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ON THE GLOBAL STABILITY OF A BETA-PLANE EQUATION

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We study the motion of an incompressible, inviscid two-dimensional fluid in a rotating frame of reference. There the fluid experiences a Coriolis force, which we assume to be linearly dependent on one of the coordinates. This is a common approximation in geophysical fluid dynamics and is referred to as the β-plane approximation. In vorticity formulation, the model we consider is then given by the Euler equation with the addition of a linear anisotropic, nondegenerate, dispersive term. This allows us to treat the problem as a quasilinear dispersive equation whose linear solutions exhibit decay in time at a critical rate.

Our main result is the global stability and decay to equilibrium of sufficiently small and localized solutions. Key aspects of the proof are the exploitation of a "double null form" that annihilates interactions between spatially coherent waves and a lemma for Fourier integral operators which allows us to control a strong weighted norm.

1. Introduction

A basic model for a fluid in a rotating frame of reference is given by the Euler–Coriolis equation

$$\begin{cases} \partial_t v + v \cdot \nabla v + f\,\Omega \wedge v + \nabla p = 0, \\ \operatorname{div} v = 0, \end{cases} \tag{1-1}$$

where $v = (v_1, v_2, v_3) : (t, x) \in \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and $p : (t, x) \in \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ are the velocity and pressure of the fluid, respectively. Here, $f\,\Omega \wedge v$ is the Coriolis force experienced in the rotating frame, with $\Omega \in \mathbb{R}^3$ being the axis of rotation and $f : \mathbb{R}^3 \to \mathbb{R}$ the strength of the effect, which depends on the spatial location (but not on time). To describe waves on the surface of the Earth, a common approximation in geophysical fluid dynamics, see [McWilliams 2006; Pedlosky 1987], consists in choosing $\Omega = (0, 0, 1)^{\intercal}$

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and assuming trivial dynamics in the vertical direction, i.e., $\partial_3 v = 0$. One can then reduce matters to a two-dimensional system

$$\begin{cases} \partial_t u + u\cdot\nabla u + (-f u_2, f u_1)^{\mathsf{T}} + \nabla p = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \operatorname{div} u = 0, \end{cases} \tag{1-2}$$

where now $u : (t,x) \in \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$, $p : (t,x)\in\mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$. A solution to the original system (1-1) is then recovered by setting $(v_1, v_2) = (u_1, u_2)$ and solving a transport equation for v_3.

Passing to a scalar equation using the vorticity $\omega := \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$ yields

$$\partial_t\omega + u\cdot\nabla\omega = -u\cdot\nabla f, \quad u = \nabla^{\perp}(-\Delta)^{-1}\omega. \tag{1-3}$$

On a rotating sphere, such as the Earth, the force f varies with the sine of the latitude. In a first rough approximation, the so-called f-plane approximation, this variation is ignored, and a fixed value f_0 is used throughout the domain. A more accurate and very common[1] model in geophysical fluid dynamics is a linear approximation to this variability, which is usually referred to as the β-plane approximation; see, e.g., [McWilliams 2006, Chapter 2; Pedlosky 1987, Chapter 3]. Assuming that the strength of the Coriolis force depends linearly on the latitude,

$$f(x,y) = f_0 + \beta(y - y_0),$$

we arrive at the so-called *β-plane equation*

$$\partial_t\omega + u\cdot\nabla\omega = \beta L_1\omega, \quad L_1 := \frac{\partial_x}{\Delta} = \frac{R_1}{|\nabla|}, \quad u = \nabla^{\perp}(-\Delta)^{-1}\omega, \tag{1-4}$$

for $\omega : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$. Here β is the parameter of linearity of the Coriolis force, which by rescaling can be assumed to be equal to 1, and R_1 stands for the Riesz transform in the first coordinate:

$$\widehat{R_1 g}(\xi) = \frac{-i\xi_1}{|\xi|}\hat{g}(\xi), \quad \hat{g}(\xi) := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix\cdot\xi} g(x)\, dx.$$

On one hand, one can view (1-4) as a perturbation of the Euler equation by a constant-coefficient differential operator and show, by arguments akin to those for the two-dimensional Euler equation, the existence of global solutions (even for large data) with at most double exponential growth in H^s, $s > 1$; see [Elgindi and Widmayer 2017, Appendix B]. On the other hand (1-4) can also be viewed as a quasilinear dispersive equation, in the sense that it is a nonlinear version of the equation $\partial_t\omega = L_1\omega$, solutions of which exhibit dispersive decay as will be shown further below.

1A. ***Main result.*** The content of this article is a treatment of the nonlinear problem (1-4), with the result that for sufficiently small and localized initial data, solutions to the Cauchy problem decay like solutions of the linear problem, and the zero solution of (1-4) is globally nonlinearly stable in a strong sense. We can state our main result as follows:

[1] Such a modeling assumption is made in various contexts: examples include rotating shallow-water equations, Rossby waves and quasigeostrophic scenarios; see [McWilliams 2006, Chapter 4; Majda 2003, Chapter 4; Pedlosky 1987, Chapter 3] among others. We also remark that in [Sukhatme and Smith 2009], equation (1-4) was viewed as part of a larger family of equations to model two-dimensional dispersive turbulence.

Theorem 1.1. *Consider the initial value problem for the β-plane equation*

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = L_1 \omega, \quad u = \nabla^{\perp}(-\Delta)^{-1}\omega, \\ \qquad\qquad \omega(0) = \omega_0. \end{cases} \tag{1-5}$$

There exist $N \gg 1$, $\varepsilon_0 > 0$, and a weighted L^2-based function space $X \subset \dot{W}^{1,1}$ on $\mathbb{R}^2$ such that for any initial data with $\|\omega_0\|_X$, $\|\omega_0\|_{H^N} \le \varepsilon_0$, there exists a unique global solution of (1-5) *which decays at the linear rate*, *namely $\|\omega(t)\|_{L^\infty} \lesssim \varepsilon_0(1+|t|)^{-1}$, and scatters.*

A more precise statement of the theorem is presented as Theorem 2.2 in Section 2, where we also illustrate its proof through a bootstrap argument in Section 2A. The key difficulty here lies in establishing a global control over a suitably chosen weighted X-norm of the profile of ω — see (2-8) on page 1592 for the precise definition — which has to be strong enough to guarantee the L^∞ decay.

1B. ***Background.*** To give some context we now present some of the key difficulties in treating the β-plane equation as a quasilinear dispersive equation. The present model features a quadratic nonlinearity and a critical decay rate of $|t|^{-1}$ at the linear level. This situation is common to many other dispersive and hyperbolic equations and a variety of different behaviors can occur even for small and Schwartz initial data. For example, one could have global solutions with linear behavior as in the case of (quasilinear) wave equations [Klainerman 1986] with a null condition, blow-up at time $T \approx e^{\frac{1}{\varepsilon_0}}$ as in the compressible Euler equations [Sideris 1985], nonlinear asymptotics in the sense of modified scattering as for nonlinear Schrödinger equations [Hayashi and Naumkin 1998; Kato and Pusateri 2011], or growth at infinity as in [Alinhac 2003].

In the present case solutions are already known to be global, so no blow-up occurs. Moreover, one can notice that there is a null structure in (1-5). More precisely, since $u = \nabla^{\perp}(-\Delta)^{-1}\omega$, the transport term $u \cdot \nabla\omega$ is depleted when two parallel frequencies interact. On the negative side, one should also notice that, when seen as a bilinear term in ω, the nonlinearity is singular because of the $(-\Delta)^{-1}$ factor. Moreover, the linear operator L_1 is anisotropic, and the impossibility of commuting the equation with rotations introduces several difficulties.

Inviscid Euler and the role of dispersion. Generally, inviscid Euler-type nonlinearities can lead to double exponential growth, as was shown by the example of [Kiselev and Šverák 2014] on a bounded domain; see also [Denisov 2015; Zlatoš 2015]. In the whole space, the question of global stability and asymptotic behavior for the Euler equation is widely open. A byproduct of Theorem 2.2 is that for sufficiently small data, instability in (1-5) is prevented by dispersion: waves with different frequencies travel with distinct velocities and their interactions lose strength over time. However, this is a much weaker effect than damping or friction. Indeed for (1-5) the same L^2-based estimates as for the inviscid Euler equation $\partial_t\omega + u \cdot \nabla\omega = 0$ hold, because of the skew symmetry (for the inner product in L^2) of the constant-coefficient right-hand-side operator L_1. Also, all Sobolev norms are preserved by the linear flow, and the same blow-up criterion as for the two-dimensional Euler equation holds.

As is shown in this article, *the dispersion produced by L_1 acts as a regularizing mechanism that globally stabilizes the fluid.* A first way of seeing improvements at the hands of dispersion is through a basic energy estimate yielding the following: assuming a linear decay rate of $|t|^{-1}$ for Du in L^∞

one obtains the slow growth of all Sobolev norms for the nonlinear problem (whereas in the absence of dispersion, or without control on the rate of dispersion, the best known bounds are double exponential — see [Elgindi and Widmayer 2017, Appendix B]). A finer understanding of the interactions in the Euler-type nonlinearity is then needed to show that decay occurs for nonlinear solutions.

In earlier work of T. Elgindi and the second author [Elgindi and Widmayer 2017], stability for the β-plane equation (1-5) for arbitrarily large times was established: it was shown that for any $M \in \mathbb{N}$ there exists a threshold $\varepsilon_M > 0$, below which initial data of size $\varepsilon \leq \varepsilon_M$ lead to solutions that decay on time scales at least ε^{-M} — for more details see [Elgindi and Widmayer 2017, Theorem 2.1]. Apart from this work, the literature on the β-plane equation is oriented towards questions of relevance in the realm of geophysical fluid dynamics. An exhaustive list is beyond the scope of this article, and beyond the expertise of its authors, so we refer the reader to the books [Drazin 2002; Majda 2003; McWilliams 2006] for some overview.

Resonance structure and (double) null form. At the basis of our approach is the formulation of the problem in a way that makes it amenable to techniques from harmonic analysis. This is done by working with the profile of the vorticity $f(t) := e^{-tL_1}\omega(t)$, and writing the Duhamel formula for solutions of (1-5) in terms of this profile f in Fourier space, in order to obtain an integral expression which can be viewed as an oscillatory integral — see the beginning of Section 2 and the formulas (2-1)–(2-2).

From this point of view the resonances of the equation, that is, roughly speaking, those sets of frequencies that do not produce oscillations, play a key role in the analysis of the nonlinear interactions. This starting point is inspired by the method of space-time resonances, as introduced in [Germain et al. 2012b]. Without entering into too much detail, for now we point out that the space-time resonant set for this equation is one-dimensional, which is the generic situation for quadratic nonlinearities in two dimensions; thus it does not provide any additional smallness, in contrast to other problems such as in [Germain et al. 2012a; 2012b]. However, as already pointed out above, a null form is available in the nonlinearity: the symbol of the quadratic interaction vanishes on parallel frequencies; see (2-1)–(2-2) in connection with (2-21). See also the models in [Pusateri and Shatah 2013; Oh and Pusateri 2015; Hani et al. 2013] for similar behaviors.

In fact, as we shall explain in detail below, even more is true for (1-5): one has a “double” null form, a quadratic (instead of linear) degree of vanishing of the symbol, as can be seen by symmetrizing the expression (2-1). This is a key insight which greatly improves the control one has over interactions close to the (space) resonant set, and for example yields much better decay estimates for the L^2 norm of $\partial_t f$ than one would normally expect.

In our proof we will also exploit the special, anisotropic, geometric structure of interactions near the (time) resonances through a TT^* argument, which was previously used in [Deng et al. 2016; 2017]. However, here we employ such an argument in a different context, not for the purpose of establishing energy estimates, but as another means of extracting more oscillations in the bilinear interactions. This allows us to prove a strong weighted bound for our solutions which in turn implies the desired decay over time.

1C. *Plan of the article.* In Section 2 we begin by setting up the problem and giving our detailed functional framework. We then state a precise formulation of Theorem 1.1 (see Theorem 2.2) and discuss its proof using a bootstrap argument. We see there that a fractional weighted estimate, see (2-16), is at the core of our efforts. By symmetrizing the formulation of the β-plane equation we obtain a "double null form". As a first application, this yields improved bounds for the first iterate (see Lemma 2.4). The rest of the article is then devoted to establishing the weighted estimate.

In Section 3 we go through preliminary reductions and a finite speed of propagation argument that limits the range of parameters we need to consider for the weighted estimate. Further reductions are then presented in Section 4. Using various localizations we balance smallness of relevant sets and repeated integration by parts to essentially reduce to a problem where only frequencies of roughly order 1 are involved. These arguments crucially rely on the improved bounds due to the double null form achieved through symmetrization.

Finally, in Section 5 we exploit a nondegeneracy property of the phase function Φ, defined in (2-1)–(2-2), via a TT^* argument, in combination with an appropriate anisotropic localization, thereby concluding the proof of the weighted estimate.

In Section 6 we collect some useful lemmata.

2. Setup

The Duhamel formulation associated to the β-plane equation (1-5) is

$$\omega(t) = e^{tL_1}\omega_0 + \int_0^t e^{(t-s)L_1} u \cdot \nabla\omega(s)\, ds.$$

Written in terms of the profile

$$f(t) := e^{-tL_1}\omega(t)$$

this reads

$$\hat{f}(t,\xi) = \hat{f}_0(\xi) + \frac{1}{(2\pi)^2}\int_0^t \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} \frac{\xi\cdot\eta^\perp}{|\eta|^2} \hat{f}(s,\xi-\eta)\hat{f}(s,\eta)\, d\eta\, ds \tag{2-1}$$

with

$$\Phi(\xi,\eta) := \frac{\xi_1}{|\xi|^2} - \frac{\xi_1-\eta_1}{|\xi-\eta|^2} - \frac{\eta_1}{|\eta|^2}. \tag{2-2}$$

From now on we will omit the time dependence of the profiles in this expression, since it is clear from the context.

We define the quadratic nonlinearity $B(f,f)$ through its Fourier transform

$$\mathcal{F}B(f,f)(t,\xi) := \int_0^t \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} \frac{\xi\cdot\eta^\perp}{|\eta|^2} \hat{f}(s,\xi-\eta)\hat{f}(s,\eta)\, d\eta\, ds, \tag{2-3}$$

so that the Duhamel formula (2-1) can be written as

$$\hat{f}(t,\xi) = \hat{f}_0(\xi) + \frac{1}{(2\pi)^2}\mathcal{F}B(f,f)(t,\xi). \tag{2-4}$$

Conserved quantities. For future reference we note that an explicit calculation using (1-2) and (1-3) shows that the L^2-norms of both u and ω are conserved along the flow of the equation:

$$\|\omega(t)\|_{L^2} = \|\omega(0)\|_{L^2} \quad \text{and} \quad \|u(t)\|_{L^2} = \|u(0)\|_{L^2}, \quad t\in\mathbb{R}.$$

As an immediate consequence we obtain that the $\dot{H}^{-1}$ norms of ω and f are controlled as well:

$$\big\||\nabla|^{-1} f\big\|_{L^2} = \big\||\nabla|^{-1}\omega\big\|_{L^2} \lesssim \|u\|_{L^2}. \tag{2-5}$$

Notation. In this article we will work with localizations in frequency, space and time. To define them, as is standard in Littlewood–Paley theory we let $\varphi : \mathbb{R} \to [0, 1]$ be an even, smooth function supported in $\big[-\frac{8}{5}, \frac{8}{5}\big]$ and equal to 1 on $\big[-\frac{5}{4}, \frac{5}{4}\big]$. With a slight abuse of notation we also let φ be the corresponding radial function on $\mathbb{R}^2$. For $k \in \mathbb{Z}$ we define $\varphi_k(x) := \varphi(2^{-k}|x|) - \varphi(2^{-k+1}|x|)$, so that the family $(\varphi_k)_{k\in\mathbb{Z}}$ forms a partition of unity,

$$\sum_{k\in\mathbb{Z}} \varphi_k(\xi) = 1, \quad \xi \neq 0.$$

We also let

$$\varphi_I(x) := \sum_{k\in I\cap\mathbb{Z}} \varphi_k \quad \text{for any } I \subset \mathbb{R}, \quad \varphi_{\le a}(x) := \varphi_{(-\infty,a]}(x), \quad \varphi_{>a}(x) = \varphi_{(a,\infty]}(x),$$

with similar definitions for $\varphi_{<a}$, $\varphi_{\ge a}$. To these cut-offs we associate frequency projections P_k through

$$P_k g := \mathcal{F}^{-1}(\varphi_k(\xi)\hat{g}(\xi))$$

and define similarly $P_I g := \mathcal{F}^{-1}(\varphi_I(\xi)\hat{g}(\xi))$, $P_{\le k} g := \mathcal{F}^{-1}(\varphi_{\le k}(\xi)\hat{g}(\xi))$, $k \in \mathbb{Z}$ etc. We will also sometimes write $\tilde{\varphi}_k = \varphi_{[k-2,k+2]}$.

To simultaneously localize in space, for $(k, j) \in \mathcal{J} := \{(k, j) \in \mathbb{Z}\times\mathbb{Z} : k + j \ge 0,\ j \ge 0\}$ we let

$$\varphi_j^{(k)}(x) := \begin{cases} \varphi_j(x), & j \ge -k+1 \text{ or } j \ge 1, \\ \varphi_{\le 0}(x), & j = 0 \quad (k \ge 0), \\ \varphi_{\le -k}(x), & j = -k \quad (k \le 0). \end{cases} \tag{2-6}$$

Notice that for any $k \in \mathbb{Z}$ we have $\sum_{j\ge -\min\{0,k\}} \varphi_j^{(k)}(x) = 1$. We then define

$$Q_{jk} g := P_{[k-2,k+2]}\varphi_j^{(k)} P_k g$$

to be the operator that localizes both in frequency and space. This will often be used to decompose our profiles into atoms

$$g = \sum_{(k,j)\in\mathcal{J}} Q_{jk} g. \tag{2-7}$$

For notational convenience we also introduce the shorthand $\langle t\rangle := \sqrt{1+t^2}$ for $t \in \mathbb{R}$.

The main norm. Apart from the usual Sobolev and Lebesgue spaces we will be using a weighted function space built on L^2 in an atomic way: with the notation $k^+ := \max\{k, 0\}$ we let

$$\|g(t)\|_X := \sup_{(k,j)\in\mathcal{J}} 2^{(k+j)(1+\delta)} 2^{4k^+} \|Q_{jk} g(t)\|_{L^2}, \quad \delta = 0.5\cdot 10^{-4}. \tag{2-8}$$

This choice of norm is motivated by our quest to control the L^∞ decay of ω through the dispersive estimate (2-9) below. The use of weighted L^2 norms in quasilinear dispersive problems is fairly standard. Here we have decided to use a fractional weight following the functional framework introduced in [Ionescu and Pausader 2014]. The particular choice of putting the same number of derivatives (the power of 2^k) as the number of weights (the power of 2^j) is dictated by the characteristics of this specific problem, including the singularity of the bilinear form in (2-3) and the "speed of propagation" of linear frequencies.

Dispersive estimate. For the linear semigroup e^{tL_1} we have the following decay estimate:

Lemma 2.1. *For $g \in \mathcal{S}(\mathbb{R}^2)$ and $k \in \mathbb{Z}$ we have*

$$\|e^{tL_1} P_k g\|_{L^\infty} \lesssim |t|^{-1} 2^{3k} \|P_k g\|_{L^1}. \tag{2-9}$$

Since the Hessian of the exponent $\xi_1|\xi|^{-2}$ on the Fourier side is $4|\xi|^{-6}$, and so in particular is nondegenerate, the proof is a standard application of the stationary phase lemma — see [Elgindi and Widmayer 2017, Proposition 4.1]. We remark that the right-hand side of (2-9) is controlled by the X-norm of g in (2-8) above.

Main theorem. In more detail, our Main Theorem, Theorem 1.1, is:

Theorem 2.2. *Let*[2] $0 < \delta \le 0.5 \cdot 10^{-4}$ *and* $N \ge 2.1 \cdot \delta^{-1}$. *Then there exists an* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \le \varepsilon_0$ *and initial data* ω_0 *with*

$$\|\omega_0\|_{H^N} + \|\omega_0\|_X \le \varepsilon, \tag{2-10}$$

(1-5) *admits a unique global solution* $\omega \in C(\mathbb{R}, H^N(\mathbb{R}^2))$. *Moreover, for all* $t \in \mathbb{R}$ *the solution satisfies the bounds*

$$\|\omega(t)\|_{H^N} \lesssim \varepsilon_0 (1+|t|)^{C\varepsilon_0}, \quad \|e^{-tL_1}\omega(t)\|_X \lesssim \varepsilon_0, \tag{2-11}$$

and, in particular, also the decay estimate

$$\|\omega(t)\|_{L^\infty} \lesssim \varepsilon_0 (1+|t|)^{-1}. \tag{2-12}$$

Finally, the solutions scatters: for any initial data ω_0 *as in* (2-10) *there exist unique* $f_{\pm\infty} \in X$ *such that*

$$\|e^{-tL_1}\omega(t) - f_{\pm\infty}\|_X \to 0 \quad \text{as } t \to \pm\infty. \tag{2-13}$$

2A. ***Proof of the main theorem.*** We will prove Theorem 2.2 through a bootstrap argument. The main ingredient is the bilinear estimate (3-1), which establishes Proposition 2.3 below. Since the equation is time reversible it suffices to consider $t > 0$. We will work with the following a priori assumptions.

A priori assumptions. We assume that for some $T > 0$ and $\varepsilon_1 = A\varepsilon_0$ with a suitably chosen constant $A > 1$ to be determined below, we have

$$\|P_k f(t)\|_{L^2} \le \varepsilon_1 \langle t\rangle^{D\varepsilon_0} 2^{-Nk^+}, \tag{2-14}$$

$$\sup_{(k,j)\in\mathcal{J}} (2^{k+j})^{1+\delta} 2^{4k^+} \|Q_{jk} f(t)\|_{L^2} \le \varepsilon_1 \tag{2-15}$$

[2] We did not optimize on the value of δ, and the related size of N, to make the proof more readable. Especially in the last part of the argument, in Sections 4 and 5, improvements on these values would be possible by tracking more carefully the various parameters involved, but due to the technicality of the proof, we have decided not to do so. It is very likely that a number N between 10 and 100 would work.

for all $t \in [0, T]$ and a suitably large $D > 0$. For small enough $T > 0$ the estimates (2-14)–(2-15) hold by virtue of (2-10) and a standard local well-posedness argument (that we omit), yielding a unique local solution such that $e^{-tL_1}\omega \in C([0, 1], H^N \cap X)$.

Weighted estimate. As a key point in this paper we will prove:

Proposition 2.3. *Assuming the a priori bounds* (2-14)–(2-15), *and with the notation* (2-3) *and* (2-8), *for all* $t \in [0, T]$ *we have*

$$\|B(f, f)(t)\|_X \lesssim \varepsilon_1^2. \tag{2-16}$$

This estimate is at the heart of our article and its proof will be carried out over the course of Sections 3–5. In fact, we will prove the stronger version (3-1) of the bilinear bound (2-16), which also implies the scattering statement (2-13) of Theorem 2.2.

Assuming Proposition 2.3 we now establish the Main Theorem.

Proof of Theorem 2.2. Our aim here is to show that the interval on which the a priori estimates (2-14)–(2-15) hold can be extended to infinity. Using a continuity argument it will suffice to prove that for $t \in [1, T]$

$$\begin{aligned} \|P_k f(t)\|_{L^2} &\le \tfrac{1}{2}\varepsilon_1 \langle t\rangle^{D\varepsilon_0} 2^{-Nk^+}, \\ \sup_{(k,j)\in\mathcal{J}} (2^{k+j})^{1+\delta} 2^{4k^+} \|Q_{jk} f(t)\|_{L^2} &\le \tfrac{1}{2}\varepsilon_1. \end{aligned} \tag{2-17}$$

Invoking the Duhamel formula (2-4) and applying Proposition 2.3 yields

$$\begin{aligned} 2^{4k^+} 2^{(k+j)(1+\delta)} \|Q_{jk} f(t)\|_{L^2} &\le 2^{4k^+} 2^{(k+j)(1+\delta)} \big(\|Q_{jk}\omega_0\|_{L^2} + \|Q_{jk} B(f, f)(t)\|_{L^2}\big) \\ &\le \varepsilon_0 + C\varepsilon_1^2 \le \tfrac{1}{2}\varepsilon_1 \end{aligned}$$

for ε_0 small enough. Combining this with the decay estimate (2-9) we also have

$$\|e^{tL_1} P_k f(t)\|_{L^\infty} \lesssim \langle t\rangle^{-1} 2^{3k} \sum_{j\ge -\min\{0,k\}} 2^j \|Q_{jk} f(t)\|_{L^2} \lesssim \langle t\rangle^{-1} (\varepsilon_0 + C\varepsilon_1^2) 2^{-4k^+} 2^{(2-\delta)k}.$$

In particular, if Du is the matrix of first derivatives of u, we have

$$\|\omega(t)\|_{L^\infty} + \|Du(t)\|_{L^\infty} \lesssim \langle t\rangle^{-1} (\varepsilon_0 + C\varepsilon_1^2) \tag{2-18}$$

for all $t \in [0, T]$. A standard energy estimate for the β-plane equation, see [Elgindi and Widmayer 2017, Lemma 3.1], gives the bound

$$\|\omega(t)\|_{H^N} \le \|\omega(0)\|_{H^N} \exp\bigg(C \int_0^t \|Du(s)\|_{L^\infty} + \|\omega(s)\|_{L^\infty}\, ds\bigg).$$

Inserting the decay estimate (2-18) and choosing appropriately the constant D, it follows that

$$\|P_k f(t)\|_{L^2} \le \varepsilon_0 \langle t\rangle^{D\varepsilon_0} 2^{-Nk^+}.$$

This gives us (2-17) and proves the bounds (2-11) and (2-12) in our Theorem 2.2.

To conclude we remark that in proving Proposition 2.3 we will actually prove the stronger version (3-1) of the bilinear bound (2-16). The estimate (3-1) then implies that $f(t)$ is a Cauchy sequence in the X space, so that (2-13) follows. □

2B. ***Symmetrization and double null form.*** By virtue of the symmetry $\Phi(\xi,\eta)=\Phi(\xi,\xi-\eta)$ we can write the bilinear term (2-3) as

$$\begin{aligned}\mathcal{F}B(f,f)(\xi) &= \int_0^t\int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}\frac{\xi\cdot\eta^\perp}{|\eta|^2}\hat f(\xi-\eta)\hat f(\eta)\,d\eta\,ds\\ &= \frac12\int_0^t\int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}\Big(\frac{\xi\cdot\eta^\perp}{|\eta|^2}+\frac{\xi\cdot(\xi-\eta)^\perp}{|\xi-\eta|^2}\Big)\hat f(\xi-\eta)\hat f(\eta)\,d\eta\,ds\\ &= \frac12\int_0^t\int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}\Big(\frac{(\xi\cdot\eta^\perp)\,\xi\cdot(\xi-2\eta)}{|\eta|^2\,|\xi-\eta|^2}\Big)\hat f(\xi-\eta)\hat f(\eta)\,d\eta\,ds.\end{aligned}$$

Here we let

$$\mathfrak{m}(\xi,\eta) := \frac12\frac{(\xi\cdot\eta^\perp)\,\xi\cdot(\xi-2\eta)}{|\eta|^2\,|\xi-\eta|^2} \tag{2-19}$$

and explicitly write the important equality

$$\begin{aligned}\mathcal{F}B(f,f) &= \int_0^t\int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}\frac{\xi\cdot\eta^\perp}{|\eta|^2}\hat f(\xi-\eta)\hat f(\eta)\,d\eta\,ds\\ &= \int_0^t\int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}\mathfrak{m}(\xi,\eta)\hat f(\xi-\eta)\hat f(\eta)\,d\eta\,ds.\end{aligned} \tag{2-20}$$

To illustrate the relevance of this symmetrization we remind the reader that we will treat the above expressions as oscillatory integrals. From this point of view, the set $\mathcal{S}=\{(\xi,\eta):\nabla_\eta\Phi=0\}$ where no oscillations in η occur in the phase $e^{is\Phi}$ (also called the space-resonant set) is one of the main obstructions to obtaining strong bounds through cancellations. In the present problem we have

$$|\nabla_\eta\Phi| = \frac{|\xi|\,|\xi-2\eta|}{|\xi-\eta|^2\,|\eta|^2}, \tag{2-21}$$

so the original multiplier $\xi\cdot\eta^\perp|\eta|^{-2}$ vanishes on $\mathcal{S}$. This is referred to as a "null structure" and allows one to (partially) compensate for the lack of oscillations; see for example [Klainerman 1986; Pusateri and Shatah 2013]. However, we highlight that in our case even more is true: *the symbol* $\mathfrak{m}$ *in* (2-20) *vanishes to second order on* $\mathcal{S}$, which is what we call a "double null form". As we will see, this offers a crucial advantage over the previous formulation with a regular null form.

Symbol bounds. Using the notation (6-4)–(6-5) we have the following basic bounds for our symbol (2-19):

$$\|\mathfrak{m}^{k,k_1,k_2}\|_{S^\infty} \lesssim 2^{k-\min\{k_1,k_2\}} \tag{2-22}$$

and

$$\begin{aligned}\|\mathfrak{m}^{k,k_1,k_2}(\xi,\eta)\varphi_r(\eta-2\xi)\|_{S^\infty} &\lesssim 2^{r-\min\{k_1,k_2\}},\\ \|\mathfrak{m}^{k,k_1,k_2}(\xi,\eta)\varphi_\ell(\xi-2\eta)\|_{S^\infty} &\lesssim 2^{\ell-\min\{k_1,k_2\}},\end{aligned}$$

as well as the more precise bound

$$\|\mathfrak{m}^{k,k_1,k_2}(\xi,\eta)\varphi_\ell(\xi-2\eta)\|_{S^\infty} \lesssim 2^{2\ell+2k-2k_1-2k_2}. \tag{2-23}$$

2C. *Estimate for $\partial_t f$.* As a first major consequence of the symmetrization in Section 2B we will establish a useful estimate for the time derivative of the profile. We will work under our main a priori assumptions (2-14)–(2-15); in order to readily have their more precise consequences (3-4)–(3-6) at our disposal we refer to them as they appear in (3-2)–(3-3).

Lemma 2.4. *Let f be given by* (2-1). *For all $m \in \{0,1,\dots\}$ and $t \in [2^m-1, 2^{m+1}] \cap [0,T]$, and under the a priori assumptions* (3-2)–(3-3), *we have*

$$\|P_k \partial_t f(t)\|_{L^2} \lesssim \varepsilon_1^2 2^k 2^{-4k_+} 2^{-2m+10\delta m}. \tag{2-24}$$

Notice that $\partial_t f(t)$ is a quadratic expression in $\omega(t)$ and is therefore expected to decay, in L^2 at least as fast as $\|\omega(t)\|_{L^\infty}$. The above lemma states that we actually have much more decay, almost t^{-2}. This is due to the favorable "double null structure" of the equations. Needless to say this estimate will be very helpful when integrating by parts in time in Duhamel's formula, which gives rise to bilinear terms involving $\partial_t f$.

Proof of Lemma 2.4. From (2-1) and (2-20) we have

$$\partial_t \hat{f}(t) = \mathcal{F}Q(f,f)(t,\xi) := \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{it\Phi(\xi,\eta)}\mathfrak{m}(\xi,\eta)\hat{f}(t,\xi-\eta)\hat{f}(t,\eta)\,d\eta.$$

We start by observing that for any $f, g \in L^2$ we have

$$\begin{aligned}\|P_k Q(P_{k_1}f, P_{k_2}g)\|_{L^2} \\ \lesssim \|\mathfrak{m}^{k,k_1,k_2}\|_{S^\infty}\cdot \sup_{t\approx 2^m} \min\bigl\{&\|P_{k_1}f\|_{L^2}\|e^{itL_1}P_{k_2}g\|_{L^\infty},\ \|e^{itL_1}P_{k_1}f\|_{L^\infty}\|P_{k_2}g\|_{L^2}, \\ &\|P_{k_1}f\|_{L^2}\|P_{k_2}g\|_{L^2}2^{\min\{k_1,k_2\}}\bigr\},\end{aligned} \tag{2-25}$$

having used Lemma 6.3. Moreover, notice that by symmetry in $\eta \leftrightarrow \xi-\eta$, when looking at $Q(P_{k_1}f, P_{k_2}f)$ we may assume that $k_2 \le k_1$ without loss of generality.

Using (2-25) and (3-6) we see that

$$\|P_k Q(P_{k_1}f, P_{k_2}f)\|_{L^2} \lesssim 2^{k-k_2}\|P_{k_1}f\|_{L^2}\|P_{k_2}f\|_{L^2}2^{k_2} \lesssim 2^k\cdot\varepsilon_1 2^{-Nk_1^+}2^{k_1}\cdot\varepsilon_1 2^{k_2},$$

so that the desired conclusion follows when $k_2 \le -2m$ or $k_1 \ge \delta m$ (we will choose $\delta(N-6) \ge 2$ in (3-7) below).

We also have

$$\|P_k Q(P_{k_1}f, P_{k_2}f)\|_{L^2} \lesssim 2^k\|\mathcal{F}P_k Q(P_{k_1}f, P_{k_2}f)\|_{L^\infty} \lesssim 2^{2k-k_2}\cdot\|P_{k_1}f\|_{L^2}\cdot\|P_{k_2}f\|_{L^2},$$

which, in view of (3-6), and after summing over k_1, k_2 with $k_2 \ge -2m$, gives the desired bound (2-24) if $k \le -2m$.

In what follows we can then assume

$$\min\{k, k_1, k_2\} \geq -2m, \quad \max\{k_1, k_2\} \leq \delta m. \tag{2-26}$$

This leaves us with a summation over (k, k_1, k_2) made by at most $O(m^3)$ terms, and we see that to obtain (2-24) it will suffice to show

$$\| P_k Q(P_{k_1} f, P_{k_2} f)\|_{L^2} \lesssim \varepsilon_1^2 2^k 2^{-4k^+} 2^{-2m+9\delta m} \tag{2-27}$$

for every fixed triple (k, k_1, k_2) satisfying (2-26). We subdivide the proof of (2-27) into two main cases: high-low and high-high interactions.

Case 1: $|k_1 - k_2| \geq 10$. In this case we have $k_1 \geq k_2 + 10$ and $|k - k_1| \leq 5$. We further decompose our inputs according to their spatial localization as in (3-17):

$$f_1 = Q_{j_1 k_1} f, \quad f_2 = Q_{j_2 k_2} f, \quad j_\nu + k_\nu \geq 0,\ \nu = 1, 2. \tag{2-28}$$

The Hölder estimate (2-25) and the a priori bounds (3-3)–(3-4) give us

$$\| P_k Q(f_1, f_2)\|_{L^2} \lesssim 2^{k-k_2} \cdot \varepsilon_1 2^{-m} \cdot \varepsilon_1 2^{-k_2} 2^{-\max\{j_1, j_2\}} \cdot 2^{-2k_1^+}.$$

Therefore, we can obtain the desired bound whenever $\max\{j_1, j_2\} \geq (1-\delta)m - 2k_2$. In the complementary case when $\max\{j_1, j_2\} \leq (1-\delta^2)m - 2k_2$ we can instead integrate by parts repeatedly in η. More precisely, using

$$|\nabla_\eta \Phi| \approx 2^{-2k_2}, \quad |D_\eta^\alpha \Phi| \lesssim 2^{-(1+|\alpha|)k_2}$$

we can apply the bound (6-6) in Lemma 6.5 with

$$K = s 2^{-2k_2}, \quad F = 2^{2k_2} \Phi, \quad \epsilon = 2^{k_2}, \quad g = \mathfrak{m}(\xi, \eta) \hat{f}_1(\xi - \eta) \hat{f}_2(\eta),$$

and obtain

$$\begin{aligned} \| P_k Q(f_1, f_2)\|_{L^2} &\lesssim 2^k \|\varphi_k(\xi) \widehat{Q}(f_1, f_2)(\xi)\|_{L^\infty_\xi} \\ &\lesssim 2^k \cdot (2^m 2^{-k_2})^{-M} (1 + 2^{k_2} 2^{\max\{j_1, j_2\}})^M \cdot 2^{k-k_2} \|f_1\|_{L^2} \|f_2\|_{L^2} \\ &\lesssim \varepsilon_1^2 2^{-5m} \|f_1\|_{L^2} \|f_2\|_{L^2}, \end{aligned}$$

where the last inequality follows by choosing M large enough. Using also (2-26) we see that this is more than sufficient to obtain (2-24).

Case 2: $|k_1 - k_2| < 10$. This case is more delicate and requires a further frequency space decomposition in the size of $|\xi - 2\eta|$. More precisely, we let

$$\mathcal{F} Q_\ell(f, g)(t, \xi) := \int_{\mathbb{R}^2} e^{it\Phi(\xi,\eta)} \mathfrak{m}(\xi, \eta) \varphi_\ell(\xi - 2\eta) \hat{f}(t, \xi - \eta) \hat{g}(t, \eta)\, d\eta.$$

Notice that this vanishes unless $\ell \leq k_1 + 20$. To obtain (2-27) it then suffices to show

$$\sum_{\ell \leq k_1 + 20} \| P_k Q_\ell(P_{k_1} f, P_{k_2} f)\|_{L^2} \lesssim \varepsilon_1^2 2^k 2^{-4k^+} 2^{-2m+9\delta m}. \tag{2-29}$$

Subcase 2.1: $\min\{k,\ell\} \leq (-1+5\delta)m + k_1$. In this case we first use the $L^2 \times L^\infty$ Hölder bound in Lemma 6.3 together with the symbol bound (2-23), and the usual a priori estimates (3-3)–(3-4), to deduce

$$2^{4k^+}\|P_k Q_\ell(P_{k_1}f, P_{k_2}f)\|_{L^2} \lesssim 2^{2\min\{k,\ell\}-k_1-k_2}\cdot \varepsilon_1 2^{(2-\delta)k_1}2^{-m}\cdot \varepsilon_1 2^{k_2}, \tag{2-30}$$

having also used (3-6). This suffices to obtain the desired bound when the sum in (2-29) is over $\ell \leq -m+k_1+5\delta m$ or when $k \leq -m+k_1+5\delta m$.

We are now left with $O(m)$ terms in the sum in (2-29), so that it suffices to show

$$2^{4k^+}\|P_k Q_\ell(P_{k_1}f, P_{k_2}f)\| \lesssim \varepsilon_1^2 2^k 2^{-2m+8\delta m}, \tag{2-31}$$

under the restrictions (2-26), $|k_1-k_2| \leq 10$ and $(-1+5\delta)m + k_1 \leq k, \ell \leq k_1+20$. We now further decompose our profiles in space, letting

$$P_k Q_\ell(P_{k_1}f, P_{k_2}f) = \sum_{j_1,j_2} P_k Q_\ell(f_1, f_2),$$

with the notation (2-28).

Subcase 2.2: $\max\{j_1,j_2\} \geq (1-4\delta)m - k_1 + \min\{\ell,k\}$. In this case we use the Hölder estimate in Lemma 6.3 with the symbol bound (2-23) to get

$$\|P_k Q_\ell(f_1,f_2)\|_{L^2} \lesssim 2^{2\min\{k,\ell\}-2k_1}\cdot \sup_{t\approx 2^m}\min\big\{\|f_1\|_{L^2}\|e^{itL_1}f_2\|_{L^\infty},\ \|e^{itL_1}f_1\|_{L^\infty}\|f_2\|_{L^2}\big\}.$$

The a priori bounds (3-3)–(3-4) then give us

$$\begin{aligned}2^{4k^+}\|P_k Q_\ell(f_1,f_2)\|_{L^2} &\lesssim 2^{2\min\{k,\ell\}-2k_1}\cdot \varepsilon_1 2^{-m}2^{(2-\delta)k_1}\cdot \varepsilon_1 2^{-k_1}2^{-\max\{j_1,j_2\}}\\ &\lesssim \varepsilon_1^2 2^k\cdot 2^{-\delta k_1}\cdot 2^{-m-k_1+\min\{k,\ell\}-\max\{j_1,j_2\}},\end{aligned}$$

which, upon summation over j_1, j_2, suffices to obtain (2-31) under the current assumptions.

Subcase 2.3: $\max\{j_1,j_2\} \leq (1-4\delta)m - k_1 + \min\{\ell,k\}$ and $\min\{k,\ell\} \geq (-1+5\delta)m + k_1$. In this last remaining case we want to resort again to repeated integration by parts through Lemma 6.5.

Before doing that, let us first look at the case $\ell \leq k+5$. Notice that if $\ell \leq -\frac{1}{2}m + \frac{3}{2}k_1 + \delta m$, then the Hölder estimate (2-30) already gives us the desired conclusion. We can then assume $\ell \geq -\frac{1}{2}m + \frac{3}{2}k_1 + \delta m$ in what follows. On the support of $P_k Q_\ell(f_1, f_2)$ we have, see (3-13),

$$|\nabla_\eta \Phi| \approx 2^\ell 2^{-3k_1}, \qquad |D_\eta^\alpha \Phi| \lesssim 2^{-(1+|\alpha|)k_1}, \quad |\alpha| \geq 2.$$

We then let

$$K = s2^\ell 2^{-3k_1}, \quad F(\eta) = \Phi(\xi,\eta)(2^\ell 2^{-3k_1})^{-1},$$

and calculate

$$|D^\alpha F| \lesssim (2^\ell 2^{-3k_1})^{-1}2^{-(1+|\alpha|)k_1} \lesssim 2^{(1-|\alpha|)\ell}, \quad |\alpha| \geq 2.$$

Choosing $\epsilon = 2^\ell$ and

$$g = \mathfrak{m}(\xi,\eta)\varphi_\ell(\xi-2\eta)\hat{f}_1(\xi-\eta)\hat{f}_2(\eta),$$

the bound (6-6) in Lemma 6.5 gives us

$$\begin{aligned}\|P_k Q_\ell(f_1, f_2)\|_{L^2} &\lesssim (2^m 2^\ell 2^{-3k_1})^{-M}(2^{-\ell}+2^{\max\{j_1,j_2\}})^M \|f_1\|_{L^2}\|f_2\|_{L^2}\\ &\lesssim 2^{-10m}\|f_1\|_{L^2}\|f_2\|_{L^2},\end{aligned}$$

which is more than enough.

Finally we look at the case $k \le \ell-5$. Recall that we may assume $k \ge -m+k_1+5\delta m$. In the present configuration we have

$$|\nabla_\eta \Phi| \approx 2^k 2^{-3k_1}, \qquad |D_\eta^\alpha \Phi| \lesssim 2^{-(2+|\alpha|)k_1} 2^k, \quad |\alpha| \ge 2.$$

We can then apply Lemma 6.5 with $K = s2^k 2^{-3k_1}$, $F(\eta) = \Phi(\xi,\eta)(2^k 2^{-3k_1})^{-1}$, $\epsilon = 2^{k_1}$, and the same choice of g as above, to obtain $\|P_k Q_\ell(f_1, f_2)\|_{L^2} \lesssim 2^{-5m}\|f_1\|_{L^2}\|f_2\|_{L^2}$. This concludes the proof of the lemma. □

3. Preliminary bounds and finite speed of propagation

Recall that our aim is to prove Proposition 2.3. We begin by localizing our time parameter on scales $\approx 2^m$, $m \in \mathbb{N}$, as follows. Given $t \in [0, T]$, we choose a suitable decomposition of the indicator function $\mathbf{1}_{[0,t]}$ by fixing functions $\tau_0, \dots, \tau_{L+1} : \mathbb{R} \to [0, 1]$, $|L - \log_2(2+t)| \le 2$, with the properties

$$\begin{gathered}\operatorname{supp}\tau_0 \subseteq [0,2], \quad \operatorname{supp}\tau_{L+1} \subseteq [t-2,t], \quad \operatorname{supp}\tau_m \subseteq [2^{m-1}, 2^{m+1}] \quad \text{for } m \in \{1,\dots,L\},\\ \sum_{m=0}^{L+1} \tau_m(s) = \mathbf{1}_{[0,t]}(s), \quad \tau_m \in C^1(\mathbb{R}), \quad \text{and} \quad \int_0^t |\tau_m'(s)|\,ds \lesssim 1 \quad \text{for } m \in \{1,\dots,L\}.\end{gathered}$$

We can then decompose

$$B(f,f) = \sum_m B_m(f,f), \quad \mathcal{F}B_m(f,f) := \int_0^t \tau_m(s) \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} \mathfrak{m}(\xi,\eta) \hat{f}(\xi-\eta)\hat{f}(\eta)\,d\eta\,ds.$$

To obtain Proposition 2.3 it will then suffice to show that for any $m = 0, 1, \dots,$

$$2^{4k^+} 2^{(k+j)(1+\delta)} \|Q_{jk} B_m(f,f)\|_{L^2} \lesssim \varepsilon_1^2 2^{-\delta^3 m}. \tag{3-1}$$

For convenience we recall here the a priori bounds (2-14)–(2-15),

$$\|P_k f(t)\|_{L^2} \le \varepsilon_1 \langle t\rangle^{p_0} 2^{-N_0 k^+}, \tag{3-2}$$

$$\sup_{(k,j)\in\mathcal{J}} (2^{k+j})^{1+\delta} 2^{4k^+} \|Q_{jk} f(t)\|_{L^2} \le \varepsilon_1, \tag{3-3}$$

where we can choose $p_0 = C\varepsilon_0 \le \delta$ for a suitable absolute constant $C > 0$. Then we also have the following consequences of (3-2)–(3-3):

$$\|e^{itL_1} Q_{jk} f(t)\|_{L^\infty} \lesssim \varepsilon_1 \langle t\rangle^{-1} 2^{-4k^+} 2^{(2-\delta)k} 2^{-\delta j}, \tag{3-4}$$

$$\|\widehat{Q_{jk} f}\|_{L^\infty} \le \|Q_{jk} f\|_{L^1} \lesssim \varepsilon_1 2^{-(1+\delta)k} 2^{-4k^+} 2^{-\delta j}. \tag{3-5}$$

Also recall that by virtue of (2-5) we have

$$2^{-k}\|P_k f\|_{L^2} \lesssim \big\||\nabla|^{-1} f\big\|_{L^2} = \big\||\nabla|^{-1}\omega\big\|_{L^2} \lesssim \|u\|_{L^2} \le \varepsilon_0. \tag{3-6}$$

In the remainder of this section we begin our proof of the weighted estimate (3-1) by treating first some ranges of parameters for which the estimates are easily seen to hold. Subsequently we present a "finite speed of propagation" argument, which invokes the idea that each frequency is expected to travel at its respective group velocity, in order to allow for a further reduction in the parameters to be considered.

3A. ***Basic cases.*** We first establish a simple lemma dealing with frequencies that are very large or very small with respect to the relevant parameters. To this end we let

$$N' := N - 6, \quad N' \ge \frac{2}{\delta}. \tag{3-7}$$

Lemma 3.1 (basic cases). *With the above notation and under the a priori assumptions* (3-2)–(3-4) *we have*

$$\sum_{\max\{k_1,k_2\} \ge \frac{k+j+\delta m}{N'}} 2^{4k^+} 2^{(1+\delta)(k+j)} \|Q_{jk} B_m(P_{k_1} f, P_{k_2} f)\|_{L^2} \lesssim 2^{-\delta^3 m} \varepsilon_1^2. \tag{3-8}$$

Moreover,

$$\sum_{\min\{k_1,k_2\} \le -1.01(k+j+\delta m)} 2^{4k^+} 2^{(1+\delta)(k+j)} \|Q_{jk} B_m(P_{k_1} f, P_{k_2} f)\|_{L^2} \lesssim 2^{-\delta^3 m} \varepsilon_1^2. \tag{3-9}$$

Proof. We begin by using an $L^2 \times L^\infty$ estimate, see Lemma 6.3, together with the symbol bound (2-22), to deduce that

$$\begin{aligned} &\|Q_{jk} B_m(P_{k_1} f, P_{k_2} f)\|_{L^2} \\ &\quad \lesssim 2^m \cdot 2^{k-\min\{k_1,k_2\}} \cdot \sup_{t \approx 2^m} \min\big\{\|P_{k_1} f\|_{L^2} \|e^{itL_1} P_{k_2} f\|_{L^\infty}, \|e^{itL_1} P_{k_1} f\|_{L^\infty} \|P_{k_2} f\|_{L^2}, \\ &\qquad\qquad \|P_{k_1} f\|_{L^2} \|P_{k_2} f\|_{L^2} 2^{\min\{k_1,k_2\}}\big\}. \end{aligned} \tag{3-10}$$

Proof of (3-8). Without loss of generality, let us assume $k_2 \le k_1$, so that the sum is over $k_1 \ge (k + j + \delta m)/N'$. Using the bound in the high Sobolev norm (3-2), the a priori decay assumption (3-4), and the estimate (3-10) above, we see that

$$\|Q_{jk} B_m(P_{k_1} f, P_{k_2} f)\|_{L^2} \lesssim 2^m \cdot 2^{k-k_2} \cdot \varepsilon_1 2^{-m} 2^{(2-\delta)k_2} 2^{-4k_2^+} \cdot \varepsilon_1 2^{p_0 m} 2^{-Nk_1}.$$

It follows that

$$\sum_{k_1 \ge \frac{k+j+\delta m}{N'}, k_2} 2^{4k^+} 2^{(1+\delta)(k+j)} \|Q_{jk} B_m(P_{k_1} f, P_{k_2} f)\|_{L^2} \lesssim 2^{(1+\delta)(k+j)} \cdot \varepsilon_1^2 2^{p_0 m} 2^{-\frac{(N-5)(k+j+\delta m)}{N'}}.$$

Since $(N-5)/N' \ge 1+\delta$ and $p_0 \le \delta$ this is sufficient.

Proof of (3-9). Again, without loss of generality we assume $k_2 \le k_1$, so that the sum is over $k_2 \le -1.01(k + j + \delta m)$. Using the estimate (3-10) above and the a priori bounds (3-3), (3-4) and (3-6), we see that

$$\|Q_{jk}B_m(P_{k_1}f, P_{k_2}f)\|_{L^2} \lesssim 2^m \cdot 2^{k-k_2} \cdot \varepsilon_1 2^{-m} 2^{(2-\delta)k_2} \cdot \varepsilon_1 2^{k_1} 2^{-4k_1^+}.$$

It follows that

$$\sum_{k_2 \le -1.01(k+j+\delta m),\, k_1} 2^{4k^+} 2^{(1+\delta)(k+j)} \|Q_{jk}B_m(P_{k_1}f, P_{k_2}f)\|_{L^2} \lesssim 2^{(1+\delta)(k+j)} \cdot \varepsilon_1^2 2^{-(1-\delta)1.01(k+j+\delta m)}$$

which is sufficient for $\delta \le \frac{1}{1000}$. □

As a consequence of the above lemma we can assume from now on that

$$\max\{k_1, k_2\} \le \tfrac{1}{2}\delta(k+j+\delta m), \quad \min\{k_1, k_2\} \ge -1.01(k+j+\delta m) \tag{3-11}$$

and, in particular,

$$\max\{k, k_1, k_2\} \le \delta(j+\delta m) + D, \tag{3-12}$$

where D is a suitably large constant. From now on we will use D to denote an absolute constant that needs to be chosen large enough in the course of our proof in order to verify several inequalities. In view of (3-11)–(3-12), when decomposing our inputs into frequencies, summations are given by at most $O((j+m)^2)$ terms.

3B. ***Finite speed of propagation.*** From (2-2) one computes

$$|\nabla_\xi \Phi| = \frac{|\eta|\,|\eta - 2\xi|}{|\xi-\eta|^2\,|\xi|^2}, \quad |\nabla_\eta \Phi| = \frac{|\xi|\,|\xi - 2\eta|}{|\xi-\eta|^2\,|\eta|^2}. \tag{3-13}$$

Notice that applying a weight x to the bilinear term $B(f, f)$ corresponds to differentiating in ξ its Fourier transform, i.e., the expression in (2-3). The main contribution from this can be expected to be the term where the ξ-derivative hits the oscillating phase, producing a factor of $s\nabla_\xi\Phi$. We then want to make this statement precise by proving that if the bilinear term $B(f, f)$ is restricted to locations $|x| \approx 2^j$, then we must have "$|x| \lesssim s|\nabla_\xi\Phi|$", that is, we should expect to have $2^j \lesssim 2^m 2^{-2\min\{k,k_2,k_2\}}$. Later on in Section 4 we will also use refinements of this statement in various scenarios.

Lemma 3.2 (finite speed of propagation). *Assume that* (3-12) *holds and*

$$j \ge m - 2\min\{k, k_1, k_2\} + D^2. \tag{3-14}$$

Then we have the bound

$$2^{4k^+} 2^{(k+j)(1+\delta)} \|Q_{jk}B_m(P_{k_1}f, P_{k_2}f)\|_{L^2} \lesssim 2^{-\delta^2(m+j)} \varepsilon_1^2. \tag{3-15}$$

Proof. We subdivide the proof into several cases and subcases.

Case 1: $k_1 \ge k_2 + 10$. In this case we must have $|k_1 - k| \le 10$ and the assumption (3-14) implies

$$j \ge m - 2k_2 + D^2. \tag{3-16}$$

Notice that in view of (3-12) we must have $j \ge \frac{1}{2}m$.

Subcase 1.1: $k \le -(1-\delta^2)j$. In this case we can use an $L^2 \times L^\infty$ estimate, see Lemma 6.3 and the symbol bound (2-22), with the a priori bounds (3-3)–(3-4), to obtain

$$\begin{aligned} 2^{(1+\delta)(k+j)} \|Q_{jk} B_m(P_{k_1} f, P_{k_2} f)\|_{L^2} &\lesssim 2^{\delta j} \|Q_{jk} B_m(P_{k_1} f, P_{k_2} f)\|_{L^2} \\ &\lesssim 2^{\delta j} \cdot 2^m \cdot 2^{k-k_2} \cdot \sup_{t \approx 2^m} \|P_{k_1} f\|_{L^2} \|e^{itL_1} P_{k_2} f\|_{L^\infty} \\ &\lesssim 2^{\delta j} \cdot 2^m \cdot 2^k \cdot \varepsilon_1 \cdot \varepsilon_1 2^{-m}, \end{aligned}$$

which suffices to obtain (3-15).

We further decompose the profiles according to their spatial localization by defining, see (2-6)–(2-7),

$$f_1 = Q_{j_1 k_1} f, \quad f_2 = Q_{j_2 k_2} f, \quad j_\nu + k_\nu \ge 0, \ \nu = 1, 2. \tag{3-17}$$

Subcase 1.2: $\min\{j_1, j_2\} \ge (1-\delta^2)j$. Here we use again an $L^2 \times L^\infty$ estimate and the a priori bounds (3-3)–(3-4):

$$\begin{aligned} &2^{4k^+} 2^{(1+\delta)(k+j)} \|Q_{jk} B_m(f_1, f_2)\|_{L^2} \\ &\qquad \lesssim 2^{4k^+} 2^{(1+\delta)(k+j)} \cdot 2^m \cdot 2^{k-k_2} \cdot \sup_{t \approx 2^m} \|f_1\|_{L^2} \|e^{itL_1} f_2\|_{L^\infty} \\ &\qquad \lesssim 2^{5k^+} 2^{(1+\delta)(k+j)} \cdot 2^m \cdot \varepsilon_1 2^{-4k_1^+} 2^{-(1+\delta)(k_1+j_1)} \cdot \varepsilon_1 2^{-m} 2^{-4k_2^+} 2^{(1-\delta)k_2} 2^{-\delta j_2}. \end{aligned}$$

Using the assumption $\min\{j_1, j_2\} \ge (1-\delta^2)j$ this can be bounded by

$$\varepsilon_1^2 2^{k^+} 2^{(1+\delta)j} \cdot 2^{-(1+\delta)j_1} \cdot 2^{-\delta j_2} \lesssim \varepsilon_1^2 2^{k^+} 2^{-\frac{4}{5}\delta j} 2^{-\delta^2 j_1} 2^{-\delta^2 j_2}.$$

Upon summing over j_1 and j_2 we obtain the bound (3-15) also in view of $k \le \frac{2}{3}\delta j + \delta^2 m + D$; see (3-11).

Subcase 1.3: $-k, \min\{j_1, j_2\} \le (1-\delta^2)j$. In this case we want to integrate by parts in ξ using the main assumption (3-14). More precisely, let us decompose according to (3-17) and inspect the formula

$$\begin{aligned} &\varphi_j^{(k)}(x) P_k B_m(f_1, f_2)(x) \\ &\qquad = \varphi_j^{(k)}(x) \int_0^t \tau_m(s) \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i[x\cdot\xi + s\Phi(\xi,\eta)]} \mathfrak{m}(\xi,\eta) \varphi_k(\xi) \hat{f}_1(\xi-\eta) \hat{f}_2(\eta) \, d\eta \, d\xi \, ds. \end{aligned} \tag{3-18}$$

Let us assume first that $j_1 \le (1-\delta^2)j$. Notice that (3-13) and the hypothesis (3-16) imply

$$\left|\nabla_\xi [x\cdot\xi + s\Phi(\xi,\eta)]\right| = |x + s\nabla_\xi \Phi| \gtrsim 2^j. \tag{3-19}$$

We then want to apply Lemma 6.5 to

$$\int_{\mathbb{R}^2} e^{i[x\cdot\xi + s\Phi(\xi,\eta)]} \mathfrak{m}(\xi,\eta) \varphi_k(\xi) \hat{f}_1(\xi-\eta) \, d\xi.$$

Let us explain this in detail since similar arguments will be used repeatedly below. We let

$$F(\xi) = 2^{-j}[x\cdot\xi + s\Phi(\xi,\eta)], \quad K \approx 2^j, \tag{3-20}$$

and have, for $|\alpha| \ge 2$,

$$|D^\alpha F| \lesssim 2^{-j} s \, |D_\xi^\alpha \Phi(\xi,\eta)| \lesssim 2^{-j+m} 2^{-(|\alpha|+1)\min\{k,k_1\}} \lesssim 2^{(1-|\alpha|)\min\{k,k_1\}}.$$

We can then choose $\epsilon = 2^{\min\{k,k_1\}}$, make the natural choice of the integrand

$$g(\xi) = \mathfrak{m}(\xi,\eta)\varphi_k(\xi)\hat{f_1}(\xi-\eta),$$

and use the bound (6-6) to obtain

$$\begin{aligned}\|Q_{jk}B_m(f_1,f_2)\|_{L^2} &\lesssim 2^{m+j}\cdot\|\hat{f_2}\|_{L^1}\cdot\frac{1}{(2^{j+\min\{k,k_1\}})^M}\sum_{|\alpha|\le M}2^{\min\{k,k_1\}|\alpha|}\|D^\alpha g\|_{L^1}\\ &\lesssim 2^{m+j}\varepsilon_1\cdot 2^{-jM}[2^{-\min\{k,k_1\}M}+2^{-kM}+2^{j_1M}]\cdot\varepsilon_1 \lesssim 2^{-10j}\varepsilon_1^2.\end{aligned}$$

For the last inequality we have used (2-22) and the fact that $\max\{-k,-k_1,j_1\}\le(1-\delta^2)j$, and chosen $M=O(\delta^{-2})$ sufficiently large. This gives (3-15) when $j_1\le(1-\delta^2)j$.

When $j_2\le(1-\delta^2)j$ we can use a similar argument. More precisely we look at the formula (3-18) and change variables to write

$$Q_{jk}B_m(f_1,f_2)(x)=\varphi_j^{(k)}(x)\int_0^t\tau_m(s)\int_{\mathbb{R}^2\times\mathbb{R}^2}\big[e^{i[x\cdot\xi+s\Phi(\xi,\eta)]}\varphi_k(\xi)\mathfrak{m}(\xi,\xi-\eta)\hat{f_2}(\xi-\eta)\,d\xi\big]\hat{f_1}(\eta)\,d\eta\,ds.$$

Notice that (3-19) still holds. Therefore we can apply Lemma 6.5 with the same phase as in (3-20) above, $\epsilon=2^{-k_2}$, and the natural choice of the integrand g, obtaining

$$\|Q_{jk}B_m(f_1,f_2)\|_{L^2}\lesssim 2^{m+j}\cdot 2^{-(j+k_2)M}\varepsilon_1^2[1+2^{(k_2+j_2)M}]\lesssim 2^{-10j}\varepsilon_1^2,$$

since $-k_2\le j_2\le(1-\delta^2)j$.

Case 2: $k_2\ge k_1+10$. This case is completely analogous to Case 1 since our main assumption is symmetric upon exchanging k_1 and k_2.

Case 3: $|k_1-k_2|\le 10$. In this case we have

$$k\le\min\{k_1,k_2\}+20,$$

and the main assumption (3-14) implies

$$j\ge m-2k+D.$$

Recall that in view of (3-12) we must have $j\ge\frac{1}{2}m$. Also, using the same estimate of Subcase 1.1 above, we may assume $k\ge-(1-\delta^2)j$.

Subcase 3.1: $\min\{j_1,j_2\}\ge(1-\delta^2)j$. This case can be treated like we have done in the analogous subcases above via an $L^\infty\times L^2$ estimate:

$$\begin{aligned}2^{4k^+}2^{(1+\delta)(k+j)}\|Q_{jk}B_m(f_1,f_2)\|_{L^2} &\lesssim 2^{4k^+}2^{(1+\delta)(k+j)}\cdot 2^m\cdot\sup_{t\approx 2^m}\|e^{itL_1}P_{k_1}f\|_{L^\infty}\|P_{k_2}f\|_{L^2}\\ &\lesssim\varepsilon_1^2 2^{(1+\delta)j}\cdot 2^{-\delta j_1}\cdot 2^{-(1+\delta)j_2}\lesssim\varepsilon_1^2 2^{-\frac{1}{2}\delta j}2^{-\delta^2 j_1}2^{-\delta^2 j_2}.\end{aligned}$$

Summing over j_1,j_2 we get the desired bound (3-15).

Subcase 3.2: $\min\{j_1,j_2\}\le(1-\delta^2)j$. In this case we can integrate by parts in ξ as previously done after (3-18), using Lemma 6.5, the lower bound (3-19) and $-k\le(1-\delta^2)j$. □

4. The weighted estimate: part I

In this section we begin the proof of the main weighted bound

$$\sup_{(k,j)\in\mathcal{J}} 2^{4k^+}2^{(k+j)(1+\delta)}\|Q_{jk}B_m(f,f)\|_{L^2} \lesssim 2^{-\delta^3 m}\varepsilon_1^2, \tag{4-1}$$

showing how this can be reduced to a similar one where the size of various important quantities can be restricted to specific ranges depending on the time variable. More precisely we will show how to restrict the size of the input and output frequencies to a range close to 1 (a range of the form $[2^{-c_1\delta m}, 2^{c_2\delta m}]$ for some constants $c_1, c_2 > 0$), the size of the phase $\Phi = \Phi(\xi, \eta)$ close to 2^{-m}, and the size of its gradients in ξ and η close to 1. In Section 5 we will then conclude our proof by treating the remaining cases.

4A. ***Main reduction of interaction frequencies.*** Here we show how to treat the contributions from input and output frequencies that are much smaller than 1, more precisely smaller than $2^{-c\delta m}$ for some $c > 0$.

Proposition 4.1. *Under the a priori assumptions* (3-3)–(3-4) *we have, for all* $(k, j) \in \mathcal{J}$,

$$\sum_{\substack{|k_1-k_2|\geq 10\\ \min\{k_1,k_2\}\leq -5\delta m+D}} 2^{4k^+}2^{(k+j)(1+\delta)}\|Q_{jk}B_m(P_{k_1}f, P_{k_2}f)\|_{L^2} \lesssim 2^{-2\delta^3 m}\varepsilon_1^2. \tag{4-2}$$

Furthermore, for all $(k, j) \in \mathcal{J}$ *we have*

$$\sum_{|k_1-k_2|\leq 10} 2^{4k^+}2^{(k+j)(1+\delta)}\|Q_{jk}B_m(P_{k_1}f, P_{k_2}f)\|_{L^2} \lesssim 2^{-2\delta^3 m}\varepsilon_1^2 \quad \text{if } k \leq -5\delta m + D. \tag{4-3}$$

Proof of Proposition 4.1. We split the proof into several scenarios, the most difficult ones being the high-high interactions.

Proof of (4-2) Because of the symmetry in k_1, k_2 we may assume $k_2 + 10 \leq k_1$, $|k - k_1| \leq 10$.

<u>Case 1</u>: $k \leq -(1-\delta^2)j$. In this case we can use an $L^2 \times L^\infty$ estimate, see Lemma 6.3 and the symbol bound (2-22), with the a priori bounds (3-3)–(3-4) to obtain

$$\begin{aligned} 2^{(1+\delta)(k+j)}\|Q_{jk}B_m(P_{k_1}f, P_{k_2}f)\|_{L^2} &\lesssim 2^{\delta j}\cdot 2^m \cdot 2^{k-k_2}\cdot \sup_{t\approx 2^m}\|P_{k_1}f\|_{L^2}\|e^{itL_1}P_{k_2}f\|_{L^\infty} \\ &\lesssim 2^{\delta j}\cdot 2^m\cdot 2^k\cdot\varepsilon_1\cdot\varepsilon_1 2^{-m}2^{(1-\delta)k_2}, \end{aligned}$$

which suffices to obtain (4-2). From now on we may assume $-k \leq (1-\delta^2)j$.

Let us now decompose the profiles according to their spatial localization, adopting the same notation as in (3-17):

$$f_1 = Q_{j_1k_1}f, \quad f_2 = Q_{j_2k_2}f, \quad j_\nu + k_\nu \geq 0,\ \nu = 1, 2. \tag{4-4}$$

<u>Case 2</u>: $j_1 \geq (1-\delta^2)j$. Here we use again an $L^2 \times L^\infty$ estimate and the a priori bounds (3-3)–(3-4):

$$\begin{aligned} &2^{4k^+}2^{(1+\delta)(k+j)}\|Q_{jk}B_m(f_1, f_2)\|_{L^2} \\ &\quad\lesssim 2^{4k^+}2^{(1+\delta)(k+j)}\cdot 2^m\cdot 2^{k-k_2}\cdot\sup_{t\approx 2^m}\|f_1\|_{L^2}\|e^{itL_1}f_2\|_{L^\infty} \\ &\quad\lesssim 2^{5k^+}2^{(1+\delta)(k+j)}\cdot 2^m\cdot\varepsilon_1 2^{-4k_1^+}2^{-(1+\delta)(k_1+j_1)}\cdot\varepsilon_1 2^{-m}2^{(1-\delta)k_2}2^{-\delta j_2}. \end{aligned}$$

Using the assumption $j_1 \geq (1-\delta^2)j$, the finite speed of propagation Lemma 3.2 to bound $j \leq m-2k_2+D$, and that $k \leq \frac{4}{5}\delta j + \delta^2 m + D$ by (3-11), we can estimate

$$2^{4k^+}2^{(1+\delta)(k+j)}\|Q_{jk}B_m(f_1,f_2)\|_{L^2} \lesssim \varepsilon_1^2 2^{k^+}2^{3\delta^2 j}\cdot 2^{-\delta^2 j_1}\cdot 2^{(1-\delta)k_2}2^{-\delta j_2}$$
$$\lesssim \varepsilon_1^2 2^{2\delta m}\cdot 2^{(1-3\delta)k_2}2^{-\delta^2 j_1}2^{-\delta j_2}.$$

Summing over j_1 and j_2 we obtain (4-2). From now on we may assume $j_1 \leq (1-\delta^2)j$.

Case 3: $j \geq k_2 - 3k_1 + m + D$. In this case we proceed in a similar way as we did in the proof of Lemma 3.2, resorting to integration by parts in ξ. We look again at the formula (3-18) and notice that $|\nabla_\xi \Phi| \approx 2^{k_2-3k_1}$; see (3-13). Then we have the same lower bound as in (3-19), that is,

$$\left|\nabla_\xi[x\cdot\xi + s\Phi(\xi,\eta)]\right| \gtrsim 2^j,$$

and we can apply Lemma 6.5 to

$$\int_{\mathbb{R}^2} e^{i[x\cdot\xi+s\Phi(\xi,\eta)]}\mathfrak{m}(\xi,\eta)\varphi_k(\xi)\hat{f}_1(\xi-\eta)\,d\xi.$$

More precisely we do this by choosing again $F(\xi) = 2^{-j}[x\cdot\xi + s\Phi(\xi,\eta)]$, $K = 2^j$, and using that for $|\alpha| \geq 2$,

$$|D^\alpha F| \lesssim 2^{-j}s\,|D_\xi^\alpha \Phi(\xi,\eta)| \lesssim 2^{-j+m}\cdot 2^{-(|\alpha|+2)\min\{k,k_1\}}2^{k_2} \lesssim 2^{(1-|\alpha|)k_1},$$

so that we can let $\epsilon = 2^{k_1}$. Using the bound (6-6), and the a priori bounds (3-3) and (3-6), we can deduce

$$\|Q_{jk}B_m(f_1,f_2)\|_{L^2} \lesssim 2^m 2^{-10j}\cdot 2^{k_1-k_2}\cdot\|\hat{f}_1\|_{L^1}\|\hat{f}_2\|_{L^1} \lesssim 2^{-5j}2^{-2k_1^+}\varepsilon_1^2,$$

which can be multiplied by the factor $2^{(j+k)(1+\delta)}$ and summed over all indices to give the desired estimate. From now on we may assume $j \leq k_2 - 3k_1 + m + D$.

Case 4: $\max\{j_1,j_2\} \geq m - 2k_2 - \delta^2 m$. We use a Hölder estimate together with the usual a priori bounds, placing the term with larger localization in L^2 and the other one in L^∞, and obtain

$$2^{4k^+}2^{(k_2-2k_1+m)(1+\delta)}\|P_k B_m(f_1,f_2)\|_{L^2}$$
$$\lesssim 2^{(k_2-2k_1+m)(1+\delta)}\cdot 2^m 2^{k_1-k_2}\cdot 2^{-m}2^{(2-\delta)k_1}\varepsilon_1\cdot 2^{-\max\{j_1,j_2\}}2^{-(1+\delta)k_2}\varepsilon_1\cdot 2^{-\delta(j_1+j_2)}$$
$$\lesssim \varepsilon_1^2 2^{2\delta m}2^{(1-3\delta)k_1}2^{k_2}2^{-\delta(j_1+j_2)}.$$

Also in view of $j \leq -2k_1 + m + D$ and (3-12) we have $k_1 \leq 2\delta m + D$; thus summing the bound above over j_1, j_2 we obtain (4-2) whenever $k_2 \leq -5\delta m$.

Case 5: $\max\{j_1,j_2\} \leq m - 2k_2 - \delta^2 m$. Notice that since $k_2 \leq k_1 - 10$ we have, see (3-13),

$$|\nabla_\eta \Phi(\xi,\eta)| \approx 2^{-2k_2}, \quad |D_\eta^\alpha \Phi(\xi,\eta)| \lesssim 2^{-k_2(|\alpha|-1)}, \quad |\alpha| \geq 2.$$

We then resort to multiple integrations by parts in η; that is, we apply Lemma 6.5 with $F = 2^{2k_2}\Phi$, $K = s2^{-2k_2}$, $\epsilon = 2^{k_2}$ and $g = \mathfrak{m}(\xi,\eta)\hat{f}_1(\xi-\eta)\hat{f}_2(\eta)$. Using the bound (6-6) we have

$$\|Q_{jk}B_m(f_1,f_2)\|_{L^2} \lesssim 2^k\|\mathcal{F}(Q_{jk}B_m(f_1,f_2))\|_{L^\infty} \lesssim 2^k 2^{-10m}2^{k_1-k_2}\|f_1\|_{L^2}\|f_2\|_{L^2},$$

which is more than sufficient to obtain (4-2) using also $j + k \leq k_2 - 2k_1 + m + D$ and (3-3)–(3-6).

Proof of (4-3). In this scenario we will make crucial use of the symmetrization argument, which gives better bounds on the null structure. In view of Lemma 3.2 (and the assumption that $k \leq -5\delta m + D$), in the current frequency configuration it is enough to show

$$\sum_{|k_1-k_2|\leq 10} 2^{(-k+m)(1+\delta)} \| P_k B_m(P_{k_1} f, P_{k_2} f)\|_{L^2} \lesssim \varepsilon_1^2 2^{-2\delta^3 m}. \tag{4-5}$$

Localization in the size of $|\xi - 2\eta|$. We now introduce a further localization in the size of $|\xi - 2\eta|$ by writing

$$\begin{gathered} \mathcal{F}B_{m,\ell}(f,g) = \int_0^t \tau_m(s) \int_{\mathbb{R}^2} W_\ell(f,g)\, d\eta\, ds, \\ W_\ell(f,g) := e^{is\Phi} \mathfrak{m}(\xi,\eta) \varphi_\ell(\xi - 2\eta) \hat{f}(\xi-\eta)\hat{g}(\eta). \end{gathered} \tag{4-6}$$

Notice that $B_{m,\ell}(P_{k_1} f, P_{k_2} f)$ vanishes if $\ell \geq k_1 + 20$. Also, notice that the symbol obeys the refined bound

$$\|\mathfrak{m}^{k,k_1,k_2} \varphi_\ell(\xi - 2\eta)\|_{S^\infty} \lesssim 2^{2\ell + 2k - 2k_1 - 2k_2}. \tag{4-7}$$

Using this bound and standard Hölder estimates, we can reduce (4-5) to proving the following:

$$\begin{gathered} 2^{(m-k)(1+\delta)} \| B_{m,\ell}(P_{k_1} f, P_{k_2} f)\|_{L^2} \lesssim \varepsilon_1^2 2^{-\delta^2 m}, \\ \text{with} \quad |k_1 - k_2| \leq 10, \quad -2m \leq \ell, k_1, k_2 \leq 4\delta m, \quad -2m \leq k \leq -5\delta m + D. \end{gathered} \tag{4-8}$$

The rest of this proof is dedicated to showing (4-8) and split into two cases, depending on which of the parameters ℓ or k is smaller.

<u>Case 1</u>: $\ell \leq k + 5$. In this case we must have $k \geq \min\{k_1, k_2\} - 15$, so that k, k_1, k_2 are all comparable to each other and smaller than $-5\delta m + D$. In particular (4-7) gives

$$\|\mathfrak{m}^{k,k_1,k_2} \varphi_\ell(\xi - 2\eta)\|_{S^\infty} \lesssim 2^{2\ell - 2k_1}. \tag{4-9}$$

We proceed in three steps.

Step 1: $\ell - k_1 \leq -\frac{4}{9} m$. In this case we use integration by parts in time. We introduce a further localization in the size of the phase Φ in the bilinear operators $B_{m,\ell}$ defined in (4-6). More precisely, we write

$$\begin{gathered} B_{m,\ell}(f,g) = B_{m,\ell,\leq p_0}(f,g) + \sum_{p > p_0} B_{m,\ell,p}(f,g), \quad p_0 := -3m, \\ B_{m,\ell,*}(f,g) := \int_0^t \tau_m(s) \int_{\mathbb{R}^2} \varphi_*(\Phi(\xi,\eta)) W_\ell(f,g)(\xi,\eta)\, d\eta\, ds, \end{gathered} \tag{4-10}$$

where W_ℓ is given in (4-6).

Notice that in analyzing the terms in (4-10) we will be dealing with a kernel of the form

$$K_{p,\ell}(\xi,\eta) := \varphi_p(\Phi(\xi,\eta)) \varphi_\ell(\xi - 2\eta) \tilde{\varphi}_k(\xi) \tilde{\varphi}_{k_1}(\xi - \eta) \tilde{\varphi}_{k_2}(\eta). \tag{4-11}$$

Since k, k_1, k_2 are all comparable and much larger than ℓ we see, using (6-3) in Lemma 6.2, that

$$\|K_{p,\ell}\|_{\mathrm{Sch}} \lesssim 2^{p + \frac{5}{2}k_1 + \frac{\ell}{2}}. \tag{4-12}$$

We can directly use this estimate to obtain the desired bound (4-8) for the term $B_{m,\ell,\le p_0}$. Since we must also have $|\Phi| \lesssim 2^{-2k_1} \lesssim 2^{5m}$, there are only $O(m)$ terms in the sum in (4-10), and it will thus suffice to prove

$$2^{(m-k)(1+\delta)}\|B_{m,\ell,p}(P_{k_1}f, P_{k_2}f)\|_{L^2} \lesssim \varepsilon_1^2 2^{-3\delta^2 m} \tag{4-13}$$

for fixed $p \in [-3m, 5m]$.

Integrating by parts in s we can write

$$\begin{aligned} B_{m,\ell,p}(P_{k_1}f, P_{k_2}f) &= I_{m,\ell,p}(P_{k_1}f, P_{k_2}f) - II_{m,\ell,p}(\partial_t P_{k_1}f, P_{k_2}f) - II_{m,\ell,p}(P_{k_1}f, \partial_t P_{k_2}f), \\ I_{m,\ell,p}(f,g) &:= \int_0^t 2^{-m}\tau_m'(s)\int_{\mathbb{R}^2}\frac{\varphi_p(\Phi(\xi,\eta))}{i\Phi(\xi,\eta)}W_\ell(f,g)(\xi,\eta)\,d\eta\,ds, \\ II_{m,\ell,p}(f,g) &:= \int_0^t \tau_m(s)\int_{\mathbb{R}^2}\frac{\varphi_p(\Phi(\xi,\eta))}{i\Phi(\xi,\eta)}W_\ell(f,g)(\xi,\eta)\,d\eta\,ds. \end{aligned} \tag{4-14}$$

For the first above term, using the a priori bounds (3-3)–(3-6), the bound on the symbol (4-9) and the bound on the kernel (4-11), we have the estimate

$$\begin{aligned} &2^{(m-k)(1+\delta)}\|I_{m,\ell,p}(P_{k_1}f, P_{k_2}f)\|_{L^2} \\ &\quad\lesssim 2^{(m-k)(1+\delta)}\cdot 2^{2\ell-2k_1}\cdot 2^{-p}\cdot \|K_{p,\ell}(\xi,\eta)\widehat{P_{k_1}f}(\xi-\eta)\|_{\mathrm{Sch}}\|P_{k_2}f\|_{L^2} \\ &\quad\lesssim 2^{(m-k)(1+\delta)}\cdot 2^{\frac{1}{2}k_1+\frac{5}{2}\ell}2^{-k_1(1+\delta)}\varepsilon_1\cdot 2^{k_2}\varepsilon_1 \\ &\quad\lesssim 2^{-(\frac{1}{2}+2\delta)k_1}2^{(1+\delta)m}2^{\frac{5}{2}\ell}\varepsilon_1^2 \lesssim \varepsilon_1^2 2^{-\frac{1}{40}m}, \end{aligned}$$

having used the assumption $\ell \le -\frac{4}{9}m + k_1$ for the last step.

For the remaining terms in (4-14) we can use a similar bound together with (2-24) to obtain

$$\begin{aligned} 2^{(m-k)(1+\delta)}\|II_{m,\ell,p}(\partial_t P_{k_1}f, P_{k_2}f)\|_{L^2} &\lesssim 2^{(m-k)(1+\delta)}.2^m.2^{\frac{1}{2}k_1+\frac{5}{2}\ell}.\|\widehat{P_{k_1}f}\|_{L^\infty}\sup_{s\approx 2^m}\|\partial_s P_{k_2}f\|_{L^2} \\ &\lesssim 2^{-(1+\delta)k}2^{(2+\delta)m}.2^{\frac{1}{2}k_1+\frac{5}{2}\ell}.\varepsilon_1 2^{-(1+\delta)k_1}.\varepsilon_1^2 2^{k_2}2^{-2m+10\delta m} \\ &\lesssim 2^{11\delta m}2^{-(\frac{1}{2}+2\delta)k_1}2^{\frac{5}{2}\ell}\varepsilon_1^3 \lesssim \varepsilon_1^3 2^{-\frac{1}{40}m}. \end{aligned}$$

The same bound can be similarly obtained for $II_{m,p}(P_{k_1}f, \partial_t P_{k_2}f)$ and this concludes the proof of (4-13) when $\ell - k_1 \le -\frac{4}{9}m$.

To deal with the remaining cases we introduce the usual spatial localizations as defined in (4-4), and aim to show

$$2^{(m-k)(1+\delta)}\sum_{j_1,j_2}\|B_{m,\ell}(f_1,f_2)\|_{L^2} \lesssim \varepsilon_1^2 2^{-2\delta^2 m},$$

under the assumptions in (4-8) and with $\ell - k_1 \ge -\frac{4}{9}m$.

Step 2: $\ell - k_1 \ge -\frac{4}{9}m$ and $\max\{j_1, j_2\} \le m + \ell - 3k_1 - \delta m$. In this case we can repeatedly integrate by parts. Indeed, in our current frequency configuration we have $|\nabla_\eta \Phi| \approx 2^\ell 2^{-3k_1}$; see (3-13). Then we can use Lemma 6.5 by letting $K = s(2^\ell 2^{-3k_1})^{-1}$, $F(\eta) = \Phi 2^\ell 2^{-3k_1}$ and $\epsilon = 2^\ell$. From (6-6), choosing M large enough, we then obtain $\|B_{m,\ell}(f_1,f_2)\|_{L^2} \lesssim 2^{-10m}\|f_1\|_{L^2}\|f_2\|_{L^2}$, which is more than sufficient to obtain (4-8).

Step 3: $\max\{j_1, j_2\} \geq m + \ell - 3k_1 - \delta m$. In this case a standard Hölder estimate, placing the input with largest position in L^2, suffices:

$$\begin{aligned}2^{(m-k)(1+\delta)}&\|B_{m,\ell}(f_1, f_2)\|_{L^2}\\ &\lesssim 2^{(m-k)(1+\delta)} \cdot 2^m \cdot 2^{2\ell-2k_1} \cdot 2^{-m} 2^{(2-\delta)k_1} 2^{-4k_1^+} \varepsilon_1 \cdot 2^{-\max\{j_1,j_2\}} 2^{-(1+\delta)k_2} \varepsilon_1 \cdot 2^{-\delta(j_1+j_2)}\\ &\lesssim 2^{2\delta m} 2^{\ell} 2^{(1-3\delta)k_1} 2^{-4k_1^+} 2^{-\delta(j_1+j_2)} \varepsilon_1^2,\end{aligned}$$

having used the a priori bounds (3-3)–(3-4), and the symbol bound (4-9). Summing over j_1, j_2 we see that this implies the desired bound (4-8) since $\min\{k, k_1, k_2\} \leq -5\delta m + D$ holds.

Remark 4.2. Notice that the bounds proved above suffice to obtain an estimate as in (4-3) for $\sum_\ell B_{m,\ell}$ instead of B_m, provided that $\ell \leq -5\delta m$, and placing no additional smallness restriction on k.

Case 2: $k \leq \ell - 5$. Here we have $k \leq -5\delta m + D$ and $|\ell - k_1| \leq 20$, and similar arguments to those of Case 1 can be used essentially by reversing the roles of k and ℓ. Note that in this case stronger bounds are available for the kernel that we need to consider; see (4-15) below. We decompose the profiles according to their spatial localization as done above and proceed as follows.

Step 1: $\max\{j_1, j_2\} \leq m + k - 3k_1 - \delta m$. Note that this case will be empty if $k < -m + 3k_1 + \delta m$ and only Step 2 below needs to be performed. In the current scenario we have $|\nabla_\eta \Phi| \approx 2^{k-3k_1}$ and $|D_\eta^\alpha \Phi| \lesssim 2^k 2^{-(|\alpha|+2)k_1}$, $|\alpha| \geq 1$. We can then use Lemma 6.5 by letting $K = s(2^k 2^{-3k_1})^{-1}$, $F(\eta) = \Phi 2^k 2^{-3k_1}$ and $\epsilon = 2^{k_1}$, obtaining

$$\|B_{m,\ell}(f_1, f_2)\|_{L^2} \lesssim 2^{-10m} \|f_1\|_{L^2} \|f_2\|_{L^2}.$$

Step 2: $\max\{j_1, j_2\} \geq m + k - 3k_1 - \delta m$. In this case we want to use integration by parts in s similarly to Step 1 of Case 1 above. From the formula for the symmetrized symbol we see that the bound (4-9) used before can be substituted by

$$\|\mathfrak{m}^{k,k_1,k_2} \varphi_\ell(\xi - 2\eta)\|_{S^\infty} \lesssim 2^{2k-2k_1}.$$

Moreover, notice that we have a bound stronger than (4-12) for the relevant kernel; that is,

$$\big\|\varphi_p(\Phi(\xi,\eta))\varphi_\ell(\xi - 2\eta)\tilde{\varphi}_k(\xi)\tilde{\varphi}_{k_1}(\xi - \eta)\tilde{\varphi}_{k_2}(\eta)\big\|_{\mathrm{Sch}} \lesssim 2^{p+k+2k_1}, \tag{4-15}$$

as per (6-3) in Lemma 6.2. Then the same arguments as in Step 1 of Case 1 above go through and give the main conclusion (4-2) when $k \leq \min\{k_1, -5\delta m\} + D$. This concludes the proof of the proposition. □

As a consequence of Proposition 4.1 we have the following:

Corollary 4.3. *In order to prove the main bound* (4-1) *it will be enough to prove the following claim*: *for all* $(k, j) \in \mathcal{J}$ *we have*

$$\begin{gathered}2^{4k_+} 2^{m-2\min\{k,k_1,k_2\}+k} \|P_k B_{m,\ell}(P_{k_1} f, P_{k_2} f)\|_{L^2} \lesssim 2^{-2\delta m} \varepsilon_1^2,\\ \textit{whenever} \quad -5\delta m \leq k, k_1, k_2, \ell \leq 4\delta m + D^2,\end{gathered} \tag{4-16}$$

where $B_{m,\ell}$ is defined as

$$\mathcal{F}B_{m,\ell}(f,g) = \int_0^t \tau_m(s) \int_{\mathbb{R}^2} W_\ell(f,g)\, d\eta\, ds, \\ W_\ell(f,g)(\xi,\eta) := e^{is\Phi(\xi,\eta)}\mathfrak{m}(\xi,\eta)\varphi_\ell(\xi-2\eta)\hat{f}(\xi-\eta)\hat{g}(\eta). \tag{4-17}$$

Proof. In view the estimates (4-2)–(4-3) in Proposition 4.1, we know that to obtain the main bound (4-1) it will suffice to show

$$\sup_{\substack{k+j\geq 0\\ k\geq -5\delta m}} 2^{4k+}2^{(k+j)(1+\delta)} \sum_{k_1,k_2\geq -5\delta m} \|Q_{jk}B_m(P_{k_1}f, P_{k_2}f)\|_{L^2} \lesssim 2^{-\delta^3 m}\varepsilon_1^2. \tag{4-18}$$

Recall that from (3-12) we have the upper bound $\max\{k,k_1,k_2\} \leq \delta(j+m)+D$. Then the finite speed of propagation Lemma 3.2 suffices to bound the sum in (4-18) whenever $j \geq m - 2\min\{k,k_1,k_2\}+D$. We may therefore restrict ourselves to $j \leq m-2\min\{k,k_1,k_2\}+D \leq (1+10\delta)m+D$, and thus also to $\max\{k,k_1,k_2\} \leq 4\delta m + D$. We then have a sum over at most $O(m^2)$ terms so that it suffices to prove the bound

$$2^{4k+}2^{(k+j)(1+\delta)}\|Q_{jk}B_m(P_{k_1}f, P_{k_2}f)\|_{L^2} \lesssim 2^{-\frac{3}{2}\delta^3 m}\varepsilon_1^2$$

for each fixed triple k,k_1,k_2 with $-5\delta m \leq k,k_1,k_2 \leq 4\delta m + D$, and $(k,j)\in\mathcal{J}$. Moreover, in view of Remark 4.2 we may also replace B_m above with $B_{m,\ell}$ and assume that $\ell \geq -5\delta m$. The claim follows since $\delta(m-2\min\{k,k_1,k_2\}+k) \leq \frac{3}{2}\delta m$. □

4B. *Further reductions.* We now turn to further reductions on the size of the phase Φ and the spatial localization of the profiles in the bilinear term $B_{m,\ell}(P_{k_1}f, P_{k_2}f)$ in (4-17). For this purpose let us write

$$B_{m,\ell}(P_{k_1}f,P_{k_2}f) = \sum_{p\in\mathbb{Z}} B_{m,\ell,p}(P_{k_1}f,P_{k_2}f) = \sum_{r,p\in\mathbb{Z}} B_{m,\ell,r,p}(P_{k_1}f,P_{k_2}f), \tag{4-19}$$

$$B_{m,\ell,p}(f,g) := \mathcal{F}\int_0^t \tau_m(s)\int_{\mathbb{R}^2}\varphi_p(\Phi(\xi,\eta))W_\ell(f,g)\,d\eta\,ds, \tag{4-20}$$

$$B_{m,\ell,r,p}(f,g) := \mathcal{F}\int_0^t \tau_m(s)\int_{\mathbb{R}^2}\varphi_p(\Phi(\xi,\eta))\varphi_r(\eta-2\xi)W_\ell(f,g)\,d\eta\,ds, \tag{4-21}$$

where W_ℓ is as in (4-17). Notice that $B_{m,\ell,p}(P_{k_1}f,P_{k_2}f)$ is trivial unless $p \leq -2\min\{k,k_1,k_2\}+D \leq 10\delta m + D$ and $r \leq \max\{k_1,k_2\}+D \leq 4\delta m + 2D^2$. Also note that a Schur-type estimate using Lemma 6.2 will give the desired bound for the sum of the terms $B_{m,\ell,p}$ when $p \leq -3m$. Similarly, it is not hard to see that one can obtain the bound (4-16) for the terms $B_{m,\ell,p,r}$ when $r \leq -3m$. Therefore the summations in (4-19) are all over at most $O(m^2)$ terms, and it suffices to prove the bound for each element in the sum.

Proposition 4.4. *With the usual notation* $f_\nu = P_{[k_\nu-2,k_\nu+2]}\varphi_{j_\nu}^{(k_\nu)}(x)P_{k_\nu}f$, $j_\nu+k_\nu\geq 0$, $\nu=1,2$, *and under the frequency restriction in* (4-16), *namely*

$$-5\delta m \leq k,k_1,k_2,\ell \leq 4\delta m + D^2,$$

we have

$$\|P_k B_{m,\ell}(f_1,f_2)\|_{L^2} \lesssim 2^{-2m}\varepsilon_1^2 \quad \text{if } \max\{j_1,j_2\} \le m+\min\{k,\ell\}-3k_1-\delta m. \tag{4-22}$$

If instead $\max\{j_1,j_2\} \ge m+\min\{k,\ell\}-3k_1-\delta m$, *then we have the following bounds*:

$$2^{4k_+}2^{m-2\min\{k,k_1,k_2\}+k}\|P_k B_{m,\ell,p}(f_1,f_2)\|_{L^2} \lesssim 2^{-3\delta m}\varepsilon_1^2 \qquad \text{if } p \ge -m+40\delta m, \tag{4-23}$$

$$2^{4k_+}2^{m-2\min\{k,k_1,k_2\}+k}\|P_k B_{m,\ell,p,r}(f_1,f_2)\|_{L^2} \lesssim 2^{-4\delta m}\varepsilon_1^2 \quad \text{if } r \le -35\delta m, \tag{4-24}$$

$$2^{4k_+}2^{m-2\min\{k,k_1,k_2\}+k}\|P_k B_{m,\ell,p,r}(f_1,f_2)\|_{L^2} \lesssim 2^{-4\delta m}\varepsilon_1^2 \quad \text{if } \min\{j_1,j_2\} \ge \tfrac{1}{2}m+60\delta m. \tag{4-25}$$

For convenience we introduce the notation

$$\underline{k} := \min\{k_1,k_2\}, \quad \bar{k} := \max\{k_1,k_2\}, \quad \underline{j} := \min\{j_1,j_2\}, \quad \bar{j} := \max\{j_1,j_2\}. \tag{4-26}$$

Proof. Each one of the bounds in the statement can be proven via similar techniques to those used in the proof of Proposition 4.1 above.

Proof of (4-22). This follows by integrating by parts in η sufficiently many times, i.e., by applying Lemma 6.5 using the fact that $|\nabla_\eta \Phi| \approx 2^{k+\ell-4k_1}$ and $|D_\eta^\alpha \Phi| \lesssim 2^{-(|\alpha|+1)\min\{k_1,k_2\}}$ on the support of the integral.

Proof of (4-23). Now we treat the term $B_{m,\ell,p}$ as defined in (4-20) analogously to what was done in (4-10) and integrate by parts in s. Similarly to (4-14) we obtain $B_{m,\ell,p}(f_1,f_2) = I_{m,p}(f_1,f_2) - II_{m,p}(\partial_t f_1, f_2) - II_{m,p}(f_1,\partial_t f_2)$, where

$$\begin{aligned} I_{m,\ell,p}(f,g) &:= \int_0^t 2^{-m}\tau_m'(s)\int_{\mathbb{R}^2} \frac{\varphi_p(\Phi(\xi,\eta))}{i\Phi(\xi,\eta)} W_\ell(f,g)(\xi,\eta)\,d\eta\,ds, \\ II_{m,\ell,p}(f,g) &:= \int_0^t \tau_m(s)\int_{\mathbb{R}^2} \frac{\varphi_p(\Phi(\xi,\eta))}{i\Phi(\xi,\eta)} W_\ell(f,g)(\xi,\eta)\,d\eta\,ds. \end{aligned} \tag{4-27}$$

For the first term in (4-27) we use Lemma 6.4 and the a priori bounds, estimating the profile with the largest spatial localization in L^2 and obtain

$$\begin{aligned} \|P_k I_{m,\ell,p}(f_1,f_2)\|_{L^2} &\lesssim 2^{-p}\cdot\|\mathfrak{m}^{k,k_1,k_2}\varphi_\ell(\xi-2\eta)\|_{S^\infty}\cdot 2^{-m}2^{-2\bar{k}^+}\varepsilon_1\cdot 2^{-\bar{j}}2^{-\underline{k}}\varepsilon_1 \\ &\lesssim 2^{-m-39\delta m}\cdot\|\mathfrak{m}^{k,k_1,k_2}\|_{S^\infty}\cdot 2^{-\underline{k}-2\bar{k}^+}2^{-\min\{k,\ell\}+3k_1}\varepsilon_1^2. \end{aligned}$$

Using the bound $\|\mathfrak{m}^{k,k_1,k_2}\|_{S^\infty} \lesssim 2^{-\underline{k}+\bar{k}}$, we see that

$$2^{m+k-2\min\{k,k_1,k_2\}}\|P_k I_{m,\ell,p}(f_1,f_2)\|_{L^2} \lesssim \varepsilon_1^2 2^{-39\delta m}\cdot 2^{-4\min\{0,k,k_1,k_2,\ell\}}2^{2\max\{0,k,k_1,k_2,\ell\}},$$

which suffices to obtain (4-23) in view of the restrictions in (4-16).

For the terms $II_{m,p}$ we use Lemma 6.4, estimating in L^2 the term involving the time derivative of the profile via (2-24), together with the bound for the symbol used above:

$$\begin{aligned} \|P_k II_{m,\ell,p}(\partial_t f_1,f_2)\|_{L^2} &\lesssim 2^m\cdot 2^{-p}\cdot\|\mathfrak{m}^{k,k_1,k_2}\varphi_\ell(\xi-2\eta)\|_{S^\infty}\cdot 2^{-m}2^{2\bar{k}}\varepsilon_1\cdot 2^{\underline{k}}2^{-2m+10\delta m}\varepsilon_1 2^{-4\bar{k}^+} \\ &\lesssim 2^{-m-30\delta m}\cdot 2^{3\bar{k}-4\bar{k}^+}\varepsilon_1^2. \end{aligned}$$

This suffices to prove (4-23).

Proof of (4-24). We now look at the bilinear term $B_{m,\ell,p,r}$ defined in (4-21) with $r \le -35\delta m \le \min\{k,k_1,k_2\} - D$, so that k, k_1, k_2 and ℓ are all comparable. In view of the previous step we may assume $p \le -m + 35\delta m$. Using the estimate (6-2) in Lemma 6.2(2) we see that

$$\left\| \varphi_p(\Phi(\xi,\eta))\,\tilde\varphi_k(\xi)\,\tilde\varphi_{k_1}(\xi-\eta)\,\tilde\varphi_{k_2}(\eta)\,\tilde\varphi_\ell(\xi-2\eta)\,\tilde\varphi_r(\eta-2\xi)\right\|_{\mathrm{Sch}} \lesssim 2^{p+\frac12 r+\frac52 k}.$$

Using this bound with Schur's test, $|\mathfrak{m}^{k,k_1,k_2}| \lesssim 2^{r-k}$, $\bar j \ge (1-\delta)m - 2k$, and the usual a priori bounds, we see that

$$\begin{aligned} 2^{4k^+}2^{m-2\min\{k,k_1,k_2\}+k}\|P_k B_{m,\ell,p,r}(f_1,f_2)\|_{L^2} &\lesssim 2^{m-k}\cdot 2^m\cdot 2^{p+\frac12 r+\frac52 k}\cdot 2^{r-k}\cdot 2^{-k}\varepsilon_1\cdot 2^{-\bar j}2^{-k}\varepsilon_1 \\ &\lesssim 2^{m+\delta m}2^{p+\frac32 r+\frac12 k}\varepsilon_1^2, \end{aligned}$$

which is sufficient to obtain (4-24).

Proof of (4-25). In view of the previous step we may assume $p \le -m + 40\delta m$ and $r \ge -35\delta m$. Just for the purpose of this proof let us define

$$K(\xi,\eta) := \varphi_p(\Phi(\xi,\eta))\,\varphi_\ell(\xi-2\eta)\,\varphi_r(\xi-2\eta)\,\tilde\varphi_k(\xi)\,\tilde\varphi_{k_1}(\xi-\eta)\,\tilde\varphi_{k_2}(\eta).$$

In view of Lemma 6.2(2) we have, recall the notation (4-26),

$$\|K(\xi,\eta)\|_{\mathrm{Sch}} + \|K(\xi,\xi-\eta)\|_{\mathrm{Sch}} \lesssim 2^{p+\frac12 \underline{k}+\frac32 \bar k}.$$

Also notice that for any kernel with $|K| \lesssim 1$ one has

$$\|K(\xi,\eta)g(\xi-\eta)\|_{\mathrm{Sch}} \lesssim \|K(\xi,\eta)\|_{\mathrm{Sch}}^{\frac12}\|g\|_{L^2}.$$

Then, using Schur's test by estimating in L^2 the profile corresponding to the larger localization $2^{\bar j}$ we can bound

$$\begin{aligned} \|P_k B_{m,\ell,p,r}(f_1,f_2)\|_{L^2} &\lesssim 2^m\cdot (2^{p+\frac12\underline{k}+\frac32\bar k})^{\frac12}\cdot \|\mathfrak{m}^{k,k_1,k_2}\|_{S^\infty}\cdot \|f_{\underline{j}}\|_{L^2}\cdot\|f_{\bar j}\|_{L^2} \\ &\lesssim 2^m\cdot 2^{\frac12 p+\frac34\bar k+\frac14\underline{k}}\cdot 2^{\bar k-\underline{k}}\cdot 2^{-\bar j-\underline{j}}\cdot 2^{-\bar k-\underline{k}-4\bar k^+}\varepsilon_1^2. \end{aligned}$$

Using the assumptions $p \le -m + 40\delta m$, $\bar j \ge (1-\delta)m - 3k_1 + \min\{k,\ell\}$ and $\underline{j} \ge \frac12 m + 60\delta m$, we see that

$$\begin{aligned} 2^{4k^+}2^{m-2\min\{k,k_1,k_2\}+k}\|P_k B_{m,\ell,p,r}(f_1,f_2)\|_{L^2} &\lesssim \varepsilon_1^2 2^{-39\delta m}.2^{k-2\min\{k,k_1,k_2\}}.2^{\frac34\bar k-\frac74\underline{k}}.2^{3k_1-\min\{k,\ell\}} \\ &\lesssim \varepsilon_1^2 2^{-39\delta m}.2^{-\frac{15}{4}\min\{0,k,k_1,k_2,\ell\}+\frac{15}{4}\max\{0,k_1,k_2\}}, \end{aligned}$$

which is sufficient for (4-25), again in view of (4-16). □

5. The weighted estimate: part II

Recall that the main weighted bound (4-1) is implied by (4-16). Combining this fact with the estimates in Proposition 4.4 we can reduce the proof of the main desired bound to showing that

$$2^{4k^+}2^{m-2\min\{k,k_1,k_2\}+k}\|P_k B_{m,\ell,\le p_0,r}(f_1,f_2)\|_{L^2} \lesssim 2^{-4\delta m}, \tag{5-1}$$

where

$$B_{m,\ell,\le p_0,r}(f,g) := \mathcal{F}^{-1}\int_0^t \tau_m(s)\int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}\varphi_{\le p_0}(\Phi(\xi,\eta))\,\mathfrak{m}(\xi,\eta)\,\varphi_\ell(\xi-2\eta)\,\varphi_r(2\xi-\eta)\,\hat f(\xi-\eta)\,\hat g(\eta)\,d\eta,$$

and whenever

$$\begin{gathered} -5\delta m\le k,k_1,k_2,\ell\le 4\delta m+D^2,\quad r\ge -35\delta m,\\ p_0:=-m+40\delta m,\\ \bar{j}:=\max\{j_1,j_2\}\ge m+\min\{k,\ell\}-3k_1-\delta m\ge m-20\delta m,\\ \underline{j}:=\min\{j_1,j_2\}\le \tfrac{1}{2}m+60\delta m. \end{gathered} \tag{5-2}$$

Remark 5.1. Intuitively speaking the reductions to the configuration (5-2) have placed us in a framework where neither integration by parts in time nor space produces any gain: $|\Phi|$ is of the order of s^{-1} and $|\nabla_\eta\Phi|$ is of order about 1, with $\bar{j}$ of the order about s. *Notice that this is not a localization to, but rather away from, the resonant set.*

Anisotropic decomposition. We now decompose the bilinear term into two pieces, according to the size of $|\xi_1-\eta_1|$:

$$B_{m,\ell,\le p_0,r}(f_1,f_2)=\mathcal{B}_{\le q_0}(f_1,f_2)+\sum_{q>q_0}\mathcal{B}_q(f_1,f_2),\quad q_0:=-\tfrac{1}{20}m,$$

$$\mathcal{B}_*(f,g):=\mathcal{F}^{-1}\int_0^t\tau_m(s)\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\varphi_{\le p_0}(\Phi(\xi,\eta))\varphi_*(\xi_1-\eta_1)m_{\ell,r}(\xi,\eta)\,\hat f(\xi-\eta)\hat g(\eta)\,d\eta, \tag{5-3}$$

$$m_{\ell,r}(\xi,\eta):=\mathfrak{m}^{k,k_1,k_2}(\xi,\eta)\varphi_\ell(\xi-2\eta)\varphi_r(2\xi-\eta);$$

see also the notation (6-5), and recall the formula (2-19) for the symbol $\mathfrak{m}$. Note that in order to simplify notation we suppress the dependence on m,ℓ,p_0,r in $\mathcal{B}_*$.

5A. *Estimate of $\mathcal{B}_{\le q_0}$*. Here we show how we can exploit the smallness in the localization in $|\xi_1-\eta_1|$ to close our bounds. The main tool here is given by improved Schur kernel bounds.

Let us introduce the notation

$$K_{q_0}(\xi,\eta):=\varphi_{\le p_0}(\Phi(\xi,\eta))\varphi_{\le q_0}(\xi_1-\eta_1)m_{\ell,r}(\xi,\eta),$$

where $m_{\ell,r}$ is as in (5-3), and so that

$$\mathcal{B}_{\le q_0}(f,g)=\mathcal{F}^{-1}\int_0^t\tau_m(s)\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}K_{q_0}(\xi,\eta)\,\hat f(\xi-\eta)\hat g(\eta)\,d\eta.$$

Proposition 5.2. *Under the assumptions* (5-2) *the following holds true*:

$$2^{4k^+}2^{m-2\min\{k,k_1,k_2\}+k}\|P_k\mathcal{B}_{\le q_0}(f_1,f_2)\|_{L^2}\lesssim 2^{-4\delta m}. \tag{5-4}$$

Proof. Observe that

$$2^{p_0}\gtrsim|\Phi(\xi,\eta)|=\left|(\xi_1-\eta_1)\left(\frac{1}{|\xi|^2}-\frac{1}{|\xi-\eta|^2}\right)-\eta_1\left(\frac{1}{|\eta|^2}-\frac{1}{|\xi|^2}\right)\right|.$$

Since on the support of the integral (5-3) we have $|\xi_1-\eta_1|\le 2^{q_0}$, we see that

$$|\eta_1|\left|\frac{1}{|\eta|^2}-\frac{1}{|\xi|^2}\right|\lesssim 2^{p_0}+2^{q_0}\left|\frac{1}{|\xi|^2}-\frac{1}{|\xi-\eta|^2}\right|\lesssim 2^{q_0+10\delta m}. \tag{5-5}$$

We then distinguish two main cases depending on the size of $|\eta_1|$ relative to $2^{\frac{1}{3}q_0+10\delta m}$. More precisely we write

$$\mathcal{B}_{\le q_0}(f,g)=\mathcal{B}^-_{\le q_0}(f,g)+\mathcal{B}^+_{\le q_0}(f,g),$$
$$\mathcal{B}^\pm_{\le q_0}(f,g):=\mathcal{F}^{-1}\int_0^t \tau_m(s)\int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}K_{q_0}(\xi,\eta)\chi_\pm(\eta_1)\,\hat f(\xi-\eta)\hat g(\eta)\,d\eta,$$
$$\chi_-(\eta_1):=\varphi_{\le\frac{1}{3}q_0+10\delta m}(\eta_1),\qquad \chi_+(\eta_1):=1-\chi_-(\eta_1).$$

Estimate of $\mathcal{B}^-_{\le q_0}$. In this case $|\eta_1|\lesssim 2^{\frac{1}{3}q_0+10\delta m}$ and we see that

$$|\xi\cdot\eta^\perp|\lesssim|(\xi_1-\eta_1)\eta_2|+|(\xi_2-\eta_2)\eta_1|\lesssim 2^{\frac{1}{3}q_0+15\delta m}.$$

This gives us an improved estimate on the symbol $\mathfrak{m}$, see (2-19), and hence on the kernel: using Lemma 6.2(2) and the restrictions (5-2) we see that

$$\|K_{q_0}(\xi,\eta)\chi_-(\eta_1)\|_{\mathrm{Sch}}+\|K_{q_0}(\xi,\xi-\eta)\chi_-(\xi_1-\eta_1)\|_{\mathrm{Sch}}\lesssim 2^{\frac{1}{3}q_0}\cdot 2^{p_0}\cdot 2^{40\delta m}.$$

We then apply Schur's test incorporating the profile with localization $\underline{j}$ in the kernel and estimating the one with largest $\bar{j}$ in L^2: using the a priori bounds (3-3) and (3-5) together with the restrictions (5-2) we have

$$\begin{aligned}\|\mathcal{B}^-_{\le q_0}(f_1,f_2)\|_{L^2}&\lesssim 2^m\cdot 2^{\frac{1}{3}q_0+p_0+40\delta m}\cdot\varepsilon_1 2^{5\delta m}\cdot\varepsilon_1 2^{-m+25\delta m}\\&\lesssim 2^{-m}\cdot 2^{-\frac{1}{60}m}\cdot 2^{110\delta m}\cdot\varepsilon_1^2.\end{aligned}$$

This is sufficient to obtain (5-4), given that the restrictions (5-2) imply $2^{m-2\min\{k,k_1,k_2\}+k}\le 2^m 2^{15\delta m}$ and $\delta\le 2\cdot 10^{-4}$.

Estimate of $\mathcal{B}^+_{\le q_0}$. In this case $|\eta_1|\gtrsim 2^{\frac{1}{3}q_0+10\delta m}$ and in view of (5-5) we must have $\big||\eta|^{-2}-|\xi|^{-2}\big|\le 2^{\frac{2}{3}q_0}$. Since $|\eta|^{-2}-|\xi|^{-2}=|\xi|^{-2}|\eta|^{-2}(\xi_2^2-\eta_2^2+\xi_1^2-\eta_1^2)$ we see that

$$|\xi_2^2-\eta_2^2|\lesssim|\xi|^2|\eta|^2 2^{\frac{2}{3}q_0}+|\xi_1^2-\eta_1^2|\lesssim 2^{\frac{1}{2}q_0+16\delta m}.$$

Therefore we know that on the support of the integral

$$|\xi_1-\eta_1|\lesssim 2^{q_0},\quad |\xi_2^2-\eta_2^2|\lesssim 2^{\frac{2}{3}q_0+16\delta m},\quad |\nabla_\xi\Phi(\xi,\eta)|,|\nabla_\eta\Phi(\xi,\eta)|\ge 2^{-50\delta m};$$

see (3-13) and the restrictions (5-2). Using these we claim that we can estimate

$$\|K_{q_0}(\xi,\eta)\chi_+(\eta_1)\|_{\mathrm{Sch}}+\|K_{q_0}(\xi,\xi-\eta)\chi_+(\xi_1-\eta_1)\|_{\mathrm{Sch}}\lesssim 2^{\frac{1}{6}q_0}\cdot 2^{p_0}\cdot 2^{70\delta m}. \tag{5-6}$$

To see why this holds true, first observe that for the support of the kernel we have

$$\mathrm{supp}(K_{q_0}(\xi,\eta))\subseteq\big\{(\xi,\eta)\in\mathbb{R}^2\times\mathbb{R}^2:\eta\in S^+(\xi)\cup S^-(\xi)\big\},$$

where

$$S^{\pm}(\xi) := \big\{\eta \in \mathbb{R}^2 : |\Phi(\xi,\eta)| \lesssim 2^{p_0},\ |\nabla_\eta \Phi(\xi,\eta)|, |\nabla_\xi \Phi(\xi,\eta)| \gtrsim 2^{-50\delta m},\\ |\eta_1 - \xi_1| \lesssim 2^{q_0},\ |\eta_2 \pm \xi_2| \lesssim 2^{\frac{1}{3}q_0 + 8\delta m}\big\}.$$

From this observation, and arguments similar to the ones in Lemma 6.2(1), it follows that

$$\sup_{\xi\in\mathbb{R}^2} \int_{\mathbb{R}^2} |K_{q_0}(\xi,\eta)\chi_+(\eta_1)|\, d\eta \lesssim 2^{p_0+60\delta m} \cdot 2^{\frac{1}{3}q_0+8\delta m},$$

having also used $|\mathfrak{m}| \lesssim 2^{10\delta m}$. The same bound can be also deduced for $K_{q_0}(\xi, \xi-\eta)\chi_+(\xi_1-\eta_1)$. Combing these bounds with the similar but cruder estimate

$$\sup_{\eta\in\mathbb{R}^2} \bigg(\int_{\mathbb{R}^2} |K_{q_0}(\xi,\eta)|\, d\xi + \int_{\mathbb{R}^2} |K_{q_0}(\xi,\xi-\eta)|\, d\xi \bigg) \lesssim 2^{p_0+65\delta m},$$

we see that (5-6) follows.

We finally use (5-6) and Schur's test to obtain

$$\begin{aligned} \|\mathcal{B}^+_{\le q_0}(f_1,f_2)\|_{L^2} &\lesssim 2^m \cdot 2^{\frac{1}{6}q_0+p_0+70\delta m} \cdot \varepsilon_1 2^{5\delta m} \cdot \varepsilon_1 2^{-m+25\delta m} \\ &\lesssim 2^{-m} 2^{-\frac{1}{120}m} \cdot 2^{140\delta m} \varepsilon_1^2. \end{aligned}$$

We can then conclude as before, since δ is small enough. This suffices to prove the desired bound (5-4) and concludes the proof of the proposition. □

5B. *Estimates of the terms $\mathcal{B}_q$.* In view of the decomposition (5-3) and Proposition 5.2, the main bound (5-1) can be reduced to showing

$$2^{4k^+} 2^{m-2\min\{k,k_1,k_2\}+k} \| P_k \mathcal{B}_q(f_1,f_2)\|_{L^2} \lesssim 2^{-5\delta m}, \quad q \ge q_0, \tag{5-7}$$

under the restrictions (5-2). This bound can in turn be reduced to the proof of the following proposition about Fourier integral operators.

Proposition 5.3. *Let*

$$p = -m + 40\delta m, \quad -\tfrac{1}{20}m \le q \le 4\delta m + D^2, \tag{5-8}$$

with $\delta \le 10^{-4}$. For any $g \in L^2$ and $s \in [2^{m-1}, 2^{m+1}]$ define the operator

$$\begin{aligned} T_{p,q}(g)(\xi) &:= \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} \varphi_{\le p}(\Phi(\xi,\eta))\varphi_q(\xi_1-\eta_1)\rho(\xi,\eta)\, g(\eta)\, d\eta, \\ \Phi(\xi,\eta) &= -L(\xi) + L(\xi-\eta) + L(\eta), \quad L(x) = \frac{x_1}{|x|^2}, \end{aligned} \tag{5-9}$$

and assume that the symbol ρ has the properties

$$\operatorname{supp}(\rho) \subseteq \big\{(\xi,\eta) \in \mathbb{R}^2\times\mathbb{R}^2 : 2^{-A\delta m} \lesssim |\xi|, |\eta| \lesssim 2^{A\delta m},\\ |\xi-\eta|, |\xi-2\eta| \gtrsim 2^{-A\delta m},\ |2\xi-\eta| \gtrsim 2^{-7A\delta m}\big\} \tag{5-10}$$

for some absolute positive constant $A \le 5$, and

$$|D^{\alpha}_{(\xi,\eta)}\rho(\xi,\eta)| \lesssim 2^{|\alpha|(\frac{1}{2}m+60\delta m)}2^{20\delta m}, \quad |\alpha| \ge 0. \tag{5-11}$$

Then $T_{p,q}$ satisfies the operator bound

$$\|T_{p,q}\|_{L^2\to L^2} \lesssim 2^{-m-100\delta m}. \tag{5-12}$$

Before proceeding with the proof of this proposition, let us explain how Proposition 5.3 implies the desired bound (5-7):

Proof of (5-7) *from Proposition 5.3.* Without loss of generality we can assume $j_1 \le j_2$. Then, according to our notation (5-3) and under the assumptions above, we can write

$$P_k\mathcal{B}_q(f_1,f_2) = \mathcal{F}^{-1}\int_{\mathbb{R}} \tau_m(s)\cdot \varepsilon_1 T_{p,q}(f_2)\,ds,$$

where we let

$$\rho(\xi,\eta) = \mathfrak{m}^{k,k_1,k_2}(\xi,\eta)\,\varphi_\ell(\xi-2\eta)\,\varphi_r(2\xi-\eta)\,\varepsilon_1^{-1}\hat{f}_1(\xi-\eta).$$

Using the a priori bound $\|\hat{f}_1\| \lesssim 2^{-k_1}\varepsilon_1$ and the restriction on $\underline{j}$ in (5-2), it is easy to see that the above $\rho(\xi,\eta)$ satisfies the hypotheses (5-11). Applying the conclusion (5-12) we can then estimate

$$\|P_k\mathcal{B}_q(f_1,f_2)\|_{L^2} \lesssim \varepsilon_1 2^m\|T_{p,q}\|_{L^2\to L^2}\|f_2\|_{L^2} \lesssim \varepsilon_1 2^m\cdot 2^{-m-100\delta m}\cdot\varepsilon_1 2^{-m+25\delta m},$$

which is sufficient to obtain (5-7) in view of the restriction (5-2). □

The proof of Proposition 5.3 will be performed in the remainder of the paper and will conclude the proof of the Main Theorem, Theorem 2.2.

5C. *Proof of Proposition 5.3.* To prove (5-12) we will use a TT^* argument which is based on a suitable nondegeneracy property of the mixed Hessian of the phase Φ. In particular, it turns out to be crucial that we can integrate by parts along the direction parallel to the level sets of Φ. We subdivide the proof into a few steps: First, in Step 1 we describe a curvature quantity that gives a measure of the aforementioned nondegeneracy. Step 2 then sets up the TT^* kernel and guides the subsequent splitting; we either use smallness of sets to get the claimed kernel bounds (Step 3) or exploit the nondegeneracy via an iterated integration by parts (Step 4).

Step 1: The curvature quantity $\widehat{\Upsilon}$. In preparation for Step 2 let us define

$$\widehat{\Upsilon}(\xi,\eta) := \nabla^2_{\xi,\eta}\Phi\left(\frac{\nabla^\perp_\xi\Phi}{|\nabla_\xi\Phi|}, \frac{\nabla^\perp_\eta\Phi}{|\nabla_\eta\Phi|}\right)(\xi,\eta). \tag{5-13}$$

We begin with the following algebraic lemma involving $\widehat{\Upsilon}$:

Lemma 5.4. *Define Γ and Θ as follows*:

$$\widehat{\Upsilon}(\xi,\eta) =: \frac{\Gamma(\xi,\eta)}{|\xi-\eta|^8\,|\nabla_\xi\Phi(\xi,\eta)|\,|\nabla_\eta\Phi(\xi,\eta)|}, \quad \Phi(\xi,\eta) =: \frac{\Theta(\xi,\eta)}{|\xi-\eta|^2}.$$

Then we have the identity

$$\tfrac{1}{2}\Gamma(\xi,\eta)-2\Theta(\xi,\eta)=3(\xi_1-\eta_1). \tag{5-14}$$

As a consequence, on the support of the operator $T_{p,q}$ the following bounds on $\widehat{\Upsilon}$ hold:

$$2^{q-6A\delta m}\lesssim|\widehat{\Upsilon}(\xi,\eta)|\lesssim 2^{q+10A\delta m}. \tag{5-15}$$

Proof. The identity (5-14) is obtained by a direct computation.

To verify (5-15) notice that

$$|\widehat{\Upsilon}(\xi,\eta)|=\frac{|\Gamma(\xi,\eta)|\,|\xi|\,|\eta|}{|\xi-\eta|^4\,|\xi-2\eta|\,|\eta-2\xi|},$$

and therefore, because of the restrictions (5-10),

$$2^{-6A\delta m}|\Gamma(\xi,\eta)|\lesssim|\widehat{\Upsilon}(\xi,\eta)|\lesssim 2^{10A\delta m}|\Gamma(\xi,\eta)|.$$

Now note that $|\Theta(\xi,\eta)|\lesssim 2^p 2^{2A\delta m}\ll 2^q\approx|\xi_1-\eta_1|$ by (5-8)–(5-9). Hence we can use (5-14) to deduce that $|\Gamma|\approx 2^q$, and the conclusion follows. ☐

Step 2: The TT^* kernel. Notice that the support of $(T_{p,q}g)(\xi)$ is contained in the ball $|\xi|\lesssim 2^{4\delta m}$. We decompose this ball into $O(2^{-2q+2(C_0+4)\delta m})$ balls of radius $R:=2^{q-C_0\delta m-D^3}$ for some absolute constant $C_0\in[50,150]$ to be determined below, depending on A. If we denote by ξ_0 the center of any such small ball and let

$$T_{p,q,\xi_0}(g)(\xi):=\varphi_{\le R}(\xi-\xi_0)T_{p,q}(g)(\xi),$$

we see that the main bound (5-12) will follow provided we can show that for every $\xi_0\in\mathbb{R}$,

$$\|T_{p,q,\xi_0}T^*_{p,q,\xi_0}\|_{L^2\to L^2}\lesssim[2^{-m-100\delta m}\cdot 2^{2q-2(C_0+4)\delta m}]^2. \tag{5-16}$$

Such a localization to a small ball in ξ will allow us to better control several remainder terms in various Taylor expansions below.

Let us write

$$T_{p,q,\xi_0}T^*_{p,q,\xi_0}g(\xi)=\int_{\mathbb{R}^2}S_{p,q,\xi_0}(\xi,\xi')g(\xi')\,d\xi',$$

where the kernel is given by

$$S_{p,q,\xi_0}(\xi,\xi')=\varphi_{\le R}(\xi-\xi_0)\varphi_{\le R}(\xi'-\xi_0)\int_{\mathbb{R}^2}e^{is[\Phi(\xi,\eta)-\Phi(\xi',\eta)]}\rho(\xi,\eta)\rho(\xi',\eta)\,\varphi_q(\xi_1-\eta_1)\\ \times\varphi_q(\xi_1'-\eta_1)\,\varphi_{\le p}(\Phi(\xi,\eta))\,\varphi_{\le p}(\Phi(\xi',\eta))\,d\eta. \tag{5-17}$$

Notice that on the support of this kernel we must have $|\xi-\xi'|\le 4R=4\cdot 2^{q-C_0\delta m-D^3}$. Also recall that the symbol ρ satisfies the properties (5-10)–(5-11). We will sometimes use the short-hand notation $S(\xi,\xi')$ for $S_{p,q,\xi_0}(\xi,\xi')$, dropping the indices where this creates no confusion.

To bound the relevant operator we will resort to an integration by parts in η in the kernel (5-17) — see Step 4. Where this integration fails we will show how to gain from the smallness of the measure of the support of the kernel (Step 3).

The integration by parts will be performed through the trivial identity

$$e^{is[\Phi(\xi,\eta)-\Phi(\xi',\eta)]} = \frac{1}{is\mathcal{D}}\frac{\nabla_\eta^\perp \Phi(\xi,\eta)}{|\nabla_\eta\Phi(\xi,\eta)|}\cdot\nabla_\eta e^{is[\Phi(\xi,\eta)-\Phi(\xi',\eta)]}, \tag{5-18}$$

with

$$\mathcal{D} := \frac{\nabla_\eta^\perp \Phi(\xi,\eta)}{|\nabla_\eta\Phi(\xi,\eta)|}\cdot\nabla_\eta[\Phi(\xi,\eta)-\Phi(\xi',\eta)]. \tag{5-19}$$

The choice of direction of integration by parts is motivated by the roughness of the symbol in the integrand in (5-17). See also the identities (5-25)–(5-26).

To see the relevance of $\widehat{\Upsilon}$ defined in (5-13) we calculate

$$\begin{aligned}\mathcal{D} &= \frac{\nabla_\eta^\perp \Phi(\xi,\eta)}{|\nabla_\eta\Phi(\xi,\eta)|}\cdot\nabla_\eta[\Phi(\xi,\eta)-\Phi(\xi',\eta)]\\ &= \frac{\nabla_\eta^\perp \Phi(\xi,\eta)}{|\nabla_\eta\Phi(\xi,\eta)|}\cdot[\nabla^2_{\xi,\eta}\Phi(\xi,\eta)(\xi-\xi')] + O(\nabla^3_{\xi,\xi,\eta}\Phi(\xi,\eta)|\xi-\xi'|^2).\end{aligned}$$

The fact that $\nabla_\xi\Phi$ does not vanish allows us write

$$\xi-\xi' = ae_1+be_2, \quad e_1 := \frac{\nabla_\xi^\perp\Phi(\xi,\eta)}{|\nabla_\xi\Phi(\xi,\eta)|}, \quad e_2 := \frac{\nabla_\xi\Phi(\xi,\eta)}{|\nabla_\xi\Phi(\xi,\eta)|}.$$

We can thus decompose $\mathcal{D}$ as

$$\mathcal{D} = a\widehat{\Upsilon}(\xi,\eta) + b\frac{\nabla_\eta^\perp \Phi(\xi,\eta)}{|\nabla_\eta\Phi(\xi,\eta)|}\nabla^2_{\xi,\eta}\Phi(\xi,\eta)\frac{\nabla_\xi\Phi(\xi,\eta)}{|\nabla_\xi\Phi(\xi,\eta)|} + O(\nabla^3_{\xi,\xi,\eta}\Phi(\xi,\eta)|\xi-\xi'|^2),$$

with $\widehat{\Upsilon}$ defined in (5-13) and satisfying the bounds (5-15). In particular

$$|\mathcal{D}| \geq |a|\,|\widehat{\Upsilon}(\xi,\eta)| - |b|\,|\nabla^2_{\xi,\eta}\Phi(\xi,\eta)| - 2^D|\nabla^3_{\xi,\xi,\eta}\Phi(\xi,\eta)|\,|\xi-\xi'|^2. \tag{5-20}$$

Observe that on the support of $S(\xi,\xi')$ we have

$$\begin{aligned}2^p \gtrsim |\Phi(\xi,\eta)-\Phi(\xi',\eta)| &\gtrsim |\nabla_\xi\Phi(\xi,\eta)\cdot(\xi-\xi')| - O(|\nabla^2_\xi\Phi(\xi,\eta)|\,|\xi-\xi'|^2)\\ &= |b|\,|\nabla_\xi\Phi(\xi,\eta)| - O(|\nabla^2_\xi\Phi(\xi,\eta)|\,|\xi-\xi'|^2).\end{aligned} \tag{5-21}$$

<u>Step 3</u>: The case $|b| \geq 2^{C_1\delta m+D}|\xi-\xi'|^2$, with $C_1 := 13A$. Using (5-21), $|\nabla_\xi\Phi(\xi,\eta)| \gtrsim 2^{-10A\delta m}$ and $|\nabla^2_{\xi\xi}\Phi(\xi,\eta)| \lesssim 2^{3A\delta m}$, we deduce that $|b| \lesssim 2^{p+10A\delta m}$ and in particular that we must have

$$|\xi-\xi'|^2 \lesssim 2^p.$$

We now use Schur's test to show how this suffices to obtain (5-16).

More generally, let us assume that the support of $S(\xi,\xi')$ is contained in the set $|\xi-\xi'| \leq L$. Using Lemma 6.2(1), the lower bounds $|\nabla_\xi\Phi(\xi,\eta)| \gtrsim 2^{-10A\delta m}$ and $|\nabla_\eta\Phi(\xi,\eta)| \gtrsim 2^{-4A\delta m}$ that hold on the support of $\rho(\xi,\eta)$, see (5-10) and (3-13), we can then estimate

$$\begin{aligned}\int_{\mathbb{R}^2}|S(\xi,\xi')|\chi_{\{|\xi-\xi'|\le L\}}\,d\xi &\lesssim \iint_{\mathbb{R}^2\times\mathbb{R}^2}\varphi_{\le p}(\Phi(\xi,\eta))|\rho(\xi,\eta)|\,\varphi_{\le p}(\Phi(\xi',\eta))|\rho(\xi',\eta)|\,\chi_{\{|\xi-\xi'|\lesssim L\}}\,d\eta\,d\xi\\ &\lesssim \int_{\mathbb{R}^2}\varphi_{\le p}(\Phi(\xi',\eta))|\rho(\xi',\eta)|\left[\int_{\mathbb{R}^2}\varphi_{\le p}(\Phi(\xi,\eta))|\rho(\xi,\eta)|\,\chi_{\{|\xi-\xi'|\lesssim L\}}\,d\xi\right]d\eta\\ &\lesssim \int_{\mathbb{R}^2}\varphi_{\le p}(\Phi(\xi',\eta))|\rho(\xi',\eta)|[2^p\cdot 2^{(10A+20)\delta m}\cdot L]\,d\eta\\ &\lesssim 2^{2p}\cdot 2^{(14A+40)\delta m}\cdot L.\end{aligned}\tag{5-22}$$

By symmetry a similar bound also holds when exchanging the roles of ξ and ξ'. Using this estimate with $L=2^{\frac{1}{2}p}$, we see that (5-16) follows from Schur's test since, under our assumptions, $\frac{5}{2}p+(14A+40)\delta m$ is less than $-2m-200\delta m+4q-4(C_0+4)\delta m$, as required.

Step 4: The case $|b|\le 2^{C_1\delta m+D}|\xi-\xi'|^2$. In this case we have $|b|\le 2^{-D}|\xi-\xi'|$, provided we choose $C_0\ge C_1+4$. Therefore $|a|\ge\frac{1}{2}|\xi-\xi'|$. Then we must also have

$$2^q|a|\ge 2^{C_0\delta m+D^2}|\xi-\xi'|^2,$$

since $|\xi-\xi'|\le 4\cdot 2^{q-C_0\delta m-D^3}$ on the support of the kernel. From (5-15) we know that $|\widehat{\Upsilon}||a|\ge 2^{q-6A\delta m-D}|a|$, and since we also have

$$|b||\nabla^2_{\xi,\eta}\Phi(\xi,\eta)|+2^D|\nabla^3_{\xi,\xi,\eta}\Phi(\xi,\eta)||\xi-\xi'|^2\le 2^{(C_1+3A)\delta m+2D}|\xi-\xi'|^2,$$

we can choose $C_0\ge C_1+9A=22A$, and invoke (5-20) to deduce

$$|\mathcal{D}|\gtrsim 2^{q-6A\delta m}|a|.$$

Notice that we can also assume that $|a|\gtrsim 2^{-\frac{3}{10}m}$, for otherwise $|a|\approx|\xi-\xi'|\lesssim 2^{-\frac{3}{10}m}$ and the bound (5-22) would give us

$$\int_\xi|S(\xi,\xi')|\chi_{\{|\xi-\xi'|\le 2^{-(3/10)m}\}}\,d\xi\lesssim 2^{2p}\cdot 2^{(14A+40)\delta m}\cdot 2^{-\frac{3}{10}m},$$

so that (5-16) would follow via Schur's test as above.

We now claim that an iterated integration by parts yields

$$|S(\xi,\xi')|\lesssim 2^{40\delta m}\big[2^{-m}|\mathcal{D}|^{-1}\max\big\{2^{\frac{1}{2}m+60\delta m},\,2^{-q},\,|\mathcal{D}|^{-1}2^{(\frac{2}{N}+1)A\delta m},\,2^{-p}|\mathcal{D}|\big\}\big]^M\tag{5-23}$$

for any positive integer M. Since $|\mathcal{D}|\gtrsim 2^{-\frac{2}{5}m}$, $p\ge -m+40\delta m$ and $q\ge-\frac{1}{20}m$, this bound clearly suffices to obtain (5-16).

To prove (5-23), we integrate by parts in η in the integral (5-17) using the identities (5-18)–(5-19): For notational convenience, we rewrite them here as

$$e^{is\Psi}=\frac{1}{is}\mathcal{X}e^{is\Psi},\quad \Psi(\xi,\xi',\eta):=\Phi(\xi,\eta)-\Phi(\xi',\eta),$$

$$\mathcal{X}(\xi,\eta):=\frac{1}{\mathcal{D}}\mathcal{V}\cdot\nabla_\eta,\quad \mathcal{X}^T(\xi,\eta):=\operatorname{div}_\eta\Big(\frac{1}{\mathcal{D}}\mathcal{V}\cdot\Big),\quad \mathcal{V}:=\frac{\nabla^\perp_\eta\Phi(\xi,\eta)}{|\nabla_\eta\Phi(\xi,\eta)|}.$$

Integrating by parts M times will then give

$$|S(\xi,\xi')| \lesssim \int_{\mathbb{R}^2} 2^{-mM}\big|(\mathcal{X}^T)^M\big[\rho(\xi,\eta)\rho(\xi',\eta)\varphi_q(\xi_1-\eta_1)\,\varphi_q(\xi_1'-\eta_1)\,\varphi_{\le p}(\Phi(\xi,\eta))\,\varphi_{\le p}(\Phi(\xi',\eta))\big]\big|\,d\eta. \tag{5-24}$$

Let us now analyze the various terms that arise in (5-24):

(a) When $\operatorname{div}_\eta \mathcal{V}$ hits the symbol $\rho(\xi,\eta)\rho(\xi',\eta)$ this produces a factor growing at most $2^{\frac{1}{2}m+60\delta m}$ in view of the assumption (5-11). This is accounted for by the first term in the curly brackets in (5-23).

(b) The terms that arise when $\operatorname{div}_\eta \mathcal{V}$ hits the cutoff $\varphi_q(\xi_1-\eta_1)\varphi_q(\xi_1'-\eta_1)$ are bounded by 2^{-q}.

(c) To deal with the terms when $\operatorname{div}_\eta \mathcal{V}$ hits the denominator $\mathcal{D}$, it suffices to observe that on the support of the kernel,

$$|D^\alpha_\eta \mathcal{D}(\xi,\eta)| \lesssim 2^{(2+|\alpha|)A\delta m}.$$

(d) For the term arising when $\operatorname{div}_\eta \mathcal{V}$ hits the cutoff $\varphi_{\le p}(\Phi(\xi',\eta))\varphi_{\le p}(\Phi(\xi,\eta))$, first notice that by construction

$$\mathcal{V}\cdot\nabla_\eta\varphi_{\le p}(\Phi(\xi,\eta)) = 0. \tag{5-25}$$

Moreover, we can calculate

$$\begin{aligned}\mathcal{V}(\xi,\eta)\cdot\nabla_\eta(\varphi_p(\Phi(\xi',\eta))) &= \mathcal{V}(\xi,\eta)\cdot\nabla_\eta\Phi(\xi',\eta)2^{-p}(\varphi')_p(\Phi(\xi',\eta))\\ &= -\mathcal{D}(\xi,\eta)\,2^{-p}(\varphi')_p(\Phi(\xi',\eta)).\end{aligned} \tag{5-26}$$

We then see that this is accounted for by the last term in the curly brackets in (5-23).

This concludes the proof of (5-23) and Proposition 5.3. The Main Theorem, Theorem 2.2, follows. □

6. Useful lemmata

A Schur lemma. We demonstrate here some bounds for integral operators defined through kernels with localizations. These bounds derive from the set-size restrictions brought about by localizations. We first recall the standard Schur's test:

Lemma 6.1. *For a kernel $K:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$, consider the corresponding operator*

$$(T_K f)(\xi) := \int_{\mathbb{R}^2} K(\xi,\eta) f(\eta)\,d\eta,$$

and assume that

$$\sup_{\xi\in\mathbb{R}^2}\int_{\mathbb{R}^2}|K(\xi,\eta)|\,d\eta \le K_1, \qquad \sup_{\eta\in\mathbb{R}^2}\int_{\mathbb{R}^2}|K(\xi,\eta)|\,d\xi \le K_2.$$

Then

$$\|T_K f\|_{L^2} \lesssim \sqrt{K_1K_2}\,\|f\|_{L^2}.$$

We will often apply the above lemma, and for this purpose define

$$\|K\|_{\mathrm{Sch}} := \left(\sup_\xi \int K(\xi,\eta)\,d\eta\right)^{\frac{1}{2}}\left(\sup_\eta\int K(\xi,\eta)\,d\xi\right)^{\frac{1}{2}}. \tag{6-1}$$

Lemma 6.2. (1) *Let* $F:\mathbb{R}^2\to\mathbb{R}$ *be smooth in a ball* $B_R(z)\subset\mathbb{R}^2,\ z\in\mathbb{R}^2,\ R>0$. *Then*

$$\int_{B_R(z)} \varphi_{\le\lambda}(F(x))\varphi_{\ge\mu}(\nabla F(x))\,dx \le 2^{-\mu}2^{\lambda}R.$$

(2) *Consider an integral operator given by the kernel*

$$K(\xi,\eta):=\varphi_p(\Phi(\xi,\eta))\varphi_\ell(\xi-2\eta)\varphi_r(\eta-2\xi)\varphi_k(\xi)\varphi_a(\xi-\eta)\varphi_b(\eta),$$

where Φ *is the phase in* (2-2). *Then we have the bound*

$$\|K\|_{\mathrm{Sch}} \lesssim 2^{p+\frac12(k+b-\ell-r)+2a}2^{\frac12\min\{\ell,r,a,b\}+\frac12\min\{\ell,r,k,a\}}, \tag{6-2}$$

so that, in particular,

$$\|K\|_{\mathrm{Sch}} \lesssim 2^{p+\frac12(k+b+2a)}.$$

As a consequence, we also see that if $\min\{k,\ell\}\le\max\{a,b\}-10$, *then, for*

$$K^\ell(\xi,\eta):=\phi_p(\Phi(\xi,\eta))\phi_\ell(\xi-2\eta)\phi_k(\xi)\phi_a(\xi-\eta)\phi_b(\eta)$$

we have the bound

$$\|K^\ell\|_{\mathrm{Sch}} \lesssim 2^{p+\frac12(k+b-\ell+3a)}2^{\frac12\min\{\ell,a,b\}+\frac12\min\{\ell,k,a\}}, \tag{6-3}$$

Proof. Point (2) is a consequence of (1) and the formulas for the gradient of Φ in (3-13), so we start by demonstrating (1).

Proof of (1). Notice that $\{x\in\mathbb{R}^2: |\nabla F(x)|\ge 2^\mu\}\subset A^1_\mu\cup A^2_\mu$, where $A^i_\mu:=\{x\in\mathbb{R}^2: |\partial_{x_i}F(x)|\ge 2^{\mu-1}\}$. Hence on $B_R(z)\cap A^1_\mu$ a well-defined change of variables is given by $(y_1,y_2)=Y(x):=(F(x_1,x_2),x_2)$. This change of variables has Jacobian determinant equal to $|\partial_{x_1}F|\gtrsim 2^\mu$, so we have

$$\begin{aligned}\int_{B_R(z)\cap A^\mu_1} \varphi_{\le\lambda}(F)\varphi_{\ge\mu}(\nabla F)(x)\,dx &\lesssim 2^{-\mu}\int_{Y(B_R(z))}\varphi_{\le\lambda}(F)\varphi_{\ge\mu}(\nabla F)(Y^{-1}(y))\,dy\\ &\lesssim 2^{-\mu}\int_{|y_2-z_2|\le R}\varphi_{\le\lambda}(y_1)\,dy \le 2^{-\mu}2^\lambda R.\end{aligned}$$

Exchanging the roles of x_1 and x_2, in complete analogy we deduce the same bound for

$$\int_{B_R(z)\cap A^\mu_2}\varphi_{\le\lambda}(F)\varphi_{\ge\mu}(\nabla F)\,dx,$$

thus proving the first claim.

Proof of (2). We estimate the two integrals in (6-1); for each it will suffice to appropriately apply (1). To this end, notice that with the localizations in $K(\xi,\eta)$ we have, see (3-13),

$$|\nabla_\eta\Phi| = \frac{|\xi|\,|\xi-2\eta|}{|\eta|^2\,|\xi-\eta|^2}\approx 2^{k+\ell}2^{-2a-2b},\quad |\nabla_\xi\Phi| = \frac{|\eta|\,|\eta-2\xi|}{|\xi|^2\,|\xi-\eta|^2}\approx 2^{b+r}2^{-2k-2a}$$

and Φ is smooth in the domains of integration.

Furthermore, for fixed ξ there exist ξ_0 and $R \lesssim \min\{2^\ell, 2^r, 2^a, 2^b\}$ such that the domain of the integral in η is contained in the ball $B_R(\xi_0)$. We then invoke (1) to obtain

$$\int_{\mathbb{R}^2} K(\xi,\eta)\,d\eta \le \int_{B_R(\xi_0)} \varphi_p(\Phi(\xi,\eta))\varphi_{2^{k+\ell-2a-2b}}(2^{-10}\nabla_\eta\Phi(\xi,\eta))\,d\eta \lesssim 2^p 2^{-k-\ell+2a+2b} 2^{\min\{\ell,r,a,b\}}.$$

Similarly, for fixed η there exists η_0 such that the domain of the integral in ξ is included in a ball of center η_0 and radius $R \lesssim \min\{2^\ell, 2^r, 2^k, 2^a\}$, which promptly yields

$$\int_{\mathbb{R}^2} K(\xi,\eta)\,d\xi \lesssim 2^p 2^{-b-r+2k+2a} 2^{\min\{\ell,r,k,a\}}.$$

Combining these gives the claim (6-2). The bound (6-3) follows since for $\min\{k,\ell\} \le \max\{a,b\} - 10$ one has $|r - \max\{a,b\}| \le 5$. $\square$

Hölder-type estimates and integration-by-parts lemmas. For simplicity of notation we define the following class of multipliers:

$$S^\infty := \{m : (\mathbb{R}^2)^2 \to \mathbb{C} : m \text{ continuous and } \|m\|_{S^\infty} := \|\mathcal{F}^{-1}m\|_{L^1} < \infty\}. \tag{6-4}$$

As we will often localize in frequency space we define, for any symbol m,

$$m^{k,k_1,k_2}(\xi,\eta) := \varphi_{[k-2,k+2]}(\xi)\varphi_{[k_1-2,k_1+2]}(\xi-\eta)\varphi_{[k_2-2,k_2+2]}(\eta)m(\xi,\eta); \tag{6-5}$$

see the notation in Section 2. Here is a basic lemma about S^∞ symbols that we will often use:

Lemma 6.3. (*i*) *We have* $S^\infty \hookrightarrow L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$. *If* $m, m' \in S^\infty$ *then* $m \cdot m' \in S^\infty$ *and*

$$\|m \cdot m'\|_{S^\infty} \lesssim \|m\|_{S^\infty}\|m'\|_{S^\infty}.$$

Moreover, if $m \in S^\infty$, $A : \mathbb{R}^2 \to \mathbb{R}^2$ *is a linear transformation,* $v \in \mathbb{R}^2$, *and* $m_{A,v}(\xi,\eta) := m(A(\xi,\eta)+v)$, *then*

$$\|m_{A,v}\|_{S^\infty} = \|m\|_{S^\infty}.$$

(*ii*) *For* $m \in S^\infty$, *consider the bilinear operator* $T_m : \mathcal{S}(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R}^2)$ *defined by*

$$T_m(f,g)(\xi) := \mathcal{F}^{-1}\int m(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta.$$

Then, for all $1 \le p, q, r \le \infty$ *satisfying the Hölder relation* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, *we have*

$$\|T_m(f,g)\|_{L^p} \lesssim \|m\|_{S^\infty}\|f\|_{L^p}\|g\|_{L^q}.$$

Proof. The properties in (i) follow directly from the definition (6-4). A direct computation unwinding the Fourier transforms shows that

$$T_m(f,g)(x) = \int_\xi e^{ix\xi}\int_\eta m(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta\,d\xi = \int_y\int_z f(x-z)g(x-y-z)\check{m}(z,y)\,dy\,dz,$$

from which the claim follows directly. $\square$

We state next a useful lemma, which allows us to use Hölder-type bounds when we integrate by parts in time.

Lemma 6.4. *Assume $t \approx 2^m$ for some $m \in \mathbb{N}$, and $p \geq -m + 2\delta m$. For $\rho \in S^\infty$, with $\|\rho\|_{S^\infty} \leq 1$, consider a bilinear operator of the form*

$$\underline{B}_p(v,w)(\xi) := \varphi_{\leq 10m}(\xi) \int_{\mathbb{R}^2} e^{it\Phi(\xi,\eta)} \chi(2^{-p}\Phi(\xi,\eta))\rho(\xi,\eta)\, \hat{v}(\xi-\eta)\hat{w}(\eta)\, d\eta,$$

where χ is a Schwartz function. Then, for any $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$,

$$\|\underline{B}_p(v,w)\|_{L^2} \lesssim \Big(\sup_{|s| \leq 2^{-p}2^{\delta m}} \|e^{i(t+s)L}v\|_{L^p} \|e^{i(t+s)L}w\|_{L^q} + \|v\|_{L^2}\|w\|_{L^2}2^{-10m}\Big).$$

Proof. Let us use

$$\chi(2^{-p}\Phi(\xi,\eta)) = c \int_{\mathbb{R}} e^{iz2^{-p}\Phi(\xi,\eta)} \check{\chi}(z)\, dz$$

to write

$$\underline{B}_p(v,w) = c \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}} e^{i(2^{-p}z+t)\Phi(\xi,\eta)} \check{\chi}(z)\, dz \Big) \rho(\xi,\eta)\hat{v}(\xi-\eta)\hat{w}(\eta)\, d\eta.$$

Using the rapid decay $|\check{\chi}| \leq (1+|z|)^{-M}$, for M large enough, we can estimate the contribution from the region $|z| \geq 2^{\delta m}$ as

$$\Big\| \int_{\mathbb{R}^2} \Big(\int_{|z|\geq 2^{\delta m}} e^{i(2^{-p}z+t)\Phi(\xi,\eta)} \check{\chi}(z)\, dz \Big) \varphi_{\leq 10m}(\xi)\rho(\xi,\eta)\hat{v}(\xi-\eta)\hat{w}(\eta)\, d\eta \Big\|_{L^2_\xi} \\ \lesssim 2^{10m}2^{-\delta M m}\|v\|_{L^2}\|w\|_{L^2} \lesssim 2^{-10m}\|v\|_{L^2}\|w\|_{L^2}.$$

We are now left with estimating

$$\Big\| \int_{\mathbb{R}^2} \int_{|z|\leq 2^{\delta m}} \check{\chi}(z)\, \varphi_{\leq 10m}(\xi)\rho(\xi,\eta)\, e^{i(2^{-p}z+t)L(\xi-\eta)}\hat{v}(\xi-\eta)e^{i(2^{-p}z+t)L(\eta)}\hat{w}(\eta)\, d\eta\, dz \Big\|_{L^2_\xi} \\ \lesssim \sup_{|z|\leq 2^{\delta m}} \Big\| \int_{\mathbb{R}^2} \rho(\xi,\eta)e^{i(2^{-p}z+t)L(\xi-\eta)}\hat{v}(\xi-\eta)e^{i(2^{-p}z+t)L(\eta)}\hat{w}(\eta)\, d\eta \Big\|_{L^2_\xi},$$

which by virtue of Lemma 6.3 and $\|\rho\|_{S^\infty} \leq 1$ is bounded by

$$\sup_{|z|\leq 2^{\delta m}} \|e^{i(t+2^{-p}z)L}v\|_{L^p}\|e^{i(2^{-p}z+t)L}w\|_{L^q}.$$

The desired conclusion follows. □

Here is a basic integration-by-parts lemma:

Lemma 6.5. *Assume that $\epsilon \in (0,1)$, $\epsilon K \geq 1$, $M \geq 1$ is an integer, and $F, g \in C^M(\mathbb{R}^n)$. Assume also that F is real-valued and satisfies*

$$|\nabla F| \geq \mathbf{1}_{\mathrm{supp}(g)}, \quad |D^\alpha F| \lesssim_M \epsilon^{1-|\alpha|} \quad \textit{for all } 2 \leq |\alpha| \leq M.$$

Then

$$\left|\int_{\mathbb{R}^n} e^{iKF} g\,dx\right| \lesssim \frac{1}{(\epsilon K)^M} \sum_{|\alpha|\le M} \epsilon^{|\alpha|} \|D^\alpha g\|_{L^1} \tag{6-6}$$

The proof is a fairly straightforward integration-by-parts argument; see Lemma 5.4 in [Ionescu and Pausader 2014].

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AIRY-TYPE EVOLUTION EQUATIONS ON STAR GRAPHS

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We define and study the Airy operator on star graphs. The Airy operator is a third-order differential operator arising in different contexts, but our main concern is related to its role as the linear part of the Korteweg–de Vries equation, usually studied on a line or a half-line. The first problem treated and solved is its correct definition, with different characterizations, as a skew-adjoint operator on a star graph, a set of lines connecting at a common vertex representing, for example, a network of branching channels. A necessary condition turns out to be that the graph is balanced, i.e., there is the same number of ingoing and outgoing edges at the vertex. The simplest example is that of the line with a point interaction at the vertex. In these cases the Airy dynamics is given by a unitary or isometric (in the real case) group. In particular the analysis provides the complete classification of boundary conditions giving momentum (i.e., L^2-norm of the solution) preserving evolution on the graph. A second more general problem solved here is the characterization of conditions under which the Airy operator generates a contraction semigroup. In this case unbalanced star graphs are allowed. In both unitary and contraction dynamics, restrictions on admissible boundary conditions occur if conservation of mass (i.e., integral of the solution) is further imposed. The above well-posedness results can be considered preliminary to the analysis of nonlinear wave propagation on branching structures.

1. Introduction

We consider the partial differential equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^3 u}{\partial x^3} + \beta \frac{\partial u}{\partial x}, \tag{1-1}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$, on half-bounded intervals $(-\infty, 0)$ or $(0, \infty)$, and more generally on collections of copies thereof, building structures commonly known as *metric star graphs*.

A metric star graph in the present setting is the structure represented by the set (see Figure 1)

$$\mathsf{E} := \mathsf{E}_- \cup \mathsf{E}_+,$$

where E_+ and E_- are finite or countable collections of semi-infinite edges e parametrized by $(-\infty, 0)$ or $(0, \infty)$ respectively. The half-lines are connected at a vertex v, where suitable boundary conditions have to be imposed in order to result in a well-posed boundary initial value problem. From a mathematical point of view the problem consists in a system of $|\mathsf{E}_-| + |\mathsf{E}_+|$ partial differential equations of the form (1-1), with possibly different coefficients α and β, coupled through the boundary condition at the vertex. Our main concern in this paper is exactly the characterization of boundary conditions yielding a well-posed dynamics for (1-1) on a metric star graph.

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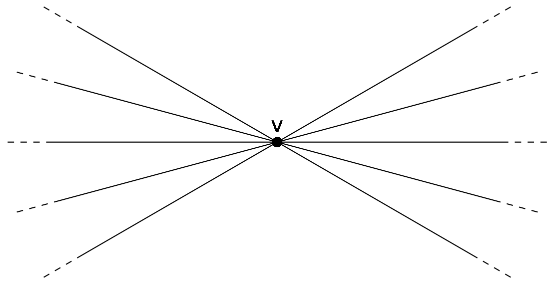

Figure 1. A balanced star graph with $|\mathsf{E}_-| = |\mathsf{E}_+| = 5$ edges.

This is a well known problem when a Schrödinger operator is considered on the graph, in which case the system is called a *quantum graph*. In this case, extended literature exists on the topic, both for the elliptic problem and for some evolution equations, like heat or wave or reaction-diffusion equations; see [Berkolaiko and Kuchment 2013; Mugnolo 2014]. The analysis has been recently extended also to the case of the nonlinear Schrödinger equation, in particular as regards the characterization of ground states and standing waves; see [Noja 2014; Cacciapuoti et al. 2017] for a review. Another dispersive nonlinear equation, the BBM equation, is treated in [Bona and Cascaval 2008; Mugnolo and Rault 2014]. A partly numerical analysis and partly theoretical analysis of some special cases of the linear Korteweg–de Vries equation on a metric graph is given in [Sobirov et al. 2015a; 2015b; 2015c]. Finally we notice that information about the linear part of the KdV equation on a star graph with special boundary conditions related to controllability problems is considered in [Ammari and Crépeau 2017]. Apart from these papers, not much seems to be known for (1-1), the linear part of the KdV equation. In this context, the solution of (1-1) represents, in the long wave or small amplitude limit, the deviation of the free surface of water from its mean level in the presence of a flat bottom. We will refer to (1-1) as the *Airy equation* and to the operator on its right-hand side as the *Airy* operator; its connection with the KdV equation is one of our main motivations for the study of this problem. Equation (1-1) appeared for the first time in [Stokes 1847] as a contribution to the understanding of solitary waves in shallow water channels observed by Russel [Lannes 2013], but some of its solutions were discussed previously by Airy (ironically, to refuse the existence of solitary waves).

One of the most fascinating features of this PDE is that it has both a dispersive and smoothing character and in fact it is well known that (1-1) is governed by a group of bounded linear operators whenever its space domain is the real line; see [Linares and Ponce 2009].

On the line, after Fourier transform, the solution of (1-1) is given by

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{ik[(x-y)-\alpha k^2-\beta]t} u_0(y)\,dy\,dk,$$

where u_0 is the initial data of (1-1). The above Fourier formula can be further managed to give in the standard $\alpha < 0$ case

$$u(t,x) = K_t * u_0(x), \quad K_t(x) = \frac{1}{\sqrt[3]{3|\alpha|t}}\operatorname{Ai}\left(\frac{x-\beta t}{\sqrt[3]{3|\alpha|t}}\right), \tag{1-2}$$

where

$$\mathrm{Ai}(x) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{iyx+i\frac{1}{3}y^3}\,dy = \frac{1}{\pi}\int_{\mathbb{R}} \cos\bigl(yx+\tfrac{1}{3}y^3\bigr)\,dy \tag{1-3}$$

is the so-called Airy function (intended with a hidden exponential convergence factor).

Translation invariance allows one to get rid of the first-order term, just changing to moving coordinates in the equation or directly in the solution, and it is not restrictive to put $\beta = 0$. In particular one has that $\|K_t\|_{L^\infty(\mathbb{R})} = \mathcal{O}(1/\sqrt[3]{t})$, from which the typical dispersive behavior of Airy equation solutions on the line follows. This estimate and other more refined dispersive properties are of utmost importance both at the linear level and in the analysis of nonlinear perturbation of the equation, such as the KdV equation; see [Linares and Ponce 2009].

Much harder is the analysis of the properties of the Airy operator when translation invariance is broken, for example in the case of a half-line or in the presence of an external potential. In the first case much work has been done in the context of the analysis of well-posedness of the KdV equation. The first problem one is faced with is the unidirectional character of the propagation, which requires some care in the definition of the correct boundary value problem. It is well known that for the standard problem ($\alpha < 0$) of the KdV equation two boundary conditions at 0 are needed on $(-\infty, 0)$ and only one suffices on $(0, +\infty)$. The issues in the definition of the correct boundary conditions of KdV equation are shared by the linear part, the Airy equation. A more complete analysis of the problem on the half-line is given in the following section.

The Airy operator with an external potential, representing the effect of an obstacle in the propagation or the result of a linearization around known stationary solutions is studied in [Miller 1997], especially as regards spectral and dispersion properties, while a different model with an inhomogeneous dispersion obtained through the introduction of a space-dependent coefficient for the third-derivative term is studied in [Craig and Goodman 1990].

In this paper we are interested in the generalization of the half-line example and from now on we will discuss only the case of a star graph in which the Airy operator has constant coefficients on every single edge (but possibly different coefficients from one edge to another) and discard any other source of inhomogeneity.

Our first goal is to properly define the Airy operator as an unbounded operator on a certain Hilbert space, in such a way that it turns out to be the generator of a C_0-semigroup. We will consider two cases in increasing order of generality. The first is the one in which the generated dynamics is unitary (in the complex case) or isometric (in the real case); the second is the case in which the generated dynamics is given by a contraction C_0-semigroup. The easiest way to understand the nature of the problem is to consider the densely defined closable and skew-symmetric Airy operator $u \mapsto A_0 u := \alpha(d^3u/dx^3) + \beta(du/dx)$ with domain $C_c^\infty(-\infty, 0)$ or $C_c^\infty(0, +\infty)$ on the Hilbert spaces $L^2(-\infty, 0)$ or $L^2(0, +\infty)$ respectively. Due to the fact that the Airy operator is of odd order, changing the sign of α it is equivalent to exchange the positive and negative half-line and so we can take $\alpha > 0$ without loss of generality.

An easy control shows that the deficiency indices of the two operators are $(2, 1)$ in the first case and $(1, 2)$ in the second case, so that iA_0 does not have any self-adjoint extension on any of the two half-lines. However, the direct sum of the operators on the two half-lines results in a symmetric operator on

$C_c^\infty((-\infty,0)\cup(0,+\infty))$ with equal deficiency indices $(3,3)$. Hence, thanks to classical von Neumann–Krein theory it admits a nine-parameter family of self-adjoint extensions generating a unitary dynamics in $L^2(\mathbb{R})$. Of course, this is not the only possibility to generate a dynamics. For instance, it could be the case that a suitable extension A of the operator on the half-line generates a nonunitary semigroup, so not conserving L^2-norm, that still consists of L^2-*contractions*. According to the Lumer–Phillips theorem, this holds true if and only if both A and its adjoint A^* are dissipative. Dissipativity in fact occurs *if the right number of correct boundary conditions are added*; for example in the standard case with $\alpha < 0$, the Dirichlet condition on $u(0)$ on the half-line $(0,+\infty)$ is sufficient, but two conditions, for example the two Dirichlet and Neumann boundary conditions on $u(0)$ and $u'(0)$, are needed on $(-\infty,0)$.

This is the ultimate reason why more or less explicit results about Airy semigroup formulas exist in the above examples and they can be fruitfully applied to some cases of nonlinear perturbation, such as the KdV equation on the half-line.

The above basic remark, which seemingly went unnoticed in the previous literature on the subject, is the starting point of the treatment of the Airy operator on the more general case of a star graph. To efficiently treat the case of star graphs, we exploit the fact that the Airy operator is antisymmetric, and we want to give first existence conditions for its skew-adjoint extensions and their classification. This can be done in principle in several ways and here we rely on a recent analysis making use (in the intermediate steps of the construction) of Krein spaces with indefinite inner products, recently developed in [Schubert et al. 2015]. As suggested by the example of the half-line, a necessary condition for skew-adjointness is that $\mathsf{E}_- = \mathsf{E}_+$, i.e., the number of incoming half lines is the same of outgoing half-lines. When this condition is met the graph is said to be *balanced*. A similar necessary condition was shown to be true in the case of the quantum momentum operator $-i(d/dx)$ on a graph; see [Carlson 1999; Exner 2013]. The complete characterization of skew-adjoint boundary conditions is more complex, and is given in Theorems 3.7 and 3.8. To explain, we introduce the space of boundary values at the vertex for the domain element of the adjoint operator A_0^*: These are given by $\big(u(0-),u'(0-),u''(0-)\big)^\top$ and $\big(u(0+),u'(0+),u''(0+)\big)^\top$, spanning respectively spaces $\mathcal{G}_-$ and $\mathcal{G}_+$ (notice that $u(0-),u(0+)$ etc. are vectors with components given by the boundary values on the single edges and "minus" and "plus" mean that they are taken on edges in E_- or E_+ respectively). The boundary form of the operator A_0 is given by

$$
\begin{aligned}
&(A_0^*u\mid v)+(u\mid A_0^*v)\\
&=\left(\begin{pmatrix}-\beta_- & 0 & -\alpha_-\\ 0 & \alpha_- & 0\\ -\alpha_- & 0 & 0\end{pmatrix}\begin{pmatrix}u(0-)\\u'(0-)\\u''(0-)\end{pmatrix}\middle|\begin{pmatrix}v(0-)\\v'(0-)\\v''(0-)\end{pmatrix}\right)_{\mathcal{G}_-}-\left(\begin{pmatrix}-\beta_+ & 0 & -\alpha_+\\ 0 & \alpha_+ & 0\\ -\alpha_+ & 0 & 0\end{pmatrix}\begin{pmatrix}u(0+)\\u'(0+)\\u''(0+)\end{pmatrix}\middle|\begin{pmatrix}v(0+)\\v'(0+)\\v''(0+)\end{pmatrix}\right)_{\mathcal{G}_+}\\
&=\left(B_-\begin{pmatrix}u(0-)\\u'(0-)\\u''(0-)\end{pmatrix}\middle|\begin{pmatrix}u(0-)\\u'(0-)\\u''(0-)\end{pmatrix}\right)_{\mathcal{G}_-}-\left(B_+\begin{pmatrix}u(0+)\\u'(0+)\\u''(0+)\end{pmatrix}\middle|\begin{pmatrix}u(0+)\\u'(0+)\\u''(0+)\end{pmatrix}\right)_{\mathcal{G}_+},
\end{aligned}
$$

where, with obvious notation, $\alpha_\pm$ and $\beta_\pm$ are vector-valued coefficients in $\mathsf{E}_\pm$ of the Airy equation on the graph. The block matrices $B_\pm$ are nondegenerate and symmetric but indefinite and endow the boundary spaces $\mathcal{G}_\pm$ with the structure of a Krein space. Correspondingly, the space $\mathcal{G}_-\oplus\mathcal{G}_+$ is endowed with the sesquilinear form

$$\omega((x,y),(u,v)) = \left(\begin{pmatrix} B_- & 0 \\ 0 & -B_+ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \begin{pmatrix} u \\ v \end{pmatrix} \right).$$

The first important characterization is that skew-adjoint extensions of A_0 are parametrized by the subspaces X of $\mathcal{G}_- \oplus \mathcal{G}_+$ which are *ω-self-orthogonal* ($X = X^\perp$ where orthogonality is with respect to the indefinite sesquilinear form ω; see Definition 3.6 for more details). An equivalent, more explicit parametrization is through relations between boundary values. Consider a linear operator $L : \mathcal{G}_- \to \mathcal{G}_+$ (here we describe the simplest case in which $\mathcal{G}_\pm$ are finite-dimensional; for the general case see Section 3) and define

$$\begin{aligned} &D(A_L) = \{u \in D(A_0^*) : L\big(u(0-), u'(0-), u''(0-)\big) = \big(u(0+), u'(0+), u''(0+)\big)\}, \\ &A_L u = -A_0^* u. \end{aligned}$$

Then A_L is a skew adjoint extension of A_0 if and only if L is $(\mathcal{G}_-, \mathcal{G}_+)$*-unitary* ($\langle Lx \mid Ly \rangle_+ = \langle x \mid y \rangle_-$ and again see Definition 3.6).

The characterization of extensions A of A_0 generating a contraction semigroup can be treated along similar lines. According to the Lumer–Phillips theorem, one has that A and A^* have to be dissipative. For a linear operator $L : \mathcal{G}_- \to \mathcal{G}_+$ define $A_L \subseteq -A_0^*$ as above. Denoting by $L^\sharp$ the $(\mathcal{G}_+, \mathcal{G}_-)$-adjoint of the operator L (see Definition 3.6) one has that A_L is the generator of a contraction semigroup if and only if L is a $(\mathcal{G}_-, \mathcal{G}_+)$-contraction (i.e., $\langle Lx \mid Lx \rangle_+ \le \langle x \mid x \rangle_-$ for all $x \in \mathcal{G}_-$) and $L^\sharp$ is a $(\mathcal{G}_+, \mathcal{G}_-)$-contraction (i.e., $\langle L^\sharp x \mid L^\sharp x \rangle_- \le \langle x \mid x \rangle_+$ for all $x \in \mathcal{G}_+$).

This condition allows for a different number of ingoing and outgoing half-lines, and it is well adapted to study realistic configurations such as a branching channel.

Notice that in both the skew-adjoint or dissipative case one can consider the possible conservation of an additional physical quantity, the *mass*, coinciding with the integral of the domain element of A. Conservation of mass (see Remark 3.10) is characterized by boundary conditions satisfying the constraint

$$\sum_{\mathsf{e} \in \mathsf{E}_-} \alpha_\mathsf{e} u''_\mathsf{e}(0-) - \sum_{\mathsf{e} \in \mathsf{E}_+} \alpha_\mathsf{e} u''_\mathsf{e}(0+) + \sum_{\mathsf{e} \in \mathsf{E}_-} \beta_\mathsf{e} u_\mathsf{e}(0-) - \sum_{\mathsf{e} \in \mathsf{E}_+} \beta_\mathsf{e} u_\mathsf{e}(0+) = 0.$$

Of course the requirement of mass conservation restricts the allowable boundary conditions, both in the skew-adjoint and the dissipative case.

In Section 4 we provide a collection of more concrete examples. Already at the level of general analysis previously done it is clear that a distinguished class of boundary conditions exists, in which the first-derivative boundary values $u'(0\pm)$ are separated. This means that they do not interact with the boundary values of u and u'' and satisfy the transmission relation $u'(0+) = Uu'(0-)$ with U unitary in the skew-adjoint case and a contraction in the dissipative case, while $u(0\pm)$ and $u''(0\pm)$ are coupled. This case is described at the beginning of Section 4. Some more special examples deserve interest. The first is the graph consisting of two half-lines. This case is interesting also because it can be interpreted as describing the presence of an obstacle or point interaction on the line. In the skew-adjoint case this perturbation does not destroy the conservation of momentum during the evolution. Moreover, there are both skew-adjoint and more generically contractive boundary conditions that conserve the mass as well.

In this sense the two half-lines case corresponds to a forcing or interaction which however can preserve in time some physical quantity. A related recent analysis is given in [Deconinck et al. 2016], where an inhomogeneous interface problem for the Airy operator on the line (in the special case $\beta = 0$) is studied by means of the Fokas unified transform method (UTM), see [Fokas 2008], and necessary and sufficient conditions for its solvability are given in terms of interface conditions (in that paper the authors prefer to distinguish between interface and boundary conditions; here this usage is not followed). It surely would be interesting to compare the interface conditions studied in the quoted paper with the ones derived in the present one both as regards the skew-adjoint case and the contraction case.

A second class of examples are given for the graph with three half-lines, which falls necessarily in the non-skew-adjoint case. A more accurate analysis of this example is relevant because of the possible application to the analysis of flow in branching or confluent channels, which has attracted some attention in recent years; see [Nachbin and Simões 2012; 2015], and the interesting early paper [Jacovkis 1991], where different models of flows are treated but much of the analysis has general value. It is not at all clear which boundary conditions should be the correct ones from a physical point of view, and the complete classification of those giving generation at least for the Airy operator is a first step to fully understand realistic situations. In general one expects, on theoretical and experimental grounds, that 1-dimensional reductions retain some memory of the geometry (for example the angles of the fork) which is not contemplated in the pure graph description, and a further step in the analysis should consist in including these effects through the introduction of further phenomenological parameters or additional terms in the equation.

The paper is so organized. In Section 2 the simplest case of a single half-line is considered, giving also some comments on the previous literature on the subject. In Section 3 the complete construction of the Airy operator on a star graph in the skew-adjoint and dissipative case is given. The exposition is self-contained as regards the preliminary definitions on graphs and operators on Krein spaces. Besides construction, some general properties of the Airy operators are further studied. In Section 4 some examples are treated.

2. The case of the half-line

As a warm-up, in the present section we consider the Airy equation on the half-line. This is an often-considered subject because of its relevance in connection with the analogous initial boundary value problem for the KdV equation, which turns out to be a rather challenging problem, especially when studied in low Sobolev regularity; for pertinent papers on the subject with information on the linear problem see [Bona et al. 2002; Colliander and Kenig 2002; Hayashi and Kaikina 2004; Holmer 2006; Faminskii 2007; Fokas et al. 2016]. For the interesting case of the interval see also [Colin and Ghidaglia 2001; Bona et al. 2003]. We recall that Korteweg and de Vries derived, under several hypotheses, the equation

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{\ell}}\left(\frac{\sigma}{3}\frac{\partial^3 \eta}{\partial x^3} + \frac{2\alpha}{3}\frac{\partial \eta}{\partial x} + \frac{1}{2}\frac{\partial \eta^2}{\partial x}\right), \tag{2-1}$$

where the unknown η is the elevation of the water surface with respect to its average depth ℓ in a shallow canal. (See [Korteweg and de Vries 1895] for an explanation of the parameters g, σ, α; there

σ is implicitly assumed to be negative. See also [Lannes 2013] for a modern and complete analysis). Renaming coefficients one obtains the KdV equation with parameter α in front of the third derivative and β in front of the first derivative. The physically relevant case of a semi-infinite channel represented by the half-line $[0, +\infty)$ and a wave maker placed at $x = 0$ is described, if dissipation is neglected, by the KdV equation in which the linear part has $\alpha < 0$ and $\beta < 0$; see [Bona et al. 2002]. So one obtains, neglecting nonlinearity, the Airy equation on the positive half-line as the linear part of the KdV equation *with the above sign of coefficients*. The first-derivative term disappears on the whole line changing to a moving reference system, but on the half-line or on a non translation-invariant domain it should be retained, and we do this. The sign of the coefficient of the third derivative is important and interacts with the choice of the half-line, left or right, where propagation occurs. Depending on the sign of α, we must impose a different number of boundary conditions: we have one and two boundary conditions on $(0, \infty)$ and $(-\infty, 0)$ respectively if $\alpha < 0$, and vice versa if $\alpha > 0$. This reflects the fact that the partial differential equation (1-1) has unidirectional nature, and is explained in the literature in several different ways. An explanation making use of the behavior of characteristic curves is recalled for example in [Deconinck et al. 2016], where the authors however notice that it has only a heuristic value. A more convincing brief discussion is given, for $\beta = 0$, in [Holmer 2006], which we reproduce now with minor modifications and considering $\alpha = \pm 1$ for simplicity, which can always be achieved by rescaling the space variable. On the left half-line, after multiplying both sides of (1-1) by u and integrating on $(-\infty, 0)$ one obtains

$$\int_{-\infty}^{0} u_t(t,x)u(t,x)\,dx = \pm u_{xx}(t,0-)u(t,0-) \mp \tfrac{1}{2}u_x^2(t,0-) + \beta\tfrac{1}{2}u^2(t,0-).$$

Integrating in time on $(0, T)$ one finally obtains the identity

$$\frac{1}{2}\int_{-\infty}^{0} u^2(T,x)\,dx - \frac{1}{2}\int_{-\infty}^{0} u^2(0,x)\,dx = \pm\int_0^T u_{xx}(t,0-)u(t,0-)\,dt \mp \frac{1}{2}\int_0^T u_x^2(t,0-)\,dt + \beta\frac{1}{2}\int_0^T u^2(t,0-)\,dt.$$

If we consider the operator with $\alpha = -1$, which is the standard Airy operator, we conclude that the boundary condition $u(t, 0-) = 0$ alone and the initial condition $u(0, x) = 0$ are compatible with a nonvanishing solution of the equation: to force a vanishing solution one has to fix the boundary value $u_x(t, 0-) = 0$ also. So uniqueness is guaranteed by both boundary conditions on u and u_x. On the contrary, the above identity for the operator with $\alpha = 1$ with the boundary condition $u(t, 0-) = 0$ and the initial condition $u(0, x) = 0$ imply $u(T, x) = 0$ for any T. So one has uniqueness in the presence of the only condition on $u(t, 0)$.

In the case of the positive half-line $(0, \infty)$ one obtains from (1-1)

$$\frac{1}{2}\int_0^{\infty} u^2(T,x)\,dx - \frac{1}{2}\int_0^{\infty} u^2(0+,x)\,dx = \mp\int_0^T u_{xx}(t,0+)u(t,0+)\,dt \pm \frac{1}{2}\int_0^T u_x^2(t,0+)\,dt - \beta\frac{1}{2}\int_0^T u^2(t,0+)\,dt$$

and for $\alpha = -1$ one has uniqueness in the presence of the boundary condition for $u(t, 0+)$ alone, while for $\alpha = +1$ uniqueness needs the specification of both $u(t, 0+)$ and $u_x(t, 0+)$. Considering the difference of two solutions corresponding to identical initial data and boundary conditions, one concludes that the same properties hold true in the case of general nonvanishing boundary conditions.

One can go in greater depth and the following lemma is the point of departure. It clarifies some of the properties of the Airy operator on the half-line and will be extended, with many consequences, to a star graph in the following sections.

Lemma 2.1. *Consider the operator*

$$u \mapsto H_0 u := \alpha \frac{d^3 u}{dx^3} + \beta \frac{du}{dx}$$

with domain

$$C_c^\infty(-\infty, 0) \quad or \quad C_c^\infty(0, +\infty)$$

on the Hilbert space

$$L^2(-\infty, 0) \quad or \quad L^2(0, +\infty)$$

respectively. Then $i H_0$ *is densely defined, closable and symmetric. However, its deficiency indices are* $(2, 1)$ *in the first case and* $(1, 2)$ *in the second case, so* $i H_0$ *does not have any self-adjoint extensions in either case.*

Proof. One checks directly that the adjoint of H_0 on $L^2(-\infty, 0)$ and $L^2(0, \infty)$ is $H_0^* = -\alpha(d^3u/dx^3) - \beta(du/dx)$ with domain $H^3(-\infty, 0)$ and $H^3(0, \infty)$, respectively.

We are going to solve the elliptic problem $(i H_0^* \mp i\,\mathrm{Id})u = 0$ in $L^2(-\infty, 0)$ and $L^2(0, \infty)$. Let us consider the sign $+$ and discuss

$$\alpha \frac{d^3 u}{dx^3} + \beta \frac{du}{dx} + u = 0$$

for $u \in L^2(-\infty, 0)$ or $L^2(0, \infty)$ without any boundary conditions. A tedious but elementary computation shows that a general solution is a linear combination of complex exponentials of the form

$$u(x) = C_1 \mathrm{e}^{-x \frac{(i\sqrt{3}A^{2/3} + 12 i\sqrt{3}\alpha\beta + A^{2/3} - 12\beta\alpha)}{12\alpha A^{1/3}}} + C_2 \mathrm{e}^{x \frac{(i\sqrt{3}A^{2/3} + 12 i\sqrt{3}\alpha\beta - A^{2/3} + 12\beta\alpha)}{12\alpha A^{1/3}}} + C_3\, \mathrm{e}^{x \frac{(A^{2/3} - 12\beta\alpha)}{6\alpha A^{1/3}}},$$

where

$$A = \left(12\sqrt{3}\sqrt{\frac{4\beta^3 + 27\alpha}{\alpha}} - 108\right)\alpha^2$$

for general constants C_1, C_2, C_3. However, carefully checking the real parts of the exponents one deduces that such functions are square integrable on $(-\infty, 0)$ and $(0, \infty)$ if and only if $C_3 = 0$ and $C_1 = C_2 = 0$, respectively, thus yielding the claim on the deficiency indices. The remaining case can be treated likewise. □

This shows that the Airy operator cannot be extended to a skew adjoint operator generating a unitary dynamics in L^2. Moreover in some sense the Airy operator "with the wrong sign" or too few boundary conditions has too much spectrum to allow for uniqueness; see [Hille and Phillips 1957, Theorem 23.7.2]. This is not however the whole story, and one can obtain more precise information and some more guiding ideas giving up a unitary evolution and asking simply for generation of a contractive semigroup. To this end we consider the Lumer–Phillips condition and its consequences; see for example [Engel and Nagel

2000, Corollary 3.17]. Again, considering only the case $\alpha = \pm 1$, which is enough, one has by integration by parts

$$\int_{-\infty}^{0} (\pm u_{xxx} + \beta u_x) v \, dx$$
$$= \int_{-\infty}^{0} u(\mp v_{xxx} - \beta v_x) \, dx \pm u_{xx}(0)v(0) + \beta u(0)v(0) \mp u_x(0)v_x(0) \pm u(0)v_{xx}(0).$$

The operator $H_{\pm,\beta}$ with domain

$$\mathcal{D}(H_{\pm,\beta}) = \{u \in H^3(-\infty, 0) : u(0) = 0,\ u_x(0) = 0\}$$

and action

$$H_{\pm,\beta} u = \pm u_{xxx} + \beta u_x$$

satisfies

$$\langle H_{\pm,\beta} u \mid u \rangle = 0$$

and it is dissipative (in fact conservative).

The operator $H^{\diamondsuit}_{\pm,\beta}$ with domain

$$\mathcal{D}(H^{\diamondsuit}_{\pm,\beta}) = \{u \in H^3(-\infty, 0) : u(0) = 0\}$$

and action

$$H^{\diamondsuit}_{\pm,\beta} u = \mp u_{xxx} - \beta u_x$$

satisfies

$$\langle H^{\diamondsuit}_{\pm,\beta} u \mid u \rangle = \pm \tfrac{1}{2} (u_x(0-))^2$$

and so $H^{\diamondsuit}_{+,\beta}$ is accretive and $H^{\diamondsuit}_{-,\beta}$ is dissipative for every $\beta \in \mathbb{R}$.

$H_{\pm,\beta}$ and $H^{\diamondsuit}_{\pm,\beta}$ are in fact adjoint one to another:

$$H^{\diamondsuit}_{\pm,\beta} = H^{*}_{\pm,\beta}. \tag{2-2}$$

In particular this means that $H_{+,\beta}$ and $H^{*}_{+,\beta}$ are both accretive and so they do not generate a continuous contraction semigroup in $L^2(-\infty, 0)$. On the contrary, $H_{-,\beta}$ and $H^{*}_{-,\beta}$ are both dissipative and generate a contraction semigroup in $L^2(-\infty, 0)$. With our convention of writing of the Airy equation, this gives well-posedness on $(-\infty, 0)$ for the standard Airy equation (1-1) with two boundary conditions (generator $H_{-,\beta}$). The specular situation occurs for $(0, \infty)$, exchanging the roles and definitions of H and $H^{\diamondsuit}$, and one has that $H^{\diamondsuit}_{+,\beta}$ generates a contraction semigroup on $L^2(0, \infty)$ and the standard Airy equation (1-1) with a single boundary condition is well-posed.

3. The case of a metric star graph

Star graphs can be regarded as the building blocks of more complicated graphs; for the purpose of investigating (local) boundary conditions, they are sufficiently generic. Therefore, in this section we are going to develop the theory of the counterpart of the operator H_0 defined on a star graph G, which indeed turns out to display some unexpected behaviors in comparison with its simpler relative introduced in Lemma 2.1.

Upon replacing an interval $(-\infty, 0)$ by $(0, \infty)$ or vice versa, we may assume all coefficients α to have the same sign on each edge e of the star graph. Throughout this paper we are going to follow the convention that all coefficients are positive.

Proposition 3.1. *Consider a quantum graph consisting of finitely or countably many half-lines* $\mathsf{E} := \mathsf{E}_- \cup \mathsf{E}_+$ *and let* $(\alpha_\mathsf{e})_{\mathsf{e}\in\mathsf{E}}, (\beta_\mathsf{e})_{\mathsf{e}\in\mathsf{E}}$ *be two sequences of real numbers with* $\alpha_\mathsf{e} > 0$ *for all* $\mathsf{e} \in \mathsf{E}$. *Consider the operator* A_0 *defined by*

$$D(A_0) := \bigoplus_{\mathsf{e}\in\mathsf{E}_-} C_c^\infty(-\infty, 0) \oplus \bigoplus_{\mathsf{e}\in\mathsf{E}_+} C_c^\infty(0, +\infty),$$

$$A_0 : (u_\mathsf{e})_{\mathsf{e}\in\mathsf{E}} \mapsto \left(\alpha_\mathsf{e} \frac{d^3 u_\mathsf{e}}{dx^3} + \beta_\mathsf{e} \frac{du_\mathsf{e}}{dx}\right)_{\mathsf{e}\in\mathsf{E}}.$$

Then iA_0 *is densely defined and symmetric on the Hilbert space*

$$L^2(\mathsf{G}) := \bigoplus_{\mathsf{e}\in\mathsf{E}_-} L^2(-\infty, 0) \oplus \bigoplus_{\mathsf{e}\in\mathsf{E}_+} L^2(0, +\infty)$$

and its defect indices are $(2|\mathsf{E}_-| + |\mathsf{E}_+|, |\mathsf{E}_-| + 2|\mathsf{E}_+|)$. *Accordingly,* A_0 *has skew-self-adjoint extensions on* $L^2(\mathsf{G})$ *if and only if* $|\mathsf{E}_+| = |\mathsf{E}_-|$.

In order to avoid some technicalities we will assume that the sequences $(\alpha_\mathsf{e})_{\mathsf{e}\in\mathsf{E}}$ and $(\beta_\mathsf{e})_{\mathsf{e}\in\mathsf{E}}$ are bounded, and furthermore, that $(1/\alpha_\mathsf{e})_{\mathsf{e}\in\mathsf{E}}$ is bounded as well.

Unlike in the case of the Laplace operator, and in spite of the relevance of related physical models, like the KdV equation, there seems to be no canonical or natural choice of boundary conditions to impose on (1-1) on a star graph. For this reason, we are going to characterize all boundary conditions within certain classes. Since (1-1) plays a role in dispersive systems in which conservation of the initial data's norm is expected, we are going to focus on those extensions that generate unitary groups, or isometric semigroups, or at least contractive semigroups.

Extensions of A_0 generating unitary groups. By Stone's theorem, generators of unitary groups are exactly the skew-self-adjoint operators. In order to determine the skew-self-adjoint extensions of A_0, take

$$u, v \in D(A_0^*) = \bigoplus_{\mathsf{e}\in\mathsf{E}_-} H^3(-\infty, 0) \oplus \bigoplus_{\mathsf{e}\in\mathsf{E}_+} H^3(0, +\infty).$$

Remark 3.2. Let $u \in D(A_0^*)$. By Sobolev's lemma and the boundedness assumption on the (α_e) and (β_e) we obtain $(u_\mathsf{e}^{(k)}(0-))_{\mathsf{e}\in\mathsf{E}_-} \in \ell^2(\mathsf{E}_-)$ and $(u_\mathsf{e}^{(k)}(0+))_{\mathsf{e}\in\mathsf{E}_+} \in \ell^2(\mathsf{E}_+)$ for $k \in \{0, 1, 2\}$.

Following the classical extension theory, see [Schmüdgen 2012, Chapter 3], we write down the boundary form to obtain

$$\begin{aligned}(A_0^*u \mid v)+&(u \mid A_0^*v)\\ &= -\sum_{\mathsf{e}\in\mathsf{E}_-} \int_{-\infty}^0 (\alpha_\mathsf{e} u_\mathsf{e}''' + \beta_\mathsf{e} u_\mathsf{e}')\bar{v}_\mathsf{e}\, dx - \sum_{\mathsf{e}\in\mathsf{E}_+} \int_0^{+\infty} (\alpha_\mathsf{e} u_\mathsf{e}''' + \beta_\mathsf{e} u_\mathsf{e}')\bar{v}_\mathsf{e}\, dx\\ &\quad - \sum_{\mathsf{e}\in\mathsf{E}_-} \int_{-\infty}^0 \overline{(\alpha_\mathsf{e} v_\mathsf{e}''' + \beta_\mathsf{e} v_\mathsf{e}')} u_\mathsf{e}\, dx - \sum_{\mathsf{e}\in\mathsf{E}_+} \int_0^{+\infty} \overline{(\alpha_\mathsf{e} v_\mathsf{e}''' + \beta_\mathsf{e} v_\mathsf{e}')} u_\mathsf{e}\, dx\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{e\in E_-} \alpha_e u_e''(x)\overline{v_e(x)}\Big|_{-\infty}^{0} + \sum_{e\in E_-}\int_{-\infty}^{0} \alpha_e u_e'' \bar{v}_e'\,dx \\
&\quad -\sum_{e\in E_-} \alpha_e \overline{v_e(x)} u_e(x)\Big|_{-\infty}^{0} + \sum_{e\in E_-}\int_{-\infty}^{0} \alpha_e u_e' \bar{v}_e''\,dx \\
&\quad -\sum_{e\in E_+} \alpha_e u_e''(x)\overline{v_e(x)}\Big|_{0}^{+\infty} + \sum_{e\in E_+}\int_{0}^{+\infty} \alpha_e u_e'' \bar{v}_e'\,dx \\
&\quad -\sum_{e\in E_+} \alpha_e \overline{v_e(x)} u_e(x)\Big|_{0}^{+\infty} + \sum_{e\in E_+}\int_{0}^{+\infty} \alpha_e u_e' \bar{v}_e''\,dx \\
&\quad -\sum_{e\in E_-}\int_{-\infty}^{0} \beta_e u_e' \bar{v}_e\,dx - \sum_{e\in E_-}\int_{-\infty}^{0} \beta_e \bar{v}_e' u_e\,dx - \sum_{e\in E_+}\int_{0}^{+\infty} \beta u_e' \bar{v}_e\,dx - \sum_{e\in E_+}\int_{0}^{+\infty} \beta_e \bar{v}_e' u_e\,dx \\
&= -\sum_{e\in E_-} \alpha_e u_e''(x)\overline{v_e(x)}\Big|_{-\infty}^{0} + \sum_{e\in E_-} \alpha_e u_e'(x)\overline{v_e(x)}\Big|_{-\infty}^{0} \\
&\quad -\sum_{e\in E_-} \alpha_e \overline{v_e(x)} u_e(x)\Big|_{-\infty}^{0} + \sum_{e\in E_+} \alpha_e u_e'(x)\overline{v_e(x)}\Big|_{0}^{-\infty} \\
&\quad -\sum_{e\in E_+} \alpha_e u_e''(x)\overline{v_e(x)}\Big|_{0}^{+\infty} \\
&\quad -\sum_{e\in E_+} \alpha_e \overline{v_e(x)} u_e(x)\Big|_{0}^{+\infty} \\
&\quad -\sum_{e\in E_-} \beta_e u_e(x)\overline{v_e(x)}\Big|_{-\infty}^{0} - \sum_{e\in E_+} \beta_e u_e(x)\overline{v_e(x)}\Big|_{0}^{+\infty} \\
&= -\sum_{e\in E_-} \alpha_e u_e''(0)\overline{v_e(0)} + \sum_{e\in E_+} \alpha_e u_e''(0)\overline{v_e(0)} - \sum_{e\in E_-} \alpha_e \overline{v_e(0)} u_e(0) + \sum_{e\in E_+} \alpha_e \overline{v_e(0)} u_e(0) \\
&\quad -\sum_{e\in E_-} \beta_e u_e(0)\overline{v_e(0)} + \sum_{e\in E_+} \beta_e u_e(0)\overline{v_e(0)} + \sum_{e\in E_-} \alpha_e u_e'(0)\overline{v_e(0)} - \sum_{e\in E_+} \alpha_e u_e'(0)\overline{v_e(0)}.
\end{aligned}
$$

Thus, abbreviating $u(0-) := (u_e(0-))_{e\in E_-}$ and similarly for the other terms, and identifying with an abuse of notation $\alpha_\pm$ and $\beta_\pm$ with the corresponding multiplication operator, i.e.,

$$
\begin{aligned}
\alpha_\pm x &:= (\alpha_e x_e)_{e\in E_\pm}, \quad x \in \ell^2(E_\pm), \\
\beta_\pm x &:= (\beta_e x_e)_{e\in E_\pm}, \quad x \in \ell^2(E_\pm),
\end{aligned}
\tag{3-1}
$$

we obtain

$$
\begin{aligned}
&(A_0^* u \mid v) + (u \mid A_0^* v) \\
&\quad = -\big(\alpha_- u''(0-) \mid v(0-)\big) + \big(\alpha_+ u''(0+) \mid v(0+)\big) - \big(\alpha_- u(0-) \mid v''(0-)\big) + \big(\alpha_+ u(0+) \mid v''(0+)\big) \\
&\qquad -\big(\beta_- u(0-) \mid v(0-)\big) + \big(\beta_+ u(0+) \mid v(0+)\big) + \big(\alpha_- u'(0-) \mid v'(0-)\big) - \big(\alpha_+ u'(0+) \mid v'(0+)\big) \\
&\quad = \left(\begin{pmatrix} -\beta_- & 0 & -\alpha_- \\ 0 & \alpha_- & 0 \\ -\alpha_- & 0 & 0 \end{pmatrix}\begin{pmatrix} u(0-) \\ u'(0-) \\ u''(0-) \end{pmatrix} \middle| \begin{pmatrix} v(0-) \\ v'(0-) \\ v''(0-) \end{pmatrix}\right)_{\mathcal{G}_-} - \left(\begin{pmatrix} -\beta_+ & 0 & -\alpha_+ \\ 0 & \alpha_+ & 0 \\ -\alpha_+ & 0 & 0 \end{pmatrix}\begin{pmatrix} u(0+) \\ u'(0+) \\ u''(0+) \end{pmatrix} \middle| \begin{pmatrix} v(0+) \\ v'(0+) \\ v''(0+) \end{pmatrix}\right)_{\mathcal{G}_+},
\end{aligned}
\tag{3-2}
$$

where

$$\mathcal{G}_- := \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_-)$$

and

$$\mathcal{G}_+ := \ell^2(\mathsf{E}_+) \oplus \ell^2(\mathsf{E}_+) \oplus \ell^2(\mathsf{E}_+).$$

(We stress the difference from G, which we let denote the quantum graph.) Consider on the graph $G(A_0^*)$ of the operator A_0^* a linear and surjective operator $F : G(A_0^*) \to \mathcal{G}_- \oplus \mathcal{G}_+$ defined by

$$F((u, A_0^* u)) := \Big(\big(u(0-), u'(0-), u''(0-)\big), \big(u(0+), u'(0+), u''(0+)\big)\Big). \tag{3-3}$$

Following the terminology in [Schubert et al. 2015, Examples 2.7], let Ω be the standard symmetric form on $G(A_0^*)$, i.e.,

$$\Omega((u, A_0^* u), (v, A_0^* v)) := \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ A_0^* u \end{pmatrix} \,\middle|\, \begin{pmatrix} v \\ A_0^* v \end{pmatrix} \right)_{L^2(\mathsf{G})}, \quad (u, A_0^* u), (v, A_0^* v) \in G(A_0^*),$$

and define a sesquilinear form $\omega : \mathcal{G}_- \oplus \mathcal{G}_+ \times \mathcal{G}_- \oplus \mathcal{G}_+ \to \mathbb{C}$ by

$$\omega((x, y), (u, v)) := \left(\begin{pmatrix} B_- & 0 \\ 0 & -B_+ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \begin{pmatrix} u \\ v \end{pmatrix} \right)_{\mathcal{G}_- \oplus \mathcal{G}_+}, \tag{3-4}$$

where $B_\pm$ is the linear block operator matrix on $\mathcal{G}_\pm$ defined by

$$B_\pm(x, x', x'') := \begin{pmatrix} -\beta_\pm & 0 & -\alpha_\pm \\ 0 & \alpha_\pm & 0 \\ -\alpha_\pm & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ x' \\ x'' \end{pmatrix}, \quad (x, x', x'') \in \mathcal{G}_\pm.$$

Observe that neither B_+ nor B_- are definite operators.

Then (3-2) can be rewritten as

$$\Omega((u, A_0^* u), (v, A_0^* v)) = \omega(F(u, A_0^* u), F(v, A_0^* v)) \quad \text{for all } u, v \in D(A_0^*). \tag{3-5}$$

Remark 3.3. Note that, since $\beta_\pm$, $\alpha_\pm$ and $1/\alpha_\pm$ are bounded, we have $B_\pm \in \mathcal{L}(\mathcal{G}_\pm)$, $B_\pm$ are injective and $B_\pm^{-1} \in \mathcal{L}(\mathcal{G}_\pm)$.

Our method is based on the notion of Krein space, i.e., of a vector space endowed with an *indefinite* inner product.

Definition 3.4. Define an (indefinite) inner product $\langle \cdot \mid \cdot \rangle_\pm : \mathcal{G}_\pm \times \mathcal{G}_\pm \to \mathbb{K}$ by

$$\langle x \mid y \rangle_\pm := (B_\pm x \mid y), \quad x, y \in \mathcal{G}_\pm.$$

Then $(\mathcal{G}_\pm, \langle \cdot \mid \cdot \rangle_\pm)$ are Krein spaces and $\langle \cdot \mid \cdot \rangle_\pm$ is nondegenerate, i.e., for $x \in \mathcal{G}_\pm$ with $\langle x \mid y \rangle_\pm = 0$ for all $y \in \mathcal{G}_\pm$ it follows that $x = 0$.

Remark 3.5. Let $\mathcal{K}$ be a vector space and $\langle \cdot \mid \cdot \rangle$ an (indefinite) inner product on $\mathcal{K}$ such that $(\mathcal{K}, \langle \cdot \mid \cdot \rangle)$ is a Krein space. Then there exists an inner product $(\cdot \mid \cdot)$ on $\mathcal{K}$ such that $(\mathcal{K}, (\cdot \mid \cdot))$ is a Hilbert space. Notions such as closedness or continuity are then defined by the underlying Hilbert space structure.

Definition 3.6. Let $\mathcal{K}_-, \mathcal{K}_+$ be Krein spaces and $\omega : \mathcal{K}_- \oplus \mathcal{K}_+ \times \mathcal{K}_- \oplus \mathcal{K}_+ \to \mathbb{C}$ be sesquilinear.

(1) A subspace X of $\mathcal{K}_- \oplus \mathcal{K}_+$ is called *ω-self-orthogonal* if

$$X = X^{\perp_\omega} := \big\{(x, y) \in \mathcal{K}_- \oplus \mathcal{K}_+ : \omega((x, y), (u, v)) = 0 \text{ for all } (u, v) \in X\big\}.$$

(2) Given a densely defined linear operator L from $\mathcal{K}_-$ to $\mathcal{K}_+$, its $(\mathcal{K}_-, \mathcal{K}_+)$*-adjoint* $L^\sharp$ is

$$D(L^\sharp) := \big\{y \in \mathcal{K}_+ : \exists z \in \mathcal{K}_- \text{ with } \langle Lx \mid y\rangle_+ = \langle x \mid z\rangle_- \text{ for all } x \in D(L)\big\},$$
$$L^\sharp y := z.$$

Clearly, $L^\sharp$ is then a linear operator from $\mathcal{K}_+$ to $\mathcal{K}_-$.

(3) A linear operator L from $\mathcal{K}_-$ to $\mathcal{K}_+$ is called a $(\mathcal{K}_-, \mathcal{K}_+)$*-contraction* if

$$\langle Lx \mid Lx\rangle_+ \le \langle x \mid x\rangle_- \quad \text{for all } x \in D(L).$$

(4) A linear operator L from $\mathcal{K}_-$ to $\mathcal{K}_+$ is called $(\mathcal{K}_-, \mathcal{K}_+)$*-unitary* if $D(L)$ is dense, $R(L)$ is dense, L is injective, and finally $L^\sharp = L^{-1}$.

If in particular $\mathcal{K}_-, \mathcal{K}_+$ are Hilbert spaces, then obviously $(\mathcal{K}_-, \mathcal{K}_+)$-adjoint/contraction/unitary operators are nothing but the usual objects of Hilbert space operator theory.

Note that if L is a $(\mathcal{K}_-, \mathcal{K}_+)$-unitary, then

$$\langle Lx \mid Ly\rangle_+ = \langle x \mid y\rangle_- \quad \text{for all } x, y \in D(L). \tag{3-6}$$

We stress that unitary operators between Krein spaces need not be bounded.

By [Schubert et al. 2015, Corollary 2.3 and Example 2.7(b)] we can now characterize skew-self-adjoint extensions A of A_0 — i.e., skew-self-adjoint restrictions of A_0^* — and therefore self-adjoint extensions of iA_0.

Theorem 3.7. *An extension A of A_0 is skew-self-adjoint if and only if there exists an ω-self-orthogonal subspace $X \subseteq \mathcal{G}_- \oplus \mathcal{G}_+$ for which $G(A) = F^{-1}(X)$, where F is the operator defined in* (3-3) *and $G(A)$ is the graph of A.*

Hence, ω-self-orthogonal subspaces X parametrize the skew-self-adjoint extensions A of A_0. A more explicit description of these objects is given next.

Theorem 3.8. (a) *Let $X \subseteq \mathcal{G}_- \oplus \mathcal{G}_+$ be a subspace such that $D := \{x \in \mathcal{G}_- : \exists y \in \mathcal{G}_+ \text{ with } (x, y) \in X\}$ is dense in $\mathcal{G}_-$ and $R := \{y \in \mathcal{G}_+ : \exists x \in \mathcal{G}_- \text{ with } (x, y) \in X\}$ is dense in $\mathcal{G}_+$, and let X be ω-self-orthogonal. Then there exists a $(\mathcal{G}_-, \mathcal{G}_+)$-unitary operator L such that $X = G(L)$.*

(b) *Let L be a $(\mathcal{G}_-, \mathcal{G}_+)$-unitary operator. Then $G(L) \subseteq \mathcal{G}_- \oplus \mathcal{G}_+$ is ω-self-orthogonal.*

Proof. (a) We first show that X is the graph of an operator. Let $(0, z) \in X$. For $(x, y) \in X$ we have

$$\omega((x, y), (0, z)) = 0;$$

i.e.,

$$0 = \langle y \mid z\rangle_+ = (B_+ y \mid z) = (y \mid B_+ z).$$

Thus, $B_+z \perp R$, and by the denseness of R we obtain $B_+z = 0$. Since B_+ is injective, we conclude $z = 0$. Thus, X is the graph of a linear operator L from $\mathcal{G}_-$ to $\mathcal{G}_+$. Let $(z, 0) \in X$. For $(x, y) \in X$ we obtain analogously

$$\omega((x, y), (z, 0)) = 0;$$

i.e.,

$$0 = \langle x \mid z\rangle_- = (x \mid B_- z).$$

Hence, $B_-z \perp D$, and by the denseness of D we have $B_-z = 0$. Since B_- is injective, $z = 0$. Thus, L is injective.

For $x, y \in D(L)$, we have $(x, Lx), (y, Ly) \in X$, so

$$\omega((x, Lx), (y, Ly)) = 0;$$

i.e.,

$$\langle Lx \mid Ly\rangle_+ = \langle x \mid y\rangle_-.$$

Let $x \in D(L)^\perp$. Then

$$(x \mid y) = 0, \quad y \in D(L).$$

Thus,

$$0 = (x \mid y) = (B_- B_-^{-1} x \mid y) = \langle B_-^{-1} x \mid y\rangle_- = \omega((B_-^{-1} x, 0), (y, Ly)), \quad y \in D(L).$$

Hence, $(B_-^{-1}x, 0) \in X^{\perp_\omega} = X = G(L)$, so $B_-^{-1}x = 0$, and therefore $x = 0$. Thus, L is densely defined. Similarly, we obtain that $R(L)$ is dense.

For $x \in D(L)$, $z \in R(L)$ we have

$$\langle Lx \mid z\rangle_+ = \langle x \mid L^{-1} z\rangle_-.$$

Thus, $R(L) \subseteq D(L^\sharp)$, and $L^\sharp z = L^{-1} z$ for all $z \in R(L)$; i.e., $L^{-1} \subseteq L^\sharp$. Let $(y, x) \in G(L^\sharp)$. Then

$$\langle Lz \mid y\rangle_+ = \langle z \mid x\rangle_-, \quad z \in D(L);$$

i.e.,

$$\omega((z, Lz), (x, y)) = 0, \quad z \in D(L).$$

Hence, $(x, y) \in X^{\perp_\omega} = X = G(L)$, and therefore $(y, x) \in G(L^{-1})$. Therefore, $L^\sharp = L^{-1}$, so L is $(\mathcal{G}_-, \mathcal{G}_+)$-unitary.

(b) Let L be a $(\mathcal{G}_-, \mathcal{G}_+)$-unitary operator from $\mathcal{G}_-$ to $\mathcal{G}_+$ and $X := G(L)$. Let $x \in D(L)$. Then

$$\langle Lx \mid Ly\rangle_+ = \langle x \mid y\rangle_-, \quad y \in D(L);$$

i.e.,

$$\omega((x, Lx), (y, Ly)) = 0, \quad y \in D(L).$$

Thus, $(x, Lx) \in X^{\perp_\omega}$, and therefore $X \subseteq X^{\perp_\omega}$.

Let now $(z, y) \in X^{\perp_\omega}$. Then, for $x \in D(L)$, we have

$$\omega((z, y), (x, Lx)) = 0,$$

and thus

$$\langle Lx \mid y\rangle_+ = \langle x \mid z\rangle_-.$$

By the definition of $L^\sharp$, we obtain $y \in D(L^\sharp)$ and $L^\sharp y = z$. Since $L^\sharp = L^{-1}$, we find $(z, y) \in G(L) = X$. Hence, $X^{\perp_\omega} \subseteq X$.

Combining both parts, we see that X is ω-self-orthogonal. □

Let L be a densely defined linear operator from $\mathcal{G}_-$ to $\mathcal{G}_+$. Then we define $A_L \subseteq -A_0^*$ by

$$G(A_L) := F^{-1}(G(L));$$

that is,

$$\begin{aligned} D(A_L) = \big\{u \in D(A_0^*) : \big(u(0-), u'(0-), u''(0-)\big) \in D(L),& \\ L\big(u(0-), u'(0-), u''(0-)\big) = \big(u(0+), u'(0+), u''(0+)\big)\big\},& \\ A_L u = -A_0^* u.& \end{aligned}$$

Theorem 3.9. *Let L be a densely defined linear operator from $\mathcal{G}_-$ to $\mathcal{G}_+$. Then A_L is the generator of a unitary group if and only if L is $(\mathcal{G}_-, \mathcal{G}_+)$-unitary.*

Proof. By Theorem 3.7 we obtain that A_L is skew-self-adjoint if and only if $G(L)$ is ω-self-orthogonal. By Theorem 3.8 we conclude that this is equivalent to L being $(\mathcal{G}_-, \mathcal{G}_+)$-unitary. Indeed, Theorem 3.8(b) yields that if L is $(\mathcal{G}_-, \mathcal{G}_+)$-unitary then $G(L)$ is ω-self-orthogonal. As in proof of Theorem 3.8(a) one shows that if $X := G(L)$ is ω-self-orthogonal then the corresponding operator L is $(\mathcal{G}_-, \mathcal{G}_+)$-unitary. □

Remark 3.10. If Theorem 3.7 applies, then Stone's theorem immediately yields that the Airy equation (1-1) on the quantum star graph G is governed by a unitary group acting on $L^2(\mathsf{G})$; hence it has a unique solution $u \in C^1(\mathbb{R}; L^2(\mathsf{G})) \cap C(\mathbb{R}; D(A))$ that continuously depends on the initial data $u_0 \in L^2(\mathsf{G})$. Because the group is unitary, the momentum $\|u\|^2_{L^2(\mathsf{G})}$ is conserved as soon as we can apply Theorem 3.7. By the above computation we also see that

$$\begin{aligned} \frac{\partial}{\partial t}\int_{\mathsf{G}} u(t,x)\,dx &= \sum_{\mathsf{e}\in\mathsf{E}_-}\int_{-\infty}^{0} \alpha_\mathsf{e} u_\mathsf{e}'''(t,x) + \beta_\mathsf{e} u_\mathsf{e}'(t,x)\,dx + \sum_{\mathsf{e}\in\mathsf{E}_-}\int_{0}^{\infty} \alpha_\mathsf{e} u_\mathsf{e}'''(t,x) + \beta_\mathsf{e} u_\mathsf{e}'(t,x)\,dx \\ &= \sum_{\mathsf{e}\in\mathsf{E}_-} \alpha_\mathsf{e} u_\mathsf{e}''(0-) - \sum_{\mathsf{e}\in\mathsf{E}_+} \alpha_\mathsf{e} u_\mathsf{e}''(0+) + \sum_{\mathsf{e}\in\mathsf{E}_-} \beta_\mathsf{e} u_\mathsf{e}(0-) - \sum_{\mathsf{e}\in\mathsf{E}_+} \beta_\mathsf{e} u_\mathsf{e}(0+). \end{aligned} \tag{3-7}$$

In other words, the solution of the system enjoys conservation of mass — just like the solution to the classical Airy equation on $\mathbb{R}$ — if and only if additionally

$$\sum_{\mathsf{e}\in\mathsf{E}_-} \alpha_\mathsf{e} u_\mathsf{e}''(0-) - \sum_{\mathsf{e}\in\mathsf{E}_+} \alpha_\mathsf{e} u_\mathsf{e}''(0+) + \sum_{\mathsf{e}\in\mathsf{E}_-} \beta_\mathsf{e} u_\mathsf{e}(0-) - \sum_{\mathsf{e}\in\mathsf{E}_+} \beta_\mathsf{e} u_\mathsf{e}(0+) = 0. \tag{3-8}$$

Remark 3.11. Since $\beta_\pm$, $\alpha_\pm$, $1/\alpha_\pm$ are bounded, the form ω introduced in (3-4) is continuous. Thus, ω-self-orthogonal subspaces are closed, so the corresponding $(\mathcal{G}_-, \mathcal{G}_+)$-unitary L is closed. We do not know whether L is in fact continuous (this holds true in Hilbert spaces, but we are not aware of any argument in Krein spaces).

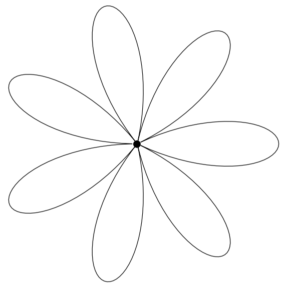

Figure 2. A flower graph on seven edges.

Remark 3.12. As already remarked at the beginning of this section, star graphs can be seen as generic building blocks of quantum graphs. Apart from their interest in scattering theory, star graphs whose edges are semi-infinite still display all relevant features for the purpose of studying extensions of operators on compact graphs. Indeed, our analysis is essentially of variational nature and therefore it only depends on the orientation of the edges and the boundary values of a function on the graph. It is therefore clear that analogous results could be formulated for graphs that include edges of finite length, too, like the *flower graph* depicted in Figure 2. Clearly, an interesting feature of flower graphs is that they are automatically balanced; i.e., the number of incoming and outgoing edges from the (only) vertex is equal: accordingly, the operator A_0 on a flower graph always admits skew-self-adjoint extensions.

Extensions of A_0 generating contraction semigroups. Let A be an extension of A_0 such that $A \subseteq -A_0^*$. We focus on generating contraction semigroups. By the Lumer–Phillips theorem and corollaries of it we have to show that A and A^* are dissipative. Since we are dealing with Hilbert spaces, A is dissipative if and only if $\operatorname{Re}(Au \mid u) \le 0$ for all $u \in D(A)$, and analogously for A^*. Recall, that for a densely defined linear operator L from $\mathcal{G}_-$ to $\mathcal{G}_+$ we defined $A_L \subseteq -A_0^*$ by $G(A_L) = F^{-1}(G(L))$.

Lemma 3.13. *Let L be a densely defined linear operator from $\mathcal{G}_-$ to $\mathcal{G}_+$. Then A_L is dissipative if and only if L is a $(\mathcal{G}_-, \mathcal{G}_+)$-contraction.*

Proof. Let $u \in D(A_L)$. Then

$$\begin{aligned}-2\operatorname{Re}(A_L u \mid u) &= \Omega((u, -A_L u), (u, -A_L u)) \\ &= \big\langle \big(u(0-), u'(0-), u''(0-)\big) \mid \big(u(0-), u'(0-), u''(0-)\big)\big\rangle_- \\ &\qquad - \big\langle L\big(u(0-), u'(0-), u''(0-)\big) \mid L\big(u(0-), u'(0-), u''(0-)\big)\big\rangle_+ .\end{aligned}$$

Thus, A_L is dissipative if and only if

$$\begin{aligned}&\big\langle L\big(u(0-), u'(0-), u''(0-)\big) \mid L\big(u(0-), u'(0-), u''(0-)\big)\big\rangle_+ \\ &\qquad \le \big\langle \big(u(0-), u'(0-), u''(0-)\big) \mid \big(u(0-), u'(0-), u''(0-)\big)\big\rangle_- \quad \text{for all } u \in D(A_L).\end{aligned}$$

By the definition of A_L we have $D(L) = \big\{\big(u(0-), u'(0-), u''(0-)\big) : u \in D(A_L)\big\}$, so the assertion follows. □

Analogously, we obtain a characterization for dissipativity of the adjoint A_L^* of A_L. Here and in the following, $L^\sharp$ denotes the $(\mathcal{G}_-, \mathcal{G}_+)$-adjoint of the operator L; see Definition 3.6.

Lemma 3.14. *Let L be a densely defined linear operator from $\mathcal{G}_-$ to $\mathcal{G}_+$. Then*

$$\begin{aligned} D(A_L^*) = \big\{u \in D(A_0^*) : \big(u(0+), u'(0+), u''(0+)\big) \in D(L^\sharp),\\ L^\sharp\big(u(0+), u'(0+), u''(0+)\big) = \big(u(0-), u'(0-), u''(0-)\big)\big\},\\ A_L^* u = A_0^* u. \end{aligned}$$

Proof. Let $u \in D(A_L)$, $v \in D(A_0^*)$. Then

$$\begin{aligned} (A_L u \mid v) = (u \mid A_0^* v) - \big\langle \big(u(0-), u'(0-), u''(0-)\big) \mid \big(v(0+), v'(0+), v''(0+)\big)\big\rangle_-\\ + \big\langle L\big(u(0-), u'(0-), u''(0-)\big) \mid \big(v(0+), v'(0+), v''(0+)\big)\big\rangle_+. \end{aligned}$$

Hence, $v \in D(A_L^*)$ if and only if $\big(v(0+), v'(0+), v''(0+)\big) \in D(L^\sharp)$ and

$$L^\sharp\big(v(0+), v'(0+), v''(0+)\big) = \big(v(0-), v'(0-), v''(0-)\big),$$

and then $A_L^* v = A_0^* v$. □

Lemma 3.15. *Let L be a densely defined linear operator from $\mathcal{G}_-$ to $\mathcal{G}_+$. Then A_L^* is dissipative if and only if $L^\sharp$ is a $(\mathcal{G}_+, \mathcal{G}_-)$-contraction.*

Proof. Let $u \in D(A_L^*)$. Then

$$\begin{aligned} 2\,\mathrm{Re}\,(A_L^* u \mid u) &= \Omega((u, A_L^* u), (u, A_L^* u))\\ &= \big\langle L^\sharp\big(u(0+), u'(0+), u''(0+)\big) \mid L^\sharp\big(u(0+), u'(0+), u''(0+)\big)\big\rangle_-\\ &\quad - \big\langle \big(u(0+), u'(0+), u''(0+)\big) \mid \big(u(0+), u'(0+), u''(0+)\big)\big\rangle_+. \end{aligned}$$

Thus, A_L^* is dissipative if and only if

$$\begin{aligned} &\big\langle L^\sharp\big(u(0+), u'(0+), u''(0+)\big) \mid L^\sharp\big(u(0+), u'(0+), u''(0^+)\big)\big\rangle_-\\ &\qquad \le \big\langle \big(u(0+), u'(0+), u''(0+)\big) \mid \big(u(0+), u'(0+), u''(0+)\big)\big\rangle_+ \quad \text{for all } u \in D(A_L). \end{aligned}$$

By the definition of A_L we have $D(L^\sharp) = \big\{\big(u(0+), u'(0+), u''(0+)\big) : u \in D(A_L)\big\}$, so the assertion follows. □

Theorem 3.16. *Let L be a densely defined linear operator from $\mathcal{G}_-$ to $\mathcal{G}_+$. Then A_L is the generator of a contraction semigroup if and only if L is a $(\mathcal{G}_-, \mathcal{G}_+)$-contraction and $L^\sharp$ is a $(\mathcal{G}_+, \mathcal{G}_-)$-contraction.*

Proof. Let A_L generate a C_0-semigroup $(T(t))_{t\ge 0}$ of contractions. By the Lumer–Phillips theorem, A_L is dissipative. Since A_L^* generates the semigroup T^* defined by $T^*(t) := T(t)^*$ $(t \ge 0)$, which is also a C_0-semigroup of contractions, the Lumer–Phillips theorem assures that A_L^* is dissipative as well. Then the Lemmas 3.13 and 3.15 yield that L and $L^\sharp$ are contractions.

Now, let L and $L^\sharp$ be contractions. Then Lemmas 3.13 and 3.15 yield that A_L and A_L^* are dissipative. Hence, A_L generates a contraction semigroup. □

Corollary 3.17. *Let* $|\mathsf{E}_-| = |\mathsf{E}_+| < \infty$, $\alpha_+ = \alpha_-$, $\beta_+ = \beta_-$ *(via some bijection between* E_- *and* E_+*). Let* L *be a densely defined linear operator from* $\mathcal{G}_-$ *to* $\mathcal{G}_+$*. Then* A_L *is the generator of a contraction semigroup if and only if* L *is* $(\mathcal{G}_-, \mathcal{G}_+)$*-contractive.*

Proof. In this setup, $\mathcal{G}_- = \mathcal{G}_+$, which are finite-dimensional and hence Pontryagin spaces. Therefore, L is $(\mathcal{G}_-, \mathcal{G}_+)$-contractive if and only if $L^\sharp$ is $(\mathcal{G}_+, \mathcal{G}_-)$-contractive by [Dritschel and Rovnyak 1990, Theorem 1.3.7]. □

Remark 3.18. Like in the case of Remark 3.10, if Theorem 3.16 applies, then the Airy equation (1-1) on the quantum star graph G has a unique solution $u \in C^1(\mathbb{R}_+; L^2(\mathsf{G})) \cap C(\mathbb{R}_+; D(A))$ that continuously depends on the initial data $u_0 \in L^2(\mathsf{G})$. Because the C_0-semigroup is contractive but not unitary, the momentum $\|u\|^2_{L^2(\mathsf{G})}$ is in general not conserved, but as in Remark 3.10 the system does enjoy conservation of mass if and only if additionally (3-8) holds.

Separating the first derivatives. The special structure of $B_\pm$ suggests to separate the boundary values of the first derivative from the ones for the function and for the second derivative. In this case, one can describe the boundary conditions also in another (equivalent but seemingly easier) way.

Note that $\alpha_\mathsf{e} > 0$ for all $\mathsf{e} \in \mathsf{E}$. We will write $\ell^2(\mathsf{E}_\pm, \alpha_\pm)$ for the weighted ℓ^2-space of sequences indexed by $\mathsf{E}_\pm$ with inner product given by

$$(x \mid y)_{\ell^2(\mathsf{E}_\pm,\alpha_\pm)} := \sum_{\mathsf{e}\in\mathsf{E}_\pm} x_\mathsf{e} \bar{y}_\mathsf{e} \alpha_\mathsf{e} = (\alpha_\pm x \mid y)$$

for all $x, y \in \ell^2(\mathsf{E}_\pm, \alpha_\pm)$, which turns them into Hilbert spaces.

For $u, v \in D(A_0^*)$ we then obtain

$$\begin{aligned}
&(A_0^* u \mid v) + (u \mid A_0^* v) \\
&\quad = \big(\alpha_- u'(0-) \mid v'(0-)\big) - \big(\alpha_+ u'(0+) \mid v'(0+)\big) \\
&\qquad + \left(\begin{pmatrix} -\beta_- & -\alpha_- \\ -\alpha_- & 0 \end{pmatrix} \begin{pmatrix} u(0-) \\ u''(0-) \end{pmatrix} \,\middle|\, \begin{pmatrix} v(0-) \\ v''(0-) \end{pmatrix} \right) - \left(\begin{pmatrix} -\beta_+ & -\alpha_+ \\ -\alpha_+ & 0 \end{pmatrix} \begin{pmatrix} u(0+) \\ u''(0+) \end{pmatrix} \,\middle|\, \begin{pmatrix} v(0+) \\ v''(0+) \end{pmatrix} \right) \\
&\quad = \big(\alpha_- u'(0-) \mid v'(0-)\big) - \big(\alpha_+ u'(0+) \mid v'(0+)\big) \\
&\qquad - \left(\begin{pmatrix} \alpha_- & 0 \\ 0 & -\alpha_+ \end{pmatrix} \begin{pmatrix} u''(0-) \\ u''(0+) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\beta_- & 0 \\ 0 & -\frac{1}{2}\beta_+ \end{pmatrix} \begin{pmatrix} u(0-) \\ u(0+) \end{pmatrix} \,\middle|\, \begin{pmatrix} v(0-) \\ v(0+) \end{pmatrix} \right) \\
&\qquad - \left(\begin{pmatrix} u(0-) \\ u(0+) \end{pmatrix} \,\middle|\, \begin{pmatrix} \alpha_- & 0 \\ 0 & -\alpha_+ \end{pmatrix} \begin{pmatrix} v''(0-) \\ v''(0+) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\beta_- & 0 \\ 0 & -\frac{1}{2}\beta_+ \end{pmatrix} \begin{pmatrix} v(0-) \\ v(0+) \end{pmatrix} \right).
\end{aligned}$$

Let $Y \subseteq \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_+)$ be a closed subspace, U a densely defined linear operator from $\ell^2(\mathsf{E}_-, \alpha_-)$ to $\ell^2(\mathsf{E}_+, \alpha_+)$, and consider

$$\begin{aligned}
D(A_{Y,U}) := \Big\{ & u \in D(A_0^*) : (u(0-), u(0+)) \in Y, \\
& \begin{pmatrix} -\alpha_- & 0 \\ 0 & \alpha_+ \end{pmatrix} \begin{pmatrix} u''(0-) \\ u''(0+) \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}\beta_- & 0 \\ 0 & \frac{1}{2}\beta_+ \end{pmatrix} \begin{pmatrix} u(0-) \\ u(0+) \end{pmatrix} \in Y^\perp, \\
& u'(0-) \in D(U),\ u'(0+) = U u'(0-) \Big\}, \\
A_{Y,U} u := {} & -A_0^* u.
\end{aligned}$$

Proposition 3.19. *Let $Y \subseteq \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_+)$ be a closed subspace and U a densely defined linear operator from $\ell^2(\mathsf{E}_-, \alpha_-)$ to $\ell^2(\mathsf{E}_+, \alpha_+)$. Then*

$$D(A_{Y,U}^*) = \Big\{u \in D(A_0^*) : (u(0-), u(0+)) \in Y, \begin{pmatrix} -\alpha_- & 0 \\ 0 & \alpha_+ \end{pmatrix} \begin{pmatrix} u''(0-) \\ u''(0+) \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}\beta_- & 0 \\ 0 & \frac{1}{2}\beta_+ \end{pmatrix} \begin{pmatrix} u(0-) \\ u(0+) \end{pmatrix} \in Y^\perp, u'(0+) \in D(U^*),\ u'(0-) = U^* u'(0+) \Big\},$$
$$A_{Y,U}^* u = A_0^* u.$$

Proof. Let $u \in D(A_{Y,U}),\ v \in D(A_0^*)$. Then

$$\begin{aligned}(A_{Y,U} u \mid v) &= (u \mid A_0^* v) + \big(\alpha_+ U u'(0-) \,\big|\, v'(0+)\big) - \big(\alpha_- u'(0-) \,\big|\, v'(0-)\big) \\ &\quad + \left(\begin{pmatrix} \alpha_- & 0 \\ 0 & -\alpha_+ \end{pmatrix} \begin{pmatrix} u''(0-) \\ u''(0+) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\beta_- & 0 \\ 0 & -\frac{1}{2}\beta_+ \end{pmatrix} \begin{pmatrix} u(0-) \\ u(0+) \end{pmatrix} \,\middle|\, \begin{pmatrix} v(0-) \\ v(0+) \end{pmatrix} \right) \\ &\quad + \left(\begin{pmatrix} u(0-) \\ u(0+) \end{pmatrix} \,\middle|\, \begin{pmatrix} \alpha_- & 0 \\ 0 & -\alpha_+ \end{pmatrix} \begin{pmatrix} v''(0-) \\ v''(0+) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\beta_- & 0 \\ 0 & -\frac{1}{2}\beta_+ \end{pmatrix} \begin{pmatrix} v(0-) \\ v(0+) \end{pmatrix} \right).\end{aligned}$$

Let now $v \in D(A_{Y,U}^*)$ such that $(A_{Y,U} u \mid v) = (u \mid A_{Y,U}^* v) = (u \mid A_0^* v)$. Choosing u such that $(u(0-), u(0+)) = (u'(0-), u'(0+)) = 0$, we obtain $(v(0-), v(0+)) \in Y$. For all $(u_-, u_+) \in Y$ there exists $u \in D(A_{Y,U})$ such that $(u(0-), u(0+)) = (u_-, u_+)$, and $(u'(0-), u'(0+)) = 0$. Thus,

$$\begin{pmatrix} \alpha_- & 0 \\ 0 & -\alpha_+ \end{pmatrix} \begin{pmatrix} v''(0-) \\ v''(0+) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\beta_- & 0 \\ 0 & -\frac{1}{2}\beta_+ \end{pmatrix} \begin{pmatrix} v(0-) \\ v(0+) \end{pmatrix} \in Y^\perp.$$

Thus, we arrive at

$$\big(\alpha_+ U u'(0-) \,\big|\, v'(0+)\big) - \big(\alpha_- u'(0-) \,\big|\, v'(0-)\big) = 0, \quad u \in D(A_{Y,U}).$$

Note that for all $x \in D(U)$ there exists $u \in D(A_{Y,U})$ such that $u'(0-) = x$. Hence, $v'(0+) \in D(U^*)$ and $U^* v'(0+) = v'(0-)$. □

Corollary 3.20. *Let $Y \subseteq \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_+)$ be a closed subspace and U a densely defined linear operator from $\ell^2(\mathsf{E}_-, \alpha_-)$ to $\ell^2(\mathsf{E}_+, \alpha_+)$. Then $A_{Y,U}$ is skew-self-adjoint if and only if U is unitary.*

Proof. This is a direct consequence of Proposition 3.19. □

Corollary 3.21. *Let $Y \subseteq \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_+)$ be a closed subspace and U a densely defined linear operator from $\ell^2(\mathsf{E}_-, \alpha_-)$ to $\ell^2(\mathsf{E}_+, \alpha_+)$. Then $A_{Y,U}$ is dissipative if and only if U is a contraction.*

Proof. Let $u \in D(A_{Y,U})$. Then

$$(A_{Y,U} u \mid u) = (u \mid A_0^* u) + \big(\alpha_+ U u'(0-) \,\big|\, U u'(0-)\big) - \big(\alpha_- u'(0-) \,\big|\, u'(0-)\big).$$

Since $A_0^* u = -A_{Y,U} u$, we obtain

$$2 \operatorname{Re} (A_{Y,U} u \mid u) = \big(\alpha_+ U u'(0-) \,\big|\, U u'(0-)\big) - \big(\alpha_- u'(0-) \,\big|\, u'(0-)\big).$$

Hence, $A_{Y,U}$ is dissipative if and only if U is a contraction. □

Corollary 3.22. *Let $Y \subseteq \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_+)$ be a closed subspace and U a densely defined linear operator from $\ell^2(\mathsf{E}_-, \alpha_-)$ to $\ell^2(\mathsf{E}_+, \alpha_+)$. Then $A^*_{Y,U}$ is dissipative if and only if U^* is a contraction.*

Proof. Let $u \in A^*_{Y,U}$. Then, similarly as for $A_{Y,U}$, we have

$$(A^*_{Y,U}u \mid u) = (u \mid -A^*_0 u) - \big(\alpha_+ u'(0+) \,\big|\, u'(0+)\big) + \big(\alpha_- U^* u'(0+) \,\big|\, U^* u'(0+)\big).$$

Since $A^*_0 u = A^*_{Y,U}u$, we obtain

$$2\,\mathrm{Re}\,(A_{Y,U}u \mid u) = \big(\alpha_- U^* u'(0+) \mid U^* u'(0+)\big) - \big(\alpha_+ u'(0+) \,\big|\, u'(0+)\big).$$

Hence, $A^*_{Y,U}$ is dissipative if and only if U^* is a contraction. □

Theorem 3.23. *Let $Y \subseteq \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_+)$ be a closed subspace and U a densely defined linear operator from $\ell^2(\mathsf{E}_-, \alpha_-)$ to $\ell^2(\mathsf{E}_+, \alpha_+)$. Then $A_{Y,U}$ generates a contraction C_0-semigroup if and only if U is a contraction and U^* is a contraction if and only if U is a contraction.*

Proof. Let $A_{Y,U}$ generate a contraction semigroup. Then $A_{Y,U}$ is dissipative by the Lumer–Phillips theorem, so Corollary 3.21 yields that U is a contraction.

Let U and U^* be contractions. Then, Corollaries 3.21 and 3.22 ensure that $A_{Y,U}$ and $A^*_{Y,U}$ are dissipative. Hence, $A_{Y,U}$ generates a semigroup of contractions. Clearly, U^* is a contraction provided U is a contraction. □

Reality of the semigroup. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on $L^2(\mathsf{G})$. We say that $(T(t))_{t\geq 0}$ is *real* if $T(t)\,\mathrm{Re}\,u = \mathrm{Re}\,T(t)u$ for all $u \in L^2(\mathsf{G})$ and $t \geq 0$. Put differently, a semigroup is real if it maps real-valued functions into real-valued functions.

For simplicity we will only consider the case of contractive semigroups here.

Proposition 3.24. *Let L be a densely defined linear operator from $\mathcal{G}_-$ to $\mathcal{G}_+$ such that L and $L^\sharp$ are contractions. Let $(T(t))_{t\geq 0}$ be the C_0-semigroup generated by A_L. Let L be real*; *i.e., for $x \in D(L)$ we have* $\mathrm{Re}\,x \in D(L)$ *and* $\mathrm{Re}\,Lx = L\,\mathrm{Re}\,x$. *Then $(T(t))_{t\geq 0}$ is real.*

Proof. By Theorem 3.16, $(T(t))_{t\geq 0}$ is a contraction semigroup. Let P be the projection from $L^2(\mathsf{G};\mathbb{C})$ to $L^2(\mathsf{G};\mathbb{R})$; i.e., $Pu := \mathrm{Re}\,u$. By [Arendt et al. 2015, Corollary 9.6] realness of the semigroup $(T(t))_{t\geq 0}$ is equivalent to the condition

$$\mathrm{Re}\,(A_L u \mid u - Pu) \leq 0 \quad \text{for all } u \in D(A_L),$$

i.e.,

$$\mathrm{Re}\,(A_L u \mid i\,\mathrm{Im}\,u) \leq 0 \quad \text{for all } u \in D(A_L).$$

Let $u \in D(A_L)$. Since L is real we obtain $\mathrm{Re}\,u, \mathrm{Im}\,u \in D(A_L)$. Since $A_L v$ is real for $v \in D(A_L)$ real we have

$$\begin{aligned}\mathrm{Re}\,(A_L(\mathrm{Re}\,u + i\,\mathrm{Im}\,u) \mid i\,\mathrm{Im}\,u) &= \mathrm{Im}\,(A_L\,\mathrm{Re}\,u \mid \mathrm{Im}\,u) + \mathrm{Re}\,(A_L\,\mathrm{Im}\,u \mid \mathrm{Im}\,u)\\ &= \mathrm{Re}\,(A_L\,\mathrm{Im}\,u \mid \mathrm{Im}\,u) \leq 0,\end{aligned}$$

since A_L is dissipative by Lemma 3.13. □

Remark 3.25. The semigroup generated by A_L is not positivity preserving, i.e., nonnegative functions need not be mapped to nonnegative functions: indeed, also in this case positivity of the semigroup is again equivalent to

$$\mathrm{Re}\,(A_L u \mid u - Pu) \le 0 \quad \text{for all } u \in D(A_L), \tag{3-9}$$

where P is now the projection of $L^2(\mathsf{G};\mathbb{C})$ onto the positive cone of $L^2(\mathsf{G};\mathbb{R})$, i.e., $Pu := (\mathrm{Re}\,u)_+$. Let u be a real-valued function: integrating by parts and neglecting without loss of generality the transmission conditions (due to locality of the operator), one sees that

$$\mathrm{Re}\,(A_L u \mid u - Pu) = -\int_{\mathsf{G}} u''' u_-\,dx = \int_{\{u\le 0\}} u''' u\,dx = -\tfrac{1}{2}|u'|^2\big|_{\partial\{u\le 0\}}.$$

Of course, wherever an H^3-function changes sign its first derivative need not vanish, so condition (3-9) cannot be satisfied.

Analogously, the semigroup is also not L^∞-contractive, i.e., the inequality $\|e^{tA_L}u\|_\infty \le \|u\|_\infty$ fails for some $u \in L^2(\mathsf{G}) \cap L^\infty(\mathsf{G})$ and some $t \ge 0$. In this case, the relevant projection onto the closed convex subset $C := \{u \in L^2(\mathsf{G}) : |u| \le 1\}$ of $L^2(\mathsf{G})$ is defined by $Pu := (|u| \wedge 1)\,\mathrm{sgn}\,u$; hence

$$u - Pu := (|u| - 1)_+\,\mathrm{sgn}\,u.$$

We also obtain realness of $(T(t))_{t\ge 0}$ in the case of separated boundary conditions.

Proposition 3.26. *Let $Y \subseteq \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_+)$ be a closed subspace and $U : \ell^2(\mathsf{E}_-,\alpha_-) \to \ell^2(\mathsf{E}_+,\alpha_+)$ be linear and contractive. Let $(T(t))_{t\ge 0}$ be the C_0-semigroup generated by $A_{Y,U}$. Assume that* $(\mathrm{Re}\,x, \mathrm{Re}\,y) \in Y$ *for all* $(x, y) \in Y$ *and U is real, i.e.,* $\mathrm{Re}\,Ux = U\,\mathrm{Re}\,x$ *for all $x \in \ell^2(\mathsf{E}_-,\alpha_-)$. Then $(T(t))_{t\ge 0}$ is real.*

Proof. By Theorem 3.23, $(T(t))_{t\ge 0}$ is a contraction semigroup. Let P be the projection from $L^2(\mathsf{G};\mathbb{C})$ to $L^2(\mathsf{G};\mathbb{R})$; i.e., $Pu := \mathrm{Re}\,u$. By [Arendt et al. 2015, Corollary 9.6] realness of the semigroup $(T(t))_{t\ge 0}$ is equivalent to

$$\mathrm{Re}\,(A_{Y,U} u \mid u - Pu) \le 0 \quad \text{for all } u \in D(A_{Y,U}),$$

i.e.,

$$\mathrm{Re}\,(A_{Y,U} u \mid i\,\mathrm{Im}\,u) \le 0 \quad \text{for all } u \in D(A_{Y,U}).$$

Let $u \in D(A_{Y,U})$. The assumptions imply $\mathrm{Re}\,u \in D(A_{Y,U})$, and therefore also $\mathrm{Im}\,u \in D(A_{Y,U})$. Since $A_{Y,U}v$ is real for $v \in D(A_{Y,U})$ real we have

$$\begin{aligned}\mathrm{Re}\,(A_{Y,U}(\mathrm{Re}\,u + i\,\mathrm{Im}\,u) \mid i\,\mathrm{Im}\,u) &= \mathrm{Im}\,(A_{Y,U}\,\mathrm{Re}\,u \mid \mathrm{Im}\,u) + \mathrm{Re}\,(A_{Y,U}\,\mathrm{Im}\,u \mid \mathrm{Im}\,u)\\ &= \mathrm{Re}\,(A_{Y,U}\,\mathrm{Im}\,u \mid \mathrm{Im}\,u) \le 0,\end{aligned}$$

since $A_{Y,U}$ is dissipative by Corollary 3.21. □

Remark 3.27. We end this section with some considerations about the boundary conditions used in the literature related to the subject.

In the classical treatise [Stoker 1957] the author, reviewing the much-studied case study of the confluence between the Ohio and Mississippi rivers, proposed that the boundary conditions suitable for branching water flows are the ones compatible with continuity and mass conservation. As we know, there are many. This assumption was essentially undisputed in the not-so-abundant literature on the subject; see also the interesting paper [Jacovkis 1991] for further information. Recently, in [Nachbin and Simões 2012; 2015] a more careful analysis has been put forth by the authors attempting a 1-dimensional reduction from a fluid dynamical model in a 2-dimensional setting; they question the above traditional point of view about the more convenient boundary conditions, in particular continuity.

We finally notice that in the few recent mathematical papers concerning the Airy or KdV equation on graphs, [Sobirov et al. 2015a; 2015b; 2015c; Ammari and Crépeau 2017; Cavalcante 2017], only very special examples of boundary conditions are considered, essentially without explanation. All of them turn out to be of the separated derivative type studied in this section.

4. Examples

The case of two half-lines. First, let us consider the case of the real line with a singular interaction at the origin, i.e., $|\mathsf{E}_-| = |\mathsf{E}_+| = 1$; see Figure 3.

We will describe the operator explicitly in the case where the first derivative is separated.

Let $Y \subseteq \mathbb{C}^2$ be a subspace and $U : \ell^2(\mathsf{E}_-, \alpha_-) \to \ell^2(\mathsf{E}_+, \alpha_+)$ be linear, i.e., $U \in \mathbb{C}$. Note that U is contractive if and only if $|U|^2\alpha_+ \le \alpha_-$.

Example 4.1. Let $Y := \{(0, 0)\}$. Then

$$D(A_{Y,U}) = \{u \in D(A_0^*) : u(0-) = u(0+) = 0,\ u'(0+) = Uu'(0-)\},$$
$$A_{Y,U}u = -A_0^*u.$$

By Corollary 3.20, $A_{Y,U}$ generates a unitary group provided $|U|^2 = \alpha_-/\alpha_+$. By Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions provided $|U|^2 \le \alpha_-/\alpha_+$. Observe that if we take $U = 0$, we are effectively reducing the Airy equation on the star graph G to decoupled Airy equations on two half-lines $(0, \infty)$ and $(-\infty, 0)$ with Dirichlet conditions (for both equations) and Neumann (on the positive half-line only) boundary conditions. The Airy equation on either of these half-lines with the above boundary conditions has been considered often in the literature; see, e.g., [Holmer 2006; Fokas et al. 2016].

Example 4.2. Let $Y := \operatorname{lin}\{(0, 1)\}$. Then

$$D(A_{Y,U}) = \{u \in D(A_0^*) : u(0-) = 0,\ u''(0+)\alpha_+ = -\tfrac{1}{2}\beta_+u(0+),\ u'(0+) = Uu'(0-)\},$$
$$A_{Y,U}u = -A_0^*u.$$

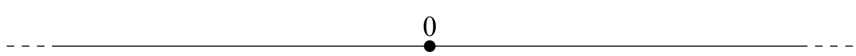

Figure 3. A graph consisting of two half-lines.

If $U = 0$, these transmission conditions can be interpreted as a reduction of the system to two decoupled half-lines: a Dirichlet condition is imposed on one of them, while a transmission condition that is the third-order counterpart of a Robin condition is imposed on the other one, along with a classical Neumann condition.

By Corollary 3.20, $A_{Y,U}$ generates a unitary group provided $|U|^2 = \alpha_-/\alpha_+$. By Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions provided $|U|^2 \le \alpha_-/\alpha_+$.

Example 4.3. Let $Y := \operatorname{lin}\{(1,0)\}$. Then

$$D(A_{Y,U}) = \{u \in D(A_0^*) : u(0+) = 0,\ u''(0-)\alpha_- = -\tfrac{1}{2}\beta_- u(0-),\ u'(0+) = Uu'(0-)\},$$
$$A_{Y,U}u = -A_0^* u.$$

If $U = 0$, these transmission conditions can be interpreted as a reduction of the system to two decoupled half-lines: Dirichlet and Neumann conditions are imposed on one of them, while the analog of a Robin condition is imposed on the other one. By Corollary 3.20, $A_{Y,U}$ generates a unitary group provided $|U|^2 = \alpha_-/\alpha_+$. By Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions provided $|U|^2 \le \alpha_-/\alpha_+$.

Due to lack of conditions on $u''(0+)$ and/or $u''(0-)$, (3-8) cannot be generally satisfied in any of the previous three cases and therefore the corresponding systems do not enjoy conservation of mass.

Example 4.4. Let $Y := \operatorname{lin}\{(1,1)\}$. Then

$$D(A_{Y,U}) = \{u \in D(A_0^*) : u(0-) = u(0+) =: u(0),$$
$$u''(0+)\alpha_+ - u''(0-)\alpha_- = \tfrac{1}{2}(\beta_- - \beta_+)u(0),\ u'(0+) = Uu'(0-)\},$$
$$A_{Y,U}u = -A_0^* u.$$

By Corollary 3.20, $A_{Y,U}$ generates a unitary group provided $|U|^2 = \alpha_-/\alpha_+$: observe that this is in particular the case if $\alpha_+ = \alpha_-$, $\beta_+ = \beta_-$ and $U = 1$, meaning that not only u, but also u' and u'' are continuous in the origin: this is the classical case considered in the literature and amounts to the free Airy equation on $\mathbb{R}$; see, e.g., the summary in [Linares and Ponce 2009, §7.1]. In view of Remark 3.10, generation of a mass-preserving unitary group still holds under the more general assumption that $U = e^{i\phi}$ for some $\phi \in [0, 2\pi)$, while in view of the prescribed transmission conditions, (3-8) cannot be satisfied unless $\beta_+ = \beta_-$. Even upon dropping the assumption that $\beta_+ = \beta_-$ we obtain the third-order counterpart of a δ-interaction, under which generation of a unitary group is still given.

On the other hand, by Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions already under the weaker assumption that $|U|^2 \le \alpha_-/\alpha_+$.

Example 4.5. Let $Y := \operatorname{lin}\{(1,-1)\}$. Then

$$D(A_{Y,U}) = \{u \in D(A_0^*) : u(0-) = -u(0+),$$
$$u''(0+)\alpha_+ + u''(0-)\alpha_- = \tfrac{1}{2}(\beta_+ - \beta_-)u(0-),\ u'(0+) = Uu'(0-)\},$$
$$A_{Y,U}u = -A_0^* u.$$

By Corollary 3.20, $A_{Y,U}$ generates a unitary group provided $|U|^2 = \alpha_-/\alpha_+$: we can regard this case as a third-order counterpart of δ'-interactions of second-order operators. By (3-8), the system enjoys conservation of mass if and only if additionally

$$2\alpha_+ u''(0+) + \tfrac{1}{2}(\beta_+ - \beta_-)u(0-) = 0, \tag{4-1}$$

which is generally not satisfied (remember that we are not considering the degenerate case $\alpha = 0$).

By Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions provided $|U|^2 \le \alpha_-/\alpha_+$.

Example 4.6. Let $Y := \mathbb{C}^2$. Then

$$D(A_{Y,U}) = \{u \in D(A_0^*) : u''(0-)\alpha_- = -\tfrac{1}{2}\beta_- u(0-),\ u''(0+)\alpha_+ = -\tfrac{1}{2}\beta_+ u(0+),\ u'(0+) = Uu'(0-)\},$$
$$A_{Y,U}u = -A_0^* u.$$

These transmission conditions amount to considering two decoupled systems, each with Robin-like conditions along with a Neumann condition on one of them. By Corollary 3.20, $A_{Y,U}$ generates a unitary group provided $|U|^2 = \alpha_-/\alpha_+$; by (3-8), the system enjoys conservation of mass if and only if additionally

$$-\tfrac{1}{2}\beta_- u(0-) + \tfrac{1}{2}\beta_+ u(0+) + \beta_- u(0-) - \beta_+ u(0+) = 0, \tag{4-2}$$

which is generally only satisfied if $\beta_- = \beta_+ = 0$. By Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions provided $|U|^2 \le \alpha_-/\alpha_+$.

Let us now describe one particular example where the first derivative is not separated and the corresponding semigroup is unitary.

Example 4.7. We consider the case $\alpha_- = \alpha_+ = 1,\ \beta_- = \beta_+ = 0$. Define the 3×3-matrix $L : \mathcal{G}_- \to \mathcal{G}_+$ by

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \end{pmatrix}.$$

An easy calculation yields

$$L^* \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} L = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix};$$

i.e., $L^* B_+ L = B_-$. Thus, for $x, y \in \mathcal{G}_-$ we have

$$\langle Lx \mid Ly \rangle_+ = (B_+ Lx \mid Ly) = (L^* B_+ Lx \mid y) = (B_- x \mid y) = \langle x \mid y \rangle_-.$$

Hence, L defines a $(\mathcal{G}_-, \mathcal{G}_+)$-unitary operator. Combining Theorems 3.8 and 3.7 we obtain that A_L is skew-self-adjoint. Note that

$$D(A_L) = \{u \in D(A_0^*) : u(0+) = u(0-) =: u(0),$$
$$u'(0+) = \sqrt{2}u(0) + u'(0-),\ u''(0+) = u(0) + \sqrt{2}u'(0-) + u''(0-)\},$$

so the transmission conditions couple the values of the function and its first and second derivatives at the boundary point. By (3-8), mass is not conserved.

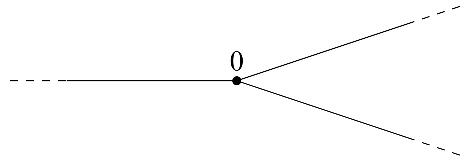

Figure 4. A graph consisting of three half-lines.

Remark 4.8. As mentioned in the Introduction, in the recent paper [Deconinck et al. 2016] the Airy equation is treated (with $\beta = 0$, which makes difference when translation invariance is broken) on a line with an interface at a point where linear transmission conditions are imposed. The authors give conditions for the solvability of the evolution problem in terms of the coefficients appearing in the transmission conditions. An interesting problem could be to compare the conditions obtained there with the ones given in the present paper in the case of two half-lines.

The case of three half-lines. Let us now describe the operator (again with separated first derivative) for the case of three half-lines; see Figure 4.

Let $|\mathsf{E}_-| = 1$ and $|\mathsf{E}_+| = 2$, which describes two confluent channels. By Proposition 3.1 it is impossible that the Airy equation is governed by a unitary group in this setting; however, we are going to discuss a few concrete cases where a semigroup of contractions is generated by the Airy operator.

Let Y be a subspace of $\ell^2(\mathsf{E}_-) \oplus \ell_+(\mathsf{E}_+) \cong \mathbb{C} \oplus \mathbb{C}^2 \cong \mathbb{C}^3$ and $U : \ell^2(\mathsf{E}_-, \alpha_-) \to \ell^2(\mathsf{E}_+, \alpha_+)$ be a linear mapping, i.e., $U \in M_{21}(\mathbb{C}) \simeq \mathbb{C}^2$: we denote for simplicity

$$U = (U_1, U_2)^\top.$$

Then U is -contractive if and only if $|U_1|^2\alpha_{+,1} + |U_2|^2\alpha_{+,2} \le \alpha_-$. Note that U^* is given by $U_1^* = \bar{U}_1(\alpha_{+,1}/\alpha_-)$ and $U_2^* = \bar{U}_2(\alpha_{+,2}/\alpha_-)$.

Example 4.9. Let $Y := \{(0, 0, 0)\}$. Then

$$\begin{aligned} D(A_{Y,U}) &= \{u \in D(A_0^*) : u(0-) = u_1(0+) = u_2(0+) = 0,\ (u_1'(0+), u_2'(0+))^\top = u'(0-)U\},\\ A_{Y,U}u &= -A_0^* u. \end{aligned}$$

As in Example 4.1, whenever $U = (0, 0)^\top$, the associated system effectively reduces to three decoupled half-lines with Dirichlet, resp. Dirichlet and Neumann boundary conditions. By Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions provided

$$|U_1|^2\alpha_{+,1} + |U_2|^2\alpha_{+,2} \le \alpha_-.$$

In view of (3-8), mass is generally not conserved regardless of U.

Example 4.10. Let $Y := \operatorname{lin}\{(1, 1, 1)\}$. Then

$$\begin{aligned} D(A_{Y,U}) = \{u \in D(A_0^*) : {} & u(0-) = u_1(0+) = u_2(0+) =: u(0),\\ & -\alpha_- u''(0-) + \alpha_{+,1}u_1''(0+) + \alpha_{+,2}u_2''(0+) = \tfrac{1}{2}(\beta_- - \beta_{+,1} - \beta_{+,2})u(0),\\ & (u_1'(0+), u_2'(0+))^\top = u'(0-)U\},\\ A_{Y,U}u = -A_0^* u. & \end{aligned}$$

Observe that the considered transmission conditions impose continuity of the values of u in the center of the star; in fact, the transmission conditions are the analog of a δ-interaction. By Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions provided

$$|U_1|^2\alpha_{+,1} + |U_2|^2\alpha_{+,2} \le \alpha_-.$$

Regardless of U, this semigroup is mass-preserving if and only if $\beta_- - \beta_{+,1} - \beta_{+,2} = 0$.

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ON s-HARMONIC FUNCTIONS ON CONES

SUSANNA TERRACINI, GIORGIO TORTONE AND STEFANO VITA

We deal with nonnegative functions satisfying

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in } C, \\ u_s = 0 & \text{in } \mathbb{R}^n \setminus C, \end{cases}$$

where $s \in (0, 1)$ and C is a given cone on $\mathbb{R}^n$ with vertex at zero. We consider the case when s approaches 1, wondering whether solutions of the problem do converge to harmonic functions in the same cone or not. Surprisingly, the answer will depend on the opening of the cone through an auxiliary eigenvalue problem on the upper half-sphere. These conic functions are involved in the study of the nodal regions in the case of optimal partitions and other free boundary problems and play a crucial role in the extension of the Alt–Caffarelli–Friedman monotonicity formula to the case of fractional diffusions.

1. Introduction

Let $n \geq 2$ and let C be an open cone in $\mathbb{R}^n$ with vertex in 0; for a given $s \in (0, 1)$, we consider the problem of the classification of nontrivial functions which are s-harmonic inside the cone and vanish identically outside, that is,

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in } C, \\ u_s \geq 0 & \text{in } \mathbb{R}^n, \\ u_s \equiv 0 & \text{in } \mathbb{R}^n \setminus C. \end{cases} \tag{1-1}$$

Here we define (see Section 2 for the details)

$$(-\Delta)^s u(x) = C(n, s)\ \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(\eta)}{|x - \eta|^{n+2s}}\, d\eta,$$

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Keywords: fractional Laplacian, conic functions, asymptotic behavior, Martin kernel.

where u is a sufficiently smooth function and

$$C(n,s)=\frac{2^{2s}s\Gamma(n/2+s)}{\pi^{n/2}\Gamma(1-s)}>0, \tag{1-2}$$

where

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t.$$

The principal value is taken at $\eta=x$; hence, though u needs not to decay at infinity, it has to keep an algebraic growth with a power strictly smaller than $2s$ in order to make the above expression meaningful. By Theorem 3.2 in [Bañuelos and Bogdan 2004], it is known that there exists a homogeneous, nonnegative and nontrivial solution to (1-1) of the form

$$u_s(x)=|x|^{\gamma_s}u_s\left(\frac{x}{|x|}\right),$$

where $\gamma_s:=\gamma_s(C)$ is a definite homogeneity degree (characteristic exponent), which depends on the cone. Moreover, such a solution is continuous in $\mathbb{R}^n$ and unique, up to multiplicative constants. We can normalize it in such a way that $\|u_s\|_{L^\infty(S^{n-1})}=1$. We consider the case when s approaches 1, wondering whether solutions of the problem do converge to a harmonic function in the same cone and, in that case, which are the suitable spaces for convergence.

Such conic s-harmonic functions appear as limiting blow-up profiles and play a major role in many free boundary problems with fractional diffusions and in the study of the geometry of nodal sets, also in the case of partition problems; see, e.g., [Allen 2012; Barrios et al. 2015; Caffarelli et al. 2017; Dipierro et al. 2017; Garofalo and Ros-Oton 2017]. Moreover, as we shall see later, they are strongly involved with the possible extensions of the Alt–Caffarelli–Friedman monotonicity formula to the case of fractional diffusion. The study of their properties and, ultimately, their classification is therefore a major achievement in this setting. The problem of homogeneous s-harmonic functions on cones has been deeply studied in [Bañuelos and Bogdan 2004; Bogdan and Byczkowski 1999; Bogdan et al. 2015; Michalik 2006]. The present paper mainly focuses on the limiting behavior as $s\nearrow 1$.

Our problem (1-1) can be linked to a specific spectral problem of local nature in the upper half-sphere; indeed let us look at the extension technique popularized by the authors in [Caffarelli and Silvestre 2007], characterizing the fractional Laplacian in $\mathbb{R}^n$ as the Dirichlet-to-Neumann map for a variable v depending on one more space dimension and satisfying

$$\begin{cases} L_s v=\operatorname{div}(y^{1-2s}\nabla v)=0 & \text{in } \mathbb{R}^{n+1}_+,\\ v(x,0)=u(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1-3}$$

Such an extension exists and is unique for a suitable class of functions u, see (2-1), and it is given by the formula

$$v(x,y)=\gamma(n,s)\int_{\mathbb{R}^n}\frac{y^{2s}u(\eta)}{(|x-\eta|^2+y^2)^{n/2+s}}\,\mathrm{d}\eta, \quad \text{where } \gamma(n,s)^{-1}:=\int_{\mathbb{R}^n}\frac{1}{(|\eta|^2+1)^{n/2+s}}\,\mathrm{d}\eta.$$

Then, the nonlocal original operator translates into a boundary derivative operator of Neumann type:

$$-\frac{C(n,s)}{\gamma(n,s)}\lim_{y\to 0} y^{1-2s}\partial_y v(x,y) = (-\Delta)^s u(x).$$

Now, let us consider an open region $\omega \subseteq S^{n-1} = \partial S^n_+$, with $S^n_+ = S^n \cap \{y > 0\}$, and define the eigenvalue

$$\lambda^s_1(\omega) = \inf\left\{ \frac{\int_{S^n_+} y^{1-2s}|\nabla_{S^n} u|^2\,\mathrm{d}\sigma}{\int_{S^n_+} y^{1-2s}u^2\,\mathrm{d}\sigma} : u \in H^1(S^n_+;\, y^{1-2s}\,\mathrm{d}\sigma)\setminus\{0\} \text{ and } u \equiv 0 \text{ in } S^{n-1}\setminus\omega \right\}.$$

Next, define the *characteristic exponent* of the cone C_ω spanned by ω (see Definition 2.1) as

$$\gamma_s(C_\omega) = \gamma_s(\lambda^s_1(\omega)), \tag{1-4}$$

where the function $\gamma_s(t)$ is defined by

$$\gamma_s(t) := \sqrt{\left(\tfrac{1}{2}(n-2s)\right)^2 + t} - \tfrac{1}{2}(n-2s).$$

Remark 1.1. There is a remarkable link between the nonnegative $\lambda^s_1(\omega)$-eigenfunctions and the $\gamma_s(\lambda^s_1(\omega))$-homogeneous L_s-harmonic functions: Let consider the spherical coordinates (r,θ) with $r > 0$ and $\theta \in S^n$. Let φ_s be the first nonnegative eigenfunction to $\lambda^s_1(\omega)$ and let v_s be its $\gamma_s(\lambda^s_1(\omega))$-homogeneous extension to $\mathbb{R}^{n+1}_+$, i.e.,

$$v_s(r,\theta) = r^{\gamma_s(\lambda^s_1(\omega))}\varphi_s(\theta),$$

which is well-defined as soon as $\gamma_s(\lambda^s_1(\omega)) < 2s$ (as we shall see, this fact is always granted). By [Rüland 2015], the operator L_s can be decomposed as

$$L_s u = \sin^{1-2s}(\theta_n)\frac{1}{r^n}\partial_r(r^{n+1+2s}\partial_r u) + \frac{1}{r^{1+2s}}L^{S^n}_s u,$$

where $y = r\sin(\theta_n)$ and the Laplace–Beltrami-type operator is defined as

$$L^{S^n}_s u = \mathrm{div}_{S^n}(\sin^{1-2s}(\theta_n)\nabla_{S^n} u)$$

with ∇_{S^n} the tangential gradient on S^n. Then, we easily get that v_s is L_s-harmonic in the upper half-space. Moreover, its trace $u_s(x) = v_s(x,0)$ is s-harmonic in the cone C_ω spanned by ω, vanishing identically outside; in other words u_s is a solution of our problem (1-1).

In a symmetric way, for the standard Laplacian, we consider the problem of γ-homogeneous functions which are harmonic inside the cone spanned by ω and vanish outside:

$$\begin{cases} -\Delta u_1 = 0 & \text{in } C_\omega, \\ u_1 \geq 0 & \text{in } \mathbb{R}^n, \\ u_1 = 0 & \text{in } \mathbb{R}^n\setminus C_\omega. \end{cases} \tag{1-5}$$

It is well known that the associated eigenvalue problem on the sphere is that of the Laplace–Beltrami operator with Dirichlet boundary conditions

$$\lambda_1(\omega) = \inf\left\{ \frac{\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2\,\mathrm{d}\sigma}{\int_{S^{n-1}} u^2\,\mathrm{d}\sigma} : u \in H^1(S^{n-1})\setminus\{0\} \text{ and } u = 0 \text{ in } S^{n-1}\setminus\omega \right\},$$

and the *characteristic exponent* of the cone C_ω is

$$\gamma(C_\omega)=\sqrt{\left(\tfrac{1}{2}(n-2)\right)^2+\lambda_1(\omega)}-\tfrac{1}{2}(n-2)=\gamma_{s|s=1}(\lambda_1(\omega)). \tag{1-6}$$

In the classical case, the characteristic exponent enjoys a number of nice properties: It is minimal on spherical caps among sets having a given measure. Moreover, for the spherical caps, the eigenvalues enjoy a fundamental convexity property with respect to the colatitude θ [Alt et al. 1984; Friedland and Hayman 1976]. The convexity plays a major role in the proof of the Alt–Caffarelli–Friedman monotonicity formula, a key tool in the free boundary theory [Caffarelli and Salsa 2005].

Since the standard Laplacian can be viewed as the limiting operator of the family $(-\Delta)^s$ as $s\nearrow 1$, some questions naturally arise:

Problem 1.2. Is it true that:

(a) $\lim_{s\to 1}\gamma_s(C)=\gamma(C)$?

(b) $\lim_{s\to 1}u_s=u_1$ uniformly on compact sets, or better, in Hölder local norms?

(c) for spherical caps of opening θ there is any convexity of the map $\theta\mapsto\lambda_1^s(\theta)$, at least, for s near 1?

We therefore addressed the problem of the asymptotic behavior of the solutions of problem (1-1) for $s\nearrow 1$, obtaining a rather unexpected result: our analysis shows high sensitivity to the opening solid angle ω of the cone C_ω, as evaluated by the value of $\gamma(C)$. In the case of wide cones, when $\gamma(C)<2$ (that is, $\theta\in(\pi/4,\pi)$ for spherical caps of colatitude θ), our solutions do converge to the harmonic homogeneous function of the cone; in the case of narrow cones, when $\gamma(C)\geq 2$ (that is, $\theta\in(0,\pi/4]$ for spherical caps), the limit of the homogeneity degree will always be 2 and the limiting profile will be something different, though related, of course, through a correction term. Similar transition phenomena have been detected in other contexts for some types of free boundary problems on cones [Allen and Lara 2015; Shahgholian 2004]. As a consequence of our main result, we will see a lack of convexity of the eigenvalue as a function of the colatitude. Our main result is the following theorem.

Theorem 1.3. *Let C be an open cone with vertex at the origin. There exist the following finite limits*:

$$\bar\gamma(C):=\lim_{s\to 1^-}\gamma_s(C)=\min\{\gamma(C),2\}, \tag{1-7}$$

$$\mu(C):=\lim_{s\to 1^-}\frac{C(n,s)}{2s-\gamma_s(C)}=\begin{cases}0 & \text{if } \gamma(C)\leq 2,\\ \mu_0(C) & \text{if } \gamma(C)\geq 2,\end{cases} \tag{1-8}$$

where $C(n,s)$ is defined in (1-2) *and*

$$\mu_0(C):=\inf\left\{\frac{\int_{S^{n-1}}|\nabla_{S^{n-1}}u|^2-2nu^2\,d\sigma}{\left(\int_{S^{n-1}}|u|\,d\sigma\right)^2}:u\in H^1(S^{n-1})\setminus\{0\}\text{ and } u=0\text{ in } S^{n-1}\setminus C\right\}.$$

Let us consider the family (u_s) of nonnegative solutions to (1-1) *such that $\|u_s\|_{L^\infty(S^{n-1})}=1$. Then, as $s\nearrow 1$, up to a subsequence, we have*:

(1) *$u_s\to\bar u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\bar u\in H^1_{\mathrm{loc}}(\mathbb{R}^n)\cap L^\infty(S^{n-1})$.*

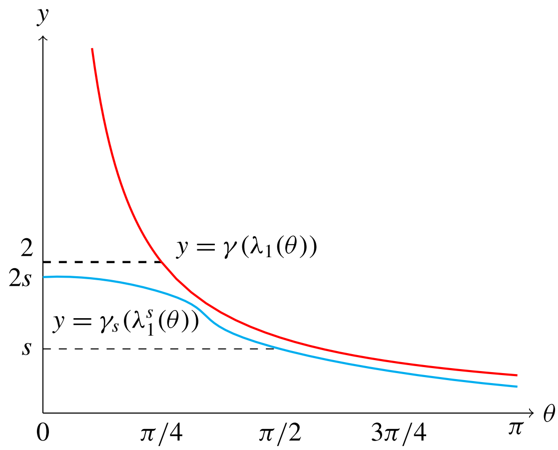

Figure 1. Characteristic exponents of spherical caps of aperture 2θ for $s < 1$ and $s = 1$.

(2) *The convergence is uniform on compact subsets of C and $\bar{u}$ is nontrivial with $\|\bar{u}\|_{L^\infty(S^{n-1})} = 1$ and is $\bar{\gamma}(C)$-homogeneous.*

(3) *The limit $\bar{u}$ solves*

$$\begin{cases} -\Delta\bar{u} = \mu(C)\int_{S^{n-1}} \bar{u}\,\mathrm{d}\sigma & \text{in } C, \\ \quad \bar{u} = 0 & \text{in } \mathbb{R}^n \setminus C. \end{cases} \tag{1-9}$$

Remark 1.4. Uniqueness of the limit $\bar{u}$ and therefore existence of the limit of u_s as $s \nearrow 1$ hold in the case of connected cones and, in any case, whenever $\gamma(C) > 2$. We will see in Remark 4.2 that under symmetry assumptions on the cone C, the limit function $\bar{u}$ is unique and hence it does not depend on the choice of the subsequence.

A nontrivial improvement of the main theorem concerns uniform bounds in Hölder spaces holding uniformly for $s \to 1$.

Theorem 1.5. *Assume the cone is $\mathcal{C}^{1,1}$. Let $\alpha \in (0, 1)$, $s_0 \in \left(\max\left\{\frac{1}{2}, \alpha\right\}, 1\right)$ and A be an annulus centered at zero. Then the family of solutions u_s to* (1-1) *is uniformly bounded in $C^{0,\alpha}(A)$ for any $s \in [s_0, 1)$.*

On the fractional Alt–Caffarelli–Friedman monotonicity formula. In the case of reaction-diffusion systems with strong competition between a number of densities which spread in space, one can observe a segregation phenomenon: as the interspecific competition rate grows, the populations tend to separate their supports in nodal sets, separated by a free boundary. For the case of standard diffusion, both the asymptotic analysis and the properties of the segregated limiting profiles are fairly well understood, we refer to [Caffarelli and Lin 2008; Conti et al. 2005; Dancer et al. 2012; Noris et al. 2010; Tavares and Terracini 2012]. Instead, when the diffusion is nonlocal and modeled by the fractional Laplacian, the only known results are contained in [Terracini et al. 2014; 2016; Terracini and Vita 2017; Wang and Wei 2016]. As shown in [Terracini et al. 2014; 2016], estimates in Hölder spaces can be obtained by the use of fractional versions of the Alt–Caffarelli–Friedman (ACF) and Almgren monotonicity formulas. For the statement, proof and applications of the original ACF monotonicity formula we refer to the book

[Caffarelli and Salsa 2005] on free boundary problems. Let us state here the fractional version of the spectral problem beyond the ACF formula used in [Terracini et al. 2014; 2016]: consider the set of 2-partitions of S^{n-1} as

$$\mathcal{P}^2 := \left\{(\omega_1,\omega_2) : \omega_i \subseteq S^{n-1} \text{ open},\ \omega_1\cap\omega_2=\varnothing,\ \bar\omega_1\cup\bar\omega_2 = S^{n-1}\right\}$$

and define the optimal partition value as

$$\nu_s^{\mathrm{ACF}} := \frac{1}{2}\inf_{(\omega_1,\omega_2)\in\mathcal{P}^2}\sum_{i=1}^{2}\gamma_s(\lambda_1^s(\omega_i)). \tag{1-10}$$

It is easy to see, by a Schwarz symmetrization argument, that ν_s^{ACF} is achieved by a pair of complementary spherical caps $(\omega_\theta,\omega_{\pi-\theta})\in\mathcal{P}^2$ with aperture 2θ and $\theta\in(0,\pi)$ (for a detailed proof of this kind of symmetrization we refer to [Terracini and Vita 2017]); that is,

$$\nu_s^{\mathrm{ACF}} = \min_{\theta\in[0,\pi]}\Gamma^s(\theta) = \min_{\theta\in[0,\pi]}\frac{\gamma_s(\theta)+\gamma_s(\pi-\theta)}{2}.$$

This gives a further motivation to our study of (1-1) for spherical caps. A classical result in [Friedland and Hayman 1976] yields $\nu^{\mathrm{ACF}}=1$ (in the case $s=1$), and the minimal value is achieved for two half-spheres; this equality is the core of the proof of the classical Alt–Caffarelli–Friedman monotonicity formula.

It was proved in [Terracini et al. 2014] that ν_s^{ACF} is linked to the threshold for uniform bounds in Hölder norms for competition-diffusion systems, as the interspecific competition rate diverges to infinity, as well as the exponent of the optimal Hölder regularity for their limiting profiles. It was also conjectured that $\nu_s^{\mathrm{ACF}}=s$ for every $s\in(0,1)$. Unfortunately, the exact value of ν_s^{ACF} is still unknown, and we only know that $0<\nu_s^{\mathrm{ACF}}\le s$; see [Terracini et al. 2014; 2016]. Actually one can easily give a better lower bound given by $\nu_s^{\mathrm{ACF}}\ge\max\left\{\frac{1}{2}s, s-\frac{1}{4}\right\}$ when $n=2$ and $\nu_s^{\mathrm{ACF}}\ge\frac{1}{2}s$ otherwise, which however is not satisfactory. As already remarked in [Allen 2012], this lack of information implies also the lack of an exact Alt–Caffarelli–Friedman monotonicity formula for the case of fractional Laplacians. Our contribution to this open problem is a byproduct of the main result, Theorem 1.3, and is depicted in Figure 2.

Corollary 1.6. *In any space dimension we have*

$$\lim_{s\to 1}\nu_s^{\mathrm{ACF}} = 1.$$

The paper is organized as follows. In Section 2 we introduce our setting and we state the relevant known properties of homogeneous s-harmonic functions on cones. After this, we will obtain local $C^{0,\alpha}$-estimates in compact subsets of C and local H^s-estimates in compact subsets of $\mathbb{R}^n$ for solutions u_s of (1-1). We will see that an important quantity which appears in these estimates and plays a fundamental role is

$$\frac{C(n,s)}{2s-\gamma_s(C)},$$

where $C(n,s)>0$ is the normalization constant given in (1-2). It will be therefore very important to bound this quantity uniformly in s. In Section 3 we analyze the asymptotic behavior of $\gamma_s(C)$ as s converges to 1, in order to understand the quantities $\bar\gamma(C)$ and $\mu(C)$. To do this, we will establish a distributional

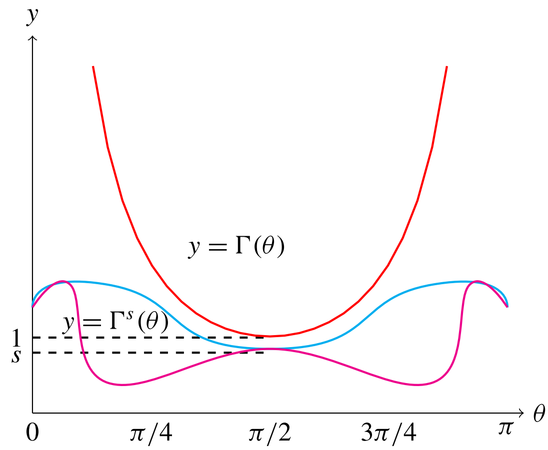

Figure 2. Possible values of $\Gamma^s(\theta) = \Gamma^s(\omega_\theta, \omega_{\pi-\theta})$ for $s < 1$ and $s = 1$ and $n = 2$.

semigroup property for the fractional Laplacian for functions which grow at infinity. In Section 4 we prove Theorem 1.3 and Corollary 1.6. Eventually, in Section 5, we prove Theorem 1.5.

2. Homogeneous s-harmonic functions on cones

In this section, we focus our attention on the local properties of homogeneous s-harmonic functions on *regular cones*. Since in the next section we will study the behavior of the characteristic exponent as s approaches 1, in this section we recall some known results related to the boundary behavior of the solution of (1-1) restricted to the unitary sphere S^{n-1} and some estimates of the Hölder and H^s seminorms.

Definition 2.1. Let $\omega \subset S^{n-1}$ be an open set, which may be disconnected. We define the *unbounded cone* with vertex in 0, spanned by ω, to be the open set

$$C_\omega = \{rx : r > 0,\ x \in \omega\}.$$

Moreover we say that $C = C_\omega$ is narrow if $\gamma(C) \geq 2$ and wide if $\gamma(C) < 2$. We call C_ω a *regular cone* if ω is connected and of class $C^{1,1}$. Let $\theta \in (0, \pi)$ and $\omega_\theta \subset S^{n-1}$ be an open spherical cap of colatitude θ. Then we denote by $C_\theta = C_{\omega_\theta}$ the *right circular cone* of aperture 2θ.

Hence, let C be a fixed unbounded open cone in $\mathbb{R}^n$ with vertex in 0 and consider

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in } C, \\ u_s \equiv 0 & \text{in } \mathbb{R}^n \setminus C \end{cases}$$

with the condition $\|u_s\|_{L^\infty(S^{n-1})} = 1$. By Theorem 3.2 in [Bañuelos and Bogdan 2004] there exists, up to a multiplicative constant, a unique nonnegative function u_s smooth in C and $\gamma_s(C)$-homogeneous, i.e.,

$$u_s(x) = |x|^{\gamma_s(C)} u_s\left(\frac{x}{|x|}\right),$$

where $\gamma_s(C) \in (0, 2s)$. As is well known, see for example [Bogdan and Byczkowski 1999; Silvestre 2005], the fractional Laplacian $(-\Delta)^s$ is a nonlocal operator well-defined in the class of integrability $\mathcal{L}^1_s := \mathcal{L}^1(\mathrm{d}x/(1+|x|)^{n+2s})$, namely the normed space of all Borel functions u satisfying

$$\|u\|_{\mathcal{L}^1_s} := \int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|)^{n+2s}}\,\mathrm{d}x < \infty. \tag{2-1}$$

Hence, for every $u \in \mathcal{L}^1_s$, $\varepsilon > 0$ and $x \in \mathbb{R}^n$ we define

$$(-\Delta)^s_\varepsilon u(x) = C(n,s)\int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y,$$

where

$$C(n,s) = \frac{2^{2s}s\Gamma(n/2+s)}{\pi^{n/2}\Gamma(1-s)} \in (0, 4\Gamma(n/2+1)].$$

and we can consider the fractional Laplacian as the limit

$$(-\Delta)^s u(x) = \lim_{\varepsilon\downarrow 0}(-\Delta)^s_\varepsilon u(x) = C(n,s)\ \text{p.v.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y.$$

We remark that $u \in \mathcal{L}^1_s$ is such that $u \in \mathcal{L}^1_{s+\delta}$ for any $\delta > 0$, which will be an important tool in this section of the paper, in order to compute high-order fractional Laplacians. Another definition of the fractional Laplacian, which can be constructed by a double change of variables as in [Di Nezza et al. 2012], is

$$(-\Delta)^s u(x) = \frac{C(n,s)}{2}\int_{\mathbb{R}^n} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,\mathrm{d}y,$$

which emphasizes that given $u \in C^2(D)\cap\mathcal{L}^1_s$, we obtain that $x \mapsto (-\Delta)^s u(x)$ is a continuous and bounded function on D for some bounded $D \subset \mathbb{R}^n$.

By [Michalik 2006, Lemma 3.3], if we consider a regular unbounded cone C symmetric with respect to a fixed axis, there exist two positive constants $c_1 = c_1(n,s,C)$ and $c_2 = c_2(n,s,C)$ such that

$$c_1|x|^{\gamma_s-s}\,\mathrm{dist}(x,\partial C)^s \le u_s(x) \le c_2|x|^{\gamma_s-s}\,\mathrm{dist}(x,\partial C)^s \tag{2-2}$$

for every $x \in C$. We remark that this result can be easily generalized to regular unbounded cones C_ω with $\omega \subset S^{n-1}$ a finite union of connected $C^{1,1}$ domains ω_i such that $\bar\omega_i \cup \bar\omega_j = \varnothing$ for $i \neq j$, since the reasoning in [Michalik 2006] relies on a boundary Harnack principle and on sharp estimates for the Green function for bounded $C^{1,1}$ domains which are not necessarily connected; for more details see [Chen and Song 1998].

Throughout the paper we will call the coefficient of homogeneity γ_s the "*characteristic exponent*", since it is strictly related to an eigenvalue partition problem.

As we already mentioned, our solutions are smooth in the interior of the cone and locally $C^{0,s}$ near the boundary $\partial C \setminus \{0\}$, see for example [Michalik 2006], but we need some quantitative estimates in order to better understand the dependence of the Hölder seminorm on the parameter $s \in (0,1)$.

Before showing the main result of Hölder regularity, we need the following estimates about the fractional Laplacian of smooth compactly supported functions; this result can be found in [Bogdan and

Byczkowski 1999, Lemma 3.5; Dávila et al. 2015, Lemma 5.1], but here we compute the formula with a deep attention on the dependence of the constant with respect to $s \in (0, 1)$.

Proposition 2.2. *Let $s \in (0, 1)$ and $\varphi \in C_c^2(\mathbb{R}^n)$. Then*

$$|(-\Delta)^s \varphi(x)| \le \frac{c}{(1+|x|)^{n+2s}} \quad \text{for all } x \in \mathbb{R}^n, \tag{2-3}$$

where the constant $c > 0$ depends only on n and the choice of φ.

Proof. Let $K \subset \mathbb{R}^n$ be the compact support of φ and $k = \max_{x \in K} |\varphi(x)|$. There exists $R > 1$ such that $K \subset B_{R/2}(0)$.

Let $|x| > R$. Then

$$\begin{aligned} |(-\Delta)^s \varphi(x)| &= \left| C(n,s) \int_{\mathbb{R}^n} \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+2s}} \,\mathrm{d}y \right| = \left| C(n,s) \int_K \frac{\varphi(y)}{|x-y|^{n+2s}} \,\mathrm{d}y \right| \\ &\le \frac{C(n,s)k}{|x|^{n+2s}} \int_K \frac{1}{(1-|y/x|)^{n+2s}} \,\mathrm{d}y \le \frac{C(n,s)k 2^{n+2s}|K|}{|x|^{n+2s}} \\ &\le \frac{C(n,s)k 2^{2(n+2s)}|K|}{(1+|x|)^{n+2s}} \le \frac{c}{(1+|x|)^{n+2s}}, \end{aligned}$$

where $c > 0$ depends only on n and the choice of φ.

Let now $|x| \le R$. We use the fact that any derivative of φ of first or second order is uniformly continuous in the compact set K and the fact that in $B_R(0)$ the function $(1+|x|)^{n+2s}$ has maximum given by $(1+R)^{n+2s}$. Hence there exist $0 < \delta < 1$ and a constant $M > 0$, both depending only on n and the choice of φ, such that

$$|\varphi(x+z) + \varphi(x-z) - 2\varphi(x)| \le M|z|^2 \quad \text{for all } z \in B_\delta(0).$$

Hence

$$\begin{aligned} |(-\Delta)^s \varphi(x)| &= \left| C(n,s) \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+2s}} \,\mathrm{d}y + C(n,s) \int_{B_\delta(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+2s}} \,\mathrm{d}y \right| \\ &\le 2kC(n,s) \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{1}{|x-y|^{n+2s}} \,\mathrm{d}y + \frac{C(n,s)}{2} \int_{B_\delta(0)} \frac{|\varphi(x+z) + \varphi(x-z) - 2\varphi(x)|}{|z|^{n+2s}} \,\mathrm{d}z \\ &\le 2kC(n,s)\omega_{n-1} \int_\delta^\infty r^{-1-2s} \,\mathrm{d}r + \frac{C(n,s)\omega_{n-1}M}{2} \int_0^\delta r^{1-2s} \,\mathrm{d}r \\ &= \frac{kC(n,s)\omega_{n-1}}{s\delta^{2s}} + \frac{C(n,s)\omega_{n-1}M\delta^{2-2s}}{4(1-s)} \\ &\le \frac{c}{\delta^2} + c = c\frac{(1+|x|)^{n+2s}}{(1+|x|)^{n+2s}} \le \frac{c(1+R)^{n+2}}{(1+|x|)^{n+2s}} = \frac{c}{(1+|x|)^{n+2s}}, \end{aligned}$$

where $c > 0$ depends only on n and the choice of φ. $\square$

By the previous calculations we have also the following result.

Remark 2.3. Let $s \in (0,1)$ and $\varphi \in C^2_c(\mathbb{R}^n)$. Then there exists a constant $c = c(n,\varphi) > 0$ and a radius $R = R(\varphi) > 0$ such that

$$|(-\Delta)^s \varphi(x)| \le c \frac{C(n,s)}{(1+|x|)^{n+2s}} \quad \text{for all } x \in \mathbb{R}^n \setminus B_R(0). \tag{2-4}$$

The following result provides interior estimates for the Hölder norm of our solutions.

Proposition 2.4. *Let C be a cone, $K \subset C$ be a compact set and $s_0 \in (0,1)$. Then there exist a constant $c > 0$ and $\bar\alpha \in (0,1)$, both dependent only on s_0, K, n, C, such that*

$$\|u_s\|_{C^{0,\alpha}(K)} \le c\left(1 + \frac{C(n,s)}{2s - \gamma_s(C)}\right)$$

for any $\alpha \in (0,\bar\alpha]$ and any $s \in [s_0,1)$.

By a standard covering argument, there exists a finite number of balls such that $K \subset \bigcup_{j=1}^k B_r(x_j)$ for a given radius $r > 0$ such that $\bigcup_{j=1}^k \overline{B_{2r}(x_j)} \subset C$. Thus, it is enough to prove:

Proposition 2.5. *Let $\overline{B_{2r}(\bar x)} \subset C$ be a closed ball and $s_0 \in (0,1)$. Then there exist a constant $c > 0$ and $\bar\alpha \in (0,1)$, both dependent only on $s_0, r, \bar x, n, C$, such that*

$$\|u_s\|_{C^{0,\alpha}(\overline{B_r(\bar x)})} \le c\left(1 + \frac{C(n,s)}{2s - \gamma_s(C)}\right)$$

for any $\alpha \in (0,\bar\alpha]$ and any $s \in [s_0,1)$.

In order to achieve the desired result, we need to estimate locally the value of the fractional Laplacian of u_s in a ball compactly contained in the cone C.

Lemma 2.6. *Let $\eta \in C^\infty_c(B_{2r}(\bar x))$ be a cut-off function such that $0 \le \eta \le 1$ with $\eta \equiv 1$ in $B_r(\bar x)$. Under the same assumptions as Proposition 2.5,*

$$\|(-\Delta)^s(u_s\eta)\|_{L^\infty(B_{2r}(\bar x))} \le C_0\left(1 + \frac{C(n,s)}{2s - \gamma_s(C)}\right)$$

for any $s \in [s_0,1)$, where $C_0 > 0$ depends on $s_0, n, \bar x, r, C$, and the choice of the function η.

Proof. Let $R > 1$ be such that $\overline{B_{2r}(\bar x)} \subset B_{R/2}(0)$. Hence, let us fix a point $x \in B_{2r}(\bar x)$. We can express the fractional Laplacian of $u_s\eta$ in the following way:

$$\begin{aligned}(-\Delta)^s(u_s\eta)(x) &= \eta(x)(-\Delta)^s u_s(x) + C(n,s)\int_{\mathbb{R}^n} u_s(y)\frac{\eta(x)-\eta(y)}{|x-y|^{n+2s}}\,\mathrm{d}y \\ &= C(n,s)\int_{B_R(0)} u_s(y)\frac{\eta(x)-\eta(y)}{|x-y|^{n+2s}}\,\mathrm{d}y + C(n,s)\int_{\mathbb{R}^n\setminus B_R(0)} u_s(y)\frac{\eta(x)-\eta(y)}{|x-y|^{n+2s}}\,\mathrm{d}y.\end{aligned}$$

We recall that $u_s(x) = |x|^{\gamma_s(C)} u_s(x/|x|)$ and that for any $s \in (0,1)$ the functions u_s are normalized such that $\|u_s\|_{L^\infty(S^{n-1})} = 1$. Moreover we remark that $\eta(x) - \eta(y) = \eta(x) \ge 0$ in $B_{2r}(\bar x) \times (\mathbb{R}^n \setminus B_R(0))$. Hence, using Proposition 2.2 and the fact that $\gamma_s(C) < 2s$, we obtain

$$\begin{aligned}
|(-\Delta)^s(u_s\eta)(x)| &\le C(n,s)\left|\int_{B_R(0)} u_s(y)\frac{\eta(x)-\eta(y)}{|x-y|^{n+2s}}\,\mathrm{d}y\right| + C(n,s)\left|\int_{\mathbb{R}^n\setminus B_R(0)} u_s(y)\frac{\eta(x)-\eta(y)}{|x-y|^{n+2s}}\,\mathrm{d}y\right| \\
&\le R^{\gamma_s(C)}|(-\Delta)^s\eta(x)| + C(n,s)2^{n+2s}\int_{\mathbb{R}^n\setminus B_R(0)} \frac{1}{|y|^{n+2s-\gamma_s(C)}}\,\mathrm{d}y \\
&\le \frac{cR^2}{(1+|x|)^{n+2s}} + C(n,s)2^{n+2}\omega_{n-1}\int_R^\infty r^{-1-2s+\gamma_s(C)}\,\mathrm{d}r \\
&\le \frac{cR^2}{(1+|x|)^{n+2s}} + \frac{cC(n,s)}{R^{2s-\gamma_s(C)}(2s-\gamma_s(C))} \\
&\le C_0\left(1+\frac{C(n,s)}{2s-\gamma_s(C)}\right).
\end{aligned}$$

□

Proof of Proposition 2.5. Let as before $\eta \in C_c^\infty(B_{2r}(\bar{x}))$ be a cut-off function such that $0 \le \eta \le 1$ with $\eta \equiv 1$ in $B_r(\bar{x})$. First, we remark that there exists a constant $c_0 > 0$ such that for any $s \in (0,1)$

$$\|u_s\eta\|_{L^\infty(\mathbb{R}^n)} \le c_0, \tag{2-5}$$

where c_0 depends only on $n, \bar{x}, r$. In fact, let $R > 0$ be such that $\overline{B_{2r}(\bar{x})} \subset B_R(0)$. Then, for any $x \in \mathbb{R}^n$, we have $0 \le u_s\eta(x) \le R^{\gamma_s(C)} \le R^2$. Using the bound (2-5) and the previous lemma, we can apply [Caffarelli and Silvestre 2009, Theorem 12.1] obtaining the existence of $\bar{\alpha} \in (0,1)$ and $C > 0$, both depending only on n, s_0 and the choice of $B_r(\bar{x})$ such that

$$\begin{aligned}
\|u_s\eta\|_{C^{0,\alpha}(\overline{B_r(\bar{x})})} &\le C(\|u_s\eta\|_{L^\infty(\mathbb{R}^n)} + \|(-\Delta)^s(u_s\eta)\|_{L^\infty(B_{2r}(\bar{x}))}) \\
&\le C\left(c_0 + C_0\left(1+\frac{C(n,s)}{2s-\gamma_s(C)}\right)\right)
\end{aligned}$$

for any $s \in [s_0, 1)$ and any $\alpha \in (0, \bar{\alpha}]$. Since $\eta \equiv 1$ in $B_r(\bar{x})$ we obtain the result. □

Similarly, now we need to construct some estimate related to the H^s seminorm of the solution u_s. Since the functions do not belong to $H^s(\mathbb{R}^n)$, we need to truncate the solution with some cut-off function in order to avoid the problems related to the growth at infinity. In such a way, we can use

$$[v]^2_{H^s(\mathbb{R}^n)} = \|(-\Delta)^{s/2}v\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} v(-\Delta)^s v\,\mathrm{d}x, \tag{2-6}$$

which holds for every $v \in H^s(\mathbb{R}^n)$. So, let $\eta \in C_c^\infty(B_2)$ be a radial cut-off function such that $\eta \equiv 1$ in B_1 and $0 \le \eta \le 1$ in B_2, and consider $\eta_R(x) = \eta((x-x_0)/R)$, the rescaled cut-off function defined in $B_{2R}(x_0)$ for some $R > 0$ and $x_0 \in \mathbb{R}^n$.

Proposition 2.7. *Let $s_0 \in (0,1)$ and $\eta_R \in C_c^\infty(B_{2R}(x_0))$ be previously defined. Then*

$$[u_s\eta_R]^2_{H^s(\mathbb{R}^n)} \le c\left(1+\frac{C(n,s)}{2s-\gamma_s(C)}\right)$$

for any $s \in [s_0, 1)$, where $c > 0$ is a constant that depends on x_0, R, C, s_0 and η.

Proof. Let $\eta \in C_c^\infty(B_2)$ be a radial cut-off function such that $\eta \equiv 1$ in B_1 and $0 \le \eta \le 1$ in B_2, and consider the collection of $(\eta_R)_R$ with $R > 0$ defined by $\eta_R(x) = \eta((x - x_0)/R)$ with some $x_0 \in \mathbb{R}^n$. By (2-6), for every $R > 0$ we obtain

$$[u_s\eta_R]^2_{H^s(\mathbb{R}^n)} = \|(-\Delta)^{s/2}(u_s\eta_R)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u_s\eta_R(-\Delta)^s(u_s\eta_R)\,\mathrm{d}x.$$

By the definition of the fractional Laplacian we have

$$\begin{aligned}\int_{\mathbb{R}^n} u_s\eta_R(-\Delta)^s(u_s\eta_R)\,\mathrm{d}x &= C(n,s)\int_{\mathbb{R}^n\times\mathbb{R}^n} u_s(x)\eta_R(x)\frac{u_s(x)\eta_R(x)-u_s(y)\eta_R(y)}{|x-y|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_{\mathbb{R}^n}\eta_R^2 u_s(-\Delta)^s u_s\,\mathrm{d}x + C(n,s)\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}u_s(x)u_s(y)\eta_R(x)\,\mathrm{d}y\,\mathrm{d}x\\ &= \frac{C(n,s)}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{n+2s}}u_s(x)u_s(y)\,\mathrm{d}y\,\mathrm{d}x,\end{aligned}$$

where the last equation is obtained by the symmetrization of the previous integral with respect to the variable $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Before splitting the domain of integration into different subsets, it is easy to see that

$$\begin{aligned}\eta_R(x)-\eta_R(y) &\equiv 0 \quad \text{in } B_R(x_0)\times B_R(x_0)\cup(\mathbb{R}^n\setminus B_{2R}(x_0))\times(\mathbb{R}^n\setminus B_{2R}(x_0)),\\ |\eta_R(x)-\eta_R(y)| &\equiv 1 \quad \text{in } B_R(x_0)\times(\mathbb{R}^n\setminus B_{2R}(x_0))\cup(\mathbb{R}^n\setminus B_{2R}(x_0))\times B_R(x_0),\end{aligned}$$

where all the previous balls are centered at the point x_0. Hence, given the sets

$$\begin{aligned}\Omega_1 &= B_{3R}(x_0)\times B_{3R}(x_0),\\ \Omega_2 &= B_{2R}(x_0)\times(\mathbb{R}^n\setminus B_{3R}(x_0))\cup(\mathbb{R}^n\setminus B_{3R}(x_0))\times B_{2R}(x_0),\end{aligned}$$

we have

$$\begin{aligned}&\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{n+2s}}u_s(x)u_s(y)\,\mathrm{d}y\,\mathrm{d}x\\ &\qquad\le \int_{\Omega_1}\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{n+2s}}u_s(x)u_s(y)\,\mathrm{d}y\,\mathrm{d}x + \int_{\Omega_2}\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{n+2s}}u_s(x)u_s(y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

In particular

$$\begin{aligned}\int_{\Omega_1}\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{n+2s}}u_s(x)u_s(y)\,\mathrm{d}y\,\mathrm{d}x &\le \sup_{B_{3R}(x_0)} u_s^2\int_{B_{3R}(x_0)\times B_{3R}(x_0)}\frac{\|\nabla\eta_R\|^2_{L^\infty(\mathbb{R}^n)}}{|x-y|^{n+2s-2}}\,\mathrm{d}y\,\mathrm{d}x\\ &\le \|\nabla\eta_R\|^2_{L^\infty}\sup_{B_{3R}(x_0)} u_s^2\int_{B_{3R}(0)}\mathrm{d}x\int_{B_{6R}(x)}\frac{1}{|x-y|^{n+2s-2}}\,\mathrm{d}y\\ &\le \frac{\|\nabla\eta\|^2_{L^\infty}}{R^2}\sup_{B_{3R}(x_0)} u_s^2|B_{3R}||S^{n-1}|\frac{(6R)^{2-2s}}{2(1-s)}\\ &\le C\|\nabla\eta\|^2_{L^\infty}\frac{R^{n-2s}}{2(1-s)}\max\{|x_0|^{2\gamma_s},(3R)^{2\gamma_s}\}\|u_s\|_{L^\infty(S^{n-1})},\end{aligned}$$

where in the second inequality we use the changes of variables $x - x_0$ and $y - x_0$ and the fact that $B_{3R}(0) \times B_{3R}(0) \subset B_{3R}(0) \times B_{6R}(x)$ for every $x \in B_{3R}(0)$. Similarly we have

$$\begin{aligned}\int_{\Omega_2} \frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{n+2s}} u_s(x)u_s(y)\,\mathrm{d}y\,\mathrm{d}x &\le 2\int_{B_{2R}(x_0)} u_s(x)\left(\int_{\mathbb{R}^n\setminus B_{3R}(x_0)} \frac{u_s(y)}{|x-y|^{n+2s}}\,\mathrm{d}y\right)\mathrm{d}x\\ &\le 2\int_{B_{2R}(0)} u_s(x+x_0)\left(\int_{\mathbb{R}^n\setminus B_{3R}(0)} \frac{u_s(y+x_0)}{|y|^{n+2s}(1-|x|/|y|)^{n+2s}}\,\mathrm{d}y\right)\mathrm{d}x\\ &\le 2\cdot 3^{n+2s}\int_{B_{2R}(0)} u_s(x+x_0)\left(\int_{\mathbb{R}^n\setminus B_{3R}(0)} \frac{C(|y|+|x_0|)^{\gamma_s}}{|y|^{n+2s}}\,\mathrm{d}y\right)\mathrm{d}x\\ &\le C\sup_{B_{2R}(x_0)} u_s|B_{2R}|\,|S^{n-1}|2^{\gamma_s}G(x_0,R)\end{aligned}$$

with

$$\begin{aligned}G(x_0,R) &= \begin{cases}|x_0|^{\gamma_s}/(2s-\gamma_s)(3R)^{-2s} & \text{if } |x_0|\ge 3R,\\ (3R)^{\gamma_s-2s}/(2s-\gamma_s) & \text{if } |x_0|\le 3R\end{cases}\\ &\le \frac{(3R)^{-2s}}{2s-\gamma_s}\max\{|x_0|,3R\}^{\gamma_s}.\end{aligned}$$

Finally, we obtain the desired bound for the seminorm $[u_s\eta_R]^2_{H^s(\mathbb{R}^n)}$ summing the two terms and recalling that $\|u_s\|_{L^\infty(S^{n-1})} = 1$. □

3. Characteristic exponent $\gamma_s(C)$: properties and asymptotic behavior

In this section we start the analysis of the asymptotic behavior of the homogeneity degree $\gamma_s(C)$ as s converges to 1. We get two main results: first we get a monotonicity result for the map $s \mapsto \gamma_s(C)$ for a fixed regular cone C, which ensures the existence of the limit and, using a comparison result, a bound on the possible value of the limit exponent. Secondly we study the asymptotic behavior of the quotient $C(n,s)/(2s-\gamma_s(C))$.

In order to prove the first result and compare different orders of s-harmonic functions for different power of $(-\Delta)^s$, we need to introduce some results which give a natural extension of the classic semigroup property of the fractional Laplacian for functions defined on cones which grow at infinity.

Distributional semigroup property. It is well known that if we deal with smooth functions with compact support, or more generally with functions in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, a semigroup property holds for the fractional Laplacian; i.e., $(-\Delta)^{s_1}\circ(-\Delta)^{s_2} = (-\Delta)^{s_1+s_2}$, where $s_1, s_2 \in (0,1)$ with $s_1+s_2<1$. Since we have to deal with functions in $\mathcal{L}^1_s$ that grow at infinity, we have to construct a distributional counterpart of the semigroup property in order to compute high-order fractional Laplacians for solutions of the problem given in (1-1).

First of all, we remark that a solution u_s to (1-1) for a fixed cone C belongs to $\mathcal{L}^1_s$ since $0\le u_s(x)\le |x|^{\gamma_s(C)}$ in $\mathbb{R}^n$ with $\gamma_s(C)\in(0,2s)$. Moreover, by the homogeneity one can rewrite the norm (2-1) as

$$\|u_s\|_{\mathcal{L}^1_s} = \int_{\mathbb{R}^n} \frac{u_s(x)}{(1+|x|)^{n+2s}}\,\mathrm{d}x = \int_{S^{n-1}} u_s\,\mathrm{d}\sigma\int_0^\infty \frac{\rho^{n-1+\gamma_s(C)}}{(1+\rho)^{n+2s}}\,\mathrm{d}\rho = \frac{\Gamma(n+\gamma_s(C))\Gamma(2s-\gamma_s(C))}{\Gamma(n+2s)}\int_{S^{n-1}} u_s\,\mathrm{d}\sigma.$$

In the recent paper [Dipierro et al. 2016] the authors introduced a new notion of fractional Laplacian applying to a wider class of functions which grow more than linearly at infinity. This is achieved by defining an equivalence class of functions modulo polynomials of a fixed order. However, it can hardly be applied to the solutions of (1-1) as they annihilate on a set of nonempty interior.

As shown in [Bogdan and Byczkowski 1999, Definition 3.6], if we consider a smooth function with compact support $\varphi \in C_c^\infty(\mathbb{R}^n)$(or $\varphi \in C_c^2(\mathbb{R}^n)$), we can define the distribution k^{2s} by the formula

$$(-\Delta)^s\varphi(0) = (k^{2s}, \varphi).$$

By this definition, it follows that $(-\Delta)^s\varphi(x) = k^{2s} * \varphi(x)$.

Definition 3.1 [Bogdan and Byczkowski 1999, Definition 3.7]. For $u \in \mathcal{L}_s^1$ we define the *distributional fractional Laplacian* $(-\tilde{\Delta})^s u$ by the formula

$$((-\tilde{\Delta})^s u, \varphi) = (u, (-\Delta)^s\varphi) \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$

In particular, since given an open subset $D \subset \mathbb{R}^n$ and $u \in C^2(D) \cap \mathcal{L}_s^1$, the fractional Laplacian exists as a continuous function of $x \in D$ and $(-\tilde{\Delta})^s u = (-\Delta)^s u$ as a distribution in D [Bogdan and Byczkowski 1999, Lemma 3.8], throughout the paper we will always use $(-\Delta)^s$ both for the classical and the distributional fractional Laplacian. The following is a useful tool for computing the distributional fractional Laplacian.

Lemma 3.2 [Bogdan and Byczkowski 1999, Lemma 3.3]. *Assume that*

$$\iint_{|y-x|>\varepsilon} \frac{|f(x)g(y)|}{|y-x|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y < \infty \quad \textit{and} \quad \int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x < \infty. \tag{3-1}$$

Then $((-\Delta)_\varepsilon^s f, g) = (f, (-\Delta)_\varepsilon^s g)$. *Moreover if* $f \in \mathcal{L}_s^1$ *and* $g \in C_c(\mathbb{R}^n)$, *the assumptions* (3-1) *are satisfied for every* $\varepsilon > 0$.

Before proving the semigroup property, we prove the following lemma which ensures the existence of the δ-Laplacian of the s-Laplacian for $0 < \delta < 1$.

Lemma 3.3. *Let* u_s *be a solution of* (1-1) *with* C *a regular cone. Then we have* $(-\Delta)^s u_s \in \mathcal{L}_\delta^1$ *for any* $\delta > 0$, *i.e.*,

$$\int_{\mathbb{R}^n} \frac{|(-\Delta)^s u_s(x)|}{(1+|x|)^{n+2\delta}}\,\mathrm{d}x < \infty.$$

Proof. Since the function u_s is s-harmonic in C, namely $(-\Delta)^s u_s(x) = 0$ for all $x \in C$, we can restrict the domain of integration to $\mathbb{R}^n \setminus C$.

By homogeneity and the results in [Bogdan and Byczkowski 1999], we have that the function $(-\Delta)^s u_s$ is $(\gamma_s - 2s)$-homogeneous and in particular $x \mapsto (-\Delta)^s u_s(x)$ is a continuous negative function, for every $x \in D \Subset \mathbb{R}^n \setminus C$. In order to compute the previous integral, we focus our attention on the restriction of the fractional Laplacian to the sphere S^{n-1}; in particular, we prove that there exist $\bar{\varepsilon} > 0$ and $C > 0$ such that

$$|(-\Delta)^s u_s(x)| \le \frac{C}{\operatorname{dist}(x, \partial C)^s} \quad \text{for all } x \in N_{\bar{\varepsilon}}(\partial C) \cap S^{n-1}, \tag{3-2}$$

where $N_\varepsilon(\partial C) = \{x \in \mathbb{R}^n \setminus C : \operatorname{dist}(x, \partial C) \le \varepsilon\}$ is the tubular neighborhood of ∂C.

Hence, fixing $R > 0$ small enough, consider initially $\varepsilon < R$ and $x \in S^{n-1} \cap N_\varepsilon(\partial C)$; since $u_s(y) \le |y|^{\gamma_s}$ in $\mathbb{R}^n$ and by (2-2) there exists a constant $C > 0$ such that for every $y \in C$ we have

$$u_s(y) \le C|y|^{\gamma_s - s}\,\mathrm{dist}(y, \partial C)^s,$$

it follows, defining $\delta(x) := \mathrm{dist}(x, \partial C) > 0$, that

$$\begin{aligned}|(-\Delta)^s u_s(x)| &= C(n,s)\int_{C\cap B_R(x)} \frac{u_s(y)}{|x-y|^{n+2s}}\,\mathrm{d}y + C(n,s)\int_{C\setminus B_R(x)} \frac{u_s(y)}{|x-y|^{n+2s}}\,\mathrm{d}y \\ &\le C(n,s)\int_{C\cap B_R(x)} \frac{C|y|^{\gamma_s-s}\,\mathrm{dist}(y,\partial C)^s}{|x-y|^{n+2s}}\,\mathrm{d}y + C(n,s)\int_{C\setminus B_R(x)} \frac{|y|^{\gamma_s}}{|x-y|^{n+2s}}\,\mathrm{d}y.\end{aligned}$$

Since $C \cap B_R(x) \subset B_R(x) \setminus B_{\delta(x)}(x)$, we have

$$\begin{aligned}|(-\Delta)^s u_s(x)| &\le C\int_{R\ge|x-y|\ge\delta(x)} \frac{|y|^{\gamma_s-s}}{|x-y|^{n+s}}\,\mathrm{d}y + \int_{|x-y|\ge R} \frac{(|x-y|+1)^{\gamma_s}}{|x-y|^{n+2s}}\,\mathrm{d}y \\ &\le C\int_{R\ge|x-y|\ge\delta(x)} \frac{1}{|x-y|^{n+s}}\,\mathrm{d}y + \omega_{n-1}\int_R^\infty \frac{(t+1)^{\gamma_s}}{t^{1+2s}}\,\mathrm{d}t \\ &\le C\int_{\delta(x)}^R \frac{1}{r^{1+s}}\,\mathrm{d}r + M \\ &\le C\frac{1}{\mathrm{dist}(x,\partial C)^s} + M.\end{aligned}$$

Moreover, again since $s \in (0, 1)$, considering a smaller neighborhood $N_\varepsilon(\partial C)$, we obtain that there exists a constant $\bar\varepsilon > 0$ small enough and $C > 0$ such that

$$|(-\Delta)^s u_s(x)| \le \frac{C}{\mathrm{dist}(x,\partial C)^s} \quad \text{for every } x \in N_{\bar\varepsilon}(\partial C) \cap S^{n-1}.$$

Now, fixing $\delta > 0$ and considering $\bar\varepsilon > 0$ of (3-2), we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus C} \frac{|(-\Delta)^s u_s(x)|}{(1+|x|)^{n+2\delta}}\,\mathrm{d}x &= \int_{\mathbb{R}^n\setminus C} \frac{|x|^{\gamma_s-2s}|(-\Delta)^s u_s(x/|x|)|}{(1+|x|)^{n+2\delta}}\,\mathrm{d}x \\ &= \int_0^\infty\int_{S^{n-1}\cap(\mathbb{R}^n\setminus C)} \frac{r^{\gamma_s-2s}|(-\Delta)^s u_s(z)|}{(1+r)^{n+2\delta}} r^{n-1}\,\mathrm{d}\sigma(z)\,\mathrm{d}r \\ &= \int_0^\infty \frac{r^{n-1+\gamma_s-2s}}{(1+r)^{n+2\delta}}\,\mathrm{d}r \int_{S^{n-1}\cap(\mathbb{R}^n\setminus C)} |(-\Delta)^s u_s(z)|\,\mathrm{d}\sigma.\end{aligned}$$

Since $\gamma_s \in (0, 2s)$ and $s \in (0, 1)$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus C} \frac{|(-\Delta)^s u_s(x)|}{(1+|x|)^{n+2\delta}}\,\mathrm{d}x &\le C\int_{S^{n-1}\cap N_{\bar\varepsilon}(\partial C)} |(-\Delta)^s u_s(z)|\,\mathrm{d}\sigma + C\int_{((\mathbb{R}^n\setminus C)\setminus N_{\bar\varepsilon}(\partial C))\cap S^{n-1}} |(-\Delta)^s u_s(z)|\,\mathrm{d}\sigma \\ &\le C\int_{S^{n-1}\cap N_{\bar\varepsilon}(\partial C)} \frac{1}{\mathrm{dist}(z,\partial C)^s}\,\mathrm{d}\sigma + M < \infty\end{aligned}$$

where in the second inequality we used that $z \mapsto (-\Delta)^s u_s(z)$ is continuous in every $A \Subset S^{n-1} \cap (\mathbb{R}^n \setminus C)$ and in the last one that $\mathrm{dist}(x, \partial C)^{-s} \in L^1(S^{n-1} \cap N_{\bar\varepsilon}(\partial C), \mathrm{d}\sigma)$. ☐

Proposition 3.4 (distributional semigroup property). *Let u_s be a solution of* (1-1) *with C a regular cone and consider $\delta \in (0, 1-s)$. Then*

$$(-\Delta)^{s+\delta} u_s = (-\Delta)^{\delta}[(-\Delta)^s u_s] \quad \text{in } \mathcal{D}'(C)$$

or equivalently

$$((-\Delta)^{s+\delta} u_s, \varphi) = ((-\Delta)^{\delta}[(-\Delta)^s u_s], \varphi) \quad \text{for all } \varphi \in C_c^\infty(C).$$

Proof. Since $|u_s(x)| \le |x|^{\gamma_s}$, with $\gamma_s \in (0, 2s)$, it is easy to see that $u_s \in \mathcal{L}_s^1 \cap C^2(C)$. Moreover, as we have already remarked, if $u_s \in \mathcal{L}_s^1$ then $u_s \in \mathcal{L}_{s+\delta}^1$ for every $\delta > 0$. In particular, $(-\Delta)^{s+\delta} u_s$ does exist and it is a continuous function of $x \in C$ for every $\delta \in (0, 1-s)$. By the definition of the distributional fractional Laplacian, we obtain

$$((-\Delta)^{s+\delta} u_s, \varphi) = (u_s, (-\Delta)^{s+\delta}\varphi),$$

and since for $\varphi \in C_c^\infty(C) \subset \mathcal{S}(\mathbb{R}^n)$ in the Schwarz space, the classic semigroup property holds, we obtain

$$((-\Delta)^{s+\delta} u_s, \varphi) = (u_s, (-\Delta)^s[(-\Delta)^{\delta}\varphi]).$$

On the other hand, since by Lemma 3.3 we have $(-\Delta)^s u_s \in \mathcal{L}_\delta^1$, it follows that

$$((-\Delta)_\varepsilon^{\delta}[(-\Delta)^s u_s], \varphi) = ((-\Delta)^s u_s, (-\Delta)_\varepsilon^{\delta}\varphi) \tag{3-3}$$

for every $\varepsilon > 0$. Since $(-\Delta)^s u_s \in \mathcal{L}_\delta^1$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$, the δ-Laplacian of $(-\Delta)^s u_s$ does exists in a distributional sense and hence the left-hand side in (3-3) does converge to $((-\Delta)^{\delta}[(-\Delta)^s u_s], \varphi)$ as $\varepsilon \to 0$. Moreover the right-hand side in (3-3) does converge to $((-\Delta)^s u_s, (-\Delta)^{\delta}\varphi)$ by the dominated convergence theorem, using Proposition 2.2 and Lemma 3.3, which give

$$\int_{\mathbb{R}^n} (-\Delta)^s u_s(x)(-\Delta)_\varepsilon^{\delta}\varphi(x)\,\mathrm{d}x \le \int_{\mathbb{R}^n} \frac{|(-\Delta)^s u_s(x)|}{(1+|x|)^{n+2\delta}}\,\mathrm{d}x < \infty.$$

By the previous remarks,

$$((-\Delta)^{\delta}[(-\Delta)^s u_s], \varphi) = ((-\Delta)^s u_s, (-\Delta)^{\delta}\varphi).$$

In order to conclude the proof of the distributional semigroup property, we need to show that

$$(u_s, (-\Delta)^s[(-\Delta)^{\delta}\varphi]) = ((-\Delta)^s u_s, (-\Delta)^{\delta}\varphi), \tag{3-4}$$

which is not a trivial equality, since $(-\Delta)^{\delta}\varphi \in C^\infty(\mathbb{R}^n)$ is no more compactly supported.

Let $\eta \in C_c^\infty(B_2(0))$ be a radial cut-off function such that $\eta \equiv 1$ in $B_1(0)$ and $0 \le \eta \le 1$ in $B_2(0)$, and define $\eta_R(x) = \eta(x/R)$ for $R > 0$. Obviously, since $u_s\eta_R \in C_c(\mathbb{R}^n)$ and $(-\Delta)^{\delta}\varphi \in \mathcal{L}_s^1$, by Lemma 3.2 we have

$$(u_s\eta_R, (-\Delta)_\varepsilon^s[(-\Delta)^{\delta}\varphi]) = ((-\Delta)_\varepsilon^s(u_s\eta_R), (-\Delta)^{\delta}\varphi) \tag{3-5}$$

for every ε, $R > 0$. First, for $R > 0$ fixed, we want to pass to the limit for $\varepsilon \to 0$. For the left-hand side in (3-5), we get the convergence to $(u_s\eta_R, (-\Delta)^s[(-\Delta)^{\delta}\varphi])$ since we can apply the dominated convergence theorem. In fact

$$\int_{\mathbb{R}^n} u_s\eta_R(-\Delta)_\varepsilon^s[(-\Delta)^{\delta}\varphi] \le c\int_K (-\Delta)^{s+\delta}\varphi < \infty,$$

where K denotes the support of $u_s\eta_R$. For the right-hand side in (3-5) we observe that, for any $x \in \mathbb{R}^n$,

$$(-\Delta)^s_\varepsilon(u_s\eta_R)(x) = \eta_R(x)(-\Delta)^s_\varepsilon u_s(x) + u_s(x)(-\Delta)^s_\varepsilon \eta_R(x) - I_\varepsilon(u_s, \eta_R)(x),$$

where

$$I_\varepsilon(u_s, \eta_R)(x) = C(n, s)\int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{(u_s(x) - u_s(y))(\eta_R(x) - \eta_R(y))}{|x - y|^{n+2s}}\, \mathrm{d}y.$$

Obviously the first term $((-\Delta)^s_\varepsilon u_s, \eta_R(-\Delta)^\delta\varphi)$ converges to $((-\Delta)^s u_s, \eta_R(-\Delta)^\delta\varphi)$ by the definition of the distributional s-Laplacian, since $u_s \in \mathcal{L}^1_s$ and $\eta_R(-\Delta)^\delta\varphi \in C^\infty_c(\mathbb{R}^n)$. The second term $(u_s(-\Delta)^s_\varepsilon\eta_R, (-\Delta)^\delta\varphi)$ converges to $(u_s(-\Delta)^s\eta_R, (-\Delta)^\delta\varphi)$ by dominated convergence, since

$$\int_{\mathbb{R}^n} u_s(-\Delta)^s_\varepsilon\eta_R(-\Delta)^\delta\varphi\, \mathrm{d}x \le c\int_{\mathbb{R}^n} \frac{u_s(x)}{(1 + |x|)^{n+2s}}\, \mathrm{d}x.$$

Finally, the last term $(I_\varepsilon(u_s, \eta_R), (-\Delta)^\delta\varphi)$ converges to $(I(u_s, \eta_R), (-\Delta)^\delta\varphi)$ by dominated convergence, since

$$\int_{\mathbb{R}^n} I_\varepsilon(u_s, \eta_R)(-\Delta)^\delta\varphi\, \mathrm{d}x \le C\int_{\mathbb{R}^n} |(-\Delta)^\delta\varphi|\, \mathrm{d}x,$$

which is integrable by Proposition 2.2. Finally, passing to the limit for $\varepsilon \to 0$, from (3-5) we get

$$(u_s\eta_R, (-\Delta)^s[(-\Delta)^\delta\varphi]) = ((-\Delta)^s(u_s\eta_R), (-\Delta)^\delta\varphi) \tag{3-6}$$

for every $R > 0$.

Now we want to prove (3-4), concluding this proof, by passing to the limit in (3-6) for $R \to \infty$. Since we know, by dominated convergence, that the left-hand side of (3-6) converges to $(u_s, (-\Delta)^s(-\Delta)^\delta\varphi)$ for $R \to \infty$, we focus our attention on the other side. At this point, we need to prove that for any $\varphi \in C^\infty_c(C)$,

$$\int_{\mathbb{R}^n} (-\Delta)^s(u_s\eta_R)(-\Delta)^\delta\varphi \to \int_{\mathbb{R}^n} (-\Delta)^s u_s(-\Delta)^\delta\varphi \tag{3-7}$$

as $R \to \infty$. First of all, we remark that $(-\Delta)^s(u_s\eta_R) \to (-\Delta)^s u_s$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. In fact, let $K \subset \mathbb{R}^n$ be a compact set. There exists $\bar r > 0$ such that $K \subset B_{\bar r}$. Then, considering any radius $R > \bar r$, we have $\eta_R(x) = 1$ for any $x \in K$. Hence, for any $R > \bar r$, using the fact that $u_s(x) = |x|^{\gamma_s}u_s(x/|x|)$, we obtain

$$\begin{aligned}
&\int_K |(-\Delta)^s(u_s\eta_R)(x) - (-\Delta)^s u_s(x)|\, \mathrm{d}x \\
&\qquad = \int_K \mathrm{d}x \left| C(n, s)\ \text{p.v.}\int_{\mathbb{R}^n} \frac{u_s(x)\eta_R(x) - u_s(y)\eta_R(y) + u_s(y) - u_s(x)}{|x - y|^{n+2s}}\, \mathrm{d}y \right| \\
&\qquad = C(n, s)\int_K \mathrm{d}x \left(\text{p.v.}\int_{C\setminus B_R} \frac{u_s(y)[1 - \eta_R(y)]}{|x - y|^{n+2s}}\, \mathrm{d}y\right) \\
&\qquad \le C(n, s)\int_K \mathrm{d}x \left(\text{p.v.}\int_{C\setminus B_R} \frac{|y|^{\gamma_s}}{(|y| - \bar r)^{n+2s}}\, \mathrm{d}y\right) \\
&\qquad \le C(n, s)\int_K \mathrm{d}x \left(\text{p.v.}\int_{C\setminus B_R} \frac{|y|^{\gamma_s}}{|y|^{n+2s}(1 - \bar r/R)^{n+2s}}\, \mathrm{d}y\right) \\
&\qquad = C\left(\frac{R}{R - \bar r}\right)^{n+2s} \lim_{\rho\to\infty}\int_R^\rho \frac{1}{r^{2s-\gamma_s+1}}\, \mathrm{d}r = C\left(\frac{R}{R - \bar r}\right)^{n+2s} \frac{1}{R^{2s-\gamma_s}} \to 0
\end{aligned}$$

as $R \to \infty$. Hence we obtain also pointwise convergence almost everywhere. Moreover, we can give the following expression:

$$(-\Delta)^s(u_s\eta_R)(x) = \eta_R(x)(-\Delta)^s u_s(x) + C(n,s)\ \text{p.v.}\int_{\mathbb{R}^n} u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\,\mathrm{d}y. \tag{3-8}$$

We remark that $\eta_R(x)(-\Delta)^s u_s(x) \to (-\Delta)^s u_s(x)$ and

$$\int_{\mathbb{R}^n} u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\,\mathrm{d}y \to 0$$

pointwisely. Moreover, we can dominate the first term in the following way:

$$\eta_R(x)(-\Delta)^s u_s(x) \le (-\Delta)^s u_s(x),$$

and

$$\int_{\mathbb{R}^n} (-\Delta)^s u_s(x)(-\Delta)^\delta \varphi(x)\,\mathrm{d}x < \infty$$

since $(-\Delta)^s u_s \in \mathcal{L}^1_\delta$ and using Proposition 2.2 over $\varphi \in C_c^\infty(C)$. In order to prove (3-7), we want to apply the dominated convergence theorem, and hence we need the following condition for any $R > 0$:

$$I := \left|\int_{\mathbb{R}^n} (-\Delta)^\delta\varphi(x)\left(\text{p.v.}\int_{\mathbb{R}^n} u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\,\mathrm{d}y\right)\mathrm{d}x\right| \le c.$$

Therefore, we will obtain a stronger condition, that is, the existence of a value $k > 0$ such that for any $R > 1$

$$I \le \frac{c}{R^k}.$$

We split the region of integration $\mathbb{R}^n \times \mathbb{R}^n$ into five different parts

$$\Omega_1 := (\mathbb{R}^n \setminus B_{2R}) \times \mathbb{R}^n, \quad \Omega_2 := B_{2R} \times B_{2R}, \quad \Omega_3 := (B_{2R}\setminus B_R) \times (B_{3R}\setminus B_{2R}),$$
$$\Omega_4 := (B_{2R}\setminus B_R) \times (\mathbb{R}^n \setminus B_{3R}), \quad \Omega_5 := B_R \times (\mathbb{R}^n \setminus B_{2R}).$$

First of all, we remark that

$$(-\Delta)^s\eta_R(x) = R^{-2s}(-\Delta)^s\eta(x/R)$$

and also that $\|(-\Delta)^s\eta\|_{L^\infty(\mathbb{R}^n)} < \infty$. For the first term, using the fact that $\eta_R(x) - \eta_R(y) = 0$ if $(x,y) \in (\mathbb{R}^n \setminus B_{2R}) \times (\mathbb{R}^n \setminus B_{2R})$,

$$\begin{aligned}
I_1 &:= \int_{\mathbb{R}^n\setminus B_{2R}} |(-\Delta)^\delta\varphi(x)| \left|\int_{\mathbb{R}^n} u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\,\mathrm{d}y\right|\mathrm{d}x \\
&\le \int_{\mathbb{R}^n\setminus B_{2R}} |(-\Delta)^\delta\varphi(x)| \left|\int_{B_{2R}} u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\right|\mathrm{d}x \\
&\le \int_{\mathbb{R}^n\setminus B_{2R}} |(-\Delta)^\delta\varphi(x)|\,(\sup_{B_{2R}} u_s)\,|(-\Delta)^s\eta_R(x)|\,\mathrm{d}x \\
&\le \frac{c}{R^{2s-\gamma_s}}\int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+2\delta}}\,\mathrm{d}x \le \frac{c}{R^{2s-\gamma_s}}.
\end{aligned}$$

For the second term, using the fact that $\eta_R(x)-\eta_R(y)\ge 0$ if $(x,y)\in B_{2R}\times(\mathbb{R}^n\setminus B_{2R})$, we obtain as before

$$\begin{aligned} I_2 &:= \int_{B_{2R}} |(-\Delta)^\delta \varphi(x)| \left| \int_{B_{2R}} u_s(y) \frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\,\mathrm{d}y \right| \mathrm{d}x \\ &\le \int_{B_{2R}} |(-\Delta)^\delta \varphi(x)| \,(\sup_{B_{2R}} u_s)\, |(-\Delta)^s \eta_R(x)|\,\mathrm{d}x \\ &\le \frac{c}{R^{2s-\gamma_s}} \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+2\delta}}\,\mathrm{d}x \le \frac{c}{R^{2s-\gamma_s}}. \end{aligned}$$

For the third part

$$I_3 := \int_{B_{2R}\setminus B_R} |(-\Delta)^\delta \varphi(x)| \left| \int_{B_{3R}\setminus B_{2R}} u_s(y) \frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\,\mathrm{d}y \right| \mathrm{d}x,$$

we consider the change of variables

$$\xi = \frac{x}{R} \in B_2\setminus B_1, \quad \zeta = \frac{y}{R} \in B_3 \setminus B_2.$$

Hence, using the γ_s-homogeneity of u_s and the definition of our cut-off functions, we obtain

$$I_3 \le \frac{R^{2n}}{R^{n+2s-\gamma_s}} \iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)} |(-\Delta)^\delta \varphi(R\xi)|\, u_s(\zeta) \frac{\eta(\xi)-\eta(\zeta)}{|\xi-\zeta|^{n+2s}}\,\mathrm{d}\xi\,\mathrm{d}\zeta.$$

We use the fact that $u_s\in C^{0,s}(B_3\setminus B_1)$, (see (2-2) proved in [Michalik 2006]) and the cut-off function $\eta\in \mathrm{Lip}(B_3\setminus B_1)$; that is, there exists a constant $c>0$ such that

$$\begin{aligned} |u_s(\xi)-u_s(\zeta)| &\le c|\xi-\zeta|^s, \\ |\eta(\xi)-\eta(\zeta)| &\le c|\xi-\zeta| \end{aligned} \tag{3-9}$$

for every $\xi,\zeta\in B_3\setminus B_1$. Hence,

$$\begin{aligned} I_3 &\le \frac{R^{2n}}{R^{n+2s-\gamma_s}} \iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)} |(-\Delta)^\delta \varphi(R\xi)| \frac{|u_s(\zeta)-u_s(\xi)|\,|\eta(\xi)-\eta(\zeta)|}{|\xi-\zeta|^{n+2s}}\,\mathrm{d}\xi\,\mathrm{d}\zeta \\ &\qquad + \frac{R^{2n}}{R^{n+2s-\gamma_s}} \iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)} |(-\Delta)^\delta \varphi(R\xi)|\, u_s(\xi) \frac{|\eta(\xi)-\eta(\zeta)|}{|\xi-\zeta|^{n+2s}}\,\mathrm{d}\xi\,\mathrm{d}\zeta \\ &= J_1+J_2. \end{aligned}$$

By (3-9), we obtain

$$\begin{aligned} J_1 &\le c\frac{R^{2n}}{R^{n+2s-\gamma_s}} \iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)} |(-\Delta)^\delta \varphi(R\xi)| \frac{|\xi-\zeta|^{s+1}}{|\xi-\zeta|^{n+2s}}\,\mathrm{d}\xi\,\mathrm{d}\zeta \\ &\le c\frac{R^{2n}}{R^{n+2s-\gamma_s}} \iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)} \frac{1}{(1+R|\xi|)^{n+2\delta}} \frac{1}{|\xi-\zeta|^{n+s-1}}\,\mathrm{d}\xi\,\mathrm{d}\zeta \\ &\le \frac{c}{R^{2s+2\delta-\gamma_s}} \iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)} \frac{1}{|\xi-\zeta|^{n+s-1}}\,\mathrm{d}\xi\,\mathrm{d}\zeta \le \frac{c}{R^{2s+2\delta-\gamma_s}}. \end{aligned}$$

Moreover, using the additional changes of variable

$$(\xi,\zeta)\mapsto(\xi,\xi+h),\quad (\xi,\zeta)\mapsto(\xi,\xi-h),$$

we obtain

$$\begin{aligned}
J_2 &\le \frac{R^{2n}}{R^{n+2s-\gamma_s}}\iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)} |(-\Delta)^\delta\varphi(R\xi)|\, u_s(\xi)\frac{\eta(\xi)-\eta(\zeta)}{|\xi-\zeta|^{n+2s}}\,\mathrm{d}\xi\,\mathrm{d}\zeta\\
&\le \frac{R^{2n}}{R^{n+2s-\gamma_s}}\iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)} \frac{1}{(1+R|\xi|)^{n+2\delta}}\, u_s(\xi)\frac{\eta(\xi)-\eta(\zeta)}{|\xi-\zeta|^{n+2s}}\,\mathrm{d}\xi\,\mathrm{d}\zeta\\
&\le \frac{c}{R^{2s+2\delta-\gamma_s}}\iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)} \frac{\eta(\xi)-\eta(\zeta)}{|\xi-\zeta|^{n+2s}}\,\mathrm{d}\xi\,\mathrm{d}\zeta\\
&\le \frac{c}{R^{2s+2\delta-\gamma_s}}\iint_{(B_2\setminus B_1)\times B_2} \frac{2\eta(\xi)-\eta(\xi+h)-\eta(\xi-h)}{|h|^{n+2s}}\,\mathrm{d}\xi\,\mathrm{d}h\\
&\le \frac{c}{R^{2s+2\delta-\gamma_s}}\left(c+\iint_{(B_2\setminus B_1)\times B_\varepsilon} \frac{\langle\nabla^2\eta(\xi)h,h\rangle}{|h|^{n+2s}}\,\mathrm{d}\xi\,\mathrm{d}h\right)\\
&\le \frac{c}{R^{2s+2\delta-\gamma_s}}\left(c+\iint_{(B_2\setminus B_1)\times B_\varepsilon} \frac{1}{|h|^{n+2s-2}}\,\mathrm{d}\xi\,\mathrm{d}h\right)\le \frac{c}{R^{2s+2\delta-\gamma_s}}.
\end{aligned}$$

For the fourth part

$$I_4 := \int_{B_{2R}\setminus B_R} |(-\Delta)^\delta\varphi(x)|\left|\int_{\mathbb{R}^n\setminus B_{3R}} u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\,\mathrm{d}y\right|\mathrm{d}x,$$

we consider, as before, the change of variables

$$\xi=\frac{x}{R}\in B_2\setminus B_1,\quad \zeta=\frac{y}{R}\in\mathbb{R}^n\setminus B_3.$$

Hence,

$$\begin{aligned}
I_4 &\le c\frac{R^{2n}}{R^{n+2s-\gamma_s}}\iint_{(B_2\setminus B_1)\times(\mathbb{R}^n\setminus B_3)} |(-\Delta)^\delta\varphi(R\xi)|\frac{|\zeta|^{\gamma_s}}{|\zeta-\xi|^{n+2s}}\,\mathrm{d}\xi\,\mathrm{d}\zeta\\
&\le c\frac{R^{2n}}{R^{n+2s-\gamma_s}}\iint_{(B_2\setminus B_1)\times(\mathbb{R}^n\setminus B_3)} \frac{1}{(1+R|\xi|)^{n+2\delta}}\frac{|\zeta|^{\gamma_s}}{\big|\zeta-2\zeta/|\zeta|\big|^{n+2s}}\,\mathrm{d}\xi\,\mathrm{d}\zeta\\
&\le \frac{c}{R^{2s+2\delta-\gamma_s}}\iint_{(B_2\setminus B_1)\times(\mathbb{R}^n\setminus B_3)} \frac{|\zeta|^{\gamma_s}}{|\zeta|^{n+2s}(1-2/|\zeta|)^{n+2s}}\,\mathrm{d}\xi\,\mathrm{d}\zeta\\
&\le \frac{c}{R^{2s+2\delta-\gamma_s}}\iint_{(B_2\setminus B_1)\times(\mathbb{R}^n\setminus B_3)} \frac{1}{|\zeta|^{n+2s-\gamma_s}}\,\mathrm{d}\xi\,\mathrm{d}\zeta\le\frac{c}{R^{2s+2\delta-\gamma_s}}.
\end{aligned}$$

Finally we consider the last term

$$I_5 := \int_{B_R} |(-\Delta)^\delta\varphi(x)|\left|\int_{\mathbb{R}^n\setminus B_{2R}} u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\,\mathrm{d}y\right|\mathrm{d}x.$$

Hence we obtain

$$\begin{aligned}
I_5 &\le c\int_{B_R} |(-\Delta)^\delta\varphi(x)|\left(\int_{\mathbb{R}^n\setminus B_{2R}} \frac{|y|^{\gamma_s}}{|y-x|^{n+2s}}\,\mathrm{d}y\right)\mathrm{d}x\\
&\le c\int_{B_R} |(-\Delta)^\delta\varphi(x)|\left(\int_{\mathbb{R}^n\setminus B_{2R}} \frac{|y|^{\gamma_s}}{\big|y-Ry/|y|\big|^{n+2s}}\,\mathrm{d}y\right)\mathrm{d}x\\
&\le c\int_{B_R} |(-\Delta)^\delta\varphi(x)|\left(\int_{\mathbb{R}^n\setminus B_{2R}} \frac{|y|^{\gamma_s}}{|y|^{n+2s}(1-R/|y|)^{n+2s}}\,\mathrm{d}y\right)\mathrm{d}x\\
&\le c\int_{B_R} |(-\Delta)^\delta\varphi(x)|\left(\int_{\mathbb{R}^n\setminus B_{2R}} \frac{1}{|y|^{n+2s-\gamma_s}}\,\mathrm{d}y\right)\mathrm{d}x\\
&\le c\left(\int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+2\delta}}\,\mathrm{d}x\right)\left(\int_{2R}^{\infty} \frac{1}{r^{1+2s-\gamma_s}}\,\mathrm{d}r\right)\\
&= c\left(\int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+2\delta}}\,\mathrm{d}x\right)\left(\lim_{\rho\to\infty}\int_{2R}^{\rho} \frac{1}{r^{1+2s-\gamma_s}}\,\mathrm{d}r\right) \le \frac{c}{R^{2s-\gamma_s}}.
\end{aligned}$$

Since $I \le \sum_{i=1}^5 I_i$, we obtain the desired result. □

At this point, fixing $s \in (0, 1)$, by the distributional semigroup property we can compute easily the high-order fractional Laplacian $(-\Delta)^{s+\delta}$ viewing it as the δ-Laplacian of the s-Laplacian.

Corollary 3.5. *Let C be a regular cone. For every $\delta \in (0, 1-s)$, the solution u_s of* (1-1) *is $(s+\delta)$-superharmonic in C in the sense of distribution*; *i.e.*,

$$((-\Delta)^{s+\delta}u_s, \varphi) \ge 0$$

for every test function $\varphi \in C_c^\infty(C)$ nonnegative in C.

Moreover, *u_s is also superharmonic in C in the sense of distribution*; *i.e.*,

$$(-\Delta u_s, \varphi) \ge 0$$

for every test function $\varphi \in C_c^\infty(C)$ nonnegative in C.

Proof. As said before, the facts that $u_s \in \mathcal{L}^1_{s+\delta}$ and $u_s \in C^2(A)$ for every $A \Subset C$ ensure the existence of the $(-\Delta)^{s+\delta}u_s$ and the continuity of the map $x \mapsto (-\Delta)^{s+\delta}u_s(x)$ for every $x \in A \Subset C$. Hence at this point, the only part we need to prove is the positivity of the $(s+\delta)$-Laplacian in the sense of the distribution, which is a direct consequence of the previous result. Indeed, since u_s is a solution of the problem (1-1), by Proposition 3.4 we know that for every $\varphi \in C_c^\infty(C)$ we have

$$\begin{aligned}
((-\Delta)^{s+\delta}u_s, \varphi) &= ((-\Delta)^\delta[(-\Delta)^s u_s], \varphi)\\
&= \int_C \varphi(x)\ \mathrm{p.v.}\!\int_{\mathbb{R}^n} \frac{(-\Delta)^s u_s(x) - (-\Delta)^s u_s(y)}{|x-y|^{n+2\delta}}\,\mathrm{d}y\,\mathrm{d}x,
\end{aligned}$$

where $(-\Delta)^\delta[(-\Delta)^s u_s]$ is well-defined since $(-\Delta)^s u_s \equiv 0 \in C^2(A)$ for every $A \Subset C$ and, by Lemma 3.3, $(-\Delta)^s u_s \in \mathcal{L}^1_\delta$ for every $\delta \in (0, 1-s)$.

Consider now the nonnegative test function $\varphi \geq 0$ in C. Since $(-\Delta)^s u_s(x) = 0$ for every $x \in C$, we have for every $x \in \mathbb{R}^n \setminus \overline{C}$

$$(-\Delta)^s u_s(x) = -\int_C \frac{u_s(y)}{|x-y|^{n+2s}}\,\mathrm{d}y \leq 0.$$

Similarly,

$$((-\Delta)^\delta[(-\Delta)^s u_s], \varphi) = \int_C \varphi(x) \int_{\mathbb{R}^n} \frac{-(-\Delta)^s u_s(y)}{|x-y|^{n+2\delta}}\,\mathrm{d}y\,\mathrm{d}x \geq 0,$$

since the support of φ is compact in the cone C, and so there exists $\varepsilon > 0$ such that $|x-y| > \varepsilon$ in the above integral. We have obtained that for any $\delta \in (0, 1-s)$ and any nonnegative $\varphi \in C_c^\infty(C)$

$$((-\Delta)^{s+\delta} u_s, \varphi) \geq 0.$$

Then, passing to the limit for $\delta \to 1-s$, the function u_s is superharmonic in the distributional sense

$$0 \leq \lim_{\delta\to 1-s} ((-\Delta)^{s+\delta} u_s, \varphi) = \lim_{\delta\to 1-s} (u_s, (-\Delta)^{s+\delta}\varphi) = (u_s, -\Delta\varphi) = (-\Delta u_s, \varphi). \qquad \square$$

Monotonicity of $s \mapsto \gamma_s(C)$. The following proposition is a consequence of Corollary 3.5 and it follows essentially the proof of Lemma 2 in [Bogdan et al. 2015].

Proposition 3.6. *For any fixed regular cone C with vertex in* 0, *the map $s \mapsto \gamma_s(C)$ is monotone nondecreasing in* (0, 1).

Proof. Fixing the cone C, let us denote by γ_s and $\gamma_{s+\delta}$ respectively the homogeneities of u_s and $u_{s+\delta}$. Let us suppose by way of contradiction that $\gamma_s > \gamma_{s+\delta}$ for a $\delta \in (0, 1-s)$, and let us consider the function

$$h(x) = u_{s+\delta}(x) - u_s(x) \quad \text{in } \mathbb{R}^n,$$

where u_s is the homogeneous solution of (1-1) and $u_{s+\delta}$ is the unique, up to multiplicative constants, nonnegative nontrivial homogeneous and continuous-in-$\mathbb{R}^n$ solution for

$$\begin{cases} (-\Delta)^{s+\delta} u = 0 & \text{in } C, \\ u = 0 & \text{in } \mathbb{R}^n \setminus C \end{cases}$$

of the form

$$u_{s+\delta}(x) = |x|^{\gamma_{s+\delta}} u_{s+\delta}\left(\frac{x}{|x|}\right).$$

The function h is continuous in $\mathbb{R}^n$ and $h(x) = 0$ in $\mathbb{R}^n \setminus C$. We want to prove that $h(x) \leq 0$ in $\mathbb{R}^n \setminus (C \cap B_1)$. Since $h = 0$ outside the cone, we can consider only what happens in $C \setminus B_1$. As we already quoted, we have

$$c_1(s)|x|^{\gamma_s - s}\operatorname{dist}(x, \partial C)^s \leq u_s(x) \leq c_2(s)|x|^{\gamma_s - s}\operatorname{dist}(x, \partial C)^s \tag{3-10}$$

for any $x \in \overline{C} \setminus \{0\}$, and there exist two constants $c_1(s+\delta), c_2(s+\delta) > 0$ such that

$$c_1(s+\delta)|x|^{\gamma_{s+\delta}-(s+\delta)}\operatorname{dist}(x, \partial C)^{s+\delta} \leq u_{s+\delta}(x) \leq c_2(s+\delta)|x|^{\gamma_{s+\delta}-(s+\delta)}\operatorname{dist}(x, \partial C)^{s+\delta}.$$

We can choose u_s and $u_{s+\delta}$ so that $c := c_1(s) = c_2(s+\delta)$ since they are defined up to a multiplicative constant. Then, for any $x \in C \setminus B_1$, since $|x|^{\gamma_{s+\delta}} \le |x|^{\gamma_s}$, we have

$$h(x) \le c|x|^{\gamma_s} \operatorname{dist}(x, \partial C)^s \left[\frac{\operatorname{dist}(x, \partial C)^\delta}{|x|^\delta} - 1 \right] \le 0. \tag{3-11}$$

In fact, if we take x such that $\operatorname{dist}(x, \partial C) \le 1$, then (3-11) follows by

$$\frac{\operatorname{dist}(x, \partial C)^\delta}{|x|^\delta} - 1 \le \operatorname{dist}(x, \partial C)^\delta - 1 \le 0.$$

Instead, if we consider x so that $\operatorname{dist}(x, \partial C) > 1$, then $\operatorname{dist}(x, \partial C)^\delta < |x|^\delta$ and hence (3-11) follows.

Now we want to show that there exists a point $x_0 \in C \cap B_1$ such that $h(x_0) > 0$. Let us take a point $\bar{x} \in S^{n-1} \cap C$ and let $\alpha := u_{s+\delta}(\bar{x}) > 0$ and $\beta := u_s(\bar{x}) > 0$. Hence, there exists a small $r > 0$ so that $\alpha r^{\gamma_{s+\delta}} > \beta r^{\gamma_s}$, and so, taking x_0 with $|x_0| = r$ and so that $x_0/|x_0| = \bar{x}$, we obtain $h(x_0) > 0$.

If we consider the restriction of h to $\overline{C \cap B_1}$, which is continuous on a compact set, by the previous arguments and the Weierstrass theorem, there exists a maximum point $x_1 \in C \cap B_1$ for the function h which is global in $\mathbb{R}^n$ and is strict at least in a set of positive measure. Hence,

$$(-\Delta)^{s+\delta} h(x_1) = C(n, s) \text{ p.v.} \int_{\mathbb{R}^n} \frac{h(x_1) - h(y)}{|x_1 - y|^{n+2(s+\delta)}} \, dy > 0,$$

and since $(-\Delta)^{s+\delta} h$ is a continuous function in the open cone, there exists an open set $U(x_1)$ with $\overline{U(x_1)} \subset C$ such that

$$(-\Delta)^{s+\delta} h(x) > 0 \quad \text{for all } x \in U(x_1).$$

But thanks to Corollary 3.5 we obtain a contradiction since for any nonnegative $\varphi \in C_c^\infty(U(x_1))$

$$((-\Delta)^{s+\delta} h, \varphi) = ((-\Delta)^{s+\delta} u_{s+\delta}, \varphi) - ((-\Delta)^{s+\delta} u_s, \varphi) = -((-\Delta)^{s+\delta} u_s, \varphi) \le 0. \qquad \square$$

With the same arguments as the previous proof we can show also the following useful upper bound.

Proposition 3.7. *For any fixed regular cone C with vertex in* 0 *and any* $s \in (0, 1)$, *we have* $\gamma_s(C) \le \gamma(C)$.

Proof. Seeking a contradiction, we suppose that there exists $s \in (0, 1)$ such that $\gamma_s > \gamma$. Hence we define the function

$$h(x) = u(x) - u_s(x) \quad \text{in } \mathbb{R}^n,$$

where u_s and u are respectively solutions to (1-1) and

$$\begin{cases} -\Delta u = 0 & \text{in } C, \\ u = 0 & \text{in } \mathbb{R}^n \setminus C. \end{cases} \tag{3-12}$$

We recall that these solutions are unique, up to multiplicative constants, nonnegative, nontrivial, homogeneous and continuous in $\mathbb{R}^n$ and of the form

$$u(x) = |x|^\gamma u\left(\frac{x}{|x|}\right), \quad u_s(x) = |x|^{\gamma_s} u_s\left(\frac{x}{|x|}\right)$$

for some $\gamma_s \in (0, 2s)$ and $\gamma \in (0, \infty)$. The function h is continuous in $\mathbb{R}^n$ and $h(x) = 0$ in $\mathbb{R}^n \setminus C$. We want to prove that $h(x) \le 0$ in $\mathbb{R}^n \setminus (C \cap B_1)$. Since $h = 0$ outside the cone, we can consider only what happens in $C \setminus B_1$. So, there exist two constants $c_1(s), c_2(s) > 0$ such that, for any $x \in \bar{C} \setminus \{0\}$, (3-10) holds. Moreover there exist two constants $c_1, c_2 > 0$ such that

$$c_1|x|^{\gamma-1}\,\mathrm{dist}(x, \partial C) \le u(x) \le c_2|x|^{\gamma-1}\,\mathrm{dist}(x, \partial C).$$

We can choose u_s and u so that $c := c_1(s) = c_2$ since they are defined up to a multiplicative constant. Then, for any $x \in C \setminus B_1$, since $|x|^\gamma \le |x|^{\gamma_s}$, we have

$$h(x) \le c|x|^{\gamma_s}\,\mathrm{dist}(x, \partial C)^s \left[\frac{\mathrm{dist}(x, \partial C)^{1-s}}{|x|^{1-s}} - 1 \right] \le 0$$

by the same arguments as the previous proof.

Now we want to show that there exists a point $x_0 \in C \cap B_1$ such that $h(x_0) > 0$. Let us take a point $\bar{x} \in S^{n-1} \cap C$ and let $\alpha := u(\bar{x}) > 0$ and $\beta := u_s(\bar{x}) > 0$. Hence, there exists a small $r > 0$ such that $\alpha r^\gamma > \beta r^{\gamma_s}$, and so, taking x_0 with $|x_0| = r$ and so that $x_0/|x_0| = \bar{x}$, we obtain $h(x_0) > 0$.

If we consider the restriction of h to $\overline{C \cap B_1}$, which is continuous on a compact set, by the previous arguments and the Weierstrass theorem, there exists at least a maximum point in $C \cap B_1$ for the function h which is global in $\mathbb{R}^n$. Moreover, since h cannot be constant on $C \cap B_1$ and it is of class C^2 inside the cone, there exists a global maximum $y \in C \cap B_1$ such that, up to a rotation, $\partial^2_{x_i x_i} h(y) \le 0$ for any $i = 1, \dots, n$ and $\partial^2_{x_j x_j} h(y) < 0$ for at least a coordinate direction. Hence

$$\Delta h(y) = \sum_{i=1}^n \partial^2_{x_i x_i} h(y) < 0.$$

By the continuity of Δh in the open cone, there exists an open set $U(y)$ with $\overline{U(y)} \subset C$ such that

$$\Delta h(x) < 0 \quad \text{for all } x \in U(y).$$

Since, by Corollary 3.5 for any nonnegative $\varphi \in C_c^\infty(U(y))$

$$(-\Delta u_s, \varphi) \ge 0,$$

we have

$$(\Delta h, \varphi) = (\Delta u, \varphi) - (\Delta u_s, \varphi) = (-\Delta u_s, \varphi) \ge 0,$$

and this is a contradiction. □

Asymptotic behavior of $C(n, s)/(2s - \gamma_s(C))$. Let us define for any regular cone C the limit

$$\mu(C) = \lim_{s \to 1^-} \frac{C(n, s)}{2s - \gamma_s(C)} \in [0, \infty].$$

Obviously, thanks to the monotonicity of $s \mapsto \gamma_s(C)$ in $(0, 1)$, this limit does exist, but we want to show that $\mu(C)$ cannot be infinite. At this point, this situation can happen since $2s - \gamma_s(C)$ can converge to zero and we do not have enough information about this convergence. The study of this limit depends on the cone C itself and so we will consider separately the cases of wide cones and narrow cones, which are respectively when $\gamma(C) < 2$ and when $\gamma(C) \geq 2$. In this section, we prove this result just for regular cones, while in Section 4 we will extend the existence of a finite limit $\mu(C)$ to any unbounded cone, without the monotonicity result of Proposition 3.6.

Wide cones: $\gamma(C) < 2$. We remark that, fixing a wide cone $C \subset \mathbb{R}^n$, there exist $\varepsilon > 0$ and $s_0 \in (0, 1)$, both depending on C, such that for any $s \in [s_0, 1)$

$$2s - \gamma_s(C) \geq \varepsilon > 0.$$

In fact we know that $s \mapsto \gamma_s(C)$ is monotone nondecreasing in $(0, 1)$ and $0 < \gamma_s(C) \leq \gamma(C) < 2$. Hence, defining $\bar{\gamma}(C) = \lim_{s\to 1} \gamma_s(C) \in (0, 2)$ we can choose

$$s_0 := \tfrac{1}{4}(\bar{\gamma}(C) - 2) + 1 \in \left(\tfrac{1}{2}, 1\right) \quad \text{and} \quad \varepsilon := \tfrac{1}{2}(2 - \bar{\gamma}(C)) > 0,$$

obtaining

$$2s - \gamma_s(C) \geq 2s_0 - \bar{\gamma}(C) = \varepsilon > 0.$$

As a consequence we have $\mu(C) = 0$ for any wide cone.

Narrow cones: $\gamma(C) \geq 2$. Before addressing the asymptotic analysis for any regular cone, we focus our attention on the spherical-caps ones with "small" aperture. Hence, let us fix $\theta_0 \in (0, \pi/4)$ and for any $\theta \in (0, \theta_0]$, let

$$\lambda_1(\theta) := \lambda_1(\omega_\theta) = \min_{\substack{u \in H_0^1(S^{n-1}\cap C_\theta)\\ u\neq 0}} \frac{\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 \,d\sigma}{\int_{S^{n-1}} u^2 \,d\sigma}.$$

We have that $\lambda_1(\theta) > 2n$, and hence the following problem is well defined:

$$\mu_0(\theta) := \min_{\substack{u \in H_0^1(S^{n-1}\cap C_\theta)\\ u\neq 0}} \frac{\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 - 2nu^2 \,d\sigma}{\left(\int_{S^{n-1}} |u| \,d\sigma\right)^2}. \tag{3-13}$$

This number $\mu_0(\theta)$ is strictly positive and achieved by a nonnegative $\varphi \in H_0^1(S^{n-1} \cap C_\theta) \setminus \{0\}$ which is strictly positive on $S^{n-1} \cap C_\theta$ and is obviously a solution to

$$\begin{cases} -\Delta_{S^{n-1}}\varphi = 2n\varphi + \mu_0(\theta)\int_{S^{n-1}} \varphi \,d\sigma & \text{in } S^{n-1} \cap C_\theta, \\ \varphi = 0 & \text{in } S^{n-1} \setminus C_\theta, \end{cases} \tag{3-14}$$

where $-\Delta_{S^{n-1}}$ is the Laplace–Beltrami operator on the unitary sphere S^{n-1}.

Let now v be the 0-homogeneous extension of φ to the whole of $\mathbb{R}^n$ and $r(x) := |x|$. Such a function will be a solution to

$$\begin{cases} -\Delta v = 2nv/r^2 + \mu_0(\theta)/r^2 \int_{S^{n-1}} v \,d\sigma & \text{in } C_\theta, \\ v = 0 & \text{in } \mathbb{R}^n \setminus C_\theta. \end{cases} \tag{3-15}$$

Since the spherical cap $C_\theta \cap S^{n-1}$ is an analytic submanifold of S^{n-1} and the data $(\partial C_\theta \cap S^{n-1}, 0, \partial_\nu\varphi)$ are not characteristic, by the classic theorem of Cauchy and Kovalevskaya we can extend the solution φ

of (3-14) to a function $\tilde{\varphi}$, which is defined in an enlarged cone and satisfies

$$\begin{cases} -\Delta_{S^{n-1}}\tilde{\varphi} = 2n\tilde{\varphi} + \mu_0(\theta)\int_{S^{n-1}} \varphi \,\mathrm{d}\sigma & \text{in } S^{n-1}\cap C_{\theta+\varepsilon}, \\ \tilde{\varphi} = \varphi & \text{in } S^{n-1}\cap C_\theta \end{cases}$$

for some $\varepsilon > 0$. As in (3-15), we can define $\tilde{v}$ as the 0-homogeneous extension of $\tilde{\varphi}$. Finally, we introduce the function

$$v_s(x) := r(x)^{\gamma_s^*(\theta)} v(x),$$

where the choice of the homogeneity exponent $\gamma_s^*(\theta) \in (0, 2s)$ is suggested by the following important result.

Theorem 3.8. *Let $\theta \in (0, \theta_0]$. Then there exists $s_0 = s_0(\theta) \in (0, 1)$ such that*

$$(-\Delta)^s v_s(x) \le 0 \quad \text{in } C_\theta$$

for any $s \in [s_0, 1)$.

Proof. By the $\gamma_s^*(\theta)$-homogeneity of v_s, it is sufficient to prove that

$$(-\Delta)^s v_s \le 0 \quad \text{on } C_\theta \cap S^{n-1},$$

since $x \mapsto (-\Delta)^s v_s$ is $(\gamma_s^*(\theta) - 2s)$-homogeneous. In order to ease the notation, through the following computations we will simply use γ instead of $\gamma_s^*(\theta)$ and $o(1)$ for the terms which converge to zero as s goes to 1. Hence, for $x \in S^{n-1} \cap C_\theta$, we have

$$(-\Delta)^s v_s(x) = |x|^\gamma(-\Delta)^s v(x) + v(x)(-\Delta)^s r^\gamma(x) - C(n,s)\int_{\mathbb{R}^n} \frac{(r^\gamma(x) - r^\gamma(y))(v(x) - v(y))}{|x-y|^{n+2s}}\,\mathrm{d}y.$$

First for $R > 0$,

$$\begin{aligned}
(-\Delta)^s r^\gamma(x) &= C(n,s)\int_{B_R(x)} \frac{|x|^\gamma - |y|^\gamma}{|x-y|^{n+2s}}\,\mathrm{d}y + C(n,s)\int_{\mathbb{R}^n\setminus B_R(x)} \frac{|x|^\gamma - |y|^\gamma}{|x-y|^{n+2s}}\,\mathrm{d}y \\
&= \frac{C(n,s)}{2}\int_{B_R(0)} \frac{2|x|^\gamma - |x+z|^\gamma - |x-z|^\gamma}{|z|^{n+2s}}\,\mathrm{d}z + C(n,s)\int_{\mathbb{R}^n\setminus B_R(x)} \frac{1 - |y|^\gamma}{|x-y|^{n+2s}}\,\mathrm{d}y \\
&= -\frac{C(n,s)}{2}\int_0^R \frac{\rho^2\rho^{n-1}}{\rho^{n+2s}}\,\mathrm{d}\rho \int_{S^{n-1}} \langle \nabla^2|x|^\gamma z, z\rangle\,\mathrm{d}\sigma + o(1) \\
&\qquad + C(n,s)|S^{n-1}|\int_R^\infty \frac{1}{\rho^{1+2s}}\,\mathrm{d}\rho - C(n,s)\int_{\mathbb{R}^n\setminus B_R(x)} \frac{|y|^\gamma}{|x-y|^{n+2s}}\,\mathrm{d}y \\
&= -\frac{C(n,s)}{2}\frac{R^{2-2s}}{2-2s}\int_{S^{n-1}} \langle \nabla^2|x|^\gamma z, z\rangle\,\mathrm{d}\sigma \\
&\qquad - C(n,s)\int_R^\infty \frac{\rho^{n-1+\gamma}}{\rho^{n+2s}}\int_{S^{n-1}} \left|\frac{x}{\rho} - \vartheta\right|^\gamma \mathrm{d}\sigma(\vartheta)\,\mathrm{d}\rho + o(1).
\end{aligned}$$

Since for every symmetric matrix A we have

$$\int_{S^{n-1}} \langle Az, z\rangle\,\mathrm{d}\sigma = \frac{\operatorname{tr} A}{n}\omega_{n-1},$$

where ω_{n-1} is the Lebesgue measure of the $(n-1)$-sphere S^{n-1}, we can simplify the first term since $\operatorname{tr}\nabla^2|x|^\gamma = \Delta(|x|^\gamma)$ and checking that

$$\left|\frac{x}{\rho} - \vartheta\right|^\gamma = 1 + \gamma\rho^{-1}\langle\vartheta, x\rangle + o(\rho^{-1})$$

as $\rho \to \infty$ it follows that

$$\begin{aligned}
(-\Delta)^s r^\gamma(x) &= -\frac{C(n,s)}{2}\frac{R^{2-2s}}{2-2s}\frac{\Delta(|x|^\gamma)\omega_{n-1}}{n} - C(n,s)\omega_{n-1}\int_R^\infty \frac{\rho^{n-1+\gamma}}{\rho^{n+2s}}\,\mathrm{d}\rho + o(1)\\
&= -\frac{C(n,s)\omega_{n-1}}{4n(1-s)}\gamma(n-2+\gamma)|x|^{\gamma-2}R^{2-2s} - \frac{C(n,s)}{2s-\gamma}\omega_{n-1}R^{\gamma-2s} + o(1)\\
&= -\frac{C(n,s)\omega_{n-1}}{4n(1-s)}\gamma(n-2+\gamma)R^{2-2s} - \frac{C(n,s)}{2s-\gamma}\omega_{n-1}R^{\gamma-2s} + o(1)\\
&= -\frac{C(n,s)\omega_{n-1}}{4n(1-s)}\gamma(n-2+\gamma) - \frac{C(n,s)}{2s-\gamma}\omega_{n-1} + o(1),
\end{aligned}$$

where in the last equality we choose $\gamma = \gamma_s^*(\theta)$ such that $\gamma_s^*(\theta) - 2s \to 0$ as s goes to 1.

Similarly, if $\tilde{v}$ is the 0-homogeneous extension of v in an enlarged cone, which is such that $v \geq \tilde{v}$ and $v = \tilde{v}$ on $C_\theta \cap S^{n-1}$, it follows that

$$\begin{aligned}
(-\Delta)^s v(x) &= \frac{C(n,s)}{2}\int_{|z|<1}\frac{2v(x)-v(x+z)-v(x-z)}{|z|^{n+2s}}\,\mathrm{d}z + C(n,s)\int_{|x-y|>1}\frac{v(x)-v(y)}{|x-y|^{n+2s}}\,\mathrm{d}y\\
&\leq \frac{C(n,s)}{2}\int_{|z|<1}\frac{2\tilde{v}(x)-\tilde{v}(x+z)-\tilde{v}(x-z)}{|z|^{n+2s}}\,\mathrm{d}z + C(n,s)\int_1^\infty \frac{\rho^{n-1}}{\rho^{n+2s}}\int_{S^{n-1}} v(x)-v(y)\,\mathrm{d}\sigma\,\mathrm{d}\rho\\
&= -\frac{C(n,s)}{2}\int_0^1 \frac{\rho^{n-1}\rho^2}{\rho^{n+2s}}\int_{S^{n-1}}\langle\nabla^2\tilde{v}(x)z, z\rangle\,\mathrm{d}\sigma\,\mathrm{d}\rho + o(1)\\
&= \frac{C(n,s)\omega_{n-1}}{4n(1-s)}(-\Delta)\tilde{v}(x) + o(1),
\end{aligned}$$

where we can use that $\tilde{v}$ solves

$$-\Delta\tilde{v} = 2n\tilde{v} + \mu_0\int_{S^{n-1}} v\,\mathrm{d}\sigma$$

in the enlarged cap $S^{n-1} \cap C_{\theta+\varepsilon}$. Finally,

$$\begin{aligned}
&C(n,s)\int_{\mathbb{R}^n}\frac{(|x|^\gamma - |y|^\gamma)(v(x)-v(y))}{|x-y|^{n+2s}}\,\mathrm{d}y\\
&\qquad = C(n,s)\left[\int_{|y|<1}\frac{(1-|y|^\gamma)(v(x)-v(y))}{|x-y|^{n+2s}}\,\mathrm{d}y + \int_{|y|>1}\frac{(1-|y|^\gamma)(v(x)-v(y))}{|x-y|^{n+2s}}\,\mathrm{d}y\right],
\end{aligned}$$

where the first term is $o(1)$ since

$$\begin{aligned}
\int_0^1 (1-\rho^\gamma)\rho^{n-1}\int_{S^{n-1}}\frac{v(x)-v(y)}{|x-\rho y|^{n+2s}}\,\mathrm{d}\sigma\,\mathrm{d}\rho &= \int_0^1 (1-\rho^\gamma)\rho^{n-1}\int_{S^{n-1}}(v(x)-v(y))(1+o(\rho))\,\mathrm{d}\sigma\,\mathrm{d}\rho\\
&\quad + \int_0^R (1-\rho^\gamma)\rho^{n-1}\int_{S^{n-1}}(v(x)-v(y))(n+2s)\rho\langle x, y\rangle\,\mathrm{d}\sigma\,\mathrm{d}\rho.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&C(n,s)\int_{\mathbb{R}^n}\frac{(|x|^\gamma-|y|^\gamma)(v(x)-v(y))}{|x-y|^{n+2s}}\,\mathrm{d}y\\
&\quad=C(n,s)\int_{|y|>1}\frac{(1-|y|^\gamma)(v(x)-v(y))}{|x-y|^{n+2s}}\,\mathrm{d}y+o(1)\\
&\quad=o(1)-C(n,s)\int_{|y|>1}\frac{|y|^\gamma(v(x)-v(y))}{|x-y|^{n+2s}}\,\mathrm{d}y+o(1)\\
&\quad=o(1)-C(n,s)\int_1^\infty\rho^\gamma\rho^{n-1}\int_{S^{n-1}}\frac{v(x)-v(y)}{|x-\rho y|^{n+2s}}\,\mathrm{d}\sigma\,\mathrm{d}\rho\\
&\quad=o(1)-C(n,s)\int_1^\infty\rho^{-1+\gamma-2s}\int_{S^{n-1}}(v(x)-v(y))(1+o(\rho^{-1}))\,\mathrm{d}\sigma\,\mathrm{d}\rho\\
&\qquad\qquad-C(n,s)\int_1^\infty\rho^{-1+\gamma-2s}\int_{S^{n-1}}(v(x)-v(y))(n+2s)\langle y,x\rangle\rho^{-1}\,\mathrm{d}\sigma\,\mathrm{d}\rho\\
&\quad=o(1)-\frac{C(n,s)\omega_{n-1}}{2s-\gamma}v(x)+\frac{C(n,s)}{2s-\gamma}\int_{S^{n-1}}v(y)\,\mathrm{d}\sigma.
\end{aligned}$$

Hence, recalling that $\gamma=\gamma_s^*(\theta)$, for $x\in S^{n-1}\cap C_\theta$ we have

$$\begin{aligned}
(-\Delta)^s v_s(x)&\le\left(\mu_0(\theta)\frac{C(n,s)\omega_{n-1}}{4n(1-s)}-\frac{C(n,s)}{2s-\gamma_s^*(\theta)}\right)\int_{S^{n-1}}v_s\,\mathrm{d}\sigma+\frac{C(n,s)\omega_{n-1}}{4n(1-s)}(n+\gamma_s^*(\theta))(2-\gamma_s^*(\theta))v_s\\
&\le\left(\mu_0(\theta)-\frac{C(n,s)}{2s-\gamma_s^*(\theta)}\right)\int_{S^{n-1}}v_s\,\mathrm{d}\sigma+o(1),
\end{aligned}$$

where $o(1)$ is uniform with respect to $\gamma_s^*(\theta)$ as $s\to 1$. In order to obtain a negative right-hand side, it is sufficient to choose $\gamma_s^*(\theta)<2s$ in such a way to make the denominator $2s-\gamma_s^*(\theta)$ small enough and the quotient $C(n,s)/(2s-\gamma_s^*(\theta))$ still bounded. □

The previous result suggests the following choice of the homogeneity exponent:

$$\gamma_s^*(\theta):=2s-s\frac{C(n,s)}{\mu_0(\theta)}.$$

We can finally prove the main result of this section.

Corollary 3.9. *For any regular cone C, we have $\mu(C)<\infty$.*

Proof. We will show that $\mu(\theta)<\infty$ for any $\theta\in(0,\theta_0]$. Then, fixing an unbounded regular cone C, there exists a spherical cone C_θ such that $\theta\in(0,\theta_0]$ and $C_\theta\subset C$. Since by inclusion $\gamma_s(C)<\gamma_s(\theta)$, we obtain

$$\mu(C)\le\mu(\theta)<\infty.$$

We want to show that fixing $\theta\in(0,\theta_0]$, we have $\gamma_s(\theta)\le\gamma_s^*(\theta)$ for any $s\in[s_0(\theta),1)$, where the choice of $s_0(\theta)\in(0,1)$ is given in Theorem 3.8. The proof of this fact is based on considerations done in Proposition 3.6. By way of contradiction, suppose $\gamma_s(\theta)>\gamma_s^*(\theta)$. Let

$$h(x)=v_s(x)-u_s(x).$$

The function h is continuous in $\mathbb{R}^n$ and $h(x) = 0$ in $\mathbb{R}^n \setminus C_\theta$. We want to prove that $h(x) \le 0$ in $\mathbb{R}^n \setminus (C_\theta \cap B_1)$. Since $h = 0$ outside the cone, we can consider only what happens in $C_\theta \setminus B_1$. By (3-10), there exist two constants $c_1(s), c_2(s) > 0$ such that, for any $x \in \bar{C}_\theta \setminus \{0\}$,

$$c_1(s)|x|^{\gamma_s - s} \operatorname{dist}(x, \partial C_\theta)^s \le u_s(x) \le c_2(s)|x|^{\gamma_s - s} \operatorname{dist}(x, \partial C_\theta)^s,$$

and there exist two constants $c_1, c_2 > 0$ such that

$$c_1|x|^{\gamma_s^* - 1} \operatorname{dist}(x, \partial C_\theta) \le v_s(x) \le c_2|x|^{\gamma_s^* - 1} \operatorname{dist}(x, \partial C_\theta).$$

We can choose v_s so that $c := c_1(s) = c_2$ since it is defined up to a multiplicative constant. Then, for any $x \in C_\theta \setminus B_1$, since $|x|^{\gamma_s^*} \le |x|^{\gamma_s}$, we have

$$h(x) \le c|x|^{\gamma_s} \operatorname{dist}(x, \partial C_\theta)^s \left[\frac{\operatorname{dist}(x, \partial C_\theta)^{1-s}}{|x|^{1-s}} - 1 \right] \le 0.$$

Now we want to show that there exists a point $x_0 \in C_\theta \cap B_1$ such that $h(x_0) > 0$. Let us consider for example the point $\bar{x} \in S^{n-1} \cap C_\theta$ determined by the angle $\vartheta = \theta/2$, and let $\alpha := v_s(\bar{x}) > 0$ and $\beta := u_s(\bar{x}) > 0$. Hence, there exists a small $r > 0$ such that $\alpha r^{\gamma_s^*} > \beta r^{\gamma_s}$, and so, taking x_0 with angle $\vartheta = \theta/2$ and $|x_0| = r$, we obtain $h(x_0) > 0$.

If we consider the restriction of h to $\overline{C_\theta \cap B_1}$, which is continuous on a compact set, by the previous arguments and the Weierstrass theorem, there exists a maximum point $x_1 \in C_\theta \cap B_1$ for the function h which is global in $\mathbb{R}^n$ and is strict at least in a set of positive measure. Hence,

$$(-\Delta)^s h(x_1) = C(n, s) \text{ p.v.} \int_{\mathbb{R}^n} \frac{h(x_1) - h(y)}{|x_1 - y|^{n+2s}} \, dy > 0,$$

and since $(-\Delta)^s h$ is a continuous function in the open cone, there exists an open set $U(x_1)$ with $\overline{U(x_1)} \subset C_\theta$ such that

$$(-\Delta)^s h(x) > 0 \quad \text{for all } x \in U(x_1).$$

But thanks to Theorem 3.8 we obtain a contradiction since for any nonnegative $\varphi \in C_c^\infty(U(x_1))$

$$((-\Delta)^s h, \varphi) = ((-\Delta)^s v_s, \varphi) - ((-\Delta)^s u_s, \varphi) = ((-\Delta)^s v_s, \varphi) \le 0,$$

where the last inequality holds for any $s \in [s_0(\theta), 1)$. Hence, for any $\theta \in (0, \theta_0]$

$$\mu(\theta) = \lim_{s \to 1^-} \frac{C(n, s)}{2s - \gamma_s(\theta)} \le \lim_{s \to 1^-} \frac{C(n, s)}{2s - \gamma_s^*(\theta)} = \mu_0(\theta) < \infty. \qquad \square$$

4. The limit for $s \nearrow 1$

In this section we prove the main result, Theorem 1.3, emphasizing the difference between wide and narrow cones. Then we improve the asymptotic analysis, proving uniqueness of the limit under assumptions on the geometry and the regularity of C.

Let $C \subset \mathbb{R}^n$ be an open cone and consider the minimization problem

$$\lambda_1(C) = \inf\left\{ \frac{\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 \, \mathrm{d}\sigma}{\int_{S^{n-1}} u^2 \, \mathrm{d}\sigma} : u \in H^1(S^{n-1}) \setminus \{0\} \text{ and } u = 0 \text{ in } S^{n-1} \setminus C \right\}, \tag{4-1}$$

which is strictly related to the homogeneity of the solution of (3-12) by

$$\lambda_1(C) = \gamma(C)(\gamma(C) + n - 2).$$

Moreover, if $\gamma(C) > 2$, equivalently if $\lambda_1(C) > 2n$, the problem

$$\mu_0(C) := \inf\left\{ \frac{\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 - 2nu^2 \, \mathrm{d}\sigma}{\left(\int_{S^{n-1}} |u| \, \mathrm{d}\sigma\right)^2} : u \in H^1(S^{n-1}) \setminus \{0\} \text{ and } u = 0 \text{ in } S^{n-1} \setminus C \right\} \tag{4-2}$$

is well-defined and the number $\mu_0(C)$ is strictly positive.

By a standard argument due to the variational characterization of the previous quantities, we already know the existence of a nonnegative eigenfunction $\varphi \in H^1_0(S^{n-1} \cap C) \setminus \{0\}$ associated to the minimization problem (4-1) and a nonnegative function $\psi \in H^1_0(S^{n-1} \cap C) \setminus \{0\}$ that achieves the minimum (4-2), since the numerator in (4-2) is a coercive quadratic form equivalent to the one in (4-1).

Since the cone C may be disconnected, it is well known that φ is not necessarily unique. Instead, the function ψ is unique up to a multiplicative constant, since it solves

$$\begin{cases} -\Delta_{S^{n-1}} \psi = 2n\psi + \mu_0(C) \int_{S^{n-1}} \psi \, \mathrm{d}\sigma & \text{in } S^{n-1} \cap C, \\ \psi = 0 & \text{in } S^{n-1} \setminus C. \end{cases} \tag{4-3}$$

In fact, due to the integral term in the equation, the solution ψ must be strictly positive in every connected component of C and localizing the equation in a generic component we can easily get uniqueness by the maximum principle.

The next result highlights the functional space in which the limit of the s-harmonic functions on cones for $s \to 1$ will be defined.

Proposition 4.1 [Bourgain et al. 2001, Corollary 7]. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. For $1 < p < \infty$, let $f_s \in W^{s,p}(\Omega)$, and assume that*

$$[f_s]_{W^{s,p}(\Omega)} \le C_0.$$

Then, up to a subsequence, (f_s) converges in $L^p(\Omega)$ as $s \to 1$ (and, in fact, in $W^{t,p}(\Omega)$ for all $t < 1$) to some $f \in W^{1,p}(\Omega)$.

We use a different notation than that in [Bourgain et al. 2001] since in our paper the normalization constant $C(n,s)$ is incorporated in the seminorm $[\,\cdot\,]_{H^s}$ in order to obtain a continuity of the norm $\|\cdot\|_{H^s}$ for $s \in (0,1]$.

Proof. Let C be an open cone and C_R be a regular cone with section on S^{n-1} of class $C^{1,1}$ such that $C_R \subset C$ and $\partial C_R \cap \partial C = \{0\}$.

By monotonicity of the homogeneity degree $\gamma_s(\,\cdot\,)$ with respect to the inclusion, we directly obtain $\gamma_s(C) < \gamma_s(C_R)$ and consequently, up to considering a subsequence, we obtain the existence of the finite limits

$$\bar{\gamma}(C) = \lim_{s\to 1} \gamma_s(C), \quad \mu(C) = \lim_{s\to 1} \frac{C(n,s)}{2s - \gamma_s(C)}. \tag{4-4}$$

Since $\gamma_s(C) < 2s$, we know $\bar{\gamma}(C) \le 2$ and similarly $\mu(C) \in [0,\infty)$.

Let $K \subset \mathbb{R}^n$ be a compact set and consider $x_0 \in K$ and $R > 0$ such that $K \subset B_R(x_0)$. Given $\eta \in C_c^\infty(B_2)$, a radial cut-off function such that $\eta \equiv 1$ in B_1 and $0 \le \eta \le 1$ in B_2, consider the rescaled function $\eta_K(x) = \eta((x - x_0)/R)$ which satisfies $\eta_K \equiv 1$ on K.

By Proposition 2.7, we have

$$[u_s\eta_K]^2_{H^s(B_{2R}(x_0))} \le [u_s\eta_K]^2_{H^s(\mathbb{R}^n)} \le M(n,K)\left[\frac{C(n,s)}{2(1-s)} + \frac{C(n,s)}{2s-\gamma_s}\right],$$

and similarly

$$\begin{aligned}\|u_s\eta_K\|^2_{H^s(B_{2R}(x_0))} &\le \|u_s\eta_K\|^2_{L^2(\mathbb{R}^n)} + [u_s\eta_K]^2_{H^s(\mathbb{R}^n)}\\ &\le M(n,K)\left[\frac{C(n,s)}{2(1-s)} + \frac{C(n,s)}{2s-\gamma_s} + 1\right] \le M(n,K)\left[\frac{2n}{\omega_{n-1}} + c\mu(C) + 1\right].\end{aligned}$$

By applying Proposition 4.1 with $\Omega = B_{2R}(x_0)$, we obtain that, up to a subsequence, $u_s\eta_K \to \bar{u}\eta_K$ in $L^2(B_{2R}(x_0))$ and

$$\|\bar{u}\eta_K\|^2_{H^1(B_{2R}(x_0))} \le M(n,K)$$

up to relabeling the constant $M(n,K)$.

By construction, since $\eta_K \equiv 1$ on K and $\eta_K \in [0,1]$, we obtain that $u_s \to \bar{u}$ in $L^2(K)$ and similarly

$$\|\bar{u}\|_{H^1(K)} \le \|\bar{u}\eta_K\|_{H^1(K)} \le \|\bar{u}\eta_K\|_{H^1(B_{2R}(x_0))} < \infty,$$

which gives us the local integrability in $H^1(\mathbb{R}^n)$.

By Proposition 2.4 and Corollary 3.9 we obtain, up to passing to a subsequence, a bound in $C^{0,\alpha}_{\mathrm{loc}}(C)$ for (u_s) that is uniform in s. Then, since we obtain uniform convergence on compact subsets of C, the limit must be necessary nontrivial with $\|\bar{u}\|_{L^\infty(S^{n-1})} = 1$, nonnegative and $\bar{\gamma}(C)$-homogeneous.

Let $\varphi \in C_c^\infty(C)$ be a positive smooth function compactly supported such that $\operatorname{supp}\varphi \subset B_\rho$ for some $\rho > 0$. By the definition of the distributional fractional Laplacian

$$0 = \int_{\mathbb{R}^n} \varphi(-\Delta)^s u_s \,\mathrm{d}x = \int_{\mathbb{R}^n} u_s(-\Delta)^s\varphi\,\mathrm{d}x = \int_{\mathbb{R}^n\setminus B_\rho} u_s(-\Delta)^s\varphi\,\mathrm{d}x + \int_{B_\rho} u_s(-\Delta)^s\varphi\,\mathrm{d}x.$$

Since

$$\frac{1}{|x-y|^{n+2s}} = \frac{1}{|x|^{n+2s}}\left(1 - (n+2s)\frac{y}{|x|}\int_0^1 \frac{x/|x| - ty/|x|}{\big|x/|x| - ty/|x|\big|^{n+2s+2}}\,\mathrm{d}t\right),$$

by the definition of the fractional Laplacian for regular functions, it follows that

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B_\rho} u_s(-\Delta)^s\varphi\,\mathrm{d}x\\ &\quad= C(n,s)\int_{\mathbb{R}^n\setminus B_\rho} u_s(x)\int_{\operatorname{supp}\varphi}\frac{-\varphi(y)}{|y-x|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x\\ &\quad= C(n,s)\int_{\mathbb{R}^n\setminus B_\rho}\frac{u_s(x)}{|x|^{n+2s}}\int_{\operatorname{supp}\varphi} -\varphi(y)\,\mathrm{d}y\,\mathrm{d}x + C(n,s)(n+2s)\int_{\mathbb{R}^n\setminus B_\rho}\frac{u_s(x)}{|x|^{n+2s+1}}\psi(x)\,\mathrm{d}x\end{aligned}$$

for some $\psi \in L^\infty$. Moreover, since u_s is $\gamma_s(C)$-homogeneous with $\gamma_s(C) < 2s$, we have

$$C(n,s)\int_{\mathbb{R}^n\setminus B_\rho} \frac{u_s(x)}{|x|^{n+2s}}\,\mathrm{d}x = \frac{C(n,s)}{2s-\gamma_s(C)}\rho^{\gamma_s(C)-2s}\int_{S^{n-1}} u_s(\theta)\,\mathrm{d}\sigma$$

and similarly

$$C(n,s)\left|\int_{\mathbb{R}^n\setminus B_\rho} \frac{u_s(x)}{|x|^{n+2s+1}}\psi(x)\,\mathrm{d}x\right| \le \frac{C(n,s)\|\psi\|_{L^\infty}}{2s-\gamma_s(C)+1}\rho^{\gamma_s(C)-2s-1}\int_{S^{n-1}} u_s(\theta)\,\mathrm{d}\sigma = o(1).$$

Hence, for each $s \in (0,1)$

$$\begin{aligned}\int_{B_\rho} u_s(-\Delta)^s\varphi\,\mathrm{d}x &= \int_{\mathbb{R}^n\setminus B_\rho} u_s(-\Delta)^s\varphi\,\mathrm{d}x\\ &= C(n,s)\int_{\mathbb{R}^n\setminus B_\rho} u_s(x)\int_{\operatorname{supp}\varphi}\frac{\varphi(y)}{|x-y|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x\\ &= \frac{C(n,s)}{2s-\gamma_s(C)}\int_{\operatorname{supp}\varphi}\varphi(x)\,\mathrm{d}x\int_{S^{n-1}} u_s\,\mathrm{d}\sigma + o(1),\end{aligned}$$

and passing through the limit, up to a subsequence, we obtain

$$\int_{B_\rho}\bar{u}(-\Delta)\varphi\,\mathrm{d}x = \mu(C)\int_{S^{n-1}}\bar{u}\,\mathrm{d}\sigma\int_{\operatorname{supp}\varphi}\varphi(x)\,\mathrm{d}x = \int_{B_\rho}\left(\mu(C)\int_{S^{n-1}}\bar{u}\,\mathrm{d}\sigma\right)\varphi(x)\,\mathrm{d}x,$$

which implies, integrating by parts, that

$$-\Delta\bar{u} = \mu(C)\int_{S^{n-1}}\bar{u}\,\mathrm{d}\sigma \quad \text{in } \mathcal{D}'(C).$$

Since the function $\bar{u}$ is $\bar{\gamma}(C)$-homogeneous, we get

$$-\Delta_{S^{n-1}}\bar{u} = \bar{\lambda}\bar{u} + \mu(C)\int_{S^{n-1}}\bar{u}\,\mathrm{d}\sigma \quad \text{on } S^{n-1}\cap C, \tag{4-5}$$

where $\bar{\lambda} = \bar{\gamma}(C)(\bar{\gamma}(C)+n-2)$ is the eigenvalue associated to the critical exponent $\bar{\gamma}(C) \le 2$.

Consider now a nonnegative $\varphi \in H_0^1(S^{n-1}\cap C)\setminus\{0\}$, strictly positive on $S^{n-1}\cap C$ which achieves (4-1). Then

$$-\Delta_{S^{n-1}}\varphi = \lambda_1(C)\varphi \quad \text{in } H^{-1}(S^{n-1}\cap C). \tag{4-6}$$

By testing this equation with $\bar{u}$ and integrating by parts, we obtain

$$(\lambda_1(C)-\bar{\lambda})\int_{S^{n-1}}\bar{u}\varphi\,\mathrm{d}\sigma = \mu(C)\int_{S^{n-1}}\bar{u}\,\mathrm{d}\sigma\int_{S^{n-1}}\varphi\,\mathrm{d}\sigma \ge 0, \tag{4-7}$$

which implies that in general $\gamma(C) \ge \bar{\gamma}(C)$ and $\gamma(C) = \bar{\gamma}(C)$ if and only if $\mu(C) = 0$.

Wide cones: $\gamma(C) < 2$. By the previous remark we have $\bar{\gamma}(C) < 2$ and by the definition of $\mu(C)$, it follows that $\mu(C) = 0$. Since φ is the trace on S^{n-1} of a homogeneous harmonic function on C, we obtain that $\bar{\gamma}(C) = \gamma(C)$ and $\bar{u}$ is a homogeneous nonnegative harmonic function on C such that $\|\bar{u}\|_{L^\infty(S^{n-1})} = 1$.

Narrow cones: $\gamma(C) \geq 2$. If $\bar{\gamma}(C) < 2$ we have $\mu(C) = 0$ and consequently $\lambda_1(C) = \bar{\lambda}$, which is a contradiction since $\gamma(C) \geq 2 > \bar{\gamma}(C)$. Hence, if C is a narrow cone we get $\bar{\gamma}(C) = 2$. Since $\gamma(C) = 2$ is trivial and it follows directly from the previous computations, consider now $\mu_0(C)$ as the minimum defined in (4-2), which is well-defined and strictly positive since we are focusing on the remaining case $\gamma(C) > 2$. We already remarked that it is achieved by a nonnegative $\psi \in H_0^1(S^{n-1} \cap C) \setminus \{0\}$ which is strictly positive on $S^{n-1} \cap C$ and a solution of

$$-\Delta_{S^{n-1}} \psi = 2n\psi + \mu_0(C) \int_{S^{n-1}} \psi \, \mathrm{d}\sigma \quad \text{in } H^{-1}(S^{n-1} \cap C).$$

As we already did in the previous cases, by testing this equation with $\bar{u}$ we obtain $\mu(C) = \mu_0(C)$.

By uniqueness of the limits $\bar{\gamma}(C)$ and $\mu(C)$, the result in (4-4) holds for $s \to 1$ and not just up to a subsequence. □

Remark 4.2. The possible obstruction to the existence of the limit of u_s as s converges to 1 lies in the possible lack of uniqueness of nonnegative solutions to (1-9) such that $\|\bar{u}\|_{L^\infty(S^{n-1})} = 1$. This is the reason why we need to extract subsequences in the asymptotic analysis of Theorem 1.3. More precisely, uniqueness of (4-1) implies uniqueness of the limit $\bar{u}$ in the case $\gamma(C) \leq 2$ and uniqueness of (4-2) in the case $\gamma(C) > 2$. When C is connected (4-1) is attained by a unique normalized nonnegative solution via a standard argument based upon the maximum principle. On the other hand, as we already remarked, when $\gamma(C) > 2$, problem (4-2) always admits a unique solution. Ultimately, the main obstacle in this analysis is the disconnection of the cone C when $\gamma(C) \leq 2$: in this case we cannot always ensure the uniqueness of the solution of the limit problem and even the positivity of the limit function $\bar{u}$ on every connected component of C.

The following example shows uniqueness of the limit function $\bar{u}$ due to the nonlocal nature of the fractional Laplacian under a symmetry assumption on the cone C.

Proposition 4.3. *Let $C = C_1 \cup \cdots \cup C_m$ be a union of disconnected cones such that C_1 is connected and there are orthogonal maps $\Phi_2, \ldots, \Phi_m \in O(n)$ (e.g., reflections about hyperplanes) such that $C_i = \Phi_i(C_1)$ and $\Phi_i(C) = (C)$ for $i = 2, \ldots, m$. Let (u_s) be the family of nonnegative solutions to* (1-1) *such that $\|u_s\|_{L^\infty(S^{n-1})} = 1$. Then there exists the limit of u_s as $s \nearrow 1$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and uniformly on compact subsets of C.*

Proof. We remark that, for any element of the orthogonal group $\Phi \colon \mathbb{R}^n \to \mathbb{R}^n$,

$$(-\Delta)^s (u \circ \Phi)(x) = C(n,s) \, \text{p.v.} \int_{\mathbb{R}^n} \frac{u(\Phi(x)) - u(y)}{|\Phi(x) - y|^{n+2s}} \, \mathrm{d}y = (-\Delta)^s u(\Phi(x)).$$

By the uniqueness result [Bañuelos and Bogdan 2004, Theorem 3.2] of s-harmonic functions on cones, we infer that $u_s \equiv u_s \circ \Phi_i$ for every $i = 2, \ldots, m$. Therefore, we have convergence to $\bar{u}$, which satisfies $\|\bar{u}\|_{L^\infty(S^{n-1})} = 1$ and is a solution of

$$\begin{cases} -\Delta \bar{u} = \mu(C) \int_{S^{n-1}} \bar{u} \, \mathrm{d}\sigma & \text{in } C, \\ \bar{u} \geq 0 & \text{in } C, \\ \bar{u} = 0 & \text{in } \mathbb{R}^n \setminus C \end{cases} \tag{4-8}$$

such that $\bar{u} \equiv \bar{u} \circ \Phi_i$ for every $i = 2, \ldots, m$. Finally, connectedness of C_1 yields uniqueness of such a solution also for narrow cones. □

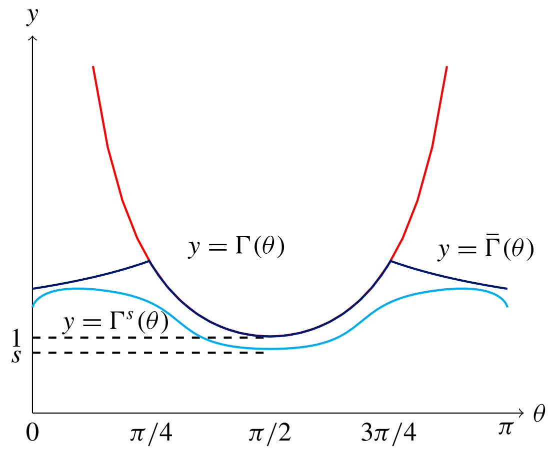

Figure 3. Values of the limit $\bar{\Gamma}(\theta) = \lim_{s\to 1} \Gamma^s(\theta)$ and $\Gamma(\theta)$ for $n = 2$.

Proof of Corollary 1.6. This corollary is an easy application of our main result, Theorem 1.3, since it is a consequence of Dini's theorem for a monotone sequence of continuous functions which converges pointwisely to a continuous function on a compact set. In fact, fixing $s \in (0, 1)$, the function $\theta \mapsto \gamma_s(\theta)$ is continuous in $[0, \pi)$ with $\gamma_s(0) = 2s$ and $\gamma_s(\pi) = 0$. Moreover this function is also monotone decreasing in $[0, \pi]$ and since there exists the limit

$$\lim_{\theta\to\pi^-} \gamma_s(\theta) = \begin{cases} \frac{1}{2}(2s-1) & \text{if } n = 2 \text{ and } s > \frac{1}{2}, \\ \gamma_s(\pi) = 0 & \text{otherwise,} \end{cases}$$

we can extend $\theta \mapsto \gamma_s(\theta)$ to a continuous function in $[0, \pi]$; see [Michalik 2006]. Nevertheless, the limit $\bar{\gamma}(\theta) = \lim_{s\to 1} \gamma_s(\theta) = \min\{\gamma(\theta), 2\}$ is continuous on $[0, \pi]$ with

$$\bar{\gamma}(\pi) = \begin{cases} \frac{1}{2} & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Eventually, for any fixed $\theta \in [0, \pi]$, the function $s \mapsto \gamma_s(\theta)$ is monotone nondecreasing in $(0, 1)$. By Dini's theorem the convergence is uniform on $[0, \pi]$. This fact obviously implies the uniform convergence

$$\Gamma^s(\theta) = \frac{\gamma_s(\theta) + \gamma_s(\pi - \theta)}{2} \to \bar{\Gamma}(\theta) = \frac{\bar{\gamma}(\theta) + \bar{\gamma}(\pi - \theta)}{2}$$

in $[0, \pi]$, and hence (see Figure 3)

$$\nu_s^{\mathrm{ACF}} = \min_{\theta\in[0,\pi]} \Gamma^s(\theta) \to \min_{\theta\in[0,\pi]} \bar{\Gamma}(\theta) = \nu^{\mathrm{ACF}}. \qquad \square$$

5. Uniform-in-s estimates in $C^{0,\alpha}$ on annuli

We have already remarked in Section 2 that, if you take a cone $C = C_\omega$ with $\omega \subset S^{n-1}$ a finite union of connected $C^{1,1}$ domains ω_i such that $\bar{\omega}_i \cup \bar{\omega}_j = \varnothing$ for $i \neq j$, by [Michalik 2006, Lemma 3.3] we have (2-2).

Hence solutions u_s to (1-1) are $C^{0,s}(S^{n-1})$ and for any fixed $\alpha \in (0,1)$, any solution u_s with $s \in (\alpha, 1)$ is $C^{0,\alpha}(S^{n-1})$; that is, there exists $L_s > 0$ such that

$$\sup_{x,y\in S^{n-1}} \frac{|u_s(x)-u_s(y)|}{|x-y|^\alpha} = L_s.$$

Let us consider an annulus $A = A_{r_1,r_2} = B_{r_2} \setminus \overline{B}_{r_1}$ with $0 < r_1 < r_2 < \infty$. We have the following result.

Lemma 5.1. *Let $\alpha \in (0,1)$, $s_0 \in \bigl(\max\{\frac{1}{2},\alpha\},1\bigr)$ and A be an annulus centered at zero. Then there exists a constant $c > 0$ such that any solution u_s to* (1-1) *with $s \in [s_0, 1)$ satisfies*

$$\sup_{x,y\in A} \frac{|u_s(x)-u_s(y)|}{|x-y|^\alpha} \le cL_s.$$

Proof. First of all we remark that

$$\sup_{x,y\in S_r^{n-1}} \frac{|u_s(x)-u_s(y)|}{|x-y|^\alpha} \le cL_s \tag{5-1}$$

for any $r \in (r_1, r_2)$. In fact, by the γ_s-homogeneity of our solutions, we have

$$\sup_{x,y\in S_r^{n-1}} \frac{|u_s(x)-u_s(y)|}{|x-y|^\alpha} = L_s r^{\gamma_s-\alpha},$$

and since $(2s_0-1)/2 \le \gamma_s(C) < 2$ for any $s \in [s_0, 1)$ by the inclusion $C \subset \mathbb{R}^n \setminus \{\text{half-line from } 0\}$, we obtain (5-1).

Now we can show what happens considering $x, y \in A$ which are not on the same sphere. We can suppose without loss of generality that $x \in S_R^{n-1}$, $y \in S_r^{n-1}$ with $r_1 < r < R < r_2$. Hence let us take the point z obtained by the intersection between S_r^{n-1} and the half-line connecting 0 and x (z may be y itself). Hence

$$\begin{aligned} |u_s(x)-u_s(y)| &\le |u_s(x)-u_s(z)| + |u_s(z)-u_s(y)| \\ &\le u_s\Bigl(\frac{x}{|x|}\Bigr)\bigl||x|^{\gamma_s} - |z|^{\gamma_s}\bigr| + cL_s|z-y|^\alpha \\ &\le cL_s|x-y|^\alpha. \end{aligned}$$

In fact we remark that $\|u_s\|_{L^\infty(S^{n-1})} = 1$. Moreover, since $\beta = \widehat{xzy} \in (\pi/2, \pi]$, obviously $|z-y|^\alpha \le |x-y|^\alpha$. Thus by the α-Hölder continuity of $t \mapsto t^{\gamma_s}$ in (r_1, r_2) and the bounds $(2s_0-1)/2 \le \gamma_s(C) < 2$, one can find a universal constant $c > 0$ such that

$$\bigl||x|^{\gamma_s} - |z|^{\gamma_s}\bigr| \le c\bigl||x|-|z|\bigr|^\alpha \le c|x-z|^\alpha \le c|x-y|^\alpha,$$

where the last inequality holds since z is the point on S_r^{n-1} which minimizes the distance $\operatorname{dist}(x, S_r^{n-1})$. □

Proof of Theorem 1.5. Seeking a contradiction,

$$\max_{x,y\in S^{n-1}} \frac{|u_{s_k}(x)-u_{s_k}(y)|}{|x-y|^\alpha} = L_{s_k} = L_k \to \infty \quad \text{as } s_k \to 1. \tag{5-2}$$

We can consider the sequence of points $x_k, y_k \in S^{n-1}$ which realizes L_k at any step. It is easy to see that this pair belongs to $\bar{C} \cap S^{n-1}$. Moreover we can always think of x_k as the one closer to the boundary $\partial C \cap S^{n-1}$. Therefore, to have (5-2), we have $r_k = |x_k - y_k| \to 0$. Hence, without loss of generality, we can assume that x_k, y_k belong definitively to the same connected component of C and

$$\frac{|u_{s_k}(y_k) - u_{s_k}(x_k)|}{r_k^\alpha} = L_k, \qquad \frac{y_k - x_k}{r_k} \to e_1.$$

Let us define

$$u^k(x) = \frac{u_{s_k}(x_k + r_k x) - u_{s_k}(x_k)}{r_k^\alpha L_k}, \qquad x \in \Omega_k = \frac{C - x_k}{r_k}.$$

We remark that $u^k(0) = 0$ and $u^k((y_k - x_k)/r_k) = 1$.

Moreover, we can have two different situations.

Case 1: If

$$\frac{r_k}{\operatorname{dist}(x_k, \partial C)} \to 0,$$

then the limit of Ω_k is $\mathbb{R}^n$.

Case 2: If

$$\frac{r_k}{\operatorname{dist}(x_k, \partial C)} \to l \in (0, \infty],$$

then the limit of Ω_k is a half-space $\mathbb{R}^n \cap \{x_1 > 0\}$.

In any case let us define Ω_∞ to be this limit set. Let us consider the annulus $A^* := B_{3/2} \setminus \bar{B}_{1/2}$. By Lemma 5.1 and the definition of u^k, we obtain, for any k,

$$\sup_{x,y \in A_k^*} \frac{|u^k(x) - u^k(y)|}{|x - y|^\alpha} \le c, \tag{5-3}$$

where $A_k^* := (A^* - x_k)/r_k \to \mathbb{R}^n$ and the constant $c > 0$ depends only on α and A^*. Let us consider a compact subset K of Ω_∞. Since for k large enough $K \subset A_k^*$, functions u^k are $C^{0,\alpha}(K)$ uniformly in k. This is due also to the fact that they are uniformly in $L^\infty(K)$, since $|u^k(x) - u^k(0)| \le c|x|^\alpha$ on K. Hence $u^k \to \bar{u}$ uniformly on compact subsets of Ω_∞. Moreover $\bar{u}$ is globally α-Hölder continuous and it is not constant, since $\bar{u}(e_1) - \bar{u}(0) = 1$. To conclude, we will show that $\bar{u}$ is harmonic in the limit domain Ω_∞; that is, for any $\varphi \in C_c^\infty(\Omega_\infty)$

$$\int_{\Omega_\infty} \varphi(-\Delta)\bar{u}\,dx = 0,$$

and this fact will be a contradiction to the global Hölder continuity. In fact we can apply Corollary 2.3 in [Noris et al. 2010], if $\Omega_\infty = \mathbb{R}^n$ directly on the function $\bar{u}$ and if $\Omega_\infty = \mathbb{R}^n \cap \{x_1 > 0\}$; since $\bar{u} = 0$ in $\partial\Omega_\infty$, we can use the same result over its odd reflection. Hence we want to prove

$$\int_{\Omega_\infty} \varphi(-\Delta)\bar{u}\,dx = \int_{\Omega_\infty} \bar{u}(-\Delta)\varphi\,dx = \lim_{k\to\infty} \int_{B_R} u^k(-\Delta)^{s_k}\varphi\,dx = 0,$$

where B_R contains the support of φ and the second equality holds by the uniform convergences $u^k \to \bar{u}$ and $(-\Delta)^{s_k}\varphi \to (-\Delta)\varphi$ on compact subsets of Ω_∞, since φ is a smooth function compactly supported.

Moreover, since u^k is s_k-harmonic on Ω_k, and for k large enough the support of φ is contained in this domain, we have

$$\int_{\mathbb{R}^n} u^k (-\Delta)^{s_k} \varphi \,\mathrm{d}x = \int_{\mathbb{R}^n} \varphi (-\Delta)^{s_k} u^k \,\mathrm{d}x = 0.$$

In order to conclude we want

$$\lim_{k\to\infty} \int_{\mathbb{R}^n \setminus B_R} u^k (-\Delta)^{s_k} \varphi \,\mathrm{d}x = 0.$$

Hence, defining $\eta = x_k + r_k x$ and using Remark 2.3, we obtain

$$\left| \int_{\mathbb{R}^n \setminus B_R} u^k (-\Delta)^{s_k} \varphi \,\mathrm{d}x \right| \le \frac{C(n, s_k)}{L_k} r_k^{2s_k-\alpha} \int_{|\eta - x_k| > R r_k} \frac{|u_{s_k}(\eta) - u_{s_k}(x_k)|}{|\eta - x_k|^{n+2s_k}} \,\mathrm{d}\eta.$$

For k large enough, we notice that we can choose $\varepsilon > 0$ such that the set $\{\eta \in \mathbb{R}^n : R r_k < |\eta - x_k| < \varepsilon\}$ is contained in A^*. So, we can split the integral obtaining

$$\int_{|\eta-x_k|>Rr_k} \frac{|u_{s_k}(\eta) - u_{s_k}(x_k)|}{|\eta - x_k|^{n+2s_k}} \,\mathrm{d}\eta \le \int_{Rr_k<|\eta-x_k|<\varepsilon} \frac{|u_{s_k}(\eta) - u_{s_k}(x_k)|}{|\eta - x_k|^{n+2s_k}} \,\mathrm{d}\eta + \int_{|\eta-x_k|>\varepsilon} \frac{|u_{s_k}(\eta) - u_{s_k}(x_k)|}{|\eta - x_k|^{n+2s_k}} \,\mathrm{d}\eta,$$

where we have

$$\begin{aligned} \frac{C(n, s_k) r_k^{2s_k-\alpha}}{L_k} \int_{Rr_k<|\eta-x_k|<\varepsilon} \frac{|u_{s_k}(\eta) - u_{s_k}(x_k)|}{|\eta - x_k|^{n+2s_k}} \,\mathrm{d}\eta &\le C(n, s_k) r_k^{2s_k-\alpha} c\omega_{n-1} \int_{Rr_k}^{\varepsilon} t^{-1+\alpha-2s_k} \,\mathrm{d}t \\ &= \frac{C(n, s_k) c\omega_{n-1}}{2s_k - \alpha} \left(R^{\alpha-2s_k} - \frac{r_k^{2s_k-\alpha}}{\varepsilon^{2s_k-\alpha}} \right) \end{aligned}$$

and similarly

$$\begin{aligned} \frac{C(n, s_k) r_k^{2s_k-\alpha}}{L_k} \int_{|\eta-x_k|>\varepsilon} \frac{|u_{s_k}(\eta) - u_{s_k}(x_k)|}{|\eta - x_k|^{n+2s_k}} \,\mathrm{d}\eta &\le \frac{C(n, s_k) r_k^{2s_k-\alpha} c\omega_{n-1}}{L_k} \int_{\varepsilon}^{\infty} \frac{(1+t)^{\gamma_{s_k}}}{t^{1+2s_k}} \,\mathrm{d}t \\ &= \frac{C(n, s_k) r_k^{2s_k-\alpha} c\omega_{n-1}}{L_k} \left(1 + \frac{\varepsilon^{\gamma_{s_k}-2s_k}}{2s_k - \gamma_{s_k}} \right). \end{aligned}$$

Finally, recalling that $r_k \to 0,\ C(n, s_k) \to 0,\ L_k \to \infty$ and $2s_k - \alpha > 0$ taking $s_0 > \frac{1}{2}$, we obtain

$$\left| \int_{\mathbb{R}^n \setminus B_R} u^k (-\Delta)^{s_k} \varphi \,\mathrm{d}x \right| \le \left(C(n, s_k) + \frac{C(n, s_k)}{2s_k - \gamma_{s_k}} \frac{r_n^{2s_k-\alpha}}{L_k} \right) M,$$

which converges to zero as we claimed, since

$$\frac{C(n, s_k)}{2s_k - \gamma_{s_k}(C)} \to \mu(C) \in [0, \infty)$$

in any regular cone $C \subset \mathbb{R}^n$. □

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WEIGHTED LITTLE BMO AND TWO-WEIGHT INEQUALITIES FOR JOURNÉ COMMUTATORS

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We characterize the boundedness of the commutators $[b, T]$ with biparameter Journé operators T in the two-weight, Bloom-type setting, and express the norms of these commutators in terms of a weighted little bmo norm of the symbol b. Specifically, if μ and λ are biparameter A_p weights, $\nu := \mu^{1/p}\lambda^{-1/p}$ is the Bloom weight, and b is in $\text{bmo}(\nu)$, then we prove a lower bound and testing condition $\|b\|_{\text{bmo}(\nu)} \lesssim \sup \|[b, R_k^1 R_l^2] : L^p(\mu) \to L^p(\lambda)\|$, where R_k^1 and R_l^2 are Riesz transforms acting in each variable. Further, we prove that for such symbols b and any biparameter Journé operators T, the commutator $[b, T] : L^p(\mu) \to L^p(\lambda)$ is bounded. Previous results in the Bloom setting do not include the biparameter case and are restricted to Calderón–Zygmund operators. Even in the unweighted, $p = 2$ case, the upper bound fills a gap that remained open in the multiparameter literature for iterated commutators with Journé operators. As a by-product we also obtain a much simplified proof for a one-weight bound for Journé operators originally due to R. Fefferman.

1. Introduction and statement of main results

Bloom [1985] proved a two-weight version of the celebrated commutator theorem of Coifman, Rochberg and Weiss [Coifman et al. 1976]. Specifically, Bloom characterized the two-weight norm of the commutator $[b, H]$ with the Hilbert transform in terms of the norm of b in a certain weighted BMO space:

$$\|[b, H] : L^p(\mu) \to L^p(\lambda)\| \simeq \|b\|_{\text{BMO}(\nu)},$$

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where μ, λ are A_p weights, $1 < p < \infty$, and $\nu := \mu^{1/p}\lambda^{-1/p}$. Recently, this was extended to the n-dimensional case of Calderón–Zygmund operators in [Holmes et al. 2017], using the modern dyadic methods started by [Petermichl 2000] and continued in [Hytönen 2012]. The main idea in these methods is to represent continuous operators like the Hilbert transform in terms of dyadic shift operators. This theory was recently extended to biparameter singular integrals in [Martikainen 2012].

In this paper we extend the Bloom theory to commutators with biparameter Calderón–Zygmund operators, also known as Journé operators, and characterize their norms in terms of a weighted version of the little bmo space of [Cotlar and Sadosky 1996]. The main results are:

Theorem 1.1 (upper bound). *Let T be a biparameter Journé operator on $\mathbb{R}^{\vec{n}} = \mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$, as defined in Section 7A. Let μ and λ be $A_p(\mathbb{R}^{\vec{n}})$ weights, $1 < p < \infty$, and define $\nu := \mu^{1/p}\lambda^{-1/p}$. Then*

$$\|[b, T] : L^p(\mu) \to L^p(\lambda)\| \lesssim \|b\|_{\mathrm{bmo}(\nu)},$$

where $\|b\|_{\mathrm{bmo}(\nu)}$ denotes the norm of b in the weighted little $\mathrm{bmo}(\nu)$ *space on $\mathbb{R}^{\vec{n}}$.*

We make a few remarks about the proof of this result. At its core, the strategy is the same as in [Holmes et al. 2017], and may be roughly stated as:

(1) Use a representation theorem to reduce the problem from bounding the norm of $[b, T]$ to bounding the norm of $[b, \text{dyadic shift}]$.

(2) Prove the two-weight bound for $[b, \text{dyadic shift}]$ by decomposing into paraproducts.

However, the biparameter case presents some significant new obstacles. In [Holmes et al. 2017], T was a Calderón–Zygmund operator on $\mathbb{R}^n$, and the representation theorem was that of [Hytönen 2012]. In the present paper, T is a biparameter Journé operator on $\mathbb{R}^{\vec{n}} = \mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$ (see Section 7A) and we use Martikainen's representation theorem [2012] to reduce the problem to commutators $[b, \mathbb{S}_{\boldsymbol{\mathcal{D}}}]$, where $\mathbb{S}_{\boldsymbol{\mathcal{D}}}$ is now a *biparameter* dyadic shift. These can be cancellative, i.e., all Haar functions have mean zero (defined in Section 7C), or noncancellative (defined in Section 7D). The strategy is summarized in Figure 1.

The main difficulty arises from the structure of the biparameter dyadic shifts. At first glance, the cancellative shifts are "almost" compositions of two one-parameter shifts $\mathbb{S}_{\mathcal{D}_1}$ and $\mathbb{S}_{\mathcal{D}_2}$ applied in each variable — if this were so, many of the results would follow trivially by iteration of the one-parameter results. Unfortunately, there is no reason for the coefficients $a_{P_1Q_1R_1P_2Q_2R_2}$ in the biparameter shifts to "separate" into a product $a_{P_1Q_1R_1} \cdot a_{P_2Q_2R_2}$, as would be required in a composition of two one-parameter shifts. Therefore, many of the inequalities needed for biparameter shifts must be proved from scratch.

Even more difficult is the case of noncancellative shifts. As outlined in Section 7D, these are really paraproducts, and there are three possible types that arise from the representation theorem:

(1) full standard paraproducts;

(2) full mixed paraproducts;

(3) partial paraproducts.

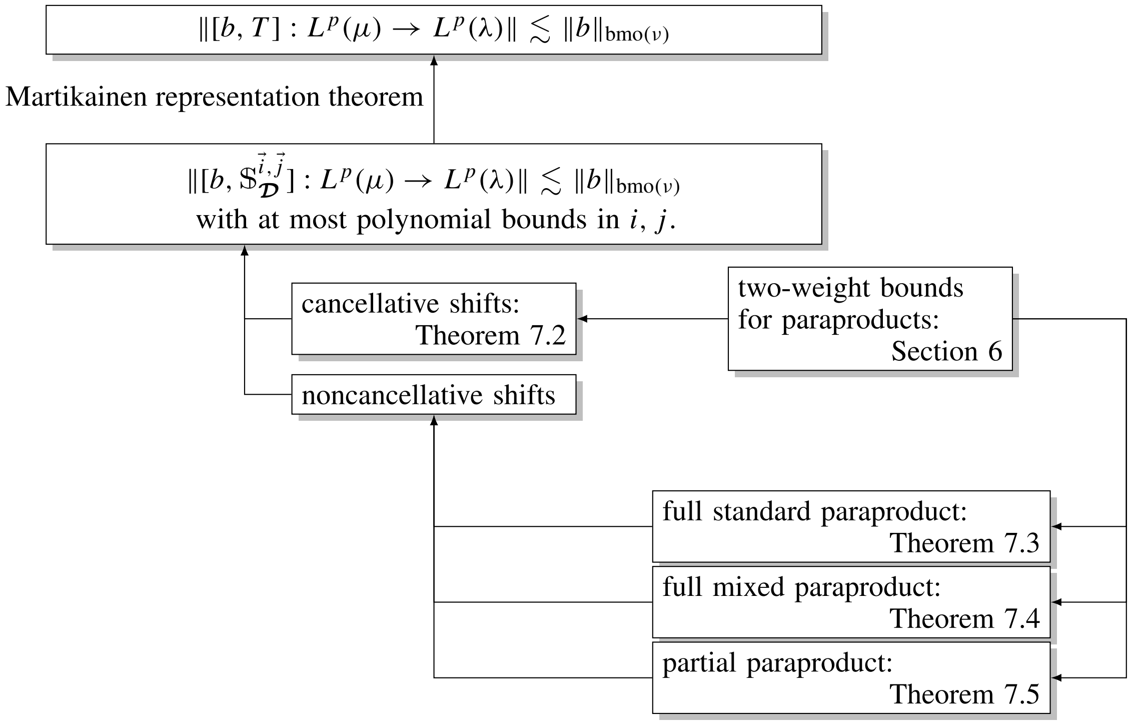

Figure 1. Strategy for Theorem 1.1.

These methods were considered previously in [Ou et al. 2016; Ou and Petermichl 2018] for the unweighted, $p = 2$ case. In [Ou et al. 2016] it was shown that

$$\|[b, T] : L^2(\mathbb{R}^{\vec{n}}) \to L^2(\mathbb{R}^{\vec{n}})\| \lesssim \|b\|_{\mathrm{bmo}(\mathbb{R}^{\vec{n}})},$$

where T is a *paraproduct-free* Journé operator. This restriction essentially means that all the dyadic shifts in the representation of T are *cancellative*, so the case of noncancellative shifts remained open. This gap was partially filled in [Ou and Petermichl 2018], which treats the case of noncancellative shifts of standard paraproduct type. So the case of general Journé operators, which includes noncancellative shifts of mixed and partial type in the representation, remained open even in the unweighted, $p = 2$ case. These types of paraproducts are notoriously difficult — see also [Martikainen and Orponen 2016] for a wonderful discussion of this issue. We fill this gap in Section 7D, where we prove two-weight bounds of the type

$$\|[b, \mathbb{S}_{\boldsymbol{\mathcal{D}}}] : L^p(\mu) \to L^p(\lambda)\| \lesssim \|b\|_{\mathrm{bmo}(\nu)},$$

where $\mathbb{S}_{\boldsymbol{\mathcal{D}}}$ is a noncancellative shift. The same is proved for cancellative shifts in Section 7C.

At the backbone of all these proofs will be the biparameter paraproducts, developed in Section 6, and a variety of biparameter square functions, developed in Section 3. For instance, in the case of the cancellative shifts, one can decompose the commutator as

$$[b, \mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}}]f = \sum [\mathsf{P}_{\mathsf{b}}, \mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}}]f + \sum [\mathsf{p}_{\mathsf{b}}, \mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}}]f + \mathcal{R}_{\vec{i},\vec{j}}f.$$

Here $\mathsf{P_b}$ runs through nine paraproducts associated with *product BMO*, and $\mathsf{p_b}$ runs through six paraproducts associated with *little bmo*, so we are dealing with fifteen paraproducts in total in the biparameter case. Some of these are straightforward generalizations of the one-parameter paraproducts, while some are more complicated "mixed" paraproducts. Two-weight bounds are proved for all these paraproducts in Section 6, building on two essential blocks: the biparameter square functions in Section 3, and the weighted H^1-BMO duality in the product setting, developed in Section 4. In fact, Section 4 is a self-contained presentation of large parts of the weighted biparameter BMO theory.

Once the paraproducts are bounded, all that is left is to bound the so-called "remainder term" $\mathcal{R}_{\vec{i},\vec{j}}f$, of the form $\Pi_{\mathbb{S}f}b - \mathbb{S}\Pi_f b$, where one can no longer appeal directly to the paraproducts. At this point, however, things become very technical, so bounding the remainder terms is no easy task. To help guide the reader, we outline below the general strategy we will employ. This applies to Theorem 7.2, and in large part to Theorems 7.3, 7.4, and 7.5:

(1) We break up the remainder term into more convenient sums of operators of the type $\mathcal{O}(b, f)$, involving both $b \in \mathrm{bmo}(\nu)$ and $f \in L^p(\mu)$. We want to show $\|\mathcal{O}(b, f) : L^p(\mu) \to L^p(\lambda)\| \lesssim \|b\|_{\mathrm{bmo}(\nu)}$. Using duality this amounts to showing that

$$|\langle \mathcal{O}(b, f), g\rangle| \lesssim \|b\|_{\mathrm{BMO}(\nu)} \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\lambda')}.$$

(2) Some of these operators $\mathcal{O}(b, f)$ involve full Haar coefficients $\hat{b}(Q_1 \times Q_2)$ of b, while others involve a Haar coefficient in one variable and averaging in the other variable, such as $\langle b, h_{Q_1} \times \mathbb{1}_{Q_2}/|Q_2|\rangle$. Since, ultimately, we wish to use some type of H^1-BMO duality, the goal will be to "separate out" b from the inner product $\langle \mathcal{O}(b, f), g\rangle$. If $\mathcal{O}(b, f)$ involves full Haar coefficients of b, we use duality with *product BMO* and obtain

$$|\langle \mathcal{O}(b, f), g\rangle| \lesssim \|b\|_{\mathrm{BMO}(\nu)} \|S_{\boldsymbol{\mathcal{D}}}\phi(f, g)\|_{L^1(\nu)},$$

where $\phi(f, g)$ is the operator we are left with after separating out b, and $S_{\boldsymbol{\mathcal{D}}}$ is the full biparameter dyadic square function. If $\mathcal{O}(b, f)$ involves terms of the form $\langle b, h_{Q_1} \times \mathbb{1}_{Q_2}/|Q_2|\rangle$, we use duality with *little bmo*, and obtain something of the form

$$|\langle \mathcal{O}(b, f), g\rangle| \lesssim \|b\|_{\mathrm{bmo}(\nu)} \|S_{\mathcal{D}_1}\phi(f, g)\|_{L^1(\nu)},$$

where $S_{\mathcal{D}_1}$ is the dyadic square function in the first variable. Obviously this is replaced with $S_{\mathcal{D}_2}$ if the Haar coefficient on b is in the second variable.

(3) Then the next goal is to show that

$$S_{\boldsymbol{\mathcal{D}}}\phi(f, g) \lesssim (\mathcal{O}_1 f)(\mathcal{O}_2 g),$$

where $\mathcal{O}_{1,2}$ will be operators satisfying a *one-weight* bound of the type $L^p(w) \to L^p(w)$. These operators will usually be a combination of the biparameter square functions in Section 3. Once we have this, we are done.

In Theorem 7.2, dealing with cancellative shifts, the crucial part is really step (1). At first glance, the remainder term $\mathcal{R}_{\vec{i},\vec{j}}f$ seems intractable using this method, since it involves average terms $\langle b\rangle_{Q_1 \times Q_2}$ instead of Haar coefficients of b. So the key here is to decompose these terms in some convenient form.

In Section 7D, dealing with noncancellative shifts, the proofs follow this strategy in spirit, but deviate as we advance through the more and more difficult operators. The main issue here is that we are really dealing with terms of the form $|\langle \mathcal{O}(a, b, f), g\rangle|$, where now the operator $\mathcal{O}$ involves a function b in the *weighted little* $\mathrm{bmo}(\nu)$, and a function a in *unweighted product* BMO. In the most difficult case of partial paraproducts, a is even more complicated because it is essentially a *sequence* of *one-parameter* unweighted BMO functions. In all these cases, the creature ϕ in the last step is really $\phi(a, f, g)$. While in the previous case involving $\phi(f, g)$ it was straightforward to see the correct operators $\mathcal{O}_{1,2}$ to achieve step (3), in this case nothing straightforward seems to work.

There are two key new ideas in these cases: one is to combine the cumbersome remainder term with a cleverly chosen third term, which will make the decompositions easier to handle. The other is to temporarily employ martingale transforms — which works for us because this does not increase the BMO norms. We briefly describe the three situations below. As above, we will be rather nonrigorous about the notation in this expository section. There is plenty of notation later, and the purpose here is just to explain the main ideas and guide the reader through the technical proofs in Section 7D:

(1) *The full standard paraproduct*: Theorem 7.3. This case only requires simple martingale transforms (a_τ and g_τ, which have all nonnegative Haar coefficients), and otherwise follows the strategy outlined above. However, we already start to see the operators $\mathcal{O}_{1,2}$ becoming strange compositions of "standard" operators and unweighted paraproducts, such as

$$S_{\boldsymbol{\mathcal{D}}}\phi \leq (M_S \Pi^*_{a_\tau} g_\tau)(S_{\boldsymbol{\mathcal{D}}} f).$$

(2) *The full mixed paraproduct*: Theorem 7.4. Here we introduce the idea of combining the remainder term $\Pi_{\mathbb{S}f} b - \mathbb{S}\Pi_f b$ with a third term T, and we analyze $(\Pi_{\mathbb{S}f} b - T)$ and $(T - \mathbb{S}\Pi_f b)$ separately. This allows us to express the remainder as

$$\sum [\mathsf{P_a}, \mathsf{p_b}] f + T^{(1,0)}_{a,b} f - T^{(0,1)}_{a,b} f,$$

a sum of *commutators of paraproduct operators*, and a new remainder term. The new remainder has no cancellation properties, so we prove separately that the $T_{a,b}$ operators satisfy

$$|\langle T_{a,b} f, g\rangle| \lesssim \|b\|_{\mathrm{bmo}(\nu)} \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\lambda')}.$$

Here is where we employ the strategy outlined earlier, combined with a martingale transform a_τ applied to a. Interestingly, this transform depends on the particular argument f of $[b, \mathbb{S}_{\boldsymbol{\mathcal{D}}}] f$. This will be absorbed in the end by the BMO norm of the symbol for $\mathbb{S}_{\boldsymbol{\mathcal{D}}}$, so ultimately the choice of f will not matter.

(3) *The partial paraproducts*: Theorem 7.5. Here we again combine the remainder terms with a third term T, and this time end up with terms of the form $\mathsf{p_b} F$, where F is a term depending on a and f. So we are done if we can show that $\|F\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu)}$. Without getting too technical about the notation, we reiterate that here a is not *one function* but rather a *sequence* a_{PQR} of one-parameter unweighted BMO functions. So the difficulty here is that the inner products look something like

$$\langle F, g\rangle = \sum \langle \Pi^*_{a_{PQR}} \tilde{f}, \tilde{g}\rangle,$$

where each summand has its own BMO function! The trick is then to write this as $\sum\langle a_{PQR}, \phi_{PQR}(f, g)\rangle$. The happy ending is that these functions a_{PQR} have uniformly bounded BMO norms, so at this point we apply unweighted one-parameter H^1- BMO duality and we are left to work with $\|S_{\mathcal{D}}\phi(f, g)\|_{L^1(\mathbb{R}^n)}$; this is manageable. In one case, we do have to work with F_τ instead, which is again obtained by applying martingale transforms chosen in terms of f — only this time to each function a_{PQR}.

Finally, we see no reason why this result cannot be generalized to k-parameter Journé operators. The main trouble in such a generalization should be strictly computational, as the number of paraproducts will blow up.

In Section 8 we recall the definition of the mixed $\mathrm{BMO}_{\mathcal{I}}$ classes in between Chang and Fefferman's product BMO and Cotlar and Sadosky's little BMO. In the same way as in [Ou et al. 2016] we deduce a corollary from Theorem 1.1:

Theorem 1.2 (upper bound, iterated, unweighted case). *Let us consider $\mathbb{R}^{\vec{d}}$, $\vec{d} = (d_1, \dots, d_t)$, with a partition $\mathcal{I} = (I_s)_{1\le s\le l}$ of $\{1, \dots, t\}$. Let $b \in \mathrm{BMO}_{\mathcal{I}}(\mathbb{R}^{\vec{d}})$ and let T_s denote a multiparameter Journé operator acting on functions defined on $\bigotimes_{k\in I_s} \mathbb{R}^{d_k}$. Then we have the estimate*

$$\|[T_1, \dots, [T_l, b], \dots]\|_{L^p(\mathbb{R}^{\vec{d}})\to L^p(\mathbb{R}^{\vec{d}})} \lesssim \|b\|_{\mathrm{BMO}_{\mathcal{I}}(\mathbb{R}^{\vec{d}})}.$$

Coming back to the Bloom setting, we prove the lower estimate below, via a modification of the unweighted one-parameter argument of Coifman, Rochberg and Weiss.

Theorem 1.3 (lower bound). *Let μ, λ be $A_p(\mathbb{R}^n \times \mathbb{R}^n)$ weights, and set $\nu = \mu^{1/p}\lambda^{-1/p}$. Then*

$$\|b\|_{\mathrm{bmo}(\nu)} \lesssim \sup_{1\leqslant k,l\leqslant n} \|[b, R_k^1 R_l^2]\|_{L^p(\mu)\to L^p(\lambda)},$$

where R_k^1 and R_l^2 are the Riesz transforms acting in the first and second variables, respectively.

This lower estimate allows us to see the tensor products of Riesz transforms as a representative testing class for all Journé operators.

We point out that in our quest to prove Theorem 1.1, we also obtain a much simplified proof of the following one-weight result for Journé operators, originally due to R. Fefferman:

Theorem 1.4 (weighted inequality for Journé operators). *Let T be a biparameter Journé operator on $\mathbb{R}^{\vec{n}} = \mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$. Then T is bounded $L^p(w) \to L^p(w)$ for all $w \in A_p(\mathbb{R}^{\vec{n}})$, $1 < p < \infty$.*

A version of Theorem 1.4 was first introduced by R. Fefferman and E. M. Stein [1982], with restrictive assumptions on the kernel. Subsequently the kernel assumptions were weakened significantly by R. Fefferman [1987], at the cost of assuming the weight belongs to the more restrictive class $A_{p/2}$. This was due to the use of his sharp function $T^\# f = M_S(f^2)^{1/2}$, where M_S is strong maximal function. Finally, he improved his own result in [Fefferman 1988], where he showed that the A_p class sufficed and obtained the full statement of Theorem 1.4. This was achieved by an involved bootstrapping argument based on his previous result [Fefferman 1987].

Our proof in Section 7E of Theorem 1.4 is significantly simpler. This may seem like a "rough sell" in light of the many pages of highly technical calculations that precede it. However, our proof of Section 7E is only based on one-weight bounds for the biparameter dyadic shifts of the form

$$\|\mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}} : L^p(w) \to L^p(w)\| \lesssim 1. \tag{1-1}$$

These had to be proved along the way, as part of our proof of the two-weight upper bound for commutators, Theorem 1.1. These one-weight bounds are useful in themselves, and their proofs are not that long: the proof for cancellative shifts, given in (7-2), is easy, and the proof for the noncancellative shifts of partial paraproduct type is given in Proposition 7.6. Once we have (1-1), the proof of Theorem 1.4 follows immediately from Martikainen's representation theorem — just as in the one-parameter case, a weighted bound for Calderón–Zygmund operators follows trivially from Hytönen's representation theorem, once one has the one-weight bounds for the one-parameter dyadic shifts.

The paper is organized as follows. In Section 2 we review the necessary background, both one-parameter and biparameter, and set up the notation. In Section 3 we set up the types of dyadic square functions we will need throughout the rest of the paper. In Section 4, we discuss the weighted and Bloom BMO spaces in the biparameter setting, and use some of these results in Section 5 to prove the lower bound result. Section 6 is dedicated to biparameter paraproducts, which will be crucial in Section 7, which proves the upper bound by an appeal to Martikainen's representation theorem [2012]. Finally, we prove Theorem 1.4.

2. Background and notation

We review some of the basic building blocks of one-parameter dyadic harmonic analysis on $\mathbb{R}^n$, followed by their biparameter versions for $\mathbb{R}^{\vec{n}} := \mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$.

2A. ***Dyadic grids on*** $\mathbb{R}^n$**.** Let $\mathcal{D}_0 := \{2^{-k}([0,1)^n + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}$ denote the standard dyadic grid on $\mathbb{R}^n$. For every $\omega = (\omega_j)_{j\in\mathbb{Z}} \in (\{0,1\}^n)^{\mathbb{Z}}$ define the shifted dyadic grid $\mathcal{D}_\omega$:

$$\mathcal{D}_\omega := \{Q \dot{+} \omega : Q \in \mathcal{D}_0\}, \quad \text{where } Q \dot{+} \omega := Q + \sum_{j:2^{-j} < l(Q)} 2^{-j}\omega_j,$$

and $l(Q)$ denotes the side length of a cube Q. The indexing parameter ω is rarely relevant in what follows: it only appears when we are dealing with $\mathbb{E}_\omega$ — expectation with respect to the standard probability measure on the space of parameters ω. In fact, an important feature of the (by now standard) methods we employ in this paper is obtaining upper bounds for dyadic operators that are independent of the choice of dyadic grid. The focus therefore is on the geometrical properties shared by all dyadic grids $\mathcal{D}$ on $\mathbb{R}^n$:

- $P \cap Q \in \{P, Q, \varnothing\}$ for every $P, Q \in \mathcal{D}$.
- The cubes $Q \in \mathcal{D}$ with $l(Q) = 2^{-k}$, for some fixed integer k, partition $\mathbb{R}^n$.

For every $Q \in \mathcal{D}$ and every nonnegative integer k we define:

- $Q^{(k)}$ — the k-th generation ancestor of Q in $\mathcal{D}$, i.e., the unique element of $\mathcal{D}$ which contains Q and has side length $2^k l(Q)$.
- $(Q)_k$ — the collection of k-th generation descendants of Q in $\mathcal{D}$, i.e., the 2^{kn} disjoint subcubes of Q with side length $2^{-k} l(Q)$.

2B. ***The Haar system on*** $\mathbb{R}^n$**.** Recall that every dyadic interval I in $\mathbb{R}$ is associated with two Haar functions,

$$h_I^0 := \frac{1}{\sqrt{|I|}}(\mathbb{1}_{I-} - \mathbb{1}_{I_+}) \quad \text{and} \quad h_I^1 := \frac{1}{\sqrt{|I|}}\,\mathbb{1}_I,$$

the first one being cancellative (it has mean 0). Given a dyadic grid $\mathcal{D}$ on $\mathbb{R}^n$, every dyadic cube $Q = I_1 \times \cdots \times I_n$, where all I_i are dyadic intervals in $\mathbb{R}$ with common length $l(Q)$, is associated with $2^n - 1$ cancellative Haar functions:

$$h_Q^\epsilon(x) := h_{I_1\times\cdots\times I_n}^{(\epsilon_1,\ldots,\epsilon_n)}(x_1,\ldots,x_n) := \prod_{i=1}^n h_{I_i}^{\epsilon_i}(x_i),$$

where $\epsilon \in \{0,1\}^n \setminus \{(1,\ldots,1)\}$ is the signature of h_Q^ϵ. To simplify notation, we assume that signatures are never the identically 1 signature, in which case the corresponding Haar function would be noncancellative. The cancellative Haar functions form an orthonormal basis for $L^2(\mathbb{R}^n)$. We write

$$f = \sum_{Q\in\mathcal{D}} \hat{f}(Q^\epsilon) h_Q^\epsilon,$$

where $\hat{f}(Q^\epsilon) := \langle f, h_Q^\epsilon\rangle$, $\langle f, g\rangle := \int_{\mathbb{R}^n} fg\,dx$, and summation over ϵ is assumed. We list here some other useful facts which will come in handy later:

- $h_P^\epsilon(x)$ is constant on any subcube $Q \in \mathcal{D}$, $Q \subsetneq P$. We denote this value by $h_P^\epsilon(Q)$.
- The average of f over a cube $Q \in \mathcal{D}$ may be expressed as

$$\langle f\rangle_Q = \sum_{P\in\mathcal{D},\,P\supsetneq Q} \hat{f}(P^\epsilon) h_P^\epsilon(Q). \tag{2-1}$$

- Then, if $Q \subsetneq R \in \mathcal{D}$,

$$\langle f\rangle_Q - \langle f\rangle_R = \sum_{P\in\mathcal{D},\,Q\subsetneq P\subset R} \hat{f}(P^\epsilon) h_P^\epsilon(Q). \tag{2-2}$$

- For $Q \in \mathcal{D}$,

$$\mathbb{1}_Q(f - \langle f\rangle_Q) = \sum_{P\in\mathcal{D},\,P\subset Q} \hat{f}(P^\epsilon) h_P^\epsilon. \tag{2-3}$$

- For two *distinct* signatures $\epsilon \neq \delta$, define the signature $\epsilon+\delta$ by letting $(\epsilon+\delta)_i$ be 1 if $\epsilon_i = \delta_i$ and 0 otherwise. Note that $\epsilon+\delta$ is distinct from both ϵ and δ, and is not the identically $\vec{1}$ signature. Then

$$h_Q^\epsilon h_Q^\delta = \frac{1}{\sqrt{Q}} h_Q^{\epsilon+\delta} \qquad \text{if } \epsilon \neq \delta \quad \text{and} \quad h_Q^\epsilon h_Q^\epsilon = \frac{\mathbb{1}_Q}{|Q|}.$$

Again to simplify notation, we assume throughout this paper that we only write $h_Q^{\epsilon+\delta}$ for *distinct* signatures ϵ and δ.

Given a dyadic grid $\mathcal{D}$, we define the dyadic square function on $\mathbb{R}^n$ by

$$S_{\mathcal{D}} f(x) := \left(\sum_{Q\in\mathcal{D}} |\hat{f}(Q^\epsilon)|^2 \frac{\mathbb{1}_Q(x)}{|Q|}\right)^{1/2}.$$

Then $\|f\|_p \simeq \|S_{\mathcal{D}} f\|_p$ for all $1 < p < \infty$. We also define the dyadic version of the maximal function:

$$M_{\mathcal{D}} f(x) = \sup_{Q \in \mathcal{D}} \langle |f| \rangle_Q \mathbb{1}_Q(x).$$

2C. $A_p(\mathbb{R}^n)$ *weights.* Let w be a weight on $\mathbb{R}^n$; i.e., w is an almost everywhere positive, locally integrable function. For $1 < p < \infty$, let $L^p(w) := L^p(\mathbb{R}^n; w(x)\,dx)$. For a cube Q in $\mathbb{R}^n$, we let

$$w(Q) := \int_Q w(x)\,dx \quad \text{and} \quad \langle w \rangle_Q := \frac{w(Q)}{|Q|}.$$

We say that w belongs to the Muckenhoupt $A_p(\mathbb{R}^n)$ class provided that

$$[w]_{A_p} := \sup_Q \langle w \rangle_Q \langle w^{1-p'} \rangle_Q^{p-1} < \infty,$$

where p' denotes the Hölder conjugate of p and the supremum above is over all cubes Q in $\mathbb{R}^n$ with sides parallel to the axes. The weight $w' := w^{1-p'}$ is sometimes called the weight "conjugate" to w, because $w \in A_p$ if and only if $w' \in A_{p'}$.

We recall the classical inequalities for the maximal and square functions

$$\|Mf\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \quad \text{and} \quad \|f\|_{L^p(w)} \simeq \|S_{\mathcal{D}} f\|_{L^p(w)}$$

for all $w \in A_p(\mathbb{R}^n)$, $1 < p < \infty$, where throughout this paper "$A \lesssim B$" denotes $A \le cB$ for some constant c which may depend on the dimensions and the weight w. In dealing with dyadic shifts, we will also need to consider the following shifted dyadic square function: given nonnegative integers i and j, define

$$S_{\mathcal{D}}^{i,j} f(x) := \left[\sum_{R \in \mathcal{D}} \left(\sum_{P \in (R)_i} |\hat{f}(P^\epsilon)| \right)^2 \left(\sum_{Q \in (R)_j} \frac{\mathbb{1}_Q(x)}{|Q|} \right) \right]^{1/2}.$$

It was shown in [Holmes et al. 2017] that

$$\|S_{\mathcal{D}}^{i,j} : L^p(w) \to L^p(w)\| \lesssim 2^{(n/2)(i+j)} \tag{2-4}$$

for all $w \in A_p(\mathbb{R}^n)$, $1 < p < \infty$.

A *martingale transform* on $\mathbb{R}^n$ is an operator of the form

$$f \mapsto f_\tau := \sum_{P \in \mathcal{D}} \tau_P^\epsilon \hat{f}(P^\epsilon) h_P^\epsilon,$$

where each τ_P^ϵ is either $+1$ or -1. Obviously $S_{\mathcal{D}} f = S_{\mathcal{D}} f_\tau$, so one can work with f_τ instead when convenient, without increasing the $L^p(w)$-norm of f.

2D. *The Haar system on* $\mathbb{R}^{\vec{n}}$. In $\mathbb{R}^{\vec{n}} := \mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$, we work with dyadic rectangles

$$\boldsymbol{\mathcal{D}} := \mathcal{D}_1 \times \mathcal{D}_2 = \{R = Q_1 \times Q_2 : Q_i \in \mathcal{D}_i\},$$

where each $\mathcal{D}_i$ is a dyadic grid on $\mathbb{R}^{n_i}$. While we unfortunately lose the nice nestedness and partitioning properties of one-parameter dyadic grids, we do have the tensor product Haar wavelet orthonormal basis for $L^2(\mathbb{R}^{\vec{n}})$, defined by

$$h_R^{\vec{\epsilon}}(x_1, x_2) := h_{Q_1}^{\epsilon_1}(x_1) \otimes h_{Q_2}^{\epsilon_2}(x_2)$$

for all $R = Q_1 \times Q_2 \in \boldsymbol{\mathcal{D}}$ and $\vec{\epsilon} = (\epsilon_1, \epsilon_2)$. We often write

$$f = \sum_{Q_1 \times Q_2} \hat{f}(Q_1^{\epsilon_1} \times Q_2^{\epsilon_2}) h_{Q_1}^{\epsilon_1} \otimes h_{Q_2}^{\epsilon_2},$$

short for summing over $Q_1 \in \mathcal{D}_1$ and $Q_2 \in \mathcal{D}_2$, and of course over all signatures, where

$$\hat{f}(Q_1^{\epsilon_1} \times Q_2^{\epsilon_2}) := \langle f, h_{Q_1}^{\epsilon_1} \otimes h_{Q_2}^{\epsilon_2} \rangle = \int_{\mathbb{R}^{\vec{n}}} f(x_1, x_2) h_{Q_1}^{\epsilon_1}(x_1) h_{Q_2}^{\epsilon_2}(x_2)\, dx_1\, dx_2.$$

While the averaging formula (2-1) has a straightforward biparameter analogue

$$\langle f \rangle_{Q_1 \times Q_2} = \sum_{\substack{P_1 \supsetneq Q_1 \\ P_2 \supsetneq Q_2}} \hat{f}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) h_{P_1}^{\epsilon_1}(Q_1) h_{P_2}^{\epsilon_2}(Q_2),$$

the expression in (2-3) takes a slightly messier form in two parameters: for any $R = Q_1 \times Q_2$

$$\begin{aligned} &\mathbb{1}_R (f - \langle f \rangle_R) \\ &= \sum_{\substack{P_1 \subset Q_1 \\ P_2 \subset Q_2}} \hat{f}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) h_{P_1}^{\epsilon_1} \otimes h_{P_2}^{\epsilon_2} + \sum_{P_2 \subset Q_2} \Big\langle f, \frac{\mathbb{1}_{Q_1}}{|Q_1|} \otimes h_{P_2}^{\epsilon_2} \Big\rangle \mathbb{1}_{Q_1} \otimes h_{P_2}^{\epsilon_2} + \sum_{P_1 \subset Q_1} \Big\langle f, h_{P_1}^{\epsilon_1} \otimes \frac{\mathbb{1}_{Q_2}}{|Q_2|} \Big\rangle h_{P_1}^{\epsilon_1} \otimes \mathbb{1}_{Q_2} \\ &= \sum_{\substack{P_1 \subset Q_1 \\ P_2 \subset Q_2}} \hat{f}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) h_{P_1}^{\epsilon_1} \otimes h_{P_2}^{\epsilon_2} + \mathbb{1}_R [m_{Q_1} f(x_2) - \langle f \rangle_R] + \mathbb{1}_R [m_{Q_2} f(x_1) - \langle f \rangle_R], \end{aligned} \tag{2-5}$$

where for any cubes $Q_i \in \mathcal{D}_i$,

$$m_{Q_1} f(x_2) := \frac{1}{|Q_1|} \int_{Q_1} f(x_1, x_2)\, dx_1 \quad \text{and} \quad m_{Q_2} f(x_1) := \frac{1}{|Q_2|} \int_{Q_2} f(x_1, x_2)\, dx_2. \tag{2-6}$$

As we shall see later, this particular expression will be quite relevant for biparameter BMO spaces.

2E. $A_p(\mathbb{R}^{\vec{n}})$ *weights.* A weight $w(x_1, x_2)$ on $\mathbb{R}^{\vec{n}}$ belongs to the class $A_p(\mathbb{R}^{\vec{n}})$ for some $1 < p < \infty$, provided that

$$[w]_{A_p} := \sup_R \langle w \rangle_R \langle w^{1-p'} \rangle_R^{p-1} < \infty,$$

where the supremum is over all *rectangles* R. These are the weights which characterize $L^p(w)$ boundedness of the *strong* maximal function

$$M_S f(x_1, x_2) := \sup_R \langle |f| \rangle_R \, \mathbb{1}_R(x_1, x_2),$$

where the supremum is again over all rectangles. As is well known, the usual weak (1, 1) inequality fails for the strong maximal function, where it is replaced by an Orlicz norm expression. In the weighted case, we have [Bagby and Kurtz 1985] for all $w \in A_p(\mathbb{R}^{\vec{n}})$,

$$w\{x \in \mathbb{R}^{\vec{n}} : M_S f(x) > \lambda\} \lesssim \int_{\mathbb{R}^{\vec{n}}} \left(\frac{|f(x)|}{\lambda} \right)^p \left(1 + \log^+ \frac{|f(x)|}{\lambda} \right)^{k-1} dw(x). \tag{2-7}$$

Moreover, w belongs to $A_p(\mathbb{R}^{\vec{n}})$ if and only if w belongs to the *one-parameter* classes $A_p(\mathbb{R}^{n_i})$ in each variable separately and uniformly:

$$[w]_{A_p(\mathbb{R}^{\vec{n}})} \simeq \max\Big\{\operatorname*{ess\,sup}_{x_1\in\mathbb{R}^{n_1}}[w(x_1,\cdot\,)]_{A_p(\mathbb{R}^{n_2})},\ \operatorname*{ess\,sup}_{x_2\in\mathbb{R}^{n_2}}[w(\,\cdot\,,x_2)]_{A_p(\mathbb{R}^{n_1})}\Big\}.$$

It also follows, as in the one-parameter case, that $w \in A_p(\mathbb{R}^{\vec{n}})$ if and only if $w' := w^{1-p'} \in A_{p'}(\mathbb{R}^{\vec{n}})$ and $L^p(w)^* \simeq L^{p'}(w')$, in the sense that

$$\|f\|_{L^p(w)} = \sup\{|\langle f, g\rangle| : g \in L^{p'}(w'),\ \|g\|_{L^{p'}(w')} \le 1\}. \tag{2-8}$$

We may also define weights $m_{Q_1}w$ and $m_{Q_2}w$ on $\mathbb{R}^{n_2}$ and $\mathbb{R}^{n_1}$, respectively, as in (2-6). As shown below, these are then also uniformly in their respective one-parameter A_p classes:

Proposition 2.1. *If $w \in A_p(\mathbb{R}^{\vec{n}})$, $1 < p < \infty$, then $m_{Q_1}w \in A_p(\mathbb{R}^{n_2})$ and $m_{Q_2}w \in A_p(\mathbb{R}^{n_1})$ for any cubes $Q_i \subset \mathbb{R}^{n_i}$, with uniformly bounded A_p constants*:

$$[m_{Q_i}w]_{A_p(\mathbb{R}^{n_j})} \le [w]_{A_p(\mathbb{R}^{\vec{n}})}$$

for all $Q_i \subset \mathbb{R}^{n_i}$, $i \in \{1, 2\}$, $i \ne j$.

Proof. Fix a cube $Q_1 \subset \mathbb{R}^{n_1}$. Then for every $x_2 \in \mathbb{R}^{n_2}$,

$$|Q_1| = \int_{Q_1} 1\,dx_1 \le \left(\int_{Q_1} w(x_1,x_2)\,dx_1\right)^{1/p}\left(\int_{Q_1} w'(x_1,x_2)\,dx_1\right)^{1/p'},$$

and so

$$(m_{Q_1}w)'(x_2) := (m_{Q_1}w)^{1-p'}(x_2) \le m_{Q_1}w'(x_2).$$

Then for all cubes $Q_2 \subset \mathbb{R}^{n_2}$,

$$\langle m_{Q_1}w\rangle_{Q_2}\langle (m_{Q_1}w)'\rangle_{Q_2}^{p-1} \le \langle w\rangle_{Q_1\times Q_2}\langle w'\rangle_{Q_1\times Q_2}^{p-1} \le [w]_{A^p(\mathbb{R}^{\vec{n}})},$$

proving the result for $m_{Q_1}w$. The other case follows symmetrically. □

Finally, we will later use a reverse Hölder property of biparameter A_p weights. This is well known to experts, but we include a proof here for completeness.

Proposition 2.2. *If $w \in A_p(\mathbb{R}^{\vec{n}})$, then there exist positive constants $C, \epsilon, \delta > 0$ (depending only on $\vec{n}$, p, and $[w]_{A_p(\mathbb{R}^{\vec{n}})}$) such that*:

(i) *For all rectangles $R \subset \mathbb{R}^{\vec{n}}$,*

$$\left(\frac{1}{|R|}\int_R w(x)^{1+\epsilon}\,dx\right)^{1/(1+\epsilon)} \le \frac{C}{|R|}\int_R w(x)\,dx.$$

(ii) *For all rectangles $R \subset \mathbb{R}^{\vec{n}}$ and all measurable subsets $E \subset R$,*

$$\frac{w(E)}{w(R)} \le C\left(\frac{|E|}{|R|}\right)^{\delta}.$$

Proof. Note first that (ii) follows easily from (i) by applying the Hölder inequality with exponents $1+\epsilon$ and $(1+\epsilon)/\epsilon$ in $w(E)=\int_E w(x)\,dx$. This gives (ii) with $\delta=\epsilon/(1+\epsilon)$.

In order to prove (i) we first recall a more general statement of the one-parameter reverse Hölder property of A_p weights (see Remark 9.2.3 in [Grafakos 2004]):

For any $1<p<\infty$ and $B>1$, there exist positive constants

$$D=D(n,p,B) \quad \textit{and} \quad \beta=\beta(n,p,B) \tag{2-9}$$

such that for all $v\in A_p(\mathbb{R}^n)$ with $[v]_{A_p(\mathbb{R}^{\vec{n}})}\le B$, the reverse Hölder condition

$$\left(\frac{1}{|Q|}\int_Q v(t)^{1+\beta}\,dt\right)^{1/(1+\beta)} \le \frac{D}{|Q|}\int_Q v(t)\,dt \tag{2-10}$$

holds for all cubes $Q\subset\mathbb{R}^n$.

It is easy to see that if a weight v satisfies the reverse Hölder condition (2-10) with constants D,β, then it also satisfies it with any constants C,ϵ with $C\ge D$ and $\epsilon\le\beta$.

Now let $w\in A_p(\mathbb{R}^{\vec{n}})$, set $B:=[w]_{A_p(\mathbb{R}^{\vec{n}})}$, and for $i\in\{1,2\}$ let $D_i:=D(n_i,p,B)$ and $\beta_i:=\beta(n_i,p,B)$ be as in (2-9). Fix a rectangle $R=Q_1\times Q_2$, a measurable subset $E\subset R$, and set

$$C^2:=\max(D_1,D_2) \quad \text{and} \quad \epsilon:=\min(\beta_1,\beta_2).$$

For almost all $x_1\in\mathbb{R}^{n_1}$, we have $w(x_1,\cdot)\in A_p(\mathbb{R}^{n_2})$ with $[w(x_1,\cdot)]_{A_p(\mathbb{R}^{n_2})}\le B$, so $w(x_1,\cdot)$ satisfies reverse Hölder with constants D_2, β_2 — and therefore also with constants $\sqrt{C},\epsilon$. So

$$\begin{aligned}\frac{1}{|R|}\int_R w(x)^{1+\epsilon}\,dx &= \frac{1}{|Q_1|}\int_{Q_1}\left(\frac{1}{|Q_2|}w(x_1,x_2)^{1+\epsilon}\,dx_2\right)dx_1\\ &\le \frac{1}{|Q_1|}\int_{Q_1}\left(\frac{\sqrt{C}}{|Q_2|}\int_{Q_2}w(x_1,x_2)\,dx_2\right)^{1+\epsilon}dx_1\\ &= \frac{C^{(1+\epsilon)/2}}{|Q_1|}\int_{Q_1}(m_{Q_2}w(x_1))^{1+\epsilon}\,dx_1.\end{aligned}$$

By Proposition 2.1, we have $m_{Q_2}w\in A_p(\mathbb{R}^{n_1})$ with $[m_{Q_2}w]_{A_p(\mathbb{R}^{n_1})}\le B$, so this weight satisfies reverse Hölder with constants D_1, β_1 — and therefore also with constants $\sqrt{C},\epsilon$. Then the last inequality above gives

$$\left(\frac{1}{|R|}\int_R w(x)^{1+\epsilon}\,dx\right)^{1/(1+\epsilon)} \le \frac{C}{|Q_1|}\int_{Q_1} m_{Q_2}w(x_1)\,dx_1=\frac{C}{|R|}\int_R w(x)\,dx. \qquad \square$$

3. Biparameter dyadic square functions

Throughout this section, fix dyadic rectangles $\boldsymbol{\mathcal{D}}:=\mathcal{D}_1\times\mathcal{D}_2$ on $\mathbb{R}^{\vec{n}}$. The dyadic square function associated with $\boldsymbol{\mathcal{D}}$ is then defined in the obvious way:

$$S_{\boldsymbol{\mathcal{D}}}f(x_1,x_2):=\left(\sum_{R\in\boldsymbol{\mathcal{D}}}|\hat{f}(R^{\vec{\epsilon}})|^2\frac{\mathbb{1}_R(x_1,x_2)}{|R|}\right)^{1/2}.$$

We also want to look at the dyadic square functions *in each variable*, namely

$$S_{\mathcal{D}_1} f(x_1,x_2) := \bigg(\sum_{Q_1\in\mathcal{D}_1} |H^{\epsilon_1}_{Q_1} f(x_2)|^2 \frac{\mathbb{1}_{Q_1}(x_1)}{|Q_1|}\bigg)^{1/2}, \quad S_{\mathcal{D}_2} f(x_1,x_2) := \bigg(\sum_{Q_2\in\mathcal{D}_2} |H^{\epsilon_2}_{Q_2}(x_1)|^2 \frac{\mathbb{1}_{Q_2}(x_2)}{|Q_2|}\bigg)^{2},$$

where for every $Q_i \in \mathcal{D}_i$ and signatures ϵ_i, we define

$$H^{\epsilon_1}_{Q_1} f(x_2) := \int_{\mathbb{R}^{n_1}} f(x_1,x_2) h^{\epsilon_1}_{Q_1}(x_1)\,dx_1, \quad H^{\epsilon_2}_{Q_2} f(x_1) := \int_{\mathbb{R}^{n_2}} f(x_1,x_2) h^{\epsilon_2}_{Q_2}(x_2)\,dx_2.$$

Then for any $w \in A_p(\mathbb{R}^{\vec{n}})$,

$$\|f\|_{L^p(w)} \simeq \|S_{\boldsymbol{\mathcal{D}}} f\|_{L^p(w)} \simeq \|S_{\mathcal{D}_1} f\|_{L^p(w)} \simeq \|S_{\mathcal{D}_2} f\|_{L^p(w)}.$$

More generally, define the shifted biparameter square function, for pairs $\vec{i} = (i_1, i_2)$ and $\vec{j} = (j_1, j_2)$ of nonnegative integers, by

$$S^{\vec{i},\vec{j}}_{\boldsymbol{\mathcal{D}}} f := \Bigg[\sum_{\substack{R_1\in\mathcal{D}_1\\ R_2\in\mathcal{D}_2}} \bigg(\sum_{\substack{P_1\in(R_1)_{i_1}\\ P_2\in(R_2)_{i_2}}} |\hat{f}(P_1^{\epsilon_1}\times P_2^{\epsilon_2})|\bigg)^2 \bigg(\sum_{\substack{Q_1\in(R_1)_{j_1}\\ Q_2\in(R_2)_{j_2}}} \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|}\bigg)\Bigg]^{1/2}. \tag{3-1}$$

We claim that

$$\|S^{\vec{i},\vec{j}}_{\boldsymbol{\mathcal{D}}} : L^p(w) \to L^p(w)\| \lesssim 2^{(n_1/2)(i_1+j_1)} 2^{(n_2/2)(i_2+j_2)} \tag{3-2}$$

for all $w \in A_p(\mathbb{R}^{\vec{n}}),\ 1 < p < \infty$. This follows by iteration of the one-parameter result in (2-4), through the following vector-valued version of the extrapolation theorem (see Corollary 9.5.7 in [Grafakos 2004]):

Proposition 3.1. *Suppose that an operator T satisfies $\|T : L^2(w) \to L^2(w)\| \le AC_n[w]_{A_2}$ for all $w \in A_2(\mathbb{R}^n)$, for some constants A and C_n, where the latter only depends on the dimension. Then*

$$\bigg\|\bigg(\sum_j |Tf_j|^2\bigg)^{1/2}\bigg\|_{L^p(w)} \le AC'_n [w]_{A_p}^{\max(1,1/(p-1))} \bigg\|\bigg(\sum_j |f_j|^2\bigg)^{1/2}\bigg\|_{L^p(w)}$$

for all $w \in A_p(\mathbb{R}^n),\ 1 < p < \infty$, and all sequences $\{f_j\} \subset L^p(w)$, where C'_n is a dimensional constant.

Proof of (3-2). Note that $(S^{\vec{i},\vec{j}}_{\boldsymbol{\mathcal{D}}} f)^2 = \sum_{R_1\in\mathcal{D}_1} (S^{i_2,j_2}_{\mathcal{D}_2} F_{R_1})^2$, where

$$F_{R_1}(x_1,x_2) := \sum_{P_2\in\mathcal{D}_2} \bigg(\sum_{P_1\in(R_1)_{i_1}} |\hat{f}(P_1^{\epsilon_1}\times P_2^{\epsilon_2})|\bigg)\bigg(\sum_{Q_1\in(R_1)_{j_1}} \frac{\mathbb{1}_{Q_1}(x_1)}{|Q_1|}\bigg)^{1/2} h^{\epsilon_2}_{P_2}(x_2).$$

Then

$$\|S^{\vec{i},\vec{j}}_{\boldsymbol{\mathcal{D}}} f\|^p_{L^p(w)} = \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \bigg(\sum_{R_1\in\mathcal{D}_1} (S^{i_2,j_2}_{\mathcal{D}_2} F_{R_1}(x_1,x_2))^2\bigg)^{p/2} w(x_1,x_2)\,dx_2\,dx_1.$$

For almost all fixed $x_1 \in \mathbb{R}^{n_1}$, we know $w(x_1,\cdot)$ is in $A_p(\mathbb{R}^{n_2})$ uniformly, so we may apply Proposition 3.1 and (2-4) to the inner integral and obtain

$$\|S^{\vec{i},\vec{j}}_{\boldsymbol{\mathcal{D}}} f\|^p_{L^p(w)} \lesssim 2^{(pn_2/2)(i_2+j_2)} \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \bigg(\sum_{R_1\in\mathcal{D}_1} |F_{R_1}(x_1,x_2)|^2\bigg)^{p/2} w(x_1,x_2)\,dx_2\,dx_1.$$

Now, we can express the integral above as

$$\int_{\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_1}} (S_{\mathcal{D}_1}^{i_1,j_1} f_\tau(x_1,x_2))^p w(x_1,x_2)\,dx_1\,dx_2 \lesssim 2^{(pn_1/2)(i_1+j_1)}\|f_\tau\|^p,$$

where

$$f_\tau = \sum_{P_1\times P_2} |\hat{f}(P_1^{\epsilon_1}\times P_2^{\epsilon_2})| h_{P_1}^{\epsilon_1}\otimes h_{P_2}^{\epsilon_2}$$

is just a biparameter martingale transform applied to f, and therefore $\|f\|_{L^p(w)} \simeq \|f_\tau\|_{L^p(w)}$ by passing to the square function. □

3A. ***Mixed square and maximal functions.*** We will later encounter mixed operators such as

$$[SM]f(x_1,x_2) := \left(\sum_{Q_1\in\mathcal{D}_1} (M_{\mathcal{D}_2}(H_{Q_1}^{\epsilon_1} f)(x_2))^2 \frac{\mathbb{1}_{Q_1}(x_1)}{|Q_1|}\right)^{1/2},$$

$$[MS]f(x_1,x_2) := \left(\sum_{Q_2\in\mathcal{D}_2} (M_{\mathcal{D}_1}(H_{Q_2}^{\epsilon_2} f)(x_1))^2 \frac{\mathbb{1}_{Q_2}(x_2)}{|Q_2|}\right)^{1/2}.$$

Next we show that these operators are bounded $L^p(w)\to L^p(w)$ for all $w\in A_p(\mathbb{R}^{\vec{n}})$. The proof only relies on the fact that the one-parameter maximal function satisfies a weighted bound. So we state the result in a slightly more general form below, replacing $M_{\mathcal{D}_2}$ and $M_{\mathcal{D}_1}$ by any one-parameter operator that satisfies a weighted bound.

Proposition 3.2. *Let T denote a (one-parameter) operator acting on functions on $\mathbb{R}^n$ that satisfies $\|T : L^2(v)\to L^2(v)\| \le C$ for all $v\in A_2(\mathbb{R}^n)$. Define the following operators on $\mathbb{R}^{\vec{n}}$:*

$$[ST]f(x_1,x_2) := \left(\sum_{Q_1\in\mathcal{D}_1} (T(H_{Q_1}^{\epsilon_1} f)(x_2))^2 \frac{\mathbb{1}_{Q_1}(x_1)}{|Q_1|}\right)^{1/2},$$

$$[TS]f(x_1,x_2) := \left(\sum_{Q_2\in\mathcal{D}_2} (T(H_{Q_2}^{\epsilon_2} f)(x_1))^2 \frac{\mathbb{1}_{Q_2}(x_2)}{|Q_2|}\right)^{1/2},$$

where T acts on $\mathbb{R}^{n_2}$ in the first operator, and on $\mathbb{R}^{n_1}$ in the second. Then $[ST]$ and $[TS]$ are bounded $L^p(w)\to L^p(w)$ for all $w\in A_p(\mathbb{R}^{\vec{n}})$.

Proof. We have

$$\begin{aligned}\|[ST]f\|_{L^p(w)}^p &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \left(\sum_{Q_1\in\mathcal{D}_1}\left(T(H_{Q_1}^{\epsilon_1})(x_2)\frac{\mathbb{1}_{Q_1}(x_1)}{\sqrt{|Q_1|}}\right)^2\right)^{p/2} w(x_1,x_2)\,dx_2\,dx_1\\ &\lesssim \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \left(\sum_{Q_1\in\mathcal{D}_1} (H_{Q_1}^{\epsilon_1})^2(x_2)\frac{\mathbb{1}_{Q_1}(x_1)}{|Q_1|}\right)^{p/2} w(x_1,x_2)\,dx_2\,dx_1\\ &= \|S_{\mathcal{D}_1} f\|_{L^p(w)}^p \lesssim \|f\|_{L^p(w)}^p,\end{aligned}$$

where the first inequality follows as before from Proposition 3.1. The proof for $[TS]$ is symmetrical. □

More generally, define *shifted* versions of these mixed operators:

$$[ST]^{i_1,j_1} f(x_1,x_2) := \bigg(\sum_{R_1\in\mathcal{D}_1}\bigg(\sum_{P_1\in(R_1)_{i_1}} T(H^{\epsilon_1}_{P_1} f)(x_2)\bigg)^2 \sum_{Q_1\in(R_1)_{j_1}} \frac{\mathbb{1}_{Q_1}(x_1)}{|Q_1|}\bigg)^{1/2},$$

$$[TS]^{i_2,j_2} f(x_1,x_2) := \bigg(\sum_{R_2\in\mathcal{D}_2}\bigg(\sum_{P_2\in(R_2)_{i_2}} T(H^{\epsilon_2}_{P_2} f)(x_1)\bigg)^2 \sum_{Q_2\in(R_2)_{j_2}} \frac{\mathbb{1}_{Q_2}(x_2)}{|Q_2|}\bigg)^{1/2}.$$

Under the same assumptions on T, it is easy to see that

$$\|[ST]^{i_1,j_1} : L^p(w)\to L^p(w)\| \lesssim 2^{(n_1/2)(i_1+j_1)} \quad\text{and}\quad \|[TS]^{i_2,j_2} : L^p(w)\to L^p(w)\| \lesssim 2^{(n_2/2)(i_2+j_2)} \tag{3-3}$$

for all $w\in A_p(\mathbb{R}^{\vec{n}})$. Specifically,

$$\|[ST]^{i_1,j_1} f\|^p_{L^p(w)} = \int |S^{i_1,j_1}_{\mathcal{D}_1} F(x_1,x_2)|^p\,dw, \quad \text{where } F(x_1,x_2) := \sum_{P_1\in\mathcal{D}_1} T(H^{\epsilon_1}_{P_1} f)(x_2) h^{\epsilon_1}_{P_1}(x_1),$$

so $\|[ST]^{i_1,j_1} f\|_{L^p(w)} \lesssim 2^{(n_1/2)(i_1+j_1)}\|F\|_{L^p(w)}$. Now,

$$\|F\|_{L^p(w)} \simeq \|S_{\mathcal{D}_1} F\|_{L^p(w)} = \|[ST] f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$

4. Biparameter weighted BMO spaces

Given a weight w on $\mathbb{R}^n$, a locally integrable function b is said to be in the weighted BMO(w) space if

$$\|b\|_{\mathrm{BMO}(w)} := \sup_Q \frac{1}{w(Q)} \int_Q |b(x) - \langle b\rangle_Q|\,dx < \infty,$$

where the supremum is over all cubes Q in $\mathbb{R}^n$. If $w = 1$, we obtain the unweighted BMO($\mathbb{R}^n$) space. The dyadic version $\mathrm{BMO}_{\mathcal{D}}(w)$ is obtained by only taking the supremum over $Q\in\mathcal{D}$ for some given dyadic grid $\mathcal{D}$ on $\mathbb{R}^n$. If $w\in A_p(\mathbb{R}^n)$ for some $1<p<\infty$, Muckenhoupt and Wheeden [1976] showed that

$$\|b\|_{\mathrm{BMO}(w)} \simeq \|b\|_{\mathrm{BMO}(w';p')} := \sup_Q \bigg(\frac{1}{w(Q)}\int_Q |b-\langle b\rangle_Q|^{p'}\,dw'\bigg)^{1/p'}, \tag{4-1}$$

where w' is the conjugate weight to w. Moreover, if $w\in A_2(\mathbb{R}^n)$, the argument in [Wu 1992] shows that $\mathrm{BMO}_{\mathcal{D}}(w) \simeq H^1_{\mathcal{D}}(w)^*$, where the dyadic Hardy space $H^1_{\mathcal{D}}(w)$ is defined by the norm

$$\|\phi\|_{H^1_{\mathcal{D}}(w)} := \|S_{\mathcal{D}}\phi\|_{L^1(w)}.$$

Then

$$|\langle b,\phi\rangle| \lesssim \|b\|_{\mathrm{BMO}_{\mathcal{D}}(w)} \|S_{\mathcal{D}}\phi\|_{L^1(w)} \quad \text{for all } w\in A_2(\mathbb{R}^n). \tag{4-2}$$

Now suppose μ and λ are $A_p(\mathbb{R}^n)$ weights for some $1<p<\infty$, and define the Bloom weight $\nu := \mu^{1/p}\lambda^{-1/p}$. As shown in [Holmes et al. 2017], we have $\nu\in A_2(\mathbb{R}^n)$, which means we may use (4-2) with ν. A two-weight John–Nirenberg theorem for the Bloom BMO space BMO(ν) is also proved in that paper, namely

$$\|b\|_{\mathrm{BMO}(\nu)} \simeq \|b\|_{\mathrm{BMO}(\mu,\lambda,p)} \simeq \|b\|_{\mathrm{BMO}(\lambda',\mu',p')},$$

where

$$\|b\|_{\mathrm{BMO}(\mu,\lambda,p)} := \sup_Q \left(\frac{1}{\mu(Q)} \int_Q |b - \langle b \rangle_Q|^p \, d\lambda \right)^{1/p},$$

$$\|b\|_{\mathrm{BMO}(\lambda',\mu',p')} := \sup_Q \left(\frac{1}{\lambda'(Q)} \int_Q |b - \langle b \rangle_Q|^{p'} \, d\mu' \right)^{1/p'}.$$

We now look at weighted BMO spaces in the product setting $\mathbb{R}^{\vec{n}} = \mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$. Suppose $w(x_1, x_2)$ is a weight on $\mathbb{R}^{\vec{n}}$. Then we have three BMO spaces:

- *Weighted little* $\mathrm{bmo}(w)$ is the space of all locally integrable functions b on $\mathbb{R}^{\vec{n}}$ such that

$$\|b\|_{\mathrm{bmo}(w)} := \sup_R \frac{1}{w(R)} \int_R |b - \langle b \rangle_R| \, dx < \infty,$$

where the supremum is over all *rectangles* $R = Q_1 \times Q_2$ in $\mathbb{R}^{\vec{n}}$. Given a choice of dyadic rectangles $\boldsymbol{\mathcal{D}} = \mathcal{D}_1 \times \mathcal{D}_2$, we define the dyadic weighted little $\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(w)$ by taking supremum over $R \in \boldsymbol{\mathcal{D}}$.

- *Weighted product* $\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(w)$ is the space of all locally integrable functions b on $\mathbb{R}^{\vec{n}}$ such that

$$\|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(w)} := \sup_\Omega \left(\frac{1}{w(\Omega)} \sum_{R \subset \Omega, R \in \boldsymbol{\mathcal{D}}} |\hat{b}(R)|^2 \frac{1}{\langle w \rangle_R} \right)^{1/2} < \infty,$$

where the supremum is over all *open sets* $\Omega \subset \mathbb{R}^{\vec{n}}$ with $w(\Omega) < \infty$.

- *Weighted rectangular* $\mathrm{BMO}_{\boldsymbol{\mathcal{D}},\mathrm{Rec}}(w)$ is defined in a similar fashion to the unweighted case — just like product BMO, but taking the supremum over rectangles instead of over open sets:

$$\|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}},\mathrm{Rec}}(w)} := \sup_R \left(\frac{1}{w(R)} \sum_{T \subset R} |\hat{b}(T^\epsilon)|^2 \frac{1}{\langle w \rangle_T} \right)^{1/2},$$

where the supremum is over all rectangles R, and the summation is over all subrectangles $T \in \boldsymbol{\mathcal{D}},\ T \subset R$.

We have the inclusions

$$\mathrm{bmo}_{\mathcal{D}}(w) \subsetneq \mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(w) \subsetneq \mathrm{BMO}_{\boldsymbol{\mathcal{D}},\mathrm{Rec}}(w).$$

Let us look more closely at some of these spaces.

4A. *Weighted product* $\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(w)$. As in the one-parameter case, we define the dyadic weighted Hardy space $\mathcal{H}^1_{\boldsymbol{\mathcal{D}}}(w)$ to be the space of all $\phi \in L^1(w)$ such that $S_{\boldsymbol{\mathcal{D}}}\phi \in L^1(w)$, a Banach space under the norm $\|\phi\|_{\mathcal{H}^1_{\boldsymbol{\mathcal{D}}}(w)} := \|S_{\boldsymbol{\mathcal{D}}}\phi\|_{L^1(w)}$. The following result exists in the literature in various forms, but we include a proof here for completeness.

Proposition 4.1. *With the notation above*, $\mathcal{H}^1_{\boldsymbol{\mathcal{D}}}(w)^* \equiv \mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(w)$. *Specifically*, *every* $b \in \mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(w)$ *determines a continuous linear functional on* $\mathcal{H}^1_{\boldsymbol{\mathcal{D}}}(w)$ *by* $\phi \mapsto \langle b, \phi \rangle$,

$$|\langle b, \phi \rangle| \lesssim \|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(w)} \|S_{\boldsymbol{\mathcal{D}}}\phi\|_{L^1(w)}, \tag{4-3}$$

and, *conversely*, *every* $L \in \mathcal{H}^1_{\boldsymbol{\mathcal{D}}}(w)^*$ *may be realized as* $L\phi = \langle b, \phi \rangle$ *for some* $b \in \mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(w)$.

Proof. To prove the first statement, let $b \in \mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(w)$ and $\phi \in \mathcal{H}^1_{\boldsymbol{\mathcal{D}}}(w)$. For every $j \in \mathbb{Z}$, define the set $U_j := \{x \in \mathbb{R}^{\vec{n}} : S_{\boldsymbol{\mathcal{D}}}\phi(x) > 2^j\}$, and the collection of rectangles $\mathcal{R}_j := \big\{R \in \boldsymbol{\mathcal{D}} : w(R \cap U_j) > \tfrac{1}{2} w(R)\big\}$. Clearly $U_{j+1} \subset U_j$ and $\mathcal{R}_{j+1} \subset \mathcal{R}_j$. Moreover,

$$\sum_{j \in \mathbb{Z}} 2^j w(U_j) \simeq \|S_{\boldsymbol{\mathcal{D}}}\phi\|_{L^1(w)}, \tag{4-4}$$

which comes from the measure-theoretical fact that for any integrable function f on a measure space $(\mathcal{X}, \mu)$, we have $\|f\|_{L^1(\mu)} \simeq \sum_{j \in \mathbb{Z}} 2^j \mu\{x \in \mathcal{X} : |f(x)| > 2^j\}$.

As shown in Proposition 2.2, there exist $C, \delta > 0$ such that $w(E)/w(R) \le C(|E|/|R|)^\delta$ for all rectangles R and measurable subsets $E \subset R$. Define then for every $j \in \mathbb{Z}$ the (open) set

$$V_j := \{x \in \mathbb{R}^{\vec{n}} : M_S \mathbb{1}_{U_j}(x) > \theta\}, \quad \text{where } \theta := \left(\frac{1}{2C}\right)^{1/\delta}.$$

First note that if $R \in \mathcal{R}_j$, then

$$\frac{1}{2} < \frac{w(R \cap U_j)}{w(R)} \le C\left(\frac{|R \cap U_j|}{|R|}\right)^\delta$$

so

$$\theta < \langle \mathbb{1}_{U_j} \rangle_R \le M_S \mathbb{1}_{U_j}(x) \quad \text{for all } x \in R.$$

Therefore

$$\bigcup_{R \in \mathcal{R}_j} R \subset V_j. \tag{4-5}$$

Using (2-7), we have

$$w(V_j) \lesssim \int_{U_j} \frac{1}{\theta^p}\left(1 + \log^+ \frac{1}{\theta}\right)^{k-1} dw \simeq w(U_j). \tag{4-6}$$

Now suppose $R \in \boldsymbol{\mathcal{D}}$ but $R \notin \bigcup_{j \in \mathbb{Z}} \mathcal{R}_j$. Then $w(R \cap \{S_{\boldsymbol{\mathcal{D}}}\phi \le 2^j\}) \ge \tfrac{1}{2} w(R)$ for all $j \in \mathbb{Z}$, and so

$$w(R \cap \{S_{\boldsymbol{\mathcal{D}}}\phi = 0\}) = w\left(\bigcap_{j=1}^{\infty} R \cap \{S_{\boldsymbol{\mathcal{D}}}\phi \le 2^{-j}\}\right) \ge \tfrac{1}{2} w(R).$$

Then $|\{S_{\boldsymbol{\mathcal{D}}}\phi = 0\}| \ge |R \cap \{S_{\boldsymbol{\mathcal{D}}}\phi = 0\}| \ge \theta |R| > 0$, and we may write

$$|\hat{\phi}(R)|^2 = \int_{\{S_{\boldsymbol{\mathcal{D}}}\phi = 0\}} |\hat{\phi}(R)|^2 \frac{\mathbb{1}_R}{|R \cap \{S_{\boldsymbol{\mathcal{D}}}\phi = 0\}|}\, dx \le \frac{1}{\theta} \int_{\{S_{\boldsymbol{\mathcal{D}}}\phi = 0\}} (S_{\boldsymbol{\mathcal{D}}}\phi)^2\, dx = 0.$$

So

$$\hat{\phi}(R) = 0 \quad \text{for all } R \in \boldsymbol{\mathcal{D}},\ R \notin \bigcup_{j \in \mathbb{Z}} \mathcal{R}_j. \tag{4-7}$$

Finally, if $R \in \bigcap_{j \in \mathbb{Z}} \mathcal{R}_j$, then

$$0 = w(R \cap \{S_{\boldsymbol{\mathcal{D}}}\phi = \infty\}) = \lim_{j \to \infty} w(R \cap \{S_{\boldsymbol{\mathcal{D}}}\phi > 2^j\}) \ge \tfrac{1}{2} w(R),$$

a contradiction. In light of this and (4-7),

$$\sum_{R\in\mathcal{D}}|\hat{b}(R)|\,|\hat{\phi}(R)| = \sum_{j\in\mathbb{Z}}\sum_{R\in\mathcal{R}_j\setminus\mathcal{R}_{j+1}}|\hat{b}(R)|\,|\hat{\phi}(R)|$$
$$\leq \sum_{j\in\mathbb{Z}}\biggl(\sum_{R\in\mathcal{R}_j\setminus\mathcal{R}_{j+1}}|\hat{b}(R)|^2\frac{1}{\langle w\rangle_R}\biggr)^{1/2}\biggl(\sum_{R\in\mathcal{R}_j\setminus\mathcal{R}_{j+1}}|\hat{\phi}(R)|^2\langle w\rangle_R\biggr)^{1/2}.$$

To estimate the first term, we simply note that

$$\sum_{R\in\mathcal{R}_j\setminus\mathcal{R}_{j+1}}|\hat{b}(R)|^2\frac{1}{\langle w\rangle_R} \leq \sum_{R\in\mathcal{R}_j}|\hat{b}(R)|^2\frac{1}{\langle w\rangle_R} \leq \sum_{R\subset V_j,\,R\in\mathcal{D}}|\hat{b}(R)|^2\frac{1}{\langle w\rangle_R} \leq \|b\|^2_{\mathrm{BMO}_{\mathcal{D}}(w)}w(V_j),$$

where the second inequality follows from (4-5). For the second term, note that any $R\in\mathcal{R}_j\setminus\mathcal{R}_{j+1}$ satisfies $R\subset V_j$ and $w(R\setminus U_{j+1})\geq\frac{1}{2}w(R)$. Then

$$\sum_{R\in\mathcal{R}_j\setminus\mathcal{R}_{j+1}}|\hat{\phi}(R)|^2\langle w\rangle_R \leq 2\sum_{R\in\mathcal{R}_j\setminus\mathcal{R}_{j+1}}|\hat{\phi}(R)|^2\frac{w(R\setminus U_{j+1})}{|R|}$$
$$= 2\int_{V_j\setminus U_{j+1}}\sum_{R\in\mathcal{R}_j\setminus\mathcal{R}_{j+1}}|\hat{\phi}(R)|^2\frac{\mathbb{1}_R}{|R|}\,dw$$
$$\leq 2\int_{V_j\setminus U_{j+1}}(S_{\mathcal{D}}\phi)^2\,dw \lesssim 2^{2j}w(V_j),$$

since $S_{\mathcal{D}}\phi\leq 2^{j+1}$ off U_{j+1}. Finally, we have by (4-6),

$$\sum_{R\in\mathcal{D}}|\hat{b}(R)|\,|\hat{\phi}(R)| \lesssim \|b\|_{\mathrm{BMO}_{\mathcal{D}}(w)}\sum_{j\in\mathbb{Z}}2^jw(V_j) \simeq \|b\|_{\mathrm{BMO}_{\mathcal{D}}(w)}\sum_{j\in\mathbb{Z}}2^jw(U_j).$$

Combining this with (4-4), we obtain (4-3).

To see the converse, let $L\in\mathcal{H}^1_{\mathcal{D}}(w)$. Then L is given by $L\phi=\langle b,\phi\rangle$ for some function b. Fix an open set Ω with $w(\Omega)<\infty$. Then

$$\biggl(\sum_{R\subset\Omega,\,R\in\mathcal{D}}|\hat{b}(R)|^2\frac{1}{\langle w\rangle_R}\biggr)^{1/2} \leq \sup_{\|\phi\|_{l^2(\Omega,w)}\leq 1}\biggl|\sum_{R\subset\Omega,\,R\in\mathcal{D}}\hat{b}(R)\hat{\phi}(R)\biggr|,$$

where $\|\phi\|^2_{l^2(\Omega,w)} := \sum_{R\subset\Omega,R\in\mathcal{D}}|\hat{\phi}(R)|^2\langle w\rangle_R$. By a simple application of Hölder's inequality,

$$\biggl|\sum_{R\subset\Omega,\,R\in\mathcal{D}}\hat{b}(R)\hat{\phi}(R)\biggr| \lesssim \|L\|_\star\|\phi\|_{\mathcal{H}^1_{\mathcal{D}}(w)} \leq \|L\|_\star(w(\Omega))^{1/2}\|\phi\|_{l^2(\Omega,w)},$$

so $\|b\|_{\mathrm{BMO}_{\mathcal{D}}(w)}\lesssim\|L\|_\star$. □

4B. ***Weighted little*** $\mathbf{bmo}_{\mathcal{D}}(\boldsymbol{w})$**.** In this case, we also want to look at each variable separately. Specifically, we look at the space $\mathrm{BMO}(w_1,x_2)$: for each $x_2\in\mathbb{R}^{n_2}$, this is the weighted BMO space over $\mathbb{R}^{n_1}$, with respect to the weight $w(\cdot, x_2)$:

$$\mathrm{BMO}(w_1, x_2) := \mathrm{BMO}(w(\cdot, x_2);\ \mathbb{R}^{n_1}) \quad \text{for each } x_2 \in \mathbb{R}^{n_2}.$$

The norm in this space is given by

$$\|b(\cdot, x_2)\|_{\mathrm{BMO}(w_1,x_2)} := \sup_{Q_1} \frac{1}{w(Q_1, x_2)} \int_{Q_1} |b(x_1, x_2) - m_{Q_1} b(x_2)|\, dx_1,$$

where

$$w(Q_1, x_2) := \int_{Q_1} w(x_1, x_2)\, dx_1 \quad \text{and} \quad m_{Q_1} b(x_2) := \frac{1}{|Q_1|} \int_{Q_1} b(x_1, x_2)\, dx_1.$$

The space $\mathrm{BMO}(w_2, x_1)$ and the quantities $w(Q_2, x_1)$ and $m_{Q_2} b(x_1)$ are defined symmetrically.

Proposition 4.2. *Let $w(x_1, x_2)$ be a weight on $\mathbb{R}^{\vec{n}} = \mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$. Then $b \in L^1_{\mathrm{loc}}(\mathbb{R}^{\vec{n}})$ is in* bmo(w) *if and only if b is in the one-parameter weighted BMO spaces* $\mathrm{BMO}(w_i, x_j)$ *separately in each variable, uniformly*:

$$\|b\|_{\mathrm{bmo}(w)} \simeq \max\Big\{ \operatorname*{ess\,sup}_{x_1 \in \mathbb{R}^{n_1}} \|b(x_1, \cdot)\|_{\mathrm{BMO}(w_2,x_1)},\ \operatorname*{ess\,sup}_{x_2 \in \mathbb{R}^{n_2}} \|b(\cdot, x_2)\|_{\mathrm{BMO}(w_1,x_2)} \Big\}.$$

Remark 4.3. In the *unweighted* case $\mathrm{bmo}(\mathbb{R}^{\vec{n}})$, if we fixed $x_2 \in \mathbb{R}^{n_2}$, we would look at $b(\cdot, x_2)$ in the space $\mathrm{BMO}(\mathbb{R}^{n_1})$ — the *same* one-parameter BMO space for all x_2. In the weighted case however, the one-parameter space for $b(\cdot, x_2)$ *changes with* x_2, because the weight $w(\cdot, x_2)$ changes with x_2.

Proof. Suppose first that $b \in \mathrm{bmo}(w)$. Then for all cubes Q_1, Q_2,

$$\begin{aligned}\|b\|_{\mathrm{bmo}(w)} &\geq \frac{1}{w(Q_1 \times Q_2)} \int_{Q_1}\int_{Q_2} |b(x_1, x_2) - \langle b \rangle_{Q_1 \times Q_2}|\, dx_2\, dx_1 \\ &\geq \frac{1}{w(Q_1 \times Q_2)} \int_{Q_1} \left| \int_{Q_2} b(x_1, x_2) - \langle b \rangle_{Q_1 \times Q_2}\, dx_2 \right| dx_1,\end{aligned}$$

so

$$\int_{Q_1} |m_{Q_2} b(x_1) - \langle b \rangle_{Q_1 \times Q_2}|\, dx_1 \leq \frac{w(Q_1 \times Q_2)}{|Q_2|} \|b\|_{\mathrm{bmo}(w)}. \tag{4-8}$$

Now fix a cube Q_2 in $\mathbb{R}^{n_2}$ and let $f_{Q_2}(x_1) := \int_{Q_2} |b(x_1, x_2) - m_{Q_2} b(x_1)|\, dx_2$. Then for any Q_1,

$$\begin{aligned}\langle f_{Q_2} \rangle_{Q_1} &\leq \frac{1}{|Q_1|} \int_{Q_1}\int_{Q_2} |b(x_1, x_2) - \langle b \rangle_{Q_1 \times Q_2}|\, dx + \frac{1}{|Q_1|} \int_{Q_1}\int_{Q_2} |m_{Q_2} b(x_1) - \langle b \rangle_{Q_1 \times Q_2}|\, dx \\ &\leq \frac{w(Q_1 \times Q_2)}{|Q_1|} \|b\|_{\mathrm{bmo}(w)} + \frac{|Q_2|}{|Q_1|} \int_{Q_1} |m_{Q_2} b(x_1) - \langle b \rangle_{Q_1 \times Q_2}|\, dx_1 \\ &\leq 2 \frac{w(Q_1 \times Q_2)}{|Q_1|} \|b\|_{\mathrm{bmo}(w)} = 2 \langle w(Q_2, \cdot) \rangle_{Q_1} \|b\|_{\mathrm{bmo}(w)},\end{aligned}$$

where the last inequality follows from (4-8). By the Lebesgue differentiation theorem,

$$f_{Q_2}(x_1) = \lim_{Q_1 \to x_1} \langle f_{Q_2} \rangle_{Q_1} \leq 2\|b\|_{\mathrm{bmo}(w)} \lim_{Q_1 \to x_1} \langle w(Q_2, \cdot) \rangle_{Q_1} = 2\|b\|_{\mathrm{bmo}(w)} w(Q_2, x_1)$$

for almost all $x_1 \in \mathbb{R}^{n_1}$, where $Q_1 \to x_1$ denotes a sequence of cubes containing x_1 with side length tending to 0.

We would like to say at this point that $\|b(x_1,\cdot)\|_{\mathrm{BMO}(w_2,x_1)} = \sup_{Q_2} 1/(w(Q_2,x_1)) f_{Q_2}(x_1)$ is uniformly (a.a. x_1) bounded. However, we must be a little careful and note that at this point we really have that for every cube Q_2 in $\mathbb{R}^{n_2}$, there is a *null set* $N(Q_2) \subset \mathbb{R}^{n_1}$ such that

$$f_{Q_2}(x_1) \le 2\|b\|_{\mathrm{bmo}(w)} w(Q_2,x_1) \quad \text{for all } x_1 \in \mathbb{R}^{n_1} \setminus N(Q_2).$$

In order to obtain the inequality we want, holding for a.a. x_1, let $N := \bigcup N(\widetilde{Q}_2)$ where $\widetilde{Q}_2$ are the cubes in $\mathbb{R}^{n_2}$ with rational side length and centers with rational coordinates. Then N is a null set and $f_{\widetilde{Q}_2}(x_1) \le 2\|b\|_{\mathrm{bmo}(w)} w(\widetilde{Q}_2,x_1)$ for all $x_1 \in \mathbb{R}^{n_1} \setminus N$. By density, this statement then holds for *all* cubes Q_2 and $x_1 \notin N$, so

$$\operatorname*{ess\,sup}_{x_1\in\mathbb{R}^{n_1}} \|b(x_1,\cdot)\|_{\mathrm{BMO}(w_2,x_1)} \le 2\|b\|_{\mathrm{bmo}(w)}.$$

The result for the other variable follows symmetrically.

Conversely, suppose

$$\|b(x_1,\cdot)\|_{\mathrm{BMO}(w_2,x_1)} \le C_1 \quad \text{for a.a. } x_1, \qquad \|b(\cdot,x_2)\|_{\mathrm{BMO}(w_1,x_2)} \le C_2 \quad \text{for a.a. } x_2.$$

Then for any $R = Q_1 \times Q_2$,

$$\begin{aligned}
\int_R |b-\langle b\rangle_R|\,dx &\le \int_{Q_1}\int_{Q_2} |b(x_1,x_2)-m_{Q_2}(x_1)|\,dx + \int_{Q_1} |Q_2|\,|m_{Q_2}b(x_1)-\langle b\rangle_{Q_1\times Q_2}|\,dx_1 \\
&\le \int_{Q_1} C_2 w(Q_2,x_1)\,dx_1 + \int_{Q_1}\int_{Q_2} |b(x_1,x_2)-m_{Q_1}b(x_2)|\,dx_2\,dx_1 \\
&\le C_2 w(R) + \int_{Q_2} C_1 w(Q_1,x_2)\,dx_2 \\
&= (C_1+C_2)w(R),
\end{aligned}$$

so

$$\|b\|_{\mathrm{bmo}(w)} \le 2\max\Bigl\{\operatorname*{ess\,sup}_{x_1\in\mathbb{R}^{n_1}} \|b(x_1,\cdot)\|_{\mathrm{BMO}(w_2,x_1)},\ \operatorname*{ess\,sup}_{x_2\in\mathbb{R}^{n_2}} \|b(\cdot,x_2)\|_{\mathrm{BMO}(w_1,x_2)}\Bigr\}. \qquad \square$$

Corollary 4.4. *Let $w \in A_2(\mathbb{R}^{\vec{n}})$ and $b \in \mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(w)$. Then*

$$|\langle b,\phi\rangle| \lesssim \|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(w)} \|S_{\mathcal{D}_i}\phi\|_{L^1(w)}$$

for all $i \in \{1,2\}$.

Proof. This follows immediately from the one-parameter result in (4-2) and the proposition above:

$$\begin{aligned}
|\langle b,\phi\rangle| &\le \int_{\mathbb{R}^{n_1}} |\langle b(x_1,\cdot),\phi(x_1,\cdot)\rangle_{\mathbb{R}^{n_2}}|\,dx_1 \\
&\lesssim \int_{\mathbb{R}^{n_1}} \|b(x_1,\cdot)\|_{\mathrm{BMO}_{\mathcal{D}_2}(w(x_1,\cdot))} \|S_{\mathcal{D}_2}\phi(x_1,\cdot)\|_{L^1(w(x_1,\cdot))}\,dx_1 \\
&\lesssim \|b\|_{\mathrm{bmo}(w)} \|S_{\mathcal{D}_2}\phi\|_{L^1(w)},
\end{aligned}$$

and similarly for $S_{\mathcal{D}_1}$. $\square$

We now look at the little bmo version of (4-1).

Proposition 4.5. *If $w \in A_p(\mathbb{R}^{\vec{n}})$ for some $1 < p < \infty$, then*

$$\|b\|_{\mathrm{bmo}(w)} \simeq \|b\|_{\mathrm{bmo}(w;p')} := \sup_R \left(\frac{1}{w(R)} \int_R |b - \langle b \rangle_R|^{p'} \, dw' \right)^{1/p'}.$$

Proof. By Proposition 4.2 and (4-1),

$$\|b\|_{\mathrm{bmo}(w)} \simeq \max\Big\{ \operatorname*{ess\,sup}_{x_1 \in \mathbb{R}^{n_1}} \|b(x_1, \cdot\,)\|_{\mathrm{BMO}(w(x_1,\cdot\,);p')}, \ \operatorname*{ess\,sup}_{x_2 \in \mathbb{R}^{n_2}} \|b(\,\cdot\,, x_2)\|_{\mathrm{BMO}(w(\,\cdot\,,x_2);p')} \Big\}.$$

Suppose first that $b \in \mathrm{bmo}(w; p')$. Note that for some function g on $\mathbb{R}^{\vec{n}}$ and a cube Q_2 in $\mathbb{R}^{n_2}$, we have

$$\int_{Q_2} |g(x_1, x_2)|^{p'} w'(x_1, x_2) \, dx_2 \ge \frac{1}{w(Q_2, x_1)^{p'-1}} \left| \int_{Q_2} g(x_1, x_2) \, dx_2 \right|^{p'}.$$

Then

$$\begin{aligned} \|b\|^{p'}_{\mathrm{bmo}(w;p')} &\ge \frac{1}{w(R)} \int_{Q_1} \frac{1}{w(Q_2, x_1)^{p'-1}} \left| \int_{Q_2} b(x_1, x_2) - \langle b \rangle_{Q_1 \times Q_2} \, dx_2 \right|^{p'} dx_1 \\ &= \frac{1}{w(R)} \int_{Q_1} |m_{Q_2} b(x_1) - \langle b \rangle_{Q_1 \times Q_2}|^{p'} \frac{|Q_2|^{p'}}{w(Q_2, x_1)^{p'-1}} \, dx_1 \\ &\ge \frac{1}{w(R)} \int_{Q_1} |m_{Q_2} b(x_1) - \langle b \rangle_{Q_1 \times Q_2}|^{p'} w'(Q_2, x_1) \, dx_1, \end{aligned}$$

where the last inequality follows from

$$\frac{|Q_2|^{p'}}{w(Q_2, x_1)^{p'-1}} = |Q_2| \frac{1}{\langle w(x_1, \cdot\,) \rangle_{Q_2}^{p'-1}} \ge |Q_2| \frac{\langle w'(x_1, \cdot) \rangle_{Q_2}}{[w(x_1, \cdot\,)]_{A_p}^{p'-1}} \simeq w'(Q_2, x_1).$$

Now fix Q_2 and consider $f_{Q_2}(x_1) := \int_{Q_2} |b(x_1, x_2) - m_{Q_2} b(x_1)|^{p'} w'(x_1, x_2) \, dx_2$. Then

$$\begin{aligned} \langle f_{Q_2} \rangle_{Q_1} &\lesssim \frac{1}{|Q_1|} \int_{Q_1} \int_{Q_2} \big(|b(x_1, x_2) - \langle b \rangle_{Q_1 \times Q_2}|^{p'} + |m_{Q_2} b(x_1) - \langle b \rangle_{Q_1 \times Q_2}|^{p'} \big) w'(x_1, x_2) \, dx_2 \, dx_1 \\ &\lesssim \frac{w(Q_1 \times Q_2)}{|Q_1|} \|b\|^{p'}_{\mathrm{bmo}(w;p')} + \frac{1}{|Q_1|} \int_{Q_1} |m_{Q_2} b(x_1) - \langle b \rangle_{Q_1 \times Q_2}|^{p'} w'(Q_2, x_1) \, dx_1 \\ &\lesssim \frac{w(Q_1 \times Q_2)}{|Q_1|} \|b\|^{p'}_{\mathrm{bmo}(w;p')}. \end{aligned}$$

Then for almost all x_1,

$$f_{Q_2}(x_1) = \lim_{Q_1 \to x_1} \langle f_{Q_2} \rangle_{Q_1} \lesssim \lim_{Q_1 \to x_1} \frac{w(Q_1 \times Q_2)}{|Q_1|} \|b\|^{p'}_{\mathrm{bmo}(w;p')} = w(Q_2, x_1) \|b\|^{p'}_{\mathrm{bmo}(w;p')}.$$

Taking again rational cubes, we obtain

$$\|b(x_1, \cdot\,)\|_{\mathrm{BMO}(w(x_1,\cdot\,);p')} = \sup_{Q_2} \left(\frac{1}{w(Q_2, x_1)} f_{Q_2}(x_1) \right)^{1/p'} \lesssim \|b\|_{\mathrm{bmo}(w;p')}$$

for almost all x_1.

Conversely, if $b \in \mathrm{bmo}(w)$, then there exist C_1 and C_2 such that

$$\|b(x_1,\cdot)\|_{\mathrm{BMO}(w(x_1,\cdot);p')} \le C_1 \quad \text{for a.a. } x_1, \qquad \text{and} \qquad \|b(\cdot,x_2)\|_{\mathrm{BMO}(w(\cdot,x_2);p')} \le C_2 \quad \text{for a.a. } x_2.$$

Then

$$\int_R |b-\langle b\rangle_R|^{p'}\,dw' \lesssim \int_{Q_1}\int_{Q_2} |b(x_1,x_2)-m_{Q_2}b(x_1)|^{p'} w'(x_1,x_2)\,dx_2\,dx_1 \\ + \int_{Q_1}\int_{Q_2} |m_{Q_2}b(x_1)-\langle b\rangle_{Q_1\times Q_2}|^{p'} w'(x_1,x_2)\,dx_2\,dx_1.$$

The first integral is easily seen to be bounded by

$$\int_{Q_1} \|b(x_1,\cdot)\|^{p'}_{\mathrm{BMO}(w(x_1,\cdot))} w(Q_2,x_1)\,dx_1 \le C_1^{p'} w(Q_1\times Q_2).$$

The second integral is equal to

$$\int_{Q_1} |m_{Q_2}b(x_1)-\langle b\rangle_{Q_1\times Q_2}|^{p'} w'(Q_2,x_1)\,dx_1 \\ \le \int_{Q_1} \frac{w'(Q_2,x_1)}{|Q_2|^{p'}}\left(\int_{Q_2} |b(x_1,x_2)-m_{Q_1}b(x_2)|\,dx_2\right)^{p'} dx_1 \\ \le \int_{Q_1} \frac{w'(Q_2,x_1)w(Q_2,x_1)^{p'-1}}{|Q_2|^{p'}} \int_{Q_2} |b(x_1,x_2)-m_{Q_1}b(x_2)|^{p'} w'(x_1,x_2)\,dx_2\,dx_1.$$

We may express the first term as $\langle w'(x_1,\cdot)\rangle_{Q_2}\langle w(x_1,\cdot)\rangle_{Q_2}^{p'-1} \lesssim [w]_{A_p}^{p'-1}$ for almost all x_1. Then, the integral is further bounded by

$$\int_{Q_2} w(Q_1,x_2)\|b(\cdot,x_2)\|_{\mathrm{BMO}(w(\cdot,x_2);p')}\,dx_2 \lesssim C_2^{p'} w(Q_1\times Q_2).$$

Finally, this gives

$$\|b\|_{\mathrm{bmo}(w;p')} \lesssim (C_1^{p'}+C_2^{p'})^{1/p'} \lesssim \max(C_1,C_2) \simeq \|b\|_{\mathrm{bmo}(w)}. \qquad \square$$

We also have a two-weight John–Nirenberg for Bloom little bmo, which follows very similarly to the proof above.

Proposition 4.6. *Let $\mu,\lambda \in A_p(\mathbb{R}^{\vec{n}})$ for $1<p<\infty$, and $\nu := \mu^{1/p}\lambda^{-1/p}$. Then*

$$\|b\|_{\mathrm{bmo}(\nu)} \simeq \|b\|_{\mathrm{bmo}(\mu,\lambda,p)} \simeq \|b\|_{\mathrm{bmo}(\lambda',\mu',p')},$$

where

$$\|b\|_{\mathrm{bmo}(\mu,\lambda,p)} := \sup_R \left(\frac{1}{\mu(R)}\int_R |b-\langle b\rangle_R|^p\,d\lambda\right)^{1/p},$$
$$\|b\|_{\mathrm{bmo}(\lambda',\mu',p')} := \sup_R \left(\frac{1}{\lambda'(R)}\int_R |b-\langle b\rangle_R|^{p'}\,d\mu'\right)^{1/p'}.$$

Note that it also easily follows that $\nu \in A_2(\mathbb{R}^{\vec{n}})$.

5. Proof of the lower bound

Proof of Theorem 1.3. To see the lower bound, we adapt the argument of Coifman, Rochberg and Weiss [Coifman et al. 1976]. Let $\{X_k(x)\}$ and $\{Y_l(y)\}$ both be orthonormal bases for the space of spherical harmonics of degree n in $\mathbb{R}^n$. Then $\sum_k |X_k(x)|^2 = c_n|x|^{2n}$ and thus

$$1 = \frac{1}{c_n}\sum_k \frac{X_k(x-x')}{|x-x'|^{2n}} X_k(x-x')$$

and similarly for Y_l.

Furthermore $X_k(x-x') = \sum_{|\alpha|+|\beta|=n} \boldsymbol{x}^{(k)}_{\alpha\beta} x^\alpha x'^\beta$ and similarly for Y_l. Remember that

$$b(x,y)\in \mathrm{bmo}(\nu) \quad\Longleftrightarrow\quad \|b\|_{\mathrm{bmo}(\nu)} = \sup_Q \frac{1}{\nu(Q)}\int_Q |b(x,y)-\langle b\rangle_Q|\,dx\,dy < \infty.$$

Here, $Q = I\times J$ and I and J are cubes in $\mathbb{R}^n$. Let us define the function

$$\Gamma_Q(x,y) = \operatorname{sign}(b(x,y)-\langle b\rangle_Q)\mathbb{1}_Q(x,y).$$

So

$$\begin{aligned}
&|b(x,y)-\langle b\rangle_Q|\,|Q|\,\mathbb{1}_Q(x,y)\\
&\qquad= (b(x,y)-\langle b\rangle_Q)|Q|\Gamma_Q(x,y)\\
&\qquad= \int_Q (b(x,y)-b(x',y'))\Gamma_Q(x,y)\,dx'\,dy'\\
&\qquad\sim \sum_{k,l}\int_Q (b(x,y)-b(x',y'))\frac{X_k(x-x')}{|x-x'|^{2n}}X_k(x-x')\frac{Y_l(y-y')}{|y-y'|^{2n}}Y_l(y-y')\Gamma_Q(x,y)\,dx'\,dy'\\
&\qquad= \sum_{k,l}\int_{\mathbb{R}^{2n}} \frac{b(x,y)-b(x',y')}{|x-x'|^{2n}|y-y'|^{2n}}X_k(x-x')Y_l(y-y')\cdot\\
&\qquad\qquad\cdot \sum_{|\alpha|+|\beta|=n}\boldsymbol{x}^{(k)}_{\alpha\beta}x^\alpha x'^\beta \sum_{|\gamma|+|\delta|=n}\boldsymbol{y}^{(l)}_{\gamma\delta}y^\gamma y'^\delta\Gamma_Q(x,y)\mathbb{1}_Q(x',y')\,dx'\,dy'.
\end{aligned}$$

Note that

$$\int_{\mathbb{R}^{2n}} \frac{b(x,y)-b(x',y')}{|x-x'|^{2n}|y-y'|^{2n}}X_k(x-x')Y_l(y-y')\cdot x'^\beta y'^\delta \mathbb{1}_Q(x',y')\,dx'\,dy' = [b, T_kT_l](x'^\beta y'^\delta\mathbb{1}_Q(x',y')).$$

Here T_k and T_l are the Calderón–Zygmund operators that correspond to the kernels

$$\frac{X_k(x)}{|x|^{2n}} \quad\text{and}\quad \frac{Y_l(y)}{|y|^{2n}}.$$

Observe that these have the correct homogeneity due to the homogeneity of the X_k and Y_l. With this notation, the above becomes

$$\begin{aligned}
&|b(x,y)-\langle b\rangle_Q|\,|Q|\,\mathbb{1}_Q(x,y)\\
&\qquad= \sum_{k,l}\sum_{|\alpha|+|\beta|=n}\sum_{|\gamma|+|\delta|=n}\boldsymbol{x}^{(k)}_{\alpha\beta}x^\alpha \boldsymbol{y}^{(l)}_{\gamma\delta}y^\gamma\Gamma_Q(x,y)[b,T_kT_l](x'^\beta y'^\delta\mathbb{1}_Q(x',y'))(x,y).
\end{aligned}$$

Now, we integrate with respect to (x, y) and the measure λ. Let us assume for a moment that both I and J are centered at 0 and thus Q is centered at 0. In this case, since Γ_Q and $\mathbb{1}_Q$ are supported in Q, there is only contribution for x, x', y, y' in Q:

$$
\begin{aligned}
|Q|\biggl(\int_Q |b(x,y)-\langle b\rangle_Q|^p \, d\lambda(x,y)\biggr)^{1/p}
&\leqslant \sum_{k,l} \sum_{|\alpha|+|\beta|=n} \sum_{|\gamma|+|\delta|=n} \bigl\| \boldsymbol{x}^{(k)}_{\alpha\beta} x^\alpha \boldsymbol{y}^{(l)}_{\gamma\delta} y^\gamma \Gamma_Q(x,y)[b, T_k T_l](x'^\beta y'^\delta \mathbb{1}_Q(x',y'))(x,y) \bigr\|_{L^p(\lambda)} \\
&\lesssim \sum_{k,l} \sum_{|\alpha|+|\beta|=n} \sum_{|\gamma|+|\delta|=n} \mathfrak{l}(I)^{|\alpha|} \mathfrak{l}(J)^{|\gamma|} \bigl\| [b, T_k T_l](x'^\beta y'^\delta \mathbb{1}_Q(x',y')) \bigr\|_{L^p(\lambda)} \\
&\lesssim \sum_{k,l} \sum_{|\alpha|+|\beta|=n} \sum_{|\gamma|+|\delta|=n} \mathfrak{l}(I)^{|\alpha|} \mathfrak{l}(J)^{|\gamma|} \|[b, T_k T_l]\|_{L^p(\mu)\to L^p(\lambda)} \|x'^\beta y'^\delta \mathbb{1}_Q(x',y')\|_{L^p(\mu)} \\
&\lesssim \sum_{k,l} \sum_{|\alpha|+|\beta|=n} \sum_{|\gamma|+|\delta|=n} \mathfrak{l}(I)^{|\alpha|} \mathfrak{l}(J)^{|\gamma|} \mathfrak{l}(I)^{|\beta|} \mathfrak{l}(J)^{|\delta|} \|[b, T_k T_l]\|_{L^p(\mu)\to L^p(\lambda)} \mu(Q)^{1/p}.
\end{aligned}
$$

We disregarded the coefficients of X and Y at the cost of a constant.

Notice that the T_k and T_l are homogeneous polynomials in Riesz transforms. Therefore the commutator $[b, T_k T_l]$ can be written as a linear combination of terms such as $M[b, R^1_i R^2_j]N$, where M and N are compositions of Riesz transforms: First write $[b, T_k T_l]$ as linear combination of terms of the form $[b, R^k_{(n)} R^l_{(n)}]$, where

$$R^k_{(n)} = \prod_s R^1_{i^{(k)}_s}$$

is a composition of n Riesz transforms acting in the first variable with the choice $i^{(k)} = (i^{(k)}_s)_{s=1}^n \in \{1, \dots, n\}^n$ for each k and similarly for $R^l_{(n)}$ acting in the second variable Then, for each term, apply $[AB, b] = A[B, b] + [A, b]B$ successively as follows. Use $A = R^1_{i_1} R^2_{j_1}$ and B of the form $R^k_{(n-1)} R^l_{(n-1)}$ and repeat. It then follows that for each k, l the commutator $[b, T_k T_l]$ can be written as a linear combination of terms such as $M[b, R^1_i R^2_j]N$, where M and N are compositions of Riesz transforms. Thus T_k and T_l are homogeneous polynomials in Riesz transforms of the same degree. We require that all commutators of the form $[b, R^1_i R^2_j]$ are bounded, and we have shown the bmo estimate for b for rectangles Q whose sides are centered at 0. We now translate b in the two directions separately and obtain what we need, by Proposition 4.6:

$$\|b\|_{\mathrm{bmo}(\nu)} \simeq \|b\|_{\mathrm{bmo}(\mu,\lambda,p)} := \sup_R \biggl(\frac{1}{\mu(R)} \int_R |b - \langle b\rangle_R|^p \, d\lambda\biggr)^{1/p} \lesssim \sup_{1\leqslant k,l\leqslant n} \|[b, R^1_k R^2_l]\|_{L^p(\mu)\to L^p(\lambda)}. \qquad \square$$

6. Biparameter paraproducts

Decomposing two functions b and f on $\mathbb{R}^n$ into their Haar series adapted to some dyadic grid $\mathcal{D}$ and analyzing the different inclusion properties of the dyadic cubes, one may express their product as

$$bf = \Pi_b f + \Pi^*_b f + \Gamma_b f + \Pi_f b,$$

where

$$\Pi_b f := \sum_{Q\in\mathcal{D}} \hat{b}(Q^\epsilon)\langle f\rangle_Q h_Q^\epsilon, \quad \Pi_b^* f := \sum_{Q\in\mathcal{D}} \hat{b}(Q^\epsilon)\hat{f}(Q^\epsilon)\frac{\mathbb{1}_Q}{|Q|}, \quad \Gamma_b f := \sum_{Q\in\mathcal{D}} \hat{b}(Q^\epsilon)\hat{f}(Q^\delta)\frac{1}{\sqrt{|Q|}}h_Q^{\epsilon+\delta}.$$

In [Holmes et al. 2017], it was shown that, when $b\in \mathrm{BMO}(\nu)$, the operators Π_b, Π_b^*, and Γ_b are bounded $L^p(\mu)\to L^p(\lambda)$.

6A. ***Product BMO paraproducts.*** In the biparameter setting $\boldsymbol{\mathcal{D}} = \mathcal{D}_1\times\mathcal{D}_2$, we have *fifteen* paraproducts. We treat them beginning with the nine paraproducts associated with product BMO. First, we have the three "pure" paraproducts, direct adaptations of the one-parameter paraproducts:

$$\Pi_b f := \sum_{Q_1\times Q_2} \hat{b}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})\langle f\rangle_{Q_1\times Q_2} h_{Q_1}^{\epsilon_1}\otimes h_{Q_2}^{\epsilon_2},$$

$$\Pi_b^* f := \sum_{Q_1\times Q_2} \hat{b}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})\hat{f}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})\frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|},$$

$$\Gamma_b f := \sum_{Q_1\times Q_2} \hat{b}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})\hat{f}(Q_1^{\delta_1}\times Q_2^{\delta_2})\frac{1}{\sqrt{|Q_1|}}\frac{1}{\sqrt{|Q_2|}}h_{Q_1}^{\epsilon_1+\delta_1}\otimes h_{Q_2}^{\epsilon_2+\delta_2} = \Gamma_b^* f.$$

Next, we have the "mixed" paraproducts. We index these based on the types of Haar functions acting on f, since the action on b is the same for all of them, namely $\hat{b}(Q_1\times Q_2)$ — this is the property which associates these paraproducts with product $\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}$: in a proof using duality, one would separate out the b function and be left with the biparameter square function $S_{\boldsymbol{\mathcal{D}}}$. They are

$$\Pi_{b;(0,1)} f := \sum_{Q_1\times Q_2} \hat{b}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})\left\langle f, h_{Q_1}^{\epsilon_1}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|}\right\rangle\frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes h_{Q_2}^{\epsilon_2},$$

$$\Pi_{b;(1,0)} f := \sum_{Q_1\times Q_2} \hat{b}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})\left\langle f, \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes h_{Q_2}^{\epsilon_2}\right\rangle h_{Q_1}^{\epsilon_1}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|} = \Pi_{b;(0,1)}^*,$$

$$\Gamma_{b;(0,1)} f := \sum_{Q_1\times Q_2} \hat{b}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})\left\langle f, h_{Q_1}^{\delta_1}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|}\right\rangle\frac{1}{\sqrt{|Q_1|}}h_{Q_1}^{\epsilon_1+\delta_1}\otimes h_{Q_2}^{\epsilon_2},$$

$$\Gamma_{b;(0,1)}^* f := \sum_{Q_1\times Q_2} \hat{b}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})\hat{f}(Q_1^{\delta_1}\times Q_2^{\epsilon_2})\frac{1}{\sqrt{|Q_1|}}h_{Q_1}^{\epsilon_1+\delta_1}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|},$$

$$\Gamma_{b;(1,0)} f := \sum_{Q_1\times Q_2} \hat{b}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})\left\langle f, \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes h_{Q_2}^{\delta_2}\right\rangle\frac{1}{\sqrt{|Q_2|}}h_{Q_1}^{\epsilon_1}\otimes h_{Q_2}^{\epsilon_2+\delta_2},$$

$$\Gamma_{b;(1,0)}^* f := \sum_{Q_1\times Q_2} \hat{b}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})\hat{f}(Q_1^{\epsilon_1}\times Q_2^{\delta_2})\frac{1}{\sqrt{|Q_2|}}\frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes h_{Q_2}^{\epsilon_2+\delta_2}.$$

Proposition 6.1. *If* $\nu := \mu^{1/p}\lambda^{-1/p}$ *for* $A_p(\mathbb{R}^{\vec{n}})$ *weights* μ *and* λ, *and* $\mathsf{P_b}$ *denotes any one of the nine paraproducts defined above*, *then*

$$\|\mathsf{P_b} : L^p(\mu)\to L^p(\lambda)\| \lesssim \|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)}, \tag{6-1}$$

where $\|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)}$ *denotes the norm of* b *in the dyadic weighted product* $\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)$ *space on* $\mathbb{R}^{\vec{n}}$.

Proof. We first outline the general strategy we use to prove (6-1). From (2-8), it suffices to take $f \in L^p(\mu)$ and $g \in L^{p'}(\lambda')$ and show that

$$|\langle \mathsf{P}_\mathsf{b} f, g\rangle| \lesssim \|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)} \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\lambda')}.$$

(1) Write $\langle \mathsf{P}_\mathsf{b} f, g\rangle = \langle b, \phi\rangle$, where ϕ depends on f and g. By (4-3), $|\langle \mathsf{P}_\mathsf{b} f, g\rangle| \lesssim \|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)} \|S_{\boldsymbol{\mathcal{D}}}\phi\|_{L^1(\nu)}$.

(2) Show that $S_{\boldsymbol{\mathcal{D}}}\phi \lesssim (\mathcal{O}_1 f)(\mathcal{O}_2 g)$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are operators satisfying a *one-weight bound* $L^p(w) \to L^p(w)$ for all $w \in A_p(\mathbb{R}^{\vec{n}})$ — these operators will usually be a combination of maximal and square functions.

(3) Then the $L^1(\nu)$-norm of $S_{\boldsymbol{\mathcal{D}}}\phi$ can be separated into the $L^p(\mu)$ and $L^{p'}(\lambda')$ norms of these operators $\mathcal{O}_i$ by a simple application of Hölder's inequality,

$$\|S_{\boldsymbol{\mathcal{D}}}\phi\|_{L^1(\nu)} \lesssim \|\mathcal{O}_1 f\|_{L^p(\mu)} \|\mathcal{O}_2 g\|_{L^{p'}(\lambda')} \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\lambda')},$$

and the result follows.

Remark also that we will not have to treat the adjoints P_b^* separately: interchanging the roles of f and g in the proof strategy above will show that P_b is also bounded $L^{p'}(\lambda') \to L^{p'}(\mu')$, which means that P_b^* is bounded $L^p(\mu) \to L^p(\lambda)$.

Let us begin with $\Pi_b f$. We write

$$\langle \Pi_b f, g\rangle = \langle b, \phi\rangle, \quad \text{where } \phi := \sum_{Q_1 \times Q_2} \langle f\rangle_{Q_1 \times Q_2} \hat{g}(Q_1^{\epsilon_1} \times Q_2^{\epsilon_2}) h_{Q_1}^{\epsilon_1} \otimes h_{Q_2}^{\epsilon_2}.$$

Then

$$(S_{\boldsymbol{\mathcal{D}}}\phi)^2 \le \sum_{Q_1 \times Q_2} \langle |f|\rangle_{Q_1 \times Q_2}^2 |\hat{g}(Q_1^{\epsilon_1} \times Q_2^{\epsilon_2})|^2 \frac{\mathbb{1}_{Q_1}}{|Q_1|} \otimes \frac{\mathbb{1}_{Q_2}}{|Q_2|} \le (M_S f)^2 \cdot (S_{\boldsymbol{\mathcal{D}}} g)^2,$$

so

$$|\langle \Pi_b f, g\rangle| \lesssim \|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)} \|M_S f\|_{L^p(\mu)} \|S_{\boldsymbol{\mathcal{D}}} g\|_{L^{p'}(\lambda')} \lesssim \|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)} \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\lambda')}.$$

Note that if we take instead $f \in L^{p'}(\lambda')$ and $g \in L^p(\mu)$, we have

$$|\langle \Pi_b f, g\rangle| \lesssim \|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)} \|M_S f\|_{L^{p'}(\lambda')} \|S_{\boldsymbol{\mathcal{D}}} g\|_{L^p(\mu)} \lesssim \|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)} \|f\|_{L^{p'}(\lambda')} \|g\|_{L^p(\mu)},$$

proving that $\|\Pi_b : L^{p'}(\lambda') \to L^{p'}(\mu')\| = \|\Pi_b^* : L^p(\mu) \to L^p(\lambda)\| \lesssim \|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)}$. For Γ_b,

$$\langle \Gamma_b f, g\rangle = \langle b, \phi\rangle, \quad \text{where } \phi := \sum_{Q_1 \times Q_2} \hat{f}(Q_1^{\epsilon_1} \times Q_2^{\epsilon_2}) \hat{g}(Q_1^{\delta_1} \times Q_2^{\delta_2}) \frac{1}{\sqrt{|Q_1|}} \frac{1}{\sqrt{|Q_2|}} h_{Q_1}^{\epsilon_1+\delta_1} \otimes h_{Q_2}^{\epsilon_2+\delta_2},$$

from which it easily follows that $S_{\boldsymbol{\mathcal{D}}}\phi \lesssim S_{\boldsymbol{\mathcal{D}}} f \cdot S_{\boldsymbol{\mathcal{D}}} g$.

Let us now look at $\Pi_{b;(0,1)}$. In this case,

$$\phi := \sum_{Q_1 \times Q_2} \left\langle f, h_{Q_1}^{\epsilon_1} \otimes \frac{\mathbb{1}_{Q_2}}{|Q_2|}\right\rangle \left\langle g, \frac{\mathbb{1}_{Q_1}}{|Q_1|} \otimes h_{Q_2}^{\epsilon_2}\right\rangle h_{Q_1}^{\epsilon_1} \otimes h_{Q_2}^{\epsilon_2}.$$

Then

$$\begin{aligned}(S_{\mathcal{D}}\phi)^2 &= \sum_{Q_1\times Q_2} \Big\langle f, h_{Q_1}^{\epsilon_1}\otimes \frac{\mathbb{1}_{Q_2}}{|Q_2|}\Big\rangle^2 \Big\langle g, \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes h_{Q_2}^{\epsilon_2}\Big\rangle^2 \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|}\\ &= \sum_{Q_1\times Q_2} \langle H_{Q_1}^{\epsilon_1} f\rangle_{Q_2}^2 \langle H_{Q_2}^{\epsilon_2} g\rangle_{Q_1}^2 \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|}\\ &\le \bigg(\sum_{Q_1} (M_{\mathcal{D}_2} H_{Q_1}^{\epsilon_1} f)^2(x_2)\frac{\mathbb{1}_{Q_1}(x_1)}{|Q_1|}\bigg)\bigg(\sum_{Q_2} (M_{\mathcal{D}_1} H_{Q_2}^{\epsilon_2} g)^2(x_1)\frac{\mathbb{1}_{Q_2}(x_2)}{|Q_2|}\bigg) = [SM]^2 f\cdot [MS]^2 g,\end{aligned}$$

where $[SM]$ and $[MS]$ are the mixed square-maximal operators in Section 3A. Boundedness of $\Pi_{b;(0,1)}$ then follows from Proposition 3.2. By the usual duality trick, the same holds for $\Pi_{b;(1,0)}$. Finally, for $\Gamma_{b;(0,1)}$,

$$\phi = \sum_{Q_1\times Q_2} \langle H_{Q_1}^{\delta_1} f\rangle_{Q_2} \frac{1}{\sqrt{|Q_1|}}\hat{g}(Q_1^{\epsilon_1+\delta_1}\times Q_2^{\epsilon_2}) h_{Q_1}^{\epsilon_1}\otimes h_{Q_2}^{\epsilon_2},$$

so $S_{\mathcal{D}}\phi \lesssim [SM]f\cdot S_{\mathcal{D}}g$. Note that $\Gamma_{b;(1,0)}$ works the same way, except we bound $S_{\mathcal{D}}\phi$ by $[MS]f\cdot S_{\mathcal{D}}g$, and the remaining two paraproducts follow by duality. □

6B. *Little bmo paraproducts.* Next, we have the six paraproducts associated with little bmo. We denote these by the small Greek letters corresponding to the previous paraproducts, and index them based on the Haar functions acting on b — in this case, separating out the b function will yield one of the square functions $S_{\mathcal{D}_i}$ in one of the variables:

$$\begin{aligned}\pi_{b;(0,1)} f &:= \sum_{Q_1\times Q_2}\Big\langle b, h_{Q_1}^{\epsilon_1}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|}\Big\rangle\Big\langle f, \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes h_{Q_2}^{\epsilon_2}\Big\rangle h_{Q_1}^{\epsilon_1}\otimes h_{Q_2}^{\epsilon_2},\\ \pi^*_{b;(0,1)} f &:= \sum_{Q_1\times Q_2}\Big\langle b, h_{Q_1}^{\epsilon_1}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|}\Big\rangle \hat{f}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})\frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes h_{Q_2}^{\epsilon_2},\\ \pi_{b;(1,0)} f &:= \sum_{Q_1\times Q_2}\Big\langle b, \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes h_{Q_2}^{\epsilon_2}\Big\rangle\Big\langle f, h_{Q_1}^{\epsilon_1}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|}\Big\rangle h_{Q_1}^{\epsilon_1}\otimes h_{Q_2}^{\epsilon_2},\\ \pi^*_{b;(1,0)} f &:= \sum_{Q_1\times Q_2}\Big\langle b, \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes h_{Q_2}^{\epsilon_2}\Big\rangle \hat{f}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2}) h_{Q_1}^{\epsilon_1}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|},\\ \gamma_{b;(0,1)} f &:= \sum_{Q_1\times Q_2}\Big\langle b, h_{Q_1}^{\delta_1}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|}\Big\rangle \hat{f}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})\frac{1}{\sqrt{|Q_1|}} h_{Q_1}^{\epsilon_1+\delta_1}\otimes h_{Q_2}^{\epsilon_2} = \gamma^*_{b;(0,1)} f,\\ \gamma_{b;(1,0)} f &:= \sum_{Q_1\times Q_2}\Big\langle b, \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes h_{Q_2}^{\delta_2}\Big\rangle \hat{f}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})\frac{1}{\sqrt{|Q_2|}} h_{Q_1}^{\epsilon_1}\otimes h_{Q_2}^{\epsilon_2+\delta_2} = \gamma^*_{b;(1,0)} f.\end{aligned}$$

Proposition 6.2. *If* $\nu := \mu^{1/p}\lambda^{-1/p}$ *for* $A_p(\mathbb{R}^{\vec{n}})$ *weights* μ *and* λ, *and* $\mathsf{p_b}$ *denotes any one of the six paraproducts defined above*, *then*

$$\|\mathsf{p_b} : L^p(\mu)\to L^p(\lambda)\| \lesssim \|b\|_{\mathrm{bmo}_{\mathcal{D}}(\nu)},$$

where $\|b\|_{\mathrm{bmo}_{\mathcal{D}}(\nu)}$ *denotes the norm of* b *in the dyadic weighted little* $\mathrm{bmo}_{\mathcal{D}}(\nu)$ *space on* $\mathbb{R}^{\vec{n}}$.

Proof. The proof strategy is the same as that of the product BMO paraproducts, with the modification that we use one of the $S_{\mathcal{D}_i}$ square functions and Corollary 4.4. For instance, in the case of $\pi_{b;(0,1)}$ we write

$$\langle \pi_{b;(0,1)} f, g\rangle = \langle b, \phi\rangle, \quad \text{where } \phi := \sum_{Q_1\times Q_2} \langle H_{Q_2}^{\epsilon_2} f\rangle_{Q_1} \hat{g}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2}) h_{Q_1}^{\epsilon_1} \otimes \frac{\mathbb{1}_{Q_2}}{|Q_2|}.$$

Then

$$\begin{aligned}(S_{\mathcal{D}_1}\phi)^2 &\le \sum_{Q_1}\biggl(\sum_{Q_2}\langle |H_{Q_2}^{\epsilon_2} f|\rangle_{Q_1}^2 \mathbb{1}_{Q_1}(x_1)\frac{\mathbb{1}_{Q_2}(x_2)}{|Q_2|}\biggr)\biggl(\sum_{Q_2}|\hat{g}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})|^2\frac{\mathbb{1}_{Q_2}(x_2)}{|Q_2|}\biggr)\frac{\mathbb{1}_{Q_1}(x_1)}{|Q_1|}\\ &\le \biggl(\sum_{Q_2} M_{\mathcal{D}_1}^2(H_{Q_2}^{\epsilon_2} f)(x_1)\frac{\mathbb{1}_{Q_2}(x_2)}{|Q_2|}\biggr)\biggl(\sum_{Q_1}\sum_{Q_2}|\hat{g}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})|^2\frac{\mathbb{1}_{Q_1}(x_1)}{|Q_1|}\otimes\frac{\mathbb{1}_{Q_2}(x_2)}{|Q_2|}\biggr)\\ &= [MS]^2 f\cdot S_{\boldsymbol{\mathcal{D}}}^2 g,\end{aligned}$$

and so

$$|\langle \pi_{b;(0,1)} f, g\rangle| \lesssim \|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)}\|S_{\mathcal{D}_1}\phi\|_{L^1(\nu)} \lesssim \|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)}\|f\|_{L^p(\mu)}\|g\|_{L^{p'}(\lambda')}.$$

The proof for $\pi_{b;(1,0)}$ is symmetrical — we take $S_{\mathcal{D}_2}\phi$, which will be bounded by $[SM]f\cdot S_{\boldsymbol{\mathcal{D}}}g$. The adjoint paraproducts $\pi^*_{b;(0,1)}$ and $\pi^*_{b;(1,0)}$ follow again by duality. Finally, for $\gamma_{b;(0,1)}$,

$$\phi := \sum_{Q_1\times Q_2}\hat{f}(Q_1^{\epsilon_1}\times Q_2^{\epsilon_2})\frac{1}{\sqrt{|Q_1|}}\hat{g}(Q_1^{\epsilon_1+\delta_1}\times Q_2^{\epsilon_2})h_{Q_1}^{\epsilon_1}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|},$$

from which it easily follows that $S_{\mathcal{D}_1}\phi \le S_{\boldsymbol{\mathcal{D}}} f\cdot S_{\boldsymbol{\mathcal{D}}} g$. The proof for $\gamma_{b;(1,0)}$ is symmetrical. □

7. Commutators with Journé operators

7A. ***Definition of Journé operators.*** We begin with the definition of biparameter Calderón–Zygmund operators, or Journé operators, on $\mathbb{R}^{\vec{n}} := \mathbb{R}^{n_1}\otimes\mathbb{R}^{n_2}$, as outlined in [Martikainen 2012]. As shown later in [Grau de la Herrán 2016], these conditions are equivalent to the original definition of [Journé 1985].

I. Structural assumptions: Given $f = f_1\otimes f_2$ and $g = g_1\otimes g_2$, where $f_i, g_i : \mathbb{R}^{n_i}\to\mathbb{C}$ satisfy $\mathrm{spt}(f_i)\cap\mathrm{spt}(g_i) = \varnothing$ for $i = 1, 2$, we assume the kernel representation

$$\langle Tf, g\rangle = \int_{\mathbb{R}^{\vec{n}}}\int_{\mathbb{R}^{\vec{n}}} K(x, y) f(y) g(x)\,dx\,dy.$$

The kernel $K : \mathbb{R}^{\vec{n}}\times\mathbb{R}^{\vec{n}}\setminus\{(x, y)\in\mathbb{R}^{\vec{n}}\times\mathbb{R}^{\vec{n}} : x_1 = y_1 \text{ or } x_2 = y_2\}\to\mathbb{C}$ is assumed to satisfy:

(1) Size condition:

$$|K(x, y)| \le C\frac{1}{|x_1 - y_1|^{n_1}}\frac{1}{|x_2 - y_2|^{n_2}}.$$

(2) Hölder conditions:

(a) If $|y_1 - y_1'| \le \frac{1}{2}|x_1 - y_1|$ and $|y_2 - y_2'| \le \frac{1}{2}|x_2 - y_2|$, then

$$\bigl|K(x, y) - K(x, (y_1, y_2')) - K(x, (y_1', y_2)) + K(x, y')\bigr| \le C\frac{|y_1 - y_1'|^\delta}{|x_1 - y_1|^{n_1+\delta}}\frac{|y_2 - y_2'|^\delta}{|x_2 - y_2|^{n_2+\delta}}.$$

(b) If $|x_1 - x_1'| \leq \frac{1}{2}|x_1 - y_1|$ and $|x_2 - x_2'| \leq \frac{1}{2}|x_2 - y_2|$, then

$$\left|K(x, y) - K((x_1, x_2'), y) - K((x_1', x_2), y) + K(x', y)\right| \leq C \frac{|x_1 - x_1'|^\delta}{|x_1 - y_1|^{n_1+\delta}} \frac{|x_2 - x_2'|^\delta}{|x_2 - y_2|^{n_2+\delta}}.$$

(c) If $|y_1 - y_1'| \leq \frac{1}{2}|x_1 - y_1|$ and $|x_2 - x_2'| \leq \frac{1}{2}|x_2 - y_2|$, then

$$\left|K(x, y) - K((x_1, x_2'), y) - K(x, (y_1', y_2)) + K((x_1, x_2'), (y_1', y_2))\right| \leq C \frac{|y_1 - y_1'|^\delta}{|x_1 - y_1|^{n_1+\delta}} \frac{|x_2 - x_2'|^\delta}{|x_2 - y_2|^{n_2+\delta}}.$$

(d) If $|x_1 - x_1'| \leq \frac{1}{2}|x_1 - y_1|$ and $|y_2 - y_2'| \leq \frac{1}{2}|x_2 - y_2|$, then

$$\left|K(x, y) - K(x, (y_1, y_2')) - K((x_1', x_2), y) + K((x_1', x_2), (y_1, y_2'))\right| \leq C \frac{|x_1 - x_1'|^\delta}{|x_1 - y_1|^{n_1+\delta}} \frac{|y_2 - y_2'|^\delta}{|x_2 - y_2|^{n_2+\delta}}.$$

(3) Mixed size and Hölder conditions:

(a) If $|x_1 - x_1'| \leq \frac{1}{2}|x_1 - y_1|$, then

$$|K(x, y) - K((x_1', x_2), y)| \leq C \frac{|x_1 - x_1'|^\delta}{|x_1 - y_1|^{n_1+\delta}} \frac{1}{|x_2 - y_2|^{n_2}}.$$

(b) If $|y_1 - y_1'| \leq \frac{1}{2}|x_1 - y_1|$, then

$$|K(x, y) - K(x, (y_1', y_2))| \leq C \frac{|y_1 - y_1'|^\delta}{|x_1 - y_1|^{n_1+\delta}} \frac{1}{|x_2 - y_2|^{n_2}}.$$

(c) If $|x_2 - x_2'| \leq \frac{1}{2}|x_2 - y_2|$, then

$$|K(x, y) - K((x_1, x_2'), y)| \leq C \frac{1}{|x_1 - y_1|^{n_1}} \frac{|x_2 - x_2'|^\delta}{|x_2 - y_2|^{n_2+\delta}}.$$

(d) If $|y_2 - y_2'| \leq \frac{1}{2}|x_2 - y_2|$, then

$$|K(x, y) - K(x, (y_1, y_2'))| \leq C \frac{1}{|x_1 - y_1|^{n_1}} \frac{|y_2 - y_2'|^\delta}{|x_2 - y_2|^{n_2+\delta}}.$$

(4) Calderón–Zygmund structure in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ separately: If $f = f_1 \otimes f_2$ and $g = g_1 \otimes g_2$ with $\operatorname{spt}(f_1) \cap \operatorname{spt}(g_1) = \varnothing$, we assume the kernel representation

$$\langle Tf, g\rangle = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1}} K_{f_2,g_2}(x_1, y_1) f_1(y_1) g_1(x_1)\, dx_1\, dy_1,$$

where the kernel $K_{f_2,g_2} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \setminus \{(x_1, y_1) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} : x_1 = y_1\}$ satisfies the size condition

$$|K_{f_2,g_2}(x_1, y_1)| \leq C(f_2, g_2) \frac{1}{|x_1 - y_1|^{n_1}},$$

and Hölder conditions:

(a) If $|x_1 - x_1'| \leq \frac{1}{2}|x_1 - y_1|$, then

$$|K_{f_2,g_2}(x_1, y_1) - K_{f_2,g_2}(x_1', y_1)| \leq C(f_2, g_2) \frac{|x_1 - x_1'|^\delta}{|x_1 - y_1|^{n_1+\delta}}.$$

(b) If $|y_1 - y_1'| \le \frac{1}{2}|x_1 - y_1|$, then

$$|K_{f_2,g_2}(x_1, y_1) - K_{f_2,g_2}(x_1, y_1')| \le C(f_2, g_2)\frac{|y_1 - y_1'|^\delta}{|x_1 - y_1|^{n_1+\delta}}.$$

We only assume the above representation and a certain control over $C(f_2, g_2)$ in the diagonal; that is,

$$C(\mathbb{1}_{Q_2}, \mathbb{1}_{Q_2}) + C(\mathbb{1}_{Q_2}, u_{Q_2}) + C(u_{Q_2}, \mathbb{1}_{Q_2}) \le C|Q_2|$$

for all cubes $Q_2 \subset \mathbb{R}^{n_2}$ and all "Q_2-adapted zero-mean" functions u_{Q_2} — that is, $\operatorname{spt}(u_{Q_2}) \subset Q_2$, $|u_{Q_2}| \le 1$, and $\int u_{Q_2} = 0$. We assume the symmetrical representation with kernel K_{f_1,g_1} in the case $\operatorname{spt}(f_2) \cap \operatorname{spt}(g_2) = \varnothing$.

II. Boundedness and cancellation assumptions:

(1) Assume $T1$, T^*1, $T_1(1)$ and $T_1^*(1)$ are in product $\mathrm{BMO}(\mathbb{R}^{\vec{n}})$, where T_1 is the partial adjoint of T, defined by

$$\langle T_1(f_1 \otimes f_2), g_1 \otimes g_2\rangle = \langle T(g_1 \otimes f_2), f_1 \otimes g_2\rangle.$$

(2) Assume

$$|\langle T(\mathbb{1}_{Q_1} \otimes \mathbb{1}_{Q_2}), \mathbb{1}_{Q_1} \otimes \mathbb{1}_{Q_2}\rangle| \le C|Q_1||Q_2|$$

for all cubes $Q_i \subset \mathbb{R}^{n_i}$ (weak boundedness).

(3) Diagonal BMO conditions: for all cubes $Q_i \subset \mathbb{R}^{n_i}$ and all zero-mean functions a_{Q_1} and b_{Q_2} that are Q_1- and Q_2-adapted, respectively, assume:

$$|\langle T(a_{Q_1} \otimes \mathbb{1}_{Q_2}), \mathbb{1}_{Q_1} \otimes \mathbb{1}_{Q_2}\rangle| \le C|Q_1||Q_2|, \quad |\langle T(\mathbb{1}_{Q_1} \otimes \mathbb{1}_{Q_2}), a_{Q_1} \otimes \mathbb{1}_{Q_2}\rangle| \le C|Q_1||Q_2|,$$
$$|\langle T(\mathbb{1}_{Q_1} \otimes b_{Q_2}), \mathbb{1}_{Q_1} \otimes \mathbb{1}_{Q_2}\rangle| \le C|Q_1||Q_2|, \quad |\langle T(\mathbb{1}_{Q_1} \otimes \mathbb{1}_{Q_2}), \mathbb{1}_{Q_1} \otimes b_{Q_2}\rangle| \le C|Q_1||Q_2|.$$

7B. ***Biparameter dyadic shifts and Martikainen's representation theorem.*** Given dyadic rectangles $\boldsymbol{\mathcal{D}} = \mathcal{D}_1 \times \mathcal{D}_2$ and pairs of nonnegative integers $\vec{i} = (i_1, i_2)$ and $\vec{j} = (j_1, j_2)$, a (cancellative) biparameter dyadic shift is an operator of the form

$$\mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}} f := \sum_{\substack{R_1 \in \mathcal{D}_1 \\ R_2 \in \mathcal{D}_2}} \sum_{\substack{P_1 \in (R_1)_{i_1} \\ P_2 \in (R_2)_{i_2}}} \sum_{\substack{Q_1 \in (R_1)_{j_1} \\ Q_2 \in (R_2)_{j_2}}} a_{P_1 Q_1 R_1 P_2 Q_2 R_2}\, \hat{f}(P_1^{\epsilon_1} \times P_2^{\epsilon_2})\, h_{Q_1}^{\delta_1} \otimes h_{Q_2}^{\delta_2}, \tag{7-1}$$

where

$$|a_{P_1 Q_1 R_1 P_2 Q_2 R_2}| \le \frac{\sqrt{|P_1||Q_1|}}{|R_1|}\frac{\sqrt{|P_2||Q_2|}}{|R_2|} = 2^{(-n_1/2)(i_1+j_1)} 2^{(-n_2/2)(i_2+j_2)}.$$

We suppress for now the signatures of the Haar functions, and assume summation over them is understood. We use the simplified notation

$$\mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}} f := \sum_{\boldsymbol{R},\boldsymbol{P},\boldsymbol{Q}}^{\vec{i},\vec{j}} a_{\boldsymbol{PQR}}\, \hat{f}(P_1 \times P_2)\, h_{Q_1} \otimes h_{Q_2}$$

for the summation above.

First note that

$$S_{\mathcal{D}}^2(\mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}} f) = \sum_{R_1\times R_2} \sum_{\substack{Q_1\in(R_1)_{j_1}\\ Q_2\in(R_2)_{j_2}}} \Biggl(\sum_{\substack{P_1\in(R_1)_{i_1}\\ P_2\in(R_2)_{i_2}}} a_{P_1Q_1R_1P_2Q_2R_2}\, \hat{f}(P_1\times P_2)\Biggr)^2 \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|}$$
$$\lesssim 2^{-n_1(i_1+j_1)}2^{-n_2(i_2+j_2)}(S_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}} f)^2,$$

where $S_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}}$ is the shifted biparameter square function in (3-1). Then, by (3-2),

$$\|\mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}} f\|_{L^p(w)} \lesssim 2^{(-n_1/2)(i_1+j_1)}2^{(-n_2/2)(i_2+j_2)}\|S_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}} f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \tag{7-2}$$

for all $w\in A_p(\mathbb{R}^{\vec{n}})$.

Next, we state Martikainen's representation theorem [2012]:

Theorem 7.1 (Martikainen). *For a biparameter singular integral operator T as defined in Section 7A, for some biparameter shifts $\mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}}$ it holds that*

$$\langle Tf, g\rangle = C_T\mathbb{E}_{\omega_1}\mathbb{E}_{\omega_2}\sum_{\vec{i},\vec{j}\in\mathbb{Z}_+^2} 2^{-\max(i_1,j_1)\delta/2}2^{-\max(i_2,j_2)\delta/2}\langle \mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}} f, g\rangle,$$

where noncancellative shifts may only appear if $(i_1, j_1)=(0,0)$ or $(i_2, j_2)=(0,0)$.

In light of this theorem, in order to prove Theorem 1.1, it suffices to prove the two-weight bound for commutators $[b,\mathbb{S}_{\boldsymbol{\mathcal{D}}}]$ with the dyadic shifts, with the requirements that the bounds be *independent* of the choice of $\boldsymbol{\mathcal{D}}$ and that they depend on $\vec{i}$ and $\vec{j}$ at most *polynomially*. We first look at the case of cancellative shifts, and then treat the noncancellative case in Section 7D.

7C. *Cancellative case.*

Theorem 7.2. *Let $\boldsymbol{\mathcal{D}}=\mathcal{D}_1\times\mathcal{D}_2$ be dyadic rectangles in $\mathbb{R}^{\vec{n}}=\mathbb{R}^{n_1}\otimes\mathbb{R}^{n_2}$ and $\mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}}$ be a cancellative dyadic shift as defined in* (7-1). *If $\mu,\lambda\in A_p(\mathbb{R}^{\vec{n}})$, $1<p<\infty$, and $\nu=\mu^{1/p}\lambda^{-1/p}$, then*

$$\|[b,\mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}}]: L^p(\mu)\to L^p(\lambda)\| \lesssim \bigl((1+\max(i_1,j_1))(1+\max(i_2,j_2))\bigr)\|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)},$$

where $\|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)}$ denotes the norm of b in the dyadic weighted little $\mathrm{bmo}(\nu)$ *space on $\mathbb{R}^{\vec{n}}$.*

Proof. We may express the product of two functions b and f on $\mathbb{R}^{\vec{n}}$ as

$$bf = \sum \mathsf{P}_{\mathsf{b}} f + \sum \mathsf{p}_{\mathsf{b}} f + \Pi_f b,$$

where P_{b} runs through the nine paraproducts associated with $\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)$ in Section 6A, and p_{b} runs through the six paraproducts associated with $\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)$ in Section 6B. Then

$$[b,\mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}}]f = \sum [\mathsf{P}_{\mathsf{b}},\mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}}]f + \sum [\mathsf{p}_{\mathsf{b}},\mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}}]f + \mathcal{R}_{\vec{i},\vec{j}} f,$$

where

$$\mathcal{R}_{\vec{i},\vec{j}} f := \Pi_{\mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}} f} b - \mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}}\Pi_f b.$$

From the two-weight inequalities for the paraproducts in Propositions 6.1 and 6.2, and the one-weight inequality for the shifts in (7-2),

$$\left\| \sum [\mathsf{P}_{\mathsf{b}}, \mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}}] + \sum [\mathsf{p}_{\mathsf{b}}, \mathbb{S}_{\boldsymbol{\mathcal{D}}}^{\vec{i},\vec{j}}] : L^p(\mu) \to L^p(\lambda) \right\| \lesssim \|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)},$$

so we are left with bounding the remainder term $\mathcal{R}_{\vec{i},\vec{j}}$. We claim that

$$\|\mathcal{R}_{\vec{i},\vec{j}} : L^p(\mu) \to L^p(\lambda)\| \lesssim \big((1+\max(i_1, j_1))(1+\max(i_2, j_2))\big)\|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)},$$

from which the result follows.

A straightforward calculation shows that

$$\mathcal{R}_{\vec{i},\vec{j}} f = \sum_{\boldsymbol{R},\boldsymbol{P},\boldsymbol{Q}}^{\vec{i},\vec{j}} a_{\boldsymbol{PQR}} \hat{f}(P_1 \times P_2)(\langle b\rangle_{Q_1\times Q_2} - \langle b\rangle_{P_1\times P_2}) h_{Q_1} \otimes h_{Q_2}.$$

We write this as a sum $\mathcal{R}_{\vec{i},\vec{j}} f = \mathcal{R}^1_{\vec{i},\vec{j}} f + \mathcal{R}^2_{\vec{i},\vec{j}} f$ by splitting the term in parentheses as

$$\langle b\rangle_{Q_1\times Q_2} - \langle b\rangle_{P_1\times P_2} = (\langle b\rangle_{Q_1\times Q_2} - \langle b\rangle_{R_1\times R_2}) + (\langle b\rangle_{R_1\times R_2} - \langle b\rangle_{P_1\times P_2}).$$

For the first term, we may apply the biparameter version of (2-2), where we keep in mind that $R_1 = Q_1^{(j_1)}$ and $R_2 = Q_2^{(j_2)}$:

$$\begin{aligned}\langle b\rangle_{Q_1\times Q_2} - \langle b\rangle_{R_1\times R_2} = &\sum_{\substack{1\le k_1\le j_1\\ 1\le k_2\le j_2}} \hat{b}(Q_1^{(k_1)} \times Q_2^{(k_2)}) h_{Q_1^{(k_1)}}(Q_1) h_{Q_2^{(k_2)}}(Q_2)\\ &+ \sum_{1\le k_1\le j_1} \Big\langle b, h_{Q_1^{(k_1)}} \otimes \frac{\mathbb{1}_{R_2}}{|R_2|}\Big\rangle h_{Q_1^{(k_1)}}(Q_1) + \sum_{1\le k_2\le j_2} \Big\langle b, \frac{\mathbb{1}_{R_1}}{|R_1|} \otimes h_{Q_2^{(k_2)}}\Big\rangle h_{Q_2^{(k_2)}}(Q_2).\end{aligned}$$

Then, we may write the operator $\mathcal{R}^1_{\vec{i},\vec{j}}$ as

$$\mathcal{R}^1_{\vec{i},\vec{j}} f = \sum_{\substack{1\le k_1\le j_1\\ 1\le k_2\le j_2}} A_{k_1,k_2} f + \sum_{1\le k_1\le j_1} B^{(0,1)}_{k_1} f + \sum_{1\le k_2\le j_2} B^{(1,0)}_{k_2} f, \tag{7-3}$$

where

$$\begin{aligned} A_{k_1,k_2} f &:= \sum_{\boldsymbol{R},\boldsymbol{P},\boldsymbol{Q}}^{\vec{i},\vec{j}} a_{\boldsymbol{PQR}} \hat{f}(P_1 \times P_2) \hat{b}(Q_1^{(k_1)} \times Q_2^{(k_2)}) h_{Q_1^{(k_1)}}(Q_1) h_{Q_2^{(k_2)}}(Q_2) h_{Q_1} \otimes h_{Q_2},\\ B^{(0,1)}_{k_1} f &:= \sum_{\boldsymbol{R},\boldsymbol{P},\boldsymbol{Q}}^{\vec{i},\vec{j}} a_{\boldsymbol{PQR}} \hat{f}(P_1 \times P_2) \Big\langle b, h_{Q_1^{(k_1)}} \otimes \frac{\mathbb{1}_{R_2}}{|R_2|}\Big\rangle h_{Q_1^{(k_1)}}(Q_1) h_{Q_1} \otimes h_{Q_2},\\ B^{(1,0)}_{k_2} f &:= \sum_{\boldsymbol{R},\boldsymbol{P},\boldsymbol{Q}}^{\vec{i},\vec{j}} a_{\boldsymbol{PQR}} \hat{f}(P_1 \times P_2) \Big\langle b, \frac{\mathbb{1}_{R_1}}{|R_1|} \otimes h_{Q_2^{(k_2)}}\Big\rangle h_{Q_2^{(k_2)}}(Q_2) h_{Q_1} \otimes h_{Q_2}.\end{aligned}$$

We show that these operators satisfy

$$\begin{aligned}
\|A_{k_1,k_2} : L^p(\mu) \to L^p(\lambda)\| &\lesssim \|b\|_{\mathrm{BMO}_{\mathcal{D}}(\nu)} \quad \text{for all } k_1, k_2,\\
\|B^{(0,1)}_{k_1} : L^p(\mu) \to L^p(\lambda)\| &\lesssim \|b\|_{\mathrm{bmo}_{\mathcal{D}}(\nu)} \quad \text{for all } k_1,\\
\|B^{(1,0)}_{k_2} : L^p(\mu) \to L^p(\lambda)\| &\lesssim \|b\|_{\mathrm{bmo}_{\mathcal{D}}(\nu)} \quad \text{for all } k_2.
\end{aligned}$$

Going back to the decomposition in (7-3), these inequalities will give

$$\|\mathcal{R}^1_{\vec{i},\vec{j}} : L^p(\mu) \to L^p(\lambda)\| \lesssim (j_1 j_2 + j_1 + j_2)\|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)}.$$

A symmetrical proof for the term $\mathcal{R}^2_{\vec{i},\vec{j}}$ coming from $(\langle b\rangle_{R_1\times R_2} - \langle b\rangle_{P_1\times P_2})$ will show that

$$\|\mathcal{R}^2_{\vec{i},\vec{j}} : L^p(\mu) \to L^p(\lambda)\| \lesssim (i_1 i_2 + i_1 + i_2)\|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)}.$$

Putting these estimates together, we obtain the desired result

$$\begin{aligned}
&\|\mathcal{R}_{\vec{i},\vec{j}} : L^p(\mu) \to L^p(\lambda)\|\\
&\qquad \lesssim (i_1 + i_2 + i_1 i_2 + j_1 + j_2 + j_1 j_2)\|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)} \lesssim (1+\max(i_1, j_1))(1+\max(i_2, j_2))\|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)}.
\end{aligned}$$

Note that we are allowed to have one of the situations $(i_1, i_2) = (0, 0)$ or $(j_1, j_2) = (0, 0)$ — but not both — and then either the term $\mathcal{R}^2_{\vec{i},\vec{j}} f$ or $\mathcal{R}^1_{\vec{i},\vec{j}} f$, respectively, will vanish.

Let us now look at the estimate for A_{k_1,k_2}. Taking again $f \in L^p(\mu)$ and $g \in L^{p'}(\lambda')$, we write $\langle A_{k_1,k_2} f, g\rangle = \langle b, \phi\rangle$, where

$$\begin{aligned}
\phi &= \sum_{\boldsymbol{R},\boldsymbol{P},\boldsymbol{Q}}^{\vec{i},\vec{j}} a_{\boldsymbol{PQR}} \hat{f}(P_1\times P_2) h_{Q_1^{(k_1)}}(Q_1) h_{Q_2^{(k_2)}}(Q_2) \hat{g}(Q_1\times Q_2) h_{Q_1^{(k_1)}} \otimes h_{Q_2^{(k_2)}}\\
&= \sum_{R_1\times R_2} \sum_{\substack{P_1\in(R_1)_{i_1}\\ P_2\in(R_2)_{i_2}}} \sum_{\substack{N_1\in(R_1)_{j_1-k_1}\\ N_2\in(R_2)_{j_2-k_2}}} \hat{f}(P_1\times P_2) \Biggl(\sum_{\substack{Q_1\in(N_1)_{k_1}\\ Q_2\in(N_2)_{k_2}}} a_{\boldsymbol{PQR}} \hat{g}(Q_1\times Q_2) h_{N_1}(Q_1) h_{N_2}(Q_2) \Biggr) h_{N_1}\otimes h_{N_2}.
\end{aligned}$$

Then

$$\begin{aligned}
S^2_{\boldsymbol{\mathcal{D}}}\phi &\lesssim \sum_{N_1\times N_2} \Biggl(\sum_{\substack{P_1\in(N_1^{(j_1-k_1)})_{i_1}\\ P_2\in(N_2^{(j_2-k_2)})_{i_2}}} |\hat{f}(P_1\times P_2)| \sum_{\substack{Q_1\in(N_1)_{k_1}\\ Q_2\in(N_2)_{k_2}}} |a_{\boldsymbol{PQR}}||\hat{g}(Q_1\times Q_2)| \frac{1}{\sqrt{|N_1|}} \frac{1}{\sqrt{|N_2|}} \Biggr)^2 \frac{\mathbb{1}_{N_1}\otimes \mathbb{1}_{N_2}}{|N_1||N_2|}\\
&\lesssim 2^{-n_1(i_1+j_1)} 2^{-n_2(i_2+j_2)} \sum_{N_1\times N_2} \Biggl(\sum_{\substack{P_1\in(N_1^{(j_1-k_1)})_{i_1}\\ P_2\in(N_2^{(j_2-k_2)})_{i_2}}} |\hat{f}(P_1\times P_2)| 2^{n_1 k_1/2} 2^{n_2 k_2/2} \langle |g|\rangle_{N_1\times N_2} \Biggr)^2 \frac{\mathbb{1}_{N_1}\otimes \mathbb{1}_{N_2}}{|N_1||N_2|}\\
&\lesssim 2^{-n_1(i_1+j_1-k_1)} 2^{-n_2(i_2+j_2-k_2)} (M_S g)^2 \sum_{R_1\times R_2} \Biggl(\sum_{\substack{P_1\in(R_1)_{i_1}\\ P_2\in(R_2)_{i_2}}} |\hat{f}(P_1\times P_2)| \Biggr)^2 \sum_{\substack{N_1\in(R_1)_{j_1-k_1}\\ N_2\in(R_2)_{j_2-k_2}}} \frac{\mathbb{1}_{N_1}\otimes \mathbb{1}_{N_2}}{|N_1||N_2|}\\
&= 2^{-n_1(i_1+j_1-k_1)} 2^{-n_2(i_2+j_2-k_2)} (M_S g)^2 (S_{\boldsymbol{\mathcal{D}}}^{(i_1,i_2),(j_1-k_1,j_2-k_2)} f)^2,
\end{aligned}$$

where the last operator is the shifted square function in (3-1). Then, from (3-2),

$$\begin{aligned}\|A_{k_1,k_2} : L^p(\mu) \to L^p(\lambda)\| &\lesssim \|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)} \|S_{\boldsymbol{\mathcal{D}}}\phi\|_{L^1(\nu)} \\ &\lesssim \|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)} 2^{(-n_1/2)(i_1+j_1-k_1)} 2^{(-n_2/2)(i_2+j_2-k_2)} \|M_S g\|_{L^{p'}(\lambda')} \|S_{\boldsymbol{\mathcal{D}}}^{(i_1,i_2),(j_1-k_1,j_2-k_2)} f\|_{L^p(\mu)} \\ &\lesssim \|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)} \|g\|_{L^{p'}(\lambda')} \|f\|_{L^p(\mu)}.\end{aligned}$$

Finally, we look at $B_{k_1}^{(0,1)}$, with the proof for $B_{k_2}^{(1,0)}$ being symmetrical. We write again $\langle B_{k_1}^{(0,1)} f, g\rangle = \langle b, \phi\rangle$, where

$$\phi = \sum_{\boldsymbol{R},\boldsymbol{P},\boldsymbol{Q}}^{\vec{i},\vec{j}} a_{\boldsymbol{PQR}} \hat{f}(P_1 \times P_2) h_{Q_1^{(k_1)}}(Q_1) \hat{g}(Q_1 \times Q_2) h_{Q_1^{(k_1)}} \otimes \frac{\mathbb{1}_{R_2}}{|R_2|}.$$

Then

$$S_{\mathcal{D}_1}^2 f \lesssim 2^{-n_1(i_1+j_1)} 2^{-n_2(i_2+j_2)} \sum_{\substack{R_1\in\mathcal{D}_1\\ N_1\in(R_1)_{j_1-k_1}}} \frac{\mathbb{1}_{N_1}}{|N_1|} \Bigg(\sum_{R_2\in\mathcal{D}_2} \sum_{\substack{P_1\in(R_1)_{i_1}\\ P_2\in(R_2)_{i_2}}} |\hat{f}(P_1\times P_2)| \sum_{Q_2\in(R_2)_{j_2}} \langle |H_{Q_2} g|\rangle_{N_1} 2^{n_1k_1/2} \frac{\mathbb{1}_{R_2}}{|R_2|} \Bigg)^2,$$

and the summation above is bounded by

$$\Bigg(\sum_{\substack{R_1\in\mathcal{D}_1\\ N_1\in(R_1)_{j_1-k_1}}} \frac{\mathbb{1}_{N_1}}{|N_1|} \sum_{R_2\in\mathcal{D}_2} \Bigg(\sum_{\substack{P_1\in(R_1)_{i_1}\\ P_2\in(R_2)_{i_2}}} |\hat{f}(P_1 \times P_2)| \Bigg)^2 \frac{\mathbb{1}_{R_2}}{|R_2|} \Bigg) \Bigg(\sum_{R_2\in\mathcal{D}_2} \Bigg(\sum_{Q_2\in(R_2)_{j_2}} M_{\mathcal{D}_1}(H_{Q_2} g) \Bigg)^2 \frac{\mathbb{1}_{R_2}}{|R_2|} \Bigg),$$

which is exactly

$$(S_{\boldsymbol{\mathcal{D}}}^{(i_1,i_2),(j_1-k_1,0)} f)^2 ([MS]^{j_2,0} g)^2.$$

From (3-2) and (3-3), we obtain exactly $\|S_{\mathcal{D}_1}\phi\|_{L^1(\nu)} \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\lambda')}$, and the proof is complete. □

7D. ***The noncancellative case.*** Following the proof in [Martikainen 2012], we are left with three types of terms to consider, all of paraproduct type,

- the full standard paraproduct, Π_a and Π_a^*,
- the full mixed paraproducts, $\Pi_{a;(0,1)}$ and $\Pi_{a;(1,0)}$,

where, in each case, a is some fixed function in *unweighted* product $\mathrm{BMO}(\mathbb{R}^{\vec{n}})$, with $\|a\|_{\mathrm{BMO}(\mathbb{R}^{\vec{n}})} \le 1$, and

- the *partial* paraproducts, defined for every i_1, $j_1 \ge 0$ as

$$\mathbb{S}_{\boldsymbol{\mathcal{D}}}^{i_1,j_1} f := \sum_{\substack{R_1\in\mathcal{D}_1\\ R_2\in\mathcal{D}_2}} \sum_{\substack{P_1\in(R_1)_{i_1}\\ Q_1\in(R_1)_{j_1}}} \hat{a}_{P_1Q_1R_1}(R_2^{\delta_2}) \hat{f}(P_1^{\epsilon_1} \times R_2^{\epsilon_2}) h_{Q_1}^{\delta_1} \times \frac{\mathbb{1}_{R_2}}{|R_2|},$$

where, for every fixed P_1, Q_1, R_1, we have $a_{P_1Q_1R_1}(x_2)$ is a $\mathrm{BMO}(\mathbb{R}^{n_2})$ function with

$$\|a_{P_1Q_1R_1}\|_{\mathrm{BMO}(\mathbb{R}^{n_2})} \le \frac{\sqrt{|P_1|}\sqrt{|Q_1|}}{|R_1|} = 2^{(-n_1/2)(i_1+j_1)},$$

and

$$\hat{a}_{P_1Q_1R_1}(R_2^{\delta_2}) := \langle a_{P_1Q_1R_1}, h_{R_2}^{\delta_2}\rangle_{\mathbb{R}^{n_2}} := \int_{\mathbb{R}^{n_2}} a_{P_1Q_1R_1}(x_2) h_{R_2}^{\delta_2}(x_2)\, dx_2.$$

The symmetrical partial paraproduct $\mathbb{S}_{\boldsymbol{\mathcal{D}}}^{i_2,j_2}$ is defined analogously.

We treat each case separately.

7D1. *The full standard paraproduct.* In this case, we are looking at the commutator $[b, \Pi_a]$, where

$$\Pi_a f := \sum_{R\in\boldsymbol{\mathcal{D}}} \hat{a}(R)\langle f\rangle_R h_R,$$

and $a \in \mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\mathbb{R}^{\vec{n}})$ with $\|a\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\mathbb{R}^{\vec{n}})} \le 1$. We prove that:

Theorem 7.3. *Let $\mu, \lambda \in A_p(\mathbb{R}^{\vec{n}})$, $1 < p < \infty$ and $\nu := \mu^{1/p}\lambda^{-1/p}$. Then*

$$\|[b, \Pi_a] : L^p(\mu) \to L^p(\lambda)\| \lesssim \|a\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\mathbb{R}^{\vec{n}})} \|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)}.$$

Proof. We remark first that

$$\Pi_a(bf) = \sum_{R\in\boldsymbol{\mathcal{D}}} \hat{a}(R)\langle bf\rangle_R h_R \quad \text{and} \quad \Pi_{\Pi_a f} b = \sum_{R\in\boldsymbol{\mathcal{D}}} \hat{a}(R)\langle b\rangle_R \langle f\rangle_R h_R,$$

so

$$\begin{aligned}\Pi_a(bf) - \Pi_{\Pi_a f} b &= \sum_{R\in\boldsymbol{\mathcal{D}}} \hat{a}(R)\big(\langle bf\rangle_R - \langle b\rangle_R\langle f\rangle_R\big)h_R \\ &= \Pi_a\Big(\sum \mathsf{P}_{\mathsf{b}} f + \sum \mathsf{p}_{\mathsf{b}} f + \Pi_f b\Big) - \Pi_{\Pi_a f} b,\end{aligned}$$

where the last equality was obtained by simply expanding bf into paraproducts. Then

$$\Pi_{\Pi_a f} b - \Pi_a \Pi_f b = \sum \Pi_a \mathsf{P}_{\mathsf{b}} f + \sum \Pi_a \mathsf{p}_{\mathsf{b}} f - \sum_{R\in\boldsymbol{\mathcal{D}}} \hat{a}(R)\big(\langle bf\rangle_R - \langle b\rangle_R\langle f\rangle_R\big)h_R.$$

Noting that

$$[b, \Pi_a]f = \sum \mathsf{P}_{\mathsf{b}}\Pi_a f + \sum \mathsf{p}_{\mathsf{b}}\Pi_a f - \sum \Pi_a\mathsf{P}_{\mathsf{b}} f - \sum \Pi_a \mathsf{p}_{\mathsf{b}} f + \Pi_{\Pi_a f} b - \Pi_a\Pi_f b,$$

we obtain

$$[b, \Pi_a]f = \sum \mathsf{P}_{\mathsf{b}}\Pi_a f + \sum \mathsf{p}_{\mathsf{b}}\Pi_a f - \sum_{R\in\boldsymbol{\mathcal{D}}} \hat{a}(R)\big(\langle bf\rangle_R - \langle b\rangle_R\langle f\rangle_R\big)h_R.$$

The first terms are easily handled:

$$\|\mathsf{P}_{\mathsf{b}}\Pi_a f\|_{L^p(\lambda)} \lesssim \|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)} \|\Pi_a f\|_{L^p(\mu)} \lesssim \|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)} \|a\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\mathbb{R}^{\vec{n}})} \|f\|_{L^p(\mu)},$$

$$\|\mathsf{p}_{\mathsf{b}}\Pi_a f\|_{L^p(\lambda)} \lesssim \|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)} \|\Pi_a f\|_{L^p(\mu)} \lesssim \|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)} \|a\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\mathbb{R}^{\vec{n}})} \|f\|_{L^p(\mu)}.$$

So we are left with the third term.

Now, for any dyadic rectangle R,

$$\langle bf\rangle_R - \langle b\rangle_R\langle f\rangle_R = \frac{1}{|R|}\int_R f(x)\mathbb{1}_R(x)(b(x) - \langle b\rangle_R)\, dx.$$

Expressing $\mathbb{1}_R(b-\langle b\rangle_R)$ as in (2-5), we obtain

$$\langle bf\rangle_R-\langle b\rangle_R\langle f\rangle_R=\frac{1}{|R|}\sum_{\substack{P_1\subset Q_1\\P_2\subset Q_2}}\hat{b}(P_1\times P_2)\hat{f}(P_1\times P_2)\\ +\frac{1}{|R|}\sum_{P_1\subset Q_1}\left\langle b,h_{P_1}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|}\right\rangle\langle f,h_{P_1}\otimes\mathbb{1}_{Q_2}\rangle+\frac{1}{|R|}\sum_{P_2\subset Q_2}\left\langle b,\frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes h_{P_2}\right\rangle\langle f,\mathbb{1}_{Q_1}\otimes h_{P_2}\rangle.$$

Therefore

$$\sum_{R\in\mathcal{D}}\hat{a}(R)\big(\langle bf\rangle_R-\langle b\rangle_R\langle f\rangle_R\big)h_R=\Lambda_{a,b}f+\lambda_{a,b}^{(0,1)}f+\lambda_{a,b}^{(1,0)}f,$$

where

$$\Lambda_{a,b}f:=\sum_{Q_1\times Q_2}\hat{a}(Q_1\times Q_2)\frac{1}{|Q_1||Q_2|}\bigg(\sum_{\substack{P_1\subset Q_1\\P_2\subset Q_2}}\hat{b}(P_1\times P_2)\hat{f}(P_1\times P_2)\bigg)h_{Q_1}\otimes h_{Q_2},$$

$$\lambda_{a,b}^{(0,1)}f:=\sum_{Q_1\times Q_2}\hat{a}(Q_1\times Q_2)\frac{1}{|Q_1||Q_2|}\bigg(\sum_{P_1\subset Q_1}\left\langle b,h_{P_1}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|}\right\rangle\langle f,h_{P_1}\otimes\mathbb{1}_{Q_2}\rangle\bigg)h_{Q_1}\otimes h_{Q_2},$$

$$\lambda_{a,b}^{(1,0)}f:=\sum_{Q_1\times Q_2}\hat{a}(Q_1\times Q_2)\frac{1}{|Q_1||Q_2|}\bigg(\sum_{P_2\subset Q_2}\left\langle b,\frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes h_{P_2}\right\rangle\langle f,\mathbb{1}_{Q_1}\otimes h_{P_2}\rangle\bigg)h_{Q_1}\otimes h_{Q_2}.$$

To analyze the term $\Lambda_{a,b}$, we write $\langle\Lambda_{a,b}f,g\rangle=\langle b,\phi\rangle$, where

$$\phi=\sum_{P_1\times P_2}\hat{f}(P_1\times P_2)\bigg(\sum_{\substack{Q_1\supset P_1\\Q_2\supset P_2}}\hat{a}(Q_1\times Q_2)\hat{g}(Q_1\times Q_2)\frac{1}{|Q_1||Q_2|}\bigg)h_{P_1}\otimes h_{P_2}\\ =\sum_{R\in\mathcal{D}}\hat{f}(R)\bigg(\sum_{T\in\mathcal{D},T\supset R}\hat{a}(T)\hat{g}(T)\frac{1}{|T|}\bigg)h_R.$$

So $|\langle\Lambda_{a,b}f,g\rangle|\lesssim\|b\|_{\mathrm{BMO}_{\mathcal{D}}(\nu)}\|S_{\mathcal{D}}\phi\|_{L^1(\nu)}$, and

$$S^2_{\mathcal{D}}\phi=\sum_{R\in\mathcal{D}}|\hat{f}(R)|^2\bigg(\sum_{T\in\mathcal{D},T\supset R}\hat{a}(T)\hat{g}(T)\frac{1}{|T|}\bigg)^2\frac{\mathbb{1}_R}{|R|}\le\sum_{R\in\mathcal{D}}|\hat{f}(R)|^2\bigg(\sum_{T\in\mathcal{D},T\supset R}\hat{a}_\tau(T)\hat{g}_\tau(T)\frac{1}{|T|}\bigg)^2\frac{\mathbb{1}_R}{|R|},$$

where $a_\tau:=\sum_{R\in\mathcal{D}}|\hat{a}(R)|h_R$ and $g_\tau:=\sum_{R\in\mathcal{D}}|\hat{g}(R)|h_R$ are martingale transforms which do not increase either the BMO norm of a, or the $L^{p'}(\lambda')$ norm of g. Now note that

$$\langle\Pi^*_{a_\tau}g_\tau\rangle_R=\sum_{T\subsetneq R}\hat{a}_\tau(T)\hat{g}_\tau(T)\frac{1}{|R|}+\sum_{T\supset R}\hat{a}_\tau(T)\hat{g}_\tau(T)\frac{1}{|T|},$$

and since all the Haar coefficients of a_τ and g_τ are nonnegative, we may write

$$\sum_{T\supset R}\hat{a}_\tau(T)\hat{g}_\tau(T)\frac{1}{|T|}\le\langle\Pi^*_{a_\tau}g_\tau\rangle_R.$$

Then

$$S^2_{\boldsymbol{\mathcal{D}}}\phi \le \sum_{R\in\boldsymbol{\mathcal{D}}} |\hat{f}(R)|^2 \langle \Pi^*_{a_\tau} g_\tau\rangle_R^2 \frac{\mathbb{1}_R}{|R|} \le (M_S \Pi^*_{a_\tau} g_\tau)^2 S^2_{\boldsymbol{\mathcal{D}}} f,$$

and

$$\begin{aligned}\|S_{\boldsymbol{\mathcal{D}}}\phi\|_{L^1(\nu)} &\le \|M_S \Pi^*_{a_\tau} g_\tau\|_{L^{p'}(\lambda')} \|S_{\boldsymbol{\mathcal{D}}} f\|_{L^p(\mu)}\\ &\lesssim \|\Pi^*_{a_\tau} g_\tau\|_{L^{p'}(\lambda')} \|f\|_{L^p(\mu)}\\ &\lesssim \|a_\tau\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\mathbb{R}^{\vec{n}})} \|g_\tau\|_{L^{p'}(\lambda')} \|f\|_{L^p(\mu)},\end{aligned}$$

which gives us the desired estimate

$$\|\Lambda_{a,b} : L^p(\mu) \to L^p(\lambda)\| \lesssim \|a\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\mathbb{R}^{\vec{n}})} \|b\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)}.$$

Finally, we analyze the term $\lambda^{(0,1)}_{a,b}$, with the last term being symmetrical. We have $\langle \lambda^{(0,1)}_{a,b} f, g\rangle = \langle b, \phi\rangle$ with

$$\phi = \sum_{P_1}\bigg(\sum_{P_2} \langle f, h_{P_1}\otimes \mathbb{1}_{P_2}\rangle \frac{1}{|P_2|} \sum_{Q_1\supset P_1} \hat{a}(Q_1\times P_2)\hat{g}(Q_1\times P_2)\frac{1}{|Q_1|}\frac{\mathbb{1}_{P_2}}{|P_2|}\bigg) h_{P_1},$$

and $|\langle \lambda^{(0,1)}_{a,b} f, g\rangle| \lesssim \|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)} \|S_{\mathcal{D}_1}\phi\|_{L^1(\nu)}$. Now

$$S^2_{\mathcal{D}_1}\phi \le \sum_{P_1}\bigg(\sum_{P_2} \langle |H_{P_1} f|\rangle_{P_2} \bigg(\sum_{Q_1\supset P_1} \hat{a}_\tau(Q_1\times P_2)\hat{g}_\tau(Q_2\times P_2)\frac{1}{|Q_1|}\bigg)\frac{\mathbb{1}_{P_2}}{|P_2|}\bigg)^2 \frac{\mathbb{1}_{P_1}}{|P_1|},$$

where we are using the same martingale transforms as above. Note that

$$\bigg\langle \Pi^*_{a_\tau} g_\tau, \frac{\mathbb{1}_{P_1}}{|P_1|}\bigg\rangle_{\mathbb{R}^{n_1}}(x_2) = \sum_{P_2} \frac{\mathbb{1}_{P_2}(x_2)}{|P_2|} \sum_{Q_1} \hat{a}_\tau(Q_1\times P_2)\hat{g}_\tau(Q_1\times P_2)\frac{|Q_1\cap P_1|}{|Q_1||P_1|},$$

and again since all terms are nonnegative:

$$\begin{aligned}S^2_{\mathcal{D}_1}\phi &\le \sum_{P_1} M^2_{\mathcal{D}_2}(H_{P_1} f)(x_2)\bigg(\sum_{Q_1\supset P_1}\sum_{P_2} \hat{a}_\tau(Q_1\times P_2)\hat{g}_\tau(Q_1\times P_2)\frac{1}{|Q_1|}\frac{\mathbb{1}_{P_2}(x_2)}{|P_2|}\bigg)^2 \frac{\mathbb{1}_{P_1}(x_1)}{|P_1|}\\ &\le \sum_{P_1} M^2_{\mathcal{D}_2}(H_{P_1} f)(x_2)\bigg(\bigg\langle \Pi^*_{a_\tau} g_\tau, \frac{\mathbb{1}_{P_1}}{|P_1|}\bigg\rangle_{\mathbb{R}^{n_1}}(x_2)\bigg)^2 \frac{\mathbb{1}_{P_1}(x_1)}{|P_1|}\\ &\le (M_{\mathcal{D}_1}(\Pi^*_{a_\tau} g_\tau)(x_1,x_2))^2 \sum_{P_1} M^2_{\mathcal{D}_2}(H_{P_1} f)(x_2)\frac{\mathbb{1}_{P_1}(x_1)}{|P_1|}\\ &= (M_{\mathcal{D}_1}(\Pi^*_{a_\tau} g_\tau)(x_1,x_2))^2 ([SM]f(x_1,x_2))^2.\end{aligned}$$

Then

$$\|S_{\mathcal{D}_1}\phi\|_{L^1(\nu)} \lesssim \|\Pi^*_{a_\tau} g_\tau\|_{L^{p'}(\lambda')} \|[SM]f\|_{L^p(\mu)} \lesssim \|a\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\mathbb{R}^{\vec{n}})} \|g\|_{L^{p'}(\lambda')} \|f\|_{L^p(\mu)},$$

and so

$$\|\lambda^{(0,1)}_{a,b} : L^p(\mu) \to L^p(\lambda)\| \lesssim \|a\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\nu)} \|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)}. \qquad \square$$

7D2. *The full mixed paraproduct.* We are now dealing with $[b, \Pi_{a;(0,1)}]$, where

$$\Pi_{a;(0,1)} f := \sum_{P_1 \times P_2} \hat{a}(P_1 \times P_2) \left\langle f, h_{P_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \right\rangle \frac{\mathbb{1}_{P_1}}{|P_1|} \otimes h_{P_2}.$$

Theorem 7.4. *Let $\mu, \lambda \in A_p(\mathbb{R}^{\vec{n}})$, $1 < p < \infty$ and $\nu := \mu^{1/p}\lambda^{-1/p}$. Then*

$$\|[b, \Pi_{a;(0,1)}] : L^p(\mu) \to L^p(\lambda)\| \lesssim \|a\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\mathbb{R}^{\vec{n}})} \|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)}.$$

Note that the case $[b, \Pi_{a;(1,0)}]$ follows symmetrically.

Proof. By the standard considerations, we only need to bound the remainder term

$$\mathcal{R}^{(0,1)}_{a,b} f := \Pi_{\Pi_{a;(0,1)} f} b - \Pi_{a;(0,1)} \Pi_f b.$$

Explicitly, these terms are

$$\Pi_{\Pi_{a;(0,1)} f} b = \sum_{P_1 \times P_2} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) \left\langle f, h_{P_1}^{\epsilon_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \right\rangle \left(\sum_{Q_1 \supsetneq P_1} \langle b \rangle_{Q_1 \times P_2} h_{Q_1}^{\delta_1}(P_1) h_{Q_1}^{\delta_1}(x_1) \right) h_{P_2}^{\epsilon_2}(x_2),$$

$$\Pi_{a;(0,1)} \Pi_f b = \sum_{P_1 \times P_2} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) \left(\sum_{Q_2 \supsetneq P_2} \hat{f}(P_1^{\epsilon_1} \times Q_2^{\delta_2}) \langle b \rangle_{P_1 \times Q_2} h_{Q_2}^{\delta_2}(P_2) \right) \frac{\mathbb{1}_{P_1}(x_1)}{|P_1|} \otimes h_{P_2}^{\epsilon_2}(x_2).$$

Consider now a third term

$$T := \sum_{P_1 \times P_2} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) \langle b \rangle_{P_1 \times P_2} \left\langle f, h_{P_1}^{\epsilon_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \right\rangle \frac{\mathbb{1}_{P_1}}{|P_1|} \otimes h_{P_2}^{\epsilon_2}.$$

Using the one-parameter formula

$$\frac{\mathbb{1}_{P_1}(x_1)}{|P_1|} = \sum_{Q_1 \supsetneq P_1} h_{Q_1}^{\delta_1}(P_1) h_{Q_1}^{\delta_1}(x_1),$$

we write T as

$$T = \sum_{P_1 \times P_2} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) \left\langle f, h_{P_1}^{\epsilon_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \right\rangle \left(\sum_{Q_1 \supsetneq P_1} \langle b \rangle_{P_1 \times P_2} h_{Q_1}^{\delta_1}(P_1) h_{Q_1}^{\delta_1}(x_1) \right) h_{P_2}^{\epsilon_2}(x_2),$$

allowing us to combine this term with $\Pi_{\Pi_{a;(0,1)} f} b$:

$$\Pi_{\Pi_{a;(0,1)} f} b - T = \sum_{P_1 \times P_2} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) \left\langle f, h_{P_1}^{\epsilon_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \right\rangle \left(\sum_{Q_1 \supsetneq P_1} (\langle b \rangle_{Q_1 \times P_2} - \langle b \rangle_{P_1 \times P_2}) h_{Q_1}^{\delta_1}(P_1) h_{Q_1}^{\delta_1}(x_1) \right) h_{P_2}^{\epsilon_2}(x_2).$$

Using (2-2), we have

$$\langle b \rangle_{Q_1 \times P_2} - \langle b \rangle_{P_1 \times P_2} = - \sum_{R_1 : P_1 \subsetneq R_1 \subset Q_1} \left\langle b, h_{R_1}^{\tau_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \right\rangle h_{R_1}^{\tau_1}(P_1),$$

and then the term in parentheses above becomes

$$- \sum_{Q_1 \supsetneq P_1} \left(\sum_{R_1 : P_1 \subsetneq R_1 \subset Q_1} \left\langle b, h_{R_1}^{\tau_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \right\rangle h_{R_1}^{\tau_1}(P_1) \right) h_{Q_1}^{\delta_1}(P_1) h_{Q_1}^{\delta_1}(x_1). \tag{7-4}$$

Next, we analyze this term depending on the relationship between R_1 and Q_1:

<u>Case 1</u>: $R_1 \subsetneq Q_1$. Then we may rewrite the sum as

$$\sum_{R_1 \supsetneq P_1} \Big\langle b, h_{R_1}^{\tau_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Big\rangle h_{R_1}^{\tau_1}(P_1) \sum_{Q_1 \supsetneq R_1} \underbrace{h_{Q_1}^{\delta_1}(P_1)}_{=h_{Q_1}^{\delta_1}(R_1)} h_{Q_1}^{\delta_1}(x_1) = \sum_{R_1 \supsetneq P_1} \Big\langle b, h_{R_1}^{\tau_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Big\rangle h_{R_1}^{\tau_1}(P_1) \frac{\mathbb{1}_{R_1}(x_1)}{|R_1|}.$$

This then leads to

$$\begin{aligned}
&\sum_{P_1 \times P_2} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) \Big\langle f, h_{P_1}^{\epsilon_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Big\rangle \bigg(\sum_{R_1 \supsetneq P_1} \Big\langle b, h_{R_1}^{\tau_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Big\rangle h_{R_1}^{\tau_1}(P_1) \frac{\mathbb{1}_{R_1}(x_1)}{|R_1|} \bigg) h_{P_2}^{\epsilon_2}(x_2) \\
&\quad = \sum_{R_1 \times P_2} \Big\langle b, h_{R_1}^{\tau_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Big\rangle \bigg(\sum_{P_1 \subsetneq R_1} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) \Big\langle f, h_{P_1}^{\epsilon_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Big\rangle h_{R_1}^{\tau_1}(P_1) \bigg) \frac{\mathbb{1}_{R_1}(x_1)}{|R_1|} \otimes h_{P_2}^{\epsilon_2}(x_2) \\
&\quad = \sum_{R_1 \times P_2} \Big\langle b, h_{R_1}^{\tau_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Big\rangle \langle \Pi_{a;(0,1)} f, h_{R_1}^{\tau_1} \otimes h_{P_2}^{\epsilon_2} \rangle \frac{\mathbb{1}_{R_1}(x_1)}{|R_1|} \otimes h_{P_2}^{\epsilon_2}(x_2) \\
&\quad = \pi_{b;(0,1)}^{*} \Pi_{a;(0,1)} f.
\end{aligned}$$

<u>Case 2a</u>: $R_1 = Q_1$ and $\tau_1 \neq \delta_1$. Then (7-4) becomes

$$-\sum_{Q_1 \supsetneq P_1} \Big\langle b, h_{Q_1}^{\tau_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Big\rangle \frac{1}{\sqrt{|Q_1|}} h_{Q_1}^{\tau_1+\delta_1}(P_1) h_{Q_1}^{\delta_1}(x_1),$$

which leads to

$$\begin{aligned}
&\sum_{Q_1 \times P_2} \Big\langle b, h_{Q_1}^{\tau_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Big\rangle \frac{1}{\sqrt{|Q_1|}} h_{Q_1}^{\delta_1}(x_1) h_{P_2}^{\epsilon_2}(x_2) \sum_{P_1 \subsetneq Q_1} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) \Big\langle f, h_{P_1}^{\epsilon_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Big\rangle h_{Q_1}^{\tau_1+\delta_1}(P_1) \\
&\qquad = \sum_{Q_1 \times P_2} \Big\langle b, h_{Q_1}^{\tau_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Big\rangle \langle \Pi_{a;(0,1)} f, h_{Q_1}^{\tau_1+\delta_1} \otimes h_{P_2}^{\epsilon_2} \rangle \frac{1}{\sqrt{|Q_1|}} h_{Q_1}^{\delta_1}(x_1) \otimes h_{P_2}^{\epsilon_2}(x_2) \\
&\qquad = \gamma_{b;(0,1)} \Pi_{a;(0,1)} f.
\end{aligned}$$

<u>Case 2b</u>: $R_1 = Q_1$ and $\tau_1 = \delta_1$. Then (7-4) becomes

$$\sum_{Q_1 \supsetneq P_1} \Big\langle b, h_{Q_1}^{\delta_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Big\rangle \frac{1}{|Q_1|} h_{Q_1}^{\delta_1},$$

which gives rise to the term

$$T_{a,b}^{(0,1)} f := \sum_{Q_1 \times P_2} \Big\langle b, h_{Q_1}^{\delta_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Big\rangle h_{Q_1}^{\delta_1}(x_1) h_{P_2}^{\epsilon_2}(x_2) \frac{1}{|Q_1|} \sum_{P_1 \subsetneq Q_1} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) \Big\langle f, h_{P_1}^{\epsilon_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Big\rangle.$$

We have proved that

$$\Pi_{\Pi_{a;(0,1)} f} b - T = -\pi_{b;(0,1)}^{*} \Pi_{a;(0,1)} f - \gamma_{b;(0,1)} \Pi_{a;(0,1)} f - T_{a,b}^{(0,1)} f.$$

Expressing T instead as

$$T = \sum_{P_1 \times P_2} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) \biggl(\sum_{Q_2 \supsetneq P_2} \hat{f}(P_1^{\epsilon_1} \times Q_2^{\delta_2}) \langle b \rangle_{P_1 \times P_2} h_{Q_2}^{\delta_2}(P_2) \biggr) \frac{\mathbb{1}_{P_1}}{|P_1|} \otimes h_{P_2}^{\epsilon_2},$$

we are able to pair it with $\Pi_{a;(0,1)} \Pi_f b$. Then, a similar analysis yields

$$T - \Pi_{a;(0,1)} \Pi_f b = \Pi_{a;(0,1)} \pi_{b;(1,0)} f + \Pi_{a;(0,1)} \gamma_{b;(1,0)} f + T_{a,b}^{(1,0)} f,$$

where

$$T_{a,b}^{(1,0)} f := \sum_{P_1 \times P_2} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) \frac{\mathbb{1}_{P_1}(x_1)}{|P_1|} \otimes h_{P_2}^{\epsilon_2}(x_2) \biggl(\sum_{Q_2 \supsetneq P_2} \Bigl\langle b, \frac{\mathbb{1}_{P_1}}{|P_1|} \otimes h_{Q_2}^{\delta_2} \Bigr\rangle \hat{f}(P_1^{\epsilon_1} \times Q_2^{\delta_2}) \frac{1}{|Q_2|} \biggr).$$

Then

$$\mathcal{R}_{a,b}^{(0,1)} f = \Pi_{a;(0,1)} \pi_{b;(1,0)} f + \Pi_{a;(0,1)} \gamma_{b;(1,0)} f - \pi_{b;(0,1)}^{*} \Pi_{a;(0,1)} f - \gamma_{b;(0,1)} \Pi_{a;(0,1)} f + T_{a,b}^{(1,0)} f - T_{a,b}^{(0,1)} f.$$

It is now obvious that the first four terms are bounded as desired, and it remains to bound the terms $T_{a,b}$.

We look at $T_{a,b}^{(0,1)}$, for which we can write $\langle T_{a,b}^{(0,1)} f, g \rangle = \langle b, \phi \rangle$, where

$$\phi = \sum_{Q_1 \times P_2} \hat{g}(Q_1^{\delta_1} \times P_2^{\epsilon_2}) \frac{1}{|Q_1|} \biggl(\sum_{P_1 \subsetneq Q_1} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) \Bigl\langle f, h_{P_1}^{\epsilon_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Bigr\rangle \biggr) h_{Q_1}^{\delta_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|}.$$

Then $|\langle T_{a,b}^{(0,1)} f, g \rangle| \lesssim \|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)} \|S_{\mathcal{D}_1} \phi\|_{L^1(\nu)}$, and

$$S_{\mathcal{D}_1}^2 \phi = \sum_{Q_1} \biggl(\sum_{P_2} \hat{g}(Q_1^{\delta_1} \times P_2^{\epsilon_2}) \biggl(\frac{1}{|Q_1|} \sum_{P_1 \subsetneq Q_1} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) \Bigl\langle f, h_{P_1}^{\epsilon_2} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Bigr\rangle \biggr) \frac{\mathbb{1}_{P_2}(x_2)}{|P_2|} \biggr)^2 \frac{\mathbb{1}_{Q_1}(x_1)}{|Q_1|}.$$

Now,

$$\Bigl\langle \Pi_{a;(0,1)} f, \frac{\mathbb{1}_{Q_1}}{|Q_1|} \otimes h_{P_2}^{\epsilon_2} \Bigr\rangle = \sum_{P_1} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) \Bigl\langle f, h_{P_1}^{\epsilon_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Bigr\rangle \frac{|P_1 \cap Q_1|}{|P_1||Q_1|}.$$

Define the martingale transform $a \mapsto a_\tau = \sum_{P_1 \times P_2} \tau_{P_1,P_2}^{\epsilon_1,\epsilon_2} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2})$, where

$$\tau_{P_1,P_2}^{\epsilon_1,\epsilon_2} = \begin{cases} l+1 & \text{if } \langle f, h_{P_1}^{\epsilon_1} \otimes \mathbb{1}_{P_2}/|P_2| \rangle \geq 0, \\ -1 & \text{otherwise.} \end{cases}$$

Note that, while this transform does depend on f, in the end it will not matter, as this will be absorbed into the product BMO norm of a_τ. Then we have

$$\frac{1}{|Q_1|} \biggl| \sum_{P_1 \subsetneq Q_1} \hat{a}(P_1^{\epsilon_1} \times P_2^{\epsilon_2}) \Bigl\langle f, h_{P_1}^{\epsilon_1} \otimes \frac{\mathbb{1}_{P_2}}{|P_2|} \Bigr\rangle \biggr| \leq \Bigl\langle \Pi_{a_\tau;(0,1)} f, \frac{\mathbb{1}_{Q_1}}{|Q_1|} \otimes h_{P_2}^{\epsilon_2} \Bigr\rangle.$$

Returning to the square function estimate, we now have

$$\begin{aligned} S_{\mathcal{D}_1}^2 \phi &\leq \sum_{Q_1} \biggl(\sum_{P_2} |\hat{g}(Q_1^{\delta_1} \times P_2^{\epsilon_2})|^2 \frac{\mathbb{1}_{P_2}(x_2)}{|P_2|} \biggr) \biggl(\sum_{P_2} \langle |H_{P_2}^{\epsilon_2} \Pi_{a_\tau;(0,1)} f| \rangle_{Q_1}^2 \mathbb{1}_{Q_1}(x_1) \frac{\mathbb{1}_{P_2}(x_2)}{|P_2|} \biggr) \frac{\mathbb{1}_{Q_1}(x_1)}{|Q_1|} \\ &\leq S_{\boldsymbol{\mathcal{D}}}^2 g \biggl(\sum_{P_2} M_{\mathcal{D}_1}^2 (H_{P_2}^{\epsilon_2} \Pi_{a_\tau;(0,1)} f)(x_1) \frac{\mathbb{1}_{P_2}(x_2)}{|P_2|} \biggr) = S_{\boldsymbol{\mathcal{D}}}^2 g ([MS] \Pi_{a_\tau;(0,1)} f)^2. \end{aligned}$$

Finally,

$$\begin{aligned}\|S_{\mathcal{D}_1}\phi\|_{L^1(\nu)} &\le \|S_{\boldsymbol{\mathcal{D}}}g\|_{L^{p'}(\lambda')}\|[MS]\Pi_{a_\tau;(0,1)}f\|_{L^p(\mu)} \\ &\lesssim \|g\|_{L^{p'}(\lambda')}\underbrace{\|\Pi_{a_\tau;(0,1)}f\|_{L^p(\mu)}}_{\lesssim\|a_\tau\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\mathbb{R}^{\vec{n}})}\|f\|_{L^p(\mu)}} \lesssim \|a\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\mathbb{R}^{\vec{n}})}\|f\|_{L^p(\mu)}\|g\|_{L^{p'}(\lambda')},\end{aligned}$$

showing that

$$\|T^{(0,1)}_{a,b} : L^p(\mu)\to L^p(\lambda)\| \lesssim \|a\|_{\mathrm{BMO}_{\boldsymbol{\mathcal{D}}}(\mathbb{R}^{\vec{n}})}\|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)}.$$

The estimate for $T^{(1,0)}_{a,b}$ follows similarly. □

7D3. *The partial paraproducts.* We work with

$$\mathbb{S}^{i_1,j_1}_{\boldsymbol{\mathcal{D}}} f := \sum_{R_1\times R_2}\sum_{\substack{P_1\in(R_1)_{i_1}\\ Q_1\in(R_1)_{j_1}}} \hat{a}_{P_1Q_1R_1}(R_2^{\epsilon_2})\hat{f}(P_1^{\epsilon_1}\times R_2^{\epsilon_2})h^{\delta_1}_{Q_1}\otimes\frac{\mathbb{1}_{R_2}}{|R_2|},$$

where i_1, j_1 are nonnegative integers, and for every P_1, Q_1, R_1,

$$a_{P_1Q_1R_1}(x_2)\in\mathrm{BMO}(\mathbb{R}^{n_2})\quad\text{with } \|a_{P_1Q_1R_1}\|_{\mathrm{BMO}(\mathbb{R}^{n_2})}\le 2^{(-n_1/2)(i_1+j_1)}.$$

Theorem 7.5. *Let $\mu,\lambda\in A_p(\mathbb{R}^{\vec{n}})$, $1<p<\infty$ and $\nu:=\mu^{1/p}\lambda^{-1/p}$. Then*

$$\|[b,\mathbb{S}^{i_1,j_1}_{\boldsymbol{\mathcal{D}}}] : L^p(\mu)\to L^p(\lambda)\|\lesssim\|b\|_{\mathrm{bmo}_{\boldsymbol{\mathcal{D}}}(\nu)}.$$

First we need the one-weight bound for the partial paraproducts:

Proposition 7.6. *For any $w\in A_p(\mathbb{R}^{\vec{n}})$, $1<p<\infty$,*

$$\|\mathbb{S}^{i_1,j_1}_{\boldsymbol{\mathcal{D}}} : L^p(w)\to L^p(w)\|\lesssim 1. \tag{7-5}$$

Proof. Let $f\in L^p(w)$ and $g\in L^{p'}(w')$, and we will show that $|\langle\mathbb{S}^{i_1,j_1}_{\boldsymbol{\mathcal{D}}}f,g\rangle|\lesssim\|f\|_{L^p(w)}\|g\|_{L^{p'}(w')}$. First,

$$\begin{aligned}\|\langle\mathbb{S}^{i_1,j_1}_{\boldsymbol{\mathcal{D}}}f,g\rangle\| &\le \sum_{R_1}\sum_{\substack{P_1\in(R_1)_{i_1}\\ Q_1\in(R_1)_{j_2}}}|\langle a_{P_1Q_1R_1},\phi_{P_1Q_1R_1}\rangle_{\mathbb{R}^{n_2}}| \\ &\le \sum_{R_1}\sum_{\substack{P_1\in(R_1)_{i_1}\\ Q_1\in(R_1)_{j_2}}}\|a_{P_1Q_1R_1}\|_{\mathrm{BMO}(\mathbb{R}^{n_2})}\|S_{\mathcal{D}_2}\phi_{P_1Q_1R_1}\|_{L^1(\mathbb{R}^{n_2})} \\ &\le 2^{(-n_1/2)(i_1+j_1)}\sum_{R_1}\sum_{\substack{P_1\in(R_1)_{i_1}\\ Q_1\in(R_1)_{j_2}}}\|S_{\mathcal{D}_2}\phi_{P_1Q_1R_1}\|_{L^1(\mathbb{R}^{n_2})},\end{aligned}$$

where for every P_1, Q_1, R_1,

$$\phi_{P_1Q_1R_1}(x_2) := \sum_{R_2}\hat{f}(P_1\times R_2)\Big\langle g, h_{Q_1}\otimes\frac{\mathbb{1}_{R_2}}{|R_2|}\Big\rangle h_{R_2}(x_2).$$

Now,

$$S^2_{\mathcal{D}_2}\phi_{P_1Q_1R_1} = \sum_{R_2} |\widehat{H_{P_1}f}(R_2)|^2 \langle |H_{Q_1}g|\rangle^2_{R_2} \frac{\mathbb{1}_{R_2}(x_2)}{|R_2|} \le (M_{\mathcal{D}_2}H_{Q_1}g)^2(x_2)(S_{\mathcal{D}_2}H_{P_1}f)^2(x_2),$$

so

$$\begin{aligned}\sum_{R_1}\sum_{\substack{P_1\in(R_1)_{i_1}\\ Q_1\in(R_1)_{j_2}}} &\|S_{\mathcal{D}_2}\phi_{P_1Q_1R_1}\|_{L^1(\mathbb{R}^{n_2})}\\
&\le \sum_{R_1}\sum_{\substack{P_1\in(R_1)_{i_1}\\ Q_1\in(R_1)_{j_2}}} \int_{\mathbb{R}^{n_2}} (M_{\mathcal{D}_2}H_{Q_1}g)(x_2)(S_{\mathcal{D}_2}H_{P_1}f)(x_2)\,dx_2\\
&= \int_{\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_1}} \sum_{R_1}\sum_{\substack{P_1\in(R_1)_{i_1}\\ Q_1\in(R_1)_{j_2}}} (M_{\mathcal{D}_2}H_{Q_1}g)(x_2)(S_{\mathcal{D}_2}H_{P_1}f)(x_2)\frac{\mathbb{1}_{R_1}(x_1)}{|R_1|}\,dx_1\,dx_2\\
&\le \int_{\mathbb{R}^{\vec{n}}} \biggl(\sum_{R_1}\biggl(\sum_{P_1\in(R_1)_{i_1}} S_{\mathcal{D}_2}H_{P_1}f(x_2)\biggr)^2 \frac{\mathbb{1}_{R_1}(x_1)}{|R_1|}\biggr)^{1/2} \biggl(\sum_{R_1}\biggl(\sum_{Q_1\in(R_1)_{j_1}} M_{\mathcal{D}_2}H_{Q_1}g(x_2)\biggr)^2 \frac{\mathbb{1}_{R_1}(x_1)}{|R_1|}\biggr)^{1/2} dx\\
&= \int_{\mathbb{R}^{\vec{n}}} [SS_{\mathcal{D}_2}]^{i_1,0} f \cdot [SM_{\mathcal{D}_2}]^{j_1,0} g w^{1/p} w^{-1/p}\,dx.\end{aligned}$$

Then, from the estimates in (3-3),

$$\begin{aligned}\|\langle \mathbb{S}^{i_1,j_1}_{\boldsymbol{\mathcal{D}}} f, g\rangle\| &\le 2^{(-n_1/2)(i_1+j_1)} \|[SS_{\mathcal{D}_2}]^{i_1,0} f\|_{L^p(w)} \|[SM_{\mathcal{D}_2}]^{j_1,0} g\|_{L^{p'}(w')}\\
&\lesssim 2^{(-n_1/2)(i_1+j_1)} 2^{(n_1 i_1/2)} \|f\|_{L^p(w)} 2^{(n_1 j_1/2)} \|g\|_{L^{p'}(w')},\end{aligned}$$

and the result follows. □

Proof of Theorem 7.5. In light of (7-5), we only need to bound the remainder term

$$\mathcal{R}^{i_1,j_1} f := \Pi_{\mathbb{S}^{i_1,j_1}_{\boldsymbol{\mathcal{D}}} f} b - \mathbb{S}^{i_1,j_1}_{\boldsymbol{\mathcal{D}}} \Pi_f b.$$

The proof is somewhat similar to that of the full mixed paraproducts, in that we combine each of these terms

$$\Pi_{\mathbb{S}^{i_1,j_1}_{\boldsymbol{\mathcal{D}}} f} b = \sum_{R_1\times R_2}\sum_{\substack{P_1\in(R_1)_{i_1}\\ Q_1\in(R_1)_{j_1}}} \hat{a}_{P_1Q_1R_1}(R_2^{\epsilon_2}) \hat{f}(P_1^{\epsilon_1}\times R_2^{\epsilon_2}) \biggl(\sum_{Q_2\supsetneq R_2} \langle b\rangle_{Q_1\times Q_2} h^{\delta_2}_{Q_2}(R_2) h^{\delta_2}_{Q_2}(x_2)\biggr) h^{\delta_1}_{Q_1}(x_1),$$

$$\mathbb{S}^{i_1,j_1}_{\boldsymbol{\mathcal{D}}} \Pi_f b = \sum_{R_1\times R_2}\sum_{\substack{P_1\in(R_1)_{i_1}\\ Q_1\in(R_1)_{j_2}}} \hat{a}_{P_1Q_1R_1}(R_2^{\epsilon_2}) \hat{f}(P_1^{\epsilon_1}\times R_2^{\epsilon_2}) \langle b\rangle_{P_1\times R_2} h^{\delta_1}_{Q_1}(x_1) \otimes \frac{\mathbb{1}_{R_2}(x_2)}{|R_2|},$$

with a third term

$$T := \sum_{R_1\times R_2}\sum_{\substack{P_1\in(R_1)_{i_1}\\ Q_1\in(R_1)_{j_2}}} \hat{a}_{P_1Q_1R_1}(R_2^{\epsilon_2}) \hat{f}(P_1^{\epsilon_1}\times R_2^{\epsilon_2}) \langle b\rangle_{Q_1\times R_2} h^{\delta_1}_{Q_1} \otimes \frac{\mathbb{1}_{R_2}}{|R_2|}.$$

As before, expanding the indicator function in T into its Haar series, we may combine T with $\Pi_{\mathbb{S}_{\mathcal{D}}^{i_1,j_1} f} b$:

$$\Pi_{\mathbb{S}_{\mathcal{D}}^{i_1,j_1} f} b - T = \sum_{R_1\times R_2} \sum_{\substack{P_1\in(R_1)_{i_1}\\ Q_1\in(R_1)_{j_2}}} \hat{a}_{P_1Q_1R_1}(R_2^{\epsilon_2}) \hat{f}(P_1^{\epsilon_1}\times R_2^{\epsilon_2}) T_b(x_2) h_{Q_1}^{\delta_1}(x_1),$$

where

$$\begin{aligned} T_b(x_2) &= \sum_{Q_2\supsetneq R_2} (\langle b\rangle_{Q_1\times Q_2} - \langle b\rangle_{Q_1\times P_2}) h_{Q_2}^{\delta_2}(R_2) h_{Q_2}^{\delta_2}(x_2) \\ &= \sum_{Q_2\supsetneq R_2} \Bigg(\sum_{P_2: R_2\subsetneq P_2\subset Q_2} \bigg\langle b, \frac{\mathbb{1}_{Q_1}}{|Q_1|} \otimes h_{P_2}^{\tau_2} \bigg\rangle h_{P_2}^{\tau_2}(R_2) \Bigg) h_{Q_2}^{\delta_2}(R_2) h_{Q_2}^{\delta_2}(x_2). \end{aligned}$$

We analyze this term depending on the relationship of P_2 with Q_2.

Case 1: $P_2 \subsetneq Q_2$. Then

$$T_b(x_2) = \sum_{P_2\supsetneq R_2} \bigg\langle b, \frac{\mathbb{1}_{Q_1}}{|Q_1|} \otimes h_{P_2}^{\tau_2} \bigg\rangle h_{P_2}^{\tau_2}(R_2) \frac{\mathbb{1}_{P_2}(x_2)}{|P_2|},$$

which gives the operator

$$\begin{aligned} &\sum_{Q_1\times P_2} \bigg\langle b, \frac{\mathbb{1}_{Q_1}}{|Q_1|} \otimes h_{P_2}^{\tau_2} \bigg\rangle h_{Q_1}^{\tau_1}(x_1) \frac{\mathbb{1}_{P_2}(x_2)}{|P_2|} \Bigg(\sum_{P_1\in(Q_1^{(j_1)})_{i_1}} \sum_{R_2\subsetneq P_2} \hat{a}_{P_1Q_1R_1}(R_2^{\epsilon_2}) \widehat{H_{P_1}^{\epsilon_1} f}(R_2^{\epsilon_2}) h_{P_2}^{\tau_2}(R_2) \Bigg) \\ &\qquad = \sum_{Q_1\times P_2} \bigg\langle b, \frac{\mathbb{1}_{Q_1}}{|Q_1|} \otimes h_{P_2}^{\tau_2} \bigg\rangle h_{Q_1}^{\tau_1}(x_1) \frac{\mathbb{1}_{P_2}(x_2)}{|P_2|} \Bigg(\sum_{P_1\in(Q_1^{(j_1)})_{i_1}} \langle \Pi^*_{a_{P_1Q_1R_1}}(H_{P_1}^{\epsilon_1} f), h_{P_2}^{\tau_2}\rangle_{\mathbb{R}^{n_2}} \Bigg) \\ &\qquad = \pi^*_{b;(1,0)} F, \end{aligned}$$

where

$$F := \sum_{Q_1} \Bigg(\sum_{P_1\in(Q_1^{(j_1)})_{i_1}} \Pi^*_{a_{P_1Q_1R_1}}(H_{P_1}^{\epsilon_1} f)(x_2) \Bigg) h_{Q_1}^{\delta_1}(x_1).$$

Now

$$\|\pi^*_{b;(1,0)} F\|_{L^p(\lambda)} \lesssim \|b\|_{\mathrm{bmo}_{\mathcal{D}}(\nu)} \|F\|_{L^p(\mu)},$$

so we are done if we can show that

$$\|F\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)}. \tag{7-6}$$

Take $g \in L^{p'}(\mu')$. Then

$$|\langle F, g\rangle| \le \sum_{Q_1} \sum_{P_1\in(Q_1^{(j_1)})_{i_1}} |\langle \Pi^*_{a_{P_1Q_1R_1}}(H_{P_1}^{\epsilon_1} f), H_{Q_1}^{\delta_1} g\rangle_{\mathbb{R}^{n_2}}|.$$

Notice that we may write

$$\langle \Pi^*_{a_{P_1Q_1R_1}}(H_{P_1}^{\epsilon_1} f), H_{Q_1}^{\delta_1} g\rangle_{\mathbb{R}^{n_2}} = \langle a_{P_1Q_1R_1}, \phi_{P_1Q_1R_1}\rangle_{\mathbb{R}^{n_2}},$$

where

$$\phi_{P_1Q_1R_1}(x_2) = \sum_{R_2} \widehat{H_{P_1}^{\epsilon_1} f}(R_2^{\delta_2}) \langle H_{Q_1}^{\delta_1} g\rangle_{R_2} h_{R_2}^{\delta_2}(x_2).$$

Then

$$
\begin{aligned}
|\langle F, g\rangle| &\leq \sum_{Q_1} \sum_{P_1 \in (Q_1^{(j_1)})_{i_1}} \|a_{P_1 Q_1 R_1}\|_{\mathrm{BMO}(\mathbb{R}^{n_2})} \|S_{\mathcal{D}_2} \phi_{P_1 Q_1 R_1}\|_{L^1(\mathbb{R}^{n_2})} \\
&\leq 2^{(-n_1/2)(i_1+j_1)} \sum_{R_1} \sum_{\substack{P_1 \in (R_1)_{i_1} \\ Q_1 \in (R_1)_{j_2}}} \int_{\mathbb{R}^{n_2}} \biggl(\sum_{R_2} |\widehat{H^{\epsilon_1}_{P_1} f}(R_2^{\delta_2})|^2 \langle |H^{\delta_1}_{Q_1} g| \rangle^2_{R_2} \frac{\mathbb{1}_{R_2}(x_2)}{|R_2|} \biggr)^{1/2} dx_2 \\
&\leq 2^{(-n_1/2)(i_1+j_1)} \int_{\mathbb{R}^{\vec{n}}} \sum_{R_1} \sum_{\substack{P_1 \in (R_1)_{i_1} \\ Q_1 \in (R_1)_{j_2}}} (M_{\mathcal{D}_2} H^{\delta_1}_{Q_1} g)(x_2) (S_{\mathcal{D}_2} H^{\epsilon_1}_{P_1} f)(x_2) \frac{\mathbb{1}_{R_1}(x_1)}{|R_1|} \, dx.
\end{aligned}
$$

The integral above is bounded by

$$
\int_{\mathbb{R}^{\vec{n}}} \biggl(\sum_{R_1} \biggl(\sum_{P_1 \in (R_1)_{i_1}} (S_{\mathcal{D}_2} H^{\epsilon_1}_{P_1} f)(x_2) \biggr)^2 \frac{\mathbb{1}_{R_1}(x_1)}{|R_1|} \biggr)^{1/2} \biggl(\sum_{R_1} \biggl(\sum_{P_1 \in (R_1)_{i_1}} (S_{\mathcal{D}_2} H^{\epsilon_1}_{P_1} f)(x_2) \biggr)^2 \frac{\mathbb{1}_{R_1}(x_1)}{|R_1|} \biggr)^{1/2} dx
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^{\vec{n}}} ([SS_{\mathcal{D}_2}]^{i_1,0} f)([SM_{\mathcal{D}_2}]^{j_1,0} g) \, dx \leq \|[SS_{\mathcal{D}_2}]^{i_1,0} f\|_{L^p(\mu)} \|[SM_{\mathcal{D}_2}]^{j_1,0} g\|_{L^{p'}(\mu')} \\
&\lesssim 2^{(n_1/2)(i_1+j_1)} \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\mu')} \quad \text{by (3-3).}
\end{aligned}
$$

The desired estimate in (7-6) is now proved.

Case 2a: $P_2 = Q_2$ and $\tau_2 \neq \delta_2$. Then

$$
T_b(x_2) = \sum_{Q_2 \supsetneq R_2} \Bigl\langle b, \frac{\mathbb{1}_{Q_1}}{|Q_1|} \otimes h^{\tau_2}_{Q_2} \Bigr\rangle \frac{1}{\sqrt{|Q_2|}} h^{\tau_2+\delta_2}_{Q_2}(R_2) h^{\delta_2}_{Q_2}(x_2),
$$

giving rise to the operator

$$
\sum_{Q_1 \times Q_2} \Bigl\langle b, \frac{\mathbb{1}_{Q_1}}{|Q_1|} \otimes h^{\tau_2}_{Q_2} \Bigr\rangle \biggl(\sum_{P_1 \in (Q_1^{(j_1)})_{i_1}} \langle \Pi^*_{a_{P_1 Q_1 R_1}} (H^{\epsilon_1}_{P_1} f), h^{\tau_2+\delta_2}_{Q_2} \rangle_{\mathbb{R}^{n_2}} \biggr) \frac{1}{\sqrt{|Q_2|}} h^{\delta_1}_{Q_1} \otimes h^{\delta_2}_{Q_2} = \gamma_{b;(1,0)} F,
$$

which is handled as in the previous case.

Case 2b: $P_2 = Q_2$ and $\tau_2 = \delta_2$. In this case, $T_b(x_2)$ gives rise to the operator

$$
T' := \sum_{Q_1 \times Q_2} \Bigl\langle b, \frac{\mathbb{1}_{Q_1}}{|Q_1|} \otimes h^{\delta_2}_{Q_2} \Bigr\rangle h^{\delta_1}_{Q_1} \otimes h^{\delta_2}_{Q_2} \sum_{P_1 \in (Q_1^{(j_1)})_{i_1}} \frac{1}{|Q_2|} \sum_{R_2 \subsetneq Q_2} \hat{a}_{P_1 Q_1 R_1}(R_2^{\epsilon_2}) \widehat{H^{\epsilon_1}_{P_1} f}(R_2^{\epsilon_2}).
$$

Now define

$$
F_\tau := \sum_{Q_1} \biggl(\sum_{P_1 \in (Q_1^{(j_1)})_{i_1}} \Pi^*_{a^\tau_{P_1 Q_1 R_1}} (H^{\epsilon_1}_{P_1} f)(x_2) \biggr) h^{\delta_1}_{Q_1}(x_1),
$$

just as we defined F before, except now to every function $a_{P_1 Q_1 R_1}$ we apply the martingale transform

$$
a_{P_1 Q_1 R_1} \mapsto a^\tau_{P_1 Q_1 R_1} = \sum_{R_2} \tau^{\epsilon_2}_{R_2} \hat{a}_{P_1 Q_1 R_1}(R_2^{\epsilon_2}) h^{\epsilon_2}_{R_2}, \quad \text{where } \tau^{\epsilon_2}_{R_2} := \begin{cases} +1 & \text{if } \widehat{H^{\epsilon_1}_{P_1} f}(R_2^{\epsilon_2}) \geq 0, \\ -1 & \text{otherwise.} \end{cases}
$$

Since this does not increase the BMO($\mathbb{R}^{n_2}$) norms of the $a_{P_1Q_1R_1}$ functions, the estimate (7-6) still holds: $\|F_\tau\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)}$.

Moreover, note that

$$\langle \Pi^*_{a^\tau_{P_1Q_1R_1}}(H^{\epsilon_1}_{P_1} f)\rangle_{Q_2} = \sum_{R_2} \underbrace{\hat{a}^\tau_{P_1Q_1R_1}(R_2^{\epsilon_2})\widehat{H^{\epsilon_1}_{P_1} f}(R_2^{\epsilon_2})}_{\geq 0} \frac{|R_2\cap Q_2|}{|R_2||Q_2|}$$

and that

$$\pi_{b;(1,0)}F_\tau = \sum_{Q_1\times Q_2} \left\langle b, \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes h^{\delta_2}_{Q_2}\right\rangle \sum_{P_1\in (Q_1^{(j_1)})_{i_1}} \langle \Pi^*_{a^\tau_{P_1Q_1R_1}}(H^{\epsilon_1}_{P_1} f)\rangle_{Q_2} h^{\delta_1}_{Q_1}\otimes h^{\delta_2}_{Q_2}.$$

Then

$$\begin{aligned} S^2_{\mathcal{D}} T' &\leq \sum_{Q_1\times Q_2} \left|\left\langle b, \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes h^{\delta_2}_{Q_2}\right\rangle\right|^2 \Biggl(\sum_{P_1\in (Q_1^{(j_1)})_{i_1}} \frac{1}{|Q_2|}\sum_{R_2\subsetneq Q_2} |\hat{a}_{P_1Q_1R_1}(R_2^{\epsilon_2})\widehat{H^{\epsilon_1}_{P_1} f}(R_2^{\epsilon_2})|\Biggr)^2 \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|} \\ &\leq \sum_{Q_1\times Q_2} \left|\left\langle b, \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes h^{\delta_2}_{Q_2}\right\rangle\right|^2 \Biggl(\sum_{P_1\in (Q_1^{(j_1)})_{i_1}} \langle \Pi^*_{a^\tau_{P_1Q_1R_1}}(H^{\epsilon_1}_{P_1} f)\rangle_{Q_2}\Biggr)^2 \frac{\mathbb{1}_{Q_1}}{|Q_1|}\otimes\frac{\mathbb{1}_{Q_2}}{|Q_2|} \\ &= S^2_{\mathcal{D}}(\pi_{b;(1,0)}F_\tau). \end{aligned}$$

Finally, this gives us

$$\begin{aligned} \|T'\|_{L^p(\lambda)} &\simeq \|S_{\mathcal{D}}T'\|_{L^p(\lambda)} \leq \|S_{\mathcal{D}}\pi_{b;(1,0)}F_\tau\|_{L^p(\lambda)} \simeq \|\pi_{b;(1,0)}F_\tau\|_{L^p(\lambda)} \lesssim \|b\|_{\mathrm{bmo}_{\mathcal{D}}(\nu)}\|F_\tau\|_{L^p(\mu)} \\ &\lesssim \|b\|_{\mathrm{bmo}_{\mathcal{D}}(\nu)}\|f\|_{L^p(\mu)}. \end{aligned}$$

This proves that $\Pi_{\mathbb{S}^{i_1,j_1}_{\mathcal{D}} f} b - T$ obeys the desired bound, and the case $T - \mathbb{S}^{i_1,j_1}_{\mathcal{D}}\Pi_f b$ is handled similarly. □

7E. ***Proof of Theorem 1.4.*** Having now proved all the one-weight inequalities for dyadic shifts, we may conclude that

$$\|\mathbb{S}^{\vec{i},\vec{j}}_{\mathcal{D}} : L^p(w)\to L^p(w)\| \lesssim 1$$

for all $w\in A_p(\mathbb{R}^{\vec{n}})$. For the cancellative shifts, this was proved in (7-2). For the noncancellative shifts, the first two types are simply paraproducts with symbol $\|a\|_{\mathrm{BMO}_{\mathcal{D}}(\mathbb{R}^{\vec{n}})} \leq 1$, while the third type, a partial paraproduct, was proved to be bounded on $L^p(w)$ in Proposition 7.6.

Theorem 1.4 now follows trivially from Martikainen's representation theorem, Theorem 7.1: Take $f\in L^p(w)$ and $g\in L^{p'}(w')$. Then

$$\begin{aligned} |\langle Tf, g\rangle| &\leq C_T \mathbb{E}_{\omega_1}\mathbb{E}_{\omega_2} \sum_{\vec{i},\vec{j}\in\mathbb{Z}^2_+} 2^{-\max(i_1,j_1)\delta/2} 2^{-\max(i_2,j_2)\delta/2} |\langle \mathbb{S}^{\vec{i},\vec{j}}_{\mathcal{D}} f, g\rangle| \\ &\lesssim \|f\|_{L^p(w)}\|g\|_{L^{p'}(w')} \sum_{\vec{i},\vec{j}\in\mathbb{Z}^2_+} 2^{-\max(i_1,j_1)\delta/2} 2^{-\max(i_2,j_2)\delta/2} \\ &\simeq \|f\|_{L^p(w)}\|g\|_{L^{p'}(w')}. \end{aligned}$$

□

8. The unweighted case of higher-order Journé commutators

Here is the definition of the BMO spaces which are in between little BMO and product BMO.

Let $b : \mathbb{R}^{\vec{d}} \to \mathbb{C}$ with $\vec{d} = (d_1, \dots, d_t)$. Take a partition $\mathcal{I} = \{I_s : 1 \le s \le l\}$ of $\{1, 2, \dots, t\}$ so that $\dot{\bigcup}_{1 \le s \le l} I_s = \{1, 2, \dots, t\}$. We say that $b \in \mathrm{BMO}_{\mathcal{I}}(\mathbb{R}^{\vec{d}})$ if for any choice $\boldsymbol{v} = (v_s)$, $v_s \in I_s$, we have b is uniformly in product BMO in the variables indexed by v_s. We call a BMO space of this type a "little product BMO". If for any $\vec{x} = (x_1, \dots, x_t) \in \mathbb{R}^{\vec{d}}$ we define $\vec{x}_{\hat{\boldsymbol{v}}}$ by removing those variables indexed by v_s, the little product BMO norm becomes

$$\|b\|_{\mathrm{BMO}_{\mathcal{I}}} = \max_{\boldsymbol{v}} \bigl\{ \sup_{\vec{x}_{\hat{\boldsymbol{v}}}} \|b(\vec{x}_{\hat{\boldsymbol{v}}})\|_{\mathrm{BMO}} \bigr\},$$

where the BMO norm is product BMO in the variables indexed by v_s.

In [Ou et al. 2016] it was proved that commutators involving tensor products of Riesz transforms in L^p are a testing class for these BMO spaces:

Theorem 8.1 (Ou, Petermichl and Strouse). *Let $\vec{j} = (j_1, \dots, j_t)$ with $1 \le j_k \le d_k$ and let for each $1 \le s \le l$, $\vec{j}^{(s)} = (j_k)_{k \in I_s}$ be associated a tensor product of Riesz transforms $\vec{R}_{s, \vec{j}^{(s)}} = \bigotimes_{k \in I_s} R_{k, j_k}$; here R_{k, j_k} are j_k-th Riesz transforms acting on functions defined on the k-th variable. We have the two-sided estimate*

$$\|b\|_{\mathrm{BMO}_{\mathcal{I}}(\mathbb{R}^{\vec{d}})} \lesssim \sup_{\vec{j}} \|[\vec{R}_{1, \vec{j}^{(1)}}, \dots, [\vec{R}_{t, \vec{j}^{(t)}}, b], \dots]\|_{L^p(\mathbb{R}^{\vec{d}}) \to L^p(\mathbb{R}^{\vec{d}})} \lesssim \|b\|_{\mathrm{BMO}_{\mathcal{I}}(\mathbb{R}^{\vec{d}})}.$$

It was also proved that the estimate self-improves to paraproduct-free Journé commutators in L^2, in the sense T is paraproduct free $T(1 \otimes \cdot\,) = T(\,\cdot \otimes 1) = T^*(1 \otimes \cdot\,) = T^*(\,\cdot \otimes 1) = 0$.

Theorem 8.2 (Ou, Petermichl and Strouse). *Let us consider $\mathbb{R}^{\vec{d}}$, $\vec{d} = (d_1, \dots, d_t)$, with a partition $\mathcal{I} = (I_s)_{1 \le s \le l}$ of $\{1, \dots, t\}$ as discussed before. Let $b \in \mathrm{BMO}_{\mathcal{I}}(\mathbb{R}^{\vec{d}})$ and let T_s denote a multiparameter paraproduct-free Journé operator acting on function defined on $\bigotimes_{k \in I_s} \mathbb{R}^{d_k}$. Then we have the estimate*

$$\|[T_1, \dots, [T_l, b], \dots]\|_{L^2(\mathbb{R}^{\vec{d}}) \to L^2(\mathbb{R}^{\vec{d}})} \lesssim \|b\|_{\mathrm{BMO}_{\mathcal{I}}(\mathbb{R}^{\vec{d}})}.$$

This estimate was generalized somewhat in [Ou and Petermichl 2018] in that the paraproduct-free condition was slightly weakened; the considerations in the present text in combination with arguments from [Dalenc and Ou 2016; Ou et al. 2016] to pass to the iterated case, readily give us the following full result, for all Journé operators and all p:

Theorem 8.3. *Let us consider $\mathbb{R}^{\vec{d}}$, $\vec{d} = (d_1, \dots, d_t)$, with a partition $\mathcal{I} = (I_s)_{1 \le s \le l}$ of $\{1, \dots, t\}$ as discussed before. Let $b \in \mathrm{BMO}_{\mathcal{I}}(\mathbb{R}^{\vec{d}})$ and let T_s denote a multiparameter Journé operator acting on functions defined on $\bigotimes_{k \in I_s} \mathbb{R}^{d_k}$. Then we have the estimate*

$$\|[T_1, \dots, [T_l, b], \dots]\|_{L^p(\mathbb{R}^{\vec{d}}) \to L^p(\mathbb{R}^{\vec{d}})} \lesssim \|b\|_{\mathrm{BMO}_{\mathcal{I}}(\mathbb{R}^{\vec{d}})}.$$

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ESTIMATES FOR EIGENVALUES OF AHARONOV–BOHM OPERATORS WITH VARYING POLES AND NON-HALF-INTEGER CIRCULATION

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We study the behavior of eigenvalues of a magnetic Aharonov–Bohm operator with non-half-integer circulation and Dirichlet boundary conditions in a planar domain. As the pole is moving in the interior of the domain, we estimate the rate of the eigenvalue variation in terms of the vanishing order of the limit eigenfunction at the limit pole. We also provide an accurate blow-up analysis for scaled eigenfunctions and prove a sharp estimate for their rate of convergence.

1. Introduction and statement of the main results

An infinitely long, thin solenoid perpendicular to the plane (x_1, x_2) at the point $a = (a_1, a_2) \in \mathbb{R}^2$ produces a point-like magnetic field as the radius of the solenoid goes to zero and the magnetic flux remains constantly equal to $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. This magnetic field is a $2\pi\alpha$-multiple of the Dirac delta at a orthogonal to the plane (x_1, x_2) and is generated by the Aharonov–Bohm vector potential

$$A_a(x) = \alpha\left(-\frac{x_2 - a_2}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\right), \quad x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{a\};$$

see, e.g., [Adami and Teta 1998; Aharonov and Bohm 1959; Melgaard et al. 2004]. We are interested in the spectral properties of Schrödinger operators with Aharonov–Bohm vector potentials, i.e., of operators

$$(i\nabla + A_a)^2 := -\Delta + 2iA_a \cdot \nabla + |A_a|^2.$$

Since curl $A_a \equiv 0$ in $\mathbb{R}^2 \setminus \{a\}$, the magnetic field is concentrated at the pole a. If the circulation α is an integer number, then the potential A_a can be gauged away by a phase transformation so that the operator $(i\nabla + A_a)^2$ becomes spectrally equivalent to the standard Laplacian. On the other hand, if $\alpha \notin \mathbb{Z}$, the vector potential A_a cannot be eliminated by gauge transformations and the spectrum of the operator is modified by the presence of the magnetic field: this produces the so-called Aharonov–Bohm effect; i.e., the magnetic potential affects charged quantum particles moving in the region $\Omega \setminus \{a\}$, even if the magnetic field $B_a = \operatorname{curl} A_a$ is zero there.

The dependence on the pole a of the spectrum of the Schrödinger operator $(i\nabla + A_a)^2$ in a bounded domain Ω was investigated in [Abatangelo and Felli 2015; 2016; Abatangelo et al. 2017; Bonnaillie-Noël et al. 2014; Noris et al. 2015; Noris and Terracini 2010] under homogeneous Dirichlet boundary conditions. In particular, in [Abatangelo and Felli 2015; 2016] sharp asymptotic estimates for eigenvalues were

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given in the case of half-integer circulation $\alpha \in \mathbb{Z} + \frac{1}{2}$ as the pole a moves towards a fixed point $\bar{a} \in \Omega$; analogous sharp estimates were derived in [Abatangelo et al. 2017] in the case $\bar{a} \in \partial\Omega$. We mention that the continuous dependence of the eigenvalue function on the position of the pole and improved regularity results under simplicity assumptions were established in [Bonnaillie-Noël et al. 2014; Léna 2015] for any value of α (in particular also for non-half-integer circulation); on the other hand, to the best of our knowledge, sharp estimates of the gap of eigenvalues have been investigated only in the case $\alpha \in \mathbb{Z} + \frac{1}{2}$; see [Abatangelo and Felli 2015; 2016; Abatangelo et al. 2017; Bonnaillie-Noël et al. 2014; Noris et al. 2015].

The case $\alpha \in \mathbb{Z} + \frac{1}{2}$ studied in the aforementioned papers presents several peculiarities which allow one to approach the problem with a perspective and a technique that are not completely adaptable to a general circulation $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Indeed, if $\alpha \in \mathbb{Z} + \frac{1}{2}$, the problem can be reduced by a gauge transformation to the case $\alpha = \frac{1}{2}$ and, in this case, the eigenfunctions of $(i\nabla + A_a)^2$ can be identified, up to a complex phase, with the antisymmetric eigenfunctions of the Laplace–Beltrami operator on the twofold covering manifold of Ω; see [Helffer et al. 1999; Noris and Terracini 2010]. As a consequence, if $\alpha = \frac{1}{2}$, the magnetic eigenfunctions have an odd number of nodal lines ending at the pole a. It has been proved in [Helffer and Hoffmann-Ostenhof 2013] that the corresponding nodal domains are related to optimal partition problems. We refer to [Bonnaillie-Noël et al. 2009] for related numerical simulations.

The special features characterizing Aharonov–Bohm operators with circulation $\frac{1}{2}$ played a crucial role in [Abatangelo and Felli 2015; 2016; Abatangelo et al. 2017; Bonnaillie-Noël et al. 2014; Noris et al. 2015; Noris and Terracini 2010]. In particular, in [Noris et al. 2015] local energy estimates for eigenfunctions near the limit pole are performed by studying an Almgren-type quotient, see [Almgren 1983], which is estimated using a representation formula by Green's functions for solutions to the corresponding Laplace problem on the twofold covering. Moreover, in [Abatangelo and Felli 2015; 2016; Abatangelo et al. 2017] a limit profile vanishing on the special directions determined by the nodal lines of limit eigenfunctions is constructed: this allows one to establish a sharp relation between the asymptotics of the eigenvalue function and the number of nodal lines, which is strongly related to the order of vanishing of the limit eigenfunction.

In this paper we will focus on the case of noninteger and non-half-integer circulation; i.e., we will assume $\alpha \in \mathbb{R} \setminus (\mathbb{Z}/2)$. A reduction to the Laplacian on the twofold covering manifold is no longer available in this case; moreover, magnetic eigenfunctions vanish at the pole a but they do not have nodal lines ending at a (see Proposition 2.1). The lack of the special features of Aharonov–Bohm operators with half-integer circulation described above requires alternative methods and produces a less precise estimate. In particular, in order to estimate the Almgren frequency function, we will give a detailed description of the behavior of eigenfunctions at the pole and we will study the dependence of the coefficients of their asymptotic expansion with respect to the moving pole a, see Lemma 2.2.

By gauge invariance, if $\alpha \in \mathbb{R} \setminus (\mathbb{Z}/2)$ it is not restrictive to assume that

$$\alpha \in (0,1) \setminus \{\tfrac{1}{2}\}. \tag{1-1}$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain. For every $a \in \Omega$, we introduce the functional space $H^{1,a}(\Omega, \mathbb{C})$ as the completion of

$$\{u \in H^1(\Omega, \mathbb{C}) \cap C^\infty(\Omega, \mathbb{C}) : u \text{ vanishes in a neighborhood of } a\}$$

with respect to the norm

$$\|u\|_{H^{1,a}(\Omega,\mathbb{C})} = \left(\|\nabla u\|^2_{L^2(\Omega,\mathbb{C}^2)} + \|u\|^2_{L^2(\Omega,\mathbb{C})} + \left\|\frac{u}{|x-a|}\right\|^2_{L^2(\Omega,\mathbb{C})}\right)^{1/2}. \tag{1-2}$$

The norm (1-2) is equivalent, under condition (1-1), to the norm

$$\left(\|(i\nabla + A_a)u\|^2_{L^2(\Omega,\mathbb{C}^2)} + \|u\|^2_{L^2(\Omega,\mathbb{C})}\right)^{1/2},$$

in view of the Hardy-type inequality proved in [Laptev and Weidl 1999], see also [Alziary et al. 2003] and [Felli et al. 2011, Lemma 3.1 and Remark 3.2],

$$\int_{D_r(a)} |(i\nabla + A_a)u|^2\,dx \geq \left(\min_{j\in\mathbb{Z}} |j-\alpha|\right)^2 \int_{D_r(a)} \frac{|u(x)|^2}{|x-a|^2}\,dx, \tag{1-3}$$

which holds for all $r>0,\ a\in\mathbb{R}^2$ and $u\in H^{1,a}(D_r(a),\mathbb{C})$. Here we denote by $D_r(a)$ the disk of center a and radius r; we will denote by $D_r := D_r(0)$ the disk with radius r centered at the origin.

It is also worth mentioning the following formulation of the magnetic Hardy inequality proved in [Alziary et al. 2003, Lemma 4.1]: for all $r_1 > r_2 > 0,\ a\in\mathbb{R}^2$, and $u\in H^{1,a}(D_{r_1}(a)\setminus D_{r_2}(a),\mathbb{C})$,

$$\int_{D_{r_1}(a)\setminus D_{r_2}(a)} |(i\nabla + A_a)u|^2\,dx \geq \left(\min_{j\in\mathbb{Z}} |j-\alpha|\right)^2 \int_{D_{r_1}(a)\setminus D_{r_2}(a)} \frac{|u(x)|^2}{|x-a|^2}\,dx. \tag{1-4}$$

We also consider the space $H^{1,a}_0(\Omega,\mathbb{C})$ as the completion of $C^\infty_c(\Omega\setminus\{a\},\mathbb{C})$ with respect to the norm $\|\cdot\|_{H^1_a(\Omega,\mathbb{C})}$, so that

$$H^{1,a}_0(\Omega,\mathbb{C}) = \left\{u\in H^1_0(\Omega,\mathbb{C}) : \frac{u}{|x-a|}\in L^2(\Omega,\mathbb{C})\right\}.$$

From classical spectral theory, for every $a\in\Omega$, the eigenvalue problem

$$\begin{cases}(i\nabla + A_a)^2 u = \lambda u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega\end{cases} \tag{E_a}$$

admits a diverging sequence of real eigenvalues $\{\lambda^a_k\}_{k\geq 1}$ with finite multiplicity; in the enumeration

$$\lambda^a_1 \leq \lambda^a_2 \leq \cdots \leq \lambda^a_j \leq \cdots,$$

we repeat each eigenvalue as many times as its multiplicity. We are interested in the behavior of the function $a\mapsto\lambda^a_j$ in a neighborhood of a fixed point $\bar a\in\Omega$. Up to a translation and a dilation, it is not restrictive to assume that $\bar a = 0\in\Omega$ and $\overline{D}_2\subset\Omega$.

Let us assume that there exists $n_0\geq 1$ such that

$$\lambda^0_{n_0} \text{ is simple,} \tag{1-5}$$

and define

$$\lambda_0 = \lambda^0_{n_0} \quad\text{and}\quad \lambda_a = \lambda^a_{n_0}$$

for any $a \in \Omega$. In [Léna 2015, Theorem 1.3] it is proved that

$$\text{if } \lambda_j^0 \text{ is simple, the function } a \mapsto \lambda_j^a \text{ is \textit{analytic} in a neighborhood of } 0. \tag{1-6}$$

In particular the function $a \mapsto \lambda_a$ is continuous and, if $a \to 0$, then $\lambda_a \to \lambda_0$; see also [Bonnaillie-Noël et al. 2014]. Let $\varphi_0 \in H_0^{1,0}(\Omega, \mathbb{C}) \setminus \{0\}$ be an $L^2(\Omega, \mathbb{C})$-normalized eigenfunction of problem (E_0) associated to the eigenvalue $\lambda_0 = \lambda_{n_0}^0$, i.e., satisfying

$$\begin{cases} (i\nabla + A_0)^2 \varphi_0 = \lambda_0 \varphi_0 & \text{in } \Omega, \\ \varphi_0 = 0 & \text{on } \partial\Omega, \\ \int_\Omega |\varphi_0(x)|^2 \, dx = 1. \end{cases} \tag{1-7}$$

From [Felli et al. 2011, Theorem 1.3] (see also Proposition 2.1) it is known that

$$\varphi_0 \text{ vanishes at } 0 \text{ with a vanishing order equal to } |\alpha - k| \text{ for some } k \in \mathbb{Z}, \tag{1-8}$$

in the sense that there exist $k \in \mathbb{Z}$ and $\beta \in \mathbb{C} \setminus \{0\}$ such that

$$r^{-|\alpha-k|} \varphi_0(r(\cos t, \sin t)) \to \beta \frac{e^{ikt}}{\sqrt{2\pi}} \quad \text{in } C^{1,\tau}([0, 2\pi], \mathbb{C}) \tag{1-9}$$

as $r \to 0^+$ for any $\tau \in (0, 1)$.

Our first result provides an estimate of the rate of convergence of $\lambda_0 - \lambda_a$ in terms of the order of vanishing of φ_0 at 0; in particular we have that higher vanishing orders imply faster convergence of eigenvalues.

Theorem 1.1. *Let $\alpha \in (0, 1) \setminus \{\frac{1}{2}\}$ and $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain such that $0 \in \Omega$. Let $n_0 \in \mathbb{N}$ be such that the n_0-th eigenvalue $\lambda_{n_0}^0 = \lambda_0$ of problem (E_0) is simple and let $\varphi_0 \in H_0^{1,0}(\Omega, \mathbb{C})$ be an associated eigenfunction satisfying* (1-7)*. Let $k \in \mathbb{Z}$ be such that $|\alpha - k|$ is the order of vanishing of φ_0 at 0 as in* (1-9)*. For $a \in \Omega$, let $\lambda_{n_0}^a = \lambda_a$ be the n_0-th eigenvalue of problem (E_a). Then*

$$|\lambda_a - \lambda_0| = O(|a|^{1+\lfloor 2|\alpha-k| \rfloor}) \quad \text{as } |a| \to 0,$$

where $\lfloor \cdot \rfloor$ denotes the floor function $\lfloor t \rfloor := \max\{k \in \mathbb{Z} : k \leq t\}$.

To prove Theorem 1.1, we will study the quotient

$$\frac{\lambda_0 - \lambda_a}{|a|^{2|\alpha-k|}} \tag{1-10}$$

as a approaches the origin along a straight line $\{tp : t > 0\}$ for any direction

$$p \in \mathbb{S}^1 := \{x \in \mathbb{R}^2 : |x| = 1\}.$$

We will prove that, for every $p \in \mathbb{S}^1$, the quotient (1-10) is bounded as $a = |a|p \to 0$. Then (1-6) and the fact that $2|\alpha - k|$ is noninteger imply that the Taylor polynomials of the function $\lambda_0 - \lambda_a$ with center 0 and degree less than or equal to $\lfloor 2|\alpha - k| \rfloor$ vanish, thus yielding the conclusion of Theorem 1.1.

In the case of half-integer circulation $\alpha = \frac{1}{2}$ the special nodal structure of the limit problem allows us to prove instead that the limit

$$\lim_{a=|a|p\to 0} \frac{\lambda_0-\lambda_a}{|a|^{2|\alpha-k|}} = \lim_{a=|a|p\to 0} \frac{\lambda_0-\lambda_a}{|a|^{|1-2k|}}$$

is different from 0 along some special directions p corresponding to tangents to the nodal lines of the limit eigenfunction. As a consequence, the leading term of the Taylor expansion of the eigenvalue variation $\lambda_0 - \lambda_a$ has order exactly $|1-2k|$. That is,

$$\lambda_0 - \lambda_a = P(a) + o(|a|^{|1-2k|}) \quad \text{as } |a| \to 0^+$$

for some homogeneous polynomial $P \not\equiv 0$ of degree $|1-2k|$; see [Abatangelo and Felli 2015, Theorem 1.2]. In [Abatangelo and Felli 2016, Theorem 2] the exact values of all coefficients of the polynomial P are determined, proving that $P(|a|(\cos t, \sin t)) = C_0|a|^{|1-2k|}\cos(|1-2k|(t-t_0))$ for some t_0 and $C_0 > 0$. In particular the leading polynomial P is harmonic.

In this paper we will also describe the behavior of the eigenfunctions as $a \to 0$, proving a blow-up result for scaled eigenfunctions and giving a sharp rate of the convergence to the limit eigenfunction φ_0. In order to state these results more precisely, we need to introduce some notation.

For every $b = (b_1, b_2) = |b|(\cos\vartheta, \sin\vartheta) \in \mathbb{R}^2 \setminus \{0\}$ with $\vartheta \in [0, 2\pi)$, we define the polar angle centered at b, $\theta_b : \mathbb{R}^2 \setminus \{b\} \to [\vartheta, \vartheta + 2\pi)$ as

$$\theta_b(b + r(\cos t, \sin t)) = t \quad \text{for all } r > 0 \text{ and } t \in [\vartheta, \vartheta + 2\pi), \tag{1-11}$$

and the function $\theta_0^b : \mathbb{R}^2 \setminus \{0\} \to [\vartheta, \vartheta + 2\pi)$ as

$$\theta_0^b(r(\cos t, \sin t)) = t \quad \text{for all } r > 0 \text{ and } t \in [\vartheta, \vartheta + 2\pi). \tag{1-12}$$

We remark that θ_b is discontinuous on the half-line starting at b with slope $\vartheta = \mathrm{Arg}(b)$, whereas θ_0^b is discontinuous on the half-line starting from 0 with the same slope; in particular, the range of θ_0^b depends on $\vartheta = \mathrm{Arg}(b)$. Hence, the difference function $\theta_0^b - \theta_b$ is regular except for the segment

$$\Gamma_b := \{tb : t \in [0, 1]\}. \tag{1-13}$$

For all $a \in \Omega$, let $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C}) \setminus \{0\}$ be an eigenfunction of problem (E_a) associated to the eigenvalue λ_a, i.e., solving

$$\begin{cases} (i\nabla + A_a)^2 \varphi_a = \lambda_a \varphi_a & \text{in } \Omega, \\ \varphi_a = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-14}$$

such that the following normalization conditions hold:

$$\int_\Omega |\varphi_a(x)|^2\,dx = 1 \quad \text{and} \quad \int_\Omega e^{i\alpha(\theta_0^a - \theta_a)(x)} \varphi_a(x)\overline{\varphi_0(x)}\,dx \text{ is a positive real number.} \tag{1-15}$$

Using (1-5), (1-7), (1-14), (1-15), and standard elliptic estimates, see, e.g., [Gilbarg and Trudinger 1983, Theorem 8.10], it is easy to prove that

$$\varphi_a \to \varphi_0 \quad \text{in } H^1(\Omega, \mathbb{C}) \text{ and in } C^2_{\mathrm{loc}}(\Omega \setminus \{0\}, \mathbb{C}), \tag{1-16}$$

and

$$(i\nabla + A_a)\varphi_a \to (i\nabla + A_0)\varphi_0 \quad \text{in } L^2(\Omega, \mathbb{C}). \tag{1-17}$$

To give a precise description of the behavior of the eigenfunction φ_a for a close to 0, we consider a homogeneous scaling of order $|a|^{|\alpha-k|}$ of φ_a along a fixed direction $p \in \mathbb{S}^1$. Theorem 1.2 below gives the convergence of scaled eigenfunctions to a nontrivial limit profile $\Psi_p \in H^{1,p}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C})$, which can be characterized as the unique solution to the problem

$$(i\nabla + A_p)^2 \Psi_p = 0 \quad \text{in } \mathbb{R}^2 \text{ in a weak } H^{1,p}\text{-sense}, \tag{1-18}$$

satisfying

$$\int_{\mathbb{R}^2 \setminus D_1} |(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p - \theta_0^p)}\psi_k)|^2 \, dx < +\infty, \tag{1-19}$$

where $\psi_k : \mathbb{R}^2 \to \mathbb{C}$ is defined as

$$\psi_k(r(\cos t, \sin t)) = r^{|\alpha-k|} \frac{e^{ikt}}{\sqrt{2\pi}}. \tag{1-20}$$

The existence and uniqueness of a limit profile satisfying (1-18) and (1-19) will be proved in Lemma 5.3. We notice that the function ψ_k in (1-20) is the unique (up to a multiplicative constant) $H^{1,0}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C})$-solution to $(i\nabla + A_0)^2 \psi_k = 0$ in $\mathbb{R}^2$ which is homogeneous of degree $|\alpha - k|$.

Theorem 1.2. *Under the same assumptions as in Theorem 1.1, for $p \in \mathbb{S}^1$ and $a = |a|p \in \Omega$, let $\varphi_a \in H^{1,a}_0(\Omega, \mathbb{C})$ be an eigenfunction of problem (E_a) associated to the eigenvalue λ_a and satisfying* (1-15). *Let moreover,*

$$\hat{\varphi}_a(x) = \frac{\varphi_a(|a|x)}{|a|^{|\alpha-k|}}.$$

Then

$$\hat{\varphi}_a \to \beta \Psi_p \quad \text{as } |a| \to 0$$

in $H^{1,p}(D_R, \mathbb{C})$ for every $R > 1$, almost everywhere in $\mathbb{R}^2$ and in $C^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{p\}, \mathbb{C})$, with $\beta \neq 0$ and $k \in \mathbb{Z}$ being as in (1-9) *and Ψ_p being as in* (1-18)–(1-19).

Finally, we describe the sharp rate of convergence (1-17), which also turns out to depend strongly on the order of vanishing of φ_0 at 0, as stated in the following theorem.

Theorem 1.3. *Under the same assumptions as in Theorems 1.1 and 1.2, for every $p \in \mathbb{S}^1$ there exists $\mathfrak{L}_p > 0$ such that*

$$|a|^{-2|\alpha-k|} \left\| (i\nabla + A_a)\varphi_a - e^{i\alpha(\theta_a - \theta_0^a)}(i\nabla + A_0)\varphi_0 \right\|^2_{L^2(\Omega,\mathbb{C})} \to |\beta|^2 \, \mathfrak{L}_p \quad \text{as } a = |a|p \to 0.$$

We observe that Theorem 1.3 extends to the case of non-half-integer circulation an analogous result obtained in [Abatangelo and Felli 2017] for half-integer circulation.

The main tools in the proof of the above-described results are energy estimates on eigenfunctions obtained by an Almgren-type monotonicity argument and blow-up analysis for scaled eigenfunctions; such a strategy was first developed in [Abatangelo and Felli 2015; Noris et al. 2015] in the half-integer case and is essentially based on the description of the behavior of limit eigenfunctions at the pole through the limit of the Almgren quotient, which is possible in both the cases of half-integer and non-half-integer circulation. On the other hand, in the implementation of this procedure for the non-half-integer case, two main points present deep differences from that of the half-integer case. First of all a reduction to the Laplacian on the twofold covering manifold is no longer available and hence a new strategy has to be developed to prove monotonicity-type formulas: this is the main goal of Section 2, where we derive precise estimates for eigenfunctions on small circles which are needed to prove Lemma 3.1 (whose analogue in the half-integer case can be directly proved using the reduction to the Laplacian on the twofold covering manifold). The second crucial difference arises in the blow-up analysis, more precisely in the construction of the limit profile, which cannot be as explicit as in the half-integer case. This exploits vanishing on the special directions of nodal lines of limit eigenfunctions. In the non-half-integer case, a nontrivial limit profile still exists (see Lemma 5.3) but its description is quite implicit: this is also the reason why the estimate we obtain here in the non-half-integer case is less precise than the estimates of [Abatangelo and Felli 2015; 2016] for half-integer α.

The paper is organized as follows. In Section 2 we perform a detailed description of the behavior of the eigenfunction φ_a near the pole a, which is crucial in Section 3 to prove an Almgren-type monotonicity formula and to derive local energy estimates for eigenfunctions uniformly with respect to the moving pole. In Section 4 we obtain some upper and lower bounds for the difference $\lambda_0 - \lambda_a$ by exploiting the Courant–Fischer minimax characterization of eigenvalues and testing the Rayleigh quotient with suitable competitor functions. Section 5 contains a blow-up analysis for scaled eigenfunctions, which allows us to prove Theorems 1.1 and 1.2. Finally, in Section 6 we prove Theorem 1.3.

Notation. We list below some notation used throughout the paper:

- For all $r > 0$ and $a \in \mathbb{R}^2$, we denote by $D_r(a) = \{x \in \mathbb{R}^2 : |x - a| < r\}$ the disk of center a and radius r.
- For all $r > 0$, we let $D_r = D_r(0)$ and $\mathbb{S}^1 = \partial D_1$.
- ds denotes the arc length on $\partial D_r(a)$.
- For every complex number $z \in \mathbb{C}$, we denote by $\bar{z}$ its complex conjugate.
- For $z \in \mathbb{C}$, we denote its real part by $\mathfrak{Re}\, z$ and its imaginary part by $\mathfrak{Im}\, z$.

2. Local asymptotics of eigenfunctions

The aim of this section is to describe the local asymptotics of eigenfunctions, showing how the coefficients of expansions depend on the pole. This goal is achieved by expanding the angular part of eigenfunctions in Fourier series with respect to the orthonormal basis of $L^2((0, 2\pi), \mathbb{C})$ given by $\{e^{ijt}/\sqrt{2\pi}\}_{j\in\mathbb{Z}}$, see (2-11), and then by estimating the Fourier coefficients (2-13) by means of Gronwall-type lemmas. These estimates will be crucial to developing the monotonicity argument of Section 3, in particular to proving Lemma 3.1 (whose analogue in the half-integer case is obtained in [Noris et al. 2015, Lemma 5.8] with techniques which are not adaptable to the non-half-integer case).

We recall from [Felli et al. 2011] the description of the asymptotics at the singularity of solutions to elliptic equations with Aharonov–Bohm potentials. In the case of Aharonov–Bohm potentials with circulation $\alpha \in (0,1)\setminus\{\frac{1}{2}\}$, such asymptotics is described in terms of eigenvalues and eigenfunctions of the following operator $\mathcal{H}$ acting on 2π-periodic functions

$$\mathcal{H}\psi = -\psi'' + 2i\alpha\psi' + \alpha^2\psi.$$

It is easy to verify that the eigenvalues of $\mathcal{H}$ are $\{(\alpha-j)^2 : j \in \mathbb{Z}\}$; each eigenvalue $(\alpha-j)^2$ has multiplicity 1 and the eigenspace associated is generated by the function $e^{ijt}/\sqrt{2\pi}$. Let us enumerate the eigenvalues $(\alpha-j)^2$ as $\{(\alpha-j)^2 : j\in\mathbb{Z}\} = \{\mu_j : j = 1,2,\dots\}$ with $\mu_1 < \mu_2 < \mu_3 < \cdots$, so that

$$\mu_1 = \min\{\alpha^2, (1-\alpha)^2\} \tag{2-1}$$

and $\mu_2 = \max\{\alpha^2,(1-\alpha)^2\}$.

Proposition 2.1 [Felli et al. 2011, Theorem 1.3]. *Let $\Omega\subset\mathbb{R}^2$ be a bounded open set containing b, $\lambda\in\mathbb{R}$, and $u\in H^{1,b}_0(\Omega,\mathbb{C})$ be a nontrivial weak solution to the problem*

$$(i\nabla + A_b)^2 u = \lambda u \quad \text{in } \Omega;$$

i.e.,

$$\int_\Omega (i\nabla + A_b)u\cdot\overline{(i\nabla + A_b)v}\,dx = \lambda\int_\Omega u\bar{v}\,dx \quad \text{for all } v\in H^{1,b}_0(\Omega,\mathbb{C}).$$

Then there exists $j\in\mathbb{Z}$ such that

$$\lim_{r\to 0^+}\frac{r\int_{D_r(b)}\big(|(i\nabla + A_b)u(x)|^2 - \lambda|u(x)|^2\big)\,dx}{\int_{\partial D_r(b)}|u|^2\,ds} = |\alpha - j|. \tag{2-2}$$

Furthermore, there exists $\beta(b,u,\lambda)\neq 0$ such that

$$r^{-|\alpha-j|}u(b + r(\cos t,\sin t)) \to \beta(b,u,\lambda)\frac{e^{ijt}}{\sqrt{2\pi}} \quad \text{in } C^{1,\tau}([0,2\pi],\mathbb{C}) \tag{2-3}$$

as $r\to 0^+$ for any $\tau\in(0,1)$.

Let us fix $n\in\mathbb{N}$, $n\geq 1$. For all $a\in\Omega$, let $\varphi^a_n\in H^{1,a}_0(\Omega,\mathbb{C})\setminus\{0\}$ be an eigenfunction of problem (E_a) associated to the eigenvalue λ^a_n, i.e., solving

$$\begin{cases}(i\nabla + A_a)^2\varphi^a_n = \lambda^a_n\varphi^a_n & \text{in } \Omega,\\ \varphi^a_n = 0 & \text{on } \partial\Omega,\end{cases} \tag{2-4}$$

such that

$$\int_\Omega |\varphi^a_n(x)|^2\,dx = 1. \tag{2-5}$$

Since $a\in\Omega\mapsto\lambda^a_n$ admits a continuous extension on $\overline{\Omega}$ as proved in [Bonnaillie-Noël et al. 2014, Theorem 1.1], we have

$$\Lambda_n = \sup_{a\in\Omega}\lambda^a_n \in (0,+\infty). \tag{2-6}$$

Moreover, from (2-4), (2-5), and (1-3) it follows that

$$\{\varphi_n^a\}_{a\in\Omega} \text{ is bounded in } H^1(\Omega,\mathbb{C}), \tag{2-7}$$

which, by (2-4) and classical elliptic regularity theory, implies that, for each $\omega \Subset \Omega \setminus \{0\}$, there exists $\rho_\omega > 0$ such that

$$\{\varphi_n^a\}_{|a|\le\rho_\omega} \text{ is bounded in } C^{2,\sigma}(\omega,\mathbb{C}) \text{ for every } \sigma\in(0,1). \tag{2-8}$$

The following lemma provides a detailed description of the behavior of the Fourier coefficients of the function $t \mapsto \varphi_n^a(a+r(\cos t,\sin t))$ as a is close to 0.

Lemma 2.2. *For $n\ge 1$ fixed and a varying in Ω, let $\varphi_n^a \in H_0^{1,a}(\Omega,\mathbb{C})\setminus\{0\}$ satisfy* (2-4) *and* (2-5). *For all $j\in\mathbb{Z}$ and $a\in\Omega$, let*

$$v_j^a(r) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \varphi_n^a(a+r(\cos t,\sin t))e^{-ijt}\,dt. \tag{2-9}$$

Then there exists $\rho_0>0$ such that, for all a with $|a|\le\rho_0$, the following properties hold:

(i) *For all $j\in\mathbb{Z}$, we have $v_j^a(r) = O(r^{|\alpha-j|})$ as $r\to 0^+$. In particular, for all $j\in\mathbb{Z}$ and for all $R>0$ such that $\{x\in\mathbb{R}^2 : |x-a|\le R\}\subset\Omega$, the value*

$$\beta_j^a = \frac{v_j^a(R)}{R^{|\alpha-j|}} + \frac{\lambda_n^a}{2|\alpha-j|}\int_0^R \left(s^{1-|\alpha-j|} - \frac{s^{1+|\alpha-j|}}{R^{2|\alpha-j|}}\right) v_j^a(s)\,ds \tag{2-10}$$

is well-defined and independent of R.

(ii) *For all $j\in\mathbb{Z}$, we have $|\beta_j^a|\le B$ for some $B>0$ independent of j and a.*

(iii) *For all $j\in\mathbb{Z}$,*

$$v_j^a(r) = r^{|\alpha-j|}\beta_j^a(1+R_{j,a}(r)) \quad \text{and} \quad (v_j^a)'(r) = |\alpha-j|\beta_j^a r^{|\alpha-j|-1}(1+\widetilde{R}_{j,a}(r)),$$

where $|R_{j,a}(r)| + |\widetilde{R}_{j,a}(r)| \le \text{const}\, r^2$ for some $\text{const}>0$ independent of j and a.

(iv) *φ_n^a can be expanded as*

$$\varphi_n^a(a+r(\cos t,\sin t)) = \frac{1}{\sqrt{2\pi}}\sum_{j\in\mathbb{Z}} r^{|\alpha-j|}\beta_j^a(1+R_{j,a}(r))e^{ijt}$$

with $R_{j,a}(r)$ as in (iii), *where the convergence of the above series is uniform on disks $D_R(a)$ for each $R\in(0,1)$.*

(v) *If we let $\boldsymbol{\nu}(t) = (\cos t,\sin t)$ and $\boldsymbol{\tau}(t) = (-\sin t,\cos t)$, then $(i\nabla + A_a)\varphi_n^a$ can be expanded as*

$$(i\nabla + A_a)\varphi_n^a(a+r(\cos t,\sin t)) \\ = \frac{1}{\sqrt{2\pi}}\sum_{j\in\mathbb{Z}} \beta_j^a r^{|\alpha-j|-1}\big(i|\alpha-j|(1+\widetilde{R}_{j,a}(r))\boldsymbol{\nu}(t) + (\alpha-j)(1+R_{j,a}(r))\boldsymbol{\tau}(t)\big)e^{ijt}$$

with $R_{j,a}(r)$, $\widetilde{R}_{j,a}(r)$ as in (iii), *where the above series converges absolutely in $L^2(D_R(a),\mathbb{C})$ and pointwise in $D_R(a)$ for each $R\in(0,1)$.*

Proof. The functions $\{e^{ijt}/\sqrt{2\pi}\}_{j\in\mathbb{Z}}$ form an orthonormal basis of $L^2((0,2\pi),\mathbb{C})$. Hence, recalling that we are assuming $\overline{D}_2\subset\Omega$, if $|a|$ is sufficiently small, φ_n^a can be expanded as

$$\varphi_n^a(a+r(\cos t,\sin t))=\sum_{j\in\mathbb{Z}}v_j^a(r)\frac{e^{ijt}}{\sqrt{2\pi}}\quad\text{in } L^2((0,2\pi),\mathbb{C}) \text{ for all } r\in(0,1], \tag{2-11}$$

where v_j^a is defined in (2-9). Equation (1-14) implies that, for every $j\in\mathbb{Z}$,

$$-(v_j^a)''(r)-\frac{1}{r}(v_j^a)'(r)+\frac{(\alpha-j)^2}{r^2}v_j^a(r)=\lambda_n^a v_j^a(r)\quad\text{for all } r\in(0,1], \tag{2-12}$$

or equivalently

$$-r^{|\alpha-j|-1}\big(r^{1-2|\alpha-j|}(r^{|\alpha-j|}v_j^a)'\big)'=\lambda_n^a v_j^a(r)\quad\text{for all } r\in(0,1].$$

Integrating twice between r and 1, we obtain, for some $c_{1,j}^a, c_{2,j}^a\in\mathbb{C}$,

$$v_j^a(r)=r^{|\alpha-j|}\left(c_{1,j}^a+\lambda_n^a\int_r^1\frac{s^{-|\alpha-j|+1}}{2|\alpha-j|}v_j^a(s)\,ds\right)+r^{-|\alpha-j|}\left(c_{2,j}^a-\lambda_n^a\int_r^1\frac{s^{|\alpha-j|+1}}{2|\alpha-j|}v_j^a(s)\,ds\right) \tag{2-13}$$

for all $r\in(0,1]$.

The convergence (2-3) in Proposition 2.1 implies that, for all a,

$$|\varphi_n^a(a+r(\cos t,\sin t))|=O(r^{\sqrt{\mu_1}})$$

as $r\to 0^+$, with μ_1 as in (2-1) (not necessarily uniformly with respect to a). Hence, for every a in a sufficiently small neighborhood of 0, there exists a constant $C(a)>0$ such that, for all $j\in\mathbb{Z}$,

$$|v_j^a(r)|\le C(a)r^{\sqrt{\mu_1}}\quad\text{for all } r\in[0,1]. \tag{2-14}$$

We deduce that each function v_j^a is bounded near 0; hence (2-13) necessarily yields

$$c_{2,j}^a=\lambda_n^a\int_0^1\frac{s^{|\alpha-j|+1}}{2|\alpha-j|}v_j^a(s)\,ds. \tag{2-15}$$

We can therefore rewrite

$$v_j^a(r)=r^{|\alpha-j|}\left(c_{1,j}^a+\frac{\lambda_n^a}{2|\alpha-j|}\int_r^1 s^{-|\alpha-j|+1}v_j^a(s)\,ds\right)+\frac{\lambda_n^a}{2|\alpha-j|}r^{-|\alpha-j|}\int_0^r s^{|\alpha-j|+1}v_j^a(s)\,ds. \tag{2-16}$$

If $\sqrt{\mu_1}+2\ge|\alpha-j|$, using (2-14) to estimate the right-hand side of (2-16) we obtain the improved estimate $|v_j^a(r)|\le C(j,a)r^{|\alpha-j|}$. Otherwise, if $\sqrt{\mu_1}+2<|\alpha-j|$, we can use (2-14) to estimate the right-hand side of (2-16) to obtain the improved estimate $|v_j^a(r)|\le C(j,a)r^{\sqrt{\mu_1}+2}$ for some constant $C(j,a)>0$ depending on a and j. By iterating the process $m+1$ times, with m the largest natural number such that $\sqrt{\mu_1}+2m<|\alpha-j|$, we obtain $|v_j^a(r)|\le C(j,a)r^{|\alpha-j|}$, possibly for a different constant $C(j,a)$. We deduce that the quantity β_j^a introduced in (2-10) is well-defined. The fact that β_j^a is independent of R is a direct consequence of (2-12) and (2-16). This proves statement (i).

Using the independence of β_j^a with respect to R, we choose $R=1$ in (2-10) and $r=1$ in (2-16) and obtain

$$\beta_j^a=c_{1,j}^a+\frac{\lambda_n^a}{2|\alpha-j|}\int_0^1 s^{-|\alpha-j|+1}v_j^a(s)\,ds, \tag{2-17}$$

so that (2-16) can be rewritten as

$$v_j^a(r) = r^{|\alpha-j|}\left(\beta_j^a - \lambda_n^a \int_0^r \frac{s^{-|\alpha-j|+1}}{2|\alpha-j|} v_j^a(s)\,ds\right) + \lambda_n^a r^{-|\alpha-j|} \int_0^r \frac{s^{|\alpha-j|+1}}{2|\alpha-j|} v_j^a(s)\,ds. \tag{2-18}$$

From (2-18) it follows that, for all $r \in (0,1]$,

$$\begin{aligned} r^{-|\alpha-j|}|v_j^a(r)| &\le |\beta_j^a| + \frac{\lambda_n^a}{2|\alpha-j|}\int_0^r s^{-|\alpha-j|}|v_j^a(s)|\,ds + \frac{\lambda_n^a}{2|\alpha-j|} r^{-2|\alpha-j|}\int_0^r s^{2|\alpha-j|}s^{-|\alpha-j|}|v_j^a(s)|\,ds \\ &\le |\beta_j^a| + \frac{\lambda_n^a}{|\alpha-j|}\int_0^r s^{-|\alpha-j|}|v_j^a(s)|\,ds. \end{aligned}$$

Hence the Gronwall lemma applied to the function $r \mapsto r^{-|\alpha-j|}|v_j^a(r)|$ yields that

$$r^{-|\alpha-j|}|v_j^a(r)| \le |\beta_j^a| e^{\lambda_n^a r/|\alpha-j|} \le C|\beta_j^a| \quad \text{for all } r \in (0,1] \text{ and } j \in \mathbb{Z}, \tag{2-19}$$

where $C = e^{\Lambda_n/\sqrt{\mu_1}}$ is independent of j, a, and r, with μ_1 and Λ_n defined in (2-1) and (2-6) respectively.

From (2-13), (2-9), and (2-8) it follows that

$$|c_{1,j}^a + c_{2,j}^a| = |v_j^a(1)| = \frac{1}{\sqrt{2\pi}}\left|\int_0^{2\pi} \varphi_n^a(a + (\cos t, \sin t))e^{-ijt}\,dt\right| \le \text{const}$$

for some const > 0 independent of j and a; moreover, from (2-15) and (2-5) we deduce that

$$|c_{2,j}^a| \le \frac{\lambda_n^a}{2|\alpha-j|}\int_0^1 s|v_j^a(s)|\,ds \le \frac{\lambda_n^a}{2|\alpha-j|\sqrt{2\pi}}\int_{D_1(a)} |\varphi_n^a|\,dx \le \text{const}$$

for some const > 0 independent of j and a. Hence

$$|c_{1,j}^a| \le \widetilde{C} \tag{2-20}$$

for some $\widetilde{C} > 0$ independent of j and a.

Let $K > 0$ be such that

$$\frac{\Lambda_n C}{2K} < \frac{1}{2}$$

with C being as in (2-19) and Λ_n being as in (2-6). Hence, from (2-6), (2-17), (2-19) and (2-20) it follows that, if $|\alpha - j| > K$, then

$$\tfrac{1}{2}|\beta_j^a| \le \left(1 - \frac{\Lambda_n C}{2K}\right)|\beta_j^a| \le |c_{1,j}^a| \le \widetilde{C}. \tag{2-21}$$

Let us choose $R_0 \in (0,1)$ such that

$$\frac{\Lambda_n C R_0^2}{2\sqrt{\mu_1}} < \frac{1}{2}.$$

From (2-10) and (2-19) it follows that, if $|\alpha - j| \le K$,

$$\begin{aligned} R_0^{-K}|v_j^a(R_0)| \ge R_0^{-|\alpha-j|}|v_j^a(R_0)| &= \left|\beta_j^a - \frac{\lambda_n^a}{2|\alpha-j|}\int_0^{R_0}\left(s^{1-|\alpha-j|} - \frac{s^{1+|\alpha-j|}}{R_0^{2|\alpha-j|}}\right)v_j^a(s)\,ds\right| \\ &\ge |\beta_j^a| - \frac{\Lambda_n C|\beta_j^a|}{2\sqrt{\mu_1}}\int_0^{R_0} 2s\,ds \\ &= |\beta_j^a| - \frac{\Lambda_n C R_0^2}{2\sqrt{\mu_1}}|\beta_j^a| \ge \tfrac{1}{2}|\beta_j^a|. \end{aligned}$$

Since, in view of (2-8), $v_j^a(R_0)$ is bounded uniformly with respect to a and j, we conclude that, for all j such that $|\alpha - j| \le K$, $|\beta_j^a|$ is bounded uniformly with respect to a and j. This, together with (2-21), yields (ii).

From (2-18) and (2-19) it follows that

$$v_j^a(r) = r^{|\alpha-j|}\beta_j^a(1 + R_{j,a}(r)), \tag{2-22}$$

where $|R_{j,a}(r)| \le \text{const}\, r^2$ for some const > 0 independent of j and a, thus proving the first estimate in (iii). Differentiating (2-18) and using the above estimate (2-22), we easily obtain

$$(v_j^a)'(r) = |\alpha - j|\beta_j^a r^{|\alpha-j|-1}(1 + \widetilde{R}_{j,a}(r)),$$

where $|\widetilde{R}_{j,a}(r)| \le \text{const}\, r^2$ for some const > 0 independent of j and a. Hence the proof of (iii) is complete.

From (2-11) and (iii) we have that the series

$$\frac{1}{\sqrt{2\pi}}\sum_{j\in\mathbb{Z}} r^{|\alpha-j|}\beta_j^a(1 + R_{j,a}(r))e^{ijt}$$

converges in $L^2((0, 2\pi), \mathbb{C})$ to $\varphi_n^a(a + r(\cos t, \sin t))$ for all $r \in (0, 1]$. In view of the estimates obtained in (ii)–(iii), the Weierstrass M-test ensures that the series is uniformly convergent in $D_R(a)$ for every $R \in (0, 1)$, thus proving (iv).

Let $f_j^a(a + r(\cos t, \sin t)) = v_j^a(r)e^{ijt}/\sqrt{2\pi}$. Since

$$(i\nabla + A_a) f_j^a(a + r(\cos t, \sin t)) = \left(i(v_j^a)'(r)\boldsymbol{\nu}(t) + (\alpha - j)\frac{v_j^a(r)}{r}\boldsymbol{\tau}(t)\right)\frac{e^{ijt}}{\sqrt{2\pi}},$$

the above estimates also imply that, for every $R \in (0, 1)$, the series of functions $\sum_j (i\nabla + A_a) f_j^a$ is convergent absolutely in $L^2(D_R(a), \mathbb{C})$ and pointwise in $D_R(a)$ to $(i\nabla + A_a)\varphi_n^a$ for every $R \in (0, 1)$. Hence (v) follows from (iii). □

Corollary 2.3. *Under the same assumptions and with the same notation as in Lemma 2.2, let* $R \in (0, 1)$. *Then, for all* $r \in (0, R)$ *and* $t \in [0, 2\pi]$,

$$\varphi_n^a(a + r(\cos t, \sin t)) = \frac{1}{\sqrt{2\pi}}(r^\alpha\beta_0^a + r^{1-\alpha}\beta_1^a e^{it}) + \mathcal{R}_a(r, t), \tag{2-23}$$

$$\begin{aligned}(i\nabla + A_a)\varphi_n^a(a + r(\cos t, \sin t)) &= \frac{1}{\sqrt{2\pi}} r^{\alpha-1}\beta_0^a\alpha(i\boldsymbol{\nu}(t) + \boldsymbol{\tau}(t)) \\ &\quad + \frac{1}{\sqrt{2\pi}} r^{-\alpha}\beta_1^a(1 - \alpha)(i\boldsymbol{\nu}(t) - \boldsymbol{\tau}(t))e^{it} + \widetilde{\mathcal{R}}_a(r, t),\end{aligned} \tag{2-24}$$

where $|\mathcal{R}_a(r, t)| \le \text{const}\, r^{1+\sqrt{\mu_1}}$ *and* $|\widetilde{\mathcal{R}}_a(r, t)| \le \text{const}\, r^{\sqrt{\mu_1}}$ *for some* const > 0 *independent of* a, r, t.

Proof. From part (iv) of Lemma 2.2 we have

$$\varphi_n^a(a + r(\cos t, \sin t)) = \frac{1}{\sqrt{2\pi}}(\beta_0^a r^\alpha + \beta_1^a r^{1-\alpha}e^{it}) + \mathcal{R}_a(r, t), \quad r \in (0, 1),\ t \in [0, 2\pi],$$

where

$$\mathcal{R}_a(r,t) = \frac{1}{\sqrt{2\pi}}\big(\beta_0^a r^\alpha R_{0,a}(r) + \beta_1^a r^{1-\alpha} R_{1,a}(r)e^{it}\big) + \frac{1}{\sqrt{2\pi}} \sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|>1}} \beta_j^a r^{|\alpha-j|}(1+R_{j,a}(r))e^{ijt}.$$

Let us fix $R \in (0,1)$. Estimates (ii)–(iii) of Lemma 2.2 imply that, for some const > 0 independent of a, r, t (possibly varying from line to line),

$$|\mathcal{R}_a(r,t)| \le \text{const}\Bigg(r^{\alpha+2} + r^{3-\alpha} + \sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\ge 1+\sqrt{\mu_1}}} r^{|\alpha-j|}\Bigg) \le \text{const}\; r^{1+\sqrt{\mu_1}}$$

for all $r \in (0, R)$, thus proving (2-23).

From part (v) of Lemma 2.2 we have

$$(i\nabla + A_a)\varphi_n^a(a + r(\cos t, \sin t)) = \frac{\alpha}{\sqrt{2\pi}}\beta_0^a r^{\alpha-1}(i\boldsymbol{\nu}(t) + \boldsymbol{\tau}(t)) + \frac{1-\alpha}{\sqrt{2\pi}}\beta_1^a r^{-\alpha}(i\boldsymbol{\nu}(t) - \boldsymbol{\tau}(t))e^{it} + \widetilde{\mathcal{R}}_a(r,t),$$

where

$$\begin{aligned}\widetilde{\mathcal{R}}_a(r,t) = {}& \frac{\alpha}{\sqrt{2\pi}}\beta_0^a r^{\alpha-1}\big(i\widetilde{R}_{0,a}(r)\boldsymbol{\nu}(t) + R_{0,a}(r)\boldsymbol{\tau}(t)\big) + \frac{1-\alpha}{\sqrt{2\pi}}\beta_1^a r^{-\alpha}\big(i\widetilde{R}_{1,a}(r)\boldsymbol{\nu}(t) - R_{1,a}(r)\boldsymbol{\tau}(t)\big)e^{it} \\ & + \frac{1}{\sqrt{2\pi}} \sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|>1}} \beta_j^a r^{|\alpha-j|-1}\big(i|\alpha-j|(1+\widetilde{R}_{j,a}(r))\boldsymbol{\nu}(t) + (\alpha-j)(1+R_{j,a}(r))\boldsymbol{\tau}(t)\big)e^{ijt}.\end{aligned}$$

From Lemma 2.2(ii)–(iii) we have that, for all $r \in (0, R)$,

$$|\widetilde{\mathcal{R}}_a(r,t)| \le \text{const}\Bigg(r^{\alpha+1} + r^{2-\alpha} + \sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\ge 1+\sqrt{\mu_1}}} |\alpha-j| r^{|\alpha-j|-1}\Bigg) \le \text{const}\; r^{\sqrt{\mu_1}}$$

for some const > 0 independent of a, r, t (possibly varying from line to line), thus proving (2-24). □

We now describe some consequences of Lemma 2.2 and Corollary 2.3, which will be needed in Section 3 to prove a monotonicity-type formula.

Lemma 2.4. *Under the same assumptions and with the same notation as in Lemma 2.2, we have*

$$\lim_{\varepsilon\to 0^+}\left\{\left|\frac{1}{2}\int_{\partial D_\varepsilon(a)} |(i\nabla + A_a)\varphi_n^a|^2 x\cdot\nu\, ds\right| + \left|\int_{\partial D_\varepsilon(a)} (i\nabla + A_a)\varphi_n^a \cdot \nu\, \overline{(i\nabla + A_a)\varphi_n^a \cdot x}\, ds\right|\right\} \le 2\alpha(1-\alpha)|a||\beta_0^a||\beta_1^a|.$$

Proof. Let $R \in (0,1)$ be fixed. From (2-24) we have that, for all $r \in (0, R)$,

$$\big|(i\nabla + A_a)\varphi_n^a(a + r(\cos t, r\sin t))\big|^2 = r^{2(\alpha-1)}|\beta_0^a|^2\frac{\alpha^2}{\pi} + r^{-2\alpha}|\beta_1^a|^2\frac{(1-\alpha)^2}{\pi} + \widehat{\mathcal{R}}_a(r,t),$$

where $|\widehat{\mathcal{R}}_a(r,t)| \le \text{const}\; r^{2\sqrt{\mu_1}-1}$ for some const > 0 independent of a, r, t. It follows that

$$\lim_{\varepsilon\to 0^+}\int_{\partial D_\varepsilon(a)} |(i\nabla + A_a)\varphi_n^a|^2 x\cdot\nu\, ds = 0.$$

Moreover, from (2-24) we have

$$
\begin{aligned}
&(i\nabla + A_a)\varphi_n^a(a+\varepsilon(\cos t,\sin t))\cdot \boldsymbol{\nu}(t)\overline{(i\nabla + A_a)\varphi_n^a(a+\varepsilon(\cos t,\sin t))\cdot (a+\varepsilon\boldsymbol{\nu}(t))}\\
&\qquad = \varepsilon^{2(\alpha-1)}|\beta_0^a|^2\frac{\alpha^2}{2\pi}(\boldsymbol{\nu}(t)+i\boldsymbol{\tau}(t))\cdot a + \varepsilon^{-2\alpha}|\beta_1^a|^2\frac{(1-\alpha)^2}{2\pi}(\boldsymbol{\nu}(t)-i\boldsymbol{\tau}(t))\cdot a\\
&\qquad\quad + 2\varepsilon^{-1}\mathfrak{Re}\left(\beta_0^a\bar\beta_1^a e^{-it}\frac{\alpha(1-\alpha)}{2\pi}(\boldsymbol{\nu}(t)-i\boldsymbol{\tau}(t))\cdot a\right) + O(\varepsilon^{2\sqrt{\mu_1}-1})
\end{aligned}
$$

as $\varepsilon\to 0^+$, and hence, taking into account that $\int_0^{2\pi} a\cdot\boldsymbol{\nu}(t)\,dt = \int_0^{2\pi} a\cdot\boldsymbol{\tau}(t)\,dt = 0$, we obtain

$$
\lim_{\varepsilon\to 0^+}\int_{\partial D_\varepsilon(a)}(i\nabla + A_a)\varphi_a\cdot\nu\overline{(i\nabla + A_a)\varphi_a\cdot x}\,ds = 2\alpha(1-\alpha)\mathfrak{Re}(\beta_0^a\bar\beta_1^a(a_1-ia_2)),
$$

from which the conclusion follows. □

Lemma 2.5. *For $n\ge 1$ fixed and a varying in Ω, let $\varphi_n^a\in H_0^{1,a}(\Omega,\mathbb{C})\setminus\{0\}$ satisfy (2-4) and (2-5). Let us assume that $\varphi_n^a\to\varphi_n^0$ in $L^2(\Omega,\mathbb{C})$ as $a\to 0$ (or respectively along a sequence $a_\ell\to 0$). Let $k\in\mathbb{Z}$ be such that $|\alpha-k|$ is the order of vanishing of φ_n^0 at 0. For all $j\in\mathbb{Z}$ and $a\in\Omega$, let v_j^a be as in (2-9) and β_j^a be as in (2-10). Then there exists $\rho_0>0$ such that, for all a with $|a|\le\rho_0$ (respectively for $a=a_\ell$ with ℓ sufficiently large), the following properties hold:*

(i) *For all $j\in\mathbb{Z}$, we have $\beta_j^a\to\beta_j^0$ as $a\to 0$ (respectively along the sequence $a_\ell\to 0$).*

(ii) *It holds that*

$$
\int_0^{2\pi}|\varphi_n^a(a+r(\cos t,\sin t))|^2\,dt = \Bigg(\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|<|\alpha-k|}} r^{2|\alpha-j|}|\beta_j^a|^2|1+R_{j,a}(r)|^2\Bigg) + r^{2|\alpha-k|}|\beta_k^a|^2(1+\widehat{R}_a(r)),
$$

where $|\widehat{R}_a(r)|\le h(r)$ for some function $h(r)$ independent of a such that $h(r)\to 0$ as $r\to 0^+$, and

$$
\varphi_n^a(a+r(\cos t,\sin t)) = \frac{1}{\sqrt{2\pi}}\Bigg(\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|<|\alpha-k|}} r^{|\alpha-j|}\beta_j^a(1+R_{j,a}(r))e^{ijt}\Bigg) + \frac{1}{\sqrt{2\pi}}r^{|\alpha-k|}\beta_k^a(e^{ikt}+R_a(r,t)),
$$

where $|R_{j,a}(r)|\le$ const r^2 for some const >0 independent of j and a, and $|R_a(r,t)|\le f(r)$ for some function $f(r)$ independent of a and t such that $f(r)\to 0$ as $r\to 0^+$.

(iii) *Let $\boldsymbol{\nu}(t)=(\cos t,\sin t)$ and $\boldsymbol{\tau}(t)=(-\sin t,\cos t)$. It holds that*

$$
\begin{aligned}
&\int_0^{2\pi}\left|(i\nabla + A_a)\varphi_n^a(a+r(\cos t,\sin t))\right|^2 dt\\
&\quad = \Bigg(\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|<|\alpha-k|}} r^{2|\alpha-j|-2}|\beta_j^a|^2|\alpha-j|^2\big(|1+R_{j,a}(r)|^2+|1+\widetilde{R}_{j,a}(r)|^2\big)\Bigg)\\
&\qquad\qquad + r^{2|\alpha-k|-2}|\beta_k^a|^2|\alpha-k|^2(1+\widetilde{R}_a(r)),
\end{aligned}
$$

where $|\widetilde{R}_a(r)| \le p(r)$ *for some function* $p(r)$ *independent of a such that* $p(r) \to 0$ *as* $r \to 0^+$, *and*

$$(i\nabla + A_a)\varphi_n^a(a+r(\cos t, \sin t)) = \frac{1}{\sqrt{2\pi}} \sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|<|\alpha-k|}} r^{|\alpha-j|-1}\beta_j^a\big(i|\alpha-j|\boldsymbol{\nu}(t) + (\alpha-j)\boldsymbol{\tau}(t) + \boldsymbol{R}_{j,a}(r)\big)e^{ijt} + \frac{1}{\sqrt{2\pi}} r^{|\alpha-k|-1}\beta_k^a\big(\big(i|\alpha-k|\boldsymbol{\nu}(t) + (\alpha-k)\boldsymbol{\tau}(t)\big)e^{ikt} + \widetilde{\boldsymbol{R}}_a(r,t)\big),$$

where $|\boldsymbol{R}_{j,a}(r,t)| \le \text{const}\ r^2$ *for some positive constant* const > 0 *independent of* j *and* a *and* $|\widetilde{\boldsymbol{R}}_a(r,t)| \le g(r)$ *for some function* $g(r)$ *independent of a and t such that* $g(r) \to 0$ *as* $r \to 0^+$.

Proof. In order to prove statement (i), we notice that (2-10) evaluated at $R = 1$ provides

$$\beta_j^a = v_j^a(1) + \frac{\lambda_n^a}{2|\alpha-j|}\int_0^1 (s^{1-|\alpha-j|} - s^{1+|\alpha-j|})v_j^a(s)\,ds. \tag{2-25}$$

From Lemma 2.2(ii)–(iii) it follows that, for $|a| \le \rho_0$ with $\rho_0 > 0$ sufficiently small,

$$|v_j^a(r)| \le C' r^{|\alpha-j|} \quad \text{for all } r \in (0,1] \text{ and } j \in \mathbb{Z} \tag{2-26}$$

for some constant $C' > 0$ independent of j, a, and r. Moreover, (2-4), (2-5), the convergence $\varphi_n^a \to \varphi_n^0$ in $L^2(\Omega,\mathbb{C})$, and standard elliptic estimates, see, e.g., [Gilbarg and Trudinger 1983, Theorem 8.10], imply

$$\varphi_n^a \to \varphi_n^0 \quad \text{in } H^1(\Omega,\mathbb{C}) \text{ and } C^2_{\text{loc}}(\Omega\setminus\{0\},\mathbb{C}) \quad \text{as } a \to 0 \text{ (or along the sequence } a_\ell \to 0). \tag{2-27}$$

From (2-25)–(2-27), and the dominated convergence theorem we obtain that, for all $j \in \mathbb{Z}$,

$$\lim_{a\to 0}\beta_j^a = v_j^0(1) + \frac{\lambda_n^0}{2|\alpha-j|}\int_0^1 (s^{1-|\alpha-j|} - s^{1+|\alpha-j|})v_j^0(s)\,ds = \beta_j^0,$$

thus proving (i).

If $k \in \mathbb{Z}$ is such that $|\alpha-k|$ is the order of vanishing of φ_n^0 at 0, from Lemma 2.2(iii) it follows that $\beta_k^0 \ne 0$ and $\beta_j^0 = 0$ for all $j \in \mathbb{Z}$ such that $|\alpha-j| < |\alpha-k|$; in particular, in view of (i), we have $\lim_{a\to 0}\beta_k^a \ne 0$ and hence $\inf_{|a|\le\rho_0}|\beta_k^a| > 0$ for ρ_0 sufficiently small. Then, from Lemma 2.2(iv) and the Parseval identity we deduce that

$$\int_0^{2\pi} |\varphi_n^a(a+r(\cos t,\sin t))|^2\,dt = \sum_{j\in\mathbb{Z}} r^{2|\alpha-j|}|\beta_j^a|^2|1+R_{j,a}(r)|^2 = \Bigg(\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|<|\alpha-k|}} r^{2|\alpha-j|}|\beta_j^a|^2|1+R_{j,a}(r)|^2\Bigg) + r^{2|\alpha-k|}|\beta_k^a|^2(1+\widehat{R}_a(r)),$$

with

$$\widehat{R}_a(r) = |R_{k,a}(r)|^2 + 2\mathfrak{Re}(R_{k,a}(r)) + \sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|>|\alpha-k|}} \frac{|\beta_j^a|^2}{|\beta_k^a|^2} r^{2|\alpha-j|-2|\alpha-k|}|1+R_{j,a}(r)|^2$$

so that the first estimate in (ii) follows from Lemma 2.2(ii)–(iii). From Lemma 2.2(iii) we also deduce that

$$\frac{1}{\sqrt{2\pi}} \sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\geq|\alpha-k|}} r^{|\alpha-j|}\beta_j^a(1+R_{j,a}(r))e^{ijt} = \frac{1}{\sqrt{2\pi}} r^{|\alpha-k|}\beta_k^a(e^{ikt}+R_a(r,t)),$$

where $|R_a(r,t)| \leq f(r)$ for some function $f(r)$ independent of a and t such that $f(r) \to 0$ as $r \to 0$. Then the second estimate in (ii) follows from Lemma 2.2(iv).

From Lemma 2.2 (v) and the Parseval identity we deduce that

$$\int_0^{2\pi} \big|(i\nabla+A_a)\varphi_n^a(a+r(\cos t,\sin t))\big|^2\,dt = \sum_{j\in\mathbb{Z}} r^{2|\alpha-j|-2}|\beta_j^a|^2\,|\alpha-j|^2\big(|1+R_{j,a}(r)|^2+|1+\widetilde{R}_{j,a}(r)|^2\big)$$

so that the first estimate in (iii) follows from Lemma 2.2(ii)–(iii) arguing as above. In a similar way, the second estimate in (iii) follows from statements (iii) and (v) of Lemma 2.2. □

Remark 2.6. In the particular case $n = n_0$ with n_0 such that (1-5) holds, the above lemma applies to the family of eigenfunctions $\varphi_a = \varphi_{n_0}^a$ satisfying (1-14) and (1-15). Indeed, in this case (1-16) holds; i.e., the eigenfunctions φ_a converge as $a \to 0^+$ so that the assumptions of Lemma 2.5 are fulfilled. In particular we deduce that, if φ_0 satisfies (1-7)–(1-9) and if φ_a is as in (1-14)–(1-15), then, for a sufficiently close to 0, the vanishing order of φ_a is less than or equal to the vanishing order of φ_0.

Lemma 2.7. *For $n \geq 1$ fixed and a varying in $\Omega \setminus \{0\}$, let $\varphi_n^a \in H_0^{1,a}(\Omega,\mathbb{C}) \setminus \{0\}$ satisfy* (2-4) *and* (2-5). *Then there exist $\sigma > 0$ and $C > 0$ such that, for all $R > 1$ and $a \in \Omega$ such that $0 < |a| < \sigma/R$,*

$$\frac{1}{|a|}\int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)} |\varphi_n^a|^2\,dx \leq C\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\,ds,$$

$$\int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)} |(i\nabla+A_a)\varphi_n^a|^2\,dx \leq \frac{C}{R^2|a|}\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\,ds.$$

Proof. Let us prove the first estimate arguing by contradiction: assume that there exist sequences $R_\ell > 1$ and $a_\ell \in \Omega$ such that $R_\ell|a_\ell| < 1/\ell$ and

$$\frac{1}{|a_\ell|}\int_{D_{(R_\ell+1)|a_\ell|}(a_\ell)\setminus D_{R_\ell|a_\ell|}(a_\ell)} |\varphi_n^{a_\ell}|^2\,dx > \ell\int_{\partial D_{R_\ell|a_\ell|}(a_\ell)} |\varphi_n^{a_\ell}|^2\,ds. \tag{2-28}$$

It is easy to verify that, up to extracting a subsequence, $\varphi_n^{a_\ell} \to \varphi_n^0$ in $L^2(\Omega,\mathbb{C})$ as $\ell \to \infty$ for some $\varphi_n^0 \in H_0^{1,0}(\Omega,\mathbb{C}) \setminus \{0\}$ satisfying

$$\begin{cases} (i\nabla+A_0)^2\varphi_n^0 = \lambda_n^0\varphi_n^0 & \text{in } \Omega,\\ \varphi_n^0 = 0 & \text{on } \partial\Omega,\\ \int_\Omega |\varphi_n^0(x)|^2\,dx = 1. \end{cases} \tag{2-29}$$

Let $k \in \mathbb{Z}$ be such that $|\alpha - k|$ is the order of vanishing of φ_n^0 at 0. Then, from Lemma 2.5 (first estimate in (ii)) it follows that, for ℓ sufficiently large,

$$\begin{aligned}\frac{1}{|a_\ell|}\int_{D_{(R_\ell+1)|a_\ell|}(a_\ell)\setminus D_{R_\ell|a_\ell|}(a_\ell)} |\varphi_n^{a_\ell}|^2\,dx &= \frac{1}{|a_\ell|}\int_{R_\ell|a_\ell|}^{(R_\ell+1)|a_\ell|} r\biggl(\int_0^{2\pi}|\varphi_n^{a_\ell}(a_\ell+r(\cos t,\sin t))|^2\,dt\biggr)dr\\ &\le \frac{2}{|a_\ell|}\int_{R_\ell|a_\ell|}^{(R_\ell+1)|a_\ell|} r\biggl(\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\le|\alpha-k|}} r^{2|\alpha-j|}|\beta_j^{a_\ell}|^2\biggr)dr\\ &\le \text{const} \sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\le|\alpha-k|}} (R_\ell|a_\ell|)^{1+2|\alpha-j|}|\beta_j^{a_\ell}|^2\end{aligned}$$

for some positive constant const > 0 independent of ℓ, while

$$\begin{aligned}\int_{\partial D_{R_\ell|a_\ell|}(a_\ell)} |\varphi_n^{a_\ell}|^2\,ds &= R_\ell|a_\ell|\int_0^{2\pi}\bigl|\varphi_n^{a_\ell}(a_\ell+R_\ell|a_\ell|(\cos t,\sin t))\bigr|^2\,dt\\ &\ge \frac{R_\ell|a_\ell|}{2}\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\le|\alpha-k|}} (R_\ell|a_\ell|)^{2|\alpha-j|}|\beta_j^{a_\ell}|^2, \end{aligned}\tag{2-30}$$

thus contradicting (2-28) as $\ell \to \infty$.

To prove the second estimate, let us assume by contradiction that there exist sequences $R_\ell > 1$ and $a_\ell \in \Omega$ such that $R_\ell|a_\ell| < 1/\ell$ and

$$\int_{D_{(R_\ell+1)|a_\ell|}(a_\ell)\setminus D_{R_\ell|a_\ell|}(a_\ell)} |(i\nabla + A_{a_\ell})\varphi_n^{a_\ell}|^2\,dx > \frac{\ell}{R_\ell^2|a_\ell|}\int_{\partial D_{R_\ell|a_\ell|}(a_\ell)} |\varphi_n^{a_\ell}|^2\,ds. \tag{2-31}$$

As above we have that, up to extracting a subsequence, $\varphi_n^{a_\ell} \to \varphi_n^0$ in $L^2(\Omega,\mathbb{C})$ as $\ell\to\infty$ for some $\varphi_n^0 \in H_0^{1,0}(\Omega,\mathbb{C})\setminus\{0\}$ satisfying (2-29). Then, from Lemma 2.5 (first estimate in (iii)) it follows that, for ℓ sufficiently large and for some positive constant const > 0 independent of ℓ,

$$\begin{aligned}&\int_{D_{(R_\ell+1)|a_\ell|}(a_\ell)\setminus D_{R_\ell|a_\ell|}(a_\ell)} |(i\nabla + A_{a_\ell})\varphi_n^{a_\ell}|^2\,dx\\ &\quad= \int_{R_\ell|a_\ell|}^{(R_\ell+1)|a_\ell|} r\biggl(\int_0^{2\pi}|(i\nabla + A_{a_\ell})\varphi_n^{a_\ell}(a_\ell+r(\cos t,\sin t))|^2\,dt\biggr)dr\\ &\quad= \int_{R_\ell|a_\ell|}^{(R_\ell+1)|a_\ell|} r\biggl(\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\le|\alpha-k|}} r^{2|\alpha-j|-2}|\beta_j^{a_\ell}|^2|\alpha-j|^2(2+o(1))\biggr)dr\\ &\quad\le 3|\alpha-k|^2\int_{R_\ell|a_\ell|}^{(R_\ell+1)|a_\ell|} r\biggl(\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\le|\alpha-k|}} r^{2|\alpha-j|-2}|\beta_j^{a_\ell}|^2\biggr)dr \le \frac{\text{const}}{R_\ell}\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\le|\alpha-k|}} (R_\ell|a_\ell|)^{2|\alpha-j|}|\beta_j^{a_\ell}|^2,\end{aligned}$$

which, in view of (2-30), contradicts (2-31) as $\ell\to\infty$. □

Remark 2.8. Arguing as in Lemma 2.7, we can also prove the following similar estimate (possibly taking a smaller σ and a larger C if necessary): for all $R > 1$ and $a \in \Omega$ such that $0 < |a| < \sigma/R$

$$\frac{1}{|a|}\int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)} |\varphi_n^a|^2\,dx \le C\int_{\partial D_{(R+1)|a|}(a)} |\varphi_n^a|^2\,ds,$$
$$\int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)} |(i\nabla + A_a)\varphi_n^a|^2\,dx \le \frac{C}{R^2|a|}\int_{\partial D_{(R+1)|a|}(a)} |\varphi_n^a|^2\,ds.$$

Lemma 2.9. *For $n \ge 1$ fixed, let φ_n^a be a solution to (2-4)–(2-5). Let $\sigma > 0$ and $C > 0$ be as in Lemma 2.7 and Remark 2.8. Then, for all $R > 2$ and $a \in \Omega$ such that $0 < |a| < \sigma/R$,*

$$\left|\int_{\partial D_{R|a|}(0)} |\varphi_n^a|^2\,ds - \int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\,ds\right| \le \frac{1+6C}{R-2}\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\,ds.$$

Proof. We note that

$$\int_{\partial D_{R|a|}(0)} |\varphi_n^a|^2\,ds - \int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\,ds = \int_{\partial\mathcal{L}_{1,R}^a} |\varphi_n^a|^2\tilde{\nu}\cdot\hat{\nu}\,ds - \int_{\partial\mathcal{L}_{2,R}^a} |\varphi_n^a|^2\tilde{\nu}\cdot(-\hat{\nu})\,ds, \tag{2-32}$$

where

$$\mathcal{L}_{1,R}^a = D_{R|a|}(0)\setminus D_{R|a|}(a), \quad \mathcal{L}_{2,R}^a = D_{R|a|}(a)\setminus D_{R|a|}(0),$$

and

$$\hat{\nu}(x) = \begin{cases} x/|x| & \text{on } \partial D_{R|a|}(0), \\ -(x-a)/|x-a| & \text{on } \partial D_{R|a|}(a), \end{cases} \qquad \tilde{\nu}(x) = \begin{cases} x/|x| & \text{on } \partial D_{R|a|}(0), \\ (x-a)/|x-a| & \text{on } \partial D_{R|a|}(a). \end{cases}$$

We note that $\hat{\nu}$ is the outer unit normal vector on $\partial\mathcal{L}_{1,R}^a$ and $-\hat{\nu}$ is the outer unit normal vector on $\partial\mathcal{L}_{2,R}^a$. By setting $\nu_1(x) = x/|x|$, we can rewrite the right-hand side of (2-32) as

$$\begin{aligned}
&\int_{\partial\mathcal{L}_{1,R}^a} |\varphi_n^a|^2(\tilde{\nu}-\nu_1)\cdot\hat{\nu}\,ds + \int_{\partial\mathcal{L}_{1,R}^a} |\varphi_n^a|^2\nu_1\cdot\hat{\nu}\,ds + \int_{\partial\mathcal{L}_{2,R}^a} |\varphi_n^a|^2(\tilde{\nu}-\nu_1)\cdot\hat{\nu}\,ds - \int_{\partial\mathcal{L}_{2,R}^a} |\varphi_n^a|^2\nu_1\cdot(-\hat{\nu})\,ds \\
&\quad = \int_{\partial\mathcal{L}_{1,R}^a} |\varphi_n^a|^2\nu_1\cdot\hat{\nu}\,ds - \int_{\partial\mathcal{L}_{2,R}^a} |\varphi_n^a|^2\nu_1\cdot(-\hat{\nu})\,ds \\
&\qquad + \int_{\partial D_{R|a|}(0)} |\varphi_n^a|^2(\tilde{\nu}-\nu_1)\cdot\hat{\nu}\,ds + \int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2(\tilde{\nu}-\nu_1)\cdot\hat{\nu}\,ds.
\end{aligned} \tag{2-33}$$

We observe that

$$(\tilde{\nu}(x)-\nu_1(x))\cdot\hat{\nu}(x) = \begin{cases} 0 & \text{on } \partial D_{R|a|}(0), \\ -1 + x\cdot(x-a)/(|x|\,|x-a|) & \text{on } \partial D_{R|a|}(a). \end{cases}$$

Moreover, since ν_1 is smooth in $\bar{\mathcal{L}}_{1,R}^a \cup \bar{\mathcal{L}}_{2,R}^a$, we can apply the divergence theorem to the first two terms in the right-hand side of (2-33), thus rewriting the right-hand side of (2-32) as

$$-\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\left(1 - \frac{x\cdot(x-a)}{|x|\,|x-a|}\right)ds + \int_{\mathcal{L}_{1,R}^a} \operatorname{div}(|\varphi_n^a|^2\nu_1)\,dx - \int_{\mathcal{L}_{2,R}^a} \operatorname{div}(|\varphi_n^a|^2\nu_1)\,dx. \tag{2-34}$$

Estimate of the first term in (2-34). Parametrizing $\partial D_{R|a|}(a)$ as $x = a + R|a|(\cos t, \sin t)$ and writing $a = |a|(\cos\theta_a, \sin\theta_a)$ for some angle $\theta_a \in [0, 2\pi)$, we get

$$\left|1 - \frac{x\cdot(x-a)}{|x|\,|x-a|}\right| = \left|1 - \frac{R + \cos(t-\theta_a)}{(R^2 + 2R\cos(t-\theta_a)+1)^{1/2}}\right| \le \frac{1}{R-1}$$

on $\partial D_{R|a|}(a)$. Therefore,

$$\left| -\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2 \left(1 - \frac{x\cdot(x-a)}{|x|\,|x-a|}\right) ds\right| \le \frac{1}{R-1}\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\, ds. \tag{2-35}$$

Estimate of the second term in (2-34). The second term in (2-34) splits into two parts:

$$\int_{\mathcal{L}^a_{1,R}} \operatorname{div}(|\varphi_n^a|^2 \nu_1)\,dx = \int_{\mathcal{L}^a_{1,R}} \frac{|\varphi_n^a|^2}{|x|}\,dx + \int_{\mathcal{L}^a_{1,R}} 2\Re\mathfrak{e}(i\varphi_n^a \overline{(i\nabla + A_a)\varphi_n^a}\cdot \nu_1)\,dx.$$

Since $D_{R|a|}(0) \subset D_{(R+1)|a|}(a)$, we have $\mathcal{L}^a_{1,R} \subseteq D_{(R+1)|a|}(a) \setminus D_{R|a|}(a)$. Let $\sigma > 0$ and $C > 0$ be as in Lemma 2.7 and Remark 2.8. Hence by Lemma 2.7 we have that, for all $R > 1$ and $a \in \Omega$ such that $0 < |a| < \sigma/R$,

$$\begin{aligned}\left|\int_{\mathcal{L}^a_{1,R}} \frac{|\varphi_n^a|^2}{|x|}\,dx\right| &\le \int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)} \frac{|\varphi_n^a|^2}{|x|}\,dx \\ &\le \frac{1}{(R-1)|a|}\int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)} |\varphi_n^a|^2\,dx \le \frac{C}{R-1}\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\,ds\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{\mathcal{L}^a_{1,R}} 2\Re\mathfrak{e}(i\varphi_n^a \overline{(i\nabla + A_a)\varphi_n^a}\cdot \nu_1)\,dx\right| \\ &\qquad\le 2\left(\int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)} |\varphi_n^a|^2\,dx\right)^{1/2}\left(\int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)} |(i\nabla + A_a)\varphi_n^a|^2\,dx\right)^{1/2} \\ &\qquad\le \frac{2C}{R}\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\,ds.\end{aligned}$$

Therefore,

$$\left|\int_{\mathcal{L}^a_{1,R}} \operatorname{div}(|\varphi_n^a|^2\nu_1)\,dx\right| \le \frac{3C}{R-1}\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\,ds \tag{2-36}$$

for all $R > 1$ and $a \in \Omega$ such that $0 < |a| < \sigma/R$.

Estimate of the third term in (2-34). The estimate of the third term can be derived in a similar way, observing that, since $D_{R|a|}(0) \supset D_{(R-1)|a|}(a)$, we have $\mathcal{L}^a_{2,R} \subseteq D_{R|a|}(a) \setminus D_{(R-1)|a|}(a)$, and using Remark 2.8 to obtain

$$\left|\int_{\mathcal{L}^a_{2,R}} \operatorname{div}(|\varphi_n^a|^2\nu_1)\,dx\right| \le \frac{3C}{R-2}\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\,ds \tag{2-37}$$

for all $R > 2$ and $a \in \Omega$ such that $0 < |a| < \sigma/R$ (by possibly changing C and σ).

Therefore combining (2-35)–(2-37) we complete the proof. □

3. Monotonicity formula

The aim of this section is to introduce an Almgren-type frequency function and to use it to obtain local estimates of the eigenfunctions in a neighborhood of order $|a|$ of the singularity. In particular, we shall prove that a suitable family of blow-up of the eigenfunctions φ_a is bounded in the magnetic Sobolev space (see Remark 3.8 ahead).

3A. ***Almgren-type frequency function.*** Arguing as in [Abatangelo and Felli 2015, Lemma 3.1], one can easily prove the *Poincaré-type inequality*

$$\frac{1}{r^2}\int_{D_r}|u|^2\,dx \le \frac{1}{r}\int_{\partial D_r}|u|^2\,ds + \int_{D_r}|(i\nabla + A_a)u|^2\,dx, \tag{3-1}$$

which holds for every $r > 0$, $a \in D_r$, and $u \in H^{1,a}(D_r, \mathbb{C})$. Furthermore, defining, for every $b \in D_1$,

$$m_b := \inf_{\substack{v\in H^{1,b}(D_1,\mathbb{C})\\ v\not\equiv 0}} \frac{\int_{D_1}|(i\nabla + A_b)v|^2\,dx}{\int_{\partial D_1}|v|^2\,ds},$$

we have that the infimum m_b is attained and $m_b > 0$. Arguing as in [Abatangelo and Felli 2015], we can prove that $b \mapsto m_b$ is continuous in D_1 and that $m_0 = \sqrt{\mu_1}$, with μ_1 as in (2-1). Therefore a standard dilation argument yields that, for any $\delta \in (0, \sqrt{\mu_1})$, there exists some sufficiently large $\Upsilon_\delta > 1$ such that, for every $r > 0$ and $a \in D_r$ such that $|a|/r < 1/\Upsilon_\delta$,

$$\frac{\sqrt{\mu_1}-\delta}{r}\int_{\partial D_r}|u|^2\,ds \le \int_{D_r}|(i\nabla + A_a)u|^2\,dx \quad \text{for all } u \in H^{1,a}(D_r, \mathbb{C}). \tag{3-2}$$

For $\lambda \in \mathbb{R}$, $b \in \mathbb{R}^2$, $u \in H^{1,b}(D_r, \mathbb{C})$ and $r > |b|$, we define the Almgren-type frequency function as

$$\mathcal{N}(u, r, \lambda, A_b) = \frac{E(u, r, \lambda, A_b)}{H(u, r)},$$

where

$$E(u, r, \lambda, A_b) = \int_{D_r}\big[|(i\nabla + A_b)u|^2 - \lambda|u|^2\big]\,dx \quad \text{and} \quad H(u, r) = \frac{1}{r}\int_{\partial D_r}|u|^2\,ds.$$

For all $1 \le n \le n_0$ and $a \in \Omega$, let $\varphi_n^a \in H_0^{1,a}(\Omega, \mathbb{C}) \setminus \{0\}$ be an eigenfunction of problem (E_a) associated to the eigenvalue λ_n^a, i.e., solving (2-4), such that

$$\int_\Omega |\varphi_n^a(x)|^2\,dx = 1 \qquad \text{and} \qquad \int_\Omega \varphi_n^a(x)\overline{\varphi_\ell^a(x)}\,dx = 0 \quad \text{if } n \ne \ell. \tag{3-3}$$

For $n = n_0$, we choose

$$\varphi_{n_0}^a = \varphi_a,$$

with φ_a as in (1-14)–(1-15). Let

$$\Lambda = \sup_{\substack{a\in\Omega\\ 1\le n\le n_0}} \lambda_n^a \in (0, +\infty).$$

We recall that Λ is finite in view of the continuity result of the eigenvalue function $a \mapsto \lambda_n^a$ in $\overline{\Omega}$ proved in [Bonnaillie-Noël et al. 2014, Theorem 1.1].

Arguing as in [Abatangelo and Felli 2015, Lemma 5.2], we can prove that there exists

$$0 < R_0 < (\Lambda(1 + 2/\sqrt{\mu_1}))^{-1/2}$$

such that $D_{R_0} \subset \Omega$ and, if $|a| < R_0$,

$$H(\varphi_n^a, r) > 0 \quad \text{for all } r \in (|a|, R_0) \text{ and } 1 \le n \le n_0. \tag{3-4}$$

Furthermore, for every $r \in (0, R_0]$ there exist $C_r > 0$ and $\alpha_r \in (0, r)$ such that

$$H(\varphi_n^a, r) \ge C_r \quad \text{for all } a \text{ with } |a| < \alpha_r \text{ and } 1 \le n \le n_0. \tag{3-5}$$

Thanks to (3-4), the function $r \mapsto N(\varphi_n^a, r, \lambda_n^a, A_a)$ is well-defined in $(|a|, R_0)$. By direct calculations, see [Noris et al. 2015] for details, we can prove that

$$\frac{d}{dr} H(\varphi_n^a, r) = \frac{2}{r} E(\varphi_n^a, r, \lambda_n^a, A_a), \tag{3-6}$$

$$\frac{d}{dr} E(\varphi_n^a, r, \lambda_n^a, A_a) = 2 \int_{\partial D_r} |(i\nabla + A_a)\varphi_n^a \cdot \nu|^2 \, ds - \frac{2}{r}\left(M_n^a + \lambda_n^a \int_{D_r} |\varphi_n^a|^2 \, dx\right) \tag{3-7}$$

where

$$M_n^a = \lim_{\varepsilon \to 0^+} \int_{\partial D_\varepsilon(a)} \left(\Re\mathfrak{e}\big((i\nabla + A_a)\varphi_n^a \cdot \nu \overline{(i\nabla + A_a)\varphi_n^a \cdot x}\big) - \tfrac{1}{2}|(i\nabla + A_a)\varphi_n^a|^2 x \cdot \nu\right) ds. \tag{3-8}$$

Lemma 2.9, together with Lemmas 2.2 and 2.4, allow us to give an estimate of the quantity M_n^a defined in (3-8). We notice that the techniques used in [Abatangelo and Felli 2015; Noris et al. 2015] to estimate the term M_n^a for $\alpha = \frac{1}{2}$ were based on the possibility of rewriting the problem as a Laplace equation on the twofold covering; hence it is not possible here to extend such proofs to the case $\alpha \notin \mathbb{Z}/2$ and a new strategy of proof is needed.

Lemma 3.1. *For $n \in \{1, \dots, n_0\}$ and $a \in \Omega$, let φ_n^a be a solution of* (2-4) *satisfying* (3-3)*. There exist $\sigma_0 > 0$ and $c_0 > 2$ such that, for every $1 \le n \le n_0$, $R > c_0$ and $a \in \Omega$ such that $|a| < \sigma_0/R$, the quantity M_n^a defined in* (3-8) *satisfies*

$$\frac{|M_n^a|}{H(\varphi_n^a, R|a|)} \le \frac{2\alpha(1-\alpha)}{R - c_0}.$$

Proof. Let us fix $n \in \{1, 2, \dots, n_0\}$ and define, for $|a|$ small and $r \in (0, 1]$,

$$\widetilde{H}(\varphi_n^a, r) = \frac{1}{r} \int_{\partial D_r(a)} |\varphi_n^a|^2 \, ds.$$

From the Parseval identity and Lemma 2.2(iv) it follows that there exists $\sigma_n > 0$ such that, for every $R > 2$ and $a \in \Omega$ such that $|a| < \sigma_n/R$,

$$\begin{aligned}
\widetilde{H}(\varphi_n^a, R|a|) &= \int_0^{2\pi} \left|\varphi_n^a(a + R|a|(\cos t, \sin t))\right|^2 dt = \sum_{j \in \mathbb{Z}} (R|a|)^{2|\alpha - j|} |\beta_j^a|^2 \left|1 + R_{j,a}(R|a|)\right|^2 \\
&\ge (R|a|)^{2\alpha} |\beta_0^a|^2 \left|1 + R_{0,a}(R|a|)\right|^2 + (R|a|)^{2(1-\alpha)} |\beta_1^a|^2 \left|1 + R_{1,a}(R|a|)\right|^2 \\
&\ge \tfrac{1}{2}\big(|\beta_0^a|^2 (R|a|)^{2\alpha} + |\beta_1^a|^2 (R|a|)^{2(1-\alpha)}\big),
\end{aligned} \tag{3-9}$$

where the β_j^a's are the coefficients defined in (2-10) for the eigenfunction φ_n^a (with n fixed). From the elementary inequality $ab \le \frac{1}{2}(a^2+b^2)$, it follows that

$$|\beta_0^a||\beta_1^a||a| = \frac{1}{R}|\beta_0^a|(R|a|)^{\alpha}|\beta_1^a|(R|a|)^{1-\alpha} \le \frac{1}{2R}\big(|\beta_0^a|^2(R|a|)^{2\alpha} + |\beta_1^a|^2(R|a|)^{2(1-\alpha)}\big). \tag{3-10}$$

Combining (3-9) and (3-10) we obtain

$$\frac{|\beta_0^a||\beta_1^a||a|}{\widetilde{H}(\varphi_n^a, R|a|)} \le \frac{1}{R}. \tag{3-11}$$

Moreover, Lemma 2.4 implies

$$|M_n^a| \le 2\alpha(1-\alpha)|\beta_0^a||\beta_1^a||a|. \tag{3-12}$$

Lemma 2.9 provides some constant c_n (independent of a and R) such that, for a possibly smaller σ_n and for all $R > 2$ and $a \in \Omega$ such that $0 < |a| < \sigma_n/R$,

$$\big|H(\varphi_n^a, R|a|) - \widetilde{H}(\varphi_n^a, R|a|)\big| \le \frac{c_n}{R-2}\widetilde{H}(\varphi_n^a, R|a|). \tag{3-13}$$

Therefore, by combining (3-11)–(3-13), we obtain

$$\frac{|M_n^a|}{H(\varphi_n^a, R|a|)} \le \frac{2\alpha(1-\alpha)}{R}\left(1 + \frac{H(\varphi_n^a, R|a|) - \widetilde{H}(\varphi_n^a, R|a|)}{\widetilde{H}(\varphi_n^a, R|a|)}\right)^{-1} \le \frac{2\alpha(1-\alpha)}{R}\frac{1}{1-\frac{c_n}{R-2}} \le \frac{2\alpha(1-\alpha)}{R-(2+c_n)}$$

for all $R > c_n + 2$ and $a \in \Omega$ such that $0 < |a| < \sigma_n/R$.

The conclusion then follows by repeating the argument for all $n \in \{1, 2, \dots, n_0\}$ and choosing

$$\sigma_0 = \min\{\sigma_n : 1 \le n \le n_0\} \quad \text{and} \quad c_0 = \max\{2 + c_n : 1 \le n \le n_0\}. \qquad \square$$

Lemma 3.2. *For $\delta \in (0, \sqrt{\mu_1}/2)$, let Υ_δ be such that* (3-2) *holds. Let R_0 be as above, $r_0 \le R_0$ and $n \in \{1, \dots, n_0\}$. If $\Upsilon_\delta|a| \le r_1 < r_2 \le r_0$ and φ_n^a is a solution to* (2-4) *satisfying* (3-3), *then*

$$\frac{H(\varphi_n^a, r_2)}{H(\varphi_n^a, r_1)} \ge e^{-\Lambda(2+\sqrt{\mu_1})r_0^2}\left(\frac{r_2}{r_1}\right)^{2(\sqrt{\mu_1}-\delta)}.$$

Proof. Combining (3-1) with (3-2) we obtain that, for every $\Upsilon_\delta|a| < r < r_0$,

$$\begin{aligned}\frac{1}{r^2}\int_{D_r}|\varphi_n^a|^2\,dx &\le \left(1 + \frac{1}{\sqrt{\mu_1}-\delta}\right)\int_{D_r}|(i\nabla + A_a)\varphi_n^a|^2\,dx \\ &\le \left(1 + \frac{2}{\sqrt{\mu_1}}\right)\int_{D_r}|(i\nabla + A_a)\varphi_n^a|^2\,dx.\end{aligned}$$

From above, (3-6) and (3-2), we have that, for every $\Upsilon_\delta|a| < r < r_0$,

$$\begin{aligned}\frac{d}{dr}H(\varphi_n^a, r) &\ge \frac{2}{r}\left(1 - \Lambda r^2\left(1 + \frac{2}{\sqrt{\mu_1}}\right)\right)\int_{D_r}|(i\nabla + A_a)\varphi_n^a|^2\,dx \\ &\ge \frac{2}{r}\left(1 - \Lambda r^2\left(1 + \frac{2}{\sqrt{\mu_1}}\right)\right)(\sqrt{\mu_1}-\delta)H(\varphi_n^a, r),\end{aligned}$$

so that, in view of (3-4),

$$\frac{d}{dr}\log H(\varphi_n^a, r) \geq \frac{2}{r}(\sqrt{\mu_1} - \delta) - 2\Lambda r(2 + \sqrt{\mu_1}).$$

Integrating between r_1 and r_2 we obtain the desired inequality. □

Lemma 3.3. *For $n \in \{1, \dots, n_0\}$ and $a \in \Omega$, let φ_n^a be a solution of* (2-4) *satisfying* (3-3). *Let R_0 be as above, σ_0 and $c_0 > 0$ be as in Lemma 3.1 and let $r_0 \leq \min\{R_0, \sigma_0\}$. For $\delta \in (0, \sqrt{\mu_1}/2)$, let $\Upsilon_\delta > 1$ be such that* (3-2) *holds. Then, there exists $c_{r_0,\delta} > 0$ such that for all $R > \max\{\Upsilon_\delta, c_0\}$, $|a| < r_0/R$, $R|a| \leq r < r_0$ and $n \in \{1, \dots, n_0\}$,*

$$e^{\Lambda r^2/(1-\Lambda r_0^2)}(\mathcal{N}(\varphi_n^a, r, \lambda_n^a, A_a) + 1) \leq e^{\Lambda r_0^2/(1-\Lambda r_0^2)}(\mathcal{N}(\varphi_n^a, r_0, \lambda_n^a, A_a) + 1) + \frac{c_{r_0,\delta}}{R - c_0}.$$

Proof. By direct calculations, using the expressions for the derivatives of the functions $H(\varphi_n^a, r)$ and $E(\varphi_n^a, r, \lambda_n^a, A_a)$ written in (3-6) and (3-7) and the Cauchy–Schwarz inequality, we obtain

$$\frac{d}{dr}\mathcal{N}(\varphi_n^a, r, \lambda_n^a, A_a) \geq -\frac{2|M_n^a|}{rH(\varphi_n^a, r)} - \frac{2\lambda_n^a}{rH(\varphi_n^a, r)}\int_{D_r} |\varphi_n^a|^2\, dx. \tag{3-14}$$

By Lemmas 3.2 and 3.1 the first term can be estimated as

$$\begin{aligned} -\frac{2|M_n^a|}{rH(\varphi_n^a, r)} &= -\frac{2|M_n^a|}{rH(\varphi_n^a, R|a|)}\frac{H(\varphi_n^a, R|a|)}{H(\varphi_n^a, r)} \\ &\geq -\frac{4\alpha(1-\alpha)}{R - c_0}e^{\Lambda(2+\sqrt{\mu_1})r_0^2}(R|a|)^{2(\sqrt{\mu_1}-\delta)}r^{-2(\sqrt{\mu_1}-\delta)-1} \end{aligned} \tag{3-15}$$

for all $R > \max\{\Upsilon_\delta, c_0\}$, $|a| < r_0/R$, $R|a| \leq r < r_0$ and $n \in \{1, \dots, n_0\}$.

For the second term, the Poincaré inequality (3-1) leads to

$$\frac{1 - \Lambda r_0^2}{r^2}\int_{D_r} |\varphi_n^a|^2\, dx \leq E(\varphi_n^a, r, \lambda_n^a, A_a) + H(\varphi_n^a, r)$$

for $r < r_0$, which implies

$$-\frac{2r\lambda_n^a}{r^2 H(\varphi_n^a, r)}\int_{D_r} |\varphi_n^a|^2\, dx \geq -\frac{2\Lambda r}{1 - \Lambda r_0^2}(\mathcal{N}(\varphi_n^a, r, \lambda_n^a, A_a) + 1) \tag{3-16}$$

for $r < r_0$. Using (3-15) and (3-16) we can estimate the right-hand side of (3-14) thus obtaining

$$\begin{aligned} &\frac{d}{dr}(e^{\Lambda r^2/(1-\Lambda r_0^2)}(\mathcal{N}(\varphi_n^a, r, \lambda_n^a, A_a) + 1)) \\ &\qquad \geq -\frac{4\alpha(1-\alpha)}{R - c_0}e^{\Lambda r_0^2/(1-\Lambda r_0^2)}e^{\Lambda(2+\sqrt{\mu_1})r_0^2}(R|a|)^{2(\sqrt{\mu_1}-\delta)}r^{-2(\sqrt{\mu_1}-\delta)-1} \end{aligned}$$

for all $R|a| \leq r < r_0$ with $R > \max\{\Upsilon_\delta, c_0\}$. Integrating between r and r_0 and using the fact that $R|a| \leq r < r_0$, we obtain the statement with

$$c_{r_0,\delta} = \frac{2\alpha(1-\alpha)}{\sqrt{\mu_1} - \delta}e^{\Lambda(2+\sqrt{\mu_1})r_0^2 + \Lambda r_0^2/(1-\Lambda r_0^2)}.$$

□

Lemma 3.4. *Let φ_a be a solution of* (1-14)–(1-15) *and let k be as in* (1-8). *For every $\delta \in (0, \sqrt{\mu_1}/2)$, there exist $r_\delta \in (0, R_0)$ and $K_\delta > \Upsilon_\delta$ such that if $R > K_\delta$, $|a| < r_\delta/R$ and $R|a| \le r < r_\delta$, then*

$$\mathcal{N}(\varphi_a, r, \lambda_a, A_a) \le |\alpha - k| + \delta.$$

Proof. From (1-16)–(1-17) it follows that, for every $r < R_0$,

$$\lim_{a\to 0} \mathcal{N}(\varphi_a, r, \lambda_a, A_a) = \mathcal{N}(\varphi_0, r, \lambda_0, A_0).$$

Moreover, from [Felli et al. 2011, Theorem 1.3] we know that, under assumption (1-9),

$$\lim_{r\to 0^+} \mathcal{N}(\varphi_0, r, \lambda_0, A_0) = |\alpha - k|.$$

Then, the proof is a direct consequence of Lemma 3.3; see [Noris et al. 2015, Lemma 7.2; Abatangelo and Felli 2015, Lemma 5.7; Abatangelo et al. 2017, Lemma 5.7] for details. □

3B. *Local energy estimates.*

Corollary 3.5. *For $\delta \in (0, \sqrt{\mu_1}/2)$ let r_δ, K_δ be as in Lemma 3.4 and α_{r_δ} be as in* (3-5). *Then there exists $C_\delta > 0$ such that*

$$H(\varphi_a, R|a|) \le H(\varphi_a, K_\delta|a|)\left(\frac{R}{K_\delta}\right)^{2(|\alpha-k|+\delta)} \quad \textit{for all } R > K_\delta \textit{ and } |a| < \frac{r_\delta}{R}, \tag{3-17}$$

$$H(\varphi_a, K_\delta|a|) \ge C_\delta |a|^{2(|\alpha-k|+\delta)} \quad \textit{for all } |a| < \min\left\{\frac{r_\delta}{K_\delta}, \alpha_{r_\delta}\right\}, \tag{3-18}$$

$$H(\varphi_a, K_\delta|a|) = O(|a|^{2(\sqrt{\mu_1}-\delta)}) \quad \textit{as } a \to 0. \tag{3-19}$$

Proof. From (3-6), the definition of $\mathcal{N}$, and Lemma 3.4 we have

$$\begin{aligned}\frac{1}{H(\varphi_a, r)}\frac{d}{dr}H(\varphi_a, r) &= \frac{2}{r}\mathcal{N}(\varphi_a, r, \lambda_a, A_a)\\ &\le \frac{2}{r}(|\alpha-k|+\delta) \quad \text{for all } K_\delta|a| \le r < r_\delta \text{ with } |a| < \frac{r_\delta}{K_\delta}\end{aligned}$$

so that estimate (3-17) follows by integration over $[K_\delta|a|, R|a|]$ and estimate (3-18) from integration over $[K_\delta|a|, r_\delta]$ and (3-5). Finally (3-19) is a direct consequence of Lemma 3.2. □

Lemma 3.6. *For $n \in \{1, \dots, n_0\}$ and $a \in \Omega$, let φ_n^a be a solution to* (2-4) *satisfying* (3-3). *Let $R_0 > 0$ be as in* (3-4). *For every $\delta \in (0, \sqrt{\mu_1}/2)$, there exist $\widetilde{K}_\delta > 1$ and $\widetilde{C}_\delta > 0$ such that, for all $R > \widetilde{K}_\delta$, $a \in \Omega$ with $R|a| < R_0$, and $n \in \{1, \dots, n_0\}$,*

$$\int_{D_{R|a|}} |(i\nabla + A_a)\varphi_n^a|^2\,dx \le \widetilde{C}_\delta (R|a|)^{2(\sqrt{\mu_1}-\delta)}, \tag{3-20}$$

$$\int_{\partial D_{R|a|}} |\varphi_n^a|^2\,ds \le \widetilde{C}_\delta (R|a|)^{2(\sqrt{\mu_1}-\delta)+1}, \tag{3-21}$$

$$\int_{D_{R|a|}} |\varphi_n^a|^2\,dx \le \widetilde{C}_\delta (R|a|)^{2(\sqrt{\mu_1}-\delta)+2}. \tag{3-22}$$

Proof. By Lemma 3.2 (choosing $r_1 = R|a|$ and $r_2 = R_0$) and the definition of H it follows that

$$\int_{\partial D_{R|a|}} |\varphi_n^a|^2\, ds = R|a| H(\varphi_n^a, R|a|) \le R|a| H(\varphi_n^a, R_0) e^{\Lambda(2+\sqrt{\mu_1})R_0^2} \left(\frac{R|a|}{R_0}\right)^{2(\sqrt{\mu_1}-\delta)}. \tag{3-23}$$

Moreover, from (2-7) and continuous trace embeddings we have $H(\varphi_n^a, R_0) = (1/R_0)\int_{\partial D_{R_0}} |\varphi_n^a|^2\, ds$ is bounded uniformly with respect to a. Hence estimate (3-23) implies (3-21).

From Lemma 3.3 it follows that the frequency $\mathcal{N}$ is bounded in $r = R|a|$ provided R is sufficiently large; hence $E(\varphi_n^a, R|a|, \lambda_n^a, A_a)$ is uniformly estimated by $H(\varphi_n^a, R|a|)$, so that (3-21) and (3-1)–(3-2) yield (3-20). Estimate (3-22) can be proved combining (3-20)–(3-21) with the Poincaré inequality (3-1). We refer to [Abatangelo and Felli 2015, Lemma 5.8] for more details in a related problem. □

Lemma 3.7. *For $a \in \Omega$ let $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C})$ be a solution of* (1-14)–(1-15). *For some fixed $\delta \in (0, \sqrt{\mu_1}/2)$, let $K_\delta > \Upsilon_\delta$ be as in Lemma 3.4. Then, for every $R > K_\delta$,*

$$\int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\, dx = O(H(\varphi_a, K_\delta|a|)) \qquad \text{as } |a| \to 0^+, \tag{3-24}$$

$$\int_{\partial D_{R|a|}} |\varphi_a|^2\, ds = O(|a| H(\varphi_a, K_\delta|a|)) \qquad \text{as } |a| \to 0^+, \tag{3-25}$$

$$\int_{D_{R|a|}} |\varphi_a|^2\, dx = O(|a|^2 H(\varphi_a, K_\delta|a|)) \qquad \text{as } |a| \to 0^+. \tag{3-26}$$

Proof. The proof follows from the boundedness of the frequency $\mathcal{N}(\varphi_a, R|a|, \lambda_a, A_a)$ established in Lemma 3.4 and by its scaling properties. For $\delta \in (0, \sqrt{\mu_1}/2)$ fixed, let $K_\delta > \Upsilon_\delta$ and r_δ be as in Lemma 3.4; hence

$$\begin{aligned} N(\varphi_a, R|a|, \lambda_a, A_a) &= \frac{\int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\, dx - \lambda_a \int_{D_{R|a|}} |\varphi_a|^2\, dx}{H(\varphi_a, R|a|)} \\ &\le |\alpha - k| + \delta \quad \text{for all } R > K_\delta \text{ and } |a| < \frac{r_\delta}{R}. \end{aligned}$$

Then, by (3-1) and (3-2) it follows that

$$\begin{aligned} \left(1 - \Lambda r_\delta^2 \left(1 + \frac{2}{\sqrt{\mu_1}}\right)\right) \int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\, dx &\le \int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\, dx - \lambda_a \int_{D_{R|a|}} |\varphi_a|^2\, dx \\ &\le H(\varphi_a, R|a|)(|\alpha - k| + \delta). \end{aligned}$$

Then (3-24) follows from (3-17). Estimates (3-25) and (3-26) follow from (3-24) and the Poincaré-type inequalities (3-1) and (3-2). □

Remark 3.8. Let us consider the blow-up family

$$\tilde{\varphi}_a(x) := \frac{\varphi_a(|a|x)}{\sqrt{H(\varphi_a, K_\delta|a|)}}, \tag{3-27}$$

with $K_\delta > \Upsilon_\delta$ as in Lemma 3.4 for some fixed $\delta \in (0, \sqrt{\mu_1}/2)$. By Lemma 3.7 it follows that, for every $p \in \mathbb{S}^1$ fixed, $r_\delta > 0$ as in Lemma 3.4, and $R > K_\delta$, the blow-up family $\{\tilde{\varphi}_a : a = |a|p,\ R|a| < r_\delta\}$ is bounded in $H^{1,p}(D_R, \mathbb{C})$.

4. Estimate on $\lambda_0 - \lambda_a$

The aim of this section is to obtain a bound (both from above and from below) of the eigenvalue variation $\lambda_a - \lambda_0$. These bounds are obtained by considering suitable competitor functions and by plugging them into the Courant–Fischer characterization of λ_a and λ_0:

$$\lambda_a = \min\left\{ \max_{u \in F\setminus\{0\}} \frac{\int_\Omega |(i\nabla + A_a)u|^2\,dx}{\int_\Omega |u|^2\,dx} : F \text{ is a linear subspace of } H_0^{1,a}(\Omega,\mathbb{C}),\ \dim F = n_0 \right\}, \tag{4-1}$$

$$\lambda_0 = \min\left\{ \max_{u \in F\setminus\{0\}} \frac{\int_\Omega |(i\nabla + A_0)u|^2\,dx}{\int_\Omega |u|^2\,dx} : F \text{ is a linear subspace of } H_0^{1,0}(\Omega,\mathbb{C}),\ \dim F = n_0 \right\}. \tag{4-2}$$

In Section 4A we construct the competitor function for λ_a. This function is obtained by modifying φ_n^0 in a small neighborhood of a. Since the asymptotics of φ_n^0 is exactly known, this allows us to obtain, in Section 4B, a sharp bound from below of $\lambda_0 - \lambda_a$. The competitor function for λ_0 is constructed in Section 4C, by modifying locally φ_n^a. The energy estimates obtained in Section 3 allow us to obtain a preliminary estimate from above of $\lambda_0 - \lambda_a$ in terms of the quantity $H(\varphi_a, K_\delta|a|)$.

Before proceeding, we find it useful to recall the following technical result, which is proved in [Abatangelo and Felli 2015, Lemma 6.1] and concerns the maximum of quadratic forms depending on the pole $a \to 0$.

Lemma 4.1. *For every $a \in \Omega$, let us consider a quadratic form*

$$Q_a : \mathbb{C}^{n_0} \to \mathbb{R}, \quad Q_a(z_1, z_2, \dots, z_{n_0}) = \sum_{j,n=1}^{n_0} M_{j,n}(a) z_j \bar{z}_n,$$

with $M_{j,n}(a) \in \mathbb{C}$ such that $M_{j,n}(a) = \overline{M_{n,j}(a)}$. Let us assume that there exist $\gamma \in (0,+\infty)$, $a \mapsto \sigma(a) \in \mathbb{R}$ with $\sigma(a) \ge 0$ and $\sigma(a) = O(|a|^{2\gamma})$ as $|a| \to 0^+$, and $a \mapsto \mu(a) \in \mathbb{R}$ with $\mu(a) = O(1)$ as $|a| \to 0^+$, such that the coefficients $M_{j,n}(a)$ satisfy the following conditions:

(i) $M_{n_0,n_0}(a) = \sigma(a)\mu(a)$.

(ii) *For all $j < n_0$, we have $M_{j,j}(a) \to M_j$ as $|a| \to 0^+$ for some $M_j \in \mathbb{R}$, $M_j < 0$.*

(iii) *For all $j < n_0$, we have $M_{j,n_0}(a) = \overline{M_{n_0,j}(a)} = O(|a|^\gamma \sqrt{\sigma(a)})$ as $|a| \to 0^+$.*

(iv) *For all $j, n < n_0$ with $j \ne n$, we have $M_{j,n}(a) = O(|a|^{2\gamma})$ as $|a| \to 0^+$.*

(v) *There exists $M \in \mathbb{N}$ such that $|a|^{(2+M)\gamma} = o(\sigma(a))$ as $|a| \to 0^+$.*

Then

$$\max_{\substack{z \in \mathbb{C}^{n_0} \\ \|z\|=1}} Q_a(z) = \sigma(a)(\mu(a) + o(1)) \quad \text{as } |a| \to 0^+,$$

where $\|z\| = \|(z_1, z_2, \dots, z_{n_0})\| = \left(\sum_{j=1}^{n_0} |z_j|^2\right)^{1/2}$.

4A. ***Construction of the test functions using φ_n^0.*** Recall that $\varphi_n^0 \in H_0^{1,0}(\Omega, \mathbb{C}) \setminus \{0\}$ is a solution of (2-4), also satisfying (2-5), with $a = 0$. Let R_0 be as in (3-4). For every $R > 1$, $a \in \Omega$ with $|a| < R_0/R$ and $1 \le n \le n_0$ we define

$$w_{n,R,a} = \begin{cases} w_{n,R,a}^{\text{int}} & \text{in } D_{R|a|}, \\ w_{n,R,a}^{\text{ext}} & \text{in } \Omega \setminus D_{R|a|}, \end{cases} \tag{4-3}$$

where

$$w_{n,R,a}^{\text{ext}} = e^{i\alpha(\theta_a - \theta_0^a)} \varphi_n^0 \quad \text{in } \Omega \setminus D_{R|a|},$$

and $w_{n,R,a}^{\text{int}}$ is the unique solution to the minimization problem

$$\min\left\{ \int_{D_{R|a|}} |(i\nabla + A_a)u|^2 \, dx : u \in H^{1,a}(D_{R|a|}, \mathbb{C}),\ u = e^{i\alpha(\theta_a - \theta_0^a)} \varphi_n^0 \text{ on } \partial D_{R|a|} \right\}.$$

We notice that $w_{n,R,a}^{\text{ext}}$ and $w_{n,R,a}^{\text{int}}$ respectively solve

$$\begin{cases} (i\nabla + A_a)^2 w_{n,R,a}^{\text{ext}} = \lambda_n^0 w_{n,R,a}^{\text{ext}} & \text{in } \Omega \setminus D_{R|a|}, \\ w_{n,R,a}^{\text{ext}} = e^{i\alpha(\theta_a - \theta_0^a)} \varphi_n^0 & \text{on } \partial(\Omega \setminus D_{R|a|}) \end{cases} \quad \text{and} \quad \begin{cases} (i\nabla + A_a)^2 w_{n,R,a}^{\text{int}} = 0 & \text{in } D_{R|a|}, \\ w_{n,R,a}^{\text{int}} = e^{i\alpha(\theta_a - \theta_0^a)} \varphi_n^0 & \text{on } \partial D_{R|a|}. \end{cases}$$

As a consequence of Proposition 2.1 we have $\varphi_n^0(x) = O(|x|^{|\alpha - j|})$ as $x \to 0$ for some $j \in \mathbb{Z}$, which implies

$$\varphi_n^0(x) = O(|x|^{\sqrt{\mu_1}}) \quad \text{as } x \to 0, \tag{4-4}$$

since $|\alpha - j| \ge \sqrt{\mu_1}$ for all $j \in \mathbb{Z}$. Furthermore (2-2) implies

$$\begin{aligned} \int_{D_r} |(i\nabla + A_0)\varphi_n^0|^2 \, dx &= \lambda_0 \int_{D_r} |\varphi_n^0|^2 \, dx + \frac{|\alpha - j| + o(1)}{r} \int_{\partial D_r} |\varphi_n^0|^2 \, ds \\ &= O(r^{2\sqrt{\mu_1}}) \quad \text{as } r \to 0^+. \end{aligned} \tag{4-5}$$

From (4-4) and (4-5) we deduce that, for every $R > 1$, $a \in \Omega$ such that $R|a| < R_0$, and $1 \le n \le n_0$,

$$\begin{gathered} \int_{D_{R|a|}} |(i\nabla + A_0)\varphi_n^0|^2 \, dx = O(|a|^{2\sqrt{\mu_1}}), \quad \int_{\partial D_{R|a|}} |\varphi_n^0|^2 \, ds = O(|a|^{2\sqrt{\mu_1}+1}), \\ \text{and} \quad \int_{D_{R|a|}} |\varphi_n^0|^2 \, dx = O(|a|^{2\sqrt{\mu_1}+2}) \quad \text{as } |a| \to 0^+. \end{gathered} \tag{4-6}$$

Using the above estimates (4-6) and the Dirichlet principle (see the proof of [Abatangelo and Felli 2015, Lemma 6.2] for details in the case of half-integer circulation), we obtain that, for every $R > 2$ and $1 \le n \le n_0$,

$$\begin{gathered} \int_{D_{R|a|}} |(i\nabla + A_a)w_{n,R,a}^{\text{int}}|^2 \, dx = O(|a|^{2\sqrt{\mu_1}}), \quad \int_{\partial D_{R|a|}} |w_{n,R,a}^{\text{int}}|^2 \, ds = O(|a|^{2\sqrt{\mu_1}+1}), \\ \text{and} \quad \int_{D_{R|a|}} |w_{n,R,a}^{\text{int}}|^2 \, dx = O(|a|^{2\sqrt{\mu_1}+2}) \quad \text{as } |a| \to 0^+. \end{gathered} \tag{4-7}$$

The above estimates can be made more precise in the case $n = n_0$ in view of (1-9): for every $R > 2$ and $a \in \Omega$ with $R|a| < R_0$,

$$\int_{D_{R|a|}} |(i\nabla + A_0)\varphi_0|^2\,dx = O(|a|^{2|\alpha-k|}), \quad \int_{\partial D_{R|a|}} |\varphi_0|^2\,ds = O(|a|^{2|\alpha-k|+1}),$$
$$\text{and} \quad \int_{D_{R|a|}} |\varphi_0|^2\,dx = O(|a|^{2|\alpha-k|+2}) \quad \text{as } |a| \to 0^+, \tag{4-8}$$

and consequently, in view of the Dirichlet principle,

$$\int_{D_{R|a|}} |(i\nabla + A_a)w^{\text{int}}_{n_0,R,a}|^2\,dx = O(|a|^{2|\alpha-k|}), \quad \int_{\partial D_{R|a|}} |w^{\text{int}}_{n_0,R,a}|^2\,ds = O(|a|^{2|\alpha-k|+1}),$$
$$\text{and} \quad \int_{D_{R|a|}} |w^{\text{int}}_{n_0,R,a}|^2\,dx = O(|a|^{2|\alpha-k|+2}) \quad \text{as } |a| \to 0^+, \tag{4-9}$$

with k as in (1-8). Furthermore, defining

$$W_a(x) := \frac{\varphi_0(|a|x)}{|a|^{|\alpha-k|}} \tag{4-10}$$

for all $R > 2$ and $a \in \Omega$ such that $R|a| < R_0$, (1-9) implies

$$W_a \to \beta\psi_k \quad \text{in } H^{1,0}(D_R, \mathbb{C}), \text{ as } |a| \to 0, \tag{4-11}$$

where ψ_k is defined in (1-20).

4B. *Estimate of the Rayleigh quotient for λ_a.*

Lemma 4.2. *There exists $\mathfrak{c} \in \mathbb{R}$ such that*

$$\lambda_0 - \lambda_a \ge \mathfrak{c}|a|^{2|\alpha-k|} \quad \textit{for all } a \in \Omega,$$

where k is as in (1-8).

Proof. The proof follows along the lines of [Abatangelo and Felli 2015, Lemma 6.7; Abatangelo et al. 2017, Lemma 7.2]. Let $w_{n,R,a}$ be defined in (4-3). Let us fix $R > 2$. By proceeding with a Gram–Schmidt process we define

$$\tilde{w}_{n,a} = \frac{\hat{w}_{n,a}}{\|\hat{w}_{n,a}\|_{L^2(\Omega,\mathbb{C})}}, \quad 1 \le n \le n_0,$$

where

$$\hat{w}_{n_0,a} = w_{n_0,R,a},$$
$$\hat{w}_{n,a} = w_{n,R,a} - \sum_{\ell=n+1}^{n_0} c^a_{\ell,n}\hat{w}_{\ell,a}, \quad 1 \le n \le n_0 - 1,$$

and

$$c^a_{\ell,n} = \frac{\int_\Omega w_{n,R,a}\bar{\hat{w}}_{\ell,a}\,dx}{\|\hat{w}_{\ell,a}\|^2_{L^2(\Omega,\mathbb{C})}}, \quad 1 \le n \le n_0 - 1, \; n+1 \le \ell \le n_0.$$

From (4-6), (4-7) and an induction argument it follows that, for all ℓ, n such that $1 \le n \le n_0 - 1$ and $n+1 \le \ell \le n_0$,

$$\|\hat{w}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})} = 1 + O(|a|^{2\sqrt{\mu_1}+2}) \quad \text{and} \quad c^a_{\ell,n} = O(|a|^{2\sqrt{\mu_1}+2}) \tag{4-12}$$

as $|a| \to 0$. Moreover, from (4-8) and (4-9) we have

$$\|\hat{w}_{n_0,a}\|^2_{L^2(\Omega,\mathbb{C})} = \|w_{n_0,R,a}\|^2_{L^2(\Omega,\mathbb{C})} = 1 + O(|a|^{2|\alpha-k|+2}) \quad \text{as } |a| \to 0, \tag{4-13}$$

and

$$c^a_{n_0,n} = O(|a|^{|\alpha-k|+\sqrt{\mu_1}+2}) \quad \text{as } |a| \to 0, \text{ for } 1 \le n \le n_0 - 1. \tag{4-14}$$

Since $\dim(\operatorname{span}\{w_{1,R,a}, \dots, w_{n_0,R,a}\}) = n_0$, we have that also $\dim(\operatorname{span}\{\tilde{w}_{1,a}, \dots, \tilde{w}_{n_0,a}\}) = n_0$, and hence from (4-1) we deduce that

$$\lambda_a \le \max_{\substack{(\alpha_1,\dots,\alpha_{n_0}) \in \mathbb{C}^{n_0} \\ \sum_{n=1}^{n_0} |\alpha_n|^2 = 1}} \int_\Omega \left| (i\nabla + A_a) \left(\sum_{n=1}^{n_0} \alpha_n \tilde{w}_{n,a} \right) \right|^2 dx,$$

which leads to

$$\lambda_a - \lambda_0 \le \max_{\substack{(\alpha_1,\dots,\alpha_{n_0}) \in \mathbb{C}^{n_0} \\ \sum_{n=1}^{n_0} |\alpha_n|^2 = 1}} \sum_{n,j=1}^{n_0} \alpha_n \bar{\alpha}_j \, p^a_{n,j}, \tag{4-15}$$

where $p^a_{n,j} = \int_\Omega (i\nabla + A_a)\tilde{w}_{n,a} \cdot \overline{(i\nabla + A_a)\tilde{w}_{j,a}} \, dx - \lambda_0 \delta_{nj}$, with $\delta_{nj} = 1$ if $n = j$ and $\delta_{nj} = 0$ otherwise. Using the estimates above we can now estimate $p^a_{n,j}$. First, using (4-8), (4-9), and (4-13)

$$\begin{aligned} p^a_{n_0,n_0} &= \frac{\lambda_0}{\int_\Omega |w_{n_0,R,a}|^2 \, dx} \left(1 - \int_\Omega |w_{n_0,R,a}|^2 \, dx \right) \\ &\quad + \frac{1}{\int_\Omega |w_{n_0,R,a}|^2 \, dx} \left(\int_{D_{R|a|}} |(i\nabla + A_a) w^{\text{int}}_{n_0,R,a}|^2 \, dx - \int_{D_{R|a|}} |(i\nabla + A_0)\varphi_0|^2 \, dx \right) \\ &= O(|a|^{2|\alpha-k|+2}) + O(|a|^{2|\alpha-k|}) \\ &= |a|^{2|\alpha-k|} O(1) \quad \text{as } |a| \to 0^+. \end{aligned}$$

Next (4-6), (4-7) and (4-12) provide for $n < n_0$

$$\begin{aligned} p^a_{n,n} &= -\lambda_0 + \frac{1}{\|\hat{w}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})}} \left(\lambda^0_n + \int_{D_{R|a|}} |(i\nabla + A_a) w^{\text{int}}_{n,R,a}|^2 \, dx - \int_{D_{R|a|}} |(i\nabla + A_0)\varphi^0_n|^2 \, dx \right) \\ &\quad + \frac{1}{\|\hat{w}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})}} \int_\Omega \left| (i\nabla + A_a) \left(\sum_{\ell=n+1}^{n_0} c^a_{\ell,n} \hat{w}_{\ell,a} \right) \right|^2 dx \\ &\quad - \frac{2}{\|\hat{w}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})}} \mathfrak{Re} \sum_{\ell=n+1}^{n_0} \left\{ \bar{c}^a_{\ell,n} \int_\Omega (i\nabla + A_a) w_{n,R,a} \cdot \overline{(i\nabla + A_a)\hat{w}_{\ell,a}} \, dx \right\} \\ &= (\lambda^0_n - \lambda_0) + o(1), \end{aligned}$$

as $|a| \to 0$. Using (4-6), (4-7), (4-8), (4-9), (4-12) and (4-14), we have that, for all $n < n_0$,

$$p^a_{n,n_0} = \bar{p}^a_{n_0,n} = O(|a|^{\sqrt{\mu_1}+|\alpha-k|}) \quad \text{as } |a| \to 0,$$

while the same estimates imply that, for all $n \neq \ell < n_0$,

$$p^a_{n,\ell} = \bar{p}^a_{\ell,n} = O(|a|^{2\sqrt{\mu_1}}) \quad \text{as } |a| \to 0.$$

Therefore, the quadratic form in (4-15) satisfies the hypothesis of Lemma 4.1 with $\sigma(a) = |a|^{2|\alpha-k|}$, $\gamma = \sqrt{\mu_1}$, $M_j = \lambda^0_j - \lambda_0 < 0$ for $j < n_0$ and $M \in \mathbb{N}$ such that $(2+M)\sqrt{\mu_1} > 2|\alpha-k|$, so that

$$\max_{\substack{(\alpha_1,\dots,\alpha_{n_0})\in\mathbb{C}^{n_0}\\ \sum_{n=1}^{n_0}|\alpha_n|^2=1}} \sum_{n,j=1}^{n_0} \alpha_n\bar{\alpha}_j\, p^a_{n,j} = |a|^{2|\alpha-k|}O(1) \quad \text{as } |a| \to 0. \qquad \square$$

We notice that Lemma 4.2 does not give any information about the sign of the constant $\mathfrak{c}$.

4C. *Construction of the test functions using φ^a_n.* Let $\varphi^a_n \in H^{1,a}_0(\Omega,\mathbb{C}) \setminus \{0\}$ satisfy (2-4) and (2-5). Let R_0 be as in (3-4), $R > 1$ and $|a| < R_0/R$. For every $1 \le n \le n_0$ we define

$$v_{n,R,a} = \begin{cases} v^{\text{int}}_{n,R,a} & \text{in } D_{R|a|},\\ v^{\text{ext}}_{n,R,a} & \text{in } \Omega \setminus D_{R|a|}, \end{cases}$$

where

$$v^{\text{ext}}_{n,R,a} = e^{i\alpha(\theta^a_0-\theta_a)}\varphi^a_n \quad \text{in } \Omega \setminus D_{R|a|},$$

and $v^{\text{int}}_{n,R,a}$ is the unique solution to the minimization problem

$$\min\left\{\int_{D_{R|a|}} |(i\nabla + A_0)u|^2\,dx : u \in H^{1,0}(D_{R|a|},\mathbb{C}),\ u = e^{i\alpha(\theta^a_0-\theta_a)}\varphi^a_n \text{ on } \partial D_{R|a|}\right\}. \tag{4-16}$$

We notice that $v^{\text{ext}}_{n,R,a}$ and $v^{\text{int}}_{n,R,a}$ respectively solve

$$\begin{cases} (i\nabla + A_0)^2 v^{\text{ext}}_{n,R,a} = \lambda^a_n v^{\text{ext}}_{n,R,a} & \text{in } \Omega \setminus D_{R|a|},\\ v^{\text{ext}}_{n,R,a} = e^{-i\alpha(\theta_a-\theta^a_0)}\varphi^a_n & \text{on } \partial(\Omega \setminus D_{R|a|}), \end{cases}$$

and

$$\begin{cases} (i\nabla + A_0)^2 v^{\text{int}}_{n,R,a} = 0 & \text{in } D_{R|a|},\\ v^{\text{int}}_{n,R,a} = e^{-i\alpha(\theta_a-\theta^a_0)}\varphi^a_n & \text{on } \partial D_{R|a|}. \end{cases} \tag{4-17}$$

The energy estimates in Lemmas 3.6 and 3.7 imply the following estimates for the functions $v^{\text{int}}_{n,R,a}$.

Lemma 4.3. *For $\delta \in (0, \sqrt{\mu_1}/2)$ fixed, let $\widetilde{K}_\delta$ be as in Lemma 3.6 and R_0 be as in* (3-4). *Let*

$$R > \max\{2, \widetilde{K}_\delta\}$$

and $1 \le n \le n_0$ be fixed. For every $a \in \Omega$ with $|a| < R_0/R$, let $v^{\text{int}}_{n,R,a}$ be defined as in (4-16). *Then*

$$\int_{D_{R|a|}} |(i\nabla + A_0)v^{\text{int}}_{n,R,a}|^2\,dx = O(|a|^{2(\sqrt{\mu_1}-\delta)})$$

$$\int_{D_{R|a|}} |v^{\text{int}}_{n,R,a}|^2\,dx = O(|a|^{2(\sqrt{\mu_1}-\delta)+2}) \quad \textit{and} \quad \int_{\partial D_{R|a|}} |v^{\text{int}}_{n,R,a}|^2\,ds = O(|a|^{2(\sqrt{\mu_1}-\delta)+1}) \tag{4-18}$$

as $|a| \to 0^+$.

Proof. The proof follows by combining the Dirichlet principle, a suitable cutting-off procedure, and Lemma 3.6 (see the proof of [Abatangelo and Felli 2015, Lemma 6.2] for details in the case of half-integer circulation). □

Lemma 4.4. *For $R > \max\{2, K_\delta\}$ fixed, with K_δ as in Lemma 3.4, let $v^{\mathrm{int}}_{n_0,R,a}$ be defined as in* (4-16). *Then*

$$\int_{D_{R|a|}} |(i\nabla + A_0)v^{\mathrm{int}}_{n_0,R,a}|^2\,dx = O(H(\varphi_a, K_\delta|a|)), \tag{4-19}$$

$$\int_{D_{R|a|}} |v^{\mathrm{int}}_{n_0,R,a}|^2\,dx = O(|a|^2 H(\varphi_a, K_\delta|a|)), \quad \int_{\partial D_{R|a|}} |v^{\mathrm{int}}_{n_0,R,a}|^2\,ds = O(|a| H(\varphi_a, K_\delta|a|)), \tag{4-20}$$

as $|a| \to 0^+$.

Proof. The proof follows from the estimates of Lemma 3.7, a suitable cutting-off procedure, and the Dirichlet principle; see (4-16). □

Remark 4.5. For all $R > 2$ and $a \in \Omega$ with $|a| < R_0/R$ we consider the blow-up family

$$Z^R_a(x) := \frac{v^{\mathrm{int}}_{n_0,R,a}(|a|x)}{\sqrt{H(\varphi_a, K_\delta|a|)}}, \tag{4-21}$$

with K_δ as in Lemma 3.4 for some fixed $\delta \in (0, \sqrt{\mu_1}/2)$. From Lemma 4.4 it follows that, for every $p \in \mathbb{S}^1$ fixed, $r_\delta > 0$ as in Lemma 3.4, and $R > \max\{K_\delta, 2\}$, the family of functions

$$\{Z^R_a : a = |a|p \in \Omega,\ |a| < r_\delta/R\}$$

is bounded in $H^{1,0}(D_R, \mathbb{C})$.

4D. *Estimate of the Rayleigh quotient for λ_0.* An estimate from above for the limit eigenvalue λ_0 in terms of the approximating eigenvalue λ_a can be obtained by choosing as test functions in (4-2) an orthonormal family constructed starting from the functions $\{v_{n,R,a}\}_{n=1,\dots,n_0}$, as done in the following.

Lemma 4.6. *For $\delta \in (0, \sqrt{\mu_1}/2)$ fixed, let r_δ, K_δ be as in Lemma 3.4 and α_{r_δ} be as in* (3-5). *Then there exists $\mathfrak{d}_\delta > 0$ such that*

$$\lambda_0 - \lambda_a \le \mathfrak{d}_\delta H(\varphi_a, K_\delta|a|)$$

for all $a \in \Omega$ such that $|a| < \min\{r_\delta/K_\delta, \alpha_{r_\delta}\}$.

Proof. In view of (3-18) it is enough to prove that $\lambda_0 - \lambda_a \le O(H(\varphi_a, K_\delta|a|))$ as $|a| \to 0^+$.

Recall the definition of $v_{n,R,a}$ given at the beginning of Section 4C. Let us fix $R > \max\{2, K_\delta, \widetilde{K}_\delta\}$, with $\widetilde{K}_\delta$ as in Lemma 3.6. As in the proof of Lemma 4.2, we use a Gram–Schmidt process; that is, we define

$$\tilde{v}_{n,a} = \frac{\hat{v}_{n,a}}{\|\hat{v}_{n,a}\|_{L^2(\Omega,\mathbb{C})}}, \quad 1 \le n \le n_0,$$

where

$$\hat{v}_{n_0,a} = v_{n_0,R,a},$$
$$\hat{v}_{n,a} = v_{n,R,a} - \sum_{\ell=n+1}^{n_0} d^a_{\ell,n}\hat{v}_{\ell,a}, \quad 1 \le n \le n_0 - 1,$$

and

$$d^a_{\ell,n} = \frac{\int_\Omega v_{n,R,a}\bar{\hat{v}}_{\ell,a}\,dx}{\|\hat{v}_{\ell,a}\|^2_{L^2(\Omega,\mathbb{C})}}, \quad 1 \le n \le n_0 - 1, \quad n+1 \le \ell \le n_0.$$

From (3-22), (4-18) and an induction argument it follows that, for every $1 \le n \le n_0 - 1$ and $n+1 \le \ell \le n_0$,

$$\|\hat{v}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})} = 1 + O(|a|^{2(\sqrt{\mu_1}-\delta)+2}) \quad \text{and} \quad d^a_{\ell,n} = O(|a|^{2(\sqrt{\mu_1}-\delta)+2}) \tag{4-22}$$

as $|a| \to 0$. Moreover, from (3-26) and (4-20), we have

$$\|\hat{v}_{n_0,a}\|^2_{L^2(\Omega,\mathbb{C})} = 1 + O(|a|^2 H(\varphi_a, K_\delta|a|)) \quad \text{as } |a| \to 0, \tag{4-23}$$

and, for $1 \le n \le n_0 - 1$,

$$d^a_{n_0,n} = O(|a|^{\sqrt{\mu_1}-\delta+2}\sqrt{H(\varphi_a, K_\delta|a|)}) \quad \text{as } |a| \to 0. \tag{4-24}$$

Since $\dim(\operatorname{span}\{v_{1,R,a}, \dots, v_{n_0,R,a}\}) = n_0$, we have that also $\dim(\operatorname{span}\{\tilde{v}_{1,a}, \dots, \tilde{v}_{n_0,a}\}) = n_0$, and hence from (4-2) we deduce that

$$\lambda_0 \le \max_{\substack{(\alpha_1,\dots,\alpha_{n_0})\in\mathbb{C}^{n_0} \\ \sum_{n=1}^{n_0}|\alpha_n|^2=1}} \int_\Omega \left|(i\nabla + A_0)\left(\sum_{n=1}^{n_0} \alpha_n \tilde{v}_{n,a}\right)\right|^2 dx,$$

which leads to

$$\lambda_0 - \lambda_a \le \max_{\substack{(\alpha_1,\dots,\alpha_{n_0})\in\mathbb{C}^{n_0} \\ \sum_{n=1}^{n_0}|\alpha_n|^2=1}} \sum_{n,j=1}^{n_0} \alpha_n \bar{\alpha}_j q^a_{n,j}, \tag{4-25}$$

where $q^a_{n,j} = \int_\Omega (i\nabla + A_0)\tilde{v}_{n,a} \cdot \overline{(i\nabla + A_0)\tilde{v}_{j,a}}\,dx - \lambda_a \delta_{nj}$. Using the results above we can now estimate $q^a_{n,j}$. First, using (4-19), (3-24), and (4-23)

$$\begin{aligned} q^a_{n_0 n_0} &= \frac{\lambda_a}{\int_\Omega |v_{n_0,R,a}|^2\,dx}\left(1 - \int_\Omega |v_{n_0,R,a}|^2\,dx\right) \\ &\qquad + \frac{1}{\int_\Omega |v_{n_0,R,a}|^2\,dx}\left(\int_{D_{R|a|}} |(i\nabla + A_0)v^{\text{int}}_{n_0,R,a}|^2\,dx - \int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\,dx\right) \\ &= H(\varphi_a, K_\delta|a|)O(1), \end{aligned}$$

as $|a| \to 0^+$. Next (4-18), (3-20), (4-22), and the fact that $\lambda^a_n \to \lambda^0_n$ as $|a| \to 0$, provide, for $n < n_0$,

$$\begin{aligned} q^a_{n,n} &= -\lambda_a + \frac{1}{\|\hat{v}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})}}\left(\lambda^a_n + \int_{D_{R|a|}} |(i\nabla + A_0)v^{\text{int}}_{n,R,a}|^2\,dx - \int_{D_{R|a|}} |(i\nabla + A_a)\varphi^a_n|^2\,dx\right) \\ &\qquad + \frac{1}{\|\hat{v}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})}} \int_\Omega \left|(i\nabla + A_0)\left(\sum_{\ell=n+1}^{n_0} d^a_{\ell,n}\hat{v}_{\ell,a}\right)\right|^2 dx \\ &\qquad - \frac{2}{\|\hat{v}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})}} \mathfrak{Re} \sum_{\ell=n+1}^{n_0} \left\{\bar{d}^a_{\ell,n} \int_\Omega (i\nabla + A_0)v_{n,R,a} \cdot \overline{(i\nabla + A_0)\hat{v}_{\ell,a}}\,dx\right\} \\ &= \lambda^0_n - \lambda_0 + o(1), \end{aligned}$$

as $|a| \to 0$. Now, using (3-20), (3-24), (4-18), (4-19), (4-22), (4-23), and (4-24), we prove that, for all $n < n_0$,

$$q^a_{n,n_0} = \bar{q}^a_{n_0,n} = O\big(|a|^{\sqrt{\mu_1}-\delta}\sqrt{H(\varphi_a, K_\delta|a|)}\big) \quad \text{as } |a| \to 0^+,$$

while the same estimates imply that, for all $n \neq \ell < n_0$,

$$q^a_{n,\ell} = \bar{q}^a_{\ell,n} = O(|a|^{2(\sqrt{\mu_1}-\delta)}), \quad \text{as } |a| \to 0^+.$$

Therefore, the quadratic form in (4-25) satisfies the hypothesis of Lemma 4.1 with $\gamma = \sqrt{\mu_1} - \delta$, $\sigma(a) = H(\varphi_a, K_\delta|a|) = O(|a|^{2\gamma})$ (by (3-19)), $M_j = \lambda^0_j - \lambda_0 < 0$ and M any natural number such that $M > 2(|\alpha - k| - \sqrt{\mu_1} + 2\delta)/(\sqrt{\mu_1} - \delta)$ by Corollary 3.5. Therefore the right-hand side in (4-25) satisfies

$$\max_{\substack{(\alpha_1,\dots,\alpha_{n_0})\in\mathbb{C}^{n_0}\\ \sum_{n=1}^{n_0}|\alpha_n|^2=1}} \sum_{n,j=1}^{n_0} \alpha_n \bar{\alpha}_j q^a_{n,j} = H(\varphi_a, K_\delta|a|)O(1),$$

as $|a| \to 0^+$. Then the conclusion follows from (4-25). □

4E. *Energy estimates.*

Corollary 4.7. *For $\delta \in (0, \sqrt{\mu_1}/2)$ fixed, let K_δ be as in Lemma 3.4. Then*

(i) $|\lambda_0 - \lambda_a| = O(1)\max\{H(\varphi_a, K_\delta|a|), |a|^{2|\alpha-k|}\}$ *as $a \to 0$;*

(ii) $|\lambda_0 - \lambda_a| = O((H(\varphi_a, K_\delta|a|))^{|\alpha-k|/(|\alpha-k|+\delta)})$ *as $a \to 0$.*

Proof. Estimate (i) is a direct consequence of Lemmas 4.2 and 4.6. Corollary 3.5 implies

$$|a|^{2|\alpha-k|} = O((H(\varphi_a, K_\delta|a|))^{|\alpha-k|/(|\alpha-k|+\delta)})$$

as $a \to 0$, so that (ii) follows from (i). □

5. Blow-up analysis

In order to obtain a more precise estimate of the order of vanishing of the eigenvalue variation $|\lambda_0 - \lambda_a|$ than Corollary 4.7, we have now to compare the order of $H(\varphi_a, K_\delta|a|)$ with $|a|^{2|\alpha-k|}$. We observe that the estimates obtained so far (in particular Corollary 3.5) are not enough to decide what is the dominant term among $H(\varphi_a, K_\delta|a|)$ and $|a|^{2|\alpha-k|}$. To this aim, our next step is a blow-up analysis for scaled eigenfunctions (3-27) along a fixed direction $p \in \mathbb{S}^1$. In order to identify the limit profile of the blow-up family (3-27), the following energy estimate of the difference between approximating and limit scaled eigenfunctions plays a crucial role.

Let $\mathcal{D}^{1,2}_0(\mathbb{R}^2, \mathbb{C})$ be the completion of $C^\infty_c(\mathbb{R}^2 \setminus \{0\}, \mathbb{C})$ with respect to the magnetic Dirichlet norm

$$\|u\|_{\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})} := \left(\int_{\mathbb{R}^2} \big|(i\nabla + A_0)u(x)\big|^2\,dx\right)^{1/2}.$$

Theorem 5.1 (energy estimates for eigenfunction variation). *Let $p \in \mathbb{S}^1$ be fixed. For some fixed $\delta \in (0, \sqrt{\mu_1}/2)$, let $K_\delta > \Upsilon_\delta$ be as in Lemma 3.4. For every $R > \max\{2, K_\delta\}$ and $a = |a|p \in \Omega$ such* *that* $|a| < R_0/R$, *let* $v_{n_0,R,a}$ *be as in Section 4C. Then*

$$\|v_{n_0,R,a}-\varphi_0\|_{H_0^{1,0}(\Omega,\mathbb{C})} \le C\big(h(p,a,R)+g(p,a,R)\big)\sqrt{H(\varphi_a,K_\delta|a|)},$$

where $C>0$ *is independent of* a, R, p,

$$h(p,a,R)=\sup_{\substack{\varphi\in\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})}=1}}\left|\int_{\partial D_R}\big(e^{i\alpha(\theta_0^p-\theta_p)}(i\nabla+A_p)\tilde{\varphi}_a-(i\nabla+A_0)Z_a^R\big)\cdot\nu\,\bar{\varphi}\,d\sigma\right|,$$

and, *for* p *and* R *fixed*,

$$h(p,a,R)=O(1)\quad\textit{and}\quad g(p,a,R)=o(1)$$

as $|a|\to 0^+$.

Proof. The proof exploits the invertibility of the differential of the function F defined below, in the spirit of [Abatangelo et al. 2017, Theorem 8.2; Abatangelo and Felli 2015, Theorem 7.2]. Let

$$F:\mathbb{C}\times H_0^{1,0}(\Omega,\mathbb{C})\to\mathbb{R}\times\mathbb{R}\times(H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C}))^\star,$$
$$(\lambda,\varphi)\mapsto\Big(\|\varphi\|^2_{H_0^{1,0}(\Omega,\mathbb{C})}-\lambda_0,\ \mathfrak{Im}\big(\textstyle\int_\Omega\varphi\bar{\varphi}_0\,dx\big),\ (i\nabla+A_0)^2\varphi-\lambda\varphi\Big).$$

In the above definition, $(H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C}))^\star$ is the real dual space of $H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C})=H_0^{1,0}(\Omega,\mathbb{C})$, which is here meant as a vector space over $\mathbb{R}$ endowed with the norm

$$\|u\|_{H_0^{1,0}(\Omega,\mathbb{C})}=\left(\int_\Omega\big|(i\nabla+A_0)u\big|^2\,dx\right)^{1/2},$$

and $(i\nabla+A_0)^2\varphi-\lambda\varphi\in(H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C}))^\star$ acts as

$${}_{(H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C}))^\star}\langle(i\nabla+A_0)^2\varphi-\lambda\varphi,u\rangle_{H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C})}=\mathfrak{Re}\left(\int_\Omega(i\nabla+A_0)\varphi\cdot\overline{(i\nabla+A_0)u}\,dx-\lambda\int_\Omega\varphi\bar{u}\,dx\right)$$

for all $\varphi\in H_0^{1,0}(\Omega,\mathbb{C})$. It is easy to prove that the function F is Fréchet-differentiable at (λ_0,φ_0), with differential $dF(\lambda_0,\varphi_0)\in\mathcal{L}(\mathbb{C}\times H_0^{1,0}(\Omega,\mathbb{C}),\mathbb{R}\times\mathbb{R}\times(H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C}))^*)$ given by

$$dF(\lambda_0,\varphi_0)(\lambda,\varphi)=\left(2\mathfrak{Re}\left(\int_\Omega(i\nabla+A_0)\varphi_0\cdot\overline{(i\nabla+A_0)\varphi}\,dx\right),\mathfrak{Im}\left(\int_\Omega\varphi\bar{\varphi}_0\,dx\right),(i\nabla+A_0)^2\varphi-\lambda_0\varphi-\lambda\varphi_0\right)$$

for every $(\lambda,\varphi)\in\mathbb{C}\times H_0^{1,0}(\Omega,\mathbb{C})$. From the simplicity assumption (1-5) it follows that $dF(\lambda_0,\varphi_0)$ is invertible; see [Abatangelo and Felli 2015, Lemma 7.1] for details.

From the definition of $v_{n_0,R,a}$, (1-17), (3-19), (3-24), (4-8), and (4-19) it follows that

$$\begin{aligned}\int_\Omega\big|(i\nabla+A_0)(v_{n_0,R,a}-\varphi_0)\big|^2\,dx&=\int_\Omega\big|e^{i\alpha(\theta_0^a-\theta_a)}(i\nabla+A_a)\varphi_a-(i\nabla+A_0)\varphi_0\big|^2\,dx\\&\quad-\int_{D_{R|a|}}\big|e^{i\alpha(\theta_0^a-\theta_a)}(i\nabla+A_a)\varphi_a-(i\nabla+A_0)\varphi_0\big|^2\,dx\\&\quad+\int_{D_{R|a|}}\big|(i\nabla+A_0)(v_{n_0,R,a}^{\mathrm{int}}-\varphi_0)\big|^2\,dx=o(1)\end{aligned}$$

as $|a| \to 0$, so that $v_{n_0,R,a} \to \varphi_0$ in $H^1_0(\Omega, \mathbb{C})$ as $|a| \to 0^+$. Then, from the invertibility of $dF(\lambda_0, \varphi_0)$ we have

$$|\lambda_a - \lambda_0| + \|v_{n_0,R,a} - \varphi_0\|_{H_0^{1,0}(\Omega,\mathbb{C})} \le \|(dF(\lambda_0,\varphi_0))^{-1}\|_{\mathcal{L}(\mathbb{R}\times\mathbb{R}\times(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star, \mathbb{C}\times H_0^{1,0}(\Omega,\mathbb{C}))} \times \|F(\lambda_a, v_{n_0,R,a})\|_{\mathbb{R}\times\mathbb{R}\times(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star}(1+o(1)) \tag{5-1}$$

as $|a| \to 0^+$. We define

$$F(\lambda_a, v_{n_0,R,a}) = (\alpha_a, \beta_a, w_a),$$

where

$$\alpha_a = \|v_{n_0,R,a}\|^2_{H_0^{1,0}(\Omega,\mathbb{C})} - \lambda_0 \in \mathbb{R},$$

$$\beta_a = \mathfrak{Im}\left(\int_\Omega v_{n_0,R,a}\bar{\varphi}_0\,dx\right) \in \mathbb{R},$$

$$w_a = (i\nabla + A_0)^2 v_{n_0,R,a} - \lambda_a v_{n_0,R,a} \in (H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star.$$

In view of (4-19), (3-24), and Corollary 4.7 we have

$$\begin{aligned}\alpha_a &= \left(\int_{D_{R|a|}} |(i\nabla + A_0)v^{\mathrm{int}}_{n_0,R,a}|^2\,dx - \int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\,dx\right) + (\lambda_a - \lambda_0)\\ &= O(H(\varphi_a, K_\delta|a|)) + O((H(\varphi_a, K_\delta|a|))^{|\alpha-k|/(|\alpha-k|+\delta)}) = o(\sqrt{H(\varphi_a, K_\delta|a|)})\end{aligned} \tag{5-2}$$

as $|a| \to 0^+$. The normalization condition for the phase in (1-15), together with (4-20), (4-8), and (3-26), yields

$$\begin{aligned}\beta_a &= \mathfrak{Im}\left(\int_{D_{R|a|}} v^{\mathrm{int}}_{n_0,R,a}\bar{\varphi}_0\,dx - \int_{D_{R|a|}} e^{i\alpha(\theta_0^a-\theta_a)}\varphi_a\bar{\varphi}_0\,dx + \int_\Omega e^{i\alpha(\theta_0^a-\theta_a)}\varphi_a\bar{\varphi}_0\,dx\right)\\ &= \mathfrak{Im}\left(\int_{D_{R|a|}} v^{\mathrm{int}}_{n_0,R,a}\bar{\varphi}_0\,dx - \int_{D_{R|a|}} e^{i\alpha(\theta_0^a-\theta_a)}\varphi_a\bar{\varphi}_0\,dx\right)\\ &= O(|a|^{2+|\alpha-k|}\sqrt{H(\varphi_a, K_\delta|a|)}) = o(\sqrt{H(\varphi_a, K_\delta|a|)})\end{aligned} \tag{5-3}$$

as $|a| \to 0^+$.

For every $a \in \Omega$, the map

$$\mathcal{T}_a : \mathcal{D}^{1,2}_0(\mathbb{R}^2, \mathbb{C}) \to \mathcal{D}^{1,2}_0(\mathbb{R}^2, \mathbb{C}), \quad \mathcal{T}_a\varphi(x) = \varphi(|a|x),$$

is an isometry of $\mathcal{D}^{1,2}_0(\mathbb{R}^2, \mathbb{C})$.

Since $H^{1,0}_0(\Omega, \mathbb{C})$ is continuously embedded into $\mathcal{D}^{1,2}_0(\mathbb{R}^2, \mathbb{C})$ by trivial extension outside Ω and $\|u\|_{\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})} = \|u\|_{H^{1,0}_0(\Omega,\mathbb{C})}$ for every $u \in H^{1,0}_0(\Omega, \mathbb{C})$, we have

$$\begin{aligned}\|w_a\|_{(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star} &= \sup_{\substack{\varphi\in H_0^{1,0}(\Omega,\mathbb{C})\\ \|\varphi\|_{H_0^{1,0}(\Omega,\mathbb{C})}=1}} \left|\mathfrak{Re}\left(\int_\Omega (i\nabla + A_0)v_{n_0,R,a}\cdot\overline{(i\nabla + A_0)\varphi}\,dx - \lambda_a\int_\Omega v_{n_0,R,a}\bar{\varphi}\,dx\right)\right|\\ &\le \sup_{\substack{\varphi\in \mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})}=1}} \left|\mathfrak{Re}\left(\int_\Omega (i\nabla + A_0)v_{n_0,R,a}\cdot\overline{(i\nabla + A_0)\varphi}\,dx - \lambda_a\int_\Omega v_{n_0,R,a}\bar{\varphi}\,dx\right)\right|.\end{aligned} \tag{5-4}$$

For every $\varphi \in \mathcal{D}^{1,2}_0(\mathbb{R}^2, \mathbb{C})$ we have

$$\begin{aligned}\int_\Omega (i\nabla + A_0)v_{n_0,R,a} \cdot \overline{(i\nabla + A_0)\varphi}\,dx - \lambda_a \int_\Omega v_{n_0,R,a}\bar\varphi\,dx \\ = \int_{\Omega\setminus D_{R|a|}} e^{i\alpha(\theta_0^a-\theta_a)}(i\nabla + A_a)\varphi_a \cdot \overline{(i\nabla + A_0)\varphi}\,dx - \lambda_a \int_{\Omega\setminus D_{R|a|}} e^{i\alpha(\theta_0^a-\theta_a)}\varphi_a\bar\varphi\,dx \\ + \int_{D_{R|a|}} (i\nabla + A_0)v_{n_0,R,a} \cdot \overline{(i\nabla + A_0)\varphi}\,dx - \lambda_a \int_{D_{R|a|}} v_{n_0,R,a}\bar\varphi\,dx. \end{aligned} \tag{5-5}$$

From scaling and integration by parts we have that, letting $\tilde\varphi_a$ be defined in (3-27),

$$\begin{aligned}\int_{\Omega\setminus D_{R|a|}} e^{i\alpha(\theta_0^a-\theta_a)}(i\nabla + A_a)\varphi_a \cdot \overline{(i\nabla + A_0)\varphi}\,dx - \lambda_a \int_{\Omega\setminus D_{R|a|}} e^{i\alpha(\theta_0^a-\theta_a)}\varphi_a\bar\varphi\,dx \\ = i\sqrt{H(\varphi_a, K_\delta|a|)} \int_{\partial D_R} \overline{\mathcal{T}_a\varphi}\, e^{i\alpha(\theta_0^p-\theta_p)}(i\nabla + A_p)\tilde\varphi_a \cdot \nu\,d\sigma, \end{aligned} \tag{5-6}$$

where $\nu = x/|x|$ is the outer unit normal vector. In a similar way we have that, defining Z_a^R as in (4-21) and using (4-17),

$$\begin{aligned}\int_{D_{R|a|}} (i\nabla + A_0)v_{n_0,R,a} \cdot \overline{(i\nabla + A_0)\varphi}\,dx - \lambda_a \int_{D_{R|a|}} v_{n_0,R,a}\bar\varphi\,dx \\ = \sqrt{H(\varphi_a, K_\delta|a|)}\Big(-i\int_{\partial D_R} (i\nabla + A_0)Z_a^R \cdot \nu\overline{\mathcal{T}_a\varphi}\,d\sigma - \lambda_a|a|^2 \int_{D_R} Z_a^R\overline{\mathcal{T}_a\varphi}\,dx\Big). \end{aligned} \tag{5-7}$$

Combining (5-4)–(5-7) and recalling that $\mathcal{T}_a$ is an isometry of $\mathcal{D}^{1,2}_0(\mathbb{R}^2, \mathbb{C})$, we obtain

$$(H(\varphi_a, K_\delta|a|))^{-\frac12}\|w_a\|_{(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star} \le h(p,a,R) + \lambda_a|a|^2 \sup_{\substack{\varphi\in\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C}) \\ \|\varphi\|_{\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})}=1}} \left|\int_{D_R} Z_a^R\bar\varphi\,dx\right|, \tag{5-8}$$

where

$$h(p,a,R) = \sup_{\substack{\varphi\in\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C}) \\ \|\varphi\|_{\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})}=1}} \left|\int_{\partial D_R} \big(e^{i\alpha(\theta_0^p-\theta_p)}(i\nabla + A_p)\tilde\varphi_a - (i\nabla + A_0)Z_a^R\big)\cdot\nu\,\bar\varphi\,d\sigma\right|.$$

From Remarks 3.8 and 4.5 it follows that, for $R > \max\{2, K_\delta\}$ and $p \in \mathbb{S}^1$ fixed,

$$\big\{\big(e^{i\alpha(\theta_0^p-\theta_p)}(i\nabla + A_p)\tilde\varphi_a - (i\nabla + A_0)Z_a^R\big)\cdot\nu\big\}_{|a|<r_\delta/R} \quad \text{is bounded in } H^{-1/2}(\partial D_R)$$

so that, for p and R fixed, $h(p,a,R) = O(1)$ as $a \to 0$. Moreover, Remark 4.5 implies that, for $R > \max\{2, K_\delta\}$ and $p \in \mathbb{S}^1$ fixed,

$$\sup_{\substack{\varphi\in\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C}) \\ \|\varphi\|_{\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})}=1}} \left|\int_{D_R} Z_a^R\bar\varphi\,dx\right| = O(1) \quad \text{as } |a| \to 0.$$

Hence the conclusion follows from (5-1), (5-2), (5-3), and (5-8). □

The previous theorem allows us to estimate the energy variation of scaled eigenfunctions and improve the results of Corollary 3.5 as follows.

Corollary 5.2. *Let $p \in \mathbb{S}^1$ be fixed. Then*

(*i*) $|a|^{2|\alpha-k|} = O(H(\varphi_a, K_\delta|a|))$ *as* $a = |a|p \to 0$;

(*ii*) *letting* $\tilde{\varphi}_a$ *and* W_a *be as in* (3-27) *and* (4-10), *for every* $R > \max\{2, K_\delta\}$ *it holds that*

$$\int_{(\Omega/|a|)\setminus D_R} \left|(i\nabla + A_p)\left(\tilde{\varphi}_a - e^{i\alpha(\theta_p-\theta_0^p)} W_a \frac{|a|^{|\alpha-k|}}{\sqrt{H(\varphi_a, K_\delta|a|)}}\right)\right|^2 dx = O(1) \quad \text{as } a = |a|p \to 0. \tag{5-9}$$

Proof. Estimate (5-9) follows from scaling and Theorem 5.1. From (5-9) it follows that

$$\begin{aligned}
&\frac{|a|^{|\alpha-k|}}{\sqrt{H(\varphi_a, K_\delta|a|)}}\left(\int_{D_{2R}\setminus D_R} \left|(i\nabla + A_0)W_a\right|^2 dx\right)^{1/2} \\
&\qquad = \frac{|a|^{|\alpha-k|}}{\sqrt{H(\varphi_a, K_\delta|a|)}}\left(\int_{D_{2R}\setminus D_R} \left|(i\nabla + A_p)(e^{i\alpha(\theta_p-\theta_0^p)} W_a)\right|^2 dx\right)^{1/2} \\
&\qquad \le O(1) + \left(\int_{D_{2R}\setminus D_R} \left|(i\nabla + A_p)\tilde{\varphi}_a(x)\right|^2 dx\right)^{1/2}
\end{aligned}$$

as $a = |a|p \to 0$. From Remark 3.8 and (4-11), the above estimate implies (i). ☐

In the following lemma we prove the existence and uniqueness of the function Ψ_p satisfying (1-18) and (1-19), which will turn out to be the limit of the blow-up family (3-27) as $a \to 0$ along the fixed direction $p \in \mathbb{S}^1$.

Lemma 5.3. *Let $p \in \mathbb{S}^1$. There exists a unique $\Psi_p \in H^{1,p}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C})$ satisfying* (1-18) *and* (1-19).

Proof. Let η be a smooth cut-off function such that $\eta \equiv 0$ in D_1 and $\eta \equiv 1$ in $\mathbb{R}^2 \setminus D_R$ for some $R > 1$. Recalling the definition of ψ_k (1-20), we have

$$\begin{aligned}
F &= (i\nabla + A_p)^2(\eta e^{i\alpha(\theta_p-\theta_0^p)}\psi_k) \\
&= -\Delta\eta e^{i\alpha(\theta_p-\theta_0^p)}\psi_k + 2i\nabla\eta\cdot(i\nabla + A_p)(e^{i\alpha(\theta_p-\theta_0^p)}\psi_k) + \eta(i\nabla + A_p)^2(e^{i\alpha(\theta_p-\theta_0^p)}\psi_k) \\
&= -\Delta\eta e^{i\alpha(\theta_p-\theta_0^p)}\psi_k + 2i\nabla\eta\cdot(i\nabla + A_p)(e^{i\alpha(\theta_p-\theta_0^p)}\psi_k) \in (\mathcal{D}^{1,2}_p(\mathbb{R}^2, \mathbb{C}))^*.
\end{aligned}$$

Here $\mathcal{D}^{1,2}_p(\mathbb{R}^2, \mathbb{C})$ is the completion of $C^\infty_c(\mathbb{R}^2 \setminus \{0\}, \mathbb{C})$ with respect to

$$\|u\|_{\mathcal{D}^{1,2}_p(\mathbb{R}^2,\mathbb{C})} = \left(\int_{\mathbb{R}^2} |(i\nabla + A_p)u(x)|^2\, dx\right)^{1/2}.$$

By the Lax–Milgram theorem, there exists a unique $g \in \mathcal{D}^{1,2}_p(\mathbb{R}^2, \mathbb{C})$ which solves

$$(i\nabla + A_p)^2 g = -F \quad \text{in } (\mathcal{D}^{1,2}_p(\mathbb{R}^2, \mathbb{C}))^*.$$

Then, $\Psi_p = g + \eta e^{i\alpha(\theta_p-\theta_0^p)}\psi_k$ satisfies (1-18) and (1-19), and the existence is proved.

The uniqueness follows from the fact that, if $\Psi_p^1, \Psi_p^2 \in H^{1,p}_{\rm loc}(\mathbb{R}^2,\mathbb{C})$ satisfy (1-18) and (1-19), then

$$(i\nabla + A_p)^2(\Psi_p^1 - \Psi_p^2) = 0 \quad \text{in } (\mathcal{D}_p^{1,2}(\mathbb{R}^2,\mathbb{C}))^*, \tag{5-10}$$

and

$$\int_{\mathbb{R}^2} |(i\nabla + A_p)(\Psi_p^1 - \Psi_p^2)|^2\,dx < +\infty,$$

which, in view of the Hardy inequality (1-3), implies

$$\int_{\mathbb{R}^2} \frac{|\Psi_p^1 - \Psi_p^2|^2}{|x-p|^2}\,dx < +\infty,$$

and hence that $\Psi_p^1 - \Psi_p^2 \in \mathcal{D}_p^{1,2}(\mathbb{R}^2,\mathbb{C})$. Therefore we can test (5-10) with $\Psi_p^1 - \Psi_p^2$ thus concluding that

$$\int_{\mathbb{R}^2} |(i\nabla + A_p)(\Psi_p^1 - \Psi_p^2)|^2\,dx = 0,$$

which implies $\Psi_p^1 \equiv \Psi_p^2$. □

We are now in a position to prove that the scaled eigenfunctions (3-27) converge to a multiple of Ψ_p as $a = |a|p \to 0$.

Lemma 5.4. *Let $p \in \mathbb{S}^1$ and $\delta \in (0, \sqrt{\mu_1}/2)$ be fixed and let $K_\delta > \Upsilon_\delta$ be as in Lemma 3.4. For $a = |a|p \in \Omega$ let $\tilde{\varphi}_a$ be as in* (3-27). *Then*

$$\tilde{\varphi}_a \to \frac{\beta}{|\beta|}\biggl(\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_p|^2\,ds}\biggr)^{1/2} \Psi_p \quad \text{as } a = |a|p \to 0$$

in $H^{1,p}(D_R,\mathbb{C})$ for every $R > 1$ and in $C^2_{\rm loc}(\mathbb{R}^2 \setminus \{p\},\mathbb{C})$, where Ψ_p is the function defined in Lemma 5.3. Moreover,

$$\lim_{a=|a|p\to 0} \frac{|a|^{|\alpha-k|}}{\sqrt{H(\varphi_a, K_\delta|a|)}} = \frac{1}{|\beta|}\biggl(\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_p|^2\,ds}\biggr)^{1/2}. \tag{5-11}$$

Proof. From Remark 3.8 and Corollary 5.2 it follows that, for every sequence $a_n = |a_n|p$ with $|a_n| \to 0$, there exist a subsequence a_{n_ℓ}, $c \in [0,+\infty)$ and $\widetilde{\Phi} \in H^{1,p}_{\rm loc}(\mathbb{R}^2,\mathbb{C})$ such that

$$\tilde{\varphi}_{a_{n_\ell}} \rightharpoonup \widetilde{\Phi} \text{ weakly in } H^{1,p}(D_R,\mathbb{C}) \text{ as } \ell \to +\infty \quad \text{and} \quad \lim_{\ell\to+\infty} \frac{|a_{n_\ell}|^{|\alpha-k|}}{\sqrt{H(\varphi_{a_{n_\ell}}, K_\delta|a_{n_\ell}|)}} = c$$

for every $R > 1$. Passing to the limit in the equation satisfied by $\tilde{\varphi}_a$, i.e., $(i\nabla + A_p)^2\tilde{\varphi}_a = \lambda_a|a|^2\tilde{\varphi}_a$ in $(1/|a|)\Omega$, we obtain that $\widetilde{\Phi}$ satisfies

$$(i\nabla + A_p)^2\widetilde{\Phi} = 0 \quad \text{in } \mathbb{R}^2. \tag{5-12}$$

Moreover, by compact trace embeddings,

$$\frac{1}{K_\delta}\int_{\partial D_{K_\delta}} |\widetilde{\Phi}|^2\,ds = 1, \tag{5-13}$$

so that $\widetilde{\Phi}$ is not identically zero. Testing the equation for $\tilde{\varphi}_a$ with $\tilde{\varphi}_a$ itself, integrating by parts and exploiting the $C^2_{\rm loc}$-convergence of $\tilde{\varphi}_a$ in $\mathbb{R}^2 \setminus \{p\}$ (which follows from classic elliptic estimates) we obtain

$\int_{D_R} |(i\nabla + A_p)\tilde{\varphi}_{a_{n_\ell}}|^2\,dx \to \int_{D_R} |(i\nabla + A_p)\widetilde{\Phi}|^2\,dx$ as $\ell \to \infty$ for every $R > 1$. Hence we conclude that, for all $R > 1$, $\tilde{\varphi}_{a_{n_\ell}} \to \widetilde{\Phi}$ strongly in $H^{1,p}(D_R, \mathbb{C})$ as $\ell \to +\infty$.

By the strong $H^{1,p}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C})$-convergence and recalling (4-11), we can pass to the limit along a_{n_ℓ} in (5-9) to obtain

$$\int_{\mathbb{R}^2\setminus D_R} \left|(i\nabla + A_p)(\widetilde{\Phi} - c\beta e^{i\alpha(\theta_p - \theta_0^p)}\psi_k)\right|^2 dx < +\infty.$$

This implies $c \neq 0$ (and hence $c > 0$), otherwise we would have $\int_{\mathbb{R}^2\setminus D_R} |(i\nabla + A_p)\widetilde{\Phi}|^2\,dx < +\infty$, which, together with (5-12), implies $\widetilde{\Phi} \equiv 0$, thus contradicting (5-13).

Then Lemma 5.3 and (5-13) provide

$$\widetilde{\Phi} = c\beta\Psi_p \quad \text{and} \quad c = \frac{1}{|\beta|}\left(\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_p|^2}\right)^{1/2}.$$

Since these limits depend neither on the sequence, nor on the subsequence, the proof is complete. □

Proof of Theorem 1.1. Let $p \in \mathbb{S}^1$. From Corollary 4.7(i) and (5-11) we conclude that

$$\lambda_0 - \lambda_a = O(|a|^{2|\alpha - k|})$$

as $a = |a|p \to 0$. Since the function $a \mapsto \lambda_a$ is analytic in a neighborhood of 0, due to the simplicity of λ_0, see [Léna 2015, Theorem 1.3], and since $2|\alpha - k|$ is noninteger, we have that the Taylor polynomials of the function $\lambda_0 - \lambda_a$ with center 0 and degree less than or equal to $\lfloor 2|\alpha - k|\rfloor$ vanish, thus yielding the conclusion. □

Proof of Theorem 1.2. It is a direct consequence of Lemma 5.4. □

6. Rate of convergence for eigenfunctions

Taking inspiration from [Abatangelo and Felli 2017], we now estimate the rate of convergence of the eigenfunctions. We then take into account the quantity

$$\left\|(i\nabla + A_a)\varphi_a - e^{i\alpha(\theta_a - \theta_0^a)}(i\nabla + A_0)\varphi_0\right\|_{L^2(\Omega,\mathbb{C})},$$

where $\varphi_a = \varphi^a_{n_0}$ satisfies (1-14), (1-15) and $\varphi_0 = \varphi^0_{n_0}$ satisfies (1-7). We split the argument in two different steps, the first considering the energy variation inside small disks of radius $R|a|$, the second considering the energy variation outside these disks.

Lemma 6.1. *Under the same assumptions as in Theorems 1.1 and 1.2, we have that, for every $p \in \mathbb{S}^1$ and $R > 1$,*

$$\lim_{a=|a|p\to 0} \frac{1}{|a|^{2|\alpha-k|}} \int_{D_{R|a|}} \left|(i\nabla + A_a)\varphi_a - e^{i\alpha(\theta_a - \theta_0^a)}(i\nabla + A_0)\varphi_0\right|^2 dx = |\beta|^2\,\mathcal{F}_p(R), \tag{6-1}$$

where

$$\mathcal{F}_p(R) = \int_{D_R} \left|(i\nabla + A_p)\Psi_p - e^{i\alpha(\theta_p - \theta_0^p)}(i\nabla + A_0)\psi_k\right|^2 dx,$$

Ψ_p is defined in Lemma 5.3 and ψ_k is as in (1-20). *Moreover,*

$$\mathfrak{L}_p := \lim_{R\to+\infty} \mathcal{F}_p(R) \in (0, +\infty).$$

Proof. We notice that, in view of (1-19), $\mathfrak{L}_p < +\infty$. The proof of (6-1) relies on a change of variables and on the convergences stated in (4-11) and in Theorem 1.2. We have

$$\begin{aligned}\lim_{R\to+\infty} \mathcal{F}_p(R) &= \int_{\mathbb{R}^2} \big|(i\nabla + A_p)\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}(i\nabla + A_0)\psi_k\big|^2\,dx \\ &= \int_{\mathbb{R}^2\setminus\Gamma_p} \big|(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)\big|^2\,dx > 0,\end{aligned}$$

where Γ_p is defined in (1-13). Indeed, suppose by contradiction that the above limit is zero. Since, for every $r_1 > r_2 > 1$ we have $\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k \in H^{1,p}(D_{r_1}(p)\setminus D_{r_2}(p),\mathbb{C})$, the Hardy inequality (1-4) implies $\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k \equiv 0$ in $\mathbb{R}^2\setminus D_1(p)$. Moreover, since $(i\nabla + A_p)^2(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k) = 0$ in $\mathbb{R}^2\setminus\Gamma_p$, a classical unique continuation principle, see, e.g., [Wolff 1992], implies $\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k \equiv 0$ in $\mathbb{R}^2\setminus\Gamma_p$ necessarily. But this is impossible since, by (1-18) and classical elliptic estimates away from p, Ψ_p is smooth in $\mathbb{R}^2\setminus\{p\}$, whereas $e^{i\alpha(\theta_p-\theta_0^p)}\psi_k$ is discontinuous on $\Gamma_p\setminus\{0\}$ since it is the product of the continuous nonzero function ψ_k and of the discontinuous function $e^{i\alpha(\theta_p-\theta_0^p)}$; see the definitions (1-11), (1-12) and (1-13). ☐

Before addressing the energy variation outside the disk, it is worthwhile introducing a preliminary result. For all $R > 2$ and $p \in \mathbb{S}^1$, let $z_{p,R}$ be the unique solution to

$$\begin{cases}(i\nabla + A_0)^2 z_{p,R} = 0 & \text{in } D_R,\\ z_{p,R} = e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p & \text{on } \partial D_R.\end{cases} \tag{6-2}$$

From Lemma 5.4 it follows that the family of functions Z_a^R introduced in (4-21) converges in $H^{1,0}(D_R,\mathbb{C})$ to some multiple of $z_{p,R}$.

Lemma 6.2. *Let $p \in \mathbb{S}^1$ and $R > 2$. For $a = |a|p \in \Omega$, let Z_a^R be as in* (4-21). *Then*

$$Z_a^R \to \frac{\beta}{|\beta|}\bigg(\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_p|^2\,ds}\bigg)^{1/2} z_{p,R}$$

in $H^{1,0}(D_R,\mathbb{C})$ as $|a| \to 0^+$.

Proof. Define

$$\gamma_{p,\delta} = \frac{\beta}{|\beta|}\bigg(\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_p|^2\,ds}\bigg)^{1/2}.$$

By (4-17) and (6-2) we have that $Z_a^R - \gamma_{p,\delta} z_{p,R}$ solves

$$\begin{cases}(i\nabla + A_0)^2(Z_a^R - \gamma_{p,\delta} z_{p,R}) = 0 & \text{in } D_R,\\ Z_a^R - \gamma_{p,\delta} z_{p,R} = e^{i\alpha(\theta_0^p-\theta_p)}(\tilde\varphi_a - \gamma_{p,\delta}\Psi_p) & \text{on } \partial D_R.\end{cases}$$

For $R > 2$, let $\eta_R : \mathbb{R}^2 \to \mathbb{R}$ be a smooth cut-off function such that

$$\eta_R \equiv 0 \quad \text{in } D_{R/2}, \qquad \eta_R \equiv 1 \quad \text{in } \mathbb{R}^2\setminus D_R, \qquad 0 \le \eta_R \le 1. \tag{6-3}$$

Then, by the Dirichlet principle and Lemma 5.4,

$$\begin{aligned}\int_{D_R} |(i\nabla + A_0)(Z_a^R - \gamma_{p,\delta} z_{p,R})|^2\,dx &\le \int_{D_R} |(i\nabla + A_0)(\eta_R e^{i\alpha(\theta_0^p - \theta_p)}(\tilde{\varphi}_a - \gamma_{p,\delta}\Psi_p))|^2\,dx \\ &\le 2\int_{D_R} |\nabla \eta_R|^2 |\tilde{\varphi}_a - \gamma_{p,\delta}\Psi_p|^2\,dx + 2\int_{D_R\setminus D_{R/2}} \eta_R^2 |(i\nabla + A_p)(\tilde{\varphi}_a - \gamma_{p,\delta}\Psi_p)|^2\,dx = o(1)\end{aligned}$$

as $a = |a|p \to 0$. Finally, the Hardy-type inequality (1-3) allows us to conclude. ☐

Lemma 6.3. *Let* $\varphi_0 \in H_0^{1,0}(\Omega, \mathbb{C})$ *be a solution to* (1-7) *satisfying* (1-5). *Let* $p \in \mathbb{S}^1$. *For* $a = |a|p \in \Omega$, *let* $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C})$ *satisfy* (1-14)–(1-15). *Then, for all* $R > \max\{2, K_\delta\}$,

$$\left\| e^{i\alpha(\theta_0^a - \theta_a)}(i\nabla + A_a)\varphi_a - (i\nabla + A_0)\varphi_0 \right\|^2_{L^2(\Omega\setminus D_{R|a|},\mathbb{C})} \le |a|^{2|\alpha - k|} G(p, a, R),$$

where $\lim_{a=|a|p\to 0} G(p, a, R) = G(p, R)$ *for some* $G(p, R)$ *such that*

$$\lim_{R\to+\infty} G(p, R) = 0. \tag{6-4}$$

Proof. Let $R > \max\{2, K_\delta\}$. From Theorem 5.1 and (5-11) we have

$$\begin{aligned}\|e^{i\alpha(\theta_0^a - \theta_a)}(i\nabla + A_a)\varphi_a - (i\nabla + A_0)\varphi_0\|_{L^2(\Omega\setminus D_{R|a|},\mathbb{C})} &\le \|v_{n_0,R,a} - \varphi_0\|_{H_0^{1,0}(\Omega,\mathbb{C})} \\ &\le C\big(h(p, a, R) + g(p, a, R)\big)|a|^{|\alpha - k|},\end{aligned}$$

where $g(p, a, R) = o(1)$ as $|a| \to 0^+$ and

$$\begin{aligned} h(p,a,R) &= \sup_{\substack{\varphi\in\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})}=1}} \left| \int_{\partial D_R} (i\nabla + A_0)(e^{i\alpha(\theta_0^p - \theta_p)}\tilde{\varphi}_a - Z_a^R)\cdot\nu\,\bar{\varphi}\,d\sigma \right| \\ &\le \mathrm{const}\left\|(i\nabla + A_0)(e^{i\alpha(\theta_0^p - \theta_p)}\tilde{\varphi}_a - Z_a^R)\cdot\nu - \gamma_{p,\delta}(i\nabla + A_0)(e^{i\alpha(\theta_0^p - \theta_p)}\Psi_p - z_{p,R})\cdot\nu\right\|_{H^{-1/2}(\partial D_R)} \\ &\qquad + \gamma_{p,\delta} \sup_{\substack{\varphi\in\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})}=1}} \left| \int_{\partial D_R} (i\nabla + A_0)(e^{i\alpha(\theta_0^p - \theta_p)}\Psi_p - z_{p,R})\cdot\nu\,\bar{\varphi}\,d\sigma \right|,\end{aligned}$$

where

$$\gamma_{p,\delta} = \frac{\beta}{|\beta|}\left(\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_p|^2\,ds}\right)^{1/2}$$

and the constant const > 0 is independent of a. From Lemmas 5.4 and 6.2 we have

$$(i\nabla + A_0)(e^{i\alpha(\theta_0^p - \theta_p)}\tilde{\varphi}_a - Z_a^R)\cdot\nu \to \gamma_{p,\delta}(i\nabla + A_0)(e^{i\alpha(\theta_0^p - \theta_p)}\Psi_p - z_{p,R})\cdot\nu$$

in $H^{-1/2}(\partial D_R)$ as $a = |a|p \to 0$. Therefore $h(p, a, R) \le f(p, a, R)$ with

$$\lim_{a=|a|p\to 0} f(p, a, R) = \gamma_{p,\delta} \sup_{\substack{\varphi\in\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})}=1}} \left| \int_{\partial D_R} (i\nabla + A_0)(e^{i\alpha(\theta_0^p - \theta_p)}\Psi_p - z_{p,R})\cdot\nu\,\bar{\varphi}\,d\sigma \right|.$$

To complete the proof is then enough to show that

$$\lim_{R\to+\infty} \sup_{\substack{\varphi\in\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})}=1}} \left|\int_{\partial D_R} (i\nabla + A_0)(e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p - z_{p,R})\cdot \nu\,\bar\varphi\,d\sigma\right| = 0. \tag{6-5}$$

Using an integration by parts we can rewrite

$$\begin{aligned}
&\left|\int_{\partial D_R} (i\nabla + A_0)(e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p - z_{p,R})\cdot \nu\,\bar\varphi\,d\sigma\right| \\
&\quad= \left|\int_{\partial D_R} e^{i\alpha(\theta_0^p-\theta_p)}(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)\cdot\nu\bar\varphi\,d\sigma + \int_{\partial D_R}(i\nabla + A_0)(\psi_k - z_{p,R})\cdot\nu\bar\varphi\,d\sigma\right| \\
&\quad= \bigg|-i\int_{\mathbb{R}^2\setminus D_R}(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)\cdot\overline{(i\nabla + A_0)\varphi}\,e^{i\alpha(\theta_0^p-\theta_p)}\,dx \\
&\qquad\qquad + i\int_{D_R}(i\nabla + A_0)(\psi_k - z_{p,R})\cdot\overline{(i\nabla + A_0)\varphi}\,dx\bigg|,
\end{aligned}$$

which implies

$$\begin{aligned}
&\sup_{\substack{\varphi\in\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})}=1}} \left|\int_{\partial D_R} (i\nabla + A_0)(e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p - z_{p,R})\cdot \nu\,\bar\varphi\,d\sigma\right| \\
&\le \left(\int_{\mathbb{R}^2\setminus D_R}\left|(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)\right|^2 dx\right)^{1/2} + \left(\int_{D_R}\left|(i\nabla + A_0)(\psi_k - z_{p,R})\right|^2 dx\right)^{1/2}.
\end{aligned} \tag{6-6}$$

The first term in the right-hand side of (6-6) goes to zero as $R\to+\infty$ because of (1-19). To estimate the second term, we consider a test function η_R satisfying (6-3) and the additional property $|\nabla\eta_R|\le 4/R$ in $D_R\setminus D_{R/2}$. Recalling that $\psi_k - z_{p,R}$ satisfies $(i\nabla + A_0)^2(\psi_k - z_{p,R}) = 0$ in D_R with the boundary condition $\psi_k - z_{p,R} = \psi_k - e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p$ on ∂D_R, the Dirichlet principle and the Hardy inequality (1-4) provide

$$\begin{aligned}
&\int_{D_R}|(i\nabla + A_0)(\psi_k - z_{p,R})|^2\,dx \\
&\qquad\le \int_{D_R}|(i\nabla + A_0)(\eta_R(\psi_k - e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p))|^2\,dx \\
&\qquad\le 2\int_{D_R}|\nabla\eta_R|^2\,|\psi_k - e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p|^2\,dx + 2\int_{\mathbb{R}^2\setminus D_{R/2}}|(i\nabla + A_0)(\psi_k - e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p)|^2\,dx \\
&\qquad\le \frac{32}{R^2}\int_{D_R\setminus D_{R/2}}|\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k|^2\,dx + 2\int_{\mathbb{R}^2\setminus D_{R/2}}|(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)|^2\,dx \\
&\qquad\le \frac{32(R+1)^2}{R^2}\int_{D_{R+1}(p)\setminus D_{(R-2)/2}(p)}\frac{|\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k|^2}{|x-p|^2}\,dx \\
&\qquad\qquad + 2\int_{\mathbb{R}^2\setminus D_{R/2}}|(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)|^2\,dx
\end{aligned}$$

$$\le \frac{32(R+1)^2}{R^2\mu_1}\int_{D_{R+1}(p)\setminus D_{(R-2)/2}(p)} |(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)|^2\,dx \\ + 2\int_{\mathbb{R}^2\setminus D_{R/2}} |(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)|^2\,dx,$$

which goes to zero again thanks to (1-19). Therefore we have obtained (6-5) and the proof is complete. □

Proof of Theorem 1.3. Let $p \in \mathbb{S}^1$ and $\varepsilon > 0$. From Lemma 6.1 and (6-4) there exists some $R_0 > \max\{2, K_\delta\}$ sufficiently large such that

$$|\mathcal{F}_p(R_0) - \mathfrak{L}_p| < \varepsilon \quad \text{and} \quad |G(p, R_0)| < \varepsilon.$$

Moreover, again from Lemmas 6.3 and 6.1 there exists $\rho > 0$ (depending on p, ε, and R_0) such that, if $a = |a|p$ and $|a| < \rho$, then

$$|G(p, a, R_0) - G(p, R_0)| < \varepsilon$$

and

$$\left| \frac{1}{|a|^{2|\alpha-k|}} \int_{D_{R_0|a|}} \left|(i\nabla + A_a)\varphi_a(x) - e^{i\alpha(\theta_a-\theta_0^a)(x)}(i\nabla + A_0)\varphi_0(x)\right|^2 dx - |\beta|^2 \mathcal{F}_p(R_0) \right| < \varepsilon.$$

Therefore, taking into account Lemma 6.3, we have that, for all $a = |a|p$ with $|a| < \rho$,

$$\begin{aligned}
&\left| |a|^{-2|\alpha-k|} \int_\Omega \left|(i\nabla + A_a)\varphi_a - e^{i\alpha(\theta_a-\theta_0^a)}(i\nabla + A_0)\varphi_0\right|^2 dx - |\beta|^2\mathfrak{L}_p \right| \\
&\le \left| |a|^{-2|\alpha-k|} \int_{D_{R_0|a|}} \left|(i\nabla + A_a)\varphi_a - e^{i\alpha(\theta_a-\theta_0^a)}(i\nabla + A_0)\varphi_0\right|^2 dx - |\beta|^2\mathcal{F}_p(R_0) \right| \\
&\qquad + |a|^{-2|\alpha-k|} \int_{\Omega\setminus D_{R_0|a|}} \left|(i\nabla + A_a)\varphi_a - e^{i\alpha(\theta_a-\theta_0^a)}(i\nabla + A_0)\varphi_0\right|^2 dx + |\beta|^2\,|\mathfrak{L}_p - \mathcal{F}_p(R_0)| \\
&< \varepsilon + G(p, a, R_0) + |\beta|^2\varepsilon \\
&\le \varepsilon + |G(p, a, R_0) - G(p, R_0)| + |G(p, R_0)| + |\beta|^2\varepsilon = (3 + |\beta|^2)\varepsilon.
\end{aligned}$$

□

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COMPLEX ROTATION NUMBERS: BUBBLES AND THEIR INTERSECTIONS

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The construction of complex rotation numbers, due to V. Arnold, gives rise to a fractal-like set "bubbles" related to a circle diffeomorphism. "Bubbles" is a complex analogue to Arnold tongues.

This article contains a survey of the known properties of bubbles, as well as a variety of open questions. In particular, we show that bubbles can intersect and self-intersect, and provide approximate pictures of bubbles for perturbations of Möbius circle diffeomorphisms.

1. Introduction

1.1. ***Complex rotation numbers: Arnold's construction.*** In what follows, $f:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ is an analytic orientation-preserving circle diffeomorphism. Its analytic extension to a small neighborhood of $\mathbb{R}/\mathbb{Z}$ in $\mathbb{C}/\mathbb{Z}$ is still denoted by f. $\mathbb{H}\subset\mathbb{C}$ is the open upper half-plane.

The following construction was suggested by V. Arnold [1983, Section 27] in 1978. Given $\omega\in\mathbb{H}/\mathbb{Z}$ and a small positive $\varepsilon\in\mathbb{R}$, one can construct a complex torus $E(f+\omega)$ as the quotient space of a cylinder Π by the action of $f+\omega$:

$$\begin{gathered}\Pi:=\{z\in\mathbb{C}/\mathbb{Z}\mid -\varepsilon<\operatorname{Im}z<\operatorname{Im}\omega+\varepsilon\},\\ E(f+\omega):=\Pi/(z\sim f(z)+\omega).\end{gathered}\tag{1}$$

For a small positive ε, the quotient space $E(f+\omega)$ is a torus, inherits a complex structure from $\mathbb{C}/\mathbb{Z}$ and does not depend on ε.

Due to the uniformization theorem, for a unique $\tau\in\mathbb{H}/\mathbb{Z}$ there exists a biholomorphism

$$H_\omega : E(f+\omega)\to\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})\tag{2}$$

such that H_ω takes $\mathbb{R}/\mathbb{Z}\subset E(f+\omega)$ to a curve homotopic to $\mathbb{R}/\mathbb{Z}\subset\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. The number $\tau(f+\omega):=\tau\in\mathbb{H}/\mathbb{Z}$, i.e., the modulus of the complex torus $E(f+\omega)$, is called the *complex rotation number* of $f+\omega$.

In the original Arnold's construction, ω was supposed to be purely imaginary. The above version of this construction was suggested by R. Fedorov. The term "complex rotation number" is due to E. Risler [1999].

The complex rotation number $\tau(f+\omega)$ depends holomorphically on $\omega\in\mathbb{H}/\mathbb{Z}$; see [Risler 1999, Section 2.1, Proposition 2].

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1.2. ***Rotation number and its properties.*** This section lists well-known results on rotation numbers; see [Katok and Hasselblatt 1995, Sections 3.11, 3.12] for more details.

Let f be an orientation-preserving circle homeomorphism, and let $F : \mathbb{R} \to \mathbb{R}$ be its lift to the real line. The limit

$$\operatorname{rot} f = \lim_{n\to\infty} \frac{F^{\circ n}(x)}{n} \mod 1$$

exists and does not depend on $x \in \mathbb{R}$. It is called the *rotation number* of the circle homeomorphism f.

Rotation number is invariant under continuous conjugations of f. It is rational, $\operatorname{rot} f = p/q$, if and only if f has a periodic orbit of period q. If $\operatorname{rot} f$ is irrational and $f \in C^2(\mathbb{R}/\mathbb{Z})$, then f is continuously conjugate to $z \mapsto z + \operatorname{rot} f$ (Denjoy theorem, see [Katok and Hasselblatt 1995, Section 3.12.1]). We will need the following, much more complicated result.

Definition. A real number ρ is called *Diophantine* if there exist $C, \beta > 0$ such that for all rationals p/q,

$$\left|\rho - \frac{p}{q}\right| \geq \frac{C}{q^{2+\beta}}.$$

Theorem 1 (M. R. Herman [1979], J.-C. Yoccoz [1984]). *If an analytic circle diffeomorphism has a Diophantine rotation number* $\operatorname{rot} f$, *then it is analytically conjugate to* $z \mapsto z + \operatorname{rot} f$.

This motivates the term "complex rotation number" for $\tau(f+\omega)$ above: while a circle diffeomorphism f is conjugate to the rotation $x \mapsto x + \operatorname{rot} f$ on $\mathbb{R}/\mathbb{Z}$, a complex-valued map $f+\omega$ is biholomorphically conjugate to the complex shift $z \mapsto z + \tau(f+\omega)$ in the cylinder $\Pi \subset \mathbb{C}/\mathbb{Z}$.

1.3. ***Steps on the graph of $\omega \mapsto \operatorname{rot}(f+\omega)$.*** Rotation number depends continuously on f in the C^0-topology. In particular, $\operatorname{rot}(f+\omega)$ depends continuously on $\omega \in \mathbb{R}/\mathbb{Z}$; clearly, it (nonstrictly) increases on ω.

Recall that a periodic orbit of a circle diffeomorphism is called *parabolic* if its multiplier is 1, and *hyperbolic* otherwise. If a circle diffeomorphism has periodic orbits, and they are all hyperbolic, then the diffeomorphism is called *hyperbolic*.

Let $I_{p/q} := \{\omega \in \mathbb{R}/\mathbb{Z} \mid \operatorname{rot}(f+\omega) = p/q\}$; from now on, we always assume that p, q are coprime. If for some value of ω the diffeomorphism $f+\omega$ has the rotation number p/q and a *hyperbolic* orbit of period q, then this orbit persists under a small perturbation of ω. In this case, $I_{p/q}$ is a segment of nonzero length. Endpoints of $I_{p/q}$ correspond to diffeomorphisms $f+\omega$ having only parabolic orbits.

In a generic case, the graph of the function $\omega \mapsto \operatorname{rot}(f+\omega)$ contains infinitely many steps, i.e., nontrivial segments $I_{p/q} \times \{p/q\}$, on rational heights.

1.4. ***Rotation numbers as boundary values of a holomorphic function.***

Question 2. Can we find a holomorphic self-map τ on $\mathbb{H}/\mathbb{Z}$ such that its boundary values on $\mathbb{R}/\mathbb{Z}$ coincide with $\omega \mapsto \rho(f+\omega)$?

The answer is No (except for the trivial case $f(x) = x + c$), because the function $\omega \mapsto \rho(f+\omega)$ is locally constant on nonempty intervals $I_{p/q}$, and this is not possible for boundary values of holomorphic functions. In more detail, note that $\mathbb{H}/\mathbb{Z}$ is biholomorphically equivalent to the punctured unit disc $D \setminus \{0\}$, so the map $1/(2\pi i)\ln z : D\setminus\{0\} \to \mathbb{H}/\mathbb{Z}$ conjugates τ to a holomorphic bounded self-map of the punctured unit disc. Clearly, 0 is a removable singularity for this self-map. The following Luzin–Privalov theorem [1925, Section 14, p. 159] shows that such an extension τ does not exist:

Theorem 3 (N. Luzin, J. Privalov). *If a holomorphic function in the unit disc D has finite nontangential limits at all points of $E \subset \partial D$, where E has a nonzero Lebesgue measure, then this function is uniquely defined by these limits.*

This motivates the next question:

Question 4. Can we find a holomorphic self-map on $\mathbb{H}/\mathbb{Z}$ such that its boundary values on $(\mathbb{R}/\mathbb{Z}) \setminus \bigcup I_{p/q}$ coincide with $\omega \mapsto \rho(f+\omega)$?

Remark. The set $(\mathbb{R}/\mathbb{Z}) \setminus \bigcup I_{p/q}$ has nonzero measure due to a result of M. R. Herman [1977, Section 6, p. 287]; so by Theorem 3, such a holomorphic extension must be unique.

The answer to this question is Yes, and this holomorphic function is the complex rotation number $\tau(f+\omega)$. The following theorem is proved in [Buff and Goncharuk 2015]; the proof is based on previous results by E. Risler [1999], V. Moldavskij [2001], Y. Ilyashenko and V. Moldavskij [2003], and N. Goncharuk [2012].

Theorem 5 (X. Buff and N. Goncharuk [2015]). *Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation-preserving analytic circle diffeomorphism. Then the holomorphic function $\tau(f + \cdot\,) : \mathbb{H}/\mathbb{Z} \to \mathbb{H}/\mathbb{Z}$ has a continuous extension $\bar\tau(f + \cdot\,) : \overline{\mathbb{H}/\mathbb{Z}} \to \overline{\mathbb{H}/\mathbb{Z}}$. Assume $\omega \in \mathbb{R}/\mathbb{Z}$:*

- *If* $\operatorname{rot}(f+\omega)$ *is irrational, then* $\bar\tau(f+\omega) = \operatorname{rot}(f+\omega)$.
- *If* $\operatorname{rot}(f+\omega)$ *is rational and $f+\omega$ has a parabolic periodic orbit, then* $\bar\tau(f+\omega) = \operatorname{rot}(f+\omega)$.
- *If* $\operatorname{rot}(f+\omega)$ *is rational and $f+\omega$ is hyperbolic on an open interval $\omega \in I \subset \mathbb{R}/\mathbb{Z}$, then $\bar\tau(f+\omega)$ depends analytically on $\omega \in I$ and $\bar\tau(f+\omega) \in \mathbb{H}/\mathbb{Z}$ for $\omega \in I$.*

The extension $\bar\tau(f+\omega)$ is also called the complex rotation number of $f+\omega$. Due to Theorem 5, it is continuous on ω, and coincides with the ordinary rotation number on $\mathbb{R}/\mathbb{Z} \setminus \bigcup I_{p/q}$.

Definition. The image of the segment $I_{p/q} = \{\omega \in \mathbb{R}/\mathbb{Z} \mid \operatorname{rot}(f+\omega) = p/q\}$ under the map $\omega \mapsto \bar\tau(f+\omega)$ is called the *p/q-bubble* of f.

Due to Theorem 5, the p/q-bubble is a union of several analytic curves in the upper half-plane with endpoints at p/q. Each analytic curve corresponds to the interval of hyperbolicity of $f+\omega$, and its endpoints correspond to $f+\omega$ with parabolic orbits.

So, each circle diffeomorphism f gives rise to a "fractal-like" set $\bar\tau(f+\omega)$ (bubbles) in the upper half-plane, containing countably many analytic curves. The picture of bubbles growing from rational points of the real axis was first described by R. Fedorov (oral communication, about 2001), and remained conjecturable until [Goncharuk 2012; Buff and Goncharuk 2015].

The possible shapes of bubbles are not known. The following question is also open.

Question 6. Is the set $\bar\tau(f+\omega)$ self-similar (i.e., is it a fractal set)?

The precise meaning of "self-similarity" in this question is not clear; conjecturably, for certain sequences of rational numbers $\{p_n/q_n\}$, the p_n/q_n-bubbles (when rescaled properly) tend to some limit shape.

1.5. ***Properties of bubbles and the Main Theorem.***

Question 7. Is $\bar{\tau}$ invariant under analytic conjugacies?

The answer is Yes:

Lemma 8. *The complex rotation number $\bar{\tau}$ is invariant under analytic conjugacies*: *for two analytically conjugate circle diffeomorphisms f_1, f_2, we have $\bar{\tau}(f_1) = \bar{\tau}(f_2)$.*

For nonhyperbolic f_1, f_2, their complex rotation numbers coincide with rotation numbers, so this lemma trivially repeats the invariance of rotation numbers under conjugacies. For hyperbolic diffeomorphisms, the proof of this lemma is implicitly contained in [Buff and Goncharuk 2015]; see also Section 5 below.

Note that in general, for conjugate f_1, f_2 and $\omega \in \overline{\mathbb{H}}/\mathbb{Z}$, the numbers $\bar{\tau}(f_1+\omega)$ and $\bar{\tau}(f_2+\omega)$ do not coincide.

Question 9. Is there an explicit formula for $\bar{\tau}(f+\omega)$?

The only case when the author can obtain an explicit formula for $\bar{\tau}(f+\omega)$ is described in the following proposition.

Let $\pi : \mathbb{C}/\mathbb{Z} \to \mathbb{C}^*$ be given by $\pi(z) := \exp(2\pi i z)$.

Proposition 10. *Let F be a Möbius map that preserves the circle $\{|w| = 1 \mid w \in \mathbb{C}\}$. Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be given by $f := \pi^{-1} \circ F \circ \pi$. Then f has only a 0-bubble, and this bubble is a vertical segment.*

Proof. First, let us compute $\tau(f+\omega)$ for $\omega \in \mathbb{H}/\mathbb{Z}$.

Put $F_\omega := e^{2\pi i\omega} F$. For $\omega \in \mathbb{H}/\mathbb{Z}$ and small $\varepsilon > 0$, let $E^*(F_\omega)$ be the quotient space of the annulus $\Pi^* := \{1 > |w| > |e^{2\pi i\omega}|\}$ via the map F_ω. Note that the map π induces a biholomorphism of $E(f+\omega)$ to $E^*(F_\omega)$. Indeed, it takes Π to the neighborhood of Π^* and conjugates $f+\omega$ to $F_\omega = \pi \circ (f+\omega) \circ \pi^{-1}$. So $\tau(f+\omega)$ is equal to the modulus of $E^*(F_\omega)$.

The map F_ω is a Möbius map that takes the unit circle to the interior of the unit disc. Let A_ω be its attractor with multiplier $\mu(\omega)$ and R_ω be its repellor. The map $(w - A_\omega)/(w - R_\omega)$ conjugates F_ω to the linear map $w \mapsto \mu(\omega)w$, and thus induces a biholomorphism of $E^*(F_\omega)$ to the complex torus $\mathbb{C}^*/(w \sim \mu(\omega)w)$. The modulus of this torus is equal to $1/(2\pi i) \ln \mu(\omega)$. Finally, $\tau(f+\omega) = 1/(2\pi i) \ln \mu(\omega)$.

Now let us study the boundary values of $\tau(f+\omega)$, i.e., $\bar{\tau}(f+\omega_0) = \lim_{\omega \to \omega_0} \tau(f+\omega)$ for $\omega_0 \in \mathbb{R}/\mathbb{Z}$.

The map F_{ω_0} is a Möbius self-map of the unit circle. If it has two hyperbolic fixed points on the unit circle (i.e., ω_0 is an interior point of I_0), then the multiplier of its attractor, $\mu(\omega_0)$, is real because F_{ω_0} preserves the unit circle. Then

$$\bar{\tau}(f+\omega_0) = \lim_{\omega \to \omega_0} \frac{1}{2\pi i} \ln \mu(\omega) = \frac{1}{2\pi i} \ln \mu(\omega_0) \in i\mathbb{R}.$$

If F_{ω_0} has one parabolic fixed point on the unit circle, then $\lim_{\omega \to \omega_0} \mu(\omega) = 1$, and $\bar{\tau}(f+\omega_0) = 0$. If F_{ω_0} has no fixed points on the unit circle (i.e., $\omega_0 \in (\mathbb{R}/\mathbb{Z}) \setminus I_0$), then it has a unique fixed point A_{ω_0} inside the unit disc and a unique fixed point R_{ω_0} outside it; the Schwarz lemma implies that the multiplier of A_{ω_0}

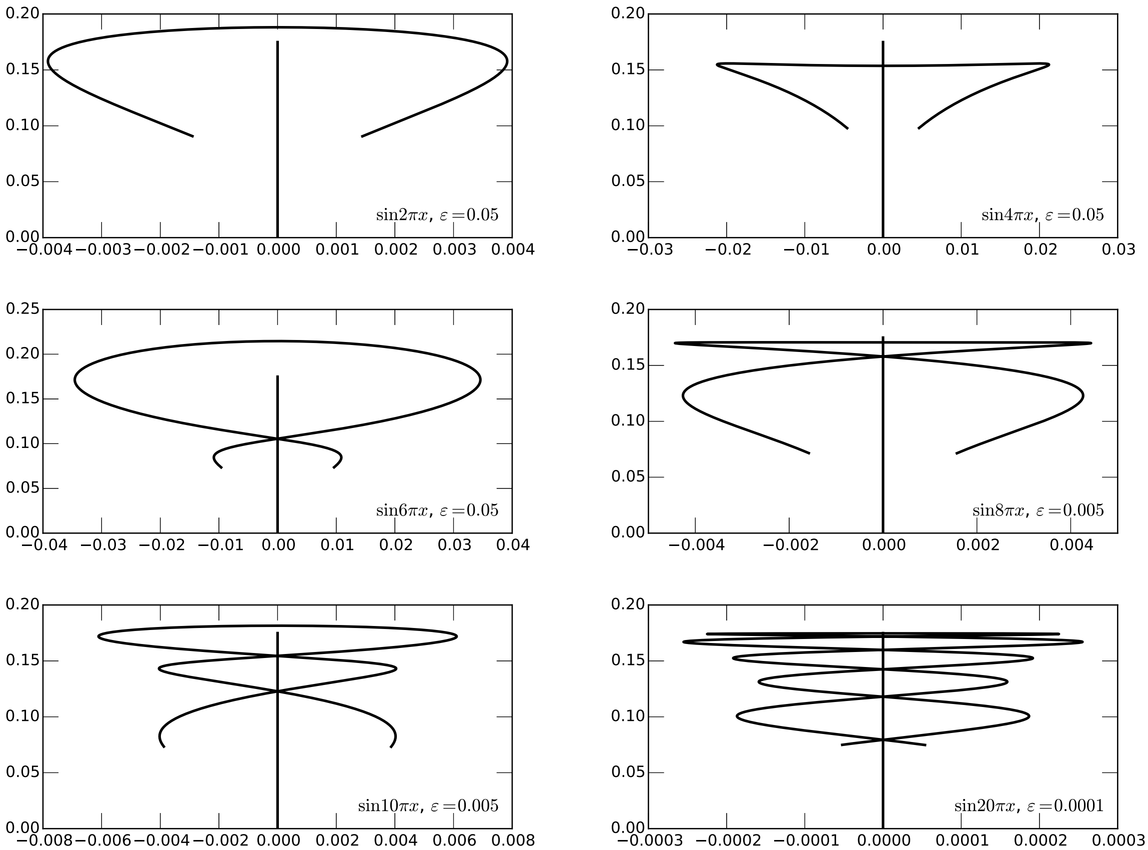

Figure 1. Infinitesimal 0-bubbles for a perturbation of the Möbius map $f = (z+0.5)/(1+0.5z)$ by the map $g = \sin 2\pi n x,\ n = 1, 2, 3, 4, 5, 10$. The pictures are rescaled horizontally. The vertical segment on each picture is the 0-bubble for f.

satisfies $|\mu(\omega_0)| = 1$, so

$$\bar{\tau}(f+\omega_0) = \lim_{\omega\to\omega_0} \frac{1}{2\pi i} \ln \mu(\omega) = \frac{1}{2\pi i} \ln \mu(\omega_0) \in \mathbb{R}/\mathbb{Z}.$$

Finally, the image of I_0 under $\bar{\tau}(f+\cdot)$ belongs to $i\mathbb{R}$, and the image of $(\mathbb{R}/\mathbb{Z}) \setminus I_0$ belongs to $\mathbb{R}/\mathbb{Z}$. We conclude that the only bubble of f is a 0-bubble, and it is a vertical segment. □

Question 11. Is there a way to compute $\bar{\tau}(f+\omega)$ approximately?

In the general case, one can try to implement the construction described in Section 5 as a computer program. The author haven't done this yet. For perturbations of Möbius maps, a simpler approach is described below.

Take a map $f + \varepsilon g$ where f is as in Proposition 10, and g is a trigonometric polynomial. Figure 1 shows infinitesimal 0-bubbles of $f + \varepsilon g$.

Definition. An *infinitesimal 0-bubble* for a perturbation $f+\varepsilon g$ of an analytic circle diffeomorphism f is the image of the segment I_0 for f under the map

$$\omega \mapsto \bar{\tau}(f+\omega)+\varepsilon \cdot \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{\tau}(f+\varepsilon g+\omega),$$

i.e., under the linear approximation to the complex rotation number.

The choice of ε is shown on each picture in Figure 1, but it does not essentially affect the shape of the infinitesimal bubble. In the lower part of bubbles, $(d/d\varepsilon)|_{\varepsilon=0}\bar{\tau}(f+\varepsilon g+\omega)$ tends to infinity. So the linear approximation is not accurate, and this part of infinitesimal bubbles is not shown on the picture.

The following proposition enables us to draw infinitesimal bubbles. Its proof follows the same scheme as the computation in [Risler 1999, Section 2.2.3]; it is postponed until the Appendix.

Proposition 12. *Let f, g be as above. Let γ be a curve in $\mathbb{C}/\mathbb{Z}$ which is close to $\mathbb{R}/\mathbb{Z}$ and passes below the attractor and above the repellor of $f+\omega$, $\omega \in I_0$. Then*

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{\tau}(f+\varepsilon g+\omega)=\int_\gamma \frac{g(z)}{f'(z)}(H'_\omega(z))^2\,dz, \tag{3}$$

where H_ω uniformizes $E(f+\omega)$. As in Proposition 10, one can compute H_ω explicitly. The derivatives in the right-hand side are with respect to z.

For any trigonometric polynomial g (say, $g(x)=\sin 2\pi n x$), the change of variable $w=\pi(z)$ turns the integral (3) into an integral of a rational function along the closed loop $\pi(\gamma)$. We then compute it explicitly via the residue theorem; for $n \geq 3$, the formulas become cumbersome and we use a computer algebra system GiNaC [Bauer et al.; Vollinga 2006] to obtain them. The infinitesimal bubbles thus obtained are shown in Figure 1.

In certain cases, intersections of infinitesimal 0-bubbles for $f+\varepsilon g$ mean that for small ε, the 0-bubbles of $f+\varepsilon g$ intersect as well; see Remark 17 below.

Question 13. Is it true that the map $\omega \mapsto \tau(f+\omega)$ is injective (so that the bubbles belong to the boundary of the set $\{\tau(f+\omega) \mid \omega \in \mathbb{H}/\mathbb{Z}\}$)?

No, see [Buff and Goncharuk 2015, Corollary 16].

Question 14. How large are the bubbles?

In [Buff and Goncharuk 2015, Main Theorem] the authors prove that the p/q-bubble (with coprime p, q) is within a disc of radius $D_f/(4\pi q^2)$ tangent to $\mathbb{R}/\mathbb{Z}$ at p/q, where D_f is the distortion of f,

$$D_f := \int_{\mathbb{R}/\mathbb{Z}} \left|\frac{f''(x)}{f'(x)}\right| dx.$$

Question 15. Can the bubbles intersect or self-intersect?

Here are several results in this direction.

Proposition 16. *If an analytic circle diffeomorphism f is sufficiently close to a rotation in C^2 metrics, its different bubbles do not intersect.*

Proof. We will use the answer to Question 14 above. Suppose that the distortion of f satisfies $D_f < 2\pi$, which holds true if f is C^2-close to a rotation. For each p/q, take the disc of radius $D_f/(4\pi q^2) < 1/(2q^2)$ tangent to $\mathbb{R}/\mathbb{Z}$ at p/q. It is easy to verify that these discs do not intersect for different p/q. As mentioned in the answer to Question 14, the bubbles are within such discs, so they do not intersect as well. □

This proposition does not imply that the bubbles of f are not self-intersecting. This article contains an affirmative answer to Question 15:

Main Theorem. (1) *There exists a circle diffeomorphism f such that its* 0*-bubble is self-intersecting.*

(2) *For each rational p/q, there exists a circle diffeomorphism f such that its* 0*-bubble intersects its p/q-bubble.*

We do not assert that these bubbles intersect transversely; it is possible that they are tangent at a common point.

Remark 17. Let

$$f = \frac{z+0.5}{1+0.5z}$$

be the Möbius map that we chose to draw infinitesimal 0-bubbles. Let $g = \sin 2\pi n x$, $n = 3, 4, 5$, or 10. Using the self-intersections of infinitesimal 0-bubbles for $f + \varepsilon g$, see Figure 1, one may show that for sufficiently small ε, the 0-bubble of $f + \varepsilon g$ is self-intersecting. This provides an alternative proof of the first part of the Main Theorem. Here we sketch this proof.

Let $l_1(\varepsilon)$ and $l_2(\varepsilon)$ be two small intersecting arcs of the infinitesimal 0-bubble for $f + \varepsilon g$. Let $a_\varepsilon, b_\varepsilon$ and $c_\varepsilon, d_\varepsilon$ be the endpoints of $l_1(\varepsilon), l_2(\varepsilon)$ respectively. It is easy to verify that the lengths of the sides and the diagonals of the quadrilateral $a_\varepsilon c_\varepsilon b_\varepsilon d_\varepsilon$ are of order ε, and $l_1(\varepsilon), l_2(\varepsilon)$ are close to these diagonals. The 0-bubble of $f + \varepsilon g$ is $o(\varepsilon)$-close to the infinitesimal 0-bubble for $f + \varepsilon g$, and thus it contains a pair of curves that are $o(\varepsilon)$-close to $l_1(\varepsilon), l_2(\varepsilon)$. This implies that the 0-bubble of $f + \varepsilon g$ is self-intersecting for small ε.

2. Main lemmas

Part 1 of the Main Theorem is based on Lemma 8 and the following lemma.

Lemma 18. *For any hyperbolic analytic circle diffeomorphism f_1 with* $\operatorname{rot} f_1 = 0$ *and any analytic circle diffeomorphism $f_2 \neq \mathrm{id}$, there exists an analytic diffeomorphism f and $\omega \in \mathbb{R}/\mathbb{Z} \setminus \{0\}$ such that f and $f + \omega$ are analytically conjugate to f_1, f_2 respectively.*

This lemma provides a nonrestrictive sufficient condition for two analytic diffeomorphisms to appear (up to analytic conjugacies) in one and the same family of the form $f + \omega$.

Part (2) of the Main Theorem also requires the following lemma, which is interesting in its own right.

Lemma 19. *For any complex number $w \in \mathbb{H}/\mathbb{Z}$ and any natural number m, there exists a hyperbolic circle diffeomorphism f having $2m$ fixed points and the complex rotation number $\bar{\tau}(f) = w$.*

Lemma 8 shows that complex rotation numbers can be used as invariants of analytic classification of families of circle diffeomorphisms; Lemma 19 is a weak version of the realization of these invariants. The following realization question is open:

Question 20. Which holomorphic self-maps of the upper half-plane are realized as $\omega \mapsto \tau(f+\omega)$ for some circle diffeomorphism f?

3. Proof of the Main Theorem modulo Lemmas 18 and 19

3.1. ***Part (1): self-intersecting 0-bubble.*** This part of the Main Theorem does not require Lemma 19.

Fix a hyperbolic circle diffeomorphism f_1 with $\operatorname{rot} f_1 = 0$. Apply Lemma 18 to f_1 and $f_2 = f_1$.

We get a circle diffeomorphism f such that f, $f+\omega$ with $\omega \neq 0 \bmod 1$ are both analytically conjugate to f_1. Due to Lemma 8, $\bar{\tau}(f) = \bar{\tau}(f_1) = \bar{\tau}(f+\omega)$. Note that $\bar{\tau}(f)$, $\bar{\tau}(f+\omega)$ belong to the 0-bubble for f because f, $f+\omega$ have zero rotation number and are hyperbolic.

So the 0-bubble for f passes twice through the point $\bar{\tau}(f_1)$. This completes the proof of the Main Theorem, part (1).

Remark. Using Lemma 19, one can also prove that the 0-bubble may self-intersect at any prescribed point $w \in \mathbb{H}/\mathbb{Z}$. To achieve this, it is sufficient to start with f_1 provided by Lemma 19 such that $\bar{\tau}(f_1) = w$.

3.2. ***Part (2): intersection of 0-bubble and p/q-bubble.*** Take a hyperbolic circle diffeomorphism f_2 with $\operatorname{rot} f_2 = p/q$. Put $w := \bar{\tau}(f_2)$. Using Lemma 19, construct a hyperbolic circle diffeomorphism f_1 with zero rotation number such that $\bar{\tau}(f_1) = w$.

Now, the two circle diffeomorphisms f_1, f_2 satisfy $\operatorname{rot} f_1 = 0$, $\operatorname{rot} f_2 = p/q$ and $\bar{\tau}(f_1) = \bar{\tau}(f_2)$.

Lemma 18 provides us with a circle diffeomorphism f such that f, $f+\omega$ are conjugate to f_1, f_2. Due to Lemma 8, $\bar{\tau}(f) = \bar{\tau}(f_1) = w$ and $\bar{\tau}(f+\omega) = \bar{\tau}(f_2) = w$. The point w belongs to the 0-bubble of f, because $\operatorname{rot} f = \operatorname{rot} f_1 = 0$ and f is hyperbolic, and it also belongs to the p/q-bubble, because $\operatorname{rot}(f+\omega) = \operatorname{rot}(f_2) = p/q$ and $f+\omega$ is hyperbolic. Finally, the 0-bubble and the p/q-bubble for f intersect at w. This completes the proof of the Main Theorem, part (2).

Remark. In a similar way one can prove that the 0-bubble and the p/q-bubble may intersect at any prescribed point $w \in \mathbb{C}/\mathbb{Z}$. This requires an analogue of Lemma 19 for circle diffeomorphisms with nonzero rational rotation numbers; the proof of this analogue repeats the proof of Lemma 19, except for some technical details.

4. Proof of Lemma 18

We say that two circle diffeomorphisms f_1, f_2 have a *Diophantine quotient* if $\operatorname{rot}(f_1 f_2^{-1}) =: \omega$ is Diophantine. Lemma 18 follows from two propositions below.

Proposition 21. *If two analytic circle diffeomorphisms f_1, f_2 have a Diophantine quotient and $\operatorname{rot}(f_1 f_2^{-1}) =: \omega$, then there exists an analytic diffeomorphism f such that f and $f+\omega$ are analytically conjugate to f_1, f_2 respectively.*

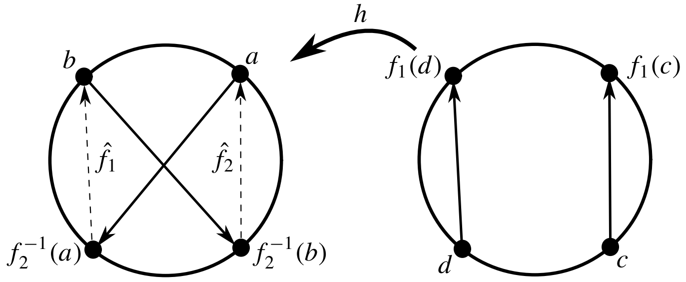

Figure 2. The choice of h that yields $\operatorname{rot}(\hat{f}_1 f_2^{-1}) = \frac{1}{2}$.

Proposition 22. *Any hyperbolic analytic circle diffeomorphism f_1 with* $\operatorname{rot} f_1 = 0$ *is analytically conjugate to a diffeomorphism that has a Diophantine quotient with a given analytic circle diffeomorphism f_2, $f_2 \neq \mathrm{id}$.*

Proof of Proposition 21 . Due to the Herman–Yoccoz theorem (see Theorem 1), in some analytic chart, $f_1 f_2^{-1}$ is the rotation by $\omega = \operatorname{rot} f_1 f_2^{-1}$. Let $\tilde{f}_1$, $\tilde{f}_2$ be the diffeomorphisms f_1, f_2 in this analytic chart; then $\tilde{f}_1 \tilde{f}_2^{-1}(z) = z + \omega$. So $\tilde{f}_1(z) = \tilde{f}_2(z) + \omega$, and we can take $f = \tilde{f}_2$. □

Proof of Proposition 22. Let $\mathcal{A}$ be the set of analytic diffeomorphisms of the form $\hat{f}_1 = h \circ f_1 \circ h^{-1}$ for all possible analytic orientation-preserving diffeomorphisms h. Then $\mathcal{A}$ is a linearly connected subset of the space of all analytic circle diffeomorphisms, because for each h_1, h_2, we can join h_1 to h_2 by a continuous family of analytic circle diffeomorphisms h^t. Now if we show that the continuous function $\hat{f}_1 \mapsto \operatorname{rot}(\hat{f}_1 f_2^{-1})$ on $\mathcal{A}$ takes two distinct values, then it takes all intermediate values, including Diophantine values.

Let us find two maps of the form $\hat{f}_1 = h \circ f_1 \circ h^{-1}$ such that $\operatorname{rot}(\hat{f}_1 f_2^{-1})$ attains values 0 and $\frac{1}{2}$:

- $\operatorname{rot}(\hat{f}_1 f_2^{-1}) = 0$. Choose h such that for some point $a \in \mathbb{R}/\mathbb{Z}$, we have $\hat{f}_1(a) = f_2(a)$. This is possible, because $f_1 \neq \mathrm{id}$ and $f_2 \neq \mathrm{id}$. Then $\hat{f}_1 f_2^{-1}(f_2(a)) = f_2(a)$, so $f_2(a)$ is a fixed point for $\hat{f}_1 f_2^{-1}$, and $\operatorname{rot}(\hat{f}_1 f_2^{-1}) = 0$.
- $\operatorname{rot}(\hat{f}_1 f_2^{-1}) = \frac{1}{2}$. Choose two points $a, b \in \mathbb{R}/\mathbb{Z}$ such that these points and their preimages under f_2 are distinct and are ordered in the following way along the circle: a, b, $f_2^{-1}(a)$, $f_2^{-1}(b)$. It is sufficient to take a not fixed and b close to a.

Choose two points $c, d \in \mathbb{R}/\mathbb{Z}$ such that these points and their images under f_1 are distinct and are ordered in the following way along the circle: c, $f_1(c)$, $f_1(d)$, d. It is sufficient to take c and d near an attracting fixed point of f_1, on the different sides with respect to it.

Choose h that takes four points c, $f_1(c)$, $f_1(d)$, d to four points $f_2^{-1}(b)$, a, b, $f_2^{-1}(a)$ (see Figure 2). Then $\hat{f}_1 = h \circ f_1 \circ h^{-1}$ satisfies $\hat{f}_1(f_2^{-1}(b)) = a$, $\hat{f}_1(f_2^{-1}(a)) = b$; hence the point a has period 2 under $\hat{f}_1 f_2^{-1}$. So $\operatorname{rot}(\hat{f}_1 f_2^{-1}) = \frac{1}{2}$.

Finally, for some h, the maps $\hat{f}_1 = h \circ f_1 \circ h^{-1}$ and f_2 have a Diophantine quotient. □

These two propositions imply Lemma 18.

The rest of the article is devoted to the proof of Lemma 19.

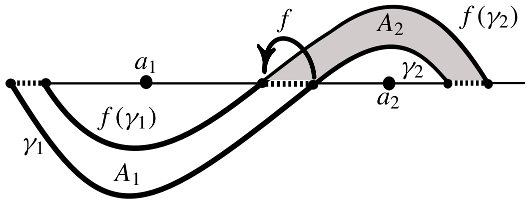

Figure 3. Construction of $\mathcal{E}(f)$.

5. Explicit construction of bubbles

Theorem 5 defines $\bar{\tau}(f+\omega)$, $\omega \in \mathbb{R}/\mathbb{Z}$, as a limit value of the map $\omega \to \tau(f+\omega)$ on the real axis. In this section, we describe $\bar{\tau}(f+\omega)$, $\omega \in I_0$, as a modulus of an explicitly constructed complex torus $\mathcal{E}(f+\omega)$.

This construction was proposed by X. Buff; see [Goncharuk 2012; Buff and Goncharuk 2015] for more details. The key idea of this construction is contained in [Risler 1999], but there it was used in different circumstances.

5.1. ***The complex torus $\mathcal{E}(f)$.*** Let f be a hyperbolic diffeomorphism. Assume that rot $f=0$.

Let a_j, $1 \le j \le 2m$, be its fixed points with multipliers λ_j. We suppose that $0<\lambda_{2j-1}<1<\lambda_{2j}$, i.e., even indices correspond to repellors, and odd indices correspond to attractors. Let $\psi_j : (\mathbb{C}, 0) \to (\mathbb{C}/\mathbb{Z}, a_j)$ be the corresponding linearization charts, i.e., $\psi_j^{-1} \circ f \circ \psi_j(z) = \lambda_j z$, $\psi_j(0) = a_j$, $\psi_j(\mathbb{R}) \subset \mathbb{R}/\mathbb{Z}$, and ψ_j preserve orientation on $\mathbb{R}$. We extend these charts by iterates of f so that the image of ψ_j contains (a_{j-1}, a_{j+1}).

Construct a simple loop $\gamma \subset \mathbb{C}/\mathbb{Z}$ (*le courbe ascendante*, in terms of [Risler 1999]) such that $f(\gamma)$ is above γ in $\mathbb{C}/\mathbb{Z}$. Namely, let $\gamma = \bigcup \gamma_j$; let γ_j have its endpoints on (a_{j-1}, a_j) and (a_j, a_{j+1}); let γ_j be the image of an arc of a circle under ψ_j; let γ_j be above $\mathbb{R}/\mathbb{Z}$ if j is even, and below $\mathbb{R}/\mathbb{Z}$ if j is odd. Since ψ_j conjugates f to $z \mapsto \lambda_j z$, the curve $f(\gamma)$ is above γ in $\mathbb{C}/\mathbb{Z}$.

Let $\widetilde{\Pi} \subset \mathbb{C}/\mathbb{Z}$ be a curvilinear cylinder between γ and $f(\gamma)$ (see Figure 3). Consider the complex torus $\mathcal{E}(f)$ which is the quotient space of a neighborhood of $\widetilde{\Pi}$ by the action of f. Due to the uniformization theorem, there exists $\tau \in \mathbb{H}/\mathbb{Z}$ and a biholomorphism $\widetilde{H}_\omega : \mathcal{E}(f) \to \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ that takes γ to a curve homotopic to $\mathbb{R}/\mathbb{Z}$. Let $\tau(\mathcal{E}(f)) := \tau$ be the modulus of $\mathcal{E}(f)$.

For rot $f = p/q$, the construction of γ should be slightly modified: ϕ_j are linearizing charts of f^q at its fixed points, γ_j are arcs of circles in charts ϕ_j, we let $\gamma = \bigcup \gamma_j$, γ winds above repelling periodic points of f and below attracting periodic points of f, and we choose γ_j so that $f(\gamma)$ is above γ in $\mathbb{C}/\mathbb{Z}$. The rest of the construction is analogous to the case of rot $f=0$.

Theorem 23 [Goncharuk 2012; Buff and Goncharuk 2015, Section 6]. *Let f be a hyperbolic circle diffeomorphism with rational rotation number*; *define $\mathcal{E}(f)$ as above. Then the modulus $\tau(\mathcal{E}(f))$ of the torus $\mathcal{E}(f)$ equals $\bar{\tau}(f)$.*

Due to the construction, $\mathcal{E}(f)$ does not depend on the analytic chart on $\mathbb{R}/\mathbb{Z}$. This implies Lemma 8.

So in order to prove Lemma 19, it is sufficient to find a circle diffeomorphism f with $2m$ fixed points such that $\tau(\mathcal{E}(f)) = w$.

5.2. ***Cutting $\mathcal{E}(f)$ by the real line.*** Let $A_j \subset \widetilde{\Pi}$ be the domain bounded by γ_j, $f(\gamma_j)$, and two segments of $\mathbb{R}/\mathbb{Z}$. Note that the complex manifold $\widetilde{A}_j := A_j/f$ is an annulus, and $\widetilde{A}_j \subset \widetilde{\Pi}/f = \mathcal{E}(f)$.

Let $\mathbb{H}^+ = \mathbb{H}$ and $\mathbb{H}^-$ be the upper and the lower half-planes of $\mathbb{C}$ respectively. From now on, we use the notation $A^{\pm}(\lambda)$ for the following standard annulus: $A^{\pm}(\lambda) := \mathbb{H}^{\pm}/(z \sim \lambda z)$. It is easy to see that its modulus is $\pi/|\log\lambda|$.

Remark 24. The linearizing chart ψ_j induces the map from $\widetilde{A}_j$ to the standard annulus $A^+(\lambda_j)$ for even j, and to $A^-(\lambda_j)$ for odd j. This follows from the fact that ψ_j conjugates f to $x \mapsto \lambda_j z$.

This gives a full description of $\mathcal{E}(f)$ in terms of multipliers and transition maps of f: $\mathcal{E}(f)$ is biholomorphically equivalent to the quotient space of the annuli $A^{\pm}(\lambda_j)$, $\operatorname{mod} A^{\pm}(\lambda_j) = \pi/|\log\lambda_j|$, by the transition maps $\psi_{j+1}^{-1} \circ \psi_j$ between linearizing charts of f.

6. Circle diffeomorphisms with prescribed complex rotation numbers

In this section, we prove Lemma 19.

6.1. ***Scheme of the proof.*** Remark 24 above shows that $\mathcal{E}(f)$ can have any modulus, which nearly implies Lemma 19. Indeed, we can obtain a complex torus of an arbitrary modulus by gluing some $2m$ annuli by some maps. We only need to show that there are no restrictions on possible multipliers and transition maps for an analytic circle diffeomorphism. This follows from Theorem 25 below.

The above arguments together with Theorem 25 show that $\mathcal{E}(f)$ can be biholomorphic to a standard torus of any modulus; however, we must also check that this biholomorphism matches the generators, as required by the definition of $\tau(\mathcal{E}(f))$; see Section 5 above. The formal proof of Lemma 19, with the explicit construction of f and the examination of generators, is contained in Section 6.3.

6.2. ***Moduli of analytic classification of hyperbolic circle diffeomorphisms.*** The following theorem is an analytic version of a smooth classification of hyperbolic diffeomorphisms due to G. R. Belitskii [1986, Proposition 2]. The proof is completely analogous, but we provide it for the sake of completeness.

Theorem 25. *Suppose that we are given a tuple of $2m$ real numbers λ_j with $0 < \lambda_{2j-1} < 1 < \lambda_{2j}$, and a tuple of analytic orientation-preserving diffeomorphisms $\psi_{j;j+1} : \mathbb{R}^+ \to \mathbb{R}^-$ such that $\psi_{j;j+1}(\lambda_j z) = \lambda_{j+1}\psi_{j;j+1}(z)$.*

Then there exists an analytic circle diffeomorphism f such that it has $2m$ fixed points with multipliers λ_j, and $\psi_{j;j+1}$ are transition maps between their linearization charts ψ_j: $\psi_{j;j+1} = \psi_{j+1}^{-1} \circ \psi_j$.

Remark. It is also true that such an f is unique up to analytic conjugacy, so the data above is the modulus of an analytic classification of hyperbolic circle diffeomorphisms. Given f, transition maps $\psi_{j;j+1}$ are uniquely defined up to the equivalence

$$(\dots\psi_{j-1;j}\dots) \sim (\dots, a_j\psi_{j-1;j}(z/a_{j-1}), \dots)$$

for some numbers $a_j > 0$; see [Belitskii 1986, Proposition 3].

Proof. Take $2m$ copies of the real axis and glue the j-th to the $(j+1)$-th copy by the map $\psi_{j;j+1} : \mathbb{R}^+ \to \mathbb{R}^-$. We get a one-dimensional C^ω-manifold homeomorphic to the circle $\mathbb{R}/\mathbb{Z}$. It is well known that such manifolds are C^ω-equivalent to $\mathbb{R}/\mathbb{Z}$. Thus there exists a tuple of C^ω charts $\psi_j : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ such that $\psi_{j;j+1} = \psi_{j+1}^{-1} \circ \psi_j$. Due to the equality $\psi_{j;j+1}(\lambda_j z) = \lambda_{j+1}\psi_{j;j+1}(z)$, the maps $\psi_j(\lambda_j \psi_j^{-1}(z))$ glue into the well-defined C^ω circle diffeomorphism f.

Let $a_j = \psi_j(0)$. Note that $f(a_j) = \psi_j(\lambda_j \psi_j^{-1}(a_j)) = \psi_j(\lambda_j \cdot 0) = \psi_j(0) = a_j$, so these points are fixed points of f.

On a segment (a_{j-1}, a_{j+1}), the map ψ_j conjugates $f = \psi_j \circ \lambda_j \psi_j^{-1}$ to $z \mapsto \lambda_j z$, so ψ_j is a linearizing chart of a fixed point a_j, and λ_j is the multiplier of f at a_j. □

6.3. ***Proof of Lemma 19, see Figure 4.*** Recall that our aim is to construct a circle diffeomorphism f with $2m$ hyperbolic fixed points and the complex rotation number w.

Consider the standard elliptic curve $E_w = \mathbb{C}/(\mathbb{Z} + w\mathbb{Z})$; let $\mathbb{R}/\mathbb{Z}$ and $w\mathbb{R}/w\mathbb{Z}$ be its first and second generators respectively. Take $2m$ arbitrary disjoint simple real-analytic loops $\nu_j \subset E_w$ along the second generator. Let $\mathcal{A}_j \subset E_w$ be the annulus between ν_j and ν_{j+1}. Let $\Delta_j = \mathcal{A}_j \cap \mathbb{R}/\mathbb{Z}$; then Δ_j joins boundaries of $\mathcal{A}_j$.

We are going to construct a circle diffeomorphism f with $2m$ fixed points, and a biholomorphism $H : \mathcal{E}(f) \to E_w$ such that $H(\widetilde{A}_j) = \mathcal{A}_j \subset E_w$, where $\widetilde{A}_j$ are the annuli in $\mathcal{E}(f)$ bounded by intervals of $\mathbb{R}/\mathbb{Z}$ as in Section 5.2. This biholomorphism H will take the class of γ in $\mathcal{E}(f)$ to the class of $\mathbb{R}/\mathbb{Z} = \bigcup \Delta_j$ in E_w. This will prove that the modulus of $\mathcal{E}(f)$ equals w.

Uniformize $\mathcal{A}_j$. For each annulus $\mathcal{A}_j$ where j is even, take $\lambda_j > 1$ such that there exists a biholomorphism $\widetilde{\Psi}_j : A^+(\lambda_j) \to \mathcal{A}_j$. For each annulus $\mathcal{A}_j$ where j is odd, take $\lambda_j < 1$ such that there exists a biholomorphism $\widetilde{\Psi}_j : A^-(\lambda_j) \to \mathcal{A}_j$. Each map $\widetilde{\Psi}_j$ extends analytically to a neighborhood of $A^\pm(\lambda_j)$ in $\mathbb{C}^*/(z \sim \lambda_j z)$, because the boundaries of $\mathcal{A}_j$ are real-analytic curves ν_j. Assume that $\widetilde{\Psi}_j^{-1}(\nu_j)$ is the left boundary of $A^\pm(\lambda_j)$; that is, $\widetilde{\Psi}_j^{-1}(\nu_j) = \mathbb{R}^-/(z \sim \lambda_j z)$. Then $\widetilde{\Psi}_j^{-1}(\nu_{j+1}) = \mathbb{R}^+/(z \sim \lambda_j z)$.

Let $\Psi_j : \overline{\mathbb{H}}^\pm \setminus \{0\} \to \mathcal{A}_j$ be the lift of $\widetilde{\Psi}_j$ to the universal cover of $A^\pm(\lambda_j)$; then $\Psi_j(\lambda_j z) = \Psi_j(z)$. For each j, choose one of the preimages $\delta_j = \Psi_j^{-1}(\Delta_j)$. Let $l_j \in \mathbb{R}^-$, $r_j \in \mathbb{R}^+$ be the left and the right endpoints of δ_j respectively. Consider the maps $\psi_{j;j+1} : \mathbb{R}^+ \to \mathbb{R}^-$,

$$\psi_{j;j+1} = \Psi_{j+1}^{-1} \circ \Psi_j,$$

where we choose the branch of Ψ_{j+1}^{-1} so that $\psi_{j;j+1}(r_j) = l_{j+1}$. Note that $\psi_{j;j+1}(\lambda_j z) = \lambda_{j+1}\psi_{j;j+1}(z)$ because $\Psi_j(\lambda_j z) = \Psi_j(z)$.

Now, the complex torus E_w is biholomorphically equivalent to the quotient space of annuli $A^\pm(\lambda_j)$ by the maps $\psi_{j;j+1}$. This, together with Remark 24, motivates the construction of f below.

Construct f and a biholomorphism $H : \mathcal{E}(f) \to E_w$. Use Theorem 25 to construct f with multipliers λ_j and transition maps $\psi_{j;j+1}$.

Let ψ_j be linearization charts of its fixed points; then $\psi_{j;j+1} = \psi_{j+1}^{-1} \circ \psi_j$. Let $\gamma, \mathcal{E}(f), A_j, \widetilde{A}_j$ be defined as in Section 5 for this circle diffeomorphism f.

Consider the tuple of maps $\Psi_j \circ \psi_j^{-1}$ on $A_j \subset \mathbb{C}/\mathbb{Z}$. These maps agree on the boundaries of A_j due to the equality

$$(\Psi_{j+1} \circ \psi_{j+1}^{-1})^{-1} \circ \Psi_j \circ \psi_j^{-1} = \psi_{j+1} \circ \Psi_{j+1}^{-1} \circ \Psi_j \circ \psi_j^{-1} = \psi_{j+1} \circ \psi_{j;j+1} \circ \psi_j^{-1} = \mathrm{id},$$

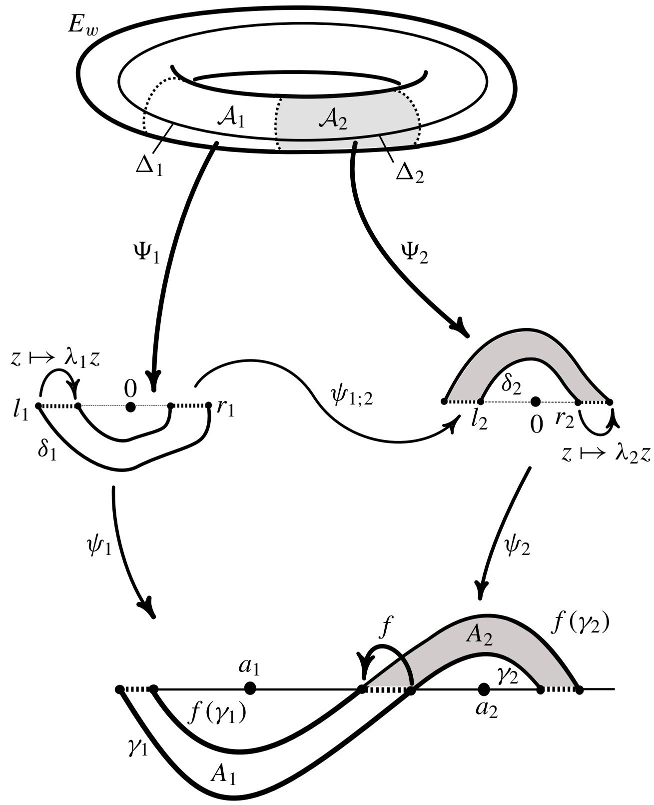

Figure 4. Proof of Lemma 19.

so they define one map on $\widetilde{\Pi}$. They descend to the map $H : \mathcal{E}(f) \to E_w$ because ψ_j conjugates f to $z \mapsto \lambda_j z$ and $\Psi_j(\lambda_j z) = \Psi_j(z)$. Clearly, $H(\widetilde{A}_j) = \mathcal{A}_j$.

H takes the class of γ in $\mathcal{E}(f)$ to the first generator of E_w. Note that the curves $\psi_j(\delta_j)$ have common endpoints $\psi_j(r_j) = \psi_{j+1}(l_{j+1})$ since $\psi_{j;j+1}(r_j) = l_{j+1}$. So $\gamma' := \bigcup \psi_j(\delta_j)$ is a loop in $\mathbb{C}/\mathbb{Z}$ that passes above the attractors $\psi_{2j-1}(0)$ and below the repellors $\psi_{2j}(0)$ of f. So γ' is homotopic to γ in an annular neighborhood of $\mathbb{R}/\mathbb{Z}$ covered by linearizing charts of fixed points; the homotopy does not pass through fixed points. Hence γ' is homotopic to γ in $\mathcal{E}(f)$, i.e., corresponds to the first generator of $\mathcal{E}(f)$.

Finally, $H(\gamma') = \bigcup \Psi_j(\delta_j) = \bigcup \Delta_j = \mathbb{R}/\mathbb{Z} \subset E_w$. This completes the proof of Lemma 19.

Appendix: Derivatives of complex rotation number

In this section we compute $(\partial/\partial\omega)\bar{\tau}(f_\omega)$ for a family of circle diffeomorphisms f_ω. In particular, this yields Proposition 12. The computation is analogous to that of [Risler 1999, Section 2.2.3].

Let f_ω be an analytic family of analytic circle diffeomorphisms. Let $G_\omega := \widetilde{H}_\omega^{-1}$, where $\widetilde{H}_\omega$ rectifies the complex torus $\mathcal{E}(f_\omega)$; see Section 5. Let $\tau(\omega) = \bar{\tau}(f_\omega)$. Then

$$f_\omega(G_\omega(z)) = G_\omega(z + \tau(\omega)) \quad \text{for } z \in G_\omega^{-1}(\gamma). \tag{4}$$

The Ahlfors–Bers theorem implies that the map G_ω, if suitably normalized, depends analytically on ω; see [Risler 1999, Section 2.1, Proposition 2].

Fix $\omega = \omega_0 \in \mathbb{R}/\mathbb{Z}$; in what follows, all derivatives with respect to ω are evaluated at $\omega = \omega_0$, and we will omit the lower indices in f_ω, G_ω etc. Here and below G', f' are derivatives with respect to z; G'_ω, f'_ω, τ'_ω are derivatives with respect to ω.

The following proposition clearly implies Proposition 12.

Proposition 26. *Let f_ω, G_ω be as above. Then*

$$\tau'_\omega = \int_\gamma \frac{f'_\omega(w)}{f'(w)}((G^{-1})'(w))^2\, dw,$$

where all derivatives are evaluated at $\omega = \omega_0$.

Proof. We may and will assume that the curve γ in the construction of $\mathcal{E}(f_\omega)$ does not depend on ω in a small neighborhood of ω_0.

Differentiate (4) with respect to ω:

$$f'_\omega|_{G(z)} + f'|_{G(z)} G'_\omega(z) = G'_\omega(z+\tau) + G'(z+\tau)\tau'_\omega.$$

Express τ'_ω using this equation and the identity $G'(z+\tau) = f'|_{G(z)} G'(z)$ (this is the derivative of (4)). We get

$$\tau'_\omega = \frac{f'_\omega|_{G(z)}}{G'(z+\tau)} + \frac{G'_\omega(z)}{G'(z)} - \frac{G'_\omega(z+\tau)}{G'(z+\tau)}.$$

Integrate this expression along $G^{-1}(\gamma)$. The second and the third summands cancel out because the function $G'_\omega(z)/G'(z)$ is holomorphic. We obtain

$$\tau'_\omega = \int_{G^{-1}(\gamma)} \frac{f'_\omega|_{G(z)}}{G'(z+\tau)}\, dz.$$

Using again $G'(z+\tau) = G'(z) f'|_{G(z)}$ and making the change of variable $w = G(z)$, we get the desired formula. □

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QUANTITATIVE STABILITY OF THE FREE BOUNDARY IN THE OBSTACLE PROBLEM

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We prove some detailed quantitative stability results for the contact set and the solution of the classical obstacle problem in $\mathbb{R}^n$ ($n \geq 2$) under perturbations of the obstacle function, which is also equivalent to studying the variation of the equilibrium measure in classical potential theory under a perturbation of the external field.

To do so, working in the setting of the whole space, we examine the evolution of the free boundary Γ^t corresponding to the boundary of the contact set for a family of obstacle functions h^t. Assuming that $h = h^t(x) = h(t, x)$ is $C^{k+1,\alpha}$ in $[-1, 1] \times \mathbb{R}^n$ and that the initial free boundary Γ^0 is regular, we prove that Γ^t is twice differentiable in t in a small neighborhood of $t = 0$. Moreover, we show that the "normal velocity" and the "normal acceleration" of Γ^t are respectively $C^{k-1,\alpha}$ and $C^{k-2,\alpha}$ scalar fields on Γ^t. This is accomplished by deriving equations for this velocity and acceleration and studying the regularity of their solutions via single- and double-layer estimates from potential theory.

1. Introduction

Motivation of the problem. Consider the classical obstacle problem; see for instance [Kinderlehrer and Nirenberg 1977; Caffarelli 1998]. If the obstacle h is perturbed into $h + t\xi$ with t small and ξ regular enough, how much does the contact set (or coincidence set) move? The best known answer to this question is in [Blank 2001], where it is proved that the new contact set is $O(t)$-close to the old one in Hausdorff distance, in the setting of a bounded domain with Dirichlet boundary condition. Some results are also proved in [Schaeffer 1975] in an analytic setting, by Nash–Moser inversion.

Our paper is concerned with getting stronger and more quantitative stability estimates, in particular obtaining closeness of the contact sets in $C^{k,\alpha}$ norms with explicitly described first and second derivatives with respect to t, which come together with an explicit asymptotic expansion of the solution itself. We believe that such results are of natural and independent interest for the obstacle problem. They are also for us motivated by an application on the analysis of Coulomb systems in statistical mechanics from [Leblé and Serfaty 2018], which relies on the present paper.

Let us get into more detail on this aspect. In potential theory, the so-called (Frostman) "equilibrium measure" for Coulomb interactions with an external "field" Q is the unique probability measure μ on $\mathbb{R}^n$ which minimizes

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} P(x-y)\, d\mu(x)\, d\mu(y) + \int_{\mathbb{R}^n} Q(x)\, d\mu(x), \tag{1-1}$$

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where P is the Newtonian potential in dimension n. If Q grows fast enough at infinity, then setting

$$u(x) = \int_{\mathbb{R}^n} P(x-y)\, d\mu(y), \tag{1-2}$$

the equilibrium measure μ is compactly supported and uniquely characterized by the fact that there exists a constant c such that

$$u \geq c - \tfrac{1}{2}Q \quad \text{and} \quad u = c - \tfrac{1}{2}Q \quad \mu - \text{a.e.};$$

see for instance [Saff and Totik 1997]. We thus find that $\mu = -\Delta u$, where u solves the classical obstacle problem in the whole space

$$\min\{-\Delta u, u - h\} = 0$$

with obstacle $h = c - \frac{1}{2}Q$ — the two problems (identifying the equilibrium measure and solving the obstacle problem) are in fact convex dual minimization problems, as seen in [Ekeland and Temam 1976]; see for instance [Serfaty 2015, Chapter 2] for a description of this correspondence. Thus, the support of the equilibrium measure is equal to the contact set wherever the obstacle is "active".

The understanding of the dependence of the equilibrium measure on the external field — which is thus equivalent to the understanding of the dependence of the solution and its contact set on the obstacle function — is crucial for the analysis of systems of particles with logarithmic or Coulomb interactions; in particular it allows one to show that the linear statistics of fluctuations of such systems converge to Gaussians. Following the method first introduced by [Johansson 1998], this relies on the computation of the Laplace transform of the fluctuations, which directly leads to considering the same system but with perturbed external field. Previously, the analysis of the perturbation of the equilibrium measure, as done in [Ameur et al. 2011], relied on Sakai's theory [1991], a complex-analytic approach which is thus only valid in two dimensions and imposed analyticity assumptions on the external field and the boundary of the coincidence set.

In that context, the evolution of the contact sets sometimes goes by the name "Laplacian growth" or "Hele-Shaw flow" or the "Hele-Shaw equation", see [Hedenmalm and Makarov 2004; 2013], and seems related to the quantum Hele-Shaw flow introduced by the physicists Wiegmann [2002] and Zabrodin. It has only been examined in dimension 2.

Setting of the study. Both for simplicity and for the applications we have in mind mentioned above, we consider global solutions to the obstacle problem in $\mathbb{R}^n$, $n \geq 2$. We note that the setting in $\mathbb{R}^2$ is slightly different than the setting in $\mathbb{R}^n$ for $n \geq 3$ due to the fact that the logarithmic Newtonian potential does not decay to zero at infinity, and this will lead us to often making parallel statements about the two. We also note that the potential u associated to the equilibrium measure in (1-2) behaves like P at infinity, since μ is a compactly supported probability measure, i.e., tends to 0 if $n \geq 3$ and behaves like $-\frac{1}{2\pi}\log|x|$ if $n = 2$. Specifying the total mass of $-\Delta u$ is equivalent to specifying the ratio of $u/{-\log|x|}$ at infinity in dimension 2, or to adding an appropriate[1] constant to u in dimension $n \geq 3$.

[1] Let u^t be defined as (1-3). For $n \geq 3$ there is a nonlinear (but monotone and continuous) relation between the mass $\int_{\mathbb{R}^n} \Delta u^t$ and value of the constant c^t. For c^t large enough, the mass is 0, and when c^t decreases, the mass increases continuously. This allows us to solve the equation with prescribed mass by varying the constant c^t.

With the above motivation, in order to consider the perturbations of the obstacle, we thus consider for each $t \in [-1, 1]$, given c^t a function of t, the function u^t solving the obstacle problem

$$\min\{-\Delta u^t, u^t - h^t\} = 0 \quad \text{in } \mathbb{R}^n, \qquad \begin{cases} \lim_{|x|\to\infty} u^t(x) = c^t & (n \geq 3), \\ \lim_{|x|\to\infty} \frac{u^t(x)}{-\log|x|} = c^t & (n = 2). \end{cases} \tag{1-3}$$

We assume $\Delta h^0 < 0$ on $\{u^0 = h^0\}$, i.e., the obstacle must be "active" in the contact set, and

$$\begin{cases} \lim_{|x|\to\infty} h^t(x) < c^t & (n \geq 3), \\ \lim_{|x|\to\infty} \frac{h^t(x)}{-\log|x|} < c^t & (n = 2), \end{cases} \tag{1-4}$$

$$h = h^t(x) = h(t, x) \in C^{k-1,\alpha}([-1, 1] \times B_{\boldsymbol{R}}), \tag{1-5}$$

while

$$c = c^t = c(t) \in C^2([-1, 1]). \tag{1-6}$$

For $n = 2$ we assume $c > 0$.

In addition, we assume

$$\Delta(h^t - h^0) \quad \text{is compactly supported in } B_{\boldsymbol{R}} \tag{1-7}$$

and

$$h^t - h^0 \to 0 \quad \text{as } |x| \to \infty \quad (n \geq 3), \tag{1-8a}$$

$$\frac{h^t - h^0}{-\log|x|} \to 0 \quad \text{as } |x| \to \infty \quad (n = 2). \tag{1-8b}$$

In particular, letting $\dot{}$ denote the derivative with respect to t, this implies

$$\dot{h}^t \to 0 \quad \text{as } |x| \to \infty \quad (n \geq 3), \tag{1-9a}$$

$$\frac{\dot{h}^t}{-\log|x|} \to 0 \quad \text{as } |x| \to \infty \quad (n = 2). \tag{1-9b}$$

Let us denote by

$$\Omega^t := \{u^t - h^t > 0\} \quad \text{and} \quad \Gamma^t := \partial\Omega^t$$

the complement of the contact set and the free boundary, respectively.

We will assume that all points of the "initial" free boundary Γ^0 are regular points in the sense of Caffarelli [1977; 1998]. In particular we assume that Ω^0 is an open set with smooth boundary.

For the analysis of the paper it is convenient to identify precisely the quantities on which the (constants in the) estimates depend. To this aim, let us fix $\boldsymbol{\rho} > 0$ and make the following quantitative assumptions.

First, we assume that, for some $\boldsymbol{U} \subset B_{\boldsymbol{R}}$, we have

$$\Delta h^0 \leq -\boldsymbol{\rho} \quad \text{in } \overline{\boldsymbol{U}} \qquad \text{and} \qquad \begin{cases} u^0 - h^0 \geq \boldsymbol{\rho} & \text{in } \mathbb{R}^n \setminus \boldsymbol{U} \quad (n \geq 3), \\ \frac{u^0 - h^0}{-\log|x|} \geq \boldsymbol{\rho} & \text{in } \mathbb{R}^n \setminus \boldsymbol{U} \quad (n = 2), \end{cases} \tag{1-10}$$

where $U \subset B_{\boldsymbol{R}}$ is some open set containing $\{u^0 = 0\}$.

Second, we assume

$$\text{all points of } \Gamma^0 \text{ can be touched from both sides by balls of radius } \boldsymbol{\rho}. \tag{1-11}$$

This is a quantitative version of the assumption that all points of Γ^0 are regular points.

Throughout the paper, if $\mathcal{C}$ is a set of parameters of the problem, we denote by $C(\mathcal{C})$ a constant depending only on $\mathcal{C}$. We define

$$\boldsymbol{\mathcal{C}} := \{n, k, \alpha, \boldsymbol{R}, \boldsymbol{U}, \boldsymbol{\rho}, \|h\|_{C^{k+1,\alpha}([-1,1]\times\bar{U})}, \|c\|_{C^2([-1,1])}\}, \tag{1-12}$$

$$\boldsymbol{\mathcal{C}^0} := \{n, k, \alpha, \boldsymbol{R}, \boldsymbol{U}, \boldsymbol{\rho}, \|h^0\|_{C^{k+1,\alpha}(\bar{U})}, c^0\}. \tag{1-13}$$

For $n = 2$ we also add to $\boldsymbol{\mathcal{C}}$ the constant $\inf_{[-1,1]} c > 0$.

Main result. Let $t_\circ > 0$ and let $\Psi = \Psi^t(x) = \Psi(t, x)$ be a 1-parameter family of diffeomorphisms $\Psi : (-t_\circ, t_\circ) \times \mathbb{R}^n \to \mathbb{R}^n$. We say that Ψ fixes the complement of $\boldsymbol{U}$ if $\Psi(x) = x$ for all $x \in \mathbb{R}^n \setminus U$.

We say Ψ is continuously differentiable if for all $t \in (-t_\circ, t_\circ)$ there exists $\dot\Psi^t \in C^0(\mathbb{R}^n; \mathbb{R}^n)$ such that

$$\|\Psi^{t+s}(x) - \Psi^t(x) - s\dot\Psi^t(x)\|_{C^0(\mathbb{R}^n;\mathbb{R}^n)} = o(s),$$
$$\|\dot\Psi^{t+s}(x) - \dot\Psi^t(x)\|_{C^0(\mathbb{R}^n;\mathbb{R}^n)} = o(1)$$

as $s \to 0$.

We say Ψ is twice continuously differentiable if, in addition, for all $t \in (-t_\circ, t_\circ)$ there exists $\ddot\Psi^t \in C^0(\mathbb{R}^n; \mathbb{R}^n)$ such that

$$\left\|\Psi^{t+s}(x) - \Psi^t(x) - s\,\dot\Psi^t(x) - \tfrac{1}{2}s^2\,\ddot\Psi^t(x)\right\|_{C^0(\mathbb{R}^n;\mathbb{R}^n)} = o(s^2),$$
$$\|\ddot\Psi^{t+s}(x) - \ddot\Psi^t(x)\|_{C^0(\mathbb{R}^n;\mathbb{R}^n)} = o(1)$$

as $s \to 0$.

Throughout the paper, given a function $f : (-t_\circ, t_\circ) \times Y \to \mathbb{R}$ we use the notation $f = f^t(x) = f(t, x)$,

$$\delta_t f^s := \frac{f^{s+t} - f^s}{t} \quad \text{and} \quad \dot f^s := \lim_{t\downarrow 0} \delta_t f^s = \partial_t f(s, y).$$

The main result of the paper is the following. In its statement, and throughout the paper, we denote by

$$\nu^t : \Gamma^t \to S^{n-1}$$

the unit normal vector to Γ^t pointing towards Ω^t.

Theorem 1.1. *Let $n \ge 2$, $k \ge 1$, $\alpha \in (0, 1)$, and u^t satisfy* (1-3), *with h and c satisfying* (1-4)–(1-8). *Assume* (1-10) *and* (1-11) *hold.*

Then, there exists $t_\circ > 0$ and a 1-parameter differentiable family of diffeomorphisms $\Psi^t \in C^{k,\alpha}(\mathbb{R}^n; \mathbb{R}^n)$ that fixes the complement of $\boldsymbol{U}$ and which satisfies, for every $t \in (-t_\circ, t_\circ)$,

$$\Psi^t(\Omega^0) = \Omega^t, \quad \Psi^t(\Gamma^0) = \Gamma^t,$$
$$\|\dot\Psi^t\|_{C^{k-1,\alpha}(\mathbb{R}^n)} \le C \qquad \text{and} \qquad (\dot\Psi^t \circ (\Psi^t)^{-1}) \cdot \nu^t = \frac{\partial_{\nu^t} V^t}{\Delta h^t} \quad \text{on } \Gamma^t, \tag{1-14}$$

where $V^t := \dot{u}^t - \dot{h}^t$ is the solution[2] to

$$\begin{cases} \Delta V^t = -\Delta \dot{h}^t & \text{in } \Omega^t, \\ V^t = 0 & \text{on } \Gamma^t, \\ \lim_{x\to\infty} V^t(x) = \dot{c}^t & (n \geq 3), \end{cases} \qquad \begin{cases} \Delta V^t = -\Delta \dot{h}^t & \text{in } \Omega^t, \\ V^t = 0 & \text{on } \Gamma^t, \\ \lim_{x\to\infty} \frac{V^t(x)}{-\log|x|} = \dot{c}^t & (n = 2). \end{cases} \tag{1-15}$$

In addition, we have

$$\dot{u}^t = \dot{h}^t + V^t \chi_{\Omega^t} \quad \text{in all } \mathbb{R}^n.$$

If moreover $k \geq 2$ then Ψ is twice differentiable and we have

$$\|\ddot{\Psi}^t\|_{C^{k-2,\alpha}(\mathbb{R}^n)} \leq C_\circ \tag{1-16}$$

and

$$\|\ddot{u}^t\|_{L^\infty(\mathbb{R}^n)} + \|\nabla \ddot{u}^t\|_{C^{k-2,\alpha}((\Omega^0\cup\Omega^t)^c \,\cup\, \overline{\Omega^0\cap\Omega^t})} \leq C_\circ. \tag{1-17}$$

The constants $t_\circ$ and $C_\circ$ depend only on[3] $\mathcal{C}$.

An informal rephrasing of Theorem 1.1 is as follows. If the moving obstacle $h(t, x)$ is $C^{k+1,\alpha}$ and $c(t)$ is C^2, then Γ^t is "twice differentiable" for t in a small neighborhood of 0. Moreover, the "normal velocity" of Γ^t and the "normal acceleration" of Γ^t are respectively $C^{k-1,\alpha}$ and $C^{k-2,\alpha}$ scalar fields on Γ^t, with the normal velocity precisely identified via a Dirichlet-to-Neumann transformation: to compute it, one finds the solution V^t to the Dirichlet problem in a exterior domain (1-15) and the normal velocity at a point of Γ^t is given by the normal derivative of V^t divided by the Laplacian of the obstacle at that point.

Open questions. It is of course natural to ask whether similar results hold for more general obstacle problems, such as those associated to fully nonlinear operators or to fractional Laplacians.

In view of our results,[4] a natural open question, which we believe to be delicate, is whether one can improve Theorem 1.1 to

$$\Psi(t, x) \in C^{k,\alpha} \quad \text{(jointly in } t \text{ and } x\text{)}.$$

Structure of the proof and organization of the paper. For the proof, we first reduce to a situation where the contact set is growing, i.e., $\Omega^t \subset \Omega^0$. We then define a coordinate system near the free boundary Γ^0, and express the "height" η^t of Γ^t in these coordinates.

In Section 3, assuming that an expansion of the type $\eta^t = \eta^0 + \dot{\eta}^0 t + \frac{1}{2}\ddot{\eta}^0 t^2 + \cdots$ holds as $t \to 0$, we derive equations for $\dot{\eta}^0$ and $\ddot{\eta}^0$, which allow us to obtain explicit formulae and Hölder regularity for these quantities via single- and double-layer potential-theoretic estimates. These regularity estimates are delicate to obtain because the relations characterizing $\dot{\eta}^0$ and $\ddot{\eta}^0$ are at first implicit and one needs to show they can be "closed" for regularity.

[2]Since we assumed $\dot{h}^t$ tends to 0 (resp. is $\ll |\log|x||$ if $n = 2$) at ∞, we have V^t is the unique solution such that $V^t + \dot{h}^t$ is bounded, coincides with $\dot{h}^t$ in the complement of Ω^t and is harmonic in Ω^t. In fact, $V^t + \dot{h}^t$ is the unique bounded harmonic extension of $\dot{h}^t$ outside of $(\Omega^t)^c$.

[3]The set of constants of the problem $\mathcal{C}$ was defined in (1-12).

[4]We establish that if $h \in C^{k+1,\alpha}$ then $\Psi^t \in C^{k,\alpha}$, $\dot{\Psi}^t \in C^{k,\alpha}$ and $\ddot{\Psi}^t \in C^{k-2,\alpha}$.

In Section 4, we show that the existence of an expansion in t for η^t, which was previously assumed, does hold. This is done by using a second set of adapted coordinates near Γ^0 (a sort of hodograph transform) and again single- and double-layer potential estimates.

Finally, in Section 5 we prove the main result by showing how to treat the general case where the contact set is not necessarily growing. In the Appendix, we collect the potential-theoretic estimates we need and some additional proofs.

2. Preliminaries

Known results. Throughout the paper it is useful to quantify the smoothness of the (boundaries of the) domains Ω^t. Let us introduce some more notation with that aim. Let U be some open set and $r > 0$. We write $\partial U \in C^{k,\alpha}_r$ if for all $x_o \in \partial U$ there are some orthonormal coordinates y_i, $1 \le i \le n$, with origin at x_o (these coordinates may vary from point to point), and a function $F_{x_o} \in C^{k,\alpha}(\bar{B}'_r)$ such that

$$U \cap \{|y'| < r,\ |y_n| < r\} = \{y_n < F_{x_o}(y')\} \cap \{|y'| < r,\ |y_n| < r\},$$

where $y' = (y_1, y_2, \dots, y_{n-1})$.

In this framework we define

$$\|\partial U\|_{C^{k,\alpha}_r} := \sup_{x_o \in \partial U} \|F_{x_o}\|_{C^{k,\alpha}(\bar{B}'_r)} < \infty, \tag{2-1}$$

where $B'_r = \{|y'| < r\} \subset \mathbb{R}^{n-1}$.

With the previous assumptions we have in our notation:

Proposition 2.1 [Caffarelli 1977; 1998; Kinderlehrer and Nirenberg 1977; Blank 2001]. *There exist universal constants $t_\circ > 0$ and C_o depending only on $\mathcal{C}$ such that the following hold*:

(i) *We have*

$$\|\Gamma^t\|_{C^{k,\alpha}_{\rho/4}} \le C_o \quad \text{for all } t \in (-t_\circ, t_\circ).$$

(ii) *For every pair $t, s \in (-t_\circ, t_\circ)$, the Hausdorff distance between Γ^t and Γ^s satisfies*

$$d_{\text{Hausdorff}}(\Gamma^t, \Gamma^s) \le C_o\, |t - s|.$$

Proposition 2.1 is contained in the results of [Blank 2001]. However, for the sake of completeness, we briefly sketch the proof in the Appendix. This is done by combining the classical results for the obstacle problem in [Caffarelli 1977; 1998; Kinderlehrer and Nirenberg 1977] and the key sharp estimate $|\Omega^t \triangle \Omega^s| \le C|t - s|$ for the symmetric difference of the positivity sets (or of the contact sets) from [Blank 2001].

Scalar parametrization of deformations (definition of η^t). By Proposition 2.1 the free boundaries Γ^t are "uniformly" $C^{k,\alpha}$ for $|t|$ small and the difference between Γ^t and Γ^s is bounded by $C|t' - t|$ in the L^∞ norm. A goal of the paper is to prove that the difference is bounded $C|t' - t|$ also in a $C^{k-1,\alpha}$ norm. To prove this type of result it is convenient to have a scalar function representing the "difference" between Γ^t and Γ^s. This has a clear meaning locally — since both Γ^t and Γ^s are graphs, and one can simply subtract the two functions that define these graphs. We next give a global analogue of this.

In an open neighborhood $U_\circ$ of Γ^0 we define coordinates

$$(z, s) : U_\circ \to \mathcal{Z} \times (-s_\circ, s_\circ),$$

where $s_\circ > 0$ and $\mathcal{Z}$ is some smooth approximation of Γ^0.

We assume the vector field

$$N := \partial_s$$

is a smooth approximation of ν^0 on Γ^0. More precisely, we assume

$$N \in C^\infty(U_\circ; \mathbb{R}^n), \qquad |N| = 1 \qquad \text{and} \qquad N \cdot \nu^t \geq (1 - \varepsilon_o) \quad \text{for } t \in (-t_\circ, t_\circ), \tag{2-2}$$

where ε_o is a constant that in the sequel will be chosen to be small enough — depending only on $\mathcal{C}$.

In this framework, Proposition 2.1 implies that for all $t \in (-t_\circ, t_\circ)$ with $t_\circ$ small enough there exists $\eta^t \in C^{k,\alpha}(\mathcal{Z})$ such that

$$\Gamma^t = \{s = \eta^t(z)\} \subset U_\circ. \tag{2-3}$$

Remark 2.2. From the data of Γ^0 we may always construct $\mathcal{Z}$ and (z, s) satisfying the previous properties — for ε_o arbitrarily small — by taking $\mathcal{Z}$ to be a smooth approximation of Γ^0 and N a smooth approximation of ν^0. Once $\mathcal{Z}$ and N are chosen, the coordinates (z, s) are then defined respectively as the projection on $\mathcal{Z}$ and the signed distance to $\mathcal{Z}$ along integral curves of N.

3. A priori estimates

Roughly speaking, the goal of this section is to show that if an expansion of the type

$$\eta^t = \eta^0 + \dot\eta^0 t + \tfrac{1}{2}\ddot\eta^0 t^2 + \cdots$$

holds, where

$$\frac{\eta^t - \eta^0}{t} \to \dot\eta^0 \quad \text{and} \quad \frac{\eta^t - \eta^0 - \dot\eta^0 t}{t^2} \to \frac{\ddot\eta^0}{2} \quad \text{as } t \to 0, \text{ in } C^0(\mathcal{Z}),$$

then $\dot\eta^0$ and $\ddot\eta^0$ must satisfy certain equations that have uniqueness of solution and a priori estimates. From these equations we obtain conditional (or a priori) estimates for $\|\dot\eta^0\|_{C^{k-1,\alpha}(\mathcal{Z})}$ and $\|\ddot\eta^0\|_{C^{k-2,\alpha}(\mathcal{Z})}$.

In the next sections, let us provisionally assume

$$\Delta(h^t - h^0) \geq 0 \quad \text{and} \quad c^t - c^0 \leq 0 \tag{3-1}$$

for all $t \geq 0$, which is not essential but simplifies the analysis: Assumption (3-1) guarantees that $\Omega^t \subset \Omega^0$ for all $t \geq 0$. Indeed, this is an immediate consequence of the characterization of

$$\tilde u^t := u^t - h^t$$

as the infimum of all nonnegative supersolutions with the same right-hand side and appropriate condition at infinity. More precisely, we have the following lemma, whose proof is standard in dimension $n \geq 3$ and which we sketch in dimension $n = 2$ in the Appendix.

Lemma 3.1. *The function $\tilde{u}^t$ can be defined as the infimum of all f satisfying $f \ge 0$, $\Delta f \le -\Delta h^t$,*

$$\lim_{x\to\infty} (f + h^t) \ge c^t \quad (n \ge 3), \tag{3-2a}$$

$$\lim_{x\to\infty} \frac{f + h^t}{-\log|x|} \ge c^t \quad (n = 2). \tag{3-2b}$$

Note that in particular $f = \tilde{u}^0$ is included since $\Delta\tilde{u}^0 = -\Delta h^0 \le -\Delta h^t$, and

$$\lim_{x\to\infty} (\tilde{u}^0 + h^t) = \lim_{x\to\infty} (\tilde{u}^0 + h^0) + \lim_{x\to\infty} (h^t - h^0) \ge c^0 \ge c^t \quad (n \ge 3),$$

$$\lim_{x\to\infty} \frac{(\tilde{u}^0 + h^t)}{-\log|x|} \ge c(t) \quad (n = 2).$$

Therefore, applying Lemma 3.1 we obtain $\tilde{u}^0 \ge \tilde{u}^t$ and

$$\Omega^t = \{\tilde{u}^t > 0\} \subset \{\tilde{u}^0 > 0\} = \Omega^0$$

for all $t > 0$. Equivalently (3-1) implies $\eta^t \ge 0$ on $\mathcal{Z}$ for $t > 0$.

Later, when we prove Theorem 1.1, we will reduce to this case by decomposing h^t as a sum of two functions, one with nonnegative Laplacian and one with nonpositive Laplacian.

Let us define

$$v^t := \delta_t \tilde{u}^0 = \frac{1}{t}(\tilde{u}^t - \tilde{u}^0). \tag{3-3}$$

The function v^t is a solution of

$$\begin{cases} \Delta v^t = -\Delta\delta_t h^0 & \text{in } \Omega^t, \\ v^t = -\frac{1}{t}\tilde{u}^0 & \text{on } \Gamma^t, \\ \lim_{x\to\infty} v^t = \delta_t c^0 & (n \ge 3), \end{cases} \qquad \begin{cases} \Delta v^t = -\Delta\delta_t h^0 & \text{in } \Omega^t, \\ v^t = -\frac{1}{t}\tilde{u}^0 & \text{on } \Gamma^t, \\ \lim_{x\to\infty} \frac{v^t(x)}{-\log|x|} = \delta_t c^0 & (n = 2). \end{cases} \tag{3-4}$$

Since $\tilde{u}^0 = |\nabla\tilde{u}^0| = 0$ on Γ^0, using the classical estimate[5]

$$\|u^0\|_{C^{1,1}(\mathbb{R}^n)} \le (n-1)\|h^0\|_{C^{1,1}(\mathbb{R}^n)},$$

we obtain

$$|\tilde{u}^0| \le C\|h\|_{C^{1,1}(\mathbb{R}^n)}\, d^2_{\text{Hausdorff}}(\Gamma^t, \Gamma^0) \le Ct^2 \quad \text{on } \Gamma^t.$$

Then, using that Ω^t grows to Ω^0 as $t \downarrow 0$ and uniform estimates for v^t we find that $v^t \to v$ as $t \downarrow 0$, where v is the solution of

$$\begin{cases} \Delta v = -\Delta\dot{h}^0 & \text{in } \Omega^0, \\ v = 0 & \text{on } \Gamma^0, \\ v(\infty) = \dot{c}^0 & (n \ge 3), \end{cases} \qquad \begin{cases} \Delta v = -\Delta\dot{h}^0 & \text{in } \Omega^0, \\ v = 0 & \text{on } \Gamma^0, \\ \lim_{x\to\infty} \frac{v(x)}{-\log|x|} = \dot{c}^0 & (n = 2). \end{cases} \tag{3-5}$$

Here $\Delta\dot{h}^0 = \lim_{t\downarrow 0} \Delta\delta_t h^0 = (\Delta\partial_t h)(0, x)$.

[5]Since u^0 is a solution of the obstacle problem in the whole $\mathbb{R}^n$ with a semiconcave obstacle h^0, we know u^0 is semiconcave with $D^2u^0 \ge -\|h\|_{C^{1,1}(\mathbb{R}^n)}\mathrm{Id}$ and the estimate follows using $\Delta u^0 = 0$, where $u^0 > h^0$.

Equation and estimate for $\dot\eta^0$. We first prove the following

Proposition 3.2. *Let $k \geq 1$. Assume that for some $t_m \downarrow 0$ there exists $\dot\eta^0 \in C^0(\mathcal{Z})$ such that*

$$\delta_{t_m}\eta^0 \to \dot\eta^0 \quad \text{in } C^0(\mathcal{Z}) \text{ as } m \to \infty.$$

Then, the limit $\dot\eta^0$ is given by

$$\dot\eta^0(z) = \left(\frac{\partial_N v}{(N\cdot\nu^0)^2\Delta h^0}\right)(z, \eta^0(z)), \tag{3-6}$$

with v as in (3-5). *As a consequence, $\dot\eta^0$ is independent of the sequence t_m and we have $\dot\eta \in C^{k-1,\alpha}(\mathcal{Z})$ with the estimate*

$$\|\dot\eta^0\|_{C^{k-1,\alpha}(\mathcal{Z})} \leq C(\boldsymbol{\mathcal{C}^0})(\|\dot h^0\|_{C^{k,\alpha}(B_R)} + |\dot c^0|). \tag{3-7}$$

Proof. We split the proof into two steps.

Step 1. We prove (3-6). Recall that since $\tilde u^t$ is a solution of a zero obstacle problem we have

$$\tilde u^t = |\nabla \tilde u^t| = 0 \quad \text{on } \Gamma^t.$$

Thus,

$$\partial_s v^t = \frac{1}{t}(\partial_s \tilde u^t - \partial_s \tilde u^0) = -\frac{\partial_s \tilde u^0}{t} \quad \text{on } \Gamma^t. \tag{3-8}$$

From (3-8) we deduce that

$$\partial_s v^{t_m}(z, \eta^{t_m}) = -\frac{1}{t_m}\partial_s \tilde u^0(z, \eta^{t_m}) = -\frac{1}{t_m}\big(\partial_s \tilde u^0(z, \eta^0) + \partial_{ss}\tilde u^0(z, \eta^0)(\eta^{t_m} - \eta^0) + o(t_m)\big), \tag{3-9}$$

where η^0 and η^{t_m} are evaluated at z (although we omit this in the notation) and where $\partial_{ss}\tilde u^0(z, \eta^0)$ is understood as the limit from the Ω^0-side. To justify the validity of the previous Taylor expansion we use that $\tilde u^0 \in C^{2,\alpha}(\overline{\Omega}^0)$; see Lemma 3.6.

Since $\tilde u^0 = |\tilde\nabla u^0| = 0$ on Γ^0 we obtain

$$\partial_{ee}\tilde u^0 = (e\cdot\nu)^2\partial_{\nu\nu}\tilde u^0 = (e\cdot\nu)^2\Delta\tilde u^0 = -(e\cdot\nu)^2\Delta h^0 \quad \text{on } \Gamma^0$$

for every vector e, where $\nu = \nu^0$ is the normal vector to Γ^0 (pointing towards Ω^0). Again, the previous second derivatives on Γ^0 mean the limits from the Ω^0-side. Hence, we have

$$\partial_s \tilde u^0(z, \eta^0(z)) = 0 \quad \text{and} \quad \partial_{ss}\tilde u^0(z, \eta^0(z)) = -((N\cdot\nu^0)^2\Delta h^0)(z, \eta^0(z)), \tag{3-10}$$

where $\partial_{ss}\tilde u^0(z, \eta^0(z))$ is from the Ω^0-side. Dividing (3-10) by t_m and taking the limit as $t_m \downarrow 0$ in (3-9) using the assumption, we obtain

$$\partial_s v(z, \eta^0(z)) = -\partial_{ss}\tilde u^0(z, \eta^0(z))\,\dot\eta^0(z) = ((N\cdot\nu^0)^2\Delta h^0)(z, \eta^0(z))\,\dot\eta^0(z), \tag{3-11}$$

where $\partial_s v(z, \eta^0(z))$ and $\partial_{ss}\tilde u^0(z, \eta^0(z))$ are from the Ω^0-side. When computing the limit that yields (3-11) we must check that

$$\partial_s v^{t_m}(z, \eta^{t_m}(z)) \to \partial_s v(z, \eta^0(z)), \tag{3-12}$$

where $\partial_s v(z, \eta^0)$ is from the Ω^0-side. To prove this, note that the equation (3-4) for v^t, since we have uniform $C^{1,\alpha}$ estimates for the boundary Γ^t, implies that $\|\nabla v^t\|_{C^{0,\alpha}(\overline{\Omega}^t)}$ is uniformly bounded (for $t > 0$ small). This implies that ∇v^t converges uniformly to ∇v in every compact set of Ω^0. Then using the uniform continuity of the derivatives of v on $\overline{\Omega}^0$ we show that

$$\lim \nabla v^{t_p}(x_p) \to \nabla v(x) \quad \text{as } p \to \infty \quad \text{whenever } t_p \downarrow 0,\ x_p \to x \text{ and } x_p \in \overline{\Omega}^{t_p}.$$

This establishes (3-12) and (3-11). Then, (3-6) follows immediately from (3-11), after recalling that $N = \partial_s$.

Step 2. We prove (3-7). Indeed, from (3-5), and using that $\Gamma^0 = \partial\Omega^0 \in C^{k,\alpha}_{\rho/4}$ with norm universally bounded, we obtain

$$\|v\|_{C^{k,\alpha}(\Omega^0)} \le C(\boldsymbol{C^0})(\|\Delta\dot{h}^0\|_{C^{k-2,\alpha}(\Omega^0)} + |\dot{c}^0|) \le C(\boldsymbol{C^0})(\|\dot{h}^0\|_{C^{k,\alpha}(\Omega^0)} + |\dot{c}^0|). \tag{3-13}$$

Now recalling that N is smooth, that $\|\nu^0\|_{C^{k-1,\alpha}(\Gamma^0)} \le C\|\Gamma^0\|_{C^{k,\alpha}_{\rho/4}} \le C$, that $-\Delta h^0 \ge \boldsymbol{\rho}$, and that $\|\eta^0\|_{C^{k,\alpha}(\mathcal{Z})} \le C$, (3-6) and (3-13) imply (3-7). □

Equation and estimate for $\ddot{\eta}^0$. In this section we estimate the second derivative in t of η at $t = 0$. It is convenient to introduce here the following notation, which we shall use throughout the paper. Given a function $f : (-t_\circ, t_\circ) \times Y \to \mathbb{R}$, recall the notation $f = f^t(y) = f(t, x)$. Let us also define

$$\delta_t^2 f^s := 2\frac{\delta_t f^s - \dot{f}^s}{t} \quad \text{and} \quad \ddot{f}^s := \lim_{t\downarrow 0} \delta_t^2 f^s = \partial_{tt} f(y, 0).$$

From now on let us consider v to be defined in all of $\mathbb{R}^n$ by extending the solution of (3-5) by 0 in $\mathbb{R}^n \setminus \Omega^0$. Note that this is consistent with $v = \lim_{t\downarrow 0} v^t$ and $v^t = \delta_t \tilde{u}^0 = 0$ in $\mathbb{R}^n \setminus \Omega^0$ (since both $\tilde{u}^t$ and $\tilde{u}^0$ vanish there).

We now introduce the function, defined in all of $\mathbb{R}^n$,

$$w^t := \delta_t v^0 = \frac{1}{t}(v^t - v) = \tfrac{1}{2}\delta_t^2 \tilde{u}^0.$$

Using (3-5) and the identity

$$\Delta v^t = \frac{1}{t}\Delta(\tilde{u}^t - \tilde{u}^0) = -\frac{1}{t}\Delta\tilde{u}^0 = \frac{1}{t}\Delta h^0 \quad \text{in } \Omega^0 \setminus \Omega^t,$$

we find, in the distributional sense,

$$\begin{cases} \Delta w^t = \frac{1}{t}\big((\partial_N v/(N\cdot\nu^0))\mathcal{H}^{n-1} \lceil_{\Gamma^0} + \big(\frac{1}{t}\Delta h^0 - \Delta\dot{h}^0\big)\, \chi_{\Omega^0\setminus\Omega^t}\big) - \frac{1}{2}\Delta\delta_t^2 h^0 \chi_{\Omega^t} & \text{in } \mathbb{R}^n, \\ w^t(\infty) = \frac{1}{2}\delta_t^2 c^0 \quad (n \ge 3), \end{cases} \tag{3-14a}$$

$$\begin{cases} \Delta w^t = \frac{1}{t}\big((\partial_N v/(N\cdot\nu^0))\mathcal{H}^{n-1} \lceil_{\Gamma^0} + \big(\frac{1}{t}\Delta h^0 - \Delta\dot{h}^0\big)\, \chi_{\Omega^0\setminus\Omega^t}\big) - \frac{1}{2}\Delta\delta_t^2 h^0 \chi_{\Omega^t} & \text{in } \mathbb{R}^n, \\ \lim_{x\to\infty} \frac{w^t}{-\log|x|} = \frac{1}{2}\delta_t^2 c^0 \quad (n = 2), \end{cases} \tag{3-14b}$$

where $\mathcal{H}$ denotes the Hausdorff measure. Indeed, note also that for $\nu = \nu^0$ we have

$$\partial_N v = (N\cdot\nu^0)\partial_\nu v \quad \text{on } \Gamma^0_{\text{out}}, \quad \text{while} \quad \partial_\nu v = 0 \quad \text{on } \Gamma^0_{\text{in}}.$$

Here, "Γ^0_{out}" refers to the limit from the Ω^0-side, while "Γ^0_{in}" refers to the limit from the $(\mathbb{R}^n \setminus \overline{\Omega}^0)$-side. Therefore, Δw^t has some mass concentrated on Γ^0 which is given by the jump in the normal derivative of v, namely,

$$\frac{1}{t}\frac{\partial_N v}{N \cdot \nu^0}\,\mathcal{H}^{n-1} \lfloor_{\Gamma^0} .$$

In the following lemma, and throughout the paper, P denotes the Newtonian potential in dimension n, namely,

$$P(x) = \frac{1}{n(n-2)|B_1|}|x|^{2-n} \quad (n \geq 3)$$

or

$$P(x) = -\frac{1}{2\pi}\log|x| \quad (n = 2).$$

Recall that $-\Delta P = \delta_{x=0}$ in the sense of distributions.

We also need to introduce the Jacobian

$$J(z,s) := |\det D\,(z,s)^{-1}|$$

of the coordinates (z, s) defined by

$$\int_A f(x)\,dx = \int_{(z,s)(A)} f(z,s)\,J(z,s)\,dz\,ds.$$

We use the following abuse of notation:

- When $f = f(x)$ we denote by $f(z,s)$ the composition $f \circ (z,s)^{-1}$.
- Conversely, when $g = g(z,s)$ we will denote by $g(x)$ the composition $g \circ (z,s)$.

Finally, let us denote by

$$\pi_1 : U_\circ \to \mathcal{Z}$$

the projection map along N, which is defined in the coordinates (z, s) by

$$(z,s) \mapsto (z,0).$$

We will need the following:

Lemma 3.3. *Given $f : \Gamma^0 \to \mathbb{R}$ continuous we have*

$$\int_{\Gamma^0} (N \cdot \nu^0)(x)\,f(x)\,d\mathcal{H}^{n-1}(x) = \int_{\mathcal{Z}} f(z,\eta^0(z))\,J(z,\eta^0(z))\,dz.$$

Proof. Let us assume without loss of generality that f is defined and continuous in the neighborhood $U_\circ$ of Γ^0. Given $\varepsilon > 0$ let

$$A^\varepsilon := \{x \in U_\circ : \eta^0(z(x)) \leq s(x) \leq \eta^0(z(x)) + \varepsilon\}.$$

Recalling that $N = \partial_s$ and that $|N| = 1$, we have

$$\int_{\Gamma^0} (N \cdot \nu^0)(x)\,f(x)\,d\mathcal{H}^{n-1}(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{A^\varepsilon} f(x)\,d\mathcal{H}^n(x).$$

On the other hand, for $(z,s)(A^\varepsilon) := \{(z,s) \in \mathcal{Z} \times (-s_\circ, s_\circ) : \eta^0(z) \le s \le \eta^0(z) + \varepsilon\}$ we have, by the definition of J,

$$\begin{aligned}\frac{1}{\varepsilon}\int_{A^\varepsilon} f(x)\, d\mathcal{H}^n(x) &= \frac{1}{\varepsilon}\int_{(z,s)(A^\varepsilon)} f(z,s)\, J(z,s)\, dz\, ds \\ &= \int_{\mathcal{Z}} dz\, \frac{1}{\varepsilon}\int_0^\varepsilon d\bar{s}\, f(z, \eta^0(z)+\bar{s})\, J(z, \eta^0(z)+\bar{s}) \\ &= \int_{\mathcal{Z}} f(z, \eta^0(z))\, J(z, \eta^0(z))\, dz + o(1)\end{aligned}$$

as $\varepsilon \downarrow 0$ and the lemma follows. □

Lemma 3.4. *Let $k \ge 2$. Assume that for some $t_m \downarrow 0$ there exist $\dot{\eta}, \ddot{\eta} \in C^0(\mathcal{Z})$ such that*

$$\delta^2_{t_m}\eta^0 = 2\frac{\eta^{t_m} - \eta^0 - \dot{\eta}^0 t_m}{t_m^2} \to \ddot{\eta}^0 \quad \text{in } C^0(\mathcal{Z})$$

as $t_m \downarrow 0$. Then,

$$w^{t_m} \xrightarrow{\text{weakly}} w \quad \text{in } \mathbb{R}^n,$$

where w can be decomposed as

$$w = w_{\text{solid}} + w_{\text{single}} + w_{\text{double}} + w_{\text{implicit}} + \textit{constant} \tag{3-15}$$

for

$$w_{\text{solid}}(x) := \int_{\mathbb{R}^n} d\mathcal{H}^n(y)(\Delta\ddot{h}^0 \chi_{\Omega_0})(y) P(x-y), \tag{3-16}$$

$$w_{\text{single}}(x) := \int_{\Gamma_0} d\mathcal{H}^{n-1}(y)\left((N\cdot\nu^0)(\dot{\eta}^0\circ\pi_1)\,\Delta\dot{h}^0 - \tfrac{1}{2}(\dot{\eta}^0\circ\pi_1)^2\frac{N\cdot\nu^0}{J}\partial_N(Jh^0)\right)(y)\, P(x-y), \tag{3-17}$$

$$w_{\text{double}}(x) := \int_{\Gamma_0} d\mathcal{H}^{n-1}(y)\left(\tfrac{1}{2}(\dot{\eta}^0\circ\pi_1)^2\frac{N\cdot\nu^0}{J}(J\Delta h^0)\right)(y)\,\partial_N P(x-y), \tag{3-18}$$

$$w_{\text{implicit}}(x) := \int_{\Gamma_0} d\mathcal{H}^{n-1}(y)\frac{\Theta}{(N\cdot\nu)}(y)\, P(x-y), \tag{3-19}$$

where $\Theta : \Gamma^0 \to \mathbb{R}$,

$$\Theta := \tfrac{1}{2}(N\cdot\nu^0)^2\Delta h^0\,(\ddot{\eta}^0\circ\pi_1). \tag{3-20}$$

Proof. Define

$$\mathcal{D}^t := \Delta w^t = \frac{1}{t}\frac{\partial_N v}{N\cdot\nu^0}\mathcal{H}^{n-1} \lceil_{\Gamma^0} - \left(\frac{1}{t^2}\Delta h^0 + \frac{1}{t}\Delta\dot{h}^0\right)\chi_{\Omega^0\setminus\Omega^t} - \tfrac{1}{2}\Delta\delta_t^2 h^0\chi_{\Omega^t}.$$

Let us show that $\mathcal{D}^{t_m} \to \mathcal{D}$ in the sense of distributions, for some distribution $\mathcal{D}$ that we compute.

Let us first write

$$\mathcal{D}^t = \mathcal{D}_1^t + \mathcal{D}_2^t,$$

where

$$\mathcal{D}_1^t := -\frac{1}{t}\Delta\dot{h}^0\,\chi_{\Omega^0\setminus\Omega^t} - \tfrac{1}{2}\Delta\delta_t^2 h^0\chi_{\Omega^t} \tag{3-21}$$

and

$$\mathcal{D}_2^t := \frac{1}{t}\left(\frac{\partial_N v}{N\cdot\nu^0}\,\mathcal{H}^{n-1}\restriction_{\Gamma^0} - \frac{1}{t}\Delta h^0\,\chi_{\Omega^0\setminus\Omega^t}\right). \tag{3-22}$$

First we clearly have, for $\phi\in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\int\phi(x)\Big(\frac{1}{t}\Delta\dot h^0\,\chi_{\Omega^0\setminus\Omega^t}\Big)(x)\,dx &= \frac{1}{t}\int_{\mathcal{Z}}\int_{\eta^0}^{\eta^t} J(z,s)(\Delta\dot h^0\,\phi)(z,s)\,dz\,ds\\ &\to \int_{\mathcal{Z}}\dot\eta^0(z)J(z,\eta^0)(\Delta\dot h^0\,\phi)(z,\eta^0)\,dz = \int_{\Gamma^0}(N\cdot\nu^0)(\dot\eta^0\circ\pi_1)\,\Delta\dot h^0\,\phi\,d\mathcal{H}^{n-1}\end{aligned}$$

as $t=t_m\downarrow 0$, where we have used Lemma 3.3, and hence

$$\mathcal{D}_1^{t_m}\xrightarrow{\text{weakly}} -(N\cdot\nu^0)(\dot\eta^0\circ\pi_1)\,\Delta\dot h^0\,\mathcal{H}^{n-1}\restriction_{\Gamma^0} - \tfrac{1}{2}\Delta\ddot h^0\chi_{\Omega^0}. \tag{3-23}$$

Next, using (3-6), we compute, for $\phi\in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\int\phi\mathcal{D}_2^t &= \frac{1}{t}\left(\int_{\mathcal{Z}} dz\frac{J(z,\eta^0)}{(N\cdot\nu^0)^2(z,\eta^0)}\partial_N v(z)\phi(z,\eta^0) - \int_{\mathcal{Z}} dz\,\frac{1}{t}\int_{\eta^0}^{\eta^t} ds(J\Delta h^0\,\phi)(z,s)\right)\\ &= \frac{1}{t}\int_{\mathcal{Z}} dz\left((J\Delta h^0\,\phi)(z,\eta^0)\dot\eta^0 - \frac{1}{t}\int_{\eta^0}^{\eta^t} ds(J\Delta h^0\,\phi)(z,s)\right)\\ &= I_1 + I_2,\end{aligned} \tag{3-24}$$

where

$$I_1 := \frac{1}{t}\int_{\mathcal{Z}} dz\left((J\Delta h^0\,\phi)(z,\eta^0)\dot\eta^0 - \frac{1}{t}\int_{\eta^0}^{\eta^0+\dot\eta^0 t} ds(J\Delta h^0\,\phi)(z,s)\right)$$

and

$$I_2 := -\frac{1}{t^2}\int_{\mathcal{Z}} dz\int_{\eta^0+\dot\eta^0 t}^{\eta^t} ds(J\Delta h^0\,\phi)(z,s).$$

On one hand, letting $s=\eta^0+\dot\eta^0 t\bar s$,

$$\begin{aligned}I_1 &= \int_{\mathcal{Z}} dz\int_0^1\dot\eta^0(z)t\,d\bar s\,\frac{\bar s}{t}\left(\frac{(J\Delta h^0\,\phi)(z,\eta^0)-(J\Delta h^0\,\phi)(z,\eta^0+\dot\eta^0 t\bar s)}{\bar s t}\right)\\ &= \int_{\mathcal{Z}} dz\int_0^1(\dot\eta^0)^2(z)\,\bar s\,d\bar s\,\partial_s(J\Delta h^0\,\phi)(z,\eta^0)+o(1)\\ &= \int_0^1\bar s\,d\bar s\int_{\mathcal{Z}} dz\,(\dot\eta^0)^2(z)\,\partial_s(J\Delta h^0\,\phi)(z,\eta^0)+o(1)\\ &= \frac{1}{2}\int_{\Gamma^0}(N\cdot\nu^0)\,d\mathcal{H}^{n-1}\,\frac{1}{J}(\dot\eta^0\circ\pi_1)^2\,\partial_N\,(J\Delta h^0\,\phi)+o(1).\end{aligned} \tag{3-25}$$

as $t=t_m\downarrow 0$, where for the last relation we used Lemma 3.3 with

$$f(x)=\Big(\frac{1}{J}(\dot\eta^0\circ\pi_1)^2\partial_N\,(J\Delta h^0\,\phi)\Big)(x),$$

noting also that $\partial_s = \partial_N$ and $(\dot\eta^0 \circ \pi_1)^2(z, \eta^0(z)) = (\dot\eta^0)^2(z)$. On the other hand,

$$\begin{aligned} I_2 &= -\frac{1}{t^2}\int_{\mathcal{Z}} dz \int_{\eta^0+\dot\eta^0 t}^{\eta^0+\dot\eta^0 t+\frac{1}{2}\ddot\eta^0 t^2} ds\,(J\,\Delta h^0\,\phi)(z,s) + o(1) \\ &= -\frac{1}{2}\int_{\mathcal{Z}} dz\,\ddot\eta^0\,(J\,\Delta h^0\,\phi)(z,\eta^0) + o(1) \\ &= -\frac{1}{2}\int_{\Gamma^0} d\mathcal{H}^{n-1}(N\cdot\nu^0)\,(\ddot\eta^0\circ\pi_1)\,\Delta h^0\phi\, + o(1) \end{aligned} \tag{3-26}$$

as $t = t_m \downarrow 0$. Therefore, $\mathcal{D}_2^{t_m} \to \mathcal{D}_2$, where

$$\int \phi\mathcal{D}_2 = \frac{1}{2}\int_{\Gamma^0} d\mathcal{H}^{n-1}(\dot\eta^0\circ\pi_1)^2\,\frac{N\cdot\nu^0}{J}\partial_N\,(J\,\Delta h^0\,\phi) - \frac{1}{2}\int_{\Gamma^0} d\mathcal{H}^{n-1}(N\cdot\nu^0)\,(\ddot\eta^0\circ\pi_1)\,\Delta h^0\phi. \tag{3-27}$$

In dimension $n \geq 3$ we have

$$w^{t_m}(\infty) = \tfrac{1}{2}\lim_{x\to\infty}\delta_{t_m}^2\tilde u^0(\infty) = \tfrac{1}{2}\delta_{t_m}^2 c^0 \to \tfrac{1}{2}\ddot c^0,$$

and thus

$$w(\infty) = \tfrac{1}{2}\ddot c^0 = \text{constant}.$$

In dimension $n = 2$ we have instead

$$\lim_{x\to\infty}\frac{w(x)}{-\log|x|} = \tfrac{1}{2}\ddot c^0$$

and this implies $2\pi\frac{1}{2}\ddot c^0 = \int_{\mathbb{R}^2}\Delta w$ and that w can be obtained (up to an additive constant) by convolving the Newtonian potential P with Δw.

Therefore, combining (3-23) and (3-27), we obtain that (3-15)–(3-19) hold. □

We may now state the final result of this section.

Proposition 3.5. *Let $k \geq 2$. Assume that for some $t_m \downarrow 0$ there exist $\dot\eta, \ddot\eta \in C^0(\mathcal{Z})$ such that*

$$\delta_{t_m}^2\eta^0 = 2\frac{\eta^{t_m} - \eta^0 - \dot\eta^0 t_m}{t_m^2} \to \ddot\eta^0 \quad \text{in } C^0(\mathcal{Z})$$

as $t_m \downarrow 0$. Assume $w \in C^1(\overline{\Omega}^0)$ and

$$\lim \nabla w^{t_m}(x_m) \to \nabla w(x) \quad \text{as } m\to\infty \quad \text{for all } x_m \to x \text{ such that } x_m \in \overline{\Omega}^{t_m}. \tag{3-28}$$

Then, $\Theta : \Gamma^0 \to \mathbb{R}$ defined by (3-20) satisfies

$$\Theta - \tfrac{1}{2}\partial_{sss}u^0(\dot\eta^0\circ\pi_1)^2 = \partial_{ss}v\,\dot\eta^0 + \partial_s w \quad \text{on } \Gamma^0_{\text{out}}. \tag{3-29}$$

Moreover, $\ddot\eta^0$ does not depend on (t_m) and

$$\|\ddot\eta^0\|_{C^{k-2,\alpha}(\mathcal{Z})} \leq C(\mathcal{C}^0)\mathcal{Q}, \tag{3-30}$$

where

$$\mathcal{Q} := \|\ddot h^0\|_{C^{k-1,\alpha}(\mathbb{R}^n)} + |\ddot c^0| + (\|\dot h^0\|_{C^{k,\alpha}(\mathbb{R}^n)} + |\dot c^0|)(\|\dot h^0\|_{L^\infty(\mathbb{R}^n)} + |\dot c^0|).$$

As for $\dot{\eta}$, the independence of t_m and regularity of $\ddot{\eta}$ will be consequences of the fact that Θ solves (3-29), for which regularity estimates and uniqueness hold. However, note that (3-29) is an implicit equation for Θ since w_{implicit} depends on Θ, which makes the analysis more involved.

To prove Proposition 3.5, we will need two auxiliary lemmas with standard proofs.

Lemma 3.6. *We have*

$$\|\tilde{u}^0\|_{C^{k+1,\alpha}(\overline{B_R\cap\Omega^t})} \le C(\mathcal{C}^0).$$

More generally, for $t\in[0,t_\circ)$, where $t_\circ = t_\circ(\mathcal{C})$, we have $\tilde{u}^t \in C^{k+1,\alpha}(\overline{\Omega}^t)$ with

$$\|\tilde{u}^t\|_{C^{k+1,\alpha}(\overline{B_R\cap\Omega^t})} \le C(\mathcal{C}).$$

Proof. Note that $\partial_i\tilde{u}^t$ solves

$$\Delta(\partial_i\tilde{u}^t) = -\Delta(\partial_i h^t) \quad \text{in } \Omega^t, \quad \text{with} \quad \partial_i\tilde{u}^t = 0 \quad \text{on } \Gamma^t = \partial\Omega^t.$$

Since $-\Delta(\partial_i h^t)\in C^{k-2,\alpha}(\mathbb{R}^n)$ and Γ^t belongs to $C^{k,\alpha}_{\rho}$, using standard Schauder estimates up to the boundary we obtain

$$\partial_i\tilde{u}^t \in C^{k,\alpha}(\overline{B_R\cap\Omega^t}),$$

and hence

$$\tilde{u}^t \in C^{k+1,\alpha}(\overline{B_R\cap\Omega^t}). \qquad \square$$

Lemma 3.7. *Let $U\subset \bar{B}_R\subset\mathbb{R}^n$ be bounded with ∂U belonging to $C^{m+2,\alpha}_r$ for some $r>0$ and $f\in C^{m,\alpha}_c(B_{2R})$, where $m\ge 0$. Let W be the solution of*

$$\begin{cases}\Delta W = f\,\chi_{\mathbb{R}^n\setminus U} & \textit{in } \mathbb{R}^n,\\ W(\infty)=0 & (n\ge 3),\end{cases}\qquad \begin{cases}\Delta W = f\,\chi_{\mathbb{R}^n\setminus U} & \textit{in } \mathbb{R}^n,\\ \lim_{x\to\infty}\frac{W(x)}{-\log|x|} = 2\pi\int_{\mathbb{R}^2} f\,\chi_{\mathbb{R}^n\setminus U} & (n=2),\end{cases}$$

which is given in dimension 2 by convolution with the logarithmic Newtonian potential.

Then,

$$\|W\|_{C^{m+2,\alpha}(\bar{B}_{2R}\setminus U)} + \|W\|_{C^{m+2,\alpha}(\bar{U})} \le C\|f\|_{C^{m,\alpha}(\bar{B}_{2R})},$$

where $C = C(n,m,\alpha,R,r,\|\partial U\|_{C^{m+2,\alpha}_r})$.

Proof. Let $\widetilde{W}$ be the solution of

$$\begin{cases}\Delta\widetilde{W} = f & \text{in } \mathbb{R}^n\setminus U,\\ \widetilde{W} = 0 & \text{on } \partial U,\\ \widetilde{W}(\infty)=0 & (n\ge 3),\end{cases}\qquad \begin{cases}\Delta\widetilde{W} = f & \text{in } \mathbb{R}^n\setminus U,\\ \widetilde{W}=0 & \text{on } \partial U,\\ \lim_{x\to\infty}\frac{\widetilde{W}(x)}{-\log|x|} = 0 & (n=2).\end{cases}$$

We consider $\widetilde{W}$ defined in all of $\mathbb{R}^n$ by extending it by 0 in U.

Note that by standard Schauder estimates up to the boundary we have

$$\|\widetilde{W}\|_{C^{m+2,\alpha}(\bar{B}_{2R}\setminus U)} \le C\|f\|_{C^{m,\alpha}(\bar{B}_{2R})}. \tag{3-31}$$

On the other hand, the difference $(\widetilde{W}-W)$ solves, in all of $\mathbb{R}^n$,

$$\begin{cases}\Delta(\widetilde{W}-W)=\partial_{\nu,\,\mathrm{out}}\widetilde{W}\,H^{n-1}\restriction_{\partial U} & \text{in } \mathbb{R}^n,\\ (\widetilde{W}-W)(\infty)=0 \quad (n\ge 3),\end{cases}$$

$$\begin{cases}\Delta(\widetilde{W}-W)=\partial_{\nu,\,\mathrm{out}}\widetilde{W}\,H^{n-1}\restriction_{\partial U} \quad \text{in } \mathbb{R}^n,\\ \lim_{x\to\infty}\frac{(\widetilde{W}-W)(x)}{-\log|x|}=-2\pi\int_{\mathbb{R}^2} f\,\chi_{\mathbb{R}^n\setminus U}=2\pi\int_{\partial U}\partial_{\nu,\,\mathrm{out}}\widetilde{W} \quad \text{at } \infty \quad (n=2).\end{cases}$$

Therefore, $\widetilde{W}-W$ is a single-layer potential and using Theorem A.1 we obtain

$$\begin{aligned}\|(\widetilde{W}-W)\|_{C^{m+2,\alpha}(\bar B_{2R}\setminus U)}+\|(\widetilde{W}-W)\|_{C^{m+2,\alpha}(\bar U)} &\le C\|\partial_{\nu,\,\mathrm{out}}\widetilde{W}\|_{C^{m+1,\alpha}(\partial U)}\\ &\le C\|\widetilde{W}\|_{C^{m+2,\alpha}(\bar B_{2R}\setminus U)}\le C\|f\|_{C^{m,\alpha}(\bar B_4)}.\end{aligned}$$

Using (3-31) and recalling that by definition $\widetilde{W}\equiv 0$ in U we obtain

$$\|W\|_{C^{m+2,\alpha}(\bar B_{2R}\setminus U)}+\|W\|_{C^{m+2,\alpha}(\bar U)}\le C\|f\|_{C^{m,\alpha}(\bar B_{2R})}. \qquad \square$$

Proof of Proposition 3.5. Step 1. We first prove (3-29).

Expanding (3-8) as in (3-9) but up to the next order, we find

$$\partial_s v^t(z,\eta^t)=-\partial_{ss}\tilde u^0(z,\eta^0)\big(\dot\eta^0+\tfrac12\ddot\eta^0 t+o(t)\big)-\tfrac12\partial_{sss}\tilde u^0(z,\eta^0)(\dot\eta^0)^2t+o(t) \tag{3-32}$$

as $t=t_m\downarrow 0$.

Here η, $\dot\eta$ and $\ddot\eta$ are evaluated at z (although we omit this in the notation) and $\partial_{ss}\tilde u^0(z,\eta^0)$ and $\partial_{sss}\tilde u^0(z,\eta^0)$ mean the limits from Ω^0. To obtain the Taylor expansion up to the third order of $\tilde u^0$ we are using that, by Lemma 3.6, $u^0\in C^{k+1,\alpha}(\overline{\boldsymbol{B_R}\cap\Omega^0})$ where $k\ge 2$. Recall here that $\{u^0=0\}=\mathbb{R}^n\setminus\Omega^0\subset \boldsymbol{U}\subset B_{\boldsymbol{R}}$.

Subtracting from both sides of (3-32) the quantity

$$\partial_s v(z,\eta^0)=-\partial_{ss}\tilde u^0(z,\eta^0)\dot\eta^0 \tag{3-33}$$

and dividing by t, we obtain

$$\frac{\partial_s v^t(z,\eta^t)-\partial_s v(z,\eta^0)}{t}=-\tfrac12\partial_{ss}\tilde u^0(z,\eta^0)\ddot\eta^0-\tfrac12\partial_{sss}u^0(z,\eta^0)(\dot\eta^0)^2+o(1). \tag{3-34}$$

Recall that by Lemma 3.4 we have $w^t\to w$ in the sense of distributions with w given by (3-15)–(3-19). Then, the assumption (3-28) allows us to compute the limit of the left-hand side in (3-34), namely,

$$\begin{aligned}\lim_{t=t_m\downarrow 0}\frac{\partial_s v^t(z,\eta^t)-\partial_s v(z,\eta^0)}{t}&=\lim_{t=t_m\downarrow 0}\frac{\partial_s v(z,\eta^t)-\partial_s v(z,\eta^0)}{t}+\frac{\partial_s v^t(z,\eta^t)-\partial_s v(z,\eta^t)}{t}\\ &=\partial_{ss}v(z,\eta^0)\,\dot\eta+\lim_{t=t_m\downarrow 0}(N(z,\eta^t)\cdot\nabla w^t(z,\eta^t)\\ &=\partial_{ss}v(z,\eta^0)\,\dot\eta+\partial_s w^t(z,\eta^0),\end{aligned} \tag{3-35}$$

where we have used the assumption (3-28).

Taking $t = t_m \downarrow 0$ in (3-34) and using (3-35) we obtain

$$-\tfrac{1}{2}\partial_{ss}\tilde{u}^0(z,\eta^0)\ddot{\eta}^0 - \tfrac{1}{2}\partial_{sss}u^0(z,\eta^0)(\dot{\eta}^0)^2 = \partial_{ss}v(z,\eta^0)\,\dot{\eta} + \partial_s w^t(z,\eta^0).$$

Recalling the definition of Θ in (3-20) and the fact that $\partial_{ss}\tilde{u}^0 = -\Delta h^0$ on Γ^0 — and in particular at (z,η^0) — we obtain (3-29).

Step 2. We use (3-29) to prove uniqueness and regularity of $\ddot{\eta}$. Recall that

$$\partial_s w = \partial_N w = \partial_N w_{\text{solid}} + \partial_N w_{\text{single}} + \partial_N w_{\text{double}} + \partial_N w_{\text{implicit}},$$

and while $\partial_N w_{\text{solid}}$, $\partial_N w_{\text{single}}$, $\partial_N w_{\text{double}}$ depend only on "known" functions — see (3-16), (3-17), (3-18) — the term $\partial_N w_{\text{implicit}}$ introduces a "implicit" dependence on Θ — see (3-19). We therefore need to "solve for Θ" in (3-29) in order to prove the uniqueness and regularity of its solutions Θ.

For this, we write

$$\partial_N w_{\text{implicit}} = (N\cdot\nu)\partial_\nu w_{\text{implicit}} + (N-(N\cdot\nu)\nu)\cdot\nabla w_{\text{implicit}} \quad \text{on } \Gamma^0_{\text{out}},$$

where $\nu = \nu^0$. Recall that by a standard result on single layer potentials — see Theorem A.1 — we have

$$(N\cdot\nu)\partial_\nu w_{\text{implicit}}(x) = \tfrac{1}{2}\Theta(x) + \widetilde{\Theta}(x) \quad \text{on } \Gamma^0_{\text{out}}, \tag{3-36}$$

where

$$\widetilde{\Theta}(x) := \int_{\Gamma_0} d\mathcal{H}^{n-1}(y)\left(-\frac{\Theta}{(N\cdot\nu)}\right)(y)\ \nu(x)\cdot\nabla P(x-y). \tag{3-37}$$

Note that the first term in the right-hand side of (3-36) is exactly the half of the first (and main) term in the left-hand side of (3-29). Using this and defining

$$\omega(x) := (N-(N\cdot\nu)\nu)(x) \quad \text{for } x \text{ on } \Gamma^0$$

we obtain

$$\tfrac{1}{2}\Theta = \tfrac{1}{2}\partial_{sss}u^0(\dot{\eta}^0\circ\pi_1)^2 + \dot{\eta}^0\,\partial_s N\cdot\nabla v + \partial_s(w_{\text{solid}}+w_{\text{single}}+w_{\text{double}}) + \omega\cdot\nabla w_{\text{implicit}} + \widetilde{\Theta} \quad \text{on } \Gamma^0_{\text{out}}. \tag{3-38}$$

Step 3. From (3-38), we may deduce optimal regularity estimates for Θ, and hence for $\ddot{\eta}^0$. To do so we will bound each of the five terms in the right-hand side of (3-38) separately.

From here on, the constant C means $C = C(n, k, \alpha, \boldsymbol{\rho}, \|h^0\|_{C^{k+1,\alpha}(\mathbb{R}^n)})$.

For the first term, we use that $h^0 \in C^{k+1,\alpha}$, and we obtain that $\Gamma^0 \in C^{k,\alpha}_{\boldsymbol{\rho}/4}$, that $\nu^0 \in C^{k-1,\alpha}(\Gamma^0)$, and that $\eta^0 \in C^{k,\alpha}(\mathcal{Z})$ with estimates — here we are using the regularity estimates on Γ^0 from Proposition 2.1. In particular,

$$\|\pi_1\|_{C^{k,\alpha}(\Gamma^0)} + \|\nu^0\|_{C^{k-1,\alpha}(\Gamma^0)} \le C. \tag{3-39}$$

Observe also that the vector field N is smooth and hence $\partial_{sss}u^0$ — the third derivative of u^0 along an integral curve of N — is regular as D^3u^0.

Therefore,

$$
\begin{aligned}
\big\|\tfrac{1}{2}\partial_{sss}u^0(\dot\eta^0\circ\pi_1)^2\big\|_{C^{k-2,\alpha}(\Gamma^0)}
&\le C\big(\|u^0\|_{C^{k+1,\alpha}(\overline{B_R\cap\Omega^0})}\|(\dot\eta^0\circ\pi_1)^2\|_{L^\infty(\Gamma^0)}+\|u^0\|_{L^\infty(B_R\cap\Omega^0)}\|(\dot\eta^0\circ\pi_1)^2\|_{C^{k-2,\alpha}(\Gamma^0)}\big)\\
&\le C\|(\dot\eta^0)^2\|_{C^{k-2,\alpha}(\mathcal{Z})}\\
&\le C\mathcal{Q}. \qquad (3\text{-}40)
\end{aligned}
$$

For the second term, we use again that N is smooth and recalling the estimate (3-13) for v and the estimate $\dot\eta$ in (3-7), we obtain

$$
\begin{aligned}
\|(\dot\eta^0\circ\pi_1)\,\partial_{ss}v\|_{C^{k-2,\alpha}(\Omega^0)} &\le C\big(\|\dot\eta^0\|_{C^{k-2,\alpha}(\mathcal{Z})}\|v\|_{L^\infty(B_R\cap\Omega^0)}+\|\dot\eta^0\|_{L^\infty(\mathcal{Z})}\|v\|_{C^{k,\alpha}(\overline{B_R\cap\Omega^0})}\big)\\
&\le C\mathcal{Q}, \qquad (3\text{-}41)
\end{aligned}
$$

where we used (3-7) and (3-13).

For the third term, we proceed as follows. From Lemma 3.7 we obtain that

$$\|\nabla w_{\text{solid}}\|_{C^{k-2,\alpha}(B_R\cap\Omega^0)}\le C\|\Delta\ddot h^0\|_{C^{k-2,\alpha}}\le C\mathcal{Q}.$$

Next, since N and J are smooth, $\Delta h^0\in C^{k-1,\alpha}$, $\Gamma^0\in C^{k,\alpha}$, and $\nu^0\in C^{k-1,\alpha}$ we obtain by Theorem A.1(i) that

$$\|w_{\text{single}}\|_{C^{k-1,\alpha}(\Omega^0)}\le C\big(\|(\dot\eta^0\circ\pi_1)\Delta\dot h^0\|_{C^{k-2,\alpha}(\Gamma_0)}+\|(\dot\eta^0\circ\pi_1)^2\|_{C^{k-2,\alpha}(\Gamma_0)}\big)\le C\mathcal{Q}$$

and by Theorem A.1(iii)

$$\|w_{\text{double}}\|_{C^{k-1,\alpha}(\Omega^0)}\le C\|(\dot\eta^0\circ\pi_1)^2\|_{C^{k-1,\alpha}(\Gamma_0)}\le C\mathcal{Q}.$$

Hence,

$$\|\partial_s(w_{\text{solid}}+w_{\text{single}}+w_{\text{double}})\|_{C^{k-2,\alpha}(\Gamma^0)}\le C\mathcal{Q}. \qquad (3\text{-}42)$$

For the term $\omega\cdot\nabla w_{\text{implicit}}$ we use that Theorem A.1(i) yields

$$\|w_{\text{implicit}}\|_{C^{k-1,\alpha}(B_R\cap\Omega^0)}\le C\|\Theta\|_{C^{k-2,\alpha}(\Gamma_0)},$$

and thus

$$\|\omega\cdot\nabla w_{\text{implicit}}\|_{C^{k-2,\alpha}(\Gamma^0)}\le C\|\omega\|_{C^{k-2,\alpha}(\Gamma^0)}\|\Theta\|_{C^{k-2,\alpha}(\Gamma^0)}. \qquad (3\text{-}43)$$

Also, recalling the definition of $\widetilde\Theta$ in (3-37) and using Theorem A.1(iii) we obtain

$$\|\widetilde\Theta\|_{C^{k-2,\alpha}(\Gamma^0)}\le C\|\Theta\|_{C^{k-3,\alpha}(\Gamma^0)}. \qquad (3\text{-}44)$$

Inserting (3-40)–(3-44) into (3-38), we obtain

$$\|\Theta\|_{C^{k-2,\alpha}(\Gamma^0)}\le C\big(\mathcal{Q}+\|\omega\|_{C^{k-2,\alpha}(\Gamma^0)}\|\Theta\|_{C^{k-2,\alpha}(\Gamma^0)}+\|\Theta\|_{C^{k-3,\alpha}(\Gamma^0)}\big).$$

Note that we may take $\|\omega\|_{C^{k-2,\alpha}(\Gamma^0)}$ arbitrarily small by taking ε_o in (2-2) small enough. Then, by a standard interpolation argument we obtain

$$\|\Theta\|_{C^{k-2,\alpha}(\Gamma^0)}\le C\mathcal{Q}. \qquad (3\text{-}45)$$

Finally we recall the definition of Θ in (3-20), use that $\nu^0 \in C^{k-1,\alpha}$, $-\Delta h^0 \geq \boldsymbol{\rho}$ and $\Delta h^0 \in C^{k-1,\alpha}$, and observe that $\pi_0^{-1} : \mathcal{Z} \to \Gamma^0$ satisfies $\|\pi_0^{-1}\|_{C^{k,\alpha}(\mathcal{Z})} \leq C$ with C universal, to obtain

$$\|\ddot{\eta}^0\|_{C^{k-2,\alpha}(\mathcal{Z})} \leq C\boldsymbol{Q}. \qquad \square$$

4. Removing the a priori assumptions

In Section 3 we assumed the existence of the limits

$$\frac{\eta^{t_m} - \eta^0}{t_m} \to \dot{\eta}^0 \quad \text{and} \quad 2\frac{\eta^{t_m} - \eta^0 - \dot{\eta}^0 t_m}{t_m^2} \to \ddot{\eta}^0 \quad \text{in } C^0(\mathcal{Z}) \tag{4-1}$$

and we have shown that $\dot{\eta}^0$ and $\ddot{\eta}^0$ must then satisfy certain equations for which uniqueness and regularity estimates were proven.

The purpose of the next section is to prove that under our assumptions, (4-1) indeed holds for every sequence $t_m \downarrow 0$.

The setup. We start by introducing a new system of coordinates in $U_\circ \cap \overline{\Omega}^0$ that are adapted to u^0.

Let us define

$$\sigma = \sigma(x) := \partial_N \tilde{u}_0(x). \tag{4-2}$$

Note that σ is defined in $U_\circ \cap \overline{\Omega}^0$ and takes positive values in that neighborhood of Γ^0 if $U_\circ$ is chosen small enough. An application of the implicit function theorem gives that (z, σ) are $C^{k,\alpha}$ coordinates in $U_\circ \cap \overline{\Omega}^0$ (up to taking a smaller neighborhood $U_\circ$). Indeed, for $\nu = \nu^0$

$$\frac{\partial\sigma}{\partial s} = \partial_{ss}\tilde{u}^0 = (N\cdot\nu)^2\partial_{\nu\nu}\tilde{u}^0 = (N\cdot\nu)^2\Delta\tilde{u}^0 = -(N\cdot\nu)^2\Delta h^0 \tag{4-3}$$

on Γ^0_{out} and where by assumption $-\Delta h^0 \geq \boldsymbol{\rho} > 0$ in a neighborhood of Γ^0. Note in addition that the new coordinates (z, σ) are indeed $C^{k,\alpha}$ since $\tilde{u}^0 \in C^{k+1,\alpha}(\overline{\Omega}^0)$.

Let us also introduce

$$\bar{\pi}_1 : U_\circ \cap \Omega^0 \to \mathcal{Z}$$

to be the projection defined in the coordinates (z, σ) by

$$(z, \sigma) \mapsto (z, 0).$$

These coordinates are clearly related to the hodograph transform of the obstacle problem introduced in [Kinderlehrer and Nirenberg 1977]. Note also that for the case of the model solution to the obstacle problem $\frac{1}{2}(x_n)_+^2$, and with $N = \boldsymbol{e}_n$ the coordinate σ would simply be x_n.

In view of Proposition 2.1 there exists $\lambda^t \in C^{k,\alpha}(\mathcal{Z})$ such that

$$\Gamma^t = \{\sigma = \lambda^t(z)\} \quad \text{for } t \in (0, t_\circ). \tag{4-4}$$

In the coordinates (z, σ) we have

$$\lambda^0 \equiv 0 \tag{4-5}$$

since $\sigma = \partial_N u^0 \equiv 0$ on Γ^0. In addition, from (3-8) and the definition of the coordinate σ we have

$$\partial_N v^t = -\frac{\partial_N \tilde{u}^0}{t} = -\frac{\sigma}{t} = -\frac{\lambda^t}{t} \circ \bar{\pi}_1 \quad \text{on } \Gamma^t;$$

hence

$$\frac{\lambda^t}{t}(z) = -\partial_N v^t(z, \lambda^t(z)). \tag{4-6}$$

Indeed to prove (4-6) we use (3-8) and the definition of the coordinate σ to obtain

$$\partial_N v^t = -\frac{\partial_N \tilde{u}^0}{t} = -\frac{\sigma}{t} = -\frac{\lambda^t}{t} \circ \bar{\pi}_1 \quad \text{on } \Gamma^t.$$

The relation (4-6) will allow us to prove uniform $C^{k-1,\alpha}$ estimates for λ^t/t, leading to the existence of the limit as $t \downarrow 0$ of λ^t/t, which will be denoted by $\dot{\lambda}^0$. Later on, we will prove uniform $C^{k-2,\alpha}$ estimates for

$$\frac{1}{2}\,\frac{\lambda^t - \dot{\lambda}^0 t}{t^2} = \frac{1}{2}\,\frac{\lambda^t/t - \dot{\lambda}^0}{t},$$

which will lead to the existence of its limit as $t \to 0$, denoted by $\ddot{\lambda}^0$. These estimates will be deduced from the equation

$$\frac{\lambda^t/t - \dot{\lambda}^0}{t} = -\frac{\partial_N v(z, \lambda^t(z)) - \partial_N v(z, 0)}{t} - \partial_N w^t(z, \lambda^t(z)), \tag{4-7}$$

obtained from (4-6) by subtracting $\dot{\lambda}^0(z) = -\partial_N v(z, 0)$ from both sides, dividing by t on both sides, and recalling that by definition $w^t = (v^t - v)/t$.

Estimate on λ^t/t. The goal of this subsection is to prove a regularity result (without a priori assumptions) on λ^t/t. We state it next.

Proposition 4.1. *For $t \in (0, t_\circ)$ we have*

$$\left\| \frac{\lambda^t}{t} \right\|_{C^{k-1,\alpha}(\mathcal{Z})} \le C(\mathcal{C}).$$

Before proving Proposition 4.1, let us state its main corollary

Corollary 4.2. *There exist $\dot{\eta}^0$ and $\dot{\lambda}^0$ such that*

$$\frac{\eta^t - \eta^0}{t} \to \dot{\eta}^0 \quad \textit{and} \quad \frac{\lambda^t}{t} \to \dot{\lambda}^0 \quad \textit{in } C^0(\mathcal{Z})$$

as $t \downarrow 0$.

Proof. Let $t_p \downarrow 0$. Note that both coordinate systems (z, s) and (z, σ) are $C^{k,\alpha}$. Hence, the estimate in Proposition 4.1 implies

$$\left\| \frac{\eta^t - \eta^0}{t} \right\|_{C^{k-1,\alpha}(\mathcal{Z})} \le C$$

and by Arzelà–Ascoli there is a subsequence t_m such that

$$\frac{\eta^{t_m} - \eta^0}{t_m} \to \ell_1 \quad \text{and} \quad \frac{\lambda^{t_m}}{t_m} \to \ell_2 \quad \text{in } C^0(\mathcal{Z})$$

for certain limit functions ℓ_1 and ℓ_2 in $C^{k-1,\alpha}(\mathcal{Z})$. Applying Proposition 3.2, we must have $\ell_1 = \dot\eta^0$, where $\dot\eta^0$ is the function given by (3-6). Then, either using the change of variables between s and σ or passing to the limit in (4-6) we obtain

$$\ell_2(z) = \dot\lambda^0(z) := \partial_N v(z, \sigma = 0).$$

Therefore, we have proven that each sequence has a subsequence converging to a limit that is independent of the sequence. In other words the limits as $t \downarrow 0$ exist and are given by $\dot\eta^0$ and $\dot\lambda^0$. □

In view of (4-6), Proposition 4.1 is an immediate consequence of the following:

Lemma 4.3. *For $t \in (0, t_\circ)$ we have*

$$\left\| \partial_N v^t(\,\cdot\,, \lambda^t(\,\cdot\,)) - \frac{1}{2}\frac{\lambda^t}{t} \right\|_{C^{k-1,\alpha}(\mathcal{Z})} \le C(\mathcal{C}) + \frac{1}{100}\left\| \frac{\lambda^t}{t} \right\|_{C^{k-1,\alpha}(\mathcal{Z})}.$$

Next we state a sequence of lemmas aimed at proving Lemma 4.3. To study the regularity of $\partial_N v^t$, let us write down (for the first time) the equation for $v^t = \frac{1}{t}(\tilde u^t - \tilde u^0)$ in all of $\mathbb{R}^n$. We have

$$\begin{cases} \Delta v^t = -\frac{\Delta h^0}{t}\chi_{\Omega^0\setminus\Omega^t} + \Delta\delta_t h^0 \chi_{\Omega^t} & \text{in } \mathbb{R}^n, \\ v^t(\infty) = \delta_t c^0 \quad (n \ge 3), \end{cases} \qquad \begin{cases} \Delta v^t = -\frac{\Delta h^0}{t}\chi_{\Omega^0\setminus\Omega^t} + \Delta\delta_t h^0 \chi_{\Omega^t} & \text{in } \mathbb{R}^n, \\ \lim_{x\to\infty} \frac{v^t(x)}{-\log|x|} = \delta_t c^0 \quad (n = 2). \end{cases} \tag{4-8}$$

Hence, we may decompose v^t as

$$v^t = v_1^t + v_2^t + \text{constant},$$

where

$$v_1^t(x) := -\int_{\mathbb{R}^n} dy \left(\frac{\Delta h^0}{t} \chi_{\Omega^0\setminus\Omega^t} \right)(y) P(x-y), \tag{4-9}$$

$$v_2^t(x) := -\int_{\mathbb{R}^n} dy\, \Delta\delta_t h^0 \chi_{\Omega^t}(y) P(x-y). \tag{4-10}$$

To prove Lemma 4.3 we will deal separately with the two contributions $\partial_N v_1$ and $\partial_N v_2$ to $\partial_N v$.

Note that $\partial_N v_1$ is an "approximate single-layer potential". To study its regularity we need the next lemma. Before giving its statement, we need to introduce some notation.

We denote by

$$\bar J(z, \sigma) := |\det D\,(z, \sigma)^{-1}|$$

the Jacobian of the coordinates (z, σ) defined by

$$\int_A f(x)\,dx = \int_{(z,\sigma)(A)} f(z, \sigma) \bar J(z, \sigma)\,dz\,d\sigma. \tag{4-11}$$

Also, for $\theta \in (0, 1)$ we define

$$\Omega_\theta^t := \{x \in U \cap \Omega^0 : \sigma(x) > \theta\lambda^t(z(x))\} \cup (\Omega^0 \setminus U),$$
$$\Gamma_\theta^t := \partial\Omega_\theta^t = \{\sigma = \theta\lambda^t(z)\},$$

and denote by ν_θ^t the unit normal to Γ_θ^t towards Ω_θ^t. Although the following lemma will be used in this subsection for $F \equiv -\Delta h^0$, we write it for general F for later use.

Lemma 4.4. *Let V be the single-layer potential*

$$V(x) = \int_{\mathbb{R}^n} dy \Big(\frac{1}{t} F\, \chi_{\Omega^t \setminus \Omega^0}\Big)(y) P(x-y). \tag{4-12}$$

We may write

$$V = \int_0^1 V^\theta \, d\theta, \tag{4-13}$$

where

$$V^\theta = \int_{\Gamma_\theta^t} \mathcal{H}^{n-1}(y) \Big(F \,\frac{\lambda^t}{t} \circ \bar{\pi}_1 \,\frac{(N \cdot \nu_\theta^t)}{\partial_{ss} u^0}\Big)(y) P(x-y)$$

and for all $\theta \in (0,1)$ we have

$$\|\nabla V^\theta\|_{C^{k-1,\alpha}(\overline{\Omega}^t)} \le C(\mathcal{C}) \Big\| F \,\frac{\lambda^t}{t} \circ \bar{\pi}_1 \Big\|_{C^{k-1,\alpha}(\Gamma_\theta^t)}. \tag{4-14}$$

Before giving the proof of the previous lemma let us give the analogue to Lemma 3.3 in the present context.

Lemma 4.5. *Given $f : \Gamma_\theta^t \to \mathbb{R}$ continuous we have*

$$\int_{\Gamma_\theta^t} \frac{(N \cdot \nu_\theta^t)}{\partial \sigma / \partial s}(x)\, f(x)\, d\mathcal{H}^{n-1}(x) = \int_Z f(z, \theta\lambda^t(z))\, \bar{J}(z, \theta\lambda^t(z))\, dz.$$

Proof. Let us assume without loss of generality that f is continuously extended in a neighborhood of Γ_0^t contained in $U_\circ \cap \overline{\Omega}^0$. Given $\varepsilon > 0$ let

$$A^\varepsilon := \{x \in U_\circ : \theta\lambda^t(z(x)) \le \sigma(x) \le \theta\lambda^t(z(x)) + \varepsilon\}.$$

Recalling that $(\partial\sigma/\partial s)\partial_\sigma = N = \partial_s$ and that $|N| = 1$, we have

$$\int_{\Gamma^0} \frac{(N \cdot \nu_\theta^t)}{\partial \sigma / \partial s}(x)\, f(x)\, d\mathcal{H}^{n-1}(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{A^\varepsilon} f(x)\, d\mathcal{H}^n(x).$$

On the other hand, for

$$(z, \sigma)(A^\varepsilon) := \{(z, \sigma) \in \mathcal{Z} \times (-\sigma_\circ, \sigma_\circ) : \theta\lambda^t(z) \le \sigma \le \theta\lambda^t(z) + \varepsilon\}$$

we have, by the definition of $\bar{J}$,

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{A^\varepsilon} f(x)\, d\mathcal{H}^n(x) &= \frac{1}{\varepsilon} \int_{(z,s)(A^\varepsilon)} f(z,s)\, \bar{J}(z,s)\, dz\, ds \\
&= \int_{\mathcal{Z}} dz\, \frac{1}{\varepsilon} \int_0^\varepsilon d\bar{s}\, f(z, \theta\lambda^t(z) + \bar{s})\, \bar{J}(z, \theta\lambda^t(z) + \bar{s}) \\
&= \int_{\mathcal{Z}} f(z, \theta\lambda^t(z))\, \bar{J}(z, \theta\lambda^t(z))\, dz + o(1)
\end{aligned}$$

as $\varepsilon \downarrow 0$ and the lemma follows. □

Proof of Lemma 4.4. The key idea in the proof is to think of an approximate single-layer potential as an average (or integral) of exact single-layer potentials. More precisely, using (4-11) we may write

$$\begin{aligned}\int \phi \Delta V &:= \frac{1}{t}\int_{\mathcal{Z}} dz \int_0^{\lambda^t(z)} d\sigma\, (F\phi \bar{J})(z,\sigma) \\ &= \int_{\mathcal{Z}} dz\, \frac{1}{t}\int_0^1 d\theta\, \lambda^t(z)\, (F\phi\bar{J})(z,\theta\lambda^t(z)) \\ &= \int_0^1 d\theta \int_{\{\sigma=\theta\lambda^t(z)\}} \frac{\lambda^t}{t}(F\phi)(y)\, \frac{(N\cdot \nu_\theta^t)(y)}{\partial\sigma/\partial s(y)}\, d\mathcal{H}^{n-1}(y),\end{aligned}$$

where we used Lemma 4.5.

Recalling that $\sigma = \partial_s u^0$, this proves (4-13).

To prove (4-14) we use that V^θ is a single-layer potential on the surface Γ_θ^t, with charge density

$$\Big(\frac{\lambda^t}{t}\circ \bar{\pi}_1\Big) F \frac{(N\cdot \nu_\theta^t)}{u_{ss}^0}.$$

Note that Proposition 2.1 yields $\|\lambda^t\|_{C^{k,\alpha}(\mathcal{Z})} \le C$ and hence $\{\sigma = \theta\lambda^t(x)\}$ is $C^{k,\alpha}$ and its normal vector ν_θ^t is $C^{k-1,\alpha}$. Recall also that $u^0 \in C^{k+1,\alpha}(\overline{\Omega}^0)$ and that $u_{ss}^0 \approx -(N\cdot\nu^0)^2\Delta h^0 > 0$ in a neighborhood of Γ^0. Then, if

$$F\,\frac{\lambda^t}{t}\circ\bar{\pi}_1 \in C^{k-1,\alpha},$$

it follows from Theorem A.1 that V^θ is $C^{k,\alpha}(\overline{\Omega}_\theta^t)$ and in particular V^θ is $C^{k,\alpha}(\overline{\Omega}^t)$ with the estimate (4-14). □

Recalling (4-9), and using Lemma 4.4 with $F = -\Delta h^0$, we may now write

$$v_1^t(x) = \int_0^1 V^\theta(x)\, d\theta, \tag{4-15}$$

where

$$V^\theta(x) := \int_{\Gamma_\theta^t} \Big(-\Delta h^0\, \frac{\lambda^t}{t}\circ\bar{\pi}_1\, \frac{(N\cdot\nu_\theta^t)}{\partial_{ss}u^0}\Big)(y) P(x-y)\, dy. \tag{4-16}$$

The following lemma is a straightforward consequence of Theorem A.1 in the Appendix.

Lemma 4.6. *Let V^θ be as in* (4-16). *We have*

$$-\partial_{\nu_\theta^t,\mathrm{out}} V^\theta = \tfrac{1}{2}(N\cdot\nu_\theta^t)\frac{-\Delta h^0}{\partial_{ss}u^0}\,\frac{\lambda^t}{t}\circ\bar{\pi}_1 + \partial_{\nu_\theta^t,0}V^\theta \quad \text{on } \Gamma_\theta^t, \tag{4-17}$$

where

$$\|\partial_{\nu_\theta^t,0}V^\theta\|_{C^{k-1,\alpha}(\Gamma_\theta^t)} \le C(\boldsymbol{\mathcal{C}})\Big\|\frac{\lambda^t}{t}\Big\|_{C^{k-2,\alpha}(\mathcal{Z})}.$$

Proof. We recall that $(N\cdot\nu_\theta^t)$, $-\Delta h^0$, $\partial_{ss}u^0 > \boldsymbol{\rho}/2 > 0$, and $\bar{\pi}_1^{-1} : \mathcal{Z}\to\Gamma_\theta^t$, $-\Delta h^0$ are all $C^{k-1,\alpha}$ functions. Then, the lemma follows from Theorem A.1(ii)–(iii). □

The next lemma will be used to control the "difference"

$$-\partial_N V^\theta(z, \eta^t(z)) - \frac{1}{2}\frac{\lambda^t}{t}(z).$$

Lemma 4.7. *Let V^θ be as in* (4-16). *We have*

$$\Big\| -\partial_N V^\theta(\,\cdot\,, \eta^t(\,\cdot\,)) - \frac{1}{2}\frac{\lambda^t}{t}\Big\|_{C^{k-1,\alpha}(\mathcal{Z})} \le C(\boldsymbol{\mathcal{C}}) + \frac{1}{100}\Big\|\frac{\lambda^t}{t}\Big\|_{C^{k-1,\alpha}(\mathcal{Z})}.$$

Proof. Step 1. We estimate the $C^{k-1,\alpha}(\mathcal{Z})$ norm of

$$I_1(z) := \partial_N V^\theta(z, \lambda^t(z)) - \partial_{N,\mathrm{out}} V^\theta(z, \theta\lambda^t(z)).$$

To do it we write this difference as

$$I_1 = t\int_\theta^1 d\bar\theta\, \partial_\sigma \partial_N V^s(x', \bar\theta\lambda^t(x'))\, \frac{\lambda^t}{t}(x').$$

Then, using Lemma 4.4 we obtain

$$\begin{aligned}\|I_1\|_{C^{k-1,\alpha}(\mathcal{Z})} &\le Ct\Big(\|V^\theta\|_{C^{k+1,\alpha}(\Omega^t_\theta)}\Big\|\frac{\lambda^t}{t}\Big\|_{L^\infty(\mathcal{Z})} + \|V^\theta\|_{L^\infty(\Omega^t_\theta)}\Big\|\frac{\lambda^t}{t}\Big\|_{C^{k-1,\alpha}(\mathcal{Z})}\Big)\\ &\le Ct\Big\|\frac{\lambda^t}{t}\Big\|_{C^{k,\alpha}(\mathcal{Z})}\Big\|\frac{\lambda^t}{t}\Big\|_{L^\infty(\mathcal{Z})} \le C\|\lambda^t\|_{C^{k,\alpha}(\mathcal{Z})}\Big\|\frac{\lambda^t}{t}\Big\|_{L^\infty(\mathcal{Z})} \le C, \end{aligned} \tag{4-18}$$

where $C = C(\boldsymbol{\mathcal{C}})$. Here we have used the fact that $\|\lambda^t/t\|_{L^\infty(\mathcal{Z})} \le C$ and information that follows from Proposition 2.1.

Step 2. We next estimate the $C^{k-1,\alpha}(\mathcal{Z})$ norm of

$$I_2(z) := \partial_{N,\mathrm{out}} V^\theta(z, \theta\lambda^t(z)) - \frac{1}{2}\frac{\lambda^t}{t}(z).$$

Using (4-17) we have

$$I_2(z) = (N - \nu^t_\theta)\cdot\nabla_{\mathrm{out}} V^\theta(z, \theta\lambda^t(z)) + \frac{1}{2}\Big((N\cdot\nu^t_\theta)\frac{-\Delta h^0}{\partial_{ss}u^0} - 1\Big)\frac{\lambda^t}{t}\circ\bar\pi_1 + \partial_{\nu^t_\theta,0} V^\theta.$$

Using the estimates from Lemma 4.6 and 4.4 we have

$$\|\nabla V^\theta\|_{C^{k-1,\alpha}(\Gamma^t_\theta)} \le \|V^\theta\|_{C^{k,\alpha}(\bar\Omega^t_\theta)} \le C\Big\|\frac{\lambda^t}{t}\Big\|_{C^{k-1,\alpha}(\mathcal{Z})}.$$

In addition,

$$|N - \nu^t_\theta| \approx 0, \quad (N\cdot\nu^t_\theta) \approx 1, \quad \text{and} \quad \frac{-\Delta h^0}{\partial_{ss}u^0} \approx 1 \quad \text{on } \Gamma^t_\theta$$

for $t \in (0, t_\circ)$, where $X \approx Y$ means that "X is arbitrarily close to Y" provided that $t_\circ$ and ε_o are chosen small enough depending only of $\boldsymbol{\mathcal{C}}$.

Therefore, using the estimate in Lemma 4.6 and an interpolation inequality, we obtain

$$\begin{aligned}\|I_2\|_{C^{k-1,\alpha}(\mathcal{Z})} &\le \frac{\varepsilon}{2}\Big\|\frac{\lambda^t}{t}\Big\|_{C^{k-1,\alpha}(\mathcal{Z})} + C\Big\|\frac{\lambda^t}{t}\Big\|_{C^{k-2,\alpha}(\mathcal{Z})}\\ &\le \varepsilon\Big\|\frac{\lambda^t}{t}\Big\|_{C^{k-1,\alpha}(\mathcal{Z})} + C\Big\|\frac{\lambda^t}{t}\Big\|_{L^\infty(\mathcal{Z})} \le \varepsilon\Big\|\frac{\lambda^t}{t}\Big\|_{C^{k-1,\alpha}(\mathcal{Z})} + C, \end{aligned} \tag{4-19}$$

where $\varepsilon > 0$ can be taken arbitrarily small by decreasing, if necessary, $t_\circ$ and ε_o.

Step 3. We conclude by the triangle inequality that

$$\left\| \partial_N V^\theta(\,\cdot\,, \eta^t(\,\cdot\,)) - \frac{1}{2}\frac{\lambda^t}{t} \right\|_{C^{k-1,\alpha}(\mathcal{Z})} \le \|I_1\|_{C^{k-1,\alpha}(\mathcal{Z})} + \|I_2\|_{C^{k-1,\alpha}(\mathcal{Z})}$$

and the lemma follows from (4-18) and (4-19), setting $\varepsilon = \frac{1}{100}$. □

Lemmas 4.4, 4.6, and 4.7 will be used to treat the term $\partial_N v_1$. As a counterpart, the next lemma will be used to treat the term $\partial_N v_2$.

Lemma 4.8. *We have*

$$\|v_2^t\|_{C^{k,\alpha}(\bar{\Omega}^t)} \le C(\mathcal{C}).$$

Proof. Recalling that $\Delta v_2^t = \Delta\delta_t h^0 \chi_{\Omega^t}$ and that $\Gamma^t = \partial\Omega^t$ are (uniformly) $C^{k,\alpha}$, it follows from Lemma 3.7 that

$$\|v_2^t\|_{C^{k,\alpha}(\bar{\Omega}^t)} \le C(\mathcal{C})\|\Delta\delta_t h^0\|_{C^{k-2,\alpha}(\mathbb{R}^n)}.$$

Using the trivial estimate

$$\|\Delta\delta_t h^0\|_{C^{k-2,\alpha}(\mathbb{R}^n)} \le \|h\|_{C^{k+1,\alpha}([-1,1]\times\mathbb{R}^n)}$$

the lemma follows. □

Proof of Lemma 4.3. We have

$$\partial_N v^t(z, \lambda^t(z)) = (\partial_N v_1^t + \partial_N v_2^t)(z, \lambda^t(z)),$$

and by (4-15)–(4-16) we have

$$\partial_N v_1^t(z, \lambda^t(z)) = \int_0^1 \partial_N V^\theta(z, \lambda^t(z))\, d\theta.$$

Hence, by the triangle inequality, and using Lemmas 4.7 and 4.8,

$$\begin{aligned}\left\| \partial_N v^t(\,\cdot\,, \lambda^t) - \frac{1}{2}\frac{\lambda^t}{t}(z) \right\|_{C^{k-1,\alpha}(\mathcal{Z})} &\le \int_0^1 d\theta \left\| \partial_N V^\theta(\,\cdot\,, \lambda^t) - \frac{1}{2}\frac{\lambda^t}{t} \right\|_{C^{k-1,\alpha}(\mathcal{Z})} + \|\partial_N v_2^t(\,\cdot\,, \lambda^t)\|_{C^{k-1,\alpha}(\mathcal{Z})} \\ &\le C + \frac{1}{100}\left\| \frac{\lambda^t}{t} \right\|_{C^{k-1,\alpha}(\mathcal{Z})} + C\|\partial_N v_2^t\|_{C^{k-1,\alpha}(\Gamma^t)} \\ &\le C + \frac{1}{100}\left\| \frac{\lambda^t}{t} \right\|_{C^{k-1,\alpha}(\mathcal{Z})},\end{aligned}$$

where $C = C(\mathcal{C})$. □

Proof of Proposition 4.1. Recall (4-6), that is, $(\lambda^t/t)(z) = -\partial_N v^t(z, \lambda^t(z))$. Subtracting $\frac{1}{2}(\lambda^t/t)(z)$ from both sides and using Lemma 4.3 we obtain

$$\frac{1}{2}\left\| \frac{\lambda^t}{t} \right\|_{C^{k-1,\alpha}(\mathcal{Z})} \le \left\| -\partial_N v^t(\,\cdot\,, \lambda^t(\,\cdot\,)) - \frac{1}{2}\frac{\lambda^t}{t} \right\|_{C^{k-1,\alpha}(\mathcal{Z})} \le C(\mathcal{C}) + \frac{1}{100}\left\| \frac{\lambda^t}{t} \right\|_{C^{k-1,\alpha}(\mathcal{Z})}$$

as desired. □

Estimate on $\frac{1}{t}(\lambda^t/t-\dot\lambda^0)$**.** The goal of this subsection is to prove the following regularity result (without a priori assumptions).

Proposition 4.9. *We have*

$$\left\|\frac{1}{t}\Big(\frac{\lambda^t}{t}-\dot\lambda^0\Big)\right\|_{C^{k-2,\alpha}(\mathcal{Z})}\le C(\mathcal{C}).$$

Before proving Proposition 4.9, let us give its main corollary.

Corollary 4.10. *There exist* $\ddot\eta^0$ *and* $\ddot\lambda^0$ *such that*

$$2\frac{\eta^t-\eta^0-t\dot\eta^0}{t^2}\to\ddot\eta^0 \quad\textit{and}\quad 2\frac{\lambda^t-t\dot\lambda^0}{t^2}\to\ddot\lambda^0 \quad \textit{in } C^0(\mathcal{Z})$$

as $t\downarrow 0$.

Proof. Let $t_p\downarrow 0$. Note that since both coordinate systems (z,s) and (z,σ) are $C^{k,\alpha}$, the estimate of Proposition 4.9 yields

$$\left\|\frac{\eta^t-\eta^0-t\dot\eta^0}{t^2}\right\|_{C^{k-2,\alpha}(\mathcal{Z})}\le C(\mathcal{C}).$$

Hence, by Arzelà–Ascoli there is a subsequence t_m such that

$$2\frac{\eta^{t_m}-\eta^0-t_m\dot\eta^0}{t_m{}^2}\to\ell_1 \quad\text{and}\quad 2\frac{\lambda^{t_m}-t_m\dot\lambda^0}{t_m{}^2}\to\ell_2 \quad\text{in } C^0(\mathcal{Z})$$

for certain limit functions ℓ_1 and ℓ_2 in $C^{k-2,\alpha}(\mathcal{Z})$.

Applying Proposition 3.5 the limit ℓ_1 must be $\ddot\eta^0$, the unique solution to (3-20)–(3-29). Using the change of variables between s and σ we obtain that there is also a unique possible limit $\ell_2(z)=\dot\lambda^0(z)$ which is independent of the subsequence.

In other words, the limits as $t\downarrow 0$ exist and they are denoted by $\ddot\eta^0$ and $\ddot\lambda^0$. □

In view of (4-7) and the regularity of $\partial_N v$, Proposition 4.9 is a consequence of the following:

Lemma 4.11. *We have*

$$\left\|\partial_N w^t(\,\cdot\,,\lambda^t(\,\cdot\,))-\frac{1}{2}\,\frac{\lambda^t/t-\dot\lambda^0}{t}\right\|_{C^{k-2,\alpha}(\mathcal{Z})}\le C(\mathcal{C})+\frac{1}{100}\left\|\frac{\lambda^t/t-\dot\lambda^0}{t}\right\|_{C^{k-2,\alpha}(\mathcal{Z})}.$$

Let us state a sequence of lemmas which will prove Lemma 4.11. To study the regularity of $\partial_N w^t$ we will use the equation for w^t in all of $\mathbb{R}^n$ that was obtained in (3-14).

As in Step 2 of the proof of Proposition 3.5 we take the decomposition

$$w^t=w_1^t+w_2^t+\text{constant},$$

where, for $n\ge 3$,

$$w_1^t(x)=\int\Big(\frac{1}{t}\Delta\dot h^0\chi_{\Omega^0\setminus\Omega^t}-\tfrac{1}{2}\Delta\delta_t^2h^0\chi_{\Omega^t}\Big)(dy)\,P(x-y),\quad w_1^t=0,$$

and

$$w_2^t(x)=-\int\frac{1}{t}\bigg(\frac{\partial_N v}{N\cdot\nu^0}\,\mathcal{H}^{n-1}\restriction_{\Gamma^0}-\frac{1}{t}\Delta h^0\,\chi_{\Omega^0\setminus\Omega^t}\bigg)(dy)\,P(x-y).$$

Respectively, for $n=2$ we define w_1^t and w_2^t as the potentials of the previous Laplacians.

The analysis of the regularity in $\overline{\Omega}^t$ of w_1^t is done using Lemmas 4.4 and 3.7, which straightforwardly imply:

Lemma 4.12. *We have*

$$\|\nabla w_1^t\|_{C^{k-2,\alpha}(\overline{\Omega}^t)} \le C(\mathcal{C}).$$

To study w_2^t let us further split it as

$$w_2^t = w_{21}^t + w_{22}^t + \text{constant},$$

where

$$w_{21}^t(x) = \int \frac{1}{t\,(N\cdot\nu^0)}\biggl(\partial_N v + (N\cdot\nu^0)^2\frac{\Delta h^0}{\partial\sigma/\partial s}\frac{\lambda^t}{t}\biggr)\mathcal{H}^{n-1}\restriction_{\Gamma^0}(dy)\,P(x-y),$$

$$w_{22}^t(x) = -\int \frac{1}{t}\biggl((N\cdot\nu^0)\frac{\Delta h^0}{\partial\sigma/\partial s}\frac{\lambda^t}{t}\,\mathcal{H}^{n-1}\restriction_{\Gamma^0} - \frac{1}{t}\Delta h^0\,\chi_{\Omega^0\setminus\Omega^t}\biggr)(dy)P(x-y).$$

The study of $\partial_N w_{21}^t$ is done by observing that w_{21}^t is a single-layer potential and using Theorem A.1. Indeed we have:

Lemma 4.13. *We have*

$$\biggl\|\partial_N w_{21}^t(\,\cdot\,,\sigma=0) - \frac{1}{2}\,\frac{\lambda^t/t-\dot\lambda^0}{t}\biggr\|_{C^{k-2,\alpha}(\mathcal{Z})} \le C(\mathcal{C}) + \frac{1}{100}\biggl\|\frac{\lambda^t/t-\dot\lambda^0}{t}\biggr\|_{C^{k-2,\alpha}(\mathcal{Z})}.$$

Proof. Let

$$f(x) := \frac{1}{t\,(N\cdot\nu^0)}\biggl(\partial_N v + (N\cdot\nu^0)^2\frac{\Delta h^0}{\partial\sigma/\partial s}\frac{\lambda^t}{t}\biggr)(x) = \frac{1}{t\,(N\cdot\nu^0)}\biggl(\dot\lambda^0 - \frac{\lambda^t}{t}\biggr)(x)$$

for $x\in\Gamma^0$. Here we have used that $\partial_N v = -\dot\lambda\circ\bar\pi_1$ and (4-3).

On one hand, by Theorem A.1(iii) we have

$$\partial_{\nu^0,\mathrm{out}} w_{21}^t = \tfrac{1}{2}f + \partial_{\nu^0,0} w_{21}^t \quad \text{on } \Gamma^0,$$

with

$$\|\partial_{\nu^0,0} w_{21}^t\|_{C^{k-2,\alpha}(\Gamma^0)} \le C\|f\|_{C^{k-3,\alpha}(\Gamma^0)}$$

and

$$\|w_{21}^t\|_{C^{k-1,\alpha}(\overline{\Omega}^0)} \le C\|f\|_{C^{k-2,\alpha}(\Gamma^0)} \quad (n\ge 3),$$
$$\|\nabla w_{21}^t\|_{C^{k-2,\alpha}(\overline{\Omega}^0)} \le C\|f\|_{C^{k-2,\alpha}(\Gamma^0)} \quad (n=2),$$

where $C=C(\mathcal{C})$. Therefore, using that $|N-\nu^0|\le\varepsilon$ we have

$$\begin{aligned}\bigl\|\partial_N w_{21}^t - \tfrac{1}{2}f\bigr\|_{C^{k-2,\alpha}(\Gamma^0)} &\le C\varepsilon\|w_{21}^t\|_{C^{k-1,\alpha}(\overline{\Omega}^0)} + \|\partial_{\nu^0,0}w_{21}^t\|_{C^{k-2,\alpha}(\Gamma^0)}\\ &\le C\varepsilon\|f\|_{C^{k-2,\alpha}(\Gamma^0)} + C\|f\|_{C^{k-3,\alpha}(\Gamma^0)},\end{aligned}$$

and the lemma follows using interpolation and choosing ε small enough. □

It thus remains to study the regularity of w_{22}^t, which we treat as an approximate double layer.

Lemma 4.14. *We have*

$$\|\nabla w_{22}^t\|_{C^{k-2,\alpha}(\bar{\Omega}^t)} \le C(\mathcal{C}).$$

Proof. We will first write our approximate double layer as an average of double layers and we will then use the regularity results for the single layers to deduce the regularity of double layers.

Let us compute

$$\begin{aligned}
-\int \phi\Delta w_{22}^t &= \int \phi(x)\frac{1}{t}\Big((N\cdot\nu^0)\frac{\Delta h^0}{\partial\sigma/\partial s}\frac{\lambda^t}{t}\,\mathcal{H}^{n-1}\restriction_{\Gamma^0} - \frac{1}{t}\Delta h^0\,\chi_{\Omega^0\setminus\Omega^t}\Big)(x)\,dx\\
&= \frac{1}{t}\int_{\mathcal{Z}} dz\,\frac{\lambda^t}{t}(z)(\bar{J}\Delta h^0\,\phi)(z,0) - \frac{1}{t^2}\int_Z dz\int_0^{\lambda^t(z)} d\sigma\,(\bar{J}\Delta h^0\,\phi)(z,\sigma)\\
&= \int_0^1 d\theta\int_{\mathcal{Z}} dz\,\frac{\lambda^t}{t}(z)\,\frac{1}{t}\Big((\bar{J}\Delta h^0\phi)(z,0) - (\bar{J}\Delta h^0\,\phi)(z,\theta\lambda^t)\Big)\\
&= -\int_0^1 d\theta\int_0^\theta d\theta'\int_{\mathcal{Z}}\Big(\frac{\lambda^t}{t}\Big)^2(z)\,dz\,\partial_\sigma(\bar{J}\Delta h^0\,\phi)(z,\theta'\lambda^t)\\
&= -\int_0^1 d\theta\int_0^\theta d\theta'\int_{\Gamma^t_{\theta'}}\Big(\frac{\lambda^t}{t}\Big)^2\circ\bar{\pi}_1\partial_\sigma(\bar{J}\Delta h^0\,\phi)\frac{(N\cdot\nu^t_{\theta'})}{\partial\sigma/\partial s},
\end{aligned}$$

where we have used Lemma 4.5. Changing the order of integration we find

$$\begin{aligned}
-\int \phi\Delta w_{22}^t &= -\int_0^1(1-\theta)\,d\theta\,\int_{\Gamma^t_\theta}\Big(\frac{\lambda^t}{t}\Big)^2\circ\bar{\pi}_1\partial_\sigma(\bar{J}\Delta h^0\,\phi)\frac{(N\cdot\nu^t_\theta)}{\partial\sigma/\partial s}\\
&= -\int_0^1(1-\theta)\,d\theta\,\int_{\Gamma^t_\theta}\Big(\frac{\lambda^t}{t}\Big)^2\circ\bar{\pi}_1\partial_s(\bar{J}\Delta h^0\,\phi)\frac{(N\cdot\nu^t_\theta)}{(\partial\sigma/\partial s)^2}.
\end{aligned}$$

Therefore, we have

$$w_{22}^t(x) = -\int_0^1(1-\theta)\,d\theta\,I_\theta(x) \tag{4-20}$$

for

$$I_\theta(x) := -\int_{\Gamma^t_\theta} d\mathcal{H}^{n-1}(y)\,\Big(\frac{\lambda^t}{t}\Big)^2\circ\bar{\pi}_1(y)\,\partial_N\big((\bar{J}\Delta h^0)(y)\,P(x-y)\big)\frac{(N\cdot\nu^t_\theta)}{(\partial\sigma/\partial s)^2}(y).$$

Note that

$$\begin{aligned}
I_\theta(x) = I_1^\theta(x)+I_2^\theta(x) =: &\int_{\Gamma^t_\theta} d\mathcal{H}^{n-1}(y)\,\Big(\Big(\frac{\lambda^t}{t}\Big)^2\circ\bar{\pi}_1\,\partial_N(\bar{J}\Delta h^0)\,\frac{(N\cdot\nu^t_\theta)}{(\partial\sigma/\partial s)^2}\Big)(y)\,P(x-y)\\
&+\operatorname{div}_x\Big(\int_{\Gamma^t_\theta} d\mathcal{H}^{n-1}(y)\,\Big(\Big(\frac{\lambda^t}{t}\Big)^2\circ\bar{\pi}_1(y)\,(\bar{J}\Delta h^0)\frac{(N\cdot\nu^t_\theta)}{(\partial\sigma/\partial s)^2}\,N\Big)(y)P(x-y)\Big).
\end{aligned}$$

Therefore, recalling that

$$\Gamma^t_\theta\in C^{k,\alpha},\quad \frac{\lambda^t}{t}\in C^{k-1,\alpha}(\mathcal{Z}),\quad \bar{\pi}_1\in C^{k,\alpha}(\Gamma^t_\theta),\quad \nu^t_\theta\in C^{k-1,\alpha}(\Gamma^t_\theta),\quad J\Delta h^0\in C^{k-1,\alpha},$$

and $\partial\sigma/\partial s = u^0_{ss}$ positive and $C^{k-1,\alpha}$ and using Theorem A.1 we obtain

$$\|\nabla I_1\|_{C^{k-2,\alpha}(\overline{\Omega}^t_\theta)} + \|\nabla I_2\|_{C^{k-2,\alpha}(\overline{\Omega}^t_\theta)} \le C(\mathcal{C}).$$

The estimate of the lemma then follows from (4-20) observing that $\overline{\Omega}^t \subset \overline{\Omega}^t_\theta$ for all $\theta \in (0, 1)$. □

Lemma 4.11 is now an immediate consequence of Lemmas 4.12, 4.13, and 4.14, and Proposition 4.9 follows.

5. Proof of the main result

In this section we conclude the proof of Theorem 1.1. If one assumes $h^{\tau+t} - h^\tau$ satisfies $\Delta(h^{\tau+t} - h^\tau) \ge 0$ and $c^{\tau+t} - c^\tau \le 0$ for $\tau, t \in (0, t_\circ)$ then Theorem 1.1 is a straightforward consequence of the results developed in Sections 2–5. Hence, the main issue that needs to be addressed is how to remove these technical sign assumptions. This is done by using a decomposition of the form

$$h^t - h^0 = \xi^t_+ + \xi^t_-, \tag{5-1}$$

where $\Delta(\xi^{\tau+t}_+ - \xi^\tau_+) \ge 0$ and $\lim_{x\to\infty}(\xi^{\tau+t}_+ - \xi^\tau_+) \ge 0$ and the same with ξ_+ replaced by ξ_- and $\ge$ replaced by $\le$. This decomposition is defined as follows. We let

$$\phi_+(z) := 1 + \frac{1 + ze^z}{e^z + e^{-z}} \quad \text{and} \quad \phi_-(z) = -1 + \frac{-1 + ze^{-z}}{e^z + e^{-z}}$$

and note that

$$\phi_+ + \phi_- = z \tag{5-2}$$

and that ϕ_+ is similar to x^+ (the positive part), while ϕ_- is similar to $-x^-$ (minus the negative part) at large scales.

Let ζ be a radial smooth cutoff function with $\zeta \equiv 1$ in $B_{\boldsymbol{R}}$ and $\zeta \equiv 0$ outside of $B_{2\boldsymbol{R}}$. For $t \in (-t_\circ, t_\circ)$ and $x \in \mathbb{R}^n$ let us define

$$\begin{aligned}
\xi^t_+(x) &:= -\int_{\mathbb{R}^n} P(x-y)\, t\phi_+\Big(\frac{1}{t}\Delta(h^t - h^0)\Big)\zeta\,(y),\\
\xi^t_-(x) &:= -\int_{\mathbb{R}^n} P(x-y)\, t\phi_-\Big(\frac{1}{t}\Delta(h^t - h^0)\Big)\zeta\,(y)
\end{aligned}$$

Note that by definition we have, for τ and t small,

$$\begin{aligned}
\Delta(\xi^{\tau+t}_+ - \xi^\tau_+) &= \Big((\tau+t)\phi_+\Big(\frac{1}{\tau+t}\Delta(h^{\tau+t} - h^0)\Big) - \tau\phi_+\Big(\frac{1}{\tau}\Delta(h^\tau - h^0)\Big)\Big)\zeta\\
&= \Big(\frac{d}{dt'}\Big|_{t'=\tau}\{t'\phi_+(\Delta\delta_{t'}h^0)\}\,t + O(t^{1+\alpha})\Big)\zeta\\
&= \Big(\phi_+(\Delta\delta_\tau h^0)t + \tau\dot{\phi}_+(\Delta\delta_\tau h^0)\,\frac{d}{dt'}\Big|_{t'=\tau}(\Delta\delta_{t'}h^0)t + O(t^{1+\alpha})\Big)\zeta\\
&= \Big(\phi_+(\Delta\delta_\tau h^0)t + \tau\dot{\phi}_+(\Delta\delta_\tau h^0)\frac{1}{\tau}O(\tau^\alpha) + O(t^{1+\alpha})\Big)\zeta\\
&\ge t(1 - C\tau^\alpha - Ct^\alpha)\zeta \ge 0,
\end{aligned} \tag{5-3}$$

where in the passage from the third to the fourth line we have used that, since $h \in C^{3,\alpha}$,

$$\frac{d}{dt'}\Big|_{t'=\tau}(\Delta\delta_{t'}h^0) = \Delta\frac{d}{dt'}\Big|_{t'=\tau}\left(\frac{h^{t'}-h^0}{t'}\right) = \Delta\left(-\frac{h^\tau - h^0}{\tau^2}+\frac{\dot h^\tau}{\tau}\right)$$
$$= \left(-\frac{\Delta\dot h^\tau + O(\tau^{1+\alpha})}{\tau^2}+\frac{\Delta\dot h^\tau}{\tau}\right) = \frac{1}{\tau}O(\tau^\alpha).$$

A similar inequality (with opposite sign) holds when $+$ is replaced by $-$. Moreover, by (5-2),

$$\Delta(\xi_+^t+\xi_-^t) = t\phi_+\Big(\frac{1}{t}\Delta(h^t-h^0)\Big)\zeta + t\phi_-\Big(\frac{1}{t}\Delta(h^t-h^0)\Big)\zeta = \Delta(h^t-h^0)$$

since $\Delta(h^{s+t}-h^s)=0$ outside of $B_{\boldsymbol{R}}$ and $\zeta = 1$ in $B_{\boldsymbol{R}}$. Therefore (5-1) follows.

Next, for $t, \bar t \in (-t_\circ, t_\circ)$ we consider the two-parameter family of solutions to obstacle problems $u^{t,\bar t}$ defined as

$$\min\{-\Delta u^{t,\bar t}, u^{t,\bar t}-h^{t,\bar t}\} = 0 \quad \text{in } \mathbb{R}^n, \qquad \begin{cases} \lim_{|x|\to\infty} u^{t,\bar t}(x) = c^{t,\bar t} & (n\ge 3),\\ \lim_{|x|\to\infty} \frac{u^{t,\bar t}(x)}{-\log|x|} = c^{t,\bar t} & (n=2),\end{cases} \tag{5-4}$$

where

$$h^{t,\bar t} := h^0 + \xi_+^t + \xi_-^{\bar t}$$

and

$$c^{t,\bar t} := t\phi_-\Big(\frac{1}{t}(c^t-c^0)\Big) + \bar t\phi_+\Big(\frac{1}{\bar t}(c^{\bar t}-c^0)\Big).$$

Note that

$$u^t = u^{t,t} \quad \text{and} \quad \eta^t = \eta^{tt}.$$

Let us define

$$\Omega^{t,\bar t} := \{u^{t,\bar t}-h^{t,\bar t} > 0\} \quad \text{and} \quad \Gamma^{t,\bar t} := \partial\Omega^{t,\bar t}$$

and let $\eta^{t,\bar t} \in C^{k,\alpha}(\mathcal{Z})$ be defined by

$$\Gamma^{t,\bar t} \;=\; \{s = \eta^{t,\bar t}(z)\} \subset U_\circ. \tag{5-5}$$

In the proof of Theorem 1.1 the following observation will be useful.

Remark 5.1. For $\boldsymbol{e} = (e^1, e^2) \in S^1$ making a small enough angle with $(1,0)$ a computation similar to (5-3) shows that

$$\Delta(h^{t+e^1\tilde t,\, \bar t + e^2\tilde t} - h^{t,\bar t}) \ge 0 \quad \text{and} \quad c^{t+e^1\tilde t,\, \bar t+e^2\tilde t} - c^{t,\bar t} \le 0 \tag{5-6}$$

for $(t,\tilde t)$ in a small neighborhood of $(0,0)$. Thanks to this observation, the results developed in Sections 2–5 can be applied to obtain, in a neighborhood of (0,0), estimates for the derivatives of $u^{t,\bar t}$ and $\eta^{t,\bar t}$ in a cone of directions $(t,\bar t)$. As a consequence, we obtain estimates for all the first and second derivatives ∂_t, $\partial_{\bar t}$, ∂_{tt}, $\partial_{\bar t\bar t}$, $\partial_{t\bar t}$ of $u^{t,\bar t}$ and $\eta^{t,\bar t}$ in a neighborhood of $(0,0)$. In particular we obtain estimates in the direction $(1,1)$ which are equivalent to estimates for $u^t = u^{t,t}$ and $\eta^t = \eta^{t,t}$.

Proof of Theorem 1.1. Step 1. Assuming that $k \geq 1$ we prove that $\eta^{t,\bar t}$ is once differentiable (jointly) in the two variables $(t, \bar t)$ in a neighborhood of $(0, 0)$ with the estimate

$$\|\partial_{\boldsymbol e}\eta^{t,\bar t}\|_{C^{k-1,\alpha}(\mathcal Z)} \leq |\boldsymbol e| C(\boldsymbol{\mathcal C}) \tag{5-7}$$

and the formula

$$\partial_{\boldsymbol e}\eta^{t,\bar t} = \left(\frac{\partial_N(\partial_{\boldsymbol e} u^{t,\bar t})}{(N\cdot\nu^{t,\bar t})^2\,\Delta h^{t,\bar t}}\right)(z, \eta^{t,\bar t}(x)), \tag{5-8}$$

which holds true for every vector $\boldsymbol e$ in the $(t, \bar t)$-plane.

Indeed, let $\boldsymbol e_1 = (1, 0)$ and let $\boldsymbol e_2$ be some different unit vector making a small enough angle with $\boldsymbol e_1$ as in Remark 5.1.

By Remark 5.1, for fixed $(t, \bar t)$ in a small enough neighborhood of $(0, 0)$ and for $i = 1, 2$, the one parameter family $(u^{t+e_i^1\tilde t,\,\bar t+e_i^2\tilde t})_{\tilde t}$ satisfies the assumptions of Sections 2–4. Applying Corollary 4.2 to it, we find that

$$\partial_{\boldsymbol e_i}\eta^{t,\bar t} := \frac{d}{d\tilde t}\bigg|_{\tilde t=0} \eta^{t+e_i^1\tilde t,\,\bar t+e_i^2\tilde t}$$

exists in the sense that the limit defining this derivative exists in $C^0(\mathcal Z)$.

Then, Proposition 3.2 yields the estimate

$$\|\partial_{\boldsymbol e_i}\eta^{t,\bar t}\|_{C^{k-1,\alpha}(\mathcal Z)} \leq C(\boldsymbol{\mathcal C})$$

and the formula

$$\partial_{\boldsymbol e_i}\eta^{t,\bar t} = \left(\frac{\partial_N\partial_{\boldsymbol e_i}(u^{t,\bar t} - h^{t,\bar t})}{(N\cdot\nu^{t,\bar t})^2\,\Delta h^{t,\bar t}}\right)(z, \eta^{t,\bar t}(x)).$$

Since $\partial_t = \partial_{\boldsymbol e_1}$ and $\partial_{\bar t}$ is a linear combination of $\partial_{\boldsymbol e_i}$ we obtain that $\eta^{t,\bar t}$ is continuously differentiable (jointly) in the two variables $(t, \bar t)$ in a neighborhood of $(0, 0)$ with the estimate (5-7) and formula (5-8).

Step 2. Applying (5-7) and formula (5-8) for $(t, \bar t)$ restricted to the "diagonal" $t = \bar t$ (still in a neighborhood of $(0, 0)$) — i.e., with $\boldsymbol e = (1, 1)$ — we obtain that η^t is differentiable with respect to t, with the estimate

$$\|\dot\eta^t\|_{C^{k-1,\alpha}(\mathcal Z)} \leq C(\boldsymbol{\mathcal C}) \tag{5-9}$$

and the formula

$$\dot\eta^t = \left(\frac{\partial_N(\dot u^t - \dot h^t)}{(N\cdot\nu^t)^2\,\Delta h^t}\right)(z, \eta^t(x)). \tag{5-10}$$

Note that (5-9) and (5-10) are identical to those of Proposition 3.2 but now they are valid under more general assumptions (we do not need to assume the sign condition that implies that the contact sets are ordered).

Step 3. Similarly we obtain

$$\|\partial_{\boldsymbol{ee}}\eta^{t,\bar t}\|_{C^{k-1,\alpha}(\mathcal Z)} \leq |\boldsymbol e|^2 C(\boldsymbol{\mathcal C}). \tag{5-11}$$

Indeed, let $\boldsymbol e_1$ and $\boldsymbol e_2$ be as in Step 1 and let $\boldsymbol e_3$ be a third vector such that the $\boldsymbol e_i$ are pairwise linearly independent and the angle of $\boldsymbol e_3$ with $(1, 0)$ is small enough.

Using again Remark 5.1, for fixed $(t, \bar t)$ in a small enough neighborhood of $(0, 0)$ and for $i = 1, 2, 3$, the one parameter family $(u^{t+e_i^1\tilde t,\, \bar t+e_i^2\tilde t})_{\tilde t}$ satisfies the assumptions of Sections 2–4. Applying Corollary 4.10 we find that

$$\partial_{\boldsymbol{e}_i\boldsymbol{e}_i}\eta^{t,\bar t} := \frac{d^2}{d\tilde t^2}\bigg|_{\tilde t=0} \eta^{t+e_i^1\tilde t,\, \bar t+e_i^2\tilde t}$$

exists in the sense that the limit defining this derivative exists in $C^0(\mathcal{Z})$.

Then, Proposition 3.2 yields the estimate

$$\|\partial_{\boldsymbol{e}_i\boldsymbol{e}_i}\eta^{t,\bar t}\|_{C^{k-1,\alpha}(\mathcal{Z})} \le C(\boldsymbol{C}).$$

Since for all $\boldsymbol{e}$ in the $(t, \bar t)$-plane $\partial_{\boldsymbol{ee}}$ is a linear combination of $\{\partial_{\boldsymbol{e}_i\boldsymbol{e}_i}\}_{i=1,2,3}$, we obtain that $\eta^{t,\bar t}$ is twice differentiable (jointly) in the two variables $(t, \bar t)$ in a neighborhood of $(0, 0)$ with the estimate (5-11).

Step 4. Applying (5-11) or $(t, \bar t)$ restricted to the "diagonal" $t = \bar t$ (still in a neighborhood of $(0, 0)$) — i.e., with $\boldsymbol{e} = (1, 1)$ — we obtain that η^t is twice differentiable with respect to t, with the estimate

$$\|\ddot\eta^t\|_{C^{k-1,\alpha}(\mathcal{Z})} \le C(\boldsymbol{C}). \tag{5-12}$$

Again note that (5-11) is identical to that of Proposition 3.5 but now it is valid under more general assumptions.

Step 5. Finally, we complete the proof of Theorem 1.1 by defining the diffeomorphisms Ψ^t from the coordinates (z, s) and the function η^t. Let $\phi \in C_c^\infty(U_\circ)$ be some function such that $\phi \equiv 1$ in a neighborhood of Γ^0. Let us define

$$\Psi^t(x) = \begin{cases} (z, s)^{-1}\big(z(x), s(x) + \eta^0(z(x)) + \phi(x)\{\eta^t(z(x)) - \eta^0(z(x))\}\big), & x \in U_\circ, \\ x, & x \in \mathbb{R}^n \setminus U_\circ. \end{cases}$$

Since we may take $U_\circ \subset \boldsymbol{U}$ we have that Ψ^t fixes the complement of $\boldsymbol{U}$. By the definition of η^t we easily show that $\Psi^t(\omega^0) = \Omega^t$ — and thus $\Psi^t(\Gamma^0) = \Gamma^t$.

It not difficult to check that (5-9), (5-10), and (5-12) yield (1-14), (1-15) and (1-16) when rewritten in terms of Ψ. On the other hand, estimate (1-17) follows from the estimates for w obtained in Step 3 of the proof of Proposition 3.5. □

Appendix: Single-layer potentials and auxiliary proofs

We recall here classical regularity properties and the formula for the jump in the normal derivative for a single-layer potential.

Theorem A.1. *Let $U \subset B_R \subset \mathbb{R}^n$ be a domain such that $\partial U \in C_r^{m,\alpha}$ for some $r > 0$, $m \in \mathbb{N}$ and $\alpha \in (0, 1)$. Given $f \in C^{m-1,\alpha}(\partial U)$ let us define*

$$w(x) := \int_{\partial U} d\mathcal{H}^{n-1}(y)\, f(y) P(x - y),$$

where P is the Newtonian potential.

We then have:

(i) $w \in C^0(\mathbb{R}^n)$, $w \in C^{m,\alpha}(\bar{U})$ *and* $w \in C^{m,\alpha}(\overline{\mathbb{R}^n \setminus U})$ *with the estimate*

$$\|w\|_{C^{m,\alpha}(\bar{U})} + \|w\|_{C^{m,\alpha}(\overline{\mathbb{R}^n\setminus U})} \le C\|f\|_{C^{m-1,\alpha}(\partial U)},$$

where C *depends only on* n, m, α, r, *and* $\|\partial U\|_{C_r^{m,\alpha}}$.

(ii) *Denoting by* $\partial_{\nu,\,\mathrm{out}} w$ *and* $\partial_{\nu,\,\mathrm{in}} w$ *the* (*outward*) *normal derivatives of* w *from outside and inside* U *respectively we have*, *for all* $x \in \partial U$,

$$\partial_{\nu,\,\mathrm{out}} w(x) = \partial_{\nu,0} w(x) - \tfrac{1}{2} f(x),$$
$$\partial_{\nu,\,\mathrm{in}} w(x) = \partial_{\nu,0} w(x) + \tfrac{1}{2} f(x),$$

where

$$\partial_{\nu,0} w(x) := \int_{\partial U} d\mathcal{H}^{n-1}(y)\, f(y)\, \nu(x)\cdot\nabla P(x-y).$$

(iii) *The linear operator* $T : f \mapsto \partial_{\nu,0} w$ *maps continuously* $C^{m-2,\alpha}(\partial U)$ *to* $C^{m-1,\alpha}(\partial U)$. *More precisely*,

$$\|\partial_{\nu,0} w\|_{C^{m-1,\alpha}(\partial U)} \le C\|f\|_{C^{m-2,\alpha}(\partial U)},$$

where C *depends only on* n, m, α, R, r, *and* $\|\partial U\|_{C_r^{m,\alpha}}$. *In particular* T *is compact in Hölder spaces.*

We provide the following for completeness.

Bibliographic references and sketch of the proof of Theorem A.1. Properties of single-layer potentials in the spirit of (i)–(iii) — and related ones for double-layer potentials — are very classical results in potential theory. They are key tools in proving the existence of solutions for the Dirichlet and Neumann problems in $C^{1,\alpha}$ domains by the method of boundary potentials (by solving in Hölder spaces Fredholm integral equations on the boundary of the domain). For more information on the topic, see for instance the classical books [Sobolev and Dawson 1964; Dautray and Lions 1990].

The proofs of (i), (ii) are given in [Dautray and Lions 1990, Section II.3]. The proof of (i) is given in full detail only for $m = 1$ but the proof for general m is similar. The result for all m is stated in [Dautray and Lions 1990, p. 303].

The compactness property of T in (iii) is in the core of the theory for solving the Dirichlet and Neumann problems by the method of boundary potentials. Indeed, by (ii), the Neumann problem $\Delta w = 0$ in U, $\partial_\nu = g$ on ∂U is equivalent to $Tf + \frac{1}{2} f = g$, where f is the charge on the boundary. Since T is compact, this equation can be solved by Fredholm's alternative;[6]see [Sobolev and Dawson 1964, Lectures 15–19].

Roughly speaking, the reason why Tf increases the order of differentiability of f by 1 is that the integral kernel ($x \in \partial U$) satisfies

$$k(x,y) := \nu(x)\cdot\nabla P(x-y) = c_n \nu(x)\frac{x-y}{|x-y|^n} = O(|x-y|^{n-2})$$

[6]In this case the orthogonality condition of Fredholm's alternative requires $\int_{\partial U} g = 0$.

as $y \to x$, $y \in \partial U$, while ∂U is an $(n-1)$-dimensional surface. The extra factor $|x-y|$ comes from $\nu(x)\cdot(x-y) = O(|x-y|^2)$ since ∂U is smooth enough. Thus, Tf behaves similarly to

$$f \mapsto \int_{\mathbb{R}^d} f(y)\frac{e\cdot y}{|y|^d}\,dy,$$

which maps $C_c^{k-1,\alpha}(\mathbb{R}^d)$ to $C^{k,\alpha}(\mathbb{R}^d)$.

Since it is not easy to find complete references for (iii), although these types of estimates are very classical, for the sake of completeness we provide next a detailed proof of a nearly optimal estimate like (iii) in the case $m=2$ (the proof for other m is more involved but similar). For the purposes of this paper the optimal estimate is not necessary — we just state the optimal result for the convenience of the reader. In our proofs, we do not need to gain a full derivative but just obtain a control in a finer Hölder norm to control the corresponding term by interpolation. Let us prove that if $\partial U \in C_r^{2,\alpha}$ then, for all $\beta \in (0,1)$,

$$\|Tf\|_{C^{0,\beta}(\partial U)} \le C\|f\|_{C^{0,\alpha}(\partial U)} \tag{A-1}$$

(note that the optimal estimate would be with $C^{1,\alpha}$ instead of $C^{0,\beta}$).

As a matter of fact we will prove the stronger (and almost sharp) estimate

$$\|Tf\|_{C^{0,\beta}(\partial U)} \le C\|f\|_{L^\infty(\partial U)}, \tag{A-2}$$

which clearly yields (A-1).

Indeed, we start by showing that

$$k(x,y) := \nu(x)\frac{x-y}{|x-y|^n} \tag{A-3}$$

satisfies

$$|k(x,y)-k(\bar{x},y)| \le C|x-\bar{x}|\,|\xi-y|^{-n+1}, \tag{A-4}$$

where ξ is a point of a curve on ∂U joining x and $\bar{x}$.

Indeed, if $\gamma \subset \partial U$ is a smooth curve joining x and $\bar{x}$ and of length comparable to $|x-\bar{x}|$ we have, at $\xi = \gamma(t)$,

$$\frac{d}{dt}k(\gamma(t),y) = \nu'(\xi)\frac{\xi-y}{|\xi-y|^n} + \nu_i(\xi)\,\gamma_j'(t)\frac{|z|^2\delta_{ij}-nz_iz_j}{|z|^{n+2}} \quad \text{for } z=\xi-y.$$

Choosing an appropriate frame, we may assume $\nu_i(\xi)=\delta_{1i}$ and $\gamma_j'(t)=C\delta_{2j}$ — since the former vector is normal to ∂U and the latter is tangent. Therefore

$$\left|\nu_i(\xi)\gamma_j'(t)\frac{|z|^2\delta_{ij}-nz_iz_j}{|z|^{n+2}}\right| = C\frac{|z_1z_2|}{|z|^{n+2}} \le C\frac{|z|^2|z|}{|z|^{n+2}} \le C|z|^{1-n} = C|\xi-y|^{1-n},$$

where we have used that the first axis is normal to ∂U and hence we have $|z_1|\le|z|^2$ — by C^2 regularity of ∂U and recalling that $z=\xi-y$ with both ξ and y on ∂U. Therefore, an application of the mean value theorem gives

$$|k(x,y)-k(\bar{x},y)| \le C|x-\bar{x}|\left|\frac{d}{dt}k(\gamma(t),y)\right| \le C|x-\bar{x}|\,|\xi-y|^{1-n}$$

and proves (A-4).

Finally, recalling that

$$|k(x, y)| \le C|x-y|^{2-n} \quad \text{and} \quad |k(\bar{x}, y)| \le C|\bar{x}-y|^{2-n}$$

and combining this with (A-4), we obtain

$$|k(x, y)-k(\bar{x}, y)| \le C|x-\bar{x}|^{\beta}\,|\xi-y|^{(1-n)\beta}\big(|x-y|^{(2-n)(1-\beta)}+|\bar{x}-y|^{(2-n)(1-\beta)}\big).$$

Therefore

$$\begin{aligned}|Tf(x)-Tf(\bar{x})| &= \left|\int_{\partial U} f(y)(k(x, y)-k(\bar{x}, y))\, d\mathcal{H}^{n-1}(z)\right| \\ &\le \int_{\partial U} |f(y)|\,|k(x, y)-k(\bar{x}, y)|\, d\mathcal{H}^{n-1}(z) \\ &\le C\|f\|_{L^\infty(\partial U)} \int_{\partial U} |x-\bar{x}|^{\beta}\,|\xi-y|^{(1-n)\beta}\big(|x-y|^{(2-n)(1-\beta)}+|\bar{x}-y|^{(2-n)(1-\beta)}\big) \\ &\le C\|f\|_{L^\infty(\partial U)}|x-\bar{x}|^{\beta},\end{aligned}$$

which proves (A-2). □

Sketch of the proof of Proposition 2.1. For the sake of clarity we give a proof assuming that, for $t>0$, we have $\Delta(h^t-h^0)\ge 0$ and $h^t-h^0\ge 0$ and thus $\Omega^t\subset\omega^0$. We give the proof in dimension $n=2$. The proof for $n\ge 3$ is similar; see [Blank 2001].

Step 1. We show that for some $t_\circ>0$ and $C_\circ$ depending only on $\boldsymbol{\mathcal{C}}$ we have

$$|\Omega^0\setminus\Omega^t| \le C(\boldsymbol{\mathcal{C}})t. \tag{A-5}$$

Indeed, from (4-8) we know that (recall that $v^t := \delta_t\tilde{u}^0$)

$$\begin{cases}\Delta v^t = -(\Delta h^0/t)\chi_{\Omega^0\setminus\Omega^t}+\Delta\delta_t h^0\chi_{\Omega^t} & \text{in } \mathbb{R}^2,\\ \lim_{x\to\infty}\frac{v^t(x)}{-\log|x|}=\delta_t c^0.\end{cases}$$

Note that by (1-10) we have $\Gamma^t\subset B_{\boldsymbol{R}}$ for $t\in[0,t_\circ)$, where $t_\circ>0$ is a small enough constant depending only on $\boldsymbol{\mathcal{C}}$. Recalling that by assumption $\Delta\delta_t h^0$ is supported in $B_{\boldsymbol{R}}$, we have

$$\delta_t c^0 = \int_{\mathbb{R}^2}\Delta v^t = \int_{\mathbb{R}^2} -\frac{\Delta h^0}{t}\chi_{\Omega^0\setminus\Omega^t}+\Delta\delta_t h^0\chi_{\Omega^t}.$$

Therefore, since $-\Delta h^0\ge\boldsymbol{\rho}$, we find

$$\frac{\rho}{t}|\Omega^0\setminus\Omega^t| \le |\delta_t c^0| + \int_{B_{\boldsymbol{R}}}|\Delta\delta_t h^0| \le C(\boldsymbol{\mathcal{C}}).$$

Step 2. We first show (i); that is, we prove that for $t_\circ$ small enough we have

$$\|\Gamma^t\|_{C^{k,\alpha}_{\rho/4}} \le C_o \quad \text{for all } t\in[0,t_\circ). \tag{A-6}$$

Indeed, by Step 1, $|\Omega^0 \setminus \Omega^t| \downarrow 0$ as $t \to 0$ and hence, for t small enough, all points of Γ^t are regular points. More precisely, for all $p \in \Gamma^t$,

$$B_\rho(p) \cap \{\tilde{u}^t = 0\} \geq c_\circ(\mathcal{C}) > 0.$$

Then, we apply:

(1) $C^{1,\alpha}$ free boundary estimates near regular points [Caffarelli 1977; 1998].

(2) $C^{1,\alpha} \Rightarrow C^{k,\alpha}$ estimates for obstacle $h \in C^{k+1,\alpha}$ [Kinderlehrer and Nirenberg 1977].

We thus obtain (A-6).

Step 3. From (A-5) and (A-6) deduce that for $t \in (0, t_\circ)$, the Hausdorff distance between Γ^t and Γ^s satisfies

$$d_{\text{Hausdorff}}(\Gamma^t, \Gamma^0) \leq C_o\, t. \qquad \square$$

Sketch of the proof of Lemma 3.1. The lemma for $n \geq 3$ is very standard. Let us prove it in the case $n = 2$.

Assume $n = 2$. We want to prove that $u^t = f_*$, where

$$f_*(x) := \inf\left\{ f(x) : f \in C(\mathbb{R}^2),\ f \geq h^t,\ \Delta f \leq 0,\ \lim_{x\to\infty} \frac{f}{-\log|x|} = c^t \right\}. \tag{A-7}$$

The admissible class in (A-7) is nonempty since the function

$$f_1(x) := c^t \min\{0, -\log|x|\} + C_1$$

is a member, provided we take $C_1 > 0$ large enough that $\log|x| + C > h^t(x)$ for all $x \in \mathbb{R}^2$ — here we are using (1-4). Hence, $f_*(x) \in [h^t(x), +\infty)$ is finite for all x.

We now check that $u^t = f_*$ is a solution of (1-3) ($n = 2$). First, as an infimum of superharmonic functions, it is superharmonic. To check that it is a subsolution of the obstacle problem, we argue by contradiction. Suppose on the contrary that there exist $r, \varepsilon, \delta > 0$ (as small as we like) and $x_\circ \in \mathbb{R}^2$ such that $f_* > \varepsilon + h^t$ in $B_r(x_\circ)$ and $f_*(x_\circ) > \delta + \fint_{\partial B_r(x_\circ)} f_*$. By changing (slightly) $x_\circ$ and making r and δ smaller, if necessary, we may assume $\delta < \varepsilon$ and

$$\operatorname{osc}_{B_r(x_\circ)} h^t \leq \varepsilon \quad \Longrightarrow \quad f_* > \sup_{B_r(x_\circ)} h^t.$$

Let $\tilde{f} \in C(B_r(x_\circ))$ be the unique harmonic function in $B_r(x_\circ)$ with Dirichlet boundary condition $\tilde{f} = \frac{1}{2}\delta + f_*$ on $\partial B_r(x_\circ)$. Note that $\tilde{f} > h^t$ in $B_r(x_\circ)$, and set

$$f(x) := \begin{cases} f_*(x), & x \notin B_r(x_\circ), \\ \min\{\tilde{f}, f_*(x)\}, & x \in B_r(x_\circ). \end{cases}$$

Then f is admissible in (A-7) and hence $f_* \leq f$. But then by the mean value formula for $\tilde{f}$ we have

$$f_*(x_\circ) \leq f(x_\circ) \leq \tilde{f}(x_\circ) = \tfrac{1}{2}\delta + \fint_{B_r(x_\circ)} f_* \leq \tfrac{1}{2}\delta + f_*(x_\circ) - \varepsilon < f_*(x_\circ),$$

a contradiction. $\square$

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