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ON WEAK WEIGHTED ESTIMATES OF THE MARTINGALE TRANSFORM AND A DYADIC SHIFT

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We consider weak-type estimates for several singular operators using the Bellman-function approach. In particular, we consider a concrete dyadic shift. We disprove the A_1 conjecture for those operators, which stayed open after Muckenhoupt and Wheeden's conjecture was disproved by Reguera and Thiele.

1. End-point estimates: notation and facts

The end-point estimates play an important part in the theory of singular integrals (weighted and unweighted). They are usually the most difficult estimates in the theory, and the most interesting of course. It is a general principle that one can extrapolate the estimate from the end-point situation to all other situations. We refer the reader to [Cruz-Uribe et al. 2011], which treats this subject of extrapolation in depth.

On the other hand, it happens quite often that the singular integral estimates exhibit a certain "blow-up" near the end point. Catching this blow-up can be a difficult task. We demonstrate this hunt for blow-ups by examples of weighted dyadic singular integrals and their behavior in $L^p(w)$. The end-point p will be naturally 1 (and sometimes slightly unnaturally 2) depending on the martingale singular operator. The singular integrals in this article are the easiest possible. They are dyadic martingale operators on the σ-algebra generated by the usual homogeneous dyadic lattice on the real line. We do not consider any nonhomogeneous situations, and this standard σ-algebra generated by a dyadic lattice $\mathcal{D}$ will be provided with Lebesgue measure.

Our goal will be to show how the Bellman-function technique gives the proof of the blow-up of the weighted estimates of the corresponding weighted dyadic singular operators. This blow-up will be demonstrated by certain estimates from below of the Bellman function of a dyadic problem.

The Bellman-function part will be reduced to the task of finding the lower estimate for the solutions of the concrete Monge–Ampère differential equation with concrete first-order terms (drift).

We will get a logarithmic blow-up not only for the martingale transform but also for a concrete dyadic shift; see our main result, Theorem 2.2.

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2. End-point estimates for the martingale transform

Our measure space throughout this article will be $(X, \mathfrak{A}, dx)$, where the σ-algebra $\mathfrak{A}$ is generated by a standard dyadic filtration $\mathcal{D} = \bigcup_k \mathcal{D}_k$ on $\mathbb{R}$. We consider the martingale transform and dyadic shifts related to this homogeneous dyadic filtration.

As always, the symbol $\langle f \rangle_I$ denotes the average value of f over the set I; i.e., $\langle f \rangle_I = (1/|I|) \int_I f\,dx$. We consider martingale differences (recall that the symbol $\mathrm{ch}(J)$ denotes the dyadic children of J)

$$\Delta_J f := \sum_{I \in \mathrm{ch}(J)} \mathbf{1}_I (\langle f \rangle_I - \langle f \rangle_J).$$

For our case of the dyadic lattice on the line we have that $|\Delta_J f|$ is constant on J, and

$$\Delta_J f = \tfrac{1}{2}[(\langle f \rangle_{J_+} - \langle f \rangle_{J_-}) \mathbf{1}_{J_+} + (\langle f \rangle_{J_-} - \langle f \rangle_{J_+}) \mathbf{1}_{J_-}].$$

In this section and in the next one we consider the dyadic A_1 and A_2 classes of weights, but we skip the word dyadic, because we consider here only dyadic operators. We consider a positive function $w(x)$, and as before we call it an A_2 weight if

$$Q := [w]_{A_2} := \sup_{J \in \mathcal{D}} \langle w \rangle_J \langle w^{-1} \rangle_J < \infty. \tag{2-1}$$

We call w an A_1 weight if

$$Q := [w]_{A_1} := \sup_{J \in \mathcal{D}} \frac{\langle w \rangle_J}{\inf_J w} < \infty. \tag{2-2}$$

By Mw we will denote the martingale maximal function of w; that is, $Mw(x) = \sup_{x \in J,\, J \in \mathcal{D}} \langle w \rangle_J$. Then $w \in A_1$ with "norm" Q means that

$$Mw \le Q \cdot w \quad a.e.,$$

and $Q = [w]_{A_1}$ is the best constant in this inequality.

Recall that the martingale transform is the operator given by $T\varphi = \sum_{J \in \mathcal{D}} \varepsilon_J \Delta_J \varphi$. It is convenient to use the Haar function h_J associated with the dyadic interval J,

$$h_J(x) := \begin{cases} 1/|J|^{1/2}, & x \in J_+, \\ -1/|J|^{1/2}, & x \in J_-. \end{cases}$$

In this notation, the martingale transform is

$$T\varphi = \sum_{J \in cD} \varepsilon_J (\varphi, h_J) h_J,$$

where we (1) always assume the sum has an unspecified but finite number of terms, and (2) $|\varepsilon_J| \le 1$.

We are interested in several weak-type estimates.

We first consider the weak estimate for the martingale transform T in the weighted space $L^1(\mathbb{R}, w\,dx)$, where $w \in A_1$. The end-point exponent is naturally $p = 1$, and we wish to understand the order of magnitude of the constant $A([w]_{A_1})$ in the weak-type inequality for the dyadic martingale transform:

$$\frac{1}{|I|}w\left\{x\in I:\sum_{J\in\mathcal{D}(I)}\varepsilon_J(\varphi,h_J)h_J(x)>\lambda\right\}\le C_{[w]_{A_1}}\frac{\langle|\varphi|w\rangle_I}{\lambda}. \tag{2-3}$$

Here φ runs over all functions such that $\operatorname{supp}\varphi\subset I$ and $\varphi\in L^1(I,w\,dx),\ w\in A_1$. This section will be devoted to the study of the "sharp" order of magnitude of constants $C_{[w]_{A_1}}$ in terms of $[w]_{A_1}$ if $[w]_{A_1}$ is large. We are primarily interested in the estimate of $C_{[w]_{A_1}}$ from below, that is, in finding the worst possible A_1 weight in terms of weak-type estimates (of course this involves also finding the worst test function φ as well).

We will prove the following result.

Theorem 2.1. *There is a positive absolute constant c and a weight $w\in A_1$ with $[w]_{A_1}$ as large as we wish such that constant $C_{[w]_{A_1}}$ from* (2-3) *satisfies*

$$C_{[w]_{A_1}}\ge c[w]_{A_1}(\log[w]_{A_1})^{1/3}.$$

In fact, we will prove a sharper result. We will consider a particular dyadic shift, and we will prove the estimate $\ge c[w]_{A_1}(\log[w]_{A_1})^{1/3}$ for one particular dyadic shift. Ours is the following dyadic singular operator on $L^1(I,w\,dx),\ I=[0,1]$:

$$S\mathbf{1}_I=0,\quad Sh_J=h_{J_-}-h_{J_+},\quad J\in D(I).$$

Our main result is the following theorem.

Theorem 2.2. *There is a positive absolute constant c and a weight $w\in A_1$ such that*

$$\|S\|_{L^1(w)\to L^{1,\infty}(w)}\ge c[w]_{A_1}(\log[w]_{A_1})^{1/3}.$$

In [Lerner et al. 2009] the following estimate from above was proved:

Theorem 2.3. *There is a positive absolute constant C such that for any weight $w\in A_1$ the constant $C_{[w]_{A_1}}$ from* (2-3) *satisfies*

$$C_{[w]_{A_1}}\le c[w]_{A_1}\log[w]_{A_1}.$$

Remark 2.4. The sharp power remained enigmatic for quite a while. Very recently it was proved that for the Hilbert transform the exponent turns out to be 1 [Lerner et al. 2017]. However, it seems to be very probable that at the end-point of the scale, all operators behave differently, and the estimate for the dyadic shift S or the martingale transform might be different from the one for the Hilbert transform. A recent preprint [Ivanisvili and Volberg 2017] shows that the sharp power is actually 1 for the martingale transform as well.

Remark 2.5. This note is based on two preprints [Nazarov et al. 2015; 2016], but Theorem 2.2 was not formulated in these preprints; however, as the attentive reader can notice, it was proved there.

2A. ***Bellman approach: the Bellman function of the weak weighted estimate of the martingale transform and its properties.*** To find the "optimal" $C_{[w]_{A_1}}$ we use again the Bellman-function technique. The idea is to reformulate the infinite-dimensional problem of optimization of $C_{[w]_{A_1}}$, that is, finding the “smallest” $C_{[w]_{A_1}}$ that works for all inequalities (2-3), in terms of the growth estimate on a certain function of only a finite number of variables (five in this case).

The Bellman function will depend on the number $Q \geq 1$ and is given by

$$\boldsymbol{B}(F, w, m, f, \lambda) := \boldsymbol{B}_Q(F, w, m, f, \lambda) := \sup \frac{1}{|I|}\omega\bigg\{x \in I : \sum_{J\subseteq I, J\in D} \varepsilon_J(\varphi, h_J)h_J(x) > \lambda\bigg\}, \tag{2-4}$$

where the sup is taken over all $\varepsilon_J,\ |\varepsilon_J| \leq 1,\ J \in D(I)$, and over all $\varphi \in L^1(I, \omega\, dx)$ such that $F := \langle|\varphi|\,\omega\rangle_I$, $f := \langle\varphi\rangle_I,\ w = \langle\omega\rangle_I,\ m \leq \inf_I \omega$, and ω are all dyadic A_1 weights such that $[w]_{A_1} \leq Q$. This function is obviously defined in the convex subdomain of $\mathbb{R}^5$

$$\Omega := \{(F, w, m, f, \lambda) \in \mathbb{R}^5 : F \geq |f|\,m,\ m \leq w \leq Qm\}. \tag{2-5}$$

Remark 2.6. We warn the reader that emotional attachment to the notation $F,\ f,\ w$ for functions should be forgotten. These symbols in this and the following sections stand for numbers.

2A1. *The properties of $\boldsymbol{B}_Q$.* <u>The first property</u>: homogeneity. By definition, it is clear that

$$s\boldsymbol{B}\left(\frac{F}{s}, \frac{w}{s}, \frac{m}{s}, f, \lambda\right) = \boldsymbol{B}(F, w, m, f, \lambda), \quad \boldsymbol{B}(tF, w, m, tf, t\lambda) = \boldsymbol{B}(F, w, m, f, \lambda).$$

Choosing $s = m$ and $t = \lambda^{-1}$ and introducing new variables

$$\alpha = \frac{F}{m\lambda}, \quad \beta = \frac{w}{m}, \quad \gamma = \frac{f}{\lambda}$$

we can see that

$$\frac{1}{m}\boldsymbol{B}(F, w, m, f, \lambda) = B\left(\frac{F}{m\lambda}, \frac{w}{m}, \frac{f}{\lambda}\right) =: B(\alpha, \beta, \gamma), \tag{2-6}$$

where $B(\alpha, \beta, \gamma) = \boldsymbol{B}(\alpha, \beta, 1, \gamma, 1)$.

Obviously B is defined in the domain

$$G := \{(\alpha, \beta, \gamma) : |\gamma| \leq \alpha,\ 1 \leq \beta \leq Q\}. \tag{2-7}$$

<u>The second property</u>: special form of concavity. We formulate this property as the following theorem.

Theorem 2.7. *Let $P,\ P_+,\ P_- \in \Omega$ and, for $0 \leq t \leq 1$,*

$$\begin{aligned} P &= (F, w, \min(m_+, m_-), f, \lambda), \\ P_+ &= (F + A, w + u, m_+, f + a, \lambda + ta), \\ P_- &= (F - A, w - u, m_-, f - a, \lambda - ta). \end{aligned}$$

Then

$$\boldsymbol{B}(P) - \tfrac{1}{2}(\boldsymbol{B}(P_+) + \boldsymbol{B}(P_-)) \geq 0. \tag{2-8}$$

At the same time, if $P,\ P_+,\ P_- \in \Omega$, and, for $0 \leq t \leq 1$,

$$\begin{aligned} P &= (F, w, \min(m_+, m_-), f, \lambda), \\ P_+ &= (F + A, w + u, m_+, f + a, \lambda - ta), \\ P_- &= (F - A, w - u, m_-, f - a, \lambda + ta), \end{aligned}$$

then

$$\boldsymbol{B}(P) - \tfrac{1}{2}(\boldsymbol{B}(P_+) + \boldsymbol{B}(P_-)) \geq 0. \tag{2-9}$$

In particular, with fixed m, and with all points being inside Ω, we get for all $t \in [0, 1]$

$$\begin{aligned}\boldsymbol{B}(F, w, m, f, \lambda) \geq \tfrac{1}{4}\big(&\boldsymbol{B}(F - dF, w - dw, m, f - d\lambda, \lambda - t d\lambda)\\ &+\boldsymbol{B}(F - dF, w - dw, m, f - d\lambda, \lambda + t d\lambda)\\ &+\boldsymbol{B}(F + dF, w + dw, m, f + d\lambda, \lambda - t d\lambda)\\ &+\boldsymbol{B}(F + dF, w + dw, m, f + d\lambda, \lambda + t d\lambda)\big).\end{aligned} \tag{2-10}$$

Remark 2.8. (1) The differential notation, i.e., dF, dw, $d\lambda$, just means small numbers. (2) In (2-10) we lose a bit of information in comparison with (2-8), (2-9), but this is exactly (2-10), which we are going to use in the future.

Before proving this theorem, let us explain a bit more about what kind of concavity is represented by inequalities (2-8), (2-9), and thus by their consequence (2-10). We can use different notation for coordinates P_+, P_-, $P_\pm := (F_\pm, w_\pm, m_\pm, f_\pm, \lambda_\pm)$. We require all P, $P_\pm$ to belong to Ω and it is evident that

$$F = \frac{F_+ + F_-}{2}, \quad w = \frac{w_+ + w_-}{2}, \quad m = m_+ \wedge m_-, \quad f = \frac{f_+ + f_-}{2}, \quad \lambda = \frac{\lambda_+ + \lambda_-}{2},$$

but also "jumps" in the fourth and the fifth coordinates must be dependent on each other, namely,

$$t\Delta f := t(f_+ - f_-) = (\lambda_+ - \lambda_-) =: \Delta\lambda \quad \text{or} \quad t\Delta f = -\Delta\lambda, \quad 0 \leq t \leq 1.$$

So the function $\boldsymbol{B}$ (as we will now see) possesses such sophisticated concavity as encoded by jumps from any point $P \in \Omega$ to P_+, $P_- \in \Omega$, where P is almost the average of $P_\pm$, but not quite: the difference is that (1) the third coordinate is not an arithmetic average of the third coordinates of $P_\pm$, but their minimum, and (2) that the jumps in the fourth and the fifth coordinates are interdependent as above.

Proof. Fix P, P_+, $P_- \in \Omega$ as in (2-8). Let φ_+, φ_-, ω_+, ω_- be functions and weights giving the supremum in $\boldsymbol{B}(P_+)$, $\boldsymbol{B}(P_-)$ respectively up to a small number $\eta > 0$. Using the fact that $\boldsymbol{B}$ does not depend on I, we assume φ_+, ω_+ are on I_+ and φ_-, ω_- are on I_-. Consider

$$\varphi(x) := \begin{cases}\varphi_+(x), & x \in I_+,\\ \varphi_-(x), & x \in I_-,\end{cases} \qquad \omega(x) := \begin{cases}\omega_+(x), & x \in I_+\\ \omega_-(x), & x \in I_-.\end{cases}$$

Notice that then

$$(\varphi, h_I) \cdot \frac{1}{\sqrt{|I|}} = \Delta_I \varphi = \tfrac{1}{2}(P_{+,4} - P_{-,4}) =: a. \tag{2-11}$$

We denote the i-th coordinate of a point P by P_i. Then it is easy to see that $P_3 = \min(P_{3,-}, P_{3,+}) = \min(\min_{I_-} \omega_-, \min_{I_+} \omega_+)$, $P_5 = \lambda$,

$$\langle |\varphi|\omega \rangle_I = F = P_1, \quad \langle \omega \rangle_I = w = P_2, \quad \langle \varphi \rangle_I = f = P_4. \tag{2-12}$$

Notice that for $x \in I_\pm$ using (2-11), we get if $\varepsilon_I = -t,\ 0 \le t \le 1$,

$$\frac{1}{|I|}\omega_\pm\Big\{x \in I_\pm : \sum_{J\subseteq I, J\in D} \varepsilon_J(\varphi, h_J)h_J(x) > \lambda\Big\} = \frac{1}{|I|}\omega_\pm\Big\{x \in I_\pm : \sum_{J\subseteq I_\pm, J\in D} \varepsilon_J(\varphi, h_J)h_J(x) > \lambda \pm ta\Big\}$$

$$= \frac{1}{2|I_\pm|}\omega_\pm\Big\{x \in I_\pm : \sum_{J\subseteq I_\pm, J\in D} \varepsilon_J(\varphi_\pm, h_J)h_J(x) > P_{\pm,5}\Big\}$$

$$\ge \tfrac{1}{2}B(P_\pm) - \eta.$$

Combining the two options for the left-hand side we obtain for $\varepsilon_I = -1$

$$\frac{1}{|I|}\omega\Big\{x \in I : \sum_{J\subseteq I, J\in D} \varepsilon_J(\varphi, h_J)h_J(x) > \lambda\Big\} \ge \tfrac{1}{2}(B(P_+) + B(P_-)) - 2\eta.$$

Let us use now the simple information (2-12): if we take the supremum in the left-hand side above over all functions φ such that $\langle|\varphi|\,\omega\rangle_I = F,\ \langle\varphi\rangle_I = f,\ \langle\omega\rangle_I = w$, and weights ω such that $\langle\omega\rangle_I = w$ in dyadic A_1 with A_1-norm at most Q, and supremum over all $\varepsilon_J = \pm s,\ s \in [0, 1]$ (only ε_I stays fixed), we get a quantity smaller than or equal to the one where we have the supremum over all functions φ such that $\langle|\varphi|\,\omega\rangle = F,\ \langle\varphi\rangle_I = f,\ \langle\omega\rangle = w$, and weights ω such that $\langle\omega\rangle = w$ in dyadic A_1 with A_1-norm at most Q, and an unrestricted supremum over all $\varepsilon_J = \pm s,\ s \in [0, 1],\ \varepsilon_I = -t,\ 0 \le t \le 1$. The latter quantity is of course $\boldsymbol{B}(F, w, m, f, \lambda)$. So we proved (2-8).

To prove (2-9) we repeat verbatim the same reasoning, only keeping now $\varepsilon_I = t,\ 0 \le t \le 1$. □

Remark 2.9. This theorem is a sort of "fancy" concavity property; the attentive reader will see that (2-8), (2-9) include a biconcavity property entirely similar to the one demonstrated by the celebrated Burkholder function. We will use the consequence of biconcavity encompassed by (2-10). This is still another concavity. Let us also remark that it can be shown that $\boldsymbol{B}$ is a supersolution of a certain degenerate elliptic equation (but this fact does not help us in estimating $\boldsymbol{B}$ below).

The third property: $\boldsymbol{B}$ decreases in m. The function $\boldsymbol{B}$ is obviously decreasing in m. In fact, if m decreases (all other coordinates being fixed) then the collection of weights increases, and the supremum increases. It is not difficult to see that $\boldsymbol{B}$ is also continuous.

The fourth property: the function B from (2-6) is concave. Recall that by (2-6)

$$B\Big(\frac{F}{\lambda}, w, \frac{f}{\lambda}\Big) = \boldsymbol{B}(F, w, 1, f, \lambda). \tag{2-13}$$

Choosing $t = 0$ in Theorem 2.7 we see that $\boldsymbol{B}(F, w, 1, f, \lambda)$ is concave when λ is fixed. This proves the fourth property, which we formulated intentionally in terms of B and not $\boldsymbol{B}$.

The fifth property: the function $t \to (1/t)B(t\alpha, t\beta, \gamma)$ is increasing. This is the combination of (2-6) and the third property above.

The sixth property: the domain of definition of B is $G = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : 1 \le \beta \le Q,\ |\gamma| \le \alpha\}$.

The seventh property: symmetry and monotonicity in γ. It is easy to see from the definition of $\boldsymbol{B}$ that it is even in its variable f. Therefore,

$$B(\alpha,\beta,\gamma)=B(\alpha,\beta,-\gamma).$$

Notice that the concavity of B (in γ) and this symmetry together imply that $\gamma\to B(\cdot\,,\cdot\,,\gamma)$ is decreasing on $\gamma\in[0,\alpha]$.

2B. *The goal and the idea of the proof.* In this section we are going to prove the following estimate from below on the function B.

Theorem 2.10. *There is an absolute positive constant c such that for some point* $(\alpha,\beta,\gamma)\in G$

$$B(\alpha,\beta,\gamma)\geq cQ(\log Q)^{1/3}\alpha. \tag{2-14}$$

Remark 2.11. It is a subtle result and it will take some space below to prove. Recall that Muckenhoupt conjectured that for the Hilbert transform H and any weight $w\in A_1$ the following two estimates hold on a unit interval I:

$$w\{x\in I:|Hf(x)|>\lambda\}\leq\frac{C}{\lambda}\int_I|f|Mw\,dx, \tag{2-15}$$

$$w\{x\in I:|Hf(x)|>\lambda\}\leq\frac{C\,[w]_{A_1}}{\lambda}\int_I|f|w\,dx, \tag{2-16}$$

Obviously if (2-15) holds then (2-16) is valid as well. It took many years to *disprove* (2-15). This was done by Maria Reguera and Christoph Thiele [Reguera 2011; Reguera and Thiele 2012]. The constructions involve a very irregular (almost a sum of delta measures) weight w, so there was a hope that such an effect cannot appear when the weight is regular in the sense that $w\in A_1$. Theorem 2.10 gives a counterexample to this hope for the case when the Hilbert transform is replaced by the martingale transform on a usual homogeneous dyadic filtration. The reader can consult [Nazarov et al. 2015] to see that for the Hilbert transform a counterexample also exists, and so (2-16) fails as well. The counterexample for the Hilbert transform is the transference of a counterexample we build here for the martingale transform. Notice that Theorem 2.10 implicitly gives a certain counterexample for the Hilbert transform.

Now a couple of words about the idea of the proof of Theorem 2.10. Ideally we would like to find the formula for B, and therefore for $\boldsymbol{B}$ because of (2-6). To proceed we rewrite the second property of $\boldsymbol{B}$ as a PDE on B. Then we try to find the boundary conditions on B on ∂G, and then we may hope to solve this PDE. Unfortunately there are many roadblocks on this path, starting with the fact that the second property of $\boldsymbol{B}$ is not a PDE; it is rather a partial differential inequality in discrete form. We will write it down as a pointwise partial differential inequality, but for that we will need a subtle result of Aleksandrov. We also can find boundary values of B; see some of them in Section 2B1 below. However, the main difficulty is that our partial differential expression is in three dimensions.

2B1. *Unweighted case.* We first consider the simplest case of $m = \omega = 1$ identically. Then we are left with function $\mathcal{B}el(F, f, \lambda) = \boldsymbol{B}(F, 1, 1, f, \lambda)$, which is defined in a convex domain $\Omega_0 \subset \mathbb{R}^3$, $\Omega_0 := \{(F, f, \lambda) \in \mathbb{R}^3 : |f| \le F\}$, and whose concavity properties are described in:

Theorem 2.12. *Let $P, P_+, P_- \in \Omega_0$,*

$$P = (F, f, \lambda), \quad P_+ = (F + A, f + a, \lambda + a), \quad P_- = (F - A, f - a, \lambda - a).$$

Then

$$\mathcal{B}el(P) - \tfrac{1}{2}(\mathcal{B}el(P_+) + \mathcal{B}el(P_-)) \ge 0. \tag{2-17}$$

At the same time, if $P, P_+, P_- \in \Omega_0$,

$$P = (F, f, \lambda), \quad P_+ = (F + A, f + a, \lambda - a), \quad P_- = (F - A, f - a, \lambda + a),$$

then

$$\mathcal{B}el(P) - \tfrac{1}{2}(\mathcal{B}el(P_+) + \mathcal{B}el(P_-)) \ge 0. \tag{2-18}$$

Let us make the change of variables $(F, f, \lambda) \to (F, y_1, y_2)$:

$$y_1 := \tfrac{1}{2}(\lambda + f), \quad y_2 := \tfrac{1}{2}(\lambda - f).$$

Define

$$M(F, y_1, y_2) := B(F, y_1 - y_2, y_1 + y_2) = \mathcal{B}el(F, f, \lambda).$$

In terms of the function M, Theorem 2.12 reads as follows:

Theorem 2.13. *The function M is defined in the domain $G := \{(F, y_1, y_2) : |y_1 - y_2| \le F\}$, and for each fixed y_2, $M(F, y_1, y_2)$ is concave in (F, y_1) and for each fixed y_1, $M(F, y_1, y_2)$ is concave in (F, y_2).*

The properties of M are strongly reminiscent of the properties of the Burkholder function.

In the unweighted situation we can find $\boldsymbol{B}$ (or M) precisely. Here is the result proved in [Reznikov et al. 2013]:

Theorem 2.14.

$$\mathcal{B}el(F, f, \lambda) = \begin{cases} 1 & \text{if } \lambda \le F, \\ 1 - (\lambda - F)^2/(\lambda^2 - f^2) & \text{if } \lambda > F. \end{cases} \tag{2-19}$$

This result means that we found a boundary value of the Bellman function $\boldsymbol{B}(F, w, m, f, \lambda)$ of the weighted problem on the part of its boundary; namely we found this function of five variables on $\{P \in \partial\Omega : w = P_2 = P_3 = m\}$:

$$\boldsymbol{B}(F, m, m, f, \lambda) = m \begin{cases} 1 & \text{if } \lambda \le F, \\ 1 - (\lambda - F)^2/(\lambda^2 - f^2) & \text{if } \lambda > F. \end{cases} \tag{2-20}$$

In terms of the function B from (2-6), we have the following boundary values of B:

$$B(\alpha, 1, \gamma) = \begin{cases} 1 & \text{if } \alpha \ge 1, \\ 1 - (1 - \alpha)^2/(1 - \gamma^2) & \text{if } 0 \le |\gamma| \le \alpha < 1. \end{cases} \tag{2-21}$$

2C. ***From discrete inequality to differential inequality via Aleksandrov's theorem.*** By the fourth property of Section 2A1 the function B is concave on its domain of definition G. By the result of Aleksandrov, see Theorem 6.9 of [Evans and Gariepy 1992], B has all second derivatives almost everywhere; this means that for a.e. $x \in G^\circ$ and all small vectors $h \in \mathbb{R}^3$,

$$B(x+h) = B(x) + \nabla B(x)\cdot h + \langle H_B(x)\cdot h, h\rangle + o(|h|^2), \tag{2-22}$$

where H_B is the Hessian matrix of B. On the other hand the second property of Section 2A1 can be rewritten in terms of B as

$$\begin{aligned} B\Big(\frac{F}{\lambda}, \beta, \frac{f}{\lambda}\Big) - \frac{1}{4}\bigg[B\Big(\frac{F-dF}{\lambda-d\lambda}, \beta-d\beta, \frac{f-d\lambda}{\lambda-d\lambda}\Big) \\ + B\Big(\frac{F-dF}{\lambda-d\lambda}, \beta-d\beta, \frac{f+d\lambda}{\lambda-d\lambda}\Big) \\ + B\Big(\frac{F+dF}{\lambda+d\lambda}, \beta+d\beta, \frac{f-d\lambda}{\lambda+d\lambda}\Big) \\ + B\Big(\frac{F+dF}{\lambda+d\lambda}, \beta+d\beta, \frac{f+d\lambda}{\lambda+d\lambda}\Big)\bigg] \ge 0. \end{aligned} \tag{2-23}$$

Here $(F/\lambda, \beta, f/\lambda) \in G^\circ$ and $(dF, d\beta, d\lambda)$ is just any small vector in $\mathbb{R}^3$.

Theorem 2.15. *For almost every point $P = (\alpha, \beta, \gamma) =: (F/\lambda, \beta, f/\lambda) \in G^\circ$ and every vector $(dF, d\beta, d\lambda) \in \mathbb{R}^3$ we have*

$$\begin{aligned} -\alpha^2 B_{\alpha\alpha}(P)\Big(\frac{dF}{F} - \frac{d\lambda}{\lambda}\Big)^2 - \beta^2 B_{\beta\beta}(P)\Big(\frac{d\beta}{\beta}\Big)^2 - (1+\gamma^2) B_{\gamma\gamma}(P)\Big(\frac{d\lambda}{\lambda}\Big)^2 \\ - 2\alpha\beta B_{\alpha\beta}(P)\Big(\frac{dF}{F} - \frac{d\lambda}{\lambda}\Big)\frac{d\beta}{\beta} + 2\beta\gamma B_{\beta\gamma}(P)\frac{d\beta}{\beta}\frac{d\lambda}{\lambda} + 2\alpha\gamma B_{\alpha\gamma}(P)\Big(\frac{dF}{F} - \frac{d\lambda}{\lambda}\Big)\frac{d\lambda}{\lambda} \\ + 2\alpha B_\alpha(P)\Big(\frac{dF}{F} - \frac{d\lambda}{\lambda}\Big)\frac{d\lambda}{\lambda} - 2\gamma B_\gamma(P)\Big(\frac{d\lambda}{\lambda}\Big)^2 \ge 0. \end{aligned} \tag{2-24}$$

Remark 2.16. We can mollify $\boldsymbol{B}$ to make it smooth and still have its "fancy concavity properties". But then we lose homogeneity and cannot reduce $\boldsymbol{B}$ to B. We can mollify $\boldsymbol{B}$ to keep its homogeneity — just choose the mollifier depending on the point — but then we loose its "fancy concavity property". In short, we have a problem with the mollification. This is why Aleksandrov's theorem is very useful now.

Proof. Fix a point $P \in G^\circ$, where Aleksandrov's identity (2-22) holds. Fix an arbitrary $(dx, dy, d\lambda) \in \mathbb{R}^3$. Let us use (2-23) by expanding the fractions

$$\frac{x \pm \varepsilon\, dx}{\lambda \pm \varepsilon d\lambda}, \quad \frac{f \pm \varepsilon d\lambda}{\lambda \pm \varepsilon d\lambda}$$

up to the second order in small parameter ε, and combining with the identity (2-22) after that. All terms with ε^0, ε^1 will disappear identically. Only the terms with ε^2 and smaller stay. After division by ε^2 we let ε tend to zero and get (2-24) for a.e. point $P \in G^\circ$. □

Of course we need something else from positive B to be able to prove that B satisfying this partial differential inequality (2-24) in the domain $G^\circ = \{P = (\alpha, \beta, \gamma) : 1 < \beta < Q,\ 0 < |\gamma| < \alpha\}$ has the estimate (2-14) from below. We actually have this "something else" in the form of an obstacle condition, which we will introduce in Section 2E.

But let us first simplify (2-24). Let us call by $\mathcal{N}$ the matrix of the quadratic form in (2-24). After a rather straightforward operation $\mathcal{N} \to \mathcal{M}_1 := A^*\mathcal{N}A$ with a certain invertible matrix A, we can write down the nonnegativity of the differential form in (2-24) as the a.e.-in-G° nonnegativity of the matrix

$$\mathcal{M}_1 := \begin{bmatrix} -\alpha^2 B_{\alpha\alpha} & -\alpha\beta B_{\alpha\beta} & \alpha\gamma B_{\alpha\gamma} + \alpha B_\alpha \\ -\alpha\beta B_{\alpha\beta} & -\beta^2 B_{\beta\beta} & \beta\gamma B_{\beta\gamma} \\ \alpha\gamma B_{\alpha\gamma} + \alpha B_\alpha & \beta\gamma B_{\beta\gamma} & -(1+\gamma^2) B_{\gamma\gamma} - 2\gamma B_\gamma \end{bmatrix} \geq 0. \tag{2-25}$$

However, we saw already that $B(\alpha, \beta, \gamma)$ is concave, which implies the nonnegativity of yet another matrix:

$$\mathcal{M}_2 := \begin{bmatrix} -\alpha^2 B_{\alpha\alpha} & -\alpha\beta B_{\alpha\beta} & -\alpha\gamma B_{\alpha\gamma} \\ -\alpha\beta B_{\alpha\beta} & -\beta^2 B_{\beta\beta} & -\beta\gamma B_{\beta\gamma} \\ -\alpha\gamma B_{\alpha\gamma} & -\beta\gamma B_{\beta\gamma} & -\gamma^2 B_{\gamma\gamma} \end{bmatrix} \geq 0. \tag{2-26}$$

Taking the half-sum of (2-25) and (2-26), we obtain the nonnegativity

$$\mathcal{M} := \begin{bmatrix} -\alpha^2 B_{\alpha\alpha} & -\alpha\beta B_{\alpha\beta} & \frac{1}{2}\alpha B_\alpha \\ -\alpha\beta B_{\alpha\beta} & -\beta^2 B_{\beta\beta} & 0 \\ \frac{1}{2}\alpha B_\alpha & 0 & -\left(\frac{1}{2}+\gamma^2\right) B_{\gamma\gamma} - \gamma B_\gamma \end{bmatrix} \geq 0. \tag{2-27}$$

It is now natural to restrict the quadratic form of this matrix on certain two-dimensional hyperplanes in the three-dimensional tangent space Tan_p of the graph $\Gamma := \{p := (P, B(P)),\ P \in G^\circ\}$ at a given point p. Namely, let us consider the quadratic form of the matrix $\mathcal{M}$ in (2-25) on vectors of the form

$$(\xi, \xi, \eta). \tag{2-28}$$

Then, using the notation

$$\psi(\alpha, \beta, \gamma) := \psi_B(\alpha, \beta, \gamma) := -\alpha^2 B_{\alpha\alpha} - 2\alpha\beta B_{\alpha\beta} - \beta^2 B_{\beta\beta}, \tag{2-29}$$

we get the a.e.-in-G° nonnegativity of the matrix

$$\begin{bmatrix} \psi(\alpha, \beta, \gamma) & \frac{1}{2}\alpha B_\alpha \\ \frac{1}{2}\alpha B_\alpha & -\left(\frac{1}{2}+\gamma^2\right) B_{\gamma\gamma} - \gamma B_\gamma \end{bmatrix} \geq 0. \tag{2-30}$$

Definition 2.17. Consider a subdomain of G,

$$G_1 := \left\{(\alpha, \beta, \gamma) \in G : |\gamma| < \tfrac{1}{2}\alpha,\ 2 < \beta < Q\right\}.$$

Fix now $(\alpha, \beta, \gamma) \in G_1$ and a parameter $t \in \left[\frac{1}{2}, 1\right]$. Replace in the previous inequality (α, β, γ) by $(t\alpha, t\beta, \gamma)$. Denote temporarily

$$P_t := (t\alpha, t\beta, \gamma), \quad (\alpha, \beta, \gamma) \in G_1, \quad \tfrac{1}{2} \leq t \leq 1.$$

Then we get for every such t and every point P_t the following inequality for all $(\xi, \eta) \in \mathbb{R}^2$:

$$\xi^2[\psi(P_t)] + \xi\eta(\alpha t B_\alpha(P_t)) + \eta^2(-\gamma B_\gamma(P_t) - \big(\tfrac{1}{2} + \gamma^2\big) B_{\gamma\gamma}(P_t)) \geq 0. \tag{2-31}$$

Consider a new function H, which is a certain averaging of B; namely, for any $P = (\alpha, \beta, \gamma) \in G_1$, let

$$H(P) = 2 \int_{1/2}^{1} B(P_t)\, dt.$$

Notice several simple facts. First of all

$$\alpha H_\alpha = 2 \int_{1/2}^{1} \alpha t B(t\alpha, t\beta, \gamma)\, dt, \quad \alpha^2 H_{\alpha\alpha} = 2 \int_{1/2}^{1} (\alpha t)^2 B_{\alpha\alpha}(t\alpha, t\beta, \gamma)\, dt.$$

Similarly, if for every function F we introduce the notation

$$\psi_F(\alpha, \beta, \gamma) := -\alpha^2 F_{\alpha\alpha} - 2\alpha\beta F_{\alpha\beta} - \beta^2 F_{\beta\beta}, \tag{2-32}$$

we get

$$\psi_H = 2 \int_{1/2}^{1} \psi_B(t\alpha, t\beta, \gamma)\, dt.$$

Now integrate (2-31) on the interval $t \in \big[\tfrac{1}{2}, 1\big]$. The previous simple observations allow us now to rewrite this as a pointwise inequality for function H on domain G_1 introduced in Definition 2.17:

$$\xi^2[\psi_H(P)] + \xi\eta(\alpha H_\alpha(P)) + \eta^2(-\gamma H_\gamma(P) - (1/2 + \gamma^2) H_{\gamma\gamma}(P)) \geq 0. \tag{2-33}$$

The reader may wonder why we are so keen to replace (2-31) by the virtually identical (2-33)? The answer is because we can give a very good pointwise estimate on $\psi_H(P),\ P \in G_1$. Unfortunately we cannot give any pointwise estimate on $\psi(P),\ P \in G$.

Now we deduce the desired pointwise estimate on ψ_H; we will use below its consequences. First, let us define

$$R := \sup \frac{B(P)}{\alpha}, \quad P = (\alpha, \beta, \gamma) \in G. \tag{2-34}$$

Our goal formulated in (2-14) is to prove $R \geq cQ(\log Q)^\varepsilon$. We are still not too close, but notice that automatically $B(P) \leq R\alpha,\ \ P = (\alpha, \beta, \gamma) \in G$.

Lemma 2.18. *If $P = (\alpha, \beta, \gamma)$ is such that $|\gamma| \leq \frac{1}{8}\alpha$ and $\beta > 100$ then*

$$\psi_H(P) = 2 \int_{1/2}^{1} \psi(t\alpha, t\beta, \gamma)\, dt \leq CR\bigg(|\gamma| + \frac{\alpha}{\beta}\bigg),$$

where C is an absolute constant.

Proof. Consider the function

$$\varphi(t) := B(t\alpha, t\beta, \gamma) \tag{2-35}$$

for a.e. $(\alpha, \beta, \gamma) \in G_1$. It is concave.

Let us first prove that

$$\int_{1/2}^{1} -\varphi''(t)\,dt \le CR\Big(|\gamma| + \frac{\alpha}{\beta}\Big). \tag{2-36}$$

This would imply

$$\int_{1/2}^{1} \psi(t\alpha, t\beta, \gamma)\,dt \le CR\Big(|\gamma| + \frac{\alpha}{\beta}\Big),$$

because by the definitions (2-29), (2-35) of ψ and φ we have

$$\psi(t\alpha, t\beta, \gamma) = -t^2\varphi''(t).$$

To prove (2-36) let us consider an auxiliary function $r(t) := \varphi(1)t - \varphi(t)$. It is defined for $t \in [\max(|\gamma|/\alpha, 1/\beta), 1]$. At 1 it vanishes, it is convex, and it attains its maximum on its left end-point $t_0 = \max(|\gamma|/\alpha, 1/\beta)$. The last statement follows from the fact that $\varphi(t)/t$ is increasing; this is the fifth property of Section 2A1 of B.

So on $[t_0, 1]$,

$$r(t) \le r(t_0) \le \varphi(1)t_0 \le R\alpha t_0 \le R\alpha\Big(\frac{|\gamma|}{\alpha} + \frac{1}{\beta}\Big). \tag{2-37}$$

As $\varphi(t)/t$ is increasing, we have $t\varphi'(t) - \varphi(t) \ge 0$, and thus $r'(1) \le 0$. Let us write down the Taylor formula for the convex function $r(t)$ in integral form, keeping in mind that $r(1) = 0$, $r'(1) \le 0$:

$$r(t_0) = (t_0 - 1)r'(1) + \int_{t_0}^{1} dt \int_{t}^{1} r''(s)\,ds.$$

Fubini's theorem, (2-37), and $r'(1) \le 0$ imply

$$\int_{t_0}^{1} (s - t_0)r''(s)\,ds \le R\alpha\Big(\frac{|\gamma|}{\alpha} + \frac{1}{\beta}\Big).$$

But $t_0 \le \frac{1}{8}$ by the assumptions of the lemma. So $\int_{1/2}^{1} r''(s)\,ds \le \frac{8}{3}R\alpha(|\gamma|/\alpha + 1/\beta)$. Hence, as $r'' = -\varphi''$,

$$\int_{1/2}^{1} -\varphi''(s)\,ds \le \tfrac{8}{3}R\alpha\Big(\frac{|\gamma|}{\alpha} + \frac{1}{\beta}\Big).$$

The proof of (2-36) is finished and this, as we saw at the beginning of the proof, gives Lemma 2.18. □

2D. ***Logarithmic blow-up.*** Recall that

$$G_3 = \big\{P \in G : |\gamma| \le \tfrac{1}{1000}\alpha,\ \beta > 100\big\}.$$

By Lemma 2.18 we conclude that for any $P = (\alpha, \beta, \gamma) \in G_3$

$$[\psi_H]\cdot\big[-\gamma H_\gamma - \big(\tfrac{1}{2} + \gamma^2\big)H_{\gamma\gamma}\big] \ge \tfrac{1}{4}\alpha^2 H_\alpha^2. \tag{2-38}$$

We will consider only points P such that

$$0 < \gamma \ll \alpha \ll \beta, \quad \alpha \le 1.$$

The absolute constants C, c will vary from line to line.

Let us temporarily take for granted the following inequality, where c_1, c_2 are absolute positive constants:

$$\alpha \le c_2 \frac{\beta}{R} \quad \Longrightarrow \quad H_\alpha(\alpha, \beta, \gamma) \ge c_1 \beta, \quad \beta \in \big(1, \tfrac{1}{2} Q\big]. \tag{2-39}$$

Using Lemma 2.18 we obtain

$$\psi_H \le C R \left(\gamma + \frac{\alpha}{\beta} \right).$$

Now we combine this inequality with inequalities and (2-39), (2-38) to obtain

$$-\gamma H_\gamma - \big(\tfrac{1}{2} + \gamma^2\big) H_{\gamma\gamma} \ge c_3 \frac{\alpha^2 \beta^2}{R(\alpha/\beta + \gamma)}. \tag{2-40}$$

Using the fact that we consider only $0 < \gamma \le \alpha \le 1$, we can rewrite (2-40) as

$$-\frac{2\gamma}{(1+2\gamma^2)} H_\gamma - H_{\gamma\gamma} \ge c_4 \frac{\alpha^2 \beta^2}{R(\alpha/\beta + \gamma)}.$$

Using the integrating factor, we get

$$-[\mu(\gamma) H_\gamma]_\gamma \ge c_5 \frac{\alpha \beta^3}{R(1 + (\beta/\alpha)\gamma)}.$$

We integrate this inequality from 0 to γ to produce (we use that $\mu(\gamma) \approx 1$ when γ is small)

$$-H_\gamma \ge c_6 \frac{\alpha^2 \beta^2}{R} \log\left(1 + \frac{\beta}{\alpha} \gamma\right). \tag{2-41}$$

From now on let us fix α as follows:

$$\alpha = c_2 \frac{\beta}{R}, \tag{2-42}$$

where c_2 is from (2-39).

We integrate (2-41) from 0 to γ and use the positivity of H to produce

$$H(\alpha, \beta, 0) - H(\alpha, \beta, \gamma) \ge c_6 \frac{\alpha^3 \beta}{R} \left[\left(1 + \frac{\beta}{\alpha}\gamma\right) \log\left(1 + \frac{\beta}{\alpha}\gamma\right) - \frac{\beta}{\alpha}\gamma \right] \ge c_7 \frac{\alpha^2 \beta^2}{R} \gamma \log\left(\frac{\beta}{\alpha}\gamma\right); \tag{2-43}$$

the last inequality holds true because $\beta/\alpha = cR$, and because from now on we will fix γ and β:

$$\beta = \frac{Q}{4}, \quad \gamma = c_8 \frac{\beta}{R}, \tag{2-44}$$

where an absolute positive constant c_8 is much smaller than c_2 from (2-42). In particular, $(\beta/\alpha)\gamma \asymp \beta = \frac{1}{4} Q$ and so it is much bigger than 1. This justifies the last inequality in (2-43). This also gives

$$\gamma \ll \alpha.$$

We just obtained the inequality

$$\frac{\alpha^2 \beta^2}{R} \gamma \log\left(\frac{\beta}{\alpha}\gamma\right) \le C(H(\alpha, \beta, 0) - H(\alpha, \beta, \gamma)). \tag{2-45}$$

Let us use the fact that $B(\alpha,\beta,\gamma)$ is concave in γ (it is concave in all three variables) and that by its definition it is even in γ. See the seventh property of Section 2A1. The same then holds for the function H, which is just some averaging of B in the first two variables. Being even in γ on $\gamma\in[-\alpha,\alpha]$ and concave, it automatically decreases for $\gamma\in[0,\alpha]$; concavity and nonnegativity of H give $H(\alpha,\beta,\gamma)\ge(1-\gamma/\alpha)H(\alpha,\beta,0)$. This allows us to estimate the right-hand side of (2-45), and we have

$$\frac{\alpha^2\beta^2}{R}\gamma\log\left(\frac{\beta}{\alpha}\gamma\right)\le C(H(\alpha,\beta,0)-H(\alpha,\beta,\gamma))\le C\frac{\gamma}{\alpha}H(\alpha,\beta,0).$$

Taking into consideration one more time that $H(\alpha,\beta,\gamma)\le R\alpha$ by the definition of R in (2-34) and by the construction of H, we get

$$\frac{\alpha^2\beta^2}{R}\gamma\log\left(\frac{\beta}{\alpha}\gamma\right)\le C(H(\alpha,\beta,0)-H(\alpha,\beta,\gamma))\le CR\gamma, \tag{2-46}$$

or

$$\frac{Q^4}{R^4}\log\left(\frac{\beta}{\alpha}\gamma\right)\le C. \tag{2-47}$$

As $\beta/\alpha=cR$ and $\gamma\asymp Q/R$, we can see that $\log((\beta/\alpha)\gamma)\ge\log(cQ)$, from which it follows that

$$R\ge cQ(\log Q)^{1/4} \tag{2-48}$$

with a positive absolute c. Theorem 2.10 gets proved with $\delta=\frac{1}{4}$.

We are left to prove (2-39).

Lemma 2.19. *Suppose $H(1,\beta,\gamma)\ge A$. Then the following holds*:

$$H_\alpha\left(\frac{A}{2R},\beta,\gamma\right)\ge\frac{A}{2}.$$

Proof. Suppose not, then $H_\alpha(A/(2R),\beta,\gamma)\le\frac{1}{2}A$. Then $H_\alpha(\alpha,\beta,\gamma)\le\frac{1}{2}A$ for all $\alpha\in[A/(2R),1]$ by the fact that H_α decreases in α as H is concave.

But

$$H(1,\beta,\gamma)-H\left(\frac{A}{2R},\beta,\gamma\right)\ge A-R\frac{A}{2R}=\frac{A}{2}$$

by the definition of R in (2-34) and the fact that H is a certain averaging of B.

On the other hand,

$$H(1,\beta,\gamma)-H\left(\frac{A}{2R},\beta,\gamma\right)=H_\alpha(\theta,\beta,\gamma)\left(1-\frac{A}{2R}\right),$$

$\theta\in[A/(2R),1]$. We obtain (combining the last inequalities)

$$\frac{A}{2}\le H(1,\beta,\gamma)-H\left(\frac{A}{2R},\beta,\gamma\right)<H_\alpha(\theta,\beta,\gamma)\le\frac{A}{2}.$$

We come to a contradiction, so the lemma is proved. □

The combination of Lemma 2.19 and (2-53) proves inequality (2-39).

2E. ***An obstacle condition on functions B and H.*** Now we want to show the following obstacle condition for B, which we already used:

$$\text{If } |\gamma| < \tfrac{1}{4}, \text{ then } B(1,\beta,\gamma) \geq \tfrac{1}{3}\beta. \tag{2-49}$$

Let $I := [0, 1]$. Given numbers $(F, \beta, m, f, \lambda)$ such that $|f| < \frac{1}{4}\lambda$, $F/m = \lambda$, $m \leq \beta \leq Qm$, it is enough to construct functions φ, ψ, w on I such that:

(1) Each of these functions has constant values on grandchildren of I.

(2) If $\varphi = \langle\varphi\rangle_I + (\varphi, h_{I_-})h_{I_-} + (\varphi, h_{I_+})h_{I_+}$, then

$$\psi = -\lambda + (\varphi, h_{I_-})h_{I_-} - (\varphi, h_{I_+})h_{I_+}.$$

(3) $\langle w\rangle_I = \beta$, $\min_I w = m$.

(4) The w-measure of the subset of I, where

$$\psi \geq 0 \tag{2-50}$$

is at least $c\beta$, where c is an absolute positive constant. Notice that (2-50) is the same as $(\varphi, h_{I_-})h_{I_-} - (\varphi, h_{I_+})h_{I_+} \geq \lambda$.

Here is the construction of such a triple (φ, ψ, w). Fix $\beta \in (1, Q]$. Put $\varphi = -a$ on I_{--}, $\varphi = b$ on I_{++}, and $\varphi = 0$ otherwise. And $w = 1$ on $I_{--} \cup I_{++}$, and $w = \beta$ otherwise. Then put

$$\psi := -\lambda + (\varphi, h_{I_-})h_{I_-} - (\varphi, h_{I_+})h_{I_+}.$$

Let $0 < a < b$ and a be close to b. Put $\lambda = \frac{1}{4}(a+b)$. Then the average of φ is $\frac{1}{4}(b-a)$. It is small with respect to λ and we can prescribe it to be any number smaller than $\frac{1}{4}\lambda$. $F = \frac{1}{4}(a+b)$, $m = 1$.

On the other hand, the function $\lambda + \psi$ (which is a martingale transform of $\varphi - \langle\varphi\rangle_I$) is at least $-(\varphi, h_{I_+})h_{I_+} \geq \frac{1}{2}b \geq \lambda$ on I_{+-}, whose w-measure is more than $\frac{1}{3}w(I)$. So

$$B\big(1, \tfrac{1}{2}(1+\beta), \gamma\big) \geq \tfrac{1}{3}\beta \tag{2-51}$$

for all sufficiently small γ.

By concavity and positivity of B we see immediately

$$B(\alpha,\beta,\gamma) \geq c\beta, \quad \alpha \geq \tfrac{1}{100}, \tag{2-52}$$

with absolute positive c and for $\beta \in \big(1, \frac{1}{2}Q\big]$.

Now, from the definition of functions H we conclude that the following obstacle condition holds for the function H:

$$H(1,\beta,\gamma) \geq \tfrac{1}{3}\beta \tag{2-53}$$

for all sufficiently small γ and for $\beta \in \big(1, \frac{1}{2}Q\big]$.

2F. ***Improving the exponent $\frac{1}{3}$.*** From (2-53) we know that (this is for all $\gamma,\ 0\le\gamma\le 1$)

$$H\big(1,\tfrac14 Q,\gamma\big)\ge\tfrac1{12}Q.$$

As $H(\alpha,\beta,\gamma)\le R\alpha$ we immediately conclude that

$$H\left(\frac{Q}{24R},\frac{Q}{4},\gamma\right)\le\frac{Q}{24}$$

(this is for all $\gamma,\ 0\le\gamma\le\alpha:=Q/(24R)$). Combined with the previous displayed inequality above this gives us (2-39),

$$H_\alpha\left(\frac{Q}{24R},\frac{Q}{4},\gamma\right)\ge\frac{Q}{24}. \tag{2-54}$$

But there may be a better point $\tilde\alpha\gg\alpha:=Q/(24R)$, where $H\big(\tilde\alpha,\frac14Q,\gamma\big)\le\frac1{24}Q$. Then automatically we have the same estimate for H_α at this point:

$$H_\alpha\big(\tilde\alpha,\tfrac14Q,\gamma\big)\ge\tfrac1{24}Q. \tag{2-55}$$

So let us consider the largest $\tilde\alpha\in[\alpha,1]$ where $\alpha=Q/(24R)$ such that the following holds:

$$\begin{aligned}&H\big(\tilde\alpha,\tfrac14Q,0\big)=\tfrac1{24}Q.\\ \text{Then}\quad &H\big(\tilde\alpha,\tfrac14Q,\gamma\big)\le\tfrac1{24}Q,\quad \gamma\in[0,\tilde\alpha].\end{aligned} \tag{2-56}$$

Two cases may occur:

<u>Case 1</u>: $\tilde\alpha\ge Q^{1/2}/(24R^{1/2})$. Then in (2-40) we can use $\tilde\alpha\ge Q^{1/2}/(24R^{1/2})$ and $\beta=\frac14Q$. We just follow (2-45) and (2-46) with these new data, but with one small change; γ in (2-46) can be between 0 and $\tilde\alpha$, so in particular, it can be chosen to be $\gamma=Q^{1/2}/(24R^{1/2})$. Then instead of (2-47) we get

$$c\frac{Q^3}{R^3}\log\left(\frac{cQ}{\tilde\alpha}\gamma\right)=c\frac{Q^3}{R^3}\log\left(\frac{cQR^{1/2}}{Q^{1/2}}\cdot\frac{cQ^{1/2}}{R^{1/2}}\right)\le C. \tag{2-57}$$

This implies

$$R\ge cQ\log^{1/3}Q. \tag{2-58}$$

<u>Case 2</u>: $\tilde\alpha\le Q^{1/2}/(24R^{1/2})$. At $\alpha_1:=\min\big(Q/(48R),\frac23\tilde\alpha\big)$ we have

$$H\big(\alpha_1,\tfrac14Q,\gamma\big)\le\tfrac1{48}Q.$$

But we saw that $\tilde\alpha\ge Q/(24R)$ by its definition. Hence, $\alpha_1=Q/(48\alpha)$. Comparing the last displayed inequality with (2-56) we conclude that

$$\begin{aligned}\tilde\alpha H_\alpha\left(\alpha_1,\frac{Q}{4},\gamma\right)&\ge(\tilde\alpha-\alpha_1)H_\alpha\left(\alpha_1,\frac{Q}{4},\gamma\right)\\&\ge H\left(\tilde\alpha,\frac{Q}{4},\gamma\right)-H\left(\alpha_1,\frac{Q}{4},\gamma\right)\ge\left(1-\frac{\gamma}{\tilde\alpha}\right)H\left(\tilde\alpha,\frac{Q}{4},0\right)-\frac{Q}{48}\\&\ge\left(1-\frac{\gamma}{\tilde\alpha}\right)H\left(\tilde\alpha,\frac{Q}{4},0\right)-\frac{Q}{48}\ge\left(1-\frac{\gamma}{\tilde\alpha}\right)\frac{Q}{24}-\frac{Q}{48}=\frac{Q}{144}\end{aligned}$$

if $\gamma \in \left[0, \frac{2}{3}\alpha_1\right]$. Hence, using that $\tilde{\alpha} \le Q^{1/2}/(24R^{1/2})$, we obtain the improved estimate on the derivative

$$\text{for all } \gamma \in \left[0, \tfrac{2}{3}\alpha_1\right], \quad H_\alpha\left(\alpha_1, \tfrac{1}{4}Q, \gamma\right) \ge cQ^{1/2}R^{1/2}. \tag{2-59}$$

Then in (2-40) we can use $\alpha := \alpha_1 = \frac{1}{48}Q$, $\beta = cQ^{1/2}R^{1/2}$, and $\gamma = \frac{2}{3}\alpha_1$.

And now we have a new estimate from below, namely (2-59). We just follow (2-45) and (2-46) with these new data, but with one small change; γ in (2-46) can be between 0 and $\alpha_1 = Q/(48R)$. Then instead of (2-47) we get

$$c\frac{Q^2}{R^2}\frac{QR}{R}\log\left(\frac{cQ}{\alpha_1}\gamma\right) \le CR;$$

so again, having $\gamma = \frac{2}{3}\alpha_1$, we obtain

$$R \ge cQ\log^{1/3} Q.$$

2G. *Our Bellman function $\boldsymbol{B}$ as a viscosity supersolution of a degenerate elliptic equation.* Let us remind the reader that we defined in (2-4) the function $\boldsymbol{B}$ on the domain Ω introduced in (2-5). We want to demonstrate in this short subsection that $\boldsymbol{B}$ is a supersolution in the viscosity sense of a certain degenerate elliptic equation.

We haven't used this before, but this knowledge might happen to be important. In particular, it may happen to be true that the reader more familiar with viscosity (super)solutions can simplify a bit our proof of Theorem 2.10, which we just finished proving. In this section D^2u denotes the Hessian matrix of u.

Definition 2.20. An equation $H(x, u, Du, D^2u) = 0$, $x \in \Omega \subset \mathbb{R}^d$, on a function u defined in a domain Ω is called degenerate elliptic if the function H satisfies the following condition: for any point $(x, u, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^d$ and any two $d \times d$ real symmetric matrices X and Y, we have that from $Y \ge X$ it follows that $H(x, u, p, X) \ge H(x, u, p, Y)$.

For example, $H(x, u, p, X) = -\operatorname{trace} X$ gives a degenerate elliptic equation $-\Delta u = 0$. Many examples of degenerate elliptic operators can be found in the first sections of [Nadirashvili et al. 2014]; our example below can be found there too.

Definition 2.21. A lower semicontinuous function u is called a viscosity supersolution of (a degenerate elliptic equation) $H(x, u, Du, D^2u) = 0$ if for every point $x_0 \in \Omega$ and for every C^2 function φ such that (1) $\varphi(x_0) = u(x_0)$ and (2) $\varphi(x) \le u(x)$ for x in a small neighborhood of x_0 inside Ω, one has the inequality $H(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \ge 0$.

To define viscosity subsolution one changes lower to upper semicontinuous, requires $\varphi(x) \ge u(x)$ for x in a small neighborhood of x_0 inside Ω, and gets the conclusion that $H(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \le 0$.

To define the degenerate elliptic equation whose viscosity supersolution is $\boldsymbol{B}$ in Ω from (2-5), we consult Theorem 2.7 and especially inequality (2-10).

Our function $H(x, u, p, X)$ will depend only on matrices X that run over 5×5 real symmetric matrices. A vector v in $\mathbb{R}^5$ is called adapted if $v = (v_1, v_2, 0, 0, v_5)$, $\|v\| = 1$. The set of adapted vectors is called $\mathcal{A}$.

Let us consider the following H_{wmt}, where the subscript stands for "weak martingale transform":

$$H_{\mathrm{wmt}}(X) := -\sup_{v\in\mathcal{A}}[(Xv, v) + X_{44}v_5^2].$$

It is very easy to check that if $Y \geq X$ are two real symmetric matrices, then $H(X) \geq H(Y)$.

Let us see that $\boldsymbol{B}$ from (2-4) satisfies all conditions of a viscosity supersolution of $H_{\mathrm{wmt}}(D^2u) = 0$ in Ω from (2-5). The lower semicontinuity of $\boldsymbol{B}$ follows easily from its definition. Now let us fix $x_0 = (F, w, m, f, \lambda)$. If a smooth φ satisfies $\varphi(x) \leq \boldsymbol{B}(x)$ in a neighborhood of this x_0, then

$$\varphi(F \pm dF, w \pm dw, m, f \pm df, \lambda + \pm d\lambda) \leq \boldsymbol{B}(F \pm dF, w \pm dw, m, f \pm df, \lambda + \pm d\lambda)$$

for all sufficiently small real numbers $dF, dw, df, d\lambda$. Of course we also have $\varphi(F, w, m, f, \lambda) = \boldsymbol{B}(F, w, m, f, \lambda)$. Automatically, (2-10) gives us now that for all sufficiently small real numbers $dF, dw, df, d\lambda$ the following holds:

$$\begin{aligned}\varphi(F, w, m, f, \lambda) \geq \tfrac{1}{4}\big(&\varphi(F - dF, w - dw, m, f - d\lambda, \lambda - d\lambda)\\ &+\varphi(F - dF, w - dw, m, f + d\lambda, \lambda - d\lambda)\\ &+\varphi(F + dF, w + dw, m, f - d\lambda, \lambda + d\lambda)\\ &+\varphi(F + dF, w + dw, m, f + d\lambda, \lambda + d\lambda)\big).\end{aligned} \tag{2-60}$$

The function φ is smooth. Let us use Taylor's formula for all terms in the right-hand side of (2-60). We can easily see that $\varphi(F, w, m, f, \lambda)$ will disappear together with all terms having the first derivatives of φ. After simple algebra, which we leave to the reader, we can see that (2-60) implies an "infinitesimal" version of itself, which holds for any triple $(dF, dw, d\lambda)$:

$$-\big(\varphi_{FF}(dF)^2+2\varphi_{Fw}dF\,dw+\varphi_{ww}(dw)^2+\varphi_{\lambda\lambda}(d\lambda)^2+2\varphi_{F\lambda}dF\,d\lambda+2\varphi_{w\lambda}dw\,d\lambda+\varphi_{ff}(d\lambda)^2\big)\geq 0. \tag{2-61}$$

The reader can immediately see by the definition of H_{wmt} that we just proved

$$H_{\mathrm{wmt}}(D^2\varphi(x_0)) \geq 0.$$

This means exactly that $\boldsymbol{B}$ is a viscosity supersolution of a degenerate elliptic equation $H_{\mathrm{wmt}}(D^2u) = 0$.

3. Random-walk interpretation

In this section we want to prepare the ground for proving our main result, Theorem 2.2. We consider again the domain (2-5), namely,

$$\Omega^s := \{(F, w, m, f, \lambda) \in \mathbb{R}^5 : F \geq |f|\,m,\ m \leq w \leq Q\,m\}. \tag{3-1}$$

We consider special random walks in this domain. From the point (F, w, m, f, λ) in Ω^s we move with equal probability $\frac{1}{4}$ to the following four points (they have to be in this same domain Ω^s):

$$(F - dF, w - dw, m_1, f - d\lambda, \lambda - td\lambda), \quad (F - dF, w - dw, m_2, f - d\lambda, \lambda + td\lambda),$$
$$(F + dF, w + dw, m_3, f + d\lambda, \lambda - td\lambda), \quad (F + dF, w + dw, m_4, f + d\lambda, \lambda + td\lambda),$$

where $\min(m_1, m_2, m_3, m_4) = m$. The symbols dF, dw, $d\lambda$ are just real numbers and $t \in [0, 1]$. The only condition on them is that we stay in Ω^s after performing this one-step random walk.

Now we make another random step from the four points listed above. We consider such random walks which also satisfy two conditions: (a) There is only finite number of steps. (b) The last step brings us to the following part of the boundary of Ω^s:

$$\partial_w \Omega^s = \{(F, w, m, f, \lambda) : w = m\}. \tag{3-2}$$

Let us call such random walks $\mathcal{W}^s$.

Every random walk is the collection of four martingales $\boldsymbol{F}$, $\boldsymbol{w}$, $\boldsymbol{f}$, Λ and the "martingale-with-respect-to-minimum" M. Martingales $\boldsymbol{f}$ and Λ are strongly dependent. The random vector $(\boldsymbol{F}, \boldsymbol{w}, M, \boldsymbol{f}, \Lambda)$ should stay in Ω^s and should finish at $\partial_w \Omega^s$. We have a natural probability measure on $\mathcal{W}^s$; the expectation will be called $\mathbb{E}$.

If we start with the point (F, w, m, f, λ) in Ω^s, let us denote by

$$(\boldsymbol{F}(\omega), \boldsymbol{w}(\omega), M(\omega), \boldsymbol{f}(\omega), \Lambda(\omega))$$

a vector function we get after the walk ends at $\partial_w \Omega^s$. In particular, $\boldsymbol{w}(\omega) = M(\omega)$ identically.

We introduce

$$\mathbb{V}(F, w, m, f, \lambda) := \mathbb{V}_Q(F, w, m, f, \lambda) := \sup \mathbb{E}\, \mathbf{1}_{\Lambda(\omega) \le 0},$$

where the supremum is taken over all walks in $\mathcal{W}^s$ started at (F, w, m, f, λ).

Theorem 3.1. *The function $V = V_Q$ satisfies all the same properties as $\boldsymbol{B}_Q$ from Section 2A1 with one change*; *instead of properties* (2-8), (2-9) *it satisfies the analog of* (2-10), *namely*,

$$\begin{aligned}\mathbb{V}(F, w, m, f, \lambda) \ge \tfrac{1}{4}\big(&\mathbb{V}(F - dF, w - dw, m, f - d\lambda, \lambda - t d\lambda)\\ &+\mathbb{V}(F - dF, w - dw, m, f - d\lambda, \lambda + t d\lambda)\\ &+\mathbb{V}(F + dF, w + dw, m, f + d\lambda, \lambda - t d\lambda)\\ &+\mathbb{V}(F + dF, w + dw, m, f + d\lambda, \lambda + t d\lambda)\big).\end{aligned} \tag{3-3}$$

The proof is the same as the proof of Theorem 2.7; it is based on the same trick of concatenation.

We can introduce the function V starting with the function $\mathbb{V}$ in the same manner as in (2-6), namely,

$$\frac{1}{m}\mathbb{V}(F, w, m, f, \lambda) = V\left(\frac{F}{m\lambda}, \frac{w}{m}, \frac{f}{\lambda}\right) =: V(\alpha, \beta, \gamma), \tag{3-4}$$

defined in the same domain $G = \{(\alpha, \beta, \gamma) : |\gamma| \le \alpha,\ 1 \le \beta \le Q\}$.

Theorem 3.2. *If $|\gamma| < \frac{1}{4}$, then $V(1, \beta, \gamma) \ge \frac{1}{3}\beta$.* (3-5)

To show this we just notice that in Section 2E we constructed a one-step random walk from $\mathcal{W}^s$ such that (3-5) is ensured by item (4) of Section 2E.

In the proof of Theorem 2.10 we used only the properties of $\boldsymbol{B}$ and B that were listed in Section 2A1 and the obstacle condition (2-49). More precisely, we never used properties (2-8), (2-9); only (2-10) was used.

But we have all those ingredients now ready for $\mathbb{V}$ and V. Therefore, we have already proved the following result.

Theorem 3.3. *There exists an absolute positive constant c_V such that*

$$\sup_{(F,w,m,f,\lambda)\in\Omega^s} \frac{|\lambda|\mathbb{V}_Q(F,w,m,f,\lambda)}{F} = \sup_{(\alpha,\beta,\gamma)\in G} \frac{V(\alpha,\beta,\gamma)}{\alpha} \geq c_V Q(\log Q)^{1/3}. \tag{3-6}$$

4. A particular martingale transform and the lower estimate of its norm from $L^1(w)$ to $L^{1,\infty}(w)$: the proof of Theorem 2.2

Let us consider a concrete dyadic shift S and prove Theorem 2.2 for it. Theorem 3.3 claims that we can choose a point $(F_0, w_0, m_0, f_0, \lambda_0) \in \Omega^s$ such that some random walk from $\mathcal{W}^s$ (in particular having finitely many steps and finishing at $\partial_w\Omega^s = \{(F, w, m, f, \lambda) \in \partial\Omega^s : w = m\}$) will have the property that

$$\mathbb{E}\mathbf{1}_{\Lambda(\omega)\leq 0} > \frac{c_V}{2} Q(\log Q)^{1/3} \frac{F_0}{\lambda_0}, \tag{4-1}$$

where $(\boldsymbol{F}(\omega), M(\omega), M(\omega), \boldsymbol{f}(\omega), \Lambda(\omega))$ are the final values of the walk. We can now establish the correspondence between ω and points of the interval $I = [0, 1]$. We assume that $(F_0, w_0, m_0, f_0, \lambda_0)$ are starting values of our "martingales" on I. But our random walk also generates by its first step certain numbers $dF, dw, d\lambda, m_1, m_2, m_3, m_4$, and $t \in [0, 1]$, $m_0 = \min(m_i)$.

We call $dF, df, d\lambda$ martingale differences, m_i, $i = 1, \dots, 4$; we call them splittings of m_0.

We associate:

- $(F_0 - dF, w_0 - dw, m_1, f_0 - d\lambda, \lambda_0 - t d\lambda)$ with values of our "martingales" on I_{--}.
- $(F_0 - dF, w_0 - dw, m_2, f_0 - d\lambda, \lambda_0 + t d\lambda)$ with values of our "martingales" on I_{-+}.
- $(F_0 + dF, w_0 + dw, m_3, f_0 + d\lambda, \lambda_0 - t d\lambda)$ with values of our "martingales" on I_{++}.
- $(F_0 + dF, w_0 + dw, m_4, f_0 + d\lambda, \lambda_0 + t d\lambda)$ with values of our "martingales" on I_{+-}.

If one or several of these points are already on $\partial_w\Omega^s$ we do not touch them anymore. For the rest of points we have the second step, which is given by new martingale differences (and new splittings, now of each of m_i, $i = 1, \dots, 4$). We continue to associate the points with now grandchildren of $I_{\sigma,\sigma'}$, $\sigma, \sigma' = \pm$. We continue this process for finitely many times, until all the points of the walk hit $\partial_w\Omega^s$, where the process stops.

By our association process we constructed functions with finitely many values, constant on some small dyadic intervals of $D(I)$. These are the functions $\varphi(x)$, $\psi(x)$, $W(x)$, $\Phi(x) = |\varphi(x)|W(x)$, $x \in I = [0, 1]$, $m_0 = \min_I W(x)$, $\langle W\rangle_J \leq Q \min_J W$ for all dyadic intervals of $J \in D(I)$. Moreover, it is easy to check by our construction that we have

$$S(-\varphi) = -\psi + \lambda_0.$$

We established the correspondence between ω and the points x of the interval $I=[0,1]$. Under this correspondence $\varphi(x)$ is $\boldsymbol{f}(\omega)$, $\psi(x)$ is $\Lambda(\omega)$, $\Phi(x)$ is $\boldsymbol{F}(\omega)$, $W(x)$ is $\boldsymbol{w}(\omega)$, and $m(\omega)$ corresponds to minimums of W on small final dyadic intervals.

Now we use (4-1). It becomes the inequality

$$W\{x\in I: S(-\varphi)(x)\geq\lambda_0\}\geq\frac{c_V}{2}Q(\log Q)^{1/3}\frac{\int|\varphi|W\,dx}{\lambda_0},$$

which proves Theorem 2.2.

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