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We define and study the Airy operator on star graphs. The Airy operator is a third-order differential opera-
tor arising in different contexts, but our main concern is related to its role as the linear part of the Korteweg–
de Vries equation, usually studied on a line or a half-line. The first problem treated and solved is its correct
definition, with different characterizations, as a skew-adjoint operator on a star graph, a set of lines connect-
ing at a common vertex representing, for example, a network of branching channels. A necessary condition
turns out to be that the graph is balanced, i.e., there is the same number of ingoing and outgoing edges at the
vertex. The simplest example is that of the line with a point interaction at the vertex. In these cases the Airy
dynamics is given by a unitary or isometric (in the real case) group. In particular the analysis provides the
complete classification of boundary conditions giving momentum (i.e., L^2-norm of the solution) preserv-
ing evolution on the graph. A second more general problem solved here is the characterization of condi-
tions under which the Airy operator generates a contraction semigroup. In this case unbalanced star graphs
are allowed. In both unitary and contraction dynamics, restrictions on admissible boundary conditions oc-
cur if conservation of mass (i.e., integral of the solution) is further imposed. The above well-posedness re-
sults can be considered preliminary to the analysis of nonlinear wave propagation on branching structures.

1. Introduction

We consider the partial differential equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^3 u}{\partial x^3} + \beta \frac{\partial u}{\partial x}, \tag{1-1}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$, on half-bounded intervals $(-\infty, 0)$ or $(0, \infty)$, and more generally on
collections of copies thereof, building structures commonly known as *metric star graphs*.

A metric star graph in the present setting is the structure represented by the set (see Figure 1)

$$\mathsf{E} := \mathsf{E}_- \cup \mathsf{E}_+,$$

where E_+ and E_- are finite or countable collections of semi-infinite edges e parametrized by $(-\infty, 0)$ or
$(0, \infty)$ respectively. The half-lines are connected at a vertex v, where suitable boundary conditions have to
be imposed in order to result in a well-posed boundary initial value problem. From a mathematical point
of view the problem consists in a system of $|\mathsf{E}_-| + |\mathsf{E}_+|$ partial differential equations of the form (1-1),
with possibly different coefficients α and β, coupled through the boundary condition at the vertex. Our
main concern in this paper is exactly the characterization of boundary conditions yielding a well-posed
dynamics for (1-1) on a metric star graph.
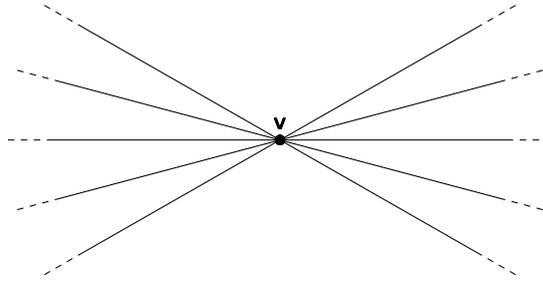
Figure 1. A balanced star graph with $|E_-| = |E_+| = 5$ edges.

This is a well known problem when a Schrödinger operator is considered on the graph, in which case the system is called a *quantum graph*. In this case, extended literature exists on the topic, both for the elliptic problem and for some evolution equations, like heat or wave or reaction-diffusion equations; see [Berkolaiko and Kuchment 2013; Mugnolo 2014]. The analysis has been recently extended also to the case of the nonlinear Schrödinger equation, in particular as regards the characterization of ground states and standing waves; see [Noja 2014; Cacciapuoti et al. 2017] for a review. Another dispersive nonlinear equation, the BBM equation, is treated in [Bona and Cascaval 2008; Mugnolo and Rault 2014]. A partly numerical analysis and partly theoretical analysis of some special cases of the linear Korteweg–de Vries equation on a metric graph is given in [Sobirov et al. 2015a; 2015b; 2015c]. Finally we notice that information about the linear part of the KdV equation on a star graph with special boundary conditions related to controllability problems is considered in [Ammari and Crépeau 2017]. Apart from these papers, not much seems to be known for (1-1), the linear part of the KdV equation. In this context, the solution of (1-1) represents, in the long wave or small amplitude limit, the deviation of the free surface of water from its mean level in the presence of a flat bottom. We will refer to (1-1) as the *Airy equation* and to the operator on its right-hand side as the *Airy* operator; its connection with the KdV equation is one of our main motivations for the study of this problem. Equation (1-1) appeared for the first time in [Stokes 1847] as a contribution to the understanding of solitary waves in shallow water channels observed by Russel [Lannes 2013], but some of its solutions were discussed previously by Airy (ironically, to refuse the existence of solitary waves).

One of the most fascinating features of this PDE is that it has both a dispersive and smoothing character and in fact it is well known that (1-1) is governed by a group of bounded linear operators whenever its space domain is the real line; see [Linares and Ponce 2009].

On the line, after Fourier transform, the solution of (1-1) is given by

$$u(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ik[(x-y)-\alpha k^2 - \beta]t} u_0(y)\, dy\, dk,$$

where u_0 is the initial data of (1-1). The above Fourier formula can be further managed to give in the standard $\alpha < 0$ case

$$u(t,x) = K_t * u_0(x), \quad K_t(x) = \frac{1}{\sqrt[3]{3|\alpha|t}} \operatorname{Ai}\left(\frac{x - \beta t}{\sqrt[3]{3|\alpha|t}}\right), \tag{1-2}$$

where

$$\mathrm{Ai}(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{iyx + i\frac{1}{3}y^3} \, dy = \frac{1}{\pi} \int_{\mathbb{R}} \cos\left(yx + \tfrac{1}{3}y^3\right) dy \qquad (1\text{-}3)$$

is the so-called Airy function (intended with a hidden exponential convergence factor).

Translation invariance allows one to get rid of the first-order term, just changing to moving coordinates in the equation or directly in the solution, and it is not restrictive to put $\beta = 0$. In particular one has that $\|K_t\|_{L^\infty(\mathbb{R})} = \mathcal{O}(1/\sqrt[3]{t})$, from which the typical dispersive behavior of Airy equation solutions on the line follows. This estimate and other more refined dispersive properties are of utmost importance both at the linear level and in the analysis of nonlinear perturbation of the equation, such as the KdV equation; see [Linares and Ponce 2009].

Much harder is the analysis of the properties of the Airy operator when translation invariance is broken, for example in the case of a half-line or in the presence of an external potential. In the first case much work has been done in the context of the analysis of well-posedness of the KdV equation. The first problem one is faced with is the unidirectional character of the propagation, which requires some care in the definition of the correct boundary value problem. It is well known that for the standard problem ($\alpha < 0$) of the KdV equation two boundary conditions at 0 are needed on $(-\infty, 0)$ and only one suffices on $(0, +\infty)$. The issues in the definition of the correct boundary conditions of KdV equation are shared by the linear part, the Airy equation. A more complete analysis of the problem on the half-line is given in the following section.

The Airy operator with an external potential, representing the effect of an obstacle in the propagation or the result of a linearization around known stationary solutions is studied in [Miller 1997], especially as regards spectral and dispersion properties, while a different model with an inhomogeneous dispersion obtained through the introduction of a space-dependent coefficient for the third-derivative term is studied in [Craig and Goodman 1990].

In this paper we are interested in the generalization of the half-line example and from now on we will discuss only the case of a star graph in which the Airy operator has constant coefficients on every single edge (but possibly different coefficients from one edge to another) and discard any other source of inhomogeneity.

Our first goal is to properly define the Airy operator as an unbounded operator on a certain Hilbert space, in such a way that it turns out to be the generator of a C_0-semigroup. We will consider two cases in increasing order of generality. The first is the one in which the generated dynamics is unitary (in the complex case) or isometric (in the real case); the second is the case in which the generated dynamics is given by a contraction C_0-semigroup. The easiest way to understand the nature of the problem is to consider the densely defined closable and skew-symmetric Airy operator $u \mapsto A_0 u := \alpha(d^3 u/dx^3) + \beta(du/dx)$ with domain $C_c^\infty(-\infty, 0)$ or $C_c^\infty(0, +\infty)$ on the Hilbert spaces $L^2(-\infty, 0)$ or $L^2(0, +\infty)$ respectively. Due to the fact that the Airy operator is of odd order, changing the sign of α it is equivalent to exchange the positive and negative half-line and so we can take $\alpha > 0$ without loss of generality.

An easy control shows that the deficiency indices of the two operators are $(2, 1)$ in the first case and $(1, 2)$ in the second case, so that iA_0 does not have any self-adjoint extension on any of the two half-lines. However, the direct sum of the operators on the two half-lines results in a symmetric operator on

$C_c^\infty((-\infty, 0) \cup (0, +\infty))$ with equal deficiency indices $(3, 3)$. Hence, thanks to classical von Neumann–Krein theory it admits a nine-parameter family of self-adjoint extensions generating a unitary dynamics in $L^2(\mathbb{R})$. Of course, this is not the only possibility to generate a dynamics. For instance, it could be the case that a suitable extension A of the operator on the half-line generates a nonunitary semigroup, so not conserving L^2-norm, that still consists of L^2-*contractions*. According to the Lumer–Phillips theorem, this holds true if and only if both A and its adjoint A^* are dissipative. Dissipativity in fact occurs *if the right number of correct boundary conditions are added*; for example in the standard case with $\alpha < 0$, the Dirichlet condition on $u(0)$ on the half-line $(0, +\infty)$ is sufficient, but two conditions, for example the two Dirichlet and Neumann boundary conditions on $u(0)$ and $u'(0)$, are needed on $(-\infty, 0)$.

This is the ultimate reason why more or less explicit results about Airy semigroup formulas exist in the above examples and they can be fruitfully applied to some cases of nonlinear perturbation, such as the KdV equation on the half-line.

The above basic remark, which seemingly went unnoticed in the previous literature on the subject, is the starting point of the treatment of the Airy operator on the more general case of a star graph. To efficiently treat the case of star graphs, we exploit the fact that the Airy operator is antisymmetric, and we want to give first existence conditions for its skew-adjoint extensions and their classification. This can be done in principle in several ways and here we rely on a recent analysis making use (in the intermediate steps of the construction) of Krein spaces with indefinite inner products, recently developed in [Schubert et al. 2015]. As suggested by the example of the half-line, a necessary condition for skew-adjointness is that $\mathsf{E}_- = \mathsf{E}_+$, i.e., the number of incoming half lines is the same of outgoing half-lines. When this condition is met the graph is said to be *balanced*. A similar necessary condition was shown to be true in the case of the quantum momentum operator $-i(d/dx)$ on a graph; see [Carlson 1999; Exner 2013]. The complete characterization of skew-adjoint boundary conditions is more complex, and is given in Theorems 3.7 and 3.8. To explain, we introduce the space of boundary values at the vertex for the domain element of the adjoint operator A_0^*: These are given by $(u(0-), u'(0-), u''(0-))^\top$ and $(u(0+), u'(0+), u''(0+))^\top$, spanning respectively spaces $\mathcal{G}_-$ and $\mathcal{G}_+$ (notice that $u(0-), u(0+)$ etc. are vectors with components given by the boundary values on the single edges and "minus" and "plus" mean that they are taken on edges in E_- or E_+ respectively). The boundary form of the operator A_0 is given by

$$(A_0^* u \mid v) + (u \mid A_0^* v)$$

$$= \left(\begin{pmatrix} -\beta_- & 0 & -\alpha_- \\ 0 & \alpha_- & 0 \\ -\alpha_- & 0 & 0 \end{pmatrix} \begin{pmatrix} u(0-) \\ u'(0-) \\ u''(0-) \end{pmatrix} \middle| \begin{pmatrix} v(0-) \\ v'(0-) \\ v''(0-) \end{pmatrix} \right)_{\mathcal{G}_-} - \left(\begin{pmatrix} -\beta_+ & 0 & -\alpha_+ \\ 0 & \alpha_+ & 0 \\ -\alpha_+ & 0 & 0 \end{pmatrix} \begin{pmatrix} u(0+) \\ u'(0+) \\ u''(0+) \end{pmatrix} \middle| \begin{pmatrix} v(0+) \\ v'(0+) \\ v''(0+) \end{pmatrix} \right)_{\mathcal{G}_+}$$

$$= \left(B_- \begin{pmatrix} u(0-) \\ u'(0-) \\ u''(0-) \end{pmatrix} \middle| \begin{pmatrix} u(0-) \\ u'(0-) \\ u''(0-) \end{pmatrix} \right)_{\mathcal{G}_-} - \left(B_+ \begin{pmatrix} u(0+) \\ u'(0+) \\ u''(0+) \end{pmatrix} \middle| \begin{pmatrix} u(0+) \\ u'(0+) \\ u''(0+) \end{pmatrix} \right)_{\mathcal{G}_+},$$

where, with obvious notation, $\alpha_\pm$ and $\beta_\pm$ are vector-valued coefficients in $\mathsf{E}_\pm$ of the Airy equation on the graph. The block matrices $B_\pm$ are nondegenerate and symmetric but indefinite and endow the boundary spaces $\mathcal{G}_\pm$ with the structure of a Krein space. Correspondingly, the space $\mathcal{G}_- \oplus \mathcal{G}_+$ is endowed with the

sesquilinear form

$$\omega((x, y), (u, v)) = \left(\left(\begin{matrix} B_- & 0 \\ 0 & -B_+ \end{matrix}\right)\left(\begin{matrix} x \\ y \end{matrix}\right) \middle| \left(\begin{matrix} u \\ v \end{matrix}\right)\right).$$

The first important characterization is that skew-adjoint extensions of A_0 are parametrized by the subspaces X of $\mathcal{G}_- \oplus \mathcal{G}_+$ which are ω-*self-orthogonal* ($X = X^\perp$ where orthogonality is with respect to the indefinite sesquilinear form ω; see Definition 3.6 for more details). An equivalent, more explicit parametrization is through relations between boundary values. Consider a linear operator $L : \mathcal{G}_- \to \mathcal{G}_+$ (here we describe the simplest case in which $\mathcal{G}_\pm$ are finite-dimensional; for the general case see Section 3) and define

$$D(A_L) = \{u \in D(A_0^*) : L\big(u(0-), u'(0-), u''(0-)\big) = \big(u(0+), u'(0+), u''(0+)\big)\},$$
$$A_L u = -A_0^* u.$$

Then A_L is a skew adjoint extension of A_0 if and only if L is $(\mathcal{G}_-, \mathcal{G}_+)$-*unitary* ($\langle Lx \mid Ly \rangle_+ = \langle x \mid y \rangle_-$ and again see Definition 3.6).

The characterization of extensions A of A_0 generating a contraction semigroup can be treated along similar lines. According to the Lumer–Phillips theorem, one has that A and A^* have to be dissipative. For a linear operator $L : \mathcal{G}_- \to \mathcal{G}_+$ define $A_L \subseteq -A_0^*$ as above. Denoting by $L^\sharp$ the $(\mathcal{G}_+, \mathcal{G}_-)$-adjoint of the operator L (see Definition 3.6) one has that A_L is the generator of a contraction semigroup if and only if L is a $(\mathcal{G}_-, \mathcal{G}_+)$-contraction (i.e., $\langle Lx \mid Lx \rangle_+ \leq \langle x \mid x \rangle_-$ for all $x \in \mathcal{G}_-$) and $L^\sharp$ is a $(\mathcal{G}_+, \mathcal{G}_-)$-contraction (i.e., $\langle L^\sharp x \mid L^\sharp x \rangle_- \leq \langle x \mid x \rangle_+$ for all $x \in \mathcal{G}_+$).

This condition allows for a different number of ingoing and outgoing half-lines, and it is well adapted to study realistic configurations such as a branching channel.

Notice that in both the skew-adjoint or dissipative case one can consider the possible conservation of an additional physical quantity, the *mass*, coinciding with the integral of the domain element of A. Conservation of mass (see Remark 3.10) is characterized by boundary conditions satisfying the constraint

$$\sum_{e \in E_-} \alpha_e u_e''(0-) - \sum_{e \in E_+} \alpha_e u_e''(0+) + \sum_{e \in E_-} \beta_e u_e(0-) - \sum_{e \in E_+} \beta_e u_e(0+) = 0.$$

Of course the requirement of mass conservation restricts the allowable boundary conditions, both in the skew-adjoint and the dissipative case.

In Section 4 we provide a collection of more concrete examples. Already at the level of general analysis previously done it is clear that a distinguished class of boundary conditions exists, in which the first-derivative boundary values $u'(0\pm)$ are separated. This means that they do not interact with the boundary values of u and u'' and satisfy the transmission relation $u'(0+) = U u'(0-)$ with U unitary in the skew-adjoint case and a contraction in the dissipative case, while $u(0\pm)$ and $u''(0\pm)$ are coupled. This case is described at the beginning of Section 4. Some more special examples deserve interest. The first is the graph consisting of two half-lines. This case is interesting also because it can be interpreted as describing the presence of an obstacle or point interaction on the line. In the skew-adjoint case this perturbation does not destroy the conservation of momentum during the evolution. Moreover, there are both skew-adjoint and more generically contractive boundary conditions that conserve the mass as well.

In this sense the two half-lines case corresponds to a forcing or interaction which however can preserve in time some physical quantity. A related recent analysis is given in [Deconinck et al. 2016], where an inhomogeneous interface problem for the Airy operator on the line (in the special case $\beta = 0$) is studied by means of the Fokas unified transform method (UTM), see [Fokas 2008], and necessary and sufficient conditions for its solvability are given in terms of interface conditions (in that paper the authors prefer to distinguish between interface and boundary conditions; here this usage is not followed). It surely would be interesting to compare the interface conditions studied in the quoted paper with the ones derived in the present one both as regards the skew-adjoint case and the contraction case.

A second class of examples are given for the graph with three half-lines, which falls necessarily in the non-skew-adjoint case. A more accurate analysis of this example is relevant because of the possible application to the analysis of flow in branching or confluent channels, which has attracted some attention in recent years; see [Nachbin and Simões 2012; 2015], and the interesting early paper [Jacovkis 1991], where different models of flows are treated but much of the analysis has general value. It is not at all clear which boundary conditions should be the correct ones from a physical point of view, and the complete classification of those giving generation at least for the Airy operator is a first step to fully understand realistic situations. In general one expects, on theoretical and experimental grounds, that 1-dimensional reductions retain some memory of the geometry (for example the angles of the fork) which is not contemplated in the pure graph description, and a further step in the analysis should consist in including these effects through the introduction of further phenomenological parameters or additional terms in the equation.

The paper is so organized. In Section 2 the simplest case of a single half-line is considered, giving also some comments on the previous literature on the subject. In Section 3 the complete construction of the Airy operator on a star graph in the skew-adjoint and dissipative case is given. The exposition is self-contained as regards the preliminary definitions on graphs and operators on Krein spaces. Besides construction, some general properties of the Airy operators are further studied. In Section 4 some examples are treated.

2. The case of the half-line

As a warm-up, in the present section we consider the Airy equation on the half-line. This is an often-considered subject because of its relevance in connection with the analogous initial boundary value problem for the KdV equation, which turns out to be a rather challenging problem, especially when studied in low Sobolev regularity; for pertinent papers on the subject with information on the linear problem see [Bona et al. 2002; Colliander and Kenig 2002; Hayashi and Kaikina 2004; Holmer 2006; Faminskii 2007; Fokas et al. 2016]. For the interesting case of the interval see also [Colin and Ghidaglia 2001; Bona et al. 2003]. We recall that Korteweg and de Vries derived, under several hypotheses, the equation

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{\ell}}\left(\frac{\sigma}{3}\frac{\partial^3 \eta}{\partial x^3} + \frac{2\alpha}{3}\frac{\partial \eta}{\partial x} + \frac{1}{2}\frac{\partial \eta^2}{\partial x}\right), \tag{2-1}$$

where the unknown η is the elevation of the water surface with respect to its average depth ℓ in a shallow canal. (See [Korteweg and de Vries 1895] for an explanation of the parameters g, σ, α; there

σ is implicitly assumed to be negative. See also [Lannes 2013] for a modern and complete analysis). Renaming coefficients one obtains the KdV equation with parameter α in front of the third derivative and β in front of the first derivative. The physically relevant case of a semi-infinite channel represented by the half-line $[0, +\infty)$ and a wave maker placed at $x = 0$ is described, if dissipation is neglected, by the KdV equation in which the linear part has $\alpha < 0$ and $\beta < 0$; see [Bona et al. 2002]. So one obtains, neglecting nonlinearity, the Airy equation on the positive half-line as the linear part of the KdV equation *with the above sign of coefficients*. The first-derivative term disappears on the whole line changing to a moving reference system, but on the half-line or on a non translation-invariant domain it should be retained, and we do this. The sign of the coefficient of the third derivative is important and interacts with the choice of the half-line, left or right, where propagation occurs. Depending on the sign of α, we must impose a different number of boundary conditions: we have one and two boundary conditions on $(0, \infty)$ and $(-\infty, 0)$ respectively if $\alpha < 0$, and vice versa if $\alpha > 0$. This reflects the fact that the partial differential equation (1-1) has unidirectional nature, and is explained in the literature in several different ways. An explanation making use of the behavior of characteristic curves is recalled for example in [Deconinck et al. 2016], where the authors however notice that it has only a heuristic value. A more convincing brief discussion is given, for $\beta = 0$, in [Holmer 2006], which we reproduce now with minor modifications and considering $\alpha = \pm 1$ for simplicity, which can always be achieved by rescaling the space variable. On the left half-line, after multiplying both sides of (1-1) by u and integrating on $(-\infty, 0)$ one obtains

$$\int_{-\infty}^{0} u_t(t, x) u(t, x)\, dx = \pm u_{xx}(t, 0-) u(t, 0-) \mp \tfrac{1}{2} u_x^2(t, 0-) + \beta \tfrac{1}{2} u^2(t, 0-).$$

Integrating in time on $(0, T)$ one finally obtains the identity

$$\frac{1}{2} \int_{-\infty}^{0} u^2(T, x)\, dx - \frac{1}{2} \int_{-\infty}^{0} u^2(0, x)\, dx$$
$$= \pm \int_{0}^{T} u_{xx}(t, 0-) u(t, 0-)\, dt \mp \frac{1}{2} \int_{0}^{T} u_x^2(t, 0-)\, dt + \beta \frac{1}{2} \int_{0}^{T} u^2(t, 0-)\, dt.$$

If we consider the operator with $\alpha = -1$, which is the standard Airy operator, we conclude that the boundary condition $u(t, 0-) = 0$ alone and the initial condition $u(0, x) = 0$ are compatible with a nonvanishing solution of the equation: to force a vanishing solution one has to fix the boundary value $u_x(t, 0-) = 0$ also. So uniqueness is guaranteed by both boundary conditions on u and u_x. On the contrary, the above identity for the operator with $\alpha = 1$ with the boundary condition $u(t, 0-) = 0$ and the initial condition $u(0, x) = 0$ imply $u(T, x) = 0$ for any T. So one has uniqueness in the presence of the only condition on $u(t, 0)$.

In the case of the positive half-line $(0, \infty)$ one obtains from (1-1)

$$\frac{1}{2} \int_{0}^{\infty} u^2(T, x)\, dx - \frac{1}{2} \int_{0}^{\infty} u^2(0+, x)\, dx$$
$$= \mp \int_{0}^{T} u_{xx}(t, 0+) u(t, 0+)\, dt \pm \frac{1}{2} \int_{0}^{T} u_x^2(t, 0+)\, dt - \beta \frac{1}{2} \int_{0}^{T} u^2(t, 0+)\, dt$$

and for $\alpha = -1$ one has uniqueness in the presence of the boundary condition for $u(t, 0+)$ alone, while for $\alpha = +1$ uniqueness needs the specification of both $u(t, 0+)$ and $u_x(t, 0+)$. Considering the difference

of two solutions corresponding to identical initial data and boundary conditions, one concludes that the same properties hold true in the case of general nonvanishing boundary conditions.

One can go in greater depth and the following lemma is the point of departure. It clarifies some of the properties of the Airy operator on the half-line and will be extended, with many consequences, to a star graph in the following sections.

Lemma 2.1. *Consider the operator*

$$u \mapsto H_0 u := \alpha \frac{d^3 u}{dx^3} + \beta \frac{du}{dx}$$

with domain

$$C_c^\infty(-\infty, 0) \quad or \quad C_c^\infty(0, +\infty)$$

on the Hilbert space

$$L^2(-\infty, 0) \quad or \quad L^2(0, +\infty)$$

respectively. Then iH_0 is densely defined, closable and symmetric. However, its deficiency indices are $(2, 1)$ in the first case and $(1, 2)$ in the second case, so iH_0 does not have any self-adjoint extensions in either case.

Proof. One checks directly that the adjoint of H_0 on $L^2(-\infty, 0)$ and $L^2(0, \infty)$ is $H_0^* = -\alpha(d^3u/dx^3) - \beta(du/dx)$ with domain $H^3(-\infty, 0)$ and $H^3(0, \infty)$, respectively.

We are going to solve the elliptic problem $(iH_0^* \mp i\,\mathrm{Id})u = 0$ in $L^2(-\infty, 0)$ and $L^2(0, \infty)$. Let us consider the sign $+$ and discuss

$$\alpha \frac{d^3 u}{dx^3} + \beta \frac{du}{dx} + u = 0$$

for $u \in L^2(-\infty, 0)$ or $L^2(0, \infty)$ without any boundary conditions. A tedious but elementary computation shows that a general solution is a linear combination of complex exponentials of the form

$$u(x) = C_1 e^{-x \frac{(i\sqrt{3}A^{2/3} + 12i\sqrt{3}\alpha\beta + A^{2/3} - 12\beta\alpha)}{12\alpha A^{1/3}}} + C_2 e^{x \frac{(i\sqrt{3}A^{2/3} + 12i\sqrt{3}\alpha\beta - A^{2/3} + 12\beta\alpha)}{12\alpha A^{1/3}}} + C_3 e^{x \frac{(A^{2/3} - 12\beta\alpha)}{6\alpha A^{1/3}}},$$

where

$$A = \left(12\sqrt{3}\sqrt{\frac{4\beta^3 + 27\alpha}{\alpha}} - 108\right)\alpha^2$$

for general constants C_1, C_2, C_3. However, carefully checking the real parts of the exponents one deduces that such functions are square integrable on $(-\infty, 0)$ and $(0, \infty)$ if and only if $C_3 = 0$ and $C_1 = C_2 = 0$, respectively, thus yielding the claim on the deficiency indices. The remaining case can be treated likewise. $\square$

This shows that the Airy operator cannot be extended to a skew adjoint operator generating a unitary dynamics in L^2. Moreover in some sense the Airy operator "with the wrong sign" or too few boundary conditions has too much spectrum to allow for uniqueness; see [Hille and Phillips 1957, Theorem 23.7.2]. This is not however the whole story, and one can obtain more precise information and some more guiding ideas giving up a unitary evolution and asking simply for generation of a contractive semigroup. To this end we consider the Lumer–Phillips condition and its consequences; see for example [Engel and Nagel

2000, Corollary 3.17]. Again, considering only the case $\alpha = \pm 1$, which is enough, one has by integration by parts

$$\int_{-\infty}^{0} (\pm u_{xxx} + \beta u_x) v \, dx$$

$$= \int_{-\infty}^{0} u(\mp v_{xxx} - \beta v_x) \, dx \pm u_{xx}(0)v(0) + \beta u(0)v(0) \mp u_x(0)v_x(0) \pm u(0)v_{xx}(0).$$

The operator $H_{\pm,\beta}$ with domain

$$\mathcal{D}(H_{\pm,\beta}) = \{u \in H^3(-\infty, 0) : u(0) = 0, \ u_x(0) = 0\}$$

and action

$$H_{\pm,\beta} u = \pm u_{xxx} + \beta u_x$$

satisfies

$$\langle H_{\pm,\beta} u \mid u \rangle = 0$$

and it is dissipative (in fact conservative).

The operator $H_{\pm,\beta}^{\Diamond}$ with domain

$$\mathcal{D}(H_{\pm,\beta}^{\Diamond}) = \{u \in H^3(-\infty, 0) : u(0) = 0\}$$

and action

$$H_{\pm,\beta}^{\Diamond} u = \mp u_{xxx} - \beta u_x$$

satisfies

$$\langle H_{\pm,\beta}^{\Diamond} u \mid u \rangle = \pm \tfrac{1}{2}(u_x(0-))^2$$

and so $H_{+,\beta}^{\Diamond}$ is accretive and $H_{-,\beta}^{\Diamond}$ is dissipative for every $\beta \in \mathbb{R}$.

$H_{\pm,\beta}$ and $H_{\pm,\beta}^{\Diamond}$ are in fact adjoint one to another:

$$H_{\pm,\beta}^{\Diamond} = H_{\pm,\beta}^{*}. \tag{2-2}$$

In particular this means that $H_{+,\beta}$ and $H_{+,\beta}^{*}$ are both accretive and so they do not generate a continuous contraction semigroup in $L^2(-\infty, 0)$. On the contrary, $H_{-,\beta}$ and $H_{-,\beta}^{*}$ are both dissipative and generate a contraction semigroup in $L^2(-\infty, 0)$. With our convention of writing of the Airy equation, this gives well-posedness on $(-\infty, 0)$ for the standard Airy equation (1-1) with two boundary conditions (generator $H_{-,\beta}$). The specular situation occurs for $(0, \infty)$, exchanging the roles and definitions of H and $H^{\Diamond}$, and one has that $H_{+,\beta}^{\Diamond}$ generates a contraction semigroup on $L^2(0, \infty)$ and the standard Airy equation (1-1) with a single boundary condition is well-posed.

3. The case of a metric star graph

Star graphs can be regarded as the building blocks of more complicated graphs; for the purpose of investigating (local) boundary conditions, they are sufficiently generic. Therefore, in this section we are going to develop the theory of the counterpart of the operator H_0 defined on a star graph G, which indeed turns out to display some unexpected behaviors in comparison with its simpler relative introduced in Lemma 2.1.

Upon replacing an interval $(-\infty, 0)$ by $(0, \infty)$ or vice versa, we may assume all coefficients α to have the same sign on each edge e of the star graph. Throughout this paper we are going to follow the convention that all coefficients are positive.

Proposition 3.1. *Consider a quantum graph consisting of finitely or countably many half-lines* $\mathsf{E} := \mathsf{E}_- \cup \mathsf{E}_+$ *and let* $(\alpha_e)_{e \in \mathsf{E}}, (\beta_e)_{e \in \mathsf{E}}$ *be two sequences of real numbers with* $\alpha_e > 0$ *for all* $e \in \mathsf{E}$. *Consider the operator* A_0 *defined by*

$$D(A_0) := \bigoplus_{e \in \mathsf{E}_-} C_c^\infty(-\infty, 0) \oplus \bigoplus_{e \in \mathsf{E}_+} C_c^\infty(0, +\infty),$$

$$A_0 : (u_e)_{e \in \mathsf{E}} \mapsto \left(\alpha_e \frac{d^3 u_e}{dx^3} + \beta_e \frac{d u_e}{dx} \right)_{e \in \mathsf{E}}.$$

Then $i A_0$ *is densely defined and symmetric on the Hilbert space*

$$L^2(\mathsf{G}) := \bigoplus_{e \in \mathsf{E}_-} L^2(-\infty, 0) \oplus \bigoplus_{e \in \mathsf{E}_+} L^2(0, +\infty)$$

and its defect indices are $(2|\mathsf{E}_-| + |\mathsf{E}_+|, |\mathsf{E}_-| + 2|\mathsf{E}_+|)$. *Accordingly,* A_0 *has skew-self-adjoint extensions on* $L^2(\mathsf{G})$ *if and only if* $|\mathsf{E}_+| = |\mathsf{E}_-|$.

In order to avoid some technicalities we will assume that the sequences $(\alpha_e)_{e \in \mathsf{E}}$ and $(\beta_e)_{e \in \mathsf{E}}$ are bounded, and furthermore, that $(1/\alpha_e)_{e \in \mathsf{E}}$ is bounded as well.

Unlike in the case of the Laplace operator, and in spite of the relevance of related physical models, like the KdV equation, there seems to be no canonical or natural choice of boundary conditions to impose on (1-1) on a star graph. For this reason, we are going to characterize all boundary conditions within certain classes. Since (1-1) plays a role in dispersive systems in which conservation of the initial data's norm is expected, we are going to focus on those extensions that generate unitary groups, or isometric semigroups, or at least contractive semigroups.

Extensions of A_0 generating unitary groups. By Stone's theorem, generators of unitary groups are exactly the skew-self-adjoint operators. In order to determine the skew-self-adjoint extensions of A_0, take

$$u, v \in D(A_0^*) = \bigoplus_{e \in \mathsf{E}_-} H^3(-\infty, 0) \oplus \bigoplus_{e \in \mathsf{E}_+} H^3(0, +\infty).$$

Remark 3.2. Let $u \in D(A_0^*)$. By Sobolev's lemma and the boundedness assumption on the (α_e) and (β_e) we obtain $(u_e^{(k)}(0-))_{e \in \mathsf{E}_-} \in \ell^2(\mathsf{E}_-)$ and $(u_e^{(k)}(0+))_{e \in \mathsf{E}_+} \in \ell^2(\mathsf{E}_+)$ for $k \in \{0, 1, 2\}$.

Following the classical extension theory, see [Schmüdgen 2012, Chapter 3], we write down the boundary form to obtain

$$(A_0^* u \mid v) + (u \mid A_0^* v)$$

$$= -\sum_{e \in \mathsf{E}_-} \int_{-\infty}^0 (\alpha_e u_e''' + \beta_e u_e') \bar{v}_e \, dx - \sum_{e \in \mathsf{E}_+} \int_0^{+\infty} (\alpha_e u_e''' + \beta_e u_e') \bar{v}_e \, dx$$

$$- \sum_{e \in \mathsf{E}_-} \int_{-\infty}^0 \overline{(\alpha_e v_e''' + \beta_e v_e')} u_e \, dx - \sum_{e \in \mathsf{E}_+} \int_0^{+\infty} \overline{(\alpha_e v_e''' + \beta_e v_e')} u_e \, dx$$

$$= -\sum_{e\in E_-} \alpha_e u_e''(x)\overline{v_e(x)}\Big|_{-\infty}^0 + \sum_{e\in E_-}\int_{-\infty}^0 \alpha_e u_e''\bar{v}_e'\,dx$$

$$-\sum_{e\in E_-} \alpha_e \overline{v_e(x)}u_e(x)\Big|_{-\infty}^0 + \sum_{e\in E_-}\int_{-\infty}^0 \alpha_e u_e'\bar{v}_e''\,dx$$

$$-\sum_{e\in E_+} \alpha_e u_e''(x)\overline{v_e(x)}\Big|_0^{+\infty} + \sum_{e\in E_+}\int_0^{+\infty} \alpha_e u_e''\bar{v}_e'\,dx$$

$$-\sum_{e\in E_+} \alpha_e \overline{v_e(x)}u_e(x)\Big|_0^{+\infty} + \sum_{e\in E_+}\int_0^{+\infty} \alpha_e u_e'\bar{v}_e''\,dx$$

$$-\sum_{e\in E_-}\int_{-\infty}^0 \beta_e u_e'\bar{v}_e\,dx -\sum_{e\in E_-}\int_{-\infty}^0 \beta_e \bar{v}_e' u_e\,dx -\sum_{e\in E_+}\int_0^{+\infty} \beta u_e'\bar{v}_e\,dx -\sum_{e\in E_+}\int_0^{+\infty} \beta_e \bar{v}_e' u_e\,dx$$

$$= -\sum_{e\in E_-} \alpha_e u_e''(x)\overline{v_e(x)}\Big|_{-\infty}^0 + \sum_{e\in E_-} \alpha_e u_e'(x)\overline{v_e(x)}\Big|_{-\infty}^0$$

$$-\sum_{e\in E_-} \alpha_e \overline{v_e(x)}u_e(x)\Big|_{-\infty}^0 + \sum_{e\in E_+} \alpha_e u_e'(x)\overline{v_e(x)}\Big|_0^{-\infty}$$

$$-\sum_{e\in E_+} \alpha_e u_e''(x)\overline{v_e(x)}\Big|_0^{+\infty}$$

$$-\sum_{e\in E_+} \alpha_e \overline{v_e(x)}u_e(x)\Big|_0^{+\infty}$$

$$-\sum_{e\in E_-} \beta_e u_e(x)\overline{v_e(x)}\Big|_{-\infty}^0 - \sum_{e\in E_+} \beta_e u_e(x)\overline{v_e(x)}\Big|_0^{+\infty}$$

$$= -\sum_{e\in E_-} \alpha_e u_e''(0)\overline{v_e(0)} + \sum_{e\in E_+} \alpha_e u_e''(0)\overline{v_e(0)} - \sum_{e\in E_-} \alpha_e \overline{v_e(0)}u_e(0) + \sum_{e\in E_+} \alpha_e \overline{v_e(0)}u_e(0)$$

$$-\sum_{e\in E_-} \beta_e u_e(0)\overline{v_e(0)} + \sum_{e\in E_+} \beta_e u_e(0)\overline{v_e(0)} + \sum_{e\in E_-} \alpha_e u_e'(0)\overline{v_e(0)} - \sum_{e\in E_+} \alpha_e u_e'(0)\overline{v_e(0)}.$$

Thus, abbreviating $u(0-) := (u_e(0-))_{e\in E_-}$ and similarly for the other terms, and identifying with an abuse of notation $\alpha_\pm$ and $\beta_\pm$ with the corresponding multiplication operator, i.e.,

$$\alpha_\pm x := (\alpha_e x_e)_{e\in E_\pm}, \quad x \in \ell^2(E_\pm), \tag{3-1}$$
$$\beta_\pm x := (\beta_e x_e)_{e\in E_\pm}, \quad x \in \ell^2(E_\pm),$$

we obtain

$$(A_0^* u \mid v) + (u \mid A_0^* v)$$
$$= -\big(\alpha_- u''(0-) \mid v(0-)\big) + \big(\alpha_+ u''(0+) \mid v(0+)\big) - \big(\alpha_- u(0-) \mid v''(0-)\big) + \big(\alpha_+ u(0+) \mid v''(0+)\big)$$
$$-\big(\beta_- u(0-) \mid v(0-)\big) + \big(\beta_+ u(0+) \mid v(0+)\big) + \big(\alpha_- u'(0-) \mid v'(0-)\big) - \big(\alpha_+ u'(0+) \mid v'(0+)\big)$$

$$= \left(\left(\begin{matrix} -\beta_- & 0 & -\alpha_- \\ 0 & \alpha_- & 0 \\ -\alpha_- & 0 & 0 \end{matrix}\right)\left(\begin{matrix} u(0-) \\ u'(0-) \\ u''(0-) \end{matrix}\right) \middle| \left(\begin{matrix} v(0-) \\ v'(0-) \\ v''(0-) \end{matrix}\right)\right)_{\mathcal{G}_-} - \left(\left(\begin{matrix} -\beta_+ & 0 & -\alpha_+ \\ 0 & \alpha_+ & 0 \\ -\alpha_+ & 0 & 0 \end{matrix}\right)\left(\begin{matrix} u(0+) \\ u'(0+) \\ u''(0+) \end{matrix}\right) \middle| \left(\begin{matrix} v(0+) \\ v'(0+) \\ v''(0+) \end{matrix}\right)\right)_{\mathcal{G}_+}, \tag{3-2}$$

where

$$\mathcal{G}_- := \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_-)$$

and

$$\mathcal{G}_+ := \ell^2(\mathsf{E}_+) \oplus \ell^2(\mathsf{E}_+) \oplus \ell^2(\mathsf{E}_+).$$

(We stress the difference from G, which we let denote the quantum graph.) Consider on the graph $G(A_0^*)$ of the operator A_0^* a linear and surjective operator $F : G(A_0^*) \to \mathcal{G}_- \oplus \mathcal{G}_+$ defined by

$$F((u, A_0^* u)) := \big((u(0-), u'(0-), u''(0-)), (u(0+), u'(0+), u''(0+))\big). \tag{3-3}$$

Following the terminology in [Schubert et al. 2015, Examples 2.7], let Ω be the standard symmetric form on $G(A_0^*)$, i.e.,

$$\Omega((u, A_0^* u), (v, A_0^* v)) := \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ A_0^* u \end{pmatrix} \,\bigg|\, \begin{pmatrix} v \\ A_0^* v \end{pmatrix} \right)_{L^2(\mathsf{G})}, \quad (u, A_0^* u), (v, A_0^* v) \in G(A_0^*),$$

and define a sesquilinear form $\omega : \mathcal{G}_- \oplus \mathcal{G}_+ \times \mathcal{G}_- \oplus \mathcal{G}_+ \to \mathbb{C}$ by

$$\omega((x, y), (u, v)) := \left(\begin{pmatrix} B_- & 0 \\ 0 & -B_+ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \,\bigg|\, \begin{pmatrix} u \\ v \end{pmatrix} \right)_{\mathcal{G}_- \oplus \mathcal{G}_+}, \tag{3-4}$$

where $B_\pm$ is the linear block operator matrix on $\mathcal{G}_\pm$ defined by

$$B_\pm(x, x', x'') := \begin{pmatrix} -\beta_\pm & 0 & -\alpha_\pm \\ 0 & \alpha_\pm & 0 \\ -\alpha_\pm & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ x' \\ x'' \end{pmatrix}, \quad (x, x', x'') \in \mathcal{G}_\pm.$$

Observe that neither B_+ nor B_- are definite operators.

Then (3-2) can be rewritten as

$$\Omega((u, A_0^* u), (v, A_0^* v)) = \omega(F(u, A_0^* u), F(v, A_0^* v)) \quad \text{for all } u, v \in D(A_0^*). \tag{3-5}$$

Remark 3.3. Note that, since $\beta_\pm$, $\alpha_\pm$ and $1/\alpha_\pm$ are bounded, we have $B_\pm \in \mathcal{L}(\mathcal{G}_\pm)$, $B_\pm$ are injective and $B_\pm^{-1} \in \mathcal{L}(\mathcal{G}_\pm)$.

Our method is based on the notion of Krein space, i.e., of a vector space endowed with an *indefinite* inner product.

Definition 3.4. Define an (indefinite) inner product $\langle \cdot \mid \cdot \rangle_\pm : \mathcal{G}_\pm \times \mathcal{G}_\pm \to \mathbb{K}$ by

$$\langle x \mid y \rangle_\pm := (B_\pm x \mid y), \quad x, y \in \mathcal{G}_\pm.$$

Then $(\mathcal{G}_\pm, \langle \cdot \mid \cdot \rangle_\pm)$ are Krein spaces and $\langle \cdot \mid \cdot \rangle_\pm$ is nondegenerate, i.e., for $x \in \mathcal{G}_\pm$ with $\langle x \mid y \rangle_\pm = 0$ for all $y \in \mathcal{G}_\pm$ it follows that $x = 0$.

Remark 3.5. Let $\mathcal{K}$ be a vector space and $\langle \cdot \mid \cdot \rangle$ an (indefinite) inner product on $\mathcal{K}$ such that $(\mathcal{K}, \langle \cdot \mid \cdot \rangle)$ is a Krein space. Then there exists an inner product $(\cdot \mid \cdot)$ on $\mathcal{K}$ such that $(\mathcal{K}, (\cdot \mid \cdot))$ is a Hilbert space. Notions such as closedness or continuity are then defined by the underlying Hilbert space structure.

Definition 3.6. Let $\mathcal{K}_-$, $\mathcal{K}_+$ be Krein spaces and $\omega : \mathcal{K}_- \oplus \mathcal{K}_+ \times \mathcal{K}_- \oplus \mathcal{K}_+ \to \mathbb{C}$ be sesquilinear.

(1) A subspace X of $\mathcal{K}_- \oplus \mathcal{K}_+$ is called *ω-self-orthogonal* if

$$X = X^{\perp_\omega} := \{(x, y) \in \mathcal{K}_- \oplus \mathcal{K}_+ : \omega((x, y), (u, v)) = 0 \text{ for all } (u, v) \in X\}.$$

(2) Given a densely defined linear operator L from $\mathcal{K}_-$ to $\mathcal{K}_+$, its *$(\mathcal{K}_-, \mathcal{K}_+)$-adjoint* $L^\sharp$ is

$$D(L^\sharp) := \{y \in \mathcal{K}_+ : \exists z \in \mathcal{K}_- \text{ with } \langle Lx \mid y \rangle_+ = \langle x \mid z \rangle_- \text{ for all } x \in D(L)\},$$
$$L^\sharp y := z.$$

Clearly, $L^\sharp$ is then a linear operator from $\mathcal{K}_+$ to $\mathcal{K}_-$.

(3) A linear operator L from $\mathcal{K}_-$ to $\mathcal{K}_+$ is called a *$(\mathcal{K}_-, \mathcal{K}_+)$-contraction* if

$$\langle Lx \mid Lx \rangle_+ \leq \langle x \mid x \rangle_- \quad \text{for all } x \in D(L).$$

(4) A linear operator L from $\mathcal{K}_-$ to $\mathcal{K}_+$ is called *$(\mathcal{K}_-, \mathcal{K}_+)$-unitary* if $D(L)$ is dense, $R(L)$ is dense, L is injective, and finally $L^\sharp = L^{-1}$.

If in particular $\mathcal{K}_-$, $\mathcal{K}_+$ are Hilbert spaces, then obviously $(\mathcal{K}_-, \mathcal{K}_+)$-adjoint/contraction/unitary operators are nothing but the usual objects of Hilbert space operator theory.

Note that if L is a $(\mathcal{K}_-, \mathcal{K}_+)$-unitary, then

$$\langle Lx \mid Ly \rangle_+ = \langle x \mid y \rangle_- \quad \text{for all } x, y \in D(L). \tag{3-6}$$

We stress that unitary operators between Krein spaces need not be bounded.

By [Schubert et al. 2015, Corollary 2.3 and Example 2.7(b)] we can now characterize skew-self-adjoint extensions A of A_0 —i.e., skew-self-adjoint restrictions of A_0^* — and therefore self-adjoint extensions of iA_0.

Theorem 3.7. *An extension A of A_0 is skew-self-adjoint if and only if there exists an ω-self-orthogonal subspace $X \subseteq \mathcal{G}_- \oplus \mathcal{G}_+$ for which $G(A) = F^{-1}(X)$, where F is the operator defined in (3-3) and $G(A)$ is the graph of A.*

Hence, ω-self-orthogonal subspaces X parametrize the skew-self-adjoint extensions A of A_0. A more explicit description of these objects is given next.

Theorem 3.8. (a) *Let $X \subseteq \mathcal{G}_- \oplus \mathcal{G}_+$ be a subspace such that $D := \{x \in \mathcal{G}_- : \exists y \in \mathcal{G}_+ \text{ with } (x, y) \in X\}$ is dense in $\mathcal{G}_-$ and $R := \{y \in \mathcal{G}_+ : \exists x \in \mathcal{G}_- \text{ with } (x, y) \in X\}$ is dense in $\mathcal{G}_+$, and let X be ω-self-orthogonal. Then there exists a $(\mathcal{G}_-, \mathcal{G}_+)$-unitary operator L such that $X = G(L)$.*

(b) *Let L be a $(\mathcal{G}_-, \mathcal{G}_+)$-unitary operator. Then $G(L) \subseteq \mathcal{G}_- \oplus \mathcal{G}_+$ is ω-self-orthogonal.*

Proof. (a) We first show that X is the graph of an operator. Let $(0, z) \in X$. For $(x, y) \in X$ we have

$$\omega((x, y), (0, z)) = 0;$$

i.e.,

$$0 = \langle y \mid z \rangle_+ = (B_+ y \mid z) = (y \mid B_+ z).$$

Thus, $B_+z \perp R$, and by the denseness of R we obtain $B_+z = 0$. Since B_+ is injective, we conclude $z = 0$. Thus, X is the graph of a linear operator L from $\mathcal{G}_-$ to $\mathcal{G}_+$. Let $(z, 0) \in X$. For $(x, y) \in X$ we obtain analogously

$$\omega((x, y), (z, 0)) = 0;$$

i.e.,

$$0 = \langle x \mid z \rangle_- = (x \mid B_-z).$$

Hence, $B_-z \perp D$, and by the denseness of D we have $B_-z = 0$. Since B_- is injective, $z = 0$. Thus, L is injective.

For $x, y \in D(L)$, we have $(x, Lx), (y, Ly) \in X$, so

$$\omega((x, Lx), (y, Ly)) = 0;$$

i.e.,

$$\langle Lx \mid Ly \rangle_+ = \langle x \mid y \rangle_-.$$

Let $x \in D(L)^\perp$. Then

$$(x \mid y) = 0, \quad y \in D(L).$$

Thus,

$$0 = (x \mid y) = (B_- B_-^{-1} x \mid y) = \langle B_-^{-1} x \mid y \rangle_- = \omega((B_-^{-1} x, 0), (y, Ly)), \quad y \in D(L).$$

Hence, $(B_-^{-1} x, 0) \in X^{\perp_\omega} = X = G(L)$, so $B_-^{-1} x = 0$, and therefore $x = 0$. Thus, L is densely defined. Similarly, we obtain that $R(L)$ is dense.

For $x \in D(L), z \in R(L)$ we have

$$\langle Lx \mid z \rangle_+ = \langle x \mid L^{-1} z \rangle_-.$$

Thus, $R(L) \subseteq D(L^\sharp)$, and $L^\sharp z = L^{-1} z$ for all $z \in R(L)$; i.e., $L^{-1} \subseteq L^\sharp$. Let $(y, x) \in G(L^\sharp)$. Then

$$\langle Lz \mid y \rangle_+ = \langle z \mid x \rangle_-, \quad z \in D(L);$$

i.e.,

$$\omega((z, Lz), (x, y)) = 0, \quad z \in D(L).$$

Hence, $(x, y) \in X^{\perp_\omega} = X = G(L)$, and therefore $(y, x) \in G(L^{-1})$. Therefore, $L^\sharp = L^{-1}$, so L is $(\mathcal{G}_-, \mathcal{G}_+)$-unitary.

(b) Let L be a $(\mathcal{G}_-, \mathcal{G}_+)$-unitary operator from $\mathcal{G}_-$ to $\mathcal{G}_+$ and $X := G(L)$. Let $x \in D(L)$. Then

$$\langle Lx \mid Ly \rangle_+ = \langle x \mid y \rangle_-, \quad y \in D(L);$$

i.e.,

$$\omega((x, Lx), (y, Ly)) = 0, \quad y \in D(L).$$

Thus, $(x, Lx) \in X^{\perp_\omega}$, and therefore $X \subseteq X^{\perp_\omega}$.

Let now $(z, y) \in X^{\perp_\omega}$. Then, for $x \in D(L)$, we have

$$\omega((z, y), (x, Lx)) = 0,$$

and thus

$$\langle Lx \mid y \rangle_+ = \langle x \mid z \rangle_-.$$

By the definition of $L^\sharp$, we obtain $y \in D(L^\sharp)$ and $L^\sharp y = z$. Since $L^\sharp = L^{-1}$, we find $(z, y) \in G(L) = X$. Hence, $X^{\perp_\omega} \subseteq X$.

Combining both parts, we see that X is ω-self-orthogonal. $\qquad\square$

Let L be a densely defined linear operator from $\mathcal{G}_-$ to $\mathcal{G}_+$. Then we define $A_L \subseteq -A_0^*$ by

$$G(A_L) := F^{-1}(G(L));$$

that is,

$$
\begin{aligned}
D(A_L) = \big\{ u \in D(A_0^*) : \big(u(0-), u'(0-), u''(0-) \big) \in D(L), \\
L\big(u(0-), u'(0-), u''(0-) \big) = \big(u(0+), u'(0+), u''(0+) \big) \big\}, \\
A_L u = -A_0^* u.
\end{aligned}
$$

Theorem 3.9. *Let L be a densely defined linear operator from $\mathcal{G}_-$ to $\mathcal{G}_+$. Then A_L is the generator of a unitary group if and only if L is $(\mathcal{G}_-, \mathcal{G}_+)$-unitary.*

Proof. By Theorem 3.7 we obtain that A_L is skew-self-adjoint if and only if $G(L)$ is ω-self-orthogonal. By Theorem 3.8 we conclude that this is equivalent to L being $(\mathcal{G}_-, \mathcal{G}_+)$-unitary. Indeed, Theorem 3.8(b) yields that if L is $(\mathcal{G}_-, \mathcal{G}_+)$-unitary then $G(L)$ is ω-self-orthogonal. As in proof of Theorem 3.8(a) one shows that if $X := G(L)$ is ω-self-orthogonal then the corresponding operator L is $(\mathcal{G}_-, \mathcal{G}_+)$-unitary. $\quad\square$

Remark 3.10. If Theorem 3.7 applies, then Stone's theorem immediately yields that the Airy equation (1-1) on the quantum star graph G is governed by a unitary group acting on $L^2(\mathsf{G})$; hence it has a unique solution $u \in C^1(\mathbb{R}; L^2(\mathsf{G})) \cap C(\mathbb{R}; D(A))$ that continuously depends on the initial data $u_0 \in L^2(\mathsf{G})$. Because the group is unitary, the momentum $\|u\|_{L^2(\mathsf{G})}^2$ is conserved as soon as we can apply Theorem 3.7. By the above computation we also see that

$$
\frac{\partial}{\partial t} \int_{\mathsf{G}} u(t, x)\, dx = \sum_{e \in E_-} \int_{-\infty}^0 \alpha_e u_e'''(t, x) + \beta_e u_e'(t, x)\, dx + \sum_{e \in E_-} \int_0^\infty \alpha_e u_e'''(t, x) + \beta_e u_e'(t, x)\, dx
$$

$$
= \sum_{e \in E_-} \alpha_e u_e''(0-) - \sum_{e \in E_+} \alpha_e u_e''(0+) + \sum_{e \in E_-} \beta_e u_e(0-) - \sum_{e \in E_+} \beta_e u_e(0+). \tag{3-7}
$$

In other words, the solution of the system enjoys conservation of mass — just like the solution to the classical Airy equation on $\mathbb{R}$ — if and only if additionally

$$
\sum_{e \in E_-} \alpha_e u_e''(0-) - \sum_{e \in E_+} \alpha_e u_e''(0+) + \sum_{e \in E_-} \beta_e u_e(0-) - \sum_{e \in E_+} \beta_e u_e(0+) = 0. \tag{3-8}
$$

Remark 3.11. Since $\beta_\pm, \alpha_\pm, 1/\alpha_\pm$ are bounded, the form ω introduced in (3-4) is continuous. Thus, ω-self-orthogonal subspaces are closed, so the corresponding $(\mathcal{G}_-, \mathcal{G}_+)$-unitary L is closed. We do not know whether L is in fact continuous (this holds true in Hilbert spaces, but we are not aware of any argument in Krein spaces).
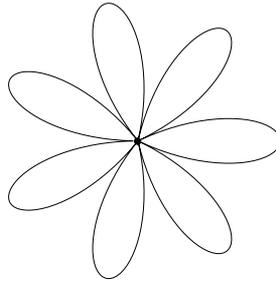
Figure 2. A flower graph on seven edges.

Remark 3.12. As already remarked at the beginning of this section, star graphs can be seen as generic building blocks of quantum graphs. Apart from their interest in scattering theory, star graphs whose edges are semi-infinite still display all relevant features for the purpose of studying extensions of operators on compact graphs. Indeed, our analysis is essentially of variational nature and therefore it only depends on the orientation of the edges and the boundary values of a function on the graph. It is therefore clear that analogous results could be formulated for graphs that include edges of finite length, too, like the *flower graph* depicted in Figure 2. Clearly, an interesting feature of flower graphs is that they are automatically balanced; i.e., the number of incoming and outgoing edges from the (only) vertex is equal: accordingly, the operator A_0 on a flower graph always admits skew-self-adjoint extensions.

Extensions of A_0 generating contraction semigroups. Let A be an extension of A_0 such that $A \subseteq -A_0^*$. We focus on generating contraction semigroups. By the Lumer–Phillips theorem and corollaries of it we have to show that A and A^* are dissipative. Since we are dealing with Hilbert spaces, A is dissipative if and only if $\operatorname{Re}(Au \mid u) \leq 0$ for all $u \in D(A)$, and analogously for A^*. Recall, that for a densely defined linear operator L from $\mathcal{G}_-$ to $\mathcal{G}_+$ we defined $A_L \subseteq -A_0^*$ by $G(A_L) = F^{-1}(G(L))$.

Lemma 3.13. *Let L be a densely defined linear operator from $\mathcal{G}_-$ to $\mathcal{G}_+$. Then A_L is dissipative if and only if L is a $(\mathcal{G}_-, \mathcal{G}_+)$-contraction.*

Proof. Let $u \in D(A_L)$. Then

$$-2\operatorname{Re}(A_L u \mid u) = \Omega((u, -A_L u), (u, -A_L u))$$
$$= \langle (u(0-), u'(0-), u''(0-)) \mid (u(0-), u'(0-), u''(0-)) \rangle_-$$
$$- \langle L(u(0-), u'(0-), u''(0-)) \mid L(u(0-), u'(0-), u''(0-)) \rangle_+.$$

Thus, A_L is dissipative if and only if

$$\langle L(u(0-), u'(0-), u''(0-)) \mid L(u(0-), u'(0-), u''(0-)) \rangle_+$$
$$\leq \langle (u(0-), u'(0-), u''(0-)) \mid (u(0-), u'(0-), u''(0-)) \rangle_- \quad \text{for all } u \in D(A_L).$$

By the definition of A_L we have $D(L) = \{(u(0-), u'(0-), u''(0-)) : u \in D(A_L)\}$, so the assertion follows. $\square$

Analogously, we obtain a characterization for dissipativity of the adjoint A_L^* of A_L. Here and in the following, $L^\sharp$ denotes the $(\mathcal{G}_-, \mathcal{G}_+)$-adjoint of the operator L; see Definition 3.6.

Lemma 3.14. *Let L be a densely defined linear operator from $\mathcal{G}_-$ to $\mathcal{G}_+$. Then*

$$D(A_L^*) = \{u \in D(A_0^*) : \big(u(0+), u'(0+), u''(0+)\big) \in D(L^\sharp),$$
$$L^\sharp\big(u(0+), u'(0+), u''(0+)\big) = \big(u(0-), u'(0-), u''(0-)\big)\},$$
$$A_L^* u = A_0^* u.$$

Proof. Let $u \in D(A_L)$, $v \in D(A_0^*)$. Then

$$(A_L u \mid v) = (u \mid A_0^* v) - \big\langle \big(u(0-), u'(0-), u''(0-)\big) \mid \big(v(0+), v'(0+), v''(0+)\big) \big\rangle_-$$
$$+ \big\langle L\big(u(0-), u'(0-), u''(0-)\big) \mid \big(v(0+), v'(0+), v''(0+)\big) \big\rangle_+.$$

Hence, $v \in D(A_L^*)$ if and only if $\big(v(0+), v'(0+), v''(0+)\big) \in D(L^\sharp)$ and

$$L^\sharp\big(v(0+), v'(0+), v''(0+)\big) = \big(v(0-), v'(0-), v''(0-)\big),$$

and then $A_L^* v = A_0^* v$. □

Lemma 3.15. *Let L be a densely defined linear operator from $\mathcal{G}_-$ to $\mathcal{G}_+$. Then A_L^* is dissipative if and only if $L^\sharp$ is a $(\mathcal{G}_+, \mathcal{G}_-)$-contraction.*

Proof. Let $u \in D(A_L^*)$. Then

$$2\operatorname{Re}(A_L^* u \mid u) = \Omega((u, A_L^* u), (u, A_L^* u))$$
$$= \big\langle L^\sharp\big(u(0+), u'(0+), u''(0+)\big) \mid L^\sharp\big(u(0+), u'(0+), u''(0+)\big) \big\rangle_-$$
$$- \big\langle \big(u(0+), u'(0+), u''(0+)\big) \mid \big(u(0+), u'(0+), u''(0+)\big) \big\rangle_+.$$

Thus, A_L^* is dissipative if and only if

$$\big\langle L^\sharp\big(u(0+), u'(0+), u''(0+)\big) \mid L^\sharp\big(u(0+), u'(0+), u''(0^+)\big) \big\rangle_-$$
$$\leq \big\langle \big(u(0+), u'(0+), u''(0+)\big) \mid \big(u(0+), u'(0+), u''(0+)\big) \big\rangle_+ \quad \text{for all } u \in D(A_L).$$

By the definition of A_L we have $D(L^\sharp) = \{\big(u(0+), u'(0+), u''(0+)\big) : u \in D(A_L)\}$, so the assertion follows. □

Theorem 3.16. *Let L be a densely defined linear operator from $\mathcal{G}_-$ to $\mathcal{G}_+$. Then A_L is the generator of a contraction semigroup if and only if L is a $(\mathcal{G}_-, \mathcal{G}_+)$-contraction and $L^\sharp$ is a $(\mathcal{G}_+, \mathcal{G}_-)$-contraction.*

Proof. Let A_L generate a C_0-semigroup $(T(t))_{t \geq 0}$ of contractions. By the Lumer–Phillips theorem, A_L is dissipative. Since A_L^* generates the semigroup T^* defined by $T^*(t) := T(t)^*$ $(t \geq 0)$, which is also a C_0-semigroup of contractions, the Lumer–Phillips theorem assures that A_L^* is dissipative as well. Then the Lemmas 3.13 and 3.15 yield that L and $L^\sharp$ are contractions.

Now, let L and $L^\sharp$ be contractions. Then Lemmas 3.13 and 3.15 yield that A_L and A_L^* are dissipative. Hence, A_L generates a contraction semigroup. □

Corollary 3.17. *Let* $|\mathsf{E}_-| = |\mathsf{E}_+| < \infty$, $\alpha_+ = \alpha_-$, $\beta_+ = \beta_-$ *(via some bijection between* E_- *and* E_+*).* *Let* L *be a densely defined linear operator from* $\mathcal{G}_-$ *to* $\mathcal{G}_+$. *Then* A_L *is the generator of a contraction semigroup if and only if* L *is* $(\mathcal{G}_-, \mathcal{G}_+)$-*contractive.*

Proof. In this setup, $\mathcal{G}_- = \mathcal{G}_+$, which are finite-dimensional and hence Pontryagin spaces. Therefore, L is $(\mathcal{G}_-, \mathcal{G}_+)$-contractive if and only if $L^\sharp$ is $(\mathcal{G}_+, \mathcal{G}_-)$-contractive by [Dritschel and Rovnyak 1990, Theorem 1.3.7]. $\square$

Remark 3.18. Like in the case of Remark 3.10, if Theorem 3.16 applies, then the Airy equation (1-1) on the quantum star graph G has a unique solution $u \in C^1(\mathbb{R}_+; L^2(\mathsf{G})) \cap C(\mathbb{R}_+; D(A))$ that continuously depends on the initial data $u_0 \in L^2(\mathsf{G})$. Because the C_0-semigroup is contractive but not unitary, the momentum $\|u\|^2_{L^2(\mathsf{G})}$ is in general not conserved, but as in Remark 3.10 the system does enjoy conservation of mass if and only if additionally (3-8) holds.

Separating the first derivatives. The special structure of $B_\pm$ suggests to separate the boundary values of the first derivative from the ones for the function and for the second derivative. In this case, one can describe the boundary conditions also in another (equivalent but seemingly easier) way.

Note that $\alpha_{\mathsf{e}} > 0$ for all $\mathsf{e} \in \mathsf{E}$. We will write $\ell^2(\mathsf{E}_\pm, \alpha_\pm)$ for the weighted ℓ^2-space of sequences indexed by $\mathsf{E}_\pm$ with inner product given by

$$(x \mid y)_{\ell^2(\mathsf{E}_\pm, \alpha_\pm)} := \sum_{\mathsf{e} \in \mathsf{E}_\pm} x_{\mathsf{e}} \bar{y}_{\mathsf{e}} \alpha_{\mathsf{e}} = (\alpha_\pm x \mid y)$$

for all $x, y \in \ell^2(\mathsf{E}_\pm, \alpha_\pm)$, which turns them into Hilbert spaces.

For $u, v \in D(A_0^*)$ we then obtain

$$(A_0^* u \mid v) + (u \mid A_0^* v)$$

$$= (\alpha_- u'(0-) \mid v'(0-)) - (\alpha_+ u'(0+) \mid v'(0+))$$

$$+ \left(\begin{pmatrix} -\beta_- & -\alpha_- \\ -\alpha_- & 0 \end{pmatrix} \begin{pmatrix} u(0-) \\ u''(0-) \end{pmatrix} \Big| \begin{pmatrix} v(0-) \\ v''(0-) \end{pmatrix} \right) - \left(\begin{pmatrix} -\beta_+ & -\alpha_+ \\ -\alpha_+ & 0 \end{pmatrix} \begin{pmatrix} u(0+) \\ u''(0+) \end{pmatrix} \Big| \begin{pmatrix} v(0+) \\ v''(0+) \end{pmatrix} \right)$$

$$= (\alpha_- u'(0-) \mid v'(0-)) - (\alpha_+ u'(0+) \mid v'(0+))$$

$$- \left(\begin{pmatrix} \alpha_- & 0 \\ 0 & -\alpha_+ \end{pmatrix} \begin{pmatrix} u''(0-) \\ u''(0+) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\beta_- & 0 \\ 0 & -\frac{1}{2}\beta_+ \end{pmatrix} \begin{pmatrix} u(0-) \\ u(0+) \end{pmatrix} \Big| \begin{pmatrix} v(0-) \\ v(0+) \end{pmatrix} \right)$$

$$- \left(\begin{pmatrix} u(0-) \\ u(0+) \end{pmatrix} \Big| \begin{pmatrix} \alpha_- & 0 \\ 0 & -\alpha_+ \end{pmatrix} \begin{pmatrix} v''(0-) \\ v''(0+) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\beta_- & 0 \\ 0 & -\frac{1}{2}\beta_+ \end{pmatrix} \begin{pmatrix} v(0-) \\ v(0+) \end{pmatrix} \right).$$

Let $Y \subseteq \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_+)$ be a closed subspace, U a densely defined linear operator from $\ell^2(\mathsf{E}_-, \alpha_-)$ to $\ell^2(\mathsf{E}_+, \alpha_+)$, and consider

$$D(A_{Y,U}) := \Big\{ u \in D(A_0^*) : (u(0-), u(0+)) \in Y,$$

$$\begin{pmatrix} -\alpha_- & 0 \\ 0 & \alpha_+ \end{pmatrix} \begin{pmatrix} u''(0-) \\ u''(0+) \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}\beta_- & 0 \\ 0 & \frac{1}{2}\beta_+ \end{pmatrix} \begin{pmatrix} u(0-) \\ u(0+) \end{pmatrix} \in Y^\perp,$$

$$u'(0-) \in D(U), \; u'(0+) = U u'(0-) \Big\},$$

$$A_{Y,U} u := -A_0^* u.$$

Proposition 3.19. *Let $Y \subseteq \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_+)$ be a closed subspace and U a densely defined linear operator from $\ell^2(\mathsf{E}_-, \alpha_-)$ to $\ell^2(\mathsf{E}_+, \alpha_+)$. Then*

$$D(A^*_{Y,U}) = \left\{ u \in D(A^*_0) : (u(0-), u(0+)) \in Y, \right.$$
$$\begin{pmatrix} -\alpha_- & 0 \\ 0 & \alpha_+ \end{pmatrix} \begin{pmatrix} u''(0-) \\ u''(0+) \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}\beta_- & 0 \\ 0 & \frac{1}{2}\beta_+ \end{pmatrix} \begin{pmatrix} u(0-) \\ u(0+) \end{pmatrix} \in Y^\perp,$$
$$\left. u'(0+) \in D(U^*), \ u'(0-) = U^* u'(0+) \right\},$$

$$A^*_{Y,U} u = A^*_0 u.$$

Proof. Let $u \in D(A_{Y,U})$, $v \in D(A^*_0)$. Then

$$(A_{Y,U} u \mid v) = (u \mid A^*_0 v) + \big(\alpha_+ U u'(0-) \mid v'(0+)\big) - \big(\alpha_- u'(0-) \mid v'(0-)\big)$$
$$+ \left(\begin{pmatrix} \alpha_- & 0 \\ 0 & -\alpha_+ \end{pmatrix} \begin{pmatrix} u''(0-) \\ u''(0+) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\beta_- & 0 \\ 0 & -\frac{1}{2}\beta_+ \end{pmatrix} \begin{pmatrix} u(0-) \\ u(0+) \end{pmatrix} \Big| \begin{pmatrix} v(0-) \\ v(0+) \end{pmatrix} \right)$$
$$+ \left(\begin{pmatrix} u(0-) \\ u(0+) \end{pmatrix} \Big| \begin{pmatrix} \alpha_- & 0 \\ 0 & -\alpha_+ \end{pmatrix} \begin{pmatrix} v''(0-) \\ v''(0+) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\beta_- & 0 \\ 0 & -\frac{1}{2}\beta_+ \end{pmatrix} \begin{pmatrix} v(0-) \\ v(0+) \end{pmatrix} \right).$$

Let now $v \in D(A^*_{Y,U})$ such that $(A_{Y,U} u \mid v) = (u \mid A^*_{Y,U} v) = (u \mid A^*_0 v)$. Choosing u such that $(u(0-), u(0+)) = (u'(0-), u'(0+)) = 0$, we obtain $(v(0-), v(0+)) \in Y$. For all $(u_-, u_+) \in Y$ there exists $u \in D(A_{Y,U})$ such that $(u(0-), u(0+)) = (u_-, u_+)$, and $(u'(0-), u'(0+)) = 0$. Thus,

$$\begin{pmatrix} \alpha_- & 0 \\ 0 & -\alpha_+ \end{pmatrix} \begin{pmatrix} v''(0-) \\ v''(0+) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\beta_- & 0 \\ 0 & -\frac{1}{2}\beta_+ \end{pmatrix} \begin{pmatrix} v(0-) \\ v(0+) \end{pmatrix} \in Y^\perp.$$

Thus, we arrive at

$$\big(\alpha_+ U u'(0-) \mid v'(0+)\big) - \big(\alpha_- u'(0-) \mid v'(0-)\big) = 0, \quad u \in D(A_{Y,U}).$$

Note that for all $x \in D(U)$ there exists $u \in D(A_{Y,U})$ such that $u'(0-) = x$. Hence, $v'(0+) \in D(U^*)$ and $U^* v'(0+) = v'(0-)$. $\qquad \square$

Corollary 3.20. *Let $Y \subseteq \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_+)$ be a closed subspace and U a densely defined linear operator from $\ell^2(\mathsf{E}_-, \alpha_-)$ to $\ell^2(\mathsf{E}_+, \alpha_+)$. Then $A_{Y,U}$ is skew-self-adjoint if and only if U is unitary.*

Proof. This is a direct consequence of [Proposition 3.19](#). $\qquad \square$

Corollary 3.21. *Let $Y \subseteq \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_+)$ be a closed subspace and U a densely defined linear operator from $\ell^2(\mathsf{E}_-, \alpha_-)$ to $\ell^2(\mathsf{E}_+, \alpha_+)$. Then $A_{Y,U}$ is dissipative if and only if U is a contraction.*

Proof. Let $u \in D(A_{Y,U})$. Then

$$(A_{Y,U} u \mid u) = (u \mid A^*_0 u) + \big(\alpha_+ U u'(0-) \mid U u'(0-)\big) - \big(\alpha_- u'(0-) \mid u'(0-)\big).$$

Since $A^*_0 u = -A_{Y,U} u$, we obtain

$$2 \operatorname{Re} (A_{Y,U} u \mid u) = \big(\alpha_+ U u'(0-) \mid U u'(0-)\big) - \big(\alpha_- u'(0-) \mid u'(0-)\big).$$

Hence, $A_{Y,U}$ is dissipative if and only if U is a contraction. $\qquad \square$

Corollary 3.22. *Let $Y \subseteq \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_+)$ be a closed subspace and U a densely defined linear operator from $\ell^2(\mathsf{E}_-, \alpha_-)$ to $\ell^2(\mathsf{E}_+, \alpha_+)$. Then $A_{Y,U}^*$ is dissipative if and only if U^* is a contraction.*

Proof. Let $u \in A_{Y,U}^*$. Then, similarly as for $A_{Y,U}$, we have

$$(A_{Y,U}^* u \mid u) = (u \mid -A_0^* u) - \big(\alpha_+ u'(0+) \mid u'(0+)\big) + \big(\alpha_- U^* u'(0+) \mid U^* u'(0+)\big).$$

Since $A_0^* u = A_{Y,U}^* u$, we obtain

$$2 \operatorname{Re}(A_{Y,U} u \mid u) = \big(\alpha_- U^* u'(0+) \mid U^* u'(0+)\big) - \big(\alpha_+ u'(0+) \mid u'(0+)\big).$$

Hence, $A_{Y,U}^*$ is dissipative if and only if U^* is a contraction. $\qquad\square$

Theorem 3.23. *Let $Y \subseteq \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_+)$ be a closed subspace and U a densely defined linear operator from $\ell^2(\mathsf{E}_-, \alpha_-)$ to $\ell^2(\mathsf{E}_+, \alpha_+)$. Then $A_{Y,U}$ generates a contraction C_0-semigroup if and only if U is a contraction and U^* is a contraction if and only if U is a contraction.*

Proof. Let $A_{Y,U}$ generate a contraction semigroup. Then $A_{Y,U}$ is dissipative by the Lumer–Phillips theorem, so Corollary 3.21 yields that U is a contraction.

Let U and U^* be contractions. Then, Corollaries 3.21 and 3.22 ensure that $A_{Y,U}$ and $A_{Y,U}^*$ are dissipative. Hence, $A_{Y,U}$ generates a semigroup of contractions. Clearly, U^* is a contraction provided U is a contraction. $\qquad\square$

Reality of the semigroup. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on $L^2(\mathsf{G})$. We say that $(T(t))_{t\geq 0}$ is *real* if $T(t) \operatorname{Re} u = \operatorname{Re} T(t)u$ for all $u \in L^2(\mathsf{G})$ and $t \geq 0$. Put differently, a semigroup is real if it maps real-valued functions into real-valued functions.

For simplicity we will only consider the case of contractive semigroups here.

Proposition 3.24. *Let L be a densely defined linear operator from $\mathcal{G}_-$ to $\mathcal{G}_+$ such that L and $L^\sharp$ are contractions. Let $(T(t))_{t\geq 0}$ be the C_0-semigroup generated by A_L. Let L be real; i.e., for $x \in D(L)$ we have $\operatorname{Re} x \in D(L)$ and $\operatorname{Re} Lx = L \operatorname{Re} x$. Then $(T(t))_{t\geq 0}$ is real.*

Proof. By Theorem 3.16, $(T(t))_{t\geq 0}$ is a contraction semigroup. Let P be the projection from $L^2(\mathsf{G}; \mathbb{C})$ to $L^2(\mathsf{G}; \mathbb{R})$; i.e., $Pu := \operatorname{Re} u$. By [Arendt et al. 2015, Corollary 9.6] realness of the semigroup $(T(t))_{t\geq 0}$ is equivalent to the condition

$$\operatorname{Re}(A_L u \mid u - Pu) \leq 0 \quad \text{for all } u \in D(A_L),$$

i.e.,

$$\operatorname{Re}(A_L u \mid i \operatorname{Im} u) \leq 0 \quad \text{for all } u \in D(A_L).$$

Let $u \in D(A_L)$. Since L is real we obtain $\operatorname{Re} u, \operatorname{Im} u \in D(A_L)$. Since $A_L v$ is real for $v \in D(A_L)$ real we have

$$\operatorname{Re}(A_L(\operatorname{Re} u + i \operatorname{Im} u) \mid i \operatorname{Im} u) = \operatorname{Im}(A_L \operatorname{Re} u \mid \operatorname{Im} u) + \operatorname{Re}(A_L \operatorname{Im} u \mid \operatorname{Im} u)$$

$$= \operatorname{Re}(A_L \operatorname{Im} u \mid \operatorname{Im} u) \leq 0,$$

since A_L is dissipative by Lemma 3.13. $\qquad\square$

Remark 3.25. The semigroup generated by A_L is not positivity preserving, i.e., nonnegative functions need not be mapped to nonnegative functions: indeed, also in this case positivity of the semigroup is again equivalent to

$$\operatorname{Re}(A_L u \mid u - Pu) \le 0 \quad \text{for all } u \in D(A_L), \tag{3-9}$$

where P is now the projection of $L^2(\mathsf{G}; \mathbb{C})$ onto the positive cone of $L^2(\mathsf{G}; \mathbb{R})$, i.e., $Pu := (\operatorname{Re} u)_+$. Let u be a real-valued function: integrating by parts and neglecting without loss of generality the transmission conditions (due to locality of the operator), one sees that

$$\operatorname{Re}(A_L u \mid u - Pu) = -\int_{\mathsf{G}} u''' u_- \, dx = \int_{\{u \le 0\}} u''' u \, dx = -\tfrac{1}{2} |u'|^2 \big|_{\partial \{u \le 0\}}.$$

Of course, wherever an H^3-function changes sign its first derivative need not vanish, so condition (3-9) cannot be satisfied.

Analogously, the semigroup is also not L^∞-contractive, i.e., the inequality $\|e^{tA_L} u\|_\infty \le \|u\|_\infty$ fails for some $u \in L^2(\mathsf{G}) \cap L^\infty(\mathsf{G})$ and some $t \ge 0$. In this case, the relevant projection onto the closed convex subset $C := \{u \in L^2(\mathsf{G}) : |u| \le 1\}$ of $L^2(\mathsf{G})$ is defined by $Pu := (|u| \wedge 1) \operatorname{sgn} u$; hence

$$u - Pu := (|u| - 1)_+ \operatorname{sgn} u.$$

We also obtain realness of $(T(t))_{t \ge 0}$ in the case of separated boundary conditions.

Proposition 3.26. *Let $Y \subseteq \ell^2(\mathsf{E}_-) \oplus \ell^2(\mathsf{E}_+)$ be a closed subspace and $U : \ell^2(\mathsf{E}_-, \alpha_-) \to \ell^2(\mathsf{E}_+, \alpha_+)$ be linear and contractive. Let $(T(t))_{t \ge 0}$ be the C_0-semigroup generated by $A_{Y,U}$. Assume that $(\operatorname{Re} x, \operatorname{Re} y) \in Y$ for all $(x, y) \in Y$ and U is real, i.e., $\operatorname{Re} Ux = U \operatorname{Re} x$ for all $x \in \ell^2(\mathsf{E}_-, \alpha_-)$. Then $(T(t))_{t \ge 0}$ is real.*

Proof. By Theorem 3.23, $(T(t))_{t \ge 0}$ is a contraction semigroup. Let P be the projection from $L^2(\mathsf{G}; \mathbb{C})$ to $L^2(\mathsf{G}; \mathbb{R})$; i.e., $Pu := \operatorname{Re} u$. By [Arendt et al. 2015, Corollary 9.6] realness of the semigroup $(T(t))_{t \ge 0}$ is equivalent to

$$\operatorname{Re}(A_{Y,U} u \mid u - Pu) \le 0 \quad \text{for all } u \in D(A_{Y,U}),$$

i.e.,

$$\operatorname{Re}(A_{Y,U} u \mid i \operatorname{Im} u) \le 0 \quad \text{for all } u \in D(A_{Y,U}).$$

Let $u \in D(A_{Y,U})$. The assumptions imply $\operatorname{Re} u \in D(A_{Y,U})$, and therefore also $\operatorname{Im} u \in D(A_{Y,U})$. Since $A_{Y,U} v$ is real for $v \in D(A_{Y,U})$ real we have

$$\operatorname{Re}(A_{Y,U}(\operatorname{Re} u + i \operatorname{Im} u) \mid i \operatorname{Im} u) = \operatorname{Im}(A_{Y,U} \operatorname{Re} u \mid \operatorname{Im} u) + \operatorname{Re}(A_{Y,U} \operatorname{Im} u \mid \operatorname{Im} u)$$
$$= \operatorname{Re}(A_{Y,U} \operatorname{Im} u \mid \operatorname{Im} u) \le 0,$$

since $A_{Y,U}$ is dissipative by Corollary 3.21. $\qquad\square$

Remark 3.27. We end this section with some considerations about the boundary conditions used in the literature related to the subject.

In the classical treatise [Stoker 1957] the author, reviewing the much-studied case study of the confluence between the Ohio and Mississippi rivers, proposed that the boundary conditions suitable for branching water flows are the ones compatible with continuity and mass conservation. As we know, there are many. This assumption was essentially undisputed in the not-so-abundant literature on the subject; see also the interesting paper [Jacovkis 1991] for further information. Recently, in [Nachbin and Simões 2012; 2015] a more careful analysis has been put forth by the authors attempting a 1-dimensional reduction from a fluid dynamical model in a 2-dimensional setting; they question the above traditional point of view about the more convenient boundary conditions, in particular continuity.

We finally notice that in the few recent mathematical papers concerning the Airy or KdV equation on graphs, [Sobirov et al. 2015a; 2015b; 2015c; Ammari and Crépeau 2017; Cavalcante 2017], only very special examples of boundary conditions are considered, essentially without explanation. All of them turn out to be of the separated derivative type studied in this section.

4. Examples

The case of two half-lines. First, let us consider the case of the real line with a singular interaction at the origin, i.e., $|E_-| = |E_+| = 1$; see Figure 3.

We will describe the operator explicitly in the case where the first derivative is separated.

Let $Y \subseteq \mathbb{C}^2$ be a subspace and $U : \ell^2(E_-, \alpha_-) \to \ell^2(E_+, \alpha_+)$ be linear, i.e., $U \in \mathbb{C}$. Note that U is contractive if and only if $|U|^2 \alpha_+ \leq \alpha_-$.

Example 4.1. Let $Y := \{(0,0)\}$. Then

$$D(A_{Y,U}) = \left\{ u \in D(A_0^*) : u(0-) = u(0+) = 0, \ u'(0+) = Uu'(0-) \right\},$$
$$A_{Y,U} u = -A_0^* u.$$

By Corollary 3.20, $A_{Y,U}$ generates a unitary group provided $|U|^2 = \alpha_-/\alpha_+$. By Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions provided $|U|^2 \leq \alpha_-/\alpha_+$. Observe that if we take $U = 0$, we are effectively reducing the Airy equation on the star graph G to decoupled Airy equations on two half-lines $(0, \infty)$ and $(-\infty, 0)$ with Dirichlet conditions (for both equations) and Neumann (on the positive half-line only) boundary conditions. The Airy equation on either of these half-lines with the above boundary conditions has been considered often in the literature; see, e.g., [Holmer 2006; Fokas et al. 2016].

Example 4.2. Let $Y := \lin\{(0,1)\}$. Then

$$D(A_{Y,U}) = \left\{ u \in D(A_0^*) : u(0-) = 0, \ u''(0+)\alpha_+ = -\tfrac{1}{2}\beta_+ u(0+), \ u'(0+) = Uu'(0-) \right\},$$
$$A_{Y,U} u = -A_0^* u.$$

$$0$$

Figure 3. A graph consisting of two half-lines.

If $U = 0$, these transmission conditions can be interpreted as a reduction of the system to two decoupled half-lines: a Dirichlet condition is imposed on one of them, while a transmission condition that is the third-order counterpart of a Robin condition is imposed on the other one, along with a classical Neumann condition.

By Corollary 3.20, $A_{Y,U}$ generates a unitary group provided $|U|^2 = \alpha_-/\alpha_+$. By Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions provided $|U|^2 \leq \alpha_-/\alpha_+$.

Example 4.3. Let $Y := \mathrm{lin}\{(1, 0)\}$. Then

$$D(A_{Y,U}) = \{u \in D(A_0^*) : u(0+) = 0, \ u''(0-)\alpha_- = -\tfrac{1}{2}\beta_- u(0-), \ u'(0+) = Uu'(0-)\},$$
$$A_{Y,U}u = -A_0^* u.$$

If $U = 0$, these transmission conditions can be interpreted as a reduction of the system to two decoupled half-lines: Dirichlet and Neumann conditions are imposed on one of them, while the analog of a Robin condition is imposed on the other one. By Corollary 3.20, $A_{Y,U}$ generates a unitary group provided $|U|^2 = \alpha_-/\alpha_+$. By Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions provided $|U|^2 \leq \alpha_-/\alpha_+$.

Due to lack of conditions on $u''(0+)$ and/or $u''(0-)$, (3-8) cannot be generally satisfied in any of the previous three cases and therefore the corresponding systems do not enjoy conservation of mass.

Example 4.4. Let $Y := \mathrm{lin}\{(1, 1)\}$. Then

$$D(A_{Y,U}) = \{u \in D(A_0^*) : u(0-) = u(0+) =: u(0),$$
$$u''(0+)\alpha_+ - u''(0-)\alpha_- = \tfrac{1}{2}(\beta_- - \beta_+)u(0), \ u'(0+) = Uu'(0-)\},$$
$$A_{Y,U}u = -A_0^* u.$$

By Corollary 3.20, $A_{Y,U}$ generates a unitary group provided $|U|^2 = \alpha_-/\alpha_+$: observe that this is in particular the case if $\alpha_+ = \alpha_-$, $\beta_+ = \beta_-$ and $U = 1$, meaning that not only u, but also u' and u'' are continuous in the origin: this is the classical case considered in the literature and amounts to the free Airy equation on $\mathbb{R}$; see, e.g., the summary in [Linares and Ponce 2009, §7.1]. In view of Remark 3.10, generation of a mass-preserving unitary group still holds under the more general assumption that $U = e^{i\phi}$ for some $\phi \in [0, 2\pi)$, while in view of the prescribed transmission conditions, (3-8) cannot be satisfied unless $\beta_+ = \beta_-$. Even upon dropping the assumption that $\beta_+ = \beta_-$ we obtain the third-order counterpart of a δ-interaction, under which generation of a unitary group is still given.

On the other hand, by Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions already under the weaker assumption that $|U|^2 \leq \alpha_-/\alpha_+$.

Example 4.5. Let $Y := \mathrm{lin}\{(1, -1)\}$. Then

$$D(A_{Y,U}) = \{u \in D(A_0^*) : u(0-) = -u(0+),$$
$$u''(0+)\alpha_+ + u''(0-)\alpha_- = \tfrac{1}{2}(\beta_+ - \beta_-)u(0-), \ u'(0+) = Uu'(0-)\},$$
$$A_{Y,U}u = -A_0^* u.$$

By Corollary 3.20, $A_{Y,U}$ generates a unitary group provided $|U|^2 = \alpha_-/\alpha_+$: we can regard this case as a third-order counterpart of δ'-interactions of second-order operators. By (3-8), the system enjoys

conservation of mass if and only if additionally

$$2\alpha_+ u''(0+) + \tfrac{1}{2}(\beta_+ - \beta_-)u(0-) = 0, \tag{4-1}$$

which is generally not satisfied (remember that we are not considering the degenerate case $\alpha = 0$).

By Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions provided $|U|^2 \le \alpha_-/\alpha_+$.

Example 4.6. Let $Y := \mathbb{C}^2$. Then

$$D(A_{Y,U}) = \{u \in D(A_0^*) : u''(0-)\alpha_- = -\tfrac{1}{2}\beta_- u(0-), \ u''(0+)\alpha_+ = -\tfrac{1}{2}\beta_+ u(0+), \ u'(0+) = Uu'(0-)\},$$
$$A_{Y,U}u = -A_0^* u.$$

These transmission conditions amount to considering two decoupled systems, each with Robin-like conditions along with a Neumann condition on one of them. By Corollary 3.20, $A_{Y,U}$ generates a unitary group provided $|U|^2 = \alpha_-/\alpha_+$; by (3-8), the system enjoys conservation of mass if and only if additionally

$$-\tfrac{1}{2}\beta_- u(0-) + \tfrac{1}{2}\beta_+ u(0+) + \beta_- u(0-) - \beta_+ u(0+) = 0, \tag{4-2}$$

which is generally only satisfied if $\beta_- = \beta_+ = 0$. By Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions provided $|U|^2 \le \alpha_-/\alpha_+$.

Let us now describe one particular example where the first derivative is not separated and the corresponding semigroup is unitary.

Example 4.7. We consider the case $\alpha_- = \alpha_+ = 1$, $\beta_- = \beta_+ = 0$. Define the 3×3-matrix $L : \mathcal{G}_- \to \mathcal{G}_+$ by

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \end{pmatrix}.$$

An easy calculation yields

$$L^* \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} L = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix};$$

i.e., $L^* B_+ L = B_-$. Thus, for $x, y \in \mathcal{G}_-$ we have

$$\langle Lx \mid Ly \rangle_+ = (B_+ Lx \mid Ly) = (L^* B_+ Lx \mid y) = (B_- x \mid y) = \langle x \mid y \rangle_-.$$

Hence, L defines a $(\mathcal{G}_-, \mathcal{G}_+)$-unitary operator. Combining Theorems 3.8 and 3.7 we obtain that A_L is skew-self-adjoint. Note that

$$D(A_L) = \{u \in D(A_0^*) : u(0+) = u(0-) =: u(0),$$
$$u'(0+) = \sqrt{2}u(0) + u'(0-), \ u''(0+) = u(0) + \sqrt{2}u'(0-) + u''(0-)\},$$

so the transmission conditions couple the values of the function and its first and second derivatives at the boundary point. By (3-8), mass is not conserved.
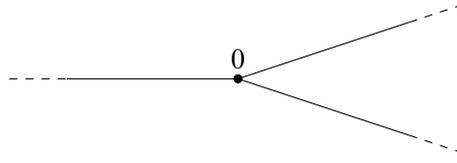
Figure 4. A graph consisting of three half-lines.

Remark 4.8. As mentioned in the Introduction, in the recent paper [Deconinck et al. 2016] the Airy equation is treated (with $\beta = 0$, which makes difference when translation invariance is broken) on a line with an interface at a point where linear transmission conditions are imposed. The authors give conditions for the solvability of the evolution problem in terms of the coefficients appearing in the transmission conditions. An interesting problem could be to compare the conditions obtained there with the ones given in the present paper in the case of two half-lines.

The case of three half-lines. Let us now describe the operator (again with separated first derivative) for the case of three half-lines; see Figure 4.

Let $|E_-| = 1$ and $|E_+| = 2$, which describes two confluent channels. By Proposition 3.1 it is impossible that the Airy equation is governed by a unitary group in this setting; however, we are going to discuss a few concrete cases where a semigroup of contractions is generated by the Airy operator.

Let Y be a subspace of $\ell^2(E_-) \oplus \ell_+(E_+) \cong \mathbb{C} \oplus \mathbb{C}^2 \cong \mathbb{C}^3$ and $U : \ell^2(E_-, \alpha_-) \to \ell^2(E_+, \alpha_+)$ be a linear mapping, i.e., $U \in M_{21}(\mathbb{C}) \simeq \mathbb{C}^2$: we denote for simplicity

$$U = (U_1, U_2)^\top.$$

Then U is -contractive if and only if $|U_1|^2 \alpha_{+,1} + |U_2|^2 \alpha_{+,2} \le \alpha_-$. Note that U^* is given by $U_1^* = \bar{U}_1(\alpha_{+,1}/\alpha_-)$ and $U_2^* = \bar{U}_2(\alpha_{+,2}/\alpha_-)$.

Example 4.9. Let $Y := \{(0, 0, 0)\}$. Then

$$D(A_{Y,U}) = \big\{u \in D(A_0^*) : u(0-) = u_1(0+) = u_2(0+) = 0, \ (u_1'(0+), u_2'(0+))^\top = u'(0-)U\big\},$$
$$A_{Y,U}u = -A_0^* u.$$

As in Example 4.1, whenever $U = (0, 0)^\top$, the associated system effectively reduces to three decoupled half-lines with Dirichlet, resp. Dirichlet and Neumann boundary conditions. By Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions provided

$$|U_1|^2 \alpha_{+,1} + |U_2|^2 \alpha_{+,2} \le \alpha_-.$$

In view of (3-8), mass is generally not conserved regardless of U.

Example 4.10. Let $Y := \mathrm{lin}\{(1, 1, 1)\}$. Then

$$D(A_{Y,U}) = \big\{u \in D(A_0^*) : u(0-) = u_1(0+) = u_2(0+) =: u(0),$$
$$-\alpha_- u''(0-) + \alpha_{+,1} u_1''(0+) + \alpha_{+,2} u_2''(0+) = \tfrac{1}{2}(\beta_- - \beta_{+,1} - \beta_{+,2})u(0),$$
$$(u_1'(0+), u_2'(0+))^\top = u'(0-)U\big\},$$
$$A_{Y,U}u = -A_0^* u.$$

Observe that the considered transmission conditions impose continuity of the values of u in the center of the star; in fact, the transmission conditions are the analog of a δ-interaction. By Theorem 3.23, $A_{Y,U}$ generates a semigroup of contractions provided

$$|U_1|^2\alpha_{+,1} + |U_2|^2\alpha_{+,2} \le \alpha_-.$$

Regardless of U, this semigroup is mass-preserving if and only if $\beta_- - \beta_{+,1} - \beta_{+,2} = 0$.

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References

[Ammari and Crépeau 2017] K. Ammari and E. Crépeau, "Feedback stabilization and boundary controllability of the Korteweg–de Vries equation on a star-shaped network", preprint, 2017. arXiv

[Arendt et al. 2015] W. Arendt, R. Chill, C. Seifert, J. Voigt, and H. Vogt, "Form methods for evolution equations", lecture notes, 2015, available at https://www.mat.tuhh.de/veranstaltungen/isem18/pdf/LectureNotes.pdf.

[Berkolaiko and Kuchment 2013] G. Berkolaiko and P. Kuchment, *Introduction to quantum graphs*, Mathematical Surveys and Monographs **186**, Amer. Math. Soc., Providence, RI, 2013. MR Zbl

[Bona and Cascaval 2008] J. L. Bona and R. C. Cascaval, "Nonlinear dispersive waves on trees", *Can. Appl. Math. Q.* **16**:1 (2008), 1–18. MR Zbl

[Bona et al. 2002] J. L. Bona, S. M. Sun, and B.-Y. Zhang, "A non-homogeneous boundary-value problem for the Korteweg–de Vries equation in a quarter plane", *Trans. Amer. Math. Soc.* **354**:2 (2002), 427–490. MR Zbl

[Bona et al. 2003] J. L. Bona, S. M. Sun, and B.-Y. Zhang, "A nonhomogeneous boundary-value problem for the Korteweg–de Vries equation posed on a finite domain", *Comm. Partial Differential Equations* **28**:7-8 (2003), 1391–1436. MR Zbl

[Cacciapuoti et al. 2017] C. Cacciapuoti, D. Finco, and D. Noja, "Ground state and orbital stability for the NLS equation on a general starlike graph with potentials", *Nonlinearity* **30**:8 (2017), 3271–3303. MR Zbl

[Carlson 1999] R. Carlson, "Inverse eigenvalue problems on directed graphs", *Trans. Amer. Math. Soc.* **351**:10 (1999), 4069–4088. MR Zbl

[Cavalcante 2017] M. Cavalcante, "The Korteweg–de Vries equation on a metric star graph", preprint, 2017. arXiv

[Colin and Ghidaglia 2001] T. Colin and J.-M. Ghidaglia, "An initial-boundary value problem for the Korteweg–de Vries equation posed on a finite interval", *Adv. Differential Equations* **6**:12 (2001), 1463–1492. MR Zbl

[Colliander and Kenig 2002] J. E. Colliander and C. E. Kenig, "The generalized Korteweg–de Vries equation on the half line", *Comm. Partial Differential Equations* **27**:11-12 (2002), 2187–2266. MR Zbl

[Craig and Goodman 1990] W. Craig and J. Goodman, "Linear dispersive equations of Airy type", *J. Differential Equations* **87**:1 (1990), 38–61. MR Zbl

[Deconinck et al. 2016] B. Deconinck, N. E. Sheils, and D. A. Smith, "The linear KdV equation with an interface", *Comm. Math. Phys.* **347**:2 (2016), 489–509. MR Zbl

[Dritschel and Rovnyak 1990] M. A. Dritschel and J. Rovnyak, "Extension theorems for contraction operators on Kreĭn spaces", pp. 221–305 in *Extension and interpolation of linear operators and matrix functions*, edited by I. Gohberg, Oper. Theory Adv. Appl. **47**, Birkhäuser, Basel, 1990. MR

[Engel and Nagel 2000] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics **194**, Springer, 2000. MR Zbl

[Exner 2013] P. Exner, "Momentum operators on graphs", pp. 105–118 in *Spectral analysis, differential equations and mathematical physics: a festschrift in honor of Fritz Gesztesy's 60th birthday*, edited by H. Holden et al., Proc. Sympos. Pure Math. **87**, Amer. Math. Soc., Providence, RI, 2013. MR Zbl

[Faminskii 2007] A. V. Faminskii, "Quasilinear evolution equations of the third order", *Bol. Soc. Parana. Mat.* (3) **25**:1-2 (2007), 91–108. MR Zbl

[Fokas 2008] A. S. Fokas, *A unified approach to boundary value problems*, CBMS-NSF Regional Conference Series in Applied Mathematics **78**, SIAM, Philadelphia, PA, 2008. MR Zbl

[Fokas et al. 2016] A. S. Fokas, A. A. Himonas, and D. Mantzavinos, "The Korteweg–de Vries equation on the half-line", *Nonlinearity* **29**:2 (2016), 489–527. MR Zbl

[Hayashi and Kaikina 2004] N. Hayashi and E. Kaikina, *Nonlinear theory of pseudodifferential equations on a half-line*, North-Holland Mathematics Studies **194**, Elsevier Science, Amsterdam, 2004. MR Zbl

[Hille and Phillips 1957] E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, American Mathematical Society Colloquium Publications **31**, Amer. Math. Soc., Providence, RI, 1957. MR Zbl

[Holmer 2006] J. Holmer, "The initial-boundary value problem for the Korteweg–de Vries equation", *Comm. Partial Differential Equations* **31**:7-9 (2006), 1151–1190. MR Zbl

[Jacovkis 1991] P. M. Jacovkis, "One-dimensional hydrodynamic flow in complex networks and some generalizations", *SIAM J. Appl. Math.* **51**:4 (1991), 948–966. MR Zbl

[Korteweg and de Vries 1895] D. J. Korteweg and G. de Vries, "On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves", *Philos. Mag.* (5) **39**:240 (1895), 422–443. MR Zbl

[Lannes 2013] D. Lannes, *The water waves problem*, Mathematical Surveys and Monographs **188**, Amer. Math. Soc., Providence, RI, 2013. Mathematical analysis and asymptotics. MR Zbl

[Linares and Ponce 2009] F. Linares and G. Ponce, *Introduction to nonlinear dispersive equations*, Springer, 2009. MR Zbl

[Miller 1997] J. R. Miller, "Spectral properties and time decay for an Airy operator with potential", *J. Differential Equations* **141**:1 (1997), 102–121. MR Zbl

[Mugnolo 2014] D. Mugnolo, *Semigroup methods for evolution equations on networks*, Springer, 2014. MR Zbl

[Mugnolo and Rault 2014] D. Mugnolo and J.-F. Rault, "Construction of exact travelling waves for the Benjamin–Bona–Mahony equation on networks", *Bull. Belg. Math. Soc. Simon Stevin* **21**:3 (2014), 415–436. MR Zbl

[Nachbin and Simões 2012] A. Nachbin and V. da Silva Simões, "Solitary waves in open channels with abrupt turns and branching points", *J. Nonlinear Math. Phys.* **19**:suppl. 1 (2012), art. id. 1240011. MR Zbl

[Nachbin and Simões 2015] A. Nachbin and V. S. Simões, "Solitary waves in forked channel regions", *J. Fluid Mech.* **777** (2015), 544–568.

[Noja 2014] D. Noja, "Nonlinear Schrödinger equation on graphs: recent results and open problems", *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **372**:2007 (2014), art. id. 20130002. MR Zbl

[Schmüdgen 2012] K. Schmüdgen, *Unbounded self-adjoint operators on Hilbert space*, Graduate Texts in Mathematics **265**, Springer, 2012. MR Zbl

[Schubert et al. 2015] C. Schubert, C. Seifert, J. Voigt, and M. Waurick, "Boundary systems and (skew-)self-adjoint operators on infinite metric graphs", *Math. Nachr.* **288**:14-15 (2015), 1776–1785. MR Zbl

[Sobirov et al. 2015a] Z. A. Sobirov, M. I. Akhmedov, O. V. Karpova, and B. Jabbarova, "Linearized KdV equation on a metric graph", *Nanosystems* **6**:6 (2015), 757–761. Zbl

[Sobirov et al. 2015b] Z. A. Sobirov, M. I. Akhmedov, and H. Uecker, "Cauchy problem for the linearized KdV equation on general metric star graphs", *Nanosystems* **6**:2 (2015), 198–204.

[Sobirov et al. 2015c] Z. A. Sobirov, H. Uecker, and M. I. Akhmedov, "Exact solutions of the Cauchy problem for the linearized KdV equation on metric star graphs", *Uzbek. Mat. Zh.* **2015**:3 (2015), 143–154. MR

[Stoker 1957] J. J. Stoker, *Water waves: the mathematical theory with applications*, Pure and Applied Mathematics **4**, Interscience, New York, 1957. MR Zbl

[Stokes 1847] G. G. Stokes, "On the theory of oscillatory waves", *Trans. of Cambridge Phil. Soc.* **8**:4 (1847), 441–473.

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