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PERTURBED INTERPOLATION FORMULAE AND APPLICATIONS

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We employ functional analysis techniques in order to deduce some versions of classical and recent interpolation results in Fourier analysis with perturbed nodes. As an application of our techniques, we obtain generalizations of Kadec's $\frac{1}{4}$-theorem for interpolation formulae in the Paley–Wiener space both in the real and complex cases, as well as versions of the recent interpolation result of Radchenko and Viazovska (*Publ. Math. Inst. Hautes Études Sci.* **129** (2019), 51–81) and the result of Cohn, Kumar, Miller, Radchenko and Viazovska (*Ann. Math* (2) **196**:3 (2022), 983–1082) for Fourier interpolation with derivatives in dimensions 8 and 24 with suitable perturbations of the interpolation nodes. We also provide several applications of the main results and techniques, relating to recent contributions in interpolation formulae and uniqueness sets for the Fourier transform.

1. Introduction

A fundamental question in analysis is that of how to recover a function f from some subset $\{f(x)\}_{x\in A}$ of its values, together with some information on its *Fourier transform* $\hat{f}:\mathbb{R}\to\mathbb{C}$, which we define to be

$$\hat{f}(\xi)=\int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\,\mathrm{d}x. \tag{1-1}$$

Perhaps the most classical result in that regard is the *Shannon–Whittaker interpolation formula*: if $\hat{f}$ is supported on an interval $[-\delta/2,\delta/2]$, then

$$f(x)=\sum_{k=-\infty}^{\infty} f(k/\delta)\,\mathrm{sinc}(\delta x-k), \tag{1-2}$$

where convergence holds both in $L^2(\mathbb{R})$ and uniformly in compact sets of $\mathbb{C}$, where we let $\mathrm{sinc}(x)=\sin(\pi x)/(\pi x)$. A major recent breakthrough in regard to the problem of determining which conditions

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on the sets $A, B \subset \mathbb{R}$ imply that a function $f \in \mathcal{S}(\mathbb{R})$ is uniquely determined by its values at A and the values of its Fourier transform at B was made in [Radchenko and Viazovska 2019], where the authors proved that, if $f : \mathbb{R} \to \mathbb{R}$ is even and Schwartz, then

$$f(x) = \sum_{k=0}^{\infty} f(\sqrt{k}) a_k(x) + \sum_{k=0}^{\infty} \hat{f}(\sqrt{k}) \hat{a}_k(x). \tag{1-3}$$

Radchenko and Viazovska's result and its techniques were somewhat inspired by Viazovska's recent solution [2017] to the sphere-packing problem in dimension 8, and her subsequent work with Cohn, Kumar, Miller and Radchenko [Cohn et al. 2017] to solve the same problem in dimension 24. Indeed, the proof of (1-3) uses such tools from the theory of *modular forms* heavily for constructing and bounding the basis functions $\{a_n\}_{n \geq 0}$.

Subsequent to the Radchenko–Viazovska result, other recent works have successfully used a similar approach in order to tackle what are now known as Fourier interpolation and Fourier uniqueness problems. Among those, we mention the following:

(1) Cohn and Gonçalves [2019] used a modular form construction in order to obtain that there are $c_j > 0$, $j \in \mathbb{N}$, so that, for each $f \in \mathcal{S}_{\text{rad}}(\mathbb{R}^{12})$ real,

$$f(0) - \sum_{j \geq 1} c_j f(\sqrt{2j}) = -\hat{f}(0) + \sum_{j \geq 1} c_j \hat{f}(\sqrt{2j}). \tag{1-4}$$

Such a formula enables the authors to prove a sharp version of a root uncertainty principle first raised by Bourgain, Clozel and Kahane [Bourgain et al. 2010] in dimension 12; see, e.g., [Gonçalves et al. 2017; 2021; 2023] for more information on this topic.

(2) On the other hand, Cohn, Kumar, Miller, Radchenko and Viazovska [Cohn et al. 2022] built upon the basic ideas of [Radchenko and Viazovska 2019] to be able to prove *universal optimality results* about the E_8 and Leech lattices in dimensions 8 and 24, respectively. In order to do so, they prove interpolation formulae in such dimensions that involve the values of $f(\sqrt{2n})$, $f'(\sqrt{2n})$, $\hat{f}(\sqrt{2n})$, $\hat{f}'(\sqrt{2n})$, where f is a radial, Schwartz function, and $n \geq n_0$, with $n_0 = 1$ if $d = 8$, and $n_0 = 2$ in case $d = 24$.

(3) Talebizadeh Sardari [2021] studied the problem of constructing interpolation formulae involving the values $f(\sqrt{r})$, $f'(\sqrt{r})$, $\hat{f}(\sqrt{r})$, $\hat{f}'(\sqrt{r})$, where f is a radial, Schwartz function, in $\mathbb{R}^2$, and r is any point in the set

$$\left\{ \left(\tfrac{4}{3}\right)^{1/4} \sqrt{n^2 + nm + m^2} : n, m \in \mathbb{Z} \right\},$$

which would correspond to a Fourier interpolation formula with derivatives over the hexagonal lattice. Such a formula was conjecture not to exist in [Cohn et al. 2022, Conjecture 7.5], and indeed that is the case: there are infinitely many linearly independent Schwartz functions that cannot be recovered by these values. This is perhaps surprising, since the hexagonal lattice is conjectured to be universally optimal in the language of [Cohn et al. 2022], which suggests this problem is not amenable to the exact same techniques in that work in dimensions 8 and 24.

(4) Finally, more recently, other developments in the theory of interpolation formulae given values on both Fourier and spatial sides have been made by Stoller [2021], who considered the problem of

recovering any function in $\mathbb{R}^d$ from its restrictions and the restrictions of its Fourier transforms to spheres of radii $\sqrt{n}$, where $n > 0$, is an integer, and for any $d > 0$. Moreover, we mention also the more recent work of Bondarenko, Radchenko and Seip [Bondarenko et al. 2023], which generalizes Radchenko and Viazovska's construction of the interpolating functions to prove interpolation formulae for some classes of functions f that take into account the values of $\hat{f}$ at $\log n/(4\pi)$, and the values of f at a sequence $\left(\rho - \frac{1}{2}\right)/i$, where ρ ranges over nontrivial zeros of some L-function with positive imaginary part.

One fundamental point to stress is that, in a suitable way, all the previously mentioned results relate some sort of *summation formula*, the most basic instance of such being the classical Poisson summation formula

$$\sum_{m\in\mathbb{Z}} f(m) = \sum_{n\in\mathbb{Z}} \hat{f}(n),$$

which is obtained in [Radchenko and Viazovska 2019] as a particular case of (1-3) by setting $x = 0$, with some *modular form* construction. In this direction, the formula (1-4) is also a manifestation of such a principle that implies rigidity between certain values of f and other values of $\hat{f}$.

The aforementioned connection between summation formulae and modular forms is classical, with the modularity of the Jacobi theta series θ being a primal example of how one relates to the other. On the other hand, this connection may be deepened through the following argument: Suppose that a summation formula of the kind

$$\sum_{a\in A} c_a f(a) = \sum_{a\in A} c_a \hat{f}(a) \tag{1-5}$$

holds for all $f \in \mathcal{S}(\mathbb{R})$ a radial function. This is seen to be equivalent, by a density argument (see, for instance, [Radchenko and Viazovska 2019, Section 6]), to (1-5) holding for $f(x) = e^{iz\pi|x|^2}$, where $z \in \mathbb{C}$ is fixed so that $\operatorname{Im}(z) > 0$. This, on the other hand, is equivalent to the function $M(z) = \sum_{a\in A} e^{i\pi z|a|^2}$ satisfying the modular relationship $(-iz)^{-d/2}M(-1/z) = M(z)$ in the upper half-space. In particular, if $A \subset \sqrt{\mathbb{Z}_+}$, then M satisfies additionally some periodicity condition, and thus a search for M can be further narrowed to a certain space of modular forms.

From a similar yet not identical point of view, however, the topics described above can also be inserted into the framework of *crystalline measures*. Indeed, if we adopt the classical definition of a crystalline measure to be a distribution with locally finite support, such that its Fourier transform possesses the same support property, we will see that the Poisson summation formula implies, for instance, that the measure $\delta_{\mathbb{Z}}$ is not only a crystalline measure, but also *self-dual*, in the sense that $\delta_{\mathbb{Z}} = \hat{\delta}_{\mathbb{Z}}$ holds in $\mathcal{S}'(\mathbb{R})$.

Outside the scope of interpolation formulae per se, we mention the works [Lev and Olevskii 2013; 2015; Meyer 2017], where the authors explore on a deeper lever structural questions on crystalline measures. In particular, Meyer [2017] exhibits examples of crystalline measures with self-duality properties, and uses modular forms to construct explicitly examples of nonzero self-dual crystalline measures μ supported on $\{\pm\sqrt{k+a} : k \in \mathbb{Z}_+\}$ for $a \in \{9, 24, 72\}$. We also mention [Kurasov and Sarnak 2020], where the authors, as a by-product of investigations of the additive structure of the spectrum of metric graphs, prove that there are exotic examples of *positive* crystalline measures other than generalized Dirac combs.

Our investigation in this paper focuses on both classical and modern results in the theory of such interpolation formulae and crystalline measures. In generic terms, we are interested in determining when, given an interpolation formula such as (1-2) or (1-3), we can *perturb* it suitably. That is, given a sequence of real numbers $\{\varepsilon_k\}_{k\in\mathbb{Z}}$, under which conditions can we recover f from the values

$$\{(f(s_n+\varepsilon_n),\ \hat{f}(\hat{s}_n+\varepsilon_n))\}_{n\in\mathbb{Z}}, \tag{1-6}$$

given that we can recover f from $\{(f(s_n),\ \hat{f}(\hat{s}_n))\}_{n\in\mathbb{Z}}$?

In this manuscript, the main idea is to study such perturbations of interpolation formulae for band-limited and Schwartz functions through functional analysis. Indeed, most of our considerations are based on the idea that, whenever an operator $T : B \to B$, where B is a Banach space, satisfies

$$\|T - I\|_{B\to B} < 1,$$

then T is, in fact, a *bijection* with continuous inverse $T^{-1} : B \to B$. In fact, in all our considerations on interpolation formulae below, some form of this principle will be employed, and other proofs and results in the paper, such as Theorem 1.6, which gives new bounds related to the Radchenko–Viazovska formula, arise naturally when trying to employ this principle in different contexts.

1A. ***Perturbations and interpolation formulae in the band-limited case.*** The question of when we are able to recover the values of a function such that its Fourier transform is supported in $\left[-\frac{1}{2},\frac{1}{2}\right]$ from its values at $n+\varepsilon_n$ is well known, having been asked in [Paley and Wiener 1934], where the authors proved that recovery — and also an associated interpolation formula — is possible as long as $\sup_n |\varepsilon_n| < \pi^{-2}$. Many results relate to the original problem of Paley and Wiener, but the most celebrated of them all is the so-called Kadec-$\frac{1}{4}$ theorem, which states that, as long as $\sup_n |\varepsilon_n| < \frac{1}{4}$, one can recover any $f \in L^2(\mathbb{R})$ which has Fourier support on $\left[-\frac{1}{2},\frac{1}{2}\right]$ from its values at $n+\varepsilon_n,\ n\in\mathbb{Z}$; see [Kadec 1964] for the original proof and [Avantaggiati et al. 2016] for a generalization.

Our first results provide one with a simpler proof of a particular range of Kadec's result. We recall, for that matter, that the Paley–Wiener space $\mathrm{PW}_\pi(\mathbb{R})$ is defined as the aforementioned space of all square-integrable functions on the real line such that $\hat{f}$ has support in the interval $\left[-\frac{1}{2},\frac{1}{2}\right]$.

Theorem 1.1. *Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of real numbers and assume $L = \sup_k |\varepsilon_k| < L_0$, where $L_0 = 0.239\ldots$ is defined to be the smallest positive solution to the equation*

$$\frac{\sin(\pi L_0)}{\pi L_0} = \frac{\pi}{3}\frac{L_0 \sin \pi L_0}{1-L_0} + \sin(\pi L_0).$$

Then any function $f \in \mathrm{PW}_\pi$ is completely determined by its values $\{f(n+\varepsilon_n)\}_{n\in\mathbb{Z}}$, and there is $C = C(L) > 0$ such that

$$\frac{1}{C}\sum_{n\in\mathbb{Z}} |f(n+\varepsilon_n)|^2 \le \|f\|_2^2 \le C\sum_{n\in\mathbb{Z}} |f(n+\varepsilon_n)|^2$$

for all $f \in \mathrm{PW}_\pi$.

Moreover, there are functions $g_n \in \mathrm{PW}_\pi(\mathbb{R})$ such that for every $f \in \mathrm{PW}_\pi$, the following identity holds:

$$f(x) = \sum_{n\in\mathbb{Z}} f(n+\varepsilon_n) g_n(x),$$

where the right-hand side converges absolutely in compact sets of $\mathbb{C}$.

The condition in Theorem 1.1 is satisfied for $L < 0.239$, which possesses only a 0.011 difference from Kadec's result. The main difference, however, is that while Kadec's proof relies on a clever expansion of the underlying functions in a different orthonormal basis, we make a less direct use of orthogonality in our considerations.

We also remark that, in the proof of Theorem 1.1, one can use complex numbers for perturbations. The difference is that we have to take into account the sine of complex numbers, and the resulting bound would be $L < 0.2125$ instead of $L < 0.239$. This only falls very mildly short of the results in [Avantaggiati et al. 2016, Theorem 3], where $L < 0.218$ is achieved in the complex setting, and our methods of proof are relatively simpler in comparison to those of that work, where the authors must enter the realm of Lamb–Oseen functions and constants.

As another application of the idea of inverting an operator, we present a couple of results related to Vaaler's interpolation formula. J. Vaaler [1985] proved, as means to study extremal problems in Fourier analysis, the following counterpart to the Shannon–Whittaker interpolation formula: Let $f \in L^2(\mathbb{R})$, and suppose that $\hat{f}$ is supported on $[-1, 1]$. Then

$$f(x) = \frac{\sin^2(\pi x)}{\pi^2} \sum_{k\in\mathbb{Z}} \left\{ \frac{f(k)}{(x-k)^2} + \frac{f'(k)}{x-k} \right\}. \tag{1-7}$$

This can be seen as a natural tradeoff: (1-2) demands that we have information at $\frac{1}{2}\mathbb{Z}$ in order to recover the functions f as stated above. On the other hand, Vaaler's result only demands information at $\mathbb{Z}$, but one must pay the price of replacing the rest of the information by values of the derivative at $\mathbb{Z}$.

The first result concerning (1-7) is a *direct* deduction of its validity from the Shannon–Whittaker formula (1-2). We state it in the following form.

Theorem 1.2 [Vaaler 1985]. *Fix a sequence $\{a_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Consider the function $f \in \mathrm{PW}_\pi$ given by*

$$f(x) = \sum_{n\in\mathbb{Z}} a_n \operatorname{sinc}(x-n)$$

for each $x \in \mathbb{R}$. Then the interpolation formula

$$f(x) = \frac{4\sin^2\left(\frac{\pi}{2}x\right)}{\pi^2} \sum_{k\in\mathbb{Z}} \left\{ \frac{a_{2k}}{(x-2k)^2} + \frac{b_{2k}}{x-2k} \right\} \tag{1-8}$$

holds, where the right-hand side converges uniformly on compact sets, and we let

$$b_k = \sum_{j\neq k} \frac{a_j}{k-j}(-1)^{k-j}.$$

It is a consequence of (1-8) that $f'(2k) = b_{2k}$ in Theorem 1.2 above. Moreover, we note that one readily obtains Vaaler's formula from (1-8) above: indeed, in order to obtain (1-7) for a square-integrable function $g \in L^2(\mathbb{R})$ with $\operatorname{supp}(\hat{g}) \subset [-1, 1]$, consider $f(x) = g\left(\frac{1}{2}x\right)$. It follows that f satisfies the hypotheses of Theorem 1.2, and substituting back allows one to conclude (1-7) from (1-8).

A main difference between our proof of Theorem 1.2 and the original proof in [Vaaler 1985] is the absence of any significant use of the Fourier transform. Differently, however, from the de Branges spaces approach in [Gonçalves 2017], we do not delve deeply into any theory of function spaces, but rather we make use of classical operators in $\ell^2(\mathbb{Z})$ such as discrete Hilbert transforms and its properties. We believe our approach might lead to derivations of other interesting interpolation formulae.

Our final contribution in the realm of interpolation formulae for band-limited function is a generalized version of Vaaler's formula (1-7) with *perturbed nodes*. We mention that, to the best of our knowledge, this result in its present form is new, as Vaaler's ideas are rigid to specific properties of integers and Fourier transforms of special functions such as $\operatorname{sinc}(x)^2$.

Theorem 1.3. *Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of real numbers and consider $L = \sup_k |\varepsilon_k|$. Suppose that $L < 0.111$. Then any function $f \in \mathrm{PW}_{2\pi}$ is completely determined by its values $\{f(n+\varepsilon_n)\}_{n\in\mathbb{Z}}$ and those of its derivative $\{f'(n+\varepsilon_n)\}_{n\in\mathbb{Z}}$, and there is $C = C(L) > 0$ such that*

$$\frac{1}{C}\sum_{n\in\mathbb{Z}}(|f(n+\varepsilon_n)|^2+|f'(n+\varepsilon_n)|^2) \le \|f\|_2^2 \le C\sum_{n\in\mathbb{Z}}|(|f(n+\varepsilon_n)|^2+|f'(n+\varepsilon_n)|^2) \tag{1-9}$$

for all $f \in \mathrm{PW}_{2\pi}$.

Moreover, there are functions $g_n, h_n \in \mathrm{PW}_{2\pi}$ so that, for all $f \in \mathrm{PW}_{2\pi}$, we have

$$f(x) = \sum_{n\in\mathbb{Z}}\{f(n+\varepsilon_n)g_n(x) + f'(n+\varepsilon_n)h_n(x)\},$$

where convergence holds absolutely.

This result and its method of proof resemble the ideas from Theorem 1.1 and its proof, with an increase in technical difficulties, such as considering higher-order analogues of the perturbed discrete Hilbert transforms we use for the proof of Theorem 1.1. We note also that some further technical changes, together with [Littmann 2006], allow one to extend the perturbation results for arbitrarily many derivatives; see Theorem 6.1 for a discussion on that.

We point the reader, for instance, to the remark following Corollary 2 in [Gonçalves 2017] together with [Lyubarskii and Seip 2002; Ortega-Cerdà and Seip 2002] for related discussion on sampling sequences with derivatives for PW_π; see also [Gonçalves and Littmann 2018] for discussions involving higher-order derivatives.

1B. ***Perturbations of symmetric interpolation formulae.*** Moving on from band-limited functions to Schwartz functions instead, we notice that the Radchenko–Viazovska result (1-3), although being a major breakthrough, is rigid in its statement: the interpolating functions are carefully tailored to interpolate at the $\{\sqrt{n}\}_{n\ge 0}$ nodes. The same sort of phenomenon happens to the result of [Cohn et al. 2022], as the construction takes into account a specific property of $\{\sqrt{2n}\}_{n\ge n_0}$ in dimensions 8 and 24.

A natural and yet unexplored question is that of determining whether formula (1-3) is rigid for its interpolation nodes or not. In other words, a natural question concerns conditions when we can replace *a single* interpolation node $\sqrt{k}$ by a suitable perturbation of it, say $\sqrt{k+\varepsilon_k}$, where $\varepsilon_k \in (-1, 1)$. To the best of our knowledge, even this simple case remained open prior to this manuscript.

Such a question inspired the following result. Perhaps surprisingly, the idea of inverting an operator T when it is reasonably close to the identity still works in this context. The next result may thus be regarded as the main result and novelty of this paper, establishing criteria when we are allowed, not only to perturb one node in the interpolation formula, but all of them *simultaneously*.

Theorem 1.4. *There is $\delta > 0$ so that, for each sequence of real numbers $\{\varepsilon_k\}_{k\geq 0}$ such that $\varepsilon_k \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, $\varepsilon_0 = 0$, $\sup_{k\geq 0} |\varepsilon_k|(1+k)^{5/4}\log^3(1+k) < \delta$, there are sequences of functions $\{\theta_j\}_{j\geq 0}$, $\{\eta_j\}_{j\geq 0}$, with*

$$|\theta_j(x)| + |\eta_j(x)| + |\hat{\theta}_j(x)| + |\hat{\eta}_j(x)| \lesssim (1+j)^{\mathcal{O}(1)}(1+|x|)^{-10}$$

and

$$f(x) = \sum_{j\geq 0} \big(f(\sqrt{j+\varepsilon_j})\theta_j(x) + \hat{f}(\sqrt{j+\varepsilon_j})\eta_j(x)\big)$$

for all $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ real-valued functions.

In other words, we can perturb each interpolation node from $\sqrt{k}$ to $\sim \sqrt{k+k^{-5/4}}$ and still obtain a valid interpolation formula converging for all Schwartz functions. In fact, one does not strictly need that $f \in \mathcal{S}(\mathbb{R})$, but only that $f, \hat{f}$ decay at least as fast as $(1+|x|)^{-M}$ for some sufficiently large $M \gg 1$.

Theorem 1.4 is related to [Cohn and Triantafillou 2021, §6]. Indeed, in that paper, they construct summation formulae of the form

$$\sum_{n=0}^{\infty} a_n f(\sqrt{n}) = \left(\frac{2}{\sqrt{N}}\right)^{d/2} \sum_{n=0}^{\infty} b_n \hat{f}(2\sqrt{n/N}),$$

where N is a suitable positive integer, where they aim to make the coefficients $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ nonnegative, in order to obtain better estimates for the linear programming bounds for the sphere-packing problem. In §7 in [Cohn and Triantafillou 2021], the authors mention that a "modular" method as carried out by them cannot achieve perturbed nodes in such an interpolation formula, which would be desirable for numerical purposes.

Theorem 1.4, on the one hand, does prove that we can make this rigid property somewhat looser when it comes to the Radchenko–Viazovska interpolation formula, but on the other hand, positivity of coefficients can by no means be guaranteed in our present case. It would be, however, interesting if one could explore further the connections between our methods and those in [Cohn and Triantafillou 2021] to obtain better bounds, but we have not pursued such a path in this work.

As an immediate corollary of Theorem 1.4, we obtain the following:

Corollary 1.5. *Let $\{\varepsilon_j\}_{j\geq 0}$ satisfy the hypotheses of Theorem 1.4. Define a continuous family of measures*

$$\mu_x = \frac{\delta_x + \delta_{-x}}{2} - \sum_{j\geq 0} \frac{\theta_j(x)}{2}\, \delta_{\pm\sqrt{j+\varepsilon_j}}.$$

Then these measures possess Fourier transforms given by

$$\hat{\mu}_x = \sum_{j\geq 0} \frac{\eta_j(x)}{2} \delta_{\pm\sqrt{j+\varepsilon_j}}.$$

In particular, these measures are nontrivial examples of ***crystalline measures*** *supported on both space and frequency on any set of the form $\{\pm x\} \cup \{\pm\sqrt{k+\varepsilon_k} : |\varepsilon_k| \ll \log^{-3}(1+k)\cdot(1+k)^{-5/4}\}$.*

This result, in particular, aligns well with the recent examples from [Bondarenko et al. 2023; Kurasov and Sarnak 2020], which indicate that crystalline measures are, if not impossible, very hard to classify. Its proof follows from the fact that μ_x is even and real-valued, so that its distributional Fourier transform will also be an even and real-valued distribution. Therefore, it suffices to test against even, real-valued functions f, and thus Theorem 1.4 gives us the asserted equality.

In order to prove Theorem 1.4, we need to find a suitable space to use the idea of inverting operators close to the identity. It turns out that, in analogy to Sobolev spaces, the weighted spaces $\ell^2_s(\mathbb{N})$ of sequences square summable against n^s are natural candidates to work with, as they are well-suited to accommodate the sequence

$$\{(f(\sqrt{k+\varepsilon_k}),\, \hat{f}(\sqrt{k+\varepsilon_k}))\}_{k\geq 0}$$

whenever $f, \hat{f}$ decay sufficiently fast. In order to prove *some* perturbation result — that is, a weaker version of Theorem 1.4 — using the spaces $\ell^2_s(\mathbb{N})$ together with the polynomial growth bounds on $\{a_n\}_{n\geq 0}$ from (1-3) is already enough.

On the other hand, the fact that we may push the perturbations up until the $k^{-5/4}$ threshold needs a suitable refinement to [Radchenko and Viazovska 2019] or even to the bound of [Bondarenko et al. 2023]. The next result, thus, represents an improvement over those in [Bondarenko et al. 2023; Radchenko and Viazovska 2019], as besides obtaining uniform bounds, we are able to introduce *exponential decay* factors to the interpolating functions.

Theorem 1.6. *Let $b_n^{\pm} = a_n \pm \hat{a}_n$, where $\{a_n\}_{n\geq 0}$ are the basis functions in* (1-3). *Then there is an absolute constant $c > 0$ such that*

$$|b_n^{\pm}(x)| \lesssim n^{1/4}\log^3(1+n)e^{-c|x|/\sqrt{n}},$$
$$|(b_n^{\pm})'(x)| \lesssim n^{3/4}\log^3(1+n)e^{-c|x|/\sqrt{n}}$$

for all positive integers $n \in \mathbb{N}$.

The proof of such a result employs a mixture of the main ideas for the uniform bounds in [Radchenko and Viazovska 2019; Bondarenko et al. 2023], with the addition of an explicit computation of the best uniform constant bounding $|x|^k|b_n^{\pm}(x) + (b_n^{\pm})'(x)|$ in terms of k and n. In order to obtain such a constant, we employ ideas from characterizations of Gelfand–Shilov spaces, as in [Chung et al. 1996].

We remark that, with a modification of the growth lemma for Fourier coefficients of 2-periodic functions, we are able to obtain a slight improvement over the growth stated in Theorem 1.6. As, however, this modification does not yield any improvement on the perturbation range stated in Theorem 1.4, we postpone a more detailed discussion about it to Corollary 4.6 below.

1C. ***Applications.*** As a by-product of our method of proof for Theorem 1.4, we are able to deduce some interesting consequences in regard to some other interpolation formulae and uniqueness results.

Indeed, it is a not-so-difficult task to adapt the ideas employed before to the contexts of interpolation formulae for *odd* functions. As remarked by Radchenko and Viazovska, the following interpolation formula is available whenever $f:\mathbb{R}\to\mathbb{R}$ is odd and belongs to the Schwartz class:

$$f(x)=d_0^+(x)\frac{f'(0)+i\hat{f}'(0)}{2}+\sum_{n\geq 1}\left(c_n(x)\frac{f(\sqrt{n})}{\sqrt{n}}-\hat{c}_n(x)\frac{\hat{f}(\sqrt{n})}{\sqrt{n}}\right),$$

where the interpolating sequence $\{c_i\}_{i\geq 0}$ possesses analogous properties to those of $\{a_i\}_{i\geq 0}$, and the function

$$d_0^+(x)=\frac{\sin(\pi x^2)}{\sinh(\pi x)}$$

is odd and real and so it vanishes together with its Fourier transform at $\pm\sqrt{n},\ n\geq 0$.

With our techniques, we are able to prove an analogous result to Theorems 1.6 and 1.4 for the odd interpolation formula. Also, with our techniques, we are able to prove a version of Cohn–Kumar–Miller–Radchenko–Viazovska interpolation results with derivatives in dimensions 8 and 24 with *perturbed* nodes in a suitable range, as polynomial growth bounds for such interpolating functions are available in [Cohn et al. 2022]; see Theorems 5.11 and 5.13 for more details.

Another interesting application of our techniques delves a little deeper into functional analysis techniques. Indeed, in order to prove that the operator that takes the set of values $\{f(\sqrt{k})\}_{k\geq 0},\ \{\hat{f}(\sqrt{k})\}_{k\geq 0}$ to the sequences

$$\{f(\sqrt{k+\varepsilon_k})\}_{k\geq 0},\quad \{\hat{f}(\sqrt{k+\varepsilon_k})\}_{k\geq 0}$$

is bounded and close to the identity on a suitable $\ell_s^2(\mathbb{N})\times\ell_s^2(\mathbb{N})$ space, we explore two main options, which are *Schur's test* and the *Hilbert–Schmidt test*. Although there is no direct relation between them, Schur's test seems to hold, in generic terms, for more operators than the Hilbert–Schmidt test, and for that reason we employ the former in our proof of Theorem 1.4. On the other hand, the Hilbert–Schmidt test has the advantage that, whenever an operator is bounded in the Hilbert–Schmidt norm, it is automatically a *compact* operator. This allows us to use many more tools derived from the theory of Fredholm operators, and, in particular, deduce a sort of interpolation/uniqueness result in the case $\varepsilon_0\neq 0$, which is excluded by Theorem 1.4 above; see Theorem 5.3 below for such an application.

The final interesting application of Theorem 1.4 and its techniques the we present is to the problem of *Fourier uniqueness for powers of integers*. In [Ramos and Sousa 2022], we have proven a preliminary result on conditions on $(\alpha,\beta),\ 0<\alpha,\beta,\ \alpha+\beta<1$, so that the only $f\in\mathcal{S}(\mathbb{R})$ such that

$$f(\pm n^{\alpha})=\hat{f}(\pm n^{\beta})=0$$

is $f\equiv 0$. In particular, we prove that, if $\alpha=\beta$, then we can take $\alpha<1-\frac{\sqrt{2}}{2}$.

By an approximation argument, a careful analysis involving Laplace transforms and the perturbation techniques and results above, we are able to reprove such a result for $\alpha=\beta$ in the $\alpha<\frac{2}{9}$ range in the case f is real and even by a completely different method than that in [Ramos and Sousa 2022]. Although the current method does not yield any improvement over [Ramos and Sousa 2022, Theorem 1], we obtain additionally some *strong annihilation* properties of such pairs, in the form of Corollary 5.10, which are novel in that context.

Still on the subject of annihilation, we obtain two other interesting results.

Theorem 1.7. *For each $s > 1$ sufficiently large, there are $\gamma > s$ and $\omega > 0$ such that both inequalities*

$$\Big(\sum_{n\geq 0}(1+n)^s[|f(\sqrt{n})|^2+|\hat{f}(\sqrt{n})|^2]\Big)^{1/2} \lesssim \|f\|_{L^2((1+|x|)^\gamma)} + \|\hat{f}\|_{L^2((1+|x|)^\gamma)}, \tag{1-10}$$

$$\|f\|_{L^2((1+|x|)^s)} + \|\hat{f}\|_{L^2((1+|x|)^s)} \lesssim \Big(\sum_{n\geq 0}(1+n)^\omega[|f(\sqrt{n})|^2+|\hat{f}(\sqrt{n})|^2]\Big)^{1/2} \tag{1-11}$$

hold for each $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ real.

Corollary 1.8. *Let $\{\varepsilon_i\}_{i\in\mathbb{N}}$ satisfy the hypotheses of Theorem 1.4. Then for $s \gg 1$ sufficiently large, both inequalities*

$$\Big(\sum_{n\geq 0}(1+n)^s[|f(\sqrt{n+\varepsilon_n})|^2+|\hat{f}(\sqrt{n+\varepsilon_n})|^2]\Big)^{1/2} \lesssim \|f\|_{L^2((1+|x|)^\gamma)}+\|\hat{f}\|_{L^2((1+|x|)^\gamma)},$$

$$\|f\|_{L^2((1+|x|)^s)}+\|\hat{f}\|_{L^2((1+|x|)^s)} \lesssim \Big(\sum_{n\geq 0}(1+n)^\omega[|f(\sqrt{n+\varepsilon_n})|^2+|\hat{f}(\sqrt{n+\varepsilon_n})|^2]\Big)^{1/2}$$

hold for each $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ real, where ω, γ are as in Theorem 1.7.

We refer the reader to discussion in Section 5C for more precise definitions about annihilating pairs

We should remark that it has been recently communicated to us by Kulikov, Nazarov and Sodin (personal communication) that they have been able to significantly strengthen the results in [Ramos and Sousa 2022]. As a particular application of their results, they are able to obtain the whole range $\alpha+\beta<1$, conjectured in [loc. cit.]. In fact, they can say quite a bit more even in the "critical" case $\alpha+\beta=1$, constructing also suitable counterexamples to these uniqueness questions. It has also been communicated to us that they have obtained strong annihilating properties in such a range as well. In spite of that, we have decided to maintain this application of our work, as it contains interesting ideas that could be applied to other uniqueness problems of similar flavor. In particular, Theorem 1.7 and Corollary 1.8 are a novelty of this present work, and seem not to be included as a consequence of the results from Kulikov, Nazarov and Sodin.

1D. ***Organization.*** We comment briefly on the overall display of our results throughout the text. In Section 2 below, we discuss generalities on background results needed for the proofs of the main theorems, going over results in the theory of band-limited functions, modular forms and functional analysis. Next, in Section 3, we prove, in this order, Theorems 1.1, 1.2 and 1.3 about band-limited perturbed interpolation formulae. We then prove, in Section 4, Theorem 1.4, by first discussing the proof of Theorem 1.6 in Section 4A. We then discuss the applications of our main results and techniques in Section 5, and finish the manuscript with Section 6, talking about some possible refinements and open problems that arise from our discussion throughout the paper.

2. Preliminaries

2A. ***Band-limited functions.*** We start by recalling some basic facts about band-limited functions. Given a function $f \in L^2(\mathbb{R})$, we say that it is *band-limited* if its Fourier transform satisfies that $\text{supp}(\hat{f}) \subset [-M, M]$ for some $M > 0$. In this case, we say that f is *band-limited to* $[-M, M]$.

It is a classical result due to Paley and Wiener that a function $f \in L^2(\mathbb{R})$ is band-limited to $[-\sigma, \sigma]$ if and only if it is the restriction of an entire function $F : \mathbb{C} \to \mathbb{C}$ to the real axis, and the function F is of exponential type $2\pi\sigma$, i.e., for each $\varepsilon > 0$, there is C_ε such that

$$|F(z)| \le C_\varepsilon e^{(2\pi\sigma+\varepsilon)|z|}$$

for all $z \in \mathbb{C}$. From now on we will abuse notation and let $F = f$ whenever there is no danger of confusion, and we may also write $f \in \mathrm{PW}_{2\pi\sigma}$ (Paley–Wiener space) to denote the space of functions with such properties.

Besides this fact, we will make use of some interpolation formulae for those functions. Namely:

(1) *Shannon–Whittaker interpolation formula*. For each $f \in L^2(\mathbb{R})$ band-limited to $\left[-\frac{1}{2}, \frac{1}{2}\right]$, the following formula holds:

$$f(x) = \sum_{n\in\mathbb{Z}} f(n)\,\mathrm{sinc}(x-n),$$

where $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$ and the sum above converges both in $L^2(\mathbb{R})$ and uniformly on compact sets of $\mathbb{C}$.

(2) *Vaaler interpolation formula*. For each $f \in L^2(\mathbb{R})$ band-limited to $[-1, 1]$, the following formula holds:

$$f(x) = \left(\frac{\sin \pi x}{\pi}\right)^2 \sum_{n\in\mathbb{Z}} \left[\frac{f(n)}{(x-n)^2} + \frac{f'(n)}{x-n}\right],$$

where the right-hand side converges both in $L^2(\mathbb{R})$ and uniformly on compact sets of $\mathbb{C}$.

For more details on these classical results, see, for instance, [Vaaler 1985; Littmann 2006; Paley and Wiener 1934; Shannon 1949; Whittaker 1915].

2B. ***Modular forms.*** In order to prove the improved estimates on the interpolation basis for the Radchenko–Viazovska interpolation result, we will need to make careful computations involving certain modular forms defining the interpolating functions. For that purpose, we gather some of the facts we will need in this subsection. For more information on the functions λ, J and the automorphy factors we just defined, we refer the reader to [Chandrasekharan 1985; Radchenko and Viazovska 2019, Section 2; Berndt and Knopp 2008; Zagier 2008].

We denote by $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the upper half-plane in $\mathbb{C}$. The special feature of this space is that the group $\mathrm{SL}_2(\mathbb{R})$ of matrices with real coefficients and determinant 1 acts naturally on it through Möbius transformations:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \quad z \in \mathbb{H} \quad \Longrightarrow \quad \gamma z = \frac{az+b}{cz+d} \in \mathbb{H}.$$

Indeed, it suffices to look at the action of the quotient $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$, since clearly the action by both matrices γ and $-\gamma$ induces the same Möbius transformation. Some elements of this group will be of special interest to us. Namely, we let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This already allows us to define the most valuable subgroup of $\mathrm{SL}_2(\mathbb{Z})$ for us: the group Γ_θ is defined then as the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by S and T^2. This group has 1 and ∞ as cusps, and its standard fundamental domain is given by

$$\mathcal{D} = \{z \in \mathbb{H} : |z| > 1, \operatorname{Re}(z) \in (-1, 1)\}.$$

With these at hand, we define *modular forms* for Γ_θ. For that purpose, we will use the following notation for the Jacobi theta series:

$$\vartheta(z, \tau) = \sum_{n \in \mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z).$$

We are interested in some of its *Nullwerte*, the so-called Jacobi theta series. These are defined in $\mathbb{H}$ by

$$\begin{aligned} \Theta_2(\tau) &= \exp\left(\tfrac{\pi}{4} i \tau\right) \vartheta\left(\tfrac{1}{2}\tau, \tau\right), \\ \Theta_3(\tau) &= \vartheta(0, \tau) (=: \theta(\tau)), \\ \Theta_4(\tau) &= \vartheta\left(\tfrac{1}{2}, \tau\right). \end{aligned}$$

These functions satisfy the identity $\Theta_3^4 = \Theta_2^4 + \Theta_4^4$. Moreover, under the action of the elements S and T of $\mathrm{SL}_2(\mathbb{Z})$, they transform as

$$\begin{aligned} (-iz)^{-1/2} \Theta_2(-1/z) &= \Theta_4(z), \quad \Theta_2(z+1) = \exp\left(\tfrac{\pi}{4} i\right) \Theta_2(z), \\ (-iz)^{-1/2} \Theta_3(-1/z) &= \Theta_3(z), \quad \Theta_3(z+1) = \Theta_4(z), \\ (-iz)^{-1/2} \Theta_4(-1/z) &= \Theta_2(z), \quad \Theta_4(z+1) = \Theta_3(z). \end{aligned} \tag{2-1}$$

These functions allow us to construct the classical lambda modular invariant given by

$$\lambda(z) = \frac{\Theta_2(z)^4}{\Theta_3(z)^4}.$$

Using $q := q(z) = e^{\pi i z}$, the lambda invariant can be alternatively rewritten as

$$\lambda(z) = 16q \times \prod_{k=1}^{\infty} \left(\frac{1 + q^{2k}}{1 + q^{2k-1}} \right)^8 = 16q - 128q^2 + 704q^3 + \cdots. \tag{2-2}$$

The function λ is also invariant under the action of elements of the subgroup $\Gamma(2) \subset \mathrm{SL}_2(\mathbb{Z})$ of all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ so that $a \equiv b \equiv 1 \mod 2$, $c \equiv d \equiv 0 \mod 2$, and $\lambda(z)$ never assumes the values 0 or 1 for $z \in \mathbb{H}$. Besides this invariance, (2-1) gives us immediately that

$$\lambda(z+1) = \frac{\lambda(z)}{\lambda(z) - 1}, \quad \lambda\left(-\frac{1}{z}\right) = 1 - \lambda(z). \tag{2-3}$$

We then define the following modular function for Γ_θ (which is a Hauptmodul for Γ_θ)

$$J(z) = \tfrac{1}{16} \lambda(z)(1 - \lambda(z)).$$

From (2-3), we obtain immediately that J is invariant under the action of elements of Γ_θ; i.e.,

$$J(z+2) = J(z), \quad J\left(-\frac{1}{z}\right) = J(z).$$

Other properties of the functions λ and J that we may eventually need will be proved throughout the text.

Finally, we mention that, for the proof in Section 4, we will need to use the so-called *θ-automorphy factor* defined, for $z \in \mathbb{H}$ and $\gamma \in \Gamma_\theta$, as

$$j_\theta(z, \gamma) = \frac{\theta(z)}{\theta(\gamma z)}.$$

We can then define a slash operator of weight $k/2$ to be

$$(f|_{k/2}\gamma)(z) = j_\theta(z, \gamma)^k f\left(\frac{az+b}{cz+d}\right),$$

where $\gamma = \left(\begin{smallmatrix} a & c \\ c & d \end{smallmatrix}\right)$. These slash operators induce other *sign* slash operators given by

$$(f|^{\varepsilon}_{k/2}\gamma) = \chi_\varepsilon(\gamma)(f|_{k/2}\gamma),$$

where we let χ_ε be the homomorphism of Γ_θ so that $\chi_\varepsilon(S) = \varepsilon$, $\chi_\varepsilon(T^2) = 1$.

2C. *Functional analysis.* We also recall some classical facts in functional analysis that will be useful throughout our proof.

As our main goal and strategy throughout this manuscript is to prove that a small perturbation of the identity is invertible, we must find ways to prove that the operators arising in our computations are bounded. To this end, we use two major criteria to prove boundedness — and therefore to prove smallness of the bounding constant. These are:

(1) *Hilbert–Schmidt test* [Brezis 2011, Chapter 6]. Let H be a (real or complex) Hilbert space, and let there be given a linear operator $T : H \to H$. If T satisfies additionally that

$$\sum_{i,j} |\langle Te_j, e_i\rangle|^2 < +\infty$$

for some orthonormal basis $\{e_i\}_{i\in\mathbb{Z}}$ of H, then the operator T is bounded. Moreover,

$$\|T\|^2_{H\to H} \le \sum_{i,j} |\langle Te_j, e_i\rangle|^2 =: \|T\|^2_{HS}.$$

(2) *Schur test* [Hedenmalm et al. 2000, Theorem 1.8]. Let $(a_{ij})_{i,j\ge 0}$ denote a (possibly infinite) matrix of complex numbers. Suppose that there are two sequences $\{v_i\}_{i\ge 0}$ and $\{w_i\}_{i\ge 0}$ of positive real numbers so that

$$\sum_{i\ge 0} |a_{ij}| w_i \le \lambda v_j, \quad \sum_{j\ge 0} |a_{ij}| v_j \le \mu q_i$$

for some positive constants $\mu, \lambda > 0$. Then the operator $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ given by $a_{ij} = \langle Te_i, e_j\rangle$ (where $\{e_i\}_{i\ge 0}$ denotes the standard orthonormal basis of $\ell^2(\mathbb{N})$) extends to a *bounded* linear operator. Moreover,

$$\|T\|_{\ell^2\to\ell^2} \le \sqrt{\mu\lambda}.$$

Both tests will play a major role in the deduction of the validity of perturbed interpolation versions of the Radchenko–Viazovska result. The main difference is that, while Schur's test generally gives one boundedness for more operators, the Hilbert–Schmidt test imposes stronger conditions on the operator. In fact, let us denote by $T \in \mathcal{HS}(H)$ the space of operators such that $\|T\|_{HS} < +\infty$. A classical consequence of this fact is that T is compact. This compactness will be used when proving that a suitable version of our interpolation results holds for small perturbations of the origin. See, for instance, [Brezis 2011, Chapter 6]

2D. ***Notation.*** We will use Vinogradov's modified notation throughout the text; that is, we write $A \lesssim B$ in the case there is an absolute constant $C > 0$ so that $A \le C \cdot B$. If the constant C depends on some set of parameters λ, we shall write $A \lesssim_\lambda B$.

On the other hand, we shall also use the big-$\mathcal{O}$ notation $f = \mathcal{O}(g)$ if there is an absolute constant C such that $|f| \le C \cdot g$, although the usage of this will be restricted mostly to sequences. We may occasionally use as well the standard Vinogradov notation $a \ll b$ to denote that there is a (relatively) *large* constant $C > 1$ such that $a \le C \cdot b$.

We shall also denote the spaces of sequences of complex numbers decaying polynomially by

$$\ell_s^2(\mathbb{Z}_+) = \Big\{(a_n)_n \in \ell^2(\mathbb{Z}_+) : |a_0|^2 + \sum_{n\in\mathbb{N}} |a_n|^2 n^{2s} < +\infty\Big\}, \tag{2-4}$$

$$\ell_s^2(\mathbb{N}) = \Big\{\{a_n\}_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}} |a_n|^2 n^{2s} < +\infty\Big\},$$

where $\mathbb{N} = \{1, 2, \dots\}$ denotes the set of natural numbers and $\mathbb{Z}_+$ denotes the nonnegative integers. We remind the reader that we always normalize the Fourier transform as in (1-1), i.e,

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i x\cdot\xi}\, \mathrm{d}x.$$

3. Perturbed interpolation formulae for band-limited functions

3A. ***Perturbed forms of the Shannon–Whittaker formula and Kadec's result.*** Fix a sequence $\varepsilon = \{\varepsilon_k\}_{k\in\mathbb{Z}}$ of real numbers such that $\sup_k |\varepsilon_k| < 1$. We wish to obtain a criterion based solely on the value of $L = \sup_n |\varepsilon_n|$ such that the sequence $\{n + \varepsilon_n\}_{n\in\mathbb{Z}}$ is completely interpolating in PW_π, i.e, for every sequence $a = \{a_n\} \in \ell^2(\mathbb{Z})$ there is a unique $f \in L^2(\mathbb{R})$ of exponential type $\tau(f) \le \pi$ that satisfies

$$f(n + \varepsilon_n) = a_n.$$

Our goal here is to obtain a simple proof of such a criterion going through new and simple ideas. We will fall short of the $\frac{1}{4}$ proven by Kadec by approximately 0.11, but it illustrates the power of our perturbation scheme and does not go through the theory of exponential bases.

In this particular case, we need to invert in $\ell^2(\mathbb{Z})$ the operator given by

$$A_\varepsilon(a)(n) = \sum_{k\in\mathbb{Z}} a_k \operatorname{sinc}(n + \varepsilon_n - k),$$

where

$$\operatorname{sinc}(x) = \frac{\sin \pi(x)}{\pi x}.$$

The fact A_ε is invertible will follow from proving that it is a close perturbation of the identity whenever L is sufficiently small.

3A1. *Auxiliary perturbations of the Hilbert transforms.* Given a sequence $a = \{a_k\}_{k\in\mathbb{Z}}$, we define the following operators, which are akin to the discrete Hilbert transform:

$$\mathcal{H}_\varepsilon(a)(n) = \sum_{k\neq n} \frac{(-1)^{n-k} a_k}{n+\varepsilon_n - k}, \quad \mathcal{H}_0(a)(n) = \sum_{k\neq n} \frac{(-1)^{n-k} a_k}{n-k}.$$

We start by comparing these two objects:

$$\begin{aligned}\mathcal{H}_0(a)(n) - \mathcal{H}_\varepsilon(a)(n) &= \sum_{k\neq n} (-1)^{n-k} a_k \left(\frac{1}{n-k} - \frac{1}{n+\varepsilon_n - k}\right) \\ &= \varepsilon_n \sum_{k\neq n} (-1)^{n-k} a_k \frac{1}{(n-k)(n+\varepsilon_n-k)}.\end{aligned}$$

This identity then gives us

$$\begin{aligned}|\mathcal{H}_0(a)(n) - \mathcal{H}_\varepsilon(a)(n)| &\le |\varepsilon_n| \sum_{k\neq n} |a_k| \frac{1}{|n-k|^2} \frac{|n-k|}{|n+\varepsilon_n-k|} \\ &\le \frac{|\varepsilon_n|}{1-|\varepsilon_n|} \sum_{k\neq n} |a_k| \frac{1}{|n-k|^2}.\end{aligned}$$

This means that, in norm, one can compare these two operators. Indeed, it is a classical result that the operator norm of $\mathcal{H}_0$ is π, and by Plancherel the operator norm of the transformation

$$\mathcal{S}(a) = \sum_{k\neq n} a_k \frac{1}{|n-k|^2}$$

is $\pi^2/3$. This in turn implies

$$\|\mathcal{H}_\varepsilon\| \le \pi + \frac{\pi^2}{3} \frac{\sup_n |\varepsilon_n|}{1-\sup_n |\varepsilon_n|}. \tag{3-1}$$

3A2. *Norm estimates of the perturbation.* It is worth noticing the estimate (3-1) is very crude, as it is meant to depend only on $L = \sup_n |\varepsilon_n|$. For instance, if $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ is a constant sequence, then the norm $\|\mathcal{H}_\varepsilon\|$ is equal to π. We also note that the fact that we obtain invertibility by means of perturbations of small norm of an invertible operator does not take into account other factors, such as cancellation.

In order to apply our perturbation scheme to the operator A_ε, we need to bound the following family of operators:

$$P_\varepsilon(a)(n) = \sum_{k\in\mathbb{Z}} a_k (\operatorname{sinc}(n+\varepsilon_n-k) - \delta_{n,k}).$$

We may rewrite them as

$$\begin{aligned}P_\varepsilon(a)(n) =& (\operatorname{sinc}(\varepsilon_n)-1)a_n + \sum_{k\neq n} a_k (\operatorname{sinc}(n+\varepsilon_n-k)) \\ =& (\operatorname{sinc}(\varepsilon_n)-1)a_n + \sum_{k\neq n} a_k \frac{(-1)^{n-k}\sin \pi\varepsilon_n}{\pi(n+\varepsilon_n-k)}.\end{aligned}$$

This implies, on the other hand,

$$P_\varepsilon(a)(n) = (\operatorname{sinc}(\varepsilon_n) - 1)a_n + \left(\frac{\sin \pi \varepsilon_n}{\pi}\right)\mathcal{H}_\varepsilon(a)(n),$$

which in turn implies that

$$\begin{aligned}\|P_\varepsilon\| &\le \sup_n |\operatorname{sinc}(\varepsilon_n) - 1| + \sup_n \left|\frac{\sin \pi\varepsilon_n}{\pi}\right| \|\mathcal{H}_\varepsilon\| \\ &\le \sup_n |\operatorname{sinc}(\varepsilon_n) - 1| + \sup_n |\sin \pi\varepsilon_n| + \frac{\pi}{3}\frac{\sup_n |\sin\pi\varepsilon_n| \sup_n |\varepsilon_n|}{1-\sup_n |\varepsilon_n|}.\end{aligned}$$

Since $A_\varepsilon = P_\varepsilon + \mathrm{Id}$, whenever

$$1 - \operatorname{sinc}(L) + |\sin \pi L| + \frac{\pi}{3}\frac{L \sin \pi L}{1-L} < 1,$$

we will have that A_ε is invertible. In particular, a routine numerical evaluation implies that $L < 0.239$ satisfies the inequality above. Let then $A_\varepsilon^{-1} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the inverse of A_ε, which is continuous by the considerations above. We know, by the Shannon–Whittaker interpolation formula (1-2) that A_ε takes $\{f(k)\}_{k\in\mathbb{Z}}$, for $f \in \mathrm{PW}_\pi$, to $\{f(k+\varepsilon_k)\}_{k\in\mathbb{Z}}$. This is enough to prove the assertion about recovery, and as such implies that

$$\sum_{n\in\mathbb{Z}} |f(n+\varepsilon_n)|^2$$

is an equivalent norm to the usual L^2-norm on PW_π, by [Young 1980, Theorem 1.13].

Moreover, by writing

$$A_\varepsilon^{-1}(b)(k) = \sum_{n\in\mathbb{Z}} b_n \cdot \rho_{k,n},$$

we have immediately

$$\sum_{n\in\mathbb{Z}} f(n+\varepsilon_n)\rho_{k,n} = f(k), \tag{3-2}$$

and $\sup_n\bigl(\sum_{k\in\mathbb{Z}} |\rho_{k,n}|^2\bigr) \lesssim 1$. If $(A_\varepsilon^{-1})^* : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ denotes the adjoint of the inverse of A_ε, then we see that for any compact set $K \subset \mathbb{C}$ there is a constant $C = C_K$ such that

$$\begin{aligned}\|(A_\varepsilon^{-1})^*(\operatorname{sinc}_z(k))\|_{\ell^2(\mathbb{Z})} &\le \|A_\varepsilon^{-1}\|_{\ell^2\to\ell^2} \|(\operatorname{sinc}_z(k))\|_{\ell^2(\mathbb{Z})} \\ &\le C\|A_\varepsilon^{-1}\|_{\ell^2\to\ell^2},\end{aligned}$$

and C does not depend on $z \in K$ and we let $\operatorname{sinc}_x(k) := \operatorname{sinc}(x-k)$. Therefore, by letting $g_n(z) = \sum_{k\in\mathbb{Z}} \rho_{k,n} \operatorname{sinc}(z-k)$, we have

$$\sup_{z\in\mathbb{R}} \left(\sum_{n\in\mathbb{Z}} |g_n(z)|^2\right)^{1/2} \lesssim 1,$$

and thus, by the previous considerations, the sum $\sum_{n\in\mathbb{Z}} f(n+\varepsilon_n) g_n(z)$ converges absolutely by Cauchy–Schwarz. As $\langle (A_\varepsilon^{-1})^*(\operatorname{sinc}_z(k)), f(n+\varepsilon_n)\rangle = \langle \operatorname{sinc}_z(k), A_\varepsilon^{-1}(f(n+\varepsilon_n))\rangle = f(z)$ by Shannon–Whittaker, this implies

$$f(z)=\sum_{n\in\mathbb{Z}} f(n+\varepsilon_n)g_n(x),$$

where the convergence happens uniformly in compact sets, as desired.

This finishes the proof of Theorem 1.1.

3B. ***From Shannon to Vaaler*: *the proof of Theorem 1.2.*** We now concentrate on proving that the usual Shannon–Whittaker interpolation formula implies Vaaler's celebrated interpolation result [1985] with derivatives.

Indeed, as proving that the interpolation formula of Theorem 1.2 converges uniformly on compact sets of $\mathbb{C}$ is a routine computation, given that $\{a_k\}_{k\in\mathbb{Z}}, \{b_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, we shall omit this part and focus on proving that the asserted equality holds.

Given a sequence $a=\{a_k\}_{k\in\mathbb{Z}}$, we define the operators

$$\mathcal{H}(a)(k)=\frac{1}{\pi}\sum_{0\neq j\in\mathbb{Z}}\frac{a_{k-j}}{j}=\frac{1}{\pi}\sum_{k\neq j\in\mathbb{Z}}\frac{a_j}{k-j},$$

$$\mathcal{H}_1(a)(k)=\frac{1}{\pi}\sum_{j\in\mathbb{Z}}\frac{a_{k-j}}{j+\frac{1}{2}}=\frac{1}{\pi}\sum_{j\in\mathbb{Z}}\frac{a_j}{k-j+\frac{1}{2}}.$$

It is known that both $\mathcal{H}$ and $\mathcal{H}_1$ are bounded operators in $\ell^2(\mathbb{Z})$, with $\mathcal{H}_1$ being also unitary with $\mathcal{H}_2$ its inverse being given by

$$\mathcal{H}_2(a)(k)=-\frac{1}{\pi}\sum_{j\in\mathbb{Z}}\frac{a_{j-k}}{j-\frac{1}{2}}=\frac{1}{\pi}\sum_{j\in\mathbb{Z}}\frac{a_j}{j-k+\frac{1}{2}}.$$

Given a function $f\in \mathrm{PW}_\pi$, as a consequence of the Shannon–Whittaker interpolation formula we obtain, for every $k\in\mathbb{Z}$, that

$$f'(k)=\sum_{j\neq k}\frac{f(j)}{k-j}(-1)^{k-j}.$$

We consider three sequences

$$a(k)=f(2k-1),\quad b(k)=f(2k),\quad c(k)=f'(2k).$$

We have, thus,

$$\begin{aligned}c(k)=f'(2k)=\sum_{j\neq 2k}\frac{f(j)}{2k-j}(-1)^{2k-j}&=\frac{1}{2}\sum_{j\neq k}\frac{f(2j)}{k-j}-\frac{1}{2}\sum_{j\in\mathbb{Z}}\frac{f(2j-1)}{k-j+\frac{1}{2}}\\&=\frac{1}{2}\sum_{j\neq k}\frac{b(j)}{k-j}-\frac{1}{2}\sum_{j\in\mathbb{Z}}\frac{a(j)}{k-j+\frac{1}{2}}=\frac{\pi}{2}\mathcal{H}(b)(k)-\frac{\pi}{2}\mathcal{H}_1(a)(k).\end{aligned}$$

This means that, for every $k\in\mathbb{Z}$,

$$\mathcal{H}_1(a)(k)=\mathcal{H}(b)(k)-\frac{2}{\pi}c(k).$$

Since $\mathcal{H}_2$ is the inverse of $\mathcal{H}_1$, this can be rewritten as

$$a(k)=(\mathcal{H}_2\circ\mathcal{H})(b)(k)-\frac{2}{\pi}\mathcal{H}_2(c)(k).$$

We know, by the Shannon–Whittaker interpolation formula, that

$$f(x)=\sum_{k\in\mathbb{Z}} f(k)\frac{\sin\pi(x-k)}{\pi(x-k)}.$$

This implies, on the other hand,

$$\begin{aligned}
f(x)&=\sum_{k\in\mathbb{Z}} f(2k)\frac{\sin\pi(x-2k)}{\pi(x-2k)}+\sum_{k\in\mathbb{Z}}\Big[(\mathcal{H}_2\circ\mathcal{H})(b)(k)-\frac{2}{\pi}\mathcal{H}_2(c)(k)\Big]\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}\\
&=\sum_{k\in\mathbb{Z}} b(k)\frac{\sin\pi x}{\pi(x-2k)}+\sum_{k\in\mathbb{Z}}(\mathcal{H}_2\circ\mathcal{H})(b)(k)\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}-\frac{2}{\pi}\sum_{k\in\mathbb{Z}}\mathcal{H}_2(c)(k)\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}\\
&=A(x)+B(x)+C(x).
\end{aligned}$$

We shall investigate each term A, B and C thoroughly in order to obtain our final result.

3B1. *Determining C.* By considering the family of functions $h_j\in\mathrm{PW}_\pi$ — which satisfy the important property $h_j(k)=0$ if $k\in 2\mathbb{Z}$ — given by

$$h_j(z)=\frac{\sin^2\big(\frac{\pi}{2}z\big)}{\pi^2(z-2j)},$$

we obtain

$$\begin{aligned}
C(x)&=-2\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\frac{f'(2j)}{\pi^2\big(j-k+\frac{1}{2}\big)}\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}\\
&=4\sum_{j\in\mathbb{Z}} f'(2j)\sum_{k\in\mathbb{Z}}\frac{1}{\pi^2((2k-1)-2j)}\frac{\sin\pi(x-(2k-1))}{\pi(x-(2k-1))}\\
&=4\sum_{j\in\mathbb{Z}} f'(2j)\sum_{k\in\mathbb{Z}} h_j(2k-1)\frac{\sin\pi(x-(2k-1))}{\pi(x-(2k-1))}\\
&=4\sum_{j\in\mathbb{Z}} f'(2j)\sum_{k\in\mathbb{Z}} h_j(k)\frac{\sin\pi(x-k)}{\pi(x-k)}.
\end{aligned}$$

Notice that one can use Fubini's theorem to justify all the changes of order of summation by the fact that $h_j\in\mathrm{PW}_\pi$. By applying the Shannon–Whittaker interpolation to h_j, we have

$$C(x)=4\sum_{j\in\mathbb{Z}} f'(2j)\frac{\sin^2\big(\frac{\pi}{2}x\big)}{\pi^2(x-2j)}.$$

3B2. *Determining B.* For the second term, we expand

$$\begin{aligned}
B(x)&=\sum_{k\in\mathbb{Z}}\mathcal{H}_2\circ\mathcal{H}(b)(k)\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}\\
&=\frac{1}{\pi}\sum_{k\in\mathbb{Z}}\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}\sum_{j}\frac{\mathcal{H}(b)(j)}{j-k+\frac{1}{2}}\\
&=\frac{1}{\pi^2}\sum_{k\in\mathbb{Z}}\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}\sum_{j}\sum_{l\neq j}\frac{b(l)}{\big(j-k+\frac{1}{2}\big)(j-l)}.
\end{aligned}$$

By Fubini's theorem, this implies

$$\begin{aligned}B(x) &= \frac{1}{\pi^2}\sum_{l\in\mathbb{Z}} b(l)\sum_{j\neq l}\frac{1}{j-l}\sum_{k\in\mathbb{Z}}\frac{1}{j-k+\frac{1}{2}}\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}\\ &= \frac{1}{\pi^2}\sum_{l\in\mathbb{Z}} b(l)\sum_{j\neq l}\frac{2}{j-l}\sum_{k\in\mathbb{Z}}\frac{1}{2j-2k+1}\frac{\sin\pi(x-2k+1)}{\pi(x-2k+1)}\\ &= \frac{1}{\pi^2}\sum_{l\in\mathbb{Z}} b(l)\sum_{j\neq l}\frac{2}{j-l}\frac{\sin^2\left(\frac{\pi}{2}x\right)}{2j-x} = \frac{\sin^2\left(\frac{\pi}{2}x\right)}{\pi^2}\sum_{l\in\mathbb{Z}} b(l)\sum_{j\neq 0}\frac{1}{j\left(j+l-\frac{1}{2}x\right)}.\end{aligned}$$

But it is a well-known fact that the summation formula

$$\sum_{j\neq 0}\frac{1}{j(j+z)} = \frac{\psi(1+z)-\psi(1-z)}{z}$$

holds, where $\psi(z) = \frac{d}{dz}\log\Gamma(z)$ is the digamma function. This implies

$$B(x) = \frac{2\sin^2\left(\frac{\pi}{2}x\right)}{\pi^2}\sum_{l\in\mathbb{Z}} b(l)\frac{\psi\left(1+l-\frac{1}{2}x\right)-\psi\left(1-l+\frac{1}{2}x\right)}{2l-x}.$$

3B3. *Determining $A+B$.* Using that $\sin(2x) = 2\sin x\cos x$, we obtain

$$A(x) = -\frac{2\sin^2\left(\frac{\pi}{2}x\right)}{\pi^2}\sum_{l\in\mathbb{Z}} b(l)\frac{\pi\cot\left(\frac{\pi}{2}x\right)}{2l-x}.$$

The digamma function satisfies the functional equations

$$\begin{aligned}\psi(1-z) &= \psi(z)+\pi\cot\pi z,\\ \psi(1+z) &= \psi(z)+1/z.\end{aligned}$$

Using these relations with $z = \frac{1}{2}x - l$ in the equations above, we obtain readily

$$A(x)+B(x) = \frac{4\sin^2\left(\frac{\pi}{2}x\right)}{\pi^2}\sum_{l\in\mathbb{Z}} b(l)\frac{1}{(x-2l)^2}.$$

3B4. *$A+B+C$.* Summing the analysis undertaken for the terms above, we have

$$f(x) = A(x)+B(x)+C(x) = \frac{4\sin^2\left(\frac{\pi}{2}x\right)}{\pi^2}\sum_{k\in\mathbb{Z}}\left\{\frac{f(2k)}{(x-2k)^2}+\frac{f'(2k)}{x-2k}\right\}.$$

This finishes the proof of Theorem 1.2.

3C. ***Perturbed interpolation formulae with derivatives.*** By the arguments in the previous section, the formula we just derived for $\mathrm{PW}_{2\pi}$, i.e.,

$$f(x) = \frac{\sin^2(\pi x)}{\pi^2}\sum_{k\in\mathbb{Z}}\left\{\frac{f(k)}{(x-k)^2}+\frac{f'(k)}{x-k}\right\},$$

converges in compact sets of $\mathbb{C}$. We fix, for shortness, the notation

$$g(x) = \frac{\sin^2(\pi x)}{\pi^2 x^2}, \quad h(x) = \frac{\sin^2(\pi x)}{\pi^2 x},$$

which means we can read Vaaler's interpolation as

$$f(x) = \sum_{k\in\mathbb{Z}} \{ f(k) g(x-k) + f'(k) h(x-k) \}.$$

Because of uniform convergence, we can differentiate term by term in the above formula. This implies

$$f'(x) = \sum_{k\in\mathbb{Z}} \{ f(k) g'(x-k) + f'(k) h'(x-k) \}.$$

We record, for completeness, the formulae for the derivatives of g and h. For $x \notin \mathbb{Z}$ we have

$$\begin{aligned} g'(x) &= \frac{2\sin(\pi x)(\pi x \cos(\pi x) - \sin(\pi x))}{\pi^2 x^3}, \\ h'(x) &= \frac{\sin(\pi x)(2\pi x \cos(\pi x) - \sin(\pi x))}{\pi^2 x^2}, \end{aligned}$$

and, for $n \in \mathbb{Z}$,

$$g(n) = h'(n) = 0, \quad g'(n) = h(n) = \delta_0.$$

Our goal now is to invert the operator $\mathcal{A} = \mathcal{A}_\varepsilon$ defined in $\ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$ by

$$\begin{aligned} \mathcal{A}_1(a,b)_n &= \sum_{k\in\mathbb{Z}} a_k \cdot g(n+\varepsilon_n - k) + \sum_{k\in\mathbb{Z}} b_k \cdot h(n+\varepsilon_n - k), \\ \mathcal{A}_2(a,b)_n &= \sum_{k\in\mathbb{Z}} a_k \cdot g'(n+\varepsilon_n - k) + \sum_{k\in\mathbb{Z}} b_k \cdot h'(n+\varepsilon_n - k), \end{aligned} \tag{3-3}$$

where $\mathcal{A}(a,b) = (\mathcal{A}_1(a,b), \mathcal{A}_2(a,b))$ for $(a,b) \in \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$. Furthermore, we wish to establish a criterion that depends only on $L = \sup |\varepsilon_n|$. For that purpose, we estimate when the operator norm of $\mathcal{A}_\varepsilon - \mathrm{Id}$ from $\ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$ to itself is small, in terms of L.

3C1. *Auxiliary perturbations for the derivative case.* Given a sequence $a = \{a_k\}_{k\in\mathbb{Z}}$, we define the operators

$$\mathcal{H}^p_\varepsilon(a)_n = \sum_{k\neq n} \frac{a_k}{(n+\varepsilon_n - k)^p},$$

and denote by $\mathcal{H}^p_0$ the operator associated to the sequence $\varepsilon_n = 0$ for all $n \in \mathbb{Z}$. In an analogous manner to the proof of Theorem 1.1, we compare

$$\begin{aligned} \mathcal{H}^p_0(a)_n - \mathcal{H}^p_\varepsilon(a)_n &= \sum_{k\neq n} a_k \left(\frac{1}{(n-k)^p} - \frac{1}{(n+\varepsilon_n - k)^p} \right) \\ &= \sum_{j=0}^{p-1} \binom{p}{j} \varepsilon_n^{p-j} \sum_{k\neq n} \frac{a_k}{(n+\varepsilon_n-k)^p (n-k)^{p-j}}. \end{aligned}$$

Therefore,

$$|\mathcal{H}_0^p(a)_n - \mathcal{H}_\varepsilon^p(a)_n| \le \sum_{j=0}^{p-1}\binom{p}{j}|\varepsilon_n|^{p-j}\sum_{k\ne n}\frac{a_k}{|n-k|^{2p-j}}\frac{|n-k|^p}{(|n-k|-|\varepsilon_n|)^p}$$
$$\le \frac{1}{(1-|\varepsilon_n|)^p}\sum_{j=0}^{p-1}\binom{p}{j}|\varepsilon_n|^{p-j}\mathcal{S}^{2p-j}(a^*)_n,$$

where

$$\mathcal{S}^q(a)_n = \sum_{k\ne n}\frac{a_k}{|n-k|^q}$$

and $a^* = (|a_n|)$. Since $\mathcal{S}^{q+1}(a^*)_n \le \mathcal{S}^q(a^*)_n$, we have

$$|\mathcal{H}_0^p(a)_n - \mathcal{H}_\varepsilon^p(a)_n| \le \frac{\mathcal{S}^{p+1}(a^*)_n}{(1-|\varepsilon_n|)^p}\sum_{j=0}^{p-1}\binom{p}{j}|\varepsilon_n|^{p-j} = \left(\frac{(1+|\varepsilon_n|)^p-1}{(1-|\varepsilon_n|)^p}\right)\mathcal{S}^{p+1}(a^*)_n.$$

This means that we have the following estimate on the norm of the perturbed operator:

$$\|\mathcal{H}_\varepsilon^p\| \le \gamma_p(L), \tag{3-4}$$

where we let

$$\gamma_p(L) = \|\mathcal{H}_0^p\| + \frac{(1+L)^p-1}{(1-L)^p}\|\mathcal{S}^{p+1}\|.$$

Now, in order to estimate the value of $\gamma_p(L)$, we resort to [Littmann 2006, Corollary 2], which gives us

$$\|\mathcal{H}_0^p\| = \frac{(2\pi)^m b_m}{m!},$$

where b_m is the maximum of $|B_m(x)|$ when $x \in [0, 1]$, and B_m denotes the m-th Bernoulli polynomial.[1] Therefore,

$$\|\mathcal{H}_0^1\| = \pi, \quad \|\mathcal{H}_0^2\| = \frac{\pi^2}{3}, \quad \|\mathcal{H}_0^3\| = \frac{\pi^3}{9\sqrt{3}}.$$

On the other hand, by Plancherel's theorem it is easy to see that

$$\|\mathcal{S}^p\| = 2\zeta(p).$$

Joining all these data into (3-4), we obtain

$$\begin{aligned}\|\mathcal{H}_\varepsilon^1\| &\le \pi + \left(\frac{L}{1-L}\right)\frac{\pi^2}{3},\\ \|\mathcal{H}_\varepsilon^2\| &\le \frac{\pi^2}{3} + 2\left(\frac{L^2+2L}{(1-L)^2}\right)\zeta(3),\\ \|\mathcal{H}_\varepsilon^3\| &\le \frac{\pi^3}{9\sqrt{3}} + \left(\frac{L^3+3L^2+3L}{(1-L)^3}\right)\frac{\pi^4}{45}.\end{aligned} \tag{3-5}$$

[1] It is worth mentioning that in [Carneiro et al. 2013, Corollary 22] the authors also obtain the same bounds.

3C2. *Norm estimates of the perturbations in the derivative case.* In order to invert the operator $\mathcal{A}_\varepsilon$, we estimate the norm of $\mathcal{P}_\varepsilon = \mathcal{A}_\varepsilon - \mathrm{Id} = (\mathcal{P}_1, \mathcal{P}_1)$, where

$$\begin{aligned} \mathcal{P}_1(a,b)_n &= \sum_{k\in\mathbb{Z}} a_k \cdot (g(n+\varepsilon_n-k)-\delta_{n,k}) + \sum_{k\in\mathbb{Z}} b_k \cdot h(n+\varepsilon_n-k), \\ \mathcal{P}_2(a,b)_n &= \sum_{k\in\mathbb{Z}} a_k \cdot g'(n+\varepsilon_n-k) + \sum_{k\in\mathbb{Z}} b_k \cdot (h'(n+\varepsilon_n-k)-\delta_{n,k}). \end{aligned} \tag{3-6}$$

By a straightforward calculation,

$$\begin{aligned} \mathcal{P}_1(a,b)_n &= (g(\varepsilon_n)-1)a_n + \frac{\sin(\pi\varepsilon_n)^2}{\pi^2}\mathcal{H}^2_\varepsilon(a)_n + h(\varepsilon_n)b_n + \frac{\sin(\pi\varepsilon_n)^2}{\pi^2}\mathcal{H}^1_\varepsilon(b)_n, \\ \mathcal{P}_2(a,b)_n &= g'(\varepsilon_n)a_n + \frac{2\sin(\pi\varepsilon_n)(\pi\varepsilon_n\cos(\pi\varepsilon_n)-\sin(\pi\varepsilon_n))}{\pi^2}\mathcal{H}^3_\varepsilon(a) \\ &\qquad + (h'(\varepsilon_n)-1)b_n + \frac{\sin(\pi\varepsilon_n)(2\pi\varepsilon_n\cos(\pi\varepsilon_n)-\sin(\pi\varepsilon_n))}{\pi^2}\mathcal{H}^2_\varepsilon(b). \end{aligned} \tag{3-7}$$

Thus,

$$\|\mathcal{P}_\varepsilon\| \le \sqrt{2}\max\{|g(L)-1|, |h'(L)-1|, |g'(L)|, |h(L)|\} + \frac{\sin(\pi L)^2}{\pi^2}\|\mathcal{G}_\varepsilon\|,$$

where $\mathcal{G}_\varepsilon = (\mathcal{G}^1_\varepsilon, \mathcal{G}^2_\varepsilon)$ and

$$\begin{aligned} \mathcal{G}^1_\varepsilon(a,b)_n &= \mathcal{H}^2_\varepsilon(a)_n + \mathcal{H}^1_\varepsilon(b)_n, \\ \mathcal{G}^2_\varepsilon(a,b)_n &= \frac{2(\pi\varepsilon_n\cos(\pi\varepsilon_n)-\sin(\pi\varepsilon_n))}{\sin(\pi\varepsilon)}\mathcal{H}^3_\varepsilon(a) + \frac{(2\pi\varepsilon_n\cos(\pi\varepsilon_n)-\sin(\pi\varepsilon_n))}{\sin(\pi\varepsilon)}\mathcal{H}^2_\varepsilon(b). \end{aligned} \tag{3-8}$$

By taking $L < \frac{1}{4}$ and using the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \frac{\|\mathcal{G}_\varepsilon\|^2}{2} &\le \max\{\|\mathcal{H}^1_\varepsilon\|, \|\mathcal{H}^2_\varepsilon\|\}^2 \\ &\qquad + \max\left\{\left(\frac{2(\pi L\cos(\pi L)-\sin(\pi L))}{\sin(\pi L)}\right)^2\|\mathcal{H}^3_\varepsilon\|^2, \left(\frac{(2\pi L\cos(\pi L)-\sin(\pi L))}{\sin(\pi L)}\right)^2\|\mathcal{H}^2_\varepsilon\|^2\right\} \\ &\le \max\{\gamma_1(L)^2, \gamma_2(L)^2\} \\ &\qquad + \max\left\{\left(\frac{2(\pi L\cos(\pi L)-\sin(\pi L))}{\sin(\pi L)}\right)^2\gamma_3(L)^2, \left(\frac{(2\pi L\cos(\pi L)-\sin(\pi L))}{\sin(\pi L)}\right)^2\gamma_2(L)^2\right\}. \end{aligned}$$

We note that we have abused the notation $\|\mathcal{G}_\varepsilon\|$ to denote the operator norm of $\mathcal{G}_\varepsilon$ when defined on $\ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$. One can further check that, for $0 \le L < \frac{1}{4}$,

$$|g(L)-1| < |h'(L)-1|, \quad |h(L)| < |g'(L)|, \quad \gamma_1(L)^2 < \gamma_2(L)^2,$$
$$\left(\frac{2(\pi L\cos(\pi L)-\sin(\pi L))}{\sin(\pi L)}\right)^2\gamma_3(L)^2 < \left(\frac{(2\pi L\cos(\pi L)-\sin(\pi L))}{\sin(\pi L)}\right)^2\gamma_2(L)^2,$$

which means, in turn,

$$\|\mathcal{G}_\varepsilon\| \le \gamma_2(L)\sqrt{2\left(1+\left(\frac{(2\pi L\cos(\pi L)-\sin(\pi L))}{\sin(\pi L)}\right)^2\right)},$$

and directly implies the estimate

$$\|\mathcal{P}_\varepsilon\| \le 1 - \frac{\sin(\pi L)(2\pi L\cos(\pi L) - \sin(\pi L))}{\pi^2 L^2} + \frac{2\sin(\pi L)(\sin(\pi L) - \pi L\cos(\pi L))}{\pi^2 L^3} + \frac{\sin(\pi L)^2}{\pi^2}\left(\frac{\pi^2}{3} + 2\left(\frac{L^2+2L}{(1-L)^2}\right)\zeta(3)\right)\sqrt{2\left(1+\left(\frac{(2\pi L\cos(\pi L)-\sin(\pi L))}{\sin(\pi L)}\right)^2\right)}.$$

By evaluating the last expression on the right-hand side above numerically, we obtain that we can go up to $L < 0.111$ and maintain $\|\mathcal{P}_\varepsilon\| < 1$. By invoking again [Young 1980, Theorem 1.13], we see immediately that

$$\sum_{n\in\mathbb{Z}} (|f(n+\varepsilon_n)|^2 + |f'(n+\varepsilon_n)|^2)$$

yields an equivalent norm for $\mathrm{PW}_{2\pi}$, as long as $\sup_n |\varepsilon_n| < 0.111$.

Moreover, as $\mathcal{A}_\varepsilon^{-1} : \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$ is bounded, the same argument as in the proof of Theorem 1.1 shows that there are $\varrho_{k,n},\ \vartheta_{k,n},\ \varrho'_{k,n},\ \vartheta'_{k,n}$ such that

$$\begin{aligned} f(k) &= \sum_{n\in\mathbb{Z}} f(n+\varepsilon_n)\varrho_{k,n} + f'(n+\varepsilon_n)\vartheta_{k,n}, \\ f'(k) &= \sum_{n\in\mathbb{Z}} f(n+\varepsilon_n)\varrho'_{k,n} + f'(n+\varepsilon_n)\vartheta'_{k,n}, \end{aligned} \tag{3-9}$$

and

$$\sup_n \left(\sum_{k\in\mathbb{Z}} \{|\varrho_{k,n}|^2 + |\vartheta_{k,n}|^2 + |\varrho'_{k,n}|^2 + |\vartheta'_{k,n}|^2\}\right) \lesssim 1.$$

By using the adjoint $(\mathcal{A}_\varepsilon^{-1})^* : \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$ in an analogous manner to that of the proof of Theorem 1.1 together with (3-9) and (1-7), we obtain the asserted existence of the functions $g_n, h_n \in \mathrm{PW}_{2\pi}$ so that

$$f(x) = \sum_{n\in\mathbb{Z}} f(n+\varepsilon_n)g_n(x) + f'(n+\varepsilon_n)h_n(x),$$

where the right-hand side converges absolutely, as desired. This proves the desired version of Vaaler's interpolation formula with perturbed nodes, given in Theorem 1.3.

4. Perturbed Fourier interpolation on the real line

4A. ***Improved estimates on the interpolation basis.*** As our goal is to obtain versions of the formula

$$f(x) = \sum_{n\ge 0} [f(\sqrt{n})a_n(x) + \hat{f}(\sqrt{n})\hat{a}_n(x)]$$

with perturbed nodes $\sqrt{k+\varepsilon_k}$ deviating from $\sqrt{k}$ as much as possible, and in order to run our argument of estimating the operator norm of a perturbation of the identity, we will need better decay estimates for the interpolating functions a_n than the ones readily available in the literature. In [Radchenko and Viazovska 2019, Section 5], the authors prove that a_n/n^2 is uniformly bounded in $n \ge 0,\ x \in \mathbb{R}$. In order to be able to make the perturbations larger, we need to improve that result substantially, as even the refined bound

$|a_n| = \mathcal{O}(n^{1/4}\log^3(1+n))$ from [Bondarenko et al. 2023] does not seem to be enough for our purposes. This first subsection is, therefore, devoted to the proof of Theorem 1.6.

A tool of major importance in our proof is the Fourier characterization of Gelfand–Shilov spaces. These are spaces where, in a nutshell, both the function and Fourier transform decay as fast as the negative exponential of a certain monomial. Several results connect these spaces with specific decay for both the function and its Fourier transform. See, e.g., [Chung et al. 1996, Theorem 2.3] for more details.

In what follows, we will use the idea behind the characterization described in [Chung et al. 1996]: from bounds for certain L^2-norms of derivatives of f and $\hat{f}$, we run an optimization procedure to obtain decay bounds in both space and frequency. This will be achieved through careful estimates involving the reproducing functions of the interpolation basis $\{a_n\}_{n\geq 0}$, which joins elements of classical analysis and estimates for modular forms.

Indeed, let $\varepsilon \in \{\pm\}$ be a sign. In [Radchenko and Viazovska 2019], the authors consider the generating functions

$$\sum_{n=0}^{\infty} g_n^{\varepsilon}(z) e^{i\pi n\tau} =: K_{\varepsilon}(\tau, z), \tag{4-1}$$

where g_n^{ε} are weakly holomorphic modular forms of weight $\frac{3}{2}$ with growth and coefficient properties so that the functions

$$b_n^{\varepsilon}(x) = \frac{1}{2}\int_{-1}^{1} g_n^{\varepsilon}(z) e^{i\pi x^2 z}\,\mathrm{d}z$$

are eigenvectors of the Fourier transform associated to the eigenvalues ε satisfying that $b_n^{\pm} = a_n \pm \hat{a}_n$ for $\{a_n\}_{n\geq 0}$ defined as in (1-3).

These functions satisfy (see [Radchenko and Viazovska 2019, Proposition 1])

$$\begin{gathered} b_m^{\varepsilon}(\sqrt{n}) = \delta_{n,m} \quad \text{if } n \geq 1, m \geq 0, \\ b_m^{+}(0) = \delta_{m,0} \quad \text{if } m \geq 0, \\ b_0^{-} = 0, \quad b_0^{+}(\sqrt{n}) = \delta_{n,0} \quad \text{if } n \geq 0, \\ b_m^{-}(0) = -2 \quad \text{if } m = k^2 \text{ for some } k \in \mathbb{Z}_{\geq 1}, \\ b_m^{-}(0) = 0 \quad \text{otherwise}. \end{gathered} \tag{4-2}$$

Moreover, we mention for completeness the following result regarding K_ε. We refer the reader to [Radchenko and Viazovska 2019] for its proof.

Proposition 4.1 [Radchenko and Viazovska 2019, Theorem 3]. *For any fixed $z \in \mathbb{H}$, there is $y_0 > 0$ so that for all $\tau \in \mathbb{H}$ with* $\mathrm{Im}(\tau) > y_0$, *the series on the left-hand side of* (4-1) *converges. Under these assumptions, we have the following equalities for the kernels*:

$$\begin{aligned} K_{+}(\tau, z) &= \frac{\theta(\tau)(1-2\lambda(\tau))\theta(z)^3 J(z)}{J(z)-J(\tau)}, \\ K_{-}(\tau, z) &= \frac{\theta(\tau)J(\tau)\theta(z)^3(1-2\lambda(z))}{J(z)-J(\tau)}, \end{aligned} \tag{4-3}$$

where θ, J and λ are as previously defined. In particular, $K_\varepsilon(\tau, z)$ are meromorphic functions with poles at $\tau \in \Gamma_\theta z$.

The authors then define the natural candidate for the generating function for the $\{b_n^\varepsilon\}_{n\geq 0}$ to be

$$F_\varepsilon(\tau,x) = \frac{1}{2}\int_{-1}^{1} K_\varepsilon(\tau,z)e^{i\pi x^2 z}\,\mathrm{d}z, \tag{4-4}$$

where the contour is the semicircle in the upper half-plane that passes through -1 and 1, which is defined, a priori, for each fixed $x\in\mathbb{R}$ and $\tau\in\{z\in\mathbb{H} : \text{for all } k\in\mathbb{Z},\ |z-2k|>1\}\supset\mathcal{D}+2\mathbb{Z}$, where $\mathcal{D}$ is the standard fundamental domain for Γ_θ. By Proposition 4.1, there holds that, whenever $\operatorname{Im}(\tau)>1$,

$$F_\varepsilon(\tau,x) = \sum_{n=0}^{\infty} b_n^\varepsilon(x)e^{i\pi n\tau}. \tag{4-5}$$

As $F_\varepsilon(\tau,x)$ admits an analytic continuation to $\mathbb{H}$ (see [Radchenko and Viazovska 2019, Proposition 2]), they are able to extend (4-5) to the entire upper half-space $\mathbb{H}$. Moreover, the following functional equations hold:

$$F_\varepsilon(\tau,x) - F_\varepsilon(\tau+2,x) = 0,$$
$$F_\varepsilon(\tau,x) + \varepsilon(-i\tau)^{-1/2}F_\varepsilon\Big(-\frac{1}{\tau},x\Big) = e^{i\pi\tau x^2} + \varepsilon(-i\tau)^{-1/2}e^{i\pi(-1/\tau)x^2}.$$

The proof of Theorem 1.6 follows the same essential philosophy as the proof of [Radchenko and Viazovska 2019, Theorem 4]: in order to bound each of the terms $b_n^\pm$, we bound, uniformly on $x\in\mathbb{R}$, the analytic function $F_\pm(\tau,x)$. Relating the two bounds is achieved by employing the idea behind the proof of the following lemma, originally attributed to Hecke (see for instance [Radchenko and Viazovska 2019, Lemma 1] and [Berndt and Knopp 2008, Lemma 2.2(ii)] for a proof).

Lemma 4.2. *Let $f:\mathbb{H}\to\mathbb{C}$ be a 2-periodic analytic function admitting an absolutely convergent Fourier expansion*

$$f(\tau) = \sum_{n\geq 0} c_n e^{i\pi n\tau}.$$

Suppose, additionally, that for some $\alpha>0$ it satisfies that $|f(\tau)|\leq C\operatorname{Im}(\tau)^{-\alpha}$ for $\operatorname{Im}(\tau)<c_0$. Then there is $\widetilde{C}>0$, depending only on C and α, such that for all $n>1/c_0$

$$|c_n| \leq \widetilde{C}n^\alpha.$$

Moreover, there is $C'>0$, depending only on C and α, such that if $n>\alpha/(\pi c_0)$, the improved estimate

$$|c_n| \leq C'\Big(\frac{e\pi}{\alpha}\Big)^\alpha n^\alpha$$

holds.

Before proving Theorem 1.6, we need one more crucial tool in our analysis. Indeed, we consider the functions

$$F_\varepsilon^k(\tau,x) := x^k F_\varepsilon(\tau,x).$$

By Lemma 4.2, if we prove that, for some $\Delta>0$,

$$|F_\varepsilon^k(\tau,x)| \leq C^k(k!)\operatorname{Im}(\tau)^{-k/2-\Delta} \tag{4-6}$$

for all $k \geq 1$, then we will have

$$\sup_{x\in\mathbb{R}} |x^k b_n^\varepsilon(x)| \leq \widetilde{C}^k n^\Delta n^{k/2}(k!).$$

As $b_n^\varepsilon = \varepsilon \hat{b}_n$, the strategy of relating norms of derivatives with Fourier decay will then imply that each of the functions b_n^ε satisfies

$$|b_n^\varepsilon(x)| \lesssim n^\Delta e^{-\theta|x|/\sqrt{n}},$$

which is the content of Theorem 1.6. Therefore, we focus on proving a suitable version of (4-6). By the functional equation for F_ε, we see that F_ε^k is a 2-periodic function on $\mathbb{H}$ that satisfies the functional equation

$$F_\varepsilon^k(\tau, x) + \varepsilon(-i\tau)^{-1/2} F_\varepsilon^k(-1/\tau, x) = x^k(e^{i\pi\tau x^2} + \varepsilon(-i\tau)^{-1/2} e^{i\pi(-1/\tau)x^2}). \tag{4-7}$$

The strategy, in analogy to that in [Radchenko and Viazovska 2019], is of splitting into cases: if $\tau \in \mathcal{D}$, then estimates for F_ε^k are available *directly* by analytic methods. Otherwise, we need to use (4-7) to obtain the bound (4-6) for all $\tau \in \mathbb{H}$.

More explicitly, we have the following:

Proposition 4.3. *There is a positive constant $C > 0$ such that, for each $k \geq 1$, the inequality*

$$|F_\varepsilon^k(\tau, x)| \leq C^k(k!)(1 + \operatorname{Im}(\tau)^{-k/2})$$

holds, whenever $\tau \in \mathcal{D}$.

This proposition can be directly compared to [Radchenko and Viazovska 2019, Lemma 4]. In fact, it is nothing but a carefully quantified version of it.

Proof of Proposition 4.3. As the proof follows thoroughly the main ideas in Lemma 4 in [Radchenko and Viazovska 2019], we will mainly focus on the points where we have to sharpen bounds.

We see directly from the definition of F_ε^k that we are allowed to consider only values of $\tau \in \mathcal{D}_1 = \mathcal{D} \cap \{\tau \in \mathbb{H} : \operatorname{Re}(\tau) \in (-1, 0)\}$. By subsequent considerations from that reduction, we see that the bound

$$|x^k F_\varepsilon(\tau, x)| \leq 10 \int_\ell |K_\varepsilon(\tau, z)| x^k (e^{-\pi x^2 \operatorname{Im}(\tau)} + |z|^{-1/2} e^{-\pi x^2 \operatorname{Im}(-1/z)}) \, |\mathrm{d}z| \tag{4-8}$$

holds, where ℓ is the path joining i to 1 on the upper half-space, defined to be

$$\ell = \left\{ w \in \mathcal{D} : \operatorname{Re}(J(w)) = \tfrac{1}{64},\ \operatorname{Im}(J(w)) > 0 \right\}. \tag{4-9}$$

An explicit computation gives us that the maximal value of

$$x^k e^{-\pi x^2 \operatorname{Im}(z)}$$

is attained at $x = (k/(2\pi \operatorname{Im}(z)))^{1/2}$. Therefore, as any $z \in \ell$ has norm bounded from above and below by absolute constants, we find that there is $C > 0$ so that

$$|F_\varepsilon^k(\tau, x)| \leq C^{k/2} \cdot \left(\frac{k}{2\pi e}\right)^{k/2} \int_\ell |K_\varepsilon(\tau, z)| \operatorname{Im}(z)^{-k/2} \, |\mathrm{d}z|. \tag{4-10}$$

We have then three regimes to consider:

Case 1: $|\tau - i| < \frac{1}{10}$. Notice that if we prove that the proposition holds for *any* $\tau \in \mathbb{H}$ so that $|\tau - i| = \frac{1}{10}$, we can use the maximum modulus principle on F^k_ε on that circle to conclude that the proposition holds inside as well. Moreover, by the functional equation (4-7), we see that the proposition holds for $\mathcal{A} = \left\{\tau \in \mathbb{H} : |\tau - i| = \frac{1}{10},\ |\tau| \le 1\right\}$ in the case it holds for the image of the circle arc $\mathcal{A}$ under the action of S. But a simple computation shows that $S\mathcal{A}$ is just another circle arc contained (up to endpoints) in $\left\{\tau \in \mathcal{D}_1 : \frac{1}{4} > |\tau - i| > \frac{1}{10}\right\}$. This shows that in order to prove the proposition for this case, it suffices to show it for the other cases.

Case 2: $|\tau - i| > \frac{1}{10}$, $\operatorname{Im}(\tau) > \frac{1}{2}$. For this case, we use the fact that $|K_\varepsilon(\tau, z)| \lesssim |\theta(z)|^3 \lesssim \operatorname{Im}(z)^{-2} e^{-\pi/\operatorname{Im}(z)}$ for $z \in \ell$, $\operatorname{Im}(\tau) > \frac{1}{2}$, with constants independent of τ. Using this bound in (4-8) yields

$$|F^k_\varepsilon(\tau, x)| \le (1 + |x|^{k+2}) e^{-c|x|} \lesssim C^k \left(\frac{k+2}{e}\right)^{k+2}$$

for some $C > 0$. Applications of Stirling's formula imply that this bound is controlled by $C_1^k(k!)$, with $C_1 > 0$ an absolute constant. This shows the result in this case.

Case 3: $|\tau - i| > \frac{1}{10}$, $\operatorname{Im}(\tau) \le \frac{1}{2}$. Again, we resort to the estimates in the proof of Lemma 4 in [Radchenko and Viazovska 2019]: there, the authors prove that

$$|K_+(\tau, z)| \lesssim \operatorname{Im}(\tau)^{-1/2} \frac{|J(\tau)|^{3/8} |J(z)|^{5/8} \operatorname{Im}(z)^{-3/2}}{|J(z) - J(\tau)|},$$
$$|K_-(\tau, z)| \lesssim \operatorname{Im}(\tau)^{-1/2} \frac{|J(\tau)|^{7/8} |J(z)|^{1/8} \operatorname{Im}(z)^{-3/2}}{|J(z) - J(\tau)|}.$$

Due to the not-so-symmetric nature of these bounds, we focus on the one for K_+, and the analysis for K_-, as well as the bounds, will be almost identical, and thus the details will be omitted.

Taking advantage of the explicit structure of the curve we are integrating over (4-9), and the fact that there is an absolute constant $C > 0$ so that $\operatorname{Im}(z)^{-1} \le C \log(1 + |J(z)|)$ and that $z \in \ell \iff J(z) = \frac{1}{64} + it,\ t \in \mathbb{R}$,

$$\begin{aligned}\int_\ell |K_+(\tau, x)| \operatorname{Im}(z)^{-k/2} |\mathrm{d}z| &\le C^{k/2} \operatorname{Im}(\tau)^{-1/2} \int_0^\infty \frac{|J(\tau)|^{3/8} t^{-3/8} \log^{(k-1)/2}(1+t)}{\sqrt{t^2 + |J(\tau)|^2}} \,\mathrm{d}t \\ &= C^{k/2} \operatorname{Im}(\tau)^{-1/2} \int_0^\infty \frac{t^{-3/8} \log^{(k-1)/2}(1 + t|J(\tau)|)}{\sqrt{1+t^2}} \,\mathrm{d}t. \end{aligned} \tag{4-11}$$

Now, the last integral in (4-11) can be estimated as follows: if $k - 1$ is even, by using that $\log(1 + ab) \le \log(1 + a) + \log(1 + b)$ whenever $a, b > 0$, the integral

$$\int_0^\infty \frac{t^{-3/8} \log^{(k-1)/2}(1 + t|J(\tau)|)}{\sqrt{1+t^2}} \,\mathrm{d}t$$

is bounded by

$$\sum_{i=0}^{(k-1)/2} \binom{(k-1)/2}{i} \log^i(1 + |J(\tau)|) \int_0^\infty \frac{t^{-3/8} \log^{(k-1)/2-i}(1+t)}{\sqrt{1+t^2}} \,\mathrm{d}t. \tag{4-12}$$

Each summand above can be easily estimated. Indeed, $\binom{(k-1)/2}{i} \le 2^{k/2}$ trivially, $\log^i(1+|J(\tau)|) \le C^i \operatorname{Im}(\tau)^{-i}$, and the integrals can be explicitly bounded in terms of gamma functions. In fact, we first split the integrals in question as

$$\left(\int_0^1 + \int_1^\infty\right) \frac{t^{-3/8}\log^{(k-1)/2-i}(1+t)}{\sqrt{1+t^2}}\,\mathrm{d}t.$$

For the first part, we simply bound the integrand by $t^{-3/8}\log(2)^{(k-1)/2-i}$, and this yields a bound uniform in k. For the second, we change variables $\log(1+t) \mapsto s$ in (4-12) above. A simple computation shows that it is bounded by

$$10\int_0^\infty e^{-3s/8} s^{(k-1)/2-i}\,\mathrm{d}s \lesssim C^k \int_0^\infty e^{-r} r^{(k-1)/2-i}\,\mathrm{d}r = C^k\Gamma\left(\frac{k-1}{2} - i + 1\right).$$

Thus, (4-12) is bounded by

$$C^k \operatorname{Im}(\tau)^{(1-k)/2}\Gamma\left(\frac{k-1}{2}\right).$$

Putting together the estimates in (4-11) and (4-10) and using Stirling's formula for the approximation of Γ, we conclude that

$$|F_\varepsilon^k(\tau,x)| \le C^k (k!) \operatorname{Im}(\tau)^{-k/2},$$

which was the content of the proposition when k is odd. In the case where k is an even number, the fact that $F_\varepsilon^j(\tau,x)^2 = F_\varepsilon^{j-1}(\tau,x)F_\varepsilon^{j+1}(\tau,x)$ allows one to use the bounds of the case where k is odd to conclude the proof. □

We are now finally able to finish the proof of Theorem 1.6.

Proof of Theorem 1.6. We first notice that F_ε^k is 2-periodic, so we lose no generality in assuming that $\tau \in \{z \in \mathbb{H} : \operatorname{Re}(z) \in [-1,1]\} = S_1$. If $\operatorname{Re}(\tau) \in [-1,1]$, then we have two cases:

Case 1: If $\tau \in \mathcal{D}$, we can use Proposition 4.3 directly, and the decay obtained by the assertion of the proposition remains unchanged.

Case 2: If $\tau \in S_1 \backslash \mathcal{D}$, the strategy is to use (4-7) to reduce it to the previous case. In fact, we define the Γ_θ-cocycle $\{\phi_A^k\}_{A\in\Gamma_\theta}$ by

$$\phi_{T^2}^k(\tau,x) = 0,$$
$$\phi_S^k(\tau,x) = x^k\big(e^{i\pi x^2\tau} + \varepsilon(-i\tau)^{-1/2} e^{i\pi x^2(-1/\tau)}\big),$$

together with the cocycle relation

$$\phi_{AB}^k = \phi_A^k + \phi_A^k|B. \tag{4-13}$$

For a fixed $\tau \in S_1 \setminus \mathcal{D}$, we associate $\tau' \in \mathcal{D}$ through the following process: Let

$$\begin{cases} \gamma_0 = \tau, \\ \gamma_i = -1/(\gamma_{i-1}) - 2n_i, \end{cases} \tag{4-14}$$

where $n_i = \lfloor \frac{1}{2}((-1/\gamma_{i-1}) + 1) \rfloor$. We define $m = m(\tau)$ to be the smallest positive integer so that $\gamma_m \in \mathcal{D}$. In this case, we let $\gamma_{m(\tau)} =: \tau'$. In other words, we have that the sequence

$$\begin{cases} \tau_0 = \tau', \\ \tau_{i+1} = -1/\tau_i + 2n_i \end{cases} \tag{4-15}$$

satisfies the hypotheses of Lemma 3 in [Radchenko and Viazovska 2019]. We therefore have that $|\tau_j| > 1$, $\mathrm{Im}(\tau_j)$ is nonincreasing and $\mathrm{Im}(\tau_j) \le 1/(2j-1)$. An inductive procedure shows us that

$$\gamma_{m-i} = -\frac{1}{\tau_i}.$$

In particular, the sequence $\{\tau_i\}_{i\ge 0}$ is in fact finite, with at most $m(\tau)$ terms. This implies that

$$m + 1 \le 4m - 2 \le 2\,\mathrm{Im}(\tau)^{-1}. \tag{4-16}$$

We will use (4-16) in the following computation with the cocycle condition. We write $\tau' = A\tau$, where $A \in \Gamma_\theta$ is of the form

$$A = ST^{2n_m}ST^{2n_{m-1}}S \cdots T^{2n_1}S.$$

As $\{\phi_A^k\}_{A\in\Gamma_\theta}$ satisfies the cocycle condition (4-13), the proof of Lemma 3 in [Radchenko and Viazovska 2019] gives us that

$$\mathrm{Im}(\tau')^{1/4}|\phi_A^k(\tau')| \le \sum_{j=1}^{m} \mathrm{Im}(\tau_j)^{1/4}|\phi_S^k(\tau_j)|.$$

By the definition of ϕ_S^k, we see that

$$|\phi_S^k(\tau_j, x)| \le C\Gamma\left(\frac{k+1}{2}\right)(\mathrm{Im}(\tau_j)^{-k/2} + |\tau_j|^{-1/2}\,\mathrm{Im}(-1/\tau_j)^{-k/2}). \tag{4-17}$$

As $\gamma_{m-i} = -1/\tau_i = \tau_{i+1} - 2n_i,\ |\tau_j| > 1$, and the sequence $\mathrm{Im}(\tau_j)$ is nonincreasing, the right-hand side of (4-17) is bounded from above by $C \cdot \Gamma((k+1)/2)\,\mathrm{Im}(\tau)^{-k/2}$. From (4-16), it follows that

$$|\phi_A^k(\tau')|\,\mathrm{Im}(\tau')^{1/4} \le C\Gamma\left(\frac{k+1}{2}\right)\mathrm{Im}(\tau)^{-k/2}\left(\sum_{j=1}^{m}\mathrm{Im}(\tau_j)^{1/4}\right).$$

If we use the aforementioned facts about $\mathrm{Im}(\tau_j)$, we will see that, in fact,

$$|\phi_A^k(\tau')|\,\mathrm{Im}(\tau')^{1/4} \le C\Gamma\left(\frac{k+1}{2}\right)\mathrm{Im}(\tau)^{-k/2}m(\tau)^{3/4}. \tag{4-18}$$

Now, using the functional equation for F_ε^k implies

$$F_\varepsilon^k - (F_\varepsilon^k)|A = \phi_A^k,$$

which then gives us

$$|F_\varepsilon^k(\tau, x)|\,|\mathrm{Im}(\tau)|^{1/4} \le |\mathrm{Im}(\tau')|^{1/4}|F_\varepsilon^k(\tau', x)| + |\phi_A^k(\tau', x)|\,|\mathrm{Im}(\tau')|^{1/4}.$$

Defining $\mathrm{Im}(\tau') =: I(\tau)$ and using Proposition 4.3 and (4-18) to estimate this expression, it follows that

$$|F_\varepsilon^k(\tau, x)| \le \mathrm{Im}(\tau)^{-k/2-1/4}\big(C^k(k!)\cdot I(\tau)^{1/4} + \Gamma((k+1)/2)m(\tau)^{3/4}\big). \tag{4-19}$$

In order to estimate (4-19), we must resort not only to the general idea of obtaining bounds for Fourier coefficients based on decay at infinity, as in Lemma 4.2, but also to the following estimate of the average values of $m(\tau)$ and $I(\tau)$, recently available by the work of Bondarenko, Radchenko and Seip. We refer the reader to Propositions 6.6 and 6.7 in [Bondarenko et al. 2023] for a proof.

Lemma 4.4. *Whenever $y \in \left(0, \frac{1}{2}\right)$, we have*

$$\int_{-1}^{1} I(x+iy)^{1/4} \lesssim 1 \quad \textit{and} \quad \int_{-1}^{1} m(x+iy)^{3/4} \lesssim \log^3(1+y^{-1}).$$

An application of Lemma 4.4 together with the bound (4-19) to the proof of the first bound in Lemma 4.2 implies

$$\sup_{x\in\mathbb{R}} |x^k b_n^{\pm}(x)| \lesssim C^k n^{1/4} n^{k/2} \log^3(1+n)(k!) \tag{4-20}$$

for $n > 1/c_0,\ k \geq 1$. Also, in the case $n \geq k/(\pi c_0)$, the sharper bound

$$\sup_{x\in\mathbb{R}} |x^k b_n^{\pm}(x)| \lesssim (C')^k n^{1/4} n^{k/2} \log^3(1+n)(k!)^{1/2} \tag{4-21}$$

holds instead. We now proceed to optimize in $k > 0$, completing the outline devised in the beginning of this section.

Indeed, let us start by optimizing (4-20). We postpone the discussion on the improved bound (4-21) to a later remark.

Notice that we may assume $|x| \geq C'\sqrt{n}$, as for if $|x| < C'\sqrt{n}$, the bound (4-20) with $k = 0$ gives us already the result, as $1 \lesssim_c e^{-c|x|/\sqrt{n}}$. If we then set $k = |x|/C'\sqrt{n}$, where $C' > 0$ will be a fixed positive constant, whose exact value shall be determined later, we have that

$$|b_n^{\pm}(x)| \lesssim n^{1/4} \log^3(1+n) \cdot \exp(k \log(Cn^{1/2}) + k\log(k) - k\log|x|).$$

The exponential term above is

$$\exp\left(\frac{|x|}{C'\sqrt{n}} \log(Cn^{1/2}) + \frac{|x|}{C'\sqrt{n}}(\log(|x|) - \log(C'\sqrt{n})) - \frac{|x|}{C'\sqrt{n}} \log|x|\right) = \exp\left(\frac{|x|}{C'\sqrt{n}} \log\left(\frac{C}{C'}\right)\right).$$

We only need to set $C' \geq 2C$ above, and this quantity will grow like $\exp(-c|x|/\sqrt{n})$. This finishes the first assertion in Theorem 1.6.

For the second one, we notice that the proof above adapts in many instances. Indeed, if we shift our attention to the function $\partial_x F_\varepsilon^k(\tau, x)$ instead, we will see that, in an almost identical fashion to that of the proof of Proposition 4.3, we are able to prove that, for all $\tau \in \mathcal{D}$,

$$|\partial_x F_\varepsilon^k(\tau, x)| \lesssim C^k (k!) \operatorname{Im}(\tau)^{-(k+1)/2}.$$

On the other hand, the partial derivative ∂_x of the cocycle $\{\phi_A^k\}_{A\in\Gamma_\theta}$ is itself a cocycle with respect to the same slash operator. Moreover, for $A = S$, the following formula holds:

$$\partial_x \phi_S^k(\tau, x) = (2\pi i) x^{k+1} (\tau e^{\pi i x^2 \tau} + i\varepsilon(-i\tau)^{-3/2} e^{\pi i x^2(-1/\tau)}).$$

In that case, using the notation from above for the elements τ', $\tau_j \in \mathbb{H}$ associated to $\tau \in \mathbb{H} \cap \{|z| \le 1\}$, we see

$$\operatorname{Im}(\tau')^{1/4} |\partial_x \phi_A^k(\tau')| \le \operatorname{Im}(\tau')^{1/4} |\partial_x \phi_S^k(\tau')| + \sum_{j=1}^{m} \operatorname{Im}(\tau_j)^{1/4} |\partial_x \phi_A^k(\tau_j)|.$$

For $j \in \{0, 1, 2, \ldots, m\}$, the definition of our new cocycle implies

$$\begin{aligned} |\partial_x \phi_S^k(\tau_j, x)| &\lesssim \Gamma\Big(\frac{k+3}{2}\Big)(|\tau_j| \operatorname{Im}(\tau_j)^{-(k+1)/2} + |\tau_j|^{-3/2} \operatorname{Im}(\tau_{j+1})^{-(k+1)/2}) \\ &\le \Gamma\Big(\frac{k+3}{2}\Big) \operatorname{Im}(\tau)^{-(k+1)/2}. \end{aligned}$$

This follows as before from the fact that $\operatorname{Im}(\tau_{j+1}) = \operatorname{Im}(\tau_j)/|\tau_j|^2 \ge \operatorname{Im}(\tau)$ and that $|\tau_j| > 1$. Analyzing the functional equations for $\partial_x F_\varepsilon^k(\tau, x)$ in the same way as before readily gives that

$$|\partial_x F_\varepsilon^k(\tau, x)| \le C^k \operatorname{Im}(\tau)^{-(k+1)/2-1/4}(k!)(I(\tau)^{1/4} + m(\tau)^{3/4}).$$

Lemma 4.4 and the considerations employed for F_ε^k apply almost verbatim here, and thus we conclude

$$|(b_n^{\pm})'(x)| \lesssim n^{3/4} \log^3(1+n) e^{-c|x|/\sqrt{n}},$$

as wished. □

As a consequence of Theorem 1.6, we are able to establish the following bound for the interpolation basis taking into account both decay and zeros.

Corollary 4.5. *Let $\{a_n\}$ be the interpolation sequence of functions from* (1-3). *Then there is $c > 0$ so that*

$$|a_n(x)| \lesssim n^{3/4} \log^3(1+n) \operatorname{dist}(|x|, \sqrt{\mathbb{N}}) e^{-c|x|/\sqrt{n}}$$

for all positive integers $n \in \mathbb{N}$.

Proof. We simply use the fundamental theorem of calculus on the a_n: Without loss of generality, we suppose $x > 0$. We then have

$$\begin{aligned} |a_n(x)| = |a_n(x) - a_n(\sqrt{m}) + \delta_{n,m}| &\le \int_{\sqrt{m}}^{x} |a_n'(x)| \,\mathrm{d}x + \delta_{n,m} \\ &\le n^{3/4} \log^3(1+n) \operatorname{dist}(x, \sqrt{\mathbb{N}}) e^{-c|x|/\sqrt{n}} + \delta_{m,n} \\ &\lesssim n^{3/4} \log^3(1+n) \operatorname{dist}(x, \sqrt{\mathbb{N}}) e^{-c|x|/\sqrt{n}}, \end{aligned}$$

as the $\delta_{m,n}$ factor is only one if $|x| \in [\sqrt{n}, \sqrt{n+1})$, where $1 \lesssim e^{-c|x|/\sqrt{n}}$. □

Remark. Although the exponential bound $n^{1/4} \log^3(1+n) e^{-c|x|/\sqrt{n}}$ suffices for our purposes, below we sketch how to deduce a slightly improved decay for the interpolation basis $\{a_n\}_{n \ge 0}$.

We again wish to optimize (4-21). If we set $k = |x|^2/C'n$, where $C' > 0$ will be chosen soon, we have

$$|b_n^{\pm}(x)| \lesssim n^{1/4} \log^3(1+n) \cdot \exp(k \log(Cn^{1/2}) + k \log(k^{1/2}) - k \log|x|).$$

This bound holds as long as $\pi n \gtrsim k \ge 1$. If instead $k < 1$, that means, $|x| \le \sqrt{C'}\sqrt{n}$, we use the bound in either (4-20) or (4-21) for $k = 0$, which yields $|b_n^{\pm}(x)| \lesssim n^{1/4} \log^3(1+n) \lesssim n^{1/4} \log^3(1+n) e^{-c|x|^2/n}$, for $c > 0$.

On the other hand, in the case $k > 1$, the first exponential term above becomes

$$\exp\biggl(\frac{|x|^2}{C'n}\log(Cn^{1/2}) + \frac{|x|^2}{C'n}(\log(|x|) - \log(\sqrt{C'n})) - \frac{|x|^2}{C'n}\log|x|\biggr) = \exp\biggl(\frac{|x|^2}{C'n}\log\biggl(\frac{C}{\sqrt{C'}}\biggr)\biggr).$$

We only need to set $C' \geq (2C)^2$ above, and this quantity will grow like $\exp(-c|x|^2/n)$.

For the remaining $|x| > \sqrt{C'n}$ case, we need to refine the analysis of the proof of Lemma 4.2 and Theorem 1.6. Indeed, it is easy to see that if $n \in (2^{-j}\alpha, 2^{1-j}\alpha)$, $j \geq 1$, then evaluating the Fourier coefficients of a 2-periodic function $f : \mathbb{H} \to \mathbb{C}$ such that $|f(\tau)| \lesssim \operatorname{Im}(\tau)^{-\alpha}(I(\tau)^{1/4} + m(\tau)^{3/4})$ for $\operatorname{Im}(\tau) \leq 1$ as

$$2c_n = \int_{-1+i\alpha/(2^j\pi n)}^{1+i\alpha/(2^j\pi n)} f(\tau)e^{-\pi i n\tau}\,\mathrm{d}\tau$$

implies

$$|c_n| \lesssim \biggl(\frac{2^j\pi e^{1/2^j}}{\alpha}\biggr)^{\alpha} n^{\alpha}\log^3(1+n).$$

Using this new bound in (4-19), we obtain that, when $n \in (2^{-j-1}k, 2^{-j}k)$,

$$|b_n^{\pm}(x)| \lesssim n^{1/4}\log^3(1+n)\cdot\exp\bigl(k(j/2 + \log(C\sqrt{n}) + \log(k^{1/2}) - \log|x|)\bigr).$$

This suggests that we take $k = |x|^2/C'2^jn$, which is admissible to the condition $n \in (2^{-j-1}k, 2^{-j}k)$ if $|x| \sim \sqrt{C'}2^jn$. A similar computation to the ones above implies that

$$|b_n^{\pm}(x)| \lesssim n^{1/4}\log^3(1+n)\exp\biggl(-c\frac{|x|^2}{2^jn}\biggr) \lesssim n^{1/4}\log^3(1+n)\exp(-c'|x|),$$

whenever $C' \gg C$. The next corollary then follows as a natural consequence.

Corollary 4.6. *Let $a_n : \mathbb{R} \to \mathbb{R}$ be the interpolating functions in the Radchenko–Viazovska interpolation formula. Then there are $c, C > 0$ so that*

$$|a_n(x)| \lesssim n^{1/4}\log^3(1+n)(e^{-c|x|^2/n}1_{|x|<Cn} + e^{-c|x|}1_{|x|>Cn})$$

for each $n \geq 1$.

Indeed, the application of Lemma 4.2 requires that we take $n \geq C$ for $C > 0$ some absolute constant. In order to prove such a result for $n \lesssim 1$, we may simply use the definition of $b_n^{\pm}$ as a Laplace transform of a the weakly holomorphic modular form $g_n^{\pm}$. Indeed, in order to extend Corollary 4.6 to $n = 0$, we write

$$a_0(x) = \hat{a}_0(x) = \frac{1}{4}\int_{-1}^{1}\theta(z)^3\,e^{\pi i x^2 z}\,\mathrm{d}z.$$

In order to prove that a_0 decays exponentially, we employ a similar technique to that of [Radchenko and Viazovska 2019, Proposition 1]. Indeed, we have

$$|\theta(z)|^3 \lesssim \operatorname{Im}(z)^{-2}\,e^{-\pi/\operatorname{Im}(z)} \quad \text{for } z \to \pm 1,$$

and moreover that $|\theta(z)| \lesssim 1$ whenever $z \in \mathbb{H},\ |z| = 1$. We also suppose without loss of generality that $x > 0$. This implies that, for $\delta > 0$,

$$|a_0(x)| \lesssim \int_0^\delta \frac{e^{-1/(2t)}}{t^2}\,\mathrm{d}t + e^{-\pi x^2 \delta} \lesssim e^{-1/(2\delta)} + e^{-\pi x^2 \delta}.$$

We then choose, for $x \gg 1,\ \delta = 1/(\sqrt{2\pi}x)$. This implies that $|a_0(x)| \lesssim e^{-(\sqrt{\pi/2})x}$, which is the desired bound. For other bounded values of n such a proof can be easily adapted.

4B. ***Proof of the main result.*** For this part, we shall use the definitions of $\ell^2_s(\mathbb{Z}_{\geq 0})$ and $\ell^2_s(\mathbb{N})$, as in (2-4) from Section 2. Let then $I : \ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+) \to \ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$ denote the identity operator. Recall the Radchenko–Viazovska interpolation result: for $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ a real-valued function,

$$f(x) = \sum_{n \geq 0} (f(\sqrt{n}) a_n(x) + \hat{f}(\sqrt{n}) \hat{a}_n(x)), \tag{4-22}$$

where $a_n : \mathbb{R} \to \mathbb{R}$ is a sequence of interpolating functions independent of the Schwartz function f. In particular,

$$f(\sqrt{k}) = \sum_{n \geq 0} (f(\sqrt{n}) a_n(\sqrt{k}) + \hat{f}(\sqrt{n}) \hat{a}_n(\sqrt{k})).$$

In fact, for any pair of sequences $(\{x_i\}_i, \{y_i\}_i)$ decaying sufficiently fast and satisfying

$$\sum_{n \in \mathbb{Z}} x_{n^2} = \sum_{n \in \mathbb{Z}} y_{n^2}, \tag{4-23}$$

the function

$$\mathfrak{G}(t) = \mathfrak{G}_{x,y}(t) = \sum_{n \geq 0} (x_n a_n(t) + y_n \hat{a}_n(t)) \tag{4-24}$$

is well-defined and satisfies $\mathfrak{G}(\sqrt{k}) = x_k,\ \widehat{\mathfrak{G}}(\sqrt{k}) = y_k$. In fact, let $(\{x_i\}_i, \{y_i\}_i) \in \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$ for $s > 0$ sufficiently large. The operator

$$T : \ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+) \to \ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$$

given by $T = (T^1, T^2)$, where

$$T^1(\{x_i\}, \{y_i\})_k = \sum_{n \geq 0} (x_n a_n(\sqrt{k}) + y_n \hat{a}_n(\sqrt{k})),$$
$$T^2(\{x_i\}, \{y_i\})_k = T^1(\{y_i\}, \{x_i\})_k,$$

has an explicit form as a consequence of (4-2). Indeed, for $k \geq 1$, we have

$$T^1(\{x_i\}, \{y_i\})_k = x_k, \quad T^2(\{x_i\}, \{y_i\}) = y_k,$$

whereas for $k = 0$, we have

$$\begin{aligned} T^1(\{x_i\}, \{y_i\})_0 &= \frac{x_0 + y_0}{2} - \sum_{n \geq 1} x_{n^2} + \sum_{n \geq 1} y_{n^2}, \\ T^2(\{x_i\}, \{y_i\})_0 &= \frac{x_0 + y_0}{2} - \sum_{n \geq 1} y_{n^2} + \sum_{n \geq 1} x_{n^2}. \end{aligned} \tag{4-25}$$

In particular, it is then easy to see that $T = I$ whenever $(\{x_i\}_i, \{y_i\}_i)$ satisfy the relation (4-23). This relation is always satisfied by sequences of the type $x_k = f(\sqrt{k})$ and $y_k = \hat{f}(\sqrt{k})$ because of the Poisson summation formula. Inspired by this fact, we define the perturbed operator associated to a sequence $\varepsilon_k > 0,\ k \in \mathbb{Z}_+$, to be

$$\widetilde{T} \text{ defined on } \ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+),$$

where $\widetilde{T} = (\widetilde{T}^1, \widetilde{T}^2)$, with

$$\begin{aligned}\widetilde{T}^1(\{x_i\}, \{y_i\})_k &= \sum_{n\geq 0}(x_n a_n(\sqrt{k+\varepsilon_k}) + y_n \hat{a}_n(\sqrt{k+\varepsilon_k})),\\ \widetilde{T}^2(\{x_i\}, \{y_i\})_k &= \widetilde{T}^1(\{y_i\}, \{x_i\})_k\end{aligned}$$

for $k \geq 1$, and $\widetilde{T}^1(\{x_i\}, \{y_i\})_0 = x_0$, $\widetilde{T}^2(\{x_i\}, \{y_i\})_0 = y_0$. At first, such an operator might not be defined in the entire space $\ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+)$ because of summability issues, but a way to avoid this trouble is to initially define the operator in the dense subspace of pairs of sequences with finitely many nonzero entries. A posteriori, we will prove the fundamental fact that this operator is *bounded* from $\ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+) \to \ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+)$, which will allow to extend it to the entirety of the space $\ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+)$. One way to see this will be provided in the proof of our main theorem, by showing that the operator norm satisfies $\|I - \widetilde{T}\|_{\ell_s^2(\mathbb{Z}_+)\times\ell_s^2(\mathbb{Z}_+)\to\ell_s^2(\mathbb{Z}_+)\times\ell_s^2(\mathbb{Z}_+)} < +\infty$. This is, incidentally, our main device to prove our result: if

$$\|I - \widetilde{T}\|_{\ell_s^2(\mathbb{Z}_+)\times\ell_s^2(\mathbb{Z}_+)\to\ell_s^2(\mathbb{Z}_+)\times\ell_s^2(\mathbb{Z}_+)} < 1,$$

then $\widetilde{T}$ is an invertible operator defined on $\ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+)$. Therefore, its inverse

$$\widetilde{T}^{-1} : \ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+) \to \ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+)$$

is well-defined and bounded. In particular, for $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ real, given the lists of values

$$\begin{aligned}&f(0),\ f(\sqrt{1+\varepsilon_1}),\ f(\sqrt{2+\varepsilon_2}),\ \ldots,\\ &\hat{f}(0),\ \hat{f}(\sqrt{1+\varepsilon_1}),\ \hat{f}(\sqrt{2+\varepsilon_2}),\ \ldots,\end{aligned}$$

there is a unique pair $(\{x_i\}_i, \{y_i\}_i) \in \ell_s^2(\mathbb{Z}_+) \times \ell_s^2(\mathbb{Z}_+)$ so that

$$\widetilde{T}(\{x_i\}, \{y_i\}) = (\{f(\sqrt{k+\varepsilon_k})\}_k, \{\hat{f}(\sqrt{k+\varepsilon_k})\}_k).$$

But we also know that

$$\widetilde{T}(\{f(\sqrt{i})\}_i, \{\hat{f}(\sqrt{i})\}_i) = T(\{f(\sqrt{i})\}_i, \{\hat{f}(\sqrt{i})\}_i) = \{f(\sqrt{k+\varepsilon_k})\}_k, \{\hat{f}(\sqrt{k+\varepsilon_k})\}_k.$$

This implies $x_j = f(\sqrt{j}),\ y_j = \hat{f}(\sqrt{j})$. By writing the k-th entry of the inverse of $\widetilde{T}$ as

$$\widetilde{T}^{-1}(\{w_i\}, \{z_i\})_k = \sum_{j\geq 0}(\gamma_{j,k} w_j + \hat{\gamma}_{j,k} z_j)$$

for two sequences $\{\gamma_{j,k}\}_{j,k\geq 0}$, $\{\hat{\gamma}_{j,k}\}_{j,k\geq 0}$ so that $|\gamma_{j,k}| + |\hat{\gamma}_{j,k}| \lesssim (j/k)^s$, we must have

$$f(\sqrt{k}) = \sum_{j\geq 0}(\gamma_{j,k} f(\sqrt{j+\varepsilon_j}) + \hat{\gamma}_{j,k}\hat{f}(\sqrt{j+\varepsilon_j})). \tag{4-26}$$

This implies, by (1-3), that we can recover f from its values and those of its Fourier transform at $\sqrt{k+\varepsilon_k}$. Moreover, as the adjoint of $\widetilde{T}^{-1}$ is also bounded from $\ell^2_s(\mathbb{Z}_+)\times\ell^2_s(\mathbb{Z}_+)$ to itself, we conclude that, for $s\gg 1$ sufficiently large and f, $\hat{f}$ both being $\mathcal{O}((1+|x|)^{-10s})$, we can use Fubini's theorem in (1-3) together with (4-26). This proves the existence of two sequences of functions $\{\theta_j\}_{j\ge 0}$, $\{\eta_j\}_{j\ge 0}$ so that

$$|\theta_j(x)|+|\eta_j(x)|+|\hat{\theta}_j(x)|+|\hat{\eta}_j(x)|\lesssim (1+j)^s(1+|x|)^{-10}$$

and

$$f(x)=\sum_{j\ge 0}\big(f(\sqrt{j+\varepsilon_j})\theta_j(x)+\hat{f}(\sqrt{j+\varepsilon_j})\eta_j(x)\big).$$

Thus, we focus on the proof of the invertibility of $\widetilde{T}$ for $s>0$ suitably chosen.

Proof of invertibility of $\widetilde{T}$. We use, for this part, the Schur test. For that, define the auxiliary infinite matrices $A=\{A_{ij}\}_{i,j>0}$ and $\hat{A}=\{\hat{A}_{ij}\}_{i,j>0}$ by

$$A_{ij}=(a_j(\sqrt{i+\varepsilon_i})-\delta_{ij})\times(i/j)^s,$$
$$\hat{A}_{ij}=\hat{a}_j(\sqrt{i+\varepsilon_i})(i/j)^s.$$

For a given vector $(x,y)\in\ell^2(\mathbb{N})\times\ell^2(\mathbb{N})$, we write then

$$B(x,y)=(A\cdot x+\hat{A}\cdot y,\ A\cdot y+\hat{A}\cdot x),$$

or, in matrix notation,

$$B=\begin{pmatrix}A & \hat{A}\\ \hat{A} & A\end{pmatrix}.$$

Furthermore, define the operator $B_0:\mathbb{C}^2\to\ell^2(\mathbb{Z}_{\ge 0})\times\ell^2(\mathbb{Z}_{\ge 0})$ by

$$B_0(r,s)=\Big(\big(r\cdot a_0(\sqrt{k+\varepsilon_k})+s\cdot\hat{a}_0(\sqrt{k+\varepsilon_k})\big)k^s,\ \big(s\cdot a_0(\sqrt{k+\varepsilon_k})+r\cdot\hat{a}_0(\sqrt{k+\varepsilon_k})\big)k^s\Big)_{k\ge 0}.$$

Notice that the operator norm of $\widetilde{T}-I$ acting on $\ell^2_s(\mathbb{Z}_+)\times\ell^2_s(\mathbb{Z}_+)$ is, by virtue of our definitions, bounded by the operator norm of B acting on $\ell^2(\mathbb{N})\times\ell^2(\mathbb{N})$ plus the norm of B_0 acting on $\mathbb{C}^2$, since

$$(\widetilde{T}-I)(x,y)_k=B_0(x_0,y_0)_k+B(x',y')_k,\quad k\ge 1,$$
$$(\widetilde{T}-I)(x,y)_0=(0,0),$$

where

$$(x',y')_n=(x_n,y_n),\quad n>0.$$

First of all, bounds for the operator B_0 are simple to obtain. In fact, by the Cauchy–Schwarz inequality

$$\|B_0(x_0,y_0)\|^2_{\ell^2(\mathbb{N})\times\ell^2(\mathbb{N})}\le 2(x_0^2+y_0^2)\bigg(\sum_{k>0}\{|a_0(\sqrt{k+\varepsilon_k})|^2+|\hat{a}_0(\sqrt{k+\varepsilon_k})|^2\}k^{2s}\bigg).$$

Since $a_0(\sqrt{k})=\hat{a}_0(\sqrt{k})=0$ for $k\ge 1$, and $a_0\in\mathcal{S}(\mathbb{R})$, for any fixed $M>0$ there is $C=C_M>0$ such that

$$\max\{|a_0(\sqrt{k+\varepsilon_k})|,\ |\hat{a}_0(\sqrt{k+\varepsilon_k})|\}\le C_M\frac{|\varepsilon_k|}{k^M}. \tag{4-27}$$

This implies the norm of B_0 is sufficiently small, assuming that we make $\sup_{k\geq 0}|\varepsilon_k|$ sufficiently small, depending on s.

We now turn to bounding the operator norm of B. By Schur's test, it suffices to find $\alpha, \beta > 0$, such that $\sqrt{\alpha\beta} \ll 1$, and positive sequences $\{p_i\}_{i\geq 0}$, $\{q_i\}_{i\geq 0}$ so that the following inequalities hold:

$$\begin{aligned}
&\sum_{j>0}(i/j)^s \times [|a_j(\sqrt{i+\varepsilon_i}) - \delta_{ij}|p_j + |\hat{a}_j(\sqrt{i+\varepsilon_i})|q_j] \leq \alpha p_i,\\
&\sum_{j>0}(i/j)^s \times [|a_j(\sqrt{i+\varepsilon_i}) - \delta_{ij}|q_j + |\hat{a}_j(\sqrt{i+\varepsilon_i})|p_j] \leq \alpha q_i,\\
&\sum_{i>0}(i/j)^s \times [|a_j(\sqrt{i+\varepsilon_i}) - \delta_{ij}|p_i + |\hat{a}_j(\sqrt{i+\varepsilon_i})|q_i] \leq \beta p_j,\\
&\sum_{i>0}(i/j)^s \times [|a_j(\sqrt{i+\varepsilon_i}) - \delta_{ij}|q_i + |\hat{a}_j(\sqrt{i+\varepsilon_i})|p_i] \leq \beta q_j.
\end{aligned} \tag{4-28}$$

Now, we make the ansatz that, for all $i > 0$, $p_i = q_i = i^\theta$, for some real number $\theta \in \mathbb{R}$. By making use of Theorem 1.6, we know that

$$|a_j(\sqrt{i+\varepsilon_i}) - \delta_{ij}| + |\hat{a}_j(\sqrt{i+\varepsilon_i})| \lesssim \frac{\varepsilon_i}{\sqrt{i}} j^{3/4}\log^3(1+j)e^{-c\sqrt{i/j}}.$$

Therefore, (4-28) reduces to verifying

$$\sum_{j>0}(i/j)^s \times j^\theta \times \frac{\varepsilon_i}{\sqrt{i}} j^{3/4}\log^3(1+j)e^{-c\sqrt{i/j}} \leq \alpha i^\theta, \tag{4-29}$$

$$\sum_{i>0}(i/j)^s \times i^\theta \times \frac{\varepsilon_i}{\sqrt{i}} j^{3/4}\log^3(1+j)e^{-c\sqrt{i/j}} \leq \beta j^\theta. \tag{4-30}$$

Estimate of (4-29). For this term, we rewrite it as

$$i^{s-1/2} \times \varepsilon_i \left(\sum_{j>0} j^{3/4-s}\log^3(1+j)e^{-c\sqrt{i/j}} j^\theta\right).$$

In order to estimate this last sum, we break it into $j < i^{1/3}$ and $j > i^{1/3}$ contributions. Therefore,

$$\begin{aligned}
&\sum_{j>0} j^{3/4-s}\log^3(1+j)e^{-c\sqrt{i/j}} j^\theta\\
&\qquad \lesssim i^{1/3} i^{\max(3/4-s+\theta,0)}\log^3(1+i^{1/3})e^{-ci^{1/3}} + \sum_{j>i^{1/3}} j^{3/4-s}\log^3(1+j)e^{-c\sqrt{i/j}} j^\theta.
\end{aligned} \tag{4-31}$$

Because of the presence of the exponential, the first term is always bounded by an absolute constant times i^θ, so we treat it as negligible. For the second term, notice that the summand is bounded by a constant times $\int_j^{j+1} x^{3/4-s+\theta}\log^3(1+x)e^{-c\sqrt{i/x}}\,\mathrm{d}x$. Indeed, the inverse of the ratio between both is bounded from below by

$$\int_j^{j+1}(x/j)^{3/4-s+\theta}\frac{\log^3(1+x)}{\log^3(1+j)}e^{c(\sqrt{i/j}-\sqrt{i/x})}\,\mathrm{d}x \geq \min\left\{\left(1+\frac{1}{j}\right)^{3/4-s+\theta}, 1\right\} \gtrsim_{\theta,s} 1. \tag{4-32}$$

Thus, we obtain that the second term on the right-hand side of (4-31) is bounded by

$$\begin{aligned}\int_{i^{1/3}}^{\infty} x^{3/4-s+\theta}\log^3(1+x)e^{-c\sqrt{i/x}}\,\mathrm{d}x &= \int_0^{i^{-1/3}}\Big(1+\frac{1}{y}\Big)^{3/4-s+\theta}\log^3\Big(1+\frac{1}{y}\Big)y^{-2}e^{-c\sqrt{iy}}\,\mathrm{d}y\\ &\lesssim_{s,\theta}\int_0^{i^{-1/3}} y^{-11/4+s-\theta}\,\log^3\Big(1+\frac{1}{y}\Big)\,e^{-c\sqrt{iy}}\,\mathrm{d}y\\ &= i^{7/4-s+\theta}\int_0^{i^{2/3}} y^{-11/4+s-\theta}\,\log^3\Big(1+\frac{i}{y}\Big)\,e^{-c\sqrt{y}}\,\mathrm{d}y\\ &\lesssim_{s,\theta} i^{7/4-s+\theta}\log^3(1+i),\end{aligned}$$

as long as $-\frac{11}{4}+s-\theta>-1$, that is, $\theta<s-\frac{7}{4}$. Thus, (4-29) is bounded under such a condition by

$$C_{s,\theta}|\varepsilon_i|i^{s-1/2}\log^3(1+i)i^{7/4-s+\theta}=i^{5/4+\theta}\log^3(1+i)|\varepsilon_i|.$$

In order for this last quantity to be less than αi^θ, we must have $|\varepsilon_i|\lesssim_{s,\theta}\alpha i^{-5/4}\log^{-3}(1+i)$. We will assume that we have this bound while estimating the second term.

Estimate of (4-30). For this term, the strategy is similar, only now the estimates become somewhat simpler by the arithmetic of the bounds given by Theorem 1.6. Indeed, (4-30) is bounded by

$$c_{s,\theta}\,j^{3/4-s}\bigg(\sum_{i>0} i^{s+\theta-7/4}\,\log^{-3}(1+i)\,e^{-c\sqrt{i/j}}\bigg).$$

Much as before, each summand above is bounded by $\int_i^{i+1} x^{s+\theta-7/4}\log^{-3}(1+x)e^{-c\sqrt{x/j}}\,\mathrm{d}x$. Thus, the expression within the parenthesis above is bounded by

$$\begin{aligned}\int_1^{\infty} x^{s+\theta-7/4}\,\log^{-3}(1+x)\,e^{-c\sqrt{x/j}}\,\mathrm{d}x &\lesssim_{s,\theta} j^{s+\theta-3/4}\int_0^{\infty} x^{s+\theta-7/4}\,\log^{-3}(1+jx)\,e^{-c\sqrt{x}}\,\mathrm{d}x\\ &\lesssim_{s,\theta} j^{s+\theta-3/4}\int_0^{\infty} x^{s+\theta-7/4}\,\log^{-3}(1+x)\,e^{-c\sqrt{x}}\,\mathrm{d}x.\end{aligned}$$

This last integral converges given that $s+\theta-\frac{7}{4}>-1\iff s+\theta>\frac{3}{4}$. In the end, we obtain that (4-30) is bounded by $c_{s,\theta}\,j^\theta$ if these conditions on s,θ hold.

Finally, we gather these two estimates to get that, if $s-\theta>\frac{7}{4}$, $s+\theta>\frac{3}{4}$ and if $\varepsilon_i<\gamma i^{-5/4}\log^{-3}(1+i)$ for $\gamma>0$ sufficiently small, then (4-29) and (4-30) are bounded by small constants times i^θ and j^θ. Notice that picking $s=10$ and $\theta>0$ sufficiently small yields that both conditions above hold true, and thus the result follows from Schur's test, as previously indicated. □

As mentioned in the beginning of this manuscript, the usage of Schur's test here was instrumental in order to expand the range of our perturbations. In fact, in Section 5A, we employ the Hilbert–Schmidt test successfully to our operator $\widetilde{T}$ and obtain that, as long as there is $\delta>0$ such that $\varepsilon_i\lesssim i^{-5/4-\delta}$, then $\widetilde{T}$ is bounded on $\ell_s^2(\mathbb{N})\times\ell_s^2(\mathbb{N})$ for s sufficiently large, but we seem to be unable to include $\frac{5}{4}$, even with a log-loss, in our considerations with the Hilbert–Schmidt method.

On the other hand, we will see in that subsection that the Hilbert–Schmidt method provides us with a way to suitably perturb the origin, a feature we could not obtain with Schur's test.

5. Applications of the main results and techniques

5A. ***Interpolation formulae perturbing the origin.*** In the main results of this manuscript, the only interpolation node that remains unchanged in every scenario is 0. One of the reasons for that is aesthetic: we are concerned mainly with even functions here, so the origin keeps a sense of symmetry. The other main reason is technical: we recall that the operator

$$T : \ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+) \to \ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$$

defined in Section 4B is the identity only when restricted to the set of pairs of sequences satisfying

$$\sum_{n\in\mathbb{Z}} x_{n^2} = \sum_{n\in\mathbb{Z}} y_{n^2}.$$

For general sequences, the first entries of this operator possess a correction factor due to the lack of Poisson summation. Indeed, the kernel of T is the set of all $(x, y) \in \ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$ such that

$$x_n = y_n = 0 \quad \text{for all } n \geq 1,$$
$$x_0 = -y_0.$$

Furthermore, the cokernel of T is the set where

$$x_0 - y_0 + 2\sum_{n\in\mathbb{N}} x_n^2 - 2\sum_{n\in\mathbb{N}} y_n^2 = 0.$$

This means $\dim(\ker(T)) = \dim(\operatorname{coker}(T)) = 1$. Therefore we can no longer prove invertibility. Nonetheless, since the kernel and cokernel of T are finite-dimensional, T is a Fredholm operator; see the comments on [Brezis 2011, p. 168] for more details.

We denote by $\mathrm{e}_n \in \ell^2_s(\mathbb{Z}_+)$ the vector consisting of $\max\{1, n\}^{-s}$ on the n-th entry, and zero otherwise. With this definition, the set

$$\{(\mathrm{e}_n, \mathbf{0}) : n \in \mathbb{Z}_+\} \cup \{(\mathbf{0}, \mathrm{e}_n) : n \in \mathbb{Z}_+\}$$

forms an orthonormal basis of $\ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$. Thus, for a general operator,

$$\|A\|^2_{HS(\ell^2_s(\mathbb{Z}_+)\times\ell^2_s(\mathbb{Z}_+))} = \sum_{n\geq 0} (\|A(\mathrm{e}_n, \mathbf{0})\|^2_{(s,s)} + \|A(\mathbf{0}, \mathrm{e}_n)\|^2_{(s,s)}),$$

where we denote by $\|\cdot\|_{(s,s)}$ the norm of $\ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$. Next we estimate the Hilbert–Schmidt norm in the case where $A = I - \widetilde{T}$.

Claim 5.1. $\|I - \widetilde{T}\|_{HS(\ell^2_s(\mathbb{Z}_+)\times\ell^2_s(\mathbb{Z}_+))} < +\infty$ *holds whenever there is* $\delta > 0$ *so that* $|\varepsilon_k| \lesssim k^{-5/4-\delta}$ *for all* $k \geq 1$.

Proof of Claim 5.1. As mentioned before, we can write the identity on $\ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$ as

$$I(\{x_i\}, \{y_i\}) = ((x_0, \mathfrak{G}(1), \mathfrak{G}(\sqrt{2}), \dots), (y_0, \widehat{\mathfrak{G}}(1), \widehat{\mathfrak{G}}(\sqrt{2}), \dots)),$$

where we define the function $\mathfrak{G}$ as in (4-24). With this notation, the operator $\widetilde{T}$ becomes

$$\widetilde{T}(\{x_i\}, \{y_i\}) = \big((x_0, \mathfrak{G}(\sqrt{1+\varepsilon_1}), \mathfrak{G}(\sqrt{2+\varepsilon_2}), \dots), (y_0, \widehat{\mathfrak{G}}(\sqrt{1+\varepsilon_1}), \widehat{\mathfrak{G}}(\sqrt{2+\varepsilon_2}), \dots)\big).$$

Therefore, evaluating at the basis vectors gives us that $(I-\widetilde{T})(\mathrm{e}_n,\mathbf{0})$ equals

$$\big((0,\max\{1,n\}^{-s}(a_n(1)-a_n(\sqrt{1+\varepsilon_1})),\max\{1,n\}^{-s}(a_n(\sqrt{2})-a_n(\sqrt{2+\varepsilon_2})),\dots),\\ (0,\max\{1,n\}^{-s}(\hat{a}_n(1)-\hat{a}_n(\sqrt{1+\varepsilon_1})),\max\{1,n\}^{-s}(\hat{a}_n(\sqrt{2})-\hat{a}_n(\sqrt{2+\varepsilon_2})),\dots)\big).$$

We readily see then that

$$\|I-\widetilde{T}\|^2_{HS(\ell^2_s(\mathbb{Z}_+)\times\ell^2_s(\mathbb{Z}_+))}\le 2^{2s+1}\sum_{n\ge0}\bigg(\sum_{k\ge1}(1+k)^{2s}(1+n)^{-2s}|a_n(\sqrt{k})-a_n(\sqrt{k+\varepsilon_k})|^2\bigg)\\ +2^{2s+1}\sum_{n\ge0}\bigg(\sum_{k\ge1}(1+k)^{2s}(1+n)^{-2s}|\hat{a}_n(\sqrt{k})-\hat{a}_n(\sqrt{k+\varepsilon_k})|^2\bigg). \tag{5-1}$$

To bound the terms involving a_0 and $\hat{a}_0$, we simply appeal to the fact these functions are of Schwartz class to use an estimate like (4-27) and obtain

$$\sum_{k\ge1}(1+k)^{2s}\big(|a_0(\sqrt{k+\varepsilon_k})-a_0(\sqrt{k})|^2+|\hat{a}_0(\sqrt{k+\varepsilon_k})-\hat{a}_0(\sqrt{k})|^2\big)\\ =\sum_{k\ge1}(1+k)^{2s}\big(|a_0(\sqrt{k+\varepsilon_k})|^2+|\hat{a}_0(\sqrt{k+\varepsilon_k})|^2\big)\le C_s\sum_{k\ge1}|\varepsilon_k|^2\frac{(1+k)^{2s}}{k^{2s+2}}.$$

From Theorem 1.6, we know that when $n>1$ there is a $c>0$ such that

$$|a_n(\sqrt{k})-a_n(\sqrt{k+\varepsilon_k})|\le\int_{\sqrt{k}}^{\sqrt{k+\varepsilon_k}}|a_n'(t)|\,\mathrm{d}t\le\frac{C\varepsilon_k}{\sqrt{k}}n^{3/4}\,\log^3(1+n)\,e^{-c\sqrt{k/n}} \tag{5-2}$$

for every $k\ge1$. Analogously, for $n>1$,

$$|\hat{a}_n(\sqrt{k})-\hat{a}_n(\sqrt{k+\varepsilon_k})|\le\frac{C\varepsilon_k}{\sqrt{k}}n^{3/4}\,\log^3(1+n)\,e^{-c\sqrt{k/n}}.$$

These estimates plus the condition $|\varepsilon_k|\le ak^{-5/4-\delta}$ for some $a>0$ imply that (5-1) may be bounded from above by a constant that depends on s times

$$a^2\sum_{n\ge1}\bigg(\sum_{k\ge1}k^{2s}k^{-5/2-2\delta}\cdot k^{-1}e^{-2c\sqrt{k/n}}\bigg)n^{3/2-2s}\log^6(1+n)+a^2\sum_{k\ge1}k^{-15/2-2\delta}. \tag{5-3}$$

The second term in the sum above is convergent, so it is not a problem. Now, in order to prove convergence of the first term, we first investigate the inner sum. A Riemann sum approach together with a change of variables shows that the first term in (5-3) is bounded by a constant times

$$(1+n)^{2s-5/2-2\delta}\log^6(1+n)\bigg(\int_0^\infty t^{2s}t^{-5/2-2\delta}\cdot t^{-1}e^{-c\sqrt{t}}\,\mathrm{d}t\bigg)=:(1+n)^{2s-5/2-2\delta}\log^6(1+n)I_{s,\delta}.$$

Clearly, the inner integral converges given that $s>\frac{5}{4}+\delta$. Putting these estimates together with (5-1), we obtain that

$$\|I-\widetilde{T}\|^2_{HS(\ell^2_s(\mathbb{Z}_+)\times\ell^2_s(\mathbb{Z}_+))}\lesssim a^2I_{s,\delta}\bigg(\sum_{n\ge0}(1+n)^{-1-2\delta}\log^6(1+n)\bigg)<+\infty,$$

as desired. □

As a direct corollary, we see that, for each $\delta > 0$, there is $a > 0$ so that, if $|\varepsilon_i| \leq a i^{-5/4-\delta}$ for every $i > 0$, then

$$\|I - \widetilde{T}\|_{HS(\ell^2_s(\mathbb{Z}_+)\times\ell^2_s(\mathbb{Z}_+))} < 1.$$

In particular, we shall make use of the fact that T is a Fredholm operator by means of such an inequality, with the aid of the following result.

Lemma 5.2 [Schechter 1967, Theorems 2.8 and 2.10]. *Let $\Phi(X, Y)$ denote the set of bounded Fredholm operators between Banach spaces X and Y. If $A \in \Phi(X, Y)$ and $K \in \mathcal{K}(X, Y)$ is a compact operator, then $A + K \in \Phi(X, Y)$ and $i(A) = i(A + K)$, where we define the* ***index*** *$i : \Phi(X, Y) \to \mathbb{N}$ by*

$$i(A) = \dim(\ker(A)) - \dim(\operatorname{coker}(A)) =: \alpha(A) - \beta(A).$$

Furthermore, if $\|K\|_{\mathrm{op}}$ is small enough, then it also holds that $\alpha(A + K) \leq \alpha(A)$.

Let us then define a new perturbed operator S, defined on $\ell^2_s(\mathbb{Z}_+) \times \ell^2_s(\mathbb{Z}_+)$, such that

$$S^1(\{x_i\}, \{y_i\})_k = \sum_{n\geq 0}(x_n a_n(\sqrt{k+\varepsilon_k}) + y_n \hat{a}_n(\sqrt{k+\varepsilon_k})),$$
$$S^2(\{x_i\}, \{y_i\})_k = S^1(\{y_i\}, \{x_i\})_k$$

for all $k \geq 0$. Notice that we may write $S - T = \widetilde{T} - I + K_0$, where K_0 has finite rank and is bounded, and thus also compact. Therefore, $S = T + (S - T) = T + (\widetilde{T} - I) + K_0$ can be written as sum of a Fredholm operator T and a compact operator $\widetilde{T} - I + K_0$. This already implies that, modulo a finite-dimensional subspace, the sequences $(\{f(\sqrt{k+\varepsilon_k})\}, \{\hat{f}(\sqrt{k+\varepsilon_k})\})$ determine the sequences $(\{f(\sqrt{k})\}, \{\hat{f}(\sqrt{k})\})$. That is, we can determine the function $f \in \mathcal{S}_{\mathrm{even}}(\mathbb{R})$ from its (Fourier) values at the set $\{\sqrt{k+\varepsilon_k}\}_{k\in\mathbb{Z}_+}$, modulo subtracting functions belonging to a finite-dimensional space.

If, however, we make $|\varepsilon_k| < \epsilon k^{-5/4-\delta}$, and $|\varepsilon_0| \leq \epsilon$, with ϵ small enough, it is now a routine calculation to conclude that the operator norms of both $I - \widetilde{T} = A$ and K_0 can be made bounded by a sum of an arbitrarily small factor plus something that will depend on a convergent series multiplied by the value of $|\varepsilon_0|$, which can made arbitrarily small by choosing ϵ properly. Thus,

$$i(S) = i(T + (S - T)) = i(T) = 0 \iff \alpha(S) = \beta(S),$$

and, moreover,

$$\alpha(S) \leq \alpha(T),$$

as the Hilbert–Schmidt norm of the difference is small. Thus, either

$$\alpha(S) = \beta(S) = 0,$$

in which case we can perfectly invert the operator S, or

$$\alpha(S) = \beta(S) = 1,$$

which implies that there is essentially *at most one* function $f_0 \in \mathcal{S}_{\mathrm{even}}(\mathbb{R})$ that vanishes at $\sqrt{k+\varepsilon_k}$. As $(\{f(\sqrt{k+\varepsilon_k})\}, \{\hat{f}(\sqrt{k+\varepsilon_k})\}) \in \operatorname{Im}(S)$ for every real $f \in \mathcal{S}_{\mathrm{even}}(\mathbb{R})$, we have proved the following result.

Theorem 5.3. *Let T, S, $\{\varepsilon_i\}_{i\geq 0}$ be as above. Then one of the following holds*:

(1) *either S is an isomorphism from $\ell^2_s(\mathbb{Z}_+)\times\ell^2_s(\mathbb{Z}_+)$ onto itself, and thus the values*

$$(\{f(\sqrt{j+\varepsilon_j})\},\{\hat f(\sqrt{j+\varepsilon_j})\})$$

determine any real-valued function $f\in\mathcal{S}_{\text{even}}(\mathbb{R})$,

(2) *or $\ker(S)$ has dimension 1, and therefore S is an isomorphism from $\ker(S)^\perp$ onto $\operatorname{Im}(S)$.*

In particular, any real-valued function $f\in\mathcal{S}_{\text{even}}(\mathbb{R})$ is uniquely determined by

$$(\{f(\sqrt{j+\varepsilon_j})\},\{\hat f(\sqrt{j+\varepsilon_j})\}),$$

together with the value of

$$\frac{\langle(\{f(\sqrt{j+\varepsilon_j})\},\{\hat f(\sqrt{j+\varepsilon_j})\}),(\{\alpha_i\},\{\beta_i\})\rangle_{(s,s)}}{\|(\{\alpha_i\},\{\beta_i\})\|^2_{(s,s)}},$$

where $(\{\alpha_i\},\{\beta_i\})\in\ker(S)$ is a generator for the kernel of S.

Notice that the first option in Theorem 5.3 yields immediately an interpolation formula, in the spirit of (4-26). For the second one, the operator is now only invertible if restricted to $\ker(S)^\perp$, and the process of recovering $f\in\mathcal{S}_{\text{even}}(\mathbb{R}:\mathbb{R})$ has to take into account the inner product with the kernel vector and the structure of the range.

5B. ***Uniqueness for small powers of integers.*** Let $\alpha\in\left(0,\frac{1}{2}\right)$. Bearing in mind the overall framework of uniqueness formulae in which Theorem 1.4 situates itself, we address the question of determining when the only function $f\in\mathcal{S}_{\text{even}}(\mathbb{R})$ that vanishes together with its Fourier transform at $\pm n^\alpha$ is the identically zero function.

Indeed, we would like to study the natural operator that sends the sequence of values at the roots of integers $(\{f(\sqrt{k})\}_k,\{\hat f(\sqrt{k})\}_k\})$ to the sequence $(\{f(n^\alpha)\}_n,\{\hat f(n^\alpha)\}_n)$. Our goal is to show that this operator is injective. In order to do that, we will first study simpler operators.

In fact, let $K_0\in\mathbb{N}$ be a fixed positive integer. Fix a set of $2K_0$ positive real numbers $t_1<t_2<\cdots<t_{2K_0}$ such that $t_1>\sqrt{K_0}$ and none of the t_j can be written as a square root of a positive integer. We fix $s>0$ sufficiently large and define the operator

$$\begin{aligned}T_{K_0}:\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N})&\to\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N}),\\ (\{x_i\}_i,\{y_i\}_i)&\mapsto\big((x_0,\mathfrak{G}(t_1),\mathfrak{G}(t_2),\dots,\mathfrak{G}(t_{2K_0}),x_{K_0+1},x_{K_0+2},\dots),\\ &\qquad(y_0,\widehat{\mathfrak{G}}(t_1),\widehat{\mathfrak{G}}(t_2),\dots,\widehat{\mathfrak{G}}(t_{2K_0}),y_{K_0+1},y_{K_0+2},\dots)\big).\end{aligned}$$

Here, we denoted by $\mathfrak{G}$ the function defined as in (4-24). Recall that $\mathfrak{G}$ depends itself on $\{x_i\}_i$, $\{y_i\}_i$, and thus, for fixed t, $\mathfrak{G}(t)$ and $\widehat{\mathfrak{G}}(t)$ are both linear functionals on $\ell^2_s(\mathbb{Z}_{\geq 0})\times\ell^2_s(\mathbb{Z}_{\geq 0})$.

Lemma 5.4. *For any $K_0\geq 1$ and $\{t_j\}_{j=1,\dots,2K_0}$ as above, the operator T_{K_0} is bounded and injective.*

Proof. We begin with the boundedness assertion. As T_{K_0} differs only in at most the first $2K_0+1$ coordinates from an iteration of the shift operator

$$s((\{x_i\}_i,\{y_i\}_i)=((0,x_0,x_1,\dots),(0,y_0,y_1,\dots)),$$

boundedness follows from boundedness of the operator that maps a pair of sequences $(\{x_i\}_i, \{y_i\}_i) \in \ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N})$ to

$$\big((x_0, \mathfrak{G}(t_1), \mathfrak{G}(t_2), \dots, \mathfrak{G}(t_{2K_0}), 0, \dots), (y_0, \widehat{\mathfrak{G}}(t_1), \widehat{\mathfrak{G}}(t_2), \dots, \widehat{\mathfrak{G}}(t_{2K_0}), 0, \dots)\big).$$

As $\mathfrak{G}, \widehat{\mathfrak{G}} \in L^\infty(\mathbb{R})$ for any pair of sequences $\{x_i\}, \{y_i\}$, with bounds depending only on the $\ell_s^2(\mathbb{N})$-norms of the sequences, it follows that this new finite-rank operator is bounded.

The injectivity part is subtler. Indeed, fix a pair of sequences $(\{x_i\}, \{y_i\}) \in \ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N})$, and suppose that $T_{K_0}(\{x_i\}, \{y_i\}) = 0$. It follows that the special function $\mathfrak{G}(t)$ is a linear combination of $a_1, \dots, a_{K_0}, \hat{a}_1, \dots, \hat{a}_{K_0}$. In order to analyze such functions, we will need to investigate further the intrinsic form of the interpolating functions a_n, and thus those of $b_n^\pm$. As the reader will see in the analysis below, we will show that the functions $\mathfrak{G} \pm \widehat{\mathfrak{G}}$ have at most K_0+1 zeros on $(\sqrt{K_0}, +\infty)$ from the assertions above. This is, indeed, the reason why we need to use $2K_0$ different values in order to prove injectivity.

Indeed, it follows from the Fourier expansion of $g_n^\pm$ near infinity and the formula

$$b_n^\pm(x) = \frac{1}{2}\int_{-1}^{1} g_n^\pm(z) e^{\pi i x^2 z}\,\mathrm{d}z \tag{5-4}$$

that, whenever $|x| > \sqrt{n}$, it can also be represented as

$$b_n^\pm(x) = \sin(\pi x^2)\int_0^\infty g_n^\pm(1+it) e^{-\pi x^2 t}\,\mathrm{d}t. \tag{5-5}$$

To see this, one shifts contours in (5-4) over the rectangular path passing through $-1, -1+iT, 1+iT$ and 1. The condition $|x| > \sqrt{n}$ comes into play in order to guarantee that one may safely send T to ∞, and the results in [Radchenko and Viazovska 2019] show that $g_n^\varepsilon(s+iR)$ grows as $e^{\pi n R}$ at infinity for fixed $s \in \mathbb{R}$. With (5-5) in mind and the facts that $a_n = (b_n^+ + b_n^-)/2$ and $\hat{a}_n = (b_n^+ - b_n^-)/2$, we see that the Fourier invariant part of $\mathfrak{G}$ may be written as

$$(\mathfrak{G} + \widehat{\mathfrak{G}})(x) = \sin(\pi x^2)\int_0^\infty \left(\sum_{j=1}^{K_0} \alpha_j g_j^+(1+it)\right) e^{-\pi x^2 t}\,\mathrm{d}t$$

for some sequence α_j of real numbers, and an analogous identity holds for $\mathfrak{G} - \widehat{\mathfrak{G}}$, with g_n^- instead. We recall that the weakly holomorphic modular forms $g_n^\pm$ satisfy that

$$\begin{aligned} g_n^+(z) &= \theta(z)^3 P_n^+(1/J(z)),\\ g_n^-(z) &= \theta(z)^3(1-2\lambda(z)) P_n^-(1/J(z)), \end{aligned}$$

where the monic polynomials P_n^-, P_n^+ are of degree n. Therefore, there are polynomials Q, R of degree $\le K_0$ such that

$$\begin{aligned} \mathfrak{G} + \widehat{\mathfrak{G}} &= \sin(\pi x^2)\int_0^\infty \theta(1+it)^3 Q\left(\frac{1}{J(1+it)}\right) e^{-\pi x^2 t}\,\mathrm{d}t,\\ \mathfrak{G} - \widehat{\mathfrak{G}} &= \sin(\pi x^2)\int_0^\infty \theta(1+it)^3(1-2\lambda(1+it)) R\left(\frac{1}{J(1+it)}\right) e^{-\pi x^2 t}\,\mathrm{d}t. \end{aligned}$$

Before moving forward, we need the following result:

Lemma 5.5. *The factors* $\theta(1+it)^3$ *and* $(1-2\lambda(1+it))$ *do not change sign for* $t \in (0,\infty)$, *and the function* $1/J(1+it)$ *is real-valued and monotonic for* $t \in (0,\infty)$.

Proof. By using (2-1), we get that

$$\theta(1+it) = \sum_{n\in\mathbb{Z}} (-1)^n e^{-\pi n^2 t} = \sum_{n\in\mathbb{Z}} e^{-4\pi n^2 t} - \sum_{n\in\mathbb{Z}} e^{-\pi(2n+1)^2 t}.$$

We now consider the function $f_t(x) = e^{-\pi(2x)^2 t}$. Then the sum above equals

$$\sum_{n\in\mathbb{Z}} f_t(n) - \sum_{n\in\mathbb{Z}} f_t\big(n+\tfrac{1}{2}\big).$$

By the Poisson summation formula, the difference above equals

$$\frac{1}{2\sqrt{t}}\Bigg(\sum_{n\in\mathbb{Z}} e^{-\pi(n/(2\sqrt{t}))^2} - \sum_{n\in\mathbb{Z}} e^{\pi i n} e^{-\pi(n/(2\sqrt{t}))^2}\Bigg) = \frac{1}{\sqrt{t}} \sum_{n \text{ odd}} e^{-\pi(n/(2\sqrt{t}))^2} \geq 0.$$

This proves the first assertion.

For the second, we simply see from (2-2) that $\lambda(1+z)$ has only nonpositive coefficients in its q-series expansion. This implies that $\lambda(1+it)$ is nonpositive for $t \in (0,\infty)$, which implies that $1-2\lambda(1+it)$ is always nonnegative.

Finally, for the third assertion, we notice that, as $J(1+z) = \frac{1}{16}\lambda(1+z)(1-\lambda(1+z))$, and thus, from the analysis above, the q-series expansion of $J(1+z)$ contains only nonpositive coefficients. Therefore, the function $1/J(1+it)$ is nonpositive for $t \in (0,\infty)$, and it is monotonically decreasing there. □

By Lemma 5.5, we get that the part of the integrand in the expressions above multiplying the $e^{-\pi x^2 t}$ factor changes sign at most K_0+1 times. Notice that we can embed both integrals in (5-6) into the framework of Laplace transforms: defining

$$\mathcal{Q}(t) = \theta(1+it)^3 Q(1/J(1+it)), \quad \mathcal{R}(t) = \theta(1+it)^3(1-2\lambda(1+it))R(1/J(1+it)),$$

we are interested in studying the positive zeros of $\mathcal{L}[\mathcal{Q}](\pi x^2)$, $\mathcal{L}[\mathcal{R}](\pi x^2)$, where

$$\mathcal{L}[\phi](s) = \int_0^\infty \phi(t) e^{-st}\,\mathrm{d}t$$

denotes the Laplace transform of ϕ evaluated at the point s. We may reduce even further our task to studying the positive zeros of $\mathcal{L}[\mathcal{Q}]$, $\mathcal{L}[\mathcal{R}]$. The following result, a version of the Descartes rule for the Laplace transform, is the tool we need to bound the number of positive zeros of such expressions as a function of the number of sign changes of the function being transformed.

Proposition 5.6 (Descartes rule for the Laplace transform). *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be a smooth function such that its Laplace transform* $\mathcal{L}[\phi]$ *converges on some open half-plane* $\mathrm{Re}(s) > s_0$. *Then the number of zeros of* $\mathcal{L}[\phi]$ *on the interval* $(s_0,+\infty)$ *is at most the number of sign changes of* ϕ.

Proof. The proof follows by induction on the number of sign changes of the function ϕ. Indeed, if $\phi \geq 0$, it follows easily that the Laplace transform satisfies $\mathcal{L}[\phi] \geq 0$, with equality if and only if $\phi \equiv 0$.

Suppose now that ϕ changes sign $n+1$ times on $(0, \infty)$. Number its zeros on the positive half-line as $a_0 < a_1 < \cdots < a_n$. Then $\mathcal{L}[\phi]$ has as many zeros as $e^{a_0 s}\mathcal{L}[\phi](s) = F(s)$. The derivative of F is then given by

$$F'(s) = -\int_0^\infty (t-a_0)\phi(t)e^{-(t-a_0)s}\,\mathrm{d}t = e^{a_0 s}\mathcal{L}[(t-a_0)\phi(s)](s).$$

Notice that the new smooth function $(t-a_0)\phi(t)$ still satisfies the same properties as ϕ, but now has exactly n sign changes. By inductive hypothesis, F' has at most n zeros, which, by the mean value theorem, implies that F has at most $n+1$ zeros. □

Using this claim for $\mathcal{Q}, \mathcal{R}$, we see that their respective Laplace transforms possess at most K_0 zeros on the interval $(\sqrt{K_0}, +\infty)$. With this information, we can already finish: From (5-6), the functions $\mathfrak{G} \pm \widehat{\mathfrak{G}}$ can only vanish at at most K_0 points on the interval $(\sqrt{K_0}, \infty)$ which are not roots of positive integers, in the case $\mathfrak{G} \not\equiv 0$. But, according to our assumption that $(\{x_i\}, \{y_i\}) \in \ker(T_{K_0})$, we have $\mathfrak{G}(t_j) = \widehat{\mathfrak{G}}(t_j) = 0,\ j = 1, \ldots, 2K_0$. By the properties we chose for the sequence t_j, $\mathfrak{G} \equiv 0$, and thus the map T_{K_0} is injective. □

We need one more result in order to infer results about uniqueness for small powers of integers. In contrast to the full perturbation case of our main theorem, we must prove that the injective operators T_{K_0} are also somewhat stable with respect to injectivity under perturbations. In order to do this, the following result is essential.

Lemma 5.7. *The range of T_{K_0} is closed.*

Proof. Suppose the sequence in $\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$ given by $\{T_{K_0}(\{x_i^j\}, \{y_i^j\})\}_{j\geq 0}$ is a Cauchy sequence. This implies that the sequence $\{\{x_i^j\}_{i=0,K_0+1,\ldots}, \{y_i^j\}_{i=0,K_0+1,\ldots}\}_{j\geq 0}$ is a Cauchy sequence, and therefore it converges to a certain limiting sequence

$$\{\{x_i\}_{i=0,K_0+1,\ldots}, \{y_i\}_{i=0,K_0+1,\ldots}\} \in \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}).$$

Define, thus, the $4K_0 \times 2K_0$ matrix A_{K_0} given by taking

$$(a_1(t_j), a_2(t_j), \ldots, a_{K_0}(t_j), \hat{a}_1(t_j), \hat{a}_2(t_j), \ldots, \hat{a}_{K_0}(t_j))$$

and

$$(\hat{a}_1(t_j), \hat{a}_2(t_j), \ldots, \hat{a}_{K_0}(t_j), a_1(t_j), a_2(t_j), \ldots, a_{K_0}(t_j))$$

to be its lines for $j = 1, \ldots, 2K_0$. We first claim that this matrix is injective. Indeed,

$$\widetilde{\mathfrak{G}}(t) = \sum_{i=1}^{K_0}(x_i a_i(t) + y_i \hat{a}_i(t))$$

vanishes, together with its Fourier transform, at $t_j,\ j = 1, \ldots, 2K_0$, where $(\{x_i\}_{i=1}^{K_0}, \{y_i\}_{i=1}^{K_0})$ belongs to $\ker(A_{K_0})$. By the proof of Lemma 5.4, this implies $x_i = y_i = 0,\ i = 1, \ldots, K_0$.

As $A_{K_0}$is injective, there is a constant $c_{K_0} > 0$ so that

$$\|A_{K_0}\boldsymbol{v}\|_{4K_0} \geq c_{K_0}\|v\|_{2K_0}, \tag{5-6}$$

where we denote by $\|\cdot\|_d$ the usual euclidean norm on a d-dimensional space. Translating to our original problem, as $\{T_{K_0}(\{x_i^j\},\{y_i^j\})\}_{j\geq 0}$ is a Cauchy sequence in $\ell_s^2(\mathbb{N})\times\ell_s^2(\mathbb{N})$,

$$\{\{x_i^j\}_{i=0,K_0+1,\dots},\{y_i^j\}_{i=0,K_0+1,\dots}\}_{j\geq 0}$$

is a convergent sequence, and thus we get that the sequences

$$\sum_{i=1}^{K_0}(x_i^k a_i(t_j)+y_i^k\hat{a}_i(t_j)),\quad j=1,\dots,2K_0,$$

are also Cauchy in $k\geq 0$. By (5-6), $(\{x_i^k\}_{i=1}^{K_0},\{y_i\}_{i=1}^{K_0})_{k\geq 0}$ is Cauchy. This implies that there is a limiting sequence $(\{x_i\},\{y_i\})\in\ell_s^2(\mathbb{N})\times\ell_s^2(\mathbb{N})$ so that

$$T_{K_0}(\{x_i^j\},\{y_i^j\})\to T_{K_0}(\{x_i\},\{y_i\})\quad\text{as } j\to\infty. \qquad\square$$

We are finally able to prove the following uniqueness result:

Corollary 5.8. *Let $\alpha\in\left(0,\frac{2}{9}\right)$. There exists $c_\alpha>0$ such that the following holds. For each $c\in(0,c_\alpha)$, if $f\in\mathcal{S}_{\text{even}}(\mathbb{R})$ is a real-valued function that vanishes together with its Fourier transform at $\pm c\cdot n^\alpha$, then $f\equiv 0$.*

Moreover, for each $n_0>1$, the same assertion as before holds under the weaker assumption that f vanishes together with $\hat{f}$ at $\pm c\cdot m^\alpha$, where $m\in\{0\}\cup(m_\alpha(n_0),+\infty)$ and $m_\alpha(n_0)=\min\{n\in\mathbb{N}: cn^\alpha>n_0\}$.

Notice that the second assertion above, albeit technical, merely means we may start the sequences of nonzero roots of f, $\hat{f}$ as far away from the origin as we wish, as long as one keeps it under a certain threshold in terms of denseness.

Proof. Fix $\alpha\in\left(0,\frac{2}{9}\right)$ and let $c>0$ be a constant, to be precisely chosen later, which is allowed to depend only on α. We start by noticing that, for each $\alpha\in\left(0,\frac{2}{9}\right)$, there is $n_0(\alpha)\geq 1$ such that whenever $n\in\mathbb{N}$ is greater than $n_0(\alpha)$, then there is $m\in\mathbb{N}$ so that for all $n\geq n_0(\alpha)$, there exists $m\in\mathbb{N}$ so that we can write $c\cdot m^\alpha=\sqrt{n+\varepsilon_n}$, where $\{\varepsilon_n\}_n$ satisfies the conditions of Theorem 1.4. Indeed, start by noticing that simply letting $m=\lceil(n/c^2)^{1/(2\alpha)}\rceil$ implies $|\sqrt{n}-cm^\alpha|\lesssim c^{1/\alpha}n^{(\alpha-1)/(2\alpha)}$.

Indeed,

$$|\sqrt{n}-cm^\alpha|=c\alpha\int_{(n/c^2)^{1/(2\alpha)}}^{\lceil(n/c^2)^{1/(2\alpha)}\rceil}t^{\alpha-1}\,\mathrm{d}t\lesssim c^{1/\alpha}\alpha n^{(\alpha-1)/(2\alpha)}. \tag{5-7}$$

In particular, if $(\alpha-1)/(2\alpha)<-\frac{5}{4}-\frac{1}{2}\iff\alpha<\frac{2}{9}$, the assertion follows. Let us single out the sequence of numbers selected above, which we index as $\{c\cdot m(n)^\alpha\}_{n\geq n_0(\alpha)}$. We then consider the operator $T_{n_0(\alpha)}$ associated to some sequence of $2n_0(\alpha)$ positive real numbers t_j, $j=1,\dots,2n_0(\alpha)$, satisfying the hypotheses of Lemma 5.4.

We claim that the *perturbed* operator

$\widetilde{T}_{n_0(\alpha)}:\ell_s^2(\mathbb{N})\times\ell_s^2(\mathbb{N})\to\ell_s^2(\mathbb{N})\times\ell_s^2(\mathbb{N})$ that takes a pair $(\{x_i\},\{y_i\})$ to

$$\begin{aligned}\big((x_0,\mathfrak{G}(t_1),\mathfrak{G}(t_2),\dots,\mathfrak{G}(t_{2n_0}),\mathfrak{G}(c\cdot m(n_0+1)^\alpha),\mathfrak{G}(c\cdot m(n_0+2)^\alpha),\dots),\\ (y_0,\widehat{\mathfrak{G}}(t_1),\widehat{\mathfrak{G}}(t_2),\dots,\widehat{\mathfrak{G}}(t_{2n_0}),\widehat{\mathfrak{G}}(c\cdot m(n_0+1)^\alpha),\widehat{\mathfrak{G}}(c\cdot m(n_0+2)^\alpha),\dots)\big)\end{aligned} \tag{5-8}$$

is injective for some $s > 0$ that depends on α. Indeed, from Lemma 5.7 there must exist a constant C_{n_0} so that

$$\|T_{n_0}\boldsymbol{v}\|_{(s,s)} \geq C_{n_0}\|\boldsymbol{v}\|_{(s,s)}$$

holds for all $\boldsymbol{v} \in \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$. But, by the same calculation as in the previous subsection, we have that

$$\|\widetilde{T}_{n_0(\alpha)} - T_{n_0(\alpha)}\|_{HS(\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N}))} < \frac{C_{n_0}}{2}$$

holds, as long as we take $\alpha < \frac{2}{9}$ and $c = c(\alpha)$ sufficiently small, because (5-7) implies we satisfy the conditions of Theorem 5.3. This implies, in particular, that

$$\|\widetilde{T}_{n_0}\boldsymbol{v}\|_{(s,s)} \geq \frac{C_{n_0}}{2}\|\boldsymbol{v}\|_{(s,s)}$$

for each $\boldsymbol{v} \in \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$, and thus the operator $\widetilde{T}_{n_0}$ is, indeed, injective, as desired.

In order to conclude, we notice that the operator

$\mathcal{T}_{n_0(\alpha)} : \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \to \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$ that takes a pair $(\{x_i\}, \{y_i\})$ to

$$\begin{aligned}\big((x_0, \mathfrak{G}(ck_1^\alpha), \mathfrak{G}(ck_2^\alpha), \dots, \mathfrak{G}(ck_{2n_0}^\alpha), \mathfrak{G}(c\cdot m(n_0+1)^\alpha), \mathfrak{G}(c\cdot m(n_0+2)^\alpha), \dots),\\ (y_0, \widehat{\mathfrak{G}}(ck_1^\alpha), \widehat{\mathfrak{G}}(ck_2^\alpha), \dots, \widehat{\mathfrak{G}}(ck_{2n_0}^\alpha), \widehat{\mathfrak{G}}(c\cdot m(n_0+1)^\alpha), \widehat{\mathfrak{G}}(c\cdot m(n_0+2)^\alpha), \dots)\big),\end{aligned} \tag{5-9}$$

for some sequence $k_j,\ j = 1, \dots, 2n_0$, of integers not belonging to the sequence $m(n)$ we selected above, is still injective. In fact, it only differs from the operator $\widetilde{T}_{n_0}$ in at most $2n_0$ entries. But, on the other hand, for $k_j = \lfloor (t_j/c)^{1/\alpha}\rfloor,\ j = 1, \dots, 2n_0$, and $c > 0$ sufficiently small, we see by Theorem 1.6 that

$$\begin{aligned}|\mathfrak{G}(ck_j^\alpha) - \mathfrak{G}(t_j)| &\leq \sum_{i=0}^{\infty}(|x_i||a_i(t_j) - a_i(ck_j^\alpha)| + |y_i||\hat{a}_i(t_j) - \hat{a}_i(ck_j^\alpha)|)\\ &\lesssim \sup_{1\leq l\leq 2n_0}|t_l - ck_l^\alpha|\left(\sum_{i=0}^{\infty}(1+i)^{5/2}(|x_i| + |y_i|)\right)\\ &\lesssim \epsilon\|(\{x_i\}, \{y_i\})\|_{(s,s)}.\end{aligned}$$

Here, note that ϵ depends on $c > 0$ and α, and tends to 0 as $c \to 0$. For $\epsilon > 0$ sufficiently small, we see from the previous argument that $\mathcal{T}_{n_0(\alpha)}$ still has closed range and is injective. Thus, by taking $c_\alpha > 0$ sufficiently small we have that the sequence $(\{f(\pm n^\alpha)\}, \{\hat{f}(\pm n^\alpha)\})$ determines uniquely the sequence $(\{f(\sqrt{n})\}, \{\hat{f}(\sqrt{n})\})$. This finishes the proof of the first assertion.

The assertion about being able to restrict the first node $c_\alpha m^\alpha$ to be as large as we want follows in the exact same way, and we thus omit it. □

One can inquire about the importance of such a result; as in [Ramos and Sousa 2022] we have shown that the uniqueness result stated in Corollary 5.8 holds for $\alpha \in \big(0, 1 - \frac{\sqrt{2}}{2}\big)$, which is significantly larger than the range stated here. Nonetheless, Corollary 5.8 gives us *automatic* results. Indeed, if one manages to prove that for all $\delta > 0$ there is $\epsilon > 0$ so that, if $|\varepsilon_k| \leq \epsilon$ for all $k \in \mathbb{N}$, then

$$\|I - \widetilde{T}\|_{\mathrm{op}} < \delta,$$

it implies automatically that we can extend the results in Corollary 5.8 to the full diagonal range $\alpha \in \big(0, \frac{1}{2}\big)$.

We also note that Corollary 5.8 is not all we can say about the problem of determining the best exponents (α, β) so that

$$f(\pm n^{\alpha}) = \hat{f}(\pm n^{\beta}) = 0, \quad f \in \mathcal{S}_{\text{even}}(\mathbb{R}) \implies f \equiv 0.$$

Indeed, we can easily go further than the diagonal case detailed above: if $\alpha, \beta \in \left(0, \frac{2}{9}\right)$ are arbitrary exponents, we notice that we can still pick $n_0 \in \mathbb{N}$ so that for each $n > n_0 = n_0(\alpha, \beta)$, there exists a pair $(m_1(n), m_2(n)) \in \mathbb{N}^2$ so that

$$|cm_1(n)^{\alpha} - \sqrt{n}| + |cm_2(n)^{\beta} - \sqrt{n}| \lesssim c^{1/\alpha}\alpha n^{(\alpha-1)/(2\alpha)} + c^{1/\beta}\beta n^{(\beta-1)/(2\beta)},$$

and the right-hand side can be made $\ll n^{-5/4-\delta}$ for some $\delta > 0$. This induces us to consider the operator

$\mathcal{T}_{n_0(\alpha,\beta)} : \ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N}) \to \ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N})$ taking pairs $(\{x_i\}, \{y_i\})$ to

$$\begin{aligned}\big((x_0, \mathfrak{G}(ck_1^{\alpha}), \mathfrak{G}(ck_2^{\alpha}), \dots, \mathfrak{G}(ck_{2n_0}^{\alpha}), \mathfrak{G}(m_1(n_0+1)^{\alpha}), \mathfrak{G}(m_1(n_0+2)^{\alpha}), \dots),\\ (y_0, \widehat{\mathfrak{G}}(cl_1^{\beta}), \widehat{\mathfrak{G}}(cl_2^{\beta}), \dots, \widehat{\mathfrak{G}}(cl_{2n_0}^{\beta}), \widehat{\mathfrak{G}}(m_2(n_0+1)^{\beta}), \widehat{\mathfrak{G}}(m_2(n_0+2)^{\beta}), \dots)\big)\end{aligned} \tag{5-10}$$

for two sequences of integers $(k_j, l_j),\ j = 1, \dots, 2n_0$, so that $|t_j - ck_j^{\alpha}| + |t_j - cl_j^{\beta}|$ is sufficiently small for all $j \in [0, 2n_0]$, where we select $t_j,\ j = 1, \dots, 2n_0$, satisfying the hypotheses of Lemma 5.4.

By the same strategy outlined in the proof of Corollary 5.8, the Hilbert–Schmidt norm as operators acting on $\ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N})$ of the difference $T_{n_0(\alpha,\beta)} - \mathcal{T}_{n_0(\alpha,\beta)}$ is arbitrarily small, as long as we make the value of $c = c(\alpha, \beta)$ smaller. As a consequence, $\mathcal{T}_{n_0}$ is also injective and its range is closed. These considerations prove, therefore, the following:

Corollary 5.9. *Let $\alpha, \beta \in \left(0, \frac{2}{9}\right)$. Then there is $c_{\alpha,\beta} > 0$ so that the following holds. For all $c \in (0, c_{\alpha,\beta})$, if $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ is a real-valued function that vanishes at $\pm cn^{\alpha}$ and its Fourier transform vanishes at $\pm cn^{\beta}$, then $f \equiv 0$.*

Moreover, for each $n_0 > 1$, the same assertion above holds under the weaker assumption that f vanishes for $\pm c \cdot m^{\alpha}$ and $\hat{f}$ vanishes for $\pm c \cdot k^{\beta}$, where $m \in \{0\} \cup (m_{\alpha,\beta}(n_0), +\infty)$, $k \in \{0\} \cup (k_{\alpha,\beta}(n_0), +\infty)$, and $m_{\alpha,\beta}(n_0), n_{\alpha,\beta}(n_0)$ are the least positive integers such that $c \cdot m^{\alpha} > n_0$ and $c \cdot k^{\beta} > n_0$, respectively.

Remark. In the end, we do not quite attain the primary goal of this section of proving Fourier uniqueness results for the sequences $(\{\pm n^{\alpha}\}, \{\pm n^{\beta}\})$, but only a slightly weaker version of it, with a small constant $c(\alpha, \beta)$ in front. The main reason for that in the proofs above is the location of the positive reals t_i: although their exact values do not matter in the end, it is crucial, in order to use Proposition 5.6, that they lie *after* the node n_0. We must therefore either force n_0 not to be too large in order not to make the norm of the matrix A_{K_0} too small, or fix them from the beginning and make the perturbations of T_{K_0} fall closer to it. In any case, this implies nontrivial use of the constant c multiplying the sequences $(\{\pm n^{\alpha}\}, \{\pm n^{\beta}\})$.

We believe that further studying operators resembling T_{K_0} above and their injectivity properties could yield better results in this regard. In order not to make this exposition even longer, we will not pursue this matter any further.

5C. ***Annihilating pairs.*** As an application of the results above, we will prove some *strong annihilating* properties of the sets $\{\pm c_\alpha n^\alpha\}_{n\in\mathbb{N}}$, $\{\pm c_\beta n^\beta\}_{n\in\mathbb{Z}}$.

Indeed, let $A, B \subset \mathbb{R}$ be two discrete sets. Inspired by the results and definitions of [Benedicks 1985; Amrein and Berthier 1977] (see also [Nazarov 1993]), we say that $(\mathbb{R}\setminus A, \mathbb{R}\setminus B)$ is a *weakly annihilating pair* for a class $\mathcal{C} \subset L^2(\mathbb{R})$ if whenever $f(A) = \hat{f}(B) = \{0\}$, $f \in \mathcal{C}$, then $f \equiv 0$.

This definition implies directly that $(\mathbb{R}\setminus\{\pm\sqrt{n}\}_{n\geq 0}, \mathbb{R}\setminus\{\pm\sqrt{n}\}_{n\geq 0})$ is a weakly annihilating pair for $\mathcal{S}_{\text{even}}(\mathbb{R};\mathbb{R})$ due to (1-3). On the other hand, under the hypotheses of Theorem 1.4, it follows directly that $(\mathbb{R}\setminus\{\pm\sqrt{n+\varepsilon_n}\}_{n\geq 0}, \mathbb{R}\setminus\{\pm\sqrt{n+\varepsilon_n}\}_{n\geq 0})$ is also weakly annihilating for $\mathcal{S}_{\text{even}}(\mathbb{R};\mathbb{R})$.

As a natural counterpart, we define a pair $(\mathbb{R}\setminus A, \mathbb{R}\setminus B)$ to be *ω-strongly annihilating* for a class $\mathcal{C} \subset L^2(\mathbb{R})$, $\omega \in \mathbb{R}$, if there is a real number $\gamma \in \mathbb{R}$ such that the inequality

$$\|f\|_{L^2((1+|x|)^\gamma)} + \|\hat{f}\|_{L^2((1+|x|)^\gamma)} \lesssim \left(\sum_{a\in A} |f(a)|^2(1+|a|)^\omega + \sum_{b\in B} |\hat{f}(b)|^2(1+|b|)^\omega\right)^{1/2}$$

holds for all $f \in \mathcal{C}$.

Our first contribution is Theorem 1.7; i.e., the pair $(\mathbb{R}\setminus\{\pm\sqrt{n}\}_{n\geq 0}, \mathbb{R}\setminus\{\pm\sqrt{n}\}_{n\geq 0})$ is ω-strongly annihilating for some $\omega > 0$.

Proof of Theorem 1.7. We start with (1-10). Indeed, consider a sequence $\{\varepsilon_n\}_{n\geq 0}$ of real numbers. We begin by observing that, for all integers $n \geq 1$, we have, by (1-3) together with Theorem 1.6,

$$|f(x) - f(\sqrt{n})| \lesssim \frac{|\varepsilon_n|}{\sqrt{n}} \sum_{m\geq 0} (1+m)^{3/4} \log^3(m+1) e^{-c\sqrt{n/m}} [|f(\sqrt{m})| + |\hat{f}(\sqrt{m})|],$$

whenever $x \in [\sqrt{n}, \sqrt{n+\varepsilon_n})$. Suppose then $|\varepsilon_n| \leq \delta(1+|n|)^{-\theta}$ holds for all $n \geq 1$, for some $\theta > 0$ and $\delta > 0$. If one uses the bound above together with the triangle inequality, an integration over the interval $[\sqrt{n}, \sqrt{n+\varepsilon_n})$ and the Cauchy–Schwarz inequality, one obtains

$$(1+n)^s |f(\sqrt{n})|^2 \lesssim \left(\int_{\sqrt{n}}^{\sqrt{n+1}} |f(y)|^2 (1+|y|)^{2(\theta+2s)+1}\,\mathrm{d}y\right) + \delta \sum_{m\geq 0} (|f(\sqrt{m})|^2 + |\hat{f}(\sqrt{m})|^2)(1+m)^{3/2} \log^6(1+m) e^{-2c\sqrt{n/m}} (1+n)^{-2\theta-1+s}.$$

If $2\theta + 1 - s > 1 \iff \theta > s/2$, we may sum the right-hand side above in $n \geq 1$ and get a *uniform* constant in $m \geq 0$. This yields

$$\sum_{n\geq 1} (1+n)^s |f(\sqrt{n})|^2 \lesssim \int_{\mathbb{R}} |f(y)|^2 (1+|y|)^{2(\theta+2s)+1}\,\mathrm{d}y + \delta \sum_{m\geq 0} (|f(\sqrt{m})|^2 + |\hat{f}(\sqrt{m})|^2)(1+m)^{3/2} \log^6(1+m).$$

An entirely analogous calculation implies the same on the level of Fourier transforms; that is,

$$\sum_{n\geq 1} (1+n)^s |\hat{f}(\sqrt{n})|^2 \lesssim \int_{\mathbb{R}} |\hat{f}(y)|^2 (1+|y|)^{2(\theta+2s)+1}\,\mathrm{d}y + \delta \sum_{m\geq 0} (|f(\sqrt{m})|^2 + |\hat{f}(\sqrt{m})|^2)(1+m)^{3/2} \log^6(1+m).$$

Summing these two bounds, if $s > \frac{3}{2}$ and $\delta \ll 1$ is sufficiently small, we obtain

$$\sum_{n\geq 1}(1+n)^s[|f(\sqrt{n})|^2+|\hat{f}(\sqrt{n})|^2] \leq C(\|f\|_{L^2((1+|x|)^\gamma)}+\|\hat{f}\|_{L^2((1+|x|)^\gamma)}), \tag{5-11}$$

which was the desired inequality, except for the $n = 0$ term. In that regard, we notice that a Sobolev embedding argument allows us to include it in the left-hand side of (5-11), which proves (1-10). Notice that we may take, for this part, any $\gamma > 5s+1$.

For (1-11), we will use once more Theorem 1.6. Indeed, it follows from that and Cauchy–Schwarz that

$$\begin{aligned}|f(x)|+|\hat{f}(x)| &\lesssim \sum_{n\geq 0}[|f(\sqrt{n})|+|\hat{f}(\sqrt{n})|](1+n)^{1/4}\log^3(1+n)e^{-c|x|/\sqrt{1+n}} \\ &\lesssim \biggl(\sum_{n\geq 0}[|f(\sqrt{n})|^2+|\hat{f}(\sqrt{n})|^2](1+n)^{5/2}\log^6(1+n)e^{-2c|x|/\sqrt{1+n}}\biggr)^{1/2}.\end{aligned}$$

Thus, we readily obtain that

$$\|f\|_{L^2((1+|x|)^s)}+\|\hat{f}\|_{L^2((1+|x|)^s)} \lesssim \biggl(\sum_{n\geq 0}[|f(\sqrt{n})|^2+|\hat{f}(\sqrt{n})|^2](1+n)^{(5+s)/2}\log^6(1+n)\biggr)^{1/2}$$

for any $s > 0$. This proves the Theorem for any $\omega > s/2+4$. □

Furthermore, as a corollary we can also obtain that the pair $(\mathbb{R}\setminus\{\pm\sqrt{n+\varepsilon_n}\}_{n\geq 0}, \mathbb{R}\setminus\{\pm\sqrt{n+\varepsilon_n}\}_{n\geq 0})$ is ω-strongly annihilating for some $\omega > 0$, which was the content of Corollary 1.8.

Proof of Corollary 1.8. Notice that the operator $\widetilde{T} : \ell^2_r(\mathbb{N})\times\ell^2_r(\mathbb{N}) \to \ell^2_r(\mathbb{N})\times\ell^2_r(\mathbb{N})$ given in Section 4B is, under our given hypotheses, *bounded and invertible* for $r \gg 1$ sufficiently large. Moreover, it takes, for each $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$, the pair $(\{f(\sqrt{n})\}_{n\in\mathbb{N}}, \{\hat{f}(\sqrt{n})\}_{n\in\mathbb{N}})$ to the pair $(\{f(\sqrt{n+\varepsilon_n})\}_{n\in\mathbb{N}}, \{\hat{f}(\sqrt{n+\varepsilon_n})\}_{n\in\mathbb{N}})$.

Therefore, if $\omega > s > r$, then the comparison of

$$\|(\{f(\sqrt{n})\}_{n\in\mathbb{N}}, \{\hat{f}(\sqrt{n})\}_{n\in\mathbb{N}})\|_{\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N})}$$

with

$$\|(\{f(\sqrt{n+\varepsilon_n})\}_{n\in\mathbb{N}}, \{\hat{f}(\sqrt{n+\varepsilon_n})\}_{n\in\mathbb{N}})\|_{\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N})}$$

holds with comparing constants *independent* of $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$. The same assertion holds with $\ell^2_\omega(\mathbb{N})\times\ell^2_\omega(\mathbb{N})$ norms instead of $\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N})$. This is enough to conclude the asserted statement. □

Finally, we conclude that, whenever c_α, c_β are sufficiently small, then $(\mathbb{R}\setminus\{\pm c_\alpha n^\alpha\}, \mathbb{R}\setminus\{\pm c_\beta n^\beta\})$ is ω-strongly annihilating for ω sufficiently large.

Corollary 5.10. *For $\alpha, \beta < \frac{2}{9}$ and c_α, c_β sufficiently small and for any $\gamma > 0$ sufficiently large, we have*

$$\|f\|_{L^2((1+|x|))}+\|\hat{f}\|_{L^2((1+|x|)^\gamma)} \lesssim \biggl(\sum_{n\geq 0}(1+n)^\omega[|f(c_\alpha n^\alpha)|^2+|\hat{f}(c_\beta n^\beta)|^2]\biggr)^{1/2},$$

whenever $\omega > (5+\gamma)/4$ and $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ is a real-valued function.

Proof. Under the hypotheses above, we know that the operator $\mathcal{T}_{n_0(\alpha,\beta)}$ from (5-10) is still injective and has closed range on $\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$ for $s \gg 1$. For that reason, the norm

$$\|(\{f(\sqrt{n})\}_{n\in\mathbb{N}}, \{\hat{f}(\sqrt{n})\}_{n\in\mathbb{N}})\|_{\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N})}$$

can be controlled by a constant independent of f times

$$\|\mathcal{T}_{n_0(\alpha,\beta)}(\{f(\sqrt{n})\}, \{\hat{f}(\sqrt{n})\})\|_{\ell^2_s(\mathbb{N})\times\ell^2_s(\mathbb{N})}.$$

However, the sequences constituting $\mathcal{T}_{n_0(\alpha,\beta)}(\{f(\sqrt{n})\}, \{\hat{f}(\sqrt{n})\})$ are *subsequences* of each entry of $(\{c_\alpha n^\alpha\}, \{c_\beta n^\beta\})$, respectively. As the weight $n \mapsto (1+n)^\omega$ is monotonic on $\mathbb{N}$, adding more terms only increases the weighted norm, and thus the conclusion follows. □

5D. ***The Cohn–Kumar-Miller-Radchenko–Viazovska result and perturbed interpolation formulae with derivatives.*** As another illustration of our main technique, we prove that the interpolation formulae with derivatives in dimension 8 and 24 from [Cohn et al. 2022] can be suitably perturbed.

Indeed, we first recall one of the main results of [Cohn et al. 2022]: let (d, n_0) be either (8, 1) or (24, 2). Then every $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ can be uniquely recovered by the sets of values

$$\{f(\sqrt{2n}),\ f'(\sqrt{2n}),\ \hat{f}(\sqrt{2n}),\ \hat{f}'(\sqrt{2n})\}, \quad n \geq n_0,$$

through the interpolation formula

$$f(x) = \sum_{n\geq n_0} f(\sqrt{2n})a_n(x) + \sum_{n\geq n_0} f'(\sqrt{2n})b_n(x) + \sum_{n\geq n_0} \hat{f}(\sqrt{2n})\hat{a}_n(x) + \sum_{n\geq n_0} \hat{f}'(\sqrt{2n})\hat{b}_n(x). \tag{5-12}$$

We also have uniform estimates on the functions $a_n, \hat{a}_n, b_n, \hat{b}_n$: indeed, there is $\tau > 0$ so that

$$\sup_{l\in\{0,1,2\}} \sup_{x\in\mathbb{R}^d} (1+|x|)^{100}(|a_n^{(l)}(x)| + |\hat{a}_n^{(l)}(x)| + |b_n^{(l)}(x)| + |\hat{b}_n^{(l)}(x)|) \lesssim n^\tau \tag{5-13}$$

for all $n \in \mathbb{N}$. Here and throughout this section, we shall denote by $g'(x)$ the derivative of the (radial) function g regarded as a one-dimensional function.

By [Cohn et al. 2022, Theorem 1.9], we know that the matrices

$$M_n(x) = \begin{pmatrix} a_n(x) & a_n'(x) & \hat{a}_n(x) & \hat{a}_n'(x) \\ b_n(x) & b_n'(x) & \hat{b}_n(x) & \hat{b}_n'(x) \\ \hat{a}_n(x) & \hat{a}_n'(x) & a_n(x) & a_n'(x) \\ \hat{b}_n(x) & \hat{b}_n'(x) & b_n(x) & b_n'(x) \end{pmatrix} \tag{5-14}$$

satisfy that $M_n(\sqrt{2m}) = \delta_{m,n} I_{4\times4}$ for $m, n \geq n_0$. As we know that the map that takes a vector of sufficiently rapidly decaying sequences

$$(\{\alpha_n\}, \{\beta_n\}, \{\tilde{\alpha}_n\}, \{\tilde{\beta}_n\})_{n\geq n_0}$$

onto the function

$$\mathfrak{f}(x) = \sum_{n\geq n_0} (\alpha_n a_n(x) + \beta_n b_n(x) + \tilde{\alpha}_n \hat{a}_n(x) + \tilde{\beta}_n \hat{b}_n(x))$$

is, in fact, injective (and moreover an isomorphism if we consider the set of all arbitrarily rapidly decaying sequences), we shall make use of this function in our estimates. Indeed, we have that the map that takes the quadruple of sequences

$$(\{\alpha_n\}, \{\beta_n\}, \{\tilde{\alpha}_n\}, \{\tilde{\beta}_n\})$$

onto

$$(\mathfrak{f}(\sqrt{2n}), \mathfrak{f}'(\sqrt{2n}), \hat{\mathfrak{f}}(\sqrt{2n}), \hat{\mathfrak{f}}'(\sqrt{2n}))_{n\geq n_0}$$

is, in fact, the identity. Another way to represent this map is as the series

$$\sum_{n\geq n_0} (\alpha_n, \beta_n, \tilde{\alpha}_n, \tilde{\beta}_n) \cdot M_n(\sqrt{2n}).$$

We define, therefore, the operator that takes the same quadruple onto

$$\big(\mathfrak{f}(\sqrt{2n+\varepsilon_n}), \mathfrak{f}'(\sqrt{2n+\varepsilon_n}), \hat{\mathfrak{f}}(\sqrt{2n+\varepsilon_n}), \hat{\mathfrak{f}}'(\sqrt{2n+\varepsilon_n})\big)_{n\geq n_0}.$$

In the alternative notation, this operator, which we shall denote by $\mathfrak{T}$, is given by

$$\sum_{n\geq n_0} (\alpha_n, \beta_n, \tilde{\alpha}_n, \tilde{\beta}_n) \cdot M_n(\sqrt{2n+\varepsilon_n}).$$

As before, we seek to prove that $\mathfrak{T}$ is invertible when defined over some space

$$\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) =: (\ell^2_s(\mathbb{N}))^4,$$

where we may take $s \gg 1$ sufficiently large. As our aim here is not to establish the sharpest possible results, but only to prove that we may prove versions of the above interpolation formula with *some* perturbed nodes; we shall make use of the Hilbert–Schmidt test, as in Section 5A above. Indeed, the same remark about the definition of the perturbed operators in the proof of Theorem 1.4 holds here as well: we first define $\mathfrak{T}$ over quadruples of sequences with finitely many nonzero terms, and then we use a priori boundedness of $I - \mathfrak{T}$ over this space to define $\mathfrak{T}$ in the whole space $(\ell^2_s(\mathbb{N}))^4$ by density. Thus, we wish to prove that

$$\|I - \mathfrak{T}\|_{HS((\ell^2_s(\mathbb{N}))^4)} < 1.$$

A simple computation with the Hilbert–Schmidt norm using (5-14) shows that this quantity is bounded by

$$\begin{aligned}\sum_{m,n>n_0} m^{2s} n^{-2s} \big(&|a_n(\sqrt{2m}) - a_n(\sqrt{2m+\varepsilon_m})|^2 + |\hat{a}_n(\sqrt{2m}) - \hat{a}_n(\sqrt{2m+\varepsilon_m})|^2 \\ &+ |a_n'(\sqrt{2m}) - a_n'(\sqrt{2m+\varepsilon_m})|^2 + |\hat{a}_n'(\sqrt{2m}) - \hat{a}_n'(\sqrt{2m+\varepsilon_m})|^2 + |b_n(\sqrt{2m}) - b_n(\sqrt{2m+\varepsilon_m})|^2 \\ &+ |\hat{b}_n(\sqrt{2m}) - \hat{b}_n(\sqrt{2m+\varepsilon_m})|^2 + |b_n'(\sqrt{2m}) - b_n'(\sqrt{2m+\varepsilon_m})|^2 + |\hat{b}_n'(\sqrt{2m}) - \hat{b}_n'(\sqrt{2m+\varepsilon_m})|^2\big).\end{aligned}$$

Notice that we have used, as in the proof of Theorems 1.4 and 5.3, the standard orthonormal basis for the space $\ell^2_s(\mathbb{N})$, which induces the additional $(m/n)^{2s}$ factor in the summand above. By (5-13) and the mean value theorem, the sum above is bounded by (an absolute constant times)

$$\sum_{m,n>0} m^{2s} n^{-2s} \times m^{-100} n^{2\tau} \varepsilon_m^2.$$

The sum above is representable as a product of a sum in m and one in n. The one in n is convergent if $s > \tau + 1$. We then fix such a value of s. For such values, the second sum is

$$\sum_{m>0} m^{2s-100}\varepsilon_m^2,$$

which converges in the case $\varepsilon_m \lesssim m^{49-s}$. For all such sequences, the difference $I - \mathfrak{T}$ is a Hilbert–Schmidt operator. Moreover, if $\varepsilon_m \le \delta m^{49-s}$ for $\delta > 0$ sufficiently small, we will have $\|I - \mathfrak{T}\|_{HS(\ell^2_s(\mathbb{N})^4)} < 1$. Summarizing, we have shown the following result:

Theorem 5.11. *There are $C_0 > 0$ and $\delta > 0$ so that the following holds: for each sequence ε_k so that $|\varepsilon_k| < \delta k^{-C_0}$, any function $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ is uniquely determined by the values*

$$\big(f(\sqrt{2n+\varepsilon_n}),\ f'(\sqrt{2n+\varepsilon_n}),\ \hat{f}(\sqrt{2n+\varepsilon_n}),\ \hat{f}'(\sqrt{2n+\varepsilon_n})\big)_{n\ge n_0}, \tag{5-15}$$

where we let $(d, n_0) = (8, 1)$ or $(24, 2)$.

In the same spirit of Section 4B, one can obtain an interpolation formula with the values (5-15) from Theorem 5.11.

We remark that, in the same way that we undertook our analysis for the Radchenko–Viazovska interpolating functions, we expect the functions a_n, b_n in [Cohn et al. 2022, Theorem 1.9] should also satisfy some exponential-like decay. This fact, although possible, should be sensibly more technically involved than Theorem 1.6, due to the more complicated nature of the construction of the interpolating functions with derivatives in dimensions 8 and 24.

5E. ***Perturbed interpolation formulae for odd functions.*** Finally, in the same spirit of the results in Section 4, we briefly comment on interpolation formulae for odd functions. Recall the following results from [Radchenko and Viazovska 2019, Section 7]:

Theorem 5.12 [Radchenko and Viazovska 2019, Theorem 7]. *There exist sequences of odd functions $d_m^{\pm} : \mathbb{R} \to \mathbb{R}$, $m \ge 0$, belonging to the Schwartz class so that*

$$\widehat{d_m^{\pm}} = (\mp i) d_m^{\pm}, \quad d_m^{\pm}(\sqrt{n}) = \delta_{n,m}\sqrt{n}, \quad n \ge 1.$$

Moreover, $\lim_{x\to 0} d_m^{+}(x)/x = \delta_{0m}$. These functions satisfy the uniform bound

$$|d_n^{\pm}(x)| \lesssim n^{5/2} \quad \text{for all } x \in \mathbb{R},\ n \ge 0,$$

and, finally, for each odd and real Schwartz function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = d_0^{+}(x)\frac{f'(0) + i\hat{f}'(0)}{2} + \sum_{n\ge 1}\left(c_n(x)\frac{f(\sqrt{n})}{\sqrt{n}} - \hat{c}_n(x)\frac{\hat{f}(\sqrt{n})}{\sqrt{n}}\right), \tag{5-16}$$

where $c_n = (d_n^{+} + d_n^{-})/2$, and the right-hand side of the sum above converges absolutely.

As a direct consequence, we see that any real, odd, Schwartz function on the real line is determined uniquely by the union of its values at $\sqrt{n}$ and the values of its Fourier transform at $\sqrt{n}$ with $f'(0)$

and $\hat{f}'(0)$. By employing the results in Section 4, we will show that we can actually recover any such function from $\{f(\sqrt{n+\varepsilon_n})\}_{n\geq 1} \cup \{\hat{f}(\sqrt{n+\varepsilon_n})\}_{n\geq 1} \cup \{f'(0)\} \cup \{\hat{f}'(0)\}$ instead.

Indeed, first of all, we start by noticing that the same techniques employed to refine the uniform estimates from [Radchenko and Viazovska 2019] can be applied to the functions $d_m^{\pm}$, as they are defined in a completely analogous way to the $b_n^{\pm}$ from Section 4. By carrying out the same kind of estimates, we are able to obtain

$$|d_n^{\pm}(x)| \lesssim n^{3/4} \log^3(1+n) e^{-c'|x|/\sqrt{n}} \quad \text{for all } x \in \mathbb{R},\ n \geq 1, \tag{5-17}$$

for some absolute constant $c' > 0$. By the same analysis of the ∂_x-partial derivative of the generating function used in Section 4A, this readily implies that the derivatives of the $d_n^{\pm}$ satisfy essentially the same decay; in fact, $|(d_n^{\pm})'(x)| \lesssim n^{5/4} \log^3(1+n) e^{-c''|x|/\sqrt{n}}$ for all $x \in \mathbb{R},\ n \geq 1$, with $c'' > 0$ another absolute constant.

We consider now the operator that takes a pair of sequences $(\{\alpha_n\}, \{\beta_n\}) \in \ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N}),\ s > 0$ to be chosen, into

$$\Big\{ \sum_{n\geq 0} (\alpha_n, \beta_n) C_n(\sqrt{m+\varepsilon_m}) \Big\}_{m\geq 0},$$

where we abbreviate

$$C_n(x) = \begin{pmatrix} c_n(x)/\sqrt{n} & \hat{c}_n(x)/\sqrt{n} \\ -\hat{c}_n(x)/\sqrt{n} & c_n(x)/\sqrt{n} \end{pmatrix}.$$

Let us denote this operator by $\mathcal{V}$. From (5-16) and the fact that the function $d_0^{+}(x) = \sin(\pi x^2)/\sinh(\pi x)$ *vanishes* together with its Fourier transform at $\pm\sqrt{n},\ n \in \mathbb{N}$, we know that the identity operator on $\ell_s^2(\mathbb{N}) \times \ell_s^2(\mathbb{N})$ may be written as

$$\Big\{ \sum_{n\geq 0} (\alpha_n, \beta_n) C_n(\sqrt{m}) \Big\}_{m\geq 0}.$$

Therefore, the techniques from Sections 4B, 5D and 5A, together with our previous considerations in this subsection, allow us to deduce the following result:

Theorem 5.13. *There is $\delta > 0$ so that, in the case $|\varepsilon_n| \leq \delta n^{-7/4}$, for each $f \in \mathcal{S}_{\mathrm{odd}}(\mathbb{R})$ real, the values*

$$(f(\sqrt{1+\varepsilon_n}), f(\sqrt{2+\varepsilon_2}), \ldots) \quad \text{and} \quad (\hat{f}(\sqrt{1+\varepsilon_n}), \hat{f}(\sqrt{2+\varepsilon_2}), \ldots)$$

allow us to recover uniquely the values $(f(1), f(\sqrt{2}), f(\sqrt{3}), \ldots)$ and $(\hat{f}(1), \hat{f}(\sqrt{2}), \hat{f}(\sqrt{3}), \ldots)$. In particular, given the values

$$\{f(\sqrt{n+\varepsilon_n})\}_{n\geq 1} \cup \{\hat{f}(\sqrt{n+\varepsilon_n})\}_{n\geq 1} \cup \{f'(0)\} \cup \{\hat{f}'(0)\},$$

we can uniquely recover any real-valued function $f \in \mathcal{S}_{\mathrm{odd}}(\mathbb{R})$.

As previously mentioned, we do not carry out the details here, for their similarities with the proof of Theorems 1.6 and 1.4.

6. Comments and remarks

In this section, we gather some remarks about the problems and techniques discussed and state some results we expect to be true.

6A. ***Asymmetric perturbations.*** In the statement of Theorem 1.4, we have assumed that the perturbations made to the Radchenko–Viazovska interpolation formula were *the same* on the function and Fourier sides for fixed j. We remark that, by the exact same proof as given above, one may obtain results with *different* perturbations: in that regard, Theorem 1.4 can be immediately reinterpreted as stating that one may recover f from the values of $f(\sqrt{n+\varepsilon_n})$, $\hat{f}(\sqrt{n+\delta_n})$, $n \geq 0$, where one assumes $\varepsilon_0 = \delta_0 = 0$, and $\sup_n (1+n)^{5/4} \log(1+n)^3 \cdot (|\varepsilon_n| + |\delta_n|)$ is sufficiently small.

Similarly, one can safely introduce four different perturbation parameters in Theorem 5.11 — one for f, one for f', one for $\hat{f}$ and a last one for $\hat{f}'$ — as long as they still satisfy the conditions predicted in that result. The same holds for Theorem 1.3, where one may select two different perturbation parameters, one for the function and another for its derivative. As these generalizations are immediate from our proofs, we chose to keep all results with one perturbation parameter, in order to simplify the exposition.

6B. ***Maximal perturbed interpolation formulae for band-limited functions.*** In Section 3, we have seen how our basic functional analysis techniques can be employed in order to deduce new interpolation formulae for band-limited functions. Although Kadec's proof also uses the basic fact that, whenever a perturbation of the identity is sufficiently small, we can basically "invert" an operator, he then proceeds to find that the set of exponentials $\{\exp(2\pi i(n+\varepsilon_n)x)\}_{n\geq 0}$ is a Riesz basis for $L^2\big(-\frac{1}{2}, \frac{1}{2}\big)$ if $\sup_n |\varepsilon_n| < \frac{1}{4}$ by means of *orthogonality* considerations. Indeed, one key strategy in his estimates is to expand in the different complete orthogonal system

$$\{1, \cos(2\pi nt), \sin((2n-1)\pi t)\}_{n\geq 1}$$

and use the properties of this expansion. Our results, as much as they do not come so close to Kadec's threshold, follow a slightly different path: instead of using the orthogonality of a different system, we choose to work directly with discrete analogues of the Hilbert transform and estimate over those. Although we do not reach — by a 0.011 margin — the sharp $\frac{1}{4}$-perturbation result, one advantage of our approach is that it yields bounds for perturbing *any* kind of interpolation formulae with derivatives. Indeed, following the line of thought of Vaaler, many authors have investigated the property of recovering the values of a function $f \in L^2(\mathbb{R})$ band-limited to $[-k/2, k/2]$ from the values of its $(k-1)$-first derivatives (see, e.g., [Littmann 2006; Gonçalves and Littmann 2018]). Our approach in Section 3 in order to prove Theorem 1.3 generalizes easily to the case of several derivatives by an easy modification. It can be summarized as follows:

Theorem 6.1. *There is* $L(k) > 0$ *so that if* $\max_{0\leq l<k} \sup_{n\in\mathbb{Z}} |\varepsilon_n^{(l)}| < L(k)$, *then any function* $f \in L^2(\mathbb{R})$ *band-limited to* $[-k/2, k/2]$ *is uniquely determined by the values of*

$$f^{(l)}(n+\varepsilon_n^{(l)}), \quad n \in \mathbb{Z},\ l = 0, 1, \ldots, k-1.$$

A natural question that connects our results to Kadec's results is about the *best* value of $L(k)$ so that Theorem 6.1 holds. We do not have evidence to back any concrete conjecture, but we find possible that the threshold $L(k) = \frac{1}{4}$ is kept for higher values of $k \in \mathbb{N}$. We speculate that, in order to prove such a result, one would need to find an appropriate hybrid of our techniques and Kadec's techniques (see for instance Section 10 in [Young 1980, Chapter 1]), taking into account properties of the discrete Hilbert transforms as well as orthogonality results.

6C. ***Theorem 1.6, optimal decay rates for interpolating functions and maximal perturbations.*** In Theorem 1.6, we have improved the uniform bound obtained in [Radchenko and Viazovska 2019] and, more recently, the sharper uniform bound of [Bondarenko et al. 2023] on the interpolating functions a_n to one that *decays* with x; namely, we have that

$$|a_n(x)| \lesssim n^{1/4} \log^3(1+n)(e^{-c|x|^2/n} 1_{|x|<Cn} + e^{-c|x|} 1_{|x|>Cn})$$

holds for all $n \in \mathbb{N}$, where $C, c > 0$ are two fixed positive constants. Although this improves the decay rates from before, the power $n^{1/4}$ found here and in [Bondarenko et al. 2023] in the growth seems likely not to be optimal; to that regard, we pose the following:

Question 1. *What is the best decay rate for a_n as in Theorem 1.6? Can one prove that* $\sup_{x\in\mathbb{R}} |a_n(x)| = \mathcal{O}((\log n)^C)$ *in n for some absolute constant $C > 0$?*

This conjectured growth seems to be the best possible, due to the recent findings of [Bondarenko et al. 2023], which show that, for each $N \gg 1$, the average

$$\frac{1}{N+1} \sum_{k \le N} |a_k(x)|^2$$

grows slower than some power of $\log N$.

Notice that, by a simple modification of the computations made in Section 4B, an affirmative answer to Question 1 yields an immediate improvement in the range of ε_i that we allow for the theorems in Section 4B. Indeed, we get automatically that $|\varepsilon_i| \lesssim i^{-1}$ is allowed in such results. On the other hand, this seems to be the best possible result one can achieve with our current methods, as the mean value theorem implies that $\sup_{x\in\mathbb{R}} |a_n'(x)| \gtrsim \sqrt{n}$.

In particular, everything indicates that one needs a new idea in order to prove the following conjecture:

Conjecture 6.2 (maximal perturbations). *Let $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ be a real-valued function. Then there is $\theta > 0$ so that, if $|\varepsilon_i| + |\delta_i| < \theta$ for all $i \in \mathbb{N}$, then f can be uniquely recovered from its values*

$$f(0),\ f(\sqrt{1+\varepsilon_1}),\ f(\sqrt{2+\varepsilon_2}),\ \dots,$$

together with the values of its Fourier transform

$$\hat{f}(0),\ \hat{f}(\sqrt{1+\delta_1}),\ \hat{f}(\sqrt{2+\delta_2}),\ \dots.$$

It might not be an easy task to prove Conjecture 6.2 even with a new idea starting from our techniques, but we believe that the following version stands a chance of being more tractable with the current methods:

Conjecture 6.3 (maximal perturbations, weak form). *Let $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ be a real-valued function. Then, for each $a > 0$, there is $\delta > 0$ so that, if $|\varepsilon_i| + |\delta_i| \le \delta k^{-a}$, then f can be uniquely recovered from its values*

$$f(0),\ f(\sqrt{1+\varepsilon_1}),\ f(\sqrt{2+\varepsilon_2}),\ \dots,$$

together with the values of its Fourier transform

$$\hat{f}(0),\ \hat{f}(\sqrt{1+\delta_1}),\ \hat{f}(\sqrt{2+\delta_2}),\ \dots.$$

In this framework, the results in Section 4B may be regarded as partial progress towards this conjecture.

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