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VOLUME COMPARISON WITH RESPECT TO SCALAR CURVATURE

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Dedicated to Nankai University
on its 100th Anniversary
(1919 - 2019)

We investigate the volume comparison with respect to scalar curvature. In particular, we show the volume comparison holds for small geodesic balls of metrics near a V-static metric. For closed manifolds, we prove the volume comparison for metrics near a strictly stable Einstein metric. As applications, we give a partial answer to a conjecture of Bray and recover a result of Besson, Courtois and Gallot, which partially confirms a conjecture of Schoen about closed hyperbolic manifolds. Applying analogous techniques, we obtain a different proof of a local rigidity result due to Dai, Wang and Wei, which shows it admits no metric with positive scalar curvature near strictly stable Ricci-flat metrics.

1. Introduction

The volume comparison theorem is a fundamental result in Riemannian geometry. It is a powerful tool in geometric analysis and frequently used in solving various problems.

The classic volume comparison theorem states that the volume of a complete manifold is upper bounded by the round sphere if its Ricci curvature is lower bounded by a corresponding positive constant. A natural question is whether we can replace the assumption on Ricci curvature by the one on scalar curvature.

In general, scalar curvature is not sufficient to control the volume. This is a straightforward consequence of a result by Corvino, Eichmair and Miao [Corvino et al. 2013]. In order to state it, we need the following fundamental concept, which was introduced in [Miao and Tam 2009].

Definition. Let $(M^n, \bar{g})$ be an n-dimensional Riemannian manifold. We say $\bar{g}$ is a V-static metric if there is a smooth function $f \not\equiv 0$ and a constant $\kappa \in \mathbb{R}$ that solve the *V-static equation*

$$\gamma_{\bar{g}}^* f = \nabla_{\bar{g}}^2 f - \bar{g}\Delta_{\bar{g}} f - f \operatorname{Ric}_{\bar{g}} = \kappa \bar{g}, \tag{1-1}$$

where $\gamma_{\bar{g}}^* : C^\infty(M) \to S_2(M)$ is the formal L^2-adjoint of $\gamma_{\bar{g}} := DR_{\bar{g}}$, the linearization of scalar curvature at $\bar{g}$. We also say a quadruple $(M, \bar{g}, f, \kappa)$ is a *V-static space*.

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Remark 1.1. A fundamental property of a V-static metric is that its scalar curvature $R_{\bar{g}}$ is a constant for M connected; see Proposition 2.1 in [Corvino et al. 2013]. By taking the trace of (1-1), we can see that f satisfies the linear elliptic equation

$$\Delta_{\bar{g}} f + \frac{R_{\bar{g}}}{n-1} f + \frac{n\kappa}{n-1} = 0. \tag{1-2}$$

In particular, f is an eigenfunction for the Laplacian if $\kappa = 0$.

Einstein metrics are in particular V-static, which can be easily seen by taking the function f to be a constant. In this sense, we can view V-static metrics as a generalization of Einstein metrics. Another class of special V-static metrics are *vacuum static metrics* when we take $\kappa = 0$. They can be used to construct an important category of solutions to *Einstein field equations* in general relativity [Qing and Yuan 2013]. The classification of V-static spaces is a crucial problem in understanding the interplay between scalar curvature and volume. For more results, please refer to [Baltazar and Ribeiro 2017; Barros et al. 2015; Corvino et al. 2013; Miao and Tam 2009; 2012].

Now we state a deformation result associated with the concept of V-static metrics.

Theorem 1.2 (Corvino, Eichmair and Miao [Corvino et al. 2013]). *Let $(M^n, \bar{g})$ be a Riemannian manifold and $\Omega \subset M$ be a precompact domain with smooth boundary. Suppose $(\Omega, \bar{g})$ is not V-static, i.e., the V-static equation (1-1) only admits the trivial solution: $f \equiv 0$ and $\kappa = 0$ in $C^\infty(\Omega) \times \mathbb{R}$. Then for any Ω_0 compactly contained in Ω, there exists a constant $\delta_0 > 0$ such that for any $(\rho, V) \in C^\infty(M) \times \mathbb{R}$ with* $\operatorname{supp}(R_{\bar{g}} - \rho) \subset \Omega_0$ *and*

$$\|R_{\bar{g}} - \rho\|_{C^1(\Omega, \bar{g})} + |V_\Omega(\bar{g}) - V| < \delta_0,$$

there exists a metric g on M such that $\operatorname{supp}(g - \bar{g}) \subset \Omega,\ R_g = \rho$ *and* $V_\Omega(g) = V$.

This result suggests that for a non-V-static domain, the information of scalar curvature is not sufficient to give a volume comparison: we can take either $V > V_\Omega(\bar{g})$ or $V < V_\Omega(\bar{g})$, but with $\rho > R_{\bar{g}}$ in Ω. In either case, we can find a metric g realizing (ρ, V) on Ω and it shows that no volume comparison holds for non-V-static domains.

However, the volume comparison with respect to scalar curvature indeed holds for some special metrics. For instance, Miao and Tam [2012] proved a rigidity result for the upper hemisphere with respect to nondecreasing scalar curvature and volume. They also showed that a similar result holds for Euclidean domains. Note that since all space forms are V-static, it is natural to ask whether all V-static spaces admit such a volume comparison result.

Inspired by the rigidity of vacuum static metrics [Qing and Yuan 2016] and related work [Miao and Tam 2012], we obtain a volume comparison theorem with respect to scalar curvature for sufficiently small geodesic balls, if appropriate boundary conditions on induced metric $g|_{T\partial B_r(p)}$ and mean curvature H_g are posed.

Theorem A. *For $n \geq 3$, suppose $(M^n, \bar{g}, f, \kappa)$ is a V-static space. For any $p \in M$ with $f(p) > 0$, there exist positive constants r_0 and ε_0 such that for any geodesic ball $B_r(p) \subset M$ with radius $r \in (0, r_0)$ and metric g on $B_r(p)$ satisfying*

- $R_g \geq R_{\bar{g}}$ *in* $B_r(p)$,
- $H_g \geq H_{\bar{g}}$ *on* $\partial B_r(p)$,
- $g|_{T\partial B_r(p)} = \bar{g}|_{T\partial B_r(p)}$,
- $\|g - \bar{g}\|_{C^2(B_r(p),\bar{g})} < \varepsilon_0$,

the following volume comparison holds:

- *if* $\kappa < 0$, *then*
$$V_\Omega(g) \leq V_\Omega(\bar{g}),$$
- *if* $\kappa > 0$, *then*
$$V_\Omega(g) \geq V_\Omega(\bar{g}),$$

with equality holding in either case if and only if the metric g *is isometric to* $\bar{g}$.

Remark 1.3. If $f(p) < 0$, we only need to replace (f, κ) by $(-f, -\kappa)$, and the reversed volume comparison follows.

Remark 1.4. If $\kappa = 0$, then V-static metrics are in particular vacuum static, and hence g is isometric to $\bar{g}$ according to [Qing and Yuan 2016]. Thus Theorem A is an extension for the rigidity of vacuum static metrics.

In general, the function f may change its sign on a closed V-static manifold. For example, we can take $f := 1 + 2x_{n+1}$ on the unit sphere $\mathbb{S}^n$, where x_{n+1} is the height-function of $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$. Hence the volume comparison may not hold in this case. However, for some special V-static spaces, the volume comparison with respect to scalar curvature might still hold for closed manifolds. Here and throughout this article, we say a manifold is *closed* if it is compact without boundary.

Schoen [1989] proposed a well-known conjecture that the Yamabe invariant of a given closed hyperbolic manifold is achieved by its canonical metric. This problem involves all possible metrics on a given hyperbolic manifold and it is obviously very difficult to solve. However, it can be shown that this conjecture is in fact equivalent to the following volume comparison problem.

Schoen's conjecture. *For* $n \geq 3$, *let* $(M^n, \bar{g})$ *be a closed hyperbolic manifold. Then for any metric* g *on* M *with*
$$R_g \geq R_{\bar{g}},$$
the volume comparison
$$V_M(g) \geq V_M(\bar{g})$$
holds.

The equivalence of the aforementioned Schoen's conjectures are known by experts. For the convenience of readers, we include a proof in the appendix.

Schoen's conjecture is known to hold for 3-manifolds due to works of Hamilton [1999] on nonsingular Ricci flow and Perelman [2002; 2003] on geometrization of 3-manifolds (also see [Agol et al. 2007] for a generalization). For higher dimensions, Besson, Courtois and Gallot [Besson et al. 1991] verified it for metrics C^2-close to the canonical metric. They also proved that the volume comparison holds without assuming g is close to $\bar{g}$ if one replaces the assumption on scalar curvature by Ricci curvature [Besson et al. 1995], which can be viewed as evidence that Schoen's conjecture holds for higher dimensions.

For the case of positive curvature, Bray proposed the following conjecture.

Bray's conjecture. *For $n \geq 3$, there is a positive constant $\varepsilon_n < 1$ such that for any complete manifold (M^n, g) with scalar curvature*

$$R_g \geq n(n-1)$$

and Ricci curvature

$$\operatorname{Ric}_g \geq \varepsilon_n (n-1) g,$$

the volume comparison

$$V_M(g) \leq V_{\mathbb{S}^n}(g_{\mathbb{S}^n})$$

holds, where $\mathbb{S}^n$ is the unit round sphere and $g_{\mathbb{S}^n}$ is the canonical metric.

Remark 1.5. Unlike Schoen's conjecture, there is an additional assumption on Ricci curvature in the positive curvature case. In fact, this assumption is necessary; see [Bray 1997] for details.

For this conjecture, Bray [1997] verified it for three dimensional manifolds and gave an estimate for ε_3. Later, Gursky and Viaclovsky [2004] showed that $\varepsilon_3 \leq \frac{1}{2}$, and Brendle [2012] proved the rigidity of volume comparison for $\varepsilon_3 = \frac{1}{2}$. For higher dimensions, Zhang [2019] gave a partial answer.

Before stating our result, we first recall the following well-known concept associated with an Einstein metric.

Definition 1.6 (stability of Einstein metrics). For $n \geq 3$, suppose $(M^n, \bar{g})$ is a closed Einstein manifold. The metric $\bar{g}$ is said to be *stable* if

$$\min \operatorname{spec}_{\mathrm{TT}}(-\Delta_E^{\bar{g}}) = \inf_{0 \not\equiv h \in S_2^{\mathrm{TT}}(M)} \frac{\int_M \langle h, -\Delta_E^{\bar{g}} h \rangle_{\bar{g}} \, dv_{\bar{g}}}{\int_M |h|_{\bar{g}}^2 \, dv_{\bar{g}}} \geq 0, \tag{1-3}$$

where $\Delta_E^{\bar{g}} := \Delta_{\bar{g}} + 2 \operatorname{Rm}_{\bar{g}}$ is the *Einstein operator* and

$$S_{2,\bar{g}}^{\mathrm{TT}}(M) := \{h \in S_2(M) : \delta_{\bar{g}} h = 0, \ \operatorname{tr}_{\bar{g}} h = 0\}$$

is the space of *transverse-traceless symmetric 2-tensors* on $(M, \bar{g})$. Moreover, $\bar{g}$ is called *strictly stable* if the inequality in (1-3) is strict.

Stability is a crucial concept in the study of Einstein manifolds. There are several equivalent way to define it, we adopt the one involving the Einstein operator for our convenience. For more information, please refer to [Besse 1987; Dai et al. 2005; 2007; Kröncke 2013].

Our main result about volume comparison for Einstein manifolds is the following:

Theorem B. *Suppose $(M^n, \bar{g})$ is a closed strictly stable Einstein manifold with*

$$\operatorname{Ric}_{\bar{g}} = (n-1)\lambda \bar{g},$$

where $\lambda \neq 0$ is a constant. There exists a constant $\varepsilon_0 > 0$ such that for any metric g on M which satisfies

$$R_g \geq n(n-1)\lambda$$

and

$$\|g - \bar{g}\|_{C^2(M,\bar{g})} < \varepsilon_0,$$

the following volume comparison holds:

- *if* $\lambda > 0$, *then*
$$V_M(g) \leq V_M(\bar{g}),$$
- *if* $\lambda < 0$, *then*
$$V_M(g) \geq V_M(\bar{g}).$$

Moreover, the equality holds in either case if and only if the metric g is isometric to $\bar{g}$.

Remark 1.7. Suppose the reference metric $\bar{g}$ is Kähler–Einstein with negative scalar curvature and all infinitesimal complex deformations of its complex structure are integrable. Applying a delicate utilization of the functional

$$K(g) = \int_M |R_g|^{n/2}\, dv_g$$

and the Yamabe functional

$$Y(g) = \frac{\int_M R_g\, dv_g}{(V_M(g))^{(n-2)/n}},$$

Dai, Wang and Wei proved that the volume comparison with respect to scalar curvature holds for metrics near $\bar{g}$; see Theorem 1.5 in [Dai et al. 2007]. In fact, their result can be extended to strictly stable Einstein metrics with negative scalar curvature.

Remark 1.8. The above volume comparison does not hold for Ricci-flat metrics: by taking $g = c^2\bar{g}$ for a constant $c > 0$, we have the scalar curvature $R_g = R_{\bar{g}} = 0$, but the volume $V_M(g)$ can be either larger or smaller than $V_M(\bar{g})$ depending on whether $c > 1$ or $c < 1$.

Remark 1.9. The stability assumption in the theorem is necessary. Macbeth constructed an example of an Einstein manifold which shows the volume comparison fails if we lack stability (personal communication, 2019). See Proposition 5.9 for more details.

Remark 1.10. Our approach in fact works for other curvatures as well. Please see [Lin and Yuan 2022] for a volume comparison theorem of Q-curvature for strictly stable positive Einstein manifolds.

It is well known that hyperbolic metrics are strictly stable as special Einstein metrics and hence Theorem B provides a partial answer to Schoen's conjectures, which recovers the following result.

Corollary A (Besson, Courtois and Gallot [Besson et al. 1991]). *For $n \geq 3$, let $(M^n, \bar{g})$ be a closed hyperbolic manifold. There exists a constant $\varepsilon_0 > 0$ such that for any metric g on M with scalar curvature*

$$R_g \geq R_{\bar{g}}$$

and

$$\|g - \bar{g}\|_{C^2(M,\bar{g})} < \varepsilon_0,$$

we have

$$V_M(g) \geq V_M(\bar{g}),$$

where equality holds if and only if the metric g is isometric to $\bar{g}$.

Similarly, the spherical metric is also strictly stable (Example 3.1.2 in [Kröncke 2013]), and we obtain a partial answer to Bray's conjecture.

Corollary B. *For $n \geq 3$, let $(\mathbb{S}^n, g_{\mathbb{S}^n})$ be the unit round sphere. There exists a constant $\varepsilon_0 > 0$ such that for any metric g on $\mathbb{S}^n$ with scalar curvature*

$$R_g \geq n(n-1)$$

and

$$\|g - g_{\mathbb{S}^n}\|_{C^2(\mathbb{S}^n, g_{\mathbb{S}^n})} < \varepsilon_0,$$

we have

$$V_{\mathbb{S}^n}(g) \leq V_{\mathbb{S}^n}(g_{\mathbb{S}^n}),$$

where equality holds if and only if the metric g is isometric to $g_{\mathbb{S}^n}$.

Remark 1.11. For metrics close to the canonical spherical metric, the assumption on Ricci curvature in Bray's conjecture holds automatically.

Remark 1.12. Corvino, Eichmair and Miao constructed a metric on the upper hemisphere which satisfies the scalar comparison but has arbitrarily large volume; see Proposition 6.2 in [Corvino et al. 2013]. In fact, by gluing a lower hemisphere, we can get a metric on the whole sphere with scalar curvature no less than $n(n-1)$ but with larger volume.

In the research of scalar curvature, a fundamental question is whether a given manifold admits a metric of positive scalar curvature. A well-known result due to Schoen and Yau [1979a; 1979b] and Gromov and Lawson [1980; 1983] is the rigidity of tori, which states that there is no metric of positive scalar curvature on tori. For an excellent survey, please refer to [Rosenberg 2007].

In [Dai et al. 2005], Dai, Wang and Wei studied the existence of metrics with positive scalar curvature on a Riemannian manifold with nonzero parallel spinors. Through investigations of variational properties for the first eigenvalue of the conformal Laplacian, they proved the local rigidity of scalar curvature near the reference metric. Note that their proof can be applied to closed strictly stable Ricci-flat manifolds.

Applying techniques similar to the argument for Theorem B, we obtain the local rigidity of strictly stable Ricci-flat manifolds, which generalizes a result of Fischer and Marsden [1975] about local rigidity of tori with a different approach than in [Dai et al. 2005]:

Theorem C (Dai, Wang and Wei [Dai et al. 2005]). *Suppose $(M^n, \bar{g})$ is a strictly stable Ricci-flat manifold. Then there exists a constant $\varepsilon_0 > 0$ such that for any metric g on M satisfying*

$$R_g \geq 0$$

and

$$\|g - \bar{g}\|_{C^2(M, \bar{g})} < \varepsilon_0,$$

we have g is homothetic to $\bar{g}$. That is, we can find a constant $c > 0$ such that $g = c^2\bar{g}$. In particular, there is no metric with positive scalar curvature near $\bar{g}$.

Remark 1.13. Note that flat tori are merely stable, since the kernel of the Einstein operator is nontrivial and in fact

$$\dim\ker\Delta_E^{\bar{g}} \geq \frac{n(n+1)}{2} - 1.$$

It will be interesting to see whether there is an example of closed stable Ricci-flat manifold which admits a metric of positive scalar curvature near the reference metric.

Remark 1.14. Similar to Theorem B, our approach can also be applied to other curvatures. Please see [Lin and Yuan 2022] for an analogous result for Q-curvature.

The article is organized as follow: In Section 2, we collect notation and conventions used frequently in this article. In Section 3, we calculate some geometric variational formulas involved in the next two sections. In Section 4, we study the volume comparison for geodesic balls in V-static spaces. In Section 5, we investigate the volume comparison for non-Ricci-flat strictly stable Einstein manifolds and the rigidity phenomenon of strictly stable Ricci-flat manifolds. In the Appendix, we present a proof for equivalence of two conjectures proposed by Schoen.

2. Notation and conventions

In this section, we collect notation frequently used and conventions adopted in this article for the convenience of readers. Please note that *all calculations are performed in the reference metric $\bar{g}$.*

Let $(\Omega^n, \bar{g})$ be an n-dimensional compact manifold possibly with C^1-boundary $\Sigma := \partial\Omega$:

(1) Indices of coordinates components:
- Greek indices run through $1, \ldots, n$;
- Latin indices run through $1, \ldots, n-1$.

(2) Connections:
- connection on Ω with respect to $\bar{g}$: $\nabla_{\bar{g}}$;
- connection on Σ with respect to $\bar{g}|_{T\Sigma}$: ∇^{Σ}.

(3) Volume forms:
- volume form on Ω with respect to $\bar{g}$: $dv_{\bar{g}}$;
- volume form on Σ with respect to $\bar{g}|_{T\Sigma}$: $d\sigma_{\bar{g}}$.

(4) Curvatures:
- Riemann curvature tensor $\mathrm{Rm}_{\bar{g}}$: $R_{\alpha\beta\gamma\delta}$;
- Ricci curvature tensor $\mathrm{Ric}_{\bar{g}}$: $R_{\beta\gamma} = \bar{g}^{\alpha\delta}R_{\alpha\beta\gamma\delta}$;
- scalar curvature $R_{\bar{g}}$: $R_{\bar{g}} = \bar{g}^{\beta\gamma}R_{\beta\gamma}$;
- second fundamental form $A_{\bar{g}}$: $A^{\bar{g}}_{ij} = \frac{1}{2}\partial_{\nu_{\bar{g}}}\bar{g}_{ij}$;
- mean curvature $H_{\bar{g}}$: $H_{\bar{g}} = \bar{g}^{ij}A^{\bar{g}}_{ij}$.

(5) Spaces:

- space of all smooth Riemannian metrics on Ω: $\mathcal{M}_\Omega$;
- space of all smooth diffeomorphisms of Ω: $\mathscr{D}(\Omega)$;
- a local slice through the metric $\bar{g}$: $\mathcal{S}_{\bar{g}}$;
- symmetric 2-tensors on Ω: $S_2(\Omega)$;
- TT-tensors on $(\Omega, \bar{g})$: $S^{\mathrm{TT}}_{2,\bar{g}}(\Omega) = \{h \in S_2(\Omega) : \delta_{\bar{g}} h = 0,\ \operatorname{tr}_{\bar{g}} h = 0\}$.

(6) Operators:

- Multiplication and inner product of symmetric 2-tensors:

$$(h \times k)_{\alpha\delta} := \bar{g}^{\beta\gamma} h_{\alpha\beta} k_{\gamma\delta} \quad \text{and} \quad \langle h, k\rangle_{\bar{g}} = h \cdot k := \bar{g}^{\alpha\delta}(h \times k)_{\alpha\delta} = \bar{g}^{\alpha\delta}\bar{g}^{\beta\gamma} h_{\alpha\beta} k_{\gamma\delta}.$$

  In particular,

$$(h^2)_{\alpha\beta} = \bar{g}^{\gamma\delta} h_{\alpha\gamma} h_{\delta\beta} \quad \text{and} \quad \operatorname{Ric}_{\bar{g}} \cdot h := R_{\beta\gamma} h^{\beta\gamma}.$$

- Riemann curvature tensor as an operator on symmetric 2-tensors:

$$(\operatorname{Rm}_{\bar{g}} \cdot h)_{\beta\gamma} := R_{\alpha\beta\gamma\delta} h^{\alpha\delta} \quad \text{and} \quad \langle \operatorname{Rm}_{\bar{g}} \cdot h, h\rangle_{\bar{g}} := R_{\alpha\beta\gamma\delta} h^{\alpha\delta} h^{\beta\gamma}.$$

- A combination involving curvature:

$$\mathscr{R}_{\bar{g}}(h, h) := \langle \operatorname{Rm}_{\bar{g}} \cdot h, h\rangle_{\bar{g}} + 2(\operatorname{Ric}_{\bar{g}} \cdot h)(\operatorname{tr}_{\bar{g}} h) - \frac{2R_{\bar{g}}}{n-1}(\operatorname{tr}_{\bar{g}} h)^2.$$

- Formal L^2-adjoint of covariant differentiation:

$$\delta_{\bar{g}} = -\operatorname{div}_{\bar{g}}, \quad (\delta_{\bar{g}} h)_\beta = -\nabla^\alpha_{\bar{g}} h_{\alpha\beta}.$$

- Einstein operator:

$$\Delta^{\bar{g}}_E h = \Delta_{\bar{g}} h + 2\operatorname{Rm}_{\bar{g}} \cdot h.$$

- Linearization of scalar curvature:

$$\gamma_{\bar{g}} h = -\Delta_{\bar{g}}(\operatorname{tr}_{\bar{g}} h) + \delta^2_{\bar{g}} h - \operatorname{Ric}_{\bar{g}} \cdot h.$$

- Formal L^2-adjoint of $\gamma_{\bar{g}}$:

$$\gamma^*_{\bar{g}} f = \nabla^2_{\bar{g}} f - \bar{g}\Delta_{\bar{g}} f - f \operatorname{Ric}_{\bar{g}}.$$

3. Geometric variational formulas

In this section, we give variational formulas for geometric functionals involved later in the argument. Throughout this section, Ω is assumed to be a compact manifold possibly with C^1-boundary $\Sigma := \partial\Omega$. In the case $\Sigma \neq \varnothing$, let

$$\{e_1, \ldots, e_{n-1},\ e_n = \nu_{\bar{g}}\}$$

be an orthonormal frame on Σ such that the $\{e_i\}_{i=1}^{n-1}$ are tangent to Σ and $\nu_{\bar{g}}$ is the outward normal vector field of Σ with respect to the metric $\bar{g}$. We also denote the induced connection on Σ by ∇^Σ.

We begin with recalling well-known variational formulas of scalar curvature; for detailed calculations, please refer to [Fischer and Marsden 1975; Yuan 2015].

Lemma 3.1. *The first and second variations of scalar curvature are*

$$DR_{\bar g}\cdot h = -\Delta_{\bar g}(\mathrm{tr}_{\bar g} h) + \delta^2_{\bar g} h - \mathrm{Ric}_{\bar g}\cdot h, \tag{3-1}$$

and

$$\begin{aligned} D^2R_{\bar g}\cdot(h,h) = -2\gamma_{\bar g}(h^2) - \Delta_{\bar g}|h|^2_{\bar g} - \tfrac12|\nabla_{\bar g} h|^2_{\bar g} - \tfrac12|d(\mathrm{tr}_{\bar g} h)|^2_{\bar g} \\ + 2\langle h, \nabla^2_{\bar g}(\mathrm{tr}_{\bar g} h)\rangle_{\bar g} - 2\langle \delta_{\bar g} h, d(\mathrm{tr}_{\bar g} h)\rangle_{\bar g} + \nabla_\alpha h_{\beta\gamma}\nabla^\beta h^{\alpha\gamma} \end{aligned} \tag{3-2}$$

for any $h \in S_2(\Omega)$.

For the mean curvature, its variations for the fixed induced boundary metric are given as follow, which was first shown in [Brendle and Marques 2011].

Lemma 3.2. *The first and second variations of mean curvature are*

$$DH_{\bar g}\cdot h = \tfrac12 h_{nn}H_{\bar g} - \nabla_i h_n{}^i + \tfrac12\nabla_n h_i{}^i \tag{3-3}$$

and

$$D^2H_{\bar g}\cdot(h,h) = \Big(-\tfrac14 h^2_{nn} + \sum_{i=1}^{n-1} h^2_{in}\Big)H_{\bar g} + h_{nn}\big(\nabla_i h_n{}^i - \tfrac12\nabla_n h_i{}^i\big) \tag{3-4}$$

for any $h \in S_2(\Omega)$ with $h|_{T\partial\Omega} \equiv 0$.

For the volume functional, we provide a proof mainly based on a technique from linear algebra, which would be useful in calculating higher order variational formulas.

Lemma 3.3. *The first and second variations of volume are*

$$DV_{\Omega,\bar g}\cdot h = \frac12\int_\Omega (\mathrm{tr}_{\bar g} h)\,dv_{\bar g} \tag{3-5}$$

and

$$D^2V_{\Omega,\bar g}\cdot(h,h) = \frac14\int_\Omega [(\mathrm{tr}_{\bar g} h)^2 - 2|h|^2_{\bar g}]\,dv_{\bar g} \tag{3-6}$$

for any $h \in S_2(\Omega)$.

Proof. Let A be an $n\times n$ symmetric matrix. Its characteristic polynomial is given by

$$\begin{aligned} p_A(\lambda) = \det(\lambda I - A) &= \sum_{k=0}^n (-1)^k\sigma_k(A)\lambda^{n-k} \\ &= \lambda^n - (\mathrm{tr}\,A)\lambda^{n-1} + \tfrac12((\mathrm{tr}\,A)^2 - \mathrm{tr}\,A^2)\lambda^{n-2} + \sum_{k=3}^n (-1)^k\sigma_k(A)\lambda^{n-k}, \end{aligned}$$

where $\sigma_k(A)$ is the k-th elementary symmetric polynomial associated to the matrix A.

We choosing normal coordinates with respect to $\bar g$ centered at an interior point $x \in \Omega$, so that $\bar g_{\alpha\beta} = \delta_{\alpha\beta}$ at x. From the linear algebra fact mentioned above, we have the expansion

$$\det(\bar g + h) = 1 + (\mathrm{tr}_{\bar g} h) + \tfrac12((\mathrm{tr}_{\bar g} h)^2 - |h|^2_{\bar g}) + O(|h|^3_{\bar g}),$$

and hence

$$\sqrt{\det(\bar g + h)} = 1 + \tfrac{1}{2}(\mathrm{tr}_{\bar g}\, h) + \tfrac{1}{8}((\mathrm{tr}_{\bar g}\, h)^2 - 2|h|^2_{\bar g}) + O(|h|^3_{\bar g}).$$

Immediately, this implies

$$DV_{\Omega,\bar g}\cdot h = \frac{1}{2}\int_\Omega (\mathrm{tr}_{\bar g}\, h)\, dv_{\bar g} \qquad \text{and} \qquad D^2 V_{\Omega,\bar g}\cdot(h,h) = \frac{1}{4}\int_\Omega ((\mathrm{tr}_{\bar g}\, h)^2 - 2|h|^2_{\bar g})\, dv_{\bar g},$$

respectively. □

In the rest of this section, we calculate variational formulas for some particularly designed functionals involving scalar curvature, mean curvature and volume.

Proposition 3.4. *For any $h \in S_2(\Omega)$ and $f \in C^\infty(\Omega)$,*

$$\int_\Omega (DR_{\bar g}\cdot h) f\, dv_{\bar g} = \int_\Omega \langle h, \gamma^*_{\bar g} f\rangle_{\bar g}\, dv_{\bar g} + \int_\Sigma [-(\partial_{\nu_{\bar g}}(\mathrm{tr}_{\bar g}\, h) + \langle \delta_{\bar g} h, \nu_{\bar g}\rangle_{\bar g}) f + (\mathrm{tr}_{\bar g}\, h)\partial_{\nu_{\bar g}} f - h(\nu_{\bar g}, \nabla_{\bar g} f)]\, d\sigma_{\bar g}.$$

Proof. It is straightforward to see that

$$\begin{aligned}\int_\Omega (DR_{\bar g}\cdot h) f\, dv_{\bar g} &= \int_\Omega (-\Delta_{\bar g}(\mathrm{tr}_{\bar g}\, h) + \delta^2_{\bar g} h - \mathrm{Ric}_{\bar g}\cdot h) f\, dv_{\bar g}\\ &= \int_\Omega \langle h, \gamma^*_{\bar g} f\rangle_{\bar g}\, dv_{\bar g} + \int_\Sigma [-(\partial_{\nu_{\bar g}}(\mathrm{tr}_{\bar g}\, h) + \langle \delta_{\bar g} h, \nu_{\bar g}\rangle_{\bar g}) f + (\mathrm{tr}_{\bar g}\, h)\partial_{\nu_{\bar g}} f - h(\nu_{\bar g}, \nabla_{\bar g} f)]\, d\sigma_{\bar g},\end{aligned}$$

using Lemma 3.1 and integration by parts. □

Proposition 3.5. *For any $h \in S_2(\Omega)$ and $f \in C^\infty(\Omega)$,*

$$\begin{aligned}&\int_\Omega (D^2 R_{\bar g}\cdot(h,h)) f\, dv_{\bar g}\\ &= \int_\Omega \big[-\tfrac{1}{2}|\nabla_{\bar g} h|^2_{\bar g} - \tfrac{1}{2}|d(\mathrm{tr}_{\bar g}\, h)|^2_{\bar g} + |\delta_{\bar g} h|^2 - 2\langle \delta_{\bar g} h, d(\mathrm{tr}_{\bar g}\, h)\rangle_{\bar g} + 2(\mathrm{tr}_{\bar g}\, h)(\delta^2_{\bar g} h) + \mathscr{R}_{\bar g}(h,h)\big] f\, dv_{\bar g}\\ &\quad + \int_\Omega \Big[2(\mathrm{tr}_{\bar g}\, h)\Big(\langle h, \gamma^*_{\bar g} f\rangle_{\bar g} - 2\langle \delta_{\bar g} h, df\rangle_{\bar g} - \frac{1}{n-1}(\mathrm{tr}_{\bar g}\, h)(\mathrm{tr}_{\bar g}(\gamma^*_{\bar g} f))\Big) - 2\langle h, \delta_{\bar g} h\otimes df\rangle_{\bar g} - \langle \gamma^*_{\bar g} f, h^2\rangle_{\bar g}\Big]\, dv_{\bar g}\\ &\quad + \int_\Sigma [\partial_{\nu_{\bar g}}|h|^2_{\bar g} + \langle \delta_{\bar g}(h^2), \nu_{\bar g}\rangle_{\bar g} + 2h(\nu_{\bar g}, \delta_{\bar g} h) + 2h(\nu_{\bar g}, \nabla_{\bar g}\,\mathrm{tr}_{\bar g}\, h) + 2(\mathrm{tr}_{\bar g}\, h)\langle \delta_{\bar g} h, \nu_{\bar g}\rangle_{\bar g}] f\, d\sigma_{\bar g}\\ &\quad + \int_\Sigma [h^2(\nu_{\bar g}, \nabla_{\bar g} f) - |h|^2_{\bar g}\partial_{\nu_{\bar g}} f - 2(\mathrm{tr}_{\bar g}\, h) h(\nu_{\bar g}, \nabla_{\bar g} f)]\, d\sigma_{\bar g},\end{aligned}$$

where

$$\mathscr{R}_{\bar g}(h,h) := \langle \mathrm{Rm}_{\bar g}\cdot h, h\rangle_{\bar g} + 2(\mathrm{Ric}_{\bar g}\cdot h)(\mathrm{tr}_{\bar g}\, h) - \frac{2R_{\bar g}}{n-1}(\mathrm{tr}_{\bar g}\, h)^2.$$

Proof. By Lemma 3.1, we have

$$\begin{aligned}\int_\Omega (D^2 R_{\bar g}\cdot(h,h)) f\, dv_{\bar g} &= \int_\Omega [-2\gamma_{\bar g}(h^2) - \Delta_{\bar g}|h|^2_{\bar g} + 2\langle h, \nabla^2_{\bar g}(\mathrm{tr}_{\bar g}\, h)\rangle_{\bar g} + \nabla_\alpha h_{\beta\gamma}\nabla^\beta h^{\alpha\gamma}] f\, dv_{\bar g}\\ &\quad + \int_\Omega \big[-2\langle \delta_{\bar g} h, d(\mathrm{tr}_{\bar g}\, h)\rangle_{\bar g} - \tfrac{1}{2}|\nabla_{\bar g} h|^2_{\bar g} - \tfrac{1}{2}|d(\mathrm{tr}_{\bar g}\, h)|^2_{\bar g}\big] f\, dv_{\bar g}.\end{aligned}$$

Integrating by parts,

$$
\begin{aligned}
&-2\int_\Omega (\gamma_{\bar g}(h^2)) f\, dv_{\bar g}\\
&\quad= -2\int_\Omega \langle \gamma_{\bar g}^* f, h^2\rangle_{\bar g}\, dv_{\bar g} - 2\int_\Sigma [(\mathrm{tr}_{\bar g}(h^2))\partial_{\nu_{\bar g}} f - f\partial_{\nu_{\bar g}}(\mathrm{tr}_{\bar g}(h^2)) - h^2(\nu_{\bar g}, \nabla f) - \langle \delta_{\bar g}(h^2), \nu_{\bar g}\rangle_{\bar g} f]\, d\sigma_{\bar g}\\
&\quad= -2\int_\Omega \langle \gamma_{\bar g}^* f, h^2\rangle_{\bar g}\, dv_{\bar g} + 2\int_\Sigma [(\partial_{\nu_{\bar g}} |h|^2_{\bar g} + \langle \delta_{\bar g}(h^2), \nu_{\bar g}\rangle_{\bar g}) f + h^2(\nu_{\bar g}, \nabla f) - |h|^2_{\bar g}\partial_{\nu_{\bar g}} f]\, d\sigma_{\bar g}
\end{aligned}
$$

and

$$
-\int_\Omega (\Delta_{\bar g}|h|^2) f\, dv_{\bar g} = -\int_\Omega (|h|^2\Delta_{\bar g} f)\, dv_{\bar g} - \int_\Sigma [f\partial_{\nu_{\bar g}}|h|^2_{\bar g} - |h|^2_{\bar g}\partial_{\nu_{\bar g}} f]\, d\sigma_{\bar g}.
$$

Also,

$$
\begin{aligned}
&2\int_\Omega \langle h, \nabla^2_{\bar g}(\mathrm{tr}_{\bar g}\, h)\rangle_{\bar g} f\, dv_{\bar g}\\
&\qquad= 2\int_\Omega [\langle \delta_{\bar g} h, d(\mathrm{tr}_{\bar g}\, h)\rangle f - \langle h, d(\mathrm{tr}_{\bar g}\, h)\otimes df\rangle_{\bar g}]\, dv_{\bar g} + 2\int_\Sigma h(\nu_{\bar g}, \nabla_{\bar g}(\mathrm{tr}_{\bar g}\, h)) f\, d\sigma_{\bar g}\\
&\qquad= 2\int_\Omega (\mathrm{tr}_{\bar g}\, h)[(\delta^2_{\bar g} h) f - 2\langle \delta_{\bar g} h, df\rangle_{\bar g} + \langle h, \nabla^2_{\bar g} f\rangle_{\bar g}]\, dv_{\bar g}\\
&\qquad\qquad + 2\int_\Sigma [(h(\nu_{\bar g}, \nabla_{\bar g}(\mathrm{tr}_{\bar g}\, h)) + (\mathrm{tr}_{\bar g}\, h)\langle \delta_{\bar g} h, \nu_{\bar g}\rangle_{\bar g}) f - (\mathrm{tr}_{\bar g}\, h) h(\nu_{\bar g}, \nabla_{\bar g} f)]\, d\sigma_{\bar g}\\
&\qquad= 2\int_\Omega (\mathrm{tr}_{\bar g}\, h)[(\delta^2_{\bar g} h) f - 2\langle \delta_{\bar g} h, df\rangle_{\bar g} + \langle h, \gamma^*_{\bar g} f\rangle_{\bar g} + (\mathrm{tr}_{\bar g}\, h)\Delta_{\bar g} f + (\mathrm{Ric}_{\bar g}\cdot h) f]\, dv_{\bar g}\\
&\qquad\qquad + 2\int_\Sigma [(h(\nu_{\bar g}, \nabla_{\bar g}(\mathrm{tr}_{\bar g}\, h)) + (\mathrm{tr}_{\bar g}\, h)\langle \delta_{\bar g} h, \nu_{\bar g}\rangle_{\bar g}) f - (\mathrm{tr}_{\bar g}\, h) h(\nu_{\bar g}, \nabla_{\bar g} f)]\, d\sigma_{\bar g}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega [\nabla_\alpha h_{\beta\gamma}\nabla^\beta h^{\alpha\gamma}] f\, dv_{\bar g}\\
&= -\int_\Omega h_\gamma{}^\beta [\nabla_\alpha\nabla_\beta h^{\alpha\gamma} f + \nabla_\beta h^{\alpha\gamma}\nabla_\alpha f]\, dv_{\bar g} + \int_\Sigma [h_{\beta\gamma}\nu_{\bar g\alpha}\nabla^\beta h^{\alpha\gamma}] f\, d\sigma_{\bar g}\\
&= -\int_\Omega h_\gamma{}^\beta [(\nabla_\beta\nabla_\alpha h^{\alpha\gamma} + R_{\alpha\beta\delta}{}^\alpha h^{\delta\gamma} + R_{\alpha\beta\delta}{}^\gamma h^{\alpha\delta}) f + \nabla_\beta h^{\alpha\gamma}\nabla_\alpha f]\, dv_{\bar g} + \int_\Sigma [h_{\beta\gamma}\nu_{\bar g\alpha}\nabla^\beta h^{\alpha\gamma}] f\, d\sigma_{\bar g}\\
&= -\int_\Omega [-\nabla_\beta h_\gamma{}^\beta\nabla_\alpha h^{\alpha\gamma} f - 2h_\gamma{}^\beta\nabla_\alpha h^{\alpha\gamma}\nabla_\beta f - h_\gamma{}^\beta h^{\alpha\gamma}\nabla_\beta\nabla_\alpha f + (\langle \mathrm{Ric}_{\bar g}, h^2\rangle_{\bar g} - \langle \mathrm{Rm}_{\bar g}\cdot h, h\rangle_{\bar g}) f]\, dv_{\bar g}\\
&\qquad + \int_\Sigma [(h_{\beta\gamma}\nu_{\bar g\alpha}\nabla^\beta h^{\alpha\gamma} - h^\beta_\gamma\nu_{\bar g\beta}\nabla_\alpha h^{\alpha\gamma}) f - h^\beta_\gamma h^{\alpha\gamma}\nu_{\bar g\beta}\nabla_\alpha f]\, d\sigma_{\bar g}\\
&= \int_\Omega [|\delta_{\bar g} h|^2_{\bar g} f - 2\langle h, \delta_{\bar g} h\otimes df\rangle_{\bar g} + \langle \nabla^2_{\bar g} f - f\,\mathrm{Ric}_{\bar g}, h^2\rangle_{\bar g} + \langle \mathrm{Rm}_{\bar g}\cdot h, h\rangle_{\bar g} f]\, dv_{\bar g}\\
&\qquad - \int_\Sigma [(\langle \delta_{\bar g}(h^2), \nu_{\bar g}\rangle_{\bar g} - 2h(\nu_{\bar g}, \delta_{\bar g} h)) f + h^2(\nu_{\bar g}, \nabla_{\bar g} f)]\, d\sigma_{\bar g}\\
&= \int_\Omega [|\delta_{\bar g} h|^2_{\bar g} f - 2\langle h, \delta_{\bar g} h\otimes df\rangle_{\bar g} + \langle \gamma^*_{\bar g} f + \bar g\Delta_{\bar g} f, h^2\rangle_{\bar g} + \langle \mathrm{Rm}_{\bar g}\cdot h, h\rangle_{\bar g} f]\, dv_{\bar g}\\
&\qquad - \int_\Sigma [(\langle \delta_{\bar g}(h^2), \nu_{\bar g}\rangle_{\bar g} - 2h(\nu_{\bar g}, \delta_{\bar g} h)) f + h^2(\nu_{\bar g}, \nabla_{\bar g} f)]\, d\sigma_{\bar g}.
\end{aligned}
$$

Combining the calculations above, we obtain

$$\begin{aligned}
&\int_\Omega (D^2 R_{\bar g}\cdot(h,h)) f\, dv_{\bar g}\\
&= \int_\Omega \big[-\tfrac12|\nabla_{\bar g} h|^2_{\bar g} - \tfrac12 |d(\mathrm{tr}_{\bar g} h)|^2_{\bar g} + |\delta_{\bar g} h|^2_{\bar g} - 2\langle \delta_{\bar g} h, d(\mathrm{tr}_{\bar g} h)\rangle_{\bar g} + \langle \mathrm{Rm}_{\bar g}\cdot h, h\rangle_{\bar g} + 2(\mathrm{tr}_{\bar g} h)(\mathrm{Ric}_{\bar g}\cdot h)\big] f\, dv_{\bar g}\\
&\quad + \int_\Omega [2(\mathrm{tr}_{\bar g} h)((\delta^2_{\bar g} h) f + \langle h, \gamma^*_{\bar g} f\rangle_{\bar g} - 2\langle \delta_{\bar g} h, df\rangle_{\bar g} + (\mathrm{tr}_{\bar g} h)\Delta_{\bar g} f) - 2\langle h, \delta_{\bar g} h\otimes df\rangle_{\bar g} - \langle \gamma^*_{\bar g} f, h^2\rangle_{\bar g}]\, dv_{\bar g}\\
&\qquad + \int_\Sigma [(\partial_{\nu_{\bar g}} |h|^2_{\bar g} + \langle \delta_{\bar g}(h^2), \nu_{\bar g}\rangle_{\bar g} + 2h(\nu_{\bar g}, \delta_{\bar g} h)) f - |h|^2_{\bar g}\partial_{\nu_{\bar g}} f + h^2(\nu_{\bar g}, \nabla_{\bar g} f)]\, d\sigma_{\bar g}\\
&\qquad\quad + 2\int_\Sigma [(h(\nu_{\bar g}, \nabla_{\bar g}(\mathrm{tr}_{\bar g} h)) + (\mathrm{tr}_{\bar g} h)\langle \delta_{\bar g} h, \nu_{\bar g}\rangle_{\bar g}) f - (\mathrm{tr}_{\bar g} h) h(\nu_{\bar g}, \nabla_{\bar g} f)]\, d\sigma_{\bar g}\\
&= \int_\Omega \big[-\tfrac12|\nabla_{\bar g} h|^2_{\bar g} - \tfrac12 |d(\mathrm{tr}_{\bar g} h)|^2_{\bar g} + |\delta_{\bar g} h|^2_{\bar g} - 2\langle \delta_{\bar g} h, d(\mathrm{tr}_{\bar g} h)\rangle_{\bar g} + 2(\mathrm{tr}_{\bar g} h)(\delta^2_{\bar g} h) + \mathscr{R}_{\bar g}(h,h)\big] f\, dv_{\bar g}\\
&\quad + \int_\Omega \Big[2(\mathrm{tr}_{\bar g} h)\Big(\langle h, \gamma^*_{\bar g} f\rangle_{\bar g} - 2\langle \delta_{\bar g} h, df\rangle_{\bar g} - \frac{1}{n-1}(\mathrm{tr}_{\bar g} h)(\mathrm{tr}_{\bar g}(\gamma^*_{\bar g} f))\Big) - 2\langle h, \delta_{\bar g} h\otimes df\rangle_{\bar g} - \langle \gamma^*_{\bar g} f, h^2\rangle_{\bar g}\Big]\, dv_{\bar g}\\
&\qquad + \int_\Sigma [\partial_{\nu_{\bar g}} |h|^2_{\bar g} + \langle \delta_{\bar g}(h^2), \nu_{\bar g}\rangle_{\bar g} + 2h(\nu_{\bar g}, \delta_{\bar g} h) + 2h(\nu_{\bar g}, \nabla_{\bar g}(\mathrm{tr}_{\bar g} h)) + 2(\mathrm{tr}_{\bar g} h)\langle \delta_{\bar g} h, \nu_{\bar g}\rangle_{\bar g}] f\, d\sigma_{\bar g}\\
&\qquad\quad + \int_\Sigma [h^2(\nu_{\bar g}, \nabla_{\bar g} f) - |h|^2_{\bar g}\partial_{\nu_{\bar g}} f - 2(\mathrm{tr}_{\bar g} h) h(\nu_{\bar g}, \nabla_{\bar g} f)]\, d\sigma_{\bar g},
\end{aligned}$$

where we used the fact that

$$\mathrm{tr}_{\bar g}(\gamma^*_{\bar g} f) = -(n-1)\Big(\Delta_{\bar g} f + \frac{R_{\bar g}}{n-1} f\Big)$$

and

$$\mathscr{R}_{\bar g}(h,h) = \langle \mathrm{Rm}_{\bar g}\cdot h, h\rangle_{\bar g} + 2(\mathrm{Ric}_{\bar g}\cdot h)(\mathrm{tr}_{\bar g} h) - \frac{2R_{\bar g}}{n-1}(\mathrm{tr}_{\bar g} h)^2. \qquad \square$$

In particular, for V-static metrics we have the following identity.

Corollary 3.6. *Suppose $(\Omega, \bar g, f, \kappa)$ is a V-static space. Then for any $h \in \ker \delta_{\bar g}$ with $h|_{T\Sigma} \equiv 0$,*

$$\begin{aligned}
\int_\Omega (D^2 R_{\bar g}\cdot(h,h)) f\, dv_{\bar g} = -\frac12 \int_\Omega &\Big[(|\nabla_{\bar g} h|^2_{\bar g} + |d(\mathrm{tr}_{\bar g} h)|^2_{\bar g} - 2\mathscr{R}_{\bar g}(h,h)) f + 2\kappa\Big(|h|^2_{\bar g} + \frac{2}{n-1}(\mathrm{tr}_{\bar g} h)^2\Big)\Big]\, dv_{\bar g}\\
&- \int_\Sigma \Big[A^{ij}_{\bar g} h_{in} h_{jn} - \Big(h^2_{nn} - 3\sum_{i=1}^{n-1} h^2_{in}\Big) H_{\bar g} + 4h_{nn}\Big(\nabla_i h_n{}^i - \frac12 \nabla_n h_i{}^i\Big)\Big] f\, d\sigma_{\bar g}\\
&- \int_\Sigma \Big[\Big(2h^2_{nn} + \sum_{i=1}^{n-1} h^2_{in}\Big)\partial_n f + 2h_{nn}\sum_{i=1}^{n-1} h_{in}\partial_i f\Big]\, d\sigma_{\bar g}.
\end{aligned}$$

Proof. Applying Proposition 3.5 with our assumptions,

$$\begin{aligned}
\int_\Omega (D^2 R_{\bar g}\cdot(h,h)) f\, dv_{\bar g} = -\frac12 \int_\Omega &\Big[(|\nabla_{\bar g} h|^2_{\bar g} + |d(\mathrm{tr}_{\bar g} h)|^2_{\bar g} - 2\mathscr{R}_{\bar g}(h,h)) f + 2\kappa\Big(|h|^2_{\bar g} + \frac{2}{n-1}(\mathrm{tr}_{\bar g} h)^2\Big)\Big]\, dv_{\bar g}\\
&+ \int_\Sigma \big[(\partial_{\nu_{\bar g}} |h|^2_{\bar g} + \langle \delta_{\bar g}(h^2), \nu_{\bar g}\rangle_{\bar g} + 2h(\nu_{\bar g}, \nabla_{\bar g}(\mathrm{tr}_{\bar g} h))) f + h^2(\nu_{\bar g}, \nabla_{\bar g} f)\\
&\qquad - |h|^2_{\bar g}\partial_{\nu_{\bar g}} f - 2(\mathrm{tr}_{\bar g} h) h(\nu_{\bar g}, \nabla_{\bar g} f)\big]\, d\sigma_{\bar g}.
\end{aligned}$$

For the boundary integral, we will rewrite it in terms of the orthonormal frame chosen for the boundary. Note that the identities

$$\Gamma_{ij}^n = -A_{ij}^{\bar{g}}, \qquad \Gamma_{jn}^k = A_j^k, \qquad \Gamma_{in}^i = H_{\bar{g}} \tag{3-7}$$

hold on Σ. Since

$$\delta_{\bar{g}} h = 0 \quad \text{and} \quad h_{ij} = 0, \quad i, j = 1, \dots, n-1,$$

we have

$$\langle \delta_{\bar{g}}(h^2), \nu_{\bar{g}} \rangle_{\bar{g}} = (\delta_{\bar{g}}(h^2))_n = -\nabla_\alpha (h_\beta{}^\alpha h_n{}^\beta) = -h_\beta{}^\alpha \nabla_\alpha h_n{}^\beta = -h_{nn} \nabla_n h_{nn} - h_n{}^i \nabla_i h_{nn} - h_n{}^i \nabla_n h_{in},$$

$$\partial_{\nu_{\bar{g}}} |h|_{\bar{g}}^2 = \nabla_n |h|_{\bar{g}}^2 = 2h_{nn} \nabla_n h_{nn} + 4h_n{}^i \nabla_n h_{in}$$

on Σ. Thus,

$$\begin{aligned}
&\partial_{\nu_{\bar{g}}} |h|_{\bar{g}}^2 + \langle \delta_{\bar{g}}(h^2), \nu_{\bar{g}} \rangle_{\bar{g}} + 2h(\nu_{\bar{g}}, \nabla_{\bar{g}}(\operatorname{tr}_{\bar{g}} h)) \\
&\qquad = h_{nn} \nabla_n h_{nn} + 3h_n{}^i \nabla_n h_{in} - h_n{}^i \nabla_i h_{nn} + 2h_{nn} \nabla_n (\operatorname{tr}_{\bar{g}} h) + 2h_n{}^i \nabla_i (\operatorname{tr}_{\bar{g}} h) \\
&\qquad = 3h_{nn} \nabla_n h_{nn} + 3h_n{}^i \nabla_n h_{in} - h_n{}^i \nabla_i h_{nn} + 2h_{nn} \nabla_n h_i{}^i + 2h_n{}^i \nabla_i^\Sigma h_{nn} \\
&\qquad = -3h_{nn} \nabla_i h_n{}^i - 3h_n{}^i \nabla_j h_i{}^j - h_n{}^i \nabla_i h_{nn} + 2h_{nn} \nabla_n h_i{}^i + 2h_n{}^i \nabla_i^\Sigma h_{nn},
\end{aligned}$$

where we used the fact that

$$\nabla_n h_{n\alpha} = -(\delta_{\bar{g}} h)_\alpha - \nabla_i h_\alpha{}^i = -\nabla_i h_\alpha{}^i.$$

Moreover, from

$$\nabla_j h_i{}^j = \partial_j h_i{}^j + \Gamma_{j\alpha}^j h_i{}^\alpha - \Gamma_{ji}^\alpha h_\alpha{}^j = A_{ij}^{\bar{g}} h_n{}^j + H_{\bar{g}} h_{in}$$

and

$$\nabla_i h_{nn} = \partial_i h_{nn} - 2\Gamma_{in}^\alpha h_{\alpha n} = \nabla_i^\Sigma h_{nn} - 2A_{ij}^{\bar{g}} h_n{}^j,$$

we obtain

$$\begin{aligned}
&\partial_{\nu_{\bar{g}}} |h|_{\bar{g}}^2 + \langle \delta_{\bar{g}}(h^2), \nu_{\bar{g}} \rangle_{\bar{g}} + 2h(\nu_{\bar{g}}, \nabla_{\bar{g}}(\operatorname{tr}_{\bar{g}} h)) \\
&\qquad = -A_{\bar{g}}^{ij} h_{in} h_{jn} - 3H_{\bar{g}} \sum_{i=1}^{n-1} h_{in}^2 + h_n{}^i \nabla_i^\Sigma h_{nn} - 3h_{nn} \nabla_i h_n{}^i + 2h_{nn} \nabla_n h_i{}^i.
\end{aligned}$$

On the other hand,

$$h^2(\nu_{\bar{g}}, \nabla_{\bar{g}} f) - |h|_{\bar{g}}^2 \partial_{\nu_{\bar{g}}} f - 2(\operatorname{tr}_{\bar{g}} h) h(\nu_{\bar{g}}, \nabla_{\bar{g}} f) = -\left(2h_{nn}^2 + \sum_{i=1}^{n-1} h_{in}^2\right) \partial_n f - h_{nn} \sum_{i=1}^{n-1} h_{in} \partial_i f.$$

Integrating by parts,

$$\begin{aligned}
&\int_\Sigma [(\partial_{\nu_{\bar{g}}} |h|_{\bar{g}}^2 + \langle \delta_{\bar{g}}(h^2), \nu_{\bar{g}} \rangle_{\bar{g}} + 2h(\nu_{\bar{g}}, \nabla_{\bar{g}}(\operatorname{tr}_{\bar{g}} h))) f + h^2(\nu_{\bar{g}}, \nabla_{\bar{g}} f) - |h|_{\bar{g}}^2 \partial_{\nu_{\bar{g}}} f - 2(\operatorname{tr}_{\bar{g}} h) h(\nu_{\bar{g}}, \nabla_{\bar{g}} f)] \, d\sigma_{\bar{g}} \\
&\quad = -\int_\Sigma \left[\left(A_{\bar{g}}^{ij} h_{in} h_{jn} + 3H_{\bar{g}} \sum_{i=1}^{n-1} h_{in}^2 \right) f + \left(2h_{nn}^2 + \sum_{i=1}^{n-1} h_{in}^2 \right) \partial_n f + 2h_{nn} \sum_{i=1}^{n-1} h_{in} \partial_i f \right] d\sigma_{\bar{g}} \\
&\qquad + \int_\Sigma (-h_{nn} \nabla_i^\Sigma h_n{}^i - 3h_{nn} \nabla_i h_n{}^i + 2h_{nn} \nabla_n h_i{}^i) f \, d\sigma_{\bar{g}}.
\end{aligned}$$

Note that

$$\begin{aligned}\nabla_i h_n{}^i &= \partial_i h_n{}^i + \Gamma^i_{i\alpha} h_n{}^\alpha - \Gamma^\alpha_{in} h_\alpha{}^i\\ &= \nabla^\Sigma_i h_n{}^i + H_{\bar g} h_{nn},\end{aligned}$$

and hence

$$\begin{aligned}\int_\Sigma [(\partial_{\nu_{\bar g}} |h|^2_{\bar g} + \langle \delta_{\bar g}(h^2), \nu_{\bar g}\rangle_{\bar g} + 2h(\nu_{\bar g}, \nabla_{\bar g}(\mathrm{tr}_{\bar g} h)))f + h^2(\nu_{\bar g}, \nabla_{\bar g} f) - |h|^2_{\bar g}\partial_{\nu_{\bar g}} f - 2(\mathrm{tr}_{\bar g} h)h(\nu_{\bar g}, \nabla_{\bar g} f)]\, d\sigma_{\bar g}\\ = -\int_\Sigma \bigg[A^{ij}_{\bar g} h_{in}h_{jn} - \Big(h^2_{nn} - 3\sum_{i=1}^{n-1} h^2_{in}\Big)H_{\bar g} + 4h_{nn}\big(\nabla_i h_n{}^i - \tfrac12 \nabla_n h_i{}^i\big)\bigg] f\, d\sigma_{\bar g}\\ - \int_\Sigma \bigg[\Big(2h^2_{nn} + \sum_{i=1}^{n-1} h^2_{in}\Big)\partial_n f + 2h_{nn}\sum_{i=1}^{n-1} h_{in}\partial_i f\bigg]\, d\sigma_{\bar g}. \quad \square\end{aligned}$$

In particular, for a special class of V-static spaces we have the following.

Corollary 3.7. *Suppose* $(M^n, \bar g)$ *is a closed Einstein manifold with*

$$\mathrm{Ric}_{\bar g} = (n-1)\lambda \bar g.$$

Then for any $h \in S^{\mathrm{TT}}_{2,\bar g}(M) \oplus (C^\infty(M)\cdot \bar g)$ *we have*

$$\int_M (D^2 R_{\bar g}\cdot(h,h))\, dv_{\bar g} = -\frac12 \int_M \Big(-\langle h, \Delta^{\bar g}_E h\rangle_{\bar g} + \frac{n^2-2}{n^2}|d(\mathrm{tr}_{\bar g} h)|^2_{\bar g} - 2(n-1)\lambda|h|^2_{\bar g}\Big)\, dv_{\bar g}.$$

Proof. According to the V-static equation (1-1), it is obvious that the Einstein manifold $(M^n, \bar g)$ is a V-static space with $f \equiv 1$ on M and $\kappa = -(n-1)\lambda$. By Corollary 3.6 we obtain

$$\int_M (D^2 R_{\bar g}\cdot(h,h))\, dv_{\bar g} = \int_M \big[-\tfrac12|\nabla_{\bar g} h|^2_{\bar g} - \tfrac12|d(\mathrm{tr}_{\bar g} h)|^2_{\bar g} + |\delta_{\bar g} h|^2_{\bar g} + \mathscr{R}_{\bar g}(h,h) + 2\lambda(\mathrm{tr}_{\bar g} h)^2 + (n-1)\lambda|h|^2_{\bar g}\big]\, dv_{\bar g}.$$

From our assumption,

$$\delta_{\bar g} h = -\frac1n d(\mathrm{tr}_{\bar g} h),$$

and hence

$$\int_M (D^2 R_{\bar g}\cdot(h,h))\, dv_{\bar g} = \int_M \Big[-\tfrac12|\nabla_{\bar g} h|^2_{\bar g} - \frac{n^2-2}{2n^2}|d(\mathrm{tr}_{\bar g} h)|^2_{\bar g} + \mathscr{R}_{\bar g}(h,h) + 2\lambda(\mathrm{tr}_{\bar g} h)^2 + (n-1)\lambda|h|^2_{\bar g}\Big]\, dv_{\bar g}.$$

Since

$$\begin{aligned}\mathscr{R}_{\bar g}(h,h) &= \langle \mathrm{Rm}_{\bar g}\cdot h, h\rangle_{\bar g} + 2(\mathrm{Ric}_{\bar g}\cdot h)(\mathrm{tr}_{\bar g} h) - \frac{2R_{\bar g}}{n-1}(\mathrm{tr}_{\bar g} h)^2\\ &= \langle \mathrm{Rm}_{\bar g}\cdot h, h\rangle_{\bar g} - 2\lambda(\mathrm{tr}_{\bar g} h)^2,\end{aligned}$$

we have

$$\int_M (D^2 R_{\bar g}\cdot(h,h))\, dv_{\bar g} = -\frac12 \int_M \Big(-\langle h, \Delta^{\bar g}_E h\rangle_{\bar g} + \frac{n^2-2}{n^2}|d(\mathrm{tr}_{\bar g} h)|^2_{\bar g} - 2(n-1)\lambda|h|^2_{\bar g}\Big)\, dv_{\bar g}. \quad \square$$

4. Volume comparison for V-static spaces

In this section, we will investigate the volume comparison for geodesic balls in generic V-static spaces.

Let Ω be an n-dimensional compact domain in a V-static space $(M^n, \bar{g}, f, \kappa)$ with C^1-boundary $\Sigma := \partial\Omega$. We define the functional

$$\mathscr{F}_{\Omega,\bar{g}}[g] := \int_\Omega R(g) f \, dv_{\bar{g}} + 2\int_\Sigma H(g) f \, d\sigma_{\bar{g}} - 2\kappa V_\Omega(g), \tag{4-1}$$

where

$$g \in \mathcal{M}_{\Omega,\Sigma,\bar{g}} := \{g \in \mathcal{M}_\Omega : g|_{T\Sigma} = \bar{g}|_{T\Sigma}\}$$

is a Riemannian metric on Ω that induces the same metric as $\bar{g}$ on the boundary Σ.

This functional is particularly designed for a given V-static space. The information of both volume and curvature is encoded in this single functional. It has excellent variational properties.

Proposition 4.1. *The V-static metric $\bar{g}$ is a critical point of the functional $\mathscr{F}_{\Omega,\bar{g}}[g]$. That is,*

$$D\mathscr{F}_{\Omega,\bar{g}} \cdot h = 0 \tag{4-2}$$

for any $h \in S_2(\Omega)$ with $h|_{T\partial\Omega} \equiv 0$.

Proof. Applying Proposition 3.4 together with Lemmas 3.2 and 3.3,

$$\begin{aligned} D\mathscr{F}_{\Omega,\bar{g}} \cdot h &= \int_\Omega (DR_{\bar{g}} \cdot h) f \, dv_{\bar{g}} + 2\int_{\partial\Omega} (DH_{\bar{g}} \cdot h) f \, d\sigma_{\bar{g}} - 2\kappa (DV_{\Omega,\bar{g}} \cdot h) \\ &= \int_\Omega [\langle h, \gamma_{\bar{g}}^* f\rangle_{\bar{g}} - \kappa(\operatorname{tr}_{\bar{g}} h)] \, dv_{\bar{g}} \\ &\qquad + \int_{\partial\Omega} [-(\partial_n(\operatorname{tr}_{\bar{g}} h) + (\delta_{\bar{g}} h)_n + 2\nabla_i h_n{}^i - \nabla_n h_i{}^i - h_{nn} H_{\bar{g}}) f - h_n{}^i \partial_i f] \, d\sigma_{\bar{g}}, \end{aligned}$$

where we used that $\operatorname{tr}_{\bar{g}} h = h_{nn}$ on $\partial\Omega$. Since

$$\nabla_i h_n{}^i = \partial_i h_n{}^i + \Gamma^i_{i\alpha} h_n{}^\alpha - \Gamma^\alpha_{in} h_\alpha{}^i = \nabla_i^\Sigma h_n{}^i + H_{\bar{g}} h_{nn},$$

we have

$$(\delta_{\bar{g}} h)_n = -\nabla_\alpha h_n{}^\alpha = -\nabla_i^\Sigma h_n{}^i - \nabla_n h_{nn} - H_{\bar{g}} h_{nn}.$$

Therefore

$$D\mathscr{F}_{\Omega,\bar{g}} \cdot h = \int_\Omega \langle h, \gamma_{\bar{g}}^* f - \kappa \bar{g}\rangle_{\bar{g}} \, dv_{\bar{g}} - \int_{\partial\Omega} [(\nabla_i^\Sigma h_n{}^i) f + h_n{}^i \partial_i f] \, d\sigma_{\bar{g}} = -\int_{\partial\Omega} \nabla_i^\Sigma (h_n{}^i f) \, d\sigma_{\bar{g}} = 0,$$

i.e., $\bar{g}$ is a critical point of $\mathscr{F}_{\Omega,\bar{g}}[g]$. □

The second variation that follows is a straightforward application of Lemmas 3.2 and 3.3 together with Corollary 3.6.

Proposition 4.2. *For any $h \in \ker \delta_{\bar{g}}$ with $h|_{T\Sigma} \equiv 0$, we have*

$$D^2\mathscr{F}_{\Omega,\bar{g}} \cdot (h,h) = -\frac{1}{2}\int_\Omega \Big[(|\nabla_{\bar{g}} h|^2_{\bar{g}} + |d(\mathrm{tr}_{\bar{g}}\, h)|^2_{\bar{g}} - 2\mathscr{R}_{\bar{g}}(h,h)) f + \frac{n+3}{n-1}(\mathrm{tr}_{\bar{g}}\, h)^2 \kappa \Big] dv_{\bar{g}}$$
$$- \int_\Sigma \Big[\Big(A^{ij}_{\bar{g}} h_{in} h_{jn} - \frac{1}{2}\Big(h^2_{nn} - 2\sum_{i=1}^{n-1} h^2_{in} \Big) H_{\bar{g}} + 2h_{nn}\Big(\nabla_i h_n{}^i - \frac{1}{2}\nabla_n h_i{}^i \Big)\Big) f \Big] d\sigma_{\bar{g}}$$
$$- \int_\Sigma \Big[\Big(2h^2_{nn} + \sum_{i=1}^{n-1} h^2_{in} \Big) \partial_n f + 2h_{nn} \sum_{i=1}^{n-1} h_{in} \partial_i f \Big] d\sigma_{\bar{g}}.$$

In general, geometric functionals are invariant under actions of diffeomorphisms and it would cause degenerations on their second variations. In order to get rid of these degenerations, we need to find a metric modulo diffeomorphisms. This is usually referred to be *gauge fixing* and it can be obtained by applying basic elliptic theory and the implicit function theorem. For manifolds with boundary, this can be achieved if one poses appropriate boundary conditions.

Lemma 4.3 [Brendle and Marques 2011, Proposition 11]. *Suppose $(\Omega^n, \bar{g})$ is a compact Riemannian manifold with boundary. Fix a real number $p > n$. Then there exists a constant $\varepsilon_1 > 0$ such that for a metric g on Ω with*

$$g|_{T\partial\Omega} = \bar{g}|_{T\partial\Omega}$$

and

$$\|g - \bar{g}\|_{W^{2,p}(\Omega,\bar{g})} < \varepsilon_1,$$

there exists a diffeomorphism $\varphi : \Omega \to \Omega$ such that $\varphi|_{\partial\Omega} = \mathrm{id}$ and $h := \varphi^* g - \bar{g} \in \ker \delta_{\bar{g}}$. Moreover,

$$\|h\|_{W^{2,p}(\Omega,\bar{g})} \le N \|g - \bar{g}\|_{W^{2,p}(\Omega,\bar{g})}$$

for some constant $N > 0$ that depends only on $(\Omega, \bar{g})$.

In particular, we take Ω to be a geodesic ball $B_r(p)$ at an interior point $p \in M$ with radius $r > 0$.

Proposition 4.4. *Suppose $(M^n, \bar{g}, \kappa, f)$ is a V-static space and $p \in M$ is an interior point. Then there is a constant $\varepsilon_1 > 0$ such that for any metric g on $B_r(p)$ satisfying*

- *$R_g \ge R_{\bar{g}}$ in $B_r(p)$,*
- *$H_g \ge H_{\bar{g}}$ on $\partial B_r(p)$,*
- *$g|_{T\partial B_r(p)} = \bar{g}|_{T\partial B_r(p)}$,*
- *$\|g - \bar{g}\|_{C^2(B_r(p),\bar{g})} < \varepsilon_1$,*

we can find a diffeomorphism $\varphi \in \mathscr{D}(B_r(p))$ such that $\varphi|_{\partial B_r(p)} = \mathrm{id}$ and

$$h := \varphi^* g - \bar{g} \in \ker \delta_{\bar{g}}$$

satisfies $|h|_{\bar{g}} < \frac{1}{2}$ in $B_r(p)$, $h|_{T\partial B_r(p)} \equiv 0$ on $\partial B_r(p)$ and

$$\|h\|_{C^2(B_r(p),\bar{g})} \le N \|g - \bar{g}\|_{C^2(B_r(p),\bar{g})}$$

for some constant $N > 0$ depending only on $(B_r(p), \bar{g})$. Additionally, we have

- $R_{\varphi^* g} \geq R_{\bar{g}}$ *in* $B_r(p)$,
- $H_{\varphi^* g} \geq H_{\bar{g}}$ *on* $\partial B_r(p)$.

Proof. The existence of a constant ε_1 and diffeomorphism φ is a straightforward application of Lemma 4.3. Furthermore, we have

- $R_{\varphi^* g} = R_g \circ \varphi \geq R_{\bar{g}}$ in $B_r(p)$,
- $H_{\varphi^* g} = H_g \circ \varphi = H_g \geq H_{\bar{g}}$ on $\partial B_r(p)$,

because of the fact that the scalar curvature $R_{\bar{g}}$ is a constant on M (see Remark 1.1) and $\varphi|_{\partial B_r(p)} = \mathrm{id}$. □

Let $\hat{g}_h = \bar{g} + h$ be a metric on $B_r(p)$, where $h \in S_2(B_r(p))$ satisfies $|h|_{\bar{g}} < \frac{1}{2}$ and $h|_{T\partial B_r(p)} \equiv 0$. From Propositions 4.1 and 4.2, the remainder of the expansion for $\mathscr{F}_{\Omega,\bar{g}}$ up to second order can be written as

$$\begin{aligned} r_{B_r(p),\bar{g}}[h] &:= \mathscr{F}_{B_r(p),\bar{g}}[\hat{g}_h] - \mathscr{F}_{B_r(p),\bar{g}}[\bar{g}] - D\mathscr{F}_{B_r(p),\bar{g}} \cdot h - \tfrac{1}{2} D^2 \mathscr{F}_{B_r(p),\bar{g}} \cdot (h,h) \\ &= \int_{B_r(p)} (R_{\hat{g}_h} - R_{\bar{g}}) f \, dv_{\bar{g}} - 2\kappa (V_{B_r(p)}(\hat{g}_h) - V_{B_r(p)}(\bar{g})) + I_{B_r(p)}[h] + I_{\partial B_r(p)}[h], \end{aligned} \tag{4-3}$$

where

$$I_{B_r(p)}[h] := \frac{1}{4} \int_{B_r(p)} \Big[(|\nabla_{\bar{g}} h|^2_{\bar{g}} + |d(\operatorname{tr}_{\bar{g}} h)|^2 - 2\mathscr{R}_{\bar{g}}(h,h)) f + \frac{n+3}{n-1} (\operatorname{tr}_{\bar{g}} h)^2 \kappa \Big] \, dv_{\bar{g}}$$

and

$$\begin{aligned} I_{\partial B_r(p)}[h] := \int_{\partial B_r(p)} \Big[2(H_{\hat{g}_h} - H_{\bar{g}}) + \frac{1}{2} A^{ij}_{\bar{g}} h_{in} h_{jn} - \frac{1}{4} \Big(h^2_{nn} - 2 \sum_{i=1}^{n-1} h^2_{in} \Big) H_{\bar{g}} + h_{nn} \Big(\nabla_i h_n{}^i - \frac{1}{2} \nabla_n h_i{}^i \Big) \Big] f \, d\sigma_{\bar{g}} \\ + \int_{\partial B_r(p)} \Big[\Big(h^2_{nn} + \frac{1}{2} \sum_{i=1}^{n-1} h^2_{in} \Big) \partial_n f + h_{nn} \sum_{i=1}^{n-1} h_{in} \partial_i f \Big] \, d\sigma_{\bar{g}}. \end{aligned}$$

The estimate for the remainder $r_{B_r(p),\bar{g}}[h]$ plays a key role in our proof. It mainly relies on estimates for lower bounds of integrals $I_{B_r(p)}$ and $I_{\partial B_r(p)}$.

The estimate for a lower bound of interior integral $I_{B_r(p)}$ is essentially due to the solution of the variational problem

$$\mu(\Omega, \bar{g}) = \inf \left\{ \frac{\int_\Omega |\nabla_{\bar{g}} h|^2_{\bar{g}} \, dv_{\bar{g}}}{\int_\Omega |h|^2_{\bar{g}} \, dv_{\bar{g}}} : h \in S_2(\Omega),\ h \not\equiv 0 \text{ and } h|_{T\partial\Omega} \equiv 0 \right\}.$$

A basic estimate was obtained by Qing and the author in [Qing and Yuan 2016, Lemma 3.7]:

Lemma 4.5. *Suppose $(M^n, \bar{g})$ is a Riemannian manifold with dimension $n \geq 3$ and $B_r(p)$ is a geodesic ball of radius r centered at any interior point $p \in M$. Then there are positive constants $\bar{r}$ and c_0 such that*

$$\mu(B_r(p), \bar{g}) \geq \frac{c_0}{r^2} \tag{4-4}$$

for all $0 < r < \bar{r}$.

From this, we are ready to obtain an estimate for a lower bound of $I_{B_r(p)}$.

Proposition 4.6. *Suppose $p \in M$ is an interior point with $f(p) > 0$. Then there is a constant $r_1 > 0$ such that*

$$f(x) > 0$$

for all $x \in \overline{B_{r_1}(p)} \subseteq M$. Furthermore, for all $r \in (0, r_1)$ and any $h \in S_2(B_r(p))$ with $h|_{T\partial B_r(p)} \equiv 0$,

$$I_{B_r(p)}[h] \geq \frac{1}{8}\Big(\inf_{B_r(p)} f\Big)\|h\|^2_{W^{1,2}(B_r(p),\bar{g})}. \tag{4-5}$$

Proof. By continuity, we can choose a constant $r_1' > 0$ such that $f(x) > 0$ for all $x \in \overline{B_{r_1'}(p)}$.

It is straightforward to see that

$$|\mathscr{R}_{\bar{g}}(h,h)| = \left|\langle \mathrm{Rm}_{\bar{g}} \cdot h, h\rangle_{\bar{g}} + 2(\mathrm{Ric}_{\bar{g}} \cdot h)(\mathrm{tr}_{\bar{g}}\, h) - \frac{2R_{\bar{g}}}{n-1}(\mathrm{tr}_{\bar{g}}\, h)^2\right| \leq \Lambda_{r_1'}|h|^2_{\bar{g}}$$

on $B_{r_1'}(p)$, where $\Lambda_{r_1'} = \Lambda(n, \bar{g}, \|\mathrm{Rm}_{\bar{g}}\|_{C^0(B_{r_1'}(p),\bar{g})})$ is a positive constant independent of h. Thus for any $r < r_1'$ and $h \in S_2(B_r(p))$ with $h|_{T\partial B_r(p)} \equiv 0$, we have

$$\begin{aligned}
I_{B_r(p)}[h] &\geq \frac{1}{4}\int_{B_r(p)} [(|\nabla_{\bar{g}} h|^2_{\bar{g}} - 2|\mathscr{R}_{\bar{g}}(h,h)|)f - 3n|\kappa|\,|h|^2_{\bar{g}}]\, dv_{\bar{g}} \\
&\geq \frac{1}{4}\int_{B_r(p)} \Big[\Big(\inf_{B_r(p)} f\Big)|\nabla_{\bar{g}} h|^2_{\bar{g}} - \Big(2\Lambda_{r_1'}\Big(\sup_{B_r(p)} f\Big) + 3n|\kappa|\Big)|h|^2_{\bar{g}}\Big]\, dv_{\bar{g}} \\
&= \frac{1}{8}\Big(\inf_{B_r(p)} f\Big)\|h\|^2_{W^{1,2}(B_r(p),\bar{g})} + \frac{1}{8}\Big(\inf_{B_r(p)} f\Big)\int_{B_r(p)} [|\nabla_{\bar{g}} h|^2_{\bar{g}} - \mu_r|h|^2_{\bar{g}}]\, dv_{\bar{g}},
\end{aligned}$$

where

$$\mu_r := \frac{4\Lambda_{r_1'}(\sup_{B_r(p)} f) + (\inf_{B_r(p)} f) + 6n|\kappa|}{\inf_{B_r(p)} f} \leq \frac{(4\Lambda_{r_1'}+1)(\sup_{B_{r_1'}(p)} f) + 6n|\kappa|}{\inf_{B_{r_1'}(p)} f} := \bar{\mu}_{r_1'}.$$

Applying Lemma 4.5, we can choose a positive constant $r_1 < r_1'$ sufficiently small such that

$$\int_{B_r(p)} |\nabla_{\bar{g}} h|^2_{\bar{g}}\, dv_{\bar{g}} \geq \bar{\mu}_{r_1'}\int_{B_r(p)} |h|^2_{\bar{g}}\, dv_{\bar{g}}$$

for all $r \in (0, r_1)$. Therefore

$$I_{B_r(p)}[h] \geq \frac{1}{8}\Big(\inf_{B_r(p)} f\Big)\|h\|^2_{W^{1,2}(B_r(p),\bar{g})}$$

for any $r \in (0, r_1)$. □

For a lower bound estimate for the boundary integral $I_{\partial B_r(p)}$ we have the following.

Proposition 4.7. *Suppose $p \in M$ is an interior point with $f(p) > 0$. Then there is a constant $r_2 > 0$ such that*

$$f(x) > 0$$

for all $x \in \overline{B_{r_2}(p)} \subseteq M$. Furthermore, for all $r \in (0, r_2)$ and any metric $\hat{g}_h := \bar{g} + h$ in $B_r(p)$ satisfying

- *$h \in S_2(B_r(p))$ with $|h|_{\bar{g}} < \frac{1}{2}$ and $h|_{T\partial B_r(p)} \equiv 0$,*
- *$H_{\hat{g}_h} \geq H_{\bar{g}}$ on $\partial B_r(p)$,*

we have

$$I_{\partial B_r(p)}[h] \geq -C_0\big(\sup_{B_r(p)} f\big)\|h\|_{C^1(B_r(p),\bar{g})}\|h\|^2_{W^{1,2}(B_r(p),\bar{g})}, \tag{4-6}$$

where $C_0 > 0$ is a constant depending only on $(B_r(p), \bar{g})$.

Proof. By continuity, we can choose a constant $r_2' > 0$ such that $f(x) > 0$ for all $x \in \overline{B_{r_2'}(p)}$.

As observed in [Brendle and Marques 2011], for all $r \in (0, r_2')$ and any metric $\hat{g}_h = \bar{g} + h$ satisfying $h \in S_2(B_r(p))$ with $|h|_{\bar{g}} < \frac{1}{2}$ and $h|_{T\partial B_r(p)} \equiv 0$, we have

$$h_{nn}(H_{\hat{g}_h} - H_{\bar{g}}) = \tfrac{1}{2}h_{nn}^2 H_{\bar{g}} - h_{nn}\big(\nabla_i h_n{}^i - \tfrac{1}{2}\nabla_n h_i{}^i\big) + F_{\bar{g}}(h)$$

due to Lemma 3.2, where the tail term $F_{\bar{g}}(h)$ satisfies

$$|F_{\bar{g}}(h)|_{\bar{g}} \leq \tilde{C}_1|h|^2_{\bar{g}}(|\nabla_{\bar{g}}h|_{\bar{g}} + |A_{\bar{g}}|_{\bar{g}}|h|_{\bar{g}}),$$

and $\tilde{C}_1 > 0$ is a constant depending only on the dimension n. From this,

$$I_{\partial B_r(p)}[h] = \int_{\partial B_r(p)} \left[(2 - h_{nn})(H_{\hat{g}} - H_{\bar{g}}) + \frac{1}{2}A_{\bar{g}}^{ij}h_{in}h_{jn} + \frac{1}{4}\left(h_{nn}^2 + 2\sum_{i=1}^{n-1}h_{in}^2\right)H_{\bar{g}}\right] f\, d\sigma_{\bar{g}}$$
$$+ \int_{\partial B_r(p)} \left[\left(h_{nn}^2 + \frac{1}{2}\sum_{i=1}^{n-1}h_{in}^2\right)\partial_n f + h_{nn}\sum_{i=1}^{n-1}h_{in}\partial_i f\right] d\sigma_{\bar{g}} + \tilde{F}_{\bar{g}}(h),$$

where the tail term $\tilde{F}_{\bar{g}}(h)$ satisfies

$$|\tilde{F}_{\bar{g}}(h)| \leq \tilde{C}_2\big(\sup_{B_r(p)} f\big) \int_{\partial B_r(p)} |h|^2_{\bar{g}}(|\nabla_{\bar{g}}h|_{\bar{g}} + |A_{\bar{g}}|_{\bar{g}}|h|_{\bar{g}})\, dv_{\bar{g}}$$

for a constant $\tilde{C}_2 > 0$ depending only on the dimension n.

For $r > 0$ sufficiently small, it is well known that the second fundamental form and mean curvature of the geodesic sphere $\partial B_r(p)$ behave similarly to round spheres in Euclidean space (see Exercise 1.123 in [Chow et al. 2006]):

$$A_{ij}^{\bar{g}} = \frac{1}{r}\bar{g}_{ij} + O(r) \qquad \text{and} \qquad H_{\bar{g}} = \frac{n-1}{r} + O(r)$$

on $\partial B_r(p)$. Thus we can choose $r_2'' \in (0, r_2')$ such that

$$A_{ij}^{\bar{g}} \geq \frac{1}{2r}\bar{g}_{ij} \qquad \text{and} \qquad H_{\bar{g}} \geq \frac{n-1}{2r}$$

for any geodesic sphere $\partial B_r(p)$ with $r < r_2''$.

For $r \in (0, r_2'')$, we have

$$I_{\partial B_r(p)}[h] \geq \frac{1}{2}\int_{\partial B_r(p)} \left[\frac{1}{4r}\left((n-1)h_{nn}^2 + 2n\sum_{i=1}^{n-1}h_{in}^2\right)f - \left(3h_{nn}^2 + n\sum_{i=1}^{n-1}h_{in}^2\right)|\nabla_{\bar{g}}f|_{\bar{g}}\right] d\sigma_{\bar{g}} + \tilde{F}_{\bar{g}}(h)$$
$$= \frac{1}{2}\int_{\partial B_r(p)} \left[3\left(\frac{n-1}{12r} - \frac{|\nabla_{\bar{g}}f|_{\bar{g}}}{f}\right)h_{nn}^2 + n\left(\frac{1}{2r} - \frac{|\nabla_{\bar{g}}f|_{\bar{g}}}{f}\right)\sum_{i=1}^{n-1}h_{in}^2\right] f\, d\sigma_{\bar{g}} + \tilde{F}_{\bar{g}}(h).$$

Since f is positively lower bounded and $|\nabla_{\bar{g}} f|_{\bar{g}}$ is upper bonded on $B_{r_2''}(p)$, we can pick a constant $r_2 \in (0, r_2'')$ such that

$$\frac{|\nabla_{\bar{g}} f|_{\bar{g}}}{f} \leq \min\left\{\frac{n-1}{12r}, \frac{1}{2r}\right\}$$

holds in $B_r(p)$ for any $r \in (0, r_2)$ and hence

$$I_{\partial B_r(p)} \geq \tilde{F}_{\bar{g}}(h) \geq -\tilde{C}_3\big(\sup_{B_r(p)} f\big)\|h\|_{C^1(\partial B_r(p),\bar{g})}\|h\|^2_{L^2(\partial B_r(p),\bar{g})}$$

for any $r \in (0, r_2)$, where $\tilde{C}_3 > 0$ is a constant depending only on n and r.

Recall the *Sobolev trace inequality*

$$\|h\|^2_{L^2(\partial B_r(p),\bar{g})} \leq \theta_0 \ \|h\|^2_{W^{1,2}(B_r(p),\bar{g})},$$

where $\theta_0 > 0$ is a constant depending only on $(B_r(p), \bar{g})$. Therefore the estimate

$$I_{\partial B_r(p)} \geq -C_0\big(\sup_{B_r(p)} f\big)\|h\|_{C^1(B_r(p),\bar{g})}\|h\|^2_{W^{1,2}(B_r(p),\bar{g})}$$

holds for any $r \in (0, r_2)$, where $C_0 := \theta_0\tilde{C}_3 > 0$ is a constant depending only on $(B_r(p), \bar{g})$. □

Now we are ready to prove the main theorem in this section.

Proof of Theorem A. Let

$$r_0 := \min\{r_1, r_2\} > 0,$$

where r_1 and r_2 are given by Propositions 4.6 and 4.7.

For all $r \in (0, r_0)$, applying Proposition 4.4, we can find a constant $\varepsilon_1 > 0$ such that for any metric g on $B_r(p) \subset M$ satisfying

- $R_g \geq R_{\bar{g}}$ in $B_r(p)$,
- $H_g \geq H_{\bar{g}}$ on $\partial B_r(p)$,
- $g|_{T\partial B_r(p)} = \bar{g}|_{T\partial B_r(p)}$,
- $\|g - \bar{g}\|_{C^2(B_r(p),\bar{g})} < \varepsilon_1$,

there is a diffeomorphism $\varphi \in \mathscr{D}(B_r(p))$ such that $\varphi|_{\partial B_r(p)} = \mathrm{id}$ and

$$h := \varphi^* g - \bar{g} \in \ker \delta_{\bar{g}}$$

satisfies $|h|_{\bar{g}} < \frac{1}{2}$ in $B_r(p)$, $h|_{T\partial B_r(p)} \equiv 0$ on $\partial B_r(p)$ and

$$\|h\|_{C^2(B_r(p),\bar{g})} \leq N\|g - \bar{g}\|_{C^2(B_r(p),\bar{g})}$$

for some constant $N > 0$ depending only on $(B_r(p), \bar{g})$. Additionally, we have

- $R_{\varphi^* g} \geq R_{\bar{g}}$ in $B_r(p)$,
- $H_{\varphi^* g} \geq H_{\bar{g}}$ on $\partial B_r(p)$.

Fix an $r \in (0, r_0)$ and assume the contrary of the claimed volume comparison:

$$\kappa(V_{B_r(p)}(g) - V_{B_r(p)}(\bar{g})) \le 0, \tag{4-7}$$

which implies

$$\kappa(V_{B_r(p)}(\varphi^* g) - V_{B_r(p)}(\bar{g})) \le 0.$$

By Propositions 4.6 and 4.7, the lower bound estimate for the remainder is

$$\begin{aligned} r_{B_r(p),\bar{g}}[h] &= \mathscr{F}_{B_r(p),\bar{g}}[\varphi^* g] - \mathscr{F}_{B_r(p),\bar{g}}[\bar{g}] - D\mathscr{F}_{B_r(p),\bar{g}} \cdot h - \tfrac{1}{2} D^2 \mathscr{F}_{B_r(p),\bar{g}} \cdot (h, h) \\ &= \int_{B_r(p)} (R_{\varphi^* g} - R_{\bar{g}}) f \, dv_{\bar{g}} - 2\kappa(V_{B_r(p)}(\varphi^* g) - V_{B_r(p)}(\bar{g})) + I_{B_r(p)}[h] + I_{\partial B_r(p)}[h] \\ &\ge \Big(\tfrac{1}{8}\big(\inf_{B_r(p)} f\big) - C_0\big(\sup_{B_r(p)} f\big)\|h\|_{C^1(B_r(p),\bar{g})}\Big)\|h\|^2_{W^{1,2}(B_r(p),\bar{g})}. \end{aligned}$$

On the other hand, if we write

$$\tau_r := \max\big\{\sup_{B_r(p)} f, \ \sup_{B_r(p)} |\nabla_{\bar{g}} f|_{\bar{g}}\big\},$$

then the upper bound of the remainder can be estimated using Taylor's formula:

$$\begin{aligned} r_{B_r(p),\bar{g}}[h] &= \tfrac{1}{6} D^3 \mathscr{F}_{B_r(p),\bar{g}+\xi h} \cdot (h, h, h) \\ &\le C_1 \tau_r \int_{B_r(p)} |h|_{\bar{g}} (|\nabla_{\bar{g}} h|^2_{\bar{g}} + |h|^2_{\bar{g}}) \, dv_{\bar{g}} + C_2 \tau_r \int_{\partial B_r(p)} |h|^2_{\bar{g}} (|\nabla_{\bar{g}} h|_{\bar{g}} + |A_{\bar{g}}|_{\bar{g}} |h|_{\bar{g}}) \, dv_{\bar{g}} \\ &\le C_1 \tau_r \|h\|_{C^0(B_r(p),\bar{g})} \|h\|^2_{W^{1,2}(B_r(p),\bar{g})} + C_3 \tau_r \|h\|_{C^1(B_r(p),\bar{g})} \|h\|^2_{L^2(\partial B_r(p),\bar{g})}, \end{aligned}$$

where $\xi \in (0, 1)$ is a constant and C_1, C_2, C_3 are positive constants depending only on $(B_r(p), \bar{g})$. Recall again the *Sobolev trace inequality*

$$\|h\|^2_{L^2(\partial B_r(p),\bar{g})} \le \theta_0 \ \|h\|^2_{W^{1,2}(B_r(p),\bar{g})},$$

where $\theta_0 > 0$ is a constant depending only on $(B_r(p), \bar{g})$. From this we obtain

$$r_{B_r(p),\bar{g}}[h] \le C_0' \tau_r \|h\|_{C^1(B_r(p),\bar{g})} \|h\|^2_{W^{1,2}(B_r(p),\bar{g})},$$

where $C_0' = C_1 + \theta_0 C_3$ is a positive constant depending only on $(B_r(p), \bar{g})$.

Combining both lower and upper bound estimates of $r_{B_r(p),\bar{g}}$, we obtain

$$\Big(\tfrac{1}{8}\big(\inf_{B_r(p)} f\big) - \Big(C_0\big(\sup_{B_r(p)} f\big) + C_0'\tau_r\Big)\|h\|_{C^1(B_r(p),\bar{g})}\Big)\|h\|^2_{W^{1,2}(B_r(p),\bar{g})} \le 0. \tag{4-8}$$

Take

$$\varepsilon_0 := \frac{1}{N} \min\Big\{\varepsilon_1, \ \tfrac{1}{8}\Big(C_0\big(\sup_{B_r(p)} f\big) + C_0'\tau_r\Big)^{-1}\big(\inf_{B_r(p)} f\big)\Big\}.$$

Then for the metric g satisfying

$$\|g - \bar{g}\|_{C^2(B_r(p),\bar{g})} < \varepsilon_0$$

we have

$$\|h\|_{C^1(B_r(p),\bar{g})} \le N\|g - \bar{g}\|_{C^2(B_r(p),\bar{g})} < N\varepsilon_0 < \tfrac{1}{8}\Big(C_0\big(\sup_{B_r(p)} f\big) + C_0'\tau_r\Big)^{-1}\big(\inf_{B_r(p)} f\big).$$

According to inequality (4-8), we see h vanishes identically on $B_r(p)$ and hence $\varphi^* g = \bar{g}$, which shows that $\varphi : B_r(p) \to B_r(p)$ has to be an isometry. Therefore the reverse of inequality (4-7) holds:

$$\kappa(V_{B_r(p)}(g) - V_{B_r(p)}(\bar{g})) \geq 0. \tag{4-9}$$

That is, the following volume comparison holds:

- if $\kappa < 0$, then
$$V_{B_r(p)}(g) \leq V_{B_r(p)}(\bar{g});$$
- if $\kappa > 0$, then
$$V_{B_r(p)}(g) \geq V_{B_r(p)}(\bar{g});$$

with equality holding in either case if and only if the metric g is isometric to $\bar{g}$. □

5. Volume comparison for closed Einstein manifolds

Suppose $(M^n, \bar{g}, f, \kappa)$ is a closed V-static manifold. Then the functional $\mathscr{F}_{M,\bar{g}}$ introduced in the previous section can be simplified as

$$\mathscr{F}_{M,\bar{g}}[g] = \int_M R(g) f \, dv_{\bar{g}} - 2\kappa V_M(g). \tag{5-1}$$

According to Proposition 4.1, the metric $\bar{g}$ is still a critical point of $\mathscr{F}_{M,\bar{g}}$. However, it is obvious that this functional is not compatible with actions of dilations, which would cause subtle issues in its second variation. Geometrically speaking, dilations introduce additional degeneracy besides actions of diffeomorphisms, since they make no essential change to the geometry of the manifold. In order to obtain volume comparison for closed manifolds, we need to construct a new functional instead, which is invariant under dilations.

Definition 5.1. Suppose $(M^n, \bar{g}, f, \kappa)$ is an n-dimensional closed V-static manifold. We define the functional

$$\mathscr{G}_{M,\bar{g}}[g] := (V_M(g))^{2/n} \int_M R(g) f \, dv_{\bar{g}} \tag{5-2}$$

for any Riemannian metric g on M.

Obviously, this functional is dilation-invariant:

$$\mathscr{G}_{M,\bar{g}}[c^2 g] = (V_M(c^2 g))^{2/n} \int_M R(c^2 g) f \, dv_{\bar{g}} = \mathscr{G}_{M,\bar{g}}[g]$$

for any constant $c \neq 0$.

Now we focus on a special type of V-static metrics: Einstein metrics. According to the V-static equation (1-1), we get

$$\gamma_{\bar{g}}^* 1 = -\operatorname{Ric}_{\bar{g}} = \kappa \bar{g}$$

by taking the function f to be constantly 1 on M. This means $(M^n, \bar{g}, 1, \kappa)$ is a V-static space if and only if the metric $\bar{g}$ is an Einstein metric with scalar curvature $R_{\bar{g}} = -n\kappa$. Moreover, if we write

$$\lambda := \frac{R_{\bar{g}}}{n(n-1)}, \tag{5-3}$$

then the Ricci curvature tensor is given by

$$\mathrm{Ric}_{\bar{g}} = (n-1)\lambda\bar{g}$$

and

$$\kappa = -(n-1)\lambda.$$

As a functional designed for V-static metrics, $\mathscr{G}_{M,\bar{g}}$ shares analogous variational properties with $\mathscr{F}_{M,\bar{g}}$.

Proposition 5.2. *Suppose $(M^n, \bar{g})$ is a closed Einstein manifold with Ricci curvature tensor*

$$\mathrm{Ric}_{\bar{g}} = (n-1)\lambda\bar{g}.$$

Then the metric $\bar{g}$ is a critical point of the functional $\mathscr{G}_{M,\bar{g}}$.

Proof. From Proposition 3.4 and Lemma 3.3,

$$\begin{aligned} D\mathscr{G}_{M,\bar{g}}\cdot h &= (V_M(\bar{g}))^{2/n}\int_M (DR_{\bar{g}}\cdot h)\,dv_{\bar{g}} + \frac{2}{n}(V_M(\bar{g}))^{(2/n)-1}(DV_{M,\bar{g}}\cdot h)\int_M R_{\bar{g}}\,dv_{\bar{g}} \\ &= (V_M(\bar{g}))^{2/n}\left[\int_M (\gamma_{\bar{g}}^* 1)\ dv_{\bar{g}} + \frac{1}{n}R_{\bar{g}}\int_M (\mathrm{tr}_{\bar{g}}\, h)\,dv_{\bar{g}}\right] \\ &= -(V_M(\bar{g}))^{2/n}\int_M \langle \mathrm{Ric}_{\bar{g}} - (n-1)\lambda\bar{g}, h\rangle_{\bar{g}}\,dv_{\bar{g}} = 0, \end{aligned}$$

for any $h \in S_2(M)$. □

For the second variation, we have the following.

Proposition 5.3. *Suppose (M^n, g) is an Einstein manifold with Ricci curvature tensor*

$$\mathrm{Ric}_{\bar{g}} = (n-1)\lambda\bar{g}.$$

Then

$$D^2\mathscr{G}_{M,\bar{g}}\cdot(h,h) = -\frac{1}{2}(V_M(\bar{g}))^{2/n}\int_M\Big[-\langle h_{\mathrm{TT}}, \Delta_E^{\bar{g}} h_{\mathrm{TT}}\rangle_{\bar{g}} + \frac{(n-1)(n+2)}{n^2}\big(|d(\mathrm{tr}_{\bar{g}}\, h)|_{\bar{g}}^2 - n\lambda(\mathrm{tr}_{\bar{g}}\, h - \overline{\mathrm{tr}_{\bar{g}}\, h})^2\big)\Big]\,dv_{\bar{g}}$$

for any $h = h_{\mathrm{TT}} + \frac{1}{n}(\mathrm{tr}_{\bar{g}}\, h)\bar{g} \in S_{2,\bar{g}}^{\mathrm{TT}} \oplus (C^\infty(M)\cdot\bar{g})$.

Proof. From Lemmas 3.1 and 3.3 and Corollary 3.7 we obtain

$$\begin{aligned} &D^2\mathscr{G}_{M,\bar{g}}\cdot(h,h) \\ &\quad= \frac{2}{n}(V_M(\bar{g}))^{(2/n)-1}(D^2V_{M,\bar{g}}\cdot(h,h))\int_M R_{\bar{g}}\,dv_{\bar{g}} + \frac{4}{n}(V_M(\bar{g}))^{(2/n)-1}(DV_{M,\bar{g}}\cdot h)\int_M (DR_{\bar{g}}\cdot h)\,dv_{\bar{g}} \\ &\qquad\quad - \frac{2(n-2)}{n^2}(V_M(\bar{g}))^{(2/n)-2}(DV_{M,\bar{g}}\cdot h)^2\int_M R_{\bar{g}}\,dv_{\bar{g}} + (V_M(\bar{g}))^{2/n}\int_M (D^2R_{\bar{g}}\cdot(h,h))\,dv_{\bar{g}} \\ &\quad= -\frac{1}{2}(V_M(\bar{g}))^{2/n}\int_M \Big(-\langle h, \Delta_E^{\bar{g}} h\rangle_{\bar{g}} + \frac{n^2-2}{n^2}|d(\mathrm{tr}_{\bar{g}}\, h)|_{\bar{g}}^2 - (n-1)\lambda(\mathrm{tr}_{\bar{g}}\, h)^2\Big)\,dv_{\bar{g}} \\ &\qquad\quad - \frac{(n-1)(n+2)}{2n}\lambda(V_M(\bar{g}))^{2/n}\int_M (\overline{\mathrm{tr}_{\bar{g}}\, h})^2\,dv_{\bar{g}}. \end{aligned}$$

Now the decomposition

$$h = h_{\text{TT}} + \frac{1}{n}(\operatorname{tr}_{\bar g} h)\bar g$$

implies

$$\begin{aligned}(D^2\mathscr{G}_{M,\bar g})\cdot(h,h) \\ = -\frac{1}{2}(V_M(\bar g))^{2/n}\int_M\Big[-\langle h_{\text{TT}}, \Delta_E^{\bar g} h_{\text{TT}}\rangle_{\bar g} + \frac{(n-1)(n+2)}{n^2}(|d(\operatorname{tr}_{\bar g} h)|^2_{\bar g} - n\lambda(\operatorname{tr}_{\bar g} h - \overline{\operatorname{tr}_{\bar g} h})^2)\Big]\,dv_{\bar g}.\end{aligned}$$ □

As a key step of the proof for our volume comparison theorem, we need to give a characterization of the second variation of the functional $\mathscr{G}_{M,\bar g}$ at $\bar g$. This is closely related to spectrum problems of two operators: one is about the Einstein operator and can be characterized by the stability of Einstein metrics, the other is about the Laplace–Beltrami operator whose eigenvalue estimate is given by the well-known *Lichnerowicz–Obata theorem*; see Theorem 5.1 in [Li 2012].

Lemma 5.4 (Lichnerowicz–Obata's eigenvalue estimate). *Suppose $(M^n, \bar g)$ is an n-dimensional closed Riemannian manifold with Ricci curvature tensor*

$$\operatorname{Ric}_{\bar g} \geq (n-1)\lambda \bar g,$$

where $\lambda > 0$ is a constant. Then for any function $u \in C^\infty(M)$ that is not identically a constant, we have

$$\int_M |du|^2\,dv_{\bar g} \geq n\lambda \int_M (u-\bar u)^2\,dv_{\bar g}, \tag{5-4}$$

where equality holds if and only if $(M^n, \bar g)$ is isometric to the round sphere $\mathbb{S}^n(r)$ with radius $r = 1/\sqrt{\lambda}$ and u is a first eigenfunction of the Laplace–Beltrami operator.

Applying this to Proposition 5.3, immediately we get the nonpositive definite property of the second variation of $\mathscr{G}_{M,\bar g}$ at $\bar g$.

Proposition 5.5. *Suppose $(M^n, \bar g)$ is a closed stable Einstein manifold with Ricci curvature tensor*

$$\operatorname{Ric}_{\bar g} = (n-1)\lambda \bar g.$$

Then

$$D^2\mathscr{G}_{M,\bar g}\cdot(h,h) \leq 0$$

for any $h \in S^{\text{TT}}_{2,\bar g}(M) \oplus (C^\infty(M)\cdot \bar g)$. Moreover, equality holds if and only if

- *$h \in \mathbb{R}\bar g \oplus \ker \Delta_E^{\bar g}$, when $(M, \bar g)$ is not isometric to the round sphere up to a rescaling of the metric,*
- *$h \in (\mathbb{R} \oplus E_{n\lambda})\bar g$, when $(M, \bar g)$ is isometric to the round sphere $\mathbb{S}^n(r)$ with radius $r = 1/\sqrt{\lambda}$,*

where

$$E_{n\lambda} := \{u \in C^\infty(\mathbb{S}^n(r)) : \Delta_{\mathbb{S}^n(r)} u + n\lambda u = 0\}$$

is the space of first eigenfunctions for the spherical metric.

Proof. Recall that the Einstein metric $\bar g$ is stable if and only if $(-\Delta_E^{\bar g})$ is a nonnegative operator. Then the conclusion follows by applying this fact and Lemma 5.4 to Proposition 5.3. □

Intuitively speaking, a *slice* is a subset of metrics in the space of all Riemannian metrics which is transverse to the orbit of diffeomorphism actions. The following refined version of the slice theorem reveals the local structure of Einstein metrics in the space of all metrics. To the best of the author's knowledge, it does not appear in the literature. We hope it can be useful in problems involving Einstein metrics. The proof is standard; please refer to [Brendle and Marques 2011; Viaclovsky 2016].

Theorem 5.6 (Ebin–Palais slice theorem). *Suppose $(M^n, \bar{g})$ is a closed n-dimensional Einstein manifold with Ricci curvature tensor*

$$\mathrm{Ric}_{\bar{g}} = (n-1)\lambda\bar{g},$$

where $\lambda \in \mathbb{R}$ is a constant. Let $\mathcal{M}$ be the space of all Riemannian metrics on M. There exists a local slice $\mathcal{S}_{\bar{g}}$ though $\bar{g}$ in $\mathcal{M}$. That is, for a fixed real number $p > n$, one can find a constant $\varepsilon_1 > 0$ such that, for any metric $g \in \mathcal{M}$ with $\|g-\bar{g}\|_{W^{2,p}(M,\bar{g})} < \varepsilon_1$, there is a diffeomorphism $\varphi \in \mathscr{D}(M)$ with $\varphi^* g \in \mathcal{S}_{\bar{g}}$. Moreover, for a smooth local slice $\mathcal{S}_{\bar{g}}$, we have the decomposition

$$S_2(M) = T_{\bar{g}}\mathcal{S}_{\bar{g}} \oplus (T_{\bar{g}}\mathcal{S}_{\bar{g}})^{\perp},$$

where the tangent space of $\mathcal{S}_{\bar{g}}$ at $\bar{g}$ and its L^2-orthogonal complement are given by

$$T_{\bar{g}}\mathcal{S}_{\bar{g}} = S^{\mathrm{TT}}_{2,\bar{g}}(M) \oplus (C^{\infty}(M)\cdot\bar{g})$$

and

$$(T_{\bar{g}}\mathcal{S}_{\bar{g}})^{\perp} = \{\mathcal{L}_{\bar{g}}(X) : \langle X, \nabla_{\bar{g}} u\rangle_{L^2(M,\bar{g})} = 0 \text{ for all } u \in C^{\infty}(M)\}$$

when $(M^n, \bar{g})$ is not isometric to the round sphere $\mathbb{S}^n(r)$ up to a scaling, and

$$T_{\bar{g}}\mathcal{S}_{\bar{g}} = S^{\mathrm{TT}}_{2,\bar{g}}(M) \oplus (E^{\perp}_{n\lambda}\cdot\bar{g})$$

and

$$(T_{\bar{g}}\mathcal{S}_{\bar{g}})^{\perp} = \{\mathcal{L}_{\bar{g}}(X) : \langle X, \nabla_{\bar{g}} u\rangle_{L^2(M,\bar{g})} = 0 \text{ for all } u \in E^{\perp}_{n\lambda}\}$$

when $(M^n, \bar{g})$ is isometric to the round sphere $\mathbb{S}^n(r)$ with $r = 1/\sqrt{\lambda}$. Here

$$E_{n\lambda} = \{u \in C^{\infty}(\mathbb{S}^n(r)) : \Delta_{\mathbb{S}^n(r)} u + n\lambda u = 0\}$$

is the space of first eigenfunctions for the spherical metric.

Now we restrict the functional $\mathscr{G}_{M,\bar{g}}$ on a local slice $\mathcal{S}_{\bar{g}}$ and denote it by

$$\mathscr{G}^{\mathcal{S}}_{M,\bar{g}} := \mathscr{G}_{M,\bar{g}}|_{\mathcal{S}}.$$

In order to investigate the local behavior of $\mathscr{G}^{\mathcal{S}}_{M,\bar{g}}$ near $\bar{g}$, we need the following *Morse lemma on Banach manifolds*.

Lemma 5.7 (Morse lemma [Fischer and Marsden 1975]). *Let $\mathcal{P}$ be a Banach manifold and $F : \mathcal{P} \to \mathbb{R}$ a C^2-function. Suppose $\mathcal{Q} \subset \mathcal{P}$ is a submanifold, $F = 0$ and $dF = 0$ on $\mathcal{Q}$ and that there is a smooth normal bundle neighborhood of $\mathcal{Q}$ such that if $\mathcal{E}_x$ is the normal complement to $T_x\mathcal{Q}$ in $T_x\mathcal{P}$ then $d^2F(x)$ is weakly negative definite on $\mathcal{E}_x$ (i.e., $d^2F(x)(v, v) \le 0$ with equality only if $v = 0$). Let $\langle\cdot,\cdot\rangle_x$ be a weak Riemannian structure with a smooth connection and assume that F has a smooth $\langle\cdot,\cdot\rangle_x$-gradient, $Y(x)$.*

Assume $DY(x)$ maps $\mathcal{E}_x$ to $\mathcal{E}_x$ and is an isomorphism for $x \in \mathcal{Q}$. Then there is a neighborhood U of $\mathcal{Q}$ such that $y \in U$ and $F(y) \geq 0$ implies $y \in \mathcal{Q}$.

Applying it to our case, we obtain the following local rigidity result.

Proposition 5.8. *Suppose $(M^n, \bar{g})$ is a strictly stable Einstein manifold and $\mathcal{S}_{\bar{g}}$ is a local slice through $\bar{g}$. Then there is a neighborhood $U_{\bar{g}}$ of $\bar{g}$ in $\mathcal{S}_{\bar{g}}$ such that for any metric $\hat{g}_s \in U_{\bar{g}}$ satisfying*

$$\mathscr{G}^{\mathcal{S}}_{M,\bar{g}}[\hat{g}_s] \geq \mathscr{G}^{\mathcal{S}}_{M,\bar{g}}[\bar{g}],$$

there is a constant $c > 0$ such that $\hat{g}_s = c^2 \bar{g}$.

Proof. Let

$$\tilde{\mathcal{Q}}_{\bar{g}} := \{g_s \in \mathcal{S}_{\bar{g}} : g_s \text{ is Einstein}\}$$

be the subset of the local slice $\mathcal{S}_{\bar{g}}$ consisting of Einstein metrics near the reference metric $\bar{g}$. By [Koiso 1980, Corollary 3.4], strict stability implies that $\bar{g}$ is rigid. That is, we can find a neighborhood $\tilde{U}_{\bar{g}} \subseteq \mathcal{S}_{\bar{g}}$ of $\bar{g}$ such that

$$\mathcal{Q}_{\bar{g}} := \tilde{\mathcal{Q}}_{\bar{g}} \cap \tilde{U}_{\bar{g}} = \{g_s \in \tilde{U}_{\bar{g}} : g_s = c^2 \bar{g},\ c > 0\}.$$

In particular, the tangent space of $\mathcal{Q}_{\bar{g}}$ at $\bar{g}$ is given by

$$T_{\bar{g}} \mathcal{Q}_{\bar{g}} = \mathbb{R}\bar{g}$$

and its L^2-orthogonal complement in $T_{\bar{g}} \mathcal{S}_{\bar{g}}$ can be expressed as

$$\mathcal{E}_{\bar{g}} := (T_{\bar{g}} \mathcal{Q}_{\bar{g}})^{\perp} = S^{\mathrm{TT}}_{2,\bar{g}}(M) \oplus (\Psi_{\bar{g}}(M) \cdot \bar{g})$$

due to Theorem 5.6, where

$$\Psi_{\bar{g}}(M) = \left\{ u \in E^{\perp}_{n\lambda} : \int_M u \ dv_{\bar{g}} = 0 \right\}$$

if $\bar{g}$ is spherical and

$$\Psi_{\bar{g}}(M) = \left\{ u \in C^{\infty}(M) : \int_M u \, dv_{\bar{g}} = 0 \right\}$$

otherwise.

Consider a weak Riemannian structure on the local slice $\mathcal{S}_{\bar{g}}$,

$$\langle\!\langle \cdot , \cdot \rangle\!\rangle_{g_s} : T_{g_s} \mathcal{S}_{\bar{g}} \times T_{g_s} \mathcal{S}_{\bar{g}} \ \to \ \mathbb{R} \quad \text{for all } \ g_s \in \mathcal{S}_{\bar{g}},$$

which is defined to be

$$\langle\!\langle h, k \rangle\!\rangle_{g_s} := \int_M [\langle \nabla_{g_s} h, \nabla_{g_s} k \rangle_{g_s} + \langle h, k \rangle_{g_s}] \, dv_{g_s} = \int_M \langle (-\Delta_{g_s} + 1) h, k \rangle_{g_s} \, dv_{g_s}$$

for any $h, k \in T_{g_s} \mathcal{S}_{\bar{g}}$. According to [Ebin 1970] it has a smooth connection. The $\langle\!\langle \cdot , \cdot \rangle\!\rangle_{g_s}$-gradient of $\mathscr{G}^{\mathcal{S}}_{M,\bar{g}}$ is given by

$$Y(g_s) = P_{g_s} (-\Delta_{g_s} + 1)^{-1} \Big[(V_M(g_s))^{2/n} \Big(\gamma^*_{g_s} f_{g_s} + \frac{1}{n} g_s (V_M(g_s))^{-(n+2)/n} \mathscr{G}_{M,\bar{g}}[g_s] \Big) \Big],$$

where P_{g_S} is the orthogonal projection on $T_{g_S}\mathcal{S}_{\bar{g}}$ and f_{g_S} is a smooth function on M with $dv_{\bar{g}} = f_{g_S}\, dv_{g_S}$. Obviously, $Y(g_S)$ is a smooth vector field on $\mathcal{S}_{\bar{g}}$. For simplicity, we write

$$Z(g_S) := (V_M(g_S))^{2/n}\Big(\gamma^*_{g_S} f_{g_S} + \frac{1}{n} g_S (V_M(g_S))^{-(n+2)/n}\, \mathscr{G}_{M,\bar{g}}[g_S]\Big).$$

It is straightforward to see that $Z(\bar{g}) = 0$ and the linearization of Z at $\bar{g}$ is given by

$$(DZ_{\bar{g}}) \cdot h = \frac{1}{2}(V_M(\bar{g}))^{2/n}\Big(\Delta^{\bar{g}}_E h_{\mathrm{TT}} + \frac{(n-1)(n+2)}{n^2}\bar{g}(\Delta_{\bar{g}} + n\lambda)(\operatorname{tr}_{\bar{g}} h - \overline{\operatorname{tr}_{\bar{g}} h})\Big) = D^2\mathscr{G}_{M,\bar{g}} \cdot (h, \cdot)$$

for any $h = h_{\mathrm{TT}} + \frac{1}{n}(\operatorname{tr}_{\bar{g}} h)\bar{g} \in \mathcal{E}_{\bar{g}}$. Thus

$$(DY_{\bar{g}}) \cdot h = P_{\bar{g}}(-\Delta_{\bar{g}} + 1)^{-1}(D^2\mathscr{G}_{M,\bar{g}} \cdot (h, \cdot))$$

and $DY_{\bar{g}}$ is an isomorphism on $\mathcal{E}_{\bar{g}}$ due to the fact that $D^2\mathscr{G}^{\mathcal{S}}_{M,\bar{g}}$ is strictly negative definite on $\mathcal{E}_{\bar{g}}$ from Proposition 5.5.

Since the functional $\mathscr{G}^{\mathcal{S}}_{M,\bar{g}}$ is dilation-invariant, applying Lemma 5.7, we can find a neighborhood $U_{\bar{g}} \subseteq \mathcal{S}_{\bar{g}}$ of $\bar{g}$ such that for any $\hat{g}_S \in U_{\bar{g}}$ satisfying

$$\mathscr{G}^{\mathcal{S}}_{M,\bar{g}}[\hat{g}_S] \geq \mathscr{G}^{\mathcal{S}}_{M,\bar{g}}[\bar{g}],$$

we have $\hat{g}_S \in \mathcal{Q}_{\bar{g}}$. That is, $\hat{g}_S = c^2\bar{g}$ for some constant $c > 0$. □

Now we can prove the volume comparison of Einstein manifolds with respect to scalar curvature.

Proof of Theorem B. According to Theorem 5.6, we can find a local slice $\mathcal{S}_{\bar{g}}$ through the reference metric $\bar{g}$. Moreover, there exists a constant $\varepsilon_0 > 0$ such that for any metric $\tilde{g}$ with

$$\|\tilde{g} - \bar{g}\|_{C^2(M,\bar{g})} < \varepsilon_0,$$

we can find a diffeomorphism $\psi \in \mathscr{D}(M)$ with the property that $\psi^*\tilde{g} \in U_{\bar{g}} \subseteq \mathcal{S}_{\bar{g}}$, where the subset $U_{\bar{g}}$ is given by Proposition 5.8.

For $\lambda \neq 0$, suppose g is a metric on M with scalar curvature

$$R_g \geq n(n-1)\lambda$$

and

$$\|g - \bar{g}\|_{C^2(M,\bar{g})} < \varepsilon_0.$$

In addition, we assume the reverse inequality of the claimed volume comparison:

$$\lambda(V_M(g) - V_M(\bar{g})) \geq 0. \tag{5-5}$$

This implies there is a diffeomorphism $\varphi \in \mathscr{D}(M)$ such that $\varphi^*g \in U_{\bar{g}} \subseteq \mathcal{S}_{\bar{g}}$ and

$$\mathscr{G}^{\mathcal{S}}_{M,\bar{g}}[\varphi^*g] = V_M(\varphi^*g)^{2/n}\int_M (R_g \circ \varphi)\, dv_{\bar{g}} \geq V_M(\bar{g})^{2/n}\int_M R_{\bar{g}}\, dv_{\bar{g}} = \mathscr{G}^{\mathcal{S}}_{M,\bar{g}}[\bar{g}],$$

due to our assumptions and the fact that $R_{\bar{g}} = n(n-1)\lambda$ is a constant. According to Proposition 5.8, there exists a constant $c > 0$ such that $\varphi^*g = c^2\bar{g}$.

From our assumptions,

$$R_{\varphi^*g} = c^{-2}R_{\bar{g}} \geq R_{\bar{g}} = n(n-1)\lambda,$$

and hence

$$\lambda(1-c) \geq 0.$$

However, inequality (5-5) suggests that

$$0 \leq \lambda(V_M(\varphi^* g) - V_M(\bar{g})) = \lambda(c^n - 1)V_M(\bar{g}),$$

which implies that $\lambda(1-c) \leq 0$. Therefore, we conclude $c = 1$ and hence $\varphi^* g = \bar{g}$. That is, (M^n, g) is isometric to $(M^n, \bar{g})$, and this concludes the theorem. □

With analogous techniques, we can prove the local rigidity of Ricci-flat manifolds.

Proof of Theorem C. Similar to the proof of Theorem B, we can find a constant $\varepsilon_0 > 0$ such that for any metric $\tilde{g}$ satisfying

$$\|\tilde{g} - \bar{g}\|_{C^2(M,\bar{g})} < \varepsilon_0,$$

there exists a diffeomorphism $\varphi \in \mathscr{D}(M)$ such that $\varphi^* g \in U_{\bar{g}} \subseteq \mathcal{S}_{\bar{g}}$, where $U_{\bar{g}}$ is given in Proposition 5.8.

Suppose g is a Riemannian metric with scalar curvature

$$R_g \geq 0$$

and

$$\|g - \bar{g}\|_{C^2(M,\bar{g})} < \varepsilon_0.$$

Then there is a diffeomorphism $\varphi \in \mathscr{D}(M)$ such that

$$\mathscr{G}^{\mathcal{S}}_{M,\bar{g}}[\varphi^* g] = V_M(\varphi^* g)^{2/n} \int_M (R_g \circ \varphi)\, dv_{\bar{g}} \geq 0.$$

However,

$$\mathscr{G}^{\mathcal{S}}_{M,\bar{g}}[\bar{g}] = V_M(\bar{g})^{2/n} \int_M R_{\bar{g}}\, dv_{\bar{g}} = 0,$$

and hence there is a constant $c > 0$ such that $\varphi^* g = c^2 \bar{g}$ due to $\bar{g}$ being strictly stable Ricci-flat and Proposition 5.8. The conclusion follows. □

According to Proposition 5.3, the second variation of $\mathscr{G}_{M,\bar{g}}$ at an unstable Einstein metric $\bar{g}$ is indefinite and hence $\bar{g}$ is a saddle point instead of a local maximum. This suggests that the volume comparison may fail for unstable Einstein manifolds and counterexamples can be constructed. It is well known that a product of positive Einstein manifolds with identical Einstein constants is still Einstein but unstable; see [Kröncke 2013]. Due to this reason and its simple structure, it can be our first choice.

The following example is constructed by Macbeth (personal communication, 2019), which shows the stability assumption is necessary for our volume comparison theorem.

Proposition 5.9. *There is a family of metrics* $\{g_t\}_{t\in[0,1)}$ *on* $\mathbb{S}^2 \times \mathbb{S}^2$ *such that*

- g_0 *is the canonical product metric on* $\mathbb{S}^2 \times \mathbb{S}^2$,
- $R_{g_t} = R_{g_{\mathbb{S}^2\times\mathbb{S}^2}} = 4$ *for all* $t \in [0, 1)$,
- $V_M(g_t) > V_M(g_{\mathbb{S}^2\times\mathbb{S}^2})$ *for all* $t \in (0, 1)$.

Proof. Let
$$g_t = (1+t)^{-1} g^1_{\mathbb{S}^2} + (1-t)^{-1} g^2_{\mathbb{S}^2}$$
with $t \in [0, 1)$, where $g^i_{\mathbb{S}^2}$ is the canonical metric on the i-th $\mathbb{S}^2$ factor, $i = 1, 2$. It is easy to see that their scalar curvature is given by
$$R_{g_t} = 2(1+t) + 2(1-t) = 4$$
for all $t \in [0, 1)$. However, its volume is
$$V_{\mathbb{S}^2\times\mathbb{S}^2}(g_t) = (1-t^2)^{-1} V_{\mathbb{S}^2\times\mathbb{S}^2}(\bar{g}) > V_{\mathbb{S}^2\times\mathbb{S}^2}(\bar{g}). \qquad \square$$

It is straightforward to generalize this example to more general product cases. It would be interesting to see whether we can find an explicit example of an unstable Einstein manifold which is not of this type but where the volume comparison fails.

Appendix: Equivalence of Schoen's conjectures

In this appendix, we show that two well-known conjectures proposed by Schoen [1989] on hyperbolic manifolds actually are equivalent to each other. We believe the proof is known to experts. Unfortunately, we could not find an appropriate reference. Thus we present a proof here for interested readers.

We start with a well-known concept in conformal geometry; see [Viaclovsky 2016].

Definition A.1. For $n \geq 3$, let (M^n, g) be a connected closed n-dimensional Riemannian manifold. The *Yamabe constant* of the conformal class $[g]$ is defined to be
$$Y(M^n, [g]) := \inf_{g \in [g]} \frac{\int_M R_g \, dv_g}{(V_M(g))^{(n-2)/n}}.$$
Moreover, we can define a min-max invariant
$$Y(M^n) := \sup_{[g]} Y(M^n, [g])$$
called the *Yamabe invariant* or *σ-invariant*.

It is well known that
$$Y(M^n) \leq Y(\mathbb{S}^n)$$
for any closed smooth manifold M^n and the canonical spherical metric achieves the Yamabe invariant of $\mathbb{S}^n$. For a given closed hyperbolic manifold with dimension at least three, its hyperbolic metric is unique up to a dilation due to the well-known *Mostow rigidity theorem*; see Theorem C.0 in [Benedetti and Petronio 1992]. Similar to the spherical case, Schoen [1989] conjectures that its Yamabe invariant is achieved by the canonical hyperbolic metric.

Conjecture A (Schoen's hyperbolic Yamabe invariant conjecture). *For $n \geq 3$, suppose $(M^n, \bar{g})$ is an n-dimensional closed hyperbolic manifold. Then*
$$Y(M^n) = Y(M^n, [\bar{g}]),$$
i.e., the Yamabe invariant is achieved by its canonical hyperbolic metric.

Another conjecture about closed hyperbolic manifolds concerns volume comparison, which is also referred to as *Schoen's conjecture*.

Conjecture B (Schoen's hyperbolic volume comparison conjecture). *For $n \geq 3$, suppose $(M^n, \bar{g})$ is an n-dimensional closed hyperbolic manifold. Then for any metric g on M with scalar curvature*

$$R_g \geq R_{\bar{g}},$$

its volume satisfies

$$V_M(g) \geq V_M(\bar{g}).$$

Obviously, Conjecture A involves all metrics on the given hyperbolic manifold and in general it is difficult to solve. Conjecture B only involves the comparison of a special metric with the reference metric, which seems easier to solve than Conjecture A. However, Conjectures A and B are in fact equivalent to each other and hence they are equally difficult in this sense. The bright side of this equivalence is that we only need to solve Conjecture B, then Conjecture A will hold automatically. This seems to be a promising approach to Conjecture A.

In the rest of the appendix, we will show the equivalence of Conjectures A and B.

We first show Conjecture A implies Conjecture B. In order to do this, we need the following lemma adapted from an observation of Kobayashi [1987].

Lemma A.2. *Let (M^n, g) be a closed manifold and $Y(M^n, [g])$ be the Yamabe constant of the conformal class $[g]$. Then*

$$-\left(\int_M |R_g^-|^{n/2}\, dv_g\right)^{2/n} \leq Y(M^n, [g]) \leq \left(\int_M |R_g^+|^{n/2}\, dv_g\right)^{2/n},$$

where $R_g^+ := \max\{R_g, 0\}$ and $R_g^- := \max\{-R_g, 0\}$.

Proof. By the conformal transformation law of scalar curvature,

$$Y(M^n, [g]) = \inf_{u>0} \frac{\int_M (a|\nabla_g u|_g^2 + R_g u^2)\, dv_g}{\left(\int_M u^{2n/(n-2)}\, dv_g\right)^{(n-2)/n}},$$

where $a := 4(n-1)/(n-2)$. Then we have

$$Y(M^n, [g]) \geq \inf_{u>0} \frac{\int_M R_g u^2\, dv_g}{\left(\int_M u^{2n/(n-2)}\, dv_g\right)^{(n-2)/n}} \geq -\inf_{u>0} \frac{\int_M R_g^- u^2\, dv_g}{\left(\int_M u^{2n/(n-2)}\, dv_g\right)^{(n-2)/n}},$$

since $R_g = R_g^+ - R_g^-$. By *Hölder's inequality*,

$$\int_M R_g^- u^2\, dv_g \leq \left(\int_M |R_g^-|^{n/2}\, dv_g\right)^{2/n} \left(\int_M u^{2n/(n-2)}\, dv_g\right)^{(n-2)/n},$$

and hence

$$Y(M^n, [g]) \geq -\left(\int_M |R_g^-|^{n/2}\, dv_g\right)^{2/n}.$$

Similarly,

$$Y(M^n, [g]) \leq \frac{\int_M R_g dv_g}{(V_M(g))^{(n-2)/n}} \leq \frac{\int_M R_g^+ dv_g}{(V_M(g))^{(n-2)/n}}.$$

By *Hölder's inequality*,

$$\int_M R_g^+ dv_g \le \left(\int_M |R_g^+|^{n/2}\, dv_g\right)^{2/n} (V_M(g))^{(n-2)/n},$$

and hence

$$Y(M^n,[g]) \le \left(\int_M |R_g^+|^{n/2}\, dv_g\right)^{2/n}. \qquad \square$$

Immediately, this implies the following conformal volume comparison.

Proposition A.3. *Suppose $(M^n, \hat{g})$ is a closed Riemannian manifold with strictly negative constant scalar curvature $R_{\hat{g}}$. Then for any metric $g \in [\hat{g}]$ with scalar curvature*

$$R_g \ge R_{\hat{g}},$$

we have

$$V_M(g) \ge V_M(\hat{g}).$$

Proof. Since $R_{\hat{g}}$ is a strictly negative constant, then its Yamabe constant satisfies

$$Y(M^n,[\hat{g}]) < 0,$$

and hence $\hat{g}$ is a Yamabe metric in the conformal class $[\hat{g}]$ due to the uniqueness of the Yamabe metric of negative Yamabe constant. Thus,

$$Y(M^n,[\hat{g}]) = R_{\hat{g}}(V_M(\hat{g}))^{2/n}.$$

By Lemma A.2,

$$\left(\min_M R_g\right)(V_M(g))^{2/n} - \left(\int_M |R_g^-|^{n/2} dv_g\right)^{n/2} \le Y(M^n,[\hat{g}]) = R_{\hat{g}}(V_M(\hat{g}))^{2/n}.$$

Therefore,

$$R_{\hat{g}}(V_M(g))^{2/n} \le \left(\min_M R_g\right)(V_M(g))^{2/n} \le R_{\hat{g}}(V_M(\hat{g}))^{2/n},$$

and hence

$$V_M(g) \ge V_M(\hat{g}). \qquad \square$$

Proposition A.4. *Conjecture A* $\Longrightarrow$ *Conjecture B.*

Proof. Let $(M^n, \bar{g})$ be a closed hyperbolic manifold. Suppose g is a metric on M with scalar curvature

$$R_g \ge R_{\bar{g}}.$$

We are going to show

$$V_M(g) \ge V_M(\bar{g}),$$

assuming $\bar{g}$ achieves its Yamabe invariant $Y(M^n)$.

From Conjecture A, the Yamabe constant of the conformal class $[g]$ satisfies

$$Y(M^n,[g]) \le Y(M^n) = Y(M^n,[\bar{g}]) < 0.$$

Let $\hat{g} \in [g]$ be the unique Yamabe metric in $[g]$ which is normalized such that $R_{\hat{g}} = R_{\bar{g}}$. By Proposition A.3, we have

$$V_M(g) \geq V_M(\hat{g}).$$

On the other hand,

$$R_{\hat{g}} V_M(\hat{g})^{2/n} = Y(M^n, [g]) \leq Y(M^n) = Y(M^n, [\bar{g}]) = R_{\bar{g}} V_M(\bar{g})^{2/n},$$

which implies

$$V_M(\hat{g}) \geq V_M(\bar{g}).$$

Therefore

$$V_M(g) \geq V_M(\hat{g}) \geq V_M(\bar{g}),$$

and hence Conjecture B holds. □

Proposition A.5. *Conjecture B* $\Longrightarrow$ *Conjecture A.*

Proof. Let $(M^n, \bar{g})$ be a closed hyperbolic manifold. We will show that its Yamabe invariant satisfies

$$Y(M^n) = Y(M^n, [\bar{g}]),$$

assuming the volume comparison holds.

We first recall a classic result of Gromov and Lawson [1983, Corollary A] which states that there is no metric with nonnegative scalar curvature on a compact hyperbolic manifold. That means the Yamabe invariant satisfies

$$Y(M^n) \leq 0,$$

and there is no metric on M with identically vanishing scalar curvature. Thus for any metric g on M, the Yamabe constant of the conformal class $[g]$ is strictly negative:

$$Y(M^n, [g]) < 0.$$

Let $\hat{g}$ be the Yamabe metric in the conformal class $[g]$ with $R_{\hat{g}} = R_{\bar{g}} < 0$. According to Conjecture B,

$$V_M(\hat{g}) \geq V_M(\bar{g}).$$

Therefore, the Yamabe constant of $[g]$ satisfies

$$Y(M^n, [g]) = \frac{\int_M R_{\hat{g}}\, dv_{\hat{g}}}{(V_M(\hat{g}))^{(n-2)/2}} = R_{\hat{g}}(V_M(\hat{g}))^{2/n} \leq R_{\bar{g}}(V_M(\bar{g}))^{2/n} = Y(M^n, [\bar{g}]).$$

Since g is arbitrary, we conclude

$$Y(M^n) = \sup_{[g]} Y(M^n, [g]) = Y(M^n, [\bar{g}]),$$

and hence Conjecture A holds. □

In summary, we have the equivalence of Schoen's Conjectures A and B.

Theorem A.6. *Conjecture A* $\Longleftrightarrow$ *Conjecture B.*

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References

[Agol et al. 2007] I. Agol, P. A. Storm, and W. P. Thurston, "Lower bounds on volumes of hyperbolic Haken 3-manifolds", *J. Amer. Math. Soc.* **20**:4 (2007), 1053–1077. MR Zbl

[Baltazar and Ribeiro 2017] H. Baltazar and E. Ribeiro, Jr., "Critical metrics of the volume functional on manifolds with boundary", *Proc. Amer. Math. Soc.* **145**:8 (2017), 3513–3523. MR Zbl

[Barros et al. 2015] A. Barros, R. Diógenes, and E. Ribeiro, Jr., "Bach-flat critical metrics of the volume functional on 4-dimensional manifolds with boundary", *J. Geom. Anal.* **25**:4 (2015), 2698–2715. MR Zbl

[Benedetti and Petronio 1992] R. Benedetti and C. Petronio, *Lectures on hyperbolic geometry*, Springer, 1992. MR Zbl

[Besse 1987] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik (3) **10**, Springer, 1987. MR Zbl

[Besson et al. 1991] G. Besson, G. Courtois, and S. Gallot, "Volume et entropie minimale des espaces localement symétriques", *Invent. Math.* **103**:2 (1991), 417–445. MR Zbl

[Besson et al. 1995] G. Besson, G. Courtois, and S. Gallot, "Entropies et rigidités des espaces localement symétriques de courbure strictement négative", *Geom. Funct. Anal.* **5**:5 (1995), 731–799. MR Zbl

[Bray 1997] H. L. Bray, *The Penrose inequality in general relativity and volume comparison theorems involving scalar curvature*, Ph.D. thesis, Stanford University, 1997, available at https://www.proquest.com/docview/304386501.

[Brendle 2012] S. Brendle, "Rigidity phenomena involving scalar curvature", pp. 179–202 in *Surveys in differential geometry, XVII* (Bethlehem, PA, 2010), edited by H.-D. Cao and S.-T. Yau, International, Boston, 2012. MR Zbl

[Brendle and Marques 2011] S. Brendle and F. C. Marques, "Scalar curvature rigidity of geodesic balls in $\mathbb{S}^n$", *J. Differential Geom.* **88**:3 (2011), 379–394. MR Zbl

[Chow et al. 2006] B. Chow, P. Lu, and L. Ni, *Hamilton's Ricci flow*, Grad. Stud. Math. **77**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[Corvino et al. 2013] J. Corvino, M. Eichmair, and P. Miao, "Deformation of scalar curvature and volume", *Math. Ann.* **357**:2 (2013), 551–584. MR Zbl

[Dai et al. 2005] X. Dai, X. Wang, and G. Wei, "On the stability of Riemannian manifold with parallel spinors", *Invent. Math.* **161**:1 (2005), 151–176. MR Zbl

[Dai et al. 2007] X. Dai, X. Wang, and G. Wei, "On the variational stability of Kähler–Einstein metrics", *Comm. Anal. Geom.* **15**:4 (2007), 669–693. MR Zbl

[Ebin 1970] D. G. Ebin, "The manifold of Riemannian metrics", pp. 11–40 in *Global analysis* (Berkeley, CA, 1968), edited by S.-s. Chern and S. Smale, Proc. Symp. Pure Math. **15**, Amer. Math. Soc., Providence, RI, 1970. MR Zbl

[Fischer and Marsden 1975] A. E. Fischer and J. E. Marsden, "Deformations of the scalar curvature", *Duke Math. J.* **42**:3 (1975), 519–547. MR Zbl

[Gromov and Lawson 1980] M. Gromov and H. B. Lawson, Jr., "Spin and scalar curvature in the presence of a fundamental group, I", *Ann. of Math.* (2) **111**:2 (1980), 209–230. MR Zbl

[Gromov and Lawson 1983] M. Gromov and H. B. Lawson, Jr., "Positive scalar curvature and the Dirac operator on complete Riemannian manifolds", *Inst. Hautes Études Sci. Publ. Math.* **58** (1983), 83–196. MR Zbl

[Gursky and Viaclovsky 2004] M. J. Gursky and J. A. Viaclovsky, "Volume comparison and the σ_k-Yamabe problem", *Adv. Math.* **187**:2 (2004), 447–487. MR Zbl

[Hamilton 1999] R. S. Hamilton, "Non-singular solutions of the Ricci flow on three-manifolds", *Comm. Anal. Geom.* **7**:4 (1999), 695–729. MR Zbl

[Kobayashi 1987] O. Kobayashi, "Scalar curvature of a metric with unit volume", *Math. Ann.* **279**:2 (1987), 253–265. MR Zbl

[Koiso 1980] N. Koiso, "Rigidity and stability of Einstein metrics: the case of compact symmetric spaces", *Osaka Math. J.* **17**:1 (1980), 51–73. MR Zbl

[Kröncke 2013] K. Kröncke, *Stability of Einstein manifolds*, Ph.D. thesis, University of Potsdam, 2013, available at http://opus.kobv.de/ubp/volltexte/2014/6963.

[Li 2012] P. Li, *Geometric analysis*, Cambridge Stud. Adv. Math. **134**, Cambridge Univ. Press, 2012. MR Zbl

[Lin and Yuan 2022] Y.-J. Lin and W. Yuan, "Deformations of Q-curvature, II", *Calc. Var. Partial Differential Equations* **61**:2 (2022), art. id 74, 28 pp. MR

[Miao and Tam 2009] P. Miao and L.-F. Tam, "On the volume functional of compact manifolds with boundary with constant scalar curvature", *Calc. Var. Partial Differential Equations* **36**:2 (2009), 141–171. MR Zbl

[Miao and Tam 2012] P. Miao and L.-F. Tam, "Scalar curvature rigidity with a volume constraint", *Comm. Anal. Geom.* **20**:1 (2012), 1–30. MR Zbl

[Perelman 2002] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications", preprint, 2002. Zbl arXiv math/0211159

[Perelman 2003] G. Perelman, "Ricci flow with surgery on three-manifolds", preprint, 2003. Zbl arXiv math/0303109

[Qing and Yuan 2013] J. Qing and W. Yuan, "A note on static spaces and related problems", *J. Geom. Phys.* **74** (2013), 18–27. MR Zbl

[Qing and Yuan 2016] J. Qing and W. Yuan, "On scalar curvature rigidity of vacuum static spaces", *Math. Ann.* **365**:3-4 (2016), 1257–1277. MR Zbl

[Rosenberg 2007] J. Rosenberg, "Manifolds of positive scalar curvature: a progress report", pp. 259–294 in *Surveys in differential geometry*, *XI*: *Metric and comparison theory*, edited by J. Cheeger and K. Grove, International, 2007. MR Zbl

[Schoen 1989] R. M. Schoen, "Variational theory for the total scalar curvature functional for Riemannian metrics and related topics", pp. 120–154 in *Topics in calculus of variations* (Montecatini Terme, Italy, 1987), edited by M. Giaquinta, Lecture Notes in Math. **1365**, Springer, 1989. MR Zbl

[Schoen and Yau 1979a] R. Schoen and S. T. Yau, "Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with non-negative scalar curvature", *Ann. of Math.* (2) **110**:1 (1979), 127–142. MR Zbl

[Schoen and Yau 1979b] R. Schoen and S. T. Yau, "On the structure of manifolds with positive scalar curvature", *Manuscripta Math.* **28**:1-3 (1979), 159–183. MR Zbl

[Viaclovsky 2016] J. A. Viaclovsky, "Critical metrics for Riemannian curvature functionals", pp. 197–274 in *Geometric analysis* (Park City, UT, 2013), edited by H. L. Bray et al., IAS/Park City Math. Ser. **22**, Amer. Math. Soc., Providence, RI, 2016. MR Zbl

[Yuan 2015] W. Yuan, *The geometry of vacuum static spaces and deformations of scalar curvature*, Ph.D. thesis, University of California, Santa Cruz, 2015, available at https://www.proquest.com/docview/1710736908.

[Zhang 2019] Y. Zhang, "Scalar curvature volume comparison theorems for almost rigid sphere", preprint, 2019. arXiv 1909.00909

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