} \mathcal{I}_M^{\pm_1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm_2} \varphi_{\lambda_2, N_2}]\|_{V_{\pm_1, M}^2} \\ & \lesssim \sup_{\|\psi_{\lambda_1, N_1}\|_{V_{\pm_1, M}^2} \lesssim 1} \left| \int_{\mathbb{R}^{1+3}} \phi_{\mu, N} (\Pi_{\pm_1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm_2} \varphi_{\lambda_2, N_2} dx dt \right|. \end{aligned}$$

Therefore an application of (8-10) in Theorem 8.8 implies

$$\begin{aligned} & \|P_{\lambda_1} H_{N_1} \Pi_{\pm_1} \mathcal{I}_M^{\pm_1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm_2} \varphi_{\lambda_2, N_2}]\|_{V_{\pm_1, M}^2} \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^{\frac{\sigma}{4}} \mathbf{B}_{\min\{\frac{\sigma}{32}, \frac{1}{2a} - \frac{1}{4}\}} \|\phi\|_{F_{\mu, N}^{+,1}} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm_2, M}}, \end{aligned} \quad (8-36)$$

which gives the required bound (7-9) for the $F_{\lambda_1, N_1}^{\pm_1, M}$ component of the norm. To complete the proof of (7-9), it remains show that there exists $\epsilon > 0$ such that

$$\|\Pi_{\pm_1} \mathcal{I}_M^{\pm_1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm_2} \varphi_{\lambda_2, N_2}]\|_{Y_{\lambda_1, N_1}^{\pm_1, M}} \lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^{\frac{\sigma}{2}} \mathbf{B}_\epsilon \|\phi\|_{F_{\mu, N}^{+,1}} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm_2, M}}. \quad (8-37)$$

To this end, we consider separately the cases $\lambda_1 \ll \lambda_2$ and $\lambda_1 \gtrsim \lambda_2$. In the former region, note that an application of (8-12) in Theorem 8.8 together with the energy inequality Lemma 7.3, and the $L_{t,x}^2$ bound in Lemma 7.2, gives

$$\begin{aligned} & \|P_{\lambda_1} H_{N_1} C_d^{\pm_1} \mathcal{I}_M^{\pm_1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm_2} \varphi_{\lambda_2, N_2}]\|_{L_{t,x}^2} \\ & \lesssim d^{-\frac{1}{2}} \|P_{\lambda_1} H_{N_1} \Pi_{\pm_1} \mathcal{I}_M^{\pm_1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm_2} \varphi_{\lambda_2, N_2}]\|_{V_{\pm_1, M}^2} \\ & \lesssim d^{-\frac{1}{2}} \mu^{\frac{1}{2}} (\min\{N, N_2\})^{\frac{\sigma}{4}} \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\sigma}{32}} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm_2, M}^2}. \end{aligned} \quad (8-38)$$

On the other hand, since we are localised away from the hyperboloid we have by (7-7) together with Lemma 8.2

$$\begin{aligned} & \|P_{\lambda_1} H_{N_1} C_d^{\pm_1} \mathcal{I}_M^{\pm_1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm_2} \varphi_{\lambda_2, N_2}]\|_{L_t^{3/2} L_x^2} \lesssim d^{-1} \|P_{\lambda_1} (\phi_{\mu, N} \gamma^0 \Pi_{\pm_2} \varphi_{\lambda_2, N_2})\|_{L_t^{3/2} L_x^2} \\ & \lesssim d^{-1} \|\phi_{\mu, N}\|_{L_{t,x}^4} \|\varphi_{\lambda_2, N_2}\|_{L_t^{12/5} L_x^4} \\ & \lesssim d^{-1} \mu^{\frac{1}{2}} \lambda_2^{\frac{1}{3}} N_2 \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm_2, M}^2}. \end{aligned} \quad (8-39)$$

Repeating this argument but instead putting $\phi \in L_t^{\frac{12}{5}} L_x^4$ and $\varphi \in L_{t,x}^4$ we deduce that, since $\lambda_1 \ll \lambda_2 \approx \mu$,

$$\begin{aligned} & d \lambda_1^{-\frac{1}{3}} \|P_{\lambda_1} H_{N_1} C_d^{\pm_1} \mathcal{I}_M^{\pm_1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm_2} \varphi_{\lambda_2, N_2}]\|_{L_t^{3/2} L_x^2} \\ & \lesssim \mu^{\frac{1}{2}} \min\{N, N_2\} \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{3}} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm_2, M}^2}. \end{aligned} \quad (8-40)$$

Note that this bound is far too weak to be useful on its own, as we have $\lambda_1 \ll \lambda_2$. On the other hand, if we combine (8-38) and (8-40), and use the convexity of the L_t^p spaces, we deduce that if we let $0 < \theta < 1$

be given by

$$\frac{1}{a} = \frac{2\theta}{3} + \frac{1-\theta}{2},$$

then, as this forces $b = \frac{1}{2}(1 + \theta)$, we deduce that

$$\begin{aligned} & \lambda_1^{\frac{1}{a}-b} d^b \|P_{\lambda_1} H_{N_1} C_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}]\|_{L_t^q L_x^2} \\ & \lesssim (d \lambda_1^{-\frac{1}{3}} \|P_{\lambda_1} H_{N_1} C_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}]\|_{L_t^{3/2} L_x^2})^\theta \\ & \quad \times (d^{\frac{1}{2}} \|P_{\lambda_1} H_{N_1} C_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}]\|_{L_{t,x}^2})^{1-\theta} \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^{\theta + \frac{\sigma}{4}(1-\theta)} \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\sigma}{32}(1-\theta) - \frac{1}{3}\theta} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned}$$

Since

$$\frac{1}{2} < \frac{1}{a} < \frac{1}{2} + \frac{\sigma}{1000},$$

it is easy enough to check that $\frac{1}{32}\sigma(1 - \theta) - \frac{1}{3}\theta > 0$, and hence (8-37) holds when $\lambda_1 \ll \lambda_2$. We now consider the case $\lambda_1 \gtrsim \lambda_2$. The proof is similar to the previous case; the main difference is that we need a more refined version of the bound (8-40). To this end, by decomposing φ into cubes of size $\min\{\mu, \lambda_2\}$, we deduce that by Lemma 8.2 and Lemma 8.6, for every $\epsilon' > 0$

$$\begin{aligned} & \|P_{\lambda_1} H_{N_1} C_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}]\|_{L_t^{3/2} L_x^2} \\ & \lesssim d^{-1} \|(\|\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} P_q \varphi_{\lambda_2, N_2}\|_{L_x^2}^2)^{\frac{1}{2}}\|_{L_t^{3/2}} \\ & \lesssim d^{-1} \|\phi_{\mu, N}\|_{L_t^{12/5} L_x^4} \left(\sum_{q \in \mathcal{Q}_{\min\{\mu, \lambda_2\}}} \|P_q \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4}^2\right)^{\frac{1}{2}} \\ & \lesssim d^{-1} \mu^{\frac{1}{3}} N (\min\{\mu, \lambda_2\})^{\frac{1}{4}-\epsilon'} \lambda_2^{\frac{1}{4}+\epsilon'} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned}$$

Since $(\min\{\mu, \lambda_2\})^{\frac{1}{4}-\epsilon'} \leq \mu^{\frac{1}{6}} \lambda_2^{\frac{1}{4}-\frac{1}{6}+\epsilon'}$ (for ϵ' sufficiently small) and $\lambda_2 \lesssim \lambda_1$, by using the bound (8-39),

$$\begin{aligned} & d \lambda_1^{-\frac{1}{3}} \|P_{\lambda_1} H_{N_1} C_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}]\|_{L_t^{3/2} L_x^2} \\ & \lesssim \mu^{\frac{1}{2}} \min\{N, N_2\} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \tag{8-41}$$

Note that, unlike the bound (8-41), we have no high-frequency loss here. As in the case $\lambda_1 \ll \lambda_2$, we now combine the bound (8-36) with (8-41), and deduce by the convexity of the L_t^p norm and Lemma 7.2 that

$$\begin{aligned} & \lambda_1^{\frac{1}{a}-b} d^b \|P_{\lambda_1} H_{N_1} C_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}]\|_{L_t^q L_x^2} \\ & \lesssim (d \lambda_1^{-\frac{1}{3}} \|P_{\lambda_1} H_{N_1} C_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}]\|_{L_t^{3/2} L_x^2})^\theta \\ & \quad \times (d^{\frac{1}{2}} \|P_{\lambda_1} H_{N_1} C_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}]\|_{L_{t,x}^2})^{1-\theta} \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^{\theta + \frac{\sigma}{4}(1-\theta)} \mathbf{B}_{\min\{\frac{\sigma}{32}, \frac{1}{2a} - \frac{1}{4}\}}^{1-\theta} \|\phi\|_{F_{\mu, N}^{+,1}} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm 2, M}}. \end{aligned}$$

Since $0 < \theta \ll \sigma$, we obtain (8-37). Therefore, the bound (7-9) follows.

We now turn to the proof of the second inequality (7-10). The argument is similar to the proof of (7-9) so we will be brief. An application of the energy inequality in Lemma 7.3 together with (8-11) in Theorem 8.8 implies

$$\begin{aligned} & \|P_\mu H_N \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \|_{V_{+,1}^2} \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N_1, N_2\})^{\frac{\sigma}{4}} \mathbf{B}_{\min\{\frac{\sigma}{32}, \frac{1}{2a} - \frac{1}{4}\}} \| \psi \|_{F_{\lambda_1, N_1}^{\pm 1, M}} \| \varphi \|_{F_{\lambda_2, N_2}^{\pm 2, M}}. \end{aligned} \quad (8-42)$$

Therefore it only remains to prove that there exists $\epsilon > 0$ such that

$$\| \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \|_{Y_{\mu, N}^{+,1}} \lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^{\frac{\sigma}{2}} \mathbf{B}_\epsilon \| \psi \|_{F_{\lambda_1, N_1}^{\pm 1, M}} \| \varphi \|_{F_{\lambda_2, N_2}^{\pm 2, M}}. \quad (8-43)$$

Similar to the proof of (8-37), we consider separately the cases $\mu \ll \lambda_1$ and $\mu \gtrsim \lambda_1$. In the former case, as in (8-41), since we are localised away from the hyperboloid we have by (7-7) together with Lemma 8.2

$$\begin{aligned} & \|P_\mu H_N C_d^{\pm 1} \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \|_{L_t^{3/2} L_x^2} \\ & \lesssim d^{-1} \|P_\mu ((\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}) \|_{L_t^{3/2} L_x^2} \\ & \lesssim d^{-1} (\min\{\lambda_1, \lambda_2\})^{\frac{1}{3}} (\max\{\lambda_1, \lambda_2\})^{\frac{1}{2}} \min\{N_1, N_2\} \| \psi_{\lambda_1, N_1} \|_{V_{\pm 1, M}^2} \| \varphi_{\lambda_2, N_2} \|_{V_{\pm 2, M}^2}. \end{aligned} \quad (8-44)$$

Since $\lambda_1 \approx \lambda_2$, we can replace the max and min in (8-44) with $\lambda_1^{\frac{1}{3} + \frac{1}{2}}$. If we now combine (8-44) with the energy inequality in Lemma 7.3, the bound (8-13) in Theorem 8.8, and Lemma 7.2, we deduce that by the convexity of the L_t^p spaces that

$$\begin{aligned} & \mu^{\frac{1}{a} - b} d^b \|P_\mu H_N C_d^+ \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \|_{L_t^a L_x^2} \\ & \lesssim (d\mu^{-\frac{1}{3}} \|P_\mu C_d^+ \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \|_{L_t^{3/2} L_x^2})^\theta \\ & \quad \times (d^{\frac{1}{2}} \|P_\mu C_d^+ \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \|_{L_{t,x}^2})^{1-\theta} \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N_1, N_2\})^{\theta + \frac{\sigma}{4}(1-\theta)} \left(\frac{\lambda_1}{\mu}\right)^{\frac{\sigma}{32}(1-\theta) - \frac{5}{6}\theta} \| \psi_{\lambda_1, N_1} \|_{V_{\pm 1, M}^2} \| \varphi_{\lambda_2, N_2} \|_{V_{\pm 2, M}^2}, \end{aligned}$$

where as previously, we have

$$\frac{1}{a} = \frac{2\theta}{3} + \frac{1-\theta}{2},$$

which implies $b = \frac{1}{2}(1 + \theta)$. Since

$$\frac{1}{2} < \frac{1}{a} < \frac{1}{2} + \frac{\sigma}{1000},$$

it is easy enough to check that $\frac{1}{32}\sigma(1 - \theta) - \frac{5}{6}\theta > 0$, and hence (8-43) holds when $\mu \ll \lambda_1$. We now consider the case $\mu \gtrsim \lambda_1$. Since we now have

$$(\min\{\lambda_1, \lambda_2\})^{\frac{1}{3}} (\max\{\lambda_1, \lambda_2\})^{\frac{1}{2}} \lesssim \mu^{\frac{1}{3} + \frac{1}{2}},$$

an application of (8-44), together with (8-42), Lemma 7.2 gives

$$\begin{aligned} & \mu^{\frac{1}{a}-b} d^b \left\| P_\mu H_N C_d^+ \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{L_t^q L_x^2} \\ & \lesssim (d\mu^{-\frac{1}{3}} \left\| P_\mu C_d^+ \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{L_t^{3/2} L_x^2})^\theta \\ & \quad \times (d^{\frac{1}{2}} \left\| P_\mu C_d^+ \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{L_{t,x}^2})^{1-\theta} \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N_1, N_2\})^{\theta + \frac{\sigma}{4}(1-\theta)} \mathbf{B}_{\min\{\frac{\sigma}{32}, \frac{1}{2a} - \frac{1}{4}\}}^{1-\theta} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned}$$

Since $0 < \theta \ll \sigma$ and $1/a > 1/2$, we obtain (8-43). Therefore, the bound (7-9) follows. This completes the proof of Theorem 7.5.

References

- [Bachelot 1988] A. Bachelot, “Problème de Cauchy global pour des systèmes de Dirac–Klein–Gordon”, *Ann. Inst. H. Poincaré Phys. Théor.* **48**:4 (1988), 387–422. MR Zbl
- [Bejenaru 2017] I. Bejenaru, “Optimal bilinear restriction estimates for general hypersurfaces and the role of the shape operator”, *Internat. Math. Res. Notices* **2017**:23 (2017), 7109–7147.
- [Bejenaru and Herr 2011] I. Bejenaru and S. Herr, “Convolutions of singular measures and applications to the Zakharov system”, *J. Funct. Anal.* **261**:2 (2011), 478–506. MR Zbl
- [Bejenaru and Herr 2015] I. Bejenaru and S. Herr, “The cubic Dirac equation: small initial data in $H^1(\mathbb{R}^3)$ ”, *Comm. Math. Phys.* **335**:1 (2015), 43–82. MR Zbl
- [Bejenaru and Herr 2017] I. Bejenaru and S. Herr, “On global well-posedness and scattering for the massive Dirac–Klein–Gordon system”, *J. Eur. Math. Soc. (JEMS)* **19**:8 (2017), 2445–2467. MR Zbl
- [Bejenaru et al. 2009] I. Bejenaru, S. Herr, J. Holmer, and D. Tataru, “On the 2D Zakharov system with L^2 -Schrödinger data”, *Nonlinearity* **22**:5 (2009), 1063–1089. MR Zbl
- [Bejenaru et al. 2010] I. Bejenaru, S. Herr, and D. Tataru, “A convolution estimate for two-dimensional hypersurfaces”, *Rev. Mat. Iberoam.* **26**:2 (2010), 707–728. MR Zbl
- [Bennett and Bez 2010] J. Bennett and N. Bez, “Some nonlinear Brascamp–Lieb inequalities and applications to harmonic analysis”, *J. Funct. Anal.* **259**:10 (2010), 2520–2556. MR Zbl
- [Bennett et al. 2005] J. Bennett, A. Carbery, and J. Wright, “A non-linear generalisation of the Loomis–Whitney inequality and applications”, *Math. Res. Lett.* **12**:4 (2005), 443–457. MR Zbl
- [Bennett et al. 2006] J. Bennett, A. Carbery, and T. Tao, “On the multilinear restriction and Kakeya conjectures”, *Acta Math.* **196**:2 (2006), 261–302. MR Zbl
- [Bjorken and Drell 1964] J. D. Bjorken and S. D. Drell, *Relativistic quantum mechanics*, McGraw-Hill, New York, 1964. MR
- [Bourgain 1991] J. Bourgain, “On the restriction and multiplier problems in \mathbb{R}^3 ”, pp. 179–191 in *Geometric aspects of functional analysis (1989–90)*, edited by V. D. Milman and J. Lindenstrauss, Lecture Notes in Math. **1469**, Springer, 1991. MR Zbl
- [Bourgain 1995] J. Bourgain, “Estimates for cone multipliers”, pp. 41–60 in *Geometric aspects of functional analysis (Israel, 1992–1994)*, edited by J. Lindenstrauss and V. Milman, Oper. Theory Adv. Appl. **77**, Birkhäuser, Basel, 1995. MR Zbl
- [Bourgain 1998] J. Bourgain, “Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity”, *Internat. Math. Res. Notices* **1998**:5 (1998), 253–283. MR Zbl
- [Buschenhenke et al. 2017] S. Buschenhenke, D. Müller, and A. Vargas, “A Fourier restriction theorem for a two-dimensional surface of finite type”, *Anal. PDE* **10**:4 (2017), 817–891. MR Zbl
- [Chadam and Glassey 1974] J. M. Chadam and R. T. Glassey, “On certain global solutions of the Cauchy problem for the (classical) coupled Klein–Gordon–Dirac equations in one and three space dimensions”, *Arch. Rational Mech. Anal.* **54** (1974), 223–237. MR Zbl

- [Cho and Lee 2013] Y. Cho and S. Lee, “Strichartz estimates in spherical coordinates”, *Indiana Univ. Math. J.* **62**:3 (2013), 991–1020. MR Zbl
- [D’Ancona et al. 2007] P. D’Ancona, D. Foschi, and S. Selberg, “Null structure and almost optimal local regularity for the Dirac–Klein–Gordon system”, *J. Eur. Math. Soc. (JEMS)* **9**:4 (2007), 877–899. MR Zbl
- [Foschi and Klainerman 2000] D. Foschi and S. Klainerman, “Bilinear space-time estimates for homogeneous wave equations”, *Ann. Sci. École Norm. Sup. (4)* **33**:2 (2000), 211–274. MR Zbl
- [Ginibre et al. 1997] J. Ginibre, Y. Tsutsumi, and G. Velo, “On the Cauchy problem for the Zakharov system”, *J. Funct. Anal.* **151**:2 (1997), 384–436. MR Zbl
- [Hadac et al. 2009] M. Hadac, S. Herr, and H. Koch, “Well-posedness and scattering for the KP-II equation in a critical space”, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**:3 (2009), 917–941. MR Zbl
- [Keel and Tao 1998] M. Keel and T. Tao, “Endpoint Strichartz estimates”, *Amer. J. Math.* **120**:5 (1998), 955–980. MR Zbl
- [Klainerman and Machedon 1993] S. Klainerman and M. Machedon, “Space-time estimates for null forms and the local existence theorem”, *Comm. Pure Appl. Math.* **46**:9 (1993), 1221–1268. MR Zbl
- [Klainerman and Selberg 2002] S. Klainerman and S. Selberg, “Bilinear estimates and applications to nonlinear wave equations”, *Commun. Contemp. Math.* **4**:2 (2002), 223–295. MR Zbl
- [Klainerman et al. 2002] S. Klainerman, I. Rodnianski, and T. Tao, “A physical space approach to wave equation bilinear estimates”, *J. Anal. Math.* **87** (2002), 299–336. MR Zbl
- [Koch and Steinerberger 2015] H. Koch and S. Steinerberger, “Convolution estimates for singular measures and some global nonlinear Brascamp–Lieb inequalities”, *Proc. Roy. Soc. Edinburgh Sect. A* **145**:6 (2015), 1223–1237. MR Zbl
- [Koch and Tataru 2005] H. Koch and D. Tataru, “Dispersive estimates for principally normal pseudodifferential operators”, *Comm. Pure Appl. Math.* **58**:2 (2005), 217–284. MR Zbl
- [Koch et al. 2014] H. Koch, D. Tataru, and M. Vişan, *Dispersive equations and nonlinear waves: generalized Korteweg–de Vries, nonlinear Schrödinger, wave and Schrödinger maps*, Oberwolfach Seminars **45**, Springer, 2014. MR Zbl
- [Lee 2006a] S. Lee, “Bilinear restriction estimates for surfaces with curvatures of different signs”, *Trans. Amer. Math. Soc.* **358**:8 (2006), 3511–3533. MR Zbl
- [Lee 2006b] S. Lee, “Linear and bilinear estimates for oscillatory integral operators related to restriction to hypersurfaces”, *J. Funct. Anal.* **241**:1 (2006), 56–98. MR Zbl
- [Lee and Vargas 2008] S. Lee and A. Vargas, “Sharp null form estimates for the wave equation”, *Amer. J. Math.* **130**:5 (2008), 1279–1326. MR Zbl
- [Lee and Vargas 2010] S. Lee and A. Vargas, “Restriction estimates for some surfaces with vanishing curvatures”, *J. Funct. Anal.* **258**:9 (2010), 2884–2909. MR Zbl
- [Lee et al. 2008] S. Lee, K. M. Rogers, and A. Vargas, “Sharp null form estimates for the wave equation in \mathbb{R}^{3+1} ”, *Int. Math. Res. Not.* **2008** (2008), art. id. rnn096. MR Zbl
- [Machihara et al. 2003] S. Machihara, K. Nakanishi, and T. Ozawa, “Small global solutions and the nonrelativistic limit for the nonlinear Dirac equation”, *Rev. Mat. Iberoamericana* **19**:1 (2003), 179–194. MR Zbl
- [Moyua et al. 1999] A. Moyua, A. Vargas, and L. Vega, “Restriction theorems and maximal operators related to oscillatory integrals in \mathbb{R}^3 ”, *Duke Math. J.* **96**:3 (1999), 547–574. MR Zbl
- [Stein 1993] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series **43**, Princeton University Press, 1993. MR Zbl
- [Stein and Weiss 1971] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series **32**, Princeton University Press, 1971. MR Zbl
- [Sterbenz 2005] J. Sterbenz, “Angular regularity and Strichartz estimates for the wave equation”, *Int. Math. Res. Not.* **2005**:4 (2005), 187–231. MR Zbl
- [Sterbenz and Tataru 2010] J. Sterbenz and D. Tataru, “Energy dispersed large data wave maps in $2 + 1$ dimensions”, *Comm. Math. Phys.* **298**:1 (2010), 139–230. MR Zbl
- [Strichartz 1972] R. S. Strichartz, “Multipliers for spherical harmonic expansions”, *Trans. Amer. Math. Soc.* **167** (1972), 115–124. MR Zbl

- [Strichartz 1977] R. S. Strichartz, “Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations”, *Duke Math. J.* **44**:3 (1977), 705–714. MR Zbl
- [Tao 2001] T. Tao, “Endpoint bilinear restriction theorems for the cone, and some sharp null form estimates”, *Math. Z.* **238**:2 (2001), 215–268. MR Zbl
- [Tao 2003] T. Tao, “A sharp bilinear restrictions estimate for paraboloids”, *Geom. Funct. Anal.* **13**:6 (2003), 1359–1384. MR Zbl
- [Tao 2004] T. Tao, “Some recent progress on the restriction conjecture”, pp. 217–243 in *Fourier analysis and convexity*, Birkhäuser, Boston, 2004. MR Zbl
- [Tao and Vargas 2000a] T. Tao and A. Vargas, “A bilinear approach to cone multipliers, I: Restriction estimates”, *Geom. Funct. Anal.* **10**:1 (2000), 185–215. MR Zbl
- [Tao and Vargas 2000b] T. Tao and A. Vargas, “A bilinear approach to cone multipliers, II: Applications”, *Geom. Funct. Anal.* **10**:1 (2000), 216–258. MR Zbl
- [Tao et al. 1998] T. Tao, A. Vargas, and L. Vega, “A bilinear approach to the restriction and Keakeya conjectures”, *J. Amer. Math. Soc.* **11**:4 (1998), 967–1000. MR Zbl
- [Wang 2015] X. Wang, “On global existence of 3D charge critical Dirac–Klein–Gordon system”, *Int. Math. Res. Not.* **2015**:21 (2015), 10801–10846. MR Zbl
- [Wolff 2001] T. Wolff, “A sharp bilinear cone restriction estimate”, *Ann. of Math. (2)* **153**:3 (2001), 661–698. MR Zbl

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TIMOTHY CANDY: tcandy@math.uni-bielefeld.de
Universität Bielefeld, Fakultät für Mathematik, Bielefeld, Germany

SEBASTIAN HERR: herr@math.uni-bielefeld.de
Universität Bielefeld, Fakultät für Mathematik, Bielefeld, Germany

WELL-POSEDNESS AND SMOOTHING EFFECT FOR GENERALIZED NONLINEAR SCHRÖDINGER EQUATIONS

PIERRE-YVES BIENAIMÉ AND ABDESSLAM BOULKHEMAIR

We improve the result obtained by one of the authors, Bienaimé (2014), and establish the well-posedness of the Cauchy problem for some nonlinear equations of Schrödinger type in the usual Sobolev space $H^s(\mathbb{R}^n)$ for $s > \frac{n}{2} + 2$ instead of $s > \frac{n}{2} + 3$. We also improve the smoothing effect of the solution and obtain the optimal exponent.

1. Introduction

Consider the nonlinear Cauchy problem

$$\begin{cases} \partial_t u = i\mathcal{L}u + F(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}), & t \in \mathbb{R}, x \in \mathbb{R}^n, \\ u(x, 0) = u_0(x) \in H^s(\mathbb{R}^n), \end{cases} \quad (1)$$

where the function F is sufficiently regular in $\mathbb{C} \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n$, the operator \mathcal{L} has the form

$$\mathcal{L} = \sum_{j \leq j_0} \partial_{x_j}^2 - \sum_{j > j_0} \partial_{x_j}^2,$$

with a fixed $j_0 \in \{1, 2, \dots, n\}$, and $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, is the usual Sobolev space on \mathbb{R}^n . Thus, \mathcal{L} generalizes the Laplace operator but is not elliptic unless $j_0 = n$. Hence, such equations are generalizations of the nonlinear Schrödinger (NLS) equations.

In this paper, we continue the work undertaken in [Bienaimé 2014] and study the local existence and the smoothing effect of the solutions of the Cauchy problem (1) with essentially the following goal: to obtain the optimal index s of regularity for which (1) is well-posed. In fact, since the partial differential equation is of second order and is semilinear, the optimal condition on s should be $s > \frac{n}{2} + 1$. Unfortunately, up to now and due to issues that occur when estimating the remainder obtained after the linearization of the nonlinear equation, we have not been able to prove the desired result under such a condition. In any case, we shall return to this question in a future work. In this paper, we establish the following:

Theorem 1.1. *Assume that F vanishes to the third order at 0; that is, F and its partial derivatives up to the second order vanish at 0. Then, for every $s > \frac{n}{2} + 2$ and every initial data $u_0 \in H^s(\mathbb{R}^n)$, there exists a real number $T > 0$ such that the Cauchy problem (1) has a unique solution u which is defined on the interval $[0, T]$ and satisfies*

$$u \in C([0, T]; H^s(\mathbb{R}^n))$$

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and

$$\| \| J^{s+\frac{1}{2}} u \| \|_T \stackrel{\text{def}}{=} \sup_{\mu \in \mathbb{Z}^n} \left(\int_0^T \int_{\mathbb{R}^n} |\langle x - \mu \rangle^{-\sigma_0} J^{s+\frac{1}{2}} u(x, t)|^2 dx dt \right)^{\frac{1}{2}} < \infty,$$

where $J = (1 - \Delta)^{\frac{1}{2}}$, $\Delta = \sum_{k=1}^{k=n} \partial_{x_k}^2$ and $\sigma_0 > \frac{1}{2}$ is fixed. Moreover, given a bounded subset B of $H^s(\mathbb{R}^n)$, there exists a real number $T > 0$ such that, for every $u_0 \in B$, the associated solution u of (1) exists on the interval $[0, T]$ and the map which associates u to u_0 is Lipschitz continuous from B into the space

$$\{w \in C([0, T]; H^s(\mathbb{R}^n)) : \| \| J^{s+\frac{1}{2}} w \| \|_T < \infty\}.$$

In [Bienaimé 2014], this theorem is proved under the assumption $s > \frac{n}{2} + 3$. We also improve the result with respect to the smoothing effect of the solution since $\sigma_0 = 2$ there. Note that the assumption $\sigma_0 > \frac{1}{2}$ in the above theorem seems to be sharp; we refer for example to the survey article [Robbiano 2013] on the subject of Kato’s smoothing effect. Recall that at the origin of [Bienaimé 2014] was the significant work of C. E. Kenig, G. Ponce and L. Vega [Kenig et al. 1998], who first studied (1) with such a nonelliptic \mathcal{L} and established the local existence and the smoothing effect of the solutions assuming that F is a polynomial and $s \geq s_0$, the index s_0 being sufficiently large. Note that these authors did not give an idea about the value of s_0 , but by going back to the details of their proof, one can see that s_0 is of the order of $\frac{n}{2} + 10n + 1$. These authors also studied the case where F (is a polynomial and) vanishes to the second order at 0. However, it seems that in that case we need to work in weighted Sobolev spaces.

The Cauchy problem (1) was extensively studied in the 90s mainly when $\mathcal{L} = \Delta$, that is, in the case of the Schrödinger equation. See the introduction of [Kenig et al. 1998]. The case $\mathcal{L} \neq \Delta$ is less well-known. Nevertheless, it is motivated by several equations coming from the applications such as Ishimori-type equations or Davey–Stewartson-type systems. For more details, we refer the reader to the instructive introduction of [Kenig et al. 1998]. Let us now quote some papers which are more or less related to this subject. In [Kenig et al. 2004], the authors extended their results of 1998 to the quasilinear case assuming essentially that the corresponding dispersive operator \mathcal{L} is elliptic and nontrapping. The nonelliptic case is treated in [Kenig et al. 2006; 2005]. In [Bejenaru and Tataru 2008], the authors solved the Cauchy problem (1) for $s > \frac{n}{2} + 1$ in modified Sobolev spaces and assuming $F(u, \nabla_x u, \bar{u}, \nabla_x \bar{u})$ bilinear. More recently, in [Marzuola et al. 2012; 2014], the authors considered the quasilinear Schrödinger equation

$$i \partial_t u + \sum_{j,k} g^{j,k}(u, \nabla_x u) \partial_j \partial_k u = F(u, \nabla_x u)$$

and obtained the local well-posedness of the associated Cauchy problem for $s > \frac{n}{2} + 3$ in the quadratic case (with modified Sobolev spaces) and for $s > \frac{n}{2} + \frac{5}{2}$ in the nonquadratic case. However, they assume the smallness of the data and they do not seem to obtain the smoothing effect of the solutions.

The proof of Theorem 1.1 follows the same ideas as that of [Kenig et al. 1998; Bienaimé 2014]. Of course, the general plan is unoriginal: linearization of the nonlinear equation, then, establishing energy estimates for solutions of the linear equation, and finally, solving the nonlinear equation by means of an appropriate fixed-point theorem. Like [Bienaimé 2014], we start by applying a parilinearization, that is, a

linearization in the sense of [Bony 1981] instead of the classical linearization. This leads us to the use of the paradifferential calculus whose main interest lies in the fact that it eliminates the usual losses of regularity due to commutators. One obtains a parilinear equation and most of the proof of the theorem is concerned with the study of such an equation, that is, the well-posedness in the Sobolev spaces of the associated Cauchy problem by means of energy and smoothing effect estimates. As did Kenig, Ponce and Vega, we establish the smoothing effect estimate by using Doi's argument [1994] via Gårding's inequality, and we prove the energy estimates by following an idea of [Takeuchi 1992], that is, by constructing a nonclassical invertible pseudodifferential operator C which allows estimates for Cu if u is a solution of the parilinear equation. Finally, we solve the nonlinear Cauchy problem (1) by applying these estimates to an integrodifferential equation which is equivalent to (1) and obtain the solution as the fixed point of an appropriate contraction in an appropriate complete metric space.

Now, in order to give a more precise idea about our proof, let us indicate the differences with that given in [Bienaimé 2014]. In fact, there are three main differences:

- We simplify certain arguments of that paper; for example, we no longer need to use the general Hörmander symbol spaces $S_{\rho,\delta}^m$; we only use $S_{1,0}^m$ and $S_{0,0}^m$. Also, we only use the original paradifferential operators (see Section 2) and not the variant introduced in [Bienaimé 2014].
- The linear theorem, that is, Theorem 3.1 (see Section 3), is proved for general paradifferential operators T_{b_1} and T_{b_2} of order 0 instead of paramultiplication operators. Note also that we allow the operators C_1 and C_2 to be abstract bounded operators.
- The third difference lies in the nonlinear part (see Section 4) and is crucial for our improvement of the result of [Bienaimé 2014]: we use anisotropic Sobolev spaces and an interpolation inequality (see Proposition A.5) to estimate the remainder of the parilinearized equation.

2. Notations and preliminary results

Some notation used in the paper:

- $J^s = (1 - \Delta)^{\frac{s}{2}} = \langle D \rangle^s$ is the operator whose symbol is $\langle \xi \rangle^s = (1 + \xi^2)^{\frac{s}{2}}$.
- $D_{x_k} = -i \partial_{x_k}$, $D_x = -i \partial_x$.
- $|\alpha| = \sum_{j=1}^n \alpha_j$ if $\alpha \in \mathbb{N}^n$.
- $\Delta v = (\Delta v_1, \dots, \Delta v_n)$ and $\nabla v = (\nabla v_1, \dots, \nabla v_n)$ if $v = (v_1, \dots, v_n)$.
- $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of rapidly decreasing functions in \mathbb{R}^n .
- $\mathcal{D}(\mathbb{R}^n)$ denotes the space of smooth functions with compact support in \mathbb{R}^n .
- $\mathcal{D}'(\mathbb{R}^n)$ denotes the space of distributions in \mathbb{R}^n .
- $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions in \mathbb{R}^n .
- \hat{u} or $\mathcal{F}(u)$ denotes the Fourier transform of u .
- $H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \langle \xi \rangle^s \hat{u} \in L^2(\mathbb{R}^n)\}$ is the usual Sobolev space of regularity s .

- $\|u\|_s = \left(\int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$ denotes the norm of u in $H^s(\mathbb{R}^n)$.
- $\|u\|_E$ denotes the norm of u in the space E .
- Hörmander’s classes of symbols: if $m \in \mathbb{R}$ and $\gamma, \delta \in [0, 1]$,

$$S_{\gamma,\delta}^m = \{a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq A_{\alpha,\beta} \langle \xi \rangle^{m-\gamma|\beta|+\delta|\alpha|} \text{ for all } \alpha, \beta \in \mathbb{N}^n\}.$$

- If $\varrho > 0$ is an integer, $C^\varrho(\mathbb{R}^n)$ denotes the set of functions in \mathbb{R}^n which are bounded, of class C^m and their derivatives up to m are bounded. If $\varrho > 0$ is not an integer, $C^\varrho(\mathbb{R}^n)$ denotes the Hölder class, that is, the set of u in $C^{[\varrho]}(\mathbb{R}^n)$ such that

$$\exists C \in \mathbb{R}, \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n, \quad |\partial^\alpha u(x) - \partial^\alpha u(y)| \leq C|x - y|^{\varrho-[\varrho]}.$$

- $\text{Op } S$ denotes the set of pseudodifferential operators whose symbols belong to S .

The following statement summarizes the pseudodifferential calculus associated to Hörmander’s classes of symbols $S_{\gamma,\delta}^m$:

Theorem 2.1. *If $a \in S_{\gamma,\delta}^m$, $b \in S_{\gamma,\delta}^{m'}$, $m, m' \in \mathbb{R}$, and $0 \leq \delta < \gamma \leq 1$ or $0 \leq \delta \leq \gamma < 1$, then:*

- (i) $a(x, D)b(x, D) = c(x, D)$ with $c \in S_{\gamma,\delta}^{m+m'}$. Moreover,

$$\begin{aligned} c(x, \xi) &= \int e^{-iy \cdot \eta} a(x, \xi + \eta) b(x + y, \xi) \frac{dy d\eta}{(2\pi)^n} \\ &= \sum_{|v| < N} \frac{1}{v!} \partial_\xi^v a(x, \xi) D_x^v b(x, \xi) + \sum_{|v|=N} \frac{1}{v!} \int_0^1 (1-\theta)^{N-1} r_{v,\theta}(x, \xi) d\theta, \end{aligned}$$

where

$$r_{v,\theta}(x, \xi) = \int e^{-iy \cdot \eta} \partial_\xi^v a(x, \xi + \theta\eta) D_x^v b(x + y, \xi) \frac{dy d\eta}{(2\pi)^n},$$

and the $S_{\gamma,\delta}^{m+m'-N(\gamma-\delta)}$ seminorms of $r_{v,\theta}$ are bounded by products of seminorms of a and b uniformly in $\theta \in [0, 1]$.

- (ii) $a(x, D)^* = a^*(x, D)$ with $a^* \in S_{\gamma,\delta}^m$. Moreover,

$$\begin{aligned} a^*(x, \xi) &= \int e^{-iy \cdot \eta} \bar{a}(x + y, \xi + \eta) \frac{dy d\eta}{(2\pi)^n} \\ &= \sum_{|v| < N} \frac{1}{v!} \partial_\xi^v D_x^v \bar{a}(x, \xi) + \sum_{|v|=N} \frac{1}{v!} \int_0^1 (1-\theta)^{N-1} r_{v,\theta}^*(x, \xi) d\theta, \end{aligned}$$

where

$$r_{v,\theta}^*(x, \xi) = \int e^{-iy \cdot \eta} \partial_\xi^v D_x^v \bar{a}(x + y, \xi + \theta\eta) \frac{dy d\eta}{(2\pi)^n},$$

and the $S_{\gamma,\delta}^{m-N(\gamma-\delta)}$ seminorms of $r_{v,\theta}^*$ are bounded by seminorms of a uniformly in $\theta \in [0, 1]$.

See [Taylor 1991], for instance, for the proof. We shall also often need the following version of the Calderón–Vaillancourt theorem:

Theorem 2.2. *Let $a : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$ be a bounded function. Assume that, for all $\alpha, \beta \in \mathbb{N}^n$ such that $|\alpha| + |\beta| \leq n + 1$, there exists a constant $C_{\alpha,\beta} > 0$ such that $|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta} \in \mathbb{R}^{2n}$. Then, the pseudodifferential operator $a(x, D)$ is bounded in $L^2(\mathbb{R}^n)$ and its operator norm is estimated by*

$$\sup_{|\alpha|+|\beta|\leq n+1} \|\partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty}.$$

See [Coifman and Meyer 1978] for the proof.

The following technical lemma, which is a consequence of Theorem 2.1, will be very useful in many of our proofs:

Lemma 2.3. *Let $a \in S_{0,0}^m$, $m, \sigma \in \mathbb{R}$ and $\mu \in \mathbb{R}^n$. Then:*

- (i) *We have $\langle x - \mu \rangle^\sigma a(x, D) \langle x - \mu \rangle^{-\sigma} = a_\mu(x, D)$, where $a_\mu \in S_{0,0}^m$ and the seminorms of a_μ are bounded by seminorms of a uniformly in μ .*
- (ii) *If $\sigma \geq 0$ and if, in addition, $a(x, \xi)$ is rapidly decreasing with respect to $x - \mu$, then we have $\langle x - \mu \rangle^\sigma a(x, D) \langle x - \mu \rangle^{-\sigma} = b_\mu(x, D)$, where $b_\mu \in S_{0,0}^m$, b_μ is also rapidly decreasing in $x - \mu$ and the seminorms of b_μ are estimated uniformly in μ by expressions of the form*

$$\sup_{|\alpha|+|\beta|\leq N} \|\langle x - \mu \rangle^{2\sigma} \langle \xi \rangle^{-m} \partial_x^\alpha D_\xi^\beta a\|_{L^\infty}.$$

Here, the fact that the symbol $a(x, \xi)$ is rapidly decreasing with respect to $x - \mu$ means that, for every integer N and all multi-indices α, β , the function $\langle x - \mu \rangle^N \langle \xi \rangle^{-m} \partial_x^\alpha D_\xi^\beta a$ is bounded in $\mathbb{R}^n \times \mathbb{R}^n$, and we shall often meet such symbols in this paper.

Proof. (i) When $\sigma \geq 0$, we can use Theorem 2.1(i) and integrations by parts to obtain

$$\begin{aligned} a_\mu(x, \xi) &= \langle x - \mu \rangle^\sigma (2\pi)^{-n} \int e^{-iy \cdot \eta} a(x, \xi + \eta) \langle x + y - \mu \rangle^{-\sigma} dy d\eta \\ &= \langle x - \mu \rangle^\sigma (2\pi)^{-n} \int e^{-iy \cdot \eta} J_\eta^N [\langle \eta \rangle^{-N} a(x, \xi + \eta)] \langle y \rangle^{-N} J_y^N [\langle x + y - \mu \rangle^{-\sigma}] dy d\eta, \end{aligned}$$

where N is a large and even integer. Hence, by taking derivatives and bounding, and next by applying Peetre’s inequality,

$$\begin{aligned} |a_\mu(x, \xi)| &\leq C \|\langle \xi \rangle^{-m} a\|_{C^N} \langle x - \mu \rangle^\sigma \int \langle \eta \rangle^{-N} \langle \xi + \eta \rangle^m \langle y \rangle^{-N} \langle x + y - \mu \rangle^{-\sigma} dy d\eta \\ &\leq 2^{\frac{\sigma+|m|}{2}} C \|\langle \xi \rangle^{-m} a\|_{C^N} \langle \xi \rangle^m \int \langle \eta \rangle^{|m|-N} \langle y \rangle^{\sigma-N} dy d\eta = C' \langle \xi \rangle^m \|\langle \xi \rangle^{-m} a\|_{C^N}, \end{aligned}$$

where C and C' are constants which are independent of μ , and N is taken for example such that $N \geq |m| + \sigma + n + 1$. Of course, the derivatives of a_μ are treated in the same manner.

The case $\sigma < 0$ follows from the preceding case by considering the adjoint

$$a_\mu(x, D)^* = \langle x - \mu \rangle^{-\sigma} a(x, D)^* \langle x - \mu \rangle^\sigma$$

and by applying Theorem 2.1(ii).

(ii) By using the formula in Theorem 2.1(ii) once more, it is easy to see that, if a is rapidly decreasing with respect to $x - \mu$, then the symbol a^* is also rapidly decreasing with respect to $x - \mu$ and that, for all

$N \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}^n$, there exist $M \in \mathbb{N}$ and a nonnegative constant $C_{N,\alpha,\beta}$ which does not depend on μ such that

$$\|\langle x - \mu \rangle^N \langle \xi \rangle^{-m} \partial_x^\alpha D_\xi^\beta a^* \|_{L^\infty} \leq C_{N,\alpha,\beta} \sup_{|\alpha'|+|\beta'| \leq M} \|\langle x - \mu \rangle^N \langle \xi \rangle^{-m} \partial_x^{\alpha'} D_\xi^{\beta'} a \|_{L^\infty}.$$

Now, by following the same argument as that used in the first part, one can check that the same claim holds exactly when we replace a^* by a_μ in the above assertion; in particular, we have the estimate

$$\|\langle x - \mu \rangle^N \langle \xi \rangle^{-m} \partial_x^\alpha D_\xi^\beta a_\mu \|_{L^\infty} \leq C_{N,\alpha,\beta} \sup_{|\alpha'|+|\beta'| \leq M} \|\langle x - \mu \rangle^N \langle \xi \rangle^{-m} \partial_x^{\alpha'} D_\xi^{\beta'} a \|_{L^\infty},$$

and since we can write obviously $b_\mu(x, \xi) = \langle x - \mu \rangle^{2\sigma} a_\mu(x, \xi)$, this achieves the proof of the lemma. \square

When studying the nonlinear equation, the following result is important in order to explain the assumption made on the nonlinearity F .

Lemma 2.4. *For all $s \geq 0$ and all $\sigma > \frac{n}{2}$, there exists a constant $C > 0$ such that, for all $v \in H^s(\mathbb{R}^n)$, the sequence $\mu \mapsto \|\langle x - \mu \rangle^{-\sigma} v\|_s$ is in $\ell^2(\mathbb{Z}^n)$ and*

$$\sum_{\mu} \|\langle x - \mu \rangle^{-\sigma} v\|_s^2 \leq C \|v\|_s^2.$$

In particular, if $s > \frac{n}{2}$, $u, v \in H^s(\mathbb{R}^n)$ and χ is a smooth and rapidly decreasing function, then, $\mu \mapsto \|\chi(x - \mu)uv\|_s$ is in $\ell^1(\mathbb{Z}^n)$ and

$$\sum_{\mu} \|\chi(x - \mu)uv\|_s \leq C \|u\|_s \|v\|_s.$$

Proof. The case $s = 0$ is obvious and follows from the fact that $\sum_{\mu} \langle x - \mu \rangle^{-2\sigma}$ is a bounded function. The case where s is a positive integer reduces to the case $s = 0$ by taking derivatives via Leibniz formula. The general case is obtained by interpolation. Indeed, since the map $v \mapsto \langle x - \mu \rangle^{-\sigma} v$ is linear and bounded from H^s into $\ell^2(\mathbb{Z}^n, H^s)$ for integral indices $s = s_1, s_2$, it will be also bounded from $H^{s'}$ into $\ell^2(\mathbb{Z}^n, H^{s'})$ for any real s' between s_1 and s_2 . This follows from the fact that

$$[\ell^2(\mathbb{Z}^n, H^{s_1}), \ell^2(\mathbb{Z}^n, H^{s_2})]_{\theta} = \ell^2(\mathbb{Z}^n, [H^{s_1}, H^{s_2}]_{\theta})$$

for $0 < \theta < 1$. See for example [Bergh and Löfström 1976, Theorem 5.1.2, page 107].

The second part is a consequence of the first one and the fact that $H^s(\mathbb{R}^n)$ is an algebra if $s > \frac{n}{2}$. \square

Let us now recall some results on paradifferential operators.

Definition 2.5. We define the class Σ_{ρ}^m where $m \in \mathbb{R}$ and $\rho \geq 0$ to be the class of symbols $a(x, \xi)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ which are C^∞ in ξ and C^ρ in x , in the sense that

$$\text{for all } \alpha \in \mathbb{N}^n, \quad |\partial_\xi^\alpha a(x, \xi)| \langle \xi \rangle^{-m+|\alpha|} \in C^\rho(\mathbb{R}^n \times \mathbb{R}^n),$$

C^ϱ being replaced by L^∞ when $\varrho = 0$. If $a \in \Sigma_\varrho^m$, then m is the order of a and ϱ is its regularity. Following J.-M. Bony, we associate to a symbol a in Σ_ϱ^m the paradifferential operator $T_{a,\chi}$ defined by the expression

$$\widehat{T_{a,\chi}u}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} \chi(\xi - \eta, \eta) \mathcal{F}_1(a)(\xi - \eta, \eta) \hat{u}(\eta) d\eta,$$

where χ is what one calls a paratruncature, that is, a C^∞ function in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the following properties:

- (i) There exists $\varepsilon > 0$ such that $\varepsilon < 1$ and $\chi(\xi, \eta) = 0$ if $|\xi| \geq \varepsilon|\eta|$, $\xi, \eta \in \mathbb{R}^n$.
- (ii) There exist $\varepsilon' > 0$, $\varepsilon'' > 0$ such that $\varepsilon' < \varepsilon$ and $\chi(\xi, \eta) = 1$ if $|\xi| \leq \varepsilon'|\eta|$ and $|\eta| \geq \varepsilon''$.
- (iii) For all $\alpha \in \mathbb{N}^{2n}$, there exists $A_\alpha > 0$ such that for all $\zeta \in \mathbb{R}^{2n}$, we have $\langle \zeta \rangle^{|\alpha|} |\partial^\alpha \chi(\zeta)| \leq A_\alpha$.

The first important result on paradifferential operators is that, even if one can show that $T_{a,\chi} = \tilde{a}(x, D)$ with some $\tilde{a} \in S_{1,1}^m$, they are bounded in the Sobolev spaces in the usual manner. In fact, we have:

Theorem 2.6. *Assume that χ satisfies only the first and third properties among the above ones. Then, for every real s , the operator $T_{a,\chi}$ is bounded from $H^s(\mathbb{R}^n)$ into $H^{s-m}(\mathbb{R}^n)$ and its operator norm is estimated by a seminorm of a in Σ_ϱ^m . In particular, if $a = a(x) \in L^\infty(\mathbb{R}^n)$, then, for every real s , the operator $T_{a,\chi}$ is bounded in $H^s(\mathbb{R}^n)$ with an operator norm bounded by a constant times $\|a\|_{L^\infty}$.*

Proof. See [Bony 1981; Meyer 1981; Taylor 1991]. □

Concerning the dependence with respect to the paratruncature χ , one can say the following:

Theorem 2.7. *If $\varrho > 0$ and χ_1, χ_2 are paratruncatures, then the operator $T_{a,\chi_1} - T_{a,\chi_2}$ is bounded from $H^s(\mathbb{R}^n)$ into $H^{s-m+\varrho}(\mathbb{R}^n)$ and its operator norm is estimated by a seminorm of a in Σ_ϱ^m .*

Proof. See [Bony 1981; Meyer 1981; Taylor 1991]. □

This result shows that the dependence of $T_{a,\chi}$ on χ is less important than that on a . It also explains why the remainders in the paradifferential theory are only ϱ -regularizing. From now on, we shall write T_a instead of $T_{a,\chi}$ unless it is needed.

Note also that a possible choice of the paratruncature that we shall often use in the sequel is given by

$$\chi(\xi, \eta) = \chi_1(\xi/|\eta|)(1 - \psi_1(\eta)),$$

where $\psi_1, \chi_1 \in C^\infty(\mathbb{R}^n)$, $\psi_1 = 1$ in a neighbourhood of 0, $\psi_1 = 0$ out of $B(0, \varepsilon'')$, and $\chi_1 = 1$ on $B(0, \varepsilon')$, $\text{supp}(\chi) \subset B(0, \varepsilon)$, with ε and ε' satisfying $0 < \varepsilon' < \varepsilon < 1$. In this case, $T_{a,\chi} = \tilde{a}(x, D)$ with the following expression of \tilde{a} :

$$\tilde{a}(x, \xi) = (1 - \psi_1(\xi)) |\xi|^n \int_{\mathbb{R}^n} \mathcal{F}^{-1}(\chi_1)(|\xi|(x - y)) a(y, \xi) dy. \tag{2}$$

The following lemma gives some properties of \tilde{a} which will be needed in the sequel and often used implicitly.

Lemma 2.8. *Let $\varrho \geq 0$ and $a \in \Sigma_\varrho^m$. Then, \tilde{a} is smooth and*

$$|\partial_\xi^\beta \partial_x^\alpha \tilde{a}(x, \xi)| \leq A_{\alpha, \beta} \langle \xi \rangle^{m-|\beta|} \quad \text{if } |\alpha| \leq \varrho, \tag{3}$$

$$|\partial_\xi^\beta \partial_x^\alpha \tilde{a}(x, \xi)| \leq A_{\alpha, \beta} \langle \xi \rangle^{m-|\beta|+|\alpha|-\varrho} \quad \text{if } |\alpha| > \varrho, \tag{4}$$

where $A_{\alpha, \beta}$ are nonnegative constants; more precisely, the $A_{\alpha, \beta}$ can be estimated by seminorms of a in Σ_ϱ^m . In particular, $\tilde{a} \in S_{1,1}^m$.

Moreover, if θ is a smooth function with support in some compact subset of \mathbb{R}^n and $\theta_\mu(x) = \theta(x - \mu)$, $\mu \in \mathbb{Z}^n$, then, for all $N \in \mathbb{N}$, we have

$$\langle x - \mu \rangle^N |\partial_\xi^\beta \partial_x^\alpha \widetilde{\theta_\mu a}(x, \xi)| \leq A_{\alpha, \beta, N} \langle \xi \rangle^{m-|\beta|} \quad \text{if } |\alpha| \leq \varrho, \tag{5}$$

$$\langle x - \mu \rangle^N |\partial_\xi^\beta \partial_x^\alpha \widetilde{\theta_\mu a}(x, \xi)| \leq A_{\alpha, \beta, N} \langle \xi \rangle^{m-|\beta|+|\alpha|-\varrho} \quad \text{if } |\alpha| > \varrho, \tag{6}$$

where the $A_{\alpha, \beta, N}$ do not depend on μ and are estimated by seminorms of a in Σ_ϱ^m .

Proof. For the first part we refer to [Meyer 1981; Taylor 1991]. The second part follows from the first one by using, for example, for even N the decomposition

$$\langle x - \mu \rangle^N = \sum_\alpha \frac{(x - y)^\alpha}{\alpha!} \partial_y^\alpha \langle y - \mu \rangle^N$$

together with the expression (2). □

When dealing with nonlinear terms, we shall frequently use the following classical result:

Proposition 2.9. *If F is a C^∞ (or sufficiently regular) function in \mathbb{C}^m , $F(0) = 0$ and u_1, \dots, u_m are functions in $H^s(\mathbb{R}^n)$, $s > \frac{n}{2}$, then, $F(u_1, \dots, u_m) \in H^s(\mathbb{R}^n)$ and we have precisely*

$$\|F(u_1, \dots, u_m)\|_s \leq C(\|(u_1, \dots, u_m)\|_{L^\infty}) \|(u_1, \dots, u_m)\|_s,$$

where $\xi \mapsto C(\xi)$ is a nonnegative and nondecreasing function.

An important property of the paradifferential operators consists in the fact that they are necessary to write down Bony’s linearization formula, a formula that we recall here.

Theorem 2.10 (Bony’s linearization formula). *For all real functions $u_1, \dots, u_m \in H^{\frac{n}{2}+\varrho}(\mathbb{R}^n)$, $\varrho > 0$, and every function F of m real variables which is C^∞ (or sufficiently regular) and vanishes in 0, we have*

$$F(u_1, \dots, u_m) = \sum_{i=1}^{i=m} T_{\partial_{u_i}} F u_i + r \quad \text{with } r \in H^{\frac{n}{2}+2\varrho}(\mathbb{R}^n).$$

Proof. See [Bony 1981; Meyer 1981; Meyer 1982]. □

The remainder r in the above formula depends of course on (u_1, \dots, u_m) . The following result essentially shows that r is a locally Lipschitz function of (u_1, \dots, u_m) . More precisely:

Theorem 2.11. *If $u = (u_1, \dots, u_m) \in H^s(\mathbb{R}^n, \mathbb{R}^m)$, $s = \frac{n}{2} + \varrho$, $\varrho > 0$, let us denote by $r(u)$ the remainder in Bony’s formula. For all $u, v \in H^s(\mathbb{R}^n, \mathbb{R}^m)$, we have then*

$$\|r(u) - r(v)\|_{s+\varrho} \leq \theta(\|u\|_s, \|v\|_s)\|u - v\|_s,$$

where $\theta(\|u\|_s, \|v\|_s)$ is bounded if u and v vary in a bounded subset of $H^s(\mathbb{R}^n, \mathbb{R}^m)$.

Proof. See [Bienaimé 2014]. □

Remark. In the case of our equation, that is (1), even if u has complex values, we shall be able to apply Bony’s formula to the nonlinear expression $F(u, \nabla u, \bar{u}, \nabla \bar{u})$ where $u \in H^{\frac{n}{2}+1+\varrho}(\mathbb{R}^n)$. Indeed, we can write

$$F(u, \nabla u, \bar{u}, \nabla \bar{u}) = G(\operatorname{Re}(u), \nabla \operatorname{Re}(u), \operatorname{Im}(u), \nabla \operatorname{Im}(u))$$

where $G(x_1, x_2, y_1, y_2) = F(x_1 + iy_1, x_2 + iy_2, x_1 - iy_1, x_2 - iy_2)$ which is a function from \mathbb{R}^{2n+2} into \mathbb{C} . We apply then Bony’s formula to G and obtain that

$$F(u, \nabla u, \bar{u}, \nabla \bar{u}) = T_{\partial_{x_1}G} \operatorname{Re}(u) + T_{\partial_{x_2}G} \nabla \operatorname{Re}(u) + T_{\partial_{y_1}G} \operatorname{Im}(u) + T_{\partial_{y_2}G} \nabla \operatorname{Im}(u) + r(u).$$

At last, by using the fact that $\operatorname{Re}(u) = \frac{u+\bar{u}}{2}$, $\operatorname{Im}(u) = \frac{u-\bar{u}}{2i}$, $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, and then the linearity of T_b with respect to b , we obtain the formula used in this paper:

$$F(u, \bar{u}, \nabla u, \nabla \bar{u}) = T_{\partial_u}Fu + T_{\partial_{\bar{u}}}F\bar{u} + T_{\partial_{\nabla u}}F\nabla u + T_{\partial_{\nabla \bar{u}}}F\nabla \bar{u} + r(u)$$

with $r(u) \in H^{\frac{n}{2}+2\varrho}(\mathbb{R}^n)$ if $u \in H^{\frac{n}{2}+1+\varrho}(\mathbb{R}^n)$.

We shall also often need the following result similar to Lemma 2.3:

Lemma 2.12. *Let $a \in \Sigma_0^0(\mathbb{R}^n)$, $\theta \in \mathcal{D}(\mathbb{R}^n)$, $\theta_\mu(x) = \theta(x - \mu)$, $\mu \in \mathbb{R}^n$ and $s \in \mathbb{R}$, and consider the paradifferential operator $T_{\theta_\mu a} = T_{\theta_\mu a, \chi}$ (where the paratruncature χ does not necessarily satisfy the second property of Definition 2.5). Then, for all $\sigma \geq 0$, the operator $\langle x - \mu \rangle^\sigma T_{\theta_\mu a} \langle x - \mu \rangle^\sigma$ is bounded in $H^s(\mathbb{R}^n)$ and there exist $N \in \mathbb{N}$ and a nonnegative constant C such that, for every $\mu \in \mathbb{R}^n$,*

$$\|\langle x - \mu \rangle^\sigma T_{\theta_\mu a} \langle x - \mu \rangle^\sigma\|_{\mathcal{L}(H^s)} \leq C \sup_{|\alpha| \leq N} \|\langle \xi \rangle^{|\alpha|} \partial_\xi^\alpha a\|_{L^\infty}.$$

Proof. First, one can assume that σ is an integer and even an even integer. Let us denote by a_μ the symbol $\theta_\mu a$ and consider first the operator $T_{a_\mu} \langle x - \mu \rangle^\sigma$. Recall that $T_{a_\mu} = \tilde{a}_\mu(x, D)$ with

$$\tilde{a}_\mu(x, \xi) = (1 - \psi_1(\xi))|\xi|^n \int_{\mathbb{R}^n} \mathcal{F}^{-1}(\chi_1)(|\xi|(x - y))a_\mu(y, \xi) dy. \tag{7}$$

where $\psi_1, \chi_1 \in C^\infty(\mathbb{R}^n)$, $\psi_1 = 1$ in a neighbourhood of 0, $\psi_1 = 0$ out of $B(0, \varepsilon'')$, and $\chi_1 = 1$ on $B(0, \varepsilon')$, $\operatorname{supp}(\chi) \subset B(0, \varepsilon)$, with ε and ε' satisfying $0 < \varepsilon' < \varepsilon < 1$. Hence, we can write for arbitrary

$u \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} T_{a_\mu} \langle x - \mu \rangle^\sigma u(x) &= (2\pi)^{-n} \int e^{ix\xi} \tilde{a}_\mu(x, \xi) \mathcal{F}(\langle x - \mu \rangle^\sigma u)(\xi) d\xi \\ &= (2\pi)^{-n} \int e^{ix\xi} \tilde{a}_\mu(x, \xi) \langle D_\xi + \mu \rangle^\sigma \hat{u}(\xi) d\xi \\ &= (2\pi)^{-n} \int \langle D_\xi - \mu \rangle^\sigma [e^{ix\xi} \tilde{a}_\mu(x, \xi)] \hat{u}(\xi) d\xi \\ &= (2\pi)^{-n} \sum_\alpha \frac{1}{\alpha!} D_x^\alpha [\langle x - \mu \rangle^\sigma] \int e^{ix\xi} \partial_\xi^\alpha \tilde{a}_\mu(x, \xi) \hat{u}(\xi) d\xi, \end{aligned}$$

where we have applied integrations by parts and the Leibniz formula. So, we have proved that

$$T_{a_\mu} \langle x - \mu \rangle^\sigma = \sum_\alpha \frac{1}{\alpha!} D_x^\alpha [\langle x - \mu \rangle^\sigma] (\partial_\xi^\alpha \tilde{a}_\mu)(x, D),$$

where the sum is of course finite. Now, let us consider the operator $(\partial_\xi^\alpha \tilde{a}_\mu)(x, D)$ and let us remark that, for example,

$$\begin{aligned} \partial_{\xi_k} \tilde{a}_\mu(x, \xi) &= (1 - \psi_1(\xi)) |\xi|^n \int_{\mathbb{R}^n} \mathcal{F}^{-1}(\chi_1)(|\xi|(x - y)) \partial_{\xi_k} a_\mu(y, \xi) dy \\ &\quad - (1 - \psi_1(\xi)) |\xi|^n \int_{\mathbb{R}^n} \mathcal{F}^{-1}(\chi_2)(|\xi|(x - y)) a_\mu(y, \xi) \frac{\xi_k}{|\xi|^2} dy \\ &\quad - \partial_k \psi_1(\xi) |\xi|^n \int_{\mathbb{R}^n} \mathcal{F}^{-1}(\chi_1)(|\xi|(x - y)) a_\mu(y, \xi) dy, \end{aligned}$$

where $\chi_2(\eta) = \sum_{j=1}^n \eta_j \partial_j \chi_1(\eta)$. This shows that

$$(\partial_{\xi_k} \tilde{a}_\mu)(x, D) = \sum_{l=1}^3 T_{\theta_\mu a^l, \chi^l},$$

where the a^l are symbols in Σ_0^{-1} and the χ^l are paratruncatures which satisfy the first and third properties of Definition 2.5. By induction, $(\partial_\xi^\alpha \tilde{a}_\mu)(x, D)$ is then a finite sum of operators of the same form as $T_{a_\mu} = T_{a_\mu, \chi}$ (of order $\leq -|\alpha|$), and note also that the seminorms of the associated symbols are bounded uniformly in μ by a seminorm of a . Hence, $T_{a_\mu} \langle x - \mu \rangle^\sigma$ is a finite sum of operators of the form $P(x - \mu) T_{a_\mu}$, where P is a polynomial (of degree $\leq \sigma$), and consequently the problem is reduced to the study of the operator $\langle x - \mu \rangle^\sigma T_{a_\mu}$ only. Now, the symbol of the latter can be written as

$$\begin{aligned} \langle x - \mu \rangle^\sigma \tilde{a}_\mu(x, \xi) &= \sum_{|\alpha| \leq \sigma} \frac{1}{\alpha!} (1 - \psi_1(\xi)) |\xi|^n \int_{\mathbb{R}^n} (x - y)^\alpha \mathcal{F}^{-1}(\chi_1)(|\xi|(x - y)) \partial_y^\alpha [\langle y - \mu \rangle^\sigma] a_\mu(y, \xi) dy \\ &= \sum_{|\alpha| \leq L} \frac{1}{\alpha!} (1 - \psi_1(\xi)) |\xi|^n \int_{\mathbb{R}^n} \mathcal{F}^{-1}(\chi_1^\alpha)(|\xi|(x - y)) \theta^\alpha(y - \mu) a^\alpha(y, \xi) dy, \end{aligned}$$

where χ_1^α and θ^α are similar to χ_1 and θ respectively, and $a^\alpha \in \Sigma_0^{-|\alpha|}$ with seminorms bounded by those of a . Hence, $\langle x - \mu \rangle^\sigma T_{a_\mu}$ is a finite sum of operators of the same form as T_{a_μ} whose symbols have seminorms bounded uniformly in μ by a seminorm of a . Eventually, the lemma follows from Theorem 2.6. □

Let us also recall the Gårding inequality which will be used crucially to prove the smoothing effect estimate.

Theorem 2.13 (sharp Gårding inequality for systems). *Let $a(x, \xi)$ be a $k \times k$ matrix whose elements are in $S_{1,0}^m$ and which satisfies*

$$\langle (a(x, \xi) + a^*(x, \xi))\zeta, \zeta \rangle \geq 0$$

for all $\zeta \in \mathbb{C}^k$ and all (x, ξ) such that $|\xi| \geq A_0$, where a^* denotes the adjoint matrix of a and $\langle \cdot, \cdot \rangle$ is the usual hermitian scalar product of \mathbb{C}^k . Then, there exist a nonnegative constant A and an integer N such that, for all $u \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^k)$, we have

$$\operatorname{Re} \langle a(x, D)u, u \rangle \geq -A \sup_{|\alpha|+|\beta| \leq N} \|\langle \xi \rangle^{|\beta|-m} \partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty} \|u\|_{\frac{m-1}{2}}^2,$$

where A depends only on n, k and A_0 .

Proof. See [Taylor 1991; Tataru 2002] for example. □

3. The parilinear equation

In this section, we solve the Cauchy problem for the parilinear equation, that is, the linear equation obtained from (1) by applying Bony’s linearization formula (Theorem 2.10).

Recall that Q_μ is the cube $\mu + [0, 1]^n$, $\mu \in \mathbb{Z}^n$ and that Q_μ^* is a larger cube with side length 2, for example, $\mu + [-\frac{1}{2}, \frac{3}{2}]^n$.

Theorem 3.1. *Given $s \in \mathbb{R}$, consider the following linear Cauchy problem:*

$$\begin{cases} \partial_t u = i\mathcal{L}u + T_{b_1} \cdot \nabla_x u + T_{b_2} \cdot \nabla_x \bar{u} + C_1 u + C_2 \bar{u} + f(x, t), \\ u(x, 0) = u_0 \in H^s(\mathbb{R}^n). \end{cases} \tag{8}$$

We assume that C_1 and C_2 are bounded operators in $H^s(\mathbb{R}^n)$ and in $H^{s+2}(\mathbb{R}^n)$, that $b_1, b_2 \in \Sigma_\varrho^m$, $\varrho > 0$, and more precisely that

$$\begin{aligned} b_k(x, \xi) &= \sum_{\mu \in \mathbb{Z}^n} \alpha_{k,\mu} \varphi_{k,\mu}(x, \xi), & \sum_{\mu} |\alpha_{k,\mu}| &\leq A_k, \quad k = 1, 2, \\ \operatorname{supp}(x \mapsto \varphi_{k,\mu}(x, \xi)) &\subseteq Q_\mu^*, & \sup_{|\beta| \leq N_0} \|\langle \xi \rangle^{|\beta|} \partial_\xi^\beta \varphi_{k,\mu}\|_{C^0} &\leq 1, \end{aligned} \tag{9}$$

and $\|C_k\|_{\mathcal{L}(H^s)}, \|C_k\|_{\mathcal{L}(H^{s+2})} \leq A_k$, $k = 1, 2$, N_0 being a large and fixed integer. We further assume that $b_2(x, \xi)$ is even in ξ and that $f \in L_{\text{loc}}^1(\mathbb{R}, H^s(\mathbb{R}^n))$. Then, problem (8) has a unique solution u which is in $C(\mathbb{R}, H^s(\mathbb{R}^n))$ and satisfies, for all $T > 0$,

$$\sup_{-T \leq t \leq T} \|u(t)\|_s^2 \leq A(\|u_0\|_s^2 + I_T(J^s f, J^s u)), \tag{10}$$

$$\|J^{s+\frac{1}{2}} u\|_T^2 \leq A(\|u_0\|_s^2 + I_T(J^s f, J^s u)), \tag{11}$$

where the constant A depends only on $n, s, \varrho, T, A_1,$ and $A_2,$ and the expression $I_T(v, w)$ is a finite sum of terms of the form

$$\sup_{\mu \in \mathbb{Z}^n} \int_{-T}^T |\langle G_\mu v, w \rangle| dt$$

with $G_\mu \in \text{Op } S_{0,0}^0$ and the seminorms of its symbol (up to N_0) are uniformly bounded by a constant that depends only on s, n, ϱ, A_1 and $A_2.$

Recall that $\|u\|_T^2 = \sup_\mu \int_{-T}^T \int_{\mathbb{R}^n} \langle x - \mu \rangle^{-2\sigma_0} |u(x, t)|^2 dt dx,$ where $\sigma_0 > \frac{1}{2}$ is fixed.

Proof. Let us start by noting that the uniqueness is an obvious matter. Indeed, if u_1 and u_2 are solutions of (8), then, $u_1 - u_2$ is a solution of (8) with $u_0 = 0$ and $f = 0,$ and the conclusion follows from (10).

As for the existence, as is customary with linear differential equations, it will follow from the a priori estimates (10) and (11) by using more or less standard arguments of functional analysis, and the proof of Theorem 3.1 will consist essentially in establishing them.

Another useful remark is that it will be sufficient to prove the theorem in $C(\mathbb{R}_+, H^s(\mathbb{R}^n))$ instead of $C(\mathbb{R}, H^s(\mathbb{R}^n))$ and the estimates (10) and (11) on $[0, T]$ instead of $[-T, T].$ In fact, if the theorem is proved on $\mathbb{R}_+,$ one can apply it to $v(t) = u(-t),$ which satisfies a Cauchy problem of the same type as (8). The result is then that $v(-t)$ will extend u to \mathbb{R}_- and satisfy (8) on $\mathbb{R}_-,$ in addition to the fact that the estimates (10) and (11) are also extended to $[-T, 0].$

So, let us assume that $u \in C([0, T]; H^s(\mathbb{R}^n))$ is a solution of the Cauchy problem (8).

In what follows, it will be quite convenient to use the notation

$$v_N(\varphi) = \sup_{1 \leq j \leq N} \sup_{|\alpha|+|\beta| \leq N} \|\langle \xi \rangle^{|\beta|} \partial_x^\alpha \partial_\xi^\beta \varphi\|_{L^\infty}^j,$$

and note that such a quantity is not a norm in general but it is well-defined for $\varphi \in S_{1,0}^0.$ Note also that, if $M \geq 1, v_N(\varphi)^M \leq v_{NM}(\varphi),$ a remark that will be often used implicitly.

In fact, the inequalities (10) and (11) will be deduced from the following ones:

Proposition 3.2. *Assume that the functions $\varphi_{k,\mu}$ defining the b_k are $C^\infty;$ that is, $\varphi_{k,\mu} \in S_{1,0}^0, k = 1, 2.$ Then, there exist a positive real number A and an integer N such that, for all $R \geq 1,$ there exists a pseudodifferential operator $\mathbf{C} \in \text{Op } S_{0,0}^0$ such that, for all $T > 0,$ any solution $u \in C([0, T]; H^s(\mathbb{R}^n))$ of the Cauchy problem (8) satisfies*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|\mathbf{C}u(t)\|_s^2 \\ & \leq \|\mathbf{C}u_0\|_s^2 + 2 \int_0^T |\langle \mathbf{C}J^s f, \mathbf{C}J^s u \rangle| dt + A \sup_{k,\mu} v_N(\varphi_{k,\mu}) \left(RT \sup_{0 \leq t \leq T} \|u(t)\|_s^2 + \frac{1}{R} \|J^{s+\frac{1}{2}}u\|_T^2 \right). \end{aligned}$$

Moreover, regarding the operator $\mathbf{C},$ we have the following precise bounds for $v \in H^s(\mathbb{R}^n) :$

$$\begin{aligned} \|\mathbf{C}v\|_s & \leq A \sup_\mu v_N(\varphi_{1,\mu}) \|v\|_s, \\ \|v\|_s & \leq A \sup_\mu v_N(\varphi_{1,\mu}) \|\mathbf{C}v\|_s + \frac{A}{R} \sup_\mu v_N(\varphi_{1,\mu})^2 \|v\|_s. \end{aligned}$$

Proposition 3.3. *Under the same assumptions as above and with the same elements A, R, C and N , there exist also pseudodifferential operators $\psi_j(x, D) \in \text{Op } S_{1,0}^0$, $j = 1, 2, 3, 4$, such that, for all $T > 0$, any solution $u \in C([0, T]; H^s(\mathbb{R}^n))$ of the Cauchy problem (8) satisfies*

$$\begin{aligned} \|J^{s+\frac{1}{2}}u\|_T^2 &\leq A(1+T+T \sup_{\mu,k} v_N(\varphi_{k,\mu})) \sup_{[0,T]} \|u\|_s^2 + A \sum_{j=1}^4 \sup_{\mu \in \mathbb{Z}^n} \int_0^T |\langle \psi_j(x-\mu, D) J^s f, J^s u \rangle| dt, \\ \|J^{s+\frac{1}{2}}Cu\|_T^2 &\leq A(1+T+T \sup_{\mu,k} v_N(\varphi_{k,\mu})) \sup_{[0,T]} \|Cu\|_s^2 + A \sum_{j=1}^4 \sup_{\mu} \int_0^T |\langle \psi_j(x-\mu, D) C J^s f, C J^s u \rangle| dt \\ &\quad + A \sup_{k,\mu} v_N(\varphi_{k,\mu}) \left(RT \sup_{0 \leq t \leq T} \|u(t)\|_s^2 + \frac{1}{R} \|J^{s+\frac{1}{2}}u\|_T^2 \right). \end{aligned}$$

Admitting these propositions (see Sections 5 and 6 for their proofs), let us go on and finish the proof of Theorem 3.1. In order to apply the above inequalities we have to regularize the b_k , $k = 1, 2$, by setting

$$\varphi_{k,\mu,m}(x, \xi) = m^n \int_{\mathbb{R}^n} \chi(m(x-y)) \varphi_{k,\mu}(y, \xi) dy \quad \text{and} \quad b_{k,m} = \sum_{\mu} \alpha_{k,\mu} \varphi_{k,\mu,m},$$

where χ is a nonnegative C^∞ function with support in the unit ball and whose integral is equal to 1. Note that $\varphi_{k,\mu,m}$ has its support (with respect to x) in a compact set which is slightly larger than Q_μ^* but this has no effect on the proofs. Since we can write

$$\partial_t u = i\mathcal{L}u + T_{b_{1,m}} \cdot \nabla_x u + T_{b_{2,m}} \cdot \nabla_x \bar{u} + C_1 u + C_2 \bar{u} + f_m,$$

where

$$f_m = f + T_{b_1-b_{1,m}} \cdot \nabla u + T_{b_2-b_{2,m}} \cdot \nabla \bar{u},$$

we can apply Proposition 3.2 to obtain

$$\begin{aligned} \sup_{[0,T]} \|C_m u\|_s^2 &\leq \|C_m u_0\|_s^2 + 2 \int_0^T |\langle C_m J^s f_m, C_m J^s u \rangle| dt \\ &\quad + A \sup_{k,\mu} v_N(\varphi_{k,\mu,m}) \left(RT \sup_{[0,T]} \|u\|_s^2 + \frac{1}{R} \|J^{s+\frac{1}{2}}u\|_T^2 \right), \end{aligned}$$

where the operator C is denoted here by C_m to indicate its dependence on m . Now, clearly, we have

$$v_N(\varphi_{k,\mu,m}) \leq Am^{N^2} \sup_{1 \leq j \leq N} \sup_{|\beta| \leq N} \|\langle \xi \rangle^{|\beta|} \partial_\xi^\beta \varphi_{k,\mu}\|_{L^\infty}^j \leq Am^{N^2}.$$

Hence,

$$\begin{aligned} \sup_{[0,T]} \|C_m u\|_s^2 &\leq \|C_m u_0\|_s^2 + 2 \int_0^T |\langle C_m J^s f, C_m J^s u \rangle| dt + 2 \int_0^T |\langle C_m J^s T_{b_1-b_{1,m}} \nabla u, C_m J^s u \rangle| dt \\ &\quad + 2 \int_0^T |\langle C_m J^s T_{b_2-b_{2,m}} \nabla \bar{u}, C_m J^s u \rangle| dt + Am^{N^2} \left(RT \sup_{[0,T]} \|u\|_s^2 + \frac{1}{R} \|J^{s+\frac{1}{2}}u\|_T^2 \right), \end{aligned}$$

and the problem now is to estimate the third and fourth terms in the right-hand side of this inequality. This is done in the following lemma.

Lemma 3.4. *Let \tilde{u} stand for u or \bar{u} , and $\sigma = \inf\{\varrho, 1\}$. Then, there exists a constant A such that, for all $k \in \{1, 2\}$, $m \geq 1$, $R \geq 1$ and $m' \geq m$,*

$$\int_0^T |\langle \mathbf{C}_m J^s T_{b_k - b_{k,m}} \nabla \tilde{u}, \mathbf{C}_m J^s u \rangle| dt \leq \left(\frac{Am^{2N^2}}{m'^\sigma} + \frac{Am'^{3N^2}}{R} \right) \| \| J^{s+\frac{1}{2}} u \| \|_T^2 + Am'^{3N^2} T \sup_{[0,T]} \|u\|_s^2 + \frac{A}{m^\sigma} \| \| J^{s+\frac{1}{2}} \mathbf{C}_m u \| \|_T^2.$$

See the Appendix for the proof of this lemma. Applying this lemma yields

$$\sup_{[0,T]} \| \mathbf{C}_m u \|_s^2 \leq \| \mathbf{C}_m u_0 \|_s^2 + 2 \int_0^T |\langle \mathbf{C}_m J^s f, \mathbf{C}_m J^s u \rangle| dt + \frac{A}{m^\sigma} \| \| J^{s+\frac{1}{2}} \mathbf{C}_m u \| \|_T^2 + \left(\frac{Am^{2N^2}}{m'^\sigma} + \frac{Am'^{3N^2}}{R} \right) \| \| J^{s+\frac{1}{2}} u \| \|_T^2 + Am'^{3N^2} RT \sup_{[0,T]} \|u\|_s^2,$$

an inequality that we can improve, thanks to Proposition 3.3, as follows:

$$\begin{aligned} \sup_{[0,T]} \| \mathbf{C}_m u \|_s^2 &\leq \| \mathbf{C}_m u_0 \|_s^2 + 2 \int_0^T |\langle \mathbf{C}_m J^s f, \mathbf{C}_m J^s u \rangle| dt \\ &\quad + \frac{A}{m^\sigma} \sum_{j=1}^4 \sup_{\mu} \int_0^T |\langle \psi_j(x - \mu, D) \mathbf{C}_m J^s f, \mathbf{C}_m J^s u \rangle| dt + \frac{A(1 + Tm^N)}{m^\sigma} \sup_{[0,T]} \| \mathbf{C}_m u \|_s^2 \\ &\quad + \left(\frac{Am^{2N^2}}{m'^\sigma} + \frac{Am'^{3N^2}}{R} \right) \| \| J^{s+\frac{1}{2}} u \| \|_T^2 + Am'^{3N^2} RT \sup_{[0,T]} \|u\|_s^2 \\ &\leq \| \mathbf{C}_m u_0 \|_s^2 + 2 \int_0^T |\langle \mathbf{C}_m J^s f, \mathbf{C}_m J^s u \rangle| dt \\ &\quad + \frac{A}{m^\sigma} \sum_{j=1}^4 \sup_{\mu} \int_0^T |\langle \psi_j(x - \mu, D) \mathbf{C}_m J^s f, \mathbf{C}_m J^s u \rangle| dt + \frac{A(1 + Tm^N)}{m^\sigma} \sup_{[0,T]} \| \mathbf{C}_m u \|_s^2 \\ &\quad + \left(\frac{Am^{2N^2}}{m'^\sigma} + \frac{Am'^{3N^2}}{R} \right) \sum_{j=1}^4 \sup_{\mu} \int_0^T |\langle \psi_j(x - \mu, D) J^s f, J^s u \rangle| dt \\ &\quad + \left(\frac{Am^{2N^2}}{m'^\sigma} + \frac{Am'^{3N^2}}{R} \right) (1 + Tm^N) \sup_{[0,T]} \|u\|_s^2 + Am'^{3N^2} RT \sup_{[0,T]} \|u\|_s^2. \end{aligned}$$

Next, by taking m such that, for example, $m^\sigma \geq 4A$ and T such that $Tm^N \leq 1$, we get

$$\begin{aligned} \sup_{[0,T]} \| \mathbf{C}_m u \|_s^2 &\leq 2 \| \mathbf{C}_m u_0 \|_s^2 + 4 \int_0^T |\langle \mathbf{C}_m J^s f, \mathbf{C}_m J^s u \rangle| dt + \sum_{j=1}^4 \sup_{\mu} \int_0^T |\langle \psi_j(x - \mu, D) \mathbf{C}_m J^s f, \mathbf{C}_m J^s u \rangle| dt \\ &\quad + \left(\frac{2Am^{2N^2}}{m'^\sigma} + \frac{2Am'^{3N^2}}{R} \right) \sum_{j=1}^4 \sup_{\mu} \int_0^T |\langle \psi_j(x - \mu, D) J^s f, J^s u \rangle| dt \\ &\quad + \left(\frac{Am^{2N^2}}{m'^\sigma} + \frac{Am'^{3N^2}}{R} + Am'^{3N^2} RT \right) \sup_{[0,T]} \|u\|_s^2, \end{aligned}$$

and by using the second part of Proposition 3.2, we obtain

$$\begin{aligned} & \sup_{[0,T]} \|u\|_s^2 \\ & \leq Am^{2N^2} \left(m^{2N^2} \|u_0\|_s^2 + \int_0^T |\langle C_m J^s f, C_m J^s u \rangle| dt + \sum_{j=1}^4 \sup_{\mu} \int_0^T |\langle \psi_j(x-\mu, D) C_m J^s f, C_m J^s u \rangle| dt \right) \\ & \quad + \left(\frac{Am^{4N^2}}{m'^{\sigma}} + \frac{Am'^{5N^2}}{R} \right) \sum_{j=1}^4 \sup_{\mu} \int_0^T |\langle \psi_j(x-\mu, D) J^s f, J^s u \rangle| dt + C(m, m', R, T) \sup_{[0,T]} \|u\|_s^2, \end{aligned}$$

where

$$C(m, m', R, T) = \frac{Am^{4N^2}}{m'^{\sigma}} + \frac{Am'^{5N^2}}{R} + Am'^{5N^2} RT + \frac{Am^{4N^2}}{R^2}.$$

Finally, since m is fixed (and depends only on A), we take m' such that $Am^{4N^2}/m'^{\sigma} \leq \frac{1}{8}$, then we take R such that $Am'^{5N^2}/R \leq \frac{1}{8}$ and $Am^{4N^2}/R^2 \leq \frac{1}{8}$, and last we take T such that $Am'^{5N^2}RT \leq \frac{1}{8}$. With these choices, we have of course $C(m, m', R, T) \leq \frac{1}{2}$, which allows to bound $\sup_{[0,T]} \|u\|_s^2$ and to get (10) (and also (11), thanks to Proposition 3.3) with

$$\begin{aligned} I_T(v, w) = & \int_0^T |\langle C^* C v, w \rangle| dt + \sum_{j=1}^4 \sup_{\mu} \int_0^T |\langle C^* \psi_j(x-\mu, D) C v, w \rangle| dt \\ & + \sup_{\mu} \int_0^T |\langle \psi_j(x-\mu, D) v, w \rangle| dt. \end{aligned}$$

In fact, we have proved (10) and (11) only for $T = T_0$ and T_0 is sufficiently small. Let us show, if $T_0 < T$, that they hold true in the whole interval $[0, T]$ where the solution u is defined. Indeed, note first that the T_0 as determined above depends only on the constant A (so, only on n, s, ϱ, A_1 and A_2) but not on the given function (or distribution) f . Next, take a $T_1 \leq T_0$ such that $T_1 = T/n_1$, with some integer $n_1 \geq 2$. Then, if we consider the function $v(x, t) = u(x, t + T_1)$, we note that v is a solution (defined at least in $[0, T - T_1]$) of (8) with $v(0) = u(T_1)$ and $g(x, t) = f(x, t + T_1)$ instead of $f(x, t)$. It follows from the above arguments that v satisfies (10) and (11) for $T = T_0$ and hence for $T = T_1$. Since

$$\begin{aligned} \sup_{[T_1, 2T_1]} \|u\|_s^2 &= \sup_{[0, T_1]} \|v\|_s^2 \leq A(\|u(T_1)\|_s^2 + I_{T_1}(J^s g, J^s v)) \leq A(\|u(T_1)\|_s^2 + I_{2T_1}(J^s f, J^s u)) \\ &\leq A(A\|u_0\|_s^2 + AI_{T_1}(J^s f, J^s u) + I_{2T_1}(J^s f, J^s u)) \\ &\leq (A^2 + A)(\|u_0\|_s^2 + I_{2T_1}(J^s f, J^s u)), \end{aligned}$$

we obtain that u satisfies (10) and (11) for $T = 2T_1$ and with the constant $A^2 + A$ instead of A . Repeating this argument, we obtain that u satisfies (10) and (11) on $[0, n_1 T_1] = [0, T]$ and with the constant $\sum_{j=1}^{n_1} A^j \simeq A^{T/T_1}$ instead of A .

As for the existence, let us consider the approximating Cauchy problem

$$\begin{cases} \partial_t u = i\mathcal{L}u + T_{b_1} \nabla h(\varepsilon D)u + T_{b_2} \nabla h(\varepsilon D)\bar{u} + C_1 u + C_2 \bar{u} + f(x, t), \\ u(x, 0) = u_0 \in H^s(\mathbb{R}^n), \end{cases} \tag{12}$$

where h is a nonnegative C^∞ function on \mathbb{R}^n which is equal to 1 near 0 and has a compact support. It is easy to see, if $\int_0^T \|f\|_s dt < +\infty$, that the above problem has a unique solution, denoted by u_ε , which is

in $C([0, T]; H^s(\mathbb{R}^n))$. Indeed, the Cauchy problem (12) is clearly equivalent to the integral equation

$$u = e^{it\mathcal{L}}u_0 + \int_0^t e^{i(t-t')\mathcal{L}}(T_{b_1}\nabla h(\varepsilon D)u + T_{b_2}\nabla h(\varepsilon D)\bar{u} + C_1u + C_2\bar{u} + f(x, t')) dt'$$

and one can easily show that the map defined by the right-hand side of this equation is a contraction in $C([0, T_\varepsilon]; H^s(\mathbb{R}^n))$ with some $T_\varepsilon > 0$ sufficiently small, which allows one to apply the fixed-point theorem and to get a solution u_ε . Now, since T_ε does not depend on the data u_0 and f , one can extend u_ε to a solution of (12) on the whole interval $[0, T]$.

The idea now is to let ε tend to 0. This is possible because u_ε satisfies the estimates (10) and (11) and even uniformly with respect to ε . Indeed, it is sufficient to remark that the Cauchy problem (12) is of the same type as (8) because we can write

$$T_{b_k}\nabla h(\varepsilon D) = T_{b_{k,\varepsilon}}\nabla,$$

where $b_{k,\varepsilon}(x, \xi) = b_k(x, \xi)h(\varepsilon\xi)$ and $b_{k,\varepsilon}$ satisfies the assumptions of Theorem 3.1 uniformly in ε . Hence, we have in particular

$$\sup_{[0,T]} \|u_\varepsilon\|_s^2 \leq A\|u_0\|_s^2 + AI_T(J^s f, J^s u_\varepsilon),$$

and it follows from the Calderón–Vaillancourt theorem that

$$AI_T(J^s f, J^s u_\varepsilon) \leq AA' \sup_{[0,T]} \|u_\varepsilon\|_s \int_0^T \|f\|_s dt \leq \frac{1}{2} \sup_{[0,T]} \|u_\varepsilon\|_s^2 + \frac{1}{2}(AA')^2 \left(\int_0^T \|f\|_s dt \right)^2,$$

so that,

$$\sup_{[0,T]} \|u_\varepsilon\|_s \leq A\|u_0\|_s + A \int_0^T \|f\|_s dt. \tag{13}$$

Next, to check the convergence of u_ε , let us consider $v = u_\varepsilon - u_{\varepsilon'}$. It is clear that v is the solution of (12) with $u_0 = 0$ and

$$f = T_{b_1}\nabla(h(\varepsilon D) - h(\varepsilon' D))u_{\varepsilon'} + T_{b_2}\nabla(h(\varepsilon D) - h(\varepsilon' D))\bar{u}_{\varepsilon'}.$$

Therefore, it follows from (13) that

$$\sup_{[0,T]} \|v\|_s \leq A \int_0^T \|T_{b_1}\nabla(h(\varepsilon D) - h(\varepsilon' D))u_{\varepsilon'} + T_{b_2}\nabla(h(\varepsilon D) - h(\varepsilon' D))\bar{u}_{\varepsilon'}\|_s dt, \tag{14}$$

and from the boundedness of the T_{b_k} in the Sobolev spaces that

$$\sup_{[0,T]} \|v\|_s \leq A|\varepsilon - \varepsilon'| \int_0^T \|u_{\varepsilon'}\|_{s+2} dt \leq A|\varepsilon - \varepsilon'|T \sup_{[0,T]} \|u_{\varepsilon'}\|_{s+2}, \tag{15}$$

that is, thanks to (13),

$$\sup_{[0,T]} \|u_\varepsilon - u_{\varepsilon'}\|_s \leq A|\varepsilon - \varepsilon'| \left(\|u_0\|_{s+2} + \int_0^T \|f\|_{s+2} dt \right), \tag{16}$$

which proves that (u_ε) is a Cauchy sequence in $C([0, T]; H^s(\mathbb{R}^n))$ if one assumes that $u_0 \in H^{s+2}(\mathbb{R}^n)$ and $f \in L^1([0, T]; H^{s+2}(\mathbb{R}^n))$. In this case, $u_\varepsilon \rightarrow u$ in $C([0, T]; H^s(\mathbb{R}^n))$ when $\varepsilon \rightarrow 0$, and by passing to the limit in (12), we obtain that u is a solution of (8). Moreover, by passing to the limit in (13), we get

$$\sup_{[0, T]} \|u\|_s \leq A \left(\|u_0\|_s + \int_0^T \|f\|_s dt \right). \tag{17}$$

Now, if we have only $u_0 \in H^s(\mathbb{R}^n)$ and $f \in L^1([0, T]; H^s(\mathbb{R}^n))$, by density of the smooth functions, we can take sequences (u_0^j) in $H^{s+2}(\mathbb{R}^n)$ and (f^j) in $L^1([0, T]; H^{s+2}(\mathbb{R}^n))$ such that $\|u_0^j - u_0\|_s \rightarrow 0$ and $\int_0^T \|f^j - f\|_s dt \rightarrow 0$, and we can consider the solution u^j of (8) associated to the data u_0^j and f^j . Then, $u^j - u^k$ is the solution of (8) associated to the data $u_0^j - u_0^k$ and $f^j - f^k$. Hence, thanks to (17),

$$\sup_{[0, T]} \|u^j - u^k\|_s \leq A \left(\|u_0^j - u_0^k\|_s + \int_0^T \|f^j - f^k\|_s dt \right),$$

which shows that (u^j) is a Cauchy sequence in $C([0, T]; H^s(\mathbb{R}^n))$ which is then convergent to some $u \in C([0, T]; H^s(\mathbb{R}^n))$. Of course, u is a solution of (8) and satisfies the estimates (10), (11) and also (17). This achieves the proof of Theorem 3.1. □

4. The nonlinear equation

Consider the nonlinear Cauchy problem

$$\begin{cases} \partial_t u = i \mathcal{L}u + F(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}), & t \in \mathbb{R}, x \in \mathbb{R}^n, \\ u(x, 0) = u_0(x) \in H^s(\mathbb{R}^n), \end{cases} \tag{18}$$

where the function $F(u, v, \bar{u}, \bar{v})$ is sufficiently regular in $\mathbb{C} \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n$ and vanishes to the third order at 0, the operator \mathcal{L} has the form

$$\mathcal{L} = \sum_{j \leq k} \partial_{x_j}^2 - \sum_{j > k} \partial_{x_j}^2,$$

with a fixed $k \in \{1, 2, \dots, n\}$, $H^s(\mathbb{R}^n)$ is the usual Sobolev space on \mathbb{R}^n , and $s = \frac{n}{2} + 2 + \varrho$, $\varrho > 0$. Using Bony’s linearization formula, (18) is equivalent to

$$\begin{cases} \partial_t u = i \mathcal{L}u + T_{b_1} \nabla_x u + T_{b_2} \nabla_x \bar{u} + T_{a_1} u + T_{a_2} \bar{u} + R(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}), \\ u(x, 0) = u_0(x) \in H^s(\mathbb{R}^n), \end{cases} \tag{19}$$

where $R(u, \nabla_x u, \bar{u}, \nabla_x \bar{u})$ is Bony’s remainder and

$$\begin{aligned} b_1 &= \partial_v F(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}), & b_2 &= \partial_{\bar{v}} F(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}), \\ a_1 &= \partial_u F(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}), & a_2 &= \partial_{\bar{u}} F(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}). \end{aligned}$$

Recall that $R(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}) \in H^{2(s-1)-\frac{n}{2}}(\mathbb{R}^n)$ if $u \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2} + 1$. Note also that it follows from Proposition 2.9 that the b_j and a_j , $j = 1$ or 2 , are in $H^{s-1}(\mathbb{R}^n)$ if $u \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2} + 1$, and that

$$\|b_j\|_{s-1} \leq C(\|u\|_{L^\infty}, \|\nabla u\|_{L^\infty})\|u\|_s, \quad \|a_j\|_{s-1} \leq C(\|u\|_{L^\infty}, \|\nabla u\|_{L^\infty})\|u\|_s, \quad j = 1, 2.$$

Moreover, by introducing the notation

$$\begin{aligned} b_1^0 &= \partial_v F(u_0, \nabla_x u_0, \bar{u}_0, \nabla_x \bar{u}_0), & b_2^0 &= \partial_{\bar{v}} F(u_0, \nabla_x u_0, \bar{u}_0, \nabla_x \bar{u}_0), \\ a_1^0 &= \partial_u F(u_0, \nabla_x u_0, \bar{u}_0, \nabla_x \bar{u}_0), & a_2^0 &= \partial_{\bar{u}} F(u_0, \nabla_x u_0, \bar{u}_0, \nabla_x \bar{u}_0), \end{aligned}$$

the above Cauchy problem is in fact equivalent to

$$\begin{cases} \partial_t u = i\mathcal{L}u + T_{b_1^0} \nabla_x u + T_{b_2^0} \nabla_x \bar{u} + T_{a_1^0} u + T_{a_2^0} \bar{u} + \tilde{R}(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}), \\ u(x, 0) = u_0(x) \in H^s(\mathbb{R}^n), \end{cases} \tag{20}$$

where

$$\tilde{R}(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}) = T_{b_1 - b_1^0} \nabla_x u + T_{b_2 - b_2^0} \nabla_x \bar{u} + T_{a_1 - a_1^0} u + T_{a_2 - a_2^0} \bar{u} + R(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}). \tag{21}$$

Clearly, the last Cauchy problem is of the same type as (8), which is studied in Theorem 3.1, and in fact we are going to apply that theorem to

$$\begin{cases} \partial_t u = i\mathcal{L}u + T_{b_1^0} \nabla_x u + T_{b_2^0} \nabla_x \bar{u} + T_{a_1^0} u + T_{a_2^0} \bar{u} + f, \\ u(x, 0) = u_0(x) \in H^s(\mathbb{R}^n). \end{cases} \tag{22}$$

This is possible because b_1^0 and b_2^0 satisfy the assumptions of Theorem 3.1. Indeed, it follows from the Taylor formula and the assumptions on F that one can write for example

$$b_1^0 = \partial_v F(z_0) = u_0 G_1(z_0) + \nabla_x u_0 G_2(z_0) + \bar{u}_0 G_3(z_0) + \nabla_x \bar{u}_0 G_4(z_0), \tag{23}$$

where $z_0 = (u_0, \nabla_x u_0, \bar{u}_0, \nabla_x \bar{u}_0)$ and G_1, G_2, G_3 and G_4 are sufficiently regular and vanish at 0. Since $s - 1 > \frac{n}{2}$, we know that the $G_i(z_0)$ are in $H^{s-1}(\mathbb{R}^n)$ and it follows from (23) and Lemma 2.4 that b_1^0 satisfies the assumption (9) of Theorem 3.1; that is, one can write

$$b_1^0 = \sum_{\mu} \alpha_{1,\mu} \varphi_{1,\mu},$$

where $\alpha_{1,\mu} = \|q_{\mu} b_1^0\|_{H^{s-1}}$, $\varphi_{1,\mu} = q_{\mu} b_1^0 / \alpha_{1,\mu}$, and $\sum_{\mu} q_{\mu} = 1$ is a smooth partition of unity with $q_{\mu}(x) = q(x - \mu)$ and $\text{supp}(q) \subset Q_0^*$. Note that we have precisely the bound

$$\begin{aligned} \sum_{\mu} \|q_{\mu} b_1^0\|_{H^{s-1}} &\leq C (\|u_0\|_{H^{s-1}} \|G_1(z_0)\|_{H^{s-1}} + \|\nabla_x u_0\|_{H^{s-1}} \|G_2(z_0)\|_{H^{s-1}} \\ &\quad + \|\bar{u}_0\|_{H^{s-1}} \|G_3(z_0)\|_{H^{s-1}} + \|\nabla_x \bar{u}_0\|_{H^{s-1}} \|G_4(z_0)\|_{H^{s-1}}), \end{aligned}$$

with some positive constant C . Of course, the same is true for b_2^0 . Moreover, since a_1^0 and a_2^0 are bounded (they are in $H^{s-1}(\mathbb{R}^n)$), the paramultiplication operators $T_{a_1^0}$ and $T_{a_2^0}$ are bounded in $H^s(\mathbb{R}^n)$.

Now, by application of Theorem 3.1 to (22), let us consider the unique solution of (22) with $f = 0$ and denote it by $U(t)u_0$.

Next, for $T > 0$, let us define the norms $\lambda_1(w), \lambda_2(w), \lambda_3(w)$ and $\lambda(w)$ by

$$\lambda_1(w) = \sup_{[0,T]} \|w\|_s, \quad \lambda_2(w) = \|J^{s+\frac{1}{2}} w\|_T, \quad \lambda_3(w) = \sup_{[0,T]} \|\partial_t w\|_{s-2}, \quad \lambda(w) = \max_{1 \leq i \leq 3} \lambda_i(w),$$

the space Z by

$$Z = \{w \in C([0, T]; H^s(\mathbb{R}^n)) : w(x, 0) = u_0(x) \text{ and } \lambda(w) \leq K\},$$

where the positive constant K is to be determined later, and, for $w \in C([0, T]; H^s(\mathbb{R}^n))$, the operator Υ by

$$\Upsilon w(t) = U(t)u_0 + \int_0^t U(t-t')\tilde{R}(w(t'), \nabla_x w(t'), \bar{w}(t'), \nabla_x \bar{w}(t')) dt'.$$

Let us first remark that Υw satisfies

$$\begin{cases} \partial_t \Upsilon w = i\mathcal{L}\Upsilon w + T_{b_1^0} \nabla_x \Upsilon w + T_{b_2^0} \nabla_x \overline{\Upsilon w} + T_{a_1^0} \Upsilon w + T_{a_2^0} \overline{\Upsilon w} + \tilde{R}(w, \nabla_x w, \bar{w}, \nabla_x \bar{w}), \\ \Upsilon w(0) = u_0, \end{cases} \tag{24}$$

and that a fixed point of Υ will be a solution of (20), hence, a solution of (18). So, in what follows, we are going to study $\lambda(\Upsilon w)$ in order to prove that Υ has a fixed point in the complete metric space (Z, λ) . Let us also note that since the life time T will be small, we can assume from now on that $T \leq 1$.

We start by applying Theorem 3.1 to (24). It follows from (10) and (11) that

$$\max\{\lambda_1(\Upsilon w)^2, \lambda_2(\Upsilon w)^2\} \leq A(\|u_0\|_s^2 + I_T(J^s \tilde{R}, J^s \Upsilon w)), \tag{25}$$

where, for simplicity, $\tilde{R} = \tilde{R}(w, \nabla_x w, \bar{w}, \nabla_x \bar{w})$ and $I_T(u, v)$ is a finite sum of terms of the form

$$\sup_{\mu \in \mathbb{Z}^n} \int_0^T |\langle G_\mu u, v \rangle| dt,$$

where $G_\mu \in \text{Op } S_{0,0}^0$ and the seminorms of its symbol are uniformly bounded with respect to μ . Recall that the constant A depends only on n, s and u_0 and we remark right now a fact that will be useful later: if we let u_0 vary in a bounded subset of $H^s(\mathbb{R}^n)$, it follows from the linear theory that we can take the constant A in the above inequality that depends only on that bounded set. The same remark holds for $\sup_\mu \|G_\mu\|_{\mathcal{L}(L^2)}$ or the seminorms of the operators G_μ uniformly in μ .

Thus, we have to estimate uniformly in μ the sum

$$\begin{aligned} & \int_0^T |\langle G_\mu J^s T_{b_1-b_1^0} \nabla_x w, J^s \Upsilon w \rangle| dt + \int_0^T |\langle G_\mu J^s T_{b_2-b_2^0} \nabla_x \bar{w}, J^s \Upsilon w \rangle| dt \\ & + \int_0^T |\langle G_\mu J^s T_{a_1-a_1^0} w, J^s \Upsilon w \rangle| dt + \int_0^T |\langle G_\mu J^s T_{a_2-a_2^0} \bar{w}, J^s \Upsilon w \rangle| dt \\ & + \int_0^T |\langle G_\mu J^s R(w, \nabla_x w, \bar{w}, \nabla_x \bar{w}), J^s \Upsilon w \rangle| dt. \end{aligned} \tag{26}$$

First, let us consider the third term. It follows from the preceding remark, the Cauchy–Schwarz inequality, the Calderón–Vaillancourt theorem and Theorem 2.6 that

$$\int_0^T |\langle G_\mu J^s T_{a_1-a_1^0} w, J^s \Upsilon w \rangle| dt \leq A \|a_1 - a_1^0\|_{L^\infty} \int_0^T \|w\|_s \|\Upsilon w\|_s dt,$$

and from Proposition 2.9 that

$$\|a_1 - a_1^0\|_{L^\infty} \leq C(\|w\|_s)\|w\|_s + C(\|u_0\|_s)\|u_0\|_s \leq C(K)K + C(\|u_0\|_s)\|u_0\|_s \leq 2C(K)K.$$

Hence,

$$\int_0^T |(G_\mu J^s T_{a_1 - a_1^0} w, J^s \Upsilon w)| dt \leq ATC(K)\lambda_1(w)\lambda_1(\Upsilon w) \leq ATC(K)\lambda(w)\lambda(\Upsilon w), \quad (27)$$

with a modified constant $C(K)$.

The fourth term of (26) is treated in the same manner.

Now, let us estimate the first term of (26). Using a smooth partition of unity $1 = \sum_{v \in \mathbb{Z}^n} \chi_v$, with $\chi_v(x) = \chi(x - v)$ and χ having a compact support, we can write

$$\begin{aligned} &\langle G_\mu J^s T_{b_1 - b_1^0} \nabla_x w, J^s \Upsilon w \rangle \\ &= \sum_v \langle J^{-\frac{1}{2}} G_\mu J^s T_{\chi_v(b_1 - b_1^0)} \nabla_x w, J^{s+\frac{1}{2}} \Upsilon w \rangle \\ &= \sum_v \langle G_{\mu,v} \langle x - v \rangle^{\sigma_0} T_{\chi_v(b_1 - b_1^0)} \langle x - v \rangle^{\sigma_0} H_\mu \langle x - v \rangle^{-\sigma_0} J^{s+\frac{1}{2}} w, \langle x - v \rangle^{-\sigma_0} J^{s+\frac{1}{2}} \Upsilon w \rangle, \end{aligned}$$

where

$$G_{\mu,v} = \langle x - v \rangle^{\sigma_0} J^{-\frac{1}{2}} G_\mu J^s \langle x - v \rangle^{-\sigma_0}, \quad H_v = \langle x - v \rangle^{-\sigma_0} J^{-s-\frac{1}{2}} \nabla \langle x - v \rangle^{\sigma_0}.$$

Next, it follows from the pseudodifferential composition formula and from Lemma 2.3 that $G_{\mu,v}$ is in $\text{Op } S_{0,0}^{s-\frac{1}{2}}$, H_v is in $\text{Op } S_{1,0}^{\frac{1}{2}-s}$, and that their seminorms are uniformly bounded with respect to μ and v . Going back to the first term of (26), these considerations in addition to Lemma 2.12 allow us to estimate it as follows:

$$\begin{aligned} &\int_0^T |\langle G_\mu J^s T_{b_1 - b_1^0} \nabla_x w, J^s \Upsilon w \rangle| dt \\ &\leq \sum_v \int_0^T \|G_{\mu,v} \langle x - v \rangle^{\sigma_0} T_{\chi_v(b_1 - b_1^0)} \langle x - v \rangle^{\sigma_0} H_v\|_{\mathcal{L}(L^2)} \left\| \frac{J^{s+\frac{1}{2}} w}{\langle x - v \rangle^{\sigma_0}} \right\|_0 \left\| \frac{J^{s+\frac{1}{2}} \Upsilon w}{\langle x - v \rangle^{\sigma_0}} \right\|_0 dt \\ &\leq A \sum_v \int_0^T \|\langle x - v \rangle^{\sigma_0} T_{\chi_v(b_1 - b_1^0)} \langle x - v \rangle^{\sigma_0}\|_{\mathcal{L}(H^{s-\frac{1}{2}})} \left\| \frac{J^{s+\frac{1}{2}} w}{\langle x - v \rangle^{\sigma_0}} \right\|_0 \left\| \frac{J^{s+\frac{1}{2}} \Upsilon w}{\langle x - v \rangle^{\sigma_0}} \right\|_0 dt \\ &\leq A \sum_v \int_0^T \|\chi_v(b_1 - b_1^0)\|_{L^\infty} \left\| \frac{J^{s+\frac{1}{2}} w}{\langle x - v \rangle^{\sigma_0}} \right\|_0 \left\| \frac{J^{s+\frac{1}{2}} \Upsilon w}{\langle x - v \rangle^{\sigma_0}} \right\|_0 dt \\ &\leq A \sum_v \sup_{[0,T]} \|\chi_v(b_1 - b_1^0)\|_{L^\infty} \|J^{s+\frac{1}{2}} w\|_T \|J^{s+\frac{1}{2}} \Upsilon w\|_T. \end{aligned}$$

Now, it follows from the Taylor formula that we can write

$$\begin{aligned} b_1 - b_1^0 &= \partial_v F(z) - \partial_v F(z_0) \\ &= (w - u_0)G_1(z_0, z) + \nabla_x(w - u_0)G_2(z_0, z) + (\bar{w} - \bar{u}_0)G_3(z_0, z) + \nabla_x(\bar{w} - \bar{u}_0)G_4(z_0, z), \end{aligned}$$

where, for simplicity, $z_0 = (u_0, \nabla_x u_0, \bar{u}_0, \nabla_x \bar{u}_0)$ and $z = (w, \nabla_x w, \bar{w}, \nabla_x \bar{w})$, and the G_k are functions of the form

$$\int_0^1 F_k(z_0 + \tau(z - z_0)) d\tau,$$

where F_k is a second-order partial derivative of F . Next, it follows from the assumption on F that $G_k(0, 0) = 0$ for all k , from which one deduces easily that

$$\|\chi_\nu(b_1 - b_1^0)\|_{L^\infty} \leq C(\|(z_0, z)\|_{L^\infty}) \|\chi_\nu(z_0, z)\|_{L^\infty} \tilde{\chi}_\nu(z_0, z) \|_{L^\infty},$$

where $\tilde{\chi}_\nu$ is similar to χ_ν , and, by using the Sobolev injection, that is, Proposition A.5(i), that

$$\begin{aligned} \|\chi_\nu(b_1 - b_1^0)\|_{L^\infty} &\leq C(\|(z_0, z)\|_{L^\infty}) \|\chi_\nu(z_0, z)\|_{H^\sigma([0, T]; H^{s'})} \|\tilde{\chi}_\nu(z_0, z)\|_{H^\sigma([0, T]; H^{s'})} \\ &\leq C(K) \|\chi_\nu(u_0, w)\|_{H^\sigma([0, T]; H^{s'+1})} \|\tilde{\chi}_\nu(u_0, w)\|_{H^\sigma([0, T]; H^{s'+1})}, \end{aligned}$$

where $\sigma > \frac{1}{2}$ and $s' > \frac{n}{2}$. Thus, to obtain the summability in ν of $\|\chi_\nu(b_1 - b_1^0)\|_{L^\infty}$, it is sufficient to prove that $\|\chi_\nu(u_0, w)\|_{H^\sigma([0, T]; H^{s'+1})}$ is square summable in ν . To this end and to get an explicit bound for the sum, let us apply the interpolation inequality of Proposition A.5. This yields, by taking $\frac{1}{2} < \sigma < 1$,

$$\begin{aligned} \|\chi_\nu(u_0, w)\|_{H^\sigma([0, T]; H^{s'+1})} &\leq A \|\chi_\nu(u_0, w)\|_{L^2([0, T]; H^{s'+2})}^{1-\sigma} \|\chi_\nu(u_0, w)\|_{H^1([0, T]; H^{s''})}^\sigma \\ &\leq A (\|\chi_\nu(u_0, w)\|_{L^2([0, T]; H^{s'+2})} + \|\chi_\nu(u_0, w)\|_{L^2([0, T]; H^{s'+2})}^{1-\sigma} \|\chi_\nu \partial_t w\|_{L^2([0, T]; H^{s''})}^\sigma), \end{aligned}$$

where s'' is such that $(1 - \sigma)(s' + 2) + \sigma s'' = s' + 1$, that is, $s'' = s' + 2 - \frac{1}{\sigma}$. One can choose s' and σ such that $s'' = s - 2$, that is, such that $s' = s - 4 + \frac{1}{\sigma}$. In fact, if $\sigma = \frac{1}{2} + \varepsilon$, then $s' = \frac{n}{2} + \varrho - 4\varepsilon/(1 + 2\varepsilon)$, which is larger than $\frac{n}{2}$ if ε is small enough. With such a choice, we also have $s' + 2 < s$, so, the expressions $\|\chi_\nu(u_0, w)\|_{L^2([0, T]; H^{s'+2})}$ and $\|\chi_\nu \partial_t w\|_{L^2([0, T]; H^{s''})}$ are both square summable in ν , which shows that $\|\chi_\nu(u_0, w)\|_{H^\sigma([0, T]; H^{s'+1})}$ is itself square summable in ν and that, by applying Hölder's inequality,

$$\begin{aligned} \sum_\nu \|\chi_\nu(u_0, w)\|_{H^\sigma([0, T]; H^{s'+1})}^2 &\leq A \sum_\nu \|\chi_\nu(u_0, w)\|_{L^2([0, T]; H^{s'})}^2 + A \left(\sum_\nu \|\chi_\nu(u_0, w)\|_{L^2([0, T]; H^{s'})}^2 \right)^{1-\sigma} \left(\sum_\nu \|\chi_\nu \partial_t w\|_{L^2([0, T]; H^{s-2})}^2 \right)^\sigma \\ &\leq A (T\lambda_1(w)^2 + (T\lambda_1(w)^2)^{1-\sigma} (T\lambda_3(w)^2)^\sigma) \leq AT\lambda(w)^2, \end{aligned}$$

where, of course, the constant A changes from one inequality to the other. Consequently,

$$\sum_\nu \|\chi_\nu(b_1 - b_1^0)\|_{L^\infty} \leq AC(K)T\lambda(w)^2,$$

which allows us finally to bound the first term of (26) as follows:

$$\begin{aligned} \int_0^T |(G_\mu J^s T_{b_1 - b_1^0} \nabla_x w, J^s \Upsilon w)| dt &\leq AC(K)T\lambda(w)^2 \lambda_2(w) \lambda_2(\Upsilon w) \\ &\leq AC(K)K^2 T\lambda(w)\lambda(\Upsilon w). \end{aligned} \tag{28}$$

The second term of (26) is treated in the same manner.

Let us now consider the last term of (26). As above, let z stand for $(w, \nabla_x w, \bar{w}, \nabla_x \bar{w})$. As $z \in H^{s-1}(\mathbb{R}^n) = H^{\frac{n}{2}+1+\varrho}(\mathbb{R}^n)$, it follows from Bony’s formula, that is, Theorem 2.10, that $R(z) \in H^{2(s-1)-\frac{n}{2}}(\mathbb{R}^n) = H^{s+\varrho}(\mathbb{R}^n)$ and that

$$\|R(z)\|_{s+\varrho} \leq C(K)\|z\|_{s-1} \leq C(K)\|w\|_s.$$

Hence,

$$\begin{aligned} \int_0^T |\langle G_\mu J^s R(z), J^s \Upsilon w \rangle| dt &\leq A \int_0^T \|R(z)\|_s \|\Upsilon w\|_s dt \leq AC(K) \int_0^T \|w\|_s \|\Upsilon w\|_s dt \\ &\leq AC(K)T\lambda_1(w)\lambda_1(\Upsilon w) \leq AC(K)T\lambda(w)\lambda(\Upsilon w). \end{aligned} \tag{29}$$

Thus, we have bounded all the terms of (26), which leads to the estimate

$$\max\{\lambda_1(\Upsilon w), \lambda_2(\Upsilon w)\} \leq A\|u_0\|_s + \sqrt{AC(K)T\lambda(w)\lambda(\Upsilon w)}, \tag{30}$$

where the constants A and $C(K)$ have changed of course.

It remains to estimate $\lambda_3(\Upsilon w)$. Recall that Υw satisfies the Cauchy problem (24). Hence, applying Theorem 2.6 yields

$$\begin{aligned} \|\partial_t \Upsilon w\|_{s-2} &\leq \|\Upsilon w\|_s + A(\|b_1^0\|_{L^\infty} + \|b_2^0\|_{L^\infty})\|\Upsilon w\|_{s-1} \\ &\quad + A(\|a_1^0\|_{L^\infty} + \|a_2^0\|_{L^\infty})\|\Upsilon w\|_{s-2} + A(\|b_1 - b_1^0\|_{L^\infty} + \|b_2 - b_2^0\|_{L^\infty})\|w\|_{s-1} \\ &\quad + A(\|a_1 - a_1^0\|_{L^\infty} + \|a_2 - a_2^0\|_{L^\infty})\|w\|_{s-2} + \|R(z)\|_{s-2} \\ &\leq A\|\Upsilon w\|_s + A(\|b_1 - b_1^0\|_{L^\infty} + \|b_2 - b_2^0\|_{L^\infty} \\ &\quad + \|a_1 - a_1^0\|_{L^\infty} + \|a_2 - a_2^0\|_{L^\infty})\|w\|_s + \|R(z)\|_{s-2}. \end{aligned} \tag{31}$$

Now, as before, it follows from Proposition A.5 that

$$\|b_j - b_j^0\|_{L^\infty} \leq A\|b_j - b_j^0\|_{H^\sigma([0,T];H^{s'})} \leq A\|b_j - b_j^0\|_{L^2([0,T];H^{s'+1})}^{1-\sigma} \|b_j - b_j^0\|_{H^1([0,T];H^{s''})}^\sigma,$$

where $j = 1, 2$, $\sigma > \frac{1}{2}$, $s' > \frac{n}{2}$ and s'' is such that $(1 - \sigma)(s' + 1) + \sigma s'' = s'$. In fact, we can take $s'' = s - 3$, which corresponds to $s' = s + \frac{1}{\sigma} - 4 = \frac{n}{2} + \varrho + \frac{1}{\sigma} - 2$; so, $s' < s - 2$ and if σ is close enough to $\frac{1}{2}$, then, $s' > \frac{n}{2}$. Therefore, with such a choice, we have

$$\|b_j - b_j^0\|_{L^\infty} \leq A\|b_j - b_j^0\|_{L^2([0,T];H^{s-1})} + A\|b_j - b_j^0\|_{L^2([0,T];H^{s-1})}^{1-\sigma} \|\partial_t b_j\|_{L^2([0,T];H^{s-3})}^\sigma.$$

Next, applying Proposition 2.9 yields

$$\begin{aligned} \|b_j - b_j^0\|_{L^2([0,T];H^{s-1})}^2 &= \int_0^T \|b_j - b_j^0\|_{s-1}^2 dt \\ &\leq \int_0^T (C(\|z\|_{L^\infty})\|z\|_{s-1} + C(\|z_0\|_{L^\infty})\|z_0\|_{s-1})^2 dt \leq C(K)^2 T\lambda_1(w)^2, \end{aligned}$$

and

$$\begin{aligned} \|\partial_t b_j\|_{L^2([0,T];H^{s-3})}^2 &= \int_0^T \|(\partial_v F)'(z)\partial_t z\|_{s-3}^2 dt \leq A \int_0^T \|(\partial_v F)'(z)\|_{s-2}^2 \|\partial_t z\|_{s-3}^2 dt \\ &\leq A \int_0^T \|(\partial_v F)'(z)\|_{s-2}^2 \|\partial_t w\|_{s-2}^2 dt \leq ATC(K)^2 \lambda_3(w)^2, \end{aligned}$$

which imply that

$$\|b_j - b_j^0\|_{L^\infty} \leq AC(K)\sqrt{T}\lambda_1(w) + AC(K)\sqrt{T}\lambda_1(w)^{1-\sigma}\lambda_3(w)^\sigma \leq AC(K)\sqrt{T}\lambda(w).$$

Of course, the same inequality holds for $\|a_j - a_j^0\|_{L^\infty}$, $j = 1, 2$. Note that we have applied the following classical lemma:

Lemma 4.1. *If $s > \frac{n}{2}$ and $|r| \leq s$, then $H^r(\mathbb{R}^n) \cdot H^s(\mathbb{R}^n) \subset H^r(\mathbb{R}^n)$ with continuous injection.*

Finally, it follows from Theorem 2.10 and Theorem 2.11 that

$$\begin{aligned} \|R(z)\|_{s-2} &= \|R(z)\|_{\frac{n}{2}+\varrho} \leq \|R(z) - R(z_0)\|_{\frac{n}{2}+\varrho} + \|R(z_0)\|_{\frac{n}{2}+\varrho} \\ &\leq C_1(\|z\|_{\frac{n}{2}+\varrho}, \|z_0\|_{\frac{n}{2}+\varrho})\|z - z_0\|_{\frac{n}{2}+\varrho} + C_2(\|z_0\|_{\frac{n}{2}+\varrho})\|z_0\|_{\frac{n}{2}+\varrho} \\ &\leq C_1(\|w\|_{\frac{n}{2}+\varrho+1}, \|u_0\|_{\frac{n}{2}+\varrho+1})\|w - u_0\|_{\frac{n}{2}+\varrho+1} + C_2(\|u_0\|_{\frac{n}{2}+\varrho+1})\|u_0\|_{\frac{n}{2}+\varrho+1} \\ &\leq C(K)\|w - u_0\|_{\frac{n}{2}+\varrho+1} + A\|u_0\|_{\frac{n}{2}+\varrho+1}, \end{aligned}$$

and, using once again Proposition A.5, we obtain

$$\begin{aligned} \sup_{[0,T]} \|w - u_0\|_{s'} &\leq A\|w - u_0\|_{H^\sigma([0,T];H^{s'})} \leq A\|w - u_0\|_{L^2([0,T];H^{s'+1})}^{1-\sigma} \|w - u_0\|_{H^1([0,T];H^{s''})}^\sigma \\ &\leq A\|w - u_0\|_{L^2([0,T];H^{s'+1})} + A\|w - u_0\|_{L^2([0,T];H^{s'+1})}^{1-\sigma} \|\partial_t w\|_{L^2([0,T];H^{s''})}^\sigma \\ &\leq A\|w - u_0\|_{L^2([0,T];H^s)} + A\|w - u_0\|_{L^2([0,T];H^s)}^{1-\sigma} \|\partial_t w\|_{L^2([0,T];H^{s-2})}^\sigma \\ &\leq A\sqrt{T}\lambda_1(w) + A\sqrt{T}\lambda_1(w)^{1-\sigma}\lambda_3(w)^\sigma \leq A\sqrt{T}\lambda(w), \end{aligned}$$

where $s' = \frac{n+\varrho}{2} + 1 < s$, $\sigma > \frac{1}{2}$, $s'' = \frac{n+\varrho}{2} + 2 - \frac{1}{\sigma}$ and $s'' \leq s - 2$ if σ is close to $\frac{1}{2}$. Hence,

$$\sup_{[0,T]} \|R(z)\|_{s-2} \leq A\|u_0\|_{\frac{n+\varrho}{2}+1} + AC(K)\sqrt{T}\lambda(w).$$

Thus, we have bounded all the terms of (31) and the result is that

$$\begin{aligned} \lambda_3(\Upsilon w) &\leq A\lambda_1(\Upsilon w) + AC(K)\sqrt{T}\lambda(w)\lambda_1(w) + A\|u_0\|_{\frac{n+\varrho}{2}+1} + AC(K)\sqrt{T}\lambda(w) \\ &\leq A\|u_0\|_s + \sqrt{AC(K)T\lambda(w)\lambda(\Upsilon w)} + AC(K)\sqrt{T}\lambda(w), \end{aligned} \tag{32}$$

where, of course, we have used (30). Therefore,

$$\begin{aligned} \lambda(\Upsilon w) &\leq A\|u_0\|_s + \sqrt{AC(K)T\lambda(w)\lambda(\Upsilon w)} + AC(K)\sqrt{T}\lambda(w) \\ &\leq A\|u_0\|_s + \frac{1}{2}AC(K)T\lambda(w) + \frac{1}{2}\lambda(\Upsilon w) + AC(K)\sqrt{T}\lambda(w), \end{aligned}$$

which leads to

$$\lambda(\Upsilon w) \leq 2A\|u_0\|_s + AC(K)T\lambda(w) + 2AC(K)\sqrt{T}\lambda(w),$$

that is, an estimate which is, by changing the constants and taking $T \leq 1$, of the form

$$\lambda(\Upsilon w) \leq A\|u_0\|_s + AC(K)\sqrt{T}\lambda(w). \tag{33}$$

This is the main nonlinear estimate. In fact, when $u_0 \neq 0$, by taking $K = 2A\|u_0\|_s$ for example, and then, $T > 0$ such that

$$T \leq \left(\frac{A\|u_0\|_s}{AC(K)K}\right)^2 = \left(\frac{1}{2AC(K)}\right)^2,$$

it follows from (33) that $\lambda(\Upsilon w) \leq K$ when $\lambda(w) \leq K$, that is, $\Upsilon(Z) \subset Z$. When $u_0 = 0$, it suffices to take $K > 0$ and $T \leq 1/A^2C(K)^2$ to obtain the same result.

Let us now show that $\Upsilon : Z \rightarrow Z$ is a contraction mapping. In fact, the arguments are similar to the above ones and we shall be brief. If $w_1, w_2 \in Z$, then $W = \Upsilon w_1 - \Upsilon w_2$ satisfies the Cauchy problem

$$\begin{cases} \partial_t W = i\mathcal{L}W + T_{b_1^0}\nabla_x W + T_{b_2^0}\nabla_x \bar{W} + T_{a_1^0}W + T_{a_2^0}\bar{W} + \tilde{R}(z_1) - \tilde{R}(z_2), \\ W(0) = 0, \end{cases} \tag{34}$$

where, as before, $z_j = (w_j, \nabla_x w_j, \bar{w}_j, \nabla_x \bar{w}_j)$, $j = 1, 2$. Applying Theorem 3.1 to (34) gives

$$\max\{\lambda_1(W)^2, \lambda_2(W)^2\} \leq AI_T(J^s(\tilde{R}(z_1) - \tilde{R}(z_2)), J^s W), \tag{35}$$

and, consequently, we have to estimate uniformly in μ the integral

$$\int_0^T |(G_\mu J^s(\tilde{R}(z_1) - \tilde{R}(z_2)), J^s W)| dt.$$

It follows from (21) that

$$\begin{aligned} \tilde{R}(z_1) - \tilde{R}(z_2) &= T_{b_1(z_1)-b_1^0}\nabla(w_1 - w_2) + T_{b_1(z_1)-b_1(z_2)}\nabla w_2 \\ &\quad + T_{b_2(z_1)-b_2^0}\nabla(\bar{w}_1 - \bar{w}_2) + T_{b_2(z_1)-b_2(z_2)}\nabla \bar{w}_2 \\ &\quad + T_{a_1(z_1)-a_1^0}(w_1 - w_2) + T_{a_1(z_1)-a_1(z_2)}w_2 \\ &\quad + T_{a_2(z_1)-a_2^0}(\bar{w}_1 - \bar{w}_2) + T_{a_2(z_1)-a_2(z_2)}\bar{w}_2 \\ &\quad + R(z_1) - R(z_2), \end{aligned} \tag{36}$$

and we have to estimate the integral corresponding to each term of the above sum. Let us first consider the terms of the third line in (36). By the same argument as that used to obtain (27), we have

$$\int_0^T |(G_\mu J^s(T_{a_1(z_1)-a_1^0}(w_1 - w_2) + T_{a_1(z_1)-a_1(z_2)}w_2), J^s W)| dt \leq ATC(K)\lambda(w_1 - w_2)\lambda(W),$$

where we also applied Proposition 2.9 for the second term. Of course, we have the same estimate for the integral corresponding to the terms of the fourth line in (36).

As for the terms of the first line in (36), applying an argument similar to that yielding (28), one obtains

$$\begin{aligned} \int_0^T & \left| \langle G_\mu J^s (T_{b_1(z_1)-b_1^0} \nabla(w_1 - w_2) + T_{b_1(z_1)-b_1(z_2)} \nabla w_2), J^s W \rangle \right| dt \\ & \leq A, C(K)T(\lambda(w_1)^2 \lambda_2(w_1 - w_2) + \lambda(w_1 - w_2)(\lambda(w_1) + \lambda(w_2))\lambda_2(w_2))\lambda_2(W) \\ & \leq AC(K)T(\lambda(w_1)^2 + \lambda(w_1)\lambda(w_2) + \lambda(w_2)^2)\lambda(w_1 - w_2)\lambda(W) \\ & \leq AC(K)K^2T\lambda(w_1 - w_2)\lambda(W), \end{aligned}$$

and the same estimate holds for the terms of the second line in (36).

Last, for the terms of the fifth line in (36), applying Theorem 2.11 and estimating as in (29), we obtain

$$\begin{aligned} \int_0^T & \left| \langle G_\mu J^s (R(z_1) - R(z_2)), J^s W \rangle \right| dt \leq A \int_0^T \|z_1 - z_2\|_{s-1} \|W\|_s dt \leq AC(K) \int_0^T \|w_1 - w_2\|_s \|W\|_s dt \\ & \leq AC(K)T\lambda_1(w_1 - w_2)\lambda_1(W) \\ & \leq AC(K)T\lambda(w_1 - w_2)\lambda(W). \end{aligned}$$

Summing up and going back to (35), we can conclude that

$$\max\{\lambda_1(W)^2, \lambda_2(W)^2\} \leq AC(K)T\lambda(w_1 - w_2)\lambda(W).$$

It remains to estimate $\lambda_3(W)$. Using the fact that W satisfies the Cauchy problem (34) and an argument similar to that yielding (32), we obtain

$$\begin{aligned} \lambda_3(W) & \leq A\lambda_1(W) + AC(K)\sqrt{T}(\lambda(w_1)\lambda_1(w_1 - w_2) + \lambda(w_1 - w_2)\lambda_1(w_2)) + AC(K)\sqrt{T}\lambda(w_1 - w_2) \\ & \leq A\lambda_1(W) + AC(K)\sqrt{T}(\lambda(w_1) + \lambda(w_2))\lambda(w_1 - w_2) + AC(K)\sqrt{T}\lambda(w_1 - w_2) \\ & \leq \sqrt{AC(K)T\lambda(w_1 - w_2)\lambda(W)} + AC(K)\sqrt{T}\lambda(w_1 - w_2). \end{aligned}$$

Summing up, we have obtained

$$\lambda(W) \leq \sqrt{AC(K)T\lambda(w_1 - w_2)\lambda(W)} + AC(K)\sqrt{T}\lambda(w_1 - w_2).$$

Hence,

$$\lambda(W) \leq \frac{1}{2}AC(K)T\lambda(w_1 - w_2) + \frac{1}{2}\lambda(W) + AC(K)\sqrt{T}\lambda(w_1 - w_2);$$

that is,

$$\lambda(W) = \lambda(\Upsilon w_1 - \Upsilon w_2) \leq AC(K)\sqrt{T}\lambda(w_1 - w_2),$$

with modified constants. This clearly implies, if T is taken small enough, that $\Upsilon : Z \rightarrow Z$ is a contraction mapping and, thus, it has a unique fixed point u in Z which is a solution of (18). In fact, this is the solution of (18) in $C([0, T], H^s(\mathbb{R}^n))$ because the above method gives the local uniqueness and we obtain eventually the full uniqueness by applying a classical bootstrap argument. This proves the first part of Theorem 1.1.

The second part of Theorem 1.1 concerns the continuity of the solution operator $u_0 \mapsto u$ and we start its proof by remarking that this operator maps bounded subsets of $H^s(\mathbb{R}^n)$ into bounded subsets of $C([0, T], H^s(\mathbb{R}^n))$. In fact, if B is a bounded subset of $H^s(\mathbb{R}^n)$, as remarked at the beginning of this

section, the constant A and the bounds of the seminorms of the operators G_μ can be taken to depend only on B ; that is, if $u_0 \in B$, the estimates proven above and satisfied by Υ can be rewritten as

$$\lambda(\Upsilon w) \leq A(B)\|u_0\|_s + A(B)C(K)\sqrt{T}\lambda(w), \tag{37}$$

$$\lambda(\Upsilon w_1 - \Upsilon w_2) \leq A(B)C(K)\sqrt{T}\lambda(w_1 - w_2), \tag{38}$$

where $A(B)$ depends only on n, s and B , which implies that the constants K and T can be chosen depending only on B . Hence, for all $u_0 \in B$, the associated solutions u are all defined on the same interval $[0, T]$ and are all in the ball of radius K . As for the continuity, let B be a bounded subset of $H^s(\mathbb{R}^n)$, $u_0, u_0^* \in B$, u, u^* the respective associated solutions and $w = u - u^*$. Then, w satisfies the Cauchy problem

$$\begin{cases} \partial_t w = i\mathcal{L}w + Du - D^*u^* + \tilde{R} - \tilde{R}^* = i\mathcal{L}w + Dw + (D - D^*)u^* + \tilde{R} - \tilde{R}^*, \\ w(x, 0) = u_0(x) - u_0^*(x), \end{cases} \tag{39}$$

where

$$\begin{aligned} Dw &= T_{b_1^0}\nabla w + T_{b_2^0}\nabla\bar{w} + T_{a_1^0}w + T_{a_2^0}\bar{w}, & D^*w &= T_{b_1^{0,*}}\nabla w + T_{b_2^{0,*}}\nabla\bar{w} + T_{a_1^{0,*}}w + T_{a_2^{0,*}}\bar{w}, \\ \tilde{R} &= \tilde{R}(u, \nabla u, \bar{u}, \nabla\bar{u}) & \tilde{R}^* &= \tilde{R}(u^*, \nabla u^*, \bar{u}^*, \nabla\bar{u}^*). \end{aligned}$$

Of course, the b_j^0, a_j^0 correspond to u_0 whereas the $b_j^{0,*}, a_j^{0,*}$ correspond to u_0^* . Applying Theorem 3.1 gives us the inequality

$$\max\{\lambda_1(w)^2, \lambda_2(w)^2\} \leq A(B)\|u_0 - u_0^*\|_s^2 + A(B)I_T(J^s((D - D^*)u^* + \tilde{R} - \tilde{R}^*), J^s w). \tag{40}$$

As it can be seen easily by going back to (21), we can write

$$\begin{aligned} \tilde{R} - \tilde{R}^* &= T_{b_1(u)-b_1^0}\nabla w + T_{b_1(u)-b_1(u^*)}\nabla u^* + T_{b_1^{0,*}-b_1^0}\nabla u^* \\ &\quad + T_{b_2(u)-b_2^0}\nabla\bar{w} + T_{b_2(u)-b_2(u^*)}\nabla\bar{u}^* + T_{b_2^{0,*}-b_2^0}\nabla\bar{u}^* \\ &\quad + T_{a_1(u)-a_1^0}w + T_{a_1(u)-a_1(u^*)}u^* + T_{a_1^{0,*}-a_1^0}u^* \\ &\quad + T_{a_2(u)-a_2^0}\bar{w} + T_{a_2(u)-a_2(u^*)}\bar{u}^* + T_{a_2^{0,*}-a_2^0}\bar{u}^* \\ &\quad + R(u, \nabla u, \bar{u}, \nabla\bar{u}) - R(u^*, \nabla u^*, \bar{u}^*, \nabla\bar{u}^*), \end{aligned} \tag{41}$$

and we also have

$$(D - D^*)u^* = T_{b_1^0-b_1^{0,*}}\nabla u^* + T_{b_2^0-b_2^{0,*}}\nabla\bar{u}^* + T_{a_1^0-a_1^{0,*}}u^* + T_{a_2^0-a_2^{0,*}}\bar{u}^*.$$

Using the same arguments as before to estimate the integrals corresponding to each of the above terms yields

$$\max\{\lambda_1(w)^2, \lambda_2(w)^2\} \leq A(B)\|u_0 - u_0^*\|_s^2 + A_1(B)C_1(K)T(\lambda(w)\|u_0 - u_0^*\|_s + \lambda(w)^2), \tag{42}$$

which becomes, after a change of the constants and assuming $T \leq 1$,

$$\max\{\lambda_1(w), \lambda_2(w)\} \leq A(B)\|u_0 - u_0^*\|_s + A(B)C(K)\sqrt{T}\lambda(w). \tag{43}$$

Next, using (39) and similar arguments, one can easily get

$$\lambda_3(w) \leq A(B)C(K)(\|u_0 - u_0^*\|_s + \lambda_1(w)),$$

which becomes, after use of (43) and a possible change of the constants,

$$\lambda_3(w) \leq A(B)C(K)(\|u_0 - u_0^*\|_s + \sqrt{T}\lambda(w)).$$

Hence,

$$\lambda(w) \leq A(B)C(K)\|u_0 - u_0^*\|_s + A(B)C(K)\sqrt{T}\lambda(w), \quad (44)$$

which, by taking $T \leq (1/2A(B)C(K))^2$ (for example), leads to the Lipschitz estimate

$$\lambda(w) = \lambda(u - u^*) \leq 2A(B)C(K)\|u_0 - u_0^*\|_s, \quad (45)$$

and this achieves the proof of Theorem 1.1.

5. Proof of Proposition 3.2

We shall only give the main steps for the convenience of the reader and refer to [Bienaimé 2014] for the full details.

Let us start by remarking that it is sufficient to treat the case $s = 0$. Indeed, if $v = J^s u$ and $v_0 = J^s u_0$, it is easy to see that u is a solution of (8) if and only if v satisfies

$$\begin{cases} \partial_t v = i\mathcal{L}v + T_{b_1} \cdot \nabla_x v + T_{b_2} \cdot \nabla_x \bar{v} + \tilde{C}_1 v + \tilde{C}_2 \bar{v} + \tilde{f}(x, t), \\ v(x, 0) = v_0 \in L^2(\mathbb{R}^n), \end{cases} \quad (46)$$

where $\tilde{f} = J^s f$ and $\tilde{C}_k = J^s C_k J^{-s} + [J^s, T_{b_k} \cdot \nabla_x] J^{-s}$, $k = 1$ or 2 , and, thanks to the paradifferential calculus, the \tilde{C}_k are bounded operators in $L^2(\mathbb{R}^n)$.

The idea of proof is that of [Kenig et al. 1998], inspired by [Takeuchi 1992], and consists in constructing a pseudodifferential operator \mathcal{C} which is bounded and invertible in $L^2(\mathbb{R}^n)$ and estimating $\sup_{[0, T]} \|\mathcal{C}u\|_0$ instead of estimating directly $\sup_{[0, T]} \|u\|_0$. Since $\frac{d}{dt} \langle \mathcal{C}u, \mathcal{C}u \rangle = \langle \mathcal{C} \partial_t u, \mathcal{C}u \rangle + \langle \mathcal{C}u, \mathcal{C} \partial_t u \rangle$ and u is a solution of (8), we obtain that

$$\begin{aligned} \frac{d}{dt} \|\mathcal{C}u\|_0^2 &= \langle i\mathcal{C}\mathcal{L}u, \mathcal{C}u \rangle + \langle \mathcal{C}T_{b_1} \nabla u, \mathcal{C}u \rangle + \langle \mathcal{C}T_{b_2} \nabla \bar{u}, \mathcal{C}u \rangle \\ &\quad + \langle \mathcal{C}C_1 u, \mathcal{C}u \rangle + \langle \mathcal{C}C_2 \bar{u}, \mathcal{C}u \rangle + \langle \mathcal{C}f, \mathcal{C}u \rangle \\ &\quad + \langle \mathcal{C}u, i\mathcal{C}\mathcal{L}u \rangle + \langle \mathcal{C}u, \mathcal{C}T_{b_1} \nabla u \rangle + \langle \mathcal{C}u, \mathcal{C}T_{b_2} \nabla \bar{u} \rangle \\ &\quad + \langle \mathcal{C}u, \mathcal{C}C_1 u \rangle + \langle \mathcal{C}u, \mathcal{C}C_2 \bar{u} \rangle + \langle \mathcal{C}u, \mathcal{C}f \rangle, \end{aligned} \quad (47)$$

and since

$$\langle i\mathcal{L}\mathcal{C}u, \mathcal{C}u \rangle + \langle \mathcal{C}u, i\mathcal{L}\mathcal{C}u \rangle = 0,$$

we have finally

$$\begin{aligned} \frac{d}{dt} \|\mathcal{C}u\|_0^2 &= 2 \operatorname{Re} \langle (i[\mathcal{C}, \mathcal{L}] + \mathcal{C}T_{b_1} \nabla)u, \mathcal{C}u \rangle + 2 \operatorname{Re} \langle \mathcal{C}T_{b_2} \nabla \bar{u}, \mathcal{C}u \rangle \\ &\quad + 2 \operatorname{Re} \langle \mathcal{C}u, \mathcal{C}f \rangle + 2 \operatorname{Re} (\langle \mathcal{C}C_1 u, \mathcal{C}u \rangle + \langle \mathcal{C}C_2 \bar{u}, \mathcal{C}u \rangle). \end{aligned}$$

The idea of [Kenig et al. 1998] is precisely to choose C so that the operator $i[C, \mathcal{L}] + CT_{b_1} \nabla$ will be small in some sense. Here, we will make a refinement by writing $b_1 = b'_1 + ib''_1$ with real b'_1, b''_1 , and by considering the operator $i[C, \mathcal{L}] + iCT_{b'_1} \nabla$ instead. This has been already used by [Bienaimé 2014] and essentially allows one to construct a *real* operator C , that is, with the property $\overline{Cu} = C\bar{u}$, which will be convenient in certain arguments. Now, clearly,

$$|2 \operatorname{Re}(\langle C C_1 u, C u \rangle + \langle C C_2 \bar{u}, C u \rangle)| \leq 2(A_1 + A_2) \|C\|_{\mathcal{L}(L^2)}^2 \|u\|_0^2,$$

and integrating on $[0, T']$, $T' \leq T$, yields

$$\begin{aligned} \|Cu(T')\|_2^2 &\leq \|Cu_0\|_0^2 + 2 \left| \operatorname{Re} \int_0^{T'} \langle (i[C, \mathcal{L}] + iCT_{b'_1} \nabla)u, Cu \rangle dt \right| + 2 \left| \operatorname{Re} \int_0^{T'} \langle CT_{b'_1} \nabla u, Cu \rangle dt \right| \\ &\quad + 2 \left| \operatorname{Re} \int_0^{T'} \langle CT_{b_2} \nabla \bar{u}, Cu \rangle dt \right| + 2 \left| \operatorname{Re} \int_0^{T'} \langle Cu, Cf \rangle dt \right| \\ &\quad + 2(A_1 + A_2) \|C\|_{\mathcal{L}(L^2)}^2 \int_0^{T'} \|u(t)\|_0^2 dt, \end{aligned} \tag{48}$$

and our task will be to estimate appropriately each of the terms in the right-hand side of this inequality. The most difficult one is

$$\left| \int_0^T \langle (i[C, \mathcal{L}] + iCT_{b'_1} \nabla)u, Cu \rangle dt \right|$$

and C will be constructed so that this term will be small with respect to some parameters to be defined later. To this end, let us denote by c the symbol of C and define

$$p(x, \xi) = -2\xi^\# \cdot \nabla_x c(x, \xi) - c(x, \xi) \tilde{b}''_1(x, \xi) \cdot \xi, \tag{49}$$

where $\xi^\# = (\xi_1, \dots, \xi_{j_0}, -\xi_{j_0+1}, \dots, -\xi_n)$ and \tilde{b}''_1 is such that $T_{b'_1} = \tilde{b}''_1(x, D)$; see (2). The problem lies essentially in the fact that $p(x, \xi)$ is not the true principal symbol of the pseudodifferential (or paradifferential) operator $i[C, \mathcal{L}] + iCT_{b'_1} \nabla$ since C will be merely in the class $\operatorname{Op}S_{0,0}^0$. Nevertheless, the constructed C will allow us to obtain good estimates.

Set $c(x, \xi) = \exp(\gamma(x, \xi))$ and $\gamma(x, \xi) = \sum_{\mu \in \mathbb{Z}^n} \alpha_{1,\mu} \gamma_\mu(x, \xi)$, where the $\alpha_{1,\mu}$ are the coefficients of b_1 in its decomposition with respect to the $\varphi_{1,\mu}$, see (9), and the $\gamma_\mu(x, \xi)$ are defined a little later. Note here that one can assume the $\alpha_{1,\mu}$ real (and even nonnegative) without loss of generality. We can then write

$$p(x, \xi) = c(x, \xi) \sum_{\mu} \alpha_{1,\mu} (-2\xi^\# \cdot \nabla_x \gamma_\mu(x, \xi) - \tilde{\varphi}_{1,\mu}(x, \xi) \cdot \xi),$$

and this suggests considering the function

$$\eta_\mu(x, \xi) = \frac{1}{2} \int_0^\infty \operatorname{Im}(\tilde{\varphi}_{1,\mu})(x + s\xi^\#, \xi) \cdot \xi ds.$$

One can show that such a function is smooth and satisfies, for all multi-indices α, β ,

$$|\partial_x^\alpha \partial_\xi^\beta \eta_\mu(x, \xi)| \leq A_{\alpha,\beta} \sup_{\beta' \leq \beta} \|\langle \xi \rangle^{|\beta'|} \partial_x^\alpha \partial_\xi^{\beta'} \varphi_{1,\mu}\|_{L^\infty} \langle x - \mu \rangle^{|\beta|} \langle \xi \rangle^{-|\beta|}, \tag{50}$$

and, moreover,

$$-2\xi^\# \cdot \nabla_x \eta_\mu(x, \xi) - \text{Im}(\tilde{\varphi}_{1,\mu})(x, \xi) \cdot \xi = 0. \tag{51}$$

See [Kenig et al. 1998; Bienaimé 2014] for the proof. To get an even function, we replace η_μ by

$$\zeta_\mu(x, \xi) = \frac{1}{2}(\eta_\mu(x, \xi) + \eta_\mu(x, -\xi)),$$

which satisfies the same properties as η_μ , and then set

$$\gamma_\mu(x, \xi) = \theta\left(\frac{|\xi|}{R}\right)\psi\left(\frac{R(x-\mu)}{\langle \xi \rangle}\right)\zeta_\mu(x, \xi),$$

where θ and ψ are smooth (real) functions on \mathbb{R} such that $\theta(t) = 1$ if $t \geq 2$, $\theta(t) = 0$ if $t \leq 1$, $\psi(x) = 1$ if $|t| \leq 1$, $\psi = 0$ outside some compact set and R is a large parameter that will be fixed later. One can easily check that $\gamma_\mu \in S_{0,0}^0$ and that its seminorms are uniformly bounded with respect to μ and R . The following lemma gives the main properties of the operator C and its symbol

$$c(x, \xi) = \exp(\gamma(x, \xi)) = \exp\left(\sum_\mu \alpha_{1,\mu} \gamma_\mu(x, \xi)\right).$$

Lemma 5.1. (i) *The symbol $c(x, \xi)$ is real and even in ξ .*

(ii) *The symbol $c(x, \xi)$ is in the class $S_{0,0}^0$. More precisely, for all $\alpha, \beta \in \mathbb{N}^n$,*

$$|\partial_x^\alpha \partial_\xi^\beta c(x, \xi)| \leq \frac{A_{\alpha,\beta}}{R^{|\beta|}} \sup_{1 \leq j \leq |\alpha|+|\beta|} \sup_\mu \sup_{\alpha' \leq \alpha; \beta' \leq \beta} \|\langle \xi \rangle^{|\beta'|} \partial_x^{\alpha'} \partial_\xi^{\beta'} \varphi_{1,\mu}\|_{L^\infty}^j \leq \frac{A_{\alpha,\beta}}{R^{|\beta|}} \sup_\mu v_{|\alpha|+|\beta|}(\varphi_{1,\mu}).$$

(iii) *There exist $N \in \mathbb{N}$ and $A > 0$ such that, for all $R \geq 1$ and all $v \in L^2(\mathbb{R}^n)$,*

$$\begin{aligned} \|Cv\|_0 &\leq A \sup_\mu v_N(\varphi_{1,\mu}) \|v\|_0, \\ \|v\|_0 &\leq A \sup_\mu v_N(\varphi_{1,\mu}) \|Cv\|_s + \frac{A}{R} \sup_\mu v_N(\varphi_{1,\mu})^2 \|v\|_s. \end{aligned}$$

(iv) *The symbol*

$$p(x, \xi) = -2\xi^\# \cdot \nabla_x c(x, \xi) - c(x, \xi) \tilde{b}_1''(x, \xi) \cdot \xi$$

is in $S_{0,0}^0$ and its seminorms (of order $\leq M$) are estimated by $AR \sup_\mu v_{M+1}(\varphi_{1,\mu})$.

Even if here the function $\varphi_{1,\mu}$ is more general, the proof follows the same lines as that of [Bienaimé 2014, Lemmas 3.5 and 3.6] and we refer to it. These properties are sufficient to allow us to get the following estimates:

Lemma 5.2. *Let $b(x, \xi)$ be a symbol satisfying*

$$b(x, \xi) = \sum_{\mu \in \mathbb{Z}^n} \alpha_\mu \varphi_\mu(x, \xi), \quad \varphi_\mu \in S_{1,0}^0, \quad \sum_\mu |\alpha_\mu| \leq A_0, \tag{52}$$

$x \mapsto \varphi_\mu(x, \xi)$ is rapidly decreasing in $x - \mu$,

and let \tilde{u} stand for u or \bar{u} . Then, there exist $N \in \mathbb{N}$ and $A > 0$ such that, for all $T > 0$, $T' \in [0, T]$, $R \geq 1$ and every $H = h(x, D)$ in $\text{Op } S_{0,0}^0$, the following estimates hold true:

- (i) $\int_0^{T'} | \langle (C T_b \nabla - (c\tilde{b})(x, D) \nabla) \tilde{u}, Hu \rangle | dt \leq \frac{A}{R} \|h\|_{C^N} \sup_{\mu} \nu_N(\varphi_{1,\mu}) \sup_{\mu} \|\varphi_{\mu}\|_{C^N} \|J^{\frac{1}{2}} u\|_T^2.$
- (ii) $\int_0^{T'} | \langle (i[C, \mathcal{L}] + i C T_{b'_1} \nabla) u, Hu \rangle | dt \leq A \|h\|_{C^N} \sup_{\mu} \nu_N(\varphi_{1,\mu}) \left(RT \sup_{[0,T]} \|u\|_0^2 + \frac{1}{R} \|J^{\frac{1}{2}} u\|_T^2 \right).$
- (iii) $\int_0^{T'} | \langle [C, J^s T_b J^{-s} \nabla] \tilde{u}, Hu \rangle | dt \leq A \|h\|_{C^N} \sup_{\mu} \nu_N(\varphi_{1,\mu}) \sup_{\mu} \|\varphi_{\mu}\|_{C^N} \left(T \sup_{[0,T]} \|u\|_0^2 + \frac{\|J^{\frac{1}{2}} u\|_T^2}{R} \right).$

Remark. The case $s \neq 0$ in (iii) is needed in the Appendix.

Proof. Using the pseudodifferential calculus, we can write the symbol $e(x, \xi)$ of the operator $E = C T_b \nabla - (c\tilde{b})(x, D) \nabla$ as $e = \sum_{\mu} \alpha_{\mu} e_{\mu}$, where e_{μ} is given by

$$e_{\mu}(x, \xi) = \frac{1}{(2\pi)^n} \sum_{j=1}^n \int_0^1 \int e^{-iy\eta} \partial_{\xi_j} c(x, \xi + t\eta) \partial_{x_j} \tilde{\varphi}_{\mu}(x + y, \xi) \cdot \xi dy d\eta dt, \tag{53}$$

and we first remark that $e_{\mu} \in \text{Op } S_{0,0}^1$ and that using the fast decrease of $\tilde{\varphi}_{\mu}(x, \xi)$ in $x - \mu$ and integrations by parts yields the fact that $e_{\mu}(x, \xi)$ is itself rapidly decreasing in $x - \mu$. Next, setting $E_{\mu} = e_{\mu}(x, D)$, we can write

$$\begin{aligned} \langle E\tilde{u}, Hu \rangle &= \sum_{\mu} \alpha_{\mu} \langle E_{\mu} \tilde{u}, Hu \rangle = \sum_{\mu} \alpha_{\mu} \langle H^* E_{\mu} \tilde{u}, u \rangle \\ &= \sum_{\mu} \alpha_{\mu} \langle \langle x - \mu \rangle^{\sigma_0} \tilde{H} \langle x - \mu \rangle^{-\sigma_0} \langle x - \mu \rangle^{\sigma_0} \tilde{E}_{\mu} \langle x - \mu \rangle^{\sigma_0} \tilde{u}, u_{\mu} \rangle, \end{aligned}$$

where $\tilde{H} = J^{-\frac{1}{2}} H^* J^{\frac{1}{2}}$, $\tilde{E}_{\mu} = J^{-\frac{1}{2}} E_{\mu} J^{-\frac{1}{2}}$ and $u_{\mu} = \langle x - \mu \rangle^{-\sigma_0} J^{\frac{1}{2}} u$. Now, it follows from the pseudodifferential calculus (Theorem 2.1) that \tilde{H} and \tilde{E}_{μ} are in $\text{Op } S_{0,0}^0$ and that we can estimate the seminorms of \tilde{H} and \tilde{E}_{μ} by those of H and E_{μ} respectively. Moreover, it is easy to see that the symbol of \tilde{E}_{μ} inherits the fast decrease in $x - \mu$ which implies, by virtue of Lemma 2.3, that the operator $\langle x - \mu \rangle^{\sigma_0} \tilde{E}_{\mu} \langle x - \mu \rangle^{\sigma_0}$ is also in $\text{Op } S_{0,0}^0$ and that its seminorms are estimated by those of E_{μ} uniformly in μ . The same property holds for the operator $\langle x - \mu \rangle^{\sigma_0} \tilde{H} \langle x - \mu \rangle^{-\sigma_0}$, as it follows also from Lemma 2.3. This allows us to apply the Calderón–Vaillancourt theorem to obtain

$$\begin{aligned} \int_0^{T'} | \langle E\tilde{u}, Hu \rangle | dt &\leq \sum_{\mu} |\alpha_{\mu}| \int_0^{T'} \| \langle x - \mu \rangle^{\sigma_0} \tilde{H} \langle x - \mu \rangle^{-\sigma_0} \|_{\mathcal{L}(L^2)} \| \langle x - \mu \rangle^{\sigma_0} \tilde{E}_{\mu} \langle x - \mu \rangle^{\sigma_0} \|_{\mathcal{L}(L^2)} \| u_{\mu} \|_0^2 dt \\ &\leq A \|h\|_{C^{N_1}} \sup_{\mu} \sum_{|\alpha|+|\beta| \leq N_1} \| \langle x - \mu \rangle^{2\sigma_0} \partial_x^{\alpha} \partial_{\xi}^{\beta} e_{\mu} \|_{L^{\infty}} \| J^{\frac{1}{2}} u \|_T^2 \\ &\leq \frac{A}{R} \|h\|_{C^{N_1}} \sup_{\mu} \nu_{N_2}(\varphi_{1,\mu}) \sup_{\mu} \|\varphi_{\mu}\|_{C^{N_2}} \| J^{\frac{1}{2}} u \|_T^2, \tag{54} \end{aligned}$$

which proves (i).

To prove (ii), note first that the symbol of $i[\mathbf{C}, \mathcal{L}]$ is given by

$$-2\xi^\# \cdot \nabla_x \mathbf{c}(x, \xi) + (\mathcal{L}_x \mathbf{c})(x, \xi)$$

and that of $i\mathbf{C}T_{b_1'}\nabla$ can be written as

$$i\mathbf{c}(x, \xi)\tilde{b}_1''(x, \xi).i\xi + \frac{1}{(2\pi)^n} \sum_{j=1}^n \int_0^1 \int e^{-iy\eta} \partial_{\xi_j} \mathbf{c}(x, \xi + t\eta) \partial_{x_j} \tilde{b}_1''(x + y, \xi).i\xi \, dy \, d\eta \, dt.$$

Thus, the symbol of the operator $i[\mathbf{C}, \mathcal{L}] + i\mathbf{C}T_{b_1'}\nabla$ is given by

$$(\mathcal{L}_x \mathbf{c})(x, \xi) + p(x, \xi) + ie(x, \xi),$$

where $p(x, \xi)$ is given by (49), $e = \sum_{\mu} \alpha_{\mu} e_{\mu}$ and $e_{\mu}(x, \xi)$ is given by (53) with $\alpha_{\mu} = \alpha_{1,\mu}$ and $\varphi_{\mu} = \text{Im}(\varphi_{1,\mu})$. Hence, applying Lemma 5.1 and the Calderón–Vaillancourt theorem yields the estimate

$$\int_0^{T'} |((\mathcal{L}_x \mathbf{c})(x, D) + p(x, D))u, Hu| \, dt \leq ART \|h\|_{C^{N_1}} \sup_{\mu} \nu_{N_1}(\varphi_{1,\mu})^2 \sup_{[0,T]} \|u\|_0^2,$$

and applying part (i) gives the estimate

$$\int_0^{T'} |(ie(x, D)u, Hu)| \, dt \leq \frac{A}{R} \|h\|_{C^{N_2}} \sup_{\mu} \nu_{N_2}(\varphi_{1,\mu})^2 \|J^{\frac{1}{2}}u\|_T^2,$$

which proves (ii).

To prove (iii), we first treat the case $s = 0$ and note that the symbol of $[\mathbf{C}, T_b \nabla] = \mathbf{C}T_b \nabla - T_b \nabla \mathbf{C}$ can be written simply as $e(x, \xi) - e_0(x, \xi)$, where $e(x, \xi)$ is the symbol of the operator E studied in (i) and

$$e_0(x, \xi) = \frac{1}{(2\pi)^n} \sum_{j=1}^n \int_0^1 \int e^{-iy\eta} \partial_{\xi_j} (\tilde{b}(x, \xi + t\eta).(\xi + t\eta)) \partial_{x_j} \mathbf{c}(x + y, \xi) \, dy \, d\eta \, dt.$$

Since $\partial_{\xi_j}(\tilde{b}(x, \xi).\xi)$ is of order 0, the symbol $e_0(x, \xi)$ is in fact in $S_{0,0}^0$ and the seminorms of e_0 are estimated by a product of seminorms of \tilde{b} and \mathbf{c} . Hence, by using the decomposition of b as above, we get

$$\int_0^{T'} |\langle e_0(x, D)\tilde{u}, Hu \rangle| \, dt \leq AT \|h\|_{C^{N_1}} \sup_{\mu} \|\varphi_{\mu}\|_{C^{N_2}} \sup_{\mu} \nu_{N_2}(\varphi_{1,\mu}) \sup_{[0,T]} \|u\|_0^2,$$

which, together with (54), yields (iii) in the case $s = 0$. If $s \neq 0$, it follows from the pseudodifferential and paradifferential calculi that $J^s T_b J^{-s} = T_{b^\#}$, where $b^\# = \sum_{\mu} \alpha_{\mu} \psi_{\mu}$ and ψ_{μ} is given by

$$\psi_{\mu}(x, \xi) = \frac{1}{(2\pi)^n} \int e^{-iy\eta} \langle \xi + \eta \rangle^s \varphi_{\mu}(x + y, \xi) \langle \xi \rangle^{-s} \, dy \, d\eta,$$

which implies that ψ_{μ} is also rapidly decreasing in $x - \mu$ and that it is in $S_{1,0}^0$ with seminorms estimated by those of φ_{μ} . This shows that the case $s \neq 0$ follows from the case $s = 0$ and achieves the proof of Lemma 5.2. □

Lemma 5.3. *Let b be as in the preceding lemma. Then, there exist $N \in \mathbb{N}$ and $A > 0$ such that, for all $T > 0$, $T' \in [0, T]$ and $R \geq 1$, the following estimates hold true:*

(i) *If $b(x, \xi)$ is even in ξ , then*

$$\int_0^{T'} |\langle \mathbf{C} T_b \nabla \bar{u}, \mathbf{C} u \rangle| dt \leq A \sup_{\mu} v_N(\varphi_{1,\mu}) \sup_{\mu} \|\varphi_{\mu}\|_{C^N} \left(T \sup_{0 \leq t \leq T} \|u\|_0^2 + \frac{1}{R} \|J^{\frac{1}{2}} u\|_T \right).$$

(ii) *If b is real, then*

$$\left| \operatorname{Re} \int_0^{T'} \langle \mathbf{C} T_b \nabla u, \mathbf{C} u \rangle dt \right| \leq A \sup_{\mu} v_N(\varphi_{1,\mu}) \sup_{\mu} \|\varphi_{\mu}\|_{C^N} \left(T \sup_{0 \leq t \leq T} \|u\|_0^2 + \frac{1}{R} \|J^{\frac{1}{2}} u\|_T \right).$$

Proof. Since \mathbf{C} is real, we can write

$$\langle \mathbf{C} T_b \nabla \bar{u}, \mathbf{C} u \rangle = \langle T_b \nabla \mathbf{C} \bar{u}, \mathbf{C} u \rangle + \langle [\mathbf{C}, T_b \nabla] \bar{u}, \mathbf{C} u \rangle = \langle T_b \nabla \overline{\mathbf{C} u}, \mathbf{C} u \rangle + \langle [\mathbf{C}, T_b \nabla] \bar{u}, \mathbf{C} u \rangle.$$

Now, the integral corresponding to $\langle [\mathbf{C}, T_b \nabla] \bar{u}, \mathbf{C} u \rangle$ is treated by Lemma 5.2(iii). As for the other term, we note that it is of the form $\langle T_b \nabla \bar{v}, v \rangle$, so it suffices to study such a term. Since $b(x, \xi)$ is even in ξ , we have

$$\langle T_b \nabla \bar{v}, v \rangle = \overline{\langle v, T_b \nabla \bar{v} \rangle} = \langle \bar{v}, T_{\bar{b}} \nabla v \rangle = \langle (T_{\bar{b}} \nabla)^* \bar{v}, v \rangle,$$

and it follows from the pseudodifferential (or paradifferential) calculus that

$$(T_{\bar{b}} \nabla)^* = -T_b \nabla + E_1, \tag{55}$$

where E_1 is of type $S_{1,0}^0$ and its seminorms (up to some finite order) are estimated by those of b . Hence,

$$\langle T_b \nabla \bar{v}, v \rangle = -\langle T_b \nabla \bar{v}, v \rangle + \langle E_1 \bar{v}, v \rangle,$$

and $\langle T_b \nabla \bar{v}, v \rangle = \frac{1}{2} \langle E_1 \bar{v}, v \rangle$, that is, $\langle T_b \nabla \overline{\mathbf{C} u}, \mathbf{C} u \rangle = \frac{1}{2} \langle E_1 \overline{\mathbf{C} u}, \mathbf{C} u \rangle$, and (i) follows just by applying the Calderón–Vaillancourt theorem and Lemma 5.1.

To prove (ii), we write as before

$$\langle \mathbf{C} T_b \nabla u, \mathbf{C} u \rangle = \langle T_b \nabla \mathbf{C} u, \mathbf{C} u \rangle + \langle [\mathbf{C}, T_b \nabla] u, \mathbf{C} u \rangle,$$

and then apply Lemma 5.2(iii) to reduce the problem to the study of $\operatorname{Re} \langle T_b \nabla \mathbf{C} u, \mathbf{C} u \rangle$. Now, it follows from (55) and the fact that b is real that we have

$$2 \operatorname{Re} \langle T_b \nabla \mathbf{C} u, \mathbf{C} u \rangle = \langle T_b \nabla \mathbf{C} u, \mathbf{C} u \rangle + \langle \mathbf{C} u, T_b \nabla \mathbf{C} u \rangle = \langle (T_b \nabla + (T_b \nabla)^*) \mathbf{C} u, \mathbf{C} u \rangle = \langle E_1 \mathbf{C} u, \mathbf{C} u \rangle$$

and the proof ends like that of (i). The lemma is thus proved. □

It is clear now that applying Lemmas 5.1, 5.2 and 5.3 to the inequality (48) yields Proposition 3.2.

6. Proof of Proposition 3.3

By the same argument as that used in the beginning of the proof of Proposition 3.2, it is sufficient to establish the first estimate in the case $s = 0$.

The proof follows the same ideas as that of [Kenig et al. 1998; Bienaimé 2014]. The difference is that here the T_{b_k} , $k = 1, 2$, are general paradifferential operators of order 0 instead of merely multiplication or paramultiplication operators.

Since

$$\begin{aligned} \partial_t u &= i\mathcal{L}u + T_{b_1} \cdot \nabla u + T_{b_2} \cdot \nabla \bar{u} + C_1 u + C_2 \bar{u} + f, \\ \partial_t \bar{u} &= -i\mathcal{L}\bar{u} + T_{\bar{b}_1} \cdot \nabla \bar{u} + T_{\bar{b}_2} \cdot \nabla u + \bar{C}_1 \bar{u} + \bar{C}_2 u + \bar{f}, \end{aligned}$$

where the operators \bar{C}_k are defined by $\bar{C}_k u = \overline{C_k \bar{u}}$, one starts by remarking that the vector unknown $w = \begin{pmatrix} u \\ \bar{u} \end{pmatrix}$ satisfies the system

$$\partial_t w = iHw + Bw + Cw + F, \tag{56}$$

where

$$H = \begin{pmatrix} \mathcal{L} & 0 \\ 0 & -\mathcal{L} \end{pmatrix}, \quad B = \begin{pmatrix} T_{b_1} \nabla & T_{b_2} \nabla \\ T_{\bar{b}_2} \nabla & T_{\bar{b}_1} \nabla \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_2 \\ \bar{C}_1 & \bar{C}_2 \end{pmatrix}, \quad F = \begin{pmatrix} f \\ \bar{f} \end{pmatrix},$$

and the idea then is to estimate the expression $\langle \Psi w, w \rangle$ by means of Gårding’s inequality for systems via Doi’s argument. Here,

$$\Psi = \begin{pmatrix} \Psi_0 & 0 \\ 0 & -\Psi_0 \end{pmatrix},$$

and Ψ_0 is an appropriate pseudodifferential operator in $\text{Op}S_{1,0}^0$ to be chosen a little later. By using (56), one gets easily

$$\begin{aligned} \partial_t \langle \Psi w, w \rangle &= \langle \Psi \partial_t w, w \rangle + \langle \Psi w, \partial_t w \rangle \\ &= \langle (i[\Psi, H] + B^* \Psi + \Psi B + C^* \Psi + \Psi C)w, w \rangle + \langle \Psi F, w \rangle + \langle \Psi w, F \rangle, \end{aligned} \tag{57}$$

and, as one can check also easily, the principal symbol of the first-order operator

$$i[\Psi, H] + B^* \Psi + \Psi B + C^* \Psi + \Psi C$$

is given by

$$M(x, \xi) = \begin{pmatrix} 2\xi^\# \cdot \nabla_x \psi_0(x, \xi) - 2\xi \cdot \text{Im}(\tilde{b}_1)(x, \xi) \psi_0(x, \xi) & 2i\xi \cdot \tilde{b}_2(x, \xi) \psi_0(x, \xi) \\ -2i\xi \cdot \tilde{b}_2(x, \xi) \psi_0(x, \xi) & 2\xi^\# \cdot \nabla_x \psi_0(x, \xi) - 2\xi \cdot \text{Im}(\tilde{b}_1)(x, \xi) \psi_0(x, \xi) \end{pmatrix},$$

where ψ_0 denotes the symbol of Ψ_0 . Now, for ψ_0 , we shall make the following choice which follows the idea of [Doi 1994]. Define

$$\begin{aligned} p(x, \xi) &= \langle \xi \rangle^{-1} \sum_{j=1}^n \xi_j^\# h(x_j) \quad \text{with } h(t) = \int_0^t \langle s \rangle^{-2\sigma_0} ds, \\ p_\mu(x, \xi) &= p(x - \mu, \xi) + A_0 \sum_{\mu' \in \mathbb{Z}^n} (|\alpha_{1, \mu'}| + |\alpha_{2, \mu'}|) p(x - \mu', \xi), \\ \psi_0(x, \xi) &= \psi_\mu(x, \xi) = \exp(-p_\mu(x, \xi)). \end{aligned}$$

Here, the $\alpha_{1, \mu'}$ and $\alpha_{2, \mu'}$ are the coefficients of b_1 and b_2 in their decompositions with respect to the $\varphi_{1, \mu'}$ and $\varphi_{2, \mu'}$ respectively, see (9), A_0 is a large constant that will be determined later and $\mu \in \mathbb{Z}^n$

is fixed for the moment. However, from now on, we shall write Ψ_μ and ψ_μ instead of Ψ_0 and ψ_0 to emphasize the dependance on μ . First, note that p_μ and ψ_μ are in $S_{1,0}^0$ and that their seminorms are uniformly bounded with respect to μ . Next, with these notations, the symbol $M(x, \xi)$ can be rewritten as

$$M(x, \xi) = 2\psi_\mu(x, \xi) \begin{pmatrix} -\xi^\# \cdot \nabla_x p_\mu(x, \xi) - \xi \cdot \text{Im}(\tilde{b}_1)(x, \xi) & i\xi \cdot \tilde{b}_2(x, \xi) \\ -i\xi \cdot \tilde{b}_2(x, \xi) & -\xi^\# \cdot \nabla_x p_\mu(x, \xi) - \xi \cdot \text{Im}(\tilde{b}_1)(x, \xi) \end{pmatrix}.$$

Consider now the matrix $Z(x, \xi) = -M(x, \xi) - V(x, \xi)$, where

$$V(x, \xi) = \frac{2\psi_\mu(x, \xi)|\xi|^2}{\langle \xi \rangle \langle x - \mu \rangle^{2\sigma_0}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$Z(x, \xi)$ is a matrix of symbols in $S_{1,0}^1$ and, in order to apply Gårding's inequality, we are going to show that, for large ξ , it is a nonnegative matrix, that is, $\langle Z(x, \xi)v, v \rangle \geq 0$ for all $v \in \mathbb{C}^2$. In fact, $Z(x, \xi)$ is of the form

$$2\psi_\mu(x, \xi) \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \alpha \end{pmatrix},$$

where

$$\alpha = \xi^\# \cdot \nabla_x p_\mu(x, \xi) - \frac{|\xi|^2}{\langle \xi \rangle \langle x - \mu \rangle^{2\sigma_0}} + \xi \cdot \text{Im}(\tilde{b}_1)(x, \xi) \quad \text{and} \quad \beta = -i\xi \cdot \tilde{b}_2(x, \xi),$$

and it is sufficient to show that the two eigenvalues $\alpha \pm |\beta|$ of $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \alpha \end{pmatrix}$ are nonnegative, or, equivalently, that $\alpha \geq |\beta|$, that is,

$$\xi^\# \cdot \nabla_x p_\mu(x, \xi) - \frac{|\xi|^2}{\langle \xi \rangle \langle x - \mu \rangle^{2\sigma_0}} + \xi \cdot \text{Im}(\tilde{b}_1)(x, \xi) \geq |-i\xi \cdot \tilde{b}_2(x, \xi)|. \tag{58}$$

Now, the main reason for the choice of the symbol p_μ is that it allows to get the following inequality:

$$\begin{aligned} \xi^\# \cdot \nabla_x p_\mu(x, \xi) &= \xi^\# \cdot \nabla_x p(x - \mu, \xi) + A_0 \sum_{\mu' \in \mathbb{Z}^n} (|\alpha_{1,\mu'}| + |\alpha_{2,\mu'}|) \xi^\# \cdot \nabla_x p(x - \mu', \xi) \\ &= \sum_{j=1}^n \frac{\xi_j^2}{\langle \xi \rangle \langle x_j - \mu_j \rangle^{2\sigma_0}} + A_0 \sum_{\mu' \in \mathbb{Z}^n} (|\alpha_{1,\mu'}| + |\alpha_{2,\mu'}|) \sum_{j=1}^n \frac{\xi_j^2}{\langle \xi \rangle \langle x_j - \mu'_j \rangle^{2\sigma_0}} \\ &\geq \frac{|\xi|^2}{\langle \xi \rangle \langle x - \mu \rangle^{2\sigma_0}} + A_0 \sum_{\mu' \in \mathbb{Z}^n} (|\alpha_{1,\mu'}| + |\alpha_{2,\mu'}|) \frac{|\xi|^2}{\langle \xi \rangle \langle x - \mu' \rangle^{2\sigma_0}}; \end{aligned} \tag{59}$$

that is,

$$\xi^\# \cdot \nabla_x p_\mu(x, \xi) - \frac{|\xi|^2}{\langle \xi \rangle \langle x - \mu \rangle^{2\sigma_0}} \geq A_0 \sum_{\mu' \in \mathbb{Z}^n} (|\alpha_{1,\mu'}| + |\alpha_{2,\mu'}|) \frac{|\xi|^2}{\langle \xi \rangle \langle x - \mu' \rangle^{2\sigma_0}}. \tag{60}$$

Besides, we have

$$\tilde{b}_k(x, \xi) = \sum_{\mu' \in \mathbb{Z}^n} \alpha_{k,\mu'} \tilde{\varphi}_{k,\mu'}(x, \xi), \quad k = 1, 2,$$

and it follows from Lemma 2.8 that

$$\langle x - \mu' \rangle^{2\sigma_0} |\tilde{\varphi}_{k,\mu'}(x, \xi)| \leq A(n),$$

with a constant $A(n)$ which depends only on the dimension. Hence,

$$|i\xi \tilde{b}_k(x, \xi)| \leq A(n) \sum_{\mu' \in \mathbb{Z}^n} |\alpha_{k,\mu'}| \frac{|\xi|}{\langle x - \mu' \rangle^{2\sigma_0}} \leq \sqrt{2}A(n) \sum_{\mu' \in \mathbb{Z}^n} |\alpha_{k,\mu'}| \frac{|\xi|^2}{\langle \xi \rangle \langle x - \mu' \rangle^{2\sigma_0}}, \quad k = 1, 2,$$

if $|\xi| \geq 1$, which, together with (60), implies (58) by taking $A_0 \geq \sqrt{2}A(n)$. Thus, the matrix symbol $Z(x, \xi)$ is nonnegative, and since it is also hermitian, $Z(x, \xi) + Z(x, \xi)^*$ is also nonnegative and we can apply Gårding’s inequality for systems:

$$\operatorname{Re}\langle Z(x, D)w, w \rangle \geq -A\left(1 + \sup_{|\alpha|+|\beta|\leq N} \sup_{k,\mu'} \|\langle \xi \rangle^{|\beta|} \partial_x^\alpha \partial_\xi^\beta \varphi_{k,\mu'}\|_{L^\infty}\right) \|w\|_0^2, \tag{61}$$

where the constant A depends only on A_1, A_2 and the dimension n and the integer N depends only on the dimension n . Now, going back to (57), we can rewrite it as

$$\partial_t \langle \Psi w, w \rangle = \langle (-Z(x, D) - V(x, D) + E)w, w \rangle + \langle \Psi F, w \rangle + \langle \Psi w, F \rangle,$$

where E is a bounded operator in $L^2(\mathbb{R}^n)$, and integrating it on $[0, T]$ yields

$$\begin{aligned} \int_0^T \langle V(x, D)w, w \rangle dt &= \langle \Psi w(0), w(0) \rangle - \langle \Psi w(T), w(T) \rangle \\ &\quad - \int_0^T \langle Z(x, D)w, w \rangle dt + \int_0^T \langle Ew, w \rangle dt + \int_0^T \langle \Psi F, w \rangle dt + \int_0^T \langle \Psi w, F \rangle dt. \end{aligned}$$

Taking the real part, using (61) and estimating, we obtain

$$\begin{aligned} \operatorname{Re} \int_0^T \langle V(x, D)w, w \rangle dt &\leq A \sup_{[0,T]} \|w\|_0^2 + AT\left(1 + \sup_{k,\mu'} \nu_N(\varphi_{k,\mu'})\right) \sup_{[0,T]} \|w\|_0^2 \\ &\quad + \left| \int_0^T \langle \Psi F, w \rangle dt \right| + \left| \int_0^T \langle \Psi w, F \rangle dt \right|, \end{aligned}$$

and since $\psi_\mu(x, \xi) \geq \exp(-A)$ and, for $|\xi| \geq 1$,

$$V(x, \xi) \geq e^{-A} \frac{\langle \xi \rangle}{\langle x - \mu \rangle^{2\sigma_0}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

a second application of Gårding’s inequality gives us

$$\begin{aligned} \operatorname{Re} \int_0^T \langle J^{\frac{1}{2}} \langle x - \mu \rangle^{-2\sigma_0} J^{\frac{1}{2}} w, w \rangle dt &\leq A \sup_{[0,T]} \|w\|_0^2 (1 + T + T \sup_{k,\mu'} \nu_N(\varphi_{k,\mu'})) \\ &\quad + \left| \int_0^T \langle \Psi F, w \rangle dt \right| + \left| \int_0^T \langle \Psi w, F \rangle dt \right|, \end{aligned}$$

with a modified constant A . Since we can write

$$\langle \Psi F, w \rangle = \langle \Psi_\mu f, u \rangle - \overline{\langle \Psi_\mu f, u \rangle}$$

and a similar expression for $\langle \Psi w, F \rangle$, by going back to u , we get eventually

$$\int_0^T \|\langle x-\mu \rangle^{-\sigma_0} J^{\frac{1}{2}} u\|_0^2 dt \leq A \sup_{[0,T]} \|u\|_0^2 (1+T+T \sup_{k,\mu'} \nu_N(\varphi_{k,\mu'})) + \int_0^T |\langle \Psi_\mu f, u \rangle| dt + \int_0^T |\langle \bar{\Psi}_\mu f, u \rangle| dt + \int_0^T |\langle \Psi_\mu^* f, u \rangle| dt + \int_0^T |\langle \bar{\Psi}_\mu^* f, u \rangle| dt,$$

which yields the first part of Proposition 3.3 by taking the supremum over all $\mu \in \mathbb{Z}^n$.

As for the second estimate of Proposition 3.3, we first remark that, since C is real, Cu satisfies

$$\partial_t Cu = i\mathcal{L}Cu + T_{b_1'} \cdot \nabla Cu + T_{b_2} \cdot \nabla \bar{C}u + C_1 Cu + C_2 \bar{C}u + \tilde{f},$$

where $k = 1, 2$, $b_1 = b_1' + ib_1''$ with real b_1', b_1'' , and

$$\tilde{f} = (i[C, \mathcal{L}] + CT_{ib_1''} \nabla)u + [C, T_{b_1'} \cdot \nabla]u + [C, T_{b_2} \cdot \nabla]\bar{u} + [C, C_1]u + [C, C_2]\bar{u} + Cf.$$

Hence, we can apply the first estimate of Proposition 3.3 to Cu obtaining

$$\|J^{s+\frac{1}{2}} Cu\|_T^2 \leq A(1+T+T \sup_{k,\mu} \nu_N(\varphi_{k,\mu})) \sup_{[0,T]} \|Cu\|_s^2 + \sum_{j=1}^4 \sup_{\mu} \int_0^T |\langle \Psi_{j,\mu} J^s \tilde{f}, J^s Cu \rangle| dt, \tag{62}$$

where $\Psi_{j,\mu} = \psi_j(x - \mu, D)$. Thus, we are led to estimate essentially the terms

$$\int_0^T \left| \langle J^s (i[C, \mathcal{L}] + CT_{ib_1''} \nabla)u, \Psi_{j,\mu}^* J^s Cu \rangle \right| dt + \int_0^T \left| \langle J^s [C, T_{b_1'} \cdot \nabla]u, \Psi_{j,\mu}^* J^s Cu \rangle \right| dt + \int_0^T \left| \langle J^s [C, T_{b_2} \cdot \nabla]\bar{u}, \Psi_{j,\mu}^* J^s Cu \rangle \right| dt.$$

Indeed, since the operators $\Psi_{j,\mu} J^s [C, C_1] J^{-s}$ and $\Psi_{j,\mu} J^s [C, C_2] J^{-s}$ are bounded in L^2 (and so is $J^s C J^{-s}$), the corresponding terms are easily estimated by

$$AT \sup_{\mu} \nu_N(\varphi_{1,\mu}) \sup_{0 \leq t \leq T} \|u(t)\|_s^2.$$

We need now for the other terms the following simple lemma:

Lemma 6.1. *If $a \in S_{0,0}^m$, then, for any real s ,*

$$J^s a(x, D) J^{-s} = a(x, D) + e(x, D),$$

where $e \in S_{0,0}^{m-1}$ and the seminorms of e are bounded by those of a .

Proof. It suffices to apply the pseudodifferential calculus and to remark that

$$e(x, \xi) = \frac{1}{(2\pi)^n} \sum_{j=1}^n \int_0^1 \int e^{-iy\eta} \partial_{\xi_j} (\langle \xi + t\eta \rangle^s) \partial_{x_j} a(x+y, \xi) \langle \xi \rangle^{-s} dy d\eta dt. \quad \square$$

We apply the lemma successively with

$$\begin{aligned} a(x, D) &= i[C, \mathcal{L}] + C T_{ib_1'} \nabla, \\ a(x, D) &= [C, T_{b_1'} \cdot \nabla], \\ a(x, D) &= [C, T_{b_2} \cdot \nabla]. \end{aligned}$$

Since here $m = 1$, we obtain that at each time the operator $e(x, D)$ is bounded in L^2 and that its operator norm is estimated by the seminorms of a . Next, it follows from the pseudodifferential calculus that $\Psi_{j,\mu}^* \in \text{Op}S_{1,0}^0$ and their seminorms are uniformly bounded with respect to μ , and, consequently, also that $\Psi_{j,\mu}^* J^s C J^{-s} \in \text{Op}S_{0,0}^0$ and their seminorms are uniformly estimated by those of C . Hence, the integrals corresponding to the operators $e(x, D)$ are easily estimated by

$$ART \sup_{k,\mu} v_N(\varphi_{k,\mu}) \sup_{0 \leq t \leq T} \|u(t)\|_S^2.$$

Thus, it remains to estimate the sum

$$\begin{aligned} &\int_0^T \left| \langle (i[C, \mathcal{L}] + C T_{ib_1'} \nabla) J^s u, \Psi_{j,\mu}^* J^s C u \rangle \right| dt \\ &\quad + \int_0^T \left| \langle [C, T_{b_1'} \cdot \nabla] J^s u, \Psi_{j,\mu}^* J^s C u \rangle \right| dt + \int_0^T \left| \langle [C, T_{b_2} \cdot \nabla] J^s \bar{u}, \Psi_{j,\mu}^* J^s C u \rangle \right| dt, \end{aligned}$$

to which we apply Lemma 5.2 with $S = \Psi_{j,\mu}^* J^s C J^{-s}$. We obtain eventually

$$\begin{aligned} &\sum_{j=1}^4 \sup_{\mu} \int_0^T \left| \langle \Psi_{j,\mu} J^s \tilde{f}, J^s C u \rangle \right| dt \\ &\quad \leq \sum_{j=1}^4 \sup_{\mu} \int_0^T \left| \langle \Psi_{j,\mu} J^s C f, J^s C u \rangle \right| dt + A \sup_{k,\mu} v_N(\varphi_{k,\mu}) \left(RT \sup_{[0,T]} \|u\|_S^2 + \frac{1}{R} \|J^{s+\frac{1}{2}} u\|_T^2 \right), \end{aligned}$$

which, together with (62), implies the second estimate of Proposition 3.3.

Appendix

Proof of Lemma 3.4. We need the following general estimate:

Lemma A.1. *Let b satisfy*

$$\begin{aligned} b(x, \xi) &= \sum_{\mu \in \mathbb{Z}^n} \alpha_{\mu} \varphi_{\mu}(x, \xi), \quad \sum_{\mu} |\alpha_{\mu}| \leq A_0, \\ \text{supp}(x \mapsto \varphi_{\mu}(x, \xi)) &\subseteq Q_{\mu}^*, \quad \sup_{\mu} \sup_{|\beta| \leq N_0} \|\langle \xi \rangle^{|\beta|} \partial_{\xi}^{\beta} \varphi_{\mu}\|_{L^{\infty}} < \infty, \end{aligned} \tag{63}$$

where N_0 is a sufficiently large integer, and let \tilde{u} stand for u or \bar{u} . Then, there exist $N \in \mathbb{N}$ and $A > 0$ such that, for all $T > 0$ and every $S_1 = s_1(x, D)$, $S_2 = s_2(x, D)$ in $\text{Op} S_{0,0}^0$, we have

$$\int_0^T \left| \langle S_1 J^s T_b J^{-s} \nabla \tilde{u}, S_2 u \rangle \right| dt \leq A \|s_1\|_{C^N} \|s_2\|_{C^N} \sup_{\mu} \sup_{|\beta| \leq N} \|\langle \xi \rangle^{|\beta|} \partial_{\xi}^{\beta} \varphi_{\mu}\|_{L^{\infty}} \|J^{\frac{1}{2}} u\|_T^2.$$

Proof. One can write

$$\begin{aligned} & \langle S_1 J^s T_b J^{-s} \nabla \tilde{u}, S_2 u \rangle \\ &= \sum_{\mu} \alpha_{\mu} \langle S_1 J^s T_{\varphi_{\mu}} J^{-s} \nabla \tilde{u}, S_2 u \rangle = \sum_{\mu} \alpha_{\mu} \langle S_2^* S_1 J^s T_{\varphi_{\mu}} J^{-s} \nabla \tilde{u}, u \rangle \\ &= \sum_{\mu} \alpha_{\mu} \langle \langle x - \mu \rangle^{\sigma_0} J^{-\frac{1}{2}} S_2^* S_1 J^s T_{\varphi_{\mu}} J^{-s} \nabla J^{-\frac{1}{2}} \langle x - \mu \rangle^{\sigma_0} \langle x - \mu \rangle^{-\sigma_0} J^{\frac{1}{2}} \tilde{u}, \langle x - \mu \rangle^{-\sigma_0} J^{\frac{1}{2}} u \rangle \\ &= \sum_{\mu} \alpha_{\mu} \langle S_{\mu} \langle x - \mu \rangle^{\sigma_0} T_{\varphi_{\mu}} \langle x - \mu \rangle^{\sigma_0} J_{\mu} \tilde{u}_{\mu}, u_{\mu} \rangle \end{aligned}$$

where

$$S_{\mu} = \langle x - \mu \rangle^{\sigma_0} J^{-\frac{1}{2}} S_2^* S_1 J^s \langle x - \mu \rangle^{-\sigma_0}, \quad J_{\mu} = \langle x - \mu \rangle^{-\sigma_0} J^{-s} \nabla J^{-\frac{1}{2}} \langle x - \mu \rangle^{\sigma_0}, \quad u_{\mu} = \langle x - \mu \rangle^{-\sigma_0} J^{\frac{1}{2}} u.$$

Now, it follows from the pseudodifferential calculus (Theorem 2.1) and from Lemma 2.3 that S_{μ} and J_{μ} are in $\text{Op } S_{0,0}^{s-\frac{1}{2}}$ and $\text{Op } S_{0,0}^{\frac{1}{2}-s}$ respectively, and that we can estimate their seminorms uniformly in μ . Next, it follows from Lemma 2.12 that the operator norm of $\langle x - \mu \rangle^{\sigma_0} T_{\varphi_{\mu}} \langle x - \mu \rangle^{\sigma_0}$ acting in $H^{s-\frac{1}{2}}(\mathbb{R}^n)$ is estimated by $\sup_{|\beta| \leq N} \|\langle \xi \rangle^{|\beta|} \partial_{\xi}^{\beta} \varphi_{\mu}\|_{L^{\infty}}$ uniformly in μ . Hence, the application of the Cauchy–Schwarz inequality and the Calderón–Vaillancourt theorem allows us to obtain

$$\begin{aligned} & \int_0^T |\langle S_1 J^s T_b J^{-s} \nabla \tilde{u}, S_2 u \rangle| dt \\ & \leq \sum_{\mu} |\alpha_{\mu}| \|S_{\mu}\|_{\mathcal{L}(H^{s-1/2}, L^2)} \|\langle x - \mu \rangle^{\sigma_0} T_{\varphi_{\mu}} \langle x - \mu \rangle^{\sigma_0}\|_{\mathcal{L}(H^{s-1/2})} \|J_{\mu}\|_{\mathcal{L}(L^2, H^{s-1/2})} \int_0^T \|u_{\mu}\|_0^2 dt \\ & \leq A \|s_1\|_{C^N} \|s_2\|_{C^N} \sup_{\mu} \sup_{|\beta| \leq N} \|\langle \xi \rangle^{|\beta|} \partial_{\xi}^{\beta} \varphi_{\mu}\|_{L^{\infty}} \|J^{\frac{1}{2}} u\|_T^2, \end{aligned}$$

which proves the lemma. □

Now, let us write $T_{b_k - b_{k,m}} = T_{b_k - b_{k,m'}} + T_{b_{k,m'} - b_{k,m}}$ and apply Lemma A.1 first to $b = b_k - b_{k,m'}$ with $S_1 = S_2 = C_m$. We obtain

$$\begin{aligned} & \int_0^T |\langle C_m J^s T_{b_k - b_{k,m'}} \nabla \tilde{u}, C_m J^s u \rangle| dt \\ &= \int_0^T |\langle C_m J^s T_{b_k - b_{k,m'}} J^{-s} \nabla \tilde{v}, C_m v \rangle| dt \\ &\leq A \sup_{\mu} v_N(\varphi_{1,\mu,m})^2 \sup_{\mu} \sup_{|\beta| \leq N} \|\langle \xi \rangle^{|\beta|} \partial_{\xi}^{\beta} (\varphi_{k,\mu} - \varphi_{k,\mu,m'})\|_{L^{\infty}} \|J^{\frac{1}{2}} v\|_T^2, \\ &\leq A \frac{m^{2N^2}}{m^{\sigma}} \sup_{\mu} \sup_{|\beta| \leq N} \|\langle \xi \rangle^{|\beta|} \partial_{\xi}^{\beta} \varphi_{k,\mu}\|_{C^{\sigma}} \|J^{\frac{1}{2}+s} u\|_T^2 \leq A \frac{m^{2N^2}}{m^{\sigma}} \|J^{\frac{1}{2}+s} u\|_T^2, \end{aligned}$$

where $v = J^s u$ and $\sigma = \inf\{\varrho, 1\}$. As for the study of the other term, we write

$$\begin{aligned} & \langle C_m J^s T_{b_{k,m'} - b_{k,m}} \nabla \tilde{u}, C_m J^s u \rangle \\ &= \langle C_m J^s T_{b_{k,m'} - b_{k,m}} J^{-s} \nabla \tilde{v}, C_m v \rangle \\ &= \langle J^s T_{b_{k,m'} - b_{k,m}} J^{-s} \nabla C_m \tilde{v}, C_m v \rangle + \langle [C_m, J^s T_{b_{k,m'} - b_{k,m}} J^{-s} \nabla] \tilde{v}, C_m v \rangle, \quad (64) \end{aligned}$$

and then apply Lemma 5.2(iii) to the second term in (64) to obtain

$$\begin{aligned} \int_0^T | \langle [C_m, J^s T_{b_{k,m'}-b_{k,m}} J^{-s} \nabla] \tilde{v}, C_m v \rangle | dt & \\ & \leq A \sup_{\mu} v_N(\varphi_{1,\mu,m})^2 \sup_{\mu} \|\varphi_{k,\mu,m'} - \varphi_{k,\mu,m}\|_{C^N} \left(T \sup_{[0,T]} \|v\|_0^2 + \frac{1}{R} \|J^{\frac{1}{2}} v\|_T^2 \right) \\ & \leq A m'^{2N^2} (m'^N + m^N) \left(T \sup_{[0,T]} \|u\|_s^2 + \frac{1}{R} \|J^{s+\frac{1}{2}} u\|_T^2 \right) \\ & \leq A m'^{2N^2+N} \left(T \sup_{[0,T]} \|u\|_s^2 + \frac{1}{R} \|J^{s+\frac{1}{2}} u\|_T^2 \right). \end{aligned}$$

Finally, recalling that $C_m \bar{u} = \overline{C_m u}$ and applying Lemma A.1 to the first term in (64) with $S_1 = S_2 = \text{Id}$, we get

$$\begin{aligned} \int_0^T | \langle J^s T_{b_{k,m'}-b_{k,m}} J^{-s} \nabla C_m \tilde{v}, C_m v \rangle | dt & \\ & \leq A \sup_{\mu} \sup_{|\beta| \leq N} \| \langle \xi \rangle^{|\beta|} \partial_{\xi}^{\beta} (\varphi_{k,\mu,m'} - \varphi_{k,\mu,m}) \|_{L^{\infty}} \|J^{\frac{1}{2}} C_m v\|_T^2 \\ & \leq A \left(\sup_{\mu} \sup_{|\beta| \leq N} \| \langle \xi \rangle^{|\beta|} \partial_{\xi}^{\beta} (\varphi_{k,\mu,m'} - \varphi_{k,\mu,m}) \|_{L^{\infty}} + \sup_{\mu} \sup_{|\beta| \leq N} \| \langle \xi \rangle^{|\beta|} \partial_{\xi}^{\beta} (\varphi_{k,\mu} - \varphi_{k,\mu,m}) \|_{L^{\infty}} \right) \|J^{\frac{1}{2}} C_m v\|_T^2 \\ & \leq \left(\frac{A}{m'^{\sigma}} + \frac{A}{m^{\sigma}} \right) \sup_{\mu} \sup_{|\beta| \leq N} \| \langle \xi \rangle^{|\beta|} \partial_{\xi}^{\beta} \varphi_{k,\mu} \|_{C^{\sigma}} \|J^{\frac{1}{2}} C_m v\|_T^2 \\ & \leq \left(\frac{A}{m'^{\sigma}} + \frac{A}{m^{\sigma}} \right) \|J^{\frac{1}{2}} C_m v\|_T^2 \leq \frac{A}{m^{\sigma}} \|J^{\frac{1}{2}} C_m v\|_T^2. \end{aligned}$$

It remains to compare $\|J^{\frac{1}{2}} C_m v\|_T^2 = \|J^{\frac{1}{2}} C_m J^s u\|_T^2$ with $\|J^{s+\frac{1}{2}} C_m u\|_T^2$. Of course, one can write $J^{\frac{1}{2}} C_m J^s u = J^{s+\frac{1}{2}} J^{-s} C_m J^s u$ and it follows from Lemma 6.1 that $J^{-s} C_m J^s - C_m = E_m$ is in $\text{Op } S_{0,0}^{-1}$ and the seminorms of E_m are bounded by those of C_m . Hence, since $J^{s+\frac{1}{2}} E_m J^{-s}$ is in $\text{Op } S_{0,0}^{-\frac{1}{2}}$,

$$\begin{aligned} \|J^{s+\frac{1}{2}} E_m u\|_T^2 &= \sup_{\mu} \int_0^T \int |\langle x - \mu \rangle^{-\sigma_0} J^{s+\frac{1}{2}} E_m u|^2 dx dt \\ &\leq \int_0^T \int |J^{s+\frac{1}{2}} E_m u|^2 dx dt \\ &\leq A \sup_{\mu} v_N(\varphi_{1,\mu,m})^2 \int_0^T \int |J^s u|^2 dx dt \leq A T m^{2N^2} \sup_{[0,T]} \|u\|_s^2 \end{aligned}$$

and

$$\|J^{\frac{1}{2}} C_m v\|_T^2 \leq 2 \|J^{s+\frac{1}{2}} C_m u\|_T^2 + 2 A T m^{2N^2} \sup_{[0,T]} \|u\|_s^2,$$

which implies that

$$\int_0^T | \langle J^s T_{b_{k,m'}-b_{k,m}} J^{-s} \nabla C_m \tilde{v}, C_m v \rangle | dt \leq \frac{A}{m^{\sigma}} \|J^{s+\frac{1}{2}} C_m u\|_T^2 + A T m^{2N^2} \sup_{[0,T]} \|u\|_s^2,$$

where, of course, the constant A has changed. Summing up, we have proven that

$$\int_0^T | \langle C_m J^s T_{b_k - b_{k,m}} \nabla \tilde{u}, C_m J^s u \rangle | dt \leq \frac{Am^{2N^2}}{m^\sigma} \| J^{s+\frac{1}{2}} u \|_T^2 + Am'^{2N^2+N} \left(T \sup_{[0,T]} \| u \|_s^2 + \frac{1}{R} \| J^{s+\frac{1}{2}} u \|_T^2 \right) + \frac{A}{m^\sigma} \| J^{s+\frac{1}{2}} C_m u \|_T^2; \tag{65}$$

that is, we have proven Lemma 3.4.

Anisotropic Sobolev spaces. There are several notions of anisotropic Sobolev space in the literature. However, we have not been able to find a reference with the results we need in this paper. Therefore, we are going to define our spaces and next prove the results we need.

We denote by (x, y) the variable in $\mathbb{R}^n \times \mathbb{R}^{n'}$ and by (ξ, η) its Fourier dual variable.

Definition A.2. If $s, s' \in \mathbb{R}$, we denote by $H^{s,s'}(\mathbb{R}^n \times \mathbb{R}^{n'})$ the space of tempered distributions u in $\mathbb{R}^n \times \mathbb{R}^{n'}$ such that the integral

$$\int_{\mathbb{R}^n \times \mathbb{R}^{n'}} \langle \xi \rangle^{2s} \langle \eta \rangle^{2s'} |\hat{u}(\xi, \eta)|^2 d\xi d\eta \tag{66}$$

is finite.

We call this space an anisotropic Sobolev space. Note that this is different, for example, from the classical space $H^{r,s}$ of [Lions and Magenes 1968]. Clearly, $H^{s,s'}(\mathbb{R}^n \times \mathbb{R}^{n'})$ is a Hilbert space when it is provided with the obvious scalar product. We also denote by $\|u\|_{s,s'}$ the norm of u in this space and, of course, it is equal to the square root of (66).

Additionally, note that the space $H^{s,s'}(\mathbb{R}^n \times \mathbb{R}^{n'})$ in the above definition coincides with the space $H^s(\mathbb{R}^n, H^{s'}(\mathbb{R}^{n'}))$ and, by symmetry, with $H^{s'}(\mathbb{R}^{n'}, H^s(\mathbb{R}^n))$.

In this paper, we need the following two results on anisotropic Sobolev spaces. The first one is the Sobolev injection:

Proposition A.3. *If $s > \frac{n}{2}$ and $s' > \frac{n'}{2}$, then $H^{s,s'}(\mathbb{R}^n \times \mathbb{R}^{n'}) \subset L^\infty(\mathbb{R}^n \times \mathbb{R}^{n'})$ with continuous injection.*

Proof. If $u \in H^{s,s'}$, then

$$\hat{u}(\xi, \eta) = \langle \xi \rangle^{-s} \langle \eta \rangle^{-s'} \cdot \langle \xi \rangle^s \langle \eta \rangle^{s'} \hat{u}(\xi, \eta);$$

hence, $\hat{u} \in L^2 \cdot L^2 \subset L^1$ and $\|u\|_{L^\infty} \leq C \|\hat{u}\|_{L^1} \leq C' \|u\|_{s,s'}$, where C and C' are constants which are independent of u . □

The other result is an interpolation inequality:

Proposition A.4. *If $s = (1 - \theta)s_1 + \theta s_2$ and $s' = (1 - \theta)s'_1 + \theta s'_2$, where $\theta \in [0, 1]$, $s_1, s_2, s'_1, s'_2 \in \mathbb{R}$, then, for any $u \in H^{s_1, s'_1}(\mathbb{R}^n \times \mathbb{R}^{n'}) \cap H^{s_2, s'_2}(\mathbb{R}^n \times \mathbb{R}^{n'})$, we have*

$$\|u\|_{s,s'} \leq \|u\|_{s_1, s'_1}^{1-\theta} \|u\|_{s_2, s'_2}^\theta.$$

Proof. Indeed, we have

$$\begin{aligned} \|u\|_{s,s'}^2 &= \int_{\mathbb{R}^n \times \mathbb{R}^{n'}} \langle \xi \rangle^{2(1-\theta)s_1 + 2\theta s_2} \langle \eta \rangle^{2(1-\theta)s'_1 + 2\theta s'_2} |\hat{u}(\xi, \eta)|^2 d\xi d\eta \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^{n'}} (|\langle \xi \rangle^{s_1} \langle \eta \rangle^{s'_1} \hat{u}(\xi, \eta)|)^{2(1-\theta)} (|\langle \xi \rangle^{s_2} \langle \eta \rangle^{s'_2} \hat{u}(\xi, \eta)|)^{2\theta} d\xi d\eta \\ &\leq \left(\int_{\mathbb{R}^n \times \mathbb{R}^{n'}} \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s'_1} |\hat{u}(\xi, \eta)|^2 d\xi d\eta \right)^{1-\theta} \left(\int_{\mathbb{R}^n \times \mathbb{R}^{n'}} \langle \xi \rangle^{2s_2} \langle \eta \rangle^{2s'_2} |\hat{u}(\xi, \eta)|^2 d\xi d\eta \right)^\theta \\ &= \|u\|_{s_1, s'_1}^{2(1-\theta)} \|u\|_{s_2, s'_2}^{2\theta}, \end{aligned}$$

where we have applied Hölder’s inequality. □

Actually, we need the above results for anisotropic Sobolev spaces on domains Ω in $\mathbb{R}^n \times \mathbb{R}^{n'}$, and since the theory of such spaces is less simple, we shall restrict ourselves to the case that arises in this paper, that is, the case $\Omega = I \times \mathbb{R}^n$ where I is a bounded interval in \mathbb{R} , and only to the case $s \geq 0$. First, let us set, by definition,

$$H^{s,s'}(\Omega) = H^s(I, H^{s'}(\mathbb{R}^n)),$$

in the sense that $u(x, y)$ is in $H^{s,s'}(\Omega)$ if and only if

$$\partial_x^\alpha J_y^{s'} u \in L^2(\Omega) \quad \text{for } |\alpha| \leq s$$

and

$$\int_{I \times I \times \mathbb{R}^n} \frac{|\partial_x^\alpha J_y^{s'} u(x, y) - \partial_x^\alpha J_y^{s'} u(x', y)|^2}{|x - x'|^{1+2\sigma}} dx dx' dy < \infty \quad \text{if } 0 < \sigma = s - [s] < 1.$$

Of course, the norm in this space is defined by

$$\|u\|_{s,s',\Omega}^2 = \sum_{|\alpha| \leq s} \|\partial_x^\alpha J_y^{s'} u\|_{L^2(\Omega)}^2 \quad \text{if } s \in \mathbb{N},$$

and

$$\|u\|_{s,s',\Omega}^2 = \sum_{|\alpha| \leq [s]} \|\partial_x^\alpha J_y^{s'} u\|_{L^2(\Omega)}^2 + \int_{I \times I \times \mathbb{R}^n} \frac{|\partial_x^\alpha J_y^{s'} u(x, y) - \partial_x^\alpha J_y^{s'} u(x', y)|^2}{|x - x'|^{1+2\sigma}} dx dx' dy \quad \text{otherwise.}$$

Now, we can prove for $H^{s,s'}(\Omega)$ the results analogous to the above ones.

Proposition A.5. (i) *If $s > \frac{1}{2}$ and $s' > \frac{n}{2}$, then $H^{s,s'}(\Omega) \subset L^\infty(\Omega)$ with continuous injection.*

(ii) *If $s = (1 - \theta)s_1 + \theta s_2$ and $s' = (1 - \theta)s'_1 + \theta s'_2$, where $\theta \in [0, 1]$, $s_1 \geq 0$, $s_2 \geq 0$, $s'_1, s'_2 \in \mathbb{R}$, then there exists a constant C such that, for any $u \in H^{s_1, s'_1}(\Omega) \cap H^{s_2, s'_2}(\Omega)$, we have*

$$\|u\|_{s,s',\Omega} \leq C \|u\|_{s_1, s'_1, \Omega}^{1-\theta} \|u\|_{s_2, s'_2, \Omega}^\theta.$$

Proof. Since we cannot use directly the Fourier transformation, the idea is to construct a bounded linear extension operator

$$P_\Omega : H^{s,s'}(\Omega) \rightarrow H^{s,s'}(\mathbb{R} \times \mathbb{R}^n), \tag{67}$$

that is, it satisfies $P_\Omega u|_\Omega = u$, for all $u \in H^{s,s'}(\Omega)$. Indeed, assume that such a P_Ω exists. Then, for $u \in H^{s,s'}(\Omega)$ with $s > \frac{1}{2}$ and $s' > \frac{n}{2}$,

$$\|u\|_{L^\infty(\Omega)} = \|P_\Omega u\|_{L^\infty(\Omega)} \leq \|P_\Omega u\|_{L^\infty(\mathbb{R} \times \mathbb{R}^n)} \leq C \|P_\Omega u\|_{s,s'} \leq C' \|u\|_{s,s',\Omega},$$

where we have applied Proposition A.3 and the boundedness of P_Ω , and this proves (i).

Furthermore, under the assumptions of (ii), we have

$$\|u\|_{s,s',\Omega} = \|P_\Omega u\|_{s,s',\Omega} \leq \|P_\Omega u\|_{s,s',\mathbb{R} \times \mathbb{R}^n},$$

and it is a classical fact that there exists a constant C such that, for all $v \in H^s(\mathbb{R}^d)$,

$$\sum_{|\alpha| \leq [s]} \|\partial^\alpha v\|_{L^2(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\partial^\alpha v(x) - \partial^\alpha v(x')|^2}{|x - x'|^{d+2\sigma}} dx dx' \leq C \|v\|_s^2;$$

now, applying this inequality to $v(x) = J_y^{s'} P_\Omega u(x, y)$, with $d = 1$, and integrating with respect to y gives

$$\|P_\Omega u\|_{s,s',\mathbb{R} \times \mathbb{R}^n}^2 \leq C \|P_\Omega u\|_{s,s'}^2.$$

Finally, applying Proposition A.4 and the boundedness of P_Ω yields

$$\|u\|_{s,s',\Omega} \leq \sqrt{C} \|P_\Omega u\|_{s,s'} \leq \sqrt{C} \|P_\Omega u\|_{s_1,s'_1}^{1-\theta} \|P_\Omega u\|_{s_2,s'_2}^\theta \leq C' \|u\|_{s_1,s'_1,\Omega}^{1-\theta} \|u\|_{s_2,s'_2,\Omega}^\theta,$$

which establishes (ii).

It remains to construct P_Ω as in (67). In fact, the classical theory of Sobolev spaces already provides a bounded linear extension operator

$$P_I : H^s(I) \rightarrow H^s(\mathbb{R}) \tag{68}$$

such that $P_I u|_I = u$ for all $u \in H^s(I)$. If $u \in H^{s,s'}(\Omega)$, let us set

$$P_\Omega u(x, y) = (P_I)_x J_y^{s'} u(x, y).$$

Clearly, this defines a linear operator such that $P_\Omega u|_\Omega = u$. Let us show the boundedness of $P_\Omega : H^{s,s'}(\Omega) \rightarrow H^{s,s'}(\mathbb{R} \times \mathbb{R}^n)$. It follows from the definition that $x \mapsto J_y^{s'} u(x, y)$ is in the Sobolev space $H^s(I)$ for almost all $y \in \mathbb{R}^n$. Hence, $x \mapsto (P_I)_x J_y^{s'} u(x, y)$ is in $H^s(\mathbb{R})$ for almost all $y \in \mathbb{R}^n$ and there exists a constant C which depends neither on u nor on y such that

$$\|(P_I)_x J_y^{s'} u(x, y)\|_{H^s(\mathbb{R})} \leq C \|J_y^{s'} u(x, y)\|_{H^s(I)} \quad \text{for a.e. } y \in \mathbb{R}^n.$$

Since $(P_I)_x J_y^{s'} u = J_y^{s'} P_\Omega u$, this inequality can be written more explicitly as

$$\begin{aligned} \int_{\mathbb{R}} |J_x^s J_y^{s'} P_\Omega u(x, y)|^2 dx &\leq C^2 \sum_{|\alpha| \leq [s]} \int_I |\partial_x^\alpha J_y^{s'} u(x, y)|^2 dx \\ &+ C^2 \int_{I \times I} \frac{|\partial_x^\alpha J_y^{s'} u(x, y) - \partial_x^\alpha J_y^{s'} u(x', y)|^2}{|x - x'|^{1+2\sigma}} dx dx' \quad \text{for a.e. } y \in \mathbb{R}^n. \end{aligned}$$

Integrating over \mathbb{R}^n with respect to y gives

$$\|P_{\Omega}u\|_{s,s'}^2 \leq C^2 \|u\|_{s,s',\Omega}^2,$$

which proves the boundedness of P_{Ω} and achieves the proof of the proposition. \square

References

- [Bejenaru and Tataru 2008] I. Bejenaru and D. Tataru, “Large data local solutions for the derivative NLS equation”, *J. Eur. Math. Soc. (JEMS)* **10**:4 (2008), 957–985. MR Zbl
- [Bergh and Löfström 1976] J. Bergh and J. Löfström, *Interpolation spaces: an introduction*, Grundlehren der Mathematischen Wissenschaften **223**, Springer, 1976. MR Zbl
- [Bienaimé 2014] P.-Y. Bienaimé, “Existence locale et effet régularisant précisés pour des équations non linéaires de type Schrödinger”, *Rev. Mat. Iberoam.* **30**:3 (2014), 751–798. MR Zbl
- [Bony 1981] J.-M. Bony, “Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires”, *Ann. Sci. École Norm. Sup. (4)* **14**:2 (1981), 209–246. MR Zbl
- [Coifman and Meyer 1978] R. R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque **57**, Société Mathématique de France, Paris, 1978. MR Zbl
- [Doi 1994] S.-i. Doi, “On the Cauchy problem for Schrödinger type equations and the regularity of solutions”, *J. Math. Kyoto Univ.* **34**:2 (1994), 319–328. MR Zbl
- [Kenig et al. 1998] C. E. Kenig, G. Ponce, and L. Vega, “Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations”, *Invent. Math.* **134**:3 (1998), 489–545. MR Zbl
- [Kenig et al. 2004] C. E. Kenig, G. Ponce, and L. Vega, “The Cauchy problem for quasi-linear Schrödinger equations”, *Invent. Math.* **158**:2 (2004), 343–388. MR Zbl
- [Kenig et al. 2005] C. E. Kenig, G. Ponce, C. Rolvung, and L. Vega, “Variable coefficient Schrödinger flows for ultrahyperbolic operators”, *Adv. Math.* **196**:2 (2005), 373–486. MR Zbl
- [Kenig et al. 2006] C. E. Kenig, G. Ponce, C. Rolvung, and L. Vega, “The general quasilinear ultrahyperbolic Schrödinger equation”, *Adv. Math.* **206**:2 (2006), 402–433. MR Zbl
- [Lions and Magenes 1968] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications, II*, Travaux et Recherches Mathématiques **18**, Dunod, Paris, 1968. MR Zbl
- [Marzuola et al. 2012] J. L. Marzuola, J. Metcalfe, and D. Tataru, “Quasilinear Schrödinger equations, I: Small data and quadratic interactions”, *Adv. Math.* **231**:2 (2012), 1151–1172. MR Zbl
- [Marzuola et al. 2014] J. L. Marzuola, J. Metcalfe, and D. Tataru, “Quasilinear Schrödinger equations, II: Small data and cubic nonlinearities”, *Kyoto J. Math.* **54**:3 (2014), 529–546. MR Zbl
- [Meyer 1981] Y. Meyer, “Remarques sur un théorème de J.-M. Bony”, pp. 1–20 in *Proceedings of the Seminar on Harmonic Analysis* (Pisa, 1980), 1981. Issue as a supplement to *Rend. Circ. Mat. Palermo (2)*. MR Zbl
- [Meyer 1982] Y. Meyer, “Nouvelles estimations pour les solutions d’équations aux dérivées partielles non linéaires”, exposé 6, pp. 1–12 in *Goulaouic–Meyer–Schwartz Seminar*, 1981/1982, École Polytech., Palaiseau, 1982. MR Zbl
- [Robbiano 2013] L. Robbiano, “Kato smoothing effect for Schrödinger operator”, pp. 355–369 in *Studies in phase space analysis with applications to PDEs*, edited by M. Cicognani et al., Progr. Nonlinear Differential Equations Appl. **84**, Springer, 2013. MR Zbl
- [Takeuchi 1992] J. Takeuchi, “Le problème de Cauchy pour certaines équations aux dérivées partielles du type de Schrödinger, VIII: Symétrisations indépendantes du temps”, *C. R. Acad. Sci. Paris Sér. I Math.* **315**:10 (1992), 1055–1058. MR Zbl
- [Tataru 2002] D. Tataru, “On the Fefferman–Phong inequality and related problems”, *Comm. Partial Differential Equations* **27**:11–12 (2002), 2101–2138. MR Zbl
- [Taylor 1991] M. E. Taylor, *Pseudodifferential operators and nonlinear PDE*, Progress in Mathematics **100**, Birkhäuser, Boston, 1991. MR Zbl

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PIERRE-YVES BIENAIMÉ: pierre-yves.bienaime@univ-nantes.fr

Laboratoire de Mathématiques Jean Leray, CNRS UMR6629, Université de Nantes, Nantes, France

ABDESSLAM BOULKHEMAIR: boulkhemair-a@univ-nantes.fr

Laboratoire de Mathématiques Jean Leray, CNRS UMR6629, Université de Nantes, Nantes, France

THE SHAPE OF LOW ENERGY CONFIGURATIONS OF A THIN ELASTIC SHEET WITH A SINGLE DISCLINATION

HEINER OLBERMANN

We consider a geometrically fully nonlinear variational model for thin elastic sheets that contain a single disclination. The free elastic energy contains the thickness h as a small parameter. We give an improvement of a recently proved energy scaling law, removing the next-to-leading-order terms in the lower bound. Then we prove the convergence of (almost-)minimizers of the free elastic energy towards the shape of a radially symmetric cone, up to Euclidean motions, weakly in the spaces $W^{2,2}(B_1 \setminus B_\rho; \mathbb{R}^3)$ for every $0 < \rho < 1$, as the thickness h is sent to 0.

1. Introduction

1.1. Setup and previous work. The present article continues a program [Müller and Olbermann 2014a; Olbermann 2016; 2017] to explore thin elastic sheets with a single disclination from the variational point of view. The free energy that we consider consists of two parts: (1) the nonconvex membrane energy, which penalizes the difference between the metric that is induced by the deformation and the reference metric, which is the metric of the (singular) cone; (2) the bending energy, which penalizes curvature. The bending energy contains a factor h^2 , where the small parameter h is to be thought of as the thickness of the sheet (see (1) below for the definition). Choosing the cone as configuration, one gets infinite energy: While the membrane term vanishes, the bending energy is infinite for this choice. Energetically, there is a competition of the membrane and the bending terms; neither will vanish for configurations of low energy.

Intuitively, it seems quite clear how configurations of low energy should look: they should be identical to the cone far away from the disclination, and near the disclination, there should be some smoothing of the cone, at a scale h (the only length scale in the problem). For such configurations, one gets an energy of $C^*h^2 \log(1/h)$ plus terms of order h^2 , where C^* is an explicitly known constant; see Lemma 4 below. It is natural to conjecture that such a scaling behavior should indeed hold true for minimizers. However, a proof of an ansatz-free lower bound with the same scaling is much more difficult than the straightforward construction for the upper bound. In the literature, lower bounds for this setting have been ansatz based [Lidmar et al. 2003; Seung and Nelson 1988; Yavari and Goriely 2013], or have assumed radial symmetry [Müller and Olbermann 2014a].

The idea underlying the recent proofs of ansatz-free lower bounds [Olbermann 2016; 2017] is to control the Gauss curvature (or a linearization thereof) by interpolation between the membrane and the bending term energy. The control over the Gauss curvature allows for a certain control over the Gauss map (or

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the deformation gradient). This information in turn yields lower bounds for the bending energy, using an inequality of Sobolev/isoperimetric type. For the corresponding result from [Olbermann 2017], see (2) below. This lower bound does not quite achieve the conjectured scaling behavior, in that there exist next-to-leading-order terms $O(h^2 \log \log(1/h))$ which are not present in the upper bound.

Here, we are going to improve the results from [Olbermann 2017] in two ways: First, we give an improved lower bound for the elastic energy, which proves the conjecture that the minimum of the energy is given by $C^*h^2 \log(1/h) + O(h^2)$. The observation that allows for this improvement is that it is unnecessary to use interpolation to control the Gauss curvature and the Gauss map (or rather, the linearized Gauss curvature and the deformation gradient). It is enough to use the membrane energy alone to obtain the necessary control, and make more efficient use of the Sobolev/isoperimetric inequality.

Second, we use this improved lower bound to show a statement about the *shape* of configurations that satisfy the energy bounds. We prove that (almost-)minimizers converge to the conical deformation, up to Euclidean motions. It is remarkable that that much information about deformations of small energy can be obtained, considering that we are dealing with a highly nonconvex variational problem. Hitherto, such results had only been achieved for situations in which the energy scales were $O(h^2)$ or less [Frieesecke et al. 2002; Pakzad 2004; Hornung 2011a]. The results of these papers will also play an important role in our proof.

1.2. Statement of results. Let $B_1 := \{x \in \mathbb{R}^2 : |x| < 1\}$ be the sheet in the reference configuration. The singular cone may be described by the mapping $y^\Delta : B_1 \rightarrow \mathbb{R}^3$,

$$y^\Delta(x) = \sqrt{1 - \Delta^2}x + \Delta|x|e_3.$$

Here, $0 < \Delta < 1$ is the height of the singular cone, and is determined by the deficit of the disclination at the origin. The reference metric on B_1 is given by

$$\begin{aligned} g_\Delta(x) &= Dy^\Delta(x)^T Dy^\Delta(x) \\ &= (1 - \Delta^2)\text{Id}_{2 \times 2} + \Delta^2 \hat{x} \otimes \hat{x} \\ &= \text{Id}_{2 \times 2} - \Delta^2 \hat{x}^\perp \otimes \hat{x}^\perp, \end{aligned}$$

where $\hat{x} = x/|x|$ and $\hat{x}^\perp = (-x_2, x_1)/|x|$. The induced metric of a deformation $y \in W^{2,2}(B_1; \mathbb{R}^3)$ is

$$g_y = Dy^T Dy.$$

The free elastic energy $I_{h,\Delta} : W^{2,2}(B_1; \mathbb{R}^3) \rightarrow \mathbb{R}$ is defined by

$$I_{h,\Delta}(y) = \int_{B_1} (|g_y - g_\Delta|^2 + h^2 |D^2 y|^2) d\mathcal{L}^2, \quad (1)$$

where $d\mathcal{L}^2$ denotes two-dimensional Lebesgue measure. In [Olbermann 2017], we proved the existence of a constant $C = C(\Delta) > 0$ such that

$$2\pi \Delta^2 h^2 \left(\log \frac{1}{h} - 2 \log \log \frac{1}{h} - C \right) \leq \min_{y \in W^{2,2}(B_1; \mathbb{R}^3)} I_{h,\Delta}(y) \leq 2\pi \Delta^2 h^2 \left(\log \frac{1}{h} + C \right). \quad (2)$$

Our first aim in the present article is to improve the lower bound for the free elastic energy. The improvement consists in getting rid of the $\log \log(1/h)$ terms on the left-hand side:

Theorem 1. *There exist positive constants C_1, C_2, C_3 that only depend on Δ with the following properties.*

First,

$$2\pi \Delta^2 h^2 \left(\log \frac{1}{h} - C_1 \right) \leq \min_{y \in W^{2,2}(B_1; \mathbb{R}^3)} I_{h,\Delta}(y) \leq 2\pi \Delta^2 h^2 \left(\log \frac{1}{h} + C_2 \right) \tag{3}$$

for all $h \in (0, \frac{1}{3})$. Furthermore, if y satisfies

$$I_{h,\Delta}(y) \leq 2\pi \Delta^2 h^2 \left(\log \frac{1}{h} + C_2 \right), \tag{4}$$

then

$$\int_{B_1 \setminus B_R} |D^2 y|^2 \, d\mathcal{L}^2 \leq 2\pi \Delta^2 \log \frac{1}{R} + C_3 \quad \text{for all } R \in (3h, 1), \tag{5}$$

$$\int_{B_1} |g_y - g_\Delta|^2 \, d\mathcal{L}^2 \leq C_3 h^2. \tag{6}$$

As a consequence of Theorem 1, we will be able to prove convergence of (almost)-minimizers of the functional (1) towards the singular cone as $h \rightarrow 0$:

Theorem 2. *Let $y^h \in W^{2,2}(B_1; \mathbb{R}^3)$ be a sequence with*

$$I_{h,\Delta}(y^h) \leq 2\pi \Delta^2 h^2 \left(\log \frac{1}{h} + C_2 \right).$$

Then there exists a subsequence $y^{h(k)}$ and a Euclidean motion T such that for every $0 < \rho < 1$,

$$y^{h(k)} \rightharpoonup T y^\Delta \quad \text{in } W^{2,2}(B_1 \setminus B_\rho; \mathbb{R}^3). \tag{7}$$

1.3. Scientific context. In the proof of Theorem 1 we show a certain focusing of the elastic energy near the disclination. Phenomena with such elastic energy focusing are also observed in many other settings. In particular, crumpled elastic sheets display networks of vertices and ridges. The investigation of these “sharp” structures in the physics community started in the mid-1990s. For a historical account and an exhaustive list of references, see the very recommendable overview article [Witten 2007]. There has been quite some activity in the analysis of ridge-like structures in particular; see [Lobkovsky 1996; DiDonna and Witten 2001; Lobkovsky and Witten 1997; Kramer and Witten 1997; Lobkovsky et al. 1995; Venkataramani 2004]. Energy focusing in conical shapes has been investigated in [Ben Amar and Pomeau 1997; Cerda et al. 1999; Cerda and Mahadevan 1998; 2005]. Disclinations in thin elastic sheets are particularly interesting as a modeling device for icosahedral elastic structures. This is a popular model for virus capsids [Seung and Nelson 1988; Lidmar et al. 2003] or carbon nanocones [Romanov 2003], the structure one obtains when inserting a single five-valent vertex into a graphene sheet (of otherwise six-valent vertices). The disclinations are located at the vertices of the elastic icosahedra.

In the mathematical literature on thin elastic sheets, there have been two strands of investigation: On the one hand, there are the rigorous derivations of elastic plate models from three-dimensional finite elasticity by means of Γ -convergence; see [Friesecke et al. 2002; 2006; Lewicka et al. 2010]. On the

other hand, there has been quite some effort to investigate the qualitative properties of low-energy states in the variational formulation of elasticity, obtained through an analysis of the scaling of the free elastic energy with respect to the relevant parameters in the model; see, e.g., [Bella and Kohn 2014a; 2014b; Ben Belgacem et al. 2002; Kohn and Nguyen 2013]. The present paper belongs of course to the latter group. In more detail, rigorous scaling laws similar to the ones we prove here have been derived for a single fold [Conti and Maggi 2008] and for the so-called *d-cone* [Müller and Olbermann 2014b; Brandman et al. 2013]. The variational problems considered in these references however are of a very special kind: the constraints on the shape of the elastic sheet are quite restrictive, and the lower bounds use these constraints in an essential way; see [Olbermann 2017] for a detailed discussion. This is not the case for our setting, whence our method of proof, which we developed in [Olbermann 2016; 2017] and which we refine here, is completely different.

1.4. Connection to convex integration and rigidity results. The Nash–Kuiper theorem [Nash 1954; Kuiper 1955a; 1955b] states that given a two-dimensional Riemannian manifold (M, g) , a *short*¹ immersion $y_0 : M \rightarrow \mathbb{R}^3$, and $\varepsilon > 0$, there exists an isometric immersion $y_1 \in C^1(M; \mathbb{R}^3)$ such that $\|y_1 - y_0\|_{C^0} < \varepsilon$. This is relevant in our context, since the leading-order term in the energy (1) measures the distance of the deformation y from an isometric immersion with respect to the target metric g_Δ . By the Nash–Kuiper theorem, there exists a vast amount of deformations y that have arbitrarily small membrane energy. A priori, these are all good candidates for energy minimization. One needs a principle that shows that all of these deformations are associated with large bending energy. The energy scaling law from Theorem 1 shows that none of these maps can beat the upper bound construction energetically. Theorem 2 shows the “stronger” statement that maps with low energy cannot look anything like the approximations of C^1 isometric immersions that appear in the proof of the Nash–Kuiper theorem.

The Nash–Kuiper result is an instance of *convex integration*, a concept that has been developed systematically by Gromov [1986]. In particular, the theorem states that solutions to isometric immersion problems are highly nonunique if one requires only C^1 -regularity. In stark contrast, there is the uniqueness in the Weyl problem: given a sufficiently smooth metric g on S^2 with positive Gauss curvature, there exists a *unique* isometric immersion $y : S^2 \rightarrow \mathbb{R}^3$ of C^2 -regularity. Such uniqueness is often called *rigidity*. The dichotomy of convex integration versus rigidity also appears in other contexts, such as the Monge–Ampère equation [Lewicka and Pakzad 2017] and the incompressible Euler equation [Constantin et al. 1994; Isett 2016].

Concerning the uniqueness of solutions in the Weyl problem, the proof is due to Pogorelov [1973]. In fact, he proved that solutions are unique up to Euclidean motions in the class of immersions of *bounded extrinsic curvature*. The latter is the class of immersions for which the pull-back of the volume form on S^2 under the Gauss map is a well-defined signed Radon measure. For smooth maps, this is just the measure $K \, dA$, where K is the Gauss curvature and dA is the volume element. We see that control over the Gauss curvature excludes constructions in the style of Nash–Kuiper. This is also the basic concept

¹An immersion $y : M \rightarrow \mathbb{R}^3$ is short with respect to the metric g on M if for every curve $\gamma : [0, 1] \rightarrow M$, the length of $y \circ \gamma$ is shorter (measured with the Euclidean metric on \mathbb{R}^3) than γ (measured with g).

underlying our proof (with the modification that we consider a linearized version of Gauss curvature). We believe that this hints at a link between questions about rigidity of surfaces and variational problems in the theory of thin elastic sheets.

Notation. For a closed line segment $\{a + t(b - a) : t \in [0, 1]\} \subset \mathbb{R}^2$, we write $[a, b]$. For a semiclosed line segment $\{a + t(b - a) : t \in (0, 1]\} \subset \mathbb{R}^2$, we write $(a, b]$. Throughout the text, we will assume the deficit of the disclination $0 < \Delta < 1$ to be fixed. A statement such as “ $f \leq Cg$ ” is shorthand for “there exists a constant $C > 0$ that only depends on Δ such that $f \leq Cg$ ”. The value of C may change within the same line.

For $r > 0$, we let $B_r = \{x \in \mathbb{R}^2 : |x| < r\}$. The two-sphere $\{x \in \mathbb{R}^3 : |x| = 1\}$ is denoted by S^2 .

The one-dimensional Hausdorff measure is denoted by \mathcal{H}^1 .

The pairing between a Radon measure μ and a continuous function f will be denoted by $\langle \mu, f \rangle$.

2. Proof of Theorem 1

As in [Olbermann 2017], the proof of the energy scaling law rests on two observations. First, by the weak formulation of the Hessian determinant,

$$\sum_{i=1}^3 \det D^2 y_i = (y_{,1} \cdot y_{,2})_{,12} - \frac{1}{2}(|y_{,1}|^2)_{,22} - \frac{1}{2}(|y_{,2}|^2)_{,11} \quad \text{for } y \in C^2(B_1; \mathbb{R}^3), \tag{8}$$

we get that the quantity $\sum_{i=1}^3 \det D^2 y_i$ is close to $\sum_{i=1}^3 \det D^2 y_i^\Delta = \pi \Delta^2 \delta_0$ (the latter equation holding in the sense of distributions), where δ_0 denotes the Radon measure defined by $\langle \delta_0, f \rangle = f(0)$. The expression $\sum_{i=1}^3 \det D^2 y_i$ is best thought of as the “linearized Gauss curvature”: for a metric of the form $g_y = \text{Id}_{2 \times 2} + \varepsilon G$, the Gauss curvature is

$$K = \varepsilon \sum_{i=1}^3 \det D^2 y_i + O(\varepsilon^2).$$

Second, the following Sobolev/isoperimetric inequality translates estimates for integrals of the Hessian determinant into lower bounds for boundary integrals of the tangential part of the second derivative.

Lemma 3. For $v \in C^2(\bar{B}_1)$ and $0 \leq r \leq 1$,

$$\int_{\partial B_r} |D^2 v| d\mathcal{H}^1 \geq \left(4\pi \left| \int_{B_r} \det D^2 v \, dx \right| \right)^{1/2}. \tag{9}$$

This inequality has been used in the literature in a number of places; see, e.g., [Müller 1990]. The proof of the statement above (including the sharp constant) can be found in [Olbermann 2017].

The main observation that allows for an improvement of the lower bound from [Olbermann 2017] is that we may get a lower bound for the quantity on the left-hand side in (9) from the smallness of the membrane energy directly by integrating a suitable test function against the membrane term $g_y - g_\Delta$. In our previous paper we obtained such an estimate by interpolation instead, which also uses the control over the bending energy. This is unnecessary, and gives slightly worse estimates.

The following calculation indicates how to use the smallness of the membrane term to obtain estimates on integrals of the linearized curvature. Let $\Phi \in L^1(B_1)$ be such that $D^2\Phi$ is a vector-valued Radon measure with support in B_1 . In this case, we have $\Phi \in C^0(B_1)$, see [Demengel 1984, Theorem 3.3], and for all $y \in C^2(B_1; \mathbb{R}^3)$ we have

$$\begin{aligned} & \int_{B_1} \left(\sum_{i=1}^3 \det D^2 y_i(x) \right) \Phi(x) \, d\mathcal{L}^2 - \pi \Delta^2 \langle \delta_0, \Phi \rangle \\ &= \int_{B_1} \left((y_{,1} \cdot y_{,2} - y_{,1}^\Delta \cdot y_{,2}^\Delta) \Phi_{,12} - \frac{1}{2}(|y_{,1}|^2 - |y_{,1}^\Delta|^2) \Phi_{,22} - \frac{1}{2}(|y_{,2}|^2 - |y_{,2}^\Delta|^2) \Phi_{,11} \right) \, d\mathcal{L}^2 \\ &= -\frac{1}{2} \int_{B_1} (g_y - g_\Delta) : \text{cof } D^2\Phi \, d\mathcal{L}^2. \end{aligned} \tag{10}$$

Here,

$$\text{cof } D^2\Phi = \begin{pmatrix} \Phi_{,22} & -\Phi_{,12} \\ -\Phi_{,21} & \Phi_{,11} \end{pmatrix}$$

denotes the cofactor matrix of $D^2\Phi$. Note that cof is linear on 2×2 matrices, and hence $\text{cof } D^2\Phi$ is a well-defined Radon measure under our assumptions. After these preliminary remarks, we construct the upper bound in the statement of Theorem 1. It is obtained by a simple mollification of y^Δ on a ball of size h centered at the origin.

Lemma 4. *We have*

$$\inf_{y \in W^{2,2}(B_1; \mathbb{R}^3)} I_{h,\Delta}(y) \leq 2\pi \Delta^2 h^2 \left(\log \frac{1}{h} + C \right),$$

where $C = C(\Delta)$ does not depend on h .

Proof. This is the same upper bound construction as in [Olbermann 2017] (see Lemma 2 in that reference), and we will be brief. We choose $\eta \in C^\infty([0, \infty))$ with $\eta = 0$ on $[0, \frac{1}{2}]$, $\eta = 1$ on $[1, \infty)$, and $|\eta'| \leq C$, $|\eta''| \leq C$. We set

$$y_h(x) = \eta\left(\frac{|x|}{h}\right) y^\Delta(x).$$

One easily shows

$$\begin{aligned} |g_{y_h} - g_\Delta| &\leq C & \text{and} & & |D^2 y_h| &\leq Ch^{-1} & \text{on } B_h, \\ g_{y_h} - g_\Delta &= 0 & \text{and} & & |D^2 y_h(x)| &= \frac{\Delta}{|x|} & \text{on } B_1 \setminus B_h. \end{aligned}$$

This implies

$$\begin{aligned} & \int_{B_1} |g_{y_h} - g_\Delta|^2 \, d\mathcal{L}^2 \leq \int_{B_h} C \, d\mathcal{L}^2 \leq Ch^2, \\ & \int_{B_1} |D^2 y_h|^2 \, d\mathcal{L}^2 \leq \int_{B_1 \setminus B_h} \frac{\Delta^2}{|x|^2} \, d\mathcal{L}^2 + \int_{B_h} \frac{C}{h^2} \, d\mathcal{L}^2 = 2\pi \Delta^2 \int_h^1 \frac{dr}{r} + C = 2\pi \Delta^2 \log \frac{1}{h} + C. \end{aligned}$$

This implies the claim of the lemma. □

Proof of Theorem 1. The upper bound is proved by Lemma 4; hence we may choose C_2 to be the constant from that lemma. Now it suffices to show the following: there exist C_1, C_3 such that if $y \in W^{2,2}(B_1; \mathbb{R}^3)$ satisfies (4), then also the lower bound in (3) and (5), (6) hold true.

Let $y \in W^{2,2}(B_1; \mathbb{R}^3)$ satisfy (4). By density of C^2 in $W^{2,2}$, we may assume $y \in C^2(B_1; \mathbb{R}^3)$ for a proof of the remaining statements. Let $0 < r < 1$. Using Lemma 3, we have for $i = 1, 2, 3$

$$\frac{1}{2\pi} \int_{\partial B_r} |D^2 y_i| \, d\mathcal{H}^1 \geq \left(\frac{1}{\pi} \left| \int_{B_r} \det D^2 y_i \, d\mathcal{L}^2 \right| \right)^{1/2}.$$

Applying Jensen’s inequality, we get

$$\frac{1}{2\pi r} \int_{\partial B_r} |D^2 y_i|^2 \, d\mathcal{H}^1 \geq \left(\frac{1}{2\pi r} \int_{\partial B_r} |D^2 y_i| \, d\mathcal{H}^1 \right)^2.$$

Combining these two estimates, we obtain

$$\int_{\partial B_r} |D^2 y_i|^2 \, d\mathcal{H}^1 \geq \frac{2}{r} \left| \int_{B_r} \det D^2 y_i \, d\mathcal{L}^2 \right|.$$

By the triangle inequality,

$$\int_{\partial B_r} |D^2 y|^2 \, d\mathcal{H}^1 \geq \frac{2}{r} \left| \int_{B_r} \sum_i \det D^2 y_i \, d\mathcal{L}^2 \right|. \tag{11}$$

Now choose $h_0 = h_0(y) \in [h, 2h]$ such that

$$\int_{\partial B_{h_0}} |g_y - g_\Delta|^2 \, d\mathcal{H}^1 \leq h^{-1} \int_{B_1} |g_y - g_\Delta|^2 \, d\mathcal{L}^2. \tag{12}$$

Choosing $R \in (h_0 + h, 1)$ and integrating (11) over the range $r \in [h_0, R]$, we get

$$\begin{aligned} \int_{B_R \setminus B_{h_0}} |D^2 y|^2 \, d\mathcal{L}^2 &\geq 2 \left| \int_{h_0}^R \frac{1}{r} \left(\int_{B_r} \sum_i \det D^2 y_i \, d\mathcal{L}^2 \right) \, dr \right| \\ &= 2 \left| \int_{h_0}^R \left(\int_{B_1} \frac{\chi_{B_r}(x)}{r} \sum_i \det D^2 y_i(x) \, d\mathcal{L}^2(x) \right) \, dr \right| \\ &= 2 \left| \int_{B_1} \Phi \left(\sum_i \det D^2 y_i \right) \, d\mathcal{L}^2 \right|, \end{aligned} \tag{13}$$

where we have used Fubini’s theorem to change the order of integration, and have defined the test function

$$\Phi(x) := \int_{h_0}^R \frac{1}{r} \chi_{B_r}(x) \, dr = \begin{cases} \log(R/h_0) & \text{if } |x| \leq h_0, \\ \log(R/|x|) & \text{if } h_0 < |x| \leq R, \\ 0 & \text{else.} \end{cases}$$

We add and subtract the term $2\pi \Delta^2 \langle \delta_0, \Phi \rangle$, use the triangle inequality and obtain

$$\int_{B_R \setminus B_{h_0}} |D^2 y|^2 \, d\mathcal{L}^2 \geq 2\pi \Delta^2 \log \frac{R}{h_0} - 2 \left| \pi \Delta^2 \langle \delta_0, \Phi \rangle - \int_{B_1} \Phi \left(\sum_i \det D^2 y_i \right) \, d\mathcal{L}^2 \right|. \tag{14}$$

Now we set

$$A(R) := \int_{B_1} \Phi \left(\sum_i \det D^2 y_i \right) d\mathcal{L}^2 - \pi \Delta^2 \langle \delta_0, \Phi \rangle = -\frac{1}{2} \int_{B_1} (g_y - g_\Delta) : \text{cof } D^2 \Phi d\mathcal{L}^2, \tag{15}$$

where we have used (10) in the second line. An explicit computation yields

$$D\Phi(x) = -\frac{x}{|x|^2} \chi_{B_R \setminus B_{h_0}}(x),$$

$$D^2\Phi(x) = (-\text{Id}_{2 \times 2} + 2\hat{x} \otimes \hat{x}) |x|^{-2} \chi_{B_R \setminus B_{h_0}}(x) + |x|^{-1} \hat{x} \otimes \hat{x} (\mathcal{H}^1 \llcorner \partial B_R - \mathcal{H}^1 \llcorner \partial B_{h_0}).$$

Inserting these computations in (15), we have

$$|A(R)| \leq \int_{B_R \setminus B_{h_0}} \frac{|g_y - g_\Delta|}{|x|^2} d\mathcal{L}^2 + \frac{1}{2R} \int_{\partial B_R} |g_y - g_\Delta| d\mathcal{H}^1 + \frac{1}{2h_0} \int_{\partial B_{h_0}} |g_y - g_\Delta| d\mathcal{H}^1. \tag{16}$$

By Cauchy–Schwarz,

$$\begin{aligned} \int_{B_R \setminus B_{h_0}} \frac{|g_y - g_\Delta|}{|x|^2} d\mathcal{L}^2 &\leq \left(\int_{B_R \setminus B_{h_0}} |g_y - g_\Delta|^2 d\mathcal{L}^2 \right)^{1/2} \left(\int_{B_R \setminus B_{h_0}} |x|^{-4} d\mathcal{L}^2 \right)^{1/2} \\ &\leq \left(\int_{B_R \setminus B_{h_0}} |g_y - g_\Delta|^2 d\mathcal{L}^2 \right)^{1/2} \sqrt{2\pi} h_0^{-1}, \\ \int_{\partial B_R} |g_y - g_\Delta| d\mathcal{H}^1 &\leq C \sqrt{R} \left(\int_{\partial B_R} |g_y - g_\Delta|^2 d\mathcal{H}^1 \right)^{1/2}, \\ \int_{\partial B_{h_0}} |g_y - g_\Delta| d\mathcal{H}^1 &\leq C \sqrt{h_0} \left(\int_{\partial B_{h_0}} |g_y - g_\Delta|^2 d\mathcal{H}^1 \right)^{1/2}. \end{aligned} \tag{17}$$

Now choose $R_0 \in [R - h, R]$ such that

$$\int_{\partial B_{R_0}} |g_y - g_\Delta|^2 d\mathcal{H}^1 \leq h^{-1} \int_{B_1} |g_y - g_\Delta|^2 d\mathcal{L}^2.$$

Together with (12) and (17), inequality (16) becomes

$$|A(R_0)| \leq C \frac{E_m(y)^{1/2}}{h_0},$$

where $E_m(y)$ is the membrane energy,

$$E_m(y) := \int_{B_1} |g_y - g_\Delta|^2 d\mathcal{L}^2.$$

The lower bound for the bending energy (13) becomes

$$\int_{B_{R_0} \setminus B_{h_0}} |D^2 y|^2 d\mathcal{L}^2 \geq 2\pi \Delta^2 \log \frac{R_0}{h_0} - C \frac{E_m(y)^{1/2}}{h_0}. \tag{18}$$

We use (18) with $R \uparrow 1$ to estimate the membrane energy by

$$E_m(y) \leq 2\pi \Delta^2 h^2 \left(\log \frac{1}{h} + C_2 \right) - 2\pi \Delta^2 h^2 \log \frac{1}{h_0} + Ch^2 \frac{E_m(y)^{1/2}}{h_0} \leq C(h^2 + hE_m(y)^{1/2}). \tag{19}$$

Using Young’s inequality $ab \leq \frac{1}{2}((\varepsilon a)^2 + (b/\varepsilon)^2)$, with $\varepsilon = C^{-1}$, we have

$$ChE_m(y)^{1/2} \leq \frac{1}{2}E_m(y) + Ch^2,$$

and inserting this in (19), we get

$$E_m(y) \leq Ch^2,$$

which proves (6). Furthermore, inserting this in (18), we have

$$\int_{B_{R_0} \setminus B_{h_0}} |D^2 y|^2 \, d\mathcal{L}^2 \geq 2\pi \Delta^2 \log \frac{R_0}{h} - C.$$

Sending $R \rightarrow 1$, this proves the lower bound in (3). Furthermore,

$$\begin{aligned} \int_{B_1 \setminus B_R} |D^2 y|^2 \, d\mathcal{L}^2 &\leq h^{-2}(I_{h,\Delta}(y) - E_m(y)) - \int_{h_0}^{R_0} |D^2 y|^2 \, d\mathcal{L}^2 \\ &\leq 2\pi \Delta^2 \left(\log \frac{1}{h} + C_2 \right) - 2\pi \Delta^2 \log \frac{R_0}{h} \\ &\leq 2\pi \Delta^2 \log \frac{1}{R} + C, \end{aligned}$$

which proves (5). This completes the proof of the theorem. □

3. Proof of Theorem 2

3.1. Isometric immersions of a singular cone. The plan of the proof is as follows: The crucial inequality (5) shows that on a fixed annulus $B_1 \setminus B_R$, the $W^{2,2}$ norm of a sequence of deformations y_h satisfying $I_{h,\Delta}(y_h) \leq 2\pi \Delta^2 h^2 (\log 1/h + C)$ is bounded as $h \rightarrow 0$. One gets weak convergence of a subsequence in $W^{2,2}$ to a limit deformation that is an isometric immersion with respect to g_Δ (since the membrane energy of the limit function vanishes by $E_m(y_h) \leq Ch^2 \rightarrow 0$). We may apply the results on $W^{2,2}$ isometric immersions from [Hornung 2011a; Pakzad 2004] to the limit, which means that the limit deformation is developable. Using our energy estimates, we can show that in fact, it must be identical to the singular cone y^Δ up to a Euclidean motion.

The fact that flat surfaces are locally developable is a classical result from the differential geometry of surfaces. For functions in $W^{2,2}$, this statement has been proved in [Pakzad 2004; Hornung 2011a; 2011b]:

Theorem 5 [Hornung 2011a, Theorem 2]. *Let $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary. Let $y \in W^{2,2}(\Omega; \mathbb{R}^3)$ with $Dy^T Dy = \text{Id}_{2 \times 2}$ almost everywhere. Then $y \in C^1(\Omega)$ and there exists a set L_y of mutually disjoint closed line segments in $\bar{\Omega}$ with endpoints on $\partial\Omega$ with the following property. For every $x \in \Omega$, exactly one of the following alternatives hold: either $D^2 y = 0$ in a neighborhood of x , or there exists $L \in L_y$ with $x \in L$ and Dy is constant on L .*

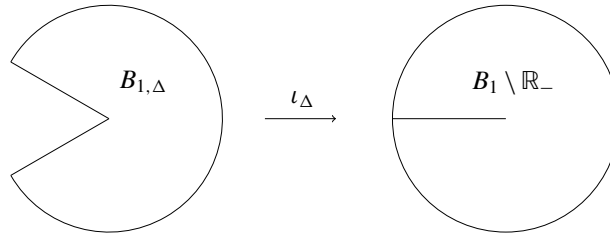


Figure 1. The domain $B_{1,\Delta}$ and the map $\iota_\Delta : B_{1,\Delta} \rightarrow B_1 \setminus \mathbb{R}_-$.

Lemma 6. *Let y be as in Theorem 5, and $\tilde{\Omega} \subseteq \Omega$. Let \tilde{y} be the restriction of y to $\tilde{\Omega}$. Then for every $\tilde{L} \in L_{\tilde{y}}$ there exists exactly one $L \in L_y$ such that $\tilde{L} \subseteq L$. In particular, L_y is unique.*

Proof. From the properties of L_y , it is clear that there can be at most one L with the stated property. Suppose there is $\tilde{L} \in L_{\tilde{y}}$ such that there does not exist $L \in L_y$ with $\tilde{L} \subseteq L$. Choose $x_0 \in \tilde{L} \setminus \partial\tilde{\Omega}$, and choose $r > 0$ such that $B(x_0, 2r) \subset \tilde{\Omega}$. For every $x \in \tilde{L} \cap B(x_0, r)$ the following holds true:

There does not exist a neighborhood of x on which D^2y vanishes. Hence there exists a line segment $L_x \in L_y$ that intersects \tilde{L} only in x such that Dy is constant on L_x . Hence $Dy(z) = Dy(x) = Dy(x_0)$ for all $z \in L_x$.

Since the line segments $\{L_x : x \in \tilde{L} \cap B(x_0, r)\}$ are mutually disjoint and their endpoints are outside $B(x_0, r)$, we have that there exists a neighborhood U of x_0 that is covered by the union of these line segments,

$$U \subset \bigcup_{x \in \tilde{L} \cap B(x_0, r)} L_x.$$

This implies that Dy is constant on U , a contradiction. □

We will need a variant of Theorem 5 for functions whose domain is a singular cone.

To be able to use Theorem 5, we are going to consider the cone in a flat reference configuration. Let $\arccos : [-1, 1] \rightarrow [0, \pi]$ denote the inverse of $\cos : [0, \pi] \rightarrow [-1, 1]$. Define

$$B_{1,\Delta} := \left\{ x = (x_1, x_2) \in B_1 \setminus \{0\} : 0 \leq \arccos \frac{x_1}{|x|} < \sqrt{1 - \Delta^2} \pi \right\}.$$

Let $\mathbb{R}_- := \{(x_1, 0) : x_1 \leq 0\}$, and let $\varphi : \mathbb{R}^2 \setminus \mathbb{R}_- \rightarrow \mathbb{R}$ be the angular coordinate satisfying $x = |x|(\cos \varphi(x), \sin \varphi(x))$ with values in $(-\pi, \pi)$. We define the map $\iota \equiv \iota_\Delta : \mathbb{R}^2 \setminus \mathbb{R}_- \rightarrow B_1$ by

$$\iota(x) = \left(|x| \cos \frac{\varphi(x)}{\sqrt{1 - \Delta^2}}, |x| \sin \frac{\varphi(x)}{\sqrt{1 - \Delta^2}} \right).$$

For a sketch of $B_{1,\Delta}$ and ι_Δ , see Figure 1.

On $\iota(B_{1,\Delta}) = B_1 \setminus \mathbb{R}_-$, the map ι has a well-defined inverse, which we denote by

$$j : B_1 \setminus \mathbb{R}_- \rightarrow B_{1,\Delta}.$$

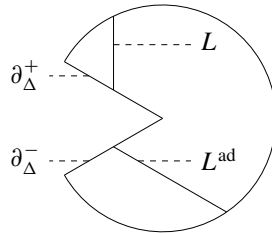


Figure 2. The subsets $\partial_{\Delta}^+, \partial_{\Delta}^-$ of the boundary and adjoint line segments L, L^{ad} .

Furthermore, let $\phi_{\Delta} := (1 - \sqrt{1 - \Delta^2})2\pi$ and let the rotation $S_{\Delta} \in SO(2)$ be defined by

$$S_{\Delta} = \begin{pmatrix} \cos \phi_{\Delta} & -\sin \phi_{\Delta} \\ \sin \phi_{\Delta} & \cos \phi_{\Delta} \end{pmatrix}.$$

Finally, let

$$\partial_{\Delta} := \partial B_{1,\Delta} \setminus (\partial B_1 \cup \{0\}).$$

Note that ∂_{Δ} has two connected components, one contained in the upper half-plane and one in the lower half-plane. We will denote them by ∂_{Δ}^+ and ∂_{Δ}^- respectively; see Figure 2. The rotation matrix S_{Δ} has been chosen such that $S_{\Delta}\partial_{\Delta}^+ = \partial_{\Delta}^-$.

We define

$$W_{\text{iso}}^{2,2}(B_{1,\Delta}) := \left\{ Y \in W_{\text{loc}}^{2,2}(\bar{B}_{1,\Delta} \setminus \{0\}; \mathbb{R}^3) : g_Y = \text{Id}_{2 \times 2}, Y(S_{\Delta}x) = Y(x) \text{ and } DY(S_{\Delta}x) = DY(x)S_{\Delta} \text{ for every } x \in \partial_{\Delta}^+ \right\} \quad (20)$$

This definition is chosen such that if $y \in W_{\text{loc}}^{2,2}(\bar{B}_1 \setminus \{0\}; \mathbb{R}^3)$ with $Dy^T Dy = g_{\Delta}$, then $y \circ \iota \in W_{\text{iso}}^{2,2}(B_{1,\Delta})$.

To $Y \in W_{\text{iso}}^{2,2}(B_{1,\Delta})$, we may apply Theorem 5 with $\Omega = B_{1,\Delta} \setminus B_{\rho}$ to obtain a set $L_Y^{(\rho)}$ of line segments with the properties stated there. For $\rho < \rho'$ we have by the uniqueness of the line segments stated in Lemma 6 that every line segment in $L_Y^{(\rho')}$ is contained in exactly one line segment of $L_Y^{(\rho)}$.

Hence, by sending $\rho \rightarrow 0$, we get a set of (relatively) closed mutually disjoint line segments in $\bar{B}_{1,\Delta} \setminus \{0\}$, denoted by L_Y .

If a line segment in L_Y has only one endpoint in $\bar{B}_{1,\Delta} \setminus \{0\}$, then we say by slight abuse of terminology that one of its endpoints is the origin.

Remark 7. We note in passing that with obvious modifications of the previous construction, one may extend Theorem 5 to maps with conical singularities, i.e., to maps $y \in W_{\text{loc}}^{2,2}(\bar{\Omega} \setminus \{x_0\}; \mathbb{R}^3)$ with $x_0 \in \Omega$ and $Dy^T Dy = \text{Id}_{2 \times 2}$ almost everywhere.

Next, we are going to define an ‘‘adjoint’’ line segment L^{ad} to any $L \in L_Y$ with an endpoint $x \in \partial_{\Delta}$. Note that for such L , there exists $v \in \partial B_1$ and $q > 0$ such that

$$L = \{x + tv : t \in [0, q]\}.$$

First let us assume $x \in \partial_{\Delta}^+$. By the definition of $W_{\text{iso}}^{2,2}(B_{1,\Delta})$ in (20), we have that $x' := S_{\Delta}x \in \partial_{\Delta}^-$, and $DY(x') = DY(x)S_{\Delta}$. Moreover, there has to exist $L^{ad} \in L_Y$ with $x' \in L^{ad}$, and

$$L^{ad} = \{x' + tS_{\Delta}v : t \in \mathbb{R}\} \cap \bar{B}_{1,\Delta}.$$

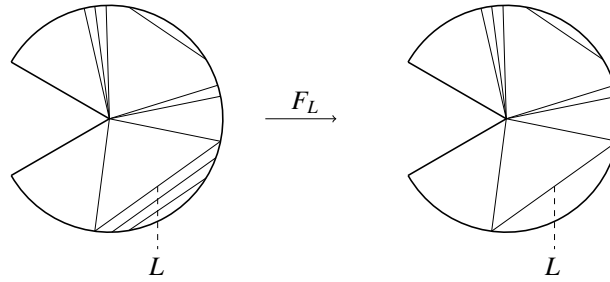


Figure 3. In the left panel, we have the segments that belong to L_Y , and $L \in L_Y$ is a bad line segment. We can flatten the deformation Y on the side of L whose closure does not contain the origin, and obtain a deformation $F_L(Y)$, such that $L_{F_L(Y)}$ consists of those line segments in L_Y that are on the same side of L as the origin; see the right panel.

This defines L^{ad} for $x \in \partial_\Delta^+$; for $x \in \partial_\Delta^-$, we define it analogously, replacing S_Δ by S_Δ^- . For a sketch of the construction, see Figure 2.

From now on, the line segments in L_Y for which one of the endpoints is 0 will be called “good”, and line segments in the complement of the set of good line segments will be called “bad”. The sets of good and bad line segments will be denoted by $L_Y^{(g)}, L_Y^{(b)}$ respectively. For any bad line segment, we can lower the elastic energy by “flattening” the deformation Y on one side of the line segment. This is the idea behind the following lemma. For a sketch of this operation, see Figure 3.

Lemma 8. *For every $Y \in W_{\text{iso}}^{2,2}(B_{1,\Delta})$, there exists $Y_\infty \in W_{\text{iso}}^{2,2}(B_{1,\Delta})$ with the following properties:*

- (i) $L_{Y_\infty}^{(b)} = \emptyset$ and $L_{Y_\infty}^{(g)} = L_Y^{(g)}$.
- (ii) For $0 < \rho < 1$, we have

$$\int_{B_{1,\Delta} \setminus B_\rho} |D^2 Y_\infty : ((Dt)^{-1} \otimes (Dt)^{-1})|^2 d\mathcal{L}^2 \leq \int_{B_{1,\Delta} \setminus B_\rho} |D^2 Y : ((Dt)^{-1} \otimes (Dt)^{-1})|^2 d\mathcal{L}^2, \quad (21)$$

with equality for all $0 < \rho < 1$ if and only if $Y = Y_\infty$.

Proof. For any $L \in L_Y^{(b)}$, we may define a modified map $F_L(Y) \in W_{\text{iso}}^{2,2}(B_{1,\Delta})$ as follows. On L , we have $Y = A_L x + b_L$ for some $A_L \in \mathbb{R}^{3 \times 2}$ and $b_L \in \mathbb{R}^3$. We note that $B_{1,\Delta} \setminus L$ has exactly two connected components. Let E_L denote the connected component whose closure does not contain the origin. First let us assume that none of the endpoints of L is in ∂_Δ . Then we define $F_L(Y) \in W_{\text{iso}}^{2,2}(B_{1,\Delta})$ by

$$F_L(Y)(x) = \begin{cases} A_L x + b_L & \text{if } x \in E_L, \\ Y(x) & \text{else.} \end{cases} \quad (22)$$

If one of the endpoints of L is in ∂_Δ , then we set

$$F_L(Y)(x) = \begin{cases} A_L x + b_L & \text{if } x \in E_L, \\ A_{L^{\text{ad}}} x + b_{L^{\text{ad}}} & \text{if } x \in E_{L^{\text{ad}}}, \\ Y(x) & \text{else.} \end{cases}$$

Note that this definition indeed satisfies $F_L(Y) \in W_{\text{iso}}^{2,2}(B_{1,\Delta})$. Obviously, we have $D^2(F_L(Y)) = 0$ on E_L (and on $E_{L^{\text{ad}}}$) and hence, for all $0 < \rho < 1$, we have

$$\int_{B_{1,\Delta} \setminus B_\rho} |D^2 F_L(Y)(Dt)^{-1}|^2 d\mathcal{L}^2 \leq \int_{B_{1,\Delta} \setminus B_\rho} |D^2 Y(Dt)^{-1}|^2 d\mathcal{L}^2. \tag{23}$$

We must distinguish two cases in (23): If $L_{F_L(Y)} \subsetneq L_Y$, then $F_L(Y) \neq Y$ and we must have $|D^2 Y| > 0$ on a subset of positive measure of E_L . Hence, inequality must hold in (23) for some ρ , since we have

$$\sqrt{1 - \Delta^2} \text{Id}_{2 \times 2} \leq (Dt)^{-1} \leq \text{Id}_{2 \times 2} \tag{24}$$

in the sense of positive definite matrices. Equality in (23) only holds in the case $F_L(Y) = Y$.

On $L_Y^{(b)}$, we may define an order relation by $L < L'$ if $E_L \subsetneq E_{L'}$. Since bad line segments are mutually disjoint, we have that either $L < L'$, $L > L'$ or $E_L \cap E_{L'} = \emptyset$. Hence, there exists an at most countable sequence L_1, L_2, \dots of maximal bad line segments. If for two maximal line segments L, L' we have $L' = L^{\text{ad}}$ then we exclude exactly one of them from that sequence. Now we define a sequence $Y_k \in W_{\text{iso}}^{2,2}(B_{1,\Delta})$ by

$$Y_k = F_{L_k} \circ \dots \circ F_{L_1}(Y). \tag{25}$$

By (23) and (24), $D^2 Y_k$ is bounded in L^2 . Thus the sequence converges weakly in $W^{2,2}(B_{1,\Delta} \setminus B_\rho; \mathbb{R}^3)$ for every $0 < \rho < 1$ to a limit $Y_\infty \in W_{\text{iso}}^{2,2}(B_{1,\Delta})$ such that L_{Y_∞} does not contain any bad line segments, and $L_{Y_\infty}^{(g)} = L_Y^{(g)}$. The claim (21) follows from (23) and the comment after that equation. \square

Remark 9. Letting Y, Y_∞ as in Lemma 8, we have that DY_∞ is constant on every line segment $(0, x)$ for $x \in \partial B_{1,\Delta} \cap \partial B_1$, and

$$Y_\infty \circ j \in W_{\text{loc}}^{2,2}(\bar{B}_1 \setminus \{0\}; \mathbb{R}^3), \quad g_{Y_\infty \circ j} = g_\Delta.$$

Furthermore,

$$\int_{B_1 \setminus B_\rho} |D^2(Y_\infty \circ j)|^2 d\mathcal{L}^2 \leq \int_{B_1 \setminus B_\rho} |D^2(Y \circ j)|^2 d\mathcal{L}^2 \quad \text{for every } 0 < \rho < 1.$$

Proof. The first statement in the remark follows from the fact that for every $x' \in B_{1,\Delta}$, we have that either there exists $x \in \partial B_{1,\Delta} \cap \partial B_1$ such that $x' \in (0, x] \in L_{Y_\infty}^{(g)}$ or there exists a sector containing x' that has empty intersection with every $L \in L_{Y_\infty}$, and hence DY_∞ vanishes in the whole sector.

The second and third statements follow immediately from $Y_\infty \in W_{\text{iso}}^{2,2}(B_{1,\Delta})$. It remains to prove the inequality. Let $\nu = Y_{,1} \wedge Y_{,2} / |Y_{,1} \wedge Y_{,2}|$ be the unit normal. By $DY^T DY = \text{Id}_{2 \times 2}$, we have $D^2 Y \perp DY$. Hence

$$\begin{aligned} |D^2(Y \circ j)|^2 &= |D^2 Y : (Dj \otimes Dj) + DY D^2 j|^2 \\ &= |D^2 Y : (Dj \otimes Dj)|^2 + |DY D^2 j|^2 \\ &= |D^2 Y : (Dj \otimes Dj)|^2 + |D^2 j|^2, \end{aligned} \tag{26}$$

where we used $DY \in O(2, 3)$ in the last equality. Now the inequality follows from (21) and a change of variables in the integrals. \square

3.2. Proof of Theorem 2. Given $0 < R < 1$, we may assume that $h \ll R$. Choose $R_0(h) \in [R - h, R]$ such that

$$\int_{\partial B_{R_0(h)}} |g_{y_h} - g_\Delta|^2 d\mathcal{H}^1 \leq h^{-1} \int_{B_1} |g_{y_h} - g_\Delta|^2 d\mathcal{L}^2.$$

By Theorem 1, we have

$$\int_{B_1 \setminus B_R} |D^2 y_h|^2 d\mathcal{L}^2 \leq \int_{B_1 \setminus B_{R_0}} |D^2 y_h|^2 d\mathcal{L}^2 \leq 2\pi \Delta^2 \log \frac{1}{R} + C, \tag{27}$$

where C depends neither on h nor on R . This proves the boundedness of y_h in $W^{2,2}(B_1 \setminus B_R; \mathbb{R}^3)$ and implies that there exists $\hat{y}_R \in W^{2,2}(B_1 \setminus B_R; \mathbb{R}^3)$ such that (for a subsequence)

$$y_h \rightharpoonup \hat{y}_R \quad \text{in } W^{2,2}(B_1 \setminus B_R; \mathbb{R}^3).$$

After taking a suitable diagonal sequence for $R = 1/j$, $j = 2, 3, \dots$, we may assume that $\hat{y}_R \in W^{2,2}_{\text{loc}}(B_1 \setminus \{0\}; \mathbb{R}^3)$ is independent of R . We denote this function by y^* . By Theorem 1, we have

$$\int_{B_1} |g_{y^*} - g_\Delta|^2 d\mathcal{L}^2 = 0;$$

i.e., y^* is an isometry with respect to g_Δ .

By (27), we have

$$\int_{B_1 \setminus B_R} |D^2 y^*|^2 d\mathcal{L}^2 \leq 2\pi \Delta^2 h^2 \log \frac{1}{R} + C. \tag{28}$$

Let $Y : B_{1,\Delta} \rightarrow \mathbb{R}^3$ be defined by

$$Y := y^* \circ \iota.$$

Recalling the definitions from Section 3.1, we have $Y \in W^{2,2}_{\text{iso}}(B_{1,\Delta})$. By an application of Lemma 8 and Remark 9, we obtain $Y_\infty \in W^{2,2}_{\text{iso}}(B_{1,\Delta})$ such that DY_∞ is constant on every line segment $(0, x)$ with $x \in \partial B_{1,\Delta} \cap \partial B_1$. Now we set $y^\infty := Y_\infty \circ j$, and obtain that Dy^∞ is constant on every line segment $(0, x]$ with $x \in \partial B_1$. Hence there exists a curve $\gamma : \partial B_1 \rightarrow S^2$ satisfying $|\gamma'| = \sqrt{1 - \Delta^2}$ such that

$$y^\infty(x) = x\gamma\left(\frac{x}{|x|}\right). \tag{29}$$

Using this expression, explicit computation yields

$$\int_{\partial B_\rho} |D^2 y^\infty|^2 d\mathcal{H}^1 = \frac{1}{\rho} \int_{\partial B_1} |D^2 y^\infty|^2 d\mathcal{H}^1. \tag{30}$$

By Remark 9 and (28), we have that for every $0 < \rho < 1$,

$$\int_{B_1 \setminus B_\rho} |D^2 y^\infty|^2 d\mathcal{L}^2 \leq 2\pi \Delta^2 \log \frac{1}{\rho} + C. \tag{31}$$

Combining (30) and (31), we see that for every $0 < \rho < 1$, we have

$$\int_{\partial B_\rho} |D^2 y^\infty|^2 d\mathcal{H}^1 \leq \frac{2\pi \Delta^2}{\rho},$$

and the constant C in (31) is in fact 0.

By $g_{y^\infty} = g_\Delta$, we have

$$\sum_{i=1}^3 \det D^2 y_i^\infty = \pi \Delta^2 \delta_0$$

distributionally. We may now estimate using Lemma 3, for any $0 < \rho < 1$,

$$\begin{aligned} \pi \Delta^2 &= \int_{B_\rho} \sum_{i=1}^3 \det D^2 y_i^\infty \, d\mathcal{L}^2 \leq \sum_i \left| \int_{B_\rho} \det D^2 y_i^\infty \, d\mathcal{L}^2 \right| \\ &\leq \frac{1}{4\pi} \sum_i \left(\int_{\partial B_\rho} |D^2 y_i^\infty(x) \cdot \hat{x}^\perp| \, d\mathcal{H}^1(x) \right)^2 \\ &\leq \frac{1}{4\pi} \sum_i 2\pi\rho \left(\int_{\partial B_\rho} |D^2 y_i^\infty(x) \cdot \hat{x}^\perp|^2 \, d\mathcal{H}^1(x) \right) \\ &\leq \frac{\rho}{2} \int_{\partial B_\rho} |D^2 y^\infty(x) \cdot \hat{x}^\perp|^2 \, d\mathcal{H}^1(x) \leq \pi \Delta^2. \end{aligned} \tag{32}$$

Here, to obtain the third from the second line, we used Jensen’s inequality. By this chain of estimates, all the inequalities must have been equalities, and we have

$$\sum_i \left(\int_{\partial B_\rho} |D^2 y_i^\infty(x) \cdot \hat{x}^\perp| \, d\mathcal{H}^1(x) \right)^2 = \sum_i 2\pi\rho \left(\int_{\partial B_\rho} |D^2 y_i^\infty(x) \cdot \hat{x}^\perp|^2 \, d\mathcal{H}^1(x) \right)$$

and thus

$$|D^2 y_i^\infty(x) \cdot \hat{x}^\perp|^2 = \text{constant} \quad \text{for } x \in \partial B_\rho, \, i \in \{1, 2, 3\}. \tag{33}$$

Additionally, (32) implies

$$|D^2 y^\infty(x) \cdot \hat{x}^\perp|^2 = \frac{\Delta^2}{\rho^2} \quad \text{for } x \in \partial B_\rho. \tag{34}$$

By (29), we have $D^2 y^\infty(x) = |x|^{-1}(\gamma + \gamma'') \otimes \hat{x}^\perp \otimes \hat{x}^\perp$. Combining this with (33), we get

$$(\gamma + \gamma'') \cdot e_i = \text{constant on } \partial B_1$$

for $i = 1, 2, 3$. We write $c_i = (\gamma + \gamma'') \cdot e_i$, and have $D^2 y_i^\infty(x) = (c_i/|x|)\hat{x}^\perp \otimes \hat{x}^\perp$, which implies

$$y_i^\infty(x) = c_i|x| + a_i \cdot x + b_i \quad \text{for } i = 1, 2, 3,$$

for some $a_i \in \mathbb{R}^2, b_i \in \mathbb{R}$. By (33) we obtain

$$|D^2 y^\infty(x)|^2 = \frac{\sum_i c_i^2}{|x|^2} = \frac{\Delta^2}{|x|^2},$$

and thus $\sum_i c_i^2 = \Delta^2$. By $g_{y^\infty} = g_\Delta$, we have

$$\text{Id}_{2 \times 2} - \Delta^2 \hat{x}^\perp \otimes \hat{x}^\perp = (c \otimes \hat{x} + a)^T (c \otimes x + a) = |c|^2 \hat{x} \otimes \hat{x} + (c \cdot a) \otimes \hat{x} + \hat{x} \otimes (c \cdot a) + a^T a.$$

This yields

$$(1 - \Delta^2)\text{Id}_{2 \times 2} = (c \cdot a) \otimes \hat{x} + \hat{x} \otimes (c \cdot a) + a^T a,$$

which can only hold true for all $\hat{x} \in \partial B_1$ if $c \cdot a = 0$ and $a^T a = (1 - \Delta^2) \text{Id}_{2 \times 2}$. This implies

$$R := \left(\frac{a}{\sqrt{1 - \Delta^2}}, \frac{c}{\Delta} \right) \in O(3)$$

is an orthogonal matrix, and we have

$$y^\infty(x) = R(\sqrt{1 - \Delta^2}x + \Delta e_3|x|) + b.$$

It remains to show that $y^\infty = y^*$. To see this, note that $y^\infty \circ \iota = Y_\infty$ satisfies

$$\{(0, x] : x \in \partial B_{1,\Delta} \cap \partial B_1\} = L_{Y_\infty}^{(g)} = L_Y^{(g)},$$

where the second equality holds by Lemma 8. This implies that for every $x \in B_{1,\Delta}$ there exists an $L \in L_Y^{(g)}$ with $x \in L$. This in turn implies that $L_Y^{(b)} = \emptyset$ (since the line segments in L_Y are pairwise disjoint). By Lemma 8, the latter yields $Y = Y_\infty$. Composing with j on both sides of this last equation, we obtain $y^* = y^\infty$. This completes the proof of the theorem. \square

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References

- [Bella and Kohn 2014a] P. Bella and R. V. Kohn, “Metric-induced wrinkling of a thin elastic sheet”, *J. Nonlinear Sci.* **24**:6 (2014), 1147–1176. MR Zbl
- [Bella and Kohn 2014b] P. Bella and R. V. Kohn, “Wrinkles as the result of compressive stresses in an annular thin film”, *Comm. Pure Appl. Math.* **67**:5 (2014), 693–747. MR Zbl
- [Ben Amar and Pomeau 1997] M. Ben Amar and Y. Pomeau, “Crumpled paper”, *Proc. Roy. Soc. London Ser. A* **453**:1959 (1997), 729–755. MR Zbl
- [Ben Belgacem et al. 2002] H. Ben Belgacem, S. Conti, A. DeSimone, and S. Müller, “Energy scaling of compressed elastic films: three-dimensional elasticity and reduced theories”, *Arch. Ration. Mech. Anal.* **164**:1 (2002), 1–37. MR Zbl
- [Brandman et al. 2013] J. Brandman, R. V. Kohn, and H.-M. Nguyen, “Energy scaling laws for conically constrained thin elastic sheets”, *J. Elasticity* **113**:2 (2013), 251–264. MR Zbl
- [Cerda and Mahadevan 1998] E. Cerda and L. Mahadevan, “Conical surfaces and crescent singularities in crumpled sheets”, *Phys. Rev. Lett.* **80**:11 (1998), 2358–2361.
- [Cerda and Mahadevan 2005] E. Cerda and L. Mahadevan, “Confined developable elastic surfaces: cylinders, cones and the *elastica*”, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **461**:2055 (2005), 671–700. MR Zbl
- [Cerda et al. 1999] E. Cerda, S. Chaieb, F. Melo, and L. Mahadevan, “Conical dislocations in crumpling”, *Nature* **401** (1999), 46–49.
- [Constantin et al. 1994] P. Constantin, W. E, and E. S. Titi, “Onsager’s conjecture on the energy conservation for solutions of Euler’s equation”, *Comm. Math. Phys.* **165**:1 (1994), 207–209. MR Zbl
- [Conti and Maggi 2008] S. Conti and F. Maggi, “Confining thin elastic sheets and folding paper”, *Arch. Ration. Mech. Anal.* **187**:1 (2008), 1–48. MR Zbl
- [Demengel 1984] F. Demengel, “Fonctions à hessien borné”, *Ann. Inst. Fourier (Grenoble)* **34**:2 (1984), 155–190. MR Zbl
- [DiDonna and Witten 2001] B. A. DiDonna and T. A. Witten, “Anomalous strength of membranes with elastic ridges”, *Phys. Rev. Lett.* **87**:20 (2001), art. id. 206105.

- [Friesecke et al. 2002] G. Friesecke, R. D. James, and S. Müller, “A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity”, *Comm. Pure Appl. Math.* **55**:11 (2002), 1461–1506. MR Zbl
- [Friesecke et al. 2006] G. Friesecke, R. D. James, and S. Müller, “A hierarchy of plate models derived from nonlinear elasticity by gamma-convergence”, *Arch. Ration. Mech. Anal.* **180**:2 (2006), 183–236. MR Zbl
- [Gromov 1986] M. Gromov, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **9**, Springer, 1986. MR Zbl
- [Hornung 2011a] P. Hornung, “Approximation of flat $W^{2,2}$ isometric immersions by smooth ones”, *Arch. Ration. Mech. Anal.* **199**:3 (2011), 1015–1067. MR Zbl
- [Hornung 2011b] P. Hornung, “Fine level set structure of flat isometric immersions”, *Arch. Ration. Mech. Anal.* **199**:3 (2011), 943–1014. MR Zbl
- [Isett 2016] P. Isett, “A proof of Onsager’s conjecture”, preprint, 2016. arXiv
- [Kohn and Nguyen 2013] R. V. Kohn and H.-M. Nguyen, “Analysis of a compressed thin film bonded to a compliant substrate: the energy scaling law”, *J. Nonlinear Sci.* **23**:3 (2013), 343–362. MR Zbl
- [Kramer and Witten 1997] E. M. Kramer and T. A. Witten, “Stress condensation in crushed elastic manifolds”, *Phys. Rev. Lett.* **78**:7 (1997), 1303–1306.
- [Kuiper 1955a] N. H. Kuiper, “On C^1 -isometric imbeddings, I”, *Indag. Math.* **17** (1955), 545–556. MR Zbl
- [Kuiper 1955b] N. H. Kuiper, “On C^1 -isometric imbeddings, II”, *Indag. Math.* **17** (1955), 683–689. MR Zbl
- [Lewicka and Pakzad 2017] M. Lewicka and M. R. Pakzad, “Convex integration for the Monge-Ampère equation in two dimensions”, *Anal. PDE* **10**:3 (2017), 695–727. MR Zbl
- [Lewicka et al. 2010] M. Lewicka, M. G. Mora, and M. R. Pakzad, “Shell theories arising as low energy Γ -limit of 3d nonlinear elasticity”, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **9**:2 (2010), 253–295. MR Zbl
- [Lidmar et al. 2003] J. Lidmar, L. Mirny, and D. R. Nelson, “Virus shapes and buckling transitions in spherical shells”, *Phys. Rev. E* (3) **68**:5 (2003), art. id. 051910.
- [Lobkovsky 1996] A. E. Lobkovsky, “Boundary layer analysis of the ridge singularity in a thin plate”, *Phys. Rev. E* (3) **53**:4, part B (1996), 3750–3759. MR
- [Lobkovsky and Witten 1997] A. E. Lobkovsky and T. A. Witten, “Properties of ridges in elastic membranes”, *Phys. Rev. E* (3) **55**:2 (1997), 1577–1589.
- [Lobkovsky et al. 1995] A. Lobkovsky, S. Gentges, H. Li, D. Morse, and A. Witten, “Scaling properties of stretching ridges in a crumpled elastic sheet”, *Science* **270**:5241 (1995), 1482–1485.
- [Müller 1990] S. Müller, “Higher integrability of determinants and weak convergence in L^1 ”, *J. Reine Angew. Math.* **412** (1990), 20–34. MR Zbl
- [Müller and Olbermann 2014a] S. Müller and H. Olbermann, “Almost conical deformations of thin sheets with rotational symmetry”, *SIAM J. Math. Anal.* **46**:1 (2014), 25–44. MR Zbl
- [Müller and Olbermann 2014b] S. Müller and H. Olbermann, “Conical singularities in thin elastic sheets”, *Calc. Var. Partial Differential Equations* **49**:3-4 (2014), 1177–1186. MR Zbl
- [Nash 1954] J. Nash, “ C^1 isometric imbeddings”, *Ann. of Math. (2)* **60** (1954), 383–396. MR Zbl
- [Olbermann 2016] H. Olbermann, “Energy scaling law for the regular cone”, *J. Nonlinear Sci.* **26**:2 (2016), 287–314. MR Zbl
- [Olbermann 2017] H. Olbermann, “Energy scaling law for a single disclination in a thin elastic sheet”, *Arch. Ration. Mech. Anal.* **224**:3 (2017), 985–1019. MR Zbl
- [Pakzad 2004] M. R. Pakzad, “On the Sobolev space of isometric immersions”, *J. Differential Geom.* **66**:1 (2004), 47–69. MR Zbl
- [Pogorelov 1973] A. V. Pogorelov, *Extrinsic geometry of convex surfaces*, Translations of Mathematical Monographs **35**, American Mathematical Society, Providence, RI, 1973. MR Zbl
- [Romanov 2003] A. E. Romanov, “Mechanics and physics of disclinations in solids”, *Eur. J. Mech. A Solids* **22**:5 (2003), 727–741. MR Zbl

- [Seung and Nelson 1988] H. S. Seung and D. R. Nelson, “Defects in flexible membranes with crystalline order”, *Phys. Rev. A* **38**:2 (1988), 1005–1018.
- [Venkataramani 2004] S. C. Venkataramani, “Lower bounds for the energy in a crumpled elastic sheet: a minimal ridge”, *Nonlinearity* **17**:1 (2004), 301–312. MR Zbl
- [Witten 2007] T. A. Witten, “Stress focusing in elastic sheets”, *Rev. Modern Phys.* **79**:2 (2007), 643–675. MR Zbl
- [Yavari and Goriely 2013] A. Yavari and A. Goriely, “Riemann–Cartan geometry of nonlinear disclination mechanics”, *Math. Mech. Solids* **18**:1 (2013), 91–102. MR Zbl

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HEINER OLBERMANN: heiner.olbermann@math.uni-leipzig.de
Universität Leipzig, Leipzig, Germany

THE THIN-FILM EQUATION CLOSE TO SELF-SIMILARITY

CHRISTIAN SEIS

We study well-posedness and regularity of the multidimensional thin-film equation with linear mobility in a neighborhood of the self-similar Smyth–Hill solutions. To be more specific, we perform a von Mises change of dependent and independent variables that transforms the thin-film free boundary problem into a parabolic equation on the unit ball. We show that the transformed equation is well-posed and that solutions are smooth and even analytic in time and angular direction. The latter gives the analyticity of level sets of the original equation, and thus, in particular, of the free boundary.

1. Introduction and main results

1.1. The background. We are concerned with a thin-film equation in arbitrary space dimensions. Our interest is in the simplest case of linear mobility; that is, we consider the partial differential equation

$$\partial_t u + \nabla \cdot (u \nabla \Delta u) = 0 \tag{1}$$

in \mathbb{R}^N . In this model, u describes the thickness of a viscous thin liquid film on a flat substrate. We will focus on what is usually referred to as the complete wetting regime, in which the liquid-solid contact angle at the film boundary is presumed to be zero. Notice that in the three-dimensional physical space, the dimension N of the substrate is 2.

Equation (1) belongs to the following family of thin-film equations:

$$\partial_t u + \nabla \cdot (m(u) \nabla \Delta u) = 0, \tag{2}$$

where the mobility factor is given by $m(u) = u^3 + \beta^{n-3} u^n$ with β being the slippage length. The nonlinearity exponent $n > 0$ depends on the slip condition at the solid-liquid interface: $n = 3$ models no-slip conditions and $n = 2$ models Navier-slip conditions. The case $n = 1$ is a further relaxation and the linear mobility considered here is obtained to leading order in the limit $u \rightarrow 0$.

The evolution described in (2) was originally derived as a long-wave approximation from the free-surface problem related to the Navier–Stokes equations and suitable model reductions; see, e.g., [Oron et al. 1997]. At the same time, it can be obtained as the Wasserstein gradient flow of the surface-tension energy [Otto 1998; Giacomelli and Otto 2001; Matthes et al. 2009] and serves thus as the natural dissipative model for surface-tension-driven transport of viscous liquids over solid substrates.

The analytical treatment of the equation is challenging and the mathematical understanding is far from being satisfactory. As a fourth-order problem, the thin-film equation lacks a maximum principle.

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Moreover, the parabolicity degenerates where u vanishes and, as a consequence, for compactly supported initial data (“droplets”), the solution remains compactly supported [Bernis 1996; Bertsch et al. 1998]. The thin-film equation features thus a free boundary given by $\partial\{u > 0\}$, which in physical terms is the contact line connecting the phases liquid, solid and vapor. Nonetheless, by using estimates for the surface energy and compactness arguments Bernis and Friedman [1990] established the existence of weak nonnegative solutions over a quarter of a century ago. The regularity of these solutions could be slightly improved with the help of certain entropy-type estimates [Beretta et al. 1995; Bertozzi and Pugh 1996; Dal Passo et al. 1998], but this regularity is not sufficient for proving general uniqueness results. To gain an understanding of the thin-film equation and its qualitative features, it is thus natural to find and study special solutions first. In the past ten years, most of the attention has been focused on the one-dimensional setting, for instance, near stationary solutions [Giacomelli et al. 2008; Giacomelli and Knüpfer 2010], traveling waves [Giacomelli et al. 2014; Gnann 2016], and self-similar solutions [Gnann 2015; Belgacem et al. 2016]. The only regularity and well-posedness result in higher dimensions available so far is due to John [2015], who analyzed the equation around stationary solutions. For completeness, we remark the thin-film equation is also studied with nonzero contact angles; see, e.g., [Otto 1998; Giacomelli and Otto 2001; 2003; Knüpfer 2011; 2015; Knüpfer and Masmoudi 2015; Belgacem et al. 2016; Degtyarev 2017]. The latter of these works is particularly interesting as it deals with the multidimensional situation.

In the present paper, we will conduct a study similar to John’s and investigate the qualitative behavior of solutions close to self-similarity. A family of self-similar solutions to (1) is given by

$$u_*(t, x) = \frac{\alpha_N}{t^{N/(N+4)}} \left(\sigma_M - \frac{|x|^2}{t^{2/(N+4)}} \right)_+^2,$$

where $\alpha_N = 1/(8(N+4)(N+2))$ and σ_M is a positive number that is determined by the mass constraint

$$\int u_* dx = M,$$

and the subscript plus sign denotes the positive part of a quantity; i.e., $(\cdot)_+ = \max\{0, \cdot\}$. These solutions were first found by Smyth and Hill [1988] in the one-dimensional case and then rediscovered in [Ferreira and Bernis 1997]. As in related parabolic settings, the Smyth–Hill solutions play a distinguished role in the theory of the thin-film equation as they are believed to describe the leading-order large-time asymptotic behavior of *any* solution — a fact that is currently known only for strong [Carrillo and Toscani 2002] and minimizing movement [Matthes et al. 2009] solutions. Besides that, these particular solutions are considered to feature the same regularity properties as any “typical” solution, at least for large times. Thus, under suitable assumptions on the initial data, we expect the solutions of (1) to be smooth up to the boundary of their support. (Notice that this behavior is exclusive for the linear-mobility thin-film equation; see [Giacomelli et al. 2013].)

In the present work we consider solutions that are in some suitable sense close to the self-similar Smyth–Hill solution. Instead of working with (1) directly, we will perform a certain von Mises change of dependent and independent variables, which has the advantage that it freezes the free boundary $\partial\{u > 0\}$ to the unit ball. We will mainly address the following four questions:

- (1) Is there some uniqueness principle available for the transformed equation?
- (2) Are solutions smooth?
- (3) Can we deduce some regularity for the moving interface $\partial\{u > 0\}$?
- (4) Do solutions depend smoothly on the initial data?

We will provide positive answers to all of these questions. In fact, applying a perturbation procedure we will show that the transformed equation is well-posed in a sufficiently small neighborhood of u_* . We will furthermore show that the unique solution is smooth in time and space. In fact, our results show that solutions to the transformed equation are analytic in time and in direction tangential to the free boundary. The latter in particular implies that all level sets and thus also the free boundary line corresponding to the original solutions are analytic. We finally prove analytic dependence on the initial data.

The fact that solutions depend differentiably (or even better) on the initial data will be of great relevance in a companion study on fine large-time asymptotic expansions. Indeed, in [Seis \geq 2018], we investigate the rates at which solutions converge to the self-similarity at any order. Optimal rates of convergence were already found by Carrillo and Toscani [2002] and Matthes, McCann and Savaré [Matthes et al. 2009], and these rates are saturated by spatial translations of the Smyth–Hill solutions. Jointly with McCann [McCann and Seis 2015] we diagonalized the differential operator obtained after formal linearization around the self-similar solution. The goal of [Seis \geq 2018] is to translate the spectral information obtained in [McCann and Seis 2015] into large-time asymptotic expansions for the nonlinear problem. For this, it is necessary to rigorously linearize the equation, the framework for which is obtained in the current paper. This strategy was recently successfully applied to the porous-medium equation near the self-similar Barenblatt solutions [Seis 2014; 2015]. The present work parallels in parts [Seis 2015] as well as the pioneering work by Koch [1999] and the further developments by Kienzler [2016] and John [2015].

1.2. Global transformation onto fixed domain. In this subsection it is our goal to transform the thin-film equation (1) into a partial differential equation that is posed on a fixed domain and that appears to be more suitable for a regularity theory than the original equation. The first step is a customary change of variables that translates the self-similarly spreading Smyth–Hill solution into a stationary solution. This is, for instance, achieved by setting

$$\hat{x} = \frac{1}{\sqrt{\sigma_M}} \frac{x}{t^{1/(N+4)}}, \quad \hat{t} = \frac{1}{N+4} \log t, \quad \hat{u} = \frac{N+4}{\sigma_M^2} t^{N/(N+4)} u,$$

with the effect that the Smyth–Hill solution becomes

$$\gamma \hat{u}_*(\hat{x})^{1/2} = \frac{1}{2}(1 - |\hat{x}|^2)_+, \quad \gamma = \sqrt{2(N+2)}, \tag{3}$$

and the thin-film equation (1) turns into the *confined* thin-film equation

$$\partial_{\hat{t}} \hat{u} + \hat{\nabla} \cdot (\hat{u} \hat{\nabla} \hat{\Delta} \hat{u}) - \hat{\nabla} \cdot (\hat{x} \hat{u}) = 0. \tag{4}$$

It is easily checked that \hat{u}_* is indeed a stationary solution to (4) and mass is no longer spreading over all of \mathbb{R}^N . Instead, the confinement term pushes all mass towards the stationary \hat{u}_* at the origin. To simplify the notation in the following, we will drop the hats immediately!

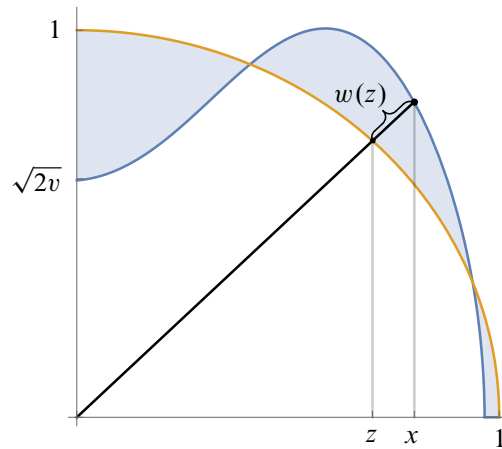


Figure 1. The definition of the (z, w) -coordinates.

Now that the Smyth–Hill solution has become stationary, we will perform a change of dependent and independent variables that parametrizes the solution as a graph over u_* . This type of a change of variables is sometimes referred to as a von Mises transformation. It is convenient to temporarily introduce the variable $v = \gamma u^{1/2}$, so that $\sqrt{2v_*}$ maps the unit ball onto the upper half-sphere. The new variables are obtained by projecting the point $(x, \sqrt{2v(x)})$ orthogonally onto the graph of $\sqrt{2v_*}$: noting that $\sqrt{2v(x) + |x|^2}$ is the hypotenuse of the triangle with the edges $(0, 0)$, $(0, |x|)$ and $(|x|, \sqrt{2v(x)})$, the projection point has the coordinates $(z, \sqrt{2v_*(z)})$ with

$$z = \frac{x}{\sqrt{2v(x) + |x|^2}}.$$

We define the new dependent variable w as the distance of the point $(x, \sqrt{2v(x)})$ from the sphere; that is,

$$1 + w = \sqrt{2v + |x|^2}, \quad (5)$$

which gives that $x = (1 + w)z$. This change of variables is illustrated in Figure 1.

The transformation of the thin-film equation (4) under this change of variables leads to straightforward but tedious computations that we conveniently defer to the Appendix. The new variable w obeys the equation

$$\partial_t w + \mathcal{L}^2 w + N \mathcal{L} w = f[w] \quad (6)$$

on $B_1(0)$, where

$$\mathcal{L} w = -\rho^{-1} \nabla \cdot (\rho^2 \nabla w) = -\rho \Delta w + 2z \cdot \nabla w$$

is precisely the linear operator that is obtained by linearizing the porous-medium equation $\partial_t u - \Delta u^{3/2} = 0$ about the Barenblatt solution; see, e.g., [Seis 2014; 2015]. Before specifying the particular form of the nonlinearity $f[w]$, let us notice that the linear operator $\mathcal{L}^2 + N \mathcal{L}$ corresponding to the thin-film dynamics was previously found by McCann and the author [McCann and Seis 2015] by a formal computation. Its relation to the porous-medium linear operator \mathcal{L} is not surprising but reflects the deep relation between

both equations. Indeed, as first exploited by Carrillo and Toscani [2002], the dissipation rate of the porous-medium entropy is just the surface energy that drives the thin-film dynamics. On a more abstract level, this observation can be expressed by the so-called energy-information relation, first formulated by Matthes, McCann and Savaré [Matthes et al. 2009], which connects the Wasserstein gradient flow structures of both equations [Otto 1998; 2001; Giacomelli and Otto 2001].

Let us finally discuss the right-hand side of (6). We can split $f[w]$ according to $f[w] = f^1[w] + f^2[w] + f^3[w]$, where

$$\begin{aligned} f^1[w] &= p \star R_1[w] \star ((\nabla w)^{2\star} + \nabla w \star \nabla^2 w), \\ f^2[w] &= p \star R_1[w] \star \rho((\nabla^2 w)^{2\star} + \nabla w \star \nabla^3 w), \\ f^3[w] &= p \star R_2[w] \star \rho^2((\nabla^2 w)^{3\star} + \nabla \tilde{w} \star \nabla^2 \tilde{w} \star \nabla^3 w + (\nabla w)^{2\star} \star \nabla^4 w), \end{aligned}$$

and

$$R_i[w] = \frac{(\nabla w)^{k\star}}{(1 + w + z \cdot \nabla w)^{k+i}}$$

for some $k \in \mathbb{N}_0$. We will see in Section 3 below that the particular form of the nonlinearity is irrelevant for the perturbation argument. We have thus introduced a slightly condensed notation to simplify the terms in the nonlinearity: we write $f \star g$ to denote any arbitrary linear combination of the tensors (vectors, matrices) f and g . For instance, $\nabla^{m_1} \tilde{w} \star \nabla^{m_2} \tilde{w}$ is an arbitrary linear combination of products of derivatives of orders m_1 and m_2 . The iterated application of the \star is abbreviated as $f^{j\star} = f \star \dots \star f$ if the latter product has j factors. The conventions $f^{1\star} = 1 \star f$ and $f^{0\star} = 1$ apply. We furthermore use p as an arbitrary representative of a (tensor-valued) polynomial in z . We have only kept track of those ρ prefactors, which will be of importance later on.

1.3. The intrinsic geometry and function spaces. In our analysis of the linear equation associated to (6), i.e.,

$$\partial_t w + \mathcal{L}^2 w + N \mathcal{L} w = f \tag{7}$$

for some general f , we will make use of the framework developed earlier in [Koch 1999; Seis 2015] for the second-order equation

$$\partial_t w + \mathcal{L} w = f. \tag{8}$$

The underlying point of view in there is the fact that the previous equation can be interpreted as a heat flow on a weighted manifold, i.e., a Riemannian manifold to which a new volume element (typically a positive multiple of the one induced by the metric tensor) is assigned. The theories for heat flows on a weighted manifold parallel those on Riemannian manifolds in many respects; see [Grigor’yan 2006]. For instance, a Calderón–Zygmund theory is available for (8). The crucial idea in [Koch 1999; Seis 2015] is to trade the Euclidean distance on $B_1(0)$ for the geodesic distance induced by the heat flow interpretation. In this way, we equip the unit ball with a non-Euclidean Carnot–Carathéodory metric, see, e.g., [Bellaïche and Risler 1996], which has the advantage that the parabolicity of the linear equation can be restored. The same strategy has been applied in similar settings in [Daskalopoulos and Hamilton 1998; Koch 1999; Kienzler 2016; John 2015; Denzler et al. 2015; Seis 2015].

We define

$$d(z, z') = \frac{|z - z'|}{\sqrt{\rho(z)} + \sqrt{\rho(z')} + \sqrt{|z - z'|}}$$

for any $z, z' \in B_1(0)$. Notice that d is not a metric as it lacks a proper triangle inequality. This semidistance is in fact equivalent to the geodesic distance induced on the (weighted) Riemannian manifold associated with the heat flow (8); see [Seis 2015, Proposition 4.2]. We define open balls with respect to the metric d by

$$B_r^d(z) = \{z' \in \overline{B_1(0)} : d(z, z') < r\},$$

and set $Q_r^d(z) = (\frac{1}{2}r^4, r^4) \times B_r^d(z)$ and also $Q(T) = (T, T + 1) \times B_1(0)$. Properties of intrinsic balls and volumes will be cited in Section 2.1 below.

With these preparations, we are in the position to introduce the (semi)norms

$$\begin{aligned} \|w\|_{X(p)} &= \sum_{(\ell, k, |\beta|) \in \mathcal{E}} \sup_{\substack{z \in \overline{B_1(0)} \\ 0 < r \leq 1}} \frac{r^{4k+|\beta|-1}}{\theta(r, z)^{2\ell-|\beta|+1}} |Q_r^d(z)|^{-1/p} \|\rho^\ell \partial_t^k \partial_z^\beta w\|_{L^p(Q_r^d(z))} \\ &\quad + \sum_{(\ell, k, |\beta|) \in \mathcal{E}} \sup_{T \geq 1} T \|\rho^\ell \partial_t^k \partial_z^\beta w\|_{L^p(Q(T))}, \\ \|f\|_{Y(p)} &= \sup_{\substack{z \in \overline{B_1(0)} \\ 0 < r \leq 1}} \frac{r^3}{\theta(r, z)} |Q_r^d(z)|^{-1/p} \|f\|_{L^p(Q_r^d(z))} + \sup_{T \geq 1} T \|f\|_{L^p(Q(T))} \end{aligned}$$

for $p \geq 1$, where

$$\mathcal{E} = \{(0, 1, 0), (0, 0, 2), (1, 0, 3), (2, 0, 4)\}.$$

These norms (or Whitney measures) induce the function spaces $X(p)$ and $Y(p)$, respectively, in the obvious way.

1.4. Statement of the results. In view of the particular form of the nonlinearity it is apparent that any well-posedness theory for the transformed equation (6) requires an appropriate control of w and ∇w to prevent the denominators in $R_i[w]$ from degenerating. This is achieved when the Lipschitz norm $\|w\|_{W^{1,\infty}} = \|w\|_{L^\infty} + \|\nabla w\|_{L^\infty}$ is sufficiently small. A suitable function space for existence and uniqueness is provided by $X(p) \cap L^\infty(W^{1,\infty})$ under minimal assumptions on the initial data. Here we have used the convention that $L^q(X) = L^q((0, T); X)$ for some (possibly infinite) $T > 0$. Our first main result is:

Theorem 1 (existence and uniqueness). *Let $p > N + 4$ be given. There exists $\varepsilon, \varepsilon_0 > 0$ such that for every $g \in W^{1,\infty}$ with*

$$\|g\|_{W^{1,\infty}} \leq \varepsilon_0$$

there exists a solution w to the nonlinear equation (6) with initial datum g and w is unique among all solutions with $\|w\|_{L^\infty(W^{1,\infty})} + \|w\|_{X(p)} \leq \varepsilon$. Moreover, this solution satisfies the estimate

$$\|w\|_{L^\infty(W^{1,\infty})} + \|w\|_{X(p)} \lesssim \|g\|_{W^{1,\infty}}.$$

Theorem 1 contains the first (conditional) uniqueness result for the multidimensional thin-film equation in a neighborhood of self-similar solutions. Since any solution to the thin-film equation is expected to converge towards the self-similar Smyth–Hill solution, our result can be considered as a uniqueness result for large times. Notice that the smallness of the Lipschitz norm of w can be translated back into the closeness of u to the stationary u_* . Indeed, $\|w\|_{W^{1,\infty}} \ll 1$ can be equivalently expressed as

$$\|u - u_*\|_{L^\infty(\mathcal{P}(u))} + \|\nabla u + x\|_{L^\infty(\mathcal{P}(v))} \ll 1$$

if $\mathcal{P}(u) = \{u > 0\}$ is the positivity set of u . Recall that $\nabla u_* = -x$ inside $B_1(0)$.

Our second result addresses the regularity of the unique solution found above.

Theorem 2. *Let w be the solution from Theorem 1. Then w is smooth and analytic in time and angular direction.*

It is clear that the smoothness of w immediately translates into the smoothness of u up to the boundary of its support. Moreover, the analyticity result particularly implies the analyticity of the level sets of u . Indeed, the level set of u at height $\lambda \geq 0$ is given by

$$\left\{ (t, x) : w(t, r, \phi) = \sqrt{r^2 + \frac{2\sqrt{\lambda}}{\gamma}} - 1 \right\}$$

if r and ϕ are the radial and angular coordinates, respectively. As a consequence, the temporal and tangential analyticity of w translates into the analyticity of the level sets of u . Notice that the zero-level set is nothing but the free boundary $\partial\{u = 0\}$, and thus, Theorem 2 proves the analyticity of the free boundary of solutions near self-similarity.

In the forthcoming paper [Seis \geq 2018], we will use the gained regularity in time for a construction of invariant manifolds that characterize the large-time asymptotic behavior at any order.

The proof of existence and uniqueness in Theorem 1 follows from a fixed-point argument and a maximal regularity theory for the linear equation. Analyticity and regularity are essentially consequences of an argument first introduced by Angenent [1990] and later improved by Koch and Lamm [2012].

1.5. Notation. One word about constants. In the major part of the subsequent analysis, we will not keep track of constants in inequalities but prefer to use the sloppy notation $a \lesssim b$ if $a \leq Cb$ for some universal constant C . Sometimes, however, we have to include constants like $e^{\pm Ct}$ when dealing with exponential growth or decay rates. In such cases, C will always be a positive constant which is generic in the sense that it will not depend on t or z , for instance. This constant might change from line to line, which allows us to write things like $e^{2Ct} \lesssim e^{Ct}$ even for large t .

2. The linear problem

Our goal in this section is the study of the initial value problem for the linear equation (7). In fact, our analysis also applies to the slightly more general equation

$$\begin{cases} \partial_t w + \mathcal{L}_\sigma^2 w + \mathcal{L}_\sigma w = f & \text{in } (0, \infty) \times B_1(0), \\ w(0, \cdot) = g & \text{in } B_1(0), \end{cases} \tag{9}$$

where \mathcal{L}_σ corresponds to the linearized porous-medium equation considered in [Seis 2014; 2015], defined by

$$\mathcal{L}_\sigma w = -\rho^{-\sigma} \nabla \cdot (\rho^{\sigma+1} \nabla w) = -\rho \Delta w + (\sigma + 1)z \cdot \nabla w$$

for any smooth function $w \in C^\infty(\bar{B})$, and $n \geq 0$ is arbitrary. The constant σ is originally chosen greater than -1 , but we will restrict our attention to the case $\sigma > 0$ for convenience.

Our notion of a weak solution is the following:

Definition 3 (weak solution). Let $0 < T \leq \infty$ and $f \in L^2((0, T); L^2_\sigma)$, $g \in L^2_\sigma$. We call w a weak solution to (9) if $w \in L^2((0, T); L^2_\sigma)$ with $\mathcal{L}w \in L^2((0, T); L^2_\sigma)$ solves

$$\begin{aligned} - \int_0^T \int \partial_t \zeta w \, d\mu_\sigma \, dt + \int_0^T \int \mathcal{L}_\sigma \zeta \mathcal{L}_\sigma w \, d\mu_\sigma \, dt + n \int_0^T \int \nabla \zeta \cdot \nabla w \, d\mu_{\sigma+1} \, dt \\ = \int_0^T \int \zeta f \, d\mu_\sigma \, dt + \int \zeta(0, \cdot) g \, d\mu_\sigma \end{aligned} \quad (10)$$

for all $\zeta \in C^\infty([0, \infty) \times \overline{B_1(0)})$ with $\text{spt } \zeta \subset [0, T) \times \overline{B_1(0)}$.

Here we have used the notation L^2_σ for Lebesgue space $L^2(\mu_\sigma)$ if μ_σ is the absolute continuous measure defined by

$$d\mu_\sigma = \rho^\sigma \, dx.$$

The Hilbert-space theory for (9) is relatively easy and will be developed in Section 2.3 below. In order to perform a perturbation argument on the nonlinear equation, however, we need to control the solution w in the Lipschitz norm. The function spaces $X(p)$ and $Y(p)$ introduced earlier are suitable for such an argument. In fact, our objective in this section is the following result for the linear equation (9).

Theorem 4. *Let $p > N + 4$ be given. Assume that $g \in W^{1,\infty}$. Then there exists a unique weak solution w to (9), and this solution satisfies the a priori estimate*

$$\|w\|_{W^{1,\infty}} + \|w\|_{X(p)} \lesssim \|f\|_{Y(p)} + \|g\|_{W^{1,\infty}}.$$

As mentioned earlier, a change from the Euclidean distance to a Carnot–Carathéodory distance suitable for the second-order operator \mathcal{L}_σ will be crucial for our subsequent analysis. In the following subsection we will recall some basic properties of the corresponding intrinsic volumes and balls, which were derived earlier in [Seis 2015]. Section 2.2 intends to provide some tools that allow us to switch from the spherical setting to the Cartesian one. Energy estimates are established in Section 2.3. In Section 2.4 we treat the homogeneous problem and derive Gaussian estimates. A bit of Calderón–Zygmund theory is provided in Section 2.5. Finally, Section 2.6 contains the theory for the inhomogeneous equation.

2.1. Intrinsic balls and volumes. In the following, we will collect some definitions and properties that are related to our choice of geometry and that will become relevant in the subsequent analysis of the linearized equation. Details and derivations can be found in [Seis 2015, Chapter 4].

It can be shown that the intrinsic balls $B_r^d(z)$ are equivalent to Euclidean balls in the sense that there exists a constant $C < \infty$ such that

$$B_{C^{-1}r\theta(r,z)}(z) \subset B_r^d(z) \subset B_{Cr\theta(r,z)}(z) \tag{11}$$

for any $z \in \overline{B_1(0)}$. Here θ is defined by

$$\theta(r, z) = r \vee \sqrt{\rho(z)}.$$

For the local estimates, it will be crucial to notice that

$$\sqrt{\rho(z_0)} \lesssim r \implies \sqrt{\rho(z)} \lesssim r \quad \text{for all } z \in B_r^d(z_0), \tag{12}$$

$$\sqrt{\rho(z_0)} \gg r \implies \rho(z) \sim \rho(z_0) \quad \text{for all } z \in B_r^d(z_0). \tag{13}$$

In particular, it holds that $\theta(r, \cdot) \sim \theta(r, z_0)$ in $B_r^d(z_0)$. Moreover, if $\sqrt{\rho(z_0)} \lesssim r$, and $z_0 \neq 0$, then (11) implies

$$B_{C^{-1}r^2}\left(\frac{z_0}{|z_0|}\right) \cap \overline{B_1(0)} \subset B_r^d(z_0) \subset B_{Cr^2}\left(\frac{z_0}{|z_0|}\right) \cap \overline{B_1(0)}. \tag{14}$$

We will sometimes write $|A|_\sigma = \mu_\sigma(A)$ for measurable sets A . The volume of an intrinsic ball can be calculated as follows:

$$|B_r^d(z)|_\sigma \sim r^N \theta(r, z)^{N+2\sigma}. \tag{15}$$

In particular, it holds that

$$\frac{|B_r^d(z)|_\sigma}{|B_r^d(z')|_\sigma} \lesssim \left(1 + \frac{d(z, z')}{r}\right)^{2N+2\sigma}. \tag{16}$$

2.2. Preliminary results. By the symmetry of \mathcal{L}_σ and the Cauchy–Schwarz inequality in L^2_σ , we have the interpolation

$$\|\nabla \xi\|_{\sigma+1}^2 = \int \xi \mathcal{L}_\sigma \xi \, d\mu_\sigma \leq \|\xi\|_\sigma \|\mathcal{L}_\sigma \xi\|_\sigma. \tag{17}$$

For further reference, we also quote the maximal regularity estimate

$$\|\nabla w\|_\sigma + \|\nabla^2 w\|_{\sigma+2} \lesssim \|\mathcal{L}_\sigma w\|_\sigma; \tag{18}$$

see [Seis 2015, Lemma 4.6].

Close to the boundary, the operator \mathcal{L}_σ can be approximated with the linear operator studied in [Kienzler 2016],

$$\tilde{\mathcal{L}}_\sigma w = -z_N^{-\sigma} \nabla \cdot (z_N^{\sigma+1} \nabla w) = -z_N \Delta w - (\sigma + 1) \partial_N w.$$

This operator is considered on the half-space $\mathbb{R}_+^N = \{z \in \mathbb{R}^N : z > 0\}$. Defining $\tilde{\mu}_\sigma = z_N^\sigma dz$ for any $\sigma > -1$ and using the notation $\|\cdot\|_\sigma$ for the norm on $L^2(\tilde{\mu}_\sigma)$ (slightly abusing notation), we have, analogously to (18), that

$$\|\nabla w\|_\sigma + \|\nabla^2 w\|_{\sigma+2} \lesssim \|\tilde{\mathcal{L}}_\sigma w\|_\sigma. \tag{19}$$

Our first lemma shows how the second-order elliptic equation can be transformed onto a problem on the half-space.

Lemma 5. *Suppose that*

$$\mathcal{L}_\sigma w = \xi$$

for some w such that $\text{spt}(w) \subset \overline{B_1(0)} \cap B_\varepsilon(e_N)$ for some $\varepsilon > 0$. Let $\Phi(z) = \sqrt{1 - |z'|^2}e_N - z$ for $z \in \overline{B_1(0)}$. If ε is sufficiently small, then Φ is a diffeomorphism on $\overline{B_1(0)} \cap B_\varepsilon(e_N)$. Moreover, \tilde{w} defined by $\tilde{w}(\Phi(z)) = w(z)$ solves the equation

$$\tilde{\mathcal{L}}_\sigma \tilde{w} = \tilde{\xi},$$

where $A = A(\tilde{z}) \in \mathbb{R}^{N \times N}$ and $b = b(\tilde{z}) \in \mathbb{R}^N$ are smooth functions with

$$\tilde{\xi} = \xi \circ \Phi^{-1} + A : \tilde{\nabla}^2 \tilde{w} + b \cdot \tilde{\nabla} \tilde{w}$$

with $|A(\tilde{z})| \lesssim |\tilde{z}|^2$, $|b(\tilde{z})| \lesssim |\tilde{z}|$.

Proof. It is clear that Φ is a diffeomorphism from a small ball around $z = e_N$ into a small neighborhood around the origin in \mathbb{R}_+^N . Moreover, a direct calculation and Taylor expansion show that

$$\begin{aligned} \Delta w &= \tilde{\Delta} \tilde{w} + \frac{2\tilde{z}'}{\sqrt{1 - |\tilde{z}'|^2}} \cdot \tilde{\nabla}' \tilde{\partial}_N \tilde{w} + \frac{|\tilde{z}'|^2}{1 - |\tilde{z}'|^2} \tilde{\partial}_N^2 \tilde{w} - \left(\frac{N-1}{\sqrt{1 - |\tilde{z}'|^2}} + \frac{|\tilde{z}'|^2}{(1 - |\tilde{z}'|^2)^{3/2}} \right) \tilde{\partial}_N \tilde{w} \\ &= \tilde{\Delta} \tilde{w} + A_1(\tilde{z}) : \tilde{\nabla}^2 \tilde{w} + b_1(\tilde{z}) \cdot \tilde{\nabla} \tilde{w}, \\ z \cdot \nabla w &= -\sqrt{1 - |\tilde{z}'|^2} \tilde{\partial}_N \tilde{w} + \tilde{z}' \cdot \tilde{\nabla}' \tilde{w} + \left(\tilde{z}_N - \frac{|\tilde{z}'|^2}{\sqrt{1 - |\tilde{z}'|^2}} \right) \tilde{\partial}_N \tilde{w} \\ &= -\tilde{\partial}_N \tilde{w} + b_2(\tilde{z}) \cdot \tilde{\nabla} \tilde{w}, \\ \rho(z) &= \sqrt{1 - |\tilde{z}'|^2} \tilde{z}_N - \frac{1}{2} \tilde{z}_N^2 = \tilde{z}_N + c(\tilde{z}), \end{aligned}$$

where $|A_1(\tilde{z})| \lesssim |\tilde{z}|$, $|b_1(\tilde{z})| \lesssim 1$, $|b_2(\tilde{z})| \lesssim |\tilde{z}|$, $|c(\tilde{z})| \lesssim |\tilde{z}|^2$. We easily infer the statement. □

A helpful tool in the derivation of the L_σ^2 maximal regularity estimates for our parabolic problem (9) will be the following estimate for the Cartesian problem.

Lemma 6. *Suppose \tilde{w} is a smooth solution of the equation*

$$\tilde{\mathcal{L}}_\sigma \tilde{w} = \tilde{\xi}$$

for some smooth $\tilde{\xi}$. Then

$$\|\tilde{\nabla}^2 \tilde{w}\|_\sigma + \|\tilde{\nabla}^3 \tilde{w}\|_{\sigma+2} + \|\tilde{\nabla}^4 \tilde{w}\|_{\sigma+4} \lesssim \|\tilde{\nabla} \tilde{\xi}\|_\sigma + \|\tilde{\nabla}^2 \tilde{\xi}\|_{\sigma+2}.$$

Proof. We start with the derivation of higher-order tangential regularity. Since $\tilde{\mathcal{L}}_\sigma$ commutes with $\tilde{\partial}_i$ for any $i \in \{1, \dots, N-1\}$, differentiation yields $\tilde{\mathcal{L}}_\sigma \tilde{\partial}_i \tilde{w} = \tilde{\partial}_i \tilde{\xi}$, and thus via (19),

$$\|\tilde{\nabla} \tilde{\partial}_i \tilde{w}\|_\sigma + \|\tilde{\nabla}^2 \tilde{\partial}_i \tilde{w}\|_{\sigma+2} \lesssim \|\tilde{\nabla} \tilde{\xi}\|_\sigma.$$

We take second-order derivatives in tangential direction and rewrite the resulting equation as $\tilde{\mathcal{L}}_{\sigma+2} \tilde{\partial}_{ij} \tilde{w} = \tilde{\partial}_{ij} \tilde{\xi} - 2\tilde{\partial}_{ijN} \tilde{w}$, where $i, j \in \{1, \dots, N - 1\}$. From the above estimate and (19) we obtain

$$\|\tilde{\nabla} \tilde{\partial}_{ij} \tilde{w}\|_{\sigma+2} + \|\tilde{\nabla}^2 \tilde{\partial}_{ij} \tilde{w}\|_{\sigma+4} \lesssim \|\tilde{\nabla} \tilde{\xi}\|_{\sigma} + \|\tilde{\nabla}^2 \tilde{\xi}\|_{\sigma+2}.$$

Transversal derivatives do not commute with $\tilde{\mathcal{L}}_{\sigma}$. Instead, it holds that $\tilde{\partial}_N \tilde{\mathcal{L}}_{\sigma} = \tilde{\mathcal{L}}_{\sigma+1} \tilde{\partial}_N - \tilde{\Delta}'$. A double differentiation in transversal direction yields thus $\tilde{\mathcal{L}}_{\sigma+2} \tilde{\partial}_N^2 \tilde{w} = \tilde{\partial}_N^2 \tilde{\xi} + 2\tilde{\Delta}' \tilde{\partial}_N \tilde{w}$. We invoke (19) and obtain with the help of the previous estimate

$$\|\tilde{\nabla} \tilde{\partial}_N^2 \tilde{w}\|_{\sigma+2} + \|\tilde{\nabla}^2 \tilde{\partial}_N^2 \tilde{w}\|_{\sigma+4} \lesssim \|\tilde{\nabla} \tilde{\xi}\|_{\sigma} + \|\tilde{\nabla}^2 \tilde{\xi}\|_{\sigma+2}.$$

Finally, the control of $\tilde{\partial}_N^2 \tilde{w}$ follows by using the transversally differentiated equation in the sense of $(\sigma + 2) \tilde{\partial}_N^2 \tilde{w} = -\tilde{\partial}_N \tilde{\xi} - \tilde{z}_N \tilde{\Delta} \tilde{\partial}_N \tilde{w} - \tilde{\Delta}' \tilde{w}$ and the previous bounds. □

2.3. Energy estimates. In this subsection, we derive the basic well-posedness result, maximal regularity estimates and local estimates in the Hilbert-space setting. We start with existence and uniqueness.

Lemma 7. *Let $0 < T \leq \infty$ and $f \in L^2((0, T); L^2_{\sigma})$, $g \in L^2_{\sigma}$. Then there exists a unique weak solution to (9). Moreover, it holds that*

$$\sup_{(0,T)} \|w\|_{\sigma}^2 + \int_0^T \|\nabla w\|_{\sigma}^2 dt + \int_0^T \|\nabla^2 w\|_{\sigma+2}^2 dt \lesssim \int_0^T \|f\|_{\sigma}^2 dt + \|g\|_{\sigma}^2.$$

Proof. Existence of weak solutions can be proved, for instance, by using an implicit Euler scheme. Indeed, thanks to (18), it is easily seen that for any $h > 0$ and $f \in L^2_{\sigma}$ the elliptic problem

$$\frac{1}{h} w + (\mathcal{L}_{\sigma}^2 + n\mathcal{L}_{\sigma}) w = f$$

has a unique solution w satisfying $\mathcal{L}_{\sigma} w \in L^2_{\sigma}$; see [Seis 2014, Appendix] for the analogous second-order problem. This solution satisfies the a priori estimate

$$\frac{1}{\sqrt{h}} \|w\|_{\sigma} + \|\mathcal{L}_{\sigma} w\|_{\sigma} \lesssim \|f\|_{\sigma}.$$

With these insights, it is an exercise to construct time-discrete solutions to (9), and standard compactness arguments allow for passing to the limit, both in the equation and in the estimate. In view of the linearity of the equation, uniqueness follows immediately. □

Our next result is a maximal regularity estimate for the homogeneous problem.

Lemma 8. *Let w be a solution to the initial value problem (9) with $g = 0$ and $f \in L^2((0, T); L^2_{\sigma})$ for some $0 < T \leq \infty$. Then the mappings $t \mapsto \|w(t)\|_{\sigma}$ and $t \mapsto \|\nabla w(t)\|_{\sigma+1}$ are continuous on $[0, T]$ and $\nabla^2 w, \nabla^3 w, \rho \nabla^4 w \in L^2((0, T); L^2_{\sigma})$ with*

$$\|\partial_t w\|_{L^2(L^2_{\sigma})} + \|\nabla^2 w\|_{L^2(L^2_{\sigma})} + \|\nabla^3 w\|_{L^2(L^2_{\sigma+2})} + \|\nabla^4 w\|_{L^2(L^2_{\sigma+4})} \lesssim \|f\|_{L^2(L^2_{\sigma})}.$$

In the statement of the lemma, we have written $L^2(L^2_{\sigma})$ for $L^2((0, T); L^2_{\sigma})$.

Proof. We perform a quite formal argument that can be made rigorous by using the customary approximation procedures. Choosing $\zeta = \chi_{[t_1, t_2]} \partial_t w$ as a test function in (10), we have the identity

$$\begin{aligned} \int_{t_1}^{t_2} \|\partial_t w\|_\sigma^2 dt + \frac{1}{2} \|\mathcal{L}_\sigma w(t_2)\|_\sigma^2 + \frac{1}{2} n \|\nabla w(t_2)\|_{\sigma+1} \\ = \int_{t_1}^{t_2} \int f \partial_t w d\mu_\sigma dt + \frac{1}{2} \|\mathcal{L}_\sigma w(t_1)\|_\sigma^2 + \frac{1}{2} n \|\nabla w(t_1)\|_{\sigma+1}. \end{aligned}$$

Combining this bound with the estimate from Lemma 7, we deduce that the mappings $t \mapsto \|w(t)\|_\sigma$ and $t \mapsto \|\nabla w(t)\|_{\sigma+1}$ are continuous. Moreover,

$$\int_0^T \|\partial_t w\|_\sigma^2 dt \lesssim \int_0^T \|f\|_\sigma^2 dt$$

because $g = 0$. A similar estimate holds for $\mathcal{L}_\sigma w$ by virtue of Lemma 7, and the statement thus follows upon proving

$$\|\nabla^2 w\|_\sigma + \|\nabla^3 w\|_{\sigma+2} + \|\nabla^4 w\|_{\sigma+4} \lesssim \|\xi\|_\sigma + \|\nabla \xi\|_\sigma + \|\nabla^2 \xi\|_{\sigma+2} \tag{20}$$

for any solution of the elliptic problem $\mathcal{L}_\sigma w = \xi$, because the right-hand side is bounded by $\|f\|_\sigma$ thanks to (18) and Lemma 7. It is not difficult to obtain estimates in the interior of $B_1(0)$. For instance, since

$$\mathcal{L}_{\sigma+2} \partial_i w = \partial_i \xi - z_i \Delta w + 2z \cdot \nabla \partial_i w - (\sigma + 1) \partial_i w$$

is bounded in $L_{\sigma+2}^2$ and because $L_\sigma^2 \subset L_{\sigma+2}^2$, an application of (18) yields

$$\|\nabla^2 w\|_{\sigma+2} + \|\nabla^3 w\|_{\sigma+4} \lesssim \|\partial_i \xi\|_\sigma + \|\nabla w\|_\sigma + \|\nabla^2 w\|_{\sigma+2} \lesssim \|\xi\|_\sigma + \|\nabla \xi\|_\sigma. \tag{21}$$

Since $\rho \sim 1$ in the interior of $B_1(0)$, this estimate gives the desired control of the second- and third-order derivatives in the interior of $B_1(0)$. Fourth-order derivatives can be estimated similarly.

To derive estimates at the boundary of $B_1(0)$, it is convenient to locally flatten the boundary. For this purpose, we localize the equation with the help of a smooth cut-off function η that is supported in a small ball centered at a given boundary point, say e_N ,

$$\mathcal{L}_\sigma(\eta w) = \eta \xi - 2\rho \nabla \eta \cdot \nabla w - \rho \Delta \eta w + (\sigma + 1) z \cdot \nabla \eta w =: \tilde{\xi}.$$

A short computation shows that

$$\|\tilde{\xi}\|_\sigma + \|\nabla \tilde{\xi}\|_\sigma + \|\nabla^2 \tilde{\xi}\|_{\sigma+2} \lesssim \|\xi\|_\sigma + \|\nabla \xi\|_\sigma + \|\nabla^2 \xi\|_{\sigma+2},$$

where we have used (18) and (21) and the Hardy–Poincaré inequality $\|w\|_\sigma \lesssim \|\nabla w\|_{\sigma+1}$ from [Seis 2014, Lemma 3]. (For this, notice that we can assume that w has zero average because solutions to $\mathcal{L}_\sigma w = \xi$ are unique up to constants.) Establishing (20) for this localized equation is now a straightforward calculation based on the transformation from Lemma 5 and the a priori estimate from Lemma 6. A covering argument concludes the proof. \square

A crucial step in the derivation of the Gaussian estimates is the following local estimate.

Lemma 9. *Let $0 < \hat{\varepsilon} < \varepsilon < 1$ and $0 < \delta < \hat{\delta} < 1$ be given. Let w be a solution to the inhomogeneous equation (9). Then the following holds for any $z_0 \in \overline{B_1(0)}$, $\tau \geq 0$ and $0 < r \lesssim 1$:*

$$\begin{aligned} & \iint_Q (\partial_t w)^2 d\mu_\sigma dt + \frac{\theta(r, z_0)^4}{r^4} \iint_Q |\nabla^2 w|^2 d\mu_\sigma dt \\ & \quad + \frac{\theta(r, z_0)^2}{r^2} \iint_Q |\nabla^3 w|^2 d\mu_{\sigma+2} dt + \iint_Q |\nabla^4 w|^2 d\mu_{\sigma+4} dt \\ & \lesssim \iint_{\hat{Q}} f^2 d\mu_\sigma dt + \frac{1}{r^8} \iint_{\hat{Q}} w^2 + r^2 \theta(r, z_0)^2 |\nabla w|^2 d\mu_\sigma dt, \end{aligned}$$

where $Q = (\tau + \varepsilon r^2, \tau + r^2) \times B_{\delta r}^d(z_0)$ and $\hat{Q} = (\tau + \hat{\varepsilon} r^2, \tau + r^2) \times B_{\hat{\delta} r}^d(z_0)$.

Proof. Because (9) is invariant under time shifts, we may set $\tau = 0$. We start by recalling that

$$\mathcal{L}_\sigma(\eta w) = \eta \mathcal{L}_\sigma w - 2\rho \nabla \eta \cdot \nabla w + (\mathcal{L}_\sigma \eta) w$$

for any two functions η and w , and thus, via iteration,

$$\mathcal{L}_\sigma^2(\eta w) = \eta \mathcal{L}_\sigma^2 w - 2\rho \nabla \eta \cdot \nabla \mathcal{L}_\sigma w + \mathcal{L}_\sigma \eta \mathcal{L}_\sigma w - 2\mathcal{L}_\sigma(\rho \nabla \eta \cdot \nabla w) + \mathcal{L}_\sigma((\mathcal{L}_\sigma \eta) w).$$

In the sequel, we will choose η as a smooth cut-off function that is supported in the intrinsic space-time cylinder \hat{Q} , and is constantly 1 in the smaller cylinder Q . For such cut-off functions, it holds that $|\partial_t^k \partial_z^\beta \eta| \lesssim r^{-2k-|\beta|} \theta(r, z_0)^{-|\beta|}$. (Here and in the following, the dependency on $\varepsilon, \hat{\varepsilon}, \delta$ and $\hat{\delta}$ is neglected in the inequalities.) Then ηw solves the equation

$$\begin{aligned} \partial_t(\eta w) + \mathcal{L}_\sigma^2(\eta w) + n \mathcal{L}_\sigma(\eta w) &= \eta f + \partial_t \eta w - 4\mathcal{L}_\sigma(\rho \nabla \eta \cdot \nabla w) - 4\rho \nabla(\rho \nabla \eta) : \nabla^2 w \\ &\quad - 2\rho z \cdot \nabla \eta \Delta w - 2(\sigma + 1 + n)\rho \nabla \eta \cdot \nabla w + 2\mathcal{L}_\sigma(\rho \nabla \eta) \cdot \nabla w \\ &\quad + \mathcal{L}_\sigma((\mathcal{L}_\sigma \eta) w) + \mathcal{L}_\sigma \eta \mathcal{L}_\sigma w + n(\mathcal{L}_\sigma \eta) w. \end{aligned}$$

For simplicity, we denote the right-hand side by \tilde{f} . Testing against ηw and using the symmetry and nonnegativity properties of \mathcal{L}_σ and the fact that $\eta w = 0$ initially, we obtain the estimate

$$\iint (\mathcal{L}_\sigma(\eta w))^2 d\mu_\sigma dt \leq \iint \eta w \tilde{f} d\mu_\sigma dt.$$

A tedious but straightforward computation then yields

$$\begin{aligned} & \left| \int \eta w \tilde{f} d\mu_\sigma \right| \\ & \lesssim \left(r^2 \|\chi f\|_\sigma + \|\mathcal{L}_\sigma(\eta w)\|_\sigma + \frac{1}{r^2} \|\chi w\|_\sigma + \frac{\theta_0}{r} \|\chi \nabla w\|_\sigma \right) \left(\frac{1}{r^2} \|\chi w\|_\sigma + \frac{\theta_0}{r} \|\chi \nabla w\|_\sigma \right), \end{aligned} \tag{22}$$

where $\chi = \chi_{\text{spt}(\eta)}$ and $\theta_0 = \theta(r, z_0)$, which in turn implies

$$\begin{aligned} & \iint_Q |\nabla w|^2 d\mu_\sigma dt + \iint_Q |\nabla^2 w|^2 d\mu_{\sigma+2} dt \\ & \lesssim r^4 \iint_{\hat{Q}} f^2 d\mu_\sigma dt + \frac{1}{r^4} \iint_{\hat{Q}} w^2 + r^2 \theta_0^2 |\nabla w|^2 d\mu_\sigma dt \end{aligned} \tag{23}$$

via (18) and Young’s inequality. We will show the argument for (22) for the leading-order terms only. For instance, from the symmetry of \mathcal{L}_σ and the fact that $|\rho\nabla\eta| \lesssim \theta_0/r$, we deduce that

$$\left| \int \eta w \mathcal{L}_\sigma(\rho\nabla\eta \cdot \nabla w) d\mu_\sigma \right| \lesssim \frac{\theta_0}{r} \|\mathcal{L}_\sigma(\eta w)\|_\sigma \|\chi\nabla w\|_\sigma.$$

Similarly, by integration by parts we calculate

$$\left| \int \eta w \rho \nabla(\rho\nabla\eta) : \nabla^2 w d\mu_\sigma \right| \lesssim \frac{\theta_0^2}{r^2} \int |\nabla(\eta w)| |\nabla w| d\mu_\sigma + \frac{1}{r^2} \int \eta |w| |\nabla w| d\mu_\sigma,$$

and conclude observing that $|\nabla(\eta w)| \lesssim \theta_0^{-1} r^{-1} |w| + |\nabla w|$. The remaining terms of \tilde{f} can be estimated similarly.

To gain control over the third-order derivatives of ηw , we test the equation with $\mathcal{L}_\sigma(\eta w)$. With the help of the symmetry and nonnegativity properties of \mathcal{L}_σ , we obtain the estimate

$$\iint |\nabla \mathcal{L}_\sigma(\eta w)|^2 d\mu_{\sigma+1} dt \leq \iint \mathcal{L}_\sigma(\eta w) \tilde{f} d\mu_\sigma dt.$$

We have to find suitable estimates for the inhomogeneity term. Because we have to make use of the previous bound (23), we have to shrink the cylinders Q and \hat{Q} such that the new function η is supported in the set where the old η was constantly 1. We claim that

$$\iint_Q |\nabla \mathcal{L}_\sigma w|^2 d\mu_{\sigma+1} dt \lesssim r^2 \iint_{\hat{Q}} f^2 d\mu_\sigma dt + \frac{1}{r^6} \iint_{\hat{Q}} w^2 + r^2 \theta_0^2 |\nabla w| d\mu_\sigma dt. \tag{24}$$

Again, we will provide the argument for the leading-order terms only. We use the symmetry of \mathcal{L}_σ , the bounds on derivatives of η and the scaling of ρ , see (12) and (13), to estimate

$$\begin{aligned} \left| \int \mathcal{L}_\sigma(\eta w) \mathcal{L}_\sigma(\rho\nabla\eta \cdot \nabla w) d\mu_\sigma \right| &= \left| \int \nabla \mathcal{L}_\sigma(\eta w) \cdot \nabla(\rho\nabla\eta \cdot \nabla w) d\mu_{\sigma+1} \right| \\ &\lesssim \|\nabla \mathcal{L}_\sigma(\eta w)\|_{\sigma+1} \left(\frac{\theta_0}{r^2} \|\chi\nabla w\|_\sigma + \frac{1}{r} \|\chi\nabla^2 w\|_{\sigma+2} \right). \end{aligned}$$

Similarly,

$$\left| \int \mathcal{L}_\sigma(\eta w) \rho \nabla(\rho\nabla\eta) : \nabla^2 w d\mu_\sigma \right| \lesssim \frac{1}{r^2} \|\mathcal{L}_\sigma(\eta w)\|_\sigma \|\chi\nabla^2 w\|_{\sigma+2}.$$

The estimates of the remaining terms have a similar flavor. We deduce (24) with the help of Young’s inequality and (23).

The estimate (24) is beneficial as it allows us to estimate \tilde{f} in L^2_σ . This time, it is enough to study the term that involves the third-order derivatives of w . We rewrite

$$\begin{aligned} \mathcal{L}_\sigma(\rho\partial_i\eta\partial_i w) &= \rho\partial_i\eta\mathcal{L}_\sigma\partial_i w - 2\rho\nabla(\rho\partial_i\eta) \cdot \nabla\partial_i w + \mathcal{L}_\sigma(\rho\partial_i\eta)\partial_i w, \\ \mathcal{L}_\sigma\partial_i w &= \partial_i\mathcal{L}_\sigma w - z_i\Delta w - (\sigma + 1)\partial_i w, \end{aligned}$$

and estimate

$$\begin{aligned} \|\mathcal{L}(\rho\partial_i\eta\partial_i w)\|_\sigma &\lesssim \frac{1}{r\theta_0}\|\chi\mathcal{L}_\sigma\partial_i w\|_{\sigma+2} + \frac{1}{r^2}\|\chi\nabla^2 w\|_{\sigma+2} + \frac{\theta_0}{r^3}\|\chi\nabla w\|_\sigma \\ &\lesssim \frac{1}{r}\|\chi\partial_i\mathcal{L}_\sigma w\|_{\sigma+1} + \frac{1}{r^2}\|\chi\nabla^2 w\|_{\sigma+2} + \frac{\theta_0}{r^3}\|\chi\nabla w\|_\sigma. \end{aligned}$$

Upon redefining Q and \hat{Q} as in the derivation of (24), an application of (23) and (24) then yields

$$\int_{\varepsilon r^2}^{r^2} \|\mathcal{L}(\rho\partial_i\eta\partial_i w)\|_\sigma^2 dt \lesssim \iint_{\hat{Q}} f^2 d\mu_\sigma dt + \frac{1}{r^8} \iint_{\hat{Q}} w^2 + r^2\theta_0^2|\nabla w|^2 d\mu_\sigma dt.$$

The remaining terms of \tilde{f} can be estimated in a similar way. Applying the energy estimate from Lemma 8 to the evolution equation for ηw , we thus deduce

$$\begin{aligned} \iint_Q (\partial_t w)^2 d\mu_\sigma dt + \iint_Q |\nabla^2 w|^2 d\mu_\sigma dt + \iint_Q |\nabla^3 w|^2 d\mu_{\sigma+2} dt + \iint_Q |\nabla^4 w|^2 d\mu_{\sigma+4} dt \\ \lesssim \iint_{\hat{Q}} f^2 d\mu_\sigma dt + \frac{1}{r^8} \iint_{\hat{Q}} w^2 + r^2\theta_0^2|\nabla w|^2 d\mu_\sigma dt. \end{aligned} \quad (25)$$

Notice that the above bound on the second-order derivatives and (23) together imply

$$\frac{\theta_0^4}{r^4} \iint_Q |\nabla^2 w|^2 d\mu_\sigma dt \lesssim \iint_{\hat{Q}} f^2 d\mu_\sigma dt + \frac{1}{r^8} \iint_{\hat{Q}} w^2 + r^2\theta_0^2|\nabla w|^2 d\mu_\sigma dt.$$

Similarly, we can produce the factor θ_0^2/r^2 in front of the integral containing the third-order derivatives. Indeed, because $\nabla\mathcal{L}_\sigma w = \mathcal{L}_{\sigma+1}\nabla w + z\Delta w - \nabla^2 w z + (\sigma + 1)\nabla w$, the bound (24) yields

$$\begin{aligned} \iint_Q |\mathcal{L}_{\sigma+1}\nabla w|^2 d\mu_{\sigma+1} dt \\ \lesssim r^2 \iint_{\hat{Q}} f^2 d\mu_\sigma dt + \frac{1}{r^6} \iint_{\hat{Q}} w^2 + r^2\theta_0^2|\nabla w|^2 d\mu_\sigma dt + \iint_{\hat{Q}} |\nabla^2 w|^2 d\mu_{\sigma+1} dt + \iint_{\hat{Q}} |\nabla w|^2 d\mu_{\sigma+1} dt. \end{aligned}$$

The second-order term on the right-hand side is controlled with the help of the Cauchy–Schwarz inequality, (23) and (25). The first-order term is of higher order as a consequence of (23). It remains to invoke (18) to the effect that

$$\iint_Q |\nabla^3 w|^2 d\mu_{\sigma+3} dt \lesssim r^2 \iint_{\hat{Q}} f^2 d\mu_\sigma dt + \frac{1}{r^6} \iint_{\hat{Q}} w^2 + r^2\theta_0^2|\nabla w|^2 d\mu_\sigma dt.$$

Combining the latter with (25) yields the statement of the lemma. □

2.4. Estimates for the homogeneous equation. In this subsection, we study the initial value problem for the homogeneous equation

$$\begin{cases} \partial_t w + \mathcal{L}_\sigma^2 w + n\mathcal{L}_\sigma w = 0 & \text{in } (0, \infty) \times B_1(0), \\ w(0, \cdot) = g & \text{in } B_1(0). \end{cases} \quad (26)$$

Our first goal is a pointwise higher-order regularity estimate.

Lemma 10. *Let $0 < \varepsilon < 1$ and $0 < \delta < 1$ be given. Let w be a solution to the homogeneous equation (26). If $\varepsilon, \delta \in (0, 1)$ and δ is sufficiently small, then the following holds for any $z_0 \in \overline{B_1(0)}$, $\tau \geq 0$ and $0 < r \lesssim 1$:*

$$|\partial_t^k \partial_z^\beta w(t, z)|^2 \lesssim \frac{r^{-8k-2|\beta|} \theta(r, z_0)^{-2|\beta|}}{r^4 |B_r^d(z_0)|_\sigma} \int_\tau^{\tau+r^4} \int_{B_r^d(z_0)} w^2 + r^2 \theta(r, z_0)^2 |\nabla w|^2 d\mu_\sigma dt$$

for any $(t, z) \in (\tau + \varepsilon r^4, \tau + r^4] \times B_{\delta r}^d(z_0)$.

Proof. The lemma is a consequence of the local higher-order regularity estimate

$$\iint_Q (\partial_t^k \partial_z^\beta w)^2 d\mu_\sigma dt \lesssim r^{-8k-2|\beta|} \theta(r, z_0)^{-2|\beta|} \iint_{\hat{Q}} w^2 + r^2 \theta(r, z_0)^2 |\nabla w|^2 d\mu_\sigma dt, \tag{27}$$

where Q and \hat{Q} are defined as in Lemma 9, and a Morrey estimate in the weighted space $L^2(\mu_\sigma)$; see, e.g., [Seis 2015, Lemma 4.9]. Notice that (27) is trivial for $(k, |\beta|) \in \{(0, 1), (0, 2)\}$. In the following, we write $\theta_0 = \theta(r, z_0)$.

To prove (27) for general choices of k and β , it is convenient to consider separately the two cases $\sqrt{\rho(z_0)} \lesssim r$ and $\sqrt{\rho(z_0)} \gg r$. The second case is relatively simple: Since $\rho \sim \rho(z_0)$ by (13) in both Q and \hat{Q} , we deduce (27) in the cases $(k, |\beta|) \in \{(1, 0), (0, 2), (0, 3), (0, 4)\}$ directly from Lemma 9 (with $f = 0$). To gain control on higher-order derivatives, we differentiate with respect to z_i ,

$$\begin{aligned} \partial_t \partial_i w + \mathcal{L}_\sigma^2 \partial_i w + n \mathcal{L}_\sigma \partial_i w \\ = -z_i \Delta \mathcal{L}_\sigma w - (\sigma + 1) \partial_i \mathcal{L}_\sigma w - \mathcal{L}_\sigma(z_i \Delta w) - (\sigma + 1) \mathcal{L}_\sigma \partial_i w - n z_i \Delta w - n(\sigma + 1) \partial_i w. \end{aligned}$$

Denoting by \tilde{f} the right-hand side of this identity, applying Lemma 9 yields the estimate

$$\iint_Q |\nabla^4 \partial_i w|^2 d\mu_{\sigma+4} dt \lesssim \iint_{\hat{Q}} \tilde{f}^2 d\mu_\sigma dt + \frac{1}{r^8} \iint_{\hat{Q}} (\partial_i w)^2 + r^2 \theta_0^2 |\nabla \partial_i w|^2 d\mu_\sigma dt.$$

We invoke the previously derived bound and the fact that $\rho \sim \rho(z_0)$ to conclude the statement in the case $(k, |\beta|) = (0, 5)$. Higher-order derivatives are controlled similarly via iteration.

The proof in the case $\rho(z_0) \lesssim r$ is lengthy and tedious. As similar results have been recently obtained in [John 2015; Kienzler 2016; Seis 2015] and most the involved tools have been already applied earlier in this paper, we will only outline the argument in the following. Thanks to (14), it is enough to study the situation where $z_0 \in \partial B_1(0)$, and upon shrinking δ , we may assume that Φ constructed in Lemma 5 is a diffeomorphism from $B_{\delta r}^2(z_0)$ onto a subset of the half-space. Under Φ , the homogeneous equation (26) transforms into

$$\partial_t \tilde{w} + \tilde{\mathcal{L}}_\sigma^2 \tilde{w} + n \tilde{\mathcal{L}}_\sigma \tilde{w} = \tilde{f},$$

where \tilde{f} is of higher-order at the boundary. Because $\tilde{\mathcal{L}}_\sigma$ commutes with tangential derivatives $\tilde{\partial}_i$ for $i \in \{1, \dots, N - 1\}$, control on higher-order tangential derivatives is deduced from Lemma 9. To obtain control on vertical derivatives, we recall that $\tilde{\partial}_N \tilde{\mathcal{L}}_\sigma = \tilde{\mathcal{L}}_{\sigma+1} \tilde{\partial}_N - \tilde{\Delta}'$. Arguing as in the proof of Lemma 6 gives the desired estimates. Again, bounds on higher-order derivatives and mixed derivatives are obtained by iteration. □

For the proof of the Gaussian estimates and the Whitney-measure estimates for the homogeneous problem, it is convenient to introduce a family of auxiliary functions $\chi_{a,b} : \overline{B_1(0)} \times \overline{B_1(0)} \rightarrow \mathbb{R}$, given by

$$\chi_{a,b}(z, z_0) = \frac{a\hat{d}(z, z_0)^2}{\sqrt{b^2 + \hat{d}(z, z_0)^2}},$$

where $a, b \in \mathbb{R}$ are given parameters, and

$$\hat{d}(z, z_0)^2 = \frac{|z - z_0|^2}{\sqrt{\rho(z)^2 + \rho(z_0)^2 + |z - z_0|^2}} \sim d(z, z_0)^2.$$

It can be verified by a short computation that $\rho|\nabla_z \hat{d}^2|^2 \lesssim \hat{d}$ and $\rho|\nabla_z^2 \hat{d}^2| \lesssim 1$, with the consequence that

$$\sqrt{\rho(z)}|\nabla_z \chi_{a,b}(z, z_0)| \lesssim |a|, \tag{28}$$

$$\rho(z)|\nabla_z^2 \chi_{a,b}(z, z_0)| \lesssim \frac{|a|}{|b|} \tag{29}$$

uniformly in $z, z_0 \in \overline{B_1(0)}$. Because g is conformally flat with $g \sim \rho^{-1}(dz)^2$, the gradient ∇_g on (\mathcal{M}, g) obeys the scaling $\nabla_g \sim \rho\nabla$, and thus (28) can be rewritten as $\sqrt{g}(\nabla_g \chi, \nabla_g \chi) \lesssim |a|$ (where we have dropped the indices and z_0). The latter implies that $\chi = \chi_{a,b}(\cdot, z_0)$ is Lipschitz with respect to the intrinsic topology; that is,

$$|\chi_{a,b}(z, z_0) - \chi_{a,b}(z', z_0)| \lesssim |a| d(z, z'). \tag{30}$$

We derive some new weighted energy estimates.

Lemma 11. *Let w be the solution to the homogeneous equation (26). Let $a, b \in \mathbb{R}$ and $z_0 \in \overline{B_1(0)}$ be given. Define $\chi = \chi_{a,b}(\cdot, z_0)$. Then there exists a constant $C > 0$ such that for any $T > 0$ it holds that*

$$\sup_{[0,T]} \int e^{2\chi} w^2 d\mu_\sigma + \int_0^T \int e^{2\chi} |\nabla w|^2 + (\mathcal{L}_\sigma(e^\chi w))^2 d\mu_\sigma dt \lesssim e^{C(a^2/b^2 + a^4)T} \int e^{2\chi} g^2 d\mu_\sigma.$$

Proof. The quantity $e^\chi w$ evolves according to

$$\begin{aligned} & \partial_t(e^\chi w) + \mathcal{L}_\sigma^2(e^\chi w) + n\mathcal{L}_\sigma(e^\chi w) \\ &= -2\rho\nabla e^\chi \cdot \nabla \mathcal{L}_\sigma w + \mathcal{L}_\sigma e^\chi \mathcal{L}_\sigma w - 2\mathcal{L}_\sigma(\rho\nabla e^\chi \cdot \nabla w) + \mathcal{L}_\sigma((\mathcal{L}_\sigma e^\chi)w) - 2n\rho\nabla e^\chi \cdot \nabla w + n(\mathcal{L}_\sigma e^\chi)w. \end{aligned}$$

Denoting the right-hand side by \tilde{f} and testing with $e^\chi w$ yields

$$\frac{d}{dt} \frac{1}{2} \int (e^\chi w)^2 d\mu_\sigma + \int (\mathcal{L}_\sigma(e^\chi w))^2 d\mu_\sigma + n \int |\nabla(e^\chi w)|^2 d\mu_{\sigma+1} = \int e^\chi w \tilde{f} d\mu_\sigma, \tag{31}$$

where we have used once more the symmetry of \mathcal{L}_σ . We claim that the term on the right can be estimated as follows:

$$\int e^\chi w \tilde{f} d\mu_\sigma \lesssim \varepsilon(\|\mathcal{L}_\sigma(e^\chi w)\|_\sigma^2 + \|\nabla(e^\chi w)\|_{\sigma+1}) + \left(1 + \frac{a^2}{b^2} + a^4\right) \|e^\chi w\|_\sigma^2, \tag{32}$$

where ε is some small constant that allows us to absorb the first two terms in the left-hand side of the energy estimate above. Indeed, multiple integrations by parts and the bounds (28) and (29) yield that the left-hand side of (31) is bounded by

$$\begin{aligned} &|a| \|\mathcal{L}_\sigma \zeta\|_\sigma \|\nabla \zeta\|_{\sigma+1} + a^2 \|\nabla \zeta\|_{\sigma+1}^2 + \left(|a| + \frac{a^2}{|b|} + |a|^3\right) \|\nabla \zeta\|_{\sigma+1} \|\zeta\|_\sigma \\ &\quad + a^2 \|\nabla \zeta\|_{\sigma+1} \|\zeta\|_{\sigma-1} + \left(\frac{|a|}{|b|} + a^2\right) \|\mathcal{L}_\sigma \zeta\|_\sigma \|\zeta\|_\sigma + \left(1 + \frac{a^2}{b^2} + a^4\right) \|\zeta\|_\sigma^2 \\ &\quad + \left(\frac{a^2}{|b|} + |a|^3\right) \|\zeta\|_{\sigma-1} \|\zeta\|_\sigma + |a| \|\mathcal{L}_\sigma \zeta\|_\sigma \|\zeta\|_{\sigma-1} + a^2 \|\zeta\|_{\sigma-1}^2, \end{aligned}$$

where we have set $\zeta = e^\chi w$. We next claim that

$$\|\zeta\|_{\sigma-1} \lesssim \|\zeta\|_\sigma + \|\nabla \zeta\|_{\sigma+1}. \tag{33}$$

Indeed, recall the Hardy–Poincaré inequality

$$\left\| \zeta - \int \zeta \, d\mu_{\tilde{\sigma}-1} \right\|_{\tilde{\sigma}-1} \lesssim \|\nabla \zeta\|_{\sigma+1},$$

see [Seis 2014, Lemma 3], which holds true for any $\tilde{\sigma} \geq \sigma$, because $\sigma > 0$. In particular, $\|\zeta\|_{\sigma-1} \lesssim \|\nabla \zeta\|_{\sigma+1} + \left| \int \zeta \, d\mu_{\sigma-1} \right|$. Notice that for any $\alpha \in (0, \tilde{\sigma})$, it holds that

$$\left| \int \zeta \, d\mu_{\tilde{\sigma}-1} \right| = \left| \int \zeta \rho^\alpha \, d\mu_{\tilde{\sigma}-1-\alpha} \right| \lesssim \left(\int \zeta^2 \rho \, d\mu_{\tilde{\sigma}-1-\alpha} \right)^{1/2} \lesssim \|\zeta\|_{\tilde{\sigma}-1+\alpha}$$

by Jensen’s inequality because $\mu_{\tilde{\sigma}-1+\alpha}$ is a finite measure. Applying the previous two estimates iteratively yields (33). Hence, combining (33) and the interpolation inequality (17) with the bound on the inhomogeneity and using Young’s inequality yields (32).

Now (31) and (32) imply for ε sufficiently small that

$$\frac{d}{dt} \int (e^\chi w)^2 \, d\mu_\sigma + \int (\mathcal{L}_\sigma(e^\chi w))^2 \, \mu_\sigma \lesssim \left(1 + \frac{a^2}{b^2} + a^4\right) \int (e^\chi w)^2 \, d\mu_\sigma.$$

In view of the bound (18) we have the estimate $\|\nabla(e^\chi w)\|_\sigma \lesssim \|\mathcal{L}_\sigma(e^\chi w)\|_\sigma$. Therefore, invoking the product rule of differentiation

$$\|e^\chi \nabla w\|_\sigma \leq \|\nabla(e^\chi w)\|_\sigma + \|e^\chi w \nabla \chi\|_\sigma \lesssim \|\mathcal{L}_\sigma(e^\chi w)\|_\sigma + |a| \|e^\chi w\|_{\sigma-1}.$$

Observe that (33) and (17) imply

$$|a| \|e^\chi w\|_{\sigma-1} \lesssim (|a| + a^2) \|e^\chi w\|_\sigma + \|\mathcal{L}_\sigma(e^\chi w)\|_\sigma.$$

Combining the previous estimates with a Grönwall argument yields the statement of the lemma. \square

The following estimate is a major step towards Gaussian estimates.

Lemma 12. *Let w be the solution to the homogeneous equation (26). Let $a, b \in \mathbb{R}$ be given. Then there exists a constant $C > 0$ such that for all $z, z_0 \in \overline{B_1(0)}$, $0 < r \lesssim 1$, $t \in (\frac{1}{2}r^4, r^4)$, $k \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^N$ it holds that*

$$|\partial_t^k \partial_z^\beta w(t, z)| \lesssim \frac{r^{-4k-|\beta|} \theta(r, z)^{-|\beta|}}{|B_r^d(z)|_\sigma^{1/2}} e^{C(a^2/b^2+a^4)t-\chi_{a,b}(z,z_0)} \|e^{\chi_{a,b}(\cdot, z_0)} g\|_\sigma.$$

Proof. For simplicity, we write $\chi = \chi_{a,b}(\cdot, z_0)$ and $\theta = \theta(r, z)$. From Lemma 10 (with $z_0 = z$ and $\tau = 0$) we deduce the estimate

$$|\partial_t^k \partial_z^\beta w(t, z)|^2 \lesssim \frac{r^{-8k-2|\beta|} \theta^{-2|\beta|}}{r^4 |B_r^d(z)|_\sigma} \sup_{B_r^d(z)} e^{-2\chi} \int_0^{r^4} \int_{B_r^d(z)} e^{2\chi} w^2 + r^2 \theta^2 e^{2\chi} |\nabla w|^2 d\mu_\sigma dt \quad (34)$$

for all $t \in (\frac{1}{2}r^4, r^4)$. We first observe that the Lipschitz estimate (30) implies

$$\sup_{B_r^d(z)} e^{-\chi} \lesssim e^{-\chi(z)+a^4 r^4}.$$

To estimate the integral expression in (34), we distinguish the cases $\sqrt{\rho(z)} \leq r$ and $\sqrt{\rho(z)} \geq r$. In the first case, we apply Lemma 11 and obtain

$$r^4 \left(\sup_{[0,r^4]} \int e^{2\chi} w^2 d\mu_\sigma + \int_0^{r^4} \int e^{2\chi} |\nabla w|^2 d\mu_\sigma dt \right) \lesssim r^4 e^{C(a^2/b^2+a^4)r^4} \int e^{2\chi} g^2 d\mu_\sigma$$

for some $C > 0$. In the second case, we only focus on second term, i.e., the gradient term. The argument for the first term remains unchanged. Because $\rho \sim \rho(z)$ in the domain of integration, see (13), it holds that

$$\int_0^{r^4} \int_{B_r^d(z)} r^2 \theta^2 e^{2\chi} |\nabla w|^2 d\mu_\sigma dt \lesssim r^2 a^2 \int_0^{r^4} \int e^{2\chi} w^2 d\mu_\sigma dt + r^2 \int_0^{r^4} \int |\nabla(e^\chi w)|^2 d\mu_{\sigma+1} dt,$$

where we have used (28). By using (17) and Young’s inequality, we further estimate

$$r^2 \int |\nabla(e^\chi w)|^2 d\mu_{\sigma+1} \lesssim r^4 \int (\mathcal{L}_\sigma(e^\chi w))^2 d\mu_\sigma + \int (e^\chi w)^2 d\mu_\sigma,$$

which in turn yields

$$\int_0^{r^4} \int_{B_r^d(z)} r^2 \theta^2 e^{2\chi} |\nabla w|^2 d\mu_\sigma dt \lesssim r^4 (1 + r^2 a^2) e^{C(a^2/b^2+a^4)r^4} \int e^{2\chi} g^2 d\mu_\sigma$$

via Lemma 11. Notice that we can eliminate the factor $r^2 a^2$ in the previous expression upon enlarging the constant C . Substituting the previous bounds into (34) yields the statement of the lemma. \square

For large times, we have exponential decay as established in the lemma that follows.

Lemma 13. *Let w be the solution of the initial value problem (9) with $f = 0$. Then for any $k \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0^N$, $t \geq \frac{1}{2}$ and $z \in \overline{B_1(0)}$ it holds that*

$$\left| \partial_t^k \partial_z^\beta \left(w(t, z) - \int g \mu_\sigma \right) \right| \lesssim e^{-\lambda_1 t} \|\nabla g\|_{\sigma+1}.$$

Proof. The proof is an easy consequence of Lemma 10 and a spectral gap estimate for \mathcal{L}_σ . Indeed, applying Lemma 10 with $t = \tau + \frac{1}{4}$, $\varepsilon = \frac{1}{4}$, $r = 1$ and $\tau \geq \frac{1}{4}$ to $w - c$, where $c = \int w \, d\mu_\sigma$ is a constant of the evolution, we obtain the estimate

$$|\partial_t^k \partial_z^\beta (w(t, z) - c)| \lesssim \int_{t-1/4}^{t+3/4} \int (w - c)^2 + |\nabla w|^2 \, d\mu_\sigma \, dt.$$

Thanks to the Hardy–Poincaré inequality [Seis 2014, Lemma 3] and because $\mu_{\sigma+1} \lesssim \mu_\sigma$, we can drop the term $(w - c)^2$ in the integrand. To prove the statement of the lemma, we thus have to establish the estimate

$$\int_{t-1/4}^{t+3/4} |\nabla w|^2 \, d\mu_\sigma \, dt \lesssim e^{-2\lambda_1 t} \int |\nabla g|^2 \, d\mu_{\sigma+1}. \tag{35}$$

For this purpose, we test the homogeneous equation with w and invoke the symmetry and nonnegativity properties of \mathcal{L}_σ and obtain the energy estimate

$$\frac{d}{dt} \frac{1}{2} \int |\nabla w|^2 \, d\mu_{\sigma+1} + \int (\mathcal{L}_\sigma w)^2 \, d\mu_\sigma \leq 0.$$

On the one hand, integration in time over $[t - \frac{1}{4}, t + \frac{3}{4}]$ and the a priori estimate (18) yield

$$\int_{t-1/4}^{t+3/4} |\nabla w|^2 \, d\mu_\sigma \, dt \lesssim \int |\nabla w(t - \frac{1}{4})|^2 \, d\mu_{\sigma+1}. \tag{36}$$

On the other hand, the smallest nonzero eigenvalue λ_1 of \mathcal{L}_σ yields the spectral gap estimate

$$\int (\mathcal{L}_\sigma w)^2 \, d\mu_\sigma = \int \nabla w \cdot \nabla \mathcal{L}_\sigma w \, d\mu_{\sigma+1} \geq \lambda_1 \int |\nabla w|^2 \, d\mu_{\sigma+1},$$

which we combine with the energy estimate from above to get

$$\int |\nabla w(t - \frac{1}{4})|^2 \, d\mu_{\sigma+1} \lesssim e^{-2\lambda_1 t} \int |\nabla g|^2 \, d\mu_{\sigma+1}.$$

Plugging this estimate into (36) yields (35) as desired. □

We are now in the position to prove the desired maximal regularity estimate for the homogeneous problem. Let us start with the latter.

Proposition 14. *Let w be the solution to the homogeneous equation (26). Then*

$$\begin{aligned} \|w\|_{L^\infty} &\lesssim \|g\|_{L^\infty}, \\ \|w\|_{X(p)} + \|\nabla w\|_{L^\infty} &\lesssim \|\nabla g\|_{L^\infty}. \end{aligned}$$

Proof. Thanks to the exponential decay estimates from Lemma 13, it is enough to focus on the norms for small times, $T \leq 1$. We fix $z_0 \in \overline{B_1(0)}$ for a moment and let $r \lesssim 1$ and $t \in (\frac{1}{2}r^4, r^4)$ be arbitrarily given. As before, we set $\theta_0 = \theta(r, z_0)$. Because $w - g(z_0)$ is a solution to the homogeneous equation with initial value $g - g(z_0)$, an application of Lemma 12 with $a = -\frac{1}{r}$ and $b = r$ yields the estimate

$$|\partial_t^k \partial_z^\beta|_{z=z_0} (w(t, z) - g(z_0))| \lesssim \frac{r^{-4k-|\beta|} |\theta_0^{-|\beta|}}{|B_r^d(z_0)|_\sigma^{1/2}} \|e^{\chi_{-1/r, r}(\cdot, z_0)} (g - g(z_0))\|_\sigma. \tag{37}$$

Notice that the function χ drops out in the exponential prefactor because $\chi(z_0, z_0) = 0$. We claim that

$$\|e^{\chi_{-1/r,r}(\cdot, z_0)}(g - g(z_0))\|_\sigma \lesssim \min\{\|g\|_{L^\infty}, r\theta_0\|\nabla g\|_{L^\infty}\} |B_r^d(z_0)|_\sigma^{1/2}. \tag{38}$$

The proof of this estimate has been already displayed earlier; see, e.g., the proof of Proposition 4.2 in [Seis 2014]. For the convenience of the reader, we recall the simple argument. Notice first that $|g(z) - g(z_0)| \lesssim \min\{\|g\|_{L^\infty}, |z - z_0|\|\nabla g\|_{L^\infty}\}$. On every annulus

$$A_j = B_{jr}^{\hat{d}}(z_0) \setminus B_{(j-1)r}^{\hat{d}}(z_0)$$

it holds that $\chi_{-1/r,r}(z, z_0) \leq -(j - 1)/\sqrt{2}$ as can be verified by an elementary computation, and thus, for $s \in \{0, 1\}$, we have

$$\int_{A_j} e^{2\chi_{-1/r,r}(z, z_0)} |z - z_0|^{2s} d\mu_\sigma(z) \lesssim j^{2s} r^{2s} \theta(jr, z_0)^{2s} e^{-\sqrt{2}j} |A_j|_\sigma$$

as a consequence of (11). Clearly, $\theta(jr, z_0) \leq j\theta_0$. We notice that $A_j = \emptyset$ for each $j \gg \frac{1}{r}$. On the other hand, thanks to the volume formula (15), it holds that

$$|A_j|_\sigma \lesssim j^{2(N+\sigma)} |B_r^d(z_0)|_\sigma.$$

It remains to notice that the annuli $\{A_j\}_{j \in \mathbb{N}}$ cover $\overline{B_1(0)}$ and deduce that

$$\|e^{\chi_{-1/r,r}(\cdot, z_0)} | \cdot - z_0|^s\|_\sigma \lesssim (r\theta_0)^s |B_r^d(z_0)|_\sigma^{1/2} \left(\sum_{j \in \mathbb{N}} e^{-\sqrt{2}j} j^\kappa \right)^{1/2}$$

for some $\kappa = \kappa(s) > 0$. Because the series is convergent, we have thus proved the bound in (38).

We now combine (37) and (38) to the effect of

$$\begin{aligned} r^{4k+|\beta|} \theta_0^{|\beta|} |\partial_t^k \partial_z^\beta |_{z=z_0} (w(t, z) - g(z_0))| &\lesssim \|g\|_{L^\infty}, \\ r^{4k+|\beta|-1} \theta_0^{|\beta|-1} |\partial_t^k \partial_z^\beta |_{z=z_0} (w(t, z) - g(z_0))| &\lesssim \|\nabla g\|_{L^\infty}. \end{aligned}$$

We obtain the uniform bounds on w and ∇w in the time interval $[0, 1]$ by setting $(k, |\beta|) = (0, 0)$ in the first and $(k, |\beta|) = (0, 1)$ in the second estimate. (Recall that we use Lemma 13 to extend the estimates to times $t \geq 1$.) To control in $X(p)$, we choose $(k, |\beta|) \in \{(1, 0), (0, 2), (0, 3), (0, 4)\}$, raise the second of the above estimates to the power p and average over $Q_r(z)$. For instance, if $(k, |\beta|) = (0, 2)$, this leads to

$$\frac{r^p}{|Q_r^d(z)|} \iint_{Q_r^d(z)} \theta(r, z_0)^p |\nabla^2 w(t, z_0)|^p dz_0 dt \lesssim \|\nabla g\|_{L^\infty}^p.$$

If view of (12) and (13), it holds that $\theta(r, z_0) \sim \theta(r, z)$ uniformly in $B_r^d(z)$, and thus, from maximizing in r and z we obtain

$$\sup_{\substack{z \in \overline{B_1(0)} \\ 0 < r \lesssim 1}} r \theta(r, z) |Q_r^d(z)|^{-1/p} \|\nabla^2 w\|_{L^p(Q_r^d(z))} \lesssim \|\nabla g\|_{L^\infty}.$$

Higher-order derivatives are bounded analogously. □

Gaussian estimates are contained in the following statement.

Proposition 15. *There exists a unique function $G : (0, \infty) \times \overline{B_1(0)} \times \overline{B_1(0)} \rightarrow \mathbb{R}$ with the following properties:*

- (1) *If w is the solution to the homogeneous equation (26), then for any $k \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0^N$ and $(t, z) \in (0, \infty) \times \overline{B_1(0)}$*

$$\partial_t^k \partial_z^\beta w(t, z) = \int \partial_t^k \partial_z^\beta G(t, z, z') g(z') d\mu_\sigma.$$

- (2) *The function G is symmetric in the last two variables; that is,*

$$G(t, z, z') = G(t, z', z)$$

for all $(t, z, z') \in (0, \infty) \times \overline{B_1(0)} \times \overline{B_1(0)}$.

- (3) *For any $z' \in B_1(0)$, $G' = G(\cdot, \cdot, z')$ solves the homogeneous equation*

$$\partial_t G' + \mathcal{L}_\sigma^2 G' + \mathcal{L}_\sigma G' = 0.$$

Moreover,

$$\rho^\sigma G' \xrightarrow{t \downarrow 0} \delta_{z'} \quad \text{in the sense of distributions.}$$

- (4) *It holds that*

$$|\partial_t^k \partial_z^\beta G(t, z, z')| \lesssim \frac{\sqrt[4]{t}^{-4k-|\beta|} \theta(\sqrt[4]{t}, z)^{-|\beta|}}{|B_{\sqrt[4]{t}}^d(z)|_\sigma^{1/2} |B_{\sqrt[4]{t}}^d(z')|_\sigma^{1/2}} e^{-C(d(z, z')/\sqrt[4]{t})^{4/3}}$$

for all $(t, z, z') \in (0, 1] \times \overline{B_1(0)} \times \overline{B_1(0)}$ and any $k \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^N$.

- (5) *It holds that*

$$|\partial_t^k \partial_z^\beta (G(t, z, z') - |B_1(0)|_\sigma^{-1})| \lesssim e^{-\lambda_1 t}$$

for all $(t, z, z') \in [1, \infty) \times \overline{B_1(0)} \times \overline{B_1(0)}$ and any $k \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^N$.

The estimates in the fourth statement are usually referred to as ‘‘Gaussian estimates’’.

Remark 16. In the fourth statement we may freely interchange the balls centered at z by balls centered at z' and vice versa. Likewise, we can substitute $\theta(\sqrt[4]{t}, z)$ by $\theta(\sqrt[4]{t}, z')$. This is a consequence of (16).

The proof of this proposition is (almost) exactly the one of [Seis 2014, Proposition 4.3]. We display the argument for completeness and the convenience of the reader.

Proof. We first notice that the linear mapping $L_\sigma^2 \ni g \mapsto \partial_t^k \partial_z^\beta w(t, z) \in \mathbb{R}$ is bounded for every fixed $(t, z) \in (0, \infty) \times \overline{B_1(0)}$ and $(k, \beta) \in \mathbb{N}_0 \times \mathbb{N}_0^N$. Indeed, for small times, boundedness is a consequence of Lemma 12 (with $a = 0$), and for large times, boundedness follows from successively applying Lemmas 13 and 12 (with $a = 0$), namely $|\partial_t^k \partial_z^\beta w(t, z)| \lesssim \|w(\frac{1}{2})\|_\sigma + \|\nabla w(\frac{1}{2})\|_{\sigma+1} \lesssim \|g\|_\sigma$. Riesz’ representation theorem thus provides us with the existence of a unique function $G_{k,\beta}(t, z, \cdot) \in L_\sigma^2$ such that

$$\partial_t^k \partial_z^\beta w(t, z) = \int G_{k,\beta}(t, z, z') g(z') d\mu_\sigma(z').$$

Setting $G = G_{0,0}$, uniqueness implies $G_{k,\beta} = \partial_t^k \partial_z^\beta G$. Notice that G inherits the symmetry in z and z' from the symmetry of the linear operator $\mathcal{L}_\sigma^2 + \mathcal{L}_\sigma$ via the symmetry of the associated semigroup operator $e^{(\mathcal{L}_\sigma^2 + \mathcal{L}_\sigma)t}$.

We now turn to the proof of the Gaussian estimates. We shall write $\chi = \chi_{a,b}(\cdot, z_0)$ for some fixed $z_0 \in \overline{B_1(0)}$ and set $\theta = \theta(r, z)$. We first notice that by Lemma 12, for $r \sim \sqrt[4]{t}$, we have

$$|B_r^d(z)|_\sigma^{1/2} e^{\chi(z)} |w(\frac{1}{2}t, z)| \lesssim e^{C(a^2/b^2 + a^4)t} \|e^\chi g\|_\sigma,$$

and thus, the mapping \mathcal{A} defined by

$$(\mathcal{A}h)(z) = |B_r^d(z)|_\sigma^{1/2} e^{\chi(z)} \int G(\frac{1}{2}t, z, z') e^{-\chi(z')} h(z') d\mu_\sigma(z')$$

for $z \in \overline{B_1(0)}$, is a bounded linear mapping from L_σ^2 to L^∞ with

$$\|\mathcal{A}\|_{L_\sigma^2 \rightarrow L^\infty} \lesssim e^{C(a^2/b^2 + a^4)t}.$$

By the symmetry of the Green's function, it holds that

$$\begin{aligned} \int \mathcal{A}h\xi d\mu_\sigma &= \iint |B_r^d(z)|_\sigma^{1/2} e^{\chi(z)} G(\frac{1}{2}t, z', z) e^{-\chi(z')} h(z') d\mu_\sigma(z') d\mu_\sigma(z) \\ &= \int e^{-\chi} w_\xi(\frac{1}{2}t) h d\mu_\sigma \end{aligned}$$

if w_ξ denotes the solution to the homogeneous equation with initial value $g_\xi = |B_r^d(\cdot)|_\sigma^{1/2} e^\chi \xi$, and if $\xi \in L_\sigma^1$ is such that $g_\xi \in L_\sigma^2$. In particular, the action of the dual $\mathcal{A}^* : (L^\infty)^* \rightarrow L_\sigma^2$ on such functions ξ is given by $\mathcal{A}^*\xi = e^{-\chi} w_\xi(\frac{1}{2}t)$. Because $\|\mathcal{A}\|_{L_\sigma^2 \rightarrow L^\infty} = \|\mathcal{A}^*\|_{(L^\infty)^* \rightarrow L_\sigma^2}$, we then have the estimate

$$\|e^{-\chi} w_\xi(\frac{1}{2}t)\|_\sigma \lesssim e^{C(a^2/b^2 + a^4)t} \|\xi\|_{L_\sigma^1}.$$

An application of Lemma 12 with a replaced by $-a$ then yields that

$$\left| \int \partial_t^k \partial_z^\beta G(t, z, \cdot) |B_r^d(\cdot)|_\sigma^{1/2} e^\chi \xi d\mu_\sigma \right| \lesssim \frac{r^{-4k - |\beta| \theta - |\beta|}}{|B_r^d(z)|_\sigma^{1/2}} e^{C(a^2/b^2 + a^4)t + \chi(z)} \|\xi\|_{L_\sigma^1}.$$

By approximation, it is clear that this estimate holds for any $\xi \in L_\sigma^1$. Thanks to the duality $L^\infty = (L_\sigma^1)^*$, we thus have

$$|\partial_t^k \partial_z^\beta G(t, z, z')| \lesssim \frac{r^{-4k - |\beta| \theta - |\beta|}}{|B_r^d(z)|_\sigma^{1/2} |B_r^d(z')|_\sigma^{1/2}} e^{C(a^2/b^2 + a^4)t + \chi(z) - \chi(z')}.$$

The term $-\chi(z')$ drops out of the exponent upon choosing $z' = z_0$. To conclude the argument for the Gaussian estimates, we distinguish two cases: First, if $\sqrt[4]{t} \geq d(z, z_0)$, then

$$1 \lesssim e^{-C(d(z, z_0)/\sqrt[4]{t})^{4/3}},$$

and thus the statement follows with $a = 0$. Otherwise, if $\sqrt[4]{t} \leq d(z, z_0)$, we choose $a = -\ell$ for some $\ell > 0$ and $b \sim d = d(z, z_0)$ so that the exponent becomes

$$\left(\frac{\ell^2}{d^2} + \ell^4\right)t - \ell d$$

modulo constant prefactors. We optimize the last two terms in ℓ by choosing $\ell \sim (d/t)^{1/3}$. It is easily checked that the exponent is bounded by an expression of the form $1 - (d/\sqrt[4]{t})^{4/3}$, which yields the desired result.

The remaining properties are immediate consequences of the preceding analysis. □

2.5. Calderón–Zygmund estimates. We will see at the beginning of the next subsection that the kernel representation of solutions of the homogeneous problem carries over to the ones of the inhomogeneous problem. This observation is commonly referred to as Duhamel’s principle. To study regularity in the inhomogeneous problem, the detailed knowledge of the Gaussian kernel provided by Proposition 15 is very helpful. A major step in the analysis of Whitney measures is the translation of the energy estimates from weighted L^2 to standard L^p spaces. We are thus led to the study of singular integrals in the spirit of Calderón and Zygmund and the theory of Muckenhoupt weights.

Out of the Euclidean setting, a good framework for these studies is provided by spaces of homogeneous type, see [Coifman and Weiss 1971], which are metric measure spaces, i.e., metric spaces endowed with a doubling Borel measure.¹ The theory of singular integrals in spaces of homogeneous type was elaborated by Koch [1999; 2004; 2008]. For the Euclidean theory, we refer to [Stein 1970; 1993].

Let us recall some pieces of the abstract theory. Let (X, D) be a metric space endowed with a doubling Borel measure μ . A linear operator T on $L^q(X, \mu)$ with $q \in (1, \infty)$ is called a *Calderón–Zygmund operator* if T can be written as

$$Tf(x) = \int_X K(x, y) f(y) d\mu(y)$$

for all $x \in (\text{spt } f)^c$ and $f \in L^\infty(X, \mu) \cap L^q(X, \mu)$, where $K : X \times X \rightarrow \mathbb{R}$ is a measurable kernel such that

$$\begin{aligned} y \mapsto K(x, y) &\in L^1_{\text{loc}}(X \setminus \{x\}, \mu), \\ x \mapsto K(x, y) &\in L^1_{\text{loc}}(X \setminus \{y\}, \mu), \end{aligned}$$

satisfying the following boundedness and *Calderón–Zygmund cancellation conditions*:

$$\sup_{x \neq y} V(x, y) |K(x, y)| \lesssim 1 \tag{39}$$

and

$$\sup_{x \neq y} \sup_{x' \neq y'} V(x, y) \wedge V(x', y') |K(x, y) - K(x', y')| \lesssim \left(\frac{D(x, x') + D(y, y')}{D(x, y) + D(x', y')}\right)^\delta \tag{40}$$

¹In fact, Coifman and Weiss introduced the notion of spaces of homogeneous type with quasimetrics instead of metrics.

for some $\delta \in (0, 1]$. Here we have used the notation

$$V(x, y) = \mu\left(B_{D(x,y)}^D\left(\frac{x+y}{2}\right)\right).$$

It is worth noting that the doubling property of μ implies that we could equivalently have chosen to center the above balls at x or y .

Finally, we call ω a *p-Muckenhoupt weight* if

$$\sup_B \left(\frac{1}{\mu(B)} \int_B \omega \, d\mu\right) \left(\frac{1}{\mu(B)} \int_B \omega^{-1/(p-1)} \, d\mu\right)^{p-1} < \infty.$$

The class of *p-Muckenhoupt weights* is denoted by $A_p(X, D, \mu)$.

The theory of singular integrals asserts that any Calderón–Zygmund operator T extends to a bounded operator on any $L^p(X, \mu)$ with $p \in (1, \infty)$; i.e.,

$$\|Tf\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)}.$$

Moreover, if $\omega \in A_p$ is a Muckenhoupt weight, then T is also bounded on $L^p(\mu \llcorner \omega)$, where $d(\mu \llcorner \omega) = \omega d\mu$.

In order to establish L^p maximal regularity estimates for our problem at hand, we have to study singular integrals of the form

$$T_{\ell,k,\beta} f(t, z) = \int_0^\infty \int K_{\ell,k,\beta}((t, z), (t', z')) f(t', z') \, d\mu_\sigma(z') \, dt',$$

where $K_{\ell,k,\beta}((t, z), (t', z')) = \chi_{(0,t)}(t') \rho(z)^\ell \partial_t^k \partial_z^\beta G(t - t', z, z')$. In fact, we will see that $T_{\ell,k,\beta}$ is a Calderón–Zygmund operator on the product space $(0, \infty) \times B_1(0)$ provided that ℓ, k , and β are such that

$$(\ell, k, |\beta|) \in \mathcal{E} = \{(0, 1, 0), (0, 0, 2), (1, 0, 3), (2, 0, 4)\}. \tag{41}$$

We will accordingly refer to any tuple (ℓ, k, β) in the above class as a *Calderón–Zygmund exponent*.

The product space $X = (0, \infty) \times B_1(0)$ will be endowed with the metric

$$D((t, z), (t', z')) = \sqrt[4]{|t - t'| + d(z, z')^4},$$

which reflects the parabolic scaling of the linear differential operator, and the product measure $\mu = \lambda^1 \otimes \mu_\sigma$, with λ^1 denoting the one-dimensional Lebesgue. Because d is doubling, so is D , and thus the metric measure space (X, D, μ) is of *homogeneous type* in the sense of [Coifman and Weiss 1971] and is thus suitable for Calderón–Zygmund theory. Notice also that the volume tensor $V((t, z), (t', z'))$ simplifies to

$$V((t, z), (t', z')) \sim D((t, z), (t', z'))^4 \left| B_{D((t,z),(t',z'))}^d\left(\frac{z+z'}{2}\right) \right|_\sigma. \tag{42}$$

Without proof, we state the following lemma:

Lemma 17. *If (ℓ, k, β) is such that (41) holds, then $T_{\ell,k,\beta}$ is a Calderón–Zygmund operator.*

The proof is almost identical to the one in the porous-medium setting; see Lemmas 4.20 and 4.21 in [Seis 2015]. We will thus refrain from repeating the argument and refer the interested reader to the quoted paper.

2.6. The inhomogeneous problem. In this subsection, we consider the inhomogeneous problem with zero initial datum,

$$\begin{cases} \partial_t w + \mathcal{L}_\sigma^2 w + n\mathcal{L}_\sigma w = f & \text{in } (0, \infty) \times B_1(0), \\ w(0, \cdot) = 0 & \text{in } B_1(0). \end{cases} \tag{43}$$

Our first observation is that the kernel representation from Proposition 15 carries over to the inhomogeneous setting.

Lemma 18 (Duhamel’s principle). *If $f \in L^2(L_\sigma^2)$ and w is the solution to the initial value problem (43), then*

$$w(t, z) = \int_0^t \int G(t - t', z, z') f(t', z') d\mu_\sigma(z') dt'$$

for all $(t, z) \in (0, \infty) \times \overline{B_1(0)}$.

Proof. The statement follows from the fact that G is a fundamental solution, see Proposition 15(3). \square

Proposition 19. *Let w be the solution to the initial value problem (43). Then, for any $p \in (1, \infty)$ it holds that*

$$\|\partial_t w\|_{L^p} + \|\nabla w\|_{L^p} + \|\nabla^2 w\|_{L^p} + \|\rho \nabla^3 w\|_{L^p} + \|\rho^2 \nabla^4 w\|_{L^p} \lesssim \|f\|_{L^p}. \tag{44}$$

Proof. The purpose of this lemma is to carry the energy estimates from Lemma 8 over to the standard L^p setting. This is achieved by applying the abstract theory recalled in the previous subsection. In fact, as a consequence of Lemma 18, any function $\rho^\ell \partial_t^k \partial_z^\beta w$ has the kernel representation

$$T_{\ell,k,\beta} f(t, z) = \int_0^\infty \int K_{\ell,k,\beta}((t, z), (t', z')) f(t', z') d\mu_\sigma(z') dt',$$

where

$$K_{\ell,k,\beta}((t, z), (t', z')) = \chi_{(0,t)}(t') \rho(z)^\ell \partial_t^k \partial_z^\beta G(t - t', z, z').$$

If (ℓ, k, β) are Calderón–Zygmund exponents (41), by Lemma 17, the energy estimates from Lemma 8 carry over to any $L^p(L^p(\mu_\sigma))$ space with $p \in (1, \infty)$. Moreover, if ν is a Muckenhoupt weight in $A_p(B_1, d, \mu_\sigma)$, then the operators $T_{\ell,k,\beta}$ are bounded on $L^p(L^p(\mu_\sigma \llcorner \nu))$. Notice that this is the case for weights of the form $\nu = \rho^\gamma$ precisely if $-(\sigma + 1) < \gamma < (p - 1)(\sigma + 1)$. In particular, choosing $\gamma = -\sigma$, we see that $T_{\ell,k,\beta}$ is bounded on $L^p = L^p(L^p)$ for any $p \in (1, \infty)$ because $\sigma > 0$. This is the statement of the proposition apart from the term $\|\nabla w\|_{L^p}$. The control of this term can be deduced, for instance, from the analogous estimates for the porous-medium equation, see Proposition 4.23 in [Seis 2015], applied to $\partial_t w + n\mathcal{L}_\sigma w = f - \mathcal{L}_\sigma^2 w$. \square

In the following, we consider the larger cylinders

$$\widehat{Q}_r^d(z_0) := (\tfrac{1}{4}r^4, r^4) \times B_{2r}^d(z_0) \quad \text{and} \quad \widehat{Q}(T) = (\tfrac{1}{4}T, T) \times \overline{B_1(0)}.$$

Lemma 20. (1) *Suppose that $\text{spt } f \subset \widehat{Q}_r^d(z_0)$ for some $z_0 \in \overline{B_1(0)}$ and $0 < r \lesssim 1$. Then for any (ℓ, k, β) satisfying (41) and any $p \in (1, \infty)$, it holds that*

$$r^4 |Q_r^d(z_0)|^{-1/p} \|\rho^\ell \partial_t^k \partial_z^\beta w\|_{L^p(Q_r^d(z_0))} \lesssim \|f\|_{Y(p)}.$$

(2) Suppose that $\text{spt } f \subset \widehat{Q}(T)$ for some $T \geq 1$. Then it holds for any $p \in (1, \infty)$ that

$$\sum_{(\ell, k, |\beta|) \in \mathcal{E}} T \|\rho^\ell \partial_t^k \partial_z^\beta w\|_{L^p(Q(T))} \lesssim \|f\|_{Y(p)}.$$

Proof. We will only prove the first statement. The argument for the second one is very similar. The desired estimate is an immediate consequence of Proposition 19. Indeed, the latter implies

$$\|\rho^\ell \partial_t^k \partial_z^\beta w\|_{L^p(Q_r^d(z_0))} \lesssim \|f\|_{L^p(\widehat{Q}_r^d(z_0))}.$$

If now $\{Q_{r_i}^d(z_i)\}_{i \in I}$ is a finite cover of $\widehat{Q}_r^d(z_0)$ with radii $r_i \sim r$ and such that $\sum_i |Q_{r_i}^d(z_i)| \lesssim |\widehat{Q}_r^d(z_0)|$, then

$$\|f\|_{L^p(\widehat{Q}_r^d(z_0))} \leq \sum_{i \in I} \|f\|_{L^p(Q_{r_i}^d(z_i))} \lesssim \frac{1}{r^4} |\widehat{Q}_r^d(z_0)|^{1/p} \|f\|_{Y(p)}.$$

Notice that $|\widehat{Q}_r^d(z_0)| \lesssim |Q_r^d(z_0)|$, because $\mu_\sigma \otimes \lambda^1$ is doubling, which concludes the proof. \square

In view of the definition of the $X(p)$ norm, the estimates on the second- and third-order spatial derivative derived in the previous lemma are not strong enough for balls $B_r^d(z_0)$ that are relatively far away from the boundary in the sense that $\sqrt{\rho(z_0)} \gg r$. Estimates in such balls, as well as uniform bounds on w and ∇w , are derived in the lemma that follows.

Lemma 21. (1) Suppose that $\text{spt } f \subset \widehat{Q}_r^d(z_0)$ for some $z_0 \in \overline{B_1(0)}$ and $0 < r \lesssim 1$ and let $p > N + 4$. Then it holds for any $0 < t \lesssim r^4$ that

$$|w(t, z_0)| + |\nabla w(t, z_0)| \lesssim \|f\|_{Y(p)}.$$

If moreover $\sqrt{\rho(z_0)} \gg r$, then it holds that

$$r\theta(r, z_0) |Q_r^d(z_0)|^{-1/p} \|\nabla^2 w\|_{L^p(Q_r^d(z_0))} + r^2\theta(r, z_0)^2 |Q_r^d(z_0)|^{-1/p} \|\nabla^3 w\|_{L^p(Q_r^d(z_0))} \lesssim \|f\|_{Y(p)}.$$

(2) Suppose that $\text{spt } f \subset \widehat{Q}(T)$ for some $T \geq 1$. Then it holds for any $p > 1 + \frac{1}{2}N$ that

$$\|w\|_{L^\infty(Q(T))} + \|\nabla w\|_{L^\infty(Q(T))} \lesssim \|f\|_{Y(p)}.$$

Proof. (1) As a consequence of Lemma 18 and Hölder’s inequality, we have

$$|\partial_z^\beta w(t, z)| \leq \left(\int_0^{r^4} \|\partial_z^\beta G(\tau, z, \cdot)\|_{L_{q\sigma}^q}^q d\tau \right)^{1/q} \|f\|_{L^p}, \tag{45}$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta \in \mathbb{N}_0^N$. The statements thus follow from suitable estimates for the kernel functions. From Proposition 15 we recall that

$$|\partial_z^\beta G(\tau, z, z')| \lesssim \sqrt[4]{\tau}^{-|\beta|} \theta(\sqrt[4]{\tau}, z)^{-|\beta|} |B_{\sqrt[4]{\tau}}^d(z)|_\sigma^{-1} e^{-C(d(z, z')/\sqrt[4]{\tau})^{4/3}}. \tag{46}$$

Let $\{B_j^d(\sqrt[4]{\tau}(z))\}_{j \in J}$ be a finite cover of $\overline{B_1}$. Then

$$\int e^{-qC(d(z, z')/\sqrt[4]{\tau})^{4/3}} d\mu_{q\sigma}(z') \leq \sum_{j \in J} e^{-qC(j-1)^{4/3}} |B_j^d(\sqrt[4]{\tau}(z))|_{q\sigma}.$$

Notice that by the virtue of (15),

$$|B_{j\sqrt[4]{\tau}}^d(z)|_{q\sigma} \lesssim j^{2N} |B_{\sqrt[4]{\tau}}^d(z)|_{q\sigma} \sim j^{2N} |B_{\sqrt[4]{\tau}}^d(z)|^{1-q} |B_{\sqrt[4]{\tau}}^d(z)|_{\sigma}^q,$$

which in turn implies

$$\int e^{-qC(d(z,z')/\sqrt[4]{\tau})^{4/3}} d\mu_{q\sigma}(z') \lesssim \left(\sum_{j \in J} e^{-q(j-1)^{4/3}} j^{2N} \right) |B_{\sqrt[4]{\tau}}^d(z)|^{1-q} |B_{\sqrt[4]{\tau}}^d(z)|_{\sigma}^q.$$

The sum is converging and can thus be absorbed in the (suppressed) constant. We now integrate (46) over time and space and obtain

$$\int_0^{r^4} \|\partial_z^\beta G(\tau, z, \cdot)\|_{L_{q\sigma}^q}^q d\tau \lesssim \int_0^{r^4} \sqrt[4]{\tau}^{-|\beta|q} \theta(\sqrt[4]{\tau}, z)^{-|\beta|q} |B_{\sqrt[4]{\tau}}^d(z)|^{1-q} d\tau \tag{47}$$

for any $z \in \bar{B}_1$.

First, if $\sqrt{\rho(z)} \lesssim r$, then by (12), estimate (47) turns into

$$\int_0^{r^4} \|\partial_z^\beta G(\tau, z, \cdot)\|_{L_{q\sigma}^q}^q d\tau \lesssim \int_0^{r^4} \sqrt[4]{\tau}^{-q|\beta|-(q-1)N} d\tau \lesssim r^{4-2q|\beta|-2(q-1)N},$$

provided that $N + 2 < (2 - |\beta|)p$, which is consistent with the assumptions in the lemma only if $|\beta| \in \{0, 1\}$. It remains to notice that

$$r^{4-2(q-1)N} \lesssim r^{4q} |Q_r^d(z)|^{1-q}$$

by virtue of (15). From this and (45), we easily derive the first estimate in the first part of the lemma in the case where $\sqrt{\rho(z)} \lesssim r$.

Second, if $\sqrt{\rho(z)} \gg r$, then (47) becomes

$$\begin{aligned} \int_0^{r^4} \|\partial_z^\beta G(\tau, z, \cdot)\|_{L_{q\sigma}^q}^q d\tau &\lesssim \sqrt{\rho(z)}^{-|\beta|q-(q-1)N} \int_0^{r^4} \sqrt[4]{\tau}^{-|\beta|q-N(q-1)} d\tau \\ &\lesssim \sqrt{\rho(z)}^{-|\beta|q-(q-1)N} r^{4-|\beta|q-(q-1)N}, \end{aligned}$$

provided that $N + 4 < (4 - |\beta|)p$, which is consistent with the assumptions only if $|\beta| \in \{0, 1, 2, 3\}$. Now we notice that

$$\sqrt{\rho(z)}^{-|\beta|q-(q-1)N} r^{4-|\beta|q-(q-1)N} \lesssim r^{(4-|\beta|)q} \theta^{-|\beta|q} |Q_r^d(z)|^{1-q},$$

using (15) again. It is not difficult to see that the latter estimates in combination with (45) imply remaining estimates in the first part of the lemma.

(2) By Duhamel’s principle in Lemma 18 and the fact that f is concentrated on $\hat{Q}(T)$, we have for any $(t, z) \in Q(T)$ and $\beta \in \mathbb{N}_0^N$ that

$$\begin{aligned} |\partial_z^\beta w(t, z)| &\leq \int_{T/4}^{t-1} \int |\partial_z^\beta G(t-t', z, z')| |f(t', z')| d\mu_\sigma(z') dt' \\ &\quad + \int_{t-1}^t \int |\partial_z^\beta G(t-t', z, z')| |f(t', z')| d\mu_\sigma(z') dt', \end{aligned}$$

with the convention that the first integral is zero if $\frac{1}{4}T \geq t - 1$. If it is nonzero, we use Proposition 15(5) and estimate the latter by

$$\int_{T/4}^{t-1} \int |f| d\mu_\sigma dt' \lesssim T^{1-1/p} \|f\|_{L^p(\hat{Q}(T))} \leq T \|f\|_{L^p(\hat{Q}(T))}.$$

Similarly, applying the same strategy as in part (1) above, we bound the second term by

$$\|\partial_z^\beta G(\cdot, z, \cdot)\|_{L^q((0,1);L^q_\sigma)} \|f\|_{L^p(\hat{Q}(T))} \lesssim \|f\|_{L^p(\hat{Q}(T))}.$$

The statement thus follows by choosing $|\beta| \in \{0, 1\}$. □

We need some estimates for the off-diagonal parts.

Lemma 22. (1) *Suppose that $\text{spt } f \subset [0, r^4) \times \overline{B_1(0)} \setminus \hat{Q}_r^d(z_0)$ for some $z_0 \in \overline{B_1(0)}$ and $0 < r \lesssim 1$. Then it holds for any $p \in (1, \infty)$ that*

$$\|w\|_{L^\infty(Q_r^d(z_0))} + \|\nabla w\|_{L^\infty(Q_r^d(z_0))} + \sum_{(\ell,k,|\beta|) \in \mathcal{E}} \frac{r^{4k+|\beta|}}{\theta(r, z_0)^{2\ell-|\beta|}} \|\rho^\ell \partial_t^k \partial_z^\beta w\|_{L^p(Q_r^d(z_0))} \lesssim \|f\|_{Y(p)}.$$

(2) *Suppose that $\text{spt } f \subset [\frac{1}{2}, \frac{1}{4}T] \times \overline{B_1(0)}$ for some $T \geq 2$. Then it holds for any $p \in (1, \infty)$ that*

$$\|w\|_{L^\infty(Q(T))} + \|\nabla w\|_{L^\infty(Q(T))} + \sum_{(\ell,k,|\beta|) \in \mathcal{E}} T \|\rho^\ell \partial_t^k \partial_z^\beta w\|_{L^p(Q(T))} \lesssim \|f\|_{Y(p)}.$$

Proof. (1) We begin our proof with a helpful elementary estimate: if θ and C are given positive constants, then there exists a new constant \tilde{C} such that

$$\sqrt[4]{t-t'}^{-\theta} e^{-C(d(z,z')/\sqrt[4]{t-t'})^{4/3}} \lesssim r^{-\theta} e^{-\tilde{C}(d(z,z')/r)^{4/3}} \tag{48}$$

for all $(t, z) \in Q_r^d(z_0)$ and $(t', z') \in [0, r^4) \times \overline{B_1} \setminus \hat{Q}_r^d(z_0)$. The argument for (48) runs as follows: To simplify the notation slightly, we write $\tau = t - t'$ and $d = d(z, z')$. If $z' \in B_{2r}^d(z_0)$, then necessarily $t' \notin (\frac{1}{4}r^4, r^4)$, and therefore $\tau \gtrsim r^4$. It follows that

$$\sqrt[4]{\tau}^{-\theta} e^{-C(d/\sqrt[4]{\tau})^{4/3}} \leq \sqrt[4]{\tau} \lesssim r^{-\theta} e^{-C(d/r)^{4/3}},$$

because $d(z, z') \leq d(z, z_0) + d(z_0, z') \leq 3r$. Otherwise, if $z' \notin B_{2r}^d(z_0)$, it holds that $2r \leq d(z', z_0) \leq d(z, z') + r$, and thus $\sqrt[4]{\tau} \lesssim r \lesssim d$. Using the fact that $\tau \mapsto \sqrt[4]{\tau}^{-\theta} e^{-C(d/\sqrt[4]{\tau})^{4/3}}$ is increasing for $0 < \tau \lesssim d^4$, we then estimate

$$\sqrt[4]{\tau}^{-\theta} e^{-C(d/\sqrt[4]{\tau})^{4/3}} \lesssim r^{-\theta} e^{-\tilde{C}(d/r)^{4/3}}.$$

This completes the proof of (48).

In the following, C will be a uniform constant whose value may change from line to line.

Because $f = 0$ in $Q_r^d(z_0)$, Duhamel’s principle (Lemma 18) and the Gaussian estimates from Proposition 15 imply

$$\begin{aligned} & |\partial_t^k \partial_z^\beta w(t, z)| \\ & \lesssim \int_0^t \int \frac{\sqrt[4]{\tau}^{-4k-|\beta|} \theta(\sqrt[4]{\tau}, z)^{-|\beta|}}{|B_{\sqrt[4]{\tau}}^d(z)|_\sigma} e^{-C(d(z, z')/\sqrt[4]{\tau})^{4/3}} |f(t - \tau, z')| d\mu_\sigma(z') d\tau \\ & \lesssim \int_0^{r^4} \int \left(\frac{\sqrt[4]{\tau}}{\sqrt[4]{\tau} + \sqrt{\rho(z)}} \right)^{|\beta|+N+2\sigma} \sqrt[4]{\tau}^{-4k-2|\beta|-2N-2\sigma} e^{-C(d(z, z')/\sqrt[4]{\tau})^{4/3}} |f(t - \tau, z')| d\mu_\sigma(z') d\tau. \end{aligned}$$

As a consequence of (48), Remark 16 and the monotonicity of the function $s \mapsto s/(s + c)$ for any fixed $c > 0$, we may substitute any $\sqrt[4]{\tau}$ by r and find

$$\begin{aligned} & \frac{r^{4k+|\beta|-1}}{\theta(r, z)^{2\ell-|\beta|+1}} |\rho(z)^\ell \partial_t^k \partial_z^\beta w(t, z)| \\ & \lesssim \frac{1}{|B_r^d(z)|_\sigma} \int_0^{r^4} \int e^{-C(d(z, z')/r)^{4/3}} r^{-1} \theta(r, z')^{-1} |f(t', z')| d\mu_\sigma(z') dt'. \end{aligned}$$

We consider now a finite family of balls $\{B_r^d(z_0)\}_{i \in I}$ covering $\overline{B_1(0)}$. Since $d(z, z_i) \leq d(z, z') + r$ for any $z' \in B_r^d(z_i)$ and

$$\sum_{i \in I} e^{-C(d(z, z_i)/r)^{4/3}} < \infty$$

uniformly in r and z , we further estimate the right-hand side of the last inequality by

$$\sup_{\tilde{z} \in B_1} \frac{1}{|B_r^d(\tilde{z})|_\sigma} \int_0^{r^4} \int_{B_r^d(\tilde{z})} r^{-1} \theta(r, z')^{-1} |f(t', z')| d\mu_\sigma(z') dt'. \tag{49}$$

We claim that this term is controlled by $\|f\|_{Y(p)}$. To see this, we fix $\tilde{z} \in B_1(0)$ and let $r_j = (\sqrt{\frac{2}{3}})^j r$. Applying a non-Euclidean version of Vitali’s covering lemma, see Lemma 2.2.2 in [Koch 1999], we find a finite family of balls $\{B_{r_j}^d(z_{ij})\}_{i \in I_j}$ covering $B_r^d(\tilde{z})$ and such that

$$\sum_{i \in I_j} |B_{r_j}^d(z_{ij})|_\sigma \lesssim |B_r^d(\tilde{z})|_\sigma \tag{50}$$

uniformly in j, r , and \tilde{z} . Then $(0, r^4) \times B_r^d(\tilde{z})$ is contained in the countable union $\bigcup_{j \in \mathbb{N}_0} \bigcup_{i \in I_j} Q_{r_j}^d(z_{ij})$. Invoking Hölder’s inequality we thus find

$$\begin{aligned} & \int_0^{r^4} \int_{B_r^d(\tilde{z})} r^{-1} \theta(r, z')^{-1} |f(t', z')| d\mu_\sigma(z') dt' \\ & \leq \sum_{j \in \mathbb{N}_0} \sum_{i \in I_j} \left(\sqrt{\frac{2}{3}} \right)^j r_j^{-1} \|\theta(r, \cdot)^{-1}\|_{L_{q\sigma}^q(Q_{r_j}^d(z_{ij}))} \|f\|_{L^p(Q_{r_j}^d(z_{ij}))}, \end{aligned}$$

where, as usual, $\frac{1}{p} + \frac{1}{q} = 1$. Notice that $\mu_{q\sigma}$ is a finite measure for any $q \in (1, \infty)$. From (12) and (13) we deduce that $\theta(r_j, z_{ij}) \sim \theta(r_j, z') \leq \theta(r, z')$ for any $z' \in B_j^d(z_{ij})$, and thus, as a consequence of (15),

$$\|\theta(r, \cdot)^{-1}\|_{L_{q\sigma}^q(Q_{r_j}^d(z_{ij}))} \lesssim r_j^4 \theta(r_j, z_{ij})^{-1} |Q_{r_j}^d(z_{ij})|^{-1/p} |B_{r_j}^d(z_{ij})|_\sigma.$$

Combining the previous two estimates, using (50) and the convergence of the geometric series finally yields that the term in (49) is bounded by $\|f\|_{Y(p)}$. We have thus proved that

$$\frac{r^{4k+|\beta|-1}}{\theta(r, z)^{2\ell-|\beta|+1}} |\rho(z)^\ell \partial_t^k \partial_z^\beta w(t, z)| \lesssim \|f\|_{Y(p)}.$$

We easily deduce the statement of the lemma.

(2) To prove the second statement, we use Lemma 18 and Proposition 15(5) to estimate

$$|\partial_t^k \partial_z^\beta w(t, z)| \lesssim \int_{1/2}^{T/4} e^{-(t-t')\lambda_1} \int |f(t', z')| d\mu_\sigma(z') dt'$$

for any $(t, z) \in Q(T)$ and any $k \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^N$ with $k + |\beta| \geq 1$. Let $M \in \mathbb{Z}$ be such that $2^M \leq \frac{1}{4}T < 2^{M+1}$. We then split and compute

$$\begin{aligned} |\partial_t^k \partial_z^\beta w(t, z)| &\lesssim \sum_{m=0}^M \int_{2^{m-1}}^{2^m} e^{-(t-t')\lambda_1} \int |f| d\mu_\sigma dt + \int_{T/8}^{T/4} e^{-(t-t')\lambda_1} \int |f| d\mu_\sigma dt' \\ &\lesssim \sum_{m=0}^M e^{-(t-2^m)\lambda_1} \|f\|_{L^p(Q(2^m))} + e^{-(T/4)\lambda_1} \|f\|_{L^p(Q(T/4))} \\ &\lesssim e^{-(T/4)\lambda_1} \|f\|_{Y(p)}. \end{aligned}$$

We easily infer all estimates but the uniform bound on w . To gain control on $\|w\|_{L^\infty}$, we argue similarly and get

$$|w(t, z)| \lesssim \sum_{m=0}^{M+1} \int_{2^{m-1}}^{2^m} \int |f| d\mu_\sigma dt' \lesssim \left(\sum_{m=0}^{M+1} \frac{1}{(2^p)^m} \right) \|f\|_{Y(p)}.$$

The desired estimate follows from the convergence of the geometric series. □

A combination of the results in this subsection yields the maximal regularity estimate for the inhomogeneous problem (43).

Proposition 23. *Suppose that $p > N + 4$. Let w be a solution to the homogeneous problem (43). Then*

$$\|w\|_{L^\infty(W^{1,\infty})} + \|w\|_{X(p)} \lesssim \|f\|_{Y(p)}.$$

Proof. The statement follows immediately from Lemmas 20, 21 and 22 and the superposition principle for linear equations: For small times, we split f into $\eta f + (1 - \eta)f$ with η being a smooth cut-off function such that $\eta = 1$ on $Q_r^d(z_0)$ and $\eta = 0$ outside $\hat{Q}_r^d(z_0)$ for some arbitrarily fixed $r \lesssim 1$ and $z_0 \in \overline{B_1(0)}$. For large times, we make a hard temporal cut-off by splitting f into $\chi f + (1 - \chi)f$, where χ is the characteristic function on $\hat{Q}(T)$. Notice that to estimate the large times, it is enough to study such f 's that are zero in the initial time interval $(0, \frac{1}{2})$. For details, we refer to [Seis 2015]. □

3. The nonlinear problem

Our goal in this section is the derivation of Theorems 1 and 2. The existence of a unique solution to the nonlinear problem is a consequence of a fixed-point argument. We need the following lemma:

Lemma 24. *Let w_1 and w_2 be two functions satisfying*

$$\|w_i\|_{L^\infty(W^{1,\infty})} + \|w_i\|_{X(p)} \leq \varepsilon, \quad i = 1, 2, \tag{51}$$

for some small $\varepsilon > 0$. Then

$$\|f[w_1] - f[w_2]\|_{Y(p)} \lesssim \varepsilon (\|w_1 - w_2\|_{L^\infty(W^{1,\infty})} + \|w_1 - w_2\|_{X(p)}).$$

Proof. For notational convenience, we write $f_i^j = f^j[w_i]$ for any $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$. We will also just write w instead of w_1 or w_2 if the index doesn't matter. We remark that by the virtue of (51), it holds that

$$\begin{aligned} |R_k[w_1] - R_k[w_2]| &\lesssim \|w_1 - w_2\|_{L^\infty(W^{1,\infty})}, \\ |R_k[w]| &\lesssim 1 \end{aligned}$$

for any value of k .

The estimates of the differences of the f_i^j is very similar. We focus on the leading-order terms, i.e., $f_1^3 - f_2^3$. Using (51) and the previous bounds on the R_k , we first notice that

$$\begin{aligned} |f_1^3 - f_2^3| &\lesssim \rho^2 \|w_1 - w_2\|_{L^\infty(W^{1,\infty})} (|\nabla^2 w|^3 + |\nabla^2 w| |\nabla^3 w| + |\nabla w| |\nabla^4 w|) \\ &\quad + \rho^2 |\nabla^2 w_1 - \nabla^2 w_2| (|\nabla^2 w|^2 + |\nabla w| |\nabla^3 w|) \\ &\quad + \rho^2 |\nabla^3 w_1 - \nabla^3 w_2| |\nabla w| |\nabla^2 w| + \rho^2 |\nabla^4 w_1 - \nabla^4 w_2| |\nabla w|^2. \end{aligned}$$

The control of the individual terms is derived very similarly. There are a few obvious cases; for instance the last term, which is simply controlled by using (51):

$$\|\rho^2 |\nabla^4 w_1 - \nabla^4 w_2| |\nabla w|^2\|_{Y(p)} \leq \|w\|_{L^\infty(W^{1,\infty})}^2 \|w_1 - w_2\|_{X(p)} \leq \varepsilon \|w_1 - w_2\|_{X(p)}.$$

For most of the remaining terms, we have to make use of the following interpolation inequality

$$\|\nabla^i \zeta\|_{L_r^m} \lesssim \|\zeta\|_{L^\infty}^{m-i} \|\nabla^m \zeta\|_{L_r^p}^i,$$

provided that $mp = ir$ for some integers $i < m$, which has been proved in Appendix A of [Seis 2015]. For instance, setting $\zeta = \eta \nabla w$ for some smooth cut-off function η satisfying $\eta = 1$ in $B_r^d(z_0)$ and $\eta = 0$ outside $B_{2r}^d(z_0)$, we have

$$\|\rho^2 |\nabla^2 w|^3\|_{L^p(B_r^d(z_0))} = \|\nabla^2 w\|_{L_{2r}^{3p}(B_r^d(z_0))}^3 \leq \|\nabla \zeta\|_{L_{2r}^{3p}}^3.$$

Applying the above interpolation inequality and using the fact that η varies on the scale $r\theta(r, z_0)$ and $\rho \lesssim \theta(r, z_0)^2$ in $B_{2r}^d(z_0)$, see (12) and (13), we then get

$$\begin{aligned} \|\rho^2 |\nabla^2 w|^3\|_{L^p(B_r^d(z_0))} &\lesssim \|\nabla w\|_{L^\infty}^2 \left(\|\rho^2 \nabla^4\|_{L^p(B_{2r}^d(z_0))} + \frac{\theta}{r} \|\rho \nabla^3 w\|_{L^p(B_{2r}^d(z_0))} \right. \\ &\quad \left. + \frac{\theta^2}{r^2} \|\nabla^2 w\|_{L^p(B_{2r}^d(z_0))} + \frac{\theta}{r^3} \|\nabla w\|_{L^p(B_{2r}^d(z_0))} \right), \end{aligned}$$

where $\theta = \theta(r, z_0)$. Integrating in time over $(\frac{1}{2}r^4, r^4)$, multiplying by r^3/θ and using (51) then yields

$$\sup_{r, z_0} \frac{r^3}{\theta} |Q_r^d(z_0)|^{-1/p} \|\rho^2 |\nabla^2 w|^3\|_{L^p(Q_r^d(z_0))} \lesssim \varepsilon \|w\|_{X(p)}.$$

This type of estimate can be used, for instance, to bound the first term in the above estimate for $f_1^3 - f_2^3$ for small times. The remaining terms and the large-time parts of the $Y(p)$ norm can be controlled in a similar way. □

We are now in the position to prove Theorems 1 and 2.

Proof of Theorems 1 and 2. To simplify the notation in the following, we denote by $\bar{X}(p)$ the intersection $X(p) \cap L^\infty(W^{1,\infty})$ and set $\|\cdot\|_{\bar{X}(p)} = \|\cdot\|_{X(p)} + \|\cdot\|_{L^\infty(W^{1,\infty})}$. Let ε and ε_0 be two positive constants. We denote by M_ε the set of all functions w in $\bar{X}(p)$ such that $\|w\|_{\bar{X}(p)} \leq \varepsilon$ and by N_{ε_0} the set of all functions g such that $\|g\|_{W^{1,\infty}} \leq \varepsilon_0$. We divide the proof into several steps.

Step 1: existence and uniqueness. For $w \in M_\varepsilon$ and $g \in N_{\varepsilon_0}$ given, we denote by $\tilde{w} := I(w, g)$ the unique solution to the linear problem (43) with inhomogeneity $f = f[w]$. By Theorem 4, we have the estimate $\|\tilde{w}\|_{\bar{X}(p)} \lesssim \|f[w]\|_{Y(p)} + \|g\|_{W^{1,\infty}}$. Applying Lemma 24 with $w_1 = w$ and $w_2 = 0$ and using the assumptions on w and g , we find that $\|\tilde{w}\|_{\bar{X}(p)} \leq C(\varepsilon^2 + \varepsilon_0)$ for some positive constant C . We choose ε and ε_0 small enough so that $C\varepsilon^2 \leq \frac{1}{2}\varepsilon$ and $C\varepsilon_0 \leq \frac{1}{2}\varepsilon$, with the consequence that $\tilde{w} \in M_\varepsilon$. This reasoning implies that for any fixed $g \in N_{\varepsilon_0}$, the function $\tilde{w}(\cdot, g)$ maps the set M_ε into itself. Moreover, given w_1 and w_2 in M_ε , we find by linearity and Lemma 24 that

$$\|I(w_1, g) - I(w_2, g)\|_{\bar{X}(p)} \lesssim \|f[w_1] - f[w_2]\|_{Y(p)} \lesssim \varepsilon \|w_1 - w_2\|_{\bar{X}(p)}.$$

Thus, choosing ε even smaller, if necessary, the previous estimate shows that $I(\cdot, g)$ is a contraction on M_ε . By Banach’s fixed-point argument, there exists thus a unique $w^* \in M_\varepsilon$ such that $w^* = I(w^*, g)$. In particular, w^* solves the nonlinear equation. From the previous choice of ε , we moreover deduce that $\|w^*\|_{\bar{X}(p)} \lesssim \|g\|_{W^{1,\infty}}$.

Step 2: analytic dependence on initial data. In order to show that w^* depends analytically on g , we will apply the analytic implicit function theorem; see [Deimling 1985, Theorem 15.3]. Because the nonlinearity $f = f[w]$ is a rational function of w and ∇w , and thus analytic away from its poles, the contraction map I is an analytic function on $M_\varepsilon \times N_{\varepsilon_0}$. We consider the map $J : M_\varepsilon \times N_{\varepsilon_0} \rightarrow M_\varepsilon$ defined by $J(w, g) = w - I(w, g)$. Because I is analytic, so is J . It holds that $I(0, 0) = 0$ and $D_w I(0, 0) = \text{id}$. From the analytic implicit function theorem we deduce the existence of two constants $\tilde{\varepsilon} < \varepsilon$ and $\tilde{\varepsilon}_0 < \varepsilon_0$ and of an analytic map $A : N_{\tilde{\varepsilon}_0} \rightarrow M_{\tilde{\varepsilon}}$ with $A(0) = 0$ and such that $J(w, g) = 0$ if and only if $A(g) = w$. From the uniqueness of the fixed point and the definition of J we then conclude that the map $g \mapsto w^*$ is analytic from $N_{\tilde{\varepsilon}_0}$ to $M_{\tilde{\varepsilon}}$.

Step 3: analytic dependence on time and tangential coordinates. Let us now change from Euclidean to spherical coordinates. For $z = (z_1, \dots, z_N)^T \in \bar{B}_1(0)$, we find radius $s \in [0, 1]$ and an angle vector $\phi = (\phi_1, \dots, \phi_{N-1})^T \in \mathcal{A}_{N-1} := [0, \pi]^{N-2} \times [0, 2\pi]$ such that $z_n = s(\prod_{i=1}^{n-1} \sin \phi_i) \cos \phi_n$ for $n \leq N-1$

and $z_N = s \prod_{i=1}^{N-1} \sin \phi_i$. By a slight abuse of notation, we write $w(t, z) = w(t, s, \phi)$. For $\lambda \in \mathbb{R}$ and $\psi \in \mathcal{A}_{N-1}$ we define

$$w_{\lambda, \psi}^* := w^* \circ \Xi_{\lambda, \psi}, \quad \Xi_{\lambda, \psi}(t, s, \phi) := (\lambda t, s, \phi + t\psi).$$

A short computation reveals that $w_{\lambda, \psi}$ solves the equation

$$\partial_t w_{\lambda, \psi}^* + \mathcal{H}_\sigma w_{\lambda, \psi}^* = f_{\lambda, \psi}[w_{\lambda, \psi}^*],$$

where

$$f_{\lambda, \psi}[w] := \lambda f[w] + (1 - \lambda)\mathcal{H}_\sigma w + \psi \cdot \nabla_\phi w, \quad \mathcal{H}_\sigma = \mathcal{L}_\sigma^2 + n\mathcal{L}_\sigma.$$

Clearly, $f_{1,0} = f$. Similarly as above, we denote by $I_{\lambda, \psi}(w, g)$ the solution to the linear equation with inhomogeneity $f_{\lambda, \psi}[w]$ and initial datum g . We furthermore set $J_{\lambda, \psi}(w, g) := w - I_{\lambda, \psi}(w, g)$. It is obvious that $J_{1,0}(0, 0) = 0$ and $D_w J_{1,0}(0, 0) = \text{id}$. Applying the analytic implicit function theorem once more, we find positive constants $\delta, \hat{\varepsilon} < \tilde{\varepsilon}, \hat{\varepsilon}_0 < \tilde{\varepsilon}_0$ and an analytic function $A_{\lambda, \psi}(g) = A(\lambda, \psi, g)$ from $B_\delta^{\mathbb{R}}(1) \times B_\delta^{\mathbb{R}^{N-1}}(0) \times N_{\hat{\varepsilon}_0}$ to $M_{\hat{\varepsilon}}$ such that $J_{\lambda, \psi}(A_{\lambda, \psi}(g), g) = 0$. In particular, the above uniqueness result gives that $A_{\lambda, \psi}(g) = A(g) \circ \Xi_{\lambda, \psi}$. We conclude that $w_{\lambda, \psi} \in \bar{X}(p)$ depends analytically on λ and ψ in a neighborhood of $(1, 0) \in \mathbb{R} \times \mathbb{R}^{N-1}$. In particular, there exists a constant Λ dependent only on N such that for any $k \in \mathbb{N}_0$ and $\beta' \in \mathbb{N}_0^{N-1}$, it holds that

$$\|\partial_\lambda^k \partial_\psi^{\beta'}|_{(\lambda, \psi)=(1,0)} w_{\lambda, \psi}\|_{\bar{X}(p)} \lesssim \Lambda^{-k-|\beta'|} k! \beta'! \|g\|_{W^{1,\infty}}.$$

It remains to notice that

$$\partial_\lambda^k \partial_\psi^{\beta'}|_{(\lambda, \psi)=(1,0)} w_{\lambda, \psi}(t, z) = t^{k+|\beta'|} \partial_t^k \partial_\phi^{\beta'} w(t, r, \phi)$$

to deduce

$$t^{k+|\beta'|} |\partial_t^k \partial_\phi^{\beta'} \nabla w(t, r, \phi)| \lesssim \Lambda^{-k-|\beta'|} k! \beta'! \|g\|_{W^{1,\infty}}. \tag{52}$$

Step 4: regularity in transversal direction. The derivation of the transversal regularity relies on the analyticity bounds established above together with the Morrey estimate

$$\begin{aligned} \|v\|_{L^\infty(Q_r^d(z))} &\lesssim |Q_r^d(z)|_\sigma^{-1/p} \|v\|_{L_\sigma^p(Q_r^d(z))} \\ &\quad + r\theta |Q_r^d(z)|_\sigma^{-1/p} \|\nabla v\|_{L_\sigma^p(Q_r^d(z))} + r^4 |Q_r^d(z)|_\sigma^{-1/p} \|\partial_t v\|_{L_\sigma^p(Q_r^d(z))}, \end{aligned} \tag{53}$$

which holds for any $p > N$ uniformly in r and z . The proof of this estimate proceeds analogously to the Euclidean case; see, e.g., [Evans and Gariepy 1992, Chapter 4.5]. We omit the argument.

In the following discussion, we keep r and z fixed and we set $\theta = \theta(r, z)$. For $b \in \{2, 3\}$, we choose $\sigma = (b - 1)p$ and apply (53) to the effect that

$$\begin{aligned} \|\nabla_\psi^{4-b} \partial_s^b w\|_{L^\infty(Q_r^d(z))} &\lesssim |Q_r^d(z)|_{(b-1)p}^{-1/p} \|\rho^{b-1} \nabla_\psi^{4-b} \partial_s^b w\|_{L^p(Q_r^d(z))} \\ &\quad + r\theta |Q_r^d(z)|_{(b-1)p}^{-1/p} \|\rho^{b-1} \nabla_\psi^{4-b} \partial_r^b \nabla w\|_{L^p(Q_r^d(z))} \\ &\quad + r^4 |Q_r^d(z)|_{(b-1)p}^{-1/p} \|\rho^{b-1} \nabla_\psi^{4-b} \partial_r^b \partial_t w\|_{L^p(Q_r^d(z))}. \end{aligned}$$

We recall from (15) that $|Q_r^d(z)|_\sigma \sim \theta^{2\sigma}|Q_r^d(z)|$ and that $\sqrt{\rho(\tilde{z})} \lesssim \theta$ for any $\tilde{z} \in B_r^d(z)$ by virtue of (13). Therefore,

$$\begin{aligned} \|\nabla_\psi^{4-b} \partial_s^b w\|_{L^\infty(Q_r^d(z))} &\lesssim \theta^{4-2b} |Q_r^d(z)|^{-1/p} \|\rho^{b-2} \nabla_\psi^{4-b} \partial_s^b w\|_{L^p(Q_r^d(z))} \\ &\quad + r\theta^{3-2b} |Q_r^d(z)|^{-1/p} \|\rho^{b-1} \nabla_\psi^{4-b} \partial_r^b \nabla w\|_{L^p(Q_r^d(z))} \\ &\quad + r^4 \theta^{4-2b} |Q_r^d(z)|^{-1/p} \|\rho^{b-2} \nabla_\psi^{4-b} \partial_r^b \partial_t w\|_{L^p(Q_r^d(z))}. \end{aligned}$$

With the help of the analyticity estimates (52), we easily deduce that

$$r\theta \|t^2 \nabla_\psi^2 \partial_s^2 w\|_{L^\infty(Q_r^d(z))} + (r\theta)^2 \|t \nabla_\psi \partial_s^3 w\|_{L^\infty(Q_r^d(z))} \lesssim \|g\|_{W^{1,\infty}}. \tag{54}$$

An analogous argument yields the corresponding control of the time derivatives, namely

$$\frac{r^3}{\theta} \|\partial_t w\|_{L^\infty(Q_r^d(z))} \lesssim \|g\|_{W^{1,\infty}}. \tag{55}$$

In order to deduce control over the fourth-order vertical derivatives, we rewrite the nonlinear equation (6) as

$$\rho \partial_r^2 (\rho \partial_r^2 w) = f[w] - \partial_t w + \text{l.o.t.}$$

The terms on the right-hand side are all uniformly controlled thanks to (52), (54) and (55). Similarly, we may write

$$-\rho^{-1} \partial_r (\rho^2 \partial_r v) = h$$

for some h such that $t^\kappa h \in L^\infty$ for some $\kappa > 0$, and where $v = -\rho^{-1} \partial_r (\rho^2 \partial_r w)$. This identity can be integrated so that

$$\partial_r v = \rho^{-2} \int_r^1 \rho h \, d\tilde{r}.$$

The expression on the right is differentiable with

$$\partial_r^2 v = 2\rho^{-3} r \int_r^1 \rho h \, d\tilde{r} - \rho^{-1} h.$$

We deduce that $\rho t^\kappa \partial_r^2 v \in L^\infty$ and thus $\rho^2 t^\kappa \partial_r^4 w \in L^\infty$.

This argument can be iterated and yields the smoothness of w . □

Appendix: Derivation of the transformed equation

Let us write $z = \Phi_t(x)$. We will first verify that Φ defines a diffeomorphism. For this purpose, we compute the derivatives of Φ in terms of x and v ,

$$\partial_i \Phi^j = \frac{\delta_{ij}}{(2v + |x|^2)^{1/2}} - \frac{x_j (\partial_i v + x_i)}{(2v + |x|^2)^{3/2}}.$$

Recalling the elementary formula $\det(I - a \otimes b) = I - a \cdot b$ for any two vectors a and b , we compute that

$$\det \nabla \phi(x) = \frac{2v - x \cdot \nabla v}{(2v + |x|^2)^{N/2+1}}.$$

If v is close to the Smyth–Hill solution in the sense that

$$\|v - v_*\|_{L^\infty(\mathcal{P}(v))} + \|\nabla v + x\|_{L^\infty(\mathcal{P}(v))} \leq \varepsilon$$

for some small ε , we find that $2v - x \cdot \nabla v \geq 1 - 3\varepsilon$ and $2v + |x|^2 \geq 1 - 2\varepsilon$, which implies that the Jacobi determinant is finite if ε is sufficiently small.

Let us express the derivative of Φ in terms of the new variables z and w . Differentiating (5) yields

$$\partial_i v + x_i = (1 + w)\nabla w \cdot \partial_i \Phi = \partial_i w - \frac{z \cdot \nabla w}{1 + w}(\partial_i v + x_i),$$

and thus

$$\partial_i v + x_i = \frac{1 + w}{1 + w + z \cdot \nabla w} \partial_i w.$$

Plugging this and (5) into the expression for the derivatives of Φ , we find

$$\partial_i \Phi^j = \frac{\delta_{ij}}{1 + w} - \frac{z_j \partial_i w}{(1 + w)(1 + w + z \cdot \nabla w)}.$$

Under the assumption that w is such that

$$\|w\|_{L^\infty} + \|\nabla w\|_{L^\infty} \leq \varepsilon$$

for some small ε , we see by a calculation similar as the one above that Φ is a diffeomorphism.

We will now compute how the change of variables acts on the confined thin-film equation (3). For notational convenience, we set

$$\rho(z) = \frac{1}{2}(1 - |z|^2),$$

and $\tilde{w} = 1 + w$, with the effect that

$$\rho \tilde{w}^2 = v = \gamma u^{1/2}. \tag{56}$$

For an arbitrary function $f = f(z)$, it thus holds that

$$\partial_i (f(\Phi)) = \frac{\partial_i f}{\tilde{w}} - \frac{(z \cdot \nabla f) \partial_i \tilde{w}}{\tilde{w}(\tilde{w} + z \cdot \nabla \tilde{w})}. \tag{57}$$

Now, differentiating (56) with respect to x_i yields

$$\begin{aligned} \gamma^2 \partial_i u &= \frac{1}{\tilde{w}} \partial_i (\rho^2 \tilde{w}^4) - \frac{\partial_i \tilde{w}}{\tilde{w}(\tilde{w} + z \cdot \nabla \tilde{w})} z \cdot \nabla (\rho^2 \tilde{w}^4) \\ &= -2\rho \tilde{w}^3 z_i + 2 \frac{\rho \tilde{w}^3 \partial_i \tilde{w}}{\tilde{w} + z \cdot \nabla \tilde{w}}. \end{aligned}$$

Differentiating with respect to x_i again, we obtain that

$$\begin{aligned} \frac{\gamma^2}{2} \partial_i^2 u &= -(\rho - z_i^2) \tilde{w}^2 + \frac{\tilde{w}^2}{\tilde{w} + z \cdot \nabla \tilde{w}} \rho^{-1} \partial_i (\rho^2 \partial_i \tilde{w}) \\ &\quad - \frac{\tilde{w}^2}{\tilde{w} + z \cdot \nabla \tilde{w}} \rho z \cdot \nabla \left(\frac{(\partial_i \tilde{w})^2}{\tilde{w} + z \cdot \nabla \tilde{w}} \right) + (\rho + |z|^2) \frac{\tilde{w}^2 (\partial_i \tilde{w})^2}{(\tilde{w} + z \cdot \nabla \tilde{w})^2}. \end{aligned}$$

Hence, summing over i and rearranging terms yields

$$\frac{\gamma^2}{2} \Delta u \times \frac{\tilde{w} + z \cdot \nabla \tilde{w}}{\tilde{w}^2} = (1 - (N + 2)\rho)(\tilde{w} + z \cdot \nabla \tilde{w}) - \mathcal{L}\tilde{w} + (1 - \rho) \frac{|\nabla \tilde{w}|^2}{\tilde{w} + z \cdot \nabla \tilde{w}} - \rho z \cdot \nabla \left(\frac{|\nabla \tilde{w}|^2}{\tilde{w} + z \cdot \nabla \tilde{w}} \right).$$

With the help of the \star -notation, the (nonlinear) term in the second line of the above identity can be rewritten as

$$p \star \sum_{k=1}^2 \frac{(\nabla \tilde{w})^{(k-1)\star}}{(\tilde{w} + z \cdot \nabla \tilde{w})^k} ((\nabla \tilde{w})^{2\star} + \rho \nabla \tilde{w} \star \nabla^2 \tilde{w}).$$

In what follows, it should become clear why this way of writing drastically simplifies the notation.

With the help of (57), we compute

$$\partial_i \left(\left(\frac{\tilde{w}^2}{\tilde{w} + z \cdot \nabla \tilde{w}} f \right) (\Phi) \right) = \frac{\tilde{w}}{\tilde{w} + z \cdot \nabla \tilde{w}} \left(\partial_i f - z \cdot \nabla \left(\frac{\partial_i \tilde{w} f}{\tilde{w} + z \cdot \nabla \tilde{w}} \right) \right)$$

for any function $f = f(z)$, and thus

$$\begin{aligned} \frac{\gamma^2}{2} (\partial_i \Delta u - x_i) \times \frac{\tilde{w} + z \cdot \nabla \tilde{w}}{\tilde{w}} &= -N \partial_i \tilde{w} - \partial_i \mathcal{L}\tilde{w} \\ &+ p \star \frac{(\nabla \tilde{w})^{2\star} + \nabla \tilde{w} \star \nabla^2 \tilde{w} + \rho (\nabla^2 \tilde{w})^{2\star} + \rho \nabla \tilde{w} \star \nabla^3 \tilde{w}}{\tilde{w} + z \cdot \nabla \tilde{w}} \\ &+ p \star \frac{(\nabla \tilde{w})^{3\star} + (\nabla \tilde{w})^{2\star} \star \nabla^2 \tilde{w} + \rho \nabla \tilde{w} \star (\nabla^2 \tilde{w})^{2\star} + \rho (\nabla \tilde{w})^{2\star} \star \nabla^3 \tilde{w}}{(\tilde{w} + z \cdot \nabla \tilde{w})^2} \\ &+ p \star \frac{(\nabla \tilde{w})^{4\star} + (\nabla \tilde{w})^{3\star} \star \nabla^2 \tilde{w} + \rho (\nabla \tilde{w})^{2\star} \star (\nabla^2 \tilde{w})^{2\star} + \rho (\nabla \tilde{w})^{3\star} \star \nabla^3 \tilde{w}}{(\tilde{w} + z \cdot \nabla \tilde{w})^3} \\ &+ p \star \frac{(\nabla \tilde{w})^{5\star} + \rho (\nabla \tilde{w})^{4\star} \star \nabla^2 \tilde{w} + \rho (\nabla \tilde{w})^{3\star} \star (\nabla^2 \tilde{w})^{2\star}}{(\tilde{w} + z \cdot \nabla \tilde{w})^4}. \end{aligned}$$

We notice that the nonlinearity belongs to the class

$$p \star \sum_{k=1}^4 \frac{(\nabla \tilde{w})^{(k-1)\star}}{(\tilde{w} + z \cdot \nabla \tilde{w})^k} \star ((\nabla \tilde{w})^{2\star} + \nabla \tilde{w} \star \nabla^2 \tilde{w} + \rho (\nabla^2 \tilde{w})^{2\star} + \rho \nabla \tilde{w} \star \nabla^3 \tilde{w}).$$

Similarly to the above, we compute for an arbitrary function $f = f(z)$ that

$$\partial_i \left(\left(\frac{\tilde{w}^5}{\tilde{w} + z \cdot \nabla \tilde{w}} f \right) (\Phi) \right) = \frac{\tilde{w}^4}{\tilde{w} + z \cdot \nabla \tilde{w}} \left(\partial_i f + 3 \frac{\partial_i \tilde{w} f}{\tilde{w} + z \cdot \nabla \tilde{w}} - z \cdot \nabla \left(\frac{\partial_i \tilde{w} f}{\tilde{w} + z \cdot \nabla \tilde{w}} \right) \right),$$

and thus

$$\begin{aligned} \frac{\gamma^4}{2} \nabla \cdot (u \nabla \Delta u - ux) \times \frac{\tilde{w} + z \cdot \nabla \tilde{w}}{\rho \tilde{w}^4} &= (N + \mathcal{L}) \mathcal{L}\tilde{w} \\ &+ p \star \tilde{R}_{-1}[\tilde{w}] \star ((\nabla \tilde{w})^{2\star} + \nabla \tilde{w} \star \nabla^2 \tilde{w}) \\ &+ p \star \tilde{R}_{-1}[\tilde{w}] \star \rho ((\nabla^2 \tilde{w})^{2\star} + \nabla \tilde{w} \star \nabla^3 \tilde{w}) \\ &+ p \star \tilde{R}_{-2}[\tilde{w}] \star \rho^2 ((\nabla^2 \tilde{w})^{3\star} + \nabla \tilde{w} \star \nabla^2 \tilde{w} \star \nabla^3 \tilde{w} + (\nabla \tilde{w})^{2\star} \star \nabla^4 \tilde{w}), \end{aligned}$$

where $\tilde{R}_i[\tilde{w}] = r_i(\nabla\tilde{w}, \tilde{w} + z \cdot \nabla\tilde{w})$ for some rational functions r_i that are homogeneous of degree i , i.e., $r_i(sa, sb) = s^i r_i(a, b)$.

We finally turn to the computation of the time derivative. For this notice first that

$$\partial_t \Phi_t(x) = -\frac{\gamma^2}{2} \frac{z}{\rho\tilde{w}^4} \partial_t u,$$

and thus, a short computation shows that

$$\frac{\gamma^2}{2} \partial_t u = \frac{\rho\tilde{w}^4}{\tilde{w} + z \cdot \nabla\tilde{w}} \partial_t \tilde{w}.$$

After a rescaling of time $t \rightarrow \gamma^2 t$, and recalling that $\tilde{w} = 1 + w$, we find the transformed equation (6).

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References

- [Angenent 1990] S. B. Angenent, “Nonlinear analytic semiflows”, *Proc. Roy. Soc. Edinburgh Sect. A* **115**:1-2 (1990), 91–107. MR Zbl
- [Belgacem et al. 2016] F. B. Belgacem, M. V. Gnann, and C. Kuehn, “A dynamical systems approach for the contact-line singularity in thin-film flows”, *Nonlinear Anal.* **144** (2016), 204–235. MR Zbl
- [Bellaïche and Risler 1996] A. Bellaïche and J.-J. Risler (editors), *Sub-Riemannian geometry*, Progress in Mathematics **144**, Birkhäuser, Basel, 1996. MR Zbl
- [Beretta et al. 1995] E. Beretta, M. Bertsch, and R. Dal Passo, “Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation”, *Arch. Rational Mech. Anal.* **129**:2 (1995), 175–200. MR Zbl
- [Bernis 1996] F. Bernis, “Finite speed of propagation and continuity of the interface for thin viscous flows”, *Adv. Differential Equations* **1**:3 (1996), 337–368. MR Zbl
- [Bernis and Friedman 1990] F. Bernis and A. Friedman, “Higher order nonlinear degenerate parabolic equations”, *J. Differential Equations* **83**:1 (1990), 179–206. MR Zbl
- [Bertozzi and Pugh 1996] A. L. Bertozzi and M. Pugh, “The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions”, *Comm. Pure Appl. Math.* **49**:2 (1996), 85–123. MR Zbl
- [Bertsch et al. 1998] M. Bertsch, R. Dal Passo, H. Garcke, and G. Grün, “The thin viscous flow equation in higher space dimensions”, *Adv. Differential Equations* **3**:3 (1998), 417–440. MR Zbl
- [Carrillo and Toscani 2002] J. A. Carrillo and G. Toscani, “Long-time asymptotics for strong solutions of the thin film equation”, *Comm. Math. Phys.* **225**:3 (2002), 551–571. MR Zbl
- [Coifman and Weiss 1971] R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Mathematics **242**, Springer, 1971. MR Zbl
- [Dal Passo et al. 1998] R. Dal Passo, H. Garcke, and G. Grün, “On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions”, *SIAM J. Math. Anal.* **29**:2 (1998), 321–342. MR Zbl
- [Daskalopoulos and Hamilton 1998] P. Daskalopoulos and R. Hamilton, “Regularity of the free boundary for the porous medium equation”, *J. Amer. Math. Soc.* **11**:4 (1998), 899–965. MR Zbl
- [Degtyarev 2017] S. Degtyarev, “Classical solvability of the multidimensional free boundary problem for the thin film equation with quadratic mobility in the case of partial wetting”, *Discrete Contin. Dyn. Syst.* **37**:7 (2017), 3625–3699. MR Zbl

- [Deimling 1985] K. Deimling, *Nonlinear functional analysis*, Springer, 1985. MR Zbl
- [Denzler et al. 2015] J. Denzler, H. Koch, and R. J. McCann, *Higher-order time asymptotics of fast diffusion in Euclidean space: a dynamical systems approach*, Mem. Amer. Math. Soc. **1101**, Amer. Math. Soc., Providence, RI, 2015. MR Zbl
- [Evans and Gariepy 1992] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, FL, 1992. MR Zbl
- [Ferreira and Bernis 1997] R. Ferreira and F. Bernis, “Source-type solutions to thin-film equations in higher dimensions”, *European J. Appl. Math.* **8**:5 (1997), 507–524. MR Zbl
- [Giacomelli and Knüpfer 2010] L. Giacomelli and H. Knüpfer, “A free boundary problem of fourth order: classical solutions in weighted Hölder spaces”, *Comm. Partial Differential Equations* **35**:11 (2010), 2059–2091. MR Zbl
- [Giacomelli and Otto 2001] L. Giacomelli and F. Otto, “Variational formulation for the lubrication approximation of the Hele-Shaw flow”, *Calc. Var. Partial Differential Equations* **13**:3 (2001), 377–403. MR Zbl
- [Giacomelli and Otto 2003] L. Giacomelli and F. Otto, “Rigorous lubrication approximation”, *Interfaces Free Bound.* **5**:4 (2003), 483–529. MR Zbl
- [Giacomelli et al. 2008] L. Giacomelli, H. Knüpfer, and F. Otto, “Smooth zero-contact-angle solutions to a thin-film equation around the steady state”, *J. Differential Equations* **245**:6 (2008), 1454–1506. MR Zbl
- [Giacomelli et al. 2013] L. Giacomelli, M. V. Gnann, and F. Otto, “Regularity of source-type solutions to the thin-film equation with zero contact angle and mobility exponent between $\frac{3}{2}$ and 3”, *European J. Appl. Math.* **24**:5 (2013), 735–760. MR Zbl
- [Giacomelli et al. 2014] L. Giacomelli, M. V. Gnann, H. Knüpfer, and F. Otto, “Well-posedness for the Navier-slip thin-film equation in the case of complete wetting”, *J. Differential Equations* **257**:1 (2014), 15–81. MR Zbl
- [Gnann 2015] M. V. Gnann, “Well-posedness and self-similar asymptotics for a thin-film equation”, *SIAM J. Math. Anal.* **47**:4 (2015), 2868–2902. MR Zbl
- [Gnann 2016] M. V. Gnann, “On the regularity for the Navier-slip thin-film equation in the perfect wetting regime”, *Arch. Ration. Mech. Anal.* **222**:3 (2016), 1285–1337. MR Zbl
- [Grigor’yan 2006] A. Grigor’yan, “Heat kernels on weighted manifolds and applications”, pp. 93–191 in *The ubiquitous heat kernel*, edited by J. Jorgenson and L. Walling, Contemp. Math. **398**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl
- [John 2015] D. John, “On uniqueness of weak solutions for the thin-film equation”, *J. Differential Equations* **259**:8 (2015), 4122–4171. MR Zbl
- [Kienzler 2016] C. Kienzler, “Flat fronts and stability for the porous medium equation”, *Comm. Partial Differential Equations* **41**:12 (2016), 1793–1838. MR Zbl
- [Knüpfer 2011] H. Knüpfer, “Well-posedness for the Navier slip thin-film equation in the case of partial wetting”, *Comm. Pure Appl. Math.* **64**:9 (2011), 1263–1296. MR Zbl
- [Knüpfer 2015] H. Knüpfer, “Well-posedness for a class of thin-film equations with general mobility in the regime of partial wetting”, *Arch. Ration. Mech. Anal.* **218**:2 (2015), 1083–1130. MR Zbl
- [Knüpfer and Masmoudi 2015] H. Knüpfer and N. Masmoudi, “Darcy’s flow with prescribed contact angle: well-posedness and lubrication approximation”, *Arch. Ration. Mech. Anal.* **218**:2 (2015), 589–646. MR Zbl
- [Koch 1999] H. Koch, *Non-Euclidean singular integrals and the porous medium equation*, habilitation thesis, Universität Heidelberg, 1999, available at www.math.uni-bonn.de/people/koch/habil.ps.gz.
- [Koch 2004] H. Koch, “Partial differential equations and singular integrals”, pp. 59–122 in *Dispersive nonlinear problems in mathematical physics*, edited by P. D’Ancona and V. Georgev, Quad. Mat. **15**, Dept. Math., Seconda Univ. Napoli, Caserta, 2004. MR Zbl
- [Koch 2008] H. Koch, “Partial differential equations with non-Euclidean geometries”, *Discrete Contin. Dyn. Syst. Ser. S* **1**:3 (2008), 481–504. MR Zbl
- [Koch and Lamm 2012] H. Koch and T. Lamm, “Geometric flows with rough initial data”, *Asian J. Math.* **16**:2 (2012), 209–235. MR Zbl
- [Matthes et al. 2009] D. Matthes, R. J. McCann, and G. Savaré, “A family of nonlinear fourth order equations of gradient flow type”, *Comm. Partial Differential Equations* **34**:10–12 (2009), 1352–1397. MR Zbl

- [McCann and Seis 2015] R. J. McCann and C. Seis, “The spectrum of a family of fourth-order nonlinear diffusions near the global attractor”, *Comm. Partial Differential Equations* **40**:2 (2015), 191–218. MR Zbl
- [Oron et al. 1997] A. Oron, S. H. Davis, and S. G. Bankoff, “Long-scale evolution of thin liquid films”, *Rev. Mod. Phys.* **69** (1997), 931–980.
- [Otto 1998] F. Otto, “Lubrication approximation with prescribed nonzero contact angle”, *Comm. Partial Differential Equations* **23**:11-12 (1998), 2077–2164. MR Zbl
- [Otto 2001] F. Otto, “The geometry of dissipative evolution equations: the porous medium equation”, *Comm. Partial Differential Equations* **26**:1-2 (2001), 101–174. MR Zbl
- [Seis 2014] C. Seis, “Long-time asymptotics for the porous medium equation: the spectrum of the linearized operator”, *J. Differential Equations* **256**:3 (2014), 1191–1223. MR Zbl
- [Seis 2015] C. Seis, “Invariant manifolds for the porous medium equation”, preprint, 2015. arXiv
- [Seis \geq 2018] C. Seis, “Invariant manifolds for the thin film equation”, work in preparation.
- [Smyth and Hill 1988] N. F. Smyth and J. M. Hill, “High-order nonlinear diffusion”, *IMA J. Appl. Math.* **40**:2 (1988), 73–86. MR Zbl
- [Stein 1970] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series **30**, Princeton University Press, 1970. MR Zbl
- [Stein 1993] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series **43**, Princeton University Press, 1993. MR Zbl

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CHRISTIAN SEIS: seis@wwu.de

Institut für Analysis und Numerik, Westfälische Wilhelms-Universität Münster, Münster, Germany

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