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HIGHER RANK QUANTUM-CLASSICAL CORRESPONDENCE

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For a compact Riemannian locally symmetric space $\Gamma\backslash G/K$ of arbitrary rank we determine the location of certain Ruelle–Taylor resonances for the Weyl chamber action. We provide a Weyl-lower bound on an appropriate counting function for the Ruelle–Taylor resonances and establish a spectral gap which is uniform in Γ if G/K is irreducible of higher rank. This is achieved by proving a quantum-classical correspondence, i.e., a one-to-one correspondence between horocyclically invariant Ruelle–Taylor resonant states and joint eigenfunctions of the algebra of invariant differential operators on G/K.

1. Introduction

Ruelle resonances for an Anosov flow provide a fundamental spectral invariant that reflects not only many important dynamical properties of the flow but also geometric and topological properties of the underlying manifold. Very recently the concept of resonances was extended to higher rank $\mathbb{R}^n$-Anosov actions and led to the notion of *Ruelle–Taylor resonances*[1] which were shown to be a discrete subset $\sigma_{\mathrm{RT}} \subset \mathbb{C}^n$ [Bonthonneau et al. 2020]. It was furthermore shown in that same paper that the leading resonances (i.e., those with vanishing real part) are related to mixing properties of the considered Anosov action. In particular, it was shown that if the action is weakly mixing in an arbitrary direction of the abelian group $\mathbb{R}^n$, then $0 \in \mathbb{C}^n$ is the only leading resonance. Furthermore, the resonant states at zero give rise to equilibrium measures that share properties of Sinai–Ruelle–Bowen (SRB) measures of Anosov flows.

Apart from the leading resonances the spectrum of Ruelle–Taylor resonances has so far not been studied if $n \geq 2$. In particular, when $n \geq 2$, it was not known whether there are other resonances than the resonance at zero. Neither was it known whether there is a spectral gap, i.e., whether the real parts of the resonances are bounded away from zero. In this article we shed some light on these questions by examining the Ruelle–Taylor resonances for the class of Weyl chamber flows via harmonic analysis.

Let us briefly introduce the setting: Let G be a real connected noncompact semisimple Lie group with finite center and Iwasawa decomposition $G = KAN$. Let $\mathfrak{a}$ be the Lie algebra of A and M the centralizer of A in K. Then A is isomorphic to $\mathbb{R}^n$, where n is the real rank of G, and acts on G/M from the right. Hence A also acts on the compact manifold $\mathcal{M} := \Gamma\backslash G/M$, where $\Gamma \leq G$ is a cocompact torsion-free lattice. It can be easily seen that this action is an Anosov action with hyperbolic splitting $T\mathcal{M} = E_0 \oplus E_s \oplus E_u$ which can be described explicitly in terms of associated vector bundles (see Section 2A for a general

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[1] They were named Ruelle–Taylor resonances because the notion of the Taylor spectrum for commuting operators is a crucial ingredient of their definition.

definition of Anosov actions and Proposition 3.1 for the description of the hyperbolic splitting for Weyl chamber flows). Furthermore, if $\Sigma \subseteq \mathfrak{a}^*$ is the set of restricted roots with simple system Π and positive system Σ^+ then the positive Weyl chamber is given by $\mathfrak{a}_+ = \{H \in \mathfrak{a} \mid \alpha(H) > 0 \ \forall \alpha \in \Pi\}$.

The *Ruelle–Taylor resonances* of this Anosov action are defined as follows: For $H \in \mathfrak{a}$ let X_H be the vector field on $\mathcal{M}$ defined by the right A-action. Then

$$\sigma_{\mathrm{RT}} := \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \exists u \in \mathcal{D}'_{E_u^*}(\mathcal{M}) \setminus \{0\} \text{ s.t. } (X_H + \lambda(H))u = 0 \ \forall H \in \mathfrak{a}\},$$

where $\mathcal{D}'_{E_u^*}(M)$ is the set of distributions with wavefront set contained in the annihilator $E_u^* \subseteq T^*\mathcal{M}$ of $E_0 \oplus E_u$. The distributions $u \in \mathcal{D}'_{E_u^*}(M)$ satisfying $(X_H + \lambda(H))u = 0$ for all $H \in \mathfrak{a}$ are called *resonant states* of λ and the dimension of the space of all such distributions is called the multiplicity $m(\lambda)$ of the resonance λ. It has been shown in [Bonthonneau et al. 2020] that $\sigma_{\mathrm{RT}} \subset \mathfrak{a}^*_{\mathbb{C}}$ is discrete and that all resonances have finite multiplicity. It also follows from that work that the real part of the resonances are located in a certain cone $\overline{{}_-\mathfrak{a}^*} \subset \mathfrak{a}^*$ which is the negative dual cone of the positive Weyl chamber $\mathfrak{a}_+$ (see Section 2B for a precise definition).

In this article we will prove that there is a bijection between a certain subset of the Ruelle–Taylor resonant states and certain joint eigenfunctions of the invariant differential operators on the locally symmetric space $\Gamma\backslash G/K$. Before explaining this correspondence in more detail we state two results on the spectrum of Ruelle–Taylor resonances that we can conclude from the correspondence.

The first result says that, for any Weyl chamber flow, there exist infinitely many Ruelle–Taylor resonances by providing a Weyl-lower bound on an appropriate counting function.

Theorem 1.1. *Let ρ be the half-sum of the positive restricted roots, let W be the Weyl group* (*see Section 2B for a precise definition*) *and, for $t > 0$, let*

$$N(t) := \sum_{\lambda \in \sigma_{\mathrm{RT}},\, \mathrm{Re}(\lambda) = -\rho,\, \|\mathrm{Im}(\lambda)\| \le t} m(\lambda).$$

Then, for $d := \dim(G/K)$,

$$N(t) \ge |W| \operatorname{Vol}(\Gamma\backslash G/K)(2\sqrt{\pi})^{-d} \frac{1}{\Gamma(d/2+1)} t^d + \mathcal{O}(t^{d-1}).$$

More generally, let $\Omega \subseteq \mathfrak{a}^*$ be open and bounded such that $\partial\Omega$ has finite $(n-1)$-dimensional Hausdorff measure. Then

$$\sum_{\lambda \in \sigma_{\mathrm{RT}},\, \mathrm{Re}(\lambda) = -\rho,\, \mathrm{Im}(\lambda) \in t\Omega} m(\lambda) \ge |W| \operatorname{Vol}(\Gamma\backslash G/K)(2\pi)^{-d} \operatorname{Vol}(\mathrm{Ad}(K)\Omega) t^d + \mathcal{O}(t^{d-1}).$$

The second result guarantees a uniform spectral gap.

Theorem 1.2. *Let G be a real semisimple Lie group with finite center. Then, for any cocompact torsion-free discrete subgroup $\Gamma \subset G$, there is a neighborhood $\mathcal{G} \subset \mathfrak{a}^*$ of 0 such that*

$$\sigma_{\mathrm{RT}} \cap (\mathcal{G} \times i\mathfrak{a}^*) = \{0\}.$$

If G furthermore has Kazhdan's property (*T*) (*e.g., if G is simple of higher rank*), *then the spectral gap $\mathcal{G}$ can be taken uniformly in Γ and only depends on the group G.*

Let us now explain in some detail the spectral correspondence that is the key to the above results.

We define the space of *first band resonant states* as those resonant states that are in addition horocyclically invariant:

$$\mathrm{Res}^0_X(\lambda) := \{u \in \mathcal{D}'_{E_u^*}(\mathcal{M}) \mid (X_H + \lambda(H))u = 0 \text{ and } \mathcal{X}u = 0\ \forall H \in \mathfrak{a} \text{ and } \mathcal{X} \in C^\infty(\mathcal{M}, E_u)\},$$

and we call a Ruelle–Taylor resonance a *first band resonance* if and only if $\mathrm{Res}^0_X(\lambda) \neq 0$. By working with horocycle operators and vector-valued Ruelle–Taylor resonances we will be able to show that all resonances with real part in a certain neighborhood of zero in $\mathfrak{a}^*$ are always first band resonances (see Proposition 3.7). As the Weyl chamber flow is generated by mutually commuting Hamilton flows, we consider the set of Ruelle–Taylor resonances as a *classical spectrum*.

Let us briefly describe the *quantum* side: In the rank 1 case the quantization of the geodesic flow is given by the Laplacian on G/K. In the higher rank case we have to consider the algebra of G-invariant differential operators on G/K which we denote by $\mathbb{D}(G/K)$. As an abstract algebra this is a polynomial algebra with n algebraically independent operators, among them the Laplace operator. These operators descend to $\Gamma\backslash G/K$ and we can define the joint eigenspace

$${}^\Gamma E_\lambda = \{f \in C^\infty(\Gamma\backslash G/K) \mid Df = \chi_\lambda(D) f\ \forall D \in \mathbb{D}(G/K)\},$$

where χ_λ is a character of $\mathbb{D}(G/K)$ parametrized by $\lambda \in \mathfrak{a}^*_{\mathbb{C}}/W$ with the Weyl group W. Here χ_ρ is the trivial character (see Section 2D). Let σ_Q denote the corresponding *quantum spectrum* $\{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid {}^\Gamma E_\lambda \neq \{0\}\}$.

We have the following correspondence between the classical first band resonant states and the joint quantum eigenspace:

Theorem 1.3. *Let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ be outside the exceptional set*

$$\mathcal{A} := \left\{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \;\middle|\; \frac{2\langle \lambda + \rho, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in -\mathbb{N}_{>0} \text{ for some } \alpha \in \Sigma^+\right\}.$$

Then there is a bijection between the finite-dimensional vector spaces

$$\pi_* : \mathrm{Res}^0_X(\lambda) \to {}^\Gamma E_{-\lambda-\rho},$$

where π_* is the push-forward of distributions along the canonical projection $\pi : \Gamma\backslash G/M \to \Gamma\backslash G/K$.

Using this one-to-one correspondence we can then use results about the quantum spectrum to obtain obstructions and existence results on the Ruelle–Taylor resonances. Notably we use results of Duistermaat, Kolk and Varadarajan [Duistermaat et al. 1979] on the spectrum σ_Q, but we also deduce refined information on the quantum spectrum. Here we use L^p-bounds for spherical functions obtained from asymptotic expansions [van den Ban and Schlichtkrull 1987] and L^p-bounds for matrix coefficients based on work by Cowling [1979] and Oh [2002]. Theorems 1.1 and 1.2 as stated above give only a rough version of the information on the Ruelle–Taylor resonances that we can actually obtain. As the full results require some further notation we refrain from stating them in the introduction and refer to Theorem 5.1. We also refer to Figure 6 for a visualization of the structure of first band resonances for the case $G = \mathrm{SL}(3, \mathbb{R})$.

Methods and related results. The key ingredient to the quantum-classical correspondence is that we can in a first step relate the horocyclically invariant first band resonant states with distributional vectors in some principal series representations. Then we can apply the Poisson transform of [Kashiwara et al. 1978] to get a bijection onto the quantum eigenspace ${}^{\Gamma}E_{-\lambda-\rho}$. The prototype of such a quantum-classical correspondence was first established by Dyatlov, Faure and Guillarmou [Dyatlov et al. 2015] in the case of manifolds of constant curvature or in other words for the rank 1 group $G = \mathrm{SO}(n, 1)$. Certain central ideas have, however, already been present for $G = \mathrm{SO}(2, 1)$ in the work of Cosentino [2005] and Flaminio and Forni [2003]. In the rank 1 setting there exist several generalizations of the quantum classical correspondence of [Dyatlov et al. 2015], for example, to convex cocompact manifolds of constant curvature [Guillarmou et al. 2018; Hadfield 2020], general compact locally symmetric spaces of rank 1 [Guillarmou et al. 2021] and vector bundles [Küster and Weich 2020; 2021].

Besides the correspondence between the classical Ruelle resonant states and the quantum Laplace eigenvalues there are several other approaches in the literature establishing exact relations between the Laplace spectrum and the geodesic flow. One approach is to relate the Laplace spectrum to divisors of zeta functions. Such relations have been obtained for rank 1 locally symmetric spaces on various levels of generality by Bunke, Olbrich, Patterson and Perry: $G = \mathrm{SO}(n, 1)$ and Γ convex cocompact [Bunke and Olbrich 1997; 1999; Patterson and Perry 2001]; G real rank 1 and Γ cocompact [Bunke and Olbrich 1995].

A third approach to an exact quantum-classical correspondence is to relate the Laplace spectrum to a transfer operator which represents a time discretized dynamics of the geodesic flow. This type of correspondence was notably studied for hyperbolic surfaces with cusps; see [Bruggeman and Pohl 2023; Bruggeman et al. 2015; Lewis and Zagier 2001] for results for $G = \mathrm{SL}(2, \mathbb{R})$ and Γ discrete subgroups of increasing generality. We refer in particular to the expository article [Pohl and Zagier 2020] and the introduction of [Bruggeman and Pohl 2023] for a current state of the art of these techniques. A very first step towards generalizations of this approach to higher rank has been recently achieved in [Pohl 2020] for the Weyl chamber flow on products of Schottky surfaces by the construction of symbolic dynamics and transfer operators.

Note that in [Dyatlov et al. 2015] not only was the first band of Ruelle resonances related to the Laplace spectrum, but a complete band structure has been established and the higher bands can be related to the Laplace spectrum on divergence-free symmetric tensors. In the present article we do not study the higher bands. This will presumably be a very hard question for general semisimple groups G (note that in [Dyatlov et al. 2015] it was crucial at several points that $N \cong \mathbb{R}^{n-1}$ is abelian for $G = \mathrm{SO}(n, 1)$). However, it might be tractable for some concrete groups with simple enough root spaces such as $G = \mathrm{SL}(3, \mathbb{R})$. For geodesic flows the phenomenon of such a band structure is quite universal and known in the case of compact locally symmetric spaces of rank 1 [Küster and Weich 2021] and also for geodesic flows on manifolds of pinched negative curvature [Cekić and Guillarmou 2021; Faure and Tsujii 2013; 2021].

As mentioned above an important application of Ruelle resonances for Anosov flows are mixing results. More precisely, the existence of a spectral gap in addition with resolvent estimates imply mixing of the flow. For Weyl chamber flows this relation of gaps and mixing rates is not yet established but conjectured to be true. From this perspective, Theorem 1.2 is related to the work of Katok and Spatzier [1994] who showed exponential mixing for the Weyl chamber action in every direction of the closure of the positive Weyl chamber if G has property (T). However it is not known whether their result remains true if the property (T) assumption is dropped. Our result above (Theorem 1.2) ensures a Γ-dependent gap in any case but as mentioned above the precise relation to mixing rates is not yet established.

Finally, Weyl laws for Ruelle resonances of geodesic flows can also be established in variable curvature (or more generally contact Anosov flows) in various settings [Datchev et al. 2014; Faure and Sjöstrand 2011; Faure and Tsujii 2023]. In particular, in the very recent article by Faure and Tsujii [2021] the Weyl law also follows because a "first band" of resonances can be related to a quantum operator. The methods in their work are, however, completely different and are based on microlocal analysis rather then global harmonic analysis.

2. Preliminaries

2A. *Ruelle–Taylor resonances for higher rank Anosov actions.* In this section we recall the main properties of Ruelle–Taylor resonances for higher rank Anosov actions from [Bonthonneau et al. 2020]. Let $\mathcal{M}$ be a compact Riemannian manifold, let $A \simeq \mathbb{R}^n$ be an abelian group and let $\tau : A \to \mathrm{Diffeo}(\mathcal{M})$ be a smooth locally free group action. If $\mathfrak{a} := \mathrm{Lie}(A)$ we define the generating map

$$X : \mathfrak{a} \to C^\infty(\mathcal{M}, T\mathcal{M}), \quad H \mapsto X_H := \frac{d}{dt}\Big|_{t=0} \tau(\exp(tH)).$$

Note that $[X_{H_1}, X_{H_2}] = 0$ for $H_i \in \mathfrak{a}$. For $H \in \mathfrak{a}$ we denote by $\varphi_t^{X_H}$ the flow of the vector field X_H. The action is called *Anosov* if there exists $H \in \mathfrak{a}$ and a continuous $\varphi_t^{X_H}$-invariant splitting

$$T\mathcal{M} = E_0 \oplus E_u \oplus E_s,$$

where $E_0 := \mathrm{span}\{X_H : H \in \mathfrak{a}\}$ is of dimension n because the action is locally free and there exist $C > 0$ and $\nu > 0$ such that, for each $x \in \mathcal{M}$,

$$\begin{aligned} &\text{for all } w \in E_s(x),\ t \geq 0, \quad \|d\varphi_t^{X_H}(x)w\| \leq Ce^{-\nu t}\|w\|, \\ &\text{for all } w \in E_u(x),\ t \leq 0, \quad \|d\varphi_t^{X_H}(x)w\| \leq Ce^{-\nu |t|}\|w\|, \end{aligned}$$

where the norm on $T\mathcal{M}$ is given by the Riemannian metric on $\mathcal{M}$. Such an $H \in \mathfrak{a}$ is called *transversally hyperbolic*. We call the set

$$\mathcal{W} := \{H' \in \mathfrak{a} \mid H' \text{ is transversally hyperbolic with the same splitting as } H\}$$

the *positive Weyl chamber containing* H.

Let $E \to \mathcal{M}$ be the complexification of a Euclidean bundle over $\mathcal{M}$, and denote by $\mathrm{Diff}^1(\mathcal{M}, E)$ the set of first-order differential operators with smooth coefficients acting on sections of E. Then a linear map $\boldsymbol{X} : \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}, E)$ such that $\boldsymbol{X}_{H_1}\boldsymbol{X}_{H_2} = \boldsymbol{X}_{H_2}\boldsymbol{X}_{H_1}$ for all $H_i \in \mathfrak{a}$ is called an *admissible lift* of the generic map X if

$$\boldsymbol{X}_H(fs) = (X_H f)s + f\boldsymbol{X}_H s \tag{1}$$

for $s \in C^\infty(\mathcal{M}, E)$, $f \in C^\infty(\mathcal{M})$ and $H \in \mathfrak{a}$.

For a fixed positive Weyl chamber $\mathcal{W}$, the set of *Ruelle–Taylor resonances* can be defined as

$$\sigma_{\mathrm{RT}}(\boldsymbol{X}) := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \exists u \in \mathcal{D}'_{E_u^*}(\mathcal{M}, E) \setminus \{0\} \text{ s.t. } (\boldsymbol{X}_H + \lambda(H))u = 0 \ \forall H \in \mathfrak{a}\},$$

where $\mathcal{D}'_{E_u^*}(\mathcal{M}, E)$ is the set of distributional sections of the bundle E with wavefront set contained in E_u^*. Here E_u^* is defined as the annihilator of $E_0 \oplus E_u$ in $T^*\mathcal{M}$. The vector space of *Ruelle–Taylor resonant states* for a resonance $\lambda \in \sigma_{\mathrm{RT}}(\boldsymbol{X})$ is defined by

$$\mathrm{Res}_{\boldsymbol{X}}(\lambda) := \{u \in \mathcal{D}'_{E_u^*}(\mathcal{M}, E) \mid (\boldsymbol{X}_H + \lambda(H))u = 0 \ \forall H \in \mathfrak{a}\}.$$

Remark 2.1. The original definition of Ruelle–Taylor resonances and resonant states is stated via Koszul complexes; see [Bonthonneau et al. 2020, Section 3]. More precisely, λ is a resonance if and only if the corresponding Koszul complex is not exact and the resonant states are the cohomologies of this complex. The space of resonant states that we are considering is just the zeroth cohomology. However, it turns out that the Koszul complex is not exact if and only if the zeroth cohomology is nonvanishing, i.e., the two notions coincide; see [Bonthonneau et al. 2020, Theorem 4].

It is known that the resonances have the following properties.

Proposition 2.2 (see [Bonthonneau et al. 2020, Theorems 1 and 4]). *The set $\sigma_{\mathrm{RT}}(\boldsymbol{X})$ of Ruelle–Taylor resonances is a discrete subset of $\mathfrak{a}_{\mathbb{C}}^*$ contained in*

$$\{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \mathrm{Re}(\lambda(H)) \leq C_{L^2}(H) \ \forall H \in \mathcal{W}\}$$

with $C_{L^2}(H) = \inf\{C > 0 \mid \|e^{-t\boldsymbol{X}_H}\|_{L^2 \to L^2} \leq e^{Ct} \ \forall t > 0\}$, where $e^{-t\boldsymbol{X}_H} : L^2(\mathcal{M}, E) \to L^2(\mathcal{M}, E)$ is the semigroup with generator $-\boldsymbol{X}_H$. Moreover, for each $\lambda \in \sigma_{\mathrm{RT}}(\boldsymbol{X})$, the space $\mathrm{Res}_{\boldsymbol{X}}(\lambda)$ of resonant states is finite-dimensional.

2B. ***Semisimple Lie groups.*** In this section we fix the notation for the present article. Let G be a real semisimple connected noncompact Lie group with finite center and Iwasawa decomposition $G = KAN$. Furthermore, let $M := Z_K(A)$ be the centralizer of A in K and $G = KAN_-$ be the opposite Iwasawa decomposition. We denote by $\mathfrak{g}$, $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{n}_-$, $\mathfrak{k}$ and $\mathfrak{m}$ the corresponding Lie algebras. For $g \in G$, let $H(g)$ be the logarithm of the A-component in the Iwasawa decomposition. We have a K-invariant inner product on $\mathfrak{g}$ that is induced by the Killing form and the Cartan involution. We have the orthogonal Bruhat decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$ into root spaces $\mathfrak{g}_\alpha$ with respect to the $\mathfrak{a}$-action via the adjoint action ad. Here $\Sigma \subseteq \mathfrak{a}^*$ is the set of restricted roots. Denote by W the Weyl group of the root system of restricted roots. Let n be the real rank of G and $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ (resp. Σ^+) the simple (resp. positive) system in Σ determined by the choice of the Iwasawa decomposition. Let $m_\alpha := \dim_{\mathbb{R}} \mathfrak{g}_\alpha$ and $\rho := \frac{1}{2} \Sigma_{\alpha \in \Sigma^+} m_\alpha \alpha$. Denote by w_0 the longest Weyl group element, i.e., the unique element in W mapping Π to $-\Pi$. Let $\mathfrak{a}_+ := \{H \in \mathfrak{a} \mid \alpha(H) > 0 \ \forall \alpha \in \Pi\}$ be the positive Weyl chamber and $\mathfrak{a}_+^*$ the corresponding cone in $\mathfrak{a}^*$ via the identification $\mathfrak{a} \leftrightarrow \mathfrak{a}^*$ through the Killing form $\langle \cdot, \cdot \rangle$ restricted to $\mathfrak{a}$. We denote by ${}_+\mathfrak{a}^*$ the dual cone $\{\lambda \in \mathfrak{a}^* \mid \lambda(H) > 0 \ \forall H \in \overline{\mathfrak{a}_+} \setminus \{0\}\}$ and by $\overline{{}_+\mathfrak{a}^*}$ its closure $\{\lambda \in \mathfrak{a}^* \mid \lambda(H) \geq 0 \ \forall H \in \mathfrak{a}_+\} = \mathbb{R}_{\geq 0}\Pi$. Hence if ω_j is the dual basis of α_j then $\overline{{}_+\mathfrak{a}^*} = \{\lambda \in \mathfrak{a}^* \mid \langle \lambda, \omega_j \rangle \geq 0 \ \forall j = 1, \ldots, n\}$. Furthermore, we write $\overline{{}_-\mathfrak{a}^*} := -\overline{{}_+\mathfrak{a}^*}$. If $\overline{A^+} := \exp(\overline{\mathfrak{a}_+})$, then we have the Cartan decomposition $G = K\overline{A^+}K$.

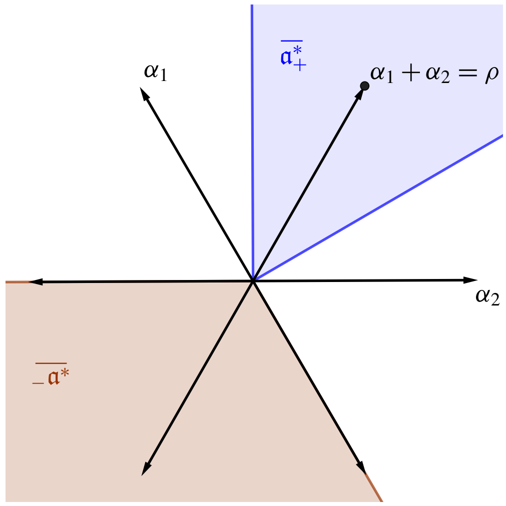

Figure 1. The root system for the special case $G=\mathrm{SL}_3(\mathbb{R})$: There are three positive roots $\Sigma^+=\{\alpha_1,\alpha_2,\alpha_1+\alpha_2\}$. As all root spaces are one-dimensional the special element $\rho=\frac{1}{2}\Sigma_{\alpha\in\Sigma^+}m_\alpha\alpha$ equals $\alpha_1+\alpha_2$.

Example 2.3. If $G=\mathrm{SL}_n(\mathbb{R})$, then we choose $K=\mathrm{SO}(n)$, A as the set of diagonal matrices of positive entries with determinant 1, and N as the set of upper triangular matrices with 1's on the diagonal. Then $\mathfrak{a}$ is the abelian Lie algebra of diagonal matrices and the set of restricted roots is $\Sigma=\{\varepsilon_i-\varepsilon_j \mid i\neq j\}$, where $\varepsilon_i(\lambda)$ is the i-th diagonal entry of λ. The positive system corresponding to the Iwasawa decomposition is $\Sigma^+=\{\varepsilon_i-\varepsilon_j \mid i<j\}$ with simple system $\Pi=\{\alpha_i=\varepsilon_i-\varepsilon_{i+1}\}$. The positive Weyl chamber is

$$\mathfrak{a}_+=\{\mathrm{diag}(\lambda_1,\dots,\lambda_n)\mid \lambda_1>\dots>\lambda_n\}$$

and the dual cone is

$$\overline{{}_+\mathfrak{a}}=\{\mathrm{diag}(\lambda_1,\dots,\lambda_n)\in\mathfrak{a}\mid \lambda_1+\dots+\lambda_k\geq 0\ \forall k\}.$$

See Figure 1 for a visualization in the special case $G=\mathrm{SL}_3(\mathbb{R})$. The Weyl group is the symmetric group S_n acting by permutation of the diagonal entries.

2C. ***Principal series representations.*** The concept of a principal series representation is an important tool in representation theory of semisimple Lie groups. It can be described using different pictures. We start with the *induced picture*: Pick $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ and (τ,V_τ) an irreducible unitary representation of M. We define

$$V^{\tau,\lambda}:=\{f:G\to V_\tau \text{ cont.}\mid f(gman)=e^{-(\lambda+\rho)\log a}\tau(m)^{-1}f(g),\ g\in G,\ m\in M,\ a\in A,\ n\in N\}$$

endowed with the norm $\|f\|^2=\int_K\|f(k)\|^2\,dk$ where dk is the normalized Haar measure on K. Recall that ρ is the half-sum of positive roots. The group G acts on $V^{\tau,\lambda}$ by the left regular representation. The completion $H^{\tau,\lambda}$ of $V^{\tau,\lambda}$ with respect to the norm is called the *induced picture of the* (*nonunitary*) *principal series representation* with respect to (τ,λ). We also write $\pi_{\tau,\lambda}$ for this representation. If τ is the trivial representation then we write H^λ and π_λ and call it the *spherical principal series* with respect to λ.

Note that for equivalent irreducible unitary representations τ_1 and τ_2 of M the corresponding principal series representations are equivalent as representations as well. In particular, the Weyl group W acts on the unitary dual of M by $w\tau(m) = \tau(w^{-1}mw)$, where $w \in W$ is given by a representative in the normalizer of A in K, and therefore $H^{\lambda, w\tau}$ is well defined up to equivalence.

A different way to view the principal series representation is the so-called *compact picture*. Although we don't need this description here, we want to introduce it in order to give a larger overview of these representations. It is given by restricting the function $f : G \to V_\tau$ to K, i.e., a dense subspace is given by

$$\{f : K \to V_\tau \text{ cont.} \mid f(km) = \tau(m)^{-1} f(k),\ k \in K,\ m \in M\}$$

with the same norm as above. In this picture the G-action is given by

$$\pi_{\tau,\lambda}(g) f(k) = e^{-(\lambda+\rho)H(g^{-1}k)} f(k_{KAN}(g^{-1}k)), \quad g \in G,\ \ k \in K,$$

where k_{KAN} is the K-component in the Iwasawa decomposition $G = KAN$. Furthermore, recall from Section 2B that $H(g) \in \mathfrak{a}$ was defined as the logarithm of the Iwasawa A component.

For the example $G = \mathrm{PSL}_2(\mathbb{R})$, the compact picture allows us to describe this representation explicitly without using the Iwasawa decomposition: since $K = \mathrm{PSO}(2) \simeq S^1 \subseteq \mathbb{R}^2$, the representation $H^{1,\lambda\alpha} = H^{\lambda\alpha}$ with $\lambda \in \mathbb{C}$ is given by $L^2(S^1)$ with the action $\pi_\lambda(g) f(\omega) = \|g^{-1}\omega\|^{-2\lambda-1} f(g^{-1}\omega/\|g^{-1}\omega\|)$.

2D. *Invariant differential operators.* Let $\mathbb{D}(G/K)$ be the algebra of *G-invariant differential operators* on G/K, i.e., differential operators commuting with the left translation by elements $g \in G$. Then we have an algebra isomorphism $\mathrm{HC} : \mathbb{D}(G/K) \to \mathrm{Poly}(\mathfrak{a}^*)^W$ from $\mathbb{D}(G/K)$ to the W-invariant complex polynomials on $\mathfrak{a}^*$ which is called the *Harish-Chandra homomorphism*; see [Helgason 1984, Chapter II Theorem 5.18]. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, let χ_λ be the character of $\mathbb{D}(G/K)$ defined by $\chi_\lambda(D) := \mathrm{HC}(D)(\lambda)$. Obviously, $\chi_\lambda = \chi_{w\lambda}$ for $w \in W$. Furthermore, the χ_λ exhaust all characters of $\mathbb{D}(G/K)$; see [Helgason 1984, Chapter III Lemma 3.11]. We define the space of joint eigenfunctions

$$E_\lambda := \{f \in C^\infty(G/K) \mid Df = \chi_\lambda(D) f\ \forall D \in \mathbb{D}(G/K)\}.$$

We will only work with the subspace of functions of moderate growth

$$E^*_\lambda := \{f \in E_\lambda \mid \exists c \in \mathbb{R} : |f(kaK)| \le Ce^{c\|\log a\|}\ \forall k \in K,\ a \in A\}.$$

Note that E_λ and E^*_λ are G-invariant.

2E. *Poisson transform.* The representation of G on E^*_λ can be described via the *Poisson transform*: If $(H^{\tau,\lambda})^{-\infty}$ denotes the distributional vectors in the principal series, then the Poisson transform $\mathcal{P}_\lambda$ maps $(H^{-\lambda})^{-\infty}$ into E^*_λ G-equivariantly. It is given by $\mathcal{P}_\lambda f(xK) = \int_K f(k) e^{-(\lambda+\rho)H(x^{-1}k)}\, dk$ if f is a sufficiently regular function in the compact picture of the principal series. If f is given in the induced picture, then $\mathcal{P}_\lambda f(xK)$ is simply $\int_K f(xk)\, dk$. Since K/M can be seen as the boundary of G/K at infinity, the Poisson transform produces a joint eigenfunction for a given boundary value; see [van den Ban and Schlichtkrull 1987] for more details.

It is important to know for which values of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the Poisson transform is a bijection. By [van den Ban and Schlichtkrull 1987, Theorem 12.2] we have that $\mathcal{P}_\lambda : (H^{-\lambda})^{-\infty} \to E_\lambda^*$ is a bijection if

$$-\frac{2\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} \notin \mathbb{N}_{>0} \quad \text{for all } \alpha \in \Sigma^+. \tag{2}$$

In particular, $\mathcal{P}_\lambda$ is a bijection if $\operatorname{Re}\lambda \in \overline{\mathfrak{a}_+^*}$.

2F. *L^p-bounds for elementary spherical functions.* One can show that in each joint eigenspace E_λ there is a unique left K-invariant function which has the value 1 at the identity; see [Helgason 1984, Chapter IV Corollary 2.3]. We denote the corresponding bi-K-invariant function on G by ϕ_λ and call it the *elementary spherical function*. Therefore, $\phi_\lambda = \phi_\mu$ if and only if $\lambda = w\mu$ for some $w \in W$. It is given by the Poisson transform of the constant function with value 1 in the compact picture, i.e., $\phi_\lambda(g) = \int_K e^{-(\lambda+\rho)H(g^{-1}k)}\, dk$.

The aim of this section is to establish the following proposition (see Figure 2 for a visualization) that will be needed to obtain a spectral gap in Theorem 4.10.

Proposition 2.4. *Let $p \in [2, \infty[$. Then the elementary spherical function ϕ_λ is in $L^{p+\varepsilon}(G)$ (where the L^p-space is defined via a Haar measure on G) for every $\varepsilon > 0$ if and only if* $\operatorname{Re}\lambda \in (1-2p^{-1})\operatorname{conv}(W\rho)$, *where* $\operatorname{conv}(W\rho)$ *is the convex hull of the finite set $W\rho$.*

Proof. First of all note that we only have to consider $\operatorname{Re}\lambda \in \overline{\mathfrak{a}_+^*}$ since $\phi_\lambda = \phi_\mu$ if and only if $\lambda = w\mu$ for some $w \in W$. In this case $\operatorname{Re}\lambda \in (1-2p^{-1})\operatorname{conv}(W\rho)$ is equivalent to $\operatorname{Re}\lambda \in (1-2p^{-1})\rho + \overline{{}_-\mathfrak{a}^*}$; see [Helgason 1984, Chapter IV Lemma 8.3].

With this remark, one implication of the proposition is a straightforward consequence of standard estimates for elementary spherical functions: Suppose that $\operatorname{Re}\lambda \in \overline{\mathfrak{a}_+^*}$ and $\operatorname{Re}\lambda \in (1-2p^{-1})\rho + \overline{{}_-\mathfrak{a}^*}$. Then we have the following bound on ϕ_λ (see [Knapp 1986, Chapter VII Property 7.15]):

$$|\phi_\lambda(a)| \le Ce^{(\operatorname{Re}\lambda-\rho)(\log a)}(1+\rho(\log a))^d, \quad a \in A^+,$$

where C and d are constants ≥ 0. By the integral formula for $G = K\overline{A^+}K$ (see [Helgason 1984, Chapter I Theorem 5.8]) and the bi-K-invariance of ϕ_λ, we have

$$\begin{aligned}\int_G |\phi_\lambda(g)|^{p+\varepsilon}\, dg &= \int_{\mathfrak{a}_+} |\phi_\lambda(\exp H)|^{p+\varepsilon} \prod_{\alpha\in\Sigma^+} \sinh(\alpha(H))^{m_\alpha}\, dH \\ &\le \int_{\mathfrak{a}_+} (Ce^{(\operatorname{Re}\lambda-\rho)H}(1+\rho(H))^d)^{p+\varepsilon} e^{2\rho(H)}\, dH\end{aligned}$$

for a suitable Lebesgue measure on $\mathfrak{a}$. Because $\operatorname{Re}\lambda \in (1-2p^{-1})\rho + \overline{{}_-\mathfrak{a}^*}$, we have

$$(p+\varepsilon)(\operatorname{Re}\lambda - \rho)(H) \le -(2+2\varepsilon p^{-1})\rho(H).$$

Hence

$$\int_G |\phi_\lambda(g)|^{p+\varepsilon}\, dg \le C^{p+\varepsilon} \int_{\mathfrak{a}_+} (1+\rho(H))^{d(p+\varepsilon)} e^{-2\varepsilon p^{-1}\rho(H)}\, dH,$$

and we see that the latter is indeed finite by coordinizing $\mathfrak{a}_+$ by $x_j \leftrightarrow \alpha_j(H)$ with $x_j > 0$. Then dH is a multiple of dx and $\rho(H) = \sum x_j\rho_j$ with $\rho_j > 0$. Therefore, $\phi_\lambda \in L^{p+\varepsilon}(G)$.

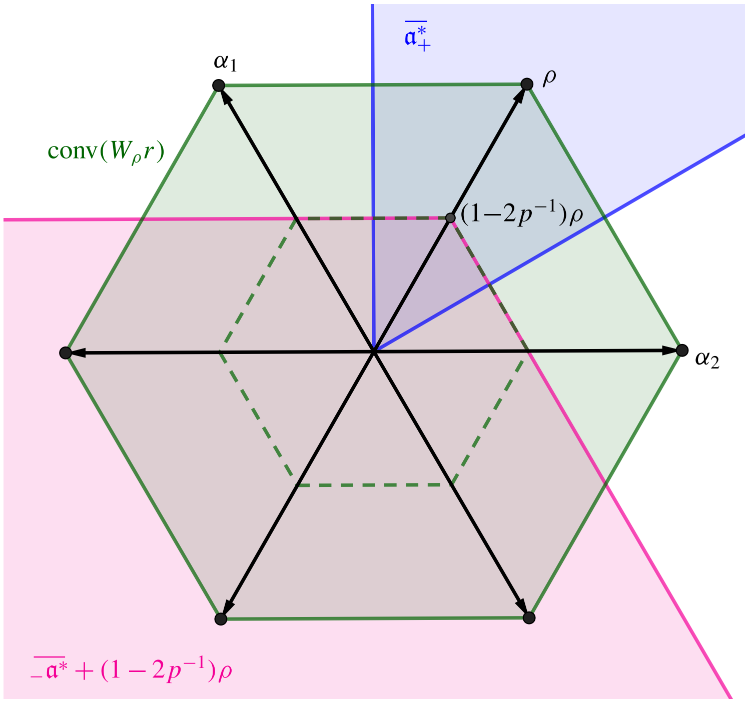

Figure 2. Visualization of the regions appearing in Proposition 2.4 for the special case $G = \mathrm{SL}_3(\mathbb{R})$: The green dashed region is the boundary of $(1-2p^{-1})\,\mathrm{conv}(W\rho)$. Its intersection with the positive Weyl chamber $\overline{\mathfrak{a}_+^*}$ (blue cone) equals $(1-2p^{-1})\rho + \overline{_-\mathfrak{a}^*}$ intersected with $\overline{\mathfrak{a}_+^*}$.

The opposite implication will be proved by combining the proof of [Knapp 1986, Theorem 8.48] with [van den Ban and Schlichtkrull 1987]: according to [van den Ban and Schlichtkrull 1987, Corollary 16.2], the elementary spherical function ϕ_λ has a converging expansion

$$\phi_\lambda(\exp H) = \sum_{\xi \in X(\lambda)} p_\xi(\lambda, H) e^{\xi(H)}, \qquad H \in \mathfrak{a}_+, \tag{3}$$

where

$$X(\lambda) = \{w\lambda - \rho - \mu \mid w \in W,\ \mu \in \mathbb{N}_0\Pi\}$$

and the $p_\xi(\lambda, \cdot\,)$ are polynomials of degree $\leq |W|$. The series converges absolutely on $\mathfrak{a}_+$ and uniformly on each subchamber $\{H \in \mathfrak{a}_+ \mid \alpha_i(H) \geq \varepsilon_i > 0\}$. The main ingredient of the proof of Proposition 2.4 is the fact that (see [van den Ban and Schlichtkrull 1987, Theorem 10.1])

$$p_{\lambda-\rho}(\lambda, \cdot\,) \neq 0. \tag{4}$$

Now, if $\phi_\lambda \in L^{p+\epsilon}(G)$, the proof of [Knapp 1986, Theorem 8.48] shows that

$$\mathrm{Re}\langle \lambda - (1-2(p+\epsilon)^{-1})\rho, \omega_j\rangle < 0.$$

Hence $\mathrm{Re}\,\lambda - (1-2p^{-1})\rho \in \overline{_-\mathfrak{a}^*}$. $\square$

2G. ***Positive definite functions and unitary representations.*** In this section we recall the correspondence between positive semidefinite elementary spherical functions and irreducible unitary spherical representations. Recall first that a continuous function $f : G \to \mathbb{C}$ is called *positive semidefinite* if the matrix $(f(x_i^{-1}x_j))_{i,j}$ for all $x_1, \dots, x_k \in G$ is positive semidefinite. If f is positive semidefinite, then f is bounded by $f(1)$ and one has $f(x^{-1}) = \overline{f(x)}$. Moreover, we can define a unitary representation π_f associated to f in the following way: if R denotes the right regular representation of G, then π_f is the completion of the space spanned by $R(x)f$ with respect to the inner product defined by $\langle R(x)f, R(y)f\rangle := f(y^{-1}x)$, which is positive definite. G acts unitarily on this space by the right regular representation. If $f(g) = \langle \pi(g)v, v\rangle$ is a matrix coefficient of a unitary representation π, then f is positive semidefinite and π_f is contained in π.

Secondly, recall that a unitary representation is called *spherical* if it contains a nonzero K-invariant vector. Denote by $\widehat{G}_{\mathrm{sph}}$ the subset of the unitary dual consisting of spherical representations. We then have a one-to-one correspondence between positive semidefinite elementary spherical functions and $\widehat{G}_{\mathrm{sph}}$ given by $\phi_\lambda \mapsto \pi_{\phi_\lambda}$; see [Helgason 1984, Chapter IV Theorem 3.7]. The preimage of an irreducible unitary spherical representation π with normalized K-invariant vector v_K is given by $g \mapsto \langle \pi(g)v_K, v_K\rangle$. If the set $\widehat{G}_{\mathrm{sph}}$ is endowed with the Fell topology (see [Bekka et al. 2008, Appendix F.2]) and we use the topology of convergence on compact sets on the set of elementary spherical functions, then the above correspondence is a homeomorphism as is easily seen from the definitions.

2H. ***Associated vector bundles.*** In order to define the Weyl chamber flow not only on the base manifold but also on vector bundles we recall the definition of the associated vector bundle $\mathcal{V}_\tau$ over a homogeneous space G/M for a unitary finite-dimensional representation (τ, V_τ) of M. Its total space is given by $\mathcal{V}_\tau = G \times_\tau V_\tau = (G \times V_\tau)/_\sim$, where $(gm, v) \sim (g, \tau(m)v)$ with $g \in G$, $m \in M$ and $v \in V_\tau$. The equivalence classes are denoted by $[g, v]$ and the projection to G/M is $[g, v] \mapsto gM$. A section s of this bundle can be identified with a function $\bar{s} : G \to V_\tau$ satisfying $\bar{s}(gm) = \tau(m)^{-1}\bar{s}(g)$. We will use this identification throughout this article. We also have a G-action on $\mathcal{V}_\tau$ defined by $g[g', v] := [gg', v]$. Therefore, we also have the left regular action on smooth sections of $\mathcal{V}_\tau$:

$$(gs)(g'M) := g(s(g^{-1}g'M)), \quad s \in C^\infty(G/M, \mathcal{V}_\tau).$$

Identifying s with $\bar{s}$, this actions reads $\overline{gs}(g') = \bar{s}(g^{-1}g')$.

A special case of an associated vector bundle is the tangent bundle $T(G/M) = G \times_{\mathrm{Ad}|_M} (\mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-)$. Hence vector fields $\mathfrak{X}$ can be identified with smooth functions $\overline{\mathfrak{X}} : G \to \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-$ satisfying

$$\overline{\mathfrak{X}}(gm) = \mathrm{Ad}(m)^{-1}\overline{\mathfrak{X}}(g).$$

Therefore, we have a canonical connection ∇ on $\mathcal{V}_\tau$ given by

$$\overline{\nabla_{\mathfrak{X}} s}(g) = \frac{d}{dt}\Big|_{t=0} \bar{s}(g \exp(t\overline{\mathfrak{X}}(g))),$$

where s is a smooth section identified with a $\bar{s} : G \to V_\tau$ and $\mathfrak{X}$ is a vector field of G/M identified with $\overline{\mathfrak{X}}$ as above. This connection will be used to lift the Weyl chamber flow to $\mathcal{V}_\tau$.

3. Ruelle–Taylor resonances for the Weyl chamber action

We keep the notation from Section 2B. Let Γ be a discrete, torsion-free, cocompact subgroup of G. Then the biquotient $\mathcal{M} = \Gamma\backslash G/M$ is a smooth compact Riemannian manifold where the Riemannian structure is induced by the inner product on $\mathfrak{g}$. More precisely, the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ is given by the quotient $\Gamma\backslash(G \times_{\mathrm{Ad}|_M} (\mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-))$, and the norm of some $\Gamma[g, Y]$ with $g \in G$ and $Y \in \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-$ is given by the norm of $Y \in \mathfrak{g}$. We have a well-defined right A-action on $\mathcal{M}$:

$$(\Gamma g M)a := \Gamma g a M, \quad a \in A, \;\; g \in G.$$

Therefore, we have an $\mathfrak{a}$-action by smooth vector fields:

$${}_\Gamma X : \mathfrak{a} \to C^\infty(\mathcal{M}, T\mathcal{M}), \quad {}_\Gamma X_H f(\Gamma g M) = \frac{d}{dt}\Big|_{t=0} f(\Gamma g e^{tH} M),$$

which we call the *Weyl chamber action*.

For later use we denote by $X : \mathfrak{a} \to \mathrm{Diff}^1(G/M)$ the corresponding action on G/M.

Proposition 3.1. *The A-action on $\mathcal{M}$ is Anosov. More precisely, each $H \in \mathfrak{a}_+$ is transversally hyperbolic with the splitting $E_0 = \Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{a})$, $E_s = \Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{n})$ and $E_u = \Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{n}_-))$. Moreover, for fixed $H_0 \in \mathfrak{a}_+$, the dynamically defined positive Weyl chamber*

$$\mathcal{W} = \{H \in \mathfrak{a} \mid H \text{ is transversally hyperbolic with the same splitting as } H_0\}$$

equals $\mathfrak{a}_+$. Hence the two notions of positive Weyl chambers agree.

Proof. Pick $\Gamma[g, X_\alpha] \in \Gamma\backslash(G \times_M \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-)$ and assume that X_α is in the root space $\mathfrak{g}_\alpha$. Then we calculate

$$\begin{aligned} d\varphi_t^{\Gamma X_H}(\Gamma g M)\Gamma[g, X_\alpha] &= \frac{d}{ds}\Big|_{s=0} \varphi_t^{\Gamma X_H}(\Gamma g e^{sX_\alpha} M) \\ &= \frac{d}{ds}\Big|_{s=0} \Gamma e^{sX_\alpha} e^{tH} M \\ &= \frac{d}{ds}\Big|_{s=0} \Gamma g e^{tH} e^{s\,\mathrm{Ad}(e^{-tH})X_\alpha} M \\ &= \Gamma[g e^{tH}, \mathrm{Ad}(e^{-tH})X_\alpha] \\ &= \Gamma[g e^{tH}, e^{-t\alpha(H)} X_\alpha]. \end{aligned}$$

Hence we have exponential decay if $\alpha \in \Sigma^+$ and exponential growth if $\alpha \in -\Sigma^+$. The general statement is obtained from the observation that $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta \perp \mathfrak{a}$ for $\alpha \neq \beta$ in Σ. □

3A. ***Lifted Weyl chamber action.*** In order to define horocycle operators we generalize the Weyl chamber action to associated vector bundles. Let (τ, V_τ) be a finite-dimensional unitary representation of M, that is, a complexification of an orthogonal representation. Then we have defined the associated vector bundle $\mathcal{V}_\tau = G \times_\tau V_\tau$ over G/M; see Section 2H.

The quotient bundle $\Gamma\backslash\mathcal{V}_\tau$ is the complexification of a Euclidean vector bundle over $\mathcal{M}$, where the Euclidean structure is induced by the inner product on V_τ. We identify smooth sections s of this bundle with smooth functions $\bar{s} : G \to V_\tau$ with $\bar{s}(\gamma g m) = \tau(m^{-1})\bar{s}(g)$ for all $\gamma \in \Gamma$, $g \in G$ and $m \in M$.

The canonical connection ∇ descends to a connection ${}_\Gamma\nabla : C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau) \to C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau \otimes T^*\mathcal{M})$ and we have

$$\overline{{}_\Gamma\nabla s(\mathfrak{X})}(g) := \overline{{}_\Gamma\nabla_{\mathfrak{X}} s}(g) = \frac{d}{dt}\Big|_{t=0} \bar{s}(g\exp(t\overline{\mathfrak{X}}(g))), \tag{5}$$

where s is a smooth section identified as above and $\mathfrak{X}$ is a vector field of $\mathcal{M}$ identified with a smooth function $\overline{\mathfrak{X}} : G \to \mathfrak{a}\oplus\mathfrak{n}\oplus\mathfrak{n}_-$ which is left Γ-invariant and right M-equivariant.

Definition 3.2. The *lifted Weyl chamber action* is defined as follows:

$${}_\Gamma\boldsymbol{X}^\tau : \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau), \quad {}_\Gamma\boldsymbol{X}^\tau_H := {}_\Gamma\nabla_{\mathfrak{X}_H},$$

where $\mathfrak{X}_H$ is the vector field identified with the constant mapping $G \to \mathfrak{a} \subseteq \mathfrak{a}\oplus\mathfrak{n}\oplus\mathfrak{n}_-$, and $g \mapsto H$.

The fact that ${}_\Gamma\nabla$ is a covariant derivative implies that ${}_\Gamma\boldsymbol{X}^\tau$ is an admissible lift of the Weyl chamber action in the sense of (1).

For later use we denote by $\boldsymbol{X}^\tau : \mathfrak{a} \to \mathrm{Diff}^1(G/M, \mathcal{V}_\tau)$ the corresponding action on G/M.

We can find a nontrivial tube domain in $\mathfrak{a}^*_{\mathbb{C}}$ which is independent of τ and contains all Ruelle–Taylor resonances for the lifted Weyl chamber action.

Proposition 3.3. *The set of Ruelle–Taylor resonances $\sigma_{\mathrm{RT}}({}_\Gamma\boldsymbol{X}^\tau)$ is contained in $\overline{{}_-\mathfrak{a}^*} + i\mathfrak{a}^*$.*

Proof. By Proposition 2.2 we have

$$\sigma_{\mathrm{RT}}({}_\Gamma\boldsymbol{X}^\tau) \subseteq \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \mathrm{Re}(\lambda(H)) \le C^\tau_{L^2}(H)\ \forall H \in \mathfrak{a}_+\}.$$

Hence it remains to show that $C^\tau_{L^2}(H) := \inf\{C > 0 \mid \|e^{-t\,{}_\Gamma\boldsymbol{X}^\tau_H}\|_{L^2\to L^2} \le e^{Ct}\ \forall t > 0\} = 0$ for all $H \in \mathfrak{a}_+$. We show the stronger statement that $e^{-t\,{}_\Gamma\boldsymbol{X}^\tau_H}$ is unitary.

Since M commutes with A, we have a well-defined action of A on $\Gamma\backslash\mathcal{V}_\tau$ given by $(\Gamma[g, v])a = \Gamma[ga, v]$. This action gives rise to an A-action on sections of the bundle $\Gamma\backslash\mathcal{V}_\tau$ defined via $(af)(x) = f(xa)a^{-1}$ with $f \in C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau)$, $x \in \mathcal{M}$ and $a \in A$. If we identify f with a equivariant function $\bar{f} : G \to V_\tau$, then $\overline{(af)}(g) = \bar{f}(ga)$. Let $d\Gamma g$ be the normalized right G-invariant Radon measure on $\Gamma\backslash G$. Then the L^2-norm of f is given by $\|f\|^2_{L^2} = \int_{\Gamma\backslash G} \|\bar{f}(g)\|^2_{V_\tau}\, d\Gamma g$, and it follows that the A-action continued to $L^2(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau)$ is unitary. By definition, $e^{-t\,{}_\Gamma\boldsymbol{X}^\tau_H} f = \exp(-tH) f$ for $f \in L^2(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau)$, and therefore $e^{-t\,{}_\Gamma\boldsymbol{X}^\tau_H}$ is unitary. ☐

3B. ***First band resonances and horocycle operators.*** In analogy to the rank 1 setting we make the following definition; see [Küster and Weich 2021, Definition 2.11] and [Guillarmou et al. 2021, Definition 3.1] in the scalar case.

Definition 3.4. We call $\lambda \in \sigma_{\mathrm{RT}}({}_\Gamma\boldsymbol{X}^\tau)$ a *first band resonance* and write $\lambda \in \sigma^0_{\mathrm{RT}}({}_\Gamma\boldsymbol{X}^\tau)$ if the vector space

$$\mathrm{Res}^0_{{}_\Gamma\boldsymbol{X}^\tau}(\lambda) = \{u \in \mathrm{Res}_{{}_\Gamma\boldsymbol{X}^\tau}(\lambda) \mid {}_\Gamma\nabla_{\mathfrak{X}} u = 0\ \forall \mathfrak{X} \in C^\infty(\mathcal{M}, E_u)\}$$

of *first band resonant states* is nontrivial.

The goal of this section is to prove that, in a certain neighborhood of 0 in $\mathfrak{a}^*_{\mathbb{C}}$, each Ruelle–Taylor resonance is a first band resonance and $\mathrm{Res}^0_{\Gamma X^\tau}(\lambda) = \mathrm{Res}_{\Gamma X^\tau}(\lambda)$. This will be done by introducing so-called horocycle operators as follows.

Recall that $T\mathcal{M} = \Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-)$ and the bundle $\Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{n})$ can be written as $\bigoplus_{\alpha\in\Sigma^+} \Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{g}_\alpha)$, and similarly for $\mathfrak{n}_-$. Therefore, the cotangent bundle $T^*\mathcal{M}$ is the Whitney sum $\Gamma\backslash(G \times_{\mathrm{Ad}^*|_M} \mathfrak{a}^*) \oplus \bigoplus_{\alpha\in\Sigma} \Gamma\backslash(G \times_{\mathrm{Ad}^*|_M} \mathfrak{g}^*_\alpha)$. Let us denote the coadjoint action of M on the complexification of $\mathfrak{g}^*_\alpha$ by τ_α. Note that τ_α is unitary with respect to the inner product induced by the Killing form and the Cartan involution. We can now define

$$\mathrm{pr}_\alpha : (T^*\mathcal{M})_{\mathbb{C}} \to \Gamma\backslash\mathcal{V}_{\tau_\alpha}$$

by fiber-wise restriction to the subbundle $\Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{g}_\alpha)$. This induces a map

$$\widetilde{\mathrm{pr}}_\alpha : C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau \otimes (T^*\mathcal{M})_{\mathbb{C}}) \to C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_{\tau\otimes\tau_\alpha}).$$

Definition 3.5. If ${}_\Gamma\nabla^{\mathbb{C}} : C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau) \to C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau \otimes (T^*\mathcal{M})_{\mathbb{C}})$ denotes the complexification of the canonical connection ${}_\Gamma\nabla$, then the *horocycle operator* $\mathcal{U}_\alpha$ for $\alpha \in \Sigma$ is defined as the composition

$$\mathcal{U}_\alpha := \widetilde{\mathrm{pr}}_\alpha \circ {}_\Gamma\nabla^{\mathbb{C}} : C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau) \to C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_{\tau\otimes\tau_\alpha}).$$

Note that we have the explicit formula

$$\overline{\mathcal{U}_\alpha s}(g)(Y) = \frac{d}{dt}\Big|_{t=0} \bar{s}(g\exp(tY)), \quad s \in C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau), \quad Y \in \mathfrak{g}_\alpha, \tag{6}$$

if we again use the identification of sections of some associated vector bundle with left Γ-invariant and right M-equivariant functions indicated by $\bar{\cdot}$ and the identification $V_\tau \otimes \mathfrak{g}^*_\alpha \simeq \mathrm{Hom}(\mathfrak{g}_\alpha, V_\tau)$.

We should point out that the space of first band resonant states can be rewritten with the horocycle operators as

$$\mathrm{Res}^0_{\Gamma X^\tau}(\lambda) = \{u \in \mathrm{Res}_{\Gamma X^\tau}(\lambda) \mid \mathcal{U}_{-\alpha} u = 0 \ \forall \alpha \in \Sigma^+\}. \tag{7}$$

Note that in the case of constant curvature manifolds (i.e., the real hyperbolic case $G = \mathrm{PSO}(n, 1)$ of rank 1) there is only one positive root and our definition reduces to the original one due to Dyatlov and Zworski; see [Dyatlov et al. 2015, p. 931]. Furthermore, our definition extends the definition of the horocycle operators for arbitrary G of rank 1; see [Küster and Weich 2021].

The horocycle operators fulfill the following important commutation relation.

Lemma 3.6. *For all $H \in \mathfrak{a}$,*

$${}_\Gamma X_H^{\tau\otimes\tau_\alpha}\mathcal{U}_\alpha - \mathcal{U}_\alpha {}_\Gamma X_H^\tau = \alpha(H)\mathcal{U}_\alpha.$$

Proof. Using the formulas (5) and (6) we obtain

$$\overline{{}_\Gamma X_H^{\tau\otimes\tau_\alpha}\mathcal{U}_\alpha - \mathcal{U}_\alpha {}_\Gamma X_H^\tau}(g)(Y) = \frac{d}{dt_1}\Big|_{t_1=0} \frac{d}{dt_2}\Big|_{t_2=0} \bar{s}(g\exp(t_1 H)\exp(t_2 Y)) - \bar{s}(g\exp(t_1 Y)\exp(t_2 H)),$$

and the latter equals

$$\frac{d}{dt}\Big|_{t=0} \bar{s}(g\exp(t[H, Y])).$$

Since $[H, Y] = \alpha(H)Y$ for $Y \in \mathfrak{g}_\alpha$, the claim follows. $\square$

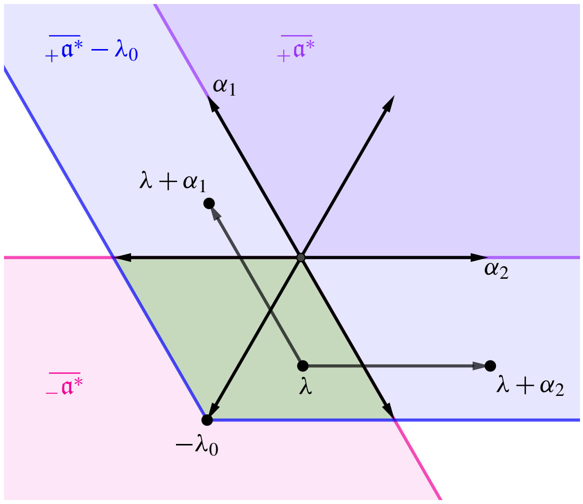

Figure 3. For $G = \mathrm{SL}_3(\mathbb{R})$ the green region depicts the real part of the region where every resonance is a first band resonance; see Proposition 3.7.

We can now prove the main result of this section.

Proposition 3.7. *The horocycle operators can be extended continuously as linear operators to distributional sections*, *i.e.*,

$$\mathcal{U}_\alpha : \mathcal{D}'(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau) \to \mathcal{D}'(\mathcal{M}, \Gamma\backslash\mathcal{V}_{\tau\otimes\tau_\alpha}).$$

Moreover, *for* $\lambda \in \sigma_{\mathrm{RT}}({}_\Gamma\boldsymbol{X}^\tau)$, *the horocycle operator* $\mathcal{U}_{-\alpha}$ *maps*

$$\mathrm{Res}_{{}_\Gamma\boldsymbol{X}^\tau}(\lambda) \quad \textit{into} \quad \mathrm{Res}_{{}_\Gamma\boldsymbol{X}^{\tau\otimes\tau_{-\alpha}}}(\lambda+\alpha).$$

In particular, *each* $\lambda \in \sigma_{\mathrm{RT}}({}_\Gamma\boldsymbol{X}^\tau)$ *with* $\mathrm{Re}\,\lambda \in \bigcap_{\alpha\in\Pi} \overline{{}_-\mathfrak{a}^*} \setminus (\overline{{}_-\mathfrak{a}^*} - \alpha)$ *is a first band resonance and* $\mathrm{Res}_{{}_\Gamma\boldsymbol{X}^\tau}(\lambda) = \mathrm{Res}^0_{{}_\Gamma\boldsymbol{X}^\tau}(\lambda)$ *holds.*

See Figure 3 for a visualization for $G = \mathrm{SL}_3(\mathbb{R})$.

Proof. Since the horocycle operators are differential operators, we obtain a continuation to distributional sections and Lemma 3.6 still holds. Let $u \in \mathrm{Res}_{{}_\Gamma\boldsymbol{X}^\tau}(\lambda)$, i.e., $u \in \mathcal{D}'(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau)$ with $\mathrm{WF}(u) \subseteq E_u^*$ and ${}_\Gamma\boldsymbol{X}_H^\tau u = -\lambda(H)u$. Since differential operators do not increase the wavefront set, we have $\mathrm{WF}(\mathcal{U}_{-\alpha}u) \subseteq E_u^*$. Furthermore,

$${}_\Gamma\boldsymbol{X}_H^{\tau\otimes\tau_{-\alpha}}\mathcal{U}_{-\alpha}u = -\alpha(H)\mathcal{U}_{-\alpha}u + \mathcal{U}_{-\alpha}{}_\Gamma\boldsymbol{X}_H^\tau u = -(\lambda+\alpha)(H)\mathcal{U}_{-\alpha}u$$

by Lemma 3.6. Hence $\mathcal{U}_{-\alpha}u \in \mathrm{Res}_{{}_\Gamma\boldsymbol{X}^{\tau\otimes\tau_{-\alpha}}}(\lambda+\alpha)$.

For the "in particular" part recall that $\mathrm{Res}_{{}_\Gamma\boldsymbol{X}^{\tau'}}(\lambda') = 0$ for each unitary representation τ' of M and $\mathrm{Re}(\lambda') \notin \overline{{}_-\mathfrak{a}^*}$ (see Proposition 3.3), and $\mathrm{Res}^0_{{}_\Gamma\boldsymbol{X}^\tau}(\lambda) = \{u \in \mathrm{Res}_{{}_\Gamma\boldsymbol{X}^\tau}(\lambda) \mid \mathcal{U}_{-\alpha}u = 0\ \forall\alpha\in\Sigma^+\}$. □

Note that $\bigcap_{\alpha\in\Pi} \overline{{}_-\mathfrak{a}^*} \setminus (\overline{{}_-\mathfrak{a}^*} - \alpha) = \overline{{}_-\mathfrak{a}^*} \cap ({}_+\mathfrak{a}^* - \lambda_0)$, where $\lambda_0 = \sum_{\alpha\in\Pi}\alpha$. Indeed, let $\lambda = \sum_{\alpha\in\Pi} c_\alpha\alpha \in \mathfrak{a}^*$. Then $\lambda \in \overline{{}_-\mathfrak{a}^*}$ if and only if $c_\alpha \le 0$ for all $\alpha \in \Pi$, $\lambda \in \overline{{}_-\mathfrak{a}^*} - \alpha$ if and only if $c_\alpha \le -1$ and $c_\beta \le 0$ for all $\beta \in \Pi\setminus\{\alpha\}$, and $\lambda \in {}_+\mathfrak{a}^*$ if and only if $c_\alpha > 0$ for all $\alpha \in \Pi$. Combining these statements implies the claim.

3C. ***First band resonant states and principal series representation.*** In this section we identify first band resonant states with certain Γ-invariant vectors in a corresponding principal series representation. The proof follows the line of arguments given in [Küster and Weich 2021, Section 2] in the rank 1 case. This will allow us to apply the Poisson transform and obtain a quantum-classical correspondence.

By analogy to [Küster and Weich 2021, Definition 2.1], we define

$$\mathcal{R}(\lambda) := \{s \in \mathcal{D}'(G/M, \mathcal{V}_\tau) \mid (X_H^\tau + \lambda(H))s = 0,\ \nabla_{\mathfrak{X}_-} s = 0\ \forall \mathfrak{X}_- \in C^\infty(G/M, G \times_{\mathrm{Ad}|_M} \mathfrak{n}_-),\ H \in \mathfrak{a}\}.$$

The following lemma allows us to first study the representation of G in $\mathcal{R}(\lambda)$ and take Γ-invariants afterwards.

Lemma 3.8. *The space* $\mathrm{Res}^0_{\Gamma X^\tau}(\lambda)$ *is isomorphic to the space of* Γ*-invariants of* $\mathcal{R}(\lambda)$*, where the isomorphism is defined by considering* Γ*-invariant sections as sections of the bundle* $\Gamma\backslash\mathcal{V}_\tau$*.*

Proof. The only part to observe is that each $s \in \mathcal{R}(\lambda)$ automatically has $\mathrm{WF}(s) \subseteq G \times_{\mathrm{Ad}|_M} \mathfrak{n}^*$. This holds because $G \times_{\mathrm{Ad}^*|_M} \mathfrak{n}^*$ is the joint characteristic set of X_H^τ and $\mathfrak{X}_-$; see [Küster and Weich 2021, Lemma 2.5] for details. □

We will now show that the smooth sections in $\mathcal{R}(\lambda)$ correspond to smooth vectors in the principal series representation for the opposite Iwasawa decomposition.

Lemma 3.9. *The smooth sections* $\mathcal{R}(\lambda) \cap C^\infty(G/M, \mathcal{V}_\tau)$ *in* $\mathcal{R}(\lambda)$ *can be identified* G*-equivariantly with*

$$W = \{\bar{s} : G \to V_\tau \text{ smooth} \mid \bar{s}(gman_-) = e^{-\lambda \log a}\tau(m)^{-1}\bar{s}(g),\ m \in M,\ a \in A,\ n_- \in N_-\}.$$

The identification is obtained by considering sections $s \in \mathcal{R}(\lambda)$ *as right* M*-equivariant functions* $\bar{s} : G \to V_\tau$*.*

Proof. The M-equivariance is clear so it remains to show the transformation properties under A and N_-. The property $(X_H^\tau + \lambda(H))s = 0$ amounts to $(d/dt)|_{t=0}\bar{s}(ge^{tH}) = -\lambda(H)\bar{s}(g)$ for every $g \in G$ and $H \in \mathfrak{a}$. Hence the function $\varphi(t) = \bar{s}(ge^{tH})$ satisfies

$$\varphi'(r) = \frac{d}{dt}\Big|_{t=0}\varphi(ge^{rH}e^{tH}) = -\lambda(H)\bar{s}(ge^{rH}) = -\lambda(H)\varphi(r).$$

Therefore, $\bar{s}(ge^{tH}) = \varphi(t) = e^{-t\lambda(H)}\bar{s}(g)$. This proves the right A-equivariance.

For the N_--invariance, let $Y \in \mathfrak{n}_-$ and consider $\varphi(t) = \bar{s}(ge^{tY})$. For $r \in \mathbb{R}$, let $g_r = ge^{rY} \in G$. Since $[g_r, Y] \in G \times_{\mathrm{Ad}|_M} \mathfrak{n}_-$ is in the fiber over $g_r M \in G/M$, there is a smooth section

$$\mathfrak{X}_r \in C^\infty(G/M, G \times_{\mathrm{Ad}|_M} \mathfrak{n}_-)$$

such that $\mathfrak{X}_r(g_r M) = [g_r, Y]$. In particular, the corresponding right M-equivariant function $\overline{\mathfrak{X}}_r : G \to \mathfrak{n}_-$ satisfies $\overline{\mathfrak{X}}_r(g_r) = Y$. It follows that

$$0 = \overline{\nabla_{\mathfrak{X}_r} s}(g_r) = \frac{d}{dt}\Big|_{t=0}\bar{s}(g_r e^{t\overline{\mathfrak{X}}_r(g_r)}) = \frac{d}{dt}\Big|_{t=0}\bar{s}(ge^{rY}e^{tY}) = \varphi'(r).$$

Hence φ is constant. This completes the proof. □

Note that the space W from Lemma 3.9 is already very close to the definition of the induced picture of the principal series representation (see Section 2C). The only difference is that in W we have a right invariance with respect to N_- instead of N. This can be easily fixed using a conjugation with the longest Weyl group element and leads to the main result of this section:

Proposition 3.10. *With the longest Weyl group element w_0 (see Section 2B), we have an isomorphism*

$$\mathrm{Res}^0_{\Gamma X^\tau}(\lambda) \to {}^\Gamma(H^{w_0\tau, w_0(\lambda+\rho)})^{-\infty},$$

where ${}^\Gamma(H^{w_0\tau, w_0(\lambda+\rho)})^{-\infty}$ denotes the Γ-invariant distributional vectors in the principal series representation $\pi_{w_0\tau, w_0(\lambda+\rho)}$.

Proof. Pick $k_0 \in K$ normalizing $\mathfrak{a}$ such that the action of $\mathrm{Ad}(k_0)$ on $\mathfrak{a}$ is the longest Weyl group element w_0. We consider the map $I\bar{s}(g) := \bar{s}(gk_0)$. Then I commutes with the left action by G and one calculates that

$$I\bar{s}(gman) = e^{-(w_0\lambda)\log a}(w_0\tau)(m)^{-1} I\bar{s}(g), \quad g \in G,\ m \in M,\ a \in A,\ n \in N.$$

Hence we have an intertwiner between W and smooth vectors in $H^{w_0\tau, w_0(\lambda+\rho)}$ which extends to distributional sections. By Lemma 3.9 we conclude that $\mathcal{R}(\lambda) \simeq (H^{w_0\tau, w_0(\lambda+\rho)})^{-\infty}$ as G-representations. Taking Γ-invariants and using Lemma 3.8 completes the proof. □

3D. ***Quantum-classical correspondence.*** In the previous section we identified the first band resonant states $\mathrm{Res}^0_{\Gamma X^\tau}(\lambda)$ with Γ-invariant distributional vectors in the principal series $(H^{w_0\tau, w_0(\lambda+\rho)})^{-\infty}$. If we restrict ourselves to the scalar case $\tau = 1$, then the Poisson transform $\mathcal{P}_{-w_0(\lambda+\rho)}$ defines a map from ${}^\Gamma(H^{w_0(\lambda+\rho)})^{-\infty}$ to ${}^\Gamma E_{-w_0(\lambda+\rho)}$, as $\mathcal{P}_{-w_0(\lambda+\rho)}$ provides a G-equivariant map from $(H^{w_0(\lambda+\rho)})^{-\infty}$ to $E_{-w_0(\lambda+\rho)}$ (see Section 2E). Hence we can identify eigendistributions of the classical motion with quantum states and we call this identification *quantum-classical correspondence*. More precisely, we have the following result, which immediately gives Theorem 1.3.

Proposition 3.11. *If $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ satisfies $2\langle\lambda+\rho, \alpha\rangle/\langle\alpha, \alpha\rangle \notin -\mathbb{N}_{>0}$ for all $\alpha \in \Sigma^+$, then we have a bijection*

$$\mathrm{Res}^0_{\Gamma X}(\lambda) \to {}^\Gamma E_{-w_0(\lambda+\rho)} = {}^\Gamma E_{-(\lambda+\rho)}.$$

In particular, $\lambda \in \sigma^0_{\mathrm{RT}}({}_\Gamma X)$ if and only if ${}^\Gamma E_{-(\lambda+\rho)} \neq 0$. Furthermore, the isomorphism is given by the push-forward π_* of distributions along the canonical projection ${}_\Gamma\pi : \Gamma\backslash G/M \to \Gamma\backslash G/K$.

Proof. In view of Section 2E, the Poisson transform is a bijection from $(H^{w_0(\lambda+\rho)})^{-\infty} \to E^*_{-\lambda-\rho}$. Restricted to Γ-invariant distributional vectors it is still injective with image ${}^\Gamma E_{-\lambda-\rho}$ since Γ is cocompact, and therefore ${}^\Gamma E_{-\lambda-\rho} = {}^\Gamma E^*_{-\lambda-\rho}$.

It remains to show that the isomorphism is the push-forward along the canonical projection. To this end let $s \in \mathcal{R}(\lambda)$ be smooth and let $\pi : G/M \to G/K$ be the canonical projection. Then the isomorphism $\mathcal{R}(\lambda) \to (H^{w_0(\lambda+\rho)})^{-\infty}$ carries s to $\tilde{s} : G \to \mathbb{C}$ with $\tilde{s}(g) = s(gk_0)$, where $k_0 \in K$ is as in the proof of Proposition 3.10. It follows that

$$\mathcal{P}_{-w_0(\lambda+\rho)}\tilde{s}(gK) = \int_K \tilde{s}(gk)\, dk = \int_K s(gkk_0)\, dk = \int_K s(gk)\, dk$$

since K is unimodular. On the other hand, for $f \in C_c^\infty(G/K)$, we have

$$(\pi_* s)(f) = s(f \circ \pi) = \int_{G/M} s(gM) f(gK)\, dgM = \int_{G/K} \left(\int_{K/M} s(gkM)\, dkM \right) f(gK)\, dgK$$

if we normalize the Haar measure on M and choose compatible invariant measures on G/K and K/M. Hence $\pi_* s = \mathcal{P}_{-w_0(\lambda+\rho)} \tilde{s}$ for $s \in \mathcal{R}(\lambda) \cap C^\infty(G/M)$. Using the density of smooth compactly supported functions in $\mathcal{R}(\lambda)$ [Küster and Weich 2021, Corollary 2.9] we obtain the equality for the whole space $\mathcal{R}(\lambda)$. As before we now restrict to Γ-invariant distributions identified with distributions on $\Gamma\backslash G/M$ and $\Gamma\backslash G/K$ to complete the proof. □

4. Quantum spectrum

In this section we analyze the quantum spectrum of the locally symmetric space $\Gamma\backslash G/K$. Recall the definition of the joint eigenspace

$$E_\lambda = \{ f \in C^\infty(G/K) \mid Df = \chi_\lambda(D) f \ \forall D \in \mathbb{D}(G/K) \}$$

for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. For the definition of χ_λ see Section 2B. Since $D \in \mathbb{D}(G/K)$ is G-invariant, it descends to a differential operator ${}_\Gamma D$ on the locally symmetric space $\Gamma\backslash G/K$. Therefore, the left Γ-invariant functions of E_λ (denoted by ${}^\Gamma E_\lambda$) can be identified with joint eigenfunctions on $\Gamma\backslash G/K$ for each ${}_\Gamma D$:

$${}^\Gamma E_\lambda = \{ f \in C^\infty(\Gamma\backslash G/K) \mid {}_\Gamma D f = \chi_\lambda(D) f \ \forall D \in \mathbb{D}(G/K) \}.$$

This leads to the following definition.

Definition 4.1. The *quantum spectrum* of $\Gamma\backslash G/K$ is defined as

$$\sigma_Q := \sigma_Q(\Gamma\backslash G/K) := \{ \lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid {}^\Gamma E_\lambda \neq 0 \}.$$

We now use the quantum-classical correspondence and the Weyl law from [Duistermaat et al. 1979].

Proof of Theorem 1.1. From [Duistermaat et al. 1979, Theorem 8.9] we have, for each set $\Omega \subset \mathfrak{a}^*$ as in Theorem 1.1,

$$\sum_{\lambda \in \sigma_Q \cap i\mathfrak{a}^*,\ \operatorname{Im}\lambda \in t\Omega} \dim({}^\Gamma E_\lambda) |W\lambda|^{-1} = \operatorname{Vol}(\Gamma\backslash G/K)(2\pi)^{-d} \operatorname{Vol}(\operatorname{Ad}(K)\Omega) t^d + \mathcal{O}(t^{d-1}),$$

where $\operatorname{Vol}(\Gamma\backslash G/K)$ is the volume of the compact Riemannian manifold $\Gamma\backslash G/K$ with Riemannian structure induced by the Killing form and $\operatorname{Vol}(\operatorname{Ad}(K)\Omega)$ is the volume of the set $\operatorname{Ad}(K)\Omega \subseteq \operatorname{Ad}(K)\mathfrak{a}$ with respect to the Killing form restricted to $\operatorname{Ad}(K)\mathfrak{a}$. Replacing Ω by $\Omega \setminus \bigcup_{\alpha \in \Sigma^+} \alpha^\perp$ we deduce that

$$\sum_{\lambda \in \sigma_Q \cap i\mathfrak{a}^*,\ \operatorname{Im}\lambda \in t\Omega \cap \bigcup \alpha^\perp} \dim({}^\Gamma E_\lambda) = \mathcal{O}(t^{d-1})$$

since $\operatorname{Vol}(\operatorname{Ad}(K)\alpha^\perp) = 0$. Therefore,

$$\sum_{\lambda \in \sigma_Q \cap i\mathfrak{a}^*,\ \operatorname{Im}\lambda \in t\Omega} \dim({}^\Gamma E_\lambda) = |W| \operatorname{Vol}(\Gamma\backslash G/K)(2\pi)^{-d} \operatorname{Vol}(\operatorname{Ad}(K)\Omega) t^d + \mathcal{O}(t^{d-1})$$

since W acts freely on the Weyl chambers. To complete the proof we observe that $\sigma_{\mathrm{RT}}({}_\Gamma\boldsymbol{X}) \supseteq \sigma^0_{\mathrm{RT}}({}_\Gamma\boldsymbol{X})$ and $m(\lambda) \geq \dim(\mathrm{Res}^0_{{}_\Gamma\boldsymbol{X}}(\lambda)) = \dim({}^\Gamma E_{-\lambda-\rho})$ for $\lambda \in i\mathfrak{a}^*$. □

As $\chi_\lambda = \chi_{w\lambda}$ for $w \in W$ it is obvious that σ_Q is W-invariant. The following properties of σ_Q were derived by Duistermaat, Kolk and Varadarajan [Duistermaat et al. 1979]. We include the proof for the convenience of the reader.

Proposition 4.2 (see [Duistermaat et al. 1979, Propositions 2.4 and 3.4, Corollary 3.5]). *If $\lambda \in \sigma_Q$, then the corresponding spherical function ϕ_λ is positive semidefinite. Moreover, there is some $w \in W$ such that $w\lambda = -\bar{\lambda}$ and $\operatorname{Re}\lambda \in \operatorname{conv}(W\rho)$. In particular, $\langle \operatorname{Re}\lambda, \operatorname{Im}\lambda\rangle = 0$ and $\|\operatorname{Re}\lambda\| \leq \|\rho\|$.*

Proof. Pick $u \in {}^\Gamma E_\lambda$, regarded as a right K-invariant element of $L^2(\Gamma\backslash G)$, normalized such that $\langle u, u\rangle_{L^2(\Gamma\backslash G)} = 1$. With the right regular representation R on $L^2(\Gamma\backslash G)$, define $\Phi(g) := \langle R(g)u, u\rangle$. Being a matrix coefficient the function Φ is positive semidefinite. We will show that Φ is the elementary spherical function ϕ_λ. By right K-invariance of u and unitarity of R we get that Φ is K-biinvariant. $\Phi(1) = 1$ is obvious. Smoothness follows from the fact that u is smooth. Furthermore,

$$D\Phi(g) = \langle R(g)Du, u\rangle = \chi_\lambda(D)\Phi(g)$$

by left invariance of D. We conclude that Φ is the elementary spherical function for χ_λ, i.e., $\Phi = \phi_\lambda$.

Since ϕ_λ is positive semidefinite we have $\phi_\lambda(g) = \overline{\phi_\lambda(g^{-1})}$ by definition of positive definiteness, and $\overline{\phi_\lambda(g^{-1})} = \phi_{-\bar{\lambda}}(g)$ by the integral representation (see Section 2G). Therefore, $\phi_\lambda = \phi_{-\bar{\lambda}}$, implying that $w\lambda = -\bar{\lambda}$ for some $w \in W$. It easily follows that

$$\langle \operatorname{Re}\lambda, \operatorname{Im}\lambda\rangle = \langle w\operatorname{Re}\lambda, w\operatorname{Im}\lambda\rangle = \langle -\operatorname{Re}\lambda, \operatorname{Im}\lambda\rangle = 0.$$

Moreover, ϕ_λ is bounded which holds if and only if $\operatorname{Re}\lambda \in \operatorname{conv}(W\rho)$; see [Helgason 1984, Chapter IV Theorem 8.1]. Since $\{\mu \in \mathfrak{a}^* \mid \|\mu\| \leq \|\rho\|\}$ is convex and contains $W\rho$, the last assertion follows. □

Remark 4.3. In the rank 1 case Proposition 4.2 implies, for $\lambda \in \sigma_Q$, that $\lambda \in \mathfrak{a}^*$ with $\|\lambda\| \leq \|\rho\|$ or that $\lambda \in i\mathfrak{a}^*$. In this particular case, this can be obtained not only from Proposition 4.2 but also from the positivity of the Laplacian on $\Gamma\backslash G/K$. In the higher rank setting the algebra $\mathbb{D}(G/K)$ contains more operators; more precisely it is a polynomial algebra in n variables. Using the properties of the Harish-Chandra isomorphism HC one can obtain that $-\bar{\lambda} \in W\lambda$ from the self/skew-adjointness of the operators in $\mathbb{D}(G/K)$.

Remark 4.4. Proposition 4.2 implies the following obstructions for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ to be in σ_Q.

(1) If $\operatorname{Re}\lambda = 0$, then we get no obstructions on $\operatorname{Im}\lambda$ since $w\lambda = -\bar{\lambda}$ is satisfied with $w = 1$.

(2) If $\operatorname{Re}\lambda \neq 0$, then $\operatorname{Im}\lambda$ is singular, i.e., $\operatorname{Im}\lambda \in \alpha^\perp$ for some $\alpha \in \Sigma$, since $\operatorname{Im}\lambda$ nonsingular implies $w = 1$ as W acts simply transitively on open Weyl chambers.

(3) If $\operatorname{Re}\lambda$ is regular, i.e., $\langle \operatorname{Re}\lambda, \alpha\rangle \neq 0$ for all $\alpha \in \Sigma$, we denote by $\tilde{w}_0$ the unique Weyl group element mapping the Weyl chamber containing $\operatorname{Re}\lambda$ to its negative. Then $\lambda \in \mathrm{Eig}_{-1}(\tilde{w}_0) + i\,\mathrm{Eig}_{+1}(\tilde{w}_0) \subseteq \mathfrak{a}^*_{\mathbb{C}}$, where $\mathrm{Eig}_{\pm1}$ denotes the eigenspace for ±1. If -1 is contained in W, then $\operatorname{Im}\lambda = 0$. In particular, this is true in the rank 1 case but need not hold in general as is seen below.

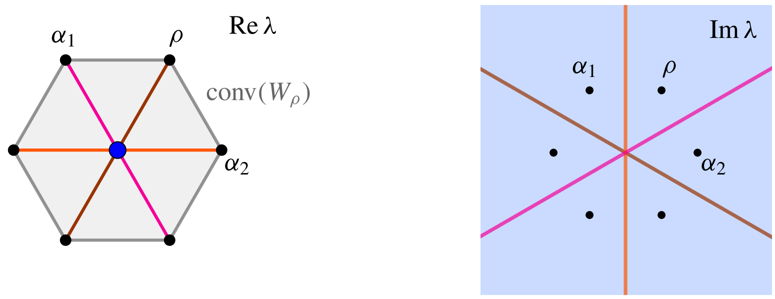

Figure 4. The situation for $\mathrm{SL}_3(\mathbb{R})$ as obtained from Remark 4.4: if $\lambda \in \sigma_Q$ then $\operatorname{Re}\lambda$ is either equal to zero (blue dot in the left picture) or lies on one of the pink, orange or brown lines depicted on the left. Furthermore, $\operatorname{Im}\lambda$ has to lie in the respective region depicted on the right, i.e., if $\operatorname{Re}\lambda = 0$, then $\operatorname{Im}\lambda$ can take any value (blue shaded plane); if $\operatorname{Re}\lambda$ lies on the orange line, then $\operatorname{Im}\lambda$ has to lie on the orange line; and so on.

Let us calculate $\dim \operatorname{Eig}_{+1}(w_0) = \dim \operatorname{Eig}_{+1}(\tilde{w}_0)$ in order to control the amount of freedom for $\operatorname{Im}\lambda$. Let $d_\pm := \dim \operatorname{Eig}_{\pm 1}(w_0)$. Then $n = d_+ + d_-$ and $\operatorname{Tr}(w_0) = d_+ - d_-$. Choosing the basis Π we observe $\operatorname{Tr}(w_0) = -\#\{\alpha \in \Pi \mid w_0\alpha = -\alpha\} \leq 0$. Thus, $d_\pm = \frac{1}{2}(n \pm \operatorname{Tr}(w_0))$, so that $d_+ \leq \frac{1}{2}n$. We obtain the following traces and dimensions for the irreducible root systems from the classification:

type	A_n, n even	A_n, n odd	B_n	C_n	D_n, n even	D_n, n odd	E_6	E_7	E_8	F_4	G_2
$-\operatorname{Tr}(w_0)$	0	1	n	n	n	$n-2$	2	7	8	4	2
d_+	$\frac{1}{2}n$	$\frac{1}{2}(n-1)$	0	0	0	1	2	0	0	0	0

Example 4.5. For $G = \mathrm{SL}_n(\mathbb{R})$, an element $\lambda \in \mathfrak{a}^* \simeq \mathfrak{a}$ is regular if and only if the diagonal entries are pairwise distinct. An element $\lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \in \sigma_Q$ with $\operatorname{Re}\lambda \in \overline{\mathfrak{a}_+^*}$ satisfies $\operatorname{Re}\lambda_k = -\operatorname{Re}\lambda_{n+1-k}$ and $\operatorname{Im}\lambda_k = \operatorname{Im}\lambda_{n+1-k}$ for all $k = 1, \dots, n$ since the longest Weyl group element is the permutation $(1 \leftrightarrow n)(2 \leftrightarrow n-1)\cdots$.

More specifically, for $G = \mathrm{SL}_3(\mathbb{R})$, the only Weyl group elements with eigenvalue equal to -1 are the reflections at hyperplanes perpendicular to the roots. Hence $\lambda \in \sigma_Q$ implies $\operatorname{Re}\lambda \in [-1, 1]\alpha$ and $\operatorname{Im}\lambda \in \alpha^\perp$ for some $\alpha \in \Sigma$ or $\lambda \in i\mathfrak{a}^*$. The obstructions for λ to be in σ_Q described by Remark 4.4 are less concrete and are visualized in Figure 4.

Let us formulate the condition that ϕ_λ is positive semidefinite in a different way.

Proposition 4.6. *The elementary spherical function ϕ_λ is positive semidefinite if and only if the subrepresentation generated by the K-invariant vector in the principal series representation $H^{w\lambda}$ is unitarizable and irreducible for some $w \in W$. Equivalently, $H^{-w\bar{\lambda}}$ has a unitarizable irreducible spherical quotient.*

Proof. By Casselman's embedding theorem, π_{ϕ_λ} is a subrepresentation of $H^{\tau,\nu}$ for some $\tau \in \widehat{M}$ and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$; see, e.g., [Knapp 1986, Theorem 8.37]. More precisely, the $(\mathfrak{g}, K)$-module of K-finite vectors are equivalent. Since the only principal series representations containing K-invariant vectors are the spherical ones, we obtain $\tau = 1$. Since infinitesimally equivalent admissible representations of G have the same set of K-finite matrix coefficients (see [Knapp 1986, Corollary 8.8]), we conclude $\phi_\lambda = \phi_\nu$, i.e., $w\lambda = \nu$.

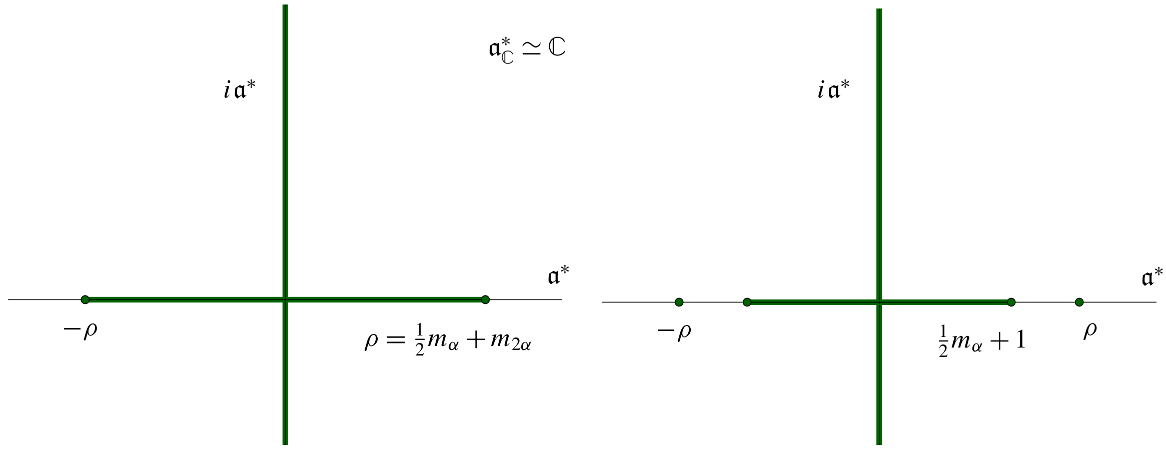

Figure 5. Spherical dual in the rank 1 case. The picture on the left describes the real and complex hyperbolic case $m_{2\alpha} \leq 1$. The picture on the right describes the quaternionic case $m_{2\alpha} \geq 2$. In the latter case note that there is a spectral gap separating ρ.

Conversely assume that the subrepresentation generated by the K-invariant vector in the principal series representation $H^{w\lambda}$ is unitarizable and irreducible. Again by the aforementioned result the matrix coefficient $\phi_{w\lambda} = \phi_\lambda$ of $H^{w\lambda}$ is a matrix coefficient of the unitary representation obtained by the unitary structure as well. Hence ϕ_λ is positive semidefinite. Transition to the dual representation implies the second equivalence. $\square$

Remark 4.7. Although the unitary dual is classified for many groups, it is difficult to deduce which elementary spherical functions are positive semidefinite. This is due to the fact that most classifications are not obtained in terms of quotients of the spherical principal series but use different descriptions of admissible representations. However, for rank 1 groups everything is classified (see [Helgason 1984, p. 484]): if α denotes the unique reduced root in Σ^+, then ϕ_λ is positive semidefinite if and only if $\lambda \in i\mathfrak{a}^*$ or $\lambda \in \mathfrak{a}^*$ and $|\langle \lambda, \alpha \rangle| \leq \langle \rho, \alpha \rangle$ for $2\alpha \notin \Sigma$ (i.e., in the real hyperbolic case) and $|\langle \lambda, \alpha \rangle| \leq \left(\frac{1}{2}m_\alpha + 1\right)\langle \alpha, \alpha \rangle$ for $2\alpha \in \Sigma$ or $\lambda = \pm\rho$. See Figure 5 for a visualization.

4A. ***Property (T).*** In this section we review some facts about Kazhdan's property (T) which will lead to a more precise description of the location of σ_Q. Recall that a locally compact group has *property* (T) if and only if the trivial representation is an isolated point in the unitary dual of the group with respect to the Fell topology; see [Bekka et al. 2008] for a general reference. It is well known that each real simple Lie group of real rank ≥ 2 has property (T); see [Bekka et al. 2008, Theorem 1.6.1]. Since the mapping $\lambda \mapsto \phi_\lambda$ is continuous and the correspondence between positive semidefinite elementary spherical functions and irreducible unitary spherical representations is a homeomorphism (see Section 2G), we obtain that in some neighborhood of ρ no elementary spherical function is positive semidefinite. We will use a more quantitative description introduced by Oh [2002, Section 7.1]. Therefore, we denote by $p_K(G)$ the smallest real number such that the K-finite matrix coefficients of π are in $L^q(G)$ for any $q > p_K(G)$ and nontrivial $\pi \in \widehat{G}$.

Remark 4.8. (1) Since each matrix coefficient of $\pi \in \widehat{G}$ is bounded, it is contained in L^q for each $q > p$ if it is in L^p. Hence

$$p_K(G) = \inf\{p \mid \text{all } K\text{-finite matrix coefficients of } \pi \text{ are in } L^p(G)\ \forall \pi \in \widehat{G} \setminus \{1\}\}.$$

(2) $p_K(G) \geq 2$.

(3) By [Cowling 1979] together with [Oh 2002] we have $p_K(G) < \infty$ if and only if G has property (T).

In many examples one knows the number $p_K(G)$ explicitly or at least its upper bounds.

Example 4.9 (see [Oh 2002, Section 7]). (1) $p_K(\mathrm{SL}_n(k)) = 2(n-1)$ for $n \geq 3$ and $k = \mathbb{R}, \mathbb{C}$.

(2) $p_K(Sp_{2n}(\mathbb{R})) = 2n$ for $n \geq 2$.

(3) $p_K(G)$ is bounded above by an explicit value for split classical groups of higher rank.

We can now prove the following theorems.

Theorem 4.10. *Let G be a noncompact real semisimple Lie group with finite center and $\Gamma \leq G$ be a discrete, cocompact, torsion-free subgroup. Then*

$$\operatorname{Re} \sigma_Q(\Gamma \backslash G/K) \subseteq (1 - 2p_K(G)^{-1}) \operatorname{conv}(W\rho) \cup W\rho.$$

Proof. Let $\lambda \in \sigma_Q(\Gamma \backslash G/K)$. By Proposition 4.2, ϕ_λ is positive semidefinite so that the irreducible unitary representation π_{ϕ_λ} is defined (see Section 2G), and ϕ_λ is a matrix coefficient of this representation. By the definition of $p_K(G)$ we have $\phi_\lambda \in L^{p_K(G)+\epsilon}(G)$ for all $\epsilon > 0$ or π_{ϕ_λ} is the trivial representation. By Proposition 2.4 we get $\operatorname{Re}\lambda \in (1 - 2p_K(G)^{-1}) \operatorname{conv}(W\rho)$ in the first case. The latter case occurs if and only if $\phi_\lambda \equiv 1$, i.e., $\lambda \in W\rho$. ☐

Theorem 4.11. *Let G be a noncompact real semisimple Lie group with finite center and $\Gamma \leq G$ be a discrete, cocompact, torsion-free subgroup. Then there is a neighborhood $\mathcal{G}$ of ρ in $\mathfrak{a}^*$ such that*

$$\sigma_Q(\Gamma \backslash G/K) \cap (\mathcal{G} \times i\mathfrak{a}^*) = \{\rho\}.$$

Proof. Without loss of generality we assume that G has trivial center, otherwise replace G by $G/Z(G)$. Then G is a product of simple Lie groups $G_1, \dots, G_l$ such that $G_1, \dots, G_k$, $k \leq l$, are of rank 1. With the obvious notation let $\lambda = (\lambda_1, \dots, \lambda_l) \in (\mathfrak{a}_1)^*_{\mathbb{C}} \oplus \dots \oplus (\mathfrak{a}_l)^*_{\mathbb{C}}$ be in σ_Q. By Proposition 4.2 we have $w\lambda = -\bar{\lambda}$ for some $w \in W$. Since the Weyl group W is the product of the Weyl groups, $\lambda_i \in \mathfrak{a}_i^*$ are real for $i \leq k$ if $\operatorname{Re}\lambda_i \neq 0$. The elementary spherical function ϕ_λ is the product of elementary spherical functions $\phi^{G_i}_{\lambda_i}$ for the factors G_i. Again by Proposition 4.2 we know that ϕ_λ is positive semidefinite and therefore each $\phi^{G_i}_{\lambda_i}$ is positive semidefinite. The same line of arguments as in the proof of Theorem 4.10 implies that $\operatorname{Re}\lambda_i \in (1 - 2p_K(G_i)^{-1}) \operatorname{conv}(W_i\rho_i) \cup W_i\rho_i$ for $i > k$. Since the G_i, $i > k$, have property (T), we conclude that there is a neighborhood U of ρ in $\mathfrak{a}^*$ such that

$$\sigma_Q \cap (U \times i\mathfrak{a}^*) \subseteq \mathfrak{a}_1^* \times \dots \times \mathfrak{a}_k^* \times \{\rho_{k+1}\} \times \dots \times \{\rho_l\}.$$

Discreteness of σ_Q implies the theorem. ☐

5. Main Theorem

In this section we present the main theorem of the article and deduce Theorem 1.2 from it. See Figure 6 for a visualization for $G = \mathrm{SL}_3(\mathbb{R})$.

Theorem 5.1. *Let G be a noncompact real semisimple Lie group with finite center and $\Gamma \leq G$ be a discrete, cocompact, torsion-free subgroup. Define*

$$\mathcal{A} := \left\{ \lambda \in \mathfrak{a}^*_{\mathbb{C}} \;\middle|\; \frac{2\langle \lambda + \rho, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in -\mathbb{N}_{>0} \text{ for some } \alpha \in \Sigma^+ \right\},$$
$$\mathcal{B} := \{ \lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid w\lambda = -\bar{\lambda} \text{ for some } w \in W \},$$
$$\mathcal{F} := \{ \lambda \in \mathfrak{a}^* \mid \lambda + \alpha \notin \overline{{}_{-}\mathfrak{a}^*} \text{ for all } \alpha \in \Pi \}.$$

Then we have the inclusions

$$\sigma_{\mathrm{RT}}(_{\Gamma}\boldsymbol{X}) \cap (\mathcal{F} \times i\mathfrak{a}^*) \subseteq \sigma^0_{\mathrm{RT}}(_{\Gamma}\boldsymbol{X})$$

and

$$\sigma^0_{\mathrm{RT}}(_{\Gamma}\boldsymbol{X}) \cap (\mathfrak{a}^*_{\mathbb{C}} \setminus \mathcal{A}) \subseteq -\sigma_Q(\Gamma\backslash G/K) - \rho \subseteq \mathcal{B} \cap (((1 - 2p_K(G)^{-1}) \operatorname{conv}(W\rho) \cup W\rho) + i\mathfrak{a}^*) - \rho.$$

Proof. This is immediate from Propositions 3.7, 3.11 and 4.2, and Theorem 4.10. □

Proof of Theorem 1.2. It follows from Theorem 5.1 that $(\mathfrak{a}^*_+ - \rho) \cap \mathcal{F} \cap (-\mathcal{G} - \rho)$ can be chosen as the neighborhood, where $\mathcal{G}$ is obtained by Theorem 4.11. If G has property (T), then $p_K(G)$ is finite and $\mathcal{G}$ can be replaced by the complement of the Γ-independent set $(1 - 2p_K(G)^{-1}) \operatorname{conv}(W\rho)$. □

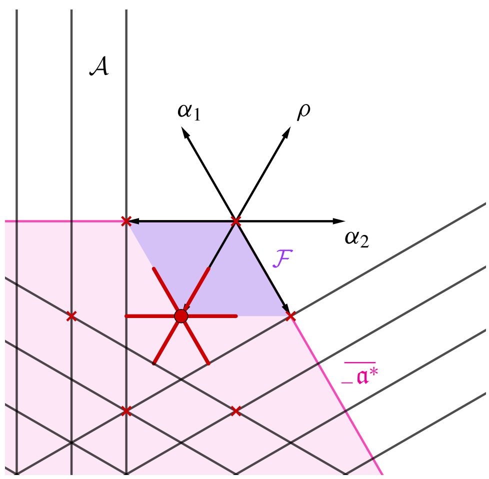

Figure 6. Visualization of the real part of $\mathfrak{a}^*_{\mathbb{C}}$ for $G = \mathrm{SL}_3(\mathbb{R})$: The pink region is where Ruelle–Taylor resonances can a priori be located in view of the results of [Bonthonneau et al. 2020]. The red points and lines depict the region where first band resonances can occur: $\left(\mathcal{B} \cap \frac{1}{2} \operatorname{conv}(W\rho) \cup W\rho\right) - \rho$. The purple shaded region illustrates the real parts in which only first band resonances can occur. Further first band resonances might occur inside the exceptional set $\mathcal{A}$ depicted by the black lines.

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GROWTH OF HIGH L^p NORMS FOR EIGENFUNCTIONS AN APPLICATION OF GEODESIC BEAMS

YAIZA CANZANI AND JEFFREY GALKOWSKI

This work concerns L^p norms of high energy Laplace eigenfunctions: $(-\Delta_g-\lambda^2)\phi_\lambda=0$, $\|\phi_\lambda\|_{L^2}=1$. Sogge (1988) gave optimal estimates on the growth of $\|\phi_\lambda\|_{L^p}$ for a general compact Riemannian manifold. Here we give general dynamical conditions guaranteeing quantitative improvements in L^p estimates for $p>p_c$, where p_c is the critical exponent. We also apply results of an earlier paper (Canzani and Galkowski 2018) to obtain quantitative improvements in concrete geometric settings including all product manifolds. These are the first results giving quantitative improvements for estimates on the L^p growth of eigenfunctions that only require dynamical assumptions. In contrast with previous improvements, our assumptions are local in the sense that they depend only on the geodesics passing through a shrinking neighborhood of a given set in M. Moreover, we give a structure theorem for eigenfunctions which saturate the quantitatively improved L^p bound. Modulo an error, the theorem describes these eigenfunctions as finite sums of quasimodes which, roughly, approximate zonal harmonics on the sphere scaled by $1/\sqrt{\log\lambda}$.

1. Introduction

Let (M,g) be a smooth, compact, Riemannian manifold of dimension n and consider normalized Laplace eigenfunctions, i.e., solutions to

$$(-\Delta_g-\lambda^2)\phi_\lambda=0,\quad \|\phi_\lambda\|_{L^2(M)}=1.$$

This article studies the growth of L^p norms of the eigenfunctions ϕ_λ as $\lambda\to\infty$. Since the work of Sogge [1988], it has been known that there is a change of behavior in the growth of L^p norms for eigenfunctions at the *critical exponent* $p_c:=2(n+1)/(n-1)$. In particular,

$$\|\phi_\lambda\|_{L^p(M)}\le C\lambda^{\delta(p)},\quad \delta(p):=\begin{cases}\frac{n-1}{2}-\frac{n}{p}, & p_c\le p,\\ \frac{n-1}{4}-\frac{n-1}{2p}, & 2\le p\le p_c.\end{cases}\tag{1-1}$$

For $p\ge p_c$, (1-1) is saturated by the zonal harmonics on the round sphere S^n. On the other hand, for $p\le p_c$, these bounds are saturated by the highest weight spherical harmonics on S^n, also known as Gaussian beams. In a very strong sense, the authors showed in [Canzani and Galkowski 2021, page 4] that any eigenfunction saturating (1-1) for $p>p_c$ behaves like a zonal harmonic, while Blair and Sogge [2015b; 2017] showed that for $p<p_c$ such eigenfunctions behave like Gaussian beams. In the case $p\le p_c$, Blair and Sogge [2015a; 2018; 2019] have made substantial progress on improved L^p estimates on manifolds with nonpositive curvature.

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Keywords: Eigenfunctions, high energy, L^p norms, microlocal, second microlocal.

This article concerns the behavior of L^p norms for high p; that is, for $p > p_c$. While there has been a great deal of work on L^p norms of eigenfunctions [Hezari and Rivière 2016; Koch et al. 2007; Sogge et al. 2011; Sogge and Zelditch 2002; 2016; Tacy 2018; 2019; Toth and Zelditch 2002; 2003], this article departs from the now standard approaches. We both adapt the geodesic beam methods developed by the authors in [Canzani and Galkowski 2023; 2019; 2021; Canzani et al. 2018; Galkowski 2018; 2019; Galkowski and Toth 2018; 2020] and develop a new second microlocal calculus used to understand the number of points at which $|u_\lambda|$ can be large (see Section 1A for details on the new ideas here). By doing this, we give general dynamical conditions guaranteeing quantitative improvements over (1-1) for $p > p_c$. In order to work in compact subsets of phase space, we semiclassically rescale our problem. Let $h = \lambda^{-1}$, and, abusing notation slightly, write $\phi_\lambda = \phi_h$, so that

$$(-h^2\Delta_g - 1)\phi_h = 0, \quad \|\phi_h\|_{L^2(M)} = 1.$$

We also work with the semiclassical Sobolev spaces $H_h^s(M)$, with $s \in \mathbb{R}$, defined by the norm

$$\|u\|^2_{H_h^s(M)} := \langle (-h^2\Delta_g + 1)^s u, u\rangle_{L^2(M)}.$$

We start by stating a consequence of our main theorem. Let Ξ denote the collection of maximal unit speed geodesics for (M, g). For m a positive integer, $r > 0$, $t \in \mathbb{R}$, and $x \in M$, define

$$\Xi_x^{m,r,t} := \{\gamma \in \Xi : \gamma(0) = x \text{ and there exists at least } m \text{ conjugate points to } x \text{ in } \gamma(t-r, t+r)\},$$

where we count conjugate points with multiplicity. Next, for a set $V \subset M$, write

$$\mathcal{C}_V^{m,r,t} := \bigcup_{x \in V} \{\gamma(t) : \gamma \in \Xi_x^{m,r,t}\}.$$

Note that if $r_t \to 0^+$ as $|t| \to \infty$, then saying $y \in \mathcal{C}_x^{n-1,r_t,t}$ for t large indicates that y behaves like a point that is maximally conjugate to x. This is the case for every point x on the sphere when y is either equal to x or its antipodal point. The following result applies under the assumption that points are not maximally conjugate and obtains quantitative improvements.

Theorem 1.1. *Let $p > p_c$ and $U \subset M$, and assume there exist $t_0 > 0$ and $a > 0$ such that*

$$\inf_{x_1, x_2 \in U} d(x_1, \mathcal{C}_{x_2}^{n-1,r_t,t}) \geq r_t \quad \textit{for } t \geq t_0,$$

with $r_t = \frac{1}{a}e^{-at}$. Then, there exist $C > 0$ and $h_0 > 0$ such that, for $0 < h < h_0$ and $u \in \mathcal{D}'(M)$,

$$\|u\|_{L^p(U)} \leq Ch^{-\delta(p)}\left(\frac{\|u\|_{L^2(M)}}{\sqrt{\log h^{-1}}} + \frac{\sqrt{\log h^{-1}}}{h}\|(-h^2\Delta_g - 1)u\|_{H_h^{(n-3)/2-n/p}(M)}\right).$$

The assumption in Theorem 1.1 rules out maximal conjugacy of any two points $x, y \in U$ uniformly up to time ∞, and we expect it to hold for a dense set of metrics on any smooth manifold M with $U = M$. Since Theorem 1.1 includes the case of manifolds without conjugate points, it generalizes the work of Hassell and Tacy [2015], where it was shown that logarithmic improvements in L^p norms for $p > p_c$ are possible on manifolds with nonpositive curvature. One family of examples where the assumptions of Theorem 1.1 hold is that of product manifolds [Canzani and Galkowski 2021, Lemma 1.1], i.e.,

$(M_1 \times M_2, g_1 \oplus g_2)$, where the (M_i, g_i) are nontrivial compact Riemannian manifolds. Note that this family of examples includes manifolds with large numbers of conjugate points, e.g., $S^2 \times M$ for any nontrivial M.

The proof of Theorem 1.1 gives a great deal of information about eigenfunctions which saturate L^p bounds ($p > p_c$). Indeed, its proof yields Theorem 3.8 (see Section 3G), which describes the profile of these functions modulo an error in L^p. It shows that, under the assumptions of Theorem 1.1, an eigenfunction can saturate the *logarithmically improved* L^∞ norm near at most *boundedly many* points (it actually shows the same for the L^p norm when $p > p_c$). That is, for $\varepsilon > 0$, there is $N_\varepsilon > 0$ such that

$$\#\left\{\alpha \in \mathcal{I}(h) : \|u\|_{L^\infty(B(x_\alpha, R(h)))} \geq \frac{\varepsilon h^{(1-n)/2}\sqrt{t_0}}{\sqrt{\log h^{-1}}}\|u\|_{L^2(M)},\ B(x_\alpha, R(h)) \cap U \neq \varnothing\right\} \leq N_\varepsilon, \tag{1-2}$$

where $\{x_\alpha\}_{\alpha \in \mathcal{I}(h)}$ is a maximal $R(h) := h^{1/2-\delta}$ separated collection of points with $\delta > 0$.

Moreover, modulo an error small in L^p, near each of these points the eigenfunction u can be decomposed as a sum of quasimodes which are similar to the highest weight spherical harmonics scaled by $h^{(n-1)/4}/\sqrt{\log h^{-1}}$ whose number is nearly proportional to $h^{(1-n)/2}$. Indeed, Theorem 3.8 (see Section 3G) shows that there is a collection of geodesic tubes $\{\mathcal{T}_j\}_{j \in \mathcal{L}(\varepsilon, u)} \subset S^*M$ of radius $R(h)$ (see Definition 1.3) with indices in the set $\mathcal{L}(\varepsilon, u) = \bigcup_{i=1}^{C} \mathcal{J}_i$ and with pairwise disjoint tubes $\mathcal{T}_k \cap \mathcal{T}_\ell = \varnothing$ for $k, \ell \in \mathcal{J}_i$ with $k \neq \ell$, such that

$$u = u_e + \frac{1}{\sqrt{\log h^{-1}}} \sum_{j \in \mathcal{L}(\varepsilon, u)} v_j.$$

Here, u_e should be understood as an error term satisfying, for all $p \leq q \leq \infty$,

$$\|u_e\|_{L^q} \leq \varepsilon h^{-\delta(q)} (\log h^{-1})^{-1/2} \|u\|_{L^2}.$$

Each v_j is microsupported in the geodesic tube $\mathcal{T}_j$ and is a quasimode with

$$\|(-h^2\Delta_g - 1)v_j\|_{L^2} \leq C\varepsilon^{-1} h R(h)^{(n-1)/2} \|u\|_{L^2} \quad \text{and} \quad \|v_j\|_{L^2} \leq C\varepsilon^{-1} R(h)^{(n-1)/2} \|u\|_{L^2}. \tag{1-3}$$

While similar to highest weight spherical harmonics (also known as Gaussian beams), they are not as tightly localized to a geodesic segment and do not have Gaussian profiles. We refer to these quasimodes as *geodesic beams* (see Remark 3.2 and Figure 1 for an illustration).

Furthermore, in Theorem 3.8 we prove that near each point x_α on which u nearly saturates the L^p bound, i.e., for α that belongs to the set displayed in (1-2), we have

$$c\varepsilon^2 R(h)^{1-n} \leq |\mathcal{L}(\varepsilon, u, \alpha)| \leq C R(h)^{1-n}, \tag{1-4}$$

where $\mathcal{L}(\varepsilon, u, \alpha) := \{j \in \mathcal{L}(\varepsilon, u) : \pi_M(\mathcal{T}_j) \cap B(x_\alpha, 3R(h)) \neq \varnothing\}$ and $\pi_M : S^*M \to M$ is the natural projection. Since $\dim S^*_{x_\alpha} M = n - 1$, this means that at points x_α at which u nearly saturates its L^p norm there must be a full measure set of directions on which u is microsupported. In addition, we also prove that the collection of geodesic beams v_j on which u has its microsupport carries a positive portion of the total L^2 mass:

$$\sum_{j \in \mathcal{L}(\varepsilon, u, \alpha)} \|v_j\|_{L^2}^2 \geq c^2 \varepsilon^2 \|u\|_{L^2}^2.$$

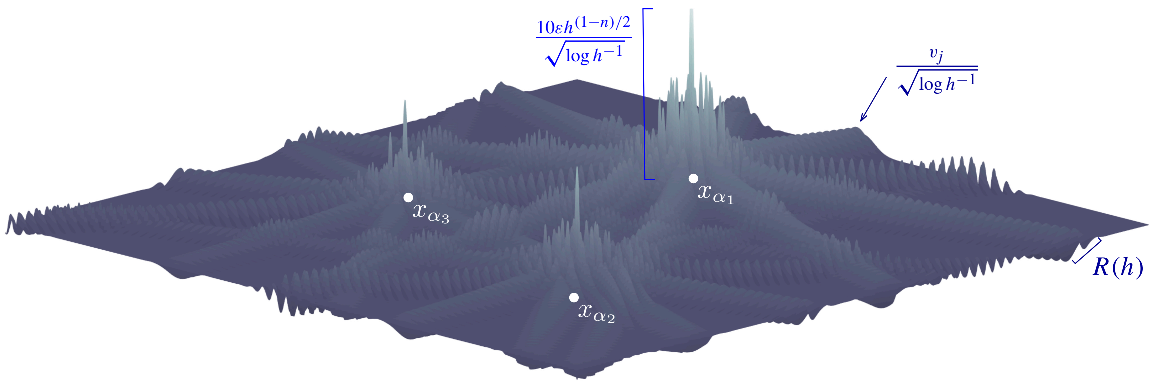

Figure 1. The figure illustrates a function u that saturates the L^∞ bound at three points $x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}$ viewed as a superposition of geodesic beams v_j. Each ridge corresponds to a beam v_j and is microsupported on a tube $\mathcal{T}_j$ of radius $R(h)$.

Note that, together with (1-3) and (1-4), this implies that most of the geodesic beams carry mass *exactly* proportional to $R(h)^{(n-1)/2}\|u\|_{L^2}$, and hence that the mass is nearly uniform over all possible directions. For the precise statement of these estimates, see Section 3G.

Remark 1.2. Note that we do *not* use the bound (1-2) to prove our main theorem. Instead, this decomposition is a consequence of the proof of Theorem 1.1, which, in principle describes much more about the profile of eigenfunctions (see the outline of the proof in Section 1A for more details).

The proofs of Theorems 1.1 and 3.8 hinge on a much more general theorem, Theorem 1.4, which does not require global geometric assumptions on (M, g). As far as the authors are aware, Theorem 1.4 is the first result giving quantitative estimates for the L^p growth of eigenfunctions that *only* requires dynamical assumptions. We emphasize that, in contrast with previous improvements on Sogge's L^p estimates, the assumptions in Theorem 1.4 are purely dynamical and, moreover, are local in the sense that they depend only on the geodesics passing through a shrinking neighborhood of a given set in M. Moreover, the techniques do not require long-time wave parametrices.

Theorem 1.4 controls $\|u\|_{L^p(U)}$ using an assumption on the maximal volume of long geodesics joining any two given points in U. For our proof, it is necessary to control the number of points in U where the L^∞ norm of u can be large (see Step 4 in Section 1A). This is a very delicate and technical part of the argument, as the points in question may be approaching one another at rates $\sim h^\delta$ as $h \to 0^+$ with $0 < \delta < \frac{1}{2}$. To state our theorem, we need to introduce a few geometric objects. First, consider the Hamiltonian function $p \in C^\infty(T^*M\backslash\{0\})$,

$$p(x, \xi) = |\xi|_g - 1,$$

and let $\varphi_t : T^*M \setminus 0 \to T^*M \setminus 0$ denote the Hamiltonian flow for p at time t, which coincides with the geodesic flow in this case. We also define the *maximal expansion rate* and the *Ehrenfest time* at frequency h^{-1}, respectively, as

$$\Lambda_{\max} := \limsup_{|t|\to\infty} \frac{1}{|t|} \log \sup_{S^*M} \|d\varphi_t(x,\xi)\| \quad \text{and} \quad T_e(h) := \frac{\log h^{-1}}{2\Lambda_{\max}}, \tag{1-5}$$

where $\|\cdot\|$ denotes the norm in any metric on $T(T^*M)$. Note that $\Lambda_{\max} \in [0,\infty)$, and if $\Lambda_{\max} = 0$ we may replace it by an arbitrarily small positive constant. We next describe a cover of S^*M by geodesic tubes.

For each $\rho_0 \in S^*M$, the cosphere bundle to M, let $H_{\rho_0} \subset M$ be a hypersurface such that $\rho_0 \in SN^*H_{\rho_0}$, the unit conormal bundle to H_{ρ_0}. Then, let

$$\mathcal{H}_{\rho_0} \subset T^*_{H_{\rho_0}}M = \{(x,\xi) \in T^*M : x \in H_{\rho_0}\}$$

be a hypersurface containing $SN^*H_{\rho_0}$. Next, for $q \in \mathcal{H}_{\rho_0}$ and $\tau > 0$, we define the tube through q of radius $R(h) > 0$ and "length" $\tau + R(h)$ as

$$\begin{aligned} \Lambda_q^\tau(R(h)) &:= \bigcup_{|t|\le \tau + R(h)} \varphi_t(B_{\mathcal{H}_{\rho_0}}(q, R(h))), \\ B_{\mathcal{H}_{\rho_0}}(q,R(h)) &:= \{\rho \in \mathcal{H}_{\rho_0} : d(\rho,q) \le R(h)\}, \end{aligned} \tag{1-6}$$

where d is the distance induced by the Sasaki metric on T^*M (see e.g., [Blair 2010, Chapter 9] for a description of the Sasaki metric). Note that the tube runs along the geodesic through $q \in \mathcal{H}_{\rho_0}$. Similarly, for $A \subset S^*M$, we define $\Lambda_A^\tau(R(h))$ in the same way, replacing q with A in (1-6).

Definition 1.3. Let $A \subset S^*M$, $r > 0$, and $\{\rho_j(r)\}_{j=1}^{N_r} \subset A$ for some $N_r > 0$. We say the collection of tubes $\{\Lambda_{\rho_j}^\tau(r)\}_{j=1}^{N_r}$ is a (τ, r) *cover* of a set $A \subset S^*M$ provided

$$\Lambda_A^\tau\big(\tfrac{1}{2}r\big) \subset \bigcup_{j=1}^{N_r} \mathcal{T}_j, \quad \mathcal{T}_j := \Lambda_{\rho_j}^\tau(r).$$

Given a (τ, r) cover $\{\mathcal{T}_j\}_{j\in\mathcal{J}}$ for S^*M, for each $x \in M$ we define

$$\mathcal{J}_x := \{j \in \mathcal{J} : \pi(\mathcal{T}_j) \cap B(x,r) \neq \varnothing\}.$$

We are now ready to state Theorem 1.4, where we give *explicit dynamical conditions* guaranteeing quantitative improvements in L^p norms.

Theorem 1.4. *There exists $\tau_M > 0$ such that for all $p > p_c$ and $\varepsilon_0 > 0$ the following holds. Let $U \subset M$ and $0 < \delta_1 < \delta_2 < \frac{1}{2}$, and let $h^{\delta_2} \le R(h) \le h^{\delta_1}$ for all $h > 0$. Let $1 \le T(h) \le (1-2\delta_2)T_e(h)$ and let $t_0 > 0$ be h-independent. Let $\{\mathcal{T}_j\}_{j\in\mathcal{J}}$ be a $(\tau, R(h))$ cover for S^*M for some $0 < \tau < \tau_M$.*

Suppose that, for any pair of points $x_1, x_2 \in U$, the tubes over x_1 can be partitioned into a disjoint union

$$\mathcal{J}_{x_1} = \mathcal{B}_{x_1,x_2} \sqcup \mathcal{G}_{x_1,x_2},$$

where

$$\bigcup_{j\in\mathcal{G}_{x_1,x_2}} \varphi_t(\mathcal{T}_j) \cap S^*_{B(x_2,R(h))}M = \varnothing, \quad t \in [t_0, T(h)].$$

Then, there are $h_0 > 0$ and $C > 0$ such that, for all $u \in \mathcal{D}'(M)$ and $0 < h < h_0$,

$$\|u\|_{L^p(U)} \le Ch^{-\delta(p)}\left(\frac{\sqrt{t_0}}{\sqrt{T(h)}} + \Big[\sup_{x_1,x_2\in U} |\mathcal{B}_{x_1,x_2}|R(h)^{n-1}\Big]^{(6+\varepsilon_0)^{-1}(1-p_c/p)}\right) \times \left(\|u\|_{L^2} + \frac{T(h)}{h}\|(-h^2\Delta_g - 1)u\|_{H_h^{(n-3)/2-n/p}}\right). \tag{1-7}$$

In order to interpret (1-7), note that we think of the tubes $\mathcal{G}_{x_1,x_2}$ and $\mathcal{B}_{x_1,x_2}$ as good (or nonlooping) and bad (or looping), respectively. Then, observe that

$$|\mathcal{B}_{x_1,x_2}|R(h)^{n-1} \sim \operatorname{vol}\Big(\bigcup_{j\in\mathcal{B}_{x_1,x_2}} \mathcal{T}_j \cap S^*_{x_1}M\Big)$$

and that $\bigcup_{j\in\mathcal{B}_{x_1,x_2}} \mathcal{T}_j$ is the set of directions over x_1 which may loop through x_2 in time $T(h)$. Therefore, if the volume of points in $S^*_{x_1}M$ looping through x_2 is bounded by $T(h)^{-(3+\varepsilon_0)(1-p_c/p)^{-1}}$, (1-7) provides $T(h)^{-1/2}$ improvements over the standard L^p bounds. We expect these nonlooping-type assumptions to be valid for a dense set of metrics on any smooth manifold M.

Theorem 1.4 can be used to obtain improved L^p resolvent bounds [Cuenin 2020, Theorem 2.21] which, as shown there, are stable by certain rough perturbations. These estimates in turn can be used to construct complex geometric optics solutions and solve certain inverse problems [Dos Santos Ferreira et al. 2013].

One can check using a similar argument to that in [Canzani and Galkowski 2021, Lemma 5.1 (see also Theorem 5, Section 1.5.3)] that in certain integrable situations

$$\Big(\sup_{x_1,x_2\in U} |\mathcal{B}_{x_1,x_2}|R(h)^{n-1}\Big)^{(6+\varepsilon_0)^{-1}(1-p_c/p)} \le \frac{C}{\sqrt{T(h)}},$$

with $T(h) \gg \log h^{-1}$ and U a nontrivial open subset of M, thus producing $o((\log h^{-1})^{-1/2})$ improvements on the L^p norms over U after an application of Theorem 1.4. One example of such an integrable system is the spherical pendulum where U can be taken to be any set that lies at a positive distance from the poles.

For other examples, where one can understand these types of good and bad tubes, we refer the reader to [Canzani and Galkowski 2023], where they are used to understand averages and L^∞ norms under various assumptions on M, including that it has Anosov geodesic flow or nonpositive curvature. Since our results do not require parametrices for the wave group, we expect that the arguments leading to Theorem 1.4 will provide *polynomial* improvements over Sogge's estimates on manifolds where Egorov-type theorems hold for longer than logarithmic times.

Note that Theorem 1.4 addresses L^p norms with $p_c < p \le \infty$, while the authors' previous work in [Canzani and Galkowski 2021] considers $p = \infty$ alone. Moreover, for $p = \infty$, the estimate in Theorem 1.4 is actually *weaker* than those in that previous work in that it requires an assumption about geodesics passing near two distinct points, while those in that previous work require only a nonrecurrent assumption on geodesics passing through a small neighborhood of a single point. This is because describing the L^p norm for $p < \infty$ requires understanding the behavior at many points simultaneously, while the L^∞ norm cares only about a single point with maximal growth.

Remark 1.5. The proofs below could be adapted to the case of quasimodes for real principal type semiclassical pseudodifferential operators of Laplace type. That is, to operators with principal symbol p satisfying both that $\partial_\xi p \neq 0$ on $\{p = 0\}$ and that $\{p = 0\} \cap T_x^*M$ has positive definite second fundamental form. This is the case, for example, for Schrödinger operators away from the forbidden region. However, for concreteness and simplicity of exposition, we have chosen to consider only the Laplace operator.

1A. ***Outline of the proof of Theorem 1.4.*** Our method for proving Theorem 1.4 differs from the standard approaches for treating L^p norms in two major ways: it hinges on adapting the geodesic beam techniques constructed by the authors [Canzani and Galkowski 2021] and on the development of a new second microlocal calculus. We now give a detailed sketch of the argument used in this proof.

To simplify the presentation in this outline, we suppose u is a Laplace eigenfunction and $U = M$, and sketch the proof of Theorem 1.4.

Step 1: We first write $u = \sum_j \chi_{\mathcal{T}_j} u$, where the $\mathcal{T}_j$ are as in Definition 1.3 and $\chi_{\mathcal{T}_j}$ is a microlocal cutoff to $\mathcal{T}_j$ which approximately commutes with $P = -h^2\Delta_g - 1$; see Section 3A. We also cover M by balls $\{B(x_\alpha, R)\}_{\alpha\in\mathcal{I}}$ such that $\mathcal{I}$ consists of a union of boundedly many collections of disjoint balls. We next organize the tubes $\mathcal{T}_j$ by the L^2 mass of $\chi_{\mathcal{T}_j} u$, writing

$$\mathcal{A}_k := \{j : 2^{-(k+1)}\|u\|_{L^2} \leq \|\chi_{\mathcal{T}_j} u\|_{L^2} \leq 2^{-k}\|u\|_{L^2}\};$$

see Section 3B. For each k, we then organize the balls $B(x_\alpha, R)$ by the L^∞ norm of $\sum_{j\in\mathcal{A}_k} \chi_{\mathcal{T}_j} u$, writing

$$\mathcal{I}_{k,m} := \left\{\alpha \in \mathcal{I} : 2^{m-k-1}\|u\|_{L^2} \leq h^{(n-1)/2} R^{(1-n)/2} \Big\| \sum_{j\in\mathcal{A}_k} \chi_{\mathcal{T}_j} u \Big\|_{L^\infty(B(x_\alpha,R))} \leq 2^{m-k}\|u\|_{L^2}\right\}; \tag{1-8}$$

see Section 3C. The reason for this choice comes from the geodesic beam estimate (see [Canzani and Galkowski 2021])

$$\Big\| \sum_{j\in\mathcal{A}_k} \chi_{\mathcal{T}_j} u \Big\|_{L^\infty(B(x_\alpha,R))} \leq C h^{(1-n)/2} R^{(n-1)/2} \sum_{j\in\mathcal{A}_k(\alpha)} \|\chi_{\mathcal{T}_j} u\|_{L^2}, \tag{1-9}$$

where $\mathcal{A}_k(\alpha)$ denotes those tubes $\mathcal{T}_j$ such that $j \in \mathcal{A}_k$ and $\mathcal{T}_j$ passes over the ball $B(x_\alpha, R)$. Because of the definition of $\mathcal{A}_k$, we have that 2^m is a lower bound for the number of tubes in $\mathcal{A}_k(\alpha)$ for $\alpha \in \mathcal{I}_{k,m}$; see (3-20).

With this bookkeeping completed, we record the estimate on the L^p norm:

$$\|u\|_{L^p} \leq C \sum_k \bigg(\sum_m \Big\| \sum_{j\in\mathcal{A}_{k,m}} \chi_{\mathcal{T}_j} u \Big\|^p_{L^p(\bigcup_{\alpha\in\mathcal{I}_{k,m}} B(x_\alpha,R))} \bigg)^{1/p}, \tag{1-10}$$

where $\mathcal{A}_{k,m} = \bigcup_{\alpha\in\mathcal{I}_{k,m}} \mathcal{A}_k(\alpha)$, i.e., those tubes in $\mathcal{A}_k$ which pass over a ball in $\mathcal{I}_{k,m}$.

Step 2: We control each L^p norm in (1-10) by using interpolation between the L^∞ estimate analogous to (1-9) and the standard L^{p_c} estimate:

$$\Big\| \sum_{j\in\mathcal{A}_{k,m}} \chi_{\mathcal{T}_j} u \Big\|_{L^{p_c}} \leq C h^{-1/p_c} \Big\| \sum_{j\in\mathcal{A}_{k,m}} \chi_{\mathcal{T}_j} u \Big\|_{L^2} \leq C h^{-1/p_c} 2^{-k} |\mathcal{A}_{k,m}|^{1/2} \|u\|_{L^2}.$$

In Section 3D, we start by handling the "easy" piece where the L^∞ norm is smaller than $T(h)^{-N}h^{-(n-1)/2}$ for some very large N. This piece can be neglected since the standard interpolation estimate shows that it has L^p norm $\ll h^{-\delta(p)}/\sqrt{T(h)}\|u\|_{L^2}$.

Next, in Section 3E, we write $\mathcal{A}_{k,m} = \mathcal{G}_{k,m} \sqcup \mathcal{B}_{k,m}$, where $\bigcup_{j\in\mathcal{G}_{k,m}} \mathcal{T}_j$ is non-self-looping in the sense that

$$\bigcup_{t\in[t_0,T(h)]} \varphi_t\big(\textstyle\bigcup_{j\in\mathcal{G}_{k,m}}\mathcal{T}_j\big) \bigcap \textstyle\bigcup_{j\in\mathcal{G}_{k,m}}\mathcal{T}_j = \varnothing.$$

Using non-self-looping estimates from [Canzani and Galkowski 2021] (see also Lemma 3.6) and summing carefully, we are able to show that

$$C\sum_k\bigg(\sum_m\Big\|\sum_{j\in\mathcal{G}_{k,m}}\chi_{\mathcal{T}_j}u\Big\|^p_{L^p(\bigcup_{\alpha\in\mathcal{I}_{k,m}}B(x_\alpha,R))}\bigg)^{1/p} \le \frac{h^{-\delta(p)}}{\sqrt{T(h)}}\|u\|_{L^2}.$$

This is done in Section 3E2.

Our final task is to estimate the sum over the bad tubes. For this, we again use the geodesic beam estimate to control the L^∞ norm of $\sum_{j\in\mathcal{B}_{k,m}}\chi_{\mathcal{T}_j}u$ by the maximal number, $|\mathcal{B}^{\max}_{k,m}|$, of "bad" tubes passing over a ball $B(x_\alpha, R)$ with $\alpha\in\mathcal{I}_{k,m}$. In addition, we control the L^2 norm of this sum by $|\mathcal{B}_{k,m}|^{1/2}2^{-k}$. The numbers of "bad" tubes are estimated in the next step.

Step 3: We first estimate $|\mathcal{B}^{\max}_{k,m}|$ using the dynamical hypothesis. In particular, we check that

$$|\mathcal{B}^{\max}_{k,m}| \le |\mathcal{I}_{k,m}|\,|\mathcal{B}_{x_1,x_2}|.$$

This estimate comes from imagining the worst case scenario that every tube connecting some ball $B(x_\alpha, R)$ with $\alpha\in\mathcal{I}_{k,m}$ to another ball $B(x_\beta, R)$ with $\beta\in\mathcal{I}_{k,m}$ lies in $\mathcal{A}_k$ and that no such tube connects $B(x_\alpha, R)$ to $B(x_\beta, R)$ and $B(x_{\beta'}, R)$ for $\beta\neq\beta'$; see (3-46). Using a similar argument, we can see that

$$|\mathcal{B}_{k,m}| \le |\mathcal{I}_{k,m}|^2 \sup_{x_1,x_2}|\mathcal{B}_{x_1,x_2}|;$$

see (3-39). Thus, it remains only to estimate $|\mathcal{I}_{k,m}|$.

Step 4: To estimate the size of $\mathcal{I}_{k,m}$, we need to estimate the number of balls on which the combination of beams $w_{k,m} := \sum_{j\in\mathcal{A}_{k,m}}\chi_{\mathcal{T}_j}u$ with L^2 mass 2^{-k} has L^∞ norm $2^{m-k}R^{(n-1)/2}h^{(1-n)/2}\|u\|_{L^2}$.

To do this, we aim to understand both the minimal amount of L^2 mass needed for an eigenfunction to have a certain (large) L^∞ norm and where that mass must be located in phase space. The standard Hörmander-type L^∞ bound (as presented in [Koch et al. 2007]) answers the first question: for $x\in M$,

$$h^{(n-1)/2}|w(x)| \le C(\|w\|_{L^2} + h^{-1}\|Pw\|_{L^2}). \tag{1-11}$$

To answer the second question, we need to understand to what extent this inequality can be microlocalized. Because of the invariance of eigenfunctions under the geodesic flow we localize to the coisotropic submanifolds

$$\Gamma_x := \bigcup_{|t|\le 1}\varphi_t(T^*_xM). \tag{1-12}$$

We want three properties for X_{Γ_x}, our localizers to Γ_x; see (3-25) for the precise requirements and Theorem 6.3 for their construction. First, they should localize tightly (h^ρ with $\rho \sim 1$) to Γ_x. Second, they must nearly maintain the value of a function at x:

$$w(x) = (X_{\Gamma_x} w)(x) + O(h^\infty). \tag{1-13}$$

Third, they must preserve quasimodes for P so that, using the inequality (1-11), we have

$$h^{(n-1)/2} |(X_{\Gamma_x} w_{k,m})(x)| \le C \|X_{\Gamma_x} w_{k,m}\|_{L^2}. \tag{1-14}$$

Thus, from (1-13) and (1-14) it follows that, for $\alpha \in \mathcal{I}_{k,m}$, there is $\tilde{x}_\alpha \in B(x_\alpha, R)$ with

$$R^{(n-1)/2} 2^{m-k} \|u\|_{L^2} \le h^{(n-1)/2} \|X_{\Gamma_{\tilde{x}_\alpha}} w_{k,m}\|_{L^\infty(B(x_\alpha, R))} \le \|X_{\Gamma_{\tilde{x}_\alpha}} w_{k,m}\|_{L^2}. \tag{1-15}$$

Note that we use Γ_x as defined above, as opposed to the flowout of $S_x^* M$, precisely so that (1-13) is possible.

Finally, we will bound $|\mathcal{I}_{k,m}|$ by summing (1-15) over all balls in $\mathcal{I}_{k,m}$ to obtain

$$R^{n-1} 2^{2(m-k)} |\mathcal{I}_{k,m}| \le \sum_{\alpha \in \mathcal{I}_{k,m}} \|X_{\Gamma_{\tilde{x}_\alpha}} w_{k,m}\|_{L^2}^2. \tag{1-16}$$

We produce an upper bound on (1-16) of the form

$$\sum_{\alpha \in \mathcal{I}_{k,m}} \|X_{\Gamma_{\tilde{x}_\alpha}} w_{k,m}\|_{L^2}^2 \le \|w_{k,m}\|_{L^2}^2. \tag{1-17}$$

This follows from Proposition 6.6 (see the analysis leading to (3-31)) and controls the minimal L^2 mass necessary for $w_{k,m}$ to have a large value at *all* the points in $\mathcal{I}_{k,m}$. We view this estimate as an uncertainty principle type of result in which we prove that, for $\tilde{x}_\alpha \ne \tilde{x}_\beta$, localization to $\Gamma_{\tilde{x}_\alpha}$ and $\Gamma_{\tilde{x}_\beta}$ are incompatible in the sense that

$$\|X_{\Gamma_{\tilde{x}_\alpha}} X_{\Gamma_{\tilde{x}_\beta}}\|_{L^2 \to L^2} \ll 1, \tag{1-18}$$

with uniform estimates in $d(\tilde{x}_\alpha, \tilde{x}_\beta)$. Combining (1-16) with (1-17) yields the bound needed on $|\mathcal{I}_{k,m}|$ to finish the analysis in the proof of Theorem 1.4. This is done in Section 3E1.

Remark 1.6 (uncertainty principle). Note that, if the function $w_{k,m}$ could be localized simultaneously on all the manifolds $\Gamma_{\tilde{x}_\alpha}$, then we would have

$$\sum_{\alpha \in \mathcal{I}_{k,m}} \|X_{\Gamma_{\tilde{x}_\alpha}} w_{k,m}\|_{L^2}^2 \ge c |\mathcal{I}_{k,m}| \|w_{k,m}\|_{L^2}^2 \gg \|w_{k,m}\|_{L^2}^2.$$

This contradicts (1-17). Hence, if one more carefully quantifies this argument by assigning weights to the localized masses $\|X_{\Gamma_{\tilde{x}_\alpha}} w_{k,m}\|_{L^2}$, we can understand how much of the L^2 mass of $w_{k,m}$ can be localized to many $\Gamma_{\tilde{x}_\alpha}$. This is a type of uncertainty principle. Since (1-15) shows that $\Gamma_{\tilde{x}_\alpha}$ must carry mass in order for $w_{k,m}(\tilde{x}_\alpha)$ to be large, this can be thought of as an estimate on how much a "single unit" of L^2 mass can be used to produce a large L^∞ norm at multiple points.

Remark 1.7 (zonal harmonics). Another way to think of the estimate (1-18) is on the round sphere S^2, where the natural enemy is a zonal harmonic Z_x at a point $x \in S^2$. Recall that the zonal harmonic Z_x is localized h close to Γ_x, in the sense that in a fixed size neighborhood of x,

$$X_{\Gamma_x} Z_x = Z_x + O(h^\infty).$$

The estimate (1-18), or more precisely Corollary 6.5, can be used to give lower bounds on

$$\Big\| \sum_{x_\alpha \in \mathcal{I}} Z_{x_\alpha} \Big\|_{L^2}^2 = \sum_{x_\alpha \in \mathcal{I}} \|Z_{x_\alpha}\|_{L^2}^2 + \sum_{x_\alpha \neq x_\beta} \langle X^*_{\Gamma_{x_\beta}} X_{\Gamma_{x_\alpha}} Z_{x_\alpha}, Z_{x_\beta} \rangle_{L^2},$$

where $d(x_\alpha, x_\beta) > R$ for $\alpha \neq \beta$. Equation (1-18) shows that, for $\alpha \neq \beta$,

$$\|X^*_{\Gamma_{x_\beta}} X_{\Gamma_{x_\alpha}}\| \ll 1$$

and hence quantifies the amount of cancellation in such a sum. This cancellation is easy to see with $d(x_\alpha, x_\beta) > c > 0$, but becomes much more subtle when this distance is small.

Remark 1.8 (second microlocal calculus). In order to build the localizers X_{Γ_x} satisfying (1-13) and (1-14), we develop a new second microlocal calculus associated to a Lagrangian foliation L over a coisotropic submanifold $\Gamma \subset T^*M$. In the case of the Γ_x defined in (1-12), the leaves of L will be given by $\varphi_t(T^*_x M)$ for a fixed time t. The calculus allows for simultaneous h^ρ localization (with ρ close to 1) along a leaf of L and along Γ. Because of this and the fact that $T^*_x M$ is one such leaf, we can find localizers with the property (1-13). We note that other works on L^p norms, especially [Blair and Sogge 2015b; 2017], use localizers to $h^{1/2}$ neighborhoods of geodesic segments. However, when two cutoffs X_1 and X_2 localizing at scale $h^{1/2}$ have overlapping support, we always have

$$\|X_1 X_2\|_{L^2 \to L^2} \sim 1,$$

and hence (1-18) does not hold. Therefore, in our framework it is necessary to localize in some directions at scales below $h^{1/2}$ and hence to develop a special calculus associated to the pairs (L, Γ). The calculus, which is developed in Section 5, can be seen as an interpolation between those in [Dyatlov and Zahl 2016; Sjöstrand and Zworski 1999].

Outline of the paper. In Section 2, we construct the covers of S^*M and T^*M consisting of tubes and balls, respectively, which are necessary in the rest of the article. Section 3 contains the proof of Theorems 1.4 and 3.8. This proof uses the anisotropic calculus developed in Section 5 and the almost-orthogonality results from Section 6. Section 4 contains the necessary dynamical arguments to prove Theorem 1.1 using Theorem 1.4.

2. Tube lemmas

The next few lemmas are aimed at constructing (τ, r)-good covers and partitions of various subsets of T^*M; see also [Canzani and Galkowski 2021, Section 3.2]. Before we proceed, we recall the symbol classes $S^m_\delta(T^*M)$; see also, e.g., [Zworski 2012, Chapters 4, 9]. We say that $a \in C^\infty(T^*M)$ is in $S^m_\delta(T^*M)$ if, for all $\alpha, \beta \in \mathbb{N}^d$, there is $C_{\alpha\beta} > 0$ such that, for $0 < h < 1$,

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \le C_{\alpha\beta} h^{-\delta(|\alpha|+|\beta|)} \langle \xi \rangle^{m-|\beta|}, \quad \langle \xi \rangle := (1 + |\xi|^2)^{1/2}.$$

We sometimes write $S_\delta(T^*M) = S^0_\delta(T^*M)$, and we write $a \in S^m_\delta(T^*M; A)$ if $a \in C^\infty(T^*M; A)$ is also in $S^m_\delta(T^*M)$.

Definition 2.1 (good covers and partitions). Let $A \subset T^*M$, $r > 0$, and $\{\rho_j(r)\}_{j=1}^{N_r} \subset A$ be a collection of points for some $N_r > 0$. Let $\mathfrak{D}$ be a positive integer. We say that the collection of tubes $\{\Lambda^\tau_{\rho_j}(r)\}_{j=1}^{N_r}$ is a $(\mathfrak{D}, \tau, r)$-*good cover* of $A \subset T^*M$ provided it is a (τ, r) cover of A and there exists a partition $\{\mathcal{J}_\ell\}_{\ell=1}^{\mathfrak{D}}$ of $\{1, \dots, N_r\}$ such that for every $\ell \in \{1, \dots, \mathfrak{D}\}$,

$$\Lambda^\tau_{\rho_j}(3r) \cap \Lambda^\tau_{\rho_i}(3r) = \varnothing, \quad i, j \in \mathcal{J}_\ell, \ \ i \ne j.$$

In addition, for $0 \le \delta \le \frac{1}{2}$ and $R(h) \ge 8h^\delta$, we say that a collection $\{\chi_j\}_{j=1}^{N_h} \subset S_\delta(T^*M; [0, 1])$ is a δ-*good partition for A associated to a* $(\mathfrak{D}, \tau, R(h))$-*good cover* if $\{\chi_j\}_{j=1}^{N_h}$ is bounded in S_δ and

$$\operatorname{supp} \chi_j \subset \Lambda^\tau_{\rho_j}(R(h)) \quad \text{and} \quad \sum_{j=1}^{N_h} \chi_j \ge 1 \quad \text{on } \Lambda_A^{\tau/2}\big(\tfrac{1}{2}R(h)\big).$$

Remark 2.2. We show below that for any compact Riemannian manifold M, there are $\mathfrak{D}_M, R_0, \tau_0 > 0$, depending only on (M, g), such that, for $0 < \tau < \tau_0$ and $0 < r < R_0$, there exists a $(\mathfrak{D}_M, \tau, r)$-good cover for S^*M.

We start by constructing a useful cover of any Riemannian manifold with bounded curvature.

Lemma 2.3. *Let $\widetilde{M}$ be a compact Riemannian manifold. There exist $\mathfrak{D}_n > 0$, depending only on n, and $R_0 > 0$, depending only on n and a lower bound for the sectional curvature of $\widetilde{M}$, so that the following holds: for $0 < r < R_0$, there exists a finite collection of points $\{x_\alpha\}_{\alpha \in \mathcal{I}} \subset \widetilde{M}$, $\mathcal{I} = \{1, \dots, N_r\}$, and a partition $\{\mathcal{I}_i\}_{i=1}^{\mathfrak{D}_n}$ of $\mathcal{I}$ such that*

$$\widetilde{M} \subset \bigcup_{\alpha \in \mathcal{I}} B(x_\alpha, r), \quad B(x_{\alpha_1}, 3r) \cap B(x_{\alpha_2}, 3r) = \varnothing \quad \textit{for } \alpha_1, \alpha_2 \in \mathcal{I}_i, \ \ \alpha_1 \ne \alpha_2,$$

$$\{x_\alpha\}_{\alpha \in \mathcal{I}} \text{ is a maximal } \tfrac{1}{2}r\text{-separated set in } \widetilde{M}.$$

Proof. Let $\{x_\alpha\}_{\alpha \in \mathcal{I}}$ be a maximal $\frac{1}{2}r$-separated set in $\widetilde{M}$. Fix $\alpha_0 \in \mathcal{I}$ and suppose $B(x_{\alpha_0}, 3r) \cap B(x_\alpha, 3r) \ne \varnothing$ for all $\alpha \in \mathcal{K}_{\alpha_0} \subset \mathcal{I}$. Then, for all $\alpha \in \mathcal{K}_{\alpha_0}$, we have $B\big(x_\alpha, \frac{1}{2}r\big) \subset B(x_{\alpha_0}, 8r)$. In particular,

$$\sum_{\alpha \in \mathcal{K}_{\alpha_0}} \operatorname{vol}\big(B\big(x_\alpha, \tfrac{1}{2}r\big)\big) \le \operatorname{vol}(B(x_{\alpha_0}, 8r)).$$

Now, there exist $R_0 > 0$, depending on n and a lower bound on the sectional curvature of $\widetilde{M}$, and $\mathfrak{D}_n > 0$, depending only on n, such that, for all $0 < r < R_0$,

$$\operatorname{vol}(B(x_{\alpha_0}, 8r)) \le \operatorname{vol}(B(x_\alpha, 14r)) \le \mathfrak{D}_n \operatorname{vol}\big(B\big(x_\alpha, \tfrac{1}{2}r\big)\big). \tag{2-1}$$

Hence, it follows from (2-1) that

$$\sum_{\alpha\in\mathcal{K}_{\alpha_0}} \operatorname{vol}\bigl(B\bigl(x_\alpha, \tfrac{1}{2}r\bigr)\bigr) \le \operatorname{vol}(B(\rho_{\alpha_0}, 8r)) \le \frac{\mathfrak{D}_n}{|\mathcal{K}_{\alpha_0}|} \sum_{\alpha\in\mathcal{K}_{\alpha_0}} \operatorname{vol}\bigl(B\bigl(x_\alpha, \tfrac{1}{2}r\bigr)\bigr).$$

In particular, $|\mathcal{K}_{\alpha_0}| \le \mathfrak{D}_n$.

At this point we have proved that each of the balls $B(x_\alpha, 3r)$ intersects at most $\mathfrak{D}_n - 1$ other balls. We now construct the sets $\mathcal{I}_1, \dots, \mathcal{I}_{\mathfrak{D}_n}$ using a greedy algorithm. We will say that the index α_1 *intersects* the index α_2 if

$$B(x_{\alpha_1}, 3r) \cap B(x_{\alpha_2}, 3r) \ne \varnothing.$$

We place the index $1 \in \mathcal{I}_1$. Then suppose we have placed the indices $\{1, \dots, \alpha\}$ in $\mathcal{I}_1, \dots, \mathcal{I}_{\mathfrak{D}_n}$ so each of the $\mathcal{I}_i$ consists of disjoint indices. Then, since $\alpha + 1$ intersects at most $\mathfrak{D}_n - 1$ indices, it is disjoint from $\mathcal{I}_i$ for some i. We add the index α to $\mathcal{I}_i$. By induction we obtain the partition $\mathcal{I}_1, \dots, \mathcal{I}_{\mathfrak{D}_n}$.

Now, suppose that there exists $x \in \widetilde{M}$ such that $x \notin \bigcup_{\alpha\in\mathcal{I}} B(x_\alpha, r)$. Then, $\min_{\alpha\in\mathcal{I}} d(x, x_\alpha) \ge r$, a contradiction of the $\frac{1}{2}r$ maximality of x_α. □

In order to construct our microlocal partition, we first fix a smooth hypersurface $H \subset M$, and choose Fermi normal coordinates $x = (x_1, x')$ in a neighborhood of $H = \{x_1 = 0\}$. We write $(\xi_1, \xi') \in T_x^*M$ for the dual coordinates. Let

$$\Sigma_H := \bigl\{(x, \xi) \in S_H^*M \bigm| |\xi_1| \ge \tfrac{1}{2}\bigr\}. \tag{2-2}$$

We then consider

$$\mathcal{H}_{\Sigma_H} := \bigl\{(x, \xi) \in T_H^*M \bigm| |\xi_1| \ge \tfrac{1}{2},\ \tfrac{1}{2} < |\xi|_{g(x)} < \tfrac{3}{2}\bigr\}. \tag{2-3}$$

Then $\mathcal{H}_{\Sigma_H}$ is transverse to the geodesic flow and there is $0 < \tau_{\mathrm{inj}H} < 1$ such that the map

$$\Psi : [-\tau_{\mathrm{inj}H}, \tau_{\mathrm{inj}H}] \times \mathcal{H}_{\Sigma_H} \to T^*M, \quad \Psi(t, \rho) := \varphi_t(\rho), \tag{2-4}$$

is injective. Our next lemma shows that there is $\mathfrak{D}_n > 0$ depending only on n such that one can construct a $(\mathfrak{D}_n, \tau, r)$-good cover of Σ_H.

Lemma 2.4. *There exist $\mathfrak{D}_n > 0$ depending only on n and $R_0 = R_0(n, H) > 0$ such that, for $0 < r_1 < R_0$, $0 < r_0 \le \frac{1}{2}r_1$, there exist points $\{\rho_j\}_{j=1}^{N_{r_1}} \subset \Sigma_H$ and a partition $\{\mathcal{J}_i\}_{i=1}^{\mathfrak{D}_n}$ of $\{1, \dots, N_{r_1}\}$ such that, for all $0 < \tau < \frac{1}{2}\tau_{\mathrm{inj}H}$,*

$$\begin{aligned} &\Lambda^\tau_{\Sigma_H}(r_0) \subset \bigcup_{j=1}^{N_{r_1}} \Lambda^\tau_{\rho_j}(r_1), \\ &\Lambda^\tau_{\rho_j}(3r_1) \cap \Lambda^\tau_{\rho_\ell}(3r_1) = \varnothing, \end{aligned} \qquad \text{for } j, \ell \in \mathcal{J}_i, \quad j \ne \ell.$$

Proof. We first apply Lemma 2.3 to $\widetilde{M} = \Sigma_H$ to obtain $R_0 > 0$ depending only on n and the sectional curvature of H and that of M near H such that, for $0 < r_1 < R_0$, there exist $\{\rho_j\}_{j=1}^{N_{r_1}} \subset \Sigma_H$ and a partition

$\{\mathcal{J}_i\}_{i=1}^{\mathfrak{D}_n}$ of $\{1,\dots,N_{r_1}\}$ such that

$$\Sigma_H \subset \bigcup_{j=1}^{N_{r_1}} B(\rho_j, r_1), \quad B(\rho_j, 3r_1)\cap B(\rho_\ell, 3r_1)=\varnothing \quad \text{for } j,\ell\in\mathcal{J}_i,\quad j\neq \ell,$$

$$\{\rho_j\}_{j=1}^{N_{r_1}} \text{ is a maximal } \tfrac{1}{2}r_1\text{-separated set in } \Sigma_H.$$

Now, suppose that $j,\ell\in\mathcal{J}_i$ and

$$\Lambda^\tau_{\rho_\ell}(3r_1)\cap\Lambda^\tau_{\rho_j}(3r_1)\neq\varnothing.$$

Then, there exist

$$q_\ell\in B(\rho_\ell,3r_1)\cap\mathcal{H}_{\Sigma_H},\quad q_j\in B(\rho_j,3r_1)\cap\mathcal{H}_{\Sigma_H},$$

and $t_\ell, t_j\in[-\tau,\tau]$ such that $\varphi_{t_\ell-t_j}(q_\ell)=q_j$. Here, $\mathcal{H}_\Sigma$ is the hypersurface defined in (2-3). In particular, for $\tau<\frac{1}{2}\tau_{\mathrm{inj}H}$, this implies that $q_\ell=q_j$, $t_\ell=t_j$, and hence $B(\rho_\ell,3r_1)\cap B(\rho_j,3r_1)\neq\varnothing$, a contradiction.

Now, suppose $r_0\le r_1$ and that there exists $\rho\in\Lambda^\tau_{\Sigma_H}(r_0)$ so that $\rho\notin\bigcup_{j=1,\dots,N_{r_1}}\Lambda^\tau_{\rho_j}(r_1)$. Then, there are $|t|<\tau+r_0$ and $q\in\mathcal{H}_{\Sigma_H}$ such that

$$\rho=\varphi_t(q),\quad d(q,\Sigma_H)<r_0,\quad \min_{j=1,\dots,N_{r_1}} d(q,\rho_j)\ge r_1.$$

In particular, there exists $\tilde\rho\in\Sigma_H$ with $d(q,\tilde\rho)<r_0$ such that for all $j=1,\dots,N_{r_1}$,

$$d(\tilde\rho,\rho_j)\ge d(q,\rho_j)-d(q,\tilde\rho)>r_1-r_0.$$

This contradicts the maximality of $\{\rho_j\}_{j=1}^{N_{r_1}}$ if $r_0\le\frac{1}{2}r_1$. □

We proceed to build a δ-good partition of unity associated to the cover we constructed in Lemma 2.4. The key feature in this partition is that it is invariant under the geodesic flow. Indeed, the partition is built so that its quantization commutes with the operator $P=-h^2\Delta-I$ in a neighborhood of Σ_H.

Proposition 2.5. *There exist $\tau_1=\tau_1(\tau_{\mathrm{inj}H})>0$ and $\varepsilon_1=\varepsilon_1(\tau_1)>0$, and given $0<\delta<\frac{1}{2}$ and $0<\varepsilon\le\varepsilon_1$ there exists $h_1>0$ such that, for any $0<\tau\le\tau_1$ and $R(h)\ge 2h^\delta$, the following holds.*

There exist $C_1>0$ such that for all $0<h\le h_1$ and every $(\tau,R(h))$ cover of Σ_H there exists a partition of unity

$$\chi_j\in S_\delta\cap C^\infty_c(T^*M;[-C_1h^{1-2\delta},1+C_1h^{1-2\delta}])$$

on $\Lambda^\tau_{\Sigma_H}\big(\frac{1}{2}R(h)\big)$ for which

$$\operatorname{supp}\chi_j\subset\Lambda^{\tau+\varepsilon}_{\rho_j}(R(h)),\quad \mathrm{MS}_h([P,\mathrm{Op}_h(\chi_j)])\cap\Lambda^\tau_{\Sigma_H}(\varepsilon)=\varnothing,\quad \sum_j\chi_j\equiv 1 \ \ \textit{on}\ \ \Lambda^\tau_{\Sigma_H}\big(\tfrac{1}{2}R(h)\big),$$

$\{\chi_j\}_j$ is bounded in S_δ, and $[-h^2\Delta_g,\mathrm{Op}_h(\chi_j)]$ is bounded in Ψ_δ.

Proof. The proof is identical to that of [Canzani and Galkowski 2021, Proposition 3.4]. Although the claim that $\sum_j\chi_j\equiv 1$ on $\Lambda^\tau_{\Sigma_H}\big(\frac{1}{2}R(h)\big)$ does not appear in its statement, it is included in its proof. □

3. Proof of Theorem 1.4

For each $q \in S^*M$, choose a hypersurface $H_q \subset M$ with $q \in SN^*H_q$ and $\tau_{\mathrm{inj}\,H_q} > \frac{1}{2}\,\mathrm{inj}(M)$, where $\tau_{\mathrm{inj}\,H_q}$ is defined in (2-4) and $\mathrm{inj}(M)$ is the injectivity radius of M. We next use Lemma 2.4 to generate a cover of Σ_{H_q}. Lemma 2.4 yields the existence of $\mathfrak{D}_n > 0$ depending only on n and $R_0 = R_0(n, H_q) > 0$ such that the following holds: Since by assumption $R(h) \le h^{\delta_1}$, there is $h_0 > 0$ such that $h^{\delta_2} \le R(h) \le R_0$ for all $0 < h < h_0$. Also, set $r_1 := R(h)$ and $r_0 := \frac{1}{2}R(h)$. Then, by Lemma 2.4 there exist

$$N_{R(h)} = N_{R(h)}(q, R(h)) > 0, \quad \{\rho_j\}_{j\in\mathcal{J}_q} \subset \Sigma_{H_q} with \mathcal{J}_q = \{1, \dots, N_{R(h)}\},$$

and a partition $\{\mathcal{J}_{q,i}\}_{i=1}^{\mathfrak{D}_n}$ of $\mathcal{J}_q$, such that, for all $0 < \tau < \frac{1}{2}\tau_{\mathrm{inj}H_q}$,

$$\Lambda^\tau_{\Sigma_{H_q}}\big(\tfrac{1}{2}R(h)\big) \subset \bigcup_{j\in\mathcal{J}_q} \Lambda^\tau_{\rho_j}(R(h)), \tag{3-1}$$

$$\bigcup_{i=1}^{\mathfrak{D}_n} \mathcal{J}_{q,i} = \mathcal{J}_q, \tag{3-2}$$

$$\Lambda^\tau_{\rho_{j_1}}(3R(h)) \cap \Lambda^\tau_{\rho_{j_2}}(3R(h)) = \varnothing \quad \text{for } j_1, j_2 \in \mathcal{J}_{q,i}, \quad j_1 \ne j_2. \tag{3-3}$$

By (3-1) there is an h-independent open neighborhood of q, $V_q \subset S^*M$, covered by tubes as in Lemma 2.4. Since S^*M is compact, we may choose $\{q_\ell\}_{\ell=1}^L$ with L independent of h such that $S^*M \subset \bigcup_{\ell=1}^L V_{q_\ell}$. In particular, if $0 < \tau \le \min_{1\le\ell\le L} \tau_{H_{q_\ell}}$ and for each $\ell \in \{1, \dots, L\}$ we let

$$\mathcal{T}_{q_\ell, j} = \Lambda^\tau_{\rho_j}(R(h)),$$

then there is $\mathfrak{D}_M > 0$ such that

$$\bigcup_{\ell=1}^L \{\mathcal{T}_{q_\ell, j}\}_{j\in\mathcal{J}_{q_\ell}}$$

is a $(\mathfrak{D}_M, \tau, R(h))$-good cover for S^*M. Let $\{\psi_{q_\ell}\}_{\ell=1}^L \subset C_c^\infty(T^*M)$ satisfy

$$\operatorname{supp} \psi_{q_\ell} \subset \left\{(x, \xi) \in T^*M \setminus \{0\} \;\middle|\; \left(x, \frac{\xi}{|\xi|_g}\right) \in V_{q_\ell}\right\} \quad \text{for all } \ell = 1, \dots, L,$$

$$\sum_{\ell=1}^L \psi_{q_\ell} \equiv 1 \text{ in an } h\text{-independent neighborhood of } S^*M.$$

We split the analysis of u in two parts: near and away from the characteristic variety $\{p = 0\} = S^*M$. In what follows we use C to denote a positive constant that may change from line to line.

3A. ***It suffices to study u near the characteristic variety.*** In this section we reduce the study of $\|u\|_{L^p(U)}$ to an h-dependent neighborhood of the characteristic variety $\{p = 0\} = S^*M$. We will use repeatedly the following result.

Lemma 3.1. *For all $\varepsilon > 0$ and all $p \ge 2$, there exists $C > 0$ such that*

$$\|u\|_{L^p} \le Ch^{n(1/p-1/2)} \|u\|_{H_h^{n(1/2-1/p)+\varepsilon}}. \tag{3-4}$$

Proof. By [Galkowski 2019, Lemma 6.1] (or more precisely its proof), for any $\varepsilon > 0$, there exists $C_\varepsilon \geq 1$ so that $\|\mathrm{Id}\|_{H_h^{n/2+\varepsilon} \to L^\infty} \leq C_\varepsilon h^{-n/2}$. By complex interpolation of $\mathrm{Id} : L^2 \to L^2$ and $\mathrm{Id} : H_h^{n/2+\varepsilon} \to L^\infty$ with $\theta = 2/p$, we obtain $\|\mathrm{Id}\|_{H_h{}^{(n/2+\varepsilon)(1-\theta)} \to L^p} \leq C_\varepsilon^{1-\theta} h^{-n(1-\theta)/2}$, and this yields (3-4). □

Observe that

$$u = \sum_{\ell=1}^{L} \mathrm{Op}_h(\psi_{q_\ell})u + \Bigl(1 - \sum_{\ell=1}^{L} \mathrm{Op}_h(\psi_{q_\ell})\Bigr)u.$$

Since $1 - \sum_{\ell=1}^{L} \psi_{q_\ell} = 0$ in an h-independent neighborhood of $S^*M = \{p = 0\}$, by the standard elliptic parametrix construction (e.g., [Dyatlov and Zworski 2019, Appendix E]) there is $E \in \Psi^{-2}(M)$ with

$$1 - \sum_{\ell=1}^{L} \mathrm{Op}_h(\psi_{q_\ell}) = EP + O(h^\infty)_{\Psi^{-\infty}}. \tag{3-5}$$

Next, combining (3-5) with Lemma 3.1 and using that $h^{n(1/p-1/2)} = h^{-\delta(p)+1/2}h^{-1}$, we have

$$\begin{aligned}\Bigl\|\Bigl(1 - \sum_{\ell=1}^{L} \mathrm{Op}_h(\psi_{q_\ell})\Bigr)u\Bigr\|_{L^p} &\leq Ch^{n(1/p-1/2)}\|EPu\|_{H_h^{n(1/2-1/p)+\varepsilon}} + O(h^\infty)\|u\|_{L^2} \\ &\leq Ch^{-\delta(p)+1/2}h^{-1}\|Pu\|_{H_h^{n(1/2-1/p)+\varepsilon-2}} + O(h^\infty)\|u\|_{L^2}. \end{aligned} \tag{3-6}$$

It remains to understand the terms $\mathrm{Op}_h(\psi_{q_\ell})u$. Since there are finitely many such terms,

$$\Bigl\|\sum_{\ell=1}^{L} \mathrm{Op}_h(\psi_{q_\ell})u\Bigr\|_{L^p} \leq \sum_{\ell=1}^{L} \|\mathrm{Op}_h(\psi_{q_\ell})u\|_{L^p}, \tag{3-7}$$

and we consider each term $\|\mathrm{Op}_h(\psi_{q_\ell})u\|_{L^p}$ individually.

By Proposition 2.5, for each $\ell = 1, \dots, L$, there exist $\tau_1(q_\ell) > 0$ and $\varepsilon_1(q_\ell) > 0$ and a family of cutoffs $\{\tilde{\chi}_{\mathcal{T}_{q_\ell,j}}\}_{j \in \mathcal{J}_{q_\ell}}$ with $\tilde{\chi}_{\mathcal{T}_{q_\ell,j}}$ supported in $\Lambda_{\rho_j}^{\tau+\varepsilon_1(q_\ell)}(R(h))$ such that, for all $0 < \tau < \tau_1(q_\ell)$,

$$\sum_{j \in \mathcal{J}_{q_\ell}} \tilde{\chi}_{\mathcal{T}_{q_\ell,j}} \equiv 1 \quad \text{on } \Lambda^\tau_{\Sigma_{H_{q_\ell}}}\bigl(\tfrac{1}{2}R(h)\bigr). \tag{3-8}$$

Let $\tau_0(q_\ell)$ be as in [Canzani and Galkowski 2021, Theorem 10]. Then, set

$$\tau_M := \min_{1 \leq \ell \leq L} \bigl\{\tfrac{1}{4}\,\mathrm{inj}(M),\ \tau_0(q_\ell),\ \tau_1(q_\ell),\ \tfrac{1}{2}\tau_{\mathrm{inj}\,H_{q_\ell}}\bigr\}.$$

From now on we work with tubes $\mathcal{T}_{q_\ell,j} = \Lambda^\tau_{\rho_j}(R(h))$ for some $0 < \tau < \tau_M$. Next, we localize u near and away from $\Lambda^\tau_{\Sigma_{H_{q_\ell}}}(h^\delta)$:

$$\mathrm{Op}_h(\psi_{q_\ell})u = \sum_{j \in \mathcal{J}_{q_\ell}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}})\,\mathrm{Op}_h(\psi_{q_\ell})u + \Bigl(1 - \sum_{j \in \mathcal{J}_{q_\ell}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}})\Bigr)\,\mathrm{Op}_h(\psi_{q_\ell})u.$$

Remark 3.2. We refer to functions of the form $\mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}})u$ as *geodesic beams*. One can check using Proposition 2.5 that if u solves $Pu = O(h)_{L^2}$, then the geodesic beams solve

$$P\mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}})u = O(h)_{H_h^k}$$

for any k and are localized to an $R(h)$ neighborhood of a length ~ 1 segment of a geodesic.

In particular, by (3-8), $\frac{1}{2}R(h) \ge \frac{1}{2}h^{\delta_2}$, and [Canzani and Galkowski 2021, Lemma 3.6], there is $E \in h^{-\delta_2}\Psi^{\mathrm{comp}}_{\delta_2}$ so that

$$\Big(1 - \sum_{j\in\mathcal{J}_{q_\ell}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}})\Big)\,\mathrm{Op}_h(\psi_{q_\ell}) = EP + O_{\Psi^{-\infty}}(h^\infty). \tag{3-9}$$

Since $h^{n(1/p-1/2)-\delta_2} = h^{-\delta(p)+1/2-\delta_2}h^{-1}$, combining (3-9) with Lemma 3.1 yields

$$\Big\|\Big(1 - \sum_{j\in\mathcal{J}_{q_\ell}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}})\Big)\,\mathrm{Op}_h(\psi_{q_\ell})u\Big\|_{L^p} \le Ch^{-\delta(p)-1/2-\delta_2}\|Pu\|_{H_h^{n(1/2-1/p)+\varepsilon-2}} + O(h^\infty)\|u\|_{L^2}. \tag{3-10}$$

Combining (3-6), (3-7), and (3-10), we have proved that for $U \subset M$,

$$\begin{aligned}\|u\|_{L^p(U)} \le \sum_{\ell=1}^{L} \Big\|\sum_{j\in\mathcal{J}_{q_\ell}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}})\,\mathrm{Op}_h(\psi_{q_\ell})u\Big\|_{L^p(U)}&\\ + Ch^{-\delta(p)+1/2-\delta_2}h^{-1}\|Pu\|_{H_h^{n(1/2-1/p)+\varepsilon-2}} + O(h^\infty)\|u\|_{L^2}.&\end{aligned} \tag{3-11}$$

3B. ***Filtering tubes by L^2 mass.*** By (3-11) it only remains to control terms of the form

$$\Big\|\sum_{j\in\mathcal{J}_{q_\ell}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}})\,\mathrm{Op}_h(\psi_{q_\ell})u\Big\|_{L^p},$$

where u is localized to V_{q_ℓ} within the characteristic variety S^*M and, more importantly, to the tubes $\mathcal{T}_{q_\ell,j}$. We fix ℓ and, abusing notation slightly, write

$$\psi := \psi_{q_\ell}, \quad \mathcal{J} = \mathcal{J}_{q_\ell}, \quad \mathcal{T}_j = \mathcal{T}_{q_\ell,j}, \quad \tilde{\chi}_{\mathcal{T}_j} := \tilde{\chi}_{\mathcal{T}_{q_\ell,j}}, \quad v := \sum_{j\in\mathcal{J}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})\,\mathrm{Op}_h(\psi)u. \tag{3-12}$$

Let $T = T(h) \ge 1$. For each $j \in \mathcal{J}$ let

$$\chi_{\mathcal{T}_j} \in C_c^\infty(T^*M; [0,1]) \cap S_\delta \tag{3-13}$$

be a smooth cut-off function with $\operatorname{supp}\chi_{\mathcal{T}_j} \subset \mathcal{T}_j$ and $\chi_{\mathcal{T}_j} \equiv 1$ on $\operatorname{supp}\tilde{\chi}_{\mathcal{T}_j}$, and such that $\{\chi_j\}_j$ is bounded in S_δ. We shall work with the modified norm

$$\|u\|_{P,T} := \|u\|_{L^2} + \frac{T}{h}\|Pu\|_{L^2}.$$

Note that this norm is the natural norm for obtaining $T^{-1/2}$ improved estimates in L^p bounds since the fact that u is an $o(T^{-1}h)$ quasimode implies, roughly, that u is an accurate solution to $(hD_t + P)u = 0$ for times $t \le T$. For each integer $k \ge -1$, we consider the set

$$\mathcal{A}_k = \Big\{ j \in \mathcal{J} : \frac{1}{2^{k+1}}\|u\|_{P,T} \le \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})u\|_{L^2} + h^{-1}\|\mathrm{Op}_h(\chi_{\mathcal{T}_j})Pu\|_{L^2} \le \frac{1}{2^k}\|u\|_{P,T} \Big\}. \tag{3-14}$$

It follows that $\mathcal{A}_k$ consists of those tubes $\mathcal{T}_j$ with L^2 mass comparable to 2^{-k}.

Remark 3.3. Note that if $A \in \Psi_\delta$ and $\mathrm{MS}_h(A) \subset \{\chi_{\mathcal{T}_j} \equiv 1\}$, then the elliptic estimate implies

$$\|Av\|_{L^2} \le C\|\mathrm{Op}_h(\chi_{\mathcal{T}_j})v\|_{L^2} + O(h^\infty)\|v\|_{L^2}.$$

In particular, if $j \in \mathcal{A}_k$ and $\mathrm{MS}_h(A) \subset \{\chi_{\mathcal{T}_j} \equiv 1\}$, then

$$\|Au\|_{L^2} + h^{-1}\|APu\|_{L^2} \le C2^{-k}\|u\|_{P,T} + O(h^\infty)\|u\|_{P,T}.$$

Observe that since $|\chi_{\mathcal{T}_j}| \le 1$, for h small enough depending on finitely many seminorms of χ_j, $\|\mathrm{Op}_h(\chi_{\mathcal{T}_j})\|_{L^2\to L^2} \le 2$. In particular, this together with $T \ge 1$ implies that

$$\mathcal{J} = \bigcup_{k\ge -1} \mathcal{A}_k. \tag{3-15}$$

Lemma 3.4. *There exists $C_n > 0$ so that for all $k \ge -1$*

$$|\mathcal{A}_k| \le C_n 2^{2k}. \tag{3-16}$$

Proof. According to (3-2), the collection $\{\mathcal{T}_j\}_{j\in\mathcal{J}}$ can be partitioned into $\mathfrak{D}_n$ sets of disjoint tubes. Thus, we have $\sum_{j\in\mathcal{J}} |\chi_{\mathcal{T}_j}|^2 \le \mathfrak{D}_n$ and there is $C_n > 0$ depending only on n such that

$$\Big\|\sum_{j\in\mathcal{J}} \mathrm{Op}_h(\chi_{\mathcal{T}_j})^* \,\mathrm{Op}_h(\chi_{\mathcal{T}_j})\Big\|_{L^2\to L^2} \le C_n.$$

In particular,

$$\sum_{j\in\mathcal{J}} \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})u\|^2_{L^2} \le C_n\|u\|^2_{L^2},$$
$$\sum_{j\in\mathcal{J}} \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})Pu\|^2_{L^2} \le C_n\|Pu\|^2_{L^2}.$$

Therefore,

$$|\mathcal{A}_k|2^{-2k-2}\|u\|^2_{P,T} \le 2\bigg(\sum_{j\in\mathcal{A}_k} \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})u\|^2_{L^2} + h^{-2}\,\|\mathrm{Op}_h(\chi_{\mathcal{T}_j})Pu\|^2_{L^2}\bigg) \le C_n\|u\|^2_{P,T}. \qquad \square$$

Next, let

$$w_k := \sum_{j\in\mathcal{A}_k} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})\mathrm{Op}_h(\psi)u. \tag{3-17}$$

Then, by (3-12) and (3-15) we have

$$v = \sum_{k=-1}^{\infty} w_k. \tag{3-18}$$

The goal is therefore to control $\|w_k\|_{L^p(U)}$ for each k since the triangle inequality yields

$$\|v\|_{L^p(U)} \le \sum_{k=-1}^{\infty} \|w_k\|_{L^p(U)}.$$

3C. ***Filtering tubes by L^∞ weight on shrinking balls.*** By Lemma 2.3, there are points $\{x_\alpha\}_{\alpha\in\mathcal{I}}\subset M$ such that there exists a partition $\{\mathcal{I}_i\}_{i=1}^{\mathfrak{D}_n}$ of $\mathcal{I}$ such that

$$M\subset\bigcup_{\alpha\in\mathcal{I}}B(x_\alpha,R(h)),$$
$$B(x_{\alpha_1},3R(h))\cap B(x_{\alpha_2},3R(h))=\varnothing\quad\text{for }\alpha_1,\alpha_2\in\mathcal{I}_i,\ \ \alpha_1\neq\alpha_2.$$

Then, for $m\in\mathbb{Z}$, define

$$\mathcal{I}_{k,m}:=\left\{\alpha\in\mathcal{I}_U:2^{m-1}\le h^{(n-1)/2}R(h)^{(1-n)/2}2^k\frac{\|w_k\|_{L^\infty(B(x_\alpha,R(h)))}}{\|u\|_{P,T}}\le 2^m\right\},\tag{3-19}$$

where $\mathcal{I}_U:=\{\alpha\in\mathcal{I}:B(x_\alpha,R(h))\cap U\neq\varnothing\}$. For each $k\in\mathbb{Z}_+$ and $\alpha\in\mathcal{I}$, consider the sets

$$\mathcal{A}_k(\alpha):=\{j\in\mathcal{A}_k:\pi_M(\mathcal{T}_j)\cap B(x_\alpha,2R(h))\neq\varnothing\},$$

where $\pi_M:T^*M\to M$ is the standard projection. The indices in $\mathcal{A}_k$ are those that correspond to tubes with mass comparable to $\frac{1}{2^k}\|u\|_{P,T}$, while indices in $\mathcal{A}_k(\alpha)$ correspond to tubes of mass $\frac{1}{2^k}\|u\|_{P,T}$ that run over the ball $B(x_\alpha,2R(h))$. In particular, we claim that Lemma 3.4 and [Canzani and Galkowski 2021, Lemma 3.7] yield the existence of $C_n,c_M>0$ such that

$$c_M2^m\le|\mathcal{A}_k(\alpha)|\le C_n2^{2k}\quad\text{for }\alpha\in\mathcal{I}_{k,m}.\tag{3-20}$$

The upper bound follows directly from Lemma 3.4, while, to obtain the lower bound, we first observe that for $\alpha\in\mathcal{I}_{k,m}$,

$$2^{m-1}h^{(1-n)/2}R(h)^{(n-1)/2}2^{-k}\|u\|_{P,T}\le\|w_k\|_{L^\infty(B(x_\alpha,R(h)))}.\tag{3-21}$$

In addition, (3-14) and [Canzani and Galkowski 2021, Lemma 3.7] imply that there exist $c_M>0$, $\tau_M>0$, and $C_n>0$, depending on M and n respectively, such that for all $N>0$ there exists $C_N>0$ with

$$\begin{aligned}\|w_k\|_{L^\infty(B(x_\alpha,R(h)))}&\\ &\le\frac{C_nR(h)^{(n-1)/2}}{\tau_M^{1/2}h^{(n-1)/2}}\sum_{j\in\mathcal{A}_k(\alpha)}\|\mathrm{Op}_h(\tilde\chi_{\mathcal{T}_j})\,\mathrm{Op}_h(\psi)u\|_{L^2}+h^{-1}\|\mathrm{Op}_h(\tilde\chi_{\mathcal{T}_j})P\,\mathrm{Op}_h(\psi)u\|_{L^2}+C_Nh^N\|u\|_{P,T}\\ &\le c_M^{-1}h^{-(n-1)/2}R(h)^{(n-1)/2}2^{-k}\|u\|_{P,T}|\mathcal{A}_k(\alpha)|+C_Nh^N\|u\|_{P,T},\end{aligned}$$

which, combined with (3-21), proves the lower bound in (3-20). To obtain the second bound we used Remark 3.3. To simplify notation, let

$$\mathcal{A}_{k,m}:=\bigcup_{\alpha\in\mathcal{I}_{k,m}}\mathcal{A}_k(\alpha).\tag{3-22}$$

Note that for each $\alpha\in\mathcal{I}_{k,m}$, there is $\tilde x_\alpha\in B(x_\alpha,R(h))$ such that

$$|w_k(\tilde x_\alpha)|\ge 2^{m-1}h^{(1-n)/2}R(h)^{(n-1)/2}2^{-k}\|u\|_{P,T}.\tag{3-23}$$

We finish this section with a result that controls the size of $\mathcal{I}_{k,m}$ in terms of that of $\mathcal{A}_{k,m}$. Let

$$\tfrac12(\delta_2+1)<\rho<1,\tag{3-24}$$

$0 < \varepsilon < \delta$, $\tilde{\chi} \in C_c^\infty((-1,1))$, and define the operator

$$\chi_{h,\tilde{x}_\alpha} u(x) := \tilde{\chi}\Big(\frac{1}{\varepsilon} h^{-\rho} d(x, \tilde{x}_\alpha)\Big)\Big[\mathrm{Op}_h\Big(\tilde{\chi}\Big(\frac{1}{\varepsilon}(|\xi|_g - 1)\Big)\Big)u\Big](x).$$

In Lemma 6.2 we prove that $\chi_{h,\tilde{x}_\alpha} \in \Psi^{-\infty}_{\Gamma_{\tilde{x}_\alpha}, L_{\tilde{x}_\alpha}, \rho}$, where

$$\Omega_{\tilde{x}_\alpha} = \{\xi \in T^*_{\tilde{x}_\alpha} M : |1 - |\xi|_{g(\tilde{x}_\alpha)}| < \delta\}, \qquad \Gamma_{\tilde{x}_\alpha} = \bigcup_{|t| < \frac{1}{2}\operatorname{inj}(M)} \varphi_t(\Omega_{\tilde{x}_\alpha}),$$

and $\Psi^{-\infty}_{\Gamma_{\tilde{x}_\alpha}, L_{\tilde{x}_\alpha}, \rho}$ is a class of smoothing pseudodifferential operators that allows for localization to h^ρ neighborhoods of $\Gamma_{\tilde{x}_\alpha}$ and is compatible with localization to h^ρ neighborhoods of the foliation $L_{\tilde{x}_\alpha}$ of $\Gamma_{\tilde{x}_\alpha}$ generated by $\Omega_{\tilde{x}_\alpha}$.

In Theorem 6.3 for $\varepsilon > 0$ we explain how to build a cut-off operator $X_{\tilde{x}_\alpha} \in \Psi^{-\infty}_{\Gamma_{\tilde{x}_\alpha}, L_{\tilde{x}_\alpha}, \rho}$ such that

$$\begin{cases} \chi_{h,\tilde{x}_\alpha} X_{\tilde{x}_\alpha} = \chi_{h,\tilde{x}_\alpha} + O(h^\infty)_{\Psi^{-\infty}}, \\ \mathrm{WF}_h{}'([P, X_{\tilde{x}_\alpha}]) \cap \big\{(x,\xi) : x \in B\big(\tilde{x}_\alpha, \tfrac{1}{2}\operatorname{inj} M\big),\ \xi \in \Omega_x\big\} = \varnothing, \end{cases} \tag{3-25}$$

where $\operatorname{inj} M$ denotes the injectivity radius of M. Moreover, $X_{\tilde{x}_\alpha}$ acts microlocally in the sense that if $a, b \in S(T^*M)$ with $\operatorname{supp} a \cap \operatorname{supp} b = \varnothing$, then

$$\mathrm{Op}_h(a) X_{\tilde{x}_\alpha} \mathrm{Op}_h(b) = O(h^\infty)_{\Psi^{-\infty}}. \tag{3-26}$$

Lemma 3.5. *Let $\frac{1}{2}(\delta_2 + 1) < \rho \le 1$. There exists $C > 0$ such that for every $k \ge -1$ and $m \in \mathbb{Z}$ the following holds*: *if*

$$|\mathcal{A}_{k,m}| \le C\, 2^{2m} R(h)^{n-1} (h^{\rho - 1/2} R(h)^{-1/2})^{-2n(n-1)/(3n+1)},$$

then

$$|\mathcal{I}_{k,m}| \le C |\mathcal{A}_{k,m}| 2^{-2m} R(h)^{1-n}. \tag{3-27}$$

Proof. We claim that by (3-17), for $\alpha \in \mathcal{I}_{k,m}$,

$$\chi_{h,\tilde{x}_\alpha} w_k = \chi_{h,\tilde{x}_\alpha} w_{k,m} + O(h^\infty \|u\|_{L^2}) \quad \text{and} \quad w_{k,m} := \sum_{j \in \mathcal{A}_{k,m}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j}) \mathrm{Op}_h(\psi) u. \tag{3-28}$$

Indeed, it suffices to show that $\chi_{h,\tilde{x}_\alpha} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j}) \mathrm{Op}_h(\psi) u = O(h^\infty \|u\|_{L^2})$ for $\alpha \in \mathcal{I}_{k,m}$ and $j \notin \mathcal{A}_{k,m}$. Note that for such indices $\pi_M(\mathcal{T}_j) \cap B(\tilde{x}_\alpha, 2R(h)) = \varnothing$, while

$$\operatorname{supp} \tilde{\chi}\Big(\frac{1}{\varepsilon} h^{-\rho} d(x, \tilde{x}_\alpha)\Big) \subset B(\tilde{x}_\alpha, C\varepsilon h^\rho) \subset B\big(x_\alpha, \tfrac{3}{2} R(h)\big)$$

for some $C > 0$ and all h small enough.

Our next goal is to produce a lower bound for $|\mathcal{A}_{k,m}|$ in terms of $|\mathcal{I}_{k,m}|$ by using the lower bound (3-23) on $\|\chi_{h,\tilde{x}_\alpha} w_{k,m}\|_{L^\infty}$ for indices $\alpha \in \mathcal{I}_{k,m}$. By (3-25), we have

$$\chi_{h,\tilde{x}_\alpha} w_{k,m} = \chi_{h,\tilde{x}_\alpha} X_{\tilde{x}_\alpha} w_{k,m} + O(h^\infty)_{L^\infty}$$

for $\alpha \in \mathcal{I}_{k,m}$.

Next, note that since $\mathrm{MS}_h(\tilde{\chi}_{\mathcal{T}_j}) \subset \{||\xi|_g - 1| \ll \varepsilon\}$, using (3-26) we have

$$\begin{aligned}
\mathrm{Op}_h&\Big(\tilde{\chi}\Big(\frac{1}{\varepsilon}(|\xi|_g-1)\Big)i\Big)X_{\tilde{x}_\alpha}w_{k,m}\\
&= \mathrm{Op}_h\Big(\tilde{\chi}\Big(\frac{1}{\varepsilon}(|\xi|_g-1)\Big)\Big)X_{\tilde{x}_\alpha}\,\mathrm{Op}_h\Big(\tilde{\chi}\Big(\frac{10}{\varepsilon}(|\xi|_g-1)\Big)\Big)w_{k,m} + O(h^\infty\|u\|_{P,T})_{L^\infty}\\
&= X_{\tilde{x}_\alpha}w_{k,m} + O(h^\infty\|u\|_{P,T})_{L^\infty}.
\end{aligned}$$

In particular, using this with (3-23) and (3-28),

$$\begin{aligned}
2^{m-1}h^{(1-n)/2}R(h)^{(n-1)/2}2^{-k}\|u\|_{P,T} &\le \|\chi_{h,\tilde{x}_\alpha}w_k\|_{L^\infty}\\
&\le \Big\|\mathrm{Op}_h\Big(\tilde{\chi}\Big(\frac{1}{\varepsilon}(|\xi|_g-1)\Big)\Big)X_{\tilde{x}_\alpha}w_{k,m}\Big\|_{L^\infty}+O(h^\infty)\|u\|_{P,T}\\
&= \|X_{\tilde{x}_\alpha}w_{k,m}\|_{L^\infty}+O(h^\infty)\|u\|_{P,T}.
\end{aligned}\tag{3-29}$$

Therefore, applying the standard L^∞ bound for quasimodes of the Laplacian (see, e.g., [Zworski 2012, Theorem 7.12]) and using, by (3-25), that $X_{\tilde{x}_\alpha}$ nearly commutes with P on $B\big(\tilde{x}_\alpha, \frac{1}{2}\operatorname{inj} M\big)$, we have

$$\begin{aligned}
2^{m-1}R(h)^{(n-1)/2}2^{-k}\|u\|_{P,T} &\le C(\|X_{\tilde{x}_\alpha}w_{k,m}\|_{L^2}+h^{-1}\|PX_{\tilde{x}_\alpha}w_{k,m}\|_{L^2(B)}) + O(h^\infty\|u\|_{P,T}).\\
&\le C(\|X_{\tilde{x}_\alpha}w_{k,m}\|_{L^2}+h^{-1}\|X_{\tilde{x}_\alpha}Pw_{k,m}\|_{L^2}) + O(h^\infty\|u\|_{P,T}).
\end{aligned}\tag{3-30}$$

Note that we have canceled the factor $h^{(1-n)/2}$ which appears both in (3-29) and the standard L^∞ bounds for quasimodes. Using that $h^{2\rho-1}R(h)^{-1} = o(1)$, Proposition 6.6 proves that, for all $\widetilde{\mathcal{I}} \subset \mathcal{I}_{k,m}$ and $v \in L^2(M)$,

$$\sum_{\alpha\in\widetilde{\mathcal{I}}}\|X_{\tilde{x}_\alpha}v\|^2_{L^2} \le C(1+a_h|\widetilde{\mathcal{I}}|^{(3n+1)/(2n)})\|v\|^2_{L^2},$$

where $a_h = (h^{\rho-1/2}R(h)^{-1/2})^{n-1}$. As a consequence, (3-30) gives

$$\begin{aligned}
|\widetilde{\mathcal{I}}|R(h)^{n-1}2^{-2k}2^{2(m-1)}\|u\|^2_{P,T} &\le C\Big(\sum_{\alpha\in\widetilde{\mathcal{I}}}\|X_{\tilde{x}_\alpha}w_{k,m}\|^2_{L^2} + h^{-2}\sum_{\alpha\in\widetilde{\mathcal{I}}}\|X_{\tilde{x}_\alpha}Pw_{k,m}\|^2_{L^2}\Big)\\
&\le C(1+a_h|\widetilde{\mathcal{I}}|^{(3n+1)/(2n)})(\|w_{k,m}\|^2_{L^2}+h^{-2}\|Pw_{k,m}\|^2_{L^2})\\
&\le C(1+a_h|\widetilde{\mathcal{I}}|^{(3n+1)/(2n)})2^{-2k}|\mathcal{A}_{k,m}|\|u\|^2_{P,T}.
\end{aligned}$$

The last inequality follows from the definition of $w_{k,m}$ together with the definition of $\mathcal{A}_k$ in (3-14).

In particular, we have proved that there is $C > 0$ such that for all $\widetilde{\mathcal{I}} \subset \mathcal{I}_{k,m}$,

$$|\widetilde{\mathcal{I}}|R(h)^{n-1}2^{2m} \le C\max(1, a_h|\widetilde{\mathcal{I}}|^{(3n+1)/(2n)})|\mathcal{A}_{k,m}|.\tag{3-31}$$

Now, suppose that $a_h|\mathcal{I}_{k,m}|^{(3n+1)/(2n)} \ge 1$. Then, there exists $\widetilde{\mathcal{I}} \subset \mathcal{I}_{k,m}$ such that $a_h|\widetilde{\mathcal{I}}|^{(3n+1)/(2n)} = 1$. In particular, $|\widetilde{\mathcal{I}}|R(h)^{n-1}2^{2m} \le C|\mathcal{A}_{k,m}|$. This implies that if

$$|\mathcal{A}_{k,m}| \le \frac{1}{C}a_h^{-(2n)/(3n+1)}R(h)^{n-1}2^{2m},$$

then $a_h|\mathcal{I}_{k,m}|^{(3n+1)/(2n)} \le 1$, and so by (3-31),

$$|\mathcal{I}_{k,m}|R(h)^{n-1}2^{2m} \le C|\mathcal{A}_{k,m}|. \qquad \square$$

Note that for $w_{k,m}$ defined as in (3-28),

$$\|w_k\|^p_{L^p(U)} \le \mathfrak{D}_n \sum_{m=-\infty}^{\infty} \|w_k\|^p_{L^p(U_{k,m})} = \mathfrak{D}_n \sum_{m=-\infty}^{\infty} \|w_{k,m}\|^p_{L^p(U_{k,m})} + O(h^\infty \|u\|_{P,T}), \tag{3-32}$$

where

$$U_{k,m} := \bigcup_{\alpha \in \mathcal{I}_{k,m}} B(x_\alpha, R(h)). \tag{3-33}$$

Finally, we split the study of $\|w_k\|_{L^p(U)}$ into two regimes: tubes with low or high L^∞ mass. Fix $N > 0$ large, to be determined later. (Indeed, we will see that it suffices to take $N > \frac{1}{2}(1 - p_c/p)^{-1}$.) Then, we claim that, for each $k \ge -1$,

$$\|w_k\|^p_{L^p(U)} \le \mathfrak{D}_n \sum_{m=-\infty}^{m_{1,k}} \|w_{k,m}\|^p_{L^p(U_{k,m})} + \mathfrak{D}_n \sum_{m=m_{1,k}+1}^{m_{2,k}} \|w_{k,m}\|^p_{L^p(U_{k,m})} + O(h^\infty \|u\|_{P,T}), \tag{3-34}$$

where $m_{1,k}$ and $m_{2,k}$ are defined by

$$\begin{aligned} 2^{m_{1,k}} &= \min\biggl(\frac{2^k R(h)^{(1-n)/2}}{T^N}, c_n 2^{2k}, c_0 R(h)^{1-n} \biggr), \\ 2^{m_{2,k}} &= \min(c_n 2^{2k}, c_0 R(h)^{1-n}), \end{aligned}$$

where c_0 and c_n are described in what follows. Indeed, note that the bound (3-20) yields that 2^m is bounded by $|\mathcal{A}_k(\alpha)|$ for all $\alpha \in \mathcal{I}_{k,m}$, and the latter is controlled by $c_0 R(h)^{n-1}$ for some $c_0 > 0$, depending only on (M, g). Also, note that by (3-20) the $w_{k,m}$ are only defined for m satisfying $2^m \le c_n 2^{2k}$. These observations justify that the second sum in (3-34) runs only up to $m_{2,k}$.

3D. ***Control of the low L^∞ mass term, $m \le m_{1,k}$.*** We first estimate the small m term in (3-34). The estimates here essentially amount to interpolation between L^{p_c} and L^∞. From the definition of $\mathcal{I}_{k,m}$ in (3-19), together with $\frac{1}{2}(1-n)(p-p_c) - 1 = -p\delta(p)$ and using Sogge's L^{p_c} estimate

$$\begin{aligned} \|w_{k,m}\|_{L^{p_c}(U_{k,m})} &\le C h^{-1/p_c} (\|w_{k,m}\|_{L^2} + h^{-1} \|P w_{k,m}\|_{L^2}) \\ &\le C h^{-1/p_c} \|u\|_{P,T}, \end{aligned}$$

we obtain

$$\begin{aligned} \sum_{m=-\infty}^{m_{1,k}} \|w_{k,m}\|^p_{L^p(U_{k,m})} &\le C \sum_{m=-\infty}^{m_{1,k}} \|w_{k,m}\|^{p-p_c}_{L^\infty(U_{k,m})} \|w_{k,m}\|^{p_c}_{L^{p_c}(U_{k,m})} \\ &\le C h^{-p\delta(p)} R(h)^{(n-1)(p-p_c)/2} 2^{-k(p-p_c)} \sum_{m=-\infty}^{m_{1,k}} 2^{m(p-p_c)} \|u\|^p_{P,T} \\ &\le C h^{-p\delta(p)} R(h)^{(n-1)(p-p_c)/2} 2^{(m_{1,k}-k)(p-p_c)} \|u\|^p_{P,T}. \end{aligned}$$

It follows that

$$\sum_{k \ge -1} \biggl(\sum_{m=-\infty}^{m_{1,k}} \|w_{k,m}\|^p_{L^p(U_{k,m})} \biggr)^{1/p} \le C h^{-\delta(p)} R(h)^{(n-1)(1-p_c/p)/2} \|u\|_{P,T} \sum_{k \ge -1} 2^{(m_{1,k}-k)(1-p_c/p)}. \tag{3-35}$$

Finally, define k_1 and k_2 such that

$$2^{k_1} = \frac{R(h)^{(1-n)/2}}{c_n T^N} \quad \text{and} \quad 2^{k_2} = c_0 R(h)^{(1-n)/2} T^N. \tag{3-36}$$

If $k \le k_1$, then $2^{m_{1,k}} = c_n 2^{2k}$, so there exists $C_{n,p} > 0$ such that

$$\sum_{k=-1}^{k_1} 2^{(m_{1,k}-k)(1-p_c/p)} \le C_{n,p} \frac{R(h)^{(1-n)(1-p_c/p)/2}}{T^{N(1-p_c/p)}}.$$

If $k_1 \le k \le k_2$, then $2^{m_{1,k}} = 2^k R(h)^{(1-n)/2}/T^N$. Therefore, since $|k_2 - k_1| \le cN \log T$ for some $c > 0$, there exists $C > 0$ such that

$$\sum_{k=k_1}^{k_2} 2^{(m_{1,k}-k)(1-p_c/p)} \le CN \log T \frac{R(h)^{(1-n)(1-p_c/p)/2}}{T^{N(1-p_c/p)}}.$$

Last, if $k \ge k_2$, then $2^{m_{1,k}} = c_0 R(h)^{1-n}$, so there exists $C_p > 0$ such that

$$\sum_{k=k_2}^{\infty} 2^{(m_{1,k}-k)(1-p_c/p)} \le C_p \frac{R(h)^{(1-n)(1-p_c/p)/2}}{T^{N(1-p_c/p)}}.$$

Putting these three bounds together with (3-35), we obtain

$$\sum_{k \ge -1} \left(\sum_{m=-\infty}^{m_{1,k}} \|w_{k,m}\|^p_{L^p(U_{k,m})} \right)^{1/p} \le C h^{-\delta(p)} \frac{N \log T}{T^{N(1-p_c/p)}} \|u\|_{P,T}. \tag{3-37}$$

3E. ***Control of the high L^∞ mass term, $m \le m_{1,k}$.*** In this section we estimate the large m term in (3-34). To do this we write

$$\mathcal{A}_{k,m} = \mathcal{G}_{k,m} \sqcup \mathcal{B}_{k,m},$$

where the set of "good" tubes $\bigcup_{j \in \mathcal{G}_{k,m}} \mathcal{T}_j$ is $[t_0, T]$ non-self-looping and the number of "bad" tubes $|\mathcal{B}_{k,m}|$ is small. To do this, let

$$\mathcal{B}_U(\alpha, \beta) := \left\{ j \in \bigcup_k \mathcal{A}_k(\alpha) : \bigcup_{t=t_0}^{T} \varphi_t(\mathcal{T}_j) \cap S^*_{B(x_\beta, 2R(h))} M \neq \varnothing \right\}. \tag{3-38}$$

Then, we define

$$\mathcal{B}_{k,m} := \bigcup_{\alpha, \beta \in \mathcal{I}_{k,m}} \mathcal{B}_U(\alpha, \beta) \cap \mathcal{A}_k(\alpha).$$

Let $\mathcal{G}_{k,m} := \mathcal{A}_{k,m} \setminus \mathcal{B}_{k,m}$. Then, by construction, $\bigcup_{j \in \mathcal{G}_{k,m}} \mathcal{T}_j$ is $[t_0, T]$ non-self-looping and we have

$$|\mathcal{B}_{k,m}| \le c |\mathcal{I}_{k,m}|^2 |\mathcal{B}_U| \tag{3-39}$$

for some $c > 0$, where

$$|\mathcal{B}_U| := \sup\{ |\mathcal{B}_U(\alpha, \beta)| : \alpha, \beta \in \mathcal{I} \}. \tag{3-40}$$

That is, $|\mathcal{B}_U|$ is the maximum number of loops of length in $[t_0, T]$ joining any two points in U.

Then, define

$$w^{\mathcal{G}}_{k,m} := \sum_{j\in\mathcal{G}_{k,m}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})\,\mathrm{Op}_h(\psi)u \quad\text{and}\quad w^{\mathcal{B}}_{k,m} := \sum_{j\in\mathcal{B}_{k,m}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})\,\mathrm{Op}_h(\psi)u. \tag{3-41}$$

Next, consider

$$\biggl(\sum_{m=m_{1,k}}^{m_{2,k}} \|w_{k,m}\|^p_{L^p(U_{k,m})}\biggr)^{1/p} \le \biggl(\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{G}}_{k,m}\|^p_{L^p(U_{k,m})}\biggr)^{1/p} + \biggl(\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{B}}_{k,m}\|^p_{L^p(U_{k,m})}\biggr)^{1/p}. \tag{3-42}$$

3E1. *Bound on the looping piece.* We start by estimating the "bad" piece

$$\sum_{k\ge -1}\biggl(\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{B}}_{k,m}\|^p_{L^p(U_{k,m})}\biggr)^{1/p}.$$

Observe that if $2^{m_{1,k}} = \min(c_0R(h)^{1-n}, c_n2^{2k})$, then $m_{1,k} = m_{2,k}$ and we need not consider this part of the sum. Therefore, the high L^∞ mass term has

$$2^{m_{1,k}} = \frac{2^kR(h)^{(1-n)/2}}{T^N} \tag{3-43}$$

and $k_1 \le k \le k_2$. Hence, for $m_{1,k} < m \le m_{2,k}$, Lemma 3.4 gives that there is $C_n > 0$ with

$$|\mathcal{A}_{k,m}| \le C_n2^{2k} \le C_nR(h)^{n-1}2^{2m}T^{2N}.$$

Furthermore, since $R(h) \ge h^{\delta_2}$ with $\delta_2 < \frac{1}{2}$, (3-24) yields that there is $\varepsilon = \varepsilon(n,N) > 0$ such that $h^{\rho-1/2}R(h)^{-1/2} < h^\varepsilon$, and hence, since $T = O(\log h^{-1})$,

$$|\mathcal{A}_{k,m}| = o(R(h)^{n-1}2^{2m}(h^{\rho-1/2}R(h)^{-1/2})^{-2n(n-1)/(3n+1)}).$$

In particular, a consequence of Lemma 3.5 is the existence of $h_0 > 0$ and $C > 0$ such that

$$|\mathcal{I}_{k,m}| \le CR(h)^{1-n}2^{-2m}|\mathcal{A}_{k,m}| \tag{3-44}$$

$$\le CR(h)^{1-n}2^{2k-2m} \tag{3-45}$$

for all $0 < h \le h_0$, where we have used again Lemma 3.4 to bound $|\mathcal{A}_{k,m}|$.

Next, note that for each point in $\mathcal{I}_{k,m}$ there are at most $c|\mathcal{I}_{k,m}||\mathcal{B}_U|$ tubes in $\mathcal{B}_{k,m}$ touching it. Therefore, we may apply [Canzani and Galkowski 2021, Lemma 3.7] to obtain $C > 0$ such that

$$\|w^{\mathcal{B}}_{k,m}\|_{L^\infty(U_{k,m})} \le Ch^{(1-n)/2}R(h)^{(n-1)/2}|\mathcal{I}_{k,m}||\mathcal{B}_U|2^{-k}\|u\|_{P,T}. \tag{3-46}$$

Using (3-46) and interpolating between L^∞ and L^{p_c} we obtain

$$\|w^{\mathcal{B}}_{k,m}\|^p_{L^p(U_{k,m})} \le Ch^{-p\delta(p)}(R(h)^{(n-1)/2}|\mathcal{I}_{k,m}||\mathcal{B}_U|2^{-k}\|u\|_{P,T})^{p-p_c}\|w^{\mathcal{B}}_{k,m}\|^{p_c}_{L^2(U_{k,m})}. \tag{3-47}$$

In addition, since combining (3-14) with (3-39) yields

$$\|w^{\mathcal{B}}_{k,m}\|_{L^2(U_{k,m})} \le C|\mathcal{B}_{k,m}|^{1/2}2^{-k}\|u\|_{P,T} \le C2^{-k}|\mathcal{I}_{k,m}||\mathcal{B}_U|^{1/2}\|u\|_{P,T},$$

the bounds in (3-47) and (3-45) together with the definition of $m_{1,k}$ in (3-43) yield

$$\begin{aligned}\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{B}}_{k,m}\|^p_{L^p(U_{k,m})} &\le Ch^{-p\delta(p)}R(h)^{(n-1)(p-p_c)/2}\sum_{m=m_{1,k}}^{m_{2,k}} |\mathcal{I}_{k,m}|^p|\mathcal{B}_U|^{p-p_c/2}2^{-kp}\|u\|^p_{P,T}\\ &\le Ch^{-p\delta(p)}R(h)^{(n-1)(-p-p_c)/2}2^{kp}|\mathcal{B}_U|^{p-p_c/2}\|u\|^p_{P,T}\sum_{m=m_{1,k}}^{m_{2,k}} 2^{-2mp}\\ &\le Ch^{-p\delta(p)}R(h)^{(n-1)(p-p_c)/2}|\mathcal{B}_U|^{p-p_c/2}T^{2Np}\,2^{-kp}\|u\|^p_{P,T}.\end{aligned}$$

Then, with k_1 and k_2 defined as in (3-36), we have

$$\begin{aligned}\sum_{k=k_1}^{k_2}\bigg(\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{B}}_{k,m}\|^p_{L^p(U_{k,m})}\bigg)^{1/p} &\le Ch^{-\delta(p)}R(h)^{(n-1)(1-p_c/p)/2}|\mathcal{B}_U|^{1-p_c/(2p)}T^{2N}\|u\|_{P,T}\sum_{k=k_1}^{k_2}2^{-k}\\ &\le Ch^{-\delta(p)}(R(h)^{n-1}|\mathcal{B}_U|)^{1-p_c/(2p)}T^{3N}\|u\|_{P,T}.\end{aligned}$$

Finally, since we only need to consider $k_1 \le k \le k_2$,

$$\sum_{k\ge -1}\bigg(\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{B}}_{k,m}\|^p_{L^p(U_{k,m})}\bigg)^{1/p} \le Ch^{-\delta(p)}(R(h)^{n-1}|\mathcal{B}_U|)^{1-p_c/(2p)}T^{3N}\|u\|_{P,T}. \tag{3-48}$$

3E2. *Bound on the non-self-looping piece.* In this section we aim to control the "good" piece,

$$\sum_{k\ge -1}\bigg(\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{G}}_{k,m}\|^p_{L^p(U_{k,m})}\bigg)^{1/p}. \tag{3-49}$$

So far all L^p bounds appearing have been $\ll h^{(1-n)/2}/\sqrt{T}$. The reason for this is that the bounds were obtained by interpolation with an L^∞ estimate which is substantially stronger than $h^{(1-n)/2}/\sqrt{T}$.

We now estimate the number of non-self-looping tubes $\mathcal{T}_j$ with $j\in\mathcal{A}_k$. That is, tubes on which the L^2 mass of u is comparable to $2^{-k}\|u\|_{P,T}$.

Lemma 3.6. *Let $k\in\mathbb{Z}$, $k\ge -1$, and $t_0>1$. Suppose that $\mathcal{G}\subset\mathcal{A}_k$ is such that*

$$\bigcup_{j\in\mathcal{G}}\mathcal{T}_j \text{ is } [t_0,T] \text{ non-self-looping}.$$

Then, there exists a constant $C_n>0$, depending only on n, such that $|\mathcal{G}|\le (C_nt_0/T)2^{2k}$.

Proof. Using that $\mathcal{G}\subset\mathcal{A}_k$, we have

$$|\mathcal{G}|\frac{\|u\|^2_{P,T}}{2^{2(k+1)}} \le 2\sum_{j\in\mathcal{G}}(\|\mathrm{Op}_h(\chi_{\mathcal{T}_j})u\|^2_{L^2}+h^{-2}\|\mathrm{Op}_h(\chi_{\mathcal{T}_j})Pu\|^2_{L^2}). \tag{3-50}$$

Since $\{\mathcal{T}_j\}_{j\in\mathcal{G}}$ is $(\mathfrak{D}_n,\tau,R(h))$-good, there are $\{\mathcal{G}_i\}_{i=1}^{\mathfrak{D}_n}\subset\mathcal{G}$, such that, for each $i=1,\dots,\mathfrak{D}_n$,

$$\mathcal{T}_j\cap\mathcal{T}_k=\varnothing,\quad j,k\in\mathcal{G}_i,\quad j\ne k.$$

By [Canzani and Galkowski 2021, Lemma 4.1] with $t_\ell = t_0$ and $T_\ell = T$ for all ℓ,

$$\sum_{j\in\mathcal{G}} \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})u\|_{L^2}^2 \le \sum_{i=1}^{\mathfrak{D}_n}\sum_{j\in\mathcal{G}_i} \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})u\|_{L^2}^2 \le \frac{\mathfrak{D}_n 4t_0}{T}\|u\|_{P,T}^2. \tag{3-51}$$

On the other hand, since $\sum_{j\in\mathcal{G}_i} \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})\|^2 \le 2$ for each i,

$$\sum_{j\in\mathcal{G}} \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})Pu\|_{L^2}^2 \le 2\mathfrak{D}_n\|Pu\|_{L^2}^2. \tag{3-52}$$

Combining (3-50), (3-51), and (3-52) yields

$$|\mathcal{G}|\frac{\|u\|_{P,T}^2}{2^{2(k+1)}} \le \frac{8\mathfrak{D}_n t_0}{T}\|u\|_{P,T}^2 + \frac{4\mathfrak{D}_n}{h^2}\|Pu\|_{L^2}^2 \le \frac{8\mathfrak{D}_n t_0 + 4\mathfrak{D}_n/T}{T}\|u\|_{P,T}^2. \qquad \square$$

We may now proceed to estimate the L^p norm of the nonlooping piece (3-49). The first step is to notice that we only need to sum up to $m \le m_{3,k}$, where $m_{3,k}$ is defined by

$$2^{m_{3,k}} := \min\left(\frac{C_n t_0 2^{2k}}{c_M T}, c_0 R(h)^{1-n}\right),$$

$c_M > 0$ is as defined in (3-20), and $C_n > 0$ is the constant in Lemma 3.6. To see this, first observe that, using (3-19), (3-44), and (3-46), for each $\alpha \in \mathcal{I}_{k,m}$,

$$\begin{aligned}\|w_{k,m}^{\mathcal{G}}\|_{L^\infty(B(x_\alpha,R(h)))} &\le \|w_{k,m}\|_{L^\infty(B(x_\alpha,R(h)))} + \|w_{k,m}^{\mathcal{B}}\|_{L^\infty(B(x_\alpha,R(h)))}\\ &\le C(2^m + |\mathcal{I}_{k,m}||\mathcal{B}_U|)2^{-k}h^{(1-n)/2}R(h)^{(n-1)/2}\|u\|_{P,T}\\ &\le C(1+R(h)^{1-n}2^{-3m}|\mathcal{A}_{k,m}||\mathcal{B}_U|)2^{m-k}h^{(1-n)/2}R(h)^{(n-1)/2}\|u\|_{P,T}.\end{aligned} \tag{3-53}$$

Furthermore, since $|\mathcal{G}_{k,m}| \ge |\mathcal{A}_{k,m}| - |\mathcal{I}_{k,m}|^2|\mathcal{B}_U|$ and $\mathcal{G}_{k,m}$ is $[t_0, T]$ non-self-looping, Lemma 3.6 yields the existence of $C_n > 0$ such that

$$|\mathcal{A}_{k,m}| - |\mathcal{I}_{k,m}|^2|\mathcal{B}_U| \le C_n\frac{t_0}{T}2^{2k}.$$

Next, since $m_{1,k} \le m \le m_{2,k}$, we may apply Lemma 3.5 to bound $|\mathcal{I}_{k,m}|$ as in (3-44) to obtain that for some $C > 0$,

$$|\mathcal{A}_{k,m}|(1 - CR(h)^{2(1-n)}2^{-4m}|\mathcal{A}_{k,m}||\mathcal{B}_U|) \le C_n\frac{t_0}{T}2^{2k}. \tag{3-54}$$

In addition, provided

$$|\mathcal{B}_U|R(h)^{n-1} \ll T^{-6N}, \tag{3-55}$$

we have that, for $m \ge m_{1,k}$ and $k_1 \le k \le k_2$,

$$R(h)^{2(1-n)}2^{-4m}|\mathcal{A}_{k,m}||\mathcal{B}_U| \le R(h)^{2(1-n)}2^{-4m+2k}|\mathcal{B}_U| \le 2^{-2k}T^{4N}|\mathcal{B}_U| \le R(h)^{n-1}T^{6N}|\mathcal{B}_U| \ll 1, \tag{3-56}$$

where we used that, by (3-20), $|A_{k,m}|$ is controlled by 2^{2k} to get the first inequality, that $m \ge m_{1,k}$ to get the second, and that $k \ge k_1$ to get the third. Combining (3-54) and the bound in (3-56) we obtain $|\mathcal{A}_{k,m}| \le C_n t_0 2^{2k}/T$, and so, by (3-20), $2^m \le C_n t_0 2^{2k}/(c_M T)$. As claimed, this shows that to deal with (3-49) we only need to sum up to $m \le m_{3,k}$.

The next step is to use interpolation to control the first sum in (3-49) by

$$\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{G}}_{k,m}\|^p_{L^p(U_{k,m})} = \sum_{m=m_{1,k}}^{m_{3,k}} \|w^{\mathcal{G}}_{k,m}\|^p_{L^p(U_{k,m})}. \tag{3-57}$$

We claim that (3-53) yields

$$\|w^{\mathcal{G}}_{k,m}\|_{L^\infty(B(x_\alpha,R(h)))} \le C2^{m-k}h^{(1-n)/2}R(h)^{(n-1)/2}\|u\|_{P,T}. \tag{3-58}$$

Indeed, using the bound (3-55) on $|\mathcal{B}_U|$, that $|\mathcal{A}_{k,m}|$ is controlled by 2^{2k}, that $m \ge m_{1,k}$ as in (3-43), and that $k_1 \le k \le k_2$, we have

$$R(h)^{1-n}2^{-3m}|\mathcal{A}_{k,m}||\mathcal{B}_U| \ll R(h)^{2(1-n)}2^{-3m+2k}T^{-6N} \le T^{-2N}.$$

Note that

$$\begin{aligned}
\|w^{\mathcal{G}}_{k,m}\|_{L^{p_c}(U_{k,m})} &\le Ch^{-1/p_c}(\|w^{\mathcal{G}}_{k,m}\|_{L^2}+h^{-1}\|Pw^{\mathcal{G}}_{k,m}\|_{L^2})\\
&\le Ch^{-1/p_c}\biggl(\|w^{\mathcal{G}}_{k,m}\|_{L^2}+h^{-1}\biggl\|\sum_{j\in\mathcal{G}_{k,m}}[P,\mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})]w^{\mathcal{G}}_{k,m}\biggr\|_{L^2}+h^{-1}\biggl\|\sum_{j\in\mathcal{G}_{k,m}}\mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})Pu\biggr\|_{L^2}\biggr)\\
&\le Ch^{-1/p_c}2^{-k}|\mathcal{G}_{k,m}|^{1/2}\|u\|_{P,T}+O(h^\infty\|u\|_{P,T}),
\end{aligned}$$

where the last line follows from the definition of $\mathcal{A}_{k,m}$, the fact that $[P,\mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})] \in h\Psi_\delta$ with its microsupport contained in $\operatorname{supp}\tilde{\chi}_{\mathcal{T}_j}$, and Remark 3.3. Finally, by Lemma 3.6, $|\mathcal{G}_{k,m}| \le (C_n t_0/T)2^{2k}$, and hence

$$\|w^{\mathcal{G}}_{k,m}\|_{L^{p_c}(U_{k,m})} \le C\sqrt{\frac{t_0}{T}}h^{-1/p_c}\|u\|_{P,T} + O(h^\infty\|u\|_{P,T}).$$

Using this together with interpolation and (3-58) we obtain

$$\begin{aligned}
\|w^{\mathcal{G}}_{k,m}\|^p_{L^p(U_{k,m})} &\le \|w^{\mathcal{G}}_{k,m}\|^{p-p_c}_{L^\infty(U_{k,m})}\|w^{\mathcal{G}}_{k,m}\|^{p_c}_{L^{p_c}(U_{k,m})}\\
&\le Ch^{-p\delta(p)}(R(h)^{(n-1)/2}2^{m-k})^{p-p_c}\frac{t_0^{p_c/2}}{T^{p_c/2}}\|u\|^p_{P,T} + O(h^\infty\|u\|^p_{P,T}).
\end{aligned}\tag{3-59}$$

Using this, we estimate (3-57):

$$\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{G}}_{k,m}\|^p_{L^p(U_{k,m})} \le Ch^{-p\delta(p)}(R(h)^{(n-1)/2}2^{(m_{3,k}-k)})^{p-p_c}\|u\|^p_{P,T}\frac{t_0^{p_c/2}}{T^{p_c/2}} + O(h^\infty\|u\|^p_{P,T}). \tag{3-60}$$

Then, summing in k, and again using that only $k_1 \le k \le k_2$ contribute,

$$\begin{aligned}
&\sum_{k=-1}^{\infty}\biggl(\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{G}}_{k,m}\|^p_{L^p(U_m)}\biggr)^{1/p}\\
&\qquad \le Ch^{-\delta(p)}\|u\|_{P,T}\frac{t_0^{p_c/(2p)}}{T^{p_c/(2p)}}\sum_{k=k_1}^{k_2}(R(h)^{(n-1)/2}2^{(m_{3,k}-k)})^{1-p_c/p} + O(h^\infty\|u\|_{P,T})\\
&\qquad \le Ch^{-\delta(p)}\frac{t_0^{1/2}}{T^{1/2}}\|u\|_{P,T}+O(h^\infty\|u\|_{P,T}).
\end{aligned}\tag{3-61}$$

Note that the sum over k in (3-61) is controlled by the value of k for which $C_n t_0 2^{2k}/(c_M T) = c_0 R(h)^{1-n}$, since the sum is geometrically increasing before such k and geometrically decreasing afterward.

3F. ***Wrapping up the proof of Theorem 1.4.*** Combining (3-37), (3-48), and (3-61) with (3-42) and (3-34), and taking $N > \frac{1}{2}(1-p_c/p)^{-1}$ provided $R(h)^{n-1}|\mathcal{B}_U| \le CT^{-6N}$ for some $C>0$, we obtain

$$\|v\|_{L^p(U)} \le \sum_{k=-1}^{\infty} \|w_k\|_{L^p(U)} \le Ch^{-\delta(p)}\biggl(\frac{t_0^{1/2}}{T^{1/2}} + (R(h)^{n-1}|\mathcal{B}_U|)^{1-p_c/(2p)}T^{3N}\biggr)\|u\|_{P,T}$$

as requested in (3-55). Since this estimate holds only when $|\mathcal{B}_U|R(h)^{n-1} \le CT^{-6N}$, we replace T by $T_0 := \min\bigl\{\frac{1}{C}(R(h)^{n-1}|\mathcal{B}_U|)^{-1/6N}, T\bigr\}$, so that

$$\begin{aligned}\|v\|_{L^p(U)} &\le Ch^{-\delta(p)}\biggl(\frac{t_0^{1/2}}{T_0{}^{1/2}} + (R(h)^{n-1}|\mathcal{B}_U|)^{1-p_c/(2p)}T_0^{3N}\biggr)\|u\|_{P,T}\\ &\le Ch^{-\delta(p)}\biggl(\frac{t_0^{1/2}}{T^{1/2}} + t_0^{1/2}(R(h)^{n-1}|\mathcal{B}_U|)^{1/(12N)} + (R(h)^{n-1}|\mathcal{B}_U|)^{(1-p_c/p)/2}\biggr)\|u\|_{P,T}\\ &\le Ch^{-\delta(p)}\biggl(\frac{t_0^{1/2}}{T^{1/2}} + (R(h)^{n-1}|\mathcal{B}_U|)^{1/(12N)}\biggr)\|u\|_{P,T},\end{aligned} \tag{3-62}$$

where the constant C is adjusted from line to line.

Next, combining (3-62) with (3-11) and the definition of v in (3-12), we obtain

$$\|u\|_{L^p(U)} \le Ch^{-\delta(p)}\biggl(\frac{t_0^{1/2}}{T^{1/2}} + (R(h)^{n-1}|\mathcal{B}_U|)^{1/(12N)}\biggr)\|u\|_{P,T} + Ch^{-\delta(p)+1/2-\delta_2}h^{-1}\|Pu\|_{H_h^{n(1/2-1/p)+\varepsilon-2}}.$$

Putting $\varepsilon = \frac{1}{2}$ and setting $N = \frac{1}{2}\bigl(1+\frac{1}{6}\varepsilon_0\bigr)(1-p_c/p)^{-1}$, estimate (1-7) will follow once we relate $|\mathcal{B}_U|$ for a given $(\tau, R(h))$ cover to $|\mathcal{B}_U|$ for the $(\mathfrak{D}, \tau, R(h))$ cover used in our proof.

Finally, to finish the proof of Theorem 1.4, we need to show that for any $(\tau, R(h))$ cover $\{\mathcal{T}_j\}_j$ of S^*M, up to a constant depending only on M, $|\mathcal{B}_U|$ can be bounded by $|\widetilde{\mathcal{B}}_U|$ where $\widetilde{\mathcal{B}}_U$ is defined as in (3-40) using a $(\widetilde{\mathfrak{D}}, \tau, R(h))$-good cover $\{\widetilde{\mathcal{T}}_k\}_k$ of S^*M.

Lemma 3.7. *There exists $C_M > 0$ depending only on M such that if $\{\mathcal{T}_j\}_{j\in\mathcal{J}}$ and $\{\widetilde{\mathcal{T}}_k\}_{k\in\mathcal{K}}$ are a $(\tau, R(h))$ cover of S^*M and a $(\widetilde{\mathfrak{D}}, \tau, R(h))$-good cover of S^*M, respectively, and $|\mathcal{B}_U|$ and $|\widetilde{\mathcal{B}}_U|$ are defined as in* (3-40) *for the covers $\{\mathcal{T}_j\}_{j\in\mathcal{J}}$ and $\{\widetilde{\mathcal{T}}_k\}_{k\in\mathcal{K}}$, respectively, then*

$$|\widetilde{\mathcal{B}}_U| \le C_M\widetilde{\mathfrak{D}}|\mathcal{B}_U|.$$

Proof. Fix α, β such that $x_\alpha, x_\beta \in U$. Suppose that $j \in \mathcal{B}_U(\alpha,\beta)$, where $\mathcal{B}_U(\alpha,\beta)$ is as in (3-38). Then, there is $k \in \widetilde{\mathcal{B}}_U(\alpha,\beta)$ such that $\widetilde{\mathcal{T}}_k \cap \mathcal{T}_j \ne \varnothing$. Now, fix $j \in \mathcal{J}$ and let

$$\mathcal{C}_j := \{k \in \mathcal{K} : \mathcal{T}_j \cap \widetilde{\mathcal{T}}_k \ne \varnothing\}.$$

We claim that there is $c_M > 0$ such that for each $k \in \mathcal{C}_j$,

$$\widetilde{\mathcal{T}}_k \subset \Lambda_{\rho_j}^{c_M\tau}(c_M R(h)). \tag{3-63}$$

Assuming (3-63) for now, there exists $C_M > 0$ such that

$$|\mathcal{C}_j| \le \widetilde{\mathfrak{D}}\frac{\operatorname{vol}(\Lambda_{\rho_j}^{c_M\tau}(c_M R(h))}{\inf_{k\in\mathcal{K}}\operatorname{vol}(\widetilde{\mathcal{T}}_k)} \le \widetilde{\mathfrak{D}}C_M.$$

Thus, for each $j \in \mathcal{B}_U(\alpha,\beta)$, there are at most $C_M\widetilde{\mathfrak{D}}$ elements in $\widetilde{\mathcal{B}}_U(\alpha,\beta)$, and hence

$$|\mathcal{B}_U(\alpha,\beta)| \geq \frac{|\widetilde{\mathcal{B}}_U(\alpha,\beta)|}{C_M\widetilde{\mathfrak{D}}}$$

as claimed.

We now prove (3-63). Let $q \in \widetilde{\mathcal{T}}_k$. Then, there are $\rho_k', \rho_j', q' \in S^*M$ and $t_k, t_j, s \in [\tau - R(h), \tau + R(h)]$ such that

$$d(\rho_k,\rho_k') < R(h), \quad d(\rho_j,\rho_j') < R(h), \quad d(\rho_k, q') < R(h),$$
$$\varphi_{t_k}(\rho_k') = \varphi_{t_j}(\rho_j'), \quad \varphi_s(q') = q.$$

In particular, $d(q',\rho_k') < 2R(h)$, so there is $c_M > 0$ such that $d(\varphi_{t_k}(\rho_k'), \varphi_{t_k-s}(q)) < c_M R(h)$. Applying φ_{-t_j}, and adjusting c_M in a way depending only on M, we have $d(\rho_j', \varphi_{t_k-t_j-s}(q)) < c_M R(h)$. In particular, adjusting c_M again, $d(\rho_j, \varphi_{t_k-t_j-s}(q)) < c_M R(h)$ and the claim follows. ☐

3G. ***Profiles of near-saturating functions.*** As explained in the introduction, our next theorem describes the profiles of functions which extremize the improved bounds from Theorem 1.4.

Theorem 3.8. *Let $p > p_c$, $T(h) \to \infty$, and $\delta > 0$. Let $0 < \delta_1 < \delta_2 < \frac{1}{2}$, $h^{\delta_2} \leq R(h) \leq h^{\delta_1}$, and $\{x_\alpha\}_{\alpha\in\mathcal{I}(h)} \subset M$ be a maximal $R(h)$-separated set. Let $\mathcal{B}_U$ be as in* (3-40), *and suppose that*

$$|\mathcal{B}_U| R(h)^{n-1} T(h)^{3p/(p-p_c)+\delta} = o(1)$$

and $u \in \mathcal{D}'(M)$ with

$$\|Pu\|_{H_h^{(n-3)/2}} = o\Big(\frac{h}{T}\|u\|_{L^2}\Big). \tag{3-64}$$

For $\varepsilon > 0$, set

$$\mathcal{S}_U(h,\varepsilon,u) := \bigg\{\alpha \in \mathcal{I}(h) : \|u\|_{L^\infty(B(x_\alpha,R(h)))} \geq \frac{\varepsilon h^{(1-n)/2}\sqrt{t_0}}{\sqrt{T(h)}}\|u\|_{L^2(M)},\ B(x_\alpha,R(h)) \cap U \neq \varnothing\bigg\}.$$

Then, there are $c, C > 0$ such that, for all $\varepsilon > 0$, there are $N_\varepsilon > 0$ and $h_0 > 0$ such that $|\mathcal{S}_U(h,\varepsilon,u)| \leq N_\varepsilon$ for all $0 < h \leq h_0$.

Moreover, there is a collection of geodesic tubes $\{\mathcal{T}_j\}_{j\in\mathcal{L}(\varepsilon,u)}$ of radius $R(h)$ (see Definition 1.3) with indices satisfying $\mathcal{L}(\varepsilon,u) = \bigcup_{i=1}^C \mathcal{J}_i$ and $\mathcal{T}_k \cap \mathcal{T}_\ell = \varnothing$ for $k,\ell \in \mathcal{J}_i$ with $k \neq \ell$, such that

$$u = u_e + \frac{1}{\sqrt{T(h)}} \sum_{j\in\mathcal{L}(\varepsilon,u)} v_j,$$

where v_j is microsupported in $\mathcal{T}_j$, $|\mathcal{L}(\varepsilon,u)| \leq C\varepsilon^{-2}R(h)^{1-n}$, and, for all $p \leq q \leq \infty$,

$$\|u_e\|_{L^q} \leq \varepsilon h^{-\delta(q)}(T(h))^{-1/2}\|u\|_{L^2},$$
$$\|v_j\|_{L^2} \leq C\varepsilon^{-1}R(h)^{(n-1)/2}\|u\|_{L^2}, \quad \|Pv_j\|_{L^2} \leq C\varepsilon^{-1}R(h)^{(n-1)/2}h\|u\|_{L^2}.$$

Finally, with $\mathcal{L}(\varepsilon,u,\alpha) := \{j \in \mathcal{L}(\varepsilon,u) : \pi(\mathcal{T}_j) \cap B(x_\alpha, 3R(h)) \neq \varnothing\}$, for every $\alpha \in \mathcal{S}_U(h,\varepsilon,u)$,

$$c\varepsilon^2 R(h)^{1-n} \leq |\mathcal{L}(\varepsilon,u,\alpha)| \leq CR(h)^{1-n} \quad \text{and} \quad \sum_{j\in\mathcal{L}(\varepsilon,u,\alpha)} \|v_j\|_{L^2}^2 \geq c^2\varepsilon^2\|u\|_{L^2}^2.$$

The proof of Theorem 3.8 is completed in the following three subsections.

3G1. *Proof of the bound on* $|\mathcal{S}_U(h,\varepsilon,u)|$. We claim that there is $c>0$ such that, for $\alpha\in S_U(h,\varepsilon,u)$,

$$\frac{c\varepsilon\sqrt{t_0}}{\sqrt{T}}h^{-1/p}\|u\|_{P,T}\le\|u\|_{L^p(B(x_\alpha,2R(h)))}. \tag{3-65}$$

To see (3-65), first let $\chi_0,\chi_1\in C_c^\infty(-2,2)$ with $\chi_0\equiv 1$ on $\left[-\frac{3}{2},\frac{3}{2}\right]$ and $\chi_1\equiv 1$ on $\operatorname{supp}\chi_0$ and note that, by Lemma 3.1, the elliptic parametrix construction for P, and (3-64),

$$\|(1-\chi_0(-h^2\Delta_g))u\|_{L^p}\le Ch^{-\delta(p)-1/2}\|Pu\|_{H_h^{(n-3)/2}}=o\left(\frac{h^{-\delta(p)+1/2}}{T}\right)\|u\|_{L^2}. \tag{3-66}$$

Therefore, for $\alpha\in S_U(h,\varepsilon,u)$, we have

$$\|\chi_0(-h^2\Delta_g)u\|_{L^\infty(B(x_\alpha,R(h)))}\ge\frac{\varepsilon h^{(1-n)/2}}{2\sqrt{T}}\|u\|_{L^2(M)} \tag{3-67}$$

for h small enough. Next, set $\chi_{\alpha,h}(x):=\chi_0(R(h)^{-1}d(x,x_\alpha))$ and note

$$\chi_1(-h^2\Delta_g)\chi_{\alpha,h}\chi_0(-h^2\Delta_g)u=\chi_{\alpha,h}\chi_0(-h^2\Delta_g)u+O(h^\infty\|u\|_{L^2})_{C^\infty}.$$

Then, by (3-67) and [Zworski 2012, Theorem 7.15],

$$\begin{aligned}\frac{\varepsilon h^{(1-n)/2}}{2\sqrt{T}}\|u\|_{L^2(M)}&\le\|\chi_0(-h^2\Delta_g)u\|_{L^\infty(B(x_\alpha,R(h)))}\\&=\|\chi_{\alpha,h}\chi_0(-h^2\Delta_g)u\|_{L^\infty(B(x_\alpha,R(h)))}\\&=\|\chi_1(-h^2\Delta_g)\chi_{\alpha,h}\chi_0(-h^2\Delta_g)u\|_{L^\infty(B(x_\alpha,R(h)))}+O(h^\infty)\|u\|_{L^2}\\&\le Ch^{-n/p}(\|\chi_0(-h^2\Delta_g)u\|_{L^p(B(x_\alpha,2R(h)))}+O(h^\infty)\|u\|_{L^2}).\end{aligned} \tag{3-68}$$

Combining (3-68) and (3-66) yields the claim in (3-65). It then follows from Theorem 1.4 that, if $\{\alpha_i\}_{i=1}^N\subset S_U(h,\varepsilon,u)$ with $B(x_{\alpha_i},2R(h))\cap B(x_{\alpha_j},2R(h))=\varnothing$ for $i\neq j$, we have

$$N^{1/p}\frac{c\varepsilon\sqrt{t_0}}{\sqrt{T}}h^{-1/p}\|u\|_{P,T}\le\|u\|_{L^p}\le Ch^{-1/p}\|u\|_{L^2}\le Ch^{-1/p}\frac{\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T}.$$

Then, $N^{1/p}\le C\varepsilon^{-1}$. Since at most $\mathfrak{D}_n$ balls $B(x_\alpha,2R(h))$ intersect, $|S_U(h,\varepsilon,u)|\le C\mathfrak{D}_n\varepsilon^{-p}$.

3G2. *Preliminaries for the decomposition of* u. Let $q\in\mathbb{R}$ such that $p\le q\le\infty$. Below, all implicit constants are uniform for $p\le q\le\infty$. As above, it suffices to prove the statement for v as in (3-12) instead of u. Then, we write $v=\sum_{k=-1}^\infty w_k$ as in (3-18). For $V\subset U$, by the same analysis that led to (3-34),

$$\|w_k\|^q_{L^q(V)}\le\mathfrak{D}_n\sum_{m=-\infty}^{m_{2,k}}\|w_{k,m}\|^q_{L^q(V\cap U_{k,m})}+O(h^\infty)\|u\|_{P,T},$$

where $w_{k,m}$ is as in (3-28). Then, by (3-37) with $N=\frac{1}{2}q/(q-p_c)+\frac{1}{6}\delta$,

$$\sum_{k\ge-1}\left(\sum_{m=-\infty}^{m_1,k}\|w_{k,m}\|^q_{L^q(U_{k,m})}\right)^{1/q}\le Ch^{-\delta(q)}\frac{\log T}{T^{1/2+\delta(q-p_c)/(6q)}}\|u\|_{P,T} \tag{3-69}$$

for h small enough. Then, splitting $w_{k,m}=w^{\mathcal{B}}_{k,m}+w^{\mathcal{G}}_{k,m}$, as in (3-41), we have by (3-48) that

$$\sum_{k\ge -1}\Bigg(\sum_{m=m_{1,k}}^{m_{2,k}}\|w^{\mathcal{B}}_{k,m}\|^q_{L^q(U_{k,m})}\Bigg)^{1/q}\le Ch^{-\delta(q)}(R(h)^{n-1}|\mathcal{B}_U|)^{1-p_c/(2q)}T^{3q/(2(q-p_c))+\delta/2}\|u\|_{P,T}. \tag{3-70}$$

Define k_1^ε and k_2^ε, respectively, by

$$2^{2k_1^\varepsilon}=\frac{C^{-2}\mathfrak{D}_n^{-2}\varepsilon^2R(h)^{1-n}c_MT}{4C_nt_0}\quad\text{and}\quad 2^{2k_2^\varepsilon}=\frac{C^{2}\mathfrak{D}_n^{2}\varepsilon^{-2}R(h)^{1-n}c_MT}{4C_nt_0}, \tag{3-71}$$

where C is as in (3-61). Then, define $\mathcal{K}(\varepsilon):=\{k:k_1^\varepsilon\le k\le k_2^\varepsilon\}$ and note that, since $2^{(k_2^\varepsilon-k_1^\varepsilon)}=C^2\mathfrak{D}_n^2\varepsilon^{-2}$, $|\mathcal{K}(\varepsilon)|\le\log_2(4C^2\mathfrak{D}_n^2\varepsilon^{-2})=:K_\varepsilon$. Using (3-59) and summing over $k\notin\mathcal{K}(\varepsilon)$, it follows that

$$\sum_{k\notin\mathcal{K}(\varepsilon)}\Bigg(\sum_{m=m_{1,k}}^{m_{3,k}}\|w^{\mathcal{G}}_{k,m}\|^q_{L^q(U_{k,m})}\Bigg)^{1/q}\le\frac{\varepsilon}{4\mathfrak{D}_n}\frac{h^{-\delta(q)}\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T}. \tag{3-72}$$

Next, for $k\in\mathcal{K}(\varepsilon)$, let

$$\mathcal{M}(k,\varepsilon):=\{m:m^\varepsilon_{3,k}\le m\le m_{3,k}\},\qquad m^\varepsilon_{3,k}:=m_{3,k}-\frac{q}{q-p_c}\log_2(\varepsilon^{-1}2C\mathfrak{D}_n),$$

and note $|\mathcal{M}(k,\varepsilon)|\le(q/(q-p_c))\log_2(\varepsilon^{-1}2C\mathfrak{D}_n):=M_\varepsilon$. Using (3-59) and summing over $k\in\mathcal{K}(\varepsilon)$ and $m\notin\mathcal{M}(k,\varepsilon)$, it follows that

$$\begin{aligned}\sum_{k\in\mathcal{K}(\varepsilon)}\Bigg(\sum_{m\notin\mathcal{M}(k,\varepsilon)}\|w^{\mathcal{G}}_{k,m}\|^q_{L^q(U_{k,m})}\Bigg)^{1/q}&\\ &\le Ch^{-\delta(q)}\frac{t_0^{p_c/(2q)}}{T^{p_c/(2q)}}\sum_{k\in\mathcal{K}(\varepsilon)}(R(h)^{(n-1)/2}2^{m^\varepsilon_{3,k}-k})^{1-p_c/q}\|u\|_{P,T}+O(h^\infty\|u\|_{P,T})\\ &\le\frac{\varepsilon}{4\mathfrak{D}_n}\frac{h^{-\delta(q)}t_0^{1/2}}{T^{1/2}}\|u\|_{P,T}.\end{aligned} \tag{3-73}$$

Let

$$\mathcal{N}_{k,m}(\varepsilon):=\Bigg\{\alpha\in\mathcal{I}_{k,m}:\|w^{\mathcal{G}}_{k,m}\|_{L^\infty(B(x_\alpha,R(h)))}\ge\frac{\varepsilon}{4\mathfrak{D}_nM_\varepsilon K_\varepsilon}\frac{h^{(1-n)/2}\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T}\Bigg\}. \tag{3-74}$$

We claim

$$\mathcal{S}_U(h,\varepsilon,u)\subset\bigcup_{k\in\mathcal{K}(\varepsilon)}\bigcup_{m\in\mathcal{M}(k,\varepsilon)}\mathcal{N}_{k,m}(\varepsilon). \tag{3-75}$$

To prove (3-75), suppose $\alpha\notin\bigcup_{k\in\mathcal{K}(\varepsilon)}\bigcup_{m\in\mathcal{M}(k,\varepsilon)}\mathcal{N}_{k,m}(\varepsilon)$. Then, using (3-69) with $q=\infty$ and $N=\frac{1}{2}+\frac{\delta}{6}$,

$$\frac{1}{\mathfrak{D}_n}\|v\|_{L^\infty(B(x_\alpha,R(h)))}\le\frac{Ch^{(1-n)/2}\log T}{T^{1/2+\delta/6}}\|u\|_{P,T}+\sum_{k\ge -1}\sum_{m=m_{1,k}}^{m_{2,k}}\|w_{k,m}\|_{L^\infty(U_{k,m})}. \tag{3-76}$$

Next, for the second term in the right-hand side of (3-76), we write the decomposition

$$\sum_{k\ge -1}\sum_{m=m_{1,k}}^{m_{2,k}}\|w^{\mathcal{B}}_{k,m}\|_{L^\infty(U_{k,m})}+\sum_{k\notin\mathcal{K}(\varepsilon)}\sum_{m=m_{1,k}}^{m_{3,k}}\|w^{\mathcal{G}}_{k,m}\|_{L^\infty(U_{k,m})}+\sum_{k\in\mathcal{K}(\varepsilon)}\sum_{m=m_{1,k}}^{m_{2,k}}\|w^{\mathcal{G}}_{k,m}\|_{L^\infty(U_{k,m})}. \tag{3-77}$$

Note that in the term with the sum over $k \notin \mathcal{K}(\varepsilon)$ we only sum over $m \le m_{3,k}$ for the same reason as in (3-57). We bound the three terms in (3-77) using (3-70), (3-72), (3-73), and (3-74) with $q = \infty$ and $N = \frac{1}{2} + \frac{\delta}{6}$. Combining with (3-76) this yields

$$\frac{1}{\mathfrak{D}_n}\|v\|_{L^\infty(B(x_\alpha,R(h)))} \le Ch^{(1-n)/2}\|u\|_{P,T}\left(\frac{\log T}{T^{1/2+\delta/6}} + R(h)^{n-1}|\mathcal{B}_U|T^{3/2+\delta/2} + \frac{3\varepsilon}{4\mathfrak{D}_n}\frac{\sqrt{t_0}}{\sqrt{T}} + O(h^\infty)\right).$$

Thus, if $\alpha \notin \bigcup_{k\in\mathcal{K}(\varepsilon)}\bigcup_{m\in\mathcal{M}(k,\varepsilon)}\mathcal{N}_{k,m}(\varepsilon)$, then $\|v\|_{L^\infty(B(x_\alpha,R(h)))} \le \varepsilon h^{(1-n)/2}\frac{\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T}$ for h small enough. In particular, $\alpha \notin \mathcal{S}_U(h,\varepsilon,u)$. This proves the claim (3-75).

3G3. *Decomposition of u.* We next decompose u as described in the theorem. First, put

$$\begin{aligned} u_{e,1} &:= \sum_{k\ge -1}\sum_{m=-\infty}^{m_{1,k}} w_{k,m} + \sum_{k\ge -1}\sum_{m=m_{1,k}}^{m_{2,k}} w^{\mathcal{B}}_{k,m} + \sum_{k\notin\mathcal{K}(\varepsilon)}\sum_{m=m_{1,k}}^{m_{3,k}} w^{\mathcal{G}}_{k,m} + \sum_{k\in\mathcal{K}(\varepsilon)}\sum_{m\notin\mathcal{M}(k,\varepsilon)} w^{\mathcal{G}}_{k,m},\\ u_{\text{big}} &:= \sum_{k\in\mathcal{K}(\varepsilon)}\sum_{m\in\mathcal{M}(k,\varepsilon)} w^{\mathcal{G}}_{k,m}, \end{aligned}$$

and $u_{e,2} := u - u_{\text{big}} - u_{e,1}$. Note that

$$\begin{aligned} \|u_{e,1}\|_{L^q} &\le \frac{3\varepsilon}{4}h^{-\delta(q)}\frac{\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T},\\ \|u_{e,2}\|_{L^q} &\le Ch^{-\delta(q)+1/2-\delta_2}h^{-1}\|Pu\|_{H_h^{(n-3)/2}}, \end{aligned}$$

where we use (3-70), (3-72), (3-73), (3-76), and (3-77) to obtain the first estimate, and (3-11) to obtain the second. These two estimates prove the claim on $\|u_e\|_{L^q}$ after combining them with (3-64). Next, observe that

$$u_{\text{big}} = \sum_{j\in\mathcal{L}(\varepsilon)} u_j, \quad u_j := \mathrm{Op}_h(\tilde\chi_{\mathcal{T}_j})\,\mathrm{Op}_h(\psi)u, \quad \text{and} \quad \mathcal{L}(\varepsilon) := \bigcup_{k\in\mathcal{K}(\varepsilon)}\bigcup_{m\in\mathcal{M}(k,\varepsilon)} \mathcal{G}_{k,m}.$$

We claim that the statement of the theorem holds with $v_j = \sqrt{T}u_j$. Note that the v_j are manifestly microsupported inside $\mathcal{T}_j$.

Let $\alpha \in \mathcal{S}_U(h,\varepsilon,u)$. Then by definition,

$$\|u_{\text{big}}\|_{L^\infty(B(x_\alpha,R(h)))} \ge \frac{\varepsilon}{4}h^{(1-n)/2}\frac{\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T}. \tag{3-78}$$

Note that for all $j \in \mathcal{L}(\varepsilon)$, the estimate

$$\|\mathrm{Op}_h(\tilde\chi_{\mathcal{T}_j})\,\mathrm{Op}_h(\psi)u\|_{L^2} + h^{-1}\|\mathrm{Op}_h(\tilde\chi_{\mathcal{T}_j})\,\mathrm{Op}_h(\psi)Pu\|_{L^2} \le 2^{-k_1^\varepsilon+1}\|u\|_{P,T} \tag{3-79}$$

follows from the definition of $\mathcal{A}_k$ in (3-14) and the fact that $\chi_{\mathcal{T}_j} \equiv 1$ on $\operatorname{supp}\tilde\chi_{\mathcal{T}_j}$. To see that u_j is a quasimode, we use the definition of $\mathcal{A}_k$ again, together with Proposition 2.5, and obtain

$$\|Pu_j\|_{L^2} \le \|[-h^2\Delta_g, \mathrm{Op}_h(\tilde\chi_{\mathcal{T}_j})]u_j\|_{L^2} + \|\mathrm{Op}_h(\tilde\chi_{\mathcal{T}_j})Pu\|_{L^2} \le C2^{-k_1^\varepsilon}h\|u\|_{P,T}. \tag{3-80}$$

The definition of k_1^ε together with (3-79) and (3-80) give the required bounds on v_j and Pv_j.

Next, define

$$\mathcal{L}(\varepsilon, u, \alpha) := \{j \in \mathcal{L} : \pi_M(\mathcal{T}_j) \cap B(x_\alpha, 3R(h)) \neq \varnothing\},$$

and note that by [Canzani and Galkowski 2021, Lemma 3.7],

$$\begin{aligned}
&\|u_{\text{big}}\|_{L^\infty(B(x_\alpha, R(h)))} \\
&\le Ch^{(1-n)/2}R(h)^{(n-1)/2}\sum_{j\in\mathcal{L}(\varepsilon,u,\alpha)}\|\mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})\,\mathrm{Op}_h(\psi)u\|_{L^2}+h^{-1}\|\mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})\,\mathrm{Op}_h(\psi)Pu\|_{L^2}+O(h^\infty)\|u\|_{L^2} \\
&\le Ch^{(1-n)/2}R(h)^{(n-1)/2}2^{-k_1^\varepsilon}|\mathcal{L}(\varepsilon,u,\alpha)|\,\|u\|_{P,T}+O(h^\infty)\|u\|_{P,T}. \qquad (3\text{-}81)
\end{aligned}$$

(Note that in [Canzani and Galkowski 2021, Lemma 3.7], the number $\tau|H_p r_H(\rho_\gamma)|$ appears in the prefactor. In our circumstance, one can check that $|H_p r_H(\rho_\gamma)| = 2$ and $\tau > 0$ is a number uniformly bounded below by $c\,\mathrm{inj}(M)$ for some $c > 0$.) Therefore, combining (3-78) with (3-81) yields

$$\varepsilon\frac{\sqrt{t_0}}{\sqrt{T}} \le CR(h)^{(n-1)/2}2^{-k_1^\varepsilon}|\mathcal{L}(\varepsilon,\alpha,u)| + O(h^\infty).$$

Moreover, $\bigcup_{j\in\mathcal{L}(\varepsilon,u)}\mathcal{T}_j$ is $[t_0, T]$ non-self-looping and so by Lemma 3.6, $|\mathcal{L}(\varepsilon,u)| \le (C_n t_0/T)2^{2k_2^\varepsilon}$. Using the definition of k_1^ε and k_2^ε in (3-71), we have, for h small enough,

$$c\varepsilon^2R(h)^{1-n} = \varepsilon\frac{\sqrt{t_0}}{\sqrt{T}}R(h)^{(1-n)/2}2^{k_1^\varepsilon} \le |\mathcal{L}(\varepsilon,u,\alpha)| \le |\mathcal{L}(\varepsilon,u)| \le \frac{C_n t_0}{T}2^{2k_2^\varepsilon} \le C\varepsilon^{-2}R(h)^{1-n},$$

which yields the upper bound on $|\mathcal{L}(\varepsilon,u)|$ and the lower bound on $|\mathcal{L}(\varepsilon,u,\alpha)|$. Note that the upper bound on $|\mathcal{L}(\varepsilon,u,\alpha)|$ follows from the fact that the total number of tubes over $B(x_\alpha, 3R(h))$ is bounded by $CR(h)^{1-n}$. Next, we note that the fact that at most $\mathfrak{D}_n$ tubes $\mathcal{T}_j$ overlap implies

$$\sum_{j\in\mathcal{L}(\varepsilon,u,\alpha)}\|\mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})\,\mathrm{Op}_h(\psi)Pu\|_{L^2}^2 \le C\|Pu\|_{L^2}^2 + O(h^\infty\|u\|_{L^2}).$$

Therefore, using the first inequality in (3-81) again, applying Cauchy–Schwarz, and using that there is $C > 0$ such that $|\mathcal{L}(\varepsilon,u,\alpha)| \le CR(h)^{1-n}$, we have

$$\begin{aligned}
\frac{\varepsilon}{4}\frac{\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T} &\le CR(h)^{(n-1)/2}|\mathcal{L}(\varepsilon,u,\alpha)|^{1/2}\Big(\sum_{j\in\mathcal{L}(\varepsilon,u,\alpha)}\|u_j\|_{L^2}^2\Big)^{1/2} + Ch^{-1}\|Pu\|_{L^2} + O(h^\infty)\|u\|_{L^2} \\
&\le C\Big(\sum_{j\in\mathcal{L}(\varepsilon,u,\alpha)}\|u_j\|_{L^2}^2\Big)^{1/2} + o(T^{-1}\|u\|_{L^2}). \qquad (3\text{-}82)
\end{aligned}$$

Here, the $o(T^{-1}\|u\|_{L^2})$ term comes from using (3-64). In particular, for h small enough,

$$c\frac{\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T} \le \Big(\sum_{j\in\mathcal{L}(\varepsilon,u,\alpha)}\|u_j\|^2\Big)^{1/2}.$$

This completes the proof of Theorem 3.8. □

4. Proof of Theorem 1.1

In order to finish the proof of Theorem 1.1, we need to verify that the hypotheses of Theorem 1.4 hold with $T(h) = b\log h^{-1}$ for some $b > 0$, such that, for all $x_1, x_2 \in U$, there is some splitting $\mathcal{J}_{x_1} = \mathcal{G}_{x_1,x_2} \cup \mathcal{B}_{x_1,x_2}$ of the set of tubes over $x_1 \in M$ with a set of "bad" tubes $\mathcal{B}_{x_1,x_2}$ satisfying

$$(|\mathcal{B}_{x_1,x_2}|R(h)^{n-1})^{(1-p_c/p)/(6+\varepsilon_0)} \le T(h)^{-1/2}$$

and $\varepsilon_0 > 0$. Fix $x_1, x_2 \in U$ and let $F_1, F_2 : T^*M \to \mathbb{R}^{n+1}$ be smooth functions such that, for $i = 1, 2$,

$$\begin{gathered} S^*_{x_i}M = F_i^{-1}(0), \quad \tfrac{1}{2}d(q, S^*_{x_i}M) \le |F_i(q)| \le 2d(q, S^*_{x_i}M), \quad \max_{|\alpha|\le 2}(|\partial^\alpha F_i(q)|) \le 2, \\ dF_i(q) \text{ has a right inverse } R_{F_i}(q) \text{ with } \|R_{F_i}(q)\| \le 2. \end{gathered} \tag{4-1}$$

Define also $\psi_i : \mathbb{R} \times T^*M \to \mathbb{R}^{n+1}$ by $\psi_i(t, \rho) = F_i \circ \varphi_t(\rho)$.

To find $\mathcal{B}_{x_1,x_2}$, we apply the arguments from [Canzani and Galkowski 2023, Sections 2, 4]. In particular, fix $a > 0$ and let $r_t := a^{-1}e^{-a|t|}$, $\Lambda > \Lambda_{\max}$, and $\Lambda_{\max}$ be as in (1-5). Suppose that $d(x_2\,, \mathcal{C}_{x_1}^{n-1,r_{t_0},t_0}) > r_{t_0}$. Then for $\rho_0 \in S^*_{x_1}M$ with $d(S^*_{x_2}M, \varphi_{t_0}(\rho_0)) < r_{t_0}$, we have by [Canzani and Galkowski 2023, Lemma 4.1] that there exists $\boldsymbol{w} \in T_{\rho_0}S^*_{x_1}M$ such that

$$d(\psi_2)_{(t_0,\rho_0)} : \mathbb{R}\partial_t \times \mathbb{R}\boldsymbol{w} \to T_{\psi_2(t_0,\rho_0)}\mathbb{R}^{n+1}$$

has a left inverse $L_{(t_0,\rho_0)}$ satisfying

$$\|L_{(t_0,\rho_0)}\| \le C_M \max(ae^{C_M(a+\Lambda)|t_0|}, 1),$$

Next, let $\{\Lambda^\tau_{\rho_j}(r_1)\}$ be a $(\mathfrak{D}_M, \tau, r_1)$-good cover for S^*M. We apply [Canzani and Galkowski 2023, Proposition 2.2] to construct $\mathcal{B}_{x_1,x_2}$ and $\mathcal{G}_{x_1,x_2}$.

Remark 4.1. We must point out that we are applying the proof of that proposition rather than the proposition as stated. The only difference here is that the loops we are interested in go from a point x_1 to a point x_2, where x_1 and x_2 are not necessarily equal. This does not affect the proof.

We use [Canzani and Galkowski 2023, Proposition 2.2] to see that there exist $\alpha_1 = \alpha_1(M) > 0$, $\alpha_2 = \alpha_2(M, a)$, and $\boldsymbol{C}_0 = \boldsymbol{C}_0(M, a)$ such that the following holds. Let $r_0, r_1, r_2 > 0$ satisfy

$$r_0 < r_1, \quad r_1 < \alpha_1 r_2, \quad r_2 \le \min\{R_0, 1, \alpha_2 e^{-\gamma T}\}, \quad r_0 < \tfrac{1}{3}e^{-\Lambda T} r_2, \tag{4-2}$$

where $\gamma = 5\Lambda + 2a$ and $\Lambda > \Lambda_{\max}$ where $\Lambda_{\max}$ is as in (1-5). Then, for all balls $B \subset S^*_{x_1}M$ of radius $R_0 > 0$, there is a family of points $\{\rho_j\}_{j\in\mathcal{B}_B} \subset S^*_{x_1}M$ such that

$$|\mathcal{B}_B| \le \boldsymbol{C}_0\mathfrak{D}_n r_2 \frac{R_0^{n-1}}{r_1^{n-1}} T\, e^{4(2\Lambda+a)T},$$

and for $j \in \mathcal{G}_B := \{j \in \mathcal{J}_{x_1} : B(\rho_j, 2r_1) \cap B \neq \varnothing\} \setminus \mathcal{B}_B\}$,

$$\bigcup_{t\in[t_0,T]} \varphi_t(\Lambda^\tau_{\rho_j}(r_1)) \cap \Lambda^\tau_{S^*_{x_2}M}(r_1) = \varnothing.$$

We proceed to apply [Canzani and Galkowski 2023, Proposition 2.2]. There is $c_M r^{1-n} \geq N_r > 0$ such that, for all $x_1 \in M$, we can cover $S^*_{x_1}M$ by N_r balls. Let $0 < R_0 < 1$ and $\{B_i\}_{i=1}^{N_{R_0}}$ be such a cover. Fix $0 < \varepsilon < \varepsilon_1 < \frac{1}{4}$ and set

$$r_0 := h^{\varepsilon_1}, \quad r_1 := h^{\varepsilon}, \quad r_2 := \frac{2}{\alpha_1} h^{\varepsilon}.$$

Let $T(h) = b \log h^{-1}$ with

$$0 < b < \frac{1}{4\Lambda_{\max}} < \frac{1-2\varepsilon_1}{2\Lambda_{\max}}$$

to be chosen later. Then, the assumptions in (4-2) hold provided

$$h^{\varepsilon} < \min\left\{\tfrac{1}{2}\alpha_1\alpha_2 e^{-\gamma T}, \tfrac{1}{2}\alpha_1 R_0\right\} \quad \text{and} \quad h^{\varepsilon_1-\varepsilon} < \frac{2}{3\alpha_1} e^{-\Lambda T}.$$

In particular, if we set $\alpha_3 = \frac{1}{2}\alpha_1\alpha_2$ and $\alpha_4 = \frac{2}{3}\alpha_1^{-1}$, the assumptions in (4-2) hold provided $h < \left(\frac{1}{2}\alpha_1 R(h)\right)^{1/\varepsilon}$ and

$$T(h) < \min\left\{\frac{\varepsilon}{\gamma}\log h^{-1} + \frac{\log \alpha_3}{\gamma}, \frac{\varepsilon_1 - \varepsilon}{\Lambda}\log h^{-1} + \frac{\log(\alpha_4)}{\Lambda}\right\}. \tag{4-3}$$

Fix $b > 0$ and $h_0 > 0$ such that $b < \frac{1}{12}\min(\varepsilon, \varepsilon_1 - \varepsilon)/(2\Lambda + a)$ and (4-3) is satisfied for all $h < h_0$. Note that this implies that $b = b(M, a)$ and $h_0 = h_0(M, a)$. Let $\mathcal{B}_{x_1,x_2} := \bigcup_{i=1}^{N_{R_0}} \mathcal{B}_{B_i}$. For $j \in \mathcal{G}_{x_1,x_2} := \mathcal{J}_{x_1} \setminus \mathcal{B}_{x_1,x_2}$, we then have

$$\bigcup_{t \in [t_0, T]} \varphi_t(\Lambda^{\tau}_{\rho_j}(r_1)) \cap \Lambda^{\tau}_{S^*_{x_2}M}(r_1) = \varnothing.$$

Moreover, shrinking h_0 in a way depending only on (M, a, ε), we have, for $0 < h < h_0$,

$$r_1^{n-1} |\mathcal{B}_{x_1,x_2}| \leq C_M \boldsymbol{C}_0 \mathfrak{D}_n r_2 T e^{4(2\Lambda + a)T} \leq h^{\varepsilon/3}.$$

Therefore, putting $R(h) = r_1 = h^{\varepsilon}$ and $T = T(h) = b \log h^{-1}$ in Theorem 1.4 proves Theorem 1.1.

5. Anisotropic pseudodifferential calculus

In this section, we develop the second microlocal calculi necessary to understand "effective sharing" of L^2 mass between two nearby points. That is, to answer the question: how much L^2 mass is necessary to produce high L^∞ growth at two nearby points? To that end, we develop a calculus associated to the coisotropic

$$\Gamma_x := \bigcup_{|t| < \frac{1}{2}\operatorname{inj}(M)} \varphi_t(\Omega_x), \quad \Omega_x := \{\xi \in T^*_x M : |1 - |\xi|_g| < \delta\},$$

which allows for localization to the Lagrangian leaves $\varphi_t(\Omega_x)$. In Section 6B we will see, using a type of uncertainty principle, that the calculi associated to two distinct points, $x_\alpha, x_\beta \in M$, are incompatible in the sense that, despite the fact that Γ_{x_α} and Γ_{x_β} intersect in a dimension 2 submanifold, for operators X_{x_α} and X_{x_β} localizing to Γ_{x_α} and Γ_{x_β}, respectively,

$$\|X_{x_\alpha} X_{x_\beta}\|_{L^2 \to L^2} \ll \|X_{x_\alpha}\|_{L^2 \to L^2} \|X_{x_\beta}\|_{L^2 \to L^2}.$$

Let $\Gamma \subset T^*M$ be a coisotropic submanifold and $L = \{L_q\}_{q\in\Gamma}$ be a family of Lagrangian subspaces $L_q \subset T_q\Gamma$ that is integrable in the sense that if U is a neighborhood of Γ, and V and W are smooth vector fields on T^*M such that $V_q, W_q \in L_q$ for all $q \in \Gamma$, then $[V, W]_q \in L_q$ for all $q \in \Gamma$. The aim of this section is to introduce a calculus of pseudodifferential operators associated to (L, Γ) that allows for localization to h^ρ neighborhoods of Γ with $0 \le \rho < 1$ and is compatible with localization to h^ρ neighborhoods of the foliation of Γ generated by L. This calculus is close in spirit to those developed in [Dyatlov and Zahl 2016; Sjöstrand and Zworski 1999]. To see the relationships between these calculi, note that the calculus in [Dyatlov and Zahl 2016] allows for localization to any leaf of a Lagrangian foliation defined over an open subset of T^*M, while that in [Sjöstrand and Zworski 1999] allows for localization to a single hypersurface. The calculus developed in this paper is designed to allow localization along leaves of a Lagrangian foliation defined only over a coisotropic submanifold of T^*M. In the case that the coisotropic is a whole open set, this calculus is the same as the one developed in [Dyatlov and Zahl 2016]. Similarly, in the case that the coisotropic is a hypersurface and no Lagrangian foliation is prescribed, the calculus becomes that developed in [Sjöstrand and Zworski 1999].

Definition 5.1. Let Γ be a coisotropic submanifold and L a Lagrangian foliation on Γ. Fix $0 \le \rho < 1$ and let k be a positive integer. We say that $a \in S^k_{\Gamma,L,\rho}$ if $a \in C^\infty(T^*M)$, a is supported in an h-independent compact set, and

$$V_1 \cdots V_{\ell_1} W_1 \cdots W_{\ell_2} a = O(h^{-\rho\ell_2} \langle h^{-\rho} d(\Gamma, \cdot)\rangle^{k-\ell_2}), \tag{5-1}$$

where $W_1, \dots, W_{\ell_2}$ are any vector fields on T^*M, $V_1, \dots, V_{\ell_1}$ are vector fields on T^*M with

$$(V_1)_q, \dots, (V_{\ell_1})_q \in L_q$$

for $q \in \Gamma$, and $q \mapsto d(\Gamma, q)$ is the distance from q to Γ induced by the Sasaki metric on T^*M.

We also define symbol classes associated to only to the coisotropic submanifold Γ.

Definition 5.2. Let Γ be a coisotropic submanifold. We say that $a \in S^k_{\Gamma,\rho}$ if $a \in C^\infty(T^*M)$, a is supported in an h-independent compact set, and

$$V_1 \cdots V_{\ell_1} W_1 \cdots W_{\ell_2} a = O(h^{-\rho\ell_2} \langle h^{-\rho} d(\Gamma, \cdot)\rangle^{k-\ell_2})$$

where $V_1, \dots, V_{\ell_1}$ are tangent vector fields to Γ and $W_1, \dots, W_{\ell_2}$ are any vector fields.

5A. ***Model case.*** The goal of this section is to define the quantization of symbols in $S^k_{\Gamma_0,L_0,\rho}$, where Γ_0 and L_0 are a model pair of coisotropic and Lagrangian foliation defined below. The model coisotropic submanifold of dimension $2n - r$ is

$$\Gamma_0 := \{(x', x'', \xi', \xi'') \in \mathbb{R}^r \times \mathbb{R}^{n-r} \times \mathbb{R}^r \times \mathbb{R}^{n-r} : x' = 0\}$$

with Lagrangian foliation

$$L_0 := \{L_{0,q}\}_{q\in\Gamma_0}, \quad L_{0,q} = \operatorname{span}\{\partial_{\xi_i}, i = 1, \dots, n\} \subset T_q\Gamma_0.$$

Note that in this model case the distance from a point (x, ξ) to Γ_0 is controlled by $|x'|$. Therefore, $a \in S^k_{\Gamma_0, L_0, \rho}$ if and only if a is supported in an h-independent compact set and, for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$, there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a| \le C_{\alpha,\beta} h^{-\rho|\alpha|} \langle h^{-\rho}|x'|\rangle^{k-|\alpha|}.$$

In the model case, it will be convenient to define $\tilde{a} \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi \times \mathbb{R}^r_\lambda)$ such that

$$a(x, \xi) = \tilde{a}(x, \xi, h^{-\rho} x'),$$

and, for all $(\alpha', \alpha'', \beta, \gamma) \in \mathbb{N}^r \times \mathbb{N}^{n-r} \times \mathbb{N}^n \times \mathbb{N}^r$, there exists $C_{\alpha,\beta,\gamma} > 0$ such that

$$|\partial_{x'}^{\alpha'} \partial_{x''}^{\alpha''} \partial_\xi^\beta \partial_\lambda^\gamma \tilde{a}(x, \xi, \lambda)| \le C_{\alpha,\beta,\gamma} h^{-\rho|\alpha''|} \langle \lambda \rangle^{k-|\gamma|-|\alpha''|}. \tag{5-2}$$

Similarly, if $a \in S^k_{\Gamma_0,\rho}$, then, for $(\alpha', \alpha'', \beta, \gamma) \in \mathbb{N}^r \times \mathbb{N}^{n-r} \times \mathbb{N}^n \times \mathbb{N}^r$, there exists $C_{\alpha,\beta,\gamma} > 0$ such that

$$|\partial_{x'}^{\alpha'} \partial_{x''}^{\alpha''} \partial_\xi^\beta \partial_\lambda^\gamma \tilde{a}(x, \xi, \lambda)| \le C_{\alpha,\beta,\gamma} \langle \lambda \rangle^{k-|\gamma|}. \tag{5-3}$$

Definition 5.3. The symbols associated with this submanifold are as follows: We say $a \in \widetilde{S}^k_{\Gamma_0, L_0, \rho}$ if $a \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi \times \mathbb{R}^r_\lambda)$ satisfies (5-2) and a is supported in an h-independent compact set in (x, ξ). If we have the improved estimates (5-3) then we say that $a \in \widetilde{S}^k_{\Gamma_0,\rho}$.

Remark 5.4. While there is no ρ in the definition of $\widetilde{S}^k_{\Gamma_0,\rho}$, we keep it in the notation for consistency.

Let $a \in \widetilde{S}^k_{\Gamma_0, L_0, \rho}$. We then define

$$[\widetilde{\mathrm{Op}}_h(a)]u(x) := \frac{1}{(2\pi h)^n} \int e^{i\langle x-y, \xi\rangle/h} a(x, \xi, h^{-\rho} x') u(y)\, dy\, d\xi.$$

Since $a \in \widetilde{S}^k_{\Gamma_0, L_0, \rho}$ is compactly supported in x, there exists $C > 0$ such that on the support of the integrand $|\lambda| \le C h^{-\rho}$, and hence $h \le C h^{1-\rho} \langle \lambda \rangle^{-1}$. This will be important when computing certain asymptotic expansions.

Lemma 5.5. *Let $k \in \mathbb{R}$ and $a \in \widetilde{S}^k_{\Gamma_0, L_0, \rho}$. Then,*

$$\|\widetilde{\mathrm{Op}}_h(a)\|_{L^2 \to L^2} \le C \sup_{\mathbb{R}^{2n}} |a(x, \xi, h^{-\rho} x')| + O(h^{-\rho \max(k,0) + (1-\rho)/2}).$$

Proof. Define $T_\delta : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$T_\delta u(x) := h^{n\delta/2} u(h^\delta x). \tag{5-4}$$

Then T_δ is unitary and, for $a \in \widetilde{S}^k_{\Gamma_0, L_0, \rho}$,

$$\widetilde{\mathrm{Op}}_h(a) u = T^{-1}_{(1+\rho)/2} \operatorname{Op}_1(a_h) T_{(1+\rho)/2} u, \quad a_h(x, \xi) := a(h^{(1+\rho)/2} x, h^{(1-\rho)/2} \xi, h^{(1-\rho)/2} x').$$

Then, for all $\alpha, \beta \in \mathbb{N}^n$, there exists $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a_h| \le C_{\alpha,\beta} h^{(1-\rho)(|\alpha|+|\beta|)/2} \langle h^{(1-\rho)/2} x' \rangle^{k-|\alpha|}.$$

Now, since $a_h \in S_{(1-\rho)/2}$, by [Zworski 2012, Theorem 4.23] there is a universal constant $M > 0$ with

$$\|\widetilde{\mathrm{Op}}_1(a_h)\|_{L^2\to L^2} \le C \sum_{|\alpha|\le Mn} \sup_{\mathbb{R}^{2n}} |\partial^\alpha a_h| \le C \sup|a| + C_a h^{-\max(\rho k,0)+(1-\rho)/2}.$$

(To see that [Zworski 2012, Theorem 4.23] applies equally well to the left quantization, we apply the change of quantization formula [Zworski 2012, Theorem 4.13] and the boundedness of $e^{i\langle QD,D\rangle/2}$ on symbol classes [Zworski 2012, Theorem 4.17].) ☐

Lemma 5.6. *Suppose that $a \in \widetilde{S}^{k_1}_{\Gamma_0,L_0,\rho}$ and $b \in \widetilde{S}^{k_2}_{\Gamma_0,L_0,\rho}$. Then*

$$\widetilde{\mathrm{Op}}_h(a)\widetilde{\mathrm{Op}}_h(b) = \widetilde{\mathrm{Op}}_h(c) + O(h^\infty)_{L^2\to L^2},$$

where $c \in \widetilde{S}^{k_1+k_2}_{\Gamma_0,L_0,\rho}$ satisfies

$$c = ab + O(h^{1-\rho})_{\widetilde{S}^{k_1+k_2-1}_{\Gamma_0,L_0,\rho}}. \tag{5-5}$$

In particular,

$$c \sim \sum_j \sum_{|\alpha|=j} \frac{i^j}{j!}((hD_{x''})^{\alpha''}(hD_{x'} + h^{1-\rho}D_\lambda)^{\alpha'} b) D_\xi^\alpha a. \tag{5-6}$$

If instead $a \in \widetilde{S}^{k_1}_{\Gamma_0,\rho}$ and $b \in \widetilde{S}^{k_2}_{\Gamma_0,\rho}$, then the remainder in (5-5) *lies in $h^{1-\rho}\widetilde{S}^{k_1+k_2-1}_{\Gamma_0,\rho}$.*

Proof. With T_δ as in (5-4), we have $\widetilde{\mathrm{Op}}_h(a)\widetilde{\mathrm{Op}}_h(b) = T^{-1}_{\rho/2}\,\mathrm{Op}_h(a_h)\,\mathrm{Op}_h(b_h)T_{\rho/2}$, where

$$a_h = a(h^{\rho/2}x, h^{-\rho/2}\xi, h^{-\rho/2}x') \quad \text{and} \quad b_h = b(h^{\rho/2}x, h^{-\rho/2}\xi, h^{-\rho/2}x').$$

Now, for all $\alpha, \beta \in \mathbb{N}^n$, there exists $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a_h| \le C_{\alpha,\beta} h^{-\rho(|\alpha|+|\beta|)/2} \langle h^{-\rho/2}x'\rangle^{k_1-|\alpha|} \quad \text{and} \quad |\partial_x^\alpha \partial_\xi^\beta b_h| \le C_{\alpha,\beta} h^{-\rho(|\alpha|+|\beta|)/2} \langle h^{-\rho/2}x'\rangle^{k_2-|\alpha|}.$$

In particular, using that a and b are compactly supported, we have that $a_h \in h^{-\max(\rho k_1,0)} S_{\rho/2}$ and $b_h \in h^{-\max(\rho k_2,0)} S_{\rho/2}$, and hence [Zworski 2012, Theorems 4.14, 4.17] apply. In particular, if we let $M > 0$ and $\tilde{k} := \max(k_1,0) + \max(k_2,0)$, we obtain $\mathrm{Op}_h(a_h)\,\mathrm{Op}_h(b_h) = \mathrm{Op}_h(c_h)$, where, for any $N > 0$,

$$\begin{aligned} c_h(x,\xi) &= \sum_{j=0}^{N-1} \sum_{|\alpha|=j} \frac{h^j i^j}{j!} (D_\xi^\alpha a_h(x,\xi))(D_x^\alpha b_h(x,\xi)) + O(h^{-\rho\tilde{k}+N(1-\rho)})_{S_{\rho/2}} \\ &= \sum_{j=0}^{N-1} \sum_{|\alpha|=j} \sum_{\alpha'+\alpha''=\alpha} \frac{h^{(1-\rho)j} i^j}{j!} (D_\xi^\alpha a)_h [(h^\rho D_{x''})^{\alpha''}(h^\rho D_{x'} + D_\lambda)^{\alpha'} b]_h + O(h^{-\rho\tilde{k}+N(1-\rho)})_{S_{\rho/2}}. \end{aligned}$$

Choosing

$$N = \max\biggl(k_1 + k_2, \frac{\rho\tilde{k} + M}{1-\rho}\biggr),$$

the remainder is $O(h^M)_{S_{\rho/2}}$. Moreover, since a and b were compactly supported, we may assume, introducing an h^∞ error, that the remainder is supported in $\{(x,\xi) : |(x,\xi)| \le Ch^{-\rho/2}\}$. Putting

$$c = \sum_{j=0}^{N-1} \sum_{|\alpha|=j} \sum_{\alpha'+\alpha''=\alpha} \frac{i^j}{j!} (D_\xi^\alpha a)[(hD_{x''})^{\alpha''}(hD_{x'} + h^{1-\rho}D_\lambda)^{\alpha'} b],$$

we thus have $T^{-1}_{\rho/2}\,\mathrm{Op}_h(c_h)T_{\rho/2} = \widetilde{\mathrm{Op}}_h(c) + O(h^M)_{\mathcal{D}'\to C^\infty}$ as claimed. ☐

Lemma 5.7. *Suppose that $a \in \widetilde{S}^{m_1}_{\Gamma_0,L_0,\rho}$ and $b \in \widetilde{S}^{m_2}_{\Gamma_0,L_0,\rho}$. Then,*

$$[\widetilde{\mathrm{Op}}_h(a), \widetilde{\mathrm{Op}}_h(b)] = -ih^{1-\rho}\widetilde{\mathrm{Op}}_h(c) + O(h^\infty)_{L^2\to L^2},$$

where $c \in \widetilde{S}^{m_1+m_2-2}_{\Gamma_0,L_0,\rho}$ satisfies

$$c = h^\rho \sum_{i=1}^n (\partial_{\xi_i} a \partial_{x_i} b - \partial_{\xi_i} b \partial_{x_i} a) + \sum_{i=1}^r (\partial_{\xi_i} a \partial_{\lambda_i} b - \partial_{\lambda_i} a \partial_{\xi_i} b) + O(h^{1-\rho})_{\widetilde{S}^{m_1+m_2-2}_{\Gamma_0,L_0,\rho}}.$$

If instead $a \in \widetilde{S}^{m_1}_{\Gamma_0,\rho}$ and $b \in \widetilde{S}^{m_2}_{\Gamma_0,\rho}$, then the remainder lies in $h^{1-\rho}\widetilde{S}^{m_1+m_2-2}_{\Gamma_0,\rho}$. Moreover, if $a \in S^{\mathrm{comp}}(\mathbb{R}^{2n})$ is independent of λ and $\partial_{\xi'} a = e(x,\xi)x'$ with $e(x,\xi) : \mathbb{R}^r \to \mathbb{R}^r$ for all (x,ξ), then

$$[\widetilde{\mathrm{Op}}_h(a), \widetilde{\mathrm{Op}}_h(b)] = -ih\widetilde{\mathrm{Op}}_h(c) + O(h^\infty)_{\Psi^{-\infty}}$$

with $c = H_a b + \sum_{i=1}^r (e\lambda)_i \partial_{\lambda_i} b + O(h^{1-\rho})_{\widetilde{S}^{m_2-1}_{\Gamma_0,L_0,\rho}}$. Similarly, the same conclusion holds if $b \in \widetilde{S}^{m_2}_{\Gamma_0,\rho}$ with the error term in c being $O(h^{1-\rho})_{\widetilde{S}^{m_2-1}_{\Gamma_0,\rho}}$.

Proof. In each case, we need only apply formula (5-6). □

5B. *Reduction to normal form.* In order to define the quantization of symbols in $S_{\Gamma,L,\rho}$ for general (Γ, L), we first explain how to reduce the problem to the model case (Γ_0, L_0).

Lemma 5.8. *Let L be a Lagrangian foliation over a coisotropic submanifold $\Gamma \subset \mathbb{R}^{2n}$ of dimension $2n-r$. Then, there is a neighborhood U_0 of (x_0,ξ_0) and a symplectomorphism $\kappa : U_0 \to V_0 \subset T^*\mathbb{R}^n$ for each $(x_0,\xi_0) \in \Gamma$ such that*

$$\kappa(\Gamma \cap U_0) = \Gamma_0 \cap V_0 \qquad \text{and} \qquad (\kappa_*)_q L_q = L_{0,q} \quad \text{for } q \in \Gamma \cap U_0.$$

Proof. We first put Γ in normal form. That is, we build symplectic coordinates (y,η) such that

$$\Gamma = \{(y,\eta) : y_1 = \cdots = y_r = 0\}. \tag{5-7}$$

First, assume $r = 1$ and let $f_1 \in C^\infty(T^*M)$ define Γ. By Darboux's theorem (see e.g., [Zworski 2012, Theorem 12.1]) there are symplectic coordinates such that $y_1 = f_1$, and the proof of (5-7) is complete for $r = 1$.

Next, assume that we can put any coisotropic of codimension $r-1$ in normal form. Let $f_1, \dots, f_r \in C^\infty(T^*M)$ define Γ. Then, for $X \in T\Gamma$ and $i = 1, \dots, r$,

$$\sigma(X, H_{f_i}) = df_i(X) = 0.$$

In addition, since Γ is coisotropic, $(T\Gamma)^\perp \subset T\Gamma$, and so $H_{f_i} \in T\Gamma$ for all $i = 1, \dots, r$. In particular,

$$\{f_i, f_j\} = H_{f_i} f_j = df_j(H_{f_i}) = 0 \quad \text{on } \Gamma.$$

Now, using Darboux' theorem, choose symplectic coordinates $(y,\eta) = (y_1, y', \eta_1, \eta')$ such that $y_1 = f_1$ and $(x_0,\xi_0) \mapsto (0,0)$. Then, $\partial_{\eta_1} f_j = \{f_j, y_1\} = 0$ on Γ for $j = 2, \dots, r$. Next, we will observe that $\Gamma = \{(y,\eta) : y_1 = f_2 = \cdots = f_r = 0\}$ and dy_1 and $\{df_j\}_{j=2}^r$ are independent. Thus, since $\partial_{\eta_1} f_j = 0$ on Γ,

$$\Gamma = \{(y,\eta) : y_1 = 0,\ f_j(0, y', 0, \eta') = 0,\ j = 2, \dots, r\}.$$

Now, $\{y_1=\eta_1=0\}\cap\Gamma$ is a coisotropic submanifold of codimension $r-1$ in $T^*\{y_1=0\}$. Hence, by induction, there are symplectic coordinates $(y_2,\dots,y_n,\eta_2,\dots,\eta_n)$ on $T^*\{y_1=0\}$ such that

$$\Gamma\cap\{y_1=\eta_1=0\}=\{y_1=\eta_1=0,\ y_2=\cdots=y_r=0\}.$$

In particular,

$$\{(y',\eta'):f_j(0,y',0,\eta')=0,\ j=2,\dots,r\}=\{y_2=\cdots=y_r=0\}.$$

Thus, extending $(y_2,\dots,y_n,\eta_2,\dots,\eta_n)$ to be independent of (y_1,η_1) puts Γ in the form (5-7).

Next, we adjust the coordinates to be adapted to L along Γ. First, define $\tilde{y}_i:=y_i$ for $i=1,\dots,r$. Then, since $L\subset T\Gamma$, for every $i=1,\dots,r$, we have that $d\tilde{y}_i(X)|_\Gamma$ is well defined for $X\in L$ and $d\tilde{y}_i(X)|_\Gamma=0$. Next, since L is integrable, the Frobenius theorem [Lee 2013, Theorem 19.21] shows that there are coordinates $(\tilde{y}_{r+1},\dots,\tilde{y}_n,\tilde{\xi}_1,\dots,\tilde{\xi}_n)$ on Γ, defined in a neighborhood of $(0,0)$, such that L is the annihilator of $d\tilde{y}$. Since we know that for every $X\in L$,

$$\sigma(X,H_{\tilde{y}_i})=d\tilde{y}_i(X)=0$$

and L is Lagrangian, we conclude that $H_{\tilde{y}_i}\in L$. In particular, since L is the annihilator of $d\tilde{y}$,

$$\{\tilde{y}_i,\tilde{y}_j\}=H_{\tilde{y}_i}\tilde{y}_j=d\tilde{y}_j(H_{\tilde{y}_i})=0.$$

Now, extend $(\tilde{y}_{r+1},\dots,\tilde{y}_n,\tilde{\xi}_1,\dots,\tilde{\xi}_n)$ outside Γ to be independent of $(\tilde{y}_1,\dots,\tilde{y}_r)$. Then, $\{\tilde{y}_i,\tilde{y}_j\}=0$ in a neighborhood of (x_0,ξ_0), and hence, by Darboux's theorem, there are functions $\{\tilde{\eta}_j\}_{j=1}^n$ such that $\{\tilde{y}_i,\tilde{\eta}_j\}=\delta_{ij}$ and $\{\tilde{\eta}_i,\tilde{\eta}_j\}=0$. In particular, in the $(\tilde{y},\tilde{\eta})$ coordinates, $\Gamma=\{(\tilde{y},\tilde{\eta}):\tilde{y}_1=\cdots=\tilde{y}_r=0\}$ and $d\tilde{y}(L)|_\Gamma=0$. In particular, $L=\operatorname{span}\{\partial\tilde{\eta}_i\}$ as claimed. □

In order to create a well-defined global calculus of pseudodifferential operators associated to (Γ,L), we will need to show invariance under conjugation by Fourier integral operators (FIOs) preserving the pair (L_0,Γ_0).

Proposition 5.9. *Suppose that U_0 and V_0 are neighborhoods of $(0,0)$ in $T^*\mathbb{R}^n$ and $\kappa:U_0\to V_0$ is a symplectomorphism such that*

$$\kappa(0,0)=(0,0),\quad \kappa(\Gamma_0\cap U_0)=\Gamma_0\cap V_0,\quad \kappa_*|_{\Gamma_0}L_0=L_0|_{\Gamma_0}. \tag{5-8}$$

Next, let T be a semiclassically elliptic FIO microlocally defined in a neighborhood of

$$((0,0),(0,0))\in T^*\mathbb{R}^n\times T^*\mathbb{R}^n$$

quantizing κ. Then, for $a\in\widetilde{S}^k_{\Gamma_0,L_0,\rho}$, there are $b\in\widetilde{S}^k_{\Gamma_0,L_0,\rho}$ and $c\in\widetilde{S}^{k-1}_{\Gamma_0,L_0,\rho}$ such that

$$T^{-1}\widetilde{\mathrm{Op}}_h(a)T=\widetilde{\mathrm{Op}}_h(b)\quad\textit{and}\quad b=a\circ K_\kappa+h^{1-\rho}c,$$

where $K_\kappa:T^*\mathbb{R}^n\times\mathbb{R}^r\to T^*\mathbb{R}^n\times\mathbb{R}^r$ is defined by

$$K_\kappa(y,\eta,\mu)=\left(\kappa(y,\eta),\pi_{x'}(\kappa(y,\eta))\frac{|\mu|}{|y'|}\right),$$

and $\pi_{x'}:T^*\mathbb{R}^n\to\mathbb{R}^r$ is the projection onto the first r-spatial coordinates. In addition, if $a\in\widetilde{S}^k_{\Gamma_0,\rho}$, then $c\in\widetilde{S}^{k-1}_{\Gamma_0,\rho}$ and $b\in\widetilde{S}^k_{\Gamma_0,\rho}$.

To prove Proposition 5.9, we follow [Sjöstrand and Zworski 1999]. First, observe that the proposition holds with $\kappa = \mathrm{Id}$ since then T is a standard pseudodifferential operator. In addition, the proposition also holds whenever, for a given $j \in \{1, \dots, n\}$, we work with

$$\kappa(y, \eta) := (y_1, \dots, y_{j-1}, -y_j, y_{j+1}, \dots, y_n, \eta_1, \dots, \eta_{j-1}, -\eta_j, \eta_{j+1}, \dots, \eta_n).$$

Indeed, this follows from the fact that in this case an FIO quantizing κ is

$$Tu(x) = u(x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_n),$$

and so the conclusion of the proposition follows from a direct computation together with the identity case. Thus, we may assume that

$$\kappa(y, \eta) = (x, \xi) \quad \Rightarrow \quad x_i y_i \geq 0, \ \ i = 1, \dots, n. \tag{5-9}$$

Lemma 5.10. *Let κ be a symplectomorphism satisfying* (5-8) *and* (5-9). *Then, there is a piecewise smooth family of symplectomorphisms* $[0, 1] \ni t \mapsto \kappa_t$ *such that κ_t satisfies* (5-8), (5-9), $\kappa_0 = \mathrm{Id}$, *and* $\kappa_1 = \kappa$.

Proof. In what follows we assume that $\kappa(y, \eta) = (x, \xi)$ but reorder the coordinates: $(y', y'', \eta', \eta'') \in T^*\mathbb{R}^n$ is written as $(y', \eta', y'', \eta'') \in \mathbb{R}^{2r} \times \mathbb{R}^{2(n-r)}$. Let ξ' and $\kappa'' = (x''(y', \eta), \xi''(y', \eta))$ with

$$\kappa|_{\Gamma_0} : (0, \eta', y'', \eta'') \mapsto (0, \xi'(y'', \eta), \kappa''(y'', \eta)).$$

Now, since $(\kappa_*)|_{\Gamma_0} L_0 = L_0$, we have, for $i = 1, \dots, n$,

$$\kappa_* \partial_{\eta_i} = \frac{\partial x_j}{\partial \eta_i} \partial_{x_j} + \frac{\partial \xi_j}{\partial \eta_i} \partial_{\xi_j} \in L_0, \tag{5-10}$$

and hence

$$\partial_\eta x|_{\Gamma_0} \equiv 0. \tag{5-11}$$

Next, since κ preserves Γ_0, $\{\kappa^* x_i\}_{i=1}^r$ defines Γ_0, and $\mathrm{span}\{d\kappa^* x_i|_{\Gamma_0}\}_{i=1}^r = \mathrm{span}\{dy_i|_{\Gamma_0}\}_{i=1}^r$, we have

$$\mathrm{span}\{H_{\kappa^* x_i}|_{\Gamma_0}\}_{i=1}^r = \mathrm{span}\{H_{y_i}|_{\Gamma_0}\}_{i=1}^r.$$

By Jacobi's theorem, $\kappa_* H_{\kappa^* x^i} = H_{x_i}$. Therefore,

$$(\kappa|_{\Gamma_0})_* (\mathrm{span}\{H_{y_i}\}_{i=1}^r|_{\Gamma_0}) = \mathrm{span}\{H_{x_i}\}_{i=1}^r|_{\Gamma_0},$$

and we conclude from (5-10) that $\xi''|_{\Gamma_0}$ is independent of η', and hence that κ'' is independent of η'. In particular, κ'' is a symplectomorphism on $T^*\mathbb{R}^{n-r}$. This also implies that, for each fixed (y'', η''), the map $\eta' \mapsto \xi'(y'', \eta', \eta'')$ is a diffeomorphism. Writing

$$\kappa''(y'', \eta'') = (x''(y'', \eta''), \xi''(y'', \eta'')),$$

we have by (5-11) that $\partial_{\eta''} x'' = 0$, and hence $x'' = x''(y'')$. Now, since κ'' is symplectic,

$$(\partial_{\eta''} \xi'' d\eta'' + \partial_{y''} \xi'' dy'') \wedge \partial_{y''} x'' dy'' = d\eta'' \wedge dy'',$$

and so we conclude that

$$(\partial_{y''} x'')^t \partial_{\eta''} \xi'' = \mathrm{Id}, \quad (\partial_{y''} x'')^t \partial_{y''} \xi'' \text{ is diagonal.} \tag{5-12}$$

The first equality in (5-12) gives that $\partial_{\eta''}\xi''$ is a function of y'' only, and hence there exists a function $F = F(y'')$ such that

$$\xi''(y'', \eta'') = [(\partial x''(y''))^t]^{-1}(\eta'' - F(y'')).$$

Therefore, calculating on $\eta'' = F(y'')$, the second statement in (5-12) implies that $-\partial_{y''}F(y'')\,dy''\wedge dy'' = 0$. In particular, $d(F(y'')\cdot dy'') = 0$. It follows from the Poincaré lemma that, shrinking the neighborhood of $(0, 0)$ to be simply connected if necessary, $F(y'')\cdot dy'' = d\psi(y'')$ for some function $\psi = \psi(y'')$. Hence,

$$\kappa''(y'', \eta'') = (x''(y''), [(dx''(y''))^t]^{-1}(\eta'' - \partial\psi(y''))). \tag{5-13}$$

Now, every symplectomorphism of the form (5-13) preserves L_0. Hence, we can deform κ'' to the identity by putting $\psi_t = t\psi$ and deforming x'' to the identity. Since the assumption in (5-9) implies $\partial_{y''}x'' > 0$, this can be done simply by taking $x''_t = (1-t)\operatorname{Id} + tx''$. Putting $\kappa''_t = (x''_t, \xi''_t)$, there is κ''_t such that $\kappa''_0 = \operatorname{Id}$ and $\kappa''_1 = \kappa''$. Now, composing κ with

$$(y', \eta'; y'', \eta'') \mapsto (y', \eta'; (\kappa''_t)^{-1}(y'', \eta'')),$$

we reduce to the case that $\kappa'' = \operatorname{Id}$. In particular, we need only consider the case in which

$$\kappa(y', \eta', y'', \eta'') = (f(y, \eta)y', \xi'(y'', \eta) + h_0(y, \eta)y', (y'', \eta'') + h_1(y, \eta)y'), \tag{5-14}$$

where $f(y, \eta) \in \mathbb{GL}_r$, $h_0(y, \eta)$ is an $r \times r$ matrix, and $h_1(y, \eta)$ is an $2(n-r) \times r$ matrix. Next, we claim that the projection map from graph(κ) to $\mathbb{R}^{2n}$ defined as $(x, \xi; y, \eta) \mapsto (x, \eta)$ is a local diffeomorphism. To see this, note that, for $|y'|$ small, the map $(x'', \eta'') \mapsto (y'', \xi'')$ is a diffeomorphism, that, for each fixed (y'', η''), the map $\eta' \mapsto \xi'$ is a diffeomorphism, and that $\det \partial_{y'}x'|_{\Gamma_0} \neq 0$. Thus, κ has a generating function ϕ:

$$\kappa : (\partial_\eta\phi(x, \eta), \eta) \mapsto (x, \partial_x\phi(x, \eta))$$

such that

$$\det \partial^2_{x\eta}\phi(0, 0) \neq 0 \quad \text{and} \quad \partial_{\eta'}\phi(0, x'', \eta) = 0.$$

Now, writing $\kappa = (\kappa', \kappa'')$, we have $\kappa'' = \operatorname{Id}$ at $x' = 0$. Therefore,

$$\partial_{\eta''}\phi(0, x'', \eta) = x'' \quad \text{and} \quad \partial_{x''}\phi(0, x'', \eta) = \eta'',$$

and we have $\phi(0, x'', \eta) = \langle x'', \eta''\rangle + C$ for some $C \in \mathbb{R}$. We may choose $C = 0$ to obtain

$$\phi(x, \eta) = \langle x'', \eta''\rangle + g(x, \eta)x' \tag{5-15}$$

for some $g : \mathbb{R}^{2n} \to \mathbb{M}_{1\times r}$. Finally, since $\kappa(0, 0) = (0, 0)$ and $\partial^2_{x\eta}\phi$ is nondegenerate, we have that $\partial_{x'}\phi(0, 0) = g(0, 0) = 0$ and $\partial_{\eta'}g$ is nondegenerate. In fact (5-9) implies that, as a quadratic form,

$$\partial_{\eta'}g > 0. \tag{5-16}$$

Observe next that every ϕ satisfying (5-15) for some g satisfying (5-16) and $g(0, 0) = 0$ generates a canonical transformation satisfying (5-14) and (5-9). In particular, the symplectomorphism satisfies (5-8). Thus, we can deform from the identity by putting $g_t = (1-t)\eta' + tg$. □

Finally, we proceed with the proof of Proposition 5.9.

Proof of Proposition 5.9. Let κ_t be as in Lemma 5.10. That is, a piecewise smooth deformation from $\kappa_0 = \mathrm{Id}$ to $\kappa_1 = \kappa$ such that κ_t preserves Γ_0 and $(\kappa_t)_*|_{\Gamma_0}$ preserves L_0. Let T_t be a piecewise smooth family of elliptic FIOs defined microlocally near (0, 0), quantizing κ_t, and satisfying

$$hD_tT_t + T_tQ_t = 0 \quad \text{and} \quad T_0 = \mathrm{Id}. \tag{5-17}$$

Here, Q_t is a smooth family of pseudodifferential operators with symbol q_t satisfying $\partial_t\kappa_t = (\kappa_t)_* H_{q_t}$. (Such an FIO exists, for example, by [Zworski 2012, Chapter 10], and q_t exists by [Zworski 2012, Thoerems 11.3, 11.4].) Next, define

$$A_t := T_t^{-1}\widetilde{\mathrm{Op}}_h(a)T_t.$$

Note that $T^{-1}\widetilde{\mathrm{Op}}(a)T = T^{-1}T_1T_1^{-1}\widetilde{\mathrm{Op}}(a)T_1T_1^{-1}T + O(h^\infty)_{\Psi^{-\infty}}$. Hence, since the proposition follows by direct calculation when $\kappa = \mathrm{Id}$, we may assume that $T = T_1$.

In that case, our goal is to find a symbol b such that $A_1 = \mathrm{Op}_h(b)$. First, observe that (5-17) implies that $hD_tT_t^{-1} - Q_tT_t^{-1} = 0$ and so

$$hD_tA_t = [Q_t, A_t] \quad \text{and} \quad A_0 = \widetilde{\mathrm{Op}}_h(a).$$

We will construct $b_t \in \widetilde{S}^k_{\Gamma_0,L_0,\rho}$ such that $B_t := \widetilde{\mathrm{Op}}_h(b_t)$ satisfies

$$hD_tB_t = [Q_t, B_t] + O(h^\infty)_{\Psi^{-\infty}} \quad \text{and} \quad B_0 = \widetilde{\mathrm{Op}}_h(a). \tag{5-18}$$

This would yield that $B_t - A_t = O(h^\infty)_{L^2\to L^2}$ and the argument would then be finished by setting $b = b_1$. Indeed, that $B_t - A_t = O(h^\infty)_{L^2\to L^2}$ would follow from the fact that, by (5-18),

$$hD_t(T_tB_tT_t^{-1}) = O(h^\infty)_{\Psi^{-\infty}},$$

and hence, since $T_0 = \mathrm{Id}$ and $B_0 = \widetilde{\mathrm{Op}}_h(a)$, we have $T_tB_tT_t^{-1} - \widetilde{\mathrm{Op}}_h(a) = O(h^\infty)_{\Psi^{-\infty}}$. Combining this with the fact that both T_t and T_t^{-1} are bounded on H_h^k completes the proof.

To find b_t as in (5-18), note that since κ_t preserves Γ_0 and L_0, $\partial_t\kappa_t = H_{q_t}$ and H_{q_t} is tangent to L_0 on Γ_0. Therefore, $\partial_{\eta'}q_t = 0$ on $y' = 0$, and so there exists $r_t(y, \eta)$ such that $\partial_{\eta'}q_t(y, \eta) = r_t(y, \eta)y'$. Hence, by Lemma 5.7, for any $b \in \widetilde{S}^k_{\Gamma_0,L_0,\rho}$,

$$[Q_t, \widetilde{\mathrm{Op}}_h(b)] = -ih\widetilde{\mathrm{Op}}_h(f) + O(h^\infty)_{\Psi^{-\infty}} \quad \text{and} \quad f = H_{q_t}b + \sum_{j=1}^{r}(r_t\lambda)_j(\partial_\lambda b)_j + O(h^{1-\rho})_{\widetilde{S}^{k-2}_{\Gamma_0,L_0,\rho}}.$$

Then, letting $b_t^0 := a \circ K_{\kappa_t} \in \widetilde{S}^k_{\Gamma_0,L_0,\rho}$ and $B_t^0 = \widetilde{\mathrm{Op}}_h(b_t^0)$ yields

$$hD_tB_t^0 = -ih\widetilde{\mathrm{Op}}_h(H_{q_t}b_t^0 + (r_t\mu)\cdot\partial_\mu b_t^0) = [Q_t, B_t^0] + h^{2-\rho}\widetilde{\mathrm{Op}}_h(e_t^0),$$

where $e_t^0 \in \widetilde{S}^{k-2}_{\Gamma_0,L_0,\rho}$. This follows from the fact that if we set $\mu(y) = y'h^{-\rho}$, then

$$\partial_t(b_t^0(y, \eta, \mu(y))) = H_{q_t}b_t^0(y, \eta, \mu(y)) + \partial_\mu b_t^0(y, \eta, \mu(y))H_{q_t}(\mu(y))$$

and $H_{q_t}\mu(y) = r_t(y, \eta)\mu(y)$.

Iterating this procedure and solving away successive errors finishes the proof of Proposition 5.9. If $a \in \widetilde{S}^k_{\Gamma_0,\rho}$, then we need only use that $\partial_{\xi'}q_t = r_tx'$ and we obtain the remaining results. $\square$

Our next lemma follows [Sjöstrand and Zworski 1999, Lemma 4.1] and gives a characterization of our second microlocal calculus in terms of the action of an operator. In what follows, given operators A and B, we define the operator ad_A by $\mathrm{ad}_A B = [A, B]$.

Lemma 5.11 (Beal's criteria). *Let $A_h : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ and $k \in \mathbb{Z}$. Then, $A_h = \widetilde{\mathrm{Op}}_h(a)$ for some $a \in \widetilde{S}^k_{\Gamma_0, L_0, \rho}$ if and only if, for any $\alpha, \beta \in \mathbb{R}^n$, there exists $C > 0$ with*

$$\|\mathrm{ad}^{\alpha}_{h^{-\rho}x}\, \mathrm{ad}^{\beta}_{hD_x} A_h u\|_{|\beta|-\min(k,0)} \le C h^{(1-\rho)(|\alpha|+|\beta|)} \|u\|_{\max(k,0)},$$

where $\|u\|_r := \|u\|_{L^2} + \|h^{-\rho r}|x'|^r u\|_{L^2}$ for $r \ge 0$. Similarly, $A_h = \mathrm{Op}_h(a)$ for some $a \in \widetilde{S}^k_{\Gamma_0,\rho}$ if and only if

$$\|\mathrm{ad}^{\alpha'}_{h^{-\rho}x'}\, \mathrm{ad}^{\alpha''}_{x''}\, \mathrm{ad}^{\beta'}_{hD_{x'}}\, \mathrm{ad}^{\beta''}_{hD_{x''}} A_h u\|_{|\beta'|-\min(k,0)} \le C h^{(1-\rho)(|\alpha'|+|\beta'|)+|\alpha''|+|\beta''|} \|u\|_{\max(k,0)}.$$

Proof. The fact that $A_h = \widetilde{\mathrm{Op}}_h(a)$ for some $a \in \widetilde{S}^k_{\Gamma_0,L_0,\rho}$ implies the estimates above follow directly from the model calculus. Let U_h be the unitary (on L^2) operator, $U_h u(x) = h^{n/2} u(hx)$, and note that

$$\|U_h^{-1} u\|_r = \|u\|_{L^2} + \|h^{(1-\rho)r}|x'|^r u\|_{L^2}.$$

Then, consider $\tilde{A}_h := U_h A_h U_h^{-1}$. For fixed h, we can use Beal's criteria (see e.g., [Zworski 2012, Theorem 8.3]) to see that there is a_h such that $\tilde{A}_h = a_h(x, D)$. Define a such that $a(hx, \xi; h) = a_h(x, \xi)$, and hence $A_h = \mathrm{Op}_h(a)$. Note that, for $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \tilde{A}_h \psi, \phi\rangle = \frac{1}{(2\pi)^n} \iint e^{i\langle x,\xi\rangle} a_h(x,\xi) \hat{\psi}(\xi) \overline{\phi(x)}\, dx\, d\xi, \tag{5-19}$$

where $\hat{\psi}(\xi) = (\mathcal{F}\psi)(\xi) = \int e^{-i\langle y,\xi\rangle} \psi(y)\, dy$. Next, define

$$B_h := U_h\, \mathrm{ad}^{\alpha}_{h^{-\rho}x} (\mathrm{ad}^{\beta}_{hD_x}(A_h)) U_h^{-1}.$$

Since $D_x U_h = U_h h D_x$ and $U_h^{-1} D_x = h D_x U_h^{-1}$, we have

$$B_h = \mathrm{ad}^{\alpha}_{h^{1-\rho}x}\, \mathrm{ad}^{\beta}_{D_x} \tilde{A}_h = (-i)^{|\alpha|+|\beta|} h^{(1-\rho)|\alpha|} b_h(x, D),$$

where $b_h(x,\xi) = (-\partial_\xi)^\alpha \partial_x^\beta a_h(x,\xi)$. Our goal is then to understand the behavior of $b_h(x,\xi)$ in terms of h and $\langle h^{1-\rho} x'\rangle$. Let τ_{x_0} and $\hat{\tau}_{\xi_0}$ be the physical and frequency shift operators

$$\tau_{x_0} u(x) = u(x - x_0) \quad \text{and} \quad \hat{\tau}_{\xi_0} u(x) = e^{i\langle x,\xi_0\rangle} u(x)$$

with $\mathcal{F}\hat{\tau}_{\xi_0} = \tau_{\xi_0}\mathcal{F}$ and $\mathcal{F}\tau_{x_0} = \hat{\tau}_{-x_0}$. In addition, write $\|u\|_{(-r)} := \|\langle h^{1-\rho}x'\rangle^{-r} u\|_{L^2}$ for the dual norm to $\|u\|_{(r)} := \|U_h^{-1} u\|_r$.

Assume that $k \ge 0$. Then, the definition of B_h combined with the assumptions yields

$$|\langle B \tau_{x_0}\hat{\tau}_{\xi_0}\psi, \tau_{y_0}\hat{\tau}_{\eta_0}\phi\rangle| \le h^{(1-\rho)(|\alpha|+|\beta|)} \|\tau_{x_0}\hat{\tau}_{\xi_0}\psi\|_{(k)} \|\tau_{y_0}\hat{\tau}_{\eta_0}\phi\|_{-|\beta|}. \tag{5-20}$$

In addition, note that, for fixed $\psi, \phi \in \mathcal{S}$,

$$\|\tau_{x_0}\hat{\tau}_{\xi_0}\psi\|_{(k)} \sim \langle h^{1-\rho}(x_0)'\rangle^k \quad \text{and} \quad \|\tau_{y_0}\hat{\tau}_{\eta_0}\psi\|_{(-|\beta|)} \sim \langle h^{1-\rho}(y_0)'\rangle^{-|\beta|}.$$

Therefore, (5-20) leads to

$$|\langle B \tau_{x_0}\hat{\tau}_{\xi_0}\psi, \tau_{y_0}\hat{\tau}_{\eta_0}\phi\rangle| \le C h^{(1-\rho)(|\alpha|+|\beta|)} \langle h^{1-\rho}(x_0)'\rangle^k \langle h^{1-\rho}(y_0)'\rangle^{-|\beta|}. \tag{5-21}$$

On the other hand, we have by (5-19) that

$$|\langle B\tau_{x_0}\hat{\tau}_{\xi_0}\psi, \tau_{y_0}\hat{\tau}_{\eta_0}\phi\rangle| = \frac{h^{(1-\rho)|\alpha|}}{(2\pi)^n}\left|\iint e^{i\langle x,\xi\rangle} b_h(x,\xi)\hat{\psi}(\xi-\xi_0)e^{-i\langle x_0,\xi-\xi_0\rangle - i\langle \eta_0, x-y_0\rangle}\bar{\phi}(x-y_0)\,dx\,d\xi\right|$$
$$= h^{(1-\rho)|\alpha|}|\mathcal{F}((\tau_{y_0,\xi_0}\chi)b_h)(\eta_0-\xi_0, x_0-y_0)|, \tag{5-22}$$

where $\chi(x,\xi) = e^{i\langle x,\xi\rangle}\hat{\psi}(\xi)\bar{\phi}(x)$. Combining (5-22) with (5-21) we then have

$$|\mathcal{F}((\tau_{y_0,\xi_0}\chi)\partial_\xi^\alpha\partial_x^\beta a_h)(\eta_0-\xi_0, x_0-y_0)| \le Ch^{(1-\rho)|\beta|}\langle h^{1-\rho}(x_0)'\rangle^k\langle h^{1-\rho}(y_0)'\rangle^{-|\beta|}.$$

Next, note that χ can be replaced by any fixed function in C_c^∞ by taking ψ and ϕ with $\hat{\psi}(\xi)\phi(x)\neq 0$ on $\operatorname{supp}\chi$. Putting $\zeta = \eta_0 - \xi_0$ and $z = x_0 - y_0$, we obtain that, for every $\tilde{\alpha}, \tilde{\beta} \in \mathbb{N}^n$,

$$|\mathcal{F}(\partial_\xi^{\tilde{\alpha}}\partial_x^{\tilde{\beta}}(\tau_{y_0,\xi_0}\chi)\partial_\xi^\alpha\partial_x^\beta a_h)(\zeta,z)| \le Ch^{(1-\rho)|\beta|}\langle h^{1-\rho}(x_0)'\rangle^k\langle h^{1-\rho}(x_0-z)'\rangle^{-|\beta|}.$$

Hence,

$$|z^{\tilde{\alpha}}\zeta^{\tilde{\beta}}\mathcal{F}((\tau_{y_0,\xi_0}\chi)\partial_\xi^\alpha\partial_x^\beta a_h)(\zeta,z)| \le Ch^{(1-\rho)|\beta|}\langle h^{1-\rho}(x_0)'\rangle^k\langle h^{1-\rho}(x_0-z)'\rangle^{-|\beta|}.$$

In particular, for every $N > 0$,

$$|\mathcal{F}((\tau_{y_0,\xi_0}\chi)\partial_\xi^\alpha\partial_x^\beta a_h)(\zeta,z)| \le Ch^{(1-\rho)|\beta|}\langle h^{1-\rho}(x_0)'\rangle^{k-|\beta|}\langle\zeta\rangle^{-N}\langle z\rangle^{-N},$$

and, as a consequence, we obtain

$$\partial_\xi^\alpha\partial_x^\beta a_h(x,\xi) = \partial_\xi^\alpha\partial_x^\beta(a(hx,\xi)) = O(h^{(1-\rho)|\beta|}\langle h^{1-\rho}x'\rangle^{k-|\beta|}).$$

This gives the first claim of the lemma for $k \ge 0$. For $k \le 0$, we consider $\langle h^{-\rho}x'\rangle^{-k}A$ and use the composition formulae. A nearly identical argument yields the second claim. □

5C. *Definition of the second microlocal class.* With Proposition 5.9 in place, we are now in a position to define the class of operators with symbols in $S^k_{\Gamma,L,\rho}$.

Definition 5.12. Let $\Gamma \subset U \subset T^*M$ be a coisotropic submanifold, U an open set, and L a Lagrangian foliation on Γ. A *chart for* (Γ, L) is a symplectomorphism

$$\kappa : U_0 \to V, \quad U_0 \subset U, \;\; V \subset T^*\mathbb{R}^n,$$

such that $\kappa(U_0\cap\Gamma) \subset V\cap\Gamma_0$ and $\kappa_{*,q}L_q = (L_0)_{\kappa(q)}$ for $q \in \Gamma\cap U$.

We now define the pseudodifferential operators associated to (Γ, L).

Definition 5.13. Let M be a smooth, compact manifold and $U \subset T^*M$ open, $\Gamma \subset U$ a coisotropic submanifold, L a Lagrangian foliation on Γ, and $\rho \in [0,1)$. We say that $A : \mathcal{D}'(M) \to C_c^\infty(M)$ is a *semiclassical pseudodifferential operator with symbol class* $S^k_{\Gamma,L,\rho}(U)$ (and write $A \in \Psi^k_{\Gamma,L,\rho}(U)$) if there are charts $\{\kappa_\ell\}_{\ell=1}^N$ for (Γ, L) and symbols $\{a_\ell\}_{\ell=1}^N \subset \widetilde{S}^k_{\Gamma,L,\rho}(U)$ such that A can be written in the form

$$A = \sum_{\ell=1}^N T'_\ell\,\widetilde{\mathrm{Op}}_h(a_\ell)\,T_\ell + O(h^\infty)_{\mathcal{D}'\to C^\infty}, \tag{5-23}$$

where T_ℓ and T'_ℓ are FIOs quantizing κ_ℓ and κ_ℓ^{-1} for $\ell = 1, \dots, N$.

We say that A is a *semiclassical pseudodifferential operator with symbol class* $S^k_{\Gamma,\rho}(U)$ (and write $A \in \Psi^k_{\Gamma,\rho}(U)$) if there are symbols $\{a_\ell\}_{\ell=1}^N \subset \widetilde{S}^k_{\Gamma,\rho}(U)$ such that A can be written in the form (5-23).

Lemma 5.14. *Suppose that* $\kappa : U \to T^*\mathbb{R}^n$ *is a chart for* (Γ, L), T *quantizes* κ, *and* T' *quantizes* κ^{-1}. *If* $A \in \Psi^k_{\Gamma,L,\rho}(U)$, *then there is* $a \in \widetilde{S}^k_{\Gamma,L,\rho}(U)$ *with* $\operatorname{supp} a(\,\cdot\,,\,\cdot\,,\lambda) \subset \kappa(U)$ *such that*

$$TAT' = \widetilde{\mathrm{Op}}_h(a) + O(h^\infty)_{\mathcal{D}'\to C^\infty}.$$

Moreover, if A *is given by* (5-23), *then*

$$a \circ K_\kappa = \sigma(T'T)\sum_{\ell=1}^N \sigma(T'_\ell T_\ell)\,(a_\ell \circ K_{\kappa_\ell}) + O(h^{1-\rho})_{\widetilde{S}^{k-1}_{\Gamma,L,\rho}}.$$

Proof. Note that we can write

$$TAT' = \sum_{\ell=1}^N TT'_\ell \widetilde{\mathrm{Op}}_h(a_\ell)T_\ell T' + O(h^\infty)_{\mathcal{D}'\to C^\infty}.$$

Next, note that TT'_ℓ quantizes $\kappa\circ\kappa_\ell^{-1}$ and that $T_\ell T'$ quantizes $\kappa_\ell\circ\kappa^{-1}$. Letting F_ℓ be a microlocally unitary FIO quantizing $\kappa_\ell\circ\kappa^{-1}$, we have that F_ℓ satisfies the hypotheses of Proposition 5.9 and we can write

$$T_\ell T' = C_L F_\ell \quad \text{and} \quad TT'_\ell = F_\ell^{-1} C_R$$

with $C_L, C_R \in \Psi(M)$ satisfying $\sigma(C_R C_L) = (\sigma(T'_\ell T_\ell)\circ\kappa_\ell^{-1})(\sigma(T'T)\circ\kappa_\ell^{-1})$. Therefore,

$$\begin{aligned} TT'_\ell\widetilde{\mathrm{Op}}_h(a_\ell)T_\ell T' &= F_\ell^{-1}C_R\widetilde{\mathrm{Op}}_h(a_\ell)C_L F_\ell = \mathrm{Op}_h(b_\ell) + (h^\infty)_{\mathcal{D}'\to C^\infty},\\ b_\ell &= (\sigma(C_R C_L)\circ\kappa_\ell\circ\kappa^{-1})(a_\ell\circ K_{\kappa_\ell\circ\kappa^{-1}}) + O(h^{1-\rho})_{\widetilde{S}^{k-1}_{\Gamma,L,\rho}}. \end{aligned}$$

□

Lemma 5.15. *Let* $\Gamma \subset U \subset T^*M$ *be a coisotropic submanifold*, U *be an open set*, *and* L *be a Lagrangian foliation on* Γ. *There is a principal symbol map*

$$\sigma_{\Gamma,L} : \Psi^k_{\Gamma,L,\rho}(U) \to S^k_{\Gamma,L,\rho}(U)/h^{1-\rho}S^{k-1}_{\Gamma,L,\rho}(U)$$

such that, *for* $A \in \Psi^{k_1}_{\Gamma,L,\rho}(U)$ *and* $B \in \Psi^{k_2}_{\Gamma,L,\rho}(U)$,

$$\sigma_{\Gamma,L}(AB) = \sigma_{\Gamma,L}(A)\sigma_{\Gamma,L}(B) \quad \textit{and} \quad \sigma_{\Gamma,L}([A,B]) = -ih\{\sigma_{\Gamma,L}(A), \sigma_{\Gamma,L}(B)\}. \tag{5-24}$$

Furthermore, *the sequence*

$$0 \to h^{1-\rho}\Psi^{k-1}_{\Gamma,L,\rho}(U) \xrightarrow{\iota} \Psi^k_{\Gamma,L,\rho}(U) \xrightarrow{\sigma_{\Gamma,L}} S^k_{\Gamma,L,\rho}(U)/h^{1-\rho}S^{k-1}_{\Gamma,L,\rho}(U) \to 0$$

is exact. The same holds with σ_Γ, $\Psi_{\Gamma,\rho}$, *and* $S^k_{\Gamma,\rho}$.

Proof. For A as in (5-23), we define

$$\sigma_{\Gamma,L}(A) = \sum_{\ell=1}^N \sigma(T_\ell T'_\ell)(\tilde{a}_\ell\circ\kappa),$$

where $\tilde{a}_\ell(x,\xi) := a_\ell(x,\xi,h^{-\rho}x')$. The fact that σ is well defined then follows from Lemma 5.14, and the formulae (5-24) follow from Lemma 5.6.

To see that the sequence is exact, we only need to check that if $A \in \Psi^k_{\Gamma,L,\rho}$ and $\sigma_{\Gamma,L}(A) = 0$, then $A \in h^{1-\rho}\Psi^{k-1}_{\Gamma,L,\rho}$. To do this, we may assume that $\mathrm{WF}_h{}'(A) \subset U$ such that there is a chart (κ, U) for (Γ, L). Let T be a microlocally unitary FIO quantizing κ and suppose that $\sigma_{\Gamma,L}(A) \in h^{1-\rho}S^{k-1}_{\Gamma,L,\rho}$. Then, by the first part of Lemma 5.14, we know $TAT^{-1} = \widetilde{\mathrm{Op}}_h(a) + O(h^\infty)$ for some $a \in \widetilde{S}^k_{\Gamma,L,\rho}$. Then, by the second part of Lemma 5.14, since $\sigma_{\Gamma,L}(A) \in h^{1-\rho}S^{k-1}_{\Gamma,L,\rho}$, we have that $a \in h^{1-\rho}\widetilde{S}^{k-1}_{\Gamma,L,\rho}$ and, in particular, $A \in h^{1-\rho}\Psi^{k-1}_{\Gamma,L,\rho}$. $\square$

Note that if $A \in \Psi^{\mathrm{comp}}(M)$, then $A \in \Psi^0_{\Gamma,L,\rho}$ and $\sigma(A) = \sigma_\Gamma(A)$. Furthermore, if $A \in \Psi^k_{\Gamma,\rho}$, then $A \in \Psi^k_{\Gamma,L,\rho}$ and $\sigma_\Gamma(A) = \sigma_{\Gamma,L}(A)$.

Lemma 5.16. *Let $\Gamma \subset U \subset T^*M$ be a coisotropic submanifold, U be an open set, and L be a Lagrangian foliation on Γ. There is a noncanonical quantization procedure*

$$\mathrm{Op}_h^{\Gamma,L} : S^k_{\Gamma,L,\rho}(U) \to \Psi^k_{\Gamma,L,\rho}(U)$$

such that, for all $A \in \Psi^k_{\Gamma,L,\rho}(U)$, there is $a \in S^k_{\Gamma,L,\rho}(U)$ such that $\mathrm{Op}_h^{\Gamma,L}(a) = A + O(h^\infty)_{\mathcal{D}' \to C^\infty}$ and

$$\sigma_{\Gamma,L} \circ \mathrm{Op}_h^{\Gamma,L} : S^k_{\Gamma,L,\rho}(U) \to S^k_{\Gamma,L,\rho}(U)/h^{1-\rho}S^{k-1}_{\Gamma,L,\rho}(U)$$

is the natural projection map.

Proof. Let $\{(\kappa_\ell, U_\ell)\}_{\ell=1}^N$ be charts for (Γ, L) such that $\{U_\ell\}_{\ell=1}^N$ is a locally finite cover for U, T_ℓ and T'_ℓ quantize κ_ℓ and κ_ℓ^{-1}, respectively, and $\sigma(T'_\ell T_\ell) \in C_c^\infty(U_\ell)$ is a partition of unity on U. Let $a \in S^k_{\Gamma,L,\rho}(U)$. Then, define $a_\ell \in \widetilde{S}^k_{\Gamma_0,L_0,\rho}$ such that $a_\ell(x, \xi, h^{-\rho}x') := (\chi_\ell a) \circ \kappa^{-1}(x, \xi)$, where $\chi_\ell \equiv 1$ on $\operatorname{supp} \sigma(T'_\ell T_\ell)$. We then define the quantization map

$$\mathrm{Op}_h^{\Gamma,L}(a) := \sum_{\ell=1}^N T'_\ell \widetilde{\mathrm{Op}}_h(a_\ell) T_\ell.$$

The fact that $\sigma_{\Gamma,L} \circ \mathrm{Op}_h^{\Gamma,L}$ is the natural projection follows immediately. Now, fix $A \in \Psi^k_{\Gamma,L,\rho}(U)$. Put $a_0 = \sigma_{\Gamma,L}(A)$. Then, $A = \mathrm{Op}_h^{\Gamma,L}(a_0) + h^{1-\rho}A_1$, where $A_1 \in \Psi^{k-1}_{\Gamma,L,\rho}$. We define $a_k = \sigma_{\Gamma,L}(A_k)$ inductively for $k \geq 1$ by

$$h^{(k+1)(1-\rho)}A_{k+1} = A - \sum_{k=0}^k h^{k(1-\rho)} \mathrm{Op}_h^{\Gamma,L}(a_k).$$

Then, letting $a \sim \sum_k h^{k(1-\rho)}a_k$, we have $A = \mathrm{Op}_h^{\Gamma,L}(a) + O(h^\infty)_{\mathcal{D}' \to C^\infty}$ as claimed. $\square$

Remark 5.17. Note that $E := \sum_{\ell=1}^N T_\ell T'_\ell$ is an elliptic pseudodifferential operator with symbol 1. Therefore, there is $E' \in \Psi^0$ with $\sigma(E') = 1$ such that $E'EE' = \mathrm{Id}$. Replacing T_ℓ by $E'T_\ell$ and T'_ℓ by $T'_\ell E'$, we may (and will) ask for $\sum_{\ell=1}^N T_\ell T'_\ell = \mathrm{Id}$, and so $\mathrm{Op}_h^{\Gamma,L}(1) = \mathrm{Id}$.

Lemma 5.18. *Let $\Gamma \subset U \subset T^*M$ be a coisotropic submanifold. If $A \in \Psi^k_{\Gamma,\rho}(U)$ and $P \in \Psi^m(U)$ with symbol p such that, for every $q \in \Gamma$, we have $H_p(q) \in T_q\Gamma$. Then,*

$$\frac{i}{h}[P, A] = \mathrm{Op}_h^\Gamma(H_p a) + O(h^{1-\rho})_{\Psi^{k-1}_{\Gamma,\rho}},$$

where $a(x, \xi; h) = \sigma_\Gamma(A)(x, \xi, h^{-\rho}x')$.

Proof. Suppose that $\mathrm{WF}_h'(A) \subset U_\ell$ for $U_\ell \subset U$ open, and suppose that $\kappa : U_\ell \to T^*\mathbb{R}^n$ is a chart for (Γ, L). Note that we may assume that $\mathrm{WF}_h(A)' \subset U_\ell$ and then use a partition of unity to cover U with a family $\{U_\ell\}_\ell$. Therefore, there exist a Fourier integral operator T that is microlocally elliptic on U_ℓ and quantizes κ and $a \in \widetilde{S}^k_{\Gamma,\rho}$ such that $A = T^{-1}\widetilde{\mathrm{Op}}_h(a)T + O(h^\infty)_{\mathcal{D}'\to C^\infty}$. Then, on $\mathrm{WF}_h'(A)$,

$$T[P, A]T^{-1} = [TPT^{-1}, \widetilde{\mathrm{Op}}_h(a)] + O(h^\infty)_{\mathcal{D}'\to C^\infty}.$$

Now, $TPT^{-1} = \mathrm{Op}_h(p \circ \kappa^{-1}) + O(h)_{\Psi^{m-1}}$. Hence, a direct computation using Lemma 5.7 gives

$$[TPT^{-1}, \widetilde{\mathrm{Op}}_h(a)] = -ih\widetilde{\mathrm{Op}}_h(c) + O(h^{2-\rho})_{\widetilde{\Psi}^{k-2}_{\Gamma_0,\rho}}$$

with $c(x, \xi, h^{-\rho}x') = H_{p\circ\kappa^{-1}}(a(x, \xi, h^{-\rho}x')) \in S^{k-1}_{\Gamma,\rho}(U_\ell)$. In particular,

$$[P, A] = -ihT^{-1}\widetilde{\mathrm{Op}}_h(c)T + O(h^{2-\rho})_{\Psi^{k-2}_{\Gamma,\rho}}.$$

Therefore, $[P, A] \in h\Psi^{k-1}_{\Gamma,\rho}$ with symbol $\sigma_\Gamma(ih^{-1}[P, A]) = H_p(a(x, \xi, h^{-\rho}x'))$. □

6. An uncertainty principle for coisotropic localizers

The first goal of this section is to build a family of cut-off operators X_y with $y \in M$ that act as the identity on the shrinking ball $B(y, h^\rho)$ and such that they commute with P in a fixed-size neighborhood of y. This is the content of Section 6A. The second goal is to control $\|X_{y_1}X_{y_2}\|_{L^2\to L^2}$ in terms of the distance $d(y_1, y_2)$ as this distance shrinks to 0. We do this in Section 6B. Finally, in Section 6C, we study the consequences of these estimates for the almost-orthogonality of X_{y_i}.

In order to localize to the ball $B(y, h^\rho)$ in a way compatible with microlocalization we need to make sense of

$$\chi_y(x) = \tilde{\chi}\Big(\frac{1}{\varepsilon}h^{-\rho}d(x, y)\Big), \qquad \tilde{\chi} \in C_c^\infty((-1, 1)),$$

as an operator in some anisotropic pseudodifferential calculus. As a function, χ_y is in the symbol class $S^{-\infty}_{\Gamma_y,L_y}$, where Γ_y and L_y are the coisotropic submanifold and Lagrangian foliation defined as follows: fix $\delta > 0$, to be chosen small later, and, for each $x \in M$, let

$$\Gamma_y := \bigcup_{|t|<\frac{1}{2}\mathrm{inj}(M)} \varphi_t(\Omega_y), \qquad \Omega_y := \{\xi \in T^*_yM : |1 - |\xi|_g| < \delta\}. \tag{6-1}$$

In this section, we construct localizers to Γ_y adapted to the Laplacian and study the incompatibility between localization to Γ_{y_1} and Γ_{y_2} as a function of the distance between $y_1, y_2 \in M$. Let $y \in M$. In what follows we work with the Lagrangian foliation L_y of Γ_y given by

$$L_y = \{L_{y,\tilde{q}}\}_{\tilde{q}\in\Gamma_y}, \qquad L_{y,\tilde{q}} = (\varphi_t)_*(T_qT^*_yM),$$

where $\tilde{q} = \varphi_t(q)$ for some $|t| < \frac{1}{2}\mathrm{inj}(M)$ and $q \in \Omega_y$.

Remark 6.1. In fact, it will be enough for us to show that $\chi_y(x)\tilde{\chi}(\delta^{-1}(|hD|_g - 1)) \in \Psi_{\Gamma_y,L_y,\rho}$ since we will be working near the characteristic variety for the Laplacian.

6A. ***Coisotropic cutoffs adapted to the Laplacian.***

Lemma 6.2. *Let* $y \in M$, $0 < \varepsilon < \delta$, $0 \le \rho < 1$, $\tilde{\chi} \in C_c^\infty((-1,1))$, *and define the operator* $\chi_{h,y}$ *by*

$$\chi_{h,y}u(x) := \tilde{\chi}\Big(\frac{1}{\varepsilon}h^{-\rho}d(x,y)\Big)\Big[\mathrm{Op}_h\Big(\tilde{\chi}\Big(\frac{1}{\varepsilon}(|\xi|_g-1)\Big)\Big)u\Big](x). \tag{6-2}$$

Then, $\chi_{h,y} \in \Psi^{-\infty}_{\Gamma_y,L_y,\rho}$.

Proof. We will use Lemma 5.11 to prove the claim. First, observe that we may work in a single chart for (Γ_y, L_y) by using a partition of unity. Therefore, suppose that $B \in \Psi^0$ and $\kappa : U_0 \to T^*\mathbb{R}^n$ is a chart for (Γ_y, L_y), $V_0 \Subset U_0$, and T is an FIO quantizing κ that is microlocally unitary on V_0. Furthermore, since $\kappa_* L_y = L_0$, we may assume that $\kappa(U_0 \cap T_y^*M) \subset T_0^*\mathbb{R}^n$. Denote the microlocal inverse of T by T'. Then, observe that, for A and B with wavefront set in V_0,

$$\mathrm{ad}_A(TBT') = T\,\mathrm{ad}_{T'AT}(B)T' + O(h^\infty)_{\mathcal{D}'\to C^\infty}.$$

By a partition of unity, we will work as though $\chi_{h,y}$ were microsupported in U_0. We then consider, for all $N > 0$ and $\alpha, \beta \in \mathbb{N}^n$,

$$\begin{aligned} &h^{-2N\rho}|x'|^{2N}\,\mathrm{ad}^\alpha_{h^{-\rho}x}\,\mathrm{ad}^\beta_{hD_x}(T\chi_{h,y}T') \\ &\qquad = h^{-2\rho N}T(T'|x'|^2T)^N\,\mathrm{ad}^\alpha_{h^{-\rho}T'xT}\,(\mathrm{ad}^\beta_{T'hD_xT}\,(\chi_{h,y}))T' + O(h^\infty)_{\mathcal{D}'\to C^\infty}. \end{aligned}$$

In order to prove the requisite estimates, we will first view $\chi_{h,y}$ as an element of the model microlocal class. In particular, we work with $x \in M$ written in geodesic normal coordinates centered at y, so that

$$\chi_{h,y}u(x) = \tilde{\chi}\Big(\frac{1}{\varepsilon}h^{-\rho}|x|\Big)\Big[\mathrm{Op}_h\Big(\tilde{\chi}\Big(\frac{1}{\varepsilon}(|\xi|_g-1)\Big)\Big)u\Big](x).$$

Then,

$$\chi_{h,y} = \widetilde{\mathrm{Op}}_h\Big(\frac{1}{\varepsilon}\tilde{\chi}(\lambda)\Big)\,\mathrm{Op}_h\Big(\tilde{\chi}\Big(\frac{1}{\varepsilon}(|\xi|-1)\Big)\Big)$$

is an element of $\widetilde{\Psi}^{-\infty}_{\Gamma_0,L_0,\rho}$ with $r = n$, and so we can apply Lemma 5.7 to compute $\mathrm{ad}_A(\chi_{h,y})$ for $A \in \Psi^{-\infty}(M)$. In particular,

$$\mathrm{ad}_{T'hD_xT}(\chi_{h,y}) = \widetilde{\mathrm{Op}}_h(c) + O(h^\infty), \tag{6-3}$$

where $c \in h^{1-\rho}\widetilde{S}^{-\infty}_{\Gamma_0,L_0,\rho}$ is supported on $\{(x,\xi,\lambda) : |x| \le \varepsilon h^\rho,\ |\lambda| \le \varepsilon\}$. Now, suppose $c \in \widetilde{S}^{-\infty}_{\Gamma_0,L_0,\rho}$ is supported on $\{(x,\xi,\lambda) : |x| \le \varepsilon h^\rho,\ |\lambda| \le \varepsilon\}$ and $B \in \Psi^{-\infty}$ with $\sigma(B)(0,\xi) = 0$. Then, again using Lemma 5.7 and the fact that $\partial_{\xi'}\sigma(B)|_{x'=0} = 0$,

$$\mathrm{ad}_B(\widetilde{\mathrm{Op}}_h(c)) = \widetilde{\mathrm{Op}}_h(c') + O(h^\infty), \tag{6-4}$$

where $c' \in h\widetilde{S}^{-\infty}_{\Gamma_0,L_0,\rho}$ is supported on $\{(x,\xi,\lambda) : |x| \le \varepsilon h^\rho,\ |\lambda| \le \varepsilon\}$.

Now, note that since $\kappa(T_y^*M) \subset T_0^*\mathbb{R}^n$, then, for all $i = 1, \dots, n$, we have that $B = T'x_iT$ has symbol $\sigma(B) = [b(x,\xi)x]_i$ for some $b \in C^\infty(T^*M; \mathbb{M}_{n\times n})$. Therefore, (6-3) and (6-4) yield

$$\mathrm{ad}^\alpha_{h^{-\rho}T'xT}(\mathrm{ad}^\beta_{T'hD_xT}(\chi_{h,y})) = h^{(1-\rho)(|\alpha|+|\beta|)}\widetilde{\mathrm{Op}}_h(c') + O(h^\infty),$$

where $c' \in \widetilde{S}^{-\infty}_{\Gamma_0, L_0,\rho}$ is supported on $\{(x, \xi, \lambda) : |x| \le \varepsilon h^\rho,\ |\lambda| \le \varepsilon\}$. Finally, using again that $T' x_i T$ has symbol $[b(x, \xi)x]_i$, we have that (6-4) gives

$$\|h^{-2N\rho}|x'|^{2N} \operatorname{ad}^{\alpha}_{h^{-\rho}T'xT}(\operatorname{ad}^{\beta}_{T'hD_xT}(\chi_{h,y}))\|_{L^2\to L^2} \le Ch^{(1-\rho)(|\alpha|+|\beta|)}. \qquad \square$$

We next construct a pseudodifferential cutoff, $X_y \in \Psi^{-\infty}_{\Gamma_y,\rho}$, which is microlocally the identity near S^*_yM and which essentially commutes with $P = -h^2\Delta_g - 1$ near y. In particular, we will have

$$\chi_{h,y}X_y = \chi_{h,y} + O(h^\infty)_{\Psi^{-\infty}}.$$

When considering the value of a quasimode u that is h^ρ close to the point y, this will allow us to effectively work with $X_y u$ instead.

Theorem 6.3. *Let $y \in M$, $0 < \varepsilon < \delta$, and $0 \le \rho < 1$. Then, there exists $X_y \in \Psi^{-\infty}_{\Gamma_y,\rho} \subset \Psi^{-\infty}_{\Gamma_y,L_y,\rho}$ satisfying*

(1) *If $\chi_{h,y}$ is defined as in* (6-2), *then*

$$\chi_{h,y}X_y = \chi_{h,y} + O(h^\infty)_{\Psi^{-\infty}}. \tag{6-5}$$

(2) $\mathrm{WF}_h{}'([P, X_y]) \cap \bigl\{(x, \xi) : x \in B\bigl(y, \tfrac{1}{2}\operatorname{inj}(M)\bigr),\ \xi \in \Omega_x\bigr\} = \varnothing$.

Proof. First, we note that we will actually prove that $X_y \in \Psi^{-\infty}_{\Gamma_y,\rho}$, and so the result will follow since $\Psi^{-\infty}_{\Gamma_y,\rho} \subset \Psi^{-\infty}_{\Gamma_y,L_y,\rho}$. Let $\mathcal{H} \subset T^*M$ be transverse to the Hamiltonian flow H_p such that $\Omega_y \subset \mathcal{H}$. Next, let $\varkappa \in C^\infty_c((-2, 2))$ with $\varkappa \equiv 1$ on $[-1, 1]$, and define $\varkappa_0 \in C^\infty_c(\mathcal{H})$ by

$$\varkappa_0 = \varkappa(h^{-\rho}d(x, y))\varkappa\Bigl(\frac{2}{\delta}(1 - |\xi|_g)\Bigr),$$

where δ is as in the definition of Ω_y. Let $\psi \in C^\infty_c(T^*M)$ with

$$\psi \equiv 1 \quad \text{on } B\bigl(y, \tfrac{1}{2}\operatorname{inj}(M)\bigr) \cap \{|\xi|_g < 2\}, \qquad \operatorname{supp}\psi \subset B\bigl(y, \tfrac{3}{4}\operatorname{inj}(M)\bigr).$$

Then, let χ_0 be defined locally by $H_p\chi_0 \equiv 0$ and $\chi_0|_{\mathcal{H}} = \varkappa_0$ such that $\chi_0 \in S^{-\infty}_{\Gamma_y,\rho}$. That is, $\chi_0(\varphi_t(q)) = \psi(\varphi_t(q))\chi_0(q)$ for $|t| < \operatorname{inj}(M)$ and $q \in \mathcal{H}$. Next, observe that by Lemma 5.7 there is $e_0 \in S^{-\infty}_{\Gamma_y,\rho}$ such that

$$-\frac{i}{h}[P, \operatorname{Op}^{\Gamma_y}_h(\chi_0)] = h^{1-\rho}\operatorname{Op}^{\Gamma_y}_h(e_0), \quad \operatorname{supp} e_0 \cap B\bigl(y, \tfrac{1}{2}\operatorname{inj}(M)\bigr) \subset \bigcup_{|t|<\frac{3}{4}\operatorname{inj}(M)} \varphi_t(\mathcal{H} \cap \operatorname{supp}\partial\varkappa_0).$$

(Here and below $\partial\varkappa_0$ denotes the gradient of $\varkappa_0$.) Suppose that there exist $\chi_{k-1}, e_{k-1} \in S^{-\infty}_{\Gamma_y,\rho}$ such that

$$-\frac{i}{h}[P, \operatorname{Op}^{\Gamma_y}_h(\chi_{k-1})] = h^{k(1-\rho)}\operatorname{Op}_h(e_{k-1}), \quad \operatorname{supp} e_{k-1} \cap B\bigl(y, \tfrac{1}{2}\operatorname{inj}(M)\bigr) \subset \bigcup_{|t|<\frac{3}{4}\operatorname{inj}(M)} \varphi_t(\mathcal{H} \cap \operatorname{supp}\partial\varkappa_0).$$

Then, define $\tilde{\chi}_k \in S^{-\infty}_{\Gamma_y,\rho}$ by solving locally $H_p\tilde{\chi}_k = e_{k-1}$ and $\tilde{\chi}_k|_{\mathcal{H}} = 0$. Note that then

$$\operatorname{supp}\tilde{\chi}_k \cap B\bigl(y, \tfrac{1}{2}\operatorname{inj}(M)\bigr) \subset \bigcup_{|t|<\frac{3}{4}\operatorname{inj}(M)} \varphi_t(\mathcal{H} \cap \operatorname{supp}\partial\varkappa_0)$$

and

$$h^{-k(1-\rho)}\sigma\Bigl(\frac{i}{h}[P, \operatorname{Op}^{\Gamma_y}_h(\chi_{k-1} + h^{k(1-\rho)}\tilde{\chi}_k)]\Bigr) = H_p\tilde{\chi}_k - e_{k-1} = 0.$$

In particular, with $\chi_k := \chi_{k-1} + h^{k(1-\rho)}\tilde{\chi}_k$, we obtain $-\frac{i}{h}[P, \mathrm{Op}_h^{\Gamma_y}(\chi_k)] = h^{(k+1)(1-\rho)}\,\mathrm{Op}_h(e_k)$ with $e_k \in S^{-\infty}_{\Gamma_y,\rho}$ and

$$\operatorname{supp} e_k \cap B\big(y, \tfrac{1}{2}\operatorname{inj}(M)\big) \subset \bigcup_{|t|<\frac{3}{4}\operatorname{inj}(M)} \varphi_t(\mathcal{H} \cap \operatorname{supp} \partial\varkappa_0).$$

Setting

$$X_y = \mathrm{Op}_h^{\Gamma_y}(\chi_\infty) \quad \text{and} \quad \chi_\infty \sim \bigg(\chi_0 + \sum_k (\chi_{k+1} - \chi_k)\bigg),$$

we have that X_y satisfies the second claim and, moreover, $\chi_\infty \equiv 1$ on

$$\bigcup_{|t|\le\frac{1}{4}\operatorname{inj}(M)} \varphi_t\big(\mathcal{H} \cap \{d(x, y) < h^\rho\} \cap \big\{||\xi|_g - 1| < \tfrac{1}{2}\delta\big\}\big).$$

To see the first claim, observe that, for $\varepsilon > 0$ small enough,

$$B(y, \varepsilon h^\rho) \cap \{||\xi|_g - 1| < \delta\} \subset \bigcup_{|t|\le\frac{1}{4}\operatorname{inj}(M)} \varphi_t\big(\mathcal{H} \cap \{d(x, y) < h^\rho\} \cap \big\{||\xi|_g - 1| < \tfrac{1}{2}\delta\big\}\big),$$

and hence, by Lemma 5.6,

$$\chi_{h,y} X_y = \chi_{h,y}\,\mathrm{Op}_h^{\Gamma,L}(1) + O(h^\infty)_{\Psi^{-\infty}} = \chi_{h,y} + O(h^\infty)_{\Psi^{-\infty}}. \qquad \square$$

6B. ***An uncertainty principle for coisotropic localizers.*** Let $\Gamma(t) \subset T^*\mathbb{R}^n$, $t \in (-\varepsilon_0, \varepsilon_0)$, be a smooth family of coisotropic submanifolds of dimension $n+1$ with

$$\Gamma(0) = \{(0, x_n, \xi', \xi_n) : x_n \in \mathbb{R},\ \xi' \in \mathbb{R}^{n-1},\ \xi_n \in \mathbb{R}\}.$$

Assume that for each t, we define $\Gamma(t)$ by the functions $\{q_i(t)\}_{i=1}^{n-1} \subset C^\infty(\mathbb{R}^{2n})$ with $q_i(0) = x_i$ (note that $q_i(t)$ should be thought of as a function in $C^\infty(\mathbb{R}^{2n})$ parametrized by t). Moreover, assume that there are $c, C > 0$ such that for $i = 1, \dots, n-1$,

$$|\{q_i(t), x_i\}| \ge c|t| \quad \text{on } \Gamma(0) \cap \Gamma(t), \quad |t| > 0, \tag{6-6}$$

and, for all $i, j = 1, \dots, n-1$ and all $t \in (-\varepsilon_0, \varepsilon_0)$,

$$\{q_i(t), q_j(t)\} = 0, \quad \{q_i(t), \xi_n\} = 0, \quad |\{q_i(t), x_j\}| \le Ct^2 \quad \text{on } \Gamma(0) \cap \Gamma(t), \quad i \ne j. \tag{6-7}$$

The main goal of this section is to prove the following proposition.

Proposition 6.4. *Let $0 < \rho < 1$ and $\{\Gamma(t) : t \in (-\varepsilon_0, \varepsilon_0)\}$ be as above. Suppose that $X(t) \in \Psi^{-\infty}_{\Gamma(t),\rho}$ for all $t \in (-\varepsilon_0, \varepsilon_0)$ and that there is $\varepsilon > 0$ such that $h^{\rho-\varepsilon} \le |t| < \varepsilon_0$. Then,*

$$\|X(0)X(t)\|_{L^2\to L^2} \le Ch^{(n-1)(2\rho-1)/2}t^{(1-n)/2}.$$

Proof. We begin by finding a convenient chart for $\Gamma(t)$. By Darboux's theorem (see, e.g., [Zworski 2012, Theorem 12.1]), there is a smooth family of symplectomorphisms $\kappa_t : V_1 \to V_2$ such that, for $j = 1, \dots, n-1$,

$$\kappa_t^*(q_j(t)) = y_j \quad \text{and} \quad \kappa_t^*\xi_n = \eta_n, \tag{6-8}$$

where V_1 and V_2 are simply connected neighborhoods of 0. Note that $\kappa_t(\Gamma(0)) = \Gamma(t)$ with this setup, so κ_t^{-1} is a chart for $\Gamma(t)$. By [Zworski 2012, Theorem 11.4], the symplectomorphism κ_t can be extended to a family of symplectomorphisms on $T^*\mathbb{R}^n$ that is the identity outside a compact set, and such that there is a smooth family of symbols $p_t \in C^\infty(T^*\mathbb{R}^n)$ satisfying $\partial_t \kappa_t = (\kappa_t)_* H_{p_t}$.

Now, let $U(t) : L^2 \to L^2$ solve

$$(hD_t + \mathrm{Op}_h(p_t))U(t) = 0, \quad U(0) = \mathrm{Id}.$$

Then, $U(t)$ is microlocally unitary from V_1 to V_2 in the sense that if $a \in C_c^\infty(V_1)$ and $b \in C_c^\infty(V_2)$ then

$$[U(t)]^* U(t)\,\mathrm{Op}_h(a) = \mathrm{Op}_h(a) + O(h^\infty)_{\Psi^{-\infty}} \quad \text{and} \quad U(t)[U(t)]^*\,\mathrm{Op}_h(b) = \mathrm{Op}_h(b) + O(h^\infty)_{\Psi^{-\infty}},$$

and $U(t)$ quantizes κ_t. Moreover,

$$U(t) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{i(\phi(t,x,\eta) - \langle y,\eta\rangle)/h} b(t,x,\eta;h)\,d\eta,$$

where $b(t,\cdot) \in S^{\mathrm{comp}}(T^*\mathbb{R}^n)$ and the phase function $\phi(t,\cdot) \in C^\infty(T^*\mathbb{R}^n;\mathbb{R})$ satisfies

$$\partial_t \phi + p_t(x, \partial_x \phi) = 0, \quad \phi(0,x,\eta) = \langle x,\eta\rangle$$

for all $t \in (-\varepsilon_0, \varepsilon_0)$. Since $U(t)$ is microlocally unitary, it is enough to estimate the operator

$$A(t) := X(0)X(t)U(t).$$

First, note that since $X(t) \in \Psi^{-\infty}_{\Gamma(t),\rho}$ and $U(t)$ quantizes κ_t, there exists $a(t) \in \widetilde{S}^{-\infty}_{\Gamma_0,\rho}$ with $t \in (-\varepsilon_0, \varepsilon_0)$ such that $X(t) = U(t)\widetilde{\mathrm{Op}}_h(a(t))[U(t)]^* + O(h^\infty)_{L^2\to L^2}$, and so

$$A(t) = \widetilde{\mathrm{Op}}_h(a(0))U(t)\widetilde{\mathrm{Op}}_h(a(t)) + O(h^\infty)_{L^2\to L^2}.$$

Fix $N > n-1$ and let $\chi = \chi(\lambda) \in \widetilde{S}^{-N}_{\Gamma_0,\rho}$ be such that $|\chi(\lambda)| \ge c\langle\lambda\rangle^{-N}$. Now, since $a(t) \in \widetilde{S}^{-\infty}_{\Gamma_0,\rho}$, by the elliptic parametrix construction there are $e_L(t), e_R(t) \in \widetilde{S}^{-\infty}_{\Gamma_0,\rho}$ such that

$$\widetilde{\mathrm{Op}}_h(e_L(t))\widetilde{\mathrm{Op}}_h(\chi) = \widetilde{\mathrm{Op}}_h(a(t)) + O(h^\infty)_{L^2\to L^2}, \quad \widetilde{\mathrm{Op}}_h(\chi)\widetilde{\mathrm{Op}}_h(e_R(t)) = \widetilde{\mathrm{Op}}_h(a(t)) + O(h^\infty)_{L^2\to L^2}$$

for all $t \in (-\varepsilon_0, \varepsilon_0)$. Note that we are implicitly using the fact that $a(t)$ is compactly supported in (x,ξ) to handle the fact that χ is not compactly supported in (x,ξ). Thus,

$$A(t) = \widetilde{\mathrm{Op}}_h(e_L(0))\widetilde{\mathrm{Op}}_h(\chi)U(t)\widetilde{\mathrm{Op}}_h(\chi)\widetilde{\mathrm{Op}}_h(e_R(t)) + O(h^\infty)_{L^2\to L^2}.$$

Since $\widetilde{\mathrm{Op}}_h(e_L(t))$ and $\widetilde{\mathrm{Op}}_h(e_R(t))$ are L^2 bounded uniformly in $t \in (-\varepsilon_0, \varepsilon_0)$, we estimate

$$\tilde{A}(t) := \widetilde{\mathrm{Op}}_h(\chi)U(t)\widetilde{\mathrm{Op}}_h(\chi).$$

In fact, we estimate $B(t) := \tilde{A}(t)(\tilde{A}(t))^*$ by considering its kernel:

$$\begin{aligned} B(t;x,y) &= \int U(t)(x,w)U(t)^*(w,y)\chi(h^{-\rho}x')\chi(h^{-\rho}y')\chi(h^{-\rho}w')^2\,dw \\ &= \frac{1}{(2\pi h)^{2n}} \int e^{i\Phi(t,x,w,y,\eta,\xi)/h} b(t,x,\eta)\bar{b}(t,y,\xi)\chi(h^{-\rho}x')\chi(h^{-\rho}y')\chi(h^{-\rho}w')^2\,dw\,d\eta\,d\xi \end{aligned}$$

with $\Phi(t,x,w,y,\eta,\xi) = \phi(t,x,\eta) - \phi(t,y,\xi) + \langle w, \xi-\eta\rangle$. First, performing stationary phase in (w_n,η_n) yields

$$\begin{aligned} B(t;x,y) &= \frac{1}{(2\pi h)^{2n-1}} \int F(t,x,w',\xi_n)\overline{F(t,y,w',\xi_n)}\,dw'\,d\xi_n,\\ F(t,x,w',\xi_n) &:= \int e^{i(\phi(t,x,\eta',\xi_n)-\langle w',\eta'\rangle)/h} b_1(t,x,\eta',\xi_n)\chi(h^{-\rho}x')\chi(h^{-\rho}w')^2\,d\eta' \end{aligned}$$

for some $b_1 \in S^{\mathrm{comp}}(T^*\mathbb{R}^n)$. Next, note that since $\phi(0,x,\eta) = \langle x,\eta\rangle$,

$$\phi(t,x,\eta) - \langle x,\eta\rangle = t\tilde{\phi}(t,x,\eta)$$

with $\tilde{\phi}$ such that, for every multi-index α, there exists $C_\alpha > 0$ with $|\partial^\alpha_{t,x,\eta}\tilde{\phi}| \le C_\alpha$.

Next, we claim that there exists $C > 0$ such that

$$\|(\partial^2_{\eta'}\tilde{\phi}(t,x,\eta))^{-1}\| \le C \quad \text{if } (x,\eta)\in\Gamma(0),\ \ \partial_{\eta'}\phi(t,x,\eta) = 0. \tag{6-9}$$

We postpone the proof of (6-9) and proceed to finish the proof of the lemma.

To continue the proof, note that, modulo an $O(h^{N\varepsilon})$ error, we may assume that the integrand of $B(t;x,y)$ is supported in $\{(x,y,w') : |x'| \le h^{\rho-\varepsilon},\ |y'| \le h^{\rho-\varepsilon},\ |w'| \le h^{\rho-\varepsilon}\}$ and $h^{\rho-\varepsilon} \le |t|$. Therefore, the bound in (6-9) continues to hold on the support of the integrand. By (6-9) and

$$\partial^2_{\eta'}(\phi(t,x,\eta) - t\tilde{\phi}(t,x,\eta)) = 0, \tag{6-10}$$

there is a unique critical point $\eta'_c(t,x,w',\xi_n)$ for the map $\eta' \mapsto \phi(t,x,\eta',\xi_n) - \langle w',\eta'\rangle$, in an $O(1)$ neighborhood of η'_c. Indeed, since $|\partial^3_{\eta'}\phi| \le Ct$,

$$\partial_{\eta'}\phi = t(\langle \partial^2_{\eta'}\tilde{\phi}(t,x,\eta'_c,\xi_n)(\eta'-\eta'_c), \eta'-\eta'_c\rangle + O(|\eta-\eta'_c|^3)).$$

In particular, η'_c is the unique solution to $\partial_{\eta'}\phi(t,x,\eta'_c,\xi_n) - w' = 0$.

Next, again using (6-10), by applying the method of stationary phase in η' to F with small parameter h/t, we obtain

$$\begin{aligned} B(t;x,y) &= \frac{1}{(2\pi h)^n t^{n-1}} \int e^{i\Phi_1(t,x,w',y,\xi_n)/h} B_1(t;x,y,w',\eta'_c,\xi)\,dw'\,d\xi_n,\\ \Phi_1(t,x,w',y,\xi_n) &:= \Psi(t,x,w',\xi_n) - \Psi(t,y,w',\xi_n),\\ \Psi(t,x,w',\xi_n) &:= \phi(t,x,\eta'_c(t,x,w',\xi_n),\xi_n) - \langle w',\eta'_c(t,x,w',\xi_n)\rangle,\\ B_1(t;x,y,w',\eta',\xi) &:= b_2(t,x,\eta',\xi_n)\bar{b}(t,y,\xi',\xi_n)\chi(h^{-\rho}x')\chi(h^{-\rho}y')\chi(h^{-\rho}w')^2 \end{aligned}$$

for some $b_2 \in S^{\mathrm{comp}}(T^*\mathbb{R}^n)$. Next, observe that

$$\begin{aligned} \partial_{x_n}\partial_{\xi_n}\Psi(t,x,w',\xi_n) &= \partial_{x_n}\partial_{\xi_n}(\langle x'-w',\eta'_c\rangle + x_n\xi_n + O(t)_{C^\infty}\\ &= \langle x'-w', \partial_{x_n}\partial_{\xi_n}\eta'_c\rangle + 1 + O(t)\\ &= 1 + O(t) + O(h^\rho) = 1 + O(t), \end{aligned}$$

where in the last line we use the fact that $|t| \ge h^{\rho-\varepsilon}$, and therefore, there exist $c > 0$ and a function $g = g(x',y,w',\xi_n)$ such that $|\partial_{\xi_n}\Phi_1| \ge c|x_n - g|$. In particular, integration by parts in ξ_n (with the operator

$L=(h^2+\partial_{\xi_n}\Phi_1 hD_{\xi_n})/(h^2+|\partial_{\xi_n}\Phi_1|^2))$ shows that for any $N>0$ there is $C_N>0$ such that

$$|B(t;x,y)|\le C_N h^{-n}t^{1-n}h^{\rho(n-1)}\chi(h^{-\rho}y')\chi(h^{-\rho}x')\frac{h^{2N}+h^N|x_n-g|^N}{(h^2+|x_n-g|^2)^N}.$$

Applying Schur's lemma together with the fact that there exists $C>0$ such that, for all t,

$$\sup_x\int|B(t;x,y)|\,dy+\sup_y\int|B(t;x,y)|\,dx\le Ch^{(2\rho-1)(n-1)}t^{1-n}$$

yields that $\|B(t)\|_{L^2\to L^2}\le Ch^{(2\rho-1)(n-1)}t^{1-n}$ for all $t\in(-\varepsilon_0,\varepsilon_0)$, and hence

$$\|X(0)X(t)\|_{L^2\to L^2}\le Ch^{(n-1)(2\rho-1)/2}t^{(1-n)/2},$$

as claimed.

Proof of the bound in (6-9). Let $\phi_t(x,\eta):=\phi(t,x,\eta)$ and $\varphi_t(x,y,\eta):=\phi_t(x,\eta)-\langle y,\eta\rangle$. Then we have $C_{\varphi_t}=\{(x,y,\eta):\partial_\eta\phi_t(x,\eta)=y\}$, and so

$$\Lambda_{\varphi_t}=\{(x,\ \partial_x\phi_t(x,\eta),\ \partial_\eta\phi_t(x,\eta),\ -\eta)\}\subset T^*\mathbb{R}^n\times T^*\mathbb{R}^n.$$

In particular, since Λ_{φ_t} is the twisted graph of κ_t, we have that κ_t is characterized by

$$\kappa_t(\partial_\eta\phi_t(x,\eta),\eta)=(x,\partial_x\phi_t(x,\eta)).$$

Furthermore, since $\kappa_t(\Gamma(0))=\Gamma(t)$, we have

$$\Gamma(t)=\{(x,\xi):\kappa_t(y,\eta)=(x,\xi),\ y=\partial_\eta\phi_t(x,\eta),\ \xi=\partial_x\phi_t(x,\eta),\ (y,\eta)\in\Gamma(0)\}.$$

Then, using $\kappa_t^*\xi_n=\eta_n$,

$$\Gamma(t)=\{(x,\xi):\xi'=\partial_{x'}\phi_t(x,\eta),\ \partial_{\eta'}\phi_t(x,\eta)=0,\ \xi_n=\eta_n,\ \eta\in\mathbb{R}^n\}.$$

Next, let $\tilde p:=(\tilde x,\tilde\eta)\in\Gamma(0)$ be such that $\partial_{\eta'}\phi_t(\tilde x,\tilde\eta)=0$. Without loss of generality, in what follows we assume that $\tilde x_n=0$. Letting $\Gamma_0(t):=\Gamma(t)|_{\{x_n=0\}}$ we have that

$$\Gamma_0(t)=\{(x,\xi):\xi'=\partial_{x'}\phi_t(x,\eta),\ \partial_{\eta'}\phi_t(x,\eta)=0,\ x_n=0,\ \xi_n=\eta_n,\ \eta\in\mathbb{R}^n\}.$$

In particular, letting $\tilde\xi:=(\partial_{x'}\phi_t(\tilde p),\tilde\eta_n)$ and $\tilde p_0:=(\tilde x,\tilde\xi)$, we have $\tilde p_0\in\Gamma_0(t)\cap\Gamma_0(0)$ and

$$\begin{aligned}T_{\tilde p_0}\Gamma_0(t)=\Big\{(\delta_x,\delta_\xi):\delta_{\xi'}&=\partial_x\partial_{x'}\phi_t(\tilde p)\delta_x+\partial_\eta\partial_{x'}\phi_t(\tilde p)\delta_\eta,\\ &\partial_x\partial_{\eta'}\phi_t(\tilde p)\delta_x+\partial_\eta\partial_{\eta'}\phi_t(\tilde p)\delta_\eta=0,\ \delta_{x_n}=0,\ \delta_{\xi_n}=\delta_{\eta_n},\ \delta_\eta\in\mathbb{R}^n\Big\}.\end{aligned}$$

Next, we note that $\partial_{x_n}\in T_{\tilde p_0}\Gamma(t)$ and $H_{q_i(t)}\in T_{\tilde p_0}\Gamma(t)$ for all $i=1,\dots,n-1$. Therefore, since $\partial_{x_n}q_i(t)=0$, we also know that $H'_{q_i(t)}:=(\partial_{\xi'}q_i(t),0,-\partial_{x'}q_i(t),0)\in T_{\tilde p_0}\Gamma_0(t)$ for all $i=1,\dots,n-1$. We claim that there exists $C>0$ such that, for all $v=(\delta_{x'},0,\delta_{\xi'},0)\in\operatorname{span}\{H'_{q_i(t)}\}_{i=1}^{n-1}\subset T_{\tilde p_0}\Gamma(t)$, we have

$$\|\delta_{x'}\|\ge Ct\|\delta_{\xi'}\|. \tag{6-11}$$

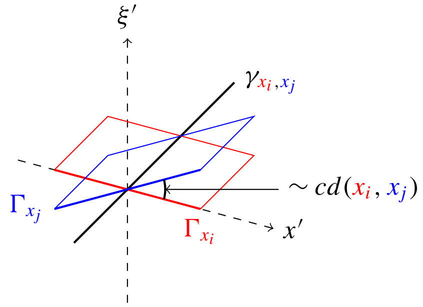

Figure 2. A pictorial representation of the coisotropics involved in Corollary 6.5, where γ_{x_i,x_j} is the geodesic from x_i to x_j. Localization to both Γ_{x_i} and Γ_{x_j} implies localization in the nonsymplectically orthogonal directions x' and ξ'. The uncertainty principle then rules this behavior out.

Suppose that the claim in (6-11) holds. Then, note that for each such v, since $\delta_{x_n} = 0$ and $\delta_{\xi_n} = 0$, we have that there is $\delta_{\eta'} \in \mathbb{R}^{n-1}$ such that

$$\delta_{\xi'} = \partial^2_{x'}\phi_t(\tilde{p})\delta_{x'} + \partial^2_{\eta' x'}\phi_t(\tilde{p})\delta_{\eta'}, \quad \partial^2_{x'\eta'}\phi_t(\tilde{p})\delta_{x'} + \partial^2_{\eta'}\phi_t(\tilde{p})\delta_{\eta'} = 0.$$

Using that $\partial^2_{x'\eta'}\phi_t(\tilde{p}) = \mathrm{Id} + O(t)$ and $\partial^2_{x'}\phi_t(\tilde{p}) = O(t)$, we conclude that

$$\partial^2_{\eta'}\phi_t(\tilde{p})[\partial^2_{\eta' x'}\phi_t(\tilde{p})]^{-1}\delta_{\xi'} = (\partial^2_{\eta'}\phi_t(\tilde{p})[\partial^2_{\eta' x'}\phi_t(\tilde{p})]^{-1}\partial^2_{x'}\phi_t(\tilde{p}) - \partial^2_{x'\eta'}\phi_t(\tilde{p}))\delta_{x'},$$

and so

$$\partial^2_{\eta'}\phi_t(\tilde{p})(\mathrm{Id} + O(t))\delta_{\xi'} = (-\mathrm{Id} + O(t))\delta_{x'}. \tag{6-12}$$

Let $H'_{q_i(t)} = (\delta^{(i)}_{x'}, 0, \delta^{(i)}_{\xi'}, 0)$. Since $\tilde{p}_0 \in \Gamma(t) \cap \Gamma(0)$, assumptions (6-6) and (6-7) yield that the vectors $\{\delta^{(i)}_{x'}\}_{i=1}^{n-1}$ are linearly independent. Indeed, setting $e_i := (\delta_{ij})_{j=1}^{n-1} \in \mathbb{R}^{n-1}$,

$$\delta^{(i)}_{x'} = \partial_{\xi_i} q_i(t) e_i + O(t^2), \quad |\partial_{\xi_i} q_i(t)| \geq Ct \tag{6-13}$$

for t small enough. Furthermore, (6-12) then yields that the $\{\delta^{(i)}_{\xi'}\}_{i=1}^{n-1}$ are linearly independent. Then, combining (6-12) with (6-11) yields (6-9) as claimed. □

To finish it only remains to prove (6-11). Let $v = (\delta_{x'}, 0, \delta_{\xi'}, 0) \in \mathrm{span}\{H'_{q_i(t)}\}_{i=1}^{n-1}$. Then, there is $a \in \mathbb{R}^{n-1}$ such that $\delta_{x'} = \sum_{i=1}^{n-1} a_i \delta^{(i)}_{x'}$ and $\delta_{\xi'} = \sum_{i=1}^{n-1} a_i \delta^{(i)}_{\xi'}$. Next, note that by (6-13) we have $\|\delta_{x'}\| \geq \|a\|(Ct + O(t^2))$. Since $\|\delta_{\xi'}\| \leq C_0\|a\|$ for some $C_0 > 0$, the claim in (6-11) follows. □

For each $x \in M$, let Γ_x be as in (6-1). (See Figure 2 for a schematic representation of these two coisotropic submanifolds.) Then we have the following result.

Corollary 6.5. *Let $0 < \rho < 1$, $0 < \varepsilon < \rho$, and $\gamma(t) : (-\varepsilon_0, \varepsilon_0) \to M$ be a unit speed geodesic. Then, for $X(t) \in \Psi^{-\infty}_{\Gamma_{\gamma(t)},\rho}$ and h such that $h^{\rho-\varepsilon} \leq |t| < \varepsilon_0$,*

$$\|X(0)X(t)\|_{L^2 \to L^2} \leq Ch^{(n-1)(2\rho-1)/2} t^{(1-n)/2}.$$

Proof. To do this, we study the geometry of the flow-out coisotropics $\Gamma_{\gamma(t)}$. Namely, we prove that $\Gamma_{\gamma(t)}$ is defined by some functions $\{q_i(t)\}_{i=1}^n$ with $q_i(0) = x_i$ that satisfy (6-6) and (6-7). We then apply Proposition 6.4 to $\Gamma(t) = \kappa^{-1}(\Gamma_{\gamma(t)})$ for a suitable symplectomorphism κ.

Fix coordinates $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ on M such that $\gamma(t) = (0, t)$ and

$$\partial^2_{\xi'} |\xi|_{g(x)}\big|_{x=0,\xi=(0,1)} = \mathrm{Id}\,.$$

For each $t \in (-\varepsilon_0, \varepsilon_0)$, let $\mathcal{H}_t$ be the submanifold transverse to the Hamiltonian vector field H_p defined by

$$\mathcal{H}_t := \{(x', t, \xi', \xi_n) : 2\xi_n > |\xi|_g,\ |x'| \le \delta_0\},$$

where $\delta_0 > 0$ is chosen such that $\Gamma_{\gamma(t)} \cap \mathcal{H}_t = \{(0, t, \xi', \xi_n) : 2\xi_n > |\xi|_g,\ ||\xi|_g - 1| < \delta\}$.

In particular, as a subset of $\{||\xi|_g - 1| < \delta\}$, we define $\Gamma_{\gamma(t)} \cap \mathcal{H}_t$ by the coordinate functions $\{x_i\}_{i=1}^{n-1}$. For each $t \in (-\varepsilon_0, \varepsilon_0)$ let $\tilde{q}_i(t) : \mathcal{H}_t \to \mathbb{R}$ be given by $\tilde{q}_i(t) = x_i$ for $i = 1, \dots, n-1$. Then, define $\{q_i(t)\}_{i=1}^{n-1}$ on T^*M by

$$H_p q_i(t) = 0, \quad q_i(t)|_{\mathcal{H}_t} = \tilde{q}_i(t).$$

For all t, we note that $H_p(H_{q_i(t)} q_j(t)) = 0$ and

$$\{q_i(t), q_j(t)\}|_{\mathcal{H}_t} = \partial_{\xi_n} q_i(t) \partial_{x_n} q_j(t) - \partial_{\xi_n} q_j(t) \partial_{x_n} q_i(t) + \widetilde{H}_{q_i(t)} q_j(t),$$

where $\widetilde{H}$ is the Hamiltonian vector field in $T^*\{x_n = t\}$. In particular, since $\partial_{\xi_n} \tilde{q}_i(t) = 0$ and $\widetilde{H}_{q_i(t)}$ is tangent to $\mathcal{H}_t$, we have $\{q_i(t), q_j(t)\}|_{\mathcal{H}_t} = 0$. Hence, $\{q_i(t), q_j(t)\} \equiv 0$, $\{q_i(t), p\} = 0$, $q_i(0) = x_i$, and $\{q_i(t)\}_{i=1}^{n-1}$ define $\Gamma_{\gamma(t)}$. Next, observe that there exists $s \in \mathbb{R}$ such that, for each $i = 1, \dots, n-1$, $q_i(0)(x, \xi) = x_i(\varphi_s(x, \xi))$ with $\varphi_s(x, \xi) \in \mathcal{H}_0$. Since $\partial_{\xi_n} p \neq 0$ on $\mathcal{H}_0$, for E near 0 there exist a_E and e_E such that

$$p(x, \xi) - E = e_E(x, \xi)(\xi_n - a_E(x, \xi')) \tag{6-14}$$

with $e_E > c$ for some constant $c > 0$. In particular, $\varphi_s = e^{sH_p}$ is a reparametrization of $\tilde{\varphi}_s := e^{s(H_{\xi_n - a_E})}$ on $\{p = E\}$, and we have that, for $(x, \xi) \in \{p = E\}$ and all $i = 1, \dots, n-1$,

$$q_i(0)(x, \xi) = x_i(\tilde{\varphi}_{-x_n}(x, \xi)) = x_i + x_n \partial_{\xi_i} a_E(x, \xi') + O(x_n^2)_{C^\infty}.$$

In particular, on $\mathcal{H}_t \cap \{p = E\}$, using this together with the fact that since $H_{q_j(t)}$ is tangent to $\{p = E\}$ and $x_n = t$, $\partial_{\xi_n} q_i(t) = \partial_{\xi_n} \tilde{q}_i(t) = 0$, we have

$$\begin{aligned}
\{q_j(t), q_i(0)\}|_{\mathcal{H}_t \cap \{p=E\}} &= \partial_{\xi_n} q_j(t) \partial_{x_n} q_i(0) - \partial_{x_n} q_j(t) \partial_{\xi_n} q_i(0) + \widetilde{H}_{q_j(t)} q_i(0) \\
&= \partial_{\xi_n} \tilde{q}_j(t) \partial_{x_n} q_i(0) - \partial_{x_n} q_j(t) O(t^2) + \widetilde{H}_{\tilde{q}_j(t)} q_i(0) \\
&= O(t^2) + \partial_{\xi_j}(t \partial_{\xi_i} a_E)(0, \xi').
\end{aligned}$$

Now, since $\partial^2_\xi p|_{T\{p=E\}} > 0$ and, for all $i, j = 1, \dots, n$,

$$\partial_{\xi_i \xi_j} p = \partial_{\xi_j} \partial_{\xi_i} e_E(\xi_n - a_E) + \partial_{\xi_i} e_E(\delta_{nj} - \partial_{\xi_j} a_E) + \partial_{\xi_j} e_E(\delta_{ni} - \partial_{\xi_i} a_E) - e_E \partial_{\xi_j} \partial_{\xi_i} a_E, \tag{6-15}$$

we have, as quadratic forms, $\partial^2_\xi p|_{T\{p=E\}} = -e_E \partial^2_\xi a_E|_{T\{p=E\}}$. Indeed, if $V = \sum_j V^j \partial_{\xi_j} \in T\{p = E\}$, then

$$0 = V(p - E)|_{p=E} = e_E V(\xi_n - a_E) + (V e_E)(\xi_n - E)|_{p=E} = e_E V(\xi_n - a_E),$$

and therefore, since $e_E \neq 0$, we have $V(\xi_n - a_E)|_{p=E} = 0$. Next, observe that, on $\{p = E\}$,

$$\partial_{\xi_i} e_E \sum_j (\delta_{nj} - \partial_{\xi_j} a_E) V^j = \partial_{\xi_i} e_E (V(\xi_n - a_E)) = 0.$$

In particular, the first three terms in (6-15) vanish on $T\{p = E\}$.

Hence, since $\partial^2_{\xi'} p|_{x=0,\xi=(0,1)} = \mathrm{Id}$, we have that $\partial^2_{\xi'} a_E(0, \xi') < 0$ is a multiple of the identity at $x = 0$, $\xi' = 0$, and $p = E$. Next, observe that

$$\Gamma_{\gamma(0)} \cap \Gamma_{\gamma(t)} \subset \{(0, s, 0, \xi_n) : s \in \mathbb{R},\ \xi' = 0\}.$$

Therefore, there are $c, C > 0$ with

$$c\delta_{ij} t + O(t^2) \leq \left|\{q_i(t), q_j(0)\}|_{\mathcal{H}_t \cap \{p=E\} \cap \Gamma_{\gamma(0)} \cap \Gamma_{\gamma(t)}}\right| \leq C\delta_{ij} t + O(t^2)$$

on $\Gamma_{\gamma(0)} \cap \Gamma_{\gamma(t)}$. Then, $c\delta_{ij} t + O(t^2) \leq \left|\{q_i(t), q_j(0)\}|_{\{p=E\}}\right| \leq C\delta_{ij} t + O(t^2)$ by invariance under H_p. Since E small is arbitrary, this holds on $\Gamma_{\gamma(0)} \cap \Gamma_{\gamma(t)}$.

Now, by Darboux's theorem, there is a symplectomorphism κ such that, for all $i = 1, \dots, n-1$, $\kappa^* q_i(0) = x_i$ and $\kappa^* p = \xi_n$. In particular, $\kappa^{-1}(\Gamma_{\gamma(0)}) \subset \Gamma(0) = \{(0, x_n, \xi', \xi_n) : x_n \in \mathbb{R},\ \xi \in \mathbb{R}^{n-1} \times \mathbb{R}\}$ and, abusing notation slightly by relabeling $q_i(t) = \kappa^* q_i(t)$, we have that (6-6) and (6-7) hold. In particular, Proposition 6.4 applies to $\Gamma(t) = \kappa^{-1}(\Gamma_{\gamma(t)})$.

Now, let U be a microlocally unitary quantization of κ and $X(t) \in \Psi^{-\infty}_{\Gamma_{\gamma(t)},\rho}$. Then, $U^{-1} X(t) U \in \Psi^{-\infty}_{\Gamma(t),\rho}$ and hence the corollary is proved. □

6C. ***Almost orthogonality for coisotropic cutoffs.*** In this section, we finally prove an estimate which shows that coisotropic cutoffs associated with Γ_{x_i} for many x_i are almost orthogonal. This, together with the fact that these cutoffs respect pointwise values near x_i, is what allows us to control the number of points at which a quasimode may be large.

Proposition 6.6. *Let $\{B(x_i, R)\}_{i=1}^{N(h)}$ be a $(\mathfrak{D}, R)$-good cover for M, and $X_i \in \Psi^{-\infty}_{\Gamma_{x_i},\rho}$, $i = 1, \dots, N(h)$, with uniform symbol estimates. Then, there are $C > 0$ and $h_0 > 0$ such that, for all $0 < h < h_0$, $\mathcal{J} \subset \{1, \dots, N(h)\}$ and $u \in L^2(M)$, we have*

$$\sum_{j \in \mathcal{J}} \|X_j u\|^2_{L^2} \leq C(1 + (h^{2\rho-1} R^{-1})^{(n-1)/2} |\mathcal{J}|^{(3n+1)/(2n)} (1 + (h^{2\rho-1} R^{-1})^{(n-1)/4})) \|u\|^2_{L^2}. \tag{6-16}$$

Proof. To prove this bound we will decompose the sum in (6-16) as

$$\sum_{i \in \mathcal{J}} \|X_i u\|^2_{L^2} = \Big\| \sum_{i \in \mathcal{J}} X_i u \Big\|^2_{L^2} - \Big\langle \sum_{\substack{i,j \in \mathcal{J} \\ i \neq j}} X_j^* X_i u, u \Big\rangle. \tag{6-17}$$

First, we note that by Corollary 6.5, (once with $X(0) = X_j^*$ and $X(t) = X_i$, and once with $X(0) = X_j$ and $X(t) = X_i^*$) there exists $C > 0$ such that, for $i \neq j$,

$$\|X_j^* X_i\| + \|X_j X_i^*\| \leq C h^{(n-1)(\rho-1/2)} d(x_i, x_j)^{(1-n)/2}.$$

Therefore, by the Cotlar–Stein lemma,

$$\Big\|\sum_{j\in\mathcal{J}} X_j\Big\| \le \sup_{j\in\mathcal{J}}\Big(\|X_j\| + \sum_{i\in\mathcal{J}\setminus\{j\}} \|X_j^*X_i\|^{1/2} + \|X_jX_i^*\|^{1/2}\Big)$$
$$\le 2 + Ch^{(n-1)(\rho-1/2)/2}\sup_{j\in\mathcal{J}}\sum_{i\in\mathcal{J}\setminus\{j\}} d(x_i,x_j)^{(1-n)/4}.$$

To estimate the sum, observe that there exists $C>0$ such that, for any $j\in\mathcal{J}$ and any positive integer k,

$$\frac{2^{kn}}{C} \le \#\{i : 2^kR \le d(x_i,x_j) \le 2^{k+1}R\} \le C2^{(k+1)n}.$$

In particular, there is $C>0$ such that, for any $j\in\mathcal{J}$,

$$\sum_{i\in\mathcal{J}\setminus\{j\}} d(x_i,x_j)^{(1-n)/4} \le C\sum_{k=0}^{\frac{1}{n}\log_2|\mathcal{J}|} 2^{kn}(2^kR)^{(1-n)/4} \le C|\mathcal{J}|^{(3n+1)/(4n)}R^{(1-n)/4}. \tag{6-18}$$

Therefore, we shall bound the first term in (6-17) using

$$\Big\|\sum_{j\in\mathcal{J}} X_j\Big\| \le C + Ch^{(n-1)(\rho-1/2)/2}R^{(1-n)/4}|\mathcal{J}|^{(3n+1)/(4n)}. \tag{6-19}$$

We next proceed to control the second term in (6-17). Let $\widetilde{X}_j\in\Psi^{-\infty}_{\Gamma_{x_j},\rho}$ such that

$$\widetilde{X}_jX_j = X_j + O(h^\infty)_{L^2\to L^2}.$$

By the Cotlar–Stein Lemma,

$$\Big\|\sum_{\substack{i,j\in\mathcal{J}\\ i\ne j}} X_j^*X_i\Big\| \le \sup_{\substack{k,\ell\in\mathcal{J}\\ k\ne\ell}}\sum_{\substack{i,j\in\mathcal{J}\\ i\ne j}} \|X_k^*\widetilde{X}_\ell X_\ell X_j^*\widetilde{X}_j^*X_i\|^{1/2} + \|X_\ell^*\widetilde{X}_kX_kX_i^*\widetilde{X}_i^*X_j\|^{1/2} + O(h^\infty|\mathcal{J}|^2). \tag{6-20}$$

By Corollary 6.5 there exists $C>0$ such that, for $k\ne\ell,\ i\ne j$,

$$\|X_k^*\widetilde{X}_\ell X_\ell X_j^*\widetilde{X}_j^*X_i\| \le Ch^{(n-1)(2\rho-1)}\min\{1, h^{(n-1)(2\rho-1)/2}d(x_j,x_\ell)^{-(n-1)/2}\}(d(x_k,x_\ell)d(x_j,x_i))^{(1-n)/2}.$$

Using that

$$\sup_{\substack{k,\ell\in\mathcal{J}\\ k\ne\ell}} d(x_k,x_\ell)^{(1-n)/4} \le R^{(1-n)/4},$$

adding in (6-20), and combining with the bound in (6-18), we get

$$\Big\|\sum_{\substack{i,j\in\mathcal{J}\\ i\ne j}} X_j^*X_i\Big\| \le Ch^{(n-1)(2\rho-1)/2}(1+h^{(n-1)(2\rho-1)/4}|\mathcal{J}|^{(3n+1)/(4n)}R^{(1-n)/4})|\mathcal{J}|^{(3n+1)/(4n)}R^{(1-n)/2}. \tag{6-21}$$

In particular, combining (6-19) and (6-21) into (6-17) we obtain

$$\sum_{i\in\mathcal{J}}\|X_iu\|^2 \le C(1+h^{(n-1)(\rho-1/2)}R^{(1-n)/2}|\mathcal{J}|^{(3n+1)/(2n)} + h^{3(n-1)(2\rho-1)/4}R^{3(1-n)/4}|\mathcal{J}|^{(3n+1)/(2n)})\|u\|^2_{L^2}$$
$$\le C(1+h^{(n-1)(\rho-1/2)}R^{(1-n)/2}(1+(h^{2\rho-1}R^{-1})^{(n-1)/4})|\mathcal{J}|^{(3n+1)/(2n)})\|u\|^2_{L^2}. \qquad\square$$

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PERTURBED INTERPOLATION FORMULAE AND APPLICATIONS

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We employ functional analysis techniques in order to deduce some versions of classical and recent interpolation results in Fourier analysis with perturbed nodes. As an application of our techniques, we obtain generalizations of Kadec's $\frac{1}{4}$-theorem for interpolation formulae in the Paley–Wiener space both in the real and complex cases, as well as versions of the recent interpolation result of Radchenko and Viazovska (*Publ. Math. Inst. Hautes Études Sci.* **129** (2019), 51–81) and the result of Cohn, Kumar, Miller, Radchenko and Viazovska (*Ann. Math* (2) **196**:3 (2022), 983–1082) for Fourier interpolation with derivatives in dimensions 8 and 24 with suitable perturbations of the interpolation nodes. We also provide several applications of the main results and techniques, relating to recent contributions in interpolation formulae and uniqueness sets for the Fourier transform.

1. Introduction

A fundamental question in analysis is that of how to recover a function f from some subset $\{f(x)\}_{x\in A}$ of its values, together with some information on its *Fourier transform* $\hat{f}:\mathbb{R}\to\mathbb{C}$, which we define to be

$$\hat{f}(\xi)=\int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\,\mathrm{d}x. \tag{1-1}$$

Perhaps the most classical result in that regard is the *Shannon–Whittaker interpolation formula*: if $\hat{f}$ is supported on an interval $[-\delta/2,\delta/2]$, then

$$f(x)=\sum_{k=-\infty}^{\infty} f(k/\delta)\,\mathrm{sinc}(\delta x-k), \tag{1-2}$$

where convergence holds both in $L^2(\mathbb{R})$ and uniformly in compact sets of $\mathbb{C}$, where we let $\mathrm{sinc}(x)=\sin(\pi x)/(\pi x)$. A major recent breakthrough in regard to the problem of determining which conditions

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on the sets $A, B \subset \mathbb{R}$ imply that a function $f \in \mathcal{S}(\mathbb{R})$ is uniquely determined by its values at A and the values of its Fourier transform at B was made in [Radchenko and Viazovska 2019], where the authors proved that, if $f : \mathbb{R} \to \mathbb{R}$ is even and Schwartz, then

$$f(x) = \sum_{k=0}^{\infty} f(\sqrt{k}) a_k(x) + \sum_{k=0}^{\infty} \hat{f}(\sqrt{k}) \hat{a}_k(x). \tag{1-3}$$

Radchenko and Viazovska's result and its techniques were somewhat inspired by Viazovska's recent solution [2017] to the sphere-packing problem in dimension 8, and her subsequent work with Cohn, Kumar, Miller and Radchenko [Cohn et al. 2017] to solve the same problem in dimension 24. Indeed, the proof of (1-3) uses such tools from the theory of *modular forms* heavily for constructing and bounding the basis functions $\{a_n\}_{n\ge 0}$.

Subsequent to the Radchenko–Viazovska result, other recent works have successfully used a similar approach in order to tackle what are now known as Fourier interpolation and Fourier uniqueness problems. Among those, we mention the following:

(1) Cohn and Gonçalves [2019] used a modular form construction in order to obtain that there are $c_j > 0$, $j \in \mathbb{N}$, so that, for each $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^{12})$ real,

$$f(0) - \sum_{j\ge 1} c_j f(\sqrt{2j}) = -\hat{f}(0) + \sum_{j \ge 1} c_j \hat{f}(\sqrt{2j}). \tag{1-4}$$

Such a formula enables the authors to prove a sharp version of a root uncertainty principle first raised by Bourgain, Clozel and Kahane [Bourgain et al. 2010] in dimension 12; see, e.g., [Gonçalves et al. 2017; 2021; 2023] for more information on this topic.

(2) On the other hand, Cohn, Kumar, Miller, Radchenko and Viazovska [Cohn et al. 2022] built upon the basic ideas of [Radchenko and Viazovska 2019] to be able to prove *universal optimality results* about the E_8 and Leech lattices in dimensions 8 and 24, respectively. In order to do so, they prove interpolation formulae in such dimensions that involve the values of $f(\sqrt{2n})$, $f'(\sqrt{2n})$, $\hat{f}(\sqrt{2n})$, $\hat{f}'(\sqrt{2n})$, where f is a radial, Schwartz function, and $n \ge n_0$, with $n_0 = 1$ if $d = 8$, and $n_0 = 2$ in case $d = 24$.

(3) Talebizadeh Sardari [2021] studied the problem of constructing interpolation formulae involving the values $f(\sqrt{r})$, $f'(\sqrt{r})$, $\hat{f}(\sqrt{r})$, $\hat{f}'(\sqrt{r})$, where f is a radial, Schwartz function, in $\mathbb{R}^2$, and r is any point in the set

$$\left\{ \left(\tfrac{4}{3}\right)^{1/4} \sqrt{n^2 + nm + m^2} : n, m \in \mathbb{Z} \right\},$$

which would correspond to a Fourier interpolation formula with derivatives over the hexagonal lattice. Such a formula was conjecture not to exist in [Cohn et al. 2022, Conjecture 7.5], and indeed that is the case: there are infinitely many linearly independent Schwartz functions that cannot be recovered by these values. This is perhaps surprising, since the hexagonal lattice is conjectured to be universally optimal in the language of [Cohn et al. 2022], which suggests this problem is not amenable to the exact same techniques in that work in dimensions 8 and 24.

(4) Finally, more recently, other developments in the theory of interpolation formulae given values on both Fourier and spatial sides have been made by Stoller [2021], who considered the problem of

recovering any function in $\mathbb{R}^d$ from its restrictions and the restrictions of its Fourier transforms to spheres of radii $\sqrt{n}$, where $n > 0$, is an integer, and for any $d > 0$. Moreover, we mention also the more recent work of Bondarenko, Radchenko and Seip [Bondarenko et al. 2023], which generalizes Radchenko and Viazovska's construction of the interpolating functions to prove interpolation formulae for some classes of functions f that take into account the values of $\hat{f}$ at $\log n/(4\pi)$, and the values of f at a sequence $\left(\rho - \frac{1}{2}\right)/i$, where ρ ranges over nontrivial zeros of some L-function with positive imaginary part.

One fundamental point to stress is that, in a suitable way, all the previously mentioned results relate some sort of *summation formula*, the most basic instance of such being the classical Poisson summation formula

$$\sum_{m\in\mathbb{Z}} f(m) = \sum_{n\in\mathbb{Z}} \hat{f}(n),$$

which is obtained in [Radchenko and Viazovska 2019] as a particular case of (1-3) by setting $x = 0$, with some *modular form* construction. In this direction, the formula (1-4) is also a manifestation of such a principle that implies rigidity between certain values of f and other values of $\hat{f}$.

The aforementioned connection between summation formulae and modular forms is classical, with the modularity of the Jacobi theta series θ being a primal example of how one relates to the other. On the other hand, this connection may be deepened through the following argument: Suppose that a summation formula of the kind

$$\sum_{a\in A} c_a f(a) = \sum_{a\in A} c_a \hat{f}(a) \tag{1-5}$$

holds for all $f \in \mathcal{S}(\mathbb{R})$ a radial function. This is seen to be equivalent, by a density argument (see, for instance, [Radchenko and Viazovska 2019, Section 6]), to (1-5) holding for $f(x) = e^{iz\pi|x|^2}$, where $z \in \mathbb{C}$ is fixed so that $\operatorname{Im}(z) > 0$. This, on the other hand, is equivalent to the function $M(z) = \sum_{a\in A} e^{i\pi z|a|^2}$ satisfying the modular relationship $(-iz)^{-d/2}M(-1/z) = M(z)$ in the upper half-space. In particular, if $A \subset \sqrt{\mathbb{Z}_+}$, then M satisfies additionally some periodicity condition, and thus a search for M can be further narrowed to a certain space of modular forms.

From a similar yet not identical point of view, however, the topics described above can also be inserted into the framework of *crystalline measures*. Indeed, if we adopt the classical definition of a crystalline measure to be a distribution with locally finite support, such that its Fourier transform possesses the same support property, we will see that the Poisson summation formula implies, for instance, that the measure $\delta_{\mathbb{Z}}$ is not only a crystalline measure, but also *self-dual*, in the sense that $\delta_{\mathbb{Z}} = \hat{\delta}_{\mathbb{Z}}$ holds in $\mathcal{S}'(\mathbb{R})$.

Outside the scope of interpolation formulae per se, we mention the works [Lev and Olevskii 2013; 2015; Meyer 2017], where the authors explore on a deeper lever structural questions on crystalline measures. In particular, Meyer [2017] exhibits examples of crystalline measures with self-duality properties, and uses modular forms to construct explicitly examples of nonzero self-dual crystalline measures μ supported on $\{\pm\sqrt{k+a} : k \in \mathbb{Z}_+\}$ for $a \in \{9, 24, 72\}$. We also mention [Kurasov and Sarnak 2020], where the authors, as a by-product of investigations of the additive structure of the spectrum of metric graphs, prove that there are exotic examples of *positive* crystalline measures other than generalized Dirac combs.

Our investigation in this paper focuses on both classical and modern results in the theory of such interpolation formulae and crystalline measures. In generic terms, we are interested in determining when, given an interpolation formula such as (1-2) or (1-3), we can *perturb* it suitably. That is, given a sequence of real numbers $\{\varepsilon_k\}_{k\in\mathbb{Z}}$, under which conditions can we recover f from the values

$$\{(f(s_n+\varepsilon_n),\ \hat{f}(\hat{s}_n+\varepsilon_n))\}_{n\in\mathbb{Z}}, \tag{1-6}$$

given that we can recover f from $\{(f(s_n),\ \hat{f}(\hat{s}_n))\}_{n\in\mathbb{Z}}$?

In this manuscript, the main idea is to study such perturbations of interpolation formulae for band-limited and Schwartz functions through functional analysis. Indeed, most of our considerations are based on the idea that, whenever an operator $T : B \to B$, where B is a Banach space, satisfies

$$\|T - I\|_{B\to B} < 1,$$

then T is, in fact, a *bijection* with continuous inverse $T^{-1} : B \to B$. In fact, in all our considerations on interpolation formulae below, some form of this principle will be employed, and other proofs and results in the paper, such as Theorem 1.6, which gives new bounds related to the Radchenko–Viazovska formula, arise naturally when trying to employ this principle in different contexts.

1A. ***Perturbations and interpolation formulae in the band-limited case.*** The question of when we are able to recover the values of a function such that its Fourier transform is supported in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ from its values at $n+\varepsilon_n$ is well known, having been asked in [Paley and Wiener 1934], where the authors proved that recovery — and also an associated interpolation formula — is possible as long as $\sup_n |\varepsilon_n| < \pi^{-2}$. Many results relate to the original problem of Paley and Wiener, but the most celebrated of them all is the so-called Kadec-$\frac{1}{4}$ theorem, which states that, as long as $\sup_n |\varepsilon_n| < \frac{1}{4}$, one can recover any $f \in L^2(\mathbb{R})$ which has Fourier support on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ from its values at $n+\varepsilon_n,\ n\in\mathbb{Z}$; see [Kadec 1964] for the original proof and [Avantaggiati et al. 2016] for a generalization.

Our first results provide one with a simpler proof of a particular range of Kadec's result. We recall, for that matter, that the Paley–Wiener space $\mathrm{PW}_\pi(\mathbb{R})$ is defined as the aforementioned space of all square-integrable functions on the real line such that $\hat{f}$ has support in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Theorem 1.1. *Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of real numbers and assume $L = \sup_k |\varepsilon_k| < L_0$, where $L_0 = 0.239\ldots$ is defined to be the smallest positive solution to the equation*

$$\frac{\sin(\pi L_0)}{\pi L_0} = \frac{\pi}{3}\frac{L_0 \sin \pi L_0}{1-L_0} + \sin(\pi L_0).$$

Then any function $f \in \mathrm{PW}_\pi$ is completely determined by its values $\{f(n+\varepsilon_n)\}_{n\in\mathbb{Z}}$, and there is $C = C(L) > 0$ such that

$$\frac{1}{C}\sum_{n\in\mathbb{Z}} |f(n+\varepsilon_n)|^2 \le \|f\|_2^2 \le C \sum_{n\in\mathbb{Z}} |f(n+\varepsilon_n)|^2$$

for all $f \in \mathrm{PW}_\pi$.

Moreover, there are functions $g_n \in \mathrm{PW}_\pi(\mathbb{R})$ such that for every $f \in \mathrm{PW}_\pi$, the following identity holds:

$$f(x) = \sum_{n\in\mathbb{Z}} f(n+\varepsilon_n) g_n(x),$$

where the right-hand side converges absolutely in compact sets of $\mathbb{C}$.

The condition in Theorem 1.1 is satisfied for $L < 0.239$, which possesses only a 0.011 difference from Kadec's result. The main difference, however, is that while Kadec's proof relies on a clever expansion of the underlying functions in a different orthonormal basis, we make a less direct use of orthogonality in our considerations.

We also remark that, in the proof of Theorem 1.1, one can use complex numbers for perturbations. The difference is that we have to take into account the sine of complex numbers, and the resulting bound would be $L < 0.2125$ instead of $L < 0.239$. This only falls very mildly short of the results in [Avantaggiati et al. 2016, Theorem 3], where $L < 0.218$ is achieved in the complex setting, and our methods of proof are relatively simpler in comparison to those of that work, where the authors must enter the realm of Lamb–Oseen functions and constants.

As another application of the idea of inverting an operator, we present a couple of results related to Vaaler's interpolation formula. J. Vaaler [1985] proved, as means to study extremal problems in Fourier analysis, the following counterpart to the Shannon–Whittaker interpolation formula: Let $f \in L^2(\mathbb{R})$, and suppose that $\hat{f}$ is supported on $[-1, 1]$. Then

$$f(x) = \frac{\sin^2(\pi x)}{\pi^2} \sum_{k\in\mathbb{Z}} \left\{ \frac{f(k)}{(x-k)^2} + \frac{f'(k)}{x-k} \right\}. \tag{1-7}$$

This can be seen as a natural tradeoff: (1-2) demands that we have information at $\frac{1}{2}\mathbb{Z}$ in order to recover the functions f as stated above. On the other hand, Vaaler's result only demands information at $\mathbb{Z}$, but one must pay the price of replacing the rest of the information by values of the derivative at $\mathbb{Z}$.

The first result concerning (1-7) is a *direct* deduction of its validity from the Shannon–Whittaker formula (1-2). We state it in the following form.

Theorem 1.2 [Vaaler 1985]. *Fix a sequence $\{a_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Consider the function $f \in \mathrm{PW}_\pi$ given by*

$$f(x) = \sum_{n\in\mathbb{Z}} a_n \operatorname{sinc}(x-n)$$

for each $x \in \mathbb{R}$. Then the interpolation formula

$$f(x) = \frac{4\sin^2\left(\frac{\pi}{2}x\right)}{\pi^2} \sum_{k\in\mathbb{Z}} \left\{ \frac{a_{2k}}{(x-2k)^2} + \frac{b_{2k}}{x-2k} \right\} \tag{1-8}$$

holds, where the right-hand side converges uniformly on compact sets, and we let

$$b_k = \sum_{j\neq k} \frac{a_j}{k-j}(-1)^{k-j}.$$

It is a consequence of (1-8) that $f'(2k) = b_{2k}$ in Theorem 1.2 above. Moreover, we note that one readily obtains Vaaler's formula from (1-8) above: indeed, in order to obtain (1-7) for a square-integrable function $g \in L^2(\mathbb{R})$ with $\operatorname{supp}(\hat{g}) \subset [-1, 1]$, consider $f(x) = g\left(\frac{1}{2}x\right)$. It follows that f satisfies the hypotheses of Theorem 1.2, and substituting back allows one to conclude (1-7) from (1-8).

A main difference between our proof of Theorem 1.2 and the original proof in [Vaaler 1985] is the absence of any significant use of the Fourier transform. Differently, however, from the de Branges spaces approach in [Gonçalves 2017], we do not delve deeply into any theory of function spaces, but rather we make use of classical operators in $\ell^2(\mathbb{Z})$ such as discrete Hilbert transforms and its properties. We believe our approach might lead to derivations of other interesting interpolation formulae.

Our final contribution in the realm of interpolation formulae for band-limited function is a generalized version of Vaaler's formula (1-7) with *perturbed nodes*. We mention that, to the best of our knowledge, this result in its present form is new, as Vaaler's ideas are rigid to specific properties of integers and Fourier transforms of special functions such as $\operatorname{sinc}(x)^2$.

Theorem 1.3. *Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of real numbers and consider $L = \sup_k |\varepsilon_k|$. Suppose that $L < 0.111$. Then any function $f \in \mathrm{PW}_{2\pi}$ is completely determined by its values $\{f(n+\varepsilon_n)\}_{n\in\mathbb{Z}}$ and those of its derivative $\{f'(n+\varepsilon_n)\}_{n\in\mathbb{Z}}$, and there is $C = C(L) > 0$ such that*

$$\frac{1}{C}\sum_{n\in\mathbb{Z}}(|f(n+\varepsilon_n)|^2+|f'(n+\varepsilon_n)|^2) \le \|f\|_2^2 \le C\sum_{n\in\mathbb{Z}}|(|f(n+\varepsilon_n)|^2+|f'(n+\varepsilon_n)|^2) \tag{1-9}$$

for all $f \in \mathrm{PW}_{2\pi}$.

Moreover, there are functions $g_n, h_n \in \mathrm{PW}_{2\pi}$ so that, for all $f \in \mathrm{PW}_{2\pi}$, we have

$$f(x) = \sum_{n\in\mathbb{Z}}\{f(n+\varepsilon_n)g_n(x) + f'(n+\varepsilon_n)h_n(x)\},$$

where convergence holds absolutely.

This result and its method of proof resemble the ideas from Theorem 1.1 and its proof, with an increase in technical difficulties, such as considering higher-order analogues of the perturbed discrete Hilbert transforms we use for the proof of Theorem 1.1. We note also that some further technical changes, together with [Littmann 2006], allow one to extend the perturbation results for arbitrarily many derivatives; see Theorem 6.1 for a discussion on that.

We point the reader, for instance, to the remark following Corollary 2 in [Gonçalves 2017] together with [Lyubarskii and Seip 2002; Ortega-Cerdà and Seip 2002] for related discussion on sampling sequences with derivatives for PW_π; see also [Gonçalves and Littmann 2018] for discussions involving higher-order derivatives.

1B. ***Perturbations of symmetric interpolation formulae.*** Moving on from band-limited functions to Schwartz functions instead, we notice that the Radchenko–Viazovska result (1-3), although being a major breakthrough, is rigid in its statement: the interpolating functions are carefully tailored to interpolate at the $\{\sqrt{n}\}_{n\ge 0}$ nodes. The same sort of phenomenon happens to the result of [Cohn et al. 2022], as the construction takes into account a specific property of $\{\sqrt{2n}\}_{n\ge n_0}$ in dimensions 8 and 24.

A natural and yet unexplored question is that of determining whether formula (1-3) is rigid for its interpolation nodes or not. In other words, a natural question concerns conditions when we can replace *a single* interpolation node $\sqrt{k}$ by a suitable perturbation of it, say $\sqrt{k+\varepsilon_k}$, where $\varepsilon_k \in (-1, 1)$. To the best of our knowledge, even this simple case remained open prior to this manuscript.

Such a question inspired the following result. Perhaps surprisingly, the idea of inverting an operator T when it is reasonably close to the identity still works in this context. The next result may thus be regarded as the main result and novelty of this paper, establishing criteria when we are allowed, not only to perturb one node in the interpolation formula, but all of them *simultaneously*.

Theorem 1.4. *There is $\delta > 0$ so that, for each sequence of real numbers $\{\varepsilon_k\}_{k\geq 0}$ such that $\varepsilon_k \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, $\varepsilon_0 = 0$, $\sup_{k\geq 0} |\varepsilon_k|(1+k)^{5/4}\log^3(1+k) < \delta$, there are sequences of functions $\{\theta_j\}_{j\geq 0}$, $\{\eta_j\}_{j\geq 0}$, with*

$$|\theta_j(x)| + |\eta_j(x)| + |\hat{\theta}_j(x)| + |\hat{\eta}_j(x)| \lesssim (1+j)^{\mathcal{O}(1)}(1+|x|)^{-10}$$

and

$$f(x) = \sum_{j\geq 0}\big(f(\sqrt{j+\varepsilon_j})\theta_j(x) + \hat{f}(\sqrt{j+\varepsilon_j})\eta_j(x)\big)$$

for all $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ real-valued functions.

In other words, we can perturb each interpolation node from $\sqrt{k}$ to $\sim \sqrt{k+k^{-5/4}}$ and still obtain a valid interpolation formula converging for all Schwartz functions. In fact, one does not strictly need that $f \in \mathcal{S}(\mathbb{R})$, but only that $f, \hat{f}$ decay at least as fast as $(1+|x|)^{-M}$ for some sufficiently large $M \gg 1$.

Theorem 1.4 is related to [Cohn and Triantafillou 2021, §6]. Indeed, in that paper, they construct summation formulae of the form

$$\sum_{n=0}^{\infty} a_n f(\sqrt{n}) = \left(\frac{2}{\sqrt{N}}\right)^{d/2} \sum_{n=0}^{\infty} b_n \hat{f}(2\sqrt{n/N}),$$

where N is a suitable positive integer, where they aim to make the coefficients $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ nonnegative, in order to obtain better estimates for the linear programming bounds for the sphere-packing problem. In §7 in [Cohn and Triantafillou 2021], the authors mention that a "modular" method as carried out by them cannot achieve perturbed nodes in such an interpolation formula, which would be desirable for numerical purposes.

Theorem 1.4, on the one hand, does prove that we can make this rigid property somewhat looser when it comes to the Radchenko–Viazovska interpolation formula, but on the other hand, positivity of coefficients can by no means be guaranteed in our present case. It would be, however, interesting if one could explore further the connections between our methods and those in [Cohn and Triantafillou 2021] to obtain better bounds, but we have not pursued such a path in this work.

As an immediate corollary of Theorem 1.4, we obtain the following:

Corollary 1.5. *Let $\{\varepsilon_j\}_{j\geq 0}$ satisfy the hypotheses of Theorem 1.4. Define a continuous family of measures*

$$\mu_x = \frac{\delta_x + \delta_{-x}}{2} - \sum_{j\geq 0} \frac{\theta_j(x)}{2}\,\delta_{\pm\sqrt{j+\varepsilon_j}}.$$

Then these measures possess Fourier transforms given by

$$\hat{\mu}_x = \sum_{j\geq 0} \frac{\eta_j(x)}{2}\delta_{\pm\sqrt{j+\varepsilon_j}}.$$

In particular, these measures are nontrivial examples of ***crystalline measures*** *supported on both space and frequency on any set of the form $\{\pm x\} \cup \{\pm\sqrt{k+\varepsilon_k} : |\varepsilon_k| \ll \log^{-3}(1+k)\cdot(1+k)^{-5/4}\}$.*

This result, in particular, aligns well with the recent examples from [Bondarenko et al. 2023; Kurasov and Sarnak 2020], which indicate that crystalline measures are, if not impossible, very hard to classify. Its proof follows from the fact that μ_x is even and real-valued, so that its distributional Fourier transform will also be an even and real-valued distribution. Therefore, it suffices to test against even, real-valued functions f, and thus Theorem 1.4 gives us the asserted equality.

In order to prove Theorem 1.4, we need to find a suitable space to use the idea of inverting operators close to the identity. It turns out that, in analogy to Sobolev spaces, the weighted spaces $\ell_s^2(\mathbb{N})$ of sequences square summable against n^s are natural candidates to work with, as they are well-suited to accommodate the sequence

$$\{(f(\sqrt{k+\varepsilon_k}),\, \hat{f}(\sqrt{k+\varepsilon_k}))\}_{k\geq 0}$$

whenever f, $\hat{f}$ decay sufficiently fast. In order to prove *some* perturbation result — that is, a weaker version of Theorem 1.4 — using the spaces $\ell_s^2(\mathbb{N})$ together with the polynomial growth bounds on $\{a_n\}_{n\geq 0}$ from (1-3) is already enough.

On the other hand, the fact that we may push the perturbations up until the $k^{-5/4}$ threshold needs a suitable refinement to [Radchenko and Viazovska 2019] or even to the bound of [Bondarenko et al. 2023]. The next result, thus, represents an improvement over those in [Bondarenko et al. 2023; Radchenko and Viazovska 2019], as besides obtaining uniform bounds, we are able to introduce *exponential decay* factors to the interpolating functions.

Theorem 1.6. *Let $b_n^{\pm} = a_n \pm \hat{a}_n$, where $\{a_n\}_{n\geq 0}$ are the basis functions in* (1-3). *Then there is an absolute constant $c > 0$ such that*

$$|b_n^{\pm}(x)| \lesssim n^{1/4} \log^3(1+n) e^{-c|x|/\sqrt{n}},$$
$$|(b_n^{\pm})'(x)| \lesssim n^{3/4} \log^3(1+n) e^{-c|x|/\sqrt{n}}$$

for all positive integers $n \in \mathbb{N}$.

The proof of such a result employs a mixture of the main ideas for the uniform bounds in [Radchenko and Viazovska 2019; Bondarenko et al. 2023], with the addition of an explicit computation of the best uniform constant bounding $|x|^k |b_n^{\pm}(x) + (b_n^{\pm})'(x)|$ in terms of k and n. In order to obtain such a constant, we employ ideas from characterizations of Gelfand–Shilov spaces, as in [Chung et al. 1996].

We remark that, with a modification of the growth lemma for Fourier coefficients of 2-periodic functions, we are able to obtain a slight improvement over the growth stated in Theorem 1.6. As, however, this modification does not yield any improvement on the perturbation range stated in Theorem 1.4, we postpone a more detailed discussion about it to Corollary 4.6 below.

1C. ***Applications.*** As a by-product of our method of proof for Theorem 1.4, we are able to deduce some interesting consequences in regard to some other interpolation formulae and uniqueness results.

Indeed, it is a not-so-difficult task to adapt the ideas employed before to the contexts of interpolation formulae for *odd* functions. As remarked by Radchenko and Viazovska, the following interpolation formula is available whenever $f : \mathbb{R} \to \mathbb{R}$ is odd and belongs to the Schwartz class:

$$f(x) = d_0^+(x)\frac{f'(0) + i\hat{f}'(0)}{2} + \sum_{n\geq 1}\left(c_n(x)\frac{f(\sqrt{n})}{\sqrt{n}} - \hat{c}_n(x)\frac{\hat{f}(\sqrt{n})}{\sqrt{n}}\right),$$

where the interpolating sequence $\{c_i\}_{i\geq 0}$ possesses analogous properties to those of $\{a_i\}_{i\geq 0}$, and the function

$$d_0^+(x) = \frac{\sin(\pi x^2)}{\sinh(\pi x)}$$

is odd and real and so it vanishes together with its Fourier transform at $\pm\sqrt{n},\ n \geq 0$.

With our techniques, we are able to prove an analogous result to Theorems 1.6 and 1.4 for the odd interpolation formula. Also, with our techniques, we are able to prove a version of Cohn–Kumar–Miller–Radchenko–Viazovska interpolation results with derivatives in dimensions 8 and 24 with *perturbed* nodes in a suitable range, as polynomial growth bounds for such interpolating functions are available in [Cohn et al. 2022]; see Theorems 5.11 and 5.13 for more details.

Another interesting application of our techniques delves a little deeper into functional analysis techniques. Indeed, in order to prove that the operator that takes the set of values $\{f(\sqrt{k})\}_{k\geq 0},\ \{\hat{f}(\sqrt{k})\}_{k\geq 0}$ to the sequences

$$\{f(\sqrt{k+\varepsilon_k})\}_{k\geq 0}, \quad \{\hat{f}(\sqrt{k+\varepsilon_k})\}_{k\geq 0}$$

is bounded and close to the identity on a suitable $\ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N})$ space, we explore two main options, which are *Schur's test* and the *Hilbert–Schmidt test*. Although there is no direct relation between them, Schur's test seems to hold, in generic terms, for more operators than the Hilbert–Schmidt test, and for that reason we employ the former in our proof of Theorem 1.4. On the other hand, the Hilbert–Schmidt test has the advantage that, whenever an operator is bounded in the Hilbert–Schmidt norm, it is automatically a *compact* operator. This allows us to use many more tools derived from the theory of Fredholm operators, and, in particular, deduce a sort of interpolation/uniqueness result in the case $\varepsilon_0 \neq 0$, which is excluded by Theorem 1.4 above; see Theorem 5.3 below for such an application.

The final interesting application of Theorem 1.4 and its techniques the we present is to the problem of *Fourier uniqueness for powers of integers*. In [Ramos and Sousa 2022], we have proven a preliminary result on conditions on $(\alpha, \beta),\ 0 < \alpha, \beta,\ \alpha + \beta < 1$, so that the only $f \in \mathcal{S}(\mathbb{R})$ such that

$$f(\pm n^{\alpha}) = \hat{f}(\pm n^{\beta}) = 0$$

is $f \equiv 0$. In particular, we prove that, if $\alpha = \beta$, then we can take $\alpha < 1 - \frac{\sqrt{2}}{2}$.

By an approximation argument, a careful analysis involving Laplace transforms and the perturbation techniques and results above, we are able to reprove such a result for $\alpha = \beta$ in the $\alpha < \frac{2}{9}$ range in the case f is real and even by a completely different method than that in [Ramos and Sousa 2022]. Although the current method does not yield any improvement over [Ramos and Sousa 2022, Theorem 1], we obtain additionally some *strong annihilation* properties of such pairs, in the form of Corollary 5.10, which are novel in that context.

Still on the subject of annihilation, we obtain two other interesting results.

Theorem 1.7. *For each $s>1$ sufficiently large, there are $\gamma>s$ and $\omega>0$ such that both inequalities*

$$\Big(\sum_{n\ge 0}(1+n)^s[|f(\sqrt{n})|^2+|\hat f(\sqrt{n})|^2]\Big)^{1/2}\lesssim \|f\|_{L^2((1+|x|)^\gamma)}+\|\hat f\|_{L^2((1+|x|)^\gamma)}, \tag{1-10}$$

$$\|f\|_{L^2((1+|x|)^s)}+\|\hat f\|_{L^2((1+|x|)^s)}\lesssim \Big(\sum_{n\ge 0}(1+n)^\omega[|f(\sqrt{n})|^2+|\hat f(\sqrt{n})|^2]\Big)^{1/2} \tag{1-11}$$

hold for each $f\in\mathcal{S}_{\text{even}}(\mathbb{R})$ real.

Corollary 1.8. *Let $\{\varepsilon_i\}_{i\in\mathbb{N}}$ satisfy the hypotheses of Theorem 1.4. Then for $s\gg 1$ sufficiently large, both inequalities*

$$\Big(\sum_{n\ge 0}(1+n)^s[|f(\sqrt{n+\varepsilon_n})|^2+|\hat f(\sqrt{n+\varepsilon_n})|^2]\Big)^{1/2}\lesssim \|f\|_{L^2((1+|x|)^\gamma)}+\|\hat f\|_{L^2((1+|x|)^\gamma)},$$

$$\|f\|_{L^2((1+|x|)^s)}+\|\hat f\|_{L^2((1+|x|)^s)}\lesssim \Big(\sum_{n\ge 0}(1+n)^\omega[|f(\sqrt{n+\varepsilon_n})|^2+|\hat f(\sqrt{n+\varepsilon_n})|^2]\Big)^{1/2}$$

hold for each $f\in\mathcal{S}_{\text{even}}(\mathbb{R})$ real, where ω,γ are as in Theorem 1.7.

We refer the reader to discussion in Section 5C for more precise definitions about annihilating pairs

We should remark that it has been recently communicated to us by Kulikov, Nazarov and Sodin (personal communication) that they have been able to significantly strengthen the results in [Ramos and Sousa 2022]. As a particular application of their results, they are able to obtain the whole range $\alpha+\beta<1$, conjectured in [loc. cit.]. In fact, they can say quite a bit more even in the "critical" case $\alpha+\beta=1$, constructing also suitable counterexamples to these uniqueness questions. It has also been communicated to us that they have obtained strong annihilating properties in such a range as well. In spite of that, we have decided to maintain this application of our work, as it contains interesting ideas that could be applied to other uniqueness problems of similar flavor. In particular, Theorem 1.7 and Corollary 1.8 are a novelty of this present work, and seem not to be included as a consequence of the results from Kulikov, Nazarov and Sodin.

1D. *Organization.* We comment briefly on the overall display of our results throughout the text. In Section 2 below, we discuss generalities on background results needed for the proofs of the main theorems, going over results in the theory of band-limited functions, modular forms and functional analysis. Next, in Section 3, we prove, in this order, Theorems 1.1, 1.2 and 1.3 about band-limited perturbed interpolation formulae. We then prove, in Section 4, Theorem 1.4, by first discussing the proof of Theorem 1.6 in Section 4A. We then discuss the applications of our main results and techniques in Section 5, and finish the manuscript with Section 6, talking about some possible refinements and open problems that arise from our discussion throughout the paper.

2. Preliminaries

2A. *Band-limited functions.* We start by recalling some basic facts about band-limited functions. Given a function $f\in L^2(\mathbb{R})$, we say that it is *band-limited* if its Fourier transform satisfies that $\operatorname{supp}(\hat f)\subset[-M,M]$ for some $M>0$. In this case, we say that f is *band-limited to* $[-M,M]$.

It is a classical result due to Paley and Wiener that a function $f \in L^2(\mathbb{R})$ is band-limited to $[-\sigma, \sigma]$ if and only if it is the restriction of an entire function $F : \mathbb{C} \to \mathbb{C}$ to the real axis, and the function F is of exponential type $2\pi\sigma$, i.e., for each $\varepsilon > 0$, there is C_ε such that

$$|F(z)| \leq C_\varepsilon e^{(2\pi\sigma+\varepsilon)|z|}$$

for all $z \in \mathbb{C}$. From now on we will abuse notation and let $F = f$ whenever there is no danger of confusion, and we may also write $f \in \mathrm{PW}_{2\pi\sigma}$ (Paley–Wiener space) to denote the space of functions with such properties.

Besides this fact, we will make use of some interpolation formulae for those functions. Namely:

(1) *Shannon–Whittaker interpolation formula*. For each $f \in L^2(\mathbb{R})$ band-limited to $\left[-\frac{1}{2}, \frac{1}{2}\right]$, the following formula holds:

$$f(x) = \sum_{n\in\mathbb{Z}} f(n)\,\mathrm{sinc}(x-n),$$

where $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$ and the sum above converges both in $L^2(\mathbb{R})$ and uniformly on compact sets of $\mathbb{C}$.

(2) *Vaaler interpolation formula*. For each $f \in L^2(\mathbb{R})$ band-limited to $[-1, 1]$, the following formula holds:

$$f(x) = \left(\frac{\sin \pi x}{\pi}\right)^2 \sum_{n\in\mathbb{Z}} \left[\frac{f(n)}{(x-n)^2} + \frac{f'(n)}{x-n}\right],$$

where the right-hand side converges both in $L^2(\mathbb{R})$ and uniformly on compact sets of $\mathbb{C}$.

For more details on these classical results, see, for instance, [Vaaler 1985; Littmann 2006; Paley and Wiener 1934; Shannon 1949; Whittaker 1915].

2B. ***Modular forms.*** In order to prove the improved estimates on the interpolation basis for the Radchenko–Viazovska interpolation result, we will need to make careful computations involving certain modular forms defining the interpolating functions. For that purpose, we gather some of the facts we will need in this subsection. For more information on the functions λ, J and the automorphy factors we just defined, we refer the reader to [Chandrasekharan 1985; Radchenko and Viazovska 2019, Section 2; Berndt and Knopp 2008; Zagier 2008].

We denote by $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the upper half-plane in $\mathbb{C}$. The special feature of this space is that the group $\mathrm{SL}_2(\mathbb{R})$ of matrices with real coefficients and determinant 1 acts naturally on it through Möbius transformations:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \quad z \in \mathbb{H} \quad \Longrightarrow \quad \gamma z = \frac{az+b}{cz+d} \in \mathbb{H}.$$

Indeed, it suffices to look at the action of the quotient $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$, since clearly the action by both matrices γ and $-\gamma$ induces the same Möbius transformation. Some elements of this group will be of special interest to us. Namely, we let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This already allows us to define the most valuable subgroup of $\mathrm{SL}_2(\mathbb{Z})$ for us: the group Γ_θ is defined then as the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by S and T^2. This group has 1 and ∞ as cusps, and its standard fundamental domain is given by

$$\mathcal{D} = \{z \in \mathbb{H} : |z| > 1, \mathrm{Re}(z) \in (-1, 1)\}.$$

With these at hand, we define *modular forms* for Γ_θ. For that purpose, we will use the following notation for the Jacobi theta series:

$$\vartheta(z, \tau) = \sum_{n \in \mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z).$$

We are interested in some of its *Nullwerte*, the so-called Jacobi theta series. These are defined in $\mathbb{H}$ by

$$\begin{aligned} \Theta_2(\tau) &= \exp\left(\tfrac{\pi}{4} i \tau\right) \vartheta\left(\tfrac{1}{2}\tau, \tau\right), \\ \Theta_3(\tau) &= \vartheta(0, \tau) (=: \theta(\tau)), \\ \Theta_4(\tau) &= \vartheta\left(\tfrac{1}{2}, \tau\right). \end{aligned}$$

These functions satisfy the identity $\Theta_3^4 = \Theta_2^4 + \Theta_4^4$. Moreover, under the action of the elements S and T of $\mathrm{SL}_2(\mathbb{Z})$, they transform as

$$\begin{aligned} (-iz)^{-1/2} \Theta_2(-1/z) &= \Theta_4(z), \quad \Theta_2(z+1) = \exp\left(\tfrac{\pi}{4} i\right) \Theta_2(z), \\ (-iz)^{-1/2} \Theta_3(-1/z) &= \Theta_3(z), \quad \Theta_3(z+1) = \Theta_4(z), \\ (-iz)^{-1/2} \Theta_4(-1/z) &= \Theta_2(z), \quad \Theta_4(z+1) = \Theta_3(z). \end{aligned} \tag{2-1}$$

These functions allow us to construct the classical lambda modular invariant given by

$$\lambda(z) = \frac{\Theta_2(z)^4}{\Theta_3(z)^4}.$$

Using $q := q(z) = e^{\pi i z}$, the lambda invariant can be alternatively rewritten as

$$\lambda(z) = 16q \times \prod_{k=1}^{\infty} \left(\frac{1 + q^{2k}}{1 + q^{2k-1}} \right)^8 = 16q - 128q^2 + 704q^3 + \cdots. \tag{2-2}$$

The function λ is also invariant under the action of elements of the subgroup $\Gamma(2) \subset \mathrm{SL}_2(\mathbb{Z})$ of all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ so that $a \equiv b \equiv 1 \bmod 2$, $c \equiv d \equiv 0 \bmod 2$, and $\lambda(z)$ never assumes the values 0 or 1 for $z \in \mathbb{H}$. Besides this invariance, (2-1) gives us immediately that

$$\lambda(z+1) = \frac{\lambda(z)}{\lambda(z) - 1}, \quad \lambda\left(-\frac{1}{z}\right) = 1 - \lambda(z). \tag{2-3}$$

We then define the following modular function for Γ_θ (which is a Hauptmodul for Γ_θ)

$$J(z) = \tfrac{1}{16} \lambda(z)(1 - \lambda(z)).$$

From (2-3), we obtain immediately that J is invariant under the action of elements of Γ_θ; i.e.,

$$J(z+2) = J(z), \quad J\left(-\frac{1}{z}\right) = J(z).$$

Other properties of the functions λ and J that we may eventually need will be proved throughout the text.

Finally, we mention that, for the proof in Section 4, we will need to use the so-called *θ-automorphy factor* defined, for $z \in \mathbb{H}$ and $\gamma \in \Gamma_\theta$, as

$$j_\theta(z, \gamma) = \frac{\theta(z)}{\theta(\gamma z)}.$$

We can then define a slash operator of weight $k/2$ to be

$$(f|_{k/2}\gamma)(z) = j_\theta(z, \gamma)^k f\left(\frac{az+b}{cz+d}\right),$$

where $\gamma = \left(\begin{smallmatrix} a & c \\ c & d \end{smallmatrix}\right)$. These slash operators induce other *sign* slash operators given by

$$(f|^{\varepsilon}_{k/2}\gamma) = \chi_\varepsilon(\gamma)(f|_{k/2}\gamma),$$

where we let χ_ε be the homomorphism of Γ_θ so that $\chi_\varepsilon(S) = \varepsilon$, $\chi_\varepsilon(T^2) = 1$.

2C. ***Functional analysis.*** We also recall some classical facts in functional analysis that will be useful throughout our proof.

As our main goal and strategy throughout this manuscript is to prove that a small perturbation of the identity is invertible, we must find ways to prove that the operators arising in our computations are bounded. To this end, we use two major criteria to prove boundedness — and therefore to prove smallness of the bounding constant. These are:

(1) *Hilbert–Schmidt test* [Brezis 2011, Chapter 6]. Let H be a (real or complex) Hilbert space, and let there be given a linear operator $T : H \to H$. If T satisfies additionally that

$$\sum_{i,j} |\langle Te_j, e_i\rangle|^2 < +\infty$$

for some orthonormal basis $\{e_i\}_{i\in\mathbb{Z}}$ of H, then the operator T is bounded. Moreover,

$$\|T\|^2_{H\to H} \le \sum_{i,j} |\langle Te_j, e_i\rangle|^2 =: \|T\|^2_{HS}.$$

(2) *Schur test* [Hedenmalm et al. 2000, Theorem 1.8]. Let $(a_{ij})_{i,j\ge 0}$ denote a (possibly infinite) matrix of complex numbers. Suppose that there are two sequences $\{v_i\}_{i\ge 0}$ and $\{w_i\}_{i\ge 0}$ of positive real numbers so that

$$\sum_{i\ge 0} |a_{ij}|w_i \le \lambda v_j, \quad \sum_{j\ge 0} |a_{ij}|v_j \le \mu q_i$$

for some positive constants $\mu, \lambda > 0$. Then the operator $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ given by $a_{ij} = \langle Te_i, e_j\rangle$ (where $\{e_i\}_{i\ge 0}$ denotes the standard orthonormal basis of $\ell^2(\mathbb{N})$) extends to a *bounded* linear operator. Moreover,

$$\|T\|_{\ell^2\to\ell^2} \le \sqrt{\mu\lambda}.$$

Both tests will play a major role in the deduction of the validity of perturbed interpolation versions of the Radchenko–Viazovska result. The main difference is that, while Schur's test generally gives one boundedness for more operators, the Hilbert–Schmidt test imposes stronger conditions on the operator. In fact, let us denote by $T \in \mathcal{HS}(H)$ the space of operators such that $\|T\|_{HS} < +\infty$. A classical consequence of this fact is that T is compact. This compactness will be used when proving that a suitable version of our interpolation results holds for small perturbations of the origin. See, for instance, [Brezis 2011, Chapter 6]

2D. ***Notation.*** We will use Vinogradov's modified notation throughout the text; that is, we write $A \lesssim B$ in the case there is an absolute constant $C > 0$ so that $A \le C \cdot B$. If the constant C depends on some set of parameters λ, we shall write $A \lesssim_\lambda B$.

On the other hand, we shall also use the big-$\mathcal{O}$ notation $f = \mathcal{O}(g)$ if there is an absolute constant C such that $|f| \le C \cdot g$, although the usage of this will be restricted mostly to sequences. We may occasionally use as well the standard Vinogradov notation $a \ll b$ to denote that there is a (relatively) *large* constant $C > 1$ such that $a \le C \cdot b$.

We shall also denote the spaces of sequences of complex numbers decaying polynomially by

$$\begin{aligned} \ell^2_s(\mathbb{Z}_+) &= \Big\{(a_n)_n \in \ell^2(\mathbb{Z}_+) : |a_0|^2 + \sum_{n\in\mathbb{N}} |a_n|^2 n^{2s} < +\infty\Big\}, \\ \ell^2_s(\mathbb{N}) &= \Big\{\{a_n\}_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}} |a_n|^2 n^{2s} < +\infty\Big\}, \end{aligned} \tag{2-4}$$

where $\mathbb{N} = \{1, 2, \dots\}$ denotes the set of natural numbers and $\mathbb{Z}_+$ denotes the nonnegative integers. We remind the reader that we always normalize the Fourier transform as in (1-1), i.e,

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i x\cdot\xi}\, \mathrm{d}x.$$

3. Perturbed interpolation formulae for band-limited functions

3A. ***Perturbed forms of the Shannon–Whittaker formula and Kadec's result.*** Fix a sequence $\varepsilon = \{\varepsilon_k\}_{k\in\mathbb{Z}}$ of real numbers such that $\sup_k |\varepsilon_k| < 1$. We wish to obtain a criterion based solely on the value of $L = \sup_n |\varepsilon_n|$ such that the sequence $\{n + \varepsilon_n\}_{n\in\mathbb{Z}}$ is completely interpolating in PW_π, i.e, for every sequence $a = \{a_n\} \in \ell^2(\mathbb{Z})$ there is a unique $f \in L^2(\mathbb{R})$ of exponential type $\tau(f) \le \pi$ that satisfies

$$f(n + \varepsilon_n) = a_n.$$

Our goal here is to obtain a simple proof of such a criterion going through new and simple ideas. We will fall short of the $\frac{1}{4}$ proven by Kadec by approximately 0.11, but it illustrates the power of our perturbation scheme and does not go through the theory of exponential bases.

In this particular case, we need to invert in $\ell^2(\mathbb{Z})$ the operator given by

$$A_\varepsilon(a)(n) = \sum_{k\in\mathbb{Z}} a_k \operatorname{sinc}(n + \varepsilon_n - k),$$

where

$$\operatorname{sinc}(x) = \frac{\sin \pi(x)}{\pi x}.$$

The fact A_ε is invertible will follow from proving that it is a close perturbation of the identity whenever L is sufficiently small.

3A1. *Auxiliary perturbations of the Hilbert transforms.* Given a sequence $a = \{a_k\}_{k\in\mathbb{Z}}$, we define the following operators, which are akin to the discrete Hilbert transform:

$$\mathcal{H}_\varepsilon(a)(n) = \sum_{k\neq n} \frac{(-1)^{n-k} a_k}{n+\varepsilon_n - k}, \quad \mathcal{H}_0(a)(n) = \sum_{k\neq n} \frac{(-1)^{n-k} a_k}{n-k}.$$

We start by comparing these two objects:

$$\begin{aligned}\mathcal{H}_0(a)(n) - \mathcal{H}_\varepsilon(a)(n) &= \sum_{k\neq n} (-1)^{n-k} a_k \left(\frac{1}{n-k} - \frac{1}{n+\varepsilon_n-k}\right) \\ &= \varepsilon_n \sum_{k\neq n} (-1)^{n-k} a_k \frac{1}{(n-k)(n+\varepsilon_n-k)}.\end{aligned}$$

This identity then gives us

$$\begin{aligned}|\mathcal{H}_0(a)(n) - \mathcal{H}_\varepsilon(a)(n)| &\leq |\varepsilon_n| \sum_{k\neq n} |a_k| \frac{1}{|n-k|^2} \frac{|n-k|}{|n+\varepsilon_n-k|} \\ &\leq \frac{|\varepsilon_n|}{1-|\varepsilon_n|} \sum_{k\neq n} |a_k| \frac{1}{|n-k|^2}.\end{aligned}$$

This means that, in norm, one can compare these two operators. Indeed, it is a classical result that the operator norm of $\mathcal{H}_0$ is π, and by Plancherel the operator norm of the transformation

$$\mathcal{S}(a) = \sum_{k\neq n} a_k \frac{1}{|n-k|^2}$$

is $\pi^2/3$. This in turn implies

$$\|\mathcal{H}_\varepsilon\| \leq \pi + \frac{\pi^2}{3} \frac{\sup_n |\varepsilon_n|}{1-\sup_n |\varepsilon_n|}. \tag{3-1}$$

3A2. *Norm estimates of the perturbation.* It is worth noticing the estimate (3-1) is very crude, as it is meant to depend only on $L = \sup_n |\varepsilon_n|$. For instance, if $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ is a constant sequence, then the norm $\|\mathcal{H}_\varepsilon\|$ is equal to π. We also note that the fact that we obtain invertibility by means of perturbations of small norm of an invertible operator does not take into account other factors, such as cancellation.

In order to apply our perturbation scheme to the operator A_ε, we need to bound the following family of operators:

$$P_\varepsilon(a)(n) = \sum_{k\in\mathbb{Z}} a_k (\operatorname{sinc}(n+\varepsilon_n-k) - \delta_{n,k}).$$

We may rewrite them as

$$\begin{aligned}P_\varepsilon(a)(n) =& (\operatorname{sinc}(\varepsilon_n) - 1) a_n + \sum_{k\neq n} a_k (\operatorname{sinc}(n+\varepsilon_n-k)) \\ =& (\operatorname{sinc}(\varepsilon_n) - 1) a_n + \sum_{k\neq n} a_k \frac{(-1)^{n-k} \sin \pi\varepsilon_n}{\pi(n+\varepsilon_n-k)}.\end{aligned}$$

This implies, on the other hand,

$$P_\varepsilon(a)(n) = (\operatorname{sinc}(\varepsilon_n) - 1)a_n + \left(\frac{\sin \pi \varepsilon_n}{\pi}\right)\mathcal{H}_\varepsilon(a)(n),$$

which in turn implies that

$$\begin{aligned}\|P_\varepsilon\| &\leq \sup_n |\operatorname{sinc}(\varepsilon_n) - 1| + \sup_n \left|\frac{\sin \pi\varepsilon_n}{\pi}\right| \|\mathcal{H}_\varepsilon\| \\ &\leq \sup_n |\operatorname{sinc}(\varepsilon_n) - 1| + \sup_n |\sin \pi\varepsilon_n| + \frac{\pi}{3}\frac{\sup_n |\sin \pi\varepsilon_n| \sup_n |\varepsilon_n|}{1 - \sup_n |\varepsilon_n|}.\end{aligned}$$

Since $A_\varepsilon = P_\varepsilon + \mathrm{Id}$, whenever

$$1 - \operatorname{sinc}(L) + |\sin \pi L| + \frac{\pi}{3}\frac{L \sin \pi L}{1 - L} < 1,$$

we will have that A_ε is invertible. In particular, a routine numerical evaluation implies that $L < 0.239$ satisfies the inequality above. Let then $A_\varepsilon^{-1} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the inverse of A_ε, which is continuous by the considerations above. We know, by the Shannon–Whittaker interpolation formula (1-2) that A_ε takes $\{f(k)\}_{k\in\mathbb{Z}}$, for $f \in \mathrm{PW}_\pi$, to $\{f(k+\varepsilon_k)\}_{k\in\mathbb{Z}}$. This is enough to prove the assertion about recovery, and as such implies that

$$\sum_{n\in\mathbb{Z}} |f(n+\varepsilon_n)|^2$$

is an equivalent norm to the usual L^2-norm on PW_π, by [Young 1980, Theorem 1.13].

Moreover, by writing

$$A_\varepsilon^{-1}(b)(k) = \sum_{n\in\mathbb{Z}} b_n \cdot \rho_{k,n},$$

we have immediately

$$\sum_{n\in\mathbb{Z}} f(n+\varepsilon_n)\rho_{k,n} = f(k), \tag{3-2}$$

and $\sup_n\bigl(\sum_{k\in\mathbb{Z}} |\rho_{k,n}|^2\bigr) \lesssim 1$. If $(A_\varepsilon^{-1})^* : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ denotes the adjoint of the inverse of A_ε, then we see that for any compact set $K \subset \mathbb{C}$ there is a constant $C = C_K$ such that

$$\begin{aligned}\|(A_\varepsilon^{-1})^*(\operatorname{sinc}_z(k))\|_{\ell^2(\mathbb{Z})} &\leq \|A_\varepsilon^{-1}\|_{\ell^2\to\ell^2} \|(\operatorname{sinc}_z(k))\|_{\ell^2(\mathbb{Z})} \\ &\leq C\|A_\varepsilon^{-1}\|_{\ell^2\to\ell^2},\end{aligned}$$

and C does not depend on $z \in K$ and we let $\operatorname{sinc}_x(k) := \operatorname{sinc}(x-k)$. Therefore, by letting $g_n(z) = \sum_{k\in\mathbb{Z}} \rho_{k,n} \operatorname{sinc}(z-k)$, we have

$$\sup_{z\in\mathbb{R}} \left(\sum_{n\in\mathbb{Z}} |g_n(z)|^2\right)^{1/2} \lesssim 1,$$

and thus, by the previous considerations, the sum $\sum_{n\in\mathbb{Z}} f(n+\varepsilon_n)g_n(z)$ converges absolutely by Cauchy–Schwarz. As $\langle (A_\varepsilon^{-1})^*(\operatorname{sinc}_z(k)), f(n+\varepsilon_n)\rangle = \langle \operatorname{sinc}_z(k), A_\varepsilon^{-1}(f(n+\varepsilon_n))\rangle = f(z)$ by Shannon–Whittaker, this implies

$$f(z) = \sum_{n\in\mathbb{Z}} f(n+\varepsilon_n) g_n(x),$$

where the convergence happens uniformly in compact sets, as desired.

This finishes the proof of Theorem 1.1.

3B. ***From Shannon to Vaaler*: *the proof of Theorem 1.2.*** We now concentrate on proving that the usual Shannon–Whittaker interpolation formula implies Vaaler's celebrated interpolation result [1985] with derivatives.

Indeed, as proving that the interpolation formula of Theorem 1.2 converges uniformly on compact sets of $\mathbb{C}$ is a routine computation, given that $\{a_k\}_{k\in\mathbb{Z}}, \{b_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, we shall omit this part and focus on proving that the asserted equality holds.

Given a sequence $a = \{a_k\}_{k\in\mathbb{Z}}$, we define the operators

$$\mathcal{H}(a)(k) = \frac{1}{\pi} \sum_{0\neq j\in\mathbb{Z}} \frac{a_{k-j}}{j} = \frac{1}{\pi} \sum_{k\neq j\in\mathbb{Z}} \frac{a_j}{k-j},$$

$$\mathcal{H}_1(a)(k) = \frac{1}{\pi} \sum_{j\in\mathbb{Z}} \frac{a_{k-j}}{j+\frac{1}{2}} = \frac{1}{\pi} \sum_{j\in\mathbb{Z}} \frac{a_j}{k-j+\frac{1}{2}}.$$

It is known that both $\mathcal{H}$ and $\mathcal{H}_1$ are bounded operators in $\ell^2(\mathbb{Z})$, with $\mathcal{H}_1$ being also unitary with $\mathcal{H}_2$ its inverse being given by

$$\mathcal{H}_2(a)(k) = -\frac{1}{\pi} \sum_{j\in\mathbb{Z}} \frac{a_{j-k}}{j-\frac{1}{2}} = \frac{1}{\pi} \sum_{j\in\mathbb{Z}} \frac{a_j}{j-k+\frac{1}{2}}.$$

Given a function $f \in \mathrm{PW}_\pi$, as a consequence of the Shannon–Whittaker interpolation formula we obtain, for every $k \in \mathbb{Z}$, that

$$f'(k) = \sum_{j\neq k} \frac{f(j)}{k-j} (-1)^{k-j}.$$

We consider three sequences

$$a(k) = f(2k-1), \quad b(k) = f(2k), \quad c(k) = f'(2k).$$

We have, thus,

$$\begin{aligned} c(k) = f'(2k) = \sum_{j\neq 2k} \frac{f(j)}{2k-j} (-1)^{2k-j} &= \frac{1}{2} \sum_{j\neq k} \frac{f(2j)}{k-j} - \frac{1}{2} \sum_{j\in\mathbb{Z}} \frac{f(2j-1)}{k-j+\frac{1}{2}} \\ &= \frac{1}{2} \sum_{j\neq k} \frac{b(j)}{k-j} - \frac{1}{2} \sum_{j\in\mathbb{Z}} \frac{a(j)}{k-j+\frac{1}{2}} = \frac{\pi}{2} \mathcal{H}(b)(k) - \frac{\pi}{2} \mathcal{H}_1(a)(k). \end{aligned}$$

This means that, for every $k \in \mathbb{Z}$,

$$\mathcal{H}_1(a)(k) = \mathcal{H}(b)(k) - \frac{2}{\pi} c(k).$$

Since $\mathcal{H}_2$ is the inverse of $\mathcal{H}_1$, this can be rewritten as

$$a(k) = (\mathcal{H}_2 \circ \mathcal{H})(b)(k) - \frac{2}{\pi} \mathcal{H}_2(c)(k).$$

We know, by the Shannon–Whittaker interpolation formula, that

$$f(x)=\sum_{k\in\mathbb{Z}} f(k)\frac{\sin\pi(x-k)}{\pi(x-k)}.$$

This implies, on the other hand,

$$\begin{aligned}
f(x)&=\sum_{k\in\mathbb{Z}} f(2k)\frac{\sin\pi(x-2k)}{\pi(x-2k)}+\sum_{k\in\mathbb{Z}}\Big[(\mathcal{H}_2\circ\mathcal{H})(b)(k)-\frac{2}{\pi}\mathcal{H}_2(c)(k)\Big]\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}\\
&=\sum_{k\in\mathbb{Z}} b(k)\frac{\sin\pi x}{\pi(x-2k)}+\sum_{k\in\mathbb{Z}}(\mathcal{H}_2\circ\mathcal{H})(b)(k)\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}-\frac{2}{\pi}\sum_{k\in\mathbb{Z}}\mathcal{H}_2(c)(k)\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}\\
&=A(x)+B(x)+C(x).
\end{aligned}$$

We shall investigate each term A, B and C thoroughly in order to obtain our final result.

3B1. *Determining C.* By considering the family of functions $h_j\in \mathrm{PW}_\pi$ — which satisfy the important property $h_j(k)=0$ if $k\in 2\mathbb{Z}$ — given by

$$h_j(z)=\frac{\sin^2\left(\frac{\pi}{2}z\right)}{\pi^2(z-2j)},$$

we obtain

$$\begin{aligned}
C(x)&=-2\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\frac{f'(2j)}{\pi^2\left(j-k+\frac{1}{2}\right)}\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}\\
&=4\sum_{j\in\mathbb{Z}} f'(2j)\sum_{k\in\mathbb{Z}}\frac{1}{\pi^2((2k-1)-2j)}\frac{\sin\pi(x-(2k-1))}{\pi(x-(2k-1))}\\
&=4\sum_{j\in\mathbb{Z}} f'(2j)\sum_{k\in\mathbb{Z}} h_j(2k-1)\frac{\sin\pi(x-(2k-1))}{\pi(x-(2k-1))}\\
&=4\sum_{j\in\mathbb{Z}} f'(2j)\sum_{k\in\mathbb{Z}} h_j(k)\frac{\sin\pi(x-k)}{\pi(x-k)}.
\end{aligned}$$

Notice that one can use Fubini's theorem to justify all the changes of order of summation by the fact that $h_j\in\mathrm{PW}_\pi$. By applying the Shannon–Whittaker interpolation to h_j, we have

$$C(x)=4\sum_{j\in\mathbb{Z}} f'(2j)\frac{\sin^2\left(\frac{\pi}{2}x\right)}{\pi^2(x-2j)}.$$

3B2. *Determining B.* For the second term, we expand

$$\begin{aligned}
B(x)&=\sum_{k\in\mathbb{Z}}\mathcal{H}_2\circ\mathcal{H}(b)(k)\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}\\
&=\frac{1}{\pi}\sum_{k\in\mathbb{Z}}\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}\sum_j\frac{\mathcal{H}(b)(j)}{j-k+\frac{1}{2}}\\
&=\frac{1}{\pi^2}\sum_{k\in\mathbb{Z}}\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}\sum_j\sum_{l\neq j}\frac{b(l)}{\left(j-k+\frac{1}{2}\right)(j-l)}.
\end{aligned}$$

By Fubini's theorem, this implies

$$\begin{aligned} B(x) =& \frac{1}{\pi^2}\sum_{l\in\mathbb{Z}} b(l)\sum_{j\neq l}\frac{1}{j-l}\sum_{k\in\mathbb{Z}}\frac{1}{j-k+\frac{1}{2}}\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)} \\ =& \frac{1}{\pi^2}\sum_{l\in\mathbb{Z}} b(l)\sum_{j\neq l}\frac{2}{j-l}\sum_{k\in\mathbb{Z}}\frac{1}{2j-2k+1}\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)} \\ =& \frac{1}{\pi^2}\sum_{l\in\mathbb{Z}} b(l)\sum_{j\neq l}\frac{2}{j-l}\frac{\sin^2\left(\frac{\pi}{2}x\right)}{2j-x} = \frac{\sin^2\left(\frac{\pi}{2}x\right)}{\pi^2}\sum_{l\in\mathbb{Z}} b(l)\sum_{j\neq 0}\frac{1}{j\left(j+l-\frac{1}{2}x\right)}. \end{aligned}$$

But it is a well-known fact that the summation formula

$$\sum_{j\neq 0}\frac{1}{j(j+z)} = \frac{\psi(1+z)-\psi(1-z)}{z}$$

holds, where $\psi(z) = \frac{d}{dz}\log\Gamma(z)$ is the digamma function. This implies

$$B(x) = \frac{2\sin^2\left(\frac{\pi}{2}x\right)}{\pi^2}\sum_{l\in\mathbb{Z}} b(l)\frac{\psi\left(1+l-\frac{1}{2}x\right)-\psi\left(1-l+\frac{1}{2}x\right)}{2l-x}.$$

3B3. *Determining* $A+B$. Using that $\sin(2x) = 2\sin x\cos x$, we obtain

$$A(x) = -\frac{2\sin^2\left(\frac{\pi}{2}x\right)}{\pi^2}\sum_{l\in\mathbb{Z}} b(l)\frac{\pi\cot\left(\frac{\pi}{2}x\right)}{2l-x}.$$

The digamma function satisfies the functional equations

$$\begin{aligned} \psi(1-z) &= \psi(z)+\pi\cot\pi z, \\ \psi(1+z) &= \psi(z)+1/z. \end{aligned}$$

Using these relations with $z = \frac{1}{2}x - l$ in the equations above, we obtain readily

$$A(x)+B(x) = \frac{4\sin^2\left(\frac{\pi}{2}x\right)}{\pi^2}\sum_{l\in\mathbb{Z}} b(l)\frac{1}{(x-2l)^2}.$$

3B4. $A+B+C$. Summing the analysis undertaken for the terms above, we have

$$f(x) = A(x)+B(x)+C(x) = \frac{4\sin^2\left(\frac{\pi}{2}x\right)}{\pi^2}\sum_{k\in\mathbb{Z}}\left\{\frac{f(2k)}{(x-2k)^2}+\frac{f'(2k)}{x-2k}\right\}.$$

This finishes the proof of Theorem 1.2.

3C. ***Perturbed interpolation formulae with derivatives.*** By the arguments in the previous section, the formula we just derived for $\mathrm{PW}_{2\pi}$, i.e.,

$$f(x) = \frac{\sin^2(\pi x)}{\pi^2}\sum_{k\in\mathbb{Z}}\left\{\frac{f(k)}{(x-k)^2}+\frac{f'(k)}{x-k}\right\},$$

converges in compact sets of $\mathbb{C}$. We fix, for shortness, the notation

$$g(x)=\frac{\sin^2(\pi x)}{\pi^2x^2},\quad h(x)=\frac{\sin^2(\pi x)}{\pi^2 x},$$

which means we can read Vaaler's interpolation as

$$f(x)=\sum_{k\in\mathbb{Z}}\{f(k)g(x-k)+f'(k)h(x-k)\}.$$

Because of uniform convergence, we can differentiate term by term in the above formula. This implies

$$f'(x)=\sum_{k\in\mathbb{Z}}\{f(k)g'(x-k)+f'(k)h'(x-k)\}.$$

We record, for completeness, the formulae for the derivatives of g and h. For $x\notin\mathbb{Z}$ we have

$$\begin{aligned} g'(x)&=\frac{2\sin(\pi x)(\pi x\cos(\pi x)-\sin(\pi x))}{\pi^2x^3},\\ h'(x)&=\frac{\sin(\pi x)(2\pi x\cos(\pi x)-\sin(\pi x))}{\pi^2x^2},\end{aligned}$$

and, for $n\in\mathbb{Z}$,

$$g(n)=h'(n)=0,\quad g'(n)=h(n)=\delta_0.$$

Our goal now is to invert the operator $\mathcal{A}=\mathcal{A}_\varepsilon$ defined in $\ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z})$ by

$$\begin{aligned}\mathcal{A}_1(a,b)_n&=\sum_{k\in\mathbb{Z}}a_k\cdot g(n+\varepsilon_n-k)+\sum_{k\in\mathbb{Z}}b_k\cdot h(n+\varepsilon_n-k),\\ \mathcal{A}_2(a,b)_n&=\sum_{k\in\mathbb{Z}}a_k\cdot g'(n+\varepsilon_n-k)+\sum_{k\in\mathbb{Z}}b_k\cdot h'(n+\varepsilon_n-k),\end{aligned}\tag{3-3}$$

where $\mathcal{A}(a,b)=(\mathcal{A}_1(a,b),\mathcal{A}_2(a,b))$ for $(a,b)\in\ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z})$. Furthermore, we wish to establish a criterion that depends only on $L=\sup|\varepsilon_n|$. For that purpose, we estimate when the operator norm of $\mathcal{A}_\varepsilon-\mathrm{Id}$ from $\ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z})$ to itself is small, in terms of L.

3C1. *Auxiliary perturbations for the derivative case.* Given a sequence $a=\{a_k\}_{k\in\mathbb{Z}}$, we define the operators

$$\mathcal{H}^p_\varepsilon(a)_n=\sum_{k\neq n}\frac{a_k}{(n+\varepsilon_n-k)^p},$$

and denote by $\mathcal{H}^p_0$ the operator associated to the sequence $\varepsilon_n=0$ for all $n\in\mathbb{Z}$. In an analogous manner to the proof of Theorem 1.1, we compare

$$\begin{aligned}\mathcal{H}^p_0(a)_n-\mathcal{H}^p_\varepsilon(a)_n&=\sum_{k\neq n}a_k\left(\frac{1}{(n-k)^p}-\frac{1}{(n+\varepsilon_n-k)^p}\right)\\ &=\sum_{j=0}^{p-1}\binom{p}{j}\varepsilon_n^{p-j}\sum_{k\neq n}\frac{a_k}{(n+\varepsilon_n-k)^p(n-k)^{p-j}}.\end{aligned}$$

Therefore,

$$|\mathcal{H}_0^p(a)_n - \mathcal{H}_\varepsilon^p(a)_n| \le \sum_{j=0}^{p-1} \binom{p}{j} |\varepsilon_n|^{p-j} \sum_{k \ne n} \frac{a_k}{|n-k|^{2p-j}} \frac{|n-k|^p}{(|n-k|-|\varepsilon_n|)^p}$$
$$\le \frac{1}{(1-|\varepsilon_n|)^p} \sum_{j=0}^{p-1} \binom{p}{j} |\varepsilon_n|^{p-j} \mathcal{S}^{2p-j}(a^*)_n,$$

where

$$\mathcal{S}^q(a)_n = \sum_{k \ne n} \frac{a_k}{|n-k|^q}$$

and $a^* = (|a_n|)$. Since $\mathcal{S}^{q+1}(a^*)_n \le \mathcal{S}^q(a^*)_n$, we have

$$|\mathcal{H}_0^p(a)_n - \mathcal{H}_\varepsilon^p(a)_n| \le \frac{\mathcal{S}^{p+1}(a^*)_n}{(1-|\varepsilon_n|)^p} \sum_{j=0}^{p-1} \binom{p}{j} |\varepsilon_n|^{p-j} = \left(\frac{(1+|\varepsilon_n|)^p - 1}{(1-|\varepsilon_n|)^p} \right) \mathcal{S}^{p+1}(a^*)_n.$$

This means that we have the following estimate on the norm of the perturbed operator:

$$\|\mathcal{H}_\varepsilon^p\| \le \gamma_p(L), \tag{3-4}$$

where we let

$$\gamma_p(L) = \|\mathcal{H}_0^p\| + \frac{(1+L)^p - 1}{(1-L)^p} \|\mathcal{S}^{p+1}\|.$$

Now, in order to estimate the value of $\gamma_p(L)$, we resort to [Littmann 2006, Corollary 2], which gives us

$$\|\mathcal{H}_0^p\| = \frac{(2\pi)^m b_m}{m!},$$

where b_m is the maximum of $|B_m(x)|$ when $x \in [0, 1]$, and B_m denotes the m-th Bernoulli polynomial.[1] Therefore,

$$\|\mathcal{H}_0^1\| = \pi, \quad \|\mathcal{H}_0^2\| = \frac{\pi^2}{3}, \quad \|\mathcal{H}_0^3\| = \frac{\pi^3}{9\sqrt{3}}.$$

On the other hand, by Plancherel's theorem it is easy to see that

$$\|\mathcal{S}^p\| = 2\zeta(p).$$

Joining all these data into (3-4), we obtain

$$\begin{aligned} \|\mathcal{H}_\varepsilon^1\| &\le \pi + \left(\frac{L}{1-L} \right) \frac{\pi^2}{3}, \\ \|\mathcal{H}_\varepsilon^2\| &\le \frac{\pi^2}{3} + 2\left(\frac{L^2+2L}{(1-L)^2} \right) \zeta(3), \\ \|\mathcal{H}_\varepsilon^3\| &\le \frac{\pi^3}{9\sqrt{3}} + \left(\frac{L^3+3L^2+3L}{(1-L)^3} \right) \frac{\pi^4}{45}. \end{aligned} \tag{3-5}$$

[1] It is worth mentioning that in [Carneiro et al. 2013, Corollary 22] the authors also obtain the same bounds.

3C2. *Norm estimates of the perturbations in the derivative case.* In order to invert the operator $\mathcal{A}_\varepsilon$, we estimate the norm of $\mathcal{P}_\varepsilon = \mathcal{A}_\varepsilon - \mathrm{Id} = (\mathcal{P}_1, \mathcal{P}_1)$, where

$$\begin{aligned}\mathcal{P}_1(a,b)_n &= \sum_{k\in\mathbb{Z}} a_k\cdot(g(n+\varepsilon_n-k)-\delta_{n,k}) + \sum_{k\in\mathbb{Z}} b_k\cdot h(n+\varepsilon_n-k),\\ \mathcal{P}_2(a,b)_n &= \sum_{k\in\mathbb{Z}} a_k\cdot g'(n+\varepsilon_n-k) + \sum_{k\in\mathbb{Z}} b_k\cdot(h'(n+\varepsilon_n-k)-\delta_{n,k}).\end{aligned}\tag{3-6}$$

By a straightforward calculation,

$$\begin{aligned}\mathcal{P}_1(a,b)_n &= (g(\varepsilon_n)-1)a_n + \frac{\sin(\pi\varepsilon_n)^2}{\pi^2}\mathcal{H}^2_\varepsilon(a)_n + h(\varepsilon_n)b_n + \frac{\sin(\pi\varepsilon_n)^2}{\pi^2}\mathcal{H}^1_\varepsilon(b)_n,\\ \mathcal{P}_2(a,b)_n &= g'(\varepsilon_n)a_n + \frac{2\sin(\pi\varepsilon_n)(\pi\varepsilon_n\cos(\pi\varepsilon_n)-\sin(\pi\varepsilon_n))}{\pi^2}\mathcal{H}^3_\varepsilon(a)\\ &\qquad + (h'(\varepsilon_n)-1)b_n + \frac{\sin(\pi\varepsilon_n)(2\pi\varepsilon_n\cos(\pi\varepsilon_n)-\sin(\pi\varepsilon_n))}{\pi^2}\mathcal{H}^2_\varepsilon(b).\end{aligned}\tag{3-7}$$

Thus,

$$\|\mathcal{P}_\varepsilon\| \le \sqrt{2}\max\{|g(L)-1|, |h'(L)-1|, |g'(L)|, |h(L)|\} + \frac{\sin(\pi L)^2}{\pi^2}\|\mathcal{G}_\varepsilon\|,$$

where $\mathcal{G}_\varepsilon = (\mathcal{G}^1_\varepsilon, \mathcal{G}^2_\varepsilon)$ and

$$\begin{aligned}\mathcal{G}^1_\varepsilon(a,b)_n =& \mathcal{H}^2_\varepsilon(a)_n + \mathcal{H}^1_\varepsilon(b)_n,\\ \mathcal{G}^2_\varepsilon(a,b)_n =& \frac{2(\pi\varepsilon_n\cos(\pi\varepsilon_n)-\sin(\pi\varepsilon_n))}{\sin(\pi\varepsilon)}\mathcal{H}^3_\varepsilon(a) + \frac{(2\pi\varepsilon_n\cos(\pi\varepsilon_n)-\sin(\pi\varepsilon_n))}{\sin(\pi\varepsilon)}\mathcal{H}^2_\varepsilon(b).\end{aligned}\tag{3-8}$$

By taking $L < \frac{1}{4}$ and using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\frac{\|\mathcal{G}_\varepsilon\|^2}{2} &\le \max\{\|\mathcal{H}^1_\varepsilon\|, \|\mathcal{H}^2_\varepsilon\|\}^2\\ &\qquad + \max\left\{\left(\frac{2(\pi L\cos(\pi L)-\sin(\pi L))}{\sin(\pi L)}\right)^2\|\mathcal{H}^3_\varepsilon\|^2, \left(\frac{(2\pi L\cos(\pi L)-\sin(\pi L))}{\sin(\pi L)}\right)^2\|\mathcal{H}^2_\varepsilon\|^2\right\}\\ &\le \max\{\gamma_1(L)^2, \gamma_2(L)^2\}\\ &\qquad + \max\left\{\left(\frac{2(\pi L\cos(\pi L)-\sin(\pi L))}{\sin(\pi L)}\right)^2\gamma_3(L)^2, \left(\frac{(2\pi L\cos(\pi L)-\sin(\pi L))}{\sin(\pi L)}\right)^2\gamma_2(L)^2\right\}.\end{aligned}$$

We note that we have abused the notation $\|\mathcal{G}_\varepsilon\|$ to denote the operator norm of $\mathcal{G}_\varepsilon$ when defined on $\ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z})$. One can further check that, for $0 \le L < \frac{1}{4}$,

$$|g(L)-1| < |h'(L)-1|, \quad |h(L)| < |g'(L)|, \quad \gamma_1(L)^2 < \gamma_2(L)^2,$$
$$\left(\frac{2(\pi L\cos(\pi L)-\sin(\pi L))}{\sin(\pi L)}\right)^2\gamma_3(L)^2 < \left(\frac{(2\pi L\cos(\pi L)-\sin(\pi L))}{\sin(\pi L)}\right)^2\gamma_2(L)^2,$$

which means, in turn,

$$\|\mathcal{G}_\varepsilon\| \le \gamma_2(L)\sqrt{2\left(1+\left(\frac{(2\pi L\cos(\pi L)-\sin(\pi L))}{\sin(\pi L)}\right)^2\right)},$$

and directly implies the estimate

$$\|\mathcal{P}_\varepsilon\| \le 1 - \frac{\sin(\pi L)(2\pi L\cos(\pi L) - \sin(\pi L))}{\pi^2 L^2} + \frac{2\sin(\pi L)(\sin(\pi L) - \pi L\cos(\pi L))}{\pi^2 L^3} + \frac{\sin(\pi L)^2}{\pi^2}\left(\frac{\pi^2}{3} + 2\left(\frac{L^2+2L}{(1-L)^2}\right)\zeta(3)\right)\sqrt{2\left(1 + \left(\frac{(2\pi L\cos(\pi L) - \sin(\pi L))}{\sin(\pi L)}\right)^2\right)}.$$

By evaluating the last expression on the right-hand side above numerically, we obtain that we can go up to $L < 0.111$ and maintain $\|\mathcal{P}_\varepsilon\| < 1$. By invoking again [Young 1980, Theorem 1.13], we see immediately that

$$\sum_{n\in\mathbb{Z}} (|f(n+\varepsilon_n)|^2 + |f'(n+\varepsilon_n)|^2)$$

yields an equivalent norm for $\mathrm{PW}_{2\pi}$, as long as $\sup_n |\varepsilon_n| < 0.111$.

Moreover, as $\mathcal{A}_\varepsilon^{-1} : \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$ is bounded, the same argument as in the proof of Theorem 1.1 shows that there are $\varrho_{k,n},\ \vartheta_{k,n},\ \varrho'_{k,n},\ \vartheta'_{k,n}$ such that

$$\begin{aligned} f(k) &= \sum_{n\in\mathbb{Z}} f(n+\varepsilon_n)\varrho_{k,n} + f'(n+\varepsilon_n)\vartheta_{k,n}, \\ f'(k) &= \sum_{n\in\mathbb{Z}} f(n+\varepsilon_n)\varrho'_{k,n} + f'(n+\varepsilon_n)\vartheta'_{k,n}, \end{aligned} \tag{3-9}$$

and

$$\sup_n \left(\sum_{k\in\mathbb{Z}} \{|\varrho_{k,n}|^2 + |\vartheta_{k,n}|^2 + |\varrho'_{k,n}|^2 + |\vartheta'_{k,n}|^2\} \right) \lesssim 1.$$

By using the adjoint $(\mathcal{A}_\varepsilon^{-1})^* : \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$ in an analogous manner to that of the proof of Theorem 1.1 together with (3-9) and (1-7), we obtain the asserted existence of the functions $g_n, h_n \in \mathrm{PW}_{2\pi}$ so that

$$f(x) = \sum_{n\in\mathbb{Z}} f(n+\varepsilon_n)g_n(x) + f'(n+\varepsilon_n)h_n(x),$$

where the right-hand side converges absolutely, as desired. This proves the desired version of Vaaler's interpolation formula with perturbed nodes, given in Theorem 1.3.

4. Perturbed Fourier interpolation on the real line

4A. ***Improved estimates on the interpolation basis.*** As our goal is to obtain versions of the formula

$$f(x) = \sum_{n\ge 0} [f(\sqrt{n})a_n(x) + \hat{f}(\sqrt{n})\hat{a}_n(x)]$$

with perturbed nodes $\sqrt{k+\varepsilon_k}$ deviating from $\sqrt{k}$ as much as possible, and in order to run our argument of estimating the operator norm of a perturbation of the identity, we will need better decay estimates for the interpolating functions a_n than the ones readily available in the literature. In [Radchenko and Viazovska 2019, Section 5], the authors prove that a_n/n^2 is uniformly bounded in $n \ge 0,\ x \in \mathbb{R}$. In order to be able to make the perturbations larger, we need to improve that result substantially, as even the refined bound

$|a_n| = \mathcal{O}(n^{1/4} \log^3(1+n))$ from [Bondarenko et al. 2023] does not seem to be enough for our purposes. This first subsection is, therefore, devoted to the proof of Theorem 1.6.

A tool of major importance in our proof is the Fourier characterization of Gelfand–Shilov spaces. These are spaces where, in a nutshell, both the function and Fourier transform decay as fast as the negative exponential of a certain monomial. Several results connect these spaces with specific decay for both the function and its Fourier transform. See, e.g., [Chung et al. 1996, Theorem 2.3] for more details.

In what follows, we will use the idea behind the characterization described in [Chung et al. 1996]: from bounds for certain L^2-norms of derivatives of f and $\hat{f}$, we run an optimization procedure to obtain decay bounds in both space and frequency. This will be achieved through careful estimates involving the reproducing functions of the interpolation basis $\{a_n\}_{n\geq 0}$, which joins elements of classical analysis and estimates for modular forms.

Indeed, let $\varepsilon \in \{\pm\}$ be a sign. In [Radchenko and Viazovska 2019], the authors consider the generating functions

$$\sum_{n=0}^{\infty} g_n^{\varepsilon}(z) e^{i\pi n\tau} =: K_{\varepsilon}(\tau, z), \tag{4-1}$$

where g_n^{ε} are weakly holomorphic modular forms of weight $\frac{3}{2}$ with growth and coefficient properties so that the functions

$$b_n^{\varepsilon}(x) = \frac{1}{2}\int_{-1}^{1} g_n^{\varepsilon}(z) e^{i\pi x^2 z}\,\mathrm{d}z$$

are eigenvectors of the Fourier transform associated to the eigenvalues ε satisfying that $b_n^{\pm} = a_n \pm \hat{a}_n$ for $\{a_n\}_{n\geq 0}$ defined as in (1-3).

These functions satisfy (see [Radchenko and Viazovska 2019, Proposition 1])

$$\begin{gathered}
b_m^{\varepsilon}(\sqrt{n}) = \delta_{n,m} \quad \text{if } n \geq 1,\, m \geq 0, \\
b_m^{+}(0) = \delta_{m,0} \quad \text{if } m \geq 0, \\
b_0^{-} = 0, \quad b_0^{+}(\sqrt{n}) = \delta_{n,0} \quad \text{if } n \geq 0, \\
b_m^{-}(0) = -2 \quad \text{if } m = k^2 \text{ for some } k \in \mathbb{Z}_{\geq 1}, \\
b_m^{-}(0) = 0 \quad \text{otherwise}.
\end{gathered} \tag{4-2}$$

Moreover, we mention for completeness the following result regarding K_{ε}. We refer the reader to [Radchenko and Viazovska 2019] for its proof.

Proposition 4.1 [Radchenko and Viazovska 2019, Theorem 3]. *For any fixed $z \in \mathbb{H}$, there is $y_0 > 0$ so that for all $\tau \in \mathbb{H}$ with* $\mathrm{Im}(\tau) > y_0$, *the series on the left-hand side of* (4-1) *converges. Under these assumptions, we have the following equalities for the kernels*:

$$\begin{aligned}
K_{+}(\tau, z) &= \frac{\theta(\tau)(1-2\lambda(\tau))\theta(z)^3 J(z)}{J(z) - J(\tau)}, \\
K_{-}(\tau, z) &= \frac{\theta(\tau) J(\tau)\theta(z)^3(1-2\lambda(z))}{J(z) - J(\tau)},
\end{aligned} \tag{4-3}$$

where θ, J and λ are as previously defined. In particular, $K_{\varepsilon}(\tau, z)$ are meromorphic functions with poles at $\tau \in \Gamma_{\theta} z$.

The authors then define the natural candidate for the generating function for the $\{b_n^\varepsilon\}_{n\geq 0}$ to be

$$F_\varepsilon(\tau, x) = \frac{1}{2}\int_{-1}^{1} K_\varepsilon(\tau, z) e^{i\pi x^2 z}\,\mathrm{d}z, \tag{4-4}$$

where the contour is the semicircle in the upper half-plane that passes through -1 and 1, which is defined, a priori, for each fixed $x \in \mathbb{R}$ and $\tau \in \{z \in \mathbb{H} : \text{for all } k \in \mathbb{Z},\ |z - 2k| > 1\} \supset \mathcal{D} + 2\mathbb{Z}$, where $\mathcal{D}$ is the standard fundamental domain for Γ_θ. By Proposition 4.1, there holds that, whenever $\operatorname{Im}(\tau) > 1$,

$$F_\varepsilon(\tau, x) = \sum_{n=0}^{\infty} b_n^\varepsilon(x) e^{i\pi n\tau}. \tag{4-5}$$

As $F_\varepsilon(\tau, x)$ admits an analytic continuation to $\mathbb{H}$ (see [Radchenko and Viazovska 2019, Proposition 2]), they are able to extend (4-5) to the entire upper half-space $\mathbb{H}$. Moreover, the following functional equations hold:

$$F_\varepsilon(\tau, x) - F_\varepsilon(\tau + 2, x) = 0,$$
$$F_\varepsilon(\tau, x) + \varepsilon(-i\tau)^{-1/2} F_\varepsilon\Big(-\frac{1}{\tau}, x\Big) = e^{i\pi\tau x^2} + \varepsilon(-i\tau)^{-1/2} e^{i\pi(-1/\tau)x^2}.$$

The proof of Theorem 1.6 follows the same essential philosophy as the proof of [Radchenko and Viazovska 2019, Theorem 4]: in order to bound each of the terms $b_n^\pm$, we bound, uniformly on $x \in \mathbb{R}$, the analytic function $F_\pm(\tau, x)$. Relating the two bounds is achieved by employing the idea behind the proof of the following lemma, originally attributed to Hecke (see for instance [Radchenko and Viazovska 2019, Lemma 1] and [Berndt and Knopp 2008, Lemma 2.2(ii)] for a proof).

Lemma 4.2. *Let $f : \mathbb{H} \to \mathbb{C}$ be a 2-periodic analytic function admitting an absolutely convergent Fourier expansion*

$$f(\tau) = \sum_{n\geq 0} c_n e^{i\pi n\tau}.$$

Suppose, additionally, that for some $\alpha > 0$ it satisfies that $|f(\tau)| \leq C \operatorname{Im}(\tau)^{-\alpha}$ for $\operatorname{Im}(\tau) < c_0$. Then there is $\widetilde{C} > 0$, depending only on C and α, such that for all $n > 1/c_0$

$$|c_n| \leq \widetilde{C} n^\alpha.$$

Moreover, there is $C' > 0$, depending only on C and α, such that if $n > \alpha/(\pi c_0)$, the improved estimate

$$|c_n| \leq C' \Big(\frac{e\pi}{\alpha}\Big)^\alpha n^\alpha$$

holds.

Before proving Theorem 1.6, we need one more crucial tool in our analysis. Indeed, we consider the functions

$$F_\varepsilon^k(\tau, x) := x^k F_\varepsilon(\tau, x).$$

By Lemma 4.2, if we prove that, for some $\Delta > 0$,

$$|F_\varepsilon^k(\tau, x)| \leq C^k (k!) \operatorname{Im}(\tau)^{-k/2-\Delta} \tag{4-6}$$

for all $k \geq 1$, then we will have

$$\sup_{x\in\mathbb{R}} |x^k b_n^\varepsilon(x)| \leq \widetilde{C}^k n^\Delta n^{k/2}(k!).$$

As $b_n^\varepsilon = \varepsilon \hat{b}_n$, the strategy of relating norms of derivatives with Fourier decay will then imply that each of the functions b_n^ε satisfies

$$|b_n^\varepsilon(x)| \lesssim n^\Delta e^{-\theta|x|/\sqrt{n}},$$

which is the content of Theorem 1.6. Therefore, we focus on proving a suitable version of (4-6). By the functional equation for F_ε, we see that F_ε^k is a 2-periodic function on $\mathbb{H}$ that satisfies the functional equation

$$F_\varepsilon^k(\tau, x) + \varepsilon(-i\tau)^{-1/2} F_\varepsilon^k(-1/\tau, x) = x^k(e^{i\pi\tau x^2} + \varepsilon(-i\tau)^{-1/2} e^{i\pi(-1/\tau)x^2}). \tag{4-7}$$

The strategy, in analogy to that in [Radchenko and Viazovska 2019], is of splitting into cases: if $\tau \in \mathcal{D}$, then estimates for F_ε^k are available *directly* by analytic methods. Otherwise, we need to use (4-7) to obtain the bound (4-6) for all $\tau \in \mathbb{H}$.

More explicitly, we have the following:

Proposition 4.3. *There is a positive constant $C > 0$ such that, for each $k \geq 1$, the inequality*

$$|F_\varepsilon^k(\tau, x)| \leq C^k (k!)(1 + \mathrm{Im}(\tau)^{-k/2})$$

holds, whenever $\tau \in \mathcal{D}$.

This proposition can be directly compared to [Radchenko and Viazovska 2019, Lemma 4]. In fact, it is nothing but a carefully quantified version of it.

Proof of Proposition 4.3. As the proof follows thoroughly the main ideas in Lemma 4 in [Radchenko and Viazovska 2019], we will mainly focus on the points where we have to sharpen bounds.

We see directly from the definition of F_ε^k that we are allowed to consider only values of $\tau \in \mathcal{D}_1 = \mathcal{D} \cap \{\tau \in \mathbb{H} : \mathrm{Re}(\tau) \in (-1, 0)\}$. By subsequent considerations from that reduction, we see that the bound

$$|x^k F_\varepsilon(\tau, x)| \leq 10 \int_\ell |K_\varepsilon(\tau, z)| x^k (e^{-\pi x^2 \mathrm{Im}(\tau)} + |z|^{-1/2} e^{-\pi x^2 \mathrm{Im}(-1/z)})\, |\mathrm{d}z| \tag{4-8}$$

holds, where ℓ is the path joining i to 1 on the upper half-space, defined to be

$$\ell = \left\{ w \in \mathcal{D} : \mathrm{Re}(J(w)) = \tfrac{1}{64},\ \mathrm{Im}(J(w)) > 0 \right\}. \tag{4-9}$$

An explicit computation gives us that the maximal value of

$$x^k e^{-\pi x^2 \mathrm{Im}(z)}$$

is attained at $x = (k/(2\pi\, \mathrm{Im}(z)))^{1/2}$. Therefore, as any $z \in \ell$ has norm bounded from above and below by absolute constants, we find that there is $C > 0$ so that

$$|F_\varepsilon^k(\tau, x)| \leq C^{k/2} \cdot \left(\frac{k}{2\pi e}\right)^{k/2} \int_\ell |K_\varepsilon(\tau, z)|\, \mathrm{Im}(z)^{-k/2}\, |\mathrm{d}z|. \tag{4-10}$$

We have then three regimes to consider:

Case 1: $|\tau - i| < \frac{1}{10}$. Notice that if we prove that the proposition holds for *any* $\tau \in \mathbb{H}$ so that $|\tau - i| = \frac{1}{10}$, we can use the maximum modulus principle on F_ε^k on that circle to conclude that the proposition holds inside as well. Moreover, by the functional equation (4-7), we see that the proposition holds for $\mathcal{A} = \left\{\tau \in \mathbb{H} : |\tau - i| = \frac{1}{10},\ |\tau| \le 1\right\}$ in the case it holds for the image of the circle arc $\mathcal{A}$ under the action of S. But a simple computation shows that $S\mathcal{A}$ is just another circle arc contained (up to endpoints) in $\left\{\tau \in \mathcal{D}_1 : \frac{1}{4} > |\tau - i| > \frac{1}{10}\right\}$. This shows that in order to prove the proposition for this case, it suffices to show it for the other cases.

Case 2: $|\tau - i| > \frac{1}{10}$, $\operatorname{Im}(\tau) > \frac{1}{2}$. For this case, we use the fact that $|K_\varepsilon(\tau, z)| \lesssim |\theta(z)|^3 \lesssim \operatorname{Im}(z)^{-2} e^{-\pi/\operatorname{Im}(z)}$ for $z \in \ell$, $\operatorname{Im}(\tau) > \frac{1}{2}$, with constants independent of τ. Using this bound in (4-8) yields

$$|F_\varepsilon^k(\tau, x)| \le (1 + |x|^{k+2}) e^{-c|x|} \lesssim C^k \left(\frac{k+2}{e}\right)^{k+2}$$

for some $C > 0$. Applications of Stirling's formula imply that this bound is controlled by $C_1^k(k!)$, with $C_1 > 0$ an absolute constant. This shows the result in this case.

Case 3: $|\tau - i| > \frac{1}{10}$, $\operatorname{Im}(\tau) \le \frac{1}{2}$. Again, we resort to the estimates in the proof of Lemma 4 in [Radchenko and Viazovska 2019]: there, the authors prove that

$$|K_+(\tau, z)| \lesssim \operatorname{Im}(\tau)^{-1/2} \frac{|J(\tau)|^{3/8} |J(z)|^{5/8} \operatorname{Im}(z)^{-3/2}}{|J(z) - J(\tau)|},$$
$$|K_-(\tau, z)| \lesssim \operatorname{Im}(\tau)^{-1/2} \frac{|J(\tau)|^{7/8} |J(z)|^{1/8} \operatorname{Im}(z)^{-3/2}}{|J(z) - J(\tau)|}.$$

Due to the not-so-symmetric nature of these bounds, we focus on the one for K_+, and the analysis for K_-, as well as the bounds, will be almost identical, and thus the details will be omitted.

Taking advantage of the explicit structure of the curve we are integrating over (4-9), and the fact that there is an absolute constant $C > 0$ so that $\operatorname{Im}(z)^{-1} \le C \log(1 + |J(z)|)$ and that $z \in \ell \iff J(z) = \frac{1}{64} + it,\ t \in \mathbb{R}$,

$$\begin{aligned}\int_\ell |K_+(\tau, x)| \operatorname{Im}(z)^{-k/2} \, |\mathrm{d}z| &\le C^{k/2} \operatorname{Im}(\tau)^{-1/2} \int_0^\infty \frac{|J(\tau)|^{3/8} t^{-3/8} \log^{(k-1)/2}(1 + t)}{\sqrt{t^2 + |J(\tau)|^2}} \, \mathrm{d}t \\ &= C^{k/2} \operatorname{Im}(\tau)^{-1/2} \int_0^\infty \frac{t^{-3/8} \log^{(k-1)/2}(1 + t|J(\tau)|)}{\sqrt{1 + t^2}} \, \mathrm{d}t. \qquad (4\text{-}11)\end{aligned}$$

Now, the last integral in (4-11) can be estimated as follows: if $k - 1$ is even, by using that $\log(1 + ab) \le \log(1 + a) + \log(1 + b)$ whenever $a, b > 0$, the integral

$$\int_0^\infty \frac{t^{-3/8} \log^{(k-1)/2}(1 + t|J(\tau)|)}{\sqrt{1 + t^2}} \, \mathrm{d}t$$

is bounded by

$$\sum_{i=0}^{(k-1)/2} \binom{(k-1)/2}{i} \log^i(1 + |J(\tau)|) \int_0^\infty \frac{t^{-3/8} \log^{(k-1)/2-i}(1 + t)}{\sqrt{1 + t^2}} \, \mathrm{d}t. \qquad (4\text{-}12)$$

Each summand above can be easily estimated. Indeed, $\binom{(k-1)/2}{i} \leq 2^{k/2}$ trivially, $\log^i(1+|J(\tau)|) \leq C^i \operatorname{Im}(\tau)^{-i}$, and the integrals can be explicitly bounded in terms of gamma functions. In fact, we first split the integrals in question as

$$\left(\int_0^1 + \int_1^\infty\right) \frac{t^{-3/8}\log^{(k-1)/2-i}(1+t)}{\sqrt{1+t^2}}\,\mathrm{d}t.$$

For the first part, we simply bound the integrand by $t^{-3/8}\log(2)^{(k-1)/2-i}$, and this yields a bound uniform in k. For the second, we change variables $\log(1+t) \mapsto s$ in (4-12) above. A simple computation shows that it is bounded by

$$10\int_0^\infty e^{-3s/8}s^{(k-1)/2-i}\,\mathrm{d}s \lesssim C^k\int_0^\infty e^{-r}r^{(k-1)/2-i}\,\mathrm{d}r = C^k\Gamma\left(\frac{k-1}{2} - i + 1\right).$$

Thus, (4-12) is bounded by

$$C^k \operatorname{Im}(\tau)^{(1-k)/2}\Gamma\left(\frac{k-1}{2}\right).$$

Putting together the estimates in (4-11) and (4-10) and using Stirling's formula for the approximation of Γ, we conclude that

$$|F_\varepsilon^k(\tau, x)| \leq C^k(k!)\operatorname{Im}(\tau)^{-k/2},$$

which was the content of the proposition when k is odd. In the case where k is an even number, the fact that $F_\varepsilon^j(\tau, x)^2 = F_\varepsilon^{j-1}(\tau, x)F_\varepsilon^{j+1}(\tau, x)$ allows one to use the bounds of the case where k is odd to conclude the proof. □

We are now finally able to finish the proof of Theorem 1.6.

Proof of Theorem 1.6. We first notice that F_ε^k is 2-periodic, so we lose no generality in assuming that $\tau \in \{z \in \mathbb{H} : \operatorname{Re}(z) \in [-1, 1]\} = S_1$. If $\operatorname{Re}(\tau) \in [-1, 1]$, then we have two cases:

Case 1: If $\tau \in \mathcal{D}$, we can use Proposition 4.3 directly, and the decay obtained by the assertion of the proposition remains unchanged.

Case 2: If $\tau \in S_1 \backslash \mathcal{D}$, the strategy is to use (4-7) to reduce it to the previous case. In fact, we define the Γ_θ-cocycle $\{\phi_A^k\}_{A\in\Gamma_\theta}$ by

$$\begin{aligned} \phi_{T^2}^k(\tau, x) &= 0, \\ \phi_S^k(\tau, x) &= x^k(e^{i\pi x^2\tau} + \varepsilon(-i\tau)^{-1/2}e^{i\pi x^2(-1/\tau)}), \end{aligned}$$

together with the cocycle relation

$$\phi_{AB}^k = \phi_A^k + \phi_A^k|B. \tag{4-13}$$

For a fixed $\tau \in S_1 \setminus \mathcal{D}$, we associate $\tau' \in \mathcal{D}$ through the following process: Let

$$\begin{cases} \gamma_0 = \tau, \\ \gamma_i = -1/(\gamma_{i-1}) - 2n_i, \end{cases} \tag{4-14}$$

where $n_i = \lfloor \frac{1}{2}((-1/\gamma_{i-1})+1) \rfloor$. We define $m = m(\tau)$ to be the smallest positive integer so that $\gamma_m \in \mathcal{D}$. In this case, we let $\gamma_{m(\tau)} =: \tau'$. In other words, we have that the sequence

$$\begin{cases} \tau_0 = \tau', \\ \tau_{i+1} = -1/\tau_i + 2n_i \end{cases} \tag{4-15}$$

satisfies the hypotheses of Lemma 3 in [Radchenko and Viazovska 2019]. We therefore have that $|\tau_j| > 1$, $\mathrm{Im}(\tau_j)$ is nonincreasing and $\mathrm{Im}(\tau_j) \le 1/(2j-1)$. An inductive procedure shows us that

$$\gamma_{m-i} = -\frac{1}{\tau_i}.$$

In particular, the sequence $\{\tau_i\}_{i\ge 0}$ is in fact finite, with at most $m(\tau)$ terms. This implies that

$$m+1 \le 4m-2 \le 2\,\mathrm{Im}(\tau)^{-1}. \tag{4-16}$$

We will use (4-16) in the following computation with the cocycle condition. We write $\tau' = A\tau$, where $A \in \Gamma_\theta$ is of the form

$$A = ST^{2n_m}ST^{2n_{m-1}}S\cdots T^{2n_1}S.$$

As $\{\phi_A^k\}_{A\in\Gamma_\theta}$ satisfies the cocycle condition (4-13), the proof of Lemma 3 in [Radchenko and Viazovska 2019] gives us that

$$\mathrm{Im}(\tau')^{1/4}|\phi_A^k(\tau')| \le \sum_{j=1}^{m} \mathrm{Im}(\tau_j)^{1/4}|\phi_S^k(\tau_j)|.$$

By the definition of ϕ_S^k, we see that

$$|\phi_S^k(\tau_j, x)| \le C\Gamma\left(\frac{k+1}{2}\right)(\mathrm{Im}(\tau_j)^{-k/2} + |\tau_j|^{-1/2}\,\mathrm{Im}(-1/\tau_j)^{-k/2}). \tag{4-17}$$

As $\gamma_{m-i} = -1/\tau_i = \tau_{i+1} - 2n_i,\ |\tau_j| > 1$, and the sequence $\mathrm{Im}(\tau_j)$ is nonincreasing, the right-hand side of (4-17) is bounded from above by $C\cdot\Gamma((k+1)/2)\,\mathrm{Im}(\tau)^{-k/2}$. From (4-16), it follows that

$$|\phi_A^k(\tau')|\,\mathrm{Im}(\tau')^{1/4} \le C\Gamma\left(\frac{k+1}{2}\right)\mathrm{Im}(\tau)^{-k/2}\left(\sum_{j=1}^{m}\mathrm{Im}(\tau_j)^{1/4}\right).$$

If we use the aforementioned facts about $\mathrm{Im}(\tau_j)$, we will see that, in fact,

$$|\phi_A^k(\tau')|\,\mathrm{Im}(\tau')^{1/4} \le C\Gamma\left(\frac{k+1}{2}\right)\mathrm{Im}(\tau)^{-k/2}m(\tau)^{3/4}. \tag{4-18}$$

Now, using the functional equation for F_ε^k implies

$$F_\varepsilon^k - (F_\varepsilon^k)|A = \phi_A^k,$$

which then gives us

$$|F_\varepsilon^k(\tau, x)|\,|\mathrm{Im}(\tau)|^{1/4} \le |\mathrm{Im}(\tau')|^{1/4}|F_\varepsilon^k(\tau', x)| + |\phi_A^k(\tau', x)|\,|\mathrm{Im}(\tau')|^{1/4}.$$

Defining $\mathrm{Im}(\tau') =: I(\tau)$ and using Proposition 4.3 and (4-18) to estimate this expression, it follows that

$$|F_\varepsilon^k(\tau, x)| \le \mathrm{Im}(\tau)^{-k/2-1/4}\big(C^k(k!)\cdot I(\tau)^{1/4} + \Gamma((k+1)/2)m(\tau)^{3/4}\big). \tag{4-19}$$

In order to estimate (4-19), we must resort not only to the general idea of obtaining bounds for Fourier coefficients based on decay at infinity, as in Lemma 4.2, but also to the following estimate of the average values of $m(\tau)$ and $I(\tau)$, recently available by the work of Bondarenko, Radchenko and Seip. We refer the reader to Propositions 6.6 and 6.7 in [Bondarenko et al. 2023] for a proof.

Lemma 4.4. *Whenever $y \in \left(0, \frac{1}{2}\right)$, we have*

$$\int_{-1}^{1} I(x+iy)^{1/4} \lesssim 1 \quad \textit{and} \quad \int_{-1}^{1} m(x+iy)^{3/4} \lesssim \log^3(1+y^{-1}).$$

An application of Lemma 4.4 together with the bound (4-19) to the proof of the first bound in Lemma 4.2 implies

$$\sup_{x\in\mathbb{R}} |x^k b_n^{\pm}(x)| \lesssim C^k n^{1/4} n^{k/2} \log^3(1+n)(k!) \tag{4-20}$$

for $n > 1/c_0,\ k \geq 1$. Also, in the case $n \geq k/(\pi c_0)$, the sharper bound

$$\sup_{x\in\mathbb{R}} |x^k b_n^{\pm}(x)| \lesssim (C')^k n^{1/4} n^{k/2} \log^3(1+n)(k!)^{1/2} \tag{4-21}$$

holds instead. We now proceed to optimize in $k > 0$, completing the outline devised in the beginning of this section.

Indeed, let us start by optimizing (4-20). We postpone the discussion on the improved bound (4-21) to a later remark.

Notice that we may assume $|x| \geq C'\sqrt{n}$, as for if $|x| < C'\sqrt{n}$, the bound (4-20) with $k = 0$ gives us already the result, as $1 \lesssim_c e^{-c|x|/\sqrt{n}}$. If we then set $k = |x|/C'\sqrt{n}$, where $C' > 0$ will be a fixed positive constant, whose exact value shall be determined later, we have that

$$|b_n^{\pm}(x)| \lesssim n^{1/4} \log^3(1+n) \cdot \exp(k \log(Cn^{1/2}) + k\log(k) - k \log|x|).$$

The exponential term above is

$$\exp\biggl(\frac{|x|}{C'\sqrt{n}} \log(Cn^{1/2}) + \frac{|x|}{C'\sqrt{n}}(\log(|x|) - \log(C'\sqrt{n})) - \frac{|x|}{C'\sqrt{n}} \log|x|\biggr) = \exp\biggl(\frac{|x|}{C'\sqrt{n}} \log\Bigl(\frac{C}{C'}\Bigr)\biggr).$$

We only need to set $C' \geq 2C$ above, and this quantity will grow like $\exp(-c|x|/\sqrt{n})$. This finishes the first assertion in Theorem 1.6.

For the second one, we notice that the proof above adapts in many instances. Indeed, if we shift our attention to the function $\partial_x F_\varepsilon^k(\tau, x)$ instead, we will see that, in an almost identical fashion to that of the proof of Proposition 4.3, we are able to prove that, for all $\tau \in \mathcal{D}$,

$$|\partial_x F_\varepsilon^k(\tau, x)| \lesssim C^k (k!) \operatorname{Im}(\tau)^{-(k+1)/2}.$$

On the other hand, the partial derivative ∂_x of the cocycle $\{\phi_A^k\}_{A\in\Gamma_\theta}$ is itself a cocycle with respect to the same slash operator. Moreover, for $A = S$, the following formula holds:

$$\partial_x \phi_S^k(\tau, x) = (2\pi i) x^{k+1} (\tau e^{\pi i x^2 \tau} + i\varepsilon(-i\tau)^{-3/2} e^{\pi i x^2(-1/\tau)}).$$

In that case, using the notation from above for the elements τ', $\tau_j \in \mathbb{H}$ associated to $\tau \in \mathbb{H} \cap \{|z| \le 1\}$, we see

$$\operatorname{Im}(\tau')^{1/4} |\partial_x \phi_A^k(\tau')| \le \operatorname{Im}(\tau')^{1/4} |\partial_x \phi_S^k(\tau')| + \sum_{j=1}^{m} \operatorname{Im}(\tau_j)^{1/4} |\partial_x \phi_A^k(\tau_j)|.$$

For $j \in \{0, 1, 2, \dots, m\}$, the definition of our new cocycle implies

$$\begin{aligned} |\partial_x \phi_S^k(\tau_j, x)| &\lesssim \Gamma\Big(\frac{k+3}{2}\Big)(|\tau_j| \operatorname{Im}(\tau_j)^{-(k+1)/2} + |\tau_j|^{-3/2} \operatorname{Im}(\tau_{j+1})^{-(k+1)/2}) \\ &\le \Gamma\Big(\frac{k+3}{2}\Big) \operatorname{Im}(\tau)^{-(k+1)/2}. \end{aligned}$$

This follows as before from the fact that $\operatorname{Im}(\tau_{j+1}) = \operatorname{Im}(\tau_j)/|\tau_j|^2 \ge \operatorname{Im}(\tau)$ and that $|\tau_j| > 1$. Analyzing the functional equations for $\partial_x F_\varepsilon^k(\tau, x)$ in the same way as before readily gives that

$$|\partial_x F_\varepsilon^k(\tau, x)| \le C^k \operatorname{Im}(\tau)^{-(k+1)/2-1/4} (k!) (I(\tau)^{1/4} + m(\tau)^{3/4}).$$

Lemma 4.4 and the considerations employed for F_ε^k apply almost verbatim here, and thus we conclude

$$|(b_n^{\pm})'(x)| \lesssim n^{3/4} \log^3(1+n) e^{-c|x|/\sqrt{n}},$$

as wished. □

As a consequence of Theorem 1.6, we are able to establish the following bound for the interpolation basis taking into account both decay and zeros.

Corollary 4.5. *Let $\{a_n\}$ be the interpolation sequence of functions from* (1-3). *Then there is $c > 0$ so that*

$$|a_n(x)| \lesssim n^{3/4} \log^3(1+n) \operatorname{dist}(|x|, \sqrt{\mathbb{N}}) e^{-c|x|/\sqrt{n}}$$

for all positive integers $n \in \mathbb{N}$.

Proof. We simply use the fundamental theorem of calculus on the a_n: Without loss of generality, we suppose $x > 0$. We then have

$$\begin{aligned} |a_n(x)| = |a_n(x) - a_n(\sqrt{m}) + \delta_{n,m}| &\le \int_{\sqrt{m}}^{x} |a_n'(x)| \, \mathrm{d}x + \delta_{n,m} \\ &\le n^{3/4} \log^3(1+n) \operatorname{dist}(x, \sqrt{\mathbb{N}}) e^{-c|x|/\sqrt{n}} + \delta_{m,n} \\ &\lesssim n^{3/4} \log^3(1+n) \operatorname{dist}(x, \sqrt{\mathbb{N}}) e^{-c|x|/\sqrt{n}}, \end{aligned}$$

as the $\delta_{m,n}$ factor is only one if $|x| \in [\sqrt{n}, \sqrt{n+1})$, where $1 \lesssim e^{-c|x|/\sqrt{n}}$. □

Remark. Although the exponential bound $n^{1/4} \log^3(1+n) e^{-c|x|/\sqrt{n}}$ suffices for our purposes, below we sketch how to deduce a slightly improved decay for the interpolation basis $\{a_n\}_{n \ge 0}$.

We again wish to optimize (4-21). If we set $k = |x|^2/C'n$, where $C' > 0$ will be chosen soon, we have

$$|b_n^{\pm}(x)| \lesssim n^{1/4} \log^3(1+n) \cdot \exp(k \log(Cn^{1/2}) + k \log(k^{1/2}) - k \log|x|).$$

This bound holds as long as $\pi n \gtrsim k \ge 1$. If instead $k < 1$, that means, $|x| \le \sqrt{C'}\sqrt{n}$, we use the bound in either (4-20) or (4-21) for $k = 0$, which yields $|b_n^{\pm}(x)| \lesssim n^{1/4} \log^3(1+n) \lesssim n^{1/4} \log^3(1+n) e^{-c|x|^2/n}$, for $c > 0$.

On the other hand, in the case $k > 1$, the first exponential term above becomes

$$\exp\left(\frac{|x|^2}{C'n}\log(Cn^{1/2}) + \frac{|x|^2}{C'n}(\log(|x|) - \log(\sqrt{C'n})) - \frac{|x|^2}{C'n}\log|x|\right) = \exp\left(\frac{|x|^2}{C'n}\log\left(\frac{C}{\sqrt{C'}}\right)\right).$$

We only need to set $C' \geq (2C)^2$ above, and this quantity will grow like $\exp(-c|x|^2/n)$.

For the remaining $|x| > \sqrt{C'n}$ case, we need to refine the analysis of the proof of Lemma 4.2 and Theorem 1.6. Indeed, it is easy to see that if $n \in (2^{-j}\alpha, 2^{1-j}\alpha)$, $j \geq 1$, then evaluating the Fourier coefficients of a 2-periodic function $f : \mathbb{H} \to \mathbb{C}$ such that $|f(\tau)| \lesssim \mathrm{Im}(\tau)^{-\alpha}(I(\tau)^{1/4} + m(\tau)^{3/4})$ for $\mathrm{Im}(\tau) \leq 1$ as

$$2c_n = \int_{-1+i\alpha/(2^j\pi n)}^{1+i\alpha/(2^j\pi n)} f(\tau)e^{-\pi i n\tau}\,\mathrm{d}\tau$$

implies

$$|c_n| \lesssim \left(\frac{2^j\pi e^{1/2^j}}{\alpha}\right)^{\alpha} n^{\alpha}\log^3(1+n).$$

Using this new bound in (4-19), we obtain that, when $n \in (2^{-j-1}k, 2^{-j}k)$,

$$|b_n^{\pm}(x)| \lesssim n^{1/4}\log^3(1+n)\cdot\exp\bigl(k(j/2 + \log(C\sqrt{n}) + \log(k^{1/2}) - \log|x|)\bigr).$$

This suggests that we take $k = |x|^2/C'2^jn$, which is admissible to the condition $n \in (2^{-j-1}k, 2^{-j}k)$ if $|x| \sim \sqrt{C'}2^jn$. A similar computation to the ones above implies that

$$|b_n^{\pm}(x)| \lesssim n^{1/4}\log^3(1+n)\exp\left(-c\frac{|x|^2}{2^jn}\right) \lesssim n^{1/4}\log^3(1+n)\exp(-c'|x|),$$

whenever $C' \gg C$. The next corollary then follows as a natural consequence.

Corollary 4.6. *Let $a_n : \mathbb{R} \to \mathbb{R}$ be the interpolating functions in the Radchenko–Viazovska interpolation formula. Then there are $c, C > 0$ so that*

$$|a_n(x)| \lesssim n^{1/4}\log^3(1+n)(e^{-c|x|^2/n}1_{|x|<Cn} + e^{-c|x|}1_{|x|>Cn})$$

for each $n \geq 1$.

Indeed, the application of Lemma 4.2 requires that we take $n \geq C$ for $C > 0$ some absolute constant. In order to prove such a result for $n \lesssim 1$, we may simply use the definition of $b_n^{\pm}$ as a Laplace transform of a the weakly holomorphic modular form $g_n^{\pm}$. Indeed, in order to extend Corollary 4.6 to $n = 0$, we write

$$a_0(x) = \hat{a}_0(x) = \frac{1}{4}\int_{-1}^{1}\theta(z)^3\, e^{\pi i x^2 z}\,\mathrm{d}z.$$

In order to prove that a_0 decays exponentially, we employ a similar technique to that of [Radchenko and Viazovska 2019, Proposition 1]. Indeed, we have

$$|\theta(z)|^3 \lesssim \mathrm{Im}(z)^{-2}\, e^{-\pi/\mathrm{Im}(z)} \quad \text{for } z \to \pm 1,$$

and moreover that $|\theta(z)| \lesssim 1$ whenever $z \in \mathbb{H}$, $|z| = 1$. We also suppose without loss of generality that $x > 0$. This implies that, for $\delta > 0$,

$$|a_0(x)| \lesssim \int_0^\delta \frac{e^{-1/(2t)}}{t^2}\,\mathrm{d}t + e^{-\pi x^2\delta} \lesssim e^{-1/(2\delta)} + e^{-\pi x^2 \delta}.$$

We then choose, for $x \gg 1$, $\delta = 1/(\sqrt{2\pi}x)$. This implies that $|a_0(x)| \lesssim e^{-(\sqrt{\pi/2})x}$, which is the desired bound. For other bounded values of n such a proof can be easily adapted.

4B. ***Proof of the main result.*** For this part, we shall use the definitions of $\ell^2_s(\mathbb{Z}_{\geq 0})$ and $\ell^2_s(\mathbb{N})$, as in (2-4) from Section 2. Let then $I : \ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+) \to \ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$ denote the identity operator. Recall the Radchenko–Viazovska interpolation result: for $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ a real-valued function,

$$f(x) = \sum_{n\geq 0} (f(\sqrt{n})a_n(x) + \hat{f}(\sqrt{n})\hat{a}_n(x)), \tag{4-22}$$

where $a_n : \mathbb{R} \to \mathbb{R}$ is a sequence of interpolating functions independent of the Schwartz function f. In particular,

$$f(\sqrt{k}) = \sum_{n\geq 0} (f(\sqrt{n})a_n(\sqrt{k}) + \hat{f}(\sqrt{n})\hat{a}_n(\sqrt{k})).$$

In fact, for any pair of sequences $(\{x_i\}_i, \{y_i\}_i)$ decaying sufficiently fast and satisfying

$$\sum_{n\in\mathbb{Z}} x_{n^2} = \sum_{n\in\mathbb{Z}} y_{n^2}, \tag{4-23}$$

the function

$$\mathfrak{G}(t) = \mathfrak{G}_{x,y}(t) = \sum_{n\geq 0} (x_n a_n(t) + y_n \hat{a}_n(t)) \tag{4-24}$$

is well-defined and satisfies $\mathfrak{G}(\sqrt{k}) = x_k$, $\widehat{\mathfrak{G}}(\sqrt{k}) = y_k$. In fact, let $(\{x_i\}_i, \{y_i\}_i) \in \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$ for $s > 0$ sufficiently large. The operator

$$T : \ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+) \to \ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$$

given by $T = (T^1, T^2)$, where

$$\begin{aligned} T^1(\{x_i\}, \{y_i\})_k &= \sum_{n\geq 0} (x_n a_n(\sqrt{k}) + y_n \hat{a}_n(\sqrt{k})),\\ T^2(\{x_i\}, \{y_i\})_k &= T^1(\{y_i\}, \{x_i\})_k, \end{aligned}$$

has an explicit form as a consequence of (4-2). Indeed, for $k \geq 1$, we have

$$T^1(\{x_i\}, \{y_i\})_k = x_k, \quad T^2(\{x_i\}, \{y_i\}) = y_k,$$

whereas for $k = 0$, we have

$$\begin{aligned} T^1(\{x_i\}, \{y_i\})_0 &= \frac{x_0 + y_0}{2} - \sum_{n\geq 1} x_{n^2} + \sum_{n\geq 1} y_{n^2},\\ T^2(\{x_i\}, \{y_i\})_0 &= \frac{x_0 + y_0}{2} - \sum_{n\geq 1} y_{n^2} + \sum_{n\geq 1} x_{n^2}. \end{aligned} \tag{4-25}$$

In particular, it is then easy to see that $T = I$ whenever $(\{x_i\}_i, \{y_i\}_i)$ satisfy the relation (4-23). This relation is always satisfied by sequences of the type $x_k = f(\sqrt{k})$ and $y_k = \hat{f}(\sqrt{k})$ because of the Poisson summation formula. Inspired by this fact, we define the perturbed operator associated to a sequence $\varepsilon_k > 0,\ k \in \mathbb{Z}_+$, to be

$$\widetilde{T} \text{ defined on } \ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+),$$

where $\widetilde{T} = (\widetilde{T}^1, \widetilde{T}^2)$, with

$$\begin{aligned}\widetilde{T}^1(\{x_i\}, \{y_i\})_k &= \sum_{n\geq 0}(x_n a_n(\sqrt{k+\varepsilon_k}) + y_n \hat{a}_n(\sqrt{k+\varepsilon_k})),\\ \widetilde{T}^2(\{x_i\}, \{y_i\})_k &= \widetilde{T}^1(\{y_i\}, \{x_i\})_k\end{aligned}$$

for $k \geq 1$, and $\widetilde{T}^1(\{x_i\}, \{y_i\})_0 = x_0$, $\widetilde{T}^2(\{x_i\}, \{y_i\})_0 = y_0$. At first, such an operator might not be defined in the entire space $\ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+)$ because of summability issues, but a way to avoid this trouble is to initially define the operator in the dense subspace of pairs of sequences with finitely many nonzero entries. A posteriori, we will prove the fundamental fact that this operator is *bounded* from $\ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+) \to \ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+)$, which will allow to extend it to the entirety of the space $\ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+)$. One way to see this will be provided in the proof of our main theorem, by showing that the operator norm satisfies $\|I - \widetilde{T}\|_{\ell_s^2(\mathbb{Z}_+)\times\ell_s^2(\mathbb{Z}_+)\to\ell_s^2(\mathbb{Z}_+)\times\ell_s^2(\mathbb{Z}_+)} < +\infty$. This is, incidentally, our main device to prove our result: if

$$\|I - \widetilde{T}\|_{\ell_s^2(\mathbb{Z}_+)\times\ell_s^2(\mathbb{Z}_+)\to\ell_s^2(\mathbb{Z}_+)\times\ell_s^2(\mathbb{Z}_+)} < 1,$$

then $\widetilde{T}$ is an invertible operator defined on $\ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+)$. Therefore, its inverse

$$\widetilde{T}^{-1} : \ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+) \to \ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+)$$

is well-defined and bounded. In particular, for $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ real, given the lists of values

$$\begin{aligned}&f(0),\ f(\sqrt{1+\varepsilon_1}),\ f(\sqrt{2+\varepsilon_2}),\ \ldots,\\ &\hat{f}(0),\ \hat{f}(\sqrt{1+\varepsilon_1}),\ \hat{f}(\sqrt{2+\varepsilon_2}),\ \ldots,\end{aligned}$$

there is a unique pair $(\{x_i\}_i, \{y_i\}_i) \in \ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+)$ so that

$$\widetilde{T}(\{x_i\}, \{y_i\}) = (\{f(\sqrt{k+\varepsilon_k})\}_k, \{\hat{f}(\sqrt{k+\varepsilon_k})\}_k).$$

But we also know that

$$\widetilde{T}(\{f(\sqrt{i})\}_i, \{\hat{f}(\sqrt{i})\}_i) = T(\{f(\sqrt{i})\}_i, \{\hat{f}(\sqrt{i})\}_i) = \{f(\sqrt{k+\varepsilon_k})\}_k, \{\hat{f}(\sqrt{k+\varepsilon_k})\}_k.$$

This implies $x_j = f(\sqrt{j}),\ y_j = \hat{f}(\sqrt{j})$. By writing the k-th entry of the inverse of $\widetilde{T}$ as

$$\widetilde{T}^{-1}(\{w_i\}, \{z_i\})_k = \sum_{j\geq 0}(\gamma_{j,k} w_j + \hat{\gamma}_{j,k} z_j)$$

for two sequences $\{\gamma_{j,k}\}_{j,k\geq 0},\ \{\hat{\gamma}_{j,k}\}_{j,k\geq 0}$ so that $|\gamma_{j,k}| + |\hat{\gamma}_{j,k}| \lesssim (j/k)^s$, we must have

$$f(\sqrt{k}) = \sum_{j\geq 0}(\gamma_{j,k} f(\sqrt{j+\varepsilon_j}) + \hat{\gamma}_{j,k} \hat{f}(\sqrt{j+\varepsilon_j})). \tag{4-26}$$

This implies, by (1-3), that we can recover f from its values and those of its Fourier transform at $\sqrt{k+\varepsilon_k}$. Moreover, as the adjoint of $\widetilde{T}^{-1}$ is also bounded from $\ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+)$ to itself, we conclude that, for $s \gg 1$ sufficiently large and $f, \hat{f}$ both being $\mathcal{O}((1+|x|)^{-10s})$, we can use Fubini's theorem in (1-3) together with (4-26). This proves the existence of two sequences of functions $\{\theta_j\}_{j\geq 0}, \{\eta_j\}_{j\geq 0}$ so that

$$|\theta_j(x)| + |\eta_j(x)| + |\hat{\theta}_j(x)| + |\hat{\eta}_j(x)| \lesssim (1+j)^s(1+|x|)^{-10}$$

and

$$f(x) = \sum_{j\geq 0} \bigl(f(\sqrt{j+\varepsilon_j})\theta_j(x) + \hat{f}(\sqrt{j+\varepsilon_j})\eta_j(x) \bigr).$$

Thus, we focus on the proof of the invertibility of $\widetilde{T}$ for $s > 0$ suitably chosen.

Proof of invertibility of $\widetilde{T}$. We use, for this part, the Schur test. For that, define the auxiliary infinite matrices $A = \{A_{ij}\}_{i,j>0}$ and $\hat{A} = \{\hat{A}_{ij}\}_{i,j>0}$ by

$$\begin{aligned} A_{ij} &= (a_j(\sqrt{i+\varepsilon_i}) - \delta_{ij}) \times (i/j)^s, \\ \hat{A}_{ij} &= \hat{a}_j(\sqrt{i+\varepsilon_i})(i/j)^s. \end{aligned}$$

For a given vector $(x, y) \in \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$, we write then

$$B(x, y) = (A\cdot x + \hat{A}\cdot y, A\cdot y + \hat{A}\cdot x),$$

or, in matrix notation,

$$B = \begin{pmatrix} A & \hat{A} \\ \hat{A} & A \end{pmatrix}.$$

Furthermore, define the operator $B_0 : \mathbb{C}^2 \to \ell^2(\mathbb{Z}_{\geq 0}) \times \ell^2(\mathbb{Z}_{\geq 0})$ by

$$B_0(r, s) = \Bigl(\bigl(r\cdot a_0(\sqrt{k+\varepsilon_k}) + s\cdot \hat{a}_0(\sqrt{k+\varepsilon_k})\bigr)k^s,\ \bigl(s\cdot a_0(\sqrt{k+\varepsilon_k}) + r\cdot \hat{a}_0(\sqrt{k+\varepsilon_k})\bigr)k^s \Bigr)_{k\geq 0}.$$

Notice that the operator norm of $\widetilde{T} - I$ acting on $\ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+)$ is, by virtue of our definitions, bounded by the operator norm of B acting on $\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$ plus the norm of B_0 acting on $\mathbb{C}^2$, since

$$\begin{aligned} (\widetilde{T} - I)(x, y)_k &= B_0(x_0, y_0)_k + B(x', y')_k, \quad k \geq 1, \\ (\widetilde{T} - I)(x, y)_0 &= (0, 0), \end{aligned}$$

where

$$(x', y')_n = (x_n, y_n), \quad n > 0.$$

First of all, bounds for the operator B_0 are simple to obtain. In fact, by the Cauchy–Schwarz inequality

$$\|B_0(x_0, y_0)\|^2_{\ell^2(\mathbb{N})\times\ell^2(\mathbb{N})} \leq 2(x_0^2 + y_0^2)\biggl(\sum_{k>0} \{|a_0(\sqrt{k+\varepsilon_k})|^2 + |\hat{a}_0(\sqrt{k+\varepsilon_k})|^2\}k^{2s}\biggr).$$

Since $a_0(\sqrt{k}) = \hat{a}_0(\sqrt{k}) = 0$ for $k \geq 1$, and $a_0 \in \mathcal{S}(\mathbb{R})$, for any fixed $M > 0$ there is $C = C_M > 0$ such that

$$\max\{|a_0(\sqrt{k+\varepsilon_k})|, |\hat{a}_0(\sqrt{k+\varepsilon_k})|\} \leq C_M \frac{|\varepsilon_k|}{k^M}. \tag{4-27}$$

This implies the norm of B_0 is sufficiently small, assuming that we make $\sup_{k\geq 0}|\varepsilon_k|$ sufficiently small, depending on s.

We now turn to bounding the operator norm of B. By Schur's test, it suffices to find $\alpha, \beta > 0$, such that $\sqrt{\alpha\beta} \ll 1$, and positive sequences $\{p_i\}_{i\geq 0}$, $\{q_i\}_{i\geq 0}$ so that the following inequalities hold:

$$\begin{aligned}
&\sum_{j>0}(i/j)^s \times [|a_j(\sqrt{i+\varepsilon_i})-\delta_{ij}|p_j + |\hat{a}_j(\sqrt{i+\varepsilon_i})|q_j] \leq \alpha p_i,\\
&\sum_{j>0}(i/j)^s \times [|a_j(\sqrt{i+\varepsilon_i})-\delta_{ij}|q_j + |\hat{a}_j(\sqrt{i+\varepsilon_i})|p_j] \leq \alpha q_i,\\
&\sum_{i>0}(i/j)^s \times [|a_j(\sqrt{i+\varepsilon_i})-\delta_{ij}|p_i + |\hat{a}_j(\sqrt{i+\varepsilon_i})|q_i] \leq \beta p_j,\\
&\sum_{i>0}(i/j)^s \times [|a_j(\sqrt{i+\varepsilon_i})-\delta_{ij}|q_i + |\hat{a}_j(\sqrt{i+\varepsilon_i})|p_i] \leq \beta q_j.
\end{aligned} \tag{4-28}$$

Now, we make the ansatz that, for all $i > 0$, $p_i = q_i = i^\theta$, for some real number $\theta \in \mathbb{R}$. By making use of Theorem 1.6, we know that

$$|a_j(\sqrt{i+\varepsilon_i})-\delta_{ij}| + |\hat{a}_j(\sqrt{i+\varepsilon_i})| \lesssim \frac{\varepsilon_i}{\sqrt{i}} j^{3/4}\log^3(1+j)e^{-c\sqrt{i/j}}.$$

Therefore, (4-28) reduces to verifying

$$\sum_{j>0}(i/j)^s \times j^\theta \times \frac{\varepsilon_i}{\sqrt{i}} j^{3/4}\log^3(1+j)e^{-c\sqrt{i/j}} \leq \alpha i^\theta, \tag{4-29}$$

$$\sum_{i>0}(i/j)^s \times i^\theta \times \frac{\varepsilon_i}{\sqrt{i}} j^{3/4}\log^3(1+j)e^{-c\sqrt{i/j}} \leq \beta j^\theta. \tag{4-30}$$

Estimate of (4-29). For this term, we rewrite it as

$$i^{s-1/2} \times \varepsilon_i \Biggl(\sum_{j>0} j^{3/4-s}\log^3(1+j)e^{-c\sqrt{i/j}} j^\theta\Biggr).$$

In order to estimate this last sum, we break it into $j < i^{1/3}$ and $j > i^{1/3}$ contributions. Therefore,

$$\begin{aligned}
&\sum_{j>0} j^{3/4-s}\log^3(1+j)e^{-c\sqrt{i/j}} j^\theta\\
&\qquad \lesssim i^{1/3} i^{\max(3/4-s+\theta,0)}\log^3(1+i^{1/3})e^{-ci^{1/3}} + \sum_{j>i^{1/3}} j^{3/4-s}\log^3(1+j)e^{-c\sqrt{i/j}} j^\theta.
\end{aligned} \tag{4-31}$$

Because of the presence of the exponential, the first term is always bounded by an absolute constant times i^θ, so we treat it as negligible. For the second term, notice that the summand is bounded by a constant times $\int_j^{j+1} x^{3/4-s+\theta}\log^3(1+x)e^{-c\sqrt{i/x}}\,\mathrm{d}x$. Indeed, the inverse of the ratio between both is bounded from below by

$$\int_j^{j+1} (x/j)^{3/4-s+\theta}\frac{\log^3(1+x)}{\log^3(1+j)} e^{c(\sqrt{i/j}-\sqrt{i/x})}\,\mathrm{d}x \geq \min\biggl\{\Bigl(1+\frac{1}{j}\Bigr)^{3/4-s+\theta}, 1\biggr\} \gtrsim_{\theta,s} 1. \tag{4-32}$$

Thus, we obtain that the second term on the right-hand side of (4-31) is bounded by

$$\begin{aligned}\int_{i^{1/3}}^{\infty} x^{3/4-s+\theta}\log^3(1+x)e^{-c\sqrt{i/x}}\,\mathrm{d}x &= \int_0^{i^{-1/3}} \Big(1+\frac{1}{y}\Big)^{3/4-s+\theta}\log^3\Big(1+\frac{1}{y}\Big)y^{-2}e^{-c\sqrt{iy}}\,\mathrm{d}y \\ &\lesssim_{s,\theta} \int_0^{i^{-1/3}} y^{-11/4+s-\theta}\,\log^3\Big(1+\frac{1}{y}\Big)\,e^{-c\sqrt{iy}}\,\mathrm{d}y \\ &= i^{7/4-s+\theta}\int_0^{i^{2/3}} y^{-11/4+s-\theta}\,\log^3\Big(1+\frac{i}{y}\Big)\,e^{-c\sqrt{y}}\,\mathrm{d}y \\ &\lesssim_{s,\theta} i^{7/4-s+\theta}\log^3(1+i),\end{aligned}$$

as long as $-\frac{11}{4}+s-\theta>-1$, that is, $\theta<s-\frac{7}{4}$. Thus, (4-29) is bounded under such a condition by

$$C_{s,\theta}|\varepsilon_i|i^{s-1/2}\log^3(1+i)i^{7/4-s+\theta} = i^{5/4+\theta}\log^3(1+i)|\varepsilon_i|.$$

In order for this last quantity to be less than αi^θ, we must have $|\varepsilon_i|\lesssim_{s,\theta}\alpha i^{-5/4}\log^{-3}(1+i)$. We will assume that we have this bound while estimating the second term.

Estimate of (4-30). For this term, the strategy is similar, only now the estimates become somewhat simpler by the arithmetic of the bounds given by Theorem 1.6. Indeed, (4-30) is bounded by

$$c_{s,\theta}\,j^{3/4-s}\bigg(\sum_{i>0} i^{s+\theta-7/4}\,\log^{-3}(1+i)\,e^{-c\sqrt{i/j}}\bigg).$$

Much as before, each summand above is bounded by $\int_i^{i+1} x^{s+\theta-7/4}\log^{-3}(1+x)e^{-c\sqrt{x/j}}\,\mathrm{d}x$. Thus, the expression within the parenthesis above is bounded by

$$\begin{aligned}\int_1^{\infty} x^{s+\theta-7/4}\,\log^{-3}(1+x)\,e^{-c\sqrt{x/j}}\,\mathrm{d}x &\lesssim_{s,\theta} j^{s+\theta-3/4}\int_0^{\infty} x^{s+\theta-7/4}\,\log^{-3}(1+jx)\,e^{-c\sqrt{x}}\,\mathrm{d}x \\ &\lesssim_{s,\theta} j^{s+\theta-3/4}\int_0^{\infty} x^{s+\theta-7/4}\,\log^{-3}(1+x)\,e^{-c\sqrt{x}}\,\mathrm{d}x.\end{aligned}$$

This last integral converges given that $s+\theta-\frac{7}{4}>-1 \iff s+\theta>\frac{3}{4}$. In the end, we obtain that (4-30) is bounded by $c_{s,\theta}j^\theta$ if these conditions on s,θ hold.

Finally, we gather these two estimates to get that, if $s-\theta>\frac{7}{4}$, $s+\theta>\frac{3}{4}$ and if $\varepsilon_i<\gamma i^{-5/4}\log^{-3}(1+i)$ for $\gamma>0$ sufficiently small, then (4-29) and (4-30) are bounded by small constants times i^θ and j^θ. Notice that picking $s=10$ and $\theta>0$ sufficiently small yields that both conditions above hold true, and thus the result follows from Schur's test, as previously indicated. □

As mentioned in the beginning of this manuscript, the usage of Schur's test here was instrumental in order to expand the range of our perturbations. In fact, in Section 5A, we employ the Hilbert–Schmidt test successfully to our operator $\widetilde{T}$ and obtain that, as long as there is $\delta>0$ such that $\varepsilon_i\lesssim i^{-5/4-\delta}$, then $\widetilde{T}$ is bounded on $\ell_s^2(\mathbb{N})\times\ell_s^2(\mathbb{N})$ for s sufficiently large, but we seem to be unable to include $\frac{5}{4}$, even with a log-loss, in our considerations with the Hilbert–Schmidt method.

On the other hand, we will see in that subsection that the Hilbert–Schmidt method provides us with a way to suitably perturb the origin, a feature we could not obtain with Schur's test.

5. Applications of the main results and techniques

5A. *Interpolation formulae perturbing the origin.* In the main results of this manuscript, the only interpolation node that remains unchanged in every scenario is 0. One of the reasons for that is aesthetic: we are concerned mainly with even functions here, so the origin keeps a sense of symmetry. The other main reason is technical: we recall that the operator

$$T : \ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+) \to \ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$$

defined in Section 4B is the identity only when restricted to the set of pairs of sequences satisfying

$$\sum_{n \in \mathbb{Z}} x_{n^2} = \sum_{n \in \mathbb{Z}} y_{n^2}.$$

For general sequences, the first entries of this operator possess a correction factor due to the lack of Poisson summation. Indeed, the kernel of T is the set of all $(x, y) \in \ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$ such that

$$\begin{aligned} x_n &= y_n = 0 \quad \text{for all } n \geq 1, \\ x_0 &= -y_0. \end{aligned}$$

Furthermore, the cokernel of T is the set where

$$x_0 - y_0 + 2 \sum_{n \in \mathbb{N}} x_n^2 - 2 \sum_{n \in \mathbb{N}} y_n^2 = 0.$$

This means $\dim(\ker(T)) = \dim(\operatorname{coker}(T)) = 1$. Therefore we can no longer prove invertibility. Nonetheless, since the kernel and cokernel of T are finite-dimensional, T is a Fredholm operator; see the comments on [Brezis 2011, p. 168] for more details.

We denote by $\mathrm{e}_n \in \ell^2_s(\mathbb{Z}_+)$ the vector consisting of $\max\{1, n\}^{-s}$ on the n-th entry, and zero otherwise. With this definition, the set

$$\{(\mathrm{e}_n, \mathbf{0}) : n \in \mathbb{Z}_+\} \cup \{(\mathbf{0}, \mathrm{e}_n) : n \in \mathbb{Z}_+\}$$

forms an orthonormal basis of $\ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$. Thus, for a general operator,

$$\|A\|^2_{HS(\ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+))} = \sum_{n \geq 0} (\|A(\mathrm{e}_n, \mathbf{0})\|^2_{(s,s)} + \|A(\mathbf{0}, \mathrm{e}_n)\|^2_{(s,s)}),$$

where we denote by $\|\cdot\|_{(s,s)}$ the norm of $\ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$. Next we estimate the Hilbert–Schmidt norm in the case where $A = I - \widetilde{T}$.

Claim 5.1. $\|I - \widetilde{T}\|_{HS(\ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+))} < +\infty$ *holds whenever there is* $\delta > 0$ *so that* $|\varepsilon_k| \lesssim k^{-5/4-\delta}$ *for all* $k \geq 1$.

Proof of Claim 5.1. As mentioned before, we can write the identity on $\ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$ as

$$I(\{x_i\}, \{y_i\}) = ((x_0, \mathfrak{G}(1), \mathfrak{G}(\sqrt{2}), \dots), (y_0, \widehat{\mathfrak{G}}(1), \widehat{\mathfrak{G}}(\sqrt{2}), \dots)),$$

where we define the function $\mathfrak{G}$ as in (4-24). With this notation, the operator $\widetilde{T}$ becomes

$$\widetilde{T}(\{x_i\}, \{y_i\}) = \big((x_0, \mathfrak{G}(\sqrt{1+\varepsilon_1}), \mathfrak{G}(\sqrt{2+\varepsilon_2}), \dots), (y_0, \widehat{\mathfrak{G}}(\sqrt{1+\varepsilon_1}), \widehat{\mathfrak{G}}(\sqrt{2+\varepsilon_2}), \dots)\big).$$

Therefore, evaluating at the basis vectors gives us that $(I-\widetilde{T})(\mathrm{e}_n, \mathbf{0})$ equals

$$\big((0, \max\{1,n\}^{-s}(a_n(1)-a_n(\sqrt{1+\varepsilon_1})), \max\{1,n\}^{-s}(a_n(\sqrt{2})-a_n(\sqrt{2+\varepsilon_2})), \dots), \\ (0, \max\{1,n\}^{-s}(\hat{a}_n(1)-\hat{a}_n(\sqrt{1+\varepsilon_1})), \max\{1,n\}^{-s}(\hat{a}_n(\sqrt{2})-\hat{a}_n(\sqrt{2+\varepsilon_2})), \dots)\big).$$

We readily see then that

$$\|I-\widetilde{T}\|^2_{HS(\ell^2_s(\mathbb{Z}_+)\times\ell^2_s(\mathbb{Z}_+))} \le 2^{2s+1}\sum_{n\ge 0}\bigg(\sum_{k\ge 1}(1+k)^{2s}(1+n)^{-2s}|a_n(\sqrt{k})-a_n(\sqrt{k+\varepsilon_k})|^2\bigg) \\ + 2^{2s+1}\sum_{n\ge 0}\bigg(\sum_{k\ge 1}(1+k)^{2s}(1+n)^{-2s}|\hat{a}_n(\sqrt{k})-\hat{a}_n(\sqrt{k+\varepsilon_k})|^2\bigg). \tag{5-1}$$

To bound the terms involving a_0 and $\hat{a}_0$, we simply appeal to the fact these functions are of Schwartz class to use an estimate like (4-27) and obtain

$$\sum_{k\ge 1}(1+k)^{2s}\big(|a_0(\sqrt{k+\varepsilon_k})-a_0(\sqrt{k})|^2+|\hat{a}_0(\sqrt{k+\varepsilon_k})-\hat{a}_0(\sqrt{k})|^2\big) \\ = \sum_{k\ge 1}(1+k)^{2s}\big(|a_0(\sqrt{k+\varepsilon_k})|^2+|\hat{a}_0(\sqrt{k+\varepsilon_k})|^2\big) \le C_s\sum_{k\ge 1}|\varepsilon_k|^2\frac{(1+k)^{2s}}{k^{2s+2}}.$$

From Theorem 1.6, we know that when $n>1$ there is a $c>0$ such that

$$|a_n(\sqrt{k})-a_n(\sqrt{k+\varepsilon_k})| \le \int_{\sqrt{k}}^{\sqrt{k+\varepsilon_k}} |a_n'(t)|\,\mathrm{d}t \le \frac{C\varepsilon_k}{\sqrt{k}} n^{3/4}\,\log^3(1+n)\,e^{-c\sqrt{k/n}} \tag{5-2}$$

for every $k\ge 1$. Analogously, for $n>1$,

$$|\hat{a}_n(\sqrt{k})-\hat{a}_n(\sqrt{k+\varepsilon_k})| \le \frac{C\varepsilon_k}{\sqrt{k}} n^{3/4}\,\log^3(1+n)\,e^{-c\sqrt{k/n}}.$$

These estimates plus the condition $|\varepsilon_k|\le ak^{-5/4-\delta}$ for some $a>0$ imply that (5-1) may be bounded from above by a constant that depends on s times

$$a^2\sum_{n\ge 1}\bigg(\sum_{k\ge 1}k^{2s}k^{-5/2-2\delta}\cdot k^{-1}e^{-2c\sqrt{k/n}}\bigg)n^{3/2-2s}\log^6(1+n)+a^2\sum_{k\ge 1}k^{-15/2-2\delta}. \tag{5-3}$$

The second term in the sum above is convergent, so it is not a problem. Now, in order to prove convergence of the first term, we first investigate the inner sum. A Riemann sum approach together with a change of variables shows that the first term in (5-3) is bounded by a constant times

$$(1+n)^{2s-5/2-2\delta}\log^6(1+n)\bigg(\int_0^\infty t^{2s}t^{-5/2-2\delta}\cdot t^{-1}e^{-c\sqrt{t}}\,\mathrm{d}t\bigg) =: (1+n)^{2s-5/2-2\delta}\log^6(1+n)I_{s,\delta}.$$

Clearly, the inner integral converges given that $s>\frac{5}{4}+\delta$. Putting these estimates together with (5-1), we obtain that

$$\|I-\widetilde{T}\|^2_{HS(\ell^2_s(\mathbb{Z}_+)\times\ell^2_s(\mathbb{Z}_+))} \lesssim a^2 I_{s,\delta}\bigg(\sum_{n\ge 0}(1+n)^{-1-2\delta}\log^6(1+n)\bigg) < +\infty,$$

as desired. $\square$

As a direct corollary, we see that, for each $\delta > 0$, there is $a > 0$ so that, if $|\varepsilon_i| \le a i^{-5/4-\delta}$ for every $i > 0$, then

$$\|I - \widetilde{T}\|_{HS(\ell^2_s(\mathbb{Z}_+)\times\ell^2_s(\mathbb{Z}_+))} < 1.$$

In particular, we shall make use of the fact that T is a Fredholm operator by means of such an inequality, with the aid of the following result.

Lemma 5.2 [Schechter 1967, Theorems 2.8 and 2.10]. *Let $\Phi(X, Y)$ denote the set of bounded Fredholm operators between Banach spaces X and Y. If $A \in \Phi(X, Y)$ and $K \in \mathcal{K}(X, Y)$ is a compact operator, then $A + K \in \Phi(X, Y)$ and $i(A) = i(A + K)$, where we define the **index** $i : \Phi(X, Y) \to \mathbb{N}$ by*

$$i(A) = \dim(\ker(A)) - \dim(\operatorname{coker}(A)) =: \alpha(A) - \beta(A).$$

Furthermore, if $\|K\|_{\mathrm{op}}$ is small enough, then it also holds that $\alpha(A + K) \le \alpha(A)$.

Let us then define a new perturbed operator S, defined on $\ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$, such that

$$S^1(\{x_i\}, \{y_i\})_k = \sum_{n\ge 0}(x_n a_n(\sqrt{k+\varepsilon_k}) + y_n \hat{a}_n(\sqrt{k+\varepsilon_k})),$$
$$S^2(\{x_i\}, \{y_i\})_k = S^1(\{y_i\}, \{x_i\})_k$$

for all $k \ge 0$. Notice that we may write $S - T = \widetilde{T} - I + K_0$, where K_0 has finite rank and is bounded, and thus also compact. Therefore, $S = T + (S - T) = T + (\widetilde{T} - I) + K_0$ can be written as sum of a Fredholm operator T and a compact operator $\widetilde{T} - I + K_0$. This already implies that, modulo a finite-dimensional subspace, the sequences $(\{f(\sqrt{k+\varepsilon_k})\}, \{\hat{f}(\sqrt{k+\varepsilon_k})\})$ determine the sequences $(\{f(\sqrt{k})\}, \{\hat{f}(\sqrt{k})\})$. That is, we can determine the function $f \in \mathcal{S}_{\mathrm{even}}(\mathbb{R})$ from its (Fourier) values at the set $\{\sqrt{k+\varepsilon_k}\}_{k\in\mathbb{Z}_+}$, modulo subtracting functions belonging to a finite-dimensional space.

If, however, we make $|\varepsilon_k| < \epsilon k^{-5/4-\delta}$, and $|\varepsilon_0| \le \epsilon$, with ϵ small enough, it is now a routine calculation to conclude that the operator norms of both $I - \widetilde{T} = A$ and K_0 can be made bounded by a sum of an arbitrarily small factor plus something that will depend on a convergent series multiplied by the value of $|\varepsilon_0|$, which can made arbitrarily small by choosing ϵ properly. Thus,

$$i(S) = i(T + (S - T)) = i(T) = 0 \iff \alpha(S) = \beta(S),$$

and, moreover,

$$\alpha(S) \le \alpha(T),$$

as the Hilbert–Schmidt norm of the difference is small. Thus, either

$$\alpha(S) = \beta(S) = 0,$$

in which case we can perfectly invert the operator S, or

$$\alpha(S) = \beta(S) = 1,$$

which implies that there is essentially *at most one* function $f_0 \in \mathcal{S}_{\mathrm{even}}(\mathbb{R})$ that vanishes at $\sqrt{k+\varepsilon_k}$. As $(\{f(\sqrt{k+\varepsilon_k})\}, \{\hat{f}(\sqrt{k+\varepsilon_k})\}) \in \operatorname{Im}(S)$ for every real $f \in \mathcal{S}_{\mathrm{even}}(\mathbb{R})$, we have proved the following result.

Theorem 5.3. *Let T, S, $\{\varepsilon_i\}_{i\ge 0}$ be as above. Then one of the following holds*:

(1) *either S is an isomorphism from $\ell^2_s(\mathbb{Z}_+)\times\ell^2_s(\mathbb{Z}_+)$ onto itself, and thus the values*

$$(\{f(\sqrt{j+\varepsilon_j})\},\{\hat f(\sqrt{j+\varepsilon_j})\})$$

determine any real-valued function $f\in\mathcal{S}_{\mathrm{even}}(\mathbb{R})$,

(2) *or $\ker(S)$ has dimension 1, and therefore S is an isomorphism from $\ker(S)^{\perp}$ onto $\mathrm{Im}(S)$.*

In particular, any real-valued function $f\in\mathcal{S}_{\mathrm{even}}(\mathbb{R})$ is uniquely determined by

$$(\{f(\sqrt{j+\varepsilon_j})\},\{\hat f(\sqrt{j+\varepsilon_j})\}),$$

together with the value of

$$\frac{\langle(\{f(\sqrt{j+\varepsilon_j})\},\{\hat f(\sqrt{j+\varepsilon_j})\}),(\{\alpha_i\},\{\beta_i\})\rangle_{(s,s)}}{\|(\{\alpha_i\},\{\beta_i\})\|^2_{(s,s)}},$$

where $(\{\alpha_i\},\{\beta_i\})\in\ker(S)$ is a generator for the kernel of S.

Notice that the first option in Theorem 5.3 yields immediately an interpolation formula, in the spirit of (4-26). For the second one, the operator is now only invertible if restricted to $\ker(S)^{\perp}$, and the process of recovering $f\in\mathcal{S}_{\mathrm{even}}(\mathbb{R}:\mathbb{R})$ has to take into account the inner product with the kernel vector and the structure of the range.

5B. *Uniqueness for small powers of integers.* Let $\alpha\in\left(0,\frac{1}{2}\right)$. Bearing in mind the overall framework of uniqueness formulae in which Theorem 1.4 situates itself, we address the question of determining when the only function $f\in\mathcal{S}_{\mathrm{even}}(\mathbb{R})$ that vanishes together with its Fourier transform at $\pm n^{\alpha}$ is the identically zero function.

Indeed, we would like to study the natural operator that sends the sequence of values at the roots of integers $(\{f(\sqrt{k})\}_k,\{\hat f(\sqrt{k})\}_k)$ to the sequence $(\{f(n^{\alpha})\}_n,\{\hat f(n^{\alpha})\}_n)$. Our goal is to show that this operator is injective. In order to do that, we will first study simpler operators.

In fact, let $K_0\in\mathbb{N}$ be a fixed positive integer. Fix a set of $2K_0$ positive real numbers $t_1<t_2<\cdots<t_{2K_0}$ such that $t_1>\sqrt{K_0}$ and none of the t_j can be written as a square root of a positive integer. We fix $s>0$ sufficiently large and define the operator

$$\begin{aligned}
T_{K_0}:\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N})&\to\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N}),\\
(\{x_i\}_i,\{y_i\}_i)&\mapsto\big((x_0,\mathfrak{G}(t_1),\mathfrak{G}(t_2),\dots,\mathfrak{G}(t_{2K_0}),x_{K_0+1},x_{K_0+2},\dots),\\
&\qquad(y_0,\widehat{\mathfrak{G}}(t_1),\widehat{\mathfrak{G}}(t_2),\dots,\widehat{\mathfrak{G}}(t_{2K_0}),y_{K_0+1},y_{K_0+2},\dots)\big).
\end{aligned}$$

Here, we denoted by $\mathfrak{G}$ the function defined as in (4-24). Recall that $\mathfrak{G}$ depends itself on $\{x_i\}_i$, $\{y_i\}_i$, and thus, for fixed t, $\mathfrak{G}(t)$ and $\widehat{\mathfrak{G}}(t)$ are both linear functionals on $\ell^2_s(\mathbb{Z}_{\ge 0})\times\ell^2_s(\mathbb{Z}_{\ge 0})$.

Lemma 5.4. *For any $K_0\ge 1$ and $\{t_j\}_{j=1,\dots,2K_0}$ as above, the operator T_{K_0} is bounded and injective.*

Proof. We begin with the boundedness assertion. As T_{K_0} differs only in at most the first $2K_0+1$ coordinates from an iteration of the shift operator

$$s((\{x_i\}_i,\{y_i\}_i)=((0,x_0,x_1,\dots),(0,y_0,y_1,\dots)),$$

boundedness follows from boundedness of the operator that maps a pair of sequences $(\{x_i\}_i, \{y_i\}_i) \in \ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N})$ to

$$\big((x_0, \mathfrak{G}(t_1), \mathfrak{G}(t_2), \dots, \mathfrak{G}(t_{2K_0}), 0, \dots), (y_0, \widehat{\mathfrak{G}}(t_1), \widehat{\mathfrak{G}}(t_2), \dots, \widehat{\mathfrak{G}}(t_{2K_0}), 0, \dots)\big).$$

As $\mathfrak{G}, \widehat{\mathfrak{G}} \in L^\infty(\mathbb{R})$ for any pair of sequences $\{x_i\}, \{y_i\}$, with bounds depending only on the $\ell_s^2(\mathbb{N})$-norms of the sequences, it follows that this new finite-rank operator is bounded.

The injectivity part is subtler. Indeed, fix a pair of sequences $(\{x_i\}, \{y_i\}) \in \ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N})$, and suppose that $T_{K_0}(\{x_i\}, \{y_i\}) = 0$. It follows that the special function $\mathfrak{G}(t)$ is a linear combination of $a_1, \dots, a_{K_0}, \hat{a}_1, \dots, \hat{a}_{K_0}$. In order to analyze such functions, we will need to investigate further the intrinsic form of the interpolating functions a_n, and thus those of $b_n^\pm$. As the reader will see in the analysis below, we will show that the functions $\mathfrak{G} \pm \widehat{\mathfrak{G}}$ have at most K_0+1 zeros on $(\sqrt{K_0}, +\infty)$ from the assertions above. This is, indeed, the reason why we need to use $2K_0$ different values in order to prove injectivity.

Indeed, it follows from the Fourier expansion of $g_n^\pm$ near infinity and the formula

$$b_n^\pm(x) = \frac{1}{2} \int_{-1}^{1} g_n^\pm(z) e^{\pi i x^2 z} \, \mathrm{d}z \tag{5-4}$$

that, whenever $|x| > \sqrt{n}$, it can also be represented as

$$b_n^\pm(x) = \sin(\pi x^2) \int_0^\infty g_n^\pm(1+it) e^{-\pi x^2 t} \, \mathrm{d}t. \tag{5-5}$$

To see this, one shifts contours in (5-4) over the rectangular path passing through $-1, -1+iT, 1+iT$ and 1. The condition $|x| > \sqrt{n}$ comes into play in order to guarantee that one may safely send T to ∞, and the results in [Radchenko and Viazovska 2019] show that $g_n^\varepsilon(s+iR)$ grows as $e^{\pi n R}$ at infinity for fixed $s \in \mathbb{R}$. With (5-5) in mind and the facts that $a_n = (b_n^+ + b_n^-)/2$ and $\hat{a}_n = (b_n^+ - b_n^-)/2$, we see that the Fourier invariant part of $\mathfrak{G}$ may be written as

$$(\mathfrak{G} + \widehat{\mathfrak{G}})(x) = \sin(\pi x^2) \int_0^\infty \left(\sum_{j=1}^{K_0} \alpha_j g_j^+(1+it) \right) e^{-\pi x^2 t} \, \mathrm{d}t$$

for some sequence α_j of real numbers, and an analogous identity holds for $\mathfrak{G} - \widehat{\mathfrak{G}}$, with g_n^- instead. We recall that the weakly holomorphic modular forms $g_n^\pm$ satisfy that

$$g_n^+(z) = \theta(z)^3 P_n^+(1/J(z)),$$
$$g_n^-(z) = \theta(z)^3 (1 - 2\lambda(z)) P_n^-(1/J(z)),$$

where the monic polynomials P_n^-, P_n^+ are of degree n. Therefore, there are polynomials Q, R of degree $\le K_0$ such that

$$\mathfrak{G} + \widehat{\mathfrak{G}} = \sin(\pi x^2) \int_0^\infty \theta(1+it)^3 Q\left(\frac{1}{J(1+it)}\right) e^{-\pi x^2 t} \, \mathrm{d}t,$$
$$\mathfrak{G} - \widehat{\mathfrak{G}} = \sin(\pi x^2) \int_0^\infty \theta(1+it)^3 (1 - 2\lambda(1+it)) R\left(\frac{1}{J(1+it)}\right) e^{-\pi x^2 t} \, \mathrm{d}t.$$

Before moving forward, we need the following result:

Lemma 5.5. *The factors* $\theta(1+it)^3$ *and* $(1-2\lambda(1+it))$ *do not change sign for* $t \in (0,\infty)$, *and the function* $1/J(1+it)$ *is real-valued and monotonic for* $t \in (0,\infty)$.

Proof. By using (2-1), we get that

$$\theta(1+it) = \sum_{n\in\mathbb{Z}} (-1)^n e^{-\pi n^2 t} = \sum_{n\in\mathbb{Z}} e^{-4\pi n^2 t} - \sum_{n\in\mathbb{Z}} e^{-\pi(2n+1)^2 t}.$$

We now consider the function $f_t(x) = e^{-\pi(2x)^2 t}$. Then the sum above equals

$$\sum_{n\in\mathbb{Z}} f_t(n) - \sum_{n\in\mathbb{Z}} f_t\big(n+\tfrac{1}{2}\big).$$

By the Poisson summation formula, the difference above equals

$$\frac{1}{2\sqrt{t}}\bigg(\sum_{n\in\mathbb{Z}} e^{-\pi(n/(2\sqrt{t}))^2} - \sum_{n\in\mathbb{Z}} e^{\pi i n} e^{-\pi(n/(2\sqrt{t}))^2}\bigg) = \frac{1}{\sqrt{t}} \sum_{n \text{ odd}} e^{-\pi(n/(2\sqrt{t}))^2} \geq 0.$$

This proves the first assertion.

For the second, we simply see from (2-2) that $\lambda(1+z)$ has only nonpositive coefficients in its q-series expansion. This implies that $\lambda(1+it)$ is nonpositive for $t \in (0,\infty)$, which implies that $1-2\lambda(1+it)$ is always nonnegative.

Finally, for the third assertion, we notice that, as $J(1+z) = \frac{1}{16}\lambda(1+z)(1-\lambda(1+z))$, and thus, from the analysis above, the q-series expansion of $J(1+z)$ contains only nonpositive coefficients. Therefore, the function $1/J(1+it)$ is nonpositive for $t \in (0,\infty)$, and it is monotonically decreasing there. □

By Lemma 5.5, we get that the part of the integrand in the expressions above multiplying the $e^{-\pi x^2 t}$ factor changes sign at most K_0+1 times. Notice that we can embed both integrals in (5-6) into the framework of Laplace transforms: defining

$$\mathcal{Q}(t) = \theta(1+it)^3 Q(1/J(1+it)), \quad \mathcal{R}(t) = \theta(1+it)^3(1-2\lambda(1+it))R(1/J(1+it)),$$

we are interested in studying the positive zeros of $\mathcal{L}[\mathcal{Q}](\pi x^2)$, $\mathcal{L}[\mathcal{R}](\pi x^2)$, where

$$\mathcal{L}[\phi](s) = \int_0^\infty \phi(t) e^{-st}\,\mathrm{d}t$$

denotes the Laplace transform of ϕ evaluated at the point s. We may reduce even further our task to studying the positive zeros of $\mathcal{L}[\mathcal{Q}]$, $\mathcal{L}[\mathcal{R}]$. The following result, a version of the Descartes rule for the Laplace transform, is the tool we need to bound the number of positive zeros of such expressions as a function of the number of sign changes of the function being transformed.

Proposition 5.6 (Descartes rule for the Laplace transform). *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be a smooth function such that its Laplace transform* $\mathcal{L}[\phi]$ *converges on some open half-plane* $\mathrm{Re}(s) > s_0$. *Then the number of zeros of* $\mathcal{L}[\phi]$ *on the interval* $(s_0, +\infty)$ *is at most the number of sign changes of* ϕ.

Proof. The proof follows by induction on the number of sign changes of the function ϕ. Indeed, if $\phi \geq 0$, it follows easily that the Laplace transform satisfies $\mathcal{L}[\phi] \geq 0$, with equality if and only if $\phi \equiv 0$.

Suppose now that ϕ changes sign $n+1$ times on $(0,\infty)$. Number its zeros on the positive half-line as $a_0 < a_1 < \cdots < a_n$. Then $\mathcal{L}[\phi]$ has as many zeros as $e^{a_0 s}\mathcal{L}[\phi](s) = F(s)$. The derivative of F is then given by

$$F'(s) = -\int_0^\infty (t-a_0)\phi(t)e^{-(t-a_0)s}\,\mathrm{d}t = e^{a_0 s}\mathcal{L}[(t-a_0)\phi(s)](s).$$

Notice that the new smooth function $(t-a_0)\phi(t)$ still satisfies the same properties as ϕ, but now has exactly n sign changes. By inductive hypothesis, F' has at most n zeros, which, by the mean value theorem, implies that F has at most $n+1$ zeros. □

Using this claim for $\mathcal{Q}, \mathcal{R}$, we see that their respective Laplace transforms possess at most K_0 zeros on the interval $(\sqrt{K_0}, +\infty)$. With this information, we can already finish: From (5-6), the functions $\mathfrak{G} \pm \widehat{\mathfrak{G}}$ can only vanish at at most K_0 points on the interval $(\sqrt{K_0}, \infty)$ which are not roots of positive integers, in the case $\mathfrak{G} \not\equiv 0$. But, according to our assumption that $(\{x_i\}, \{y_i\}) \in \ker(T_{K_0})$, we have $\mathfrak{G}(t_j) = \widehat{\mathfrak{G}}(t_j) = 0,\ j = 1, \ldots, 2K_0$. By the properties we chose for the sequence t_j, $\mathfrak{G} \equiv 0$, and thus the map T_{K_0} is injective. □

We need one more result in order to infer results about uniqueness for small powers of integers. In contrast to the full perturbation case of our main theorem, we must prove that the injective operators T_{K_0} are also somewhat stable with respect to injectivity under perturbations. In order to do this, the following result is essential.

Lemma 5.7. *The range of T_{K_0} is closed.*

Proof. Suppose the sequence in $\ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N})$ given by $\{T_{K_0}(\{x_i^j\}, \{y_i^j\})\}_{j\geq 0}$ is a Cauchy sequence. This implies that the sequence $\{\{x_i^j\}_{i=0,K_0+1,\ldots}, \{y_i^j\}_{i=0,K_0+1,\ldots}\}_{j\geq 0}$ is a Cauchy sequence, and therefore it converges to a certain limiting sequence

$$\{\{x_i\}_{i=0,K_0+1,\ldots}, \{y_i\}_{i=0,K_0+1,\ldots}\} \in \ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N}).$$

Define, thus, the $4K_0 \times 2K_0$ matrix A_{K_0} given by taking

$$(a_1(t_j), a_2(t_j), \ldots, a_{K_0}(t_j), \hat{a}_1(t_j), \hat{a}_2(t_j), \ldots, \hat{a}_{K_0}(t_j))$$

and

$$(\hat{a}_1(t_j), \hat{a}_2(t_j), \ldots, \hat{a}_{K_0}(t_j), a_1(t_j), a_2(t_j), \ldots, a_{K_0}(t_j))$$

to be its lines for $j = 1, \ldots, 2K_0$. We first claim that this matrix is injective. Indeed,

$$\widetilde{\mathfrak{G}}(t) = \sum_{i=1}^{K_0}(x_i a_i(t) + y_i \hat{a}_i(t))$$

vanishes, together with its Fourier transform, at $t_j,\ j = 1, \ldots, 2K_0$, where $(\{x_i\}_{i=1}^{K_0}, \{y_i\}_{i=1}^{K_0})$ belongs to $\ker(A_{K_0})$. By the proof of Lemma 5.4, this implies $x_i = y_i = 0,\ i = 1, \ldots, K_0$.

As A_{K_0} is injective, there is a constant $c_{K_0} > 0$ so that

$$\|A_{K_0}\boldsymbol{v}\|_{4K_0} \geq c_{K_0}\|v\|_{2K_0}, \tag{5-6}$$

where we denote by $\|\cdot\|_d$ the usual euclidean norm on a d-dimensional space. Translating to our original problem, as $\{T_{K_0}(\{x_i^j\},\{y_i^j\})\}_{j\geq 0}$ is a Cauchy sequence in $\ell_s^2(\mathbb{N})\times\ell_s^2(\mathbb{N})$,

$$\{\{x_i^j\}_{i=0,K_0+1,\dots},\{y_i^j\}_{i=0,K_0+1,\dots}\}_{j\geq 0}$$

is a convergent sequence, and thus we get that the sequences

$$\sum_{i=1}^{K_0}(x_i^k a_i(t_j)+y_i^k\hat{a}_i(t_j)),\quad j=1,\dots,2K_0,$$

are also Cauchy in $k\geq 0$. By (5-6), $(\{x_i^k\}_{i=1}^{K_0},\{y_i\}_{i=1}^{K_0})_{k\geq 0}$ is Cauchy. This implies that there is a limiting sequence $(\{x_i\},\{y_i\})\in\ell_s^2(\mathbb{N})\times\ell_s^2(\mathbb{N})$ so that

$$T_{K_0}(\{x_i^j\},\{y_i^j\})\to T_{K_0}(\{x_i\},\{y_i\})\quad\text{as } j\to\infty. \qquad\square$$

We are finally able to prove the following uniqueness result:

Corollary 5.8. *Let $\alpha\in\left(0,\frac{2}{9}\right)$. There exists $c_\alpha>0$ such that the following holds. For each $c\in(0,c_\alpha)$, if $f\in\mathcal{S}_{\text{even}}(\mathbb{R})$ is a real-valued function that vanishes together with its Fourier transform at $\pm c\cdot n^\alpha$, then $f\equiv 0$.*

Moreover, for each $n_0>1$, the same assertion as before holds under the weaker assumption that f vanishes together with $\hat{f}$ at $\pm c\cdot m^\alpha$, where $m\in\{0\}\cup(m_\alpha(n_0),+\infty)$ and $m_\alpha(n_0)=\min\{n\in\mathbb{N}:cn^\alpha>n_0\}$.

Notice that the second assertion above, albeit technical, merely means we may start the sequences of nonzero roots of f, $\hat{f}$ as far away from the origin as we wish, as long as one keeps it under a certain threshold in terms of denseness.

Proof. Fix $\alpha\in\left(0,\frac{2}{9}\right)$ and let $c>0$ be a constant, to be precisely chosen later, which is allowed to depend only on α. We start by noticing that, for each $\alpha\in\left(0,\frac{2}{9}\right)$, there is $n_0(\alpha)\geq 1$ such that whenever $n\in\mathbb{N}$ is greater than $n_0(\alpha)$, then there is $m\in\mathbb{N}$ so that for all $n\geq n_0(\alpha)$, there exists $m\in\mathbb{N}$ so that we can write $c\cdot m^\alpha=\sqrt{n+\varepsilon_n}$, where $\{\varepsilon_n\}_n$ satisfies the conditions of Theorem 1.4. Indeed, start by noticing that simply letting $m=\lceil (n/c^2)^{1/(2\alpha)}\rceil$ implies $|\sqrt{n}-cm^\alpha|\lesssim c^{1/\alpha}n^{(\alpha-1)/(2\alpha)}$.

Indeed,

$$|\sqrt{n}-cm^\alpha|=c\alpha\int_{(n/c^2)^{1/(2\alpha)}}^{\lceil (n/c^2)^{1/(2\alpha)}\rceil}t^{\alpha-1}\,\mathrm{d}t\lesssim c^{1/\alpha}\alpha n^{(\alpha-1)/(2\alpha)}. \tag{5-7}$$

In particular, if $(\alpha-1)/(2\alpha)<-\frac{5}{4}-\frac{1}{2}\iff\alpha<\frac{2}{9}$, the assertion follows. Let us single out the sequence of numbers selected above, which we index as $\{c\cdot m(n)^\alpha\}_{n\geq n_0(\alpha)}$. We then consider the operator $T_{n_0(\alpha)}$ associated to some sequence of $2n_0(\alpha)$ positive real numbers t_j, $j=1,\dots,2n_0(\alpha)$, satisfying the hypotheses of Lemma 5.4.

We claim that the *perturbed* operator

$\widetilde{T}_{n_0(\alpha)}:\ell_s^2(\mathbb{N})\times\ell_s^2(\mathbb{N})\to\ell_s^2(\mathbb{N})\times\ell_s^2(\mathbb{N})$ that takes a pair $(\{x_i\},\{y_i\})$ to

$$\begin{aligned}\big((x_0,\mathfrak{G}(t_1),\mathfrak{G}(t_2),\dots,\mathfrak{G}(t_{2n_0}),\mathfrak{G}(c\cdot m(n_0+1)^\alpha),\mathfrak{G}(c\cdot m(n_0+2)^\alpha),\dots),\\ (y_0,\widehat{\mathfrak{G}}(t_1),\widehat{\mathfrak{G}}(t_2),\dots,\widehat{\mathfrak{G}}(t_{2n_0}),\widehat{\mathfrak{G}}(c\cdot m(n_0+1)^\alpha),\widehat{\mathfrak{G}}(c\cdot m(n_0+2)^\alpha),\dots)\big)\end{aligned} \tag{5-8}$$

is injective for some $s>0$ that depends on α. Indeed, from Lemma 5.7 there must exist a constant C_{n_0} so that

$$\|T_{n_0}\boldsymbol{v}\|_{(s,s)} \geq C_{n_0}\|\boldsymbol{v}\|_{(s,s)}$$

holds for all $\boldsymbol{v}\in \ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N})$. But, by the same calculation as in the previous subsection, we have that

$$\|\widetilde{T}_{n_0(\alpha)} - T_{n_0(\alpha)}\|_{HS(\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N}))} < \frac{C_{n_0}}{2}$$

holds, as long as we take $\alpha<\frac{2}{9}$ and $c=c(\alpha)$ sufficiently small, because (5-7) implies we satisfy the conditions of Theorem 5.3. This implies, in particular, that

$$\|\widetilde{T}_{n_0}\boldsymbol{v}\|_{(s,s)} \geq \frac{C_{n_0}}{2}\|\boldsymbol{v}\|_{(s,s)}$$

for each $\boldsymbol{v}\in \ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N})$, and thus the operator $\widetilde{T}_{n_0}$ is, indeed, injective, as desired.

In order to conclude, we notice that the operator

$\mathcal{T}_{n_0(\alpha)}:\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N})\to\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N})$ that takes a pair $(\{x_i\},\{y_i\})$ to

$$\begin{aligned}\big((x_0, \mathfrak{G}(ck_1^\alpha), \mathfrak{G}(ck_2^\alpha), \dots, \mathfrak{G}(ck_{2n_0}^\alpha), \mathfrak{G}(c\cdot m(n_0+1)^\alpha), \mathfrak{G}(c\cdot m(n_0+2)^\alpha), \dots),\\ (y_0, \widehat{\mathfrak{G}}(ck_1^\alpha), \widehat{\mathfrak{G}}(ck_2^\alpha), \dots, \widehat{\mathfrak{G}}(ck_{2n_0}^\alpha), \widehat{\mathfrak{G}}(c\cdot m(n_0+1)^\alpha), \widehat{\mathfrak{G}}(c\cdot m(n_0+2)^\alpha), \dots)\big),\end{aligned} \tag{5-9}$$

for some sequence $k_j,\ j=1,\dots,2n_0$, of integers not belonging to the sequence $m(n)$ we selected above, is still injective. In fact, it only differs from the operator $\widetilde{T}_{n_0}$ in at most $2n_0$ entries. But, on the other hand, for $k_j=\lfloor (t_j/c)^{1/\alpha}\rfloor,\ j=1,\dots,2n_0$, and $c>0$ sufficiently small, we see by Theorem 1.6 that

$$\begin{aligned}|\mathfrak{G}(ck_j^\alpha)-\mathfrak{G}(t_j)| &\leq \sum_{i=0}^{\infty}(|x_i|\,|a_i(t_j)-a_i(ck_j^\alpha)| + |y_i|\,|\hat{a}_i(t_j)-\hat{a}_i(ck_j^\alpha)|)\\ &\lesssim \sup_{1\leq l\leq 2n_0}|t_l - ck_l^\alpha|\left(\sum_{i=0}^{\infty}(1+i)^{5/2}(|x_i|+|y_i|)\right)\\ &\lesssim \epsilon\|(\{x_i\},\{y_i\})\|_{(s,s)}.\end{aligned}$$

Here, note that ϵ depends on $c>0$ and α, and tends to 0 as $c\to 0$. For $\epsilon>0$ sufficiently small, we see from the previous argument that $\mathcal{T}_{n_0(\alpha)}$ still has closed range and is injective. Thus, by taking $c_\alpha>0$ sufficiently small we have that the sequence $(\{f(\pm n^\alpha)\},\{\hat{f}(\pm n^\alpha)\})$ determines uniquely the sequence $(\{f(\sqrt{n})\},\{\hat{f}(\sqrt{n})\})$. This finishes the proof of the first assertion.

The assertion about being able to restrict the first node $c_\alpha m^\alpha$ to be as large as we want follows in the exact same way, and we thus omit it. □

One can inquire about the importance of such a result; as in [Ramos and Sousa 2022] we have shown that the uniqueness result stated in Corollary 5.8 holds for $\alpha\in\big(0,1-\frac{\sqrt{2}}{2}\big)$, which is significantly larger than the range stated here. Nonetheless, Corollary 5.8 gives us *automatic* results. Indeed, if one manages to prove that for all $\delta>0$ there is $\epsilon>0$ so that, if $|\varepsilon_k|\leq\epsilon$ for all $k\in\mathbb{N}$, then

$$\|I-\widetilde{T}\|_{\mathrm{op}}<\delta,$$

it implies automatically that we can extend the results in Corollary 5.8 to the full diagonal range $\alpha\in\big(0,\frac{1}{2}\big)$.

We also note that Corollary 5.8 is not all we can say about the problem of determining the best exponents (α, β) so that

$$f(\pm n^{\alpha}) = \hat{f}(\pm n^{\beta}) = 0, \quad f \in \mathcal{S}_{\text{even}}(\mathbb{R}) \implies f \equiv 0.$$

Indeed, we can easily go further than the diagonal case detailed above: if $\alpha, \beta \in \left(0, \frac{2}{9}\right)$ are arbitrary exponents, we notice that we can still pick $n_0 \in \mathbb{N}$ so that for each $n > n_0 = n_0(\alpha, \beta)$, there exists a pair $(m_1(n), m_2(n)) \in \mathbb{N}^2$ so that

$$|cm_1(n)^{\alpha} - \sqrt{n}| + |cm_2(n)^{\beta} - \sqrt{n}| \lesssim c^{1/\alpha} \alpha n^{(\alpha-1)/(2\alpha)} + c^{1/\beta} \beta n^{(\beta-1)/(2\beta)},$$

and the right-hand side can be made $\ll n^{-5/4-\delta}$ for some $\delta > 0$. This induces us to consider the operator

$\mathcal{T}_{n_0(\alpha,\beta)} : \ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N}) \to \ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N})$ taking pairs $(\{x_i\}, \{y_i\})$ to

$$\begin{aligned}\big((x_0, \mathfrak{G}(ck_1^{\alpha}), \mathfrak{G}(ck_2^{\alpha}), \dots, \mathfrak{G}(ck_{2n_0}^{\alpha}), \mathfrak{G}(m_1(n_0+1)^{\alpha}), \mathfrak{G}(m_1(n_0+2)^{\alpha}), \dots),\\ (y_0, \widehat{\mathfrak{G}}(cl_1^{\beta}), \widehat{\mathfrak{G}}(cl_2^{\beta}), \dots, \widehat{\mathfrak{G}}(cl_{2n_0}^{\beta}), \widehat{\mathfrak{G}}(m_2(n_0+1)^{\beta}), \widehat{\mathfrak{G}}(m_2(n_0+2)^{\beta}), \dots)\big)\end{aligned} \tag{5-10}$$

for two sequences of integers $(k_j, l_j),\ j = 1, \dots, 2n_0$, so that $|t_j - ck_j^{\alpha}| + |t_j - cl_j^{\beta}|$ is sufficiently small for all $j \in [0, 2n_0]$, where we select $t_j,\ j = 1, \dots, 2n_0$, satisfying the hypotheses of Lemma 5.4.

By the same strategy outlined in the proof of Corollary 5.8, the Hilbert–Schmidt norm as operators acting on $\ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N})$ of the difference $T_{n_0(\alpha,\beta)} - \mathcal{T}_{n_0(\alpha,\beta)}$ is arbitrarily small, as long as we make the value of $c = c(\alpha, \beta)$ smaller. As a consequence, $\mathcal{T}_{n_0}$ is also injective and its range is closed. These considerations prove, therefore, the following:

Corollary 5.9. *Let $\alpha, \beta \in \left(0, \frac{2}{9}\right)$. Then there is $c_{\alpha,\beta} > 0$ so that the following holds. For all $c \in (0, c_{\alpha,\beta})$, if $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ is a real-valued function that vanishes at $\pm cn^{\alpha}$ and its Fourier transform vanishes at $\pm cn^{\beta}$, then $f \equiv 0$.*

Moreover, for each $n_0 > 1$, the same assertion above holds under the weaker assumption that f vanishes for $\pm c \cdot m^{\alpha}$ and $\hat{f}$ vanishes for $\pm c \cdot k^{\beta}$, where $m \in \{0\} \cup (m_{\alpha,\beta}(n_0), +\infty)$, $k \in \{0\} \cup (k_{\alpha,\beta}(n_0), +\infty)$, and $m_{\alpha,\beta}(n_0)$, $n_{\alpha,\beta}(n_0)$ are the least positive integers such that $c \cdot m^{\alpha} > n_0$ and $c \cdot k^{\beta} > n_0$, respectively.

Remark. In the end, we do not quite attain the primary goal of this section of proving Fourier uniqueness results for the sequences $(\{\pm n^{\alpha}\}, \{\pm n^{\beta}\})$, but only a slightly weaker version of it, with a small constant $c(\alpha, \beta)$ in front. The main reason for that in the proofs above is the location of the positive reals t_i: although their exact values do not matter in the end, it is crucial, in order to use Proposition 5.6, that they lie *after* the node n_0. We must therefore either force n_0 not to be too large in order not to make the norm of the matrix A_{K_0} too small, or fix them from the beginning and make the perturbations of T_{K_0} fall closer to it. In any case, this implies nontrivial use of the constant c multiplying the sequences $(\{\pm n^{\alpha}\}, \{\pm n^{\beta}\})$.

We believe that further studying operators resembling T_{K_0} above and their injectivity properties could yield better results in this regard. In order not to make this exposition even longer, we will not pursue this matter any further.

5C. ***Annihilating pairs.*** As an application of the results above, we will prove some *strong annihilating* properties of the sets $\{\pm c_\alpha n^\alpha\}_{n\in\mathbb{N}}$, $\{\pm c_\beta n^\beta\}_{n\in\mathbb{Z}}$.

Indeed, let $A, B \subset \mathbb{R}$ be two discrete sets. Inspired by the results and definitions of [Benedicks 1985; Amrein and Berthier 1977] (see also [Nazarov 1993]), we say that $(\mathbb{R}\setminus A, \mathbb{R}\setminus B)$ is a *weakly annihilating pair* for a class $\mathcal{C} \subset L^2(\mathbb{R})$ if whenever $f(A) = \hat{f}(B) = \{0\},\ f \in \mathcal{C}$, then $f \equiv 0$.

This definition implies directly that $(\mathbb{R}\setminus\{\pm\sqrt{n}\}_{n\ge 0}, \mathbb{R}\setminus\{\pm\sqrt{n}\}_{n\ge 0})$ is a weakly annihilating pair for $\mathcal{S}_{\text{even}}(\mathbb{R};\mathbb{R})$ due to (1-3). On the other hand, under the hypotheses of Theorem 1.4, it follows directly that $(\mathbb{R}\setminus\{\pm\sqrt{n+\varepsilon_n}\}_{n\ge 0}, \mathbb{R}\setminus\{\pm\sqrt{n+\varepsilon_n}\}_{n\ge 0})$ is also weakly annihilating for $\mathcal{S}_{\text{even}}(\mathbb{R};\mathbb{R})$.

As a natural counterpart, we define a pair $(\mathbb{R}\setminus A, \mathbb{R}\setminus B)$ to be *ω-strongly annihilating* for a class $\mathcal{C} \subset L^2(\mathbb{R}),\ \omega \in \mathbb{R}$, if there is a real number $\gamma \in \mathbb{R}$ such that the inequality

$$\|f\|_{L^2((1+|x|)^\gamma)} + \|\hat{f}\|_{L^2((1+|x|)^\gamma)} \lesssim \left(\sum_{a\in A} |f(a)|^2(1+|a|)^\omega + \sum_{b\in B} |\hat{f}(b)|^2(1+|b|)^\omega\right)^{1/2}$$

holds for all $f \in \mathcal{C}$.

Our first contribution is Theorem 1.7; i.e., the pair $(\mathbb{R}\setminus\{\pm\sqrt{n}\}_{n\ge 0}, \mathbb{R}\setminus\{\pm\sqrt{n}\}_{n\ge 0})$ is ω-strongly annihilating for some $\omega > 0$.

Proof of Theorem 1.7. We start with (1-10). Indeed, consider a sequence $\{\varepsilon_n\}_{n\ge 0}$ of real numbers. We begin by observing that, for all integers $n \ge 1$, we have, by (1-3) together with Theorem 1.6,

$$|f(x) - f(\sqrt{n})| \lesssim \frac{|\varepsilon_n|}{\sqrt{n}} \sum_{m\ge 0} (1+m)^{3/4} \log^3(m+1) e^{-c\sqrt{n/m}} [|f(\sqrt{m})| + |\hat{f}(\sqrt{m})|],$$

whenever $x \in [\sqrt{n}, \sqrt{n+\varepsilon_n})$. Suppose then $|\varepsilon_n| \le \delta(1+|n|)^{-\theta}$ holds for all $n \ge 1$, for some $\theta > 0$ and $\delta > 0$. If one uses the bound above together with the triangle inequality, an integration over the interval $[\sqrt{n}, \sqrt{n+\varepsilon_n})$ and the Cauchy–Schwarz inequality, one obtains

$$\begin{aligned}(1+n)^s|f(\sqrt{n})|^2 \lesssim &\left(\int_{\sqrt{n}}^{\sqrt{n+1}} |f(y)|^2(1+|y|)^{2(\theta+2s)+1}\,\mathrm{d}y\right)\\ &+ \delta \sum_{m\ge 0} (|f(\sqrt{m})|^2 + |\hat{f}(\sqrt{m})|^2)(1+m)^{3/2} \log^6(1+m) e^{-2c\sqrt{n/m}} (1+n)^{-2\theta-1+s}.\end{aligned}$$

If $2\theta + 1 - s > 1 \iff \theta > s/2$, we may sum the right-hand side above in $n \ge 1$ and get a *uniform* constant in $m \ge 0$. This yields

$$\begin{aligned}\sum_{n\ge 1} (1+n)^s |f(\sqrt{n})|^2 \lesssim &\int_{\mathbb{R}} |f(y)|^2(1+|y|)^{2(\theta+2s)+1}\,\mathrm{d}y\\ &+ \delta \sum_{m\ge 0} (|f(\sqrt{m})|^2 + |\hat{f}(\sqrt{m})|^2)(1+m)^{3/2} \log^6(1+m).\end{aligned}$$

An entirely analogous calculation implies the same on the level of Fourier transforms; that is,

$$\begin{aligned}\sum_{n\ge 1} (1+n)^s |\hat{f}(\sqrt{n})|^2 \lesssim &\int_{\mathbb{R}} |\hat{f}(y)|^2(1+|y|)^{2(\theta+2s)+1}\,\mathrm{d}y\\ &+ \delta \sum_{m\ge 0} (|f(\sqrt{m})|^2 + |\hat{f}(\sqrt{m})|^2)(1+m)^{3/2} \log^6(1+m).\end{aligned}$$

Summing these two bounds, if $s > \frac{3}{2}$ and $\delta \ll 1$ is sufficiently small, we obtain

$$\sum_{n\geq 1}(1+n)^s[|f(\sqrt{n})|^2+|\hat{f}(\sqrt{n})|^2] \leq C(\|f\|_{L^2((1+|x|)^\gamma)}+\|\hat{f}\|_{L^2((1+|x|)^\gamma)}), \tag{5-11}$$

which was the desired inequality, except for the $n = 0$ term. In that regard, we notice that a Sobolev embedding argument allows us to include it in the left-hand side of (5-11), which proves (1-10). Notice that we may take, for this part, any $\gamma > 5s + 1$.

For (1-11), we will use once more Theorem 1.6. Indeed, it follows from that and Cauchy–Schwarz that

$$\begin{aligned}|f(x)|+|\hat{f}(x)| &\lesssim \sum_{n\geq 0}[|f(\sqrt{n})|+|\hat{f}(\sqrt{n})|](1+n)^{1/4}\log^3(1+n)e^{-c|x|/\sqrt{1+n}} \\ &\lesssim \biggl(\sum_{n\geq 0}[|f(\sqrt{n})|^2+|\hat{f}(\sqrt{n})|^2](1+n)^{5/2}\log^6(1+n)e^{-2c|x|/\sqrt{1+n}}\biggr)^{1/2}.\end{aligned}$$

Thus, we readily obtain that

$$\|f\|_{L^2((1+|x|)^s)}+\|\hat{f}\|_{L^2((1+|x|)^s)} \lesssim \biggl(\sum_{n\geq 0}[|f(\sqrt{n})|^2+|\hat{f}(\sqrt{n})|^2](1+n)^{(5+s)/2}\log^6(1+n)\biggr)^{1/2}$$

for any $s > 0$. This proves the Theorem for any $\omega > s/2 + 4$. □

Furthermore, as a corollary we can also obtain that the pair $(\mathbb{R}\setminus\{\pm\sqrt{n+\varepsilon_n}\}_{n\geq 0}, \mathbb{R}\setminus\{\pm\sqrt{n+\varepsilon_n}\}_{n\geq 0})$ is ω-strongly annihilating for some $\omega > 0$, which was the content of Corollary 1.8.

Proof of Corollary 1.8. Notice that the operator $\widetilde{T} : \ell^2_r(\mathbb{N}) \times \ell^2_r(\mathbb{N}) \to \ell^2_r(\mathbb{N}) \times \ell^2_r(\mathbb{N})$ given in Section 4B is, under our given hypotheses, *bounded and invertible* for $r \gg 1$ sufficiently large. Moreover, it takes, for each $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$, the pair $(\{f(\sqrt{n})\}_{n\in\mathbb{N}}, \{\hat{f}(\sqrt{n})\}_{n\in\mathbb{N}})$ to the pair $(\{f(\sqrt{n+\varepsilon_n})\}_{n\in\mathbb{N}}, \{\hat{f}(\sqrt{n+\varepsilon_n})\}_{n\in\mathbb{N}})$.

Therefore, if $\omega > s > r$, then the comparison of

$$\|(\{f(\sqrt{n})\}_{n\in\mathbb{N}}, \{\hat{f}(\sqrt{n})\}_{n\in\mathbb{N}})\|_{\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N})}$$

with

$$\|(\{f(\sqrt{n+\varepsilon_n})\}_{n\in\mathbb{N}}, \{\hat{f}(\sqrt{n+\varepsilon_n})\}_{n\in\mathbb{N}})\|_{\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N})}$$

holds with comparing constants *independent* of $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$. The same assertion holds with $\ell^2_\omega(\mathbb{N}) \times \ell^2_\omega(\mathbb{N})$ norms instead of $\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$. This is enough to conclude the asserted statement. □

Finally, we conclude that, whenever c_α, c_β are sufficiently small, then $(\mathbb{R}\setminus\{\pm c_\alpha n^\alpha\}, \mathbb{R}\setminus\{\pm c_\beta n^\beta\})$ is ω-strongly annihilating for ω sufficiently large.

Corollary 5.10. *For $\alpha, \beta < \frac{2}{9}$ and c_α, c_β sufficiently small and for any $\gamma > 0$ sufficiently large, we have*

$$\|f\|_{L^2((1+|x|))}+\|\hat{f}\|_{L^2((1+|x|)^\gamma)} \lesssim \biggl(\sum_{n\geq 0}(1+n)^\omega[|f(c_\alpha n^\alpha)|^2+|\hat{f}(c_\beta n^\beta)|^2]\biggr)^{1/2},$$

whenever $\omega > (5+\gamma)/4$ and $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ is a real-valued function.

Proof. Under the hypotheses above, we know that the operator $\mathcal{T}_{n_0(\alpha,\beta)}$ from (5-10) is still injective and has closed range on $\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$ for $s \gg 1$. For that reason, the norm

$$\|(\{f(\sqrt{n})\}_{n\in\mathbb{N}}, \{\hat{f}(\sqrt{n})\}_{n\in\mathbb{N}})\|_{\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N})}$$

can be controlled by a constant independent of f times

$$\|\mathcal{T}_{n_0(\alpha,\beta)}(\{f(\sqrt{n})\}, \{\hat{f}(\sqrt{n})\})\|_{\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N})}.$$

However, the sequences constituting $\mathcal{T}_{n_0(\alpha,\beta)}(\{f(\sqrt{n})\}, \{\hat{f}(\sqrt{n})\})$ are *subsequences* of each entry of $(\{c_\alpha n^\alpha\}, \{c_\beta n^\beta\})$, respectively. As the weight $n \mapsto (1+n)^\omega$ is monotonic on $\mathbb{N}$, adding more terms only increases the weighted norm, and thus the conclusion follows. □

5D. ***The Cohn–Kumar-Miller-Radchenko–Viazovska result and perturbed interpolation formulae with derivatives.*** As another illustration of our main technique, we prove that the interpolation formulae with derivatives in dimension 8 and 24 from [Cohn et al. 2022] can be suitably perturbed.

Indeed, we first recall one of the main results of [Cohn et al. 2022]: let (d, n_0) be either $(8, 1)$ or $(24, 2)$. Then every $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ can be uniquely recovered by the sets of values

$$\{f(\sqrt{2n}),\, f'(\sqrt{2n}),\, \hat{f}(\sqrt{2n}),\, \hat{f}'(\sqrt{2n})\}, \quad n \ge n_0,$$

through the interpolation formula

$$f(x) = \sum_{n\ge n_0} f(\sqrt{2n})a_n(x) + \sum_{n\ge n_0} f'(\sqrt{2n})b_n(x) + \sum_{n\ge n_0} \hat{f}(\sqrt{2n})\hat{a}_n(x) + \sum_{n\ge n_0} \hat{f}'(\sqrt{2n})\hat{b}_n(x). \tag{5-12}$$

We also have uniform estimates on the functions $a_n, \hat{a}_n, b_n, \hat{b}_n$: indeed, there is $\tau > 0$ so that

$$\sup_{l\in\{0,1,2\}} \sup_{x\in\mathbb{R}^d} (1+|x|)^{100}(|a_n^{(l)}(x)| + |\hat{a}_n^{(l)}(x)| + |b_n^{(l)}(x)| + |\hat{b}_n^{(l)}(x)|) \lesssim n^\tau \tag{5-13}$$

for all $n \in \mathbb{N}$. Here and throughout this section, we shall denote by $g'(x)$ the derivative of the (radial) function g regarded as a one-dimensional function.

By [Cohn et al. 2022, Theorem 1.9], we know that the matrices

$$M_n(x) = \begin{pmatrix} a_n(x) & a_n'(x) & \hat{a}_n(x) & \hat{a}_n'(x) \\ b_n(x) & b_n'(x) & \hat{b}_n(x) & \hat{b}_n'(x) \\ \hat{a}_n(x) & \hat{a}_n'(x) & a_n(x) & a_n'(x) \\ \hat{b}_n(x) & \hat{b}_n'(x) & b_n(x) & b_n'(x) \end{pmatrix} \tag{5-14}$$

satisfy that $M_n(\sqrt{2m}) = \delta_{m,n} I_{4\times 4}$ for $m, n \ge n_0$. As we know that the map that takes a vector of sufficiently rapidly decaying sequences

$$(\{\alpha_n\}, \{\beta_n\}, \{\tilde{\alpha}_n\}, \{\tilde{\beta}_n\})_{n\ge n_0}$$

onto the function

$$\mathfrak{f}(x) = \sum_{n\ge n_0} (\alpha_n a_n(x) + \beta_n b_n(x) + \tilde{\alpha}_n \hat{a}_n(x) + \tilde{\beta}_n \hat{b}_n(x))$$

is, in fact, injective (and moreover an isomorphism if we consider the set of all arbitrarily rapidly decaying sequences), we shall make use of this function in our estimates. Indeed, we have that the map that takes the quadruple of sequences

$$(\{\alpha_n\}, \{\beta_n\}, \{\tilde{\alpha}_n\}, \{\tilde{\beta}_n\})$$

onto

$$(\mathfrak{f}(\sqrt{2n}), \mathfrak{f}'(\sqrt{2n}), \hat{\mathfrak{f}}(\sqrt{2n}), \hat{\mathfrak{f}}'(\sqrt{2n}))_{n\geq n_0}$$

is, in fact, the identity. Another way to represent this map is as the series

$$\sum_{n\geq n_0} (\alpha_n, \beta_n, \tilde{\alpha}_n, \tilde{\beta}_n) \cdot M_n(\sqrt{2n}).$$

We define, therefore, the operator that takes the same quadruple onto

$$\left(\mathfrak{f}(\sqrt{2n+\varepsilon_n}), \mathfrak{f}'(\sqrt{2n+\varepsilon_n}), \hat{\mathfrak{f}}(\sqrt{2n+\varepsilon_n}), \hat{\mathfrak{f}}'(\sqrt{2n+\varepsilon_n})\right)_{n\geq n_0}.$$

In the alternative notation, this operator, which we shall denote by $\mathfrak{T}$, is given by

$$\sum_{n\geq n_0} (\alpha_n, \beta_n, \tilde{\alpha}_n, \tilde{\beta}_n) \cdot M_n(\sqrt{2n+\varepsilon_n}).$$

As before, we seek to prove that $\mathfrak{T}$ is invertible when defined over some space

$$\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) =: (\ell^2_s(\mathbb{N}))^4,$$

where we may take $s \gg 1$ sufficiently large. As our aim here is not to establish the sharpest possible results, but only to prove that we may prove versions of the above interpolation formula with *some* perturbed nodes; we shall make use of the Hilbert–Schmidt test, as in Section 5A above. Indeed, the same remark about the definition of the perturbed operators in the proof of Theorem 1.4 holds here as well: we first define $\mathfrak{T}$ over quadruples of sequences with finitely many nonzero terms, and then we use a priori boundedness of $I-\mathfrak{T}$ over this space to define $\mathfrak{T}$ in the whole space $(\ell^2_s(\mathbb{N}))^4$ by density. Thus, we wish to prove that

$$\|I-\mathfrak{T}\|_{HS((\ell^2_s(\mathbb{N}))^4)} < 1.$$

A simple computation with the Hilbert–Schmidt norm using (5-14) shows that this quantity is bounded by

$$\begin{aligned}\sum_{m,n>n_0} m^{2s}n^{-2s}\big(&|a_n(\sqrt{2m})-a_n(\sqrt{2m+\varepsilon_m})|^2+|\hat{a}_n(\sqrt{2m})-\hat{a}_n(\sqrt{2m+\varepsilon_m})|^2\\ &+|a_n'(\sqrt{2m})-a_n'(\sqrt{2m+\varepsilon_m})|^2+|\hat{a}_n'(\sqrt{2m})-\hat{a}_n'(\sqrt{2m+\varepsilon_m})|^2+|b_n(\sqrt{2m})-b_n(\sqrt{2m+\varepsilon_m})|^2\\ &+|\hat{b}_n(\sqrt{2m})-\hat{b}_n(\sqrt{2m+\varepsilon_m})|^2+|b_n'(\sqrt{2m})-b_n'(\sqrt{2m+\varepsilon_m})|^2+|\hat{b}_n'(\sqrt{2m})-\hat{b}_n'(\sqrt{2m+\varepsilon_m})|^2\big).\end{aligned}$$

Notice that we have used, as in the proof of Theorems 1.4 and 5.3, the standard orthonormal basis for the space $\ell^2_s(\mathbb{N})$, which induces the additional $(m/n)^{2s}$ factor in the summand above. By (5-13) and the mean value theorem, the sum above is bounded by (an absolute constant times)

$$\sum_{m,n>0} m^{2s}n^{-2s} \times m^{-100}n^{2\tau}\varepsilon_m^2.$$

The sum above is representable as a product of a sum in m and one in n. The one in n is convergent if $s > \tau + 1$. We then fix such a value of s. For such values, the second sum is

$$\sum_{m>0} m^{2s-100} \varepsilon_m^2,$$

which converges in the case $\varepsilon_m \lesssim m^{49-s}$. For all such sequences, the difference $I - \mathfrak{T}$ is a Hilbert–Schmidt operator. Moreover, if $\varepsilon_m \leq \delta m^{49-s}$ for $\delta > 0$ sufficiently small, we will have $\|I - \mathfrak{T}\|_{HS(\ell_s^2(\mathbb{N})^4)} < 1$. Summarizing, we have shown the following result:

Theorem 5.11. *There are $C_0 > 0$ and $\delta > 0$ so that the following holds: for each sequence ε_k so that $|\varepsilon_k| < \delta k^{-C_0}$, any function $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ is uniquely determined by the values*

$$\bigl(f(\sqrt{2n+\varepsilon_n}),\ f'(\sqrt{2n+\varepsilon_n}),\ \hat{f}(\sqrt{2n+\varepsilon_n}),\ \hat{f}'(\sqrt{2n+\varepsilon_n})\bigr)_{n \geq n_0}, \tag{5-15}$$

where we let $(d, n_0) = (8, 1)$ or $(24, 2)$.

In the same spirit of Section 4B, one can obtain an interpolation formula with the values (5-15) from Theorem 5.11.

We remark that, in the same way that we undertook our analysis for the Radchenko–Viazovska interpolating functions, we expect the functions a_n, b_n in [Cohn et al. 2022, Theorem 1.9] should also satisfy some exponential-like decay. This fact, although possible, should be sensibly more technically involved than Theorem 1.6, due to the more complicated nature of the construction of the interpolating functions with derivatives in dimensions 8 and 24.

5E. ***Perturbed interpolation formulae for odd functions.*** Finally, in the same spirit of the results in Section 4, we briefly comment on interpolation formulae for odd functions. Recall the following results from [Radchenko and Viazovska 2019, Section 7]:

Theorem 5.12 [Radchenko and Viazovska 2019, Theorem 7]. *There exist sequences of odd functions $d_m^{\pm} : \mathbb{R} \to \mathbb{R}$, $m \geq 0$, belonging to the Schwartz class so that*

$$\widehat{d_m^{\pm}} = (\mp i) d_m^{\pm}, \quad d_m^{\pm}(\sqrt{n}) = \delta_{n,m}\sqrt{n}, \quad n \geq 1.$$

Moreover, $\lim_{x \to 0} d_m^+(x)/x = \delta_{0m}$. These functions satisfy the uniform bound

$$|d_n^{\pm}(x)| \lesssim n^{5/2} \quad \text{for all } x \in \mathbb{R},\ n \geq 0,$$

and, finally, for each odd and real Schwartz function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = d_0^+(x) \frac{f'(0) + i\hat{f}'(0)}{2} + \sum_{n \geq 1} \biggl(c_n(x) \frac{f(\sqrt{n})}{\sqrt{n}} - \hat{c}_n(x) \frac{\hat{f}(\sqrt{n})}{\sqrt{n}} \biggr), \tag{5-16}$$

where $c_n = (d_n^+ + d_n^-)/2$, and the right-hand side of the sum above converges absolutely.

As a direct consequence, we see that any real, odd, Schwartz function on the real line is determined uniquely by the union of its values at $\sqrt{n}$ and the values of its Fourier transform at $\sqrt{n}$ with $f'(0)$

and $\hat{f}'(0)$. By employing the results in Section 4, we will show that we can actually recover any such function from $\{f(\sqrt{n+\varepsilon_n})\}_{n\geq 1} \cup \{\hat{f}(\sqrt{n+\varepsilon_n})\}_{n\geq 1} \cup \{f'(0)\} \cup \{\hat{f}'(0)\}$ instead.

Indeed, first of all, we start by noticing that the same techniques employed to refine the uniform estimates from [Radchenko and Viazovska 2019] can be applied to the functions $d_m^{\pm}$, as they are defined in a completely analogous way to the $b_n^{\pm}$ from Section 4. By carrying out the same kind of estimates, we are able to obtain

$$|d_n^{\pm}(x)| \lesssim n^{3/4} \log^3(1+n) e^{-c'|x|/\sqrt{n}} \quad \text{for all } x \in \mathbb{R},\ n \geq 1, \tag{5-17}$$

for some absolute constant $c' > 0$. By the same analysis of the ∂_x-partial derivative of the generating function used in Section 4A, this readily implies that the derivatives of the $d_n^{\pm}$ satisfy essentially the same decay; in fact, $|(d_n^{\pm})'(x)| \lesssim n^{5/4} \log^3(1+n) e^{-c''|x|/\sqrt{n}}$ for all $x \in \mathbb{R},\ n \geq 1$, with $c'' > 0$ another absolute constant.

We consider now the operator that takes a pair of sequences $(\{\alpha_n\}, \{\beta_n\}) \in \ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N}),\ s > 0$ to be chosen, into

$$\Big\{ \sum_{n\geq 0} (\alpha_n, \beta_n) C_n(\sqrt{m+\varepsilon_m}) \Big\}_{m\geq 0},$$

where we abbreviate

$$C_n(x) = \begin{pmatrix} c_n(x)/\sqrt{n} & \hat{c}_n(x)/\sqrt{n} \\ -\hat{c}_n(x)/\sqrt{n} & c_n(x)/\sqrt{n} \end{pmatrix}.$$

Let us denote this operator by $\mathcal{V}$. From (5-16) and the fact that the function $d_0^{+}(x) = \sin(\pi x^2)/\sinh(\pi x)$ *vanishes* together with its Fourier transform at $\pm\sqrt{n},\ n \in \mathbb{N}$, we know that the identity operator on $\ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N})$ may be written as

$$\Big\{ \sum_{n\geq 0} (\alpha_n, \beta_n) C_n(\sqrt{m}) \Big\}_{m\geq 0}.$$

Therefore, the techniques from Sections 4B, 5D and 5A, together with our previous considerations in this subsection, allow us to deduce the following result:

Theorem 5.13. *There is $\delta > 0$ so that, in the case $|\varepsilon_n| \leq \delta n^{-7/4}$, for each $f \in \mathcal{S}_{\mathrm{odd}}(\mathbb{R})$ real, the values*

$$(f(\sqrt{1+\varepsilon_n}), f(\sqrt{2+\varepsilon_2}), \dots) \quad \textit{and} \quad (\hat{f}(\sqrt{1+\varepsilon_n}), \hat{f}(\sqrt{2+\varepsilon_2}), \dots)$$

allow us to recover uniquely the values $(f(1), f(\sqrt{2}), f(\sqrt{3}), \dots)$ and $(\hat{f}(1), \hat{f}(\sqrt{2}), \hat{f}(\sqrt{3}), \dots)$. In particular, given the values

$$\{f(\sqrt{n+\varepsilon_n})\}_{n\geq 1} \cup \{\hat{f}(\sqrt{n+\varepsilon_n})\}_{n\geq 1} \cup \{f'(0)\} \cup \{\hat{f}'(0)\},$$

we can uniquely recover any real-valued function $f \in \mathcal{S}_{\mathrm{odd}}(\mathbb{R})$.

As previously mentioned, we do not carry out the details here, for their similarities with the proof of Theorems 1.6 and 1.4.

6. Comments and remarks

In this section, we gather some remarks about the problems and techniques discussed and state some results we expect to be true.

6A. ***Asymmetric perturbations.*** In the statement of Theorem 1.4, we have assumed that the perturbations made to the Radchenko–Viazovska interpolation formula were *the same* on the function and Fourier sides for fixed j. We remark that, by the exact same proof as given above, one may obtain results with *different* perturbations: in that regard, Theorem 1.4 can be immediately reinterpreted as stating that one may recover f from the values of $f(\sqrt{n+\varepsilon_n})$, $\hat{f}(\sqrt{n+\delta_n})$, $n \geq 0$, where one assumes $\varepsilon_0 = \delta_0 = 0$, and $\sup_n (1+n)^{5/4} \log(1+n)^3 \cdot (|\varepsilon_n| + |\delta_n|)$ is sufficiently small.

Similarly, one can safely introduce four different perturbation parameters in Theorem 5.11 — one for f, one for f', one for $\hat{f}$ and a last one for $\hat{f}'$ — as long as they still satisfy the conditions predicted in that result. The same holds for Theorem 1.3, where one may select two different perturbation parameters, one for the function and another for its derivative. As these generalizations are immediate from our proofs, we chose to keep all results with one perturbation parameter, in order to simplify the exposition.

6B. ***Maximal perturbed interpolation formulae for band-limited functions.*** In Section 3, we have seen how our basic functional analysis techniques can be employed in order to deduce new interpolation formulae for band-limited functions. Although Kadec's proof also uses the basic fact that, whenever a perturbation of the identity is sufficiently small, we can basically "invert" an operator, he then proceeds to find that the set of exponentials $\{\exp(2\pi i(n+\varepsilon_n)x)\}_{n\geq 0}$ is a Riesz basis for $L^2\left(-\frac{1}{2}, \frac{1}{2}\right)$ if $\sup_n |\varepsilon_n| < \frac{1}{4}$ by means of *orthogonality* considerations. Indeed, one key strategy in his estimates is to expand in the different complete orthogonal system

$$\{1, \cos(2\pi nt), \sin((2n-1)\pi t)\}_{n\geq 1}$$

and use the properties of this expansion. Our results, as much as they do not come so close to Kadec's threshold, follow a slightly different path: instead of using the orthogonality of a different system, we choose to work directly with discrete analogues of the Hilbert transform and estimate over those. Although we do not reach — by a 0.011 margin — the sharp $\frac{1}{4}$-perturbation result, one advantage of our approach is that it yields bounds for perturbing *any* kind of interpolation formulae with derivatives. Indeed, following the line of thought of Vaaler, many authors have investigated the property of recovering the values of a function $f \in L^2(\mathbb{R})$ band-limited to $[-k/2, k/2]$ from the values of its $(k-1)$-first derivatives (see, e.g., [Littmann 2006; Gonçalves and Littmann 2018]). Our approach in Section 3 in order to prove Theorem 1.3 generalizes easily to the case of several derivatives by an easy modification. It can be summarized as follows:

Theorem 6.1. *There is $L(k) > 0$ so that if $\max_{0\leq l<k} \sup_{n\in\mathbb{Z}} |\varepsilon_n^{(l)}| < L(k)$, then any function $f \in L^2(\mathbb{R})$ band-limited to $[-k/2, k/2]$ is uniquely determined by the values of*

$$f^{(l)}(n+\varepsilon_n^{(l)}), \quad n \in \mathbb{Z},\ l = 0, 1, \ldots, k-1.$$

A natural question that connects our results to Kadec's results is about the *best* value of $L(k)$ so that Theorem 6.1 holds. We do not have evidence to back any concrete conjecture, but we find possible that the threshold $L(k) = \frac{1}{4}$ is kept for higher values of $k \in \mathbb{N}$. We speculate that, in order to prove such a result, one would need to find an appropriate hybrid of our techniques and Kadec's techniques (see for instance Section 10 in [Young 1980, Chapter 1]), taking into account properties of the discrete Hilbert transforms as well as orthogonality results.

6C. ***Theorem 1.6, optimal decay rates for interpolating functions and maximal perturbations.*** In Theorem 1.6, we have improved the uniform bound obtained in [Radchenko and Viazovska 2019] and, more recently, the sharper uniform bound of [Bondarenko et al. 2023] on the interpolating functions a_n to one that *decays* with x; namely, we have that

$$|a_n(x)| \lesssim n^{1/4} \log^3(1+n)(e^{-c|x|^2/n} 1_{|x|<Cn} + e^{-c|x|} 1_{|x|>Cn})$$

holds for all $n \in \mathbb{N}$, where $C, c > 0$ are two fixed positive constants. Although this improves the decay rates from before, the power $n^{1/4}$ found here and in [Bondarenko et al. 2023] in the growth seems likely not to be optimal; to that regard, we pose the following:

Question 1. *What is the best decay rate for a_n as in Theorem 1.6? Can one prove that $\sup_{x\in\mathbb{R}} |a_n(x)| = \mathcal{O}((\log n)^C)$ in n for some absolute constant $C > 0$?*

This conjectured growth seems to be the best possible, due to the recent findings of [Bondarenko et al. 2023], which show that, for each $N \gg 1$, the average

$$\frac{1}{N+1} \sum_{k \le N} |a_k(x)|^2$$

grows slower than some power of $\log N$.

Notice that, by a simple modification of the computations made in Section 4B, an affirmative answer to Question 1 yields an immediate improvement in the range of ε_i that we allow for the theorems in Section 4B. Indeed, we get automatically that $|\varepsilon_i| \lesssim i^{-1}$ is allowed in such results. On the other hand, this seems to be the best possible result one can achieve with our current methods, as the mean value theorem implies that $\sup_{x\in\mathbb{R}} |a_n'(x)| \gtrsim \sqrt{n}$.

In particular, everything indicates that one needs a new idea in order to prove the following conjecture:

Conjecture 6.2 (maximal perturbations). *Let $f \in \mathcal{S}_{\mathrm{even}}(\mathbb{R})$ be a real-valued function. Then there is $\theta > 0$ so that, if $|\varepsilon_i| + |\delta_i| < \theta$ for all $i \in \mathbb{N}$, then f can be uniquely recovered from its values*

$$f(0),\ f(\sqrt{1+\varepsilon_1}),\ f(\sqrt{2+\varepsilon_2}),\ \dots,$$

together with the values of its Fourier transform

$$\hat{f}(0),\ \hat{f}(\sqrt{1+\delta_1}),\ \hat{f}(\sqrt{2+\delta_2}),\ \dots.$$

It might not be an easy task to prove Conjecture 6.2 even with a new idea starting from our techniques, but we believe that the following version stands a chance of being more tractable with the current methods:

Conjecture 6.3 (maximal perturbations, weak form). *Let $f \in \mathcal{S}_{\mathrm{even}}(\mathbb{R})$ be a real-valued function. Then, for each $a > 0$, there is $\delta > 0$ so that, if $|\varepsilon_i| + |\delta_i| \le \delta k^{-a}$, then f can be uniquely recovered from its values*

$$f(0),\ f(\sqrt{1+\varepsilon_1}),\ f(\sqrt{2+\varepsilon_2}),\ \dots,$$

together with the values of its Fourier transform

$$\hat{f}(0),\ \hat{f}(\sqrt{1+\delta_1}),\ \hat{f}(\sqrt{2+\delta_2}),\ \dots.$$

In this framework, the results in Section 4B may be regarded as partial progress towards this conjecture.

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NONEXISTENCE OF THE BOX DIMENSION FOR DYNAMICALLY INVARIANT SETS

NATALIA JURGA

One of the key challenges in the dimension theory of smooth dynamical systems lies in establishing whether or not the Hausdorff, lower and upper box dimensions coincide for invariant sets. For sets invariant under conformal dynamics, these three dimensions always coincide. On the other hand, considerable attention has been given to examples of sets invariant under nonconformal dynamics whose Hausdorff and box dimensions do not coincide. These constructions exploit the fact that the Hausdorff and box dimensions quantify size in fundamentally different ways, the former in terms of covers by sets of varying diameters and the latter in terms of covers by sets of fixed diameters. In this article we construct the first example of a dynamically invariant set with distinct lower and upper box dimensions. Heuristically, this says that if size is quantified in terms of covers by sets of equal diameters, a dynamically invariant set can appear bigger when viewed at certain resolutions than at others.

1. Introduction

The dimension theory of dynamical systems is the study of the complexity of sets and measures which remain invariant under dynamics, from a dimension theoretic point of view. This branch of dynamical systems has its foundations in the seminal work [Bowen 1979] on the dimension of quasicircles and [Ruelle 1982] on the dimension of conformal repellers, and has since developed into an independent field of research which continues to receive noteworthy attention in the literature [Bárány et al. 2019; Cao et al. 2019; Das and Simmons 2017]. For an overview of this extensive field, see the monographs [Barreira 2008; Pesin 1997] and the surveys [Barreira and Gelfert 2011; Chen and Pesin 2010; Schmeling 2001].

The most common ways of measuring the dimension of invariant sets are through the Hausdorff dimension and the lower and upper box dimensions, which quantify the complexity of the set in related but subtly distinct ways. Roughly speaking, the Hausdorff dimension measures how efficiently the set can be covered by sets of arbitrarily small size, whereas the lower and upper box dimensions measure this in terms of covers by sets of uniform size, along the scales for which this can be done in the most and least efficient way, respectively. Given a subset E of a separable metric space X, the lower and upper box dimensions are defined by

$$\underline{\dim}_{\mathrm{B}} E = \liminf_{\delta\to 0} \frac{\log N_\delta(E)}{-\log\delta} \quad \text{and} \quad \overline{\dim}_{\mathrm{B}} E = \limsup_{\delta\to 0} \frac{\log N_\delta(E)}{-\log\delta},$$

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respectively, where $N_\delta(E)$ denotes the smallest number of sets of diameter $\delta > 0$ required to cover E. If the lower and upper box dimensions coincide we call the common value the box dimension, written $\dim_B$, otherwise we say that the box dimension does not exist.

For any subset $E \subseteq X$,

$$\dim_H E \le \underline{\dim}_B E \le \overline{\dim}_B E, \tag{1}$$

where $\dim_H$ denotes the Hausdorff dimension. A priori each inequality may or may not be strict. However, when E is invariant under a smooth mapping f, the additional structure imposed by the dynamical invariance of E means that certain properties of f can either force some degree of homogeneity or, on the contrary, inhomogeneity across the set, forcing equalities or strict inequalities in (1), respectively. Characterising which properties of f imply or preclude equalities in (1) is one of the key challenges in dimension theory.

A common feature in the dimension theory of smooth *conformal* dynamics is the coincidence of the Hausdorff and lower and upper box dimensions for invariant sets. For example, in the setting of smooth expanding maps, the following result pertains to a more general result which was obtained independently by Gatzouras and Peres [1997] and Barreira [1996], generalising previous results of Falconer [1989].

Theorem 1.1 [Barreira 1996; Gatzouras and Peres 1997]. *Suppose $f : M \to M$ is a C^1 map of a Riemannian manifold M and that $\Lambda = f(\Lambda)$ is a compact set such that $f^{-1}(\Lambda) \cap U \subset \Lambda$ for some open neighbourhood U of Λ. Additionally, assume that*

- *f is **conformal**: for each $x \in M$, the derivative $d_x f$ is a scalar multiple of an isometry,*
- *f is **expanding** on Λ: there exist constants $C > 0$ and $\lambda > 1$ such that, for all $x \in \Lambda$ and u in the tangent space $T_x M$,*
$$\|d_x f^n u\| \ge C\lambda^n \|u\|.$$

Then, for any compact set $F = f(F) \subset \Lambda$,

$$\underline{\dim}_B F = \overline{\dim}_B F = \dim_H F.$$

Similar results hold in the setting of smooth diffeomorphisms. For example, if $f : M \to M$ is a topologically transitive C^1 diffeomorphism with a basic set Λ and f is conformal on Λ, then we have $\dim_H \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda$ [Barreira 1996; Pesin 1997], and an analogous statement holds for the dimensions of the intersections of Λ with its local stable and unstable manifolds [Palis and Viana 1988; Takens 1988].

In contrast, in the realm of smooth *nonconformal* dynamical systems, coincidence of the Hausdorff and box dimensions is no longer a universal trait of invariant sets. Indeed, examples of invariant sets with distinct Hausdorff and box dimensions have attracted enormous attention [Bedford 1984; Kenyon and Peres 1996; Lalley and Gatzouras 1992; McMullen 1984; Neunhäuserer 2002; Pollicott and Weiss 1994] and discussion in surveys [Barreira and Gelfert 2011; Chen and Pesin 2010; Fraser 2021]. This type of dimension gap result exploits the fact that the Hausdorff dimension quantifies the size of the set in terms of covers by sets of varying diameters rather than fixed diameters which are used by the box dimension. Indeed invariant sets of certain nonconformal dynamics will contain long, thin and well-aligned copies of itself, meaning that covering by sets of varying diameter is often more efficient, inducing this type of dimension gap. However, surprisingly there seems to be no mention in the literature of the possibility of a dynamically invariant set with *distinct lower and upper box dimensions*. Our main result demonstrates the existence of such sets.

Theorem 1.2. *There exist integers $n > m \geq 2$ and a compact subset of the torus $F \subset \mathbb{T}^2$ such that F is invariant, $F = T(F)$ under the expanding toral endomorphism*

$$T(x, y) = (mx \mod 1, ny \mod 1)$$

and

$$\underline{\dim}_{\mathrm{B}} F < \overline{\dim}_{\mathrm{B}} F.$$

In particular, the box dimension of F does not exist.

Since $n > m$, we have that T is a nonconformal map. Well-known examples from the literature, such as Bedford–McMullen carpets [Fraser 2021], demonstrate that equality of the Hausdorff and box dimensions is not guaranteed in Theorem 1.1 if the assumption of conformality is dropped. Furthermore, Theorem 1.2 indicates that the lower and upper box dimensions need not coincide either in Theorem 1.1 if the assumption of conformality is dropped. This is arguably a more striking type of dimension gap since, while it is easy to see that sets invariant under nonconformal dynamics may cease to be homogeneous in space, which is captured by the possibility of distinct Hausdorff and box dimensions, one would expect the dynamical invariance to at least force homogeneity in scale, but our result demonstrates that this too can fail. In particular Theorem 1.2 describes that, when measuring size in terms of covers by sets of equal diameter, a dynamically invariant set can sometimes appear bigger and at other times appear smaller depending on the "resolution" we are viewing it at. We highlight that our construction is also significantly more involved than standard examples of invariant sets with distinct Hausdorff and box dimensions, such as Bedford–McMullen carpets.

The dynamics of T on the invariant set F, which will be constructed in Section 2, has two key features which in conjunction induce distinct box dimensions. Firstly, the nonconformality of T causes the box dimensions of F to be sensitive to the length of time it takes for an orbit of T to move from a subset $A \subset F$ which is "entropy maximising" for the dynamics of T to a subset B which is "entropy maximising" for the dynamics of the projection $x \mapsto mx \mod 1$ of T. Secondly, the dynamics on F, which can be modelled by a topologically mixing *coded subshift* [Blanchard and Hansel 1986] on an appropriate symbolic space, has the property that the length of time it takes an orbit of T to move from A to B is highly dependent on how long the orbit has spent in A. In particular, the dynamics fails to satisfy most forms of specification [Kwietniak et al. 2016]. The resolution at which F is viewed determines how long the orbits of points of interest (for the dimension estimates at that particular resolution) spend in A, and combined with the properties mentioned above this forces distinct box dimensions.

Finally, we discuss some connections between Theorem 1.2 and the literature on self-affine and sub-self-affine sets. Let $\{S_i : \mathbb{R}^d \to \mathbb{R}^d\}_{i=1}^N$ be a collection of affine contractions, i.e., $S_i(\,\cdot\,) = A_i(\,\cdot\,) + \boldsymbol{t}_i$ for each $1 \leq i \leq N$, where $A_i \in GL(d, \mathbb{R})$ with Euclidean norm $\|A_i\| < 1$ and $\boldsymbol{t}_i \in \mathbb{R}^d$. We call $\{S_i\}_{i=1}^N$ an affine iterated function system. A *sub-self-affine set* [Käenmäki and Vilppolainen 2010] is a nonempty, compact set $E \subset \mathbb{R}^d$ such that

$$E \subseteq \bigcup_{i=1}^{N} S_i(E). \tag{2}$$

If (2) is an equality then E is called a *self-affine set*, in particular every self-affine set is an example of a sub-self-affine set. Every affine iterated function system admits a unique self-affine set. However, there are infinitely many sub-self-affine sets which are not self-affine. Indeed, the unique self-affine set is the image of the full shift $\{1, \ldots, N\}^{\mathbb{N}}$ under an appropriate projection induced from the family $\{S_i\}_{i=1}^{N}$, whereas sub-self-affine sets are in one-to-one correspondence with the projections of subshifts of the full shift. Under suitable "separation conditions" on $\{S_i\}_{i=1}^{N}$, any sub-self-affine set E satisfies $f(E) \subseteq E$ for an appropriate piecewise expanding map f given by the inverses of the contractions. The set F which will be constructed in Section 2 to prove Theorem 1.2 is a sub-self-affine set (which is not self-affine) for the affine iterated function system induced from the inverse branches of T.

The dimension theory of self-affine sets has been an active topic of research since the 1980s and substantial progress has been made in recent years. Sub-self-affine sets were introduced by Käenmäki and Vilppolainen [2010] as natural analogues of sub-self-similar sets which were studied earlier by Falconer [1995]. It is known by the results of Falconer [1988] and Käenmäki and Vilppolainen [2010] that the box dimension of a generic sub-self-affine sets exists, moreover this has been verified for large explicit families of planar self-affine sets [Bárány et al. 2019]. However, the following question was open until now.

Question 1.3. Does the box dimension of every (sub-)self-affine set exist?

The version of the above question for self-affine sets is a folklore open question within the fractal geometry community, to which the answer is widely conjectured to be affirmative. In contrast, a corollary of our main result is that the answer to Question 1.3 for general sub-self-affine sets is negative.

Corollary 1.4. *There exist sub-self-affine sets whose box dimension does not exist.*

Organisation of paper. In Section 2 we construct the set F and its underlying subshift Σ and offer some heuristic reasoning behind Theorem 1.2. Section 3 contains entropy estimates. In Section 4 we introduce the scales for the lower and upper box dimension computations and prove Theorem 1.2. Section 5 contains some questions for further investigation.

2. Construction of a $(\times m, \times n)$-invariant set

Fix $m = 2$ and $n = 12$. Let

$$\Delta = \{(a, b) : 1 \le a \le 2,\ 1 \le b \le 12,\ a, b \in \mathbb{N}\}.$$

For any $(a, b) \in \Delta$, define the contraction $S_{(a,b)} : [0, 1]^2 \to [0, 1]^2$ as

$$S_{(a,b)}(x, y) = \left(\tfrac{1}{2}x + \tfrac{1}{2}(a-1),\ \tfrac{1}{12}y + \tfrac{1}{12}(b-1)\right).$$

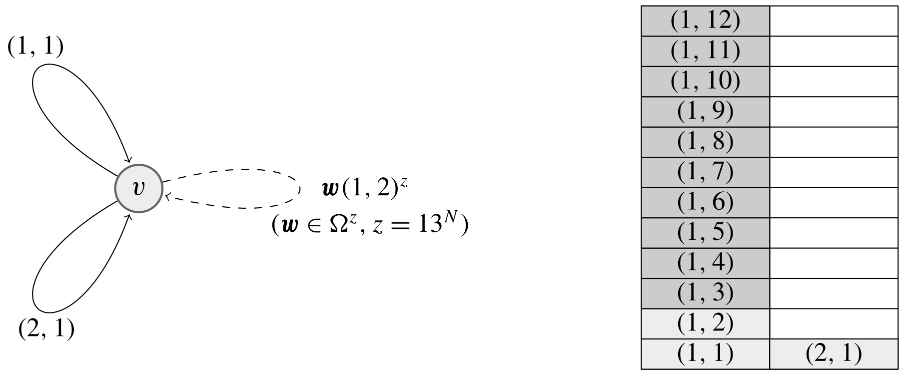

Figure 1. Left: The presentation G of Σ. The dashed loop indicates that, for each $N \in \mathbb{N}$ and $\boldsymbol{w} \in \Omega^{13^N}$, there is a path of length $2 \cdot 13^N$ which begins and ends at v such that its sequence of labels reads $\boldsymbol{w}(1,2)^{13^N}$. Right: Images of $[0,1]^2$ under $S_{(a,b)}$ for each (a,b) that labels some edge in G. The darker of the shaded rectangles correspond to $S_{(a,b)}([0,1]^2)$ for $(a,b) \in \Omega$.

These are the partial inverses of T. If $\mathtt{i}, \mathtt{j} \in \Delta^{\mathbb{N}}$ with $\mathtt{i} \neq \mathtt{j}$, we let $\mathtt{i} \wedge \mathtt{j}$ denote the longest common prefix to $\mathtt{i}$ and $\mathtt{j}$ and denote its length by $|\mathtt{i} \wedge \mathtt{j}|$. We equip $\Delta^{\mathbb{N}}$ with the metric

$$d(\mathtt{i}, \mathtt{j}) = \begin{cases} 1/2^{|\mathtt{i} \wedge \mathtt{j}|} & \text{if } \mathtt{i} \neq \mathtt{j}, \\ 0 & \text{if } \mathtt{i} = \mathtt{j}. \end{cases}$$

The set F that satisfies Theorem 1.2 will be the projection of a set $\Sigma \subseteq \Delta^{\mathbb{N}}$ under the continuous and surjective (but not injective) coding map $\Pi : \Delta^{\mathbb{N}} \to [0,1]^2$ given by

$$\Pi((a_1, b_1)(a_2, b_2) \cdots) := \lim_{n \to \infty} S_{(a_1,b_1)\cdots(a_n,b_n)}(0),$$

where $S_{(a_1,b_1)\cdots(a_n,b_n)}$ denotes the composition $S_{(a_1,b_1)} \circ \cdots \circ S_{(a_n,b_n)}$.

Let $\Omega = \{(1,i)\}_{i=3}^{12}$. For each $N \in \mathbb{N}$, let Ω^N denote words of length N with symbols in Ω, and $\Omega^{\mathbb{N}}$ the set of infinite sequences with symbols in Ω. Given any $(a,b) \in \Delta$, we denote by $(a,b)^n$ the word $(a,b)(a,b) \cdots (a,b)$ of length n. Define $\mathcal{C}$ to be the collection of words

$$\mathcal{C} := \{(1,1), (2,1)\} \cup \bigcup_{N=1}^{\infty} \bigcup_{\boldsymbol{w} \in \Omega^{13^N}} \{\boldsymbol{w}(1,2)^{13^N}\}$$

and

$$B := \{\boldsymbol{u}\boldsymbol{u}_1\boldsymbol{u}_2\boldsymbol{u}_3 \cdots : \boldsymbol{u}_i \in \mathcal{C} \text{ for all } i \in \mathbb{N},\ \boldsymbol{u} \text{ is a suffix of some word in } \mathcal{C}\}. \tag{3}$$

Then we define the sequence space $\Sigma = \bar{B}$.[1] Equivalently B can be understood as the set of all infinite sequences which label a one-sided infinite path on the directed graph G in Figure 1. G is called the presentation of Σ.

[1]The set of accumulation points $\Sigma \setminus B$ will turn out to be unimportant for our analysis, but for the reader's convenience we provide a description of this set in (4).

It is easy to check that $\sigma(\Sigma) = \Sigma$, where $\sigma : \Sigma \to \Sigma$ denotes the left shift map. In particular, Σ is an example of a *coded subshift*, meaning a subshift which can be expressed as the closure of the space of all infinite paths on a path-connected (possibly infinite) graph, which were first introduced by Blanchard and Hansel [1986]. Note that whenever this graph is finite, its coded subshift is necessarily sofic, and that any $(\times m, \times n)$-invariant set which can be modelled by a sofic shift has a well-defined box dimension which can be explicitly computed [Fraser and Jurga 2020; Kenyon and Peres 1996]. Finally, we set $F = \Pi(\Sigma)$, noting that $F = T(F)$ since $\sigma(\Sigma) = \Sigma$ and $\Pi \circ \sigma = T \circ \Pi$. From this it is easy to see that F is a sub-self-affine set for the iterated function system $\{S_{(a,b)} : (a, b) \in \Delta\}$.

While of course it will be necessary to cover the entirety of F and obtain bounds on the size of this cover at different scales, the proof of Theorem 1.2 will essentially boil down to the asymptotic difference that emerges between

(a) the size of the cover — by squares of side 12^{-13^N} — of the intersection of F with the collection of rectangles $\{S_{\boldsymbol{i}}([0, 1]^2) : \boldsymbol{i} \in \Omega^{13^N}\}$, and

(b) the size of the cover — by squares of side $12^{-13^{N-1/2}}$ — of the intersection of F with the collection of rectangles $\{S_{\boldsymbol{i}}([0, 1]^2) : \boldsymbol{i} \in \Omega^{13^{N-1/2}}\}$.

Roughly speaking, F occupies a large proportion of the width of each rectangle $S_{\boldsymbol{i}}([0, 1]^2)$ in case (a). Such a rectangle has width 2^{-13^N} and height 12^{-13^N} (which equals the sidelength of squares in the cover). For any $\boldsymbol{i} \in \Omega^{13^N}$ and $\boldsymbol{j} \in \{(1, 1), (2, 1)\}^{13^N(\log 12/\log 2-2)}$, we have that $\boldsymbol{i}(1, 2)^{13^N}\boldsymbol{j}$ constitutes a legal word in Σ and each $S_{\boldsymbol{i}(1,2)^{13^N}\boldsymbol{j}}([0, 1]^2)$ has width roughly 12^{-13^N} (which equals the sidelength of squares in the cover), therefore $S_{\boldsymbol{i}}([0, 1]^2)$ requires roughly $2^{13^N(\log 12/\log 2-2)}$ squares to cover it. Importantly, this is a positive power of 12^{13^N}, which indicates "growth" in dimension.

In case (b), F occupies a very thin proportion of the width of each rectangle $S_{\boldsymbol{i}}([0, 1]^2)$. Each such rectangle has width $2^{-13^{N-1/2}}$ and height $12^{-13^{N-1/2}}$ (which is equal to the sidelength of squares in this cover). Any $\mathtt{i} \in \Sigma$ which begins with a word in $\Omega^{13^{N-1/2}}$ can be written as $\mathtt{i} = \boldsymbol{i}\boldsymbol{j}\mathtt{j}$ for $\boldsymbol{i} \in \Omega^{13^{N-1/2}}$, $\boldsymbol{j} = (1, b_1)\cdots(1, b_{13^N})$ and some infinite word $\mathtt{j} \in \Sigma$. In particular, any point in $F \cap S_{\boldsymbol{i}}([0, 1]^2)$ belongs to $S_{\boldsymbol{i}\boldsymbol{j}}([0, 1]^2)$ which has width *less than* $12^{-13^{N-1/2}}$. In particular, only one square of sidelength $12^{-13^{N-1/2}}$ is required to cover $S_{\boldsymbol{i}}([0, 1]^2)$, meaning no further "growth" in dimension at this scale.

Notation. For any $N \in \mathbb{N}$, we let Σ_N denote the subwords of sequences in Σ of length N. Finite words in $\bigcup_{N=1}^{\infty} \Sigma_N$ will be denoted in bold using notation such as $\boldsymbol{i}$ or $\boldsymbol{j}$, whereas infinite words in Σ will be denoted using typewriter notation such as $\mathtt{i}$ and $\mathtt{j}$. For infinite sequences $\mathtt{i} = (a_1, b_1)(a_2, b_2)\cdots$ and integers $n \geq 1$, we write $\mathtt{i}|n$ for the truncation of $\mathtt{i}$ to its first n symbols: $\mathtt{i}|n = (a_1, b_1)\cdots(a_n, b_n)$. The same notation is used for the truncation of a finite word $\boldsymbol{i} = (a_1, b_1)\cdots(a_m, b_m)$ to its first n symbols: $\boldsymbol{i}|n = (a_1, b_1)\cdots(a_n, b_n)$ when $m \geq n$. For any finite word $\boldsymbol{i} = (a_1, b_1)\cdots(a_n, b_n)$, its length is denoted by $|\boldsymbol{i}| = n$. Given any $(a, b) \in \Delta$, we write $(a, b)^\infty$ for the infinite word $(a, b)(a, b)\cdots$. For any finite word $\boldsymbol{i}$, we denote the cylinder set by $[\boldsymbol{i}] := \{\mathtt{i} \in \Sigma : \mathtt{i}|n = \boldsymbol{i}\}$. We let $\varnothing$ denote the empty word.

To avoid a profusion of constants, we write $A \lesssim B$ if $A \leq cB$ for some universal constant $c > 0$. We write $A \lesssim_\varepsilon B$ if $A \leq c_\varepsilon B$ for all $\varepsilon > 0$, where the constant c_ε depends on ε. We write $A \gtrsim B$ if $B \lesssim A$ and write $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$, and we define the notation $A \gtrsim_\varepsilon B$ and $A \approx_\varepsilon B$ analogously.

3. Entropy estimates

In this section we obtain estimates on the entropy of important subsets of Σ. Let $\mathcal{G}_N$ be the words in Σ_N which label a path that starts and ends at the vertex v of the graph G in Figure 1. Define

$$h(\mathcal{G}) := \limsup_{N\to\infty} \frac{1}{N} \log \#\mathcal{G}_N,$$

where $\#\mathcal{G}_N$ denotes the cardinality of $\mathcal{G}_N$.

Lemma 3.1. $$h(\mathcal{G}) \leq \log 4.$$

Proof. Fix $N \in \mathbb{N}$. Given a word in $\mathcal{G}_N$, let c denote the number of symbols belonging to Ω and a denote the number of symbols belonging to $\{(1, 1), (2, 1)\}$, noting that

(a) $2c + a = N$ and

(b) $c = \sum_{i=1}^{j} 13^{n_i}$ for some integers $n_1, \ldots, n_j$.

Fix $0 \leq a \leq N$ and let $\mathcal{S}_c$ be the set of possible ways that $c = \frac{1}{2}(N - a)$ can be written as an ordered sum $c = \sum_{i=1}^{j} 13^{n_i}$. By ordered sum, we mean that if $(n'_1, \ldots, n'_j)$ is a permutation of $(n_1, \ldots, n_j)$ such that $(n'_1, \ldots, n'_j) \neq (n_1, \ldots, n_j)$, then $\sum_{i=1}^{j} 13^{n_i}$ is considered a distinct way of writing c as a sum of powers of 13. Observe that $j \leq \frac{1}{13}c$ (for example, consider writing $c = 13 \cdot \frac{1}{13}c$ when c is a multiple of 13).

We begin by bounding $\#\mathcal{S}_c \leq 2^{c/13-1}$. Recall that any $n \in \mathbb{N}$ can be expressed in 2^{n-1} ways as an ordered sum of one or more positive integers. Moreover, $\#\mathcal{S}_c$ is clearly bounded above by the number of ways that $\frac{1}{13}c$ can be decomposed into an ordered sum $\sum_{i=1}^{\ell} p_i$ for some positive integers $p_1, \ldots, p_\ell$. Hence $\#\mathcal{S}_c \leq 2^{c/13-1}$.

Now let us return to considering a word in $\mathcal{G}_N$. Following each substring of symbols from Ω, there is a tail of the same length consisting of $(1, 2)$'s. The a symbols from $\{(1, 1), (2, 1)\}$ can either be placed directly after any of these tails or at the beginning of the word. Therefore assuming that the string contains $c = \frac{1}{2}(N - a)$ symbols from Ω in blocks of lengths $13^{n_1}, \ldots, 13^{n_j}$ — so that $c = \sum_{i=1}^{j} 13^{n_i}$ — it follows that there are $\binom{a+j}{j}$ ways in which the a symbols from $\{(1, 1), (2, 1)\}$ can be distributed. Bounding this above by the central binomial term and using the bounds $\binom{2K}{K} \leq 4^K$ and $j \leq \frac{1}{13}c$ we obtain $\binom{a+j}{j} \leq 2^{a+(N-a)/(2\cdot 13)}$. Hence

$$\begin{aligned}\#\mathcal{G}_N \leq \sum_{a=0}^{N} \#\mathcal{S}_{(N-a)/2} 2^{a+(N-a)/(2\cdot 13)} 10^{(N-a)/2} 2^a &\leq \sum_{a=0}^{N} 2^{2a+(N-a)(2/13+\log_2 10)/2} \\ &= \frac{2^{2(N+1)} - 2^{(2/13+\log_2 10)(N+1)/2}}{2^2 - 2^{(2/13+\log_2 10)/2}} \lesssim 4^N\end{aligned}$$

since $\frac{1}{2}\left(\frac{2}{13} + \log_2 10\right) < 2$, completing the proof of the lemma. $\square$

Let $\mathcal{I}_N$ be the words in Σ_N which label a path that ends at v in the graph G in Figure 1. Clearly $\mathcal{G}_N \subseteq \mathcal{I}_N$. Writing $\mathcal{I}^* = \bigcup_{N=1}^{\infty} \mathcal{I}_N$ and $\Omega^* = \bigcup_{N=1}^{\infty} \Omega^N$, observe that

$$\Sigma \setminus B = \{\boldsymbol{u}\mathtt{w} : \boldsymbol{u} \in \mathcal{I}^* \cup \varnothing,\ \mathtt{w} \in \Omega^{\mathbb{N}}\} \cup \{\boldsymbol{w}(1, 2)^{\infty} : \boldsymbol{w} \in \Omega^* \cup \varnothing\}. \tag{4}$$

Define

$$h(\mathcal{I}) = \limsup_{N\to\infty} \frac{1}{N} \log \#\mathcal{I}_N.$$

Lemma 3.2. $$h(\mathcal{I}) \leq \log 4.$$

Proof. Fix $N \in \mathbb{N}$. Note that any word in $\mathcal{I}_N \setminus \mathcal{G}_N$ is either of the form

(a) $(1,2)^z \boldsymbol{g}$ for $\boldsymbol{g} \in \mathcal{G}_{N-z}$ or

(b) $\boldsymbol{w}(1,2)^z \boldsymbol{g}$ for $z = 13^k$ for some $k \in \mathbb{N}$ and $\boldsymbol{w} \in \Omega^w$, where $0 < w < z$ and $\boldsymbol{g} \in \mathcal{G}_{N-z-w}$.

Fix any $\varepsilon > 0$. The number of words of the form (a) is

$$\sum_{z=1}^{N} \#\mathcal{G}_{N-z} \lesssim_\varepsilon e^{N(h(\mathcal{G})+\varepsilon)} = (4e^\varepsilon)^N.$$

The number of words of the form (b) is

$$\begin{aligned}\sum_{z=13^k<N} \sum_{w=1}^{\min\{z-1,N-z\}} 10^w \#\mathcal{G}_{N-z-w} &\lesssim_\varepsilon \sum_{z=13^k<N} \sum_{w=1}^{\min\{z-1,N-z\}} 10^w (4e^\varepsilon)^{N-z-w} \\ &\lesssim \sum_{z=13^k<N} \left(\tfrac{10}{4}\right)^{\min\{z-1,N-z\}} (4e^\varepsilon)^{N-z}.\end{aligned}$$

Since

$$\sum_{z=13^k<N/2} \left(\tfrac{10}{4}\right)^{\min\{z-1,N-z\}} (4e^\varepsilon)^{N-z} = \sum_{z=13^k<N/2} 10^{z-1} 4^{N-2z+1} e^{\varepsilon(N-z)} \lesssim_\varepsilon (4e^{2\varepsilon})^N$$

and

$$\sum_{N/2\leq z=13^k<N} \left(\tfrac{10}{4}\right)^{\min\{z-1,N-z\}} (4e^\varepsilon)^{N-z} = \sum_{N/2\leq z=13^k<N} (10e^\varepsilon)^{N-z} \lesssim_\varepsilon (10e^{2\varepsilon})^{N/2} < 4^N$$

for sufficiently small ε, we have that

$$\#\mathcal{I}_N \lesssim_\varepsilon (4e^{2\varepsilon})^N.$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete. $\square$

4. Dimension estimates

In this section, we introduce the sequences of scales which will be used for the lower and upper box dimension estimates and prove Theorem 1.2. We also show how the proof of Theorem 1.2 can be used to construct an infinitely generated self-affine set whose box dimension does not exist.

Let $\delta > 0$. We let $k(\delta)$ denote the unique positive integer satisfying $12^{-k(\delta)} \leq \delta < 12^{1-k(\delta)}$ and $l(\delta)$ denote the unique positive integer satisfying $2^{-l(\delta)} \leq \delta < 2^{1-l(\delta)}$, noting that $k(\delta) < l(\delta)$ for sufficiently small δ. By definition $l(\delta) = \lceil -\log\delta/\log 2 \rceil$ and $k(\delta) = \lceil -\log\delta/\log 12 \rceil$.

Define the projection $\pi : \Delta^{\mathbb{N}} \to \{1,2\}^{\mathbb{N}}$ by $\pi((a_1,b_1)(a_2,b_2)\cdots) = (a_1 a_2 \cdots)$. For $\boldsymbol{i} \in \Sigma_k$ and $l > k$, define

$$M(\boldsymbol{i}, l) = \#\pi(\boldsymbol{j} \in \Sigma_l : \boldsymbol{j}|k = \boldsymbol{i}). \tag{5}$$

Our general covering strategy at each scale δ can now be described as follows. For each $\boldsymbol{i} \in \Sigma_{k(\delta)}$, observe that $S_{\boldsymbol{i}}([0,1]^2)$ is a rectangle of height $1/12^{k(\delta)} \approx \delta$. In particular, $N_\delta(\Pi(\Sigma)) \approx \sum_{\boldsymbol{i}\in\Sigma_{k(\delta)}} N_\delta(\Pi([\boldsymbol{i}]))$. For each $\boldsymbol{j} \in \Sigma_{l(\delta)}$, we note that $S_{\boldsymbol{j}}([0,1]^2)$ has width $1/2^{l(\delta)} \approx \delta$. Therefore, for each $\boldsymbol{i} \in \Sigma_{k(\delta)}$, we cover each projected cylinder $\Pi([\boldsymbol{i}])$ independently by considering how many level $l(\delta)$ columns contain part of the set $\Pi(\Sigma)$ inside $\Pi([\boldsymbol{i}])$. Since by definition the number of such columns is given by $M(\boldsymbol{i}, l(\delta))$, we obtain

$$N_\delta(\Pi(\Sigma)) \approx \sum_{\boldsymbol{i} \in \Sigma_{k(\delta)}} N_\delta(\Pi([\boldsymbol{i}])) \approx \sum_{\boldsymbol{i} \in \Sigma_{k(\delta)}} M(\boldsymbol{i}, l(\delta)).$$

Define the null sequence $\{\delta_N\}_{N\in\mathbb{N}}$ by $\delta_N = 1/12^{13^N}$, noting that $l(\delta_N) = \lceil 13^N \log 12/\log 2\rceil$ and $k(\delta_N) = 13^N$. Also define the null sequence $\{\delta'_N\}_{N\in\mathbb{N}}$ by $\delta'_N = 1/12^{13^{N-1/2}}$, noting that $k(\delta'_N) = \lceil 13^{N-1/2}\rceil$ and $l(\delta'_N) = \lceil 13^{N-1/2} \log 12/\log 2\rceil$.

In this section we will prove that

$$\limsup_{N\to\infty} \frac{\log N_{\delta_N}(\Pi(\Sigma))}{-\log \delta_N} > \liminf_{N\to\infty} \frac{\log N_{\delta'_N}(\Pi(\Sigma))}{-\log \delta'_N}. \tag{6}$$

Theorem 1.2 will follow from (6) since it implies that $\overline{\dim}_{\mathrm{B}}\,\Pi(\Sigma) > \underline{\dim}_{\mathrm{B}}\,\Pi(\Sigma)$.

Lemma 4.1 (scales with large dimension).

$$\limsup_{N\to\infty} \frac{\log N_{\delta_N}(\Pi(\Sigma))}{-\log \delta_N} \geq \frac{\log 10}{\log 12} + \log 2\left(\frac{1}{\log 2} - \frac{2}{\log 12}\right).$$

Proof. For all $\boldsymbol{w} \in \Omega^{k(\delta_N)}$ and $\boldsymbol{u} \in \{(1,1),(2,1)\}^{l(\delta_N)-2k(\delta_N)}$, we have that $\boldsymbol{w}(1,2)^{k(\delta_N)}\boldsymbol{u} \in \Sigma_{l(\delta_N)}$. In particular, for any $\boldsymbol{w} \in \Omega^{k(\delta_N)}$,

$$M(\boldsymbol{w}, l(\delta_N)) = 2^{l(\delta_N)-2k(\delta_N)} \approx 2^{(\log 12/\log 2 - 2)13^N}, \tag{7}$$

noting that $\log 12/\log 2 > 2$. Hence

$$\begin{aligned} N_{\delta_N}(\Pi(\Sigma)) \geq N_{\delta_N}\Bigg(\bigcup_{\boldsymbol{w}\in\Omega^{k(\delta_N)}} \Pi([\boldsymbol{w}])\Bigg) &\approx \sum_{\boldsymbol{w}\in\Omega^{k(\delta_N)}} N_{\delta_N}(\Pi([\boldsymbol{w}])) \\ &\approx \sum_{\boldsymbol{w}\in\Omega^{k(\delta_N)}} M(\boldsymbol{w}, l(\delta_N)) \approx 10^{13^N} 2^{(\log 12/\log 2 - 2)13^N}. \end{aligned}$$

Hence for some uniform constant $c > 0$,

$$\begin{aligned} \frac{\log N_{\delta_N}(\Pi(\Sigma))}{-\log \delta_N} &\geq \frac{13^N \log 10}{13^N \log 12} + \frac{13^N\left(\frac{\log 12}{\log 2} - 2\right)\log 2}{13^N \log 12} + \frac{\log c}{-13^N \log 12} \\ &= \frac{\log 10}{\log 12} + \log 2\left(\frac{1}{\log 2} - \frac{2}{\log 12}\right) + \frac{\log c}{-13^N \log 12}. \end{aligned}$$

The result follows by letting $N \to \infty$. □

Lemma 4.2 (scales with small dimension).

$$\liminf_{N\to\infty} \frac{\log N_{\delta'_N}(\Pi(\Sigma))}{-\log \delta'_N} \leq \frac{\frac{1}{\sqrt{13}}\log 10 + \left(1 - \frac{1}{\sqrt{13}}\right)\log 4}{\log 12} + \log 2\left(\frac{1}{\log 2} - \frac{1 + \frac{1}{\sqrt{13}}}{\log 12}\right).$$

Proof. Let $\varepsilon > 0$. Recall that for all $N \in \mathbb{N}$, we have $-\log \delta'_N = 13^{N-1/2} \log 12$, $k(\delta'_N) = \lceil 13^{N-1/2} \rceil$ and $l(\delta'_N) = \lceil 13^{N-1/2} \log 12/\log 2 \rceil$. Recall that $\Sigma = \bar{B}$, where B is the set of all infinite sequences which label a one-sided infinite path on the graph G given in Figure 1, and where the set of points $\bar{B} \setminus B$ are characterised in (4). Therefore, any word $\boldsymbol{i} \in \Sigma_{k(\delta'_N)}$ has one of the following forms:

(a) $\boldsymbol{i} = \boldsymbol{u}$ for $\boldsymbol{u} \in \mathcal{I}_{k(\delta'_N)}$.

(b) $\boldsymbol{i} = \boldsymbol{u}\boldsymbol{w}$ for $\boldsymbol{u} \in \mathcal{I}_u$ and $\boldsymbol{w} \in \Omega^w$, where $u + w = k(\delta'_N)$.

(c) $\boldsymbol{i} = \boldsymbol{w}$ for $\boldsymbol{w} \in \Omega^{k(\delta'_N)}$.

(d) $\boldsymbol{i} = \boldsymbol{w}(1,2)^z$ for $\boldsymbol{w} \in \Omega^w$, where $w + z = k(\delta'_N)$.

(e) $\boldsymbol{i} = \boldsymbol{u}\boldsymbol{w}(1,2)^z$ for $\boldsymbol{u} \in \mathcal{I}_u$ and $\boldsymbol{w} \in \Omega^w$, where $u + w + z = k(\delta'_N)$ and $z \le w$.

Let $Y_a \subset \Sigma_{k(\delta'_N)}$ be the set of words which are of the form (a), and let $X_a \subset \Sigma$ be the subset

$$X_a := \{\mathtt{i} \in \Sigma : \mathtt{i}|k(\delta'_N) \in Y_a\}.$$

Define X_b, X_c, X_d, X_e and Y_b, Y_c, Y_d, Y_e analogously. We note that these sets are not all mutually exclusive, for example $Y_a \cap Y_e \neq \varnothing$, but this will not affect our bounds.

Upper bound on $N_{\delta'_N}(\Pi(X_a))$. For any $\boldsymbol{j} \in \{(1,1),(2,1)\}^{l(\delta'_N)-k(\delta'_N)}$ and $\boldsymbol{u} \in \mathcal{I}_{k(\delta'_N)}$, we have $\boldsymbol{u}\boldsymbol{j} \in \Sigma^{l(\delta'_N)}$. Therefore, for each $\boldsymbol{u} \in \mathcal{I}_{k(\delta'_N)}$,

$$M(\boldsymbol{u}, l(\delta'_N)) = 2^{l(\delta'_N)-k(\delta'_N)} \approx 2^{13^{N-1/2}(\log 12/\log 2 - 1)}. \tag{8}$$

Hence

$$N_{\delta'_N}(\Pi(X_a)) \approx \sum_{\boldsymbol{u} \in Y_a} N_{\delta'_N}(\Pi([\boldsymbol{u}])) \approx \sum_{\boldsymbol{u} \in \mathcal{I}_{k(\delta'_N)}} M(\boldsymbol{u}, l(\delta'_N)) \lesssim_\varepsilon (4e^\varepsilon)^{13^{N-1/2}} 2^{13^{N-1/2}(\log 12/\log 2-1)}$$

by Lemma 3.2 and (8). Since $\varepsilon > 0$ was chosen arbitrarily and $-\log \delta'_N = 13^{N-1/2} \log 12$, we deduce that

$$\liminf_{N \to \infty} \frac{\log N_{\delta'_N}(\Pi(X_a))}{-\log \delta'_N} \le \frac{\log 4}{\log 12} + \log 2\left(\frac{1}{\log 2} - \frac{1}{\log 12}\right). \tag{9}$$

Upper bound on $N_{\delta'_N}(\Pi(X_c))$. Suppose $\mathtt{i} \in X_c$, so that $\mathtt{i}|k(\delta'_N) = \boldsymbol{w} \in \Omega^{k(\delta'_N)}$. By definition of Σ, either $\mathtt{i} \in \Omega^{\mathbb{N}}$ or $\mathtt{i}$ begins with $\boldsymbol{u}(1,2)^z$ for some $\boldsymbol{u} \in \Omega^*$, where $|\boldsymbol{u}| \ge k(\delta'_N) = \lceil 13^{N-1/2} \rceil$ and $z \ge 13^N$. For N sufficiently large,

$$z + |\boldsymbol{u}| \ge 13^N + 13^{N-1/2} > 13^{1/2} 13^{N-1/2} > \left\lceil \frac{\log 12}{\log 2} 13^{N-1/2} \right\rceil = l(\delta'_N).$$

In particular, for any $\boldsymbol{w} \in \Omega^{k(\delta'_N)}$,

$$M(\boldsymbol{w}, l(\delta'_N)) = 1. \tag{10}$$

By (10),

$$N_{\delta'_N}(\Pi(X_c)) \approx \sum_{\boldsymbol{w} \in Y_c} N_{\delta'_N}(\Pi([\boldsymbol{w}])) \approx \sum_{\boldsymbol{w} \in \Omega^{k(\delta'_N)}} M(\boldsymbol{w}, l(\delta'_N)) = 10^{k(\delta'_N)} \approx 10^{13^{N-1/2}}.$$

Therefore, since $-\log\delta'_N = 13^{N-1/2}\log 12$,

$$\liminf_{N\to\infty}\frac{\log N_{\delta'_N}(\Pi(X_c))}{-\log\delta'_N}\le\frac{\log 10}{\log 12}. \tag{11}$$

Upper bound on $N_{\delta'_N}(\Pi(X_d))$. For $x>0$ we let $\mathcal{T}(x)$ denote the smallest power of 13 which is greater than or equal to x. Suppose $\mathtt{i}\in X_d$, so that $\mathtt{i}|k(\delta'_N)=\boldsymbol{w}(1,2)^z$ for $\boldsymbol{w}\in\Omega^w$, where $w+z=k(\delta'_N)$. Either $\mathtt{i}=\boldsymbol{w}(1,2)^\infty$ or $\mathtt{i}$ begins with $\boldsymbol{w}(1,2)^{z'}\boldsymbol{j}$ for some $\boldsymbol{j}\in\Sigma_1\setminus\{(1,2)\}$ and

$$z'\ge\mathcal{T}(\max\{w,z\})=\mathcal{T}(\max\{w,k(\delta'_N)-w\})=13^N,$$

where the final equality is because, for sufficiently large N,

$$\max\{w,k(\delta'_N)-w\}\ge\tfrac{1}{2}k(\delta'_N)=\tfrac{1}{2}\lceil 13^{N-1/2}\rceil>13^{N-1}.$$

Moreover, for sufficiently large N,

$$w+z'\ge 13^N>\left\lceil 13^{N-1/2}\frac{\log 12}{\log 2}\right\rceil=l(\delta'_N).$$

In particular, for any $\boldsymbol{w}(1,2)^z\in Y_d$,

$$M(\boldsymbol{w}(1,2)^z,l(\delta'_N))=1. \tag{12}$$

By (12),

$$\begin{aligned}N_{\delta'_N}(\Pi(X_d))&\approx\sum_{\boldsymbol{i}\in Y_d}N_{\delta'_N}(\Pi([\boldsymbol{i}]))\\&\approx\sum_{w=1}^{k(\delta'_N)-1}\sum_{\boldsymbol{w}\in\Omega^w}M(\boldsymbol{w}(1,2)^{k(\delta'_N)-w},l(\delta'_N))\\&\lesssim_\varepsilon(10e^\varepsilon)^{k(\delta'_N)}\approx(10e^\varepsilon)^{13^{N-1/2}}.\end{aligned}$$

Since $\varepsilon>0$ was arbitrary and $-\log\delta'_N=13^{N-1/2}\log 12$,

$$\liminf_{N\to\infty}\frac{\log N_{\delta'_N}(\Pi(X_d))}{-\log\delta'_N}\le\frac{\log 10}{\log 12}. \tag{13}$$

Upper bound on $N_{\delta'_N}(\Pi(X_b))$. Suppose $\mathtt{i}\in X_b$, so that $\mathtt{i}|k(\delta'_N)=\boldsymbol{u}\boldsymbol{w}$ for $\boldsymbol{u}\in\mathcal{I}_u$ and $\boldsymbol{w}\in\Omega^w$, where $u+w=k(\delta'_N)$. Either $\mathtt{i}=\boldsymbol{u}\mathtt{j}$, where $\mathtt{j}\in\Omega^{\mathbb{N}}$, or $\mathtt{i}$ begins with $\boldsymbol{u}\boldsymbol{v}(1,2)^z$, where $\boldsymbol{v}\in\Omega^z$ and we have $z=|\boldsymbol{v}|=\mathcal{T}(|\boldsymbol{v}|)\ge\mathcal{T}(w)$. In particular, for any $\boldsymbol{u}\boldsymbol{w}\in Y_b$,

$$\begin{aligned}M(\boldsymbol{u}\boldsymbol{w},l(\delta'_N))&\le 2^{l(\delta'_N)-|\boldsymbol{u}|-|\boldsymbol{v}|-z}\\&=2^{l(\delta'_N)-|\boldsymbol{u}|-2z}\\&=2^{l(\delta'_N)-k(\delta'_N)+w-2z}\\&\le 2^{l(\delta'_N)-k(\delta'_N)+w-2\cdot\mathcal{T}(w)}.\end{aligned} \tag{14}$$

By (14) and Lemma 3.2,

$$
\begin{aligned}
&N_{\delta'_N}(\Pi(X_b))\\
&\quad\approx \sum_{\boldsymbol{i}\in Y_b} N_{\delta'_N}(\Pi([\boldsymbol{i}])) \approx \sum_{w=1}^{k(\delta'_N)-1} \sum_{\boldsymbol{u}\in\mathcal{I}^{k(\delta'_N)-w}} \sum_{\boldsymbol{w}\in\Omega^w} M(\boldsymbol{uw}, l(\delta'_N))\\
&\quad\lesssim_\varepsilon \sum_{w=1}^{k(\delta'_N)-1} 10^w (4e^\varepsilon)^{k(\delta'_N)-w} 2^{l(\delta'_N)-k(\delta'_N)+w-2\mathcal{T}(w)}\\
&\quad\le \sum_{w=1}^{13^{N-1}} \left(\frac{10\cdot 2}{4e^\varepsilon\cdot 2^2}\right)^w (4e^\varepsilon)^{k(\delta'_N)} 2^{l(\delta'_N)-k(\delta'_N)} + \sum_{w=13^{N-1}+1}^{k(\delta'_N)-1} \left(\frac{10\cdot 2}{4e^\varepsilon\cdot 2^{2\sqrt{13}}}\right)^w (4e^\varepsilon)^{k(\delta'_N)} 2^{l(\delta'_N)-k(\delta'_N)},
\end{aligned}
\tag{15}
$$

where in the last line of (15) we have used the trivial lower bound $\mathcal{T}(x)\ge x$ in the first sum and, in the second sum, that, for all $13^{N-1}+1\le x\le k(\delta'_N)-1=\lceil 13^{N-1/2}\rceil-1$,

$$\sqrt{13}x \le \sqrt{13}(13^{N-1/2}-1)\le 13^N=\mathcal{T}(x). \tag{16}$$

For sufficiently small $\varepsilon>0$, the first sum of the last line of (15) can be bounded above by

$$\sum_{w=1}^{13^{N-1}} \left(\frac{10\cdot 2}{4e^\varepsilon\cdot 2^2}\right)^w (4e^\varepsilon)^{k(\delta'_N)} 2^{l(\delta'_N)-k(\delta'_N)} \lesssim_\varepsilon 10^{13^{N-1}} (4e^{2\varepsilon})^{k(\delta'_N)-13^{N-1}} 2^{l(\delta'_N)-k(\delta'_N)-13^{N-1}}.$$

For sufficiently small $\varepsilon>0$,

$$\frac{10\cdot 2}{4e^\varepsilon\cdot 2^{2\sqrt{13}}}=\frac{5}{e^\varepsilon 4^{\sqrt{13}}}<1;$$

hence the second sum of the last line of (15) can be bounded above by

$$
\begin{aligned}
\sum_{w=13^{N-1}+1}^{k(\delta'_N)-1} \left(\frac{10\cdot 2}{4e^\varepsilon\cdot 2^{2\sqrt{13}}}\right)^w (4e^\varepsilon)^{k(\delta'_N)} 2^{l(\delta'_N)-k(\delta'_N)} &\lesssim_\varepsilon 10^{13^{N-1}} (4e^{2\varepsilon})^{k(\delta'_N)-13^{N-1}} 2^{l(\delta'_N)-k(\delta'_N)+13^{N-1}-2\cdot 13^{N-1/2}}\\
&< 10^{13^{N-1}} (4e^{2\varepsilon})^{k(\delta'_N)-13^{N-1}} 2^{l(\delta'_N)-k(\delta'_N)-13^{N-1}}.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
N_{\delta'_N}(\Pi(X_b)) &\lesssim_\varepsilon 10^{13^{N-1}} (4e^{2\varepsilon})^{k(\delta'_N)-13^{N-1}} 2^{l(\delta'_N)-k(\delta'_N)-13^{N-1}}\\
&\approx 10^{13^{N-1}} (4e^{2\varepsilon})^{13^{N-1/2}-13^{N-1}} 2^{(\log 12/\log 2-1)13^{N-1/2}-13^{N-1}}.
\end{aligned}
$$

Since $\varepsilon>0$ was chosen arbitrarily and $-\log\delta'_N=13^{N-1/2}\log 12$,

$$\liminf_{N\to\infty}\frac{\log N_{\delta'_N}(\Pi(X_b))}{-\log\delta'_N} \le \frac{\frac{1}{\sqrt{13}}\log 10+\left(1-\frac{1}{\sqrt{13}}\right)\log 4}{\log 12}+\log 2\left(\frac{1}{\log 2}-\frac{1+\frac{1}{\sqrt{13}}}{\log 12}\right). \tag{17}$$

Upper bound on $N_{\delta'_N}(\Pi(X_e))$. If $\boldsymbol{uw}(1,2)^z\in Y_e$ with $|\boldsymbol{w}|=w$ and $|\boldsymbol{u}|=u$, then since $u+w\le k(\delta'_N)=\lceil 13^{N-1/2}\rceil$ we have

$$l(\delta'_N)-2w-u\ge l(\delta'_N)-2\lceil 13^{N-1/2}\rceil > l(\delta'_N)-\left\lceil\frac{\log 12}{\log 2}13^{N-1/2}\right\rceil=0.$$

In particular,

$$M(\boldsymbol{u}\boldsymbol{w}(1,2)^z, l(\delta'_N)) = 2^{l(\delta'_N)-2w-u}. \tag{18}$$

By (18) and Lemma 3.2,

$$\begin{aligned}
N_{\delta'_N}(\Pi(X_e)) \approx \sum_{\boldsymbol{i}\in Y_e} N_{\delta'_N}(\Pi([\boldsymbol{i}])) &\approx \sum_{w=13^r\le 13^{N-1}} \sum_{u=1}^{k(\delta'_N)-w-1} \sum_{\boldsymbol{u}\in\mathcal{I}^u} \sum_{\boldsymbol{w}\in\Omega^w} M(\boldsymbol{u}\boldsymbol{w}(1,2)^{k(\delta'_N)-u-w}, l(\delta'_N)) \\
&\lesssim_\varepsilon \sum_{w=13^r\le 13^{N-1}} \sum_{u=1}^{k(\delta'_N)-w-1} (4e^\varepsilon)^u 10^w 2^{l(\delta'_N)-2w-u} \\
&\lesssim_\varepsilon \sum_{w=13^r\le 13^{N-1}} \left(\frac{10\cdot 2}{4e^{2\varepsilon}\cdot 2^2}\right)^w \left(\frac{4e^{2\varepsilon}}{2}\right)^{k(\delta'_N)} 2^{l(\delta'_N)} \\
&\lesssim_\varepsilon 10^{13^{N-1}} (4e^{3\varepsilon})^{k(\delta'_N)-13^{N-1}} 2^{l(\delta'_N)-k(\delta'_N)-13^{N-1}} \\
&\approx 10^{13^{N-1}} (4e^{3\varepsilon})^{13^{N-1/2}-13^{N-1}} 2^{(\log 12/\log 2-1)13^{N-1/2}-13^{N-1}}.
\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily and $-\log\delta'_N = 13^{N-1/2}\log 12$,

$$\liminf_{N\to\infty} \frac{\log N_{\delta'_N}(\Pi(X_e))}{-\log\delta'_N} \le \frac{\frac{1}{\sqrt{13}}\log 10 + \left(1-\frac{1}{\sqrt{13}}\right)\log 4}{\log 12} + \log 2\left(\frac{1}{\log 2} - \frac{1+\frac{1}{\sqrt{13}}}{\log 12}\right). \tag{19}$$

Since the upper bounds in (17) and (19) are strictly greater than the upper bounds in (9), (11) and (13) the proof is complete. □

Proof of Theorem 1.2. $\Pi(\Sigma)$ is invariant under the smooth expanding map

$$T(x, y) = (mx \bmod 1, ny \bmod 1).$$

Note that to four decimal places

$$\frac{\log 10}{\log 12} + \log 2\left(\frac{1}{\log 2} - \frac{2}{\log 12}\right) \approx 1.3687$$

and

$$\frac{\frac{1}{\sqrt{13}}\log 10 + \left(1-\frac{1}{\sqrt{13}}\right)\log 4}{\log 12} + \log 2\left(\frac{1}{\log 2} - \frac{1+\frac{1}{\sqrt{13}}}{\log 12}\right) \approx 1.3038.$$

By Lemmas 4.1 and 4.2,

$$\overline{\dim}_{\mathrm{B}}\,\Pi(\Sigma) \ge \limsup_{N\to\infty} \frac{\log N_{\delta_N}(\Pi(\Sigma))}{-\log\delta_N} > \liminf_{N\to\infty} \frac{\log N_{\delta'_N}(\Pi(\Sigma))}{-\log\delta'_N} \ge \underline{\dim}_{\mathrm{B}}\,\Pi(\Sigma).$$

In particular, the box dimension of $\Pi(\Sigma)$ does not exist. □

Remark 4.3. Lemmas 4.1 and 4.2 can also be used to demonstrate the existence of infinitely generated self-affine sets whose box dimensions are distinct. Consider the countable family of affine contractions

$$\{S_{(1,1)}\} \cup \{S_{(1,2)}\} \cup \bigcup_{N=1}^{\infty} \bigcup_{\boldsymbol{w}\in\Omega^{13^N}} \{S_{\boldsymbol{w}(1,2)^{13^N}}\}$$

which generates the infinitely generated self-affine set $E = \Pi(\widetilde{\Sigma})$, where

$$\widetilde{\Sigma} := \{\boldsymbol{u}_1\boldsymbol{u}_2 \cdots : \boldsymbol{u}_i \in \mathcal{C} \text{ for all } i \in \mathbb{N}\}.$$

Since $E \subset F$, we have that $\underline{\dim}_{\mathrm{B}}\, E \le \underline{\dim}_{\mathrm{B}}\, F$. On the other hand, for all $N \in \mathbb{N}$, $\boldsymbol{w} \in \Omega^{k(\delta_N)}$ and $\boldsymbol{u} \in \{(1,1),(2,1)\}^{l(\delta_N)-2k(\delta_N)}$,

$$[\boldsymbol{w}(1,2)^{13^N}\boldsymbol{u}] \cap \widetilde{\Sigma} \ne \varnothing.$$

Therefore by bounding $N_{\delta_N}(E)$ in the same way as in Lemma 4.1 we deduce that $\underline{\dim}_{\mathrm{B}}\, E < \overline{\dim}_{\mathrm{B}}\, E$.

5. Further questions

Here we suggest possible directions for future work.

Question 5.1. Does there exist an expanding repeller whose box dimension does not exist? Namely, does there exist a smooth expanding map $f : M \to M$ of a Riemannian manifold M and compact set $\Lambda = f(\Lambda)$ such that $\Lambda = \{x \in U : f^n(x) \in U,\ \forall n \in \mathbb{N}\}$ for some open neighbourhood U of Λ?

Question 5.2. Given a smooth diffeomorphism $f : M \to M$, does the box dimension of its basic set (or intersections of the basic set with local stable and unstable manifolds) always exist?

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DECOUPLING INEQUALITIES FOR SHORT GENERALIZED DIRICHLET SEQUENCES

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We study decoupling theory for functions on $\mathbb{R}$ with Fourier transform supported in a neighborhood of short Dirichlet sequences $\{\log n\}_{n=N+1}^{N+N^{1/2}}$, as well as sequences with similar convexity properties. We utilize the wave packet structure of functions with frequency support near an arithmetic progression.

1. Introduction

We study decoupling theory for functions $f:\mathbb{R}\to\mathbb{C}$ with Fourier support near certain convex sequences. As a model case of decoupling, consider the truncated parabola $\mathbb{P}^1=\{(t,t^2):|t|\le 1\}$. Let $R\ge 1$ be a large parameter and write $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$ as a disjoint union of caps $\theta=\mathcal{N}_{R^{-1}}(\mathbb{P}^1)\cap(I\times\mathbb{R})$, where I is an $R^{-1/2}$-interval. The decoupling inequality of [Bourgain and Demeter 2015] says that if $2\le p\le 6$, then for any $\varepsilon>0$ there exists C_ε such that

$$\Big\|\sum_\theta f_\theta\Big\|_{L^p(\mathbb{R}^2)}\le C_\varepsilon R^\varepsilon\Big(\sum_\theta \|f_\theta\|^2_{L^p(\mathbb{R}^2)}\Big)^{\frac12}$$

whenever $f_\theta:\mathbb{R}^2\to\mathbb{C}$ are Schwartz functions satisfying $\operatorname{supp}\widehat{f_\theta}\subset\theta$.

This paper explores analogues between decoupling for $\mathbb{P}^1$ and short Dirichlet sequences $\{\log n\}_{n=N+1}^{N+N^{1/2}}$, as well as sequences with similar convexity properties described in the following definition.

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Definition 1.1. Let $N \geq 2$. We call $\{a_n\}_{n=1}^N$ a generalized Dirichlet sequence (with parameter N) if it satisfies the property

$$a_2 - a_1 \in \left[\frac{1}{4N}, \frac{4}{N}\right], \quad (a_{i+2} - a_{i+1}) - (a_{i+1} - a_i) \in \left[\frac{1}{4N^2}, \frac{4}{N^2}\right]. \tag{1}$$

We will call $\{a_n\}_{n=1}^{N^{1/2}}$ satisfying (1) an $N^{1/2}$- short generalized Dirichlet sequence.

For simplicity, we say short (generalized) Dirichlet sequence to mean $N^{1/2}$-short (generalized) Dirichlet sequence, unless otherwise specified. Note that the reflected short Dirichlet sequence,

$$\{-\log(N + N^{-\frac{1}{2}} - n + 1)\}_{n=1}^{N^{1/2}},$$

satisfies (1).

Now we describe our decoupling set-up. From now on $C, c > 0$ will denote absolute constants that may vary from line to line. For convenience of reading, we may regard C, c as 1. For $1 \leq L \leq cN^{1/2}$ and each $j = 1, \ldots, N^{1/2}/L$, define

$$I_j = \bigcup_{i=(j-1)L+1}^{jL} B_{L^2/N^2}(a_i),$$

where $B_{L^2/N^2}(a_i)$ means the L^2/N^2 interval centered at a_i. Let Ω be the L^2/N^2-neighborhood of $\{a_n\}_{n=1}^{N^{1/2}}$. We consider the partition

$$\Omega = \bigsqcup_j I_j. \tag{2}$$

We choose the L^2/N^2-neighborhood of $\{a_n\}_{n=1}^{N^{1/2}}$ because every I_j is essentially an L^2/N^2-neighborhood of an arithmetic progression, which we call a fat AP. To see this we calculate, for $1 \leq n \leq N^{1/2} - L$,

$$a_{n+L} - a_n - L(a_{n+1} - a_n) = \sum_{m=1}^{L} (a_{n+m} - a_{n+m-1} - (a_{n+1} - a_n)) \sim \sum_{m=1}^{L} \frac{m-1}{N^2} \sim \frac{L^2}{N^2}.$$

So indeed I_j lies in a CL^2/N^2-neighborhood of an L-term AP with common difference $a_{(j-1)L+1} - a_{(j-1)L}$ and starting point $a_{(j-1)L}$. Also, note that the common differences for distinct I_j are cL/N^2-separated.

We denote the partition $\{I_j\}_{j=1}^{N^{1/2}/L}$ by $\mathcal{I}$. The first main result of this paper is the following decoupling theorem for $\Omega = \bigsqcup_{I \in \mathcal{I}} I$.

Theorem 1.2. *Let Ω and $\mathcal{I}$ be defined as in the last paragraphs. Then for $2 \leq p \leq 6$ and every $\varepsilon > 0$*

$$\Big\| \sum_{I \in \mathcal{I}} f_I \Big\|_{L^p(\mathbb{R})} \lesssim_\varepsilon N^\varepsilon \Big(\sum_{I \in \mathcal{I}} \|f_I\|_{L^p(\mathbb{R})}^2 \Big)^{\frac{1}{2}} \tag{3}$$

for functions f_I with supp $\widehat{f_I} \subset I$.

The range of p is sharp in the sense that (3) cannot hold for $p > 6$, which can be seen by taking $\widehat{f_I}$ to be a smooth bump with height 1 adapted to I for every I. Indeed for this choice of f_I, we have $\left|\sum_I f_I\right| \sim (L^2/N^2)N^{1/2}$ on $B_{cN^{1/2}}(0)$, and $\|f_I\|_{L^p(\mathbb{R})} \sim \|\widehat{f_I}\|_{L^{p'}(\mathbb{R})} \sim (L(L^2/N^2))^{1/p'}$, where $1/p + 1/p' = 1$.

So

$$\Big\|\sum_{I\in\mathcal{I}} f_I\Big\|_{L^p(\mathbb{R})} \gtrsim \frac{L^2}{N^2} N^{\frac12}(N^{\frac12})^{\frac1p}, \quad \Big(\sum_{I\in\mathcal{I}} \|f_I\|^2_{L^p(\mathbb{R})}\Big)^{\frac12} \sim \Big(\frac{N^{\frac12}}{L}\Big)^{\frac12}\Big(\frac{L^3}{N^2}\Big)^{1-\frac1p}.$$

Then (3) would imply

$$\Big(\frac{N^{\frac12}}{L}\Big)^{\frac12-\frac3p} \lesssim_\varepsilon N^\varepsilon,$$

and hence $p \le 6$. We shall compare Theorem 1.2 with the $\ell^2 L^p$ decoupling inequality of the parabola in [Bourgain and Demeter 2015], which has the same critical exponent 6. Indeed we will see many similarities between short generalized Dirichlet sequences and $\mathbb{P}^1$ from a Fourier analytic point of view.

The notion of strict convexity of a sequence $\{a_n\}$ in $\mathbb{R}$ will parallel the role of curvature of the parabola in decoupling. Some key geometric aspects in the proof of decoupling for $\mathbb{P}^1$ are identifying caps θ as approximate $R^{-1/2}\times R^{-1}$ rectangles, which give rise to dual tubes θ^* of dimension $R^{1/2}\times R$, and noting that θ are separated in angle and so are θ^*. The $|f_\theta|$ are roughly constant on translates of θ^*.

In the $\{a_n\}_{n=1}^{N^{1/2}}$ setting, corresponding to f_θ we have f_{I_j} which are functions $f_{I_j}:\mathbb{R}\to\mathbb{C}$ satisfying $\operatorname{supp}\widehat{f}_{I_j}\subset I_j$. We may identify the (L^2/N^2)-neighborhood of I as approximately an (L^2/N^2)-neighborhood of an arithmetic progression (called a fat AP), giving rise to dual I^* defined in Definition 2.1, which are also fat APs, and note that distinct I are separated in step-size of the corresponding arithmetic progressions (and the same for I^*). The $|f_I|$ are also roughly constant on translates of I^* [Bourgain 1991; 1993].

Bourgain [1991; 1993] made use of this locally constant property to connect a conjecture of Montgomery with the Kakeya conjecture. To prove a decoupling inequality we need to identify another geometric analogy, the "ball", which is roughly the smallest set restricting to which in the physical space essentially preserves the frequency support.

For the R^{-1}-neighborhood of the parabola, the "ball" is a ball B_R of radius of R. We will define the "ball" $P(L)$ in the short generalized Dirichlet sequence setting in Section 3B. $P(L)$ will be a fat AP which sometimes degenerates to a Euclidean ball. With these notions of caps, tubes, and balls in the short generalized Dirichlet sequence setting, we are able to exploit the wave packet structure of a function with frequency support on $I\in\mathcal{I}$, and prove a bilinear Kakeya-type estimate (Proposition 3.3) and a bilinear restriction-type estimate (Proposition 3.5) that look almost identical to those in the parabola setting. The choice of $N^{1/2}$ plays an important role in making this resemblance possible, which we will discuss at the end of Section 7.

The proof of Theorem 1.2 is based on the high-low decomposition method in [Guth et al. 2022]. We do not intend to get a logarithmic decoupling constant as in that work, but we want to prove a refined decoupling inequality as in [Guth et al. 2020], which creates some technical differences.

The partition $\Omega=\bigsqcup_{I\in\mathcal{I}} I$ is maximal in the sense that if $\Omega=\bigsqcup I'$, where I' is the union of more than CL many adjacent intervals, then I' is no longer essentially a fat AP. Because of this, we will call $\Omega=\bigsqcup_{I\in\mathcal{I}} I$ the canonical partition and refer to Theorem 1.2 as decoupling for the canonical partition, or simply decoupling. In the spirit of small cap decoupling as in [Demeter et al. 2020], we may also consider the "small cap" decoupling for short generalized Dirichlet sequences. Now we let $L_1 \in [1, L]$ be an integer, and we partition Ω into L_1 consecutive intervals J_j:

$$\Omega = \bigsqcup_{j=1}^{N^{1/2}/L_1} J_j = \bigsqcup_{j=1}^{N^{1/2}/L_1} \Bigg(\bigcup_{i=(j-1)L_1+1}^{jL_1} B_{L^2/N^2}(a_i) \Bigg). \tag{4}$$

We let $\mathcal{J}$ denote the partition $\{J_j\}_{j=1}^{N^{1/2}/L_1}$. The next decoupling result in this paper is small-cap-type decoupling inequalities.

Theorem 1.3. *Let $1 \le L_1 \le L \le N^{1/2}$, and $\{J\}_{J\in\mathcal{J}}$ be defined as in the paragraph above. Suppose $p \ge 4$. Then, for every $\varepsilon > 0$,*

$$\Big\| \sum_{J\in\mathcal{J}} f_J \Big\|_{L^p(\mathbb{R})} \lesssim_\varepsilon N^\varepsilon \Bigg(\frac{N^{\frac12-\frac2p} L^{\frac2p}}{L_1^{1-\frac2p}} + \Big(\frac{N^{\frac12}}{L_1}\Big)^{\frac12-\frac1p} \Bigg) \Big(\sum_{J\in\mathcal{J}} \|f_J\|_{L^p(\mathbb{R})}^p \Big)^{\frac1p} \tag{5}$$

for a function $f_J : \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} \widehat{f_J} \subset J$.

Inequality (5) is sharp up to $C_\varepsilon N^\varepsilon$ for every fixed p, L, L_1 satisfying the condition in Theorem 1.3. The first factor in front of $\big(\sum_{J\in\mathcal{J}} \|f_J\|_{L^p(\mathbb{R})}^p\big)^{1/p}$ is sharp because of the example $\widehat{f_J}$ equals a smooth bump adapted to J with height 1 for every $J \in \mathcal{J}$. The calculation is similar to the one in the paragraph below Theorem 1.2. The second factor is sharp because of the example $\widehat{f_J}$ equals a random sign times a smooth bump adapted to a ball of radius L^2/N^2 inside J with height 1 for every $J \in \mathcal{J}$, where the random signs are chosen so that $\int_{\mathbb{R}} \big|\sum_J f_J\big|^p \sim \int_{\mathbb{R}} \big(\sum_J |f_J|^2\big)^{p/2}$ by Khintchine's inequality.

The structure of the proof of Theorem 1.3 is similar to that of Theorem 3.1 in [Demeter et al. 2020], consisting of three ingredients: refined decoupling for the canonical partition, refined flat decoupling, and an incidence estimate. Refined decoupling for the canonical partition is a refined version of Theorem 1.2, which we will prove in Sections 4, 5, and 6 in order to derive Theorem 1.2. We show the other two counterparts in Section 8.

1A. L^p *estimates for short generalized Dirichlet polynomials.*

A straight corollary of Theorem 1.3 is essentially sharp L^p estimates for short generalized Dirichlet polynomials $\sum_{n=1}^{N^{1/2}} b_n e^{ita_n}$.

Corollary 1.4. *Let $\{a_n\}_{n=1}^{N^{1/2}}$ be a short generalized Dirichlet sequence. Suppose $p \ge 4$ and $N \le T \le N^2$. We have for every $\varepsilon > 0$*

$$\Big\| \sum_{n=1}^{N^{1/2}} b_n e^{ita_n} \Big\|_{L^p(B_T)} \lesssim_\varepsilon N^\varepsilon (N^{\frac12} + T^{\frac1p} N^{\frac14-\frac1{2p}}) \|b_n\|_{\ell^p} \tag{6}$$

for every B_T and every $\{b_n\}_{n=1}^{N^{1/2}} \subset \mathbb{C}$.

If we let $L \in [1, N^{1/2}]$ be the integer such that $N^2/L^2 = T$, then Corollary 1.4 follows from Theorem 1.3 with that L, and $L_1 = 1$, applied to functions $f_J(t) = b_n e^{ita_n}\phi(t)$ for every J, where ϕ is a Schwartz function adapted to B_T with Fourier support inside $B_{T^{-1}}(0)$.

The inequality (6) is sharp up to $C_\varepsilon N^\varepsilon$. This is from discrete versions of the examples described below Theorem 1.3, taken with $L_1 = 1$: $b_n = 1$ for every n, and b_n equal to random signs.

We will in fact prove a more general version of Theorem 1.3 which allows us to get essentially sharp (ℓ^q, L^p) estimates for $\sum_{n=1}^{N^{1/2}} b_n e^{ita_n}$ in the range $p \geq 4$, $\frac{1}{p} + \frac{3}{q} \leq 1$. See Theorem 8.5 and Corollary 8.2.

After this work was done we learned from James Maynard a general transference method, which can in particular transfer the L^p estimate on a short generalized Dirichlet polynomial to a 2-dimensional L^p estimate on an exponential sum with frequency support near a convex curve in $\mathbb{R}^2$. This allows us to derive Corollary 1.4 directly from the small cap decoupling inequalities for the parabola in [Demeter et al. 2020]. We provide that particular argument in detail in the Appendix.

The starting point of this paper was to see whether decoupling methods could be used to make progress on Montgomery's conjecture on Dirichlet polynomials [1971; 1994]. Our investigation led us in a different direction, proving decoupling inequalities for short generalized Dirichlet sequences.

Conjecture 1.5 (Montgomery's conjecture). *For every $p \geq 2$ and every $\varepsilon > 0$ we have*

$$\Big\| \sum_{n=N+1}^{2N} b_n n^{it} \Big\|_{L^p(B_T)} \leq C_\varepsilon T^\varepsilon N^{\frac{1}{2}} (N^{\frac{p}{2}} + T)^{\frac{1}{p}} \|b_n\|_{\ell^\infty} \tag{7}$$

for every ball B_T of radius T and every $\{b_n\}_{n=N+1}^{2N} \subset \mathbb{C}$.

Conjecture 1.5 is widely open. In fact it has significant implications which are also hard conjectures. It is shown in [Montgomery 1971] that Conjecture 1.5 implies the density conjecture for the Riemann zeta function. Bourgain [1991; 1993] observed that a stronger version of Conjecture 1.5 on large value estimate of Dirichlet polynomials implies the Kakeya maximal operator conjecture in all dimensions. Conjecture 1.5 itself also implies a weaker statement that a Kakeya set has full Minkowski dimension; see [Green 2002].

Our Corollary 1.4 proves some L^p estimates for "short" Dirichlet polynomials which do not directly connect to Montgomery's conjecture. In fact we believe to make progress on Montgomery's conjecture significant new ideas are needed.

On the other hand, combining Theorem 1.2 with flat decoupling we obtain $\ell^2 L^p$ decoupling inequalities for generalized Dirichlet sequences (with N many terms instead of $N^{1/2}$), and the decoupling inequalities we get are essentially sharp for the class of generalized Dirichlet sequences. As a corollary we have essentially sharp (ℓ^2, L^p) estimates on generalized Dirichlet polynomials, but the Dirichlet polynomial $\sum_{n=N+1}^{2N} b_n e^{it\log n}$ has more structure and admits better estimates. This has to do with examples of generalized Dirichlet sequences containing a $cN^{1/2}$-term AP with common difference $CN^{-1/2}$, which $\{\log n\}_{n=N+1}^{2N}$ cannot contain by a number theory argument. We discuss these in detail in Section 7.

The paper is structured as follows. In Section 2 we will illustrate the wave packet structure of functions with frequency support in a fat AP. In Section 3 we prove a bilinear Kakeya-type estimate and a bilinear restriction-type estimate for functions with frequency support in a neighborhood of a short generalized Dirichlet sequence $\{a_n\}_{n=1}^{N^{1/2}}$. Sections 4, 5, and 6 are dedicated to proving Theorem 1.2. Section 4 introduces a refined decoupling inequality for the canonical partition (Theorem 4.4), which implies Theorem 1.2, and which we will actually prove. Section 5 sets up a high-low frequency decomposition for square functions at different scales, and in Section 6 we finish the proof of Theorem 4.4. Section 7 discusses the decoupling problem for (N-term) generalized Dirichlet sequences. In Section 8 we prove

Theorem 1.3. The Appendix is about the transference method for one-dimensional exponential sum estimates like (6).

Notation. C will denote a positive absolute constant that may vary from line to line, and it may be either small or large. $A \lesssim B$ means $A \le CB$, and $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. We will also use $\mathcal{O}(A)$ to denote a quantity that is less than or equal to CA. $A \lesssim_q B$ will mean $A \le C_q B$ for some constant depending on q. Similarly $\mathcal{O}_q(A)$ denotes a quantity that is less than or equal to $C_q A$. There will be a parameter N and $A \lessapprox B$ denotes $A \lesssim_\varepsilon N^\varepsilon B$ for every $\varepsilon > 0$.

2. Locally constant property

We set up some notation and describe the locally constant property related to fat APs in this section.

Definition 2.1. We let $P_v^\delta(a)$ denote the δ-neighborhood of the arithmetic progression on $\mathbb{R}$ which contains a and has common difference v. We call $P_v^\delta(x_0) \cap B_R(x_0)$, or simply $P_v^\delta \cap B_R$, a fat AP with thickness δ, common difference v, and diameter R. We will call $P_{v^{-1}}^{R^{-1}} \cap B_{\delta^{-1}}$ a fat AP dual to $P_v^\delta \cap B_R$.

To exploit the locally constant property of a function with frequency support in a fat AP, we first construct a family of functions $\psi_k : \mathbb{R} \to \mathbb{C}$ adapted to a fat AP (in the frequency space).

Lemma 2.2. *For every $x_0 \in \mathbb{R}$, $\delta \le v/2$, $M \ge 1$, and $k \ge 1$ there exists a function $\psi_k : \mathbb{R} \to \mathbb{C}$ with the property*

$$\widehat{\psi}_k(\xi) = 1 \quad \text{on } P_v^\delta(x_0) \cap B_{Mv}(x_0), \qquad \operatorname{supp} \widehat{\psi}_k \subset P_v^{2\delta}(x_0) \cap B_{8^k Mv}(x_0), \tag{8}$$

and ψ_k decays at order k outside of the dual fat AP $P_{v^{-1}}^{(Mv)^{-1}}(0) \cap B_{\delta^{-1}}(0)$:

$$(M\delta) 1_{P_{v^{-1}}^{(Mv)^{-1}}(0) \cap B_{\delta^{-1}}(0)} \lesssim_k |\psi_k(x)| \lesssim_k M\delta \left(1 + \frac{d(x, v^{-1}\mathbb{Z})}{(Mv)^{-1}}\right)^{-k} \left(1 + \frac{d(x, B_{\delta^{-1}}(0))}{\delta^{-1}}\right)^{-k}. \tag{9}$$

We say such a ψ_k is adapted to the fat AP $P_v^\delta(x_0) \cap B_{Mv}(x_0)$ in the frequency space with order of decay k.

Proof. Since translation in frequency space corresponds to modulation in the physical space, we may assume $x_0 = 0$.

We start with the Dirichlet kernel

$$D_M(x) = \sum_{|j| \le M} e^{2\pi i j x} = \frac{\sin((2M+1)\pi x)}{\sin(\pi x)}.$$

We define $\widetilde{D}_1(x) = D_M(x)$. Then we define $\widetilde{D}_k(x)$ inductively by

$$\widetilde{D}_k(x) = d_k^{-1} \widetilde{D}_{k-1}(x) D_{8^{k-1}M/2}(x),$$

where $d_k = \|\widehat{D}_{8^{k-1}M/2}\|_{L^1(\mathbb{R})}$ is the total measure of $\widehat{D}_{8^{k-1}M/2}$. Equivalently we can define $\widetilde{D}_k$ explicitly as

$$\widetilde{D}_k = \tilde{d}\, D_M \prod_{1 \le s \le k-2} D_{8^s M/2}$$

for some suitable constant $\tilde{d} > 0$.

Since $\widetilde{D}_1 = D_M$ has the property

$$\widehat{\widetilde{D}}_1(\xi) = \sum_{|j|\le M} \delta_0(\xi - j),$$

by induction we can show that

$$\widehat{\widetilde{D}}_k(\xi) = \sum_{|j|\le M} \delta_0(\xi - j) + \sum_{M<|j|\le 8^k M/4} b_{j,k}\delta_0(\xi - j)$$

for some $0 \le b_{j,k} \le 1$. From the explicit expression of the Dirichlet kernel we see that $\widetilde{D}_1$ decays at order 1 outside of $P_1^{M^{-1}}(0)$:

$$|\widetilde{D}_0(x)| = |D_M(x)| \lesssim \frac{M}{1 + d(x,\mathbb{Z})/M^{-1}}.$$

By induction on k we obtain $\widetilde{D}_k$ decays at order k outside of $P_1^{M^{-1}}(0)$:

$$|\widetilde{D}_k(x)| \lesssim_k M\left(1 + \frac{d(x,\mathbb{Z})}{M^{-1}}\right)^{-k}. \tag{10}$$

Now let $\phi(x)$ be a Schwartz function such that $\widehat{\phi}$ is a smooth bump adapted to $B_1(0)$:

$$\widehat{\phi}(\xi) = 1 \quad \text{on } B_1(0), \quad \operatorname{supp}\widehat{\phi} \subset B_2(0).$$

Let $\phi_{\delta^{-1}}(x)$ be the function $\phi(\delta x)$. Note that $\phi_{\delta^{-1}}$ decays rapidly outside of $B_{\delta^{-1}}(0)$. Let ψ_k be given by

$$\widehat{\psi}_k := \widehat{\phi}_{\delta^{-1}} * \widehat{\widetilde{D}}_k(v^{-1}\xi)/v = \sum_{|j|\le M} \widehat{\phi}_{\delta^{-1}}(\xi - jv) + \sum_{M<|j|\le 8^k \frac{M}{4}} b_{j,k}\widehat{\phi}_{\delta^{-1}}(\xi - jv).$$

From this definition we immediately see property (8) holds. Writing ψ_k as

$$\psi_k(x) = \phi_{\delta^{-1}}(x)\widetilde{D}_k(vx)$$

we observe from (10) and the rapid decay of $\phi_{\delta^{-1}}$ outside $B_{\delta^{-1}}(0)$ that (9) holds. $\square$

For every fat AP $P = P_{v^{-1}}^{(Mv)^{-1}}(x_0) \cap B_{\delta^{-1}}(x_0)$ with $\delta \le v$, and every $k \ge 100$, let $W_{P,k}$ be the weight function

$$W_{P,k}(x) = \left(1 + \frac{d(x, x_0 + v^{-1}\mathbb{Z})}{(Mv)^{-1}}\right)^{-k}\left(1 + \frac{d(x, B_{\delta^{-1}}(x_0))}{\delta^{-1}}\right)^{-k}.$$

We will use the notation

$$\int_{W_{P,k}} f(x)\,dx := \int_{\mathbb{R}} f(x)W_{P,k}(x)\,dx,$$

$$\fint_{W_{P,k}} f(x)\,dx := \frac{1}{\|W_{P,k}\|_{L^1(\mathbb{R})}} \int_{\mathbb{R}} f(x)W_{P,k}(x)\,dx,$$

$$\|f\|_{L^p_{\#}(W_{P,k})} := \left(\fint_{W_{P,k}} |f|^p(x)\,dx\right)^{\frac{1}{p}}.$$

For measurable sets $E \subset \mathbb{R}$ we use similar notation for average integrals and L^p norms:

$$\fint_E f(x)\,dx := \frac{1}{|E|}\int_E f(x)\,dx,$$
$$\|f\|_{\text{Ł}^p(E)} := \left(\fint_E |f|^p(x)\,dx\right)^{\frac{1}{p}}.$$

For a fat AP P, consider another fat AP $P' \subset P$. Let $P' \subset P$ denote an indexing set of translates of P' which form an $\mathcal{O}(1)$-overlapping tiling of P. Then we have the pointwise inequality

$$1_P(x) \lesssim_k \sum_{P'\subset P} W_{P',k}(x) \lesssim_k W_{P,k}(x). \tag{11}$$

If we look at translated copies P'' of P, we have

$$\sum_{P''\subset\mathbb{R}} W_{P'',k}(x) W_{P,k}(P'') \lesssim_k W_{P,k}(x). \tag{12}$$

Here $\sum_{P''\subset\mathbb{R}}$ means summing over a tiling (with $\mathcal{O}(1)$ overlap) of $\mathbb{R}$ by P'', and $W_{P,k}(P'')$ is defined to be $W_{P,k}(\sup P'')$, which is comparable to $W_{P,k}(x)$ for any $x \in P''$.

Proposition 2.3 (locally constant property). *Suppose f satisfies* $\operatorname{supp}\widehat{f} \subset P_v^\delta \cap B_{Mv}$. *Then for every dual fat AP $P = P_{v^{-1}}^{(Mv)^{-1}} \cap B_{\delta^{-1}}$ and every $1 \le q < p < \infty$ we have*

$$\|f\|_{\text{Ł}^p(W_{P,k})} \lesssim_{p,q,k} \|f\|_{\text{Ł}^q(W_{P,\frac{qk}{p}})} \quad \textit{if } \frac{qk}{p} \ge 100,$$
$$\|f\|_{L^\infty(P)} \lesssim_k \|f\|_{\text{Ł}^1(W_{P,k})}.$$

Proof. We first prove the second inequality. Fix $k \ge 100$. From (8) we have

$$f(x) = f * \psi_k(x) = \int_{\mathbb{R}} f(y)\psi_k(x-y)\,dy,$$

where ψ_k is the function in Lemma 2.2 adapted to $P_v^\delta \cap B_{Mv}$ in the frequency space with order of decay k. Therefore for $x \in P$ we have

$$\begin{aligned}
|f(x)| &\le \int_{\mathbb{R}} |f(y)||\psi_k(x-y)|\,dy \\
&\le \int_{\mathbb{R}} |f(y)| \sup_{x\in P} |\psi_k(x-y)|\,dy \\
&\lesssim_k \delta M \int_{\mathbb{R}} |f(y)| W_{P,k}(y)\,dy \sim_k \fint_{W_{P,k}} |f(y)|\,dy.
\end{aligned}$$

For the third inequality we used (9). Now we prove the first inequality in the proposition. We claim that from (12) (applied with k replaced by qk/p) and the assumption $q < p$ we only need to show

$$\|f\|_{\text{Ł}^p(P)} \lesssim_{p,q,k} \|f\|_{\text{Ł}^q(W_{P,k})}. \tag{13}$$

Indeed if (13) holds, then

$$\begin{aligned}\int_{W_{P,k}} |f|^p &\lesssim_k \sum_{P'\subset\mathbb{R}} \int_{P'} |f|^p W_{P,k}(P') \\ &\lesssim_{p,q,k} |P|^{1-\frac{p}{q}} \sum_{P'\subset\mathbb{R}} W_{P,k}(P')\Big(\int_{W_{P',qk/p}} |f|^q\Big)^{\frac{p}{q}} \\ &\le |P|^{1-\frac{p}{q}}\Big(\int_{\mathbb{R}} |f(x)|^q \sum_{P'\subset\mathbb{R}} W_{P,k}(P')^{\frac{q}{p}} W_{P',\frac{qk}{p}}(x)\,dx\Big)^{\frac{p}{q}} \\ &\lesssim_{p,q,k} |P|^{1-\frac{p}{q}}\Big(\int |f(x)|^q \sum_{P'\subset\mathbb{R}} W_{P,\frac{qk}{p}}(P') W_{P',\frac{qk}{p}}(x)\,dx\Big)^{\frac{p}{q}} \\ &\lesssim_{p,q,k} |P|^{1-\frac{p}{q}}\Big(\int |f|^q W_{P,\frac{qk}{p}}\Big)^{\frac{p}{q}} \quad \text{(by (12))},\end{aligned}$$

which is exactly the first inequality in the proposition. To show (13) we observe that the second inequality in the proposition together with Hölder's inequality implies that

$$\|f\|_{L\!\!\!-^p(P)} \le \|f\|_{L^\infty(P)} \lesssim_{p,q,k} \|f\|_{L\!\!\!-^q(W_{P,\frac{qk}{p}})},$$

which is (13). ☐

3. Bilinear Kakeya-type and restriction-type estimates

Kakeya and restriction-type estimates are closely related to decoupling, and we will use the bilinear version of them in the proof of Theorem 1.2, but first we need to introduce a more general decoupling set-up for the purpose of induction.

3A. ***General set-up.*** To prove Theorem 1.2 we will do a broad-narrow argument which involves rescaling of a segment of $\{a_n\}_{n=1}^{N^{1/2}}$. To properly set up our induction hypothesis we consider the following more general class of generalized Dirichlet sequences.

Definition 3.1 (generalized Dirichlet sequence). Let $\theta \in (0, 1]$ and $N \ge 2$. We call $\{a_n\}_{n=1}^{N}$ a generalized Dirichlet sequence (with parameters N, θ) if it satisfies the property

$$a_2 - a_1 \in \Big[\frac{1}{4N}, \frac{4}{N}\Big], \quad (a_{i+2} - a_{i+1}) - (a_{i+1} - a_i) \in \Big[\frac{\theta}{4N^2}, \frac{4\theta}{N^2}\Big]. \tag{14}$$

We will call $\{a_n\}_{n=1}^{N^{1/2}}$ satisfying (1) an $N^{1/2}$-short generalized Dirichlet sequence (with parameters N, θ).

As before we write "short" for "$N^{1/2}$-short" for simplicity. Comparing with Definition 1.1 we see an extra parameter θ which measures the convexity of the sequence. From now on we use Definition 3.1 for the definition of generalized Dirichlet sequence.

We shall also incorporate θ in our decoupling set-up. Let $\{a_n\}_{n=1}^{N^{1/2}}$ be a short generalized Dirichlet sequence with parameter $\theta \in (0, 1]$. From the spacing property (14) of $\{a_n\}_{n=1}^{N^{1/2}}$ we see that, for every $1 \le j \le N^{1/2}/L$, $\{a_n\}_{n=(j-1)L+1}^{jL}$ is essentially contained in an $L^2\theta/N^2$-neighborhood of an arithmetic progression. Indeed, if we define $v_j = a_{(j-1)L+2} - a_{(j-1)L+1}$, then $\{a_n\}_{n=(j-1)L+1}^{jL}$ is contained in the $CL^2\theta/N^2$-neighborhood of the arithmetic progression containing a_{jL} with common difference v_j, that is,

$$\{a_n\}_{n=(j-1)L+1}^{jL} \subset P_{v_j}^{CL^2\theta/N^2}(a_{jL}) \cap B_{CL/N}(a_{jL}).$$

Now we let Ω be the $\theta L^2/N^2$-neighborhood of $\{a_n\}_{n=1}^{N^{1/2}}$. For $1 \le L \le cN^{1/2}$ and each $j = 1, \dots, N^{1/2}/L$, define

$$I_j = \bigcup_{i=(j-1)L+1}^{jL} B_{\theta L^2/N^2}(a_i).$$

We denote the collection of I_j by $\mathcal{I}$, and consider the partition

$$\Omega = \bigsqcup_{I \in \mathcal{I}} I.$$

This will be our new decoupling set-up for the canonical partition, and from now on the notation here supersedes that in the Introduction. For small-cap-type decoupling we postpone the description of the corresponding general set-up to Section 8.

3B. ***Analogies between $\{a_n\}_{n=1}^{N^{1/2}}$ and $\mathbb{P}^1$.*** For $I = I_j \in \mathcal{I}$, we let

$$\widetilde{I}_j := P_{v_j}^{CL^2\theta/N^2}(a_{jL}) \cap B_{CL/N}(a_{jL}),$$

with C large enough so that

$$I = I_j \subset \widetilde{I}_j = \widetilde{I}.$$

Here $v_j = a_{(j-1)L+2} - a_{(j-1)L+1}$ and $v_j \sim N^{-1}$.

For each $I \in \mathcal{I}$, we denote by $P_I(x)$ the fat AP dual to $\widetilde{I}$ and centered at x, that is,

$$P_I(x) := P_{v_j^{-1}}^{CN/L}(x) \cap B_{CN^2/(L^2\theta)}(x) \tag{15}$$

if $I = I_j$, and we simply write P_I if stressing the center x is unnecessary. For $I = I_j$, we also write v_I to denote v_j. We let $P(L, y)$ denote a larger fat AP

$$P(L, y) := P_{v_1^{-1}}^{CN^{3/2}/L^2}(y) \cap B_{CN^2/(L^2\theta)}(y), \tag{16}$$

and we simply write $P(L)$ if stressing the center y is unnecessary. If $L \lesssim N^{1/4}$ we have $N^{3/2}/L^2 \ge N$ and in that case $P(L)$ is a ball $B_{CN^2/(L^2\theta)}$. Comparing (15) and (16), we see $P(L)$ has a larger thickness size $CN^{3/2}/L^2$. We will see shortly (Lemma 3.2 and the paragraph following it) that $CN^{3/2}/L^2$ is the smallest thickness that allows us to fit a P_I in any fixed $P(L)$ for every $I \in \mathcal{I}$.

The starting point of this paper is to make use of an analogy between the extension operator on $\{a_n\}_{n=1}^{N^{1/2}}$

$$\{b_n\}_{n=1}^{N^{1/2}} \mapsto \sum_{n=1}^{N^{1/2}} b_n e^{ita_n}$$

and the extension operator on the truncated parabola $\mathbb{P}^1$

$$f \mapsto \int_{[-1,1]} f(\xi)\, e^{i(x\xi + t\xi^2)}\, d\xi.$$

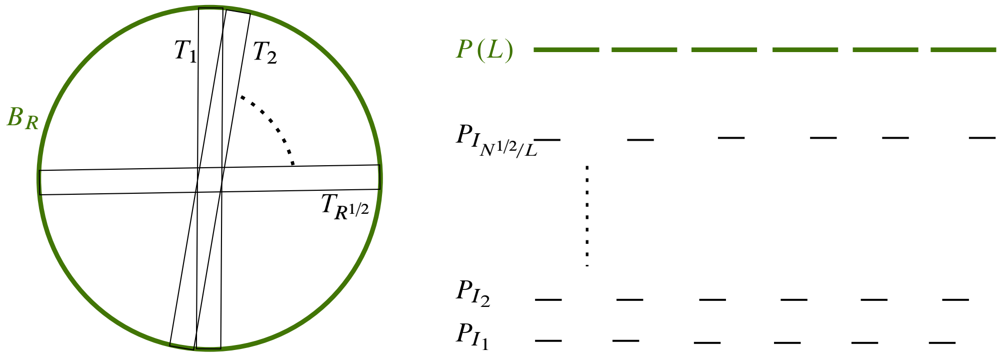

Figure 1. The ball $B_R \subset \mathbb{R}^2$ contains the union of tubes T_i having the same center, each of which is dual to θ_i, where $\bigsqcup_i \theta_i$ partitions $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$. On the right, we see analogous dual fat APs, one P_{I_i} per I_i which partition Ω into L consecutive intervals. We see that $P(L)$ contains the union of the P_{I_i} which have the same starting point.

We list the correspondence between objects in this paper and in the parabola setting. For simplicity we assume $\theta = 1$ in the following list:

(1) The parameter $L \in [1, N^{1/2}]$ is the length of the "cap" that we are looking at, and that determines a canonical neighborhood Ω with width L^2/N^2. The corresponding parameter in the parabola setting is R, which determines the length ($R^{-1/2}$) of the cap and a canonical neighborhood with width R^{-1}.

(2) The $\widetilde{I}$, P_I defined above is analogous to the cap and tube in the context of parabola decoupling. Let Θ be a partition of $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$, the R^{-1}-neighborhood of the truncated parabola $\mathbb{P}^1$ (over $[-1, 1]$), into $R^{-1/2} \times R^{-1}$ caps θ. The dual object of θ is a tube T of dimension $R^{1/2} \times R$.

(3) $P(L)$ is defined to be the smallest fat AP with the property that, for a function F with frequency support on Ω, "restricting" F in the physical space to $P(L)$ will essentially preserve its frequency support. The corresponding object for the parabola is B_R, a ball of radius R.

See Figure 1 which illustrates the analogous properties of tubes T with the ball B_R and fat APs P_I with $P(L)$. Bourgain [1991; 1993] made use of the first two analogies. The new ingredient we need is the third analogy, which gives an appropriate notion of ball in the short generalized Dirichlet sequence setting. It is very important that we define $P(L)$ to be the smallest fat AP with such a property. If we naively use B_{N^2/L^2} as the ball $P(L)$, the whole argument that follows will break down.

To make the third point precise, we prove the following lemma. We introduce one more notation. For a general fat AP $P = P_v^\delta(x_0) \cap B_{Mv}(x_0)$ and $s > 0$, sP will denote the fat AP $P_v^{s\delta}(x_0) \cap B_{sMv}(x_0)$.

Lemma 3.2. *Fix a $P(L)$. For every $I \in \mathcal{I}$ and every P_I with $P_I \cap P(L) \neq \varnothing$, P_I is contained in $2P(L)$.*

Proof. In fact for every j, the difference of differences hypothesis in (14) implies that $|v_j - v_1| \lesssim N^{-3/2}\theta$. It follows that $|v_j^{-1} - v_1^{-1}| \lesssim N^{1/2}\theta$. Therefore $P_I \cap P(L) \neq \varnothing$ implies

$$d(P_I, P(L)) \lesssim (N^{\frac{1}{2}}\theta)\frac{N^2/(L^2\theta)}{N} = \frac{N^{\frac{3}{2}}}{L^2}, \tag{17}$$

which implies $P_I \subset 2P(L)$ if C is large enough in the definition of $P(L)$. $\square$

To put it in another way, the proof above shows that, for every $x \in \mathbb{R}$,

$$\bigcup_{I \in \mathcal{I}} P_I(x) \subset P(L, x) \tag{18}$$

if C is large enough in the definition of $P(L)$. Since the inequality (17) is sharp up to a constant, the choice of $CN^{3/2}/L^2$ as the thickness in the definition of $P(L)$ makes (18) and Lemma 3.2 barely hold.

We note that the above lemma holds if we replace I, $\mathcal{I}$, P_I, $P(L)$ by θ, Θ, T, B_R respectively.

3C. ***Transversality and bilinear Kakeya-type estimate.*** We say $I, J \in \mathcal{I}$ are transversal if $|v_I^{-1} - v_J^{-1}| \gtrsim N^{1/2}\theta$, or equivalently, if $d(I, J) \gtrsim N^{-1/2}$ on $\mathbb{R}$. We now prove a bilinear Kakeya-type estimate for two transversal families of P_I.

Proposition 3.3 (bilinear Kakeya-type estimate). *Suppose $g_1 = \sum_I a_I 1_{P_I}$ and $g_2 = \sum_J b_J 1_{P_J}$, where a_I, b_J are positive real numbers, $I, J \in \mathcal{I}$ and P_I are transversal to P_J. Then*

$$\fint_{P(L)} g_1 g_2 \lesssim \fint_{2P(L)} g_1 \fint_{2P(L)} g_2. \tag{19}$$

For comparison we state the bilinear Kakeya-type estimates for $R^{1/2} \times R$ tubes in $\mathbb{R}^2$.

Proposition 3.4. *Suppose $g_1 = \sum_i a_i 1_{T_i}$ and $g_2 = \sum_j b_j 1_{T_j}$, where a_i, b_j are positive real numbers, T_i, T_j are $R^{1/2} \times R$ tubes and every T_i is transversal to every T_j (in the sense that the angle between T_i, T_j is $\gtrsim 1$). Then*

$$\fint_{B_R} g_1 g_2 \lesssim \fint_{2B_R} g_1 \fint_{2B_R} g_2.$$

Proof of Proposition 3.3. For simplicity of notation we assume $C = 1$ in (15), (16). For general C the argument works the same way. Since

$$\fint_{P(L)} g_1 g_2 \le \sum_{I,J:\, P_I \cap P(L) \neq \varnothing,\, P_J \cap P(L) \neq \varnothing} a_I b_J |P(L)|^{-1} |P_I \cap P_J|$$

it suffices to show that for I, J transversal we have

$$|P_I \cap P_J| \lesssim \frac{|P_I|^2}{|P(L)|}. \tag{20}$$

We consider two cases $L \ge C_1 N^{1/4}$ and $L \le C_1 N^{1/4}$ separately, where C_1 is a sufficiently large constant that will be chosen.

<u>Case 1</u>: $L \ge C_1 N^{1/4}$. Without loss of generality we assume P_I, P_J both start at the origin (meaning that the first term of the underlying AP is 0). Let $P_{I,k}$ denote the k-th interval in P_I. If V_I, V_J are the common difference of P_I, P_J respectively, then from the transversality assumption we have $|V_I - V_J| \sim N^{1/2}\theta$. So for some integer

$$K \sim \frac{N/L}{N^{1/2}\theta} = \frac{N^{1/2}}{L\theta}$$

we have

$$d(P_{I,k}, P_{J,k}) \le \frac{N}{L} \quad \text{if } 1 \le k \le K$$

and

$$d(P_{I,k}, P_{J,k}) \in \Big[\frac{N}{L}, N\Big] \quad \text{if } K \le k \lesssim \frac{N}{N^{\frac{1}{2}}\theta} = \frac{N^{\frac{1}{2}}}{\theta}.$$

Since $L \ge C_1 N^{1/4}$ we know that if C_1 is sufficiently large then $N^{1/2}/\theta N = N^{3/2}/\theta$ is larger than $N^2/(L^2\theta)$, which is the diameter of P_I. Therefore we have

$$|P_I \cap P_J| \lesssim \frac{N^{\frac{1}{2}}}{L\theta}\frac{N}{L} = \frac{N^{\frac{3}{2}}}{L^2\theta} = \frac{|P_I|^2}{|P(L)|}.$$

<u>Case 2</u>: $L \le C_1 N^{1/4}$. From the first case we know that

$$|P_I \cap P_J \cap B_{CN^{3/2}/\theta}| \lesssim \frac{N^{\frac{3}{2}}}{L^2\theta}.$$

Therefore by the triangle inequality we have

$$|P_I \cap P_J| \lesssim \frac{N^{\frac{3}{2}}}{L^2\theta}\frac{N^2/(L^2\theta)}{N^{\frac{3}{2}}/\theta} = \frac{N^2}{L^4\theta} = \frac{|P_I|^2}{|P(L)|}.$$

Here we recall that $P(L)$ degenerates to the Euclidean ball $B_{N^2/(L^2\theta)}$ if $L \le N^{1/4}$. So we have shown (20) and hence (19). □

3D. ***Bilinear restriction-type estimate.*** To prove a bilinear restriction estimate, we will use the above bilinear Kakeya estimate and induction on L. First we identify where the (square of the) square function $\sum_{I\in\mathcal{I}} |f_I|^2$ is locally constant on. Note that $\operatorname{supp} \widehat{|f_I|^2} \subset I - I \subset P_{v_I}^{CL^2\theta/N^2}(0) \cap B_{CL/N}(0)$. Since $|v_I - v_1| \lesssim N^{-3/2}\theta$ for every $I \in \mathcal{I}$, we have

$$\bigcup_{I\in\mathcal{I}} (I - I) \subset P_{v_1}^{CL\theta/N^{3/2}} \cap B_{CL/N}.$$

Therefore $\sum_I |f_I|^2$ is locally constant on dual fat AP of the form $P_{v_1^{-1}}^{CN/L} \cap B_{CN^{3/2}/(L\theta)}$. Observe that if we define $L_1 = (N^{1/2}L)^{1/2}$, then

$$P_{v_1^{-1}}^{CN/L} \cap B_{CN^{3/2}/(L\theta)} = P_{v_1^{-1}}^{CN^{3/2}/L_1^2} \cap B_{CN^2/(L_1^2\theta)} = CP(L_1).$$

Now suppose I', I'' are unions of I in $\mathcal{I}$, and I', I'' are transversal in the sense that $d(I', I'') \gtrsim N^{-1/2}$ on $\mathbb{R}$. Then we have the following bilinear restriction estimate. The proof closely resembles the multilinear Kakeya implies multilinear restriction proof in [Bennett et al. 2006].

Proposition 3.5 (bilinear restriction-type estimate). *Suppose* $\operatorname{supp} \widehat{F}_1 \subset I'$ *and* $\operatorname{supp} \widehat{F}_2 \subset I''$. *Then we have*

$$\fint_{P(L)} |F_1|^2|F_2|^2 \lesssim_\varepsilon N^\varepsilon |P(L)|^{-2} \int_{\mathbb{R}} |F_1|^2 \int_{\mathbb{R}} |F_2|^2. \tag{21}$$

Before proving the proposition, we remark that under the conditions of Proposition 3.5, the seemingly stronger inequality

$$\fint_{P(L)} |F_1|^2|F_2|^2 \lesssim_\varepsilon N^\varepsilon |P(L)|^{-2} \int_{\mathbb{R}} |F_1|^2 W_{P(L),100} \int_{\mathbb{R}} |F_2|^2 W_{P(L),100} \tag{22}$$

holds. This is essentially by applying Proposition 3.5 to the functions $F_1\psi$, $F_2\psi$, where ψ is from Lemma 2.2 and is adapted to the fat AP dual to $P(L)$, with order of decay 100.

Proof of Proposition 3.5. We define $\mathrm{BR}(L)$ to be the smallest constant such that

$$\fint_{P(L)} |F_1|^2|F_2|^2 \le \mathrm{BR}(L)|P(L)|^{-2}\int_{\mathbb{R}} |F_1|^2 \int_{\mathbb{R}} |F_2|^2$$

holds for all F_1, F_2 with $\operatorname{supp}\widehat{F}_1 \subset I'$ and $\operatorname{supp}\widehat{F}_2 \subset I''$. We let $\mathrm{BK}(L)$ be the smallest constant such that

$$\fint_{P(L)} g_1 g_2 \le \mathrm{BK}(L)|P(L)|^{-2}\int_{\mathbb{R}} g_1 \int_{\mathbb{R}} g_2$$

holds for all $g_1 = \sum a_I 1_{P_I}$ and $g_2 = \sum b_J 1_{P_J}$, where a_I, b_J are positive real numbers and $I, J \in \mathcal{I}$ with $I \subset I'$, $J \subset I''$. Equivalently, we have

$$|P_I|^{-2}\fint_{P(L)}\Big(\sum_{I\subset I'} g_{1,I} * 1_{P_I(0)}\Big)\Big(\sum_{J\subset I''} g_{2,J} * 1_{P_J(0)}\Big) \le \mathrm{BK}(L)|P(L)|^{-2}\Big(\int_{\mathbb{R}}\sum_I g_{1,I}\Big)\Big(\int_{\mathbb{R}}\sum_J g_{2,J}\Big) \tag{23}$$

for all finite measures $g_{1,I}$, $g_{2,J}$ which are linear combinations of Dirac measures with nonnegative coefficients. By a density argument (linear combinations of Dirac measures are dense in the weak* topology on $C_0(\mathbb{R})^*$), (23) also holds for all finite measures $g_{1,I}$, $g_{2,J}$. In particular, (23) holds for all nonnegative L^1 functions $g_{1,I}$, $g_{2,J}$.

We have shown in Proposition 3.3 that

$$\mathrm{BK}(L) \lesssim 1.$$

Now we want to show $\mathrm{BR}(L) \lesssim_\varepsilon N^\varepsilon$. First we prove

$$\mathrm{BR}(L) \lesssim \mathrm{BR}(L_1)\,\mathrm{BK}(L). \tag{24}$$

From the definition of BR and local L^2 orthogonality (Lemma 3.6 below) we have

$$\begin{aligned}\fint_{P(L)} |F_1F_2|^2 &\lesssim \fint_{P(L)} \|F_1F_2\|^2_{\text{Ł}^2(P(L_1,x))}\,dx\\ &\lesssim \mathrm{BR}(L_1)\fint_{P(L)} \|F_1\|^2_{\text{Ł}^2(W_{P(L_1,x),200})}\|F_2\|^2_{\text{Ł}^2(W_{P(L_1,x),200})}\\ &\lesssim \mathrm{BR}(L_1)\fint_{P(L)}\Big(\sum_{I\subset I'}\|F_{1,I}\|^2_{\text{Ł}^2(W_{P(L_1,x),200})}\Big)\Big(\sum_{J\subset I''}\|F_{2,J}\|^2_{\text{Ł}^2(W_{P(L_1,x),200})}\Big).\end{aligned}$$

We claim that

$$\fint_{P(L)}\sum_{I,J}\|F_{1,I}\|^2_{\text{Ł}^2(W_{P(L_1,x),200})}\|F_{2,J}\|^2_{\text{Ł}^2(W_{P(L_1,x),200})} \lesssim \mathrm{BK}(L)|P(L)|^{-2}\|F_1\|^2_{L^2(\mathbb{R})}\|F_2\|^2_{L^2(\mathbb{R})}, \tag{25}$$

which together with previous arguments will imply (24). Since $\sum_{P(L_1)\subset\mathbb{R}} W_{P(L_1,x),200}(P(L_1)) \lesssim 1$, it suffices to show that

$$\fint_{P(L)}\sum_{I,J}\|F_{1,I}\|^2_{\text{Ł}^2(P(L_1,x))}\|F_{2,J}\|^2_{\text{Ł}^2(P(L_1,x))} \lesssim_k \mathrm{BK}(L)|P(L)|^{-2}\|F_1\|^2_{L^2(\mathbb{R})}\|F_2\|^2_{L^2(\mathbb{R})}.$$

We choose $\psi_{I,200}$ adapted to $P_I(0)$ in the frequency space with order of decay 200 as in Lemma 2.2. Let $\phi_I := \check{\psi}_{I,200}/|P_I|$. If we define $G_{1,I} = (\widehat{F}_{1,I}/\widehat{\phi}_I)^{\vee}$, then due to the support property of $\widehat{F}_{1,I}$ we have pointwise

$$|\widehat{G}_{1,I}| \sim |\widehat{F}_{1,I}|. \tag{26}$$

Also by definition we have $F_{1,I} = G_{1,I} * \phi_I$. We define $G_{2,J} = (\widehat{F}_{2,J}/\widehat{\phi}_J)^{\vee}$ for $F_{2,J}$ in the same way. Now for $y \in \mathbb{R}$ such that $x + y \in P(L_1, x)$, we have

$$|F_{1,I}(x+y)|^2 = |(G_{1,I} * \phi_I)(x+y)|^2 \lesssim (|G_{1,I}|^2 * |\phi_I|)(x+y) \lesssim |G_{1,I}|^2 * 1_{CP_I}/|P_I|,$$

where we used Jensen's inequality for the first inequality. Therefore we have

$$\|F_{1,I}\|^2_{\not{L}^2(P(L_1,x))} \lesssim |G_{1,I}|^2 * 1_{CP_I}/|P_I|.$$

and similarly

$$\|F_{2,J}\|^2_{\not{L}^2(P(L_1,x))} \lesssim |G_{2,J}|^2 * 1_{CP_J}/|P_I|.$$

Hence using (23) we obtain

$$\begin{aligned}
⨍_{P(L)} \sum_{I,J} \|F_{1,I}\|^2_{\not{L}^2(P(L_1,x))} \|F_{2,J}\|^2_{\not{L}^2(P(L_1,x))} &\lesssim |P_I|^{-2} \sum_{I,J} ⨍_{P(L)} (|G_{1,I}|^2 * 1_{CP_I})(|G_{2,J}|^2 * 1_{CP_J}) \\
&\lesssim \mathrm{BK}(L)|P(L)|^{-2} \Big(\int_{\mathbb{R}} \sum_I |G_{1,I}|^2\Big)\Big(\int_{\mathbb{R}} \sum_J |G_{2,J}|^2\Big) \\
&\lesssim \mathrm{BK}(L)|P(L)|^{-2} \Big(\int_{\mathbb{R}} \sum_I |F_{1,I}|^2\Big)\Big(\int_{\mathbb{R}} \sum_J |F_{2,J}|^2\Big) \\
&\lesssim \mathrm{BK}(L)|P(L)|^{-2} \Big(\int_{\mathbb{R}} |F_1|^2\Big)\Big(\int_{\mathbb{R}} |F_2|^2\Big),
\end{aligned}$$

where the second-to-last inequality is due to (26). So we have proved (25) and therefore (24). Now we prove $\mathrm{BR}(L) \lesssim_\varepsilon N^\varepsilon$. Define $L_m = (L_{m-1}N^{1/2})^{1/2}$. Fix an $\varepsilon > 0$. We define M to be the smallest integer such that $L_M \gtrsim N^{1/2-\varepsilon}$. So $M \lesssim_\varepsilon 1$. Plugging in $\mathrm{BK}(L_m) \lesssim 1$ and applying (24) repeatedly we get

$$\mathrm{BR}(L) \le C^M \,\mathrm{BR}(L_M).$$

Since $\mathrm{BR}(L_M) \lesssim_\varepsilon N^{C\varepsilon}$ for some universal constant C (because of the locally constant property Proposition 2.3) we conclude $\mathrm{BR}(L) \lesssim_\varepsilon N^{C\varepsilon}$, which is what we want. □

The L^4 bilinear restriction inequality for the parabola in $\mathbb{R}^2$ has a more straightforward proof exploiting the fact that $\#\{(\theta_3, \theta_4) : d(\theta_3, \theta_4) \gtrsim 1,\ \mathcal{N}_{R^{-1/2}}(\theta_3 + \theta_4) \cap \mathcal{N}_{R^{-1/2}}(\theta_1 + \theta_2)\} \lesssim 1$ for every fixed θ_1, θ_2, with $d(\theta_1, \theta_2) \gtrsim 1$, where θ_i are $R^{-1/2} \times R^{-1}$ caps that cover the compact parabola [Cordoba 1977; Fefferman 1973]. However, it is not obvious whether a similar property would hold for $\mathcal{I}$ in our setting, so we took the approach in [Bennett et al. 2006] instead.

Now we give a proof of the local L^2 orthogonality used in the proof above. We denote $(LN^{1/2})^{1/2}$ by L'. So $P(L') = P(L_1) = P^{CN/L}_{v_1^{-1}} \cap B_{CN^{3/2}/(L\theta)}$.

Lemma 3.6 (local L^2 orthogonality). *For every f_I with* supp $\widehat{f_I} \subset I$ *we have*

$$\Big\|\sum_{I\in\mathcal{I}} f_I\Big\|^2_{L^2(W_{P(L'),k})} \lesssim_k \sum_{I\in\mathcal{I}} \|f_I\|^2_{L^2(W_{P(L'),k})} \tag{27}$$

Proof. Due to (12) it suffices to prove

$$\Big\|\sum_{I\in\mathcal{I}} f_I\Big\|^2_{L^2(P(L'))} \lesssim_k \sum_{I\in\mathcal{I}} \|f_I\|^2_{L^2(W_{P(L'),k})}.$$

We choose ψ_k adapted to $P(L')^* := P_{v_1}^{CL\theta/N^{3/2}}(0) \cap B_{CL/N}(0)$ in the frequency space with order of decay k as in Lemma 2.2. Here $P(L')^*$ is dual to $P(L')$. Since supp $\widehat{\psi}_k \subset 8^k P(L')^*$, and $\{I + 8^k P(L')^*\}_{I\in\mathcal{I}}$ is $\mathcal{O}_k(1)$-overlapping, we conclude

$$\begin{aligned}\Big\|\sum_{I\in\mathcal{I}} f_I\Big\|^2_{L^2(P(L'))} &\lesssim_k |P(L')|\Big\|\sum_{I\in\mathcal{I}} f_I\psi_k\Big\|^2_{L^2(\mathbb{R})}\\ &\lesssim_k |P(L')|\sum_{I\in\mathcal{I}}\|f_I\psi_k\|^2_{L^2(\mathbb{R})} \lesssim_k \sum_{I\in\mathcal{I}}\|f_I\|^2_{L^2(W_{P(L'),k})}.\end{aligned}$$ □

4. Decoupling for the canonical partition

We focus on proving Theorem 1.2 in Sections 4, 5, and 6, and in these three sections decoupling will refer to decoupling for the canonical partition.

Recall that $\{a_n\}_{n=1}^{N^{1/2}}$ satisfies

$$a_{i+1} - a_i \sim \frac{1}{N}, \quad (a_{i+2} - a_{i+1}) - (a_{i+1} - a_i) \sim \frac{\theta}{N^2}, \tag{28}$$

where, here, $\sim$ means within a factor of 4. The parameter θ is in $(0, 1]$, Ω is the $L^2\theta/N^2$-neighborhood of $\{a_n\}_{n=1}^{N^{1/2}}$, and

$$\Omega = \bigsqcup_{I\in\mathcal{I}} I,$$

where each I is an $L^2\theta/N^2$-neighborhood of L consecutive terms in $\{a_n\}_{n=1}^{N^{1/2}}$.

We restate Theorem 1.2 but for all short generalized Dirichlet sequences with $\theta \in (0, 1]$.

Theorem 4.1. *Let Ω and $\mathcal{I}$ be defined as in the last paragraphs. Then for $2 \le p \le 6$ and every $\varepsilon > 0$ we have*

$$\Big\|\sum_{I\in\mathcal{I}} f_I\Big\|_{L^p(\mathbb{R})} \lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1)\Big(\sum_{I\in\mathcal{I}}\|f_I\|^2_{L^p(\mathbb{R})}\Big)^{\frac{1}{2}} \tag{29}$$

for functions f_I with supp $\widehat{f_I} \subset I$.

Comparing (29) with (3) we see an extra factor $\log^C(\theta^{-1}+1)$. This factor appears as a consequence of dyadic pigeonholing in our proof.

4A. *Local decoupling and refined decoupling inequalities.* We first formulate a local decoupling inequality which implies (in fact is equivalent to) the global decoupling inequality (29).

Proposition 4.2. *Let $p \geq 2$. Suppose that, for some $k \geq 100$,*

$$\Big\|\sum_{I\in\mathcal{I}} f_I\Big\|_{L^p(P(L))} \lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1)\Big(\sum_{I\in\mathcal{I}} \|f_I\|^2_{L^p(W_{P(L),k})}\Big)^{\frac{1}{2}} \tag{30}$$

holds for every f_I with supp $\widehat{f_I} \subset I$. *Then* (29) *is true.*

Proof. Suppose (30) holds for some $k \geq 100$. Since $\sum_{P(L)\subset\mathbb{R}} W_{P(L),k} \lesssim_k 1$ and $p \geq 2$, by Minkowski's inequality, we have

$$\begin{aligned}\Big\|\sum_I f_I\Big\|^p_{L^p(\mathbb{R})} \leq \sum_{P(L)\subset\mathbb{R}} \int_{P(L)} |f|^p &\lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1) \sum_{P(L)} \Big(\sum_I \|f_I\|^2_{L^p(W_{P(L),k})}\Big)^{\frac{p}{2}} \\ &\lesssim N^\varepsilon \log^C(\theta^{-1}+1)\Big(\sum_I \|f_I\|^2_{L^p(\sum_{P(L)} W_{P(L),k})}\Big)^{\frac{p}{2}} \\ &\lesssim N^\varepsilon \log^C(\theta^{-1}+1)\Big(\sum_I \|f_I\|^2_{L^p(\mathbb{R})}\Big)^{\frac{p}{2}},\end{aligned}$$

which is (29). □

The following local decoupling inequality will imply Theorem 4.1 by Proposition 4.2.

Theorem 4.3 (local decoupling). *Suppose $2 \leq p \leq 6$. Then*

$$\Big\|\sum_{I\in\mathcal{I}} f_I\Big\|_{L^p(P(L))} \lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1)\Big(\sum_{I\in\mathcal{I}} \|f_I\|^2_{L^p(W_{P(L),100})}\Big)^{\frac{1}{2}} \tag{31}$$

for f_I with supp $\widehat{f_I} \subset I$.

Theorem 4.3 is a consequence of the following refined decoupling theorem, which we focus on proving in the next two sections. The analogous result for the parabola can be found in [Guth et al. 2020; Demeter et al. 2020]. We will show how Theorem 4.4 implies Theorem 4.3 in Section 6E.

Theorem 4.4 (refined decoupling). *Suppose $2 \leq p \leq 6$. For every $P(L)$ and every $X \subset P(L)$, we have*

$$\Big\|\sum_I f_I\Big\|_{L^p(X)} \lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1)\Big(\sup_{x\in X}\sum_I \|f_I\|^2_{\text{Ł}^2(W_{P_I(x),100})}\Big)^{\frac{1}{2}-\frac{1}{p}}\Big(\sum_I \|f_I\|^2_{L^2(W_{P(L),100})}\Big)^{\frac{1}{p}} \tag{32}$$

for f_I with supp $\widehat{f_I} \subset I$.

We remark that Theorem 4.4 implies that for every $X \subset P$, where P is a fat AP larger than $P(L)$ in the sense that $P(L) \subset P$ for at least one $P(L)$, and, for $2 \leq p \leq 6$,

$$\Big\|\sum_I f_I\Big\|_{L^p(X)} \lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1)\Big(\sup_{x\in X}\sum_I \|f_I\|^2_{\text{Ł}^2(W_{P_I(x),100})}\Big)^{\frac{1}{2}-\frac{1}{p}}\Big(\sum_I \|f_I\|^2_{L^2(W_{P,100})}\Big)^{\frac{1}{p}} \tag{33}$$

for f_I with supp $\widehat{f_I} \subset I$. Indeed, (33) follows from taking (32) to the p-th power and summing over $X \cap P(L)$ with $P(L) \subset P$.

4B. ***Induction scheme for proving Theorem 4.4.*** We fix p, L and let $\mathrm{Dec}(N,\theta)=\mathrm{Dec}_p(N,L,\theta)$ denote the smallest constant such that

$$\Big\|\sum_I f_I\Big\|_{L^p(X)} \le \mathrm{Dec}(N,\theta)\Big(\sup_{x\in X}\sum_I \|f_I\|^2_{L^2(W_{P_I(x),100})}\Big)^{\frac12-\frac1p}\Big(\sum_I \|f_I\|^2_{L^2(W_{P(L),100})}\Big)^{\frac1p} \tag{34}$$

holds for every sequence $\{a_n\}_{n=1}^{N^{1/2}}$ satisfying (14), every $P(L)$, every $X\subset P(L)$, and every f_I with $\operatorname{supp}\widehat{f_I}\subset I$. For a specific choice of the short generalized Dirichlet sequence $\{a_n\}_{n=1}^{N^{1/2}}$ satisfying (14) we will call the smallest constant the refined decoupling constant of $\{a_n\}_{n=1}^{N^{1/2}}$ such that (34) holds for every $X\subset P(L)$, and every f_I with $\operatorname{supp}\widehat{f_I}\subset I$. Note that $\mathrm{Dec}_p(N,L,\theta)$ is the supremum of all refined decoupling constants of sequences $\{a_n\}_{n=1}^{N^{1/2}}$ satisfying (14).

We will deduce Theorem 4.4, which now is equivalent to $\mathrm{Dec}(N,\theta)\lesssim_\varepsilon N^\varepsilon\log^C(\theta^{-1}+1)$, from the following main proposition.

Proposition 4.5. *For every $\varepsilon>0$ and every $1\le K\le N^{\varepsilon/2}$ satisfying $N^{1/2}/K\ge L$,*

$$\mathrm{Dec}(N,\theta)\lesssim_\varepsilon \sup_{\theta'\in[\theta/4,\theta]}\mathrm{Dec}\Big(\frac{N}{K^2},\frac{\theta'}{K^2}\Big)+K^DN^\varepsilon\log^D(\theta^{-1}+1). \tag{35}$$

Here D is an absolute constant.

We postpone the proof of Proposition 4.5 to Section 6. Here we show how it implies Theorem 4.4.

Proof of Theorem 4.4 assuming Proposition 4.5. For some sufficiently large S_0 we have $\mathrm{Dec}(N,\theta)\le C_sN^s\le C_sN^s\log^D(\theta^{-1}+1)$ for $s\ge S_0$. Now suppose $\mathrm{Dec}(N,\theta)\le C_sN^s\log^D(\theta^{-1}+1)$ for some $s\le S_0$. Then from (35) we have, for every $\varepsilon>0$ and K with $N^{1/2}/K\ge L$,

$$\begin{aligned}\mathrm{Dec}(N,\theta)&\le C_\varepsilon\Big(\sup_{\theta'\in[\theta/4,\theta]}C_s\Big(\frac{N}{K^2}\Big)^s\log^D(K^2(\theta')^{-1}+1)+K^DN^\varepsilon\log^D(\theta^{-1}+1)\Big).\\&\le C_\varepsilon\Big(CC_s\Big(\frac{N}{K^2}\Big)^s\log^D(K^2\theta^{-1}+1)+K^DN^\varepsilon\log^D(\theta^{-1}+1)\Big)\\&\le C_\varepsilon\Big(CC_s\Big(\frac{N}{K^2}\Big)^s(C\log^D(\theta^{-1}+1)+C\log^D(K^2))+K^DN^\varepsilon\log^D(\theta^{-1}+1)\Big).\end{aligned}$$

If we choose ε to be $s/2$ and let $N^s/K^{2s}=K^DN^\varepsilon=K^DN^{s/2}$, that is, $K=N^{s/(2(2s+D))}$, then for some constant C_s' depending only on s,

$$\mathrm{Dec}(N,\theta)\le C_s'N^{s(1-\frac{1}{2s+D})}(\log^D(\theta^{-1}+1)+\log^D N)$$

if $N^{1/2}N^{-s/(2(2s+D))}\ge L$. If $N^{1/2}N^{-s/(2(2s+D))}\le L$, then $|\mathcal{I}|\lesssim N^{s/(2(2s+D))}$ and by the triangle inequality and Cauchy–Schwarz inequality we have

$$\mathrm{Dec}(N,\theta)\lesssim N^{\frac{s}{2(2s+D)}}.$$

We can assume that D is large enough such that $\max\{2,S_0\}\le D$. Then $1/(2s+D)\sim D^{-1}$ and $K\le N^{\varepsilon/2}$, so for some absolute constant $c>0$,

$$\mathrm{Dec}(N,\theta)\lesssim_s N^{s(1-c)}\log^D(\theta^{-1}+1).$$

Therefore we conclude

$$\mathrm{Dec}(N,\theta) \lesssim_\varepsilon N^\varepsilon \log^D(\theta^{-1}+1)$$

for every $\varepsilon > 0$, since $S_0(1-c)^m$ is arbitrarily small for large enough m. □

4C. *Two applications.* Before ending this section, we record two applications of Theorem 4.1. Technically these are corollaries of the $\ell^2 L^6$ decoupling inequality for the parabola in [Bourgain and Demeter 2015], by deriving the corresponding (ℓ^2, L^6) estimate on short generalized Dirichlet polynomials using the method described in the Appendix.

First we may estimate approximate solutions to the equation $a_{n_1}+a_{n_2}+a_{n_3} = a_{n_4}+a_{n_5}+a_{n_6}$ for a short generalized Dirichlet sequence $\{a_n\}_{n=1}^{N^{1/2}}$. The number of exact solutions of such equations for general convex sequences was studied in [Iosevich et al. 2006].

Corollary 4.6. *Let $\{a_n\}_{n=1}^{N^{1/2}}$ be a short generalized Dirichlet sequence with parameter $\theta \in (0,1]$. Then*

$$\#\{(a_{n_1},\dots,a_{n_6}) : 1 \le n_i \le N^{\frac{1}{2}}, |(a_{n_1}+a_{n_2}+a_{n_3})-(a_{n_4}+a_{n_5}+a_{n_6})| \le \theta/N^2\} \lesssim_\varepsilon \log^C(\theta^{-1}+1) N^{\frac{3}{2}+\varepsilon}. \quad (36)$$

This estimate is sharp up to $C_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1)$ due to $N^{3/2}$ many diagonal solutions.

In particular if we take $a_n = \log(n+N+1)$ in the above corollary, then $\theta \sim 1$ and (36) reads

$$\#\{(n_1,\dots,n_6) : N+1 \le n_i \le N+N^{\frac{1}{2}}, |n_1n_2n_3 - n_4n_5n_6| \lesssim N\} \lesssim_\varepsilon N^{\frac{3}{2}+\varepsilon}. \quad (37)$$

We note that the triple products $n_1n_2n_3$ with $N+1 \le n_1,n_2,n_3 \le N+N^{1/2}$ lies in the interval $[N^3, N^3+CN^{5/2}]$. So (37) implies that the triple products $\{n_1n_2n_3 : N+1 \le n_1,n_2,n_3 \le N+N^{1/2}\}$ are roughly evenly distributed in $[N^3, N^3+CN^{5/2}]$ with cN separation. Indeed if we split the interval $[N^3, N^3+CN^{5/2}]$ into intervals of length cN and let E_λ denote the number of cN-intervals which contain at least λ many triple products $n_1n_2n_3$, then (37) says that

$$\lambda^2 E_\lambda \le C_\varepsilon N^{\frac{3}{2}+\varepsilon}.$$

Consequently if we choose $\lambda \ge 10C_\varepsilon N^\varepsilon$, then we have $\lambda E_\lambda \le \frac{9}{10}N^{3/2}$, and λE_λ is the number of triple products $n_1n_2n_3$ that lie in a cN-interval which contains at least λ many triple products. The total number of triple products is $N^{3/2}$ so we can conclude most of the triple products lie in cN-intervals, each of which contains few triple products.

Proof of Corollary 4.6. We let ϕ be a Schwartz function whose Fourier transform is given by a smooth bump function adapted to $B_{\theta/N^2}(0)$:

$$\widehat{\phi} = 1 \quad \text{on } B_{\theta/N^2}(0), \quad \operatorname{supp}\widehat{\phi} \subset B_{2\theta/N^2}(0), \ \ 0 \le \widehat{\phi} \le 1, \quad \widehat{\phi} \text{ is even.}$$

Applying Theorem 4.1 with $p=6$, $L=1$ we obtain

$$\begin{aligned}\int_{\mathbb{R}} \Big| \sum_{n=1}^{N^{1/2}} e^{ia_nx}\phi(x)\Big|^6 &\lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1)\Big(\sum_{n=1}^{N^{1/2}} \|e^{ia_nx}\phi(x)\|^2_{L^6(\mathbb{R})}\Big)^3 \\ &\lesssim N^\varepsilon \log^C(\theta^{-1}+1) N^{\frac{3}{2}}\theta^5 N^{-10}. \end{aligned} \quad (38)$$

We expand the left-hand side of (38) as

$$\begin{aligned}\int_{\mathbb{R}}\Big|\sum_{n=1}^{N^{1/2}} e^{ia_n x}\phi(x)\Big|^6\,dx &= \sum_{n_1,\dots,n_6}\int_{\mathbb{R}} e^{i(a_{n_1}+a_{n_2}+a_{n_3}-a_{n_4}-a_{n_5}-a_{n_6})x}|\phi|^6\,dx\\ &= \sum_{n_1,\dots,n_6}\widehat{|\phi|^6}(a_{n_1}+a_{n_2}+a_{n_3}-a_{n_4}-a_{n_5}-a_{n_6}).\end{aligned}$$

Since $\widehat{\phi}$ is even we know that ϕ is real-valued and hence $\widehat{|\phi|^6}=\widehat{\phi}*\cdots*\widehat{\phi}$ is nonnegative and $\widehat{|\phi|^6}\gtrsim \theta^5 N^{-10}$ on $B_{c\theta/N^2}(0)$ for some small absolute constant $c>0$. Therefore

$$\begin{aligned}&\int_{\mathbb{R}}\Big|\sum_{n=1}^{N^{1/2}} e^{ia_n x}\phi(x)\Big|^6\\ &\qquad\gtrsim \theta^5 N^{-10}\#\{(a_{n_1},\dots,a_{n_6}) : 1\le n_i\le N^{\frac12}, |(a_{n_1}+a_{n_2}+a_{n_3})-(a_{n_4}+a_{n_5}+a_{n_6})|\le \theta/N^2\}.\end{aligned}$$

Combining the above estimate and (38) we obtain (36). □

Another application of Theorem 4.1 is estimating the size of the intersection of an AP with a generalized Dirichlet sequence.

Corollary 4.7. *Let $\{a_n\}_{n=1}^N$ be a generalized Dirichlet sequence with parameter $\theta\in(0,1]$ and let $a=N^{-\alpha}$ with $\alpha\in[0,2]$. Then*

$$|\{a_n\}_{n=1}^{n=N}\cap a\mathbb{Z}|\lesssim\begin{cases}N^\alpha & \text{if } \alpha\in\big[0,\tfrac12\big],\\ C_\varepsilon N^\varepsilon\log^C(\theta^{-1}+1)N^{\frac13+\frac{\alpha}{3}} & \text{if } \alpha\in\big[\tfrac12,2\big].\end{cases}$$

When $\theta=1$, Corollary 4.7 is sharp for $\alpha\in\big[0,\frac12\big]$ (see Lemma 7.3), but we do not know if it is sharp for $\alpha\in\big[\frac12,2\big]$. Corollary 4.7 has a slight connection to a conjecture of Rudin which states in an N-term AP we can find at most $\mathcal{O}(N^{1/2})$ many squares (numbers of the form n^2 for some $n\in\mathbb{Z}$). The best result so far seems to be in [Bombieri and Zannier 2002], which proves at most $\mathcal{O}(N^{3/5}\log^{\mathcal{O}(1)}N)$ many squares can be found in an N-term AP. We note that $\{n^2/N^2\}_{n=N+1}^{2N}$ is a generalized Dirichlet sequence. However we shall not expect to solve Rudin's conjecture exploiting only the convexity of the sequence $\{n^2 : n\in\mathbb{N}\}$, as shown by the example given in Lemma 7.3.

Proof of Corollary 4.7. The case $\alpha\in\big[0,\frac12\big]$ is trivial as $\{a_n\}_{n=1}^N$ is contained in a ball of radius $\lesssim 1$ and $a\mathbb{Z}$ has at most $\lesssim a^{-1}=N^\alpha$ many terms in such a ball. Now we suppose $\alpha\in\big[\frac12,2\big]$. It suffices to show that, for a short generalized Dirichlet sequence $\{a_n\}_{n=1}^{N^{1/2}}$, $H:=|\{n : 1\le n\le N^{1/2},\ a_n\in a\mathbb{Z}\}|$ satisfies

$$H\lesssim_\varepsilon C_\varepsilon\log^C(\theta^{-1}+1)N^{\frac{\alpha}{3}-\frac16+\varepsilon}.$$

We consider the function

$$f(x)=\sum_{n:1\le n\le N^{1/2},\,a_n\in a\mathbb{Z}} e^{2\pi i t a_n}.$$

<u>Case 1</u>: $\alpha\in[1,2]$. We apply Theorem 4.3 with $p=6$, $L=1$ and $P(L)=P(L,0)$. Since $|f|\ge H/10$ on $\mathcal{N}_{cN^{1/2}}(a^{-1}\mathbb{Z})$ with $c\gtrsim 1$, we obtain

$$H\bigg(\frac{N^2\theta^{-1}}{N^\alpha}N^{\frac12}\bigg)^{\frac16}\lesssim_\varepsilon N^\varepsilon\log^C(\theta^{-1}+1)H^{\frac12}(N^2\theta^{-1})^{\frac16},$$

where we used that $P(L)$ is approximately an $N^2\theta^{-1}$ interval. Simplifying the above displayed math,

$$H \lesssim_\varepsilon C_\varepsilon \log^C(\theta^{-1}+1) N^{\frac{\alpha}{3}-\frac{1}{6}+\varepsilon}.$$

Case 2: $\alpha \in \left[\frac{1}{2}, 1\right]$. We apply Theorem 4.3 with $p = 6$, $L = N^{1-\alpha}$ and $P(L) = P(L, 0)$. Since $|f| \geq H/10$ on $\mathcal{N}_{cN^{1/2}}(a^{-1}\mathbb{Z})$ with $c \gtrsim 1$, we obtain

$$H\left(\frac{N^{2\alpha}2\theta^{-1}}{N^\alpha} N^{\frac{1}{2}}\right)^{\frac{1}{6}} \lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1) H^{\frac{1}{2}} (N^{2\alpha}2\theta^{-1})^{\frac{1}{6}},$$

that is,

$$H \lesssim_\varepsilon C_\varepsilon \log^C(\theta^{-1}+1) N^{\frac{\alpha}{3}-\frac{1}{6}+\varepsilon}. \qquad \square$$

5. High-low frequency decomposition for the square function

The proof of Proposition 4.5 is based on the method in [Guth et al. 2022], which uses a high-low frequency decomposition for the square function. Such a decomposition is also used in [Guth et al. 2019] to study incidence estimates for tubes. We set up the preliminaries in this section and prove Proposition 4.5 in Section 6. We begin in Section 5A with an overview of the argument, at a symbolic and heuristic level, and refer readers to Section 2 of [Guth et al. 2022] for a more detailed description of the intuition behind this method.

5A. ***Overview of the argument.*** Let $2 \leq p \leq 6$. We will present a heuristic overview of the high-low proof of Theorem 4.4 (which is our goal to prove via Proposition 4.5). By a pigeonholing argument, we may assume that there is a parameter $\alpha > 0$ so that

$$\int_X \Big|\sum_I f_I\Big|^p \sim \alpha^p |U_\alpha|,$$

where $U_\alpha = \{x \in X : |\sum_I f_I(x)| \sim \alpha\}$. A “broad/narrow” argument (written in our context in Section 6A) roughly allows us to reduce to the case that, on most of U_α, $|\sum_I f_I|$ is bounded by a bilinear expression $|\sum_{I_1\subset I'} f_{I_1} \sum_{I_2\subset I''} f_{I_2}|^{1/2}$ where I', I'' are transverse, meaning $d(I', I'') \gtrsim N^{-1/2}$. The high-low frequency proof of decoupling involves upgrading the bilinear restriction theorem (Proposition 3.5) to the refined decoupling theorem (Theorem 4.4).

We split U_α into $\leq \varepsilon^{-1}$ many sets on which we know certain square functions are high- or low-frequency dominated. Consider scales $1 \leq L \leq L_{m+1} \leq L_m \leq N^{1/2}$, where $L_m/L_{m+1} \leq N^\varepsilon$. Define the (square of the) square functions $g_m = \sum_{I_m} |f_{I_m}|^2$, $g_{m+1} = \sum_{I_{m+1}} |f_{I_{m+1}}|^2$, where I_m, I_{m+1} are unions of L_m, L_{m+1} many consecutive intervals in Ω, respectively. Also write $g = \sum_I |f_I|^2$. Suppose that on most of U_α, $g_{m+1}(x) \lesssim g(x)$. Observe the pointwise inequality that, for $x \in U_\alpha$ satisfying $g_{m+1}(x) \lesssim g(x)$,

$$\begin{aligned}
\alpha \sim \Big|\sum_{I_{m+1}} f_{I_{m+1}}(x)\Big| &\lesssim \sum_{I_{m+1}:|f_{I_{m+1}}(x)|>N^\varepsilon\frac{g(x)}{\alpha}} |f_{I_{m+1}}(x)| + \Big|\sum_{I_{m+1}:|f_{I_{m+1}}(x)|\leq N^\varepsilon\frac{g(x)}{\alpha}} f_{I_{m+1}}(x)\Big| \\
&\lesssim \frac{\alpha}{N^\varepsilon g(x)} \sum_{I_{m+1}:|f_{I_{m+1}}(x)|>N^\varepsilon\frac{g(x)}{\alpha}} |f_{I_{m+1}}(x)|^2 + \Big|\sum_{I_{m+1}:|f_{I_{m+1}}(x)|\leq N^\varepsilon\frac{g(x)}{\alpha}} f_{I_{m+1}}(x)\Big| \\
&\lesssim \frac{\alpha}{N^\varepsilon g(x)} g_{m+1}(x) + \Big|\sum_{I_{m+1}:|f_{I_{m+1}}(x)|\leq N^\varepsilon\frac{g(x)}{\alpha}} f_{I_{m+1}}(x)\Big|.
\end{aligned}$$

This type of reasoning means that on most of U_α we may perform a wave packet decomposition of f at scale I_{m+1} and replace f with a version which only preserves the "small" wave packets, ensuring the property that $\|f_{I_{m+1}}\|_{L^\infty(\mathbb{R})} \lessapprox \|g\|_{L^\infty(X)}/\alpha$.

Case 1: high-dominance. Suppose that on most of U_α, $g_m(x) \lesssim |g_m * \check{\eta}_{\geq L_{m+1}/N}(x)|$, where $\eta_{\geq L_{m+1}/N}$ is a smooth bump function with support in $L_{m+1}/N \leq |\omega| \leq 2$, on most of U_α. A combination of a broad-narrow argument, Proposition 3.5, the locally constant property, and the assumption of high-frequency dominance of g_m leads to the inequality

$$\alpha^4 |U_\alpha| \lesssim \int |g_m * \check{\eta}_{\geq L_{m+1}/N}|^2.$$

Next, by Plancherel's theorem, we analyze the integral on the right-hand side. A geometric argument shows that the supports of the $\widehat{|f_{I_m}|^2}$ from $\widehat{g}_m$ are sparsely overlapping on the support of $\eta_{\geq L_{m+1}/N}$. This allows us to bound the right-hand side of the previous displayed inequality by

$$C_\varepsilon N^\varepsilon \sum_{I_m} \int |f_{I_m}|^4,$$

which is bounded by $C_\varepsilon N^{10\varepsilon} \sum_{I_{m+1}} \int |f_{I_{m+1}}|^4$ using Cauchy–Schwarz. Finally, use the good L^∞ bound for each $f_{I_{m+1}}$ from the pruning of the wave packets to get

$$\sum_{I_{m+1}} \int |f_{I_{m+1}}|^4 \lessapprox \frac{\|g\|^2_{L^\infty(X)}}{\alpha^2} \sum_{I_{m+1}} \int |f_{I_{m+1}}|^2.$$

A pigeonholing argument may be used to show that without loss of generality, we may assume that $\|g\|_{L^\infty(X)} \lesssim \alpha^2$. By L^2 orthogonality, the integral on the right-hand side of the previous displayed line equals $\sum_I \int |f_I|^2$. The conclusion of the argument in this case is then

$$|U_\alpha| \lessapprox \frac{\|g\|^2_{L^\infty(X)}}{\alpha^6} \sum_I \|f_I\|^2_{L^2(\mathbb{R})} \lesssim \frac{\|g\|^{\frac{p}{2}-1}_{L^\infty(X)}}{\alpha^p} \sum_I \int |f_I|^2,$$

which is a version of the statement of Theorem 4.4.

Case 2: low-dominance. The remaining case is if $g_m(x) \leq |g_m * \check{\eta}_{<L_{m+1}/N}(x)|$ on most of U_α. A local L^2-orthogonality argument shows that $|g_m * \check{\eta}_{<L_{m+1}/N}(x)|$ is bounded by $g_{m+1} * |\check{\eta}_{<L_{m+1}/N}|(x)$, which by the locally constant heuristic, is roughly the same as $g_{m+1}(x)$. We conclude in this case that, on most of U_α, $g_m(x) \lesssim g_{m+1}(x) \lesssim g(x)$. This is the same type of assumption we made before consider the cases, except at the scale L_m instead of L_{m+1}. This allows us to reinitiate the argument beginning with the assumption that $g_m(x) \lesssim g(x)$ in place of $g_{m+1}(x) \lesssim g(x)$.

In the case that we are "low"-dominated for ε^{-1} many scales, then

$$|U_\alpha| \approx |\{x \in U_\alpha : g_1(x) \lesssim g(x)\}|,$$

where g_1 is a square function corresponding to partitions of Ω into I_1, which are N^ε many adjacent intervals. Since $|\sum_I f_I(x)| \lesssim N^\varepsilon g_1(x)$ by Cauchy–Schwarz, the statement of Theorem 4.4 becomes trivial. In the next sections, we set up the argument in full technical detail.

5B. ***Wave-packet decomposition.*** We start with a few definitions. Write $f = \sum_{I\in\mathcal{I}} f_I$, where f_I will always denote a function with frequency support in I.

Fix $2 \le p \le 6$ and $\varepsilon > 0$. For $m \in \mathbb{N}$, let $L_m = N^{1/2}N^{-\varepsilon m}$. Without loss of generality we assume $L_M = L$ for some $M \in \mathbb{N}$. So $M \lesssim_\varepsilon 1$. For every $1 \le m \le M$ we let $\mathcal{I}_m$ be the partition of Ω into $N^{1/2}/L_m$ many I_m, each of which is the union of L_m-consecutive intervals in Ω. L_m can be thought of as scales.

Note that

$$I_m \subset P_{v_m}^{CL_m^2\theta/N^2} \cap B_{CL_m/N},$$

where $v_m \sim \frac{1}{N}$. We denote the right-hand side as $\widetilde{I}_m$:

$$\widetilde{I}_m := P_{v_m}^{CL_m^2\theta/N^2} \cap B_{CL_m/N}.$$

Let $\mathcal{P}_{I_m}$ be a tiling of $\mathbb{R}$ by P_{I_m}. For each I_m, we will now construct a partition of unity $\{\phi_{I_m}\}_{P_{I_m}\in\mathcal{P}_{I_m}}$ which will be used to perform the wave packet decomposition

$$f_{I_m} = \sum_{P_{I_m}} \phi_{P_{I_m}} f_{I_m}.$$

We regard each summand $\phi_{P_{I_m}} f_{I_m}$ as a wave packet. Specifically, we let ψ_{I_m} be adapted to $\widetilde{I}_m - \widetilde{I}_m$, which is of the form $P_{v_0}^{CL_m^2\theta/N^2}(0) \cap B_{CL_m/N}(0)$, in the frequency space as in Lemma 2.2, with order of decay 200 outside of the dual fat AP P_{I_m}. For each $P_{I_m} \in \mathcal{P}_{I_m}$, define

$$\phi_{P_{I_m}} := \|\psi_{I_m}^2\|_{L^1(\mathbb{R})}^{-1} \int_{P_{I_m}} |\psi_{I_m}(x-y)|^2\,dy. \tag{39}$$

Proposition 5.1 (wave-packet decomposition). *$\{\phi_{P_{I_m}}\}_{P_{I_m}\in\mathcal{P}_{L_m}}$ forms a partition of unity, that is, $\sum \phi_{P_{I_m}} = 1$, $\phi_{P_{I_m}} \ge 0$. Each $\phi_{P_{I_m}}$ is a translated copy of the others, and*

$$\operatorname{supp}\widehat{\phi}_{P_{I_m}} \subset 8^{400}(\widetilde{I}_m - \widetilde{I}_m), \quad 1_{P_{I_m}} \lesssim \phi_{P_{I_m}} \lesssim W_{P_{I_m},200}. \tag{40}$$

Proof. By definition we see that $\phi_{P_{I_m}}$ forms a partition of unity, and each $\phi_{P_{I_m}}$ is a translated copy of the others. Also it follows from the definition that

$$1_{P_{I_m}} \lesssim |\phi_{P_{I_m}}|.$$

Note that $\phi_{P(L_m)}$ equals $\|\psi_{I_m}^2\|_{L^1(\mathbb{R})}^{-1}|\psi_{I_m}|^2 * 1_{P_{I_m}}$. Therefore ψ_{I_m} decays at order 200 outside $P_{I_m}(0)$ implies that $\phi_{P(L_m)}$ decays at order 400 outside P_{I_m}, and in particular

$$|\phi_{P_{I_m}}| \lesssim W_{P_{I_m},200}.$$

The support property $\operatorname{supp}\widehat{\phi}_{P_{I_m}} \subset 8^{400}(\widetilde{I}_m - \widetilde{I}_m)$ follows from the fact that

$$\widehat{\phi}_{P_{I_m}} = \|\psi_{I_m}^2\|_{L^1(\mathbb{R})}^{-1}\widehat{|\psi_{I_m}|^2}\ \widehat{1}_{P_{I_m}}$$

and from Lemma 2.2. □

5C. ***A pruning process and modified square functions.*** Now we define "square functions" (squared) at scales L_m, which differ from the usual square functions by a pruning process of wave packets and taking spatial averages. The pruning process will depend on two parameters α and r, which can be thought of as the values of $|f|$ and $\sum_{I_M}|f_{I_M}|^2=\sum_I|f_I|^2$ which dominate the L^p norm of f. We define $\lambda=\lambda(\alpha,r)$ by

$$\lambda=\widetilde{C}_\varepsilon N^\varepsilon\frac{r}{\alpha},\tag{41}$$

where $\widetilde{C}_\varepsilon$ is a sufficiently large constant depending on ε which will be chosen later in the proof of Lemma 5.4.

We first do the pruning process (with parameters α, r), which inductively removes wave packets at each scale whose height exceeds λ. As we shall see (Lemma 5.4), those wave packets do not play a dominant role in the L^p norm of f. This process produces a family of functions f_{m,I_m}, $f_{m,I_{m-1}}$, f_m that depend on α, r, which is implicit in the notation. We will write $f_{m,I_m,\alpha,r}$, $f_{m,I_{m-1},\alpha,r}$, $f_{m,\alpha,r}$ to emphasize such dependence when necessary.

Let $\mathcal{P}_{I_M,\lambda}=\{P_{I_M}\in\mathcal{P}_{L_M}:\|\phi_{P_{I_M}}f_{I_M}\|_{L^\infty(\mathbb{R})}\le\lambda\}$, and define

$$f_{M,I_M}:=\sum_{P_{I_M}\in\mathcal{P}_{I_M,\lambda}}\phi_{P_{I_m}}f_{I_M},\quad f_M:=\sum_{I_M}f_{M,I_M}.$$

We let $f_{M,I_{M-1}}=\sum_{I_M\subset I_{M-1}}f_{M,I_M}$. Now we define f_m and f_{m,I_m} inductively for $m=1,\dots,M-1$ by

$$f_{m,I_m}:=\sum_{P_{I_m}\in\mathcal{P}_{I_m,\lambda}}\phi_{P_{I_m}}f_{m+1,I_m},\quad f_m:=\sum_{I_m}f_{m,I_m},\tag{42}$$

where $f_{m+1,I_m}=\sum_{I_{m+1}\subset I_m}f_{m+1,I_{m+1}}$ and $\mathcal{P}_{I_m,\lambda}=\{P_{I_m}\in\mathcal{P}_{I_m}:\|\phi_{P_{I_m}}f_{m+1,I_m}\|_{L^\infty(\mathbb{R})}\le\lambda\}$. For notational convenience we also define $f_{M+1}=f$ and $f_{M+1,I_M}:=f_{I_M}=f_I$.

We note that

(i) $f_m=\sum_{I_m}f_{m,I_m}=\sum_{I_{m-1}}f_{m,I_{m-1}}$,

(ii) $\operatorname{supp}\widehat{f}_{m,I_m}\subset C\widetilde{I}_m$,

(iii) $\operatorname{supp}\widehat{f}_{m,I_{m-1}}\subset C\widetilde{I}_{m-1}$,

(iv) $|f_{m,I_m}|\le|f_{m+1,I_m}|$ pointwise.

Item (i) follows from the definitions, and (iv) holds because $\{\phi_{P_{I_m}}\}_{P_{I_m}}$ is a partition of unity. To see (ii) and (iii) we may induct on m and note that

$$\bigcup_{I_m\subset I_{m-1}}C\widetilde{I}_m\subset 2\widetilde{I}_{m-1}$$

when N is sufficiently large depending on ε.

To define the "square function" g_m at scale L_m we introduce ρ_{I_m}, which is an L^1-normalized nonnegative function adapted to $P_{I_m}(0)$ with decay order 100

$$|P_{I_m}|^{-1}1_{P_{I_m(0)}}(x)\lesssim\rho_{I_m}(x)\lesssim\frac{W_{P_{I_m}(0),100}(x)}{\|W_{P_{I_m}(0),100}\|_{L^1(\mathbb{R})}},\tag{43}$$

and $\operatorname{supp}\widehat{\rho}_{I_m}\subset C(\widetilde{I}_m-\widetilde{I}_m)$. Such a function can be constructed by taking $|\psi|^2/\|\psi^2\|_{L^1}$ for ψ adapted to $\widetilde{I}_m$ with decay order 100 as in Lemma 2.2.

Finally we define the "square function" by

$$g_m := \sum_{I_m} |f_{m+1,I_m}|^2 * \rho_{I_m}$$

for $1 \le m \le M-1$ and for $m = M$ we define

$$g_M := \sum_{I_M} |f_{I_M}|^2 * \rho_{I_M}.$$

We note here that g_m for $1 \le m \le M-1$ implicitly depends on α, r, and we will write $g_{m,\alpha,r}$ to emphasize such dependence when necessary; g_M does not depend on α, r.

5D. ***High-low decomposition.*** To set up a high-low frequency decomposition for g_m, we let $\eta_m(\xi)$ be an even smooth bump function that equals to 1 on $B_{L_{m+1}/N}(0)$ and vanishes outside $B_{2L_{m+1}/N}(0)$ for every $1 \le m \le M-1$. We also assume that η_m are rescalings of each other.

Define, for $1 \le m \le M-1$,

$$g_m^\ell := g_m * \check{\eta}_m \quad \text{and} \quad g_m^h := g_m - g_m^\ell,$$

which are low- and high-frequency parts of g_m. Both g_m^ℓ and g_m^h satisfy some proprieties. We discuss them in the following two lemmas.

Lemma 5.2 (low lemma). *For $1 \le m \le M-1$, we have the pointwise inequality*

$$|g_m^\ell| \lesssim g_{m+1}.$$

Proof. By definition

$$g_m^\ell = \left(\sum_{I_m} |f_{m+1,I_m}|^2\right) * \rho_{I_m} * \check{\eta}_m = \left(\sum_{I_m} |f_{m+1,I_m}|^2\right) * \check{\eta}_m * \rho_{I_m}.$$

Using Plancherel's theorem,

$$\begin{aligned} |f_{m+1,I_m}|^2 * \check{\eta}_m(x) &= \int |f_{m+1,I_m}(y)|^2 \check{\eta}_m(x-y)\,dy \\ &= \int (\widehat{f}_{m+1,I_m} * \widehat{\overline{f}}_{m+1,I_m})(\xi)\, e^{2\pi i x\xi} \eta_m(\xi)\, d\xi \\ &= \sum_{I_{m+1}, I'_{m+1} \subset I_m} \int (\widehat{f}_{m+1,I_{m+1}} * \widehat{\overline{f}}_{m+1,I'_{m+1}})(\xi)\, e^{2\pi i x\xi} \eta_m(\xi)\, d\xi. \end{aligned} \tag{44}$$

We note that $\widehat{f}_{m+1,I_{m+1}} * \widehat{\overline{f}}_{m+1,I'_{m+1}}$ is supported in $C\widetilde{I}_{m+1} - C\widetilde{I}'_{m+1}$ and $\widetilde{I}_{m+1}$ is of the form $P^{CL^2\theta/N^2}_{v_{I_{m+1}}} \cap B_{CL_{m+1}/N}$. Since η_m is supported on $B_{2L_{m+1}/N}(0)$ we conclude that for every fixed I_{m+1} there are only $\mathcal{O}(1)$ many I'_{m+1} such that the integral in (44) is nonzero, and for those I'_{m+1} we write $I'_{m+1} \sim I_{m+1}$. We let $\psi_{I_{m+1}}$ be adapted to $C(\widetilde{I}_{m+1} - \widetilde{I}_{m+1})$ as in Lemma 2.2 with order of decay 200. Then, using Cauchy–Schwarz in the first two inequalities, we have

$$\begin{aligned} ||f_{m+1,I_m}|^2 * \check{\eta}_m(x)| &= \sum_{I_{m+1} \subset I_m} \sum_{I'_{m+1} \sim I_{m+1}} f_{m+1,I_{m+1}} \overline{f}_{m+1,I'_{m+1}} * \check{\eta}_m \\ &\le \sum_{I_{m+1} \subset I_m} \sum_{I'_{m+1} \sim I_{m+1}} (|f_{m+1,I_{m+1}}|^2 * |\check{\eta}_m|)^{\frac12} (|f_{m+1,I'_{m+1}}|^2 * |\check{\eta}_m|)^{\frac12} \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{I_{m+1}\subset I_m} |f_{m+1,I_{m+1}}|^2 * |\check{\eta}_m| \\
&\lesssim \sum_{I_{m+1}\subset I_m} |f_{m+1,I_{m+1}}|^2 * |\check{\psi}_{I_{m+1}}| * |\check{\eta}_m| \\
&\le \sum_{I_{m+1}\subset I_m} |f_{m+2,I_{m+1}}|^2 * |\check{\psi}_{I_{m+1}}| * |\check{\eta}_m|,
\end{aligned}$$

where the last inequality is because of $|f_{m+1,I_{m+1}}| \le |f_{m+2,I_{m+1}}|$ pointwise. Now to finish the proof, it suffices to observe that

$$|\check{\eta}_m| * |\check{\psi}_{I_{m+1}}| * \rho_{I_m} \lesssim \rho_{I_{m+1}},$$

since $|\check{\eta}_m|$ decays rapidly outside $B_{N/L_{m+1}}(0)$, $|\check{\psi}_{I_{m+1}}|$ decays at order 200 outside $P_{I_{m+1}}(0)$, ρ_{I_m} decays at order 100 outside $P_{I_m}(0)$, and $B_{L_{m+1}/N}(0) + P_{I_m}(0) \subset CP_{I_{m+1}}(0)$. $\square$

Recall that

$$P(L_m) = P_{v_1^{-1}}^{CN^{3/2}/L_m^2} \cap B_{CN^2/(L_m^2\theta)}$$

(which degenerates to $B_{CN^2/(L_m^2\theta)}$ if $L_m \le CN^{1/4}$) as defined in (16). Let $\phi_{P(L_M)}$ be a function such that

$$\operatorname{supp}\widehat{\phi_{P(L_M)}} \subset P_{v_1}^{CL_M^2\theta/N^2}(0) \cap B_{CL_M^2/N^{3/2}}(0) \subset \bigcap_{I\in\mathcal{I}} (\tilde{I}-\tilde{I}), \quad \text{where } 1_{P(L_M)} \lesssim |\phi_{P(L_M)}| \lesssim W_{P(L_M),200}.$$

To construct such a function we can take a ψ in Lemma 2.2 adapted to certain fat AP and apply a translation in the physical space to it.

Lemma 5.3 (high lemma). *For $1 \le m \le M-1$ we have*

$$\int |g_m^h|^2 W_{P(L_M),100} \lesssim N^\varepsilon \int \sum_{I_m} |f_{m+1,I_m}|^4 W_{P(L_M),100}.$$

Proof. Because of (12), it suffices to show for every $P(L_M)$

$$\int_{P(L_M)} |g_m^h|^2 \lesssim N^\varepsilon \int \sum_{I_m} |f_{m+1,I_m}|^4 W_{P(L_M),100}.$$

Calculate

$$\int |g_m^h|^2 W_{P(L_M),100} \lesssim \int |g_m^h \phi_{P(L_M)}|^2 = \int \Big| \sum_{I_m} (\widehat{|f_{m+1,I_m}|^2}) \hat{\rho}_{I_m}(1-\eta_m) * \widehat{\phi_{P(L_M)}} \Big|^2.$$

Note that

$$\operatorname{supp}((\widehat{|f_{m+1,I_m}|^2})\hat{\rho}_{I_m}(1-\eta_m) * \widehat{\phi_{P(L_M)}}) \subset C(\tilde{I}_m - \tilde{I}_m) \setminus B_{L_{m+1}/(2N)}(0).$$

Indeed, the high-frequency cutoff $(1-\eta_m)$ removes the ball $B_{L_{m+1}/N}(0)$. The support of $\widehat{\phi_{P(L_M)}}$ is contained in a ball of radius $\le \frac{1}{2}L_M^2/N^{3/2}$ (if the C in the definition of $P(L)$ as in (16) is large enough), so convolution with $\widehat{\phi_{P(L_M)}}$ shrinks the high-frequency cutoff by an amount smaller than $L_{m+1}/(2N)$. The structure of $\tilde{I}_m - \tilde{I}_m$ is unchanged by convolution by $\widehat{\phi_{P(L_M)}}$ because the thickness of $\tilde{I}_m$ is $\sim L_m/N$

and $\frac{1}{2}L_M^2/N^{3/2} \le L_{m+1}/(2N) \le N^{-\varepsilon}L_m/N$. We claim that at every point on $\mathbb{R}$, the collection of sets $\{C(\widetilde{I}_m - \widetilde{I}_m) \setminus B_{L_{m+1}/(2N)}(0)\}_{I_m}$ has at most $\mathcal{O}(N^\varepsilon)$ overlap. Assuming this claim, by the Cauchy–Schwarz inequality we obtain

$$\int |g_m^h|^2 W_{P(L_M),100} \lesssim N^\varepsilon \int \sum_{I_m} \bigl|(\widehat{|f_{m+1,I_m}|^2})\widehat{\rho}_{I_m}(1-\eta_m) * \widehat{\phi_{P(L_M)}}\bigr|^2.$$

So we have

$$\begin{aligned}\int |g_m^h|^2 W_{P(L_M),100} &\lesssim N^\varepsilon \sum_{I_m} \int \bigl||f_{m+1,I_m}|^2 * \rho_{I_m} * \widecheck{(1-\eta_m)}\bigr|^2 |\phi_{P(L_M)}|^2 \\ &\lesssim N^\varepsilon \sum_{I_m} \biggl(\int \bigl||f_{m+1,I_m}|^2 * \rho_{I_m}\bigr|^2 |\phi_{P(L_M)}|^2 + \int \bigl||f_{m+1,I_m}|^2 * \rho_{I_m} * |\check{\eta}_m|\bigr|^2 |\phi_{P(L_M)}|^2\biggr) \\ &\lesssim N^\varepsilon \sum_{I_m} \biggl(\int |f_{m+1,I_m}|^4 (|\phi_{P(L_M)}|^2 * \rho_{I_m}) + \int |f_{m+1,I_m}|^4 (|\phi_{P(L_M)}|^2 * \rho_{I_m} * |\check{\eta}_m|)\biggr),\end{aligned}$$

where we used Cauchy–Schwarz and that ρ_{I_m} and $\check{\eta}_m$ have L^1 norms ~ 1 to justify

$$\bigl||f_{m+1,I_m}|^2 * \rho_{I_m}\bigr|^2 \lesssim |f_{m+1,I_m}|^4 * \rho_{I_m}, \quad \bigl||f_{m+1,I_m}|^2 * \rho_{I_m} * |\check{\eta}_m|\bigr|^2 \lesssim |f_{m+1,I_m}|^4 * \rho_{I_m} * |\check{\eta}_m|.$$

Noting that $|\phi_{P(L_M)}|^2 * \rho_{I_m} \lesssim W_{P(L_M),100}$ and $|\phi_{P(L_M)}|^2 * \rho_{I_m} * |\check{\eta}_m| \lesssim W_{P(L_M),100}$, we conclude

$$\int |g_m^h|^2 W_{P(L_M),100} \lesssim N^\varepsilon \sum_{I_m} \int |f_{m+1,I_m}|^4 W_{P(L_M),100}.$$

Now we prove the claim. Recall that $\widetilde{I}_m$ is a fat AP of the form $P_{v_{I_m}}^{CL^2\theta/N^2} \cap B_{CL_m/N}$, where $v_{I_m} \sim N^{-1}$. Suppose $x \in C(\widetilde{I}_m - \widetilde{I}_m) \setminus B_{L_{m+1}/(2N)}(0)$ and $x \in C(\widetilde{I}'_m - \widetilde{I}'_m) \setminus B_{L_{m+1}/N}(0)$ for distinct $\widetilde{I}_m$ and $\widetilde{I}'_m$. We denote the common difference of $\widetilde{I}_m$ and $\widetilde{I}'_m$ by v and v' respectively. Recalling that v_{I_m} are $C\theta L_m/N^2$ separated, and the maximal separation is $C(N^{1/2}/L_m)(\theta L_m/N^2) = C\theta/N^{3/2}$, we have

$$\theta L_m/N^2 \lesssim |v - v'| \lesssim \theta/N^{\frac{3}{2}}.$$

Suppose $x \in B_{CL_m^2\theta/N^2}(kv)$ and $x \in B_{CL_m^2\theta/N^2}(k'v')$ for some $k, k' \in \mathbb{N}$. Then since $x \notin B_{L_{m+1}/(2N)}(0)$, $L_{m+1} \lesssim k, k' \lesssim L_m$. By definition $L_m = N^\varepsilon L_{m+1} \le N^{1/2-\varepsilon}$, so we have

$$L_{m+1}\frac{\theta L_m}{N^2} \gtrsim N^{-\varepsilon}\frac{\theta L_m^2}{N^2}, \quad L_m\frac{\theta}{N^{\frac{3}{2}}} \le \frac{\theta}{N^{1+\varepsilon}} \le \frac{1}{N^{1+\varepsilon}}.$$

It follows that $|k - k'| \lesssim 1$ and

$$\text{either} \quad |v - v'| \lesssim N^\varepsilon \theta \frac{L_m}{N^2} \quad \text{or} \quad |v - v'| \gtrsim \frac{1}{N^{\frac{3}{2}-\varepsilon}}.$$

The second case cannot happen if N is sufficiently large (depending on ε). Since common differences v are $\mathcal{O}(\theta L_m/N^2)$-separated, we conclude that there are at most $\mathcal{O}(N^\varepsilon)$ many $\widetilde{I}'_m$ such that $x \in C(\widetilde{I}'_m - \widetilde{I}'_m) \setminus B_{L_{m+1}/(2N)}(0)$. ☐

5E. ***The sets $\Omega_{m,\alpha,r}$ and $U_{\alpha,r}$.*** The last part of our high-low decomposition set-up is to partition $P(L_M)$ into $\Omega_{m,\alpha,r}$, for a fixed pair (α,r). For $1\le m\le M-1$ we define $\Omega_{m,\alpha,r}$ to be

$$\Omega_{m,\alpha,r}:=\{x\in P(L_M): g_m(x)\le 2|g_m^h(x)|, g_{m+1}(x)\le 2|g_{m+1}^{\ell}(x)|,\dots,g_{M-1}(x)\le 2|g_{M-1}^{\ell}(x)|\}.$$

Here $g_k=g_{k,\alpha,r}$. Also define $\Omega_{0,\alpha,r}$ to be

$$\Omega_{0,\alpha,r}:=\{x\in P(L_M): g_1(x)\le 2|g_1^{\ell}(x)|, g_2(x)\le 2|g_2^{\ell}(x)|,\dots,g_{M-1}(x)\le 2|g_{M-1}^{\ell}(x)|\}.$$

Clearly

$$P(L_M)=\bigcup_{0\le m\le M-1}\Omega_{m,\alpha,r}$$

for every α,r. For notational convenience we let $\Omega_{M,\alpha,r}=P(L_M)$.

We define $U_{\alpha',r'}$ by

$$U_{\alpha',r'}:=\{x\in P(L_M): r'/2<g_M(x)\le 2r',\ \alpha'/2<|f(x)|\le 2\alpha'\}. \tag{45}$$

Recall that $g_M=\sum_{I_M}|f_{I_M}|^2*\rho_{I_M}$ is defined without the pruning process so in particular it does not depend on the pruning parameters α,r.

We prove the following lemma, which shows that, on $U_{\alpha,r}\cap\Omega_{m,\alpha,r}$, $|f_m-f_{m,\alpha,r}|$ is very small so that $|f_m|\sim|f_{m,\alpha,r}|$. We define $f_0=f_1$ for notational convenience. Also recall we have defined $f_{M+1}=f$ and $f_{M+1,I_M}=f_{I_M}=f_I$.

Lemma 5.4. *If the constant $\widetilde{C}_\varepsilon$ in the definition of λ is large enough depending on ε, then for every α,r, every $1\le m\le M-1$, and any subset $\mathcal{S}$ of the partition $\mathcal{I}_m=\{I_m\}$, we have*

$$\left|\sum_{I_m\in\mathcal{S}}f_{I_m}-\sum_{I_m\in\mathcal{S}}f_{m,\alpha,r,I_m}\right|\le\frac{\alpha}{100}$$

on $U_{\alpha,r}\cap\Omega_{m,\alpha,r}$, and also on $U_{\alpha,r}\cap\Omega_{0,\alpha,r}$ if $m=1$. In particular if $\widetilde{C}_\varepsilon$ in the definition of λ is large enough depending on ε, then for every α,r, every $0\le m\le M-1$,

$$|f_{m,\alpha,r}|\in\left[\frac{\alpha}{4},4\alpha\right],$$

on $U_{\alpha,r}\cap\Omega_{m,\alpha,r}$.

Proof. Fix α,r. In the following proof g_k means $g_{k,\alpha,r}$, and f_{k,I_k}, $f_{k,I_{k-1}}$, f_k mean $f_{k,I_k,\alpha,r}$, $f_{k,I_{k-1},\alpha,r}$, $f_{k,\alpha,r}$ respectively. First suppose $1\le m\le M-1$. By the definition of $\Omega_{m,\alpha,r}$ and Lemma 5.2 we know that on $U_{\alpha,r}\cap\Omega_{m,\alpha,r}$,

$$g_{m+1}\lesssim g_{m+2}\lesssim\cdots\lesssim g_M\lesssim r.$$

We also have by the Cauchy–Schwarz inequality $g_m\lesssim_\varepsilon N^\varepsilon g_{m+1}$. Recall that $M\lesssim_\varepsilon 1$ so we have, for $m\le k\le M$,

$$g_k\lesssim_\varepsilon N^\varepsilon r\quad\text{on } U_{\alpha,r}\cap\Omega_{m,\alpha,r}.$$

Let m' be an integer between m and M and let $I_{m'} \in \mathcal{I}_{m'}$. By the definition of $f_{m',I_{m'}}$ and $f_{m'+1,I_{m'}}$ we have for $x \in U_{\alpha,r} \cap \Omega_{m,\alpha,r}$

$$\begin{aligned}
|f_{m',I_{m'}}(x) - f_{m'+1,I_{m'}}(x)| &= \Big| \sum_{P_{I_{m'}} \notin \mathcal{P}_{I_{m'},\lambda}} \phi_{P_{I_{m'}}}(x) f_{m'+1,I_{m'}}(x) \Big| \\
&\lesssim \sum_{P_{I_{m'}} \notin \mathcal{P}_{I_{m'},\lambda}} |\phi^{\frac12}_{P_{I_{m'}}}(x) f_{m'+1,I_{m'}}(x)| \phi^{\frac12}_{P_{I_{m'}}}(x) \\
&\lesssim \sum_{P_{I_{m'}} \notin \mathcal{P}_{I_{m'},\lambda}} \lambda^{-1} \|\phi_{P_{I_{m'}}} f_{m'+1,I_{m'}}\|_{L^\infty(\mathbb{R})} \|\phi^{\frac12}_{P_{I_{m'}}} f_{m'+1,I_{m'}}\|_{L^\infty(\mathbb{R})} \phi^{\frac12}_{P_{I_{m'}}}(x) \\
&\lesssim \lambda^{-1} \sum_{P_{I_{m'}} \notin \mathcal{P}_{I_{m'},\lambda}} \|\phi^{\frac12}_{P_{I_{m'}}} f_{m'+1,I_{m'}}\|^2_{L^\infty(\mathbb{R})} \phi^{\frac12}_{P_{I_{m'}}}(x) \\
&\lesssim \lambda^{-1} \sum_{P_{I_{m'}} \notin \mathcal{P}_{I_{m'},\lambda}} \sum_{\widetilde{P}_{I_{m'}}} \|\phi_{P_{I_{m'}}} f^2_{m'+1,I_{m'}}\|_{L^\infty(\widetilde{P}_{I_{m'}})} \phi^{\frac12}_{P_{I_{m'}}}(x) \\
&\lesssim \lambda^{-1} \sum_{P_{I_{m'}}} \sum_{\widetilde{P}_{I_{m'}}} \|\phi_{P_{I_{m'}}}\|_{L^\infty(\widetilde{P}_{I_{m'}})} \|f^2_{m'+1,I_{m'}}\|_{\cancel{L}^1(W_{\widetilde{P}_{I_{m'}}})} \phi^{\frac12}_{P_{I_{m'}}}(x),
\end{aligned}$$

where we used $\phi_{P_{I_{m'}}} \lesssim \phi^{1/2}_{P_{I_{m'}}}$. We also used the locally constant property Proposition 2.3 for the last inequality. If we use $\phi_{I_{m'}}(\widetilde{P}_{I_{m'}})$ to denote $\phi_{I_{m'}}(\sup \widetilde{P}_{I_{m'}})$, which is comparable to $\phi_{I_{m'}}(y)$ for any $y \in \widetilde{P}_{I_{m'}}$, then we have

$$\begin{aligned}
|f_{m',I_{m'}}(x) - f_{m'+1,I_{m'}}(x)| &\lesssim \lambda^{-1} |P_{I_{m'}}|^{-1} \sum_{P_{I_{m'}}} \sum_{\widetilde{P}_{I_{m'}}} \Big(\int W_{\widetilde{P}_{I_{m'}}} \phi_{P_{I_{m'}}}(\widetilde{P}_{I_{m'}}) |f_{m'+1,I_{m'}}|^2 \Big) \phi^{\frac12}_{P_{I_{m'}}}(x) \\
&\lesssim \lambda^{-1} |P_{I_{m'}}|^{-1} \sum_{\widetilde{P}_{I_{m'}}} \Big(\int W_{\widetilde{P}_{I_{m'}}} |f_{m'+1,I_{m'}}|^2 \Big) \phi^{\frac12}_{\widetilde{P}_{I_{m'}}}(P_{I_{m'}}(x)) \\
&\lesssim \lambda^{-1} |P_{I_{m'}}|^{-1} \int |f_{m'+1,I_{m'}}|^2(y) \sum_{\widetilde{P}_{I_{m'}}} W_{\widetilde{P}_{I_{m'}}}(y) \phi^{\frac12}_{\widetilde{P}_{I_{m'}}}(P_{I_{m'}}(x))\, dy \\
&\lesssim \lambda^{-1} |P_{I_{m'}}|^{-1} \int |f_{m'+1,I_{m'}}|^2(y) \phi^{\frac12}_{P_{I_{m'}}(x)}(y)\, dy.
\end{aligned}$$

Noting that $|P_{I_{m'}}|^{-1} \phi^{1/2}_{P_{I_{m'}}(x)}(y) \lesssim \rho_{I_{m'}}(x-y)$, we get

$$|f_{m',I_{m'}}(x) - f_{m'+1,I_{m'}}(x)| \lesssim \lambda^{-1} |f_{m'+1,I_{m'}}|^2 * \rho_{I_{m'}}(x).$$

Summing the above over $I_{m'} \subset \bigcup_{I_m \in \mathcal{S}} I_m$ we conclude

$$\begin{aligned}
\Big| \sum_{I_{m'} \subset \bigcup_{I_m \in \mathcal{S}} I_m} f_{m',I_{m'}}(x) - \sum_{I_{m'} \subset \bigcup_{I_m \in \mathcal{S}} I_m} f_{m'+1,I_{m'}}(x) \Big| &\le \lambda^{-1} \sum_{I_{m'} \in \mathcal{I}_{m'}} |f_{m'+1,I_{m'}}|^2 * \rho_{I_{m'}}(x) \\
&= \lambda^{-1} g_{m'}(x) \lesssim_\varepsilon N^\varepsilon \frac{r}{\lambda}.
\end{aligned}$$

Therefore if we choose the constant $\widetilde{C}_\varepsilon$ in the definition of $\lambda = \widetilde{C}_\varepsilon N^\varepsilon \frac{r}{\alpha}$ to be large enough depending on ε, then we have, for $x \in U_{\alpha,r} \cap \Omega_{m,\alpha,r}$,

$$\sum_{m \le m' \le M} \Bigg| \sum_{I_{m'} \subset \bigcup_{I_m \in \mathcal{S}} I_m} f_{m',I_{m'}}(x) - \sum_{I_{m'} \subset \bigcup_{I_m \in \mathcal{S}} I_m} f_{m'+1,I_{m'}}(x) \Bigg| \le \frac{\alpha}{100}.$$

Since by definition $\sum_{I_{m'} \subset \bigcup_{I_m \in \mathcal{S}} I_m} f_{m',I_{m'}} = \sum_{I_{m'-1} \subset \bigcup_{I_m \in \mathcal{S}} I_m} f_{m',I_{m'-1}}$, we have by the triangle inequality that

$$\Bigg| \sum_{I_m \in \mathcal{S}} f_{I_m} - \sum_{I_m \in \mathcal{S}} f_{m,I_m} \Bigg| \le \frac{\alpha}{100}.$$

The case $m = 0$ follows from the above argument for $m = 1$ as by definition $f_0 = f_1$. □

From now on we will assume that $\widetilde{C}_\varepsilon$ is chosen large enough such that the conclusion of Lemma 5.4 holds.

6. Proof of Proposition 4.5

We prove Proposition 4.5 in this section, and consequently Theorem 4.4. We also give the proof of Theorem 4.3 assuming Theorem 4.4 in the last subsection. Still fix $2 \le p \le 6$, $\varepsilon > 0$, and $P(L_M) \subset \mathbb{R}$.

Suppose $1 \le K \le N^{\varepsilon/2}$ and $N^{1/2}/K \ge L$. Let $\mathcal{I}'$ be a partition of $\mathcal{N}_{N^{-1}K^{-1}}(\{a_n\}_{n=1}^{N^{1/2}})$ into K many I', which is a union of $N^{1/2}/K$ consecutive intervals in $\mathcal{N}_{N^{-1}K^{-1}}(\{a_n\}_{n=1}^{N^{1/2}})$. We call $I', I'' \in \mathcal{I}'$ nonadjacent if there exist at least two other $I''' \in \mathcal{I}'$ between I' and I'' on the real line. Alternatively, we can list $I' \in \mathcal{I}'$ as I'_j so that I'_{j+1} is on the right side of I'_j on the real line for every j. Then we define $I'_j, I'_{j'}$ to be nonadjacent if $|j - j'| \ge 3$. In displayed math we write "nonadj." as the shorthand for nonadjacent.

For f with $\operatorname{supp} \widehat{f} \subset \Omega$, we let $f_{I'}$ denote the projection of f to I' in the frequency space. So $f_{I'} = \sum_{I_M \subset I'} f_{I_M}$.

6A. ***Broad-narrow decomposition.*** The following lemma is a broad-narrow analysis on f with some complication. For parameters $\alpha, r > 0$ and m, $0 \le m \le M - 1$, define

$$f_{m,\alpha,r,I'} := \sum_{I_m \subset I'} f_{m,\alpha,r,I_m},$$

where we recall that f_{m,α,r,I_m} is defined in (42).

Lemma 6.1. *For every $X \subset P(L_M)$, there exist some α, r with $\alpha \ge r^{1/2}$ and some m such that $0 \le m \le M - 1$ and*

$$\int_X |f|^p \lesssim_\varepsilon \sum_{I' \in \mathcal{I}'} \int_X |f_{I'}|^p + (\log N \log(\theta^{-1} + 1))^C \frac{K^C}{\alpha^{4-p}} \max_{\substack{I', I'' \\ \text{nonadj.}}} \int_{X \cap U_{\alpha,r} \cap \Omega_{m,\alpha,r}} |f_{m,\alpha,r,I'}|^2 |f_{m,\alpha,r,I''}|^2$$
$$+ \Big(\sup_{x \in X} \sum_I \|f_I\|^2_{L^2(W_{P_I(x),100})} \Big)^{\frac{p}{2}-1} \Big(\sum_I \|f_I\|^2_{L^2(W_{P(L),100})} \Big). \quad (46)$$

First we prove a technical lemma which is a pointwise broad-narrow analysis.

By taking all parameters to have dyadic values, we may assume that for each I_m, $0 \le m \le M$, and any I', either $I_m \subset I'$ or $I_m \cap I' = \varnothing$.

Lemma 6.2. *For every $\alpha, r > 0$ and $0 \le m \le M-1$,*

$$|f_{m,\alpha,r}(x)|^2 \lesssim \max_{I'} |f_{I'}(x)|^2 + K^C \max_{\substack{I', I'' \\ \text{nonadj.}}} |f_{m,\alpha,r,I'}(x)|\,|f_{m,\alpha,r,I''}(x)|$$

for every $x \in X \cap U_{\alpha,r} \cap \Omega_{m,\alpha,r}$.

Proof. Let $x \in X \cap U_{\alpha,r} \cap \Omega_{m,\alpha,r}$. If there exist $I', I'' \in \mathcal{I}'$ nonadjacent such that $|f_{m,\alpha,r,I'}|, |f_{m,\alpha,r,I''}| \ge \frac{1}{100K}|f_{m,\alpha,r}(x)|$, then we have

$$|f_{m,\alpha,r}(x)|^2 \lesssim K^2 \max_{\substack{I', I'' \\ \text{nonadj.}}} |f_{m,\alpha,r,I'}(x)|\,|f_{m,\alpha,r,I''}(x)|. \tag{47}$$

Now we assume there do not exist $I', I'' \in \mathcal{I}'$ nonadjacent with $|f_{m,\alpha,r,I'}|, |f_{m,\alpha,r,I''}| \ge \frac{1}{100K}|f_{m,\alpha,r}(x)|$. Note that $f_{m,\alpha,r}(x) = \sum_{I'} f_{m,\alpha,r,I'}(x)$ and the number of I' is bounded by K. So if we choose $I''' \in \mathcal{I}'$ with $|f_{m,\alpha,r,I'''}(x)| = \max_{I' \in \mathcal{I}'} |f_{m,\alpha,r,I'}(x)|$, then

$$|f_{m,\alpha,r,I'''}(x)| \ge \tfrac{1}{2}|f_{m,\alpha,r}(x)|. \tag{48}$$

By Lemma 5.4 we have $|f_{m,\alpha,r}(x)| \in [\alpha/4, 4\alpha]$, and $|f_{m,\alpha,r,I'''}(x) - f_{I'''}(x)| \le \frac{\alpha}{100}$. Therefore by the triangle inequality and (48) we obtain

$$|f_{I'''}(x)| \gtrsim \alpha \sim |f_{m,\alpha,r}(x)|.$$

This combined with (47) proves the lemma. □

Proof of Lemma 6.1. Since $P(L_M) = \bigsqcup_{\alpha,r:\,\text{dyadic}} U_{\alpha,r}$, we have

$$\int_X |f|^p \le \sum_{\alpha,r:\,\text{dyadic}} \int_{X \cap U_{\alpha,r}} |f|^p.$$

Without loss of generality we assume

$$\Bigl(\sup_{x\in X} \sum_I \|f_I\|^2_{L^2(W_{P_I(x),100})}\Bigr)^{\frac{1}{2}-\frac{1}{p}} \Bigl(\sum_I \|f_I\|^2_{L^2(W_{P(L),100})}\Bigr)^{\frac{1}{p}} = 1. \tag{49}$$

Then $X \cap U_{\alpha,r} = \varnothing$ if $\max\{\alpha, r\} \ge CN^C\theta^{-C}$ for some sufficiently large constant C. Also

$$\Bigl(\int_{X \cap (\bigcup_{\min\{\alpha,r\} \le C^{-1}N^{-C}\theta^C} U_{\alpha,r})} |f|^p\Bigr)^{\frac{1}{p}} \lesssim 1$$

if C is sufficiently large. So now we write

$$\int_X |f|^p \le \sum_{\alpha,r} \int_{X \cap U_{\alpha,r}} |f|^p + C, \tag{50}$$

where the number of pairs (α, r) in the summation is $\mathcal{O}(\log N \log(\theta^{-1}+1))^2$, since the number of dyadic numbers between $C^{-1}N^{-C}\theta^C$ and $CN^C\theta^{-C}$ is $\mathcal{O}(\log N + \log(\theta^{-1}+1)) = \mathcal{O}(\log N \log(\theta^{-1}+1))$.

We also observe that by Hölder's inequality and Fubini's theorem we have

$$\int_{X\cap\bigcup_{\alpha\le r^{1/2}}U_{\alpha,r}}|f|^p\lesssim\int_X\Big(\sum_I|f_I|^2*\rho_I\Big)^{\frac{p}{2}}\lesssim\Big\|\sum_I|f_I|^2*\rho_I\Big\|_{L^\infty(X)}^{\frac{p}{2}-1}\Big(\sum_I\|f_I\|^2_{L^2(W_{P(L),100})}\Big).$$

Since

$$\Big\|\sum_I|f_I|^2*\rho_I\Big\|_{L^\infty(X)}\le\sup_{x\in X}\sum_I|f_I|^2*\rho_I(x)\lesssim\sup_{x\in X}\sum_I\|f_I\|^2_{\bar{L}^2(W_{P_I(x),100})},$$

we obtain

$$\int_{X\cap\bigcup_{\alpha\le r^{1/2}}U_{\alpha,r}}|f|^p\lesssim\Big(\sup_{x\in X}\sum_I\|f_I\|^2_{\bar{L}^2(W_{P_I(x),100})}\Big)^{\frac{p}{2}-1}\Big(\sum_I\|f_I\|^2_{L^2(W_{P(L),100})}\Big)=1.$$

So in summary

$$\int_X|f|^p\lesssim\sum_{\alpha,r:\alpha\ge r^{1/2}}\int_{X\cap U_{\alpha,r}}|f|^p+1. \tag{51}$$

Next we further decompose $X\cap U_{\alpha,r}$ into $\bigcup_m(X\cap U_{\alpha,r}\cap\Omega_{m,\alpha,r})$:

$$\int_{X\cap U_{\alpha,r}}|f|^p\le\sum_{m=0}^{M-1}\int_{X\cap U_{\alpha,r}\cap\Omega_{m,\alpha,r}}|f|^p.$$

By Lemma 5.4 we have, for $0\le m\le M-1$,

$$\int_{X\cap U_{\alpha,r}\cap\Omega_{m,\alpha,r}}|f|^p\sim\int_{X\cap U_{\alpha,r}\cap\Omega_{m,\alpha,r}}|f_{m,\alpha,r}|^p.$$

It then follows from Lemmas 6.2 and 5.4 that

$$\begin{aligned}\int_X|f|^p&\\ &\lesssim 1+\sum_{\alpha,r:\alpha\ge r^{1/2}}\sum_{m=0}^{M-1}\Big(\sum_{I'\in\mathcal{I}'}\int_{X\cap U_{\alpha,r}\cap\Omega_{m,\alpha,r}}|f_{I'}|^p+\frac{K^C}{\alpha^{4-p}}\max_{\substack{I',I''\\ \text{nonadj.}}}\int_{X\cap U_{\alpha,r}\cap\Omega_{m,\alpha,r}}|f_{m,\alpha,r,I'}|^2|f_{m,\alpha,r,I''}|^2\Big)\\ &\lesssim 1+C_\varepsilon\sum_{I'\in\mathcal{I}'}\int_X|f_{I'}|^p+\sum_{\alpha,r:\alpha\ge r^{1/2}}\sum_m\frac{K^C}{\alpha^{4-p}}\max_{\substack{I',I''\\ \text{nonadj.}}}\int_{X\cap U_{\alpha,r}\cap\Omega_{m,\alpha,r}}|f_{m,\alpha,r,I'}|^2|f_{m,\alpha,r,I''}|^2,\end{aligned}$$

where we used $M\lesssim_\varepsilon 1$ in the last inequality. Recall that the number of pairs (α,r) in the summation is $\mathcal{O}(\log N\log(\theta^{-1}+1))^2$ (see (50)); by the pigeonhole principle we have

$$\begin{aligned}&\sum_{\alpha,r:\alpha\ge r^{1/2}}\sum_m\frac{K^C}{\alpha^{4-p}}\max_{\substack{I',I''\\ \text{nonadj.}}}\int_{X\cap U_{\alpha,r}\cap\Omega_{m,\alpha,r}}|f_{m,\alpha,r,I'}|^2|f_{m,\alpha,r,I''}|^2\\ &\qquad\lesssim_\varepsilon(\log N\log(\theta^{-1}+1))^2\frac{K^C}{\alpha^{4-p}}\max_{\substack{I',I''\\ \text{nonadj.}}}\int_{X\cap U_{\alpha,r}\cap\Omega_{m,\alpha,r}}|f_{m,\alpha,r,I'}|^2|f_{m,\alpha,r,I''}|^2\end{aligned}$$

for some α,r with $\alpha\ge r^{1/2}$, $0\le m\le M-1$, which completes the proof. □

Now fix $X \subset P(L_M)$. We have identified a pair (α, r) from Lemma 6.1, and we fix that pair of α, r and suppress the dependence on α, r from now on in the notation. In particular write $g_m = g_{m,\alpha,r}$, $\Omega_m = \Omega_{m,\alpha,r}$, $f_{m,I'} = f_{m,\alpha,r,I'}$ and $f_{m,I_m} = f_{m,\alpha,r,I_m}$ where α, r are those chosen in Lemma 6.1.

We estimate the broad and narrow parts separately, which together with Lemma 6.1 will imply Proposition 4.5.

6B. *Narrow part.*

Proposition 6.3. *For every $I' \in \mathcal{I}'$ we have*

$$\int_X |f_{I'}|^p \lesssim \Big(\sup_{\theta' \in [\theta/4, \theta]} \operatorname{Dec}\Big(\frac{N}{K^2}, \frac{\theta'}{K^2}\Big)^p\Big)\Big(\sup_{x \in X} \sum_{I \subset I'} \|f_I\|^2_{\text{Ł}^2(W_{P_I(x),100})}\Big)^{\frac{p}{2}-1}\Big(\sum_{I \subset I'} \|f_I\|^2_{L^2(W_{P(L),100})}\Big). \tag{52}$$

Proof. In this proof, the notation cA with $c \in \mathbb{R}$, $A \subset \mathbb{R}$ will denote the set $\{ca : a \in A\}$.

We first prove (52) for $I' = \mathcal{N}_{L^2\theta/N^2}(\{a_n\}_{n=1}^{N^{1/2}/K})$. Note that $K^2 I' = \mathcal{N}_{K^2 L^2\theta/N^2}(\{K^2 a_n\})_{n=1}^{N^{1/2}/K}$, and if we let $\tilde{a} = K^2 a_n$, $\widetilde{N} = N/K^2$ and $\tilde{\theta} = \theta/K^2$, then

$$\tilde{a} - \tilde{a} \in \Big[\frac{K^2}{4N}, \frac{4K^2}{N}\Big] = \Big[\frac{1}{4\widetilde{N}}, \frac{4}{\widetilde{N}}\Big], \quad (\tilde{a} - \tilde{a}) - (\tilde{a} - \tilde{a}) \in \Big[\frac{K^2\theta}{4N^2}, \frac{4K^2\theta}{N^2}\Big] = \Big[\frac{\tilde{\theta}}{4\widetilde{N}^2}, \frac{4\tilde{\theta}}{\widetilde{N}^2}\Big],$$

and $K^2 I' = \mathcal{N}_{L^2\tilde{\theta}/\widetilde{N}^2}(\{\tilde{a}\}_{n=1}^{\widetilde{N}^{1/2}})$.

We define $\widetilde{P}(L)$, $\widetilde{P}_{K^2 I}$ by (16), (15) respectively with N, L, θ, v_j replaced by $\widetilde{N}, L, \tilde{\theta}, K^2 v_j$. Then for any x_0 we have $\widetilde{P}_{K^2 I}(K^{-2}x_0) = K^{-2}P_I(x_0)$, and $\widetilde{P}(L, K^{-2}x_0) \subset K^{-2}P(L, x_0)$. Now by the change of variable formula,

$$\int_X |f_{I'}(x)|^p\, dx = K^2 \int_{K^{-2}X} |f_{I'}(K^2 x)|^p\, dx.$$

We have $\operatorname{supp} \widehat{f_{I'}(K^2\cdot)} \subset K^2 I' = \mathcal{N}_{L^2\tilde{\theta}/\widetilde{N}^2}(\{\tilde{a}\}_{n=1}^{\widetilde{N}^{1/2}})$. Let $\tilde{f}(x)$ denote the function $f_{I'}(K^2 x)$. Therefore by the definition of the refined decoupling constant for $\mathcal{N}_{L^2\tilde{\theta}/\widetilde{N}^2}(\{\tilde{a}\}_{n=1}^{\widetilde{N}^{1/2}})$, and (33) (as $\widetilde{P}(L, K^{-2}x_0) \subset K^{-2}P(L, x_0)$), we have

$$\int_{K^{-2}X} |\tilde{f}(x)|^p\, dx \le \operatorname{Dec}(\widetilde{N}, \tilde{\theta})^p \Big(\sup_{x \in X} \sum_{I \subset I'} \|\tilde{f}\|^2_{\text{Ł}^2(W_{\widetilde{P}_{K^2 I}(K^{-2}x),100})}\Big)^{\frac{p}{2}-1}\Big(\sum_{I \subset I'} \|\tilde{f}\|^2_{L^2(W_{K^{-2}P(L),100})}\Big).$$

By the change of variable formula,

$$\|\tilde{f}\|_{\text{Ł}^2(W_{\widetilde{P}_{K^2 I}(K^{-2}x),100})} \lesssim \|f_I\|_{\text{Ł}^2(W_{P_I(x),100})},$$
$$\|\tilde{f}\|^2_{L^2(W_{K^{-2}P(L),100})} \lesssim K^{-2}\|f_I\|^2_{L^2(W_{P(L),100})}.$$

So we conclude

$$\int_X |f_{I'}|^p \lesssim \operatorname{Dec}\Big(\frac{N}{K^2}, \frac{\theta}{K^2}\Big)^p \Big(\sup_{x \in X} \sum_{I \subset I'} \|f_I\|^2_{\text{Ł}^2(W_{P_I(x),100})}\Big)^{\frac{p}{2}-1}\Big(\sum_{I \subset I'} \|f_I\|^2_{L^2(W_{P(L),100})}\Big).$$

Now we consider a general $I' \in \mathcal{I}'$. Suppose a_l is the first term in $I' \cap \{a_n\}_{n=1}^{N^{1/2}}$, and let $v_l = a_{l+1} - a_l$. Because of (14) we have $v_l \in [v_1, 2v_1]$. So we may choose $K_l \in [K/\sqrt{2}, K]$ such that

$$K_l^2 v_l \in \left[\frac{1}{4\widetilde{N}}, \frac{4}{\widetilde{N}}\right].$$

Then

$$K_l^2((a_{n+1}-a_n)-(a_n-a_{n-1})) \in \left[\frac{\theta K_l^2}{4N^2}, \frac{4\theta K_l^2}{N^2}\right] = \left[\frac{\tilde{\theta}'}{4\widetilde{N}^2}, \frac{4\tilde{\theta}_l}{\widetilde{N}^2}\right]$$

for some $\tilde{\theta}_l \in [\tilde{\theta}/4, \tilde{\theta}]$. Let $\theta_l = K^2\tilde{\theta}_l$, which lies in $[\theta/4, 4\theta]$. So by a change of variable argument again we have

$$\int_X |f_{I'}|^p \lesssim \operatorname{Dec}\left(\frac{N}{K^2}, \frac{\theta_l}{K^2}\right)^p \left(\sup_{x\in X} \sum_{I\subset I'} \|f_I\|^2_{L^2(W_{P_I(x),100})}\right)^{\frac{p}{2}-1} \left(\sum_{I\subset I'} \|f_I\|^2_{L^2(W_{P(L),100})}\right).$$

Therefore we have shown (52) for every $I' \in \mathcal{I}'$. □

The proof of Proposition 6.3 actually shows that (52) holds for every f with frequency support in Ω (not only alternately spaced f) and every $X \subset P(L)$.

6C. *Broad part.*

Proposition 6.4. *For $1 \le m \le M-1$ and $I', I'' \in \mathcal{I}'$ nonadjacent we have*

$$\int_{X\cap U_{\alpha,r}\cap\Omega_m} |f_{m,I'}|^2 |f_{m,I''}|^2 \lesssim_\varepsilon N^{C\varepsilon} K^C \left(\frac{r}{\alpha}\right)^2 \left(\sum_{I\in\mathcal{I}} \|f_I\|^2_{L^2(W_{P(L),100})}\right). \tag{53}$$

Proof. Fix a $P(L'_m)$ such that $P(L'_m) \cap X \cap U_{\alpha,r} \cap \Omega_m \neq \varnothing$. Recall that $L'_m = (L_m N^{1/2})^{1/2}$ as defined in Section 3. Suppose the distance between I' and I'' is $1/K'$. Since I', I'' are nonadjacent, we have $1/K \le 1/K' \lesssim 1$. Let $\tilde{f}(x)$ denote the function $f_{m,I'}((K')^2 x)$, and $\tilde{f}(x)$ denote the function $f_{m,I''}((K')^2 x)$. Then supp $\hat{\tilde{f}}_1 \subset (K')^2 I'$, supp $\hat{\tilde{f}}_2 \subset (K')^2 I''$, and $d((K')^2 I', (K')^2 I'') \gtrsim 1$. By (22) and a change of variable argument similar to that in the proof of Proposition 6.3, we have

$$\int_{(K')^{-2}P(L'_m)} |\tilde{f}|^2 |\tilde{f}|^2 \lesssim_\varepsilon N^\varepsilon (K')^C |P(L'_m)|^{-1} \int |\tilde{f}|^2 W_{(K')^{-2}P(L'_m),200} \int |\tilde{f}|^2 W_{(K')^{-2}P(L'_m),200}.$$

By the local L^2 orthogonality Lemma 3.6, we further obtain

$$\begin{aligned}&\int_{(K')^{-2}P(L'_m)} |\tilde{f}|^2 |\tilde{f}|^2 \\ &\lesssim_\varepsilon N^\varepsilon (K')^C |P(L'_m)|^{-1} \int \sum_{I_m\subset I'} |(\tilde{f})_{(K')^2 I_m}|^2 W_{(K')^{-2}P(L'_m),200} \int \sum_{I_m\subset I''} |(\tilde{f})_{(K')^2 I_m}|^2 W_{(K')^{-2}P(L'_m),200}.\end{aligned}$$

Here the notation cA with $c \in \mathbb{R}$, $A \subset \mathbb{R}$ denotes the set $\{ca : a \in A\}$. Applying the change of variable $x \mapsto (K')^{-2}x$ to both sides of the above inequality, and using $K' \le K$, we get

$$\int_{P(L'_m)} |f_{m,I'}|^2 |f_{m,I''}|^2 \lesssim_\varepsilon N^\varepsilon K^C |P(L'_m)|^{-1} \int \sum_{I_m\subset I'} |f_{m,I_m}|^2 W_{P(L'_m),200} \int \sum_{I_m\subset I''} |f_{m,I_m}|^2 W_{P(L'_m),200}.$$

By Hölder's inequality,

$$\int_{P(L'_m)} |f_{m,I'}|^2 |f_{m,I''}|^2 \lesssim_e N^\varepsilon K^C \int \Big(\sum_{I_m} |f_{m,I_m}|^2\Big)^2 W_{P(L'_m),200},$$

and due to $|f_{m,I_m}| \le |f_{m+1,I_m}|$ we further have

$$\int_{P(L'_m)} |f_{m,I'}|^2 |f_{m,I''}|^2 \lesssim_\varepsilon N^\varepsilon K^C \int \Big(\sum_{I_m} |f_{m+1,I_m}|^2\Big)^2 W_{P(L'_m),200}.$$

Now applying Proposition 2.3 we obtain

$$\begin{aligned}\int_{P(L'_m)} |f_{m,I'}|^2 |f_{m,I''}|^2 &\lesssim_\varepsilon N^\varepsilon K^C |P(L'_m)|^{-1} \Big(\int \Big(\sum_{I_m} |f_{m+1,I_m}|^2\Big) W_{P(L'_m),100}\Big)^2 \\ &\lesssim N^\varepsilon K^C \int_{P(L'_m)} g_m^2.\end{aligned}$$

Note that from the definition of Ω_m and the definition of $g_m := \sum_{I_m} |f_{m+1,I_m}|^2 * \rho_{I_m}$ we have $x \in \Omega_m$ implies $|g_m(x)| \sim \sup_{y \in P(L'_m(x))} |g_m(y)| \lesssim |g_m^h(x)|$. Therefore we have (by Proposition 2.3)

$$\int_{P(L'_m)} g_m^2 \lesssim |P(L'_m)| |g_m^h(x)|^2 \lesssim \int |g_m^h|^2 W_{P(L'_m),100},$$

where $x \in P(L'_m) \cap \Omega_m$. Summing over disjoint $P(L'_m)$ that intersect $X \cap U_{\alpha,r} \cap \Omega_m$ we obtain

$$\int_{X \cap U_{\alpha,r} \cap \Omega_m} |f_{m,I'}|^2 |f_{m,I''}|^2 \lesssim_\varepsilon N^\varepsilon K^C \int |g_m^h|^2 W_{P(L_M),100} \lesssim N^{2\varepsilon} K^C \int \sum_{I_m} |f_{m+1,I_m}|^4 W_{P(L_M),100},$$

where the last inequality is due to Lemma 5.3. By Hölder's inequality and the definition of $f_{m+1,I_{m+1}}$ we have

$$\begin{aligned}\int \sum_{I_m} |f_{m+1,I_m}|^4 W_{P(L_M),100} &\lesssim N^{C\varepsilon} \int \sum_{I_{m+1}} |f_{m+1,I_{m+1}}|^4 W_{P(L_M),100} \\ &\lesssim N^{C\varepsilon} \Big(\frac{r}{\alpha}\Big)^2 \int \sum_{I_{m+1}} |f_{m+1,I_{m+1}}|^2 W_{P(L_M),100}.\end{aligned}$$

By the pointwise inequality $|f_{m+1,I_{m+1}}| \le |f_{m+2,I_{m+1}}|$ and local L^2 orthogonality (Lemma 3.6),

$$\begin{aligned}\int \sum_{I_{m+1}} |f_{m+1,I_{m+1}}|^2 W_{P(L_M),100} &\lesssim \int \sum_{I_{m+2}} |f_{m+2,I_{m+1}}|^2 W_{P(L_M),100} \\ &\lesssim \int \sum_{I_{m+2}} |f_{m+2,I_{m+2}}|^2 W_{P(L_M),100}.\end{aligned}$$

Continuing this process we obtain

$$\int \sum_{I_{m+1}} |f_{m+1,I_{m+1}}|^2 W_{P(L_M),100} \lesssim_\varepsilon \int \sum_{I_M} |f_{M,I_M}|^2 W_{P(L_M),100}. \tag{54}$$

Recalling that $|f_{M,I_M}|\le|f_{I_M}|=|f_I|$ we conclude

$$\int_{X\cap U_{\alpha,r}\cap\Omega_m}|f_{m,I'}|^2|f_{m,I''}|^2\lesssim_\varepsilon N^{C\varepsilon}K^C\left(\frac{r}{\alpha}\right)^2\int\sum_I|f_I|^2W_{P(L_M),100}. \qquad\square$$

Proposition 6.5. *For $I',I''\in\mathcal{I}'$ nonadjacent we have*

$$\int_{X\cap U_{\alpha,r}\cap\Omega_0}|f_{0,I'}|^{\frac{p}{2}}|f_{0,I''}|^{\frac{p}{2}}\lesssim_\varepsilon N^\varepsilon\left(\sup_{x\in X}\sum_I\|f_I\|^2_{L^2(W_{P_I(x),100})}\right)^{\frac{p}{2}-1}\left(\sum_I\|f_I\|^2_{L^2(W_{P(L),100})}\right).$$

Proof. By the Cauchy–Schwarz inequality we have

$$\begin{aligned}\int_{X\cap U_{\alpha,r}\cap\Omega_0}|f_{0,I'}|^{\frac{p}{2}}|f_{0,I''}|^{\frac{p}{2}}&\lesssim N^\varepsilon\int_{X\cap U_{\alpha,r}\cap\Omega_0}\left(\sum_{I_1}|f_{1,I_1}|^2\right)^{\frac{p}{2}}\\&\lesssim N^\varepsilon\sup_{x\in X\cap\Omega_0}\left(\sum_{I_1}|f_{1,I_1}|^2\right)^{\frac{p}{2}-1}\int\sum_{I_1}|f_{1,I_1}|^2W_{P(L_M),100}.\end{aligned}$$

We have shown in the proof of Proposition 6.4 (inequality (54)) that

$$\int\sum_{I_1}|f_{1,I_1}|^2W_{P(L_M),100}\lesssim_\varepsilon\int\sum_I|f_I|^2W_{P(L),100}.$$

So it suffices to show

$$\sup_{x\in X\cap\Omega_0}\left(\sum_{I_1}|f_{1,I_1}|^2\right)\lesssim_\varepsilon\sup_{x\in X}\sum_I\|f_I\|^2_{L^2(W_{P_I(x),100})}. \tag{55}$$

From the locally constant property (Proposition 2.3) we have

$$\sum_{I_1}|f_{1,I_1}|^2(x)\lesssim\sum_{I_1}|f_{1,I_1}|^2*\rho_{I_1}(x)\lesssim\sum_{I_1}|f_{2,I_1}|^2*\rho_{I_1}(x)=g_1(x)$$

(recall that ρ_{I_1} is an L^1 normalized nonnegative function adapted to $P_{I_1}(0)$ satisfying (43)), and by Lemma 5.2 we have, for $x\in X\cap\Omega_0$, $g_1(x)\lesssim_\varepsilon g_M(x)$. So we conclude

$$\sup_{x\in X\cap\Omega_0}\sum_{I_1}|f_{1,I_1}|^2(x)\lesssim_\varepsilon\sup_{x\in X\cap\Omega_0}g_M(x)\lesssim\sup_{x\in X}\sum_I\|f_I\|^2_{L^2(W_{P_I(x),100})}. \qquad\square$$

6D. ***Proof of Proposition 4.5.*** Let $X\subset P(L)$. We choose α,r as in Lemma 6.1. Note that

$$r\le2\left\|\sum_I|f_I|^2\right\|_{L^\infty(X)}$$

since otherwise $X\cap U_{\alpha,r}=\varnothing$. So

$$r\le2\left\|\sum_I|f_I|^2\right\|_{L^\infty(X)}\lesssim\sup_{x\in X}\sum_{I\subset I'}\|f_I\|^2_{L^2(W_{P_I(x),100})}.$$

Also $\alpha \geq r^{1/2}$ implies that $r^{3-p/2}/\alpha^{6-p} \leq 1$ as $p \leq 6$. Therefore combining Propositions 6.3, 6.4, and 6.5 and Lemma 6.1 we obtain

$$\int_X |f|^p \lesssim_\varepsilon \left(\sup_{\theta' \in [\theta/4, 4\theta]} \mathrm{Dec}\left(\frac{N}{K^2}, \frac{\theta'}{K^2} \right)^p + \log^C(\theta^{-1}+1) N^{C\varepsilon} K^C \right) \\ \times \left(\sup_{x \in X} \sum_I \|f_I\|^2_{L^2(W_{P_I(x)},100)} \right)^{\frac{p}{2}-1} \left(\sum_I \|f_I\|^2_{L^2(W_{P(L)},100)} \right). \quad (56)$$

6E. *Proof of Theorem 4.3.* Finally, in this section we show how Theorem 4.4 implies Theorem 4.3. Let $f = \sum_I f_I$. Taking $X = P(L)$ in (32) we see that

$$\|f\|_{L^p(P(L))} \lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1) \left(\sup_{x \in P(L)} \sum_I \|f_I\|^2_{L^2(W_{P_I(x)},100)} \right)^{\frac{1}{2}-\frac{1}{p}} \left(\sum_I \|f_I\|^2_{L^2(W_{P(L)},100)} \right)^{\frac{1}{p}}.$$

To prove Theorem 4.1 we will do dyadic pigeonholing on the L^2-norm of wave packets of f, using Proposition 5.1. More precisely we write

$$f = \sum_I f_I = \sum_I \sum_{P_I} \phi_{P_I} f_I = \sum_{\lambda:\ \text{dyadic}} \sum_{I, P_I : \|\phi_{P_I} f_I\|_{L^2(W_{P_I},100)} \in [\lambda/2, \lambda)} \phi_{P_I} f_I.$$

Without loss of generality we assume $\left(\sum_I \|f_I\|^2_{L^p(W_{P(L)},100)}\right)^{1/2} = 1$. Then

$$\left\| \sum_{I, P_I : \|\phi_{P_I} f_I\|_{L^2(W_{P_I},100)} \notin [N^{-C}\theta^C, N^C \theta^{-C}]} \phi_{P_I} f_I \right\|_{L^p(P(L))} \lesssim 1$$

for sufficiently large C. Therefore there exists a λ such that

$$\|f\|_{L^p(P(L))} \lesssim C_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1) \left\| \sum_{I, P_I : \|\phi_{P_I} f_I\|_{L^2(W_{P_I},100)} \in [\lambda/2, \lambda)} \phi_{P_I} f_I \right\|_{L^p(P(L))} + 1.$$

By a further dyadic pigeonholing argument on I, we may assume, for every I, either

$$\#\{P_I : \|\phi_{P_I} f_I\|_{L^2(W_{P(L)},100)} \in [\lambda/2, \lambda)\} = 0$$

or

$$\#\{P_I : \|\phi_{P_I} f_I\|_{L^2(W_{P(L)},100)} \in [\lambda/2, \lambda)\} \in [A/2, A) \quad \text{for some constant } A.$$

We denote by $\#I$ the number of I such that $\#\{P_I : \|\phi_{P_I} f_I\|_{L^2(W_{P(L)},100)} \in [\lambda/2, \lambda)\} \in [A/2, A)$. For simplicity of notation we will also drop writing the condition $\|\phi_{P_I} f_I\|_{L^2(W_{P(L)},100)} \in [\lambda/2, \lambda)$ in the summation. Now apply Theorem 4.4 to get

$$\left\| \sum_{I, P_I} \phi_{P_I} f_I \right\|_{L^p(P(L))} \lesssim_\varepsilon \log^C(\theta^{-1}+1) N^\varepsilon \left(\sup_{x \in P(L)} \sum_I \left\| \sum_{P_I} \phi_{P_I} f_I \right\|^2_{L^2(W_{P_I(x)},100)} \right)^{\frac{1}{2}-\frac{1}{p}} \\ \times \left(\sum_I \left\| \sum_{P_I} \phi_{P_I} f_I \right\|^2_{L^2(W_{P(L)},100)} \right)^{\frac{1}{p}}. \quad (57)$$

To estimate the first factor on the right-hand side of (57) we note that, for every $x \in P(L)$,

$$\sum_I \Big\| \sum_{P_I} \phi_{P_I} f_I \Big\|^2_{\text{Ł}^2(W_{P_I(x),100})} \lesssim \sum_I \sum_{P_I} \|\phi_{P_I} f_I\|^2_{\text{Ł}^2(W_{P_I(x),100})} \lesssim (\#I)\lambda^2 |P_I|^{-1}$$

because of $\big(\sum_{P_I} \phi_{P_I}(y)\big)^2 \lesssim \sup_{P_I} \phi^2_{P_I}(y) \le \sum_{P_I} \phi^2_{P_I}(y)$ and (12). Therefore

$$\sup_{x \in P(L)} \sum_I \Big\| \sum_{P_I} \phi_{P_I} f_I \Big\|^2_{\text{Ł}^2(W_{P_I(x),100})} \lesssim (\#I)\lambda^2 |P_I|^{-1}.$$

To estimate the second factor on the right-hand side of (57) we calculate

$$\sum_I \Big\| \sum_{P_I} \phi_{P_I} f_I \Big\|^2_{L^2(W_{P(L),100})} \lesssim \sum_I \sum_{P_I} \|\phi_{P_I} f_I\|^2_{L^2(W_{P(L),100})} \lesssim (\#I)\lambda^2 A.$$

To summarize, (57) implies that

$$\Big\| \sum_{I,P_I} \phi_{P_I} f_I \Big\|_{L^p(P(L))} \lesssim_\varepsilon \log^C(\theta^{-1}+1) N^\varepsilon |P_I|^{\frac{1}{p}-\frac{1}{2}} (\#I)^{\frac{1}{2}} A^{\frac{1}{p}} \lambda.$$

Now by Hölder's inequality we have

$$\begin{aligned}
\Big(\sum_I \|f_I\|^2_{L^p(W_{P(L),100})} \Big)^{\frac{1}{2}} &\ge \bigg(\sum_I \Big(\sum_{P_I} \|\phi^{\frac{1}{2}}_{P_I} f_I\|^p_{L^p(W_{P(L),100})} \Big)^{\frac{2}{p}} \bigg)^{\frac{1}{2}} \\
&\gtrsim \bigg(\sum_I \Big(\sum_{P_I} \|\phi_{P_I} f_I\|^p_{L^2(W_{P(L),100})} |P_I|^{1-\frac{p}{2}} \Big)^{\frac{2}{p}} \bigg)^{\frac{1}{2}} \\
&\gtrsim |P_I|^{\frac{1}{p}-\frac{1}{2}} (\#I)^{\frac{1}{2}} A^{\frac{1}{p}} \lambda.
\end{aligned}$$

Hence we have (31).

7. A decoupling inequality for generalized Dirichlet sequences

In this section we focus only on generalized Dirichlet sequences with parameter $\theta = 1$. That is, we say $\{a_n\}_{n=1}^N$ is a generalized Dirichlet sequence if it satisfies (14) with $\theta = 1$. We will present a decoupling inequality for generalized Dirichlet sequences, by combining Theorem 4.1 and the flat decoupling (Proposition 7.2 below). Then we show that for certain choices of the generalized Dirichlet sequences $\{a_n\}_{n=1}^N$ the decoupling inequality that we obtain in this way is sharp (up to $C_\varepsilon N^\varepsilon$).

More precisely, for $1 \le L \le N^{1/2}$, we let Ω' denote the L^2/N^2-neighborhood of $\{a_n\}_{n=1}^N$, and let $\{J\}_{J\in\mathcal{J}}$ be a partition of Ω' into $\Omega' \cap B_{N^{-1/2}}$, where $B_{N^{-1/2}}$ runs over a tiling of $\mathbb{R}$ by balls of radius $N^{-1/2}$. So there are about $N^{1/2}$ many J and each J contains $\mathcal{O}(N^{1/2})$ many consecutive intervals in Ω'. For each J we let $\mathcal{I}_J$ be the partition of J into I, which is a union of L many consecutive intervals in Ω'.

We have the following decoupling inequality for the partition $\Omega' = \bigsqcup_{J\in\mathcal{J}} \bigsqcup_{I\in\mathcal{I}_J} I$.

Theorem 7.1. *For* $2 \le p \le 6$, *we have*

$$\|f\|_{L^p(\mathbb{R})} \lesssim_\varepsilon N^{\frac{1}{4}-\frac{1}{2p}+\varepsilon}\Big(\sum_{J\in\mathcal{J}}\sum_{I\in\mathcal{I}_J}\|f_I\|^2_{L^p(\mathbb{R})}\Big)^{\frac{1}{2}} \tag{58}$$

for every $f:\mathbb{R}\to\mathbb{C}$ *with* supp $\widehat{f}\subset\Omega'$. *There exists a choice of* $\{a_n\}_{n=1}^N$ (*satisfying* (14) *with* $\theta=1$) *such that the above estimate is sharp up to an* N^ε *factor.*

7A. *Proof of* (58). From Theorem 4.1 we have, for every $J\in\mathcal{J}$ and $2\le p\le 6$,

$$\|f_J\|_{L^p(\mathbb{R})} \lesssim_\varepsilon N^{\varepsilon}\Big(\sum_{I\in\mathcal{I}_J}\|f_I\|^2_{L^p(\mathbb{R})}\Big)^{\frac{1}{2}}. \tag{59}$$

Next we decouple f_J into f_I using the flat decoupling:

Proposition 7.2. *Let* $\mathcal{U}$ *denote the partition*

$$[0,M)=\bigsqcup_{m=0}^{M-1}[m,m+1).$$

Then for $p\ge 2$ *we have*

$$\|f\|_{L^p(\mathbb{R})} \lesssim_p M^{\frac{1}{2}-\frac{1}{p}}\Big(\sum_{U\in\mathcal{U}}\|f_U\|^2_{L^p(\mathbb{R})}\Big)^{\frac{1}{2}}$$

for every $f:\mathbb{R}\to\mathbb{C}$ *with* supp $\widehat{f}\subset[0,M)$.

Flat decoupling inequality is well known (see for example Proposition 2.4 in [Demeter et al. 2020]) but we include a proof here for the sake of completeness.

Proof. Fix $p\ge 2$. It suffices to prove that

$$\|f\|_{L^p(B_1)} \lesssim M^{\frac{1}{2}-\frac{1}{p}}\Big(\sum_{U\in\mathcal{U}}\|f_U\|^2_{L^p(W_{B_1,100})}\Big)^{\frac{1}{2}}$$

for f with supp $\widehat{f}\subset[0,M)$. We calculate

$$\begin{aligned}
\|f\|^p_{L^p(B_1)} &\le \|f\|^{p-2}_{L^\infty(B_1)}\|f\|^2_{L^2(B_1)}\\
&\lesssim \Big(\sum_U \|f_U\|_{L^\infty(B_1)}\Big)^{p-2}\Big(\sum_U\|f_U\|^2_{L^2(W_{B_1,100})}\Big)\\
&\lesssim \Big(\sum_U \|f_U\|_{L^p(W_{B_1,100})}\Big)^{p-2}\Big(\sum_U\|f_U\|^2_{L^p(W_{B_1,100})}\Big)\\
&\lesssim M^{\frac{p-2}{2}}\Big(\sum_U \|f_U\|^2_{L^p(W_{B_1,100})}\Big)^{\frac{p-2}{2}}\Big(\sum_U\|f_U\|^2_{L^p(W_{B_1,100})}\Big)\\
&\lesssim M^{\frac{p-2}{2}}\Big(\sum_U \|f_U\|^2_{L^p(W_{B_1,100})}\Big)^{\frac{p}{2}}.
\end{aligned}$$

Here we used the locally constant property similar to Proposition 2.3 and local L^2 orthogonality similar to Lemma 3.6. ☐

Now we prove the decoupling inequality in Theorem 7.1.

Proof of (58) *in Theorem 7.1.* Combining (59) with Proposition 7.2 we obtain

$$\|f\|_{L^p(\mathbb{R})} \lesssim_\varepsilon N^\varepsilon \Big(\sum_{J\in\mathcal{J}} \|f_J\|^2_{L^p(\mathbb{R})}\Big)^{\frac{1}{2}} \lesssim N^{\frac{1}{4}-\frac{1}{2p}+\varepsilon} \Big(\sum_{J\in\mathcal{J}} \sum_{I\in\mathcal{I}_J} \|f_I\|^2_{L^p(\mathbb{R})}\Big)^{\frac{1}{2}}$$

for f with supp $\widehat{f} \subset \Omega'$. □

7B. ***An example and sharpness of* (58).** To prove the sharpness part, we construct a sequence $\{a_n\}_{n=1}^N$ satisfying (14) (with $\theta = 1$) and for which (58) is sharp. We will use the function

$$g(x) = \frac{4x + (N^{\frac{1}{2}} - \sqrt{N-4x})^2}{4N}$$

to define the sequence. For $n = 0, \dots, \frac{N}{8}$, let

$$a_n = g(n).$$

Distinguish the subsequence a_{n_k} where $n_k = kN^{1/2} - k^2$.

Lemma 7.3. *There is an absolute constant $N_0 > 0$ such that for every $N \ge N_0$, the sequence $\{a_n\}_{n=1}^{N/8}$ constructed above satisfies property* (14) (*with* $\theta = 1$). *Furthermore*, *there is an absolute constant $c > 0$ so that*

$$\left\{ \frac{j}{N^{\frac{1}{2}}} : j = 1, \dots, \lfloor cN^{\frac{1}{2}} \rfloor \right\}$$

is a subsequence of $\{a_n\}_{n=1}^{N/8}$.

Proof. First we verify the presence of the subsequence: Let n_k and a_{n_k} be as above. Calculate directly that

$$\begin{aligned} a_{n_k} = g(n_k) &= \frac{4n_k + (N^{\frac{1}{2}} - \sqrt{N-4n_k})^2}{4N} \\ &= \frac{4(kN^{\frac{1}{2}} - k^2) + (N^{\frac{1}{2}} - \sqrt{N - 4(kN^{\frac{1}{2}} - k^2)})^2}{4N} \\ &= \frac{4(kN^{\frac{1}{2}} - k^2) + (N^{\frac{1}{2}} - (N^{\frac{1}{2}} - 2k))^2}{4N} \\ &= \frac{4kN^{\frac{1}{2}} - 4k^2 + 4k^2}{4N} = \frac{k}{N^{\frac{1}{2}}}. \end{aligned}$$

This calculation holds as long as $k \le N^{1/2}/2$. Also note that $n_k = kN^{1/2} - k^2$ is increasing as a function of k as long as $k \le N^{1/2}/2$, so the n_k define a subsequence $a_{n_0}, \dots, a_{n_K}$ where $K = \lfloor N^{1/2}/2 \rfloor$.

To verify property (14), it suffices to check that for N large enough

$$a_1 - a_0 \in \left[\frac{1}{2N}, \frac{2}{N}\right] \tag{60}$$

and that

$$(a_{n+1} - a_n) - (a_n - a_{n-1}) \in \left[\frac{1}{4N^2}, \frac{4}{N^2}\right] \tag{61}$$

whenever $1 \le n \le \frac{N}{8} - 1$, since (60) together with (61) will imply $a_2 - a_1 \in \left[\frac{1}{4N}, \frac{4}{N}\right]$ for N large enough.

First we check (60). Note that $a_0 = 0$ and

$$a_1 = g(1) = \frac{4 + (N^{\frac{1}{2}} - \sqrt{N-4})^2}{4N}.$$

Then

$$a_1 - a_0 = \frac{1}{4N}\left(4 + \frac{16}{(N^{\frac{1}{2}} + \sqrt{N-4})^2}\right) \in \left[\frac{1}{2N}, \frac{2}{N}\right]$$

if N is large enough.

Next we check (61). First calculate

$$\begin{aligned} g(x+1) - g(x) &= \frac{4 + (N^{\frac{1}{2}} - \sqrt{N-4x-4})^2 - (N^{\frac{1}{2}} - \sqrt{N-4x})^2}{4N} \\ &= \frac{4 + 2N^{\frac{1}{2}}(\sqrt{N-4x} - \sqrt{N-4x-4}) - 4}{4N} \\ &= \frac{\sqrt{N-4x} - \sqrt{N-4x-4}}{2N^{\frac{1}{2}}} = \frac{2}{N^{\frac{1}{2}}(\sqrt{N-4x} + \sqrt{N-4x-4})}. \end{aligned}$$

Use this formula to calculate the difference

$$\begin{aligned} &(a_{n+1} - a_n) - (a_n - a_{n-1}) \\ &\quad = \frac{2}{N^{\frac{1}{2}}}\left(\frac{1}{\sqrt{N-4n} + \sqrt{N-4n-4}} - \frac{1}{\sqrt{N-4n+4} + \sqrt{N-4n}}\right) \\ &\quad = \frac{2}{N^{\frac{1}{2}}} \frac{\sqrt{N-4n+4} - \sqrt{N-4n-4}}{(\sqrt{N-4n} + \sqrt{N-4n-4})(\sqrt{N-4n+4} + \sqrt{N-4n})} \\ &\quad = \frac{16}{N^{\frac{1}{2}}(\sqrt{N-4n} + \sqrt{N-4n-4})(\sqrt{N-4n+4} + \sqrt{N-4n})(\sqrt{N-4n+4} + \sqrt{N-4n-4})}. \end{aligned}$$

As long as $n \leq \frac{N}{8}$, and N is sufficiently large, this lies in $\left[\frac{1}{4N}, \frac{4}{N}\right]$ and we are done. □

Now we can finish the sharpness part of Theorem 7.1.

Proof of the sharpness part of Theorem 7.1. For $N \geq N_0$, we take $\{a_n\}_{n=1}^{N/8}$ to be the sequence constructed in Lemma 7.3, extended arbitrarily to $\{a_n\}_{n=1}^{N}$ so that (14) is satisfied with $\theta = 1$. We take $f = \sum_I f_I$ to be the function

$$\phi_{N^2/L^2}(x) \sum_{n=1}^{\lfloor cN^{1/2} \rfloor} e^{ixa_n},$$

where c is the constant in Lemma 7.3, and $\phi_{N^2/L^2}(x)$ is an L^∞-normalized Schwartz function whose Fourier transform is a smooth bump adapted to $B_{L^2/N^2}(0)$. Then we have

$$\|f\|_{L^p(\mathbb{R})} \gtrsim N^{\frac{1}{2}} \left(\frac{N^{\frac{3}{2}}}{L^2}\right)^{\frac{1}{p}}$$

since $|f(x)| \sim N^{1/2}$ on $P^C_{N^{1/2}}(0) \cap B_{CN^2/L^2}(0)$. Since $|f_I| = \phi_{N^2/L^2}$, we have

$$\Big(\sum_{J\in\mathcal{J}}\sum_{I\in\mathcal{I}_J}\|f_I\|^2_{L^p(\mathbb{R})}\Big)^{\frac12} \sim N^{\frac14}\Big(\frac{N^2}{L^2}\Big)^{\frac1p}.$$

Therefore (58) is sharp up to N^ε. □

7C. *Some discussions.* If we take $L = 1$ and $p = 4$ in Theorem 7.1, we get

$$\Big\|\sum_{n=1}^N b_n e^{ia_n x}\Big\|_{L^4(B_{N^2})} \lesssim_\varepsilon N^{\frac12+\frac18+\varepsilon}\|b_n\|_{\ell^2}. \tag{62}$$

On the other hand, for the Dirichlet polynomial we have, by unique factorization in $\mathbb{Z}$ and local L^2 orthogonality, that

$$\Big\|\sum_{n=N+1}^{2N} b_n e^{ix\log n}\Big\|_{L^4(B_{N^2})} = \Big\|\sum_{m=N+1}^{2N}\sum_{n=N+1}^{2N} b_m b_n e^{ix\log(nm)}\Big\|^{\frac12}_{L^2(B_{N^2})} \lesssim_\varepsilon N^{\frac12+\varepsilon}\|b_n\|_{\ell^2}. \tag{63}$$

Comparing (62) with (63) we see that while we can construct a generalized Dirichlet sequence that contains an AP with about $N^{1/2}$ many terms and common difference $N^{-1/2}$ so that (62) is sharp for that sequence, the Dirichlet sequence $\{\log n\}_{n=N+1}^{2N}$ does not contain such an (N^{-2}-approximate) AP and therefore allows a better estimate (63).

However we notice that the example $D_0(x) = \sum_{j=1}^{cN^{1/2}} e^{ixj/N^{1/2}}$ does not exclude the possibility that Montgomery's conjecture may hold for generalized Dirichlet polynomials. By Montgomery's conjecture for generalized Dirichlet polynomials we mean, for every $\varepsilon > 0$,

$$\Big\|\sum_{n=1}^N b_n e^{ixa_n}\Big\|_{L^p(B_T)} \lesssim_\varepsilon T^\varepsilon N^{\frac12}(N^{\frac p2}+T)^{\frac1p}\|b_n\|_{\ell^\infty} \tag{64}$$

for every generalized Dirichlet sequence $\{a_n\}_{n=1}^N$ with $\theta = 1$. Indeed we know $|D_0(x)| \gtrsim N^{1/2}$ on $P^C_{N^{1/2}}(0)$, so

$$\|D_0\|_{L^p(B_T)} \gtrsim T^{\frac1p}N^{\frac12-\frac1{2p}}.$$

On the right-hand side of (7) we have $C_\varepsilon T^\varepsilon N^{1/2}(N^{p/2}+T)^{1/p} \geq N^{1/2}T^{1/p}$. So there is no contradiction to (64). Note that if we apply Hölder's inequality $\|b_n\|_{\ell^2} \leq N^{1/2}\|b_n\|_{\ell^\infty}$ to (63) then we obtain

$$\Big\|\sum_{n=N+1}^{2N} b_n e^{ix\log n}\Big\|_{L^4(B_{N^2})} \lesssim_\varepsilon N^{1+\varepsilon}\|b_n\|_{\ell^\infty},$$

which is exactly (7) with $p = 4$, $T = N^2$. However although we know (62) is sharp (up to $C_\varepsilon N^\varepsilon$) for our example $D_0(x)$, the Hölder step $\|b_n\|_{\ell^2} \leq N^{1/2}\|b_n\|_{\ell^\infty}$ is not sharp because $D_0(x)$ has only $N^{1/2}$ many nonzero coefficients.

On the other hand we may construct a periodic generalized Dirichlet polynomial

$$f=\sum_{n=1}^{N} e^{it\frac{(N+n)}{N^2}},$$

which contradicts (64) for $p>4,\ T>N^{2+\varepsilon_0}$ with any $\varepsilon>0$. We notice that $|f|\gtrsim N$ on $\mathcal{N}_C(N^2\mathbb{Z})$. So

$$\|f\|_{L^p(B_T)}\gtrsim N\left(\frac{T}{N^2}\right)^{\frac{1}{p}}=N^{1-\frac{2}{p}}T^{\frac{1}{p}}.$$

Under the condition $p>4$ we have

$$N^{1-\frac{2}{p}}T^{\frac{1}{p}}\gtrsim_{\varepsilon_0} N^{\varepsilon_1}N^{\frac{1}{2}}T^{\frac{1}{p}}$$

for some $\varepsilon_1>0$ depending on p. Under the condition $T>N^{2+\varepsilon_0}$ we have

$$N^{1-\frac{2}{p}}T^{\frac{1}{p}}>N^{\varepsilon_2}N$$

for some $\varepsilon_2>0$ depending on p. Therefore when $p>4$ and $T>N^{2+\varepsilon_0}$ with any $\varepsilon_0>0$, (64) fails for the generalized Dirichlet polynomial f.

At the end of this section we discuss briefly what makes $N^{1/2}$ special. Suppose we consider the sequence $\{a_n\}_{n=1}^{N^\alpha}$ for some $\alpha\in\left(\frac{1}{2},1\right]$, and $\{a_n\}_{n=1}^{N}$ is a generalized Dirichlet sequence with $\theta=1$. For simplicity we will omit constants C in the following discussion. Still we look at (L^2/N^2)-neighborhood of $\{a_n\}_{n=1}^{N^\alpha}$ with $L\geq 1$. For $L\geq N^{1/2}$, the (L^2/N^2)-neighborhood is essentially the same as the $(1/N)$-neighborhood (as long as $L\leq N$), which is an interval of length about 1. So the induction scheme in this paper fails to work for $L\geq N^{1/2}$.

Another difficulty is about the “bush” structure of $\bigcup_I(I-I)$ in the frequency space. To illustrate this, we let $L\leq N^{1/2}$, and define I, P_I as before, that is, I is the (L^2/N^2)-neighborhood of an L-segment $\{a_n\}_{n=(j-1)L+1}^{jL}$ of the sequence $\{a_n\}_{n=1}^{N^\alpha}$, and P_I denotes a fat AP of the form $P_{v_I^{-1}}^{CN/L}\cap B_{CN^2/L^2}$, where $v_I=a_{(j-1)L+2}-a_{(j-1)L+1}$ (see (15)). So now there are N^α/L many I, $v_I\sim 1/N$ are L/N^2 separated, and the maximal separation of v_I is $1/N^{2-\alpha}$. For $\alpha>\frac{1}{2}$, we no longer have an essentially linear decaying pattern of the bush $\bigcup_I(I-I)$ if $L\geq N^{1-\alpha}$, which is exploited in the proof of Lemma 5.3. To be precise, we consider the function $\sum_I 1_{I-I}(t)$, which counts the number of overlap of the sets $I-I$ at t. If $\alpha\leq\frac{1}{2}$ then we can verify that

$$\left|\sum_I 1_{I-I}(t)\right|\lesssim\frac{N/L}{|t|}\quad\text{when }\frac{1}{N}\lesssim|x|\lesssim\frac{L}{N}. \tag{65}$$

See Figure 2 for a rough graph of the function $\sum_I 1_{I-I}(t)$. However if $\alpha>\frac{1}{2}$ then we no longer have (65). This is because $\frac{1}{2}$ is the largest value for α such that for every $L\leq N^{1/2}$, the k-th intervals in all $I-I$ are within about N^{-1} distance from each other for every $1\leq k\leq L$. For comparison, we note that for $R^{-1/2}\times R^{-1}$ caps θ that tile the R^{-1}-neighborhood of the truncated parabola, the bush $\{\theta-\theta\}$ has a

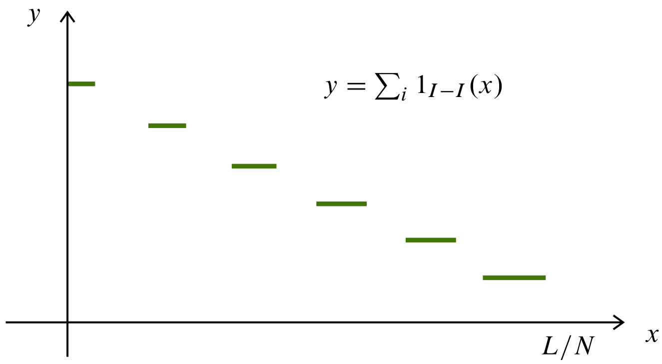

Figure 2. The overlap number of the $I-I$ has a linear decay pattern provided $L/N^{2-\alpha} \lesssim N^{-1}$. This condition is guaranteed as long as $\alpha \le \frac{1}{2}$. Controlling the overlap number of the $I-I$ outside of a certain neighborhood of the origin is a central step in Lemma 5.3.

similar linear decay pattern:

$$\left|\sum_{\theta} 1_{\theta-\theta}(x)\right| \lesssim \frac{R^{-\frac{1}{2}}}{|x|} \quad \text{when } R^{-1} \lesssim |x| \lesssim R^{-\frac{1}{2}}.$$

On the physical side, how P_I interact also becomes more complicated when $\alpha > 2$. One important property we used in the $\alpha = \frac{1}{2}$ case is that the maximal separation of v_I^{-1} (which is about $N^{1/2}$) is less than the thickness of P_I (which is about N/L) for every $1 \le L \le N^{1/2}$. However for $\alpha > \frac{1}{2}$, the maximal separation is about $N^{1-\alpha}$ which is greater than the thickness N/L for $L \ge N^{1-\alpha}$. In particular this makes the pattern of the intersection $P_I \cap P_J$ more complicated and the notion of transversal less clear.

8. Small-cap-type decoupling

In this section we prove Theorem 1.3, which is about small-cap-type decoupling inequalities in the spirit of [Demeter et al. 2020].

First we restate Theorem 1.3 but with the more general definition of generalized Dirichlet sequence. Let $\{a_n\}_{n=1}^{N^{1/2}}$ be a short generalized Dirichlet sequence with parameter $\theta \in (0,1]$ as defined in Definition 3.1. Let L, L_1 be two integers such that $1 \le L_1 \le L \le N^{1/2}$. Denote by Ω the $\theta L^2/N^2$-neighborhood of $\{a_n\}_{n=1}^{N^{1/2}}$. We let $\{J\}_{J\in\mathcal{J}} = \{J_k\}_{k=0}^{\lfloor N^{1/2}/L_1\rfloor}$ be the partition of Ω into unions of L_1 many consecutive intervals, that is,

$$J_k = \bigcup_{i=1}^{L_1} B_{\theta L^2/N^2}(a_{kL_1+i}).$$

Let $\{I\}_{I\in\mathcal{I}}$ be the partition of Ω into unions of L many consecutive intervals, which we called the canonical partition.

A more general version of Theorem 1.3 is the following, which we prove in the rest of this section.

Theorem 8.1. *Let $\{J\}_{J\in\mathcal{J}}$ be defined as in the above paragraph. Suppose $p \geq 4$, $\frac{1}{q}+\frac{3}{p}\leq 1$. If either of the two conditions*

(a) $L_1 = 1$,

(b) $p = q$,

is satisfied, then, for every $\varepsilon > 0$,

$$\Big\|\sum_{J\in\mathcal{J}} f_J\Big\|_{L^p(\mathbb{R})} \lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1)\bigg(\frac{N^{\frac{1}{2}-\frac{1}{2q}-\frac{3}{2p}}L^{\frac{2}{p}}}{L_1^{1-\frac{1}{p}-\frac{1}{q}}}+\Big(\frac{N^{\frac{1}{2}}}{L_1}\Big)^{\frac{1}{2}-\frac{1}{q}}\bigg)\Big(\sum_{J\in\mathcal{J}}\|f_J\|^q_{L^p(\mathbb{R})}\Big)^{\frac{1}{q}} \tag{66}$$

for all functions $f_J:\mathbb{R}\to\mathbb{C}$ with $\operatorname{supp}\widehat{f_J}\subset J$.

As a corollary we have a more general version of Corollary 1.4.

Corollary 8.2. *Let $\{a_n\}_{n=1}^{N^{1/2}}$ be a short generalized Dirichlet sequence with parameter $\theta\in(0,1]$. Suppose $p\geq 4$, $\frac{1}{q}+\frac{3}{p}\leq 1$, and $N\theta^{-1}\leq T\leq N^2\theta^{-1}$. We have, for every $\varepsilon>0$,*

$$\Big\|\sum_{n=1}^{N^{1/2}} b_n e^{ita_n}\Big\|_{L^p(B_T)} \lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1)(N^{\frac{1}{2}(1+\frac{1}{p}-\frac{1}{q})}\theta^{-\frac{1}{p}}+T^{\frac{1}{p}}N^{\frac{1}{4}-\frac{1}{2q}})\|b_n\|_{\ell^q} \tag{67}$$

for every B_T and every $\{b_n\}_{n=1}^{N^{1/2}}\subset\mathbb{C}$,

To prove results of the form (66), we may use the small cap decoupling method for $\mathbb{P}^1$ developed in [Demeter et al. 2020], which is based on refined decoupling for the canonical partition, refined flat decoupling and an incidence estimate for tubes with spacing conditions. We have three analogous results in the short generalized Dirichlet sequence setting. Theorem 4.4 is the analogy of the refined canonical cap decoupling for $\mathbb{P}^1$. Now we state and prove the other two.

8A. ***An incidence estimate for fat APs.*** We start with the incidence estimate. First we introduce some notation. Suppose P, P' are fat APs such that $P = P_I(y)$ and $P' = P_{I'}(y')$ for some $I, I'\in\mathcal{I}$. We say P, P' are parallel if $I = I'$. For a collection $\mathcal{P}=\{P\}$ of fat APs, we say $x\in\mathbb{R}$ is an r-rich point if r many P contain it.

Proposition 8.3. *Let $1\leq L_1\leq L\leq N^{1/2}$ and let $\{J\}_{J\in\mathcal{J}}$, $\{I\}_{I\in\mathcal{I}}$ be defined as in the beginning of Section 8. Suppose we have a collection of fat AP $\mathcal{P}=\{P\}$ inside a fixed $P(L)$, where each $P=P_I$ for some $I\in\mathcal{I}$. Assume for every $J\in\mathcal{J}$ and every $P_J\subset P(L)$, P_J contains either M or 0 parallel $P\in\mathcal{P}$. Denote by Q_r the set of r-rich points of $\mathcal{P}$. Suppose $Q_r\neq\varnothing$. Then one of the two cases below happens*:

(1) *There exists a dyadic $s\in[1,\min\{L, N^{1/2}/L\}]$ and $M_s\in\mathbb{N}$ such that*

$$|Q_r| \lessapprox \frac{M_s}{sr^2}(\#P)|P|, \tag{68}$$

$$r \lessapprox \frac{M_s N^{1/2}}{s^2 L}, \tag{69}$$

$$M_s \lessapprox sM\max\Big\{1, s\frac{L_1}{L}\Big\}. \tag{70}$$

(2) *We have*

$$|Q_r| \le |P(L)|, \tag{71}$$

$$r \lessapprox (\#P)\frac{|P|}{|P(L)|}. \tag{72}$$

Here $\#P$ denotes the cardinality of $\mathcal{P}$.

Proof. For each dyadic $1 \le s \le \min\{L, N^{1/2}/L\}$, we let η_s denote a smooth bump with height 1 adapted to the annulus $|\xi| \sim (L/s)v$ in the frequency space, and let η_0 denote a smooth bump with height 1 adapted to $P_{v_1}^{C\theta L^2/N^2}(0) \cap B_{CL^2/N^{3/2}}(0)$ (which degenerates to $B_{C\theta L^2/N^2}(0)$ when $L \le N^{1/4}$) such that

$$\eta_0 + \sum_{\substack{1 \le s \le \min\{L, N^{1/2}/L\},\\ s:\ \text{dyadic}}} \eta_s = 1 \quad \text{on } \bigcup_I (I - I).$$

For each $P \in \mathcal{P}$ we let $v_P(x)$ be a positive smooth function (with height 1) adapted to P in the physical space with frequency support in $C(I-I)$, where $P = P_I$. If we define $g = \sum_P v_P$, then we can write

$$g = g * \check{\eta}_0 + \sum_{1 \le s \le \min\{L, N^{1/2}/L\}} g * \check{\eta}_s.$$

Fix $s \in [1, \min\{L, N^{1/2}/L\}]$. There exists a collection of fat APs $\mathcal{I}_s$ consisting of $I_s = P_{v_{I_s}}^{C\theta s L^2/N^2}(0) \cap B_{CL/N}(0)$ with the properties that $v_{I_s} \sim N^{-1}$ and v_{I_s} are $\sim s\theta L/N^2$ separated such that for every $I \in \mathcal{I}$, $I - I$ is contained in one and only one $I_s \in \mathcal{I}_s$. In fact we may let $v_{I_s} = v_I$ for any I with $(I-I) \subset I_s$. The cardinality of $\mathcal{I}_s$ is $N^{1/2}/(sL)$. For $I_s \in \mathcal{I}_s$ we let $\mathcal{P}_{I_s}$ be the tiling of $\mathbb{R}$ by fat APs of the form $P_{v_{I_s}^{-1}}^{\theta C s N/L} \cap B_{CN^2/(L^2\theta)}$. For every $P = P_I \in \mathcal{P}$ there exists a unique $I_s \in \mathcal{I}_s$ and $P_s \in \mathcal{P}_{I_s}$ such that $I - I \subset I_s$ and $P \subset P_s$. For every $1 \le M \lessapprox s^2$, we define $\mathcal{P}_{s,M}$ be the subcollection of $\mathcal{P}$ consisting of P such that P_s contains $\sim M$ many $P' \in \mathcal{P}$. For $1 \le s \le \min\{L, N^{1/2}/L\}$ let

$$g_{s,M} = \sum_{P \in \mathcal{P}_{s,M}} v_P * \check{\eta}_s.$$

By the pigeonhole principle, for every $x \in Q_r$ there either exist an s and M_s such that $g(x) \lessapprox |g_{s,M_s}(x)|$ or $g(x) \lessapprox |g_0(x)|$. Again by the pigeonhole principle either we can find s, M_s such that, for x in a subset E of Q_r with measure $\gtrapprox |Q_r|$,

$$g(x) \lessapprox |g_{s,M_s}(x)|$$

or, for x in a subset E of Q_r with measure $\gtrapprox |Q_r|$,

$$g(x) \lessapprox |g_0(x)|.$$

We consider these two cases separately.

Case 1: Suppose $g(x) \lessapprox |g_{s,M_s}(x)|$ for x in a subset E of Q_r with measure $\gtrapprox |Q_r|$. We write

$$g_{s,M_s} = \sum_{I_s} \sum_{P_{I_s}} \sum_{P \subset P_{I_s},\, P \in \mathcal{P}_{s,M_s}} v_P * \check{\eta}_s =: \sum_{I_s} \sum_{P_{I_s}} g_{P_{I_s}}.$$

Here the sum over P_{I_s} is over $P_{I_s} \in \mathcal{P}_{I_s}$ such that $g_{P_{I_s}}$ is nonzero.

We note that $\sum_{P_{I_s}} g_{P_{I_s}}$ with I_s varying are almost orthogonal (meaning that the Fourier support of them has $\mathcal{O}(1)$-overlap). This is because $\operatorname{supp} \widehat{g}_{P_{I_s}} \subset \big(\bigcup_{I \subset I_s}(I - I)\big) \cap \{\xi : |\xi| \sim \frac{Lv}{s}\}$, and for every distinct $I_s, I_s' \in \mathcal{I}_s$, and every $I, I' \in \mathcal{I}$ with $I \subset I_s$, $I' \subset I_s'$, the distance $d_{I,I'}$ between the $\frac{L}{s}$-th terms in I and I' satisfies

$$\frac{\theta L^2}{N^2} = \frac{s\theta L}{N^2}\frac{L}{s} \lesssim d_{I,I'} \lesssim \frac{N^{\frac{1}{2}}\theta}{N^2}\frac{L}{s} \lesssim \frac{1}{N}.$$

Therefore $\operatorname{supp} \widehat{\sum_{P_{I_s}} g_{P_{I_s}}}$ are $\mathcal{O}(1)$-overlapping.

Hence

$$|Q_r| r^2 \lessapprox \int_E g^2 \lessapprox \int_{\mathbb{R}} |g_{s,M_s}|^2 \lesssim \sum_{I_s} \int_{\mathbb{R}} \Big|\sum_{P_{I_s}} g_{P_{I_s}}\Big|^2.$$

We note that for $P \subset P_{I_s}$,

$$|v_P * \check{\eta}_s| \lesssim \frac{1}{s} W_{P_{I_s},100},$$

so

$$\int_{\mathbb{R}} \Big|\sum_{P_{I_s}} g_{P_{I_s}}\Big|^2 \lesssim \int_{\mathbb{R}} \Big(\sum_{P_{I_s}} \sum_{P \subset P_{I_s},\, P \in \mathcal{P}_{s,M_s}} \frac{1}{s} W_{P_{I_s},100}\Big)^2 \lesssim \sum_{P_{I_s}} \frac{M_s^2}{s^2} |P_{I_s}|.$$

Hence

$$|Q_r| r^2 \lessapprox \sum_{I_s} \sum_{P_{I_s}} |P_{I_s}| \Big(\frac{M_s}{s}\Big)^2.$$

Since $|P_{I_s}|/s \sim |P|$ and $\sum_{I_s} \sum_{P_{I_s}} M_s \le (\#P)$, we obtain

$$r^2 |Q_r| \lessapprox (\#P)|P| \frac{M_s}{s},$$

which is (68).

Now we show (69). We choose $x \in E$. Then

$$r \lesssim g(x) \lessapprox |g_{s,M_s}(x)| \le \sum_{I_s} \sum_{P_{I_s}} |g_{P_s}(x)| \lesssim |\mathcal{I}_s| \frac{M_s}{s} \lesssim \frac{N^{\frac{1}{2}}}{sL} \frac{M_s}{s}.$$

Finally we prove (70). When $s \le L/L_1$, every P_{I_s} is contained in a single P_J and therefore can contain $\lesssim M$ parallel $P \in \mathcal{P}$. For every P_{I_s}, there are $\lesssim s$ many $I \in \mathcal{I}$ such that there could exist P_I such that $P_I \subset P_{I_s}$, so we conclude P_{I_s} contain $\lesssim sM$ many $P \in \mathcal{P}$. When $s \ge L/L_1$, every P_{I_s} is contained in at most sL_1/L many P_J and therefore can contain $\lesssim sMsL_1/L$ many $P \in \mathcal{P}$. Hence we obtain (70).

Case 2: Suppose $g(x) \lessapprox |g_0(x)|$ for x in a subset of Q_r with measure $\gtrapprox |Q_r|$. Inequality (71) is trivial since $Q_r \subset P(L)$. To show (72) we choose $x \in E$. Then

$$r \lesssim g(x) \lessapprox |g_0(x)| \lesssim (\#P) \frac{|P|}{|P(L)|},$$

where the last inequality is because

$$|g_0(x)| = |g * \check{\eta}_0(x)| \le \|g\|_{L^1} \|\check{\eta}_0\|_{L^\infty} \lesssim (\#P)|P| \frac{1}{|P(L)|} = (\#P) \frac{|P|}{|P(L)|}. \qquad \square$$

8B. ***Refined flat decoupling for fat APs.*** Next we have the following refined flat decoupling inequality for fat APs.

Proposition 8.4. *Suppose $2 \le q \le p$, and let $\{J\}_{J\in\mathcal{J}}$, $\{I\}_{I\in\mathcal{I}}$ be defined as in the beginning of Section 8. Fix $I \in \mathcal{I}$. Write $f_I = \sum_{P_I\in\mathcal{P}_I} f_{I,P_I}$ for the wave packet decomposition of f_I. Suppose that $\widetilde{P}_I \subset \mathcal{P}_I$ is a collection of P_I for which f_{I,P_I} are nonzero, $\|f_{I,P_I}\|_{L^\infty(\mathbb{R})}$ are roughly constant, and for every $J \subset I$, and every P_J (in a tiling of $\mathbb{R}$), P_J contains either $\sim M$ or 0 wave packets f_{I,P_I} (in the sense that $P_I \subset P_J$). Then*

$$\Big\| \sum_{P_I\in\widetilde{\mathcal{P}}_I} f_{I,P_I} \Big\|_{L^p(\mathbb{R})} \lesssim M^{\frac{1}{p}-\frac{1}{2}} \Big(\frac{L}{L_1}\Big)^{1-\frac{1}{p}-\frac{1}{q}} \Big(\sum_{J\subset I} \|f_J\|^q_{L^p(\mathbb{R})} \Big)^{\frac{1}{q}}. \tag{73}$$

Proof. Fix a P_J that contains $\sim M$ many wave packets f_{I,P_I}. We first show

$$\Big\| \sum_{P_I\in\widetilde{\mathcal{P}}_I} f_{I,P_I} \Big\|_{L^p(P_J)} \lesssim M^{\frac{1}{p}-\frac{1}{2}} \Big(\frac{L}{L_1}\Big)^{1-\frac{1}{p}-\frac{1}{q}} \Big(\sum_{J\subset I} \|f_J\|^q_{L^p(W_{P_J,100})} \Big)^{\frac{1}{q}}. \tag{74}$$

Assume $\|f_{I,P_I}\|_{L^\infty(\mathbb{R})} \sim H$ for every nonzero f_{I,P_I}, $P_I \in \widetilde{\mathcal{P}}_I$. By assumption we have

$$\Big\| \sum_{P_I\in\widetilde{\mathcal{P}}_I} f_{I,P_I} \Big\|_{L^p(P_J)} \lesssim H(M|P_I|)^{\frac{1}{p}}.$$

On the other hand by local L^2 orthogonality we have

$$H(M|P_I|)^{\frac{1}{2}} \lesssim \Big\| \sum_{P_I\in\widetilde{\mathcal{P}}_I} f_{I,P_I} \Big\|_{L^2(P_J)} \le \|f_I\|_{L^2(P_J)} \lesssim \Big(\sum_{J\subset I} \|f_J\|^2_{L^2(W_{P_J,100})} \Big)^{\frac{1}{2}}$$

(where we used that $\big|\sum_{P_I\in\widetilde{\mathcal{P}}_I} f_{I,P_I}\big| \le |f_I|$), and by Hölder's inequality the right-hand side is bounded by

$$\Big(\frac{L}{L_1}\Big)^{\frac{1}{2}-\frac{1}{q}} |P_J|^{\frac{1}{2}-\frac{1}{p}} \Big(\sum_{J\subset I} \|f_J\|^q_{L^p(W_{P_J,100})} \Big)^{\frac{1}{q}}.$$

Noting that $|P_I|/|P_J| = L_1/L$, we conclude

$$\begin{aligned}\Big\| \sum_{P_I\in\widetilde{\mathcal{P}}_I} f_{I,P_I} \Big\|_{L^2(P_J)} &\lesssim H(M|P_I|)^{\frac{1}{2}} (M|P_I|)^{\frac{1}{p}-\frac{1}{2}} \\ &\lesssim M^{\frac{1}{p}-\frac{1}{2}} \Big(\frac{L}{L_1}\Big)^{1-\frac{1}{p}-\frac{1}{q}} \Big(\sum_{J\subset I} \|f_J\|^q_{L^p(W_{P_J,100})} \Big)^{\frac{1}{q}}.\end{aligned}$$

So (74) holds. Since $q \le p$, (73) follows from (74) by raising (74) to the p-th power, summing over P_J in a tiling of $\mathbb{R}$, and applying Minkowski's inequality (see Proposition 4.2). □

8C. ***Proof of Theorem 8.1.*** Now we are ready to prove Theorem 8.1. We first show a bilinear version of Theorem 8.1 and then conclude Theorem 8.1 by a broad-narrow argument. Still let $\{J\}_{J\in\mathcal{J}}$ be defined as in the beginning of Section 8. We say two subcollections of $\mathcal{J}$, $\mathcal{J}_1$ and $\mathcal{J}_2$, are transversal if $d(J_1, J_2) \gtrsim N^{-1/2}$ for every $J_1 \in \mathcal{J}_1$, $J_2 \in \mathcal{J}_2$.

Theorem 8.5. *Suppose* $4 \le q \le p \le 6$, $\frac{1}{q} + \frac{3}{p} \le 1$. *If either of the two conditions*

(a) $L_1^{1/2-1/q} \le L^{1-3/p-1/q}$,

(b) $p = q$,

is satisfied, then, for every $\varepsilon > 0$,

$$\Big\| \prod_{i\in\{1,2\}} \Big|\sum_{J\in\mathcal{J}_i} f_J\Big|^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R})} \lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1)\bigg(\frac{N^{\frac{1}{2}-\frac{1}{2q}-\frac{3}{2p}} L^{\frac{2}{p}}}{L_1^{1-\frac{1}{p}-\frac{1}{q}}} + \Big(\frac{N^{\frac{1}{2}}}{L_1}\Big)^{\frac{1}{2}-\frac{1}{q}}\bigg) \prod_{i\in\{1,2\}} \Big(\sum_{J\in\mathcal{J}_i} \|f_J\|^q_{L^p(\mathbb{R})}\Big)^{\frac{1}{2q}} \tag{75}$$

for all transversal subcollections $\mathcal{J}_1, \mathcal{J}_2$ *of* $\mathcal{J}$, *and all functions* $f_J : \mathbb{R} \to \mathbb{C}$ *with* $\operatorname{supp} \widehat{f_J} \subset J$.

Proof. By a local-to-global argument similar to Proposition 4.2, to show (75), it suffices to show, for a sufficiently large k and for every ball $B_{N^2/(\theta L^2)}$,

$$\Big\| \prod_{i\in\{1,2\}} \Big|\sum_{J\in\mathcal{J}_i} f_J\Big|^{\frac{1}{2}} \Big\|_{L^p(B_{N^2/(\theta L^2)})} \lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1)\bigg(\frac{N^{\frac{1}{2}-\frac{1}{2q}-\frac{3}{2p}} L^{\frac{2}{p}}}{L_1^{1-\frac{1}{p}-\frac{1}{q}}} + \Big(\frac{N^{\frac{1}{2}}}{L_1}\Big)^{\frac{1}{2}-\frac{1}{q}}\bigg) \prod_{i\in\{1,2\}} \Big(\sum_{J\in\mathcal{J}_i} \|f_J\|^q_{L^p(W_{B_{N^2/(\theta L^2)},k})}\Big)^{\frac{1}{2q}}. \tag{76}$$

We will assume that f_J has been replaced by $f_J\psi_{B_{N^2/(\theta L^2)}}$, where $\psi_{B_{N^2/(\theta L^2)}}$ is a Schwartz function satisfying $|\psi_{B_{N^2/(\theta L^2)}}| \sim 1$ on $B_{N^2/(\theta L^2)}$, $\psi_{B_{N^2/(\theta L^2)}}$ decays rapidly away from $B_{N^2/(\theta L^2)}$, and $\operatorname{supp} \widehat{\psi}_{B_{N^2/(\theta L^2)}} \subset (-\theta L^2/N^2, \theta L^2/N^2)$. Then $f_J\psi_{B_{N^2/(\theta L^2)}}$ has Fourier support which is contained in a $(\theta L^2/N^2)$-neighborhood of J. The arguments which follow apply equally well to the $\theta L^2/N^2$ neighborhoods of J (which are contained in $2J$) as they do to J. Note also that $\|f_J\psi_{B_{N^2/(\theta L^2)}}\|_{L^p(\mathbb{R})} \lesssim_k \|f_J\|_{L^p(W_{B_{N^2/(\theta L^2)},k})}$, so abusing notation by letting f_J mean $f_J\psi_{B_{N^2/(\theta L^2)}}$ from here on in the proof, the inequality

$$\Big\| \prod_{i\in\{1,2\}} \Big|\sum_{J\in\mathcal{J}_i} f_J\Big|^{\frac{1}{2}} \Big\|_{L^p(B_{N^2/(\theta L^2)})} \lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1)\bigg(\frac{N^{\frac{1}{2}-\frac{1}{2q}-\frac{3}{2p}} L^{\frac{2}{p}}}{L_1^{1-\frac{1}{p}-\frac{1}{q}}} + \Big(\frac{N^{\frac{1}{2}}}{L_1}\Big)^{\frac{1}{2}-\frac{1}{q}}\bigg) \prod_{i\in\{1,2\}} \Big(\sum_{J\in\mathcal{J}_i} \|f_J\|^q_{L^p(\mathbb{R})}\Big)^{\frac{1}{2q}} \tag{77}$$

implies (76). Now we fix a $B_{N^2/\theta L^2}$ and prove (77). Write $F_1 = \sum_{J\in\mathcal{J}_1} f_J$ and $F_2 = \sum_{J\in\mathcal{J}_2} f_J$. For $i \in \{1,2\}$ we write $F_i = \sum_{P\in\mathcal{P}_i} F_{i,P}$ for the wave packet decomposition with respect to $\{I\}_{I\in\mathcal{I}}$. So

$$F_i = \sum_{I\in\mathcal{I}} F_{i,I} = \sum_{I\in\mathcal{I}} \sum_{P_I} F_{i,I,P_I} =: \sum_{P\in\mathcal{P}_i} F_{i,P}.$$

Write $\mathcal{I}_1 = \{I \in \mathcal{I} : I \subset \cup_{J\in\mathcal{J}_1} J\}$ and $\mathcal{I}_2 = \{I \in \mathcal{I} : I \subset \cup_{J\in\mathcal{J}_2} J\}$. Let $F = F_1 + F_2$. By a dyadic pigeonholing argument and rescaling which we detail in Proposition 8.6 directly following this proof, we may assume that, for every nonzero $F_{i,P}$, $\|F_{i,P}\|_{L^\infty} \sim 1$. We assume $\mathcal{P}_i$ contains only nonzero $F_{i,P}$.

By a further dyadic pigeonholing argument we may assume that for every P_J (in a tiling of $\mathbb{R}$), P_J either contains M_i or 0 many wave packets F_{i,I,P_I}, where $J \subset I$, for $i \in \{1,2\}$. Lastly, by one more dyadic pigeonholing argument we may assume that, for each $i \in \{1,2\}$, $\|F_I\|_{L^p(\mathbb{R})}$ are comparable for nonzero F_I with $I \in \mathcal{I}_i$. For dyadic $1 \le r_1, r_2 \le N^{1/2}/L$ we let Q_{r_1,r_2} denote the collection of $P(L')$ (in the tiling of $P(L)$) that intersect $\sim r_1$ many $P \in \mathcal{P}_1$, and $\sim r_2$ many $P \in \mathcal{P}_2$. Recall that $L' = (N^{1/2}L)^{1/2}$. From the refined decoupling inequality (Theorem 4.4) we have

$$\|(F_1F_2)^{\frac12}\|_{L^6(Q_{r_1,r_2})} \le \|F_1\|^{\frac12}_{L^6(Q_{r_1,r_2})}\|F_2\|^{\frac12}_{L^6(Q_{r_1,r_2})} \lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1) r_1^{\frac16} r_2^{\frac16} \prod_{i\in\{1,2\}}\Bigl(\sum_{I\in\mathcal{I}_i}\int |F_I|^2\Bigr)^{\frac{1}{12}}.$$

On the other hand from bilinear restriction (Proposition 3.5) we have for every $P(L') \subset Q_{r_1,r_2}$

$$\|(F_1F_2)^{\frac12}\|_{L^4(P(L'))} \lesssim_\varepsilon N^\varepsilon r_1^{\frac14} r_2^{\frac14} |P(L')|^{\frac14}$$

and thus

$$\|(F_1F_2)^{\frac12}\|_{L^4(Q_{r_1,r_2})} \lesssim_\varepsilon N^\varepsilon r_1^{\frac14} r_2^{\frac14} |Q_{r_1,r_2}|^{\frac14}.$$

Therefore by the interpolation inequality we obtain

$$\|(F_1F_2)^{\frac12}\|_{L^p(Q_{r_1,r_2})} \lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1) r_1^{\frac1p} r_2^{\frac1p} |Q_{r_1,r_2}|^{\frac3p-\frac12} \prod_{i\in\{1,2\}}\Bigl(\sum_{I\in\mathcal{I}_i}\|F_I\|^2_{L^2}\Bigr)^{\frac14-\frac1p}. \tag{78}$$

We assumed each nonzero wave packet $F_{i,P}$ satisfies $\|F_{i,P}\|_{L^\infty} \sim 1$, so

$$\sum_{I\in\mathcal{I}_i}\|F_I\|^2_{L^2} \sim (\#P_i)|P| \sim \sum_{I\in\mathcal{I}_i}\|F_I\|^p_{L^p},$$

where $\#P_i$ denotes the total number of nonzero wave packets in F_i, that is, $|\mathcal{P}_i|$. Hence we may rewrite (78) as

$$\begin{aligned}&\|(F_1F_2)^{\frac12}\|_{L^p(Q_{r_1,r_2})}\\ &\quad\lesssim_\varepsilon N^\varepsilon \log^C(\theta^{-1}+1)|Q_{r_1,r_2}|^{\frac3p-\frac12}\prod_{i\in\{1,2\}}\biggl(r_i^{\frac2p}\Bigl(\sum_{I\in\mathcal{I}_i}\|F_I\|^q_{L^p}\Bigr)^{\frac1q}((\#P_i)|P|)^{\frac12-\frac3p}(\#I_i)^{\frac1p-\frac1q}\biggr)^{\frac12},\end{aligned}$$

where $\#I_i$ denotes the total number of $I \in \mathcal{I}_i$ such that F_I is nonzero. By Proposition 8.4 we have (note that in (73) the left-hand side involves pigeonholed wave packets while the right-hand side includes all wave packets)

$$\sum_{I\in\mathcal{I}_i}\|F_I\|^q_{L^p} \lesssim M_i^{\frac qp-\frac q2}\Bigl(\frac{L}{L_1}\Bigr)^{q-\frac qp-1}\Bigl(\sum_{J\in\mathcal{J}_i}\|f_J\|^q_{L^p}\Bigr). \tag{79}$$

Therefore we conclude

$$\begin{aligned}&\|(F_1F_2)^{\frac12}\|_{L^p(Q_{r_1,r_2})}\\ &\quad\lesssim_\varepsilon N^\varepsilon\log^C(\theta^{-1}+1)|Q_{r_1,r_2}|^{\frac3p-\frac12}\\ &\qquad\times\prod_{i\in\{1,2\}}\biggl(r_i^{\frac2p}((\#P_i)|P|)^{\frac12-\frac3p}(\#I_i)^{\frac1p-\frac1q}M_i^{\frac1p-\frac12}\Bigl(\frac{L}{L_1}\Bigr)^{1-\frac1p-\frac1q}\Bigl(\sum_{J\in\mathcal{J}_i}\|f_J\|^q_{L^p}\Bigr)^{\frac1q}\biggr)^{\frac12}.\end{aligned} \tag{80}$$

So (75) follows if we may show for $i \in \{1,2\}$,

$$|Q_{r_1,r_2}|^{\frac{3}{p}-\frac{1}{2}} r_i^{\frac{2}{p}} ((\#P_i)|P|)^{\frac{1}{2}-\frac{3}{p}} (\#I_i)^{\frac{1}{p}-\frac{1}{q}} M_i^{\frac{1}{p}-\frac{1}{2}} \left(\frac{L}{L_1}\right)^{1-\frac{1}{p}-\frac{1}{q}} \lessapprox \frac{N^{\frac{1}{2}-\frac{1}{2q}-\frac{3}{2p}} L^{\frac{2}{p}}}{L_1^{1-\frac{1}{p}-\frac{1}{q}}} + \left(\frac{N^{\frac{1}{2}}}{L_1}\right)^{\frac{1}{2}-\frac{1}{q}}. \quad (81)$$

We show (81) using Proposition 8.3. Fix $i \in \{1,2\}$. We split the proof into two cases depending on which case happens in Proposition 8.3 when applied to $\{P\}_{P\in\mathcal{P}_i}$ with $r = r_i$.

<u>Case 1</u>: (1) in Proposition 8.3 happens. Let s, M_s be the s, M_s given in case (1) of Proposition 8.3. By (68) we have

$$\text{LHS of (81)} \lessapprox r_i^{1-\frac{4}{p}} s^{\frac{1}{2}-\frac{3}{p}} M_s^{\frac{3}{p}-\frac{1}{2}} (\#I_i)^{\frac{1}{p}-\frac{1}{q}} M_i^{\frac{1}{p}-\frac{1}{2}} \left(\frac{L}{L_1}\right)^{1-\frac{1}{p}-\frac{1}{q}}.$$

<u>Case 1.1</u>: $s \le L/L_1$. Then (70) reads $M_s \lesssim sM_i$. Note that we have

$$(\#I) \gtrsim r_i$$

since we have assumed $\|F_{i,P}\|_{L^\infty} \sim 1$. Therefore by (69) and (70) we have

$$\begin{aligned}
\text{LHS of (81)} &\lessapprox \left(\frac{M_s N^{\frac{1}{2}}}{s^2 L}\right)^{1-\frac{3}{p}-\frac{1}{q}} s^{\frac{1}{2}-\frac{3}{p}} M_s^{\frac{3}{p}-\frac{1}{2}} M_i^{\frac{1}{p}-\frac{1}{2}} \left(\frac{L}{L_1}\right)^{1-\frac{1}{p}-\frac{1}{q}} \\
&= M_s^{\frac{1}{2}-\frac{1}{q}} \left(\frac{N^{\frac{1}{2}}}{L}\right)^{1-\frac{3}{p}-\frac{1}{q}} \left(\frac{L}{L_1}\right)^{1-\frac{1}{p}-\frac{1}{q}} s^{-\frac{3}{2}+\frac{3}{p}+\frac{2}{q}} M_i^{\frac{1}{p}-\frac{1}{2}} \\
&\lesssim (sM_i)^{\frac{1}{2}-\frac{1}{q}} \left(\frac{N^{\frac{1}{2}}}{L}\right)^{1-\frac{3}{p}-\frac{1}{q}} \left(\frac{L}{L_1}\right)^{1-\frac{1}{p}-\frac{1}{q}} s^{-\frac{3}{2}+\frac{3}{p}+\frac{2}{q}} M_i^{\frac{1}{p}-\frac{1}{2}} \\
&= M_i^{\frac{1}{p}-\frac{1}{q}} s^{-1+\frac{3}{p}+\frac{1}{q}} \left(\frac{N^{\frac{1}{2}}}{L}\right)^{1-\frac{3}{p}-\frac{1}{q}} \left(\frac{L}{L_1}\right)^{1-\frac{1}{p}-\frac{1}{q}}.
\end{aligned}$$

Since $p \ge q$, $\frac{1}{q} + \frac{3}{p} \le 1$, and $s, M_i \ge 1$, we conclude

$$\text{LHS of (81)} \lessapprox \left(\frac{N^{\frac{1}{2}}}{L}\right)^{1-\frac{3}{p}-\frac{1}{q}} \left(\frac{L}{L_1}\right)^{1-\frac{1}{p}-\frac{1}{q}} = \frac{N^{\frac{1}{2}-\frac{1}{2q}-\frac{3}{2p}} L^{\frac{2}{p}}}{L_1^{1-\frac{1}{p}-\frac{1}{q}}}.$$

<u>Case 1.2</u>: $s \ge L/L_1$. This is the case where we see the two conditions in Theorem 8.5. Now (70) reads $M_s \lesssim s^2 M_i L_1/L$. By $(\#I) \gtrsim r_i$ and (69) we have

$$\begin{aligned}
\text{LHS of (81)} &\lessapprox r_i^{1-\frac{3}{p}-\frac{1}{p}} s^{\frac{1}{2}-\frac{3}{p}} M_s^{\frac{3}{p}-\frac{1}{2}} M_i^{\frac{1}{p}-\frac{1}{2}} \left(\frac{L}{L_1}\right)^{1-\frac{1}{p}-\frac{1}{q}} \\
&\lesssim \left(\frac{M_s}{s^2}\frac{N^{\frac{1}{2}}}{L}\right)^{1-\frac{3}{p}-\frac{1}{q}} s^{\frac{1}{2}-\frac{3}{p}} M_s^{\frac{3}{p}-\frac{1}{2}} M_i^{\frac{1}{p}-\frac{1}{2}} \left(\frac{L}{L_1}\right)^{1-\frac{1}{p}-\frac{1}{q}} \\
&= M_s^{\frac{1}{2}-\frac{1}{q}} \left(\frac{N^{\frac{1}{2}}}{L}\right)^{1-\frac{3}{p}-\frac{1}{q}} \left(\frac{L}{L_1}\right)^{1-\frac{1}{p}-\frac{1}{q}} s^{-\frac{3}{2}+\frac{3}{p}+\frac{2}{q}} M_i^{\frac{1}{p}-\frac{1}{2}}.
\end{aligned}$$

Plugging in (70) we obtain

$$\text{LHS of (81)} \lessapprox \left(s^2 M_i \frac{L_1}{L}\right)^{\frac{1}{2}-\frac{1}{q}} \left(\frac{N^{\frac{1}{2}}}{L}\right)^{1-\frac{3}{p}-\frac{1}{q}} \left(\frac{L}{L_1}\right)^{1-\frac{1}{p}-\frac{1}{q}} s^{-\frac{3}{2}+\frac{3}{p}+\frac{2}{q}} M_i^{\frac{1}{p}-\frac{1}{2}}$$
$$= M_i^{\frac{1}{p}-\frac{1}{q}} s^{-\frac{1}{2}+\frac{3}{p}} \left(\frac{N^{\frac{1}{2}}}{L}\right)^{1-\frac{3}{p}-\frac{1}{q}} \left(\frac{L}{L_1}\right)^{\frac{1}{2}-\frac{1}{p}}.$$

Since $M_i \geq 1$ and $q \leq p$, we conclude

$$\text{LHS of (81)} \lessapprox s^{-\frac{1}{2}+\frac{3}{p}} \left(\frac{N^{\frac{1}{2}}}{L}\right)^{1-\frac{3}{p}-\frac{1}{q}} \left(\frac{L}{L_1}\right)^{\frac{1}{2}-\frac{1}{p}}.$$

If we use $s \leq L$, then

$$s^{-\frac{1}{2}+\frac{3}{p}} \left(\frac{N^{\frac{1}{2}}}{L}\right)^{1-\frac{3}{p}-\frac{1}{q}} \left(\frac{L}{L_1}\right)^{\frac{1}{2}-\frac{1}{p}} \leq L^{-\frac{1}{2}+\frac{3}{p}} \left(\frac{N^{\frac{1}{2}}}{L}\right)^{1-\frac{3}{p}-\frac{1}{q}} \left(\frac{L}{L_1}\right)^{\frac{1}{2}-\frac{1}{p}}.$$

We may then verify that

$$L^{-\frac{1}{2}+\frac{3}{p}} \left(\frac{N^{\frac{1}{2}}}{L}\right)^{1-\frac{3}{p}-\frac{1}{q}} \left(\frac{L}{L_1}\right)^{\frac{1}{2}-\frac{1}{p}} \leq \frac{N^{\frac{1}{2}-\frac{1}{2q}-\frac{3}{2p}} L^{\frac{2}{p}}}{L_1^{1-\frac{1}{p}-\frac{1}{q}}}$$

if and only if

$$L_1^{\frac{1}{2}-\frac{1}{q}} \leq L^{1-\frac{3}{p}-\frac{1}{q}}.$$

On the other hand if we use $s \leq N^{1/2}/L$, then

$$\text{LHS of (81)} \lessapprox \left(\frac{N^{\frac{1}{2}}}{L}\right)^{-\frac{1}{2}+\frac{3}{p}} \left(\frac{N^{\frac{1}{2}}}{L}\right)^{1-\frac{3}{p}-\frac{1}{q}} \left(\frac{L}{L_1}\right)^{\frac{1}{2}-\frac{1}{p}} = \left(\frac{N^{\frac{1}{2}}}{L}\right)^{\frac{1}{2}-\frac{1}{q}} \left(\frac{L}{L_1}\right)^{\frac{1}{2}-\frac{1}{p}}.$$

The last line equals

$$\left(\frac{N^{\frac{1}{2}}}{L}\right)^{\frac{1}{2}-\frac{1}{p}} \left(\frac{L}{L_1}\right)^{\frac{1}{2}-\frac{1}{p}}$$

if $p = q$. In conclusion we have shown (81) holds in this case if either condition (a) or (b) is satisfied.

Case 2: (2) in Proposition 8.3 happens. By (71), (72) we have

$$\text{LHS of (81)} \lessapprox |P(L)|^{\frac{3}{p}-\frac{1}{2}} \left(\frac{(\#P_i)|P|}{|P(L)|}\right)^{\frac{2}{p}} (\#I_i)^{\frac{1}{p}-\frac{1}{q}} M_i^{\frac{1}{p}-\frac{1}{2}} \left(\frac{L}{L_1}\right)^{1-\frac{1}{p}-\frac{1}{q}} ((\#P_i)|P|)^{\frac{1}{2}-\frac{3}{p}}. \tag{82}$$

Note that we have

$$(\#P_i) \lesssim (\#I_i) M_i \frac{|P(L)|}{|P_J|} \sim (\#I_i) M_i \frac{|P(L)|}{|P|} \frac{L_1}{L}$$

since the right-hand side is the maximal number of P one can fit into a $P(L)$ under the assumption that each P_J can contain $\lesssim M_i$ many $P \in \mathcal{P}_i$. Substituting the above for M_i in (82) and simplifying the algebra we obtain

$$\text{LHS of (81)} \lessapprox (\#I_i)^{\frac{1}{2}-\frac{1}{q}} \left(\frac{L}{L_1}\right)^{\frac{1}{2}-\frac{1}{q}}.$$

Since $\#I_i \le N^{1/2}/L$ and $q \ge 2$, we conclude

$$\text{LHS of (81)} \lessapprox \left(\frac{N^{\frac{1}{2}}}{L_1}\right)^{\frac{1}{2}-\frac{1}{q}}.$$

Hence (81) holds in this case.

In conclusion we have shown (81) and therefore (77) and (75). □

The following proposition shows that it was justified in the proof of Theorem 8.5 to treat functions $\widetilde{F}_i = \sum_{P\in\mathcal{P}_i} F_{i,P}$ whose wave packets with respect to $\{I\}_{I\in\mathcal{I}}$ satisfied certain extra assumptions. Here, each wave packet $F_{i,P}$ equals $\phi_{P_I} f_I$ for some $I \in \mathcal{I}$ and some P_I, as in the definition of wave packet decomposition from Section 5B, except we assume the extra condition that ϕ_{P_I} decays at a rate of $10^3\varepsilon^{-2}$ away from P_I.

Write $\mathcal{I}_i = \left\{I \in \mathcal{I} : I \subset \bigcup_{J\in\mathcal{J}_i} J\right\}$. For each $I \in \mathcal{I}_i$, write

$$f_I = \sum_{P_I\in\mathcal{P}_I} f_{I,P_I},$$

where $f_{I,P_I} = \phi_{P_I} f_I$ and $\mathcal{P}_I$ denotes the collection of translates of P_I which tile $\mathbb{R}$, from the definition of wave packet decomposition. Fix collections $\mathcal{P}_J$ of translates of P_J which tile $\mathbb{R}$ and with the property that $P_J \cap P_I$ is either P_I or $\varnothing$ whenever $J \subset I$. Note that the set $\mathcal{P}_J$ does not vary for $J \subset I$.

Proposition 8.6 (pigeonholing of the wave packets). *Assume the hypotheses of Theorem 8.5. There exist subsets $\widetilde{\mathcal{I}}_i \subset \mathcal{I}_i$ and $\widetilde{\mathcal{P}}_I \subset \mathcal{P}_I$ as well as integers M_i, H_i with the following properties*:

$$\Big\| \prod_{i\in\{1,2\}} |F_i|^{\frac{1}{2}} \Big\|_{L^p(B_{N^2/(\theta L^2)})} \lesssim \log(\theta^{-1}+1)(\log N)^2 \Big\| \prod_{i\in\{1,2\}} |\widetilde{F}_i|^{\frac{1}{2}} \Big\|_{L^p(B_{N^2/(\theta L^2)})} + N^{-50}(\textit{RHS of } (77)),$$

where $\widetilde{F}_i = \sum_{I\in\widetilde{\mathcal{I}}_i} \sum_{P_I\in\widetilde{\mathcal{P}}_I} f_{I,P_I}$,

$$\#\{P_I \in \widetilde{\mathcal{P}}_I : P_I \subset P_J\} \sim M_i \quad \textit{or} \quad = 0 \quad \textit{for all} \quad P_J \in \mathcal{P}_J,\ J \subset I \in \widetilde{\mathcal{I}}_i, \tag{83}$$

$$\#\widetilde{\mathcal{P}}_I \sim \#\widetilde{\mathcal{P}}_{I'} \quad \textit{for all } I, I' \in \widetilde{\mathcal{I}}_i, \tag{84}$$

$$\|f_{I,P_I}\|_{L^\infty(\mathbb{R})} \sim H_i \quad \textit{for all } I \in \widetilde{I}_i \textit{ and } P_I \in \widetilde{\mathcal{P}}_I. \tag{85}$$

It follows that, for $\widetilde{F}_I = \sum_{P_I\in\widetilde{\mathcal{P}}_I} f_{I,P_I}$ with $I \in \widetilde{\mathcal{I}}_i$, $\|\widetilde{F}_I\|^p_{L^p(\mathbb{R})}$ is within a factor of $C_\varepsilon N^\varepsilon$ of $H_i^p \#\{P_I \in \widetilde{\mathcal{P}}_i\}|P_I| + N^{-500} \max_{J\in\mathcal{J}_i} \|f_J\|^p_{L^p(\mathbb{R})}$.

The collection $\mathcal{P}_i$ from the proof of Theorem 8.5 is the union of the $\widetilde{\mathcal{P}}_I$, where $I \in \widetilde{\mathcal{I}}_i$.

Proof. First we will show that $\||F_1F_2|^{1/2}\|_{L^p(B_{N^2/(\theta L^2)})} \lesssim \||F_1\widetilde{F}_2|^{1/2}\|_{L^p(B_{N^2/(\theta L^2)})}$ plus the remainder term. The argument showing $\||F_1\widetilde{F}_2|^{1/2}\|_{L^p(B_{N^2/(\theta L^2)})} \lesssim \||\widetilde{F}_1\widetilde{F}_2|^{1/2}\|_{L^p(B_{N^2/(\theta L^2)})}$ plus the remainder term is analogous, so we omit it. Split F_2 into

$$F_2 = \sum_{I\in\mathcal{I}_2}\sum_{P_I\in\mathcal{P}_I^c} f_{I,P_I} + \sum_{I\in\mathcal{I}_2}\sum_{P_I\in\mathcal{P}_I^f} f_{I,P_I}, \tag{86}$$

where the close set is

$$\mathcal{P}_I^c := \{P_I \in \mathcal{P}_I : P_I \cap N^{10} B_{N^2/(\theta L^2)} \neq \varnothing\}$$

and the far set is

$$\mathcal{P}_I^f := \{P_I \in \mathcal{P}_I : P_I \cap N^{10} B_{N^2/(\theta L^2)} = \varnothing\}.$$

Using Hölder's inequality, Cauchy–Schwarz, and Minkowski's inequality with $q \le p$, we have

$$\begin{aligned}
&\Big\| \Big| F_1 \sum_{I\in\mathcal{I}_2}\sum_{P_I\in\mathcal{P}_I^f} f_{I,P_I}\Big|^{\frac12}\Big\|_{L^p(B_{N^2/(\theta L^2)})} \\
&\lesssim \Big(\frac{N}{L_1}\Big)^{1-\frac1q} \Big\| \Big| \sum_{J_1\in\mathcal{J}_1} |f_{J_1}|^q \sum_{I\in\mathcal{I}_2}\sum_{J_2\subset I}\Big|\sum_{P_I\in\mathcal{P}_I^f}\phi_{P_I} f_{J_2}\Big|^q\Big|^{\frac{1}{2q}}\Big\|_{L^p(B_{N^2/(\theta L^2)})} \\
&\le \Big(\frac{N}{L_1}\Big)^{1-\frac1q} \Big\| \Big| \sum_{J_1\in\mathcal{J}_1} |f_{J_1}|^q\Big|^{\frac1q}\Big\|^{\frac12}_{L^p(B_{N^2/(\theta L^2)})} \Big\|\Big| \sum_{I\in\mathcal{I}_2}\sum_{J_2\subset I}\Big|\sum_{P_I\in\mathcal{P}_I^f}\phi_{P_I} f_{J_2}\Big|^q\Big|^{\frac1q}\Big\|^{\frac12}_{L^p(B_{N^2/(\theta L^2)})} \\
&\le \Big(\frac{N}{L_1}\Big)^{1-\frac1q} \Big(\sum_{J_1\in\mathcal{J}_1} \|f_{J_1}\|^q_{L^p(B_{N^2/(\theta L^2)})}\Big)^{\frac{1}{2q}} \Big(\sum_{I\in\mathcal{I}_2}\sum_{J_2\subset I}\Big\|\sum_{P_I\in\mathcal{P}_I^f}\phi_{P_I} f_{J_2}\Big\|^q_{L^p(B_{N^2/(\theta L^2)})}\Big)^{\frac{1}{2q}} \\
&\le \Big(\frac{N}{L_1}\Big)^{1-\frac1q} \Big(\sum_{J_1\in\mathcal{J}_1} \|f_{J_1}\|^q_{L^p(B_{N^2/(\theta L^2)})}\Big)^{\frac{1}{2q}} \max_{I\in\mathcal{I}_2}\Big\|\sum_{P_I\in\mathcal{P}_I^f}\phi_{P_I}\Big\|^{\frac12}_{L^\infty(B_{N^2/(\theta L^2)})}\Big(\sum_{J_2\in\mathcal{J}_2}\|f_{J_2}\|^q_{L^p(B_{N^2/(\theta L^2)})}\Big)^{\frac{1}{2q}} \\
&\le \frac{1}{N^{100}} \prod_{i\in\{1,2\}} \Big(\sum_{J_i\in\mathcal{J}_i}\|f_{J_i}\|^q_{L^p(B_{N^2/(\theta L^2)})}\Big)^{\frac{1}{2q}}.
\end{aligned}$$

This takes care of the *far* portion of F_2.

For each $I \in \mathcal{I}_2$, the close set has cardinality $\#\mathcal{P}_I^c \le N^{11}$. Let

$$H_2 = \max_{I\in\mathcal{I}_2}\max_{P_I\in\mathcal{P}_I^c} \|f_{I,P_I}\|_{L^\infty(\mathbb{R})}. \tag{87}$$

By Proposition 2.3 and Hölder's inequality,

$$H_2 \le \max_{I\in\mathcal{I}_2}\|f_I\|_{L^\infty(\mathbb{R})} \lesssim N\Big(\sum_{J_2\in\mathcal{J}_2}\|f_{J_2}\|^q_{L^p(\mathbb{R})}\Big)^{\frac1q}. \tag{88}$$

Split the *close* part of F_2 into

$$\sum_{I\in\mathcal{I}_2}\sum_{P_I\in\mathcal{P}_I^c} f_{I,P_I} = \sum_{I\in\mathcal{I}_2}\sum_{\theta N^{-10^3}\le\lambda\le 1}\sum_{P_I\in\mathcal{P}_{I,\lambda}^c} f_{I,P_I} + \sum_{I\in\mathcal{I}_2}\sum_{P_I\in\mathcal{P}_{I,s}^c} f_{I,P_I}, \tag{89}$$

where λ is a dyadic number in the range $[\theta N^{-10^3}, 1]$,

$$\mathcal{P}_{I,\lambda}^c := \Big\{P_I \in \mathcal{P}_I^c : \|f_{I,P_I}\|_{L^\infty(\mathbb{R})} \in \Big(\frac{\lambda H_2}{2}, \lambda H_2\Big]\Big\},$$

and

$$\mathcal{P}_{I,s}^c := \Big\{P_I \in \mathcal{P}_I^c : \|f_{I,P_I}\|_{L^\infty(\mathbb{R})} \le \frac{\theta}{2} N^{-10^3} H_2\Big\}.$$

Handle the *small* term from (89) by

$$\begin{aligned}
&\Big\|\Big|F_1 \sum_{I\in\mathcal{I}_2}\sum_{P_I\in\mathcal{P}_{I,s}^c} f_{I,P_I}\Big|^{\frac12}\Big\|_{L^p(B_{N^2/(\theta L^2)})}\\
&\quad\le \Big(\frac{N^2}{L_1 L}\Big)^{\frac12-\frac{1}{2q}} \Big\|\Big|\sum_{J_1\in\mathcal{J}_1}|f_{J_1}|^q \sum_{I\in\mathcal{I}_2}\Big|\sum_{P_I\in\mathcal{P}_{I,s}^c} f_{I,P_I}\Big|^q\Big|^{\frac{1}{2q}}\Big\|_{L^p(B_{N^2/(\theta L^2)})}\\
&\quad\le \Big(\frac{N^2}{L_1 L}\Big)^{\frac12-\frac{1}{2q}} \Big(\sum_{J_1\in\mathcal{J}_1}\|f_{J_1}\|_{L^p(B_{N^2/(\theta L^2)})}^q\Big)^{\frac{1}{2q}}\\
&\qquad\qquad \times\Big(\#\mathcal{I}_2^{\frac1q}\max_{I\in\mathcal{I}_2}\#\mathcal{P}_I^c \max_{P_I\in\mathcal{P}_{I,s}^c}\|f_{I,P_I}\|_{L^\infty(B_{N^2/(\theta L^2)})}|B_{N^2/(\theta L^2)}|^{\frac2p}\Big)^{\frac12}\\
&\quad\le \Big(\sum_{J_1\in\mathcal{J}_1}\|f_{J_1}\|_{L^p(B_{N^2/(\theta L^2)})}^q\Big)^{\frac{1}{2q}} \Big(\Big(\frac{N}{L}\Big)^{\frac1q} N^{11}\theta N^{-10^3} H_2 |B_{N^2/(\theta L^2)}|^{\frac2p}\Big)^{\frac12}\\
&\quad\le N^{-150}\Big(\sum_{J_1\in\mathcal{J}_1}\|f_{J_1}\|_{L^p(B_{N^2/(\theta L^2)})}^q\Big)^{\frac{1}{2q}} H_2^{\frac12} \lesssim N^{-100}\prod_{i\in\{1,2\}}\Big(\sum_{J\in\mathcal{J}_i}\|f_J\|_{L^p(\mathbb{R})}^q\Big)^{\frac{1}{2q}}.
\end{aligned}$$

Decompose the remaining term from (89) using the fact that for $J \subset I \in \mathcal{I}_2,\ P_J \in \mathcal{P}_J$, the number $\#\{P_I \in \mathcal{P}_{I,\lambda}^c : P_I \subset P_J\}$ is in $\{0, \dots, L/L_1\}$ (and does not depend on the specific $J \subset I$), which allows us to write

$$\sum_{\theta N^{-10^3}\le\lambda\le 1}\sum_{I\in\mathcal{I}_2}\sum_{P_I\in\mathcal{P}_{I,\lambda}^c} f_{I,P_I} = \sum_{\theta N^{-10^3}\le\lambda\le 1}\sum_{1\le 2^k\le L/L_1}\sum_{I\in\mathcal{I}_2}\sum_{P_I\in\mathcal{P}_{I,\lambda,k}^c} f_{I,P_I}, \tag{90}$$

where, for $J \subset I$,

$$\mathcal{P}_J^k = \{P_J \in \mathcal{P}_J : \#\{P_I \in \mathcal{P}_{I,\lambda}^c : P_I \subset P_J\} \sim 2^k\},$$
$$\mathcal{P}_{I,\lambda,k}^c = \bigcup_{P_J\in\mathcal{P}_J^k}\{P_I \in \mathcal{P}_{I,\lambda}^c : P_I \subset P_J\}.$$

Finally, note that the number of $P_J \in \mathcal{P}_J$ which intersect $N^{10}B_{N^2/(\theta L^2)}$ is bounded by $N^{10}L_1 \le N^{11}$. Further decompose the right-hand side from (90) as

$$\sum_{\theta N^{-10^3}\le\lambda\le 1}\sum_{1\le 2^k\le L/L_1}\sum_{1\le 2^j\le N^{11}}\sum_{I\in\mathcal{I}_{2,\lambda}^{k,j}}\sum_{P_I\in\mathcal{P}_{I,\lambda,k}^c} f_{I,P_I}, \tag{91}$$

where, for $J \subset I$, $\mathcal{I}_{2,\lambda}^{k,j} = \{I \in \mathcal{I}_2 : \#\mathcal{P}_J^k \sim 2^j\}$.

Because 2^k, 2^j, and λ are dyadic numbers, by the pigeonhole principle, there is a choice of (k, j, λ) so that

$$\Bigg\|\Big|F_1 \sum_{\theta N^{-10^3}\le\lambda\le 1}\ \sum_{1\le 2^k\le L/L_1}\ \sum_{1\le 2^j\le N^{11}}\ \sum_{I\in\mathcal{I}_{2,k,j}}\ \sum_{P_I\in\mathcal{P}^c_{I,\lambda,k}} f_{I,P_I}\Big|^{\frac12}\Bigg\|_{L^p(B_{N^2/(\theta L^2)})} \lesssim \log(\theta^{-1}+1)(\log N)^2 \Bigg\|\Big|F_1 \sum_{I\in\mathcal{I}^{k,j}_{2,\lambda}}\ \sum_{P_I\in\mathcal{P}^c_{I,\lambda,k}} f_{I,P_I}\Big|^{\frac12}\Bigg\|_{L^p(B_{N^2/(\theta L^2)})}.$$

Let $\widetilde{\mathcal{I}}_2 = \mathcal{I}^{k,j}_{2,\lambda}$ and for each $I\in\widetilde{\mathcal{I}}_2$, let $\widetilde{\mathcal{P}}_I = \mathcal{P}^c_{I,\lambda,k}$. It follows from Proposition 2.3 and properties of weight functions ϕ_{P_I} and $W_{P_I} = W_{P_I,600}$ that, for each $I\in\widetilde{\mathcal{I}}_2$,

$$\begin{aligned}
\|\widetilde{F}_I\|^p_{L^p(\mathbb{R})} &= \sum_{P'_I\in\mathcal{P}_I}\Big\|\sum_{P_I\in\widetilde{\mathcal{P}}_I} f_{I,P_I}\Big\|^p_{L^p(P'_I)}\\
&\le \sum_{P'_I\in\mathcal{P}_I}\int_{P'_I}\Big|\sum_{P_I\in\widetilde{\mathcal{P}}_I}\phi_{P_I}\Big|^p\|f_I\|^p_{L^\infty(P'_I)} \le \sum_{P'_I\in\mathcal{P}_I}|P'_I|\max_{P_I\in\widetilde{\mathcal{P}}_I}\phi_{P_I}(P'_I)^p\|f_I\|^p_{L^\infty(P'_I)}\\
&\le \sum_{P'_I\in\mathcal{P}_I}|P'_I|\sum_{P_I\in\widetilde{\mathcal{P}}_I}\phi_{P_I}(P'_I)^p\|f_I\|^p_{L^\infty(P'_I)} \lesssim \sum_{P'_I\in\mathcal{P}_I}\sum_{P_I\in\widetilde{\mathcal{P}}_I}|P'_I|\|\phi_{P_I}f_I\|^p_{L^\infty(P'_I)}\\
&\lesssim \sum_{P_I\in\widetilde{\mathcal{P}}_I}\sum_{P'_I\in\mathcal{P}_I}\int|\phi_{P_I}f_I|^pW_{P'_I} \sim \sum_{P_I\in\widetilde{\mathcal{P}}_I}\int|\phi_{P_I}f_I|^p.
\end{aligned}$$

The assumption that ϕ_{P_I} decays at order $10^3\varepsilon^{-2}$ allows us to write, for each $I\in\widetilde{\mathcal{I}}_2$ and $P_I\in\widetilde{\mathcal{P}}_I$,

$$\begin{aligned}
\Big|\int|\phi_{P_I}f_I|^p - \int_{N^\varepsilon P_I}|\phi_{P_I}f_I|^p\Big| &\le C_\varepsilon N^{-1000}\|f_I\|^p_{L^p(\mathbb{R})}\\
&\le C_\varepsilon N^{-500}\max_{J\subset I}\|f_J\|^p_{L^p(\mathbb{R})}
\end{aligned}$$

and

$$\int_{N^\varepsilon P_I}|\phi_{P_I}f_I|^p \le C_\varepsilon N^\varepsilon|P_I|B_2^p \lesssim C_\varepsilon N^\varepsilon\int|f_{I,P_I}|^p,$$

which proves the final property about $\|f_{I,P_I}\|_{L^p(\mathbb{R})}$ from the proposition. □

Proof of Theorem 8.1 using Theorem 8.5. The proof resembles Section 5.1 in [Demeter et al. 2020]. First we fix (p,q) with $4\le p\le 6$, and either $\frac1q+\frac3p=1$ or $p=q$. Note that under such assumption we always have $p\ge q$ and $q\ge 2$. Recall that Ω is the $(\theta L^2/N^2)$-neighborhood of $\{a_n\}_{n=1}^{N^{1/2}}$, which is a union of $N^{1/2}$ many intervals of length $C\theta L^2/N^2$. We let τ denote the union of l many consecutive intervals in Ω, and write $\ell(\tau)=l$, so in this notation $\ell(I)=L$ and $\ell(J)=L_1$. Let $F=\sum_{J\in\mathcal{J}} f_J$, and denote by F_τ the Fourier projection of F to τ, that is, $(1_\tau\widehat{F})^{\vee}$. Fix $K>1$. We have the inequality

$$|F(x)| \le \sum_{\ell(\tau)=\frac{N^{1/2}}{K}}|F_\tau(x)| \le C\max_{\ell(\tau)=\frac{N^{1/2}}{K}}|F_\tau(x)| + K^C\max_{\substack{\ell(\tau_1)=\ell(\tau_2)=\frac{N^{1/2}}{K}\\ d(\tau_1,\tau_2)\gtrsim\frac{1}{KN^{1/2}}}}|F_{\tau_1}F_{\tau_2}|^{\frac12}.$$

Iterating this (for the first term) we obtain

$$\|F\|_{L^p(\mathbb{R})}^p \lesssim C^m \sum_{\ell(\tau)=L} \|F_\tau\|_{L^p(\mathbb{R})}^p + C^m K^C \sum_{\substack{l=\frac{N^{1/2}}{K^a} \text{ for } a\in\mathbb{Z} \\ KL\le l\le N^{1/2}}} \sum_{\tau:\ell(\tau)=l} \sum_{\substack{\tau_1,\tau_2\subset\tau \\ \ell(\tau_1)=\ell(\tau_2)=K^{-1}l \\ d(\tau_1,\tau_2)\gtrsim K^{-1}l}} \|(F_{\tau_1}F_{\tau_2})^{\frac12}\|_{L^p(\mathbb{R})}^p. \tag{92}$$

Here m satisfies $N^{1/2}/K^m = L$.

By Proposition 7.2 and Hölder's inequality we have

$$\sum_{\ell(\tau)=L} \|F_\tau\|_{L^p(\mathbb{R})}^p \lesssim \sum_{\ell(\tau)=L} \left(\frac{L}{L_1}\right)^{\frac{p}{2}-1} \left(\sum_{J\subset\tau} \|F_J\|_{L^p(\mathbb{R})}^2\right)^{\frac{p}{2}} \le \sum_{\ell(\tau)=L} \left(\frac{L}{L_1}\right)^{p-1-\frac{p}{q}} \left(\sum_{J\subset\tau} \|F_J\|_{L^p(\mathbb{R})}^q\right)^{\frac{p}{q}}.$$

Since $\frac1q+\frac3p\le 1$ and $L\le N^{1/2}$, we have

$$\frac{N^{\frac12-\frac{1}{2q}-\frac{3}{2p}} L^{\frac2p}}{L_1^{1-\frac1p-\frac1q}} \ge \frac{L^{1-\frac1q-\frac1p}}{L_1^{1-\frac1p-\frac1q}}.$$

Therefore,

$$\sum_{\ell(\tau)=L} \|F_\tau\|_{L^p(\mathbb{R})}^p \lesssim \left(\frac{L}{L_1}\right)^{p-1-\frac{p}{q}} \left(\sum_{J\in\mathcal{J}} \|F_J\|_{L^p(\mathbb{R})}^q\right)^{\frac{p}{q}} \le \left(\frac{N^{\frac12-\frac{1}{2q}-\frac{3}{2p}} L^{\frac2p}}{L_1^{1-\frac1p-\frac1q}}\right)^p \left(\sum_{J\in\mathcal{J}} \|F_J\|_{L^p(\mathbb{R})}^q\right)^{\frac{p}{q}}. \tag{93}$$

Now we estimate the second term on the right-hand side of (92). Let $s=N^{1/2}/l$. Then using the change of variable $x\mapsto s^2x$ as in the proof of Proposition 6.3, and by Theorem 8.5 we have

$$\|(F_{\tau_1}F_{\tau_2})^{\frac12}\|_{L^p(\mathbb{R})} \lesssim_\varepsilon N^\varepsilon \log^C(\tilde\theta^{-1}+1)\left(\frac{\widetilde{N}^{\frac12-\frac{1}{2q}-\frac{3}{2p}}\widetilde{L}^{\frac2p}}{\widetilde{L}_1^{1-\frac1p-\frac1q}} + \left(\frac{\widetilde{N}^{\frac12}}{\widetilde{L}_1}\right)^{\frac12-\frac1q}\right)\left(\sum_{J\subset\tau} \|f_J\|_{L^p(\mathbb{R})}^q\right)^{\frac1q},$$

where $\widetilde{N}=N/s^2,\ \tilde\theta=\theta/s^2,\ \widetilde{L}_1=L_1,\ \widetilde{L}=L$. Plugging in the expressions for $\widetilde{N},\tilde\theta,\widetilde{L}_1,\widetilde{L}$ we obtain

$$\begin{aligned}&\|(F_{\tau_1}F_{\tau_2})^{\frac12}\|_{L^p(\mathbb{R})}\\ &\quad\lesssim_\varepsilon N^\varepsilon\log^C(\theta^{-1}+1)\left(s^{-1+\frac1q+\frac3p}\frac{N^{\frac12-\frac{1}{2q}-\frac{3}{2p}}L^{\frac2p}}{L_1^{1-\frac1p-\frac1q}} + s^{-\frac12+\frac1q}\left(\frac{N^{\frac12}}{L_1}\right)^{\frac12-\frac1q}\right)\left(\sum_{J\subset\tau}\|f_J\|_{L^p(\mathbb{R})}^q\right)^{\frac1q}.\end{aligned} \tag{94}$$

We let $K=N^{\varepsilon'}$ for some $\varepsilon'>0$ which will be chosen depending on ε. Then from (93) and (94) we conclude

$$\begin{aligned}\|F\|_{L^p(\mathbb{R})} &\lesssim_{\varepsilon,\varepsilon'} N^{\varepsilon+C\varepsilon'}\log^C(\theta^{-1}+1)\Bigg(\Bigg(\sum_{\substack{s=K^a \text{ for } a\in\mathbb{Z}\\ 1\le s\le\frac{N^{1/2}}{KL}}} s^{-1+\frac1q+\frac3p}\Bigg)\frac{N^{\frac12-\frac{1}{2q}-\frac{3}{2p}}L^{\frac2p}}{L_1^{1-\frac1p-\frac1q}}\\ &\qquad + \Bigg(\sum_{\substack{s=K^a \text{ for } a\in\mathbb{Z}\\ 1\le s\le\frac{N^{1/2}}{KL}}} s^{-\frac12+\frac1q}\Bigg)\left(\frac{N^{\frac12}}{L_1}\right)^{\frac12-\frac1q}\Bigg)\left(\sum_{J\in\mathcal{J}}\|f_J\|_{L^p(\mathbb{R})}^q\right)^{\frac1q}\\ &\lesssim_{\varepsilon,\varepsilon'} N^{\varepsilon+C\varepsilon'}\log^C(\theta^{-1}+1)\left(\frac{N^{\frac12-\frac{1}{2q}-\frac{3}{2p}}L^{\frac2p}}{L_1^{1-\frac1p-\frac1q}} + \left(\frac{N^{\frac12}}{L_1}\right)^{\frac12-\frac1q}\right)\left(\sum_{J\in\mathcal{J}}\|f_J\|_{L^p(\mathbb{R})}^q\right)^{\frac1q}.\end{aligned}$$

Therefore we have shown Theorem 8.1 under condition (a) and the extra condition $\frac{1}{q}+\frac{3}{p}=1,\ p\le 6$, or under condition (b) with the extra condition $p\le 6$.

First assume (a) and we want to remove the condition $\frac{1}{q}+\frac{3}{p}=1,\ p\le 6$. First we note that it suffices to show (66) for every (p,q) with $p\ge 4,\ \frac{1}{q}+\frac{3}{p}=1$. This is because for a general (p,q) with $p\ge 4$, $\frac{1}{q}+\frac{3}{p}\le 1$ we may consider (66) with (p,q) replaced by (p,q_0), where $\frac{1}{q_0}+\frac{3}{p}=1$. Then (66) with (p,q) follows from Hölder's inequality applied in the index J to the right-hand side of (66) with (p,q_0), since $|\mathcal{J}|\lesssim N^{1/2}/L_1$. Second we note that it suffices to show (66) for every (p,q) with $4\le p\le 6$, $\frac{1}{q}+\frac{3}{p}=1$. This is because when $p\ge 6$, we always have

$$\frac{N^{\frac{1}{2}-\frac{1}{2q}-\frac{3}{2p}}L^{\frac{2}{p}}}{L_1^{1-\frac{1}{p}-\frac{1}{q}}}\ge\left(\frac{N^{\frac{1}{2}}}{L_1}\right)^{\frac{1}{2}-\frac{1}{q}}$$

and (66) reduces to

$$\Big\|\sum_{J\in\mathcal{J}}f_J\Big\|_{L^p(\mathbb{R})}\lesssim_\varepsilon N^\varepsilon\log^C(\theta^{-1}+1)\frac{N^{\frac{1}{2}-\frac{1}{2q}-\frac{3}{2p}}L^{\frac{2}{p}}}{L_1^{1-\frac{1}{p}-\frac{1}{q}}}\Big(\sum_{J\in\mathcal{J}}\|f_J\|^q_{L^p(\mathbb{R})}\Big)^{\frac{1}{q}}.$$

So (66) with $q>6,\ \frac{1}{q}+\frac{3}{p}=1$ follows from interpolating (66) with $(p,q)=(6,2)$, and with $(p,q)=(\infty,1)$. (For the interpolation of decoupling inequalities, see Exercise 9.21 of [Demeter 2020].) When $p=\infty, q=1$, (66) becomes the triangle inequality which holds trivially. Hence we have shown Theorem 8.1 under condition (a).

Now assume (b) and we want to remove the condition $p\le 6$. As in the previous paragraph, when $p\ge 6$ we always have

$$\frac{N^{\frac{1}{2}-\frac{1}{2p}-\frac{3}{2p}}L^{\frac{2}{p}}}{L_1^{1-\frac{1}{p}-\frac{1}{p}}}\ge\left(\frac{N^{\frac{1}{2}}}{L_1}\right)^{\frac{1}{2}-\frac{1}{p}},$$

and therefore (66) with $q>6,\ p=q$ follows from interpolating (see Exercise 9.21 of [Demeter 2020]) (66) with $(p,q)=(6,6)$, and with $(p,q)=(\infty,\infty)$. So Theorem 8.1 holds under condition (b) as well. □

Appendix

Corollary 1.4 can be derived from small-cap decoupling inequalities for the parabola in [Demeter et al. 2020]. This is through a transference method which we learned from James Maynard. We record a detailed proof here. The same argument would also imply Corollary 8.2 if the corresponding ℓ^qL^p small cap decoupling inequalities for the parabola are known.

We first recall the small-cap decoupling inequalities in [Demeter et al. 2020].

Theorem A.1 [Demeter et al. 2020]. *Suppose $\alpha\in\left[\frac{1}{2},1\right]$, and let $\Gamma=\{\gamma\}$ be the partition of $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$ into R^α many $R^{-\alpha}\times R^{-1}$ rectangles γ. Assume $p=2+\frac{2}{\alpha}$. Then for every $\varepsilon>0$ we have*

$$\Big\|\sum_{\gamma\in\Gamma}f_\gamma\Big\|_{L^p(\mathbb{R}^2)}\lesssim_\varepsilon R^{\alpha(\frac{1}{2}-\frac{1}{p})+\varepsilon}\Big(\sum_\gamma\|f_\gamma\|^p_{L^p(\mathbb{R}^2)}\Big)^{\frac{1}{p}}\tag{95}$$

for every $f_\gamma:\mathbb{R}^2\to\mathbb{C}$ with supp $\widehat{f_\gamma}\subset\gamma$.

Theorem A.1 continues to hold, by essentially the same proof, with $\mathbb{P}^1$ replaced by a C^2 curve of the form $\{(x, g(x)) : x \in [0,1]\}$, with $g'(0) = 0$, $g''(x) \sim 1$ for $x \in [0,1]$. See for example Section 7 of [Bourgain and Demeter 2015] (whose argument we think actually requires a bit more regularity of the curve than C^2), or the appendix of [Guth et al. 2022]. Additionally we may interpolate (see Exercise 9.21 of [Demeter 2020]) between (95) and the elementary inequalities

$$\Big\|\sum_{\gamma\in\Gamma} f_\gamma\Big\|_{L^2(\mathbb{R}^2)} \lesssim \Big(\sum_\gamma \|f_\gamma\|^2_{L^2(\mathbb{R}^2)}\Big)^{\frac{1}{2}}$$

$$\Big\|\sum_{\gamma\in\Gamma} f_\gamma\Big\|_{L^\infty(\mathbb{R}^2)} \lesssim R^\alpha(\sup_\gamma \|f_\gamma\|_{L^\infty(\mathbb{R}^2)})$$

to obtain the following version of Theorem A.1.

Theorem A.2 [Demeter et al. 2020]. *Suppose G is a C^2 convex curve of the form $\{(x, g(x)) : x \in [0,1]\}$, where $g'(0) = 0$, $g''(x) \sim 1$ for $x \in [0,1]$. Suppose $\alpha \in \left[\frac{1}{2}, 1\right]$, and let $\Gamma = \{\gamma\}$ be the partition of $\mathcal{N}_{R^{-1}}(G)$ into R^α many $R^{-\alpha} \times R^{-1}$ rectangles γ. Assume $p \geq 2$. Then for every $\varepsilon > 0$ we have*

$$\Big\|\sum_{\gamma\in\Gamma} f_\gamma\Big\|_{L^p(\mathbb{R}^2)} \lesssim_\varepsilon R^\varepsilon\big(R^{\alpha(\frac{1}{2}-\frac{1}{p})} + R^{\alpha(1-\frac{1}{p})-(1+\alpha)\frac{1}{p}}\big)\Big(\sum_\gamma \|f_\gamma\|^p_{L^p(\mathbb{R}^2)}\Big)^{\frac{1}{p}} \tag{96}$$

for every $f_\gamma : \mathbb{R}^2 \to \mathbb{C}$ with $\operatorname{supp} \widehat{f_\gamma} \subset \gamma$.

For the rest of this section we work under the assumption of Corollary 1.4. In particular $\theta = 1$. For simplicity we assume $a_1 = 0$, and $v := a_2 - a_1 = N^{-1}$. Let $1 \leq L \leq N^{1/2}$. It suffices to show (67) for $4 \leq p \leq 6$ and we assume that (since the $p > 6$ case follows from interpolating between $p = 6$ and $p = \infty$).

By (14) we may write $a_n = (n-1)/N + e_n$, where $e_n = a_n - (n-1)/N \sim (n-1)^2/N^2$. For every $t \in \mathbb{R}$ we may write it as $t_1 + t_2$, where $t_1 \in 2\pi N\mathbb{Z}$ and $t_2 \in [0, 2\pi N)$. Without loss of generality we assume $2\pi N$ divides T, so $(2\pi)^{-1}N^{-1}T \in \mathbb{Z}$. Now we may write

$$\begin{aligned}\int_0^T \Big|\sum_{n=1}^{N^{1/2}} b_n e^{ita_n}\Big|^p dt &= \sum_{t_1\in 2\pi N\mathbb{Z}\cap[0,T-2\pi N]} \int_0^{2\pi N} \Big|\sum_{n=1}^{N^{1/2}} b_n e^{i(t_1+t_2)(\frac{n-1}{N}+e_n)}\Big|^p dt_2 \\ &= \sum_{t_1\in 2\pi N\mathbb{Z}\cap[0,T-2\pi N]} \int_0^{2\pi N} \Big|\sum_{n=1}^{N^{1/2}} b_n e^{i(t_1e_n+t_2\frac{n-1}{N}+t_2e_n)}\Big|^p dt_2.\end{aligned}$$

We write $e(n) = e_n$ and let $e : [1, N^{1/2}] \to \mathbb{R}$ be the piecewise linear function such that, for every $n \in \mathbb{Z} \cap [1, N^{1/2} - 1]$, $e(x)$ is linear on $[n, n+1]$ and $e(n) = e_n$. Since $e_{n+1} - e_n \sim n/N^2$, we have $|e'(x)| \lesssim 1/N^{3/2}$ for $x \in [1, N^{1/2}] \setminus \mathbb{Z}$.

By Abel's summation formula we have

$$\begin{aligned}\Big|\sum_{n=1}^{N^{1/2}} b_n e^{i\left(t_1 e_n + t_2\frac{n-1}{N} + t_2 e_n\right)}\Big| &\le \Big|\sum_{n=1}^{N^{1/2}} b_n e^{i\left(t_1 e_n + t_2\frac{n-1}{N}\right)}\Big| + \int_1^{N^{1/2}} \Big|\sum_{n=1}^{\lfloor u\rfloor} b_n e^{i\left(t_1 e_n + t_2\frac{n-1}{N}\right)}\Big| |t_2 e'(u)|\,du \\ &\lesssim \Big|\sum_{n=1}^{N^{1/2}} b_n e^{i\left(t_1 e_n + t_2\frac{n-1}{N}\right)}\Big| + \frac{1}{N^{\frac12}}\int_1^{N^{1/2}} \Big|\sum_{n=1}^{\lfloor u\rfloor} b_n e^{i\left(t_1 e_n + t_2\frac{n-1}{N}\right)}\Big|\,du. \qquad (97)\end{aligned}$$

The last inequality uses $t_2 \lesssim N$.

We first estimate

$$A := \sum_{t_1\in 2\pi N\mathbb{Z}\cap[0,T-2\pi N]} \int_0^{2\pi N} \Big|\sum_{n=1}^{N^{1/2}} b_n e^{i\left(t_1 e_n + t_2\frac{n-1}{N}\right)}\Big|^p\,dt_2.$$

Since $e_n \lesssim \frac{1}{N}$ for every $1 \le n \le N^{1/2}$, $\sum_{n=1}^{N^{1/2}} b_n e^{i(t_1 e_n + t_2(n-1)/N)}$ is locally constant on intervals of length N in t_1, that is, for every $y \in \mathbb{R}$,

$$\sup_{t_1\in[y,y+2\pi N]} \Big|\sum_{n=1}^{N^{1/2}} b_n e^{i\left(t_1 e_n + t_2\frac{n-1}{N}\right)}\Big| \lesssim \left(\int_{\mathbb{R}} \Big|\sum_{n=1}^{N^{1/2}} b_n e^{i\left(t_1 e_n + t_2\frac{n-1}{N}\right)}\Big|^p W_{[y,y+2\pi N],100}(t_1)\,dt_1\right)^{\frac1p}.$$

We note that the above is also a special case of Proposition 2.3, applied to a fat AP that is just a single interval. Since $\sum_{y\in 2\pi N\mathbb{Z}\cap[0,T-2\pi N]} W_{[y,y+2\pi N],100}(t_1) \lesssim W_{[0,T],100}(t_1)$, we have

$$A \lesssim \frac{1}{N}\int_{\mathbb{R}}\int_0^{2\pi N} \Big|\sum_{n=1}^{N^{1/2}} b_n e^{i\left(t_1 e_n + t_2\frac{n-1}{N}\right)}\Big|^p\,dt_2\, W_{[0,T],100}(t_1)\,dt_1. \qquad (98)$$

We consider two cases, $T \ge N^{3/2}$ and $T \le N^{3/2}$.

<u>Case 1</u>: $T \ge N^{3/2}$. We observe that $\sum_{n=1}^{N^{1/2}} b_n e^{i(t_1 e_n + t_2(n-1)/N)}$ is $2\pi N$-periodic in t_2, so we have

$$A \lesssim \frac{1}{N}\frac{N^{\frac32}}{T}\int_{\mathbb{R}}\int_0^{TN^{-1/2}} \Big|\sum_{n=1}^{N^{1/2}} b_n e^{i\left(t_1 e_n + t_2\frac{n-1}{N}\right)}\Big|^p\,dt_2\, W_{[0,T],100}(t_1)\,dt_1$$

By a change of variable $t_1 \mapsto N^1 t_1$, $t_2 \mapsto N^{1/2} t_2$, we obtain

$$A \lesssim N^{\frac12}\frac{N^{\frac32}}{T}\int_{\mathbb{R}}\int_{\mathbb{R}} \Big|\sum_{n=1}^{N^{1/2}} b_n e^{i\left(t_1 e_n N + t_2\frac{n-1}{N^{1/2}}\right)}\Big|^p W_{B_{TN^{-1}}(0),100}(t_1,t_2)\,dt_2\,dt_1.$$

Now we let $g(x)$ be a C^2 strictly convex function defined on $[0,1]$ such that $|g((n-1)/N^{1/2}) - e_n N| \le N^{-1}/4$ for $n = 1,\dots,N^{1/2}$. (See Lemma A.3 below.) Since $N^{-1} \le T^{-1}N$, we have for every n, the ball of radius $T^{-1}N/4$ centered at $((n-1)/N^{1/2}, e_n N)$ fits in exactly one of the γ in the partition of the $T^{-1}N$ neighborhood of $G = \{(x,g(x)) : x\in[0,1]\}$ by $N^{-1/2}\times T^{-1}N$ rectangles. Under our assumption that $T \in [N^{3/2}, N^2]$ we have $\log(N^{-1/2})/\log(T^{-1}N) \in \left[\frac12, 1\right]$. Therefore we may apply Theorem A.2

with $R = TN^{-1}$, $R^\alpha = N^{1/2}$ to the curve G, which yields for every $T \in [N^{3/2}, N^2]$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\sum_{n=1}^{N^{1/2}} b_n e^{i\left(t_1 e_n N + t_2 \frac{n-1}{N^{1/2}}\right)}\right|^p W_{B_{TN^{-1}}(0),100}(t_1,t_2)\,dt_2\,dt_1 \lesssim_\varepsilon N^\varepsilon \left(T^{\frac{1}{p}} N^{\frac{1}{2}-\frac{2}{p}} + T^{\frac{2}{p}} N^{\frac{1}{4}-\frac{5}{2p}}\right)^p \|b_n\|_{\ell^p}^p. \tag{99}$$

Hence

$$A \lesssim_\varepsilon N^\varepsilon \left(N^{\frac{1}{2}} + T^{\frac{1}{p}} N^{\frac{1}{4}-\frac{1}{2p}}\right)^p \|b_n\|_{\ell^p}^p.$$

<u>Case 2</u>: $T \le N^{3/2}$. From (98) and a change of variable we have

$$A \lesssim N^{\frac{1}{2}} \int_{\mathbb{R}}\int_0^{2\pi N^{1/2}} \left|\sum_{n=1}^{N^{1/2}} b_n e^{i\left(t_1 e_n N + t_2 \frac{n-1}{N^{1/2}}\right)}\right|^p dt_2\, W_{[0,TN^{-1}],100}(t_1)\,dt_1.$$

Since $T \le N^{3/2}$, we may bound the right-hand side trivially by

$$N^{\frac{1}{2}} \int_{\mathbb{R}}\int_{\mathbb{R}}\left|\sum_{n=1}^{N^{1/2}} b_n e^{i\left(t_1 e_n N + t_2 \frac{n-1}{N^{1/2}}\right)}\right|^p W_{B_{N^{1/2}}(0),100}(t_1,t_2)\,dt_2\,dt_1,$$

so by (99) with $T = N^{3/2}$ we have

$$A \lesssim_\varepsilon N^\varepsilon N^{\frac{1}{2}} \left(N^{\frac{3}{2p}} N^{\frac{1}{2}-\frac{2}{p}} + N^{\frac{3}{2}\frac{2}{p}} N^{\frac{1}{4}-\frac{5}{2p}}\right)^p \|b_n\|_{\ell^p}^p.$$

Since $p \ge 4$ we may verify

$$N^{\frac{3}{2p}} N^{\frac{1}{2}-\frac{2}{p}} \ge N^{\frac{3}{2}\frac{2}{p}} N^{\frac{1}{4}-\frac{5}{2p}}.$$

Hence

$$A \lesssim_\varepsilon N^\varepsilon (N^{\frac{1}{2}})^p \|b_n\|_{\ell^p}^p.$$

In conclusion we have shown

$$A \lesssim_\varepsilon N^\varepsilon \left(N^{\frac{1}{2}} + T^{\frac{1}{p}} N^{\frac{1}{4}-\frac{1}{2p}}\right)^p \|b_n\|_{\ell^p}^p. \tag{100}$$

Next we estimate the second term in (97). We define

$$B := \sum_{t_1 \in 2\pi N\mathbb{Z}\cap[0,N^2/L^2-2\pi N]} \int_0^{2\pi N} \left|\frac{1}{N^{\frac{1}{2}}}\int_1^{N^{1/2}} \left|\sum_{n=1}^{\lfloor u\rfloor} b_n e^{i\left(t_1 e_n + t_2 \frac{n-1}{N}\right)}\right| du\right|^p dt_2.$$

By Minkowski's inequality we have

$$B^{\frac{1}{p}} \le \frac{1}{N^{\frac{1}{2}}}\int_1^{N^{1/2}} \left(\sum_{t_1 \in 2\pi N\mathbb{Z}\cap[0,T-2\pi N]} \int_0^{2\pi N} \left|\sum_{n=1}^{\lfloor u\rfloor} b_n e^{i\left(t_1 e_n + t_2 \frac{n-1}{N}\right)}\right|^p dt_2\right)^{\frac{1}{p}} du.$$

Then applying (100) to the expression in the brackets we obtain

$$\begin{aligned} B^{\frac{1}{p}} &\lesssim_\varepsilon N^\varepsilon \frac{1}{N^{\frac{1}{2}}}\int_0^{N^{1/2}} \left(N^{\frac{1}{2}} + T^{\frac{1}{p}} N^{\frac{1}{4}-\frac{1}{2p}}\right)\|b_n\|_{\ell^p}\,du \\ &= N^\varepsilon \left(N^{\frac{1}{2}} + T^{\frac{1}{p}} N^{\frac{1}{4}-\frac{1}{2p}}\right)\|b_n\|_{\ell^p}. \end{aligned}$$

Combining the estimates for A and B we conclude

$$\Big\|\sum_{n=1}^{N^{1/2}} b_n e^{ita_n}\Big\|_{L^p(B_T)} \lesssim_\varepsilon N^\varepsilon (N^{\frac{1}{2}} + T^{\frac{1}{p}} N^{\frac{1}{4}-\frac{1}{2p}})\|b_n\|_{\ell^p}.$$

We used the following lemma in the proof above.

Lemma A.3. *Suppose $\{a_n\}_{n=1}^{N^{1/2}}$ is a short generalized Dirichlet sequence with $\theta = 1$, $a_2 - a_1 = N^{-1}$, $a_1 = 0$. Let $e_n = a_n - (n-1)/N$. Then, for every $c > 0$, there exists a C^2 curve $g : [0,1] \to \mathbb{R}$ with $g''(x) \sim 1$ for $x \in [0,1]$ such that $|g((n-1)/N^{1/2}) - e_n N| \le cN^{-1}$ for every $n = 1, \dots, N^{1/2}$.*

Proof. We first define $g_0 : [0,1] \to \mathbb{R}$ to be a C^1 piecewise quadratic polynomial with $g_0'(0) = 0$ such that g_0 restricted to $[n/N^{1/2}, (n+1)/N^{1/2}]$ is a quadratic polynomial for every $n = 0, \dots, N^{1/2}-1$, and

$$g_0\Big(\frac{n-1}{N^{\frac{1}{2}}}\Big) = e_n N.$$

Since

$$\frac{N(e_{n+1} - 2e_n + e_{n-1})}{N^{-1}} \sim 1,$$

we have $g_0'' \sim 1$ on $[0,1] \setminus N^{-1/2}\mathbb{Z}$, and consequently $\|g_0\|_{L^\infty([0,1])} \lesssim 1$ because $g_0'(0) = 0$. Now we let $g = g_0 * \phi$ be the $c'N^{-1}$ mollification of g_0. Here ϕ is an L^1-normalized smooth bump adapted to $B_{c'N^{-1}}(0)$ and $c' > 0$ is sufficiently small depending on c. Then we have, for every $x \in [0,1]$,

$$g''(x) = \int_{\mathbb{R}} g_0''(y)\phi(x-y)\,dy \sim 1,$$

and

$$\Big|g\Big(\frac{n-1}{N^{\frac{1}{2}}}\Big) - e_n N\Big| \le \int_{\mathbb{R}} \Big|g_0(y) - g_0\Big(\frac{n-1}{N^{\frac{1}{2}}}\Big)\Big| \phi\Big(\frac{n-1}{N^{\frac{1}{2}}} - y\Big)\,dy \le c'N^{-1} \sup_{y\in[0,1]} |g_0'| \le cN^{-1}$$

if $c' = c/(\|g_0'\|_{L^\infty([0,1])} + 1)$. □

We can use the same approach to transfer an L^p estimate for a longer generalized Dirichlet polynomial to an L^p estimate on an exponential sum with frequency support near a C^2 convex curve.

Suppose $\{a_n\}_{n=1}^N$ is a generalized Dirichlet sequence with $\theta = 1$, $a_2 - a_1 = 1/N$, $a_1 = 0$, and let $\alpha \in \big(\frac{1}{2}, 1\big]$. As before we write $e_n = (n-1)/N \sim ((n-1)^2)/N^2$. The same calculation as above shows that

$$\int_{[0,T]} \Big|\sum_{n=1}^{N^\alpha} b_n e^{ita_n}\Big|^p dt \lesssim \sum_{t_1 \in 2\pi N\mathbb{Z} \cap [0, T-2\pi N]} \int_0^{2\pi N} \Big|\sum_{n=1}^{N^\alpha} b_n e^{i(t_1 e_n + t_2 \frac{n-1}{N} + t_2 e_n)}\Big|^p dt_2.$$

One difficulty that appears is that we cannot treat $e^{it_2 e_n}$ as an error term as before. This is because when we apply the partial summation formula we get

$$\Big|\sum_{n=1}^{N^\alpha} b_n e^{i(t_1 e_n + t_2 \frac{n-1}{N} + t_2 e_n)}\Big| \lesssim \Big|\sum_{n=1}^{N^\alpha} b_n e^{i(t_1 e_n + t_2 \frac{n-1}{N})}\Big| + \frac{1}{N^{1-\alpha}} \int_1^{N^\alpha} \Big|\sum_{n=1}^{\lfloor u \rfloor} b_n e^{i(t_1 e_n + t_2 \frac{n-1}{N})}\Big|\,du.$$

However now $N^{1-\alpha} > N^{\alpha}$ and we cannot estimate the second term on the right-hand side as before using the estimate for the first term and Minkowski's inequality. We could still find a C^2 convex curve such that $((n-1)/N + e_n, e_n)$ lies in an N^{-1}-neighborhood of it, but the extra e_n doesn't allow us to use the $2\pi N$-periodicity in the t_2-variable.

Another difficulty we find is the integrand is locally constant on intervals of length $N^{2-2\alpha}$ in the t_1-variable, and since $N < N^{2-2\alpha}$, that prevents us from transferring the discrete summation into $\sum_{t_1 \in 2\pi N\mathbb{Z} \cap [0, T-2\pi N]}$ into $\int_{[0,T]}$. We may though transfer the discrete sum into an integral over a fat AP $\int_{P_{2\pi N}^{N^{2-2\alpha}} \cap B_{[0,T]}}$, and that might suggest some new decoupling problems in $\mathbb{R}^2$ that might be helpful for estimating longer generalized Dirichlet polynomials.

Finally we remark that for the Dirichlet sequence $\{\log n\}_{n=N+1}^{2N}$, we may implement this transference method to higher-order approximations of $\log n$. For examples we can write

$$\left|\sum_{n=N+1}^{N+N^{\alpha}} b_n e^{it\log n}\right| = \left|\sum_{n=1}^{N^{\alpha}} b_{n+N} e^{it\log\left(1+\frac{n}{N}\right)}\right| = \left|\sum_{n=1}^{N^{\alpha}} b_{n+N} e^{it\left(\frac{n}{N} - \frac{n^2}{2N^2} + e_n'\right)}\right|,$$

where

$$e_n' := \log\left(1+\frac{n}{N}\right) - \frac{n}{N} + \frac{n^2}{2N^2} \sim \frac{n^3}{N^3}.$$

If we write $t = t_1 + t_2 + t_3$ with $t_1 \in 2\pi N^2\mathbb{Z},\ t_2 \in 2\pi N\mathbb{Z},\ t_3 \in [0, 2\pi N)$, then we could transfer L^p estimates on $\sum_{n=N+1}^{N+N^{\alpha}} b_n e^{it\log n}$ to 3-dimensional L^p estimates on exponential sums with frequency supported on a nondegenerate curve in $\mathbb{R}^3$. More generally one can exploit more terms in the Taylor expansion and get higher-dimensional estimates. We do not know how much this would help with estimates on Dirichlet polynomials using decoupling techniques.

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GLOBAL WELL-POSEDNESS OF VLASOV–POISSON-TYPE SYSTEMS IN BOUNDED DOMAINS

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In this paper we prove global existence of classical solutions to the Vlasov–Poisson and ionic Vlasov–Poisson models in bounded domains. On the boundary, we consider the specular reflection boundary condition for the Vlasov equation and either homogeneous Dirichlet or Neumann conditions for the Poisson equations.

1. Introduction

Here, we investigate the well-posedness of Vlasov–Poisson models in bounded domains. These models describe the evolution of particles in a plasma, which is an ionised gas mostly constituted of two species of charged particles: ions and electrons. Due to the significant difference in size between those two species, the former being much larger and slower than the latter, it is classical to decouple their dynamics.

On the one hand, when investigating the behaviour of electrons it is reasonable to assume that the ions are stationary. Assuming the plasma has low density and that the velocity of the particles is significantly lower than the speed of light — i.e., neglecting electron-electron collisions and magnetic forces — one can model the evolution of the distribution function of the electrons $f = f(t, x, v)$, which represents at time t the probability of finding an electron at position x with velocity v, by the Vlasov–Poisson system

$$(VP) := \begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0 & \text{in } (0,+\infty) \times \Omega \times \mathbb{R}^d, \\ E = -\nabla U, \quad \Delta U = -\rho & \text{in } (0,+\infty) \times \Omega, \\ f|_{t=0} = f_0 & \text{in } \Omega \times \mathbb{R}^d, \end{cases} \tag{1}$$

where $\rho = \int_{\mathbb{R}^d} f \,\mathrm{d}v$ is the macroscopic density. In this model, the Vlasov equation describes the transport of the electrons under the influence of the electric field E, while the Poisson equation models how the electric potential U is generated by the distribution of the electrons. We shall always assume that the initial distribution f_0 is nonnegative and normalized:

$$f_0 \geq 0, \quad \iint_{\Omega \times \mathbb{R}^d} f_0 \,\mathrm{d}x \,\mathrm{d}v = 1. \tag{2}$$

On the other hand, when investigating the behaviour of the ions in the plasma, it is common in physics literature to assume that the electrons are close to thermal equilibrium. Indeed, although electron-electron collisions are neglected in the model above because of their rarity, they become relevant in the ion's timescale and it is reasonable to assume that the distribution of the electrons is the thermal equilibrium of

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a collisional kinetic model. The Vlasov–Poisson model for massless electrons (VPME) — sometimes called ionic Vlasov–Poisson — which is a celebrated model for the evolution of ions, can then be derived asymptotically as the ratio of mass between electrons and ions grows small. For more details on the massless limit we refer to [Bardos et al. 2018], and for a more thorough introduction of this model we refer, e.g., to [Griffin-Pickering and Iacobelli 2021c]. The VPME system consists of a Vlasov equation coupled with a nonlinear Poisson equation, which models how the electric potential is generated by the distribution of the ions and the Maxwell–Boltzmann distribution of the electrons. It reads

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + E\cdot\nabla_v f = 0 & \text{in } (0,+\infty)\times\Omega\times\mathbb{R}^d, \\ E = -\nabla U, \quad \Delta U = e^U - \rho - 1 & \text{in } (0,+\infty)\times\Omega, \\ f|_{t=0} = f_0 & \text{in } \Omega\times\mathbb{R}^d, \end{cases} \tag{3}$$

with the same assumptions on f_0 given in (2).

We consider a bounded $C^{2,1}$ domain $\Omega = \{x\in\mathbb{R}^d : \xi(x) < 0\}$ in $\mathbb{R}^d$, where $\xi:\mathbb{R}^d\to\mathbb{R}$ is a $C^{2,1}$ function, and its boundary $\partial\Omega = \{x\in\mathbb{R}^d : \xi(x) = 0\}$. We assume that Ω is uniformly convex, which means that for some $C_\Omega > 0$ we have

$$v\cdot\nabla^2\xi(x)\cdot v \geq C_\Omega|v|^2 \quad \text{for all } (x,v)\in\bar{\Omega}\times\mathbb{R}^d, \tag{4}$$

where $\nabla^2\xi$ denotes the Hessian matrix of $\xi : v\cdot\nabla^2\xi(x)\cdot v = \sum_{i,j} v_i v_j \partial_{ij}\xi(x)$. We also assume that the normal vectors are well defined on the boundary, i.e., $\nabla\xi(x)\neq 0$ for any x such that $|\xi(x)|\ll 1$. The outward unit normal vector is then defined, for $x\in\partial\Omega$, as $n(x) = \nabla\xi(x)/|\nabla\xi(x)|$.

On the boundary of Ω we need to prescribe the behaviour of the particles in the Vlasov equation as well as the behaviour of the electric potential U in the Poisson equation. For the Vlasov part, the boundary condition takes the form of a balance between the incoming and outgoing traces of f in the phase-space. Namely, if we introduce the sets

$$\gamma_\pm = \{(x,v) : x\in\partial\Omega,\ \pm v\cdot n(x) > 0\}$$

and write $\gamma_\pm f$ for the restriction of the trace of f to $\gamma_\pm$, then the boundary condition takes the form

$$\gamma_- f(t,x,v) = \mathcal{B}[\gamma_+ f](t,x,v) \quad \text{on } (0,+\infty)\times\gamma_-.$$

In this paper, we will focus on the specular reflection boundary condition:

$$\gamma_- f(t,x,v) = \gamma_+ f(t,x,\mathcal{R}_x v) \quad \text{on } (0,+\infty)\times\gamma_- \tag{5}$$

with

$$\mathcal{R}_x v = v - 2(v\cdot n(x))n(x).$$

This means that we assume the boundary is a surface with no asperities and the particles bounce on this surface in a billiard-like fashion.

For the Poisson equation on the electric potential U, we will consider either the homogeneous Dirichlet condition

$$U(t,x) = 0 \quad \text{on } (0,+\infty)\times\partial\Omega \tag{6}$$

or the Neumann boundary condition

$$\partial_n U(t,x) = h \quad \text{on } (0,+\infty)\times\partial\Omega, \tag{7}$$

where $\partial_n U = n(x)\cdot\nabla U$ is the normal derivative of U at $x\in\partial\Omega$. Note that we will require h to satisfy a compatibility condition in order for the system to be well-posed. The Dirichlet boundary condition arises when one assumes that the boundary is a perfect conductor and that it is grounded for the homogeneous case that we consider. On the other hand, the Neumann boundary condition comes down to specifying the value of the electric field E everywhere on the surface. We refer, e.g., to [Jackson 1999, Chapter 1.9] for a more detailed physical interpretation of these conditions.

In the case $\Omega=\mathbb{R}^3$, the Cauchy theory for the Vlasov–Poisson system (1) is well developed. In particular, existence and uniqueness of global-in-time classical C^1 solutions were established in the 90s [Horst 1993; Lions and Perthame 1991; Pfaffelmoser 1992; Schaeffer 1991], and there is also an extended literature on weaker notions of solutions; see, e.g., [Ambrosio et al. 2017; Arsenev 1975; DiPerna and Lions 1988; Horst and Hunze 1984]. The bounded domain case is more challenging due to the fact that singularities may form at the boundary and propagate inside the domain even in one dimension [Guo 1995]. In the case of the half-space, the existence of global classical solutions was proved for the specular reflection condition (5) and both Neumann and Dirichlet conditions on the electric potential [Guo 1994; Hwang and Velázquez 2009]. Note that while Y. Guo proved well-posedness [Guo 1994] by adapting the velocity moments method of P.L. Lions and B. Perthame [Lions and Perthame 1991], H.J. Hwang and J.J.L. Velázquez [Hwang and Velázquez 2009] took a different approach by adapting the ideas of Pfaffelmoser [1992] to the half-space case. In both approaches, the key difficulty is the analysis of the trajectories of the particles, governed by the Vlasov equation, near the singular set γ_0, see (8), where these transport dynamics are degenerate. Hwang and Velázquez [2010] also refined their approach in order to consider uniformly convex domains of class C^5 by means of local changes of coordinates near the singular set γ_0 that allow them to efficiently estimate the effect of the curvature of the boundary on the transport dynamics. In our paper we develop a more global analysis of the trajectories of the particles, without local coordinates, in order to lower the regularity of the boundary to $C^{2,1}$, which is optimal for our notion of classical solutions. Note that weaker notions of solutions have also been investigated in bounded domains; see, for instance, [Abdallah 1994; Alexandre 1993; Fernández-Real 2018; Mischler 2000; Weckler 1995].

For the VPME system, the Cauchy theory is much less developed due to the difficulties arising from the additional nonlinearity of the Poisson equation. In the case $\Omega=\mathbb{R}^3$, global-in-time weak solutions were first constructed by F. Bouchut [Bouchut 1991] and, in one dimension, D. Han-Kwan and M. Iacobelli constructed global weak solutions for measure data with bounded first moment [Han-Kwan and Iacobelli 2017]. More recently, M. Griffin-Pickering and M. Iacobelli proved the global well-posedness of VPME in the torus in dimensions 2 and 3 [Griffin-Pickering and Iacobelli 2021b], and in the whole space in dimension 3 [Griffin-Pickering and Iacobelli 2021a]. They proved existence of strong solutions for measure initial data with bounded moments — strong in the sense that if the initial data is C^1 then the solution they construct is a classical C^1 solution — and uniqueness of solution with bounded density in the spirit of Loeper's uniqueness result for Vlasov–Poisson [2006]. In this paper we present the first result of well-posedness for VPME in bounded domains.

2. Main results

Let us first consider the Vlasov–Poisson system (1). We will prove existence and uniqueness of classical solutions in a bounded domain Ω in dimension 3 with the boundary conditions mentioned above. One of the key difficulties is to control the behaviour of the solution f near the grazing set

$$\gamma_0 := \{(x, v) : x \in \partial\Omega,\ v \cdot n(x) = 0\} \tag{8}$$

where the Vlasov equation with specular reflections is degenerate. In order to avoid having singularities at the initial time we shall assume flatness of the initial distribution near the grazing set. Furthermore, we will also assume compactness of support in velocity and regularity of f_0. Namely, we consider initial distributions f_0 satisfying (2) and

$$f_0 \in C^{1,\mu}(\bar{\Omega} \times \mathbb{R}^3), \quad \mu \in (0, 1), \tag{9}$$

$$\operatorname{supp} f_0 \Subset \bar{\Omega} \times \mathbb{R}^3, \tag{10}$$

$$f_0(x, v) = \text{constant} \quad \text{for all } (x, v) \text{ such that } \alpha(0, x, v) \le \delta_0, \tag{11}$$

where $\alpha(0, x, v)$ is the kinetic distance defined in Definition 3.1 and Lemma 3.3, which measures a distance to the grazing set γ_0, and by $A \Subset B$ we mean that A is a compact subset of B. Before stating our existence result for the Vlasov–Poisson system we also need a compatibility condition on h in the Neumann boundary condition case (7) in order to ensure well-posedness. This condition can be derived by integrating the boundary condition over $\partial\Omega$ and using Green's formula and the Poisson equation:

$$\int_{\partial\Omega} h(x)\,\mathrm{d}x = -\iint_{\Omega\times\mathbb{R}^3} f_0(x, v)\,\mathrm{d}x\,\mathrm{d}v = -1. \tag{12}$$

Finally, we introduce the following notation: for $\mathcal{A} \subseteq \Omega$ or $\Omega \times \mathbb{R}^3$ and $T > 0$, we write

$$C_t^1 C^{1,\mu}([0, T] \times \mathcal{A}) := C^1([0, T]; C^0(\mathcal{A})) \cap L^\infty((0, T); C^{1,\mu}(\mathcal{A})).$$

We now state our main result for the Vlasov–Poisson system.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a $C^{2,1}$ uniformly convex domain, let f_0 satisfy* (2) *and* (9)–(11), *and consider the Vlasov–Poisson system* (1) *with the specular reflection condition* (5) *for the Vlasov equation, and either the Dirichlet boundary condition* (6) *or the Neumann boundary condition* (7) *with h satisfying* (12) *for the Poisson equation. Then there exists a unique classical solution $f \in C_t^1 C^{1,\mu}((0, \infty) \times \Omega \times \mathbb{R}^3)$ and $E \in C_t^1 C^{1,\mu}((0, \infty) \times \Omega)^3$. Moreover, the solution f has a compact support in velocity for all $t \ge 0$.*

Apart from our analysis of the trajectories of transport, our strategy of proof is somewhat classical. We begin in any dimension d with an approximation of the Vlasov–Poisson system by a sequence of linear equations (27). We show, using our analysis of the trajectories of transport, that given a fixed electric field E the Vlasov equation has a solution in the appropriate functional space. Conversely, by classical elliptic regularity, we know that given a regular enough density ρ the Poisson equation in Ω will have a regular solution. This yields sequences of solutions (f^n) and (E^n) to the linear Vlasov and Poisson equations. We then assume boundedness of the velocity uniformly in time, namely that, for all $t \in [0, T]$, the quantity

$$Q^n(t) = \sup\{|v| : (x, v) \in \operatorname{supp} f^n(s),\ 0 \le s \le t\}$$

is bounded by some $K(T) > 0$. Under this assumption we show convergence of the sequences (f^n) and (E^n) to a solution of the Vlasov–Poisson system. Then, we restrict ourself to the case $d = 3$ and remove the assumption of uniformly bounded velocities by proving that if the velocities are initially bounded (10), then $Q(t) = \lim_{n\to\infty} Q^n(t)$ is bounded for all $t \in [0, T]$ via a Pfaffelmoser-type argument. Finally, we conclude the proof of Theorem 2.1 with the global-in-time existence by showing that the bound on $Q(t)$ holds as $T \to \infty$ and prove uniqueness of solution adapting the idea of P.L. Lions and B. Perthame [Lions and Perthame 1991] albeit in an L^1 framework.

This strategy of construction of a solution via an iterative sequence also applies to the VPME case. However, the nonlinearity of the Poisson equation in (3) will remain in the iterative sequence, and therefore classical elliptic regularity will not provide the desired regularity estimates on the force field E. In order to derive such estimates we adopt a calculus of variation approach, identifying the solution of the Poisson equation with the minimiser of an energy functional which will take into account the boundary conditions. Furthermore, we introduce a splitting of the electric potential into a singular part $\hat{U}$ — solution to a linear Poisson equation — and a regular part $\overline{U}$ — solution to a nonlinear elliptic PDE — in the spirit of [Griffin-Pickering and Iacobelli 2021b; Han-Kwan and Iacobelli 2017]. The purpose of this splitting is to isolate the difficulties due to the nonlinearity from those that we can handle via classical elliptic regularity theory. The estimates we derive, which are stated in Proposition 5.1 for the Dirichlet case and Proposition 5.2 for the Neumann case, are crucial for our analysis and can be useful in the study of singular limits for VPME, as done in [Griffin-Pickering and Iacobelli 2020]. In the Neumann boundary condition case this will naturally lead to a compatibility condition on h, which reads as follows:

$$h < 0, \qquad \int_{\partial\Omega} |h| \, \mathrm{d}\sigma(x) < 1 + |\Omega|. \tag{13}$$

We can now state our main result in the VPME case:

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a $C^{2,1}$ uniformly convex domain, let f_0 satisfy* (2) *and* (9)–(11), *and consider the Vlasov–Poisson model for massless electrons* (3) *with the specular reflection condition* (5) *for the Vlasov equation, and either the Dirichlet boundary condition* (6) *or the Neumann boundary condition* (7) *with h satisfying* (13) *for the Poisson equation. Then there exists a unique classical solution $f \in C^1_t C^{1,\mu}((0,\infty) \times \Omega \times \mathbb{R}^3)$ and $E \in C^1_t C^{1,\mu}((0,\infty) \times \Omega)^3$. Moreover, the solution f has a compact support in velocity for all $t \ge 0$.*

For the sake of clarity, we have decided to devote the core of our paper to the Vlasov–Poisson system and treat the VPME case independently in the last section. Since the general method is the same, we will only highlight in that section the differences between the two cases and the modifications required for the VPME case.

The paper is thus organised as follows. In Section 3 we develop our global analysis of the trajectories of particles governed by the Vlasov equation in any dimension d, which culminates in the Velocity Lemma (Lemma 3.3). In Section 4 we focus on the Vlasov–Poisson case and show well-posedness via an approximating sequence of linear problems as explained above and a Pfaffelmoser-like argument in dimension 3 to show boundedness of the velocities. Finally, in Section 5 we derive elliptic regularity estimates for the nonlinear Poisson equation of (3) in any dimension d and outline the proof of well-posedness for the VPME system.

3. Velocity lemma

Consider a uniformly convex $C^{2,1}$ domain Ω and a field $E \in C^{0,1}([0, T] \times \overline{\Omega})^d$ satisfying $E(t, x) \cdot n(x) > 0$ for all $t \in [0, T]$ and $x \in \partial\Omega$. Note that the latter condition on E holds for any strong solution to the Poisson equation thanks to the Hopf lemma in the Dirichlet case, and by assumption in the Neumann case.

The characteristic curves associated to the Vlasov equation with specular reflections

$$(X_s, V_s) = (X(s; t, x, v), V(s; t, x, v))$$

are governed by the following ODE system:

$$\partial_s X_s = V_s, \qquad X(t; t, x, v) = x, \tag{14}$$

$$\partial_s V_s = E(s, X_s), \qquad V(t; t, x, v) = v, \tag{15}$$

$$V_{s^+} = V_{s^-} - 2(n(X_\tau) \cdot V_{s^-}) n(X_s) \quad \text{for all } s \text{ s.t. } X_s \in \partial\Omega. \tag{16}$$

These trajectories evolve in the phase-space $\overline{\Omega} \times \mathbb{R}^d$. The purpose of this section is to characterise their distance to the grazing set γ_0, and to deduce some control on the number of bounces on $\partial\Omega$ that such trajectories undergo in a finite time interval. To that end we begin by defining a notion of kinetic distance.

Definition 3.1. Consider $\delta > 0$ and define the neighbourhood of the boundary

$$\partial\Omega_\delta := \{x \in \overline{\Omega} : \mathrm{dist}(x, \partial\Omega) < \delta\}.$$

We say that $\alpha : [0, T] \times \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}_+$ is a δ-kinetic distance if α is in $C^0([0, T] \times \overline{\Omega} \times \mathbb{R}^d)$ and satisfies,

$$\text{for all } (t, x, v) \in [0, T] \times \partial\Omega_\delta \times \mathbb{R}^d, \quad [\alpha(t, x, v) = 0] \iff [(x, v) \in \gamma_0]. \tag{17}$$

We then have the following lemma of isolation of the grazing set.

Lemma 3.2. *Consider $E \in C^{0,1}([0, T] \times \overline{\Omega})^d$ and the flow of transport $(X(s; t, x, v), V(s; t, x, v))$ given by* (14)–(16). *If there exists a δ-kinetic distance α such that, for all $(t, x, v) \in [0, T] \times \partial\Omega_\delta \times \mathbb{R}^d$ and $s \in [0, T]$,*

$$C_s^- \alpha(t, x, v) \le \alpha(s, X(s; t, x, v), V(s; t, x, v)) \le C_s^+ \alpha(t, x, v) \tag{18}$$

with $C_s^\pm = C_s^\pm(t - s, x, v) > 0$, then the grazing set γ_0 is isolated in the sense that a trajectory

$$s \to (X(s; t, x, v), V(s; t, x, v))$$

can only reach γ_0 if it starts in γ_0.

Proof. If δ is large enough that $\partial\Omega_\delta = \overline{\Omega}$, then the isolation of the grazing set is a direct consequence of (17) and (18) since the former states that α only cancels on γ_0, while the latter states that α cannot cancel along a trajectory (X_s, V_s) unless it is initially null. When $\partial\Omega_\delta \subsetneq \overline{\Omega}$, then there may exist $(x, v) \notin \partial\Omega_\delta \times \mathbb{R}^d \supset \gamma_0$ such that $\alpha(0, x, v) = 0$. In order to prove Lemma 3.2 it is enough to show that a trajectory (X_s, V_s) starting in $\Omega \times \mathbb{R}^d$ cannot reach γ_0 without going through $(\partial\Omega_\delta \times \mathbb{R}^d) \setminus \gamma_0$. This follows immediately from the continuity of $s \to X(s; t, x, v)$ given by (14)–(16) with $E \in C^{0,1}([0, T] \times \overline{\Omega})^d$. □

We now turn to the main result of this section: the Velocity Lemma which states the existence of a kinetic distance.

Lemma 3.3 (Velocity Lemma). *We consider a $C^{2,1}$ uniformly convex domain Ω and a field E in $C^{0,1}([0, T] \times \Omega)^d$ such that $E(t, x) \cdot \nabla\xi(x) \geq C_0 > 0$ for all $t \in [0, T]$ and $x \in \partial\Omega$. We define*

$$\alpha(s, x, v) = \tfrac{1}{2}(v \cdot \nabla\xi(x))^2 + (v \cdot \nabla^2\xi(x) \cdot v + E(s, x) \cdot \nabla\xi(x))|\xi(x)|. \tag{19}$$

Then there exists $\delta > 0$ such that α is a δ-kinetic distance satisfying (18) *with*

$$C_s^{\pm} = \exp(\pm C_0[(|v| + 1)|s - t| + \|E\|_{L^\infty}(s - t)^2])$$

and $C_0 = C_0(\|\xi\|_{C^{2,1}}, \|E\|_{C^{0,1}})$.

Proof. Since E is continuous and $E(t, x) \cdot \nabla\xi(x) \geq C_0 > 0$ for all $t \in [0, T]$ and $x \in \partial\Omega$, there exists $\delta > 0$ such that $E(t, x) \cdot \nabla\xi(x) > 0$ for all $x \in \partial\Omega_\delta$. In that neighbourhood we have, using (4) and the continuity of $\nabla^2\xi$, that, for all $v \in \mathbb{R}^d$ and $t \in [0, T]$,

$$v \cdot \nabla^2\xi(x) \cdot v + E(t, x) \cdot \nabla\xi(x) > 0,$$

and therefore α only cancels if both $v \cdot \nabla\xi(x) = 0$ and $\xi(x) = 0$, i.e., if $(x, v) \in \gamma_0$; hence α is a δ-kinetic distance.

We will prove that α satisfies (18) by a Grönwall argument, differentiating α along a trajectory. To that end, note that if we write $b = \xi(X_s)$, then $\partial_s b = V_s \cdot \nabla_x \xi(X_s)$ and $\partial^2_{ss} b = V_s \cdot \nabla^2\xi(X_s) \cdot V_s + E(s, X_s) \cdot \nabla\xi(X_s)$, so we have

$$\alpha(s, X_s, V_s) = \tfrac{1}{2}(\partial_s b)^2 - b\partial^2_{ss} b,$$

and we easily compute $\frac{\mathrm{d}}{\mathrm{d}s}\alpha = -b\partial^3_{sss} b$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\alpha(s, X_s, V_s) = |\xi(X_s)|\big(&V_s \cdot (V_s \cdot \nabla^3\xi(X_s) \cdot V_s) + 3E(s, X_s) \cdot \nabla^2\xi(X_s) \cdot V_s \\ &+ (\partial_s E(s, X_s) + V_s \cdot \nabla_x E(s, X_s)) \cdot \nabla\xi(X_s)\big).\end{aligned}$$

Our regularity assumptions on E and ξ yield on the one hand

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}s}\alpha(s, X_s, V_s)\right| &\leq C|\xi(X_s)|(|V_s|^3\|\nabla^3\xi\|_{L^\infty} + |V_s|\|E\|_{C^{0,1}_x}\|\xi\|_{C^{1,1}} + \|\partial_s E\|_{L^\infty}\|\nabla\xi\|_{L^\infty}) \\ &\leq C|\xi(X_s)|(|V_s|^3 + |V_s| + 1),\end{aligned}$$

and on the other hand, since Ω is uniformly convex by (4),

$$\alpha(s, X_s, V_s) \geq C|\xi(X_s)|(C_\Omega|V_s|^2 + 1) \tag{20}$$

with $C_\Omega > 0$. Hence

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\alpha(s, X_s, V_s)\right| \le C(|V_s|+1)\alpha(s, X_s, V_s),$$

and Grönwall's lemma concludes the proof with (15). □

Remark 3.4. Note that if we only assume that Ω is convex but not uniformly convex, i.e., $C_\Omega \ge 0$ in (4), then α given by (19) is still a δ-kinetic distance. However, for $s \le t$, the constant $C_s^\pm$ in (18) given by the Grönwall argument is controlled by

$$\begin{aligned} C_s^\pm &\le \exp\left(\pm C\left[(|v|+1)|s-t| + \|E\|_{L^\infty}(s-t)^2 + \int_s^t |V_\tau|\,\mathrm{d}\tau\right]\right) \\ &\le \exp\left(\pm C\left[(|v|+1)|s-t| + \|E\|_{L^\infty}(s-t)^2 + \frac{(|v|+\|E\|_{L^\infty}(s-t))^4}{4\|E\|_{L^\infty}} - \frac{|v|^4}{4\|E\|_{L^\infty}}\right]\right). \end{aligned}$$

We will see in Section 4C that this bound is not enough to close our proof of existence of classical solutions to the Vlasov–Poisson systems. We believe that the nonuniformly convex domain case actually requires a much finer characterisation of the isolation of the grazing set.

The Velocity Lemma is an essential part of this proof because it is directly related to the number of reflections on the boundary that a trajectory $s \to (X(s; t, x, v), V(s; t, x, v))$ undergoes within a given time interval, and consequently with the fact that said trajectory is uniquely defined. Morally, the closer you are to the grazing set the more reflections can happen. We characterise this relation in the following lemma in which we establish an upper bound on the number of reflections along a trajectory within a given time interval in terms of the distance of that trajectory to the grazing set.

Lemma 3.5. *Under the assumptions of Lemma 3.3, for any $(t, x, v) \in [0, T] \times \Omega \times \mathbb{R}^d$, the trajectory $s \to (X(s; t, x, v), V(s; t, x, v))$ is uniquely defined and the number of reflections k that it undergoes within an interval of time $s \in (t-\Delta, t)$ is bounded above*:

$$k \le \Delta C_1 \frac{(|v|+\Delta\|E\|_{L^\infty})^2 + \|E\|_{L^\infty}}{\sqrt{\alpha(t, x, v)}} e^{C_0[(|v|+1)\Delta + \|E\|_{L^\infty}\Delta^2]}$$

with $C_1 = C_1(\Omega) > 0$ and C_0 given by Lemma 3.3.

Proof. The fact that the trajectory is uniquely defined follows directly from the upper bound on the number of reflections. Indeed, for any $(t, x, v) \in [0, T] \times (\Omega \times \mathbb{R}^d) \setminus \gamma_0$, if the trajectory $(X(s; t, x, v), V(s; t, x, v))$ undergoes a finite number of reflections in a finite time, then we can construct the trajectory by the composition of a finite number of transports given by the ODE system (14)–(15) and specular reflections (16). The velocity component $s \to V(s; t, x, v)$ will be piecewise continuous since E is in $C^{0,1}([0, T] \times \overline{\Omega})^d$ with discontinuities at the times of the reflections, and since specular reflections do not affect the norm of the velocity, we see that $s \to |V(s; t, x, v)|$ will be continuous. Furthermore, the position component $s \to X(s; t, x, v)$ will be continuous and piecewise C^1.

Let us now fix $(t, x, v) \in [0, T] \times (\overline{\Omega} \times \mathbb{R}^d) \setminus \gamma_0$ and prove the upper bound on the number of reflections. Since $E \in L^\infty([0, T] \times \Omega)$, the norm of the velocity $V(s; t, x, v)$ is uniformly bounded on $(t-\Delta, t)$ as

$$|V(s; t, x, v)| \le M = |v| + \Delta\|E\|_{L^\infty}.$$

Let us consider two consecutive reflection times $t-\Delta \le s_{i+1} < s_i \le t$ such that $X_{s_i} = X(s_i; t, x, v) \in \partial\Omega$, $X_{s_{i+1}} = X(s_{i+1}; t, x, v) \in \partial\Omega$, and, for all $s \in (s_{i+1}, s_i)$, we have $X_s \notin \partial\Omega$. Since the velocity is bounded and by continuity of the transport flow on (s_{i+1}, s_i), we have immediately

$$\frac{|X_{s_i} - X_{s_{i+1}}|}{s_i - s_{i+1}} < M. \tag{21}$$

Furthermore, integrating (14) and (15) over (s_{i+1}, s_i) we get

$$X_{s_i} = X_{s_{i+1}} + \int_{s_{i+1}}^{s_i} \left(V_{s_{i+1}} + \int_{s_{i+1}}^{\tau} E(u, X_u)\,\mathrm{d}u \right) \mathrm{d}\tau = X_{s_{i+1}} + (s_i - s_{i+1}) V_{s_{i+1}} + \int_{s_{i+1}}^{s_i} \int_{s_{i+1}}^{\tau} E(u, X_u)\,\mathrm{d}u\,\mathrm{d}\tau,$$

which yields, multiplying by $\nabla\xi(X_{s_{i+1}})$,

$$|V_{s_{i+1}} \cdot \nabla\xi(X_{s_{i+1}})| \le \left| \frac{X_{s_i} - X_{s_{i+1}}}{s_i - s_{i+1}} \cdot \nabla\xi(X_{s_{i+1}}) \right| + \tfrac{1}{2} \|E\|_{L^\infty} (s_i - s_{i+1}). \tag{22}$$

We know that the norm of $(X_{s_{i+1}} - X_{s_i})/(s_{i+1} - s_i)$ is bounded; we are now interested in its direction. To that end, we write the Taylor expansion of $\xi \in C^{2,1}$:

$$\begin{aligned} \xi(X_{s_i}) = \xi(X_{s_{i+1}}) &+ (X_{s_i} - X_{s_{i+1}}) \cdot \nabla\xi(X_{s_{i+1}}) \\ &+ \frac{1}{2} \int_0^1 (X_{s_i} - X_{s_{i+1}}) \cdot \nabla^2\xi(X_{s_{i+1}} + t(X_{s_i} - X_{s_{i+1}})) \cdot (X_{s_i} - X_{s_{i+1}})\,\mathrm{d}t. \end{aligned} \tag{23}$$

Since ξ cancels both at $X_{s_{i+1}}$ and $X_{s_i} \in \partial\Omega$, we get

$$|(X_{s_i} - X_{s_{i+1}}) \cdot \nabla\xi(X_{s_{i+1}})| \le \tfrac{1}{2} |X_{s_i} - X_{s_{i+1}}|^2 \|\nabla^2\xi\|_{L^\infty}.$$

Together with (22) and (21) this yields

$$|V_{s_{i+1}} \cdot \nabla\xi(X_{s_{i+1}})| \le \tfrac{1}{2} \|\nabla^2\xi\|_{L^\infty} M^2 (s_i - s_{i+1}) + \tfrac{1}{2} \|E\|_{L^\infty} (s_i - s_{i+1}).$$

Furthermore, by definition of α from (19), $\alpha(s_{i+1}, X_{s_{i+1}} V_{s_{i+1}}) = |V_{s_{i+1}} \cdot \nabla\xi(X_{s_{i+1}})|^2$; hence we get

$$|s_i - s_{i+1}| \ge C \frac{\sqrt{\alpha(s_{i+1}, X_{s_{i+1}} V_{s_{i+1}})}}{M^2 + \|E\|_{L^\infty}}$$

with $C = C(\Omega)$. The result then follows from the Velocity Lemma (Lemma 3.3) since $k \le \sup_i \Delta/(s_i - s_{i+1})$ by construction. $\square$

4. The Vlasov–Poisson system

We will construct a solution to the Vlasov–Poisson equation as a limit of an iterative sequence defined as follows:

For any $n \ge 0$, we consider an initial data f_0^n satisfying

$$f_0^n \in C^{1,\mu}(\overline{\Omega} \times \mathbb{R}^d), \quad f_0^n \ge 0, \tag{24}$$

$$\operatorname{supp} f_0^n \Subset \overline{\Omega} \times \mathbb{R}^d, \tag{25}$$

$$f_0^n(x, v) = \text{constant} \quad \text{for all } (x, v) \text{ such that } \alpha_n(0, x, v) \le \delta_0, \tag{26}$$

where $\delta_0 > 0$ is fixed, $\mu \in (0, 1)$, and α_n is the δ-kinetic distance defined in (19) using the field E^n defined below, initiated with the stationary field $E^0(x) = -\nabla U^0$ and $\Delta U^0 = -\int f_0 \,\mathrm{d}v$ with Dirichlet (6) or Neumann (7)-(12) boundary conditions. Note that the fields E^n are indeed regular enough for this kinetic distance to exist; see Corollary 4.2.

We then define the sequences f^n and E^n for $n \geq 1$ as

$$\begin{cases} \partial_t f^n + v \cdot \nabla_x f^n + E^{n-1} \cdot \nabla_v f^n = 0 & \text{in } (0, T] \times \Omega \times \mathbb{R}^d, \\ E^n(t, x) = -\nabla U^n, \quad \Delta U^n = -\int_{\mathbb{R}^d} f^n \,\mathrm{d}v & \text{in } (0, T] \times \Omega, \\ f^n(0, x, v) = f_0^{n-1}(x, v) & \text{in } \Omega \times \mathbb{R}^d, \end{cases} \tag{27}$$

with the specular reflection boundary condition (5) for every f_n and either the Dirichlet (6) or the Neumann boundary condition (7) for every U_n, with $h \in C^{1,\mu}(\partial\Omega)$ satisfying (12) in the latter case.

4A. *Well-posedness of the linear problem.* We prove well-posedness of (27) in two steps. First we consider the Vlasov equation with a fixed electric field $E \in C_t^0 C^{1,\mu}([0, T] \times \Omega)^d$ and prove existence and uniqueness of a solution $f \in C_t^1 C^{1,\mu}([0, T] \times \Omega \times \mathbb{R}^d)$ in Theorem 4.1. Secondly, classical elliptic PDE theory yields the converse, namely the existence and uniqueness of a solution $E \in C_t^1 C^{1,\mu}([0, T] \times \Omega)^d$ to the Poisson equation for a fixed $\rho \in C_t^1 C^{0,\mu}([0, T] \times \Omega)$. Combining these two results, we finally state in Corollary 4.2 the well-posedness of (27).

Theorem 4.1. *Consider a $C^{2,1}$ domain Ω and a fixed electric field $E \in C_t^0 C^{1,\mu}([0, T] \times \overline{\Omega})^d$, $\mu \in (0, 1)$, satisfying $E(t, x) \cdot \nabla\xi(x) \geq C_0 > 0$ for all $t \in [0, T]$ and $x \in \partial\Omega$. For all f_0 satisfying* (24)–(26) *with the kinetic distance associated with the field E, there is a unique solution $f \in C_t^1 C^{1,\mu}([0, T] \times \Omega \times \mathbb{R}^d)$ to the linear Vlasov equation*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0 & \textit{in } (t, x, v) \in (0, T] \times \overline{\Omega} \times \mathbb{R}^d, \\ \gamma_- f(t, x, v) = \gamma_+ f(t, x, \mathcal{R}_x v) & \textit{on } (t, x, v) \in (0, T] \times \gamma_-, \\ f|_{t=0} = f_0 & \textit{in } \Omega \times \mathbb{R}^d. \end{cases} \tag{28}$$

Proof. By assumption on E and Lemmas 3.5 and 3.2 we know that α defined in (19) is a δ-kinetic distance, that the grazing set is isolated, and that the flow of transport $(X(s; t, x, v), V(s; t, x, v))$ is uniquely defined. Therefore, there is a unique solution f to our system, which is given by the push-forward of the initial distribution along the flow of transport as expressed by the representation formula

$$f(t, x, v) = f_0(X(0; t, x, v), V(0; t, x, v)). \tag{29}$$

The key subject of this proof is then the regularity of f. Combining the representation formula with the flatness assumption (26) we see that $f(t, x, v)$ is constant if

$$\alpha(0, X(0; t, x, v), V(0; t, x, v)) \leq \delta_0,$$

and the Velocity Lemma (Lemma 3.3) then means that $f(t, x, v)$ is constant if

$$\alpha(t, x, v) \leq \delta_0 e^{C_0[(|v|+1)t + \|E\|_{L^\infty} t^2]} \leq \delta_0(T)$$

with

$$\delta_0(T) := \delta_0 e^{C_0[(Q+1)T + \|E\|_{L^\infty} T^2]}, \quad Q = \sup\{|v|,\ v \in \operatorname{supp}(f_0)\}. \tag{30}$$

As a consequence, it is enough to study the regularity of f away from a neighbourhood of γ_0 where it is constant, i.e., on the set

$$\mathcal{O} = \{(t, x, v) \in [0, T] \times \Omega \times \mathbb{R}^d : \alpha(t, x, v) \geq \delta_0(T)\}. \tag{31}$$

For any $(t, x, v) \in \mathcal{O}$, we know from Lemma 3.2 that the number of reflections k that the trajectory $s \to (X(s; t, x, v), V(s; t, x, v))$ undergoes in the interval of time $[0, t]$ is bounded by

$$k \leq k_\delta(T) = TC_1 \frac{(Q + T\|E\|_{L^\infty})^2 + \|E\|_{L^\infty}}{\sqrt{\delta_0(T)}} e^{C_0[(Q+1)t + \|E\|_{L^\infty} t^2]}. \tag{32}$$

As a consequence the trajectory can be expressed as at most $k_\delta(T)$ compositions of transports inside the domain, governed by the ODEs (14)-(15), and specular reflections on the boundary. By classical ODE theory we know that, since $E \in C_t^0 C^{1,\mu}([0, T] \times \overline{\Omega})^d$, the flow of transport inside the domain will be $C_t^1 C^{1,\mu}([0, T] \times \Omega \times \mathbb{R}^d))$. Moreover, at the boundary, by assumption we have $n(x) \in C^{1,1}(\partial\Omega)$ so the specular reflection operator $\mathcal{R}_x v = v - 2(v \cdot n(x))n(x)$ is $C^{1,1}(\Gamma_+)$. Thus, the entire flow $(X(s; t, x, v), V(s; t, x, v))$ is $C_t^1 C^{1,\mu}([0, T] \times \Omega \times \mathbb{R}^d)$ for all $(t, x, v) \in \mathcal{O}$. □

We conclude this section with the well-posedness of the sequences f^n and E^n.

Corollary 4.2. *Consider a $C^{2,1}$ domain Ω and an initial datum f_0 satisfying* (24)–(26) *with $\mu \in (0, 1)$. Then the sequences f^n and E^n given by* (27) *are globally defined on* $(0, T) \times \Omega \times \mathbb{R}^d$ *and, moreover, we have, for any $T > 0$,*

$$f^n \in C_t^1 C^{1,\mu}([0, T] \times \Omega \times \mathbb{R}^d), \qquad E^n \in C_t^1 C^{1,\mu}([0, T] \times \Omega)^d,$$
$$\|f^n\|_{L^\infty(\Omega\times\mathbb{R}^d)} = \|f_0\|_{L^\infty(\Omega\times\mathbb{R}^d)}, \quad \|f^n\|_{L^1(\Omega\times\mathbb{R}^d)} = \|f_0\|_{L^1(\Omega\times\mathbb{R}^d)}.$$

Proof. We recall that by standard Elliptic regularity theory, see e.g., [Gilbarg and Trudinger 1998, Chapter 6; Nardi 2014], if $f^n \in C_t^1 C^{1,\mu}([0, T] \times \Omega \times \mathbb{R}^d)$ and $\partial\Omega$ is $C^{2,1}$ then the field E^n given by (27) is in $C_t^1 C^{1,\mu}([0, T] \times \Omega)^d$. The well-posedness and the regularity of f^n and E^n follow directly by induction using Theorem 4.1 and this classical elliptic regularity result. The conservation of the L^∞ norm follows from the representation formula (29) and the conservation of the L^1 norm follows from integrating the Vlasov equation in (27) over $\Omega \times \mathbb{R}^d$. □

4B. ***Compactness and convergence with bounded velocity.*** In this section we will prove compactness of the sequences f^n and E^n under the assumption of bounded velocity support. To that end, we introduce

$$Q^n(t) = \sup\{|v| : (x, v) \in \operatorname{supp} f^n(s),\ 0 \leq s \leq t\}, \tag{33}$$

and we shall assume in this section that, for all $t \in [0, T]$, we have $Q^n(t) < K = K(T)$ uniformly in n.

Proposition 4.3. *Consider a $C^{2,1}$ domain Ω and an initial datum f_0 satisfying* (24)–(26) *with $\mu \in (0, 1)$. Assume that, for all $n \in \mathbb{N}$, we have $Q^n(t) \leq K$. Then, for some $n_0 > 0$, the sequences f^n and E^n given by* (27) *satisfy, for all $n \geq n_0$,*

$$\|E^n\|_{C_t^1 C_x^{1,\mu}} \leq C(T) \quad \textit{and} \quad \|f^n\|_{C_t^1 C_{x,v}^{1,\mu}} \leq C(T),$$

where $C(T)$ depends only on T, K, and $\|f_0\|_{L^\infty}$.

Proof. From the uniform bound on $Q^n(t)$ and the conservation of the L^∞ norm in Corollary 4.2, we have

$$\|\rho^n(t)\|_{L^\infty(\Omega)} = \sup_\Omega \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v \le \|f(t)\|_{L^\infty(\Omega\times\mathbb{R}^d)} Q^n(t)^d \le K^d \|f_0\|_{L^\infty(\Omega\times\mathbb{R}^d)}.$$

Hence, by classical elliptic regularity, E^n is log-Lipschitz uniformly in n. This allows for uniform estimates on ρ^{n+1} in $C^{0,\gamma}(\Omega)$ for some $\gamma < 1$. Indeed, we can consider $v, w \in \mathbb{R}^d$, $x, y \in \Omega$, and $s_*^{n+1} \in [0,t]$ such that, for all $s \in (s_*^{n+1}, t)$, trajectories $X^{n+1}(s;t,x,v)$ and $X^{n+1}(s;t,y,v)$ do not undergo any reflections on the boundary. Then, introducing

$$Y^{n+1}(s) = |X^{n+1}(s;t,x,v) - X^{n+1}(s;t,y,w)| + |V^{n+1}(s;t,x,v) - V^{n+1}(s;t,y,w)|$$

we have, using the characteristic equations,

$$\begin{aligned}|\dot{Y}^{n+1}(s)| &\le |V^{n+1}(s;t,x,v) - V^{n+1}(s;t,y,w)| + |E^n(s, X(s;t,x,v)) - E^n(s; X(s;t,y,w))|\\ &\le Y^{n+1}(s) - C|X^{n+1}(s;t,x,v) - X^{n+1}(s;t,y,w)| \log(|X^{n+1}(s;t,x,v) - X^{n+1}(s;t,y,w)|)\\ &\le C Y^{n+1}(s)|\log Y^n(s)|,\end{aligned}$$

and hence

$$Y^{n+1}(s) \le e^{\log Y^{n+1}(t) e^{-C(t-s)}} \le (Y^{n+1}(t))^\gamma \le |x-y|^\gamma + |v-w|^\gamma$$

for all $\gamma \le e^{-C(t-s_*^{n+1})}$. We have proved that the flow $(X^{n+1}(s;t,x,v), V^{n+1}(s;t,x,v))$ is uniformly bounded in some $C^{0,\gamma}(\Omega\times\mathbb{R}^d)$ for $s \in (s_*^{n+1}, t)$.

Analogously to the proof of Theorem 4.1, we introduce the set

$$\mathcal{O}_n = \{(t,x,v) \in [0,T]\times\Omega\times\mathbb{R}^d : \alpha_n(x,v) \ge \delta(T)\} \tag{34}$$

with $\delta(T)$ given by (30), and we know that f^{n+1} is constant on $(\overline{\Omega}\times\mathbb{R}^d)\setminus\mathcal{O}_n$ so it is enough to study its regularity on $\mathcal{O}_n$.

Note that $\delta(T)$ does not depend on n thanks to the uniform bounds on $Q^n(t)$ and $\|E^n\|_{L^\infty}$. Similarly, for any $(t,x,v) \in \mathcal{O}_n$, we can also bound the number of reflections within the interval $(0,t)$ uniformly in n with $k_\delta(T)$ given by (32). Hence the flow $(X^{n+1}(s;t,x,v), V^{n+1}(s;t,x,v))$ can be expressed as at most $k_\delta(T)$ compositions of transports in $C^{0,\gamma}(\Omega\times\mathbb{R}^d)$ and specular reflections, and hence the flow is in $C^{0,\gamma'}(\Omega\times\mathbb{R}^d)$ with $\gamma' \le \gamma^{k_\delta(T)} \le e^{-k_\delta(T)CT}$. Combined with the representation formula (29) this yields a uniform bound of f^{n+1}, and in turn ρ^{n+1}, in $C^{0,\gamma'}(\Omega\times\mathbb{R}^d)$ and $C^{0,\gamma'}(\Omega)$, respectively.

Consequently, E^{n+1} will be in $C^{1,\gamma'}(\Omega)^d$ by classical elliptic regularity, which means the system (27) at rank $n{+}2$ satisfies the assumptions of Theorem 4.1, and our proposition then follows by iteration. Indeed, at rank $n{+}2$ we have $\rho^{n+2} \in C^{1,\gamma'}(\Omega)$ which yields $E^{n+2} \in C^{1,\mu}(\Omega)^d$ — note that the limiting factor for the regularity of E^{n+2} is the regularity of the domain Ω — and finally we get $\rho^{n+3} \in C^{1,\mu}(\Omega)$. □

Remark 4.4. Note that the limiting factor for the regularity is, *in fine*, the specular reflection operator which stops the flow of transport from reaching any regularity above $C^{1,1}$.

Using the uniform bounds derived previously, we now state the following convergence result.

Proposition 4.5. *Consider a $C^{2,1}$ domain Ω and an initial datum f_0 satisfying* (24)–(26) *with $\mu \in (0,1)$. Assume that, for all $n \in \mathbb{N}$, we have $Q^n(t) \le K$. Then, as $n \to \infty$,*

$$(f^n, E^n) \to (f, E) \quad \text{in } C_t^\nu C^{1,\mu'}([0,T] \times \Omega \times \mathbb{R}^d) \times C_t^\nu C^{1,\mu'}([0,T] \times \Omega)^d$$

with $0 < \nu < 1$ and $0 < \mu' < \mu$. Moreover the limits f and E are in $C_t^1 C^{1,\mu}([0,T] \times \Omega \times \mathbb{R}^d)$ and $C_t^1 C^{1,\mu}([0,T] \times \Omega)^d$, respectively, and (f, E) is a solution to the Vlasov–Poisson system (1)–(5) *with either the Dirichlet* (6) *or the Neumann* (7)–(12) *boundary condition.*

Unlike the previous results of this section, this proof does not rely on any geometrical considerations but rather on some standard functional analysis using Proposition 4.3; we refer to [Hwang and Velázquez 2010, Proposition 3] for details.

4C. *Global bound on $Q(t)$ in dimension 3.* We now restrict ourselves to the 3-dimensional case. The purpose of this section is to remove the assumption of bounded velocity support by proving that if f_0 is compactly supported in velocity (10), i.e., $Q(0) < K$, then

$$Q(t) = \sup\{|v| : (x, v) \in \operatorname{supp} f(s),\ 0 \le s \le t\} \tag{35}$$

is uniformly bounded on $[0, T]$.

Proposition 4.6. *Consider a $C^{2,1}$ domain Ω, an initial datum f_0 satisfying* (24)–(26) *with $\mu \in (0,1)$, and the solution $(f, E) \in C_t^1 C^{1,\mu}([0,T] \times \Omega \times \mathbb{R}^3) \times C_t^1 C^{1,\mu}([0,T] \times \Omega)^3$ of the Vlasov–Poisson system given by Proposition 4.5. Then there exists $K(T) < \infty$ depending only on T and f_0 such that*

$$Q(t) \le K(T) \quad \text{for all } t \in [0, T]. \tag{36}$$

Proof. We prove this proposition via an estimation of the acceleration of the velocity along a characteristic trajectory. To that end let us fix a trajectory $(\hat{X}(t), \hat{V}(t))$. We wish to control, for some $\Delta > 0$,

$$\begin{aligned}\int_{t-\Delta}^t |E(s, \hat{X}(s))| \,\mathrm{d}s &\le \int_{t-\Delta}^t \iint_{\Omega\times\mathbb{R}^3} \frac{f(s,y,w)}{|y - \hat{X}(s)|^2} \,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}s + C\Delta \|h\|_{L^\infty} \\ &\le \int_{t-\Delta}^t \iint_{\Omega\times\mathbb{R}^3} \frac{f(t,x,v)}{|X(s;t,x,v) - \hat{X}(s)|^2} \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s + C\Delta \|h\|_{L^\infty},\end{aligned} \tag{37}$$

where we have used the fact that the evolution of the characteristic trajectories is Hamiltonian, and hence $\mathrm{d}y\,\mathrm{d}w = \mathrm{d}X(t;s,y,w)\,\mathrm{d}V(t;s,y,w) = \mathrm{d}x\,\mathrm{d}v$.

Following the approach of Pfaffelmoser [1992], we will split this integral into three parts. We introduce three parameters $\eta, \beta, \gamma > 0$ to be determined later, and define $P = Q^{\eta}$‘, $R = Q^{\beta}$‘, and $\Delta = c_0 Q^{-\gamma}$, with $c_0 > 0$ fixed. We then split the domain of integration as follows:

$$G = \{(s,x,v) \in [t-\Delta, t] \times \Omega \times \mathbb{R}^3 : |v| < P \text{ or } |v - \hat{V}(t)| < P\}, \tag{38}$$

$$B = \{(s,x,v) \in ([t-\Delta, t] \times \Omega \times \mathbb{R}^3) \setminus G : |X(s;t,x,v) - \hat{X}(s)| < \varepsilon_0(v)\}, \tag{39}$$

$$U = \{(s,x,v) \in ([t-\Delta, t] \times \Omega \times \mathbb{R}^3) \setminus G : |X(s;t,x,v) - \hat{X}(s)| > \varepsilon_0(v)\}, \tag{40}$$

with

$$\varepsilon_0(v) = \max\left\{\frac{R}{|v|^3}, \frac{R}{|v - \hat{V}(t)|^3}\right\}. \tag{41}$$

We shall now handle each part of the integration individually. Throughout this section, we are mostly interested in bounding each part of the integral with respect to powers of Q, and since the constants in these bounds will not play a role, we introduce the notation $a \lesssim b$ to denote $a \le Cb$ for some constant C independent of Q. Note that we will also commonly use the notation G, B, or U to mean subsets of $\Omega \times \mathbb{R}^d$ when the time parameter s is fixed, and also write G as a subset of $\mathbb{R}^d$ for fixed (s, x).

Remark 4.7. Note that our definition of ε_0 differs from that of Hwang and Velázquez [2010] which morally includes the term $R/|v - \hat{V}^+(t)|^3$ in the maximum (41), where $\hat{V}^+(t)$ is the specular reflection of $\hat{V}(t)$ at the last time of reflection $s_0 \in [t - \Delta, t]$.

This difference is directly related to the fact that we develop in the paper a global analysis of the trajectories, whereas Hwang and Velázquez developed a localised analysis. In the global framework, the main argument to control the evolution of the velocity is the balance between the number of reflections and the impact of these reflections on the direction of the velocity. Heuristically, the more reflections happen within the interval $(t - \Delta, t)$, the closer the trajectory is to the grazing, i.e., the more tangential the trajectory is at the points of reflections, and hence the lesser the impact of the specular reflection on the direction of the velocity. In this context, it is enough to compare v with $\hat{V}(t)$ so we do not need to add a comparison with $\hat{V}_+(t)$ in the definition of ε_0.

4C1. *The good set.* For the integral over the good set G, the key argument is the following pointwise control of E: for any $(s, x) \in (0, T) \times \Omega$ and $\lambda > 0$,

$$|E(s,x)| \lesssim \int_\Omega \frac{\rho(s,y)}{|x-y|^2}\,\mathrm{d}y \lesssim \|\rho(s)\|_{L^\infty} \int_{|x-y|<\lambda} \frac{\mathrm{d}y}{|y-x|^2} + \|\rho(s)\|_{L^{5/3}} \left(\int_{|y-x|>\lambda} \frac{\mathrm{d}y}{|y-x|^5} \right)^{2/5}$$
$$\lesssim \|\rho(s)\|_{L^\infty} \lambda + \|\rho(s)\|_{L^{5/3}} \lambda^{-4/5}.$$

Choosing λ such that $\|\rho(s)\|_{L^\infty}\lambda = \|\rho(s)\|_{L^{5/3}}\lambda^{-4/5}$ yields

$$|E(s,x)| \lesssim \|\rho(s)\|_{L^\infty}^{4/9} \|\rho(s)\|_{L^{5/3}}^{5/9}.$$

Moreover, the norm $\|\rho(s)\|_{L^{5/3}}$ is related to the kinetic energy of the system: for any $\lambda > 0$,

$$\rho(s,x) \le \int_{|v|<\lambda} f\,\mathrm{d}v + \lambda^{-2} \int_{|v|>\lambda} |v|^2 f\,\mathrm{d}v \lesssim \|f_0\|_{L^\infty} \lambda^3 + \lambda^{-2} \int |v|^2 f\,\mathrm{d}v,$$

and we know that the kinetic energy is bounded:

$$\iint |v|^2 f(t)\,\mathrm{d}v\,\mathrm{d}x := K(t) \le K < \infty$$

by conservation of the total energy of the system; see e.g., [Glassey 1996, Chapter 4]. Hence choosing $\lambda = K^{1/5}$ yields $\|\rho(s)\|_{L^{5/3}} \lesssim K^{3/5}$. Therefore we can bound E pointwise as

$$|E(s,x)| \lesssim \|\rho\|_{L^\infty}^{4/9} \lesssim \left(\int_{\mathbb{R}^3} f(s,x,v)\,\mathrm{d}v \right)^{4/9} \lesssim \left(\|f_0\|_{L^\infty} \int_{|v|<Q} \mathrm{d}v \right)^{4/9} \lesssim Q^{4/3}. \tag{42}$$

We use this bound on E to derive a bound on $w = V(s; t, x, v)$ when $|v| < P$:

$$|w| = |v| + \int_s^t |E(u, X(u))|\,\mathrm{d}u \le P + Q^{4/3}(t-s).$$

Choosing $s \in (t-\Delta, t)$ and $\gamma \geq \frac{4}{3} - \eta \geq 0$ we get $|w| \lesssim P$. This yields a control on the restriction of ρ to the good set G:

$$\rho_G(s, y) := \int_{w \in G} f(s, y, w)\,\mathrm{d}w \lesssim \|f_0\|_{L^\infty} P^3.$$

Let us now use these estimates to control the integral over the good set in (37) with an approach similar to that of the pointwise bound of E above. Since the integral with respect to s will not play a role in this bound we first control the rest: for any $\lambda > 0$,

$$\begin{aligned}
\iint_G \frac{f(s, y, w)}{|y - \hat{X}(s)|^2}\,\mathrm{d}y\,\mathrm{d}w &\lesssim \int_\Omega \frac{\rho_G(s, y)}{|y - \hat{X}(s)|^2}\,\mathrm{d}y \\
&\lesssim \|\rho_G(s)\|_{L^\infty} \int_{|y - \hat{X}(s)| < \lambda} \frac{1}{|y - \hat{X}(s)|^2}\,\mathrm{d}y + \|\rho_G\|_{L^{5/3}} \int_{|y - \hat{X}(s)| > \lambda} \frac{1}{|y - \hat{X}(s)|^2}\,\mathrm{d}y \\
&\lesssim \|\rho_G(s)\|_{L^\infty} \lambda + \|\rho_G\|_{L^{5/3}} \lambda^{-4/5} \\
&\lesssim \|\rho_G(s)\|_{L^\infty}^{4/9} \|\rho_G\|_{L^{5/3}}^{5/9},
\end{aligned}$$

where we chose λ such that

$$\|\rho_G(s)\|_{L^\infty} \lambda = \|\rho_G\|_{L^{5/3}} \lambda^{-4/5}.$$

Note that $\rho_G(s, y) \leq \rho(s, y)$ by positivity of f, and hence we have immediately $\|\rho_G\|_{L^{5/3}} \leq \|\rho\|_{L^{5/3}} \leq K$, the kinetic energy. Finally, we get

$$\iiint_G \frac{f(s, y, w)}{|y - \hat{X}(s)|^2}\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}s \lesssim \Delta P^{4/3}.$$

4C2. *The bad set.* The control of the integral over the bad set B in (37) follows rather immediately from our choice of ε_0:

$$\begin{aligned}
\iiint_B \frac{f(t, x, v)}{|X(s; t, x, v) - \hat{X}(s)|^2}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s &\lesssim \int_{t-\Delta}^t \int_{v \notin G} \|f\|_{L^\infty} \varepsilon_0(v)\,\mathrm{d}v\,\mathrm{d}s \\
&\lesssim \Delta \int_{v \notin G} \max\left\{ \frac{R}{|v|^3}, \frac{R}{|v - \hat{V}(t)|^3}, \frac{R}{|v - \hat{V}^+(t)|^3} \right\} \mathrm{d}v \\
&\lesssim \Delta R \int_P^Q \frac{1}{r}\,\mathrm{d}r \lesssim \Delta R \ln \frac{Q}{P}.
\end{aligned}$$

4C3. *The ugly set.* For the ugly set U, the time-integration of (37) is essential. Let us fix some (t, x, v) and define $\mathcal{W}(s)$ as

$$\mathcal{W}(s) = V(s; t, x, v) - v.$$

We easily deduce from our transport dynamics (14)–(16) the following system of ODEs for the evolution of $\mathcal{W}(s)$:

$$\dot{\mathcal{W}}(s) = E(s, X(s; t, x, v)), \qquad \mathcal{W}(t) = 0, \tag{43}$$

$$\mathcal{W}(\tau) = \mathcal{R}_{X(\tau; t, x, v)} V(\tau; t, x, v) - v \quad \text{for all } \tau \text{ such that } X(\tau; t, x, v) \in \partial\Omega. \tag{44}$$

Note that the reflection condition (44) does not preserve the norm of $\mathcal{W}(\tau)$, as such it is not comparable to specular reflections and the norm $|\mathcal{W}(s)|$ will not be a continuous function of s. Nevertheless, we can bound the jump of $|\mathcal{W}(s)|$ at a reflection time τ using the Velocity Lemma (Lemma 3.3). Indeed, recall that, for any $\tau \in (t-\Delta, t)$, if (x, v) belongs to $\partial\Omega_\delta$ as defined in Lemma 3.2, then the Velocity Lemma yields

$$\begin{aligned}|V(\tau;t,x,v)\cdot n(X(\tau;t,x,v))| &\le c_\Omega\sqrt{\alpha(\tau, X(\tau;t,x,v), V(\tau;t,x,v))}\\ &\le c_\Omega(\alpha(t,x,v)e^{C(|v|+1)(t-\tau)+\|E\|_{L^\infty}(t-\tau)^2})^{1/2}\\ &\le c_\Omega(\alpha(t,x,v)e^{C[(Q+1)\Delta+Q^{4/3}\Delta^2]})^{1/2}\end{aligned}$$

with $c_\Omega = \|\nabla\xi\|_{L^\infty(\partial\Omega)}$, and hence

$$\begin{aligned}|\mathcal{W}(\tau^+)| &= |V(\tau^-;t,x,v) - v - 2(V(\tau^-;t,x,v)\cdot n(X(\tau;t,x,v)))n(X(\tau;t,x,v))|\\ &\le |\mathcal{W}(\tau^-)| + 2c_\Omega(\alpha(t,x,v)e^{C[(Q+1)\Delta+Q^{4/3}\Delta^2]})^{1/2}.\end{aligned} \tag{45}$$

Moreover, from the uniform bound on E given in (42) we know that $\mathcal{W}$ is Lipschitz between reflection times so that, assuming there are k reflections within the interval $(t-\Delta, t)$, we have

$$|\mathcal{W}(s)| \lesssim Q^{4/3}\Delta + 2k(\alpha(t,x,v)e^{C[(Q+1)\Delta+Q^{4/3}\Delta^2]})^{1/2}. \tag{46}$$

If $x \notin \partial\Omega_\delta$, then the coefficient before $|\xi(x)|$ in (19) could take negative values and $\alpha(t,x,v)$ could cancel even though $(x,v)\notin\gamma_0$. However, if $\tau_1 \in (t-\Delta, t)$ is the first reflection time of the backwards trajectory $(X(s;t,x,v), V(s;t,x,v))$, then, by continuity of $X(s)$, the trajectory will reach $\partial\Omega_\delta$ at some time $s = t-\tilde{\delta}_1 \in (\tau_1, t)$ before it reflects on the boundary. In the interval $[t-\tilde{\delta}_1, t]$ there are no reflections, so the evolution of $|\mathcal{W}(s)|$ is bounded by the uniform estimate of the electric field (42). As a consequence, the inequality (45) still holds with $\alpha(t-\tilde{\delta}, X(t-\tilde{\delta};t,x,v), V(t-\tilde{\delta};t,x,v))$ instead of $\alpha(t,x,v)$. Moreover, the same argument holds at any reflection time $\tau_i \in (t-\Delta, t)$, namely there exists a $\tilde{\delta}_i > 0$ such that, if $s \in (\tau_i - \tilde{\delta}_i, \tau_i)$ then $X(s;t,x,v) \in \partial\Omega_\delta$ and the isolation of the grazing set ensures that

$$\begin{aligned}&\alpha(\tau_i-\tilde{\delta}_i, X(\tau_i-\tilde{\delta}_i;t,x,v), V(\tau_i-\tilde{\delta}_i;t,x,v))\\ &\qquad\le \alpha(t-\tilde{\delta}, X(t-\tilde{\delta};t,x,v), V(t-\tilde{\delta};t,x,v))e^{C[(Q+1)(\tau_i-\tilde{\delta}_i-t+\tilde{\delta})+Q^{4/3}(\tau_i-\tilde{\delta}_i-t+\tilde{\delta})^2]}\\ &\qquad\le \alpha(t-\tilde{\delta}, X(t-\tilde{\delta};t,x,v), V(t-\tilde{\delta};t,x,v))e^{C[(Q+1)\Delta+Q^{4/3}\Delta^2]}.\end{aligned}$$

Since (45) is established at a reflection time τ, this control of the kinetic distance holds and (46) follows. Therefore, we define an extension of the kinetic distance as

$$\alpha(t,x,v) := \begin{cases}\alpha(t,x,v) & \text{for all } x \in \partial\Omega_\delta,\\ \alpha(t-\tilde{\delta}, X(t-\tilde{\delta};t,x,v), V(t-\tilde{\delta};t,x,v)) & \text{for all } x \notin \partial\Omega_\delta,\end{cases}$$

and (46) holds without the need to distinguish the cases $x \in \partial\Omega_\delta$ and $x \notin \partial\Omega_\delta$.

Now, we know from Lemma 3.5 that $\alpha(t,x,v)$ yields an upper bound on the number of reflections k. More precisely, with $|v| + \Delta\|E\|_{L^\infty} \sim Q$ (see the proof of Lemma 3.5) and using (42) we have, assuming without loss of generality that $Q \ge 1$,

$$k \lesssim \frac{Q^2\Delta}{\sqrt{\alpha(t,x,v)}}e^{C[(Q+1)\Delta+Q^{4/3}\Delta^2]}$$

from which we deduce the following bound on $|\mathcal{W}(s)|$:

$$|\mathcal{W}(s)| \lesssim Q^{4/3}\Delta + Q^2 \Delta e^{C[(Q+1)\Delta + Q^{4/3}\Delta^2]}.$$

In order for the exponential term to be uniformly bounded in Q, we want ΔQ to decay when Q grows large. Therefore, we choose $\gamma > 1$ and get the estimate

$$|\mathcal{W}(s)| \lesssim c_0 Q^{2-\gamma} \lesssim c_0 P Q^{2-\gamma-\eta} \leq \tfrac{1}{4} P \tag{47}$$

if we choose $1 < \gamma \leq 2-\eta$ with $\eta < \frac{3}{4}$ and the appropriate choice of $c_0 > 0$ in the definition of $\Delta = c_0 Q^{-\gamma}$.

Now that we've established this bound on $|\mathcal{W}(s)|$, the control of the integral over the ugly set in (37) follows rather classically. As a consequence, we shall skip the details which can be found, e.g., in [Schaeffer 1991; Glassey 1996, Section 4.4] and only outline the following steps of the proof. First, note that one can define $\hat{\mathcal{W}}(s) = \hat{V}(s) - \hat{V}(t)$ and derive the same estimate so that we have

$$\begin{aligned}|V(s;t,x,v) - \hat{V}(s)| &\geq |v - \hat{V}(t)| - |V(s;t,x,v) - v| - |\hat{V}(t) - \hat{V}(s)| \\ &\geq |v - \hat{V}(t)| - \tfrac{1}{2}P \geq \tfrac{1}{2}|v - \hat{V}(t)|,\end{aligned}$$

where the last bound follows from the fact that $(s,x,v) \in U$. We now define $Z(s)$ as

$$Z(s) = X(s;t,x,v) - \hat{X}(s)$$

and have immediately $|\dot{Z}(s)| = |V(s;t,x,v) - \hat{V}(s)| \geq \frac{1}{2}|v - \hat{V}(t)|$. Moreover, one can compare $Z(s)$ with the linear approximation $\bar{Z}(s) = Z(s_0) + \dot{Z}(s_0)(s - s_0)$ with $s_0 \in [t-\Delta, t]$ minimizing $|Z(s)|$, and using the uniform bound on $\ddot{Z}(s)$ which follows from (42) one gets

$$|Z(s)| \gtrsim |s - s_0|\,|v - \hat{V}(t)|.$$

Furthermore, considering $(s,x,v) \in U$ for which $\varepsilon_0(v) = \frac{R}{|v|^3}$, the substitution $s \to \tau = |s - s_0|\,|v - \hat{V}(t)|$ yields

$$\int_{t-\Delta}^{t} \frac{\mathrm{d}s}{|Z(s)|^2} \leq \int_{t-\Delta}^{t} \frac{\mathrm{d}s}{|s-s_0|^2 |v - \hat{V}(t)|^2} \leq \frac{1}{|v - \hat{V}(t)|}\left(\int_0^{\varepsilon_0(v)} \frac{1}{\varepsilon_0^2}\,\mathrm{d}\tau + \int_{\varepsilon_0}^{+\infty} \frac{1}{\tau^2}\,\mathrm{d}\tau\right) \lesssim \frac{|v|^3}{R|v - \hat{V}(t)|}.$$

Similarly, for $(s,x,v) \in U$ such that $\varepsilon_0(v) = \frac{R}{|v-\hat{V}(t)|^3}$, one gets

$$\int_{t-\Delta}^{t} \frac{\mathrm{d}s}{|Z(s)|^2} \lesssim \frac{|v - \hat{V}(t)|^3}{R|v - \hat{V}(t)|},$$

so that, for any $(s,x,v) \in U$,

$$\int_{t-\Delta}^{t} \frac{\mathrm{d}s}{|Z(s)|^2} \lesssim \frac{1}{R|v - \hat{V}(t)|}(\min\{|v|, |v - \hat{V}(t)|\})^3 \lesssim \frac{|v|^2}{R}.$$

Using the boundedness of the kinetic energy, we finally derive the estimate

$$\iiint_U \frac{f(t,x,v)}{|X(s;t,x,v) - \hat{X}(s)|^2}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s \lesssim \frac{1}{R}\iint_U |v|^2 f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \lesssim \frac{1}{R}.$$

4C4. *Conclusion of the Pfaffelmoser argument.* Collecting the estimates above, we have

$$\frac{1}{\Delta}\int_{t-\Delta}^{t} |E(s,\hat{X}(s))|\,\mathrm{d}s \lesssim P^{4/3} + R\ln\frac{Q}{P} + \frac{1}{\Delta R} \lesssim Q^{4\eta/3} + Q^{\beta}\ln Q^{1-\eta} + Q^{\gamma-\beta}.$$

To optimize the order of Q on the right-hand side we can take $R = Q^{\beta}\ln^{-1/2} Q$ and choose $\eta = \frac{6}{11}$, $\beta = \frac{8}{11}$, and $\gamma = \frac{16}{11}$, which yields

$$\frac{1}{\Delta}\int_{t-\Delta}^{t} |E(s,\hat{X}(s))|\,\mathrm{d}s \lesssim Q^{8/11}\ln^{1/2} Q.$$

We see in particular that the right-hand side is sublinear in Q so we have proved that

$$\frac{Q(t)-Q(t-\Delta)}{\Delta} \lesssim Q^{8/11}(t)\ln^{1/2} Q(t), \tag{48}$$

and the boundedness of Q follows from a classical iteration procedure; see [Glassey 1996, Section 4.5]. □

4D. ***Proof of Theorem 2.1.*** In the previous sections, we have constructed solutions to the Vlasov–Poisson system in the sense of Theorem 2.1 on a time interval $[0, T]$ in dimension 3. Note, indeed, that due to the Hopf lemma the electric field in the Dirichlet case satisfies $E(t,x)\cdot n(x) < 0$ for all $t \in [0, T]$ and $x \in \partial\Omega$; therefore the Velocity Lemma (Lemma 3.3) applies. To conclude the proof of the theorem we only need to show uniqueness and that this construction holds as $T \to \infty$.

Global solutions: First of all, recall that thanks to the flatness assumption (11) and the isolation of the grazing set given by Lemma 3.3 we can restrict our analysis to the set $\mathcal{O}$ defined in (31), i.e., away from the grazing set, where the number of reflections that any trajectory undergoes within a finite interval of time is uniformly bounded as a consequence of Lemma 3.5. By convergence of $E^n \to E$, the characteristic flow $(X^n(s;t,x,v), V^n(s;t,x,v))$ converges uniformly to $(X(s;t,x,v), V(s;t,x,v))$, and therefore $Q^n(t) \to Q(t)$ uniformly on $[0, T]$; see, e.g., [Hwang and Velázquez 2009, Proposition 5] for details. Therefore, there exists $n_0 > 0$ such that, for all $n \geq n_0$, the assumption $Q^n(t) < K$ in Proposition 4.3 can be removed as it follows from Proposition 4.6 and our choice of initial data. Hence in order to prove that our construction holds as $T \to \infty$ it is enough to show that $K(T)$ given by Proposition 4.6 is finite for any $T > 0$, which follows classically from our bound (48) on the discrete derivative of Q.

Uniqueness: Let us consider (f^1, E^1) and (f^2, E^2), two solutions of the Vlasov–Poisson system in the sense of Theorem 2.1 with the same initial condition f_0. We easily see that the difference $f^1 - f^2$ satisfies

$$\partial_t(f^1-f^2) + v\cdot\nabla_x(f^1-f^2) + E^1\cdot\nabla_v(f^1-f^2) = (E^1-E^2)\cdot\nabla_v f^2.$$

We then consider the characteristic flow $(X^1(s;t,x,v), V^1(s;t,x,v))$ associated with the force field E^1, which is indeed characteristic for the transport operator on the left-hand side of the equation above, and integrate along this flow to write

$$(f^1-f^2)(t,x,v) = \int_0^t (E^1-E^2)(s, X^1(s;t,x,v))\cdot\nabla_v f^2(s, X^1(s;t,x,v), V^1(s;t,x,v))\,\mathrm{d}s,$$

since $(f^1 - f^2)(0, x, v) = 0$ by assumption. Using classical estimates on the Poisson kernel in a $C^{2,1}$ domain Ω we have, for all (s, x),

$$|(E^1 - E^2)(s, x)| \lesssim \int_\Omega \frac{|\rho^1(s, y) - \rho^2(s, y)|}{|x - y|^2} \, \mathrm{d}y, \tag{49}$$

and hence, with Fubini and the fact that the characteristic flow is Hamiltonian,

$$\|(f^1 - f^2)(t)\|_{L^1(\Omega\times\mathbb{R}^3)} \le \int_0^t \int_\Omega |\rho^1(s, y) - \rho^2(s, y)| \iint_{\Omega\times\mathbb{R}^3} \frac{|\nabla_v f^2(s, x, v)|}{|x - y|^2} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}y \, \mathrm{d}s.$$

On the one hand, we have

$$\begin{aligned} \iint_{\Omega\times\mathbb{R}^3} \frac{|\nabla_v f^2(s, x, v)|}{|x - y|^2} \, \mathrm{d}x \, \mathrm{d}v &\le \int_{|y-x|<1} \frac{|\nabla_v f^2(t, y)|}{|x - y|^2} \, \mathrm{d}y + \int_{|y-x|\ge 1} \frac{|\nabla_v f^2(t, y)|}{|x - y|^2} \, \mathrm{d}y \\ &\lesssim \|\nabla_v f^2(t)\|_{L^\infty(\Omega; L^1(\mathbb{R}^3))} + \|\nabla_v f^2(t)\|_{L^1(\Omega\times\mathbb{R}^3)}, \end{aligned}$$

where both norms of $\nabla_v f^2$ are uniformly bounded on $[0, T]$ by assumption. On the other hand, we have

$$\int_\Omega |\rho^1(s, y) - \rho^2(s, y)| \, \mathrm{d}y \le \|(f^1 - f^2)(s)\|_{L^1(\Omega\times\mathbb{R}^3)},$$

and hence

$$\|(f^1 - f^2)(t)\|_{L^1(\Omega\times\mathbb{R}^3)} \le C(T) \int_0^t \|(f^1 - f^2)(s)\|_{L^1(\Omega\times\mathbb{R}^3)} \, \mathrm{d}s,$$

and uniqueness follows from Grönwall's lemma, which concludes the proof of Theorem 2.1. □

5. The ionic Vlasov–Poisson system

We now turn to the VPME system (3). The general strategy of proof for Theorem 2.2 is the same as for Theorem 2.1 in the Vlasov–Poisson system case, with the exception of the elliptic regularity estimate of Corollary 4.2 which does not apply to the nonlinear Poisson equation of (3). Therefore, we will prove analogous elliptic estimates in the next two sections as stated in Proposition 5.1 for the Dirichlet case and Proposition 5.2 for the Neumann case, and in Section 5C we will rather briefly present the proof of Theorem 2.2, focusing on the differences between the VP and the VPME case in order to avoid unnecessary repetitions.

5A. *The Dirichlet case.*

Proposition 5.1. *For any $\rho \ge 0$ in $C^{0,\alpha}(\Omega)$, the nonlinear Poisson equation with Dirichlet boundary condition*

$$\begin{cases} \Delta U = e^U - \rho - 1, & x \in \Omega, \\ U(x) = 0, & x \in \partial\Omega, \end{cases} \tag{50}$$

has a unique solution $U \in H_0^1(\Omega)$. Furthermore, this solution is in $C^{2,\alpha}(\Omega)$ and satisfies $\partial_n U(x) < 0$ for all $x \in \partial\Omega$.

Proof. In the spirit of [Griffin-Pickering and Iacobelli 2021b, Proposition 3.5], we prove existence of a solution in $H_0^1(\Omega)$ via a calculus of variation approach, by finding a minimiser for the energy functional

$$\phi \to \mathcal{E}_D[\phi] := \int_\Omega \Big(\frac{1}{2}|\nabla \phi|^2 + e^\phi - \phi - \rho\phi \Big)\, \mathrm{d}x$$

in $H_0^1(\Omega)$, and proving that this minimiser solves the Euler–Lagrange equation associated with $\mathcal{E}_D$, which is (50). Uniqueness then follows from the strict convexity of $\mathcal{E}_D$.

First of all, for any $\phi \in H_0^1(\Omega)$ let us write $\phi_+(x) = \max(0, \phi(x))$. Since $\rho \geq 0$, we have $-\rho\phi \geq -\rho\phi_+$ and $e^\phi - \phi \geq e^{\phi_+} - \phi_+$ because, for all $x \leq 0$, we have $e^x - x \geq e^0 - 0 = 1$. As a consequence,

$$\mathcal{E}_D[\phi] = \int_\Omega \Big(\frac{1}{2}|\nabla\phi_+|^2 + \frac{1}{2}|\nabla(\phi-\phi_+)|^2 + e^\phi - \phi - \rho\phi \Big)\, \mathrm{d}x \geq \int_\Omega \Big(\frac{1}{2}|\nabla\phi_+|^2 + e^{\phi_+} - \phi_+ - \rho\phi_+ \Big)\, \mathrm{d}x = \mathcal{E}_D(\phi_+).$$

Hence if ϕ minimises $\mathcal{E}_D$ on $H_0^1(\Omega)$ then $\phi = \phi_+$. Moreover, by the strong maximum principle, if $\rho \not\equiv 0$ then $\phi > 0$ in Ω and cancels at the boundary, which implies $\partial_n \phi < 0$ on $\partial\Omega$.

Secondly, we show existence of a minimiser. Let us consider a minimising sequence (ϕ^k), i.e., a sequence in $H_0^1(\Omega)$ such that

$$\mathcal{E}_D[\phi^k] \to \inf_{\phi \in H_0^1(\Omega)} \mathcal{E}_D[\phi] =: m.$$

Note that since $\mathcal{E}_D[\phi_+] \leq \mathcal{E}_D[\phi]$ we can assume without loss of generality that $\phi^k \geq 0$. We want to show that the sequence ϕ^k is uniformly bounded in $H_0^1(\Omega)$. To that end, we first notice that $\mathcal{E}_D(0) = |\Omega|$, and hence, for k large enough,

$$\mathcal{E}_D[\phi^k] \leq |\Omega|.$$

Furthermore, since $\phi^k \geq 0$ we have that $e^{\phi^k} - \phi^k \geq (\phi^k)^2$, and since $\rho \in L^\infty(\Omega)$ we can fix $C > 0$ such that $\frac{1}{2}(\phi^k)^2 \geq \rho\phi^k - C$. As a result, we have

$$|\Omega| \geq \mathcal{E}_D[\phi^k] \geq \int_\Omega \Big(\frac{1}{2}|\nabla\phi^k|^2 + \frac{1}{2}(\phi^k)^2 - C \Big)\, \mathrm{d}x = \frac{1}{2}\|\phi^k\|_{H_0^1(\Omega)} - C|\Omega|,$$

and hence the sequence (ϕ^k) is equibounded in $H_0^1(\Omega)$. The rest of the proof of existence of a solution to (50) follows exactly the proof of [Griffin-Pickering and Iacobelli 2021b, Proposition 3.5], which is a similar result in the torus. For the sake of completeness we outline the main arguments here. The boundedness of (ϕ^k) in $H_0^1(\Omega)$ implies that, up to a subsequence, $\phi^k \to U$ a.e. with U a minimiser of $\mathcal{E}_D$. Finally, one can show that U solves the Euler–Lagrange equation associated with $\mathcal{E}_D$, namely (50), by investigating the limit as $\eta \to 0$ of $(\mathcal{E}_D[U + \eta\varphi] - \mathcal{E}_D[U])/\eta$ for any $\varphi \in C_c^\infty(\overline{\Omega})$.

We now turn to the regularity of U. We split U into a regular part $\hat{U}$ and a singular part $\overline{U}$, solutions to

$$\begin{cases} \Delta\hat{U} = e^{\overline{U}+\hat{U}} - 1, \\ \hat{U}|_{\partial\Omega} = 0, \end{cases} \qquad \begin{cases} \Delta\overline{U} = -\rho, \\ \overline{U}|_{\partial\Omega} = 0. \end{cases} \tag{51}$$

By classical elliptic PDE theory on a $C^{2,1}$ domain Ω (see, e.g., [Evans 1998, Chapter 6]), we know that, for $\rho \in C^{0,\alpha}(\Omega)$, there is a unique solution $\overline{U} \in C_c^{2,\alpha}(\overline{\Omega}) \subset H_0^1(\Omega)$, and consequently we also have a unique $\hat{U} = U - \overline{U} \in H_0^1(\Omega)$. We are now interested in the regularity of $\hat{U}$.

First of all, we know that $\hat{U}$ is the unique minimiser in $H_0^1(\Omega)$ of

$$\phi \to \hat{\mathcal{E}}_D[\phi] := \int_\Omega \Big(\frac{1}{2}|\nabla\phi|^2 + e^{\overline{U}+\phi} - \phi\Big)\,\mathrm{d}x,$$

where the Dirichlet Poisson potential $\overline{U}$ is uniformly bounded, i.e., there exists $M_1 > 0$ such that $-M_1 \le \overline{U}(x) \le M_1$ for all $x \in \overline{\Omega}$. By minimality, this means on the one hand

$$\hat{\mathcal{E}}_D[\hat{U}] \le \hat{\mathcal{E}}_D[0] = \int e^{\overline{U}}\,\mathrm{d}x = e^{M_1}|\Omega| < \infty.$$

On the other hand, using the Poincaré inequality,

$$\hat{\mathcal{E}}_D[\hat{U}] \ge \frac{1}{2}\int_\Omega (|\nabla\hat{U}|^2 + e^{\overline{U}+\hat{U}})\,\mathrm{d}x - C\|\nabla\hat{U}\|_{L^2(\Omega)} \ge \int_\Omega e^{\hat{U}+\overline{U}}\,\mathrm{d}x - C^*$$

for some $C^* > 0$. Combining the two estimates we get a bound of $e^{\hat{U}}$ in $L^1(\Omega)$:

$$\int_\Omega e^{\hat{U}}\,\mathrm{d}x \le e^{M_1}(e^{M_1}|\Omega| + C^*).$$

Furthermore, by construction, for any test function $\phi \in H_0^1(\Omega)$, we know that $\hat{U}$ satisfies

$$\int_\Omega (\nabla\hat{U}\cdot\nabla\phi + (e^{\overline{U}+\hat{U}} - 1)\phi)\,\mathrm{d}x = 0.$$

In particular, for any $n \in \mathbb{N}$, writing $\hat{U}_n := \hat{U} \wedge n$ we can take $\phi_n = e^{\hat{U}_n} - 1$ which is indeed in $H_0^1(\Omega) \cap L^\infty(\Omega)$, and hence

$$\int_\Omega (|\nabla\hat{U}|^2 e^{\hat{U}} \mathbb{1}_{\hat{U}\le n} + e^{\overline{U}} e^{\hat{U}+\hat{U}_n} - e^{\hat{U}+\overline{U}} - e^{\hat{U}_n} + 1)\,\mathrm{d}x = 0.$$

Since $|\nabla\hat{U}|^2 e^{\hat{U}} + 1 \ge 0$, this yields

$$\int_\Omega e^{\overline{U}} e^{\hat{U}+\hat{U}_n}\,\mathrm{d}x \le \int_\Omega (e^{\hat{U}+\overline{U}} + e^{\hat{U}_n})\,\mathrm{d}x. \tag{52}$$

Moreover, by construction $e^{\hat{U}_n}$ is increasing and converges to $e^{\hat{U}}$, and hence the monotone convergence theorem yields

$$\|e^{\hat{U}}\|_{L^2(\Omega)}^2 = \int_\Omega e^{2\hat{U}}\,\mathrm{d}x \le \frac{1+e^{M_1}}{e^{-M_1}}\int_\Omega e^{\hat{U}}\,\mathrm{d}x := C_0\|e^{\hat{U}}\|_{L^1(\Omega)}.$$

We may iterate this argument: for any $k \in \mathbb{N}$, we take $\phi = e^{k\hat{U}_n} - 1$, write $C_0 = (1+e^{M_1})e^{M_1}$, and get

$$\|e^{\hat{U}}\|_{L^k(\Omega)}^k = \int_\Omega e^{k\hat{U}}\,\mathrm{d}x \le C_0\int_\Omega e^{(k-1)\hat{U}}\,\mathrm{d}x \le \cdots \le C_0^{k-1}\|e^{\hat{U}}\|_{L^1(\Omega)}.$$

Thus, for any $k \in \mathbb{N}$, we have $\Delta\hat{U} = e^{\hat{U}+\overline{U}} - 1 \in L^k(\Omega)$ with $\|e^{\hat{U}+\overline{U}} - 1\|_{L^k(\Omega)} \le Ce^{5M_1}$, so by standard elliptic regularity $\hat{U} \in W^{2,k}(\Omega)$; see e.g., [Gilbarg and Trudinger 1998, Section 6.3]. We can take k large enough for Sobolev embeddings to yield $\hat{U} \in C^{1,\alpha}(\Omega)$ with

$$\|\nabla\hat{U}\|_{C^{0,\alpha}(\Omega)} \le Ce^{5M_1}, \tag{53}$$

which in turns implies $e^U - \rho - 1 \in C^{0,\alpha}(\Omega)$ since $\bar{U} \in C_c^{2,\alpha}(\bar{\Omega})$. Standard elliptic regularity for (50) then yields $U \in C_c^{2,\alpha}(\bar{\Omega})$ and concludes the proof since we already have $\partial_n U < 0$ on $\partial\Omega$ from the fact that U minimises $\mathcal{E}_D$. □

5B. *The Neumann case.*

Proposition 5.2. *For any $\rho \geq 0$ in $C^{0,\alpha}(\Omega)$ with $\int_\Omega \rho \, \mathrm{d}x = 1$, consider the nonlinear Poisson equation with Neumann boundary condition*

$$\begin{cases} \Delta U = e^U - \rho - 1, & x \in \Omega, \\ \partial_n U(x) = h, & x \in \partial\Omega, \end{cases} \tag{54}$$

with $h < 0$ in $C^{1,1}(\partial\Omega)$ satisfying (13). *Then there is a unique solution $U \in H^1(\Omega)$ to this problem. Furthermore, this solution is in $C^{2,\alpha}(\Omega)$.*

Note that one could remove the assumption $\int_\Omega \rho \, \mathrm{d}x = 1$ as long as ρ remains integrable on Ω. The condition that h satisfies (13) would then be

$$h < 0, \qquad \int_{\partial\Omega} |h| \, \mathrm{d}\sigma(x) < \int_\Omega \rho \, \mathrm{d}x + |\Omega|.$$

Proof. Analogously to the Dirichlet case, we will prove existence of a solution to (54) in $H^1(\Omega)$ by finding the minimiser in $H^1(\Omega)$ of

$$\mathcal{E}_N[U] := \int_\Omega \Big(\frac{1}{2} |\nabla U|^2 + e^U - U - \rho U \Big) \, \mathrm{d}x - \int_{\partial\Omega} U h \, \mathrm{d}\sigma(x),$$

and uniqueness will follow from the strict convexity of $\mathcal{E}_N$.

Let us consider a minimising sequence (ϕ^k) in $H^1(\Omega)$ and prove that it is equibounded. An upper bound is immediate since $\mathcal{E}_N[0] = |\Omega|$, and hence for k large enough $\mathcal{E}_N[\phi^k] \leq |\Omega|$. Note however that, unlike the Dirichlet case, we cannot easily compare $\mathcal{E}_N[\max(\phi, 0)]$ with $\mathcal{E}_N[\phi]$. Instead, we will show equiboundedness of the positive and negative parts of ϕ^k which we denote $\phi^k_\pm = \pm \max(0, \pm\phi^k)$ and that will yield the equiboundedness of ϕ^k since

$$\mathcal{E}_N[\phi^k] = \mathcal{E}_N[\phi^k_+] + \mathcal{E}_N[\phi^k_-] \quad \text{and} \quad \|\phi^k\|_{H^1(\Omega)} = \|\phi^k_+\|_{H^1(\Omega)} + \|\phi^k_-\|_{H^1(\Omega)}.$$

On the one hand, we have

$$-\int_{\partial\Omega} h \phi^k_+ \, \mathrm{d}\sigma(x) \geq 0,$$

and, with the same arguments as in the Dirichlet case, we have $e^{\phi^k_+} - \phi^k_+ - \rho\phi^k_+ \geq \frac{1}{2}(\phi^k_+)^2 - C$ for some $C > 0$, and hence

$$|\Omega| \geq \mathcal{E}_N[\phi^k_+] \geq \int_\Omega \Big(\frac{1}{2} |\nabla \phi^k_+|^2 + \frac{1}{2} (\phi^k_+)^2 - C \Big) \, \mathrm{d}x \geq \tfrac{1}{2} \|\phi^k_+\|_{H^1(\Omega)} - C|\Omega|,$$

which is the equiboundedness of ϕ^k_+.

On the other hand, we write $\phi_-^k = C_k + \psi_k$ with $C_k = \int_\Omega \phi_-^k \,\mathrm{d}x \le 0$ and $\psi_k = \phi_-^k - C_k \in H^1(\Omega)$ with $\int_\Omega \psi_k \,\mathrm{d}x = 0$, which yields

$$\mathcal{E}_N[\phi_-^k] = \int_\Omega \Big(\frac{1}{2}|\nabla\psi_k|^2 + e^{C_k+\psi_k} - (\rho+1)\psi_k\Big)\,\mathrm{d}x - \int_{\partial\Omega} h\psi_k \,\mathrm{d}\sigma(x) - C_k\bigg(1 + |\Omega| + \int_{\partial\Omega} h\,\mathrm{d}\sigma(x)\bigg).$$

Assumption (13) immediately yields, for some $c_0 > 0$,

$$-C_k\bigg(1 + |\Omega| + \int_{\partial\Omega} h\,\mathrm{d}\sigma(x)\bigg) \ge c_0|C_k|. \tag{55}$$

Further, using the Poincaré inequality and the Sobolev trace theorem we have

$$\begin{aligned}
\int_\Omega \Big(\frac{1}{2}|\nabla\psi_k|^2 + e^{C_k+\psi_k} - (\rho+1)\psi_k\Big)\,\mathrm{d}x - \int_{\partial\Omega} h\psi_k \,\mathrm{d}\sigma(x)& \\
&\ge \frac{1}{2}\int_\Omega |\nabla\psi_k|^2\,\mathrm{d}x - C\int_\Omega |\psi_k|\,\mathrm{d}x - C\int_{\partial\Omega} |\psi_k|\,\mathrm{d}\sigma(x)\\
&\ge \frac{1}{2}\int_\Omega |\nabla\psi_k|^2\,\mathrm{d}x - C\|\nabla\psi_k\|_{L^2(\Omega)}\\
&\ge \frac{1}{2}\int_\Omega |\nabla\psi_k|^2\,\mathrm{d}x - C^*
\end{aligned}$$

for some $C^* > 0$. Together with (55) and the upper bound $\mathcal{E}_N(\phi_k) \le |\Omega|$ we have

$$\|\nabla\phi_-^k\|^2_{L^2(\Omega)} + c_0|C_k| \le C^* + |\Omega|.$$

Finally, using the Poincaré inequality again we get boundedness of ϕ_-^k in $H^1(\Omega)$ since

$$\|\phi_-^k\|_{L^2(\Omega)} \le \|\phi_-^k - C_k\|_{L^2(\Omega)} + |C_k|\,|\Omega| \le \|\nabla\phi_-^k\|_{L^2(\Omega)} + C,$$

and the equiboundedness of ϕ_k in $H^1(\Omega)$ follows. This implies, up to a subsequence, convergence a.e. of ϕ^k towards U, the unique minimiser of $\mathcal{E}_N$ in $H^1(\Omega)$. Let us now check that the Euler–Lagrange equation associated with $\mathcal{E}_N$ is indeed (54). For any $\eta > 0$ and $\phi \in C^\infty(\bar\Omega)$, since U is the minimiser of $\mathcal{E}_N$ we have $\mathcal{E}_N[U + \eta\phi] \ge \mathcal{E}_N[U]$ and

$$\begin{aligned}
0 &\le \lim_{\eta\to 0}\frac{1}{\eta}(\mathcal{E}_N[U+\eta\phi] - \mathcal{E}_N[U])\\
&\le \lim_{\eta\to 0}\bigg[\int_\Omega\bigg(\nabla U\cdot\nabla\phi + \frac{1}{2}\eta|\nabla\phi|^2 + e^U\Big(\frac{e^{\eta\phi}-1}{\eta}\Big) - \phi - \rho\phi\bigg)\,\mathrm{d}x - \int_{\partial\Omega}\phi h\,\mathrm{d}\sigma(x)\bigg]\\
&\le \int_\Omega [\nabla U\cdot\nabla\phi + (e^u - 1 - \rho)\phi]\,\mathrm{d}x - \int_{\partial\Omega}\phi h\,\mathrm{d}\sigma(x).
\end{aligned}$$

This holds for any $\phi \in C^\infty(\bar\Omega)$, so it is true in particular for $\tilde\phi = -\phi$, and hence, for all $\phi \in C^\infty(\bar\Omega)$, U satisfies

$$\int_\Omega [\nabla U\cdot\nabla\phi + (e^U - 1 - \rho)\phi]\,\mathrm{d}x - \int_{\partial\Omega}\phi h\,\mathrm{d}\sigma(x) = 0,$$

which means that U is indeed the unique weak solution in $H^1(\Omega)$ of (54).

For the regularity of U, we follow the strategy of the Dirichlet case. We split U into a regular part $\hat{U}$ and a singular part $\overline{U}$, solutions to

$$\begin{cases} \Delta\hat{U} = e^{\overline{U}+\hat{U}} - 1 & \text{in } \Omega, \\ \partial_n \hat{U} = h_1 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} \Delta\overline{U} = -\rho & \text{in } \Omega, \\ \partial_n \overline{U} = h_2 & \text{on } \partial\Omega, \end{cases} \tag{56}$$

with

$$\begin{cases} h_2 < 0, & \fint_{\partial\Omega} h_2 \,\mathrm{d}\sigma(x) = -1, \\ h_1 \le 0, & \fint_{\partial\Omega} h_1 \,\mathrm{d}\sigma(x) = \fint_{\partial\Omega} h \,\mathrm{d}\sigma(x) + 1, \end{cases}$$

and naturally $h_1 + h_2 = h$ on $\partial\Omega$. By standard elliptic regularity theory, see e.g., [Nardi 2014], there exists a unique (up to an additive constant) solution $\overline{U} \in C^{2,\alpha}(\Omega)$ for all $\alpha \in (0, 1)$, which in turns yields existence and uniqueness (for a fixed $\overline{U}$) of $\hat{U} \in H^1(\Omega)$ which satisfies, for all $\phi \in H^1(\Omega)$,

$$\int_\Omega (\nabla\hat{U} \cdot \nabla\phi + (e^{\hat{U}+\overline{U}} - 1)\phi) \,\mathrm{d}x - \int_{\partial\Omega} h_1 \phi \,\mathrm{d}\sigma(x) = 0.$$

Next, we write $\hat{U}_n = \hat{U} \wedge n$ for any $n \in \mathbb{N}$ and take $\phi = e^{\hat{U}_n} \in H^1(\Omega) \cap L^\infty(\Omega)$ as a test function. Since $h_1 \le 0$ we get a similar estimate as (52) in the Dirichlet case:

$$\int_\Omega (e^{\overline{U}} e^{\hat{U}+\hat{U}_n} - e^{\hat{U}_n}) \,\mathrm{d}x = -\int_\Omega |\nabla\hat{U}|^2 e^{\hat{U}} \mathbb{1}_{\hat{U} \le n} \,\mathrm{d}x - \int_{\partial\Omega} |h_1| e^{\hat{U}_n} \,\mathrm{d}\sigma(x) \le 0,$$

and hence, for $M_1 > 0$ such that $-M_1 \le \overline{U}(x) \le M_1$ for all $x \in \overline{\Omega}$, we have by monotone convergence

$$\|e^{\hat{U}}\|^2_{L^2(\Omega)} \le e^{M_1} \|e^{\hat{U}}\|_{L^1(\Omega)}.$$

We can iterate this estimate, choosing $\phi = e^{k\hat{U}_n}$ as a test function for any $k \in \mathbb{N}$, and derive

$$\|e^{\hat{U}}\|^k_{L^k(\Omega)} \le e^{M_1} \|e^{\hat{U}}\|^{k-1}_{L^{k-1}(\Omega)} \le \cdots \le e^{(k-1)M_1} \|e^{\hat{U}}\|_{L^1(\Omega)}.$$

Furthermore, we can take $\phi = 1 \in H^1(\Omega)$ as a test function as well which yields

$$\int_\Omega e^{\hat{U}} \,\mathrm{d}x \le e^{M_1} \int_\Omega (e^{\hat{U}+\overline{U}} - 1) \,\mathrm{d}x \le e^{M_1} \|h_1\|_{L^1(\partial\Omega)}.$$

As a consequence, $e^{\hat{U}} \in L^k(\Omega)$ for all $k \in \mathbb{N}$ which yields $e^{\overline{U}+\hat{U}} - 1 \in L^k(\Omega)$ with $\|e^{\overline{U}+\hat{U}} - 1\|_{L^k(\Omega)} \le Ce^{3M_1}$. Similarly to the Dirichlet case, standard elliptic regularity yields $\hat{U} \in W^{2,k}(\Omega)$ and we can take k large enough for Sobolev embeddings to yield $\hat{U} \in C^{1,\alpha}(\Omega)$ with

$$\|\nabla\hat{U}\|_{C^{0,\alpha}(\Omega)} \le Ce^{3M_1}, \tag{57}$$

and since $\overline{U} \in C^{2,\alpha}(\overline{\Omega})$ this means $e^U - 1 - \rho \in C^{0,\alpha}(\Omega)$ and Proposition 5.2 follows by standard elliptic regularity theory. □

5C. ***Proof of Theorem 2.2.*** Using the elliptic regularity estimates established above for both the Dirichlet and the Neumann case, we can now prove well-posedness of the VPME system in the sense of Theorem 2.2 following the same arguments as in the Vlasov–Poisson case.

More precisely, we notice that the electric field $E = -\nabla U$ with U a solution to (50) or (54) is in $C_t^0 C^{1,\mu}([0, T] \times \overline{\Omega})^d$ and satisfies $E \cdot n(x) = -\partial_n U > 0$ for all $x \in \partial\Omega$, so the results of Section 3 apply, namely there exists a δ-kinetic distance α and we have a Velocity Lemma. Moreover, if we define an iterative sequence (27) with the nonlinear Poisson equation (50) or (54), then Theorem 4.1 holds and the elliptic regularity estimates proved above ensure that both Corollary 4.2 and Proposition 4.3 hold. Therefore, we have constructed a classical solution of the VPME system under the assumption of uniformly bounded velocities $Q(t) < K(T)$ for all $t \in [0, T]$ with $Q(t)$ given by (35). In order to remove this assumption we decompose the electric field as $E = \hat{E} + \bar{E}$ with the regular part given by $\hat{E} = -\nabla \hat{U}$ and the singular part given by $\bar{E} = -\nabla \bar{U}$, where $\hat{U}$ and $\bar{U}$ are defined either by (51) or (56).

On the one hand, since $|\Omega| < \infty$ and by classical elliptic regularity, see e.g., [Gilbarg and Trudinger 1998, Chapter 6; Nardi 2014],

$$\|\bar{U}\|_{L^\infty(\Omega)} \leq C(1 + \|\rho\|_{C^{0,\alpha}(\Omega)}),$$

where C depends on $\|h\|_{C^{1,\alpha}(\partial\Omega)}$. Using (53) or (57) this yields a uniform control of $\hat{E}$ in $L^\infty(\Omega)$ as

$$\|\hat{E}(t, \cdot)\|_{L^\infty(\Omega)} \leq C \exp((1 + \|\rho\|_{C^{0,\alpha}(\Omega)})).$$

Therefore, in order to bound the maximum velocity $Q(t)$, one only needs to consider $\bar{E}$ for which the analysis developed in Section 4C applies, so we have an equivalent of Proposition 4.6 for VPME. Finally, we conclude the proof of Theorem 2.2 by the same argument as in Section 4D in order to show global existence and uniqueness, with the following estimate instead of (49):

$$\begin{aligned} |(E^1 - E^2)(s, x)| &\leq |(\bar{E}^1 - \bar{E}^2)(s, x)| + |(\hat{E}^1 - \hat{E}^2)(s, x)| \\ &\lesssim \int_\Omega \frac{|\rho^1(s, y) - \rho^2(s, y)|}{|x - y|^2} \, \mathrm{d}y + \|\hat{E}^1(t, \cdot) - \hat{E}^2(t, \cdot)\|_{L^\infty(\Omega)}. \end{aligned}$$

□

Appendix: Numerical simulations

In this appendix, we present some numerical simulations of the trajectories of particles in the linear Vlasov equation (28) in order to illustrate the results of Section 3.

We consider Ω to be the unit disk in dimension 2 and introduce an electric field E. In order to stay close to a Vlasov–Poisson framework, we will consider a field given by a Poisson equation with a nice density ρ. More precisely, we consider a Gaussian bell function centred at $x_0 = (0.5, 0)$,

$$\rho(x) = C \exp\left(-\frac{1}{1 - 4|x - x_0|^2}\right) \mathbb{1}_{|x - x_0| < 1/2},$$

and the stationary electric field E given by $E = -\nabla U$ and $\Delta U = -\rho$ with Dirichlet boundary conditions $U|_{\partial\Omega} = 0$. This field can be explicitly written as a convolution with the Green function of the ball; see, e.g., [Evans 1998, Section 2.2.4]. We illustrate ρ and E in Figure 1 and note that E is indeed outgoing on $\partial\Omega$ in the sense that $E \cdot n(x) > 0$ for all $x \in \partial\Omega$ as assumed in Section 3.

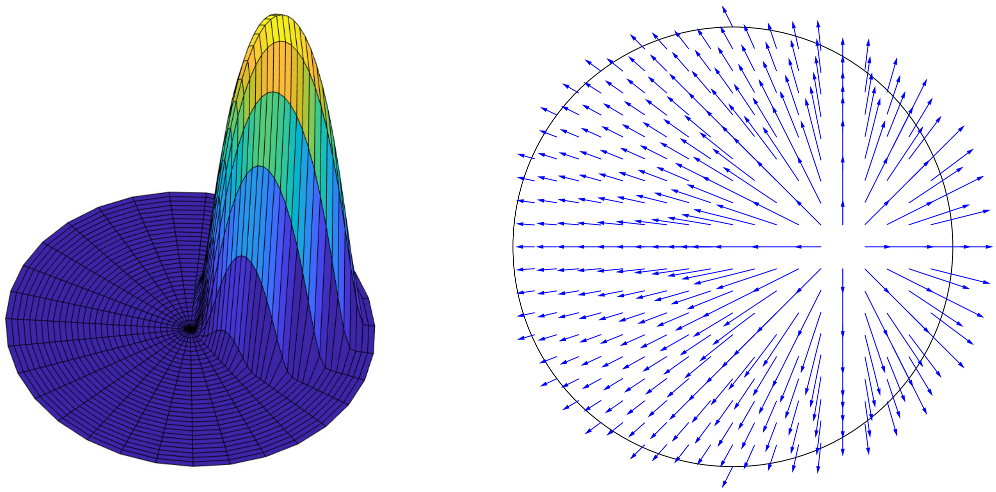

Figure 1. Choice of density ρ and associated field E.

Given this fixed field E, we can compute the trajectory of a particle by solving the system of ODEs (14)–(16). For instance, if we consider a particle starting at $x = (-0.9, 0)$ with velocity direction $(-0.2, 1)$, then its trajectory in the domain Ω will be given by Figure 2.

On that figure, we also plot the norm of the velocity $s \to |V(s; 0, x, v)|$, which decreases when the particle moves against the direction of the field and increases when it follows the field. In particular, we see that when the trajectory moves towards the right of the disk, where the field is strongest, it may change direction if the norm of the velocity is too small, as is the case in Figure 2. This can be interpreted as the particle slowing down to the point where the electric field becomes stronger than the natural inertia of the particle, and hence the change of direction. We also notice on this plot that the norm of the velocity is a continuous function of s that is piecewise smooth (for the regular field E we consider in this example) with singularities at the points of reflection, as expected.

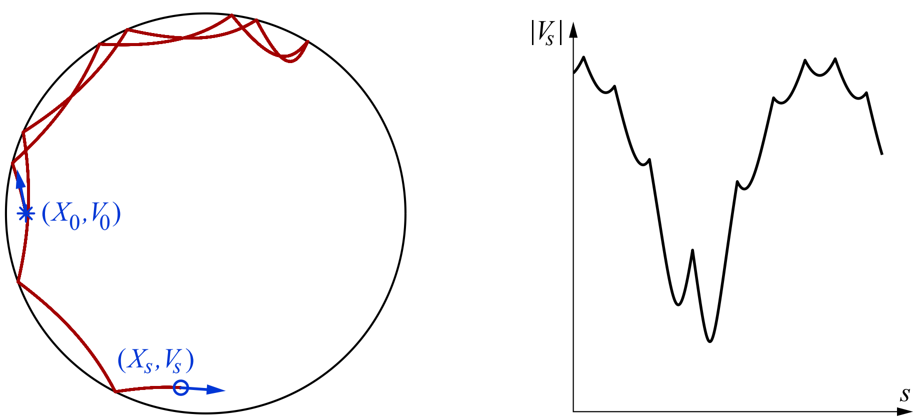

Figure 2. Trajectory in the disk from $x = (-0.9, 0)$, $v \propto (-0.2, 1)$, and evolution of speed.

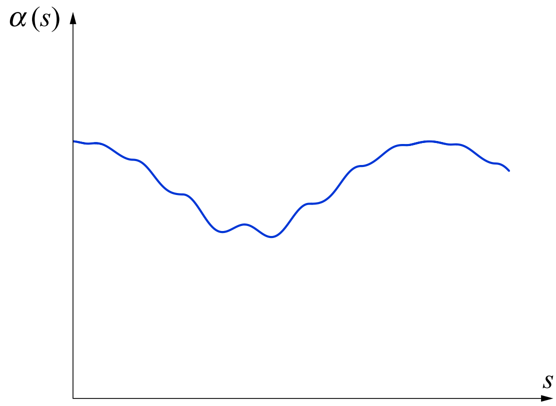

Figure 3. Kinetic distance along a trajectory.

We would like to illustrate the Velocity Lemma (Lemma 3.3), which states that if a trajectory starts close to the grazing set γ_0 then it remains close to γ_0 through time, although the size of the neighbourhood increases exponentially fast; see (18). However, since any neighbourhood of γ_0 is a subset of the four-dimensional phase-space, we cannot really illustrate this result on the plot of the trajectory (even though if a trajectory is close to γ_0 then necessarily X_s is close to $\partial\Omega$ by construction, but that is not a sufficient condition). Instead, let us look at the kinetic distance α given by (19). In our example, we characterise the unit disk via the function $\xi(x) = \frac{1}{2}(|x|^2 - 1)$ for which the kinetic distance α can be written as

$$\alpha(x, v) = \tfrac{1}{2}(v \cdot x)^2 + (1 - |x|)(|v|^2 + E(x) \cdot x).$$

Note that since the electric field is stationary, the kinetic distance does not depend directly on t. Introducing $\alpha(s) = \alpha(X(s; 0, x, v), V(s; 0, x, v))$ for a fixed $(x, v) \in \Omega \times \mathbb{R}^2$, we plot in Figure 3 the evolution of the kinetic distance along the trajectory illustrated in Figure 2.

The Velocity Lemma (Lemma 3.3) gives a uniform exponential bound on α: for all $(x, v) \in \overline{\Omega} \times \mathbb{R}^2$ and $s \in (0, t)$,

$$\alpha(x, v)e^{-C_0[(|v|+1)s+\|E\|_{L^\infty}s^2]} \le \alpha(X(s; 0, x, v), V(s; 0, x, v)) \le \alpha(x, v)e^{C_0[(|v|+1)s+\|E\|_{L^\infty}s^2]}$$

for some $C_0 = C_0(\xi, E) > 0$. Since the constant C_0 is not given explicitly by our Velocity Lemma we will not illustrate this exponential bound.

To conclude this appendix we now consider other examples of trajectories and their associated kinetic distances in order to illustrate the behaviours that these trajectories can exhibit and the associated variations of their kinetic distances. In Figure 4 we represent 6 trajectories, all starting with the same vertical direction of velocity (0, 1) (with a greater initial norm than the one of Figure 2, which is why there is no change of direction in plots 4 to 6 when the trajectory travels through the right side of the domain) and initial position on the x-axis with coordinate -0.2, -0.4, -0.6, -0.8, -0.9, and -0.95, respectively. As in Figure 2, we represent with a blue star the position at $s = 0$ and with a blue circle the position at the end time, with colours matching that of Figure 5. These trajectories illustrate in particular the isolation of

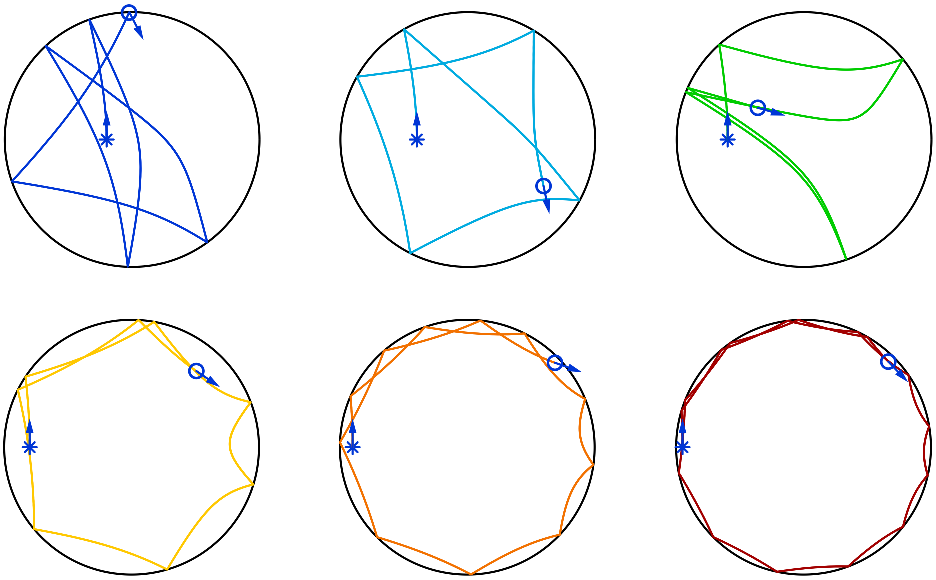

Figure 4. Examples of trajectories on the disk; (top) examples 1–3, (bottom) examples 4–6.

grazing. We see indeed that the trajectories 4 to 6, which start relatively close to the grazing set, remain in a neighbourhood of $\partial\Omega$, a neighbourhood which grows smaller as (x, v) grows closer to the grazing set γ_0.

We also plot the evolution of the kinetic distance along these trajectories in Figure 5. Note that since they all start with the same velocity (and initial position on the same axis) it is easy to identify which curve corresponds to which trajectory by the initial value of the kinetic distance which decreases as the initial position grows near the boundary.

One may observe many phenomena in this last illustration. For instance, we see that the kinetic distances of the last two trajectories, which start rather close to grazing, vary little through time. Note that this also applies to the trajectory of Figure 2, which morally would fit between the fourth and fifth

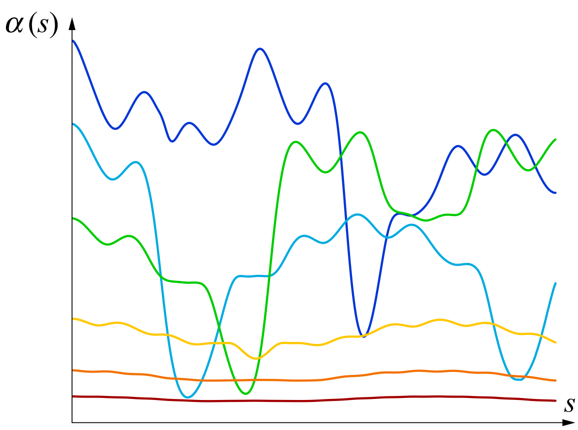

Figure 5. Kinetic distances as functions of s.

trajectories of Figure 4. On the other hand, the kinetic distance of the trajectories that start with a position far from $\partial\Omega$ show significant variations, and we see in particular that, at their lowest, their value is close to that of the last two trajectories. This illustrates the fact that if x is close to $\partial\Omega$ and $|v| \ll 1$, then $\alpha(x, v)$ will be small even if v is not tangential. The exponential bounds given by the Velocity Lemma (Lemma 3.3) naturally allow for such behaviour since the exponential coefficients are uniform in $x \in \overline{\Omega}$ and only depend on the norm of v.

Finally, let us emphasize that for these illustrations we have chosen a very smooth electric field E and a smooth domain Ω with constant curvature. Naturally, if the field E is less regular, and if one consider a more general uniformly convex domain Ω, then one may observe a much wider variety of behaviours for the trajectories of the linear Vlasov equation (28). On the importance of curvature, let us recall that we are not yet able to prove a strong enough isolation of the grazing set when the domain is not uniformly convex — i.e., when the curvature may cancel pointwise or on portions of the boundary — in order to conclude the Pfaffelmoser argument of Section 4C since the exponential controls of our Velocity Lemma are significantly worse in the nonuniformly convex case, as explained in Remark 3.4.

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