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GAUSSIAN ANALYTIC FUNCTIONS OF BOUNDED MEAN OSCILLATION

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We consider random analytic functions given by a Taylor series with independent, centered complex Gaussian coefficients. We give a new sufficient condition for such a function to have bounded mean oscillation. Under a mild regularity assumption this condition is optimal. We give as a corollary a new bound for the norm of a random Gaussian Hankel matrix. Finally, we construct some *exceptional* Gaussian analytic functions which in particular disprove the conjecture that a random analytic function with bounded mean oscillation always has *vanishing* mean oscillation.

1. Introduction

Functions with random Fourier (or Taylor) coefficients play an important role in harmonic and complex analysis, e.g., in the proof of de Leeuw, Kahane, and Katznelson [de Leeuw et al. 1977] that Fourier coefficients of continuous functions can majorize any sequence in ℓ^2. A well-known phenomenon is that series with independent random coefficients are much "nicer" than an arbitrary function would be. For example, a theorem of [Paley and Zygmund 1930, Chapter 5, Proposition 10] (see also [Kahane 1985]) states that a Fourier series with square summable coefficients and random signs almost surely represents a *subgaussian* function on the circle.

In this paper we choose to focus on one particularly nice model of random analytic functions, the *Gaussian analytic functions* (GAFs). A GAF is given by a random Taylor series

$$G(z) = \sum_{n=0}^{\infty} a_n \xi_n z^n, \tag{1}$$

where $\{\xi_n\}_{n\geq 0}$ is a sequence of independent standard complex Gaussian random variables (i.e., with density $\frac{1}{\pi}e^{-|z|^2}$ with respect to the Lebesgue measure on the complex plane $\mathbb{C}$) and where $\{a_n\}_{n\geq 0}$ is a sequence of nonnegative constants. Many of the results we cite can be extended to more general probability distributions, and it is likely that our results can be similarly generalized, but we will not pursue this here. For recent accounts of random Taylor series, many of which focus on the distributions of their zeros, see for example [Hough et al. 2009; Nazarov and Sodin 2010]. A classical book on this and related subjects is [Kahane 1985].

We are interested in properties of the sequence $\{a_n\}$ that imply various regularity and finiteness properties of the function G represented by the series (1). One of the central spaces of analytic functions

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is H^p, those functions F on the unit disk $\mathbb{D}$ that satisfy

$$\sup_{0<r<1} \int_0^1 |F(\mathrm{Re}(\theta))|^p \, \mathrm{d}\theta < \infty,$$

where $e(\theta) = e^{2\pi i\theta}$ for $\theta \in \mathbb{R}$ (see [Duren 1970] for background). This is a class of analytic functions whose *nontangential* boundary values on $\mathbb{T} = \{z : |z| = 1\}$ exist Lebesgue a.e. and are in $L^p(\mathbb{T})$ [Duren 1970, Theorem 2.2]. An important early effort is the aforementioned paper [Paley and Zygmund 1930], in which it was established that G is almost surely in $\bigcap_{0<p<\infty} H^p$ if and only if $\{a_n\} \in \ell^2$. One should compare this result with the well-known fact that a nonrandom analytic function belongs to H^2 if and only if the sequence of its Taylor coefficients is square summable. The related question of when G is almost surely in H^∞, the *bounded* analytic functions on the unit disk, is substantially more involved (see [Marcus and Pisier 1978]).

To fix ideas, let us make for a moment a few simplifying assumptions about the coefficients $\{a_n\}$ of the series (1). We assume $a_0 = 0$, and denote by

$$\sigma_k^2 = \sum_{n=2^k}^{2^{k+1}-1} a_n^2, \quad k \in \{0, 1, 2, \dots\},$$

the total variance of the dyadic blocks of coefficients. We say that the sequence $\{a_n\}$ (or equivalently G) is *dyadic-regular* if the sequence $\{\sigma_k\}$ is *decreasing* as $k \to \infty$. It is known (see [Kahane 1985, Chapters 7 and 8]) that if G is dyadic-regular, then G is almost surely in H^∞ if and only if

$$\sum_{k=0}^{\infty} \sigma_k < \infty, \quad \text{i.e., } \{\sigma_k\} \in \ell^1. \tag{2}$$

Moreover, if the series in (2) converges, then G is almost surely *continuous* on the closed disk $\overline{\mathbb{D}}$. Hence, a bounded random series *gains* additional regularity.

For a space S of analytic functions on the unit disk, let S_G be the set of coefficients $\{a_n\}$ for which a GAF $G \in S$ almost surely. If $S \subsetneq T$ and $S_G = T_G$, then we say that GAFs have a *regularity boost* from T to S, e.g., $C_G = H_G^\infty$. This regularity boost can be viewed as a manifestation of a general probabilistic principle: a Borel probability measure on a complete metric space tends to be concentrated on a *separable* subset of that space.[1]

Clearly there is a gap between (2) and the Paley–Zygmund condition $\{\sigma_k\} \in \ell^2$. A well-known function space that lies *strictly* between H^∞ and $\bigcap_{0<p<\infty} H^p$ is the space of analytic functions of *bounded mean oscillation* or BMOA (e.g., see [Girela 2001, Equation (5.4)]). For an interval $I \subseteq \mathbb{R}/\mathbb{Z}$ and any $f \in L^1(\mathbb{T})$, put

$$M_I(f) := \fint_I \left| f(e(\theta)) - \fint_I f \right| \mathrm{d}\theta, \quad \text{where } \fint_I f := \frac{1}{|I|} \int_I f(e(\theta)) \, \mathrm{d}\theta. \tag{3}$$

[1] Under the continuum hypothesis, by the main theorem of [Marczewski and Sikorski 1948], any Borel probability measure on a metric space with the cardinality of the continuum is supported on a separable subset.

Define the seminorm on H^1

$$\|F\|_* = \sup_{I \subseteq \mathbb{R}/\mathbb{Z}} M_I(F). \tag{4}$$

The restriction of $F \in H^1$ is necessary for F to have nontangential boundary values in L^1 on the unit disk. On the subspace of H^1 in which $F(0) = 0$, this becomes a norm. We may take BMOA to be the (closed) subspace of H^1 for which $\|\cdot\|_*$ is finite.

Fefferman and Stein [1972] show the space BMOA is the dual space of H^1 with respect to the bilinear form on analytic functions of the unit disk given by

$$(F, G) = \lim_{r \to 1} \int_0^1 F(\mathrm{Re}(\theta))\overline{G(\mathrm{Re}(\theta))}\, \mathrm{d}\theta,$$

and in many aspects it serves as a convenient "replacement" for the space H^∞. However, BMOA is not separable (see [Girela 2001, Corollary 5.4]).

One of our main results is the following.

Theorem 1.1. *A dyadic-regular Gaussian analytic function G that satisfies the Paley–Zygmund condition $\{\sigma_k\} \in \ell^2$ almost surely belongs to* VMOA, *the space of analytic functions of* ***vanishing*** *mean oscillation.*

The space VMOA is the closure of polynomials (or continuous functions) in the norm $\|\cdot\|_*$, and hence it is separable. It can alternatively be characterized as the subspace of H^1 for which $\lim_{|I| \to 0} M_I^1(F) = 0$. In fact, we show that a dyadic-regular GAF with square-summable coefficients almost surely belongs to a subspace of VMOA, which we attribute to Sledd [1981].

1A. *The Sledd Space* SL. Sledd [1981] introduced a function space, which is contained in BMOA and is much more amenable to analysis. Define the seminorm for $F \subset H^1$

$$\|F\|_{S(T)}^2 = \sup_{|x|=1} \sum_{n=0}^{\infty} |T_n \star F(x)|^2, \tag{5}$$

where $\star$ denotes convolution on $\mathbb{T}$ and $\{T_n\}$ is a certain sequence of compactly supported bump functions in Fourier space, so that $\hat{T}_n = 1$ for modes from $[2^n, 2^{n+1}]$ (see (15) for the explicit definition of $\{T_n\}$). We let SL denote the subspace of H^1 with finite $\|\cdot\|_{S(T)}$ norm; [Sledd 1981] showed that SL $\subsetneq$ BMOA.[2] Sledd proved the following result.

Theorem I [Sledd 1981, Theorem 3.2]. *If $\{\sqrt{k}\sigma_k\} \in \ell^2$, then $G \in$* VMOA *almost surely.*

Remark 1.2. Sledd proved the result for series with random signs, but his method works also in our setting. In fact his theorem shows that G is almost surely in VMOA $\cap$ SL.

We extend the analysis of the $\|\cdot\|_{S(T)}$ seminorm, and in particular find a better sufficient condition for the finiteness of $\|G\|_{S(T)}$.

[2] The function $I_F = \sum_{n=0}^{\infty} |T_n \star F(x)|^2$ is essentially what appears in Littlewood–Paley theory. For each $\frac{2}{3} < p < \infty$, finiteness of the p-norm of I_F is equivalent to being in H^p; see [Stein 1966, Theorem 5]. Thus, in some sense SL could be viewed as a natural point in the hierarchy of H^p spaces.

Theorem 1.3. *If* $\sum_{k=1}^{\infty} \sup_{n \geq k}\{\sigma_n^2\} < \infty$, *then* $G \in \mathrm{SL}$ *almost surely.*

In particular, if G is dyadic-regular and $\{\sigma_k\} \in \ell^2$, then $G \in \mathrm{SL}$. The latter condition is necessary for G to have well-defined boundary values, and so we see that under the monotonicity assumption, a GAF G which has boundary values in L^2 is in BMOA . We also note that the condition in Theorem 1.3 is strictly weaker than the one in Theorem I (see Lemma 4.9).

The Sledd space SL is nonseparable (see Proposition 3.3). The proof of Theorem I is based on a stronger condition than $\|G\|_{S(T)} < \infty$, that in addition implies that a function is in the space $\mathrm{SL} \cap \mathrm{VMOA}$.[3] We show that this is unnecessary, as a GAF which is in SL has a regularity boost.

Theorem 1.4. *If* $G \in \mathrm{SL}$ *almost surely*, *then* $G \in \mathrm{VMOA}$ *almost surely.*

Theorems 1.4 and 1.3 imply Theorem 1.1.

This could raise suspicion that there is also a regularity boost from BMOA to VMOA, which is perhaps the most natural separable subspace of BMOA. Indeed, [Sledd 1981] asks whether it is possible to construct a non-VMOA random analytic function in BMOA.

1B. ***Exceptional Gaussian analytic functions.*** Sledd [1981, Theorem 3.5] gives a construction of a random analytic function with square summable coefficients which is not in BMOA, and moreover is not *Bloch* (this construction can be easily adapted to GAFs). The Bloch space, $\mathcal{B}$, contains all analytic functions F on the unit disk for which

$$\|F\|_{\mathcal{B}} := \sup_{|z| \leq 1} ((1 - |z|^2)|F'(z)|) < \infty. \tag{6}$$

See [Anderson et al. 1974; Girela 2001] for more background on this space. Gao [2000] provides a complete characterization of which sequences of coefficients $\{a_n\}$ give GAFs in $\mathcal{B}$.

The space $\mathcal{B}$ is nonseparable, suggesting that GAFs in $\mathcal{B}$ could concentrate on a much smaller space. Finding this space is a natural open question and does not seem obvious from the characterization in [Gao 2000]. It is known that $\mathrm{BMOA} \subset \mathcal{B}$ (see, e.g., [Girela 2001, Corollary 5.2]), and, a priori, it could be that GAFs which are in $H^2 \cap \mathcal{B}$ are automatically in BMOA. However, our following result disproves this, and also answers the aforementioned question of Sledd.

Theorem 1.5. *We have*

$$\mathrm{SL}_G \subsetneq \mathrm{VMOA}_G \subsetneq \mathrm{BMOA}_G \subsetneq (H^2 \cap \mathcal{B})_G. \tag{7}$$

Remark 1.6. From Theorem 1.3 and standard results on boundedness of Gaussian processes, we may add that $H_G^\infty \subsetneq \mathrm{SL}_G$. From the example in [Sledd 1981], it also follows that $(H^2 \cap \mathcal{B})_G \subsetneq H_G^2$.

We leave open the question of the existence of a natural separable subspace S of BMOA such that $\mathrm{BMOA}_G = S_G$.

[3] Specifically, [Sledd 1981] shows that under the condition in Theorem I, $\sum_{n=0}^{\infty} \sup_{|x|=1} |T_n \star F(x)|^2$ is finite, which implies $F \in \mathrm{SL} \cap \mathrm{VMOA}$.

1C. ***Some previously known results.*** Billard [1963] (see also [Kahane 1985, Chapter 5]) proved that a random analytic function with independent symmetric coefficients is almost surely in H^∞ if and only if it almost surely extends continuously to the closed unit disk.

A complete characterization of Gaussian analytic functions which are bounded on the unit disk was found by Marcus and Pisier [1978] in terms of rearrangements of the covariance function (see also [Kahane 1985, Chapter 15]). Moreover, they show the answer is the same for Steinhaus and Rademacher random series (where the common law of all $\{\xi_n\}$ is taken uniform on the unit circle and on $\{\pm 1\}$, respectively). Their criterion can be seen to be equivalent to the finiteness of Dudley's entropy integral for the process of boundary values of G on the unit circle.

The best existing sufficient conditions that we know for the sequence $\{a_n\}$ to belong to BMOA_G are due to [Sledd 1981]. The more recent paper of [Wulan 1994] treats a more general problem, which in the particular case of VMOA gives another proof of Theorem I.

1D. ***Norms of random Hankel matrices.*** A *Hankel* matrix A is any $n \times n$ matrix with the structure $A_{ij} = (c_{i+j-2})$ for some sequence $\{c_k\}_0^\infty$. The function $\phi(z) = \sum_{k=0}^\infty c_k z^{k+1}$ is referred to as the *symbol* of A. We will consider the case that $n \in \mathbb{N}$, and we will also consider the infinite case. We denote by B the Hankel operator with the same symbol on ℓ^2, which may well be unbounded. Then by a combination of results of Fefferman and Nehari (see [Peller 2003, Chapter 1] and [Holland and Walsh 1986, Part III]), there is an absolute constant M such that

$$\frac{1}{M}\|\phi\|_* \le \|B\| \le M\|\phi\|_*, \tag{8}$$

with $\|B\|$ the operator norm of B.

If we take $c_m = a_{m+1}\xi_{m+1}$ for all $m \ge 0$ with $\{\xi_m\}$ i.i.d. $N_{\mathbb{C}}(0,1)$ and with $a_m \ge 0$ for all m, then ϕ is exactly the GAF G. Moreover, by combining Theorem 3.1, Remark 3.7 and Lemma 4.8, we have that there is an absolute constant $C > 0$ such that

$$\mathbb{E}\|\phi\|_*^2 \le C \sum_{k=1}^\infty \sup_{m \ge k}\{\sigma_m^2\}.$$

Note that for any $n \times n$ Hankel matrix A with symbol $\phi(z) = \sum_{k=0}^\infty c_k z^{k+1}$, if B is the infinite Hankel operator with *finite* symbol $\phi_n(z) = \sum_{k=0}^{2n} c_k z^{k+1}$, then $\|A\| \le \|B\|$ as A is the $n \times n$ upper-left corner of B. Hence, using (8),

$$\|A\| \le \|B\| \le M\|\phi_n\|_*,$$

and we arrive at the following corollary.

Theorem 1.7. *There is an absolute constant $C > 0$ such that if A is an $n \times n$ Hankel matrix with symbol G* (*see* (1)) *and L is the smallest integer greater than or equal to* $\log_2(2n)$, *then*

$$\mathbb{E}\|A\|^2 \le C \sum_{k=0}^{L} \sup_{k \le m \le L} \sigma_m^2.$$

We emphasize that by virtue of (8) the problem of estimating the norm of a random Gaussian Hankel matrix is essentially equivalent to the problem of estimating the $\|\cdot\|_*$ norm of a random Gaussian polynomial.

This is particularly relevant as random Hankel and Toeplitz matrices[4] have appeared many times in the literature and have numerous applications to various statistical problems. See the discussion in [Bryc et al. 2006] for details. The particular case of Hankel matrices with symbol $G = \sum_{k=0}^{\infty} \mathrm{Re}(\xi_k) z^{k+1}$, i.e., Hankel matrices with i.i.d. Gaussian antidiagonals, is particularly well studied. In that case, [Meckes 2007] and [Nekrutkin 2013] give proofs that $\mathbb{E}\|A\| \le c\sqrt{n \log n}$. Finer results for the symmetric Toeplitz case are available in [Sen and Virág 2013].

Furthermore, Meckes [2007] gives a matching lower bound, and his method can be applied to show that (deterministically)

$$\|A\| \ge \sup_{|z|=1} \left| \sum_{j=0}^{2(n-1)} \left(1 - \frac{|n-1-j|}{n}\right) a_j \xi_j z^j \right|.$$

Fernique's theorem [Kahane 1985, Chapter 15, Theorem 5] can then be used to show that Theorem 1.7 is sharp up to multiplicative numerical constant, at least when $a_j = j^{-\alpha}$ for $\alpha \in \mathbb{R}$.

Some results for more general random symbols exist; in particular, [Adamczak 2010, Theorem 4] shows that in the setting of Theorem 1.7,

$$\mathbb{E}\|A\|^2 \le C(\log n) \sum_{m=0}^{L} \sigma_m^2, \tag{9}$$

which is always larger than the bound in Theorem 1.7; in the case that σ_m^2 is monotonically decreasing and summable, (9) differs substantially from the condition in Theorem 1.7. Note that in Theorem 1.7, the entries of the $n \times n$ Hankel matrix are independent standard *complex* Gaussian random variables, whereas [Adamczak 2010, Theorem 4] holds for non-Gaussian symbols as well.

Organization. In Section 2, we give some background theory for working with GAFs and random series. In Section 3, we give some further properties of the space SL and we give some equivalent characterizations for $G \in \mathrm{SL}$. We also prove Theorem 1.4. In Section 4, we give a sufficient condition for G to be in SL; in particular, we prove Theorem 1.3. Finally, in Section 5 we construct exceptional GAFs, and we show the inclusions in (7) are strict.

Notation. We use the expression *numerical constant* and *absolute constant* to refer to fixed real numbers without dependence on any parameters. We make use of the notation $\lesssim$ and $\gtrsim$ and $\asymp$. In particular, we say that $f(a,b,c,\dots) \lesssim g(a,b,c,\dots)$ if there is an absolute constant $C > 0$ such that $f(a,b,c,\dots) \le Cg(a,b,c,\dots)$ for all $a,b,c,\dots$. We use $f \asymp g$ to mean $f \lesssim g$ and $f \gtrsim g$.

[4] A Toeplitz matrix A has the form $A_{ij} = w_{i-j}$ for some $(w_k)_{-\infty}^{\infty}$. The symbol for such a matrix is again $\sum w_k z^k$. By reordering the rows, it can be seen that a Toeplitz matrix with symbol $\sum_{-n}^{n} w_k z^k$ has the same norm as the Hankel matrix with symbol $\sum_{0}^{2n} w_{k-n} z^k$.

2. Preliminaries

Some of our proofs will rely on the so-called contraction principle.

Proposition 2.1 (contraction principle). *For any finite sequence (x_i) in a topological vector space V, any continuous convex $F : V \to [0, \infty]$, any i.i.d., symmetrically distributed random variables (ϵ_i), and any (α_i) real numbers in $[-1, 1]$:*

(i) $\mathbb{E} F\big(\sum_i \alpha_i \epsilon_i x_i\big) \le \mathbb{E} F\big(\sum_i \epsilon_i x_i\big)$.

(ii) *If F is a seminorm, then $\mathbb{P}\big[F\big(\sum_i \alpha_i \epsilon_i x_i\big) \ge t\big] \le 2\mathbb{P}\big[F\big(\sum_i \epsilon_i x_i\big) \ge t\big]$ for all $t > 0$.*

This is essentially [Ledoux and Talagrand 1991, Theorem 4.4], although we have changed the formulation slightly. For convenience we sketch the proof.

Proof. The mapping

$$(\alpha_1, \alpha_2, \ldots, \alpha_N) \mapsto \mathbb{E} F\left(\sum_i \alpha_i \epsilon_i x_i\right)$$

is convex. Therefore it attains its maximum on $[-1, 1]^N$ at an extreme point, i.e., an element of $\{\pm 1\}^N$. By the symmetry of the distributions, for all such extreme points, the value of the expectation is $\mathbb{E} F\big(\sum_i \epsilon_i x_i\big)$, which completes the proof of the first part.

For the second part, we may without loss of generality assume that $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_N \ge \alpha_{N+1} = 0$ by relabeling the variables and using the symmetry of the distributions of $\{\epsilon_i\}$. Letting $S_n = \sum_{k=1}^n \epsilon_i x_i$ for any $1 \le n \le N$, we can use summation by parts to express

$$\sum_i \alpha_i \epsilon_i x_i = \sum_i \alpha_i (S_i - S_{i-1}) = \sum_i (\alpha_i - \alpha_{i+1}) S_i.$$

Hence, as F is a seminorm,

$$F\left(\sum_i \alpha_i \epsilon_i x_i\right) \le \alpha_1 \max_{1 \le i \le N} F(S_i) \le \max_{1 \le i \le N} F(S_i).$$

Using the reflection principle, it now follows that for any $t \ge 0$,

$$\mathbb{P}\Big[\max_{1 \le i \le N} F(S_i) \ge t\Big] \le 2\mathbb{P}[F(S_N) \ge t],$$

which completes the proof (see [Ledoux and Talagrand 1991, Theorem 4.4] for details). □

We also need the following standard Gaussian concentration inequality.

Proposition 2.2. *Suppose that $X = (X_j)_1^n$ are i.i.d. standard complex Gaussian variables, and suppose $F : \mathbb{C}^n \to \mathbb{R}$ is a 1-Lipschitz function with respect to the Euclidean metric. Then $\mathbb{E}|F(X)| < \infty$ and, for all $t \ge 0$,*

$$\mathbb{P}[F(X) - \mathbb{E} F(X) > t] \le e^{-t^2}.$$

Proof. This follows from the real case (see [Ledoux and Talagrand 1991, (1.5)]). The real and imaginary Gaussian random variables have variance $\frac{1}{2}$, for which reason the exponent is e^{-t^2}. □

Approximation of seminorms. Let $\|\cdot\|$ be a densely defined seminorm on H^2 which dominates the H^2 norm. We will say that $\|\cdot\|$ is *approximable* if there exists a sequence of polynomials $\{p_n\}$ with $\sup_{n,j} \|z^j \star p_n(z)\| \le 1$ such that for all $F \in H^2$,

$$\sup_n \|F \star p_n\| < \infty \iff \|F\| < \infty \qquad \text{and} \qquad \sup_n \|F \star p_n\| = 0 \iff \|F\| = 0. \tag{10}$$

Let V be the quotient space of $\{F \in H^2 : \|F\| < \infty\}$ by the space $\{F \in H^2 : \|F\| = 0\}$. Then both $\|\cdot\|$ and $\sup_n \|\cdot \star p_n\|$ make V into Banach spaces with equivalent topologies, by the hypotheses. Hence (10) is equivalent to

$$\text{there exists } C > 0 \text{ such that } \frac{1}{C} \sup_n \|F \star p_n\| \le \|F\| \le C \sup_n \|F \star p_n\| \text{ for all } F \in H^2, \tag{11}$$

as the inclusion map from one of these Banach spaces to the other is continuous and hence bounded.

Remark 2.3. While approximable seminorms could be formulated in greater generality, we work in the H^2 setting to appeal to general concentration of measure theory.

We say that G is an H^2-GAF if $\{a_k\} \in \ell^2$.

Proposition 2.4. *Let G be an H^2-GAF. Let $\|\cdot\|$ be any approximable seminorm. Then the following are equivalent*:

(i) $\|G\| < \infty$ *a.s.*

(ii) $\mathbb{E}\|G\| < \infty$.

(iii) $\mathbb{E}\|G\|^2 < \infty$.

Remark 2.5. We remark that these equivalences hold in great generality for a Gaussian measure in a *separable* Banach space, due to a theorem of Fernique [Ledoux 1996, Theorem 4.1]. As the spaces BMOA and $\mathcal{B}$ are not separable, we instead will appeal to this notion of approximable.

Remark 2.6. A priori it is not clear that a seminorm being finite is a measurable event with respect to the product σ-algebra generated by the Taylor coefficients of G. However, for an approximable seminorm, measurability is implied by the equivalence in (10), since $\sup_n \|G \star p_n\|$ is clearly measurable; cf. [Kahane 1985, Chapter 5, Proposition 1].

Proof. The implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are trivial, and so it only remains to show that (i) $\Longrightarrow$ (iii).

Let $\{p_m\}$ be the polynomials making $\|\cdot\|$ approximable. Define

$$G_m = G \star p_m, \quad \text{where } G(z) = \sum_{k=0}^{\infty} a_k \xi_k z^k.$$

Without loss of generality, we may assume that $\|a\|_{\ell_2}^2 = \sum_{k=0}^{\infty} a_k^2 = 1$. For any $m \in \mathbb{N}$, let $k_m = \deg(p_m)$, and define the function on $\mathbb{C}^{k_m}$

$$F_m(x) = F_m(x_0, x_1, \dots, x_{k_m}) = \left\| \left(\sum_{j=0}^{k_m} a_j x_j z^j \right) \star p_m(z) \right\|.$$

Then for any complex vectors $x = (x_j)_0^{k_m}$ and $y = (y_j)_0^{k_m}$, by changing coordinates one at a time and using $\sup_{n,j} \|z^j \star p_n(z)\| \le 1$,

$$|F_m(x) - F_m(y)| \le \sum_{j=0}^{k_m} a_j |x_j - y_j| \le \|a\|_{\ell_2} \|x - y\|_{\ell_2}.$$

For any $\ell \in \mathbb{N}$, the function $\max_{1\le m\le \ell} F_m(x)$ is again 1-Lipschitz. So define for any $\ell \in \mathbb{N}$

$$H_\ell := \max_{1\le m\le \ell} \|G_m\|.$$

Therefore, by Proposition 2.2, we have that, for all $t \ge 0$ and all $\ell \in \mathbb{N}$,

$$\mathbb{P}[|H_\ell - \mathbb{E} H_\ell| \ge t] = \mathbb{P}\Big[\big|F_m(\xi_0, \xi_1, \ldots, \xi_{k_m}) - \mathbb{E}(\max_{1\le m\le \ell} F_m(\xi_0, \xi_1, \ldots, \xi_{k_m}))\big| \ge t\Big] \le 2e^{-t^2}. \tag{12}$$

Hence there is an absolute constant $C > 0$ such that for all $\ell \in \mathbb{N}$,

$$|\mathrm{med}(H_\ell) - \mathbb{E}(H_\ell)| \le C, \tag{13}$$

where $\mathrm{med}(X)$ denotes any median of the random variable X.

Suppose that $\|G\| < \infty$ a.s. By (10), $\sup_m \|G_m\| = \sup_\ell H_\ell < \infty$ a.s. Therefore there is a constant $M > 0$ such that $\mathbb{P}(\sup_\ell H_\ell > M) < \frac{1}{2}$, and so $\mathrm{med}(H_\ell) \le M$ for all $\ell \in \mathbb{N}$. By monotone convergence and (13),

$$\mathbb{E} \sup_m \|G_m\| = \mathbb{E} \sup_\ell H_\ell = \sup_\ell \mathbb{E} H_\ell \le M + C.$$

Using (11), there is another absolute constant C such that

$$\mathbb{E}\|G\| \le C\,\mathbb{E} \sup_m \|G_m\| < \infty.$$

Using (11) and (12), $\mathrm{Var}(\sup_m \|G_m\|) < \infty$, and therefore

$$\mathbb{E}\|G\|^2 \le C\,\mathbb{E}(\sup_m \|G_m\|^2) \le \mathrm{Var}(\sup_m \|G_m\|) + (\mathbb{E} \sup_m \|G_m\|)^2 < \infty. \qquad \square$$

Both $\|\cdot\|_*$ and $\|\cdot\|_{\mathcal{B}}$ are approximable with $\{p_n\}$ given by the analytic part of the Fejér kernel

$$K_n^A(z) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) z^k.$$

See [Holland and Walsh 1986, Theorems 1 and 4]. In fact, it is elementary to observe the following.

Lemma 2.7. *For any* $f \in \mathrm{H}^1(\mathbb{T})$, $\sup_n \|K_n^A \star f\|_* = \|f\|_*$ *and* $\sup_n \|K_n^A \star f\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$.

Proof. We show the first of these claimed identities. For any fixed interval $I \subseteq \mathbb{R}/\mathbb{Z}$,

$$\lim_{n\to\infty} M_I(f \star K_n^A) = M_I(f),$$

and hence, $\sup_n \|K_n^A \star f\|_* \ge \|f\|_*$. On the other hand, for any fixed $\omega \in \mathbb{T}$, $f_\omega := z \mapsto f(\bar{\omega} z)$ has that $\|f_\omega\|_* = \|f\|_*$. Hence by comparing to the Fejér kernel (see (14)), which is positive, for any $n \ge 0$,

$$\|f \star K_n^A\|_* = \|f \star K_n\|_* = \left\| \int f_{e(\theta)} K_n(e(\theta))\, d\theta \right\|_* \le \sup_{\omega\in\mathbb{T}} \|f_\omega\|_* = \|f\|_*,$$

where the inequality follows as $\|\cdot\|_*$ is convex and the Fejér kernel $K_n(z)$ is the density of a probability measure on $\mathbb{T}$. $\square$

Corollary 2.8. *Let F be an H^2-GAF. Then $\|F\|_* < \infty$ a.s. if and only if $\mathbb{E}\|F\|_* < \infty$, and $\|F\|_{\mathcal{B}} < \infty$ a.s. if and only if $\mathbb{E}\|F\|_{\mathcal{B}} < \infty$.*

We also have that the probability that a GAF is in BMOA, VMOA, or $\mathcal{B}$ is either 0 or 1.

Proposition 2.9. *For any H^2-GAF G, the events $\{G \in \mathrm{BMOA}\}$, $\{G \in \mathrm{VMOA}\}$, $\{G \in \mathcal{B}\}$ all have probability* 0 *or* 1.

Proof. Take the decomposition $G = G_{\le n} + G_{>n}$, where $G_{\le n}$ is the n-th Taylor polynomial of G at 0. Then as $G_{\le n}$ is a polynomial, $\|G_{\le n}\|_* < \infty$ almost surely. Hence $\|G\|_* < \infty$ if and only if $\|G_{>n}\|_* < \infty$, up to null events. Therefore, $\|G\|_* < \infty$ differs from a tail event of $\{\xi_n : 1 \le n < \infty\}$ by a null event, and so the statement follows from the Kolmogorov 0-1 law. The same proof shows that $\mathbb{P}[G \in \mathcal{B}] \in \{0, 1\}$.

For VMOA, as $G_{\le n}$ is a polynomial,

$$\lim_{|I|\to 0} \sup_I M_I^1(G_{\le n}) = 0 \quad \text{a.s.},$$

and the same reasoning as above gives the 0-1 law. $\square$

3. The Sledd space

Let K_n for $n \in \mathbb{N}$ be the n-th Fejér kernel, which for $|z| = 1$ is given by

$$K_n(z) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) z^k = \frac{1}{n+1} \cdot \frac{|1 - z^{n+1}|^2}{|1-z|^2}. \tag{14}$$

This kernel has the two familiar properties: $\|K_n\|_1 = 1$ and $K_n(z) \le 4/(n+1) \cdot (1/|1-z|^2)$.

For a function $F : \mathbb{T} \to \mathbb{C}$ with a Laurent expansion on $\mathbb{T}$, let $\hat{F} : \mathbb{Z} \to \mathbb{C}$ be its Fourier coefficients, i.e., let $\hat{F}(k)$ be the k-th coefficient of its Laurent expansion.

We let T_n be the dyadic trapezoidal kernel

$$\begin{aligned} T_0(z) &= 1 + \tfrac{1}{2} z + \tfrac{1}{2} z^{-1} \\ T_n &= 2K_{2^{n+2}} - K_{2^{n+1}} + K_{2^{n-1}} - 2K_{2^n}, \quad n \ge 1. \end{aligned} \tag{15}$$

The kernel T_n satisfies that $\hat{T}_n$ is supported in $[2^{n-1}, 2^{n+2})$, has $|\hat{T}_n(K)| \le 1$ everywhere, has $\hat{T}_n(K) = 1$ for $K \in [2^n, 2^{n+1}]$, and satisfies

$$\sum_{n=0}^{\infty} \hat{T}_n(K) = 1$$

for all integers $K \ge 0$. Further, $\|T_n\|_1 \le 6$ for all $n \ge 0$. Also,

$$|T_n(z)| \le 20 \cdot 2^{-n} |1-z|^{-2}. \tag{16}$$

Recall that in terms of the kernels $\{T_n\}$, we defined the seminorm (in (5)) as

$$\|F\|_{S(T)}^2 = \sup_{|x|=1} \sum_{n=0}^{\infty} |T_n \star F(x)|^2. \tag{17}$$

In [Sledd 1981], it is shown that this norm is related to $\|\cdot\|_*$ in the following way.

Theorem 3.1. *If $F \in H^1$, then there is an absolute constant $C > 1$ such that*

$$\|F\|_* \leq C\|F\|_{S(T)}.$$

Sledd also gives a sufficient condition for F to be in VMOA, though we observe that there is a stronger one that follows directly from Theorem 3.1.

Theorem 3.2. *If $F \in H^1$ and if*

$$\lim_{k\to\infty} \sup_{|x|=1} \sum_{n=k}^{\infty} |T_n \star F(x)|^2 = 0,$$

then $F \in$ VMOA.

Proof. The space VMOA is the closure of continuous functions in the BMOA norm. Hence it suffices to find, for any $\epsilon > 0$, a decomposition $G = G_1 + G_2$ with G_1 continuous and $\|G_2\|_{\mathrm{BMOA}} \leq \epsilon$. For any $\epsilon > 0$, we may by hypothesis pick k sufficiently large such that

$$\sup_{|x|=1} \sum_{n=k}^{\infty} |T_n \star G(x)|^2 \leq \epsilon.$$

Using Theorem 3.1, it follows that if we take the decomposition

$$G = G_1 + G_2, \qquad \text{where} \quad G_1 = \sum_{n=0}^{k-1} T_n \star G \quad \text{and} \quad G_2 = \sum_{n=k}^{\infty} T_n \star G,$$

then G_1 is a polynomial and is in particular continuous. From the properties of the Fourier support of $\{T_n\}$,

$$T_n \star G_2 = \begin{cases} T_n \star G & \text{if } n \geq k+2, \\ \sum_{p=k}^{k+3} T_n \star T_p \star G & \text{if } k-2 \leq n \leq k+1, \\ 0 & \text{if } n \leq k-3. \end{cases} \tag{18}$$

Thus we have, for any $n \in [k-2, k+1]$ by using $\|T_n\|_1 \leq 6$ and convexity of the square, that

$$\|T_n \star G_2\|_\infty^2 \lesssim \sup_{n\geq k} \|T_n \star G\|_\infty^2 \leq \sup_{|x|=1} \sum_{n=k}^{\infty} |T_n \star G(x)|^2 \leq \epsilon.$$

Applying Theorem 3.1 to G_2 and using the properties derived in (18),

$$\|G_2\|_*^2 \lesssim \sup_{|x|=1} \sum_{n=0}^{\infty} |T_n \star G_2(x)|^2 = \sup_{|x|=1} \sum_{n=k}^{\infty} |T_n \star G_2(x)|^2 \leq \sup_{|x|=1} \sum_{n=k+2}^{\infty} |T_n \star G(x)|^2 + \sum_{n=k-2}^{k+1} \|T_n \star G_2\|_\infty^2 \lesssim \epsilon. \qquad \square$$

Proposition 3.3. *The Sledd space* SL *is nonseparable.*

Sketch of the proof. We sketch the construction of an uncountable family of analytic functions in SL whose pairwise distances in $\|\cdot\|_{S(T)}$ are uniformly bounded below. Put

$$G_j(z) = \frac{1}{2^j+1} z^{2^{j+1}} K_{2^j}(ze(1/j)), \quad j \geq 1.$$

Notice that $\hat{G}_j$ is supported in $[2^j, 2^{j+2}]$ and that G_j has the following properties:

(1) $|G_j(e(-1/j))| = 1$.

(2) $|G_j(e(\theta))| \leq 1$ for all θ.

(3) $|G_j(e(-1/j+\theta))| \lesssim 2^{-j}$ when $c2^{-j/2} \leq |\theta| \leq \pi$.

For any $A \subset 5\mathbb{N}$ let $H_A = \sum_{n\in A} G_n$. By the above properties all these functions belong to SL and are uniformly separated from each other. □

Remark 3.4. The construction above gives an example of functions in SL which are not continuous on the boundary of the disk.

GAFs and the Sledd space. We shall be interested in applying Sledd's condition to GAFs, for which purpose it is possible to make some simplifications. For any $n \geq 0$, let R_n be the kernel defined by

$$\hat{R}_n(K) = \begin{cases} 1 & \text{if } K \in [2^n, 2^{n+1}), \\ 0 & \text{otherwise.} \end{cases}$$

In short, for a GAF, (and more generally any random series with symmetric independent coefficients) we may replace the trapezoidal kernel T_n by R_n; specifically:

Theorem 3.5. *Suppose G is an H^2-GAF. Then the following are equivalent*:

(i) $\lim_{k\to\infty} \sup_{|x|=1} \sum_{n=k}^{\infty} |T_n \star G(x)|^2 = 0$ *a.s.*

(ii) $\lim_{k\to\infty} \mathbb{E}\big[\sup_{|x|=1} \sum_{n=k}^{\infty} |T_n \star G(x)|^2\big] = 0.$

(iii) $\lim_{k\to\infty} \mathbb{E}\big[\sup_{|x|=1} \sum_{n=k}^{\infty} |R_n \star G(x)|^2\big] = 0.$

(iv) $\lim_{k\to\infty} \sup_{|x|=1} \sum_{n=k}^{\infty} |R_n \star G(x)|^2 = 0$ *a.s.*

Proof of Theorem 3.5. We begin with the equivalence of (ii) and (iii), and the implication that (iii) implies (ii). For any $n \geq 0$ and any $j \in \{1, 2, 3, 4\}$ define $R_{n,j} = T_n \star R_{n+j-1}$. Then $T_n = \sum_{j=1}^{4} R_{n,j}$. Using convexity, we can bound

$$\sup_{|x|=1} \sum_{n=k}^{\infty} |T_n \star G(x)|^2 \lesssim \sum_{j=1}^{4} \sup_{|x|=1} \sum_{n=k}^{\infty} |R_{n,j} \star G(x)|^2.$$

Since $\hat{R}_{n,j}$ is supported in $[2^n, 2^{n+1})$ and has $\|\hat{R}_{n,j}\|_\infty \leq 1$, the contraction principle implies that, for any $0 \leq k \leq m < \infty$,

$$\mathbb{E} \sup_{|x|=1} \sum_{n=k}^{m} |R_{n,j} \star G(x)|^2 \leq \mathbb{E} \sup_{|x|=1} \sum_{n=k}^{m} |R_n \star G(x)|^2 \leq \mathbb{E} \sup_{|x|=1} \sum_{n=k}^{\infty} |R_n \star G(x)|^2.$$

Sending $m \to \infty$ and using monotone convergence implies that

$$\mathbb{E} \sup_{|x|=1} \sum_{n=k}^{\infty} |R_{n,j} \star G(x)|^2 \leq \mathbb{E} \sup_{|x|=1} \sum_{n=k}^{\infty} |R_n \star G(x)|^2,$$

from which the desired convergence follows.

Conversely, to see that (ii) implies (iii), we begin by bounding

$$\sup_{|x|=1} \sum_{n=k}^{\infty} |R_n \star G(x)|^2 \leq \sum_{j=1}^{4} \sup_{|x|=1} \sum_{\substack{n \geq k \\ n \in 4\mathbb{N}+j}}^{\infty} |R_n \star G(x)|^2.$$

Then by the contraction principle and monotone convergence, for any $j \in \{1, 2, 3, 4\}$,

$$\mathbb{E} \sup_{|x|=1} \sum_{\substack{n \geq k \\ n \in 4\mathbb{N}+j}}^{\infty} |R_n \star G(x)|^2 \leq \mathbb{E} \sup_{|x|=1} \sum_{\substack{n \geq k \\ n \in 4\mathbb{N}+j}}^{\infty} |T_n \star G(x)|^2 \leq \mathbb{E} \sup_{|x|=1} \sum_{n \geq k}^{\infty} |T_n \star G(x)|^2,$$

which completes the proof of the desired implication.

We turn to showing the equivalence of (i) and (ii). From Markov's inequality, (ii) implies that

$$\sup_{|x|=1} \sum_{n=k}^{\infty} |T_n \star G(x)|^2 \xrightarrow[k \to \infty]{\mathbb{P}} 0.$$

As the sequence $\sup_{|x|=1} \sum_{n=k}^{\infty} |T_n \star G(x)|^2$ is monotone and therefore always converges, it follows that it converges almost surely to 0.

Define for each $k \in \mathbb{N}$ the seminorms

$$\|\cdot\|_{S(R),k} : H^1 \to [0, \infty], \quad \text{where } \|f\|_{S(R),k}^2 := \sup_{|x|=1} \sum_{n=k}^{\infty} |R_n \star f(x)|^2,$$

$$\|\cdot\|_{S(T),k} : H^1 \to [0, \infty], \quad \text{where } \|f\|_{S(T),k}^2 := \sup_{|x|=1} \sum_{n=k}^{\infty} |T_n \star f(x)|^2.$$

In preparation to use Proposition 2.4, we make the following claim.

Claim 3.6. *The seminorms $\{\|\cdot\|_{S(R),k}, \|\cdot\|_{S(T),k}\}$ are approximable.*

We shall return to the proof of this claim after completing the proof of Theorem 3.5. We now show the equivalence of (iii) and (iv). The proof of the equivalence of (i) and (ii) is the same. From (iii) it follows from Markov's inequality that

$$\sup_{|x|=1} \sum_{n=k}^{\infty} |R_n \star G(x)|^2 \xrightarrow[k \to \infty]{\mathbb{P}} 0.$$

By monotonicity $\sup_{|x|=1} \sum_{n=k}^{\infty} |R_n \star G(x)|^2$ converges almost surely, and so it converges almost surely to 0. From (iv) and by Claim 3.6, there exists a k_0 such that

$$\mathbb{E} \sup_{|x|=1} \sum_{n=k_0}^{\infty} |R_n \star G(x)|^2 < \infty.$$

As a consequence, it is possible to take $k_0 = 0$. By dominated convergence,

$$\lim_{k\to\infty} \mathbb{E}\left[\sup_{|x|=1} \sum_{n=k}^{\infty} |R_n \star G(x)|^2\right] = 0. \qquad \square$$

Proof of Claim 3.6. Let p_m be the polynomial of degree $2^{m+1} - 1$ whose nonzero coefficients are all 1. Then, for any $m > k$,

$$\|p_m \star f\|^2_{S(R),k} = \sup_{|x|=1} \sum_{n=k}^{m} |R_n \star f(x)|^2 \xrightarrow[m\to\infty]{} \|f\|^2_{S(R),k}.$$

Let $q_m(z)$ be the sum of the analytic part of $\sum_{k=0}^{m} T_k(z)$. Then, for analytic f in the disk,

$$q_m \star f = \sum_{k=0}^{m} T_k \star f.$$

Moreover, using (15) the sum $\sum_{k=0}^{m} T_k$ can be represented by a sum of a finite number of Fejér kernels with cardinality bounded independent of m. Therefore there is an absolute constant $C > 0$ such that, for all m,

$$\|q_m \star f\|_\infty \le \left\| \sum_{k=0}^{m} T_k \right\|_1 \|f\|_\infty \le C\|f\|_\infty. \tag{19}$$

Using that $\hat{q}_m(j) = 1$, for $0 \le j \le 2^m - 1$,

$$\|q_m \star f\|^2_{S(T),k} \ge \sup_{|x|=1} \sum_{n=k}^{m-2} |T_n \star f(x)|^2 \xrightarrow[m\to\infty]{} \|f\|^2_{S(T),k},$$

and so if $\sup_m \|q_m \star f\|^2_{S(T),k} < \infty$, this means $\|f\|^2_{S(T),k} < \infty$ also. Conversely, if $\|f\|^2_{S(T),k} < \infty$, then $\sup_{n\ge k} \|T_n \star f\|_\infty < \infty$, and hence, with the same C as in (19),

$$\max_{m-1\le n\le m+2} \|q_m \star T_n \star f\|_\infty \le C\|f\|_{S(T),k}.$$

So

$$\|q_m \star f\|^2_{S(T),k} \le \sup_{|x|=1} \sum_{n=k}^{m-2} |T_n \star f(x)|^2 + \sum_{n=m-1}^{m+2} \|q_m \star T_n \star f\|^2_\infty \le (1+4C^2)\|f\|^2_{S(T),k} < \infty. \qquad \square$$

Remark 3.7. In reviewing the proof of Theorem 3.5, one also sees that under the same assumptions the following are equivalent:

(i) $\sup_{|x|=1} \sum_{n=0}^{\infty} |T_n \star G(x)|^2 < \infty$ a.s.
(ii) $\mathbb{E}\big[\sup_{|x|=1} \sum_{n=0}^{\infty} |T_n \star G(x)|^2\big] < \infty$.
(iii) $\mathbb{E}\big[\sup_{|x|=1} \sum_{n=0}^{\infty} |R_n \star G(x)|^2\big] < \infty$.
(iv) $\sup_{|x|=1} \sum_{n=0}^{\infty} |R_n \star G(x)|^2 < \infty$ a.s.

Moreover, the proof gives that there is an absolute constant $C > 0$ such that

$$\frac{1}{C}\mathbb{E}\|G\|^2_{S(R)} \le \mathbb{E}\|G\|^2_{S(T)} \le C\,\mathbb{E}\|G\|^2_{S(R)}.$$

Finally, we show that for a GAF, finiteness of $\|G\|_{S(R)}$ in fact implies $G \in \mathrm{VMOA}$.

Theorem 3.8. *If G is an H^2-GAF for which*

$$\|G\|_{S(R)}^2 = \sup_{|x|=1} \sum_{n=0}^{\infty} |R_n \star G(x)|^2 < \infty \quad \text{a.s.},$$

then

$$\lim_{k\to\infty} \sup_{|x|=1} \sum_{n=k}^{\infty} |R_n \star G(x)|^2 = 0 \quad \text{a.s.}$$

Furthermore, $\|G\|_{S(R)} < \infty$ *implies G is in* VMOA.

We will need the following result [Kahane 1985, Chapter 5, Proposition 12].

Proposition 3.9. *Let $u_1, u_2, \dots$ be a sequence of continuous functions on the unit circle such that* $\limsup_{k\to\infty} \|u_k\|_\infty > 0$, *and let $\theta_1, \theta_2, \dots$ be a sequence of independent random variables uniformly distributed on* $[0, 1]$. *Then almost surely there exists a $t \in \mathbb{R}/\mathbb{Z}$ such that* $\limsup_{k\to\infty} |u_k(e(t-\theta_k))| > 0$.

Proof of Theorem 3.8. Let $v_n := |R_n \star G|^2$ for all $n \geq 1$. Suppose to the contrary that

$$V := \lim_{k\to\infty} \sup_{|x|=1} \sum_{n=k}^{\infty} v_n(x)$$

is not almost surely 0. Then as V is tail-measurable, there is a $\delta \in (0, 1)$ so that $V > \delta$ a.s. By monotonicity, it follows that, for all k,

$$\sup_{|x|=1} \sum_{n=k}^{\infty} v_n(x) > \delta \quad \text{a.s.}$$

Furthermore, deterministically,

$$\lim_{m\to\infty} \sup_{|x|=1} \sum_{n=k}^{m} v_n(x) = \sup_{|x|=1} \sum_{n=k}^{\infty} v_n(x).$$

By continuity of measure,

$$\lim_{m\to\infty} \mathbb{P}\left(\sup_{|x|=1} \sum_{n=k}^{m} v_n(x) > \delta\right) = \mathbb{P}\left(\lim_{m\to\infty} \sup_{|x|=1} \sum_{n=k}^{m} v_n(x) > \delta\right) = 1.$$

Thus there is a sequence $m_1 < m_1' < m_2 < m_2' < \cdots$ such that if $u_k := \sum_{n=m_k}^{m_k'} v_n$, then

$$\mathbb{P}(\|u_k\|_\infty > \delta) > \delta.$$

By Borel–Cantelli,

$$\mathbb{P}\big(\limsup_{k\to\infty} \|u_k\|_\infty > \delta\big) = 1.$$

Let θ_k be i.i.d. uniform variables on $[0, 1]$ which are also independent of G. Therefore by conditioning on G and using Proposition 3.9 there is almost surely a $t \in \mathbb{R}/\mathbb{Z}$ such that

$$\limsup_{k\to\infty} v_k(e(t-\theta_k)) > 0.$$

Because $\{v_n(xe(\theta_k))\}$ has the same distribution as $\{v_n(x)\}$, it follows there is almost surely a $s \in \mathbb{R}/\mathbb{Z}$ such that

$$\limsup_{k\to\infty} v_k(e(s)) > 0.$$

Therefore $\|G\|^2_{S(R)} \geq V = \infty$ a.s., which concludes the first part of the proof.

Using Theorem 3.2, Theorem 3.5 and Remark 3.7, the second conclusion follows. $\square$

4. Sufficient condition for a GAF to be Sledd

In this section we will give a sufficient condition on the coefficients of the GAF to be in SL. Recall that a standard complex Gaussian random variable is one with density on $\mathbb{C}$ given by $\frac{1}{\pi}e^{-|z|^2}$. A vector (H_1, H_2) is a centered complex Gaussian vector if it has the same distribution as a linear image of the i.i.d. standard complex Gaussian random variables $(\xi_j : j \in \mathbb{N})$, or equivalently if it is the linear image of some pair of independent standard complex Gaussian random variables (Z_1, Z_2). We begin with the following preliminary calculation.

Lemma 4.1. *Let (H_1, H_2) be a centered complex Gaussian vector with $\mathbb{E}|H_1|^2 = \mathbb{E}|H_2|^2 = 1$ and $|\mathbb{E}[H_1\overline{H}_2]| = \rho \in [0, 1]$. Then for all $|\lambda| < (1-\rho^2)^{-1/2}$,*

$$\mathbb{E}e^{\lambda(|H_1|^2-|H_2|^2)} = \frac{1}{1-\lambda^2(1-\rho^2)}.$$

Proof. We may assume without loss of generality that $\mathbb{E}[H_1\overline{H}_2] = \rho \geq 0$. Hence, we may write

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = A\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

where $Z = (Z_1, Z_2)$ are independent standard complex normals, considered as a column vector. Therefore,

$$|H_1|^2 - |H_2|^2 = Z^*A^*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}AZ.$$

It follows that

$$\mathbb{E}e^{\lambda(|H_1|^2-|H_2|^2)} = \frac{1}{\det\bigl(\mathrm{Id} - \lambda A^*\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)A\bigr)} = \frac{1}{1-\lambda^2(1-\rho^2)}. \qquad \square$$

We shall apply this equality to the complex Gaussian process $Q_n(\theta) := R_n \star G(e(\theta))$. Then

$$\sigma_n^2 = \mathbb{E}|Q_n|^2 \quad \text{and define} \quad \rho_n := \rho_n(\theta_1-\theta_2) := \sigma_n^{-2}|\mathbb{E}[Q_n(\theta_1)\,\overline{Q_n(\theta_2)}]| \in [0, 1].$$

In the case that $\sigma_n^2 = 0$, we may take any value in $[0, 1]$ for ρ_n. From Lemma 4.1, we have, for any $|\lambda|^2 < (1-\rho_n^2)^{-1}\sigma_n^{-4}$,

$$\mathbb{E}\exp(\lambda(|Q_n(\theta_1)|^2 - |Q_n(\theta_2)|^2)) = \frac{1}{1-\lambda^2(1-\rho_n^2)\sigma_n^4}. \tag{20}$$

While we would like to use $\sigma_n^4(1-\rho_n^2(\theta_1-\theta_2))$ as a distance, it does not obviously satisfy the triangle inequality, for which reason we introduce

$$\Delta_n(\theta) := \mathbb{E}\bigl||Q_n(\theta)|^2 - |Q_n(0)|^2\bigr|, \tag{21}$$

which defines a pseudometric on $\mathbb{R}/\mathbb{Z}$ through $\Delta_n(\theta_1, \theta_2) := \Delta_n(\theta_1 - \theta_2)$. While Δ_n may not obviously control the tails of $|Q_n(\theta)|^2$, we observe the following lemma.

Lemma 4.2. *There is a numerical constant $C > 1$ such that, for all choices of $\{a_k\}$ and any $n \geq 0$ and all $\theta \in [0, 1]$,*

$$\frac{1}{C}\sigma_n^2\sqrt{1-\rho_n^2(\theta)} \leq \Delta_n(\theta) \leq C\sigma_n^2\sqrt{1-\rho_n^2(\theta)}.$$

Proof. From (20), it follows that

$$\mathbb{E}(|Q_n(\theta)|^2 - |Q_n(0)|^2)^2 = 2\sigma_n^4(1-\rho_n^2),$$
$$\mathbb{E}(|Q_n(\theta)|^2 - |Q_n(0)|^2)^4 = 24\sigma_n^8(1-\rho_n^2)^2.$$

Hence by Cauchy–Schwarz,

$$\Delta_n^2(\theta) \leq 2\sigma_n^4(1-\rho_n^2).$$

On the other hand, by the Paley–Zygmund inequality,

$$(|Q_n(\theta)|^2 - |Q_n(0)|^2)^2 \geq \sigma_n^4(1-\rho_n^2)$$

with probability at least $\frac{1}{4}\cdot\frac{2}{24}$ which gives a lower bound for Δ_n of the same order. □

We now define two pseudometrics on [0, 1] in terms of $\{\Delta_n\}$:

$$\begin{aligned} d_\infty(\theta_1, \theta_2) &:= d_\infty(\theta_1 - \theta_2) := \sup_{n\geq 0} \Delta_n(\theta_1 - \theta_2), \\ d_2^2(\theta_1, \theta_2) &:= d_2^2(\theta_1 - \theta_2) := \sum_{n\geq 0} \Delta_n^2(\theta_1 - \theta_2). \end{aligned} \tag{22}$$

Using Lemma 4.1, we can also now give a tail bound for differences of

$$F(\theta) := \sum_{n=0}^{\infty} |Q_n(\theta)|^2. \tag{23}$$

Lemma 4.3. *Let $\theta_1, \theta_2 \in [0, 1]$. There is a numerical constant $C > 1$ such that, for all $t \geq 0$,*

$$\mathbb{P}[F(\theta_1) - F(\theta_2) \geq t] \leq \exp\biggl(-C \min\biggl\{\frac{t}{d_\infty(\theta_1-\theta_2)}, \frac{t^2}{d_2^2(\theta_1-\theta_2)}\biggr\}\biggr).$$

Proof. The desired tail bound follows from estimating the Laplace transform of $F(\theta_1) - F(\theta_2)$. Specifically we use the following estimate.

Lemma 4.4. *Suppose that there are $\lambda_0, \sigma > 0$ and X a real-valued random variable for which*

$$\mathbb{E}e^{\lambda X} \leq e^{\lambda^2\sigma^2/2} \quad \textit{for } \lambda^2 \leq \lambda_0^2.$$

Then, for all $t \geq 0$,

$$\mathbb{P}[X \geq t] \leq \exp\biggl(-\min\biggl\{\frac{\lambda_0 t}{2}, \frac{t^2}{2\sigma^2}\biggr\}\biggr).$$

Proof. Applying Markov's inequality, for any $t \geq 0$ and $0 < \lambda \leq \lambda_0$,

$$\mathbb{P}[X \geq t] \leq \exp(-\lambda t + \lambda^2\sigma^2/2).$$

Taking $\lambda = t/\sigma^2$, if possible, gives one of the bounds. Otherwise, for $\lambda_0 \leq t/\sigma^2$, taking $\lambda = \lambda_0$ gives the other bound. ☐

We return to estimating the Laplace transform of $F(\theta_1) - F(\theta_2)$. Recalling (20), for any

$$|\lambda|^2 < \lambda_\star^2 := \inf_{n\in\mathbb{N}} (1-\rho_n^2)^{-1}\sigma_n^{-4} \leq \frac{C^2}{d_\infty(\theta_1-\theta_2)^2},$$

where C is the numerical constant from Lemma 4.2, we have

$$\mathbb{E}\exp(\lambda(F(\theta_1) - F(\theta_2))) = \prod_{n=1}^{\infty} \frac{1}{1-\lambda^2(1-\rho_n^2)\sigma_n^4}. \tag{24}$$

Therefore, for all $|\lambda|^2 < \lambda_\star^2/2$,

$$\mathbb{E}\exp(\lambda(F(\theta_1) - F(\theta_2))) \leq \prod_{n=1}^{\infty} \frac{1}{1-\lambda^2(1-\rho_n^2)\sigma_n^4} \leq \exp\biggl(2\lambda^2\sum_{n=1}^{\infty}(1-\rho_n^2)\sigma_n^4\biggr), \tag{25}$$

using $(1-x)^{-1} \leq e^{2x}$ for $0 \leq x \leq \frac{1}{2}$. The desired conclusion now follows from Lemmas 4.2 and 4.4. ☐

We now recall the technique of Talagrand for controlling the supremum of processes. We let $T = [0, 1]$. Define, for any metric d on T and any $\alpha \geq 1$,

$$\gamma_\alpha(d) = \inf \sup_{t\in T} \sum_{k\geq 0} d(t, C_k) 2^{k/\alpha}, \tag{26}$$

where the infimum is taken over all choices of finite subsets $(C_k)_{k\geq 0}$ of T with cardinality $|C_k| = 2^{2^k}$ for $k \geq 1$ and $|C_0| = 1$.

Theorem 4.5 (see [Talagrand 2001, Theorem 1.3]). *Let d_∞ and d_2 be two pseudometrics on T and let $(X_t)_{t\in T}$ be a process so that*

$$\mathbb{P}[|X_s - X_t| \geq u] \leq 2\exp\biggl(-\min\biggl\{\frac{u}{d_\infty(s,t)}, \frac{u^2}{d_2^2(s,t)}\biggr\}\biggr).$$

Then there is a universal constant $C > 0$ such that

$$\mathbb{E}\sup_{s,t\in T} |X_s - X_t| \leq C(\gamma_1(d_\infty) + \gamma_2(d_2)).$$

Hence, as an immediate corollary of this theorem and of Lemma 4.3, we have:

Corollary 4.6. *There is a numerical constant $C > 0$ such that*

$$\mathbb{E}\sup_\theta F(\theta) \leq C(\gamma_1(d_\infty) + \gamma_2(d_2)) + \sqrt{\textstyle\sum a_n^2}.$$

Finally, we give some estimates on the quantities γ_1 and γ_2 for the metrics we consider. We begin with an elementary observation that shows $\Delta_n(\theta)$ must decay for sufficiently small angles (when $|\theta| \leq 2^{-n}$).

Lemma 4.7. *There is a numerical constant $C > 1$ such that, for all $\theta \in [-1, 1]$,*

$$1 - \rho_n^2(\theta) \le C 2^{2n} |\theta|^2 \quad \textit{and} \quad \Delta_n(\theta) \le C\sigma_n^2 2^n |\theta|.$$

Proof. We begin by observing that ρ_n can always be bounded by

$$\rho_n \ge \sigma_n^{-2} \sum_{k=2^n}^{2^{n+1}-1} |a_k|^2 \cos(2\pi k(\theta)) \ge 1 - 2\pi^2 2^{2n+2}\theta^2.$$

The proof now follows from Lemma 4.2. □

We now show that $\mathbb{E}\|G\|_{S(R)}^2$ has the desired control. For any $k \ge 0$, let

$$\tau_k^2 = \sup_{n \ge k} \sigma_n^2.$$

Lemma 4.8. *There is an absolute constant $C > 0$ such that*

$$\mathbb{E}\|G\|_{S(R)}^2 \le C \sum \tau_k^2.$$

This lemma proves Theorem 1.3.

Proof. From Corollary 4.6,

$$\mathbb{E}\|G\|_{S(R)}^2 \lesssim \gamma_1(d_\infty) + \gamma_2(d_2) + \sum \tau_k^2.$$

We will choose C_k to be the dyadic net $\{\ell 2^{-2^k} : 1 \le \ell \le 2^{2^k}\}$. Then using Lemma 4.7 it follows that for any $t \in [0, 1]$,

$$\begin{aligned} d_\infty(t, C_k) = d_\infty(2^{-2^k}) &\lesssim \sup_{n \ge 0}\{\Delta_n(2^{-2^k})\sigma_n^2\} \lesssim \sup_{n \ge 0}\{2^{-(n-2^k)_-}\sigma_n^2\}, \\ d_2^2(t, C_k) = d_2^2(2^{-2^k}) &\lesssim \sum_{n=0}^{\infty} \Delta_n^2(2^{-2^k})\sigma_n^4 \lesssim \sum_{n=0}^{\infty}\{2^{-2(n-2^k)_-}\sigma_n^4\}. \end{aligned} \tag{27}$$

In the previous equations, $x_- := -\min\{x, 0\}$.

This leads to the following estimates on γ_1 and γ_2:

$$\gamma_1(d_1) \le \sum_{k=0}^{\infty} \Big\{\sup_{n \ge 0}\{2^{-(n-2^k)_-}\sigma_n^2\}\Big\} \cdot 2^k, \tag{28}$$

$$\gamma_2(d_2) \le \sum_{k=0}^{\infty} \sqrt{\left\{\sum_{n \ge 0} 2^{-2(n-2^k)_-}\sigma_n^4\right\}} \cdot 2^{k/2}. \tag{29}$$

We show that

$$\gamma_1(d_1) + \gamma_2(d_2) \lesssim \sum_{k=0}^{\infty} \tau_k^2. \tag{30}$$

To control $\gamma_1(d_1)$, we begin by applying Cauchy condensation:

$$\gamma_1(d_1) \lesssim \sum_{k=0}^{\infty} \Big\{\sup_{n \ge 0}\{2^{-(n-k)_-}\sigma_n^2\}\Big\}. \tag{31}$$

We then estimate

$$\sup_{n\geq 0}\{2^{-(n-k)_-}\sigma_n^2\} \leq \sum_{n=0}^{k} 2^{n-k}\sigma_n^2 + \tau_k^2.$$

Applying this bound and changing the order of summation for the first, it follows that $\gamma_1(d_1) \lesssim \sum_k \tau_k^2$.

To control $\gamma_2(d_2)$, we again begin by applying Cauchy condensation which results in

$$\gamma_2(d_2) \leq \sum_{k=0}^{\infty} \sqrt{\left\{\sum_{n\geq 0} 2^{-2(n-k)_-}\sigma_n^4\right\} \cdot \frac{1}{k}}. \tag{32}$$

We then split the sum to get

$$\gamma_2(d_2) \leq \sum_{k=0}^{\infty} \sqrt{\left\{\sum_{0\leq n\leq k} 2^{2(n-k)}\tau_n^4\right\} \cdot \frac{1}{k}} + \sum_{k=0}^{\infty} \sqrt{\left\{\sum_{n\geq k} \tau_n^4\right\} \cdot \frac{1}{k}}. \tag{33}$$

To the first term we apply the subadditivity of $\sqrt{\cdot}$, which produces

$$\sum_{k=0}^{\infty} \sqrt{\left\{\sum_{0\leq n\leq k} 2^{2(n-k)}\tau_n^4\right\} \cdot \frac{1}{k}} \lesssim \sum_{k=0}^{\infty} \frac{1}{\sqrt{k}} \cdot \left\{\sum_{0\leq n\leq k} 2^{n-k}\tau_n^2\right\} \lesssim \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} \cdot \tau_n^2,$$

where the second sum follows from changing the order of summation. To the second term in (33) we again apply Cauchy condensation:

$$\sum_{k=0}^{\infty} \sqrt{\sum_{n\geq k} \tau_n^4 \cdot \frac{1}{k}} \lesssim \sum_{k=0}^{\infty} \sqrt{\left\{\sum_{j\geq k} \tau_{2^j}^4 \cdot 2^j\right\}} \cdot 2^{k/2} \lesssim \sum_{k=0}^{\infty} \left\{\sum_{j\geq k} \tau_{2^j}^2 \cdot 2^{j/2}\right\} \cdot 2^{k/2} \lesssim \sum_{j=0}^{\infty} \tau_{2^j}^2 \cdot 2^j,$$

where the penultimate inequality follows from subadditivity of $\sqrt{\cdot}$ and the final inequality follows by changing the order of summation. From another application of Cauchy condensation, (30) follows. ☐

We remark that sequences for which $\sum_{k=0}^{\infty} \tau_k^2 = \infty$ but which are square summable necessarily have some amount of lacunary behavior.

Lemma 4.9. *Suppose $\sum_{k=0}^{\infty} \tau_k^2 = \infty$ but $\sum_{n=0}^{\infty} \sigma_n^2 < \infty$. Then for any $C > 1$ there is a sequence $\{j_k\}$ tending to infinity with $j_{k+1}/j_k > C$ for all k such that*

$$\sum_{k=1}^{\infty} \sigma_{j_k}^2 \cdot j_k = \infty.$$

Proof. Using Cauchy condensation, we have that, for any $m \in \mathbb{N}$ with $m > 1$,

$$\sum_{j=1}^{\infty} \tau_{m^j}^2 \cdot m^j = \infty = \sum_{k=0}^{\infty} \tau_k^2.$$

Let $\{j_k^*\}$ be the subsequence of $\{m^j\}$ at which $\tau_{m^j} > \tau_{m^{j+1}}$. Picking j_k as an ℓ in $[j_k^*, mj_k^*)$ that maximizes σ_ℓ^2 produces the desired result, after possibly passing to the subsequence $\{j_{2k}\}$ or $\{j_{2k+1}\}$. ☐

5. Exceptional GAFs

In this section, we construct GAFs with exceptional properties. In particular, we show the strict inclusions in (7).

5A. ***H^2–Bloch GAFs are not always BMO GAFs.*** Both lacunary and regularly varying H^2-GAFs are VMOA. Sledd [1981, Theorem 3.5] constructs an example of an H^2 random series that is not Bloch, and so is not BMOA. This leaves open the possibility that once an H^2-GAF is Bloch, it additionally is BMO. We give an example that shows there are H^2-GAFs that are Bloch but not BMO.

Recall (3), that for an interval $I \subseteq \mathbb{R}/\mathbb{Z}$, any $p \geq 1$, and any $\mathrm{L}^p(\mathbb{T})$ function f,

$$M_I^p(f) := \fint_I \left| f(e(\theta)) - \fint_I f \right|^p \mathrm{d}\theta, \quad \text{where} \fint_I f(e(\theta))\,\mathrm{d}\theta := \frac{1}{|I|}\int_I f(e(\theta))\,\mathrm{d}\theta.$$

Lemma 5.1. *For every $R > 0$, there exists $n_0 = n_0(R)$ such that for any $n > n_0$ there is a polynomial $f(z) := \sum_{k=n}^m a_k \xi_k z^k$ with the following properties*:

(i) $\sum_k a_k^2 = 1$.

(ii) $\mathbb{E}\|f\|_* \geq R$.

(iii) $\mathbb{E}\|f\|_{\mathcal{B}} \leq C$, *where $C > 0$ is an absolute constant.*

We can then use this lemma to construct the desired GAF.

Theorem 5.2. *There exists an H^2, Bloch, non-*BMOA *GAF.*

Proof. Let $\{\beta_i\}$ and $\{R_i\}$ be two positive sequences with $\{\beta_i\} \in \ell_1$ and $\beta_i R_i \to \infty$. Let f_i be a sequence of independent random Gaussian polynomials given by Lemma 5.1 having

$$\mathbb{E}\|f_i\|_* \geq R_i \quad \text{and} \quad \mathbb{E}\|f_i\|_{\mathcal{B}} \leq C.$$

The function $f = \sum_i \beta_i f_i$ satisfies, for all $\theta \in \mathbb{R}/\mathbb{Z}$,

$$\mathbb{E}|f(e(\theta))|^2 = \sum_{i=1}^{\infty} \beta_i^2 < \infty,$$

and so f is in L^2. The Bloch norm satisfies

$$\mathbb{E}\|f\|_{\mathcal{B}} \leq \sum_{i=1}^{\infty} \mathbb{E}\beta_i \|f_i\|_{\mathcal{B}} < \infty.$$

Finally, by the contraction principle (Proposition 2.1),

$$\mathbb{E}\|f\|_* \geq \beta_i \mathbb{E}\|f_i\|_* \geq \beta_i R_i \to \infty,$$

as $i \to \infty$. Therefore $\|f\|_* = \infty$ a.s. by Corollary 2.8. □

Remark 5.3. It is possible to choose the polynomial $\{f_i\}$ to have disjoint Fourier support, although it is not necessary for the proof, as we have picked them to be independent.

Proof of Lemma 5.1.

Construction of f. Let $r \in \mathbb{N}$ be some parameter to be fixed later (sufficiently large). Let

$$\{\lambda_{i,j} : i, j \in \{1, 2, \ldots, r\}\} \cup \{1\}$$

be real numbers that are linearly independent over the rationals and that satisfy

$$\lambda_{i,j} \in [2^i, 2^i + 4^{-r}]. \tag{34}$$

By Kronecker's theorem, for every $\omega \in \left\{0, \frac{1}{2}\right\}^{r\times r}$ there is an $m = m(\omega)$ such that

$$|\{m\lambda_{i,j}\} - \omega_{i,j}| \le 4^{-r} \quad \text{for all } i, j = 1, \ldots, r, \tag{35}$$

where as usual $\{x\} = x - \lfloor x \rfloor$ is the fractional value.

Let $n_0 = 4^r(\max_\omega m(\omega) + 1)$, and let $n > n_0$ be arbitrary. Define

$$a_k = \begin{cases} \frac{1}{r} & \text{if } k = \lfloor n\lambda_{i,j} \rfloor \text{ for some } i, j = 1, \ldots, r, \\ 0 & \text{otherwise.} \end{cases}$$

For brevity, write $\zeta_{i,j} = \xi_{\lfloor n\lambda_{i,j} \rfloor}$ for any $i, j = 1, \ldots, r$. Note that the $\zeta_{i,j}$ are independent and we can write

$$f(z) = \frac{1}{r} \sum_{i,j=1}^{r} \zeta_{i,j} z^{\lfloor n\lambda_{i,j} \rfloor}. \tag{36}$$

Lower bound for $\mathbb{E}\|f\|_*$. Define a random variable $\omega \in \left\{0, \frac{1}{2}\right\}^{r\times r}$ by

$$\omega_{i,j} = \begin{cases} 0 & \text{if } \operatorname{Re}\zeta_{i,j} \ge 0, \\ \frac{1}{2} & \text{if } \operatorname{Re}\zeta_{i,j} < 0. \end{cases}$$

Let I be the interval of length $1/n$ centered around $m(\omega)/n$.

We will show that $\mathbb{E}M_I^2(f)$ is large. To do so, we give an effective approximation for $\operatorname{Re} f$ on I.

Define

$$g(\theta) := \sum_{i=1}^{r} \Xi_i \cos(2\pi \cdot 2^i n\theta) \quad \text{where } \Xi_i := \frac{1}{r}\sum_{j=1}^{r} |\operatorname{Re}\zeta_{i,j}|.$$

Notice that g is $1/n$-periodic and therefore

$$M_I^2(g) = \fint_I |g(\theta)|^2 \, d\theta = \frac{1}{2}\sum_{i=1}^{r} \Xi_i^2.$$

Hence $\mathbb{E}m_I^2(g) \ge Cr$ for some absolute constant $C > 0$, and so it remains to approximate f by g.

Claim 5.4. *There is a sine trigonometric polynomial h such that with*

$$E = E(\theta) := \operatorname{Re} f\left(e\left(\frac{m(\omega)}{n} + \theta\right)\right) - g(\theta) - h(n\theta)$$

and, for $|\theta| \le 1/n$,

$$|E(\theta)| \le 3 \cdot 4^{-r} \sum_{i,j} |\zeta_{i,j}|.$$

Proof. By (35),

$$d\Big(\frac{m(\omega)}{n}\lfloor n\lambda_{i,j}\rfloor - \omega_{i,j}, \mathbb{Z}\Big) \le 4^{-r} + \frac{m(\omega)}{n} \le 2\cdot 4^{-r}.$$

By (34), $\lfloor n\lambda_{i,j}\rfloor \in [n2^i, n2^i + n4^{-r}]$, and so for $|\theta| \le 1/n$,

$$|\theta\lfloor n\lambda_{i,j}\rfloor - \theta n 2^i| \le 4^{-r}.$$

Combining these two estimates, for $|\theta| \le 1/n$ and for all i, $j = 1, 2, \dots, r$,

$$\left|\mathrm{Re}\Big(\zeta_{i,j}e\Big(\Big(\frac{m(\omega)}{n}+\theta\Big)\lfloor n\lambda_{i,j}\rfloor\Big)\Big) - \mathrm{Re}(\zeta_{i,j}e(\omega_{i,j}+2^i n\theta))\right| \le 3\cdot 4^{-r}|\zeta_{i,j}|.$$

Using that $e\big(\theta+\frac{1}{2}\big) = -e(\theta)$, the claim follows by applying the previous estimate term by term to (36). $\square$

We now bound the oscillation $M_I^2(f)$ as follows. Using $⨍_I g = ⨍_I h = 0$ and the orthogonality of g and h on I,

$$\begin{aligned}\left[⨍_I \left|f(e(\theta)) - ⨍_I f\right|^2 d\theta\right]^{1/2} &\ge \left[⨍_I \left|\mathrm{Re} f(e(\theta)) - ⨍_I \mathrm{Re} f\right|^2 d\theta\right]^{1/2} \\ &\ge \left[⨍_I |g(\theta)+h(\theta)|^2\, d\theta\right]^{1/2} - \left[⨍_I |E(\theta)|^2\, d\theta\right]^{1/2} \\ &\ge \left[⨍_I |g(\theta)|^2\, d\theta\right]^{1/2} - 3\cdot 4^{-r}\sum_{i,j}|\zeta_{i,j}| \\ &\ge \left[\frac{1}{2}\sum_{i=1}^{r}\Xi_i^2\right]^{1/2} - 3\cdot 4^{-r}\sum_{i,j}|\zeta_{i,j}|\end{aligned}$$

Using [Girela 2001, Proposition 4.1], there is a constant $C_2 \ge 1$ such that

$$\mathbb{E}\|f\|_* \ge \frac{1}{C_2}\mathbb{E}\left(\left[⨍_I \left|f(e(\theta)) - ⨍_I f\right|^2 d\theta\right]^{1/2}\right) \ge \frac{\sqrt{r}}{C} - C\cdot 4^{-r} r^2$$

for some sufficiently large absolute constant $C > 0$.

Upper bound for $\mathbb{E}\|f\|_{\mathcal{B}}$. We begin by computing

$$f'(z) = \frac{1}{r}\sum_{i,j}\zeta_{i,j}\lfloor n\lambda_{i,j}\rfloor z^{\lfloor n\lambda_{i,j}\rfloor - 1},$$

with the sum over all i, j in $\{1, 2, \dots, r\}$. Let $\Theta_i = \frac{1}{r}\sum_j |\zeta_{i,j}|$. Then, for $t = |z| < 1$,

$$\begin{aligned}(1-|z|)|f'(z)| \le \frac{1-t}{r}\sum_{i,j}|\zeta_{i,j}| n 2^{i+1} t^{n2^i - 1} &\le \big(\max_i \Theta_i\big)(1-t)\sum_i n2^{i+1}t^{n2^i-1} \\ &\le 2\big(\max_i \Theta_i\big)(1-t)\sum_{i=1}^{\infty} t^{i-1} \le 2\big(\max_i \Theta_i\big).\end{aligned}$$

Using the Pythagorean property of subgaussian norms [Vershynin 2018, Proposition 2.6.1], the random variables Θ_i have subgaussian norm $C/\sqrt{i}$, and hence using standard estimates,

$$\sup_{r\in\mathbb{N}}\Bigl[\mathbb{E}\max_{i=1,2,\dots,r}\Theta_i\Bigr]<\infty.$$

This concludes the proof of Lemma 5.1. □

5B. ***BMO GAFs are not always VMO GAFs.*** We answer a question of [Sledd 1981], showing that there are GAFs which are in BMOA but not in VMOA. We begin by defining a new seminorm on BMOA

$$\|f\|_{*,n}:=\sup_{I:2^{-n}\le|I|\le 2^{-(n-1)}} M_I^1(f),$$

where the supremum is over intervals $I\subset\mathbb{R}/\mathbb{Z}$.

Lemma 5.5. *There is a constant $c>0$ and an $m\in\mathbb{N}$ such that, for all integers $n\ge m$ and for all polynomials p with coefficients supported in* $[2^n,2^{n+1}]$,

$$\|p\|_*\ge\|p\|_{*,n-m}\ge c\|p\|_\infty\ge c\|p\|_*.$$

Proof. The first inequality is trivial. The last inequality is [Girela 2001, Proposition 2.1]. Thus it only remains to prove the second inequality. Recall T_n, the dyadic trapezoidal kernel from (15), which satisfies for all $n\in\mathbb{N}$ that

$$\hat{T}_n(j)=1\quad\text{for }\ j\in[2^n,2^{n+1}],\qquad \hat{T}_n(0)=0,\qquad\text{and}\qquad \|T_n\|_\infty\le 10\cdot 2^n$$

(see (14) and (15) — this follows by bounding $\|K_n\|_\infty=n+1$ and using the positivity of K). From the condition on the support of the coefficients, $p\star T_n=p$. As the constant coefficient of T_n vanishes, $1\star T_n=0$, and therefore we have the identity that for any $I\subseteq\mathbb{R}/\mathbb{Z}$ and any $\phi\in\mathbb{R}/\mathbb{Z}$,

$$\begin{aligned}p(e(\phi))&=\Bigl(\Bigl(p-\fint_I p\Bigr)\star T_n\Bigr)(e(\phi))\\&=\int_0^1\Bigl(p(e(\theta))-\fint_I p\Bigr)T_n(e(\phi-\theta))\,d\theta.\end{aligned}\tag{37}$$

Fix $m\in\mathbb{N}$. Let I be the interval around ϕ of length $2\cdot 2^{m-n}$. Then, for $n\ge m+1$,

$$\begin{aligned}|p(e(\phi))|&\le\int_{I\cup I^c}\Bigl|p(e(\theta))-\fint_I p\Bigr||T_n(e(\phi-\theta))|\,d\theta\\&\le\|T_n\|_\infty\int_I\Bigl|p(e(\theta))-\fint_I p\Bigr|\,d\theta+2\|p\|_\infty\int_{I^c}|T_n(e(\phi-\theta))|\,d\theta.\end{aligned}\tag{38}$$

The first summand we control as follows (using the length of $|I|$ and $\|T_n\|_\infty\le 10\cdot 2^n$):

$$\begin{aligned}\|T_n\|_\infty\int_I\Bigl|p(e(\theta))-\fint_I p\Bigr|\,d\theta&\le 20\cdot 2^m\fint_I\Bigl|p(e(\theta))-\fint_I p\Bigr|\,d\theta\\&\le 20\cdot 2^m\|p\|_{*,n-m-1}.\end{aligned}\tag{39}$$

For the second summand, using (16),

$$\begin{aligned}2\|p\|_\infty \int_{I^c} |T_n(e(\phi-\theta))|\,d\theta &\le 4\|p\|_\infty \int_{2^{m-n}}^{1/2} |T_n(e(\theta))|\,d\theta \\ &\le 80\cdot 2^{-n}\|p\|_\infty \int_{2^{m-n}}^{1/2} |1-e(\theta)|^{-2}\,d\theta \\ &\le 20\cdot 2^{-n}\|p\|_\infty \int_{2^{m-n}}^{\infty} \theta^{-2}\,d\theta \\ &\le 20\cdot 2^{-m}\|p\|_\infty. \end{aligned} \tag{40}$$

Applying (39) and (40) to (38),

$$\|p\|_\infty \le 20\cdot 2^m\|p\|_{*,n-m-1} + 20\cdot 2^{-m}\|p\|_\infty.$$

Taking $m=5$, we conclude

$$\|p\|_\infty \le 2^{11}\|p\|_{*,n-6}. \qquad \square$$

The previous lemma allows us to estimate $\|\cdot\|_*$ for polynomials supported on dyadic blocks efficiently in terms of the supremum norm. Hence, we record the following simple observation.

Lemma 5.6. *For any $n \ge 2$, let f_n be the Gaussian polynomial*

$$f_n(z) = \frac{1}{\sqrt{n\log n}} \sum_{k=n}^{2n-1} \xi_k z^k.$$

Then there is an absolute constant $C>0$ such that

$$C^{-1} < \mathbb{E}\|f_n\|_\infty < C.$$

Further, for all $t \ge 0$,

$$\mathbb{P}[|\|f_n\|_\infty - \mathbb{E}\|f_n\|_\infty| > t] \le 2e^{-(\log n)t^2}.$$

Proof. Observe that the family $\{f_n(e(k/n)) : 0 \le k < n\}$ consists of i.i.d. complex Gaussian random variables of variance $1/\log n$. Hence,

$$\mathbb{E}\|f_n\|_\infty \ge \mathbb{E}\max_{0\le k<n} |f_n(e(k/n))| \ge C$$

for some constant $C>0$ (see [Vershynin 2018, Exercise 2.5.11]). Conversely, there is an absolute constant such that for any polynomial p of degree $2n$ (e.g., see [Rakhmanov and Shekhtman 2006]),

$$\|p\|_\infty \le C \max_{0\le k\le 4n} |p(e(k/(4n)))|.$$

Hence using that each $f_n(e(k/(4n)))$ is complex Gaussian of variance $1/\log n$, we conclude that there is another constant $C>0$ so that

$$\mathbb{E}\|f_n\|_\infty \le C$$

(see [Vershynin 2018, Exercise 2.5.10]). The concentration is a direct consequence of Proposition 2.2. $\square$

Let $\{n_k\}$ be a monotonically increasing sequence of positive integers, to be chosen later. Let f_k be independent Gaussian polynomials as in Lemma 5.6 with $n = 2^{n_k}$. Let $\{a_k\}$ be a nonnegative sequence. Define $g = \sum_{k=1}^{\infty} a_k f_k$. Under the condition that $\sum_{k=1}^{\infty} a_k^2/n_k < \infty$, we have that g is an H^2-GAF.

Lemma 5.7. *Let $n_1 = 1$ and define $n_{k+1} = 3^{n_k}$ for all $k \geq 0$. If the sequence $\{a_k\}$ is bounded, then g is in* BMOA *almost surely. Furthermore, if $\lim_{k\to\infty} a_k = 0$, then g is in* VMOA *almost surely.*

Proof. Without loss of generality we may assume all $a_k \leq 1$. Observe that

$$\|g\|_* = \sup_{\ell\in\mathbb{N}} \|g\|_{*,\ell}.$$

Therefore, if $\sup_{\ell\in\mathbb{N}} \|g\|_{*,\ell} < \infty$ a.s., then g is in BMOA. If, furthermore, $\lim_{\ell\to\infty} \|g\|_{*,\ell} = 0$ a.s., then g is in VMOA almost surely.

Put $g_j = a_j f_j$ for all $j \in \mathbb{N}$. Fix $\ell \in \mathbb{N}$ and let k be such that $n_{k-1} - m \leq \ell \leq n_k - m$, where m is the constant from Lemma 5.5, and take the decomposition $g = g_{<k-1} + g_{k-1} + g_k + g_{>k}$. Then

$$\|g_{>k}\|_{*,\ell} \leq 2^{\ell/2}\|g_{>k}\|_2 \leq 2^{n_k/2}\|g_{>k}\|_2,$$

which follows from Cauchy–Schwarz applied to $M_I^1(g_{>k})$ for an interval $|I| \geq 2^{-\ell}$. On the other hand,

$$\|g_{<k}\|_{*,\ell} \leq 2^{-\ell+1}\|g'_{<k}\|_\infty \leq 2^{-\ell+1}2^{n_{k-2}+1}\|g_{<k}\|_\infty \leq 2^{-n_{k-1}+n_{k-2}+m+2}\|g_{<k}\|_\infty,$$

where the penultimate inequality is Bernstein's inequality for polynomials. We conclude that

$$\|g\|_{*,\ell} \leq 2^{-n_{k-1}+n_{k-2}+m+2}\|g_{<k}\|_\infty + 2\|g_k\|_\infty + 2\|g_{k-1}\|_\infty + 2^{n_k/2}\|g_{>k}\|_2.$$

Using Lemma 5.6 and Borel–Cantelli,

$$D := \sup_k \|f_k\|_\infty < \infty \quad \text{a.s.}$$

In particular,

$$\|g_{<k}\|_\infty \leq kD.$$

Meanwhile, the family $\{\|f_j\|_2^2 \cdot 2 \cdot 2^{n_j}\log(2^{n_j})\}$ consists of independent χ^2 random variables with 2^{n_j+1} degrees of freedom, respectively. Hence,

$$R := \sup_j\{\|g_j\|_2\sqrt{n_j}\} < \infty \quad \text{a.s.}$$

Therefore,

$$\|g_{>k}\|_2^2 = \sum_{j>k}\|g_j\|_2^2 \leq R^2\sum_{j>k}\frac{1}{n_j} \leq \frac{3R^2}{n_{k+1}}.$$

Finally, we have

$$\|g\|_{*,\ell} \leq 2^{-n_{k-1}+n_{k-2}+m+2}kD + 2(a_{k-1}+a_k)D + \sqrt{3}\cdot 2^{n_k/2}\frac{R}{\sqrt{n_{k+1}}}.$$

By our choice of n_k (recalling $k = k(\ell)$), the last expression is *uniformly* bounded in ℓ almost surely. In addition, if $a_k \to 0$, then

$$\limsup_{\ell\to\infty} \|g\|_{*,\ell} \leq \limsup_{k\to\infty} 2(a_{k-1}+a_k)D = 0. \qquad \square$$

Remark 5.8. A more careful analysis of $\|f_{>k}\|_{*,\ell}$ reveals that it suffices to have $n_{k+1}/n_k > c > 1$ for some c to bound $\|f_{>k}\|_{*,\ell}$ uniformly over all ℓ. We will not pursue this here.

We now turn to proving the existence of the desired GAF.

Theorem 5.9. *There is a BMO GAF which is almost surely not a VMO GAF.*

Proof. We let g be as in Lemma 5.7 with $a_k = 1$ for all k. By making t sufficiently small and using the contraction principle (Proposition 2.1), Lemma 5.5 and Lemma 5.6, for all $k \in \mathbb{N}$,

$$2\mathbb{P}(\|g\|_{*,n_k-m} > t) \geq \mathbb{P}(\|f_k\|_{*,n_k-m} > t) \geq \mathbb{P}(\|f_k\|_\infty > ct) \geq \tfrac{1}{2}.$$

Therefore by the reverse Fatou lemma,

$$\mathbb{P}\Big(\limsup_{k\to\infty} \|g\|_{*,n_k-m} > t\Big) \geq \limsup_{k\to\infty} \mathbb{P}(\|g\|_{*,n_k-m} > t) \geq \tfrac{1}{4}.$$

This implies, by Proposition 2.9, that g is not in VMOA a.s. □

Finally, we show there is a VMO GAF which is not Sledd.

Lemma 5.10. *There is an absolute constant $c > 0$ such that for all $\epsilon > 0$ there is an $n_0(\epsilon)$ sufficiently large such that, for all $n \geq n_0$ and for all intervals $I \subset \mathbb{R}/\mathbb{Z}$ with $|I| = \epsilon$,*

$$\mathbb{P}\Big(\textit{there exists } J \subset I \textit{ an interval with } |J| = c/n \textit{ such that } \min_{x\in J} |f_n(x)| > \tfrac{1}{4}\Big) \geq \tfrac{1}{3},$$

where f_n is as in Lemma 5.6.

Proof. We again use the observation that the family $\{f_n(e(k/n)) : 0 \leq k < n\}$ consists of i.i.d. complex Gaussian random variables of variance $1/\log n$. Let I be an interval as in the statement of the lemma. Let I' be the middle third of that interval. Then for any n, there are at least $n\epsilon/4$ many k such that k/n are in I'. For any such k and any t,

$$\mathbb{P}[|f_n(e(k/n))| > t] = e^{-(\log n)t^2}.$$

Hence, if we define n_0 so that $n_0^{3/4}\epsilon = 4\log(3)$, then for all $n \geq n_0$,

$$\mathbb{P}\Big[\text{for all } k : k/n \in I', |f_n(e(k/n))| \leq \tfrac{1}{2}\Big] \leq e^{-n^{3/4}\epsilon/4} \leq \tfrac{1}{3}.$$

Using Bernstein's inequality and Lemma 5.6, there is an absolute constant such that

$$\|f_n'\|_\infty \leq 2n\|f_n\|_\infty \leq Cn,$$

except with probability $1/n$. Hence, if we let J be the interval of length c/n around a point in I' where $|f_n(e(k/n))| > \frac{1}{2}$, then $\min_{x\in J}|f_n(x)| \geq \frac{1}{4}$ except with probability $\frac{2}{3}$. □

Theorem 5.11. *There exists a GAF that is almost surely in* VMOA *and which is almost surely not Sledd.*

Proof. We let g be as in Lemma 5.7 with $a_k \to 0$ to be defined, so that g is almost surely in VMOA.

We define a nested sequence of random intervals $\{J_\ell\}$. Let $J_0 = \mathbb{R}/\mathbb{Z}$. Define a subsequence n_{k_ℓ} inductively by letting n_{k_ℓ} be the smallest integer bigger than $n_0(c/n_{k_{\ell-1}})$ for $\ell > 1$ and $n_0(c)$ for $\ell = 1$. Let $a_{n_{k_\ell}} = 1/\sqrt{\ell}$, and let $a_j = 0$ if j is not in $\{n_{k_\ell}\}$.

We say that an interval J_ℓ *succeeds* if there is a subinterval J' of length $c/n_{k_{\ell-1}}$ such that $\min_{x\in J'}|f_{k_\ell}| > \frac{1}{4}$. If the interval J_ℓ succeeds, we let $J_{\ell+1} = J'$, and otherwise we let $J_{\ell+1}$ be the interval of length $c/n_{k_{\ell-1}}$ that shares a left endpoint with J_ℓ. The nested intervals $\bar{J}_\ell$ decrease to a point x, and

$$\|f\|^2_{S(R)} \geq \sum_{\ell=1}^{\infty} \frac{1}{\ell} |f_{k_\ell}(x)|^2 \geq \sum_{\ell=1}^{\infty} \frac{1}{16\ell} \mathbf{1}\{J_\ell \text{ succeeds}\}.$$

From Lemma 5.10, the family $\{\mathbf{1}\{J_\ell \text{ succeeds}\}\}$ consists of independent Bernoulli random variables with parameter at least $\frac{1}{3}$. Then by [Kahane 1985, Chapter 3, Theorem 6], this series is almost surely infinite. □

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