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RIGIDITY OF MINIMIZERS IN NONLOCAL PHASE TRANSITIONS

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We obtain the classification of certain global bounded solutions for semilinear nonlocal equations of the type

$$\Delta^s u = W'(u) \quad \text{in } \mathbb{R}^n, \quad \text{with } s \in \left(\tfrac{1}{2}, 1\right),$$

where W is a double-well potential.

1. Introduction

We extend to the case of the fractional Laplacian Δ^s with $s \in \left(\frac{1}{2}, 1\right)$ the results from [Savin 2009; 2017] concerning a conjecture of De Giorgi about the classification of certain global bounded solutions for semilinear equations of the type

$$\Delta u = W'(u),$$

where W is a double-well potential.

We consider the Ginzburg–Landau energy functional with nonlocal interactions

$$J(u, \Omega) = \frac{1}{4} \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}}\, dx\, dy + \int_{\Omega} W(u)\, dx,$$

with $|u| \le 1$. Here W is a double-well potential with minima at 1 and -1 satisfying

$$W \in C^2([-1, 1]), \quad W(-1) = W(1) = 0, \quad W > 0 \text{ on } (-1, 1),$$
$$W'(-1) = W'(1) = 0, \quad W''(-1) > 0, \quad W''(1) > 0.$$

The classical double-well potential W to have in mind is

$$W(s) = \tfrac{1}{4}(1 - s^2)^2.$$

Physically $u \equiv -1$ and $u \equiv 1$ represent the stable "phases". A critical function for the energy J corresponds to a phase transition with nonlocal interaction between these states, and it satisfies the Euler–Lagrange equation

$$\Delta^s u = W'(u),$$

where $\Delta^s u$ is defined as

$$\Delta^s u(x) = \text{PV} \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|y - x|^{n+2s}}\, dy.$$

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Our main result provides the classification of minimizers with asymptotically flat level sets.

Theorem 1.1. *Let u be a global minimizer of J in $\mathbb{R}^n$ with $s \in \left(\frac{1}{2}, 1\right)$. If the 0 level set $\{u = 0\}$ is asymptotically flat at ∞, then u is one-dimensional.*

The hypothesis that $\{u = 0\}$ is asymptotically flat means that there exist sequences of positive numbers θ_k, l_k and unit vectors ξ_k with $l_k \to \infty$, $\theta_k l_k^{-1} \to 0$, such that

$$\{u = 0\} \cap B_{l_k} \subset \{|x \cdot \xi_k| < \theta_k\}.$$

By saying that u is one-dimensional we understand that u depends only on one direction ξ; i.e., $u = g(x \cdot \xi)$.

A more quantitative version of Theorem 1.1 is given in Theorem 6.1.

In a subsequent work [Savin 2018] we will treat also the case $s = \frac{1}{2}$, which requires some modifications of the methods presented in this paper. We remark that Theorem 1.1 when $s \in \left(0, \frac{1}{2}\right)$ was obtained recently by Dipierro, Serra and Valdinoci [2016].

It is known that blowdowns of the level set $\{u = 0\}$ have different behavior depending on the value of s. If $s \geq \frac{1}{2}$, there are sequences $\varepsilon_k \{u = 0\}$ with $\varepsilon_k \to 0$ that converge uniformly on compact sets to a minimal surface and, if $s < \frac{1}{2}$ they converge to an s-nonlocal minimal surface. This follows from a Γ-convergence result together with a uniform density estimate of level sets of minimizers which were obtained by the author and Valdinoci in [Savin and Valdinoci 2012; 2014]; see for example Corollary 1.7 in the latter paper.

From the classification of global minimal surfaces in low dimensions we find that the level sets of minimizers of J are always asymptotically flat at ∞ in dimension $n \leq 7$ if $s \geq \frac{1}{2}$, and we obtain the following corollary of Theorem 1.1.

Theorem 1.2. *A global minimizer of J is one-dimensional in dimension $n \leq 7$ if $s \in \left(\frac{1}{2}, 1\right)$.*

Another consequence of Theorem 1.1 is the following version of De Giorgi's conjecture to the fractional Laplace case.

Theorem 1.3. *Let $u \in C^2(\mathbb{R}^n)$ be a solution of*

$$\Delta^s u = W'(u), \tag{1-1}$$

with $s \in \left(\frac{1}{2}, 1\right)$, such that

$$|u| \leq 1, \quad \partial_n u > 0, \quad \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1. \tag{1-2}$$

Then u is one-dimensional if $n \leq 8$.

Theorems 1.2 and 1.3 without the limit assumption in (1-2) have been proved in two and three dimensions using stability inequality methods. In dimension $n = 3$ and for $s \geq \frac{1}{2}$ they have been established by Cabre and Cinti [2014], and in dimension $n = 2$ for all $s \in (0, 1)$ by Sire and Valdinoci [2009]; see also [Cabré and Cinti 2010; Cabré and Sire 2015; Cabré and Solà-Morales 2005]. The case $n = 3$ and $s \in \left(0, \frac{1}{2}\right)$ was also addressed recently by S. Dipierro, A. Farina, and E. Valdinoci [Dipierro et al. 2018].

It is not difficult to show that the ± 1 limit assumption implies that u is a global minimizer in $\mathbb{R}^n$; see for example Theorem 1 in [Palatucci et al. 2013]. Since $\{u = 0\}$ is a graph, it is asymptotically flat in dimension $n \leq 8$ and Theorem 1.1 applies.

Similarly we see that if the 0 level set is a graph in the x_n-direction which has a one-sided linear bound at ∞ then the conclusion is true in any dimension.

Theorem 1.4. *If u satisfies (1-1), (1-2),*

$$\{u = 0\} \subset \{x_n < C(1 + |x'|)\},$$

and $s \in \left(\frac{1}{2}, 1\right)$ then u is one-dimensional.

Our proof of Theorem 1.1 follows closely the one for the classical Laplacian given in [Savin 2017]. The main steps consist in (1) finding some appropriate families of radial subsolutions, (2) applying a version of the weak Harnack inequality and (3) a Γ-convergence result. Some new technicalities are present in our setting due to the nonlocal nature of the equation. For example in the improvement-of-flatness property Theorem 6.1, we need to impose a geometric restriction to the level set $\{u = 0\}$ possibly outside the flat cylinder $\mathcal{C}(l, \theta)$.

It turns out that when $s \in \left(\frac{1}{2}, 1\right)$, the level sets of u satisfy a local curvature estimate. For example, at a point of $\{u = 0\}$ which has a large ball of radius R tangent from one side, we can estimate its curvatures in terms of R^{-1} (see Lemma 4.3). In the borderline case $s = \frac{1}{2}$ the curvature bound requires a logarithmic correction and the same methods no longer apply.

We prove Theorem 1.1 by making use of the extension property of the fractional Laplacian of [Caffarelli and Silvestre 2007]. Precisely we consider the extension $U(x, y)$ of $u(x)$ in $\mathbb{R}^{n+1}_+$ such that

$$\operatorname{div}(y^a \nabla U) = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \qquad U(x, 0) = u(x), \qquad a := 1 - 2s \in (-1, 1),$$

and then

$$\Delta^s u(x) = c_{n,s} \lim_{y \to 0^+} y^a U_y(x, y),$$

with $c_{n,s}$ a constant that depends only on n and s. Then global minimizers of $J(u)$ in $\mathbb{R}^n$ with $|u| \leq 1$ correspond to global minimizers of the "extension energy" $\mathcal{J}(U)$ with $|U| \leq 1$, where

$$\mathcal{J}(U) := \frac{c_{n,s}}{2} \int |\nabla U|^2 y^a \, dx \, dy + \int W(u) \, dx.$$

After dividing by a constant and relabeling W, we may fix $c_{n,s}$ to be 1. We obtain an improvement-of-flatness property for the level sets of minimizers of $\mathcal{J}$ which are defined in large balls $\mathcal{B}_R^+$; see Theorem 6.1. We remark that the principal use of the extension is to make the various subsolution computations easier to handle and it is not essential to the method of proof.

The paper is organized as follows. In Sections 2 and 3 we introduce some notation and then construct a family of axial subsolutions. In Section 4 we provide certain "viscosity solution" properties of the level set $\{u = 0\}$. In Section 5 we obtain a Harnack inequality of the 0 level set and in Section 6 we prove Theorem 6.1.

2. Notation and preliminaries

We introduce the following notation:

We denote points in $\mathbb{R}^n$ as $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. The ball of center z and radius r is denoted by $B_r(z)$,

$$B_r(z) := \{x \in \mathbb{R}^n : |x - z| < r\}, \quad B_r := B_r(0).$$

The cylinder with base l and height θ is denoted by $\mathcal{C}(l, \theta) \subset \mathbb{R}^n$,

$$\mathcal{C}(l, \theta) := \{x : |x'| \le l,\ |x_n| \le \theta\}.$$

Points in the extension variables $\mathbb{R}^{n+1}_+$ are denoted by (x, y) with $y > 0$, and the ball of radius r as $\mathcal{B}_r^+$,

$$\mathcal{B}_r^+ := \{(x, y) \in \mathbb{R}^{n+1}_+ : |(x, y)| < r\} \subset \mathbb{R}^{n+1}.$$

Given a function $U(x, y)$, we define u to be its trace on $\{y = 0\}$,

$$u(x) = U(x, 0).$$

Also let

$$a := 1 - 2s \in (-1, 0),$$

and

$$\Delta_a U := \Delta U + a \frac{U_y}{y} = y^{-a} \operatorname{div}(y^a \nabla U),$$

$$\partial_y^{1-a} U(x) := \lim_{y \to 0^+} y^a U_y(x, y) = \frac{1}{1-a} \lim_{y \to 0^+} y^{a-1}(U(x, y) - U(x, 0)).$$

We define the energy $\mathcal{J}$ as

$$\mathcal{J}(U, \mathcal{B}_R^+) := \frac{1}{2} \int_{\mathcal{B}_R^+} |\nabla U|^2 y^a \, dx \, dy + \int_{B_r} W(u) \, dx,$$

and a critical function U for $\mathcal{J}$ satisfies the Euler–Lagrange equation

$$\Delta_a U = 0, \quad \partial_y^{1-a} U = W'(u). \tag{2-1}$$

In [Palatucci et al. 2013, Theorem 2], see also [Cabré and Sire 2014], they proved the existence and uniqueness up to translations of a global minimizer of $\mathcal{J}$ in two dimensions which is increasing in the first variable and which has limits ± 1 at infinity. Precisely there exists a unique $G : \mathbb{R}^2_+ \to (-1, 1)$ that solves (2-1) such that $G(t, y)$ is increasing in the t-variable and its trace $g(t) := G(t, 0)$ satisfies

$$g(0) = 0, \quad \lim_{t \to \pm\infty} g(t) = \pm 1.$$

Moreover, g and g' have the asymptotic behavior

$$1 - |g| \sim \min\{1, |t|^{-2s}\}, \quad g' \sim \min\{1, |t|^{-1-2s}\},$$

and since $a \in (-1, 0)$ we have $\mathcal{J}(G, \mathbb{R}^2_+) < \infty$.

Since $\Delta_a G_t = 0$ and $G_t \geq 0$, we easily conclude that

$$|\nabla G| \leq C \min\{1, r^{-1}\}, \quad G_t \geq c\, r^{-1-2s}, \tag{2-2}$$

where r denotes the distance to the origin in the (t, y)-plane.

In Theorem 6.1 we show that the only global minimizer of $\mathcal{J}$ that has asymptotically flat level sets on $y = 0$ is $G(x_n, y)$ up to translations and rotations.

For simplicity of notation we assume that W is uniformly convex outside the interval $[g(-1), g(1)]$.

Constants that depend on n, s, W, G are called universal constants, and we denote them by C, c. In the course of the proofs, the values of C, c may change from line to line when there is no possibility of confusion. If the constants depend on other parameters, say θ, ρ, then we denote them by $C(\theta, \rho)$ etc.

3. Two-dimensional barriers

We construct two families of comparison functions G_R and Ψ_R which are perturbations of the solution G.

Lemma 3.1 (radial supersolutions). *For all large R, there exist continuous functions $G_R : \mathbb{R}^2 \to (-1, 1]$ and universal constants $\delta > 0$ small, C large such that*

(1) $G_R = 1$ *outside* $\mathcal{B}^+_{R^{1-\delta}} \cup ((-\infty, 0] \times [0, R^{1-\delta}])$,

(2) $G_R(t, y)$ *is nondecreasing in* t, *and* $\partial_t G_R = 0$ *outside* $\mathcal{B}^+_{R^{1-\delta}}$,

(3) $$|G_R - G| \leq \frac{C}{R} \quad \text{in } \mathcal{B}^+_4,$$

(4) $$\Delta_a G_R + \frac{2(n-1)}{R} |\nabla G_R| \leq 0,$$

and on $y = 0$,

$$\partial_y^{1-a} G_R < W'(G_R) \quad \text{if } t \notin [-1, 1].$$

The inequalities in (4) are understood in the viscosity sense.

Notice that by (2-2), property (3) implies

$$G_R(t, y) \leq G\left(t + \frac{C'}{R}, y\right) \quad \text{in } \mathcal{B}^+_4.$$

We remark that property (3) and the inequality above hold in any ball $\mathcal{B}^+_K$, for a fixed large constant K, provided that we replace C/R, C'/R by $C(K)/R$, $C'(K)/R$.

Proof. We begin with the following claim whose proof we provide at the end.

Claim. For each $\alpha \in (1, 1-a)$ there exists H a homogeneous function of degree α such that

$$H \geq r^\alpha, \quad \Delta_a H \leq -r^{\alpha-2}, \quad |\nabla H| \leq C r^{\alpha-1}, \quad \partial_y^{1-a} H \leq C|t|^{\alpha-(1-a)}.$$

Here r denotes the distance to the origin and $C = C(\alpha)$ depends on the universal constants and α.

Fix such an α and define

$$H_R := \min\left\{G + \frac{C_0}{R}(H + C_1), 1\right\}, \tag{3-1}$$

with C_0, C_1 large constants to be specified later.

We define G_R as the infimum over all left translations of H_R; i.e.,

$$G_R(t, y) = \inf_{l \geq 0} H_R(t + l, y).$$

Since $|G| < 1$ we have $H_R > -1$, and $H_R = 1$ outside $\mathcal{B}^+_{R^{1-\delta}}$ provided that δ is chosen sufficiently small such that $(1-\delta)\alpha > 1$. Properties (1) and (2) are clearly satisfied.

Notice that H is increasing in a band $[C, \infty) \times [0, 4]$ and we obtain that H_R is increasing in $[-4, \infty) \times [0, 4]$. This gives $G_R = H_R$ in $\mathcal{B}^+_4$ and property (3) is satisfied.

The properties of H and (2-2) imply that in the set $\{H_R < 1\}$ we have

$$|\nabla H_R| \leq C \min\{1, r^{-1}\} + C C_0 R^{-1} r^{\alpha - 1},$$

and

$$\Delta_a H_R \leq -C_0 R^{-1} r^{\alpha-2}.$$

Then the first inequality in (4) holds for H_R provided that C_0 is chosen sufficiently large, and therefore holds also for G_R as the infimum over translations of H_R.

On $y = 0$ in the set $\{H_R < 1\}$ we have

$$\partial_y^{1-a} H_R = \partial_y^{1-a} G + C_0 R^{-1} \partial_y^{1-a} H \leq W'(G) + C R^{-1} |t|^{\alpha - (1-a)}.$$

From the behavior of g and g' for large t, we see that the minimum of $H_R(t, 0)$ occurs at some $t = q_R \sim -R^{1/(2s+\alpha)} \ll -1$ and

$$\|(H_R - G)(t, 0)\|_{L^\infty([q_R, \infty))} \to 0 \quad \text{as } R \to \infty.$$

Since $W'' \geq c$ outside $[g(-1), g(1)]$ we find that when $t \in [q_R, \infty) \setminus [-1, 1]$ and $\{H_R < 1\}$ we have

$$W'(H_R) - W'(G) \geq \tfrac{1}{2} c (H_R - G) \geq c' R^{-1}(|t|^\alpha + C_1);$$

thus, if C_1 is sufficiently large,

$$\partial_y^{1-a} H_R < W'(H_R) \quad \text{in } [q_R, \infty) \setminus [-1, 1].$$

Now the second inequality of (4) is satisfied by G_R as the infimum of left translations of H_R. □

Proof of Claim. We find H as a perturbation of the function $C y^\alpha$ near $y = 0$. Notice that y^{1-a} is Δ_a-harmonic; thus y^α is Δ_a-superharmonic for $\alpha < 1 - a$. However, $C y^\alpha$ does not satisfy the first and last properties given in the claim.

We write H in polar coordinates as $H = r^\alpha h(\theta)$, with h an even function with respect to $\frac{\pi}{2}$, and then

$$r^{2-\alpha} \Delta_a H = h'' + \alpha(\alpha + a) h + a \cot\theta \; h', \tag{3-2}$$

$$\partial_y^{1-a} H = r^{\alpha - (1-a)} \, \partial_\theta^{1-a} h. \tag{3-3}$$

For all small σ, the function

$$h_\sigma = \sigma + \theta^{1-a} - \theta^2$$

gives a negative right-hand side in (3-2) when θ belongs to a small fixed interval $[0, c]$. We choose first M large and then σ small such that the graphs of $M h_\sigma$ and $(\sin\theta)^\alpha$ become tangent by above at some point in the interval $[0, c]$. We "glue" parts of the two graphs in a single graph of a $C^{1,1}$ function $\tilde{h}$. Now it is easy to check that all properties hold by taking h to be a large multiple of $\tilde{h}$. □

From the construction of H_R, G_R we see that both of them decrease with R as we increase R.

Next we construct a similar family Ψ_R with a slightly slower decay in R than G_R. This allows us to have more flexibility in the choice of the two-dimensional profiles of explicit supersolutions. In the next lemma we compare two such profiles Ψ_R and $G_{\bar{R}}$ when R and $\bar{R} \gg R$ have different orders of magnitude. This is an important tool in the proof of the key Propositions 4.6 and 4.7 from next section, where two explicit supersolutions need to be compared in a certain region.

Lemma 3.2. *There exist functions G_R and Ψ_R that satisfy the properties (1)–(4) of Lemma 3.1 for some δ, C universal such that*

$$G_R(t + R^{-\sigma}, y) \geq \Psi_{R^{1-\sigma}}(t, y),$$

with $\sigma \in \left(0, \frac{\delta}{3}\right)$ small universal.

Proof. Denote by $G_{R,\alpha}$ the function constructed in Lemma 3.1.

We choose $G_R := G_{R,\alpha}$, $\Psi_R := G_{R,\beta}$ for some fixed α, β such that $1 < \beta < \alpha < 1 - a$. We take

$$\delta = \min\{\delta(\alpha), \delta(\beta)\} \quad \text{and} \quad C = \max\{C(\alpha), C(\beta)\}$$

and then Lemma 3.1 holds for both G_R and Ψ_R with the same constants δ and C.

We show that

$$H_{R,\alpha}(t + R^{-\sigma}, y) \geq H_{R^{1-\sigma},\beta}(t, y),$$

with $H_{R,\alpha}$ defined as in (3-1), and the lemma follows by taking the infimum over the left translations.

In the inequality above it suffices to restrict to the set where $\{H_{R,\alpha} < 1\}$. We have

$$H_R \geq G + R^{-1}(c_1 r^\alpha + c_2)$$

for some constants c_1, c_2 depending on α. After a translation of $R^{-\sigma}$ we obtain, see (2-2),

$$H_R(t + R^{-\sigma}, y) \geq G(t, y) + c R^{-\sigma} \min\{1, r^{-1-2s}\} + \tfrac{1}{2} R^{-1}(c_1 r^\alpha + c_2).$$

When $r \geq 1$ we use the inequality $a + b \geq a^\mu b^{1-\mu}$ for $\mu > 0$ small, and we find

$$H_R(t + R^{-\sigma}, y) \geq G(t, y) + c(\alpha) R^{-\eta}(r^\gamma + 1), \tag{3-4}$$

with

$$\gamma = \alpha(1-\mu) - \mu(1+2s), \quad \eta = 1 - \mu + \sigma\mu$$

(and $\eta > \sigma$). We choose μ small and then σ such that $\gamma > \beta$ and $\eta < 1-\sigma$. Then the right-hand side of (3-4) is greater than

$$G + R^{\sigma-1}(C_1(\beta)r^\beta + C_2(\beta)) \geq H_{R^{1-\sigma},\beta}$$

for all large R, and the lemma is proved. □

Remark 3.3. Using the monotonicity of Ψ_r with respect to r, we have

$$G_R(s + R^{-\sigma}, y) \geq \Psi_r(s, y) \quad \text{for all } r \geq R^{1-\sigma}.$$

4. Estimates for $\{u = 0\}$

We now derive properties of the level sets of solutions to

$$\Delta_a U = 0, \quad \partial_y^{1-a} U = W'(U), \tag{4-1}$$

which are defined in large domains.

In the next lemma we find axial approximations to the two-dimensional solution G.

Lemma 4.1 (axial approximations). *Let $G_R : \mathbb{R}^2_+ \to (-1, 1]$ be the function constructed in Lemma 3.2. Then its axial rotation in $\mathbb{R}^{n+1}$*

$$\Phi_R(x, y) := G_R(|x| - R, y)$$

satisfies

(1) $\Phi_R = 1$ *outside* $\mathcal{B}^+_{R+R^{1-\delta}}$,

(2) $$\Delta_a \Phi_R \leq 0 \quad \textit{in } \mathbb{R}^{n+1}_+,$$

and

$$\partial_y^{1-a} \Phi_R < W'(\Phi_R) \quad \textit{when } |x| - R \notin [-1, 1].$$

Let $\phi_R(x) = \Phi_R(x, 0)$ denote the trace of Φ_R on $\{y = 0\}$. Notice that ϕ_R is radially increasing, and $\{\phi_R = 0\}$ is a sphere which is in a C/R-neighborhood of the sphere of radius R.

Proof. We have

$$\Delta_a \Phi_R(x, y) = \Delta_a G_R(s, y) + \frac{n-1}{R+s} \partial_s G_R(s, y), \quad s = |x| - R,$$

$$\partial_y^{1-a} \Phi_R(x, 0) = \partial_y^{1-a} G_R(s, 0).$$

The conclusion follows from Lemma 3.2 since $\partial_s G_R = 0$ when $|s| \geq R^{1-\delta}$ and $R + s > \frac{1}{2}R$ when $|s| < R^{1-\delta}$. □

Definition 4.2. We denote by $\Phi_{R,z}$ the translation of Φ_R by z; i.e.,

$$\Phi_{R,z}(x, y) := \Phi_R(x - z, y) = G_R(|x - z| - R, y).$$

Similarly we define $\Psi_{R,z}$ to be the axial rotation of the other two-dimensional solution Ψ_R given in Lemma 3.2,

$$\Psi_{R,z}(x, y) := \Psi_R(|x - z| - R, y).$$

Clearly $\Psi_{R,0}$ satisfies properties (1), (2) of Lemma 4.1.

We recall that we use ϕ, ψ to denote the traces of Φ and Ψ.

Sliding the graph of Φ_R. Assume that u is less than ϕ_{R,x_0} in $B_{2R}(x_0)$. By the maximum principle we obtain $U < \Phi_{R,z}$ with $z = x_0$ in $\mathcal{B}_{2R}(x_0, 0)$ (and therefore globally). We translate the function Φ_R above by moving continuously the center z, and let's assume that it touches U by above, say for simplicity when $z = 0$; i.e., the strict inequality becomes equality for some contact point (x^*, y^*). From Lemma 4.1 we know that Φ_R is a strict supersolution away from $\{y = 0\}$, and moreover the contact point must satisfy $y^* = 0$, $|x^*| - R \in [-1, 1]$; that is, it belongs to the annular region $B_{R+1} \setminus B_{R-1}$ in the n-dimensional subspace $\{y = 0\}$.

Lemma 4.3 (estimates near a contact point). *Assume that the graph of Φ_R touches by above the graph of U at a point $(x^*, 0, u(x^*))$ with $x^* \in B_{R+1} \setminus B_{R-1}$. Let $\pi(x^*)$ be the projection of x^* onto the sphere ∂B_R. Then in $\mathcal{B}_1(\pi(x^*), 0)$:*

(1) *$\{u = 0\}$ is a smooth hypersurface in $\mathbb{R}^n$ with curvatures bounded by C/R which stays in a C/R neighborhood of ∂B_R.*

(2)
$$|U - G(x \cdot \nu - R, y)| \le \frac{C}{R}, \quad \nu := \frac{\pi(x^*)}{R}.$$

Proof. Assume for simplicity that x^* is on the positive x_n-axis and therefore $\pi(x^*) = Re_n$, $|x^* - Re_n| \le 1$. By Lemma 4.1 we have

$$U \le \Phi_R \le G\left(|x| - R + \frac{C}{R}, y\right) \le G\left(x_n - R + \frac{C'}{R}, y\right) =: V \quad \text{in } \mathcal{B}_3(Re_n).$$

Both U and V solve (4-1), and

$$(V - U)(x^*, 0) \le \frac{C''}{R}.$$

Since $V - U \ge 0$ satisfies

$$\Delta_a(V - U) = 0, \quad \partial_y^{1-a}(V - U) = b(x)(V - U),$$
$$b(x) := \int_0^1 W''(tu(x) + (1 - t)v(x))\, dt,$$

we obtain

$$|V - U| \le \frac{C}{R} \quad \text{in } \mathcal{B}_{5/2}(Re_n)$$

from the Harnack inequality with Neumann condition for Δ_a. Moreover since b has bounded Lipschitz norm and $s > \frac{1}{2}$ we obtain $U - V \in C_x^{2,\alpha}$ for some $\alpha > 0$, and

$$\|U - V\|_{C_x^{2,\alpha}(\mathcal{B}_2(Re_n))} \le \frac{C}{R}$$

by local Schauder estimates. This easily implies the lemma. □

Remark 4.4. If instead of $\mathcal{B}_1((\pi(x^*), 0))$ we write the conclusion in $\mathcal{B}_K((\pi(x^*), 0))$ for some large, fixed constant K, then we need to replace C/R by $C(K)/R$. Here $C(K)$ represents a constant which depends also on K.

Next we obtain estimates near a point on $\{u = 0\}$ which admits a one-sided tangent ball of large radius R.

Lemma 4.5. *Assume that U is defined in $\mathcal{B}^+_{2R}$, satisfies (4-1), and that*

(a) $B_R(-Re_n) \subset \{u < 0\}$ *is tangent to* $\{u = 0\}$ *at* 0,

(b) *there is* $x_0 \in B_{R/2}(-Re_n)$ *such that* $u(x_0) \le -1 + c$ *for some* $c > 0$ *small.*

Then:

(1) $\{u = 0\}$ *is smooth in* B_1 *and has curvatures bounded by* C/R.

(2) $|U - G(x_n, y)| \le C/R$ *in* $\mathcal{B}_1$.

Proof. Assume first that $u < \phi_{R/8,z}$ for $z = -Re_n$.

We translate the graph of $\Phi_{R/8,z}$ by moving z continuously upward on the x_n axis. We stop when the translating graph becomes tangent by above to the graph of U for the first time. Denote by $(x^*, 0, u(x^*))$ the contact point and by z^* the final center z and by $\pi(x^*)$ the projection of x^* onto $\partial B_{R/8}(z^*)$.

By Lemma 4.3, $\{u = 0\}$ must be in a C_1/R neighborhood of $\partial B_{R/8}(z^*) \cap B_1(\pi(x^*))$ for some C_1 universal. This implies

$$z^* = te_n \quad \text{with } t \in \left[-\frac{R}{8} - \frac{C_1}{R}, -\frac{R}{8} + \frac{C_1}{R}\right].$$

Moreover, $\pi(x^*) \in B_{C_2}$ for some C_2 large universal, since otherwise $\pi(x^*)$ is at a distance greater than

$$\frac{1}{R}\frac{C_2^2}{8} > \frac{C_1}{R}$$

in the interior of the ball $B_R(-Re_n)$; hence $\{u = 0\}$ must intersect this ball and we reach a contradiction.

Now we apply Lemma 4.3 and Remark 4.4 at $\pi(x^*)$ and obtain the conclusion of the lemma.

It remains to show that $u < \phi_{R/8,-Re_n}$. By hypothesis (b) and the Harnack inequality we see that u is still sufficiently close to -1 in a whole ball $B_{R_0}(x_0)$ for some large universal R_0, and therefore $u < \phi_{R_0/2,x_0}$ provided that c is sufficiently small. Now we deform $\Phi_{R_0/2,x_0}$ by a continuous family of functions $\Phi_{r,z}$ and first we move z continuously from x_0 to $-Re_n$ and then we increase the radius r from R_0 to $\frac{1}{8}R$. By Lemma 4.3, the graphs of these functions cannot touch the graph of U by above and we obtain the desired inequality. With this the lemma is proved. □

In the next proposition we prove a localized version of Lemma 4.5.

Proposition 4.6. *Assume that U satisfies the equation in $\mathcal{B}_{R^{1-\sigma}}$ with σ small, universal as in Lemma 3.2, and*

(a) $B_R(-Re_n) \cap B_{R^{1/2-\sigma}} \subset \{u < 0\}$ *is tangent to* $\{u = 0\}$ *at* 0,

(b) *all balls of radius* $\frac{1}{4}R^{1-\sigma}$ *which are tangent by below to* $\partial B_R(-Re_n)$ *at some point in* $B_{R^{1/2-\sigma}}$ *are included in* $\{u < 0\}$,

(c) *there is* $x_0 \in B_{R^{1-\sigma}/4}(-\frac{1}{2}R^{1-\sigma}e_n)$ *such that* $u(x_0) \le -1 + c$.

Then in B_1 we have that $\{u = 0\}$ is smooth and has curvatures bounded by C/R.

Proof. As in Lemma 4.5, we slide the graph of $\Phi_{R/8,z}$ in the e_n-direction until it touches the graph of U, except that now we restrict only to the region

$$\mathcal{C}_R := \{|x'| \le \tfrac{1}{2} R^{1/2-\sigma}, |x_n| \le \tfrac{1}{2} R^{1-\sigma}, |y| \le \tfrac{1}{2} R^{1-\sigma}\}. \tag{4-2}$$

In order to repeat the argument above we need to show that the first contact point is an interior point and it occurs in $\mathcal{C}_{R/2}$. For this it suffices to prove that

$$U < \Phi_{R/8,z_0} \quad \text{in } \mathcal{C}_R \setminus \mathcal{C}_{R/2}, \qquad z_0 := \left(-\frac{R}{8} + \frac{C_1}{R}\right) e_n. \tag{4-3}$$

We estimate U by using the functions $\Psi_{R,z}$ given in Definition 4.2. Notice that Lemma 4.3 holds if we replace Φ_R by Ψ_R.

Now we slide the graphs $\Psi_{r,z}$, with $r := \frac{1}{4} R^{1-\sigma}$ and $|z'| \le R^{1/2-\sigma}$, $z_n = -2r$, upward in the e_n-direction. We use hypotheses (b), (c) and as in the proof of Lemma 4.5 we find $\Psi_{r,z} > U$ as long as $B_r(z)$ is at distance greater than Cr^{-1} from $\partial B_R(-Re_n)$. We obtain

$$U(x) < \Psi_r(d_1(x) + Cr^{-1}, y), \tag{4-4}$$

where $d_1(x)$ is the signed distance to $\partial B_R(-Re_n)$. From Remark 3.3 we have

$$\Psi_r(s, y) \le G_{R/8}\big(s + \big(\tfrac{1}{8} R\big)^{-3\sigma}, y\big).$$

We obtain

$$U(x, y) < G_{R/8}(d_1(x) + 2R^{-3\sigma}, y). \tag{4-5}$$

Let $d_2(x)$ represent the distance to $\partial B_{R/8}(z_0)$. Then in the region $\mathcal{C}_R \setminus \mathcal{C}_{R/2}$ we have either

(i) $|x'| \ge \frac{1}{2}\big(\frac{1}{2} R\big)^{1/2-\sigma}$ and then

$$d_2(x) - d_1(x) \ge -\frac{C_1}{R} + \frac{1}{R}|x'|^2 \ge 2R^{-3\sigma}, \tag{4-6}$$

or

(ii) $\min\{|x_n|, y\} \ge \frac{1}{8} R^{1-\sigma}$ and then both $(d_2(x), y)$ and $(d_1(x) + 2R^{-\sigma}, y)$ are outside $B^+_{1-\delta} \subset \mathbb{R}^2$; thus $G_{R/8}$ has the same value at these two points.

From (4-5) we find

$$U(x, y) < G_{R/8}(d_2(x), y) \quad \text{in } \mathcal{C}_R \setminus \mathcal{C}_{R/2}, \tag{4-7}$$

and (4-3) is proved. □

Next we consider the case in which the 0 level set of u is tangent by above at the origin to the graph of a quadratic polynomial.

Proposition 4.7. *Let U satisfy the equation in $\mathcal{B}_{R^{1-\sigma}}$ and hypothesis (c) of Proposition 4.6. Assume the surface*

$$\Gamma := \left\{ x_n = \sum_1^{n-1} \frac{a_i}{2} x_i^2 + b' \cdot x' \right\} \cap B_{R^{1/2-\sigma}} \quad \textit{with } |b'| \le \varepsilon, \;\; |a_i| \le \varepsilon^{-2} R^{-1},$$

is tangent to $\{u=0\}$ at 0 for some small ε that satisfies $\varepsilon \geq R^{-\sigma/2}$, and assume further that all balls of radius $\frac{1}{2}R^{1-\sigma}$ which are tangent to Γ by below are included in $\{u<0\}$. Then

$$\sum_1^{n-1} a_i \leq CR^{-1}.$$

Proposition 4.7 states that the blowdown of $\{u=0\}$ satisfies the minimal surface equation in some viscosity sense. Indeed, if we take $\varepsilon = R^{-\sigma/2}$, then the set $R^{\sigma-1}\{u=0\}$ cannot be touched at 0 in an $R^{-1/2}$ neighborhood of the origin by a surface with curvatures bounded by $\frac{1}{2}$ and mean curvature greater than $CR^{-\sigma}$.

Proof. We argue as in the proof of Proposition 4.6 except that now we replace $\partial B_R(-Re_n)$ by Γ and $\partial B_{R/8}(z_0)$ by

$$\Gamma_2 := \left\{ x_n = \sum_1^{n-1} \frac{a_i}{2} x_i^2 + b' \cdot x' + \frac{C_1}{R} - \frac{1}{R}|x'|^2 \right\}.$$

We claim that

$$U(x,y) < G_{R/8}(d_2(x), y) \quad \text{in } \mathcal{C}_R \setminus \mathcal{C}_{R/2}, \tag{4-8}$$

where d_2 represents the signed distance to the Γ_2 surface and $\mathcal{C}_R$ is defined in (4-2). Using the surfaces $\Psi_{r,z}$ as comparison functions we obtain as in (4-4), (4-5) above that

$$U(x,y) < G_{R/8}(d_1(x) + C'r^{-1}, y) \quad \text{in } \mathcal{C}_R,$$

with $d_1(x)$ representing the signed distance to Γ. Notice that (4-6) is valid in our setting. Now we argue as in (4-7) and obtain the desired claim (4-8).

Next we show that $G_{R/8}(d_2(x), y)$ is a supersolution away from the set $\{|d_2| \leq 1, y=0\}$ provided that

$$\sum_1^{n-1} a_i \geq MR^{-1}$$

for some M large, universal to be made precise later. The boundary inequality on $\{y=0\}$ is clearly satisfied and on $\{y>0\}$ we have

$$\Delta_a G_{R/8}(d_2(x), y) = \Delta_a G_{R/8}(s,y) + H(x)\, \partial_s G_{R/8}(s,y), \quad s := d_2(x), \tag{4-9}$$

where $H(x)$ represents the mean curvature at x of the parallel surface to Γ_2, and Δ_a on the right-hand side is with respect to the variables (s,y). If $|s| > R^{1-\delta}$ then $\partial_s G_{R/8} = 0$, and if $|s| \leq R^{1-\delta}$ we show below that $H<0$, and in both cases we obtain $\Delta_a G_{R/8} \leq 0$.

Let κ_i, $i = 1, \dots, n-1$, be the principal curvatures of Γ_2 at the projection of x onto Γ_2. Notice that at this point the slope of the tangent plane to Γ_2 is less than 4ε; hence we have

$$|\kappa_i| \leq 2\varepsilon^{-2}R^{-1} \leq 2R^{\sigma-1}, \quad \sum \kappa_i \leq -\sum a_i + C\varepsilon^2 \max|a_i| \leq -\tfrac{1}{2}MR^{-1}.$$

When $|d_2| \leq R^{1-\delta}$, we obtain $d_2\kappa_i = o(1)$, $d_2\kappa_i^2 = o(R^{-1})$ since $\sigma < \frac{\delta}{3}$; hence

$$H(x) = \sum \frac{\kappa_i}{1-d_2\kappa_i} = \sum \left(\kappa_i + \frac{d_2\kappa_i^2}{1-d_2\kappa_i} \right) \leq -\tfrac{1}{4}MR^{-1}. \tag{4-10}$$

Now we translate the graph of $G_{R/8}(d_2, y)$ along the e_n-direction until it touches the graph of U by above. Precisely, we consider the graphs of $G_R(d_2(x-te_n), y)$ with $t \le 0$ and start with t negative so that the function is identically 1 in $\mathcal{C}_R$. Then we increase t continuously until this graph becomes tangent by above to the graph of U in $\mathcal{C}_R$. Since $u(0)=0$, a contact point must occur for some $t \le 0$ and, by (4-8), this point is interior to $\mathcal{C}_{R/2}$ and lies on $y=0$. Let $(x^*, 0, u(x^*))$ be the first contact point where a translate $G_{R/8}(d_2(x-t^*e_n), y)$ touches U by above. We show that we reach a contradiction if M is chosen sufficiently large.

Define V as

$$V(x,y) := G\left(d_2(x-t^*e_n)+\frac{C}{R}, y\right) \ge G_{R/8}(d_2(x-t^*e_n), y) \ge U(x,y).$$

Notice that

$$\partial_y^{1-a} V = W'(V), \quad (V-U)(x^*,0) \le \frac{C}{R}.$$

In $\mathcal{B}_1(x^*)$ we use the computation (4-9) above for V together with (4-10) and obtain

$$\Delta_a V \le -cMR^{-1} \quad \text{in } \mathcal{B}_1(x^*).$$

The function $Q := (V-U)/(cMR^{-1}) \ge 0$ satisfies in $\mathcal{B}_1(x^*)$

$$\Delta_a Q \le -1, \quad |\partial_y^{1-a} Q| \le CQ, \quad Q(x^*,0) \le C'M^{-1}.$$

By the maximum principle

$$Q(x,y) \ge \mu^2 + \mu\, y^{1-a} - \frac{1}{2(n+1)}(|x-x^*|^2 + y^2)$$

for some μ small universal, and we reach a contradiction at $(x^*,0)$ if M is sufficiently large. □

5. Harnack inequality

We use Proposition 4.6 to prove a Harnack-inequality property for flat level sets; see Theorem 5.1 below. The key step in the proof is to control the x_n-coordinate of the level set $\{u=0\}$ in a set of large measure in the x'-variables.

Notation. We denote by $\mathcal{C}(l,\theta)$ the cylinder

$$\mathcal{C}(l,\theta) := \{|x'| \le l, |x_n| \le \theta\}.$$

Theorem 5.1 (Harnack inequality for minimizers). *Let U be a minimizer of J in $\mathcal{B}_q$ and assume that*

$$0 \in \{u=0\} \cap \mathcal{C}(l,l) \subset \mathcal{C}(l,\theta),$$

and that all balls of radius $q := (l^2\theta^{-1})^{1-\sigma/2}$ which are tangent to $\mathcal{C}(l,\theta)$ by below and above are included in $\{u<0\}$ and $\{u>0\}$ respectively.

Given $\theta_0 > 0$ there exist $\omega > 0$ small depending on n, W, and $\varepsilon_0(\theta_0) > 0$ depending on n, W and θ_0 such that if

$$\theta l^{-1} \le \varepsilon_0(\theta_0), \quad \theta_0 \le \theta,$$

then

$$\{u=0\}\cap\mathcal{C}(\bar l,\bar l)\subset\mathcal{C}(\bar l,\bar\theta),\quad \bar l:=\tfrac{l}{4},\ \bar\theta:=(1-\omega)\theta,$$

and all balls of radius $\bar q:=(\bar l^2\bar\theta^{-1})^{1-\sigma/2}$ *which are tangent to* $\mathcal{C}(\bar l,\bar\theta)$ *by below or above do not intersect* $\{u=0\}$.

The fact that u is a minimizer of J is only used in a final step of the proof. This hypothesis can be replaced by x_n-monotonicity for u, or more generally by the monotonicity of u in a given direction which is not perpendicular to e_n.

Definition 5.2. For a small $a>0$, we denote by $\mathcal{D}_a$ the set of points on

$$\{u=0\}\cap\mathcal{C}\big(\tfrac{3}{4}l,\theta\big)$$

which have a paraboloid of opening $-a$ and vertex $y=(y',y_n)$

$$P_{a,y}:=\big\{x_n=-\tfrac{a}{2}|x'-y'|^2+y_n\big\}$$

tangent by below in $\mathcal{C}(l,\theta)$, and with $P_{a,y}$ below the lateral boundary of $\mathcal{C}(l,\theta)$. In other words we allow only those polynomials $P_{a,y}$ which exit $\mathcal{C}(l,\theta)$ through the "bottom".

We denote by $D_a\subset\mathbb{R}^{n-1}$ the projection of $\mathcal{D}_a$ into $\mathbb{R}^{n-1}$ along the e_n-direction.

By Proposition 4.6 we see that as long as

$$l^{-1}\ge a\ge l^{-2-\eta}\quad\text{and}\quad l\ge C(\theta_0)\tag{5-1}$$

for some η small universal (depending on σ), $\{u=0\}$ has the following property (P):

(P) In a neighborhood of any point of $\mathcal{D}_a$, the set $\{u=0\}$ is a graph in the e_n-direction of a C^2 function with second derivatives bounded by Λa with Λ a universal constant.

Indeed, since $a\le l^{-1}$, at a point $z\in\mathcal{D}_a$ the corresponding paraboloid at z has a tangent ball of radius

$$R:=ca^{-1}\le l^{2+\eta}$$

by below. Since $|z'|\le\frac{3}{4}l$ we see that $\{u=0\}\cap B_{l/4}(z)$ has a tangent ball $B_R(x_0)$ by below at z and hypothesis (a) of Proposition 4.6 holds since

$$\tfrac{l}{4}\ge R^{1/2-\sigma}.$$

The assumption that all balls of radius $q\ge c(\theta_0)l^{2-\sigma}\ge R^{1-\sigma}$ tangent by below to $C(l,\theta)$ are included in $\{u<0\}$ gives that all balls tangent to $\partial B_R(x_0)\cap B_{l/4}(z)$ by below are also included in $\{u<0\}$; hence hypothesis (b) of Proposition 4.6 holds.

Since u is a minimizer, in any sufficiently large ball in $\{u<0\}$ we have points that satisfy $u<-1+c$ and hypothesis (c) holds as well. In conclusion Proposition 4.6 applies and property (P) holds.

Since $\{u=0\}$ satisfies property (P), it satisfies a general version of the weak Harnack inequality which we proved in [Savin 2017]. In particular we are in the setting of Propositions 6.2 and 6.4 (see also Remark 6.7) in that paper.

This means that for any $\mu > 0$ small, there exists $M(\mu)$ depending on μ and universal constants such that if

$$\{u = 0\} \cap (B'_{l/2} \times [-\theta, (\omega-1)\theta]) \neq \varnothing, \quad \omega := (32M)^{-1}, \tag{5-2}$$

then, by Proposition 6.2 in [Savin 2017], we obtain

$$\mathcal{H}^{n-1}(D_a \cap B'_{l/2}) \geq (1-\mu)\mathcal{H}^{n-1}(B'_{l/2}), \quad \text{with } a := M\,\omega\,\theta l^{-2}, \tag{5-3}$$

and

$$\mathcal{D}_a \cap \{|x'| \leq \tfrac{l}{2}\} \subset \{x_n \leq (8M\omega - 1)\theta\} = \{x_n \leq -\tfrac{3}{4}\theta\}. \tag{5-4}$$

We can apply that proposition since the interval I of allowed openings of the paraboloids satisfies, see (5-1),

$$I = [\omega\,\theta l^{-2}, M\omega\,\theta l^{-2}] \subset [l^{-2-\eta}, l^{-1}],$$

provided that $l \geq C(\mu, \theta_0)$ and $\varepsilon_0 \leq c$.

Next we let $\mathcal{D}^*_a$ denote the set of points on

$$\mathcal{D}^*_a := \{u = 0\} \cap \left(\{|x'| \leq \tfrac{l}{2}\} \times [-\tfrac{\theta}{2}, \theta]\right) \tag{5-5}$$

which admit a tangent paraboloid of opening a by above which exit $\mathcal{C}(l, \theta)$ through the "top". Also we denote by $D^*_a \subset \mathbb{R}^{n-1}$ the projection of $\mathcal{D}^*_a$ along e_n. Then according to Proposition 6.4 in [Savin 2017], (applied "upside down") we have

$$\mathcal{H}^{n-1}(D^*_{\tilde a} \cap B'_{l/2}) \geq \mu_0\,\mathcal{H}^{n-1}(B'_{l/2}), \quad \text{with } \tilde a = 8\theta l^{-2}, \tag{5-6}$$

for some μ_0 universal.

We choose μ in (5-2)–(5-4) universal as

$$\mu := \tfrac{1}{2}\mu_0.$$

According to (5-3), (5-6) this gives

$$\mathcal{H}^{n-1}(D_a \cap D^*_{\tilde a}) \geq \tfrac{1}{2}\mu_0 \mathcal{H}^{n-1}(B'_{l/2}). \tag{5-7}$$

Notice that by (5-4), (5-5) the sets $\mathcal{D}_a$ and $\mathcal{D}^*_{\tilde a}$ are disjoint.

At this point we would reach a contradiction to (5-2) if $\{u = 0\}$ were assumed to be a graph in the e_n-direction. Instead we use (5-7) and show that U cannot be a minimizer.

Proof of Theorem 5.1. It suffices to show that

$$\{u = 0\} \cap \mathcal{C}\left(\tfrac{l}{2}, \tfrac{l}{2}\right) \subset C\left(\tfrac{l}{2}, (1-\omega)\theta\right).$$

Then the existence of the balls of size $q \ll l^2\theta^{-1}$ (included in $\{u < 0\}$ and $\{u > 0\}$ respectively) tangent to $C\left(\frac{l}{4}, (1-\omega)\theta\right)$ follows easily as we restrict from the cylinder of size $\frac{l}{2}$ to the one of size $\frac{l}{4}$, and the conclusion is satisfied since $\tilde q \leq q$.

Assume by contradiction that (5-2) holds, and therefore (5-3), (5-7) hold as well. For each $x \in \mathcal{D}_a$ the set $\{u = 0\}$ has a tangent ball of radius $ca^{-1} \geq cl$ by below. Moreover, the normal to this ball at the contact points in the e_n-direction makes a small angle which is bounded by $c\theta l^{-1} \le c\varepsilon_0$. According to Lemma 4.5 part (2) and Remark 4.4, we conclude that for any fixed constant K we have

$$\max_{(t,y)\in\mathcal{B}_K^+} |U(x', x_n + t, y) - G(t, y)| \le \rho, \tag{5-8}$$

with $\rho = \rho(K, \varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$.

We denote the two-dimensional half disk of radius r in the (x_n, y)-variables centered at $z \in \mathbb{R}^n$ as

$$\mathcal{B}_{r,z}^+ := \{(z', z_n + t, y) : |(t, y)| \le r,\ y \ge 0\}.$$

From above we find for all $x \in \mathcal{D}_a$, or similarly if $x \in \mathcal{D}_a^*$, we have

$$J(U, \mathcal{B}_{K,x}^+) \ge \mathcal{J}(G, \mathcal{B}_K^+) - \bar{\rho}, \tag{5-9}$$

with $\bar{\rho} = \bar{\rho}(K, \varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$.

If $x' \in D_a \cap D_{\tilde{a}}^*$ then by (5-4), (5-5) the two points $x^1 = (x', x_n^1) \in \mathcal{D}_a$ and $x^2 = (x', x_n^2) \in \mathcal{D}_{\tilde{a}}^*$ satisfy $x_n^2 - x_n^1 \ge \frac{1}{4}\theta \ge \frac{1}{4}\theta_0$. By (5-8) this means that the two disks $\mathcal{B}_{K,x^i}$ are disjoint provided that ρ is small; thus

$$\mathcal{J}(U, \mathcal{B}_{l/2,(x',0)}^+) \ge 2(\mathcal{J}(G, \mathcal{B}_K^+) - \bar{\rho}) \quad \text{if } x' \in D_a \cap D_{\tilde{a}}^*.$$

We integrate in x' and use also (5-3), (5-7), (5-9) to obtain

$$\mathcal{J}(U, A_{l/2}) \ge \big(1 + \tfrac{1}{2}\mu_0\big)(\mathcal{J}(G, \mathcal{B}_K^+) - \bar{\rho})\, \mathcal{H}^{n-1}(B'_{l/2}),$$

with

$$A_{l/2} := \mathcal{C}\big(\tfrac{l}{2}, \tfrac{l}{2}\big) \times \big[0, \tfrac{l}{2}\big].$$

We choose first K large and then ε_0 small such that $\bar{\rho}$ is sufficiently small so that

$$\mathcal{J}(U, A_{l/2}) \ge \big(1 + \tfrac{1}{4}\mu_0\big)\, \mathcal{J}(G, \mathbb{R}_+^2)\, \mathcal{H}^{n-1}(B'_{l/2}).$$

This contradicts Lemma 5.3 below provided that ε_0 is taken sufficiently small. □

The next lemma is a Γ-convergence result and it is a consequence of the minimality of U in $A_{l/2}$.

Lemma 5.3.

$$\mathcal{J}(U, A_{l/2}) \le \mathcal{J}(G, \mathbb{R}_+^2)\, \mathcal{H}^{n-1}(B'_{l/2}) + \gamma(\varepsilon_0)\, l^{n-1}, \tag{5-10}$$

with $\gamma(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$.

Proof. We interpolate between U and $V(x, y) := G(x_n, y)$ as

$$H = (1 - \varphi)U + \varphi V.$$

Here φ is a cutoff Lipschitz function such that $\varphi = 0$ outside $A_{l/2}$, $\varphi = 1$ in $\mathcal{R}$ and $|\nabla\varphi| \le 8/(1 + y)$ in $A_{l/2} \setminus \mathcal{R}$, where $\mathcal{R}$ is the cone

$$\mathcal{R} := \big\{(x, y) : \max\{|x'|, |x_n|\} \le \tfrac{l}{2} - 1 - 2y\big\}.$$

By the minimality of U we have

$$\mathcal{J}(U, A_{l/2}) \le \mathcal{J}(H, A_{l/2}) = \mathcal{J}(V, \mathcal{R}) + \mathcal{J}(H, A_{l/2} \setminus \mathcal{R}).$$

Since

$$\mathcal{J}(V, \mathcal{R}) \le \mathcal{J}(V, A_{l/2}) \le \mathcal{J}(G, \mathbb{R}^2_+)\, \mathcal{H}^{n-1}(B'_{l/2}),$$

we need to show that

$$\mathcal{J}(H, A_{l/2} \setminus \mathcal{R}) \le \gamma\, l^{n-1}, \tag{5-11}$$

with γ arbitrarily small. We have

$$\begin{aligned}\mathcal{J}(H, A_{l/2} \setminus \mathcal{R})& \\ \le 4 \int_{A_{l/2}\setminus\mathcal{R}} \big(|\nabla\varphi|^2 (V-U)^2 + |\nabla(V-U)|^2\big) y^a \, dx\, dy + \int_D W(u) + W(v) + C(v-u)^2 \, dx,& \end{aligned}\tag{5-12}$$

with $D := \mathcal{C}\big(\tfrac{l}{2}, \tfrac{l}{2}\big) \setminus \mathcal{C}\big(\tfrac{l}{2} - 1, \tfrac{l}{2} - 1\big)$.

We use that $|U|, |V| \le 1$, $|\nabla U|, |\nabla V| \le C/(1+y)$ and we see that in (5-12) the first integral in the region where $y \ge C\gamma^{1/a}$ is bounded by

$$\int_{C\gamma^{1/a}}^{l/2} C_1 (1+y)^{-2} (1+y)\, y^a \, dy \le \frac{\gamma}{4}.$$

Next we notice that u and v are sufficiently close to each other in $\mathcal{C}\big(\tfrac{l}{2}, \tfrac{l}{2}\big)$ away from a thin strip around $x_n = 0$. Indeed, we can use barrier functions as in Proposition 4.6, see (4-4), and bound u by above and below in terms of the function $\psi_{l/2}$ and distance to the hyperplanes $x_n = \pm\theta$. This implies

$$W(u), W(v), |v-u| \le \gamma \quad \text{in } \mathcal{C}\big(\tfrac{l}{2}, \tfrac{l}{2}\big) \text{ if } |x_n| \ge C(\gamma) + \theta,$$

with $C(\gamma)$ large, depending on the universal constants and γ. For the extensions U and V, this gives

$$|V-U|, |\nabla(V-U)| \le C_2 \gamma \quad \text{in } A_{l/2} \text{ if } |x_n| \ge C'(\gamma) + \theta \text{ and } y \le C\gamma^{1/a},$$

with C_2 universal. Now (5-11) easily follows from (5-12). □

6. Improvement of flatness

We state the improvement-of-flatness property of minimizers.

Theorem 6.1 (improvement of flatness). *Let U be a minimizer of J in $\mathcal{B}_q$ and assume that*

$$0 \in \{u = 0\} \cap \mathcal{C}(l, l) \subset \mathcal{C}(l, \theta),$$

and that all balls of radius $q := (l^2\theta^{-1})^{1-\sigma/2}$ which are tangent to $\mathcal{C}(l, \theta)$ by below and above are included in $\{u < 0\}$ and $\{u > 0\}$ respectively.

Given $\theta_0 > 0$ there exist $\eta > 0$ small depending on n, and $\varepsilon_1(\theta_0) > 0$ depending on n, W and θ_0 such that if

$$\theta l^{-1} \le \varepsilon_1(\theta_0), \quad \theta_0 \le \theta,$$

then

$$\{u = 0\} \cap \mathcal{C}_\xi(\bar{l}, \bar{l}) \subset C_\xi(\bar{l}, \bar{\theta}), \quad \bar{l} := \eta l,\ \bar{\theta} := \eta^{3/2}\theta,$$

and all balls of radius $\bar{q} := (\bar{l}^2\bar{\theta}^{-1})^{1-\sigma/2}$ *which are tangent to* $\mathcal{C}_\xi(\bar{l}, \bar{\theta})$ *by below and above are included in* $\{u < 0\}$ *and* $\{u > 0\}$ *respectively.*

Here $\xi \in \mathbb{R}^n$ *is a unit vector and* $C_\xi(\bar{l}, \bar{\theta})$ *represents the cylinder with axis* ξ, *base* $\bar{l}$ *and height* $\bar{\theta}$.

As a consequence of this flatness theorem we obtain our main theorem.

Theorem 6.2. *Let* U *be a global minimizer of* $\mathcal{J}$. *Suppose that the* 0 *level set* $\{u = 0\}$ *is asymptotically flat at* ∞. *Then the* 0 *level set is a hyperplane and* u *is one-dimensional.*

Proof. Without loss of generality assume $u(0) = 0$. Fix $\theta_0 > 0$, and $\varepsilon \le \varepsilon_1(\theta_0)$. We choose k sufficiently large such that, after increasing θ_k if necessary we have $\theta_k l_k^{-1} = \varepsilon$. We can apply Theorem 6.1 since $q = (l_k\varepsilon^{-1})^{1-\sigma/2} \ll l_k$, and we obtain that $\{u = 0\}$ is trapped in a flatter cylinder. We apply Theorem 6.1 repeatedly until the height of the cylinder becomes less than θ_0. We conclude that $\{u = 0\}$ is trapped in a cylinder with flatness less than ε and height θ_0. We let first $\varepsilon \to 0$ and then $\theta_0 \to 0$ and obtain the desired conclusion. □

Proof of Theorem 6.1. The proof is by compactness and it follows from Theorem 5.1 and Proposition 4.7. Assume by contradiction that there exist U_k, θ_k, l_k, ξ_k such that u_k is a minimizer of J, $u_k(0) = 0$, and the level set $\{u_k = 0\}$ stays in the flat cylinder $\mathcal{C}(l_k, \theta_k)$ with $\theta_k \ge \theta_0$, $\theta_k l_k^{-1} \to 0$ as $k \to \infty$ for which the conclusion of Theorem 6.1 doesn't hold.

Let A_k be the rescaling of the 0 level sets given by

$$(x', x_n) \in \{u_k = 0\} \mapsto (z', z_n) \in A_k,$$
$$z' = x' l_k^{-1}, \quad z_n = x_n\theta_k^{-1}.$$

Claim 1. A_k has a subsequence that converges uniformly on $|z'| \le \frac{1}{2}$ to a set $A_\infty = \{(z', w(z')), |z'| \le \frac{1}{2}\}$, where w is a Holder continuous function. In other words, given ε, all but a finite number of the A_k's from the subsequence are in an ε neighborhood of A_∞.

Proof of Claim 1. Fix z_0', $|z_0'| \le \frac{1}{2}$ and suppose $(z_0', z_k) \in A_k$. We apply Theorem 5.1 for the function u_k in the cylinder

$$\{|x' - l_k z_0'| < \tfrac{1}{2} l_k\} \times \{|x_n - \theta_k z_k| < 2\theta_k\}$$

in which the set $\{u_k = 0\}$ is trapped. Thus, there exists an increasing function $\varepsilon_0(\theta) > 0$, $\varepsilon_0(\theta) \to 0$ as $\theta \to 0$, such that $\{u_k = 0\}$ is trapped in the cylinder

$$\{|x' - l_k z_0'| < \tfrac{1}{8} l_k\} \times \{|x_n - \theta_k z_k| < 2(1-\omega)\theta_k\}$$

provided that $4\theta_k l_k^{-1} \le \varepsilon_0(2\theta_k)$. Rescaling back we find that

$$A_k \cap \{|z' - z_0'| \le \tfrac{1}{8}\} \subset \{|z_n - z_k| \le 2(1-\omega)\}.$$

We apply the Harnack inequality repeatedly and we find that

$$A_k \cap \{|z' - z'_0| \le 2^{-2m-1}\} \subset \{|z_n - z_k| \le 2(1-\omega)^m\} \tag{6-1}$$

provided that

$$\theta_k l_k^{-1} \le 4^{-m-1} \varepsilon_0 (2(1-\omega)^m \theta_k).$$

Since these inequalities are satisfied for all k large we conclude that (6-1) holds for all but a finite number of k's. Now the claim follows from Arzelà–Ascoli theorem. □

Claim 2. The function w is harmonic (in the viscosity sense).

Proof of Claim 2. The proof is by contradiction. Fix a quadratic polynomial

$$z_n = P(z') = \tfrac{1}{2} z'^T M z' + \xi \cdot z', \quad \|M\| < \delta^{-1}, \ |\xi| < \delta^{-1},$$

such that $\operatorname{tr} M > \delta$ and $P(z') + \delta |z'|^2$ touches the graph of w, say, at 0 for simplicity, and stays below w in $|z'| < 8\delta$ for some small δ. Notice that at all points in the cylinder $|z'| < 2\delta$, the quadratic polynomial above admits a tangent paraboloid by below of opening $-\delta^{-2}$ which is below $z_n = -2$ when $|z'| \ge 6\delta$.

Thus, for all k large we find points (z_k', z_{kn}) close to 0 such that $P(z') + \text{const}$ touches A_k by below at (z_k', z_{kn}) and stays below it in $|z' - z_k'| < \delta$.

This implies that, after eventually a translation, there exists a surface

$$\Gamma := \left\{ x_n = \frac{\theta_k}{l_k^2} \frac{1}{2} x'^T M x' + \frac{\theta_k}{l_k} \xi_k \cdot x' \right\}, \quad |\xi_k| < 2\delta^{-1},$$

that touches $\{u_k = 0\}$ at the origin and stays below it in $\mathcal{C}(\delta l_k, 2\theta_k)$. Moreover in the cylinder $\mathcal{C}\left(\frac{1}{2} l_k, 2\theta_k\right)$ the surface Γ admits at all points with $|x'| \le \delta l$ a tangent ball by below of radius $\delta^2 l_k^2 \theta_k^{-1} \gg q$. In view of our hypothesis we conclude that $\Gamma \cap B_{\delta l_k}$ admits at all its points a tangent ball of radius q by below which is included in $\{u < 0\}$.

We contradict Proposition 4.7 by choosing R as

$$R^{-1} := C^{-1} \delta \, \theta_k l_k^{-2},$$

with C the constant from Proposition 4.7 and with $\varepsilon = \delta^2$. Then for all large k we have

$$\theta_k l_k^{-1} |\xi_k| \le \varepsilon, \quad \theta_k l_k^{-2} \|M\| \le \varepsilon^{-2} R^{-1}, \quad \delta l_k \ge R^{1/2-\sigma}, \quad q \ge R^{1-\sigma},$$

and Proposition 4.7 applies. We obtain $\operatorname{tr} M \le \delta$ and we have reached a contradiction. □

Since w is harmonic, there exists $0 < \eta$ small depending only on n such that

$$|w - \xi \cdot z'| < \tfrac{1}{2} \eta^{3/2} \quad \text{for } |z'| < 2\eta\ ,$$

and the parabolas of opening $-C$ tangent by below (and above) to

$$z_n = \xi \cdot z' \pm \tfrac{1}{2} \eta^{3/2}$$

in the cylinder $|z'| < 2\eta$ lie below (or above) to the graph of w.

Rescaling back and using the fact that the A_k's converge uniformly to the graph of w and that $\bar{q} < q$ we easily conclude that u_k satisfies the conclusion of Theorem 6.1 for k large enough, and we have reached a contradiction. □

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