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HIGHER RANK QUANTUM-CLASSICAL CORRESPONDENCE

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For a compact Riemannian locally symmetric space $\Gamma\backslash G/K$ of arbitrary rank we determine the location of certain Ruelle–Taylor resonances for the Weyl chamber action. We provide a Weyl-lower bound on an appropriate counting function for the Ruelle–Taylor resonances and establish a spectral gap which is uniform in Γ if G/K is irreducible of higher rank. This is achieved by proving a quantum-classical correspondence, i.e., a one-to-one correspondence between horocyclically invariant Ruelle–Taylor resonant states and joint eigenfunctions of the algebra of invariant differential operators on G/K.

1. Introduction

Ruelle resonances for an Anosov flow provide a fundamental spectral invariant that reflects not only many important dynamical properties of the flow but also geometric and topological properties of the underlying manifold. Very recently the concept of resonances was extended to higher rank $\mathbb{R}^n$-Anosov actions and led to the notion of *Ruelle–Taylor resonances*[1] which were shown to be a discrete subset $\sigma_{\mathrm{RT}} \subset \mathbb{C}^n$ [Bonthonneau et al. 2020]. It was furthermore shown in that same paper that the leading resonances (i.e., those with vanishing real part) are related to mixing properties of the considered Anosov action. In particular, it was shown that if the action is weakly mixing in an arbitrary direction of the abelian group $\mathbb{R}^n$, then $0 \in \mathbb{C}^n$ is the only leading resonance. Furthermore, the resonant states at zero give rise to equilibrium measures that share properties of Sinai–Ruelle–Bowen (SRB) measures of Anosov flows.

Apart from the leading resonances the spectrum of Ruelle–Taylor resonances has so far not been studied if $n \geq 2$. In particular, when $n \geq 2$, it was not known whether there are other resonances than the resonance at zero. Neither was it known whether there is a spectral gap, i.e., whether the real parts of the resonances are bounded away from zero. In this article we shed some light on these questions by examining the Ruelle–Taylor resonances for the class of Weyl chamber flows via harmonic analysis.

Let us briefly introduce the setting: Let G be a real connected noncompact semisimple Lie group with finite center and Iwasawa decomposition $G = KAN$. Let $\mathfrak{a}$ be the Lie algebra of A and M the centralizer of A in K. Then A is isomorphic to $\mathbb{R}^n$, where n is the real rank of G, and acts on G/M from the right. Hence A also acts on the compact manifold $\mathcal{M} := \Gamma\backslash G/M$, where $\Gamma \leq G$ is a cocompact torsion-free lattice. It can be easily seen that this action is an Anosov action with hyperbolic splitting $T\mathcal{M} = E_0 \oplus E_s \oplus E_u$ which can be described explicitly in terms of associated vector bundles (see Section 2A for a general

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[1] They were named Ruelle–Taylor resonances because the notion of the Taylor spectrum for commuting operators is a crucial ingredient of their definition.

definition of Anosov actions and Proposition 3.1 for the description of the hyperbolic splitting for Weyl chamber flows). Furthermore, if $\Sigma \subseteq \mathfrak{a}^*$ is the set of restricted roots with simple system Π and positive system Σ^+ then the positive Weyl chamber is given by $\mathfrak{a}_+ = \{H \in \mathfrak{a} \mid \alpha(H) > 0 \ \forall \alpha \in \Pi\}$.

The *Ruelle–Taylor resonances* of this Anosov action are defined as follows: For $H \in \mathfrak{a}$ let X_H be the vector field on $\mathcal{M}$ defined by the right A-action. Then

$$\sigma_{\mathrm{RT}} := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \exists u \in \mathcal{D}'_{E_u^*}(\mathcal{M}) \setminus \{0\} \text{ s.t. } (X_H + \lambda(H))u = 0 \ \forall H \in \mathfrak{a}\},$$

where $\mathcal{D}'_{E_u^*}(M)$ is the set of distributions with wavefront set contained in the annihilator $E_u^* \subseteq T^*\mathcal{M}$ of $E_0 \oplus E_u$. The distributions $u \in \mathcal{D}'_{E_u^*}(M)$ satisfying $(X_H + \lambda(H))u = 0$ for all $H \in \mathfrak{a}$ are called *resonant states* of λ and the dimension of the space of all such distributions is called the multiplicity $m(\lambda)$ of the resonance λ. It has been shown in [Bonthonneau et al. 2020] that $\sigma_{\mathrm{RT}} \subset \mathfrak{a}_{\mathbb{C}}^*$ is discrete and that all resonances have finite multiplicity. It also follows from that work that the real part of the resonances are located in a certain cone $\overline{{}_-\mathfrak{a}^*} \subset \mathfrak{a}^*$ which is the negative dual cone of the positive Weyl chamber $\mathfrak{a}_+$ (see Section 2B for a precise definition).

In this article we will prove that there is a bijection between a certain subset of the Ruelle–Taylor resonant states and certain joint eigenfunctions of the invariant differential operators on the locally symmetric space $\Gamma\backslash G/K$. Before explaining this correspondence in more detail we state two results on the spectrum of Ruelle–Taylor resonances that we can conclude from the correspondence.

The first result says that, for any Weyl chamber flow, there exist infinitely many Ruelle–Taylor resonances by providing a Weyl-lower bound on an appropriate counting function.

Theorem 1.1. *Let ρ be the half-sum of the positive restricted roots, let W be the Weyl group (see Section 2B for a precise definition) and, for $t > 0$, let*

$$N(t) := \sum_{\lambda \in \sigma_{\mathrm{RT}},\ \mathrm{Re}(\lambda) = -\rho,\ \|\mathrm{Im}(\lambda)\| \leq t} m(\lambda).$$

Then, for $d := \dim(G/K)$,

$$N(t) \geq |W| \operatorname{Vol}(\Gamma\backslash G/K)(2\sqrt{\pi})^{-d} \frac{1}{\Gamma(d/2+1)} t^d + \mathcal{O}(t^{d-1}).$$

More generally, let $\Omega \subseteq \mathfrak{a}^*$ be open and bounded such that $\partial\Omega$ has finite $(n-1)$-dimensional Hausdorff measure. Then

$$\sum_{\lambda \in \sigma_{\mathrm{RT}},\ \mathrm{Re}(\lambda) = -\rho,\ \mathrm{Im}(\lambda) \in t\Omega} m(\lambda) \geq |W| \operatorname{Vol}(\Gamma\backslash G/K)(2\pi)^{-d} \operatorname{Vol}(\mathrm{Ad}(K)\Omega) t^d + \mathcal{O}(t^{d-1}).$$

The second result guarantees a uniform spectral gap.

Theorem 1.2. *Let G be a real semisimple Lie group with finite center. Then, for any cocompact torsion-free discrete subgroup $\Gamma \subset G$, there is a neighborhood $\mathcal{G} \subset \mathfrak{a}^*$ of 0 such that*

$$\sigma_{\mathrm{RT}} \cap (\mathcal{G} \times i\mathfrak{a}^*) = \{0\}.$$

If G furthermore has Kazhdan's property (T) (e.g., if G is simple of higher rank), then the spectral gap $\mathcal{G}$ can be taken uniformly in Γ and only depends on the group G.

Let us now explain in some detail the spectral correspondence that is the key to the above results.

We define the space of *first band resonant states* as those resonant states that are in addition horocyclically invariant:

$$\mathrm{Res}^0_X(\lambda) := \{u \in \mathcal{D}'_{E_u^*}(\mathcal{M}) \mid (X_H + \lambda(H))u = 0 \text{ and } \mathcal{X}u = 0 \ \forall H \in \mathfrak{a} \text{ and } \mathcal{X} \in C^\infty(\mathcal{M}, E_u)\},$$

and we call a Ruelle–Taylor resonance a *first band resonance* if and only if $\mathrm{Res}^0_X(\lambda) \neq 0$. By working with horocycle operators and vector-valued Ruelle–Taylor resonances we will be able to show that all resonances with real part in a certain neighborhood of zero in $\mathfrak{a}^*$ are always first band resonances (see Proposition 3.7). As the Weyl chamber flow is generated by mutually commuting Hamilton flows, we consider the set of Ruelle–Taylor resonances as a *classical spectrum*.

Let us briefly describe the *quantum* side: In the rank 1 case the quantization of the geodesic flow is given by the Laplacian on G/K. In the higher rank case we have to consider the algebra of G-invariant differential operators on G/K which we denote by $\mathbb{D}(G/K)$. As an abstract algebra this is a polynomial algebra with n algebraically independent operators, among them the Laplace operator. These operators descend to $\Gamma\backslash G/K$ and we can define the joint eigenspace

$${}^\Gamma E_\lambda = \{f \in C^\infty(\Gamma\backslash G/K) \mid Df = \chi_\lambda(D) f \ \forall D \in \mathbb{D}(G/K)\},$$

where χ_λ is a character of $\mathbb{D}(G/K)$ parametrized by $\lambda \in \mathfrak{a}^*_{\mathbb{C}}/W$ with the Weyl group W. Here χ_ρ is the trivial character (see Section 2D). Let σ_Q denote the corresponding *quantum spectrum* $\{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid {}^\Gamma E_\lambda \neq \{0\}\}$.

We have the following correspondence between the classical first band resonant states and the joint quantum eigenspace:

Theorem 1.3. *Let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ be outside the exceptional set*

$$\mathcal{A} := \left\{ \lambda \in \mathfrak{a}^*_{\mathbb{C}} \,\middle|\, \frac{2\langle \lambda + \rho, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in -\mathbb{N}_{>0} \text{ for some } \alpha \in \Sigma^+ \right\}.$$

Then there is a bijection between the finite-dimensional vector spaces

$$\pi_* : \mathrm{Res}^0_X(\lambda) \to {}^\Gamma E_{-\lambda-\rho},$$

where π_* is the push-forward of distributions along the canonical projection $\pi : \Gamma\backslash G/M \to \Gamma\backslash G/K$.

Using this one-to-one correspondence we can then use results about the quantum spectrum to obtain obstructions and existence results on the Ruelle–Taylor resonances. Notably we use results of Duistermaat, Kolk and Varadarajan [Duistermaat et al. 1979] on the spectrum σ_Q, but we also deduce refined information on the quantum spectrum. Here we use L^p-bounds for spherical functions obtained from asymptotic expansions [van den Ban and Schlichtkrull 1987] and L^p-bounds for matrix coefficients based on work by Cowling [1979] and Oh [2002]. Theorems 1.1 and 1.2 as stated above give only a rough version of the information on the Ruelle–Taylor resonances that we can actually obtain. As the full results require some further notation we refrain from stating them in the introduction and refer to Theorem 5.1. We also refer to Figure 6 for a visualization of the structure of first band resonances for the case $G = \mathrm{SL}(3, \mathbb{R})$.

Methods and related results. The key ingredient to the quantum-classical correspondence is that we can in a first step relate the horocyclically invariant first band resonant states with distributional vectors in some principal series representations. Then we can apply the Poisson transform of [Kashiwara et al. 1978] to get a bijection onto the quantum eigenspace ${}^{\Gamma}E_{-\lambda-\rho}$. The prototype of such a quantum-classical correspondence was first established by Dyatlov, Faure and Guillarmou [Dyatlov et al. 2015] in the case of manifolds of constant curvature or in other words for the rank 1 group $G = \mathrm{SO}(n, 1)$. Certain central ideas have, however, already been present for $G = \mathrm{SO}(2, 1)$ in the work of Cosentino [2005] and Flaminio and Forni [2003]. In the rank 1 setting there exist several generalizations of the quantum classical correspondence of [Dyatlov et al. 2015], for example, to convex cocompact manifolds of constant curvature [Guillarmou et al. 2018; Hadfield 2020], general compact locally symmetric spaces of rank 1 [Guillarmou et al. 2021] and vector bundles [Küster and Weich 2020; 2021].

Besides the correspondence between the classical Ruelle resonant states and the quantum Laplace eigenvalues there are several other approaches in the literature establishing exact relations between the Laplace spectrum and the geodesic flow. One approach is to relate the Laplace spectrum to divisors of zeta functions. Such relations have been obtained for rank 1 locally symmetric spaces on various levels of generality by Bunke, Olbrich, Patterson and Perry: $G = \mathrm{SO}(n, 1)$ and Γ convex cocompact [Bunke and Olbrich 1997; 1999; Patterson and Perry 2001]; G real rank 1 and Γ cocompact [Bunke and Olbrich 1995].

A third approach to an exact quantum-classical correspondence is to relate the Laplace spectrum to a transfer operator which represents a time discretized dynamics of the geodesic flow. This type of correspondence was notably studied for hyperbolic surfaces with cusps; see [Bruggeman and Pohl 2023; Bruggeman et al. 2015; Lewis and Zagier 2001] for results for $G = \mathrm{SL}(2, \mathbb{R})$ and Γ discrete subgroups of increasing generality. We refer in particular to the expository article [Pohl and Zagier 2020] and the introduction of [Bruggeman and Pohl 2023] for a current state of the art of these techniques. A very first step towards generalizations of this approach to higher rank has been recently achieved in [Pohl 2020] for the Weyl chamber flow on products of Schottky surfaces by the construction of symbolic dynamics and transfer operators.

Note that in [Dyatlov et al. 2015] not only was the first band of Ruelle resonances related to the Laplace spectrum, but a complete band structure has been established and the higher bands can be related to the Laplace spectrum on divergence-free symmetric tensors. In the present article we do not study the higher bands. This will presumably be a very hard question for general semisimple groups G (note that in [Dyatlov et al. 2015] it was crucial at several points that $N \cong \mathbb{R}^{n-1}$ is abelian for $G = \mathrm{SO}(n, 1)$). However, it might be tractable for some concrete groups with simple enough root spaces such as $G = \mathrm{SL}(3, \mathbb{R})$. For geodesic flows the phenomenon of such a band structure is quite universal and known in the case of compact locally symmetric spaces of rank 1 [Küster and Weich 2021] and also for geodesic flows on manifolds of pinched negative curvature [Cekić and Guillarmou 2021; Faure and Tsujii 2013; 2021].

As mentioned above an important application of Ruelle resonances for Anosov flows are mixing results. More precisely, the existence of a spectral gap in addition with resolvent estimates imply mixing of the flow. For Weyl chamber flows this relation of gaps and mixing rates is not yet established but conjectured to be true. From this perspective, Theorem 1.2 is related to the work of Katok and Spatzier [1994] who showed exponential mixing for the Weyl chamber action in every direction of the closure of the positive Weyl chamber if G has property (T). However it is not known whether their result remains true if the property (T) assumption is dropped. Our result above (Theorem 1.2) ensures a Γ-dependent gap in any case but as mentioned above the precise relation to mixing rates is not yet established.

Finally, Weyl laws for Ruelle resonances of geodesic flows can also be established in variable curvature (or more generally contact Anosov flows) in various settings [Datchev et al. 2014; Faure and Sjöstrand 2011; Faure and Tsujii 2023]. In particular, in the very recent article by Faure and Tsujii [2021] the Weyl law also follows because a "first band" of resonances can be related to a quantum operator. The methods in their work are, however, completely different and are based on microlocal analysis rather then global harmonic analysis.

2. Preliminaries

2A. ***Ruelle–Taylor resonances for higher rank Anosov actions.*** In this section we recall the main properties of Ruelle–Taylor resonances for higher rank Anosov actions from [Bonthonneau et al. 2020]. Let $\mathcal{M}$ be a compact Riemannian manifold, let $A \simeq \mathbb{R}^n$ be an abelian group and let $\tau : A \to \mathrm{Diffeo}(\mathcal{M})$ be a smooth locally free group action. If $\mathfrak{a} := \mathrm{Lie}(A)$ we define the generating map

$$X : \mathfrak{a} \to C^\infty(\mathcal{M}, T\mathcal{M}), \qquad H \mapsto X_H := \frac{d}{dt}\Big|_{t=0} \tau(\exp(tH)).$$

Note that $[X_{H_1}, X_{H_2}] = 0$ for $H_i \in \mathfrak{a}$. For $H \in \mathfrak{a}$ we denote by $\varphi_t^{X_H}$ the flow of the vector field X_H. The action is called *Anosov* if there exists $H \in \mathfrak{a}$ and a continuous $\varphi_t^{X_H}$-invariant splitting

$$T\mathcal{M} = E_0 \oplus E_u \oplus E_s,$$

where $E_0 := \mathrm{span}\{X_H : H \in \mathfrak{a}\}$ is of dimension n because the action is locally free and there exist $C > 0$ and $\nu > 0$ such that, for each $x \in \mathcal{M}$,

$$\begin{aligned} &\text{for all } w \in E_s(x),\ \ t \geq 0, \quad \|d\varphi_t^{X_H}(x)w\| \leq Ce^{-\nu t}\|w\|, \\ &\text{for all } w \in E_u(x),\ \ t \leq 0, \quad \|d\varphi_t^{X_H}(x)w\| \leq Ce^{-\nu |t|}\|w\|, \end{aligned}$$

where the norm on $T\mathcal{M}$ is given by the Riemannian metric on $\mathcal{M}$. Such an $H \in \mathfrak{a}$ is called *transversally hyperbolic*. We call the set

$$\mathcal{W} := \{H' \in \mathfrak{a} \mid H' \text{ is transversally hyperbolic with the same splitting as } H\}$$

the *positive Weyl chamber containing* H.

Let $E \to \mathcal{M}$ be the complexification of a Euclidean bundle over $\mathcal{M}$, and denote by $\mathrm{Diff}^1(\mathcal{M}, E)$ the set of first-order differential operators with smooth coefficients acting on sections of E. Then a linear map $\boldsymbol{X} : \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}, E)$ such that $\boldsymbol{X}_{H_1}\boldsymbol{X}_{H_2} = \boldsymbol{X}_{H_2}\boldsymbol{X}_{H_1}$ for all $H_i \in \mathfrak{a}$ is called an *admissible lift* of the generic map X if

$$\boldsymbol{X}_H(fs) = (X_H f)s + f\boldsymbol{X}_H s \tag{1}$$

for $s \in C^\infty(\mathcal{M}, E)$, $f \in C^\infty(\mathcal{M})$ and $H \in \mathfrak{a}$.

For a fixed positive Weyl chamber $\mathcal{W}$, the set of *Ruelle–Taylor resonances* can be defined as

$$\sigma_{\mathrm{RT}}(\boldsymbol{X}) := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \exists u \in \mathcal{D}'_{E_u^*}(\mathcal{M}, E) \setminus \{0\} \text{ s.t. } (\boldsymbol{X}_H + \lambda(H))u = 0\ \forall H \in \mathfrak{a}\},$$

where $\mathcal{D}'_{E_u^*}(\mathcal{M}, E)$ is the set of distributional sections of the bundle E with wavefront set contained in E_u^*. Here E_u^* is defined as the annihilator of $E_0 \oplus E_u$ in $T^*\mathcal{M}$. The vector space of *Ruelle–Taylor resonant states* for a resonance $\lambda \in \sigma_{\mathrm{RT}}(\boldsymbol{X})$ is defined by

$$\mathrm{Res}_{\boldsymbol{X}}(\lambda) := \{u \in \mathcal{D}'_{E_u^*}(\mathcal{M}, E) \mid (\boldsymbol{X}_H + \lambda(H))u = 0\ \forall H \in \mathfrak{a}\}.$$

Remark 2.1. The original definition of Ruelle–Taylor resonances and resonant states is stated via Koszul complexes; see [Bonthonneau et al. 2020, Section 3]. More precisely, λ is a resonance if and only if the corresponding Koszul complex is not exact and the resonant states are the cohomologies of this complex. The space of resonant states that we are considering is just the zeroth cohomology. However, it turns out that the Koszul complex is not exact if and only if the zeroth cohomology is nonvanishing, i.e., the two notions coincide; see [Bonthonneau et al. 2020, Theorem 4].

It is known that the resonances have the following properties.

Proposition 2.2 (see [Bonthonneau et al. 2020, Theorems 1 and 4]). *The set $\sigma_{\mathrm{RT}}(\boldsymbol{X})$ of Ruelle–Taylor resonances is a discrete subset of $\mathfrak{a}_{\mathbb{C}}^*$ contained in*

$$\{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \mathrm{Re}(\lambda(H)) \le C_{L^2}(H)\ \forall H \in \mathcal{W}\}$$

with $C_{L^2}(H) = \inf\{C > 0 \mid \|e^{-t\boldsymbol{X}_H}\|_{L^2 \to L^2} \le e^{Ct}\ \forall t > 0\}$, where $e^{-t\boldsymbol{X}_H} : L^2(\mathcal{M}, E) \to L^2(\mathcal{M}, E)$ is the semigroup with generator $-\boldsymbol{X}_H$. Moreover, for each $\lambda \in \sigma_{\mathrm{RT}}(\boldsymbol{X})$, the space $\mathrm{Res}_{\boldsymbol{X}}(\lambda)$ of resonant states is finite-dimensional.

2B. ***Semisimple Lie groups.*** In this section we fix the notation for the present article. Let G be a real semisimple connected noncompact Lie group with finite center and Iwasawa decomposition $G = KAN$. Furthermore, let $M := Z_K(A)$ be the centralizer of A in K and $G = KAN_-$ be the opposite Iwasawa decomposition. We denote by $\mathfrak{g}$, $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{n}_-$, $\mathfrak{k}$ and $\mathfrak{m}$ the corresponding Lie algebras. For $g \in G$, let $H(g)$ be the logarithm of the A-component in the Iwasawa decomposition. We have a K-invariant inner product on $\mathfrak{g}$ that is induced by the Killing form and the Cartan involution. We have the orthogonal Bruhat decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$ into root spaces $\mathfrak{g}_\alpha$ with respect to the $\mathfrak{a}$-action via the adjoint action ad. Here $\Sigma \subseteq \mathfrak{a}^*$ is the set of restricted roots. Denote by W the Weyl group of the root system of restricted roots. Let n be the real rank of G and $\Pi = \{\alpha_1, \dots, \alpha_n\}$ (resp. Σ^+) the simple (resp. positive) system in Σ determined by the choice of the Iwasawa decomposition. Let $m_\alpha := \dim_{\mathbb{R}} \mathfrak{g}_\alpha$ and $\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$. Denote by w_0 the longest Weyl group element, i.e., the unique element in W mapping Π to $-\Pi$. Let $\mathfrak{a}_+ := \{H \in \mathfrak{a} \mid \alpha(H) > 0\ \forall \alpha \in \Pi\}$ be the positive Weyl chamber and $\mathfrak{a}_+^*$ the corresponding cone in $\mathfrak{a}^*$ via the identification $\mathfrak{a} \leftrightarrow \mathfrak{a}^*$ through the Killing form $\langle \cdot, \cdot \rangle$ restricted to $\mathfrak{a}$. We denote by ${}_+\mathfrak{a}^*$ the dual cone $\{\lambda \in \mathfrak{a}^* \mid \lambda(H) > 0\ \forall H \in \overline{\mathfrak{a}_+} \setminus \{0\}\}$ and by $\overline{{}_+\mathfrak{a}^*}$ its closure $\{\lambda \in \mathfrak{a}^* \mid \lambda(H) \ge 0\ \forall H \in \mathfrak{a}_+\} = \mathbb{R}_{\ge 0}\Pi$. Hence if ω_j is the dual basis of α_j then $\overline{{}_+\mathfrak{a}^*} = \{\lambda \in \mathfrak{a}^* \mid \langle \lambda, \omega_j \rangle \ge 0\ \forall j = 1, \dots, n\}$. Furthermore, we write $\overline{{}_-\mathfrak{a}^*} := -\overline{{}_+\mathfrak{a}^*}$. If $\overline{A^+} := \exp(\overline{\mathfrak{a}_+})$, then we have the Cartan decomposition $G = K\overline{A^+}K$.

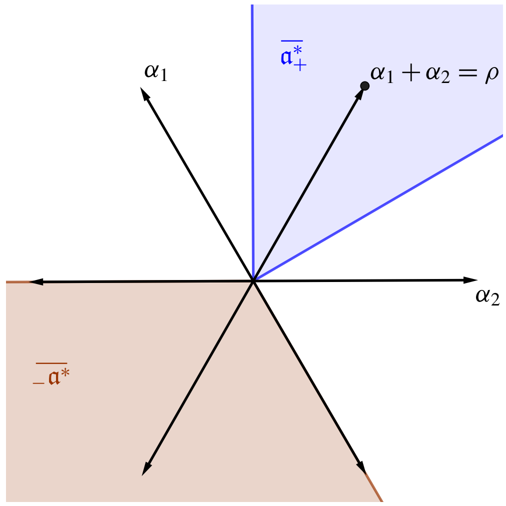

Figure 1. The root system for the special case $G=\mathrm{SL}_3(\mathbb{R})$: There are three positive roots $\Sigma^+=\{\alpha_1,\alpha_2,\alpha_1+\alpha_2\}$. As all root spaces are one-dimensional the special element $\rho=\frac{1}{2}\Sigma_{\alpha\in\Sigma^+}m_\alpha\alpha$ equals $\alpha_1+\alpha_2$.

Example 2.3. If $G=\mathrm{SL}_n(\mathbb{R})$, then we choose $K=\mathrm{SO}(n)$, A as the set of diagonal matrices of positive entries with determinant 1, and N as the set of upper triangular matrices with 1's on the diagonal. Then $\mathfrak{a}$ is the abelian Lie algebra of diagonal matrices and the set of restricted roots is $\Sigma=\{\varepsilon_i-\varepsilon_j\mid i\neq j\}$, where $\varepsilon_i(\lambda)$ is the i-th diagonal entry of λ. The positive system corresponding to the Iwasawa decomposition is $\Sigma^+=\{\varepsilon_i-\varepsilon_j\mid i<j\}$ with simple system $\Pi=\{\alpha_i=\varepsilon_i-\varepsilon_{i+1}\}$. The positive Weyl chamber is

$$\mathfrak{a}_+=\{\mathrm{diag}(\lambda_1,\dots,\lambda_n)\mid\lambda_1>\cdots>\lambda_n\}$$

and the dual cone is

$$\overline{{}_+\mathfrak{a}}=\{\mathrm{diag}(\lambda_1,\dots,\lambda_n)\in\mathfrak{a}\mid\lambda_1+\cdots+\lambda_k\geq 0\ \forall k\}.$$

See Figure 1 for a visualization in the special case $G=\mathrm{SL}_3(\mathbb{R})$. The Weyl group is the symmetric group S_n acting by permutation of the diagonal entries.

2C. ***Principal series representations.*** The concept of a principal series representation is an important tool in representation theory of semisimple Lie groups. It can be described using different pictures. We start with the *induced picture*: Pick $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ and (τ,V_τ) an irreducible unitary representation of M. We define

$$V^{\tau,\lambda}:=\{f:G\to V_\tau\text{ cont.}\mid f(gman)=e^{-(\lambda+\rho)\log a}\tau(m)^{-1}f(g),\ g\in G,\ m\in M,\ a\in A,\ n\in N\}$$

endowed with the norm $\|f\|^2=\int_K\|f(k)\|^2\,dk$ where dk is the normalized Haar measure on K. Recall that ρ is the half-sum of positive roots. The group G acts on $V^{\tau,\lambda}$ by the left regular representation. The completion $H^{\tau,\lambda}$ of $V^{\tau,\lambda}$ with respect to the norm is called the *induced picture of the* (*nonunitary*) *principal series representation* with respect to (τ,λ). We also write $\pi_{\tau,\lambda}$ for this representation. If τ is the trivial representation then we write H^λ and π_λ and call it the *spherical principal series* with respect to λ.

Note that for equivalent irreducible unitary representations τ_1 and τ_2 of M the corresponding principal series representations are equivalent as representations as well. In particular, the Weyl group W acts on the unitary dual of M by $w\tau(m) = \tau(w^{-1}mw)$, where $w \in W$ is given by a representative in the normalizer of A in K, and therefore $H^{\lambda,w\tau}$ is well defined up to equivalence.

A different way to view the principal series representation is the so-called *compact picture*. Although we don't need this description here, we want to introduce it in order to give a larger overview of these representations. It is given by restricting the function $f : G \to V_\tau$ to K, i.e., a dense subspace is given by

$$\{f : K \to V_\tau \text{ cont.} \mid f(km) = \tau(m)^{-1} f(k),\ k \in K,\ m \in M\}$$

with the same norm as above. In this picture the G-action is given by

$$\pi_{\tau,\lambda}(g) f(k) = e^{-(\lambda+\rho)H(g^{-1}k)} f(k_{KAN}(g^{-1}k)), \quad g \in G,\ \ k \in K,$$

where k_{KAN} is the K-component in the Iwasawa decomposition $G = KAN$. Furthermore, recall from Section 2B that $H(g) \in \mathfrak{a}$ was defined as the logarithm of the Iwasawa A component.

For the example $G = \mathrm{PSL}_2(\mathbb{R})$, the compact picture allows us to describe this representation explicitly without using the Iwasawa decomposition: since $K = \mathrm{PSO}(2) \simeq S^1 \subseteq \mathbb{R}^2$, the representation $H^{1,\lambda\alpha} = H^{\lambda\alpha}$ with $\lambda \in \mathbb{C}$ is given by $L^2(S^1)$ with the action $\pi_\lambda(g) f(\omega) = \|g^{-1}\omega\|^{-2\lambda-1} f(g^{-1}\omega/\|g^{-1}\omega\|)$.

2D. ***Invariant differential operators.*** Let $\mathbb{D}(G/K)$ be the algebra of *G-invariant differential operators* on G/K, i.e., differential operators commuting with the left translation by elements $g \in G$. Then we have an algebra isomorphism $\mathrm{HC} : \mathbb{D}(G/K) \to \mathrm{Poly}(\mathfrak{a}^*)^W$ from $\mathbb{D}(G/K)$ to the W-invariant complex polynomials on $\mathfrak{a}^*$ which is called the *Harish-Chandra homomorphism*; see [Helgason 1984, Chapter II Theorem 5.18]. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, let χ_λ be the character of $\mathbb{D}(G/K)$ defined by $\chi_\lambda(D) := \mathrm{HC}(D)(\lambda)$. Obviously, $\chi_\lambda = \chi_{w\lambda}$ for $w \in W$. Furthermore, the χ_λ exhaust all characters of $\mathbb{D}(G/K)$; see [Helgason 1984, Chapter III Lemma 3.11]. We define the space of joint eigenfunctions

$$E_\lambda := \{f \in C^\infty(G/K) \mid Df = \chi_\lambda(D) f\ \forall D \in \mathbb{D}(G/K)\}.$$

We will only work with the subspace of functions of moderate growth

$$E^*_\lambda := \{f \in E_\lambda \mid \exists c \in \mathbb{R} : |f(kaK)| \le Ce^{c\|\log a\|}\ \forall k \in K,\ a \in A\}.$$

Note that E_λ and E^*_λ are G-invariant.

2E. ***Poisson transform.*** The representation of G on E^*_λ can be described via the *Poisson transform*: If $(H^{\tau,\lambda})^{-\infty}$ denotes the distributional vectors in the principal series, then the Poisson transform $\mathcal{P}_\lambda$ maps $(H^{-\lambda})^{-\infty}$ into E^*_λ G-equivariantly. It is given by $\mathcal{P}_\lambda f(xK) = \int_K f(k) e^{-(\lambda+\rho)H(x^{-1}k)}\, dk$ if f is a sufficiently regular function in the compact picture of the principal series. If f is given in the induced picture, then $\mathcal{P}_\lambda f(xK)$ is simply $\int_K f(xk)\, dk$. Since K/M can be seen as the boundary of G/K at infinity, the Poisson transform produces a joint eigenfunction for a given boundary value; see [van den Ban and Schlichtkrull 1987] for more details.

It is important to know for which values of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the Poisson transform is a bijection. By [van den Ban and Schlichtkrull 1987, Theorem 12.2] we have that $\mathcal{P}_\lambda : (H^{-\lambda})^{-\infty} \to E_\lambda^*$ is a bijection if

$$-\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \notin \mathbb{N}_{>0} \quad \text{for all } \alpha \in \Sigma^+. \tag{2}$$

In particular, $\mathcal{P}_\lambda$ is a bijection if $\operatorname{Re}\lambda \in \overline{\mathfrak{a}_+^*}$.

2F. *L^p-bounds for elementary spherical functions.* One can show that in each joint eigenspace E_λ there is a unique left K-invariant function which has the value 1 at the identity; see [Helgason 1984, Chapter IV Corollary 2.3]. We denote the corresponding bi-K-invariant function on G by ϕ_λ and call it the *elementary spherical function*. Therefore, $\phi_\lambda = \phi_\mu$ if and only if $\lambda = w\mu$ for some $w \in W$. It is given by the Poisson transform of the constant function with value 1 in the compact picture, i.e., $\phi_\lambda(g) = \int_K e^{-(\lambda+\rho)H(g^{-1}k)}\,dk$.

The aim of this section is to establish the following proposition (see Figure 2 for a visualization) that will be needed to obtain a spectral gap in Theorem 4.10.

Proposition 2.4. *Let $p \in [2,\infty[$. Then the elementary spherical function ϕ_λ is in $L^{p+\varepsilon}(G)$ (where the L^p-space is defined via a Haar measure on G) for every $\varepsilon > 0$ if and only if* $\operatorname{Re}\lambda \in (1-2p^{-1})\operatorname{conv}(W\rho)$, *where* $\operatorname{conv}(W\rho)$ *is the convex hull of the finite set $W\rho$.*

Proof. First of all note that we only have to consider $\operatorname{Re}\lambda \in \overline{\mathfrak{a}_+^*}$ since $\phi_\lambda = \phi_\mu$ if and only if $\lambda = w\mu$ for some $w \in W$. In this case $\operatorname{Re}\lambda \in (1-2p^{-1})\operatorname{conv}(W\rho)$ is equivalent to $\operatorname{Re}\lambda \in (1-2p^{-1})\rho + \overline{{}_-\mathfrak{a}^*}$; see [Helgason 1984, Chapter IV Lemma 8.3].

With this remark, one implication of the proposition is a straightforward consequence of standard estimates for elementary spherical functions: Suppose that $\operatorname{Re}\lambda \in \overline{\mathfrak{a}_+^*}$ and $\operatorname{Re}\lambda \in (1-2p^{-1})\rho + \overline{{}_-\mathfrak{a}^*}$. Then we have the following bound on ϕ_λ (see [Knapp 1986, Chapter VII Property 7.15]):

$$|\phi_\lambda(a)| \le Ce^{(\operatorname{Re}\lambda-\rho)(\log a)}(1+\rho(\log a))^d, \quad a \in A^+,$$

where C and d are constants ≥ 0. By the integral formula for $G = K\overline{A^+}K$ (see [Helgason 1984, Chapter I Theorem 5.8]) and the bi-K-invariance of ϕ_λ, we have

$$\begin{aligned}\int_G |\phi_\lambda(g)|^{p+\varepsilon}\,dg &= \int_{\mathfrak{a}_+} |\phi_\lambda(\exp H)|^{p+\varepsilon} \prod_{\alpha\in\Sigma^+} \sinh(\alpha(H))^{m_\alpha}\,dH \\ &\le \int_{\mathfrak{a}_+} (Ce^{(\operatorname{Re}\lambda-\rho)H}(1+\rho(H))^d)^{p+\varepsilon} e^{2\rho(H)}\,dH\end{aligned}$$

for a suitable Lebesgue measure on $\mathfrak{a}$. Because $\operatorname{Re}\lambda \in (1-2p^{-1})\rho + \overline{{}_-\mathfrak{a}^*}$, we have

$$(p+\varepsilon)(\operatorname{Re}\lambda-\rho)(H) \le -(2+2\varepsilon p^{-1})\rho(H).$$

Hence

$$\int_G |\phi_\lambda(g)|^{p+\varepsilon}\,dg \le C^{p+\varepsilon}\int_{\mathfrak{a}_+} (1+\rho(H))^{d(p+\varepsilon)} e^{-2\varepsilon p^{-1}\rho(H)}\,dH,$$

and we see that the latter is indeed finite by coordinizing $\mathfrak{a}_+$ by $x_j \leftrightarrow \alpha_j(H)$ with $x_j > 0$. Then dH is a multiple of dx and $\rho(H) = \sum x_j\rho_j$ with $\rho_j > 0$. Therefore, $\phi_\lambda \in L^{p+\varepsilon}(G)$.

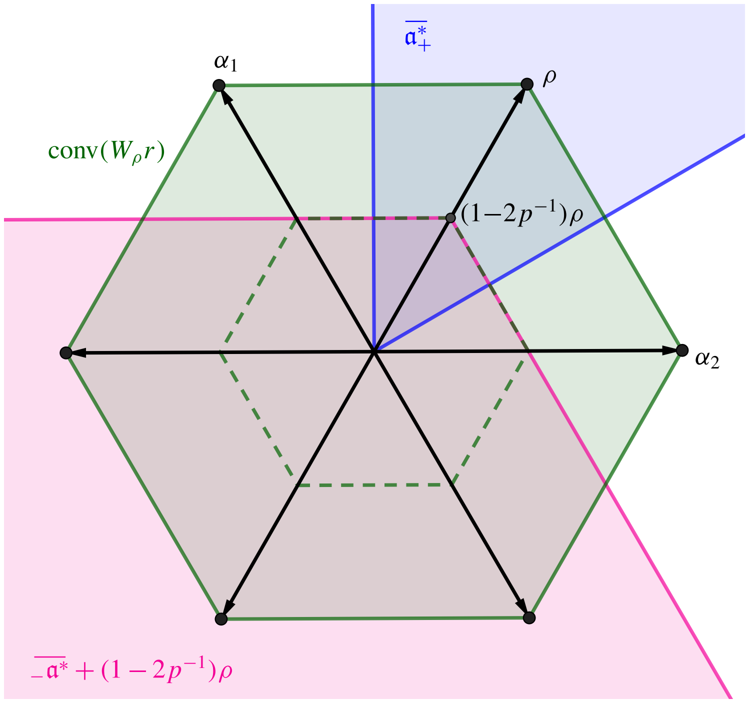

Figure 2. Visualization of the regions appearing in Proposition 2.4 for the special case $G = \mathrm{SL}_3(\mathbb{R})$: The green dashed region is the boundary of $(1-2p^{-1})\,\mathrm{conv}(W\rho)$. Its intersection with the positive Weyl chamber $\overline{\mathfrak{a}_+^*}$ (blue cone) equals $(1-2p^{-1})\rho + \overline{_-\mathfrak{a}^*}$ intersected with $\overline{\mathfrak{a}_+^*}$.

The opposite implication will be proved by combining the proof of [Knapp 1986, Theorem 8.48] with [van den Ban and Schlichtkrull 1987]: according to [van den Ban and Schlichtkrull 1987, Corollary 16.2], the elementary spherical function ϕ_λ has a converging expansion

$$\phi_\lambda(\exp H) = \sum_{\xi \in X(\lambda)} p_\xi(\lambda, H) e^{\xi(H)}, \qquad H \in \mathfrak{a}_+, \tag{3}$$

where

$$X(\lambda) = \{w\lambda - \rho - \mu \mid w \in W,\ \mu \in \mathbb{N}_0\Pi\}$$

and the $p_\xi(\lambda,\cdot)$ are polynomials of degree $\leq |W|$. The series converges absolutely on $\mathfrak{a}_+$ and uniformly on each subchamber $\{H \in \mathfrak{a}_+ \mid \alpha_i(H) \geq \varepsilon_i > 0\}$. The main ingredient of the proof of Proposition 2.4 is the fact that (see [van den Ban and Schlichtkrull 1987, Theorem 10.1])

$$p_{\lambda-\rho}(\lambda,\cdot) \neq 0. \tag{4}$$

Now, if $\phi_\lambda \in L^{p+\epsilon}(G)$, the proof of [Knapp 1986, Theorem 8.48] shows that

$$\mathrm{Re}\langle \lambda - (1-2(p+\epsilon)^{-1})\rho, \omega_j\rangle < 0.$$

Hence $\mathrm{Re}\,\lambda - (1-2p^{-1})\rho \in \overline{_-\mathfrak{a}^*}$. $\square$

2G. ***Positive definite functions and unitary representations.*** In this section we recall the correspondence between positive semidefinite elementary spherical functions and irreducible unitary spherical representations. Recall first that a continuous function $f : G \to \mathbb{C}$ is called *positive semidefinite* if the matrix $(f(x_i^{-1}x_j))_{i,j}$ for all $x_1, \dots, x_k \in G$ is positive semidefinite. If f is positive semidefinite, then f is bounded by $f(1)$ and one has $f(x^{-1}) = \overline{f(x)}$. Moreover, we can define a unitary representation π_f associated to f in the following way: if R denotes the right regular representation of G, then π_f is the completion of the space spanned by $R(x)f$ with respect to the inner product defined by $\langle R(x)f, R(y)f\rangle := f(y^{-1}x)$, which is positive definite. G acts unitarily on this space by the right regular representation. If $f(g) = \langle \pi(g)v, v\rangle$ is a matrix coefficient of a unitary representation π, then f is positive semidefinite and π_f is contained in π.

Secondly, recall that a unitary representation is called *spherical* if it contains a nonzero K-invariant vector. Denote by $\widehat{G}_{\mathrm{sph}}$ the subset of the unitary dual consisting of spherical representations. We then have a one-to-one correspondence between positive semidefinite elementary spherical functions and $\widehat{G}_{\mathrm{sph}}$ given by $\phi_\lambda \mapsto \pi_{\phi_\lambda}$; see [Helgason 1984, Chapter IV Theorem 3.7]. The preimage of an irreducible unitary spherical representation π with normalized K-invariant vector v_K is given by $g \mapsto \langle \pi(g)v_K, v_K\rangle$. If the set $\widehat{G}_{\mathrm{sph}}$ is endowed with the Fell topology (see [Bekka et al. 2008, Appendix F.2]) and we use the topology of convergence on compact sets on the set of elementary spherical functions, then the above correspondence is a homeomorphism as is easily seen from the definitions.

2H. ***Associated vector bundles.*** In order to define the Weyl chamber flow not only on the base manifold but also on vector bundles we recall the definition of the associated vector bundle $\mathcal{V}_\tau$ over a homogeneous space G/M for a unitary finite-dimensional representation (τ, V_τ) of M. Its total space is given by $\mathcal{V}_\tau = G \times_\tau V_\tau = (G \times V_\tau)/_\sim$, where $(gm, v) \sim (g, \tau(m)v)$ with $g \in G$, $m \in M$ and $v \in V_\tau$. The equivalence classes are denoted by $[g, v]$ and the projection to G/M is $[g, v] \mapsto gM$. A section s of this bundle can be identified with a function $\bar{s} : G \to V_\tau$ satisfying $\bar{s}(gm) = \tau(m)^{-1}\bar{s}(g)$. We will use this identification throughout this article. We also have a G-action on $\mathcal{V}_\tau$ defined by $g[g', v] := [gg', v]$. Therefore, we also have the left regular action on smooth sections of $\mathcal{V}_\tau$:

$$(gs)(g'M) := g(s(g^{-1}g'M)), \quad s \in C^\infty(G/M, \mathcal{V}_\tau).$$

Identifying s with $\bar{s}$, this actions reads $\overline{gs}(g') = \bar{s}(g^{-1}g')$.

A special case of an associated vector bundle is the tangent bundle $T(G/M) = G \times_{\mathrm{Ad}|_M} (\mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-)$. Hence vector fields $\mathfrak{X}$ can be identified with smooth functions $\overline{\mathfrak{X}} : G \to \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-$ satisfying

$$\overline{\mathfrak{X}}(gm) = \mathrm{Ad}(m)^{-1}\overline{\mathfrak{X}}(g).$$

Therefore, we have a canonical connection ∇ on $\mathcal{V}_\tau$ given by

$$\overline{\nabla_{\mathfrak{X}} s}(g) = \frac{d}{dt}\Big|_{t=0} \bar{s}(g \exp(t\overline{\mathfrak{X}}(g))),$$

where s is a smooth section identified with a $\bar{s} : G \to V_\tau$ and $\mathfrak{X}$ is a vector field of G/M identified with $\overline{\mathfrak{X}}$ as above. This connection will be used to lift the Weyl chamber flow to $\mathcal{V}_\tau$.

3. Ruelle–Taylor resonances for the Weyl chamber action

We keep the notation from Section 2B. Let Γ be a discrete, torsion-free, cocompact subgroup of G. Then the biquotient $\mathcal{M} = \Gamma\backslash G/M$ is a smooth compact Riemannian manifold where the Riemannian structure is induced by the inner product on $\mathfrak{g}$. More precisely, the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ is given by the quotient $\Gamma\backslash(G \times_{\mathrm{Ad}|_M} (\mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-))$, and the norm of some $\Gamma[g, Y]$ with $g \in G$ and $Y \in \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-$ is given by the norm of $Y \in \mathfrak{g}$. We have a well-defined right A-action on $\mathcal{M}$:

$$(\Gamma g M)a := \Gamma g a M, \quad a \in A, \quad g \in G.$$

Therefore, we have an $\mathfrak{a}$-action by smooth vector fields:

$${}_{\Gamma}X : \mathfrak{a} \to C^\infty(\mathcal{M}, T\mathcal{M}), \quad {}_{\Gamma}X_H f(\Gamma g M) = \frac{d}{dt}\Big|_{t=0} f(\Gamma g e^{tH} M),$$

which we call the *Weyl chamber action*.

For later use we denote by $X : \mathfrak{a} \to \mathrm{Diff}^1(G/M)$ the corresponding action on G/M.

Proposition 3.1. *The A-action on $\mathcal{M}$ is Anosov. More precisely, each $H \in \mathfrak{a}_+$ is transversally hyperbolic with the splitting $E_0 = \Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{a})$, $E_s = \Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{n})$ and $E_u = \Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{n}_-))$. Moreover, for fixed $H_0 \in \mathfrak{a}_+$, the dynamically defined positive Weyl chamber*

$$\mathcal{W} = \{H \in \mathfrak{a} \mid H \textit{ is transversally hyperbolic with the same splitting as } H_0\}$$

equals $\mathfrak{a}_+$. Hence the two notions of positive Weyl chambers agree.

Proof. Pick $\Gamma[g, X_\alpha] \in \Gamma\backslash(G \times_M \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-)$ and assume that X_α is in the root space $\mathfrak{g}_\alpha$. Then we calculate

$$\begin{aligned} d\varphi_t^{\Gamma X_H}(\Gamma g M)\Gamma[g, X_\alpha] &= \frac{d}{ds}\Big|_{s=0} \varphi_t^{\Gamma X_H}(\Gamma g e^{sX_\alpha} M) \\ &= \frac{d}{ds}\Big|_{s=0} \Gamma e^{sX_\alpha} e^{tH} M \\ &= \frac{d}{ds}\Big|_{s=0} \Gamma g e^{tH} e^{s\,\mathrm{Ad}(e^{-tH})X_\alpha} M \\ &= \Gamma[g e^{tH}, \mathrm{Ad}(e^{-tH})X_\alpha] \\ &= \Gamma[g e^{tH}, e^{-t\alpha(H)} X_\alpha]. \end{aligned}$$

Hence we have exponential decay if $\alpha \in \Sigma^+$ and exponential growth if $\alpha \in -\Sigma^+$. The general statement is obtained from the observation that $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta \perp \mathfrak{a}$ for $\alpha \neq \beta$ in Σ. □

3A. *Lifted Weyl chamber action.* In order to define horocycle operators we generalize the Weyl chamber action to associated vector bundles. Let (τ, V_τ) be a finite-dimensional unitary representation of M, that is, a complexification of an orthogonal representation. Then we have defined the associated vector bundle $\mathcal{V}_\tau = G \times_\tau V_\tau$ over G/M; see Section 2H.

The quotient bundle $\Gamma\backslash\mathcal{V}_\tau$ is the complexification of a Euclidean vector bundle over $\mathcal{M}$, where the Euclidean structure is induced by the inner product on V_τ. We identify smooth sections s of this bundle with smooth functions $\bar{s} : G \to V_\tau$ with $\bar{s}(\gamma g m) = \tau(m^{-1})\bar{s}(g)$ for all $\gamma \in \Gamma$, $g \in G$ and $m \in M$.

The canonical connection ∇ descends to a connection ${}_\Gamma\nabla : C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau) \to C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau \otimes T^*\mathcal{M})$ and we have

$$\overline{{}_\Gamma\nabla s(\mathfrak{X})}(g) := \overline{{}_\Gamma\nabla_{\mathfrak{X}} s}(g) = \frac{d}{dt}\Big|_{t=0} \bar{s}(g\exp(t\overline{\mathfrak{X}}(g))), \tag{5}$$

where s is a smooth section identified as above and $\mathfrak{X}$ is a vector field of $\mathcal{M}$ identified with a smooth function $\overline{\mathfrak{X}} : G \to \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-$ which is left Γ-invariant and right M-equivariant.

Definition 3.2. The *lifted Weyl chamber action* is defined as follows:

$${}_\Gamma\boldsymbol{X}^\tau : \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau), \quad {}_\Gamma\boldsymbol{X}^\tau_H := {}_\Gamma\nabla_{\mathfrak{X}_H},$$

where $\mathfrak{X}_H$ is the vector field identified with the constant mapping $G \to \mathfrak{a} \subseteq \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-$, and $g \mapsto H$.

The fact that ${}_\Gamma\nabla$ is a covariant derivative implies that ${}_\Gamma\boldsymbol{X}^\tau$ is an admissible lift of the Weyl chamber action in the sense of (1).

For later use we denote by $\boldsymbol{X}^\tau : \mathfrak{a} \to \mathrm{Diff}^1(G/M, \mathcal{V}_\tau)$ the corresponding action on G/M.

We can find a nontrivial tube domain in $\mathfrak{a}^*_{\mathbb{C}}$ which is independent of τ and contains all Ruelle–Taylor resonances for the lifted Weyl chamber action.

Proposition 3.3. *The set of Ruelle–Taylor resonances $\sigma_{\mathrm{RT}}({}_\Gamma\boldsymbol{X}^\tau)$ is contained in $\overline{{}_-\mathfrak{a}^*} + i\mathfrak{a}^*$.*

Proof. By Proposition 2.2 we have

$$\sigma_{\mathrm{RT}}({}_\Gamma\boldsymbol{X}^\tau) \subseteq \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \mathrm{Re}(\lambda(H)) \le C^\tau_{L^2}(H)\ \forall H \in \mathfrak{a}_+\}.$$

Hence it remains to show that $C^\tau_{L^2}(H) := \inf\{C > 0 \mid \|e^{-t\,{}_\Gamma\boldsymbol{X}^\tau_H}\|_{L^2\to L^2} \le e^{Ct}\ \forall t > 0\} = 0$ for all $H \in \mathfrak{a}_+$. We show the stronger statement that $e^{-t\,{}_\Gamma\boldsymbol{X}^\tau_H}$ is unitary.

Since M commutes with A, we have a well-defined action of A on $\Gamma\backslash\mathcal{V}_\tau$ given by $(\Gamma[g,v])a = \Gamma[ga,v]$. This action gives rise to an A-action on sections of the bundle $\Gamma\backslash\mathcal{V}_\tau$ defined via $(af)(x) = f(xa)a^{-1}$ with $f \in C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau)$, $x \in \mathcal{M}$ and $a \in A$. If we identify f with a equivariant function $\bar{f} : G \to V_\tau$, then $\overline{(af)}(g) = \bar{f}(ga)$. Let $d\Gamma g$ be the normalized right G-invariant Radon measure on $\Gamma\backslash G$. Then the L^2-norm of f is given by $\|f\|^2_{L^2} = \int_{\Gamma\backslash G} \|\bar{f}(g)\|^2_{V_\tau}\, d\Gamma g$, and it follows that the A-action continued to $L^2(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau)$ is unitary. By definition, $e^{-t\,{}_\Gamma\boldsymbol{X}^\tau_H} f = \exp(-tH)f$ for $f \in L^2(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau)$, and therefore $e^{-t\,{}_\Gamma\boldsymbol{X}^\tau_H}$ is unitary. □

3B. *First band resonances and horocycle operators.* In analogy to the rank 1 setting we make the following definition; see [Küster and Weich 2021, Definition 2.11] and [Guillarmou et al. 2021, Definition 3.1] in the scalar case.

Definition 3.4. We call $\lambda \in \sigma_{\mathrm{RT}}({}_\Gamma\boldsymbol{X}^\tau)$ a *first band resonance* and write $\lambda \in \sigma^0_{\mathrm{RT}}({}_\Gamma\boldsymbol{X}^\tau)$ if the vector space

$$\mathrm{Res}^0_{{}_\Gamma\boldsymbol{X}^\tau}(\lambda) = \{u \in \mathrm{Res}_{{}_\Gamma\boldsymbol{X}^\tau}(\lambda) \mid {}_\Gamma\nabla_{\mathfrak{X}} u = 0\ \forall \mathfrak{X} \in C^\infty(\mathcal{M}, E_u)\}$$

of *first band resonant states* is nontrivial.

The goal of this section is to prove that, in a certain neighborhood of 0 in $\mathfrak{a}_{\mathbb{C}}^*$, each Ruelle–Taylor resonance is a first band resonance and $\mathrm{Res}^0_{{}_\Gamma X^\tau}(\lambda) = \mathrm{Res}_{{}_\Gamma X^\tau}(\lambda)$. This will be done by introducing so-called horocycle operators as follows.

Recall that $T\mathcal{M} = \Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}_-)$ and the bundle $\Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{n})$ can be written as $\bigoplus_{\alpha\in\Sigma^+} \Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{g}_\alpha)$, and similarly for $\mathfrak{n}_-$. Therefore, the cotangent bundle $T^*\mathcal{M}$ is the Whitney sum $\Gamma\backslash(G \times_{\mathrm{Ad}^*|_M} \mathfrak{a}^*) \oplus \bigoplus_{\alpha\in\Sigma} \Gamma\backslash(G \times_{\mathrm{Ad}^*|_M} \mathfrak{g}_\alpha^*)$. Let us denote the coadjoint action of M on the complexification of $\mathfrak{g}_\alpha^*$ by τ_α. Note that τ_α is unitary with respect to the inner product induced by the Killing form and the Cartan involution. We can now define

$$\mathrm{pr}_\alpha : (T^*\mathcal{M})_{\mathbb{C}} \to \Gamma\backslash\mathcal{V}_{\tau_\alpha}$$

by fiber-wise restriction to the subbundle $\Gamma\backslash(G \times_{\mathrm{Ad}|_M} \mathfrak{g}_\alpha)$. This induces a map

$$\widetilde{\mathrm{pr}}_\alpha : C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau \otimes (T^*\mathcal{M})_{\mathbb{C}}) \to C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_{\tau\otimes\tau_\alpha}).$$

Definition 3.5. If ${}_\Gamma\nabla^{\mathbb{C}} : C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau) \to C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau \otimes (T^*\mathcal{M})_{\mathbb{C}})$ denotes the complexification of the canonical connection ${}_\Gamma\nabla$, then the *horocycle operator* $\mathcal{U}_\alpha$ for $\alpha \in \Sigma$ is defined as the composition

$$\mathcal{U}_\alpha := \widetilde{\mathrm{pr}}_\alpha \circ {}_\Gamma\nabla^{\mathbb{C}} : C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau) \to C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_{\tau\otimes\tau_\alpha}).$$

Note that we have the explicit formula

$$\overline{\mathcal{U}_\alpha s}(g)(Y) = \frac{d}{dt}\Big|_{t=0} \bar{s}(g\exp(tY)), \quad s \in C^\infty(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau), \quad Y \in \mathfrak{g}_\alpha, \tag{6}$$

if we again use the identification of sections of some associated vector bundle with left Γ-invariant and right M-equivariant functions indicated by $\bar{\cdot}$ and the identification $V_\tau \otimes \mathfrak{g}_\alpha^* \simeq \mathrm{Hom}(\mathfrak{g}_\alpha, V_\tau)$.

We should point out that the space of first band resonant states can be rewritten with the horocycle operators as

$$\mathrm{Res}^0_{{}_\Gamma X^\tau}(\lambda) = \{u \in \mathrm{Res}_{{}_\Gamma X^\tau}(\lambda) \mid \mathcal{U}_{-\alpha} u = 0 \ \forall \alpha \in \Sigma^+\}. \tag{7}$$

Note that in the case of constant curvature manifolds (i.e., the real hyperbolic case $G = \mathrm{PSO}(n, 1)$ of rank 1) there is only one positive root and our definition reduces to the original one due to Dyatlov and Zworski; see [Dyatlov et al. 2015, p. 931]. Furthermore, our definition extends the definition of the horocycle operators for arbitrary G of rank 1; see [Küster and Weich 2021].

The horocycle operators fulfill the following important commutation relation.

Lemma 3.6. *For all $H \in \mathfrak{a}$,*

$${}_\Gamma X_H^{\tau\otimes\tau_\alpha} \mathcal{U}_\alpha - \mathcal{U}_\alpha {}_\Gamma X_H^\tau = \alpha(H)\mathcal{U}_\alpha.$$

Proof. Using the formulas (5) and (6) we obtain

$$\overline{{}_\Gamma X_H^{\tau\otimes\tau_\alpha} \mathcal{U}_\alpha - \mathcal{U}_\alpha {}_\Gamma X_H^\tau}(g)(Y) = \frac{d}{dt_1}\Big|_{t_1=0} \frac{d}{dt_2}\Big|_{t_2=0} \bar{s}(g\exp(t_1H)\exp(t_2Y)) - \bar{s}(g\exp(t_1Y)\exp(t_2H)),$$

and the latter equals

$$\frac{d}{dt}\Big|_{t=0} \bar{s}(g\exp(t[H, Y])).$$

Since $[H, Y] = \alpha(H)Y$ for $Y \in \mathfrak{g}_\alpha$, the claim follows. ☐

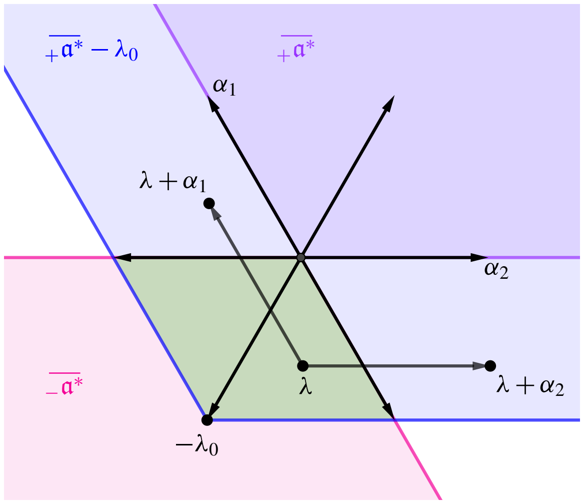

Figure 3. For $G = \mathrm{SL}_3(\mathbb{R})$ the green region depicts the real part of the region where every resonance is a first band resonance; see Proposition 3.7.

We can now prove the main result of this section.

Proposition 3.7. *The horocycle operators can be extended continuously as linear operators to distributional sections*, *i.e.*,

$$\mathcal{U}_\alpha : \mathcal{D}'(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau) \to \mathcal{D}'(\mathcal{M}, \Gamma\backslash\mathcal{V}_{\tau\otimes\tau_\alpha}).$$

Moreover, *for* $\lambda \in \sigma_{\mathrm{RT}}({}_\Gamma \boldsymbol{X}^\tau)$, *the horocycle operator* $\mathcal{U}_{-\alpha}$ *maps*

$$\mathrm{Res}_{{}_\Gamma \boldsymbol{X}^\tau}(\lambda) \quad \textit{into} \quad \mathrm{Res}_{{}_\Gamma \boldsymbol{X}^{\tau\otimes\tau_{-\alpha}}}(\lambda+\alpha).$$

In particular, *each* $\lambda \in \sigma_{\mathrm{RT}}({}_\Gamma \boldsymbol{X}^\tau)$ *with* $\mathrm{Re}\,\lambda \in \bigcap_{\alpha\in\Pi} \overline{{}_-\mathfrak{a}^*} \setminus (\overline{{}_-\mathfrak{a}^*} - \alpha)$ *is a first band resonance and* $\mathrm{Res}_{{}_\Gamma \boldsymbol{X}^\tau}(\lambda) = \mathrm{Res}^0_{{}_\Gamma \boldsymbol{X}^\tau}(\lambda)$ *holds.*

See Figure 3 for a visualization for $G = \mathrm{SL}_3(\mathbb{R})$.

Proof. Since the horocycle operators are differential operators, we obtain a continuation to distributional sections and Lemma 3.6 still holds. Let $u \in \mathrm{Res}_{{}_\Gamma \boldsymbol{X}^\tau}(\lambda)$, i.e., $u \in \mathcal{D}'(\mathcal{M}, \Gamma\backslash\mathcal{V}_\tau)$ with $\mathrm{WF}(u) \subseteq E_u^*$ and ${}_\Gamma \boldsymbol{X}_H^\tau u = -\lambda(H)u$. Since differential operators do not increase the wavefront set, we have $\mathrm{WF}(\mathcal{U}_{-\alpha}u) \subseteq E_u^*$. Furthermore,

$${}_\Gamma \boldsymbol{X}_H^{\tau\otimes\tau_{-\alpha}} \mathcal{U}_{-\alpha} u = -\alpha(H)\mathcal{U}_{-\alpha}u + \mathcal{U}_{-\alpha}\,{}_\Gamma \boldsymbol{X}_H^\tau u = -(\lambda+\alpha)(H)\mathcal{U}_{-\alpha}u$$

by Lemma 3.6. Hence $\mathcal{U}_{-\alpha}u \in \mathrm{Res}_{{}_\Gamma \boldsymbol{X}^{\tau\otimes\tau_{-\alpha}}}(\lambda+\alpha)$.

For the "in particular" part recall that $\mathrm{Res}_{{}_\Gamma \boldsymbol{X}^{\tau'}}(\lambda') = 0$ for each unitary representation τ' of M and $\mathrm{Re}(\lambda') \notin \overline{{}_-\mathfrak{a}^*}$ (see Proposition 3.3), and $\mathrm{Res}^0_{{}_\Gamma \boldsymbol{X}^\tau}(\lambda) = \{u \in \mathrm{Res}_{{}_\Gamma \boldsymbol{X}^\tau}(\lambda) \mid \mathcal{U}_{-\alpha}u = 0\ \forall \alpha \in \Sigma^+\}$. $\square$

Note that $\bigcap_{\alpha\in\Pi} \overline{{}_-\mathfrak{a}^*} \setminus (\overline{{}_-\mathfrak{a}^*} - \alpha) = \overline{{}_-\mathfrak{a}^*} \cap ({}_+\mathfrak{a}^* - \lambda_0)$, where $\lambda_0 = \sum_{\alpha\in\Pi} \alpha$. Indeed, let $\lambda = \sum_{\alpha\in\Pi} c_\alpha \alpha \in \mathfrak{a}^*$. Then $\lambda \in \overline{{}_-\mathfrak{a}^*}$ if and only if $c_\alpha \le 0$ for all $\alpha \in \Pi$, $\lambda \in \overline{{}_-\mathfrak{a}^*} - \alpha$ if and only if $c_\alpha \le -1$ and $c_\beta \le 0$ for all $\beta \in \Pi \setminus \{\alpha\}$, and $\lambda \in {}_+\mathfrak{a}^*$ if and only if $c_\alpha > 0$ for all $\alpha \in \Pi$. Combining these statements implies the claim.

3C. ***First band resonant states and principal series representation.*** In this section we identify first band resonant states with certain Γ-invariant vectors in a corresponding principal series representation. The proof follows the line of arguments given in [Küster and Weich 2021, Section 2] in the rank 1 case. This will allow us to apply the Poisson transform and obtain a quantum-classical correspondence.

By analogy to [Küster and Weich 2021, Definition 2.1], we define

$$\mathcal{R}(\lambda) := \{s \in \mathcal{D}'(G/M, \mathcal{V}_\tau) \mid (\mathbf{X}_H^\tau + \lambda(H))s = 0,\ \nabla_{\mathfrak{X}_-} s = 0\ \forall \mathfrak{X}_- \in C^\infty(G/M, G \times_{\mathrm{Ad}|_M} \mathfrak{n}_-),\ H \in \mathfrak{a}\}.$$

The following lemma allows us to first study the representation of G in $\mathcal{R}(\lambda)$ and take Γ-invariants afterwards.

Lemma 3.8. *The space* $\mathrm{Res}^0_{\Gamma \mathbf{X}^\tau}(\lambda)$ *is isomorphic to the space of* Γ*-invariants of* $\mathcal{R}(\lambda)$, *where the isomorphism is defined by considering* Γ*-invariant sections as sections of the bundle* $\Gamma\backslash\mathcal{V}_\tau$.

Proof. The only part to observe is that each $s \in \mathcal{R}(\lambda)$ automatically has $\mathrm{WF}(s) \subseteq G \times_{\mathrm{Ad}|_M} \mathfrak{n}^*$. This holds because $G \times_{\mathrm{Ad}^*|_M} \mathfrak{n}^*$ is the joint characteristic set of $\mathbf{X}_H^\tau$ and $\mathfrak{X}_-$; see [Küster and Weich 2021, Lemma 2.5] for details. □

We will now show that the smooth sections in $\mathcal{R}(\lambda)$ correspond to smooth vectors in the principal series representation for the opposite Iwasawa decomposition.

Lemma 3.9. *The smooth sections* $\mathcal{R}(\lambda) \cap C^\infty(G/M, \mathcal{V}_\tau)$ *in* $\mathcal{R}(\lambda)$ *can be identified* G*-equivariantly with*

$$W = \{\bar{s} : G \to V_\tau \text{ smooth} \mid \bar{s}(gman_-) = e^{-\lambda \log a}\tau(m)^{-1}\bar{s}(g),\ m \in M,\ a \in A,\ n_- \in N_-\}.$$

The identification is obtained by considering sections $s \in \mathcal{R}(\lambda)$ *as right* M*-equivariant functions* $\bar{s} : G \to V_\tau$.

Proof. The M-equivariance is clear so it remains to show the transformation properties under A and N_-. The property $(\mathbf{X}_H^\tau + \lambda(H))s = 0$ amounts to $(d/dt)|_{t=0}\bar{s}(ge^{tH}) = -\lambda(H)\bar{s}(g)$ for every $g \in G$ and $H \in \mathfrak{a}$. Hence the function $\varphi(t) = \bar{s}(ge^{tH})$ satisfies

$$\varphi'(r) = \frac{d}{dt}\Big|_{t=0}\varphi(ge^{rH}e^{tH}) = -\lambda(H)\bar{s}(ge^{rH}) = -\lambda(H)\varphi(r).$$

Therefore, $\bar{s}(ge^{tH}) = \varphi(t) = e^{-t\lambda(H)}\bar{s}(g)$. This proves the right A-equivariance.

For the N_--invariance, let $Y \in \mathfrak{n}_-$ and consider $\varphi(t) = \bar{s}(ge^{tY})$. For $r \in \mathbb{R}$, let $g_r = ge^{rY} \in G$. Since $[g_r, Y] \in G \times_{\mathrm{Ad}|_M} \mathfrak{n}_-$ is in the fiber over $g_r M \in G/M$, there is a smooth section

$$\mathfrak{X}_r \in C^\infty(G/M, G \times_{\mathrm{Ad}|_M} \mathfrak{n}_-)$$

such that $\mathfrak{X}_r(g_r M) = [g_r, Y]$. In particular, the corresponding right M-equivariant function $\overline{\mathfrak{X}}_r : G \to \mathfrak{n}_-$ satisfies $\overline{\mathfrak{X}}_r(g_r) = Y$. It follows that

$$0 = \overline{\nabla_{\mathfrak{X}_r} s}(g_r) = \frac{d}{dt}\Big|_{t=0}\bar{s}(g_r e^{t\overline{\mathfrak{X}}_r(g_r)}) = \frac{d}{dt}\Big|_{t=0}\bar{s}(ge^{rY}e^{tY}) = \varphi'(r).$$

Hence φ is constant. This completes the proof. □

Note that the space W from Lemma 3.9 is already very close to the definition of the induced picture of the principal series representation (see Section 2C). The only difference is that in W we have a right invariance with respect to N_- instead of N. This can be easily fixed using a conjugation with the longest Weyl group element and leads to the main result of this section:

Proposition 3.10. *With the longest Weyl group element w_0 (see Section 2B), we have an isomorphism*

$$\mathrm{Res}^0_{\Gamma X^\tau}(\lambda) \to {}^\Gamma(H^{w_0\tau, w_0(\lambda+\rho)})^{-\infty},$$

where ${}^\Gamma(H^{w_0\tau, w_0(\lambda+\rho)})^{-\infty}$ denotes the Γ-invariant distributional vectors in the principal series representation $\pi_{w_0\tau, w_0(\lambda+\rho)}$.

Proof. Pick $k_0 \in K$ normalizing $\mathfrak{a}$ such that the action of $\mathrm{Ad}(k_0)$ on $\mathfrak{a}$ is the longest Weyl group element w_0. We consider the map $I\bar{s}(g) := \bar{s}(gk_0)$. Then I commutes with the left action by G and one calculates that

$$I\bar{s}(gman) = e^{-(w_0\lambda)\log a}(w_0\tau)(m)^{-1} I\bar{s}(g), \quad g \in G,\ m \in M,\ a \in A,\ n \in N.$$

Hence we have an intertwiner between W and smooth vectors in $H^{w_0\tau, w_0(\lambda+\rho)}$ which extends to distributional sections. By Lemma 3.9 we conclude that $\mathcal{R}(\lambda) \simeq (H^{w_0\tau, w_0(\lambda+\rho)})^{-\infty}$ as G-representations. Taking Γ-invariants and using Lemma 3.8 completes the proof. ☐

3D. ***Quantum-classical correspondence.*** In the previous section we identified the first band resonant states $\mathrm{Res}^0_{\Gamma X^\tau}(\lambda)$ with Γ-invariant distributional vectors in the principal series $(H^{w_0\tau, w_0(\lambda+\rho)})^{-\infty}$. If we restrict ourselves to the scalar case $\tau = 1$, then the Poisson transform $\mathcal{P}_{-w_0(\lambda+\rho)}$ defines a map from ${}^\Gamma(H^{w_0(\lambda+\rho)})^{-\infty}$ to ${}^\Gamma E_{-w_0(\lambda+\rho)}$, as $\mathcal{P}_{-w_0(\lambda+\rho)}$ provides a G-equivariant map from $(H^{w_0(\lambda+\rho)})^{-\infty}$ to $E_{-w_0(\lambda+\rho)}$ (see Section 2E). Hence we can identify eigendistributions of the classical motion with quantum states and we call this identification *quantum-classical correspondence*. More precisely, we have the following result, which immediately gives Theorem 1.3.

Proposition 3.11. *If $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ satisfies $2\langle\lambda+\rho, \alpha\rangle/\langle\alpha, \alpha\rangle \notin -\mathbb{N}_{>0}$ for all $\alpha \in \Sigma^+$, then we have a bijection*

$$\mathrm{Res}^0_{\Gamma X}(\lambda) \to {}^\Gamma E_{-w_0(\lambda+\rho)} = {}^\Gamma E_{-(\lambda+\rho)}.$$

In particular, $\lambda \in \sigma^0_{\mathrm{RT}}({}_\Gamma X)$ if and only if ${}^\Gamma E_{-(\lambda+\rho)} \neq 0$. Furthermore, the isomorphism is given by the push-forward π_* of distributions along the canonical projection ${}_\Gamma\pi : \Gamma\backslash G/M \to \Gamma\backslash G/K$.

Proof. In view of Section 2E, the Poisson transform is a bijection from $(H^{w_0(\lambda+\rho)})^{-\infty} \to E^*_{-\lambda-\rho}$. Restricted to Γ-invariant distributional vectors it is still injective with image ${}^\Gamma E_{-\lambda-\rho}$ since Γ is cocompact, and therefore ${}^\Gamma E_{-\lambda-\rho} = {}^\Gamma E^*_{-\lambda-\rho}$.

It remains to show that the isomorphism is the push-forward along the canonical projection. To this end let $s \in \mathcal{R}(\lambda)$ be smooth and let $\pi : G/M \to G/K$ be the canonical projection. Then the isomorphism $\mathcal{R}(\lambda) \to (H^{w_0(\lambda+\rho)})^{-\infty}$ carries s to $\tilde{s} : G \to \mathbb{C}$ with $\tilde{s}(g) = s(gk_0)$, where $k_0 \in K$ is as in the proof of Proposition 3.10. It follows that

$$\mathcal{P}_{-w_0(\lambda+\rho)}\tilde{s}(gK) = \int_K \tilde{s}(gk)\, dk = \int_K s(gkk_0)\, dk = \int_K s(gk)\, dk$$

since K is unimodular. On the other hand, for $f \in C_c^\infty(G/K)$, we have

$$(\pi_* s)(f) = s(f \circ \pi) = \int_{G/M} s(gM) f(gK)\, dgM = \int_{G/K} \left(\int_{K/M} s(gkM)\, dkM \right) f(gK)\, dgK$$

if we normalize the Haar measure on M and choose compatible invariant measures on G/K and K/M. Hence $\pi_* s = \mathcal{P}_{-w_0(\lambda+\rho)} \tilde{s}$ for $s \in \mathcal{R}(\lambda) \cap C^\infty(G/M)$. Using the density of smooth compactly supported functions in $\mathcal{R}(\lambda)$ [Küster and Weich 2021, Corollary 2.9] we obtain the equality for the whole space $\mathcal{R}(\lambda)$. As before we now restrict to Γ-invariant distributions identified with distributions on $\Gamma\backslash G/M$ and $\Gamma\backslash G/K$ to complete the proof. $\square$

4. Quantum spectrum

In this section we analyze the quantum spectrum of the locally symmetric space $\Gamma\backslash G/K$. Recall the definition of the joint eigenspace

$$E_\lambda = \{ f \in C^\infty(G/K) \mid Df = \chi_\lambda(D) f \ \forall D \in \mathbb{D}(G/K) \}$$

for $\lambda \in \mathfrak{a}_\mathbb{C}^*$. For the definition of χ_λ see Section 2B. Since $D \in \mathbb{D}(G/K)$ is G-invariant, it descends to a differential operator ${}_\Gamma D$ on the locally symmetric space $\Gamma\backslash G/K$. Therefore, the left Γ-invariant functions of E_λ (denoted by ${}^\Gamma E_\lambda$) can be identified with joint eigenfunctions on $\Gamma\backslash G/K$ for each ${}_\Gamma D$:

$${}^\Gamma E_\lambda = \{ f \in C^\infty(\Gamma\backslash G/K) \mid {}_\Gamma D f = \chi_\lambda(D) f \ \forall D \in \mathbb{D}(G/K) \}.$$

This leads to the following definition.

Definition 4.1. The *quantum spectrum* of $\Gamma\backslash G/K$ is defined as

$$\sigma_Q := \sigma_Q(\Gamma\backslash G/K) := \{ \lambda \in \mathfrak{a}_\mathbb{C}^* \mid {}^\Gamma E_\lambda \neq 0 \}.$$

We now use the quantum-classical correspondence and the Weyl law from [Duistermaat et al. 1979].

Proof of Theorem 1.1. From [Duistermaat et al. 1979, Theorem 8.9] we have, for each set $\Omega \subset \mathfrak{a}^*$ as in Theorem 1.1,

$$\sum_{\lambda \in \sigma_Q \cap i\mathfrak{a}^*,\, \operatorname{Im}\lambda \in t\Omega} \dim({}^\Gamma E_\lambda) |W\lambda|^{-1} = \operatorname{Vol}(\Gamma\backslash G/K)(2\pi)^{-d} \operatorname{Vol}(\operatorname{Ad}(K)\Omega) t^d + \mathcal{O}(t^{d-1}),$$

where $\operatorname{Vol}(\Gamma\backslash G/K)$ is the volume of the compact Riemannian manifold $\Gamma\backslash G/K$ with Riemannian structure induced by the Killing form and $\operatorname{Vol}(\operatorname{Ad}(K)\Omega)$ is the volume of the set $\operatorname{Ad}(K)\Omega \subseteq \operatorname{Ad}(K)\mathfrak{a}$ with respect to the Killing form restricted to $\operatorname{Ad}(K)\mathfrak{a}$. Replacing Ω by $\Omega \setminus \bigcup_{\alpha\in\Sigma^+} \alpha^\perp$ we deduce that

$$\sum_{\lambda \in \sigma_Q \cap i\mathfrak{a}^*,\, \operatorname{Im}\lambda \in t\Omega \cap \bigcup \alpha^\perp} \dim({}^\Gamma E_\lambda) = \mathcal{O}(t^{d-1})$$

since $\operatorname{Vol}(\operatorname{Ad}(K)\alpha^\perp) = 0$. Therefore,

$$\sum_{\lambda \in \sigma_Q \cap i\mathfrak{a}^*,\, \operatorname{Im}\lambda \in t\Omega} \dim({}^\Gamma E_\lambda) = |W| \operatorname{Vol}(\Gamma\backslash G/K)(2\pi)^{-d} \operatorname{Vol}(\operatorname{Ad}(K)\Omega) t^d + \mathcal{O}(t^{d-1})$$

since W acts freely on the Weyl chambers. To complete the proof we observe that $\sigma_{\mathrm{RT}}({}_\Gamma\boldsymbol{X}) \supseteq \sigma^0_{\mathrm{RT}}({}_\Gamma\boldsymbol{X})$ and $m(\lambda) \geq \dim(\mathrm{Res}^0_{{}_\Gamma\boldsymbol{X}}(\lambda)) = \dim({}^\Gamma E_{-\lambda-\rho})$ for $\lambda \in i\mathfrak{a}^*$. □

As $\chi_\lambda = \chi_{w\lambda}$ for $w \in W$ it is obvious that σ_Q is W-invariant. The following properties of σ_Q were derived by Duistermaat, Kolk and Varadarajan [Duistermaat et al. 1979]. We include the proof for the convenience of the reader.

Proposition 4.2 (see [Duistermaat et al. 1979, Propositions 2.4 and 3.4, Corollary 3.5]). *If $\lambda \in \sigma_Q$, then the corresponding spherical function ϕ_λ is positive semidefinite. Moreover, there is some $w \in W$ such that $w\lambda = -\bar{\lambda}$ and $\mathrm{Re}\,\lambda \in \mathrm{conv}(W\rho)$. In particular, $\langle \mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda\rangle = 0$ and $\|\mathrm{Re}\,\lambda\| \leq \|\rho\|$.*

Proof. Pick $u \in {}^\Gamma E_\lambda$, regarded as a right K-invariant element of $L^2(\Gamma\backslash G)$, normalized such that $\langle u, u\rangle_{L^2(\Gamma\backslash G)} = 1$. With the right regular representation R on $L^2(\Gamma\backslash G)$, define $\Phi(g) := \langle R(g)u, u\rangle$. Being a matrix coefficient the function Φ is positive semidefinite. We will show that Φ is the elementary spherical function ϕ_λ. By right K-invariance of u and unitarity of R we get that Φ is K-biinvariant. $\Phi(1) = 1$ is obvious. Smoothness follows from the fact that u is smooth. Furthermore,

$$D\Phi(g) = \langle R(g)Du, u\rangle = \chi_\lambda(D)\Phi(g)$$

by left invariance of D. We conclude that Φ is the elementary spherical function for χ_λ, i.e., $\Phi = \phi_\lambda$.

Since ϕ_λ is positive semidefinite we have $\phi_\lambda(g) = \overline{\phi_\lambda(g^{-1})}$ by definition of positive definiteness, and $\overline{\phi_\lambda(g^{-1})} = \phi_{-\bar{\lambda}}(g)$ by the integral representation (see Section 2G). Therefore, $\phi_\lambda = \phi_{-\bar{\lambda}}$, implying that $w\lambda = -\bar{\lambda}$ for some $w \in W$. It easily follows that

$$\langle \mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda\rangle = \langle w\,\mathrm{Re}\,\lambda, w\,\mathrm{Im}\,\lambda\rangle = \langle -\mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda\rangle = 0.$$

Moreover, ϕ_λ is bounded which holds if and only if $\mathrm{Re}\,\lambda \in \mathrm{conv}(W\rho)$; see [Helgason 1984, Chapter IV Theorem 8.1]. Since $\{\mu \in \mathfrak{a}^* \mid \|\mu\| \leq \|\rho\|\}$ is convex and contains $W\rho$, the last assertion follows. □

Remark 4.3. In the rank 1 case Proposition 4.2 implies, for $\lambda \in \sigma_Q$, that $\lambda \in \mathfrak{a}^*$ with $\|\lambda\| \leq \|\rho\|$ or that $\lambda \in i\mathfrak{a}^*$. In this particular case, this can be obtained not only from Proposition 4.2 but also from the positivity of the Laplacian on $\Gamma\backslash G/K$. In the higher rank setting the algebra $\mathbb{D}(G/K)$ contains more operators; more precisely it is a polynomial algebra in n variables. Using the properties of the Harish-Chandra isomorphism HC one can obtain that $-\bar{\lambda} \in W\lambda$ from the self/skew-adjointness of the operators in $\mathbb{D}(G/K)$.

Remark 4.4. Proposition 4.2 implies the following obstructions for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ to be in σ_Q.

(1) If $\mathrm{Re}\,\lambda = 0$, then we get no obstructions on $\mathrm{Im}\,\lambda$ since $w\lambda = -\bar{\lambda}$ is satisfied with $w = 1$.

(2) If $\mathrm{Re}\,\lambda \neq 0$, then $\mathrm{Im}\,\lambda$ is singular, i.e., $\mathrm{Im}\,\lambda \in \alpha^\perp$ for some $\alpha \in \Sigma$, since $\mathrm{Im}\,\lambda$ nonsingular implies $w = 1$ as W acts simply transitively on open Weyl chambers.

(3) If $\mathrm{Re}\,\lambda$ is regular, i.e., $\langle \mathrm{Re}\,\lambda, \alpha\rangle \neq 0$ for all $\alpha \in \Sigma$, we denote by $\tilde{w}_0$ the unique Weyl group element mapping the Weyl chamber containing $\mathrm{Re}\,\lambda$ to its negative. Then $\lambda \in \mathrm{Eig}_{-1}(\tilde{w}_0) + i\,\mathrm{Eig}_{+1}(\tilde{w}_0) \subseteq \mathfrak{a}^*_{\mathbb{C}}$, where $\mathrm{Eig}_{\pm 1}$ denotes the eigenspace for ± 1. If -1 is contained in W, then $\mathrm{Im}\,\lambda = 0$. In particular, this is true in the rank 1 case but need not hold in general as is seen below.

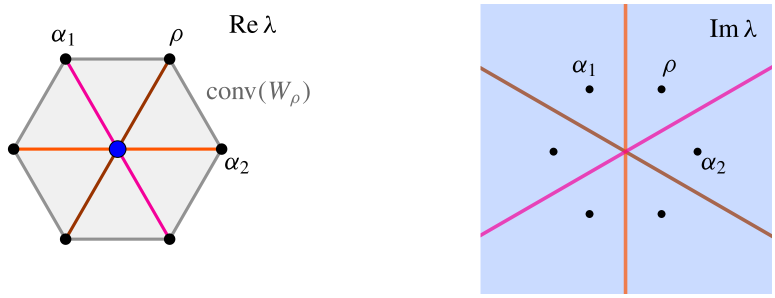

Figure 4. The situation for $\mathrm{SL}_3(\mathbb{R})$ as obtained from Remark 4.4: if $\lambda \in \sigma_Q$ then $\operatorname{Re}\lambda$ is either equal to zero (blue dot in the left picture) or lies on one of the pink, orange or brown lines depicted on the left. Furthermore, $\operatorname{Im}\lambda$ has to lie in the respective region depicted on the right, i.e., if $\operatorname{Re}\lambda = 0$, then $\operatorname{Im}\lambda$ can take any value (blue shaded plane); if $\operatorname{Re}\lambda$ lies on the orange line, then $\operatorname{Im}\lambda$ has to lie on the orange line; and so on.

Let us calculate $\dim \operatorname{Eig}_{+1}(w_0) = \dim \operatorname{Eig}_{+1}(\tilde{w}_0)$ in order to control the amount of freedom for $\operatorname{Im}\lambda$. Let $d_\pm := \dim \operatorname{Eig}_{\pm 1}(w_0)$. Then $n = d_+ + d_-$ and $\operatorname{Tr}(w_0) = d_+ - d_-$. Choosing the basis Π we observe $\operatorname{Tr}(w_0) = -\#\{\alpha \in \Pi \mid w_0\alpha = -\alpha\} \leq 0$. Thus, $d_\pm = \frac{1}{2}(n \pm \operatorname{Tr}(w_0))$, so that $d_+ \leq \frac{1}{2}n$. We obtain the following traces and dimensions for the irreducible root systems from the classification:

type	A_n, n even	A_n, n odd	B_n	C_n	D_n, n even	D_n, n odd	E_6	E_7	E_8	F_4	G_2
$-\operatorname{Tr}(w_0)$	0	1	n	n	n	$n-2$	2	7	8	4	2
d_+	$\frac{1}{2}n$	$\frac{1}{2}(n-1)$	0	0	0	1	2	0	0	0	0

Example 4.5. For $G = \mathrm{SL}_n(\mathbb{R})$, an element $\lambda \in \mathfrak{a}^* \simeq \mathfrak{a}$ is regular if and only if the diagonal entries are pairwise distinct. An element $\lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \in \sigma_Q$ with $\operatorname{Re}\lambda \in \overline{\mathfrak{a}_+^*}$ satisfies $\operatorname{Re}\lambda_k = -\operatorname{Re}\lambda_{n+1-k}$ and $\operatorname{Im}\lambda_k = \operatorname{Im}\lambda_{n+1-k}$ for all $k = 1, \dots, n$ since the longest Weyl group element is the permutation $(1 \leftrightarrow n)(2 \leftrightarrow n-1)\cdots$.

More specifically, for $G = \mathrm{SL}_3(\mathbb{R})$, the only Weyl group elements with eigenvalue equal to -1 are the reflections at hyperplanes perpendicular to the roots. Hence $\lambda \in \sigma_Q$ implies $\operatorname{Re}\lambda \in [-1, 1]\alpha$ and $\operatorname{Im}\lambda \in \alpha^\perp$ for some $\alpha \in \Sigma$ or $\lambda \in i\mathfrak{a}^*$. The obstructions for λ to be in σ_Q described by Remark 4.4 are less concrete and are visualized in Figure 4.

Let us formulate the condition that ϕ_λ is positive semidefinite in a different way.

Proposition 4.6. *The elementary spherical function ϕ_λ is positive semidefinite if and only if the subrepresentation generated by the K-invariant vector in the principal series representation $H^{w\lambda}$ is unitarizable and irreducible for some $w \in W$. Equivalently, $H^{-w\bar{\lambda}}$ has a unitarizable irreducible spherical quotient.*

Proof. By Casselman's embedding theorem, π_{ϕ_λ} is a subrepresentation of $H^{\tau,\nu}$ for some $\tau \in \widehat{M}$ and $\nu \in \mathfrak{a}_\mathbb{C}^*$; see, e.g., [Knapp 1986, Theorem 8.37]. More precisely, the $(\mathfrak{g}, K)$-module of K-finite vectors are equivalent. Since the only principal series representations containing K-invariant vectors are the spherical ones, we obtain $\tau = 1$. Since infinitesimally equivalent admissible representations of G have the same set of K-finite matrix coefficients (see [Knapp 1986, Corollary 8.8]), we conclude $\phi_\lambda = \phi_\nu$, i.e., $w\lambda = \nu$.

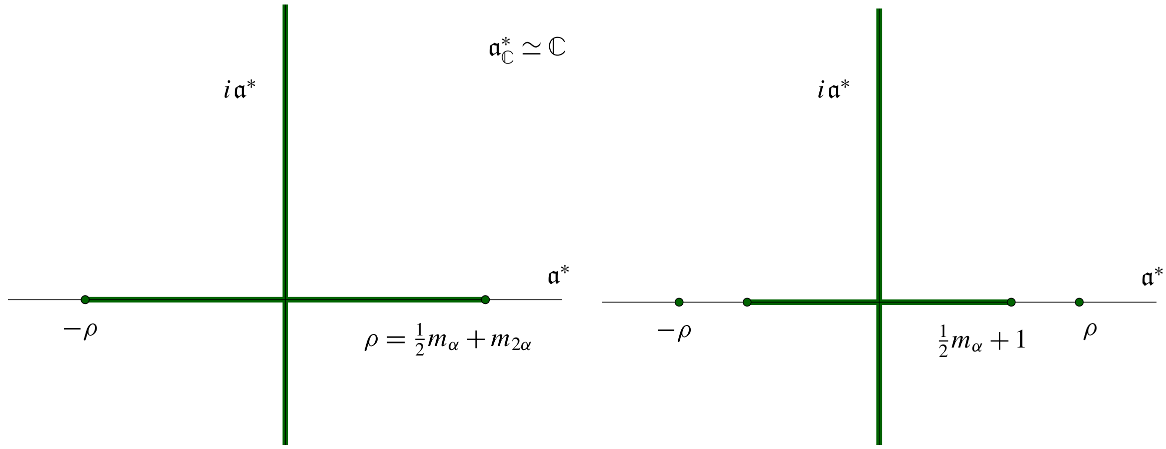

Figure 5. Spherical dual in the rank 1 case. The picture on the left describes the real and complex hyperbolic case $m_{2\alpha} \leq 1$. The picture on the right describes the quaternionic case $m_{2\alpha} \geq 2$. In the latter case note that there is a spectral gap separating ρ.

Conversely assume that the subrepresentation generated by the K-invariant vector in the principal series representation $H^{w\lambda}$ is unitarizable and irreducible. Again by the aforementioned result the matrix coefficient $\phi_{w\lambda} = \phi_\lambda$ of $H^{w\lambda}$ is a matrix coefficient of the unitary representation obtained by the unitary structure as well. Hence ϕ_λ is positive semidefinite. Transition to the dual representation implies the second equivalence. $\square$

Remark 4.7. Although the unitary dual is classified for many groups, it is difficult to deduce which elementary spherical functions are positive semidefinite. This is due to the fact that most classifications are not obtained in terms of quotients of the spherical principal series but use different descriptions of admissible representations. However, for rank 1 groups everything is classified (see [Helgason 1984, p. 484]): if α denotes the unique reduced root in Σ^+, then ϕ_λ is positive semidefinite if and only if $\lambda \in i\mathfrak{a}^*$ or $\lambda \in \mathfrak{a}^*$ and $|\langle \lambda, \alpha\rangle| \leq \langle \rho, \alpha\rangle$ for $2\alpha \notin \Sigma$ (i.e., in the real hyperbolic case) and $|\langle \lambda, \alpha\rangle| \leq \left(\frac{1}{2}m_\alpha + 1\right)\langle \alpha, \alpha\rangle$ for $2\alpha \in \Sigma$ or $\lambda = \pm\rho$. See Figure 5 for a visualization.

4A. ***Property (T).*** In this section we review some facts about Kazhdan's property (T) which will lead to a more precise description of the location of σ_Q. Recall that a locally compact group has *property* (T) if and only if the trivial representation is an isolated point in the unitary dual of the group with respect to the Fell topology; see [Bekka et al. 2008] for a general reference. It is well known that each real simple Lie group of real rank ≥ 2 has property (T); see [Bekka et al. 2008, Theorem 1.6.1]. Since the mapping $\lambda \mapsto \phi_\lambda$ is continuous and the correspondence between positive semidefinite elementary spherical functions and irreducible unitary spherical representations is a homeomorphism (see Section 2G), we obtain that in some neighborhood of ρ no elementary spherical function is positive semidefinite. We will use a more quantitative description introduced by Oh [2002, Section 7.1]. Therefore, we denote by $p_K(G)$ the smallest real number such that the K-finite matrix coefficients of π are in $L^q(G)$ for any $q > p_K(G)$ and nontrivial $\pi \in \widehat{G}$.

Remark 4.8. (1) Since each matrix coefficient of $\pi \in \widehat{G}$ is bounded, it is contained in L^q for each $q > p$ if it is in L^p. Hence

$$p_K(G) = \inf\{p \mid \text{all } K\text{-finite matrix coefficients of } \pi \text{ are in } L^p(G)\ \forall \pi \in \widehat{G} \setminus \{1\}\}.$$

(2) $p_K(G) \geq 2$.

(3) By [Cowling 1979] together with [Oh 2002] we have $p_K(G) < \infty$ if and only if G has property (T).

In many examples one knows the number $p_K(G)$ explicitly or at least its upper bounds.

Example 4.9 (see [Oh 2002, Section 7]). (1) $p_K(\mathrm{SL}_n(k)) = 2(n-1)$ for $n \geq 3$ and $k = \mathbb{R}, \mathbb{C}$.

(2) $p_K(Sp_{2n}(\mathbb{R})) = 2n$ for $n \geq 2$.

(3) $p_K(G)$ is bounded above by an explicit value for split classical groups of higher rank.

We can now prove the following theorems.

Theorem 4.10. *Let G be a noncompact real semisimple Lie group with finite center and $\Gamma \leq G$ be a discrete, cocompact, torsion-free subgroup. Then*

$$\operatorname{Re} \sigma_Q(\Gamma\backslash G/K) \subseteq (1 - 2p_K(G)^{-1}) \operatorname{conv}(W\rho) \cup W\rho.$$

Proof. Let $\lambda \in \sigma_Q(\Gamma\backslash G/K)$. By Proposition 4.2, ϕ_λ is positive semidefinite so that the irreducible unitary representation π_{ϕ_λ} is defined (see Section 2G), and ϕ_λ is a matrix coefficient of this representation. By the definition of $p_K(G)$ we have $\phi_\lambda \in L^{p_K(G)+\epsilon}(G)$ for all $\epsilon > 0$ or π_{ϕ_λ} is the trivial representation. By Proposition 2.4 we get $\operatorname{Re} \lambda \in (1 - 2p_K(G)^{-1}) \operatorname{conv}(W\rho)$ in the first case. The latter case occurs if and only if $\phi_\lambda \equiv 1$, i.e., $\lambda \in W\rho$. □

Theorem 4.11. *Let G be a noncompact real semisimple Lie group with finite center and $\Gamma \leq G$ be a discrete, cocompact, torsion-free subgroup. Then there is a neighborhood $\mathcal{G}$ of ρ in $\mathfrak{a}^*$ such that*

$$\sigma_Q(\Gamma\backslash G/K) \cap (\mathcal{G} \times i\mathfrak{a}^*) = \{\rho\}.$$

Proof. Without loss of generality we assume that G has trivial center, otherwise replace G by $G/Z(G)$. Then G is a product of simple Lie groups $G_1, \dots, G_l$ such that $G_1, \dots, G_k$, $k \leq l$, are of rank 1. With the obvious notation let $\lambda = (\lambda_1, \dots, \lambda_l) \in (\mathfrak{a}_1)^*_{\mathbb{C}} \oplus \cdots \oplus (\mathfrak{a}_l)^*_{\mathbb{C}}$ be in σ_Q. By Proposition 4.2 we have $w\lambda = -\bar{\lambda}$ for some $w \in W$. Since the Weyl group W is the product of the Weyl groups, $\lambda_i \in \mathfrak{a}_i^*$ are real for $i \leq k$ if $\operatorname{Re} \lambda_i \neq 0$. The elementary spherical function ϕ_λ is the product of elementary spherical functions $\phi^{G_i}_{\lambda_i}$ for the factors G_i. Again by Proposition 4.2 we know that ϕ_λ is positive semidefinite and therefore each $\phi^{G_i}_{\lambda_i}$ is positive semidefinite. The same line of arguments as in the proof of Theorem 4.10 implies that $\operatorname{Re} \lambda_i \in (1-2p_K(G_i)^{-1}) \operatorname{conv}(W_i\rho_i) \cup W_i\rho_i$ for $i > k$. Since the G_i, $i > k$, have property (T), we conclude that there is a neighborhood U of ρ in $\mathfrak{a}^*$ such that

$$\sigma_Q \cap (U \times i\mathfrak{a}^*) \subseteq \mathfrak{a}_1^* \times \cdots \times \mathfrak{a}_k^* \times \{\rho_{k+1}\} \times \cdots \times \{\rho_l\}.$$

Discreteness of σ_Q implies the theorem. □

5. Main Theorem

In this section we present the main theorem of the article and deduce Theorem 1.2 from it. See Figure 6 for a visualization for $G = \mathrm{SL}_3(\mathbb{R})$.

Theorem 5.1. *Let G be a noncompact real semisimple Lie group with finite center and $\Gamma \leq G$ be a discrete, cocompact, torsion-free subgroup. Define*

$$\mathcal{A} := \left\{ \lambda \in \mathfrak{a}_{\mathbb{C}}^* \;\middle|\; \frac{2\langle \lambda + \rho, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in -\mathbb{N}_{>0} \text{ for some } \alpha \in \Sigma^+ \right\},$$
$$\mathcal{B} := \{ \lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid w\lambda = -\bar{\lambda} \text{ for some } w \in W \},$$
$$\mathcal{F} := \{ \lambda \in \mathfrak{a}^* \mid \lambda + \alpha \notin \overline{{}_{-}\mathfrak{a}^*} \text{ for all } \alpha \in \Pi \}.$$

Then we have the inclusions

$$\sigma_{\mathrm{RT}}({}_{\Gamma}\boldsymbol{X}) \cap (\mathcal{F} \times i\mathfrak{a}^*) \subseteq \sigma_{\mathrm{RT}}^0({}_{\Gamma}\boldsymbol{X})$$

and

$$\sigma_{\mathrm{RT}}^0({}_{\Gamma}\boldsymbol{X}) \cap (\mathfrak{a}_{\mathbb{C}}^* \setminus \mathcal{A}) \subseteq -\sigma_Q(\Gamma \backslash G/K) - \rho \subseteq \mathcal{B} \cap (((1 - 2p_K(G)^{-1}) \operatorname{conv}(W\rho) \cup W\rho) + i\mathfrak{a}^*) - \rho.$$

Proof. This is immediate from Propositions 3.7, 3.11 and 4.2, and Theorem 4.10. □

Proof of Theorem 1.2. It follows from Theorem 5.1 that $(\mathfrak{a}_+^* - \rho) \cap \mathcal{F} \cap (-\mathcal{G} - \rho)$ can be chosen as the neighborhood, where $\mathcal{G}$ is obtained by Theorem 4.11. If G has property (T), then $p_K(G)$ is finite and $\mathcal{G}$ can be replaced by the complement of the Γ-independent set $(1 - 2p_K(G)^{-1}) \operatorname{conv}(W\rho)$. □

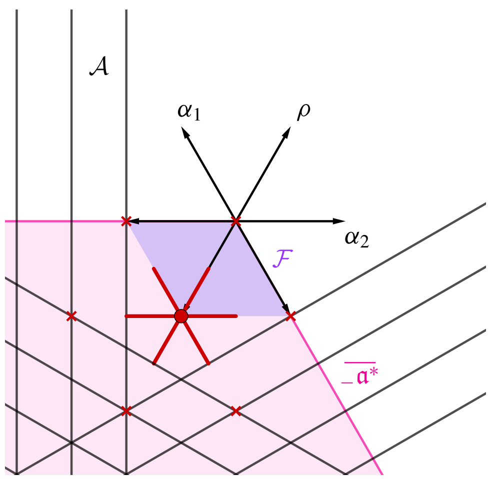

Figure 6. Visualization of the real part of $\mathfrak{a}_{\mathbb{C}}^*$ for $G = \mathrm{SL}_3(\mathbb{R})$: The pink region is where Ruelle–Taylor resonances can a priori be located in view of the results of [Bonthonneau et al. 2020]. The red points and lines depict the region where first band resonances can occur: $\left(\mathcal{B} \cap \frac{1}{2} \operatorname{conv}(W\rho) \cup W\rho\right) - \rho$. The purple shaded region illustrates the real parts in which only first band resonances can occur. Further first band resonances might occur inside the exceptional set $\mathcal{A}$ depicted by the black lines.

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