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DOLGOPYAT'S METHOD AND THE FRACTAL UNCERTAINTY PRINCIPLE

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We show a fractal uncertainty principle with exponent $\frac{1}{2}-\delta+\varepsilon,\ \varepsilon>0$, for Ahlfors–David regular subsets of $\mathbb{R}$ of dimension $\delta\in(0,1)$. This is an improvement over the volume bound $\frac{1}{2}-\delta$, and ε is estimated explicitly in terms of the regularity constant of the set. The proof uses a version of techniques originating in the works of Dolgopyat, Naud, and Stoyanov on spectral radii of transfer operators. Here the group invariance of the set is replaced by its fractal structure. As an application, we quantify the result of Naud on spectral gaps for convex cocompact hyperbolic surfaces and obtain a new spectral gap for open quantum baker maps.

1. Introduction

A *fractal uncertainty principle* (FUP) states that no function can be localized close to a fractal set in both position and frequency. Its most basic form is

$$\|\mathbb{1}_{\Lambda(h)}\mathcal{F}_h\mathbb{1}_{\Lambda(h)}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}=\mathcal{O}(h^{\beta})\quad\text{as } h\to 0, \tag{1-1}$$

where $\Lambda(h)$ is the h-neighborhood of a bounded set $\Lambda\subset\mathbb{R}$, β is called the *exponent* of the uncertainty principle, and $\mathcal{F}_h$ is the semiclassical Fourier transform:

$$\mathcal{F}_h u(\xi)=(2\pi h)^{-\frac{1}{2}}\int_{\mathbb{R}} e^{-\frac{ix\xi}{h}}u(x)\,dx. \tag{1-2}$$

We additionally assume that Λ is an Ahlfors–David regular set (see Definition 1.1) of dimension $\delta\in(0,1)$ with some regularity constant $C_R>1$. Using the bounds $\|\mathcal{F}_h\|_{L^2\to L^2}=1,\ \|\mathcal{F}_h\|_{L^1\to L^\infty}\le h^{-\frac{1}{2}}$, the Lebesgue volume bound $\mu_L(\Lambda(h))\le Ch^{1-\delta}$, and Hölder's inequality, it is easy to obtain (1-1) with $\beta=\max\bigl(0,\frac{1}{2}-\delta\bigr)$.

Fractal uncertainty principles were applied by Dyatlov and Zahl [2016], Dyatlov and Jin [2017], and Bourgain and Dyatlov [2016] to the problem of essential spectral gap in quantum chaos: *which open quantum chaotic systems have exponential decay of local energy at high frequency*? A fractal uncertainty principle can be used to show local energy decay $\mathcal{O}(e^{-\beta t})$, as was done for convex cocompact hyperbolic quotients in [Dyatlov and Zahl 2016] and for open quantum baker's maps in [Dyatlov and Jin 2017]. Here Λ is related to the set of all trapped classical trajectories of the system and (1-1) needs to be replaced by a more general statement, in particular allowing for a different phase in (1-2). The volume bound $\beta=\frac{1}{2}-\delta$ corresponds to the Patterson–Sullivan gap or more generally, the *pressure gap*. See Sections 4–5 below for a more detailed discussion.

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A natural question is: can one obtain (1-1) with $\beta > \max\big(0, \frac{1}{2} - \delta\big)$, and if so, how does the size of the improvement depend on δ and C_R? Partial answers to this question have been obtained in the papers mentioned above:

- Dyatlov and Zahl [2016] obtained FUP with $\beta > 0$ when $\big|\delta - \frac{1}{2}\big|$ is small depending on C_R, and gave the bound $\beta > \exp(-\boldsymbol{K}(1 + \log^{14} C_R))$, where $\boldsymbol{K}$ is a global constant.
- Bourgain and Dyatlov [2016] proved FUP with $\beta > 0$ in the entire range $\delta \in (0, 1)$, with no explicit bounds on the dependence of β on δ, C_R.
- Dyatlov and Jin [2017] showed that discrete Cantor sets satisfy FUP with $\beta > \max\big(0, \frac{1}{2} - \delta\big)$ in the entire range $\delta \in (0, 1)$ and obtained quantitative lower bounds on the size of the improvement — see Section 5 below.

Our main result, Theorem 1, shows that FUP holds with $\beta > \frac{1}{2} - \delta$ in the case $\delta \in (0, 1)$, and gives bounds on $\beta - \frac{1}{2} + \delta$ which are polynomial in C_R and thus stronger than the ones in [Dyatlov and Zahl 2016]. Applications include

- an essential spectral gap for convex cocompact hyperbolic surfaces of size $\beta > \frac{1}{2} - \delta$, recovering and making quantitative the result of [Naud 2005], see Section 4;
- an essential spectral gap of size $\beta > \max\big(0, \frac{1}{2} - \delta\big)$ for open quantum baker's maps, extending the result of [Dyatlov and Jin 2017] to matrices whose sizes are not powers of the base, see Section 5. (For the case $\delta > \frac{1}{2}$ we use the results of [Bourgain and Dyatlov 2016] rather than Theorem 1.)

1A. ***Statement of the result.*** We recall the following definition of Ahlfors–David regularity, which requires that a set (or a measure) has the same dimension δ at all points and on a range of scales:

Definition 1.1. Let $X \subset \mathbb{R}$ be compact, μ_X be a finite measure supported on X, and $\delta \in [0, 1]$. We say that (X, μ_X) is *δ-regular* up to scale $h \in [0, 1)$ with regularity constant $C_R \geq 1$ if

- for each interval I of size $|I| \geq h$, we have $\mu_X(I) \leq C_R |I|^\delta$;
- if additionally $|I| \leq 1$ and the center of I lies in X, then $\mu_X(I) \geq C_R^{-1} |I|^\delta$.

Our fractal uncertainty principle has a general form which allows for two different sets X, Y of different dimensions in (1-1), replaces the Lebesgue measure by the fractal measures μ_X, μ_Y, and allows a general nondegenerate phase and amplitude in (1-2):

Theorem 1. *Assume that (X, μ_X) is δ-regular, and (Y, μ_Y) is δ'-regular, up to scale $h \in (0, 1)$ with constant C_R, where $0 < \delta, \delta' < 1$, and $X \subset I_0$, $Y \subset J_0$ for some intervals I_0, J_0. Consider an operator $\mathcal{B}_h : L^1(Y, \mu_Y) \to L^\infty(X, \mu_X)$ of the form*

$$\mathcal{B}_h f(x) = \int_Y \exp\left(\frac{i\Phi(x, y)}{h}\right) G(x, y) f(y)\, d\mu_Y(y), \tag{1-3}$$

where $\Phi(x, y) \in C^2(I_0 \times J_0; \mathbb{R})$ satisfies $\partial^2_{xy}\Phi \neq 0$ and $G(x, y) \in C^1(I_0 \times J_0; \mathbb{C})$.

Then there exist constants $C, \varepsilon_0 > 0$ such that

$$\|\mathcal{B}_h\|_{L^2(Y,\mu_Y) \to L^2(X,\mu_X)} \leq C h^{\varepsilon_0}. \tag{1-4}$$

Here ε_0 depends only on δ, δ', C_R,

$$\varepsilon_0 = (5C_R)^{-80\left(\frac{1}{\delta(1-\delta)}+\frac{1}{\delta'(1-\delta')}\right)}, \tag{1-5}$$

and C additionally depends on I_0, J_0, Φ, G.

Remarks. (1) Theorem 1 implies the Lebesgue-measure version of FUP, (1-1), with exponent $\beta = \frac{1}{2} - \delta + \varepsilon_0$. Indeed, assume that (Λ, μ_Λ) is δ-regular up to scale h with constant C_R. Put $X := \Lambda(h)$ and let μ_X be $h^{\delta-1}$ times the restriction of the Lebesgue measure to X. Then (X, μ_X) is δ-regular up to scale h with constant $30C_R^2$; see Lemma 2.2. We apply Theorem 1 with $(Y, \mu_Y) := (X, \mu_X)$, $G \equiv 1$, and $\Phi(x, y) = -xy$; then

$$\|\mathbb{1}_{\Lambda(h)}\mathcal{F}_h\mathbb{1}_{\Lambda(h)}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \frac{h^{\frac{1}{2}-\delta}}{\sqrt{2\pi}}\|\mathcal{B}_h\|_{L^2(X,\mu_X)\to L^2(X,\mu_X)} \le Ch^{\frac{1}{2}-\delta+\varepsilon_0}.$$

(2) Definition 1.1 is slightly stronger than [Bourgain and Dyatlov 2016, Definition 1.1] (where "up to scale h" should be interpreted as "on scales h to 1") because it imposes an upper bound on $\mu_L(I)$ when $|I| > 1$. However, this difference is insignificant as long as X is compact. Indeed, if $X \subset [-R, R]$ for some integer $R > 0$, then using upper bounds on μ_L on intervals of size 1 we get $\mu_L(I) \le \mu_L(X) \le 2RC_R \le 2RC_R|I|^\delta$ for each interval I of size $|I| > 1$.

(3) The restriction $\delta, \delta' > 0$ is essential. Indeed, if $\delta' = 0$, $Y = \{0\}$, μ_Y is the delta measure, and $f \equiv 1$, $G \equiv 1$, then

$$\|\mathcal{B}_h f\|_{L^2(X,\mu_X)} = \sqrt{\mu_X(X)}.$$

The restriction $\delta, \delta' < 1$ is technical; however, in the application to Lebesgue-measure FUP this restriction is not important since $\beta = \frac{1}{2} - \delta + \varepsilon_0 < 0$ when δ is close to 1.

(4) The constants in (1-5) are far from sharp. However, the dependence of ε_0 on C_R cannot be removed entirely. Indeed, [Dyatlov and Jin 2017] gives examples of Cantor sets for which the best exponent ε_0 in (1-4) decays polynomially as $C_R \to \infty$; see Proposition 3.17 of that paper. See also Sections 5B–5C.

1B. ***Ideas of the proof.*** The proof of Theorem 1 is inspired by the method originally developed by Dolgopyat [1998] and its application to essential spectral gaps for convex cocompact hyperbolic surfaces by Naud [2005]. In fact, Theorem 1 implies a quantitative version of Naud's result; see Section 4. More recently, Dolgopyat's method has been applied to the spectral-gap problem by Petkov and Stoyanov [2010], Stoyanov [2011; 2012], Oh and Winter [2016], and Magee, Oh and Winter [Magee et al. 2017].

We give a sketch of the proof, assuming for simplicity that $G \equiv 1$. For $f \in L^2(Y, \mu_Y)$, we have

$$\|\mathcal{B}_h f\|_{L^2(X,\mu_X)} \le \sqrt{\mu_X(X)\mu_Y(Y)} \cdot \|f\|_{L^2(Y,\mu_Y)}, \tag{1-6}$$

applying Hölder's inequality and the bound $\|\mathcal{B}_h\|_{L^1(X,\mu_X)\to L^\infty(Y,\mu_Y)} \le 1$. However, under a mild assumption on the differences between the phases $\Phi(x, y)$ for different x, y, the resulting estimate is not sharp, as illustrated by the following example where $X = Y = \{1, 2\}$, $\mu_X(j) = \mu_Y(j) = \frac{1}{2}$ for $j = 1, 2$, and $\omega_{j\ell} := \Phi(j, \ell)/h$:

Lemma 1.2. *Assume that* $\omega_{j\ell} \in \mathbb{R}$, $j, \ell = 1, 2$, *satisfy*

$$\tau := \omega_{11} + \omega_{22} - \omega_{12} - \omega_{21} \notin 2\pi\mathbb{Z}. \tag{1-7}$$

For $f_1, f_2 \in \mathbb{C}$, *put*

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \frac{1}{2} \begin{pmatrix} \exp(i\omega_{11}) & \exp(i\omega_{12}) \\ \exp(i\omega_{21}) & \exp(i\omega_{22}) \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Assume that $(f_1, f_2) \neq 0$. *Then*

$$|u_1|^2 + |u_2|^2 < |f_1|^2 + |f_2|^2. \tag{1-8}$$

Remark. Note that (1-8) cannot be replaced by either of the statements

$$|u_1| + |u_2| < |f_1| + |f_2|, \quad \max(|u_1|, |u_2|) < \max(|f_1|, |f_2|).$$

Indeed, the first statement fails when $f_1 = 0$, $f_2 = 1$. The second one fails if $\omega_{11} = \omega_{12}$ and $f_1 = f_2 = 1$. This explains why we use L^2 norms in the iteration step, Lemma 3.2.

Proof. We have

$$\tfrac{1}{2}(|u_1|^2 + |u_2|^2) \le \max(|u_1|^2, |u_2|^2) \le \big(\tfrac{1}{2}(|f_1| + |f_2|)\big)^2 \le \tfrac{1}{2}(|f_1|^2 + |f_2|^2). \tag{1-9}$$

Assume that (1-8) does not hold. Then the inequalities in (1-9) have to be equalities, which implies that $|u_1| = |u_2|$, $|f_1| = |f_2| > 0$, and for $a = 1, 2$,

$$\exp(i(\omega_{a1} - \omega_{a2})) f_1 \bar{f_2} \ge 0.$$

The latter statement contradicts (1-7). □

To get the improvement h^{ε_0} in (1-4), we use the nonsharpness of (1-6) on many scales:

- We fix a large integer $L > 1$ depending on δ, C_R and discretize X and Y on scales $1, L^{-1}, \dots, L^{-K}$, where $h \sim L^{-K}$. This results in two trees of intervals V_X, V_Y, with vertices of height k corresponding to intervals of length $\sim L^{-k}$.

- For each interval J in the tree V_Y, we consider the function

$$F_J(x) = \frac{1}{\mu_Y(J)} \exp\left(-\frac{i\Phi(x, y_J)}{h}\right) \mathcal{B}_h(\mathbb{1}_J f)(x),$$

where y_J is the center of J. The function F_J oscillates on scale $h/|J|$. Thus both F_J and the rescaled derivative $h|J|^{-1}F_J'$ are controlled in uniform norm by $\|f\|_{L^1(Y,\mu_Y)}$. We express this fact using the spaces $\mathcal{C}_\theta$ introduced in Section 2B.

- If $J_1, \dots, J_B \in V_Y$ are the children of J, then F_J can be written as a convex combination of $F_{J_1}, \dots, F_{J_B}$ multiplied by some phase factors $e^{i\Psi_b}$; see (3-12). We then employ an iterative procedure which estimates a carefully chosen norm of F_J via the norms of $F_{J_1}, \dots, F_{J_B}$. Each step in this procedure gives a gain $1 - \varepsilon_1 < 1$ in the norm, and after K steps we obtain a gain polynomial in h.

- To obtain a gain at each step, we consider two intervals $I \in V_X$, $J \in V_Y$ such that $|I| \cdot |J| \sim Lh$, take their children $I_1, \dots, I_A$ and $J_1, \dots, J_B$, and argue similarly to Lemma 1.2 to show that the triangle inequality for $e^{i\Psi_1} J_1, \dots, e^{i\Psi_B} J_B$ cannot be sharp on all the intervals $I_1, \dots, I_A$.

• To do the latter, we take two pairs of children I_a, I'_a (with generic points in I_a, I'_a denoted by $x_a, x_{a'}$) and J_b, J'_b. Due to the control on the derivatives of F_{J_b}, the differences $|F_{J_b}(x_a) - F_{J_b}(x_{a'})|$ and $|F_{J_{b'}}(x_a) - F_{J_{b'}}(x_{a'})|$ are bounded by $(Lh)^{-1}|J| \cdot |x_a - x_{a'}|$. On the other hand, the phase shift τ from (1-7) equals

$$\tau = \Psi_b(x_a) + \Psi_{b'}(x_{a'}) - \Psi_{b'}(x_a) - \Psi_b(x_{a'}) \sim h^{-1}(x_a - x_{a'})(y_b - y_{b'}).$$

Choosing a, a', b, b' such that $|x_a - x_{a'}| \sim L^{-\frac{2}{3}}|I|$, $|y_b - y_{b'}| \sim L^{-\frac{2}{3}}|J|$, and recalling that $|I| \cdot |J| \sim Lh$, we see that $\tau \sim L^{-\frac{1}{3}}$ does not lie in $2\pi\mathbb{Z}$ and it is larger than $(Lh)^{-1}|J| \cdot |x_a - x_{a'}| \sim L^{-\frac{2}{3}}$. This gives the necessary improvement on each step. Keeping track of the parameters in the argument, we obtain the bound (1-5) on ε_0.

This argument has many similarities with the method of Dolgopyat mentioned above. In particular, an inductive argument using L^2 norms appears for instance in [Naud 2005, Lemma 5.4], which also features the spaces $\mathcal{C}_\theta$. The choice of children $I_a, I_{a'}, J_b, J_{b'}$ in the last step above is similar to the nonlocal integrability condition (NLIC); see for instance [Naud 2005, Sections 2 and 5.3]. However, our inductive Lemma 3.2 avoids the use of Dolgopyat operators and dense subsets, see for instance [Naud 2005, p.138], instead relying on strict convexity of balls in Hilbert spaces, see Lemma 2.7.

Moreover, the strategy of obtaining an essential spectral gap for hyperbolic surfaces in the present paper is significantly different from that of [Naud 2005]. The latter uses zeta-function techniques to reduce the spectral-gap question to a spectral radius bound of a Ruelle transfer operator of the Bowen–Series map associated to the surface. The present paper instead relies on microlocal analysis of the scattering resolvent in [Dyatlov and Zahl 2016] to reduce the gap problem to a fractal uncertainty principle, thus decoupling the dynamical aspects of the problem from the combinatorial ones. The role of the group invariance of the limit set, used in [Naud 2005], is played here by its δ-regularity, proved by Sullivan [1979], and words in the group are replaced by vertices in the discretizing tree.

1C. *Structure of the paper.*

- In Section 2, we establish basic properties of Ahlfors–David regular sets (Section 2A), introduce the functional spaces used (Section 2B), and show several basic identities and inequalities (Section 2C).
- In Section 3, we prove Theorem 1.
- In Section 4, we apply Theorem 1 and the results of [Dyatlov and Zahl 2016] to establish an essential spectral gap for convex cocompact hyperbolic surfaces.
- In Section 5, we apply Theorem 1 and the results of [Dyatlov and Jin 2017; Bourgain and Dyatlov 2016] to establish an essential spectral gap for open quantum baker's maps.

2. Preliminaries

2A. ***Regular sets and discretization.*** An *interval* in $\mathbb{R}$ is a subset of the form $I = [c, d]$, where $c < d$. Define the center of I by $\frac{1}{2}(c + d)$ and the size of I by $|I| = d - c$.

Let μ be a finite measure on $\mathbb{R}$ with compact support. Fix an integer $L \ge 2$. Following [Dyatlov and Zahl 2016, Section 6.4], we describe the *discretization of μ with base L*. For each $k \in \mathbb{Z}$, let V_k be the set of all intervals $I = [c, d]$ which satisfy the following conditions:

- $c, d \in L^{-k}\mathbb{Z}$.
- For each $q \in L^{-k}\mathbb{Z}$ with $c \le q < d$, we have $\mu([q, q + L^{-k}]) > 0$.
- $\mu_X([c - L^{-k}, c]) = \mu([d, d + L^{-k}]) = 0$.

In other words, V_k is obtained by partitioning $\mathbb{R}$ into intervals of size L^{-k}, throwing out intervals of zero measure μ, and merging consecutive intervals.

We define the set of vertices of the discretization as

$$V := \bigsqcup_{k \in \mathbb{Z}} V_k,$$

and define the *height function* by putting $H(I) := k$ if $I \in V_k$. (It is possible that V_k intersect for different k, so formally speaking, a vertex is a pair (k, I), where $I \in V_k$.) We say that $I \in V_k$ is a *parent* of $I' \in V_{k+1}$, and I' is a *child* of I, if $I' \subset I$. It is easy to check that the resulting structure has the following properties:

- Any two distinct intervals $I, I' \in V_k$ are at least L^{-k} apart.
- $\mu(\mathbb{R} \setminus \bigsqcup_{I \in V_k} I) = 0$ for all k.
- Each $I \in V_k$ has exactly one parent.
- If $I \in V_k$ and $I_1, \dots, I_n \in V_{k+1}$ are the children of I, then

$$0 < \mu(I) = \sum_{j=1}^{n} \mu(I_j). \tag{2-1}$$

For regular sets, the discretization has the following additional properties:

Lemma 2.1. *Let $L \ge 2$, $K > 0$ be integers and assume (X, μ_X) is δ-regular up to scale L^{-K} with regularity constant C_R, where $0 < \delta < 1$. Then the discretization of μ_X with base L has the following properties*:

(1) *Each $I \in V$ with $0 \le H(I) \le K$ satisfies, for $C'_R := (3C_R^2)^{\frac{1}{1-\delta}}$,*

$$L^{-H(I)} \le |I| \le C'_R L^{-H(I)}, \tag{2-2}$$

$$C_R^{-1} L^{-\delta H(I)} \le \mu_X(I) \le C_R (C'_R)^{\delta} L^{-\delta H(I)}. \tag{2-3}$$

(2) *If I' is a child of $I \in V$ and $0 \le H(I) < K$, then*

$$\frac{\mu_X(I')}{\mu_X(I)} \ge \frac{L^{-\delta}}{C'_R}. \tag{2-4}$$

(3) *Assume that*

$$L \ge (4C_R)^{\frac{6}{\delta(1-\delta)}}. \tag{2-5}$$

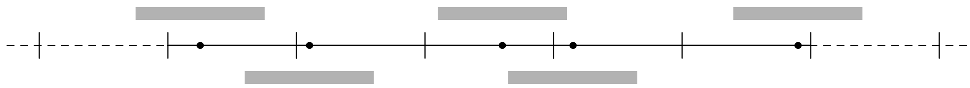

Figure 1. An illustration of the proof of the upper bound in (2-2). The ticks mark points in $L^{-k}\mathbb{Z}$, the solid interval is I, the dots mark the points x_q, and the shaded intervals are I_q. The intervals of length L^{-k} adjacent to I have zero measure μ_X.

Then for each $I \in V$ with $0 \le H(I) < K$, there exist two children I', I'' of I such that

$$\tfrac{1}{2} C_R^{-\frac{2}{\delta}} L^{-H(I)-\frac{2}{3}} \le |x' - x''| \le 2L^{-H(I)-\frac{2}{3}} \quad \textit{for all } x' \in I',\ x'' \in I''.$$

Remark. Parts (1) and (2) of the lemma state that the tree of intervals discretizing μ_X is approximately regular. Part (3), which is used at the end of Section 3B, states that once the base of discretization L is large enough, each interval I in the tree has two children which are $\sim L^{-H(I)-\frac{2}{3}}$ apart from each other. A similar statement would hold if $\frac{2}{3}$ were replaced by any number in $(0, 1)$.

Proof. (1) Put $k := H(I)$. The lower bound on $|I|$ follows from the construction of the discretization. To show the upper bound, assume that $I = [c, d]$ and $d - c = ML^{-k}$. For each $q \in L^{-k}\mathbb{Z}$ with $c \le q < d$, we have $\mu_X([q, q + L^{-k}]) > 0$; thus there exists $x_q \in [q, q + L^{-k}] \cap X$. Let I_q be the interval of size L^{-k} centered at x_q; see Figure 1. Then

$$\mu_X\Big(\bigcup_q I_q\Big) \le \mu_X([c - L^{-k}, d + L^{-k}]) = \mu_X(I) \le C_R(ML^{-k})^\delta.$$

On the other hand, each point is covered by at most three intervals I_q; therefore

$$MC_R^{-1}L^{-k\delta} \le \sum_q \mu_X(I_q) \le 3\mu_X\Big(\bigcup_q I_q\Big).$$

Together these two inequalities imply $M \le C_R'$, giving (2-2).

The upper bound on $\mu_X(I)$ follows from (2-2). To show the lower bound, take $x \in I \cap X$ and let I' be the interval of size L^{-k} centered at x. Then $\mu_X(I' \setminus I) = 0$; therefore $\mu_X(I) \ge \mu_X(I') \ge C_R^{-1}L^{-\delta k}$.

(2) This follows directly from (2-3) and the fact that $C_R^2(C_R')^\delta \le C_R'$.

(3) Put $k := H(I)$. Take $x \in I \cap X$ and let J be the interval of size $L^{-k-\frac{2}{3}}$ centered at x. Let $I_1, \dots, I_n$ be all the intervals in V_{k+1} which intersect J; they all have to be children of I. Let $x_1, \dots, x_n$ be the centers of $I_1, \dots, I_n$. Define

$$T := L^{k+\frac{2}{3}} \max_{j,\ell} |x_j - x_\ell|.$$

By (2-2), we have $|I_j| \le C_R' L^{-k-1}$ and thus $T \le 1 + C_R' L^{-\frac{1}{3}}$. On the other hand, the union of $I_1, \dots, I_n$ is contained in an interval of size $TL^{-k-\frac{2}{3}} + C_R' L^{-k-1}$. Therefore

$$C_R^{-1} L^{-\delta(k+\frac{2}{3})} \le \mu_X(J) \le \sum_{j=1}^n \mu_X(I_j) \le C_R(TL^{-k-\frac{2}{3}} + C_R' L^{-k-1})^\delta.$$

This implies $T \ge C_R^{-\frac{2}{\delta}} - C_R' L^{-\frac{1}{3}}$.

Now, put $I' := I_j$, $I'' = I_\ell$, where j, ℓ are chosen so that $T = L^{k+\frac{2}{3}}|x_j - x_\ell|$. Then for each $x' \in I'$, $x'' \in I''$, we have by (2-5)

$$\tfrac{1}{2}C_R^{-\frac{2}{\delta}} \le C_R^{-\frac{2}{\delta}} - 2C'_R L^{-\frac{1}{3}} \le L^{k+\frac{2}{3}}|x' - x''| \le 1 + 2C'_R L^{-\frac{1}{3}} \le 2. \qquad \square$$

We finally have the following estimates on the Lebesgue measure of neighborhoods of a δ-regular set, which are used in Sections 4–5:

Lemma 2.2. *Assume that (Λ, μ_Λ) is δ-regular up to scale $h \in (0, 1)$ with constant C_R. Let $X := \Lambda(h) = \Lambda + [-h, h]$ be the h-neighborhood of Λ and define the measure μ_X by*

$$\mu_X(A) := h^{\delta-1}\mu_L(X \cap A), \quad A \subset \mathbb{R}, \tag{2-6}$$

where μ_L denotes the Lebesgue measure. Then (X, μ_X) is δ-regular up to scale h with constant $C'_R := 30C_R^2$.

Proof. We follow [Dyatlov and Zahl 2016, Lemma 7.4]. Let $I \subset \mathbb{R}$ be an interval with $|I| \ge h$. Let $x_1, \dots, x_N \in \Lambda \cap I(h)$ be a maximal set of $2h$-separated points. Denote by I'_n the interval of size h centered at x_n. Since I'_n are disjoint and their union is contained in $I(2h)$, which is an interval of size $|I| + 4h \le 5|I|$, we have

$$N \cdot C_R^{-1}h^\delta \le \sum_{n=1}^{N} \mu_\Lambda(I'_n) \le \mu_\Lambda(I(2h)) \le 5C_R|I|^\delta. \tag{2-7}$$

Next, let I_n be the interval of size $6h$ centered at x_n. Then $X \cap I$ is contained in the union of I_n and thus

$$\mu_L(X \cap I) \le \sum_{n=1}^{N} \mu_L(I_n) = 6hN. \tag{2-8}$$

Together (2-7) and (2-8) give the required upper bound

$$\mu_X(I) = h^{\delta-1}\mu_L(X \cap I) \le 30C_R^2|I|^\delta.$$

Now, assume additionally that $|I| \le 1$ and I is centered at a point in X. Let $y_1, \dots, y_M \in \Lambda \cap I$ be a maximal set of h-separated points. Denote by I_m the interval of size $2h$ centered at y_m. Then $\Lambda \cap I$ is contained in the union of I_m; therefore

$$C_R^{-1}|I|^\delta \le \mu_\Lambda(I) = \mu_\Lambda(\Lambda \cap I) \le \sum_{m=1}^{M} \mu_\Lambda(I_m) \le M \cdot 2C_R h^\delta. \tag{2-9}$$

Next, let I'_m be the interval of size h centered at y_m. Then $I'_m \subset X$ are nonoverlapping and each $I'_m \cap I$ has size at least $\frac{1}{2}h$; therefore

$$\mu_L(X \cap I) \ge \sum_{m=1}^{M} \mu_L(I'_m \cap I) \ge \tfrac{1}{2}Mh. \tag{2-10}$$

Combining (2-9) and (2-10) gives the required lower bound

$$\mu_X(I) = h^{\delta-1}\mu_L(X \cap I) \ge \frac{1}{4C_R^2}|I|^\delta. \qquad \square$$

2B. ***Functional spaces.*** For a constant $\theta > 0$ and an interval I, let $\mathcal{C}_\theta(I)$ be the space $C^1(I)$ with the norm

$$\|f\|_{\mathcal{C}_\theta(I)} := \max\bigl(\sup_I |f|, \theta|I|\cdot\sup_I |f'|\bigr).$$

The following lemma shows that multiplications by functions of the form $\exp(i\psi)$ have norm 1 when mapping $\mathcal{C}_\theta(I)$ into the corresponding space for a sufficiently small subinterval of I:

Lemma 2.3. *Consider intervals*

$$I' \subset I, \quad |I'| \le \tfrac{1}{4}|I|. \tag{2-11}$$

Assume that $\psi \in C^\infty(I;\mathbb{R})$ *and* $\theta > 0$ *are such that*

$$4\theta|I'|\cdot\sup_{I'}|\psi'| \le 1. \tag{2-12}$$

Then for each $f \in \mathcal{C}_\theta(I)$, *we have* $\|\exp(i\psi)f\|_{\mathcal{C}_\theta(I')} \le \|f\|_{\mathcal{C}_\theta(I)}$ *and*

$$\theta|I'|\cdot\sup_{I'}|(\exp(i\psi)f)'| \le \tfrac{1}{2}\|f\|_{\mathcal{C}_\theta(I)}. \tag{2-13}$$

Proof. The left-hand side of (2-13) is bounded from above by

$$\theta|I'|\cdot\bigl(\sup_{I'}|\psi' f| + \sup_{I'}|f'|\bigr).$$

From (2-12), (2-11) we get

$$\theta|I'|\cdot\sup_{I'}|\psi' f| \le \tfrac{1}{4}\|f\|_{\mathcal{C}_\theta(I)}, \quad \theta|I'|\cdot\sup_{I'}|f'| \le \tfrac{1}{4}\|f\|_{\mathcal{C}_\theta(I)},$$

which finishes the proof of (2-13). The bound (2-13) implies $\|\exp(i\psi)f\|_{\mathcal{C}_\theta(I')} \le \|f\|_{\mathcal{C}_\theta(I)}$. ☐

The following is a direct consequence of the mean value theorem:

Lemma 2.4. *Let* $f \in \mathcal{C}_\theta(I)$. *Then for all* $x, x' \in I$, *we have*

$$|f(x) - f(x')| \le \frac{|x - x'|}{\theta|I|}\cdot\|f\|_{\mathcal{C}_\theta(I)}. \tag{2-14}$$

2C. ***A few technical lemmas.*** The following is a two-dimensional analog of the mean value theorem:

Lemma 2.5. *Let* $I = [c_1, d_1]$ *and* $J = [c_2, d_2]$ *be two intervals and* $\Phi \in C^2(I \times J;\mathbb{R})$. *Then there exists* $(x_0, y_0) \in I \times J$ *such that*

$$\Phi(c_1, c_2) + \Phi(d_1, d_2) - \Phi(c_1, d_2) - \Phi(d_1, c_2) = |I|\cdot|J|\cdot\partial^2_{xy}\Phi(x_0, y_0).$$

Proof. Replacing $\Phi(x, y)$ by $\Phi(x, y) - \Phi(c_1, y) - \Phi(x, c_2) + \Phi(c_1, c_2)$, we may assume that $\Phi(c_1, y) = 0$ and $\Phi(x, c_2) = 0$ for all $x \in I$, $y \in J$. By the mean value theorem, we have $\Phi(d_1, d_2) = |I|\cdot\partial_x\Phi(x_0, d_2)$ for some $x_0 \in I$. Applying the mean value theorem again, we have $\partial_x\Phi(x_0, d_2) = |J|\cdot\partial^2_{xy}\Phi(x_0, y_0)$ for some $y_0 \in J$, finishing the proof. ☐

Lemma 2.6. *Assume that* $\tau \in \mathbb{R}$ *and* $|\tau| \le \pi$. *Then* $|e^{i\tau} - 1| \ge \frac{2}{\pi}|\tau|$.

Proof. We have

$$|e^{i\tau}-1| = 2\sin\left(\tfrac{1}{2}|\tau|\right).$$

It remains to use that $\sin x \geq \frac{2}{\pi}x$ when $0 \leq x \leq \frac{\pi}{2}$, which follows from the concavity of $\sin x$ on that interval. □

The next lemma, used several times in Section 3B, is a quantitative version of the fact that balls in Hilbert spaces are strictly convex:

Lemma 2.7. *Assume that $\mathcal{H}$ is a Hilbert space, $f_1, \dots, f_n \in \mathcal{H}$, $p_1, \dots, p_n \geq 0$, and $p_1 + \dots + p_n = 1$. Then*

$$\Big\| \sum_{j=1}^n p_j f_j \Big\|_{\mathcal{H}}^2 = \sum_{j=1}^n p_j \|f_j\|_{\mathcal{H}}^2 - \sum_{1\leq j<\ell\leq n} p_j p_\ell \|f_j - f_\ell\|_{\mathcal{H}}^2. \tag{2-15}$$

If moreover for some $\varepsilon, R \geq 0$

$$\sum_{j=1}^n p_j \|f_j\|_{\mathcal{H}}^2 = R, \quad \Big\| \sum_{j=1}^n p_j f_j \Big\|_{\mathcal{H}}^2 \geq (1-\varepsilon)R, \quad p_{\min} := \min_j p_j \geq 2\sqrt{\varepsilon} \tag{2-16}$$

then for all j

$$\tfrac{1}{2}\sqrt{R} \leq \|f_j\|_{\mathcal{H}} \leq 2\sqrt{R}. \tag{2-17}$$

Proof. The identity (2-15) follows by a direct computation. To show (2-17), note that by (2-15) and (2-16) for each j, ℓ

$$\|f_j - f_\ell\|_{\mathcal{H}}^2 \leq \frac{\varepsilon R}{p_{\min}^2} \leq \tfrac{1}{4}R.$$

Put

$$f_{\max} := \max_j \|f_j\|_{\mathcal{H}}, \quad f_{\min} := \min_j \|f_j\|_{\mathcal{H}}.$$

Then

$$f_{\max} - f_{\min} \leq \tfrac{1}{2}\sqrt{R}, \quad f_{\min} \leq \sqrt{R} \leq f_{\max},$$

which implies (2-17). □

Lemma 2.8. *Assume that $\alpha_j, p_j \geq 0$, $j = 1, \dots, n$, $p_1 + \dots + p_n = 1$, and for some $\varepsilon, R \geq 0$*

$$\sum_{j=1}^n p_j \alpha_j \geq (1-\varepsilon)R, \quad \max_j \alpha_j \leq R, \quad p_{\min} := \min_j p_j \geq 2\varepsilon.$$

Then for all j,

$$\alpha_j \geq \tfrac{1}{2}R.$$

Proof. We have

$$\sum_{j=1}^n p_j(R - \alpha_j) \leq \varepsilon R.$$

All the terms in the sum are nonnegative; therefore for all j

$$R - \alpha_j \leq \frac{\varepsilon R}{p_{\min}} \leq \tfrac{1}{2}R.$$ □

3. Proof of Theorem 1

3A. ***The iterative argument.*** In this section, we prove the following statement which can be viewed as a special case of Theorem 1. Its proof relies on an inductive bound, Lemma 3.2, which is proved in Section 3B. In Section 3C, we deduce Theorem 1 from Proposition 3.1, in particular removing the condition (3-1).

Proposition 3.1. *Let $\delta, \delta' \in (0,1)$, $C_R > 1$, $I_0, J_0 \subset \mathbb{R}$ be some intervals, $G \in C^1(I_0 \times J_0; \mathbb{C})$, and the phase function $\Phi \in C^2(I_0 \times J_0; \mathbb{R})$ satisfy*

$$\tfrac{1}{2} < |\partial^2_{xy}\Phi(x,y)| < 2 \quad \textit{for all } (x,y) \in I_0 \times J_0. \tag{3-1}$$

Choose constants $C'_R > 0$ and $L \in \mathbb{N}$ such that

$$C'_R = (2C_R)^{\frac{2}{1-\max(\delta,\delta')}}, \qquad L \geq (2C'_R(6C_R)^{\frac{1}{\delta}+\frac{1}{\delta'}})^6. \tag{3-2}$$

Fix $K_0 \in \mathbb{N}_0$ and put $h := L^{-K}$ for some $K \in \mathbb{N}_0$, $K \geq 2K_0$. Assume that (X, μ_X) is δ-regular, and (Y, μ_Y) is δ'-regular, up to scale L^{K_0-K} with regularity constant C_R, and $X \subset I_0$, $Y \subset J_0$. Put

$$\varepsilon_1 := 10^{-5}(C_R^{\frac{1}{\delta}+\frac{1}{\delta'}} C'_R)^{-4} L^{-5}, \quad \varepsilon_0 := -\frac{\log(1-\varepsilon_1)}{2\log L}. \tag{3-3}$$

Then for some C depending only on $K_0, G, \mu_X(X), \mu_Y(Y)$, and $\mathcal{B}_h$ defined in (1-3),

$$\|\mathcal{B}_h f\|_{L^2(X,\mu_X)} \leq C h^{\varepsilon_0} \|f\|_{L^2(Y,\mu_Y)} \quad \textit{for all } f \in L^2(Y,\mu_Y). \tag{3-4}$$

Remark. Proposition 3.1 has complicated hypotheses in order to make it useful for the proof of Theorem 1. However, the argument is essentially the same in the following special case which could simplify the reading of the proof below: $\delta = \delta'$, $G \equiv 1$, $\Phi(x,y) = xy$, $K_0 = 0$. Note that in this case $\mathcal{B}_h$ is related to the semiclassical Fourier transform (1-2).

To start the proof of Proposition 3.1, we extend Φ to a function in $C^2(\mathbb{R}^2;\mathbb{R})$ such that (3-1) still holds, and extend G to a function in $C^1(\mathbb{R}^2;\mathbb{C})$ such that $G, \partial_x G$ are uniformly bounded. Following Section 2A, consider the discretizations of μ_X, μ_Y with base L, denoting by V_X, V_Y the sets of vertices and by H the height functions.

Fix $f \in L^2(Y,\mu_Y)$. For each $J \in V_Y$, let y_J denote the center of J and define the function of $x \in \mathbb{R}$

$$F_J(x) = \frac{1}{\mu_Y(J)} \int_J \exp\left(\frac{i(\Phi(x,y) - \Phi(x,y_J))}{h}\right) G(x,y) f(y)\, d\mu_Y(y). \tag{3-5}$$

In terms of the operator $\mathcal{B}_h$ from (1-3), we may write

$$F_J(x) = \frac{1}{\mu_Y(J)} \exp\left(-\frac{i\Phi(x,y_J)}{h}\right) \mathcal{B}_h(\mathbb{1}_J f)(x). \tag{3-6}$$

Put

$$\theta := \frac{1}{8(C'_R)^2} \tag{3-7}$$

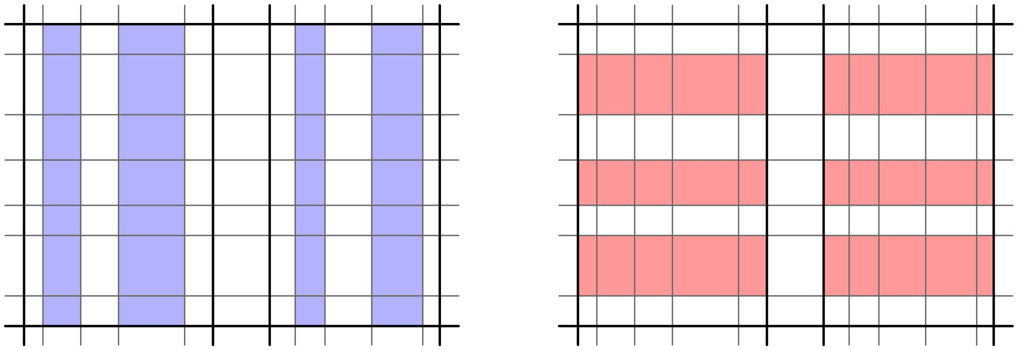

Figure 2. An illustration of (3-8) in the case $K = 1$. The vertical lines mark the endpoints of intervals in V_X and the horizontal lines mark the endpoints of intervals in V_Y. The thick lines correspond to intervals of height 0 and the thin lines to intervals of height 1. The shaded rectangles have the form $I \times J$, $I \in V_X$, $J \in V_Y$, where E_J is constant on I, and the shaded rectangles on the left/right correspond to the left-/right-hand sides of (3-9) for $H(J) = 0$.

and for $J \in V_Y$ define the piecewise constant function $E_J \in L^\infty(X, \mu_X)$ using the space $\mathcal{C}_\theta(I)$ defined in Section 2B:

$$E_J(x) = \|F_J\|_{\mathcal{C}_\theta(I)}, \quad \text{where } x \in I \in V_X,\ H(I) + H(J) = K. \tag{3-8}$$

See Figure 2. Note that $|F_J(x)| \le E_J(x)$ for μ_X-almost every x.

The L^2 norms of the functions E_J satisfy the following key bound, proved in Section 3B, which gives an improvement from one scale to the next. The use of the L^2 norm of E_J as the monotone quantity is convenient for several reasons. On one hand, the averaging provided by the L^2 norm means it is only necessary to show an improvement on F_J in sufficiently many places; more precisely we will show in (3-11) that such improvement happens on at least one child of each interval $I \in V(X)$ with $H(I) + H(J) = K - 1$. On the other hand, such improvement is obtained by a pointwise argument which also uses that the F_{J_b} are slowly varying on each interval I with $H(I) + H(J) = K - 1$ (see Lemma 3.7); this motivates the use of $\mathcal{C}_\theta(I)$ norms in the definition of E_{J_b}.

Lemma 3.2. *Let $J \in V_Y$ with $K_0 \le H(J) < K - K_0$ and $J_1, \dots, J_B \in V_Y$ be the children of J. Then, with ε_1 defined in* (3-3),

$$\|E_J\|^2_{L^2(X,\mu_X)} \le (1 - \varepsilon_1) \sum_{b=1}^{B} \frac{\mu_Y(J_b)}{\mu_Y(J)} \|E_{J_b}\|^2_{L^2(X,\mu_X)}. \tag{3-9}$$

Iterating Lemma 3.2, we obtain:

Proof of Proposition 3.1. First of all, we show that for all $J \in V_Y$ with $H(J) = K - K_0$, and some constant C_0 depending on $G, \mu_X(X)$ and defined below, we have

$$\|E_J\|^2_{L^2(X,\mu_X)} \le C_0 \frac{\|f\|^2_{L^2(J,\mu_Y)}}{\mu_Y(J)}. \tag{3-10}$$

Indeed, take $I \in V_X$ such that $H(I) = K_0$. By (2-2) and (3-1), for all $y \in J$

$$\frac{1}{h}\sup_{x\in I}|\partial_x\Phi(x,y)-\partial_x\Phi(x,y_J)| \le \frac{2}{h}|J| \le 2C'_R L^{K_0}$$

and thus by (2-2) and (3-7)

$$\frac{4\theta|I|}{h}\sup_{x\in I}|\partial_x\Phi(x,y)-\partial_x\Phi(x,y_J)| \le 1.$$

Arguing similarly to Lemma 2.3, we obtain for all $y \in J$

$$\left\|\exp\left(\frac{i(\Phi(x,y)-\Phi(x,y_J))}{h}\right)G(x,y)\right\|_{\mathcal{C}_\theta(I)} \le C_G := \max(\sup|G|, \sup|\partial_x G|).$$

Using Hölder's inequality in (3-5), we obtain

$$E_J|_I = \|F_J\|_{\mathcal{C}_\theta(I)} \le \frac{C_G}{\mu_Y(J)}\int_J |f(y)|\,d\mu_Y(y) \le \frac{C_G\|f\|_{L^2(J,\mu_Y)}}{\sqrt{\mu_Y(J)}}$$

and (3-10) follows by integration in x, where we put $C_0 := C_G^2\mu_X(X)$.

Now, arguing by induction on $H(J)$ with (3-10) as the base case and (3-9) as the inductive step, we obtain for all $J \in V_Y$ with $K_0 \le H(J) \le K - K_0$,

$$\|E_J\|^2_{L^2(X,\mu_X)} \le C_0(1-\varepsilon_1)^{K-K_0-H(J)}\,\frac{\|f\|^2_{L^2(J,\mu_Y)}}{\mu_Y(J)}.$$

In particular, for all $J \in V_Y$ with $H(J) = K_0$, we have by (3-6)

$$\left\|\frac{\mathcal{B}_h(\mathbb{1}_J f)}{\mu_Y(J)}\right\|^2_{L^2(X,\mu_X)} = \|F_J\|^2_{L^2(X,\mu_X)} \le \|E_J\|^2_{L^2(X,\mu_X)} \le C_1 h^{2\varepsilon_0}\frac{\|f\|^2_{L^2(J,\mu_Y)}}{\mu_Y(J)},$$

where $C_1 := C_0(1-\varepsilon_1)^{-2K_0}$. Using the identity

$$\mathcal{B}_h f = \mu_Y(Y)\sum_{\substack{J\in V_Y\\ H(J)=K_0}}\frac{\mu_Y(J)}{\mu_Y(Y)}\cdot\frac{\mathcal{B}_h(\mathbb{1}_J f)}{\mu_Y(J)}$$

and (2-15), we estimate

$$\|\mathcal{B}_h f\|^2_{L^2(X,\mu_X)} \le C_1\mu_Y(Y)h^{2\varepsilon_0}\|f\|^2_{L^2(Y,\mu_Y)}$$

and (3-4) follows with $C := C_G(1-\varepsilon_1)^{-K_0}\sqrt{\mu_X(X)\mu_Y(Y)}$. □

3B. ***The inductive step.*** In this section we prove Lemma 3.2. Let $J \in V_Y$ satisfy $K_0 \le H(J) < K - K_0$ and $J_1, \dots, J_B$ be the children of J. It suffices to show that for all $I \in V_X$ with $H(I) + H(J) = K - 1$ we have

$$\|E_J\|^2_{L^2(I,\mu_X)} \le (1-\varepsilon_1)\sum_{b=1}^{B}\frac{\mu_Y(J_b)}{\mu_Y(J)}\|E_{J_b}\|^2_{L^2(I,\mu_X)}. \tag{3-11}$$

Indeed, summing (3-11) over I, we obtain (3-9).

Fix $I \in V_X$ with $H(I) + H(J) = K - 1$ and let $I_1, \dots, I_A$ be the children of I. Define

$$p_a := \frac{\mu_X(I_a)}{\mu_X(I)}, \quad q_b := \frac{\mu_Y(J_b)}{\mu_Y(J)}.$$

Note that $p_a, q_b \geq 0$ and $p_1 + \dots + p_A = q_1 + \dots + q_B = 1$.

The functions F_J and F_{J_b} are related by the following formula:

$$F_J = \sum_{b=1}^{B} q_b \exp(i\Psi_b) F_{J_b}, \quad \Psi_b(x) := \frac{\Phi(x, y_{J_b}) - \Phi(x, y_J)}{h}. \tag{3-12}$$

That is, F_J is a convex combination of $F_{J_1}, \dots, F_{J_B}$ multiplied by the phase factors $\exp(i\Psi_b)$. At the end of this subsection we exploit cancellation between these phase factors to show (3-11). However there are several preparatory steps necessary. Before we proceed with the proof, we show the version of (3-11) with no improvement:

Lemma 3.3. *We have*

$$\|E_J\|^2_{L^2(I,\mu_X)} \leq \sum_{b=1}^{B} q_b \|E_{J_b}\|^2_{L^2(I,\mu_X)}. \tag{3-13}$$

Proof. By (2-2), (3-1), and (3-7), we have for all a, b

$$4\theta |I_a| \cdot \sup_I |\Psi_b'| \leq \frac{8\theta |I_a| \cdot |J|}{h} \leq 1. \tag{3-14}$$

Moreover, by (2-2) and (3-2) we have $|I_a| \leq \frac{1}{4}|I|$. Applying Lemma 2.3, we obtain

$$\|\exp(i\Psi_b) F_{J_b}\|_{C_\theta(I_a)} \leq \|F_{J_b}\|_{C_\theta(I)}.$$

By (3-12) and (2-15) we then have

$$\|F_J\|^2_{C_\theta(I_a)} \leq \left(\sum_{b=1}^{B} q_b \|F_{J_b}\|_{C_\theta(I)} \right)^2 \leq \sum_{b=1}^{B} q_b \|F_{J_b}\|^2_{C_\theta(I)}. \tag{3-15}$$

By (3-8), we have for all a, b

$$E_J|_{I_a} = \|F_J\|_{C_\theta(I_a)}, \quad E_{J_b}|_I = \|F_{J_b}\|_{C_\theta(I)}. \tag{3-16}$$

Now, summing both sides of (3-15) over a with weights $\mu_X(I_a)$, we obtain (3-13). □

The rest of this section is dedicated to the proof of (3-11), studying the situations in which the bound (3-13) is almost sharp and ultimately reaching a contradiction. The argument is similar in spirit to Lemma 1.2. In fact we can view Lemma 1.2 as the special degenerate case when $A = B = 2$, $p_a = q_b = \frac{1}{2}$, the intervals I_a are replaced by points x_a, $F_{J_b} \equiv f_b$ are constants, $u_a = F_J(x_a)$, and $\omega_{ab} = \Psi_b(x_a)$. The general case is more technically complicated. In particular we use Lemma 2.7 to deal with general convex combinations. We also use δ-regularity in many places, for instance to show that the coefficients p_a, q_b are bounded away from zero and to get the phase factor cancellations in (3-30) at the end of the proof. The reading of the argument below may be simplified by making the illegal choice $\varepsilon_1 := 0$.

We henceforth assume that (3-11) does not hold. Put

$$R := \sum_{b=1}^{B} q_b \|F_{J_b}\|^2_{\mathcal{C}_\theta(I)}. \tag{3-17}$$

By (3-16), the failure of (3-11) can be rewritten as

$$\sum_{a=1}^{A} p_a \|F_J\|^2_{\mathcal{C}_\theta(I_a)} > (1-\varepsilon_1)R. \tag{3-18}$$

We note for future use that p_a, q_b are bounded below by (2-4):

$$p_{\min} := \min_a p_a \geq \frac{L^{-\delta}}{C'_R}, \quad q_{\min} := \min_b q_b \geq \frac{L^{-\delta'}}{C'_R}. \tag{3-19}$$

We first deduce from (3-17) and the smallness of ε_1 an upper bound on each $\|F_{J_b}\|_{\mathcal{C}_\theta(I)}$ in terms of the averaged quantity R:

Lemma 3.4. *We have for all b,*

$$\|F_{J_b}\|_{\mathcal{C}_\theta(I)} \leq 2\sqrt{R}. \tag{3-20}$$

Proof. The first inequality in (3-15) together with (3-18) implies

$$\left(\sum_{b=1}^{B} q_b \|F_{J_b}\|_{\mathcal{C}_\theta(I)}\right)^2 \geq \sum_{a=1}^{A} p_a \|F_J\|^2_{\mathcal{C}_\theta(I_a)} \geq (1-\varepsilon_1)R. \tag{3-21}$$

By (3-3) and (3-19) we have $q_{\min} \geq 2\sqrt{\varepsilon_1}$. Applying (2-17) to $f_b := \|F_{J_b}\|_{\mathcal{C}_{\theta(I)}}$ with (3-17) and (3-21), we obtain (3-20). $\square$

We next obtain a version of (3-18) which gives a lower bound on the size of F_J, rather than on the norm $\|F_J\|_{\mathcal{C}_\theta(I_a)}$:

Lemma 3.5. *There exist $x_a \in I_a$, $a = 1, \dots, A$, such that*

$$\sum_{a=1}^{A} p_a |F_J(x_a)|^2 > (1-2\varepsilon_1)R. \tag{3-22}$$

Proof. By Lemma 2.3 and (3-14), we have

$$\theta |I_a| \cdot \sup_{I_a} |(\exp(i\Psi_b)F_{J_b})'| \leq \tfrac{1}{2}\|F_{J_b}\|_{\mathcal{C}_\theta(I)}.$$

It follows by (3-12) and the triangle inequality that for all a,

$$\|F_J\|_{\mathcal{C}_\theta(I_a)} \leq \max\left(\sup_{I_a} |F_J|, \frac{1}{2}\sum_{b=1}^{B} q_b \|F_{J_b}\|_{\mathcal{C}_\theta(I)}\right). \tag{3-23}$$

By (3-15) we have

$$\sup_{I_a} |F_J|^2 \leq \|F_J\|^2_{\mathcal{C}_\theta(I_a)} \leq R. \tag{3-24}$$

Therefore by (3-23) and the second inequality in (3-15)

$$\|F_J\|^2_{\mathcal{C}_\theta(I_a)} \le \tfrac{1}{2}\big(R + \sup_{I_a} |F_J|^2\big).$$

Summing this inequality over a with weights p_a, we see that (3-18) implies

$$\sum_{a=1}^{A} p_a \sup_{I_a} |F_J|^2 > (1-2\varepsilon_1)R,$$

which gives (3-22). ☐

Now, choose x_a as in Lemma 3.5 and put

$$F_{ab} := F_{J_b}(x_a) \in \mathbb{C}, \quad \omega_{ab} := \Psi_b(x_a) \in \mathbb{R}.$$

Note that by (3-12)

$$F_J(x_a) = \sum_{b=1}^{B} q_b \exp(i\omega_{ab}) F_{ab}.$$

Using (2-15) for $f_b = \exp(i\omega_{ab})F_{ab}$ and (3-22), we obtain

$$\sum_{a,b} p_a q_b |F_{ab}|^2 > (1-2\varepsilon_1)R + \sum_{\substack{a,b,b' \\ b<b'}} p_a q_b q_{b'} \big|\exp(i(\omega_{ab}-\omega_{ab'}))F_{ab} - F_{ab'}\big|^2. \tag{3-25}$$

From the definition (3-17) of R, we have for all a

$$\sum_{b=1}^{B} q_b |F_{ab}|^2 \le R. \tag{3-26}$$

Therefore, the left-hand side of (3-25) is bounded above by R. Using (3-19), we then get for all a, b, b' the following approximate equality featuring the phase terms ω_{ab}:

$$\begin{aligned} \big|\exp(i(\omega_{ab}-\omega_{ab'}))F_{ab} - F_{ab'}\big| &< \sqrt{\frac{2\varepsilon_1 R}{p_{\min} q_{\min}^2}} \\ &\le 2(C_R')^2 L^{\delta+\delta'} \sqrt{\varepsilon_1 R}. \end{aligned} \tag{3-27}$$

Using the smallness of ε_1, we obtain from here a lower bound on $|F_{ab}|$:

Lemma 3.6. *For all a, b we have*

$$|F_{ab}| \ge \tfrac{1}{2}\sqrt{R}. \tag{3-28}$$

Proof. By (3-3) and (3-19), we have $p_{\min} \ge 4\varepsilon_1$. Applying Lemma 2.8 to $\alpha_a = \sum_b q_b |F_{ab}|^2$ and using (3-25) and (3-26), we obtain for all a

$$\sum_{b=1}^{B} q_b |F_{ab}|^2 \ge \tfrac{1}{2}R. \tag{3-29}$$

We now argue similarly to the proof of (2-17). Fix a and let $F_{a,\min} = \min_b |F_{ab}|$, $F_{a,\max} = \max_b |F_{ab}|$. By (3-29) we have $F_{a,\max} \geq \sqrt{R/2}$. On the other hand the difference $F_{a,\max} - F_{a,\min}$ is bounded above by (3-27). By (3-3) we then have

$$F_{a,\min} \geq \sqrt{\tfrac{1}{2}R} - 2(C_R')^2 L^{\delta+\delta'} \sqrt{\varepsilon_1 R} \geq \tfrac{1}{2}\sqrt{R}. \qquad \square$$

We next estimate the discrepancy between the values F_{ab} for fixed b and different a, using the fact that we control the norm $\|F_{J_b}\|_{\mathcal{C}_\theta(I)}$ and thus the derivative of F_{J_b}:

Lemma 3.7. *For all a, a', b we have*

$$|F_{ab} - F_{a'b}| \leq \frac{2\sqrt{R}\,|x_a - x_{a'}|}{\theta |I|} \leq \frac{2\sqrt{R}}{\theta} L^{H(I)} \cdot |x_a - x_{a'}|.$$

Proof. This follows immediately by combining Lemma 2.4, Lemma 3.4, and (2-2). $\square$

Armed with the bounds obtained above, we are now ready to reach a contradiction and finish the proof of Lemma 3.2, using the discrepancy of the phase shifts ω_{ab} and the lower bound on $|\partial^2_{xy}\Phi|$ from (3-1).

Using part (3) of Lemma 2.1 and (3-2), choose a, a', b, b' such that

$$\tfrac{1}{2} C_R^{-\frac{2}{\delta}} L^{-\frac{2}{3}} \leq L^{H(I)} \cdot |x_a - x_{a'}| \leq 2L^{-\frac{2}{3}},$$
$$\tfrac{1}{2} C_R^{-\frac{2}{\delta}'} L^{-\frac{2}{3}} \leq L^{H(J)} \cdot |y_b - y_{b'}| \leq 2L^{-\frac{2}{3}}.$$

Recall that $x_a \in I_a$ is chosen in Lemma 3.5 and $y_b := y_{J_b}$ is the center of J_b. By Lemma 2.5, we have for some $(\tilde{x}, \tilde{y}) \in I \times J$,

$$\tau := \omega_{ab} + \omega_{a'b'} - \omega_{a'b} - \omega_{ab'} = \frac{(x_a - x_{a'})(y_b - y_{b'})}{h} \partial^2_{xy}\Phi(\tilde{x}, \tilde{y}).$$

By (3-1) and (3-2) and since $h = L^{-K}$, $H(I) + H(J) = K - 1$, we have

$$\tfrac{1}{8} C_R^{-\frac{2}{\delta} - \frac{2}{\delta'}} L^{-\frac{1}{3}} \leq |\tau| \leq 8L^{-\frac{1}{3}} \leq \pi.$$

Therefore, by Lemma 2.6 the phase factor $e^{i\tau}$ is bounded away from 1, which combined with (3-28) gives a lower bound on the discrepancy:

$$|F_{ab}| \cdot |e^{i\tau} - 1| \geq \frac{|\tau|\sqrt{R}}{\pi} \geq \frac{C_R^{-\frac{2}{\delta} - \frac{2}{\delta'}}}{8\pi} L^{-\frac{1}{3}} \sqrt{R}. \tag{3-30}$$

On the other hand we can estimate the same discrepancy from above by (3-27), Lemma 3.7, and the triangle inequality:

$$\begin{aligned}
|F_{ab}| \cdot |e^{i\tau} - 1| &= |e^{i(\omega_{ab} - \omega_{ab'})} F_{ab} - e^{i(\omega_{a'b} - \omega_{a'b'})} F_{ab}| \\
&\leq |e^{i(\omega_{ab} - \omega_{ab'})} F_{ab} - F_{ab'}| + |F_{ab'} - F_{a'b'}| + |e^{i(\omega_{a'b} - \omega_{a'b'})} F_{a'b} - F_{a'b'}| + |F_{ab} - F_{a'b}| \\
&< 4(C_R')^2 L^{\delta+\delta'} \sqrt{\varepsilon_1 R} + 8\theta^{-1} L^{-\frac{2}{3}} \sqrt{R}.
\end{aligned}$$

Comparing this with (3-30) and dividing by $\sqrt{R}$, we obtain

$$\frac{C_R^{-\frac{2}{\delta}-\frac{2}{\delta'}}}{8\pi}L^{-\frac{1}{3}} < 4(C_R')^2 L^{\delta+\delta'}\sqrt{\varepsilon_1} + 8\theta^{-1}L^{-\frac{2}{3}}.$$

This gives a contradiction with the following consequences of (3-2) and (3-3):

$$8\theta^{-1}L^{-\frac{2}{3}} \le \frac{C_R^{-\frac{2}{\delta}-\frac{2}{\delta'}}}{16\pi}L^{-\frac{1}{3}}, \quad 4(C_R')^2 L^{\delta+\delta'}\sqrt{\varepsilon_1} \le \frac{C_R^{-\frac{2}{\delta}-\frac{2}{\delta'}}}{16\pi}L^{-\frac{1}{3}}.$$

3C. ***Proof of Theorem 1.*** We now show how to reduce Theorem 1 to Proposition 3.1. The idea is to split G into pieces using a partition of unity. On each piece, by appropriate rescaling we keep the regularity constant C_R and reduce to the case (3-1) and $h = L^{-K}$ for some fixed L satisfying (3-2) and some integer $K > 0$.

To be more precise, let (X,μ_X), (Y,μ_Y), δ, δ', I_0, J_0, Φ, G satisfy the hypotheses of Theorem 1. Using a partition of unity, we write G as a finite sum

$$G = \sum_\ell G_\ell, \quad G_\ell \in C^1(I_0\times J_0;\mathbb{C}),\ \operatorname{supp} G_\ell \subset I_\ell \times J_\ell, \tag{3-31}$$

where $I_\ell \subset I_0$, $J_\ell \subset J_0$ are intervals such that for some $m = m(\ell) \in \mathbb{Z}$,

$$2^{m-1} < |\partial^2_{xy}\Phi| < 2^{m+1} \quad \text{on } I_\ell \times J_\ell.$$

It then suffices to show (1-4), where G is replaced by one of the functions G_ℓ. By changing Φ outside of the support of G (which does not change the operator $\mathcal{B}_h$), we then reduce to the case when

$$2^{m-1} < |\partial^2_{xy}\Phi| < 2^{m+1} \quad \text{on } I_0 \times J_0 \tag{3-32}$$

for some $m \in \mathbb{Z}$.

We next rescale $\mathcal{B}_h$ to an operator $\widetilde{\mathcal{B}}_{\tilde h}$ satisfying the hypotheses of Proposition 3.1. Fix the smallest $L \in \mathbb{Z}$ satisfying (3-2). Choose $K \in \mathbb{Z}$ and $\sigma \in [1,\sqrt{L})$ such that

$$\sigma^2 = 2^m \frac{\tilde h}{h}, \quad \tilde h := L^{-K}. \tag{3-33}$$

Put for all intervals I, J

$$\begin{aligned}\widetilde X &:= \sigma X \subset \widetilde I_0 := \sigma I_0, \quad \mu_{\widetilde X}(\sigma I) := \sigma^\delta \mu_X(I),\\ \widetilde Y &:= \sigma Y \subset \widetilde J_0 := \sigma J_0, \quad \mu_{\widetilde Y}(\sigma J) := \sigma^{\delta'} \mu_Y(J).\end{aligned}$$

Then $(\widetilde X, \mu_{\widetilde X})$ is δ-regular, and $(\widetilde Y, \mu_{\widetilde Y})$ is δ'-regular, up to scale σh with regularity constant C_R. Consider the unitary operators

$$\begin{aligned}U_X &: L^2(X,\mu_X) \to L^2(\widetilde X, \mu_{\widetilde X}), \quad U_X f(\tilde x) = \sigma^{-\frac{\delta}{2}} f(\sigma^{-1}\tilde x),\\ U_Y &: L^2(Y,\mu_Y) \to L^2(\widetilde Y, \mu_{\widetilde Y}), \quad U_Y f(\tilde y) = \sigma^{-\frac{\delta'}{2}} f(\sigma^{-1}\tilde y).\end{aligned}$$

Then the operator $\widetilde{\mathcal{B}}_{\tilde{h}} := U_X \mathcal{B}_h U_Y^{-1} : L^2(\widetilde{Y}, \mu_{\widetilde{Y}}) \to L^2(\widetilde{X}, \mu_{\widetilde{X}})$ has the form (1-3):

$$\widetilde{\mathcal{B}}_{\tilde{h}} f(\tilde{x}) = \int_{\widetilde{Y}} \exp\left(\frac{i\widetilde{\Phi}(\tilde{x},\tilde{y})}{\tilde{h}}\right) \widetilde{G}(\tilde{x},\tilde{y}) f(\tilde{y})\, d\mu_{\widetilde{Y}}(\tilde{y}),$$

where

$$\widetilde{\Phi}(\tilde{x},\tilde{y}) = 2^{-m}\sigma^2 \Phi(\sigma^{-1}\tilde{x}, \sigma^{-1}\tilde{y}), \quad \widetilde{G}(\tilde{x},\tilde{y}) = \sigma^{-\frac{\delta}{2}-\frac{\delta'}{2}} G(\sigma^{-1}\tilde{x}, \sigma^{-1}\tilde{y}).$$

By (3-32) the function $\widetilde{\Phi}$ satisfies (3-1). Fix smallest $K_0 \in \mathbb{N}_0$ such that $\sigma h \le L^{K_0-K}$, that is,

$$L^{K_0} \ge \frac{2^m}{\sigma}.$$

Without loss of generality, we may assume that h is small enough depending on L, m so that $K \ge 2K_0$. Then Proposition 3.1 applies to $\widetilde{\mathcal{B}}_{\tilde{h}}$ and gives

$$\|\mathcal{B}_h\|_{L^2(Y,\mu_Y)\to L^2(X,\mu_X)} = \|\widetilde{\mathcal{B}}_{\tilde{h}}\|_{L^2(\widetilde{Y},\mu_{\widetilde{Y}})\to L^2(\widetilde{X},\mu_{\widetilde{X}})} \le C\tilde{h}^{\varepsilon_0} \le C(2^{-m}L)^{\varepsilon_0} h^{\varepsilon_0}$$

for ε_0 defined in (1-5) and some constant C depending only on $\delta, \delta', C_R, I_0, J_0, \Phi, G$. This finishes the proof of Theorem 1.

4. Application: spectral gap for hyperbolic surfaces

We now discuss applications of Theorem 1 to spectral gaps. We start with the case of hyperbolic surfaces, referring the reader to [Borthwick 2016; Dyatlov and Zahl 2016] for the terminology used here.

Let $M = \Gamma\backslash\mathbb{H}^2$ be a convex cocompact hyperbolic surface, $\Lambda_\Gamma \subset \mathbb{S}^1$ be its limit set, $\delta \in [0,1)$ be the dimension of Λ_Γ, and μ be the Patterson–Sullivan measure, which is a probability measure supported on Λ_Γ; see for instance [Borthwick 2016, Section 14.1]. Since Λ_Γ is closed and is not equal to the entire $\mathbb{S}^1$, we may cut the circle $\mathbb{S}^1$ to turn it into an interval and treat Λ_Γ as a compact subset of $\mathbb{R}$. Then (Λ_Γ, μ) is δ-regular up to scale 0 with some constant C_R; see for instance [Borthwick 2016, Lemma 14.13]. The regularity constant C_R depends continuously on the surface, as explained in the case of three-funnel surfaces in [Dyatlov and Zahl 2016, Proposition 7.7].

The main result of this section is the following essential spectral gap for M. We formulate it here in terms of the scattering resolvent of the Laplacian. Another formulation is in terms of a zero-free region for the Selberg zeta function past the first pole; see for instance [Dyatlov and Zahl 2016]. See below for a discussion of previous work on spectral gaps.

Theorem 2. *Consider the meromorphic scattering resolvent*

$$R(\lambda) = \left(-\Delta_M - \tfrac{1}{4} - \lambda^2\right)^{-1} : \begin{cases} L^2(M) \to L^2(M), & \operatorname{Im}\lambda > 0, \\ L^2_{\mathrm{comp}}(M) \to L^2_{\mathrm{loc}}(M), & \operatorname{Im}\lambda \le 0. \end{cases}$$

Assume that $0 < \delta < 1$. *Then* M *has an essential spectral gap of size*

$$\beta = \tfrac{1}{2} - \delta + (13C_R)^{-\frac{320}{\delta(1-\delta)}}; \tag{4-1}$$

that is, $R(\lambda)$ has only finitely many poles in $\{\operatorname{Im}\lambda > -\beta\}$ and it satisfies the cutoff estimates for each $\psi \in C_0^\infty(M)$, $\varepsilon > 0$ and some constant C_0 depending on ε

$$\|\psi R(\lambda)\psi\|_{L^2\to L^2} \le C(\psi,\varepsilon)|\lambda|^{-1-2\min(0,\operatorname{Im}\lambda)+\varepsilon}, \quad \operatorname{Im}\lambda \in [-\beta, 1],\ |\operatorname{Re}\lambda| \ge C_0.$$

Proof. We use the strategy of [Dyatlov and Zahl 2016]. By Theorem 3 of that paper, it suffices to show the following fractal uncertainty principle: for each $\rho \in (0,1)$,

$$\beta_0 := \tfrac{1}{2} - \delta + (150C_R^2)^{-\frac{160}{\delta(1-\delta)}},$$

and each cutoff function $\chi \in C^\infty(\mathbb{S}^1\times\mathbb{S}^1)$ supported away from the diagonal, there exists a constant C depending on M, χ, ρ such that for all $h \in (0,1)$

$$\|\mathbb{1}_{\Lambda_\Gamma(h^\rho)} B_{\chi,h} \mathbb{1}_{\Lambda_\Gamma(h^\rho)}\|_{L^2(\mathbb{S}^1)\to L^2(\mathbb{S}^1)} \le C h^{\beta_0 - 2(1-\rho)}, \tag{4-2}$$

where $\Lambda_\Gamma(h^\rho) \subset \mathbb{S}^1$ is the h^ρ neighborhood of Λ_Γ and the operator $B_{\chi,h}$ is defined by (here $|x-y|$ is the Euclidean distance between $x, y \in \mathbb{S}^1 \subset \mathbb{R}^2$)

$$B_{\chi,h} f(x) = (2\pi h)^{-\frac{1}{2}} \int_{\mathbb{S}^1} |x-y|^{\frac{2i}{h}} \chi(x,y) f(y)\, dy.$$

To show (4-2), we first note that by Lemma 2.2, (Y, μ_Y) is δ-regular up to scale h with constant $30C_R^2$, where $Y = \Lambda_\Gamma(h)$ and μ_Y is $h^{\delta-1}$ times the restriction of the Lebesgue measure to Y. We lift $\chi(x,y)$ to a compactly supported function on $\mathbb{R}^2$ (splitting it into pieces using a partition of unity) and write

$$B_{\chi,h}\mathbb{1}_{\Lambda_\Gamma(h)} f(x) = (2\pi)^{-\frac{1}{2}} h^{\frac{1}{2}-\delta} \mathcal{B}_h f(x),$$

where $\mathcal{B}_h$ has the form (1-3) with $G(x,y) = \chi(x,y)$ and (with $|x-y|$ still denoting the Euclidean distance between $x, y \in \mathbb{S}^1$)

$$\Phi(x,y) = 2\log|x-y|.$$

The function Φ is smooth and satisfies the condition $\partial^2_{xy}\Phi \neq 0$ on the open set $\mathbb{S}^1\times\mathbb{S}^1 \setminus \{x=y\}$ which contains the support of G; see for instance [Bourgain and Dyatlov 2016, Section 4.3]. Applying Theorem 1 with $(X, \mu_X) := (Y, \mu_Y)$, we obtain

$$\|\mathbb{1}_{\Lambda_\Gamma(h)} B_{\chi,h} \mathbb{1}_{\Lambda_\Gamma(h)}\|_{L^2(\mathbb{S}^1)\to L^2(\mathbb{S}^1)} \le C h^{\beta_0}.$$

Similarly we have

$$\|\mathbb{1}_{\Lambda_\Gamma(h)+t} B_{\chi,h} \mathbb{1}_{\Lambda_\Gamma(h)+s}\|_{L^2(\mathbb{S}^1)\to L^2(\mathbb{S}^1)} \le C h^{\beta_0}, \quad t, s \in [-1, 1],$$

where $X + t$ is the result of rotating $X \subset \mathbb{S}^1$ by angle t. Covering $\Lambda_\Gamma(h^\rho)$ with at most $10h^{\rho-1}$ rotations of the set $\Lambda_\Gamma(h)$, see for instance the proof of [Bourgain and Dyatlov 2016, Proposition 4.2], and using triangle inequality, we obtain (4-2), finishing the proof. ☐

We now briefly discuss previous results on spectral gaps for hyperbolic surfaces:

- The works [Patterson 1976; Sullivan 1979] imply that $R(\lambda)$ has no poles with $\operatorname{Im}\lambda > \delta - \frac{1}{2}$. On the other hand, the fact that $R(\lambda)$ is the L^2 resolvent of the Laplacian in $\{\operatorname{Im}\lambda > 0\}$ shows that it has only has finitely many poles in this region. Together these two results give the essential spectral gap $\beta = \max\big(0, \frac{1}{2} - \delta\big)$. Thus Theorem 2 gives no new results when δ is much larger than $\frac{1}{2}$.

- Using the method developed by Dolgopyat [1998], Naud [2005] showed an essential spectral gap of size $\beta > \frac{1}{2} - \delta$ when $\delta > 0$. Oh and Winter [2016] showed that the size of the gap is uniformly controlled for towers of congruence covers in the arithmetic case.

- Dyatlov and Zahl [2016] introduced the fractal-uncertainty-principle approach to spectral gaps and used it together with tools from additive combinatorics to give an estimate of the size of the gap in terms of C_R in the case when δ is very close to $\frac{1}{2}$.

- Bourgain and Dyatlov [2016] showed that each convex cocompact hyperbolic surface has an essential spectral gap of some size $\beta = \beta(\delta, C_R) > 0$. Their result is new in the case $\delta > \frac{1}{2}$ and is thus complementary to the results mentioned above, as well as to Theorem 2.

More generally, spectral gaps have been studied for noncompact manifolds with hyperbolic trapped sets. (See for instance [Nonnenmacher 2011, Section 2.1] for a definition.) In this setting the Patterson–Sullivan gap $\frac{1}{2} - \delta$ generalizes to the *pressure gap* $-P\left(\frac{1}{2}\right)$ which has been established by Ikawa [1988], Gaspard and Rice [1989], and Nonnenmacher and Zworski [2009]. An improved gap $\beta > -P\left(\frac{1}{2}\right)$ has been proved in several cases; see in particular [Petkov and Stoyanov 2010; Stoyanov 2011; 2012]. We refer the reader to [Nonnenmacher 2011] for an overview of results on spectral gaps for general hyperbolic trapped sets.

5. Application: spectral gap for open quantum maps

In this section, we discuss applications of the fractal uncertainty principle to the spectral properties of open quantum maps. Following the notation in [Dyatlov and Jin 2017] we consider an open quantum baker's map B_N determined by a triple $(M, \mathcal{A}, \chi)$, where $M \in \mathbb{N}$ is called the base, $\mathcal{A} \subset \mathbb{Z}_M = \{0, 1, \dots, M-1\}$ is called the alphabet, and $\chi \in C_0^\infty((0,1);[0,1])$ is a cutoff function. The map B_N is a sequence of operators $B_N : \ell_N^2 \to \ell_N^2$, $\ell_N^2 = \ell^2(\mathbb{Z}_N)$, defined for every positive $N \in M\mathbb{Z}$ by

$$B_N = \mathcal{F}_N^* \begin{pmatrix} \chi_{N/M}\, \mathcal{F}_{N/M}\, \chi_{N/M} & & \\ & \ddots & \\ & & \chi_{N/M}\, \mathcal{F}_{N/M}\, \chi_{N/M} \end{pmatrix} I_{\mathcal{A},M}, \tag{5-1}$$

where $\mathcal{F}_N$ is the unitary Fourier transform given by the $N \times N$ matrix $(1/\sqrt{N})(e^{-\frac{2\pi i j \ell}{N}})_{j\ell}$, $\chi_{N/M}$ is the multiplication operator on $\ell_{N/M}^2$ discretizing χ, and $I_{\mathcal{A},M}$ is the diagonal matrix with ℓ-th diagonal entry equal to 1 if $\lfloor \ell/(N/M) \rfloor \in \mathcal{A}$ and 0 otherwise.

An important difference from [Dyatlov and Jin 2017] is that in the present paper we allow N to be any multiple of M, while they required that N be a power of M. To measure the size of N, we let k be the unique integer such that $M^k \le N < M^{k+1}$, i.e., $k = \lfloor \log N / \log M \rfloor$. Denote by δ the dimension of the Cantor set corresponding to M and $\mathcal{A}$, given by

$$\delta = \frac{\log |\mathcal{A}|}{\log M}.$$

The main result of this section is the following spectral gap, which was previously established in [Dyatlov and Jin 2017, Theorem 1] for the case when N is a power of M:

Theorem 3. *Assume that* $0<\delta<1$; *that is*, $1<|\mathcal{A}|<M$. *Then there exists*

$$\beta=\beta(M,\mathcal{A})>\max\bigl(0,\tfrac{1}{2}-\delta\bigr) \tag{5-2}$$

such that, *with* $\mathrm{Sp}(B_N)\subset\{\lambda\in\mathbb{C}:|\lambda|\le 1\}$ *denoting the spectrum of* B_N,

$$\limsup_{N\to\infty,\,N\in M\mathbb{Z}}\max\{|\lambda|:\lambda\in\mathrm{Sp}(B_N)\}\le M^{-\beta}. \tag{5-3}$$

The main component of the proof is a fractal uncertainty principle. For the case $N=M^k$, the following version of it was used in [Dyatlov and Jin 2017]:

$$\|\mathbb{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbb{1}_{\mathcal{C}_k}\|_{\ell^2_N\to\ell^2_N}\le CN^{-\beta}, \tag{5-4}$$

where $\mathcal{C}_k$ is the discrete Cantor set given by

$$\mathcal{C}_k:=\left\{\sum_{j=0}^{k-1}a_jM^j:a_0,\dots,a_{k-1}\in\mathcal{A}\right\}\subset\mathbb{Z}_N. \tag{5-5}$$

For general $N\in M\mathbb{Z}\cap[M^k,M^{k+1})$, we define a similar discrete Cantor set in $\mathbb{Z}_N$ by

$$\mathcal{C}_k(N):=\{b_j(N):j\in\mathcal{C}_k\}\subset\mathbb{Z}_N,\quad b_j(N):=\left\lceil\frac{jN}{M^k}\right\rceil. \tag{5-6}$$

In fact, in our argument we only need $b_j(N)$ to be some integer in $[jN/M^k,(j+1)N/M^k)$.

The uncertainty principle then takes the following form:

Theorem 4. *Assume that* $0<\delta<1$. *Then there exists*

$$\beta=\beta(M,\mathcal{A})>\max\bigl(0,\tfrac{1}{2}-\delta\bigr) \tag{5-7}$$

such that for some constant C *and all* N,

$$\|\mathbb{1}_{\mathcal{C}_k(N)}\mathcal{F}_N\mathbb{1}_{\mathcal{C}_k(N)}\|_{\ell^2_N\to\ell^2_N}\le CN^{-\beta}. \tag{5-8}$$

In Section 5A below, we show that Theorem 4 implies Theorem 3. We prove Theorem 4 in Sections 5C and 5D using Ahlfors–David regularity of the Cantor set, which is verified in Section 5B.

5A. ***Fractal uncertainty principle implies spectral gap.*** We first show that Theorem 4 implies Theorem 3. The argument is essentially the same as in [Dyatlov and Jin 2017, Section 2.3], relying on the following generalization of Proposition 2.5 from that paper:

Proposition 5.1 (localization of eigenstates). *Fix* $\nu>0$, $\rho\in(0,1)$, *and assume that for some* $k\in\mathbb{N}$, $N\in M\mathbb{Z}\cap[M^k,M^{k+1})$, $\lambda\in\mathbb{C}$, $u\in\ell^2_N$, *we have*

$$B_Nu=\lambda u,\quad|\lambda|\ge M^{-\nu}.$$

Define

$$X_\rho := \bigcup \{\mathcal{C}_k(N) + m : m \in \mathbb{Z},\ |m| \le (M+2)N^{1-\rho}\} \subset \mathbb{Z}_N.$$

Then

$$\|u\|_{\ell^2_N} \le M^\nu |\lambda|^{-\rho k}\, \|\mathbb{1}_{X_\rho} u\|_{\ell^2_N} + \mathcal{O}(N^{-\infty})\|u\|_{\ell^2_N}, \tag{5-9}$$

$$\|u - \mathcal{F}_N^* \mathbb{1}_{X_\rho} \mathcal{F}_N u\|_{\ell^2_N} = \mathcal{O}(N^{-\infty})\|u\|_{\ell^2_N}, \tag{5-10}$$

where the constants in $\mathcal{O}(N^{-\infty})$ depend only on ν, ρ, χ.

Proof. Following [Dyatlov and Jin 2017, (2.7)], let $\Phi = \Phi_{M,\mathcal{A}}$ be the expanding map defined by

$$\Phi : \bigsqcup_{a\in\mathcal{A}} \left(\frac{a}{M}, \frac{a+1}{M}\right) \to (0,1), \quad \Phi(x) = Mx - a,\ x \in \left(\frac{a}{M}, \frac{a+1}{M}\right). \tag{5-11}$$

Put

$$\tilde{k} := \lceil \rho k \rceil \in \{1, \dots, k\}. \tag{5-12}$$

With $d(\cdot,\cdot)$ denoting the distance function on the circle as in [Dyatlov and Jin 2017, Section 2.1], define

$$\mathcal{X}_\rho := \{x \in [0,1] : d(x, \Phi^{-\tilde{k}}([0,1])) \le N^{-\rho}\}.$$

Then (5-9), (5-10) follow from the long time Egorov theorem [Dyatlov and Jin 2017, Proposition 2.4] (whose proof never used that N is a power of M) similarly to Proposition 2.5 of the same paper, as long as we show the following analog of [Dyatlov and Jin 2017, (2.30)]:

$$\ell \in \{0, \dots, N-1\}, \quad \frac{\ell}{N} \in \mathcal{X}_\rho \quad \Longrightarrow \quad \ell \in X_\rho. \tag{5-13}$$

To see (5-13), note that (with the intervals considered in $\mathbb{R}/\mathbb{Z}$)

$$\Phi^{-\tilde{k}}([0,1]) \subset \bigcup_{j\in\mathcal{C}_k} \left(\frac{j - M^{k-\tilde{k}}}{M^k}, \frac{j + M^{k-\tilde{k}}}{M^k}\right).$$

Assume that $\ell \in \{0, \dots, N-1\}$ and $\ell/N \in \mathcal{X}_\rho$. Then there exists $j \in \mathcal{C}_k$ such that

$$d\left(\frac{\ell}{N}, \frac{j}{M^k}\right) \le N^{-\rho} + M^{-\tilde{k}} \le (M+1)N^{-\rho}.$$

It follows that

$$d\left(\frac{\ell}{N}, \frac{b_j(N)}{N}\right) \le (M+2)N^{-\rho}$$

and thus $\ell \in X_\rho$ as required. □

Now, we assume that Theorem 4 holds and prove Theorem 3. Using the triangle inequality as in the proof of [Dyatlov and Jin 2017, Proposition 2.6], we obtain

$$\begin{aligned}\|\mathbb{1}_{X_\rho} \mathcal{F}_N^* \mathbb{1}_{X_\rho}\|_{\ell^2_N \to \ell^2_N} &\le (2M+5)^2 N^{2(1-\rho)} \|\mathbb{1}_{\mathcal{C}_k(N)} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k(N)}\|_{\ell^2_N \to \ell^2_N} \\ &\le C N^{2(1-\rho)-\beta}.\end{aligned} \tag{5-14}$$

Here C denotes a constant independent of N.

Assume that $\lambda \in \mathbb{C}$ is an eigenvalue of B_N such that $|\lambda| \geq M^{-\beta}$ and $u \in \ell^2_N$ is a normalized eigenfunction of B_N with eigenvalue λ. By (5-9), (5-10), and (5-14)

$$\begin{aligned} 1 = \|u\|_{\ell^2_N} &\leq M^{\beta}|\lambda|^{-\rho k}\|\mathbb{1}_{X_\rho}u\|_{\ell^2_N} + \mathcal{O}(N^{-\infty}) \\ &\leq M^{\beta}|\lambda|^{-\rho k}\|\mathbb{1}_{X_\rho}\mathcal{F}_N^*\mathbb{1}_{X_\rho}\mathcal{F}_N u\|_{\ell^2_N} + \mathcal{O}(N^{-\infty}) \\ &\leq C|\lambda|^{-\rho k}N^{2(1-\rho)-\beta} + \mathcal{O}(N^{-\infty}). \end{aligned} \tag{5-15}$$

It follows that $|\lambda|^{\rho k} \leq CN^{-\beta+2(1-\rho)}$ or equivalently

$$|\lambda| \leq C^{\frac{1}{\rho}k} M^{\frac{2(1-\rho)-\beta}{\rho}}.$$

This implies

$$\limsup_{N\to\infty} \max\{|\lambda| : \lambda \in \operatorname{Sp}(B_N)\} \leq \max\{M^{-\beta}, M^{\frac{2(1-\rho)-\beta}{\rho}}\}.$$

Letting $\rho \to 1$, we conclude the proof of Theorem 3.

5B. ***Regularity of discrete Cantor sets.*** Theorem 4 will be deduced from Theorem 1 and the results of [Bourgain and Dyatlov 2016]. To apply these, we establish Ahlfors–David regularity of the Cantor set $\mathcal{C}_k(N) \subset \mathbb{Z}_N = \{0, \dots, N-1\}$ in the following discrete sense.

Definition 5.2. We say that $X \subset \mathbb{Z}_N$ is δ-regular with constant C_R if

- for each interval J of size $|J| \geq 1$, we have $\#(J \cap X) \leq C_R|J|^\delta$, and
- for each interval J with $1 \leq |J| \leq N$ which is centered at a point in X, we have $\#(J \cap X) \geq C_R^{-1}|J|^\delta$.

Definition 5.2 is related to Definition 1.1 as follows:

Lemma 5.3. *Let $X \subset \mathbb{Z}_N$. Define $\widetilde{X} := N^{-1}X \subset [0,1]$ which supports the measure*

$$\mu_{\widetilde{X}}(A) := N^{-\delta} \cdot \#(\widetilde{X} \cap A), \quad A \subset \mathbb{R}. \tag{5-16}$$

Then X is δ-regular with constant C_R in the sense of Definition 5.2 if and only if $(\widetilde{X}, \mu_{\widetilde{X}})$ is δ-regular up to scale N^{-1} with constant C_R in the sense of Definition 1.1.

Proof. This follows directly from the two definitions. □

We first establish the regularity of the discrete Cantor set $\mathcal{C}_k$ defined in (5-5):

Lemma 5.4. *The set $\mathcal{C}_k \subset \mathbb{Z}_{M^k}$ is δ-regular with constant $C_R = 2M^{2\delta}$.*

Proof. We notice that for all integers $k' \in [0,k]$ and $j' \in \mathbb{Z}$

$$\#\big(\mathcal{C}_k \cap [j'M^{k'}, (j'+1)M^{k'})\big) = \begin{cases} |\mathcal{A}|^{k'} = M^{\delta k'}, & j' \in \mathcal{C}_{k-k'}, \\ 0, & j' \notin \mathcal{C}_{k-k'}. \end{cases} \tag{5-17}$$

Let J be an interval in $\mathbb{R}$, with $1 \leq |J| \leq N = M^k$. Choose an integer $k' \in [0, k-1]$ such that $M^{k'} \leq |J| \leq M^{k'+1}$. Then there exists some $j' \in \mathbb{Z}$ such that

$$J \subset [j'M^{k'+1}, (j'+2)M^{k'+1}).$$

Therefore by (5-17)

$$\#(\mathcal{C}_k \cap J) \le 2M^{\delta(k'+1)} \le 2M^{\delta}|J|^{\delta} \le C_R|J|^{\delta}.$$

On the other hand, if $|J| > N$ then

$$\#(\mathcal{C}_k \cap J) \le \#(\mathcal{C}_k) = N^{\delta} \le |J|^{\delta}.$$

This gives the required upper bound on $\#(\mathcal{C}_k \cap J)$.

Now, assume that $1 \le |J| \le N$ and J is centered at some $j \in \mathcal{C}_k$. Choose k' as before. If $k' = 0$ then

$$\#(\mathcal{C}_k \cap J) \ge 1 \ge M^{-\delta}|J|^{\delta} \ge C_R^{-1}|J|^{\delta}.$$

We henceforth assume that $1 \le k' \le k-1$. Let $j' \in \mathcal{C}_{k-k'+1}$ be the unique element such that $j'M^{k'-1} \le j < (j'+1)M^{k'-1}$. Since $M \ge 2$, we have $|J| \ge M^{k'} \ge 2M^{k'-1}$ and thus

$$[j'M^{k'-1}, (j'+1)M^{k'-1}] \subset [j - M^{k'-1}, j + M^{k'-1}] \subset J.$$

Therefore by (5-17)

$$\#(\mathcal{C}_k \cap J) \ge M^{\delta(k'-1)} \ge M^{-2\delta}|J|^{\delta} \ge C_R^{-1}|J|^{\delta}.$$

This gives the required lower bound on $\#(\mathcal{C}_k \cap J)$, finishing the proof. □

We now establish regularity of the dilated Cantor set $\mathcal{C}_k(N)$:

Proposition 5.5. *Assume that $M^k \le N < M^{k+1}$ and let $\mathcal{C}_k(N) \subset \mathbb{Z}_N$ be given by* (5-6). *Then $\mathcal{C}_k(N)$ is δ-regular with constant $C_R = 8M^{3\delta}$.*

Proof. For any interval J, we have

$$\#(\mathcal{C}_k(N) \cap J) = \#\{j \in \mathcal{C}_k : b_j(N) \in J\} = \#\Big\{j \in \mathcal{C}_k : \frac{M^k}{N} b_j(N) \in \frac{M^k}{N} J\Big\}.$$

By our choice of $b_j(N)$, we have $(M^k/N)b_j(N) \in [j, j+1)$. Therefore

$$\#\Big(\mathcal{C}_k \cap \frac{M^k}{N} J\Big) - 1 \le \#(\mathcal{C}_k(N) \cap J) \le \#\Big(\mathcal{C}_k \cap \frac{M^k}{N} J\Big) + 1.$$

We apply Lemma 5.4 to see that for any interval J with $|J| \ge 1$

$$\#(\mathcal{C}_k(N) \cap J) \le 2M^{2\delta}|J|^{\delta} + 1 \le 3M^{2\delta}|J|^{\delta} \le C_R|J|^{\delta}.$$

Now, assume that J is an interval with $8^{\frac{1}{\delta}} M^3 \le |J| \le N$ centered at $b_j(N)$ for some $j \in \mathcal{C}_k$. Then $(M^k/N)J$ contains the interval of size $\frac{1}{2M}|J|$ centered at j. Therefore, by Lemma 5.4

$$\#(\mathcal{C}_k(N) \cap J) \ge \frac{1}{2M^{2\delta}} \left(\frac{|J|}{2M}\right)^{\delta} - 1 \ge \frac{|J|^{\delta}}{8M^{3\delta}} \ge C_R^{-1}|J|^{\delta}.$$

Finally, if J is an interval with $1 \le |J| \le 8^{\frac{1}{\delta}} M^3$ centered at a point in $\mathcal{C}_k(N)$, then

$$\#(\mathcal{C}_k(N) \cap J) \ge 1 \ge C_R^{-1}|J|^{\delta}.$$

□

5C. *Fractal uncertainty principle for $\delta \le \frac{1}{2}$.* The proof of Theorem 4 in the case $\delta \le \frac{1}{2}$ relies on the following corollary of Theorem 1:

Proposition 5.6. *Let $X, Y \subset \mathbb{Z}_N$ be δ-regular with constant C_R and $0 < \delta < 1$. Then*

$$\|\mathbb{1}_X \mathcal{F}_N \mathbb{1}_Y\|_{\ell^2_N \to \ell^2_N} \le CN^{-(\frac{1}{2}-\delta+\varepsilon_0)}, \tag{5-18}$$

where C only depends on δ, C_R and

$$\varepsilon_0 = (5C_R)^{-\frac{160}{\delta(1-\delta)}}. \tag{5-19}$$

Proof. Put $h := N^{-1}$, $\widetilde{X} := hX$, $\widetilde{Y} := hY$, and define the measures $\mu_{\widetilde{X}}, \mu_{\widetilde{Y}}$ by (5-16). By Lemma 5.3, $(\widetilde{X}, \mu_{\widetilde{X}})$ and $(\widetilde{Y}, \mu_{\widetilde{Y}})$ are δ-regular up to scale h with constant C_R. Consider the operator

$$\mathcal{B}_h : L^1(\widetilde{Y}, \mu_{\widetilde{Y}}) \to L^\infty(\widetilde{X}, \mu_{\widetilde{X}})$$

defined by

$$\mathcal{B}_h f(x) = \int_{\widetilde{Y}} \exp\left(-\frac{2\pi i x y}{h}\right) f(y)\, d\mu_{\widetilde{Y}}(y)$$

and note that it has the form (1-3) with $\Phi(x, y) = -2\pi x y$, $G \equiv 1$. By Theorem 1

$$\|\mathcal{B}_h\|_{L^2(\widetilde{Y},\mu_{\widetilde{Y}}) \to L^2(\widetilde{X},\mu_{\widetilde{X}})} \le Ch^{\varepsilon_0}.$$

Comparing the formula

$$\mathcal{B}_h f\left(\frac{j}{N}\right) = N^{-\delta} \sum_{\ell \in Y} \exp\left(-\frac{2\pi i j \ell}{N}\right) f\left(\frac{\ell}{N}\right), \quad j \in X,$$

with the definition of the discrete Fourier transform $\mathcal{F}_N$, we see that

$$\|\mathbb{1}_X \mathcal{F}_N \mathbb{1}_Y\|_{\ell^2_N \to \ell^2_N} = N^{\delta - \frac{1}{2}} \|\mathcal{B}_h\|_{L^2(\widetilde{Y},\mu_{\widetilde{Y}}) \to L^2(\widetilde{X},\mu_{\widetilde{X}})},$$

which finishes the proof. $\square$

Combining Propositions 5.5 and 5.6, we get (5-8) for

$$\beta = \tfrac{1}{2} - \delta + (40M^{3\delta})^{-\frac{160}{\delta(1-\delta)}} \tag{5-20}$$

which finishes the proof of Theorem 4 for $\delta \le \frac{1}{2}$.

5D. *Fractal uncertainty principle for $\delta > \frac{1}{2}$.* For $\delta > \frac{1}{2}$, Theorem 1 does not in general give an improvement over the trivial gap $\beta = 0$. Instead, we shall use the following reformulation of [Bourgain and Dyatlov 2016, Theorem 4]:

Proposition 5.7. *Let $0 \le \delta < 1$, $C_R \ge 1$, $N \ge 1$ and assume that $\widetilde{X}, \widetilde{Y} \subset [-1, 1]$ and $(\widetilde{X}, \mu_{\widetilde{X}})$ and $(\widetilde{Y}, \mu_{\widetilde{Y}})$ are δ-regular up to scale N^{-1} with constant C_R in the sense of Definition 1.1, for some finite measures $\mu_{\widetilde{X}}, \mu_{\widetilde{Y}}$ supported on $\widetilde{X}, \widetilde{Y}$.*

Then there exist $\beta_0 > 0$, C_0 depending only on δ, C_R such that for all $f \in L^2(\mathbb{R})$,

$$\operatorname{supp} \hat{f} \subset N \cdot \widetilde{Y} \quad \Longrightarrow \quad \|f\|_{L^2(\widetilde{X})} \le C_0 N^{-\beta_0} \|f\|_{L^2(\mathbb{R})}. \tag{5-21}$$

Here $\hat f$ denotes the Fourier transform of f:

$$htf(\xi)=\mathcal{F}f(\xi)=\int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)\,dx. \tag{5-22}$$

Proposition 5.7 implies the following discrete fractal uncertainty principle:

Proposition 5.8. *Let $X, Y \subset \mathbb{Z}_N$ be δ-regular with constant C_R and $0 \le \delta < 1$. Then*

$$\|\mathbb{1}_X \mathcal{F}_N \mathbb{1}_Y\|_{\ell^2_N \to \ell^2_N} \le CN^{-\beta}, \tag{5-23}$$

where $C, \beta > 0$ only depend on δ, C_R.

Proof. Put $h := N^{-1}$,

$$\widetilde{X} := hX + [-h,h], \quad \widetilde{Y} := hY + [-h,h],$$

and define the measures $\mu_{\widetilde{X}}, \mu_{\widetilde{Y}}$ on $\widetilde{X}, \widetilde{Y}$ by (2-6). By Lemmas 5.3 and 2.2, $(\widetilde{X}, \mu_{\widetilde{X}})$ and $(\widetilde{Y}, \mu_{\widetilde{Y}})$ are δ-regular up to scale h with constant $30C_R^2$. Applying Proposition 5.7, we obtain for some constants $\beta_0 > 0,\ C_0$ depending only on δ, C_R and all $f \in L^2(\mathbb{R})$

$$\operatorname{supp} htf \subset N\cdot\widetilde{Y} \quad \Longrightarrow \quad \|f\|_{L^2(\widetilde{X})} \le C_0 N^{-\beta_0}\|f\|_{L^2(\mathbb{R})}. \tag{5-24}$$

To pass from (5-24) to (5-23), fix a cutoff function χ such that for some constant $c > 0$

$$\chi \in C_0^\infty\big((-\tfrac{1}{2},\tfrac{1}{2})\big), \quad \|\chi\|_{L^2} = 1, \quad \inf_{[0,1]} |\mathcal{F}^{-1}\chi| \ge c.$$

This is possible since for any $\chi \in C_0^\infty(\mathbb{R})$ which is not identically 0, $\mathcal{F}^{-1}\chi$ extends to an entire function and thus has no zeros on $\{\operatorname{Im} z = s\}$ for all but countably many choices of $s \in \mathbb{R}$. Choosing such s we see that $\mathcal{F}^{-1}(e^{-s\xi}\chi(\xi))$ has no real zeros.

Now, take arbitrary $u \in \ell^2_N$. Consider the function $f \in L^2(\mathbb{R})$ defined by

$$\hat f(\xi) = \sum_{\ell \in Y} u(\ell)\chi(\xi - \ell).$$

Then $\operatorname{supp} \hat f \subset N\cdot\widetilde{Y}$ and $\|f\|_{L^2(\mathbb{R})} \le \|u\|_{\ell^2_N}$, so by (5-24)

$$\|f\|_{L^2(\widetilde{X})} \le C_0 N^{-\beta_0}\|u\|_{\ell^2_N}. \tag{5-25}$$

On the other hand, for all $j \in \mathbb{Z}_N$, we have for all $j \in X$

$$\frac{1}{\sqrt{N}} f\left(\frac{j}{N}\right) = \mathcal{F}_N^* \mathbb{1}_Y u(j)\cdot(\mathcal{F}^{-1}\chi)\left(\frac{j}{N}\right). \tag{5-26}$$

Consider the nonoverlapping collection of intervals

$$I_j := \left[\frac{j}{N} - \frac{1}{2N}, \frac{j}{N} + \frac{1}{2N}\right] \subset \widetilde{X}, \quad j \in X.$$

Using that $(|f|^2)' = 2\operatorname{Re}(\bar{f} f')$, we have

$$|\mathcal{F}_N^* \mathbb{1}_Y u(j)|^2 \le \frac{1}{c^2 N}\left| f\left(\frac{j}{N}\right)\right|^2 \le C \int_{I_j} |f(x)|^2\, dx + \frac{C}{N}\int_{I_j} |f(x)|\cdot|f'(x)|\, dx,$$

where C denotes some constant depending only on δ, C_R, χ. Summing over $j \in X$ and using the Cauchy–Schwarz inequality, we obtain

$$\|\mathbb{1}_X \mathcal{F}_N^* \mathbb{1}_Y u\|^2_{\ell^2_N} \le C\|f\|^2_{L^2(\tilde{X})} + \frac{C}{N}\|f\|_{L^2(\tilde{X})}\cdot\|f'\|_{L^2(\mathbb{R})}.$$

Since $\operatorname{supp}\hat{f} \subset [-N, N]$, we have $\|f'\|_{L^2(\mathbb{R})} \le 10N\|f\|_{L^2(\mathbb{R})} \le 10N\|u\|_{\ell^2_N}$ and thus by (5-25)

$$\|\mathbb{1}_X \mathcal{F}_N^* \mathbb{1}_Y u\|^2_{\ell^2_N} \le CN^{-\beta_0}\|u\|^2_{\ell^2_N},$$

which gives (5-23) with $\beta = \frac{1}{2}\beta_0$. □

Combining Propositions 5.5 and 5.8, we obtain (5-8) for $\frac{1}{2} \le \delta < 1$, finishing the proof of Theorem 4.

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