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RIESZ TRANSFORM AND VERTICAL OSCILLATION IN THE HEISENBERG GROUP

KATRIN FÄSSLER AND TUOMAS ORPONEN

We study the L^2-boundedness of the 3-dimensional (Heisenberg) Riesz transform on intrinsic Lipschitz graphs in the first Heisenberg group $\mathbb{H}$. Inspired by the notion of vertical perimeter, recently defined and studied by Lafforgue, Naor, and Young, we first introduce new scale and translation invariant coefficients $\mathrm{osc}_\Omega(B(q,r))$. These coefficients quantify the vertical oscillation of a domain $\Omega \subset \mathbb{H}$ around a point $q \in \partial\Omega$, at scale $r > 0$. We then proceed to show that if Ω is a domain bounded by an intrinsic Lipschitz graph Γ, and

$$\int_0^\infty \mathrm{osc}_\Omega(B(q,r))\,\frac{dr}{r} \le C < \infty, \quad q \in \Gamma,$$

then the Riesz transform is L^2-bounded on Γ. As an application, we deduce the boundedness of the Riesz transform whenever the intrinsic Lipschitz parametrisation of Γ is an ϵ better than $\frac{1}{2}$-Hölder continuous in the vertical direction.

We also study the connections between the vertical oscillation coefficients, the vertical perimeter, and the natural Heisenberg analogues of the β-numbers of Jones, David, and Semmes. Notably, we show that the L^p-vertical perimeter of an intrinsic Lipschitz domain Ω is controlled from above by the p-th powers of the L^1-based β-numbers of $\partial\Omega$.

1. Introduction

1A. ***A Euclidean introduction to the Heisenberg Riesz transform.*** A fundamental singular integral operator (SIO) in $\mathbb{R}^d$ is the $(d-1)$-*dimensional Riesz transform*, formally defined by the convolution

$$R_{d-1}\nu(x) = \nu * \frac{x}{|x|^d}.$$

Here $x/|x|^d$ is the $(d-1)$-*dimensional Riesz kernel* which is, up to a constant, the gradient of the fundamental solution of the Laplacian. Through this connection to the Laplace equation, the operator R_{d-1} has many applications to problems concerning analytic and harmonic functions. For instance, whenever R_{d-1} is bounded on $L^2(\mu)$ for a $(d-1)$-regular measure μ, then the support of μ is nonremovable for Lipschitz harmonic functions (or bounded analytic functions in the plane); see [Tolsa 2014] for an in depth introduction to this topic and many more references.

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A second application of the SIO R_{d-1} is the *method of layer potentials* employed to solve the Dirichlet problem

$$\begin{cases} \Delta u(x) = 0, & x \in \Omega, \\ u|_{\partial\Omega} = g \end{cases} \tag{1.1}$$

on domains $\Omega \subset \mathbb{R}^d$ with Lipschitz boundaries, and with, say, $g \in L^2(\mathcal{H}^{d-1}|_{\partial\Omega})$. As the name suggests, a key component in the method of layer potentials is the study of the *boundary layer potential*

$$D\nu(x) = \text{p.v.}\, \frac{1}{\omega_d} \int_{\partial\Omega} \frac{(y-x) n_{\partial\Omega}(y)}{|y-x|^d} \, d\nu(y).$$

The boundedness of the operator D on $L^2(\mathcal{H}^{d-1}|_{\partial\Omega})$ can be derived from the boundedness of R_{d-1}; see [Fabes et al. 1978; Verchota 1984].

By now, the L^2-boundedness properties of the operator R_{d-1} are well understood. According to a result of David and Semmes [1991], generalising the earlier works of Calderón [1977] and Coifman, McIntosh, and Meyer [Coifman et al. 1982], R_{d-1} is bounded on $L^2(\mathcal{H}^{d-1}|_S)$ whenever $S \subset \mathbb{R}^d$ is *uniformly* $(d-1)$*-rectifiable*. More recently, Nazarov, Tolsa, and Volberg [Nazarov et al. 2014a] proved a converse: if $S \subset \mathbb{R}^d$ is $(d-1)$-regular, then the uniform rectifiability of S is necessary for the boundedness of R_{d-1} on $L^2(\mathcal{H}^{d-1}|_S)$. These results have been used to show that a compact $(d-1)$-set is removable for Lipschitz harmonic functions if and only if it is purely $(d-1)$-unrectifiable [Mattila and Paramonov 1995; Nazarov et al. 2014b] and that the Dirichlet problem (1.1) is solvable in Lipschitz domains with L^2-boundary values [Verchota 1984].

The work here is motivated by aspirations to extend parts of the theory above to the case of a basic hypoelliptic and nonelliptic operator, the *sub-Laplacian* (also known as the *Kohn Laplacian*)

$$\Delta_{\mathbb{H}} = X^2 + Y^2$$

in $\mathbb{R}^3$. Here X and Y are the vector fields

$$X = \partial_x - \tfrac{1}{2} y \partial_t \quad \text{and} \quad Y = \partial_y + \tfrac{1}{2} x \partial_t. \tag{1.2}$$

A first step is to understand the L^2-boundedness of an associated *Riesz transform* operator, which we will soon define.

Whereas the operators X, Y, $\Delta_{\mathbb{H}}$ do not interact particularly nicely with Euclidean translations, they do commute with the following *left translations* $\tau_p : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\tau_p(q) := \big(x + x',\, y + y',\, t + t' + \tfrac{1}{2}(xy' - x'y)\big),$$

where $p = (x, y, t) \in \mathbb{R}^3$ and $q = (x', y', t') \in \mathbb{R}^3$. This suggests that it is natural to study questions about $\Delta_{\mathbb{H}}$ in the setting of the first *Heisenberg group* $\mathbb{H} = (\mathbb{R}^3, \cdot)$, where the group law "$\cdot$" is *defined* so that X and Y are (left) invariant:

$$p \cdot q := \tau_p(q).$$

It was shown by Folland [1975] that the operator $\Delta_{\mathbb{H}}$ has a fundamental solution $G : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$, whose formula is given by

$$G(p) = \frac{c}{((x^2+y^2)^2+16t^2)^{1/2}} =: \frac{c}{\|p\|_{\mathrm{Kor}}^2}, \quad p = (x, y, t) \in \mathbb{H} \setminus \{0\}.$$

Here $c > 0$ is a constant and $\|p\|_{\mathrm{Kor}} := ((x^2+y^2)^2+16t^2)^{1/4}$. This quantity is known as the *Korányi norm* of the point $p \in \mathbb{H}$, and it induces a metric d_{Kor} on $\mathbb{H}$ via the relation

$$d_{\mathrm{Kor}}(p, q) = \|q^{-1} \cdot p\|_{\mathrm{Kor}}. \tag{1.3}$$

The distance d_{Kor} is invariant under the left translations, that is, $d_{\mathrm{Kor}}(p \cdot q_1, p \cdot q_2) = d_K(q_1, q_2)$ for all $p, q_1, q_2 \in \mathbb{H}$.

In analogy with the $(d-1)$-dimensional Riesz transform discussed above, one may now consider the SIO R formally defined by

$$R\nu(p) := \nu * \nabla_{\mathbb{H}} G(p).$$

Here $\nabla_{\mathbb{H}}$ stands for the *horizontal gradient* $\nabla_{\mathbb{H}} G = (XG, YG)$, and the convolution should be understood in the Heisenberg sense:

$$f * g(p) = \int f(q) g(q^{-1} \cdot p)\, dq.$$

The main open question is the following:

Question 1. For which locally finite Borel measures μ on $\mathbb{H}$ (equivalently $\mathbb{R}^3$) is the operator R bounded on $L^2(\mu)$?

Here the boundedness on $L^2(\mu)$ is defined in the standard way via ϵ-truncations; see Section 4 for the precise definition.

1B. ***Previous work.*** To the best of our knowledge, the Heisenberg Riesz transform R was first mentioned in [Chousionis and Mattila 2014], where the following removability question was raised and studied: Which subsets of $\mathbb{H}$ (more generally, of Heisenberg groups of arbitrary dimensions) are removable for Lipschitz harmonic functions? The notions of *Lipschitz* and *harmonic* should be interpreted in the Heisenberg sense: We call a function $u : \mathbb{H} \to \mathbb{R}$ *harmonic* if it solves the sub-Laplace equation $\Delta_{\mathbb{H}} u = 0$. A function $f : \mathbb{H} \to \mathbb{R}$ is *Lipschitz* if $|f(p) - f(q)| \le L d_{\mathrm{Kor}}(p, q)$ for some $L \ge 1$ and all $p, q \in \mathbb{H}$.

It was shown in [Chousionis and Mattila 2014, Theorem 3.13] that the critical exponent for the removability problem in $\mathbb{H}$ is 3 (keeping in mind that $\dim_{\mathrm{H}}(\mathbb{H}, d_{\mathrm{Kor}}) = 4$). More precisely, sets with vanishing 3-dimensional measure are removable, while sets of Hausdorff dimension exceeding 3 are not. In [Chousionis and Mattila 2014, Section 5], the authors formulate (essentially) Question 1 and suggest its connection to the removability problem.

The connection was formalised by Chousionis and the authors in the following theorem:

Theorem 1.4 [Chousionis et al. 2019a, Theorem 1.2]. *If μ is a* 3*-regular measure on* $\mathbb{H}$ (see (1.5) below), *and R is bounded on $L^2(\mu)$, then* $\operatorname{spt}\mu$ *is nonremovable for Lipschitz harmonic functions in* $\mathbb{H}$.

In [Chousionis et al. 2019a], we also proved the first nontrivial results on the L^2-boundedness of R (and a class of other SIOs). To discuss these results, and also the ones in the present paper, we need the concept of *intrinsic Lipschitz functions and graphs*. A *vertical subgroup* $\mathbb{W} \subset \mathbb{H}$ is, from a geometric point of view, any 2-dimensional subspace of $\mathbb{R}^3$ containing the t-axis. The *complementary horizontal subgroup of* $\mathbb{W}$ is the line $\mathbb{V} = \mathbb{W}^{\perp}$ in the xy-plane.

We give the definition of *intrinsic Lipschitz functions* $\phi : \mathbb{W} \to \mathbb{V}$ and the associated *intrinsic Lipschitz graphs* $\Gamma_{\phi} \subset \mathbb{H}$ in Section 2C. These objects were introduced by Franchi, Serapioni, and Serra Cassano [Franchi et al. 2006], and they appear to be fundamental building blocks in the theory of *high-dimensional* rectifiability in the Heisenberg group; see for example [Chousionis et al. 2019b; Mattila et al. 2010]. In particular, intrinsic Lipschitz graphs $\Gamma \subset \mathbb{H}$ are closed 3-regular sets, which means that the measure $\mu = \mathcal{H}^3|_{\Gamma}$ satisfies

$$\mu(B(p,r)) \sim r^3, \quad p \in \operatorname{spt}\mu, \ \ 0 < r \le \operatorname{diam}(\operatorname{spt}\mu). \tag{1.5}$$

In another paper of Franchi, Serapioni, and Serra Cassano [Franchi et al. 2011], a Rademacher-type theorem was established for intrinsic Lipschitz functions: without delving into detail, we just mention that if $\phi : \mathbb{W} \to \mathbb{V}$ is intrinsic Lipschitz, then for Lebesgue almost every $w \in \mathbb{W}$ there exists an *intrinsic gradient* for ϕ, denoted by $\nabla^{\phi}\phi(w)$.

Recall that in $\mathbb{R}^d$, Calderón [1977] and Coifman, McIntosh, and Meyer [Coifman et al. 1982] proved that R_{d-1} is bounded on $L^2(\mathcal{H}^{d-1}|_{\Gamma})$ if $\Gamma \subset \mathbb{R}^d$ is a Lipschitz graph. In analogy, one can ask:

Question 2. Assume that $\Gamma \subset \mathbb{H}$ is an intrinsic Lipschitz graph. Is R bounded on $L^2(\mathcal{H}^3|_{\Gamma})$?

We are not convinced enough to upgrade the question to a conjecture. In [Chousionis et al. 2019a], we obtained a positive answer under a extra regularity:

Theorem 1.6 [Chousionis et al. 2019a, Theorem 1.1]. *Assume $\alpha > 0$ and that $\phi \in C^{1,\alpha}(\mathbb{W})$ has compact support. Then R is bounded on $L^2(\mathcal{H}^3|_{\Gamma_{\phi}})$.*

The assumption $\phi \in C^{1,\alpha}(\mathbb{W})$ means that the intrinsic gradient of ϕ exists everywhere and satisfies an intrinsic version of α-Hölder regularity (which is weaker than Euclidean α-Hölder regularity). The assumption implies, see [Chousionis et al. 2019a, Proposition 4.1], that the affine approximation of Γ_{ϕ} at $p \in \Gamma$ improves at a geometric rate as one zooms into p.

1C. ***New results.*** A novelty of the current paper is to prove the L^2-boundedness of R in some scenarios where there is no *pointwise decay* for the quality of affine approximation of Γ. As a basic example, Theorem 4.1 below applies to graphs of the form

$$\Gamma = \Gamma_{\mathbb{R}^2} \times \mathbb{R} \subset \mathbb{H},$$

where $\Gamma_{\mathbb{R}^2}$ is a (Euclidean) Lipschitz graph in $\mathbb{R}^2$. It turns out that a key feature of these graphs is the following. The two complementary domains $\Omega_1, \Omega_2 \subset \mathbb{H} \setminus \Gamma$ have zero *vertical oscillation*: for $j \in \{1, 2\}$, every vertical line $\ell \subset \mathbb{H}$ satisfies

$$\ell \subset \Omega_j \quad \text{or} \quad \ell \cap \Omega_j = \varnothing. \tag{1.7}$$

The condition (1.7) is qualitative, not to mention exceedingly restrictive, so we looked for a way to quantify and relax it. For these purposes, we introduce the *vertical oscillation coefficients* $\mathrm{osc}_\Omega(B(p,r))$. Given a domain $\Omega \subset \mathbb{H}$ and a point $p \in \partial\Omega$, the number $\mathrm{osc}_\Omega(B(p,r))$ quantifies, in a scale and translation invariant way, how far Ω is (locally) from satisfying (1.7). The definition of the coefficients $\mathrm{osc}_\Omega(B(p,r))$ was inspired by the notion of *vertical perimeter* recently introduced in [Lafforgue and Naor 2014, Section 4] and further studied in [Naor and Young 2018]; see Remark 3.2 for the definition. We postpone further details on the vertical oscillation coefficients to Section 3.

Here is the main theorem of the paper.

Theorem 1.8. *Let $\Gamma \subset \mathbb{H}$ be an intrinsic Lipschitz graph, and let Ω be one of the components of $\mathbb{H} \setminus \Gamma$. Assume that there is a finite constant $C > 0$ such that*

$$\int_0^\infty \mathrm{osc}_\Omega(B(p,r))\,\frac{dr}{r} \le C, \quad p \in \partial\Omega. \tag{1.9}$$

Then R is bounded on $L^2(\mathcal{H}^3|_\Gamma)$.

In general, we do not know how reasonable the assumption (1.9) is. It follows easily from the Rademacher theorem for intrinsic Lipschitz functions (and Corollary 3.34 below) that $\mathrm{osc}_\Omega(B(p,r)) \to 0$ for $\mathcal{H}^3$ almost every $p \in \Gamma$ as $r \searrow 0$. But we have no quantitative estimates for $\mathrm{osc}_\Omega(B(p,r))$ if nothing better than intrinsic Lipschitz regularity is assumed of Γ; see Section 6 for a concrete question in this vein. However, we can complement Theorem 1.8 with the following application.

Theorem 1.10. *Let $\phi : \mathbb{W} \to \mathbb{R}$ be an intrinsic Lipschitz function that satisfies the following Hölder regularity in the vertical direction*:

$$|\phi(y,t) - \phi(y,s)| \le H|t-s|^{(1+\tau)/2}, \quad |s-t| \le 1 \tag{1.11}$$

and

$$|\phi(y,t) - \phi(y,s)| \le H|t-s|^{(1-\tau)/2}, \quad |s-t| > 1, \tag{1.12}$$

where $H \ge 1$ and $0 < \tau \le 1$. Then R is bounded on $L^2(\mathcal{H}^3|_{\Gamma_\phi})$.

It is well known that intrinsic Lipschitz functions are always $\frac{1}{2}$-Hölder continuous in the vertical direction. So, Theorem 1.10 states that an ϵ of additional regularity in this one direction yields the L^2-boundedness of R on Γ_ϕ.

1D. ***Vertical oscillation and β-numbers.*** A fundamental concept in the theory of quantitative rectifiability in $\mathbb{R}^n$ is the *β-number*, first introduced in [Jones 1990], then further developed in [David and Semmes 1991], and later applied by too many authors to begin acknowledging here. It is no surprise that suitable variants of the β-numbers (see Section 3A for definitions) can also be used to study quantitative rectifiability questions in $\mathbb{H}$, as well as higher dimensional Heisenberg groups. A few papers already doing so are [Chousionis and Li 2017; Chousionis et al. 2019a; 2019b; Fässler et al. 2020; Juillet 2010; Li and Schul 2016a; 2016b]. Since we here introduce new coefficients related to the theory of quantitative rectifiability in $\mathbb{H}$, it is natural to ask: is there a connection to β-numbers? We investigate this matter in Sections 3A and 6B.

We only mention the key results here briefly and informally. First, the vertical oscillation coefficients of Ω are bounded from above by the (L^1-based) β-numbers of $\partial\Omega$ — at least if $\partial\Omega$ is 3-regular. This is the content of Corollary 3.34. Second, if $\partial\Omega$ is 3-regular, and if the β-numbers associated to $\partial\Omega$ satisfy an *L^p-Carleson packing condition*, see (6.4), then the L^p-variant of the vertical perimeter of Ω inside balls $B(q, r)$, $q \in \partial\Omega$, is bounded by the usual (horizontal) perimeter of Ω in $B(q, r)$. This is Corollary 6.5.

This result should be contrasted with the work of Naor and Young in higher dimensional Heisenberg groups: in [Naor and Young 2018, Proposition 41], they prove that if $\Omega \subset \mathbb{H}^n$, $n \geq 2$, is an intrinsic Lipschitz domain, then the L^2-vertical perimeter of Ω in balls centred at $\partial\Omega$ is automatically bounded by the horizontal perimeter — without any reference to β-numbers. Then, at the very end of [Naor and Young 2018], see also Remark 4 in the same work, the authors mention showing in a forthcoming paper [Naor and Young 2022] that a similar inequality fails for the L^2-vertical perimeter in $\mathbb{H}^1 = \mathbb{H}$, but holds for the L^p-vertical perimeter for some $p > 2$ (specifically, the authors mention $p = 4$).[1] If this is the case, then, according to Corollary 6.5, one cannot expect the β-numbers of intrinsic Lipschitz graphs to satisfy an L^2-Carleson packing condition. This is in contrast to the situation in $\mathbb{R}^n$, where the β-numbers on Lipschitz graphs do satisfy an L^2-Carleson packing condition; see [David and Semmes 1991, (C3)].

2. Preliminaries

In this section, we collect essential notions related to the algebraic and metric structures of the first Heisenberg group $\mathbb{H}$, and we recall the definition and basic properties of intrinsic Lipschitz graphs over vertical planes in $\mathbb{H}$. For a more thorough introduction to these subjects, we refer the reader to [Capogna et al. 2007; Serra Cassano 2016].

2A. *Right- and left-invariant vector fields.* Recall from the Introduction that X and Y denote the standard left-invariant vector fields on $\mathbb{H}$ defined in (1.2). We will also work with their right invariant counterparts

$$\widetilde{X} = \partial_x + \tfrac{1}{2} y \partial_t \quad \text{and} \quad \widetilde{Y} = \partial_y - \tfrac{1}{2} x \partial_t.$$

We define the left and right (horizontal) gradients of $\phi \in \mathcal{C}^1(\mathbb{R}^3)$ as the 2-vectors

$$\nabla_{\mathbb{H}}\phi = (X\phi, Y\phi) \quad \text{and} \quad \widetilde{\nabla}_{\mathbb{H}}\phi = (\widetilde{X}\phi, \widetilde{Y}\phi).$$

For $V = (V_1, V_2) \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^2)$, we define the left and right divergences as the functions

$$\operatorname{div}_{\mathbb{H}} V := XV_1 + YV_2 \in \mathcal{C}^0(\mathbb{R}^3) \quad \text{and} \quad \widetilde{\operatorname{div}}_{\mathbb{H}} V := \widetilde{X}V_1 + \widetilde{Y}V_2 \in \mathcal{C}^0(\mathbb{R}^3).$$

For $V, W \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^2)$, we define the *inner product*

$$\langle V, W \rangle := V_1 W_1 + V_2 W_2 \in \mathcal{C}^1(\mathbb{R}^3).$$

Finally, we denote the left and right sub-Laplacians as

$$\Delta_{\mathbb{H}} := XX + YY \quad \text{and} \quad \widetilde{\Delta_{\mathbb{H}}} := \widetilde{X}\widetilde{X} + \widetilde{Y}\widetilde{Y}.$$

[1] While the present paper was under review, the paper [Naor and Young 2022] appeared, and indeed contains the results mentioned here.

2B. ***Metric structure.*** Various left-invariant distance functions on $\mathbb{H}$ are commonly used in the literature, for instance the standard sub-Riemannian distance or the Korányi metric given in (1.3). The choice of metric that we are going to use is motivated by the divergence theorem (Theorem 4.3), which holds for the spherical Hausdorff measure $\mathcal{S}^3$ with respect to the metric

$$d : \mathbb{H} \times \mathbb{H} \to [0, +\infty), \quad d(p, q) := \|q^{-1} \cdot p\|, \tag{2.1}$$

where

$$\|(x, y, t)\| := \max\{|(x, y)|, 2\sqrt{|t|}\}.$$

However, every left-invariant metric on $\mathbb{H}$ that is continuous with respect to the Euclidean topology on $\mathbb{R}^3$ and homogeneous with respect to the one-parameter family of *Heisenberg dilations* $(\delta_\lambda)_{\lambda>0}$,

$$\delta_\lambda : \mathbb{H} \to \mathbb{H}, \quad \delta_\lambda(x, y, t) := (\lambda x, \lambda y, \lambda^2 t),$$

is bi-Lipschitz equivalent to the metric d; this applies in particular to the Korányi distance d_{Kor}. Unless otherwise stated, all metric concepts such as balls $B(p, r)$, diameters, and Hausdorff measures will be defined using the metric d.

2C. ***Intrinsic Lipschitz graphs.*** Let $\mathbb{W}$ be a vertical subgroup with complementary horizontal subgroup $\mathbb{V}$; recall from the paragraph after Theorem 1.4 that, in this paper, the *complementary horizontal subgroup of* $\mathbb{W}$ is the orthogonal complement of $\mathbb{W}$ in $\mathbb{R}^3$. Any point $p \in \mathbb{H}$ can be written as $p = w \cdot v$ for uniquely given $w \in \mathbb{W}$ and $v \in \mathbb{V}$. We write $w =: \pi_{\mathbb{W}}(p)$ and call it the *vertical projection of p to* $\mathbb{W}$; similarly, we denote the *horizontal projection* by $v = \pi_{\mathbb{V}}(p)$. These projections have been studied in connection with uniform rectifiability problems in the Heisenberg group; see for example [Chousionis et al. 2019b; Fässler et al. 2020].

Definition 2.2. A function $\phi : \mathbb{W} \to \mathbb{V}$ is *intrinsic L-Lipschitz* if

$$\|\pi_{\mathbb{V}}(\Phi(w')^{-1}\Phi(w))\| \le L\|\pi_{\mathbb{W}}(\Phi(w')^{-1}\Phi(w))\|, \quad \text{for all } w, w' \in \mathbb{W}, \tag{2.3}$$

where $\Phi : \mathbb{W} \to \mathbb{H}$ denotes the *graph map* $\Phi(w) = w \cdot \phi(w)$. The *intrinsic graph* of ϕ is

$$\Gamma_\phi := \{w \cdot \phi(w) : w \in \mathbb{W}\} = \Phi(\mathbb{W}).$$

The term *intrinsic* refers to the fact that if ϕ is an intrinsic L-Lipschitz function, then, for all $p \in \mathbb{H}$ and $r > 0$, also $\tau_p(\delta_r(\Gamma_\phi))$ is an intrinsic graph of an intrinsic L-Lipschitz function. According to [Chousionis et al. 2019b, Remark 2.6], an intrinsic L-Lipschitz graph over an arbitrary vertical plane can be mapped to an intrinsic L-Lipschitz graph over the (y, t)-plane by an isometry of the form

$$R_\theta : \mathbb{H} \to \mathbb{H}, \quad R_\theta(x, y, t) := (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta, t).$$

Since moreover the (complexified) kernel of the Heisenberg Riesz transform satisfies

$$(XG - iYG) \circ R_\theta = e^{i\theta}(XG - iYG),$$

we may without loss of generality assume in the following that $\mathbb{W}$ is the (y,t)-plane and $\mathbb{V}$ is the x-axis. For this choice, we have

$$\pi_{\mathbb{V}}(x,y,t) = (x,0,0) \quad \text{and} \quad \pi_{\mathbb{W}}(x,y,t) = \left(0, y, t + \tfrac{1}{2}xy\right), \qquad \text{for all } (x,y,t) \in \mathbb{H}.$$

Moreover, the map $(x,0,0) \mapsto x$ provides an isometric isomorphism between $(\mathbb{V}, \cdot, d)$ and $(\mathbb{R}, +, |\cdot|)$, and under this identification of $\mathbb{V}$ with $\mathbb{R}$, the intrinsic Lipschitz condition (2.3) is equivalent to

$$|\phi(0,y,t) - \phi(0,y',t')| \leq L\|\pi_{\mathbb{W}}(\Phi(0,y',t')^{-1}\Phi(0,y,t))\|, \quad \text{for all } (y,t),(y',t') \in \mathbb{R}^2.$$

The subgroup $(\mathbb{W}, \cdot)$ is isomorphic to $(\mathbb{R}^2, +)$, and the map $(0,y,t) \mapsto (y,t)$ pushes the measure $\mathcal{H}^3|_{\mathbb{W}}$ forward to $c\mathcal{L}^2$ on $\mathbb{R}^2$, for a constant $0 < c < \infty$. As mentioned in the Introduction, an intrinsic Lipschitz function $\phi : \mathbb{W} \to \mathbb{V}$ possesses an *intrinsic gradient* $\nabla^{\phi}\phi$ at $\mathcal{H}^3$ almost every point of $\mathbb{W}$. In analogy with the behaviour of Euclidean Lipschitz functions, if $\phi : \mathbb{W} \to \mathbb{V}$ is intrinsic Lipschitz, then

$$\|\nabla^{\phi}\phi\|_{L^{\infty}(\mathcal{H}^3|_{\mathbb{W}})} < \infty,$$

by [Citti et al. 2014, Proposition 4.4]. More information about intrinsic gradients is collected for instance in [Chousionis et al. 2019b, Section 4.2; Serra Cassano 2016].

3. Vertical oscillation coefficients

In this section, we define and study the main new concept of the paper, the *vertical oscillation coefficients*. These coefficients are derived from the recent notion of *vertical perimeter*, due to [Lafforgue and Naor 2014, Definition 4.2] (see also [Naor and Young 2018, (28)]).

Definition 3.1 (vertical perimeter). Let $\Omega, U \subset \mathbb{H}$ be Lebesgue measurable sets, and let $s > 0$ be a scale. The *vertical perimeter of* Ω *relative to* U *at scale* s is the quantity

$$\mathrm{v}_{\Omega}(U)(s) := \int_U |\chi_{\Omega}(p) - \chi_{\Omega}(p \cdot (0,0,s^2))|\, dp.$$

Here and in the following, dp refers to integration with respect to Lebesgue measure $\mathcal{L}^3$ on $\mathbb{R}^3$, which agrees up to a multiplicative constant with $\mathcal{H}^4$.

Remark 3.2. Having first defined the vertical perimeter $\mathrm{v}_{\Omega}(U)(s)$ at a fixed scale $s > 0$, [Lafforgue and Naor 2014, (70)] and [Naor and Young 2018, Section 2.2] proceed to define the L^2*-vertical perimeter of* Ω as the $L^2(ds/s)$-norm of the function $s \mapsto \mathrm{v}_{\Omega}(\mathbb{H})/s$. More generally, for $p \geq 1$ and an open set $U \subset \mathbb{H}$, one can consider (as in [Naor and Young 2018, (68)] for example) the L^p*-vertical perimeter of* Ω *in* U:

$$\wp_{\Omega,p}(U) := \left\| s \mapsto \frac{\mathrm{v}_{\Omega}(U)(s)}{s} \right\|_{L^p(ds/s)} = \left(\int_0^{\infty} \left(\frac{\mathrm{v}_{\Omega}(U)(s)}{s} \right)^p \frac{ds}{s} \right)^{1/p}.$$

It would be interesting to know if the L^p-vertical perimeter of Ω — for some $p \geq 1$ — can be related to the boundedness of the Heisenberg Riesz transform on $L^2(\mathcal{H}^3|_{\partial\Omega})$.

Definition 3.3 (vertical oscillation coefficients). Let $\Omega \subset \mathbb{H}$ be a Lebesgue measurable (typically open) set, and let $B(p,r) \subset \mathbb{H}$ be a ball. We define

$$\operatorname{osc}_{\Omega}(B(p,r)) := \frac{1}{r^4} \fint_0^r \mathrm{v}_{\Omega}(B(p,r))(s)\, ds.$$

Next we examine the basic properties of the oscillation coefficients.

Lemma 3.4. *There is an absolute constant $C \geq 1$ such that $\operatorname{osc}_{\Omega}(B(p,r)) \leq C$ for all Lebesgue measurable sets $\Omega \subset \mathbb{H}$ and all balls $B(p,r) \subset \mathbb{H}$. The vertical oscillation coefficients are approximately monotone in the following sense: if $B(p_1,r_1) \subset B(p_2,r_2) \subset \mathbb{H}$ are two balls with $r_2 \leq C_1 r_1$, then*

$$\operatorname{osc}_{\Omega}(B(p_1,r_1)) \lesssim_{C_1} \operatorname{osc}_{\Omega}(B(p_2,r_2)). \tag{3.5}$$

Finally, the vertical oscillation coefficients are invariant with respect to dilations and left translations in the following sense:

$$\operatorname{osc}_{\delta_t(q\cdot\Omega)}(B(\delta_t(q\cdot p),tr)) = \operatorname{osc}_{\Omega}(B(p,r)), \quad t>0, \;\; q \in \mathbb{H}. \tag{3.6}$$

Proof. To prove the first claim, observe that $\mathrm{v}_{\Omega}(B(p,r))(s) \leq 2\mathcal{H}^4(B(p,r)) \sim r^4$ for all $0 \leq s \leq r$, so

$$\operatorname{osc}_{\Omega}(B(p,r)) \lesssim \fint_0^r \frac{r^4}{r^4}\, ds = 1.$$

The approximate monotonicity property (3.5) follows immediately from the inequality $\mathrm{v}_{\Omega}(B(p_1,r_1))(s) \leq \mathrm{v}_{\Omega}(B(p_2,r_2))(s)$, valid for all $s>0$.

The left-invariance $\operatorname{osc}_{q\cdot\Omega}(B(q\cdot p,r)) = \operatorname{osc}_{\Omega}(B(p,r))$ of the vertical oscillation coefficients follows from the evident left-invariance of the vertical perimeter, so we assume that $p=q=0$ and prove that

$$\operatorname{osc}_{\delta_t(\Omega)}(B(0,tr)) = \operatorname{osc}_{\Omega}(B(0,r)), \quad t>0.$$

To see this, we start by expanding

$$\begin{aligned}\operatorname{osc}_{\delta_t(\Omega)}(B(0,tr)) &= \frac{1}{(tr)^5}\int_0^{tr} \mathrm{v}_{\delta_t(\Omega)}(B(0,tr))(s)\, ds \\ &= \frac{1}{(tr)^5}\int_0^{tr}\int_{B(0,tr)} |\chi_{\delta_t(\Omega)}(p) - \chi_{\delta_t(\Omega)}(p\cdot(0,0,s^2))|\, dp\, ds.\end{aligned}$$

Then, we make the change of variables $p \mapsto \delta_t(q)$, and finally $s \mapsto ut$:

$$\operatorname{osc}_{\delta_t(\Omega)}(B(0,tr)) = \frac{1}{r^5}\int_0^r \int_{B(0,r)} |\chi_{\Omega}(q) - \chi_{\Omega}(q\cdot(0,0,u^2))|\, dq\, du = \operatorname{osc}_{\Omega}(B(0,r)). \qquad \square$$

Remark 3.7. The previous lemma says that $\operatorname{osc}_{\Omega}(B(p,r)) \lesssim 1$ no matter what Ω looks like. If Ω is the sub- or supergraph of an intrinsic Lipschitz function satisfying better than $\frac{1}{2}$-Hölder regularity in the vertical direction, then the oscillation coefficients of Ω have geometric decay. A more precise statement can be found in Lemma 5.6.

In connection with singular integrals, the vertical oscillation coefficients will enter through the next lemma.

Lemma 3.8. *Let $\Omega \subset \mathbb{H}$ be a Lebesgue measurable set. Let $p \in \mathbb{H}$, $r > 0$, and let $\psi \in \mathcal{C}^1(\mathbb{R}^3)$ with* $\operatorname{spt}\psi \subset B(p,r)$. *Then*

$$\left|\frac{1}{r^4}\int_{\Omega} \partial_t \psi(p)\,dp\right| \lesssim \|\partial_t\psi\|_\infty \operatorname{osc}_{\Omega}(B(p,10r)), \tag{3.9}$$

where $\partial_t\psi$ is the derivative of ψ with respect to the third variable.

Proof. We start by reducing to the case $B(p,r)=B(0,1)$. So, assume that (3.9) holds for every Lebesgue measurable set Ω and all $\psi \in \mathcal{C}^1(\mathbb{R}^3)$ with $\operatorname{spt}\psi \subset B(0,1)$ and with $\operatorname{osc}_{\Omega}(B(0,10))$ on the right-hand side. Then, if $\psi \in \mathcal{C}^1(\mathbb{R}^3)$ with $\operatorname{spt}\psi \subset B(p,r)$, we consider the function $\psi_{p,r} = \psi \circ \tau_p \circ \delta_r \in \mathcal{C}^1(\mathbb{R}^3)$ with $\operatorname{spt}\psi_{p,r} \subset B(0,1)$. It follows that

$$\begin{aligned}\left|\frac{1}{r^4}\int_{\Omega} \partial_t\psi(q)\,dq\right| &= \left|\int_{\delta_{1/r}(p^{-1}\cdot\Omega)} \partial_t\psi(p\cdot\delta_r(q))\,dq\right| \\ &= \left|\int_{\delta_{1/r}(p^{-1}\cdot\Omega)} r^{-2}\partial_t\psi_{p,r}(q)\,dq\right| \\ &\lesssim \frac{\|\partial_t\psi_{p,r}\|_\infty}{r^2}\operatorname{osc}_{\delta_{1/r}(p^{-1}\cdot\Omega)}(B(0,10)) \\ &= \|\partial_t\psi\|_\infty \operatorname{osc}_{\Omega}(B(p,10r)),\end{aligned}$$

using Lemma 3.4 in the last equation.

It remains to prove the case $B(p,r)=B(0,1)$, so fix $\psi \in \mathcal{C}^1(\mathbb{R}^3)$ with $\operatorname{spt}\psi \subset B(0,1)$. By Fubini's theorem, we can write

$$\int_{\Omega} \partial_t\psi(q)\,dq = \int_{\mathcal{L}}\int_{\ell} \partial_t\psi(q)\chi_{\Omega}(q)\,d\mathcal{H}^1_E(q)\,d\eta(\ell), \tag{3.10}$$

where $\mathcal{L}$ stands for the collection of vertical lines, η is the two-dimensional Lebesgue measure on $\mathbb{R}^2$ (which is used to parametrise $\mathcal{L}$), and $\mathcal{H}^1_E$ denotes the 1-dimensional Hausdorff measure with respect to the Euclidean distance. Next, we note that if $\ell \in \mathcal{L}$ is a fixed line, then

$$\int_{\ell} \partial_t\psi(q)\,d\mathcal{H}^1_E(q) = 0. \tag{3.11}$$

Now, let $Q := [-5,5]^2 \times [-2,-1] \subset B(0,10)$. We note that whenever $\ell \in \mathcal{L}$ is a line with nonzero contribution in (3.10), we have $\ell \cap B(0,1) \neq \varnothing$, and in particular

$$\mathcal{H}^1_E(\ell \cap Q) = 1.$$

Then, use (3.10)–(3.11) to write

$$\begin{aligned}\left|\int_{\Omega} \partial_t\psi(q)\,dq\right| &= \left|\int_{\mathcal{L}}\int_{\ell\cap Q}\int_{\ell} \partial_t\psi(q)[\chi_{\Omega}(q)-\chi_{\Omega}(p)]\,d\mathcal{H}^1_E(q)\,d\mathcal{H}^1_E(p)\,d\eta(\ell)\right| \\ &\le \|\partial_t\psi\|_\infty \int_{\mathcal{L}}\int_{\ell\cap Q}\int_{\ell\cap B(0,1)} |\chi_{\Omega}(q)-\chi_{\Omega}(p)|\,d\mathcal{H}^1_E(q)\,d\mathcal{H}^1_E(p)\,d\eta(\ell).\end{aligned}$$

Next, for $\ell \in \mathcal{L}$ and $p \in \ell \cap Q$ fixed, we make the change of variable $q \mapsto p \cdot (0, 0, s)$ in the innermost integral: since $q \in \ell \cap B(0, 1)$ and $p \in \ell \cap Q$, we note that $s \in [0, 3]$. This leads to

$$\begin{aligned}\left|\int_\Omega \partial_t \psi(q)\, dq\right| &\le \|\partial_t \psi\|_\infty \int_{\mathcal{L}} \int_{\ell\cap Q} \int_0^3 |\chi_\Omega(p\cdot(0,0,s)) - \chi_\Omega(p)|\, ds\, d\mathcal{H}^1_E(p)\, d\eta(\ell) \\ &\le \|\partial_t \psi\|_\infty \int_0^3 \int_{\mathcal{L}} \int_{\ell\cap B(0,10)} |\chi_\Omega(p\cdot(0,0,s)) - \chi_\Omega(p)|\, d\mathcal{H}^1_E(p)\, d\eta(\ell)\, ds \\ &\lesssim \|\partial_t \psi\|_\infty \int_0^{\sqrt{3}} \mathrm{v}_\Omega(B(0,10))(u) u\, du \lesssim \|\partial_t \psi\|_\infty \operatorname{osc}_\Omega(B(0,10)).\end{aligned}$$

In the final inequality, the factor "u" was simply estimated by $\sqrt{3}$. □

3A. *Vertical oscillation vs. vertical β-numbers.* Given a set $E \subset \mathbb{H}$ and a ball $B(q, r) \subset \mathbb{H}$, we recall from [Chousionis et al. 2019b, Definition 3.3] the following *vertical β-number of E in $B(q, r)$*, $q \in E$,

$$\beta_{E,\infty}(B(q,r)) := \inf_{\mathbb{W},z} \sup_{x\in B(q,r)\cap E} \frac{\operatorname{dist}(x, z\cdot\mathbb{W})}{r},$$

where the inf runs over all vertical subgroups $\mathbb{W} \subset \mathbb{H}$ and all points $z \in \mathbb{H}$. More generally, one can consider the L^p-variants

$$\beta_{E,p}(B(q,r)) := \inf_{\mathbb{W},z} \left(\frac{1}{r^3} \int_{B(q,r)\cap E} \left(\frac{\operatorname{dist}(x, z\cdot\mathbb{W})}{r}\right)^p d\mathcal{H}^3(x)\right)^{1/p}, \quad 1 \le p < \infty,$$

assuming that E has locally finite 3-dimensional measure. If E happens to be 3-regular, then the $\beta_{E,p}$-numbers are essentially monotone in p:

$$\beta_{E,p_1}(B(q,r)) \lesssim \beta_{E,p_2}(B(q,r)), \quad q \in E, \ 1 \le p_1 \le p_2 \le \infty.$$

The next theorem shows that the vertical oscillation coefficients of Ω are always bounded by the $\beta_{E,\infty}$-numbers of $\partial\Omega$, and also *almost* bounded from above by the $\beta_{E,1}$-numbers of $\partial\Omega$. After this statement concerning general domains Ω, we will give a corollary to domains with 3-regular boundaries: in this case the word *almost* above can be omitted.

Theorem 3.12. *Let $\Omega \subset \mathbb{H}$ be an open set such that $\partial\Omega$ has locally finite 3-dimensional measure, and let $r > 0$. Then, for any $p \in \partial\Omega$ and $0 < s \le r$,*

$$\frac{\mathrm{v}_\Omega(B(p,r))(s)}{r^4} \lesssim_\epsilon \inf_{\mathbb{W},z}\left[\frac{1}{r^3}\int_{B(p,15r)\cap\partial\Omega} \frac{d(q, z\cdot\mathbb{W})}{15r}\, d\mathcal{H}^3(q) + \epsilon\left(\sup_{q\in B(p,15r)\cap\partial\Omega} \frac{d(q, z\cdot\mathbb{W})}{15r}\right)\right] \tag{3.13}$$

for any nondecreasing function $\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\epsilon(\delta) \to 0$ as $\delta \to 0$.

The same estimate for the vertical oscillation coefficient $\operatorname{osc}_\Omega(B(p, r))$ follows immediately by taking the average over $s \in (0, r]$ on the left-hand side; we will however need the sharper result later, in Section 6B. Note also that the quantity on the right-hand side of (3.13) looks like

$$\beta_{\partial\Omega,1}(B(p, 15r)) + \epsilon[\beta_{\partial\Omega,\infty}(B(p, 15r))],$$

but can be sometimes larger, as only one choice of $z, \mathbb{W}$ is made on the right-hand side of (3.13). The quantities on both sides of the inequality (3.13) are invariant under scaling and translation, so we may assume that $p = 0$ and $r = 1$. We start the proof of Theorem 3.12 with the following simple lemma.

Lemma 3.14. *Let $\Omega \subset \mathbb{H}$ be an open set. Let $H \subset \mathbb{H}$ be a vertical half-space, that is, a half-space bounded by the translate of some vertical subgroup. Then*

$$\mathrm{v}_\Omega(B(0,1))(s) \leq 2\mathcal{H}^4([\Omega \Delta H] \cap B(0,3)), \quad 0 < s \leq 1.$$

Proof. Let $0 \leq s \leq 1$. Note that $\chi_H(q) = \chi_H(q \cdot (0,0,s^2))$ for all $q \in \mathbb{H}$. Hence,

$$\begin{aligned}\mathrm{v}_\Omega(B(0,1))(s) &\leq \int_{B(0,1)} |\chi_\Omega(q) - \chi_H(q) + \chi_H(q \cdot (0,0,s^2)) - \chi_\Omega(q \cdot (0,0,s^2))|\,dq \\ &\leq 2\int_{B(0,3)} |\chi_\Omega(q) - \chi_H(q)|\,dq = 2\mathcal{H}^4([\Omega \Delta H] \cap B(0,3)). \qquad \square\end{aligned}$$

Now, to conclude the proof of Theorem 3.12, it suffices to show $\left(\text{after scaling } \Omega \text{ by } \frac{1}{3}\right)$ that there exists a vertical half-space $H \subset \mathbb{H}$ such that

$$\mathcal{H}^4([\Omega \Delta H] \cap B(0,1)) \lesssim_\epsilon \inf_{\mathbb{W},z}\left[\int_{B(0,5)\cap\partial\Omega} d(q, z\cdot\mathbb{W})\,d\mathcal{H}^3(q) + \epsilon\left(\sup_{q\in B(0,5)\cap\partial\Omega} d(q, z\cdot\mathbb{W})\right)\right]. \tag{3.15}$$

Further, to prove (3.15), we may assume that if $P := z \cdot \mathbb{W}$ is a vertical plane minimising the right-hand side in (3.15), then $d(q, P) \leq \delta := 10^{-10}$ for all $q \in B(0,5) \cap \partial\Omega$. Indeed, (3.15) is clear if this fails (with implicit constant $\sim 1/\epsilon(\delta)$). In particular, since $0 = p \in \partial\Omega$, we may write $P = z' \cdot \mathbb{W}$ with $d(0, z') \leq \delta$. By left-translating both P and Ω by the inverse of z' and then rotating suitably around the t-axis, we may suppose that $P = \{(0, y, t) : y, t \in \mathbb{R}\}$ and

$$\sup_{q\in B(0,4)\cap\partial\Omega} d(q, P) \leq \delta. \tag{3.16}$$

In other words, (3.16) holds for a suitable rotation of $(z')^{-1} \cdot \Omega$, but we keep denoting this set by Ω. We will no longer need the information $0 \in \partial\Omega$ in the sequel. Now, with this new notation, it suffices to prove (3.15) with $[\Omega \Delta H] \cap B(0, 1.1)$ on the left-hand side and, say, $B(0,4) \cap \partial\Omega$ on the right-hand side.

We will, in fact, show that there exists a vertical half-space $H \subset \mathbb{H}$ such that

$$\mathcal{H}^4([\Omega \Delta H] \cap B(0, 1.1)) \lesssim \int_{B(0,4)\cap\partial\Omega} d(q, P)\,d\mathcal{H}^3(q). \tag{3.17}$$

So, the L^1-based β-number of $\partial\Omega$ dominates the vertical oscillation of Ω under the a priori assumption that the L^∞-based β-number is sufficiently small. We now choose H. We denote the (closed) half-spaces bounded by P by

$$\mathbb{H}_+ := \{(x,y,t) : x \geq 0\} \quad \text{and} \quad \mathbb{H}_- := \{(x,y,t) : x \leq 0\}.$$

Write U_+, U_- for the connected components of $B(0,4) \setminus P(\delta)$, where $P(\delta)$ is the closed δ-neighbourhood of P, with

$$U_+ \subset \mathbb{H}_+ \quad \text{and} \quad U_- \subset \mathbb{H}_-.$$

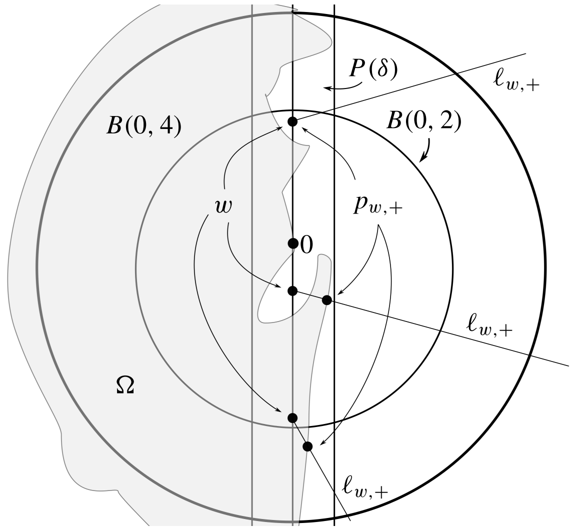

Figure 1. Various concepts in the proof of Theorem 3.12. Scenario (a) is depicted.

By (3.16), we may infer that either $U_+ \subset \Omega$ or $U_+ \cap \Omega = \varnothing$, and similarly either $U_- \subset \Omega$ or $U_- \cap \Omega = \varnothing$. The definition of H depends on which of these cases occur:

(a) If $U_- \subset \Omega$ and $U_+ \cap \Omega = \varnothing$, let $H := \mathbb{H}_-$.

(b) If $U_- \cap \Omega = \varnothing$ and $U_+ \subset \Omega$, let $H := \mathbb{H}_+$.

(c) If $U_+, U_- \subset \Omega$, let H be any vertical half-space containing $B(0,4)$.

(d) If $U_+ \cap \Omega = \varnothing = U_- \cap \Omega$, let H be any vertical half-space with $H \cap B(0,4) = \varnothing$.

The point of these choices is that always

$$[\Omega \Delta H] \cap B(0,4) \subset P(\delta), \tag{3.18}$$

as one may easily verify.

We claim that (3.17) holds for the choice of H above. To see this, we will need additional notation. For $w \in P$, let

$$\ell_w := \{w \cdot (x,0,0) : x \in \mathbb{R}\}$$

be the left-translate of the x-axis passing through w. We also define the half-lines

$$\ell_{w,+} := \ell_w \cap \mathbb{H}_+ \quad \text{and} \quad \ell_{w,-} := \ell_w \cap \mathbb{H}_-,$$

see Figure 1. To prove (3.17), we study separately the parts of $[\Omega \Delta H] \cap B(0,1.1)$ inside $\mathbb{H}_-$ and $\mathbb{H}_+$. These investigations are symmetrical, so we restrict our attention to $\mathbb{H}_+$. For notational convenience, we write $B(0,s) \cap \mathbb{H}_+ := B_+(0,s)$ in the sequel. We will apply the general integration estimate

$$\mathcal{H}^4(A) \sim \int_P \mathcal{H}^1(A \cap \ell_w)\, dw, \qquad A \subset \mathbb{H} \text{ Borel}. \tag{3.19}$$

Here "dw" refers to the 3-dimensional Hausdorff measure on P, which coincides (up to a constant) with the Lebesgue measure on P. In order to establish formula (3.19), recall that $\mathcal{H}^4$ agrees up to a multiplicative constant with the 3-dimensional Lebesgue measure and the transformation $\Phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{H}$, $\Phi((w_1, w_2), s) = (0, w_1, w_2) \cdot (s, 0, 0)$ has Jacobian determinant equal to 1. Hence,

$$\mathcal{H}^4(A) \sim \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \chi_A(\Phi(w, s))\, ds\, dw. \tag{3.20}$$

Next, for every $w \in P$, the map $s \mapsto \Phi(w, s) = w \cdot (s, 0, 0)$ is an isometry between $(\mathbb{R}, |\cdot|)$ and (ℓ_w, d), and thus we find that

$$\int_{-\infty}^{\infty} \chi_A(\Phi(w, s))\, ds = \int_{\ell_w} \chi_A(q)\, d\mathcal{H}^1(q) = \mathcal{H}^1(A \cap \ell_w). \tag{3.21}$$

These facts together prove (3.19). Applied to the set $A = [\Omega \Delta H] \cap B_+(0, 1.1)$, this formula then yields

$$\mathcal{H}^4([\Omega \Delta H] \cap B_+(0, 1.1)) \lesssim \int_{P \cap B(0,2.2)} \mathcal{H}^1([\Omega \Delta H] \cap \ell_{w,+} \cap B(0, 1.1))\, dw. \tag{3.22}$$

Here, the integration is restricted to $P \cap B(0, 2.2)$ as $\Phi(w, s)$, $w \in P$, can lie in $B(0, 1.1)$ only if $|s| \leq 1.1$, and in that case $d(\Phi(w, s), 0) \geq d(w, 0) - d(0, (s, 0, 0)) > 1.1$ if $w \in P \setminus B(0, 2.2)$; in other words, the lines ℓ_w with $w \in P \setminus B(0, 2.2)$ avoid $B(0, 1.1)$. Now, we fix $w \in P \cap B(0, 2.2)$, and we will establish a suitable pointwise bound for the integrand in (3.22). To this end:

- If $\ell_{w,+} \cap \partial[\Omega \Delta H] \cap B(0, 4) = \varnothing$, set $p_{w,+} := w$.
- If $\ell_{w,+} \cap \partial[\Omega \Delta H] \cap B(0, 4) \neq \varnothing$, let

  $$p_{w,+} := \max[\ell_{w,+} \cap \partial[\Omega \Delta H] \cap B(0, 4)],$$

  where the max refers to the only natural ordering on $\ell_{w,+}$.

Then, by (3.18), we have in both cases

$$p_{w,+} \in \ell_{w,+} \cap P(\delta) \subset P(\delta) \cap B(0, 3), \quad w \in P \cap B(0, 2.2). \tag{3.23}$$

(If w is sufficiently close to $\partial B(0, 2.2)$, then it may happen that $\ell_{w,+} \cap P(\delta) \not\subset B(0, 2.2)$, see Figure 1. However, $\delta > 0$ has been chosen so small that the second inclusion in (3.23) holds.) Next we define

$$h_+(w) := \operatorname{dist}(p_{w,+}, P), \quad w \in P \cap B(0, 2.2).$$

The *suitable pointwise bound* for the integrand in (3.22) is

$$\mathcal{H}^1([\Omega \Delta H] \cap \ell_{w,+} \cap B(0, 1.1)) \leq h_+(w), \quad w \in P \cap B(0, 2.2). \tag{3.24}$$

In proving (3.24), we may evidently assume that

$$[\Omega \Delta H] \cap \ell_{w,+} \cap B(0, 1.1) \neq \varnothing. \tag{3.25}$$

Now, to prove (3.24), we will first argue that also

$$[\Omega \Delta H]^c \cap \ell_{w,+} \cap B(0, 4) \neq \varnothing. \tag{3.26}$$

This will follow immediately once we manage to argue that

$$U_+ \subset [\Omega \Delta H]^c, \tag{3.27}$$

since evidently $\ell_{w,+} \cap U_+ \neq \varnothing$. The proof of (3.27) depends on the scenario (a)–(d):

(a) Here $U_+ \cap \Omega = \varnothing$ and $H = \mathbb{H}_-$, so $U_+ \subset \Omega^c \cap H^c \subset [\Omega \Delta H]^c$.

(b) Here $U_+ \subset \Omega$ and $H = \mathbb{H}_+$, so $U_+ \subset \Omega \cap H \subset [\Omega \Delta H]^c$.

(c) Here $U_+ \subset \Omega$ and $B(0,4) \subset H$, so $U_+ \subset \Omega \cap H \subset [\Omega \Delta H]^c$.

(d) Here $U_+ \cap \Omega = \varnothing$ and $H \cap B(0,4) = \varnothing$, so $U_+ \subset \Omega^c \cap H^c \subset [\Omega \Delta H]^c$.

We have now established (3.27), and hence (3.26). Combining (3.25)–(3.26), we see that

$$p_{w,+} = \max[\ell_{w,+} \cap \partial[\Omega \Delta H] \cap B(0,4)]$$

is well-defined, and moreover

$$[\Omega \Delta H] \cap \ell_{w,+} \cap B(0, 1.1) \subset [w, p_{w,+}], \tag{3.28}$$

where $[w, p_{w,+}]$ stands for the (horizontal) line segment connecting w and $p_{w,+}$. The point $p_{w,+}$ can be uniquely expressed as $p_{w,+} = w \cdot v_+$, where $v_+ = (x_+, 0, 0)$ for some $x_+ \geq 0$. Thus we find by the definition of the metric d that

$$x_+ \leq \|\bar{w}^{-1} w v_+\| = d(w v_+, \bar{w}) = d(p_{w,+}, \bar{w}), \quad \text{for all } \bar{w} \in P.$$

On the other hand, it holds that $d(p_{w,+}, w) = x_+$. Hence

$$h_+(w) = \operatorname{dist}(p_{w,+}, P) = d(p_{w,+}, w) = \mathcal{H}^1([w, p_{w,+}]), \tag{3.29}$$

where the last identity follows from the fact that $x \mapsto w \cdot (x, 0, 0)$ is an isometry from $(\mathbb{R}, |\cdot|)$ to (ℓ_w, d). We can now infer (3.24) from (3.28) and (3.29).

Before proceeding further, we record that the function $h_+ : P \cap B(0, 2.2) \to \mathbb{R}$ is Borel, in fact even upper semicontinuous. To see this, note that $p_{w,+}$ is always contained in the compact set

$$K := (P \cup \partial[\Omega \Delta H]) \cap \overline{B(0,3)}$$

for $w \in P \cap B(0, 2.2)$, and, consequently, also $h_+(P \cap B(0, 2.2))$ is contained in the compact set $K' := \{\operatorname{dist}(p, P) : p \in K\} \subset \mathbb{R}$. If h_+ was not upper semicontinuous, there would exist $w \in P \cap B(0, 2.2)$, $\epsilon > 0$, and a sequence $(w_n)_n \subseteq P \cap B(0, 2.2)$ with

$$\lim_{n \to \infty} w_n = w \quad \text{and} \quad \lim_{n \to \infty} h_+(w_n) > h_+(w).$$

We may assume that the limit on the right exists by the compactness of K'. Reducing to a further subsequence if necessary, we may assume that the sequence of points $p_{w_n,+} = w_n \cdot (h_+(w_n), 0, 0)$ converges to a point $p = w \cdot v \in K$. Moreover,

$$h_+(w) < \lim_{k \to \infty} h_+(w_n) = \lim_{k \to \infty} \operatorname{dist}(p_{w_n,+}, P) = \operatorname{dist}(p, P). \tag{3.30}$$

Since $p \in \ell_{w,+} \cap \partial[\Omega \Delta H] \cap B(0,4)$ (note that $p \notin P$ by (3.30)), this contradicts the maximality in the definition of $p_{w,+}$, and the proof of the upper semicontinuity of h_+ is complete.

We now resume the proof of our goal (3.17). Combining (3.18) and (3.24), we have now established that

$$\mathcal{H}^4([\Omega \Delta H] \cap B_+(0,1.1)) \lesssim \int_{P \cap B(0,2.2)} h_+(w)\,dw = \int_{P \cap B(0,2.2)} \operatorname{dist}(p_{w,+}, P)\,dw. \tag{3.31}$$

Noting that $p_{w,+} \in \partial\Omega \cap B(0,4)$ if $\operatorname{dist}(p_{w,+}, P) \neq 0$, this conclusion is not too far from (3.17) anymore. To arrive at (3.17) from (3.31), we use the vertical projection $\pi := \pi_P$ to the subgroup P, introduced in Section 2C. The most central features of π, for now, are that $\pi^{-1}\{w\} = \ell_w$ for $w \in P$ and that π does not increase the 3-dimensional Hausdorff measure (too much): there exists a constant $C \geq 1$ such that

$$\mathcal{H}^3(\pi(A)) \leq C\mathcal{H}^3(A), \quad A \subset \mathbb{H}. \tag{3.32}$$

For a proof, see [Chousionis et al. 2019b, Lemma 3.6]. To apply these facts, let $F : P \cap B(0,2.2) \to \mathbb{H}$ be the map $F(w) := p_{w,+}$. It follows from the discussion leading to (3.29) that $F(w) = w \cdot (h_+(w), 0, 0)$ and hence F is a Borel function. We deduce that the push-forward measure $\nu := F_\sharp(\mathcal{H}^3|_{B(0,2.2)\cap P})$, defined by $\nu(A) := \mathcal{H}^3(B(0,2.2) \cap P \cap F^{-1}(A))$, is a Borel measure on $\mathbb{H}$, and we have the integration formula

$$\int_{B(0,2.2)\cap P} \operatorname{dist}(p_{w,+}, P)\,dw = \int_{\mathbb{H}} \operatorname{dist}(q, P)\,d\nu(q), \tag{3.33}$$

see for instance [Mattila 1995, Theorem 1.19]. Clearly $\nu(\mathbb{H} \setminus F(P \cap B(0,2.2))) = 0$, which shows that $\operatorname{spt} \nu \subseteq \overline{F(P \cap B(0,2.2))}$. Moreover,

$$\nu \ll \mathcal{H}^3|_{\overline{F(P\cap B(0,2.2))}}$$

with bounded density, because $F^{-1}(A) \subset \pi(A)$ for all $A \subset \mathbb{H}$, and hence

$$\nu(A) = \mathcal{H}^3([B(0,2.2) \cap P] \cap F^{-1}(A)) \leq \mathcal{H}^3(\pi(A)) \leq C\mathcal{H}^3(A), \quad A \subset \mathbb{H},$$

using (3.32). Finally, we observe that

$$\overline{F(P \cap B(0,2.2))} \subseteq \overline{B(0,3)} \cap (P \cup \partial[\Omega \Delta H]) \subseteq B(0,4) \cap (P \cup \partial\Omega).$$

The last inclusion follows from the generalities $\partial[A \cup B], \partial[A \cap B] \subset \partial A \cup \partial B$:

$$\partial[\Omega \Delta H] \subset \partial[\Omega \cap H^c] \cup \partial[\Omega^c \cap H] \subset \partial\Omega \cup \partial H.$$

In cases (a) and (b) we have $\partial H = P$, while in cases (c) and (d) the boundary of H does not intersect $B(0,4)$. Combining these observations with (3.33), we find

$$\int_{B(0,2.2)\cap P} \operatorname{dist}(p_{w,+}, P)\,dw \lesssim \int_{B(0,4)\cap\partial\Omega} \operatorname{dist}(q, P)\,d\mathcal{H}^3(q).$$

Hence the right-hand side of (3.31) is bounded by a constant times the right-hand side of (3.17). The proof of (3.17), and of Theorem 3.12, is complete. ☐

We conclude the section by strengthening Theorem 3.12 in the case when $\partial\Omega$ is 3-regular.

Corollary 3.34. *Assume that $\Omega \subset \mathbb{H}$ is an open set such that $\partial\Omega$ is 3-regular. Then*

$$\frac{\mathrm{v}_{\Omega}(B(p,r))(s)}{r^4} \lesssim \beta_{\partial\Omega,1}(B(p,30r)), \quad p \in \partial\Omega, \ \ 0 < s \le r.$$

Proof. As usual, we may assume that $p = 0 \in \partial\Omega$ and $r = 1$. The proof is based on the general observation that if $E \subset \mathbb{H}$ is 3-regular and $P \subset \mathbb{H}$ is a vertical plane with $P \cap B(0,2) \neq \varnothing$, then

$$\operatorname{dist}(q, P) \lesssim \left(\int_{B(0,2)\cap E} d(x, P)\, d\mathcal{H}^3(x) \right)^{1/4}, \quad q \in E \cap B(0,1). \tag{3.35}$$

In Euclidean space, the analogous argument can be found for example in [David and Semmes 1991, (5.4)]. To prove (3.35), denote the right-hand side by $\beta^{1/4}$, and assume to reach a contradiction that there exists a point $q \in B(0,1) \cap E$ with $d(q, P) \ge C\beta^{1/4}$ for some large constant $C \ge 1$. We record that this implies that $\frac{1}{4}C\beta^{1/4} \le 1$, since we assumed $P \cap B(0,2) \neq \varnothing$. Also, clearly

$$\operatorname{dist}(y, P) \ge \tfrac{1}{2}C\beta^{1/4}, \quad y \in E \cap B\big(q, \tfrac{1}{4}C\beta^{1/4}\big) \subset B(0,2).$$

By 3-regularity,

$$(C\beta^{1/4})^3 \lesssim \mathcal{H}^3\big(B\big(q, \tfrac{1}{4}C\beta^{1/4}\big) \cap E\big) \le \frac{2}{C\beta^{1/4}} \int_{B(q, C\beta^{1/4}/4)\cap E} d(x, P)\, d\mathcal{H}^3(x) \le \frac{2\beta^{3/4}}{C},$$

and a contradiction is hence reached for $C \ge 1$ large enough.

From (3.35) (with "1" and "2" replaced by "15" and "30", respectively), choosing $P = z \cdot \mathbb{W}$ to be the best-approximating vertical plane for $\beta_{\partial\Omega,1}(B(0,30))$, we may now infer that

$$\inf_{\mathbb{W},z} \left[\int_{B(0,30)\cap\partial\Omega} d(q, z\cdot\mathbb{W})\, d\mathcal{H}^3(q) + \Big(\sup_{q \in B(0,15)\cap\partial\Omega} d(q, z\cdot\mathbb{W}) \Big)^4 \right] \lesssim \beta_{\partial\Omega,1}(B(0,30)).$$

In combination with Theorem 3.12 applied to $\epsilon(\delta) := \delta^4$, this inequality completes the proof. □

4. Boundedness of the Riesz transform

4A. ***Definitions and restating the main theorem.*** We now begin to relate the vertical oscillation coefficients to the boundedness of the 3-dimensional Riesz transform in $\mathbb{H}$. For technical convenience, we replace the vectorial kernel $\nabla_{\mathbb{H}} G = (XG, YG)$ from the Introduction with the complex kernel

$$K(p) = XG(p) - iYG(p),$$

where $G(p) = c\|p\|_{\mathrm{Kor}}^{-2}$ is still the fundamental solution to the sub-Laplace equation $\Delta_{\mathbb{H}} u = 0$. For the time being, we will only need to know that K is smooth outside the origin and -3-homogeneous with respect to the dilations δ_r:

$$K(\delta_r(q)) = r^{-3} K(q), \quad q \in \mathbb{H} \setminus \{0\}.$$

It follows that $|K(q)| \lesssim \|q\|^{-3}$ for $q \in \mathbb{H} \setminus \{0\}$. To the kernel K we associate the ϵ-truncated SIOs

$$\mathcal{R}_\epsilon(\mu)(p) := \int_{\{q \in \mathbb{H} : \|q^{-1}\cdot p\| \ge \epsilon\}} K(q^{-1}\cdot p)\, d\mu(q),$$

where μ is any complex measure on $\mathbb{H}$ with finite total variation.

Let μ be a locally finite Borel measure on $\mathbb{H}$. We say that $\mathcal{R}$ *is bounded on* $L^2(\mu)$ if the operators $\mathcal{R}_\epsilon$ are bounded on $L^2(\mu)$ uniformly in $\epsilon > 0$:

$$\|\mathcal{R}_\epsilon(f\mu)\|_{L^2(\mu)} \leq A\|f\|_{L^2(\mu)}, \quad f \in L^1(\mu) \cap L^2(\mu), \ \ \epsilon > 0.$$

The measures μ relevant here are 3-regular measures on intrinsic Lipschitz graphs. For intrinsic Lipschitz graphs $\Gamma \subset \mathbb{H}$ as in Theorem 1.8, we will directly prove the $L^2(\mu)$-boundedness of $\mathcal{R}$ for the particular measure

$$\mu := \mathcal{S}^3|_\Gamma,$$

where $\mathcal{S}^3$ is the 3-dimensional spherical Hausdorff measure defined using the metric d from (2.1). This choice makes it more straightforward to use the divergence theorem, but is otherwise arbitrary. In particular, once the $L^2(\mathcal{S}^3|_\Gamma)$-boundedness of $\mathcal{R}$ has been established, then it is easy to check (or see [Chousionis et al. 2019a, Lemma 3.1]) that $\mathcal{R}$ is bounded on $L^2(\mu)$ with respect to any 3-regular measure μ supported on Γ — in particular $\mathcal{H}^3|_\Gamma$.

Here is more precisely the result we will prove.

Theorem 4.1. *Let $\mathbb{W} \subset \mathbb{H}$ be a vertical subgroup, which we identify with $\{(y,t) : y, t \in \mathbb{R}\}$. Let $\phi : \mathbb{W} \to \mathbb{R}$ be an intrinsic Lipschitz function, let*

$$\Omega := \{(x,y,t) : x > \phi(\pi_{\mathbb{W}}(x,y,t))\}$$

be the supergraph of ϕ, and assume that

$$\int_0^\infty \operatorname{osc}_\Omega(B(p,r))\,\frac{dr}{r} \leq C < \infty, \quad p \in \Gamma.$$

Then $\mathcal{R}$ is bounded on $L^2(\mathcal{S}^3|_{\Gamma_\phi})$.

It is easy to check that $\mathbb{H} \setminus \Gamma_\phi$ has exactly two connected components, namely the supergraph Ω above and the subgraph $\Omega' := \{(x,y,t) : x < \phi(\pi_{\mathbb{W}}(x,y,t))\}$. Since

$$\operatorname{osc}_\Omega(B(p,r)) = \operatorname{osc}_{\mathbb{H}\setminus\Omega}(B(p,r)) = \operatorname{osc}_{\Omega'}(B(p,r)), \quad p \in \Gamma, \ \ r > 0,$$

fixing the complementary component in Theorem 4.1 does not render the statement less general than that of Theorem 1.8 in the Introduction.

4B. ***Test functions and the divergence theorem.*** We will prove Theorem 4.1 by verifying the conditions of Christ's local $T(b)$ theorem [1990]. We first introduce some more notation. From now on the intrinsic Lipschitz graph $\Gamma := \Gamma_\phi$ will be fixed as in Theorem 4.1, and we write $\mu := \mathcal{S}^3|_\Gamma$. We define the complex-valued function ν on Γ as

$$\nu(w\cdot\phi(w)) := \nu_1(w\cdot\phi(w)) + i\nu_2(w\cdot\phi(w)) := \frac{1}{\sqrt{1+(\nabla^\phi\phi(w))^2}} + i\frac{-\nabla^\phi\phi(w)}{\sqrt{1+(\nabla^\phi\phi(w))^2}}, \tag{4.2}$$

where $\nabla^\phi\phi$ is the intrinsic gradient of ϕ. Since ϕ is intrinsic Lipschitz, $\nu(p)$ exists for μ almost every $p \in \Gamma$, because $\nabla^\phi\phi(w)$ exists for $\mathcal{S}^3$ almost every $w \in \mathbb{W}$, and the graph map $\Phi(w) = w\cdot\phi(w)$ preserves $\mathcal{S}^3$

null sets by the area formula for intrinsic Lipschitz functions, see [Citti et al. 2014, Theorem 1.6]. By similar reasoning, $\nu \in L^\infty(\mu)$.

We also define the $\mathbb{R}^2$-valued map

$$\nu_H(q) = (\nu_1(q), \nu_2(q)) = \left(\frac{1}{\sqrt{1 + (\nabla^\phi \phi(w))^2}}, \frac{-\nabla^\phi \phi(w)}{\sqrt{1 + (\nabla^\phi \phi(w))^2}} \right) \in \mathbb{R}^2, \quad q = w \cdot \phi(w).$$

Then, by [Citti et al. 2014, Corollary 4.2], ν_H is the inward-pointing horizontal normal of the intrinsic supergraph $\Omega = \{(x, y, t) : x > \phi(\pi_{\mathbb{W}}(x, y, t))\}$, expressed in the frame $\{X, Y\}$. With this notation, we have the following divergence theorem due to Franchi, Serapioni, and Serra Cassano [Franchi et al. 2001].

Theorem 4.3 (divergence theorem). *Let $V \in \mathcal{C}^1_c(\mathbb{R}^3, \mathbb{R}^2)$, and let $\Gamma = \Gamma_\phi$ be an intrinsic Lipschitz graph as above. Then*

$$-\int_\Omega \operatorname{div}_{\mathbb{H}} V(p)\, dp = c \int_\Gamma \langle V, \nu_H \rangle \, d\mathcal{S}^3,$$

where $\Omega = \{(x, y, t) : x > \phi(\pi_{\mathbb{W}}(x, y, t))\}$ and $c > 0$ is an absolute constant.

Remark 4.4. The divergence theorem in [Franchi et al. 2001] looks a little different than Theorem 4.3 above, so a few remarks are in order. First, the sub- and supergraphs of intrinsic Lipschitz graphs are $\mathbb{H}$-Caccioppoli sets by [Franchi et al. 2011, Theorem 4.18], so [Franchi et al. 2001, Corollary 7.6] gives the formula

$$-\int_\Omega \operatorname{div}_{\mathbb{H}} V(p)\, dp = c \int_{\partial_{*,\mathbb{H}}\Omega} \langle V, \nu_H \rangle \, d\mathcal{S}^3, \quad V \in \mathcal{C}^1_c(\mathbb{R}^3, \mathbb{R}^2).$$

Here $\partial_{*,\mathbb{H}}\Omega$ stands for the measure theoretic boundary of Ω; see [Franchi et al. 2001, Definition 7.4]. But for domains Ω bounded by intrinsic Lipschitz graphs Γ, the measure theoretic boundary of Ω equals the topological boundary $\partial\Omega = \Gamma$: the inclusion $\Gamma \subset \partial_{*,\mathbb{H}}\Omega$ follows from basic definitions, and the inclusion $\partial_{*,\mathbb{H}}\Omega \subset \Gamma$ follows from [Franchi et al. 2001, Lemma 7.5 (i)].

We now use the complex function ν to specify a collection of accretive test functions. Let $\psi : \mathbb{H} \to [0, 1]$ be a smooth function with $\chi_{B(0,1/2)} \le \psi \le \chi_{B(0,1)}$, and let

$$\psi_{B(p,r)}(q) := \psi(\delta_{1/r}(p^{-1} \cdot q))$$

be a rescaled version of ψ with $\operatorname{spt} \psi_{B(p,r)} \subset B(p, r)$. We record that

$$|\nabla_{\mathbb{H}} \psi_{B(p,r)}| \lesssim \frac{1}{r} \chi_{B(p,r)} \quad \text{and} \quad |\partial_t \psi_{B(p,r)}| \lesssim \frac{1}{r^2} \chi_{B(p,r)}. \tag{4.5}$$

We set

$$b_{B(p,r)} := \psi_{B(p,r)} \nu, \quad p \in \Gamma, \quad r > 0.$$

Then, recalling the formula (4.2) for ν, we note

$$\|b_{B(p,r)}\|_{L^\infty(\mu)} \lesssim 1 \quad \text{and} \quad \operatorname{Re}\left(\int b_{B(p,r)}\, d\mu \right) \gtrsim \mu(B(p, r))$$

for all $B(p,r)$ with $p \in \Gamma$ and $r > 0$. According to Main Theorem 10 in [Christ 1990], the $L^2(\mu)$ boundedness of $\mathcal{R}$ will follow once we verify the testing conditions

$$\|\mathcal{R}_\epsilon(b_B\mu)\|_{L^\infty(\mu)} \le C \quad \text{and} \quad \|\mathcal{R}_\epsilon^*(b_B\mu)\|_{L^\infty(\mu)} \le C \tag{4.6}$$

for all balls $B = B(p,r)$ centred on Γ, with $C \ge 1$ independent of $\epsilon > 0$. Here $\mathcal{R}_\epsilon^*$ is the adjoint of $\mathcal{R}_\epsilon$ with kernel

$$K^*(p) = K(p^{-1}).$$

In fact, it will be technically more convenient to verify the testing conditions (4.6) for *smooth truncations* of $\mathcal{R}$. By a smooth truncation, we mean the operator $\mathcal{R}_{s,\epsilon}$ associated to the kernel

$$K_\epsilon := \varphi_\epsilon K, \tag{4.7}$$

where φ is smooth and radially symmetric with

$$\chi_{\mathbb{H}\setminus B(0,2)} \le \varphi \le \chi_{\mathbb{H}\setminus B(0,1)},$$

and $\varphi_\epsilon(p) := \varphi(\delta_{1/\epsilon}(p))$ for $p \in \mathbb{H}$. For future reference, we remark that

$$|\nabla_{\mathbb{H}}\varphi_\epsilon| \lesssim \frac{1}{\epsilon}\chi_{B(0,2\epsilon)\setminus B(0,\epsilon)} \quad \text{and} \quad |\partial_t\varphi_\epsilon| \lesssim \frac{1}{\epsilon^2}\chi_{B(0,2\epsilon)\setminus B(0,\epsilon)}. \tag{4.8}$$

Also, if $\epsilon = 2^{-N}$ for some $N \in \mathbb{N}$, then φ_ϵ can be expanded as a series:

$$\varphi_\epsilon = \varphi_{2^{-N}} = \sum_{j \le N}(\varphi_{2^{-j}} - \varphi_{2^{-j+1}}) =: \sum_{j \le N}\eta_j, \tag{4.9}$$

noting that η_j is supported on the annulus $B(0,2^{-j+2}) \setminus B(0,2^{-j})$. We will assume without loss of generality that ϵ has this form in the sequel.

Now, instead of (4.6), we will check that

$$\|\mathcal{R}_{s,\epsilon}(b_B\mu)\|_{L^\infty(\mu)} \le C \quad \text{and} \quad \|\mathcal{R}_{s,\epsilon}^*(b_B\mu)\|_{L^\infty(\mu)} \le C \tag{4.10}$$

for all balls B centred on Γ, and for some constant $C \ge 1$ independent of $\epsilon > 0$. It is easy to check that

$$|\mathcal{R}_{s,\epsilon}(f)(p) - \mathcal{R}_\epsilon(f)(p)| \lesssim M_\mu(f)(p)$$

for all $f \in L^\infty(\mu)$ and $p \in \Gamma$, where M_μ is the Hardy–Littlewood maximal function

$$M_\mu f(p) = \sup_{r>0} \fint_{B(p,r)} |f(q)|\,d\mu(q).$$

Since $\|M_\mu(b_B\mu)\|_{L^\infty(\mu)} \lesssim \|b_B\|_{L^\infty(\mu)} \lesssim 1$, we see that (4.10) implies (4.6).

4C. ***Initial reductions for verifying the testing conditions.*** We start by verifying the first condition in (4.10), that is, proving

$$|\mathcal{R}_{s,\epsilon}(b_B\mu)(p)| \le C, \quad p \in \Gamma. \tag{4.11}$$

The arguments concerning the second testing condition in (4.10) will be very similar. To prove (4.11), we make a few reductions, which show that it suffices to verify (4.11) for $p = 0 \in \Gamma$ and for a ball B with $\operatorname{dist}(0, B) \leq \operatorname{diam}(B) = 1$.

As a first step, we argue that it suffices to consider $p \in \Gamma$ with

$$\operatorname{dist}(p, B) \leq \operatorname{diam}(B). \tag{4.12}$$

Indeed, (4.11) follows from standard kernel estimates if $\operatorname{dist}(p, B) > \operatorname{diam}(B)$. To see this, we write $B = B(p_0, r)$ and fix $p \in \Gamma$ with $\operatorname{dist}(p, p_0) \geq 2r$. Then $d(p, q) \geq r$ for all $q \in B$, and consequently

$$|\mathcal{R}_{s,\epsilon}(b_B)(p)| \lesssim \|b_B\|_{L^\infty(\mu)} \int_B \frac{d\mu(q)}{d(p,q)^3} \lesssim \frac{\mu(B)}{r^3} \sim 1.$$

So, in the sequel we may assume that (4.12) holds.

Next, we argue that it suffices to consider the case $p = 0 \in \Gamma$. Indeed, note first that

$$\tilde{\mu} := \mathcal{S}^3|_{p^{-1}\cdot\Gamma} = (\tau_{p^{-1}})_\sharp \mathcal{S}^3|_\Gamma = (\tau_{p^{-1}})_\sharp \mu.$$

Then, write

$$\tilde{b}_{p^{-1}\cdot B} := \psi_{p^{-1}\cdot B}\nu_{p^{-1}\cdot\Gamma},$$

where $\nu_{p^{-1}\cdot\Gamma}$ is the analogue of ν (recall (4.2)) for the left-translated intrinsic Lipschitz graph $p^{-1} \cdot \Gamma$. In particular,

$$\nu_{p^{-1}\cdot\Gamma}(p^{-1} \cdot q) = \nu(q), \quad q \in \Gamma,$$

so that

$$\tilde{b}_{p^{-1}\cdot B}(p^{-1} \cdot q) = \psi_B(q)\nu(q) = b_B(q), \quad q \in \Gamma.$$

Using this equation, we infer that

$$\begin{aligned}
\mathcal{R}_{s,\epsilon}(\tilde{b}_{p^{-1}\cdot B}\tilde{\mu})(0) &= \int_{p^{-1}\cdot\Gamma} K_\epsilon(q^{-1})\tilde{b}_{p^{-1}\cdot B}(q)\, d\mathcal{S}^3(q) \\
&= \int K_\epsilon(q^{-1})\tilde{b}_{p^{-1}\cdot B}(q)\, d[(\tau_{p^{-1}})_\sharp\mu](q) \\
&= \int_\Gamma K_\epsilon((p^{-1}\cdot q)^{-1})\tilde{b}_{p^{-1}\cdot B}(p^{-1}\cdot q)\, d\mathcal{S}^3(q) \\
&= \int_\Gamma K_\epsilon(q^{-1}\cdot p) b_B(q)\, d\mathcal{S}^3(q) = \mathcal{R}_{s,\epsilon}(b_B\mu)(p).
\end{aligned}$$

This shows that, to find a bound for $\mathcal{R}_{s,\epsilon}(b_B\mu)(p)$, it suffices to do so for $\mathcal{R}_{s,\epsilon}(\tilde{b}_{p^{-1}\cdot B}\tilde{\mu})(0)$. But the intrinsic Lipschitz graph $p^{-1} \cdot \Gamma$ has all the same properties as we assumed from Γ in Theorem 4.1: the intrinsic Lipschitz constants do not change, nor do the bounds for the vertical oscillation numbers, recalling Lemma 3.4, so we may assume that $p = 0 \in \Gamma$.

Finally, we argue that we may assume $\operatorname{diam}(B) = 1$. For this purpose, we first note that

$$r^3 \cdot \delta_{r\sharp}\mu = \mathcal{S}^3|_{\delta_r(\Gamma)} =: \tilde{\mu}. \tag{4.13}$$

Indeed, if $A \subset \delta_r(\Gamma)$, then $\delta_{1/r}(A) \subset \Gamma$, hence

$$r^3 \cdot (\delta_{r\sharp}\mu)(A) = r^3 \mathcal{S}^3(\Gamma \cap \delta_{1/r}(A)) = \mathcal{S}^3(\delta_r(\Gamma) \cap A) = \tilde{\mu}(A),$$

which proves (4.13). Now, let $r := \operatorname{diam}(B)$, and let $\tilde{b}_{\delta_{1/r}(B)} := \psi_{\delta_{1/r}(B)} \cdot \nu_{\delta_{1/r}(\Gamma)}$, where $\nu_{\delta_{1/r}(\Gamma)}$ stands for the analogue of ν for the dilated intrinsic Lipschitz graph $\delta_{1/r}(\Gamma)$. In particular, it is easy to check that

$$\tilde{b}_{\delta_{1/r}(B)}(\delta_{1/r}(q)) = b_B(q), \quad q \in \Gamma.$$

We also record the equation

$$K_\epsilon(\delta_r(q)) = \varphi_\epsilon(\delta_r(q))K(\delta_r(q)) = r^{-3} \cdot \varphi_{\epsilon/r}(q)K(q) = r^{-3}K_{\epsilon/r}(q),$$

using the definition of the kernel K_ϵ from (4.7) and the -3-homogeneity of K. Then we may use (4.13) and the equations above to get

$$\begin{aligned}
\mathcal{R}_{s,\epsilon/r}(\tilde{b}_{\delta_{1/r}(B)}\tilde{\mu})(0) &= \int_{\delta_{1/r}(\Gamma)} K_{\epsilon/r}(q^{-1})\tilde{b}_{\delta_{1/r}(B)}(q)\, d\mathcal{S}^3(q) \\
&= r^{-3} \int_\Gamma K_{\epsilon/r}(q^{-1})\tilde{b}_{\delta_{1/r}(B)}(q)\, d\delta_{(1/r)\sharp}\mu(q) \\
&= r^{-3} \int_\Gamma K_{\epsilon/r}([\delta_{1/r}(q)]^{-1})\tilde{b}_{\delta_{1/r}(B)}(\delta_{1/r}(q))\, d\mathcal{S}^3(q) \\
&= \int_\Gamma K_\epsilon(q^{-1})b_B(q)\, d\mathcal{S}^3(q) = \mathcal{R}_{s,\epsilon}(b_B\mu)(0).
\end{aligned}$$

So, to estimate $\mathcal{R}_{s,\epsilon}(b_B\mu)(0)$ it suffices to estimate $\mathcal{R}_{s,\epsilon/r}(\tilde{b}_{\delta_{1/r}(B)}\tilde{\mu})(0)$. But, arguing as in the previous reduction, $\delta_{1/r}(\Gamma)$ is an intrinsic Lipschitz graph with the same properties as Γ. So in the sequel we assume that $\operatorname{diam}(B) = 1$.

Summarising, we have reduced the proof of (4.11) to the case

$$p = 0 \in \Gamma \quad \text{and} \quad \operatorname{dist}(0, B) \le \operatorname{diam}(B) = 1. \tag{4.14}$$

4D. ***Verifying the testing conditions.*** With the above reductions in mind, we start the proof of (4.11). We record that

$$K(q^{-1}) = -\widetilde{X}G(q) + i\widetilde{Y}G(q), \quad q \in \mathbb{H} \setminus \{0\}, \tag{4.15}$$

as a straightforward computation shows. Hence, we may write

$$\begin{aligned}
&\mathcal{R}_{s,\epsilon}(b_B\mu)(0) \\
&\quad = \int_\Gamma \varphi_\epsilon(q)(-\widetilde{X}G(q) + i\widetilde{Y}G(q))b_B(q)\, d\mathcal{S}^3(q) \\
&\quad = -\int_\Gamma \langle \psi_B(q)\varphi_\epsilon(q)\widetilde{\nabla}_{\mathbb{H}}G(q), \nu_H(q)\rangle\, d\mathcal{S}^3(q) + i\int_\Gamma \langle \psi_B(q)\varphi_\epsilon(q)(\widetilde{Y}G(q), -\widetilde{X}G(q)), \nu_H(q)\rangle\, d\mathcal{S}^3(q) \\
&\quad =: I_1 + iI_2,
\end{aligned}$$

recalling the notation from Section 2A. In order to evaluate I_1 and I_2, we will apply the divergence theorem (Theorem 4.3) to the vector fields

$$V_1 := (\psi_B \varphi_\epsilon \widetilde{X} G, \psi_B \varphi_\epsilon \widetilde{Y} G) \in \mathcal{C}_c^\infty(\mathbb{R}^3, \mathbb{R}^2) \quad \text{and} \quad V_2 := (\psi_B \varphi_\epsilon \widetilde{Y} G, -\psi_B \varphi_\epsilon \widetilde{X} G) \in \mathcal{C}_c^\infty(\mathbb{R}^3, \mathbb{R}^2),$$

respectively.

4D1. *Estimate for I_1.* After an application of Theorem 4.3, I_1 becomes

$$\begin{aligned} I_1 &= -c \int_\Omega \operatorname{div}_{\mathbb{H}}(\psi_B(q)\varphi_\epsilon(q)\widetilde{\nabla}_{\mathbb{H}} G(q))\, dq \\ &= -c \int_\Omega \langle \nabla_{\mathbb{H}}(\psi_B\varphi_\epsilon)(q), \widetilde{\nabla}_{\mathbb{H}} G(q)\rangle\, dq - c\int_\Omega (\psi_B\varphi_\epsilon)(q) \operatorname{div}_{\mathbb{H}} \widetilde{\nabla}_{\mathbb{H}} G(q)\, dq \\ &=: -cI_1^1 - cI_1^2. \end{aligned}$$

For I_1^1, we infer from (4.5), (4.8), and the product rule that

$$|\nabla_{\mathbb{H}}(\psi_B\varphi_\epsilon)| \lesssim \frac{1}{\epsilon}\chi_{B(0,2\epsilon)\setminus B(0,\epsilon)} + \chi_B.$$

Since moreover $|\widetilde{\nabla}_{\mathbb{H}} G(q)| \lesssim \|q\|^{-3}$ (this follows from (4.15) for instance), we get

$$\left| \int_\Omega \langle \nabla_{\mathbb{H}}(\psi_B\varphi_\epsilon)(q), \widetilde{\nabla}_{\mathbb{H}} G(q)\rangle\, dq \right| \lesssim \frac{1}{\epsilon}\int_{B(0,2\epsilon)\setminus B(0,\epsilon)} \|q\|^{-3} dq + \int_B \|q\|^{-3}\, dq \lesssim 1. \tag{4.16}$$

To handle the term I_1^2, we observe the following general relationship between left and right divergence:

$$\operatorname{div}_{\mathbb{H}}(V_1, V_2) = \widetilde{\operatorname{div}}_{\mathbb{H}}(V_1, V_2) + \partial_t(-yV_1 + xV_2), \quad (V_1, V_2) \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^2). \tag{4.17}$$

It follows that

$$I_1^2 = \int_\Omega (\psi_B\varphi_\epsilon)(q) \widetilde{\operatorname{div}}_{\mathbb{H}} \widetilde{\nabla}_{\mathbb{H}} G(q)\, dq + \int_\Omega (\psi_B\varphi_\epsilon)(q)\partial_t(-y\widetilde{X}G(q) + x\widetilde{Y}G(q))\, dq.$$

Here

$$\widetilde{\operatorname{div}}_{\mathbb{H}} \widetilde{\nabla}_{\mathbb{H}} G(q) = \widetilde{\Delta}_{\mathbb{H}} G(q) = 0, \quad q \in \operatorname{spt}\varphi_\epsilon,$$

since G is simultaneously the fundamental solution for both operators $\Delta_{\mathbb{H}}$ and $\widetilde{\Delta}_{\mathbb{H}}$. So the first term vanishes. Consequently,

$$I_1^2 =: \int_\Omega (\psi_B\varphi_\epsilon)(q)\partial_t \widetilde{K}(q)\, dq = \int_\Omega \partial_t(\psi_B\varphi_\epsilon\widetilde{K})(q)\, dq - \int_\Omega \partial_t(\psi_B\varphi_\epsilon)(q)\widetilde{K}(q)\, dq, \tag{4.18}$$

where $\widetilde{K}$ is the -2-homogeneous kernel

$$\widetilde{K}(z,t) = -y\widetilde{X}G(z,t) + x\widetilde{Y}G(z,t) = \frac{8t|z|^2}{\|(z,t)\|_{\mathrm{Kor}}^6}, \quad z = (x,y).$$

The main term in (4.18) is the first one, because the second one can be treated in the same fashion as I_1^1 above. Indeed, simply notice from (4.5), (4.8), and the product rule that

$$|\partial_t(\psi_B\varphi_\epsilon)(q)| \lesssim \frac{1}{\epsilon^2}\chi_{B(0,2\epsilon)\setminus B(0,\epsilon)} + \chi_B,$$

so that

$$\begin{aligned}\left|\int_\Omega \partial_t(\psi_B\varphi_\epsilon)(q)\widetilde{K}(q)\,dq\right| &\lesssim \frac{1}{\epsilon^2}\int_{B(0,2\epsilon)\setminus B(0,\epsilon)}|\widetilde{K}(q)|\,dq + \int_B |\widetilde{K}(q)|\,dq\\ &\lesssim \frac{1}{\epsilon^4}\mathcal{H}^4(B(0,2\epsilon)) + 1 \sim 1.\end{aligned}$$

Finally, the first term in (4.18) is handled using (4.9) and Lemma 3.8 (noting that $\operatorname{spt}(\psi_B\eta_j\widetilde{K}) \subset B(0,s)$ for any $s \in [2^{-j+2}, 2^{-j+3}]$) to yield

$$\begin{aligned}\left|\int_\Omega \partial_t(\psi_B\varphi_\epsilon\widetilde{K})(q)\,dq\right| &\le \sum_{j\le N}\left|\int_\Omega \partial_t(\psi_B\eta_j\widetilde{K})(q)\,dq\right|\\ &\lesssim \sum_{j\le N} 2^{-4j}\|\partial_t(\psi_B\eta_j\widetilde{K})\|_\infty \int_{2^{-j+2}}^{2^{-j+3}} \operatorname{osc}_\Omega(B(0,10s))\,\frac{ds}{s}.\end{aligned}$$

From the product rule, noting that

- $\operatorname{spt}\eta_j \subset B(0,2^{-j+2}) \setminus B(0,2^{-j})$,
- $\operatorname{spt}\psi_B \subset B \subset B(0,2)$ by (4.14),
- $\widetilde{K}$ is -2-homogeneous, and
- $\partial_t\widetilde{K}$ is -4-homogeneous,

we see that

$$\|\partial_t(\psi_B\eta_j\widetilde{K})\|_\infty \lesssim \begin{cases} 2^{4j}, & j \ge -1,\\ 0, & j < -1.\end{cases}$$

To verify the last bullet point, one can simply compute that $\partial_t\widetilde{K}$ is the kernel

$$\partial_t\widetilde{K}(z,t) = 8\frac{|z|^2(|z|^4 - 32t^2)}{\|(z,t)\|_{\mathrm{Kor}}^{10}}, \quad z = (x,y).$$

Summarising the estimate above, we have now shown that

$$\begin{aligned}|I_1| &\lesssim 1 + \sum_{-1\le j\le N}\int_{2^{-j+2}}^{2^{-j+3}} \operatorname{osc}_\Omega(B(0,10s))\,\frac{ds}{s}\\ &\lesssim 1 + \int_0^\infty \operatorname{osc}_\Omega(B(0,s))\,\frac{ds}{s}\\ &\le 1 + C.\end{aligned}$$

4D2. *Estimate for I_2.* We move to the term

$$\begin{aligned} I_2 &= \int_\Gamma \langle \psi_B(q)\varphi_\epsilon(q)(\widetilde{Y}G(q), -\widetilde{X}G(q)), \nu_H(q)\rangle \, d\mathcal{S}^3(q) \\ &= -c\int_\Omega \mathrm{div}_{\mathbb{H}}(\psi_B\varphi_\epsilon(\widetilde{Y}G, -\widetilde{X}G))(q)\, dq \\ &= -c\int_\Omega \langle \nabla_{\mathbb{H}}(\psi_B\varphi_\epsilon)(q), (\widetilde{Y}G(q), -\widetilde{X}G(q))\rangle \, dq - c\int_\Omega (\psi_B\varphi_\epsilon)(q)\, \mathrm{div}_{\mathbb{H}}(\widetilde{Y}G, -\widetilde{X}G)(q)\, dq \\ &=: -cI_2^1 - cI_2^2, \end{aligned}$$

where the divergence theorem was applied. The term I_2^1 can be handled precisely as I_1^1 above; see (4.16). So we concentrate on the term I_2^2. Once again, due to the presence of the right-invariant vector fields $\widetilde{X}$ and $\widetilde{Y}$, it is useful to consider the right divergence instead of the left one. Recalling (4.17) and setting $p = (x, y, t)$, we write

$$\begin{aligned} \mathrm{div}_{\mathbb{H}}(\widetilde{Y}G, -\widetilde{X}G)(p) &= \widetilde{\mathrm{div}}_{\mathbb{H}}(\widetilde{Y}G, -\widetilde{X}G)(p) + \partial_t(-y\widetilde{Y}G - x\widetilde{X}G)(p) \\ &= (\widetilde{X}\widetilde{Y}G - \widetilde{Y}\widetilde{X}G)(p) + \partial_t\hat{K}(p) \\ &= -\partial_t G(p) + \partial_t \hat{K}(p). \end{aligned}$$

Here $\hat{K}$ is yet another -2-homogeneous kernel with explicit expression

$$\hat{K}(z,t) = \frac{2|z|^4}{\|(z,t)\|_{\mathrm{Kor}}^6}, \qquad (z,t) \in \mathbb{H} \setminus \{0\}.$$

In other words,

$$I_2^2 = -\int_\Omega (\psi_B\varphi_\epsilon)(q)\partial_t G(q)\, dq + \int_\Omega (\psi_B\varphi_\epsilon)(q)\partial_t\hat{K}(q)\, dq. \tag{4.19}$$

From this point on, the treatment of both terms can be continued as on line (4.18) above. The only facts we needed about the kernel $\widetilde{K}$ there was that it is -2-homogeneous and its t-derivative is -4-homogeneous. These properties are also satisfied for G and $\hat{K}$. In fact, the t-derivatives are given by

$$\partial_t G(z,t) = \frac{16t}{\|(z,t)\|_{\mathrm{Kor}}^6} \quad \text{and} \quad \partial_t\hat{K}(z,t) = -\frac{96|z|^4 t}{\|(z,t)\|_{\mathrm{Kor}}^{10}}.$$

Continuing as in (4.18), and afterwards, we obtain

$$|I_2^2| \lesssim 1 + \int_0^\infty \mathrm{osc}_\Omega(B(0,s))\, \frac{ds}{s} \le 1 + C.$$

This concludes the proof of (4.11) as we have shown that

$$\|\mathcal{R}_{s,\epsilon}(b_B\mu)\|_{L^\infty(\mu)} \le C. \tag{4.20}$$

4D3. *The adjoint.* To prove Theorem 4.1, it remains to establish the bound analogous to (4.20) for the adjoint $\mathcal{R}_{s,\epsilon}^*$. Arguing as in Section 4C, we may assume that the conditions in (4.14) are in force. In other words, it suffices to show that

$$|\mathcal{R}_{s,\epsilon}^*(b_B\mu)(0)| \le C,$$

where $B \subset \mathbb{H}$ is a ball with $\operatorname{dist}(0, B) \leq 1 = \operatorname{diam}(B)$, and $0 \in \Gamma$. By definition,

$$\begin{aligned}\mathcal{R}^*_{s,\epsilon}(b_B\mu)(0) &= \int_\Gamma \varphi_\epsilon(q)(XG(q) - iYG(q))b_B(q)\, d\mathcal{S}^3(q)\\ &= \int_\Gamma \langle (\psi_B\varphi_\epsilon)(q)\nabla_{\mathbb{H}}G(q), \nu_H(q)\rangle\, d\mathcal{S}^3 + i\int_\Gamma \langle (\psi_B\varphi_\epsilon)(q)(-YG, XG)(q), \nu_H(q)\rangle\, d\mathcal{S}^3(q)\\ &=: J_1 + iJ_2.\end{aligned}$$

The situation is now similar to, but slightly simpler than, the one we have already treated. After we apply the divergence theorem and use the product rule, we have

$$J_1 = -c\int_\Omega \langle \nabla_{\mathbb{H}}(\psi_B\varphi_\epsilon)(q), \nabla_{\mathbb{H}}G(q)\rangle\, dq - c\int_\Omega (\psi_B\varphi_\epsilon)(q)\operatorname{div}_{\mathbb{H}}\nabla_{\mathbb{H}}G(q)\, dq.$$

The second term vanishes, as $\operatorname{div}_{\mathbb{H}}\nabla_{\mathbb{H}}G(q) = \Delta_{\mathbb{H}}G(q) = 0$ for $q \in \operatorname{spt}\varphi_\epsilon$. The first term can be estimated as in (4.16).

Concerning J_2, the divergence theorem gives

$$J_2 = -c\int_\Omega \langle \nabla_{\mathbb{H}}(\psi_B\varphi_\epsilon)(q), (-YG, XG)(q)\rangle\, dq - c\int_\Omega (\psi_B\varphi_\epsilon)(q)\operatorname{div}_{\mathbb{H}}(-YG, XG)(q)\, dq.$$

Once more, the first term is estimated using the argument from (4.16). In the second term, we find that

$$\operatorname{div}_{\mathbb{H}}(-YG, XG)(q) = -XYG(q) + YXG(q) = -\partial_t G(q), \quad q \in \mathbb{H}\setminus\{0\}.$$

From this point on, the estimates are the same as for the term I_2^2 above; see (4.19). We have now established that

$$\|\mathcal{R}^*(b_B\mu)\|_{L^\infty(\mu)} \leq C,$$

and the proof of Theorem 4.1 is complete. □

5. Application: intrinsic Lipschitz graphs with extra vertical regularity

In this section, we prove Theorem 1.10, which we restate below.

Theorem 5.1. *Let $\phi : \mathbb{W} \to \mathbb{R}$ be an intrinsic Lipschitz function which satisfies the following Hölder regularity in the vertical direction*:

$$|\phi(y, t) - \phi(y, s)| \leq H|t - s|^{(1+\tau)/2}, \quad |s - t| \leq 1 \tag{5.2}$$

and

$$|\phi(y, t) - \phi(y, s)| \leq H|t - s|^{(1-\tau)/2}, \quad |s - t| > 1, \tag{5.3}$$

where $H \geq 1$ and $0 < \tau \leq 1$. Then R is bounded on $L^2(\mathcal{H}^3|_{\Gamma_\phi})$.

As a corollary, we recover the main theorem of [Chousionis et al. 2019a] for the Riesz transform.

Corollary 5.4. *Let $\mathbb{W} \subset \mathbb{H}$ be a vertical plane, let $\alpha > 0$, and let $\phi : \mathbb{W} \to \mathbb{V}$ be a compactly supported $C^{1,\alpha}(\mathbb{W})$ in the sense of* [Chousionis et al. 2019a]. *Then R is bounded on $L^2(\mathcal{H}^3|_{\Gamma_\phi})$.*

Proof. By [Chousionis et al. 2019a, Proposition 4.2], an intrinsic $C^{1,\alpha}$-function ϕ satisfies (5.2) with exponent $\tau = \alpha$. Since ϕ is continuous and compactly supported, (5.3) is also satisfied if the constant H is chosen large enough. To apply Theorem 5.1, we still need to argue that ϕ is intrinsic Lipschitz: this is the content of [Chousionis et al. 2019a, Remark 2.18]. □

Besides the compact support assumption, a notable difference between Theorem 5.1 and the main theorem of [Chousionis et al. 2019a] is that the intrinsic $C^{1,\alpha}$-condition implies extra regularity in both vertical and horizontal directions. The conditions (5.2)–(5.3), on the other hand, imply nothing about the horizontal behaviour of ϕ. To emphasise this, we give another corollary of Theorem 5.1.

Corollary 5.5. *Let $\phi_0 : \mathbb{R} \to \mathbb{R}$ be a (Euclidean) Lipschitz function, and let $\phi(0, y, t) := \phi_0(y)$. Then $\mathcal{R}$ is bounded on $L^2(\mu)$, where μ is $\mathcal{H}^3$ restricted to Γ_ϕ.*

Proof. We first note that ϕ is intrinsic Lipschitz because

$$|\phi(0, y, t) - \phi(0, y', t')| \lesssim |y - y'| \le \|\pi_{\mathbb{W}}(\Phi(0, y', t')^{-1} \cdot \Phi(0, y, t))\|,$$

where $\Phi(0, y, t) = (0, y, t) \cdot (\phi(0, y, t), 0, 0)$ is the graph map parametrising Γ_ϕ. Conditions (5.2)–(5.3) are trivially satisfied, so the claim follows from Theorem 5.1. □

5A. *Proof of Theorem 5.1.* The proof is based on the following lemma.

Lemma 5.6. *Assume $\phi : \mathbb{W} := \{(0, y, t) : y, t \in \mathbb{R}\} \to \mathbb{R}$ is intrinsic Lipschitz and satisfies* (5.2)–(5.3). *Then*

$$\operatorname{osc}_\Omega(B(p, r)) \lesssim H^4 \min\{r^\tau, r^{-\tau}\}, \quad p \in \Gamma_\phi, \ 0 < r < \infty, \tag{5.7}$$

where $\Omega = \{(x, y, t) : x > \phi(\pi_{\mathbb{W}}(x, y, t))\}$, and the implicit constants depend on the intrinsic Lipschitz constants of ϕ.

By Theorem 4.1, the lemma above will prove Theorem 5.1.

Proof of Lemma 5.6. The plan is to first use (5.2) to establish the bound

$$\operatorname{osc}_\Omega(B(p, r)) \lesssim H^4 r^\tau, \quad p \in \Gamma_\phi, \ 0 < r \le 1. \tag{5.8}$$

The second bound in (5.7) will follow by a similar argument from (5.3) for $r > 1$.

Write $\Gamma := \Gamma_\phi$, and fix $0 < r \le 1$ and $0 < s \le r$. We claim

$$\mathrm{v}_\Omega(B(p, r))(s) = \int_{B(p,r) \cap \Gamma(Hr^{1+\tau})} |\chi_\Omega(q) - \chi_\Omega(q \cdot (0, 0, s^2))| \, dq, \tag{5.9}$$

where $\Gamma(Hr^{1+\tau})$ denotes the $(Hr^{1+\tau})$-neighbourhood of Γ. To prove this, it suffices to show that if $q \in B(p, r)$ with $\operatorname{dist}(q, \Gamma) > Hr^{1+\tau}$, then

$$\chi_\Omega(q) = \chi_\Omega(q \cdot (0, 0, s^2)).$$

Indeed, assume to the contrary that $q = (x, y, t) \in B(p, r)$ can be found with $\operatorname{dist}(q, \Gamma) > Hr^{1+\tau}$ and $\chi_\Omega(q) \neq \chi_\Omega(q \cdot (0, 0, s^2))$. This has two consequences: First, in particular,

$$\begin{aligned}|x - \phi(\pi_{\mathbb{W}}(x, y, t))| &= d((x, 0, 0), \phi(\pi_{\mathbb{W}}(q)))\\ &= d(\pi_{\mathbb{W}}(q) \cdot (x, 0, 0), \pi_{\mathbb{W}}(q) \cdot \phi(\pi_{\mathbb{W}}(q)))\\ &= d(q, \Phi(\pi_{\mathbb{W}}(q))) > Hr^{1+\tau},\end{aligned}$$

where $\Phi(w) = w \cdot \phi(w)$ is the graph map parametrising Γ. Second, there exists $0 \leq u \leq s$ such that $(x, y, t + u^2) = q \cdot (0, 0, u^2) \in \Gamma$, so in particular,

$$x = \phi(\pi_{\mathbb{W}}(q \cdot (0, 0, u^2))).$$

Combining the information above,

$$|\phi(\pi_{\mathbb{W}}(x, y, t + u^2)) - \phi(\pi_{\mathbb{W}}(x, y, t))| > Hr^{1+\tau}.$$

Spelling out the definition of $\pi_{\mathbb{W}}$, this is equivalent to

$$Hr^{1+\tau} < \left|\phi\big(0, y, t + u^2 + \tfrac{1}{2}xy\big) - \phi\big(0, y, t + \tfrac{1}{2}xy\big)\right| \leq Hu^{1+\tau} \leq Hs^{1+\tau} \leq Hr^{1+\tau}.$$

We have reached a contradiction, and hence proved (5.9).

It follows from (5.9) that

$$\operatorname{osc}_\Omega(B(p, r)) = \frac{1}{r^4} \int_0^r \mathrm{v}_\Omega(B(p, r))(s)\, ds \lesssim \frac{\mathcal{H}^4(B(p, r) \cap \Gamma(Hr^{1+\tau}))}{r^4}.$$

To conclude the proof, we find a maximal $Hr^{1+\tau}$-separated set $S \subset B(p, 2Hr) \cap \Gamma$; note that this step uses the assumption $r \leq 1$, so that $r^{1+\tau} \leq r$. Since Γ is 3-regular,

$$\operatorname{card} S \lesssim r^{-3\tau}. \tag{5.10}$$

On the other hand, the balls $B(q, 10Hr^{1+\tau})$, $q \in S$, cover $B(p, r) \cap \Gamma(Hr^{1+\tau})$, whence

$$\operatorname{osc}_\Omega(B(p, r)) \lesssim \frac{\mathcal{H}^4(B(p, r) \cap \Gamma(Hr^{1+\tau}))}{r^4} \lesssim (\operatorname{card} S)\frac{(Hr^{1+\tau})^4}{r^4} \lesssim H^4 r^\tau.$$

This proves (5.8).

To prove the second bound in (5.7), one fixes $r \geq 1$ and proceeds as above, using (5.3) instead of (5.2). One first obtains

$$\mathrm{v}_\Omega(B(p, r))(s) = \int_{B(p,r) \cap \Gamma(Hr^{1-\tau})} |\chi_\Omega(q) - \chi_\Omega(q \cdot (0, 0, s^2))|\, dq$$

This leads to $\operatorname{osc}_\Omega(B(p, r)) \lesssim \mathcal{H}^4(B(p, r) \cap \Gamma(Hr^{1-\tau}))/r^4$. Since $r \geq 1$, one has $r^{1-\tau} \leq r$. One finally chooses a maximal $Hr^{1-\tau}$-separated set $S \subset B(p, 2Hr) \cap \Gamma$, and finds that (5.10) gets replaced by $\operatorname{card} S \lesssim r^{3\tau}$. This gives $\operatorname{osc}_\Omega(B(p, r)) \lesssim H^4 r^{-\tau}$, as desired. □

6. Problems and remarks

6A. ***Carleson packing conditions for the vertical oscillation coefficients?*** Theorem 1.8 guarantees the L^2-boundedness of R on intrinsic Lipchitz graphs $\Gamma = \partial\Omega \subset \mathbb{H}$ satisfying the uniform condition

$$\int_0^\infty \mathrm{osc}_\Omega(B(p,r))\,\frac{dr}{r} \lesssim 1, \quad p \in \partial\Omega. \tag{6.1}$$

A comparison with analogous results in Euclidean space, in particular those in [David and Semmes 1991], suggests that it might be possible to relax (6.1) to a *Carleson packing condition* for the vertical oscillation coefficients, such as

$$\int_{\partial\Omega\cap B(p_0,R)} \int_0^R \mathrm{osc}_\Omega(B(p,r))^\eta\,\frac{dr}{r}\,d\mathcal{H}^3(p) \lesssim R^3, \quad p_0 \in \partial\Omega,\ \ 0 < R \le \operatorname{diam}\partial\Omega. \tag{Car(η)}$$

Here $\eta \ge 1$ is a parameter, and evidently the condition (Car(η)) gets weaker as η increases. Two questions now arise:

Question 3. For which parameters $\eta \ge 1$ — if any — does the following hold? Assume that $\Gamma = \partial\Omega \subset \mathbb{H}$ is an intrinsic Lipschitz graph satisfying (Car(η)). Then R is bounded on $L^2(\mathcal{H}^3|_\Gamma)$.

Question 4. For which parameters $\eta \ge 1$ — if any — does the following hold? Every intrinsic Lipschitz graph $\Gamma \subset \mathbb{H}$ satisfies (Car(η)).

We have no further insight on either of the questions at the moment. We conjecture that every intrinsic Lipschitz graph $\Gamma \subset \mathbb{H}$ satisfies (Car(η)) for $\eta \ge 4$.

6B. ***A connection between vertical perimeter and β-numbers.*** Let $\Omega \subset \mathbb{H}$ be an open set with 3-regular boundary, and let $1 \le p < \infty$. Recall from Remark 3.2 that the L^p-vertical perimeter of Ω in a ball $B(q,r)$, $q \in \partial\Omega$, is the quantity

$$\wp_{\Omega,p}(B(q,r)) := \left(\int_0^\infty \left(\frac{\mathrm{v}_\Omega(B(q,r))(s)}{s}\right)^p \frac{ds}{s}\right)^{1/p}.$$

Given Corollary 3.34, it is reasonable to expect an inequality between $\wp_{\Omega,p}$ and some quantity defined via the vertical β-numbers $\beta_{\partial\Omega,1}$. Such an inequality is given by the following proposition.

Proposition 6.2. *Let $\Omega \subset \mathbb{H}$ be a nonempty open set with 3-regular boundary, and let $p_0 \in \partial\Omega$ and $0 < R \le \operatorname{diam}\partial\Omega$. Then*

$$\wp_{\Omega,p}(B(p_0,R)) \lesssim R^3 + \int_{\partial\Omega\cap B(p_0,CR)} \left(\int_0^R \beta_{\partial\Omega,1}(B(q,Cr))^p\,\frac{dr}{r}\right)^{1/p} d\mathcal{H}^3(q),$$

where $C \ge 1$ is an absolute constant.

Proof. Fix $0 < r \le R$. We start by arguing that

$$\frac{\mathrm{v}_\Omega(B(p_0,R))(r)}{r} \lesssim \int_{\partial\Omega\cap B(p_0,CR)} \beta_{\partial\Omega,1}(B(p,Cr))\,d\mathcal{H}^3(p). \tag{6.3}$$

To this end, let $\mathcal{B}_r$ be a finite family of balls of radius r covering $B(p_0, R)$ such that the concentric balls of radius $r/2$ are disjoint. Note that if $\operatorname{dist}(B, \partial\Omega) > 2r$, then

$$|\chi_\Omega(q) - \chi_\Omega(q \cdot (0,0,r^2))| = 0, \quad q \in B,$$

because $d(q, q \cdot (0,0,r^2)) = 2r$ with our choice of metric d; recall (2.1). Whenever $B \in \mathcal{B}_r$ with $\operatorname{dist}(B, \partial\Omega) \leq 2r$, we pick some ball $\hat{B} \supset B$ which is centred on $\partial\Omega$ and has radius at most $5r$. By the 3-regularity of the boundary, we then have

$$\mathcal{H}^3(\hat{B} \cap \partial\Omega) \sim r^3, \quad B \in \mathcal{B}_r, \ \operatorname{dist}(B, \partial\Omega) \leq 2r.$$

Then, by the bounded overlap of the balls $\hat{B}$ and applying Corollary 3.34, we can estimate

$$\begin{aligned}
\frac{\mathrm{v}_\Omega(B(p_0,R))(r)}{r} &= \int_{B(p_0,R)} \frac{|\chi_\Omega(q) - \chi_\Omega(q \cdot (0,0,r^2))|}{r}\, dq \\
&\leq \sum_{\substack{B \in \mathcal{B}_r \\ \operatorname{dist}(B,\partial\Omega) \leq 2r}} \int_B \frac{|\chi_\Omega(q) - \chi_\Omega(q \cdot (0,0,r^2))|}{r}\, dq \\
&\lesssim \sum_{\substack{B \in \mathcal{B}_r \\ \operatorname{dist}(B,\partial\Omega) \leq 2r}} \frac{\mathrm{v}_\Omega(\hat{B})(r)}{r^4} \mathcal{H}^3(\hat{B} \cap \partial\Omega) \\
&\lesssim \sum_{\substack{B \in \mathcal{B}_r \\ \operatorname{dist}(B,\partial\Omega) \leq 2r}} \beta_{\partial\Omega,1}(30\hat{B}) \mathcal{H}^3(\hat{B} \cap \partial\Omega) \\
&\lesssim \int_{B(p_0,CR)} \beta_{\partial\Omega,1}(B(q,Cr))\, d\mathcal{H}^3(q).
\end{aligned}$$

This is (6.3). Applying Minkowski's integral inequality, we infer the bound

$$\begin{aligned}
\left(\int_0^R \left(\frac{\mathrm{v}_\Omega(B(p_0,R))(r)}{r} \right)^p \frac{dr}{r} \right)^{1/p} &\lesssim \left(\int_0^R \left(\int_{\partial\Omega \cap B(p_0,CR)} \beta_{\partial\Omega,1}(B(q,Cr))\, d\mathcal{H}^3(q) \right)^p \frac{dr}{r} \right)^{1/p} \\
&\leq \int_{\partial\Omega \cap B(p_0,CR)} \left(\int_0^R \beta_{\partial\Omega,1}(B(q,Cr))^p \frac{dr}{r} \right)^{1/p} d\mathcal{H}^3(q).
\end{aligned}$$

Finally, it remains to note

$$\left(\int_R^\infty \left(\frac{\mathrm{v}_\Omega(B(p_0,R))(r)}{r} \right)^p \frac{dr}{r} \right)^{1/p} \lesssim \left(\int_R^\infty \frac{R^{4p}}{r^{p+1}}\, dr \right)^{1/p} \sim R^3,$$

and the proposition follows by combining the two estimates above. □

As an immediate corollary, we infer that if the $\beta_{\partial\Omega,1}$-numbers satisfy a Carleson packing condition similar to ($\mathrm{Car}(\eta)$), namely

$$\int_{\partial\Omega \cap B(p_0,R)} \int_0^R \beta_{\partial\Omega,1}(B(q,r))^p \frac{dr}{r}\, d\mathcal{H}^3(q) \lesssim R^3, \quad p_0 \in \partial\Omega, \ 0 < R \leq \operatorname{diam} \partial\Omega, \tag{6.4}$$

then the L^p-vertical perimeter is bounded by (a constant times) the horizontal perimeter.

Corollary 6.5. *Let* $1 \le p < \infty$. *Assume that* $\Omega \subset \mathbb{H}$ *is a nonempty open set with* 3*-regular boundary*, *and assume that* (6.4) *holds. Then*

$$\wp_{\Omega,p}(B(q,r)) \lesssim r^3, \quad q \in \partial\Omega, \ \ 0 < r \le \operatorname{diam} \partial\Omega.$$

Proof. Apply Proposition 6.2, then Hölder's inequality, and finally (6.4). □

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A WESS–ZUMINO–WITTEN TYPE EQUATION IN THE SPACE OF KÄHLER POTENTIALS IN TERMS OF HERMITIAN–YANG–MILLS METRICS

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We prove that the solution of a Wess–Zumino–Witten type equation from a domain D in $\mathbb{C}^m$ to the space of Kähler potentials can be approximated uniformly by Hermitian–Yang–Mills metrics on certain vector bundles. The key is a new version of Berndtsson's theorem on the positivity of direct image bundles.

1. Introduction

Let L be a positive line bundle over a compact complex manifold X of dimension n, and let h be a positively curved metric on L with curvature ω. The space of Kähler potentials is

$$\mathcal{H}_\omega = \{\phi \in C^\infty(X, \mathbb{R}) : \omega + i\partial\bar{\partial}\phi > 0\},$$

and for a positive integer k we denote by $\mathcal{H}_k$ the space of inner products on $H^0(X, L^k)$. Starting from a question asked by Yau [1987] and the work of Tian [1990], Zelditch [1998], Catlin [1999], and many others, it is well known that a given Kähler potential $\phi \in \mathcal{H}_\omega$ can be approximated by $\phi_k \in \mathcal{H}_\omega$ associated with $\mathcal{H}_k$ as $k \to \infty$. Furthermore, Mabuchi [1987], Semmes [1992], and Donaldson [1999] discovered that $\mathcal{H}_\omega$ carries a Riemannian metric which allows one to talk about geometry, especially geodesics, of $\mathcal{H}_\omega$. Thanks to Phong and Sturm [2006], Chen and Sun [2012], Berndtsson [2018], and Darvas, Lu, and Rubinstein [Darvas et al. 2020], geodesics in $\mathcal{H}_\omega$ can be approximated by geodesics in $\mathcal{H}_k$ as $k \to \infty$. More generally, one may wonder if harmonic maps into $\mathcal{H}_\omega$ can also be approximated by harmonic maps associated with $\mathcal{H}_k$. A version of this was confirmed by Rubinstein and Zelditch [2010] when X is toric, and the maps take values in toric Kähler metrics; see also [Song and Zelditch 2007; 2010].

Here we focus on a Wess–Zumino–Witten (WZW) type equation for a map from $D \subset \mathbb{C}^m$ to $\mathcal{H}_\omega$, and we show that the solution to such an equation can be approximated by Hermitian–Yang–Mills metrics on certain direct image bundles. We will also see how this result recovers some of those mentioned in the first paragraph.

We first explain how to derive this WZW equation. Recall that the tangent space $T_\phi \mathcal{H}_\omega$ at $\phi \in \mathcal{H}_\omega$ can be canonically identified with $C^\infty(X, \mathbb{R})$, and following [Donaldson 1999; Mabuchi 1987; Semmes 1992],

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the Mabuchi metric g_M on $\mathcal{H}_\omega$ is

$$g_M(\xi,\eta)=\int_X \xi\eta\omega_\phi^n, \quad \text{for } \phi\in\mathcal{H}_\omega \text{ and } \xi,\eta\in T_\phi\mathcal{H}_\omega.$$

Let D be a bounded smooth strongly pseudoconvex domain in $\mathbb{C}^m$. A map $\Phi: D\to\mathcal{H}_\omega$ will be identified as $\Phi: D\times X\to\mathbb{R}$ with $\Phi(z,\cdot)\in\mathcal{H}_\omega$ for $z\in D$. A map $\Phi: D\to\mathcal{H}_\omega$ is said to be harmonic if it is a critical point of the functional $E(\Phi)=\int_D|\Phi_*|^2\,dV$, where dV is the Euclidean volume form on D, Φ_* is the differential of Φ, and $|\Phi_*|$ is the Hilbert–Schmidt norm of Φ_*, measured by the Mabuchi metric g_M and the Euclidean metric of D. A straightforward computation gives the harmonic map equation

$$\sum_{j=1}^m|\nabla\Phi_{z_j}|^2-2\Phi_{z_j\bar z_j}=0, \tag{1}$$

where $\{z_j\}$ are coordinates on D and $|\nabla\Phi_{z_j}(z)|^2$ is computed using the metric $\omega_{\Phi(z)}$. On the other hand, there is a perturbed functional $\mathscr{E}$, whose Euler–Lagrange equation is also of interest. The construction of this perturbed functional is similar to that of [Donaldson 1999, Section 5] (see also [Witten 1983]), where one-dimensional D were considered. In order to define $\mathscr{E}$, we first define a three-form θ on $\mathcal{H}_\omega$: for $\phi\in\mathcal{H}_\omega$ and $\xi_1,\xi_2,\xi_3\in T_\phi\mathcal{H}_\omega$,

$$\theta(\xi_1,\xi_2,\xi_3):=g_M(\{\xi_1,\xi_2\}_{\omega_\phi},\xi_3)=\int_X\{\xi_1,\xi_2\}_{\omega_\phi}\xi_3\omega_\phi^n, \tag{2}$$

where $\{\cdot,\cdot\}_{\omega_\phi}$ is the Poisson bracket determined by the symplectic form ω_ϕ. This three-form θ is d-closed by Lemma 4.5 below, and by Lemma 4.4 there is a two-form α on $\mathcal{H}_\omega$ such that $d\alpha=\theta$. For a map $\Phi: D\to\mathcal{H}_\omega$, we define

$$\mathscr{E}(\Phi):=E(\Phi)+4i\sum_j\int_D\alpha(\Phi_{\bar z_j},\Phi_{z_j})\,dV.$$

We will show in Lemma 4.6 that the Euler–Lagrange equation of $\mathscr{E}$ is

$$\sum_{j=1}^m|\nabla\Phi_{z_j}|^2-2\Phi_{z_j\bar z_j}+i\{\Phi_{\bar z_j},\Phi_{z_j}\}_{\omega_\Phi}=0. \tag{3}$$

Following [Witten 1983] and [Donaldson 1999], we call (3) the WZW equation for a map $\Phi: D\to\mathcal{H}_\omega$.

Donaldson [1999] showed, when $m=1$, that the WZW equation is equivalent to a homogeneous complex Monge–Ampère equation. We have the following extended equivalence for $m\geq 1$ by a similar computation. Let $\pi: D\times X\to X$ be the projection onto X. Then the extended equivalence is

$$\Phi \text{ solves (3)} \iff (i\partial\bar\partial\Phi+\pi^*\omega)^{n+1}\wedge\Big(i\sum_{j=1}^m dz_j\wedge d\bar z_j\Big)^{m-1}=0. \tag{4}$$

This suggests that the proper generality of the WZW equation is for maps from a Kähler manifold D to $\mathcal{H}_\omega$. Nevertheless, in this paper we restrict to $D\subset\mathbb{C}^m$.

The next step is to construct a solution of the WZW equation, and then we will show it can be approximated by the solutions of Hermitian–Yang–Mills equations.

Definition 1.1. We will say that a function $u : D \times X \to [-\infty, \infty)$ is ω-subharmonic on graphs if, for any holomorphic map f from an open subset of D to X, we have that $\psi(f(z)) + u(z, f(z))$ is subharmonic, where ψ is a local potential of ω.

This definition does not depend on the choice of ψ since any two local potentials differ by a pluriharmonic function. (This definition has its origin in the works of Slodkowski [1988; 1990a; 1990b], and Coifman and Semmes [1993]; however, they focus on functions u defined on $D \times V$ with a vector space V where $u(z, \cdot)$ are norms or quasinorms, whereas we consider simply functions on $D \times X$. There is also a notion of k-subharmonicity, see [Błocki 2005], but it is not equivalent to subharmonicity on graphs.)

Let v be a real-valued smooth function on $\partial D \times X$ and $\partial D \ni z \mapsto v(z, \cdot) = v_z \in \mathcal{H}_\omega$. We simply write $v \in C^\infty(\partial D, \mathcal{H}_\omega)$. Consider the Perron family

$$G_v := \Big\{ u \in \operatorname{usc}(D \times X) : u \text{ is } \omega\text{-subharmonic on graphs}, \limsup_{D \ni z \to \zeta \in \partial D} u(z, x) \le v(\zeta, x) \Big\}.$$

As we will later see, the upper envelope $V = \sup\{u : u \in G_v\}$ is a weak solution of the WZW equation from D to $\mathcal{H}_\omega$. The above setup is for $\mathcal{H}_\omega$. As for $\mathcal{H}_k$, we recall first the two maps that connect $\mathcal{H}_\omega$ and $\mathcal{H}_k$. The Hilbert map $H_k : \mathcal{H}_\omega \to \mathcal{H}_k$ is

$$H_k(\phi)(s, s) = \int_X h^k(s, s) e^{-k\phi} \omega^n, \quad \text{for } \phi \in \mathcal{H}_\omega \text{ and } s \in H^0(X, L^k).$$

In the other direction, the Fubini–Study map $FS_k : \mathcal{H}_k \to \mathcal{H}_\omega$ is given by

$$FS_k(G)(x) = \frac{1}{k} \log \sup_{\substack{s \in H^0(X, L^k) \\ G(s,s) \le 1}} h^k(s, s)(x), \quad \text{for } G \in \mathcal{H}_k \text{ and } x \in X.$$

Now following the definitions from [Coifman and Semmes 1993], let $\mathcal{N}_k^*$ be the set of norms on $H^0(X, L^k)^*$. Then a norm function $D \ni z \mapsto U_z \in \mathcal{N}_k^*$ is said to be subharmonic if $\log U_z(f(z))$ is subharmonic for any holomorphic section $f : W \subset D \to H^0(X, L^k)^*$. The second Perron family we consider is

$$G_v^k := \Big\{ D \ni z \to U_z \in \mathcal{N}_k^* \text{ is subharmonic}, \limsup_{D \ni z \to \zeta \in \partial D} U_z^2(s) \le H_k^*(v_\zeta)(s, s) \text{ for any } s \in H^0(X, L^k)^* \Big\},$$

where $H_k^*(v)$ is the inner product dual to $H_k(v)$. We note a remarkable theorem about the upper envelope $V^k = \sup\{U : U \in G_v^k\}$ from [Coifman and Semmes 1993], which shows that V^k is not only a norm but an inner product (see [Slodkowski 1990a, Corollary 2.7] for a different proof); moreover, it solves the Hermitian–Yang–Mills equation (see also [Donaldson 1992])

$$\begin{cases} \Lambda\Theta(V^k) = 0, \\ V^k|_{\partial D} = H_k^*(v). \end{cases}$$

Here we view V^k as a Hermitian metric on the bundle $\bar{D} \times H^0(X, L^k)^* \to \bar{D}$, and $\Theta(V^k)$ is its curvature. Further, Λ is the trace with respect to the Euclidean metric of D, so in general $\Lambda\Theta(V^k)$ takes values in endomorphisms of $H^0(X, L^k)^*$. Denoting the dual metric by $(V^k)^*$, our main result is that the upper envelope V of G_v is the limit of Hermitian–Yang–Mills metrics.

Theorem 1.2. *$FS_k((V^k)^*)$ converges to V uniformly on $D \times X$, as $k \to \infty$.*

Now we turn to the interpretation of the upper envelope V and its relation to the WZW equation. The next theorem shows that V solves the WZW equation under a regularity assumption.

Theorem 1.3. *If the upper envelope V of G_v is in $C^2(D \times X)$, then*

$$(i\partial\bar{\partial}V + \pi^*\omega)^{n+1} \wedge \left(i\sum_{j=1}^{m} dz_j \wedge d\bar{z}_j\right)^{m-1} = 0.$$

As a result, Theorems 1.2 and 1.3 together show that the solution of the WZW equation can be approximated by the Hermitian–Yang–Mills metrics. (The equation in Theorem 1.3 is similar to the complex Hessian equation, which has been studied extensively in [Błocki 2005; Collins and Picard 2019; Dinew and Kołodziej 2014; Dinew et al. 2019; Lu and Nguyen 2015; 2019], and we hope to return to it in the future.)

The C^2 assumption in Theorem 1.3 is somewhat artificial. At this point, we are able to show V is continuous by Corollary 3.3, and it is desirable to prove higher regularity of V either through PDE techniques or pluripotential theory which we will pursue in a different paper. The guiding example is when $m = 1$. In that case the WZW equation is the much studied complex Monge–Ampère equation; it is known that V is not smooth in general (see [Darvas 2014; Darvas and Lempert 2012; Lempert and Vivas 2013]), and $C^{1,1}$ is the best one can hope for (see [Błocki 2012; Chen 2000; Chu et al. 2017]).

We mention briefly works related to our result. If $m = 1$ and $D \subset \mathbb{C}$ is an annulus, and v is invariant under rotation of the annulus, then Theorems 1.2 and 1.3 recover the geodesic approximation result of Phong and Sturm [2006] and Berndtsson [2018]. When X is toric, these theorems are reduced to the harmonic approximation of Rubinstein and Zelditch [2010], except that C^2 convergence is proved in their paper (see also [Song and Zelditch 2007; 2010]).

The proof of Theorem 1.2 hinges on Theorem 2.1, a result regarding the positivity of direct image bundles. Although Berndtsson's theorem [2009] has played a crucial role in approximation theorems similar to Theorem 1.2 (for example [Berman and Keller 2012; Berndtsson 2018; Darvas and Wu 2019; Darvas et al. 2020]), when it comes to approximating by Hermitian–Yang–Mills metrics, a subharmonic analogue of Berndtsson's theorem is desired. It is Theorem 2.1, where we prove a version of positivity of direct image bundles for weights that are subharmonic on graphs. This is perhaps the crux of this paper. A corresponding result on Stein manifolds can be proved easily following the proof of Theorem 2.1.

The WZW equation (3) is the harmonic map (1) perturbed with Poisson's bracket, which is closely related to the geometry of $\mathcal{H}_\omega$, an infinite-dimensional nonpositively curved manifold. Since the theory of harmonic maps into nonpositively curved manifolds is well developed by Eells and Sampson [1964], Hamilton [1975], and many others, a possible future direction is to see if one can combine the classical results with those of this paper to study $\mathcal{H}_\omega$. Yet another possible but remote direction is to use slope stability in the Donaldson–Uhlenbeck–Yau theorem to study the K-stability by Theorem 1.2. This is a vast subject and we only mention a few papers that are closer to our study. See [Chen et al. 2015a; 2015b; 2015c; Dervan and Keller 2019; Donaldson 1985; Li 2012; Székelyhidi 2014; Uhlenbeck and Yau 1986; Zhang 2021].

Before we end this introduction, a few words about the structure of this paper. In Section 2, the subharmonic version of positivity of direct image bundles is proved, except we put off a technical lemma to Section 5. Section 3 is devoted to Theorem 1.2 and Section 4 to Theorem 1.3. In Section 6, we draw parallels with [Darvas and Wu 2019].

2. Positivity of direct image bundles

Consider a Hermitian holomorphic line bundle $(E, g) \to X^n$ over a compact complex manifold, and assume the curvature η of the metric g is positive. For two sections $s, t \in H^0(X, E \otimes K_X)$, we write locally

$$s = \sigma \otimes s', \quad t = \tau \otimes t',$$

where $\sigma, \tau \in E$ and $s', t' \in K_X$. (Such an expression is possible as long as one of the bundles is of rank 1. In the current case, E and K_X are both line bundles.) We extend the metric g to acting on sections of $E \otimes K_X$ by setting $g(s, t) = g(\sigma, \tau) s' \wedge \bar{t}'$, which is an (n, n)-form. It is not hard to see this (n, n)-form is globally defined on X.

We define a variant of the Hilbert map: $\mathrm{Hilb}_{E\otimes K_X}(u)$, for a function $u : D \times X \to \mathbb{R}$, is given by

$$\mathrm{Hilb}_{E\otimes K_X}(u)(s, s) = \int_X g(s, s) e^{-u(z,\cdot)}$$

with $s \in H^0(X, E \otimes K_X)$. Since the integrand on the right is already an (n, n)-form, the integral makes sense. In the following, suitable assumptions will be made on u to make sure the integral converges. Then the map $z \mapsto \mathrm{Hilb}_{E\otimes K_X}(u)$ is a Hermitian metric on the bundle $D \times H^0(X, E \otimes K_X) \to D$. The main result of this section is the following positivity theorem.

Theorem 2.1. *If u is bounded and upper semicontinuous (usc) on $D \times X$, and η-subharmonic on graphs, then the dual metric $\mathrm{Hilb}^*_{E\otimes K_X}(u)$ is a subharmonic norm function.*

The following approximation lemma is somewhat technical and we postpone its proof to Section 5.

Lemma 2.2. *Let u be a bounded usc function on $D \times X$ which is η-subharmonic on graphs. Then, for D' relatively compact open in D, there exist $\varepsilon_j \searrow 0$ and $u_j \in C^\infty(D' \times X)$ decreasing to u, where u_j is $(1 - \varepsilon_j)\eta$-subharmonic on graphs. Namely, for any holomorphic map f from an open subset of D' to X, $\Delta(\psi(f(z)) + u_j(z, f(z))) \geq \varepsilon_j \Delta(\psi(f(z))$, where $\eta = i\partial\bar{\partial}\psi$ locally.*

Proof of Theorem 2.1. Since being a subharmonic norm function is a local property, we focus on D', a relatively compact open set in D. Take ε_j and u_j as in Lemma 2.2. Assuming the theorem holds for such a u_j (namely, the dual metric $\mathrm{Hilb}^*_{E\otimes K_X}(u_j)$ is a subharmonic norm function), it follows that $\mathrm{Hilb}^*_{E\otimes K_X}(u)$ is also a subharmonic norm function because $\mathrm{Hilb}^*_{E\otimes K_X}(u_j)$ decreases to $\mathrm{Hilb}^*_{E\otimes K_X}(u)$ as $j \to \infty$.

As a result, we only need to prove the theorem for $u \in C^\infty(D' \times X)$ with the property that there exists $\varepsilon > 0$ such that for any holomorphic function f from an open subset of D' to X,

$$\Delta(\psi(f(z)) + u(z, f(z))) \geq \varepsilon \Delta(\psi(f(z)), \quad \text{where } \eta = i\partial\bar{\partial}\psi \text{ locally.} \tag{5}$$

In a coordinate system $\Omega \subset \mathbb{C}^n$ on X, we will not write out the coordinate map. We will use Greek letters μ, λ for indices of coordinates on X, and Roman letters i, j for indices of coordinates on D; moreover, f^μ means the μ-th component of f, whereas $\psi_{\mu\bar\lambda}$, $u_{i\bar i}$, and $u_{i\bar\lambda}$ mean partial derivatives $\partial^2\psi/\partial x_\mu \partial\bar x_\lambda$, $\partial^2 u/\partial z_i\partial\bar z_i$, and $\partial^2 u/\partial z_i\partial\bar x_\lambda$, respectively. In this coordinate system $\Omega\subset\mathbb{C}^n$ on X, we first show that the matrix $(\psi_{\mu\bar\lambda}+u_{\mu\bar\lambda})(z_0, x_0)$ is positive definite, for any given $(z_0, x_0)\in D'\times\Omega$. Inequality (5) about $\psi+u$ is unchanged after a translation in coordinates of $D'\times\Omega$, so we can assume $(z_0, x_0) = (0, 0)$. In terms of local coordinates, inequality (5) becomes

$$\varepsilon\sum_{i,\lambda,\mu}\psi_{\mu\bar\lambda}\frac{\partial f^\mu}{\partial z_i}\frac{\partial\bar f^\lambda}{\partial\bar z_i}\le\sum_{i,\lambda,\mu}\psi_{\mu\bar\lambda}\frac{\partial f^\mu}{\partial z_i}\frac{\partial\bar f^\lambda}{\partial\bar z_i}+\sum_i u_{i\bar i}+\sum_{i,\lambda}u_{i\bar\lambda}\frac{\partial\bar f^\lambda}{\partial\bar z_i}+\sum_{i,\mu}u_{\bar i\mu}\frac{\partial f^\mu}{\partial z_i}+\sum_{i,\lambda,\mu}u_{\mu\bar\lambda}\frac{\partial f^\mu}{\partial z_i}\frac{\partial\bar f^\lambda}{\partial\bar z_i}. \quad (6)$$

Fix $(\xi_1, \xi_2, \dots, \xi_n)\in\mathbb{C}^n$. For N a positive number, we consider $f(z) = N(\xi_1, \xi_2, \dots, \xi_n)z_1$; note that $f(z)$ is in Ω by restricting z in a small neighborhood of 0 in D'. With such a choice of f, we deduce from (6) that

$$\begin{aligned}\varepsilon\sum_{\lambda,\mu}\psi_{\mu\bar\lambda}(0,0)\xi_\mu\bar\xi_\lambda N^2 &\le \sum_{\lambda,\mu}\psi_{\mu\bar\lambda}(0,0)\xi_\mu\bar\xi_\lambda N^2+\sum_i u_{i\bar i}(0,0)\\ &\quad+\sum_\lambda u_{1\bar\lambda}(0,0)\bar\xi_\lambda N+\sum_\mu u_{\bar 1\mu}(0,0)\xi_\mu N+\sum_{\lambda,\mu}u_{\mu\bar\lambda}(0,0)\xi_\mu\bar\xi_\lambda N^2. \quad (7)\end{aligned}$$

For larger N, we have to restrict f to a smaller domain in D', but since inequality (7) is evaluated at $(0, 0)$, it holds for any N. Divide (7) by N^2 and send N to infinity, to obtain $(\psi_{\mu\bar\lambda}+u_{\mu\bar\lambda})(0,0)\ge\varepsilon(\psi_{\mu\bar\lambda})(0,0)$ as matrices, and hence $(\psi_{\mu\bar\lambda}+u_{\mu\bar\lambda})(0,0)$ is positive definite.

Let $L^2(X, E\otimes K_X)$ be the space of measurable sections s whose L^2-norm $\int_X g(s,s)e^{-u(z,\cdot)}$ is finite. Since different z will give rise to comparable L^2-norms, the space $L^2(X, E\otimes K_X)$ does not change with z, and so we have a Hermitian Hilbert bundle $D'\times L^2(X, E\otimes K_X)\to D'$ which has $D'\times H^0(X, E\otimes K_X)\to D'$ as a subbundle. Denote the curvature of the subbundle by $\Theta = \sum\Theta_{j\bar k}dz_j\wedge dz_{\bar k}$. This setup is almost identical to [Berndtsson 2009, Theorem 1.1], where the author observed that the second fundamental form of the subbundle $D'\times H^0(X, E\otimes K_X)\to D'$ can be controlled by L^2-estimates. Following the computations in Section 3 of the same work, we deduce

$$\sum_j(\Theta_{j\bar j}s, s)\ge\int_X K(z,\cdot)g(s,s)e^{-u(z,\cdot)}, \quad (8)$$

where $s\in H^0(X, E\otimes K_X)$ and $K : D'\times X\to\mathbb{R}$ is a smooth function, given in local coordinates on X by

$$K=\sum_j\biggl(u_{j\bar j}-\sum_{\lambda,\mu}(\psi+u)^{\bar\lambda\mu}u_{j\bar\lambda}u_{\bar j\mu}\biggr);$$

here $(\psi+u)^{\bar\lambda\mu}$ stands for the inverse matrix of $(\psi+u)_{\bar\lambda\mu}$; cf. [Berndtsson 2009, Formula (3.1)].

We claim that $K\ge 0$. Fix $(z_0, x_0)\in D'\times X$ with a coordinate system Ω around x_0. First notice that ψ is independent of z, so if we denote $\psi(x)+u(z,x)$ by $\phi(z,x)$, then

$$K=\sum_j\biggl(\phi_{j\bar j}-\sum_{\lambda,\mu}\phi_{j\bar\lambda}\phi^{\bar\lambda\mu}\phi_{\bar j\mu}\biggr).$$

Since the matrix $(\phi_{\mu\bar{\lambda}})$ is positive definite, we can assume local coordinates in Ω are such that $(\phi_{\mu\bar{\lambda}})$ is the identity matrix at (z_0, x_0), and therefore $K(z_0, x_0) = \sum_j (\phi_{j\bar{j}} - \sum_\lambda |\phi_{j\bar{\lambda}}|^2)(z_0, x_0)$. For a holomorphic function f from an open subset of D' to Ω, the subharmonicity of $\phi(z, f(z))$ reads as

$$\sum_i \phi_{i\bar{i}} + \sum_{i,\lambda} \phi_{i\bar{\lambda}} \frac{\partial \bar{f}^\lambda}{\partial \bar{z}_i} + \sum_{i,\mu} \phi_{\bar{i}\mu} \frac{\partial f^\mu}{\partial z_i} + \sum_{i,\lambda,\mu} \phi_{\mu\bar{\lambda}} \frac{\partial f^\mu}{\partial z_i} \frac{\partial \bar{f}^\lambda}{\partial \bar{z}_i} \geq 0. \tag{9}$$

Without loss of generality, we assume $(z_0, x_0) = (0, 0)$ and choose $f^\lambda = -\sum_i \phi_{i\bar{\lambda}}(0, 0) z_i$ in (9) with z small so that $f(z)$ is in Ω. Inequality (9) becomes $\sum_j \bigl(\phi_{j\bar{j}} - \sum_\lambda |\phi_{j\bar{\lambda}}|^2\bigr)(0, 0) \geq 0$. Therefore $K \geq 0$. See also the remark after Lemma 4.1 for a slightly different proof of this claim and an invariant meaning of K.

As a result, (8) implies $\sum_j (\Theta_{j\bar{j}} s, s) \geq 0$, and hence the curvature of the dual metric $\mathrm{Hilb}^*_{E\otimes K_X}(u)$ satisfies the opposite inequality; according to [Coifman and Semmes 1993, Theorem 4.1] this implies $\mathrm{Hilb}^*_{E\otimes K_X}(u)$ is a subharmonic norm function. □

Now we replace (E, g) by $(L^k \otimes K_X^*, h^k \otimes \omega^n)$, which is positively curved for large k since

$$\Theta(h^k \otimes \omega^n) = k\omega + \mathrm{Ric}\,\omega.$$

We have the following proposition regarding the metric $H_k(u)$ on the bundle $D \times H^0(X, L^k)$.

Proposition 2.3. *Suppose u is a bounded usc function on $D \times X$ and with some $\varepsilon \in (0, 1)$ we have that u is $(1-\varepsilon)\omega$-subharmonic on graphs. Then there exists $k_0 = k_0(\varepsilon, \omega)$, independent of u, such that, for $k \geq k_0$, the dual metric $H_k^*(u)$ is a subharmonic norm function.*

Proof. In order to use Theorem 2.1, we must check if ku is $(k\omega + \mathrm{Ric}\,\omega)$-subharmonic on graphs. Suppose that $\omega = i\partial\bar{\partial}\psi$ and $\mathrm{Ric}\,\omega = i\partial\bar{\partial}\phi$ locally. Then we want to see if $k\psi(f(z)) + \phi(f(z)) + ku(z, f(z))$ is subharmonic for any holomorphic map f. Note that

$$k\psi + \phi + ku = k(1-\varepsilon)\psi + ku + \varepsilon k\psi + \phi,$$

and $k(1-\varepsilon)\psi(f(z)) + ku(z, f(z))$ is subharmonic by assumption. On the other hand, there exists k_0 depending on ε and ω such that $\varepsilon k\psi + \phi$ is plurisubharmonic (psh) for $k \geq k_0$. Therefore, for $k \geq k_0$, ku is $(k\omega + \mathrm{Ric}\,\omega)$-subharmonic on graphs. By Theorem 2.1, the metric $\mathrm{Hilb}^*_{L^k}(ku)$ is a subharmonic norm function for $k \geq k_0$. The proposition follows since $\mathrm{Hilb}_{L^k}(ku) = H_k(u)$. □

3. Approximation by Hermitian–Yang–Mills metrics

Recall that D is in $\mathbb{C}^m$ and $(L, h) \to X^n$ is a positive line bundle with curvature ω.

Lemma 3.1. *Let u be an usc function on $D \times X$ and ω-subharmonic on graphs. Then for any fixed $z \in D$, $u(z, x)$ is ω-psh on X, and for any fixed $x \in X$, $u(z, x)$ is subharmonic on D.*

This can be seen as a special case of an abstract theorem in [Slodkowski 1990a, Section 1], whose proof we translate to our setting.

Proof. By choosing the holomorphic map f constant in the definition of ω-subharmonic on graphs, it follows immediately that $u(z, x)$ is subharmonic in z.

For a fixed $z_0 \in D$, we want to show $x \mapsto \psi(x) + u(z_0, x)$ is psh in a coordinate system $\Omega \subset \mathbb{C}^n$ on X, where ψ is a local potential of ω. Let P be the complex line $\{\lambda e_1 : \lambda \in \mathbb{C}, e_1 = (1, 0, \ldots, 0) \in \mathbb{C}^n\}$. Without loss of generality, it suffices to prove that, for $\lambda e_1 \in P \cap \Omega$, the function $\lambda \mapsto \psi(\lambda e_1) + u(0, \lambda e_1)$ is subharmonic. Let U be a disc in $P \cap \Omega$, and we simply write $U = \{\lambda \in \mathbb{C} : |\lambda - a| < R\}$. Let $h(\lambda)$ be harmonic on U and continuous up to the boundary. We will be done if

$$\psi(ae_1) + u(0, ae_1) + h(a) \leq \max_{\lambda \in \partial U} (u\psi(\lambda e_1) + u(0, \lambda e_1) + h(\lambda)).$$

Suppose the inequality is not true. By [Slodkowski 1986, Lemma 4.5] with $\partial U \subset \bar{U}$ as the two compact sets in that lemma, there is an $\mathbb{R}$-linear function $l : \mathbb{C} \to \mathbb{R}$ and $b \in U$ such that, if we write

$$v(z, \lambda) = \psi(\lambda e_1) + u(z, \lambda e_1) + h(\lambda) + l(\lambda), \tag{10}$$

then

$$v(0, b) > v(0, \lambda), \quad \text{for } \lambda \in U - \{b\}.$$

Now define $W(z, \lambda_1, \ldots, \lambda_m) := v(z, \lambda_1) + \cdots + v(z, \lambda_m)$ in a neighborhood of $(0, b^*) := (0, b, \ldots, b)$ in $\mathbb{C}^m \times \mathbb{C}^m$. As $W(0, b^*) > W(0, \lambda_1, \ldots, \lambda_m)$ for $(\lambda_1, \ldots, \lambda_m) \neq b^*$, there exists a ball $B \subset \mathbb{C}^m$ of radius r centered at b^* such that

$$W(0, b^*) > \max_{\{0\} \times \partial B} W.$$

Since W is usc, there exists $\varepsilon > 0$ such that $W(z, \lambda_1, \ldots, \lambda_m) < W(0, b^*)$, for $|z| \leq \varepsilon$ and $(\lambda_1, \ldots, \lambda_m) \in \partial B$. Let $S = (r/\varepsilon)\,\mathrm{Id}_{\mathbb{C}^m}$. We have $W(z, b^* + S(z)) < W(0, b^*)$ for $|z| = \varepsilon$, which contradicts the maximum principle because $W(z, b^* + S(z)) = \sum_{i=1}^m v(z, b + (r/\varepsilon)z_i)$ is subharmonic by (10). □

Although in the Introduction the boundary data v is in $C^\infty(\partial D, \mathcal{H}_\omega)$, we will prove a lemma for a broader class of boundary data ν. Let $\nu : \partial D \times X \to \mathbb{R}$ be a continuous map such that $\nu_z(\,\cdot\,) := \nu(z, \,\cdot\,) \in \mathrm{PSH}(X, \omega)$ for $z \in \partial D$. Let

$$G_\nu = \Big\{ u \in \mathrm{usc}(D \times X) : u \text{ is } \omega\text{-subharmonic on graphs}, \limsup_{D \ni z \to \zeta \in \partial D} u(z, x) \leq \nu(\zeta, x) \Big\}.$$

In order to study the properties of the upper envelope $\mathcal{V}$ of G_ν, we introduce a closely related family. With the projection $\pi : D \times X \to X$, let

$$F_\nu := \Big\{ u : u \in \mathrm{PSH}(D \times X, \pi^*\omega), \limsup_{D \ni z \to \zeta \in \partial D} u(z, x) \leq \nu(\zeta, x) \Big\}.$$

The upper envelope of F_ν extends to a solution $\mathcal{U} \in C(\bar{D} \times X)$ of

$$\begin{cases} (\pi^*\omega + i\partial\bar\partial\mathcal{U})^{n+m} = 0 & \text{on } D \times X, \\ \pi^*\omega + i\partial\bar\partial\mathcal{U} \geq 0 & \text{on } D \times X, \\ \mathcal{U}|_{\partial D \times X} = \nu; \end{cases}$$

see for example [Boucksom 2012; Darvas and Wu 2019]. In addition, we also need the solution h to the Dirichlet problem

$$\begin{cases} \sum_j h_{j\bar{j}} + \Delta_\omega h + 2n = 0 \quad \text{on } D \times X, \\ h|_{\partial D \times X} = \nu. \end{cases}$$

Lemma 3.2. *If we denote the upper envelopes of G_ν and F_ν by $\mathcal{V}$ and $\mathcal{U}$, respectively, then $\mathcal{U} \le \mathcal{V} \le h$ and*

$$\lim_{(z,x)\to(z_0,x_0)\in\partial D\times X} \mathcal{V}(z,x) = \nu(z_0,x_0).$$

Moreover, if ν is negative, then so is $\mathcal{V}$.

Proof. Unraveling the definitions of F_ν and G_ν, we see $F_\nu \subset G_\nu$, so $\mathcal{U} \le \mathcal{V}$. For any $u \in G_\nu$, $u(z,\cdot)$ is ω-psh for fixed z by Lemma 3.1, hence $\Delta_\omega u + 2n \ge 0$; in addition, $u(\cdot\,,x)$ is subharmonic for fixed x. By the maximum principle, $u \le h$ and hence $\mathcal{V} \le h$ also. $\mathcal{U}$ and h are both equal to ν on $\partial D \times X$, and so is $\mathcal{V}$.

For a fixed $x_0 \in X$, let $H_0(z)$ be the harmonic function on D with boundary values $\nu(z,x_0)$. For $u \in G_\nu$, we have $u(z,x_0) \le H_0(z)$, and therefore $\mathcal{V}(z,x_0) \le H_0(z)$. The second statement follows at once. □

With Proposition 2.3 at hand, we can start to prove Theorem 1.2. The following envelope will be used in the proof: for an usc function F on X, we introduce

$$P(F) := \sup\{h \in \mathrm{PSH}(X,\omega) \mid h \le F\} \in \mathrm{PSH}(X,\omega);$$

see [Berman 2019; Ross and Witt Nyström 2017].

Proof of Theorem 1.2. Without loss of generality, we will assume $v \le 0$. Fix $\delta > 1$, and for $z \in \partial D$, define $v_z^\delta = P(\delta v_z)$. By [Darvas and Wu 2019, Lemma 4.9], $\partial D \times X \ni (z,x) \mapsto v_z^\delta(x)$ is continuous. Let V^δ be the upper envelope of G_{v^δ}. By Lemma 3.2, $V^\delta \le 0$, and so $u \le 0$ for $u \in G_{v^\delta}$. The next step is to have a better upper bound for $u \in G_{v^\delta}$. To that end, we can look instead at $\max\{u,c\}$, which is still in G_{v^δ} as long as the constant $c \le \min v^\delta$. Since $\max\{u,c\}$ is bounded, we will assume u is bounded. Moreover, u/δ is ω/δ-subharmonic on graphs. According to Proposition 2.3, there exists $k_0 = k_0(\delta)$ such that for $k \ge k_0$, $H_k^*(u/\delta)$ is a subharmonic norm function. Because $\limsup_{\partial D} H_k^*(u/\delta) \le H_k^*(v)$, it follows that $H_k^*(u/\delta) \in G_v^k$ and therefore $H_k^*(u/\delta) \le V^k$ on D and $FS_k(H_k(u/\delta)) \le FS_k((V^k)^*)$. By Lemma 3.1, we have $\omega + i\partial\bar\partial u/\delta \ge (1-1/\delta)\omega$, (the operator $i\partial\bar\partial$ here is with respect to variables in X). The Ohsawa–Takegoshi extension theorem implies (see [Darvas et al. 2020, Theorem 2.11] or [Darvas and Wu 2019, Lemma 4.10]) that there exist $C > 0$ and $k_0(\delta)$ such that, for $k \ge k_0$,

$$\frac{1}{\delta}u - \frac{C}{k} \le FS_k \circ H_k\Big(\frac{1}{\delta}u\Big) \le FS_k((V^k)^*).$$

Since $\delta v \le 0$, both V^δ and u are negative by Lemma 3.2, and as a result we have $u - C/k \le FS_k((V^k)^*)$; this statement is true for any $u \in G_{v^\delta}$, so we actually have $V^\delta - C/k \le FS_k((V^k)^*)$. In addition, since $v_z + (\delta-1)\inf_{\partial D\times X}(v_z)$ is a competitor in $P(\delta v_z)$,

$$V + (\delta-1)\inf_{\partial D\times X}(v) \le V^\delta.$$

Putting things together, we conclude

$$V + (\delta - 1) \inf_{\partial D \times X} (v) - \frac{C}{k} \le FS_k((V^k)^*), \quad \text{for } k \ge k_0(\delta). \tag{11}$$

Next we claim that $FS_k((V_z^k)^*)(x)$ is ω-subharmonic on graphs. Some preparation is needed. Let s be a nonvanishing holomorphic section of L^k over an open set $Y \subset X$. Let $e^{-k\phi} := h^k(s, s)$ and $s_k^* : Y \to (L^k)^*$ be defined by $s_k^*(x)(\cdot) = h^k(\cdot, e^{k\phi(x)/2} s(x))$ for $x \in Y$. Suppose $\hat{s}_k^* : Y \to H^0(X, L^k)^*$ is the pointwise evaluation map of s_k^*, namely $\hat{s}_k^*(x)(\sigma) := s_k^*(x)(\sigma(x))$ for $\sigma \in H^0(X, L^k)$. Then we have the following formula, which is taken from [Darvas and Wu 2019, Lemma 4.1]:

$$FS_k((V_z^k)^*)(x) = \frac{2}{k} \log[V_z^k(\hat{s}_k^*(x))], \qquad x \in Y. \tag{12}$$

Meanwhile, for $\sigma \in H^0(X, L^k)$, we have $e^{k\phi(x)/2} \hat{s}_k^*(x)(\sigma) = \sigma(x)/s(x)$ is holomorphic, so $e^{k\phi/2} \hat{s}_k^*$ is holomorphic. Hence for any holomorphic map g from an open subset of D to X,

$$\Delta(\phi(g(z)) + FS_k((V_z^k)^*)(g(z))) = \Delta\Big(\frac{1}{k} \log[V_z^k((e^{k\phi/2} \hat{s}_k^*) \circ g(z))]^2\Big). \tag{13}$$

By [Coifman and Semmes 1993, Theorem 4.1], the Hermitian–Yang–Mills metric V_z^k is a subharmonic norm function, so the last term of (13) is nonnegative, which means $FS_k((V^k)^*)$ is ω-subharmonic on graphs as we claimed. Further, according to the Tian–Catlin–Zelditch asymptotic theorem or by [Darvas and Wu 2019, Lemma 4.10], we have an easier but cruder estimate

$$FS_k((V_z^k)^*|_{\partial D}) = FS_k(H_k(v)) \le v + O(\log k/k),$$

so

$$FS_k((V^k)^*) \in G_{v + O(\log k/k)}$$

and

$$FS_k((V^k)^*) \le V + O(\log k/k).$$

This last inequality together with (11) concludes the proof. □

It is natural to ask if V belongs to G_v. A standard approach to show that the envelope belongs to a family is to take upper regularization, and the case at hand is very similar to [Coifman and Semmes 1993, Lemma 11.11], where upper regularization is taken in the z-variables. The reason it works in their lemma is because their function in the x-variables is a norm, but ours is not and regularization does not seem to work. However, with Theorem 1.2 one can easily show $V \in G_v$. It would be interesting to prove $V \in G_v$ directly without using Theorem 1.2; after all, G_v and V can be defined on any Kähler manifold (X, ω) without reference to a line bundle.

Corollary 3.3. *The upper envelope V is continuous, and $V \in G_v$.*

Proof. The first statement is a direct consequence of Theorem 1.2. As to the second statement, let ψ be a local potential of ω and f a holomorphic map from an open subset of D to X. For any $u \in G_v$, $\psi(f(z)) + u(z, f(z))$ is subharmonic; hence $\psi(f(z)) + V(z, f(z))$, the supremum over $u \in G_v$, is also subharmonic since V is continuous. By Lemma 3.2, it follows that $V \in G_v$. □

4. The WZW equation

We will prove Theorem 1.3 and compute the Euler–Lagrange equation of $\mathscr{E}$ in this section. We begin with an observation. Suppose u is a C^2 function on $D \times X$ and ψ is a local potential of ω. Consider the complex Hessian of $u+\psi$ with respect to a fixed coordinate z_j in D and local coordinates x in X where ψ is defined,

$$\begin{pmatrix} (u+\psi)_{z_j\bar{z}_j} & (u+\psi)_{z_j\bar{x}_1} & \cdots & (u+\psi)_{z_j\bar{x}_n} \\ (u+\psi)_{x_1\bar{z}_j} & (u+\psi)_{x_1\bar{x}_1} & \cdots & (u+\psi)_{x_1\bar{x}_n} \\ \vdots & \vdots & \ddots & \vdots \\ (u+\psi)_{x_n\bar{z}_j} & (u+\psi)_{x_n\bar{x}_1} & \cdots & (u+\psi)_{x_n\bar{x}_n} \end{pmatrix}, \tag{14}$$

which we will denote by $(u+\psi)_j$. Then

$$\begin{aligned}(i\partial\bar{\partial}u+\pi^*\omega)^{n+1} &\wedge \Bigl(i\sum_{j=1}^m dz_j\wedge d\bar{z}_j\Bigr)^{m-1} \\ &= (n+1)!(m-1)!\sum_{j=1}^m \det(u+\psi)_j\Bigl(\bigwedge_{k=1}^m i\,dz_k\wedge d\bar{z}_k\wedge\bigwedge_{l=1}^n i\,dx_l\wedge d\bar{x}_l\Bigr). \end{aligned}\tag{15}$$

Lemma 4.1. *Suppose u is a C^2 function on $D\times X$ and $\omega+i\partial\bar{\partial}u(z,\cdot)>0$ on X for all $z\in D$. Then u is ω-subharmonic on graphs if and only if*

$$(i\partial\bar{\partial}u+\pi^*\omega)^{n+1}\wedge\Bigl(i\sum_{j=1}^m dz_j\wedge d\bar{z}_j\Bigr)^{m-1}\geq 0.$$

Proof. Let ψ be a local potential of ω and denote the complex Hessian of $u+\psi$ with respect to z_j and x by $(u+\psi)_j$, as in the matrix (14). Due to (15), we will focus on $\sum_{j=1}^m \det(u+\psi)_j$.

Let f be a holomorphic function from an open subset of D to X. Then in a coordinate system on X,

$$\Delta(\psi(f(z))+u(z,f(z)))=\sum_{i,\lambda,\mu}\psi_{\mu\bar{\lambda}}\frac{\partial f^\mu}{\partial z_i}\frac{\partial\bar{f}^\lambda}{\partial\bar{z}_i}+\sum_i u_{i\bar{i}}+\sum_{i,\lambda}u_{i\bar{\lambda}}\frac{\partial\bar{f}^\lambda}{\partial\bar{z}_i}+\sum_{i,\mu}u_{\bar{i}\mu}\frac{\partial f^\mu}{\partial z_i}+\sum_{i,\lambda,\mu}u_{\mu\bar{\lambda}}\frac{\partial f^\mu}{\partial z_i}\frac{\partial\bar{f}^\lambda}{\partial\bar{z}_i}.$$

If we denote the matrix $(\psi_{\mu\bar{\lambda}}+u_{\mu\bar{\lambda}})$ by A and the column vector $(u_{i\bar{\lambda}})$ by B_i, then the right side of the equation above can be written as

$$\sum_i\Bigl(\Bigl\langle A\frac{\partial f}{\partial z_i},\frac{\partial f}{\partial z_i}\Bigr\rangle+\Bigl\langle B_i,\frac{\partial f}{\partial z_i}\Bigr\rangle+\overline{\Bigl\langle B_i,\frac{\partial f}{\partial z_i}\Bigr\rangle}+u_{i\bar{i}}\Bigr),\tag{16}$$

where the angled inner product is the usual Euclidean inner product and $\partial f/\partial z_i$ is the column vector $(\partial f^\mu/\partial z_i)$. The matrix form can be further written as

$$\sum_i\Bigl(\Bigl\|\sqrt{A}\frac{\partial f}{\partial z_i}+\sqrt{A}^{-1}B_i\Bigr\|^2-\|\sqrt{A}^{-1}B_i\|^2+u_{i\bar{i}}\Bigr).\tag{17}$$

Notice that

$$\begin{aligned}\sum_i(-\|\sqrt{A}^{-1}B_i\|^2+u_{i\bar{i}}) &= \sum_i(u_{i\bar{i}}-\langle A^{-1}B_i,B_i\rangle)=\sum_i\Big(u_{i\bar{i}}-\sum_{\lambda,\mu}u_{i\bar{\lambda}}(\psi+u)^{\bar{\lambda}\mu}u_{\bar{i}\mu}\Big)\\ &= \sum_i\frac{\det(u+\psi)_i}{\det(\psi_{\mu\bar{\lambda}}+u_{\mu\bar{\lambda}})},\end{aligned} \tag{18}$$

where the last equality can be deduced from Schur's formula for determinants of block matrices as follows (see also [Semmes 1992; Berndtsson 2009] for a different computation). We examine the complex Hessian of $u+\psi$,

$$(u+\psi)_j=\begin{pmatrix}(u+\psi)_{z_j\bar{z}_j} & (u+\psi)_{z_j\bar{x}_1} & \cdots & (u+\psi)_{z_j\bar{x}_n}\\ (u+\psi)_{x_1\bar{z}_j} & (u+\psi)_{x_1\bar{x}_1} & \cdots & (u+\psi)_{x_1\bar{x}_n}\\ \vdots & \vdots & \ddots & \vdots\\ (u+\psi)_{x_n\bar{z}_j} & (u+\psi)_{x_n\bar{x}_1} & \cdots & (u+\psi)_{x_n\bar{x}_n}\end{pmatrix},$$

and find that the Schur complement of the trailing $n\times n$ minor $((u+\psi)_{\mu\bar{\lambda}})$ is precisely

$$u_{j\bar{j}}-\sum_{\lambda,\mu}u_{j\bar{\lambda}}(u+\psi)^{\bar{\lambda}\mu}u_{\bar{j}\mu},$$

which is also equal to $\det(u+\psi)_j/\det((u+\psi)_{\mu\bar{\lambda}})$ by Schur's formula; see [Horn and Zhang 2005].

Now u is ω-subharmonic on graphs if and only if (17) is nonnegative for any holomorphic maps f, and it is equivalent to the last summation in (18) being nonnegative. The lemma follows from the positivity of the matrix $(\psi_{\mu\bar{\lambda}}+u_{\mu\bar{\lambda}})$ and (15). □

From (15) and (18), the function K in the proof of Theorem 2.1 has the invariant expression

$$K=\frac{m!n!}{(m-1)!(n+1)!}\frac{(\pi^*\omega+i\partial\bar{\partial}u)^{n+1}\wedge\big(i\sum_{j=1}^m dz_j\wedge d\bar{z}_j\big)^{m-1}}{(\omega+i\partial\bar{\partial}u)^n\wedge\big(i\sum_{j=1}^m dz_j\wedge d\bar{z}_j\big)^m},$$

and one can see $K\geq 0$ if u is ω-subharmonic on graphs. See also [Campana et al. 2019, Section 4.1, Formula (85)].

Proof of Theorem 1.3. By (15), the equation

$$(i\partial\bar{\partial}V+\pi^*\omega)^{n+1}\wedge\Big(i\sum_{j=1}^m dz_j\wedge d\bar{z}_j\Big)^{m-1}=0$$

is equivalent to $\sum_j\det(\psi+V)_j=0$, so we will prove the latter equation.

By Corollary 3.3, the function V is ω-subharmonic on graphs, and hence $V(z,x)$ is ω-psh on X by Lemma 3.1. Take a coordinate chart Ω of X. Then for $\varepsilon>0$ and $x\in\Omega$, the function $V(z,x)+\varepsilon|x|^2$ satisfies the assumption of Lemma 4.1, so $\sum_i\det(\psi+V+\varepsilon|x|^2)_i\geq 0$ and $\sum_i\det(\psi+V)_i\geq 0$.

Suppose $\sum_i\det(\psi+V)_i$ is positive at a point p in $D\times X$. We may assume $\det(\psi+V)_1$ is positive at p. We digress here to prove the following lemma.

Lemma 4.2. *Let A be an $(n+1)\times(n+1)$ Hermitian matrix partitioned as*

$$A = \begin{pmatrix} a_{11} & a_{12} \; \cdots \; a_{1(n+1)} \\ a_{21} & \\ \vdots & B \\ a_{(n+1)1} & \end{pmatrix},$$

where B has size $n \times n$. If $\det A > 0$ *and the matrix $B \geq 0$, then the matrix $A > 0$.*

Proof. The semipositivity of B implies that A has at least n nonnegative eigenvalues, and actually it has at least n positive eigenvalues since A is invertible. The last eigenvalue of A must also be positive because $\det A > 0$. □

By the above lemma, the matrix $(\psi + V)_1$ is actually positive at p. Its $n \times n$ trailing minor $(\psi + V)_{\mu\bar{\lambda}}(p)$ is also positive. Since V is assumed to be C^2, we can find a neighborhood N of p in $D \times X$ such that the matrix $(\psi + V)_{\mu\bar{\lambda}} > \delta$ in N, for some positive number δ. By possibly shrinking N, we also have $\sum_i \det(\psi + V)_i > 0$ in N.

For the last step in the proof of Theorem 1.3, choose a smooth cutoff function ρ supported in N with $-\delta/2 \leq (\rho_{\mu\bar{\lambda}}) \leq \delta/2$ and such that $\sum_i \det(\psi + V + \rho)_i > 0$ in N. We see the function $V + \rho$ satisfies the assumption of Lemma 4.1 on N, and hence $V + \rho$ is ω-subharmonic on graphs and is in G_v, which contradicts $V = \sup G_v$. Therefore, $\sum_j \det(\psi + V)_j = 0$. □

As in the Introduction, θ on $\mathcal{H}_\omega$ is given by

$$\theta(\xi_1, \xi_2, \xi_3) := g_M(\{\xi_1, \xi_2\}_{\omega_\phi}, \xi_3) = \int_X \{\xi_1, \xi_2\}_{\omega_\phi} \xi_3 \omega_\phi^n, \tag{19}$$

where $\phi \in \mathcal{H}_\omega$ and $\xi_1, \xi_2, \xi_3 \in T_\phi \mathcal{H}_\omega$. We have $\{\xi_1, \xi_2\}_{\omega_\phi} \omega_\phi^n = n d\xi_1 \wedge d\xi_2 \wedge \omega_\phi^{n-1}$, and using integration by parts we deduce that

$$\int_X \{\xi_1, \xi_2\}_{\omega_\phi} \xi_3 \omega_\phi^n = \int_X \xi_1 \{\xi_2, \xi_3\}_{\omega_\phi} \omega_\phi^n,$$

and therefore θ is indeed skew-symmetric and a three-form. Moreover, θ is smooth in the sense that, for smooth vector fields X_1, X_2, X_3, the function $\theta(X_1, X_2, X_3) : \mathcal{H}_\omega \to \mathbb{R}$ is smooth. The rest of this section is devoted to proving that the three-form θ is d-closed on $\mathcal{H}_\omega$ and showing the derivation of the Euler–Lagrange equation of $\mathscr{E}$.

The exterior derivative and the Poincaré lemma over a Banach manifold are discussed in detail in [Abraham et al. 1988, Supplement 6.4A], and although $\mathcal{H}_\omega$ is a Fréchet manifold, we can still derive the following two lemmas by similar approaches. See [Hamilton 1982] for a discussion of Fréchet manifolds.

We define first the exterior derivative on $\mathcal{H}_\omega$. Given a smooth k-form β on $\mathcal{H}_\omega$ and tangent vectors $\xi_0, \ldots, \xi_k$ at $T_\phi \mathcal{H}_\omega$, in order to define

$$d\beta(\xi_0, \ldots, \xi_k),$$

we extend ξ_i to vector fields on $\mathcal{H}_\omega$, which are constant in the canonical trivialization $T\mathcal{H}_\omega \approx \mathcal{H}_\omega \times C^\infty(X)$. Still denoting the constant vector fields by ξ_i, the exterior derivative is given by the well-known formula

$$d\beta(\xi_0, \ldots, \xi_k) = \sum_{j=0}^{k} (-1)^j L_{\xi_j}(\beta(\xi_0, \ldots, \hat{\xi}_j, \ldots, \xi_k)) + \sum_{i<j} (-1)^{i+j} \beta(L_{\xi_i} \xi_j, \xi_0, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_k),$$

where $\hat{\xi}_j$ means ξ_j is to be omitted and L_{ξ_i} is the Lie derivative along ξ_i. Since the flow that ξ_i generates is simply the translation $t \mapsto \phi + t\xi_i$, the Lie derivative $L_{\xi_i}\xi_j$ equals 0. We summarize the discussion in the following lemma.

Lemma 4.3. *Let β be a smooth k-form on $\mathcal{H}_\omega$, and let $\xi_0, \dots, \xi_k$ be vector fields on $\mathcal{H}_\omega$ which are constant in the canonical trivialization $T\mathcal{H}_\omega \approx \mathcal{H}_\omega \times C^\infty(X)$. Then*

$$d\beta(\xi_0, \dots, \xi_k) = \sum_{j=0}^{k} (-1)^j L_{\xi_j}(\beta(\xi_0, \dots, \hat{\xi}_j, \dots, \xi_k)) \tag{20}$$

$$= \sum_{j=0}^{k} (-1)^j \frac{d}{dt}\Big|_{t=0} \beta(\xi_0, \dots, \hat{\xi}_j, \dots, \xi_k)(\phi + t\xi_j), \tag{21}$$

where $\hat{\xi}_j$ means ξ_j is to be omitted. (*This formula is true if $\mathcal{H}_\omega \subset C^\infty(X)$ is replaced by an open subset of a Fréchet space.*)

Lemma 4.4. *If β is a d-closed smooth k-form on $\mathcal{H}_\omega$, then there exists a $(k-1)$-form $H\beta$ on $\mathcal{H}_\omega$ such that $d(H\beta) = \beta$.*

Proof. The proof is similar to the finite-dimensional case. Recall that $\mathcal{H}_\omega$ is convex and that $0 \in \mathcal{H}_\omega$. Given $\xi_1, \dots, \xi_{k-1} \in T_\phi\mathcal{H}_\omega$, we define the $(k-1)$-form $H\beta$ by

$$H\beta(\xi_1, \dots, \xi_{k-1}) = \int_0^1 t^{k-1} \beta(\phi, \xi_1, \dots, \xi_{k-1})(t\phi)\, dt.$$

Here $\phi, \xi_1, \dots, \xi_{k-1}$ are regarded as constant vector fields through $T\mathcal{H}_\omega \approx \mathcal{H}_\omega \times C^\infty(X)$.

To find $d(H\beta)(\xi_1, \dots, \xi_k)$, let us compute

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} H\beta(\xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(\phi + t\xi_j) \\
&\quad = \lim_{h\to 0} \frac{H\beta(\xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(\phi + h\xi_j) - H\beta(\xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(\phi)}{h} \\
&\quad = \lim_{h\to 0} \int_0^1 t^{k-1} \frac{\beta(\phi + h\xi_j, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi + th\xi_j) - \beta(\phi, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi)}{h}\, dt \\
&\quad = \lim_{h\to 0} \int_0^1 \Bigg(t^{k-1} \frac{\beta(\phi, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi + th\xi_j) - \beta(\phi, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi)}{h} \\
&\qquad\qquad + t^{k-1} \beta(\xi_j, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi + th\xi_j) \Bigg)\, dt.
\end{aligned}$$

As $(t, h) \mapsto \beta(\phi, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi + th\xi_j)$ and $(t, h) \mapsto \beta(\xi_j, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi + th\xi_j)$ are smooth, we can exchange the limit and integral and obtain

$$\begin{aligned}
&\int_0^1 t^{k-1} \frac{d}{dh}\Big|_{h=0} \beta(\phi, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi + th\xi_j) + t^{k-1} \beta(\xi_j, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi)\, dt \\
&\qquad = \int_0^1 t^k \frac{d}{dh}\Big|_{h=0} \beta(\phi, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi + h\xi_j) + (-1)^{j-1} t^{k-1} \beta(\xi_1, \dots, \xi_k)(t\phi)\, dt.
\end{aligned}$$

As a result, by Lemma 4.3,

$$\begin{aligned}
&d(H\beta)(\xi_1,\dots,\xi_k)\\
&\quad=\sum_{j=1}^{k}(-1)^{j+1}\int_0^1 t^k\frac{d}{dh}\Big|_{h=0}\beta(\phi,\xi_1,\dots,\hat{\xi}_j,\dots,\xi_k)(t\phi+h\xi_j)+(-1)^{j-1}t^{k-1}\beta(\xi_1,\dots,\xi_k)(t\phi)\,dt.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&H(d\beta)(\xi_1,\dots,\xi_k)\\
&\quad=\int_0^1 t^k(d\beta)(\phi,\xi_1,\dots,\xi_k)(t\phi)\,dt\\
&\quad=\int_0^1 t^k\bigg(\sum_{j=1}^{k}(-1)^j\frac{d}{dh}\Big|_{h=0}\beta(\phi,\xi_1,\dots,\hat{\xi}_j,\dots,\xi_k)(t\phi+h\xi_j)+\frac{d}{dh}\Big|_{h=0}\beta(\xi_1,\dots,\xi_k)(t\phi+h\phi)\bigg)\,dt,
\end{aligned}$$

where the last equality is due to Lemma 4.3. Therefore

$$\begin{aligned}
[d(H\beta)+H(d\beta)](\xi_1,\dots,\xi_k)&=\int_0^1 kt^{k-1}\beta(\xi_1,\dots,\xi_k)(t\phi)+t^k\frac{d}{dh}\Big|_{h=0}\beta(\xi_1,\dots,\xi_k)(t\phi+h\phi)\,dt\\
&=\int_0^1\frac{d}{dt}(t^k\beta(\xi_1,\dots,\xi_k)(t\phi))\,dt=\beta(\xi_1,\dots,\xi_k),
\end{aligned}$$

and the lemma follows since $d\beta=0$. □

Lemma 4.5. *The three-form θ is d-closed.*

Proof. This is similar to the derivation of the Aubin–Yau functional and the Mabuchi energy; see e.g., [Błocki 2013, Section 4]. Consider four vector fields ξ_1,ξ_2,ξ_3,ξ_4 on $\mathcal{H}_\omega$ which are constant in the canonical trivialization $T\mathcal{H}_\omega\approx\mathcal{H}_\omega\times C^\infty(X)$. By Lemma 4.3,

$$d\theta(\xi_1,\xi_2,\xi_3,\xi_4)=\xi_1\theta(\xi_2,\xi_3,\xi_4)-\xi_2\theta(\xi_1,\xi_3,\xi_4)+\xi_3\theta(\xi_1,\xi_2,\xi_4)-\xi_4\theta(\xi_1,\xi_2,\xi_3). \tag{22}$$

Using

$$\{\xi_3,\xi_4\}_{\omega_\phi}\omega_\phi^n=nd\xi_3\wedge d\xi_4\wedge\omega_\phi^{n-1}\quad\text{and}\quad\frac{d}{dt}\Big|_{t=0}\omega_{\phi+t\xi_1}^{n-1}=(n-1)i\partial\bar{\partial}\xi_1\wedge\omega_\phi^{n-2},$$

we have

$$\begin{aligned}
\xi_1\theta(\xi_2,\xi_3,\xi_4)=\xi_1\theta(\xi_3,\xi_4,\xi_2)&=\frac{d}{dt}\Big|_{t=0}\theta(\xi_3,\xi_4,\xi_2)(\phi+t\xi_1)\\
&=\frac{d}{dt}\Big|_{t=0}\int_X\{\xi_3,\xi_4\}_{\omega_{\phi+t\xi_1}}\xi_2\omega_{\phi+t\xi_1}^n\\
&=\frac{d}{dt}\Big|_{t=0}\int_X\xi_2nd\xi_3\wedge d\xi_4\wedge\omega_{\phi+t\xi_1}^{n-1}\\
&=\int_X\xi_2nd\xi_3\wedge d\xi_4\wedge(n-1)i\partial\bar{\partial}\xi_1\wedge\omega_\phi^{n-2}\\
&=\int_X\xi_1nd\xi_3\wedge d\xi_4\wedge(n-1)i\partial\bar{\partial}\xi_2\wedge\omega_\phi^{n-2}=\xi_2\theta(\xi_1,\xi_3,\xi_4),
\end{aligned}$$

where the second to last equality is due to integration by parts. Because of the symmetry in index, all terms on the right side of (22) equal 0, and therefore $d\theta = 0$. $\square$

Since θ is d-closed, there exists a two-form α on $\mathcal{H}_\omega$ such that $d\alpha = \theta$ by Lemma 4.4. For a map $\Phi : D \to \mathcal{H}_\omega$, the derivative $\Phi_{z_j} = \frac{1}{2}(\Phi_{\operatorname{Re} z_j} - i\Phi_{\operatorname{Im} z_j})$ is a section of $\mathbb{C} \otimes T\mathcal{H}_\omega$ along Φ, and $\alpha(\Phi_{\bar{z}_j}, \Phi_{z_j})$ is a function on D. We define

$$\begin{aligned}\mathscr{E}(\Phi) &:= E(\Phi) + 4i \sum_j \int_D \alpha(\Phi_{\bar{z}_j}, \Phi_{z_j})\, dV \\ &= \int_D |\Phi_*|^2 dV + 4i \sum_j \int_D \alpha(\Phi_{\bar{z}_j}, \Phi_{z_j})\, dV,\end{aligned}$$

with dV the Euclidean volume form on D.

Lemma 4.6. *The Euler–Lagrange equation of $\mathscr{E}$ is*

$$\sum_{j=1}^m |\nabla \Phi_{z_j}|^2 - 2\Phi_{z_j\bar{z}_j} + i\{\Phi_{\bar{z}_j}, \Phi_{z_j}\}_{\omega_\Phi} = 0, \tag{23}$$

where $\nabla\Phi_{z_j}$ is the gradient of Φ_{z_j} with respect to the metric ω_Φ.

Proof. Let Ψ be a smooth map from D to $C^\infty(X)$ with compact support. The variational equation is

$$0 = \frac{d}{dt}\Big|_{t=0} \biggl(\int_D |(\Phi + t\Psi)_*|^2\, dV + 4i \sum_j \int_D \alpha((\Phi + t\Psi)_{\bar{z}_j}, (\Phi + t\Psi)_{z_j})\, dV \biggr). \tag{24}$$

An extension of the computation in [Donaldson 1999, Section 2] shows that the first term in (24) equals

$$\frac{d}{dt}\Big|_{t=0} \int_D |(\Phi + t\Psi)_*|^2\, dV = \int_D \int_X 4 \biggl(\sum_j |\nabla \Phi_{z_j}|^2 - 2 \sum_j \Phi_{z_j\bar{z}_j} \biggr) \Psi \omega_\Phi^n\, dV. \tag{25}$$

The remaining task is to compute the second term in (24).

To that end, we denote $C^\infty(X, \mathbb{C})$ by $C^\infty_{\mathbb{C}}(X)$ and introduce $A : \mathcal{H}_\omega \times C^\infty_{\mathbb{C}}(X) \times C^\infty_{\mathbb{C}}(X) \to \mathbb{C}$ as follows. If $(u, \xi), (u, \eta) \in \mathcal{H}_\omega \times C^\infty_{\mathbb{C}}(X) \approx \mathbb{C} \otimes T\mathcal{H}_\omega$, then $A(u, \xi, \eta) := \alpha((u, \xi), (u, \eta))$. Therefore, for fixed small $t \in \mathbb{R}$, $\alpha((\Phi + t\Psi)_{\bar{z}_j}, (\Phi + t\Psi)_{z_j}) = A(\Phi + t\Psi, (\Phi + t\Psi)_{\bar{z}_j}, (\Phi + t\Psi)_{z_j})$ maps from D to $\mathbb{C}$. By the chain rule,

$$\begin{aligned}&\frac{d}{dt}\Big|_{t=0} A(\Phi + t\Psi, (\Phi + t\Psi)_{\bar{z}_j}, (\Phi + t\Psi)_{z_j}) \\ &\qquad = d_1 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi) + d_2 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi_{\bar{z}_j}) + d_3 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi_{z_j}),\end{aligned} \tag{26}$$

where $d_1 A$, $d_2 A$, and $d_3 A$ are partial differentials of A. Since A is linear in the second and third variables, $d_2 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi_{\bar{z}_j}) = A(\Phi, \Psi_{\bar{z}_j}, \Phi_{z_j})$ and $d_3 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi_{z_j}) = A(\Phi, \Phi_{\bar{z}_j}, \Psi_{z_j})$. Hence the right side of (26) becomes

$$d_1 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi) + A(\Phi, \Psi_{\bar{z}_j}, \Phi_{z_j}) + A(\Phi, \Phi_{\bar{z}_j}, \Psi_{z_j}). \tag{27}$$

By similar computations,

$$\begin{aligned}\frac{\partial}{\partial \bar{z}_j} A(\Phi, \Psi, \Phi_{z_j}) &= d_1 A(\Phi, \Psi, \Phi_{z_j})(\Phi_{\bar{z}_j}) + A(\Phi, \Psi_{\bar{z}_j}, \Phi_{z_j}) + A(\Phi, \Psi, \Phi_{z_j\bar{z}_j}),\\ \frac{\partial}{\partial z_j} A(\Phi, \Phi_{\bar{z}_j}, \Psi) &= d_1 A(\Phi, \Phi_{\bar{z}_j}, \Psi)(\Phi_{z_j}) + A(\Phi, \Phi_{\bar{z}_j z_j}, \Psi) + A(\Phi, \Phi_{\bar{z}_j}, \Psi_{z_j}).\end{aligned} \tag{28}$$

So integration by parts gives

$$\begin{aligned}\int_D A(\Phi, \Psi_{\bar{z}_j}, \Phi_{z_j})\, dV &= -\int_D (d_1 A(\Phi, \Psi, \Phi_{z_j})(\Phi_{\bar{z}_j}) + A(\Phi, \Psi, \Phi_{z_j\bar{z}_j}))\, dV,\\ \int_D A(\Phi, \Phi_{\bar{z}_j}, \Psi_{z_j})\, dV &= -\int_D (d_1 A(\Phi, \Phi_{\bar{z}_j}, \Psi)(\Phi_{z_j}) + A(\Phi, \Phi_{\bar{z}_j z_j}, \Psi))\, dV.\end{aligned} \tag{29}$$

Combining (27) and (29), we find

$$\begin{aligned}&\frac{d}{dt}\Big|_{t=0} \int_D \alpha((\Phi + t\Psi)_{\bar{z}_j}, (\Phi + t\Psi)_{z_j})\, dV\\ &\qquad = \int_D d_1 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi) - d_1 A(\Phi, \Psi, \Phi_{z_j})(\Phi_{\bar{z}_j}) - d_1 A(\Phi, \Phi_{\bar{z}_j}, \Psi)(\Phi_{z_j})\, dV.\end{aligned} \tag{30}$$

For a fixed point $z_0 \in D$, let $\Psi(z_0)$, $\Phi_{\bar{z}_j}(z_0)$, and $\Phi_{z_j}(z_0)$ define three constant vector fields on $\mathcal{H}_\omega$ denoted by ξ_1, ξ_2, and ξ_3, respectively. By Lemma 4.3,

$$d\alpha(\xi_1, \xi_2, \xi_3) = \xi_1\alpha(\xi_2, \xi_3) - \xi_2\alpha(\xi_1, \xi_3) + \xi_3\alpha(\xi_1, \xi_2).$$

Meanwhile, for constant vector fields ξ_1, ξ_2, and ξ_3, the function $\xi_1\alpha(\xi_2, \xi_3)$ evaluated at $u \in \mathcal{H}_\omega$ is $d_1 A(u, \xi_2, \xi_3)(\xi_1)$. So at $\Phi(z_0) \in \mathcal{H}_\omega$,

$$\begin{aligned}d\alpha(\xi_1, \xi_2, \xi_3) =&\, d_1 A(\Phi(z_0), \xi_2, \xi_3)(\xi_1) - d_1 A(\Phi(z_0), \xi_1, \xi_3)(\xi_2) + d_1 A(\Phi(z_0), \xi_1, \xi_2)(\xi_3)\\ =&\, d_1 A(\Phi(z_0), \xi_2, \xi_3)(\xi_1) - d_1 A(\Phi(z_0), \xi_1, \xi_3)(\xi_2) - d_1 A(\Phi(z_0), \xi_2, \xi_1)(\xi_3).\end{aligned} \tag{31}$$

Hence (30) becomes

$$\int_D d\alpha(\Psi, \Phi_{\bar{z}_j}, \Phi_{z_j})\, dV = \int_D \theta(\Psi, \Phi_{\bar{z}_j}, \Phi_{z_j})\, dV = \int_D \int_X \{\Phi_{\bar{z}_j}, \Phi_{z_j}\}_{\omega_\Phi} \Psi \omega_\Phi^n\, dV. \tag{32}$$

Finally, with (25) and (32), the variational equation (24) becomes

$$0 = \int_D \int_X \biggl(4\biggl(\sum_j |\nabla \Phi_{z_j}|^2 - 2\sum_j \Phi_{z_j\bar{z}_j}\biggr) + 4i \sum_j \{\Phi_{\bar{z}_j}, \Phi_{z_j}\}_{\omega_\Phi}\biggr) \Psi \omega_\Phi^n\, dV, \tag{33}$$

and we obtain the Euler–Lagrange equation

$$\sum_j |\nabla \Phi_{z_j}|^2 - 2\sum_j \Phi_{z_j\bar{z}_j} + i \sum_j \{\Phi_{\bar{z}_j}, \Phi_{z_j}\}_{\omega_\Phi} = 0. \qquad \square$$

5. Lemma 2.2

This section is mainly devoted to the proof of Lemma 2.2, and we will follow closely the ideas in [Błocki and Kołodziej 2007]. The first two lemmas, concerning smooth approximation of continuous η-subharmonic functions, are based on the exposition in [Demailly 2012, Chapter I, Section 5E] of [Richberg 1968]. See also [Demailly 1992].

Let $\theta \in C^\infty(\mathbb{R}, \mathbb{R})$ be a nonnegative function having support in $[-1, 1]$ with $\int_{\mathbb{R}} \theta(h)\, dh = 1$ and $\int_{\mathbb{R}} h\theta(h)\, dh = 0$. For arbitrary $\xi = (\xi_1, \dots, \xi_p) \in (0, \infty)^p$, the regularized maximal function is

$$M_\xi(t_1, \dots, t_p) := \int_{\mathbb{R}^n} \max\{t_1 + h_1, \dots, t_p + h_p\} \prod_{j=1}^{n} \theta\Big(\frac{h_j}{\xi_j}\Big) \frac{dh_1}{\xi_1} \cdots \frac{dh_p}{\xi_p}.$$

Lemma 5.1. *Fix a closed smooth positive* $(1, 1)$*-form* η *on* X. *Let* $\Omega_\alpha \Subset D \times X$ *be a locally finite open cover of* $D \times X$, *let* c *be a real number, and let* $u_\alpha \in C^\infty(\overline{\Omega}_\alpha)$ *such that* $u_\alpha(z, x) + c|z|^2$ *is* η*-subharmonic on graphs. Assume that there exists a family* $\{\xi_\alpha\}$ *of positive numbers such that, for all* β *and* $(z, x) \in \partial\Omega_\beta$,

$$u_\beta(z, x) + \xi_\beta \le \max_{\alpha:(z,x)\in\Omega_\alpha} \{u_\alpha(z, x) - \xi_\alpha\}.$$

Define a function $\tilde{u}$ *on* $D \times X$ *as follows. Given* $(z, x) \in D \times X$, *let* $A = \{\alpha : (z, x) \in \Omega_\alpha\}$, $\xi_A = (\xi_\alpha)_{\alpha\in A}$, $u_A(z, x) = \{u_\alpha(z, x) : \alpha \in A\}$, *and*

$$\tilde{u}(z, x) := M_{\xi_A}(u_A(z, x)).$$

Then $\tilde{u}$ *is in* $C^\infty(D \times X)$ *and* $\tilde{u}(z, x) + c|z|^2$ *is* η*-subharmonic on graphs.*

Proof. As in the proof of [Demailly 2012, Chapter I, Lemma 5.17 and Corollary 5.19], one can deduce that for a fixed point in $D \times X$, there exist a neighborhood V and a finite set I of indices α such that $V \subset \bigcap_{\alpha\in I} \Omega_\alpha$ and on which $\tilde{u} = M_{\xi_I}(u_I)$. As a result, by [Demailly 2012, Lemma 5.18 (a)], $\tilde{u}$ is smooth on $D \times X$. Now for a holomorphic map f from an open subset of D to X, we have

$$\begin{aligned}\tilde{u}(z, f(z)) + c|z|^2 + \psi(f(z)) &= c|z|^2 + \psi(f(z)) + M_{\xi_I}(u_I(z, f(z)))\\ &= M_{\xi_I}(c|z|^2 + \psi(f(z)) + u_I(z, f(z))),\end{aligned}$$

where $\eta = i\partial\bar{\partial}\psi$ and we use [Demailly 2012, Lemma 5.18 (d)] in the last equality. Furthermore, since $c|z|^2 + \psi(f(z)) + u_\alpha(z, f(z))$ is subharmonic by assumption, so is $M_{\xi_I}(c|z|^2 + \psi(f(z)) + u_I(z, f(z)))$ by [Demailly 2012, Lemma 5.18 (a)], and therefore $\tilde{u} + c|z|^2$ is η-subharmonic on graphs. □

We introduce here some notation that will be used later. Let ρ_1 and ρ_2 be kernels (i.e., nonnegative radial smooth functions with support in the unit ball and having integral one) in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. For $\varepsilon > 0$, write $\rho_{1,\varepsilon}(\cdot) := \varepsilon^{-m}\rho_1(\cdot/\varepsilon)$, and let $\rho_{2,\varepsilon}$ be similarly defined.

The proof of the following lemma is very similar to that of [Demailly 2012, Chapter 1, Theorem 5.21].

Lemma 5.2. *Let* $u \in C(D \times X)$ *be* η*-subharmonic on graphs. For any number* $\lambda > 0$, *there exists* $\tilde{u} \in C^\infty(D \times X)$ *such that* $u \le \tilde{u} \le u + M\lambda$, *where* M *depends only on the diameter of* D *and* $\tilde{u}$ *is* $(1 + \lambda)\eta$*-subharmonic on graphs.*

Proof. Let $\{\Omega_\alpha\}$ be a locally finite open cover of $D \times X$ by relatively compact open balls contained in coordinate patches of $D \times X$. Choose concentric balls $\Omega''_\alpha \subset \Omega'_\alpha \subset \Omega_\alpha$ of radii $r''_\alpha < r'_\alpha < r_\alpha$ and center $(c_\alpha, 0)$ in the given coordinates (z, x) near $\overline{\Omega}_\alpha$, such that the Ω''_α still cover $D \times X$ and η has a local potential ψ_α in a neighborhood of $\overline{\Omega}_\alpha$. For small $\varepsilon_\alpha > 0$ and $\delta_\alpha > 0$, we set

$$u_\alpha(z, x) = ((u + \psi_\alpha) * \rho_{\varepsilon_\alpha})(z, x) - \psi_\alpha(x) + \delta_\alpha(r'^2_\alpha - |z - c_\alpha|^2 - |x|^2) \quad \text{on } \overline{\Omega}_\alpha,$$

where $*\rho_{\varepsilon_\alpha}$ is the convolution with $\rho_{\varepsilon_\alpha} := \rho_{1,\varepsilon_\alpha}\rho_{2,\varepsilon_\alpha}$. Since $\psi_\alpha(x) + u(z, x)$ is subharmonic in z and psh in x by Lemma 3.1, the functions $(\psi_\alpha + u) * \rho_{\varepsilon_\alpha}$ decrease to $\psi_\alpha + u$ as ε_α goes to 0, locally uniformly because u is continuous. For ε_α and δ_α small enough, we have $u_\alpha \le u + \frac{1}{2}\lambda$ on $\overline{\Omega}_\alpha$. Moreover, for any holomorphic map f from an open subset of D to X,

$$\begin{aligned}\Delta(u_\alpha(z, f(z)) + \psi_\alpha(f(z))) &= \Delta((u + \psi_\alpha) * \rho_{\varepsilon_\alpha})(z, f(z)) - \delta_\alpha \Delta(|z - c_\alpha|^2 + |f(z)|^2)\\ &\ge -\delta_\alpha \Delta(|z - c_\alpha|^2 + |f(z)|^2)\\ &\ge -\lambda\Delta|z|^2 - \lambda\Delta\psi_\alpha(f(z)),\end{aligned}$$

where the first inequality is due to the fact that $(u + \psi_\alpha) * \rho_{\varepsilon_\alpha}$ is subharmonic on holomorphic graphs, which can be verified easily because $(u + \psi_\alpha)$ is subharmonic on holomorphic graphs (or see the proof of Lemma 2.2 where we provide such verification). So $u_\alpha(z, x) + \lambda|z|^2$ is $(1 + \lambda)\eta$-subharmonic on graphs. Set

$$\xi_\alpha = \delta_\alpha \min\bigl\{r'^2_\alpha - r''^2_\alpha, \tfrac{1}{2}(r^2_\alpha - r'^2_\alpha)\bigr\}.$$

Choose first δ_α such that $\xi_\alpha < \frac{1}{2}\lambda$, and then ε_α so small that $u \le (u + \psi_\alpha) * \rho_{\varepsilon_\alpha}(z, x) - \psi_\alpha(x) < u + \xi_\alpha$ on $\overline{\Omega}_\alpha$. As $\delta_\alpha(r'^2_\alpha - |z - c_\alpha|^2 - |x|^2)$ is less than or equal to $-2\xi_\alpha$ on $\partial\Omega_\alpha$ and greater than ξ_α on Ω''_α, we have $u_\alpha < u - \xi_\alpha$ on $\partial\Omega_\alpha$ and $u_\alpha > u + \xi_\alpha$ on Ω''_α, so that the assumption in Lemma 5.1 is satisfied. Also, the function

$$U(z, x) := M_{\xi_A}(u_A(z, x)), \quad \text{for } A = \{\alpha : \Omega_\alpha \ni (z, x)\},$$

is in $C^\infty(D \times X)$ and $U(z, x) + \lambda|z|^2$ is $(1 + \lambda)\eta$-subharmonic on graphs. Then we have $u \le U \le u + \lambda$ by [Demailly 2012, Lemma 5.18 (b)], and the function defined by $\tilde{u} := U + \lambda|z|^2$ is what we need. ☐

The following lemma is proved in the same way as Lemmas 4 and 5 in [Błocki and Kołodziej 2007]. The only issue is keeping track of uniformity.

Lemma 5.3. *Let U, V be two open sets in $\mathbb{C}^n$ and F a biholomorphic map from U to V. Let u be usc, bounded, and subharmonic on holomorphic graphs in $D \times U$. Define the convolution*

$$u_{\delta_1,\delta_2}(z, x) = \int_{\mathbb{C}^n} \int_{\mathbb{C}^m} u(z - a, x - b)\rho_{1,\delta_1}(a)\rho_{2,\delta_2}(b)\, da\, db,$$

where ρ_{1,δ_1} and ρ_{2,δ_2} are kernels in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. On the other hand, define

$$u^F_{\delta_1,\delta_2}(z, x) = (u \circ (\mathrm{Id} \times F^{-1}))_{\delta_1,\delta_2} \circ (\mathrm{Id} \times F). \tag{34}$$

Then $(u^F_{\delta_1,\delta_2} - u_{\delta_1,\delta_2})(z, x) \to 0$ locally uniformly in z, x, and δ_1 as $\delta_2 \to 0$.

Proof. Define

$$\hat{u}_{\delta_2}(z,x) = \max_{\{z\}\times\overline{B(x,\delta_2)}} u,$$
$$\tilde{u}_{\delta_2}(z,x) = \fint_{\partial B(x,\delta_2)} u(z,b)\,db,$$
$$u_{\delta_2}(z,x) = \int_{\mathbb{C}^n} u(z,x-b)\rho_{2,\delta_2}(b)\,db,$$

where $\fint$ means the average. Their counterparts under $\mathrm{Id}\times F^{-1}$ and $\mathrm{Id}\times F$ as in (34) are denoted by $\hat{u}^F_{\delta_2}(z,x)$, $\tilde{u}^F_{\delta_2}(z,x)$, and $u^F_{\delta_2}(z,x)$, respectively.

By Lemma 3.1, $u(z,\cdot)$ is psh in U, so $\hat{u}_{\delta_2}(z,x)$ is a convex function of $\log\delta_2$. Fixing $a\geq 1$ and $r>0$, choose δ_2 so small that $0\leq(\log a)/(\log(r/\delta_2))\leq 1$. Then by convexity,

$$0\leq \hat{u}_{a\delta_2}(z,x)-\hat{u}_{\delta_2}(z,x)\leq \frac{\log a}{\log(r/\delta_2)}(\hat{u}_r(z,x)-\hat{u}_{\delta_2}(z,x)).$$

Since u is assumed to be bounded, it follows that for any $a>0$ (for the case $1>a>0$, use $1/a$ instead), $\hat{u}_{a\delta_2}(z,x)-\hat{u}_{\delta_2}(z,x)$ goes to 0 as $\delta_2\to 0$, locally uniformly in z and x. Then following the same argument as in [Błocki and Kołodziej 2007, Lemma 4], we see $\hat{u}^F_{\delta_2}-\hat{u}_{\delta_2}$ goes to 0 locally uniformly in z and x, as $\delta_2\to 0$.

Since $u(z,\cdot)$ is psh in U, $\tilde{u}_{\delta_2}(z,x)$ is convex in $\log\delta_2$. By the argument of [Błocki and Kołodziej 2007, Lemma 5] and the fact that u is bounded, we see both $\hat{u}_{\delta_2}-\tilde{u}_{\delta_2}$ and $\tilde{u}_{\delta_2}-u_{\delta_2}$ go to 0 locally uniformly in z, x, as $\delta_2\to 0$, and as a result, so does $u^F_{\delta_2}-u_{\delta_2}$. Since $(u^F_{\delta_1,\delta_2}-u_{\delta_1,\delta_2})$ is the convolution of $(u^F_{\delta_2}-u_{\delta_2})$ in z, we see at once the conclusion of the lemma. □

Proof of Lemma 2.2. Fix a finite number of charts $U_\alpha\Supset V_\alpha$ such that V_α covers X, and η has a local potential ψ_α in a neighborhood of $\overline{U}_\alpha$. For each α, let $f_\alpha: U_\alpha\to\mathbb{C}^n$ be the coordinate map, we consider the convolution $((\psi_\alpha+u)\circ f_\alpha^{-1})_{\delta_1,\delta_2}\circ f_\alpha$, which we simply denote by $(\psi_\alpha+u)_{\delta_1,\delta_2}$ on $D\times U_\alpha$. Because u added by a constant still satisfies the same assumption in Lemma 2.2, we will assume u is so negative that $(\psi_\alpha+u)_{\delta_1,\delta_2}-\psi_\alpha<-a$ for some $a>0$ and all α. At the same time, we consider the convolution of $(\psi_\alpha+u)$ under f_β, namely $((\psi_\alpha+u)\circ f_\beta^{-1})_{\delta_1,\delta_2}\circ f_\beta$, which can be written as

$$((\psi_\alpha+u)\circ f_\alpha^{-1}\circ F^{-1})_{\delta_1,\delta_2}\circ F\circ f_\alpha, \tag{35}$$

if $F^{-1}=f_\alpha\circ f_\beta^{-1}$. We denote (35) by $(\psi_\alpha+u)^F_{\delta_1,\delta_2}$ (the notation is consistent with Lemma 5.3 except we do not write out the identity map of D here). By Lemma 5.3 on $D\times(U_\alpha\cap U_\beta)$

$$(\psi_\alpha+u)_{\delta_1,\delta_2}-(\psi_\beta+u)_{\delta_1,\delta_2}=(\psi_\alpha+u)_{\delta_1,\delta_2}-(\psi_\alpha+u)^F_{\delta_1,\delta_2}+(\psi_\alpha+u-(\psi_\beta+u))^F_{\delta_1,\delta_2}\to\psi_\alpha-\psi_\beta \tag{36}$$

locally uniformly in z and x, as $\delta_2,\delta_1\to 0$.

Let χ_α be a smooth function in U_α that is 0 in V_α and -1 near ∂U_α. We have $i\partial\bar\partial\chi_\alpha\geq -C\eta$ for some constant C. For $0<\varepsilon<1$, according to (36) we can find δ_1,δ_2 small enough such that for any β and for any $(z,x)\in\overline{D}'\times\partial U_\beta$,

$$\Big((\psi_\beta+u)_{\delta_1,\delta_2}-\psi_\beta+\frac{\varepsilon}{C}\chi_\beta\Big)(z,x)<\max_{(z,x)\in\overline{D}'\times U_\alpha}\Big((\psi_\alpha+u)_{\delta_1,\delta_2}-\psi_\alpha+\frac{\varepsilon}{C}\chi_\alpha\Big)(z,x),$$

where the maximum is taken over all $\bar{D}' \times U_\alpha$ that contain (z, x). Let $\delta = \min\{\delta_1, \delta_2\}$. Then by [Demailly 2012, Chapter I, Lemma 5.17], the function

$$u_\delta^\varepsilon(z, x) := \max_{(z,x)\in\bar{D}'\times U_\alpha} \Big((\psi_\alpha + u)_{\delta,\delta} - \psi_\alpha + \frac{\varepsilon}{C}\chi_\alpha\Big)(z, x)$$

is continuous on $\bar{D}' \times X$. Notice that $u_\delta^\varepsilon(z, x) < -a$ for any $0 < \varepsilon < 1$. Since $\psi_\alpha(x) + u(z, x)$ is subharmonic in z and psh in x by Lemma 3.1, the function $(\psi_\alpha + u)_{\delta,\delta}$ is decreasing to $\psi_\alpha + u$ as $\delta \to 0$, and hence u_δ^ε is decreasing to u as $\delta \to 0$.

We already know that $\psi_\alpha + u$ is subharmonic on holomorphic graphs, and in this paragraph we will show this is also true for $(\psi_\alpha + u)_{\delta,\delta}$. Let us denote $\psi_\alpha + u$ by G momentarily: We want to show that, for any holomorphic map g from an open subset of D to U_α, the function $G_{\delta,\delta}(z, g(z))$ is subharmonic. Indeed, since G is bounded on $D \times U_\alpha$, the convolution $G_{\delta,\delta}$ is smooth and so $G_{\delta,\delta}(z, g(z))$ is usc. The map $w \mapsto G(w, g(w + a) - b)$ is subharmonic, therefore the mean-value inequality says

$$G(z - a, g(z) - b) \le \fint_{B(z-a,r)} G(w, g(w + a) - b)\, dw.$$

So,

$$\begin{aligned} G_{\delta,\delta}(z, g(z)) &\le \int_{\mathbb{C}^n}\int_{\mathbb{C}^m}\fint_{B(z-a,r)} G(w, g(w + a) - b)\, dw\rho_{1,\delta}(a)\rho_{2,\delta}(b)\, da\, db \\ &= \int_{\mathbb{C}^n}\int_{\mathbb{C}^m}\fint_{B(z,r)} G(W - a, g(W) - b)\, dW\rho_{1,\delta}(a)\rho_{2,\delta}(b)\, da\, db \\ &= \fint_{B(z,r)} G_{\delta,\delta}(W, g(W))\, dW; \end{aligned}$$

the use of Fubini's theorem is justified since G is bounded on $D \times U_\alpha$. As a result, $G_{\delta,\delta}(z, g(z))$ is subharmonic.

The fact that $(\psi_\alpha + u)_{\delta,\delta}$ is subharmonic on holomorphic graphs together with $(\chi_\alpha)_{\lambda\bar{\mu}} \ge -C(\psi_\alpha)_{\lambda\bar{\mu}}$ as matrices, shows, for any holomorphic function f from an open subset of D' to X,

$$\Delta\Big((\psi_\alpha + u)_{\delta,\delta} - \psi_\alpha + \frac{\varepsilon}{C}\chi_\alpha\Big)(z, f(z)) \ge (-1 - \varepsilon)\Delta\psi_\alpha(f(z)),$$

so u_δ^ε is $(1 + \varepsilon)\eta$-subharmonic on graphs.

So far we have shown that given $1 < p \in \mathbb{N}$, there exists $q_0 \in \mathbb{N}$ such that, for $q > q_0$, the functions $u_{1/q}^{1/p}$ are in $C(\bar{D}' \times X)$, $(1 + 1/p)\eta$-subharmonic on graphs, and decrease to u as $q \to \infty$. For simplicity, we will denote $u_{1/q}^{1/p}$ by u_q^p. Let M be the constant in Lemma 5.2. We will construct $u_{j_k}^k$ inductively with $j_k > k^2$ and $\tilde{u}_k \in C^\infty(D' \times X)$ such that

$$u_{j_k}^k + \frac{1}{j_k} \le \tilde{u}_k \le u_{j_k}^k + \frac{1}{j_k} + \frac{M}{j_k}. \tag{37}$$

Moreover, $\tilde{u}_k$ is $(1 + 1/k)(1 + 1/j_k)\eta$-subharmonic on graphs, and $u_{j_k}^k + 1/j_k + M/j_k$ is less than both $u_{j_{k-1}}^{k-1} + 1/j_{k-1}$ and $u_{j_{k-1}}^2 + 1/j_{k-1}$.

Suppose that this is true at the $(k-1)$-th step. As $u_{j_{k-1}}^{k-1} + 1/j_{k-1}$ and $u_{j_{k-1}}^2 + 1/j_{k-1}$ are both greater than u, we can find $j_k > \max\{j_{k-1}, k^2\}$ such that $u_{j_k}^k + 1/j_k + M/j_k$ is less than both $u_{j_{k-1}}^{k-1} + 1/j_{k-1}$

and $u^2_{j_{k-1}} + 1/j_{k-1}$ by continuity on the compact set $\overline{D}' \times X$. We can then find a function $\tilde{u}_k \in C^\infty(D' \times X)$ with $u^k_{j_k} + 1/j_k \le \tilde{u}_k \le u^k_{j_k} + 1/j_k + M/j_k$ and where $\tilde{u}_k$ is $(1+1/k)(1+1/j_k)\eta$-subharmonic on graphs by applying Lemma 5.2 with $\lambda = 1/j_k$. So the induction process is true at the k-th step. (One can begin the induction process with $u^2_{j_2} + 1/j_2$ with j_2 large enough such that $u^2_{j_2} + 1/j_2 < 0$.)

One can see that $\tilde{u}_k$ is decreasing to u. Since $\tilde{u}_k < 0$, we have that $(1-1/k)\tilde{u}_k$ is still decreasing to u. The function $(1-1/k)\tilde{u}_k$ is $(1-1/k^2)(1+1/j_k)\eta$-subharmonic on graphs, and, because $j_k > k^2$, is also $(1-1/k^2 j_k)\eta$-subharmonic on graphs. So the $(1-1/k)\tilde{u}_k$ are the desired approximants. □

6. A remark

In this final section, we compare results in this paper to those in [Darvas and Wu 2019], where the author and Darvas consider two other families closely related to G_v and G^k_v. For $\pi : D \times X \to X$, define

$$F_v := \Big\{ u : u \in \mathrm{PSH}(D \times X, \pi^*\omega),\ \limsup_{D \ni z \to \zeta \in \partial D} u(z,x) \le v(\zeta, x) \Big\},$$

$$F^k_v := \Big\{ D \ni z \to U_z \in \mathcal{N}^*_k \text{ is Griffiths negative, } \limsup_{D \ni z \to \zeta \in \partial D} U^2_z(s) \le H^*_k(v_\zeta)(s,s) \text{ for any } s \in H^0(X, L^k)^* \Big\},$$

where a norm function U_z is called Griffiths negative if $\log U_z(f(z))$ is psh for any holomorphic section $f : W \subset D \to H^0(X, L^k)^*$. Denote the upper envelopes of F_v and F^k_v by U and U^k, respectively. Then one result in [Darvas and Wu 2019] is that $FS_k((U^k_z)^*)$ converges to U uniformly.

The transition from the aforementioned paper to this paper is the change of plurisubharmonicity to subharmonicity, as one can see when comparing the definitions of F^k_v and G^k_v. Such a change between F_v and G_v is a little more subtle, and it can be seen as follows. Let ψ be a local potential of ω. Then a function $u \in \mathrm{PSH}(D \times X, \pi^*\omega)$ is equivalent to $\psi(x) + u(z,x)$ being psh in z and x jointly, which is also equivalent to $\psi(f(z)) + u(z, f(z))$ being psh for any holomorphic function $f : U \subset D \to X$ (see Lemma 6.1 below); therefore we see the change from F_v to G_v is again plurisubharmonicity to subharmonicity. Also notice that when $\dim D = 1$, Theorem 1.2 and the result in [Darvas and Wu 2019] are the same because $F_v = G_v$ and $F^k_v = G^k_v$.

Lemma 6.1. *Let Ω_1 and Ω_2 be open sets in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. If $u(z,\xi)$ is an usc function on $\Omega_1 \times \Omega_2$ such that $u(z, s(z))$ is psh for any holomorphic map s from an open subset of Ω_1 to Ω_2, then u is psh on $\Omega_1 \times \Omega_2$.*

Proof. We want to show that u is subharmonic on any complex line in $\Omega_1 \times \Omega_2$, and it suffices to consider the line $\mathbb{C} \ni \lambda \mapsto (\lambda z_0, \lambda \xi_0)$ where $(z_0, \xi_0) \in \Omega_1 \times \Omega_2$. In the case when z_0 and ξ_0 are both nonzero, we may assume $z_0 = (1, 0, \dots, 0)$ and $\xi_0 = (1, 0, \dots, 0)$. Let $G : \Omega_1 \to \mathbb{C}$ be the projection on the first coordinate, and let $F : \mathbb{C} \to \Omega_2$ be the injection to the first coordinate. By assumption, $u(z, F \circ G(z))$ is psh, so the function $\lambda \mapsto u(\lambda z_0, F \circ G(\lambda z_0)) = u(\lambda z_0, \lambda \xi_0)$ is subharmonic.

If $\xi_0 = 0$, then the function $\lambda \mapsto u(\lambda z_0, 0)$ is of course subharmonic. The final case is $z_0 = 0$ and $\xi_0 = (1, 0, \dots, 0)$, and we need to show the function

$$\lambda \mapsto u(0, \dots, 0; \lambda, 0, \dots, 0)$$

is subharmonic, where the semicolon “;” in the argument is to separate the variables of $\mathbb{C}^m$ and $\mathbb{C}^n$. Given $\varepsilon > 0$ and $a \in \mathbb{C}$, the function $z \mapsto u(z_1, \dots, z_m; z_1/\varepsilon + a, 0, \dots, 0)$ is psh, so its restriction to the complex line $\lambda \mapsto ((\lambda - a)\varepsilon, 0, \dots, 0)$ is subharmonic; namely, $\lambda \mapsto u((\lambda - a)\varepsilon, 0, \dots, 0; \lambda, 0, \dots, 0)$ is subharmonic. Hence,

$$u(0, \dots, 0; a, 0, \dots, 0) \le \fint_{\partial B(a,r)} u((\lambda - a)\varepsilon, 0, \dots, 0; \lambda, 0, \dots, 0)\, d\lambda,$$

for $r > 0$. By Fatou's lemma and the fact that u is usc,

$$\limsup_{\varepsilon \to 0} \int_{\partial B(a,r)} u((\lambda - a)\varepsilon, 0, \dots, 0; \lambda, 0, \dots, 0)\, d\lambda \le \int_{\partial B(a,r)} u(0, 0, \dots, 0; \lambda, 0, \dots, 0)\, d\lambda.$$

As a result,

$$u(0, \dots, 0; a, 0, \dots, 0) \le \fint_{\partial B(a,r)} u(0, \dots, 0; \lambda, 0, \dots, 0)\, d\lambda. \qquad \square$$

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THE STRONG TOPOLOGY OF ω-PLURISUBHARMONIC FUNCTIONS

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On a compact Kähler manifold (X,ω), given a model-type envelope $\psi \in \mathrm{PSH}(X,\omega)$ (i.e., a singularity type) we prove that the Monge–Ampère operator is a homeomorphism between the set of ψ-relative finite energy potentials and the set of ψ-relative finite energy measures endowed with their strong topologies given as the coarsest refinements of the weak topologies such that the relative energies become continuous. Moreover, given a totally ordered family $\mathcal{A}$ of model-type envelopes with positive total mass representing different singularity types, the sets $X_{\mathcal{A}}$ and $Y_{\mathcal{A}}$, given as the union of all ψ-relative finite energy potentials and of all ψ-relative finite energy measures with varying $\psi \in \bar{\mathcal{A}}$, respectively, have two natural strong topologies which extend the strong topologies on each component of the unions. We show that the Monge–Ampère operator produces a homeomorphism between $X_{\mathcal{A}}$ and $Y_{\mathcal{A}}$.

As an application we also prove the strong stability of a sequence of solutions of complex Monge–Ampère equations when the measures have uniformly L^p-bounded densities for $p>1$ and the prescribed singularities are totally ordered.

1. Introduction

Let (X,ω) be a compact Kähler manifold where ω is a fixed Kähler form, and let $\mathcal{H}_\omega$ denote the set of all Kähler potentials, i.e., all $\varphi \in C^\infty$ such that $\omega + dd^c\varphi$ is a Kähler form. The pioneering work of Yau [1978] shows that the Monge–Ampère operator

$$\begin{aligned} \mathrm{MA}_\omega : \mathcal{H}_{\omega,\mathrm{norm}} &\to \left\{ dV \text{ volume form} : \int_X dV = \int_X \omega^n \right\}, \\ \mathrm{MA}_\omega(\varphi) &:= (\omega + dd^c\varphi)^n, \end{aligned} \tag{1}$$

is a bijection, where for any subset $A \subset \mathrm{PSH}(X,\omega)$ of all ω-plurisubharmonic functions, we use the notation $A_{\mathrm{norm}} := \{u \in A : \sup_X u = 0\}$. Note that the assumption on the total mass of the volume forms in (1) is necessary since $\mathcal{H}_{\omega,\mathrm{norm}}$ represents all Kähler forms in the cohomology class $\{\omega\}$ and the quantity $\int_X \omega^n$ is cohomological.

In [Guedj and Zeriahi 2007] the authors extended the Monge–Ampère operator using the *nonpluripolar product* (as defined successively in [Boucksom et al. 2010]) and the bijection (1) to

$$\mathrm{MA}_\omega : \mathcal{E}_{\mathrm{norm}}(X,\omega) \to \left\{ \mu \text{ nonpluripolar positive measure} : \mu(X) = \int_X \omega^n \right\}, \tag{2}$$

where $\mathcal{E}(X,\omega) := \left\{ u \in \mathrm{PSH}(X,\omega) : \int_X \mathrm{MA}_\omega(u) = \int_X \mathrm{MA}_\omega(0) \right\}$ is the set of all ω-psh functions with full Monge–Ampère mass.

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The set $\mathrm{PSH}(X,\omega)$ is naturally endowed with the L^1-topology which we will call *weak*, but the Monge–Ampère operator in (2) is not continuous even if the set of measures is endowed with the weak topology. Thus in [Berman et al. 2019], setting $V_0 := \int_X \mathrm{MA}_\omega(0)$, strong topologies were introduced for

$$\mathcal{E}^1(X,\omega) := \{u \in \mathcal{E}(X,\omega) : E(u) > -\infty\}$$

and

$$\mathcal{M}^1(X,\omega) := \{V_0\mu : \mu \text{ is a probability measure satisfying } E^*(\mu) < +\infty\},$$

as the coarsest refinements of the weak topologies such that the Monge–Ampère energy $E(u)$ [Aubin 1984; Berman and Boucksom 2010; Boucksom et al. 2010] and the energy for probability measures E^* [Berman et al. 2013; 2019], respectively, become continuous. The map

$$\mathrm{MA}_\omega : (\mathcal{E}^1_{\mathrm{norm}}(X,\omega), \text{strong}) \to (\mathcal{M}^1(X,\omega), \text{strong}) \tag{3}$$

is then a homeomorphism. Later Darvas [2015] showed that $(\mathcal{E}^1(X,\omega), \text{strong})$ actually coincides with the metric closure of $\mathcal{H}_\omega$ endowed with the Finsler metric $|f|_{1,\varphi} := \int_X |f| \,\mathrm{MA}_\omega(\varphi)$ with $\varphi \in \mathcal{H}_\omega$, $f \in T_\varphi \mathcal{H}_\omega \simeq C^\infty(X)$ and associated distance

$$d(u,v) := E(u) + E(v) - 2E(P_\omega(u,v)),$$

where $P_\omega(u,v)$ is the rooftop envelope given basically as the largest ω-psh function bounded above by $\min(u,v)$ [Ross and Witt Nyström 2014]. This metric topology has played an important role in the last decade to characterize the existence of special metrics [Berman et al. 2020; Chen and Cheng 2021a; 2021b; Darvas and Rubinstein 2017].

It is also important and natural to solve complex Monge–Ampère equations requiring that the solutions have some prescribed behavior, for instance along a divisor.

We first recall that on $\mathrm{PSH}(X,\omega)$ there is a natural partial order $\preccurlyeq$ given as $u \preccurlyeq v$ if $u \le v + O(1)$, and the total mass through the Monge–Ampère operator respects such partial order, i.e., $V_u := \int_X \mathrm{MA}_\omega(u) \le V_v$ if $u \preccurlyeq v$ [Boucksom et al. 2010; Witt Nyström 2019]. Thus in [Darvas et al. 2018], the authors introduced the *ψ-relative* analogs of the sets $\mathcal{E}(X,\omega)$ and $\mathcal{E}^1(X,\omega)$, for $\psi \in \mathrm{PSH}(X,\omega)$ fixed, as

$$\mathcal{E}(X,\omega,\psi) := \{u \in \mathrm{PSH}(X,\omega) : u \preccurlyeq \psi \text{ and } V_u = V_v\},$$
$$\mathcal{E}^1(X,\omega,\psi) := \{u \in \mathcal{E}(X,\omega,\psi) : E_\psi(u) > -\infty\},$$

where E_ψ is the ψ-relative energy. They then proved that

$$\mathrm{MA}_\omega : \mathcal{E}_{\mathrm{norm}}(X,\omega,\psi) \to \{\mu \text{ nonpluripolar positive measure} : \mu(X) = V_\psi\} \tag{4}$$

is a bijection if and only if ψ, up to a bounded function, is a *model-type envelope*, or in other words, $\psi = (\lim_{C\to+\infty} P(\psi + C, 0))^*$ satisfies $V_\psi > 0$ (the star is for the upper semicontinuous regularization). There are plenty of these functions, for instance, to any ω-psh function ψ with analytic singularities is associated a unique model-type envelope. We denote by $\mathcal{M}$ the set of all model-type envelopes and by $\mathcal{M}^+$ those elements ψ such that $V_\psi > 0$.

Letting $\psi \in \mathcal{M}^+$, in [Trusiani 2022], we proved that $\mathcal{E}^1(X, \omega, \psi)$ can be endowed with a natural metric topology given by the complete distance $d(u, v) := E_\psi(u) + E_\psi(v) - 2E_\psi(P_\omega(u, v))$.

Analogously to E^*, we introduce in Section 5 a natural ψ-relative energy for probability measures E^*_ψ; thus the set

$$\mathcal{M}^1(X, \omega, \psi) := \{V_\psi \mu : \mu \text{ is a probability measure satisfying } E^*_\psi(\mu) < +\infty\}$$

can be endowed with its strong topology given as the coarsest refinement of the weak topology such that E^*_ψ becomes continuous.

Theorem A. *Let $\psi \in \mathcal{M}^+$. Then*

$$\mathrm{MA}_\omega : (\mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi), d) \to (\mathcal{M}^1(X, \omega, \psi), \text{strong}) \tag{5}$$

is a homeomorphism.

It is natural to wonder if one can extend the bijections (2) and (4) to bigger subsets of $\mathrm{PSH}(X, \omega)$.

Given $\psi_1, \psi_2 \in \mathcal{M}^+$ such that $\psi_1 \neq \psi_2$, the sets $\mathcal{E}(X, \omega, \psi_1)$ and $\mathcal{E}(X, \omega, \psi_2)$ are disjoint ([Darvas et al. 2018, Theorem 1.3] quoted below as Theorem 2.1), but it may happen that $V_{\psi_1} = V_{\psi_2}$. So in these situations, at least one of $\mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi_1)$ or $\mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi_2)$ must be ruled out to extend (4). However, given a totally ordered family $\mathcal{A} \subset \mathcal{M}^+$ of model-type envelopes, the map $\mathcal{A} \ni \psi \to V_\psi$ is injective (again by [Darvas et al. 2018, Theorem 1.3]), i.e.,

$$\mathrm{MA}_\omega : \bigsqcup_{\psi \in \mathcal{A}} \mathcal{E}_{\mathrm{norm}}(X, \omega, \psi) \to \{\mu \text{ nonpluripolar positive measure} : \mu(X) = V_\psi \text{ for } \psi \in \mathcal{A}\}$$

is a bijection.

In [Trusiani 2022] we introduced a complete distance $d_\mathcal{A}$ on

$$X_\mathcal{A} := \bigsqcup_{\psi \in \bar{\mathcal{A}}} \mathcal{E}^1(X, \omega, \psi),$$

where $\bar{\mathcal{A}} \subset \mathcal{M}$ is the weak closure of $\mathcal{A}$ and where we identify $\mathcal{E}^1(X, \omega, \psi_{\min})$ with a point $P_{\psi_{\min}}$ if $\psi_{\min} \in \mathcal{M} \setminus \mathcal{M}^+$ (since in this case $E_\psi \equiv 0$, see Remark 2.7). Here $\psi_{\min}$ is given as the smallest element in $\bar{\mathcal{A}}$, observing that the Monge–Ampère operator $\mathrm{MA}_\omega : \bar{\mathcal{A}} \to \mathrm{MA}_\omega(\bar{\mathcal{A}})$ is a homeomorphism when the range is endowed with the weak topology (Lemma 3.12). We call the strong topology on $X_\mathcal{A}$ the metric topology given by $d_\mathcal{A}$ since $d_{\mathcal{A}|\mathcal{E}^1(X,\omega,\psi)\times\mathcal{E}^1(X,\omega,\psi)} = d$. The precise definition of $d_\mathcal{A}$ is quite technical (in Section 2 we will recall many of its properties), but the strong topology is natural since it is the coarsest refinement of the weak topology such that $E_\cdot(\,\cdot\,)$ becomes continuous as Theorem 6.2 shows. In particular the strong topology is independent of the set $\mathcal{A}$ chosen.

Also the set

$$Y_\mathcal{A} := \bigsqcup_{\psi \in \bar{\mathcal{A}}} \mathcal{M}^1(X, \omega, \psi)$$

has a natural strong topology given as the coarsest refinement of the weak topology such that $E^*_\cdot(\,\cdot\,)$ becomes continuous.

Theorem B. *The Monge–Ampère map*

$$\mathrm{MA}_\omega : (X_{\mathcal{A},\mathrm{norm}}, d_{\mathcal{A}}) \to (Y_{\mathcal{A}}, \text{strong})$$

is a homeomorphism.

Obviously in Theorem B we define $\mathrm{MA}_\omega(P_{\psi_{\min}}) := 0$ if $V_{\psi_{\min}} = 0$.

Note that by Hartogs' lemma and Theorem 6.2 the metric subspace $X_{\mathcal{A},\mathrm{norm}}$ is complete and represents the set of all closed and positive (1, 1)-currents $T = \omega + dd^c u$ such that $u \in X_{\mathcal{A}}$, where $P_{\psi_{\min}}$ encases all currents whose potentials u are more singular than $\psi_{\min}$ if $V_{\psi_{\min}} = 0$.

Finally, as an application of Theorem B we study an example of the stability of solutions of complex Monge–Ampère equations. Other important situations will be dealt with in a future work.

Theorem C. *Let $\mathcal{A} := \{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{M}^+$ be totally ordered, and let $\{f_k\}_{k\in\mathbb{N}} \subset L^1 \setminus \{0\}$ be a sequence of nonnegative functions such that $f_k \to f \in L^1 \setminus \{0\}$ and such that $\int_X f_k \omega^n = V_{\psi_k}$ for any $k \in \mathbb{N}$. Assume also that there exists $p > 1$ such that $\|f_k\|_{L^p}$ and $\|f\|_{L^p}$ are uniformly bounded. Then $\psi_k \to \psi \in \mathcal{M}^+$ weakly, and the sequence $\{u_k\}_{k\in\mathbb{N}}$ of solutions of*

$$\mathrm{MA}_\omega(u_k) = f_k \omega^n, \quad u_k \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi_k), \tag{6}$$

converges strongly to $u \in X_{\mathcal{A}}$ (i.e., $d_{\mathcal{A}}(u_k, u) \to 0$), which is the unique solution of

$$\mathrm{MA}_\omega(u) = f \omega^n, \quad u \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi).$$

In particular, $u_k \to u$ in capacity.

The existence of the solutions of (6) follows by Theorem A in [Darvas et al. 2021a], while the fact that the strong convergence implies the convergence in capacity is our Theorem 6.3. Note also that the convergence in capacity of Theorem C was already obtained in [Darvas et al. 2021b]; see Remark 7.1.

1A. ***Structure of the paper.*** Section 2 is dedicated to introducing preliminaries, and, in particular, all necessary results presented in [Trusiani 2022]. In Section 3 we extend some known uniform estimates for $\mathcal{E}^1(X, \omega)$ to the relative setting, and we prove the key upper-semicontinuity of the relative energy functional $E_\cdot(\,\cdot\,)$ in $X_{\mathcal{A}}$. Section 4 regards the properties of the action of measures on $\mathrm{PSH}(X, \omega)$ and, in particular, their continuity. Then Section 5 is dedicated to proving Theorem A. We use a variational approach to show the bijection, then we need some further important properties of the strong topology on $\mathcal{E}^1(X, \omega, \psi)$ to conclude the proof. Section 6 is the heart of the article where we extend the results proved in the previous section to $X_{\mathcal{A}}$, and we present our main Theorem B. Finally in Section 7 we show Theorem C.

1B. ***Future developments.*** As mentioned above, in a future work we will present some strong stability results of more general solutions of complex Monge–Ampère equations with prescribed singularities than Theorem C, starting the study of a kind of *continuity method* where the singularities will also vary. As an application we will study the existence of (log) Kähler–Einstein metrics with prescribed singularities, with a particular focus on the relationships among them varying the singularities.

2. Preliminaries

We recall that given a Kähler complex compact manifold (X, ω), the set $\mathrm{PSH}(X, \omega)$ is the set of all ω-plurisubharmonic functions (ω-psh), i.e., all $u \in L^1$ given locally as the sum of a smooth function and a plurisubharmonic function such that $\omega + dd^c u \geq 0$ as a (1, 1)-current. Here $d^c := \frac{i}{2\pi}(\bar\partial - \partial)$ so that $dd^c = \frac{i}{\pi}\partial\bar\partial$. For any pair of ω-psh functions u, v, the function

$$P_\omega[u](v) := \big(\lim_{C\to\infty} P_\omega(u + C, v)\big)^* = (\sup\{w \in \mathrm{PSH}(X, \omega) : w \preccurlyeq u,\ w \leq v\})^*$$

is ω-psh, where the star is for the upper semicontinuous regularization and

$$P_\omega(u, v) := (\sup\{w \in \mathrm{PSH}(X, \omega) : w \leq \min(u, v)\})^*.$$

Then the set of all model-type envelopes is defined as

$$\mathcal{M} := \{\psi \in \mathrm{PSH}(X, \omega) : \psi = P_\omega[\psi](0)\}.$$

We also recall that $\mathcal{M}^+$ denotes the elements $\psi \in \mathcal{M}$ such that $V_\psi > 0$ where, as said in the Introduction, $V_\psi := \int_X \mathrm{MA}_\omega(\psi)$.

The class of ψ-relative full mass functions $\mathcal{E}(X, \omega, \psi)$ complies with the following characterization.

Theorem 2.1 [Darvas et al. 2018, Theorem 1.3]. *Suppose $v \in \mathrm{PSH}(X, \omega)$ such that $V_v > 0$ and v is less singular than $u \in \mathrm{PSH}(X, \omega)$. Then the following are equivalent*:

(i) $u \in \mathcal{E}(X, \omega, v)$.

(ii) $P_\omega[u](v) = v$.

(iii) $P_\omega[u](0) = P_\omega[v](0)$.

The clear inclusion $\mathcal{E}(X, \omega, v) \subset \mathcal{E}(X, \omega, P_\omega[v](0))$ may be strict, and it seems more natural in many cases to consider only functions $\psi \in \mathcal{M}$. For instance, as shown in [Darvas et al. 2018], ψ being a model-type envelope is a necessary assumption to make the equation

$$\mathrm{MA}_\omega(u) = \mu, \quad u \in \mathcal{E}(X, \omega, \psi),$$

always solvable where μ is a nonpluripolar measure such that $\mu(X) = V_\psi$. It is also worth recalling that there are plenty of elements in $\mathcal{M}$, since $P_\omega[P_\omega[\psi]] = P_\omega[\psi]$ for any $\psi \in \mathrm{PSH}(X, \omega)$ with $\int_X \mathrm{MA}_\omega(\psi) > 0$, see [Darvas et al. 2018, Theorem 3.12]. Indeed, $v \to P_\omega[v]$ may be thought of as a projection from the set of negative ω-psh functions with positive Monge–Ampère mass to $\mathcal{M}^+$.

We also retrieve the following useful result.

Theorem 2.2 [Darvas et al. 2018, Theorem 3.8]. *Let $u, \psi \in \mathrm{PSH}(X, \omega)$ such that $u \succcurlyeq \psi$. Then*

$$\mathrm{MA}_\omega(P_\omega[\psi](u)) \leq \mathbb{1}_{\{P_\omega[\psi](u) = u\}} \mathrm{MA}_\omega(u).$$

In particular, if $\psi \in \mathcal{M}$ then $\mathrm{MA}_\omega(\psi) \leq \mathbb{1}_{\{\psi = 0\}} \mathrm{MA}_\omega(0)$.

Note also, in Theorem 2.2 the equality holds if u is continuous with bounded distributional Laplacian with respect to ω as a consequence of [Di Nezza and Trapani 2021]. In particular, for any $\psi \in \mathcal{M}$, $\mathrm{MA}_\omega(\psi) = \mathbb{1}_{\{\psi=0\}} \mathrm{MA}_\omega(0)$.

2A. ***The metric space $(\mathcal{E}^1(X, \omega, \psi), d)$.*** In this subsection we assume $\psi \in \mathcal{M}^+ := \{\psi \in \mathcal{M} : V_\psi > 0\}$.

As in [Darvas et al. 2018], we also denote by $\mathrm{PSH}(X, \omega, \psi)$ the set of all ω-psh functions which are more singular than ψ, and we recall that a function $u \in \mathrm{PSH}(X, \omega, \psi)$ has *ψ-relative minimal singularities* if $|u - \psi|$ is globally bounded on X. We also use the notation

$$\mathrm{MA}_\omega(u_1^{j_1}, \dots, u_l^{j_l}) := (\omega + dd^c u_1)^{j_1} \wedge \cdots \wedge (\omega + dd^c u_l)^{j_l}$$

for $u_1, \dots, u_l \in \mathrm{PSH}(X, \omega)$ where $j_1, \dots, j_l \in \mathbb{N}$ such that $j_1 + \cdots + j_l = n$.

Definition 2.3 [Darvas et al. 2018, Section 4.2]. The *ψ-relative energy functional E_ψ* $: \mathrm{PSH}(X, \omega, \psi) \to \mathbb{R} \cup \{-\infty\}$ is defined as

$$E_\psi(u) := \frac{1}{n+1} \sum_{j=0}^{n} \int_X (u - \psi) \, \mathrm{MA}_\omega(u^j, \psi^{n-j})$$

if u has ψ-relative minimal singularities, and as

$$E_\psi(u) := \inf\{E_\psi(v) : v \in \mathcal{E}(X, \omega, \psi) \text{ with } \psi\text{-relative minimal singularities}, v \geq u\}$$

otherwise. The subset $\mathcal{E}^1(X, \omega, \psi) \subset \mathcal{E}(X, \omega, \psi)$ is defined as

$$\mathcal{E}^1(X, \omega, \psi) := \{u \in \mathcal{E}(X, \omega, \psi) : E_\psi(u) > -\infty\}.$$

When $\psi = 0$, the ψ-relative energy functional is the *Aubin–Mabuchi energy functional*, also called the *Monge–Ampère energy*; see [Aubin 1984; Mabuchi 1986].

Proposition 2.4. *The following properties from* [Darvas et al. 2018] *hold*:

(i) [Theorem 4.10] *E_ψ is nondecreasing.*

(ii) [Lemma 4.12] $E_\psi(u) = \lim_{j\to\infty} E_\psi(\max(u, \psi - j))$.

(iii) [Lemma 4.14] *E_ψ is continuous along decreasing sequences.*

(iv) [Theorem 4.10 and Corollary 4.16] *E_ψ is concave along affine curves.*

(v) [Lemma 4.13] *$u \in \mathcal{E}^1(X, \omega, \psi)$ if and only if $u \in \mathcal{E}(X, \omega, \psi)$ and $\int_X (u - \psi) \, \mathrm{MA}_\omega(u) > -\infty$.*

(vi) [Proposition 4.19] *$E_\psi(u) \geq \limsup_{k\to\infty} E_\psi(u_k)$ if $u_k, u \in \mathcal{E}^1(X, \omega, \psi)$ and $u_k \to u$ with respect to the weak topology.*

(vii) [Proposition 4.20] *Letting $u \in \mathcal{E}^1(X, \omega, \psi)$, $\chi \in \mathcal{C}^0(X)$ and $u_t := \sup\{v \in \mathrm{PSH}(X, \omega)\ v \leq u + t\chi\}^*$ for any $t > 0$, then $t \to E_\psi(u_t)$ is differentiable and its derivative is given by*

$$\frac{d}{dt} E_\psi(u_t) = \int_X \chi \, \mathrm{MA}_\omega(u_t).$$

(viii) [Theorem 4.10] *If* $u, v \in \mathcal{E}^1(X, \omega, \psi)$, *then*

$$E_\psi(u) - E_\psi(v) = \frac{1}{n+1} \sum_{j=0}^{n} \int_X (u - v) \, \mathrm{MA}_\omega(u^j, v^{n-j})$$

and the function $\mathbb{N} \ni j \to \int_X (u - v) \, \mathrm{MA}_\omega(u^j, v^{n-j})$ *is decreasing. In particular*,

$$\int_X (u - v) \, \mathrm{MA}_\omega(u) \le E_\psi(u) - E_\psi(v) \le \int_X (u - v) \, \mathrm{MA}_\omega(v).$$

(ix) [Theorem 4.10] *If* $u \le v$, *then*

$$E_\psi(u) - E_\psi(v) \le \frac{1}{n+1} \int_X (u - v) \, \mathrm{MA}_\omega(u).$$

Remark 2.5. All the properties of Proposition 2.4 are shown in [Darvas et al. 2018] assuming ψ has *small unbounded locus*, but [Trusiani 2022, Proposition 2.7] and the general integration by parts formula proved in [Xia 2019] allow us to extend these properties to the general case as described in [Trusiani 2022, Remark 2.10].

Recalling that for any $u, v \in \mathcal{E}^1(X, \omega, \psi)$ the function $P_\omega(u, v) = \sup\{w \in \mathrm{PSH}(X, \omega) : w \le \min(u, v)\}^*$ belongs to $\mathcal{E}^1(X, \omega, \psi)$ (see [Trusiani 2022, Proposition 2.13]), then we also have that the function $d : \mathcal{E}^1(X, \omega, \psi) \times \mathcal{E}^1(X, \omega, \psi) \to \mathbb{R}_{\ge 0}$ defined as

$$d(u, v) = E_\psi(u) + E_\psi(v) - 2E_\psi(P_\omega(u, v))$$

assumes finite values. Moreover, it is a complete distance as the next result shows.

Theorem 2.6 [Trusiani 2022, Theorem A]. $(\mathcal{E}^1(X, \omega, \psi), d)$ *is a complete metric space.*

We call the *strong topology* on $\mathcal{E}^1(X, \omega, \psi)$ the metric topology given by the distance d. Note that, by construction, $d(u_k, u) \to 0$ as $k \to \infty$ if $u_k \searrow u$, and $d(u, v) = d(u, w) + d(w, v)$ if $u \le w \le v$; see [Trusiani 2022, Lemma 3.1].

Moreover, as a consequence of Proposition 2.4, it follows that for any $C \in \mathbb{R}_{>0}$ the set

$$\mathcal{E}^1_C(X, \omega, \psi) := \left\{u \in \mathcal{E}^1(X, \omega, \psi) : \sup_X u \le C \text{ and } E_\psi(u) \ge -C\right\}$$

is a weakly compact convex set.

Remark 2.7. If $\psi \in \mathcal{M} \setminus \mathcal{M}^+$, then $\mathcal{E}^1(X, \omega, \psi) = \mathrm{PSH}(X, \omega, \psi)$ since $E_\psi \equiv 0$ by definition; see [Trusiani 2022, Remark 3.10]. In particular, $d \equiv 0$, and it is natural to identify $(\mathcal{E}^1(X, \omega, \psi), d)$ with a point P_ψ. Moreover, we recall that $\mathcal{E}^1(X, \omega, \psi_1) \cap \mathcal{E}^1(X, \omega, \psi_2) = \varnothing$ if $\psi_1, \psi_2 \in \mathcal{M}$, $\psi_1 \ne \psi_2$ and $V_{\psi_2} > 0$.

2B. *The space* $(X_{\mathcal{A}}, d_{\mathcal{A}})$. From now on we assume $\mathcal{A} \subset \mathcal{M}^+$ to be a totally ordered set of model-type envelopes, and we denote by $\bar{\mathcal{A}}$ its closure as a subset of $\mathrm{PSH}(X, \omega)$ endowed with the weak topology. Note that $\bar{\mathcal{A}} \subset \mathrm{PSH}(X, \omega)$ is compact by [Trusiani 2022, Lemma 2.6]. Indeed, we will prove in Lemma 3.12 that $\bar{\mathcal{A}}$ is actually homeomorphic to its image through the Monge–Ampère operator MA_ω when the set of measures is endowed with the weak topology. This yields that $\bar{\mathcal{A}}$ is also homeomorphic to a closed set contained in $\left[0, \int_X \omega^n\right]$ through the map $\psi \to V_\psi$.

Definition 2.8. We define the set

$$X_{\mathcal{A}} := \bigsqcup_{\psi\in\bar{\mathcal{A}}} \mathcal{E}^1(X,\omega,\psi)$$

if $\psi_{\min} := \inf \mathcal{A}$ satisfies $V_{\psi_{\min}} > 0$, and

$$X_{\mathcal{A}} := P_{\psi_{\min}} \sqcup \bigsqcup_{\psi'\in\bar{\mathcal{A}},\psi\neq\psi_{\min}} \mathcal{E}^1(X,\omega,\psi)$$

if $V_{\psi_{\min}} = 0$, where $P_{\psi_{\min}}$ is a singleton.

$X_{\mathcal{A}}$ can be endowed with a natural metric structure as [Trusiani 2022, Section 4] shows.

Theorem 2.9 [Trusiani 2022, Theorem B]. *$(X_{\mathcal{A}}, d_{\mathcal{A}})$ is a complete metric space such that*

$$d_{\mathcal{A}|\mathcal{E}^1(X,\omega,\psi)\times\mathcal{E}^1(X,\omega,\psi)} = d$$

for any $\psi \in \bar{\mathcal{A}} \cap \mathcal{M}^+$.

We call the *strong topology* on $X_{\mathcal{A}}$ the metric topology given by the distance $d_{\mathcal{A}}$. Note that the definition is coherent with that of Section 2A since the induced topology on $\mathcal{E}^1(X,\omega,\psi) \subset X_{\mathcal{A}}$ coincides with the strong topology given by d.

We will also need the following contraction property which is the starting point to construct $d_{\mathcal{A}}$.

Proposition 2.10 [Trusiani 2022, Lemma 4.2 and Proposition 4.3]. *Let $\psi_1, \psi_2, \psi_3 \in \mathcal{M}$ such that $\psi_1 \preccurlyeq \psi_2 \preccurlyeq \psi_3$. Then $P_\omega[\psi_1](P_\omega[\psi_2](u)) = P_\omega[\psi_1](u)$ for any $u \in \mathcal{E}^1(X,\omega,\psi_3)$ and $|P_\omega[\psi_1](u) - \psi_1| \leq C$ if $|u - \psi_3| \leq C$. Moreover, the map*

$$P_\omega[\psi_1](\,\cdot\,) : \mathcal{E}^1(X,\omega,\psi_2) \to \mathrm{PSH}(X,\omega,\psi_1)$$

has image in $\mathcal{E}^1(X,\omega,\psi_1)$ and is a Lipschitz map of constant 1 *when the sets $\mathcal{E}^1(X,\omega,\psi_i)$, $i = 1, 2$, are endowed with the d distances, i.e.,*

$$d(P_\omega[\psi_1](u), P_\omega[\psi_1](v)) \leq d(u, v)$$

for any $u, v \in \mathcal{E}^1(X,\omega,\psi_2)$.

Here we report some properties of the distance $d_{\mathcal{A}}$ and some consequences which will be useful later.

Proposition 2.11. *The following properties from* [Trusiani 2022] *hold*:

(i) [Proposition 4.14] *If $u \in \mathcal{E}^1(X,\omega,\psi_1)$ and $v \in \mathcal{E}^1(X,\omega,\psi_2)$ for $\psi_1, \psi_2 \in \bar{\mathcal{A}}$ and $\psi_1 \succcurlyeq \psi_2$, then*

$$d_A(u, v) \geq d(P_\omega[\psi_2](u), v).$$

(ii) [Lemma 4.6] *If $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{M}^+$, $\psi \in \mathcal{M}$, with $\psi_k \searrow \psi$ (resp. $\psi_k \nearrow \psi$ a.e.), $u_k \searrow u$ and $v_k \searrow v$ (resp. $u_k \nearrow u$ a.e. and $v_k \nearrow v$ a.e.), for $u_k, v_k \in \mathcal{E}^1(X,\omega,\psi_k)$ and $u, v \in \mathcal{E}^1(X,\omega,\psi)$ and $|u_k - v_k|$ is uniformly bounded, then*

$$d(u_k, v_k) \to d(u, v).$$

(iii) [Proposition 4.5] *If* $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{M}^+$, $\psi \in \mathcal{M}$, *such that* $\psi_k \to \psi$ *monotonically a.e., then for any* $\psi' \in \mathcal{M}$ *such that* $\psi' \succcurlyeq \psi_k$ *for any* $k \gg 1$ *big enough and for any strongly compact set* $K \subset (\mathcal{E}^1(X, \omega, \psi'), d)$,

$$d(P_\omega[\psi_k](\varphi_1), P_\omega[\psi_k](\varphi_2)) \to d(P_\omega[\psi](\varphi_1), P_\omega[\psi](\varphi_2))$$

uniformly on $K \times K$, *i.e., varying* $(\varphi_1, \varphi_2) \in K \times K$. *In particular, if* $\psi_k, \psi \in \bar{\mathcal{A}}$, *then*

$$d_{\mathcal{A}}(P_\omega[\psi](u), P_\omega[\psi_k](u)) \to 0,$$
$$d(P_\omega[\psi_k](u), P_\omega[\psi_k](v)) \to d(P_\omega[\psi](u), P_\omega[\psi](v))$$

monotonically for any $(u, v) \in \mathcal{E}^1(X, \omega, \psi') \times \mathcal{E}^1(X, \omega, \psi')$.

(iv) [Section 4.2] $d_{\mathcal{A}}(u_1, u_2) \geq |V_{\psi_1} - V_{\psi_2}|$ *if* $u_1 \in \mathcal{E}^1(X, \omega, \psi_1)$ *and* $u_2 \in \mathcal{E}^1(X, \omega, \psi_2)$, *and the equality holds if* $u_1 = \psi_1$ *and* $u_2 = \psi_2$ *(by definition of* $d_{\mathcal{A}}$*).*

The following lemma is a special case of [Xia 2019, Theorem 2.2]; see also [Darvas et al. 2018, Lemma 4.1].

Lemma 2.12 [Trusiani 2022, Proposition 2.7]. *Let* $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{M}^+$, $\psi \in \mathcal{M}$, *such that* $\psi_k \to \psi$ *monotonically almost everywhere. Let also* $u_k, v_k \in \mathcal{E}^1(X, \omega, \psi_k)$ *converge in capacity to* $u, v \in \mathcal{E}^1(X, \omega, \psi)$, *respectively. Then for any* $j = 0, \dots, n$,

$$\mathrm{MA}_\omega(u_k^j, v_k^{n-j}) \to \mathrm{MA}_\omega(u^j, v^{n-j})$$

weakly. Moreover, if $|u_k - v_k|$ *is uniformly bounded, then for any* $j = 0, \dots, n$,

$$(u_k - v_k)\,\mathrm{MA}_\omega(u_k^j, v_k^{n-j}) \to (u - v)\,\mathrm{MA}_\omega(u^j, v^{n-j})$$

weakly.

It is well known that the set of Kähler potentials $\mathcal{H}_\omega := \{\varphi \in \mathrm{PSH}(X, \omega) \cap C^\infty(X) : \omega + dd^c\varphi > 0\}$ is dense in $(\mathcal{E}^1(X, \omega), d)$. The same holds for $P_\omega[\psi](\mathcal{H}_\omega)$ in $(\mathcal{E}^1(X, \omega, \psi), d)$.

Lemma 2.13 [Trusiani 2022, Lemma 4.8]. *The set* $\mathcal{P}_{\mathcal{H}_\omega}(X, \omega, \psi) := P_\omega[\psi](\mathcal{H}) \subset \mathcal{P}(X, \omega, \psi)$ *is dense in* $(\mathcal{E}^1(X, \omega, \psi), d)$.

The following lemma shows that, for $u \in \mathrm{PSH}(X, \omega)$ fixed, the map $\mathcal{M}^+ \ni \psi \to P_\omega[\psi](u)$ is weakly continuous over any totally ordered set of model-type envelopes that are more singular than u.

Lemma 2.14. *Let* $u \in \mathrm{PSH}(X, \omega)$, *and let* $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{M}^+$ *be a totally ordered sequence of model-type envelopes converging to* $\psi \in \mathcal{M}$. *Assume also that* $\psi_k \preccurlyeq u$ *for any* $k \gg 1$ *big enough. Then* $P_\omega[\psi_k](u) \to P_\omega[\psi](u)$ *weakly.*

Proof. As $\{\psi_k\}_{k\in\mathbb{N}}$ is totally ordered, without loss of generality we may assume that $\psi_k \to \psi$ monotonically almost everywhere. Set $\tilde{u} := \lim_{k\to\infty} P_\omega[\psi_k](u)$. We want to prove that $\tilde{u} = P_\omega[\psi](u)$.

Suppose $\psi_k \searrow \psi$. We can immediately check that $P_\omega[\psi_k](u) \leq P_\omega[\psi_k](\sup_X u) = \psi_k + \sup_X u$, which implies $\tilde{u} \leq \psi + \sup_X u$ letting $k \to +\infty$. Thus $\tilde{u} \leq P_\omega[\psi](u)$, as the inequality $\tilde{u} \leq u$ is trivial. Moreover, since $\psi \le \psi_k$ we also have $P_\omega[\psi](u) \le P_\omega[\psi_k](u)$, which clearly yields $P_\omega[\psi](u) \le \tilde{u}$ and concludes this part.

Suppose $\psi_k \nearrow \psi$. Then the inequality $\tilde{u} \le P_\omega[\psi](u)$ is immediate. Next, combining Theorem 2.2 and Proposition 2.10, we have

$$\begin{aligned}\mathrm{MA}_\omega(P_\omega[\psi_k](u)) &= \mathrm{MA}_\omega(P_\omega[\psi_k](P_\omega[\psi](u)))\\ &\le \mathbb{1}_{\{P_\omega[\psi_k](u)=P_\omega[\psi](u)\}}\,\mathrm{MA}_\omega(P_\omega[\psi](u))\\ &\le \mathbb{1}_{\{\tilde{u}=P_\omega[\psi](u)\}}\,\mathrm{MA}_\omega(P_\omega[\psi](u)),\end{aligned}$$

where the last inequality follows from $P_\omega[\psi_k](u) \le \tilde{u} \le P_\omega[\psi](u)$. Thus, as $\mathrm{MA}_\omega(P_\omega[\psi_k](u)) \to \mathrm{MA}_\omega(\tilde{u})$ weakly by [Darvas et al. 2018, Theorem 2.3], we deduce that $\tilde{u} \in \mathcal{E}(X, \omega, \psi)$ and

$$\mathrm{MA}_\omega(\tilde{u}) \le \mathbb{1}_{\{\tilde{u}=P_\omega[\psi](u)\}}\,\mathrm{MA}_\omega(P_\omega[\psi](u)).$$

Moreover, we also have $P_\omega[\psi](u) \in \mathcal{E}(X, \omega, \psi)$. Indeed, $P_\omega[\psi](u) \le P_\omega[\psi](\sup_X u) = \psi + \sup_X$, i.e., $P_\omega[\psi](u) \preccurlyeq \psi$, while $P_\omega[\psi](u) \ge P_\omega[\psi](\psi_k - C_k) = \psi_k - C_k$ for nonnegative constants C_k and for any $k \gg 1$ big enough as u, ψ are less singular than ψ_k. Thus $P_\omega[\psi](u) \succcurlyeq \psi_k$ for any k, which yields $\int_X \mathrm{MA}_\omega(P_\omega[\psi](u)) \ge V_\psi > 0$ and gives $P_\omega[\psi](u) \in \mathcal{E}(X, \omega, \psi)$. Hence

$$\begin{aligned}0 &\le \int_X (P_\omega[\psi](u) - \tilde{u})\,\mathrm{MA}_\omega(\tilde{u})\\ &\le \int_{\{\tilde{u}=P_\omega[\psi](u)\}} (P_\omega[\psi](u) - \tilde{u})\,\mathrm{MA}_\omega(P_\omega[\psi](u)) = 0,\end{aligned}$$

which by the domination principle of [Darvas et al. 2018, Proposition 3.11] implies $\tilde{u} \ge P_\omega[\psi](u)$. ☐

3. Tools

In this section we collect some uniform estimates on $\mathcal{E}^1(X, \omega, \psi)$ for $\psi \in \mathcal{M}^+$, we recall the ψ-relative capacity and we prove the upper semicontinuity of $E_\cdot(\,\cdot\,)$ on $X_{\mathcal{A}}$.

3A. *Uniform estimates.* Let $\psi \in \mathcal{M}^+$.

We first define in the ψ-relative setting the analogs of some well-known functionals of the variational approach; see [Berman et al. 2013].

We define the *ψ-relative I- and J-functionals*,

$$I_\psi, J_\psi : \mathcal{E}^1(X, \omega, \psi) \times \mathcal{E}^1(X, \omega, \psi) \to \mathbb{R}, \quad \text{where } \psi \in \mathcal{M}^+,$$

as

$$\begin{aligned}I_\psi(u, v) &:= \int_X (u - v)(\mathrm{MA}_\omega(v) - \mathrm{MA}_\omega(u)),\\ J_\psi(u, v) &:= J_u^\psi(v) := E_\psi(u) - E_\psi(v) + \int_X (v - u)\,\mathrm{MA}_\omega(u),\end{aligned}$$

respectively; see also [Aubin 1984]. They assume nonnegative values by Proposition 2.4, and I_ψ is clearly symmetric while J_ψ is convex, again by Proposition 2.4. Moreover, the ψ-relative I- and J-functionals are related to each other by the following result.

Lemma 3.1. *Let $u, v \in \mathcal{E}^1(X, \omega, \psi)$. Then*

(i)
$$\frac{1}{n+1} I_\psi(u, v) \leq J_u^\psi(v) \leq \frac{n}{n+1} I_\psi(u, v),$$

(ii)
$$\frac{1}{n} J_u^\psi(v) \leq J_v^\psi(u) \leq n J_u^\psi(v).$$

In particular,
$$d(\psi, u) \leq n J_u^\psi(\psi) + (\|\psi\|_{L^1} + \|u\|_{L^1})$$
for any $u \in \mathcal{E}^1(X, \omega, \psi)$ such that $u \leq \psi$.

Proof. By Proposition 2.4 it follows that
$$n \int_X (u-v)\,\mathrm{MA}_\omega(u) + \int_X (u-v)\,\mathrm{MA}_\omega(v) \leq (n+1)(E_\psi(u) - E_\psi(v))$$
$$\leq \int_X (u-v)\,\mathrm{MA}_\omega(u) + n \int_X (u-v)\,\mathrm{MA}_\omega(v)$$
for any $u, v \in \mathcal{E}^1(X, \omega, \psi)$, which yields (i) and (ii).

Next, considering $v = \psi$ and assuming $u \leq \psi$ from the second inequality in (ii), we obtain
$$d(u, \psi) = -E_\psi(u) \leq n J_u^\psi(\psi) + \int_X (\psi - u)\,\mathrm{MA}_\omega(\psi),$$
which implies the assertion since $\mathrm{MA}_\omega(\psi) \leq \mathrm{MA}_\omega(0)$ by Theorem 2.2. ☐

We can now proceed to show the uniform estimates, adapting some results in [Berman et al. 2013].

Lemma 3.2 [Trusiani 2022, Lemma 3.7]. *Let $\psi \in \mathcal{M}^+$. Then there exists positive constants $A > 1$, $B > 0$ depending only on n, ω such that for any $u \in \mathcal{E}^1(X, \omega, \psi)$,*
$$-d(\psi, u) \leq V_\psi \sup_X (u - \psi) = V_\psi \sup_X u \leq A\, d(\psi, u) + B$$

Remark 3.3. As a consequence of Lemma 3.2, if $d(\psi, u) \leq C$, then $\sup_X u \leq (AC + B)/V_\psi$ while
$$-E_\psi(u) = d(\psi + (AC + B)/V_\psi, u) - (AC + B) \leq d(\psi, u) \leq C,$$
i.e., $u \in \mathcal{E}_D^1(X, \omega, \psi)$ where $D := \max(C, (AC + B)/V_\psi)$. Conversely, using the definitions and the triangle inequality, it is easy to check that $d(u, \psi) \leq C(2V_\psi + 1)$ for any $u \in \mathcal{E}_C^1(X, \omega, \psi)$.

Proposition 3.4. *Let $C \in \mathbb{R}_{>0}$. Then there exists a continuous increasing function $f_C : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ depending only on C, ω, n with $f_C(0) = 0$ such that*
$$\left| \int_X (u - v)(\mathrm{MA}_\omega(\varphi_1) - \mathrm{MA}_\omega(\varphi_2)) \right| \leq f_C(d(u, v)) \tag{7}$$
for any $u, v, \varphi_1, \varphi_2 \in \mathcal{E}^1(X, \omega, \psi)$ with $d(u, \psi), d(v, \psi), d(\varphi_1, \psi), d(\varphi_2, \psi) \leq C$.

Proof. As said in Remark 3.3, if $w \in \mathcal{E}^1(X, \omega, \psi)$ with $d(\psi, w) \le C$, then $\tilde{w} := w - (AC + B)/V_\psi$ satisfies $\sup_X \tilde{w} \le 0$ and

$$-E_\psi(\tilde{w}) = d(\psi, \tilde{w}) \le d(\psi, w) + d(w, \tilde{w}) \le C + AC + B =: D.$$

Therefore, setting $\tilde{u} := u - (AC + B)/V_\psi$ and $\tilde{v} := v - (AC + B)/V_\psi$, we can proceed exactly as in [Berman et al. 2013, Lemma 5.8] using the integration by parts formula in [Xia 2019] (see also [Boucksom et al. 2010, Theorem 1.14]) to get

$$\left| \int_X (\tilde{u} - \tilde{v})(\mathrm{MA}_\omega(\varphi_1) - \mathrm{MA}_\omega(\varphi_2)) \right| \le I_\psi(\tilde{u}, \tilde{v}) + h_D(I_\psi(\tilde{u}, \tilde{v})), \tag{8}$$

where $h_D : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is an increasing continuous function depending only on D such that $h_D(0) = 0$. Furthermore, by definition,

$$d(\psi, P_\omega(\tilde{u}, \tilde{v})) \le d(\psi, \tilde{u}) + d(\tilde{u}, P_\omega(\tilde{u}, \tilde{v})) \le d(\psi, \tilde{u}) + d(\tilde{u}, \tilde{v}) \le 3D,$$

so by the triangle inequality and (8) we have

$$\begin{aligned}\left| \int_X (u - v)(\mathrm{MA}_\omega(\varphi_1) - \mathrm{MA}_\omega(\varphi_2)) \right| \\ \le I_\psi(\tilde{u}, P_\omega(\tilde{u}, \tilde{v})) + I_\psi(\tilde{v}, P_\omega(\tilde{u}, \tilde{v})) + h_{3D}(I_\psi(\tilde{u}, P_\omega(\tilde{u}, \tilde{v}))) + h_{3D}(I_\psi(\tilde{v}, P_\omega(\tilde{u}, \tilde{v}))).\end{aligned} \tag{9}$$

On the other hand, if $w_1, w_2 \in \mathcal{E}^1(X, \omega, \psi)$ with $w_1 \ge w_2$, then by Proposition 2.4

$$I_\psi(w_1, w_2) \le \int_X (w_1 - w_2)\, \mathrm{MA}_\omega(w_2) \le (n+1) d(w_1, w_2).$$

Hence from (9) it is sufficient to set $f_C(x) := (n+1)x + 2h_{3D}((n+1)x)$ to conclude the proof since clearly $d(\tilde{u}, \tilde{v}) = d(u, v)$. □

Corollary 3.5. *Let $\psi \in \mathcal{M}^+$ and let $C \in \mathbb{R}_{>0}$. Then there exists a continuous increasing function $f_C : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ depending only on C, ω, n with $f_C(0) = 0$ such that*

$$\int_X |u - v|\, \mathrm{MA}_\omega(\varphi) \le f_C(d(u, v))$$

for any $u, v, \varphi \in \mathcal{E}^1(X, \omega, \psi)$ with $d(\psi, u), d(\psi, v), d(\psi, \varphi) \le C$.

Proof. Since $d(\psi, P_\omega(u, v)) \le 3C$, letting $g_{3C} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be the map (7) of Proposition 3.4, it follows that

$$\begin{aligned}\int_X (u - P_\omega(u, v))\, \mathrm{MA}_\omega(\varphi) &\le \int_X (u - P_\omega(u, v))\, \mathrm{MA}_\omega(P_\omega(u, v)) + g_{3C}(d(u, P_\omega(u, v))) \\ &\le (n+1) d(u, P_\omega(u, v)) + g_{3C}(d(u, v)),\end{aligned}$$

where in the last inequality we used Proposition 2.4. Hence by the triangle inequality we get

$$\begin{aligned}\int_X |u - v|\, \mathrm{MA}_\omega(\varphi) &\le (n+1) d(u, P_\omega(u, v)) + (n+1) d(v, P_\omega(u, v)) + 2g_{3C}(d(u, v)) \\ &= (n+1) d(u, v) + 2g_{3C}(d(u, v)).\end{aligned}$$

Defining $f_C(x) := (n+1)x + 2g_{3C}(x)$ concludes the proof. □

As a first important consequence we obtain that the strong convergence in $\mathcal{E}^1(X,\omega,\psi)$ implies the weak convergence.

Proposition 3.6. *Let $\psi \in \mathcal{M}^+$ and let $C \in \mathbb{R}_{>0}$. Then there exists a continuous increasing function $f_{C,\psi}:\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ depending on C,ω,n,ψ with $f_{C,\psi}(0)=0$ such that*

$$\|u-v\|_{L^1} \leq f_{C,\psi}(d(u,v))$$

for any $u,v \in \mathcal{E}^1(X,\omega,\psi)$ with $d(\psi,u), d(\psi,v) \leq C$. In particular, $u_k \to u$ weakly if $u_k \to u$ strongly.

Proof. Theorem A in [Darvas et al. 2021a] (see also Theorem 1.4 in [Darvas et al. 2018]) implies that there exists $\phi \in \mathcal{E}^1(X,\omega,\psi)$ with $\sup_X \phi = 0$ such that

$$\mathrm{MA}_\omega(\phi) = c\,\mathrm{MA}_\omega(0),$$

where $c := V_\psi/V_0 > 0$. Therefore it follows that

$$\|u-v\|_{L^1} \leq \frac{1}{c} g_{\hat{C}}(d(u,v)),$$

where $\hat{C} := \max(d(\psi,\phi), C)$ and $g_{\hat{C}}$ is the continuous increasing function with $g_{\hat{C}}(0)=0$ given by Corollary 3.5. Setting $f_{C,\psi} := \frac{1}{c} g_{\hat{C}}$ concludes the proof. □

Finally we also get the following useful estimate.

Proposition 3.7. *Let $\psi \in \mathcal{M}^+$ and let $C \in \mathbb{R}_{>0}$. Then there exists a constant $\tilde{C}$ depending only on C,ω,n such that*

$$\left| \int_X (u-v)(\mathrm{MA}_\omega(\varphi_1) - \mathrm{MA}_\omega(\varphi_2)) \right| \leq \tilde{C} I_\psi(\varphi_1,\varphi_2)^{1/2} \tag{10}$$

for any $u,v,\varphi_1,\varphi_2 \in \mathcal{E}^1(X,\omega,\psi)$ with $d(u,\psi), d(v,\psi), d(\varphi_1,\psi), d(\varphi_2,\psi) \leq C$.

Proof. As in Proposition 3.4 and with the same notation, the function $\tilde{u} := u-(AC+B)/V_\psi$ satisfies $\sup_X u \leq 0$ (by Lemma 3.2) and $-E_\psi(u) \leq C + AC + B =: D$ (and similarly for v,φ_1,φ_2). Therefore by integration by parts and using Lemma 3.8 below, it follows exactly as in [Berman et al. 2013, Lemma 3.13] that there exists a constant $\tilde{C}$ depending only on D,n such that

$$\left| \int_X (\tilde{u}-\tilde{v})(\mathrm{MA}_\omega(\tilde{\varphi}_1) - \mathrm{MA}_\omega(\tilde{\varphi}_2)) \right| \leq \tilde{C} I_\psi(\tilde{\varphi}_1,\tilde{\varphi}_2)^{1/2},$$

which clearly implies (10). □

Lemma 3.8. *Let $C \in \mathbb{R}_{>0}$. Then there exists a constant $\tilde{C}$ depending only on C,ω,n such that*

$$\int_X |u_0-\psi|(\omega+dd^c u_1)\wedge\cdots\wedge(\omega+dd^c u_n) \leq \tilde{C}$$

for any $u_0,\dots,u_n \in \mathcal{E}^1(X,\omega,\psi)$, with $d(u_j,\psi) \leq C$ for any $j=0,\dots,n$.

Proof. As in Proposition 3.4 and with the same notation, $v_j := u_j - (AC+B)/V_\psi$ satisfies $\sup_X v_j \leq 0$, and setting $v := (v_0 + \cdots + v_n)/(n+1)$ we obtain $\psi - u_0 \leq (n+1)(\psi - v)$. Thus by Proposition 2.4,

$$\int_X (\psi - v_0)\,\mathrm{MA}_\omega(v) \leq (n+1)\int_X (\psi - v)\,\mathrm{MA}_\omega(v) \leq (n+1)^2 |E_\psi(v)| \\ \leq (n+1)\sum_{j=0}^n |E_\psi(v_j)| \leq (n+1)\sum_{j=0}^n (d(\psi, u_j) + D) \leq (n+1)^2(C+D),$$

where $D := AC + B$. On the other hand, $\mathrm{MA}_\omega(v) \geq E(\omega + dd^c u_1) \wedge \cdots \wedge (\omega + dd^c u_n)$, where the constant E depends only on n. Finally we get

$$\int_X |u_0 - \psi|(\omega + dd^c u_1) \wedge \cdots \wedge (\omega + dd^c u_n) \leq D + \frac{1}{E}\int_X (\psi - v_0)\,\mathrm{MA}_\omega(v) \\ \leq D + \frac{(n+1)^2(C+D)}{E}. \qquad \square$$

3B. *ψ-relative Monge–Ampère capacity.*

Definition 3.9 [Darvas et al. 2018, Section 4.1; Darvas et al. 2021a, Definition 3.1]. Let $B \subset X$ be a Borel set, and let $\psi \in \mathcal{M}^+$. Then its *ψ-relative Monge–Ampère capacity* is defined as

$$\mathrm{Cap}_\psi(B) := \sup\left\{\int_B \mathrm{MA}_\omega(u) : u \in \mathrm{PSH}(X, \omega),\ \psi - 1 \leq u \leq \psi\right\}.$$

In the absolute setting the Monge–Ampère capacity is very useful for studying the existence and regularity of solutions of the degenerate complex Monge–Ampère equation [Kołodziej 1998], and the analog holds in the relative setting [Darvas et al. 2018, 2021a]. We refer to these articles for many properties of the Monge–Ampère capacity.

For any fixed constant A, write $\mathcal{C}_{A,\psi}$ for the set of all probability measures μ on X such that

$$\mu(B) \leq A\,\mathrm{Cap}_\psi(B)$$

for any Borel set $B \subset X$ [Darvas et al. 2018, Section 4.3].

Proposition 3.10. *Let $u \in \mathcal{E}^1(X, \omega, \psi)$ with ψ-relative minimal singularities. Then $\mathrm{MA}_\omega(u)/V_\psi \in \mathcal{C}_{A,\psi}$ for a constant $A > 0$.*

Proof. Let $j \in \mathbb{R}$ such that $u \geq \psi - j$ and assume without loss of generality that $u \leq \psi$ and $j \geq 1$. Then the function $v := j^{-1}u + (1 - j^{-1})\psi$ is a candidate in the definition of Cap_ψ, which implies that $\mathrm{MA}_\omega(v) \leq \mathrm{Cap}_\psi$. Hence, since $\mathrm{MA}_\omega(u) \leq j^n\,\mathrm{MA}(v)$, we get that $\mathrm{MA}_\omega(u) \in \mathcal{C}_{A,\psi}$ for $A = j^n$ and the result follows. $\square$

Lemma 3.11 [Darvas et al. 2018, Lemma 4.18]. *If $\mu \in \mathcal{C}_{A,\psi}$, then there is a constant $B > 0$ depending only on A, n such that*

$$\int_X (u - \psi)^2 \mu \leq B(|E_\psi(u)| + 1)$$

for any $u \in \mathrm{PSH}(X, \omega, \psi)$ such that $\sup_X u = 0$.

Similar to the case $\psi = 0$ (see [Guedj and Zeriahi 2017]), we say that a sequence $u_k \in \mathrm{PSH}(X,\omega)$ converges to $u \in \mathrm{PSH}(X,\omega)$ in ψ-relative capacity for $\psi \in \mathcal{M}$ if

$$\mathrm{Cap}_\psi(\{|u_k - u| \geq \delta\}) \to 0$$

as $k \to \infty$ for any $\delta > 0$.

By [Guedj and Zeriahi 2017, Theorem 10.37] (see also [Berman et al. 2013, Theorem 5.7]) the convergence in $(\mathcal{E}^1(X,\omega), d)$ implies the convergence in capacity. The analog holds for $\psi \in \mathcal{M}^+$, i.e., the strong convergence in $\mathcal{E}^1(X,\omega,\psi)$ implies the convergence in ψ-relative capacity. Indeed, in Proposition 5.7 we will prove the strong convergence implies the convergence in ψ'-relative capacity for any $\psi' \in \mathcal{M}^+$.

3C. *(Weak) upper semicontinuity of $u \to E_{P_\omega[u]}(u)$ over $X_{\mathcal{A}}$.* One of the main features of E_ψ for $\psi \in \mathcal{M}$ is its upper semicontinuity with respect to the weak topology. Here we prove the analog for $E_\cdot(\,\cdot\,)$ over $X_{\mathcal{A}}$.

Lemma 3.12. *The map*

$$\mathrm{MA}_\omega : \bar{\mathcal{A}} \to \mathrm{MA}_\omega(\bar{\mathcal{A}}) \subset \{\mu \textit{ a positive measure on } X\}$$

is a homeomorphism considering the weak topologies. In particular, $\bar{\mathcal{A}}$ is homeomorphic to a closed set contained in $\big[0, \int_X \mathrm{MA}_\omega(0)\big]$ through the map $\psi \to V_\psi$.

Proof. The map is well-defined and continuous by [Trusiani 2022, Lemma 2.6]. Moreover, the injectivity follows from the fact that $V_{\psi_1} = V_{\psi_2}$ for $\psi_1, \psi_2 \in \bar{\mathcal{A}}$ implies $\psi_1 = \psi_2$ using Theorem 2.1 and the fact that $\mathcal{A} \subset \mathcal{M}^+$.

Finally, to conclude the proof it is enough to prove that $\psi_k \to \psi$ weakly assuming $V_{\psi_k} \to V_\psi$, and it is clearly sufficient to show that any subsequence of $\{\psi_k\}_{k\in\mathbb{N}}$ admits a subsequence weakly convergent to ψ. Moreover, since $\bar{\mathcal{A}}$ is totally ordered and $\succcurlyeq$ coincides with $\geq$ on $\mathcal{M}$, we may assume $\{\psi_k\}_{k\in\mathbb{N}}$ is a monotonic sequence. Then, up to considering a further subsequence, ψ_k converges almost everywhere to an element $\psi' \in \bar{\mathcal{A}}$ by compactness, and Lemma 2.12 implies that $V_{\psi'} = V_\psi$, i.e., $\psi' = \psi$. ☐

In the case $\mathcal{A} := \{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{M}^+$, we say that the $u_k \in \mathcal{E}^1(X,\omega,\psi_k)$ converge weakly to $P_{\psi_{\min}}$ where $\psi_{\min} \in \mathcal{M} \setminus \mathcal{M}^+$ if $|\sup_X u_k| \leq C$ for any $k \in \mathbb{N}$ and any weak accumulation point u of $\{u_k\}_{k\in\mathbb{N}}$ satisfies $u \preccurlyeq \psi_{\min}$. This definition is the most natural since $\mathrm{PSH}(X,\omega,\psi) = \mathcal{E}^1(X,\omega,\psi_{\min})$.

Lemma 3.13. *Let $\{u_k\}_{k\in\mathbb{N}} \subset X_{\mathcal{A}}$ be a sequence converging weakly to $u \in X_{\mathcal{A}}$. If $E_{P_\omega[u_k]}(u_k) \geq C$ uniformly, then $P_\omega[u_k] \to P_\omega[u]$ weakly.*

Proof. By Lemma 3.12 the convergence requested is equivalent to $V_{\psi_k} \to V_\psi$, where we set

$$\psi_k := P_\omega[u_k], \quad \psi := P_\omega[u].$$

Moreover, by a simple contradiction argument it is enough to show that any subsequence $\{\psi_{k_h}\}_{h\in\mathbb{N}}$ admits a subsequence $\{\psi_{k_{h_j}}\}_{j\in\mathbb{N}}$ such that $V_{\psi_{k_{h_j}}} \to V_\psi$. Thus up to considering a subsequence, by abuse of notation and by the lower semicontinuity $\liminf_{k\to\infty} V_{\psi_k} \geq V_\psi$ of [Darvas et al. 2018, Theorem 2.3], we may suppose by contradiction that $\psi_k \searrow \psi'$ for $\psi' \in \mathcal{M}$ such that $V_{\psi'} > V_\psi$. In particular, $V_{\psi'} > 0$ and $\psi' \succcurlyeq \psi$. Then by Proposition 2.10 and Remark 3.3, the sequence $\{P_\omega[\psi'](u_k)\}_{k\in\mathbb{N}}$ is bounded in $(\mathcal{E}^1(X,\omega,\psi'),d)$ and it belongs to $\mathcal{E}^1_{C'}(X,\omega,\psi')$ for some $C'\in\mathbb{R}$. Therefore, up to considering a subsequence, we have that $\{u_k\}_{k\in\mathbb{N}}$ converges weakly to an element $v\in\mathcal{E}^1(X,\omega,\psi)$ (which is the element u itself when $u\neq P_{\psi_{\min}}$), while the sequence $P_\omega[\psi'](u_k)$ converges weakly to an element $w\in\mathcal{E}^1(X,\omega,\psi')$. Thus the contradiction follows from $w\leq v$ since $\psi'\succcurlyeq\psi$, $V_{\psi'}>0$ and $\mathcal{E}^1(X,\omega,\psi')\cap\mathcal{E}^1(X,\omega,\psi)=\varnothing$. $\square$

Proposition 3.14. *Let $\{u_k\}_{k\in\mathbb{N}}\subset X_{\mathcal{A}}$ be a sequence converging weakly to $u\in X_{\mathcal{A}}$. Then*

$$\limsup_{k\to\infty} E_{P_\omega[u_k]}(u_k)\leq E_{P_\omega[u]}(u). \tag{11}$$

Proof. Let $\psi_k:=P_\omega[u_k]$ and $\psi:=P_\omega[u]\in\bar{\mathcal{A}}$. We may assume $\psi_k\neq\psi_{\min}$ for any $k\in\mathbb{N}$ if $\psi=\psi_{\min}$ and $V_{\psi_{\min}}=0$.

Moreover, we can suppose that $E_{\psi_k}(u_k)$ is bounded from below, which implies that $u_k\in\mathcal{E}^1_C(X,\omega,\psi_k)$ for a uniform constant C and that $\psi_k\to\psi$ weakly by Lemma 3.13. Thus since

$$E_{\psi_k}(u_k)=E_{\psi_k}(u_k-C)+CV_{\psi_k}$$

for any $k\in\mathbb{N}$, Lemma 3.12 implies that we may assume that $\sup_X u_k\leq 0$. Furthermore, since $\mathcal{A}$ is totally ordered, it is enough to show (11) when $\psi_k\to\psi$ a.e. monotonically.

If $\psi_k\searrow\psi$, setting $v_k:=(\sup\{u_j:j\geq k\})^*\in\mathcal{E}^1(X,\omega,\psi_k)$, we easily have

$$\limsup_{k\to\infty}E_{\psi_k}(u_k)\leq\limsup_{k\to\infty}E_{\psi_k}(v_k)\leq\limsup_{k\to\infty}E_{\psi}(P_\omega[\psi](v_k))$$

using the monotonicity of E_{ψ_k} and Proposition 2.10. Hence if $\psi=\psi_{\min}$ and $V_{\psi_{\min}}=0$, then

$$E_\psi(P_\omega[\psi](v_k))=0=E_\psi(u),$$

while otherwise the conclusion follows from Proposition 2.4 since $P_\omega[\psi](v_k)\searrow u$ by construction.

If instead $\psi_k\nearrow\psi$, fix $\epsilon>0$ and for any $k\in\mathbb{N}$ let $j_k\geq k$ such that

$$\sup_{j\geq k}E_{\psi_j}(u_j)\leq E_{\psi_{j_k}}(u_{j_k})+\epsilon.$$

Thus again by Proposition 2.10, $E_{\psi_{j_k}}(u_{j_k})\leq E_{\psi_l}(P_\omega[\psi_l](u_{j_k}))$ for any $l\leq j_k$. Moreover, assuming $E_{\psi_{j_k}}(u_{j_k})$ is bounded from below, $-E_{\psi_l}(P_\omega[\psi_l](u_{j_k}))=d(\psi_l,P_\omega[\psi_l](u_{j_k}))$ is uniformly bounded in l,k, which implies that $\sup_X P_\omega[\psi_l](u_{j_k})$ is uniformly bounded by Remark 3.3 since $V_{\psi_{j_k}}\geq a>0$ for $k\gg 0$ big enough. By compactness, up to considering a subsequence, we obtain $P_\omega[\psi_l](u_{j_k})\to v_l$ weakly where $v_l\in\mathcal{E}^1(X,\omega,\psi_l)$ by the upper semicontinuity of $E_{\psi_l}(\cdot)$ on $\mathcal{E}^1(X,\omega,\psi_l)$. Hence

$$\limsup_{k\to\infty}E_{\psi_k}(u_k)\leq\limsup_{k\to\infty}E_{\psi_l}(P_\omega[\psi_l](u_{j_k}))+\epsilon=E_{\psi_l}(v_l)+\epsilon$$

for any $l\in\mathbb{N}$. Moreover, by construction, $v_l\leq P_\omega[\psi_l](u)$ since $P_\omega[\psi_l](u_{j_k})\leq u_{j_k}$ for any k such that $j_k\geq l$ and $u_{j_k}\to u$ weakly. Therefore by the monotonicity of $E_{\psi_l}(\cdot)$ and by Proposition 2.11 (ii), we conclude that

$$\limsup_{k\to\infty}E_{\psi_k}(u_k)\leq\lim_{l\to\infty}E_{\psi_l}(P_\omega[\psi_l](u))+\epsilon=E_\psi(u)+\epsilon$$

letting $l\to\infty$. $\square$

As a consequence, defining

$$X_{\mathcal{A},C} := \bigsqcup_{\psi\in\bar{\mathcal{A}}} \mathcal{E}^1_C(X,\omega,\psi),$$

we get the following compactness result.

Proposition 3.15. *Let $C, a \in \mathbb{R}_{>0}$. The set*

$$X^a_{\mathcal{A},C} := X_{\mathcal{A},C} \cap \Bigg(\bigsqcup_{\psi\in\bar{\mathcal{A}}:V_\psi\geq a} \mathcal{E}^1(X,\omega,\psi)\Bigg)$$

is compact with respect to the weak topology.

Proof. It follows directly from the definition that

$$X^a_{\mathcal{A},C} \subset \big\{u\in \mathrm{PSH}(X,\omega) : \big|\sup_X u\big| \leq C'\big\},$$

where $C' := \max(C, C/a)$. Therefore by Proposition 8.5 in [Guedj and Zeriahi 2017], $X^a_{\mathcal{A},C}$ is weakly relatively compact. Finally Proposition 3.14 and Hartogs' lemma imply that $X^a_{\mathcal{A},C}$ is also closed with respect to the weak topology, concluding the proof. ☐

Remark 3.16. The whole set $X_{\mathcal{A},C}$ may not be weakly compact. Indeed, assuming $V_{\psi_{\min}} = 0$ and letting $\psi_k \in \bar{\mathcal{A}}$ such that $\psi_k \searrow \psi_{\min}$, the functions $u_k := \psi_k - 1/\sqrt{V_{\psi_k}}$ belong to $X_{\mathcal{A},V}$ for $V = \int_X \mathrm{MA}_\omega(0)$ since $E_{\psi_k}(u_k) = -\sqrt{V_{\psi_k}}$ but $\sup_X u_k = -1/\sqrt{V_{\psi_k}} \to -\infty$.

4. The action of measures on $\mathrm{PSH}(X,\omega)$

In this section we want to replace the action on $\mathrm{PSH}(X,\omega)$ defined in [Berman et al. 2013] given by a probability measure μ with an action which assumes finite values on elements $u\in \mathrm{PSH}(X,\omega)$ with ψ-relative minimal singularities, where $\psi = P_\omega[u]$ for almost all $\psi\in\mathcal{M}$. On the other hand, for any $\psi\in\mathcal{M}$ we want there to exist many measures μ whose action over $\{u\in \mathrm{PSH}(X,\omega) : P_\omega[u]=\psi\}$ is well-defined. The problem is that μ varies among *all probability measures* while ψ varies among *all model-type envelopes*. So it may happen that μ takes mass on nonpluripolar sets and that the unbounded locus of $\psi\in\mathcal{M}$ is very nasty.

Definition 4.1. Let μ be a probability measure on X. Then μ acts on $\mathrm{PSH}(X,\omega)$ through the functional $L_\mu : \mathrm{PSH}(X,\omega) \to \mathbb{R}\cup\{-\infty\}$ defined as $L_\mu(u) = -\infty$ if μ charges $\{P_\omega[u] = -\infty\}$, as

$$L_\mu(u) := \int_X (u - P_\omega[u])\mu$$

if u has $P_\omega[u]$-relative minimal singularities and μ does not charge $\{P_\omega[u] = -\infty\}$ and otherwise as

$$L_\mu(u) := \inf\{L_\mu(v) : v\in \mathrm{PSH}(X,\omega) \text{ with } P_\omega[u]\text{-relative minimal singularities}, v\geq u\}.$$

Proposition 4.2. *The following properties hold*:

(i) *L_μ is affine, i.e., it satisfies the scaling property $L_\mu(u+c) = L_\mu(u)+c$ for any $c\in\mathbb{R}$, $u\in \mathrm{PSH}(X,\omega)$.*

(ii) *L_μ is nondecreasing on $\{u\in \mathrm{PSH}(X,\omega) : P_\omega[u]=\psi\}$ for any $\psi\in\mathcal{M}$.*

(iii) $L_\mu(u) = \lim_{j\to\infty} L_\mu(\max(u, P_\omega[u]-j))$ *for any* $u \in \mathrm{PSH}(X,\omega)$.

(iv) *If* μ *is nonpluripolar*, *then* L_μ *is convex.*

(v) *If* μ *is nonpluripolar and* $u_k \to u$ *and* $P_\omega[u_k] \to P_\omega[u]$ *weakly as* $k\to\infty$, *then*

$$L_\mu(u) \geq \limsup_{k\to\infty} L_\mu(u_k).$$

(vi) *If* $u \in \mathcal{E}^1(X,\omega,\psi)$ *for* $\psi \in \mathcal{M}^+$, *then* $L_{\mathrm{MA}_\omega(u)/V_\psi}$ *is finite on* $\mathcal{E}^1(X,\omega,\psi)$.

Proof. The first two properties follow by definition.

For the third property, setting $\psi := P_\omega[u]$, clearly $L_\mu(u) \leq \lim_{j\to\infty} L_\mu(\max(u,\psi-j))$. Conversely, for any $v \geq u$ with ψ-relative minimal singularities $v \geq \max(u,\psi-j)$ for $j \gg 0$ big enough, by (ii) we get $L_\mu(v) \geq \lim_{j\to\infty} L_\mu(\max(u,\psi-j))$ which implies (iii) by definition.

Next we prove (iv). Let $v = \sum_{l=1}^m a_l u_l$ be a convex combination of elements $u_l \in \mathrm{PSH}(X,\omega)$. Without loss of generality we may assume $\sup_X v, \sup_X u_l \leq 0$. In particular, we have $L_\mu(v), L_\mu(u_l) \leq 0$.

Suppose $L_\mu(v) > -\infty$ (otherwise it is trivial) and let $\psi := P_\omega[v]$, $\psi_l := P_\omega[u_l]$. Then for any $C \in \mathbb{R}_{>0}$ it is easy to see that

$$\sum_{l=1}^m a_l P_\omega(u_l + C, 0) \leq P_\omega(v + C, 0) \leq \psi,$$

which leads to $\sum_{l=1}^m a_l \psi_l \leq \psi$ letting $C \to \infty$. Hence (iii) yields

$$-\infty < L_\mu(v) = \int_X (v-\psi)\mu \leq \sum_{l=1}^n a_l \int_X (u_l - \psi_l)\mu = \sum_{l=1}^n a_l L_\mu(u_l).$$

Property (v) easily follows from $\limsup_{k\to\infty} \max(u_k, P_\omega[u_k]-j) \leq \max(u, P_\omega[u]-j)$ and (iii), while the last property is a consequence of Lemma 3.8. □

Next, since for any $t \in [0,1]$ and any $u, v \in \mathcal{E}^1(X,\omega,\psi)$

$$\int_X (u-v)\,\mathrm{MA}_\omega(tu+(1-t)v) = (1-t)^n \int_X (u-v)\,\mathrm{MA}_\omega(v) + \sum_{j=1}^n \binom{n}{j} t^j (1-t)^{n-j} \int_X (u-v)\,\mathrm{MA}_\omega(u^j, v^{n-j})$$
$$\geq (1-t)^n \int_X (u-v)\,\mathrm{MA}_\omega(v) + (1-(1-t)^n) \int_X (u-v)\,\mathrm{MA}_\omega(u),$$

we can proceed exactly as in [Berman et al. 2013, Proposition 3.4] (see also [Guedj and Zeriahi 2007, Lemma 2.11]), replacing V_θ with ψ, to get the following result.

Proposition 4.3. *Let* $A \subset \mathrm{PSH}(X,\omega)$ *and let* $L : A \to \mathbb{R}\cup\{-\infty\}$ *be a convex and nondecreasing function satisfying the scaling property* $L(u+c) = L(u)+c$ *for any* $c \in \mathbb{R}$.

(i) *If* L *is finite-valued on a weakly compact convex set* $K \subset A$, *then* $L(K)$ *is bounded.*

(ii) *If* $\mathcal{E}^1(X,\omega,\psi) \subset A$ *and* L *is finite-valued on* $\mathcal{E}^1(X,\omega,\psi)$, *then*

$$\sup_{\{u \in \mathcal{E}^1_C(X,\omega,\psi) : \sup_X u \leq 0\}} |L| = O(C^{1/2}) \quad \text{as } C \to \infty.$$

4A. ***When is L_μ continuous?*** The continuity of L_μ is a hard problem. However, we can characterize its continuity on some weakly compact sets as the next theorem shows.

Theorem 4.4. *Let μ be a nonpluripolar probability measure, and let $K \subset \mathrm{PSH}(X, \omega)$ be a compact convex set such that L_μ is finite on K, the set $\{P_\omega[u] : u \in K\} \subset \mathcal{M}$ is totally ordered and its closure in $\mathrm{PSH}(X, \omega)$ has at most one element in $\mathcal{M} \setminus \mathcal{M}^+$. Suppose also that there exists $C \in \mathbb{R}$ such that $|E_{P_\omega[u]}(u)| \leq C$ for any $u \in K$. Then the following properties are equivalent*:

(i) *L_μ is continuous on K.*

(ii) *The map $\tau : K \to L^1(\mu)$, $\tau(u) := u - P_\omega[u]$ is continuous.*

(iii) *The set $\tau(K) \subset L^1(\mu)$ is uniformly integrable*, *i.e.*,

$$\int_{t=m}^{\infty} \mu\{u \leq P_\omega[u] - t\} \to 0$$

as $m \to \infty$, uniformly for $u \in K$.

Proof. We first observe that if $u_k \in K$ converges to $u \in K$, then by Lemma 3.13, $\psi_k \to \psi$, where we set $\psi_k := P_\omega[u_k]$ and $\psi := P_\omega[u]$.

Then we can proceed exactly as in [Berman et al. 2013, Theorem 3.10] to get the equivalence between (i) and (ii), (ii) $\Rightarrow$ (iii) and the fact that the graph of τ is closed. It is important to emphasize that (iii) is equivalent to saying that $\tau(K)$ is *weakly* relative compact by the Dunford–Pettis theorem, i.e., with respect to the weak topology on $L^1(\mu)$ induced by $L^\infty(\mu) = L^1(\mu)^*$.

Finally, assuming that (iii) holds it remains to prove (i). So, letting $u_k, u \in K$ such that $u_k \to u$, we have to show that $\int_X \tau(u_k)\mu \to \int_X \tau(u)\mu$. Since $\tau(K) \subset L^1(\mu)$ is bounded, unless considering a subsequence, we may suppose $\int_X \tau(u_k) \to L \in \mathbb{R}$. By Fatou's lemma,

$$L = \lim_{k\to\infty} \int_X \tau(u_k)\mu \leq \int_X \tau(u)\mu. \tag{12}$$

Then for any $k \in \mathbb{N}$ the closed convex envelope

$$C_k := \overline{\mathrm{Conv}\{\tau(u_j) : j \geq k\}}$$

is weakly closed in $L^1(\mu)$ by the Hahn–Banach theorem, which implies that C_k is weakly compact since it is contained in $\tau(K)$. Thus since C_k is a decreasing sequence of nonempty weakly compact sets, there exists $f \in \bigcap_{k\geq 1} C_k$ and there exist elements $v_k \in \mathrm{Conv}(u_j : j \geq k)$ given as finite convex combinations such that $\tau(v_k) \to f$ in $L^1(\mu)$. Moreover, by the closed graph property, $f = \tau(u)$ since $v_k \to u$ as a consequence of $u_k \to u$. On the other hand, by Proposition 4.2 (iv) we get

$$\int_X \tau(v_k)\mu \leq \sum_{l=1}^{m_k} a_{l,k} \int_X \tau(u_{k_l})\mu$$

if $v_k = \sum_{l=1}^{m_k} a_{l,k} u_{k_l}$. Hence $L \geq \int_X \tau(u)\mu$, which together with (12) implies $L = \int_X \tau(u)\mu$. □

Corollary 4.5. *Let $\psi \in \mathcal{M}^+$ and $\mu \in \mathcal{C}_{A,\psi}$. Then L_μ is continuous on $\mathcal{E}^1_C(X,\omega,\psi)$ for any $C \in \mathbb{R}_{>0}$. In particular, if $\mu = \mathrm{MA}_\omega(u)/V_\psi$ for $u \in \mathcal{E}^1(X,\omega,\psi)$ with ψ-relative minimal singularities, then L_μ is continuous on $\mathcal{E}^1_C(X,\omega,\psi)$ for any $C \in \mathbb{R}_{>0}$.*

Proof. With the notation of Theorem 4.4, $\tau(\mathcal{E}^1_C(X,\omega,\psi))$ is bounded in $L^2(\mu)$ by Lemma 3.11. Hence by Holder's inequality $\tau(\mathcal{E}^1_C(X,\omega,\psi))$ is uniformly integrable and Theorem 4.4 yields the continuity of L_μ on $\mathcal{E}^1_C(X,\omega,\psi)$ for any $C \in \mathbb{R}_{>0}$.

The last assertion follows directly from Proposition 3.10. □

The following lemma will be essential to prove Theorem A and Theorem B.

Lemma 4.6. *Let $\varphi \in \mathcal{H}_\omega$ and let $\mathcal{A} \subset \mathcal{M}$ be a totally ordered subset. Set also $v_\psi := P_\omega[\psi](\varphi)$ for any $\psi \in \mathcal{A}$. Then the actions $\{V_\psi L_{\mathrm{MA}_\omega(v_\psi)/V_\psi}\}_{\psi\in\mathcal{A}}$ take finite values and they are equicontinuous on any compact set $K \subset \mathrm{PSH}(X,\omega)$ such that $\{P_\omega[u] : u \in K\}$ is a totally ordered set whose closure in $\mathrm{PSH}(X,\omega)$ has at most one element in $\mathcal{M} \setminus \mathcal{M}^+$ and such that $|E_{P_\omega[u]}(u)| \le C$ uniformly for any $u \in K$. If $\psi \in \mathcal{M} \setminus \mathcal{M}^+$, for the action $V_\psi L_{\mathrm{MA}_\omega(v_\psi)/V_\psi}$ we mean the null action. In particular, if $\psi_k \to \psi$ monotonically almost everywhere and $\{u_k\}_{k\in\mathbb{N}} \subset K$ converges weakly to $u \in K$, then*

$$\int_X (u_k - P_\omega[u_k])\,\mathrm{MA}_\omega(v_{\psi_k}) \to \int_X (u - P_\omega[u])\,\mathrm{MA}_\omega(v_\psi). \tag{13}$$

Proof. By Theorem 2.2,

$$|V_\psi L_{\mathrm{MA}_\omega(v_\psi)/V_\psi}(u)| \le \int_X |u - P_\omega[u]|\,\mathrm{MA}_\omega(\varphi)$$

for any $u \in \mathrm{PSH}(X,\omega)$ and any $\psi \in \mathcal{A}$, so the actions in the statement assume finite values. Then the equicontinuity on any weak compact set $K \subset \mathrm{PSH}(X,\omega)$ satisfying the assumptions of the lemma follows from

$$V_\psi \big| L_{\mathrm{MA}_\omega(v_\psi)/V_\psi}(w_1) - L_{\mathrm{MA}_\omega(v_\psi)/V_\psi}(w_2) \big| \le \int_X |w_1 - P_\omega[w_1] - w_2 + P_\omega[w_2]|\,\mathrm{MA}_\omega(\varphi)$$

for any $w_1, w_2 \in \mathrm{PSH}(X,\omega)$ since $\mathrm{MA}_\omega(\varphi)$ is a volume form on X and $P_\omega[w_k] \to P_\omega[w]$ if $\{w_k\}_{k\in\mathbb{N}} \subset K$ converges to $w \in K$ under our hypothesis by Lemma 3.13.

For the second assertion, if $\psi_k \searrow \psi$ (resp. $\psi_k \nearrow \psi$ almost everywhere), letting $f_k, f \in L^\infty$ such that $\mathrm{MA}_\omega(v_{\psi_k}) = f_k\,\mathrm{MA}_\omega(\varphi)$ and $\mathrm{MA}_\omega(v_\psi) = f\,\mathrm{MA}_\omega(\varphi)$ (Theorem 2.2), we have $0 \le f_k \le 1$, $0 \le f \le 1$ and $\{f_k\}_{k\in\mathbb{N}}$ is a monotone sequence. Therefore $f_k \to f$ in L^p for any $p > 1$ as $k \to \infty$, which implies

$$\int_X (u - P_\omega[u])\,\mathrm{MA}_\omega(v_{\psi_k}) \to \int_X (u - P_\omega[u])\,\mathrm{MA}_\omega(v_\psi)$$

as $k \to \infty$ since $\mathrm{MA}_\omega(\varphi)$ is a volume form. Hence (13) follows since by the first part of the proof,

$$\int_X (u_k - P_\omega[u_k] - u + P_\omega[u])\,\mathrm{MA}_\omega(v_{\psi_k}) \to 0.$$ □

5. Theorem A

In this section we fix $\psi \in \mathcal{M}^+$ and, using a variational approach, we first prove the bijectivity of the Monge–Ampère operator between $\mathcal{E}^1_{\mathrm{norm}}(X,\omega,\psi)$ and $\mathcal{M}^1(X,\omega,\psi)$, and then we prove that it is actually a homeomorphism considering the strong topologies.

5A. ***Degenerate complex Monge–Ampère equations.*** Letting μ be a probability measure and $\psi \in \mathcal{M}$, we define the functional $F_{\mu,\psi} : \mathcal{E}^1(X, \omega, \psi) \to \mathbb{R} \cup \{-\infty\}$ as

$$F_{\mu,\psi}(u) := (E_\psi - V_\psi L_\mu)(u),$$

where we recall from Section 4 that

$$\begin{aligned} L_\mu(u) &= \lim_{j\to\infty} L_\mu(\max(u, \psi - j)) \\ &= \lim_{j\to\infty} \int_X (\max(u, \psi - j) - \psi)\mu. \end{aligned}$$

$F_{\mu,\psi}$ is clearly a translation invariant functional, and $F_{\mu,\psi} \equiv 0$ for any μ if $V_\psi = 0$.

Proposition 5.1. *Let μ be a probability measure, $\psi \in \mathcal{M}^+$ and let $F := F_{\mu,\psi}$. If L_μ is continuous then F is upper semicontinuous on $\mathcal{E}^1(X, \omega, \psi)$. Moreover, if L_μ is finite-valued on $\mathcal{E}^1(X, \omega, \psi)$, then there exist $A, B > 0$ such that*

$$F(v) \le -A\, d(\psi, v) + B$$

*for any $v \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$, i.e., F is **d-coercive**. In particular, F is upper semicontinuous on $\mathcal{E}^1(X, \omega, \psi)$ and d-coercive on $\mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$ if $\mu = \mathrm{MA}_\omega(u)/V_\psi$ for $u \in \mathcal{E}^1(X, \omega, \psi)$.*

Proof. If L_μ is continuous then F is easily upper semicontinuous by Proposition 2.4.

Then, since $d(\psi, v) = -E_\psi(v)$ on $\mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$, it is easy to check that the coercivity requested is equivalent to

$$\sup_{\mathcal{E}^1_C(X,\omega,\psi)\cap\mathcal{E}^1_{\mathrm{norm}}(X,\omega,\psi)} |L_\mu| \le \frac{(1-A)}{V_\psi} C + O(1),$$

which holds by Proposition 4.3 (ii).

Next assuming $\mu = \mathrm{MA}_\omega(u)/V_\psi$, it is sufficient to check the continuity of L_μ since L_μ is finite-valued on $\mathcal{E}^1(X, \omega, \psi)$ by Proposition 4.2. We may suppose without loss of generality that $u \le \psi$. By Proposition 3.7 and Remark 3.3, for any $C \in \mathbb{R}_{>0}$, L_μ restricted to $\mathcal{E}^1_C(X, \omega, \psi)$ is the uniform limit of L_{μ_j}, where $\mu_j := \mathrm{MA}_\omega(\max(u, \psi - j))$, since $I_\psi(\max(u, \psi - j), u) \to 0$ as $j \to \infty$. Therefore L_μ is continuous on $\mathcal{E}^1_C(X, \omega, \psi)$ because of the uniform limit of continuous functionals L_{μ_j} (Corollary 4.5). □

Because of the concavity of E_ψ, if $\mu = \mathrm{MA}_\omega(u)/V_\psi$ for $u \in \mathcal{E}^1(X, \omega, \psi)$ where $V_\psi > 0$, then

$$J^\psi_u(\psi) = F_{\mu,\psi}(u) = \sup_{\mathcal{E}^1(X,\omega,\psi)} F_{\mu,\psi},$$

i.e., u is a maximizer of $F_{\mu,\psi}$. The other way around also holds as the next result shows.

Proposition 5.2. *Let $\psi \in \mathcal{M}^+$ and let μ be a probability measure such that L_μ is finite-valued on $\mathcal{E}^1(X, \omega, \psi)$. Then $\mu = \mathrm{MA}_\omega(u)/V_\psi$ for $u \in \mathcal{E}^1(X, \omega, \psi)$ if and only if u is a maximizer of $F_{\mu,\psi}$.*

Proof. As said before, it is clear that $\mu = \mathrm{MA}_\omega(u)/V_\psi$ implies that u is a maximizer of $F_{\mu,\psi}$. Conversely, if u is a maximizer of $F_{\mu,\psi}$, then by [Darvas et al. 2018, Theorem 4.22], $\mu = \mathrm{MA}_\omega(u)/V_\psi$. □

Similarly to [Berman et al. 2013] we thus define the *ψ-relative energy* for $\psi \in \mathcal{M}$ of a probability measure μ as

$$E^*_\psi(\mu) := \sup_{u \in \mathcal{E}^1(X,\omega,\psi)} F_{\mu,\psi}(u),$$

i.e., essentially as the Legendre transform of E_ψ. It takes nonnegative values ($F_{\mu,\psi}(\psi) = 0$), and it is easy to check that E^*_ψ is a convex function.

Moreover, defining

$$\mathcal{M}^1(X, \omega, \psi) := \{V_\psi \mu : \mu \text{ is a probability measure satisfying } E^*_\psi(\mu) < \infty\},$$

we note that $\mathcal{M}^1(X, \omega, \psi)$ consists only of the null measure if $V_\psi = 0$, while if $V_\psi > 0$, any probability measure μ such that $V_\psi \mu \in \mathcal{M}^1(X, \omega, \psi)$ is nonpluripolar as the next lemma shows.

Lemma 5.3. *Let $A \subset X$ be a (locally) pluripolar set. Then there exists $u \in \mathcal{E}^1(X, \omega, \psi)$ such that $A \subset \{u = -\infty\}$. In particular, if $V_\psi \mu \in \mathcal{M}^1(X, \omega, \psi)$ for $\psi \in \mathcal{M}^+$, then μ is nonpluripolar.*

Proof. By [Berman et al. 2013, Corollary 2.11], there exists $\varphi \in \mathcal{E}^1(X, \omega)$ such that $A \subset \{\varphi = -\infty\}$. Therefore setting $u := P_\omega[\psi](\varphi)$ proves the first part.

Next, let $V_\psi \mu \in \mathcal{M}^1(X, \omega, \psi)$ for $\psi \in \mathcal{M}^+$ and let μ be a probability measure, and assume by contradiction that μ takes mass on a pluripolar set A. Then by the first part of the proof there exists $u \in \mathcal{E}^1(X, \omega, \psi)$ such that $A \subset \{u = -\infty\}$. On the other hand, since $V_\psi \mu \in \mathcal{M}^1(X, \omega, \psi)$, by definition μ does not charge $\{\psi = -\infty\}$. Thus by Proposition 4.2 (iii) we obtain $L_\mu(u) = -\infty$, a contradiction. □

We now prove that the Monge–Ampère operator is a bijection between $\mathcal{E}^1(X, \omega, \psi)$ and $\mathcal{M}^1(X, \omega, \psi)$.

Lemma 5.4. *Let $\psi \in \mathcal{M}^+$ and $\mu \in \mathcal{C}_{A,\psi}$, where $A \in \mathbb{R}$. Then there exists $u \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$ maximizing $F_{\mu,\psi}$.*

Proof. By Lemma 3.11, L_μ is finite-valued on $\mathcal{E}^1(X, \omega, \psi)$, and it is continuous on $\mathcal{E}^1_C(X, \omega, \psi)$ for any $C \in \mathbb{R}$ thanks to Corollary 4.5. Therefore it follows from Proposition 5.1 that $F_{\mu,\psi}$ is upper semicontinuous and d-coercive on $\mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$. Hence $F_{\mu,\psi}$ admits a maximizer $u \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$ as an easy consequence of the weak compactness of $\mathcal{E}^1_C(X, \omega, \psi)$. □

Proposition 5.5. *Let $\psi \in \mathcal{M}^+$. Then the Monge–Ampère map* $\mathrm{MA} : \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi) \to \mathcal{M}^1(X, \omega, \psi)$, $u \to \mathrm{MA}(u)$, *is bijective. Furthermore, if $V_\psi \mu = \mathrm{MA}_\omega(u) \in \mathcal{M}^1(X, \omega, \psi)$ for $u \in \mathcal{E}^1(X, \omega, \psi)$, then any maximizing sequence $u_k \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$ for $F_{\mu,\psi}$ necessarily converges weakly to u.*

Proof. The proof is inspired by [Berman et al. 2013, Theorem 4.7].

The map is well-defined as a consequence of Proposition 5.1, i.e., $\mathrm{MA}_\omega(u) \in \mathcal{M}^1(X, \omega, \psi)$ for any $u \in \mathcal{E}^1(X, \omega, \psi)$. Moreover, the injectivity follows from [Darvas et al. 2021a, Theorem 4.8].

Let $u_k \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$ be a sequence such that $F_{\mu,\psi}(u_k) \nearrow \sup_{\mathcal{E}^1(X,\omega,\psi)} F_{\mu,\psi}$, where $\mu = \mathrm{MA}_\omega(u)/V_\psi$ is a probability measure and $u \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$. Up to considering a subsequence, we may also assume that $u_k \to v \in \mathrm{PSH}(X, \omega)$. Then, by the upper semicontinuity and d-coercivity of $F_{\mu,\psi}$ (Proposition 5.1), it follows that $v \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$ and $F_{\mu,\psi}(v) = \sup_{\mathcal{E}^1(X,\omega,\psi)} F_{\mu,\psi}$. Thus by Proposition 5.2 we get $\mu = \mathrm{MA}_\omega(v)/V_\psi$. Hence $v = u$ since $\sup_X v = \sup_X u = 0$.

Then let μ be a probability measure such that $V_\psi \mu \in \mathcal{M}^1(X, \omega, \psi)$. Again by Proposition 5.2, to prove the existence of $u \in \mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$ such that $\mu = \text{MA}_\omega(u)/V_\psi$ it is sufficient to check that $F_{\mu,\psi}$ admits a maximum over $\mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$. Moreover by Proposition 5.1, we also know that $F_{\mu,\psi}$ is d-coercive on $\mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$. Thus if there exists a constant $A > 0$ such that $\mu \in \mathcal{C}_{A,\psi}$, then Corollary 4.5 leads to the upper semicontinuity of $F_{\mu,\psi}$, which clearly implies that $V_\psi \mu = \text{MA}_\omega(u)$ for $u \in \mathcal{E}^1(X, \omega, \psi)$ since $\mathcal{E}^1_C(X, \omega, \psi) \subset \text{PSH}(X, \omega)$ is compact for any $C \in \mathbb{R}_{>0}$.

In the general case, by [Darvas et al. 2018, Lemma 4.26] (see also [Cegrell 1998]), μ is absolutely continuous with respect to $\nu \in \mathcal{C}_{1,\psi}$ using also that μ is a nonpluripolar measure (Lemma 5.3). Therefore, letting $f \in L^1(\nu)$ such that $\mu = f\nu$, we define for any $k \in \mathbb{N}$

$$\mu_k := (1 + \epsilon_k) \min(f, k)\nu,$$

where the $\epsilon_k > 0$ are chosen such that μ_k is a probability measure, noting that $(1 + \epsilon_k) \min(f, k) \to f$ in $L^1(\nu)$. Then by Lemma 5.4 it follows that $\mu_k = \text{MA}_\omega(u_k)/V_\psi$ for $u_k \in \mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$.

Moreover, by weak compactness we may also assume that $u_k \to u \in \text{PSH}(X, \omega)$, without loss of generality. Note that $u \le \psi$ since $u_k \le \psi$ for any $k \in \mathbb{N}$. Then by [Darvas et al. 2021a, Lemma 2.8] we obtain

$$\text{MA}_\omega(u) \ge V_\psi f \nu = V_\psi \mu,$$

which implies $\text{MA}_\omega(u) = V_\psi \mu$ by [Witt Nyström 2019] since u is more singular than ψ and μ is a probability measure. It remains to prove that $u \in \mathcal{E}^1(X, \omega, \psi)$.

It is not difficult to see that $\mu_k \le 2\mu$ for $k \gg 0$, thus Proposition 4.3 implies that there exists a constant $B > 0$ such that

$$\sup_{\mathcal{E}^1_C(X,\omega,\psi)} |L_{\mu_k}| \le 2 \sup_{\mathcal{E}^1_C(X,\omega,\psi)} |L_\mu| \le 2B(1 + C^{1/2})$$

for any $C \in \mathbb{R}_{>0}$. Therefore

$$J^\psi_{u_k}(\psi) = E_\psi(u_k) + V_\psi |L_{\mu_k}(u_k)| \le \sup_{C>0} (2V_\psi B(1 + C^{1/2}) - C),$$

and Lemma 3.1 yields $d(\psi, u_k) \le D$ for a uniform constant D, i.e., $u_k \in \mathcal{E}^1_{D'}(X, \omega, \psi)$ for any $k \in \mathbb{N}$ for a uniform constant D'; see Remark 3.3. Hence since $\mathcal{E}^1_{D'}(X, \omega, \psi)$ is weakly compact we obtain $u \in \mathcal{E}^1_{D'}(X, \omega, \psi)$. □

5B. *Proof of Theorem A.* We further explore the properties of the strong topology on $\mathcal{E}^1(X, \omega, \psi)$.

By Proposition 3.6, the strong convergence implies the weak convergence. Moreover, the strong topology is the coarsest refinement of the weak topology such that $E_\psi(\,\cdot\,)$ becomes continuous.

Proposition 5.6. *Let $\psi \in \mathcal{M}^+$ and $u_k, u \in \mathcal{E}^1(X, \omega, \psi)$. Then $u_k \to u$ strongly if and only if $u_k \to u$ weakly and $E_\psi(u_k) \to E_\psi(u)$.*

Proof. Assume $u_k \to u$ weakly and $E_\psi(u_k) \to E_\psi(u)$. Then $w_k := (\sup\{u_j : j \ge k\})^* \in \mathcal{E}^1(X, \omega, \psi)$ and it decreases to u. Thus by Proposition 2.4, $E_\psi(w_k) \to E_\psi(u)$ and

$$d(u_k, u) \le d(u_k, w_k) + d(w_k, u) = 2E_\psi(w_k) - E_\psi(u_k) - E_\psi(u) \to 0.$$

Conversely, assuming that $d(u_k, u) \to 0$, we immediately get that $u_k \to u$ weakly as said above; see Proposition 3.6. Moreover, $\sup_X u_k, \sup_X u \le A$ uniformly for a constant $A \in \mathbb{R}$. Thus

$$|E_\psi(u_k) - E_\psi(u)| = |d(\psi + A, u_k) - d(\psi + A, u)| \le d(u_k, u) \to 0. \qquad \square$$

We also observe that the strong convergence implies the convergence in ψ'-capacity for any $\psi' \in \mathcal{M}^+$.

Proposition 5.7. *Let $\psi \in \mathcal{M}^+$ and $u_k, u \in \mathcal{E}^1(X, \omega, \psi)$ such that $d(u_k, u) \to 0$. Then there exists a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ such that $w_j := (\sup\{u_{k_h} : h \ge j\})^*$ and $v_j := P_\omega(u_{k_j}, u_{k_{j+1}}, \dots)$ belong to $\mathcal{E}^1(X, \omega, \psi)$ and converge monotonically almost everywhere to u. In particular, $u_k \to u$ in ψ'-capacity for any $\psi' \in \mathcal{M}^+$, and $\mathrm{MA}_\omega(u_k^j, \psi^{n-j}) \to \mathrm{MA}_\omega(u^j, \psi^{n-j})$ weakly for any $j = 0, \dots, n$.*

Proof. Since the strong convergence implies the weak convergence by Proposition 5.6, it is clear that $w_k \in \mathcal{E}^1(X, \omega, \psi)$ and that it decreases to u. In particular, up to considering a subsequence we may assume that $d(u_k, w_k) \le 1/2^k$ for any $k \in \mathbb{N}$.

Next for any $j \ge k$, set $v_{k,j} := P_\omega(u_k, \dots, u_j) \in \mathcal{E}^1(X, \omega, \psi)$ and $v_{k,j}^u := P_\omega(v_{k,j}, u) \in \mathcal{E}^1(X, \omega, \psi)$. Then it follows from Proposition 2.4 and [Darvas et al. 2018, Lemma 3.7] that

$$\begin{aligned} d(u, v_{k,j}^u) &\le \int_X (u - v_{k,j}^u)\, \mathrm{MA}_\omega(v_{k,j}^u) \le \int_{\{v_{k,j}^u = v_{k,j}\}} (u - v_{k,j})\, \mathrm{MA}_\omega(v_{k,j}) \\ &\le \sum_{s=k}^{j} \int_X (w_s - u_s)\, \mathrm{MA}_\omega(u_s) \le (n+1) \sum_{s=k}^{j} d(w_s, u_s) \le \frac{n+1}{2^{k-1}}. \end{aligned}$$

Therefore by Proposition 3.15, $v_{k,j}^u$ decreases (hence converges strongly) to a function $\phi_k \in \mathcal{E}^1(X, \omega, \psi)$ as $j \to \infty$. Similarly we also observe that

$$d(v_{k,j}, v_{k,j}^u) \le \int_{\{v_{k,j}^u = u\}} (v_{k,j} - u)\, \mathrm{MA}_\omega(u) \le \int_X |v_{k,1} - u|\, \mathrm{MA}_\omega(u) \le C$$

uniformly in j by Corollary 3.5. Hence by definition, $d(u, v_{k,j}) \le C + (n+1)/2^{k-1}$, i.e., $v_{k,j}$ decreases and converges strongly as $j \to \infty$ to the function $v_k = P_\omega(u_k, u_{k+1}, \dots) \in \mathcal{E}^1(X, \omega, \psi)$, again by Proposition 3.15. Moreover, by construction, $u_k \ge v_k \ge \phi_k$ since $v_k \le v_{k,j} \le u_k$ for any $j \ge k$. Hence

$$d(u, v_k) \le d(u, \phi_k) \le \frac{n+1}{2^{k-1}} \to 0$$

as $k \to \infty$, i.e., $v_k \nearrow u$ strongly.

The convergence in ψ'-capacity for $\psi' \in \mathcal{M}^+$ is now clearly an immediate consequence. Indeed by an easy contradiction argument it is enough to prove that any arbitrary subsequence, which we will keep denoting by $\{u_k\}_{k\in\mathbb{N}}$ for the sake of simplicity, admits a further subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ converging in ψ'-capacity to u. Thus taking the subsequence satisfying $v_j \le u_{k_j} \le w_j$, where v_j, w_j are the monotonic sequences of the first part of the proposition, the convergence in ψ'-capacity follows from the inclusions

$$\{|u - u_{k_j}| > \delta\} = \{u - u_{k_j} > \delta\} \cup \{u_{k_j} - u > \delta\} \subset \{u - v_j > \delta\} \cup \{w_j - u > \delta\}$$

for any $\delta > 0$. Finally Lemma 2.12 gives the weak convergence of the measures. $\square$

We now endow the set $\mathcal{M}^1(X,\omega,\psi)=\{V_\psi\mu:\mu$ is a probability measure satisfying $E^*_\psi(\mu)<+\infty\}$ (Section 5A) with its natural strong topology given as the coarsest refinement of the weak topology such that $E^*_\psi(\,\cdot\,)$ becomes continuous and prove Theorem A.

Theorem A. *Let $\psi\in\mathcal{M}^+$. Then*

$$\mathrm{MA}_\omega:(\mathcal{E}^1_{\mathrm{norm}}(X,\omega,\psi),d)\to(\mathcal{M}^1(X,\omega,\psi),\text{strong})$$

is a homeomorphism.

Proof. The map is bijective as an immediate consequence of Proposition 5.5.

Next, letting the $u_k\in\mathcal{E}^1_{\mathrm{norm}}(X,\omega,\psi)$ converge strongly to $u\in\mathcal{E}^1_{\mathrm{norm}}(X,\omega,\psi)$, Proposition 5.7 gives the weak convergence of $\mathrm{MA}_\omega(u_k)\to\mathrm{MA}_\omega(u)$ as $k\to\infty$. Moreover, since $E^*_\psi(\mathrm{MA}_\omega(v)/V_\psi)=J^\psi_v(\psi)$ for any $v\in\mathcal{E}^1(X,\omega,\psi)$, we get

$$\begin{aligned}|E^*_\psi(\mathrm{MA}_\omega(u_k)/V_\psi)-E^*_\psi(\mathrm{MA}_\omega(u)/V_\psi)|&\\ \le|E_\psi(u_k)-E_\psi(u)|&+\left|\int_X(\psi-u_k)\,\mathrm{MA}_\omega(u_k)-\int_X(\psi-u)\,\mathrm{MA}_\omega(u)\right|\\ \le|E_\psi(u_k)-E_\psi(u)|&+\left|\int_X(\psi-u_k)(\mathrm{MA}_\omega(u_k)-\mathrm{MA}_\omega(u))\right|+\int_X|u_k-u|\,\mathrm{MA}_\omega(u).\qquad(14)\end{aligned}$$

Hence $\mathrm{MA}_\omega(u_k)\to\mathrm{MA}_\omega(u)$ strongly in $\mathcal{M}^1(X,\omega,\psi)$ since each term on the right-hand side of (14) goes to 0 as $k\to+\infty$, combining Proposition 5.6, Proposition 3.7 and Corollary 3.5, and recalling that by Proposition 3.4, $I_\psi(u_k,u)\to0$ as $k\to\infty$.

Conversely, suppose that $\mathrm{MA}_\omega(u_k)\to\mathrm{MA}_\omega(u)$ strongly in $\mathcal{M}^1(X,\omega,\psi)$, where $u_k,u\in\mathcal{E}^1_{\mathrm{norm}}(X,\omega,\psi)$. Then, letting $\{\varphi_j\}_{j\in\mathbb{N}}\subset\mathcal{H}_\omega$ such that $\varphi_j\searrow u$ [Błocki and Kołodziej 2007] and setting $v_j:=P_\omega[\psi](\varphi_j)$, by Lemma 3.1,

$$\begin{aligned}(n+1)I_\psi(u_k,v_j)&\le E_\psi(u_k)-E_\psi(v_j)+\int_X(v_j-u_k)\,\mathrm{MA}_\omega(u_k)\\&=E^*_\psi(\mathrm{MA}_\omega(u_k)/V_\psi)-E^*_\psi(\mathrm{MA}_\omega(v_j)/V_\psi)+\int_X(v_j-\psi)(\mathrm{MA}_\omega(u_k)-\mathrm{MA}_\omega(v_j)).\qquad(15)\end{aligned}$$

By construction and the first part of the proof, it follows that $E^*_\psi(\mathrm{MA}_\omega(u_k)/V_\psi)-E^*_\psi(\mathrm{MA}_\omega(v_j)/V_\psi)\to0$ as $k,j\to\infty$. Setting $f_j:=v_j-\psi$, we want to prove

$$\limsup_{k\to\infty}\int_X f_j\,\mathrm{MA}_\omega(u_k)=\int_X f_j\,\mathrm{MA}_\omega(u),$$

which would imply $\limsup_{j\to\infty}\limsup_{k\to\infty}I_\psi(u_k,v_j)=0$ since $\int_X f_j(\mathrm{MA}_\omega(u)-\mathrm{MA}_\omega(v_j))\to0$ as a consequence of Propositions 3.7 and 3.4.

We observe that $\|f_j\|_{L^\infty}\le\|\varphi_j\|_{L^\infty}$ by Proposition 2.10, and we denote by $\{f_j^s\}_{s\in\mathbb{N}}\subset C^\infty$ a sequence of smooth functions converging in capacity to f_j such that $\|f_j^s\|_{L^\infty}\le2\|f_j\|_{L^\infty}$. Here we briefly recall how to construct such a sequence. Let $\{g_j^s\}_{s\in\mathbb{N}}$ be the sequence of bounded functions converging in capacity to f_j defined as $g_j^s:=\max(v_j,-s)-\max(\psi,-s)$. We have that $\|g_j^s\|_{L^\infty}\le\|f_j\|_{L^\infty}$ and that $\max(v_j,-s),\ \max(\psi,-s)\in\mathrm{PSH}(X,\omega)$. By a regularization process (see [Błocki and Kołodziej 2007]) and a diagonal argument we can now construct a sequence $\{f_j^s\}_{j\in\mathbb{N}} \subset C^\infty$ converging in capacity to f_j such that $\|f_j^s\|_{L^\infty} \leq 2\|g_j^s\| \leq 2\|f_j\|_{L^\infty}$, where $f_j^s = v_j^s - \psi^s$ with v_j^s, ψ^s quasi-psh functions decreasing to v_j, ψ, respectively.

Then letting $\delta > 0$ we have

$$\begin{aligned}\int_X (f_j - f_j^s)\,\mathrm{MA}_\omega(u_k) &\leq \delta V_\psi + 3\|\varphi_j\|_{L^\infty}\int_{\{f_j - f_j^s > \delta\}} \mathrm{MA}_\omega(u_k)\\ &\leq \delta V_\psi + 3\|\varphi_j\|_{L^\infty}\int_{\{\psi^s - \psi > \delta\}} \mathrm{MA}_\omega(u_k)\end{aligned}$$

from the trivial inclusion $\{f_j - f_j^s > \delta\} \subset \{\psi^s - \psi > \delta\}$. Therefore

$$\begin{aligned}\limsup_{s\to\infty}\limsup_{k\to\infty}\int_X (f_j - f_j^s)\,\mathrm{MA}_\omega(u_k) &\leq \delta V_\psi + \limsup_{s\to\infty}\limsup_{k\to\infty}\int_{\{\psi^s-\psi\geq\delta\}} \mathrm{MA}_\omega(u_k)\\ &\leq \delta V_\psi + \limsup_{s\to\infty}\int_{\{\psi^s-\psi\geq\delta\}} \mathrm{MA}_\omega(u) = \delta V_\psi,\end{aligned}$$

where we used that $\{\psi^s - \psi \geq \delta\}$ is a closed set in the plurifine topology. Hence since $f_j^s \in C^\infty$ we obtain

$$\begin{aligned}\limsup_{k\to\infty}\int_X f_j\,\mathrm{MA}_\omega(u_k) &= \limsup_{s\to\infty}\limsup_{k\to\infty}\left(\int_X (f_j - f_j^s)\,\mathrm{MA}_\omega(u_k) + \int_X f_j^s\,\mathrm{MA}_\omega(u_k)\right)\\ &\leq \limsup_{s\to\infty}\int_X f_j^s\,\mathrm{MA}_\omega(u) = \int_X f_j\,\mathrm{MA}_\omega(u),\end{aligned}$$

which as said above implies $I_\psi(u_k, v_j) \to 0$ letting $k, j \to \infty$ in this order.

Next we obtain $u_k \in \mathcal{E}^1_C(X,\omega,\psi)$ for some $C \in \mathbb{N}$ big enough since $J^\psi_{u_k}(\psi) = E^*_\psi(\mathrm{MA}_\omega(u_k)/V_\psi)$, again by Lemma 3.1. In particular, up to considering a subsequence, $u_k \to w \in \mathcal{E}^1_{\mathrm{norm}}(X,\omega,\psi)$ weakly by Proposition 3.15. Observe also that by Proposition 3.7,

$$\left|\int_X (\psi - u_k)(\mathrm{MA}_\omega(v_j) - \mathrm{MA}_\omega(u_k))\right| \to 0 \tag{16}$$

as $k, j \to \infty$ in this order. Moreover, by Proposition 3.14 and Lemma 4.6,

$$\begin{aligned}&\limsup_{k\to\infty}\left(E^*_\psi(\mathrm{MA}_\omega(u_k)/V_\psi) + \int_X (\psi - u_k)(\mathrm{MA}_\omega(v_j) - \mathrm{MA}_\omega(u_k))\right)\\ &\qquad= \limsup_{k\to\infty}\left(E_\psi(u_k) + \int_X (\psi - u_k)\,\mathrm{MA}_\omega(v_j)\right) \leq E_\psi(w) + \int_X (\psi - w)\,\mathrm{MA}_\omega(v_j).\end{aligned} \tag{17}$$

Therefore combining (16) and (17) with the strong convergence of v_j to u we obtain

$$\begin{aligned}E_\psi(u) + \int_X (\psi - u)\,\mathrm{MA}_\omega(u) &= \lim_{k\to\infty} E^*_\psi(\mathrm{MA}_\omega(u_k)/V_\psi)\\ &\leq \limsup_{j\to\infty}\left(E_\psi(w) + \int_X (\psi - w)\,\mathrm{MA}_\omega(v_j)\right)\\ &= E_\psi(w) + \int_X (\psi - w)\,\mathrm{MA}_\omega(u),\end{aligned}$$

i.e., w is a maximizer of $F_{\mathrm{MA}_\omega(u)/V_\psi,\psi}$. Hence $w = u$ (Proposition 5.5), i.e., $u_k \to u$ weakly. Furthermore, again by Lemma 3.1 and Lemma 4.6,

$$\begin{aligned}\limsup_{k\to\infty}(E_\psi(v_j) - E_\psi(u_k)) &\le \limsup_{k\to\infty}\biggl(\frac{n}{n+1} I_\psi(u_k, v_j) + \biggl|\int_X (u_k - v_j)\,\mathrm{MA}_\omega(v_j)\biggr|\biggr) \\ &\le \biggl|\int_X (u - v_j)\,\mathrm{MA}_\omega(v_j)\biggr| + \limsup_{k\to\infty}\frac{n}{n+1} I_\psi(u_k, v_j). \qquad (18)\end{aligned}$$

Finally letting $j \to \infty$, since $v_j \searrow u$ strongly, we obtain $\liminf_{j\to\infty} E_\psi(u_k) \ge \lim_{j\to\infty} E_\psi(v_j) = E_\psi(u)$, which implies that $E_\psi(u_k) \to E_\psi(u)$ and that $u_k \to u$ strongly by Proposition 5.6. □

The main difference between the proof of Theorem A and the proof of the same result in the absolute setting, i.e., when $\psi = 0$, is that for fixed $u \in \mathcal{E}^1(X, \omega, \psi)$ the action

$$\mathcal{M}^1(X, \omega, \psi) \ni \mathrm{MA}_\omega(v) \to \int_X (u - \psi)\,\mathrm{MA}_\omega(v)$$

is not a priori continuous with respect to the weak topologies of measures even if we restrict the action on $\mathcal{M}^1_C(X, \omega, \psi) := \{V_\psi \mu : E^*_\psi(\mu) \le C\}$ for $C \in \mathbb{R}$, while in the absolute setting this is given by [Berman et al. 2019, Proposition 1.7], where the authors used the fact that any $u \in \mathcal{E}^1(X, \omega)$ can be approximated inside the class $\mathcal{E}^1(X, \omega)$ by a sequence of continuous functions.

6. Strong topologies

In this section we investigate the strong topology on $X_\mathcal{A}$ in detail, proving that it is the coarsest refinement of the weak topology such that $E_\cdot(\,\cdot\,)$ becomes continuous (Theorem 6.2) and proving that the strong convergence implies the convergence in ψ-capacity for any $\psi \in \mathcal{M}^+$ (Theorem 6.3), i.e., we extend all the typical properties of the L^1-metric geometry to the bigger space $X_\mathcal{A}$, justifying further the construction of the distance $d_\mathcal{A}$ [Trusiani 2022] and its naturality. Moreover, we define the set $Y_\mathcal{A}$ and prove Theorem B.

6A. *About* $(X_\mathcal{A}, d_\mathcal{A})$. First we prove that the strong convergence in $X_\mathcal{A}$ implies the weak convergence, recalling that for the weak convergence of $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ to $P_{\psi_{\min}}$, where $\psi_{\min} \in \mathcal{M}$ with $V_{\psi_{\min}} = 0$, we mean that $|\sup_X u_k| \le C$ and that any weak accumulation point of $\{u_k\}_{k\in\mathbb{N}}$ is more singular than $\psi_{\min}$.

Proposition 6.1. *Let $u_k, u \in X_\mathcal{A}$ such that $u_k \to u$ strongly. If $u \ne P_{\psi_{\min}}$, then $u_k \to u$ weakly. If instead $u = P_{\psi_{\min}}$, then the following dichotomy holds*:

(i) *$u_k \to P_{\psi_{\min}}$ weakly.*

(ii) $\limsup_{k\to\infty}|\sup_X u_k| = +\infty$.

Proof. The dichotomy for the case $u = P_{\psi_{\min}}$ follows by definition. Indeed, if $|\sup_X u_k| \le C$ and $d_\mathcal{A}(u_k, u) \to 0$ as $k \to \infty$, then $V_{\psi_k} \to V_{\psi_{\min}} = 0$ by Proposition 2.11 (iv), which implies that $\psi_k \to \psi_{\min}$ by Lemma 3.12. Hence any weak accumulation point u of $\{u_k\}_{k\in\mathbb{N}}$ satisfies $u \le \psi_{\min} + C$.

Thus, let $\psi_k, \psi \in \mathcal{A}$ such that $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ and $u \in \mathcal{E}^1(X, \omega, \psi)$ where $\psi \in \mathcal{M}^+$. Observe that

$$d(u_k, \psi_k) \le d_\mathcal{A}(u_k, u) + d(u, \psi) + d_\mathcal{A}(\psi, \psi_k) \le A$$

for a uniform constant $A > 0$ by Proposition 2.11 (iv).

On the other hand, by [Błocki and Kołodziej 2007], for any $j \in \mathbb{N}$ there exists $h_j \in \mathcal{H}_\omega$ such that $h_j \geq u$, $\|h_j - u\|_{L^1} \leq 1/j$ and $d(u, P_\omega[\psi](h_j)) \leq 1/j$. In particular, by the triangle inequality and Proposition 2.11, we have

$$\begin{aligned}\limsup_{k\to\infty} d(P_\omega[\psi_k](h_j), \psi_k) &\leq \limsup_{k\to\infty}\Big(d_{\mathcal{A}}(P_\omega[\psi_k](h_j), P_\omega[\psi](h_j)) + \frac{1}{j} + d(u, \psi) + d(\psi, \psi_k)\Big)\\ &\leq d(u, \psi) + \frac{1}{j}, \end{aligned}\tag{19}$$

Similarly, again by the triangle inequality and Proposition 2.11,

$$\limsup_{k\to\infty} d(u_k, P_\omega[\psi_k](h_j)) \leq \limsup_{k\to\infty}\Big(d_{\mathcal{A}}(P_\omega[\psi_k](h_j), P_\omega[\psi](h_j)) + \frac{1}{j} + d_{\mathcal{A}}(u, u_k)\Big) \leq \frac{1}{j} \tag{20}$$

and

$$\begin{aligned}\limsup_{k\to\infty}\|u_k - u\|_{L^1} &\leq \limsup_{k\to\infty}(\|u_k - P_\omega[\psi_k](h_j)\|_{L^1} + \|P_\omega[\psi_k](h_j) - P_\omega[\psi](h_j)\|_{L^1} + \|P_\omega[\psi](h_j) - u\|_{L^1})\\ &\leq \frac{1}{j} + \limsup_{k\to\infty}\|u_k - P_\omega[\psi_k](h_j)\|_{L^1},\end{aligned}\tag{21}$$

where we also used Lemma 2.14. In particular, we deduce that $d(\psi_k, P_\omega[\psi_k](h_j)),\ d(\psi_k, u_k) \leq C$ for a uniform constant $C \in \mathbb{R}$ from (19) and (20). Next let $\phi_k \in \mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$ be the unique solution of $\mathrm{MA}_\omega(\phi_k) = (V_{\psi_k}/V_0)\,\mathrm{MA}_\omega(0)$, and observe that by Proposition 2.4,

$$d(\psi_k, \phi_k) = -E_{\psi_k}(\phi_k) \leq \int_X (\psi_k - \phi_k)\,\mathrm{MA}_\omega(\phi_k) \leq \frac{V_{\psi_k}}{V_0}\int_X |\phi_k|\,\mathrm{MA}_\omega(0) \leq \|\phi_k\|_{L^1} \leq C',$$

since ϕ_k belongs to a compact (hence bounded) subset of $\mathrm{PSH}(X, \omega) \subset L^1$. Therefore, since $V_{\psi_k} \geq a > 0$ for $k \gg 0$ big enough, by Proposition 3.6 it follows that there exists a continuous increasing function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $f(0) = 0$ such that

$$\|u_k - P_\omega[\psi_k](h_j)\|_{L^1} \leq f(d(u_k, P_\omega[\psi_k](h_j)))$$

for any k, j big enough. Hence, combining (20) and (21), the convergence requested follows letting $k, j \to +\infty$ in this order. □

We can now prove the important characterization of the strong convergence as the coarsest refinement of the weak topology such that $E_\cdot(\,\cdot\,)$ becomes continuous.

Theorem 6.2. *Let $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ and $u \in \mathcal{E}^1(X, \omega, \psi)$ for $\{\psi_k\}_{k\in\mathbb{N}},\ \psi \in \bar{\mathcal{A}}$. If $\psi \neq \psi_{\min}$ or $V_{\psi_{\min}} > 0$, then the following are equivalent*:

(i) *$u_k \to u$ strongly.*

(ii) *$u_k \to u$ weakly and $E_{\psi_k}(u_k) \to E_\psi(u)$.*

In the case $\psi = \psi_{\min}$ and $V_{\psi_{\min}} = 0$, if $u_k \to P_{\psi_{\min}}$ weakly and $E_{\psi_k}(u_k) \to 0$, then $u_k \to P_{\psi_{\min}}$ strongly. Finally, if $d_{\mathcal{A}}(u_k, P_{\psi_{\min}}) \to 0$ as $k \to \infty$, then the following dichotomy holds:

(a) *$u_k \to P_{\psi_{\min}}$ weakly and $E_{\psi_k}(u_k) \to 0$.*

(b) *$\limsup_{k\to\infty}|\sup_X u_k| = \infty$.*

Proof. (ii) $\Rightarrow$ (i): Assume that (ii) holds where we include the case $u = P_{\psi_{\min}}$ setting $E_\psi(P_{\psi_{\min}}) := 0$. Clearly it is enough to prove that any subsequence of $\{u_k\}_{k\in\mathbb{N}}$ admits a subsequence which is $d_{\mathcal{A}}$-convergent to u. For the sake of simplicity we denote by $\{u_k\}_{k\in\mathbb{N}}$ the arbitrary initial subsequence, and since $\mathcal{A}$ is totally ordered by Lemma 3.13 we may also assume either $\psi_k \searrow \psi$ or $\psi_k \nearrow \psi$ almost everywhere. In particular, even if $u = P_{\psi_{\min}}$ we may suppose that u_k converges weakly to a proper element $v \in \mathcal{E}^1(X, \omega, \psi)$ up to considering a further subsequence by definition of the weak convergence to the point $P_{\psi_{\min}}$. In this case by abuse of notation we denote the function v, which depends on the subsequence chosen, by u. Note also that by Hartogs' lemma we have $u_k \leq \psi_k + A$ and $u \leq \psi + A$ for a uniform constant $A \in \mathbb{R}_{\geq 0}$ since $|\sup_X u_k| \leq A$.

In the case of $\psi_k \searrow \psi$, we have that $v_k := (\sup\{u_j : j \geq k\})^* \in \mathcal{E}^1(X, \omega, \psi_k)$ decreases to u. Thus $w_k := P_\omega[\psi](v_k) \in \mathcal{E}^1(X, \omega, \psi)$ decreases to u, which implies $d(u, w_k) \to 0$ as $k \to \infty$. (If $u = P_{\psi_{\min}}$, we immediately have $w_k = P_{\psi_{\min}}$.)

Moreover, by Propositions 2.4 and 2.10,

$$\begin{aligned} E_\psi(u) = \lim_{k\to\infty} E_\psi(w_k) &= AV_\psi - \lim_{k\to\infty} d(\psi + A, w_k) \\ &\geq \lim_{k\to\infty} (AV_{\psi_k} - d(\psi_k + A, v_k)) \\ &= \limsup_{k\to\infty} E_{\psi_k}(v_k) \geq \lim_{k\to\infty} E_{\psi_k}(u_k) = E_\psi(u) \end{aligned}$$

since $\psi_k + A = P_\omega[\psi_k](A)$. Hence

$$\limsup_{k\to\infty} d(v_k, u_k) = \limsup_{k\to\infty} (d(\psi_k + A, u_k) - d(v_k, \psi_k + A)) = \lim_{k\to\infty} (E_{\psi_k}(v_k) - E_{\psi_k}(u_k)) = 0.$$

Thus by the triangle inequality it is sufficient to show that $\limsup_{k\to\infty} d_{\mathcal{A}}(u, v_k) = 0$.

Next, for any $C \in \mathbb{R}$ we set $v_k^C := \max(v_k, \psi_k - C)$ and $u^C := \max(u, \psi - C)$, and we observe that $d(\psi_k + A, v_k^C) \to d(\psi + A, u^C)$ by Proposition 2.11 since $v_k^C \searrow u^C$. This implies that

$$\begin{aligned} d(v_k, v_k^C) &= d(\psi_k + A, v_k) - d(\psi_k + A, v_k^C) = AV_{\psi_k} - E_{\psi_k}(v_k) - d(\psi_k + A, v_k^C) \\ &\to AV_\psi - E_\psi(u) - d(\psi + A, u^C) = d(\psi + A, u) - d(\psi + A, u^C) = d(u, u^C). \end{aligned}$$

Thus, since $u^C \to u$ strongly, again by the triangle inequality it remains to estimate $d_{\mathcal{A}}(u, v_k^C)$. Fix $\epsilon > 0$ and $\phi_\epsilon \in \mathcal{P}_{\mathcal{H}_\omega}(X, \omega, \psi)$ such that $d(\phi_\epsilon, u) \leq \epsilon$ (by Lemma 2.13). Then letting $\varphi \in \mathcal{H}_\omega$ such that $\phi_\epsilon = P_\omega[\psi](\varphi)$ and setting $\phi_{\epsilon,k} := P_\omega[\psi_k](\varphi)$, by Proposition 2.11 we have

$$\begin{aligned} \limsup_{k\to\infty} d_{\mathcal{A}}(u, v_k^C) &\leq \limsup_{k\to\infty} (d(u, \phi_\epsilon) + d_{\mathcal{A}}(\phi_\epsilon, \phi_{\epsilon,k}) + d(\phi_{\epsilon,k}, v_k^C)) \\ &\leq \epsilon + d(\phi_\epsilon, u^C) \\ &\leq 2\epsilon + d(u, u^C), \end{aligned}$$

which concludes the first case of (ii) $\Rightarrow$ (i) by the arbitrariness of ϵ since $u^C \to u$ strongly in $\mathcal{E}^1(X, \omega, \psi)$.

Next assume that $\psi_k \nearrow \psi$ almost everywhere. In this case we may assume $V_{\psi_k} > 0$ for any $k \in \mathbb{N}$. Then $v_k := (\sup\{u_j : j \geq k\})^* \in \mathcal{E}^1(X, \omega, \psi)$ decreases to u. Moreover, setting $w_k := P_\omega[\psi_k](v_k) \in \mathcal{E}^1(X, \omega, \psi_k)$ and combining with the monotonicity of $E_{\psi_k}(\cdot)$, the upper semicontinuity of $E_\cdot(\cdot)$ (Proposition 3.14) and the contraction property of Proposition 2.10, we obtain

$$\begin{aligned} E_\psi(u) = \lim_{k\to\infty} E_\psi(v_k) = AV_\psi - \lim_{k\to\infty} d(v_k, \psi + A) &\leq \liminf_{k\to\infty}(AV_{\psi_k} - d(w_k, \psi_k + A)) \\ &= \liminf_{k\to\infty} E_{\psi_k}(w_k) \leq \limsup_{k\to\infty} E_{\psi_k}(w_k) \leq E_\psi(u), \end{aligned}$$

i.e., $E_{\psi_k}(w_k) \to E_\psi(u)$ as $k \to \infty$. As an easy consequence we get $d(w_k, u_k) = E_{\psi_k}(w_k) - E_{\psi_k}(u_k) \to 0$, thus it is sufficient to prove that

$$\limsup_{k\to\infty} d_{\mathcal{A}}(u, w_k) = 0.$$

Similar to the previous case, fix $\epsilon > 0$ and let $\phi_\epsilon = P_\omega[\psi](\varphi_\epsilon)$ for $\varphi \in \mathcal{H}_\omega$ such that $d(u, \phi_\epsilon) \leq \epsilon$. Again Propositions 2.10 and 2.11 yield

$$\begin{aligned} \limsup_{k\to\infty} d_{\mathcal{A}}(u, w_k) &\leq \epsilon + \limsup_{k\to\infty}(d_{\mathcal{A}}(\phi_\epsilon, P_\omega[\psi_k](\phi_\epsilon)) + d(P_\omega[\psi_k](\phi_\epsilon), w_k)) \\ &\leq \epsilon + \limsup_{k\to\infty}(d_{\mathcal{A}}(\phi_\epsilon, P_\omega[\psi_k](\phi_\epsilon)) + d(\phi_\epsilon, v_k)) \leq 2\epsilon, \end{aligned}$$

which concludes the first part.

(i) $\Rightarrow$ (ii) if $u \neq P_{\psi_{\min}}$, while (i) implies the dichotomy if $u = P_{\psi_{\min}}$: If $u \neq P_{\psi_{\min}}$, then Proposition 6.1 implies that $u_k \to u$ weakly and, in particular, that $|\sup_X u_k| \leq A$. Thus it remains to prove that $E_{\psi_k}(u_k) \to E_\psi(u)$.

If $u = P_{\psi_{\min}}$, then again by Proposition 6.1 it remains to show that $E_{\psi_k}(u_k) \to 0$ assuming $u_{k_h} \to P_{\psi_{\min}}$ strongly and weakly. Note that we also have $|\sup_X u_k| \leq A$ for a uniform constant $A \in \mathbb{R}$ by definition of the weak convergence to $P_{\psi_{\min}}$.

Since by an easy contradiction argument it is enough to prove that any subsequence of $\{u_k\}_{k\in\mathbb{N}}$ admits a further subsequence such that the convergence of the energies holds, without loss of generality we may assume that $u_k \to u \in \mathcal{E}^1(X, \omega, \psi)$ weakly even in the case $V_\psi = 0$ (i.e., when, with abuse of notation, $u = P_{\psi_{\min}}$).

So we want to show the existence of a further subsequence $\{u_{k_h}\}_{h\in\mathbb{N}}$ such that $E_{\psi_{k_h}}(u_{k_h}) \to E_\psi(u)$ (note that if $V_\psi = 0$, then $E_\psi(u) = 0$). It easily follows that

$$\begin{aligned} |E_{\psi_k}(u_k) - E_\psi(u)| &\leq |d(\psi_k + A, u_k) - d(\psi + A, u)| + A|V_{\psi_k} - V_\psi| \\ &\leq d_{\mathcal{A}}(u, u_k) + d(\psi_k + A, \psi + A) + A|V_{\psi_k} - V_\psi|, \end{aligned}$$

and this leads to $\lim_{k\to\infty} E_{\psi_k}(u_k) = E_\psi(u)$ by Proposition 2.11, since we have $\psi_k + A = P_\omega[\psi_k](A)$ and $\psi + A = P_\omega[\psi](A)$. Hence $E_{\psi_k}(u_k) \to E_\psi(u)$ as desired. □

Note that in Theorem 6.2, case (b) may happen (Remark 3.16), but obviously one can consider

$$X_{\mathcal{A},\mathrm{norm}} = \bigsqcup_{\psi\in\bar{\mathcal{A}}} \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$$

to exclude such pathology.

The strong convergence also implies the convergence in ψ'-capacity for any $\psi' \in \mathcal{M}^+$, as our next result shows.

Theorem 6.3. *Let $\psi_k, \psi \in \mathcal{A}$ and let $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ strongly converge to $u \in \mathcal{E}^1(X, \omega, \psi)$. Assume also that $V_\psi > 0$. Then there exists a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ such that the sequences $w_j := (\sup\{u_{k_s} : s \geq j\})^*$ and $v_j := P_\omega(u_{k_j}, u_{k_{j+1}}, \dots)$ belong to $X_{\mathcal{A}}$, satisfying $v_j \leq u_{k_j} \leq w_j$ and converging strongly and monotonically to u. In particular, $u_k \to u$ in ψ'-capacity for any $\psi' \in \mathcal{M}^+$ and $\mathrm{MA}_\omega(u_k^j, \psi_k^{n-j}) \to \mathrm{MA}_\omega(u^k, \psi^{n-j})$ weakly for any $j \in \{0, \dots, n\}$.*

Proof. We first observe that by Theorem 6.2, $u_k \to u$ weakly and $E_{\psi_k}(u_k) \to E_\psi(u)$. In particular, $\sup_X u_k$ is uniformly bounded and the sequence of ω-psh $w_k := (\sup\{u_j : j \geq k\})^*$ decreases to u.

Up to considering a subsequence we may assume either $\psi_k \searrow \psi$ or $\psi_k \nearrow \psi$ almost everywhere. We treat the two cases separately.

Assume first that $\psi_k \searrow \psi$. Since clearly $w_k \in \mathcal{E}^1(X, \omega, \psi_k)$ and $E_{\psi_k}(w_k) \geq E_{\psi_k}(u_k)$, Theorem 6.2 and Proposition 3.14 yield

$$E_\psi(u) = \lim_{k\to\infty} E_{\psi_k}(u_k) \leq \limsup_{k\to\infty} E_{\psi_k}(w_k) \leq E_\psi(u),$$

i.e., $w_k \to u$ strongly. Thus up to considering a further subsequence we can suppose that $d(u_k, w_k) \leq 1/2^k$ for any $k \in \mathbb{N}$.

Next, similar to the proof of Proposition 5.7, we define $v_{j,l} := P_\omega(u_j, \dots, u_{j+l})$ for any $j, l \in \mathbb{N}$, observing that $v_{j,l} \in \mathcal{E}^1(X, \omega, \psi_{j+l})$. Thus the function $v_{j,l}^u := P_\omega(u, v_{j,l}) \in \mathcal{E}^1(X, \omega, \psi)$ satisfies

$$\begin{aligned} d(u, v_{j,l}^u) &\leq \int_X (u - v_{j,l}^u)\,\mathrm{MA}_\omega(v_{j,l}^u) \leq \int_{\{v_{j,l}^u = v_{j,l}\}} (u - v_{j,l})\,\mathrm{MA}_\omega(v_{j,l}) \\ &\leq \sum_{s=j}^{j+l} \int_X (w_s - u_s)\,\mathrm{MA}_\omega(u_s) \leq (n+1)\sum_{s=j}^{j+l} d(w_s, u_s) \leq \frac{n+1}{2^{j-1}}, \end{aligned} \tag{22}$$

where we combined Proposition 2.4 and [Darvas et al. 2018, Lemma 3.7]. Therefore by Proposition 3.15, $v_{j,l}^u$ converges decreasingly and strongly in $\mathcal{E}^1(X, \omega, \psi)$ to a function ϕ_j which satisfies $\phi_j \leq u$.

Similarly,

$$\int_{\{P_\omega(u, v_{j,l}^u) = u\}} (v_{j,l}^u - u)\,\mathrm{MA}_\omega(u) \leq \int_X |v_{j,1}^u - u|\,\mathrm{MA}_\omega(u) < \infty$$

by Corollary 3.5, which implies that $v_{j,l}$ converges decreasingly to $v_j \in \mathcal{E}^1(X, \omega, \psi)$ such that $u \geq v_j \geq \phi_j$, since $v_j \leq u_s$ for any $s \geq j$ and $v_{j,l} \geq v_{j,l}^u$. Hence from (22) we obtain

$$d(u, v_j) \leq d(u, \phi_j) = \lim_{l\to\infty} d(u, v_{j,l}^u) \leq \frac{n+1}{2^{j-1}},$$

i.e., v_j converges increasingly and strongly to u as $j \to \infty$.

Next assume $\psi_k \nearrow \psi$ almost everywhere. In this case, $w_k \in \mathcal{E}^1(X, \omega, \psi)$ for any $k \in \mathbb{N}$, and clearly w_k converges strongly and decreasingly to u. On the other hand, letting $w_{k,k} := P_\omega[\psi_k](w_k)$ we observe by Theorem 6.2 and Proposition 3.14 that $w_{k,k} \to u$ weakly since $w_k \geq w_{k,k} \geq u_k$ and

$$E_\psi(u) = \lim_{k\to\infty} E_{\psi_k}(u_k) \leq \limsup_{k\to\infty} E_{\psi_k}(w_{k,k}) \leq E_\psi(u),$$

i.e., $w_{k,k} \to u$ strongly, again by Theorem 6.2. As in the previous case, we assume that $d(u_k, w_{k,k}) \leq 1/2^k$ up to considering a further subsequence. Therefore, setting $v_{j,l} := P_\omega(u_j, \dots, u_{j+l}) \in \mathcal{E}^1(X, \omega, \psi_j)$, $u^j := P_\omega[\psi_j](u)$ and $v_{j,l}^{u^j} := P_\omega(v_{j,l}, u^j)$ we obtain

$$d(u^j, v_{j,l}^{u^j}) \leq \int_X (u^j - v_{j,l}^{u^j}) \, \mathrm{MA}_\omega(v_{j,l}^{u^j}) \leq \sum_{s=j}^{j+l} \int_X (w_{s,s} - u_s) \, \mathrm{MA}_\omega(u_s) \leq \frac{n+1}{2^{j-1}}, \tag{23}$$

proceeding as in the previous case. This implies that $v_{j,l}^{u^j}$ and $v_{j,l}$ converge decreasingly and strongly to functions $\phi_j, v_j \in \mathcal{E}^1(X, \omega, \psi_j)$, respectively, as $l \to +\infty$ which satisfy $\phi_j \leq v_j \leq u^j$. Therefore combining (23), Proposition 2.11 and the triangle inequality we get

$$\limsup_{j\to\infty} d_{\mathcal{A}}(u, v_j) \leq \limsup_{j\to\infty} (d_{\mathcal{A}}(u, u^j) + d(u^j, \phi_j)) \leq \limsup_{j\to\infty} \left(d_{\mathcal{A}}(u, u^j) + \frac{n+1}{2^{j-1}} \right) = 0.$$

Hence v_j converges strongly and increasingly to u, so $v_j \nearrow u$ almost everywhere (Proposition 6.1) and the first part of the proof is concluded.

The convergence in ψ'-capacity and the weak convergence of the mixed Monge–Ampère measures follow exactly as in the proof of Proposition 5.7. □

We observe that the assumption $u \neq P_{\psi_{\min}}$ if $V_{\psi_{\min}} = 0$ in Theorem 6.3 is obviously necessary as the counterexample of Remark 3.16 shows. On the other hand, if $d_{\mathcal{A}}(u_k, P_{\psi_{\min}}) \to 0$, then trivially $\mathrm{MA}_\omega(u_k^j, \psi_k^{n-j}) \to 0$ weakly as $k \to \infty$ for any $j \in \{0, \dots, n\}$ as a consequence of $V_{\psi_k} \searrow 0$.

6B. *Proof of Theorem B.*

Definition 6.4. We define $Y_{\mathcal{A}}$ as

$$Y_{\mathcal{A}} := \bigsqcup_{\psi \in \bar{\mathcal{A}}} \mathcal{M}^1(X, \omega, \psi),$$

and we endow it with its natural *strong topology* given as the coarsest refinement of the weak topology such that $E^*_\cdot$ becomes continuous, i.e., $V_{\psi_k}\mu_k$ converges strongly to $V_\psi \mu$ if and only if $V_{\psi_k}\mu_k \to V_\psi \mu$ weakly and $E^*_{\psi_k}(\mu_k) \to E^*_\psi(\mu)$ as $k \to \infty$.

Observe that $Y_{\mathcal{A}} \subset \{$nonpluripolar measures of total mass belonging to $[V_{\psi_{\min}}, V_{\psi_{\max}}]\}$, where clearly $\psi_{\max} := \sup \mathcal{A}$. As stated in the Introduction, the definition is coherent with [Berman et al. 2019] since if $\psi = 0 \in \bar{\mathcal{A}}$, then the induced topology on $\mathcal{M}^1(X, \omega)$ coincides with the strong topology as defined in that paper.

We also recall that

$$X_{\mathcal{A},\mathrm{norm}} := \bigsqcup_{\psi \in \bar{\mathcal{A}}} \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi),$$

where $\mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi) := \{u \in \mathcal{E}^1(X, \omega, \psi) : \sup_X u = 0\}$ (if $V_{\psi_{\min}} = 0$, then we can assume $P_{\psi_{\min}} \in X_{\mathcal{A},\mathrm{norm}}$).

Theorem B. *The Monge–Ampère map*

$$\mathrm{MA}_\omega : (X_{\mathcal{A},\mathrm{norm}}, d_{\mathcal{A}}) \to (Y_{\mathcal{A}}, \text{strong})$$

is a homeomorphism.

Proof. The map is a bijection as a consequence of Lemma 3.12 and Proposition 5.5, where we clearly define $\mathrm{MA}_\omega(P_{\psi_{\min}}) := 0$, i.e., the null measure.

Step 1: continuity. Assume first that $V_{\psi_{\min}} = 0$ and that $d_{\mathcal{A}}(u_k, P_{\psi_{\min}}) \to 0$ as $k \to \infty$. Then clearly $\mathrm{MA}_\omega(u_k) \to 0$ weakly. Moreover, assuming $u_k \neq P_{\psi_{\min}}$ for any k, it follows from Proposition 2.4 that

$$\begin{aligned} E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) &= E_{\psi_k}(u_k) + \int_X (\psi_k - u_k)\,\mathrm{MA}_\omega(u_k) \\ &\leq \frac{n}{n+1}\int_X (\psi_k - u_k)\,\mathrm{MA}_\omega(u_k) \leq -nE_{\psi_k}(u_k) \to 0 \end{aligned}$$

as $k \to \infty$ where the convergence is given by Theorem 6.2. Hence $\mathrm{MA}_\omega(u_k) \to 0$ strongly in $Y_{\mathcal{A}}$.

We can now assume that $u \neq P_{\psi_{\min}}$.

Theorem 6.3 immediately gives the weak convergence of $\mathrm{MA}_\omega(u_k)$ to $\mathrm{MA}_\omega(u)$. Let $\varphi_j \in \mathcal{H}_\omega$ be a decreasing sequence converging to u such that $d(u, P_\omega[\psi](\varphi_j)) \leq 1/j$ for any $j \in \mathbb{N}$ [Błocki and Kołodziej 2007], and set $v_{k,j} := P_\omega[\psi_k](\varphi_j)$ and $v_j := P_\omega[\psi](\varphi_j)$. Observe also that as a consequence of Proposition 2.11 and Theorem 6.2, for any $j \in \mathbb{N}$ there exists $k_j \gg 0$ big enough such that

$$d(\psi_k, v_{k,j}) \leq d_{\mathcal{A}}(\psi_k, \psi) + d(\psi, v_j) + d_{\mathcal{A}}(v_j, v_{k,j}) \leq d(\psi, v_j) + 1 \leq C$$

for any $k \geq k_j$, where C is a uniform constant independent of $j \in \mathbb{N}$. Therefore, again combining Theorem 6.2 with Lemma 4.6 and Proposition 3.7, we obtain

$$\begin{aligned} &\limsup_{k\to\infty} |E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) - E^*_{\psi_k}(\mathrm{MA}_\omega(v_{k,j})/V_{\psi_k})| \\ &\leq \limsup_{k\to\infty}\left(|E_{\psi_k}(u_k) - E_{\psi_k}(v_{k,j})| + \left|\int_X (\psi_k - u_k)(\mathrm{MA}_\omega(u_k) - \mathrm{MA}_\omega(v_{k,j}))\right| + \left|\int_X (v_{k,j} - u_k)\,\mathrm{MA}_\omega(v_{k,j})\right|\right) \\ &\leq |E_\psi(u) - E_\psi(v_j)| + \limsup_{k\to\infty} C I_{\psi_k}(u_k, v_{k,j})^{1/2} + \int_X (v_j - u)\,\mathrm{MA}_\omega(v_j), \end{aligned} \tag{24}$$

since clearly we may assume that either $\psi_k \searrow \psi$ or $\psi_k \nearrow \psi$ almost everywhere, up to considering a subsequence. On the other hand, if $k \geq k_j$, Proposition 3.4 implies $I_{\psi_k}(u_k, v_{k,j}) \leq 2 f_{\tilde{C}}(d(u_k, v_{k,j}))$, where $\tilde{C}$ is a uniform constant independent of j, k and $f_{\tilde{C}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous increasing function such that $f_{\tilde{C}}(0) = 0$. Hence continuing the estimates in (24) we get

$$(24) \leq |E_\psi(u) - E_\psi(v_j)| + 2C f_{\tilde{C}}(d(u, v_j)) + d(v_j, u), \tag{25}$$

using also Propositions 2.4 and 2.11. Letting $j \to \infty$ in (25), it follows that

$$\limsup_{j\to\infty} \limsup_{k\to\infty} |E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) - E^*_{\psi_k}(\mathrm{MA}_\omega(v_{k,j})/V_{\psi_k})| = 0$$

since $v_j \searrow u$. Furthermore, it is easy to check that $E^*_{\psi_k}(\mathrm{MA}_\omega(v_{k,j})/V_{\psi_k}) \to E^*_\psi(\mathrm{MA}_\omega(v_j)/V_\psi)$ as $k \to \infty$ for j fixed by Lemma 4.6 and Proposition 2.11. Therefore the convergence

$$E^*_\psi(\mathrm{MA}_\omega(v_j)/V_\psi) \to E^*_\psi(\mathrm{MA}_\omega(u)/V_\psi) \tag{26}$$

as $j \to \infty$ given by Theorem A concludes this step.

Step 2: continuity of the inverse. We will assume $u_k \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi_k)$ and $u \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$ such that $\mathrm{MA}_\omega(u_k) \to \mathrm{MA}_\omega(u)$ strongly. Note that when $\psi = \psi_{\min}$ and $V_{\psi_{\min}} = 0$, the assumption does not depend on the function u chosen. Clearly this implies $V_{\psi_k} \to V_\psi$ which leads to $\psi_k \to \psi$ as $k \to \infty$ by Lemma 3.12 since $\mathcal{A} \subset \mathcal{M}^+$ is totally ordered. Hence, up to considering a subsequence, we may assume that $\psi_k \to \psi$ monotonically almost everywhere. We keep the same notation of the previous step for $v_{k,j}, v_j$. We may also suppose that $V_{\psi_k} > 0$ for any $k \in \mathbb{N}$ big enough otherwise it would be trivial.

The strategy is to proceed similarly to the proof of Theorem A, i.e., we first prove that $I_{\psi_k}(u_k, v_{k,j}) \to 0$ as $k, j \to \infty$ in this order. Then we will use this to prove that the unique weak accumulation point of $\{u_k\}_{k\in\mathbb{N}}$ is u. Finally we will deduce the convergence of the ψ_k-relative energies to conclude that $u_k \to u$ strongly thanks to Theorem 6.2.

By Lemma 3.1,

$$\begin{aligned}(n+1)^{-1} & I_{\psi_k}(u_k, v_{k,j}) \\ &\leq E_{\psi_k}(u_k) - E_{\psi_k}(v_{k,j}) + \int_X (v_{k,j} - u_k)\,\mathrm{MA}_\omega(u_k) \\ &= E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) - E^*_{\psi_k}(\mathrm{MA}_\omega(v_{k,j})/V_{\psi_k}) + \int_X (v_{k,j} - \psi_k)(\mathrm{MA}_\omega(u_k) - \mathrm{MA}_\omega(v_{k,j})) \quad (27)\end{aligned}$$

for any j, k. Moreover, by Step 1 and Proposition 2.11 we know that $E^*_{\psi_k}(\mathrm{MA}_\omega(v_{k,j})/V_{\psi_k})$ converges, as $k \to +\infty$, to 0 if $V_\psi = 0$ and to $E^*_\psi(\mathrm{MA}_\omega(v_j)/V_\psi)$ if $V_\psi > 0$. Next by Lemma 4.6,

$$\int_X (v_{k,j} - \psi_k)\,\mathrm{MA}_\omega(v_{k,j}) \to \int_X (v_j - \psi)\,\mathrm{MA}_\omega(v_j)$$

letting $k \to \infty$. So if $V_\psi = 0$, then from

$$\lim_{k\to\infty} \sup_X (v_{k,j} - \psi_k) = \sup_X (v_j - \psi) = \sup_X v_j$$

we easily get $\limsup_{k\to\infty} I_{\psi_k}(u_k, v_{k,j}) = 0$. Thus we may assume $V_\psi > 0$, and it remains to estimate $\int_X (v_{k,j} - \psi_k)\,\mathrm{MA}_\omega(u_k)$ from above.

We set $f_{k,j} := v_{k,j} - \psi_k$, and as in the proof of Theorem A we construct a sequence of smooth functions $f_j^s := v_j^s - \psi^s$ converging in capacity to $f_j := v_j - \psi$ and satisfying $\|f_j^s\|_{L^\infty} \leq 2\|f_j\|_{L^\infty} \leq 2\|\varphi_j\|_{L^\infty}$. Here v_j^s and ψ^s are sequences of ω-psh functions decreasing to v_j and ψ, respectively. Then we write

$$\int_X f_{k,j}\,\mathrm{MA}_\omega(u_k) = \int_X (f_{k,j} - f_j^s)\,\mathrm{MA}_\omega(u_k) + \int_X f_j^s\,\mathrm{MA}_\omega(u_k), \quad (28)$$

and we observe that

$$\limsup_{s\to\infty} \limsup_{k\to\infty} \int_X f_j^s\,\mathrm{MA}_\omega(u_k) = \int_X f_j\,\mathrm{MA}_\omega(u),$$

since $\mathrm{MA}_\omega(u_k) \to \mathrm{MA}_\omega(u)$ weakly, $f_j^s \in C^\infty$, f_j^s converges to f_j in capacity and $\|f_j^s\|_{L^\infty} \leq 2\|f_j\|_{L^\infty}$. We also claim that the first term on the right-hand side of (28) goes to 0 letting $k, s \to \infty$ in this order.

Indeed, for any $\delta > 0$,

$$\begin{aligned}\int_X (f_{k,j} - f_j)\,\mathrm{MA}_\omega(u_k) &\le \delta V_{\psi_k} + 2\|\varphi_j\|_{L^\infty} \int_{\{f_{k,j}-f_j>\delta\}} \mathrm{MA}_\omega(u_k)\\ &\le \delta V_{\psi_k} + 2\|\varphi_j\|_{L^\infty} \int_{\{|h_{k,j}-h_j|>\delta\}} \mathrm{MA}_\omega(u_k),\end{aligned} \tag{29}$$

where we set $h_{k,j} := v_{k,j}$, $h_j := v_j$ if $\psi_k \searrow \psi$ and $h_{k,j} := \psi_k$, $h_j := \psi$ if instead $\psi_k \nearrow \psi$ almost everywhere. Moreover, since $\{|h_{k,j} - h_j| > \delta\} \subset \{|h_{l,j} - h_j| > \delta\}$ for any $l \le k$, from (29) we obtain

$$\begin{aligned}\limsup_{k\to\infty} \int_X (f_{k,j} - f_j)\,\mathrm{MA}_\omega(u_k) &\le \delta V_\psi + \limsup_{l\to\infty} \limsup_{k\to\infty} 2\|\varphi_j\|_{L^\infty} \int_{\{|h_{l,j}-h_j|\ge\delta\}} \mathrm{MA}_\omega(u_k)\\ &\le \delta V_\psi + \limsup_{l\to\infty} 2\|\varphi_j\|_{L^\infty} \int_{\{|h_{l,j}-h_j|\ge\delta\}} \mathrm{MA}_\omega(u) = \delta V_\psi,\end{aligned}$$

where we also used that $\{|h_{l,j} - h_j| \ge \delta\}$ is a closed set in the plurifine topology since it is equal to $\{v_{l,j} - v_j \ge \delta\}$ if $\psi_l \searrow \psi$ and to $\{\psi - \psi_l \ge \delta\}$ if $\psi_l \nearrow \psi$ almost everywhere. Hence

$$\limsup_{k\to\infty} \int_X (f_{k,j} - f_j)\,\mathrm{MA}_\omega(u_k) \le 0.$$

Similarly we also get

$$\limsup_{s\to\infty} \limsup_{k\to\infty} \int_X (f_j - f_j^s)\,\mathrm{MA}_\omega(u_k) \le 0;$$

see also the proof of Theorem A.

Summarizing from (27), we obtain

$$\begin{aligned}&\limsup_{k\to\infty} (n+1)^{-1} I_{\psi_k}(u_k, v_{k,j})\\ &\le E^*_\psi(\mathrm{MA}_\omega(u)/V_\psi) - E^*_\psi(\mathrm{MA}_\omega(v_j)/V_\psi) + \int_X (v_j - \psi)\,\mathrm{MA}_\omega(u) - \int_X (v_j - \psi)\,\mathrm{MA}_\omega(v_j) =: F_j,\end{aligned} \tag{30}$$

and $F_j \to 0$ as $j \to \infty$ by Step 1 and Proposition 3.7, since $\mathcal{E}^1(X, \omega, \psi) \ni v_j \searrow u \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$, hence strongly.

Next by Lemma 3.1, $u_k \in X_{\mathcal{A},C}$ for $C \gg 1$ since $E^*(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) = J^{\psi}_{u_k}(\psi)$ and $\sup_X u_k = 0$, thus, up to considering a further subsequence, $u_k \to w \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$ weakly where $d(w, \psi) \le C$. Indeed, if $V_\psi > 0$ this follows from Proposition 3.15 while it is trivial if $V_\psi = 0$. In particular, by Lemma 4.6,

$$\int_X (\psi_k - u_k)\,\mathrm{MA}_\omega(v_{k,j}) \to \int_X (\psi - w)\,\mathrm{MA}_\omega(v_j), \tag{31}$$

$$\int_X (v_{k,j} - u_k)\,\mathrm{MA}_\omega(v_{k,j}) \to \int_X (v_j - w)\,\mathrm{MA}_\omega(v_j) \tag{32}$$

as $j \to \infty$. Therefore if $V_\psi = 0$, then combining $I_{\psi_k}(u_k, v_{k,j}) \to 0$ as $k \to \infty$ with (32) and Lemma 3.1, we obtain

$$\limsup_{k\to\infty} (-E_{\psi_k}(u_k) + E_{\psi_k}(v_{k,j})) \le \limsup_{k\to\infty} \left(\frac{n}{n+1} I_{\psi_k}(u_k, v_{k,j}) + \left| \int_X (v_{k,j} - u_k)\,\mathrm{MA}_\omega(v_{k,j}) \right| \right) = 0.$$

This implies that $d(\psi_k, u_k) = -E_{\psi_k}(u_k) \to 0$ as $k \to \infty$, i.e., that $d_{\mathcal{A}}(P_{\psi_{\min}}, u_k) \to 0$ using Theorem 6.2. We may assume from now until the end of the proof that $V_\psi > 0$.

By (31) and Proposition 3.14 it follows that

$$\limsup_{k\to\infty}\biggl(E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) + \int_X (\psi_k - u_k)(\mathrm{MA}_\omega(v_{k,j}) - \mathrm{MA}_\omega(u_k))\biggr)$$
$$= \limsup_{k\to\infty}\biggl(E_{\psi_k}(u_k) + \int_X (\psi_k - u_k)\,\mathrm{MA}_\omega(v_{k,j})\biggr) \le E_\psi(w) + \int_X (\psi - w)\,\mathrm{MA}_\omega(v_j). \tag{33}$$

On the other hand, by Proposition 3.7 and (30),

$$\limsup_{k\to\infty}\biggl|\int_X (\psi_k - u_k)(\mathrm{MA}_\omega(v_{k,j}) - \mathrm{MA}_\omega(u_k))\biggr| \le C F_j^{1/2}. \tag{34}$$

In conclusion, by the triangle inequality and combining (33) and (34) we get

$$\begin{aligned} E_\psi(u) + \int_X (\psi - u)\,\mathrm{MA}_\omega(u) &= \lim_{k\to\infty} E^*(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) \\ &\le \limsup_{j\to\infty}\biggl(E_\psi(w) + \int_X (\psi - w)\,\mathrm{MA}_\omega(v_j) + C F_j^{1/2}\biggr) \\ &= E_\omega(w) + \int_X (\psi - w)\,\mathrm{MA}_\omega(u) \end{aligned}$$

since $F_j \to 0$, i.e., $w \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$ is a maximizer of $F_{\mathrm{MA}_\omega(u)/V_\psi, \psi}$. Hence $w = u$ (Proposition 5.5), i.e., $u_k \to u$ weakly. Furthermore, similar to the case $V_\psi = 0$, Lemma 3.1 and (32) imply

$$\begin{aligned} E_\psi(v_j) - \liminf_{k\to\infty} E_{\psi_k}(u_k) &= \limsup_{k\to\infty}(-E_{\psi_k}(u_k) + E_{\psi_k}(v_{k,j})) \\ &\le \limsup_{k\to\infty}\biggl(\frac{n}{n+1} I_{\psi_k}(u_k, v_{k,j}) + \biggl|\int_X (u_k - v_{j,k})\,\mathrm{MA}_\omega(v_{k,j})\biggr|\biggr) \\ &\le \frac{n}{n+1} F_j + \biggl|\int_X (u - v_j)\,\mathrm{MA}_\omega(v_j)\biggr|. \end{aligned}$$

Finally, letting $j \to \infty$, since $v_j \to u$ strongly, we obtain $\liminf_{k\to\infty} E_{\psi_k}(u_k) \ge \lim_{j\to\infty} E_\psi(v_j) = E_\psi(u)$. Hence $E_{\psi_k}(u_k) \to E_\psi(u)$ by Proposition 3.14, which implies $d_{\mathcal{A}}(u_k, u) \to 0$ by Theorem 6.2. □

7. Stability of complex Monge–Ampère equations

As stated in the Introduction, we want to use the homeomorphism of Theorem B to deduce the strong stability of solutions of complex Monge–Ampère equations with prescribed singularities when the measures have uniformly bounded L^p density for $p > 1$.

Theorem C. *Let $\mathcal{A} := \{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{M}^+$ be totally ordered, and let $\{f_k\}_{k\in\mathbb{N}} \subset L^1$ be a sequence of nonnegative functions such that $f_k \to f \in L^1 \setminus \{0\}$ and such that $\int_X f_k \omega^n = V_{\psi_k}$ for any $k \in \mathbb{N}$. Assume also that there exists $p > 1$ such that $\|f_k\|_{L^p}$ and $\|f\|_{L^p}$ are uniformly bounded. Then $\psi_k \to \psi \in \bar{\mathcal{A}} \subset \mathcal{M}^+$,* *and the sequence of solutions of*

$$\mathrm{MA}_\omega(u_k) = f_k\omega^n, \quad u_k \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi_k), \tag{35}$$

converges strongly to $u \in X_{\mathcal{A}}$, which is the unique solution of

$$\mathrm{MA}_\omega(u) = f\omega^n, \quad u \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi). \tag{36}$$

In particular, $u_k \to u$ in capacity.

Proof. We first observe that the existence of the unique solutions of (35) follows by [Darvas et al. 2021a, Theorem A].

Moreover, letting u be any weak accumulation point for $\{u_k\}_{k\in\mathbb{N}}$ (there exists at least one by compactness), [Darvas et al. 2021a, Lemma 2.8] yields $\mathrm{MA}_\omega(u) \geq f\omega^n$ and by the convergence of f_k to f we also obtain $\int_X f\omega^n = \lim_{k\to\infty} V_{\psi_k}$. Moreover, since $u_k \leq \psi_k$ for any $k \in \mathbb{N}$, by [Witt Nyström 2019] we obtain $\int_X \mathrm{MA}_\omega(u) \leq \lim_{k\to\infty} V_{\psi_k}$. Hence $\mathrm{MA}_\omega(u) = f\omega^n$ which, in particular, means that there is a unique weak accumulation point for $\{u_k\}_{k\in\mathbb{N}}$ and that $\psi_k \to \psi$ as $k \to \infty$ since $V_{\psi_k} \to V_\psi$ (by Lemma 3.12). Then it easily follows by combining Fatou's lemma with Proposition 2.10 and Lemma 2.12 that for any $\varphi \in \mathcal{H}_\omega$,

$$\begin{aligned}\liminf_{k\to\infty} E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) &\geq \liminf_{k\to\infty}\left(E_{\psi_k}(P_\omega[\psi_k](\varphi)) + \int_X (\psi_k - P_\omega[\psi_k](\varphi)) f_k\omega^n\right)\\ &\geq E_\psi(P_\omega[\psi](\varphi)) + \int_X (\psi - P_\omega[\psi](\varphi)) f\omega^n, \end{aligned}\tag{37}$$

since $(\psi_k - P_\omega[\psi_k](\varphi)) f_k \to (\psi - P_\omega[\psi](\varphi)) f$ almost everywhere by Lemma 2.14. Thus, for any $v \in \mathcal{E}^1(X, \omega, \psi)$, letting $\varphi_j \in \mathcal{H}_\omega$ be a decreasing sequence converging to v [Błocki and Kołodziej 2007], from inequality (37) we get

$$\begin{aligned}\liminf_{k\to\infty} E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) &\geq \limsup_{j\to\infty}\left(E_\psi(P_\omega[\psi](\varphi_j)) + \int_X (\psi - P_\omega[\psi](\varphi_j)) f\omega^n\right)\\ &= E_\psi(v) + \int_X (\psi - v) f\omega^n,\end{aligned}$$

using Proposition 2.4 and the monotone convergence theorem. Hence by definition,

$$\liminf_{k\to\infty} E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) \geq E^*_\psi(f\omega^n/V_\psi). \tag{38}$$

On the other hand, since $\|f_k\|_{L^p}$ and $\|f\|_{L^p}$ are uniformly bounded for $p > 1$ and $u_k \to u$, $\psi_k \to \psi$ in L^q for any $q \in [1, +\infty)$ (see [Guedj and Zeriahi 2017, Theorem 1.48]), we also have

$$\int_X (\psi_k - u_k) f_k\omega^n \to \int_X (\psi - u) f\omega^n < +\infty,$$

which implies that $\int_X (\psi - u)\,\mathrm{MA}_\omega(u) < +\infty$, i.e., $u \in \mathcal{E}^1(X, \omega, \psi)$ by Proposition 2.4. Moreover, by Proposition 3.14 we also get

$$\limsup_{k\to\infty} E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) \leq E^*_\psi(\mathrm{MA}_\omega(u)/V_\psi),$$

which together with (38) leads us to $\mathrm{MA}_\omega(u_k) \to \mathrm{MA}_\omega(u)$ strongly in $Y_{\mathcal{A}}$ by definition (observe that $\mathrm{MA}_\omega(u_k) = f_k\omega^n \to \mathrm{MA}_\omega(u) = f\omega^n$ weakly). Hence $u_k \to u$ strongly by Theorem B while the convergence in capacity follows from Theorem 6.3. □

Remark 7.1. As said in the Introduction, the convergence in capacity of Theorem C was already obtained in [Darvas et al. 2021b, Theorem 1.4]. Indeed, under the hypotheses of Theorem C it follows from Lemma 2.12 and [Darvas et al. 2021b, Lemma 3.4] that $d_S(\psi_k, \psi) \to 0$ where d_S is the pseudometric on $\{[u] : u \in \mathrm{PSH}(X, \omega)\}$ introduced in [Darvas et al. 2021b], where the class $[u]$ is given by the partial order $\preccurlyeq$.

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SHARP POINTWISE AND UNIFORM ESTIMATES FOR $\bar\partial$

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We use weighted L^2-methods to obtain sharp pointwise estimates for the canonical solution to the equation $\bar\partial u = f$ on smoothly bounded strictly convex domains and the Cartan classical domains when f is bounded in the Bergman metric g. We provide examples to show our pointwise estimates are sharp. In particular, we show that on the Cartan classical domains Ω of rank 2 the maximum blow-up order is greater than $-\log\delta_\Omega(z)$, which was obtained for the unit ball case by Berndtsson. For example, for Ω of type IV(n) with $n \geq 3$, the maximum blow-up order is $\delta(z)^{1-n/2}$ because of the contribution of the Bergman kernel. Additionally, we obtain uniform estimates for the canonical solutions on the polydiscs, strictly pseudoconvex domains and the Cartan classical domains under stronger conditions on f.

1. Introduction

The existence and regularity of solutions to the Cauchy–Riemann equation $\bar\partial u = f$ on a bounded pseudoconvex domain Ω in $\mathbb{C}^n$ is a fundamental topic in several complex variables. Since the kernel of $\bar\partial$ is the set of holomorphic functions, a solution to the Cauchy–Riemann equation is not unique if it exists. However, let $A^2(\Omega) := L^2(\Omega)\cap\ker(\bar\partial)$ denote the Bergman space over Ω. Then the solution to $\bar\partial u = f$ with $u \perp A^2(\Omega)$ is unique, and it is called the canonical solution or L^2-minimal solution because it has minimal L^2-norm among all solutions. Hörmander [1965] showed that if Ω is bounded and pseudoconvex and $f \in L^2_{(0,1)}(\Omega)$ is $\bar\partial$-closed, then there exists a solution u that satisfies the estimate $\|u\|_{L^2} \leq C\|f\|_{L^2}$ for some constant C depending only on the diameter of Ω. In view of Hörmander's result, a natural question arises: does there exist a constant C depending only on Ω such that for any $\bar\partial$-closed $f \in L^\infty_{(0,1)}(\Omega)$ there exists a solution to $\bar\partial u = f$ with $\|u\|_\infty \leq C\|f\|_\infty$? If the answer is yes, we say the $\bar\partial$-equation can be solved with uniform estimates on Ω. An important method for solving the $\bar\partial$-equation is the integral representation for solutions. In this method, one constructs a differential form $B(z, w)$ on $\Omega\times\Omega$ which is an $(n, n-1)$ form in w such that solutions to $\bar\partial u = f$ can be written as

$$u(z) = \int_\Omega B(z, w) \wedge f(w). \tag{1-1}$$

The method of integral representation of solutions was initiated by Cauchy, Leray, Fantappié, etc. On a smoothly bounded strictly pseudoconvex domain Ω in $\mathbb{C}^n$, Henkin [1970] and Grauert and Lieb [1970] constructed integral kernels $B(z, w)$ such that u given by (1-1) satisfies $\|u\|_\infty \leq C\|f\|_\infty$. Kerzman [1971] improved the estimate by showing that $\|u\|_{C^\alpha(\bar\Omega)} \leq C_\alpha\|f\|_\infty$ for any $0 < \alpha < \frac{1}{2}$. Henkin and Romanov [1971] obtained the sharp estimate $\|u\|_{C^{1/2}(\bar\Omega)} \leq C\|f\|_\infty$. For more results on strictly pseudoconvex

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domains, the reader may consult the papers [Krantz 1976; Range and Siu 1972; 1973] and the books [Chen and Shaw 2001; Fornæss and Stensønes 1987; Range 1986].

When the class of domains under consideration is changed from strictly pseudoconvex to weakly pseudoconvex, it is no longer possible to conclude in generality the existence of uniform estimates for $\bar\partial$. Berndtsson [1993], Fornæss [1986] and Sibony [1980] constructed examples of weakly pseudoconvex domains in $\mathbb{C}^2$ and $\mathbb{C}^3$ where uniform estimates for $\bar\partial$ fail. More strikingly, Fornæss and Sibony [1993] constructed a smoothly bounded pseudoconvex domain $\Omega \subset \mathbb{C}^2$ such that $\partial\Omega$ is strictly pseudoconvex except at one point, but any solution to $\bar\partial u = f$ for some given $\bar\partial$-closed $f \in L^\infty_{(0,1)}(\Omega)$ does not belong to $L^p(\Omega)$ for any $2 < p \le \infty$. Range [1978] gave uniform estimates on bounded convex domains in $\mathbb{C}^2$ with real analytic boundaries, and in [Range 1990] gave Hölder estimates on pseudoconvex domains of finite type in $\mathbb{C}^2$. See [Laurent-Thiébaut and Leiterer 1993; Michel and Shaw 1999] for related results. Of particular interest is the unit polydisc $\mathbb{D}^n := \mathbb{D}(0,1)^n \subset \mathbb{C}^n$, which is pseudoconvex with nonsmooth boundary. When $n = 2$, Henkin [1971] showed that there exists a constant C such that $\|u\|_\infty \le C\|f\|_\infty$ for any $f \in C^1_{(0,1)}(\overline{\mathbb{D}}^2)$. Landucci [1975] obtained the same uniform estimate for the canonical solution on $\mathbb{D}^2$. Chen and McNeal [2020] and Fassina and Pan [2019] generalized Henkin's result to higher dimensions when additional regularity assumptions on f are imposed. It remains open whether uniform estimates hold on $\mathbb{D}^n$ with $n \ge 2$ when f is only assumed to be bounded. See [Dong et al. 2020; Fornæss et al. 2011; Grundmeier et al. 2022] for related results.

A class of pseudoconvex domains in $\mathbb{C}^n$ including $\mathbb{D}^n$ and the unit ball $\mathbb{B}^n$ are the so-called bounded symmetric domains, which up to biholomorphism are Cartesian product(s) of the Cartan classical domains of types I to IV and two domains of exceptional types. In [Henkin and Leiterer 1983, p. 200], the authors asked whether there exists a uniform estimate for the $\bar\partial$-equation on the Cartan classical domains of rank at least 2. Additionally, [Sergeev 1994] conjectured that the $\bar\partial$-equation cannot be solved with uniform estimates on the Cartan classical domains of type IV of dimension $n \ge 3$.

Let $g = (g_{j\bar k})^n_{j,k=1}$ be the Bergman metric on a domain Ω. For a $(0,1)$-form $f = \sum_{j=1}^n f_j\, d\bar z_j$, one defines

$$\|f\|^2_{g,\infty} := \operatorname{ess\,sup}\left\{\sum_{j,k=1}^n g^{j\bar k}(z) f_k(z)\overline{f_j(z)} : z \in \Omega\right\},$$

where $(g^{j\bar k})^\tau = (g_{j\bar k})^{-1}$; see (3-1) for details. Berndtsson used weighted L^2 estimates of Donnelly–Fefferman type to prove the following pointwise and uniform estimates.

Theorem 1.1 [Berndtsson 1997, 2001]. *There is a numerical constant C such that for any $\bar\partial$-closed $(0,1)$-form f on $\mathbb{B}^n$, the canonical solution to $\bar\partial u = f$ satisfies*

$$|u(z)| \le C\|f\|_{g,\infty} \log\frac{2}{1-|z|}, \tag{1-2}$$

and for any $\epsilon > 0$,

$$\|u\|_\infty \le \frac{C}{\epsilon}\|(1-|z|^2)^{-\epsilon} f\|_{g,\infty}. \tag{1-3}$$

The estimate (1-2) is sharp. If $f(z) := \sum_{k=1}^n z_k(|z|^2-1)^{-1}\, d\bar z_k$ then f is $\bar\partial$-closed, $\|f\|_{g,\infty} = 1$ and the canonical solution to $\bar\partial u = f$ is $u = \log(1-|z|^2) - C_n$, which shows the sharpness of (1-2).

Berndtsson [2001] also pointed out that his proof should generalize to other domains when enough information about the Bergman kernel is known. This result [Berndtsson 1997, 2001] was improved in [Schuster and Varolin 2014] via the "double twisting" method.

Motivated by Berndtsson's results (1-2) and (1-3) and the problems raised by Henkin and Leiterer [1983] and Sergeev [1994], we study sharp pointwise estimates for $\bar\partial u = f$ for any $\bar\partial$-closed $(0, 1)$-form f with $\|f\|_{g,\infty} < \infty$ and uniform estimates under stronger conditions on f. We generalize Berndtsson's results from $\mathbb{B}^n$ to smoothly bounded strictly pseudoconvex domains and the Cartan classical domains. Our main theorem, Theorem 1.2 (see also Theorem 3.3), for pointwise estimates is stated as follows.

Theorem 1.2. *Let Ω be a smoothly bounded strictly convex domain, a Cartan classical domain or the polydisc, whose Bergman kernel and metric are denoted by K and g, respectively. Then there is a constant C such that for any $\bar\partial$-closed $(0, 1)$-form f with $\|f\|_{g,\infty} < \infty$, the canonical solution to $\bar\partial u = f$ satisfies*

$$|u(z)| \le C\|f\|_{g,\infty} \int_\Omega |K(z, w)|\, dv_w, \quad z \in \Omega. \tag{1-4}$$

Remarks. (i) When Ω is a smoothly bounded strictly pseudoconvex domain, by Fefferman's asymptotic expansion for the Bergman kernel,

$$\int_\Omega |K(z, w)|\, dv_w \approx C \log \frac{1}{\delta_\Omega(z)} \approx \log K(z, z), \quad z \to \partial\Omega.$$

In this case, the estimate (1-4) is sharp. Take for example $\Omega = \mathbb{B}^n$ and $u(z) = \log K(z, z) - c$, where c is chosen so that $P[u] = 0$.

(ii) We will show in Section 3, Theorem 3.4, that if Ω is a smoothly bounded strictly pseudoconvex domain, then (1-4) holds for a solution u which may not be canonical.

(iii) When Ω is the unit polydisc $\mathbb{D}^n$, one has

$$\int_\Omega |K(z, w)|\, dv_w \approx \prod_{j=1}^n \log \frac{2}{1-|z_j|}, \quad z \to \partial\Omega.$$

(iv) When Ω is a Cartan classical domain of rank greater than or equal to 2, the blow-up order of $\int_\Omega |K(z, w)|\, dv_w$ depends on the direction in which z approaches $\partial\Omega$ and it may be larger than $-\log \delta_\Omega(z)$. For example, if $z = tI_2 \in \Omega := \mathrm{II}(2)$, then $\int_\Omega |K(z, w)|\, dv_w \approx \delta_\Omega(z)^{-1/2}$ as $t \to 1^-$. Moreover, if $z = te_1 \oplus te_2 \in \mathrm{IV}(n)$ with $e_j \in \mathcal{U}$ and $n \ge 3$, then $\int_\Omega |K(z, w)|\, dv_w \approx \delta_\Omega(z)^{1-(n/2)}$ as $t \to 1^-$. Here $\mathcal{U}$ denotes the characteristic boundary of Ω.

(v) In Section 6B, we show the estimate (1-4) is sharp on the Cartan classical domains.

Our main theorem for uniform estimates is stated as follows, as a combination of Theorems 4.1 and 4.2.

Theorem 1.3. *Let Ω be either the unit polydisc or a smoothly bounded strictly convex domain, whose Bergman kernel and metric are denoted by K and g, respectively. Then for any $p \in (1, \infty)$, there is a constant C such that for any $\bar\partial$-closed $(0, 1)$-form f, the canonical solution to $\bar\partial u = f$ satisfies*

$$\|u\|_\infty \le C \left\| f(\,\cdot\,) \left(\int_\Omega |K(\,\cdot\,, w)|\, dv_w \right)^p \right\|_{g,\infty}.$$

For Cartan classical domains, we give a uniform estimate under condition (5-2) in Theorem 5.4.

This paper is organized as follows: In Section 2, we recall and prove some properties of the Bergman kernel and metric which will be used later. In Section 3, we use L^2-methods to establish pointwise estimates on strictly pseudoconvex domains and the Cartan classical domains. In Sections 4 and 5, we obtain uniform estimates on the polydiscs, strictly pseudoconvex domains and the Cartan classical domains under various conditions on f. In Section 6, we verify the sharpness of our pointwise estimates on the Cartan classical domains; in particular, on $\mathrm{IV}(n)$ with $n \geq 3$ we show the estimate has maximum blow-up order $\delta^{1-(n/2)}(z)$.

2. Bergman kernel and metric

The Bergman space $A^2(\Omega)$ on a domain $\Omega \subset \mathbb{C}^n$ is the closed holomorphic subspace of $L^2(\Omega)$. The Bergman projection is the orthogonal projection $P_\Omega : L^2(\Omega) \to A^2(\Omega)$ given by

$$P_\Omega[f](z) = \int_\Omega K(z, w) f(w)\, dv(w),$$

where $K(z, w)$ is the Bergman kernel on Ω and dv is the Lebesgue $\mathbb{R}^{2n}$ measure. We will write $K(z)$ to denote the on-diagonal Bergman kernel $K(z, z)$. When Ω is bounded, the complex Hessian of $\log K(z)$ induces the Bergman metric $B_\Omega(z; X)$ defined by

$$B_\Omega(z; X) := \left(\sum_{j,k=1}^{n} g_{j\bar{k}} X_j \bar{X}_k \right)^{1/2}, \quad g_{j\bar{k}}(z) := \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K(z), \quad \text{for } z \in \Omega, \quad X \in \mathbb{C}^n.$$

The Bergman distance between $z, w \in \Omega$ is

$$\beta_\Omega(z, w) := \inf\left\{ \int_0^1 B_\Omega(\gamma(t); \gamma'(t))\, dt \right\},$$

where the infimum is taken over all piecewise C^1-curves $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = z$ and $\gamma(1) = w$. Throughout the paper,

$$B_a(\epsilon) := \{ z \in \Omega : \beta_\Omega(z, a) \leq \epsilon \} \tag{2-1}$$

will denote the hyperbolic ball in the Bergman metric centered at $a \in \Omega$ of radius ϵ. Additionally, $K(z, w)$, P_Ω and g will always denote the Bergman kernel, the Bergman projection on Ω and the Bergman metric, respectively.

Consider a convex domain Ω that contains no complex lines and $a \in \Omega$. Choose any $a^1 \in \partial\Omega$ such that $\tau_1(a) := |a - a^1| = \operatorname{dist}(a, \partial\Omega)$ and define $V_1 = a + \operatorname{span}(a^1 - a)^\perp$. Let $\Omega_1 = \Omega \cap V_1$ and choose any $a^2 \in \partial\Omega_1$ such that $\tau_2(a) := \|a - a^2\| = \operatorname{dist}(a, \partial\Omega_1)$. Let $V_2 = a + \operatorname{span}(a^1 - a, a^2 - a)^\perp$ and $\Omega_2 = \Omega \cap V_2$. Repeat this process to obtain $a^1, \dots, a^n$ and $w_k = (a^k - a)/\|a^k - a\|$, $1 \leq k \leq n$. Define

$$D(a; w, r) = \{ z \in \mathbb{C}^n : |\langle z - a, w_k \rangle| < r_k,\ 1 \leq k \leq n \} \tag{2-2}$$

and

$$D(a, r) = \{ z \in \mathbb{C}^n : |z_k - a_k| < r_k,\ 1 \leq k \leq n \}.$$

By [Nikolov and Pflug 2003, Theorem 2], for convex domains that contain no complex lines, the Kobayashi metric and the Bergman metric are comparable. It follows by [Nikolov and Trybuła 2015, Corollary 2] that if Ω is a convex domain with no complex lines, then for every $\epsilon > 0$ there exists constants C_1 and C_2 such that for any a,

$$D(a; w, C_1\tau(a)) \subset B_a(\epsilon) \subset D(a; w, C_2\tau(a)). \tag{2-3}$$

By [Nikolov and Pflug 2003, Theorem 1],

$$\frac{1}{4^n} \leq K(a) \prod_{j=1}^{n} \pi \tau_j^2(a) \leq \frac{(2n)!}{2^n},$$

which implies that

$$\left(\frac{C_1}{2}\right)^{2n} \leq K(a) v(B_a(\epsilon)) \leq (2n)! \left(\frac{C_2^2}{2}\right)^n.$$

For any open subset A of Ω, we define

$$\|\bar{\partial} u\|_{g,\infty,A} = \| |\bar{\partial} u(z)|_g \|_{L^\infty(A)}.$$

In the proofs of this paper, C will denote a numerical constant which may be different at each appearance. The Cauchy–Pompeiu formula gives the following useful proposition.

Proposition 2.1. *Let Ω be a bounded convex domain. For any $\epsilon > 0$ sufficiently small, there exists a constant C such that for any complex-valued C^1 function u on Ω*

$$|u(a)| \leq C \oint_{B_a(\epsilon)} |u(z)|\, dv_z + C \|\bar{\partial} u\|_{g,\infty,B_a(\epsilon)}.$$

Proof. After a complex rotation, without loss of generality, we may assume the standard basis for $\mathbb{C}^n$ is $(w_k)_{k=1}^n$, using the notation of (2-2). Let $r_k(a) = C_1\tau_k(a)$, where C_1 is the same constant as in (2-3). Define the metric

$$M_A(z; X) = \left(\sum_{k=1}^{n} \frac{|X_k|^2}{\tau_k(z)^2}\right)^{1/2}, \quad X \in \mathbb{C}^n.$$

It was proved in [McNeal 2001] (see also [McNeal 1994; Nikolov and Pflug 2003]) that

$$M_A(z; X) \approx B_\Omega(z; X), \quad X \in \mathbb{C}^n,$$

where $\approx$ is independent of z and X. So we can choose holomorphic coordinates such that

$$\frac{1}{C} D\left[\frac{1}{\tau_1^2}, \ldots, \frac{1}{\tau_n^2}\right] \leq [g_{i\bar{j}}] \leq C D\left[\frac{1}{\tau_1^2}, \ldots, \frac{1}{\tau_n^2}\right],$$

where $D[a_1, \ldots, a_n]$ is a diagonal matrix with diagonal entries $a_1, \ldots, a_n$. Therefore

$$\frac{1}{C} D[\tau_1^2, \ldots, \tau_n^2] \leq [g_{i\bar{j}}]^{-1} \leq C D[\tau_1^2, \ldots, \tau_n^2].$$

Additionally, by (2-3) and the definition of the hyperbolic ball (2-1),

$$\tau_k(a) \leq C\tau_k(z), \quad z \in D(a; w, C_1\tau(a)),$$

and the constant C is independent of a. Therefore, for $a = (a_1, \dots, a_n) \in \Omega$,

$$\begin{aligned} r_1(a)\|\bar\partial_1 u(\,\cdot\,, a_2, \dots, a_n)\|_{L^\infty(D(a_1,r_1))} &\le C \sup_{z\in D(a;w,r)} \sum_{k=1}^n r_k(z)\Big|\frac{\partial u}{\partial \bar w_k}(z)\Big| \\ &\le C \sup_{z\in D(a;w,r)} \Big(\sum_{k=1}^n \tau_k^2(z)\Big|\frac{\partial u}{\partial \bar w_k}(z)\Big|^2\Big)^{1/2} \\ &\le C\|\bar\partial u\|_{g,\infty,D(a;w,r)} \le C\|\bar\partial u\|_{g,\infty,B_a(\epsilon)}. \end{aligned}$$

By Stokes' theorem, for $0 < s_k < r_k$,

$$u(a) = \frac{1}{2\pi i}\int_{|w_1-a_1|=s_1} \frac{u(w_1, a_2, \dots, a_n)}{w_1 - a_1}\, dw_1 + \frac{1}{2\pi i}\int_{|w_1-a_1|<s_1} \frac{\partial u}{\partial \bar w_1}\frac{1}{w_1-a_1}\, dw_1 \wedge d\bar w_1.$$

By polar coordinates and (2-3),

$$\begin{aligned} |u(a)| &\le \frac{1}{\pi r_1^2}\int_{|z_1-a_1|<r_1} |u(w_1, a_2, \dots, a_n)|\, dv_{z_1} + \frac{2r_1}{3}\|\bar\partial_1 u(\,\cdot\,, a_2, \dots, a_n)\|_{L^\infty(D(a_1,r_1))} \\ &\le \frac{1}{\pi r_1^2}\int_{|w_1-a_1|<r_1} |u(w_1, a_2, \dots, a_n)|\, dv_{w_1} + C\|\bar\partial u\|_{g,\infty,B_a(\epsilon)}. \end{aligned}$$

Using the same estimate on the disc $|w_k - a_k| < s_k$ for $2 \le k \le n$,

$$\begin{aligned} |u(a)| &\le \oint_{D(a,C_1\tau(a))} |u(w_1, \dots, w_n)|\, dv_w + C\|\bar\partial u\|_{g,\infty,B_a(\epsilon)} \\ &\le C^{2n}\oint_{B_a(\epsilon)} |u(w)|\, dv_w + C\|\bar\partial u\|_{g,\infty,B_a(\epsilon)}. \end{aligned}$$ □

We remark that Proposition 2.1 also holds for smoothly bounded strictly pseudoconvex domains.

For positive real-valued functions f and g on Ω, we say $f \approx g$ for $z \in B_a(\epsilon)$ if for every $\epsilon > 0$ sufficiently small, there exists a constant $C = C(\epsilon, \Omega)$ such that

$$C^{-1} \le f(z)g(z)^{-1} \le C, \quad z \in B_a(\epsilon)$$

for all $a \in \Omega$. A similar definition holds for $f \approx g$ for $z \in \Omega$.

A domain Ω is homogeneous if it has a transitive (holomorphic) automorphism group. For convex homogeneous domains, the following results are known; see [Ishi and Yamaji 2011].

Proposition 2.2. *Let Ω be a bounded homogeneous convex domain. Then*

$$|K(z,a)| \approx K(a) \approx \frac{1}{v(B_a(\epsilon))}, \quad z \in B_a(\epsilon),$$

and for any $\epsilon > 0$, there is a C_ϵ such that for any $a \in \Omega$

$$\max_{w\in B_a(\epsilon)} \Big|\frac{K(z,w)}{K(z,a)}\Big| \le C_\epsilon, \quad z \in \Omega.$$

Let Ω be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and let $-r \in C^\infty(\overline{\Omega})$ be a strictly plurisubharmonic defining function for Ω. Define

$$X(z,w) = r(w) + \sum_{j=1}^n \frac{\partial r(w)}{\partial z_j}(z_j - w_j) + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 r(w)}{\partial z_i \partial z_j}(z_i - w_i)(z_j - w_j).$$

It was proved by Fefferman [1974] that there is a $\delta > 0$ such that

$$K(z,w) = \frac{F(z,w)}{X(z,w)^{n+1}} + G(z,w)\log X(z,w)$$

for all $(z,w) \in R_\delta(\Omega) = \{(z,w) \in \Omega \times \Omega : r(z) + r(w) + |z-w|^2 < \delta\}$, where $F, G \in C^\infty(\overline{\Omega} \times \overline{\Omega})$ and $F(z,z) > 0$ on $\partial\Omega$.

When Ω is a smoothly bounded strictly convex domain, the definition of $X(z,w)$ can be simplified. In fact, one can take

$$X(z,a) = h_a(z) = r(a) + \sum_{j=1}^n \frac{\partial r}{\partial z_j}(a)(z_j - a_j).$$

We can take $-r(z)$ to be strictly convex. Then by Taylor's theorem, one can easily see that

$$\operatorname{Re} h_a(z) \approx \operatorname{Re} X(z,w).$$

Moreover, for any $a, z \in \Omega$, we will write $\tilde z = (x_j)_{j=1}^{2n}$ and $\tilde a = (a_j)_{j=1}^{2n}$ if $z = (x_{2j-1} + i x_{2j})_{j=1}^n$ and $a = (a_{2j-1} + i a_{2j})_{j=1}^n$. If we apply Taylor's theorem on the line segment between $\tilde a$ and $\tilde z$, then there is a $\theta \in (0,1)$ such that

$$\operatorname{Re} h_a(z) = r(z) - \sum_{i,j=1}^{2n} \frac{\partial^2 r(\tilde a + \theta(\tilde z - \tilde a))}{\partial \tilde z_i \partial \tilde z_j}(\tilde z_i - \tilde a_i)(\tilde z_j - \tilde a_j) \approx r(z) + |z-a|^2.$$

Therefore, for any $a \in \Omega$,

$$|h_a(z)| \approx \frac{r(z)+r(a)}{2} + |z-a|^2 + \left|\operatorname{Im} \sum_{j=1}^n \frac{\partial r(a)}{\partial z_j}(z_j - a_j)\right|, \quad z \in \Omega.$$

In particular, this implies $h_a(z) \neq 0$. Therefore, by Fefferman's asymptotic expansion [1974] on strictly pseudoconvex domains mentioned above, we know the following.

Lemma 2.3 [Fefferman 1974]. *Let Ω be a smoothly bounded strictly convex domain. Then,*

$$|h_a(z)|^{-n-1} \approx K(a) \approx \frac{1}{v(B_a(\epsilon))}, \quad z \in B_a(\epsilon),$$

and there is a constant C such that

$$\int_\Omega |h_a(z)|^{-n-1}\, dv_z \approx \int_\Omega |K(z,a)|\, dv_z \approx \log \frac{C}{\delta_\Omega(a)}, \quad a \in \Omega,$$

where $\delta_\Omega(\cdot)$ is the Euclidean distance function to $\partial\Omega$. Moreover, for any $\epsilon > 0$, there is a constant C_ϵ such that for any $a \in \Omega$

$$\max_{w \in B_a(\epsilon)} |K(z,w) h_a(z)^{n+1}| \le C_\epsilon, \quad z \in \Omega.$$

Note. We provide some insight into the integration of $|K(z, w)|$ as a Forelli–Rudin-type integral. Roughly, one can view $\partial\Omega$ as a space of homogeneous type with Borel measure dX and quasidistance $|X(z, w)|$. Write

$$|K(z, w)| \approx (\delta(z) + t + |X(\pi(z), \pi(w))|)^{-n-1},$$

where $\pi(z)$ and $\pi(w)$ are the projections of z and w on $\partial\Omega$ along the outer normal direction and $z, w \in R_\delta$. It follows that

$$\int_{\partial\Omega} (\delta(z) + t + |X|)^{-n-1}\, dX \approx (\delta + t)^{-1}.$$

Consequently,

$$\int_\Omega |K(z, w)|\, dv(w) \approx \int_0^C (\delta(z) + t)^{-1}\, dt \approx \log \frac{1}{\delta(z)}.$$

For more information, one can consult the paper of Beatrous and the second author [Beatrous and Li 1993] and the papers of Krantz and the second author [Krantz and Li 2001a; 2001b].

Lemma 2.4. *Let Ω be either a smoothly bounded strictly pseudoconvex domain or a Cartan classical domain. Let $\phi(z) := \gamma \log K(z)$ with $\gamma > 0$. Then, for γ sufficiently small,*

$$\int_\Omega e^{\phi(z)}\, dv_z < \infty \quad \textit{and} \quad \|\bar\partial\phi\|^2_{i\partial\bar\partial\phi} \le \tfrac{1}{2}.$$

Proof. When Ω is a smoothly bounded strictly pseudoconvex domain, from Fefferman's asymptotic expansion for the Bergman kernel, one has

$$\phi(z) \approx \gamma \log \frac{1}{\delta(z)} \quad \text{and} \quad \int_\Omega e^{\phi(z)}\, dv(z) \approx \int_\Omega \Big(\frac{1}{\delta(z)}\Big)^\gamma dv \approx \int_0^1 t^{-\gamma}\, dt < \infty.$$

Notice that

$$\|\bar\partial\phi\|^2_{i\partial\bar\partial\phi} = \gamma \|\bar\partial \log K\|^2_g,$$

where g is the Bergman metric. From Fefferman's asymptotic expansion formula (see also [Donnelly 1994]), one gets the boundedness of $\|\bar\partial \log K\|^2_g$. Choose $\gamma > 0$ small enough so that $\|\bar\partial\phi\|^2_g < \frac{1}{2}$. For the Cartan classical domains, the first inequality follows from explicit formulas of the Bergman kernel [Hua 1963], and we compute the precise value of $\|\partial\phi\|_{i\partial\bar\partial\phi}$ in Section 6A. □

3. Pointwise estimates

An upper semicontinuous function ϕ defined on a domain $\Omega \subset \mathbb{C}^n$ with values in $\mathbb{R} \cup \{-\infty\}$ is called plurisubharmonic if its restriction to every complex line is subharmonic. Let $L^2(\Omega, \phi)$ denote the set of measurable functions h satisfying $\int_\Omega |h(z)|^2 e^{-\phi(z)}\, dv_z < \infty$. A C^2 function ϕ is called strongly plurisubharmonic if $i\partial\bar\partial\phi$ is strictly positive definite. Now, let Ω be a bounded pseudoconvex domain and ϕ be strongly plurisubharmonic on Ω. Then, for any $(0, 1)$-form $f = \sum_{k=1}^n f_k(z)\, d\bar z_k$, define the norm of f induced by $i\partial\bar\partial\phi$ as (see also [Błocki 2005])

$$|f|^2_{i\partial\bar\partial\phi}(z) := \sum_{j,k=1}^n \phi^{j\bar k}(z)\, \overline{f_j(z)}\, f_k(z), \tag{3-1}$$

where $(\phi^{j\bar{k}})^\tau$ equals the inverse of the complex Hessian matrix $H(\phi)$. Demailly's reformulation [1982; 2012] of Hörmander's theorem [1965] says that *for any* $\bar{\partial}$*-closed* $(0, 1)$*-form* f, *the* $L^2(\Omega, \phi)$ *minimal solution to* $\bar{\partial}u = f$ *satisfies*

$$\int_\Omega |u|^2 e^{-\phi}\, dv \le \int_\Omega |f|^2_{i\partial\bar{\partial}\phi} e^{-\phi}\, dv. \tag{3-2}$$

From this we see that when the $(0, 1)$-form f is bounded in the Bergman metric g, the canonical solution u to $\bar{\partial}u = f$ exists and the right-hand side of the estimate (3-2) is finite because it is dominated by a constant times a positive power of the Euclidean volume.

Donnelly and Fefferman [1983] (see also [Berndtsson 1993; 1996; 1997; McNeal 1996; Siu 1996]) modified Hörmander's theorem further as follows.

Theorem 3.1 (Donnelly–Fefferman-type estimate). *Let* Ω *be a bounded pseudoconvex domain in* $\mathbb{C}^n$, *and let* ψ *and* ϕ *be plurisubharmonic functions on* Ω *such that* $i\partial\bar{\partial}\phi > 0$ *and* $|\partial\phi|^2_{i\partial\bar{\partial}\phi} \le \frac{1}{2}$. *Then the* $L^2\big(\Omega, \psi + \frac{1}{2}\phi\big)$ *minimal solution* u_0 *to* $\bar{\partial}u = f$ *satisfies*

$$\int_\Omega |u_0|^2 e^{-\psi}\, dv \le 4 \int_\Omega |f|^2_{i\partial\bar{\partial}\phi} e^{-\psi}\, dv. \tag{3-3}$$

Next we prove the following lemma, using the estimates (3-2) and (3-3).

Lemma 3.2. *Let* Ω *be a bounded pseudoconvex domain and* f *be a* $\bar{\partial}$*-closed* $(0, 1)$*-form on* Ω. *Let* ψ *and* ϕ *be plurisubharmonic on* Ω *and* u_0 *and* u_1 *be the* L^2*-minimal solutions to* $\bar{\partial}u = f$ *in* $L^2\big(\Omega, \psi + \frac{1}{2}\phi\big)$ *and* $L^2(\Omega, \phi)$, *respectively. Suppose* B *is a compact subset of* Ω *and* $h \in L^\infty(\Omega)$ *with support in* B.

(i) *If* $i\partial\bar{\partial}\phi > 0$ *and* $\|\partial\phi\|^2_{i\partial\bar{\partial}\phi} \le \frac{1}{2}$ *on* Ω, *then*

$$\int_B |u_0|\, dv \le 2\left(\int_\Omega |f|^2_{i\partial\bar{\partial}\phi} e^{-\psi}\, dv\right)^{1/2} \left(\int_B e^{\psi}\, dv\right)^{1/2} \tag{3-4}$$

and

$$\left|\int_\Omega u_0 \overline{P(h)}\, dv\right| \le 2v(B)\|h\|_\infty \left(\int_\Omega |f|^2_{i\partial\bar{\partial}\phi} e^{-\psi}\, dv\right)^{1/2} \left(\int_\Omega \max_{w\in B} |K(z, w)|^2 e^{\psi(z)}\, dv_z\right)^{1/2}. \tag{3-5}$$

(ii) *Additionally*,

$$\int_B |u_1|\, dv \le 2\left(\int_\Omega |f|^2_{i\partial\bar{\partial}\phi} e^{-\phi}\, dv\right)^{1/2} \left(\int_B e^{\phi}\, dv\right)^{1/2}$$

and

$$\left|\int_\Omega u_1 \overline{P(h)}\, dv\right| \le 2v(B)\|h\|_\infty \left(\int_\Omega |f|^2_{i\partial\bar{\partial}\phi} e^{-\phi}\, dv\right)^{1/2} \left(\int_\Omega \max_{w\in B} |K(z, w)|^2 e^{\phi(z)}\, dv_z\right)^{1/2}.$$

Proof. Let χ_B denote the characteristic function on B, and let $\beta := \chi_B(u_0(z)/|u_0(z)|)$. By (3-3),

$$\left(\int_B |u_0|\, dv\right)^2 = \left|\int_\Omega u_0 \bar{\beta}\, dv\right|^2 \le \int_\Omega |u_0|^2 e^{-\psi}\, dv \int_B |\beta|^2 e^{\psi}\, dv \le 4 \int_\Omega |f|^2_{i\partial\bar{\partial}\phi} e^{-\psi}\, dv \int_B e^{\psi}\, dv,$$

which proves (3-4). Notice that

$$\left|\int_\Omega u_0\overline{P(h)}\,dv\right|^2 \le \int_\Omega |u_0|^2 e^{-\psi}\,dv \int_\Omega |P(h)|^2 e^{\psi}\,dv$$
$$\le 4\int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\psi}\,dv\cdot v^2(B)\|h\|^2_\infty \int_\Omega \max_{w\in B}|K(z,w)|^2 e^{\psi(z)}\,dv_z,$$

which proves (3-5). Part (ii) can be proved similarly using Hörmander's estimate (3-2) in place of Donnelly–Fefferman's estimate (3-3). □

Theorem 3.3 (key estimate). *Let Ω be a Cartan classical domain or a smoothly bounded strictly convex domain. Then there is a constant C such that for any $\bar\partial$-closed $(0,1)$-form f on Ω with $\|f\|_{g,\infty}<\infty$, the canonical solution to $\bar\partial u = f$ satisfies*

$$|u(z)| \le C\|f\|_{g,\infty}\int_\Omega |K(z,w)|\,dv_w, \quad z\in\Omega.$$

Proof. From the discussion after (3-2) we see that the canonical solution to $\bar\partial u = f$ exists. Suppose first that Ω is a Cartan classical domain. For an arbitrary $a\in\Omega$ and any sufficiently small $\epsilon>0$, let $\beta := \chi_{B_a(\epsilon)}(u(z)/|u(z)|)$, where $\chi_{B_a(\epsilon)}$ is the characteristic function of the hyperbolic ball $B_a(\epsilon)$. Let $\phi := \gamma\log K(z)$ be a plurisubharmonic function on Ω for some chosen γ that satisfies the condition in Lemma 2.4. Define $\psi(z) =: \psi_a(z) := -\log|K(z,a)|$. Then ψ_a is pluriharmonic and bounded on Ω. Also define the function

$$\phi_0 := \psi_a + \tfrac{1}{2}\phi,$$

and let u_0 be the $L^2(\Omega,\phi_0)$ minimal solution to the equation $\bar\partial v = f$. Then by Theorem 3.1,

$$\int_\Omega |u_0|^2 e^{-\psi}\,dv \le 4\gamma^{-1}\int_\Omega |f|^2_g e^{-\psi}\,dv \le 4\gamma^{-1}\|f\|^2_{g,\infty}\int_\Omega e^{-\psi}\,dv < \infty,$$

which implies that $u_0\in L^2(\Omega)$. So $u-u_0\in A^2(\Omega)$ and

$$\int_{B_a(\epsilon)}|u|\,dv = \int_\Omega u\bar\beta\,dv = \int_\Omega u(\overline{\beta-P(\beta)})\,dv = \int_\Omega u_0(\overline{\beta-P(\beta)})\,dv = \int_\Omega u_0\bar\beta\,dv - \int_\Omega u_0\overline{P(\beta)}\,dv.$$

By Lemma 2.4 and (3-4) in Lemma 3.2,

$$\left|\int_\Omega u_0\bar\beta\,dv\right|^2 \le 4\int_\Omega |f|^2_{i\partial\bar\partial\phi}e^{-\psi_a}\,dv\int_{B_a(\epsilon)} e^{\psi_a}\,dv$$
$$\le C\|f\|^2_{g,\infty}\int_\Omega |K(z,a)|\,dv_z\int_{B_a(\epsilon)}|K(z,a)|^{-1}\,dv_z$$
$$\le C_\epsilon\|f\|^2_{g,\infty}\int_\Omega |K(z,a)|\,dv_z\cdot v(B_a(\epsilon))K(a)^{-1}$$
$$\le C_\epsilon\|f\|^2_{g,\infty}v^2(B_a(\epsilon))\int_\Omega |K(z,a)|\,dv_z,$$

where the last two inequalities hold due to Proposition 2.2 and C_ϵ is a constant depending on ϵ. On the other hand, by (3-5) in Lemma 3.2 and Proposition 2.2 again,

$$\begin{aligned}\left|\int_\Omega u_0 \overline{P(\beta)}\,dv\right|^2 &\le C v^2(B_a(\epsilon)) \int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\psi_a}\,dv \int_\Omega \max_{w\in \overline{B_a(\epsilon)}} |K(z,w)|^2 e^{\psi_a(z)}\,dv_z \\ &\le C_\epsilon v^2(B_a(\epsilon)) \int_\Omega |f|^2_{i\partial\bar\partial\phi}(z)|K(z,a)|\,dv_z \int_\Omega |K(z,a)|^{2-1}\,dv_z \\ &\le C_\epsilon \|f\|^2_{g,\infty} v^2(B_a(\epsilon)) \biggl(\int_\Omega |K(z,a)|\,dv_z\biggr)^2.\end{aligned}$$

Combining the above estimates, one easily sees that

$$\frac{1}{v(B_a(\epsilon))}\int_{B_a(\epsilon)} |u|\,dv \le C_\epsilon \|f\|_{g,\infty} \int_\Omega |K(z,a)|\,dv_z.$$

Fix $\epsilon > 0$. By Proposition 2.1, there exists a constant C depending only on Ω such that

$$|u(a)| \le C\|f\|_{g,\infty} \int_\Omega |K(z,a)|\,dv_z.$$

If Ω is instead a smoothly bounded strictly convex domain, then let $\psi_a(z) = (n+1)\log|h_a(z)|$, repeat the argument for the Cartan classical domains and use Lemma 2.3. □

In Section 6, Proposition 6.1, we show that the estimate in Theorem 3.3 is sharp for the Cartan classical domains. When Ω is the unit ball $\mathbb{B}^n$, the *key estimate* reduces to Berndtsson's result (1-2). Now we generalize (1-2) and (1-4) to smoothly bounded strictly pseudoconvex domains.

Theorem 3.4. *Let Ω be a smoothly bounded strictly pseudoconvex domain. Then there is a constant C such that for any $\bar\partial$-closed $(0,1)$-form f on Ω with $\|f\|_{g,\infty} < \infty$, there is a solution u to $\bar\partial u = f$ such that*

$$|u(z)| \le C\|f\|_{g,\infty} \log(1+K(z)), \quad z \in \Omega.$$

Proof. Let $r(z)$ be a strongly plurisubharmonic defining function for Ω such that $r(z) \in C^\infty(\bar\Omega)$ and $r > 0$ in Ω. Consider the polynomial

$$X(z,w) := r(w) + \sum_{j=1}^n \Bigl(\frac{\partial r}{\partial w_j}\Big|_w (z_j - w_j)\Bigr) + \frac{1}{2}\sum_{j,k=1}^n \Bigl(\frac{\partial^2 r}{\partial w_j \partial w_k}\Big|_w (z_j-w_j)(z_k-w_k)\Bigr).$$

Define the region $R_\epsilon = \{(z,w) : z, w \in \Omega,\ r(z)+r(w)+|z-w|^2 < \epsilon\}$. For $(z,w) \in R_\epsilon$, Fefferman [1974] showed the Bergman kernel on Ω can be expressed as

$$K(z,w) = \frac{F(z,w)}{X(z,w)^{n+1}} + G(z,w)\log X(z,w), \tag{3-6}$$

where $G, F \in C^\infty(\bar\Omega \times \bar\Omega)$, $F(z,z) > 0$ on $(\bar\Omega\times\bar\Omega)\cap R_\epsilon$ and "log" denotes the principal branch of the logarithm defined on $\mathbb{C}\setminus(-\infty,0]$. The asymptotic expansion (3-6) implies

$$\int_\Omega |K(z,w)|\,dv_w \le C(1+\log K(z)), \quad z \in \Omega. \tag{3-7}$$

Since the boundary $\partial\Omega$ is compact, for any $\delta > 0$ there are finitely many $b^j \in \partial\Omega,\ j = 1, \dots, m$, such that $\partial\Omega \subset \cup_{j=1}^m \mathbb{B}(b^j, \delta)$. Choose smoothly bounded strictly pseudoconvex domains $\Omega^j,\ j = 1, \dots, m$, such that

$$\mathbb{B}(b^j, 3\delta) \cap \Omega \subset \Omega^j \subset \Omega \cap \mathbb{B}(b^j, 4\delta),$$

where δ is chosen small enough such that for each j, after a polynomial change of variables, each Ω^j is a strictly convex domain. Let $\{\Omega^j\}_{j=m+1}^{m+k}$ be a finite open cover of $\Omega \setminus \cup_{j=1}^m (\mathbb{B}(b^j, \delta) \cap \Omega)$ consisting of balls contained in Ω. In the argument of Theorem 3.3 by letting $\phi_0 = \gamma \log K_\Omega(z)$ (instead of $\gamma \log K_{\Omega^j}(z)$) and using (3-7), we can solve the equation $\bar\partial u^j = f$ on Ω^j with minimal solution u^j satisfying

$$|u^j(z)| \le C\|f\|_{g,\infty} \log(1 + K(z)). \tag{3-8}$$

Let $\{\eta_j\}_{j=1}^{m+k}$ be a partition of unity of $\overline{\Omega}$ subordinate to the cover $\{B(b^j, \delta)\}_{j=1}^m \cup \{\Omega^j\}_{j=m+1}^{m+k}$, and let $v(z) := \sum_{j=1}^{m+k} \eta_j(z) u^j(z)$. Then $\bar\partial v = f + h$, where $h := \sum_{j=1}^{m+k} u^j \bar\partial \eta_j$ is a $\bar\partial$-closed $(0, 1)$-form on Ω. By the integral formula in [Grauert and Lieb 1970; Henkin 1970], there is a bounded solution v_0 to the equation $\bar\partial v_0 = h$. Let $u = v - v_0$. Then $\bar\partial u = f$ and by (3-8),

$$|u(z)| \le \sum_{j=1}^{k} \eta_j(z) C\|f\|_{g,\infty} \log(1 + K(z)) + C \le C\|f\|_{g,\infty} \log(1 + K(z)). \qquad \square$$

Remark. For a smoothly bounded strictly pseudoconvex domain Ω, if the canonical solution is u_0, then for $h \in L^\infty(\Omega)$ with $\|h\|_\infty \le 1$,

$$|P[h(\,\cdot\,) \log K(\,\cdot\,)](z)| \le C(1 + \log K(z))^2.$$

In fact, letting $\omega_t = \{z \in \Omega : \delta(z) > t\}$, by Fefferman's expansion theorem on the Bergman kernel [Fefferman 1974], we know that

$$\begin{aligned}
|P[h(\,\cdot\,) \log K(\,\cdot\,)](z)| &\le \int_\Omega |h(w)| \log K(w) |K(z, w)|\, dv(w) \le \|h\|_\infty \int_\Omega \log K(w) |K(z, w)|\, dv(w) \\
&\approx \|h\|_\infty \int_0^c \int_{\partial\Omega_t} \log K(w) |K(z, w)|\, d\sigma(w)\, dt \\
&\le C\|h\|_\infty \int_0^c (-\log t) \frac{1}{\delta(z)+t}\, dt \\
&\le C\|h\|_\infty \left(\frac{1}{\delta(z)} \int_0^{\delta(z)} (-\log t\, dt) + \int_{\delta(z)}^c -\log t \frac{1}{\delta(z)+t}\, dt \right) \\
&\le C\|h\|_\infty \left(\log \frac{C}{\delta(z)} + \int_{\delta(z)}^c -\log(\delta(z)+t) \frac{1}{\delta(z)+t}\, dt \right) \le C\|h\|_\infty \left(\log \frac{C}{\delta(z)} \right)^2.
\end{aligned}$$

Combining this with Theorem 3.4 one gets

$$|u_0(z)| \le C(1 + \log K(z))^2.$$

4. Uniform estimates

In this section, we obtain uniform estimates for the equation $\bar\partial u = f$ on the unit polydisc $\mathbb{D}^n$ and strictly pseudoconvex domains by imposing conditions on f stronger than $\|f\|_{g,\infty} < \infty$.

Theorem 4.1. *For any $p \in (1, \infty)$, there is a constant C such that for any $\bar\partial$-closed $(0, 1)$-form f on $\mathbb{D}^n$, the canonical solution u to $\bar\partial u = f$ satisfies*

$$\|u\|_\infty \le C\left\| f \prod_{j=1}^{n}\Bigl(\log\Bigl(\frac{2}{1-|z_j|^2}\Bigr)\Bigr)^p\right\|_{g,\infty}.$$

Proof. Let $A_0 = 2p + \log v(\mathbb{D}^n)$. Choose $0 < \gamma < \frac{1}{2}$ such that $\phi(z) := \gamma \log K(z)$ satisfies Lemma 2.4. Choose $\alpha > 1$ such that $\alpha - \gamma = 1$. Let $K_j(z) := \pi^{-1}(1 - |z_j|^2)^{-2}$, and let

$$\phi_0(z) := \phi(z) - \sum_{j=1}^{n} p \log(A_0 + \gamma \log K_j(z)) - \alpha \log|K(z, a)|.$$

Since on $\mathbb{D}^n$, $\log|K(z, a)|$ is pluriharmonic and

$$A_0 + \gamma \log K_j = 2p + n \log\pi - \gamma \log\pi - 2\gamma \log(1 - |z_j|^2) \ge 2p,$$

we know that

$$i\partial\bar\partial\phi = \sum_{j=1}^{n}\biggl[i\partial\bar\partial(\gamma \log K_j)\biggl(1 - \frac{p}{A_0 + \gamma \log K_j}\biggr) + \pi\frac{\partial(\gamma \log K_j) \wedge \bar\partial(\gamma \log K_j)}{(A_0 + \gamma \log K_j)^2}\biggr] \ge \frac{1}{2}\sum_{j=1}^{n} i\partial\bar\partial(\gamma \log K_j).$$

Thus $|f|^2_{i\partial\bar\partial\phi_0} \le 2|f|^2_{i\partial\bar\partial\phi} = 2|f|^2_g/\gamma$. Let

$$A_p(f) = \left\| f\biggl(\prod_{j=1}^{n}(A_0 - \gamma \log\pi - 2\gamma \log(1 - |z_j|^2))\biggr)^p\right\|_{g,\infty}.$$

As in Theorem 3.3, let u_0 be the $L^2(\mathbb{D}^n, \phi_0)$ minimal solution to $\bar\partial u_0 = f$ and $\beta := e^{i\theta(z)}\chi_{B_a(\epsilon)}$, where $u(z) = |u(z)|e^{i\theta(z)}$. Then

$$\begin{aligned}
\int_{B_a(\epsilon)} |u|\,dv &\le Cv(B_a(\epsilon))\biggl(\int_{D(0,1)^n} |f|^2_g e^{-\phi_0}\,dv\biggr)^{1/2}\biggl(\int_{D(0,1)^n} |K(z, a)|^2 e^{\phi_0(z)}\,dv_z\biggr)^{1/2}\\
&\le CA_p(f)v(B_a(\epsilon))\biggl(\int_{D(0,1)^n} \frac{e^{-\phi_0}\,dv_z}{\prod_{j=1}^{n}(A_0 + \gamma \log K_j(z))^{2p}} \int_{D(0,1)^n} |K(z, a)|^2 e^{\phi_0}\,dv_z\biggr)^{1/2}\\
&= CA_p(f)v(B_a(\epsilon))\prod_{j=1}^{n}\biggl(\int_{D(0,1)} \frac{|K_j(z, a)|^\alpha K_j(z)^{-\gamma}\,dv_z}{(A_0 + \gamma \log K_j(z))^p} \int_{D(0,1)} \frac{|K_j(z, a)|^{2-\alpha} K_j(z)^\gamma\,dv_z}{(A_0 + \gamma \log K_j(z))^p}\biggr)^{1/2}\\
&\le CA_p(f)v(B_a(\epsilon)).
\end{aligned}$$

Fix $\epsilon > 0$ sufficiently small. By Proposition 2.1,

$$\begin{aligned}|u(a)| &\le CA_p(f) + C\|f\|_{g,\infty}\\ &\le C\left\| f\prod_{j=1}^{n}(2(n+p) - \log(1-|z_j|^2))^p\right\|_{g,\infty}.\end{aligned}$$

Notice that

$$2(n+p) - \log(1-|z_j|^2) \le 5(n+p)\log\frac{2}{1-|z_j|^2},$$

which completes the proof of the theorem. □

In fact, the finiteness of the right-hand side of the estimate in Theorem 4.1 is a stronger condition than f being L^∞ on $\mathbb{D}^n$. Recently, in [Dong et al. 2020], the first author, Pan and Zhang obtained uniform estimates for the canonical solution to $\bar\partial u = f$ when f is continuous up to the boundary of $\mathbb{D}^n$, and more generally the Cartesian product of smoothly bounded planar domains.

However, the situation for strictly pseudoconvex domains is quite different. The finiteness of the right-hand side of the estimate in either Theorem 4.2 or Theorem 4.3 is a much weaker condition than f being L^∞ on each smooth domain considered. In fact, we allowed f to blow-up on $\partial\Omega$ to order less than $\frac{1}{2}$.

Theorem 4.2. *Let Ω be a smoothly bounded strictly convex domain. For any $p \in (1, \infty)$ and sufficiently small $\gamma > 0$, there exists a constant C such that for any $\bar\partial$-closed $(0, 1)$-form f, the canonical solution u to $\bar\partial u = f$ satisfies*

$$\|u\|_\infty \le C\|(1 + \log v(\Omega) + \gamma\log K(z))^p f\|_{g,\infty}.$$

Proof. Choose $0 < \gamma < 1/(n+2)$ such that $\phi(z) := \gamma\log K(z)$ satisfies Lemma 2.4, and let $\alpha = \gamma + 1$. Let

$$A_0 := 2p + \gamma\log v(\Omega),$$

and let

$$\phi_0(z) = \phi(z) - (n+1)\alpha\log|h_a(z)| - p\log(A_0 + \gamma\log K(z)).$$

Notice that

$$i\partial\bar\partial\phi_0 \ge \left(1 - \frac{p}{A_0+\phi}\right)i\partial\bar\partial\phi \ge \frac{i\partial\bar\partial\phi}{2}$$

and

$$\gamma K(z) > \gamma v(\Omega)^{-1}.$$

Therefore

$$|f|^2_{i\partial\bar\partial\phi_0} \le \frac{2}{\gamma}|f|^2_g.$$

Define

$$A_p(f) := \|(A_0 + \gamma\log K(z))^p f\|_{g,\infty}.$$

Using arguments similar to those in Theorems 3.3 and 4.1 and $\alpha - \gamma = 1$,

$$\begin{aligned}
&\int_{B_a(\epsilon)} |u|\,dv \\
&\quad\le Cv(B_a(\epsilon))\biggl(\int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\phi_0}\,dv\biggr)^{1/2}\biggl(\int_\Omega |K(z,a)|^2 e^{\phi_0(z)}\,dv_z\biggr)^{1/2} \\
&\quad\le CA_p(f)v(B_a(\epsilon))\biggl(\int_\Omega \frac{e^{-\phi_0}}{(A_0+\gamma\log K(z))^{2p}}\,dv_z \int_\Omega |K(z,a)|^2 e^{\phi_0}\,dv_z\biggr)^{1/2} \\
&\quad= CA_p(f)v(B_a(\epsilon))\biggl(\int_\Omega \frac{K(z)^{-\gamma}|h_a(z)|^{\alpha(n+1)}}{(A_0+\gamma\log K(z))^{p}}\,dv_z \int_\Omega \frac{|K(z,a)|^2 K(z)^{\gamma}|h_a(z)|^{-\alpha(n+1)}}{(A_0+\gamma\log K(z))^{p}}\,dv_z\biggr)^{1/2} \\
&\quad\le CA_p(f)v(B_a(\epsilon))\biggl(\int_\Omega \frac{|K(z,a)|\,dv_z}{(A_0+\gamma\log K(z))^{p}} \int_\Omega \frac{|K(z,a)|^{2-\alpha} K(z)^{\gamma}}{(A_0+\gamma\log K(z))^{p}}\,dv_z\biggr)^{1/2} \le CA_p(f)v(B_a(\epsilon)),
\end{aligned}$$

where the last inequality follows from Fefferman's asymptotic expansion. In fact, since $n/(n+1) < 2-\alpha$, if $\Omega_t = \{z : r(z) > t\}$ where $r(z)$ is a defining function satisfying the definition of $h_a(z)$, then

$$\begin{aligned}
\int_\Omega \frac{|K(z,a)|^{2-\alpha} K(z)^{\gamma}}{(A_0+\gamma\log K(z))^{p}}\,dv_z &\le C\biggl(1+\int_0^\epsilon \int_{\partial\Omega_t} \frac{|K(z,a)|^{2-\alpha} K(z)^{\gamma}}{(A_0+\gamma\log K(z))^{p}}\,d\sigma_t(z)\,dt\biggr) \\
&\le C\biggl(1+\int_0^\epsilon \frac{t^{-(2-\alpha)(n+1)+n}t^{-\gamma(n+1)}}{(A_0-\gamma(n+1)\log t)^{p}}\,dt\biggr) \\
&\le C\biggl(1+\int_0^\epsilon \frac{1}{t(\log t)^{p}}\,dt\biggr) \le C\biggl(1-\frac{\log\epsilon}{p-1}\biggr).
\end{aligned}$$

By Proposition 2.1, for a fixed $\epsilon > 0$ sufficiently small, $|u(a)| \le CA_p(f) + \|f\|_{g,\infty} \le CA_p(f)$. □

Using an argument similar to the proof of Theorem 3.4 we get the following generalization of Theorem 4.2 to smoothly bounded strictly pseudoconvex domains.

Theorem 4.3. *Let Ω be a smoothly bounded strictly pseudoconvex domain. Then, for any $p \in (1,\infty)$, there exists a constant C such that for any $\bar\partial$-closed $(0,1)$-form f, there is a solution u to $\bar\partial u = f$ that satisfies*

$$\|u\|_\infty \le C\|(\log K(\cdot))^p f(\cdot)\|_{g,\infty}.$$

Remark. Let $f \in L^\infty_{(0,1)}(\Omega)$ be a $\bar\partial$-closed form on a smoothly bounded strictly pseudoconvex domain Ω. Henkin and Romanov's theorem [1971] states that there exists a solution $u \in C^{1/2}(\Omega)$. Theorem 3.4 implies that one can find a bounded solution when $(\log(1/\delta(z)))^p\delta(z)|f(z)|^2$ is bounded. Moreover, [Lieb and Range 1986, Theorem 2 (i)] shows that uniform estimates hold for the canonical solutions to the $\bar\partial$-equations on Ω.

5. Additional estimates for Cartan classical domains

A domain Ω is symmetric if, for all $a \in \Omega$, there is an involutive automorphism G such that a is isolated in the set of fixed points of G. All bounded symmetric domains are convex and homogeneous. E. Cartan

classical domain	rank r	multiplicity $\boldsymbol{a}$	genus p	dimension N	index k
$\mathrm{I}(m,n),\ m\le n$	m	2	$m+n$	mn	1
$\mathrm{II}(n)$	n	1	$n+1$	$\frac{1}{2}n(n+1)$	1
$\mathrm{III}(2n+\epsilon),\ \epsilon=0$ or 1	n	4	$2(2n+\epsilon-1)$	$n(2n+2\epsilon-1)$	$\frac{1}{2}$
$\mathrm{IV}(n)$	2	$n-2$	n	n	1

Table 1. Characteristics of classical domains.

proved that all bounded symmetric domains in $\mathbb{C}^N$ up to biholomorphism are the Cartesian product(s) of the following four types of Cartan classical domains and two domains of exceptional types.

Definition 5.1. A Cartan classical domain is a domain of one of the following types:

(i) $\mathrm{I}(m,n) := \{z \in M_{(m,n)}(\mathbb{C}) : I_m - zz^* > 0\},\ m \le n.$

(ii) $\mathrm{II}(n) := \{z \in \mathrm{I}(n,n) : z^\tau = z\}.$

(iii) $\mathrm{III}(n) := \{z \in \mathrm{I}(n,n) : z^\tau = -z\}.$

(iv) $\mathrm{IV}(n) := \{z \in \mathbb{C}^n : 1-2|z|^2+|s(z)|^2 > 0$ and $|s(z)| < 1\}$, where $s(z) := \sum_{j=1}^n z_j^2$ and $n > 2$.

Here $z^* := \bar{z}^\tau$ is the conjugate transpose of z.

Let Ω be a Cartan classical domain. Denote the rank, characteristic multiplicity, genus, complex dimension and kernel index of Ω by r, $\boldsymbol{a}$, p, N and k, respectively. Their values are given in Table 1.

Hua [1963] obtained explicit formulas for the Bergman kernels on the Cartan classical domains. For a domain Ω of type I, II or III,

$$K(z,w) = C_\Omega[\det(I - zw^*)]^{-pk},$$

and for a domain of type IV,

$$K(z,w) = C_n\left[1 - 2\sum_{j=1}^n z_j\bar{w}_j + s(z)\overline{s(w)}\right]^{-n}.$$

For any $z \in \Omega$,

$$\delta_\Omega(z) \le K(z)^{-1/(rpk)}.$$

Let $\lambda = pk$. By [Faraut and Korányi 1990, Theorem 3.8], one can write the Bergman kernel on a Cartan classical domain Ω as

$$K(z,w) = h(z,w)^{-\lambda} = \sum_{\boldsymbol{m}\ge 0} (\lambda)_{\boldsymbol{m}} K_{\boldsymbol{m}}(z,w),$$

where

$$\boldsymbol{m} = (m_1,\dots,m_r) \text{ and } \boldsymbol{m} \ge 0 \quad\Longleftrightarrow\quad m_1 \ge m_2 \ge \cdots \ge m_r \ge 0,$$

and

$$(\lambda)_{\boldsymbol{m}} = \frac{\Gamma_\Omega(\boldsymbol{\lambda}+\boldsymbol{m})}{\Gamma_\Omega(\boldsymbol{\lambda})}, \quad \Gamma_\Omega(\boldsymbol{s}) = c_\Omega \prod_{j=1}^r \Gamma\Big(s_j - (j-1)\frac{a}{2}\Big), \quad \boldsymbol{\lambda} = (\lambda,\dots,\lambda).$$

Here, $K_{\boldsymbol{m}}$ is the Bergman kernel for homogeneous polynomials in $\mathbb{C}^r$ of degree $|\boldsymbol{m}| = m_1 + \cdots + m_r$. For each Cartan domain Ω, there is a subgroup $\mathcal{K}(\Omega)$ of the unitary group such that for each $z \in \Omega$ there is $k \in \mathcal{K}(\Omega)$ such that $z = k\tilde{z}$ where $\tilde{z} \in \mathbb{C}^r \times \prod_{j=r+1}^{N}\{0\}$ and $K_{\boldsymbol{m}}(z, z) =: K_{\boldsymbol{m}}(\tilde{z}, \tilde{z})$.

The following Forelli–Rudin-type integral was studied in [Faraut and Korányi 1990]:

$$J_{\beta,c}(z) := \int_\Omega K(w)^\beta |K(w, z)|^{1+c-\beta}\, dv_w.$$

By the proof of Theorem 4.1 in [Faraut and Korányi 1990], one has

$$J_{\beta,c}(z) = \sum_{\boldsymbol{m}\geq 0} \frac{|(\mu)_{\boldsymbol{m}}|^2}{((1-k\beta)p)_{\boldsymbol{m}}} K_{\boldsymbol{m}}(z, z), \qquad \mu = \tfrac{1}{2}kp(1+c-\beta).$$

Using Stirling's formula, one can show (see [Faraut and Korányi 1990, (4.3)] or [Engliš and Zhang 2004, (2.9)]) that as m varies,

$$\frac{\left|\left(\frac{1}{2}pk(1-\beta)\right)_{\boldsymbol{m}}\right|^2}{((1-k\beta)p)_{\boldsymbol{m}}} \approx \frac{\left(\frac{1}{2}kp\right)^2_{\boldsymbol{m}}}{(p)_{\boldsymbol{m}}},$$

which implies that

$$J_{\beta,0}(z) \approx J_{0,0}(z) = \int_\Omega |K(z, w)|\, dv(w), \qquad \beta < \frac{1}{pk}. \tag{5-1}$$

Further computations were carried out by Faraut and Koranyi [1990].

Theorem 5.2 [Faraut and Korányi 1990]. *For any $\beta < 1/(pk)$,*

(i) *$J_{\beta,c}(z)$ is bounded for all $z \in \Omega$ if and only if $c < -(r-1)\boldsymbol{a}/(2p)$,*

(ii) *$J_{\beta,c}(z) \approx K(z)^c$ if $c > (r-1)\boldsymbol{a}/(2p)$.*

When $|c| \leq (r-1)\boldsymbol{a}/(2p)$, it is difficult to compute $J_{\beta,c}(z)$; see [Korányi 1991; Yan 1992]. Theorem 1 of [Engliš and Zhang 2004], whose parameters are chosen as $\frac{1}{2}p(1+c-\beta)$, $\frac{1}{2}p(1+c-\beta)$ and $p(1-\beta)$, is stated as follows.

Theorem 5.3. *Let Ω be a Cartan classical domain of rank 2 with characteristic boundary $\mathcal{U}$. Then for any $z = te_1 + Te_2$ with $0 \leq t \leq T < 1$ and $e_1, e_2 \in \mathcal{U}$ the following statements hold*:

(i) *If $2pc = \boldsymbol{a}$, then $J_{\beta,c}(z) \approx (1-t)^{-\boldsymbol{a}/2}(1-T)^{-\boldsymbol{a}/2}[1-\log(1-t)]$.*

(ii) *If $0 < 2pc < \boldsymbol{a}$, then $J_{\beta,c} \approx (1-t)^{-\boldsymbol{a}/2}(1-T)^{-pc}$.*

(iii) *If $c = 0$, then $J_{\beta,c}(z) \approx (1-t)^{-\boldsymbol{a}/2}[1+\log[(1-t)/(1-T)]]$.*

(iv) *If $-\boldsymbol{a} < 2pc < 0$, then $J_{\beta,c}(z) \approx (1-t)^{-pc-\boldsymbol{a}/2}$.*

(v) *If $2pc = -\boldsymbol{a}$, then $J_{\beta,c}(z) \approx 1-\log(1-t)$.*

As a consequence, when Ω is a Cartan classical domain of rank 2 and $z = te_1 + te_2$ with $0 \leq t < 1$ and $e_i \in \mathcal{U}$, one has

$$\int_\Omega |K(z, w)|\, dv_w \approx (1-t)^{-\boldsymbol{a}/2} \approx \delta_\Omega(z)^{-\boldsymbol{a}/2}.$$

On the Cartan classical domains, we impose a stronger assumption on f to get bounded solutions to $\bar\partial u = f$. The following result provides a partial answer to the problems raised by Henkin and Leiterer [1983] and Sergeev [1994].

Theorem 5.4. *Let Ω be a Cartan classical domain and $\alpha > 1 + (r-1)\boldsymbol{a}/(2p)$. Then there exists a constant C such that for any $\bar\partial$-closed $(0,1)$-form f, the canonical solution u to $\bar\partial u = f$ satisfies*

$$\|u\|_\infty \le C \left\| \int_\Omega |f|_g^2(z)|K(z,\cdot)|^\alpha \, dv_z \right\|_\infty^{1/2} + C\|f\|_{g,\infty}. \tag{5-2}$$

Proof. As in the proof of Theorem 3.3, for any $a \in \Omega$, let $\beta := \chi_{B_a(\epsilon)} u(z)/|u(z)|$, $\phi := \gamma \log K(z)$ and $\psi_a(z) := -\alpha \log|K(z,a)|$. Then

$$\int_{B_a(\epsilon)} |u| \, dv \le \int_\Omega |u_0 \bar\beta| \, dv + \int_\Omega |u_0 \overline{P(\beta)}| \, dv.$$

By Lemma 2.4 and (3-4),

$$\begin{aligned}
\left| \int_\Omega u_0 \bar\beta \, dv \right| &\le C \left(\int_\Omega |f|_g^2(z)|K(z,a)|^\alpha \, dv_z \right)^{1/2} \left(\int_{B_a(\epsilon)} |K(z,a)|^{-\alpha} \, dv_z \right)^{1/2} \\
&\le C \left(\int_\Omega |f|_g^2(z)|K(z,a)|^\alpha \, dv_z \right)^{1/2} (v(B_a(\epsilon))K(a)^{-\alpha})^{1/2} \\
&\le C v(B_a(\epsilon))^{(1+\alpha)/2} \left(\int_\Omega |f|_g^2(z)|K(z,a)|^\alpha \, dv_z \right)^{1/2}.
\end{aligned}$$

On the other hand, by (3-5),

$$\begin{aligned}
\left| \int_\Omega u_0 \overline{P(\beta)} \, dv \right| &\le C \left(\int_\Omega |f|_g^2(z)|K(z,a)|^\alpha \, dv_z \right)^{1/2} v(B_a(\epsilon)) \left(\int_\Omega \max_{w \in \overline{B_a(\epsilon)}} |K(z,w)|^2 e^{\psi_a} \, dv_z \right)^{1/2} \\
&\le C \left(\int_\Omega |f|_g^2(z)|K(z,a)|^\alpha \, dv_z \right)^{1/2} v(B_a(\epsilon)) \left(\int_\Omega |K(z,a)|^{2-\alpha} \, dv_z \right)^{1/2}.
\end{aligned}$$

If $\alpha > 1 + (r-1)\boldsymbol{a}/(2p) \ge 1$, then $|K(z,a)|^{2-\alpha}$ is integrable on Ω by Theorem 5.2. Therefore, for any $a \in \Omega$,

$$\frac{1}{v(B_a(\epsilon))} \int_{B_a(\epsilon)} |u| \, dv \le C \left(\int_\Omega |f|_g^2(z)|K(z,a)|^\alpha \, dv_z \right)^{1/2}.$$

Coupling this estimate with Proposition 2.1, we have proved u is bounded. □

6. Sharpness of the pointwise estimates

For the Cartan classical domains, we show that the logarithm of the Bergman kernel has a bounded gradient with respect to the Bergman metric and also verify that Theorem 3.3 is sharp.

6A. *Solutions with logarithmic growth.*

Example 1. Let Ω be a Cartan classical domain and $u(z) = \log K(z)$. Then $P[u](z)$ is a constant function on Ω and there exists a constant c such that $|\bar\partial u|_g^2 = c\,\mathrm{Tr}(zz^*)$.

Proof. Notice that for all $z \in \Omega$,

$$
\begin{aligned}
P[u](z) &= \int_\Omega u(w)K(z,w)\,dv_w \\
&= \int_\Omega \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta}w)K(z,e^{i\theta}w)\,d\theta\,dv_w \\
&= \int_\Omega \frac{1}{2\pi}\int_0^{2\pi} u(w)K(z,e^{i\theta}w)\,d\theta\,dv_w \\
&= \int_\Omega u(w)K(z,0)\,dv_w = \frac{1}{v(\Omega)}\int_\Omega u(w)\,dv_w,
\end{aligned}
$$

where the third equality follows by the transformation rule of the Bergman kernel, and the fourth equality follows by the mean value property of (anti)holomorphic functions.

Now we show the second part of the example. For $z \in M_{(m,n)}(\mathbb{C})$, define $V(z) := I_m - zz^*$ and let V_{uv} denote the (u,v) entry of V. Then by [Hua 1963; Lu 1997] (see also [Chen and Li 2019, Proposition 2.1]), for domains of type I, II and III,

$$
g^{j\alpha,\overline{k\beta}}(z) = \begin{cases} V_{jk}(\delta_{\alpha\beta} - \sum_{l=1}^m z_{l\alpha}\bar z_{l\beta}), & z \in \mathrm{I}(m,n), \\ V_{jk}\dfrac{V_{\alpha\beta}}{(2-\delta_{j\alpha})(2-\delta_{k\beta})}, & z \in \mathrm{II}(n), \\ \frac14 V_{jk}V_{\alpha\beta}(1-\delta_{j\alpha})(1-\delta_{k\beta}), & z \in \mathrm{III}(n). \end{cases}
$$

For matrices $E_{j\alpha} := (\delta_{ju}\delta_{\alpha v})_{u,v}$, $A := (a_{uv})_{u,v} \in M_{(n,m)}(\mathbb{C})$,

$$
E_{j\alpha}A = (\delta_{ju}a_{\alpha v})_{u,v} \quad \text{and} \quad \frac{\partial V}{\partial z_{j\alpha}} = -E_{j\alpha}z^*.
$$

Then for $z \in \mathrm{I}(m,n)$,

$$
\begin{aligned}
\frac{\partial \log\det V(z)}{\partial z_{j\alpha}} &= \mathrm{Tr}\Big(V^{-1}(z)\frac{\partial V(z)}{\partial z_{j\alpha}}\Big) = -\mathrm{Tr}(V^{-1}(z)E_{j\alpha}z^*) = -\mathrm{Tr}(E_{j\alpha}z^*V^{-1}(z)) \\
&= -\sum_u \delta_{ju}[z^*V^{-1}]_{\alpha u} = -[z^*V^{-1}]_{\alpha j}.
\end{aligned}
$$

Since $u(z) = \log(\det(V(z)))^{-(m+n)} - \log v(\mathrm{I}(m,n))$ is real-valued,

$$
\begin{aligned}
|\bar\partial u|_g^2(z) = \sum_{j,\beta,k,\alpha} g^{j\alpha,\overline{k\beta}}\frac{\partial u}{\partial z_{j\alpha}}\overline{\frac{\partial u}{\partial z_{k\beta}}} &= (m+n)^2\sum_{j,\beta,k,\alpha} V_{jk}[I - z^\tau\bar z]_{\alpha\beta}[z^*V^{-1}(z)]_{\alpha j}\overline{[z^*V^{-1}]}_{\beta k} \\
&= (m+n)^2\sum_{\alpha,k}[z^*]_{\alpha k}[(I - z^\tau\bar z)z^\tau\overline{V^{-1}}]_{\alpha k} \\
&= (m+n)^2\sum_{\alpha,k}[z^*]_{\alpha k}[z^\tau]_{\alpha k} = (m+n)^2\,\mathrm{Tr}(zz^*).
\end{aligned}
$$

For $z \in \mathrm{II}(n)$, using the symmetry of z, we know

$$\frac{\partial V(z)}{\partial z_{j\alpha}} = -\big(1-\tfrac{1}{2}\delta_{j\alpha}\big)(E_{j\alpha}+E_{\alpha j})z^* \quad \text{and} \quad z^*V^{-1}(z) = (z^*V^{-1}(z))^{\tau}.$$

Hence,

$$\begin{aligned}\frac{\partial \log\det V(z)}{\partial z_{j\alpha}} &= \mathrm{Tr}\Big(V^{-1}(z)\frac{\partial V}{\partial z_{j\alpha}}(z)\Big) = -\big(1-\tfrac{1}{2}\delta_{j\alpha}\big)\,\mathrm{Tr}(E_{j\alpha}z^*V^{-1}(z)+z^*V^{-1}(z)E_{\alpha j})\\ &= -(2-\delta_{j\alpha})\,\mathrm{Tr}(E_{j\alpha}z^*V^{-1}(z)) = -(2-\delta_{j\alpha})[z^*V^{-1}(z)]_{j\alpha}.\end{aligned}$$

Since $u(z) = \log(\det(V(z)))^{-(n+1)} - \log v(\mathrm{II}(n))$ and for z symmetric $\overline{z^*V(z)^{-1}} = V(z)^{-1}z$,

$$\begin{aligned}|\bar\partial u|_g^2(z) &= \sum_{j,\beta,k,\alpha} g^{j\alpha,\overline{k\beta}}\,\frac{\partial u}{\partial z_{j\alpha}}\,\overline{\frac{\partial u}{\partial z_{k\beta}}} = (n+1)^2\sum_{\alpha,\beta,k,j} V_{jk}V_{\alpha\beta}[z^*V^{-1}(z)]_{j\alpha}[\overline{z^*V^{-1}(z)}]_{k\beta}\\ &= (n+1)^2\sum_{j,\beta}[z^*]_{j\beta}[V(z)V^{-1}(z)z]_{j\beta} = (n+1)^2\,\mathrm{Tr}(zz^*).\end{aligned}$$

The proof for skew-symmetric $z \in \mathrm{III}(n)$ is similar to the preceding proofs.

For a Cartan classical domain $\mathrm{IV}(n)$, let $s(z) := \sum z_j^2$ and $r(z) := 1-2|z|^2+|s(z)|^2$ for $z \in \mathbb{C}^n$. By [Hua 1963], the Bergman kernel $K(z,z)$ equals $cr(z)^{-n}$. Also,

$$g^{j,\bar k}(z) = r(z)(\delta_{jk}-2z_j\bar z_k) + 2(\bar z_j - \overline{s(z)}z_j)(z_k - s(z)\bar z_k).$$

Notice that

$$(\log(r(z)^{-n}))_{z_j}(\log(r(z)^{-n}))_{z_{\bar k}} = \frac{4n^2}{r(z)^2}[z_js(\bar z)-\bar z_j][\bar z_ks(z)-z_k]$$

and

$$\begin{aligned}|\bar\partial u|_g^2(z) &= 4n^2\sum_{j,k=1}^{n}[r(z)(\delta_{jk}-2z_j\bar z_k)+2(\bar z_j - s(\bar z)z_j)(z_k-s(z)\bar z_k)]\frac{(\bar z_j - s(\bar z)z_j)(z_k-s(z)\bar z_k)}{r(z)^2}\\ &= \sum_{j,k=1}^{n}\frac{4n^2}{r(z)}(\delta_{jk}-2z_j\bar z_k)[\bar z_j-\overline{s(z)}z_j][z_k-s(z)\bar z_k] + \sum_{j,k=1}^{n}\frac{8n^2(\bar z_j-\overline{s(z)}z_j)^2(z_k-s(z)\bar z_k)^2}{r(z)^2}\\ &=: F(z)+G(z).\end{aligned}$$

Thus,

$$\begin{aligned}F(z)\frac{r}{4n^2} &= \sum_{j=1}^{n}|z_j|^2 - s\bar z_j^2 - z_j^2\bar s + |s|^2|z_j|^2 - 2\sum_{j,k=1}^{n}|z_j|^2|z_k|^2 - s|z_j|^2\bar z_k^2 - z_j^2|z_k|^2\bar s + |s|^2z_j^2\bar z_k^2\\ &= |z|^2-2|s|^2+|s|^2|z|^2-2(|z|^4-s|z|^2\bar s - s|z|^2\bar s + |s|^2s\bar s)\\ &= -2|z|^4+5|s|^2|z|^2-2|s|^2+|z|^2-2|s|^4\end{aligned}$$

and

$$G(z) = \frac{8n^2}{r^2}\Big|\sum_{j=1}^{n}(z_j-s\bar z_j)^2\Big|^2 = \frac{8n^2}{r^2}\big|s-2s|z|^2+s^2\bar s\big|^2 = 8n^2|s|^2.$$

Therefore

$$\begin{aligned}|\bar\partial u|_g^2 &= \frac{4n^2}{r}[-2|z|^4+5|s|^2|z|^2-2|s|^2+|z|^2-2|s|^4]+\frac{4n^2}{r}2|s(z)|^2r(z)\\ &=\frac{4n^2}{r}[-2|z|^4+|z|^2|s|^2+|z|^2]\\ &=4n^2|z|^2=4n^2\operatorname{Tr}(zz^*).\end{aligned}$$ □

Example 2 shows that the canonical solution to the equation $\bar\partial u = f := \bar\partial \log K(z)$ (here $\|f\|_{g,\infty}<\infty$) given by $\log K(z)-C_\Omega$ is unbounded with logarithmic growth near the boundary of the polydisc.

Example 2. Consider $f(z) := -\sum_{j=1}^n z_j(1-|z_j|^2)^{-1}\,d\bar z_j$ defined on $\mathbb{D}^n$. Then f is $\bar\partial$-closed, $\|f\|_{g,\infty}\le\frac12$ and the canonical solution to $\bar\partial u = f$ on $\mathbb{D}^n$ is

$$u(z) := \sum_{j=1}^n \log(1-|z_j|^2) - n\int_0^1 \log(1-r)\,dr. \tag{6-1}$$

Proof. We compute directly that u given by (6-1) satisfies $\bar\partial u = f$, and that

$$|f(z)|_g^2 = \frac12\sum_{j=1}^n \frac{(1-|z_j|^2)^2}{(1-|z_j|^2)^2}|z_j|^2 = \frac{|z|^2}{2}.$$

To verify that u is canonical, notice that

$$\begin{aligned}P_{\mathbb{D}^n}\Bigg[\sum_{j=1}^n \log(1-|w_j|^2)\Bigg](z) &= \frac{1}{\pi^n}\int_{\mathbb{D}^n}\prod_{j=1}^n \frac{1}{(1-\langle z_j,w_j\rangle)^2}\sum_{k=1}^n \log(1-|w_k|^2)\,dv_{w_1}\cdots dv_{w_n}\\ &=\sum_{k=1}^n \frac1\pi\int_{\mathbb{D}^n}\frac{\log(1-|w_k|^2)}{(1-\langle z_k,w_k\rangle)^2}\,dv_{w_k}\\ &=\sum_{k=1}^n 2\int_0^1 \log(1-r_k^2)r_k\,dr_k = n\int_0^1\log(1-r)\,dr.\end{aligned}$$ □

6B. *A sharp example.* The maximum blow-up order for a solution to $\bar\partial u = f$ with $\|f\|_{g,\infty}<\infty$ is $\int_\Omega |K(\cdot\,,w)|\,dv_w$. Here we provide an example to show that Theorem 3.3 is sharp on the Cartan classical domains.

Proposition 6.1. *Let Ω be a Cartan classical domain. Then there is a constant c such that for each $z\in\Omega$, there is a $\bar\partial$-closed $(0,1)$-form f_z on Ω with $\|f_z\|_{g,\infty}=1$ and the canonical solution to $\bar\partial u = f_z$ satisfies*

$$|u(z)| \ge c\int_\Omega |K(z,w)|\,dv_w.$$

Proof. For any point $z\in\Omega$, consider the functions $U_z(\cdot) := K(\cdot)^{-1}K(\cdot\,,z)$ and

$$f_z(\cdot) := \bar\partial U_z(\cdot) = K(\cdot\,,z)\bar\partial(K(\cdot)^{-1}).$$

Then, by Example 1,

$$\begin{aligned}\|f_z\|_{g,\infty} &= \|K(\cdot\,,z)K(\cdot)^{-2}\bar{\partial}(K(\cdot))\|_{g,\infty} = \|K(\cdot\,,z)K(\cdot)^{-1}\bar{\partial}(\log K(\cdot))\|_{g,\infty}\\ &\le \|K(\cdot\,,z)K(\cdot)^{-1}\|_\infty \|\bar{\partial}(\log K(\cdot))\|_{g,\infty} \le C.\end{aligned}$$

The Bergman projection of U_z is

$$P[U_z](\cdot) = \int_\Omega U_z(w)K(\cdot\,,w)\,dv_w = \int_\Omega K(w)^{-1}K(w,z)K(\cdot\,,w)\,dv_w.$$

In particular, by (5-1) with $\beta = -1$,

$$P[U_z](z) = \int_\Omega K(w)^{-1}K(w,z)K(z,w)\,dv_w = \int_\Omega K(w)^{-1}|K(w,z)|^2\,dv_w \approx \int_\Omega |K(z,w)|\,dv_w.$$

The canonical solution to $\bar{\partial}u = f$ is $u_z := U_z - P[U_z]$ and

$$|u_z(z)| = |1 - J_{-1,0}(z)| \ge c\int_\Omega |K(z,w)|\,dv_w - 1$$

for a uniform constant $c > 0$, independent of z. □

6C. ***Blow-up order greater than* log.** With the previous example and Theorem 5.3 we will provide the maximum blow-up order when Ω is a Cartan classical domain of rank 2. By Theorem 5.3, for $z = te_1 + te_2$ where $e_1, e_2 \in \mathcal{U}$,

$$\int_\Omega |K(z,w)|\,dv_w \approx (1-t)^{-a/2} \approx \delta_\Omega(z)^{-a/2} \quad \text{as } t \to 1^-.$$

When Ω is $\mathrm{IV}(n)$ with $n \ge 3$,

$$\int_\Omega |K(z,w)|\,dv_w \approx \delta_\Omega(z)^{-n/2+1}.$$

When Ω is III(4) or III(5),

$$\int_\Omega |K(z,w)|\,dv_w \approx \delta_\Omega(z)^{-2}.$$

When Ω is $\mathrm{I}(2,n)$ with $n \ge 2$,

$$\int_\Omega |K(z,w)|\,dv_w \approx \delta_\Omega(z)^{-1}.$$

When Ω is II(2),

$$\int_\Omega |K(z,w)|\,dv_w \approx \delta_\Omega(z)^{-1/2}.$$

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SOME APPLICATIONS OF GROUP-THEORETIC RIPS CONSTRUCTIONS TO THE CLASSIFICATION OF VON NEUMANN ALGEBRAS

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We study various von Neumann algebraic rigidity aspects for the property (T) groups that arise via the Rips construction developed by Belegradek and Osin (*Groups Geom. Dyn.* **2**:1 (2008), 1–12) in geometric group theory. Specifically, developing a new interplay between Popa's deformation/rigidity theory (*Int. Congr. Math, I* (2007), 445–477) and geometric group theory methods, we show that several algebraic features of these groups are completely recognizable from the von Neumann algebraic structure. In particular, we obtain new infinite families of pairwise nonisomorphic property (T) group factors, thereby providing positive evidence towards Connes' rigidity conjecture.

In addition, we use the Rips construction to build examples of property (T) II_1-factors which possess maximal von Neumann subalgebras without property (T), which answers a question raised by Y. Jiang and A. Skalski (arXiv:1903.08190 (2019), version 3).

1. Introduction

The von Neumann algebra $\mathcal{L}(G)$ associated to a countable discrete group G is called *the group von Neumann algebra* and it is defined as the bicommutant of the left regular representation of G computed inside the algebra of all bounded linear operators on the Hilbert space of the square summable functions on G. $\mathcal{L}(G)$ is a II_1-factor (has trivial center) precisely when all nontrivial conjugacy classes of G are infinite (icc), this being the most interesting for study [Murray and von Neumann 1943]. The classification of group factors is a central research theme revolving around the following fundamental question: *What aspects of the group G are remembered by $\mathcal{L}(G)$*? This is a difficult topic as algebraic group properties usually do not survive after passage to the von Neumann algebra regime. Perhaps the best illustration of this phenomenon is Connes' celebrated result [1976] asserting that all amenable icc groups give isomorphic factors. Hence genuinely different groups such as the group of all finite permutations of the positive integers, the lamplighter group, or the wreath product of the integers with itself give rise to isomorphic factors. Ergo, basic algebraic group constructions such as direct products, semidirect products, extensions, inductive limits or classical algebraic invariants such as torsion, rank, or generators and relations in general cannot be recognized from the von Neumann algebraic structure. In this case the only information on G retained by the von Neumann algebra is amenability.

When G is nonamenable, the situation is far more complex and unprecedented progress has been achieved through the emergence of Popa's deformation/rigidity theory [Popa 2007; Vaes 2010;

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Ioana 2013; 2018]. Using this completely new conceptual framework it was shown that various algebraic/analytic properties of groups and their representations can be completely recovered from their von Neumann algebras [Ozawa and Popa 2004; 2010; Ioana et al. 2013; Berbec and Vaes 2014; Chifan et al. 2016b; Drimbe et al. 2019; Chifan and Ioana 2018; Chifan and Udrea 2020]. In this direction an impressive milestone was Ioana, Popa and Vaes's discovery [Ioana et al. 2013] of the first examples of groups G that can be completely reconstructed from $\mathcal{L}(G)$, i.e., *W^*-superrigid groups*.[1] Additional examples were found subsequently in [Berbec and Vaes 2014; Berbec 2015; Chifan and Ioana 2018]. It is worth noting that the general strategies used in establishing these results share a common essential ingredient — the ability to first reconstruct from $\mathcal{L}(G)$ specific given algebraic features of G. For instance, in the examples covered in [Ioana et al. 2013; Berbec and Vaes 2014; Berbec 2015], the first step was to show that whenever $\mathcal{L}(G)\cong\mathcal{L}(H)$, the mystery group H admits a generalized wreath product decomposition exactly as G does; also in the case of [Chifan and Ioana 2018, Theorem A] again the main step was to show that H admits an amalgamated free product splitting exactly as G. These aspects motivate a fairly broad and independent study on this topic — the quest of identifying a comprehensive list of algebraic features of groups which completely pass to the von Neumann algebraic structure. While a couple of works have already appeared in this direction [Chifan et al. 2016b; Chifan and Ioana 2018; Chifan and Udrea 2020], we are still far away from having a satisfactory overview of these properties and a great deal of work remains to be done.

A striking conjecture of Connes predicts that all icc property (T) groups are W^*-superrigid. Despite the fact that this conjecture motivated to great effect a significant portion of the main developments in Popa's deformation/rigidity theory [Popa 2006b; 2006c; Ioana 2011; Ioana et al. 2013], no example of a property (T) W^*-superrigid group is currently known. The first hard evidence towards Connes' conjecture was found in [Cowling and Haagerup 1989], where it was shown that uniform lattices in $\mathrm{Sp}(n,1)$ give rise to nonisomorphic factors for different values of $n\geqslant 2$. Moreover, for any collection $\{G_k\}_k$ of uniform lattices in $\mathrm{Sp}(n_k,1)$, $n_k\geqslant 2$, the group algebras $\{\mathcal{L}(\times_{i=1}^n G_i)\}_n$ are pairwise nonisomorphic. Later on, using a completely different approach, Ozawa and Popa [2004] obtained a far-reaching generalization of this result by showing that for any collection $\{G_n\}_n$ of hyperbolic, property (T) groups (e.g., uniform lattices in $\mathrm{Sp}(n,1)$, $n\geqslant 2$ [Cowling and Haagerup 1989]) the group algebras $\{\mathcal{L}(\times_{i=1}^n G_i)\}_n$ are pairwise nonisomorphic. However, little is known beyond these two classes of examples. Moreover, the current literature offers an extremely limited account on which algebraic features that occur in a property (T) group are completely recognizable at the von Neumann algebraic level. For instance, besides the preservation of the Cowling–Haagerup constant [1989], the amenability of normalizers of infinite amenable subgroups in hyperbolic property (T) groups from [Ozawa and Popa 2010, Theorem 1], and the direct product rigidity for hyperbolic property (T) groups from [Chifan et al. 2016b, Theorem A; Chifan and Udrea 2020, Theorem A] very little is known. Therefore in order to successfully construct property (T) W^*-superrigid groups via a strategy similar to the ones used in [Ioana et al. 2013; Berbec and Vaes 2014; Berbec 2015; Chifan and Ioana 2018] we believe it is imperative to identify new algebraic features of property (T) groups that survive the passage to the von Neumann algebraic regime. Any success in this direction will potentially hint at which group theoretic methods to pursue in order to address Connes' conjecture.

[1] If H is any group such that $\mathcal{L}(G)\cong\mathcal{L}(H)$ then $H\cong G$.

In this paper we make new progress on this study by showing that many algebraic aspects of the Rips constructions developed in geometric group theory by Belegradek and Osin [2008] are entirely recoverable from the von Neumann algebraic structure. To properly introduce the result we briefly describe their construction. Using the prior Dehn filling results from [Osin 2010], Belegradek and Osin [2008, Theorem] showed that for every finitely generated group Q one can find a property (T) group N such that Q can be realized as a finite-index subgroup of $\mathrm{Out}(N)$. This canonically gives rise to an action $Q \curvearrowright^{\sigma} N$ by automorphisms such that the corresponding semidirect product group $N \rtimes_{\sigma} Q$ is hyperbolic relative to $\{Q\}$. Throughout the document the semidirect products $N \rtimes_{\sigma} Q$ will be termed Belegradek–Osin group Rips constructions. When Q is torsion-free, one can pick N to be torsion-free as well, and hence both N and $N \rtimes_{\sigma} Q$ are icc groups. Also when Q has property (T) then $N \rtimes_{\sigma} Q$ has property (T). Under all these assumptions we will denote by $\mathcal{R}ip_T(Q)$ the class of these Rips construction groups $N \rtimes_{\sigma} Q$.

The first main result of our paper concerns a fairly large class of canonical fiber products of groups in $\mathcal{R}ip_T(Q)$. Specifically, consider any two groups $N_1 \rtimes_{\sigma_1} Q$, $N_2 \rtimes_{\sigma_2} Q \in \mathcal{R}ip_T(Q)$ and form the canonical fiber product $G = (N_1 \times N_2) \rtimes_{\sigma} Q$, where $\sigma = (\sigma_1, \sigma_2)$ is the diagonal action. Notice that since property (T) is closed under extensions [Bekka et al. 2008, Section 1.7] it follows that G has property (T). Developing new interplay between geometric group theoretic methods [Rips 1982; Dahmani et al. 2017; Osin 2010; Belegradek and Osin 2008] and deformation/rigidity methods [Ioana 2011; Ioana et al. 2013; Chifan et al. 2016b; 2018; Chifan and Ioana 2018; Chifan and Udrea 2020], for a fairly large family of groups Q, we show that the semidirect product feature of G is an algebraic property completely recoverable from the von Neumann algebraic regime. In addition, we also have a complete reconstruction of the acting group Q. The precise statement is the following:

Theorem A (Theorem 5.1). *Let $Q = Q_1 \times Q_2$, where Q_i are icc, biexact, weakly amenable, property (T), torsion-free, residually finite groups. For $i = 1, 2$, let $N_i \rtimes_{\sigma_i} Q \in \mathcal{R}ip_T(Q)$ and denote by $\Gamma = (N_1 \times N_2) \rtimes_{\sigma} Q$ the semidirect product associated with the diagonal action $\sigma = \sigma_1 \times \sigma_2 : Q \curvearrowright N_1 \times N_2$. Denote by $\mathcal{M} = \mathcal{L}(\Gamma)$ the corresponding II_1-factor. Assume that Λ is any arbitrary group and $\Theta : \mathcal{L}(\Gamma) \to \mathcal{L}(\Lambda)$ is any $*$-isomorphism. Then there exist group actions by automorphisms $H \curvearrowright^{\tau_i} K_i$ such that $\Lambda = (K_1 \times K_2) \rtimes_{\tau} H$, where $\tau = \tau_1 \times \tau_2 : H \curvearrowright K_1 \times K_2$ is the diagonal action. Moreover one can find a multiplicative character $\eta : Q \to \mathbb{T}$, a group isomorphism $\delta : Q \to H$ and unitary $w \in \mathcal{L}(\Lambda)$ and $*$-isomorphisms $\Theta_i : \mathcal{L}(N_i) \to \mathcal{L}(K_i)$ such that for all $x_i \in L(N_i)$ and $g \in Q$ we have*

$$\Theta((x_1 \otimes x_2)u_g) = \eta(g)w((\Theta_1(x_1) \otimes \Theta(x_2))v_{\delta(g)})w^*. \tag{1.1}$$

Here $\{u_g : g \in Q\}$ and $\{v_h : h \in H\}$ are the canonical unitaries implementing the actions of $Q \curvearrowright \mathcal{L}(N_1) \mathbin{\bar{\otimes}} \mathcal{L}(N_2)$ and $H \curvearrowright \mathcal{L}(K_1) \mathbin{\bar{\otimes}} \mathcal{L}(K_2)$, respectively.

There are countably infinitely many groups that are residually finite, torsion-free, hyperbolic, and have property (T). A concrete such family is $\{\Lambda_k : k \geqslant 2\}$, where $\Lambda_k < \mathrm{Sp}(k, 1)$ is a uniform lattice. It is well known the Λ_k's are residually finite [Malcev 1940], (virtually) torsion-free [Selberg 1960], hyperbolic [Gromov 1987, Example B], have property (T) (see for instance, [Bekka et al. 2008, Theorem 1.5.3]) and are pairwise nonisomorphic [Cowling and Haagerup 1989]. However, there are infinitely many pairwise nonisomorphic such lattices even in the same Lie group. To see this, fix $k \geqslant 2$ together with a torsion-free, uniform lattice $\Gamma < \mathrm{Sp}(k, 1)$. Since Γ is residually finite there is a sequence of normal, finite-index, proper subgroups $\cdots \lhd \Gamma_{n+1} \lhd \Gamma_n \lhd \cdots \lhd \Gamma_1 \lhd \Gamma$ such that $\bigcap_n \Gamma_n = 1$. Being subgroups, Γ_n are clearly residually finite and torsion-free. Moreover, the finite-index condition implies that all Γ_n's are hyperbolic and have property (T). As the Γ_n's are cohopfian [Prasad 1976] and $\Gamma_n < \Gamma_m$ for every $n < m$, we have $\Gamma_n \ncong \Gamma_m$. Therefore the class $\{\Gamma_n : n \in \mathbb{N}\}$ satisfies our conditions. Finally we note that, since every hyperbolic group is finitely presented and there are only countably many such groups, one cannot built examples of larger cardinality than the ones presented above.

In conclusion, Theorem A provides explicit examples of infinitely many pairwise nonisomorphic group II_1-factors with property (T). Moreover these groups are quite different from the previous classes [Cowling and Haagerup 1989; Ozawa and Popa 2004], as they give rise to factors that are nonsolid ($\mathcal{L}(\Gamma)$ contains two commuting nonamenable subfactors $\mathcal{L}(N_1)$ and $\mathcal{L}(N_2)$), are tensor indecomposable [Das 2020, Lemma 2.3] and do not admit Cartan subalgebras (Corollary 7.2). Moreover, using the Margulis normal subgroup theorem, the factors covered by Theorem A are nonisomorphic to any factor arising from any irreducible lattices in a higher-rank semisimple Lie group (see remarks after the proof of Theorem 5.1). We also mention that Theorem A, or its strong rigidity version Theorem 6.1 (see also Corollary 6.2), provides examples of infinite families of finite-index subgroups $\Gamma_n \leqslant \Gamma$ in a given icc property (T) group Γ such that the corresponding group factors $\mathcal{L}(\Gamma_n)$ and $\mathcal{L}(\Gamma_m)$ are nonisomorphic for $n \neq m$. As the Γ_n's are measure equivalent this provides new counterexamples to D. Shlyakhtenko's question, asking whether measure equivalence of icc groups implies isomorphism of the corresponding group factors (see [Popa 2009, page 18]), which are very different in nature from the ones obtained in [Chifan and Ioana 2011; Chifan et al. 2016b]. We summarize this discussion in the next corollary.

Corollary B (Corollary 6.2). *Assume the same notation as in Theorem A.*

(1) *Let Q_1, Q_2 be uniform lattices in* $\mathrm{Sp}(n, 1)$ *with $n \geqslant 2$ and let $Q := Q_1 \times Q_2$. Also let $\cdots \leqslant Q_1^s \leqslant \cdots \leqslant Q_1^2 \leqslant Q_1^1 \leqslant Q_1$ be an infinite family of finite-index subgroups and define $Q_s := Q_1^s \times Q_2 \leqslant Q$. Then consider $N_1 \rtimes_{\sigma_1} Q, N_2 \rtimes_{\sigma_2} Q \in \mathcal{R}ip_T(Q)$ and let $\Gamma = (N_1 \times N_2) \rtimes_{\sigma_1 \times \sigma_2} Q$. Inside Γ consider the finite-index subgroups $\Gamma_s := (N_1 \times N_2) \rtimes_{\sigma_1 \times \sigma_2} Q_s$. Then the family $\{\mathcal{L}(\Gamma_s) : s \in I\}$ consists of pairwise nonisomorphic finite-index subfactors of $\mathcal{L}(\Gamma)$.*

(2) *Let Γ, Γ_n be as above. Then Γ_n is measure equivalent to Γ for all $n \in \mathbb{N}$, but $\mathcal{L}(\Gamma_n)$ is not isomorphic to $\mathcal{L}(\Gamma_m)$ for $n \neq m$.*

From a different perspective our theorem can be also seen as a von Neumann algebraic superrigidity result regarding conjugacy of actions on noncommutative von Neumann algebras. Notice that very little is known in this direction as most of the known superrigidity results concern algebras arising from actions of groups on probability spaces.

In certain ways one can view Theorem A as a first step towards providing an example of a property (T) superrigid group. While the acting group Q can be completely recovered, as well as certain aspects of the action $Q \curvearrowright N_1 \times N_2$ (e.g., trivial stabilizers) only the product feature of the "core" $\mathcal{L}(N_1 \times N_2)$ can be reconstructed at this point. While the reconstruction of N_1 and N_2 seems to be out of reach momentarily, we believe that a deeper understanding of the Rips construction, along with new von Neumann algebraic techniques are necessary to tackle this problem. We also remark that in a subsequent article it was shown that the group factors as in Theorem A have trivial fundamental group; see [Chifan et al. 2020, Theorem B].

Besides the aforementioned rigidity results we also investigate applications of group Rips constructions to the study of maximal von Neumann algebras. If $\mathcal{M}$ is a von Neumann algebra then a von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ is called *maximal* if there is no intermediate von Neumann subalgebra $\mathcal{P}$ so that $\mathcal{N} \subsetneq \mathcal{P} \subsetneq \mathcal{M}$. Understanding the structure of maximal subalgebras of a given von Neumann algebra is a rather difficult problem that is intimately related with the very classification of these objects. Despite a series of remarkable earlier successes on the study of maximal *amenable* subalgebras initiated in [Popa 1983] and continued more recently [Shen 2006; Cameron et al. 2010; Houdayer 2014; Boutonnet and Carderi 2015; 2017; Suzuki 2020; Chifan and Das 2020; Jiang and Skalski 2019a], significantly less is known for the arbitrary maximal ones. For instance Ge's question [2003, Section 3, Question 2] on the existence of nonamenable factors that possess maximal factors that are hyperfinite was settled in the affirmative only very recently by Y. Jiang and A. Skalski [2019a]. In fact in their work they proposed a more systematic approach towards the study of maximal von Neumann subalgebras within various categories, such as the von Neumann algebras with Haagerup's property or with property (T) of Kazhdan. Their investigation also naturally led to several interesting open problems [Jiang and Skalski 2019a, Section 5].

In this paper we explain how in a setting similar to [Jiang and Skalski 2019a] the Belegradek–Osin group Rips construction techniques and Olshanski-type monster groups can be used in conjunction with Galois correspondence results for II_1-factors à la [Choda 1978] to produce many maximal von Neumann subalgebras arising from group/subgroup situation. In particular, through this mix of results we are able to construct many examples of II_1 -actors with property (T) that have maximal von Neumann subalgebras without property (T), thereby answering Problem 5.5 in the first version of [Jiang and Skalski 2019a] (see Theorem 4.4). More specifically, using Olshanskii's small cancellation techniques [2009] in the setting of lacunary hyperbolic groups we explain how one can construct a property (T) monster group Q whose maximal subgroups are all isomorphic to a given rank-1 group[2] Q_m (see Section 2C). Then if one considers the Belegradek–Osin Rips construction $N \rtimes Q$ corresponding to Q then using a Galois correspondence (Lemma 4.2) one can show the following:

Theorem C (Theorem 4.4). *For every maximal rank-*1 *subgroup $Q_m < Q$ consider the subgroup $N \rtimes Q_m <$ $N \rtimes Q$. Then $\mathcal{L}(N \rtimes Q_m) \subset \mathcal{L}(N \rtimes Q)$ is a maximal von Neumann subalgebra.*

Note that since N and Q have property (T), so does $N \rtimes Q$ and therefore the corresponding II_1-factor $\mathcal{L}(N \rtimes Q)$ has property (T) by [Connes and Jones 1985]. However since $N \rtimes Q_m$ surjects onto the infinite abelian group Q_m, it does not have property (T) and hence $\mathcal{L}(N \rtimes Q_m)$ does not have property (T) either. Another solution to the problem of finding maximal subalgebras without property (T) inside factors with property (T) was also obtained independently by Jiang and Skalski [2019b]. Their beautiful solution has a different flavor from ours; even though the Galois correspondence theorem à la Choda is a common ingredient in both of the proofs. Hence we refer the reader to [Jiang and Skalski 2019b, Theorem 4.8] for another solution to the aforementioned problem.

[2]Any group that is isomorphic to a subgroup of $(\mathbb{Q}, +)$ is called rank-1.

2. Preliminaries

2A. ***Notation and terminology.*** We denote by $\mathbb{N}$ and $\mathbb{Z}$ the set of natural numbers and the integers, respectively. For any $k \in \mathbb{N}$ we denote by $\overline{1,k}$ the integers $\{1, 2, \dots, k\}$.

All von Neumann algebras in this document will be denoted by calligraphic letters, e.g., $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$, $\mathcal{N}$, etc. Given a von Neumann algebra $\mathcal{M}$ we will denote by $\mathscr{U}(\mathcal{M})$ its unitary group, by $\mathscr{P}(\mathcal{M})$ the set of all its nonzero projections, and by $\mathscr{Z}(\mathcal{M})$ its center. We also denote by $(\mathcal{M})_1$ its unit ball. All algebra inclusions $\mathcal{N} \subseteq \mathcal{M}$ are assumed unital unless otherwise specified. Given an inclusion $\mathcal{N} \subseteq \mathcal{M}$ of von Neumann algebras we denote by $\mathcal{N}' \cap \mathcal{M}$ the relative commutant of $\mathcal{N}$ in $\mathcal{M}$, i.e., the subalgebra of all $x \in \mathcal{M}$ such that $xy = yx$ for all $y \in \mathcal{N}$. We also consider the one-sided quasinormalizer $\mathscr{QN}^{(1)}_{\mathcal{M}}(\mathcal{N})$ (the semigroup of all $x \in \mathcal{M}$ for which there exist $x_1, x_2, \dots, x_n \in \mathcal{M}$ such that $\mathcal{N}x \subseteq \sum_i x_i\mathcal{N}$) and the quasinormalizer $\mathscr{QN}_{\mathcal{M}}(\mathcal{N})$ (the set of all $x \in \mathcal{M}$ for which there exist $x_1, x_2, \dots, x_n \in \mathcal{M}$ such that $\mathcal{N}x \subseteq \sum_i x_i\mathcal{N}$ and $x\mathcal{N} \subseteq \sum_i \mathcal{N}x_i$) and we notice that $\mathcal{N} \subseteq \mathscr{N}_{\mathcal{M}}(\mathcal{N}) \subseteq \mathscr{QN}_{\mathcal{M}}(\mathcal{N}) \subseteq \mathscr{QN}^{(1)}_{\mathcal{M}}(\mathcal{N})$.

All von Neumann algebras $\mathcal{M}$ considered in this article will be tracial, i.e., endowed with a unital, faithful, normal linear functional $\tau : M \to \mathbb{C}$ satisfying $\tau(xy) = \tau(yx)$ for all $x, y \in \mathcal{M}$. This induces a norm on $\mathcal{M}$ by the formula $\|x\|_2 = \tau(x^*x)^{1/2}$ for all $x \in \mathcal{M}$. The $\|\cdot\|_2$-completion of $\mathcal{M}$ will be denoted by $L^2(\mathcal{M})$. For any von Neumann subalgebra $\mathcal{N} \subseteq \mathcal{M}$ we denote by $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ the τ-preserving conditional expectation onto $\mathcal{N}$.

For a countable group G we denote by $\{u_g : g \in G\} \in \mathscr{U}(\ell^2G)$ its left regular representation given by $u_g(\delta_h) = \delta_{gh}$, where $\delta_h : G \to \mathbb{C}$ is the Dirac function at $\{h\}$. The weak operatorial closure of the linear span of $\{u_g : g \in G\}$ in $\mathscr{B}(\ell^2G)$ is the so-called group von Neumann algebra and will be denoted by $\mathcal{L}(G)$. $\mathcal{L}(G)$ is a II_1-factor precisely when G has infinite nontrivial conjugacy classes (icc). If $\mathcal{M}$ is a tracial von Neumann algebra and $G \curvearrowright^{\sigma} \mathcal{M}$ is a trace-preserving action we denote by $\mathcal{M} \rtimes_{\sigma} G$ the corresponding cross product von Neumann algebra [Murray and von Neumann 1937]. For any subset $K \subseteq G$ we denote by $P_{\mathcal{M}K}$ the orthogonal projection from the Hilbert space $L^2(\mathcal{M} \rtimes G)$ onto the closed linear span of $\{xu_g : x \in \mathcal{M}, g \in K\}$. When $\mathcal{M}$ is trivial we will denote this simply by P_K.

Given a subgroup $H \leqslant G$ we denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G. Also we will denote by $QN^{(1)}_G(H)$ the one-sided quasinormalizer of H in G; this is the semigroup of all $g \in G$ for which there exist a finite set $F \subseteq G$ such that $Hg \subseteq FH$. Similarly we denote by $QN_G(H)$ the quasinormalizer (or commensurator) of H in G, i.e., the subgroup of all $g \in G$ for which there is a finite set $F \subseteq G$ such that $Hg \subseteq FH$ and $gH \subseteq HF$. We canonically have $HC_G(H) \leqslant N_G(H) \leqslant QN_G(H) \subseteq QN^{(1)}_G(H)$. We often consider the virtual centralizer of H in G, i.e., $vC_G(H) = \{g \in G : |g^H| < \infty\}$. Notice $vC_G(H)$ is a subgroup of G that is normalized by H. When $H = G$, the virtual centralizer is the FC-radical of G. Also one can easily see from definitions that $HvC_G(H) \leqslant QN_G(H)$. For a subgroup $H \leqslant G$ we denote by $\langle\langle H \rangle\rangle$ the normal closure of H in G.

Finally, for any groups G and N and an action $G \curvearrowright^{\sigma} N$ we denote by $N \rtimes_{\sigma} G$ the corresponding semidirect product group.

2B. ***Popa's intertwining techniques.*** Over fifteen years ago, Sorin Popa introduced [2006b, Theorem 2.1 and Corollary 2.3] a powerful analytic criterion for identifying intertwiners between arbitrary subalgebras of tracial von Neumann algebras. Now this is known in the literature as *Popa's intertwining-by-bimodules technique* and has played a key role in the classification of von Neumann algebras program via Popa's deformation/rigidity theory.

Theorem 2.1 [Popa 2006b]. *Let $(\mathcal{M},\tau)$ be a separable tracial von Neumann algebra and let $\mathcal{P},\mathcal{Q}\subseteq\mathcal{M}$ be (not necessarily unital) von Neumann subalgebras. Then the following are equivalent*:

(1) *There exist $p\in\mathscr{P}(\mathcal{P})$, $q\in\mathscr{P}(\mathcal{Q})$, a $*$-homomorphism $\theta: p\mathcal{P}p\to q\mathcal{Q}q$ and a partial isometry $0\neq v\in q\mathcal{M}p$ such that $\theta(x)v=vx$ for all $x\in p\mathcal{P}p$.*

(2) *For any group $\mathcal{G}\subset\mathscr{U}(\mathcal{P})$ such that $\mathcal{G}''=\mathcal{P}$, there is no sequence $(u_n)_n\subset\mathcal{G}$ satisfying $\|E_{\mathcal{Q}}(xu_ny)\|_2\to 0$ for all $x,y\in\mathcal{M}$.*

(3) *There exist finitely many $x_i,y_i\in\mathcal{M}$ and $C>0$ such that $\sum_i\|E_{\mathcal{Q}}(x_iuy_i)\|_2^2\geqslant C$ for all $u\in\mathscr{U}(\mathcal{P})$.*

If one of the three equivalent conditions from Theorem 2.1 holds then we say that *a corner of $\mathcal{P}$ embeds into $\mathcal{Q}$ inside $\mathcal{M}$*, and write $\mathcal{P}\prec_{\mathcal{M}}\mathcal{Q}$. If we moreover have $\mathcal{P}p'\prec_{\mathcal{M}}\mathcal{Q}$ for any projection $0\neq p'\in\mathcal{P}'\cap 1_{\mathcal{P}}\mathcal{M}1_{\mathcal{P}}$ (equivalently, for any projection $0\neq p'\in\mathscr{Z}(\mathcal{P}'\cap 1_{\mathcal{P}}\mathcal{M}1_P)$), then we write $\mathcal{P}\prec^s_{\mathcal{M}}\mathcal{Q}$.

For further use we record the following result which controls the intertwiners in algebras arising from malnormal subgroups. Its proof is essentially contained in [Popa 2006b, Theorem 3.1] so it will be left to the reader.

Lemma 2.2 [Popa 2006b]. *Assume that $H\leqslant G$ is an almost malnormal subgroup and let $G\curvearrowright\mathcal{N}$ be a trace-preserving action on a tracial von Neumann algebra $\mathcal{N}$. Let $\mathcal{P}\subseteq\mathcal{N}\rtimes H$ be a von Neumann algebra such that $\mathcal{P}\nprec_{\mathcal{N}\rtimes H}\mathcal{N}$. Then for all elements $x,x_1,x_2,\dots,x_l\in\mathcal{N}\rtimes G$ satisfying $\mathcal{P}x\subseteq\sum_{i=1}^l x_i\mathcal{P}$ we must have $x\in\mathcal{N}\rtimes H$.*

We continue with the following intertwining result for group algebras which is a generalization of some previous results obtained under normality assumptions [Drimbe et al. 2019]. For the reader's convenience we also include a brief proof.

Lemma 2.3. *Assume that $H_1,H_2\leqslant G$ are groups, let $G\curvearrowright\mathcal{N}$ be a trace-preserving action on a tracial von Neumann algebra $\mathcal{N}$ and denote by $\mathcal{M}=\mathcal{N}\rtimes G$ the corresponding crossed product. Also assume that $\mathcal{A}\prec^s\mathcal{N}\rtimes H_1$ is a von Neumann algebra such that $\mathcal{A}\prec_{\mathcal{M}}\mathcal{N}\rtimes H_2$. Then one can find $h\in G$ such that $\mathcal{A}\prec_{\mathcal{M}}\mathcal{N}\rtimes(H_1\cap hH_2h^{-1})$.*

Proof. Since $\mathcal{A}\prec^s\mathcal{N}\rtimes H_1$, by [Vaes 2013, Lemma 2.6] for every $\varepsilon>0$ there exists a finite subset $S\subset G$ such that $\|P_{SH_1S}(x)-x\|_2\leqslant\varepsilon$ for all $x\in(\mathcal{A})_1$. Here for every $K\subseteq G$ we denote by P_K the orthogonal projection from $L^2(\mathcal{M})$ onto the closure of the linear span of $\mathcal{N}u_g$ with $g\in K$. Also since $\mathcal{A}\prec_{\mathcal{M}}\mathcal{N}\rtimes H_2$, by Popa's intertwining techniques there exist a scalar $0<\delta<1$ and a finite subset $T\subset G$ so that $\|P_{TH_2T}(x)\|_2\geqslant\delta$ for all $x\in(\mathcal{A})_1$. Thus, using this in combination with the previous inequality, for every $x\in\mathscr{U}(\mathcal{A})$ and every $\varepsilon>0$, there are finite subsets $S,T\subset G$ so that $\|P_{TH_2T}\circ P_{SH_1S}(x)\|_2\geqslant\delta-\varepsilon$. Since there exist finite subsets $R,U\subset G$ such that $TH_2T\cap SH_1S\subseteq U\big(\bigcup_{r\in R}H_2\cap rH_1r^{-1}\big)U$, we further get that $\|P_{U(\bigcup_{r\in R}H_2\cap rH_1r^{-1})U}(x)\|_2\geqslant\delta-\varepsilon$. Then choosing $\varepsilon>0$ sufficiently small and using Popa's intertwining techniques together with a diagonalization argument (see the proof of [Ioana et al. 2008, Theorem 4.3]) one can find $r\in R$ so that $\mathcal{A}\prec\mathcal{N}\rtimes(H_2\cap rH_1r^{-1})$, as desired. $\square$

In the sequel we need the following three intertwining lemmas, which establish that under certain conditions, intertwining in a larger algebra implies that the intertwining happens in a "smaller subalgebra".

Lemma 2.4. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{N} \subseteq \mathcal{M}$ be von Neumann algebras so that $\mathscr{N}_{\mathcal{M}}(\mathcal{A})'' = \mathcal{M}$. If $\mathcal{B} \prec_{\mathcal{M}} \mathcal{A}$ then $\mathcal{B} \prec_{\mathcal{N}} \mathcal{A}$.*

Proof. Since $\mathcal{B} \prec_{\mathcal{M}} \mathcal{A}$, by Theorem 2.1 one can find $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in \mathcal{M}$ and $c > 0$ such that $\sum_{i=1}^n \|E_{\mathcal{A}}(x_i b y_i)\|_2^2 \geqslant c$ for all $b \in \mathscr{U}(\mathcal{B})$. Since $\mathscr{N}_{\mathcal{M}}(\mathcal{A})'' = \mathcal{M}$, using basic $\|\cdot\|_2$-approximation for x_i and y_i and shrinking $c > 0$ if necessary, one can find $g_1, g_2, \dots, g_l, h_1, h_2, \dots, h_l \in \mathscr{N}_{\mathcal{M}}(\mathcal{A})$ and $c' > 0$ such that for all $b \in \mathscr{U}(\mathcal{B})$ we have

$$\sum_{i=1}^{n} \|E_{\mathcal{A}}(g_i b h_i)\|_2^2 \geqslant c' > 0. \tag{2B.1}$$

Using normalization we see that $E_{\mathcal{A}}(g_i b h_i) = E_{g_i \mathcal{A} g_i^*}(g_i b h_i) = g_i E_{\mathcal{A}}(b h_i g_i) g_i^*$. This, combined with (2B.1) and $\mathcal{A} \subseteq \mathcal{N}$, gives

$$0 < c' \leqslant \sum_{i=1}^{l} \|E_{\mathcal{A}}(b h_i g_i)\|_2^2 = \sum_{i=1}^{l} \|E_{\mathcal{A}} \circ E_{\mathcal{N}}(b h_i g_i)\|_2^2 = \sum_{i=1}^{l} \|E_{\mathcal{A}}(b E_{\mathcal{N}}(h_i g_i))\|_2^2$$

for all $b \in \mathscr{U}(\mathcal{B})$. Since $E_{\mathcal{N}}(h_i g_i) \in \mathcal{N}$, using Theorem 2.1 this clearly shows that $\mathcal{B} \prec_{\mathcal{N}} \mathcal{A}$. □

Lemma 2.5. *Let Q be a group and define $\mathrm{d}(Q) = \{(q, q) : q \in Q\}$. Let $\mathcal{A}$ be a tracial von Neumann algebra and assume $(Q \times Q) \curvearrowright^{\sigma} \mathcal{A}$ is a trace-preserving action. Let $\mathcal{B} \subseteq \mathcal{A}$ be a regular von Neumann subalgebra which is invariant under the action σ. Let $\mathcal{D} \subseteq \mathcal{A} \rtimes_{\sigma} \mathrm{d}(Q)$ be a subalgebra such that $\mathcal{D} \prec_{\mathcal{A} \rtimes_{\sigma} (Q \times Q)} \mathcal{B} \rtimes_{\sigma} \mathrm{d}(Q)$. Then $\mathcal{D} \prec_{\mathcal{A} \rtimes_{\sigma} \mathrm{d}(Q)} \mathcal{B} \rtimes_{\sigma} \mathrm{d}(Q)$.*

Proof. Define $\mathcal{M} := \mathcal{A} \rtimes_{\sigma} (Q \times Q)$, $\mathcal{N} := \mathcal{A} \rtimes_{\sigma} \mathrm{d}(Q)$, and $\mathcal{P} := \mathcal{B} \rtimes_{\sigma} \mathrm{d}(Q)$. Thus $\mathcal{P} \subset \mathcal{N} \subset \mathcal{M}$ and with this notation we establish the following:

Claim 1. *Let $(v_n)_n \subset \mathscr{U}(\mathcal{N})$ be a sequence such that $\lim_{n \to \infty} \|E_{\mathcal{P}}(a v_n b)\|_2 = 0$ for all $a, b \in \mathcal{N}$. Then*

$$\lim_{n \to \infty} \|E_{\mathcal{P}}(x v_n y)\|_2 = 0 \quad \textit{for all } x, y \in \mathcal{M}. \tag{2B.2}$$

Proof of Claim 1. Notice that $(Q \times Q) = (Q \times 1) \rtimes_{\rho} \mathrm{d}(Q)$, where $\mathrm{d}(Q) \curvearrowright^{\rho} (Q \times 1)$ is the action by conjugation. Therefore, using basic $\|\cdot\|_2$-approximations and the $\mathcal{P}$-bimodularity of the conditional expectation $E_{\mathcal{P}}$, it suffices to show (2B.2) only for $x = (u_g \otimes 1)c$ and $y = d(u_h \otimes 1)$ for all $g, h \in Q$ and $c, d \in \mathcal{A}$. Under these assumptions we see that

$$\begin{aligned} E_{\mathcal{P}}((u_g \otimes 1) c v_n d (u_h \otimes 1)) &= E_{\mathcal{P}} \circ P_{(u_g \otimes 1)\mathcal{M}(u_h \otimes 1)}((u_g \otimes 1) c v_n d (u_h \otimes 1)) \\ &= P_{\mathcal{B}(\mathrm{d}(Q) \cap (g,1)\mathrm{d}(Q)(h,1))}((u_g \otimes 1) c v_n d (u_h \otimes 1)). \end{aligned} \tag{2B.3}$$

Here, and throughout the proof, for every set $S \subseteq Q \times Q$ we denote by $P_{\mathcal{B}S}$ the orthogonal projection onto the closed subspace $\overline{\operatorname{span}}\{\mathcal{B} u_g : g \in S\}$.

To this end observe there exists an element $s \in Q$ such that

$$\mathrm{d}(Q) \cap (g, 1)\mathrm{d}(Q)(h, 1) \subseteq [\mathrm{d}(Q) \cap (g, 1)\mathrm{d}(Q)(g^{-1}, 1)]\mathrm{d}(s).$$

Moreover, a basic computation shows that $\mathrm{d}(Q)\cap(g,1)\mathrm{d}(Q)(g^{-1},1)=\mathrm{d}(C_Q(g))$, where $C_Q(g)$ is the centralizer of g in Q. Hence altogether we have $\mathrm{d}(Q)\cap(g,1)\mathrm{d}(Q)(h,1)\subseteq\mathrm{d}(C_Q(g))\mathrm{d}(s)$. Combining this with (2B.3) and using the fact that $u_g\otimes 1$ normalizes $\mathcal{B}\rtimes\mathrm{d}(C_Q(g))$ we see that

$$\begin{aligned}\|E_{\mathcal{P}}((u_g\otimes 1)cv_nd(u_h\otimes 1))\|_2 &\leqslant \|P_{\mathcal{B}(\mathrm{d}(C_Q(g))\mathrm{d}(s))}((u_g\otimes 1)cv_nd(u_h\otimes 1))\|_2\\ &= \|E_{\mathcal{B}\rtimes\mathrm{d}(C_Q(g))}((u_g\otimes 1)cv_nd(u_{hs^{-1}}\otimes u_{s^{-1}}))\|_2\\ &= \|E_{\mathcal{B}\rtimes\mathrm{d}(C_Q(g))}(cv_nd(u_{hs^{-1}g}\otimes u_{s^{-1}}))\|_2\\ &= \|E_{\mathcal{B}\rtimes\mathrm{d}(C_Q(g))}(cv_ndE_{\mathcal{N}}(u_{hs^{-1}g}\otimes u_{s^{-1}}))\|_2\\ &= \delta_{hs^{-1}g,s^{-1}}\|E_{\mathcal{B}\rtimes\mathrm{d}(C_Q(g))}(cv_nd)\|_2\leqslant\|E_{\mathcal{P}}(cv_nd)\|_2.\end{aligned}\tag{2B.4}$$

Letting $n\to\infty$ in (2B.4) and using the assumption, the claim is obtained. □

To show our lemma assume by contradiction that $\mathcal{D}\nprec_{\mathcal{N}}\mathcal{P}$. By Theorem 2.1 there is a sequence of unitaries $(v_n)_n\subset\mathcal{D}\subset\mathcal{N}$ so that $\lim_{n\to\infty}\|E_{\mathcal{P}}(av_nb)\|_2=0$ for all $a,b\in\mathcal{N}$. Using Claim 1 we get $\lim_{n\to\infty}\|E_{\mathcal{P}}(xv_ny)\|_2=0$ for all $x,y\in\mathcal{M}$, which by Theorem 2.1 again implies $\mathcal{D}\nprec_{\mathcal{M}}\mathcal{P}$, a contradiction. □

Lemma 2.6. *Let $\mathcal{C}\subseteq\mathcal{B}$ and $\mathcal{N}\subseteq\mathcal{M}$ be inclusions of tracial von Neumann algebras. If $\mathcal{A}\subseteq\mathcal{N}\bar{\otimes}\mathcal{B}$ is a von Neumann subalgebra such that $\mathcal{A}\prec_{\mathcal{M}\bar{\otimes}\mathcal{B}}\mathcal{M}\bar{\otimes}\mathcal{C}$ then $\mathcal{A}\prec_{\mathcal{N}\bar{\otimes}\mathcal{B}}\mathcal{N}\bar{\otimes}\mathcal{C}$.*

Proof. By Theorem 2.1 one can find $x_i,y_i\in\mathcal{M}\bar{\otimes}\mathcal{B}$, $i=\overline{1,k}$, and a scalar $c>0$ such that

$$\sum_{i=1}^{n}\|E_{\mathcal{M}\bar{\otimes}\mathcal{C}}(x_iay_i)\|^2\geqslant c\quad\text{for all } d\in\mathscr{U}(\mathcal{A}).\tag{2B.5}$$

Using $\|\cdot\|_2$-approximations of x_i and y_i by finite linear combinations of elements in $\mathcal{M}\bar{\otimes}_{\mathrm{alg}}\mathcal{B}$ together with the $\mathcal{M}\otimes 1$-bimodularity of $E_{\mathcal{M}\bar{\otimes}\mathcal{C}}$, after increasing k and shrinking $c>0$ if necessary, in (2B.5) we can assume without loss of generality that $x_i,y_i\in 1\otimes\mathcal{B}$. However, since $\mathcal{A}\subseteq\mathcal{N}\bar{\otimes}\mathcal{B}$, in this situation we have $E_{\mathcal{M}\bar{\otimes}\mathcal{C}}(x_iay_i)=E_{\mathcal{M}\bar{\otimes}\mathcal{C}}\circ E_{\mathcal{N}\bar{\otimes}\mathcal{B}}(x_iay_i)=E_{\mathcal{N}\bar{\otimes}\mathcal{C}}(x_iay_i)$. Thus (2B.5) combined with Theorem 2.1 give $\mathcal{A}\prec_{\mathcal{N}\bar{\otimes}\mathcal{B}}\mathcal{N}\bar{\otimes}\mathcal{C}$, as desired. □

In the sequel we need the following (minimal) technical variation of [Chifan and Ioana 2018, Lemma 2.6]. The proof is essentially the same with the one presented in that work and we leave the details to the reader.

Lemma 2.7 [Chifan and Ioana 2018, Lemma 2.6]. *Let $\mathcal{P},\mathcal{Q}\subseteq\mathcal{M}$ be inclusions of tracial von Neumann algebras. Assume that $\mathcal{QN}^{(1)}_{\mathcal{M}}(\mathcal{P})=P$ and $\mathcal{Q}$ is a II_1-factor. Suppose there is a projection $z\in\mathscr{Z}(\mathcal{P})$ such that $\mathcal{P}z\prec^s\mathcal{Q}$ and a projection $p\in\mathcal{P}z$ such that $p\mathcal{P}p=p\mathcal{Q}p$. Then one can find a unitary $u\in\mathcal{M}$ such that $u\mathcal{P}zu^*=r\mathcal{Q}r$, where $r=uzu^*\in\mathscr{P}(\mathcal{Q})$.*

The next lemma is a mild generalization of [Ioana et al. 2013, Proposition 7.1], using the same techniques (see also the proof of [Krogager and Vaes 2017, Lemma 2.3]).

Lemma 2.8. *Let Λ be an icc group, and let $\mathcal{M}=\mathcal{L}(\Lambda)$. Consider the comultiplication map $\Delta:\mathcal{M}\to\mathcal{M}\bar{\otimes}\mathcal{M}$ given by $\Delta(v_\lambda)=v_\lambda\otimes v_\lambda$ for all $\lambda\in\Lambda$. Let $\mathcal{A},\mathcal{B}\subseteq\mathcal{M}$ be* (*unital*) *$*$-subalgebras such that $\Delta(\mathcal{A})\subseteq\mathcal{M}\bar{\otimes}\mathcal{B}$. Then there exists a subgroup $\Sigma<\Lambda$ such that $\mathcal{A}\subseteq\mathcal{L}(\Sigma)\subseteq\mathcal{B}$. In particular, if $\mathcal{A}=\mathcal{B}$, then $\mathcal{A}=\mathcal{L}(\Sigma)$.*

Proof. Let $\Sigma = \{s \in \Lambda : v_s \in \mathcal{B}\}$. Since $\mathcal{B}$ is a unital $*$-subalgebra, Σ is a subgroup, and clearly $\mathcal{L}(\Sigma) \subseteq \mathcal{B}$. We argue that $\mathcal{A} \subseteq \mathcal{L}(\Sigma)$.

Fix $a \in \mathcal{A}$, and let $a = \sum_\lambda a_\lambda v_\lambda$ be its Fourier decomposition. Let $I = \{s \in \Lambda : a_s \neq 0\}$. Fix $s \in I$, and consider the normal linear functional ω on $\mathcal{M}$ given by $\omega(x) = \bar{a}_s \tau(x v_s^*)$. Note that $(\omega \otimes 1)(a) = |a_s|^2 \otimes v_s$. Since $\Delta(\mathcal{A}) \subseteq \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{B}$, we have $(\omega \otimes 1)\Delta(\mathcal{A}) \subseteq \mathbb{C} \mathbin{\bar{\otimes}} \mathcal{B}$. Thus, $v_s \in \mathcal{B}$ implies $s \in \Sigma$. Since this holds for all $s \in I$, we get $a \in \mathcal{L}(\Sigma)$, and hence we are done. □

We end this section with the following elementary result. We are grateful to the referee for suggesting a (much) shorter proof than the one we originally had, which used [Chifan and Das 2018, Proposition 2.3].

Lemma 2.9. *Let $\mathcal{M}$ be a tracial von Neumann algebra and let $\mathcal{N}$ be a II_1-factor, with $\mathcal{N} \subseteq \mathcal{M}$ a unital inclusion. If there is $p \in \mathscr{P}(\mathcal{N})$ so that $p\mathcal{N}p = p\mathcal{M}p$ then $\mathcal{N} = \mathcal{M}$.*

Proof. Shrinking p if necessary we can assume $\tau(p) = 1/n$. Let $v_1, \dots, v_n \in \mathcal{N}$ be partial isometries such that $v_i v_i^* = p$ for all i, and $\sum_{i=1}^n v_i^* v_i = 1$. Fix $x \in \mathcal{M}$. Since for every $1 \leqslant i, j \leqslant n$ we have $v_i x v_j^* \in p\mathcal{M}p = p\mathcal{N}p$, we get $x = \sum_{i,j=1}^n v_i^*(v_i x v_j^*) v_j \in \mathcal{N}$, as desired. □

2C. *Small cancellation techniques.* In this section, we recollect some geometric group theoretic preliminaries that will be used throughout this paper. We refer the reader to [Olshanskii 1991; 1993; Olshanskii et al. 2009] for more details related to the small cancellation techniques. We also refer the reader to [Lyndon and Schupp 1977] for details concerning van Kampen diagrams.

2C1. *van Kampen diagrams.* Given a word W over the alphabet set $\mathcal{S}$, we denote its length by $\|W\|$. We also write $W \equiv V$ to express the letter-for-letter equality for words W, V.

Let G be a group generated by a set of alphabets $\mathcal{S}$. A van Kampen diagram $\triangle$ over a presentation

$$G = \langle S \mid \mathcal{R} \rangle \tag{2C.1}$$

is a finite, oriented, connected, planar 2-complex endowed with a labeling function $\mathrm{Lab} : E(\triangle) \to \mathcal{S}^{\pm 1}$, where $E(\triangle)$ denotes the set of oriented edges of $\triangle$, such that $\mathrm{Lab}(e^{-1}) \equiv (\mathrm{Lab}(e))^{-1}$. Given a cell Π of $\triangle$, $\partial\Pi$ denotes its boundary. Similarly $\partial\triangle$ denotes the boundary of $\triangle$. The labels of $\partial\triangle$ and $\partial\Pi$ are defined up to cyclic permutations. We also stipulate that the label for any cell Π of $\triangle$ is equal to (up to a cyclic permutation) $R^{\pm 1}$, where $R \in \mathcal{R}$.

Using the van Kampen lemma [Lyndon and Schupp 1977, Chapter 5, Theorem 1.1], a word W over the alphabet set $\mathcal{S}$ represents the identity element in the group given by the presentation (2C.1) if and only if there exists a connected, simply connected planar diagram $\triangle$ over (2C.1) satisfying $\mathrm{Lab}(\partial\triangle) \equiv W$.

2C2. *Small cancellation over hyperbolic groups.* Let $G = \langle X \rangle$ be a finitely generated group and X be a finite generating set for G. Recall that the Cayley graph $\Gamma(G, X)$ of a group G with respect to the set of generators X is an oriented labeled 1-complex with vertex set $V(\Gamma(G, X)) = G$ and edge set $E(\Gamma(G, X)) = G \times X^{\pm 1}$. An edge $e = (g, a)$ goes from the vertex g to the vertex ga and has label a. Given a combinatorial path p in the Cayley graph $\Gamma(G, X)$, the length $|p|$ is the number of edges in p. The word length $|g|$ of an element $g \in G$ with respect to the generating set X is defined to be the length of a shortest word in X representing g in the group G, i.e., $|g| := \min_{h =_G g} \|h\|$. The formula

$d(f, g) = |g^{-1} f|$ defines a metric on the group G. The metric on the Cayley graph $\Gamma(G, X)$ is the natural extension of this metric. A word W is called a (λ, c)-quasi geodesic in $\Gamma(G, X)$ for some $\lambda > 0, c \geqslant 0$ if $\lambda\|W\| - c \leqslant |W| \leqslant \lambda\|W\| + c$. A word W is called a geodesic if it is a $(1, 0)$-quasigeodesic. A word W in the alphabet $X^{\pm 1}$ is called (λ, c)-quasigeodesic (respectively geodesic) in G if any path in the Cayley graph $\Gamma(G, X)$ labeled by W is (λ, c)-quasigeodesic (respectively geodesic). Throughout this section, $\mathcal{R}$ denotes a symmetric set of words (i.e., it is closed under taking cyclic shifts and inverses of words, and all the words are cyclically reduced) from X^*, the set of words on the alphabet X. A common initial subword of any two distinct words in $\mathcal{R}$ is called a piece. We say that $\mathcal{R}$ satisfies the $C'(\mu)$ condition if any piece contained (as a subword) in a word $R \in \mathcal{R}$ has length smaller than $\mu\|R\|$.

Definition 2.10 [Olshanskii 1993, Section 4]. A subword U of a word $R \in \mathcal{R}$ is called an ϵ-piece of the word R, for $\epsilon \geqslant 0$, if there exists a word $R' \in \mathcal{R}$ satisfying the following conditions:

(1) $R \equiv UV$ and $R' \equiv U'V'$ for some $U', V' \in \mathcal{R}$.

(2) $U' =_G YUZ$ for some $Y, Z \in X^*$, where $\|Y\|, \|Z\| \leqslant \epsilon$.

(3) $YRY^{-1} \neq_G R'$.

We say the system $\mathcal{R}$ satisfies the $C(\lambda, c, \epsilon, \mu, \rho)$-condition for some $\lambda \geqslant 1,\ c \geqslant 0,\ \epsilon \geqslant 0,\ \mu > 0,\ \rho > 0$ if:

(a) $\|R\| \geqslant \rho$ for any $R \in \mathcal{R}$.

(b) Any word $R \in \mathcal{R}$ is a (λ, c)-quasigeodesic.

(c) For any ϵ-piece U of any word $R \in \mathcal{R}$, the inequalities $\|U\|, \|U'\| < \mu\|R\|$ hold.

In practice, we will need some slight modifications of the above definition [Olshanskii 1993, Section 4].

Definition 2.11. A subword U of a word $R \in \mathcal{R}$ is called an ϵ'-piece of the word R, for $\epsilon \geqslant 0$, if:

(1) $R \equiv UVU'V'$ for some $V, U', V' \in X^*$.

(2) $U' =_G YU^{\pm}Z$ for some words $Y, Z \in X^*$, where $\|Y\|, \|Z\| \leqslant \epsilon$.

We say the system $\mathcal{R}$ satisfies the $C'(\lambda, c, \epsilon, \mu, \rho)$-condition for some $\lambda \geqslant 1,\ c \geqslant 0,\ \epsilon \geqslant 0,\ \mu > 0,\ \rho > 0$ if:

(d) $\mathcal{R}$ satisfies the $C(\lambda, c, \epsilon, \mu, \rho)$ condition.

(e) Every ϵ'-piece U of R satisfies $\|U'\| < \mu\|R\|$, where U' is as above.

Let G be a group defined by

$$G = \langle X \mid \mathcal{O} \rangle, \tag{2C.2}$$

where $\mathcal{O}$ is the set of *all* relators (not just the defining relations) of G. Given a symmetrized set of words $\mathcal{R}$ in the alphabet set X, we consider the quotient group

$$H = \langle G \mid \mathcal{R} \rangle = \langle G \mid \mathcal{O} \cup \mathcal{R} \rangle. \tag{2C.3}$$

A cell over a van Kampen diagram over (2C.3) is called an $\mathcal{R}$-cell (respectively, an $\mathcal{O}$-cell) if its boundary label is a word from $\mathcal{R}$ (respectively, $\mathcal{O}$). We always consider a van Kampen diagram over (2C.3) up to some elementary transformations. For example we do not distinguish diagrams if one can be obtained from the other by joining two distinct $\mathcal{O}$-cells having a common edge or by inverse transformations [Olshanskii 1993, Section 5].

3. Some examples of Olshanskii's monster groups in the context of lacunary hyperbolic groups

In this section, we collect some group theoretic results needed for our main theorems in Sections 4 and 5. Readers who are mainly interested in the results in Section 5 may skip ahead to Section 3C. The results in Subsections 3A and 3B shall be required for our main results in Section 4.

In order to derive our main result on the study of maximal von Neumann algebras (i.e., Theorem 4.4) we need to construct a new monster-like group in the same spirit as the famous examples from [Olshanskii 1980]. Specifically, generalizing the geometric methods from [Olshanskii 1993] to the context of lacunary hyperbolic groups [Olshanskii et al. 2009] and using techniques developed in [Khan 2020], we construct a group G such that every maximal subgroup of G is isomorphic to a subgroup of $\mathbb{Q}$, the group of rational numbers. While in our approach we explain in detail how these results are used, the main emphasis will be on the new aspects of these techniques. Therefore we recommend that the interested reader consult beforehand the aforementioned results [Olshanskii 1993; Khan 2020].

3A. *Elementary subgroups.* In this section, using methods developed in [Olshanskii 1993], we construct a group Q whose maximal (proper) subgroups are rank-1 abelian groups; see Theorem 3.12. More specifically, we study "special limits" of hyperbolic groups, called lacunary hyperbolic groups, as introduced in [Olshanskii et al. 2009].

Definition 3.1. Let $\alpha : G \to H$ be a group homomorphism and $G = \langle A\rangle,\ H = \langle B\rangle$. The injectivity radius $r_A(\alpha)$ is the radius of largest ball centered at the identity of G in the Cayley graph of G with respect to A on which the restriction of α is injective.

Definition 3.2 [Olshanskii et al. 2009, Theorem 1.2]. A finitely generated group G is called lacunary hyperbolic group if G is the direct limit of a sequence of hyperbolic groups and epimorphisms

$$G_1 \xrightarrow{\eta_1} G_2 \xrightarrow{\eta_2} \cdots \xrightarrow{\eta_{i-1}} G_i \xrightarrow{\eta_i} G_{i+1} \xrightarrow{\eta_{i+1}} G_{i+2} \xrightarrow{\eta_{i+2}} \cdots, \tag{3A.1}$$

where G_i is generated by a finite set S_i and $\eta_i(S_i) = S_{i+1}$. Also the G_i's are δ_i-hyperbolic, where $\delta_i = o(r_{S_i}(\eta_i))$, where $r_{S_i}(\eta_i)$ is the injective radius of η_i with respect to S_i.

Fix ω a nonprincipal ultrafilter. An asymptotic cone $\mathrm{Cone}^{\omega}(X, e, d)$ of a metric space (X, dist), where $e = \{e_i\}_i$, $e_i \in X$ for all i and $d = \{d_i\}_i$ is an unbounded sequence of nondecreasing positive real numbers, is the ω-limit of the spaces $(X, \mathrm{dist}/d_i)$. The sequence $d = \{d_i\}$ is called a scaling sequence. Following [Olshanskii et al. 2009, Theorem 3.3], G being a lacunary hyperbolic group is equivalent to the existence of a scaling sequence $d = \{d_i\}$ such that the asymptotic cone $\mathrm{Cone}^{\omega}(\Gamma(G, X), e, d)$ associated with the Cayley graph $\Gamma(G, X)$ for a finite generating set X of G with $e = \{\text{identity}\}$ is an $\mathbb{R}$-tree for any nonprincipal ultrafilter ω. For more details on asymptotic cones and their connection with lacunary hyperbolic groups we refer the reader to [Olshanskii et al. 2009, Section 2.3, Section 3.1].

Our construction relies heavily on the notion of *elementary* subgroups. For the readers' convenience, we collect below some preliminaries regarding elementary subgroups.

Definition 3.3. A group E is called elementary if it is virtually cyclic. Let G be a hyperbolic group and $g \in G$ be an infinite-order element. Then the elementary subgroup containing g is defined as

$$E(g) := \{x \in G : x^{-1} g^n x = g^{\pm n} \text{ for some } n = n(x) \in \mathbb{N}\}.$$

For further use we need the following result describing in depth the structure of elementary subgroups.

Lemma 3.4. (1) [Olshanskii 1991] *If E is a torsion-free elementary group then E is cyclic.*

(2) [Olshanskii 1993, Lemma 1.16] *Let E be an infinite elementary group. Then E contains normal subgroups $T \lhd E^+ \lhd E$ such that $[E : E^+] \leqslant 2$, T is finite and $E^+/T \simeq \mathbb{Z}$. If $E \neq E^+$ then $E/T \simeq D_\infty$ (the infinite dihedral group). For a hyperbolic group G, $E(g)$ is unique maximal elementary subgroup of G containing the infinite-order element $g \in G$.*

In the context of lacunary hyperbolic groups we need to introduce the following definition which generalizes Definition 3.3.

Definition 3.5. Let G be a lacunary hyperbolic group and let $g \in G$ be an infinite-order element. We define $E^{\mathcal{L}}(g) := \{x \in G : x g^n x^{-1} = g^{\pm n} \text{ for some } n = n(x) \in \mathbb{N}\}$.

For future reference we now recall the following structural result regarding torsion elements in a δ-hyperbolic group.

Theorem 3.6 [Gromov 1987, 2.2.B]. *Let $g \in G$ be a torsion element in a δ-hyperbolic group G. Then g is conjugate to an element h in G such that $|h|_G \leqslant 4\delta + 1$.*

The following elementary lemma will be used in the proof of Theorem 3.8. For convenience we include a short proof.

Lemma 3.7. *If G is a torsion-free lacunary hyperbolic group, then one can choose G_i to be torsion-free such that $G = \varinjlim G_i$.*

Proof. Fix a presentation $G = \langle S \mid \mathcal{R} \rangle$. By [Olshanskii et al. 2009, Theorem 3.3], one can choose $G_i := \langle S \mid \mathcal{R}_{c(i)} \rangle$, where $\{c(n)\}_n$ is a strictly increasing sequence such that $\mathcal{R}_{c(i)}$ consists of labels of all cycles in the ball of radius d_i (corresponding to the scaling sequence $\{d_i\}_i$ of the lacunary hyperbolic group) around the identity in $\Gamma(G, S)$. Let r_i be the injectivity radius of the quotient map $\phi_i : G_i \to G_{i+1}$. The lacunary hyperbolic condition implies that $\lim_{i \to \infty} \delta_i / r_i = 0$, where δ_i is the hyperbolic constant for the group G_i for all i. Choose i_0 such that for all $j \geqslant i_0$ we have $r_j > 9\delta_j$. We will show the G_j's are torsion-free for all $j \geqslant i_0$, which proves the lemma.

Fix any $j \geqslant i_0$. Assume by contradiction that $g \in G_j \setminus \{1\}$ is a torsion element. By Theorem 3.6 there is an element $h \in G_j \setminus \{1\}$ such that h is conjugate to g and $|h|_{G_j} \leqslant 4\delta_j + 1$. Thus h is a torsion element of G_j. Since $|h|_{G_j} \leqslant 4\delta_j + 1 < r_i$, h is a nontrivial element of G_k for all $k \geqslant j$. Thus h is a nontrivial torsion element in the limit group G, which is a contradiction! □

The next result generalizes Lemma 3.4, and provides a complete description of the structure of elementary subgroups of a torsion-free lacunary hyperbolic group. This result can be deduced from the main theorem of [Khan 2020]. For the readers' convenience, we include a short proof.

Theorem 3.8. *Let G be a torsion-free lacunary hyperbolic group and let $g \in G$ be an infinite-order element. Then $E^{\mathcal{L}}(g)$ is an abelian group of rank* 1 *(i.e., $E^{\mathcal{L}}(g)$ embeds in* $(\mathbb{Q}, +)$*).*

Proof. From the definition (3A.1) of lacunary hyperbolic group, $E^{\mathcal{L}}(g) = \varinjlim E_i(g)$ for every $e \neq g \in G$, where $E_i(g)$ is the elementary subgroup containing the element g in the hyperbolic group G_i when viewing $g \in G_i$. Since G is torsion-free, one can choose G_i to be torsion-free by Lemma 3.7. By Lemma 3.4 (1) we get that $E_i(g)$ is cyclic for all i. Observe that every surjective homomorphism between hyperbolic groups takes elementary subgroups into elementary subgroups; in particular $E_i(g)$ maps into $E_{i+1}(g)$. We now get the group $E^{\mathcal{L}}(g)$ is equal to $\varinjlim E_i(g)$ as an inductive limit of cyclic groups, which proves the theorem. □

Remark. Let G be a torsion-free lacunary hyperbolic group and let $e \neq g \in G$. Note that $C_G(g) \leqslant E^{\mathcal{L}}(g)$, where $C_G(g)$ is the centralizer of g in G.

3B. ***Maximal subgroups.*** Let $G_0 = \langle X \rangle$ be a torsion-free δ-hyperbolic group with respect to X, where $X = \{x_1, x_2, \dots, x_n\}$ is a finite generating set. Without loss of generality we assume that $E(x_i) \cap E(x_j) = \{e\}$ for $i \neq j$. We define a linear order on X by $x_i^{-1} < x_j^{-1} < x_i < x_j$, whenever $i < j$. Let $F'(X)$ denote the set of all nonempty reduced words on X. Note that the order on X induces the lexicographic order on $F'(X)$. Let $F'(X) = \{w_1, w_2, \dots\}$ be an enumeration with $w_i < w_j$ for $i < j$. Observe that $w_1 = x_1$ and $w_2 = x_2$. We now consider the set $\mathcal{S} := F'(X) \times F'(X) \setminus \{(w, w) : w \in F'(X)\}$ and enumerate the elements of $\mathcal{S}$ as $\mathcal{S} = \{(u_1, v_1), (u_2, v_2), \dots\}$.

Our next goal is to construct the chain

$$G_0 \overset{\beta_0}{\hookrightarrow} K_1 \overset{\alpha_1}{\twoheadrightarrow} G_1' \overset{\gamma_1}{\twoheadrightarrow} G_1 \overset{\beta_1}{\hookrightarrow} K_2 \overset{\alpha_2}{\twoheadrightarrow} G_2' \overset{\gamma_1}{\twoheadrightarrow} G_2 \cdots , \tag{3B.1}$$

where K_i, G_i, G_i' are hyperbolic for all i and $\eta_i := \gamma_i \circ \alpha_i \circ \beta_{i-1}$, $i \geqslant 1$, satisfies the conditions in (3A.1).

Let L be a rank-1 abelian group. Then L can be written as $L = \bigcup_{i=0}^{\infty} L_i$, where $L_i = \langle g_i \rangle_\infty$ and $g_i = g_{i+1}^{m_{i+1}}$ for some $m_{i+1} \in \mathbb{N}$. Here $\langle g_i \rangle_\infty$ denotes the infinite cyclic group generated by the infinite-order element g_i.

Since G_0 is nonelementary, there exists a smallest index $j_i \geqslant i$ such that $v_{j_i} \notin E(u_{j_i})$. For $m \in \mathbb{N}$, define

$$H_{i+1}^k := H_{i+1}^{k-1} \underset{u_k = g_{(k,i+1)}^{m_{i+1}}}{*} \langle g_{(k,i+1)} \rangle_\infty, \quad \text{where } H_{i+1}^0 = G_i \text{ and } g_{(k,i+1)} = g_{i+1} \text{ for } k = 1, 2, \dots, j_i. \tag{3B.2}$$

For $i \geqslant 0$ let K_{i+1} be $H_{i+1}^{j_i}$. Note that K_{i+1} is hyperbolic as H_{i+1}^k is hyperbolic for all k by [Mikhajlovskii and Olshanskii 1998, Theorem 3]. Choose $c_i, c_i' \in G_i$ such that $c_i, c_i' \notin E(u_k)$ for all $1 \leqslant k \leqslant j_i$ and $c_i, c_i' \notin E(v_{j_i})$. One can find such c_i and c_i' since there are infinitely many elements in a nonelementary hyperbolic group which are pairwise noncommensurable [Olshanskii 1993, Lemma 3.8]. Let $Y_i := \{g_{(k,i+1)} : 1 \leqslant k \leqslant j_i\}$. Define

$$R_k := g_{(k,i+1)} c_i^{n_{1,k}} c_i' c_i^{n_{2,k}} c_i' \cdots c_i^{n_{s_k,k}} c_i', \tag{3B.3}$$

where $n_{s,k}$, for $1 \leqslant k \leqslant j_i$ are defined as

$$n_{1,k} = 2^{k-1} n_{1,1}, \quad s_k = n_{1,k-1} \quad \text{and} \quad n_{s,k} = n_{1,k} + (s-1).$$

We also denote by $\mathcal{R}_i$ the set of all cyclic shifts of $\{R_k^{\pm 1} : 1 \leqslant k \leqslant j_i\}$.

Lemma 3.9 [Darbinyan 2017, Lemma 5.1]. *There exists a constant K such that the set of words $\mathcal{R}$ defined above by* (3B.3) *are (λ, c)-quasigeodesic in $\Gamma(G, X)$, provided $n_{1,1} \geqslant K$, $c \notin E(g_{(k,i+1)})$, and $c' \notin E(g_{(k,i+1)})$.*

We now define $\widetilde{\mathcal{R}}_{i+1}$ to be the set of words $\mathcal{R}_i$, defined as above, with $n_{1,k} \geqslant K$.

Lemma 3.10 [Darbinyan 2017, Lemma 5.2]. *For any given constant $\epsilon_i \geqslant 0$, $\mu_i > 0$, $\rho_i > 0$, the system of words $\widetilde{\mathcal{R}}_{i+1}$ (defined above) satisfies the $C'(\lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)$ condition over K_{i+1}.*

By construction there is a natural embedding $\beta_i : G_i \hookrightarrow K_{i+1}$. Let $G'_{i+1} := \langle K_{i+1} \mid \widetilde{\mathcal{R}}_{i+1}\rangle$ (where we are using the notation in Section 2C1). The factor group G'_{i+1} is hyperbolic by [Olshanskii 1993, Lemma 7.2]. Now consider the natural quotient map $\alpha_{i+1} : K_{i+1} \twoheadrightarrow G'_{i+1}$. Since $\alpha_{i+1} \circ \beta_i$ takes generators of G_i to generators of G'_{i+1}, the map $\alpha_{i+1} \circ \beta_i$ is surjective.

Consider the set

$$Z_i := \{x \in X : x \notin E(u_{j_i})\}.$$

Let $G_{i+1} := G'_{i+1}/\langle\langle \mathcal{R}(Z_i, u_{j_i}, v_{j_i}, \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)\rangle\rangle$ and let $\gamma_{i+1} : G'_{i+1} \twoheadrightarrow G_{i+1}$ be the quotient map. Here $\mathcal{R}(Z_i, u_{j_i}, v_{j_i}, \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)$ is the set of all conjugates and the cyclic shifts of some relations, where we identify the elements of Z_i with words of the form (3B.3) generated by u_{j_i} and v_{j_i}. Since the relators $\mathcal{R}_i$ are generic, we have added all the parameters to indicate these relations satisfy the small cancellation conditions with the parameters and their dependency to the specific set of words. One can choose the powers of u_{j_i} and v_{j_i} such that the small cancellation condition is satisfied by Lemmas 3.9 and 3.10. For more details on how to choose these words, we refer the reader to [Olshanskii 1993, Section 5; Darbinyan 2017, Section 5.4]. Thus it follows that the group G_{i+1} is hyperbolic by [Olshanskii 1993, Lemma 7.2] as one can choose parameters $\lambda_i, c_i, \epsilon_i, \mu_i, \rho_i$ such that $\mathcal{R}(Z_i, u_{j_i}, v_{j_i}, \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)$ satisfies the $C'(\lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)$ small cancellation condition in Definition 2.11 and the map γ_{i+1} takes generating set to generating set. In particular, $\eta_{i+1} := \gamma_{i+1} \circ \alpha_{i+1} \circ \beta_i$ is a surjective homomorphism which takes the generating set of G_i to the generating set of G_{i+1}. Let $G^L := \varinjlim G_i$. From its definition, it follows that G_{i+1} is the group generated by u_{j_i} and v_{j_i}.

We summarize the above discussion in the following statement.

Lemma 3.11. *The above construction satisfies the following properties*:

(1) *G_i is nonelementary hyperbolic group for all i.*

(2) *Either $u_i \in E(v_i)$ or the group generated by $\{u_i, v_i\}$ in G_{i+1} is equal to all of G_{i+1}.*

(3) *For each element $x \in X$, we have $E(x) = \langle y\rangle$ in G_i, where $x = y^{m_1 m_2 \cdots m_i}$. The exponents m_i are described as follows*: *a rank-1 abelian group L can be written as $L = \bigcup_{i=0}^{\infty} L_i$, where $L_i = \langle g_i\rangle_\infty$ and $g_i = g_{i+1}^{m_{i+1}}$ for some $m_{i+1} \in \mathbb{N}$.*

(4) *$G^L := \varinjlim G_i$ may be chosen to have property (T).*

Proof. Part (1) follows from [Olshanskii 1993, Lemma 7.2]. To see part (2) notice that by definition if $j_i > i$ then $v_i \in E(u_i)$ in G_i. Otherwise if $j_i = i$ then $v_i \notin E(u_i)$ in G_i and G_i is the group generated by $\{u_i, v_i\}$. Part (3) follows immediately from the fact that x is not a proper power in G_0. Finally, for part (4) notice that we may start the above construction with G_0 being a property (T) group. Then G_1 has property (T), as G_0 surjects onto G_1. By induction, each of the groups G_i in the above construction have property (T). Hence G^L has property (T). □

We are now ready to prove the main theorem of this section.

Theorem 3.12. *For any subgroup Q_m of $(\mathbb{Q}, +)$ there exists a nonelementary torsion-free lacunary hyperbolic group G such that all maximal subgroups of G are isomorphic to Q_m. Moreover, we may choose G to have property (T).*

Proof. In the above construction let $L = Q_m$, $G = G^{Q_m}$ and take $d = m_1 m_2 \cdots m_i$ in (3B.2), where $L_i = \langle g_i \rangle_\infty$ and $g_i = g_{i+1}^{m_{i+1}}$ for some $m_{i+1} \in \mathbb{N}$ and $Q_m = \bigcup_{i=1}^{\infty} L_i$. One can choose sparse enough parameters to satisfy the injectivity radius condition in (3A.1), which in turn will ensure that G is lacunary hyperbolic. The above construction also guarantees that $E^{\mathcal{L}}(g) = Q_m$ for all $g \in G \setminus \{1\}$. Suppose $P \nleqslant G$ is a maximal subgroup of G. As P is a proper subgroup, P is abelian by Lemma 3.11 (2). Now let $e \neq h \in G$. Note that as P is abelian, P is contained in the centralizer of h. Now from Definition 3.5 it follows that $g \in P \leqslant E^{\mathcal{L}}(g) (\cong Q_m) \nleqslant G$. By the maximality of P we get $P \cong Q_m$. Thus, all maximal subgroups of G are isomorphic to Q_m and hence any proper subgroup of G is isomorphic to a subgroup of Q_m.

The “moreover” part follows from part (4) of Lemma 3.11. □

We end this section with the following well-known counterexamples to von Neumann’s conjecture.

Corollary 3.13 [Olshanskii 1980; 1993]. *For every noncyclic torsion-free hyperbolic group Γ there exists a nonabelian torsion-free quotient $\overline{\Gamma}$ such that all proper subgroups of $\overline{\Gamma}$ are infinite cyclic.*

Proof. Take $Q_m = \mathbb{Z}$ in Theorem 3.12. □

3C. *Belegradek–Osin Rips construction in group theory.* Rips constructions emerged in geometric group theory with [Rips 1982] and represent a rich source of examples for various pathological properties in group theory. This type of construction was used effectively to study automorphisms of property (T) groups. In this direction Ollivier and Wise [2007] were able to construct property (T) groups whose automorphism group contain any given countable group. This answered an important older question of P. de la Harpe and A. Valette about finiteness of outer automorphism groups of property (T) groups. Using the small cancellation methods developed in [Osin 2010; Arzhantseva et al. 2007], Belegradek and Osin discovered the following version of the Rips construction in the context of relatively hyperbolic groups:

Theorem 3.14 [Belegradek and Osin 2008]. *Let H be a nonelementary hyperbolic group, Q be a finitely generated group and S a subgroup of Q. Suppose Q is finitely presented with respect to S. Then there exists a short exact sequence*

$$1 \to N \to G \xrightarrow{\epsilon} Q \to 1$$

and an embedding $\iota : Q \to G$ such that:

(1) *N is isomorphic to a quotient of H.*

(2) *G is hyperbolic relative to the proper subgroup $\iota(S)$.*

(3) $\iota \circ \epsilon = \mathrm{Id}$.

(4) *If H and Q are torsion-free then so is G.*

(5) *The canonical map $\phi : Q \hookrightarrow \mathrm{Out}(N)$ is injective and* $[\mathrm{Out}(N) : \phi(Q)] < \infty$.

This construction is extremely important for our work. We are particularly interested in the case when H is torsion-free and has property (T) and $Q = S$ and is torsion-free. In this situation Theorem 3.14 implies that G is admits a semidirect product decomposition $G = N \rtimes Q$ and it is hyperbolic relative to $\{Q\}$. Notice that the finite conjugacy radical $\mathrm{FC}(N)$ of N is invariant under the action of Q and hence $\mathrm{FC}(N)$ is an amenable normal subgroup G. Since G is relative hyperbolic, it follows that $\mathrm{FC}(N)$ is finite and hence it is trivial as G is torsion-free; in particular N is an icc group. Since G is hyperbolic relative to Q it follows that the stabilizer of any $n \in N$ in Q under the action $Q \curvearrowright^{\sigma} N$ is trivial.

We now introduce the following classes of groups that shall play an extremely important role throughout the rest of the paper.

Definition 3.15. We denote by $\mathcal{R}ip(Q)$ the class of all semidirect products $G = N \rtimes Q$ satisfying the properties of Theorem 3.14, where $Q = S$, Q and H are torsion-free and H has property (T).

Moreover, when Q has property (T), we denote the class $\mathcal{R}ip(Q)$ by $\mathcal{R}ip_T(Q)$.

Since property (T) is closed under extensions, it follows that all groups in $\mathcal{R}ip_T(Q)$ have property (T). Our rigidity results in Section 5 concern this class of groups.

In the second part of this section we recall a powerful method from geometric group theory, termed Dehn filling. We are interested specifically in the group theoretic Dehn filling constructions developed by D. Osin and his collaborators in [Osin 2010; Dahmani et al. 2017]. The next result, which is due to Osin, is a technical variation of [Osin 2010, Theorem 1.1] and [Dahmani et al. 2017, Theorem 7.9] and plays a key role in deriving some of our main rigidity theorems in Section 5 (see Theorems 5.2 and 5.3). For its proof the reader may consult [Chifan et al. 2015, Corollary 5.1].

Theorem 3.16 (Osin). *Let $H \leqslant G$ be infinite groups where H is finitely generated and residually finite. Suppose that G is hyperbolic relative to $\{H\}$. Then there exist a nonelementary hyperbolic group K and an epimorphism $\delta : G \to K$ such that $R = \ker(\delta)$ is isomorphic to a nontrivial (possibly infinite) free product $R = *_{g \in T} R_0^g$, where $T \subset G$ is a subset and $R_0^g = g R_0 g^{-1}$ for a finite-index normal subgroup $R_0 \lhd H$.*

We end this section with an application of Theorem 3.16. The result describes the structure of the normal subgroups N of $N \rtimes Q \in \mathcal{R}ip_T(Q)$. Namely, combining Theorems 3.16 and 3.14 we show that these groups are free-by-hyperbolic. This result will be essential to the proof of Theorem 5.1.

Proposition 3.17. *Let $G = N \rtimes Q \in \mathcal{R}ip_T(Q)$ and assume that Q is an infinite residually finite group. Then N is a $\mathbb{F}_{n+1}$-by-(nonelementary, hyperbolic property (T)) group, where $n \in \mathbb{N} \cup \{\infty\}$.*

Proof. Since G is hyperbolic relative to $\{Q\}$ and Q is residually finite, by Theorem 3.16 there is a nonelementary hyperbolic group K and an epimorphism $\delta : G \to K$ such that $L = \ker(\delta)$ is isomorphic to a nontrivial free product $L = *_{g \in T} Q_0^g$, where $T \subset G$ is a subset and $Q_0 \lhd Q$ is a finite-index, normal subgroup. Since $G = N \rtimes Q$ and Q_0 is normal in Q, one can assume without any loss of generality that

$T \subset N$. Next we show that $N \cap L$ infinite. If it were finite, as G is icc, it would follow that $N \cap L = 1$. As N and L are normal in G, the commutator satisfies $[N, L] \leqslant N \cap L = 1$ and hence $L \leqslant C_G(N)$. To describe this centralizer, fix $g = nq \in C_G(N)$, where $n \in N$, $q \in Q$. Thus for all $m \in N$ we have $nqm = mnq$ and hence $n\sigma_q(m) = mn$, where $\sigma_q(x) = q^{-1}xq$ for all $x \in N$. Therefore $\sigma_q = ad(n)$ and by Theorem 3.14 (5) we must have $q = 1$. This further implies that $m \in Z(N) = 1$ and hence $C_G(N) = 1$; in particular, $L = 1$, which is a contradiction. In conclusion $N \cap L \lhd N$ is an infinite normal subgroup. Using the isomorphism theorem we see that $N/(N \cap L) \cong (NL)/L$. Also from the free product description of L we see that $N \rtimes Q_0 \leqslant NL$ and hence $[G : NL] < \infty$. In particular $(NL)/L$ is a finite-index subgroup of $G/L = K$ and hence $(NL)/L$ is a (nonelementary) hyperbolic, property (T) group. To finish our proof we only need to argue that $N \cap L$ is a free group with at least two generators. Since $L = *_{g\in T} Q_0^g$, by the Kurosh theorem there exist a set $X \subset L$ and a collection of subgroups $Q_i \leqslant Q_0$, together with elements $g_i \in L$ such that $N \cap L = F(X) * (*_{i\in I} Q_i^{g_i})$; here $F(X)$ is a free group with free basis X. In particular, for every $i \in I$ the previous relation implies that $Q_i^{g_i} \leqslant N$ and writing $g_i = n_i q_i$ for some $n_i \in N$, $q_i \in Q$ we see that $Q_i^{q_i} \leqslant N$. As $Q_i^{q_i} \leqslant Q$ we conclude that $Q_i^{q_i} \leqslant N \cap Q = 1$ and hence $Q_i = 1$. Thus $N \cap L = F(X)$ and since G is icc and $N \cap L$ is normal in G, we see that $|X| \geqslant 2$, which finishes the proof. □

4. Maximal von Neumann subalgebras arising from groups Rips construction

If $\mathcal{M}$ is a von Neumann algebra then a von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ is called *maximal* if there is no intermediate von Neumann subalgebra $\mathcal{P}$ so that $\mathcal{N} \subsetneq \mathcal{P} \subsetneq \mathcal{M}$. Understanding the structure of maximal subalgebras of a given von Neumann algebra is a rather difficult problem that plays a key role in the very classification of these objects. Despite a series of earlier remarkable successes on the study of maximal *amenable* subalgebras initiated in [Popa 1983] and continued more recently [Shen 2006; Cameron et al. 2010; Houdayer 2014; Boutonnet and Carderi 2015; 2017; Suzuki 2020; Chifan and Das 2020; Jiang and Skalski 2019a], much less is known for the maximal ones. For instance Ge's question [2003, Section 3, Question 2] on the existence of nonamenable factors that possess maximal factors which are amenable was settled in the affirmative only very recently in [Jiang and Skalski 2019a]. We also remark that the study of maximal (or by duality minimal) intermediate subfactors has recently led to the discovery of a rigidity phenomenon for the intermediate subfactor lattice in the case of irreducible finite-index subfactors [Bakshi et al. 2019].

In this section we make new progress in this direction by describing several concrete collections of maximal subalgebras in the von Neumann algebras arising from the groups $\mathcal{R}ip(Q)$ introduced in the previous subsection (see Theorem 4.4 below). In particular, these examples allow construction of property (T) von Neumann algebras which have maximal von Neumann subalgebras without property (T). This answers a question raised in [Jiang and Skalski 2019a, Problem 5.5]. Our arguments rely on the usage of Galois correspondence results for von Neumann algebras à la [Choda 1978] and the classification of maximal subgroups in the monster-type groups provided in Theorem 3.12. We remark that Jiang and Skalski [2019a, Theorem 4.8] independently obtained a different solution, using different techniques.

First we need a couple of basic lemmas concerning automorphisms of groups. For the reader's convenience we include short proofs.

Lemma 4.1. *Let N be a group, let $\mathrm{Id} \neq \alpha \in \mathrm{Aut}(N)$ and denote by $N_1 = \{n \in N : \alpha(n) = n\}$ its fixed point subgroup. Then the following hold*:

(1) *Either $[N : N_1] = \infty$ or there is a subgroup $N_0 \leqslant N_1 \leqslant N$ that is normal in N with $[N : N_0] < \infty$ and such that the induced automorphism $\tilde{\alpha} \in \mathrm{Aut}(N/C_N(N_0))$ given by $\tilde{\alpha}(nC_N(N_0)) = \alpha(n)C_N(N_0)$ is the identity map*; *in particular*, *when N is icc we always have $[N : N_1] = \infty$.*

(2) *Either $[N : N_1] = \infty$, or α has finite order in $\mathrm{Aut}(N)$, or there is a $k \in \mathbb{N}$ and a subgroup $N_0 \leqslant N_1 \leqslant N$ that is normal in N with $[N : N_0] < \infty$ and such that the induced automorphism $\tilde{\alpha} \in \mathrm{Aut}(N/Z(N_0))$ given by $\tilde{\alpha}(nZ(N_0)) = \alpha(n)Z(N_0)$ has order k*; *in particular*, *when all finite-index subgroups of N have trivial center we either have $[N : N_1] = \infty$ or $\tilde{\alpha}$ has finite order.*

Proof. (1) Assume that $2 \leqslant [N : N_1] < \infty$. Then $N_0 := \bigcap_{h\in N} hN_1h^{-1} \leqslant N_1$ is a finite-index normal subgroup of N. Notice that the centralizer $C_N(N_0)$ is also normal in N. Let $n \in N$ and $n_0 \in N_0$. As N_0 is normal, we have $nn_0n^{-1} \in N_0 \leqslant N_1$ and hence $nn_0n^{-1} = \alpha(nn_0n^{-1}) = \alpha(n)n_0\alpha(n^{-1})$. This implies $n_0^{-1}n^{-1}\alpha(n)n_0 = n^{-1}\alpha(n)$ and hence $n^{-1}\alpha(n) \in C_N(N_0)$. Since α acts identically on N_0, one can see that $\alpha(C_N(N_0)) = C_N(N_0)$. Thus one can define an automorphism $\tilde{\alpha} : N/C_N(N_0) \to N/C_N(N_0)$ by letting $\tilde{\alpha}(nC_N(N_0)) = \alpha(n)C_N(N_0)$. However, the previous relations show that $\tilde{\alpha}$ is the identity map, as desired. For the remaining part of the statement, we notice that if $[N : N_1] < \infty$ and N is icc then the centralizer $C_N(N_0)$ is trivial and hence $\alpha = \mathrm{Id}$, which is a contradiction.

(2) Assume $[N : N_1] < \infty$ and α has infinite order in $\mathrm{Aut}(N)$. Also for each $i \geqslant 2$ define $N_i = \{n \in N : \alpha^i(n) = n\}$ and notice that $N_1 \leqslant N_i \leqslant N_{i+1} \leqslant N$. Since $[N : N_1] < \infty$, there is $s \in \mathbb{N}$ so that either $N_s = N_l$ for all $l \geqslant s$, or $N_s = N$. If $N_s = N$ then $\alpha^s = \mathrm{Id}$, contradicting the infinite-order assumption on α. Now assume that $N_s = N_{s+1}$. For every $n \in N_{s+1}$ we have $\alpha^s(n) = \alpha^{s+1}(n)$ and thus $\alpha(n) = n$, which is equivalent to $n \in N_1$. This shows that $N_1 = N_{s+1}$ and combining with the above we conclude that $N_1 = N_i$ for all i.

As $[N : N_1] < \infty$, we have $N_0 := \bigcap_{h\in N} hN_1h^{-1} \leqslant N_1$ is a finite-index normal subgroup of N. The automorphism α induces an automorphism $\tilde{\alpha}$ on the quotient group N/N_0 by $\tilde{\alpha}(nN_0) = \alpha(n)N_0$ for all $n \in N$. Since $[N : N_0] < \infty$, there is $k \in \mathbb{N}$ such that $\tilde{\alpha}^k = \mathrm{Id}$ on N/N_0. Thus for every $n \in N$ we have $n^{-1}\alpha^k(n) \in N_0$.

Let $n \in N$ and $n_0 \in N_0$. By normality we have $nn_0n^{-1} \in N_0 \leqslant N_1$ and hence $nn_0n^{-1} = \alpha^k(nn_0n^{-1}) = \alpha^k(n)n_0\alpha^k(n^{-1})$. This implies $n_0^{-1}n^{-1}\alpha^k(n)n_0 = n^{-1}\alpha^k(n)$ and hence $n^{-1}\alpha^k(n) \in Z(N_0)$. Since N_0 is normal in N, so is $Z(N_0)$. Since α leaves $Z(N_0)$ invariant, the map $\tilde{\alpha} : N/Z(N_0) \to N/Z(N_0)$ given by $\tilde{\alpha}(nZ(N_0)) = \alpha(n)Z(N_0)$ is an automorphism. The previous relations show that it has order k. □

Using this we will see that, in the case of icc groups, outer group actions $Q \curvearrowright N$ by automorphisms lift to outer actions $Q \curvearrowright \mathcal{L}(N)$ at the von Neumann algebra level. More precisely we have the following:

Lemma 4.2. *Let N be an icc group and let Q be a group together with an outer action $Q \curvearrowright^{\sigma} N$. Then $\mathcal{L}(N)' \cap \mathcal{L}(N \rtimes_\sigma Q) = \mathbb{C}$.*

Proof. To get $\mathcal{L}(N)' \cap \mathcal{L}(N \rtimes_\sigma Q) = \mathbb{C}$ it suffices to show that for all $g \in (N \rtimes_\sigma Q) \setminus \{e\}$ the N-conjugacy orbit $\mathcal{O}_N(g) = \{ngn^{-1} : n \in N\}$ is infinite. Suppose by contradiction there is $h = n_0q_0 \in (N \rtimes Q) \setminus \{e\}$ with

$n_0 \in N$ and $q_0 \in Q$ such that $|\mathcal{O}_N(h)| < \infty$. Hence there exists a finite-index subgroup $N_1 \leqslant N$ such that $nhn^{-1} = h$ for all $n \in N_1$. This gives that $nn_0q_0n^{-1} = n_0q_0$ and thus $n = n_0q_0nq_0^{-1}n_0^{-1} = \mathrm{ad}(n_0) \circ \sigma_{q_0}(n)$ for all $n \in N_1$. Also, since N is icc, we have $q_0 \neq e$. Let $\alpha = \mathrm{ad}(n_0) \circ \sigma_{q_0}$. Since $Q \curvearrowright N$ is outer it follows that $\mathrm{Id} \neq \alpha \in \mathrm{Aut}(N)$. Since N is icc and $[N : N_1] < \infty$, Lemma 4.1 (1) leads to a contradiction. □

With these results at hand we are now ready to deduce the main result of the section.

Notation 4.3. Fix any rank-1 group Q_m. Consider the lacunary hyperbolic groups Q from Theorem 3.12 where the maximal rank-1 subgroups of Q are isomorphic to Q_m. Also let $N \rtimes Q \in \mathcal{R}ip(Q)$ be the semidirect product obtained via the Rips construction together with the subgroups $N \rtimes Q_m < N \rtimes Q$. Throughout this section we will consider the corresponding von Neumann algebras $\mathcal{M}_m := \mathcal{L}(N \rtimes Q_m) \subset \mathcal{L}(N \rtimes Q) := \mathcal{M}$.

Assuming Notation 4.3, we now show the following:

Theorem 4.4. *$\mathcal{M}_m$ is a maximal von Neumann algebra of $\mathcal{M}$. In particular, if $N \rtimes Q \in \mathcal{R}ip_{\mathcal{T}}(Q)$ then $\mathcal{M}_m$ is a non-property* (*T*) *maximal von Neumann subalgebra of a property* (*T*) *von Neumann algebra $\mathcal{M}$.*

Proof. Let $\mathcal{P}$ be any intermediate subalgebra $\mathcal{M}_m \subseteq \mathcal{P} \subseteq \mathcal{M}$. Since $\mathcal{M}_m \subset \mathcal{M}$ is spatially isomorphic to the crossed product inclusion $\mathcal{L}(N) \rtimes Q_m \subset \mathcal{L}(N) \rtimes Q$, we have $\mathcal{L}(N) \rtimes Q_m \subseteq \mathcal{P} \subseteq \mathcal{L}(N) \rtimes Q$. By Lemma 4.2 we have $(\mathcal{L}(N) \rtimes Q_m)' \cap (\mathcal{L}(N) \rtimes Q) \subseteq \mathcal{L}(N)' \cap (\mathcal{L}(N) \rtimes Q) = \mathbb{C}$. In particular, $\mathcal{P}$ is a factor. Moreover, by the Galois correspondence theorem [Choda 1978] (see also [Chifan and Das 2020, Corollary 3.8]) there is a subgroup $Q_m \leqslant K \leqslant Q$ so that $\mathcal{P} = \mathcal{L}(N) \rtimes K$. Since by construction Q_m is a maximal subgroup of Q, we must have $K = Q_m$ or Q. Thus we get $\mathcal{P} = \mathcal{M}_m$ or $\mathcal{M}$ and the conclusion follows.

For the remaining part note that $\mathcal{M}$ has property (T) by [Connes and Jones 1985]. Also, since $N \rtimes Q_m$ surjects onto an infinite abelian group, it does not have property (T). Thus by [Connes and Jones 1985] again, $\mathcal{M}_m = \mathcal{L}(N \rtimes Q_m)$ does not have property (T) either. □

As pointed out at the beginning of the section, the above theorem provides a positive answer to [Jiang and Skalski 2019a, Problem 5.5]. Another solution to the problem of finding maximal subalgebras without property (T) inside factors with property (T) was also obtained independently by Jiang and Skalski in a more recent version of that paper. Their beautiful solution has a different flavor from ours; even though the Galois correspondence theorem à la Choda is a common ingredient in both of the proofs. Hence we refer the reader to [Jiang and Skalski 2019b, Theorem 4.8] for another solution to the aforementioned problem. Also note that while the algebras $\mathcal{M}_m$ do not have property (T), they are also nonamenable. In connection with this it would be very interesting if one could find an example of a property (T) II_1-factor which has maximal hyperfinite subfactors. This is essentially Ge's question but for property (T) factors.

In the final part of the section we show that whenever Q_ι is not isomorphic to Q_κ, the resulting maximal von Neumann subalgebras $\mathcal{M}_m$ and $\mathcal{M}_n$ are nonisomorphic. In fact we have the following more precise statement:

Theorem 4.5. *Assume that $Q_\iota, Q_\kappa < (\mathbb{Q}, +)$ and let $\Theta : \mathcal{M}_\iota \to \mathcal{M}_\kappa$ be a $*$-isomorphism. Then there exists a unitary $u \in \mathscr{U}(\mathcal{M}_\kappa)$ such that* $\mathrm{ad}(u) \circ \Theta : \mathcal{L}(N_1) \to \mathcal{L}(N_2)$ *is a $*$-isomorphism. Moreover* *there exist a group isomorphism* $\delta : Q_\iota \to Q_\kappa$ *and a* 1*-cocycle* $r : Q_\kappa \to \mathscr{U}(\mathcal{L}(N_2))$ *such that for all* $a \in \mathcal{L}(N_1)$ *and* $g \in Q_\iota$ *we have* $\mathrm{ad}(u) \circ \Theta(au_g) = \mathrm{ad}(u) \circ \Theta(a)v_{\delta(g)}r_{\delta(g)}$. *In particular*, *we have* $\mathrm{ad}(u) \circ \Theta \circ \alpha_g = \mathrm{ad}(r_{\delta(g)}) \circ \beta_{\delta(g)} \circ \mathrm{ad}(u) \circ \Theta$.

Proof. Identify $\mathcal{M}_\iota = \mathcal{L}(N_1) \rtimes Q_\iota$ and $\mathcal{M}_\kappa = \mathcal{L}(N_2) \rtimes Q_\kappa$ and let $\Theta : \mathcal{L}(N_1) \rtimes Q_\iota \to \mathcal{L}(N_2) \rtimes Q_\kappa$ be the $*$-isomorphism. Notice that since $\Theta(\mathcal{L}(N_1))$ has property (T) and Q_κ is amenable, by [Popa 2006a] we have $\Theta(\mathcal{L}(N_1)) \prec_{\mathcal{M}_\kappa} \mathcal{L}(N_2)$. Also by Lemma 4.2 we note that $\Theta(\mathcal{L}(N))$ is a regular irreducible subfactor of $\mathcal{M}_\kappa$, i.e., $\Theta(\mathcal{L}(N_1))' \cap \mathcal{M}_\kappa = \Theta(\mathcal{L}(N_1)' \cap \mathcal{M}_\iota) = \mathbb{C}1$. Similarly, $\mathcal{L}(N_2)$ is a regular irreducible subfactor of $\mathcal{M}_\kappa$ satisfying $\mathcal{L}(N_2) \prec_{\mathcal{M}_\kappa} \Theta(\mathcal{L}(N_1))$. Thus by the proof of [Ioana et al. 2008, Lemma 8.4], since Q_ι's are torsion-free, one can find a unitary $u \in \mathcal{M}_\kappa$ such that $\mathrm{ad}(u) \circ \Theta(\mathcal{L}(N_1)) = \mathcal{L}(N_2)$. So replacing Θ with $\mathrm{ad}(u) \circ \Theta$ we can assume that $\Theta(\mathcal{L}(N_1)) = \mathcal{L}(N_2)$. Hence for every $g \in Q_\iota$ we have $\Theta(\alpha_g(x))\Theta(u_g) = \Theta(u_g)\Theta(x)$ for all $x \in \mathcal{L}(N_1)$. Consider the Fourier decomposition $\Theta(u_g) = \sum_{h \in Q_\kappa} n_h v_h$, where $n_h \in \mathcal{L}(N_2)$. Using the previous relation we get $\Theta(\alpha_g(x))n_h = n_h \beta_h \Theta(x)$ for all $h \in Q_\kappa$ and $x \in \mathcal{L}(N_2)$. Thus $n_h n_h{}^* \in \mathcal{L}(N_2)' \cap \mathcal{M}_\kappa = \mathbb{C}1$ and hence there exist unitary $t_h \in \mathcal{L}(N_2)$ and scalar $s_h \in \mathbb{C}$ so that $n_h = s_h t_h$. Assume there exist $h_1 \neq h_2 \in Q_\kappa$ so that $s_{h_1}, s_{h_2} \neq 0$. This implies that $\Theta(\alpha_g(x)) = t_{h_1}\beta_{h_1}\Theta(x)t^*_{h_1} = t_{h_2}\beta_{h_2}\Theta(x)t^*_{h_2}$ for all $x \in \mathcal{L}(N_2)$. Thus $\beta_{h_1}(t^*_{h_1}t_{h_2})v_{h_1^{-1}h_2} = v^*_{h_1}t^*_{h_1}t_{h_2}v_{h_2} \in \mathcal{L}(N_2)' \cap \mathcal{M}_\kappa = \mathbb{C}1$. Therefore $h_1^{-1}h_2 = 1$ and $h_1 = h_2$, which is a contradiction. In particular there exists a unique $\delta(g) \in Q_\kappa$ so that $s_k = 0$ for all $k \in Q_\kappa \setminus \{\delta(g)\}$. Altogether these show that there is a well-defined map $\delta : Q_\iota \to Q_\kappa$ so that $\Theta(u_g) = n_{\delta(g)}v_{\delta(g)}$ for all $g \in Q_\iota$. It is easy to see that δ is a group isomorphism and the map $r : Q_\kappa \to \mathscr{U}(\mathcal{L}(N_2))$ given by $r(h) = \beta_h(n_h)$ is a 1-cocycle, i.e., $r(hk) = c_h\beta_h(c_k)$. □

Final remarks. We notice that our strategy from the proof of Theorem 4.4 can also be used to produce other examples of non-property (T) subalgebras in property (T) factors. Indeed for Q in the Rips construction one can take in fact *any* torsion-free, property (T) monster group Q in the sense of Olshanskii. If one picks any maximal subgroup $Q_0 < Q$ then, as before, the group von Neumann algebra $\mathcal{L}(N \rtimes Q_0)$ will obviously be maximal in $\mathcal{L}(N \rtimes Q)$. Notice that since $Q_0 < Q$ is maximal, Q_0 is infinite-index in Q. To see this note that if Q_0 is finite-index in Q, then Q_0 has property (T) and hence is finitely generated. Therefore Q_0 would be abelian and hence trivial, which is a contradiction. Therefore Q_0 must have infinite index in Q. In this case it is either finitely generated, in which case is abelian or it is infinitely generated. However, in both scenarios Q_0 does not have property (T) and hence neither does $N \rtimes Q_0$. Thus by [Connes and Jones 1985], $\mathcal{L}(N \rtimes Q_0)$ does not have property (T).

5. Von Neumann algebraic rigidity aspects for groups arising via Rips constructions

An impressive milestone in the classification of von Neumann algebras was the emergence over the past decade of the first examples of groups G that can be completely reconstructed from their von Neumann algebras $\mathcal{L}(G)$, i.e., W^**-superrigid groups* [Ioana et al. 2013; Berbec and Vaes 2014; Chifan and Ioana 2018]. The strategies used in establishing these results share a common key ingredient, namely, the ability to first reconstruct from $\mathcal{L}(G)$ various algebraic features of G such as its (generalized) wreath product decomposition in [Ioana et al. 2013; Berbec and Vaes 2014] and, respectively, its amalgam splitting in [Chifan and Ioana 2018, Theorem A]. This naturally leads to a broad and independent study, specifically identifying canonical group algebraic features of a group that pass to its von Neumann algebra. While several works have emerged recently in this direction [Chifan et al. 2016b; Chifan and Ioana 2018; Chifan and Udrea 2020], the surface has been only scratched and still a great deal of work remains to be done.

A difficult conjecture of Connes predicts that all icc property (T) groups are W^*-superrigid. Unfortunately, not a single example of such group is known at this time. Moreover, in the current literature there is an almost complete lack of examples of algebraic features occurring in a property (T) group that are recognizable at the von Neumann algebraic level. In this section we make progress on this problem for property (T) groups that appear as certain fiber products of Belegradek–Osin Rips-type constructions. Specifically, we have the following result:

Theorem 5.1. *Let $Q = Q_1 \times Q_2$, where Q_i are icc, torsion-free, biexact, property (T), weakly amenable, residually finite groups. For $i = 1, 2$, let $N_i \rtimes_{\sigma_i} Q \in \mathcal{R}ip_T(Q)$ and denote by $\Gamma = (N_1 \times N_2) \rtimes_\sigma Q$ the semidirect product associated with the diagonal action $\sigma = \sigma_1 \times \sigma_2 : Q \curvearrowright N_1 \times N_2$. Denote by $\mathcal{M} = \mathcal{L}(\Gamma)$ the corresponding II_1-factor. Assume that Λ is any arbitrary group and $\Theta : \mathcal{L}(\Gamma) \to \mathcal{L}(\Lambda)$ is any $*$-isomorphism. Then there exist group actions by automorphisms $H \curvearrowright^{\tau_i} K_i$ such that $\Lambda = (K_1 \times K_2) \rtimes_\tau H$, where $\tau = \tau_1 \times \tau_2 : H \curvearrowright K_1 \times K_2$ is the diagonal action. Moreover one can find a multiplicative character $\eta : Q \to \mathbb{T}$, a group isomorphism $\delta : Q \to H$, a unitary $w \in \mathcal{L}(\Lambda)$, and $*$-isomorphisms $\Theta_i : \mathcal{L}(N_i) \to \mathcal{L}(K_i)$ such that for all $x_i \in \mathcal{L}(N_i)$ and $g \in Q$ we have*

$$\Theta((x_1 \otimes x_2)u_g) = \eta(g)w((\Theta_1(x_1) \otimes \Theta(x_2))v_{\delta(g)})w^*. \tag{5.1}$$

Here $\{u_g : g \in Q\}$ and $\{v_h : h \in H\}$ are the canonical unitaries implementing the actions of $Q \curvearrowright \mathcal{L}(N_1) \mathbin{\bar\otimes} \mathcal{L}(N_2)$ and $H \curvearrowright \mathcal{L}(K_1) \mathbin{\bar\otimes} \mathcal{L}(K_2)$, respectively.

From a different perspective our theorem can be also seen as a von Neumann algebraic superrigidity result regarding conjugacy of actions on noncommutative von Neumann algebras. Notice that very little is known in this direction as well, as most of the known superrigidity results concern algebras arising from actions of groups on probability spaces.

We continue with a series of preliminary results that are essential to deriving the proof of Theorem 5.1 at the end of the section. First we present a location result for commuting diffuse property (T) subalgebras inside a von Neumann algebra arising from products of relative hyperbolic groups.

Theorem 5.2. *For $i = \overline{1, n}$ let $H_i < G_i$ be an inclusion of infinite groups such that H_i is residually finite and G_i is hyperbolic relative to H_i. Denote by $H = H_1 \times \cdots \times H_n < G_1 \times \cdots \times G_n = G$ the corresponding direct product inclusion. Let $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{L}(G)$ be two commuting von Neumann subalgebras with property (T). Then for every $i \in \overline{1, n}$ there exists $k \in \overline{1, 2}$ such that $\mathcal{N}_k \prec \mathcal{L}(\widehat{G}_i \times H_i)$, where $\widehat{G}_i := \bigtimes_{j \neq i} G_j$.*

Proof. Fix $i \in \overline{1, n}$. Since H_i is residually finite, using Theorem 3.16 there is a short exact sequence

$$1 \to \ker(\pi_i) \hookrightarrow G_i \xrightarrow{\pi_i} F_i \to 1,$$

where F_i is a nonelementary hyperbolic group and $\ker(\pi_i) = \langle H_i^0 \rangle = *_{t \in T_i} (H_i^0)^t$ for some subset $T \subset G_i$ and a finite-index normal subgroup $H_i^0 \lhd H_i$.

Following [Chifan et al. 2015, Notation 3.3] we now consider the von Neumann algebraic embedding corresponding to π_i, i.e., $\Pi_i : \mathcal{L}(G) \to \mathcal{L}(G) \bar{\otimes} \mathcal{L}(F_i)$ given by $\Pi_i(u_g) = u_g \otimes v_{\pi_i(g_i)}$ for all $g = (g_j) \in G$; here the u_g's are the canonical unitaries of $\mathcal{L}(G)$ and the v_h's are the canonical unitaries of $\mathcal{L}(F_i)$. From the hypothesis we have that $\Pi_i(\mathcal{N}_1), \Pi_i(\mathcal{N}_2) \subset \mathcal{L}(G) \bar{\otimes} \mathcal{L}(F_i) =: \widetilde{\mathcal{M}}_i$ are commuting property (T) subalgebras. Let $\mathcal{A} \subset \Pi_i(\mathcal{N}_1)$ be any diffuse amenable von Neumann subalgebra. Using [Popa and Vaes 2014, Theorem 1.4] we have either (a) $\mathcal{A} \prec_{\widetilde{\mathcal{M}}_i} \mathcal{L}(G) \bar{\otimes} 1$ or (b) $\Pi_i(\mathcal{N}_2)$ is amenable relative to $\mathcal{L}(G) \bar{\otimes} 1$ inside $\widetilde{\mathcal{M}}_i$.

Since the $\mathcal{N}_k$'s have property (T), so do the $\Pi_i(\mathcal{N}_k)$'s. Thus using part (b) above we get that $\Pi_i(\mathcal{N}_2) \prec_{\widetilde{\mathcal{M}}} \mathcal{L}(G) \bar{\otimes} 1$. On the other hand, if case (a) above were to hold for all $\mathcal{A}$'s then by [Brown and Ozawa 2008, Corollary F.14] we would get $\Pi_i(\mathcal{N}_1) \prec_{\widetilde{\mathcal{M}}_i} \mathcal{L}(G) \bar{\otimes} 1$. Therefore we can always assume that $\Pi_i(\mathcal{N}_k) \prec_{\widetilde{\mathcal{M}}_i} \mathcal{L}(G) \bar{\otimes} 1$ for $k = 1$ or 2.

Due to symmetry we only treat $k = 1$. Using [Chifan et al. 2015, Proposition 3.4] we get $\mathcal{N}_1 \prec \mathcal{L}(\ker(\Pi_i)) = \mathcal{L}(\widehat{G}_i \times \ker(\pi_i))$. Thus there exist nonzero projections $p \in \mathcal{N}_1$, $q \in \mathcal{L}(\widehat{G}_i \times \ker(\pi_i))$, a nonzero partial isometry $v \in \mathcal{M}$ and a $*$-isomorphism $\phi : p\mathcal{N}_1 p \to \mathcal{B} := \phi(p\mathcal{N}_1 p) \subset q\mathcal{L}(\widehat{G}_i \times \ker(\pi_i))q$ on the image such that

$$\phi(x)v = vx \quad \text{for all } x \in p\mathcal{N}_1 p. \tag{5.2}$$

Also notice that since $\mathcal{N}_1$ has property (T), so does $p\mathcal{N}_1 p$ and therefore $\mathcal{B} \subseteq q\mathcal{L}(\widehat{G}_i \times \ker(\pi_i))q$ is a property (T) subalgebra. Since $\ker(\pi_i) = *_{t \in T} (H_i^0)^t$, by further conjugating q in the factor $\mathcal{L}(\widehat{G}_i \times \ker(\pi_i))$ we can assume that there exists a unitary $u \in \mathcal{L}(\widehat{G}_i \times \ker(\pi_i))$ and a projection $q_0 \in \mathcal{L}(\widehat{G}_i)$ such that $\mathcal{B} \subseteq u(q_0 \mathcal{L}(\widehat{G}_i) q_0) \bar{\otimes} \mathcal{L}(\ker(\pi_i))u^*$. Using property (T) of $\mathcal{B}$ and [Ioana et al. 2008, Theorem] we further conclude that there is $t_0 \in T$ such that $\mathcal{B} \prec_{u(q_0 \mathcal{L}(\widehat{G}_i) q_0 \bar{\otimes} \mathcal{L}(\ker(\pi_i)))u^*} u(q_0 \mathcal{L}(\widehat{G}_i) q_0 \otimes \mathcal{L}((H_i^0)^{t_0}))u^*$. Composing this intertwining with ϕ we finally conclude that $\mathcal{N}_1 \prec_{\mathcal{M}} \mathcal{L}(\widehat{G}_i \times H_i^0)$, as desired. □

Theorem 5.3. *Under the same assumptions as in Theorem 5.2, for every $k \in \overline{1, n}$ one of the following must hold*:

(1) *There exists $i \in 1, 2$ such that $\mathcal{N}_i \prec_{\mathcal{M}} \mathcal{L}(\widehat{G}_k)$.*

(2) $\mathcal{N}_1 \vee \mathcal{N}_2 \prec_{\mathcal{M}} \mathcal{L}(\widehat{G}_k \times H_k)$.

Proof. From Theorem 5.2 there exists $i \in \overline{1, 2}$ such that $\mathcal{N}_i \prec \mathcal{L}(\widehat{G}_k \times H_k)$. For convenience assume that $i = 1$. Thus there exist nonzero projections $p \in \mathcal{N}_1$, $q \in \mathcal{L}(\widehat{G}_k \times H_k)$, a nonzero partial isometry $v \in \mathcal{M}$ and a $*$-isomorphism $\phi : p\mathcal{N}_1 p \to \mathcal{B} := \phi(p\mathcal{N}_1 p) \subset q\mathcal{L}(\widehat{G}_k \times H_k)q$ on the image such that

$$\phi(x)v = vx \quad \text{for all } x \in p\mathcal{N}_1 p. \tag{5.3}$$

Notice that $q \geqslant vv^* \in \mathcal{B}' \cap q\mathcal{M}q$ and $p \geqslant v^*v \in p\mathcal{N}_i p' \cap p\mathcal{M}p$. Also we can pick v such that $s(E_{\mathcal{L}(\widehat{G}_k \times H_k)}(vv^*)) = q$. Next we assume that $\mathcal{B} \prec_{L(\widehat{G}_k \times H_k)} L(\widehat{G}_k)$. Thus there exist nonzero projections $p' \in \mathcal{B}$, $q' \in \mathcal{L}(\widehat{G}_k)$, a nonzero partial isometry $w \in q'\mathcal{L}(\widehat{G}_k \times H_k)p'$ and a $*$-isomorphism $\psi : p'\mathcal{B}p' \to q'\mathcal{L}(\widehat{G}_k)q'$ on the image such that

$$\psi(x)w = wx \quad \text{for all } x \in p'\mathcal{B}p'. \tag{5.4}$$

Notice that $q \geqslant p' \geqslant ww^* \in (p'\mathcal{B}p')' \cap p'\mathcal{M}p'$ and $q' \geqslant w^*w \in \psi(p'\mathcal{B}p')' \cap q'\mathcal{M}q'$. Using (5.3) and (5.4) we see that

$$\psi(\phi(x))wv = w\phi(x)v = wvx \quad \text{for all } x \in p_0\mathcal{N}_i p_0, \tag{5.5}$$

where $p_0 \in \mathcal{N}_i$ is a projection picked so that $\phi(p_0) = p'$. Also we note that if $0 = wv$ then $0 = wvv^*$, and hence $0 = E_{\mathcal{L}(\widehat{G}_k \times H_k)}(wvv^*) = wE_{\mathcal{L}(\widehat{G}_k \times H_k)}(vv^*)$. This further implies that $0 = ws(E_{\mathcal{L}(\widehat{G}_k \times H_k)}(vv^*)) = wq = w$, which is a contradiction. Thus $wv \neq 0$ and taking the polar decomposition of wv we see that (5.5) gives (1).

Next we assume that $\mathcal{B} \not\prec_{\mathcal{L}(\widehat{G}_k \times H_k)} \mathcal{L}(\widehat{G}_k)$. Since G_k is hyperbolic relative to H_k, by Lemma 2.2 we have that for all $x, x_1x_2, \dots, x_l \in M$ such that $\mathcal{B}x \subseteq \sum_{i=1}^{l} x_i\mathcal{B}$ we must have $x \in \mathcal{L}(\widehat{G}_k \times H_k)$. Hence in particular we have $vv^* \in \mathcal{B}' \cap q\mathcal{M}q \subseteq \mathcal{L}(\widehat{G}_k \times H_k)$ and thus relation (5.3) implies that $\mathcal{B}vv^* = v\mathcal{N}_i v^* \subseteq \mathcal{L}(\widehat{G}_k \times H_k)$. Also for every $c \in \mathcal{N}_{i+1}$ we can see that

$$\begin{aligned}\mathcal{B}vcv^* &= \mathcal{B}vv^*vcv^* = v\mathcal{N}_i v^*vcv^* = vv^*vc\mathcal{N}_i v^* \\ &= vc\mathcal{N}_i v^* = vc\mathcal{N}_i v^*vv^* = vcv^*v\mathcal{N}_i v^* = vcv^*\mathcal{B}vv^* = vcv^*\mathcal{B}.\end{aligned} \tag{5.6}$$

Therefore by Lemma 2.2 again we have $vcv^* \in \mathcal{L}(\widehat{G}_k \times H_k)$ and hence $v\mathcal{N}_{i+1}v^* \subseteq \mathcal{L}(\widehat{G}_k \times H_k)$. Thus $v\mathcal{N}_i\mathcal{N}_{i+1}v^* = vv^*v\mathcal{N}_i\mathcal{N}_{i+1}v^* = v\mathcal{N}_i v^*v\mathcal{N}_{i+1}v^* \subseteq \mathcal{L}(\widehat{G}_k \times H_k)$, which by Popa's intertwining techniques implies that $\mathcal{N}_1 \vee \mathcal{N}_2 \prec \mathcal{L}(\widehat{G}_k \times H_k)$, i.e., (2) holds. □

We now proceed towards proving the main result of this section. To simplify the exposition we first introduce notation that will be used throughout the section.

Notation 5.4. Define $Q = Q_1 \times Q_2$, where Q_i are infinite, residually finite, biexact, property (T), icc groups. Then consider $\Gamma_i = N_i \rtimes Q \in \mathcal{R}ip_T(Q)$ and the semidirect product $\Gamma = (N_1 \times N_2) \rtimes_\sigma Q$ arising from the diagonal action $\sigma = \sigma_1 \times \sigma_2 : Q \to \mathrm{Aut}(N_1 \times N_2)$, i.e., $\sigma_g(n_1, n_2) = ((\sigma_1)_g(n_1), (\sigma_2)_g(n_2))$ for all $(n_1, n_2) \in N_1 \times N_2$. For further use we observe that Γ is the fiber product $\Gamma = \Gamma_1 \times_Q \Gamma_2$ and thus embeds into $\Gamma_1 \times \Gamma_2$, where Q embeds diagonally into $Q \times Q$. In the next proofs when we refer to this copy we will often denote it by $\mathrm{d}(Q)$. Also notice that Γ is a property (T) group as it arises from an extension of property (T) groups. Furthermore, $\Gamma_1, \Gamma_2 \in \mathcal{R}ip_T(Q)$ easily implies that Γ is an icc group.

For future use, use also recall the notion of the comultiplication studied in [Ioana et al. 2013; Ioana 2011]. Let Γ be a group as above, and assume that Λ is a group such that $\mathcal{L}(\Gamma) = \mathcal{L}(\Lambda) = \mathcal{M}$. Then the "comultiplication along Λ" $\Delta : \mathcal{M} \to \mathcal{M} \mathbin{\bar\otimes} \mathcal{M}$ is defined by $\Delta(v_\lambda) = v_\lambda \otimes v_\lambda$ for all $\lambda \in \Lambda$.

Theorem 5.5. *Let Γ be a group as in Notation 5.4 and assume that Λ is a group such that $\mathcal{L}(\Gamma) = \mathcal{L}(\Lambda) = \mathcal{M}$. Let $\Delta : \mathcal{M} \to \mathcal{M} \mathbin{\bar\otimes} \mathcal{M}$ be the comultiplication along Λ as in Notation 5.4. Then the following hold*:

(3) *For all $j \in \overline{1,2}$ there is $i \in \overline{1,2}$ such that $\Delta(\mathcal{L}(N_i)) \prec_{\mathcal{M} \bar\otimes \mathcal{M}} \mathcal{M} \mathbin{\bar\otimes} \mathcal{L}(N_j)$.*

(4) (a) *For all $j \in \overline{1,2}$ there is $i \in \overline{1,2}$ such that $\Delta(\mathcal{L}(Q_i)) \prec_{\mathcal{M} \bar\otimes \mathcal{M}} \mathcal{M} \mathbin{\bar\otimes} \mathcal{L}(N_j)$ or*
   (b) $\Delta(\mathcal{L}(Q)) \prec_{\mathcal{M} \bar\otimes \mathcal{M}} \mathcal{M} \mathbin{\bar\otimes} \mathcal{L}(Q)$; *moreover in this case for every $j \in \overline{1,2}$ there is $i \in \overline{1,2}$ such that $\Delta(\mathcal{L}(Q_j)) \prec_{\mathcal{M} \bar\otimes \mathcal{M}} \mathcal{M} \mathbin{\bar\otimes} \mathcal{L}(Q_i)$.*

Proof. Let $\widetilde{\mathcal{M}} = \mathcal{L}(\Gamma_1 \times \Gamma_2)$. Since $\Gamma < \Gamma_1 \times \Gamma_2$, we notice the inclusions $\Delta(\mathcal{L}(N_1)), \Delta(\mathcal{L}(N_2)) \subset \mathcal{M} \bar{\otimes} \mathcal{M} = \mathcal{L}(\Gamma \times \Gamma) \subset \mathcal{L}(\Gamma_1 \times \Gamma_2 \times \Gamma_1 \times \Gamma_2)$. Since Γ_i is hyperbolic relative to Q, using Theorem 5.3 we have either

(5) there exists $i \in 1, 2$ such that $\Delta(\mathcal{L}(N_i)) \prec_{\widetilde{\mathcal{M}} \bar{\otimes} \widetilde{\mathcal{M}}} \mathcal{M} \bar{\otimes} \mathcal{L}(\Gamma_1)$, or

(6) $\Delta(\mathcal{L}(N_1 \times N_2)) \prec_{\widetilde{\mathcal{M}} \bar{\otimes} \widetilde{\mathcal{M}}} \mathcal{M} \bar{\otimes} \mathcal{L}(\Gamma_1 \times Q)$.

Assume (5) holds. Since $\Delta(\mathcal{L}(N_i)) \subset \mathcal{M} \bar{\otimes} \mathcal{L}(\Gamma)$ then by Lemma 2.3 there is an $h \in \Gamma_1 \times \Gamma_2 \times \Gamma_1 \times \Gamma_2$ so that $\Delta(\mathcal{L}(N_i)) \prec_{\widetilde{\mathcal{M}} \bar{\otimes} \widetilde{\mathcal{M}}} \mathcal{L}(\Gamma \times (\Gamma \cap h(\Gamma_1 \times \Gamma_2 \times \Gamma_1)h^{-1})) = \mathcal{L}(\Gamma \times (\Gamma \cap \Gamma_1)) = \mathcal{M} \bar{\otimes} (\mathcal{L}((N_1 \times N_2) \rtimes \mathrm{d}(Q)) \cap (N_1 \rtimes Q \times 1)) = \mathcal{M} \bar{\otimes} \mathcal{L}(N_1)$. Note that since $\Delta(\mathcal{L}(N_i))$ is regular in $\mathcal{M} \bar{\otimes} \mathcal{M}$, using Lemma 2.4, we get that $\Delta(\mathcal{L}(N_i)) \prec_{\mathcal{M} \bar{\otimes} \mathcal{M}} \mathcal{M} \bar{\otimes} \mathcal{L}(\Gamma_1)$, thereby establishing 3).

Assume (6) holds. Since $\Delta(\mathcal{L}(N_1 \times N_2)) \subset \mathcal{L}(\Gamma \times \Gamma)$, by Lemma 2.3 there is an $h \in \Gamma_1 \times \Gamma_2 \times \Gamma_1 \times \Gamma_2$ such that

$$\begin{aligned}\Delta(\mathcal{L}(N_1 \times N_2)) &\prec \mathcal{L}(\Gamma \times (\Gamma \cap h(\Gamma_1 \times \Gamma_2 \times \Gamma_1 \times Q)h^{-1})) \\ &= \mathcal{L}(\Gamma \times (\Gamma \cap (\Gamma_1 \times h_4 Q h_4^{-1}))) \\ &= \mathcal{M} \bar{\otimes} \mathcal{L}((N_1 \times N_2) \rtimes \mathrm{d}(Q)) \cap (N_1 \rtimes Q \times h_4 Q h_4^{-1}).\end{aligned}$$

Since $h_4 \in \Gamma_2 = N_2 \rtimes Q$, we can assume that $h_4 \in N_2$. Notice that

$$\begin{aligned}((N_1 \times N_2) \rtimes \mathrm{d}(Q)) \cap (N_1 \rtimes Q \times h_4 Q h_4^{-1}) &= h_4((N_1 \times N_2) \rtimes \mathrm{d}(Q)) \cap (N_1 \rtimes Q \times Q)h_4^{-1} \\ &= h_4((N_1 \times 1) \rtimes \mathrm{d}(Q))h_4^{-1}\end{aligned}$$

and hence $\Delta(\mathcal{L}(N_1 \times N_2)) \prec_{\widetilde{\mathcal{M}} \bar{\otimes} \widetilde{\mathcal{M}}} \mathcal{M} \bar{\otimes} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q))$. Moreover using Lemma 2.5 we further have $\Delta(\mathcal{L}(N_1 \times N_2)) \prec_{\mathcal{M} \bar{\otimes} \mathcal{M}} \mathcal{M} \bar{\otimes} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q))$.

In conclusion, there exist a $*$-isomorphism on its image

$$\phi : p\Delta(\mathcal{L}(N_1 \times N_2))p \to \mathcal{B} := \phi(p\Delta(\mathcal{L}(N_1 \times N_2))p) \subseteq q\mathcal{M} \bar{\otimes} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q))$$

and $0 \neq v \in q(\mathcal{M} \bar{\otimes} \mathcal{M})p$ such that

$$\phi(x)v = vx \quad \text{for all } x \in p\Delta(\mathcal{L}(N_1 \times N_2))p. \tag{5.7}$$

Next assume that (3) doesn't hold. Thus proceeding as in the first part of the proof of Theorem 5.3, we get

$$\mathcal{B} \nprec_{\mathcal{M} \bar{\otimes} (N_1 \rtimes \mathrm{d}(Q))} \mathcal{M} \bar{\otimes} \mathcal{L}(N_1) =: \mathcal{M}_1. \tag{5.8}$$

Next we observe the inclusions

$$\begin{aligned}\mathcal{M}_1 \rtimes_{1 \otimes \sigma} \mathrm{d}(Q) &= \mathcal{M} \bar{\otimes} \mathcal{L}(N_1) \rtimes_{1 \otimes \sigma} \mathrm{d}(Q) = \mathcal{M} \bar{\otimes} \mathcal{L}(N_1 \rtimes_\sigma \mathrm{d}(Q)) \\ &\subset \mathcal{M} \bar{\otimes} \mathcal{L}((N_1 \times N_2) \rtimes_\sigma \mathrm{d}(Q)) = \mathcal{M} \bar{\otimes} \mathcal{L}(N_1) \bar{\otimes} \mathcal{L}(N_2) \rtimes \mathrm{d}(Q) = \mathcal{M}_1 \rtimes_{1 \otimes \sigma} N_2 \rtimes \mathrm{d}(Q).\end{aligned} \tag{5.9}$$

Also since Q is malnormal in $N_2 \rtimes Q$ it follows from Lemma 2.2 that $vv^* \in \mathcal{M} \bar{\otimes} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q))$ and hence $\mathcal{B}vv^* \subset \mathcal{M} \bar{\otimes} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q))$. Pick $u \in \mathcal{QN}_{p(\mathcal{M} \bar{\otimes} \mathcal{M})p}(p\Delta(\mathcal{L}(N_1 \times N_2))p)$ and using (5.7) we see that there exist $n_1, n_2, \dots, n_s \in p(\mathcal{M} \mathbin{\bar{\otimes}} \mathcal{M})p$ satisfying

$$\begin{aligned}\mathcal{B}vuv^* = \mathcal{B}vv^*vuv^* &= vp(\Delta(\mathcal{L}(N_1 \times N_2)))pv^*vnv^* = vp(\Delta(\mathcal{L}(N_1 \times N_2)))pnv^* \\ &\subseteq \sum_{i=1}^{s} vn_i p(\Delta(\mathcal{L}(N_1 \times N_2)))pv^* = \sum_{i=1}^{s} vn_i p(\Delta(\mathcal{L}(N_1 \times N_2)))pv^*vv^* \\ &= \sum_{i=1}^{s} vn_i pv^*v(\Delta(\mathcal{L}(N_1 \times N_2)))pv^* = \sum_{i=1}^{s} vn_i pv^*\mathcal{B}vv^* = \sum_{i=1}^{s} vn_i pv^*\mathcal{B}. \end{aligned} \tag{5.10}$$

Then by Lemma 2.2 again we must have $vuv^* \in \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q))$. Hence we have shown that

$$v\mathscr{QN}_{p(\mathcal{M}\bar{\otimes}\mathcal{M})p}(p\Delta(\mathcal{L}(N_1 \times N_2))p)v^* \subseteq \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q)). \tag{5.11}$$

Since $v^*v \in (p\Delta(\mathcal{L}(N_1 \times N_2))p)' \cap p(\mathcal{M} \mathbin{\bar{\otimes}} \mathcal{M})p \subset \mathscr{QN}_{p(\mathcal{M}\bar{\otimes}\mathcal{M})p}(p(\Delta(\mathcal{L}(N_1 \times N_2))))p$, (5.11) further implies

$$v\mathscr{QN}_{p(\mathcal{M}\bar{\otimes}\mathcal{M})p}(p(\Delta(\mathcal{L}(N_1 \times N_2)))p)''v^* \subseteq \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q)). \tag{5.12}$$

Here for every inclusion of von Neumann algebras $\mathcal{R} \subseteq \mathcal{T}$ and projection $p \in \mathcal{R}$ we used the formula $\mathscr{QN}_{p\mathcal{T}p}(p\mathcal{R}p)'' = p\mathscr{QN}_{\mathcal{T}}(\mathcal{R})''p$ [Popa 2006b, Lemma 3.5]. As

$$vp\Delta(\mathcal{M})pv^* \subseteq v\mathscr{QN}_{p(\mathcal{M}\bar{\otimes}\mathcal{M})p}(p(\Delta(\mathcal{L}(N_1 \times N_2)))p)''v^*,$$

we conclude that $\Delta(\mathcal{M}) \prec \mathcal{L}(N_1 \rtimes Q)$, which contradicts the fact that N_2 is infinite. Thus (3) must always hold.

Next we derive (4). Again we notice that

$$\Delta(\mathcal{L}(Q_1)), \Delta(\mathcal{L}(Q_2)) \subset \Delta(\mathcal{M}) \subset \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{M} = \mathcal{L}(\Gamma \times \Gamma) \subset \mathcal{L}(\Gamma_1 \times \Gamma_2 \times \Gamma_1 \times \Gamma_2).$$

Using Theorem 5.3 we must have either

(7) $\Delta(\mathcal{L}(Q_i)) \prec_{\widetilde{\mathcal{M}}\bar{\otimes}\widetilde{\mathcal{M}}} \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{L}(\Gamma_1)$, or

(8) $\Delta(\mathcal{L}(Q)) \prec_{\widetilde{\mathcal{M}}\bar{\otimes}\widetilde{\mathcal{M}}} \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{L}(\Gamma_1 \times Q)$.

Proceeding as in the previous case, and using Lemma 2.4, we see that (7) implies $\Delta(\mathcal{L}(Q_i)) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{L}(N_1)$, which in turn gives (4a). Also proceeding as in the previous case, and using Lemma 2.5, we see that (8) implies

$$\Delta(\mathcal{L}(\mathrm{d}(Q))) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q)). \tag{5.13}$$

To show part (4b) we will exploit (5.13). Notice that there exist nonzero projections $r \in \Delta(\mathcal{L}(Q))$, $t \in \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q))$, a nonzero partial isometry $w \in r(\mathcal{M} \mathbin{\bar{\otimes}} \mathcal{M})t$ and a $*$-isomorphism onto its image $\phi : r\Delta(\mathcal{L}(Q))r \to \mathcal{C} := \phi(r\Delta(\mathcal{L}(Q))r) \subseteq t(\mathcal{M} \mathbin{\bar{\otimes}} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q)))t$ such that

$$\phi(x)w = wx \quad \text{for all } x \in r\Delta(\mathcal{L}(Q))r. \tag{5.14}$$

Since $\mathcal{L}(Q)$ is a factor we can assume without loss of generality that $r = \Delta(r_1 \otimes r_2)$, where $r_i \in \mathcal{L}(Q_i)$. Hence $\mathcal{C} = \phi(r\Delta(\mathcal{L}(Q))r) = \phi(\Delta(r_1\mathcal{L}(Q_i)r_2)) \mathbin{\bar{\otimes}} r_2\mathcal{L}(Q_2)r_2 =: \mathcal{C}_1 \vee \mathcal{C}_2$, where $\mathcal{C}_i = \phi(\Delta(r_i\mathcal{L}(Q_i))r_i) \subseteq$

$t(\mathcal{M}\bar{\otimes}\mathcal{L}(N_1 \rtimes \mathrm{d}(Q)))t$. Notice that the $\mathcal{C}_i$'s are commuting property (T) subfactors of $\mathcal{M}\bar{\otimes}\mathcal{L}(N_1 \rtimes \mathrm{d}(Q))$. Since $N_i \rtimes Q$ is hyperbolic relative to $\{Q\}$ and seeing

$$\mathcal{C}_1 \vee \mathcal{C}_2 \subseteq \mathcal{M} \bar{\otimes} \mathcal{L}(N_i \rtimes \mathrm{d}(Q)) \subset \mathcal{L}(\Gamma_1 \times \Gamma_2 \times (N_1 \rtimes \mathrm{d}(Q))),$$

by applying Theorem 5.3 we have that there exists $i \in 1, 2$ such that

(9) $\mathcal{C}_1 \prec_{\tilde{\mathcal{M}}\bar{\otimes}\mathcal{L}(N_1\rtimes\mathrm{d}(Q))} \mathcal{L}(\Gamma_1 \times \Gamma_2)$ or

(10) $\mathcal{C}_1 \vee \mathcal{C}_2 \prec_{\tilde{\mathcal{M}}\bar{\otimes}\mathcal{L}(N_1\rtimes\mathrm{d}(Q))} \mathcal{L}(\Gamma_1 \times \Gamma_2 \times \mathrm{d}(Q))$.

Since $\mathcal{C}_1 \subset \mathcal{M}\bar{\otimes}\mathcal{M}$ then (9) and Lemma 2.6 imply $\mathcal{C}_1 \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\otimes 1$, which by [Ioana 2011, Lemma 9.2] further implies that $\mathcal{C}_1$ is atomic, which is a contradiction. Thus we must have (10). However since $\mathcal{C}_1 \vee \mathcal{C}_2 \subset \mathcal{M}\bar{\otimes}\mathcal{M}$, part (10) and Lemma 2.6 give $\mathcal{C}_1 \vee \mathcal{C}_2 \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\bar{\otimes}\mathcal{L}(\mathrm{d}(Q))$ and composing this intertwining with ϕ (as done in the proof of the first case in Theorem 5.3) we get $\Delta(\mathcal{L}(Q)) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\bar{\otimes}\mathcal{L}(\mathrm{d}(Q))$. Now we show the "moreover" part. So in particular the above intertwining shows that we can assume from the beginning that $\mathcal{C} = \mathcal{C}_1 \vee \mathcal{C}_2 \subset t(\mathcal{M}\bar{\otimes}\mathcal{L}(\mathrm{d}(Q)))t$. Since the Q_i are biexact, weakly amenable, by applying [Popa and Vaes 2014, Theorem 1.4] we must have that either $\mathcal{C}_1 \prec \mathcal{M}\bar{\otimes}\mathcal{L}(\mathrm{d}(Q_1))$ or $\mathcal{C}_2 \prec \mathcal{M}\bar{\otimes}\mathcal{L}(\mathrm{d}(Q_1))$ or $\mathcal{C}_1\vee\mathcal{C}_2$ is amenable relative to $\mathcal{M}\bar{\otimes}\mathcal{L}(\mathrm{d}(Q_1))$ inside $\mathcal{M}\bar{\otimes}\mathcal{M}$. However since $\mathcal{C}_1\vee\mathcal{C}_2$ has property (T) the last case above still gives that $\mathcal{C}_1\vee\mathcal{C}_2 \prec \mathcal{M}\bar{\otimes}\mathcal{L}(\mathrm{d}(Q_1))$, which completes the proof. □

Theorem 5.6. *Let Γ be a group as in Notation 5.4 and assume that Λ is a group such that $\mathcal{L}(\Gamma) = \mathcal{L}(\Lambda) = \mathcal{M}$. Let $\Delta : \mathcal{M} \to \mathcal{M}\bar{\otimes}\mathcal{M}$ be the "comultiplication along Λ" as in Notation 5.4. Also assume for every $j \in 1, 2$ there is $i \in 1, 2$ such that either $\Delta(\mathcal{L}(Q_i)) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\bar{\otimes}\mathcal{L}(Q_j)$ or $\Delta(\mathcal{L}(Q_i)) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\bar{\otimes}\mathcal{L}(N_j)$. Then one can find subgroups $\Phi_1, \Phi_2 \leqslant \Phi \leqslant \Lambda$ such that*:

(1) *Φ_1, Φ_2 are infinite, commuting, property (T), finite-by-icc groups.*

(2) *$[\Phi : \Phi_1\Phi_2] < \infty$ and $\mathrm{QN}^{(1)}_{\Lambda}(\Phi) = \Phi$.*

(3) *There exist $\mu \in \mathscr{U}(\mathcal{M})$, $z \in \mathscr{P}(\mathscr{Z}(\mathcal{L}(\Phi)))$, $h = \mu z \mu^* \in \mathscr{P}(\mathcal{L}(Q))$ such that*

$$\mu\mathcal{L}(\Phi)z\mu^* = h\mathcal{L}(Q)h. \tag{5.15}$$

Proof. For the proof we use an approach based upon the methods developed in [Chifan et al. 2016b; Chifan and Ioana 2018; Chifan and Udrea 2020]. For the reader's convenience we include all the details.

Since the relative commutants $\mathcal{L}(Q_j)' \cap \mathcal{M}$ and $\mathcal{L}(N_j)' \cap \mathcal{M}$ are nonamenable, in both cases using [Drimbe et al. 2019, Theorem 4.1] (see also [Ioana 2011, Theorem 3.1; Chifan et al. 2016b, Theorem 3.3]), one can find a subgroup $\Sigma < \Lambda$ with $C_\Lambda(\Sigma)$ nonamenable such that $\mathcal{L}(Q_1) \prec_{\mathcal{M}} \mathcal{L}(\Sigma)$. Thus there are $0 \neq p \in \mathscr{P}(\mathcal{L}(Q_1))$, $0 \neq f \in \mathscr{P}(\mathcal{L}(\Sigma))$, a partial isometry $0 \neq v \in f\mathcal{M}p$ and a $*$-isomorphism onto its image $\phi : p\mathcal{L}(Q_1)p \to \mathcal{B} := \phi(p\mathcal{L}(Q_1)p) \subseteq f\mathcal{L}(\Sigma)f$ so that

$$\phi(x)v = vx \quad \text{for all } x \in p\mathcal{L}(Q_1)p. \tag{5.16}$$

Notice that $vv^* \in \mathcal{B}' \cap f\mathcal{M}f$ and $v^*v \in (p\mathcal{L}(Q_1)p)' \cap p\mathcal{M}p = \mathcal{L}(Q_2)p$. Then (5.16) implies that $\mathcal{B}vv^* = v\mathcal{L}(Q_1)v^* = u_1\mathcal{L}(Q_1)v^*vu_1^*$, where $u_1 \in \mathscr{U}(\mathcal{M})$ extends v. Passing to relative commutants we get $vv^*(\mathcal{B}' \cap f\mathcal{M}f)vv^* = u_1v^*v((p\mathcal{L}(Q_1)p)' \cap p\mathcal{M}p)v^*vu_1^* = u_1v^*v(p\mathcal{L}(Q_2))v^*vu_1^*$. These relations further imply $vv^*(\mathcal{B}\vee\mathcal{B}'\cap f\mathcal{M}f)vv^*=\mathcal{B}vv^*\vee vv^*(\mathcal{B}'\cap f\mathcal{M}f)vv^*\subseteq u_1\mathcal{L}(Q)u_1^*$. As $\mathcal{L}(Q)$ is a factor, there is a new $u_2\in\mathscr{U}(\mathcal{M})$, with

$$(\mathcal{B}\vee\mathcal{B}'\cap f\mathcal{M}f)z_2\subseteq u_2\mathcal{L}(Q)u_2^*. \tag{5.17}$$

Here z_2 is the central support of vv^* in $\mathcal{B}\vee\mathcal{B}'\cap f\mathcal{M}f$ and hence $z_2\in\mathscr{Z}(\mathcal{B}'\cap f\mathcal{M}f)$ and $vv^*\leqslant z_2\leqslant f$.

Let $\Omega=C_\Lambda(\Sigma)$ and notice that $\mathcal{L}(\Omega)z_2\subseteq((fL(\Sigma)f)'\cap f\mathcal{M}f)z_2\subseteq(\mathcal{B}'\cap f\mathcal{M}f)z_2\subseteq u_2\mathcal{L}(Q)u_2^*$. Since Q is malnormal in Γ and $z_2\in(L(\Omega)f)'\cap f\mathcal{M}f$, we further have $z_2(\mathcal{L}(\Omega)f\vee((\mathcal{L}(\Omega)f)'\cap f\mathcal{M}f))z_2\subseteq u_2\mathcal{L}(Q)u_2^*$. Again since $\mathcal{L}(Q)$ is a factor, there is $\eta\in\mathscr{U}(\mathcal{M})$ so that

$$(\mathcal{L}(\Omega)f\vee((\mathcal{L}(\Omega)f)'\cap f\mathcal{M}f))z\subseteq\eta^*\mathcal{L}(Q)\eta, \tag{5.18}$$

where z is the central support of z_2 in $\mathcal{L}(\Omega)f\vee((\mathcal{L}(\Omega)f)'\cap f\mathcal{M}f)$. In particular, we have $vv^*\leqslant z_2\leqslant z\leqslant f$. Now since $f\mathcal{L}(\Sigma)f\subseteq(\mathcal{L}(\Omega)f)'\cap f\mathcal{M}f$, by (5.18) we get $(f\mathcal{L}(\Sigma)f\vee\mathcal{L}(\Omega)f)z\subseteq\eta^*\mathcal{L}(Q)\eta$ and hence

$$\eta(\mathcal{L}(\Omega)f\vee f\mathcal{L}(\Sigma)f)z\eta^*\subseteq\mathcal{L}(Q). \tag{5.19}$$

Since $vv^*\leqslant z\in(f\mathcal{L}(\Sigma)f)'\cap f\mathcal{M}f$ and $\mathcal{B}$ is a factor, the map $\phi':p\mathcal{L}(Q)p\to\eta\mathcal{B}z\eta^*\subseteq f\mathcal{L}(\Sigma)fz$ given by $\phi'(x)=\eta\phi(x)z\eta^*$ still defines a $*$-isomorphism that satisfies $\phi'(x)y=yx$ for any $x\in p\mathcal{L}(Q_1)p$, where $0\neq y=\eta zv$ is a partial isometry. Hence, $\mathcal{L}(Q_1)\prec_{\mathcal{M}}u^*f\mathcal{L}(\Sigma)fzu$. Since Q is malnormal in Γ, it follows that $\mathcal{L}(Q_1)\prec_{\mathcal{L}(Q)}\eta f\mathcal{L}(\Sigma)fz\eta^*$.

To this end, using [Chifan et al. 2016a, Proposition 2.4] and its proof, there are $0\neq a\in\mathscr{P}(\mathcal{L}(Q_1))$, $0\neq r=\eta qz\eta^*\in\eta f\mathcal{L}(\Sigma)fz\eta^*$, with $q\in\mathscr{P}(f\mathcal{L}(\Sigma)f)$, and a $*$-isomorphism onto its image $\psi: a\mathcal{L}(Q_1)a\to\mathcal{D}:=\psi(a\mathcal{L}(Q_1)a)\subseteq\eta q\mathcal{L}(\Sigma)qz\eta^*$ satisfying the following properties:

(4) The inclusion $\mathcal{D}\vee(\mathcal{D}'\cap\eta q\mathcal{L}(\Sigma)qz\eta^*)\subseteq\eta q\mathcal{L}(\Sigma)qz\eta^*$ has finite index.

(5) There is a partial isometry $0\neq w\in\mathcal{L}(Q)$ such that $\psi(x)w=wx$ for all $x\in a\mathcal{L}(Q_1)a$.

Now observe the algebras $\mathcal{D}$, $\mathcal{D}'\cap\eta q\mathcal{L}(\Sigma)qz\eta^*$ and $\eta\mathcal{L}(\Omega)qz\eta^*$ are mutually commuting. Also the prior relations show that $\mathcal{D}$ and $\eta\mathcal{L}(\Omega)qz\eta^*$ have no amenable direct summand. Since Q_1 and Q_2 are biexact, it follows that $\mathcal{D}'\cap\eta q\mathcal{L}(\Sigma)qz\eta^*$ must be purely atomic. Therefore, one can find $0\neq e\in\mathscr{P}(\mathscr{Z}(\mathcal{D}'\cap u^*q\mathcal{L}(\Omega)qzu))$ such that after cutting down by q the containment in (4) and replacing $\mathcal{D}$ by $\mathcal{D}e$ one can assume that

(4′) $\mathcal{D}\subseteq\eta q\mathcal{L}(\Sigma)qz\eta^*$ is a finite-index inclusion of nonamenable II_1-factors.

Moreover, replacing w by ew and $\psi(x)$ by $\psi(x)e$ in the intertwining in (5) still holds.

Notice that (5) implies $ww^*\in\mathcal{D}'\cap r\mathcal{L}(Q)r,\ w^*w\in a\mathcal{L}(Q_1)a'\cap a\mathcal{L}(Q)a=\mathbb{C}a\otimes\mathcal{L}(Q_2)$. Thus there exists $0\neq b\in\mathscr{P}(\mathcal{L}(Q_2))$ such that $w^*w=a\otimes b$. Pick $c\in\mathscr{U}(\mathcal{L}(Q))$ such that $w=c(a\otimes b)$. Then (5) gives

$$\mathcal{D}ww^*=w\mathcal{L}(Q_1)w^*=c(a\mathcal{L}(Q_1)a\otimes\mathbb{C}b)c^*. \tag{5.20}$$

Let $\Xi=QN_\Lambda(\Sigma)$. Then using (5.20) and (4′) above we see that

$$c(a\otimes b)\mathcal{L}(Q)(a\otimes b)c^*=ww^*\eta qz\mathscr{Q}\mathscr{N}_{\mathcal{L}(\Lambda)}(\mathcal{L}(\Sigma))''qz\eta^*ww^*=ww^*\eta qz\mathcal{L}(\Xi)qz\eta^*ww^* \tag{5.21}$$

and also

$$\begin{aligned} c(\mathbb{C}a\otimes b\mathcal{L}(Q_2)b)c^* &= (c(a\mathcal{L}(Q_1)a\otimes \mathbb{C}b)c^*)'\cap c(a\otimes b)\mathcal{L}(Q)(a\otimes b)c^* \\ &= (\mathcal{D}ww^*)'\cap ww^*\eta qz\mathcal{L}(\Xi)qz\eta^* ww^* \\ &= ww^*(\mathcal{D}'\cap \eta qz\mathcal{L}(\Xi)qz\eta^*)ww^*. \end{aligned} \tag{5.22}$$

Using (4′) and [Popa 2002, Lemma 3.1] we also have

$$\mathcal{D}\vee(\eta qz\mathcal{L}(\Sigma)zq\eta^*)'\cap \eta qz\mathcal{L}(\Xi)zq\eta^* \subseteq^f \mathcal{D}\vee\mathcal{D}'\cap\eta qz\mathcal{L}(\Xi)zq\eta^*\subseteq \eta qz\mathcal{L}(\Xi)zq\eta^*, \tag{5.23}$$

where the symbol $\subseteq^f$ above means inclusion of finite index.

Relation (5.20) also shows that

$$\begin{aligned} \mathcal{D}\vee(\eta qz\mathcal{L}(\Sigma)zq\eta^*)'\cap \eta qz\mathcal{L}(\Xi)zq\eta^* &\subseteq^f \eta qz\mathcal{L}(\Sigma)zq\eta^*\vee(\eta qz\mathcal{L}(\Sigma)zq\eta^*)'\cap\eta qz\mathcal{L}(\Xi)zq\eta^* \\ &\subseteq \eta qz\mathcal{L}(\Sigma(vC_\Lambda(\Sigma)))zq\eta^*\subseteq \eta qz\mathcal{L}(\Xi)zq\eta^*. \end{aligned} \tag{5.24}$$

Here $vC_\Lambda(\Sigma)=\{\lambda\in\Lambda : |\lambda^\Sigma|<\infty\}$ is the virtual centralizer of Σ in Λ.

Let $\Phi = QN^{(1)}_\Lambda(\Xi)$. Using (5.21) and the fact that Q is malnormal in Γ, the same argument from [Chifan and Udrea 2020, Claim 5.2, page 26, lines 1–10] shows that $\Xi\leqslant\Phi$ has finite index.

Combining (5.22), (5.20) (5.21) we notice that

$$ww^*(\mathcal{D}\vee\mathcal{D}'\cap\eta qz\mathcal{L}(\Xi)zq\eta^*)ww^* = ww^*\eta qz\mathcal{L}(\Xi)zq\eta^*ww^* = ww^*\eta qz\mathcal{L}(\Phi)zq\eta^*ww^*. \tag{5.25}$$

In particular, (5.25) shows that $\eta qz\mathcal{L}(\Xi)zq\eta^*\prec_{\eta qz\mathcal{L}(\Xi)zq\eta^*}\mathcal{D}\vee\mathcal{D}'\cap\eta qz\mathcal{L}(\Xi)zq\eta^*$ and using the finite-index condition in (5.23) we get $\eta qz\mathcal{L}(\Xi)zq\eta^*\prec_{\eta qz\mathcal{L}(\Xi)zq\eta^*}\mathcal{D}\vee(\eta qz\mathcal{L}(\Sigma)zq\eta^*)'\cap\eta qz\mathcal{L}(\Xi)zq\eta^*$. Thus, by (5.24) we further have $\eta qz\mathcal{L}(\Xi)zq\eta^*\prec_{\eta qz\mathcal{L}(\Xi)zq\eta^*}\eta qz\mathcal{L}(\Sigma(vC_\Lambda(\Sigma)))zq\eta^*$ and since $\Sigma(vC_\Lambda(\Sigma))\leqslant\Phi$ and $[\Phi:\Xi]<\infty$, using [Chifan and Ioana 2018, Lemma 2.6] we get $[\Phi:\Sigma(vC_\Lambda(\Sigma))]<\infty$.

Relation (5.21) also shows that

$$c(a\otimes b)\mathcal{L}(Q)(a\otimes b)c^* = ww^*\eta qz\mathcal{L}(\Xi)zq\eta^*ww^* = ww^*\eta qz\mathcal{L}(\Phi)zq\eta^*ww^*. \tag{5.26}$$

As Q has property (T), by [Chifan and Ioana 2018, Lemma 2.13] so do Φ and Ξ, and hence $\Sigma vC_\Lambda(\Sigma)$ as well. Let $\{\mathcal{O}_n\}_n$ be an enumeration of all the orbits in Λ under conjugation by Σ. Define $\Omega_n := \langle\mathcal{O}_1,\dots,\mathcal{O}_n\rangle$. Clearly $\Omega_n\leqslant\Omega_{n+1}$ and Σ normalizes Ω_n for all n. Notice that $\Omega_n\Sigma\leqslant\Omega_{n+1}\Sigma$ for all n and in fact $\Omega_n\Sigma\nearrow\Sigma(vC_\Lambda(\Sigma))$. Since $\Sigma(vC_\Lambda(\Sigma))$ has property (T), there exists n_0 such that $\Omega_{n_0}\Sigma=\Sigma(vC_\Lambda(\Sigma))$. In particular, there is a finite-index subgroup $\Sigma'\leqslant\Sigma$ such that $[\Sigma',\Omega_{n_0}]=1$, and hence $\Sigma',\Omega_{n_0}\leqslant^f\Sigma(vC_\Lambda(\Sigma))\leqslant^f\Phi$ are commuting subgroups. Moreover if r_1 is the central support of ww^* in $\eta zL(\Phi)qz\eta^*$ then by (5.26) we also have $\eta_0\mathcal{L}(Q)\eta_0^*\supseteq\eta qz\mathcal{L}(\Xi)qz\eta^*r_1$ for some unitary η_0. Now since the Q_i's are biexact, the same argument from [Chifan et al. 2016b] shows that the finite conjugacy radical of Φ is finite. Hence Φ is a finite-by-icc group and this canonically implies that $\Phi_1:=\Sigma'$ and $\Phi_2:=\Omega_{l_0}$ are also finite-by-icc. As Φ has property (T), so do the Φ_i's. Altogether, the above arguments and (5.26) show that there exist subgroups $\Phi_1,\Phi_2\leqslant\Phi<\Lambda$ satisfying the following properties:

(1) Φ_1, Φ_2 are infinite, commuting, property (T), finite-by-icc groups.

(2) $[\Phi:\Phi_1\Phi_2]<\infty$ and $QN^{(1)}_\Lambda(\Phi)=\Phi$.

(3) There exist $\mu \in \mathscr{U}(\mathcal{M}),\ d \in \mathscr{P}(\mathcal{L}(\Phi)),\ h = \mu d \mu^* \in \mathscr{P}(\mathcal{L}(Q))$ such that

$$\mu d \mathcal{L}(\Phi) d \mu^* = h \mathcal{L}(Q) h. \tag{5.27}$$

In the last part of the proof we show that after replacing d with its central support in $\mathcal{L}(Q)$, all the required relations in the statement still hold. Since $\mathcal{L}(Q)$ is a factor, using (5.27) one can find $\xi \in \mathscr{U}(\mathcal{M})$ such that $\xi \mathcal{L}(\Phi) t \xi^* \subseteq \mathcal{L}(Q)$, where t is the central support of d in $\mathcal{L}(Q)$. Hence $\xi \mathcal{L}(\Phi) t \xi^* \subseteq r_2 \mathcal{L}(Q) r_2$, where $r_2 = \xi t \xi^*$. Fix $e_o \leqslant t$ and $f_o \leqslant d$ projections in the factor $\mathcal{L}(\Phi)t$ such that $\tau(f_o) \geqslant \tau(e_o)$. From (5.27) we have $\mu f_o \mathcal{L}(\Phi) f_o \mu^* = l \mathcal{L}(Q) l$ and $\xi e_o \mathcal{L}(\Phi) e_o \xi^* \subseteq r_o \mathcal{L}(Q) r_o$, where $r_o = \xi e_o \xi^*$ and $l = \mu f_o \mu^*$. Let $\xi_o \in \mathcal{L}(Q)$ be a unitary such that $r_o \leqslant \xi_o l \xi_o^*$. Thus

$$\xi e_o \mathcal{L}(\Phi) e_o \xi^* \subseteq r_o \mathcal{L}(Q) r_o \subseteq \xi_o l \mathcal{L}(Q) l \xi_o^* = \xi_o \mu f_o \mathcal{L}(\Phi) f_o \mu^* \xi_o^*$$

and hence

$$\mu^* \xi_o^* \xi e_o \mathcal{L}(\Phi) e_o \subseteq f_o \mathcal{L}(\Phi) f_o \mu^* \xi_o^* \xi \subset \mathcal{L}(\Phi) \mu^* \xi_o^* \xi. \tag{5.28}$$

Next let $e_o + p_1 + p_2 + \cdots + p_s = t$, where $p_i \in \mathcal{L}(\Phi)t$ are mutually orthogonal projections such that e_o is von Neumann equivalent (in $\mathcal{L}(\Phi)t$) to p_i for all $i \in \overline{1, s-1}$ and p_s is von Neumann subequivalent to e_o. Now let u_i be unitaries in $\mathcal{L}(\Phi)t$ such that $u_i p_i u_i^* = e_o$ for all $i \in \overline{1, s-1}$ and $u_s p_s u_s^* = z_o' \leqslant e_o$. Combining this with (5.28) we get

$$\mu^* \xi_o^* \xi e_o \mathcal{L}(\Phi) p_i = \mu^* \xi_o^* \xi e_o \mathcal{L}(\Phi) u_i^* e_o u_i = \mu^* \xi_o^* \xi e_o \mathcal{L}(\Phi) e_o u_i \subseteq \mathcal{L}(\Phi) \mu^* \xi_o^* \xi u_i$$

for all $i \in \overline{1, s-1}$. Similarly, we get

$$\mu^* \xi_o^* \xi e_o \mathcal{L}(\Phi) p_s = \mu^* \xi_o^* \xi e_o \mathcal{L}(\Phi) u_s^* z_o' u_s = \mu^* \xi_o^* \xi e_o \mathcal{L}(\Phi) z_o' u_s \subseteq \mu^* \xi_o^* \xi e_o \mathcal{L}(\Phi) e_o u_s \subset \mathcal{L}(\Phi) \mu^* \xi_o^* \xi u_s.$$

Using these relations we conclude that

$$\begin{aligned} \mu^* \xi_o^* \xi e_o \mathcal{L}(\Phi) &= \mu^* \xi_o^* \xi e_o \mathcal{L}(\Phi) t = \mu^* \xi_o^* \xi e_o \mathcal{L}(\Phi) \Big(e_o + \sum_{i=1}^{s} p_i \Big) \\ &\subseteq \mu^* \xi_o^* \xi e_o \mathcal{L}(\Phi) e_o + \sum_{i=1}^{s} \mu^* \xi_o^* \xi e_o \mathcal{L}(\Phi) p_i \subseteq \mathcal{L}(\Phi) \mu^* \xi_o^* \xi + \sum_{i=1}^{s} \mathcal{L}(\Phi) \mu^* \xi_o^* \xi u_i. \end{aligned}$$

In particular, this relation shows that $\mu^* \xi_o^* \xi e_o \in \mathscr{Q}\mathscr{N}^{(1)}_{\mathcal{L}(\Lambda)}(\mathcal{L}(\Phi))$ and since $\mathscr{Q}\mathscr{N}^{(1)}_{\mathcal{L}(\Lambda)}(\mathcal{L}(\Phi))'' = \mathcal{L}(\Phi)$ by (2), we conclude that $\mu^* \xi_o^* \xi e_o \in \mathcal{L}(\Phi)$. Thus using this together with (5.28) one can check that

$$\begin{aligned} \xi e_o \mathcal{L}(\Phi) e_o \xi^* &= \xi e_o \xi^* \xi_o \mu (\mu^* \xi_o^* \xi e_o \mathcal{L}(\Phi) e_o \xi^* \xi_o \mu) \mu^* \xi_o^* \xi e_o \xi^* \\ &= \xi e \xi^* \xi_o \mu f_o \mathcal{L}(\Phi) f_o \mu^* \xi_o^* \xi e \xi^* \\ &= \xi e_o \xi^* \xi_o l \mathcal{L}(Q) l \xi_o^* \xi e_o \xi^* = r_o \mathcal{L}(Q) r_o. \end{aligned}$$

In conclusion we have proved that $\xi \mathcal{L}(\Phi) t \xi^* \subseteq r_2 \mathcal{L}(Q) r_2$ and for all $e_o \leqslant t$ and $f_o \leqslant d$ projections in the factor $\mathcal{L}(\Phi)t$ such that $\tau(f_o) \geqslant \tau(e_o)$ we have $\xi e_o \mathcal{L}(\Phi) e_o \xi^* = r_o \mathcal{L}(Q) r_o$, where $r_o \leqslant r_2 = \xi t \xi^*$. By Lemma 2.9 this clearly implies $\xi \mathcal{L}(\Phi) t \xi^* = r_2 \mathcal{L}(Q) r_2$, which finishes the proof. $\square$

Lemma 5.7. *Let Γ be a group as in Notation 5.4 and assume that Λ is a group such that $\mathcal{L}(\Gamma) = \mathcal{L}(\Lambda) = \mathcal{M}$. Also assume there exists a subgroup $\Phi < \Lambda$, a unitary $\mu \in \mathscr{U}(\mathcal{M})$ and projections $z \in \mathscr{Z}(\mathcal{L}(\Phi))$,*

$r = \mu z\mu^* \in \mathcal{L}(Q)$ such that

$$\mu\mathcal{L}(\Phi)z\mu^* = r\mathcal{L}(Q)r. \tag{5.29}$$

For every $\lambda \in \Lambda \setminus \Phi$ so that $|\Phi \cap \Phi^{\lambda}| = \infty$ we have $zu_\lambda z = 0$. In particular, there is $\lambda_o \in \Lambda \setminus \Phi$ so that $|\Phi \cap \Phi^{\lambda_o}| < \infty$.

Proof. Notice that since $Q < \Gamma = (N_1 \times N_2) \rtimes Q$ is almost malnormal, we have the following property: for every sequence $\mathcal{L}(Q) \ni x_n \to 0$ weakly and every $x, y \in M$ such that $E_{\mathcal{L}(Q)}(x) = E_{\mathcal{L}(Q)}(y) = 0$ we have

$$\|E_{\mathcal{L}(Q)}(xx_k y)\|_2 \to 0 \quad \text{as } k \to \infty. \tag{5.30}$$

Using basic approximations and the $\mathcal{L}(Q)$-bimodularity of the expectation we see that it suffices to check (5.30) only for elements of the form $x = u_n$ and $y = u_m$, where $n, m \in (N_1 \times N_2) \setminus \{1\}$. Consider the Fourier decomposition $x_n = \sum_{h\in Q} \tau(x_k u_{h^{-1}})u_h$ and notice that

$$\begin{aligned}\|E_{\mathcal{L}(Q)}(xx_k y)\|_2^2 &= \Big\|\sum_{h\in Q} \tau(x_k u_{h^{-1}})\delta_{nhm,Q}u_{nhm}\Big\|_2^2 \\ &= \Big\|\sum_{h\in Q} \tau(x_k u_{h^{-1}})\delta_{n\sigma_h(m)h,Q}u_{n\sigma_h(m)h}\Big\|_2^2 = \sum_{h\in Q,\sigma_h(m)=n^{-1}} |\tau(x_k u_{h^{-1}})|^2. \end{aligned}\tag{5.31}$$

Since the action $Q \curvearrowright N_i$ has finite stabilizers one can easily see that the set $\{h \in Q : \sigma_h(m) = n^{-1}\}$ is finite and since $x_n \to 0$ weakly, $\sum_{h\in Q,\sigma_h(m)=n^{-1}} |\tau(x_k u_{h^{-1}})|^2 \to 0$ as $k \to \infty$, which concludes the proof of (5.30). Using the conditional expectation formula for compression we see that (5.30) implies that for every sequence $\mathcal{L}(Q) \ni x_n \to 0$ weakly and every $x, y \in r\mathcal{M}r$ so that $E_{r\mathcal{L}(Q)r}(x) = E_{r\mathcal{L}(Q)r}(y) = 0$ we have $\|E_{r\mathcal{L}(Q)r}(xx_k y)\|_2 \to 0$ as $k \to \infty$. Thus using the formula (5.29) we get that for all $\mu\mathcal{L}(\Phi)z\mu^* \ni x_n \to 0$ weakly and every $x, y \in \mu z\mathcal{M}z\mu^*$ so that $E_{\mu\mathcal{L}(\Phi)z\mu^*}(x) = E_{\mu\mathcal{L}(\Phi)z\mu^*}(y) = 0$ we have $\|E_{\mu\mathcal{L}(\Phi)z\mu^*}(xx_k y)\|_2 \to 0$ as $k \to \infty$. This gives that for all $\mathcal{L}(\Phi)z \ni x_n \to 0$ weakly and every $x, y \in z\mathcal{M}z$ satisfying $E_{\mathcal{L}(\Phi)z}(x) = E_{\mathcal{L}(\Phi)z}(y) = 0$ we have

$$\|E_{\mathcal{L}(\Phi)z}(xx_k y)\|_2 \to 0 \quad \text{as } k \to \infty. \tag{5.32}$$

Fix $\lambda \in \Lambda \setminus \Phi$ so that $|\Phi \cap \Phi^{\lambda}| = \infty$. Hence there are infinite sequences $\lambda_k, \omega_n \in \Lambda$ so that $\lambda\omega_k\lambda^{-1} = \lambda_k$ for all integers k. Since $\lambda \in \Lambda \setminus \Phi$, we have $E_{\mathcal{L}(\Phi)}(u_\lambda z) = E_{\mathcal{L}(\Phi)z}(zu_{\lambda^{-1}}) = 0$. Also we have $u_{\omega_k}z \to 0$ weakly as $k \to \infty$. Using these calculations,

$$\begin{aligned}\|E_{\mathcal{L}(\Phi)}(zu_\lambda zu_{\lambda^{-1}}z)\|_2^2 &= \|E_{\mathcal{L}(\Phi)}(u_\lambda zu_{\lambda^{-1}}z)\|_2^2 = \|u_{\lambda\omega_k\lambda^{-1}}E_{\mathcal{L}(\Phi)}(u_\lambda zu_{\lambda^{-1}}z)\|_2^2 \\ &= \|E_{\mathcal{L}(\Phi)}(u_{\lambda\omega_k}zu_{\lambda^{-1}}z)\|_2^2 = \|E_{\mathcal{L}(\Phi)z}(zu_\lambda zu_{\omega_k}zu_{\lambda^{-1}}z)\|_2^2 \to 0 \quad \text{as } k \to \infty. \end{aligned}\tag{5.33}$$

Also using (5.33) the last quantity above converges to 0 as $k \to \infty$ and hence $E_{\mathcal{L}(\Phi)}(zu_\lambda zu_{\lambda^{-1}}z) = 0$, which gives that $zu_\lambda z = 0$, as desired. For the remaining part notice first that since $[\Gamma : Q] = \infty$, (5.29) implies that $[\Lambda : \Phi] = \infty$. Assume by contradiction that for all $\lambda \in \Lambda \setminus \Phi$ we have $zu_\lambda z = 0$. As $[\Lambda : \Phi] = \infty$, for every positive integer l one can construct inductively $\lambda_i \in \Lambda \setminus \Phi$ with $i \in \overline{1,l}$ such that $\lambda_i\lambda_j^{-1} \in \Lambda \setminus \Phi$ for all $i > j$ such that $i, j \in \overline{1,l}$. But this implies $0 = zu_{\lambda_i\lambda_j^{-1}}z = zu_{\lambda_i}u_{\lambda_j^{-1}}z$ and hence $u_{\lambda_i^{-1}}zu_{\lambda_i}$ are mutually orthogonal projections when $i = \overline{1,l}$. This is obviously false when l is sufficiently large. $\square$

Theorem 5.8. *Assume the same conditions as in Theorem 5.6. Then one can find subgroups $\Phi_1, \Phi_2 \leqslant \Phi \leqslant \Lambda$ so that*

(1) Φ_1, Φ_2 *are infinite, icc, property (T) groups so that* $\Phi = \Phi_1 \times \Phi_2$.

(2) $\mathrm{QN}^{(1)}_{\Lambda}(\Phi) = \Phi$.

(3) *There exists* $\mu \in \mathscr{U}(\mathcal{M})$ *such that* $\mu\mathcal{L}(\Phi)\mu^* = \mathcal{L}(Q)$.

Proof. From Theorem 5.6 there exist subgroups $\Phi_1, \Phi_2 \leqslant \Phi \leqslant \Lambda$ such that:

(1) Φ_1, Φ_2 are, infinite, commuting, finite-by-icc, property (T) groups so that $[\Phi : \Phi_1\Phi_2] < \infty$.

(2) $\mathrm{QN}^{(1)}_{\Lambda}(\Phi) = \Phi$.

(3) There exist $\mu \in \mathscr{U}(\mathcal{M})$ and $z \in \mathscr{P}(\mathscr{Z}(\mathcal{L}(\Phi)))$ with $h = \mu z \mu^* \in \mathscr{P}(\mathcal{L}(Q))$ satisfying

$$\mu\mathcal{L}(\Phi)z\mu^* = h\mathcal{L}(Q)h. \tag{5.34}$$

Next we show that in (5.34) we can pick $z \in \mathscr{Z}(\mathcal{L}(\Phi))$ maximal with the property that for every projection $t \in \mathscr{Z}(\mathcal{L}(\Phi)z^{\perp})$ we have

$$L(\Phi_i)t \nprec_{\mathcal{M}} \mathcal{L}(Q) \quad \text{for } i = 1, 2. \tag{5.35}$$

To see this let $z \in \mathcal{F}$ be a maximal family of mutually orthogonal (minimal) projections $z_i \in \mathscr{Z}(\mathcal{L}(\Phi))$ such that $\mathcal{L}(\Phi)z_i \prec_{\mathcal{M}} \mathcal{L}(Q)$. Note that since Φ has finite conjugacy radical it follows that $\mathcal{F}$ is actually finite. Next let $z \leqslant \sum z_i := a \in \mathscr{Z}(\mathcal{L}(\Phi))$ and we briefly argue that $\mathcal{L}(\Phi)a \prec^s_{\mathcal{M}} \mathcal{L}(Q)$. Indeed since $(\mathcal{L}(\Phi)a)' \cap a\mathcal{M}a = a(\mathcal{L}(\Phi)' \cap \mathcal{M})a = \mathscr{Z}(\mathcal{L}(\Phi))a$ and the latter is finite-dimensional, for every $r \in (\mathcal{L}(\Phi)a)' \cap a\mathcal{M}a$ there is $z_i \in \mathcal{F}$ such that $rz_i = z_i \neq 0$. Since $\mathcal{L}(\Phi)z_i \prec_{\mathcal{M}} \mathcal{L}(Q)$, we have $\mathcal{L}(\Phi)r \prec_{\mathcal{M}} \mathcal{L}(Q)$, as desired. Thus applying Lemma 2.7, after perturbing μ to a new unitary, we get $\mu\mathcal{L}(\Phi)a\mu^* = h_o\mathcal{L}(Q)h_o$. Finally, we show (5.35). Assume by contradiction there is $t_o \in \mathscr{Z}(\mathcal{L}(\Phi)z^{\perp})$ so that $\mathcal{L}(\Phi_i)t_o \prec_{\mathcal{M}} \mathcal{L}(Q)$ for some $i = 1, 2$. Thus there exist projections $r \in \mathcal{L}(\Phi)t_o$, $q \in \mathcal{L}(Q)$, a partial isometry $w \in \mathcal{M}$ and a $*$-isomorphism on the image $\phi : r\mathcal{L}(\Phi)r \to \mathcal{B} := \phi(r\mathcal{L}(\Phi)r) \subseteq q\mathcal{L}(Q)q$ such that $\phi(x)w = wx$. Notice that $w^*w \in t_o(\mathcal{L}(\Phi_i)' \cap \mathcal{M})t_o$ and $ww^* \in \mathcal{B}' \cap q\mathcal{M}q$. But since $Q < \Gamma$ is malnormal, it follows that $\mathcal{B}' \cap q\mathcal{M}q \subseteq q\mathcal{L}(Q)q$ and hence $ww^* \in q\mathcal{L}(Q)q$. Using this in combination with previous relations we get $wr\mathcal{L}(\Phi_i)rw^* = \mathcal{B}ww^* \subseteq \mathcal{L}(Q)$ and extending w to a unitary u we have $ur\mathcal{L}(\Phi_i)ru^* \subseteq \mathcal{L}(Q)$. Since $\mathcal{L}(Q)$ is a factor, we can further perturb the unitary u so that $u\mathcal{L}(\Phi_i)r_ou^* \subseteq \mathcal{L}(Q)$, where $r \leqslant r_o \leqslant t_o$ is the central support of r in $\mathcal{L}(\Phi_i)t_o$. Using malnormality of Q again we further get $r_o(\mathcal{L}(\Phi_i) \vee \mathcal{L}(\Phi_i)' \cap \mathcal{M})r_ou^* \subseteq \mathcal{L}(Q)$ and perturbing u we can further assume that $(\mathcal{L}(\Phi_i) \vee \mathcal{L}(\Phi_i)' \cap \mathcal{M})s_ou^* \subseteq \mathcal{L}(Q)$ where $r_o \leqslant s_o$ is the central support of r_o in $\mathcal{L}(\Phi_i) \vee \mathcal{L}(\Phi_i)' \cap \mathcal{M}$. In particular, $u(\mathcal{L}(\Phi)s_ou^* \subseteq \mathcal{L}(Q)$ and hence $\mathcal{L}(\Phi)s_o \subseteq u^*\mathcal{L}(Q)u$. Since $r \leqslant r_o \leqslant s_o$ and $r \leqslant t_o$, the previous containment implies that there is a minimal projection $s' \in \mathcal{L}(\Phi)a^{\perp}$ so that $\mathcal{L}(\Phi)s' \prec \mathcal{L}(Q)$, which contradicts the maximality assumption on $\mathcal{F}$. Finally replacing z with a in our statement, our claim follows.

Next fix $t \in \mathscr{Z}(\mathcal{L}(\Phi)z^{\perp})$. Since $\mathcal{L}(\Phi_1)t$ and $\mathcal{L}(\Phi_2)t$ are commuting property (T) von Neumann algebras, using the same arguments as in the first part of the proof of Theorem 5.5 there are two possibilities: either (i) there exists $j \in 1, 2$ such that $\mathcal{L}(\Phi_j)t \prec_{\mathcal{M}} \mathcal{L}(N_2)$ or (ii) $\mathcal{L}(\Phi)t \prec_{\mathcal{M}} \mathcal{L}(N_2 \rtimes Q)$. Next we briefly argue

(ii) is impossible. Indeed, assuming (ii), Theorem 5.2 for $n = 1$ would imply the existence of $j \in 1, 2$ so that $\mathcal{L}(\Phi_j)t \prec_{\mathcal{M}} \mathcal{L}(Q)$, which obviously contradicts the choice of z. Thus we have (i), and passing to the relative commutants we have $\mathcal{L}(N_1) \prec \mathcal{L}(\Phi_j)t' \cap t\mathcal{M}t = t(\mathcal{L}(\Phi_j)' \cap \mathcal{M})t$. Using the relationships between the Φ_j's we see that $t(\mathcal{L}(\Phi_j)' \cap \mathcal{M})t \subset t\mathcal{L}(\Phi_j) \vee \mathcal{L}(\Phi_j)' \cap \mathcal{M})t \subseteq t\mathcal{L}(\Phi_j(vC_\Lambda(\Phi_j)))t \subseteq t\mathcal{L}(\Phi)t$. In conclusion, we have

$$\mathcal{L}(N_1) \prec_{\mathcal{M}} t\mathcal{L}(\Phi)t \quad \text{for all } t \in \mathscr{P}(\mathcal{L}(\Phi)z^\perp). \tag{5.36}$$

Let $A = \{\lambda \in \Lambda : |\Phi \cap \Phi^\lambda| < \infty\}$ and $B = \{\lambda \in \Lambda : |\Phi \cap \Phi^\lambda| = \infty\}$. Note that $A \cup B = \Lambda$ and $A \neq \varnothing$. Since N_1 is infinite, for every $\lambda \in A$ we have $\mathcal{L}(N_1) \nprec_{\mathcal{M}} \mathcal{L}(\Phi \cap \Phi^\lambda)z^\perp$. Thus using (5.36) together with the same argument from the proof of [Popa and Vaes 2008, Theorem 6.16], working under $z^\perp$, we get $z^\perp E_{\mathcal{L}(\Phi)}(u_\lambda z^\perp x z^\perp) = 0$ for all $x \in \mathcal{M}$. This further implies $z^\perp u_\lambda z^\perp = 0$ for all $\lambda \in A$ and hence $u_\lambda z^\perp u_{\lambda^{-1}} \leqslant z$.

On the other hand by Lemma 5.7 for all $\lambda \in B$ we get $zu_\lambda z = 0$, and hence $u_\lambda z u_{\lambda^{-1}} \leqslant z^\perp$. So if $B \neq \varnothing$, we obviously have equality in the previous two relations, i.e., $u_\lambda z u_{\lambda^{-1}} = z^\perp$ for all $\lambda \in B$ and $u_\lambda z^\perp u_{\lambda^{-1}} = z$ for all $\lambda \in A$. These further imply there exist $a_o \in A$ and $b_0 \in B$ such that $A = a_0 C_\Lambda(z^\perp)$ and $B = b_o C_\Lambda(z)$; here $C_\Lambda(z) \leqslant \Lambda$ is the subgroup of all elements of Λ that commute with z and similarly for $C_\Lambda(z^\perp)$. Thus $\Lambda = A \cup B = a_o C_\Lambda(z^\perp) \cup b_o C_\Lambda(z)$. Thus we can assume, without loss of generality, that $[\Lambda : C_\Lambda(z)] < \infty$. But since Λ is icc this implies $z = 1$. The rest of the statement follows. □

Theorem 5.9. *In Theorem 5.5 we cannot have case* (4a).

Proof. Assume by contradiction that for all $j \in 1, 2$ there is $i \in 1, 2$ such that $\Delta(\mathcal{L}(Q_i)) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\bar{\otimes}\mathcal{L}(N_j)$. Using [Drimbe et al. 2019, Theorem 4.1] and property (T) on N_j, one can find a subgroup $\Sigma < \Lambda$ such that $\mathcal{L}(Q_i) \prec_{\mathcal{M}} \mathcal{L}(\Sigma)$ and $\mathcal{L}(N_j) \prec_{\mathcal{M}} \mathcal{L}(C_\Lambda(\Sigma))$. Since $\mu\mathcal{L}(\Phi)\mu^* = \mathcal{L}(Q)$ and Q_i are biexact, by the product rigidity results in [Chifan et al. 2016b] one can assume that there is a unitary $u \in \mathcal{L}(Q)$ such that $u\mathcal{L}(Q_1)u^* = \mathcal{L}(\Phi_1)^t$ and $u\mathcal{L}(Q_2)u^* = \mathcal{L}(\Phi_2)^{1/t}$. Thus we get $\mathcal{L}(\Phi_i) \prec_{\mathcal{M}} \mathcal{L}(\Sigma)$, and hence $[\Phi_i : g\Sigma g^{-1} \cap \Phi_i] < \infty$. So working with $g\Sigma g^{-1}$ instead of Σ, we can assume that $[\Phi_i : \Sigma \cap \Phi_i] < \infty$. In particular $\Sigma \cap \Phi_i$ is infinite and since Φ is almost malnormal in Λ, it follows that $C_\Lambda(\Sigma \cap \Phi_i) < \Phi$. Thus we have $\mathcal{L}(N_j) \prec_{\mathcal{M}} \mathcal{L}(C_\Lambda(\Sigma)) \subseteq \mathcal{L}(C_\Lambda(\Sigma \cap \Phi_i)) \subset \mathcal{L}(\Phi) = \mu^*\mathcal{L}(Q)\mu$, which is obviously a contradiction. □

Theorem 5.10. *Let Γ be a group as in Notation 5.4 and assume that Λ is a group such that $\mathcal{L}(\Gamma) = \mathcal{L}(\Lambda) = \mathcal{M}$. Let $\Delta : \mathcal{M} \to \mathcal{M}\,\bar{\otimes}\,\mathcal{M}$ be the comultiplication along Λ as in Notation 5.4. Then the following hold*:

(i) $\Delta(\mathcal{L}(N_1)), \Delta(\mathcal{L}(N_2)), \Delta(\mathcal{L}(N_1 \times N_2)) \prec^s_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{L}(N_1 \times N_2)\,\bar{\otimes}\,\mathcal{L}(N_1 \times N_2)$.

(ii) *There is a unitary $u \in \mathcal{M}\,\bar{\otimes}\,\mathcal{M}$ such that $u\Delta(\mathcal{L}(Q))u^* \subseteq \mathcal{L}(Q)\,\bar{\otimes}\,\mathcal{L}(Q)$.*

Proof. First we show (i). From Theorem 5.5 we have that for all $j \in 1, 2$ there is $j_i \in 1, 2$ such that $\Delta(\mathcal{L}(N_{j_i})) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\,\bar{\otimes}\,\mathcal{L}(N_j)$. Since $\mathscr{N}_{\mathcal{M}\bar{\otimes}\mathcal{M}}\Delta(\mathcal{L}(N_i))'' \supset \Delta(\mathcal{M})$ and $\Delta(\mathcal{M})' \cap \mathcal{M}\,\bar{\otimes}\,\mathcal{M} = \mathbb{C}1$, by [Drimbe et al. 2019, Lemma 2.4 (3)] we actually have $\Delta(\mathcal{L}(N_{j_i})) \prec^s_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\,\bar{\otimes}\,\mathcal{L}(N_j)$. Notice that for all $i \neq k$ we have $j_i \neq j_k$. Otherwise we would have $\Delta(\mathcal{L}(N_{j_i})) \prec^s_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\,\bar{\otimes}\,\mathcal{L}(N_1)$ and $\Delta(\mathcal{L}(N_{j_i})) \prec^s_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\,\bar{\otimes}\,\mathcal{L}(N_2)$, which by [Drimbe et al. 2019, Lemma 2.8 (2)] would imply

$$\Delta(\mathcal{L}(N_{j_i})) \prec^s_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\,\bar{\otimes}\,\mathcal{L}(N_1 \cap N_2) = \mathcal{M} \otimes 1,$$

which is a contradiction. Furthermore using the same arguments as in [Isono 2020, Lemma 2.6] we have $\Delta(\mathcal{L}(N_1 \times N_2)) \prec^s_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{L}(N_1 \times N_2)$. Then working on the left side of the tensor we get $\Delta(\mathcal{L}(N_1 \times N_2)) \prec^s_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{L}(N_1 \times N_2) \mathbin{\bar{\otimes}} \mathcal{L}(N_1 \times N_2)$.

Finally, notice that part (ii) is a direct consequence of Theorem 5.8. □

5A. *Proof of Theorem 5.1.*

Proof. We divide the proof into separate parts to improve the exposition.

Reconstruction of the acting group Q. To accomplish this we will use the notion of height for elements in group von Neumann algebras as introduced in [Ioana et al. 2013; Ioana 2011]. From the previous theorem recall that $u\Delta(\mathcal{L}(Q))u^* \subseteq \mathcal{L}(Q) \mathbin{\bar{\otimes}} \mathcal{L}(Q)$. Let $\mathcal{A} = u\Delta(\mathcal{L}(N_1))u^*$. Next we claim that

$$h_{Q\times Q}(u\Delta(Q)u^*) > 0. \tag{5A.1}$$

For every $x, y \in \mathcal{L}(Q) \mathbin{\bar{\otimes}} \mathcal{L}(Q)$ and every $a \in \mathcal{A} \mathbin{\bar{\otimes}} \mathcal{A}$ supported on a finite set $F \subset N = N_1 \times N_2$ we have

$$\begin{aligned}
\|E_{\mathcal{A}\bar{\otimes}\mathcal{A}}(xay)\|_2^2 &= \Big\| \sum_{q,l} \tau(xu_{q^{-1}})\tau(yu_l)E_{\mathcal{A}\bar{\otimes}\mathcal{A}}(u_q a u_{l^{-1}}) \Big\|_2^2 = \Big\| \sum_{q,l} \tau(xu_{q^{-1}})\tau(yu_l)E_{\mathcal{A}\bar{\otimes}\mathcal{A}}(\sigma_q(a)u_{ql^{-1}}) \Big\|_2^2 \\
&= \Big\| \sum_{q} \tau(xu_{q^{-1}})\tau(yu_l)\sigma_q(a) \Big\|_2^2 = \Big\| \sum_{q\in Q, n\in N^2} \tau(xu_{q^{-1}})\tau(yu_l)\tau(au_{n^{-1}})u_{\sigma_q(n)} \Big\|_2^2 \\
&= \sum_{r\in N^2} \Big| \sum_{\sigma_q(n)=r} \tau(xu_{q^{-1}})\tau(yu_l)\tau(au_{n^{-1}}) \Big|^2 \\
&\leqslant h^2_{Q\times Q}(x) \sum_{r\in N^2} \Big(\sum_{q\in Q:\sigma_{q^{-1}}(r^{-1})\in F} |\tau(yu_l)|\,|\tau(au_{\sigma_{q^{-1}}(r)})| \Big)^2 \\
&\leqslant h^2_{Q\times Q}(x)\|y\|_2^2\,\|a\|_2^2 \max_{r\in N^2} |\{q \in Q : \sigma_{q^{-1}}(r^{-1}) \in F\}|.
\end{aligned} \tag{5A.2}$$

This estimate leads to the following property: for all finite sets $K, S \subset Q$, every $a \in \operatorname{span}\{\mathcal{A} \mathbin{\bar{\otimes}} \mathcal{A}u_g : g \in K\}$ and all $\varepsilon > 0$ there exist a scalar $C > 0$ and a finite set $F \subset N^2$ such that, for all $x, y \in \mathcal{L}(Q)\bar{\otimes}\mathcal{L}(Q)$,

$$\begin{aligned}
&\|P_{\sum_{s\in S}\mathcal{A}\bar{\otimes}\mathcal{A}u_s}(xay)\|_2^2 \\
&\qquad \leqslant |K|\,|S|C(h^2_{Q\times Q}(x)\|y\|_2^2\,\|a\|_2^2 \max_{r\in N^2} |\{q \in Q : \sigma_{q^{-1}}(r^{-1}) \in F\}|) + \varepsilon\|x\|_\infty\|y\|_\infty.
\end{aligned} \tag{5A.3}$$

Note this follows directly from (5A.2) after we decompose the a and the projection $P_{\sum_{s\in S}\mathcal{A}\bar{\otimes}\mathcal{A}u_s}$.

Next we use (5A.3) to prove our claim. Fix $\varepsilon > 0$. Since $\Delta(\mathcal{A}) \not\prec \mathcal{M} \otimes 1,\ 1 \otimes \mathcal{M}$, by Theorem 2.1 one can find a finite subset $F_o \subset N^2 \setminus ((N \times 1) \cup (1 \times N))$ such that $a_{F_o} \in \mathcal{A} \mathbin{\bar{\otimes}} \mathcal{A}$ is supported on F_o and $\|a - a_{F_o}\|_2 \leqslant \varepsilon$. Since $\Delta(\mathcal{A}) \prec^s \mathcal{A} \mathbin{\bar{\otimes}} \mathcal{A}$, there is a finite $S \subseteq Q \times Q$ such that

$$\|P_{\sum_{s\in S}\mathcal{A}\bar{\otimes}\mathcal{A}u_s}(a) - a\|_2 \leqslant \varepsilon \quad \text{for all } a \in \Delta(\mathcal{A}). \tag{5A.4}$$

Assume by contradiction (5A.1) doesn't hold. Thus there is a sequence $t_n \in Q$ such that $h_{Q\times Q}(t_n) = h_{Q\times Q}(u\Delta(u_{t_n})u^*) \to 0$ as $n \to \infty$. As t_n normalizes $\Delta(\mathcal{A})$, one can see that

$$\begin{aligned}1-\varepsilon = \|t_n a t_n^*\|_2^2 - \varepsilon &\leqslant \|P_{\sum_{s\in S}\mathcal{A}\bar{\otimes}\mathcal{A}u_s}(t_n a t_n^*)\|_2^2 \leqslant \|P_{\sum_{s\in S}\mathcal{A}\bar{\otimes}\mathcal{A}u_s}(t_n a t_n^*)\|_2^2 + \varepsilon \\ &\leqslant |F_o||S|C(h^2_{Q\times Q}(t_n)\|t_n\|_2^2\|a_{F_o}\|_2^2 \max_{r\in N^2}|\{q\in Q : \sigma_{q^{-1}}(r^{-1})\in F_o\}|) + \varepsilon\|t_n\|_\infty^2 \\ &\leqslant |F_o||S|C(h^2_{Q\times Q}(t_n)\max_{r\neq 1}|\mathrm{Stab}_Q(r)||F_o|) + 2\varepsilon. \qquad (5A.5)\end{aligned}$$

Since the stabilizer sizes are uniformly bounded, we get a contradiction if $\varepsilon > 0$ is arbitrary small. Now we notice that the height condition, together with Theorem 5.8 and [Chifan and Udrea 2020, Lemmas 2.4, 2.5], already implies $h_Q(\mu\Phi\mu^*) > 0$ and by [Ioana et al. 2013, Theorem 3.1] there is a unitary $\mu_0 \in \mathcal{M}$ such that $\mathbb{T}\mu_0\Phi\mu_0^* = \mathbb{T}Q$.

Reconstruction of a core subgroup and its product feature. From Theorem 5.10, we have

$$\Delta(\mathcal{L}(N_1\times N_2)) \prec^s_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_1\times N_2).$$

Proceeding exactly as in the proof of [Chifan and Udrea 2020, Claim 4.5] we can show that $\Delta(\mathcal{A}) \subseteq \mathcal{A}\bar{\otimes}\mathcal{A}$, where $\mathcal{A} = u\mathcal{L}(N_1\times N_2)u^*$. By Lemma 2.8, there exists a subgroup $\Sigma < \Lambda$ such that $\mathcal{A} = \mathcal{L}(\Sigma)$. The last part of the proof of [Chifan and Udrea 2020, Theorem 5.2] shows that $\Lambda = \Sigma \rtimes \Phi$. In order to reconstruct the product feature of Σ, we need a couple more results.

Claim 2. *For every $i = 1, 2$ there exists $j = 1, 2$ such that*

$$\Delta(\mathcal{L}(N_j)) \prec^s \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_i). \qquad (5A.6)$$

Proof of Claim. We prove this only for $i = 1$ as the other case is similar. We also notice that since $\mathcal{N}_{\mathcal{M}\otimes\mathcal{M}}(\Delta(\mathcal{L}(N_j)))'' \supseteq \Delta(\mathcal{M})$ and $\Delta(\mathcal{M})'\cap\mathcal{M}\,\bar{\otimes}\,\mathcal{M} = \mathbb{C}1$, to establish (5A.6) we only need to show that $\Delta(\mathcal{L}(N_j)) \prec \mathcal{L}(N_1\times N_2)\bar{\otimes}\mathcal{L}(N_i)$. From above we have $\Delta(\mathcal{L}(N_1\times N_2) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{L}(N_1\times N_2)\bar{\otimes}\mathcal{L}(N_1\times N_2)$. Hence there exist nonzero projections $a_i \in \Delta(\mathcal{L}(N_i))$ and $b \in \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_1\times N_2)$, a partial isometry $v \in \mathcal{M}\,\bar{\otimes}\,\mathcal{M}$ and a $*$-isomorphism on the image

$$\Psi : a_1\otimes a_2\Delta(\mathcal{L}(N_1\times N_2))a_1\otimes a_2 \to \Psi(a_1\otimes a_2\Delta(\mathcal{L}(N_1\times N_2))a_1\otimes a_2) := \mathcal{R} \subseteq b(\mathcal{L}(N_1\times N_2)\bar{\otimes}\mathcal{L}(N_1\times N_2))b$$

such that $\Psi(x)v = vx$ for all $x \in a_1\otimes a_2\Delta(\mathcal{L}(N_1\times N_2))a_1\otimes a_2$.

Define $\mathcal{D}_i := \Psi(a_i(\Delta(\mathcal{L}(N_i)))a_i) \subseteq b\mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_1\times N_2)b$ and notice that $\mathcal{D}_1$ and $\mathcal{D}_2$ are commuting property (T) diffuse subfactors. Since the group N_2 is ($\mathbb{F}_\infty$)-by-(nonelementary hyperbolic group), by [Chifan et al. 2015; Chifan and Kida 2015] it follows that there is $j = 1, 2$ such that $\mathcal{D}_j \prec_{\mathcal{L}(N_1\times N_2)\bar{\otimes}\mathcal{L}(N_1\times N_2)} \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_1\times\mathbb{F}_\infty)$. Since $\mathbb{F}_\infty$ has Haagerup's property and $\mathcal{D}_j$ has property (T) this further implies that $\mathcal{D}_j \prec_{\mathcal{L}(N_1\times N_2)\bar{\otimes}\mathcal{L}(N_1\times N_2)} \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_1)$. Composing this intertwining with Ψ we get $\Delta(\mathcal{L}(N_j)) \prec \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_1)$, as desired.

Also, we note that $j_1 \neq j_2$. Otherwise we would have $\Delta(\mathcal{L}(N_j)) \prec^s \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_1)\cap\mathcal{L}(N_2) = \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,1$, which obviously contradicts [Ioana et al. 2013, Proposition 7.2.1]. ☐

Let $\mathcal{A} = u\mathcal{L}(N_1))u^*$. Thus, we get $\Delta(\mathcal{A}) \prec^s \mathcal{L}(N_1\times N_2)\otimes\mathcal{L}(N_i)$ for some $i = 1, 2$. This implies that for every $\varepsilon > 0$ there exists a finite set $S \subset u^*Qu$, containing e, such that $\|d - P_{S\times S}(d)\|_2 \leqslant \varepsilon$ for all

$d \in \Delta(\mathcal{A})$. However, $\Delta(\mathcal{A})$ is invariant under the action of u^*Qu, and hence arguing exactly as in [Chifan and Udrea 2020, Claim 4.5] we get $\Delta(\mathcal{A}) \subset (\mathcal{L}(\Sigma) \bar{\otimes} u\mathcal{L}(N_i)u^*)$. We now separate the argument into two different cases:

Case I: $i = 1$. In this case, $\Delta(\mathcal{A}) \subseteq \mathcal{L}(\Sigma) \bar{\otimes} \mathcal{A}$. Thus by Lemma 2.8 we get that there exists a subgroup $\Sigma_0 < \Sigma$ with $\mathcal{A} = \mathcal{L}(\Sigma_0)$. Now, $\mathcal{A}' \cap \mathcal{L}(\Sigma) = u\mathcal{L}(N_2)u^*$. Thus, $\mathcal{L}(\Sigma_0)' \cap \mathcal{L}(\Sigma) = u\mathcal{L}(N_2)u^*$. Note that Σ and Σ_0 are both icc property (T) groups. This implies $\mathcal{L}(\Sigma_0)' \cap \mathcal{L}(\Sigma) = \mathcal{L}(vC_\Sigma(\Sigma_0))$, where $vC_\Sigma(\Sigma_0)$ denotes the *virtual centralizer* of Σ_0 in Σ. Proceeding as in [Chifan et al. 2018] we can show $\Sigma = \Sigma_0 \times \Sigma_1$.

Case II: $i = 2$. Let $\mathcal{B} = u\mathcal{L}(N_2)u^*$. In this case, $\Delta(\mathcal{A}) \subseteq \mathcal{L}(\Sigma) \bar{\otimes} \mathcal{B}$. However, Lemma 2.8 then implies that $\mathcal{A} \subseteq \mathcal{B}$, which is absurd, as $\mathcal{L}(N_1)$ and $\mathcal{L}(N_2)$ are orthogonal algebras. Hence this case is impossible. □

Remarks. (1) There are several immediate consequences of Theorem 5.1. For instance one can easily see the von Neumann algebras covered by this theorem are nonisomorphic with the ones arising from any irreducible lattice in higher-rank Lie group. Indeed, if Λ is any such lattice satisfying $\mathcal{L}(\Gamma) \cong \mathcal{L}(\Lambda)$, then Theorem 5.1 would imply that Λ must contain an infinite normal subgroup of infinite index which contradicts Margulis' normal subgroup theorem.

(2) While it well known there are uncountably many nonisomorphic group II_1-factors with property (T) [Popa 2007], little is known about producing concrete examples of such families. In fact the only currently known infinite families of pairwise nonisomorphic property (T) groups factors are $\{\mathcal{L}(G_n) : n \geqslant 2\}$ for G_n uniform latices in $\mathrm{Sp}(n, 1)$ [Cowling and Haagerup 1989] and $\{\mathcal{L}(G_1 \times G_2 \times \cdots \times G_k) : k \geqslant 1\}$, where G_k is any icc property (T) hyperbolic group [Ozawa and Popa 2004]. Theorem 5.1 makes new progress in this direction by providing a new explicit infinite family of icc property (T) groups which gives rise to pairwise nonisomorphic II_1-factors. For instance, in the statement one can simply let Q_i vary in any infinite family of nonisomorphic uniform lattices in $\mathrm{Sp}(n, 1)$ for any $n \neq 2$. Unlike the other families, ours consists of factors which are not solid, do not admit tensor decompositions [Chifan et al. 2018], and do not have Cartan subalgebras [Chifan et al. 2015].

(3) We notice that Theorem 5.1 still holds if instead of $\Gamma = (N_1 \times N_2) \rtimes (Q_1 \times Q_2)$ one considers any finite-index subgroup of Γ of the form $\Gamma_{s,r} = (N_1 \times N_2) \rtimes (Q_1^s \times Q_2^r) \leqslant \Gamma$, where $Q_1^s \leqslant Q_1$ and $Q_2^r \leqslant Q_2$ are arbitrary finite-index subgroups. One can verify these groups still enjoy all the algebraic/geometric properties used in the proof of Theorem 5.1 (including the fact that $N_1 \rtimes Q_1^s$ is hyperbolic relative to Q_1^s and $N_1 \rtimes Q_2^r$ is hyperbolic relative to Q_2^r) and hence all the von Neumann algebraic arguments in the proof of Theorem 5.1 apply verbatim. The details are left to the reader.

(4) The group factors considered in Theorem 5.1 have trivial fundamental group by [Chifan et al. 2020, Theorem B]

6. Concrete examples of infinitely many pairwise nonisomorphic group II_1-factors with property (T)

In this section we present several applications of our main techniques to the structural study of property (T) group factors. An earlier result of Popa [2007] shows that the map $\Gamma \mapsto \mathcal{L}(\Gamma)$ is at most countable-to-1. Since there are uncountably many icc property (T) groups, this obviously implies the existence of uncountably many group property (T) factors which are pairwise nonisomorphic. However, currently there are still no explicit constructions of such families in the literature. In this section we make new progress in this direction by showing that the canonical fiber product of Belegradek–Osin Rips construction groups can be successfully used to provide possibly the first such examples (Corollary 6.4). In addition, our methods also yield other interesting consequences. For instance, they can be used to provide an infinite series of finite-index subfactors of a given property (T) II_1-factor that are pairwise nonisomorphic, which is also a novelty in the area (Corollary 6.2). This further gives infinitely many examples of icc, property (T) groups Γ_n measure equivalent to a fixed group Γ such that $\mathcal{L}(\Gamma_n)$ are pairwise mutually nonisomorphic. The first examples of group measure equivalent groups Γ and Λ giving rise to nonisomorphic group von Neumann algebras were given in [Chifan and Ioana 2011], thereby answering a question of Shlyakhtenko. Note that the examples in [Chifan and Ioana 2011] don't have property (T).

The following is the main von Neumann algebraic result of the section. Some of the arguments used in the proof are very similar to the ones used in the proof of Theorem 5.1 and thus we shall just refer the reader to the previous section for these. However, we will include all the details on the new aspects of the proof.

Theorem 6.1. *Let Q_1, Q_2, P_1, P_2 be icc, torsion-free, residually finite property (T) groups. Let $Q = Q_1 \times Q_2$ and $P = P_1 \times P_2$. Assume that $N_1 \rtimes Q$, $N_2 \rtimes Q \in \mathcal{R}ip_T(Q)$ and $M_1 \rtimes P$, $M_2 \rtimes P \in \mathcal{R}ip_T(P)$. Assume that $\Theta : \mathcal{L}((N_1 \times N_2) \rtimes Q) \to \mathcal{L}((M_1 \times M_2) \rtimes P)$ is a $*$-isomorphism.*

Then one can find a $*$-isomorphism, $\Theta_i : \mathcal{L}(N_i) \to \mathcal{L}(M_i)$, a group isomorphism $\delta : Q \to P$, a multiplicative character $\eta : Q \to \mathbb{T}$, and a unitary $u \in \mathscr{U}(\mathcal{L}((M_1 \times M_2) \rtimes P))$ such that for all $\gamma \in Q$, $x_i \in N_i$ we have

$$\Theta((x_1 \otimes x_2)u_\gamma) = \eta(\gamma)u(\Theta_1(x_1) \otimes \Theta_2(x_2)v_{\delta(\gamma)})u^*.$$

Proof. Let $\mathcal{M} = \mathcal{L}((M_1 \times M_2) \rtimes P)$, $\Gamma_i = N_i \rtimes Q$ and $\widetilde{\mathcal{M}} = \mathcal{L}(\Gamma_1 \times \Gamma_2)$. Note that $\Theta(\mathcal{L}(N_1))$ and $\Theta(\mathcal{L}(N_2))$ are commuting property (T) subfactors of $\mathcal{L}((M_1 \times M_2) \rtimes P)$. Hence by Theorem 5.3 we have that either

(1) exists $i \in \{1, 2\}$ such that $\Theta(\mathcal{L}(N_i)) \prec_{\widetilde{\mathcal{M}}} \mathcal{L}(\Gamma_1)$ or

(2) $\Theta(\mathcal{L}(N_1 \times N_2)) \prec_{\widetilde{\mathcal{M}}} \mathcal{L}(\Gamma_1 \times P)$.

Assume (1) holds. Then proceeding as in the first part of proof of Theorem 5.5 we have $\Theta(\mathcal{L}(N_i)) \prec_{\widetilde{\mathcal{M}}} \mathcal{L}(M_1)$. As $\mathcal{L}(M_1)$ is regular in $\mathcal{M}$, we conclude using Lemma 2.4 that $\Theta(\mathcal{L}(N_i)) \prec_{\mathcal{M}} \mathcal{L}(M_1)$.

Assume (2). Then by the same argument as in the second part of the proof of Theorem 5.5 we have $\Theta(\mathcal{L}(N_1 \times N_2)) \prec_{\widetilde{\mathcal{M}}} \mathcal{L}(M_1 \rtimes \mathrm{diag}(P))$. Thus if $\Theta(\mathcal{L}(N_i)) \nprec \mathcal{L}(M_1)$ for all $i = 1, 2$, then the same argument as in the last part of Theorem 5.5 will lead to a contradiction.

In conclusion, we have shown that for all $i = 1, 2$ there exists $j \in 1, 2$ such that $\Theta(\mathcal{L}(N_j)) \prec_{\mathcal{M}} \mathcal{L}(M_i)$. As $\Theta(\mathcal{L}(N_j))$ is regular in $\mathcal{M}$, we actually have $\Theta(\mathcal{L}(N_j)) \prec^s_{\mathcal{M}} \mathcal{L}(M_i)$. Notice that in particular this forces different i's to give rise to different j's. Indeed, otherwise we would have $\Theta(\mathcal{L}(N_j)) \prec^s_{\mathcal{M}} \mathcal{L}(M_1)$ and $\Theta(\mathcal{L}(N_j)) \prec^s_{\mathcal{M}} \mathcal{L}(M_2)$. Then by [Drimbe et al. 2019, Lemma 2.6], this would imply $\Theta(\mathcal{L}(N_j)) \prec_{\mathcal{M}} \mathcal{L}(M_1) \cap \mathcal{L}(M_2) = \mathbb{C}$, which is obviously a contradiction. Therefore we get that either

(4a) $\Theta(\mathcal{L}(N_1)) \prec^s_{\mathcal{M}} \mathcal{L}(M_1)$ and $\Theta(\mathcal{L}(N_2)) \prec^s_{\mathcal{M}} \mathcal{L}(M_2)$ or

(4b) $\Theta(\mathcal{L}(N_1)) \prec^s_{\mathcal{M}} \mathcal{L}(M_2)$ and $\Theta(\mathcal{L}(N_2)) \prec^s_{\mathcal{M}} \mathcal{L}(M_1)$.

Note that both cases imply $\Theta(\mathcal{L}(N_1)), \Theta(\mathcal{L}(N_2)) \prec^s_{\mathcal{M}} \mathcal{L}(M_1 \times M_2)$. Using [Isono 2020, Lemma 2.6], we further get

$$\Theta(\mathcal{L}(N_1 \times N_2)) \prec^s_{\mathcal{M}} \mathcal{L}(M_1 \times M_2). \tag{6.1}$$

Proceeding in a similar manner, we also have the reverse intertwining $\mathcal{L}(M_1 \times M_2) \prec^s_{\mathcal{M}} \Theta(\mathcal{L}(N_1 \times N_2))$. Since $\mathcal{L}(M_1 \times M_2)$, $\mathcal{L}(N_1 \times N_2)$ are irreducible, regular subfactors of $\mathcal{M}$, by [Ioana et al. 2008, Lemma 8.4] one can find $u \in \mathcal{U}(\mathcal{M})$ such that

$$u\mathcal{L}(M_1 \times M_2)u^* = \Theta(\mathcal{L}(N_1 \times N_2)). \tag{6.2}$$

Note that $\Theta(\mathcal{L}(Q_1)), \Theta(\mathcal{L}(Q_2))$ are commuting property (T) subfactors of $\mathcal{L}((M_1 \times M_2) \rtimes P)$. Proceeding exactly as in the first part of the proof, we conclude that either $\Theta(\mathcal{L}(Q_i)) \prec_{\tilde{\mathcal{M}}} \mathcal{L}(\Gamma_1)$ or $\Theta(\mathcal{L}(Q_1 \times Q_2)) \prec_{\tilde{\mathcal{M}}} \mathcal{L}(\Gamma_1 \rtimes P)$. As before, this further implies that either

(7) $\Theta(\mathcal{L}(Q_i)) \prec_{\mathcal{M}} \mathcal{L}(M_1)$ or

(8) $\Theta(\mathcal{L}(Q_1 \times Q_2)) \prec_{\mathcal{M}} \mathcal{L}(M_1 \rtimes \mathrm{diag}(P))$.

Assume (7). Since by (6.2) we also have $\mathcal{L}(M_1) \prec^s_{\mathcal{M}} \Theta(\mathcal{L}(N_1 \times N_2))$ and hence by [Vaes 2009, Lemma 3.7] we conclude $\Theta(\mathcal{L}(Q_i)) \prec_{\mathcal{M}} \Theta(\mathcal{L}(N_1 \times N_2))$. However, this implies Q_i is finite, which is a contradiction.

Hence, we must have (8). Proceeding as in the end of proof of Theorem 5.5, we conclude that $\Theta(\mathcal{L}(Q)) \prec_{\mathcal{M}} \mathcal{L}(P)$. Thus there exists $\Psi : p\Theta(\mathcal{L}(Q))p \to \mathcal{R} := \Psi(p\Theta(\mathcal{L}(Q))p) \subseteq q\mathcal{L}(P)q$ such that $\Psi(x)v = vx$ for all $x \in p\Theta(\mathcal{L}(Q))p$. Also note that $vv^* \in \mathcal{R}' \cap q\mathcal{M}q$ and $v^*v \in p\Theta(\mathcal{L}(Q))p' \cap p\mathcal{M}p$. Since $\mathcal{R} \subseteq q\mathcal{L}(P)q$ is diffuse and $P \leqslant (M_1 \times M_2) \rtimes P$ is a malnormal subgroup, we have $\mathscr{QN}_{q\mathcal{M}q}(\mathcal{R})'' \subseteq q\mathcal{L}(P)q$. Thus $vv^* \in q\mathcal{L}(P)q$ and hence $vp\Theta(\mathcal{L}(Q))pv^* = \mathcal{R}vv^* \subseteq q\mathcal{L}(P)q$. Extending v to a unitary v_0 in $\mathcal{M}$ we have $v_0 p\Theta(\mathcal{L}(Q))pv_0^* \subseteq \mathcal{L}(P)$. As $\mathcal{L}(P)$ and $\mathcal{L}(Q)$ are factors, after perturbing v_0 to a new unitary we may assume that

(9) $v_0\Theta(\mathcal{L}(Q))v_0^* \subseteq \mathcal{L}(P)$.

In a similar manner we have that there exists $w_0 \in \mathscr{U}(\mathcal{M})$ with

(10) $w_0\mathcal{L}(P)w_0^* \subseteq \Theta(\mathcal{L}(Q))$.

Conditions (9) and (10) imply $w_0\mathcal{L}(P)w_0^* \subseteq \Theta(\mathcal{L}(Q)) \subseteq v_0^*\mathcal{L}(P)v_0$. In particular, $v_0w_0\mathcal{L}(P)w_0^*v_0^* \subseteq \mathcal{L}(P)$. Since P is malnormal in $(M_1 \times M_2) \rtimes P$, we have $v_0w_0 \in \mathcal{L}(P)$ and hence $w_0\mathcal{L}(P)w_0^* = v_0^*\mathcal{L}(P)v_0$. Combining this with the above relations we get

(11) $w_0\mathcal{L}(P)w_0^* = \Theta(\mathcal{L}(Q))$.

Since the action $Q \curvearrowright (N_1 \times N_2)$ has trivial stabilizers, using conditions (11) and (6), arguing as in the proof of Theorem 5.1, we get $h_{w_0\mathcal{L}(P)w_0^*}(\Theta(Q)) > 0$. By [Ioana et al. 2013, Theorem 3.3] we get that there exists $w_1 \in \mathscr{U}(\mathcal{M})$ and an isomorphism $\delta : Q \to P$ such that $\Theta(u_g) = w_1 v_{\delta(g)} w_1^*$ for all $g \in Q$.

Finally, this together with relation (4), proceeding exactly as in the proof of Theorem 5.1, implies the desired conclusion. $\square$

The previous theorem can be used to provide an infinite series of finite-index subfactors of a given property (T) II_1-factor that are pairwise nonisomorphic.

Corollary 6.2. (1) *Let Q_1, Q_2 be uniform lattices in* $\mathrm{Sp}(n,1)$ *with $n \geqslant 2$ and let $Q := Q_1 \times Q_2$. Also let $\cdots \leqslant Q_1^s \leqslant \cdots \leqslant Q_1^2 \leqslant Q_1^1 \leqslant Q_1$ be an infinite family of finite-index subgroups and define $Q_s := Q_1^s \times Q_2 \leqslant Q$. Then consider $N_1 \rtimes_{\sigma_1} Q, N_2 \rtimes_{\sigma_2} Q \in \mathcal{R}ip_T(Q)$ and let $\Gamma = (N_1 \times N_2) \rtimes_{\sigma_1\times\sigma_2} Q$. Inside Γ consider the finite-index subgroups $\Gamma_s := (N_1 \times N_2) \rtimes_{\sigma_1\times\sigma_2} Q_s$. Then the family $\{\mathcal{L}(\Gamma_s) : s \in I\}$ consists of pairwise nonisomorphic finite-index subfactors of $\mathcal{L}(\Gamma)$.*

(2) *Let Γ, Γ_n be as above. Then Γ_n is measure equivalent to Γ for all $n \in \mathbb{N}$, but $\mathcal{L}(\Gamma_n)$ is not isomorphic to $\mathcal{L}(\Gamma_m)$ for $n \neq m$.*

Proof. (1) Assume $\mathcal{L}(\Gamma_s) \cong \mathcal{L}(\Gamma_l)$. Notice that Q_2, Q_1^s, Q_1^l are torsion-free, residually finite property (T) groups. Thus applying Theorem 6.1 we get in particular that $Q_s \cong Q_l$. However since Q_2, Q_1^s, and Q_1^l are icc hyperbolic, this further implies $Q_1^s \cong Q_1^l$. However, by [Prasad 1976] or the cohopfian property of one-ended hyperbolic groups, this implies $s = l$ and the proof follows.

(2) As $[\Gamma : \Gamma_n] < \infty$, Γ_n is measure equivalent to Γ, and hence Γ_n is measure equivalent to Γ_m for all $n, m \in \mathbb{N}$. The rest follows from part (1). □

Notation. Denote by $\mathcal{ST}$ denote the family of all icc, torsion-free, residually finite property (T) groups.

For further use we record the following elementary result. Its proof is left to the reader.

Proposition 6.3. *Fix Q to be an icc, torsion-free, residually finite, hyperbolic property* (*T*) *group. For instance, Q can be chosen to be a uniform lattice in* $\mathrm{Sp}(n,1)$ *for $n \geqslant 2$. Then the family $\mathcal{ST}' = \{G \times Q : G \in \mathcal{ST}\}$ consists of pairwise nonisomorphic groups.*

Finally, we present the main application of this section:

Corollary 6.4. *Let $\{Q_\iota\}_{\iota\in\mathcal{I}}$ be an infinite family of pairwise nonisomorphic groups in $\mathcal{ST}'$. Consider the semidirect products $N_{\iota_1} \rtimes_{\sigma_1} Q_\alpha, N_{\iota_2} \rtimes_{\sigma_2} Q_\iota \in \mathcal{R}ip_T(Q_\iota)$ for every $\iota \in \mathcal{I}$. Consider the canonical semidirect product $\Gamma_\iota := (N_{\iota_1} \times N_{\iota_2}) \rtimes_{\sigma_1\times\sigma_2} Q_\iota$ corresponding to the diagonal action $\sigma_1 \times \sigma_2$. Then $\{\mathcal{L}(\Gamma_\iota) : \iota \in \mathcal{I}\}$ is an infinite family of pairwise nonisomorphic group II_1-factors with property* (*T*).

Proof. This follows directly from Theorem 6.1 and Proposition 6.3 □

We strongly believe the family $\mathcal{ST}$ consists of uncountably many pairwise nonisomorphic groups. In this scenario, Corollary 6.4 would provide an explicit family of uncountably many nonisomorphic property (T) group von Neumann algebras. However, we were unable to find in the literature a reference for whether $\mathcal{ST}$ contains uncountably many nonisomorphic groups. Therefore we leave the following as an open question.

Open Problem. *Find examples of uncountably many nonisomorphic icc property* (*T*) *groups G that give nonstably isomorphic II_1-factors $\mathcal{L}(G)$.*

7. Cartan-rigidity for von Neumann algebras of groups in $\mathcal{R}ip(Q)$

In this last section we classify the Cartan subalgebras in II_1-factors associated with the groups in $\mathcal{R}ip_T(Q)$ and their free ergodic pmp actions on probability spaces (see Theorem 7.1, and Corollary 7.2). Our proofs rely in an essential way on the methods introduced in [Popa and Vaes 2014; Chifan et al. 2015], as well as on the group theoretic Dehn filling discussed in Section 3C. For convenience we include detailed proofs.

First we establish the following general intertwining result regarding crossed product algebras arising from groups in $\mathcal{R}ip(Q)$.

Theorem 7.1. *Let $Q = Q_1 \times Q_2$, where Q_i are residually finite groups. For every $i = 1, 2$, let $\Gamma_i = N_i \rtimes_{\sigma_i} Q \in \mathcal{R}ip(Q)$ and denote by $\Gamma = (N_1 \times N_2) \rtimes_\sigma Q$ the semidirect product associated with the diagonal action $\sigma = (\sigma_1, \sigma_2) : Q \to \mathrm{Aut}(N_1 \times N_2)$. Let $\mathcal{P}$ be a von Neumann algebra together with an action $\Gamma \curvearrowright \mathcal{P}$ and define $\mathcal{M} = \mathcal{P} \rtimes \Gamma$. Let $p \in \mathcal{M}$ be a projection and let $\mathcal{A} \subset p\mathcal{M}p$ be a maximal abelian self-adjoint subalgebra (masa) whose normalizer $\mathscr{N}_{p\mathcal{M}p}(\mathcal{A})'' \subseteq p\mathcal{M}p$ has finite index. Then $\mathcal{A} \prec_\mathcal{M} \mathcal{P}$.*

Proof. Since $\Gamma_i = N_i \rtimes Q$ is hyperbolic relative to a residually finite group Q, by Theorem 3.16 there exists a nonelementary hyperbolic group H_i, a subset $T_i \subseteq N_i$ with $|T_i| \geqslant 2$ and a normal subgroup $R_i \lhd Q$ of finite index such that we have a short exact sequence

$$1 \to *_{t \in T_i} R_i^t \hookrightarrow \Gamma_i \xrightarrow{\varepsilon_i} H_i \to 1.$$

In particular there are infinite groups K_1, K_2 so that $*_{t\in T_i} R_i^t = K_1 * K_2$.

Denote by $\pi_i : \Gamma \twoheadrightarrow \Gamma_i$ the canonical projection given by $\pi_i((n_1, n_2)q) = n_i q$ for all $(n_1, n_2)q \in (N_1 \times N_2) \rtimes Q = \Gamma$. Then for every $i = 1, 2$ consider the epimorphism $\rho_i = \varepsilon_i \circ \pi_i : \Gamma \to H_i$. Following [Chifan et al. 2015, Section 3], consider the $*$-embedding $\Delta^{\rho_i} : \mathcal{M} \to \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{L}(H_i) := \widetilde{\mathcal{M}}_i$ given by $\Delta^{\rho_i}(xu_g) = xu_g \otimes v_{\rho_i(g)}$ for all $x \in \mathcal{M}$, $g \in \Gamma$. Here $(u_g)_{g\in\Gamma}$ and $(v_h)_{h\in H_i}$ are the canonical group unitaries in $\mathcal{P} \rtimes \Gamma$ and $\mathcal{L}(H_i)$, respectively. As $\mathcal{A}$ is amenable, [Popa and Vaes 2014, Theorem 1.4] implies either (a) $\Delta^{\rho_i}(\mathcal{A}) \prec_{\widetilde{\mathcal{M}}_i} \mathcal{M} \otimes 1$ or (b) the normalizer $\Delta^{\rho_i}(\mathscr{N}_{p\mathcal{M}p}(\mathcal{A})'')$ is amenable relative to $\mathcal{M} \otimes 1$ inside $\widetilde{\mathcal{M}}_i$. Assume (b) holds. As $\mathscr{N}_{p\mathcal{M}p}(\mathcal{A})'' \subseteq p\mathcal{M}p$ has finite index, it follows that $\Delta^{\rho_i}(p\mathcal{M}p)$ is amenable relative to $\mathcal{M} \otimes 1$ inside $\widetilde{\mathcal{M}}_i$. However, using [Chifan et al. 2015, Proposition 3.5] this further gives that H_i is amenable, a contradiction. Thus (a) must hold and using [loc. cit., Proposition 3.4] we get $\mathcal{A} \prec_\mathcal{M} \mathcal{P} \rtimes \ker(\rho_i)$. Let $\mathcal{N} = \mathcal{P} \rtimes \ker(\rho_i)$ and using [loc. cit., Proposition 3.6] we can find a projection $0 \neq q \in \mathcal{N}$ such that a masa $\mathcal{B} \subset q\mathcal{N}q$ with $\mathcal{Q} := \mathscr{N}_{q\mathcal{N}q}(\mathcal{B})'' \subseteq q\mathcal{N}q$ has finite index. In addition one can find projections $0 \neq p_0 \in \mathcal{A}$, $0 \neq q_0' \in \mathcal{B}' \cap p\mathcal{M}p$ and a unitary $u \in \mathcal{M}$ such that $u(\mathcal{A}p_0)u^* = \mathcal{B}p_0$.

To this end, observe the restriction homomorphism $\pi_i : \ker(\rho_i) \to K_1 * K_2$ is an epimorphism with $\ker(\pi_i) = N_{\hat{i}}$. As before, consider the $*$-embedding $\Delta^{\pi_i} : \mathcal{N} \to \mathcal{N} \mathbin{\bar{\otimes}} \mathcal{L}(K_1 * K_2)$ given by $\Delta^{\pi_i}(xu_g) = xu_g \otimes v_{\pi_i(g)}$ for all $x \in \mathcal{P}$, $g \in \ker(\rho_i)$. Define $\widetilde{\mathcal{N}}_i := \mathcal{N} \mathbin{\bar{\otimes}} \mathcal{L}(\ker(\rho_i))$. Also fix $0 \neq z \in \mathscr{Z}(\mathcal{Q}' \cap q\mathcal{N}q)$. Since $\Delta^{\pi_i}(\mathcal{B}z) \subset \mathcal{N} \mathbin{\bar{\otimes}} \mathcal{L}(K_1 * K_2)$ is amenable, using [Ioana 2013; Vaes 2014] one of the following must hold: (c) $\Delta^{\pi_i}(\mathcal{Q}z)$ is amenable relative to $\mathcal{N} \otimes 1$ inside $\widetilde{\mathcal{N}}_i$; (d) $\Delta^{\pi_i}(\mathcal{Q}z) \prec_{\widetilde{\mathcal{N}}_i} \mathcal{N} \mathbin{\bar{\otimes}} \mathcal{L}(K_j)$ for some $j = 1, 2$; (e) $\Delta^{\pi_i}(\mathcal{B}z) \prec_{\widetilde{\mathcal{N}}_i} \mathcal{N} \otimes 1$.

Assume (c) holds. As $\mathcal{Q} \subseteq q\mathcal{N}q$ is finite-index so is $\mathcal{Q}z \subseteq z\mathcal{N}z$ and [Chifan et al. 2015, Lemma 2.4] implies $z\mathcal{N}z \prec^s \mathcal{Q}z$. Using [Ozawa and Popa 2010, Proposition 2.3 (3)] we get $\Delta^{\pi_i}(z\mathcal{N}z)$ is amenable relative to $\mathcal{N} \otimes 1$ inside $\widetilde{\mathcal{N}}_i$. Thus [Chifan et al. 2015, Proposition 3.5] implies that $K_1 * K_2$ is amenable, a contradiction. Assume (d) holds. By [loc. cit., Proposition 3.4] we have $\mathcal{Q}z \prec \mathcal{P} \rtimes (\pi_i)^{-1}(K_j)$ and using [Drimbe et al. 2019, Lemma 2.4 (3)] one can find a projection $0 \neq r \in \mathscr{Z}(\mathcal{Q}z' \cap z\mathcal{N}z)$ such that $\mathcal{Q}r \prec^s \mathcal{P} \rtimes (\pi_i)^{-1}(K_j)$. Since $\mathcal{Q}z \subseteq z\mathcal{N}z$ is of finite index, so is $\mathcal{Q}r \subseteq r\mathcal{N}r$ and thus $r\mathcal{N}r \prec_\mathcal{N} \mathcal{Q}r$. Therefore using [Drimbe et al. 2019, Lemma 2.4 (1)] (or [Vaes 2009, Remark 3.7]) we conclude that $\mathcal{N} \prec \mathcal{P} \rtimes (\pi_i)^{-1}(K_j)$. However, this implies that $\pi^{-1}(K_j) \leqslant \ker(\rho_i)$ is finite-index, a contradiction. Hence

(e) must hold and using [Chifan et al. 2015, Proposition 3.4] we further get $\mathcal{B}z \prec_{\mathcal{N}} \mathcal{P} \rtimes N_{\hat{i}}$. Since this holds for all z, we conclude that $\mathcal{B} \prec^{s}_{\mathcal{N}} \mathcal{P} \rtimes N_{\hat{i}}$. This combined with the prior paragraph clearly implies $\mathcal{A} \prec \mathcal{P} \rtimes N_{\hat{i}}$.

Since all the arguments above still work and the same conclusion holds if one replaces $\mathcal{A}$ by $\mathcal{A}a$ for any projection $0 \neq a \in \mathcal{A}$, one actually has $\mathcal{A} \prec^{s}_{\mathcal{M}} \mathcal{P} \rtimes N_{\hat{i}}$. Since this holds for all $i = 1, 2$, using [Drimbe et al. 2019, Lemma 2.8 (2)] one concludes that $\mathcal{A} \prec_{\mathcal{M}} \mathcal{P}$, as desired. □

Corollary 7.2. *Let Γ be a group as in the previous theorem and let $\Gamma \curvearrowright X$ be a free ergodic pmp action on a probability space. Then the following hold*:

(1) *The crossed product $L^\infty(X) \rtimes \Gamma$ has unique Cartan subalgebra.*

(2) *The group von Neumann algebra $\mathcal{L}(\Gamma)$ has no Cartan subalgebra.*

Proof. (1) Let $\mathcal{A} \subset L^\infty(X) \rtimes \Gamma =: \mathcal{M}$ be a Cartan subalgebra. By Theorem 7.1 we have $\mathcal{A} \prec_{\mathcal{M}} L^\infty(X)$ and since $L^\infty(X) \subseteq \mathcal{M}$ is Cartan then [Popa 2006a, Theorem] gives the conclusion.

(2) If $\mathcal{A} \subset \mathcal{L}(\Gamma)$ is a Cartan subalgebra then Theorem 7.1 implies $\mathcal{A} \prec \mathbb{C}1$, which contradicts that $\mathcal{A}$ is diffuse. □

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LONG TIME EXISTENCE OF YAMABE FLOW ON SINGULAR SPACES WITH POSITIVE YAMABE CONSTANT

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We establish long-time existence of the normalized Yamabe flow with positive Yamabe constant on a class of manifolds that includes spaces with incomplete cone-edge singularities. We formulate our results axiomatically so that they extend to general stratified spaces as well, provided certain parabolic Schauder estimates hold. The central analytic tool is a parabolic Moser iteration, which yields uniform upper and lower bounds on both the solution and the scalar curvature.

1. Introduction and statement of the main results

The Yamabe conjecture states that for any compact, smooth Riemannian manifold (M, g_0) without boundary there exists a constant scalar curvature metric conformal to g_0. The first proof of this conjecture was initiated by Yamabe [1960] and continued by Trudinger [1968], Aubin [1976] and Schoen [1984]. The proof is based on the calculus of variations and elliptic partial differential equations. An alternative tool for proving the conjecture is due to Hamilton [1989], who utilized the normalized Yamabe flow of a Riemannian manifold (M, g_0), which is a family $g \equiv g(t),\ t \in [0, T]$ of Riemannian metrics on M such that the following evolution equation holds:

$$\partial_t g = -(S-\rho)g, \quad \rho := \mathrm{Vol}_g(M)^{-1} \int_M S \, d\mathrm{Vol}_g. \tag{1-1}$$

Here S is the scalar curvature of g, $\mathrm{Vol}_g(M)$ is the total volume of M with respect to g and ρ is the average scalar curvature of g. The normalization by ρ ensures that the total volume does not change along the flow. Hamilton [1989] introduced the Yamabe flow and also showed its long-time existence. It preserves the conformal class of g_0 and ideally should converge to a constant scalar curvature metric, thereby establishing the Yamabe conjecture by parabolic methods.

Establishing convergence of the normalized Yamabe flow is intricate already in the setting of smooth, compact manifolds. In the case of scalar negative, scalar flat and locally conformally flat scalar positive metrics, convergence is due to Ye [1994]. The case of a nonconformally flat g_0 with positive scalar curvature is delicate and has been studied first by Schwetlick and Struwe [2003] for large energies and later by Brendle [2005; 2007] for arbitrary energies. More specifically, [Schwetlick and Struwe 2003, Section 5] as well as [Brendle 2005, p. 270; 2007, p. 544] invoke the positive mass theorem, which is where the dimensional restriction in [Schwetlick and Struwe 2003; Brendle 2005] and the spin assumption

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in [Brendle 2007, Theorem 4] come from. Assuming [Schoen and Yau 2022] to be correct, [Schwetlick and Struwe 2003; Brendle 2005; 2007] cover all closed manifolds which are not conformally equivalent to spheres.

In the noncompact setting, our understanding is limited. On complete manifolds, long-time existence has been discussed in various settings by Ma [2016], Ma and An [1999] and Schulz [2020]. On incomplete surfaces, where Ricci and Yamabe flows coincide, Giesen and Topping [2010; 2011] constructed a flow that becomes instantaneously complete.

In this work, we study the Yamabe flow on a general class of spaces that includes incomplete spaces with cone-edge (wedge) singularities or, more generally, stratified spaces with iterated cone-edge singularities. This continues a program initiated in [Bahuaud and Vertman 2014; 2019], where existence and convergence of the Yamabe flow has been established in case of negative Yamabe invariant. Here, we study the positive case and, utilizing methods of Akutagawa, Carron and Mazzeo [Akutagawa et al. 2014], we establish long-time existence of the flow under certain mild geometric assumptions. We don't attempt to prove convergence here, in view of [Schwetlick and Struwe 2003; Brendle 2005; 2007] and the fact that we don't have a substitute for the positive mass theorem in the singular setting.

Our main result (see Theorem 5.1 for the precise statement) is as follows.

Theorem 1.1. *Let* (M, g_0) *be a Riemannian manifold of dimension* $n = \dim M \geq 3$ *such that the following four assumptions* (*to be made precise below*) *hold*:

(1) *The Yamabe constant* $Y(M, g_0)$ *defined in* (1-6) *is positive.*

(2) (M, g_0) *is admissible and satisfies a Sobolev inequality as in Definition 1.2.*

(3) *Parabolic Schauder estimates of Definition 1.4 hold on* (M, g_0).

(4) *The initial scalar curvature* $S_0 \in C^{1,\alpha}(M)$ *as in Assumption 4. Moreover*, *we also require that* $S_0 \in L^{n^2/(2(n-2))}(M)$ *and that its negative part* $(S_0)_- \in L^\infty(M)$.

Under these assumptions, *a normalized Yamabe flow of* g_0 *exists within the space of admissible spaces*, *with infinite existence time.*

Examples, where the assumptions of the theorem are satisfied, include spaces with incomplete wedge singularities. More general stratified spaces with iterated cone-edge metrics are also covered, provided parabolic Schauder estimates continue to hold in that setting.

Let us point out two technical novelties of our work.

(1) We prove uniform bounds on the solution and on the scalar curvature along the normalized Yamabe flow without using the maximum principle. We have not found any such argument in the existing literature.

(2) We derive such uniform bounds starting with low initial Sobolev regularity, $S_0 \in H^1(M)$. This low initial regularity forces us to develop very intricate arguments to deal even with the chain rule. We have not seen any such argument in the existing literature on parabolic equations.

We now proceed with explaining the assumptions in detail.

Normalized Yamabe flow and Yamabe constant. Consider a Riemannian manifold (M, g_0), with g_0 normalized such that the total volume $\mathrm{Vol}_{g_0}(M)$ equals 1. The Yamabe flow (1-1) preserves the conformal class of the initial metric g_0 and, assuming $\dim M = n \geq 3$, we can write $g = u^{4/(n-2)} g_0$ for some function $u > 0$ on $M_T = M \times [0, T]$ for some upper time limit $T > 0$. Then the normalized Yamabe flow equation can be equivalently written as an equation for u:

$$\partial_t (u^{(n+2)/(n-2)}) = \tfrac{1}{4}(n+2)(\rho u^{(n+2)/(n-2)} - L_0(u)), \quad L_0 := S_0 - \frac{4(n-1)}{n-2}\Delta_0, \tag{1-2}$$

where L_0 is the conformal Laplacian of g_0, defined in terms of the scalar curvature S_0 and the Laplace Beltrami operator Δ_0 associated to the initial metric g_0. The scalar curvature of the evolving metric g is given by $S = u^{-(n+2)/(n-2)} L_0(u)$, and the volume form of $g = u^{4/(n-2)} g_0$ is given by $d\mathrm{Vol}_g = u^{2n/(n-2)}\, d\mu$, where we write $d\mu := d\mathrm{Vol}_{g_0}$ for the time-independent initial volume form. One computes that

$$\partial_t\, d\mathrm{Vol}_g = -\tfrac{1}{2} n (S - \rho)\, d\mathrm{Vol}_g. \tag{1-3}$$

Hence, the total volume of (M, g) is constant and thus equal to 1 along the flow. The average scalar curvature then takes the form

$$\rho = \int_M S\, d\mathrm{Vol}_g = \int_M L(u) u^{-(n+2)/(n-2)} u^{2n/(n-2)}\, d\mu = \int_M \frac{4(n-1)}{n-2} |\nabla u|^2 + S_0 u^2\, d\mu. \tag{1-4}$$

Explicit computations lead to the following evolution equation for the average scalar curvature:

$$\partial_t \rho = -\frac{n-2}{2} \int_M (\rho - S)^2 u^{2n/(n-2)}\, d\mu. \tag{1-5}$$

The latter evolution equation in particular implies that $\rho \equiv \rho(t)$ is nonincreasing along the flow. We conclude the exposition by defining the Yamabe constant of g_0, which incidentally provides a lower bound for ρ. Let u be a solution of (1-2). We define the $L^q(M)$ spaces with respect to the integration measure $d\mu$.

We define the first Sobolev space $H^1(M)$ as the space of all $v : M \to \mathbb{R}$ such that the first Sobolev norm, defined with respect to $d\mu$ and the pointwise norm associated to g_0, satisfies

$$\|v\|^2_{H^1(M)} := \int_M v^2\, d\mu + \int_M |\nabla v|^2\, d\mu < \infty.$$

Similarly we define $H^1(M, g)$ by using $d\mathrm{Vol}_g$ instead of $d\mu$ and the pointwise norm associated to g. If u and u^{-1} are both bounded, one easily checks that $H^1(M) = H^1(M, g)$.

We define the Yamabe invariant of g_0 as

$$\begin{aligned} Y(M, g_0) &:= \inf_{v \in H^1(M)\setminus\{0\}} \frac{1}{\|v\|^2_{L^{2n/(n-2)}(M)}} \int_M \frac{4(n-1)}{n-2} |\nabla v|^2 + S_0 v^2\, d\mu \\ &\leq \int_M \frac{4(n-1)}{n-2} |\nabla u|^2 + S_0 u^2\, d\mu = \rho \qquad \text{(by (1-4))}, \end{aligned} \tag{1-6}$$

where in the inequality we have used that for any solution u of (1-2), $\|u\|_{L^{2n/(n-2)}(M)} = d\mathrm{Vol}_g(M) \equiv 1$. How one proceeds will depend heavily on the sign of the Yamabe constant. In this paper we will assume that $Y(M, g_0) > 0$. In particular, the average curvature ρ is then positive and uniformly bounded away from zero along the normalized Yamabe flow.

Assumption 1. The Yamabe constant $Y(M, g_0)$ is positive.

A Sobolev inequality and other admissibility assumptions. The Moser iteration arguments in this paper are strongly motivated by the related work in [Akutagawa et al. 2014] on the Yamabe problem on stratified spaces. Thus, similar to that paper, we impose certain admissibility assumptions, which are naturally satisfied by certain compact stratified spaces with iterated cone-edge metrics.

Definition 1.2. Let (M, g_0) be a smooth Riemannian manifold of dimension n. We call (M, g_0) admissible if it satisfies the following conditions:

- (M, g_0) with volume form $d\mu = d\mathrm{Vol}_{g_0}$ has finite volume: $\mathrm{Vol}_{g_0}(M) < \infty$.
- For any $\varepsilon > 0$, there exist finitely many open balls $B_{2R_i}(x_i) \subset M$ such that

$$\mathrm{Vol}_{g_0}\Big(M \setminus \bigcup_i B_{R_i}(x_i)\Big) \leq \varepsilon. \tag{1-7}$$

- Smooth, compactly supported functions $C_c^\infty(M)$ are dense in $H^1(M)$.[1]
- (M, g_0) admits a Sobolev inequality of the following kind: defining $L^q(M)$ spaces with respect to $d\mu$, there exist $A_0, B_0 > 0$ such that for all $f \in H^1(M)$,

$$\|f\|^2_{L^{2n/(n-2)}(M)} \leq A_0 \|\nabla f\|^2_{L^2(M)} + B_0 \|f\|^2_{L^2(M)}. \tag{1-8}$$

The main examples we have in mind are closed manifolds[2] and regular parts of smoothly stratified spaces endowed with iterated cone-edge metrics. See [Akutagawa et al. 2014, Section 2.1] for a definition of the latter. That the Sobolev inequality holds in this case is shown in Proposition 2.2 of the same paper. Note that the list of admissibility assumptions does not contain compactness. Nor do we specify explicitly how the metric g_0 looks near the singular strata of $\overline{M}$, in the case of stratified spaces. Restrictions on the local behavior of the metric will instead be coded in L^q-data, like requiring the initial scalar curvature S_0 to be in $L^q(M)$ for suitable $q > 0$. These requirements are stated in the theorems below, and will vary from statement to statement.

Assumption 2. (M, g_0) is an admissible Riemannian manifold.

In what follows we want to relate the assumption of the Sobolev inequality (1-8) in Definition 1.2 to positivity of the Yamabe constant $Y(M, g_0)$.

Proposition 1.3. *Assume $S_0 \in L^\infty(M)$ and $Y(M, g_0) > 0$. Then* (1-8) *holds.*

Proof. Indeed, it follows directly from the definition of $Y(M, g_0)$ in (1-6) that

$$\|f\|^2_{L^{2n/(n-2)}(M)} \leq \frac{1}{Y(M, g_0)} \left(\frac{4(n-1)}{n-2} \|\nabla f\|^2_{L^2(M)} + \|S_0\|_{L^\infty(M)} \|f\|^2_{L^2(M)} \right)$$

for all $f \in H^1(M)$. This is indeed the Sobolev inequality (1-8). □

[1] This can be phrased as $H^1_0(M) = H^1(M)$. Note that this rules out M being the interior of a manifold with a codimension 1 boundary.

[2] This includes finite volume, complete manifolds, since any finite volume, complete manifold satisfying the Sobolev inequality is compact; see [Hebey 1996, Lemma 3.2, pp. 18–19 and Remark 2, pp. 56–57].

Parabolic Schauder estimates and short-time existence. Our proof requires intricate arguments involving the heat operator and its mapping properties, as seen in the previous work by the second author jointly with Bahuaud [Bahuaud and Vertman 2014; 2019] in the setting of spaces with incomplete wedge singularities. Here we axiomatize these arguments into a definition of certain parabolic Schauder estimates, having in mind further generalizations to stratified spaces.

Definition 1.4. (M, g_0) satisfies parabolic Schauder estimates if there is a scale of Banach spaces $\{C^{k,\alpha} \equiv C^{k,\alpha}(M \times [0, T])\}_{k\in\mathbb{N}_0}$ of continuous functions on $M \times [0, T]$ with

$$C^{0,\alpha} \supset C^{1,\alpha} \supset C^{2,\alpha} \supset \cdots$$

for some $\alpha \in (0, 1)$ and any $T > 0$, with the following properties:

(1) Algebraic properties of the Banach spaces:

- For any $k \in \mathbb{N}_0$, the constant function 1 exists in $C^{k,\alpha}(M \times [0, T])$.
- For any $k \in \mathbb{N}_0$ and any $u \in C^{k,\alpha}(M \times [0, T])$ uniformly bounded away from zero, we have that the inverse u^{-1} exists in $C^{k,\alpha}(M \times [0, T])$.
- For any $k \geq 2$ and $\ell \leq k$ we have $C^{k,\alpha} \cdot C^{\ell,\alpha} \subseteq C^{\ell,\alpha}$. Writing $\|\cdot\|_{\ell,\alpha}$ for the norm on $C^{\ell,\alpha}$, we have a uniform constant $C_{\ell,\alpha}$ such that for any $u \in C^{k,\alpha}$ and $v \in C^{\ell,\alpha}$,

$$\|u \cdot v\|_{\ell,\alpha} \leq C_{\ell,\alpha} \|u\|_{k,\alpha} \|v\|_{\ell,\alpha}. \tag{1-9}$$

(2) Regularity properties of the Banach spaces:

- We have the inclusions

$$\begin{aligned} C^{0,\alpha}(M \times [0, T]) &\subseteq C^0([0, T], L^2(M)), \\ C^{1,\alpha}(M \times [0, T]) &\subseteq C^0([0, T], H^1(M)), \\ C^{2,\alpha}(M \times [0, T]) &\subseteq L^\infty(M \times [0, T]). \end{aligned} \tag{1-10}$$

  Moreover, for any $u \in C^{0,\alpha}(M \times [0, T])$ and any fixed $p \in M$, the evaluation $u(p, \cdot)$ still lies in $C^{0,\alpha}$. The map $M \ni p \mapsto \|u(p, \cdot)\|_{0,\alpha}$ is again $L^2(M)$.
- If $C^{k,\alpha}([0, T]) \subset C^{k,\alpha}(M \times [0, T])$ consists of functions that are constant on M, then the spaces $C^{2k,\alpha}([0, T])$ are characterized as

$$C^{2k,\alpha}([0, T]) = \{u \in C^{0,\alpha}([0, T]) \mid \partial_t^k u \in C^{0,\alpha}([0, T])\}. \tag{1-11}$$

- For any $k \in \mathbb{N}_0$, the following maps are bounded:

$$\begin{aligned} \partial_t, \Delta_0 &: C^{k+2,\alpha}(M \times [0, T]) \to C^{k,\alpha}(M \times [0, T]), \\ \nabla &: C^{k+1,\alpha}(M \times [0, T]) \to C^{k,\alpha}(M \times [0, T]). \end{aligned} \tag{1-12}$$

(3) Weak maximum principle for elements of the Banach spaces:

- Any $u \in C^{2,\alpha}(M \times [0, T])$ satisfies a weak maximum principle; that is for any Cauchy sequence $\{q_\ell\}_{\ell\in\mathbb{N}} \subset M$ we have

$$\inf_M u = \lim_{\ell\to\infty} u(q_\ell) \implies \lim_{\ell\to\infty} (\Delta_0 u)(q_\ell) \geq 0. \tag{1-13}$$

In the case that the Cauchy sequence $\{q_\ell\}_{\ell\in\mathbb{N}}$ converges to an interior point $p \in M$, where u attains a global minimum, we have that $\Delta_0 u(p) \geq 0$.

(4) Mapping properties of the heat operator:

- The heat operator $e^{t\Delta_0}$ admits the mapping properties

$$\begin{aligned} e^{t\Delta_0} &: C^{k,\alpha}(M\times[0,T]) \to C^{k+2,\alpha}(M\times[0,T]),\\ e^{t\Delta_0} &: C^{k,\alpha}(M\times[0,T]) \to t^\alpha C^{k+1,\alpha}(M\times[0,T]),\\ e^{t\Delta_0} &: L^\infty(M\times[0,T]) \to C^{1,\alpha}(M\times[0,T]). \end{aligned} \tag{1-14}$$

If $e^{t\Delta_0}$ acts without convolution in time, then we have a bounded map

$$e^{t\Delta_0} : C^{k,\alpha}(M) \to C^{k,\alpha}(M\times[0,T]). \tag{1-15}$$

(5) Mapping properties of other solution operators:

- For any positive $a \in C^{1,\alpha}(M\times[0,T])$ uniformly bounded away from zero, there is a solution operator Q for $(\partial_t - a\cdot\Delta_0)u = f,\ u(0) = 0$, with

$$Q : C^{0,\alpha}(M\times[0,T]) \to C^{2,\alpha}(M\times[0,T]). \tag{1-16}$$

  If $a \in C^{2,\alpha}$, then additionally $Q : C^{1,\alpha} \to C^{3,\alpha}$ is bounded.

- For any positive $a \in C^{1,\alpha}(M\times[0,T])$ uniformly bounded away from zero, there is a solution operator R for $(\partial_t - a\cdot\Delta_0)u = 0,\ u(0) = f$, with

$$R : C^{2,\alpha}(M) \to C^{2,\alpha}(M\times[0,T]), \tag{1-17}$$

  where $C^{k,\alpha}(M)$ denotes the subspace of $C^{k,\alpha}(M\times[0,T])$ consisting of time-independent functions. If $a \in C^{2,\alpha}$, then additionally $R : C^{3,\alpha}(M) \to C^{3,\alpha}(M\times[0,T])$ is bounded.

Let us now discuss where such parabolic Schauder estimates hold.

Examples 1.5. (a) Parabolic Schauder estimates hold on smooth compact Riemannian manifolds without boundary by the classical estimates of [Ladyženskaja et al. 1968].

(b) By [Bahuaud and Vertman 2014; 2019], a manifold with a wedge singularity satisfies the parabolic Schauder estimates,[3] assuming that the wedge metric is feasible in the sense of [Bahuaud and Vertman 2019, Definition 2.2]. The proof is based on the microlocal heat kernel description in [Mazzeo and Vertman 2012]. Note that the choice of Banach spaces is not canonical, and instead one can use, for example, the scale of weighted Hölder spaces as in [Vertman 2021].

(c) In view of the recent work by Albin and Gell-Redman [2017], we expect the same parabolic Schauder estimates to hold on general stratified spaces with iterated cone-wedge metrics.

Assumption 3. (M, g_0) satisfies parabolic Schauder estimates.

[3]In fact, in the mapping properties of solution operators Q and R we require here less than in [Bahuaud and Vertman 2019]: in the case $a \in C^{2,\alpha}$ we only ask for $Q : C^{1,\alpha} \to C^{3,\alpha}$ and $R : C^{3,\alpha}(M) \to C^{3,\alpha}$, while in that paper, these additional mapping properties are proved for one order higher.

Using parabolic Schauder estimates, we can prove short time existence and regularity of the renormalized Yamabe flow, exactly as in [Bahuaud and Vertman 2014, Theorem 1.7 and 4.1] and by a slight adaptation of [Bahuaud and Vertman 2019, Proposition 4.8].

Theorem 1.6. *Let* (M, g_0) *satisfy parabolic Schauder estimates. Assume, moreover, that the scalar curvature* S_0 *of* g_0 *lies in* $C^{1,\alpha}(M)$. *Then the following hold*:

(1) *The Yamabe flow* (1-2) *admits, for some* $T > 0$ *sufficiently small, a solution*

$$u \in C^{2,\alpha}(M \times [0, T]) \subseteq C^0([0, T], H^1(M)) \cap L^\infty(M \times [0, T])$$

which is positive and uniformly bounded away from zero.[4]

(2) *If a solution*, $u \in C^{2,\alpha}(M \times [0, T])$, *to the Yamabe flow* (1-2) *exists for a given* $T > 0$ *and is uniformly bounded away from zero, then in fact* $u \in C^{3,\alpha}(M \times [0, T])$. *In particular, we obtain*

$$S \in C^{1,\alpha}(M \times [0, T]) \subseteq C^0([0, T], H^1(M)).$$

Proof. We shall only provide a brief proof outline. The first statement is proved by setting up a fixed point argument in the Banach space $C^{2,\alpha}(M \times [0, T])$. If $u = 1 + v \in C^{2,\alpha}(M \times [0, T])$ is a solution to (1-2), then v satisfies the equation

$$\partial_t v - (n-1)\Delta_0 v = -\tfrac{1}{4}(n-2)S_0 + \Phi(v), \tag{1-18}$$

where $\Phi : C^{2,\alpha}(M \times [0, T]) \to C^{0,\alpha}(M \times [0, T])$ is a bounded map, in view of the algebraic and regularity properties (1-12) in Definition 1.4. Moreover, Φ is quadratic in its argument, i.e., writing $\|\cdot\|_{k,\alpha}$ for the norm on $C^{k,\alpha}$ for any $k \in \mathbb{N}$, there exists a uniform $C > 0$, such that by (1-9) (compare [Bahuaud and Vertman 2014, Lemma 5.1]), for all $w, w' \in C^{2,\alpha}$,

$$\|\Phi(w)\|_{0,\alpha} \le C\|w\|_{2,\alpha}^2 \quad \text{and} \quad \|\Phi(w) - \Phi(w')\|_{0,\alpha} \le C(\|w\|_{2,\alpha} + \|w'\|_{2,\alpha})\|w - w'\|_{2,\alpha}. \tag{1-19}$$

Now a solution v of (1-18) (and hence also a solution $u = 1 + v$ of (1-2)) is obtained as a fixed point of the map

$$C^{2,\alpha}(M \times [0, T]) \ni v \mapsto e^{t(n-1)\Delta_0}\big(-\tfrac{1}{4}(n-2)S_0 + \Phi(v)\big) \in C^{2,\alpha}(M \times [0, T]), \tag{1-20}$$

which is a contraction mapping on a subset of $C^{2,\alpha}(M \times [0, T])$ for $T > 0$ sufficiently small,[5] by (1-14) in Definition 1.4. One argues exactly as in [Bahuaud and Vertman 2014, Theorem 4.1]. Note that the regularity of the scalar curvature S along the flow is then $S \in C^{0,\alpha}(M \times [0, T])$.

Note also that the fixed point argument is performed in a small ball around zero in $C^{2,\alpha}(M \times [0, T])$, and thus for $T > 0$ sufficiently small, the norm of v is small. Hence $u = 1 + v$ is positive and bounded away from zero.

The second statement improves the regularity of S. By the regularity properties (1-10) in Definition 1.4, we conclude that $\rho, \partial_t \rho \in C^{0,\alpha}([0, T])$. By (1-11), this implies that $\rho \in C^{2,\alpha}([0, T])$. We can now apply

[4] Later on, we will prove uniform lower bounds on u for any finite $T > 0$.

[5] We need to assume that $T > 0$ is sufficiently small in order to control $e^{t(n-1)\Delta_0}(S_0)$.

the mapping properties (1-16) and (1-17)[6] in Definition 1.4 to obtain a solution $u' \in C^{3,\alpha}(M \times [0, T])$ with

$$\partial_t u' - (n-1)u^{-4/(n-2)}\Delta_0 u' = \tfrac{1}{4}(n-2)(\rho u - S_0 u^{(n-6)/(n-2)}), \quad u'(0) = 1. \tag{1-21}$$

The given solution $u \in C^{2,\alpha}$ satisfies the same equation, and we can prove that $u \equiv u'$ by the weak maximum property (1-13) of elements in $C^{2,\alpha}$. Thus, indeed, $u \in C^{3,\alpha}$ and hence $S \in C^{1,\alpha}$. This is basically the argument used in [Bahuaud and Vertman 2019, Proposition 4.8]. □

Remark 1.7. If we assume $Q : C^{2,\alpha} \to C^{4,\alpha}$ and $R : C^{4,\alpha}(M) \to C^{4,\alpha}$ in Definition 1.4, as has been proved in [Bahuaud and Vertman 2019], then the condition $S_0 \in C^{2,\alpha}(M)$ implies, by similar arguments as in Theorem 1.6, that any solution u in $C^{2,\alpha}$ is actually in $C^{4,\alpha}$. This would lead to $S \in C^{2,\alpha}$, in particular, the scalar curvature would stay bounded along the flow. Here, we decided to require less in Definition 1.4, assume less regularity for S_0 and conclude boundedness of S by Moser iteration methods instead.

Regularity of the initial scalar curvature. In view of Theorem 1.6, we arrive at our final assumption on a regularity of the initial scalar curvature S_0 with respect to the scale of Banach spaces in Definition 1.4.

Assumption 4. Assuming that (M, g_0) satisfies parabolic Schauder estimates, we also ask that the initial scalar curvature S_0 be an element of $C^{1,\alpha}(M)$.

In view of Theorem 1.6, this implies that $S \in H^1(M)$. Moreover, since $u \in C^{2,\alpha}(M \times [0, T]) \subset L^\infty$ for $T > 0$ sufficiently small, norms on the Sobolev space $H^1(M)$ with respect to g_0 and norms on the Sobolev space $H^1(M, g)$ with respect to $g = u^{4/(n-2)}g_0$ are equivalent. Thus $S(t)$ lies in the Sobolev space $H^1(M, g(t))$ for any $t \in [0, T]$, which we abbreviate as

$$S \in H^1(M, g). \tag{1-22}$$

Our arguments below will use (1-22) to show that given $S_0 \in L^q(M)$ for

$$q = \frac{n^2}{2(n-2)} = \frac{n}{2} + \frac{n}{n-2} > \frac{n}{2},$$

we may conclude by Moser iteration that $S \in L^\infty(M)$ for positive times. We close this subsection with the observation that on stratified spaces, $S_0 \in L^q(M)$ for $q > \frac{1}{2}n$ and $S_0 \in L^\infty(M)$ basically carry the same geometric restriction. Indeed, consider a cone $(0, 1) \times N$ over a Riemannian manifold (N, g_0), with metric $g_0 = dx^2 \oplus x^2 g_N + h$, where h is smooth in $x \in [0, 1]$ and $|h|_{\bar{g}} = O(x)$ as $x \to 0$, and where we write $\bar{g} := dx^2 \oplus x^2 g_N$. Then

$$S_0 \sim \frac{\operatorname{scal}(g_N) - \dim N(\dim N - 1)}{x^2} + O(x^{-1}) \quad \text{as } x \to 0, \tag{1-23}$$

where the higher order term $O(x^{-1})$ comes from the perturbation h. We see that both of the assumptions $S_0 \in L^\infty(M)$ and $S_0 \in L^q(M)$ for $q > \frac{1}{2}n$ imply that the leading term of the metric g_0 is scalar-flat, i.e., $\operatorname{scal}(g_N) = \dim N(\dim N - 1)$.

[6] Here we use the assumption that u is uniformly bounded away from zero and that $1 \in C^{3,\alpha}$ by the algebraic properties of the Banach spaces.

The overarching strategy. Studies of the Yamabe flow usually follow the very rough pattern that we outline here. One first argues that (1-2) has a short-time solution. This is the step we have been concerned with in this section. This step doesn't invoke the sign of the Yamabe constant.

The next step is to show that the flow can be extended to all times. The way one does this is to assume the flow is defined for $t \in (0, T)$ for some maximal time $T < \infty$ and then derive a priori bounds on the solution u and the scalar curvature S, showing that neither of them develop singularities as $t \to T$. One can thus keep flowing past T, establishing long-time existence. This is the step we address in the rest of the paper.

2. The evolution of the scalar curvature and lower bounds

In this section we derive a lower bound on the scalar curvature S along the normalized Yamabe flow. We present an argument that requires neither the maximum principle nor the full set of assumptions in Theorem 1.1, but rather the following assumptions (provided the flow exists):

- $S \in H^1(M, g)$ along normalized Yamabe flow,
- $H^1(M)$ and $H^1(M, g)$ have equivalent norms,
- $C_c^\infty(M)$ is dense in $H^1(M)$,
- $Y(M, g_0) > 0$.

(2-1)

These properties clearly follow from Assumptions 1, 2, 3 and 4.

Lemma 2.1. *Let $g = u^{4/(n-2)} g_0$ be a family of metrics evolving according to the normalized Yamabe flow equation* (1-2) *satisfying* (2-1). *Then S evolves according to*

$$\partial_t S - (n-1)\Delta S = S(S - \rho), \tag{2-2}$$

where Δ denotes the Laplacian with respect to the time-evolving metric g. We write $S_+ := \max\{S, 0\}$ and $S_- := -\min\{S, 0\}$. Then $S_\pm$ are elements of $H^1(M, g)$ and satisfy

$$\partial_t S_+ - (n-1)\Delta S_+ \le S_+(S_+ - \rho), \tag{2-3}$$

$$\partial_t S_- - (n-1)\Delta S_- \le -S_-(S_- + \rho). \tag{2-4}$$

Remark 2.2. Equation (2-2) is to be understood in the weak sense: for any compactly supported smooth test function $\phi \in C_c^\infty(M)$ we have

$$\int_M \partial_t S \cdot \phi \, d\mathrm{Vol}_g + (n-1) \int_M (\nabla S, \nabla \phi)_g \, d\mathrm{Vol}_g = \int_M S(S-\rho) \cdot \phi \, d\mathrm{Vol}_g.$$

Similarly for the partial differential inequalities (2-3) and (2-4) and $\phi \ge 0$, we have

$$\int_M \partial_t S_\pm \cdot \phi \, d\mathrm{Vol}_g + (n-1) \int_M (\nabla S_\pm, \nabla \phi)_g \, d\mathrm{Vol}_g \le \pm \int_M S_\pm (S_\pm \mp \rho) \cdot \phi \, d\mathrm{Vol}_g.$$

By (2-1), $C_c^\infty(M)$ is dense in $H^1(M) = H^1(M, g)$. Hence we might as well assume $\phi \in H^1(M, g)$ in the weak formulation above.

Proof. Equation (2-2) is well known and can be deduced as follows. Write

$$L_g := S - 4\frac{n-1}{n-2}\Delta$$

for the conformal Laplacian of the metric g. We write $L_0 \equiv L_{g_0}$. If g and g_0 are related by $g = u^{4/(n-2)} g_0$, then L_g and L_0 are related by

$$L_g(\,\cdot\,) = u^{-(n+2)/(n-2)} L_0(u\,\cdot\,).$$

In particular, $S = L_g(1) = u^{-(n+2)/(n-2)} L_0(u)$. We differentiate this equation weakly in time and use (1-2) to replace $\partial_t u = -\frac{1}{4}(n-2)(S-\rho)u$ and get

$$\partial_t S = \tfrac{1}{4}(n+2)(S-\rho)u^{-(n+2)/(n-2)} L_0(u) - \tfrac{1}{4}(n-2)u^{-(n+2)/(n-2)} L_0((S-\rho)u).$$

Applying the transformation rule for L we may rewrite this as

$$\begin{aligned}\partial_t S &= \tfrac{1}{4}(n+2)(S-\rho)L_g(1) - \tfrac{1}{4}(n-2)L_g(S-\rho)\\ &= \tfrac{1}{4}(n+2)(S-\rho)S + (n-1)\Delta S - \tfrac{1}{4}(n-2)(S-\rho)S.\end{aligned}$$

This proves formula (2-2). In order to derive the differential inequality for S_+, consider any $\varepsilon > 0$ and define

$$\psi_\varepsilon(x) := \begin{cases}\sqrt{x^2+\varepsilon^2} - \varepsilon, & x \geq 0,\\ 0, & x < 0.\end{cases}$$

For $v \in H^1(M, g)$ it is readily checked that $\psi_\varepsilon(v) \in H^1(M, g)$ and $\lim_{\varepsilon\to 0}\psi_\varepsilon(v) = v_+$. Furthermore, in the case $x > 0$, we compute the derivatives

$$\psi_\varepsilon'(x) = \frac{x}{\sqrt{x^2+\varepsilon^2}}, \quad \psi_\varepsilon''(x) = \frac{\varepsilon^2}{(x^2+\varepsilon^2)^{3/2}}.$$

These are both bounded for a fixed $\varepsilon > 0$, so the chain rule applies. Next up we claim, for any $v \in H^1(M, g)$, in the weak sense

$$\Delta\psi_\varepsilon(v) \geq \frac{v}{\sqrt{v^2+\varepsilon^2}}\,\Delta v \equiv \psi_\varepsilon'(v)\,\Delta v. \tag{2-5}$$

This is seen as follows. Let $0 \leq \xi \in C_c^\infty(M)$ be arbitrary and compute

$$\begin{aligned}\int_M \xi\,\Delta\psi_\varepsilon(v)\,d\mathrm{Vol}_g &:= -\int_M (\nabla\xi, \nabla\psi_\varepsilon(v))_g\,d\mathrm{Vol}_g = -\int_M \frac{v}{\sqrt{v^2+\varepsilon^2}}(\nabla\xi, \nabla v)_g\,d\mathrm{Vol}_g\\ &= -\int_M \left(\nabla v, \nabla\left(\frac{v}{\sqrt{v^2+\varepsilon^2}}\xi\right)\right)_g d\mathrm{Vol}_g + \int_M \frac{\xi\varepsilon^2|\nabla v|_g^2}{(v^2+\varepsilon^2)^{3/2}}\,d\mathrm{Vol}_g\\ &\geq -\int_M \left(\nabla v, \nabla\left(\frac{v}{\sqrt{v^2+\varepsilon^2}}\xi\right)\right)_g d\mathrm{Vol}_g\\ &=: \int_M \xi\frac{v}{\sqrt{v^2+\varepsilon^2}}\Delta v\,d\mathrm{Vol}_g.\end{aligned}$$

This proves (2-5), which allows us to deduce that

$$\begin{aligned}\partial_t\psi_\varepsilon(S)-(n-1)\Delta\psi_\varepsilon(S) &\le \begin{cases}\psi_\varepsilon'(S)(\partial_t S-(n-1)\Delta S), & S\ge 0,\\ 0, & S<0,\end{cases}\\ &= \begin{cases}\psi_\varepsilon'(S)S(S-\rho), & S\ge 0,\\ 0, & S<0,\end{cases} \qquad \text{(by (2-2))}\\ &= \frac{S}{\sqrt{S^2+\varepsilon^2}}S_+(S_+-\rho).\end{aligned}$$

Letting $\varepsilon\to 0$ results in (2-3). To prove (2-4), observe that $S_-=S_+-S$. Hence

$$\begin{aligned}\partial_t S_- -(n-1)\Delta S_- &= \partial_t S_+ -(n-1)\Delta S_+ -(\partial_t S-(n-1)\Delta S)\\ &\le S_+(S-\rho)-S(S-\rho)=S_-(S-\rho),\end{aligned}$$

where we have used (2-2) and (2-3) in the inequality step. The only thing which remains to be observed is that $S_-\cdot S=S_-(S_+-S_-)=-S_-^2$. □

We can now derive lower bounds for S by studying the evolution (in-)equalities above. This is usually done by invoking the weak maximum principle for S, which is not available under the assumptions (2-1). Thus, we provide an alternative novel argument, which does not use a maximum principle and which we could not find elsewhere in the literature.

Proposition 2.3. *Let $g=u^{4/(n-2)}g_0$ be a family of metrics evolving according to the normalized Yamabe flow equation* (1-2) *satisfying* (2-1)*. Then*

$$\|S_-\|_{L^p(M,g)}(t)\le e^{tn\rho(0)/(2p)}\|(S_0)_-\|_{L^p(M)}$$

for all $2\le p\le\infty$. In particular, if $(S_0)_-\in L^\infty(M)$, then $S_-\in L^\infty$ on $[0,T]$ with uniform bounds depending only on T and S_0. Moreover, if $S_0\ge 0$, then $S\ge 0$ along the normalized Yamabe flow for all time.

Proof. The weak formulation of (2-4) is that for any $0\le\xi\in H^1(M,g)$,

$$\int_M \xi\,\partial_t S_-\,d\mathrm{Vol}_g+(n-1)\int_M(\nabla S_-,\nabla\xi)_g\,d\mathrm{Vol}_g\le-\int\xi S_-(S_-+\rho)\,d\mathrm{Vol}_g. \tag{2-6}$$

A problem when manipulating this is that the chain rule fails to hold in general, so we use the same workaround as [Akutagawa et al. 2014, pp. 10–13] (who in turn are following [Gursky 1993, pp. 349–352]). Let $L>0$, $\beta\ge 1$ and define

$$\phi_{\beta,L}(x):=\begin{cases}x^\beta, & x\le L,\\ \beta L^{\beta-1}(x-L)+L^\beta, & x>L,\end{cases} \tag{2-7}$$

$$G_{\beta,L}(x):=\int_0^x\phi_{\beta,L}'(y)^2\,dy=\begin{cases}\dfrac{\beta^2}{2\beta-1}x^{2\beta-1}, & x\le L,\\ \beta^2L^{2(\beta-1)}x-\dfrac{2\beta^2L^{2\beta-1}(\beta-1)}{2\beta-1}, & x>L.\end{cases} \tag{2-8}$$

Finally, we define $H_{\beta,L}(x) := \int_0^x G_{\beta,L}(y)\,dy$ and conclude that

$$H_{\beta,L}(x) = \begin{cases} \dfrac{\beta x^{2\beta}}{2(2\beta-1)}, & x \le L, \\ \dfrac{\beta^2 L^{2(\beta-1)}}{2}(x^2-L^2) - \dfrac{2\beta^2 L^{2\beta-1}(\beta-1)}{2\beta-1}(x-L) + \dfrac{\beta L^{2\beta}}{2(2\beta-1)}, & x > L. \end{cases}$$

The crucial features of these definitions are

$$\phi_{\beta,L}(x) \xrightarrow{L\to\infty} x^\beta, \quad G_{\beta,L}(x) \xrightarrow{L\to\infty} \frac{\beta^2}{2\beta-1}x^{2\beta-1}, \quad H_{\beta,L}(x) \xrightarrow{L\to\infty} \frac{\beta}{2(2\beta-1)}x^{2\beta}.$$

These functions are also dominated by simpler expressions. For instance, $H_{\beta,L}(x) \le \beta^2 x^{2\beta}$ holds for all $L > 0$ and $\beta \ge 1$ as follows: For $x \le L$, there is nothing to show. For $x > L$, we first observe that

$$H_{\beta,L}(x) = \frac{\beta^2}{2}L^{2(\beta-1)}x^2 - \frac{2\beta^2(\beta-1)}{2\beta-1}L^{2\beta-1}x + \frac{\beta(\beta-1)}{2}L^{2\beta}.$$

Dropping the nonpositive middle term and estimating by $x \ge L$, we find

$$H_{\beta,L}(x) \le \frac{\beta^2}{2}x^{2\beta} + \frac{\beta(\beta-1)}{2}x^{2\beta} < \beta^2 x^{2\beta}.$$

Another important property is that $\phi_{\beta,L} \in C^1(\mathbb{R}_+)$, with $\phi'_{\beta,L} \in L^\infty(\mathbb{R}_+)$ for all $L > 0$, and so we may apply the chain rule to $\phi_{\beta,L}(S_-)$. Finally, since we are assuming a C^1 time-dependence, we have $\partial_t H_{\beta,L}(S_-) = (\partial_t S_-)G_{\beta,L}(S_-)$. We will use $\xi := G_{\beta,L}(S_-)$ as a test function in (2-6). Note that by definition, $G_{\beta,L}(x)$ is linear for $x > L$ and hence $G_{\beta,L}(f) \in H^1(M,g)$ whenever $f \in H^1(M,g)$ (here we are also using that $\mathrm{Vol}(M) < \infty$). Then (2-6) implies

$$\int_M \partial_t H_{\beta,L}(S_-)\,d\mathrm{Vol}_g \le -(n-1)\int_M |\nabla\phi_{\beta,L}(S_-)|_g^2\,d\mathrm{Vol}_g - \int_M G_{\beta,L}(S_-)S_-(S_-+\rho)\,d\mathrm{Vol}_g. \tag{2-9}$$

We then use (1-3) to conclude

$$\begin{aligned} \int_M \partial_t H_{\beta,L}(S_-)\,d\mathrm{Vol}_g &= \partial_t \int_M H_{\beta,L}(S_-)\,d\mathrm{Vol}_g + \frac{n}{2}\int_M H_{\beta,L}(S_-)(S-\rho)\,d\mathrm{Vol}_g \\ &= \partial_t \int_M H_{\beta,L}(S_-)\,d\mathrm{Vol}_g - \frac{n}{2}\int_M H_{\beta,L}(S_-)(S_-+\rho)\,d\mathrm{Vol}_g, \end{aligned} \tag{2-10}$$

where the last step uses

$$SH_{\beta,L}(S_-) \equiv (S_+ - S_-)H_{\beta,L}(S_-) = -S_- H_{\beta,L}(S_-).$$

Finally, we need a Sobolev inequality given to us by the positivity of the Yamabe constant, namely for any $f \in H^1(M,g)$ we have by the definition of $Y(M,g_0)$ (note that $Y(M,g_0) = Y(M,g)$ by conformal invariance) that

$$Y(M,g_0)\|f\|^2_{L^{2n/(n-2)}(M,g)} \le 4\frac{n-1}{n-2}\|\nabla f\|^2_{L^2(M,g)} + \int_M Sf^2\,d\mathrm{Vol}_g. \tag{2-11}$$

We set $f = \phi_{\beta,L}(S_-)$. Observe that $\phi_{\beta,L}(S_-)^2 S = -\phi_{\beta,L}(S_-)^2 S_-$. Then (2-11) implies

$$\begin{aligned}(n-1)\|\nabla\phi_{\beta,L}(S_-)\|^2_{L^2(M,g)} &\geq \frac{n-2}{4}\,Y(M,g_0)\|\phi_{\beta,L}(S_-)\|^2_{L^{2n/(n-2)}(M,g)} + \frac{n-2}{4}\int_M \phi_{\beta,L}(S_-)^2 S_-\, d\mathrm{Vol}_g\\ &\geq \frac{n-2}{4}\int_M \phi_{\beta,L}(S_-)^2 S_-\, d\mathrm{Vol}_g. \end{aligned}\tag{2-12}$$

Combining (2-9), (2-10) and (2-12) yields

$$\begin{aligned}\partial_t \int_M H_{\beta,L}(S_-)\, d\mathrm{Vol}_g \leq \int_M \Big(\frac{n}{2}\,H_{\beta,L}(S_-) - G_{\beta,L}(S_-)S_- - \frac{n-2}{4}\,\phi_{\beta,L}(S_-)^2\Big) S_-\, d\mathrm{Vol}_g&\\ + \int_M \rho\Big(\frac{n}{2}\,H_{\beta,L}(S_-) - G_{\beta,L}(S_-)S_-\Big)\, d\mathrm{Vol}_g&.\end{aligned}$$

We claim the first group of terms on the right-hand side is nonpositive, which follows by a direct computation. We begin by noting that

$$\begin{aligned}&\tfrac{1}{2}nH_{\beta,L}(x) - xG_{\beta,L}(x) - \tfrac{1}{4}(n-2)\phi_{\beta,L}(x)^2\\ &\qquad= \begin{cases} \dfrac{(-1)}{4(2\beta-1)}((4\beta+n)(\beta-1)+2)x^{2\beta}, & x \leq L,\\ \dfrac{L^{2\beta}}{4}\Big(-2\beta^2\Big(\dfrac{x}{L}\Big)^2 - \dfrac{2(n-2)\beta(\beta-1)}{2\beta-1}\Big(\dfrac{x}{L}\Big) + (\beta-1)(n+2(\beta-1))\Big), & x > L.\end{cases}\end{aligned}$$

In both cases one checks that the expressions are nonpositive[7] for $\beta \geq 1$. Hence using that $G_{\beta,L}(S_-) \geq 0$ and ρ is nonincreasing by (1-5), we conclude

$$\partial_t \int_M H_{\beta,L}(S_-)\, d\mathrm{Vol}_g \leq \int_M \frac{n\rho}{2} H_{\beta,L}(S_-)\, d\mathrm{Vol}_g \leq \frac{n\rho(0)}{2}\int_M H_{\beta,L}(S_-)\, d\mathrm{Vol}_g.$$

Integrating shows

$$\int_M H_{\beta,L}(S_-)\, d\mathrm{Vol}_g(t) \leq e^{tn\rho(0)/2}\int_M H_{\beta,L}(S_-)\, d\mathrm{Vol}_g(t=0).$$

The conclusion will follow when we take the limit $L \to \infty$, which we can do for the following reason.[8] On the left-hand side we appeal to Fatou's lemma and the pointwise convergence of $H_{\beta,L}$ to find

$$\liminf_{L\to\infty}\int_M H_{\beta,L}(S_-)\, d\mathrm{Vol}_g \geq \int_M \liminf_{L\to\infty} H_{\beta,L}(S_-)\, d\mathrm{Vol}_g = \frac{\beta}{2(2\beta-1)}\int_M S_-^{2\beta}\, d\mathrm{Vol}_g.$$

The right-hand side we deal with by the dominated convergence theorem. We showed on page 488 that $H_{\beta,L}(x) \leq \beta^2 x^{2\beta}$ holds for all $L > 0$ and $\beta \geq 1$. Since we are assuming $(S_0)_- \in L^\infty(M)$, we can use $\beta^2((S_0)_-)^{2\beta}$ as a dominating integrable function to deduce

$$\liminf_{L\to\infty}\int_M H_{\beta,L}((S_0)_-)\, d\mu = \lim_{L\to\infty}\int_M H_{\beta,L}((S_0)_-)\, d\mu = \frac{\beta}{2(2\beta-1)}\int_M ((S_0)_-)^{2\beta}\, d\mu.$$

[7]For the $x \geq L$ case observe that the polynomial is negative for $x = L$, and the expression for $x > L$ clearly has a negative derivative. So the expression remains negative for $x > L$.

[8]This argument is applied several times, without writing out the details in the latter instances.

Combined we have for $\beta \geq 1$,

$$\int_M S_-^{2\beta} \, d\mathrm{Vol}_g \leq e^{tn\rho(0)/2} \int_M (S_0)_-^{2\beta} \, d\mu.$$

This gives the conclusion when writing $2\beta = p$. □

Remark 2.4. Let us again emphasize the novelty of this argument: it circumvents the maximum principle, and one only needs to know that $S \in H^1(M, g)$, as assumed in (2-1).

For completeness, let us also provide the classical and widely known argument (see [Brendle 2005]), using the weak maximum principle: we assume S satisfies (1-13), which is the case if $S \in C^{2,\alpha}(M \times [0, T])$. See Remark 1.7 for conditions which ensure this regularity of S along the flow.

Proposition 2.5. *Assume that $S \in C^0(M \times [0, T])$ satisfies the weak maximum principle* (1-13) *and that $Y(M, g_0) > 0$. Then S admits a uniform lower bound*

$$S \geq \min\left\{0, \inf_M S_0\right\}.$$

Proof. By the weak maximum principle, we have, for $S_{\min} := \inf_M S$,

$$\partial_t S_{\min} \geq S_{\min}(S_{\min} - \rho).$$

If $S_{\min}$ is negative for all times, then the right-hand side becomes positive, and we get $S_{\min} \geq \inf_M S_0$. If $S_{\min}$ is positive for all times, we can further estimate the right-hand side using $\rho \leq \rho(0)$; see (1-5). Dividing, we then get

$$\frac{\partial_t S_{\min}}{S_{\min}(\rho(0) - S_{\min})} \geq -1.$$

Integrating this differential inequality, we find

$$S_{\min}(t) \geq \frac{\rho(0)(S_0)_{\min}}{e^{\rho(0)t}(\rho(0) - (S_0)_{\min}) + (S_0)_{\min}} \geq 0.$$

If $S_{\min}$ changes sign along the flow, the statement follows by a combination of both estimates. □

3. Uniform bounds on the solution along the flow

The arguments of this section employ the assumptions

$$\begin{aligned}
&\bullet\ (M, g_0) \text{ is an admissible manifold},\\
&\bullet\ H^1(M) \text{ and } H^1(M, g) \text{ have equivalent norms},\\
&\bullet\ u \in C^0([0, T], H^1(M)) \text{ and } S \in H^1(M, g),\\
&\bullet\ Y(M, g_0) > 0,
\end{aligned} \tag{3-1}$$

provided the flow exists. These properties follow from Assumptions 1, 2, 3 and 4.

We begin with the upper bound on u, which follows easily from the lower bound on the scalar curvature S, obtained in Proposition 2.3.

Proposition 3.1. *Let $g = u^{4/(n-2)} g_0$ be a family of metrics, $u > 0$, such that* (3-1) *holds and the normalized Yamabe flow equation* (1-2) *holds weakly, with $u(0) = 1$. Assume furthermore that $(S_0)_- \in L^\infty(M)$, where S_0 is the scalar curvature of g_0. Then there exists some uniform constant $0 < C(T) < \infty$, depending only on $T > 0$ and S_0, such that $u \le C(T)$ for all $t \in [0, T]$ with $T < \infty$.*

Proof. We have by (1-1) and (1-5) that

$$\partial_t u = -\tfrac{1}{4}(n-2)(S-\rho)u \le \tfrac{1}{4}(n-2)(S_- + \rho)u \le \tfrac{1}{4}(n-2)(S_- + \rho(0))u.$$

By Proposition 2.3 we have $\|S_-\|_{L^\infty(M)} \le \|(S_0)_-\|_{L^\infty(M)}$, and hence setting

$$C := \tfrac{1}{4}(n-2)(\|(S_0)_-\|_{L^\infty(M)} + \rho(0)),$$

we conclude

$$\partial_t u \le Cu \implies u \le e^{CT} u_0 = e^{CT}. \qquad \square$$

The lower bound is more intricate and in many ways more interesting. The argument will rely on the upper bound on u and the lower bound on S. The proof will be a mixture and modification of the methods in [Akutagawa et al. 2014, pp. 20–21; Brendle 2005, pp. 221–222].

Theorem 3.2. *Let $g = u^{4/(n-2)} g_0$ be a family of metrics, $u > 0$, such that* (3-1) *holds and the normalized Yamabe flow equation* (1-2) *holds weakly, with $u(0) = 1$. Assume furthermore that $(S_0)_- \in L^\infty(M)$ and that $S_0 \in L^q(M)$ for some $q > \frac{1}{2}n$. Then there exists some uniform constant $c(T) > 0$, depending only on $T > 0$ and S_0, such that $c(T) \le u$ for all $t \in [0, T]$.*

Proof. By combining (1-2) and (1-1) we may eliminate the term $\partial_t u$ and get

$$-4\frac{n-1}{n-2}\,\Delta_0 u = (u^{4/(n-2)} S - S_0)u.$$

Using that $(S_0)_- \in L^\infty(M)$ and $u \in L^\infty(M \times [0, T])$, by Proposition 3.1 we may define

$$P := \frac{n-2}{4(n-1)}(S_0 + \|u\|^{4/(n-2)}_{L^\infty(M_T)} \|(S_0)_-\|_{L^\infty(M)}) \in L^q(M).$$

Note that P only depends on S_0 and T. Furthermore, Proposition 2.3 yields

$$(-\Delta_0 + P)u \ge 0. \tag{3-2}$$

Let us explain the proof idea. Assume we can show that there is some $\delta > 0$ such that $u^{-\delta} \in H^1(M)$ uniformly in $t \in [0, T]$. Then (3-2) implies

$$\begin{aligned}(-\Delta_0 - \delta P)u^{-\delta} &= \delta u^{-1-\delta}\Delta_0 u - \delta(1+\delta)u^{-2-\delta}|\nabla u|^2 - \delta P u^{-\delta} \\ &= -\delta u^{-1-\delta}(-\Delta_0 + P)u - \delta(1+\delta)u^{-2-\delta}|\nabla u|^2 \le 0. \end{aligned}\tag{3-3}$$

This is precisely the setting of [Akutagawa et al. 2014, Proposition 1.8], which then concludes by Moser iteration and Sobolev inequality (1-8) that

$$\|u^{-\delta}\|_{L^\infty(M)} \le C\|u^{-\delta}\|_{H^1(M)},$$

where the constant $C > 0$ depends on δP, hence only on T and S_0, but not on t. Under our temporary assumption (3-2), we thus get a uniform bound on $u^{-\delta}$, which gives a uniform lower bound on u.

Hence we only need to show that $u^{-\delta} \in H^1(M)$ uniformly. Let $\varepsilon, \delta > 0$ and (following [Akutagawa et al. 2014, pp. 20–21]) define the functions $\psi_\varepsilon(u) := (u+\varepsilon)^{-\delta}$ and $\phi_\varepsilon(u) := (u+\varepsilon)^{-1-2\delta}$. These are both in $H^1(M)$ since $u \in H^1(M)$. Using ϕ_ε as a test function in the weak formulation of (3-2), we deduce

$$-\frac{1+2\delta}{\delta^2}\|\nabla\psi_\varepsilon(u)\|^2_{L^2(M)} + \int_M Pu\phi_\varepsilon(u)\,d\mu \geq 0,$$

and, using that $u\phi_\varepsilon(u) \leq \psi_\varepsilon(u)^2$ along with the Hölder inequality, we find

$$\|\nabla\psi_\varepsilon(u)\|^2_{L^2(M)} \leq \frac{\delta^2}{1+2\delta}\|P\|_{L^q(M)}\|\psi_\varepsilon(u)^2\|_{L^{q/(q-1)}(M)}. \tag{3-4}$$

Since $q > \frac{1}{2}n$, we have $q/(q-1) < n/(n-2)$ and thus $\|\psi_\varepsilon(u)^2\|_{L^{q/(q-1)}(M)} \leq \|\psi_\varepsilon(u)\|^2_{L^{2n/(n-2)}(M)}$. By the Sobolev inequality (1-8) we know

$$\|\psi_\varepsilon(u)\|^2_{L^{2n/(n-2)}(M)} \leq A_0\|\nabla\psi_\varepsilon(u)\|^2_{L^2(M)} + B_0\|\psi_\varepsilon(u)\|^2_{L^2(M)}. \tag{3-5}$$

Next we need a Poincaré inequality. Let $B \subset M$ be a ball. Then, exactly as in [Akutagawa et al. 2014, Lemma 1.14], there exists a constant $C_B > 0$ such that

$$\|f\|^2_{L^2(M)} \leq C_B(\|\nabla f\|^2_{L^2(M)} + \|f\|^2_{L^2(B)}) \tag{3-6}$$

holds for all $f \in H^1(M)$. Plugging (3-5) and (3-6) into (3-4) yields

$$\|\nabla\psi_\varepsilon(u)\|^2_{L^2(M)} \leq \frac{\delta^2}{1+2\delta}\|P\|_{L^q(M)}((A_0+B_0C_B)\|\nabla\psi_\varepsilon(u)\|^2_{L^2(M)} + B_0C_B\|\psi_\varepsilon(u)\|^2_{L^2(B)}),$$

which is equivalent to

$$\left(1-\frac{\delta^2}{1+2\delta}\|P\|_{L^q(M)}(A_0+B_0C_B)\right)\|\nabla\psi_\varepsilon(u)\|^2_{L^2(M)} \leq \frac{\delta^2}{1+2\delta}\|P\|_{L^q(M)}B_0C_B\|\psi_\varepsilon(u)\|^2_{L^2(B)}.$$

Choosing $\delta > 0$ small enough so that the left-hand side becomes positive, we get a uniform (meaning now both t- and ε-independent) bound on $\|\nabla\psi_\varepsilon(u)\|_{L^2(M)}$ if we can get a uniform bound on $\|\psi_\varepsilon(u)\|_{L^2(B)}$. The uniform bound on $\|\psi_\varepsilon(u)\|_{L^2(B)}$ will come from the local theory for elliptic supersolutions. Observe that since u satisfies (3-2), we have $u^{2n/(n-2)}$ satisfies (by the same computation as in (3-3))

$$-\Delta_0 u^{2n/(n-2)} + \frac{2n}{n-2}Pu^{2n/(n-2)} \geq 0.$$

Let $R > 0$ be such that $B_{4R}(x) \subset M$ for some $x \in M$. Then, according to [Gilbarg and Trudinger 1983, Theorem 8.18, p. 194], the following weak Harnack inequality holds on $B_{2R}(x)$; namely there is a constant $C > 0$ independent of u but depending on g_0, R and n such that

$$\mathrm{Vol}_g(B_{2R}(x)) \equiv \|u^{2n/(n-2)}\|_{L^1(B_{2R}(x))} \leq C \inf_{B_R(x)} u^{2n/(n-2)}, \tag{3-7}$$

where in the first identification we recall that $d\mathrm{Vol}_g = u^{2n/(n-2)} d\mu$. By the admissibility of (M, g_0), the assumption (1-7) holds and we may take a collection of balls $B_{4R_i}(x_i) \subset M$, indexed by $i = 1, \dots, N < \infty$, with the property that[9]

$$\Bigl(1 - \mathrm{Vol}_{g_0}\Bigl(\bigcup_{i=1}^{N} B_{2R_i}(x_i)\Bigr)\Bigr) \|u\|_{L^\infty(M_T)}^{2n/(n-2)} < 1. \tag{3-8}$$

Let C_i be the constant in (3-7) for the ball $B_{2R_i}(x_i)$. By summing all the individual inequalities (3-7) for each $i = 1, \dots, N$, we have

$$\sum_{i=1}^{N} \mathrm{Vol}_g(B_{2R_i}(x_i)) \le \sum_{i=1}^{N} C_i \inf_{B_{R_i}(x_i)} u^{2n/(n-2)} \le NC \max_i \bigl(\inf_{B_{R_i}(x_i)} u^{2n/(n-2)} \bigr)$$

with $C := \max_i C_i$. The left-hand side can bounded from below by

$$\begin{aligned} \sum_{i=1}^{N} \mathrm{Vol}_g(B_{2R_i}(x_i)) \ge \mathrm{Vol}_g\Bigl(\bigcup_{i=1}^{N} B_{2R_i}(x_i)\Bigr) &= 1 - \mathrm{Vol}_g\Bigl(M \setminus \bigcup_{i=1}^{N} B_{2R_i}(x_i)\Bigr) \\ &\ge 1 - \mathrm{Vol}_{g_0}\Bigl(M \setminus \bigcup_{i=1}^{N} B_{2R_i}(x_i)\Bigr) \|u\|_{L^\infty(M_T)}^{2n/(n-2)} =: c, \end{aligned}$$

which is positive by choice of the balls subject to (3-8). Thus

$$0 < c \le NC \max_i \bigl(\inf_{B_{R_i}(x_i)} u^{2n/(n-2)} \bigr).$$

This shows that there has to be a ball $B_{R_i}(x_i)$ with u uniformly bounded from below by $c(T) > 0$ for $t \in [0, T]$. On this ball we thus get a uniform bound $\psi_\varepsilon(u) \ge c(T)^{-\delta}$, which gives our desired t- and ε-independent bound on $\|\psi_\varepsilon(u)\|_{L^2(B)}^2$, and thereby we have that $u^{-\delta} \in H^1(M)$ uniformly. □

Corollary 3.3. *Under the conditions of Theorem 3.2, one can find uniform constants $0 < A(T), B(T) < \infty$, depending only on $T > 0$ and initial scalar curvature S_0 (but not dependent on t), such that for all $f \in H^1(M, g)$,*

$$\|f\|_{L^{2n/(n-2)}(M,g)}^2 \le A(T)\|\nabla f\|_{L^2(M,g)}^2 + B(T)\|f\|_{L^2(M,g)}^2, \tag{3-9}$$

i.e., (1-8) *holds for the time-dependent metric but with time-independent constants.*

Proof. Due to (1-8) we have, for all $f \in H^1(M) = H^1(M, g)$,

$$\|f\|_{L^{2n/(n-2)}(M,g_0)}^2 \le A_0\|\nabla f\|_{L^2(M,g_0)}^2 + B_0\|f\|_{L^2(M,g_0)}^2.$$

Using $g = u^{4/(n-2)} g_0$, we conclude a similar estimate with respect to g:

$$\|f\|_{L^{2n/(n-2)}(M,g)}^2 \le A(T)\|\nabla f\|_{L^2(M,g)}^2 + B(T)\|f\|_{L^2(M,g)}^2, \tag{3-10}$$

where

$$A(T) := A_0 \frac{(\sup_{M_T} u)^2}{(\inf_{M_T} u)^2}, \quad B(T) := B_0 \frac{(\sup_{M_T} u)^2}{(\inf_{M_T} u)^{2n/(n-2)}}.$$

Now the statement follows, since $u, u^{-1} \in L^\infty(M \times [0, T])$ by Proposition 3.1 and Theorem 3.2. □

[9] Note that the volume of (M, g_0) is normalized to 1 and thus (3-8) corresponds to (1-7).

We shall need this Sobolev inequality (3-9) when we tackle the upper bound on the scalar curvature S in Section 4.

4. Upper bound on the scalar curvature along the flow

The arguments of this section employ the assumptions

$$\begin{aligned} &\bullet\ (M, g_0) \text{ is an admissible manifold,} \\ &\bullet\ H^1(M) \text{ and } H^1(M, g) \text{ have equivalent norms,} \\ &\bullet\ C_c^\infty(M) \text{ is dense in } H^1(M), \\ &\bullet\ \text{the Sobolev inequality (3-9) holds,} \\ &\bullet\ S \in H^1(M, g) \text{ and } Y(M, g_0) > 0, \end{aligned} \tag{4-1}$$

provided the flow exists. These properties follow from Assumptions 1, 2, 3 and 4, as in the previous section. The Sobolev inequality (3-9) holds under the same assumptions in view of Corollary 3.3. In this section we use (4-1) to show a uniform upper bound on the scalar curvature. More precisely, we will show the following result.

Theorem 4.1. *Let S evolve according to* (2-2) *with initial curvature $S_0 \in L^{n^2/(2(n-2))}(M)$ and its negative part $(S_0)_- \in L^\infty(M)$. Then, assuming* (4-1) *holds, there exists a uniform constant $0 < C(T) < \infty$, depending only on $T > 0$ and S_0, such that*

$$\|S\|_{L^\infty(M\times[T/2,T])} \leq C(T).$$

The proof proceeds in two steps. The first step is to prove an $L^{n^2/(2(n-2))}(M, g)$-norm bound on S, uniform in $t \in [0, T]$. That uniform bound rests on a chain of arguments of [Brendle 2005, Lemmas 2.2, 2.3, 2.5] (also to be found in [Schwetlick and Struwe 2003, Lemma 3.3]) that apply in our setting as well. In the second step we perform a Moser iteration argument by following [Ma et al. 2012]. Our proofs are close to those in [Brendle 2005] with some additional arguments due to lower regularity.

Lemma 4.2. *Under the conditions of Theorem 4.1, there exists for any finite $T > 0$ a uniform constant $0 < C(T) < \infty$, depending only on T and S_0, such that for all $t \in [0, T]$ we have the estimate*[10]

$$\int_0^T \left(\int_M S^{n^2/(2(n-2))} \, d\mathrm{Vol}_g \right)^{(n-2)/n} dt \leq C(T), \quad \|S\|_{L^{n/2}(M,g)} \leq C, \tag{4-2}$$

where the second constant C only depends on S_0, not on T.

Proof. It suffices to prove the statement for S_+ and S_- individually. By Proposition 2.3, the statement holds for the negative part S_-. Thus we only need to prove the claim for S_+. We may therefore assume without loss of generality that $S \geq 0$, so that $S \equiv S_+$, and use (2-3) as the evolution equation.

The claim will follow from the evolution equation (2-2), but we have to argue a bit differently depending on whether $3 \leq n \leq 4$ or $n > 4$. The idea is the same in all dimensions $n \geq 3$ however. Let

[10] Below, we will denote all uniform positive constants, depending only on T and S_0, either by $C(T)$ or C_T, unless stated otherwise.

us start with $3 \le n \le 4$. Fix any $\sigma > 0$, and set $\beta = \frac{1}{4}n$. Since $\beta \le 1$, the function $x \mapsto (x+\sigma)^\beta$ is in $C^1[0,\infty)$ with bounded derivative. Thus, we may apply the chain rule to $(S+\sigma)^\beta$ and conclude that $(S+\sigma)^\beta \in C^1([0,T]; H^1(M,g))$. We use $\beta^2/(2\beta-1)(S+\sigma)^{2\beta-1}$ as a test function with $\beta = \frac{1}{4}n$ in the weak formulation of (2-3), which yields the inequality

$$\frac{\beta^2}{2\beta-1}\int_M (S+\sigma)^{2\beta-1}\partial_t(S+\sigma)\,d\mathrm{Vol}_g + (n-1)\int_M |\nabla(S+\sigma)^\beta|^2\,d\mathrm{Vol}_g \le \frac{\beta^2}{2\beta-1}\int_M S(S-\rho)(S+\sigma)^{2\beta-1}\,d\mathrm{Vol}_g.$$

Using (1-3) yields

$$\begin{aligned}\frac{\beta}{2(2\beta-1)}\partial_t\int_M (S+\sigma)^{2\beta}\,d\mathrm{Vol}_g &+ (n-1)\int_M |\nabla(S+\sigma)^\beta|^2\,d\mathrm{Vol}_g \\ &\le \frac{\beta}{2\beta-1}\int_M \beta S(S-\rho)(S+\sigma)^{2\beta-1} - \frac{n}{4}(S-\rho)(S+\sigma)^{2\beta}\,d\mathrm{Vol}_g \\ &= -\frac{\beta^2\sigma}{2\beta-1}\int_M (S-\rho)(S+\sigma)^{2\beta-1}\,d\mathrm{Vol}_g \\ &= -\frac{\beta^2\sigma}{2\beta-1}\int_M (S+\sigma-\rho)(S+\sigma)^{2\beta-1}\,d\mathrm{Vol}_g + \frac{\sigma^2\beta^2}{2\beta-1}\int_M (S+\sigma)^{2\beta-1}\,d\mathrm{Vol}_g \\ &\le \frac{\sigma\beta^2(\sigma+\rho(0))}{2\beta-1}\int_M (S+\sigma)^{2\beta-1}\,d\mathrm{Vol}_g \le \frac{\sigma\beta^2(\sigma+\rho(0))}{2\beta-1}\int_M (S+\sigma)^{2\beta}\,d\mathrm{Vol}_g,\end{aligned}$$

where the first equality is due to $\beta = \frac{1}{4}n$, the penultimate inequality uses $\rho(0) \ge \rho(t)$ and the final inequality is due to Hölder with $p = \beta/(\beta-1)$ and $q = \beta$. We want to integrate this inequality in time. Note that any inequality of the form $\partial_t w(t) + a(t) \le bw(t)$ with $a(t) \ge 0$ yields $\partial_t w \le bw$ and hence $w(t) \le e^{bt}w(0)$. Plugging this estimate into the original differential inequality leads to $\partial_t w + a \le be^{bt}w(0)$. Integrating the latter inequality in time yields $w(t) + \int_0^t a(s)\,ds \le e^{bt}w(0)$. We therefore conclude that

$$\int_M (S+\sigma)^{n/2}\,d\mathrm{Vol}_g(T) + \frac{4(n-2)(n-1)}{n}\int_0^T\int_M |\nabla(S+\sigma)^{n/4}|^2\,d\mathrm{Vol}_g \le e^{\sigma(\sigma+\rho(0))nT/2}\int_M (S_0+\sigma)^{n/2}\,d\mu. \tag{4-3}$$

This is for any $\sigma > 0$. Sending $\sigma \to 0$ and using Fatou's lemma on the left-hand side and the monotone convergence theorem on the right-hand side yields (on dropping the nonnegative term with ∇S)

$$\int_M S^{n/2}\,d\mathrm{Vol}_g(T) \le \int_M S_0^{n/2}\,d\mu.$$

This yields our uniform $L^{n/2}(M,g)$ bound on S in (4-2). Returning to (4-3), we appeal to the Sobolev inequality (3-9) to deduce

$$\int_0^T \|(S+\sigma)^{n/4}\|^2_{L^{2n/(n-2)}(M,g)}\,dt \le \left(\frac{A(T)n}{4(n-2)(n-1)} + TB(T)\right)e^{\sigma(\sigma+\rho(0))nT/2}\int_M (S_0+\sigma)^{n/2}\,d\mu,$$

hence also

$$\int_0^T \left(\int_M |S|^{n^2/(2(n-2))} \, d\mathrm{Vol}_g \right)^{(n-2)/n} dt \le C(T).$$

This proves the claim for $3 \le n \le 4$.

For $n > 4$ the claim will follow similarly, but the above test function does not have bounded derivative for $n > 4$, and we neither know that it is in H^1 nor do we know that the chain rule applies. We therefore argue similarly to the proof of Proposition 2.3, where we introduced the functions $\phi_{\beta,L}$, $G_{\beta,L}$ and $H_{\beta,L}$. We again set $\beta = \frac{1}{4}n$. Using $G_{\beta,L}(S)$ as a test function in (2-2), we find

$$\int_M G_{\beta,L}(S)(\partial_t S) \, d\mathrm{Vol}_g + (n-1) \int_M |\nabla \phi_{\beta,L}(S)|^2 \, d\mathrm{Vol}_g \le \int_M S(S-\rho) G_{\beta,L}(S) \, d\mathrm{Vol}_g.$$

Using the evolution equation (1-3) for the volume form, we have

$$\partial_t \int_M H_{\beta,L}(S) \, d\mathrm{Vol}_g + (n-1) \int_M |\nabla \phi_{\beta,L}(S)|^2 \, d\mathrm{Vol}_g \le \int_M (S-\rho) \left(S G_{\beta,L}(S) - \frac{n}{2} H_{\beta,L}(S) \right) d\mathrm{Vol}_g. \tag{4-4}$$

One readily checks from the definitions of $G_{\beta,L}$ and $H_{\beta,L}$ in Proposition 2.3 that

$$x\, G_{\beta,L}(x) - \frac{n}{2}\, H_{\beta,L}(x) = \begin{cases} \dfrac{\beta}{2\beta-1} x^{2\beta} \left(\beta - \dfrac{n}{4} \right), & x \le L, \\ \beta^2 L^{2\beta} \left(\left(1 - \dfrac{n}{4}\right) \left(\dfrac{x}{L} \right)^2 + \dfrac{2(\beta-1)}{2\beta-1} \left(\dfrac{n}{2} - 1 \right) \dfrac{x}{L} - \dfrac{n(\beta-1)}{4\beta} \right), & x > L, \end{cases} \tag{4-5}$$

and from this one sees that $x G_{\beta,L}(x) - \frac{1}{2} n H_{\beta,L}(x) \le 0$ for $\beta = \frac{1}{4}n$ and $n \ge 4$ as follows: For $x \le L$ there is nothing to show. For $x > L$, notice that

$$\beta^2 L^{2\beta} \left(\left(1 - \frac{n}{4}\right) \left(\frac{x}{L} \right)^2 + \frac{2(\beta-1)}{2\beta-1} \left(\frac{n}{2} - 1 \right) \frac{x}{L} - \frac{n(\beta-1)}{4\beta} \right) = -\beta^2 (\beta-1) L^{2\beta} \left(\frac{x}{L} - 1 \right)^2 \le 0,$$

where we have substituted $n = 4\beta$ and recognized a square.[11] Hence, using again that ρ is nonincreasing along the flow, we conclude that the inequality

$$\partial_t \int_M H_{\beta,L}(S) \, d\mathrm{Vol}_g + (n-1) \int_M |\nabla \phi_{\beta,L}(S)|^2 \, d\mathrm{Vol}_g \le 0$$

holds for any $L \ge \rho(0)$. This is a differential inequality of the same kind as in the above $3 \le n \le 4$ case. Integrating it we deduce, for any $t \in [0, T]$,

$$\int_M H_{\beta,L}(S) \, d\mathrm{Vol}_g(T) + (n-1) \int_0^T \int_M |\nabla \phi_{\beta,L}(S)|^2 \, d\mathrm{Vol}_g \, dt \le \int_M H_{\beta,L}(S_0) \, d\mu. \tag{4-6}$$

Using $\beta = \frac{1}{4}n$ and letting $L \to \infty$, this yields, using Fatou's lemma and dominated convergence exactly as in the final step of the proof of Proposition 2.3 (neglecting the positive second summand on the left-hand

[11] This is the point where we need $n \ne 3$, since in this case $\beta - 1 < 0$ and the above expression fails to be negative for $x > L$.

side of (4-6)), the inequality

$$\|S\|_{L^{n/2}(M,g)} = \left(\int_M S^{n/2}\, d\mathrm{Vol}_g\right)^{2/n} \leq \left(\int_M |S_0|^{n/2}\, d\mu\right)^{2/n} \equiv C, \tag{4-7}$$

where the constant $C(T) > 0$ depends only on T and S_0. This yields the second estimate in (4-2) for $n > 4$. For the first estimate in (4-2), note that $\phi_{\beta,L}(S) \in H^1(M, g)$[12] and thus by (3-9) and (4-6) we deduce

$$\begin{aligned}
\int_0^T \left(\int_M |\phi_{\beta,L}(S)|^{2n/(n-2)}\, d\mathrm{Vol}_g\right)^{(n-2)/n} dt \\
&\leq A(T) \int_0^T \int_M |\nabla \phi_{\beta,L}(S)|^2\, d\mathrm{Vol}_g\, dt + B(T) \int_0^T \int_M |\phi_{\beta,L}(S)|^2\, d\mathrm{Vol}_g\, dt \\
&\leq \frac{A(T)}{n-1}\left(\int_M H_{\beta,L}(S_0)\, d\mu - \int_M H_{\beta,L}(S)\, d\mathrm{Vol}_g(T)\right) + B(T) \int_0^T \int_M |\phi_{\beta,L}(S)|^2\, d\mathrm{Vol}_g\, dt.
\end{aligned}$$

Thus, letting $L \to \infty$ we conclude, using Fatou's lemma and dominated convergence as before, with $\beta = \frac{1}{4}n$, and (4-7), that

$$\int_0^T \left(\int_M S^{n^2/(2(n-2))}\, d\mathrm{Vol}_g\right)^{(n-2)/n} dt \leq \left(\frac{nA(T)}{4(n-1)(n-2)} + B(T)T\right) \int_M |S_0|^{n/2}\, d\mu \equiv C(T), \tag{4-8}$$

where the uniform constant $C(T) > 0$ depends only on T and S_0. This proves the first estimate in (4-2) for $n > 4$. □

Lemma 4.3. *Under the conditions of Theorem 4.1, there exists for any finite $T > 0$ a uniform constant $0 < C(T) < \infty$, depending only on T and S_0, such that for all $t \in [0, T]$ we have the estimate*

$$\int_M |S|^{n^2/(2(n-2))}\, d\mathrm{Vol}_g \leq C(T).$$

Proof. As in the previous lemma we have to split the argument into cases based on the dimension. We first show the statement for $n \geq 4$. We will again use the inequality (4-4). However, while in Lemma 4.2 we set $\beta = \frac{1}{4}n$, here we will use the inequality (4-4) with $\beta = n^2/(4(n-2))$. For this choice of β the expression $xG_{\beta,L}(x) - \frac{1}{2}nH_{\beta,L}(x)$ is no longer necessarily nonpositive, and we estimate it against a new approximation function

$$f_{\beta,L}(x) := \begin{cases} \beta x^{2\beta}, & x \leq L, \\ n\beta^2 L^{2\beta-1} x, & x > L. \end{cases} \tag{4-9}$$

By (4-5) one sees that the inequality $xG_{\beta,L}(x) - \frac{1}{2}nH_{\beta,L}(x) \leq f_{\beta,L}(x)$ holds for all $\beta \geq 1$ and $L > 0$ in the case $n \geq 4$. One important aspect to notice is that $f_{\beta,L}(x)$ is linear in x for $x > L$, as opposed to quadratic in x for $H_{\beta,L}(x)$ and $xG_{\beta,L}(x)$. This will become important below. Returning to (4-4) and

[12] Note that a priori we do not know if $S^{n/4} \in H^1(M; g)$ and thus cannot directly apply the Sobolev inequality (3-9) to $S^{n/4}$. However, we do know that $\phi_{\beta,L}(S) \in H^1(M, g)$, since $\phi_{\beta,L}(x)$ is linear for $x > L$ and $S \in H^1(M, g)$ for each fixed time argument.

applying (3-9) to the term $\|\nabla\phi_{\beta,L}(S)\|^2_{L^2(M,g)}$, after some reshuffling we find

$$\partial_t\|H_{\beta,L}(S)\|_{L^1(M,g)} \le (n-1)\frac{B(T)}{A(T)}\|\phi_{\beta,L}(S)^2\|_{L^1(M,g)} - \frac{(n-1)}{A(T)}\|\phi_{\beta,L}(S)\|^2_{L^{2n/(n-2)}(M,g)} + \rho(0)\big\|SG_{\beta,L}(S) - \tfrac{1}{2}nH_{\beta,L}(S)\big\|_{L^1(M,g)} + \|Sf_{\beta,L}(S)\|_{L^1(M,g)}. \tag{4-10}$$

A straightforward computation shows, for all $\beta \ge 1$ and $L > 0$, that

$$12\beta H_{\beta,L}(x) \ge \phi_{\beta,L}(x)^2 \quad\text{and}\quad 4\beta H_{\beta,L}(x) \ge xG_{\beta,L}, \tag{4-11}$$

hold, and here is a way of seeing this: For $x \le L$ these are both obvious from the definitions, so we look at $x > L$. One first notices that

$$\phi_{\beta,L}(x)^2 = \beta^2L^{2(\beta-1)}x^2 - 2\beta(\beta-1)L^{2\beta-1}x + (\beta-1)^2L^{2\beta} \le \beta^2L^{2(\beta-1)}(x^2+2L^2) \le 3\beta^2L^{2(\beta-1)}x^2,$$

where the first inequality comes from dropping the nonpositive linear term and estimating $1 \le \beta$, and the final inequality is simply $L^2 < x^2$. We similarly estimate $H_{\beta,L}(x)$ from below for $x > L$ and find

$$\begin{aligned} H_{\beta,L}(x) &= \left(\frac{\beta^2}{2}\left(\frac{x}{L}\right)^2 - \frac{2\beta^2(\beta-1)}{2\beta-1}\left(\frac{x}{L}\right) + \frac{\beta(\beta-1)}{2}\right)L^{2\beta} \\ &\ge \left(\frac{\beta^2}{2(2\beta-1)}\left(\frac{x}{L}\right)^2 + \frac{\beta(\beta-1)}{2}\right)L^{2\beta} \ge \frac{\beta^2}{2(2\beta-1)}x^2L^{2(\beta-1)}, \end{aligned} \tag{4-12}$$

where the first inequality uses $-x/L \ge -x^2/L^2$ and the second inequality comes from dropping the nonnegative constant term. Using these two estimates one readily sees that

$$12\beta H_{\beta,L}(x) \ge \frac{2\beta}{2\beta-1}3\beta^2x^2L^{2(\beta-1)} \ge 3\beta^2x^2L^{2(\beta-1)} \ge \phi_{\beta,L}(x)^2,$$

showing half of the claim in (4-11). To see the other half, first observe that (for $x > L$) by dropping the nonpositive term in (2-8) we have $xG_{\beta,L}(x) \le \beta^2x^2L^{2(\beta-1)}$. Using (4-12) again we deduce

$$4\beta H_{\beta,L}(x) \ge \frac{2\beta}{2\beta-1}\beta^2x^2L^{2(\beta-1)} \ge \beta^2x^2L^{2(\beta-1)} \ge xG_{\beta,L}(x).$$

This finishes the proof of (4-11), so we arrive by overestimating the right-hand side of (4-10) at the inequality

$$\partial_t\|H_{\beta,L}(S)\|_{L^1(M,g)} \le C_T\|H_{\beta,L}(S)\|_{L^1(M,g)} + \|Sf_{\beta,L}(S)\|_{L^1(M,g)} - \frac{(n-1)}{A(T)}\|\phi_{\beta,L}(S)\|^2_{L^{2n/(n-2)}(M,g)}, \tag{4-13}$$

where the uniform constant $C_T > 0$ is explicitly given by

$$C_T := 12(n-1)\beta\,\frac{A(T)}{B(T)} + \rho(0)\left(\frac{n}{2} + 4\beta\right).$$

Introduce the nonnegative, real function $F_{\beta,L}$ via

$$F_{\beta,L}(x) := (xf_{\beta,L}(x))^{1/(2\beta+1)}.$$

Assume $\beta > \frac{1}{4}n$, which holds, for example, for $\beta = n^2/(4(n-2))$. Set $\alpha := n/(4\beta) < 1$ and choose any $\delta > 0$. Observe that by the Hölder inequality in the first estimate and the Young inequality in the second, we obtain

$$\begin{aligned} \|F_{\beta,L}(S)^{2\beta+1}\|_{L^1(M,g)} &\le \|F_{\beta,L}(S)\|^{2\alpha\beta}_{L^{2n\beta/(n-2)}(M,g)} \|F_{\beta,L}(S)\|^{1+2(1-\alpha)\beta}_{L^{2\beta}(M,g)} \\ &\le \delta\alpha \|F_{\beta,L}(S)^{\beta}\|^2_{L^{2n/(n-2)}(M,g)} + \delta^{-\alpha/(1-\alpha)}(1-\alpha)\|F_{\beta,L}(S)\|^{1/(1-\alpha)+2\beta}_{L^{2\beta}(M,g)}. \end{aligned} \tag{4-14}$$

These norms are finite for finite $L > 0$, as follows: The claim is clear for $S \le L$, and the delicate point is the behavior of the function for S large. For $S > L$, $F_{\beta,L}(S) \sim S^{2/(2\beta+1)}$, and (since $2\beta/(2\beta+1) \le 1$) the terms $\|F_{\beta,L}(S)^{\beta}\|_{L^{2n/(n-2)}(M,g)}$ and $\|F_{\beta,L}(S)\|_{L^{2\beta}(M,g)}$ can be controlled via $\|S\|_{L^{2n/(n-2)}(M,g)}$ and $\|S\|_{L^2(M,g)}$, respectively. These latter norms are bounded[13] because of S existing in $C^0([0,T]; H^1(M,g))$ and (3-9).

We can compare $\|F_{\beta,L}(S)^{\beta}\|^2_{L^{2n/(n-2)}(M,g)}$ and $\|\phi_{\beta,L}(S)\|^2_{L^{2n/(n-2)}(M,g)}$ since we have the following pointwise estimates. Directly from the definition we have

$$F_{\beta,L}(x)^{\beta} = \begin{cases} \beta^{\beta/(2\beta+1)}x^{\beta} \le \beta x^{\beta}, & x \le L, \\ (n\beta^2)^{\beta/(2\beta+1)}L^{\beta}\left(\dfrac{x}{L}\right)^{2\beta/(2\beta+1)} \le n\beta L^{\beta-1}x, & x > L. \end{cases}$$

Similarly, we may estimate $\phi_{\beta,L}$ from below as

$$\phi_{\beta,L}(x) = \begin{cases} x^{\beta} = x^{\beta}, & x \le L, \\ \beta L^{\beta-1}x - (\beta-1)L^{\beta} \ge xL^{\beta-1}, & x > L. \end{cases}$$

Combining these two estimates we find $n\beta\phi_{\beta,L}(x) \ge F_{\beta,L}(x)^{\beta}$. By sufficiently shrinking $\delta > 0$ (choosing $\delta \le 4(n-1)/(n^3\beta A(T))$ to be precise), we can thus ensure for all $L > 0$ that

$$\delta\alpha\|F_{\beta,L}(S)^{\beta}\|^2_{L^{2n/(n-2)}(M,g)} \le \frac{(n-1)}{A(T)}\|\phi_{\beta,L}(S)\|^2_{L^{2n/(n-2)}(M,g)},$$

and therefore deduce from (4-13) and (4-14)

$$\partial_t\|H_{\beta,L}(S)\|_{L^1(M,g)} \le C_T\|H_{\beta,L}(S)\|_{L^1(M,g)} + C'_T\|F_{\beta,L}(S)\|^{1/(1-\alpha)+2\beta}_{L^{2\beta}(M,g)}, \tag{4-15}$$

for uniform constants $C_T, C'_T > 0$, where C_T is given above,

$$C'_T := \delta^{-n/(4\beta-n)}\left(\frac{4\beta-n}{4\beta}\right) \quad \text{and} \quad \delta \le \frac{4(n-1)}{n^3\beta A(T)}.$$

The point is that both constants depend only on $T > 0$ and S_0.

We then compare $F_{\beta,L}(x)^{2\beta}$ to $H_{\beta,L}(x)$ as follows: From the definition of $F_{\beta,L}(x)$ again we find

$$F_{\beta,L}(x)^{2\beta} = \begin{cases} \beta^{2\beta/(2\beta+1)}x^{2\beta} \le \beta x^{2\beta}, & x \le L, \\ (n\beta^2)^{2\beta/(2\beta+1)}L^{2\beta}\left(\dfrac{x}{L}\right)^{4\beta/(2\beta+1)} \le n\beta^2L^{2(\beta-1)}x^2, & x > L. \end{cases}$$

[13] This is where it was necessary to estimate $xG_{\beta,L} - \frac{1}{2}nH_{\beta,L} \le f_{\beta,L}$. Otherwise, defining $F_{\beta,L}$ in terms of $xG_{\beta,L} - \frac{1}{2}nH_{\beta,L}$ would cause $F_{\beta,L}(S)$ to go as $S^{3/(2\beta+1)}$ for large L and we would not be able to guarantee that $\|F_{\beta,L}(S)^{\beta}\|^2_{L^{2n/(n-2)}(M,g)}$ is finite.

Consulting (4-12) we find

$$4n\beta H_{\beta,L}(x) \geq \frac{2\beta}{2\beta-1}\begin{cases} n\beta x^{2\beta}, & x \leq L, \\ n\beta^2 L^{2(\beta-1)} x^2, & x > L. \end{cases}$$

We therefore conclude $4n\beta H_{\beta,L}(x) \geq F_{\beta,L}(x)^{2\beta}$. Defining

$$C_T'' := \max\{(4n\beta)^{1+2/(4\beta-n)} C_T', C_T\},$$

we deduce from (4-15) that

$$\partial_t \|H_{\beta,L}(S)\|_{L^1(M,g)} \leq C_T''(1+\|H_{\beta,L}(S)\|_{L^1(M,g)}^{2/(4\beta-n)})\|H_{\beta,L}(S)\|_{L^1(M,g)}.$$

Setting $\beta = n^2/(4(n-2))$, we can rewrite this differential inequality as

$$\partial_t \log(\|H_{\beta,L}(S)\|_{L^1(M,g)}) \leq C_T''(1+\|H_{\beta,L}(S)\|_{L^1(M,g)}^{n-2/(n)}).$$

Integrating this differential inequality in time, we conclude

$$\log(\|H_{\beta,L}(S(T))\|_{L^1(M,g)}) \leq \log(\|H_{\beta,L}(S_0)\|_{L^1(M,g_0)}) + C_T''T + C_T''\int_0^T \|H_{\beta,L}(S)\|_{L^1(M,g)}^{(n-2)/n}\,dt.$$

Taking the limit $L \to \infty$ (using Fatou's lemma and dominated convergence as before in the final step of the proof of Proposition 2.3) and using Lemma 4.2, we deduce

$$\log\|S(T)^{n^2/(2(n-2))}\|_{L^1(M,g)} \leq \log\|S_0^{n^2/(2(n-2))}\|_{L^1(M,g)} + C_T''T + C_T''C(T),$$

which proves the statement for $n \geq 4$.

The above proof would almost work for $n = 3$. The problem is that $xG_{\beta,L} - \frac{1}{2}nH_{\beta,L} \leq f_{\beta,L}$ no longer holds true, and one would have a problem showing that the norms in (4-14) are finite. One solution is to redefine the approximation functions $\phi_{\beta,L}$, $G_{\beta,L}$ and $H_{\beta,L}$ to ensure $xG_{\beta,L}(x) - \frac{1}{2}nH_{\beta,L}(x)$ is dominated by a function $f_{\beta,L}$ which, for large x, behaves like at most x rather than x^2. This is a nontrivial task, because it is also important for the above argument that one can find constants (depending on n and β but not L) such that

$$C_1 H_{\beta,L}(x) \geq \phi_{\beta,L}(x)^2, \quad C_2 H_{\beta,L}(x) \geq xG_{\beta,L}(x), \quad C_3 F_{\beta,L}(x)^\beta \leq \phi_{\beta,L}(x), \quad C_4 H_{\beta,L}(x) \geq F_{\beta,L}(x)^{2\beta},$$

where $F_{\beta,L}(x) = (xf_{\beta,L}(x))^{1/(2\beta+1)}$. Consider the following family of approximation functions with $\nu \leq 1$ and $\nu \notin \{0, \frac{1}{2}\}$:

$$\tilde{\phi}_{\beta,L}(x) := \begin{cases} x^\beta, & x \leq L, \\ \dfrac{\beta}{\nu} L^{\beta-\nu} x^\nu + L^\beta\left(1 - \dfrac{\beta}{\nu}\right), & x > L, \end{cases} \tag{4-16}$$

$$\tilde{G}_{\beta,L}(x) := \int_0^x \tilde{\phi}'_{\beta,L}(y)^2\,dy = \begin{cases} \dfrac{\beta^2}{2\beta-1} x^{2\beta-1}, & x \geq L, \\ \dfrac{\beta^2 L^{2(\beta-\nu)}}{2\nu-1} x^{2\nu-1} - \dfrac{2\beta^2 L^{2\beta-1}(\beta-\nu)}{(2\nu-1)(2\beta-1)}, & x > L, \end{cases} \tag{4-17}$$

$$\tilde{H}_{\beta,L}(x) := \int_0^x \tilde{G}_{\beta,L}(y)\,dy = \begin{cases} \dfrac{\beta}{2(2\beta-1)}x^{2\beta}, & x \le L, \\ \dfrac{\beta^2 L^{2(\beta-\nu)}}{2\nu(2\nu-1)}x^{2\nu} - \dfrac{2\beta^2 L^{2\beta-1}(\beta-\nu)}{(2\nu-1)(2\beta-1)}x - C_{\beta,\nu}L^{2\beta}, & x > L, \end{cases}$$

where

$$C_{\beta,\nu} := \frac{\beta(\beta(2\beta-1)+4\nu\beta(\nu-\beta)+\nu(1-2\nu))}{2\nu(2\beta-1)(2\nu-1)}.$$

In the $n \ge 4$ case we considered these functions with $\nu = 1$. These functions have the same qualitative properties as before, namely that $\tilde{\phi}_{\beta,L} \xrightarrow{L\to\infty} x^\beta$ and $\tilde{\phi}_{\beta,L} \in C^1(\mathbb{R}_+)$ with $\tilde{\phi}'_{\beta,L} \in L^\infty(\mathbb{R}_+)$, and similarly for $\tilde{G}_{\beta,L}$ and $\tilde{H}_{\beta,L}$. We can therefore use $\tilde{G}_{\beta,L}(S)$ as a test function in (2-2) and deduce the analogue of (4-4), namely

$$\partial_t \int_M \tilde{H}_{\beta,L}(S)\,d\mathrm{Vol}_g + (n-1)\int_M |\nabla\tilde{\phi}_{\beta,L}(S)|^2\,d\mathrm{Vol}_g \le \int_M (S-\rho)\Big(S\tilde{G}_{\beta,L}(S) - \frac{n}{2}\tilde{H}_{\beta,L}(S)\Big)\,d\mathrm{Vol}_g. \quad (4\text{-}18)$$

Consider the expression $x\tilde{G}_{\beta,L}(x) - \frac{1}{2}n\tilde{H}_{\beta,L}(x)$ for $x > L$:

$$x\tilde{G}_{\beta,L}(x) - \frac{n}{2}\tilde{H}_{\beta,L}(x) = \frac{\beta^2 L^{2(\beta-\nu)}}{2\nu(2\nu-1)}\Big(2\nu - \frac{n}{2}\Big)x^{2\nu} + \frac{2\beta^2(\beta-\nu)L^{2\beta-1}}{(2\beta-1)(2\nu-1)}\Big(\frac{n}{2}-1\Big)x + \frac{n}{2}C_{\beta,\nu}L^{2\beta}.$$

From this one sees that when $0 < \nu \le \frac{1}{4}n$ and $\beta \ge \frac{1}{4}n$, the first two terms become negative. So assume from now on that $0 < \nu \le \frac{1}{4}n$ and later we will make a choice of $\beta \ge \frac{1}{4}n$. Introduce the function

$$\tilde{f}_{\beta,L}(x) := \begin{cases} \beta x^{2\beta}, & x \le L, \\ \frac{1}{2}n|C_{\beta,\nu}|L^{2\beta}, & x > L, \end{cases}$$

which has the property that $x\tilde{G} - \frac{1}{2}nH_{\beta,L} \le \tilde{f}_{\beta,L}(x)$ for all $x \ge 0$ and $L > 0$, as long as $\beta \ge \frac{1}{4}n \ge \nu$. Proceeding exactly as in the $n \ge 4$ case, we deduce

$$\begin{aligned} \partial_t \|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)} \le (n-1)\frac{B(T)}{A(T)}\|\tilde{\phi}_{\beta,L}(S)^2\|_{L^1(M,g)} - \frac{(n-1)}{A(T)}\|\tilde{\phi}_{\beta,L}(S)\|^2_{L^{2n/(n-2)}(M,g)} \\ + \rho(0)\big\|S\tilde{G}_{\beta,L}(S) - \tfrac{1}{2}n\tilde{H}_{\beta,L}(S)\big\|_{L^1(M,g)} + \|S\tilde{f}_{\beta,L}(S)\|_{L^1(M,g)}. \end{aligned} \quad (4\text{-}19)$$

We compare $\tilde{\phi}_{\beta,L}(x)^2$, $x\tilde{G}_{\beta,L}(x)$ and $\tilde{H}_{\beta,L}(x)$ as in (4-11) and conclude by similar arguments, for all $\beta \ge 1$, $L > 0$, and some L-independent constants C_1, C_2, that

$$\begin{aligned} C_1\tilde{H}_{\beta,L}(x) &\ge \tilde{\phi}_{\beta,L}(x)^2, \\ C_2\tilde{H}_{\beta,L}(x) &\ge x\tilde{G}_{\beta,L}(x). \end{aligned} \quad (4\text{-}20)$$

We now proceed as before, getting

$$\begin{aligned} &\partial_t \|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)} \\ &\qquad \le C_T\|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)} + \|S\tilde{f}_{\beta,L}(S)\|_{L^1(M,g)} - \frac{(n-1)}{A(T)}\|\tilde{\phi}_{\beta,L}(S)\|^2_{L^{2n/(n-2)}(M,g)}, \end{aligned} \quad (4\text{-}21)$$

where the uniform constant $C_T > 0$ is explicitly given by

$$C_T := C_1(n-1)\,\frac{A(T)}{B(T)} + \rho(0)\big(\tfrac{1}{2}n + C_2\big).$$

Introduce the nonnegative, real function $\tilde F_{\beta,L}$ via

$$\tilde F_{\beta,L}(x) := (x\tilde f_{\beta,L}(x))^{1/(2\beta+1)}.$$

Assume $\beta > \frac{1}{4}n$, which holds, for example, for $\beta = n^2/(4(n-2))$. Set $\alpha := n/(4\beta) < 1$ and choose any $\delta > 0$. Observe that by the Hölder inequality in the first estimate and the Young inequality in the second, we obtain

$$\begin{aligned}\|\tilde F_{\beta,L}(S)^{2\beta+1}\|_{L^1(M,g)} &\le \|\tilde F_{\beta,L}(S)\|^{2\alpha\beta}_{L^{2n\beta/(n-2)}(M,g)}\|\tilde F_{\beta,L}(S)\|^{1+2(1-\alpha)\beta}_{L^{2\beta}(M,g)} \\ &\le \delta\alpha\|\tilde F_{\beta,L}(S)^\beta\|^2_{L^{2n/(n-2)}(M,g)} + \delta^{-\alpha/(1-\alpha)}(1-\alpha)\|\tilde F_{\beta,L}(S)\|^{1/(1-\alpha)+2\beta}_{L^{2\beta}(M,g)}. \end{aligned} \tag{4-22}$$

These integrals are finite for the same reasons as in the $n \ge 4$ case.

We shall from now on set $\nu = \beta/(2\beta+1)$ and $\beta = n^2/(4(n-2))$, which translates into $\nu = \frac{9}{22}$ for $n = 3$. Notice that this choice satisfies $\nu \le \frac{1}{4}n$, so the manipulations up until now are allowed. The reason for choosing this ν is that then

$$\tilde F_{\beta,L}(x)^\beta = \begin{cases} \beta^\nu x^\beta, & x \le L, \\ \big(\frac{1}{2}n|C_{\beta,\nu}|\big)^\nu L^{\beta-\nu}x^\nu, & x > L. \end{cases}$$

This is easily comparable to $\tilde\phi_{\beta,L}(x)$. Since

$$\tilde\phi_{\beta,L}(x) \ge \frac{\beta}{\nu}L^{\beta-\nu}x^\nu$$

for $x > L$, we see that by defining

$$C_3^{-1} := \max\left\{\beta^\nu, \frac{\nu}{\beta}\left(\frac{n}{2}|C_{\beta,\nu}|\right)^\nu\right\}$$

we achieve[14] $C_3\tilde F_{\beta,L}(x)^\beta \le \tilde\phi_{\beta,L}(x)$. So if we choose

$$\delta \le \frac{(n-1)}{A(T)}\frac{4C_3^2\beta}{n},$$

then the inequality

$$\delta\alpha\|\tilde F_{\beta,L}(S)^\beta\|^2_{L^{2n/(n-2)}(M,g)} - \frac{(n-1)}{A(T)}\|\tilde\phi_{\beta,L}(S)\|^2_{L^{2n/(n-2)}(M,g)} \le 0$$

holds for all $L > 0$, and we deduce from (4-21) and (4-22) that

$$\partial_t\|\tilde H_{\beta,L}(S)\|_{L^1(M,g)} \le C_T\|\tilde H_{\beta,L}(S)\|_{L^1(M,g)} + C_T'\|\tilde F_{\beta,L}(S)\|^{1/(1-\alpha)+2\beta}_{L^{2\beta}(M,g)} \tag{4-23}$$

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[14] This is a somewhat delicate point. If one chooses ν small, it is easy to make $x\tilde G - \frac{1}{2}n\tilde H$ sublinear, but if ν is too small, $\tilde F$ will increase faster than $\tilde\phi$, ruining the comparison. On the other hand, if ν is bigger than $\frac{1}{4}n$ we see above that $x\tilde G - \frac{1}{2}n\tilde H$ becomes too large to guarantee the finiteness of the integrals in (4-22).

for uniform constants $C_T, C'_T > 0$, where C_T is given above,

$$C'_T := \delta^{-n/(4\beta-n)}\left(\frac{4\beta-n}{4\beta}\right) \quad \text{and} \quad \delta \leq \frac{(n-1)}{A(T)}\frac{4C_3^2\beta}{n}.$$

The point is that both constants depend only on $T > 0$ and S_0. The final comparison we need is that $C_4 H_{\beta,L}(x) \geq F_{\beta,L}(x)^{2\beta}$ holds for some $C_4 > 0$ independent of L, and here is a way to see that this is doable: For $x \leq L$ both functions are proportional, so there is nothing to show. Inserting $\nu = \beta/(2\beta+1)$ into the definition of $\tilde{H}_{\beta,L}(x)$ yields (for $x > L$)

$$\tilde{H}_{\beta,L}(x) = L^{2\beta}\left(\frac{4\beta^4}{2\beta-1}\left(\frac{x}{L}\right) - \frac{\beta(2\beta+1)^2}{2}\left(\frac{x}{L}\right)^{2\nu} + \beta^2\right),$$

which shows that $\tilde{H}_{\beta,L}(x)$ is dominated by a positive linear term for $x > L$, which will dominate the sublinear term $x^{2\nu}$ of $\tilde{F}_{\beta,L}(x)^{2\beta}$. Defining

$$C''_T := \max\{C_4^{1+2/(4\beta-n)} C'_T, C_T\},$$

we deduce from (4-23) that

$$\partial_t \|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)} \leq C''_T(1 + \|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)}^{2/(4\beta-n)})\|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)}.$$

The rest of the proof then follows exactly as in the $n \geq 4$ case, giving us our required bound for $n = 3$. ☐

This completes the first step on the way to Theorem 4.1, proving a uniform $L^{n^2/(2(n-2))}(M, g)$-norm bound on S. Before we can go on to prove Theorem 4.1 by a Moser iteration argument, we need the following parabolic Sobolev inequality.

Lemma 4.4. *Let $A(T)$ and $B(T)$ denote the constants of the* (*elliptic*) *Sobolev inequality* (3-9). *Then for any $f \in H^1(M, g)$ with uniform norm in $t \in [0, T]$, we have* (*writing $M_T := M \times [0, T]$*)[15]

$$\|f^2\|_{L^{(n+2)/n}(M_T,g)} \leq \frac{n}{n+2}(A(T)\|\nabla f\|^2_{L^2(M_T,g)} + B(T)\|f\|^2_{L^2(M_T,g)}) + \frac{2}{n+2}\sup_{t\in[0,T]}\|f(t)\|^2_{L^2(M,g)}. \quad (4\text{-}24)$$

Proof. The statement and the proof are close to [Ma et al. 2012, Equation 12]. We compute

$$\begin{aligned}\int_0^T\int_M f^{2(n+2)/n}\,d\mathrm{Vol}_g\,dt &= \int_0^T\int_M f^2 f^{4/n}\,d\mathrm{Vol}_g\,dt \\ &\leq \int_0^T (\|f\|^2_{L^{2n/(n-2)}(M,g)}\|f\|^{4/n}_{L^2(M,g)})\,dt \\ &\leq \int_0^T (A(T)\|\nabla f\|^2_{L^2(M,g)} + B(T)\|f\|^2_{L^2(M,g)})(\|f\|^{4/n}_{L^2(M,g)})\,dt \\ &\leq (A(T)\|\nabla f\|^2_{L^2(M_T,g)} + B(T)\|f\|^2_{L^2(M_T,g)})\sup_{t\in[0,T]}(\|f\|^{4/n}_{L^2(M,g)}),\end{aligned}$$

[15] We write $L^p(M_T, g) \equiv L^p(M_T, g \oplus dt^2)$ for any $p \geq 1$.

where in the first estimate we applied the Hölder inequality with $p=\frac{1}{2}n$ and $q=n/(n-2)$ and in the second estimate applied (3-9). Raising both sides of the inequality to the power of $n/(n+2)$ and using Young's inequality $AB \le A^p/p + B^q/q$ with $p=(n+2)/n$ and $q=\frac{1}{2}(n+2)$ we arrive at the estimate as claimed. □

Proof of Theorem 4.1. Since we assume that $(S_0)_- \in L^\infty(M)$, we have uniform bounds on S_- by Proposition 2.3. Thus it suffices to prove the statement for S_+. Therefore we may replace S by S_+, replacing the evolution equation (2-2) for S by the inequality (2-3) for S_+. Hence we continue under the assumption $S \equiv S_+ \ge 0$, subject to (2-3).

Let $\eta \in C^1([0,T],\mathbb{R}_+)$ be nondecreasing with $\eta(0)=0$ and $\|\eta\|_\infty \le 1$. We would like to use $\beta^2\eta^2 S^{2\beta-1}/(2\beta-1)$ (with $\beta>1$) as a test function in the weak formulation of (2-3). The problem is of course that the chain rule fails to hold in general, so we use the same workaround as in Proposition 2.3 and Lemma 4.3. Let $L>0$ and define $\phi_{\beta,L}$, $G_{\beta,L}$ and $H_{\beta,L}$ as before. Using $\eta(s)^2 G_{\beta,L}(S)$ as a test function in (2-3) we get

$$\int_M (\partial_s S)\eta^2 G_{\beta,L}(S)\,d\mathrm{Vol}_g + (n-1)\int_M \eta^2|\nabla\phi_{\beta,L}(S)|^2\,d\mathrm{Vol}_g \le \int_M SG_{\beta,L}(S)|S-\rho|\,d\mathrm{Vol}_g.$$

On the right-hand side we observe (by a direct computation) that

$$SG_{\beta,L}(S) \le \beta^2/(2\beta-1)\phi_{\beta,L}(S)^2.$$

We integrate this in time for any $t\in[0,T]$ and get

$$\begin{aligned}\int_0^t\!\!\int_M (\partial_s S)\eta^2 G_{\beta,L}(S)\,d\mathrm{Vol}_g\,ds &+ (n-1)\int_0^t\!\!\int_M \eta^2|\nabla\phi_{\beta,L}(S)|^2\,d\mathrm{Vol}_g\,ds \\ &\le \frac{\beta^2}{2\beta-1}\int_0^t\!\!\int_M \eta^2\phi_{\beta,L}(S)^2|S-\rho|\,d\mathrm{Vol}_g\,ds. \qquad (4\text{-}25)\end{aligned}$$

We rewrite the first term on the left-hand side of (4-25) using (1-3) as

$$\begin{aligned}\int_0^t\!\!\int_M \eta^2(\partial_s S)G_{\beta,L}(S)\,d\mathrm{Vol}_g\,ds &\equiv \int_0^t\!\!\int_M \eta^2\partial_s H_{\beta,L}(S)\,d\mathrm{Vol}_g\,ds \\ &= \int_M \eta^2 H_{\beta,L}(S)\,d\mathrm{Vol}_g(s=t) - 2\int_0^t\!\!\int_M \eta\dot\eta H_{\beta,L}(S)\,d\mathrm{Vol}_g\,ds \\ &\quad + \frac{n}{2}\int_0^t\!\!\int_M \eta^2 H_{\beta,L}(S)(S-\rho)\,d\mathrm{Vol}_g\,ds,\end{aligned}$$

where we write $\dot\eta \equiv \partial_s\eta$ and use $\eta(0)=0$. Plugging this into (4-25), we obtain

$$\begin{aligned}\int_M \eta^2 H_{\beta,L}(S)\,d\mathrm{Vol}_g(s=t) &+ (n-1)\int_0^t\!\!\int_M \eta^2|\nabla\phi_{\beta,L}(S)|^2\,d\mathrm{Vol}_g\,ds \\ &\le \int_0^t\!\!\int_M \eta^2\Big(\frac{\beta^2}{2\beta-1}\phi_{\beta,L}(S)^2 + \frac{n}{2}H_{\beta_L}(S)\Big)|S-\rho|\,d\mathrm{Vol}_g\,ds + 2\int_0^t\!\!\int_M \eta\dot\eta H_{\beta,L}(S)\,d\mathrm{Vol}_g\,ds.\end{aligned}$$

We now take the supremum over $t \in [0, T]$ and appeal to the parabolic Sobolev inequality (4-24) with $f = \eta\phi_{\beta,L}(S)$. The result is

$$\begin{aligned}
&\frac{(n-1)}{nA(T)}\Big((n+2)\|\eta^2\phi_{\beta,L}(S)^2\|_{L^{(n+2)/n}(M_T,g)} - 2\sup_{t\in[0,T]}\|\eta\phi_{\beta,L}(S)\|^2_{L^2(M,g)} - nB(T)\|\eta\phi_{\beta,L}(S)\|^2_{L^2(M_T,g)}\Big)\\
&\quad + \sup_{t\in[0,T]}\int_M \eta^2 H_{\beta,L}(S)\,d\mathrm{Vol}_g\\
&\quad \le \int_0^T\int_M \eta^2\Big(\frac{\beta^2}{2\beta-1}\phi_{\beta,L}(S)^2 + \frac{n}{2}H_{\beta_L}(S)\Big)|S-\rho|\,d\mathrm{Vol}_g\,dt + 2\int_0^T\int_M \eta\dot\eta H_{\beta,L}(S)\,d\mathrm{Vol}_g\,dt. \qquad (4\text{-}26)
\end{aligned}$$

By increasing $A(T) > 0$ if needed, we may assume (also noting that $H_{\beta,L}$ and $\phi^2_{\beta,L}$ are comparable by (4-11)) that

$$\sup_{t\in[0,T]}\int_M \eta^2 H_{\beta,L}(S)\,d\mathrm{Vol}_g - \frac{2(n-1)}{nA(T)}\sup_{t\in[0,T]}\|\eta\phi_{\beta,L}(S)\|^2_{L^2(M,g)} \ge 0$$

for all $\beta \ge 1$ and $L > 0$. We may therefore drop these terms from (4-26). Taking the limit $L \to \infty$ (using Fatou's lemma and the dominated convergence theorem) we get

$$\begin{aligned}
&(n-1)\frac{n+2}{nA(T)}\|\eta^2 S^{2\beta}\|_{L^{(n+2)/n}(M_T)} - \frac{B(T)(n-1)}{A(T)}\|\eta S^\beta\|^2_{L^2(M_T,g)}\\
&\qquad \le \Big(\frac{\beta^2}{2\beta-1} + \frac{\beta}{4n(2\beta-1)}\Big)\int_0^T\int_M \eta^2 S^{2\beta}|S-\rho|\,d\mathrm{Vol}_g\,dt + \frac{\beta}{2\beta-1}\int_0^T\int_M \eta\dot\eta S^{2\beta}\,d\mathrm{Vol}_g\,dt.
\end{aligned}$$

Introducing $C := nA(T)/((n+2)(n-1))$ we get, for any $\beta > 1$, the inequality

$$\begin{aligned}
&\|\eta^2 S^{2\beta}\|_{L^{(n+2)/n}(M_T)}\\
&\qquad \le \frac{nB(T)}{n+2}\|\eta S^\beta\|^2_{L^2(M_T)} + C\int_0^T\int_M \eta\dot\eta S^{2\beta}\,d\mathrm{Vol}_g\,dt + 2C\beta\int_0^T\int_M \eta^2 S^{2\beta}|S-\rho|\,d\mathrm{Vol}_g\,dt.
\end{aligned}$$

We apply the Hölder inequality with $p = n^2/(2(n-2))$ to the last integral on the right-hand side above. Using Lemma 4.3 to get a bound on the integral of $|S-\rho|^p$, we conclude

$$\|\eta^2 S^{2\beta}\|_{L^{(n+2)/n}(M_T)} \le \frac{nB(T)}{n+2}\|\eta S^\beta\|^2_{L^2(M_T)} + C\int_0^T\int_M \eta\dot\eta S^{2\beta}\,d\mathrm{Vol}_g\,dt + C(T)\beta\|\eta^2 S^{2\beta}\|_{L^N(M_T)}, \quad (4\text{-}27)$$

with $N := p/(p-1) = n^2/(n^2-2n+4) < (n+2)/n$. This is almost the expression we want to iterate, but the presence of $\dot\eta$ means we have to shrink our time interval in the iteration (as is standard for parabolic Moser iteration). The details (inspired by [Ma et al. 2012, pp. 889–890]) follow.

Consider the sequence $t_k := \big(\frac{1}{2} - \big(\frac{1}{2}\big)^k\big)T$ for integers $k \ge 1$. Let $M_k := M \times [t_k, T]$, $M_1 = M_T$ and $M_\infty = M \times \big[\frac{1}{2}T, T\big]$. Choose nondecreasing test functions $\eta_k \in C^1([0,T], \mathbb{R}_+)$ with $\|\eta_k\|_\infty \le 1$ such that

$$\eta_k(t) = \begin{cases} 0, & t \le t_{k-1},\\ 1, & t \ge t_k.\end{cases}$$

The choice of $\{\eta_k\}_k$ can be made subject to a bound on the derivative $0 \le \dot\eta_k \le 2^{k+1}/T$, which we henceforth assume. Using these functions in (4-27), we find

$$\begin{aligned}\|S^{2\beta}\|_{L^{(n+2)/n}(M_k)} &= \|\eta_k^2 S^{2\beta}\|_{L^{(n+2)/n}(M_k)} \le \|\eta_k^2 S^{2\beta}\|_{L^{(n+2)/n}(M_T)} \\ &\le \frac{nB(T)}{n+2}\|\eta_k S^\beta\|^2_{L^2(M_T)} + C\int_0^T\int_M \eta_k\dot\eta_k S^{2\beta}\, d\mathrm{Vol}_g\, dt + C(T)\beta\|\eta_k^2 S^{2\beta}\|_{L^N(M_T)} \\ &\le \tilde C(T)\beta 2^{k+1}\|S^{2\beta}\|_{L^N(M_{k-1})}, \end{aligned} \tag{4-28}$$

where the second inequality uses (4-27) and last step uses $\dot\eta \le 2^{k+1}/T$ together with the Hölder inequality to compare L^1- and L^N-norms. This is the equation we will be iterating. Introduce $\gamma := 2\beta N$ and $\rho := (n+2)/(nN) = (n^3+8)/n^3 > 1$. Then (4-28) reads

$$\|S\|_{L^{\rho\gamma}(M_k)} \le (\tilde C(T)\gamma 2^k)^{N/\gamma}\|S\|_{L^\gamma(M_{k-1})}.$$

Replacing γ by $\rho^m\gamma$ for $m \ge 0$ results in

$$\|S\|_{L^{\rho^{m+1}\gamma}(M_{k+m})} \le (\tilde C(T)\rho^m\gamma 2^{k+m})^{N/(\rho^m\gamma)}\|S\|_{L^{\rho^m\gamma}(M_{k+m-1})},$$

which can be iterated down to

$$\|S\|_{L^{\rho^{m+1}\gamma}(M_{k+m})} \le \prod_{i=0}^m (\tilde C(T)\rho^i\gamma 2^{k+i})^{N/(\rho^i\gamma)}\|S\|_{L^\gamma(M_{k-1})}.$$

The expression $\prod_{i=0}^m (\tilde C(T)\rho^i\gamma 2^{k+i})^{N/(\rho^i\gamma)}$ converges as $m\to\infty$, as one checks by computing the logarithm

$$\begin{aligned}\lim_{m\to\infty}\log\prod_{i=0}^m (\tilde C(T)\rho^i\gamma 2^{k+i})^{N/(\rho^i\gamma)} &= \frac{N}{\gamma}\sum_{i=0}^\infty\left(\log(2^k\tilde C(T)\gamma)\frac{1}{\rho^i} + \log(2\rho)\frac{i}{\rho^i}\right) \\ &= \frac{N}{\gamma}\left(\frac{\rho}{\rho-1}\log(\tilde C(T)\gamma 2^k) + \log(2\rho)\frac{\rho}{(\rho-1)^2}\right).\end{aligned}$$

We therefore deduce for some uniform constant $C_T > 0$

$$\|S\|_{L^\infty(M\times[T/2,T])} \le \lim_{m\to\infty}\|S\|_{L^{\rho^{m+1}\gamma}(M_{k+m})} \le C_T\|S\|_{L^\gamma(M_{k-1})} \le C_T\|S\|_{L^\gamma(M_T)}.$$

Choosing

$$\beta = \frac{n^2-2n+4}{4(n-2)} \iff \gamma = \frac{n^2}{2(n-2)},$$

we can estimate the right-hand side using Lemma 4.3 and deduce for some uniform constant $C(T) > 0$

$$\|S\|_{L^\infty(M\times[T/2,T])} \le C(T). \qquad \square$$

Remark 4.5. It is worth pointing out that we do not assume $S_0 \in L^\infty(M)$, only that $S_0 \in L^{n^2/(2(n-2))}(M)$. The above proof tells us that $S \in L^\infty(M)$ for positive times, even if the initial curvature is unbounded. This is analogous to the well-known behavior of the heat equation, where the solutions for positive times are often much more regular than the initial data.

5. Long-time existence of the normalized Yamabe flow

We can now establish our main Theorem 1.1, which explicitly reads as follows.

Theorem 5.1. *Let (M, g_0) be a Riemannian manifold of dimension $n = \dim M \geq 3$ such that the following four assumptions hold*:

(1) *The Yamabe constant $Y(M, g_0)$ is positive, i.e., Assumption 1 holds.*

(2) *(M, g_0) is admissible, i.e., Assumption 2 holds.*

(3) *Parabolic Schauder estimates (as defined in Definition 1.4) hold on (M, g_0), i.e., Assumption 3 holds.*

(4) *$S_0 \in C^{1,\alpha}(M)$, i.e., Assumption 4 holds. Moreover, we require that $S_0 \in L^{n^2/(2(n-2))}(M)$ and that its negative part $(S_0)_- \in L^\infty(M)$.*

Under these assumptions, a normalized Yamabe flow $u^{4/(n-2)} g_0$ exists with $u \in C^{3,\alpha}(M \times [0, \infty))$, with infinite existence time, and with scalar curvature $S(t) \in L^\infty(M)$ for all $t > 0$.

Proof. Short time existence of the flow with $u \in C^{3,\alpha}(M \times [0, T'])$ for some small $T' > 0$ is due to Theorem 1.6. Let $T > 0$ be the maximal existence time, so that $u \in C^{3,\alpha}(M \times [0, T))$ with locally uniform control of the Hölder norms in $[0, T)$, but with no uniform control of the norms up to $t = T$. If $T = \infty$, there is nothing to prove. Otherwise, we proceed as follows.

Proposition 2.3 yields a uniform (i.e., depending only on S_0 and the finite T) lower bound on the scalar curvature S. Proposition 3.1 and Theorem 3.2 yield uniform upper and lower bounds on the solution u, so that $u \in L^\infty(M_T)$. This in turn gives us bounds on the Sobolev constants $A(T)$ and $B(T)$ (Corollary 3.3), so we use Theorem 4.1 to argue that $S \in L^\infty(M_T)$. By the evolution equation

$$\partial_t u = -\frac{4}{n-2}(S - \rho)u,$$

we deduce $\partial_t u \in L^\infty(M_T)$. Then, arguing exactly as in [Bahuaud and Vertman 2019, Proposition 2.8], we may then restart the flow and extend the solution past T. For the purpose of self-containment, we provide the argument here.

Let us consider the linearized equation (1-18) with $u = 1 + v$,

$$\partial_t v - (n-1)\Delta_0 v = -\tfrac{1}{4}(n-2)S_0 + \Phi(v), \quad v(0) = 0, \tag{5-1}$$

where $\Phi(v) \in L^\infty(M_T)$, since $u, \partial_t u, \rho \in L^\infty(M_T)$. By the third mapping property in (1-14), we conclude that $v \in C^{1,\alpha}(M \times [0, T])$.[16] Rewrite flow equation (1-2) using $N = (n+2)/(n-2)$ as

$$\partial_t u - (n-1)u^{1-N}\Delta_0 u = \tfrac{1}{4}(n-2)(\rho\, u - S_0 u^{2-N}). \tag{5-2}$$

We will treat the right-hand side of this equation as a fixed element of $C^{0,\alpha}(M \times [0, T])$. Since $u^{1-N} \in C^{1,\alpha}(M \times [0, T])$ is positive and uniformly bounded away from zero, we may apply (1-16) and (1-17) to obtain a solution $u' \in C^{2,\alpha}(M \times [0, T])$ with initial condition $u'(0) = 1$.

[16] Note that we now have uniform control of the $C^{1,\alpha}$-norm up to $t = T$.

Note that $w := u - u'$ solves $\partial_t w - (n-1)u^{1-N}\Delta_0 w = 0$ with zero initial condition. By the weak maximum principle (1-13), $\partial_t w_{\max} \le 0$ and $\partial_t w_{\min} \ge 0$. Due to the initial condition $w(0) = 0$, we deduce $w \equiv 0$ and hence $u = u' \in C^{2,\alpha}(M \times [0,T])$. Thus $u' \in C^{2,\alpha}(M \times [0,T])$ extends $u(t)$ up to $t = T$, and we conclude

$$u \in C^{2,\alpha}(M \times [0,T]).$$

By the second statement of Theorem 1.6, we even have $u \in C^{3,\alpha}(M \times [0,T])$ and can now restart the flow as follows. Consider $u_0 = u(T) \in C^{3,\alpha}(M)$ as the initial condition for the normalized Yamabe flow. By (1-15), $e^{t\Delta_0}u_0 \in C^{3,\alpha}(M \times [0,T])$, where the heat operator acts without convolution in time.

We write $u = f + e^{t\Delta}u_0$ and plug this into the Yamabe flow equation (1-2) with rescaled time $\tau = (t - T)$. This yields an equation for f,

$$[\partial_t - (n-1)(e^{t\Delta_0}u_0)^{1-N}\Delta_0]f = Q_1(f) + Q_2(f, \partial_t f), \quad u'(0) = 0, \tag{5-3}$$

where Q_1 and Q_2 denote linear and quadratic combinations of the elements in brackets, respectively, with coefficients given by polynomials in $e^{t\Delta_0}u_0$, $\partial_t e^{t\Delta_0}u_0$ and $\Delta_0 e^{t\Delta_0}u_0$. Since these coefficients are of higher Hölder regularity $C^{1,\alpha}(M)$, we may set up a contraction mapping argument in $C^{3,\alpha}$ and thus extend u past the maximal existence time T exactly as in the proof of Theorem 1.6. This proves long-time existence. □

Corollary 5.2. *In the setting of the above theorem, we have*

$$\lim_{t\to\infty}\int_M (S-\rho)^2\, d\mathrm{Vol}_g = 0,$$

and there exists $u_\infty \in L^2(M)$ such that

$$\lim_{t\to\infty}\int_M (u - u_\infty)^2\, d\mu = 0.$$

Proof. By (1-5) we have

$$\partial_t \rho = -\frac{n-2}{2}\int_M (S-\rho)^2\, d\mathrm{Vol}_g.$$

This shows that $\rho(t)$ is monotonically decreasing, and we know it's bounded from below by $Y(M, g_0) > 0$, so $\lim_{t\to\infty}\rho(t)$ exists. Thus $\int_0^\infty \partial_t\rho(t)\,dt < \infty$, and thus $\partial_t\rho(t)$ must converge to zero as $t \to \infty$. This gives the conclusion on $\int_M (S-\rho)^2\, d\mathrm{Vol}_g$. By (1-1) we also conclude that

$$\int_M (\partial_t u^{2n/(n-2)})\, d\mu = -\frac{n}{2}\int_M (S-\rho)u^{2n/(n-2)}\, d\mu = 0,$$

and using u as a test function in (1-2) leads to

$$\frac{n+2}{2n}\int_M \partial_t u^{2n/(n-2)}\, d\mu + (n-1)\int_M |\nabla u|^2\, d\mu = \tfrac{1}{4}(n+2)\biggl(\rho(t) - \int_M u^2 S_0\, d\mu\biggr),$$

so

$$\int_M |\nabla u|^2\, d\mu \le \tfrac{1}{4}(n+2)(\rho(0) + \|(S_0)_-\|_{L^\infty(M)}),$$

where we have used $\int_M u^{2n/(n-2)}\, d\mu = 1$. This shows that u is uniformly bounded in $H^1(M)$ independent of t for all $t \geq 0$. Since the Sobolev embedding $H^1(M) \hookrightarrow L^q(M)$ is compact for $q < 2n/(n-2)$ (see [Akutagawa et al. 2014, Proposition 1.6]), we in particular get that u has a convergent subsequence in $L^2(M)$ as $t \to \infty$, and we call this limit u_∞. □

Remark 5.3. The above methods would also show that $\partial_t u^{(n+2)/(n-2)} \to 0$ in $L^1(M)$, since we may use (1-1) and the Hölder inequality to write

$$\begin{aligned}\|\partial_t u^{(n+2)/(n-2)}\|_{L^1(M)} &\leq \tfrac{1}{4}(n+2)\|(S-\rho)u^{n/(n-2)}\|_{L^2(M)}\|u^{2/(n-2)}\|_{L^2(M)} \\ &\leq \tfrac{1}{4}(n+2)\|(S-\rho)u^{n/(n-2)}\|_{L^2(M)}.\end{aligned}$$

We then use the first part of the corollary to show that the right-hand side tends to 0.

6. Future research directions and open problems

Long time existence alone does *not* guarantee regularity of the limit solution $u_\infty \in L^2(M)$. Indeed, this has to be obstructed for the following two reasons. In the case of closed manifolds, we know that the Yamabe problem is not uniquely solvable on a round sphere, but so far we have not assumed that (M, g_0) is not a sphere. In the singular setting, the Yamabe problem doesn't always have a solution, as demonstrated by Viaclovsky [2010]. We suspect that demanding

$$Y(M, g_0) < \lim_{R\to 0} Y(B_R(p), g_0),$$

for all $p \in \overline{M}$, is the required condition in our setting. Under this assumption, Akutagawa, Carron and Mazzeo are able to solve the Yamabe problem for smoothly stratified spaces in [Akutagawa et al. 2014]. For closed manifolds, this condition becomes $Y(M, g_0) < Y(\mathbb{S}^n, g_{\mathbb{S}^n})$ with the round metric $g_{\mathbb{S}^n}$, which is the assumption used by Brendle [2005] in his study of the Yamabe flow. Brendle's proof of convergence of the Yamabe flow relies on the positive mass theorem, which is not available in the singular setting.

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DISENTANGLEMENT, MULTILINEAR DUALITY AND FACTORISATION FOR NONPOSITIVE OPERATORS

ANTHONY CARBERY, TIMO S. HÄNNINEN AND STEFÁN INGI VALDIMARSSON

In a previous work we established a multilinear duality and factorisation theory for norm inequalities for pointwise weighted geometric means of positive linear operators defined on normed lattices. In this paper we extend the reach of the theory for the first time to the setting of *general* linear operators defined on normed spaces. The scope of this theory includes multilinear Fourier restriction-type inequalities. We also sharpen our previous theory of positive operators.

Our results all share a common theme: estimates on a weighted geometric mean of linear operators can be *disentangled* into quantitatively linked estimates on each operator separately. The concept of disentanglement recurs throughout the paper.

The methods we used in the previous work — principally convex optimisation — relied strongly on positivity. In contrast, in this paper we use a vector-valued reformulation of disentanglement, geometric properties (Rademacher-type) of the underlying normed spaces, and probabilistic considerations related to p-stable random variables.

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1. Introduction

In our previous work [Carbery et al. 2022] we introduced and developed a general functional-analytic principle concerning norm inequalities for pointwise weighted geometric means

$$\prod_{j=1}^{d} |T_j f_j(x)|^{\alpha_j}$$

of *positive* linear operators T_j defined on suitable spaces, where $\alpha_j \geq 0$ and $\sum_{j=1}^{d} \alpha_j = 1$. In this paper we extend our study to the situation in which the linear operators T_j are no longer assumed to be positive.

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The techniques of [Carbery et al. 2022] relied strongly on positivity, so it will be necessary to involve a new set of ideas.

In order to set the scene for this, it will be helpful to recall the main theorem of [Carbery et al. 2022], but we first we need to set up some notation. Let $(X, \mathrm{d}\mu)$ be a measure space and let $\mathcal{M}(X)$ be the class of measurable functions on X. Let $\mathcal{Y}$ be a real or complex normed space. (For example, if Y is a measure space, $\mathcal{Y}$ could be the class $\mathcal{S}(Y)$ of simple functions with an L^p-norm for some $p \geq 1$.) We say that a linear map $T : \mathcal{Y} \to \mathcal{M}(X)$ *saturates* X if, for each subset $E \subseteq X$ of positive measure, there exists a subset $E' \subseteq E$ with $\mu(E') > 0$ and an $h \in \mathcal{Y}$ such that $|Th| > 0$ a.e. on E'. For reasons explained in [Carbery et al. 2022], such a condition is needed for the result which follows to hold.

Theorem 1.1 [Carbery et al. 2022]. *Suppose that X is a σ-finite measure space and that $\mathcal{Y}_j$, for $j = 1, \dots, d$, are normed lattices. Suppose that the linear operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ are positive and that each T_j saturates X. Suppose that $0 < q \leq \infty$ and $\sum_{j=1}^d \alpha_j = 1$. Finally, suppose that*

$$\left\| \prod_{j=1}^{d} (T_j f_j)^{\alpha_j} \right\|_{L^q(X)} \leq A \prod_{j=1}^{d} \|f_j\|_{\mathcal{Y}_j}^{\alpha_j} \tag{1}$$

for all nonnegative $f_j \in \mathcal{Y}_j$, $1 \leq j \leq d$.

***Case I*:** (*disentanglement*) *If $q = 1$, then there exist nonnegative measurable functions g_j on X such that*

$$1 \leq \prod_{j=1}^{d} g_j(x)^{\alpha_j} \quad \text{a.e. on } X \tag{2}$$

and such that, for each j,

$$\int_X g_j(x) T_j f_j(x) \, \mathrm{d}\mu(x) \leq A \|f_j\|_{\mathcal{Y}_j} \tag{3}$$

for all $f_j \in \mathcal{Y}_j$, with the same constant A as in (1).

Conversely, if the T_j are positive linear operators such that there exist nonnegative measurable functions g_j on X such that (2) *holds and such that* (3) *holds for all $f_j \in \mathcal{Y}_j$, then* (1) *holds for all nonnegative $f_j \in \mathcal{Y}_j$.*

***Case II*:** (*multilinear duality*) *If $q > 1$, then for every nonnegative $G \in L^{q'}(X)$ there exist nonnegative measurable functions g_j on X such that*

$$G(x) \leq \prod_{j=1}^{d} g_j(x)^{\alpha_j} \quad \text{a.e. on } X \tag{4}$$

and such that, for each j,

$$\int_X g_j(x) T_j f_j(x) \, \mathrm{d}\mu(x) \leq A \|G\|_{L^{q'}} \|f_j\|_{\mathcal{Y}_j} \tag{5}$$

for all $f_j \in \mathcal{Y}_j$, with the same constant A as in (1).

Conversely, if the T_j are positive linear operators such that for every nonnegative $G \in L^{q'}(X)$ there exist nonnegative measurable functions g_j on X such that (4) *holds and such that* (5) *holds for all $f_j \in \mathcal{Y}_j$, then* (1) *holds for all nonnegative $f_j \in \mathcal{Y}_j$.*

Case III: (*multilinear Maurey factorisation*) *If* $0 < q < 1$, *then there exist nonnegative measurable functions* g_j *on* X *such that*[1]

$$\left\| \prod_{j=1}^{d} g_j(x)^{\alpha_j} \right\|_{q'} = 1 \tag{6}$$

and such that, *for each* j, (3) *holds for all* $f_j \in \mathcal{Y}_j$, *with the same constant* A *as in* (1).

Conversely, *if the* T_j *are positive linear operators such that there exist nonnegative measurable functions* g_j *on* X *such that* (6) *holds and such that* (3) *holds for all* $f_j \in \mathcal{Y}_j$, *then* (1) *holds for all nonnegative* $f_j \in \mathcal{Y}_j$.

Numerous illustrations and applications of this theorem were given in [Carbery et al. 2022]. It should be stressed that this result is a general one, applying to the class of positive operators broadly.

The forward parts of this result are the difficult ones; the converses follow easily by applying Hölder's inequality. When $d = 1$, Case II reduces to an elementary duality statement concerning the operator $T : \mathcal{Y} \to L^q$ and this gives rise to the sobriquet "multilinear duality" in the case of general d. The term "factorisation" relates both to the pointwise factorisation expressed by (4) and to the condition (5), which is a statement that each operator T_j factorises through a certain weighted L^1-space.

Case I, corresponding to $q = 1$, plays a special role, and indeed the remaining cases corresponding to $q \neq 1$ can be deduced from it without too much difficulty — see Section 5 for arguments of this type. We describe the case $q = 1$ as a "disentanglement" result since it disentangles a bound (1) on the pointwise combination of the T_j's into bounds (3) on each T_j separately, with the individual bounds linked via (2).

Notice that, when suitably modified, the statement of Theorem 1.1 makes perfectly good sense in principle *without* the hypothesis of positivity of the operators T_j; nevertheless, as we have mentioned, the arguments from [Carbery et al. 2022] rely very heavily on positivity. In this paper we use vector-valued techniques to develop an analogue of Theorem 1.1 which applies to general linear operators defined on normed spaces. See Theorems 1.5, 1.7, 4.3 and 5.2 below.

In what follows we shall primarily focus on the case of L^1 norms of pointwise weighted products $\prod_{j=1}^{d} |T_j f_j|^{\gamma_j}$ in our pursuit of extending Theorem 1.1 to general linear operators T_j. We return to the case of general L^q-norms of such expressions in Section 5, and there we see that it is relatively straightforward to derive the results for general q, which even in the positive case significantly generalise Theorem 1.1, from those corresponding to $q = 1$.

We next give a simple lemma. All of our main results can be framed as reversals of the implication it establishes (under various auxiliary hypotheses).

Lemma 1.2. *Let* $\mathcal{Y}_j$ *be normed spaces and let* $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ *be linear mappings for* $1 \le j \le d$. *Suppose* $\gamma_j > 0$ *are given. Assume that for some* (p_j) *with* $0 < p_j < \infty$ *we have the condition*

$$\sum_{j=1}^{d} \frac{\gamma_j}{p_j} = 1, \tag{7}$$

[1]We caution that we use the notation $\|g\|_q := \left(\int |g|^q\right)^{1/q}$ and $q' := q/(q-1)$ for $q < 0$ and for $0 < q < 1$, even though in these cases $\|\cdot\|_q$ does not define a norm.

and that there exist nonnegative measurable functions (ϕ_j) *on* X *such that*

$$\prod_{j=1}^{d} \phi_j(x)^{\gamma_j/p_j} \geq 1 \tag{8}$$

almost everywhere on X *and such that*

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x)\, \mathrm{d}\mu(x)\right)^{1/p_j} \leq A\|f_j\|_{\mathcal{Y}_j} \tag{9}$$

for all $f_j \in \mathcal{Y}_j$. *Then*

$$\int_X \prod_{j=1}^{d} |T_j f_j(x)|^{\gamma_j}\, \mathrm{d}\mu(x) \leq A^{\sum_{j=1}^{d}\gamma_j} \prod_{j=1}^{d} \|f_j\|_{\mathcal{Y}_j}^{\gamma_j} \tag{10}$$

for all $f_j \in \mathcal{Y}_j$.

Proof. Let $\theta_j = \gamma_j/p_j$. Then $\sum_{j=1}^{d}\theta_j = 1$, and, by (8), (9) and Hölder's inequality, we have

$$\begin{aligned}
\int_X \prod_{j=1}^{d} |T_j f_j(x)|^{\gamma_j}\, \mathrm{d}\mu(x) &\leq \int_X \prod_{j=1}^{d} |T_j f_j(x)|^{\gamma_j} \phi_j(x)^{\gamma_j/p_j}\, \mathrm{d}\mu(x) \\
&= \int_X \prod_{j=1}^{d} |T_j f_j(x)|^{p_j\theta_j} \phi_j(x)^{\theta_j}\, \mathrm{d}\mu(x) \\
&\leq \prod_{j=1}^{d} \left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x)\, \mathrm{d}\mu(x)\right)^{\theta_j} \\
&\leq A^{\sum_{j=1}^{d} p_j\theta_j} \prod_{j=1}^{d} \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j} = A^{\sum_{j=1}^{d}\gamma_j} \prod_{j=1}^{d} \|f_j\|_{\mathcal{Y}_j}^{\gamma_j}. \qquad \square
\end{aligned}$$

Taking $\gamma_j = q\alpha_j$ with q and $\sum_{j=1}^{d}\alpha_j = 1$ as in the preceding discussion makes a point of contact with Theorem 1.1.

Note that Lemma 1.2 has no content in the linear case $d = 1$. Our main concern will therefore be with the converse scenario in the genuinely multilinear case $d \geq 2$. The lemma delineates what we might hope for. More precisely:

Basic Question. *Let* $d \geq 2$. *Suppose* X *is a* σ*-finite measure space*, $\mathcal{Y}_j$ *are normed spaces*, $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ *are saturating linear mappings*, *and* $\gamma_j > 0$ *for* $1 \leq j \leq d$. *We suppose that* (10) *holds. For which* (p_j) (*if any*) *with* $0 < p_j < \infty$ *satisfying condition* (7) *can we conclude that there exist nonnegative* (ϕ_j) *such that conditions* (8) *and* (9) *hold*, *perhaps with a loss in the constants*?

Once again we emphasise that we ask this question in the broad context: we seek answers which do not rely upon the precise nature of the operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$, but instead which will hold universally over a wide class of linear operators. We expect that the set of admissible exponents (p_j), in addition to satisfying (7),[2] will reflect whatever geometric structures the normed spaces $\mathcal{Y}_j$ may possess.

---

[2]For a discussion of why we require this condition, see Proposition A.1 in the Appendix.

We shall give separate answers to this question in the settings of general linear operators and of positive linear operators. It transpires that in order to develop the theory for general linear operators, it first makes sense to consider a related question for positive linear operators: if in Theorem 1.1 we take the lattices $\mathcal{Y}_j$ to be L^{r_j}-spaces, are there stronger, r_j-dependent, conclusions that we can make?

The following result answers our Basic Question for positive linear operators on Lebesgue spaces, with no loss in constants. A corresponding answer in the case of general linear operators on Lebesgue spaces is given in Theorem 1.5.

Theorem 1.3. *Suppose that X and Y_j, for $j = 1, \ldots, d$, are measure spaces and that X is σ-finite. Suppose that the linear operators $T_j : \mathcal{S}(Y_j) \to \mathcal{M}(X)$ are positive and that each T_j saturates X. Suppose that $1 \le r_j \le \infty$ for all j. Finally, suppose that for some exponents $\gamma_j > 0$ we have*

$$\int_X \prod_{j=1}^{d} (T_j f_j)(x)^{\gamma_j} \,\mathrm{d}\mu(x) \le A^{\sum_{j=1}^{d} \gamma_j} \prod_{j=1}^{d} \| f_j \|_{L^{r_j}(Y_j)}^{\gamma_j} \tag{11}$$

for all nonnegative simple functions f_j on Y_j, $1 \le j \le d$.

Then for all (p_j) satisfying $0 < p_j < \infty$ for all j, $\sum_{j=1}^{d} \gamma_j/p_j = 1$ and $p_j \le r_j$ for all j, there exist nonnegative (ϕ_j) such that

$$\prod_{j=1}^{d} \phi_j(x)^{\gamma_j/p_j} \ge 1 \tag{12}$$

almost everywhere on X and such that

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x) \,\mathrm{d}\mu(x) \right)^{1/p_j} \le A \| f_j \|_{r_j} \tag{13}$$

for all $f_j \in \mathcal{S}(Y_j)$.

Remark 1. In the Appendix below we give an example of positive linear operators (T_j) satisfying (11) for which the set of (p_j) satisfying $0 < p_j < \infty$ and $\sum_{j=1}^{d} \gamma_j/p_j = 1$, and for which the conclusion of Theorem 1.3 holds, consists *precisely* of those satisfying $p_j \le r_j$ for every j. See Corollary A.7. Thus the condition $p_j \le r_j$ is sharp if we want our result to hold broadly for positive operators without further reference to their individual properties.[3]

Notice that the set

$$\left\{ (p_j) \in (0, \infty)^d : \sum_{j=1}^{d} \frac{\gamma_j}{p_j} = 1 \text{ and } p_j \le r_j \text{ for all } j \right\}$$

is nonempty if and only if $\sum_{j=1}^{d} \gamma_j/r_j \le 1$. In particular, Theorem 1.3 has no content unless $\sum_{j=1}^{d} \gamma_j/r_j \le 1$. In Corollary A.7 we demonstrate, by example, that if $\sum_{j=1}^{d} \gamma_j/r_j > 1$, then the set of (p_j) satisfying the conclusion of Theorem 1.3 may indeed be empty.

[3]For *particular* positive operators (T_j), the result may hold even when $p_j > r_j$ for some j. Indeed, let $X = Y_j = [0, 1]$ with Lebesgue measure, let $r_j = 1$ for all j and let each T_j be given by $T_j f = \int_0^1 f$, so that each $T_j f$ is constant on $[0, 1]$. Then (11) holds for all exponents $\gamma_j > 0$, with $A = 1$. If we take $\phi_j(x) = 1$ for all j, then both (12) and (13) hold for all exponents $0 < p_j < \infty$.

Under hypothesis (11), the disentangled conclusions (13) for $p_j \le \max\{r_j, \gamma_j\}$ alone, with otherwise unspecified but nontrivial (ϕ_j), are more straightforward, and can be established by methods which are not genuinely multilinear.[4] The significant feature of Theorem 1.3 is that under the hypotheses $\sum_{j=1}^d \gamma_j/p_j = 1$ and $p_j \le r_j$ for all j, we can choose (ϕ_j) also satisfying the specific quantitative lower bound (12). Similar remarks apply to our subsequent results.

We point out that the case $p_j = 1$ for all j of Theorem 1.3 directly implies Case I (and therefore Case II) of Theorem 1.1 (in the special case where the spaces $\mathcal{Y}_j$ are taken to be L^{r_j}). The case $p_j = r_j$ of Theorem 1.3 is, however, the crucial one, and in a slightly different notation can be presented as follows:

Theorem 1.4 (disentanglement for positive operators on Lebesgue spaces). *Suppose that X and Y_j, for $j = 1, \ldots, d$, are measure spaces and that X is σ-finite. Suppose that the linear operators $T_j : \mathcal{S}(Y_j) \to \mathcal{M}(X)$ are positive and that each T_j saturates X. Suppose that $1 \le p_j < \infty$ for all j, and that $\theta_j \ge 0$ are such that $\sum_{j=1}^d \theta_j = 1$. Finally, suppose that*

$$\int_X \prod_{j=1}^d (T_j f_j)(x)^{p_j\theta_j}\, \mathrm{d}\mu(x) \le B \prod_{j=1}^d \|f_j\|_{L^{p_j}(Y_j)}^{p_j\theta_j}$$

for all nonnegative simple functions f_j on Y_j, $1 \le j \le d$. Then there exist nonnegative measurable functions ϕ_j on X such that

$$\prod_{j=1}^d \phi_j(x)^{\theta_j} \ge 1$$

almost everywhere on X and such that, for each j,

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x)\, \mathrm{d}\mu(x)\right)^{1/p_j} \le B^{1/p_j} \|f_j\|_{L^{p_j}(Y_j)}$$

for all simple functions f_j on Y_j.

In analogy with the Case I of Theorem 1.1, we shall also call this result a disentanglement theorem, and it is an instance of the general disentanglement theorem for positive operators on p_j-convex spaces, which we shall present as Theorem 3.2.

As the reader will have noticed, by homogeneity we may take $B = 1$ (and $A = 1$ in earlier results) without loss. (And by playing with homogeneities the constant B^{1/p_j} can be replaced with $B^{\left(\sum_{j=1}^d p_j\theta_j\right)^{-1}}$).

In order to address our main concern in the paper — the extension of the theory to include general linear operators which are not necessarily positive — we shall consider the analogous situation under hypotheses of Rademacher-type in place of p-convexity. Our use of p-convexity and Rademacher-type proceeds in parallel with their deployment in the development of the Maurey theory; see [García-Cuerva and Rubio de Francia 1985; Albiac and Kalton 2006]. For now we state a sample theorem, which, in the case that the normed spaces $\mathcal{Y}_j$ are L^{r_j}-spaces, answers the Basic Question. We shall significantly generalise this result later; see Theorem 4.3.

[4]The range $p_j \le \max\{r_j, \gamma_j\}$ for this simpler problem is also known to be sharp, as the arguments in the Appendix confirm.

Theorem 1.5. *Suppose that X and Y_j, for $j = 1, \dots, d$, are measure spaces and that X is σ-finite. Suppose that $T_j : \mathcal{S}(Y_j) \to \mathcal{M}(X)$ are linear (not necessarily positive) operators and that each T_j saturates X. Suppose that*[5] *$1 \le r_j < \infty$ for all j. Finally, suppose that for some exponents $\gamma_j > 0$ we have*

$$\int_X \prod_{j=1}^{d} |T_j f_j(x)|^{\gamma_j} \, \mathrm{d}\mu(x) \le A^{\sum_{j=1}^{d} \gamma_j} \prod_{j=1}^{d} \|f_j\|_{L^{r_j}(Y_j)}^{\gamma_j} \tag{14}$$

for all simple functions f_j on Y_j, $1 \le j \le d$.

Then for all (p_j) such that $\sum_{j=1}^{d} \gamma_j / p_j = 1$ and

$$\begin{aligned} &0 < p_j < r_j \quad \text{for those } j \text{ for which } 1 \le r_j < 2, \\ &0 < p_j \le 2 \quad \text{for those } j \text{ for which } 2 \le r_j < \infty, \end{aligned} \tag{15}$$

there exist nonnegative ϕ_j such that

$$\prod_{j=1}^{d} \phi_j(x)^{\gamma_j / p_j} \ge 1$$

almost everywhere on X and such that

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x) \, \mathrm{d}\mu(x) \right)^{1/p_j} \lesssim_{\{\gamma_j, r_j, p_j\}} A \|f_j\|_{L^{r_j}(Y_j)}$$

for all $f_j \in \mathcal{S}(Y_j)$.

Remark 2. In the Appendix below we give an example of linear operators (T_j) satisfying (14) for which the set of (p_j) satisfying $0 < p_j < \infty$ and $\sum_{j=1}^{d} \gamma_j / p_j = 1$, and for which the conclusion of Theorem 1.5 holds, consists *precisely* of those satisfying (15). See Corollary A.8. Thus the condition (15) is sharp if we want our result to hold broadly for linear operators without further reference to their individual properties. For specific operators T_j the conclusion may nevertheless hold even if (15) is violated.

Note that the set of (p_j) satisfying $\sum_{j=1}^{d} \gamma_j / p_j = 1$ together with (15) will be nonempty if and only if

$$\begin{aligned} \textstyle\sum_{j=1}^{d} \gamma_j / \min\{r_j, 2\} < 1 \quad &\text{when at least one } r_j < 2, \\ \textstyle\sum_{j=1}^{d} \gamma_j \le 2 \quad &\text{when all } 2 \le r_j < \infty. \end{aligned}$$

In Corollary A.8 we demonstrate, by example, that if this condition is violated, the set of (p_j) satisfying the conclusion of Theorem 1.5 may indeed be empty.

The special case of this result corresponding to $p_j = 2$ for all j is singled out:

Theorem 1.6 (disentanglement for general linear operators on Lebesgue spaces). *Suppose that X and Y_j, for $j = 1, \dots, d$, are measure spaces and that X is σ-finite. Suppose that the linear operators $T_j : \mathcal{S}(Y_j) \to \mathcal{M}(X)$ saturate X. Suppose that $\theta_j > 0$ and $\sum_{j=1}^{d} \theta_j = 1$. Finally, suppose that for some*

[5] The proof will reveal that the result remains valid under the weaker assumption $0 < r_j < \infty$, provided that we accordingly modify (15) to $0 < p_j < r_j$ for those j for which $0 < r_j < 2$.

exponents $2 \le r_j < \infty$ *we have*

$$\int_X \prod_{j=1}^d |T_j f_j(x)|^{2\theta_j} \, \mathrm{d}\mu(x) \le B \prod_{j=1}^d \|f_j\|_{L^{r_j}(Y_j)}^{2\theta_j}$$

for all simple functions f_j *on* Y_j, $1 \le j \le d$. *Then there exist nonnegative measurable functions* ϕ_j *on* X *such that*

$$\prod_{j=1}^d \phi_j(x)^{\theta_j} \ge 1$$

almost everywhere on X *and such that, for each* j,

$$\left(\int_X |T_j f_j(x)|^2 \phi_j(x) \, \mathrm{d}\mu(x) \right)^{1/2} \lesssim B^{1/2} \|f_j\|_{L^{r_j}(Y_j)}$$

for all simple functions f_j *on* Y_j.

Theorem 1.6 readily upgrades to the following result (see Section 5), whose formulation can be compared to Case II of Theorem 1.1:

Theorem 1.7 (multilinear duality for general operators on Lebesgue spaces). *Suppose that* X *and* Y_j, *for* $j = 1, \dots, d$, *are measure spaces and that* X *is* σ*-finite. Suppose that the linear operators* $T_j : \mathcal{S}(Y_j) \to \mathcal{M}(X)$ *saturate* X. *Suppose that* $\alpha_j > 0$ *and* $\sum_{j=1}^d \alpha_j = 1$. *Finally, suppose that, for some exponents* $q \ge 2$ *and* $2 \le r_j < \infty$, *we have*

$$\left\| \prod_{j=1}^d |T_j f_j|^{\alpha_j} \right\|_q \le B \prod_{j=1}^d \|f_j\|_{L^{r_j}(Y_j)}^{\alpha_j}$$

for all simple functions f_j *on* Y_j, $1 \le j \le d$. *Then for every nonnegative* $G \in L^{(q/2)'}$ *there exist nonnegative measurable functions* g_j *on* X *such that*

$$\prod_{j=1}^d g_j(x)^{\alpha_j} \ge G(x)$$

almost everywhere on X *and such that, for each* j,

$$\left(\int_X |T_j f_j(x)|^2 g_j(x) \, \mathrm{d}\mu(x) \right)^{1/2} \lesssim B \|G\|_{(q/2)'} \|f_j\|_{L^{r_j}(Y_j)}$$

for all simple functions f_j *on* Y_j.

The converse statements to these three results are once again also true, and are easy to verify.

Note that in these last three results we do not assert "$\le$" but only "$\lesssim$" in the conclusions, and moreover the case $r_j = \infty$ is excluded from Theorems 1.5 and 1.7. This is ultimately because we shall need to apply Khintchine's inequality. Note also the numerology familiar from harmonic analysis, in which L^p-boundedness of a positive operator for $p > 1$ (such as a maximal operator) often corresponds to $L^{2p'}$-boundedness of a corresponding nonpositive operator (such as a singular integral operator). Even in the linear case $d = 1$, the duality statement is along the lines that $T : L^r \to L^q$ with $q, r \ge 2$ if and only if $\| |T^* g|^2 \|_{q'/2} \lesssim \| |g|^2 \|_{r'/2}$ (rather than $\|T^* g\|_{q'} \lesssim \|g\|_{r'}$).

1.1. ***Multilinear restriction and the Mizohata–Takeuchi conjecture.*** As an indication of the scope of Theorem 1.7, we consider the so-called multilinear restriction problem for the Fourier transform. For $1 \le j \le n$, let $\Gamma_j : U_j \to \mathbb{R}^n$ (with $U_j \subseteq \mathbb{R}^{n-1}$) be smooth parametrisations of compact hypersurfaces S_j in $\mathbb{R}^n$ with nonvanishing gaussian curvature. We assume that the hypersurfaces are transversal in the sense that if $\omega_j(x)$ denotes a unit normal to S_j at $x \in S_j$, then $|\omega_1(x_1) \wedge \cdots \wedge \omega_n(x_n)| \ge c > 0$ for all $x_j \in S_j$. The Fourier extension (or dual restriction) operator $\mathcal{E}_j$ for S_j is given by

$$\mathcal{E}_j f_j(x) = \int_{U_j} e^{2\pi i x\cdot\Gamma(t_j)} f_j(t_j)\,\mathrm{d}t_j.$$

It is conjectured (see [Bennett et al. 2006]) that these operators satisfy the multilinear bound

$$\int_{\mathbb{R}^n} \prod_{j=1}^n |\mathcal{E}_j f_j(x)|^{2/(n-1)}\,\mathrm{d}x \lesssim \prod_{j=1}^n \|f_j\|_{L^2(U_j)}^{2/(n-1)} \tag{16}$$

or equivalently

$$\left\| \prod_{j=1}^n |\mathcal{E}_j f_j(x)|^{1/n} \right\|_{2n/(n-1)} \lesssim \prod_{j=1}^n \|f_j\|_{L^2(U_j)}^{1/n}. \tag{17}$$

This is known up to endpoints (see [Bennett et al. 2006; Tao 2020]) but is as yet unresolved in the form stated here.

These considerations clearly fit into the framework which we were discussing above, in particular Theorem 1.7, and we therefore have the following:

Theorem 1.8 (factorisation for multilinear restriction). *The multilinear restriction bound* (17) *holds if and only if*, *for all nonnegative $G \in L^n(\mathbb{R}^n)$*, *there exist nonnegative $g_1, \dots, g_n$ such that*

$$\prod_{j=1}^n g_j(x)^{1/n} \ge G(x)$$

almost everywhere and, *for all j*,

$$\left(\int_{\mathbb{R}^n} |\mathcal{E}_j f_j(x)|^2 g_j(x)\,\mathrm{d}x \right)^{1/2} \lesssim \|G\|_n \|f_j\|_2.$$

On the other hand, the corresponding endpoint multilinear Kakeya theorem is due to Guth [2010] (see also [Carbery and Valdimarsson 2013]). He proved it by directly establishing the following fundamental factorisation result:

Theorem 1.9 [Guth 2010]. *For $1 \le j \le n$*, *let $\mathcal{T}_j$ be families of doubly infinite tubes of unit cross-section with transversal directions. For all nonnegative $G \in L^n(\mathbb{R}^n)$*, *there exist nonnegative $g_1, \dots, g_n$ such that*

$$\prod_{j=1}^n g_j(x)^{1/n} \ge G(x)$$

almost everywhere and, *for all j and $T \in \mathcal{T}_j$*,

$$\int_T g_j(x)\,\mathrm{d}x \lesssim \|G\|_n.$$

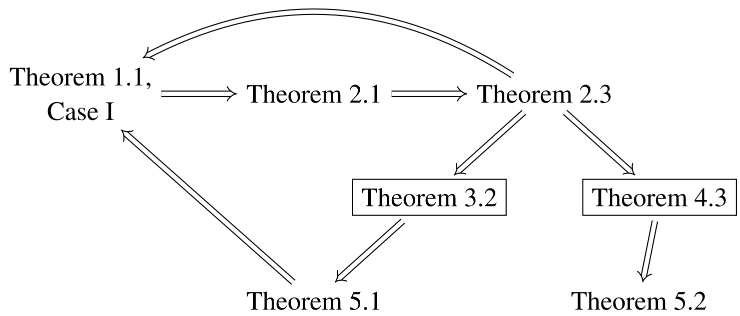

Figure 1. Taxonomy of main theorems.

Moreover, coming from entirely different considerations, there is a conjecture, often attributed to Mizohata and Takeuchi, which states:

Conjecture 1 (Mizohata–Takeuchi conjecture). *Let S be a compact hypersurface of nonvanishing gaussian curvature, with corresponding Fourier extension operator $\mathcal{E}$. Then, for any nonnegative weight w, we have*

$$\int_{\mathbb{R}^n} |\mathcal{E} f(x)|^2 w(x)\,\mathrm{d}x \lesssim \sup_T w(T) \int |f(t)|^2\,\mathrm{d}t,$$

where the sup *is taken over all doubly infinite tubes of unit cross-section with direction normal to S.*

Combining these last two statements we obtain:

Proposition 1.10. *Conditional on the Mizohata–Takeuchi conjecture, the multilinear restriction bound* (16) *holds.*

Proof. In order to establish (16), we integrate the function $\prod_{j=1}^n |\mathcal{E}_j f_j(x)|^{2/n}$ against a test function G in the unit ball of L^n. We let $\mathcal{T}_j$ consist of tubes with directions normal to S_j. We apply Guth's theorem to G obtain g_j as in Theorem 1.9. Then

$$\int_{\mathbb{R}^n} \prod_{j=1}^n |\mathcal{E}_j f_j(x)|^{2/n} G(x)\,\mathrm{d}x \le \int_{\mathbb{R}^n} \prod_{j=1}^n |\mathcal{E}_j f_j(x)|^{2/n} g_j(x)^{1/n}\,\mathrm{d}x \le \prod_{j=1}^n \biggl(\int_{\mathbb{R}^n} |\mathcal{E}_j f_j(x)|^2 g_j(x)\,\mathrm{d}x \biggr)^{1/n}$$

by Hölder's inequality. For each j we have

$$\int_{\mathbb{R}^n} |\mathcal{E}_j f_j(x)|^2 g_j(x)\,\mathrm{d}x \lesssim \biggl(\sup_{T\in\mathcal{T}_j} \int_T g_j \biggr) \int |f_j(t)|^2\,\mathrm{d}t \lesssim \|f_j\|_2^2$$

by the Mizohata–Takeuchi conjecture and the second conclusion of Theorem 1.9. Combining these estimates yields (16). □

1.2. ***Structure of the paper.*** In Section 2 we first state and prove two results, Theorems 2.1 and 2.3, both equivalent to Case I of Theorem 1.1, and then we indicate how we shall use vector-valued techniques to obtain our main theorems. In Section 3 we discuss refinements of Theorem 1.1 for positive operators to the case of p-convex lattices; the main result here is Theorem 3.2. The case of general linear operators is taken up in Section 4, and here we impose conditions of Rademacher-type; the main result in this

$$\begin{tikzcd}
& \boxed{\text{Theorem 3.2}} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{Theorem 1.3} \arrow[r, Rightarrow] & \text{Theorem 1.4} & \text{Theorem 5.1}
\end{tikzcd}$$

Figure 2. Positive operators.

$$\begin{tikzcd}
& \boxed{\text{Theorem 4.3}} \arrow[ddl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
& \text{Theorem 4.1} & \text{Theorem 5.2} \arrow[d, Rightarrow] \\
\text{Theorem 1.5} \arrow[r, Rightarrow] & \text{Theorem 1.6} \arrow[r, Rightarrow] & \text{Theorem 1.7}
\end{tikzcd}$$

Figure 3. General linear operators.

setting is Theorem 4.3. In Section 5 we establish sharp multilinear duality and Maurey-type factorisation theorems for both positive and general linear operators, in Theorems 5.1 and 5.2 respectively. The logical connections between these main results are summarised in Figure 1.

The implications between the main result for positive operators on p-convex lattices, Theorem 3.2, and its more basic manifestations Theorems 1.3 and 1.4 for L^r-spaces, are given in Figure 2.

For general linear operators on normed spaces of (nontrivial) Rademacher-type, the corresponding logical implications between the main result, Theorem 4.3 and the more basic manifestations Theorems 1.5, 1.6 and 1.7 for L^r-spaces, are given by Figure 3. Finally, in the Appendix, we consider the necessity of the conditions we have imposed on the exponents (p_j) in the Basic Question and in Theorems 1.3 and 1.5, and we show that they cannot in general be dispensed with. We also show that one cannot avoid the hypothesis of (p_j)-convexity in Theorem 3.2.

2. Vector-valued disentanglement

In this section we state and prove two results, both of which are equivalent to the disentanglement result given by Case I of Theorem 1.1. These will be crucial in the development of both the positive theory stated in terms of p-convexity and of the general linear theory using Rademacher-type. At the end of this section we describe the strategy that we will adopt in order to achieve these aims in the succeeding sections.

2.1. ***Functional form.*** We first derive an equivalent, arguably more primordial, form of Case I of Theorem 1.1, which makes no reference to saturating positive linear operators, nor to normed lattices, but instead is couched in terms of saturating families of nonnegative measurable functions on a σ-finite measure space X.

Let $(X, \mathrm{d}\mu)$ be a σ-finite measure space. Suppose that for each $1 \leq j \leq d$ we have an indexing set $\mathcal{K}_j$ and a family $\{g_{k_j}\}_{k_j \in \mathcal{K}_j}$ of nonnegative measurable functions on X. We assume that, for each j, the family

$\{g_{k_j}\}_{k_j\in\mathcal{K}_j}$ *saturates* X in the sense that, for every $E \subseteq X$ with $\mu(X) > 0$, there is a subset $E' \subseteq E$ with $\mu(E') > 0$ and a $k_j \in \mathcal{K}_j$ such that $g_{k_j} > 0$ on E'.

Theorem 2.1 (disentanglement of functions). *With* $(X, \mathrm{d}\mu)$ *and* $\{g_{k_j}\}_{k_j\in\mathcal{K}_j}$ *as above, and* $\alpha_j > 0$ *such that* $\sum_{j=1}^d \alpha_j = 1$, *assume that*

$$\int_X \prod_{j=1}^d \Big(\sum_{k_j\in\mathcal{K}_j} \beta_{k_j} g_{k_j}\Big)^{\alpha_j} \mathrm{d}\mu \le A \prod_{j=1}^d \Big(\sum_{k_j\in\mathcal{K}_j} \beta_{k_j}\Big)^{\alpha_j} \tag{18}$$

for all (finitely supported) nonnegative $\{\beta_{k_j}\}$. *Then there exist nonnegative* ϕ_j *such that*

$$\prod_{j=1}^d \phi_j(x)^{\alpha_j} \ge 1 \tag{19}$$

almost everywhere on X, *and such that, for all* j,

$$\int_X g_{k_j}(x)\phi_j(x)\,\mathrm{d}\mu(x) \le A \tag{20}$$

for all $k_j \in \mathcal{K}_j$.

Proof. Let $\mathcal{Y}_j$ be the normed lattice $l^1(\mathcal{K}_j)$ with counting measure on $\mathcal{K}_j$, whose members are denoted by $\boldsymbol{\beta}_j = \{\beta_{k_j}\}_{k_j\in\mathcal{K}_j}$. (There is no requirement on $\mathcal{K}_j$ to be countable.) Define $T_j : l^1(\mathcal{K}_j) \to \mathcal{M}(X)$ by

$$T_j(\boldsymbol{\beta}_j) := \sum_{k_j\in\mathcal{K}_j} \beta_{k_j} g_{k_j}.$$

Note that T_j are saturating positive linear operators. Then (18) becomes

$$\int_X \prod_{j=1}^d (T_j\boldsymbol{\beta}_j)^{\alpha_j}\,\mathrm{d}\mu \le A \prod_{j=1}^d \|\boldsymbol{\beta}\|_{\mathcal{Y}_j}^{\alpha_j}.$$

By Case I of Theorem 1.1, there exist ϕ_j such that (19) holds and such that

$$\int_X (T_j\boldsymbol{\beta}_j)\phi_j\,\mathrm{d}\mu \le A\|\boldsymbol{\beta}_j\|_{\mathcal{Y}_j},$$

which is the same as

$$\int_X \Big(\sum_{k_j\in\mathcal{K}_j} \beta_{k_j} g_{k_j}\Big)\phi_j \mathrm{d}\mu \le A \sum_{k_j\in\mathcal{K}_j} \beta_{k_j},$$

or, equivalently, (20). □

Theorem 2.1 can be equivalently rephrased in terms of convex families of functions as follows:

Theorem 2.2 (disentanglement of convex families of functions). *Let* $(X, \mathrm{d}\mu)$ *be a* σ*-finite measure space. Suppose that* $\sum_{j=1}^d \alpha_j = 1$ *and that each* $\alpha_j > 0$. *For each* $j \in \{1, \dots, d\}$ *let* $\mathcal{G}_j$ *be a saturating convex set of nonnegative measurable functions. Assume that*

$$\int_X \prod_{j=1}^d g_j(x)^{\alpha_j}\,\mathrm{d}\mu(x) \le A \quad \textit{for all } g_j \in \mathcal{G}_j.$$

Then there exist nonnegative ϕ_j such that

$$\prod_{j=1}^{d} \phi_j(x)^{\alpha_j} \geq 1$$

almost everywhere on X, and such that, for all j,

$$\int_X g_j(x)\phi_j(x)\,\mathrm{d}\mu(x) \leq A \quad \textit{for all } g_j \in \mathcal{G}_j.$$

Proof. The equivalence of Theorems 2.1 and 2.2 is clear from the following observation: writing $\gamma_{k_j} := \beta_{k_j} / \big(\sum_{k_j \in \mathcal{K}_j} \beta_{k_j}\big)$ and using homogeneity, assumption (18) of Theorem 2.1 can be rephrased as

$$\int_X \prod_{j=1}^{d} g_j^{\alpha_j}\,\mathrm{d}\mu \leq A \quad \text{for all } g_j \in \operatorname{conv} \mathcal{G}_j,$$

where $\operatorname{conv} \mathcal{G}_j$ is the convex hull of $\mathcal{G}_j$. □

2.2. *Vector-valued form.* The viewpoint of Theorem 2.1 lends itself more readily to applications which are far from obvious from the viewpoint of the formulation of Theorem 1.1. For some of these applications we shall need to work with quasinormed spaces rather than normed spaces $\mathcal{Y}_j$. We recall that a quasinormed space $\mathcal{Y}$ is one in which we have the quasitriangle inequality $\|x+y\|_{\mathcal{Y}} \leq K(\|x\|_{\mathcal{Y}} + \|y\|_{\mathcal{Y}})$ for some $K \geq 1$ in place of the usual triangle inequality.[6]

For example, we have:

Theorem 2.3. *Suppose that $(X, \mathrm{d}\mu)$ is a σ-finite measure space, $\mathcal{Y}_j$ are quasinormed spaces and $0 < p_j < \infty$. Suppose $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ are homogeneous of degree 1 — that is, $T_j(\lambda f_j) = \lambda T_j f_j$ for all $f_j \in \mathcal{Y}_j$ and all scalars λ. Assume that, for all j, the functions $\{|T_j f_j| : f_j \in \mathcal{Y}_j\}$ saturate X. Let $\theta_j > 0$ satisfy $\sum_{j=1}^{d} \theta_j = 1$ and suppose that we have the (p_j)-vector-valued inequality*

$$\int_X \prod_{j=1}^{d} \Big(\sum_{k=1}^{N} |T_j f_{jk}(x)|^{p_j} \Big)^{\theta_j} \mathrm{d}\mu(x) \leq A \prod_{j=1}^{d} \Big(\sum_{k=1}^{N} \|f_{jk}\|_{\mathcal{Y}_j}^{p_j} \Big)^{\theta_j} \tag{21}$$

uniformly in N. Then there exist nonnegative ϕ_j such that

$$\prod_{j=1}^{d} \phi_j(x)^{\theta_j} \geq 1$$

almost everywhere on X and such that, for each j,

$$\Big(\int_X |T_j f_j(x)|^{p_j} \phi_j(x)\,\mathrm{d}\mu(x) \Big)^{1/p_j} \leq A^{1/p_j} \|f_j\|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$.

Notice that we do not need $\mathcal{Y}_j$ to have a lattice structure, nor do we need linearity or positivity of T_j.

[6]We shall not use the quasitriangle inequality, and so the constant K will not appear explicitly in our analysis. In fact, every quasinormed space $\mathcal{Y}$ is r-normable and hence has Rademacher-type r for some $0 < r \leq 1$; see for example [Kalton 2005]. The Rademacher-type constant $R_r(\mathcal{Y})$ will instead feature.

Proof. Consider the saturating families

$$\left\{\left(\frac{|T_j f_j(x)|}{\|f_j\|_{\mathcal{Y}_j}}\right)^{p_j} : f_j \in \mathcal{Y}_j \setminus \{0\}\right\}$$

of nonnegative functions defined on X. Assumption (21) translates into (18) with $\alpha_j = \theta_j$, with the same constant A. So by Theorem 2.1 there are nonnegative ϕ_j such that (19) and (20) hold. And (20) translates into

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x)\,\mathrm{d}\mu(x)\right)^{1/p_j} \leq A^{1/p_j} \|f_j\|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$. □

To complete the assertion that Theorems 1.1 (Case I), 2.1 and 2.3 are all equivalent, we note that Theorem 2.3 implies Case I of Theorem 1.1. Indeed, the scalar-valued inequality (the hypothesis of Theorem 1.1) readily upgrades to the vector-valued inequality (the hypothesis of Theorem 2.3 with $p_j = 1$ for all j) via positivity, as follows: we have

$$\begin{aligned}\int_X \prod_{j=1}^d \Big(\sum_k |T_j f_{jk}(x)|\Big)^{\theta_j} \mathrm{d}\mu(x) &\leq \int_X \prod_{j=1}^d \Big|T_j\Big(\sum_k |f_{jk}|\Big)(x)\Big|^{\theta_j} \mathrm{d}\mu(x)\\ &\leq A \prod_{j=1}^d \Big\|\sum_k |f_{jk}|\Big\|_{\mathcal{Y}_j}^{\theta_j} \leq A \prod_{j=1}^d \Big(\sum_k \|f_{jk}\|_{\mathcal{Y}_j}\Big)^{\theta_j}.\end{aligned}$$

(Note that the use of the triangle inequality for $\mathcal{Y}_j$ here is legitimate since in the implication under consideration the spaces $\mathcal{Y}_j$ are indeed normed spaces.) Summarising, Theorems 1.1 (Case I), 2.1 and 2.3 are all equivalent.

The reader will readily verify using Hölder's inequality that the converse statements to Theorems 2.1 and 2.3 also hold.

2.3. ***Vector-valued approach to disentanglement.*** We now give a preview of how we shall employ Theorem 2.3 to establish the main disentanglement theorems of the following sections. Indeed, thanks to Theorem 2.3 (and its easy converse), given weights (θ_j) with $\sum_{j=1}^d \theta_j = 1$, exponents (p_j) with $p_j > 0$, a measure space (X, μ) and linear operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ defined on quasinormed spaces $\mathcal{Y}_j$, the following two statements are equivalent:

- (disentanglement of p_j-th powers) The norm inequality

$$\int_X \prod_{j=1}^d |T_j f_j(x)|^{p_j\theta_j}\,\mathrm{d}\mu(x) \leq A \prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j}$$

implies that there exist nonnegative ϕ_j such that $\prod_{j=1}^d \phi_j(x)^{\theta_j} \geq 1$ almost everywhere on X and such that, for each j,

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x)\,\mathrm{d}\mu(x)\right)^{1/p_j} \leq \tilde{A}^{1/p_j} \|f_j\|_{\mathcal{Y}_j}.$$

- (scalar-valued implies vector-valued inequality) The scalar-valued inequality

$$\int_X \prod_{j=1}^{d} |T_j f_j(x)|^{p_j\theta_j} \, \mathrm{d}\mu(x) \le A \prod_{j=1}^{d} \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j}$$

implies the vector-valued inequality

$$\int_X \prod_{j=1}^{d} \Big(\sum_k |T_j f_{jk}(x)|^{p_j} \Big)^{\theta_j} \mathrm{d}\mu(x) \le \tilde{A} \prod_{j=1}^{d} \Big(\sum_k \|f_{jk}\|_{\mathcal{Y}_j}^{p_j} \Big)^{\theta_j}.$$

In the following sections, we prove disentanglement theorems via this *vector-valued approach*: subject to geometric properties of the spaces $\mathcal{Y}_j$ (p-convexity for positive linear operators, Rademacher-type for general linear operators), we deduce the vector-valued inequality from the corresponding scalar-valued inequality, and thereby establish our disentanglement theorems via the equivalence we have just set out.

3. Positive operators and p-convexity

In this section we state and prove a more general form of Theorem 1.3 applying to normed lattices which enjoy p-convexity properties.

Definition 3.1 (p-convexity). Let $1 \le p < \infty$. A normed lattice $\mathcal{Y}$ is p-convex if for all finite sequences (f_j) in $\mathcal{Y}$ we have

$$\Big\| \Big(\sum_j |f_j|^p \Big)^{1/p} \Big\|_{\mathcal{Y}} \le C_p(\mathcal{Y}) \Big(\sum_j \|f_j\|_{\mathcal{Y}}^p \Big)^{1/p}.$$

The least such constant is denoted by $C_p(\mathcal{Y})$ and is called the p-convexity constant of $\mathcal{Y}$. Clearly $C_p(\mathcal{Y}) \ge 1$.

Notice that L^p is p-convex with p-convexity constant equal to 1, and that every normed lattice is 1-convex with 1-convexity constant equal to 1. If a lattice $\mathcal{Y}$ is p-convex for some $1 \le p < \infty$, then it is $\tilde{p}$-convex for all $1 \le \tilde{p} \le p$; see, for example, [Lindenstrauss and Tzafriri 1979].

Using the fact that L^r is p-convex for $1 \le p \le r$, with p-convexity constant 1, Theorem 1.3 follows directly from the next, more general result, which is the principal result of this section. This answers our Basic Question for positive linear operators defined on p-convex lattices upon taking $\gamma_j = p_j\theta_j$.

Theorem 3.2 (disentanglement theorem for positive operators on p-convex lattices). *Suppose that X is a σ-finite measure space and that $\mathcal{Y}_j$, for $j = 1, \dots, d$, are p_j-convex normed lattices for some $1 \le p_j < \infty$. Suppose that the linear operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ are positive, and that each T_j saturates X. Suppose that $\theta_j > 0$ and that $\sum_{j=1}^{d} \theta_j = 1$. Finally, suppose that*

$$\int_X \prod_{j=1}^{d} (T_j f_j)(x)^{p_j\theta_j} \, \mathrm{d}\mu(x) \le B \prod_{j=1}^{d} \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j} \tag{22}$$

for all nonnegative f_j in $\mathcal{Y}_j$, $1 \le j \le d$.

Then there exist nonnegative measurable functions ϕ_j on X such that

$$\prod_{j=1}^{d} \phi_j(x)^{\theta_j} \ge 1 \tag{23}$$

almost everywhere on X and such that for each j,

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x)\,\mathrm{d}\mu(x)\right)^{1/p_j} \le B^{1/p_j} C_{p_j}(\mathcal{Y}_j) \|f_j\|_{\mathcal{Y}_j} \tag{24}$$

for all $f_j \in \mathcal{Y}_j$.

Remark 3. The necessity of the geometric assumption that each lattice $\mathcal{Y}_j$ is p_j-convex is addressed in the Appendix — see Proposition A.9.

We establish Theorem 3.2 using the strategy described above in Section 2.3. Indeed, by the discussion there, and some playing with homogeneities, it suffices to show that under the assumptions of the theorem, the scalar-valued inequality

$$\int_X \prod_{j=1}^d |T_j f_j(x)|^{p_j\theta_j}\,\mathrm{d}\mu(x) \le B \prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j} \tag{25}$$

implies the (p_j)-vector-valued inequality

$$\int_X \prod_{j=1}^d \left(\sum_{k=1}^N |T_j f_{jk}(x)|^{p_j}\right)^{\theta_j} \mathrm{d}\mu(x) \le B \prod_{j=1}^d C_{p_j}(\mathcal{Y}_j)^{p_j\theta_j} \prod_{j=1}^d \left(\sum_{k=1}^N \|f_{jk}\|_{\mathcal{Y}_j}^{p_j}\right)^{\theta_j}, \tag{26}$$

and this is exactly what we do in the next lemma:

Lemma 3.3 (scalar-valued to vector-valued). *Suppose that $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ are positive linear operators and that $\mathcal{Y}_j$ are p_j-convex normed lattices for some $p_j \ge 1$. Then* (25) *implies* (26).

Note that when each $\mathcal{Y}_j$ is an L^{r_j}-space for $r_j \ge p_j$, the constant in (26) is precisely B since then we have $C_{p_j}(L^{r_j}) = 1$.

Proof. By homogeneity, we may assume that, for each j, $\left(\sum_{k=1}^N \|f_{jk}\|_{\mathcal{Y}_j}^{p_j}\right)^{1/p_j} = 1$.

We are seeking a bound for the left-hand side of (26), and start by linearising the expression $\left(\sum_{k=1}^N |T_j f_{jk}(x)|^{p_j}\right)^{1/p_j}$ in a pointwise manner. We do this by using classical duality for l^p spaces, together with positivity. Indeed, we have, with the sup taken over all (λ_k) with $\sum_k \lambda_k^{p_j'} = 1$,

$$\begin{aligned}\left(\sum_{k=1}^N \left|T_j f_{jk}(x)\right|^{p_j}\right)^{1/p_j} &= \sup_{(\lambda_k)} \Big|\sum_{k=1}^N \lambda_k T_j f_{jk}(x)\Big| = \sup_{(\lambda_k)} \left|T_j\left(\sum_{k=1}^N \lambda_k f_{jk}\right)(x)\right| \\ &\le \sup_{(\lambda_k)} T_j\left[\left(\sum_{k=1}^N \lambda_k^{p_j'}\right)^{1/p_j'}\left(\sum_{k=1}^N |f_{jk}|^{p_j}\right)^{1/p_j}\right](x) \\ &= T_j\left[\left(\sum_{k=1}^N |f_{jk}|^{p_j}\right)^{1/p_j}\right](x) := T_j F_j(x).\end{aligned}$$

Now we are in a position to apply (25), and we thus have

$$\int_X \prod_{j=1}^d \left(\sum_{k=1}^N |T_j f_{jk}(x)|^{p_j}\right)^{\theta_j} \mathrm{d}\mu(x) \le \int_X \prod_{j=1}^d T_j F_j(x)^{p_j\theta_j}\,\mathrm{d}\mu(x) \le B \prod_{j=1}^d \|F_j\|_{\mathcal{Y}_j}^{p_j\theta_j}.$$

We use the definition of p-convexity to obtain

$$\begin{aligned}\|F_j\|_{\mathcal{Y}_j} &= \left\| \left[\left(\sum_{k=1}^{N} |f_{jk}|^{p_j} \right)^{1/p_j} \right] \right\|_{\mathcal{Y}_j} \\ &\le C_{p_j}(\mathcal{Y}_j) \left(\sum_{k=1}^{N} \|f_{jk}\|_{\mathcal{Y}_j}^{p_j} \right)^{1/p_j} = C_{p_j}(\mathcal{Y}_j).\end{aligned}$$

Combining these inequalities establishes the lemma. □

Notice that we really use linearity of T_j in this argument; sublinearity does not suffice for it to work.

Remark 4. The essence of the vector-valued approach to disentanglement lies in upgrading a scalar-valued estimate into the corresponding vector-valued estimate. From the viewpoint of disentanglement of convex families of functions, this amounts to upgrading the estimate

$$\int_X \prod_{j=1}^{d} |g_j(x)|^{\theta_j} \, \mathrm{d}\mu(x) \le A \quad \text{for all } g_j \in \mathcal{G}_j$$

from the family

$$\mathcal{G}_j := \mathcal{G}(T_j, \mathcal{Y}_j, p_j) := \left\{ \frac{|T_j f_j|^{p_j}}{\|f_j\|_{\mathcal{Y}_j}^{p_j}} \right\}$$

to its convex hull $\operatorname{conv} \mathcal{G}_j$. Now, Lemma 3.3 loosely states that, under its assumptions, the family $\mathcal{G}_j$ is "essentially convex". Indeed, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be sets of nonnegative measurable functions and $C > 0$ be a constant. Let us write $\mathcal{F}_1 \le C\mathcal{F}_2$ if for each $f_1 \in \mathcal{F}_1$ there is $f_2 \in \mathcal{F}_2$ such that $f_1 \le Cf_2$. Assume that $T : \mathcal{Y} \to \mathcal{M}(X)$ is a positive linear operator on a p-convex normed lattice $\mathcal{Y}$ with p-convexity constant $C_p(\mathcal{Y})$. Then from the definition of p-convexity it follows that

$$\operatorname{conv} \mathcal{G}(T, \mathcal{Y}, p) \le C_p(\mathcal{Y}) \mathcal{G}(T, \mathcal{Y}, p).$$

4. General linear operators and Rademacher-type

We now consider general linear (not necessarily positive) operators. We will follow the same general lines of argument as in the previous section. The key new ingredient in this setting will be an analogue of the argument of Lemma 3.3 which converts scalar to vector inequalities, but now without a positivity hypothesis. Once again we shall first need to linearise the expression $\left(\sum_{k=1}^{N} |T_j f_{jk}(x)|^{p_j}\right)^{1/p_j}$ in a pointwise manner. We no longer have positivity at our disposal, so we shall instead use the sequence of Rademacher functions, which we denote by (ϵ_k).

Let us first suppose for simplicity that each $p_j = 2$. In this case, we have, for each j,

$$\begin{aligned}\left(\sum_{k=1}^{N} |T_j f_{jk}(x)|^2 \right)^{1/2} &= \left(\mathbb{E} \left| \sum_{k=1}^{N} \epsilon_k T_j f_{jk}(x) \right|^2 \right)^{1/2}, \\ &\sim_{\theta_j} \left(\mathbb{E} \left| \sum_{k=1}^{N} \epsilon_k T_j f_{jk}(x) \right|^{2\theta_j} \right)^{1/2\theta_j} = \left(\mathbb{E} \left| T_j \left(\sum_{k=1}^{N} \epsilon_k f_{jk} \right)(x) \right|^{2\theta_j} \right)^{1/2\theta_j}\end{aligned}$$

by Khintchine's inequality, so that

$$\int_X \prod_{j=1}^d \Big(\sum_{k=1}^N |T_j f_{jk}(x)|^2 \Big)^{\theta_j} \mathrm{d}\mu(x) \lesssim_{\{\theta_j\}} \mathbb{E} \int_X \prod_{j=1}^d \Big| T_j \Big(\sum_{k=1}^N \epsilon_{jk} f_{jk} \Big)(x) \Big|^{2\theta_j} \mathrm{d}\mu(x).$$

If we now assume (25) with $p_j = 2$ for all j, we can dominate this last expression by

$$B\, \mathbb{E} \prod_{j=1}^d \Big\| \sum_{k=1}^N \epsilon_{jk} f_{jk} \Big\|_{\mathcal{Y}_j}^{2\theta_j}.$$

If $\mathcal{Y}_j$ is assumed to be of Rademacher-type 2, that is to say

$$\Big(\mathbb{E} \Big\| \sum_{k=1}^N \epsilon_k F_k \Big\|_{\mathcal{Y}_j}^2 \Big)^{1/2} \le R_2(\mathcal{Y}_j) \Big(\sum_{k=1}^N \|F_k\|_{\mathcal{Y}_j}^2 \Big)^{1/2}$$

for some finite $R_2(\mathcal{Y}_j)$, we will obtain (using Jensen's inequality $\mathbb{E}(X^\theta) \le \mathbb{E}(X)^\theta$ for $0 < \theta < 1$)

$$\int_X \prod_{j=1}^d \Big(\sum_{k=1}^N |T_j f_{jk}(x)|^2 \Big)^{\theta_j} \mathrm{d}\mu(x) \lesssim_{\{\theta_j\}} B \prod_{j=1}^d R_2(\mathcal{Y}_j)^{2\theta_j} \prod_{j=1}^d \Big(\sum_{k=1}^N \|f_{jk}\|_{\mathcal{Y}_j}^2 \Big)^{\theta_j},$$

which is the analogue of (26) in this setting.

(Note that even in the case that each $\mathcal{Y}_j$ is an L^2-space, and so $R_2(\mathcal{Y}_j) = 1$, there is an implicit constant greater than 1 in this last conclusion, due to the use of Khintchine's inequality.)

The argument now proceeds exactly in accordance with the remarks in Section 2.3, and we arrive at:

Theorem 4.1 (disentanglement theorem for general linear operators on spaces of Rademacher type 2). *Suppose that X is a σ-finite measure space and that $\mathcal{Y}_j$, for $j = 1, \dots, d$, are normed spaces which are of Rademacher-type 2. Suppose that the linear operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ saturate X, and that $\sum_{j=1}^d \theta_j = 1$. Finally, suppose that*

$$\int \prod_{j=1}^d |T_j f_j(x)|^{2\theta_j} \mathrm{d}\mu(x) \le B \prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{2\theta_j}$$

for all f_j in $\mathcal{Y}_j$, $1 \le j \le d$.

Then there exist nonnegative measurable functions ϕ_j on X such that

$$\prod_{j=1}^d \phi_j(x)^{\theta_j} \ge 1$$

almost everywhere on X, and such that, for each j,

$$\Big(\int_X |T_j f_j(x)|^2 \phi_j(x)\, \mathrm{d}\mu(x) \Big)^{1/2} \lesssim_{\{\theta_j\}} B^{1/2} R_2(\mathcal{Y}_j) \|f_j\|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$.

The special case of this result when each $\mathcal{Y}_j$ is an L^{r_j}-space with $2 \le r_j < \infty$ is Theorem 1.6, which immediately follows from Theorem 4.1 upon using the fact (see below) that the Lebesgue space L^r with $r \ge 2$ has Rademacher-type 2.

We now need to discuss what happens when one or more of the p_j are not equal to 2. We need the notion of Rademacher-type p.

Definition 4.2 (Rademacher-type). Let $0 < p \leq 2$. A quasinormed space $\mathcal{Y}$ is of Rademacher-type p if for all finite sequences (F_k) in $\mathcal{Y}$ we have

$$\left(\mathbb{E}\left\|\sum_{k=1}^{N} \epsilon_k F_k\right\|_{\mathcal{Y}}^{p}\right)^{1/p} \leq R_p(\mathcal{Y})\left(\sum_{k=1}^{N} \|F_k\|_{\mathcal{Y}}^{p}\right)^{1/p}$$

for some finite constant $R_p(\mathcal{Y})$.

The least such constant is denoted by $R_p(\mathcal{Y})$ and is called the p-Rademacher-type constant of $\mathcal{Y}$. When $0 < r \leq 2$, the Lebesgue space L^r has Rademacher-type p for $0 < p \leq r$; when $2 < r < \infty$, L^r has Rademacher-type p for $0 < p \leq 2$. Every normed space $\mathcal{Y}$ has Rademacher-type 1. Note that by Khintchine's inequality, if a quasinormed space is of Rademacher-type p, then it is also of Rademacher-type $\tilde{p}$ for all $0 < \tilde{p} \leq p$. Observe that the one-dimensional normed space $\mathbb{R}$ (and more generally any Hilbert space) has Rademacher-type 2 with corresponding constant 1. When $0 < p < 1$, Rademacher-type p is equivalent to p-normability, i.e., the existence of a constant C such that

$$\left\|\sum_{k=1}^{N} F_k\right\|_{\mathcal{Y}} \leq C\left(\sum_{k=1}^{N} \|F_k\|_{\mathcal{Y}}^{p}\right)^{1/p}.$$

Ideally we would hope to have:

Aspiration (general disentanglement aspiration for linear operators). *Suppose that X is a σ-finite measure space and that $\mathcal{Y}_j$, for $j = 1, \ldots, d$, are quasinormed spaces which are of Rademacher-type p_j for certain $0 < p_j \leq 2$. Suppose that the linear operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ saturate X, and that $\sum_{j=1}^{d} \theta_j = 1$. Finally, suppose that*

$$\int \prod_{j=1}^{d} |T_j f_j(x)|^{p_j\theta_j}\, \mathrm{d}\mu(x) \leq B \prod_{j=1}^{d} \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j} \tag{27}$$

for all f_j in $\mathcal{Y}_j$, $1 \leq j \leq d$.

Then there exist nonnegative measurable functions ϕ_j on X such that

$$\prod_{j=1}^{d} \phi_j(x)^{\theta_j} \geq 1 \tag{28}$$

almost everywhere on X and such that, for each j,

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x)\, \mathrm{d}\mu(x)\right)^{1/p_j} \lesssim_{\{\theta_j, p_j\}} B^{1/p_j} R_{p_j}(\mathcal{Y}_j) \|f_j\|_{\mathcal{Y}_j} \tag{29}$$

for all $f_j \in \mathcal{Y}_j$.

We cannot hope for this to be true in general in situations in which some $p_j < 2$; see the Appendix. Nevertheless, we are able to prove something slightly weaker, namely that the aspiration is in fact a theorem under the stronger hypothesis that for those j with $p_j < 2$, the normed spaces $\mathcal{Y}_j$ have Rademacher-type *strictly* larger than p_j.

Theorem 4.3 (disentanglement theorem for general linear operators on spaces of nontrivial Rademacher type). *Let X be a σ-finite measure space and $\mathcal{Y}_j$ quasinormed spaces. Let $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ be linear operators. Suppose that the linear operators T_j saturate X. Let $0 < p_j \le 2$ and $\sum_{j=1}^d \theta_j = 1$. Assume that*

$$\int \prod_{j=1}^d |T_j f_j(x)|^{p_j\theta_j} \, \mathrm{d}\mu(x) \le B \prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j} \tag{30}$$

for all f_j in $\mathcal{Y}_j$, $1 \le j \le d$.

Suppose moreover that each space $\mathcal{Y}_j$ has Rademacher-type $r_j = 2$ for those j with $p_j = 2$, and has Rademacher-type $r_j > p_j$ for those j with $p_j < 2$.

Then there exist nonnegative measurable functions ϕ_j on X such that

$$\prod_{j=1}^d \phi_j(x)^{\theta_j} \ge 1$$

almost everywhere on X and such that, for each j,

$$\left(\int_X |T_j f_j(x)|^{p_j} \phi_j(x) \, \mathrm{d}\mu(x)\right)^{1/p_j} \lesssim_{\{\theta_j, p_j, r_j\}} B^{1/p_j} R_{r_j}(\mathcal{Y}_j) \|f_j\|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$.

Using the fact that the Lebesgue space L^r (with $0 < r < \infty$) has Rademacher-type $\min\{2, r\}$, and hence also Rademacher-type $\tilde{r}$ for every $0 < \tilde{r} \le \min\{2, r\}$, we immediately obtain Theorem 1.5 (and also the assertion made in the accompanying footnote).

Proof. Once again the key issue is to pass from the scalar-valued inequality (30) to the vector-valued inequality analogous to (26), and this is achieved by linearising the expression

$$\left(\sum_{k=1}^N |T_j f_{jk}(x)|^{p_j}\right)^{1/p_j}$$

for each j. When $p_j = 2$ the Rademacher functions achieve this, but they are unsuited to do so when $0 < p_j < 2$ and instead we use p-stable random variables. (For simplicity of notation, in what follows we shall assume that $p_j < 2$ for all j; the easy modifications when $p_j = 2$ for some j are left to the reader.)

We recall that for $0 < p \le 2$, a real-valued random variable γ on a probability space is called (normalised) *p-stable* if it satisfies $\mathbb{E}(e^{it\gamma}) = e^{-|t|^p}$. Note that the distribution (i.e., the pushforward measure on the real line) of a p-stable random variable is unique because the characteristic function (i.e., the Fourier transform up to a sign) of a random variable determines its distribution. These random variables enjoy the following key property:

Lemma 4.4 (key property of independent p-stable random variables). *Let $0 < q < p \le 2$. Let (γ_k) be a sequence of independent p-stable random variables. Then*

$$\left(\mathbb{E}\left|\sum_k \gamma_k a_k\right|^q\right)^{1/q} \sim_{p,q} \left(\sum_k |a_k|^p\right)^{1/p}$$

for all sequences (a_k) of scalars.

Pisier [1974] proved that this property can be upgraded to the vector-valued setting under an appropriate hypothesis of Rademacher-type:

Lemma 4.5 (Rademacher-type r implies stable-type $p < r$). *Let $0 < q < p < r \le 2$. Let $\mathcal{Y}$ be a quasinormed space of Rademacher-type r. Let (γ_k) be a sequence of independent p-stable random variables. Then*

$$\Big(\mathbb{E}\Big\|\sum_k \gamma_k f_k\Big\|_{\mathcal{Y}}^q\Big)^{1/q} \lesssim_{p,q,r} R_r(\mathcal{Y})\Big(\sum_k \|f_k\|_{\mathcal{Y}}^p\Big)^{1/p}$$

for all sequences (f_k) of vectors.

Note that we need $q < p$ in the above lemmas because p-stable random variables fail to be p-integrable. For a textbook treatment of Rademacher and p-stable random variables and Rademacher and p-stable types, see for example [Albiac and Kalton 2006, Sections 6.2, 6.4, and 7.1].

Now, for each $j = 1, \dots, d$, let (γ_{jk}) be a sequence of independent p_j-stable random variables. Then, by Lemma 4.4, we have

$$\Big(\sum_{k=1}^N |T_j f_{jk}(x)|^{p_j}\Big)^{1/p_j} \sim_{\{\theta_j\}} \Big(\mathbb{E}\Big|\sum_k \gamma_{jk} T_j f_{jk}(x)\Big|^{p_j\theta_j}\Big)^{1/p_j\theta_j}.$$

Using this linearisation we can rephrase the left-hand side of the vector-valued inequality in terms of the left-hand side of the scalar-valued inequality,

$$\begin{aligned}\int_X \prod_{j=1}^d \Big(\sum_{k=1}^N |T_j f_{jk}(x)|^{p_j}\Big)^{\theta_j} \mathrm{d}\mu(x) &\sim_{\{\theta_j\}} \mathbb{E}\int_X \prod_{j=1}^d \Big|\sum_k \gamma_{jk} T_j f_{jk}(x)\Big|^{p_j\theta_j} \mathrm{d}\mu(x)\\ &= \mathbb{E}\int_X \prod_{j=1}^d \Big|T_j\Big(\sum_k \gamma_{jk} f_{jk}\Big)(x)\Big|^{p_j\theta_j} \mathrm{d}\mu(x).\end{aligned}$$

Using the assumed scalar-valued inequality (30), we have the estimate

$$\begin{aligned}\mathbb{E}\int_X \prod_{j=1}^d \Big|T_j\Big(\sum_k \gamma_{jk} f_{jk}\Big)(x)\Big|^{p_j\theta_j} \mathrm{d}\mu(x) &\le B\mathbb{E}\prod_{j=1}^d \Big\|\sum_k \gamma_{jk} f_{jk}\Big\|_{\mathcal{Y}_j}^{p_j\theta_j}\\ &= B\prod_{j=1}^d \mathbb{E}\Big(\Big\|\sum_k \gamma_{jk} f_{jk}\Big\|_{\mathcal{Y}_j}^{p_j\theta_j}\Big).\end{aligned}$$

By Lemma 4.5, together with the assumption that each space $\mathcal{Y}_j$ has Rademacher-type $r_j > p_j$, and the fact that $\theta_j < 1$, we obtain

$$\mathbb{E}\Big(\Big\|\sum_k \gamma_{jk} f_{jk}\Big\|_{\mathcal{Y}_j}^{p_j\theta_j}\Big) \lesssim_{\theta_j,p_j,r_j} R_{r_j}(\mathcal{Y}_j)^{p_j\theta_j}\Big(\sum_k \|f_{jk}\|_{\mathcal{Y}}^{p_j}\Big)^{\theta_j}$$

for each j and therefore

$$\mathbb{E}\prod_{j=1}^d \Big\|\sum_k \gamma_{jk} f_{jk}\Big\|_{\mathcal{Y}_j}^{p_j\theta_j} \lesssim_{\theta_j,p_j,r_j} \prod_{j=1}^d R_{r_j}(\mathcal{Y}_j)^{p_j\theta_j}\Big(\sum_k \|f_{jk}\|_{\mathcal{Y}}^{p_j}\Big)^{\theta_j}.$$

Summarising, we have proved that if the quasinormed spaces $\mathcal{Y}_j$ have Rademacher-type r_j, then the scalar-valued inequality (30) implies the vector-valued inequality

$$\int_X \prod_{j=1}^d \Big(\sum_{k=1}^N |T_j f_{jk}(x)|^{p_j} \Big)^{\theta_j} \mathrm{d}\mu(x) \lesssim_{\theta_j, p_j, r_j} \prod_{j=1}^d R_{r_j}(\mathcal{Y}_j)^{p_j \theta_j} \Big(\sum_k \|f_{jk}\|_{\mathcal{Y}}^{p_j} \Big)^{\theta_j}.$$

By the remarks in Section 2.3, this suffices to establish Theorem 4.3. □

Remark 5. Since the linearisation arguments of Theorems 3.2 and 4.3 run componentwise, in the case where some of the operators are positive on p_j-convex lattices and some nonpositive on r_j-Rademacher-type normed spaces, we may obtain a hybrid of these two theorems, whose precise formulation we leave to the interested reader.

5. Multilinear duality and Maurey factorisation extended

In this section we apply the two main disentanglement theorems (Theorem 3.2 for positive linear operators and Theorem 4.3 for general linear operators) to deduce multilinear duality and multilinear Maurey factorisation theorems in the spirit of Theorem 1.1. The treatment we give is very much in parallel to the manner in which Cases II and III of Theorem 1.1 can be deduced from Case I.

Note that multilinear Maurey factorisation theorems below (Cases III of Theorems 5.1 and 5.2) in the linear case $d = 1$ recover the Maurey factorisation theorems [1974] for linear operators. We emphasise, however, that our main theorems (Theorems 3.2 and 4.3) have no linear counterparts since in the case $d = 1$ they are vacuous.

5.1. *Positive operators.* We begin with the setting of positive operators.

Theorem 5.1. *Suppose that X is a σ-finite measure space and that $\mathcal{Y}_j$, for $j = 1, \dots, d$, are p_j-convex normed lattices for some $1 \le p_j < \infty$. Suppose that the linear operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ are positive and that each T_j saturates X. Suppose that $\theta_j > 0$ and that $\sum_{j=1}^d \theta_j = 1$. Finally, suppose that for some $0 < q \le \infty$ we have*

$$\Big\| \prod_{j=1}^d (T_j f_j)^{p_j \theta_j} \Big\|_{L^q(\mathrm{d}\mu)} \le B \prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{p_j \theta_j} \tag{31}$$

for all nonnegative f_j in $\mathcal{Y}_j$, $1 \le j \le d$.

Case I*: (*disentanglement*). $q = 1$. *See Theorem 3.2.

Case II*: (*multilinear duality*) *If $q > 1$, then for every nonnegative $G \in L^{q'}(X)$ there exist nonnegative measurable functions g_j on X such that

$$G(x) \le \prod_{j=1}^d g_j(x)^{\theta_j}$$

almost everywhere, and such that

$$\Big(\int_X |T_j f_j(x)|^{p_j} g_j(x) \, \mathrm{d}\mu(x) \Big)^{1/p_j} \le B^{1/p_j} C_{p_j}(\mathcal{Y}_j) \|G\|_{q'} \|f_j\|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$.

Case III: (*multilinear Maurey factorisation*) *If* $0 < q < 1$ *then there exist nonnegative measurable functions* g_j *on* X *such that*

$$\left\| \prod_{j=1}^{d} g_j(x)^{\theta_j} \right\|_{q'} = 1$$

and such that

$$\left(\int_X |T_j f_j(x)|^{p_j} g_j(x) \, \mathrm{d}\mu(x) \right)^{1/p_j} \leq B^{1/p_j} C_{p_j}(\mathcal{Y}_j) \| f_j \|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$.

Note that Theorem 5.1 in the special case $p_j = 1$ for all j is precisely Theorem 1.1.

Proof. We begin with Case II. Suppose that

$$\left\| \prod_{j=1}^{d} (T_j f_j)^{p_j \theta_j} \right\|_{L^q(X)} \leq B \prod_{j=1}^{d} \| f_j \|_{\mathcal{Y}_j}^{p_j \theta_j}$$

for all nonnegative $f_j \in \mathcal{Y}_j$, $1 \leq j \leq d$. Then, for all nonnegative $G \in L^{q'}(X)$ with $\|G\|_{L^{q'}} = 1$, we have

$$\int_X \prod_{j=1}^{d} (T_j f_j(x))^{p_j \theta_j} G \, \mathrm{d}\mu(x) \leq \left\| \prod_{j=1}^{d} (T_j f_j)^{p_j \theta_j} \right\|_q \leq B \prod_{j=1}^{d} \| f_j \|_{\mathcal{Y}_j}^{p_j \theta_j}.$$

It is easy to see that if T_j saturates X with respect to the measure $\mathrm{d}\mu$, then it also does so with respect to $G \, \mathrm{d}\mu$. Moreover, the measure $G \, \mathrm{d}\mu$ is σ-finite. Therefore, by Theorem 3.2 applied with the measure $G \, \mathrm{d}\mu$ in place of $\mathrm{d}\mu$, there are nonnegative measurable functions γ_j such that

$$1 \leq \prod_{j=1}^{d} \gamma_j(x)^{\theta_j} \quad G \, \mathrm{d}\mu\text{-a.e. on } X,$$

and such that, for each j,

$$\left(\int_X |T_j f_j(x)|^{p_j} \gamma_j(x) G(x) \, \mathrm{d}\mu(x) \right)^{1/p_j} \leq B^{1/p_j} C_{p_j}(\mathcal{Y}_j) \| f_j \|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$. Setting $g_j = \gamma_j G$ gives the desired conclusion.

Now we turn to Case III. The main hypothesis (31) is that

$$\int_X \prod_{j=1}^{d} (T_j f_j)^{p_j \theta_j q} \, \mathrm{d}\mu \leq B^q \prod_{j=1}^{d} \| f_j \|_{\mathcal{Y}_j}^{p_j \theta_j q}$$

for all nonnegative $f_j \in \mathcal{Y}_j$, $1 \leq j \leq d$.

We introduce a new one-dimensional normed lattice $\mathcal{Y}_{d+1}$ with a nonnegative element y of unit norm. Let $T_{d+1} : \mathcal{Y}_{d+1} \to \mathcal{M}(X)$ be given by $\lambda y \to \lambda \mathbf{1}$, where $\mathbf{1}$ denotes the constant function taking the value 1 on X.

Then we have

$$\int_X \prod_{j=1}^{d+1} (T_j f_j)^{p_j \theta_j q} \, \mathrm{d}\mu \leq B^q \prod_{j=1}^{d+1} \| f_j \|_{\mathcal{Y}_j}^{p_j \theta_j q}$$

for all $f_j \in \mathcal{Y}_j$, $1 \leq j \leq d+1$, where the exponents $\theta_{d+1} > 0$ and $p_{d+1} > 0$ are at our disposal.

We want to impose the condition $\theta_{d+1} = 1/q - 1 > 0$ because, with $\tilde{\theta}_j := \theta_j q$, we then have $\sum_{j=1}^{d+1} \tilde{\theta}_j = 1$ and

$$\int_X \prod_{j=1}^{d+1} (T_j f_j)^{p_j \tilde{\theta}_j} \, \mathrm{d}\mu \le B^q \prod_{j=1}^{d+1} \| f_j \|_{\mathcal{Y}_j}^{p_j \tilde{\theta}_j}$$

for all $f_j \in \mathcal{Y}_j,\ 1 \le j \le d+1$.

By Theorem 3.2 we therefore have that there exist ψ_j, $1 \le j \le d+1$, such that

$$\prod_{j=1}^{d+1} \psi_j(x)^{\tilde{\theta}_j} = 1$$

almost everywhere, and

$$\left(\int_X |T_j f_j(x)|^{p_j} \psi_j(x) \, \mathrm{d}\mu(x) \right)^{1/p_j} \le B^{q/p_j} C_{p_j}(\mathcal{Y}_j) \| f_j \|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j,\ 1 \le j \le d+1$.

The case $j = d+1$ of this last inequality tells us that (if we choose $p_{d+1} = 1$)

$$\int_X \psi_{d+1}(x) \, \mathrm{d}\mu(x) \le B^q$$

and, since by the previous equality we have

$$\psi_{d+1}(x) = \prod_{j=1}^{d} \psi_j(x)^{-\tilde{\theta}_j/\tilde{\theta}_{d+1}} = \prod_{j=1}^{d} \psi_j(x)^{-\theta_j/\theta_{d+1}} = \prod_{j=1}^{d} \psi_j(x)^{\theta_j q'},$$

it gives

$$\left\| \prod_{j=1}^{d} \psi_j(x)^{\theta_j} \right\|_{q'} \ge B^{q/q'}.$$

If we now set $g_j = B^{-q/q'} \psi_j$ for $1 \le j \le d$ we obtain

$$\left\| \prod_{j=1}^{d} g_j(x)^{\theta_j} \right\|_{q'} \ge 1$$

and

$$\left(\int_X |T_j f_j(x)|^{p_j} g_j(x) \, \mathrm{d}\mu(x) \right)^{1/p_j} \le B^{1/p_j} C_{p_j}(\mathcal{Y}_j) \| f_j \|_{\mathcal{Y}_j}$$

for all $1 \le j \le d$, and for all $f_j \in \mathcal{Y}_j$. □

5.2. ***General linear operators.*** Next we turn to general linear operators and state a result which in particular contains Theorem 1.7. The proof follows exactly the same arguments as in Theorem 5.1, with the exception that the application of Theorem 3.2 there is now replaced by that of Theorem 4.3. (We also need for Case III to observe that the one-dimensional normed space $\mathcal{Y}_{d+1}$ which we introduce has Rademacher-type strictly greater than 1 — indeed it has Rademacher-type 2 with constant 1 as we noted earlier.) We leave the remaining details to the reader.

Theorem 5.2. *Let X be a σ-finite measure space and $\mathcal{Y}_j$ quasinormed spaces. Let $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ be linear operators. Suppose that the linear operators T_j saturate X. Let $0 < p_j \leq 2$ and $\sum_{j=1}^{d} \theta_j = 1$. Assume that for some $0 < q \leq \infty$ we have*

$$\left\| \prod_{j=1}^{d} |T_j f_j|^{p_j\theta_j} \right\|_{L^q(\mathrm{d}\mu)} \leq B \prod_{j=1}^{d} \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j}$$

for all f_j in $\mathcal{Y}_j$, $1 \leq j \leq d$.

Suppose moreover that each space $\mathcal{Y}_j$ has Rademacher-type $r_j = 2$ for those j with $p_j = 2$, and has Rademacher-type $r_j > p_j$ for those j with $p_j < 2$.

Case I*: (*disentanglement*) $q = 1$. *See Theorem 4.3.

Case II*: (*multilinear duality*) *If $q > 1$, then for every nonnegative $G \in L^{q'}(X)$ there exist nonnegative measurable functions g_j on X such that

$$G(x) \leq \prod_{j=1}^{d} g_j(x)^{\theta_j}$$

almost everywhere, and such that

$$\left(\int_X |T_j f_j(x)|^{p_j} g_j(x)\, \mathrm{d}\mu(x) \right)^{1/p_j} \lesssim_{\{\theta_j, p_j, r_j\}} B^{1/p_j} R_{r_j}(\mathcal{Y}_j) \|G\|_{q'} \|f_j\|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$.

Case III*: (*multilinear Maurey factorisation*) *If $0 < q < 1$ then there exist nonnegative measurable functions g_j on X such that

$$\left\| \prod_{j=1}^{d} g_j(x)^{\theta_j} \right\|_{q'} = 1$$

and such that

$$\left(\int_X |T_j f_j(x)|^{p_j} g_j(x)\, \mathrm{d}\mu(x) \right)^{1/p_j} \lesssim_{\{\theta_j, p_j, r_j\}} B^{1/p_j} R_{r_j}(\mathcal{Y}_j) \|f_j\|_{\mathcal{Y}_j}$$

for all $f_j \in \mathcal{Y}_j$.

There are further extensions to Case II in both Theorems 5.1 and 5.2 when we replace the role of L^q for $q > 1$ by Köthe function spaces as in [Carbery et al. 2022]. We leave the details to the interested reader.

Appendix: Why certain conditions are needed

At various points in the development of our results we have imposed conditions whose necessity might not be immediately obvious. For example, in the Basic Question we imposed the homogeneity condition (7), in Theorems 1.3 and 1.5 we imposed upper bounds on the exponents p_j, and in Theorem 3.2 we imposed p_j-convexity on the lattices $\mathcal{Y}_j$. In this final section we establish that, in all these cases, the conditions we impose are indeed needed in order for our results to have a sufficiently broad scope so as to include certain natural examples.

A.1. ***Condition* (7) *in the Basic Question.*** We first want to clarify to what extent condition (7) is needed in the formulation of the Basic Question.

Proposition A.1. *Fix $r_j \geq 1$ and $\gamma_j > 0$ for $1 \leq j \leq d$. Suppose that (p_j) is such that whenever $T_j : L^{r_j}(\mathbb{R}) \to \mathcal{M}(\mathbb{R}^d)$ are positive linear operators such that*

$$\int_{\mathbb{R}^d} \prod_{j=1}^{d} |T_j f_j(x)|^{\gamma_j} \, \mathrm{d}x \lesssim \prod_{j=1}^{d} \|f_j\|_{L^{r_j}(\mathbb{R})}^{\gamma_j} \tag{32}$$

holds, then there exists (ϕ_j) such that

$$\prod_{j=1}^{d} \phi_j(x)^{\gamma_j/p_j} \geq 1 \tag{33}$$

and

$$\left(\int_{\mathbb{R}^d} |T_j f_j(x)|^{p_j} \phi_j(x) \, \mathrm{d}x\right)^{1/p_j} \lesssim \|f_j\|_{L^{r_j}(\mathbb{R})} \tag{34}$$

hold. Then (p_j) must necessarily satisfy

$$\sum_{j=1}^{d} \frac{\gamma_j}{p_j} = 1.$$

Proof. Let $\Phi_j \in L^{\gamma_j}(\mathbb{R}) \setminus \bigcup_{\beta_j \neq \gamma_j} L^{\beta_j}(\mathbb{R})$ and $g_j \in L^{r_j'}(\mathbb{R})$ be nonzero and strictly positive. Let $T_j : L^{r_j}(\mathbb{R}) \to L^{\gamma_j}(\mathbb{R})$ be given by

$$T_j f(s) = \left(\int_{\mathbb{R}} f g_j\right) \Phi_j(s).$$

Extend T_j to $T_j : L^{r_j}(\mathbb{R}) \to \mathcal{M}(\mathbb{R}^d)$ by defining

$$(T_j f)(x_1, \ldots, x_d) := T_j f(x_j).$$

Then (32) holds with exponents (γ_j), but if we replace any γ_j by any other exponent, its left-hand side becomes infinite for all nontrivial nonnegative $f_j \in L^{r_j}(\mathbb{R})$.

By hypothesis, (p_j) is such that there exists (ϕ_j) satisfying (33) and (34) for this particular (T_j). Let $\lambda = \sum_{j=1}^{d} \gamma_j/p_j$. Then (33) gives

$$\prod_{j=1}^{d} \phi_j(x)^{\gamma_j/\lambda p_j} \geq 1,$$

and so by Lemma 1.2 we can conclude that

$$\int \prod_{j=1}^{d} |T_j f_j(x)|^{\gamma_j/\lambda} \, \mathrm{d}\mu(x) \lesssim \prod_{j=1}^{d} \|f_j\|_{L^{r_j}}^{\gamma_j/\lambda};$$

that is, (32) holds also with exponents (γ_j/λ) in place of (γ_j) for this (T_j). This is a contradiction to what we observed above, unless $\lambda = 1$. $\square$

A.2. ***Sharpness of the exponents in Theorems 1.3 and 1.5.*** As a preliminary observation, we note that the next two lemmas can be used to demonstrate the sharpness of the exponents arising in the classical Maurey–Nikishin–Stein theory of factorisation of linear operators.

Lemma A.2. *For each $1 \le r \le \infty$ and $0 < \gamma < \infty$ we can construct a positive translation-invariant bounded linear operator $T : L^r(\mathbb{G}) \to L^\gamma(\mathbb{G})$ (where $\mathbb{G} = \mathbb{T}$ or $\mathbb{R}$ with Haar measure) such that*

$$\{0 < p < \infty : \textit{for some nontrivial } \phi,\ T : L^r \to L^p(\phi) \textit{ boundedly}\} = I_{r,\gamma} := (0, \max\{\gamma, r\}].$$

This is well known. When $\gamma \le r$, we take $T = I$, and when $\gamma > r$, we take T to be a fractional integral operator (or slight variant thereof when $r = 1$).

We next consider general operators.

Lemma A.3. *For each $1 \le r < \infty$ and $0 < \gamma < \infty$ we can construct a translation-invariant bounded linear operator $T : L^r(\mathbb{G}) \to L^\gamma(\mathbb{G})$ (where $\mathbb{G} = \mathbb{T}$ or $\mathbb{R}$ with Haar measure) such that*

$$\{0 < p < \infty : \textit{for some nontrivial } \phi,\ T : L^r \to L^p(\phi) \textit{ boundedly}\} \\ = J_{r,\gamma} := \begin{cases} (0, \gamma] & \textit{when } 2 \le \gamma < r \textit{ or } \gamma \ge r, \\ (0, 2] & \textit{when } \gamma < 2 \le r, \\ (0, r) & \textit{when } \gamma < r < 2. \end{cases}$$

This is also mostly well known. The exponents $\gamma \ge r$ are covered by Lemma A.2 (in which case we can take $\mathbb{G} = \mathbb{T}$ or $\mathbb{R}$ with Haar measure), so it remains to consider the exponents $\gamma < r$ (in which case we shall take $\mathbb{G} = \mathbb{T}$). Note that, by an averaging argument, for a translation-invariant operator on a compact abelian group, $T : L^r \to L^p(\phi)$ boundedly for a nontrivial weight ϕ if and only if $T : L^r \to L^p(\phi)$ boundedly for the weight $\phi = 1$. Thus,

$$\{0 < p < \infty : \text{for some nontrivial } \phi,\ T : L^r(\mathbb{T}) \to L^p(\mathbb{T}, \phi) \text{ boundedly}\} \\ = \{0 < p < \infty : T : L^r(\mathbb{T}) \to L^p(\mathbb{T}) \text{ boundedly}\}.$$

When $r > 2$ we shall also need the following result to assist us in establishing Lemma A.3:

Lemma A.4. *Let $2 \le \gamma < \infty$. Then there is a bounded translation-invariant linear operator $T : L^2(\mathbb{T}) \to L^\gamma(\mathbb{T})$ such that for no $p > \gamma$ is T bounded from $L^\infty(\mathbb{T})$ to $L^p(\mathbb{T})$.*

For the case $\gamma = 2$ of Lemma A.4, an argument based on Rademacher functions can be found in [García-Cuerva and Rubio de Francia 1985, Chapter VI, Example 2.10(e)]. The case $\gamma > 2$ follows readily from Bourgain's solution [1989] of the $\Lambda(p)$-set problem. This result states that for each $2 < \gamma < \infty$ there is a set $E \subseteq \mathbb{Z}$ which is a $\Lambda(\gamma)$-set, but which is not a $\Lambda(\tilde{p})$-set for any $\tilde{p} > \gamma$. If T is the Fourier multiplier operator with multiplier χ_E, then T is bounded from $L^2(\mathbb{T})$ to $L^\gamma(\mathbb{T})$ (since E is a $\Lambda(\gamma)$-set) but unbounded from $L^\infty(\mathbb{T})$ to $L^p(\mathbb{T})$ for every $p > \gamma$ (since if $T : L^\infty \to L^p$ boundedly for some $p > \gamma$, then interpolating between this bound and the bound $T : L^2 \to L^\gamma$ with $\gamma > 2$ gives the bound $T : L^q \to L^{\tilde{p}}$ for some $q < \tilde{p}$ and $\tilde{p} > \gamma$, which would imply that E is a $\Lambda(\tilde{p})$-set, a contradiction). (We thank an anonymous referee for pointing out this connection to us.) Bourgain's argument gives the stronger conclusion that the operator T can also be chosen to satisfy $T^2 = T$. On the other hand, his argument is not constructive, and so we give a simple constructive proof of Lemma A.4 — which is perhaps of independent interest — in Section A.4 below.

We return to the detailed discussion of Lemma A.3.

- When $2 \le \gamma < r$ we appeal to Lemma A.4, and we take T to be a translation-invariant bounded linear operator $T : L^2 \to L^\gamma$ (and hence $T : L^r \to L^\gamma$) that is not bounded from L^∞ to L^p for any $p > \gamma$.
- When $\gamma < 2 < r$ we appeal to Lemma A.4, and we take T to be a translation-invariant bounded linear operator $T : L^2 \to L^2$ (and hence $T : L^r \to L^\gamma$) that is not bounded from L^∞ to L^p for any $p > 2$.
- When $\gamma < r$ and $r = 2$ we take T to be the identity operator.
- When $\gamma < r < 2$ we appeal to a theorem of [Zafran 1975] which states that for each $r < 2$ there is a translation-invariant bounded linear operator $T : L^r(\mathbb{T}) \to L^{r,\infty}(\mathbb{T})$ (and thus $T : L^r(\mathbb{T}) \to L^\gamma(\mathbb{T})$ for all $\gamma < r$) such that T is not bounded on L^r.

By taking tensor products we obtain corresponding multilinear examples. Indeed, by choosing operators $T_j : L^{r_j}(\mathbb{G}_j) \to L^{\gamma_j}(\mathbb{G}_j)$ as in Lemmas A.2 and A.3, and letting the measure space $(X, \mathrm{d}\mu)$ be the product $X = \mathbb{G}_1 \times \cdots \times \mathbb{G}_d$, with $\mathrm{d}\mu$ as product measure, we obtain:

Proposition A.5. *For each $1 \le r_j \le \infty$ and $0 < \gamma_j < \infty$ there is a σ-finite measure space X and there are positive linear operators $T_j : L^{r_j}(\mathbb{G}_j) \to \mathcal{M}(X)$ such that*

$$\int_X \prod_{j=1}^d |T_j f_j|^{\gamma_j} \lesssim \prod_{j=1}^d \|f_j\|_{r_j}^{\gamma_j}$$

and such that

$$\{(p_j) \in (0,\infty)^d : \textit{for each } j,\ T_j : L^{r_j} \to L^{p_j}(\phi_j) \textit{ boundedly for some nontrivial } \phi_j\} = \prod_{j=1}^d I_{r_j,\gamma_j} = \prod_{j=1}^d (0, \max\{\gamma_j, r_j\}].$$

Proposition A.6. *For each $1 \le r_j < \infty$ and $0 < \gamma_j < \infty$ there is a σ-finite measure space X and there are linear operators $T_j : L^{r_j}(\mathbb{G}_j) \to \mathcal{M}(X)$ such that*

$$\int_X \prod_{j=1}^d |T_j f_j|^{\gamma_j} \lesssim \prod_{j=1}^d \|f_j\|_{r_j}^{\gamma_j}$$

and such that

$$\{(p_j) \in (0,\infty)^d : \textit{for each } j,\ T_j : L^{r_j} \to L^{p_j}(\phi_j) \textit{ boundedly for some nontrivial } \phi_j\} = \prod_{j=1}^d J_{r_j,\gamma_j}.$$

As immediate corollaries we have:

Corollary A.7. *For each $1 \le r_j \le \infty$ and $0 < \gamma_j < \infty$ there is a σ-finite measure space X and there are positive linear operators $T_j : L^{r_j}(\mathbb{G}_j) \to \mathcal{M}(X)$ such that*

$$\int_X \prod_{j=1}^d |T_j f_j|^{\gamma_j} \lesssim \prod_{j=1}^d \|f_j\|_{r_j}^{\gamma_j}$$

and such that

$$\left\{(p_j) \in (0,\infty)^d : \sum_{j=1}^d \frac{\gamma_j}{p_j} = 1 \text{ and, for each } j,\ T_j : L^{r_j} \to L^{p_j}(\phi_j) \text{ boundedly for some nontrivial } \phi_j\right\}$$

is nonempty if and only if $\sum_{j=1}^d \gamma_j / r_j \le 1$, and, when this condition holds, equals

$$\left(\prod_{j=1}^d (0, r_j]\right) \cap \left\{(p_j) \in (0,\infty)^d : \sum_{j=1}^d \frac{\gamma_j}{p_j} = 1\right\}.$$

Corollary A.8. *For each $1 \le r_j < \infty$ and $0 < \gamma_j < \infty$ there is a σ-finite measure space X and there are linear operators $T_j : L^{r_j}(\mathbb{G}_j) \to \mathcal{M}(X)$ such that*

$$\int_X \prod_{j=1}^d |T_j f_j|^{\gamma_j} \lesssim \prod_{j=1}^d \|f_j\|_{r_j}^{\gamma_j}$$

and such that

$$\begin{aligned}\left\{(p_j) \in (0,\infty)^d : \sum_{j=1}^d \frac{\gamma_j}{p_j} = 1 \text{ and, for each } j,\ T_j : L^{r_j} \to L^{p_j}(\phi_j) \text{ boundedly for some nontrivial } \phi_j\right\} \\ = \left(\prod_{j=1}^d J_{r_j,\gamma_j}\right) \cap \left\{(p_j) \in (0,\infty)^d : \sum_{j=1}^d \frac{\gamma_j}{p_j} = 1\right\}.\end{aligned}$$

This set is nonempty if and only if we have $\sum_{j=1}^d \gamma_j / \min\{r_j, 2\} < 1$ when at least one $r_j < 2$, and $\sum_{j=1}^d \gamma_j \le 2$ when all $r_j \ge 2$. When nonempty, this set equals

$$\left(\prod_{j : r_j < 2} (0, r_j) \times \prod_{j : r_j \ge 2} (0, 2]\right) \cap \left\{(p_j) \in (0,\infty)^d : \sum_{j=1}^d \frac{\gamma_j}{p_j} = 1\right\}.$$

These two corollaries establish the assertions concerning sharpness of Theorems 1.3 and 1.5 which we made in the Introduction.

A.3. ***Disentanglement implies p-convexity.*** Here we show that the hypotheses of p-convexity are intrinsic to Theorem 3.2, since p-convexity follows from the conclusion of that result, at least in the case when the spaces $\mathcal{Y}_j$ are Köthe spaces whose duals are norming. This class includes Lorentz spaces and Orlicz spaces.

We therefore assume in what follows that each $\mathcal{Y}_j$ is a Köthe function lattice over the σ-finite measure space $(Y_j, \mathrm{d}\nu_j)$, and that we can realise the norm of any $f \in \mathcal{Y}_j$ as

$$\|f\|_{\mathcal{Y}_j} = \sup_{\|g\|_{\mathcal{Y}_j'} \le 1} \left| \int_{Y_j} f g \, \mathrm{d}\nu_j \right|.$$

We remark that a Köthe dual $\mathcal{Y}'$ is norming if and only if the pointwise convergence $f_n \uparrow f$ implies the norm convergence $\|f_n\|_{\mathcal{Y}} \to \|f\|_{\mathcal{Y}}$ for all pointwise increasing sequences (f_n) (though we shall not need this characterisation here).

Proposition A.9. *Fix $\mathcal{Y}_j$ as above, and fix $1 < p_j < \infty$ for $1 \le j \le d$. Assume that there exists a constant $C_{\{\mathcal{Y}_j\}}$ such that for all weights (θ_j) with $\theta_j > 0$ and $\sum_{j=1}^d \theta_j = 1$, all σ-finite measure spaces $(X, \mathrm{d}\mu)$, and all saturating positive linear operators $T_j : \mathcal{Y}_j \to \mathcal{M}(X)$ the estimate*

$$\int_X \prod_{j=1}^d |T_j f_j(x)|^{p_j\theta_j}\,\mathrm{d}\mu(x) \le A \prod_{j=1}^d \|f_j\|_{\mathcal{Y}_j}^{p_j\theta_j} \quad \textit{for all } f_j \in \mathcal{Y}_j$$

implies the existence of functions ϕ_j such that $\prod_{j=1}^d \phi_j(x)^{\theta_j} \ge 1$ and such that

$$\left(\int_X |T_j f_j|^{p_j}\phi_j\,\mathrm{d}\mu\right)^{1/p_j} \le C_{\{\mathcal{Y}_j\}} A^{1/p_j} \|f_j\|_{\mathcal{Y}_j}.$$

Then each space $\mathcal{Y}_j$ is p_j-convex.

Proof. Fix j. Let $g_j \in \mathcal{Y}_j'$ be of unit norm. Let $(X, \mathrm{d}\mu) := (Y_j, |g_j|\mathrm{d}\nu_j)$. We define $T_j := I_{\mathcal{Y}_j \to \mathcal{Y}_j}$.

For each $i \ne j$, we choose a nonnegative function F_i on Y_i such that $\|F_i\|_{\mathcal{Y}_i} = 1$. Since $\mathcal{Y}_i'$ is assumed to be norming, for each $\epsilon > 0$ we can choose a nonnegative function G_i on Y_i with $\|G_i\|_{\mathcal{Y}_i'} = 1$ such that $\int_{Y_i} F_i G_i\,\mathrm{d}\nu_i \ge (1-\epsilon)\|F_i\|_{\mathcal{Y}_i} = (1-\epsilon)$. We define $T_i : \mathcal{Y}_i \to \mathcal{M}(X)$ by

$$T_i f(x) = \int_{Y_i} f G_i\,\mathrm{d}\nu_i,$$

so that each $T_i f$ is a constant function on X. Note that $|T_i f_i(x)| \le \|f_i\|_{\mathcal{Y}_i}$ for all $f_i \in \mathcal{Y}_i$ and that $|T_i F_i(x)| \ge (1-\epsilon)$ for all $x \in X$.

Let $\theta_j := 1/p_j \in (0,1)$, and choose the remaining $\theta_i \in (0,1)$ in such a way that $\sum_{i=1}^d \theta_i = 1$.

With these choices, we have

$$\begin{aligned}\int_X \prod_{i=1}^d |T_i f_i(x)|^{p_i\theta_i}\,\mathrm{d}\mu(x) &\le \int_{Y_j} |f_j||g_j|\,\mathrm{d}\mu_j \prod_{i\ne j} \|f_i\|_{\mathcal{Y}_i}^{p_i\theta_i} \\ &\le \|g_j\|_{\mathcal{Y}_j'}\|f_j\|_{\mathcal{Y}_j} \prod_{i\ne j} \|f_i\|_{\mathcal{Y}_i}^{p_i\theta_i} = \prod_{i=1}^d \|f_i\|_{\mathcal{Y}_i}^{p_i\theta_i}.\end{aligned}$$

By assumption, there are (ϕ_i) such that $\prod_{i=1}^d \phi_i(x)^{\theta_i} \ge 1$ and such that, for each i,

$$\left(\int_X |T_i f_i|^{p_i}\phi_i\,\mathrm{d}\mu\right)^{1/p_i} \le C_{\{\mathcal{Y}_j\}}\|f_i\|_{\mathcal{Y}_i}.$$

Hence, by the equivalence set out in Section 2.3, we have the vector-valued inequality

$$\int_X \prod_{i=1}^d \left(\sum_{k=1}^N |T_i f_{i,k}|^{p_i}\right)^{\theta_i} \mathrm{d}\mu \le C_{\{\mathcal{Y}_j\}} \prod_{i=1}^d \left(\sum_{k=1}^N \|f_{i,k}\|_{\mathcal{Y}_i}^{p_i}\right)^{\theta_i}$$

for the same constant $C_{\{\mathcal{Y}_j\}}$.

For $i \neq j$, set $f_{i,k} = F_i$ for $k = 1$ and $f_{i,k} = 0$ for $k = 2, \dots, N$. We obtain

$$\int_{Y_j} \left(\sum_{k=1}^{N} |f_{j,k}|^{p_j} \right)^{1/p_j} |g_j| \, \mathrm{d}\nu_j \leq C_{\{\mathcal{Y}_j\}} \frac{1}{(1-\epsilon)^{d-1}} \left(\sum_{k=1}^{N} \|f_{j,k}\|_{\mathcal{Y}_j}^{p_j} \right)^{1/p_j}.$$

By assumption, the Köthe dual $\mathcal{Y}_j'$ is norming, and hence taking supremum over g_j in the unit ball of $\mathcal{Y}_j'$ and letting $\epsilon \to 0$ yields

$$\left\| \left(\sum_{k=1}^{N} |f_{j,k}|^{p_j} \right)^{1/p_j} \right\|_{\mathcal{Y}_j} \leq C_{\{\mathcal{Y}_j\}} \left(\sum_{k=1}^{N} \|f_{j,k}\|_{\mathcal{Y}_j}^{p_j} \right)^{1/p_j}.$$

This is the defining inequality of p_j-convexity. □

A.4. *Constructive proof of Lemma A.4.* Finally, we turn to our constructive proof of Lemma A.4, which represents a slight strengthening (in the particular case when the underlying group is $\mathbb{T}$) of a result found in [Figà-Talamanca and Price 1973, Theorem 4.4];[7] see the references therein for a full history.

We recall (see for example [Katznelson 2004, p. 33]) the sequence of Rudin–Shapiro polynomials P_m on $\mathbb{T}$. There is a (deterministic) sequence $a_n \in \{\pm 1\}$ such that the sequence of trigonometric polynomials defined for $m \geq 0$ by

$$P_m(x) := \sum_{n=0}^{2^m-1} a_n e^{2\pi i n x}$$

has the following properties (of which the first and the last are trivial and the second is the interesting one):

- $\|P_m\|_2 = 2^{m/2}$.
- $\|P_m\|_\infty \leq 2^{(m+1)/2}$.
- $2^{(m-1)/2} \leq \|P_m\|_q \leq 2^{(m+1)/2}$ for $1 \leq q \leq \infty$.
- $\|\widehat{P_m}\|_\infty = 1$.

For the third item, the upper bounds are clear from the second item; for the lower bounds it suffices by Hölder's inequality to show that $\|P_m\|_1 \geq 2^{(m-1)/2}$, and this follows from the first two items together with $\|P_m\|_2 \leq \|P_m\|_1^{1/2} \|P_m\|_\infty^{1/2}$.

From the first and fourth of these we deduce by Young's inequality and interpolation that, for $1 \leq r \leq 2$,

$$\|P_m * f\|_2 \leq 2^{m(1/r-1/2)} \|f\|_r.$$

Let $F_m(x) = \sum_{n=0}^{2^m-1} e^{2\pi i n x}$ so that $\|F_m\|_p \lesssim 2^{m/p'}$ for $1 < p \leq \infty$ and $\|F_m\|_1 \lesssim m$.

Observe that $P_m * F_m = P_m$, so that $\|P_m * F_m\|_q = \|P_m\|_q \gtrsim 2^{m/2}$ for all $1 \leq q \leq \infty$. Let T_m denote convolution with P_m. Using these bounds we can easily see that $\|T_m\|_{L^p \to L^q} \lesssim \|T_m\|_{L^r \to L^2}$ only when $p \geq r$. Indeed, from the upper bounds on $\|F_m\|_p$ we deduce that, for all $1 \leq p, q \leq \infty$, $\|T_m\|_{L^p \to L^q}$ is bounded below by $2^{m(1/2-1/p')}$ when $p > 1$ and $m^{-1} 2^{m/2}$ when $p = 1$.

[7] The examples in [Figà-Talamanca and Price 1973] depend in principle also on the exponent p, whereas ours is p-independent.

We now build an explicit example. We first note that $\widetilde{P}_m := e^{2\pi i 2^m x} P_m(x)$ has frequencies in $[2^m, 2^{m+1})$, and similarly for $\widetilde{F}_m(x) := e^{2\pi i 2^m x} F_m(x)$. Performing this modulation does not change any of the estimates on P_m and F_m which we had above, and we have $\widetilde{P}_m * \widetilde{F}_m = \widetilde{P}_m$ and $\widetilde{P}_m * \widetilde{F}_{m'} = 0$ for $m \neq m'$.

Fix an r with $1 \leq r \leq 2$. Let T (depending on r) be given by convolution with

$$\sum_{m=1}^{\infty} m^{-2} 2^{m/2} 2^{-m/r} \widetilde{P}_m;$$

by the bounds for P_m derived above we see that T is bounded from L^r to L^2.

Fix $p \geq 1$ and let $f_m = m^{-3} 2^{-m/p'} \widetilde{F}_m$ so that

$$\| f_m \|_p \leq m^{-3} 2^{-m/p'} \| \widetilde{F}_m \|_p \lesssim 1$$

uniformly in $m \geq 1$.

Moreover, we have

$$T f_m = m^{-5} 2^{m/2} 2^{-m/r} 2^{-m/p'} \widetilde{P}_m * \widetilde{F}_m$$

since $\widetilde{P}_m * \widetilde{F}_{m'} = 0$ for $m \neq m'$. Therefore,

$$\| T f_m \|_1 = m^{-5} 2^{m/2} 2^{-m/r} 2^{-m/p'} \| \widetilde{P}_m * \widetilde{F}_m \|_1 \sim m^{-5} 2^{-m/r} 2^{m/p}$$

for each $m \geq 1$.

Consequently,

$$\| T \|_{L^p \to L^1} \gtrsim \sup_m \| T f_m \|_1 = \infty$$

when $p < r$.

Thus, for each $1 < r \leq 2$, we have built an example of an $L^r \to L^2$-bounded translation-invariant operator T on $\mathbb{T}$ such that, for every $1 \leq p < r$, we have $\| T \|_{L^p \to L^1} = \infty$.

By duality, for each $2 \leq r < \infty$, we have an explicit example of an $L^2 \to L^r$-bounded translation-invariant operator T on $\mathbb{T}$ such that if $q > r$, we have $\| T \|_{L^\infty \to L^q} = \infty$. This establishes the constructive version of Lemma A.4.

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THE GREEN FUNCTION WITH POLE AT INFINITY APPLIED TO THE STUDY OF THE ELLIPTIC MEASURE

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In $\mathbb{R}^{d+1}_+$ or in $\mathbb{R}^n \setminus \mathbb{R}^d$ ($d < n-1$), we study the Green function with pole at infinity defined for instance by David, Engelstein, and Mayboroda. In two cases, we deduce the equivalence between the elliptic measure and the Lebesgue measure on $\mathbb{R}^d$. We further prove the A_∞-absolute continuity of the elliptic measure for operators that can be related to the two previous cases via Carleson measures, extending the range of operators for which the A_∞-absolute continuity of the elliptic measure is known.

1. Introduction

History and motivation. Over the past decades, a considerable number of articles have studied the relationship between the geometry of the boundary of a domain Ω and the L^p-solvability of the Dirichlet problem $-\Delta u = 0$ in Ω. The L^p-solvability of the Dirichlet problem for large p is equivalent to the absolute continuity of the harmonic measure, and we shall focus our presentation on the latter. The theory was pioneered in 1916 by the Riesz brothers (see [Riesz and Riesz 1920]), who established the absolute continuity of the harmonic measure for simply connected domains in the complex plane with a rectifiable boundary. The quantitative and local analogues are stated in [Lavrentev 1963] and [Bishop and Jones 1990], respectively. The development of the theory in $\mathbb{R}^n$, for $n \geq 2$, started in [Dahlberg 1977] and treated Lipschitz domains. Many works were then devoted to finding the optimal conditions on Ω and $\partial\Omega$ to guarantee the absolute continuity of the harmonic measure. It was finally understood that a quantitative version of absolute continuity of the harmonic measure holds if and only if the boundary $\partial\Omega$ is uniformly rectifiable and the domain Ω has enough access to its boundary. A nonexhaustive list of articles that lead to this conclusion includes [Azzam et al. 2016; 2017; David and Jerison 1990; Hofmann et al. 2014; Hofmann and Martell 2014; Semmes 1990], and the minimal access condition to the boundary was recently obtained in [Azzam et al. 2020].

One of the strategies for studying the absolute continuity of the harmonic measure, and by extension the L^p-solvability of the Dirichlet problem, is to make a change of variable in order to obtain an equivalent problem for simpler sets but for more complicated elliptic operators. So instead of studying $-\Delta u = 0$ on a general domain Ω, many works focused their interest on the study of elliptic operators of the form

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$L = -\operatorname{div} \mathcal{A}\nabla$ on, for instance, $\Omega = \mathbb{R}^{n-1}_+ := \{(x,t) \in \mathbb{R}^{n-1} \times (0,+\infty)\}$. Here, $\mathcal{A}$ is a matrix satisfying the ellipticity and boundedness conditions

$$\mathcal{A}(x,t)\xi \cdot \xi \geq C_L|\xi|^2 \quad \text{for } (x,t) \in \Omega \text{ and } \quad \xi \in \mathbb{R}^n, \tag{1.1}$$

$$|\mathcal{A}(x,t)\xi \cdot \zeta| \leq C_L|\xi||\zeta| \quad \text{for } (x,t) \in \Omega \text{ and } \xi, \zeta \in \mathbb{R}^n, \tag{1.2}$$

for some constant $C_L > 0$. As shown in [Caffarelli et al. 1981b; Modica and Mortola 1980], the conditions (1.1) and (1.2) are not sufficient to ensure that the elliptic measure associated to L is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{n-1}$, and thus some extra assumptions are needed on $\mathcal{A}$ to obtain our absolute continuity. Two situations that give positive results are heavily studied: the first situation focuses on t-independent matrices $\mathcal{A}$ and are studied in [Jerison and Kenig 1981] (use a Rellich identity), [Auscher et al. 2008] (perturbations), [Hofmann et al. 2015] ($\mathcal{A}$ is nonsymmetric), [Hofmann et al. 2019] (Dirichlet problem in weighted L^p), or [Hofmann et al. 2022] (the antisymmetric part of $\mathcal{A}$ can be unbounded); while in the second situation, the coefficients of $\mathcal{A}$ satisfy some conditions described with the help of Carleson measures and Carleson measure perturbations, and are considered, for instance, in [Dindoš and Pipher 2019; Dindoš et al. 2017; 2007; Fefferman et al. 1991; Hofmann and Martell 2012; Hofmann et al. 2021; Kenig and Pipher 2001].

When the domain is the complement of a thin set, for instance $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d := \{(x,t) \in \mathbb{R}^d \times \mathbb{R}^{n-d},\ t \neq 0\}$ with $d < n-1$, studying the solutions to $-\Delta u = 0$ in Ω does not make sense. Indeed, the solutions to $-\Delta u = 0$ in Ω are the same as the solutions to $-\Delta u = 0$ in $\mathbb{R}^n$, which means that the boundary $\mathbb{R}^d$ is not "seen" by the Laplacian or, in term of harmonic measure, it means that the Brownian motion has zero probability to hit the boundary $\mathbb{R}^d$. In [David et al. 2021b; 2020], G. David, S. Mayboroda, and the author developed an elliptic theory for domains with thin boundaries by using appropriate degenerate operators. If $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d$ is considered, we assume that the elliptic operator $L = -\operatorname{div} A\nabla$ satisfies

$$A(x,t)\xi \cdot \xi \geq C_L|t|^{d+1-n}|\xi|^2 \quad \text{for } (x,t) \in \Omega \text{ and } \quad \xi \in \mathbb{R}^n, \tag{1.3}$$

$$|A(x,t)\xi \cdot \zeta| \leq C_L|t|^{d+1-n}|\xi||\zeta| \quad \text{for } (x,t) \in \Omega \text{ and } \xi, \zeta \in \mathbb{R}^n, \tag{1.4}$$

for some constant $C_L > 0$. The operator L can thus be written as $-\operatorname{div} |t|^{d+1-n}\mathcal{A}\nabla$ where $\mathcal{A}$ satisfies conditions (1.1) and (1.2). Under those conditions, the elliptic measure with pole in $X \in \Omega$ associated to L, denoted by ω^X_L, is the probability measure on $\mathbb{R}^d$ so that the function u_f on Ω constructed for any $f \in C^\infty_0(\mathbb{R}^d)$ as

$$u_f(X) = \int_{\mathbb{R}^d} f(y)\, d\omega^X_L(y) \tag{1.5}$$

is a weak solution to $Lu = 0$, is continuous on $\overline{\Omega}$, and has trace on $\mathbb{R}^d$ equal to f. The articles [David and Mayboroda 2022b; David et al. 2019a; Feneuil 2022; Feneuil et al. 2021; Mayboroda and Poggi 2021; Mayboroda and Zhao 2019] tackled the absolute continuity of the elliptic measure (or L^p-solvability of the Dirichlet problem) in the case where the boundary of Ω is a low dimensional set.

We finish the subsection with the following observation made in [David et al. 2019b]. Let $L = -\operatorname{div} \mathcal{A}\nabla$ be an elliptic operator defined on $\mathbb{R}^{d+1}_+$ that satisfies (1.1)–(1.2). We define $\mathcal{A}_1$ as the top left $d \times d$

submatrix of $\mathcal{A}$, and $\mathcal{A}_2$, $\mathcal{A}_3$, a_4 so that we have the block matrix

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_3 & a_4 \end{pmatrix}. \tag{1.6}$$

For $n > d+1$, we construct the elliptic operator $\widetilde{L} = -\operatorname{div} |t|^{d+1-n} \widetilde{\mathcal{A}} \nabla$ defined on $\mathbb{R}^n \setminus \mathbb{R}^d$ as

$$\widetilde{\mathcal{A}}(x,t) := \begin{pmatrix} \mathcal{A}_1(x,|t|) & \mathcal{A}_2(x,|t|)\dfrac{t}{|t|} \\ \dfrac{t^T}{|t|}\mathcal{A}_3(x,|t|) & a_4(x,|t|) I_{n-d} \end{pmatrix}, \tag{1.7}$$

where t is seen here as a horizontal vector in $\mathbb{R}^{n-d}$, which means that $\mathcal{A}_2 t$ and $t^T \mathcal{A}_3$ are matrices of dimensions $d \times (n-d)$ and $(n-d) \times d$, respectively, and I_{n-d} is the identity matrix of order $n-d$. Then the elliptic measures on $\mathbb{R}^d$ associated to L and $\widetilde{L}$ — we call them $\omega^{(x,r)}$ and $\widetilde{\omega}^{(x,t)}$ — satisfy

$$\widetilde{\omega}^{(x,t)} = \omega^{(x,|t|)} \quad \text{for } (x,t) \in \mathbb{R}^n \setminus \mathbb{R}^d. \tag{1.8}$$

More generally, any solution u to $Lu = 0$ in $\mathbb{R}^{d+1}_+$ yields a solution $\widetilde{u}(x,t) := u(x,|t|)$ to $\widetilde{L}u = 0$ in $\mathbb{R}^n \setminus \mathbb{R}^d$. As a consequence, the construction from [Caffarelli et al. 1981b; Modica and Mortola 1980] can be adapted to provide, for any $1 \leq d < n$, examples of operators whose elliptic measures are not absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. It also means that if an operator $\widetilde{L} = -\operatorname{div} |t|^{d+1-n} \widetilde{\mathcal{A}} \nabla$ can be written as (1.7) and if the elliptic measure of the original operator $L = -\operatorname{div} \mathcal{A} \nabla$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, then the elliptic measure associated to $\widetilde{L}$ is also absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. The above construction provides, for any dimension and codimension of the boundary, a wide range of elliptic operators that satisfy the absolute continuity of the elliptic measure. However, the (relevant) solutions of those operators are radial, i.e., they depend only on the distance to the boundary $\mathbb{R}^d$ and their projection on $\mathbb{R}^d$.

The goal of this article is to go beyond the matrices that can be written as (1.7). Of course, as we shall discuss in the next subsection, we already know of some cases where the first d lines do not matter for the A_∞-absolute continuity of the elliptic measure (see [David et al. 2019a; Feneuil et al. 2021]), and we also know that the A_∞ property is stable under Carleson perturbations (see [Mayboroda and Poggi 2021]). However, we do not know, for instance, whether it is possible that the bottom right corner of $\mathcal{A}$ is not a Carleson perturbation of a submatrix of the form $b(x,|t|) I_{n-d}$.

Most of the earlier literature focused on elliptic operators that are "close" to an operator for which $|t|$ (or t in the codimension 1 case) is a solution. In this article, we show that we are justified in replacing $|t|$ by any x-independent "Green function with pole at infinity". We shall first construct the Green function with pole at infinity in the spirit of [David et al. 2021a]. The Green function (and the Green function with pole at infinity) has a deep connection with the harmonic measure (see Lemma 2.9 below); some recent works [David and Mayboroda 2022a; David et al. 2023; 2022] even started to link the geometry of $\partial\Omega$ directly to bounds on the Green function (instead of estimates on the harmonic measure). We shall thus study the Green function with pole at infinity in a few easy cases and deduce that the harmonic measure and the Lebesgue measure are comparable (hence A_∞-absolute continuous with respect to each other). Then we will use the Green function with pole at infinity as a substitute of $|t|$ in a now classical argument that establishes the stability of the A_∞-absolute continuity of the harmonic measure under some transformations on the elliptic operator. This will enlarge the class of operators for which the A_∞-absolute continuity of the harmonic measure is known, especially in the case where $d < n-1$.

Presentation of the results. In the rest of the article d is an integer in $\{0, \dots, n-1\}$. If $d = n-1$, then $\Omega = \mathbb{R}^n_+ = \mathbb{R}^{d+1}_+ = \{(x,t) \in \mathbb{R}^d \times (0, +\infty)\}$. If $d < n-1$, then $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d = \{(x,t) \in \mathbb{R}^d \times \mathbb{R}^{n-d},\ t \neq 0\}$. When we write that $0 < |t| < r$, we understand $t \in (0,r)$ if $n-d = 1$ and $t \in B(0,r) \subset \mathbb{R}^{n-d}$ otherwise.

If $L = -\operatorname{div} A\nabla$ satisfies (1.3)–(1.4), then the elliptic measure defined in (1.5) is nondegenerate, is doubling, and satisfies the change of pole property (respectively Lemmas 11.10, 11.12, and 11.16 in [David et al. 2021b]), and those conditions are the ones needed to prove the following result from [David et al. 2019a].

Theorem 1.9 (Theorem 8.9 in [David et al. 2019a]). *Let $L = -\operatorname{div} A\nabla$, where the real matrix-valued function A satisfies the ellipticity and boundedness conditions* (1.1)–(1.2). *Assume that there exists $M > 0$ such that, for any Borel set $H \subset \mathbb{R}^d$, the solution u_H defined by $u_H(X) = \omega^X_L(H)$ satisfies the Carleson measure estimate*

$$\sup_{x \in \mathbb{R}^d, r>0} \fint_{B_{\mathbb{R}^d}(x,r)} \int_{|t|<r} |t\nabla u_H|^2 \frac{dy\,dt}{|t|^{n-d}} \leq M. \tag{1.10}$$

Then the elliptic measure is A_∞ with respect to the Lebesgue measure on $\mathbb{R}^d$, i.e., for every $\epsilon > 0$ there exists a $\delta > 0$ (that depends only on ϵ, d, n, C_L, and M) such that for every ball $B := B(x,r) \subset \mathbb{R}^d$, every t that satisfies $|t| = r$, and any Borel set $E \subset B$, one has

$$\text{if } \frac{\omega^{(x,t)}_L(E)}{\omega^{(x,t)}_L(B)} < \delta, \quad \text{then } \frac{|E|}{|B|} < \epsilon. \tag{1.11}$$

For a proof when $d = n-1$, see Corollary 3.2 in [Kenig et al. 2016]. The condition (1.10) is closely related to another characterization of the A_∞-absolute continuity of the elliptic measure on $\mathbb{R}^d$ called BMO-solvability, which can be found in [Dindoš et al. 2011] for the codimension 1 case and in [Mayboroda and Zhao 2019] when $d < n-1$.

The condition (1.10) means that $(|t||\nabla u_H|)^2 |t|^{d-n}\,dt\,dx$ is a Carleson measure. In order to lighten the presentation, we introduce a notation for inequalities like (1.10). We say that a quantity f satisfies the *Carleson measure condition* if there exists $C > 0$ such that

$$\|f\|_{L^\infty} \leq C \quad \text{and} \quad \sup_{x \in \mathbb{R}^d, r>0} \fint_{B_{\mathbb{R}^d}(x,r)} \int_{|t|<r} |f|^2 \frac{dt\,dy}{|t|^{n-d}} \leq C. \tag{1.12}$$

In short, we write $f \in CM_2$ or $f \in CM_2(C)$ when we want to refer to the constant on the right side of the bound (1.10). So to conclude, in order to apply Theorem 1.9, we need to assume that there exists $K > 0$ such that for any Borel set H, the function u_H exists in $CM_2(K)$. It will also be useful to write the variant $f \in \widetilde{CM}_2(C)$ when

$$\sup_{x\in\mathbb{R}^d, r>0} \fint_{B_{\mathbb{R}^d}(x,r)} \int_{|t|<r} \Big(\sup_{|Z-(y,t)|<|t|/4} |f(Z)|^2\Big) \frac{dt\,dy}{|t|^{n-d}} \leq C. \tag{1.13}$$

To the best of the author's knowledge, in our setting of high codimensional boundaries, the most general condition on the coefficients of the matrix A that ensures the A_∞-absolute continuity of the elliptic measure with respect to the d-dimensional Hausdorff measure is given in [Feneuil et al. 2021].

Theorem 1.14 (Theorem 1.9 (1) in [Feneuil et al. 2021] for $p = 2$). *Let $L = -\operatorname{div} |t|^{d+1-n}\mathcal{A}\nabla$, where the real matrix-valued function $\mathcal{A}$ satisfies the ellipticity and boundedness conditions* (1.1)–(1.2). *Assume that $\mathcal{A}$ can be decomposed as*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{B}_3 & b\cdot I_{n-d} \end{pmatrix} + \mathcal{C}, \tag{1.15}$$

where I_{n-d} is the identity matrix, $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{B}_3$ are $d\times d$, $d\times(n-d)$, and $(n-d)\times d$ matrix-valued functions, respectively, and b is a scalar function, all of which satisfy

- $K^{-1} \leq b \leq K$,
- $|t||\nabla b| + |t||\nabla_x \mathcal{B}_3| + |t|^{n-d}\operatorname{div}_t(|t|^{d+1-n}\mathcal{B}_3) + |\mathcal{C}| \in CM_2(K)$,

for a constant $K > 0$. Then the hypothesis (1.10) *of Theorem 1.9 is satisfied* (*with a constant M that depends only on d, n, C_L, and K*) *and therefore the elliptic measure ω_L^X is A_∞-absolutely continuous with respect to the Lebesgue measure.*

Remarks. (i) In codimension 1, that is when $d = n-1$, Theorem 1.14 requires that the last line $\mathfrak{a}_{d+1}$ of the matrix $\mathcal{A}$ can be decomposed as $\mathfrak{a}_{d+1} = \mathfrak{b}_{d+1} + \mathfrak{c}_{d+1}$ with $|t||\nabla\mathfrak{b}_{d+1}| + |\mathfrak{c}_{d+1}| \in CM_2$. This condition is thus weaker than the one found in [Kenig and Pipher 2001], where one assumes that $|t||\nabla\mathcal{A}| \in CM$, and the conditions are the same if we add to that result the perturbation theory from [Hofmann and Martell 2012]. However, to the best of the author's knowledge, the first time where no conditions on the first d lines were assumed is in [David et al. 2019a; Feneuil et al. 2021].

(ii) Observe that if $\mathcal{A}$ is a $(d+1)\times(d+1)$ matrix-valued function on $\mathbb{R}^{d+1}_+$ that satisfies the assumptions of the above theorem, then the $n\times n$ matrix-valued function $\widetilde{\mathcal{A}}$ defined from $\mathcal{A}$ on $\mathbb{R}^n\setminus\mathbb{R}^d$ as in (1.7) also satisfies the assumptions of Theorem 1.14.

(iii) With the same argument as the one used in [Dindoš et al. 2007, Corollary 2.3], one can show that if $\mathcal{B}_3$ and b satisfy

$$(x,t) \mapsto \operatorname*{osc}_{B((x,t),|t|/4)} \mathcal{B}_3 + \operatorname*{osc}_{B((x,t),|t|/4)} b \in CM_2(K), \tag{1.16}$$

where $\operatorname{osc}_B f = \sup_B f - \inf_B f$, then we can find $\widehat{\mathcal{B}}_3$ and $\hat{b}$ such that

$$(x,t) \mapsto \sup_{B((x,t),|t|/4)} |\mathcal{B}_3 - \widehat{\mathcal{B}}_3| + \operatorname*{osc}_{B((x,t),|t|/4)} |b-\hat{b}| \in CM_2(K')$$

and

$$\nabla\widehat{\mathcal{B}}_3 + \nabla\hat{b} \in CM_2(K').$$

So assuming the apparently weaker condition (1.16) is enough to satisfy the assumptions of Theorem 1.14 and therefore obtain the A_∞-absolute continuity of the elliptic measure.

When $d < n-1$, the operator $L = -\operatorname{div} A\nabla$ will necessarily depend on $|t|$ as long as it satisfies (1.3)–(1.4). However, once the weight $|t|^{d+1-n}$ is removed, we can see that Theorem 1.14 does not even consider the simple case where $\mathcal{A} = |t|^{n-d-1}A$ is an arbitrary constant elliptic matrix.

Let $\mathcal{T}_3$ be an $(n-d)\times d$ matrix-valued function and $\mathcal{T}_4$ be an $(n-d)\times(n-d)$ matrix-valued function. We say that $(\mathcal{T}_3, \mathcal{T}_4)$ satisfies (H1) if

$$\mathcal{T}_3 \text{ and } \mathcal{T}_4 \text{ are } x\text{-independent,} \tag{H1}$$

and we say that $(\mathcal{T}_3, \mathcal{T}_4)$ satisfies (H2) if

$$\begin{cases} (\mathcal{T}_3)^T \nabla|t| \ \text{ is } x\text{-independent,} \\ \text{there exists } h : (0,+\infty) \mapsto \mathbb{R} \text{ such that } (\mathcal{T}_4)^T \nabla|t| = h(|t|)\nabla|t|. \end{cases} \tag{H2}$$

In addition, we say that $\mathcal{T}_4$ satisfies (H1)/(H2) if $(0, \mathcal{T}_4)$ satisfies (H1)/(H2). Note that when $d = n-1$, $\mathcal{T}_4$ is a scalar function, and (H1) and (H2) are the same hypothesis.

The condition (H2) for $\mathcal{T}_4$ is neither weaker nor stronger than (H1). Roughly, if $\mathcal{T}_4$ satisfies (H2), then $\nabla|t|$ is an eigenvalue of $\mathcal{T}_4$ and $\mathcal{T}_4$ may depend on x.

Example 1.17. If we set $v(t)$ to be a horizontal vector orthogonal to t and independent of x, for instance $v(t) = (-t_2, t_1, 0, \dots, 0)$, and $a(x)$ to be a vertical vector in $\mathbb{R}^{n-d}$ independent of t, for instance $a(x) = (\cos(x), 0, \dots, 0)^T$, then

$$\mathcal{T}_4 := I_{n-d} + \frac{1}{2|t|}a(x)v(t)$$

satisfies (H2) but not (H1). On the other hand, a matrix $\mathcal{T}_4$ which is constant will satisfy (H1) but not (H2) except if $\mathcal{T}_4$ is actually a scalar multiple of the identity matrix. Also, observe that $\mathcal{T}_4$ can go beyond $b \cdot I_{n-d}$ and still stabilize $\nabla|t|$. Remember that t is seen as a horizontal vector and hence

$$\mathcal{T}_4 := I_{n-d} + \frac{1}{2}\frac{t^T t}{|t|^2}$$

is a matrix that satisfies both (H1) and (H2) but is not the multiplication of the identity matrix by a scalar function.

Our first result states that, if the last $n-d$ lines of $\mathcal{A}$ satisfy either (H1) or (H2), then the elliptic measure and the Lebesgue measure on $\mathbb{R}^d$ are equivalent. Taking matrices as given in Example 1.17 will already allow us to obtain control of the harmonic measure for some elliptic operators not considered in the previous literature (for instance when $\mathcal{A}$ is a constant matrix where $\mathcal{T}_3 \neq 0$ and $\mathcal{T}_4$ is not a scalar multiple of the identity).

Theorem 1.18. *Let $L = -\operatorname{div} |t|^{d+1-n}\mathcal{A}\nabla$ be an elliptic operator satisfying* (1.1)–(1.2). *Assume that L is such that*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{T}_3 & \mathcal{T}_4 \end{pmatrix}, \quad \textit{where } (\mathcal{T}_3, \mathcal{T}_4) \textit{ satisfies either } \text{(H1)} \textit{ or } \text{(H2)}. \tag{1.19}$$

Then, for any $Y_0 = (y_0, t_0)$ and any Borel set $E \subset \Delta_{Y_0} := B_{\mathbb{R}^d}(y_0, |t_0|)$, we have

$$C^{-1} \frac{|E|}{|\Delta_{Y_0}|} \leq \omega^{Y_0}(E) \leq \frac{|E|}{|\Delta_{Y_0}|}, \tag{1.20}$$

where $|E|$ denotes the d-dimensional Lebesgue measure and $C > 0$ depends only on n, d, and C_L.

For our second result, we consider elliptic operators whose coefficients are close to a matrix of the form in (1.19). We shall show that for such operators the bound in Theorem 1.9 holds by adapting an $S < N$ argument (see [Kenig et al. 2000] and ensuing literature). Our contribution will be the use of the Green function as a substitute for $|t|$, a bit like in [Akman et al. 2023], but we handle the (possible) roughness of the Green function with a much simpler Caccioppoli-type argument.

Theorem 1.21. *Let $L = -\operatorname{div} |t|^{d+1-n} \mathcal{A} \nabla$ be an elliptic operator satisfying* (1.1)–(1.2), *and write the decomposition*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{B}_3 + \mathcal{C}_3 & b\mathcal{T}_4 + \mathcal{C}_4 \end{pmatrix}, \tag{1.22}$$

where b is a scalar function, $\mathcal{A}_1$ is a $d \times d$ matrix, and the dimensions of $\mathcal{A}_2, \mathcal{B}_3, \mathcal{C}_3, \mathcal{T}_4, \mathcal{C}_4$ are such that the matrices complete the $n \times n$ matrix $\mathcal{A}$. Assume that the submatrices of $\mathcal{A}$ satisfy the following:

(a) *$\mathcal{T}_4$ satisfies either* (H1) *or* (H2),

and there exists a constant $K > 0$ such that

(b) $K^{-1} \leq b \leq K$,

(c) $|\mathcal{C}_3| + |\mathcal{C}_4| \in \widetilde{CM}_2(K)$,

(d) $|t||\nabla b| + |t||\operatorname{div}_x(\mathcal{B}_3)^T| + |t|^{n-d}|\operatorname{div}_t(|t|^{d+1-n}\mathcal{B}_3)| \in CM_2(K)$.

Then the hypothesis (1.10) *of Theorem 1.9 is true and thus the elliptic measure ω_L^X is A_∞-absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.*

Remarks. (i) Theorem 1.14 is a consequence of Theorem 1.21 when $\mathcal{T}_4$ is the identity matrix.

(ii) In the above theorem, when $M = (M_{ij})_{ij}$ is an $(n-d) \times d$ matrix, then the quantity $\operatorname{div}_x M^T$ is a vector in $\mathbb{R}^{n-d}$ whose k-th component is $\sum_{j=1}^d \partial_{x_j} M_{kj}$, and similarly the quantity $\operatorname{div}_t M$ is a vector in $\mathbb{R}^d$ whose k-th component is $\sum_{i=1}^{n-d} \partial_{t_i} M_{ik}$.

(iii) When $d = n-1$, $\mathcal{T}_4$ is a scalar function, and (H2) should read "there exists $h : (0, +\infty) \mapsto \mathbb{R}$ such that $\mathcal{T}_4 \nabla t = h(t) \nabla t$ for all $t \in (0, +\infty)$", but the later just means that $\mathcal{T}_4$ is x-independent, and thus (H1) and (H2) are the same hypothesis.

(iv) We actually prove a stronger estimate than (1.10); we prove a local $S < N$ L^2-estimate which is stated in (4.10) below. We see a priori no big obstacles in our methods that will stop us from obtaining $N < S$ estimates under the assumptions of Theorem 1.21, and hence from studying the solvability of the Dirichlet problem.

In the next result, we assume a stronger condition on $\mathcal{T}_4$ which will allow us to be slightly more flexible on the bottom left corner of $\mathcal{A}$. In the next lemma, $\mathcal{B}_3$ can satisfy either $|t|^{n-d}|\mathrm{div}_t\, |t|^{d+1-n}\mathcal{B}_3| \in CM_2$ as in Theorem 1.21, or simply $|t|\,|\mathrm{div}_t\, \mathcal{B}_3| \in CM_2$.

Theorem 1.23. *Assume that $d < n-2$. Let $L = -\operatorname{div} |t|^{d+1-n}\mathcal{A}\nabla$ be an elliptic operator satisfying* (1.1)–(1.2). *Write the decomposition*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{B}_3+\widetilde{\mathcal{B}}_3+\mathcal{C}_3 & b\mathcal{T}_4+\mathcal{C}_4 \end{pmatrix}, \tag{1.24}$$

and assume that

(a) $(\mathcal{T}_4)^T\nabla|t| = \nabla|t|$,

and there exists a constant $K > 0$ such that

(b) $K^{-1} \leq b \leq K$,

(c) $|\mathcal{C}_3| + |\mathcal{C}_4| \in \widetilde{CM}_2(K)$,

(d) $|t|\,|\nabla b| + |t|\,|\mathrm{div}_x(\mathcal{B}_3)^T| + |t|^{n-d}|\mathrm{div}_t(|t|^{d+1-n}\mathcal{B}_3)| \in CM_2(K)$,

(e) $|t|\,|\mathrm{div}_x(\widetilde{\mathcal{B}}_3)^T| + |t|\,|\mathrm{div}_t\, \widetilde{\mathcal{B}}_3| \in CM_2(K)$.

Then the hypothesis (1.10) *of Theorem 1.9 is true and thus the elliptic measure ω_L^X is A_∞-absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.*

Remark 1.25. The last theorem is a bit unmotivated at the moment. One classical strategy to deal with nonflat boundaries is to make changes of variable. One can thus obtain an equivalent problem where the boundary is better (e.g., flat) but the coefficients of the operators are much worse. See for instance [Kenig and Pipher 2001] in the classical case and [David et al. 2019a] in higher codimension. That is why it is key to obtain, in the flat case, the largest possible set of operators for which the harmonic measure is A_∞-absolute continuous with respect to the Lebesgue measure. The term $\mathcal{B}_3$ is the one that we can treat if we adapt the proof of [Kenig and Pipher 2001] in higher codimension, however $\mathcal{B}_3$ is not well adapted to a change of variable and we would much prefer to use $\mathcal{B}_3$.

In [David et al. 2019a], the authors had to introduce a new (and more complicated) change of variable in order to deal with the case where the boundary is the graph of a Lipschitz function. Still, the construction is limited to graphs of Lipschitz functions with small Lipschitz constant. I claim here that we can deal with big Lipschitz constant if we can allow terms in the form of $\widetilde{\mathcal{B}}_3$ in the bottom left corner of $\mathcal{A}$, as we do in Theorem 1.23.

The full construction of the change of variable that maps the graph of an arbitrary Lipschitz function $\varphi : \mathbb{R}^d \to \mathbb{R}^{n-d}$ to $\mathbb{R}^d$ and that turns the elliptic operator from [David et al. 2019a] into one in the form of (1.24) will not be done here, since it would be too long and technical (and we do not have a new result to prove with it). We will only give a rough idea via an example. If the Lipschitz function is

$$\varphi : x \in \mathbb{R} \mapsto (ax, 0) \in \mathbb{R}^2$$

and its graph is given by $\Phi(x) = (x, \varphi(x))$, then the change of variable that maps $\mathbb{R}$ to the graph of φ constructed in [Kenig and Pipher 2001] would be

$$\rho_1(x, t_1, t_2) = (x, ax + t_1, t_2) = \Phi(x) + (0, t),$$

while the one in [David et al. 2019a] would be

$$\rho_2(x, t_1, t_2) = (x - c_1 t_1, ax + c_2 t_1, t_2) = \Phi(x) + (-c_1 t_1, c_2 t_1, t_2),$$

where $c_1 = a/\sqrt{1+a^2}$ and $c_2 = 1/\sqrt{1+a^2}$ are such that $\Phi(x)$ is orthogonal to $(-c_1 t_1, c_2 t_1, t_2)$. Our alternative is to take

$$\rho_3(x, t_1, t_2) = (x, ax + ct_1, t_2) = \Phi(x) + (0, ct_1, t_2), \quad \text{with } c = \sqrt{1+a^2},$$

which is constructed so that the distance between $\rho_3(x, t_1, t_2)$ and the graph of φ is $|t|$ (like for ρ_2 but where $\rho_3(x, t_1, t_2)$ lies in the plane $\{(x, s_1, s_2),\ (s_1, s_2) \in \mathbb{R}^2\}$, like for ρ_1). If we consider the operator $L = -\operatorname{div} \delta(X)^{-1} \nabla$, where $\delta(X)$ is the distance between X and the graph of φ, then using the change of variable ρ_3 will turn L into $L_3 = -\operatorname{div} |t|^{-1} \mathcal{A}^3 \nabla$ where

$$\mathcal{A}^3 = \begin{pmatrix} c^{-4} & -ac & 0 \\ -ac & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is in the form (1.24), but not in the form (1.15) or (1.22). Replacing an affine function φ by a Lipschitz function included in a plane will already complicate the computations, but if we change Φ to a mollified version $\Phi_{|t|}$ in the construction of ρ_3, then we pretend that it stays "fairly short". Adding the torsion (i.e., when the Lipschitz function is not anymore included in a plane) will complicate the construction even more.

The article is divided as follows. Section 2 introduces the notion of a Green function with pole at infinity and will deduce a relation between this Green function and the elliptic measure that holds whenever L satisfies the ellipticity and boundedness conditions (1.1)–(1.2). Section 3 is devoted to the study of operators of the form (1.19) and proves Theorem 1.18. In Section 4, we demonstrate Theorem 1.21 and 1.23 by establishing a local $S < N$ estimate that implies (1.10).

In the rest of the article, $A \lesssim B$ means that $A \leq CB$ for a constant C whose dependence on the parameters will be stated or will be obvious from context. In addition, $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

2. General results on the Green function with pole at infinity

In this section, we consider an elliptic operator $L = -\operatorname{div} |t|^{d+1-n} \mathcal{A} \nabla$ satisfying (1.1)–(1.2). Even if this article applies when $\Omega = \mathbb{R}^{d+1}_+$ (if $d = n-1$) or $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d$ (if $d < n-1$), the definitions and results of this section can be easily generalized to domains (and elliptic operators) that enter the scope of the elliptic theory developed in [David et al. 2021b; 2020]. In particular, we only need Ω to satisfy the Harnack chain condition and the corkscrew point condition (see [David et al. 2021b; 2020] for these definitions).

We need a bit of functional theory, which is only needed for the precise statement of Definition 2.4 and Proposition 2.5 below, and can be overlooked. The space

$$W := \left\{ u \in L^1_{\mathrm{loc}}(\Omega) \int_\Omega |\nabla u| \frac{dt\,dx}{|t|^{n-d-1}} < +\infty \right\} \tag{2.1}$$

is equipped with the seminorm $\|u\|_W := \|\nabla u\|_{L^2(\Omega)}$. Observe that $\|\cdot\|_W$ is a norm for $C_0^\infty(\Omega)$ and we write W_0 for the completion of $C_0^\infty(\Omega)$ under $\|\cdot\|_W$. We also define

$$W_0(\overline{\Omega}) := \{u \in W^{1,2}_{\mathrm{loc}}(\Omega) : u\varphi \in W_0 \;\text{ for any }\; \varphi \in C_0^\infty(\mathbb{R}^n) < +\infty\}. \tag{2.2}$$

The proof of the properties of W, W_0, and $W_0(\overline{\Omega})$ can be found in [David et al. 2021b; 2020], but let us give a few comments to build the reader's intuition. The spaces W and W_0 are the ones where we find the solutions to the Dirichlet problem $Lu = 0$ in Ω, $u = f \in H^{1/2}(\mathbb{R}^d)$ by using the Lax–Milgram theorem; here $H^{1/2}(\mathbb{R}^d) = W^{1/2,2}(\mathbb{R}^d) = B^{1/2}_{2,2}(\mathbb{R}^d)$ is the (classical) Besov space of traces. The space W_0 is the subspace of W containing the functions with zero trace. The space $W_0(\overline{\Omega})$ is a space bigger than W_0 that possess the same local properties as W_0 but does not have any control when $|(x,t)| \to \infty$.

We recall that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution to $Lu = 0$ in Ω if

$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla\varphi \frac{dt\,dx}{|t|^{n-d-1}} = 0 \quad \text{for } \varphi \in C_0^\infty(\Omega). \tag{2.3}$$

Definition 2.4. A Green function (associated to L^*) with pole at infinity is a *positive* weak solution $G := G_{L^*} \in W_0(\overline{\Omega})$ to $L^*u = -\operatorname{div} |t|^{d+1-n}\mathcal{A}^T\nabla u = 0$ in Ω.

Be aware that, in the above definition, the function G is a solution to the adjoint operator $L^* = -\operatorname{div}|t|^{d+1-n}\mathcal{A}^T\nabla$. We prefer here to associate G to the adjoint right away, because it is the appropriate tool we ultimately need for our proofs. But since L and L^* satisfy the same properties (1.1)–(1.2), we have the following.

Proposition 2.5 [David et al. 2021a, Lemma 6.5]. *A Green function with pole at infinity G enjoys the following properties*:

- $G \in C(\overline{\Omega})$, *i.e., G is continuous up to the boundary* $\mathbb{R}^d$.
- $G = 0$ *on* $\mathbb{R}^d$.
- *G is unique up to a constant. We write G_X for the only Green function with pole at infinity which satisfies $G_X(X) = 1$, and the uniqueness gives*

$$G_X(Y)G_Y(X) = 1 \quad \textit{for } X, Y \in \Omega. \tag{2.6}$$

- *Let $G^Y(X)$ be the Green function* (*associated to L^**) *with pole at Y as defined in Chapter* 10 *of* [David et al. 2021b]. *Take $Y_0 = (y_0, t_0) \in \Omega$, and define for $j \in \mathbb{N}$ the point $Y_j = (y_0, 2^j t_0)$. There exists a subsequence $j_n \to \infty$ such that*

$$\frac{G^{Y_{j_n}}}{G^{Y_{j_n}}(Y_0)} \textit{ converges uniformly on compact sets of } \overline{\Omega} \textit{ to } G_{Y_0}. \tag{2.7}$$

Proof. The first two points are a consequence of the De Giorgi–Nash–Moser estimates on weak solutions that can be found (for instance) in [David et al. 2021b, Chapter 8]. The last two points are in Lemma 6.5 from [David et al. 2021a] or in its proof. □

We assign to any point $Z = (z, s) \in \Omega$ the boundary ball

$$\Delta_Z := B(z, |s|) \subset \mathbb{R}^d. \tag{2.8}$$

We apply the comparison principle (see [Caffarelli et al. 1981a] for the codimension 1 case and [David et al. 2021b] for the higher codimension) to compare the Green function with pole at infinity and the elliptic measure.

Lemma 2.9. *Let $Y_0 = (y_0, t_0) \in \Omega$. If $X = (x, t) \in \Omega$ satisfies $x \in B(y_0, 2|t_0|)$ and $0 < |t| < 2|t_0|$, we have that*

$$C^{-1} G_{Y_0}(X) \le \left(\frac{|t|}{|t_0|}\right)^{1-d} \omega^{Y_0}(\Delta_X) \le C G_{Y_0}(X), \tag{2.10}$$

where $C > 0$ depends only on $n,\ d$, and C_L. Here $G_{Y_0} = G_{L^*,Y_0}$ is defined in Proposition 2.5 and is the Green function associated to L^* with pole at infinity, and $\omega^{Y_0} = \omega_L^{Y_0}$ is the elliptic measure associated to L with pole at Y_0 defined in (1.5).

Remark 2.11. We can use the uniqueness of the Green function (2.6) to get an estimate of $G_{Y_0}(X)$ using the elliptic measure when X is far from Y_0.

Proof. We need to invoke some results from [David et al. 2021b]. The classical case $d = n - 1$ is not included in that work but is either already known to the reader or can be found in the last section of [David et al. 2020].

Let $Y_1 = (y_0, 4t_0)$. The change of pole property [David et al. 2021b, Lemma 11.16] states that, for any Borel set $E \subset \Delta_{Y_1} = 4\Delta_{Y_0}$ and any $Y \in \Omega$ satisfying $|Y - y_0| > 8t_0$, we have

$$\omega^{Y_1}(E) \approx \frac{\omega^Y(E)}{\omega^Y(\Delta_{Y_1})}, \tag{2.12}$$

with constants that depend only on $n,\ d$, and C_L. Together with the doubling property of the elliptic measure [David et al. 2021b, Lemma 11.12] and the Harnack inequality [David et al. 2021b, Lemma 8.9], we deduce that, for the same set E, point Y, and with constants that depend on the same parameters, we have

$$\omega^{Y_0}(E) \approx \frac{\omega^Y(E)}{\omega^Y(\Delta_{Y_0})}. \tag{2.13}$$

For our second result, we want to compare the Green function and the elliptic measure. Let $g^X(Y)$ be the Green function associated to L with pole in X. Then [David et al. 2021b, Lemma 10.6] implies that

$$G^Y(X) = g^X(Y) \quad \text{for } X, Y \in \Omega. \tag{2.14}$$

Moreover, [David et al. 2021b, Lemma 11.11] gives, for $X = (x, t) \in \Omega$ and $Y \in \Omega \setminus B_{\mathbb{R}^n}(x, 2|t|)$,

$$|t|^{d-1} g^X(Y) \approx \omega^Y(\Delta_X), \tag{2.15}$$

with constants that depend only on n, d, and C_L. So the combination of (2.14) and (2.15) implies, for $X = (x,t) \in \Omega$, that

$$|t|^{d-1} G^Y(X) \approx \omega^Y(\Delta_X) \quad \text{for } Y \in \Omega \setminus B_{\mathbb{R}^n}(x, 2|t|). \tag{2.16}$$

The proof of the lemma is then pretty easy. Let Y_0 and X be as in the assumptions of the lemma. For any Y far enough from Y_0, we use (2.16) to obtain

$$\frac{G^Y(X)}{G^Y(Y_0)} \approx \frac{|t|^{1-d}\omega^Y(\Delta_X)}{|t_0|^{1-d}\omega^Y(\Delta_{Y_0})},$$

but, since the conditions on X and Y_0 imply $E := \Delta_X \subset 4\Delta_{Y_0}$, the estimate (2.13) yields

$$\frac{G^Y(X)}{G^Y(Y_0)} \approx \left(\frac{|t|}{|t_0|}\right)^{1-d} \omega^{Y_0}(\Delta_X).$$

The above bounds on $G^Y/G^Y(Y_0)$ are uniform in Y, therefore, by (2.7), those bounds are transferred to G_{Y_0}. The lemma follows. □

3. x-independent Green functions with pole at infinity

In this section, we shall make two easy observations: first, that the Green function, associated to

$$L^* = -\operatorname{div} |t|^{n-d-1} \mathcal{A}^T \nabla$$

as in Section 2, with pole at infinity is independent of x whenever $\mathcal{A}$ is x-independent; and second, if both $\mathcal{A}$ and the Green function G with pole at infinity are x-independent, then G does not depend on the first $n-d$ lines of $\mathcal{A}$. We shall invoke, in addition, the uniqueness of the Green function with pole at infinity and (2.12) to deduce that the elliptic measure and the Lebesgue measure are equivalent on $\mathbb{R}^d$ whenever the last $n-d$ lines of $\mathcal{A}$ are x-independent.

Lemma 3.1. *Let $L = -\operatorname{div} |t|^{d+1-n}\mathcal{A}\nabla$ be an elliptic operator satisfying* (1.1)–(1.2) *and where $\mathcal{A}$ is as in* (1.19). *Then the Green function* (*associated to L^**) *with pole at infinity is x-independent and satisfies, for any $Y_0 = (y_0, t_0)$ and $X = (x,t)$ in Ω,*

$$C^{-1}\frac{|t|}{|t_0|} \le G_{Y_0}(X) \le C\frac{|t|}{|t_0|}, \tag{3.2}$$

where the constants depend only on $n-d$ and C_L.

Proof. The proof is similar under either assumption, (H1) or (H2). We know that the Green function with pole at infinity has to depend on $|t|$, but it does not need to depend on x or $t/|t|$. When $(\mathcal{T}_3, \mathcal{T}_4)$ satisfies (H2), L stabilizes the space of functions that depend on $|t|$, and thus by uniqueness the Green function will depend only on $|t|$. When $(\mathcal{T}_3, \mathcal{T}_4)$ satisfies (H1), L stabilizes the space of functions that are x-independent, and hence the Green function will be independent of x.

Case 1: $(\mathcal{T}_3, \mathcal{T}_4)$ satisfies (H1). Define

$$L_0 := -\operatorname{div} |t|^{d+1-n} \mathcal{T}_4 \nabla. \tag{3.3}$$

The operator L_0 is an elliptic operator on $\mathbb{R}^{n-d} \setminus \{0\}$ satisfying the ellipticity and boundedness conditions (1.1)–(1.2) with the same constant C_L as L.

When $d < n-1$, the space $\mathbb{R}^{n-d} \setminus \{0\}$ and the operator L_0 enter the scope of the elliptic theory developed in [David et al. 2021b] or [David et al. 2020],[1] and so all the results in Section 2 hold. Of course, the study of the elliptic measure of L_0, where the boundary is reduced to the point $\{0\}$, is trivial and hence not very interesting. But using this easy case will allow us to find a good candidate for the Green function with pole at infinity for L^*. Let $\omega^X_{L_0}$ be the elliptic measure on $\{0\}$ and $G_{(L_0)^*,t_0}$ be the Green function with pole at infinity (associated to $(L_0)^*$) which takes the value 1 at t_0. Lemma 2.9 implies, for $|t| < 2|t_0|$, that

$$G_{(L_0)^*,t_0}(t) \approx \frac{|t|}{|t_0|} \omega^{Y_0}(\Delta_t) = \frac{|t|}{|t_0|} \omega^{Y_0}(\{0\}) = \frac{|t|}{|t_0|}.$$

The probability measure $\omega^X_{L_0}$ on $\{0\}$ obviously satisfies $\omega^X_{L_0}(\{0\}) = 1$, hence

$$G_{(L_0)^*,t_0}(t) \approx \frac{|t|}{|t_0|} \quad \text{for } |t| < 2|t_0|. \tag{3.4}$$

When $|t| \geq 2|t_0|$, we use (2.6) and (3.4) to write

$$G_{(L_0)^*,t_0}(t) = [G_{(L_0)^*,t}(t_0)]^{-1} \approx \left(\frac{|t_0|}{|t|}\right)^{-1} = \frac{|t|}{|t_0|}.$$

We conclude, for any $t, t_0 \in \mathbb{R}^{n-d}$, that

$$G_{(L_0)^*,t_0}(t) \approx \frac{|t|}{|t_0|}. \tag{3.5}$$

When $d = n-1$, the result (3.5) holds without the need of Lemma 2.9. The operator L_0 is defined on the half line, and there exists $f(t)$ defined on $(0, +\infty)$ such that $L_0 = \partial_t f(t) \partial_t$ and $f(t) \approx 1$ in order to satisfy the ellipticity and boundedness conditions. A simple exercise of integration shows that the Green functions with pole at infinity of $(L_0)^* = L_0$ are

$$G_{(L_0)^*}(t) = K \int_0^t \frac{dt}{f(t)} \approx C|t|, \tag{3.6}$$

where K is any positive constant, and thus (3.5) follows easily.

We set, for $Y_0 = (y_0, t_0) \in \Omega$ and $X = (x, t) \in \Omega$,

$$H_{Y_0}(X) := G_{(L_0)^*,t_0}(t). \tag{3.7}$$

[1] Maybe also when $d = n-1$, but let us not take any risks.

Because of the x-independence of H_{Y_0} and $\mathcal{T}_3$, we have for $\varphi \in C_0^\infty(\Omega)$ that

$$\int_\Omega \mathcal{A}^T \nabla H_{Y_0} \cdot \nabla\varphi \frac{dt\,dx}{|t|^{n-d-1}}$$
$$= \int_{\mathbb{R}^{n-d}} (\mathcal{T}_3)^T \nabla_t G_{(L_0)^*,t_0} \cdot \left(\int_{\mathbb{R}^d} \nabla_x \varphi\, dx\right) \frac{dt}{|t|^{n-d-1}} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^{n-d}} (\mathcal{T}_4)^T \nabla_t G_{(L_0)^*,t_0} \cdot \nabla_t \varphi \frac{dt}{|t|^{n-d-1}}\, dx.$$

The first integral on the right-hand side above is 0 because $\int_{\mathbb{R}^d} \nabla_x \varphi(x,t)\, dx = 0$ for all t. The second integral is also 0 because $G_{(L_0)^*,t_0}$ is a weak solution to $(L_0)^*$. So H_{Y_0} is a weak solution to L^*. Moreover, $H_{Y_0} \in W_0(\mathbb{R}^n)$ because $G_{(L_0)^*,t_0} \in W_0(\mathbb{R}^{n-d})$. By the uniqueness given in Proposition 2.5, we necessarily have

$$G_{Y_0}(X) = H_{Y_0}(X) := G_{(L_0)^*,t_0}(t). \tag{3.8}$$

As a consequence, G_{Y_0} is x-independent, and the conclusion (3.2) of the lemma follows from (3.5).

Case 2: $(\mathcal{T}_3, \mathcal{T}_4)$ satisfies (H2). In this case, the proof is a simple exercise of integration. By (1.1) and (1.2), we have

$$(C_L)^{-1}\big|\nabla|t|\big|^2 \le \mathcal{T}_4 \nabla|t| \cdot \nabla|t| \le C_L \big|\nabla|t|\big|^2 \quad \text{for all } t \in \mathbb{R}^{n-d} \setminus \{0\}.$$

Since $\big|\nabla|t|\big| = 1$, our assumption on $\mathcal{T}_4$ implies that

$$(C_L)^{-1} \le h(|t|) \le C_L \quad \text{for all } t \in \mathbb{R}^{n-d} \setminus \{0\}. \tag{3.9}$$

We define g_{r_0} as

$$g_{r_0} = K_{r_0} \int_0^r \frac{1}{h(r)}\, dr, \tag{3.10}$$

where K is chosen such that $g_{r_0}(r_0) = 1$. Our bounds on h yield

$$g_{r_0} \approx \frac{r}{r_0}. \tag{3.11}$$

We construct now $H_{Y_0}(X)$ for $Y_0 = (y_0, t_0) \in \Omega$ and $X = (x,t) \in \Omega$ as

$$H_{Y_0}(X) := g_{|t_0|}(|t|). \tag{3.12}$$

Observe that since H_{Y_0} depends only on $|t|$, we have

$$\begin{aligned} L^* H_{Y_0}(X) &= g'_{|t_0|}(|t|)\, \mathrm{div}_x (\mathcal{T}_3)^T \nabla|t| + \mathrm{div}_t [g'_{|t_0|}(|t|)(\mathcal{T}_4)^T \nabla|t|] \\ &= 0 + K_{|t_0|}\, \mathrm{div}_t \frac{1}{h(|t|)} h(|t|) = 0, \end{aligned}$$

thanks to the conditions (H2) and the definition (3.10). In addition, H_{Y_0} is Lipschitz by (3.10)–(3.9) and is 0 on the boundary, therefore it lies in $W_0(\overline{\Omega})$. So again by uniqueness of the Green function with pole at infinity (see Proposition 2.5), we have $G_{Y_0} = H_{Y_0}$. The conclusion (3.2) is then an easy consequence of (3.12) and (3.11). □

Remark 3.13. An interesting consequence of the above proof, for instance (3.6), is that for a general operator of the form $L = -\operatorname{div} |t|^{d+1-n} \mathcal{A} \nabla$, knowing that the elliptic measure is A_∞-absolute continuous with respect to the Lebesgue measure (or even equivalent to the Lebesgue measure) will not help us to get a lot of control on t-derivatives of the Green functions with pole at infinity. Indeed, it is possible to take h to be any arbitrary function in L^∞ that stays between 1 and 2. In this case, g_{r_0} defined in (3.10) and G_{Y_0} are only Lipschitz. In particular, the nontangential limit of $|\nabla G|$ at the boundary may not exist in any reasonable sense (only inferior and superior limits exist). It means that the estimates on the Green function obtained in [David et al. 2023; 2022] are not equivalent to the A_∞-absolute continuity of the harmonic measure without any restriction on the elliptic operator $\mathcal{L}$.

Corollary 3.14. *Let $L = -\operatorname{div} |t|^{d+1-n} \mathcal{A} \nabla$ be an elliptic operator satisfying* (1.1)–(1.2). *Assume that $\mathcal{A}$ can be written as*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ 0 & \mathcal{T}_4 \end{pmatrix}, \tag{3.15}$$

where $\mathcal{T}_4 \nabla |t| = \nabla |t|$ for all $t \in \mathbb{R}^{n-d}$.

Then, for $X = (x, t) \in \Omega$ and $Y_0 = (y_0, t_0) \in \Omega$, the Green function with pole at infinity satisfies

$$G_{Y_0}(X) = \frac{|t|}{|t_0|}. \tag{3.16}$$

Proof. Under our assumptions, $\mathcal{T}_4$ satisfies (H2) with $h(r) \equiv 1$. From the proof of Lemma 3.1, we have $G_{Y_0}(X) = g_{|t_0|}(|t|)$ where $g_{r_0}(r)$ is given by (3.10). The lemma follows. □

Proof of Theorem 1.18. Lemma 3.1 easily implies the equivalence between the harmonic measure and the surface measure. It was already done in the proof of Theorem 6.7 in [David et al. 2021a], but let us repeat it for completeness. Take $x \in \Delta_{Y_0}$. The combination of Lemma 2.9 and Lemma 3.1 requires, for any $0 < r < |t_0|$ and any $X = (x, t)$ satisfying $|t| = r$, that

$$\omega^{Y_0}(B_{\mathbb{R}^d}(x, r)) \approx G_{Y_0}(X) \left(\frac{|t|}{|t_0|} \right)^{d-1} \approx \left(\frac{|t|}{|t_0|} \right)^{d} = \frac{|B_{\mathbb{R}^d}(x, r)|}{|\Delta_{Y_0}|}. \tag{3.17}$$

In particular, the measure is absolutely continuous with respect to the d-dimensional Lebesgue measure on $\mathbb{R}^d$, and, by the Lebesgue differentiation theorem, the Poisson kernel $k^{Y_0} := d\omega^{Y_0}/d\mathcal{L}^d$ satisfies, for almost any $x \in \Delta_{Y_0}$,

$$k^{Y_0}(x) = \lim_{r \to 0} \frac{\omega^{Y_0}(B_{\mathbb{R}^d}(x, r))}{|B_{\mathbb{R}^d}(x, r)|} \approx \frac{1}{|\Delta_{Y_0}|}.$$

The theorem follows by integrating k^{Y_0} over E. □

4. Proof of Theorems 1.21 and 1.23

The proof of Theorems 1.21 and 1.23 will rely on an S vs N argument, where S is the square function (which will not be introduced here but is related to the left-hand side of (4.10)) and N is the nontangential maximal function. The importance of the two functionals S and N for the A_∞-absolute continuity of the harmonic measure was noted in [Kenig et al. 2000], and the general method to compare S and N (when Carleson measures are involved) was first found in [Kenig and Pipher 2001]. In [Kenig et al. 2016], it was observed that being able to bound the L^p-norm of S by the L^p-norm of N is enough to guarantee the absolute continuity of the harmonic measure, which is basically our Theorem 1.9. The adaptation of the methods to higher codimensional boundaries can be found in Sections 7 and 8 of [David et al. 2019a] and in [Feneuil et al. 2021].

Let $1 \leq d < n$ be integers, and let $\Omega = \mathbb{R}^n_+ := \{(x,t) \in \mathbb{R}^d \times (0,+\infty)\}$ if $d = n-1$ and $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d := \{(x,t) \in \mathbb{R}^d \times \mathbb{R}^{n-d},\ t \neq 0\}$ if $d < n-1$. The nontangential maximal functions N and $\widetilde{N}$ are defined for any continuous function v on Ω and any $x \in \mathbb{R}^d$ as

$$N(v)(x) = \sup_{(y,t)\in\gamma(x)} |v| \quad \text{and} \quad \widetilde{N}(v)(x) = \sup_{(y,t)\in\gamma(x)} \left(\fint_{|Z-(y,t)|<|t|/4} |v|^2\, dZ \right)^{1/2}, \tag{4.1}$$

where

$$\gamma(x) = \{(y,t) \in \Omega,\ |y-x| < |t|\}. \tag{4.2}$$

We shall introduce here the variants $\gamma_{10}(x) := \{(y,t) \in \Omega,\ |y-x| < 10|t|\}$ and $N_{10}(v)(x) := \sup_{\gamma_{10}(x)} |v|$. They will be used to compare $\widetilde{N}$ and N. Indeed, we have the pointwise bound $\widetilde{N}(v)(x) \leq N_{10}(v)(x)$ and it is well known (see [Stein 1993], Chapter II, Section 2.5.1) that $\|N_{10}(v)\|_2 \approx \|N(v)\|_2$. Altogether,

$$\|\widetilde{N}(v)\|_{L^2(\mathbb{R}^d)} \leq \|N_{10}(v)\|_{L^2(\mathbb{R}^d)} \approx \|N(v)\|_{L^2(\mathbb{R}^d)}. \tag{4.3}$$

We recall that the nontangential maximal functions behave well with the Carleson measure condition (1.12) and (1.13). Indeed, if v is a continuous function on Ω and $f \in CM_2(K)$, then we have the Carleson inequality

$$\int_\Omega f^2 v^2 \frac{dx\, dt}{|t|^{n-d}} \lesssim K \|N(v)\|^2_{L^2(\mathbb{R}^d)}, \tag{4.4}$$

and similarly, if $g \in \widetilde{CM}_2(K)$, then

$$\int_\Omega g^2 v^2 \frac{dx\, dt}{|t|^{n-d}} \lesssim K \|\widetilde{N}(v)\|^2_{L^2(\mathbb{R}^d)} \lesssim K \|N(v)\|^2_{L^2(\mathbb{R}^d)}. \tag{4.5}$$

Combining (4.4) and (4.5) with the Cauchy–Schwarz inequality, for all $w \in L^2_{\mathrm{loc}}(\Omega)$, one has

$$\int_\Omega f v w \frac{dx\, dt}{|t|^{n-d}} \leq C K^{1/2} \|N(v)\|_{L^2(\mathbb{R}^d)} \left(\int_\Omega w^2 \frac{dx\, dt}{|t|^{n-d}} \right)^{1/2}, \tag{4.6}$$

$$\int_\Omega g v w \frac{dx\, dt}{|t|^{n-d}} \leq C K^{1/2} \|\widetilde{N}(v)\|_{L^2(\mathbb{R}^d)} \left(\int_\Omega w^2 \frac{dx\, dt}{|t|^{n-d}} \right)^{1/2}. \tag{4.7}$$

We also introduce cut-off functions associated to tent sets. Choose a smooth function $\phi \in C_0^\infty(\mathbb{R})$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $(-1,1)$, $\phi \equiv 0$ on $(2,+\infty)$, and $|\phi'| \leq 2$. For a ball $B := B(x,r) \subset \mathbb{R}^d$, we define Ψ_B as

$$\Psi_B(y,t) = \phi\left(\frac{\operatorname{dist}(x,B)}{|t|} \right) \phi\left(\frac{|t|}{r} \right). \tag{4.8}$$

We also associate to B the tent set $T_B := \{(x,t) \in \Omega : x \in B,\ |t| \leq r\}$. The function Ψ_B is such that $\Psi \equiv 1$ on T_B and $\Psi \equiv 0$ on $\Omega \subset T_{2B}$. Note that, if a different definition of tent sets is used, we can easily change the definition of Ψ_B so that Ψ_B is adapted to the other definition of tent sets.

Theorems 1.21 and 1.23 are consequences of the following lemma.

Lemma 4.9. *If $L = -\operatorname{div} |t|^{d+1-n}\nabla$ satisfies the assumptions of either Theorem 1.21 or Theorem 1.23, then, for any ball $B = B(x,r) \subset \mathbb{R}^d$ and for any bounded weak solution u to $Lu = 0$, we have*

$$\int_\Omega |\nabla u|^2 \Psi_B^4 \frac{dt\,dx}{|t|^{n-d-2}} \leq C(1+K)\|N(u\Psi_B)\|^2_{L^2(\mathbb{R}^d)}, \tag{4.10}$$

where $C > 0$ depends only on n, d, and C_L.

Proof of Theorems 1.21 and 1.23 from Lemma 4.9. We only need to show that (4.10) implies (1.10). Take the function $u_H(X) := \omega_L^X(H)$, which is a weak solution to $Lu = 0$ bounded by 1. Pick $x \in \mathbb{R}^d$ and $r > 0$, and define $B := B(x,r) \subset \mathbb{R}^d$. The function Ψ_B is 1 on $B(x,r) \times \{t \in \mathbb{R}^{n-d},\ 0 < |t| < r\}$, so Lemma 4.9 gives

$$\int_{B(x,r)} \int_{|t|<r} |t\nabla u_H|^2 \frac{dy\,dt}{|t|^{n-d}} \lesssim \|N(u_H\Psi_B)\|^2_{L^2(\mathbb{R}^d)}.$$

The function $N(u_H\Psi_B)$is bounded by 1 and is supported on $4B$ (since $u_H\Psi_B$ is supported on T_{2B}). As a consequence, the above bound becomes

$$\int_{B_{\mathbb{R}^d}(x,r)} \int_{|t|<r} |t\nabla u_H|^2 \frac{dy\,dt}{|t|^{n-d}} \lesssim |B(x,4r)| \lesssim |B(x,r)|.$$

The bound (1.10) and thus the theorems follow. □

Proof of Lemma 4.9. The proof will be largely identical under the two kinds of assumptions that we have (the ones from Theorem 1.21 and the ones from Theorem 1.23). The proof will split at the very end (in Step 5), when we consider terms involving $\mathcal{B}_3$ and $\widetilde{\mathcal{B}}_3$ (the bottom left corner of $\mathcal{A}$), which need to be addressed in a different (yet somehow related) manner.

Our proof will follow the outline of the one of Theorem 7.10 in [David et al. 2019a], but will be significantly different on two occasions. First, in Step 3, we give a simple Caccioppoli-type argument to deal with the possible nonsmoothness of the Green function with pole at infinity, which will replace here what was $|t|$ in [David et al. 2019a]. And in Step 5, we will deal with the terms $\widetilde{\mathcal{B}}_3$, which were considered in neither [David et al. 2019a] nor [Feneuil et al. 2021].

Step 1: Carleson estimates on the cut-off functions. In order to deal with finite quantities, we need to refine our cut-off function Ψ_B. We define $\Psi_{B,\epsilon}$ as

$$\Psi_{B,\epsilon}(y,t) = \Psi_B(y,t)\phi\left(\frac{\epsilon}{|t|}\right), \tag{4.11}$$

where ϕ is the smooth function introduced above (4.8) and was already used to define Ψ_B. We first gather some properties of the cut-off function $\Psi_{B,\epsilon}$. Observe that

$$|\nabla\Psi_{B,\epsilon}(y,t)| \lesssim \frac{1}{|t|} \quad \text{for } (y,t) \in \Omega, \tag{4.12}$$

and $\nabla \Psi_{B,\epsilon}$ is supported on $E_1 \cup E_2 \cup E_3$, where

$$E_1 := \{(y,t) \in \Omega,\ \operatorname{dist}(y,B) \leq 2|t| \leq 2\operatorname{dist}(y,B)\},$$
$$E_2 := \{(y,t) \in \Omega,\ r(B) \leq |t| \leq 2r(B)\},$$

with $r(B)$ being the radius of B, and

$$E_3 := \{(y,t) \in \Omega,\ |t| \leq \epsilon \leq 2|t|\}.$$

So we deduce that

$$|t||\nabla \Psi_{B,\epsilon}(y,t)| + |t|^2 |\nabla \Psi_{B,\epsilon}(y,t)|^2 \lesssim \mathbb{1}_{E_1 \cup E_2 \cup E_3}(y,t). \tag{4.13}$$

We will need the fact that $|t||\nabla \Psi_{B,\epsilon}(y,t)|$ and $(|t||\nabla \Psi_{B,\epsilon}(y,t)|)^{1/2}$ satisfy the Carleson measure condition $\widetilde{CM}_2(M)$ for some uniform constant M which, combined with (4.4), implies, for any continuous function v, that

$$\int_\Omega |t||\nabla \Psi_{B,\epsilon}(y,t)| v^2 \frac{dt\,dx}{|t|^{n-d}} + \int_\Omega |t||\nabla \Psi_{B,\epsilon}(y,t)| v^2 \frac{dt\,dx}{|t|^{n-d}} \lesssim \|\widetilde{N}(v)\|^2_{L^2(\mathbb{R}^d)}. \tag{4.14}$$

Of course, thanks to (4.3), if (4.14) is true, then we also have the analogue estimate where $\widetilde{N}$ is replaced by N. Thanks to (4.13), the claim (4.14) will be then proven if we can show that $\mathbb{1}_{E_1 \cup E_2 \cup E_3} \in \widetilde{CM}_2(M)$, that is

$$\sup_{x \in \mathbb{R}^d,\, r>0} \fint_{B_{\mathbb{R}^d}(x,r)} \int_{|t|<r} \sup_{|Z-(y,t)|<|t|/4} |\mathbb{1}_{E_1 \cup E_2 \cup E_3}(Z)|^2 \frac{dy\,dt}{|t|^{n-d}} \lesssim 1. \tag{4.15}$$

However, (4.15) is an immediate consequence of the fact that, for each $y \in \mathbb{R}^d$,

$$\begin{aligned}\int_{t \in \mathbb{R}^{n-d}} &\sup_{|Z-(y,t)|<|t|/4} |\mathbb{1}_{E_1 \cup E_2 \cup E_3}(Z)|^2 \frac{dt}{|t|^{n-d}} \\ &\leq \int_{\operatorname{dist}(y,B)/4 \leq |t| \leq 2\operatorname{dist}(y,B)} \frac{dt}{|t|^{n-d}} + \int_{r(B)/2 \leq |t| \leq 4r(B)} \frac{dt}{|t|^{n-d}} + \int_{\epsilon/4 \leq |t| \leq 2\epsilon} \frac{dt}{|t|^{n-d}} \lesssim 1.\end{aligned}$$

The claim (4.14) follows.

Step 2: introduction of G. First, we decompose L as

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{B}_3 + \widetilde{\mathcal{B}}_3 + \mathcal{C}_3 & b\mathcal{T}_4 + \mathcal{C}_4 \end{pmatrix}, \tag{4.16}$$

so that it includes the assumptions of both Theorem 1.21 and Theorem 1.23. In particular, we have

$$\begin{gathered}|\mathcal{C}_3| + |\mathcal{C}_4| \in \widetilde{CM}_2(K), \\ |t||\nabla b| + |t||\operatorname{div}_x(\mathcal{B}_3)^T| + |t|^{n-d}|\operatorname{div}_t(|t|^{d+1-n}\mathcal{B}_3)| + |t||\operatorname{div}_x(\widetilde{\mathcal{B}}_3)^T| + |t||\operatorname{div}_t \widetilde{\mathcal{B}}_3| \in CM_2(3K).\end{gathered} \tag{4.17}$$

We set $L_0 := -\operatorname{div} |t|^{d+1-n} \mathcal{A}_0 \nabla$, where

$$\mathcal{A}_0 := \begin{pmatrix} \frac{1}{b}\mathcal{A}_1 & \frac{1}{b}\mathcal{A}_2 \\ 0 & \mathcal{T}_4 \end{pmatrix}, \tag{4.18}$$

which satisfies

$$\frac{1}{b}\mathcal{A} = \mathcal{A}_0 + \frac{1}{b}\begin{pmatrix} 0 & 0 \\ \mathcal{B}_3+\widetilde{\mathcal{B}}_3+\mathcal{C}_3 & \mathcal{C}_4 \end{pmatrix}. \tag{4.19}$$

Let $Y_0 = (y_0, t_0) \in \Omega$ be such that $|t_0| = 1$, and write G for G_{Y_0}, the Green function associated to $(L_0)^*$ with pole at infinity. The important properties of G for this proof are first that $G \in W^{1,2}_{\text{loc}}(\Omega)$ is a weak solution to $(L_0)^* u = 0$ in Ω, that is

$$\int_\Omega \mathcal{A}_0 \nabla\varphi \cdot \nabla G \,\frac{dt\,dx}{|t|^{n-d-1}} = 0 \quad \text{for any compactly supported } \varphi \in W^{1,2}(\Omega), \tag{4.20}$$

and second, that Lemma 3.1 requires that

$$G \text{ is } x\text{-independent and } G(X) \approx |t| \quad \text{for all } X = (x,t) \in \Omega. \tag{4.21}$$

Step 3: estimation of $\|\widetilde{N}(u\Psi^2_{B,\epsilon}\nabla G)\|_2$. If the goal were to only obtain (1.10), we would not need to go through the same computations, we would just have to prove

$$\|\widetilde{N}(u_H \Psi^2_{B,\epsilon}\nabla G)\|_{L^2(\mathbb{R}^d)} \lesssim |B|. \tag{4.22}$$

Since G is a weak solution to $L_0 u = 0$, Caccioppoli's inequality yields

$$\fint_{|Z-(y,t)|<|t|/4} |\nabla G|^2\,dZ \lesssim \frac{1}{|t|^2}\fint_{|Z-(y,t)|<|t|/2} |G|^2\,dZ \quad \text{for } (y,t) \in \Omega.$$

But since $G \approx |t|$ by (4.21), the above inequality becomes

$$\fint_{|Z-(y,t)|<|t|/4} |\nabla G|^2\,dZ \lesssim 1.$$

We take the supremum on $(y,t) \in \gamma(x)$ and then integrate on $x \in 100B$, and we get

$$|B| \gtrsim \|\widetilde{N}(\nabla G)\|_{L^2(100B)} \gtrsim \|\widetilde{N}(u_H \Psi^2_{B,\epsilon}\nabla G)\|_{L^2(\mathbb{R}^d)}$$

because $u_H \le 1$ by construction. The claim (4.22) follows.

However, what we really need in order to prove the inequality (4.10) is

$$\|\widetilde{N}(u\Psi^2_{B,\epsilon}\nabla G)\|_{L^2(\mathbb{R}^d)} \lesssim \|N(u\Psi_{B,\epsilon})\|_{L^2(\mathbb{R}^d)}, \tag{4.23}$$

where u is any weak solution of $Lu = 0$ which is bounded on T_{2B}. To reach this goal, we first need the following Caccioppoli inequality. Let $D \subset \mathbb{R}^n$ be a ball of radius r such that $4D \subset \Omega$ and $5D \cap \partial\Omega \neq \varnothing$. In particular, we have

$$G(X) \approx |t| \approx r \quad \text{for } X = (x,t) \in 2D \tag{4.24}$$

by (4.21). Let Ψ be a function such that $0 \le \Psi \le 1$ and $|\nabla\Psi| \lesssim 1/|t|$, and let u be a weak solution to $Lu = 0$. We claim that

$$\fint_D |\nabla G|^2 u^2 \Psi^4\,dX \lesssim \frac{1}{r^2}\fint_{2D} |u|^2\Psi^2\,dX. \tag{4.25}$$

Let Φ be such that $0 \leq \Phi \leq 1$, $\Phi \equiv 1$ on D, $\Phi \equiv 0$ outside $\frac{4}{3}D$, and $|\nabla\Phi| \leq 5r$. Then

$$\int_D |\nabla G|^2 u^2 \Psi^4 \, dX \leq T := \int_D |\nabla G|^2 u^2 \Psi^4 \Phi^2 \, dX. \tag{4.26}$$

The function G is a weak solution of $L_0 u = 0$, so, by the ellipticity of $\mathcal{A}_0$ and since the weight satisfies $|t|^{d+1-n} \approx r^{d+1-n}$ on $2D$, we have

$$\begin{aligned} T &\lesssim \iint_\Omega \mathcal{A}_0 \nabla G \cdot \nabla G u^2 \Psi^4 \Phi^2 \, \frac{dt\, dx}{|t|^{n-d-1}} \\ &= \iint_\Omega \mathcal{A}_0 \nabla[G u^2 \Psi^4 \Phi^2] \cdot \nabla G \, \frac{dt\, dx}{|t|^{n-d-1}} - 2 \iint_\Omega \mathcal{A}_0 \nabla u \cdot \nabla G \, (G u \Psi^4 \Phi^2) \, \frac{dt\, dx}{|t|^{n-d-1}} \\ &\qquad - 2 \iint_\Omega \mathcal{A}_0 \nabla \Phi \cdot \nabla G \, (G u^2 \Psi^4 \Phi) \, \frac{dt\, dx}{|t|^{n-d-1}} - 4 \iint_\Omega \mathcal{A}_0 \nabla \Psi \cdot \nabla G \, (G u^2 \Psi^3 \Phi^2) \, \frac{dt\, dx}{|t|^{n-d-1}} \\ &:= T_1 + T_2 + T_3 + T_4. \end{aligned}$$

The functions G, u, Φ, and Ψ all belong to $L^\infty(2D) \cap W^{1,2}(2D)$, so $G u^2 \Psi^4 \Phi^2$ is a valid test function and (4.20) gives that $T_1 = 0$. By the boundedness of $\mathcal{A}_0$ and Cauchy–Schwarz's inequality, the terms T_2, T_3, and T_4 can be bounded as follows. We have

$$|T_3| \lesssim T^{1/2} \Big(\iint_\Omega |\nabla \Phi|^2 G^2 u^2 \Psi^4 \frac{dt\, dx}{|t|^{n-d-1}} \Big)^{1/2} \lesssim T^{1/2} \Big(\iint_{4D/3} u^2 \Psi^4 \frac{dt\, dx}{|t|^{n-d-1}} \Big)^{1/2}$$

because $|\nabla \Phi| \lesssim 1/r \approx 1/G$ on $2D$. Similarly

$$|T_4| \lesssim T^{1/2} \Big(\iint_\Omega |\nabla \Psi| G^2 u^2 \Psi^2 \Phi^2 \frac{dt\, dx}{|t|^{n-d-1}} \Big)^{1/2} \lesssim T^{1/2} \Big(\iint_{4D/3} u^2 \Psi^2 \frac{dt\, dx}{|t|^{n-d-1}} \Big)^{1/2}$$

because $|\nabla \Psi| \lesssim 1/|t| \approx 1/G$ on $2D$. At last

$$|T_2| \lesssim T^{1/2} \Big(\iint_\Omega |\nabla u|^2 G^2 \Psi^4 \Phi^2 \frac{dt\, dx}{|t|^{n-d-1}} \Big)^{1/2} \lesssim T^{1/2} \Big(r^2 \iint_{4D/3} |\nabla u|^2 \Psi^4 \frac{dt\, dx}{|t|^{n-d-1}} \Big)^{1/2}.$$

We deduce that

$$T \lesssim T^{1/2} \Big(\iint_{4D/3} u^2 \Psi^2 \frac{dt\, dx}{|t|^{n-d-1}} + r^2 \iint_{4D/3} |\nabla u|^2 \Psi^4 \frac{dt\, dx}{|t|^{n-d-1}} \Big)^{1/2}$$

and then

$$\iint_D |\nabla G|^2 u^2 \Psi^4 \, \frac{dt\, dx}{|t|^{n-d-1}} \lesssim \iint_{4D/3} u^2 \Psi^2 \frac{dt\, dx}{|t|^{n-d-1}} + r^2 \iint_{4D/3} |\nabla u|^2 \Psi^4 \frac{dt\, dx}{|t|^{n-d-1}}. \tag{4.27}$$

We repeat the process for the last integral of the right-hand side above, using the fact that u is a weak solution to $Lu = 0$, and we obtain[2]

$$r^2 \iint_{4D/3} |\nabla u|^2 \Psi^4 \frac{dt\, dx}{|t|^{n-d-1}} \lesssim \iint_{2D} u^2 \Psi^2 \frac{dt\, dx}{|t|^{n-d-1}}.$$

[2]The estimate below can also be seen as a variant of Caccioppoli's inequality, and is a consequence of Lemma 3.1 (i) in [Feneuil et al. 2021].

We combine the last estimate with (4.27) and get that

$$\iint_D |\nabla G|^2 u^2 \Psi^4 \frac{dt\,dx}{|t|^{n-d-1}} \lesssim \iint_{2D} u^2 \Psi^2 \frac{dt\,dx}{|t|^{n-d-1}}. \tag{4.28}$$

The claim (4.25) follows after we recall that $|t| \approx r$ on $2D$.

We now apply (4.25) and have

$$\fint_{|Z-(y,t)|<|t|/4} |\nabla G|^2 u^2 \Psi_{B,\epsilon}^4 \, dZ \lesssim \frac{1}{|t|^2} \fint_{|Z-(y,t)|<|t|/2} u^2 \Psi_{B,\epsilon}^2 \, dZ \quad \text{for } (y,t) \in \Omega.$$

As a consequence, for any $x \in \mathbb{R}^d$,

$$\widetilde{N}(u\Psi_{B,\epsilon}^2 \nabla G)(x) \lesssim N_{10}(u\Psi_{B,\epsilon})(x).$$

The claim (4.23) follows from (4.3).

Step 4: proof of (4.10). We define

$$J = J_{B,\epsilon} := \int_\Omega |\nabla u|^2 \Psi_{B,\epsilon}^4 \frac{dt\,dx}{|t|^{n-d-2}},$$

and we want to show that

$$J_{B,\epsilon} \lesssim (1+K)\|N(u\Psi_{B,\epsilon})\|_{L^2(\mathbb{R}^d)}^2 + (1+K^{1/2}) J_{B,\epsilon}^{1/2} \|N(u\Psi_{B,\epsilon})\|_{L^2(\mathbb{R}^d)}, \tag{4.29}$$

where K is the constant used in the assumptions of the theorem under proof. Since $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, all the quantities in (4.29) are finite, and therefore (4.29) improves itself in

$$J_{B,\epsilon} \lesssim (1+K)\|N(u\Psi_{B,\epsilon})\|_{L^2(\mathbb{R}^d)}^2. \tag{4.30}$$

We assumed that the solution u is bounded, so the left-hand side above is uniformly bounded in ϵ. We take then the limit as ϵ goes to 0 to obtain the desired bound (4.10).

To lighten the notation, we shall write until the end of the proof Ψ for $\Psi_{B,\epsilon}$ and J for $J_{B,\epsilon}$. Since b is bounded from above (assumption (b) of both Theorems 1.21 and 1.23), $G \gtrsim |t|$ by (4.21), and $\mathcal{A}$ is elliptic by (1.1), we deduce that

$$J \lesssim I := \iint_\Omega \mathcal{A}\nabla u \cdot \nabla u \frac{\Psi^4 G}{b} \frac{dt\,dy}{|t|^{n-d-1}}.$$

Using the product rule, we insert $\Psi^4 G/b$ into the second gradient, and we obtain

$$\begin{aligned} I = &\iint_\Omega \mathcal{A}\nabla u \cdot \nabla\left(\frac{u\Psi^4 G}{b}\right) \frac{dt\,dy}{|t|^{n-d-1}} - 4\iint_\Omega \mathcal{A}\nabla u \cdot \nabla \Psi \, \frac{u\Psi^3 G}{b} \frac{dt\,dy}{|t|^{n-d-1}} \\ &+ \iint_\Omega \mathcal{A}\nabla u \cdot \nabla b \, \frac{u\Psi^4 G}{b^2} \frac{dt\,dy}{|t|^{n-d-1}} - \iint_\Omega \mathcal{A}\nabla u \cdot \nabla G \, \frac{u\Psi^4}{b} \frac{dt\,dy}{|t|^{n-d-1}} \\ := &\, I_0 + I_1 + I_2 + I_3. \end{aligned}$$

The term I_0 equals 0 because u is a weak solution to $Lu = 0$ (and the compactly supported function $u\Psi^4 G/b \in W^{1,2}(\Omega)$ is a valid test function thanks to Lemma 8.3 in [David et al. 2021b]). The terms I_1 and I_2 are bounded in a similar manner. Since $b \gtrsim 1$, $G \approx |t|$, $\mathcal{A}$ is bounded (due to (1.2)), and $0 \le \Psi \le 1$, the Cauchy–Schwarz inequality infers that

$$|I_1 + I_2| \lesssim \iint_\Omega |t|(|\nabla\Psi| + |\nabla b|)u\Psi^3|\nabla u| \frac{dt\,dy}{|t|^{n-d-1}} \lesssim J^{1/2}\biggl(\iint_\Omega |t|^2(|\nabla\Psi|^2 + |\nabla b|^2)u^2\Psi^2 \frac{dt\,dy}{|t|^{n-d}}\biggr)^{1/2}.$$

We know that $|t||\nabla b| \in CM(K)$ by assumption (4.17) and that $|t||\nabla\Psi| \in CM$ by (4.14), so the Carleson inequality (4.4) requires that

$$|I_1 + I_2| \lesssim (1 + K^{1/2}) J^{1/2} \|N(u\Psi)\|_{L^2(\mathbb{R}^d)}.$$

As for I_3, we use the decomposition of $\mathcal{A}$ given in (4.19) to obtain

$$\begin{aligned} I_3 = &-\iint_\Omega (\mathcal{C}_3 \nabla_x u + \mathcal{C}_4 \nabla_t u) \cdot \nabla_t G \frac{u\Psi^4}{b} \frac{dt\,dy}{|t|^{n-d-1}} \\ &\qquad - \iint_\Omega \mathcal{A}_0 \nabla u \cdot \nabla G (u\Psi^4) \frac{dt\,dy}{|t|^{n-d-1}} - \iint_\Omega (\mathcal{B}_3 + \widetilde{\mathcal{B}}_3) \nabla_x u \cdot \nabla_t G \frac{u\Psi^4}{b} \frac{dt\,dy}{|t|^{n-d-1}} \\ &:= I_{31} + I_{32} + I_{33}. \end{aligned}$$

Recall that $b \gtrsim 1$, and combined with the fact that $|\mathcal{C}_3| + |\mathcal{C}_4| \in \widetilde{CM}_2(K)$ and (4.7), we deduce

$$\begin{aligned} |I_{31}| &\lesssim \iint_\Omega (|\mathcal{C}_3| + |\mathcal{C}_4|)|\nabla u||u||\nabla G|\Psi^4 \frac{dt\,dy}{|t|^{n-d-1}} \lesssim J^{1/2} K^{1/2} \|\widetilde{N}(u\Psi^2\nabla G)\|_{L^2(\mathbb{R}^d)} \\ &\lesssim J^{1/2} K^{1/2} \|N(u\Psi)\|_{L^2(\mathbb{R}^d)} \end{aligned}$$

by (4.6) and then (4.23). We force $(u\Psi^4)$ into the first gradient and I_{32} becomes

$$\begin{aligned} I_{32} &= -\frac{1}{2} \iint_\Omega \mathcal{A}_0 \nabla(u^2\Psi^4) \cdot \nabla G \frac{dt\,dy}{|t|^{n-d-1}} + 2 \iint_\Omega \mathcal{A}_0 \nabla\Psi \cdot \nabla G (u^2\Psi^3) \frac{dt\,dy}{|t|^{n-d-1}} \\ &:= I_{321} + I_{322}. \end{aligned}$$

The term I_{321} equals 0 thanks to (4.20). As for I_{322}, we use the boundedness of $\mathcal{A}_0$ and the inequality $2ab \le a^2 + b^2$ to write

$$I_{322} \lesssim \iint_\Omega |\nabla\Psi||\nabla G|^2 u^2\Psi^4 \frac{dt\,dy}{|t|^{n-d-1}} + \iint_\Omega |\nabla\Psi| u^2\Psi^2 \frac{dt\,dy}{|t|^{n-d-1}},$$

and then, by (4.14) and (4.23),

$$I_{322} \lesssim \|\widetilde{N}(u\Psi^2\nabla G)\|^2_{L^2(\mathbb{R}^d)} + \|\widetilde{N}(u\Psi)\|^2_{L^2(\mathbb{R}^d)} \lesssim \|N(u\Psi)\|^2_{L^2(\mathbb{R}^d)}.$$

Step 5: bound of I_{33}, which is the only difference between Theorems 1.21 and 1.23. Recall that $\mathcal{B}_3$ and $\widetilde{\mathcal{B}}_3$ satisfy the same condition on the x-derivative, that is

$$|t||\mathrm{div}_x(\mathcal{B}_3)^T| + |t||\mathrm{div}_x(\widetilde{\mathcal{B}}_3)^T| \in CM_2(K),$$

but differ on the condition on the t-derivative, which is

$$|t|^{n-d}|\mathrm{div}_t(|t|^{d+1-n}\mathcal{B}_3)| + |t||\mathrm{div}_t\,\widetilde{\mathcal{B}}_3| \in CM_2(K).$$

The goal is to permute the gradients ∇_x and ∇_t on I_{33}. We define the part of I_{33} that contains $\mathcal{B}_3$ as

$$S := -\iint_\Omega \mathcal{B}_3 \nabla_x u \cdot \nabla_t G \frac{u\Psi^4}{b} \frac{dt\,dy}{|t|^{n-d-1}}. \tag{4.31}$$

Using integration by parts in t, S becomes

$$\begin{aligned} S &= -\frac{1}{2}\int_\Omega \mathcal{B}_3\nabla_x[u^2]\cdot\nabla_t G\frac{\Psi^4}{b}\frac{dt\,dy}{|t|^{n-d-1}} \\ &= \frac{1}{2}\int_\Omega \mathrm{div}_t(|t|^{d+1-n}\mathcal{B}_3\nabla_x[u^2])\frac{G\Psi^4}{b}dt\,dy \\ &\qquad + 2\int_\Omega \mathcal{B}_3\nabla_x[u^2]\cdot\nabla_t\Psi\frac{G\Psi^3}{b}\frac{dt\,dy}{|t|^{n-d-1}} - \int_\Omega \mathcal{B}_3\nabla_x[u^2]\cdot\nabla_t b\frac{G\Psi^4}{b^2}\frac{dt\,dy}{|t|^{n-d-1}} \\ &:= S_0 + S_1 + S_2. \end{aligned}$$

We write the term S_0 as a sum on the coefficients of $\mathcal{B}_3$, we permute the x and the t-derivatives on u^2, and then we integrate by parts in x. Recall that, in this paper, when M is a matrix-valued function, div M is a vector-valued function whose j-th entry is the divergence of the j-th column of M.

$$\begin{aligned} S_0 &:= \frac{1}{2}\sum_{1\le j\le d<i\le n}\iint_\Omega \partial_{t_i}[|t|^{d+1-n}(\mathcal{B}_3)_{ij}\partial_{x_j}u^2]\frac{G\Psi^4}{b}dt\,dy \\ &= \frac{1}{2}\iint_\Omega \mathrm{div}_t(|t|^{d+1-n}\mathcal{B}_3)\cdot\nabla_x(u^2)\frac{G\Psi^4}{b}dt\,dy + \frac{1}{2}\sum_{1\le j\le d<i\le n}\iint_\Omega (\mathcal{B}_3)_{ij}\partial_{t_i}[\partial_{x_j}u^2]\frac{G\Psi^4}{b}\frac{dt\,dy}{|t|^{n-d-1}} \\ &= \frac{1}{2}\iint_\Omega \mathrm{div}_t(|t|^{d+1-n}\mathcal{B}_3)\cdot\nabla_x(u^2)\frac{G\Psi^4}{b}dt\,dy - \frac{1}{2}\iint_\Omega \mathrm{div}_x(\mathcal{B}_3)^T\cdot\nabla_t[u^2]\frac{G\Psi^4}{b}\frac{dt\,dy}{|t|^{n-d-1}} \\ &\qquad -2\iint_\Omega(\mathcal{B}_3)^T\nabla_t[u^2]\cdot\nabla_x\Psi\frac{G\Psi^3}{b}\frac{dt\,dy}{|t|^{n-d-1}} + \frac{1}{2}\iint_\Omega(\mathcal{B}_3)^T\nabla_t[u^2]\cdot\nabla_x b\frac{G\Psi^3}{b^2}\frac{dt\,dy}{|t|^{n-d-1}} \\ &:= S_3+S_4+S_5+S_6. \end{aligned}$$

We do not have x-derivatives on G because G is x-independent; see (4.21). We deal with S_1, S_2, S_3, S_4, S_5, and S_6 in a similar manner as I_2+I_3 earlier. We have $G \lesssim |t|$, $1/b \lesssim 1$, and $\mathcal{B}_3$ is bounded, hence, if

$$f := |t||\nabla\Psi| + |t||\nabla b| + |t||\mathrm{div}_x(\mathcal{B}_3)^T| + |t|^{n-d}|\mathrm{div}_t(|t|^{d+1-n}\mathcal{B}_3)|,$$

the sum of the S_i can be bounded by

$$|S| \le \sum_{i=1}^{6}|S_i| \lesssim \iint_\Omega f|\nabla(u^2)|\Psi^3\frac{dt\,dy}{|t|^{n-d-1}} \lesssim \iint_\Omega f|\nabla u|u\Psi^3\frac{dt\,dy}{|t|^{n-d-1}}.$$

But, since $f \in CM_2(1+K)$ by (4.17) and (4.14), the Carleson estimate (4.6) yields

$$|S| \leq \sum_{i=1}^{6} |S_i| \lesssim J^{1/2}(1+K^{1/2})\|N(u\Psi)\|_{L^2(\mathbb{R}^d)}, \tag{4.32}$$

as desired. Theorem 1.21 is now proven, because $\widetilde{\mathcal{B}}_3 = 0$ in its assumption.

In order to establish Theorem 1.23, it remains to treat the part of I_{33} that contains $\widetilde{\mathcal{B}}_3$. If $\widetilde{S} := I_{33} - S$, and if, for $i \in \{0, \dots, 6\}$, $\widetilde{S}_i$ is obtained from S_i by substituting $\mathcal{B}_3$ for $\widetilde{\mathcal{B}}_3$, for $i \neq 3$, we can bound $\widetilde{S}_i$ as we bound S_i, because the assumptions on $\widetilde{\mathcal{B}}_3$ match those of $\mathcal{B}_3$. So, similarly to (4.32), we have that

$$|\widetilde{S} - \widetilde{S}_3| \lesssim J^{1/2}(1+K^{1/2})\|N(u\Psi)\|_{L^2(\mathbb{R}^d)}. \tag{4.33}$$

We do not know how to estimate $\widetilde{S}_3$, but instead we know how to estimate

$$\begin{aligned} \widetilde{S}_7 &:= \frac{1}{2} \iint_\Omega \operatorname{div}_t(\widetilde{\mathcal{B}}_3) \cdot \nabla_x (u^2) \frac{G\Psi^4}{b} \frac{dt\,dy}{|t|^{n-d-1}} \\ &= \iint_\Omega \operatorname{div}_t(\widetilde{\mathcal{B}}_3) \cdot \nabla_x u \frac{Gu\Psi^4}{b} \frac{dt\,dy}{|t|^{n-d-1}}. \end{aligned} \tag{4.34}$$

Indeed, we use $G \lesssim |t|$, $1/b \lesssim 1$, $|t||\operatorname{div}_t(\mathcal{B}_3)| \in CM_2(K)$, and the Carleson estimate (4.6), to get, similarly to the S_i's, that

$$|\widetilde{S}_7| \lesssim J^{1/2}(1+K^{1/2})\|N(u\Psi)\|_{L^2(\mathbb{R}^d)}. \tag{4.35}$$

So, in order to bound $\widetilde{S}$ and prove Theorem 1.23 we only have to write $\widetilde{S}$ as a linear combination of $\widetilde{S}_7$ and $\widetilde{S}_3$. Since we are currently under the assumptions of Theorem 1.23, Corollary 3.14 requires that $G = |t|$. With this in mind, we have

$$G|t|^{n-d-1} \operatorname{div}_t(|t|^{d+1-n}\mathcal{B}_3) = G \operatorname{div}_t(\mathcal{B}_3) + (d+1-n)(\nabla_t G)^T \mathcal{B}_3,$$

which can be reformulated as

$$\widetilde{S}_3 = \widetilde{S}_7 + (n-d-1)\widetilde{S}.$$

We conclude that

$$|\widetilde{S}| = \frac{1}{n-d-2} |(\widetilde{S}_3 - \widetilde{S}) + \widetilde{S}_7| \lesssim J^{1/2}(1+K^{1/2})\|N(u\Psi)\|_{L^2(\mathbb{R}^d)}$$

by (4.33) and (4.35). The lemma follows. □

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TALAGRAND'S INFLUENCE INEQUALITY REVISITED

DARIO CORDERO-ERAUSQUIN AND ALEXANDROS ESKENAZIS

Let $\mathcal{C}_n = \{-1, 1\}^n$ be the discrete hypercube equipped with the uniform probability measure σ_n. Talagrand's influence inequality (1994), also known as the $L_1 - L_2$ inequality, asserts that there exists $C \in (0, \infty)$ such that for every $n \in \mathbb{N}$, every function $f : \mathcal{C}_n \to \mathbb{C}$ satisfies

$$\mathrm{Var}_{\sigma_n}(f) \leqslant C \sum_{i=1}^{n} \frac{\|\partial_i f\|^2_{L_2(\sigma_n)}}{1 + \log(\|\partial_i f\|_{L_2(\sigma_n)} / \|\partial_i f\|_{L_1(\sigma_n)})}.$$

We undertake a systematic investigation of this and related inequalities via harmonic analytic and stochastic techniques and derive applications to metric embeddings. We prove that Talagrand's inequality extends, up to an additional doubly logarithmic factor, to Banach space-valued functions under the necessary assumption that the target space has Rademacher type 2 and that this doubly logarithmic term can be omitted if the target space admits an equivalent 2-uniformly smooth norm. These are the first vector-valued extensions of Talagrand's influence inequality. Moreover, our proof implies vector-valued versions of a general family of $L_1 - L_p$ inequalities, each refining the dimension independent L_p-Poincaré inequality on $(\mathcal{C}_n, \sigma_n)$. We also obtain a joint strengthening of results of Bakry–Meyer (1982) and Naor–Schechtman (2002) on the action of negative powers of the hypercube Laplacian on functions $f : \mathcal{C}_n \to E$, whose target space $(E, \|\cdot\|_E)$ has nontrivial Rademacher type via a new vector-valued version of Meyer's multiplier theorem (1984). Inspired by Talagrand's influence inequality, we introduce a new metric invariant called Talagrand type and estimate it for Banach spaces with prescribed Rademacher or martingale type, Gromov hyperbolic groups and simply connected Riemannian manifolds of pinched negative curvature. Finally, we prove that Talagrand type is an obstruction to the bi-Lipschitz embeddability of nonlinear quotients of the hypercube $\mathcal{C}_n$ equipped with the Hamming metric, thus deriving new nonembeddability results for these finite metrics. Our proofs make use of Banach space-valued Itô calculus, Riesz transform inequalities, Littlewood–Paley–Stein theory and hypercontractivity.

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Keywords: Hamming cube, Talagrand's inequality, Rademacher type, martingale type, Itô calculus, Riesz transforms, Littlewood–Paley–Stein theory, hypercontractivity, CAT(0) space, bi-Lipschitz embedding.

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1. Introduction

Let $\mathcal{C}_n = \{-1, 1\}^n$ be the discrete hypercube equipped with the uniform probability measure σ_n. If $(E, \|\cdot\|_E)$ is a complex Banach space, we will denote the vector-valued $L_p(\sigma_n)$-norm of a function $f : \mathcal{C}_n \to E$ by

$$\|f\|_{L_p(\sigma_n;E)} \stackrel{\text{def}}{=} \left(\int_{\mathcal{C}_n} \|f(\varepsilon)\|_E^p \, \mathrm{d}\sigma_n(\varepsilon)\right)^{1/p}, \quad \text{for all } p \in [1, \infty), \tag{1}$$

and $\|f\|_{L_\infty(\sigma_n;E)} \stackrel{\text{def}}{=} \max_{\varepsilon\in\mathcal{C}_n} \|f(\varepsilon)\|_E$. When $E = \mathbb{C}$, we will abbreviate $\|f\|_{L_p(\sigma_n;\mathbb{C})}$ simply as $\|f\|_{L_p(\sigma_n)}$. We will also denote by $\mathbb{E}_{\sigma_n} f$ the expectation of f with respect to σ_n. The i-th partial derivative of a function $f : \mathcal{C}_n \to E$ is given by

$$\partial_i f(\varepsilon) = \frac{f(\varepsilon) - f(\varepsilon_1, \ldots, \varepsilon_{i-1}, -\varepsilon_i, \varepsilon_{i+1}, \ldots, \varepsilon_n)}{2}, \quad \text{for all } \varepsilon \in \mathcal{C}_n. \tag{2}$$

The discrete Poincaré inequality asserts that every function $f : \mathcal{C}_n \to \mathbb{C}$ satisfies

$$\|f - \mathbb{E}_{\sigma_n} f\|_{L_2(\sigma_n)}^2 \leqslant \sum_{i=1}^n \|\partial_i f\|_{L_2(\sigma_n)}^2. \tag{3}$$

Extensions and refinements of (3) have been a central object of study in the probability and analysis literature for decades. A natural problem, first raised by Enflo [1978], is to understand for which target spaces E every function $f : \mathcal{C}_n \to E$ satisfies (3), up to a universal multiplicative factor depending only on the geometry of E but not on n or the choice of f. Recall that a Banach space $(E, \|\cdot\|_E)$ has Rademacher type s with constant $T \in (0, \infty)$ if for every $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$,

$$\int_{\mathcal{C}_n} \left\|\sum_{i=1}^n \varepsilon_i x_i\right\|_E^s \mathrm{d}\sigma_n(\varepsilon) \leqslant T^s \sum_{i=1}^n \|x_i\|_E^s. \tag{4}$$

It is evident that if a Banach space E is such that every function $f : \mathcal{C}_n \to E$ satisfies

$$\|f - \mathbb{E}_{\sigma_n} f\|_{L_2(\sigma_n;E)}^2 \leqslant C^2 \sum_{i=1}^n \|\partial_i f\|_{L_2(\sigma_n;E)}^2, \tag{5}$$

then E has Rademacher type 2 with constant C, since this condition coincides with (5) for functions of the form $f(\varepsilon) = \sum_{i=1}^n \varepsilon_i x_i$, where $x_1, \ldots, x_n \in E$. The reverse implication, i.e., the fact that Rademacher type 2 implies the vector-valued Poincaré inequality (5), was a recent breakthrough proved by Ivanisvili, van Handel and Volberg [Ivanisvili et al. 2020].

In a different direction, an important refinement of the scalar-valued discrete Poincaré inequality (3) was obtained in the celebrated work by Talagrand [1994]. Also known as the $L_1 - L_2$ inequality, Talagrand's influence inequality asserts that there exists a universal constant $C \in (0, \infty)$ such that for every $n \in \mathbb{N}$, every function $f : \mathcal{C}_n \to \mathbb{C}$ satisfies

$$\|f - \mathbb{E}_{\sigma_n} f\|_{L_2(\sigma_n)}^2 \leqslant C \sum_{i=1}^n \frac{\|\partial_i f\|_{L_2(\sigma_n)}^2}{1 + \log(\|\partial_i f\|_{L_2(\sigma_n)}/\|\partial_i f\|_{L_1(\sigma_n)})}. \tag{6}$$

Observe that (6) is a strengthening of the discrete Poincaré inequality (3) up to the value of the universal constant C, which becomes substantial for functions satisfying $\|\partial_i f\|_{L_2(\sigma_n)} \gg \|\partial_i f\|_{L_1(\sigma_n)}$. Since its conception, Talagrand's inequality has played a major role in Boolean analysis [Falik and Samorodnitsky 2007; Friedgut and Kalai 1996; Kahn et al. 1988; O'Donnell 2014; Rossignol 2006], percolation [Benaïm and Rossignol 2008; Benjamini et al. 2003; Chatterjee 2014; Garban and Steif 2015; Russo 1982] and geometric functional analysis [Paouris and Valettas 2018; Paouris et al. 2017; 2022; Tikhomirov 2018]. In particular, applying (6) to a Boolean function $f : \mathcal{C}_n \to \{0, 1\}$, one readily recovers the celebrated theorem of Kahn, Kalai and Linial [Kahn et al. 1988], quantifying the fact that in any (essentially) unbiased voting scheme, there exists a voter with disproportionately large influence over the outcome of the vote. We refer to the above references and [Cordero-Erausquin and Ledoux 2012; Ledoux 2019] for further bibliographical information on Talagrand's inequality.

The main purpose of the present paper is to investigate vector-valued versions of Talagrand's inequality (6) and other refinements and extensions of (3). These new vector-valued inequalities motivate the definition of a new bi-Lipschitz invariant for metric spaces called *Talagrand type* (Definition 10), which captures new KKL-type phenomena in embedding theory (see Theorem 13 and the ensuing discussion). We shall now present a summary of these results, which rely on a range of stochastic and harmonic analytic tools such as Banach space-valued Itô calculus, Riesz transforms and Littlewood–Paley–Stein theory, along with standard uses of hypercontractivity.

Asymptotic notation. In what follows we use the convention that for $a, b \in [0, \infty]$ the notation $a \gtrsim b$ (resp. $a \lesssim b$) means that there exists a universal constant $c \in (0, \infty)$ such that $a \geqslant cb$ (resp. $a \leqslant cb$). Moreover, $a \asymp b$ stands for $(a \lesssim b) \wedge (a \gtrsim b)$. The notations $\lesssim_\xi$, $\gtrsim_\chi$ and $\asymp_\psi$ mean that the implicit constant c depends on ξ, χ and ψ, respectively.

1.1. ***Vector-valued influence inequalities.*** In view of Enflo's problem [1978] and its recent solution in [Ivanisvili et al. 2020], it would be most natural to try and understand for which Banach spaces $(E, \|\cdot\|_E)$ there exists a constant $C = C(E) \in (0, \infty)$ such that for every $n \in \mathbb{N}$, every function $f : \mathcal{C}_n \to E$ satisfies

$$\|f - \mathbb{E}_{\sigma_n} f\|^2_{L_2(\sigma_n;E)} \leqslant C \sum_{i=1}^{n} \frac{\|\partial_i f\|^2_{L_2(\sigma_n;E)}}{1 + \log(\|\partial_i f\|_{L_2(\sigma_n;E)}/\|\partial_i f\|_{L_1(\sigma_n;E)})}. \tag{7}$$

Evidently, as (7) is a strengthening of (5), if a space $(E, \|\cdot\|_E)$ satisfies (7) then E has Rademacher type 2. Conversely, we shall prove the following theorem.

Theorem 1 (vector-valued influence inequality for spaces with Rademacher type 2). *Let $(E, \|\cdot\|_E)$ be a Banach space with Rademacher type 2. Then there exists $C = C(E) \in (0, \infty)$ such that for every $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$, every function $f : \mathcal{C}_n \to E$ satisfies*

$$\|f - \mathbb{E}_{\sigma_n} f\|^2_{L_2(\sigma_n;E)} \leqslant \frac{C}{\varepsilon} \sum_{i=1}^{n} \frac{\|\partial_i f\|^2_{L_2(\sigma_n;E)}}{1 + \log^{1-\varepsilon}(\|\partial_i f\|_{L_2(\sigma_n;E)}/\|\partial_i f\|_{L_1(\sigma_n;E)})}. \tag{8}$$

In particular, if $\sigma(f) \overset{\text{def}}{=} \max_{i\in\{1,\dots,n\}} \log\log(e + \|\partial_i f\|_{L_2(\sigma_n;E)}/\|\partial_i f\|_{L_1(\sigma_n;E)})$, *then*

$$\|f - \mathbb{E}_{\sigma_n} f\|^2_{L_2(\sigma_n;E)} \leqslant C\sigma(f) \sum_{i=1}^n \frac{\|\partial_i f\|^2_{L_2(\sigma_n;E)}}{1+\log(\|\partial_i f\|_{L_2(\sigma_n;E)}/\|\partial_i f\|_{L_1(\sigma_n;E)})}. \tag{9}$$

The proof of Theorem 1 builds upon a novel idea exploited in [Ivanisvili et al. 2020], which in turn is reminiscent of a trick due to Maurey [Pisier 1986]. It remains unclear whether one can deduce from this idea a vector-valued extension of Talagrand's inequality (6) for spaces of Rademacher type 2 and whether the doubly logarithmic error term $\sigma(f)$ on the right-hand side of (9) is needed. Let us mention that, even in the scalar-valued case, the argument of Maurey or the one of Ivanisvili, van Handel and Volberg are slightly different than standard semigroup approaches to functional inequalities, in particular to the semigroup proof of (6) from [Cordero-Erausquin and Ledoux 2012]. On the other hand, we will see that a slightly stronger condition on the Banach space allows for different approaches, relying on more intricate connections between the space and the semigroup, which will lead to the desired optimal vector-valued $L_1 - L_2$ inequality. Recall first that a Banach space $(E, \|\cdot\|_E)$ has martingale type s with constant $M \in (0,\infty)$ if for every $n \in \mathbb{N}$, every probability space $(\Omega, \mathcal{F}, \mu)$ and every filtration $\{\mathcal{F}_i\}_{i=0}^n$ of sub-σ-algebras of $\mathcal{F}$, every E-valued martingale $\{\mathcal{M}_i : \Omega \to E\}_{i=0}^n$ adapted to $\{\mathcal{F}_i\}_{i=0}^n$ satisfies

$$\|\mathcal{M}_n - \mathcal{M}_0\|^s_{L_s(\mu;E)} \leqslant M^s \sum_{i=1}^n \|\mathcal{M}_i - \mathcal{M}_{i-1}\|^s_{L_s(\mu;E)}. \tag{10}$$

Martingale type, which is a strengthening of Rademacher type, was introduced by Pisier [1975], who proved the fundamental fact that for every $s \in (1,2]$, a Banach space E has martingale type s if and only if E admits an equivalent s-uniformly smooth norm (see [Pisier 1975; 2016] for further information on these important notions).

Theorem 2 (vector-valued influence inequality for spaces with martingale type 2). *Let* $(E, \|\cdot\|_E)$ *be a Banach space with martingale type* 2. *Then there exists* $C = C(E) \in (0,\infty)$ *such that for every* $n \in \mathbb{N}$, *every function* $f : \mathcal{C}_n \to E$ *satisfies*

$$\|f - \mathbb{E}_{\sigma_n} f\|^2_{L_2(\sigma_n;E)} \leqslant C \sum_{i=1}^n \frac{\|\partial_i f\|^2_{L_2(\sigma_n;E)}}{1+\log(\|\partial_i f\|_{L_2(\sigma_n;E)}/\|\partial_i f\|_{L_1(\sigma_n;E)})}. \tag{11}$$

Theorem 2 establishes the optimal vector-valued influence inequality for spaces of martingale type 2. We will present two proofs of Theorem 2. The first one uses a clever stochastic process on the cube which was recently constructed by Eldan and Gross [2022], while the second relies on Xu's vector-valued Littlewood–Paley–Stein inequalities for superreflexive targets; see [Xu 2020]. There exist examples of exotic Banach spaces [James 1978; Pisier and Xu 1987] which have Rademacher type 2 yet fail to have martingale type 2, thus Theorem 2 does not exhaust the list of potential target spaces satisfying (7). Nevertheless, a combination of classical results of Maurey [1974], Pisier [1975] and Figiel [1976] imply that every Banach lattice of Rademacher type 2 has martingale type 2.

The influence inequalities of Theorems 1 and 2 have analogues for spaces of Rademacher and martingale type s which will be presented in Section 9.1 for the sake of simplicity of exposition.

1.2. $L_1 - L_p$ *inequalities.* For a function $f : \mathcal{C}_n \to \mathbb{C}$, denote by

$$\|\nabla f\|_{L_p(\sigma_n)} \stackrel{\text{def}}{=} \left\| \left(\sum_{i=1}^{n} (\partial_i f)^2 \right)^{1/2} \right\|_{L_p(\sigma_n)}, \quad \text{for all } p \in [1, \infty), \tag{12}$$

the L_p-norm of the gradient of f. It has already been pointed out that Talagrand's influence inequality (6) is a refinement of the discrete Poincaré inequality (3). It is therefore worth investigating whether similar strengthenings of the L_p discrete Poincaré inequality

$$\|f - \mathbb{E}_{\sigma_n} f\|_{L_p(\sigma_n)} \leqslant C_p \|\nabla f\|_{L_p(\sigma_n)} \tag{13}$$

hold true for other values of p. The fact that for every $p \in [1, \infty)$ there exists a constant $C_p \in (0, \infty)$ such that (13) holds true for every $n \in \mathbb{N}$ and $f : \mathcal{C}_n \to \mathbb{C}$ was established by Talagrand [1993].

In the vector-valued setting which is of interest here, the most common substitute of (12) for the norm of the gradient of a function $f : \mathcal{C}_n \to E$, where $(E, \|\cdot\|_E)$ is a Banach space, is

$$\|\nabla f\|_{L_p(\sigma_n;E)} \stackrel{\text{def}}{=} \left(\int_{\mathcal{C}_n} \left\| \sum_{i=1}^{n} \delta_i \partial_i f \right\|_{L_p(\sigma_n;E)}^p \mathrm{d}\sigma_n(\delta) \right)^{1/p} = \left(\int_{\mathcal{C}_n \times \mathcal{C}_n} \left\| \sum_{i=1}^{n} \delta_i \partial_i f(\varepsilon) \right\|_E^p \mathrm{d}\sigma_{2n}(\varepsilon, \delta) \right)^{1/p},$$

for all $p \in [1, \infty)$. Observe that when $E = \mathbb{C}$, for every $p \in [1, \infty)$, we have $\|\nabla f\|_{L_p(\sigma_n;\mathbb{C})} \asymp_p \|\nabla f\|_{L_p(\sigma_n)}$ by Khintchine's inequality [1923]. With this definition, the vector-valued extension

$$\|f - \mathbb{E}_{\sigma_n} f\|_{L_p(\sigma_n;E)} \leqslant C_p(n) \|\nabla f\|_{L_p(\sigma_n;E)} \tag{14}$$

of (13) is called Pisier's inequality, since Pisier [1986] established the validity of (14) for every Banach space E and $p \in [1, \infty)$ with $C_p(n) = 2e \log n$. Understanding for which Banach spaces E and $p \in [1, \infty)$ the constant $C_p(n)$ in Pisier's inequality could be replaced by a constant $C_p(E)$, independent of the dimension n, was a long-standing open problem settled in [Ivanisvili et al. 2020]. We will recall in (97) the definition of Rademacher cotype; let us simply say here that a Banach space E has finite cotype if E does not isomorphically contain the family $\{\ell_\infty^n\}_{n=1}^\infty$ with uniformly bounded distortion; see [Maurey and Pisier 1976; Pisier 2016]. In [Ivanisvili et al. 2020], the authors proved that a Banach space E with finite cotype satisfies (14) with $C_p(n)$ replaced by a universal constant $C_p(E)$, thus complementing a result of Talagrand [1993] who proved that if a space does not have finite cotype, then $C_p(n) \asymp_p \log n$.

Theorem 3 (vector-valued $L_1 - L_p$ inequality for spaces of finite cotype). *Let $(E, \|\cdot\|_E)$ be a Banach space with finite Rademacher cotype and $p \in (1, \infty)$. Then there exist $C_p = C_p(E) \in (0, \infty)$ and $\alpha_p = \alpha_p(E) \in \left(0, \frac{1}{2}\right]$ such that for every $n \in \mathbb{N}$, every function $f : \mathcal{C}_n \to E$ satisfies*

$$\|f - \mathbb{E}_{\sigma_n} f\|_{L_p(\sigma_n;E)} \leqslant C_p \frac{\|\nabla f\|_{L_p(\sigma_n;E)}}{1 + \log^{\alpha_p}(\|\nabla f\|_{L_p(\sigma_n;E)} / \|\nabla f\|_{L_1(\sigma_n;E)})}. \tag{15}$$

The proof of Theorem 3 builds upon the technique of [Ivanisvili et al. 2020]. A stronger inequality for functions on the Gauss space will be presented in Theorem 27. This approach seems insufficient to yield the optimal $\alpha_p = \frac{1}{2}$ exponent for $E = \mathbb{C}$ and all $p > 1$, yet we derive the following result using Lust-Piquard's Riesz transform inequalities [Ben Efraim and Lust-Piquard 2008; Lust-Piquard 1998].

Theorem 4 (scalar-valued $L_1 - L_p$ inequality). *For every $p \in (1, \infty)$, there exists $C_p \in (0, \infty)$ such that for every $n \in \mathbb{N}$, every function $f : \mathcal{C}_n \to \mathbb{C}$ satisfies*

$$\|f - \mathbb{E}_{\sigma_n} f\|_{L_p(\sigma_n)} \leqslant C_p \frac{\|\nabla f\|_{L_p(\sigma_n)}}{1 + \sqrt{\log(\|\nabla f\|_{L_p(\sigma_n)} / \|\nabla f\|_{L_1(\sigma_n)})}}. \tag{16}$$

1.3. ***Negative powers of the Laplacian.*** Let (Ω, μ) be a finite measure space, $(E, \|\cdot\|_E)$ be a Banach space and $p \in [1, \infty]$. If $T : L_p(\mu) \to L_p(\mu)$ is a bounded linear operator, then, by abuse of notation, we will also denote by T its natural E-valued extension

$$T \equiv T \otimes \mathrm{Id}_E : L_p(\mu; E) \to L_p(\mu; E).$$

The discrete derivatives (2) on the Hamming cube $\mathcal{C}_n$ satisfy $\partial_i^2 = \partial_i$ for every $i \in \{1, \dots, n\}$ and thus the hypercube Laplacian is defined as $\Delta \stackrel{\text{def}}{=} \sum_{i=1}^n \partial_i$. Note that for g and h on $\mathcal{C}_n$ with values in $\mathbb{C}$ and E, respectively, we have

$$\mathbb{E}_{\sigma_n}[g\, \partial_i h] = \mathbb{E}_{\sigma_n}[(\partial_i g) h] = \mathbb{E}_{\sigma_n}[(\partial_i g)(\partial_i h)], \quad \text{for all } i \in \{1, \dots, n\}. \tag{17}$$

The formula is also true if g has values in the dual E^* and the product is the duality bracket. The operator Δ is the (positive) infinitesimal generator of the discrete heat semigroup $\{P_t\}_{t \geqslant 0}$ on $\mathcal{C}_n$, that is, $P_t = e^{-t\Delta}$; see, e.g., [O'Donnell 2014]. Let us mention that functional calculus involving Δ can be easily expressed using the Walsh basis. This is the case for all Fourier multipliers appearing below which are defined by formula (106).

All available proofs of Talagrand's inequality (6) make crucial use of the hypercontractivity of $\{P_t\}_{t \geqslant 0}$ (first proven by Bonami [1970]) along with some version of "orthogonality" [Talagrand 1994] or semigroup identities [Benjamini et al. 2003; Cordero-Erausquin and Ledoux 2012] specific to the scalar case. In particular, Talagrand [1994] used Parseval's identity for the Walsh basis to express the variance of a function $f : \mathcal{C}_n \to \mathbb{C}$ as

$$\mathrm{Var}_{\sigma_n}(f) = \sum_{i=1}^n \|\Delta^{-1/2} \partial_i f\|_{L_2(\sigma_n)}^2, \tag{18}$$

and thus reduced the problem to obtaining effective estimates for $\|\Delta^{-1/2} h\|_{L_2(\sigma_n)}$. One tool which allows us to circumvent algebraic representations such as (18) (see the proof of Theorem 4 below) are one-sided Riesz transform inequalities, which can combined with certain new vector-valued estimates on negative powers of the generator of the semigroup $\{P_t\}_{t \geqslant 0}$.

Let $\alpha \geqslant 0$. We say that a Banach space E has nontrivial Rademacher type if E has Rademacher type s for some $s \in (1, 2]$. It has been proven by Naor and Schechtman [2002] that if a Banach space $(E, \|\cdot\|_E)$ has nontrivial Rademacher type, then for every $p \in (1, \infty)$ and $\alpha \in (0, \infty)$, there exists $K_p(\alpha) = K_p(\alpha, E) \in (0, \infty)$ such that for every $n \in \mathbb{N}$ and $f : \mathcal{C}_n \to E$, we have

$$\|\Delta^{-\alpha} f\|_{L_p(\sigma_n; E)} \leqslant K_p(\alpha) \|f\|_{L_p(\sigma_n; E)}. \tag{19}$$

Conversely, if (19) holds true for some p and α, then E has nontrivial Rademacher type. The proof of Theorem 4 relies on the following strengthening of Naor and Schechtman's inequality (19).

Theorem 5. *Let $(E, \|\cdot\|_E)$ be a Banach space of nontrivial Rademacher type. Then, for every $p \in (1, \infty)$ and $\alpha \in (0, \infty)$, there exists $K_p(\alpha) = K_p(\alpha, E) \in (0, \infty)$ such that for every $n \in \mathbb{N}$ and $f : \mathcal{C}_n \to E$, we have*

$$\|\Delta^{-\alpha} f\|_{L_p(\sigma_n;E)} \leqslant K_p(\alpha) \frac{\|f\|_{L_p(\sigma_n;E)}}{1 + \log^\alpha(\|f\|_{L_p(\sigma_n;E)}/\|f\|_{L_1(\sigma_n;E)})}. \tag{20}$$

We note in passing that when $E = \mathbb{C}$, $\alpha = \frac{1}{2}$ and $p = 2$, Theorem 5 had been proven in [Talagrand 1994, Proposition 2.3]. However, Talagrand's argument heavily uses orthogonality via Parseval's identity for the Walsh basis and is unlikely to work in the vector-valued setting which is of interest here.

1.4. ***Vector-valued multipliers and inequalities involving Orlicz norms.*** In his original work, Talagrand [1994] observed that (6) admits a strengthening in terms of Orlicz norms; see [Rao and Ren 1991]. Recall that if $\psi : [0, \infty) \to [0, \infty)$ is a Young function, i.e., a convex function satisfying

$$\lim_{x \to 0} \frac{\psi(x)}{x} = 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{\psi(x)}{x} = \infty, \tag{21}$$

and $(E, \|\cdot\|_E)$ is a Banach space, then the ψ-Orlicz norm of a function $f : \mathcal{C}_n \to E$ is given by

$$\|f\|_{L_\psi(\sigma_n;E)} \stackrel{\text{def}}{=} \inf\left\{t \geqslant 0 : \int_{\mathcal{C}_n} \psi(\|f(\varepsilon)\|_E/t)\, d\sigma_n(\varepsilon) \leqslant 1\right\}. \tag{22}$$

It is evident that for $\psi(t) = t^p$, we have $\|\cdot\|_{L_\psi(\sigma_n;E)} = \|\cdot\|_{L_p(\sigma_n;E)}$. More generally, for $p \in (1, \infty)$ and $r \in \mathbb{R}$ we will denote by $\|\cdot\|_{L_p(\log L)^r(\sigma_n;E)}$ the Orlicz norm corresponding to a Young function $\psi_{p,r}$ with $\psi_{p,r}(x) = x^p \log^r(e + x)$ for x large enough (to ensure convexity of $\psi_{p,r}$ when $r < 0$).

In [Talagrand 1994, Theorem 1.6], the author showed that (6) can be strengthened as follows. There exists a universal constant $C \in (0, \infty)$ such that for every $n \in \mathbb{N}$, every function $f : \mathcal{C}_n \to \mathbb{C}$ satisfies

$$\|f - \mathbb{E}_{\sigma_n} f\|^2_{L_2(\sigma_n)} \leqslant C \sum_{i=1}^n \|\partial_i f\|^2_{L_2(\log L)^{-1}(\sigma_n)}. \tag{23}$$

It is in fact true (see [Talagrand 1994, Lemma 2.5] or Lemma 17 below) that (23) formally implies (6). In this direction we can prove the following strengthening of Theorem 1.

Theorem 6. *Let $(E, \|\cdot\|_E)$ be a Banach space with Rademacher type 2. Then there exists $C = C(E) \in (0, \infty)$ such that for every $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$, every function $f : \mathcal{C}_n \to E$ satisfies*

$$\|f - \mathbb{E}_{\sigma_n} f\|^2_{L_2(\sigma_n;E)} \leqslant \frac{C}{\varepsilon} \sum_{i=1}^n \|\partial_i f\|^2_{L_2(\log L)^{-1+\varepsilon}(\sigma_n;E)}. \tag{24}$$

Furthermore, the proofs of Theorem 2 in fact yield the following improvement of (11), which extends (23) to spaces of martingale type 2.

Theorem 7. *Let $(E, \|\cdot\|_E)$ be a Banach space with martingale type 2. Then there exists $C = C(E) \in (0, \infty)$ such that for every $n \in \mathbb{N}$, every function $f : \mathcal{C}_n \to E$ satisfies*

$$\|f - \mathbb{E}_{\sigma_n} f\|^2_{L_2(\sigma_n;E)} \leqslant C \sum_{i=1}^n \|\partial_i f\|^2_{L_2(\log L)^{-1}(\sigma_n;E)}. \tag{25}$$

We now turn to Orlicz space strengthenings of Theorem 5. The scalar-valued analogue of this problem had first been studied by Feissner [1975] and was later completely settled by Bakry and Meyer [1982a; 1982b], who showed the following. For every $p \in (1, \infty)$ and $\alpha \in (0, \infty)$ there exists $K_p(\alpha) \in (0, \infty)$ such that for every $n \in \mathbb{N}$ and $f : \mathcal{C}_n \to \mathbb{C}$,

$$\|\Delta^{-\alpha} f\|_{L_p(\sigma_n)} \leqslant K_p(\alpha) \|f\|_{L_p(\log L)^{-p\alpha}(\sigma_n)}. \tag{26}$$

In [Bakry and Meyer 1982a; 1982b], inequality (26) is stated and proven for the generator of the Ornstein–Uhlenbeck semigroup on Gauss space, yet straightforward modifications of the proof show that (26) holds for the generator of a general hypercontractive semigroup. While proving (26) with the Orlicz norm on the right-hand side replaced by $L_p(\log L)^{-r}(\sigma_n)$ for $r < p\alpha$ is fairly simple (see [Bakry and Meyer 1982a, Théorème 5]), obtaining the result with the optimal Orlicz space $L_p(\log L)^{-p\alpha}(\sigma_n)$ is more delicate. In [Bakry and Meyer 1982b, Théorème 6] this is achieved via a complex interpolation scheme relying on Littlewood–Paley–Stein theory [Stein 1970] (in the form of bounds for the imaginary Riesz potentials Δ^{it}, where $t \in \mathbb{R}$). Even though such tools are generally not available for functions with values in a general Banach space of nontrivial type (see, e.g., [Guerre-Delabrière 1991; Hytönen 2007; Xu 1998]), we prove the following theorem.

Theorem 8 (vector-valued Bakry–Meyer inequality). *Let $(E, \|\cdot\|_E)$ be a Banach space of nontrivial Rademacher type. Then, for every $p \in (1, \infty)$ and $\alpha \in (0, \infty)$, there exists $K_p(\alpha) = K_p(\alpha, E) \in (0, \infty)$ such that for every $n \in \mathbb{N}$ and $f : \mathcal{C}_n \to E$, we have*

$$\|\Delta^{-\alpha} f\|_{L_p(\sigma_n;E)} \leqslant K_p(\alpha) \|f\|_{L_p(\log L)^{-p\alpha}(\sigma_n;E)}. \tag{27}$$

It will be shown in Lemma 17 below that Theorem 8 is indeed a strengthening of Theorem 5. In view of the result of [Naor and Schechtman 2002], it is evident that the assumption that the target space E has nontrivial type is both necessary and sufficient in Theorem 8. While the ingredients used in the proof of [Bakry and Meyer 1982b, Théorème 6] cannot be applied in the vector-valued setting of Theorem 8, (27) will be proven as a consequence of the scalar inequality (26) using the following vector-valued multiplier theorem.

Theorem 9. *Let $(E, \|\cdot\|_E)$ be a Banach space of nontrivial Rademacher type and consider a holomorphic function $h : \mathbb{D}_r \to \mathbb{C}$ where $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, where $r \in (0, \infty)$. Then, for every $\alpha \in (0, \infty)$ and $p \in (1, \infty)$, there exists a constant $C_h(\alpha, p) = C_h(\alpha, p, E) \in (0, \infty)$ such that for every $n \in \mathbb{N}$, every function $f : \mathcal{C}_n \to E$ satisfies*

$$\|h(\Delta^{-\alpha}) f\|_{L_p(\sigma_n;E)} \leqslant C_h(\alpha, p) \|f\|_{L_p(\sigma_n;E)}. \tag{28}$$

When $E = \mathbb{C}$, Theorem 9 is a classical result of Meyer [1984, Thèoréme 3]. The vector-valued extension presented here crucially relies on the bounds on the action of negative powers of Δ on vector-valued tail spaces obtained by Mendel and Naor [2014].

1.5. ***Talagrand metric spaces.*** The vector-valued discrete Poincaré inequality (5) is intimately connected to a metric version of Rademacher type, called Enflo type; see [Enflo 1978; Naor and Schechtman 2002]. In view of this connection, we introduce the following metric invariant, inspired by Talagrand's inequality (23).

Definition 10 (Talagrand type). Let $\psi : [0,\infty) \to [0,\infty)$ be a Young function and $p \in (0,\infty)$. We say that a metric space $(\mathcal{M}, d_{\mathcal{M}})$ has Talagrand type (p,ψ) with constant $\tau \in (0,\infty)$ if for every $n \in \mathbb{N}$, every function $f : \mathcal{C}_n \to \mathcal{M}$ satisfies

$$\int_{\mathcal{C}_n\times\mathcal{C}_n} d_{\mathcal{M}}(f(\varepsilon), f(\delta))^p \,\mathrm{d}\sigma_{2n}(\varepsilon,\delta) \leqslant \tau^p \sum_{i=1}^n \|\mathfrak{d}_i f\|^p_{L_\psi(\sigma_n)}, \tag{29}$$

where $\mathfrak{d}_i f : \mathcal{C}_n \to \mathbb{R}_+$ is given by

$$\mathfrak{d}_i f(\varepsilon) = \tfrac{1}{2} d_{\mathcal{M}}(f(\varepsilon), f(\varepsilon_1,\dots,\varepsilon_{i-1},-\varepsilon_i,\varepsilon_{i+1},\dots,\varepsilon_n)), \quad \text{for all } \varepsilon\in\mathcal{C}_n. \tag{30}$$

It is clear that if $(E, \|\cdot\|_E)$ is a Banach space then $\|\partial_i f(\varepsilon)\|_E$ coincides with $\mathfrak{d}_i f(\varepsilon)$. It can be easily seen that if a Banach space E has the property that for every $n \in \mathbb{N}$, every $f : \mathcal{C}_n \to E$ satisfies

$$\|f - \mathbb{E}_{\sigma_n} f\|^p_{L_p(\sigma_n;E)} \leqslant \tau_*^p \sum_{i=1}^n \|\partial_i f\|^p_{L_\psi(\sigma_n;E)} \tag{31}$$

for some $\tau_* \in (0,\infty)$, then E also has Talagrand type (p,ψ). Indeed, applying (31) to the function $F : \mathcal{C}_n \times \mathcal{C}_n \to E$ given by $F(\varepsilon,\delta) = f(\varepsilon) - f(\delta)$ which has $\mathbb{E}_{\sigma_{2n}} F = 0$, we get

$$\begin{aligned}\int_{\mathcal{C}_n\times\mathcal{C}_n} \|f(\varepsilon)-f(\delta)\|_E^p \,\mathrm{d}\sigma_{2n}(\varepsilon,\delta) = \|F-\mathbb{E}_{\sigma_{2n}}F\|^p_{L_p(\sigma_{2n};E)} &\leqslant \tau_*^p \sum_{i=1}^n (\|\partial_{\varepsilon_i} F\|^p_{L_\psi(\sigma_{2n};E)} + \|\partial_{\delta_i} F\|^p_{L_\psi(\sigma_{2n};E)}) \\ &= 2\tau_*^p \sum_{i=1}^n \|\partial_i f\|^p_{L_\psi(\sigma_n;E)},\end{aligned}$$

and thus E has Talagrand type (p,ψ) with constant $\tau \leqslant 2^{1/p}\tau_*$. Hence, Theorems 6 and 7 can both be translated as implications of Talagrand type from Rademacher and martingale type, respectively; see also the discussion in Section 9. It is worth investigating whether natural examples of nonlinear metric spaces (e.g., Alexandrov spaces of nonpositive or nonnegative curvature, transportation cost spaces and others) have Talagrand type. In this direction, we prove the following Talagrand type inequality for functions with values in Gromov hyperbolic groups. For $p \in [1,\infty)$ and $\delta \in [0,1]$, let $\psi_{p,\delta} : [0,\infty) \to \mathbb{R}$ be a Young function with $\psi_{p,\delta}(x) = t^p \log^{-\delta}(e+x)$ for x large enough.

Theorem 11. *There exists $\tau \in (0,\infty)$ such that for every $\varepsilon \in (0,1)$ the following holds. Every Gromov hyperbolic group G equipped with the shortest path metric on the Cayley graph with respect to a finite generating set $\mathsf{S} \subseteq \mathsf{G}$ has Talagrand type $(2, \psi_{2,1-\varepsilon})$ with constant $\tau/\sqrt{\varepsilon}$.*

The proof of Theorem 11 relies on a result of Ostrovskii [2014], according to which the Cayley graph of every Gromov hyperbolic group admits a bi-Lipschitz embedding in an arbitrary nonsuperreflexive Banach space, combined with a classical construction of James [1978].

We say that a Riemannian manifold has pinched negative curvature if its sectional curvature takes values in the interval $[-R,-r]$ for some $r, R \in (0,\infty)$ with $r < R$. After the proof of Theorem 11 in Section 7, we also prove the following result.

Theorem 12. *Let $n \in \mathbb{N}$ and (M, g) be an n-dimensional complete, simply connected Riemannian manifold with pinched negative curvature. Then, for every $\varepsilon \in (0, 1)$, $(\mathsf{M}, d_{\mathsf{M}})$ has Talagrand type $(2, \psi_{2,1-\varepsilon})$ with constant $\tau/\sqrt{\varepsilon}$ where τ depends only on n and the parameters r, R.*

Theorems 11 and 12 describe two classes of nonpositively curved spaces which satisfy a Talagrand type inequality that strengthens Enflo type 2. It remains an intriguing open problem to understand whether every CAT(0) space has this property; see also Section 9.

1.6. ***Embeddings of nonlinear quotients of the cube and Talagrand type.*** Let $(\mathcal{M}, d_{\mathcal{M}})$ and $(\mathcal{N}, d_{\mathcal{N}})$ be metric spaces. A function $f : \mathcal{M} \to \mathcal{N}$ has bi-Lipschitz distortion at most $D \geqslant 1$ if there exists $s \in (0, \infty)$ such that

$$s d_{\mathcal{M}}(x, y) \leqslant d_{\mathcal{N}}(f(x), f(y)) \leqslant s D d_{\mathcal{M}}(x, y), \quad \text{for all } x, y \in \mathcal{M}. \tag{32}$$

We will denote by $c_{\mathcal{N}}(\mathcal{M})$ the infimal bi-Lipschitz distortion of a function $f : \mathcal{M} \to \mathcal{N}$. When $\mathcal{N} = L_p(\mathbb{R})$, we will abbreviate $c_{L_p(\mathbb{R})}(\mathcal{M})$ as $c_p(\mathcal{M})$. Consider the hypercube $\mathcal{C}_n$ endowed with the Hamming metric $\rho(\varepsilon, \delta) = \|\varepsilon - \delta\|_1$. The geometric significance of Enflo type stems (partially) from the fact (see [Naor and Schechtman 2002]) that if a metric space $\mathcal{M}$ has Enflo type p with constant $T \in (0, \infty)$, then

$$c_{\mathcal{M}}(\mathcal{C}_n) \geqslant T^{-1} n^{1-1/p}. \tag{33}$$

In this section, we will establish a more delicate bi-Lipschitz nonembeddability property which is a consequence of the Talagrand type inequality (29).

Let $\mathcal{R} \subseteq \mathcal{C}_n \times \mathcal{C}_n$ be an arbitrary equivalence relation and denote by $\mathcal{C}_n/\mathcal{R}$ the set of all equivalence classes of $\mathcal{R}$ equipped with the quotient metric, which is given by

$$\rho_{\mathcal{C}_n/\mathcal{R}}([\varepsilon], [\delta]) \stackrel{\text{def}}{=} \min\{\rho(\eta_1, \zeta_1) + \cdots + \rho(\eta_k, \zeta_k)\}, \quad \text{for all } [\varepsilon], [\delta] \in \mathcal{C}_n/\mathcal{R}; \tag{34}$$

here the minimum is taken over all $k \geqslant 1$ and $\eta_1, \ldots, \eta_k, \zeta_1, \ldots, \zeta_k \in \mathcal{C}_n$ with $\eta_1 \in [\varepsilon]$, $\zeta_k \in [\delta]$ and $[\zeta_j] = [\eta_{j+1}]$ for every $j \in \{1, \ldots, k-1\}$. We shall now present an implication of Talagrand type on embeddings of nonlinear quotients[1] of the cube which strengthens the corresponding bounds that one can deduce from Enflo type. We will denote by $\partial_i \mathcal{R}$ the boundary of $\mathcal{R}$ in the direction i, that is

$$\partial_i \mathcal{R} \stackrel{\text{def}}{=} \{\varepsilon \in \mathcal{C}_n : (\varepsilon, (\varepsilon_1, \ldots, \varepsilon_{i-1}, -\varepsilon_i, \varepsilon_{i+1}, \ldots, \varepsilon_n)) \notin \mathcal{R}\}, \quad \text{for all } i \in \{1, \ldots, n\}, \tag{35}$$

and by $\mathsf{a}_p(\mathcal{R})$ the quantity

$$\mathsf{a}_p(\mathcal{R}) \stackrel{\text{def}}{=} \left(\int_{\mathcal{C}_n \times \mathcal{C}_n} \rho_{\mathcal{C}_n/\mathcal{R}}([\varepsilon], [\delta])^p \, d\sigma_{2n}(\varepsilon, \delta) \right)^{1/p}. \tag{36}$$

[1]The term "nonlinear" here is meant to emphasize the distinction between quotients of the hypercube with respect to an arbitrary equivalence relation and quotients by linear codes; see [MacWilliams and Sloane 1977] and Remark 39. Recall that if we identify $\mathcal{C}_n$ with $\mathbb{F}_2^n$, where $\mathbb{F}_2$ is the field with two elements, a linear code is an $\mathbb{F}_2$-subspace $C \subseteq \mathcal{C}_n$ and the corresponding quotient is the $\mathbb{F}_2$-vector space $\mathbb{F}_2^n/C$ endowed with the quotient metric.

Theorem 13. *Fix $p \in (0, \infty)$ and a Young function $\psi : [0, \infty) \to [0, \infty)$. If a metric space $(\mathcal{M}, d_{\mathcal{M}})$ has Talagrand type (p, ψ) with constant $\tau \in (0, \infty)$, then, for every $n \in \mathbb{N}$ and every equivalence relation $\mathcal{R} \subseteq \mathcal{C}_n \times \mathcal{C}_n$, we have*

$$c_{\mathcal{M}}(\mathcal{C}_n/\mathcal{R}) \geqslant \frac{2\tau^{-1}\mathsf{a}_p(\mathcal{R})}{\left(\sum_{i=1}^n \psi^{-1}(\sigma_n(\partial_i\mathcal{R})^{-1})^{-p}\right)^{1/p}}. \tag{37}$$

It is worth noting that in the setting of Theorem 13, if $\mathcal{M}$ has Talagrand type $(p, t \mapsto t^p)$ with constant τ (a property which is very closely related to Enflo type p, see Remark 38), then

$$c_{\mathcal{M}}(\mathcal{C}_n/\mathcal{R}) \geqslant \frac{2\tau^{-1}\mathsf{a}_p(\mathcal{R})}{\left(\sum_{i=1}^n \sigma_n(\partial_i\mathcal{R})\right)^{1/p}}. \tag{38}$$

This estimate, which generalizes (33), is substantially weaker than (37) when $\psi(t) \ll t^p$ for large values of t. In particular, this is the case for Banach spaces of Rademacher or martingale type p (see Theorems 40 and 41). It is also worth mentioning that, in view of Theorem 42 below, Theorem 13 provides nontrivial distortion lower bounds even for bi-Lipschitz embeddings into $L_1(\mu)$ spaces.

Theorem 13 is reminiscent of the celebrated theorem of Kahn, Kalai and Linial [Kahn et al. 1988], which asserts that there exists a constant $c \in (0, \infty)$ such that for every Boolean function $f : \mathcal{C}_n \to \{0, 1\}$,

$$\max_{i\in\{1,\dots,n\}} \|\partial_i f\|_{L_2(\sigma_n)}^2 \geqslant \frac{c \log n}{n} \operatorname{Var}_{\sigma_n} f = \frac{c \log n}{n} p(1-p), \tag{39}$$

where $p = \mathbb{E}_{\sigma_n} f$. Viewing f as a voting scheme, (39) asserts that if all influences $\|\partial_i f\|_{L_2(\sigma_n)}^2$ are small, then f is necessarily an unfair system in the sense that its expectation is very close to either 0 or 1. Inequality (37) puts forth a similar phenomenon in embedding theory: if all *geometric influences* $\sigma_n(\partial_i\mathcal{R})$ of the partition are small, then the quotient $\mathcal{C}_n/\mathcal{R}$ is incompatible with the geometry of the target space $\mathcal{M}$. Moreover, the quantitative improvement (37) of (38) is in direct analogy with the improvement that the KKL inequality (39) offers to the weaker estimate

$$\max_{i\in\{1,\dots,n\}} \|\partial_i f\|_{L_2(\sigma_n)}^2 \geqslant \frac{1}{n} \operatorname{Var}_{\sigma_n} f,$$

which follows readily from the Poincaré inequality (3) for any function $f : \mathcal{C}_n \to \mathbb{C}$.

Organization of the paper. In Section 2, we will present some elementary inequalities and properties of Orlicz norms which we shall use in the sequel. Section 3 contains the proof of Theorems 1 and 6 and Section 4 contains two proofs of Theorems 2 and 7, one using stochastic calculus and one Fourier analytic. In Section 5, we prove Theorems 3 and 4 and their analogue in Gauss space, and Theorem 27 by a combination of semigroup methods and Riesz transforms. Section 6 contains the proof of Theorem 9 and the derivation of Theorems 5 and 8. In Section 7 we present the proof of Theorems 11 and 12 and in Section 8 we present the proof of the nonembeddability result of Theorem 13. Finally, Section 9 contains some concluding remarks and open problems.

2. Some preliminary calculus lemmas

In this section, we present a few elementary facts related to Orlicz norms which we shall repeatedly use in the sequel. While these results are central for our proofs, they are mostly technical and therefore can be skipped on first reading. We gather them here in order to avoid digressions in the main part of the text.

Lemma 14. *Let $(E, \|\cdot\|_E)$ be a Banach space and (Ω, μ) a probability space. For every $r \in (1,\infty)$, $\gamma, \eta \in (0,\infty)$ and $\varepsilon \in [0,1)$, there exists $A = A(r,\gamma,\eta,\varepsilon) \in (0,\infty)$ such that every $h : \Omega \to E$ satisfies*

$$\int_0^\infty e^{-\eta t} \|h\|^r_{L_{1+(r-1)e^{-\gamma t}}(\mu;E)} \frac{\mathrm{d}t}{t^\varepsilon} \leqslant A \|h\|^r_{L_r(\log L)^{-1+\varepsilon}(\mu;E)}. \tag{40}$$

Proof. Since both sides only depend on the norm of h, we can assume that $E = \mathbb{C}$ and $h \geqslant 0$. Moreover, without loss of generality $\eta \leqslant 1 = \gamma$. Suppose, by homogeneity, that the right-hand side satisfies $\|h\|_{L_r(\log L)^{-1+\varepsilon}(\mu)} \leqslant 1$, which implies that

$$\int_\Omega \frac{h^r}{\log^{1-\varepsilon}(e+h)} \,\mathrm{d}\mu \leqslant 1.$$

For $k \geqslant 1$, let $h_k = h \cdot \mathbb{1}_{\{2^{k-1} < h \leqslant 2^k\}}$ and $h_0 = h \cdot \mathbb{1}_{\{h \leqslant 1\}}$, so that

$$\sum_{k=0}^\infty \frac{1}{(k+1)^{1-\varepsilon}} \int_\Omega h_k^r \,\mathrm{d}\mu \leqslant 1. \tag{41}$$

Moreover, observe that

$$\int_0^\infty e^{-\eta t} \|h\|^r_{L_{1+(r-1)e^{-t}}(\mu)} \frac{\mathrm{d}t}{t^\varepsilon} \leqslant \int_0^\infty e^{-\eta t} \|h\|^r_{L_{1+(r-1)e^{-\eta t}}(\mu)} \frac{\mathrm{d}t}{t^\varepsilon} \asymp_{r,\eta} \int_1^r \|h\|^r_{L_\nu(\mu)} \frac{\mathrm{d}\nu}{(r-\nu)^\varepsilon},$$

where the inequality follows from the monotonicity of $L_s(\mu)$-norms and the equivalence by the change of variables $\nu = 1 + (r-1)e^{-\eta t}$.

The right-hand side then satisfies

$$\begin{aligned}
\int_1^r \|h\|^r_{L_\nu(\mu)} \frac{\mathrm{d}\nu}{(r-\nu)^\varepsilon} &= \int_1^r \left(\sum_{k=0}^\infty \int_\Omega h_k^\nu \,\mathrm{d}\mu \right)^{r/\nu} \frac{\mathrm{d}\nu}{(r-\nu)^\varepsilon} \\
&\leqslant 2^r \int_1^r \left(\sum_{k=0}^\infty 2^{-(r-\nu)k} \int_\Omega h_k^r \,\mathrm{d}\mu \right)^{r/\nu} \frac{\mathrm{d}\nu}{(r-\nu)^\varepsilon} \\
&\overset{(41)}{\leqslant} 2^r \int_1^r \sum_{k=0}^\infty (k+1)^{r(1-\varepsilon)/\nu} 2^{-(r-\nu)kr/\nu} \frac{1}{(k+1)^{1-\varepsilon}} \int_\Omega h_k^r \,\mathrm{d}\mu \frac{\mathrm{d}\nu}{(r-\nu)^\varepsilon} \\
&= 2^r \sum_{k=0}^\infty \left(\int_1^r (k+1)^{r(1-\varepsilon)/\nu} 2^{-(r-\nu)kr/\nu} \frac{\mathrm{d}\nu}{(r-\nu)^\varepsilon} \right) \frac{1}{(k+1)^{1-\varepsilon}} \int_\Omega h_k^r \,\mathrm{d}\mu \\
&\overset{(41)}{\leqslant} 2^r \max_{k \geqslant 0} \left\{ \int_1^r (k+1)^{r(1-\varepsilon)/\nu} 2^{-(r-\nu)kr/\nu} \frac{\mathrm{d}\nu}{(r-\nu)^\varepsilon} \right\},
\end{aligned}$$

where the second inequality follows from Jensen's inequality for the convex function $t \mapsto t^{r/\nu}$ with weights (41). Now, by multiplying k by r, one can easily see that

$$\max_{k\geqslant 0}\left\{\int_1^r (k+1)^{r(1-\varepsilon)/\nu}2^{-(r-\nu)kr/\nu}\frac{\mathrm{d}\nu}{(r-\nu)^{\varepsilon}}\right\} \asymp_{r,\varepsilon} \max_{k\geqslant 0}\left\{\int_1^r k^{r(1-\varepsilon)/\nu}e^{-(r-\nu)k}\frac{\mathrm{d}\nu}{(r-\nu)^{\varepsilon}}\right\}$$
$$\asymp_{r,\varepsilon} \max_{k\geqslant 0}\left\{\int_{1/r}^1 k^{(1-\varepsilon)/u}e^{-(1-u)k}\frac{\mathrm{d}u}{(1-u)^{\varepsilon}}\right\},$$

where the second equivalence follows by the change of variables $u=\nu/r$ and a further change of variables in k. For $k\geqslant 0$ and $\varepsilon\in(0,1)$, write

$$\int_{1/r}^1 k^{(1-\varepsilon)/u}e^{-(1-u)k}\frac{\mathrm{d}u}{(1-u)^{\varepsilon}} = \underbrace{\int_{1/r}^{1-1/k} k^{(1-\varepsilon)/u}e^{-(1-u)k}\frac{\mathrm{d}u}{(1-u)^{\varepsilon}}}_{I_k(\varepsilon)} + \underbrace{\int_{1-1/k}^{1} k^{(1-\varepsilon)/u}e^{-(1-u)k}\frac{\mathrm{d}u}{(1-u)^{\varepsilon}}}_{J_k(\varepsilon)}$$

and notice that

$$J_k(\varepsilon)\leqslant k^{(1-\varepsilon)/(1-1/k)}\int_{1-1/k}^1\frac{\mathrm{d}u}{(1-u)^{\varepsilon}} = \frac{1}{1-\varepsilon}k^{(1-\varepsilon)k/(k-1)-(1-\varepsilon)} \asymp_{\varepsilon} 1.$$

Moreover, if $u\leqslant 1-1/k$, then

$$k^{-\varepsilon/u}\frac{1}{(1-u)^{\varepsilon}}\leqslant k^{\varepsilon(1-1/u)}<1,$$

which implies that

$$I_k(\varepsilon)\leqslant \int_{1/r}^{1-1/k} k^{1/u}e^{-(1-u)k}\,\mathrm{d}u \leqslant \int_{1/r}^{1} k^{1/u}e^{-(1-u)k}\,\mathrm{d}u \stackrel{\text{def}}{=} R_k.$$

Finally, to bound R_k, we integrate by parts and get

$$R_k=\int_{1/r}^1 k^{1/u}\left(\frac{e^{-(1-u)k}}{k}\right)'\mathrm{d}u = 1-k^{r-1}e^{-(1-1/r)k}+\frac{\log k}{k}\int_{1/r}^1 k^{1/u}e^{-(1-u)k}\frac{\mathrm{d}u}{u^2}$$
$$\leqslant 1-k^{r-1}e^{-(1-1/r)k}+\frac{r^2\log k}{k}R_k,$$

which, after rearranging, readily implies that $R_k\lesssim_r 1$ and the proof is complete. $\square$

Using Hölder's inequality, we can easily deduce the following variant of Lemma 14 which we will need to prove Theorems 29 and 30.

Lemma 15. *Let $(E,\|\cdot\|_E)$ be a Banach space and (Ω,μ) a probability space. For every $r\in(1,\infty)$, $\gamma,\eta\in(0,\infty)$ and $\varepsilon\in[0,1)$, there exists $B=B(r,\gamma,\eta,\varepsilon)\in(0,\infty)$ such that for every $\theta\in(0,1)$, every $h:\Omega\to E$ satisfies*

$$\int_0^\infty e^{-\eta t}\|h\|_{L_{1+(r-1)e^{-\gamma t}}(\mu;E)}\frac{\mathrm{d}t}{t^{\varepsilon}}\leqslant \frac{B}{\theta^{(r-1)/r}}\|h\|_{L_r(\log L)^{-r(1-\varepsilon)+\theta}(\mu;E)}. \tag{42}$$

Proof. Without loss of generality, we will again assume that $E=\mathbb{C}$, $h\geqslant 0$ and $\eta\leqslant 1=\gamma$. As in the proof of Lemma 14, a change of variables shows that

$$\int_0^\infty e^{-\eta t}\|h\|_{L_{1+(r-1)e^{-t}}(\mu)}\frac{\mathrm{d}t}{t^\varepsilon}\leqslant\int_0^\infty e^{-\eta t}\|h\|_{L_{1+(r-1)e^{-\eta t}}(\mu)}\frac{\mathrm{d}t}{t^\varepsilon}\asymp_{r,\eta}\int_1^r\|h\|_{L_\nu(\mu)}\frac{\mathrm{d}\nu}{(r-\nu)^\varepsilon}. \tag{43}$$

Fix $\theta\in(0,1)$. By Hölder's inequality, we have

$$\int_1^r\|h\|_{L_\nu(\mu)}\frac{\mathrm{d}\nu}{(r-\nu)^\varepsilon}\leqslant\left(\int_1^r\|h\|_{L_\nu(\mu)}^r\frac{\mathrm{d}\nu}{(r-\nu)^{1-r(1-\varepsilon)+\theta}}\right)^{1/r}\left(\int_1^r\frac{\mathrm{d}\nu}{(r-\nu)^{1-\theta/(r-1)}}\right)^{(r-1)/r},$$

and since $\int_1^r 1/(r-\nu)^{1-\theta/(r-1)}\,\mathrm{d}\nu\asymp_r 1/\theta$, we deduce from Lemma 14 that

$$\int_1^r\|h\|_{L_\nu(\mu)}\frac{\mathrm{d}\nu}{(r-\nu)^\varepsilon}\leqslant\frac{A}{\theta^{(r-1)/r}}\|h\|_{L_r(\log L)^{-r(1-\varepsilon)+\theta}(\mu)}$$

for some $A=A(r,\varepsilon)$. Then the proof is complete by (43). □

The following lemma will be used to prove Theorems 3 and 5.

Lemma 16. *Let $(E,\|\cdot\|_E)$ be a Banach space and (Ω,μ) a probability space. For every $r\in[1,\infty)$, $\gamma,\eta\in(0,\infty)$ and $\varepsilon\in[0,1)$, there exists $C=C(r,\gamma,\eta,\varepsilon)\in(0,\infty)$ such that every $h:\Omega\to E$ satisfies*

$$\int_0^\infty e^{-\eta t}\|h\|_{L_{1+(r-1)e^{-\gamma t}}(\mu;E)}\frac{\mathrm{d}t}{t^\varepsilon}\leqslant C\,\frac{\|h\|_{L_r(\mu;E)}}{1+\log^{1-\varepsilon}(\|h\|_{L_r(\mu;E)}/\|h\|_{L_1(\mu;E)})}. \tag{44}$$

Proof. Without loss of generality, we will again assume that $E=\mathbb{C}$, $h\geqslant 0$ and $\eta\leqslant 1=\gamma$. As in the proof of Lemma 14, a change of variables shows that

$$\int_0^\infty e^{-\eta t}\|h\|_{L_{1+(r-1)e^{-t}}(\mu)}\frac{\mathrm{d}t}{t^\varepsilon}\leqslant\int_0^\infty e^{-\eta t}\|h\|_{L_{1+(r-1)e^{-\eta t}}(\mu)}\frac{\mathrm{d}t}{t^\varepsilon}\asymp_{r,\eta}\int_1^r\|h\|_{L_\nu(\mu)}\frac{\mathrm{d}\nu}{(r-\nu)^\varepsilon}. \tag{45}$$

By Hölder's inequality, if $\theta(\nu)=(r-\nu)/(\nu(r-1))$ is such that $(1-\theta)/r+\theta/1=1/\nu$, then

$$\int_1^r\|h\|_{L_\nu(\mu)}\frac{\mathrm{d}\nu}{(r-\nu)^\varepsilon}\leqslant\|h\|_{L_r(\mu)}\int_1^r b^{\theta(\nu)}\frac{\mathrm{d}\nu}{(r-\nu)^\varepsilon}\asymp_{r,\varepsilon}\|h\|_{L_r(\mu)}\int_0^1 b^\theta\frac{\mathrm{d}\theta}{\theta^\varepsilon}, \tag{46}$$

where $b=\|h\|_{L_1(\mu)}/\|h\|_{L_r(\mu)}\in(0,1]$. Finally, if $b<1$, notice that

$$\int_0^1 b^\theta\frac{\mathrm{d}\theta}{\theta^\varepsilon}=\int_0^1 e^{-\theta\log(1/b)}\frac{\mathrm{d}\theta}{\theta^\varepsilon}=\frac{1}{\log^{1-\varepsilon}(1/b)}\int_0^{\log(1/b)}e^{-u}\frac{\mathrm{d}u}{u^\varepsilon}\lesssim_\varepsilon\frac{1}{\log^{1-\varepsilon}(1/b)}$$

and the conclusion follows from (45) and (46). □

The following lemma shows that the Orlicz norm statements of Theorems 6 and 7 indeed strengthen Theorems 1 and 2 respectively. In the special case $r=2$ and $s=1$, this has been proven in [Talagrand 1994, Lemma 2.5] and the general case treated here is similar.

Lemma 17. *Let $(E,\|\cdot\|_E)$ be a Banach space and (Ω,μ) a probability space. For every $r\in(1,\infty)$ and $s\in(0,\infty)$, there exists $D=D(r,s)\in(0,\infty)$ such that every function $h:\Omega\to E$ satisfies*

$$\|h\|_{L_r(\log L)^{-s}(\mu;E)}\leqslant D\,\frac{\|h\|_{L_r(\mu;E)}}{1+\log^{s/r}(\|h\|_{L_r(\mu;E)}/\|h\|_{L_1(\mu;E)})}. \tag{47}$$

Proof. Without loss of generality, we will again assume that $E=\mathbb{C}$ and $h \geqslant 0$. We will prove that

$$\int_\Omega \frac{h^r}{\log^s(e+h)}\,\mathrm{d}\mu \geqslant 1 \quad \Longrightarrow \quad \|h\|_{L_r(\mu)}^r \geqslant \frac{1}{D^r}\big(1+\log^s(\|h\|_{L_r(\mu;E)}/\|h\|_{L_1(\mu;E)})\big).$$

Let $a \in (0,\infty)$. We will distinguish two cases.

Case 1. Suppose that

$$\int_{\{h\geqslant a\}} \frac{h^r}{\log^s(e+h)}\,\mathrm{d}\mu \geqslant \tfrac{1}{2}.$$

Then,

$$\int_\Omega h^r\,\mathrm{d}\mu \geqslant \log^s(e+a)\int_{\{h\geqslant a\}} \frac{h^r}{\log^s(e+h)}\,\mathrm{d}\mu \geqslant \tfrac{1}{2}\log^s(e+a). \tag{48}$$

Case 2. Suppose that

$$\int_{\{h\geqslant a\}} \frac{h^r}{\log^s(e+h)}\,\mathrm{d}\mu < \tfrac{1}{2},$$

so that

$$\int_{\{h< a\}} \frac{h^r}{\log^s(e+h)}\,\mathrm{d}\mu \geqslant \tfrac{1}{2}.$$

Notice that on $\{h<a\}$, we have $h^r/\log^s(e+h) \leqslant a^{r-1}h$, which implies that $\|h\|_{L_1(\mu)} \geqslant 1/2a^{r-1}$. Hence, setting $b=\log(e\|h\|_{L_r(\mu)}/\|h\|_{L_1(\mu)})$, we get

$$b \leqslant \log(2ea^{r-1}\|h\|_{L_r(\mu)}) = (r-1)\log a + \log(2e\|h\|_{L_r(\mu)}). \tag{49}$$

Now choose $a=(e\|h\|_{L_r(\mu)}/\|h\|_{L_1(\mu)})^{1/r}$ so that $b=r\log a$. In Case 1, (48) then implies that

$$\|h\|_{L_r(\mu)}^r \geqslant \tfrac{1}{2}\log^s\big(e+(e\|h\|_{L_r(\mu)}/\|h\|_{L_1(\mu)})^{1/r}\big) \asymp_{r,s} \big(1+\log^s(\|h\|_{L_r(\mu;E)}/\|h\|_{L_1(\mu;E)})\big).$$

On the other hand, since $b=r\log a$, in Case 2, (49) gives

$$\|h\|_{L_r(\mu)}^r \geqslant \frac{1}{(2e)^r}\frac{e\|h\|_{L_r(\mu)}}{\|h\|_{L_1(\mu)}} \gtrsim_{r,s} \big(1+\log^s(\|h\|_{L_r(\mu;E)}/\|h\|_{L_1(\mu;E)})\big),$$

since $x \gtrsim_s 1+\log^s x$ for every $s, x \in (0,\infty)$. □

3. Influence inequalities under Rademacher type

In this section we shall present the proofs of Theorems 1 and 6 which rely on the novel approach introduced in [Ivanisvili et al. 2020]. For $t \in (0,\infty)$, let $\xi(t)=(\xi_1(t),\dots,\xi_n(t))$ be a random vector on $\mathcal{C}_n$ whose coordinates are independent and identically distributed with distribution given by

$$\mathbb{P}\{\xi_i(t)=1\} = \tfrac{1}{2}(1+e^{-t}) \quad \text{and} \quad \mathbb{P}\{\xi_i(t)=-1\} = \tfrac{1}{2}(1-e^{-t}), \tag{50}$$

for $i \in \{1,\dots,n\}$. Moreover, consider the normalized vector $\delta(t)=(\delta_1(t),\dots,\delta_n(t))$ with

$$\delta_i(t) \stackrel{\mathrm{def}}{=} \frac{\xi_i(t)-\mathbb{E}\xi_i(t)}{\sqrt{\operatorname{Var}\xi_i(t)}} = \frac{\xi_i(t)-e^{-t}}{\sqrt{1-e^{-2t}}}. \tag{51}$$

In the following statements, we will denote by ε a random vector independent of $\xi(t)$, uniformly distributed on $\mathcal{C}_n$. We will need the following (straightforward) refinement of [Ivanisvili et al. 2020, Theorem 1.4].

Proposition 18. *For every Banach space $(E, \|\cdot\|_E)$, $p \in [1,\infty)$, $n \in \mathbb{N}$ and $f : \mathcal{C}_n \to E$, we have*

$$\Big\| \frac{\partial}{\partial t} P_t f \Big\|_{L_p(\sigma_n;E)} \leqslant \frac{1}{\sqrt{e^{2t}-1}} \bigg(\mathbb{E} \Big\| \sum_{i=1}^n \delta_i(t) \partial_i f(\varepsilon) \Big\|_E^p \bigg)^{1/p}, \quad \text{for all } t \geqslant 0, \tag{52}$$

where the expectation on the right-hand side is with respect to ε and $\delta(t)$.

Let us mention here that we will apply the previous proposition to $P_t f$ instead of f, and use the semigroup property $P_{2t} f = P_t(P_t f)$. This is more easily done after reformulating (52) with ΔP_t in place of $\partial P_t/\partial t$. So, keeping the notation of Proposition 18, we have that

$$\|\Delta P_{2t} f\|_{L_p(\sigma_n;E)} \leqslant \frac{1}{\sqrt{e^{2t}-1}} \bigg(\mathbb{E} \Big\| \sum_{i=1}^n \delta_i(t) \partial_i P_t f(\varepsilon) \Big\|_E^p \bigg)^{1/p}, \quad \text{for all } t \geqslant 0. \tag{53}$$

Proof of Proposition 18. The crucial observation of Ivanisvili, van Handel and Volberg is that one can write, for $x \in \mathcal{C}_n$,

$$\frac{\partial}{\partial t} P_t f(x) = -\frac{1}{\sqrt{e^{2t}-1}} \mathbb{E}_{\xi(t)} \bigg[\sum_{i=1}^n \delta_i(t) \partial_i f(x\xi(t)) \bigg], \tag{54}$$

where $x\xi(t)$ denotes the point $(x_1\xi_1(t), \dots, x_n\xi_n(t))$. This formula can be proved by writing

$$P_t f(x) = \mathbb{E} f(x\xi(t)) = \sum_{\xi \in \mathcal{C}_n} \omega_t(\xi) f(x\xi),$$

where, for $\xi \in \mathcal{C}_n$, $\omega_t(\xi) = 2^{-n} \prod_{i=1}^n (1 + e^{-t}\xi_i)$; then we note that, with some abuse of notation (denoting ∂_{ξ_i} for the discrete derivative ∂_i for functions of the variable $\xi \in \mathcal{C}_n$),

$$\frac{\partial}{\partial t} \omega_t(\xi) = -\frac{e^{-t}}{1-e^{-2t}} \sum_{i=1}^n \partial_{\xi_i} [(\xi_i - e^{-t}) \omega_t(\xi)].$$

Hence, using the integration by parts formula (17) together with the fact that $\partial_{\xi_i}[f(x\xi)] = \partial_i f(x\xi)$,

$$\frac{\partial}{\partial t} P_t f(x) = -\frac{e^{-t}}{\sqrt{1-e^{-2t}}} \sum_{i=1}^n \sum_{\xi \in \mathcal{C}_n} \frac{\xi_i - e^{-t}}{\sqrt{1-e^{-2t}}} \omega_t(\xi) \partial_i f(x\xi) = -\frac{1}{\sqrt{e^{2t}-1}} \mathbb{E}_{\xi(t)} \bigg[\sum_{i=1}^n \delta_i(t) \partial_i f(x\xi(t)) \bigg],$$

and this concludes the proof of (54). Alternatively, it suffices to check the validity of formula (54) in the case of the scalar-valued Walsh basis $w_J(x) = \prod_{j \in J} x_j$, where $J \subseteq \{1, \dots, n\}$, for which $P_t w_J(x) = e^{-t|J|} w_J(x)$ and $\partial_i w_J(x) = \mathbb{1}_{i \in J}\, w_J(x)$.

Therefore, using Jensen's inequality and (54) we have

$$\sqrt{e^{2t}-1} \Big\| \frac{\partial}{\partial t} P_t f \Big\|_{L_p(\sigma_n;E)} = \bigg(\mathbb{E}_\varepsilon \Big\| \mathbb{E}_{\xi(t)} \sum_{i=1}^n \delta_i(t) \partial_i f(\varepsilon\xi(t)) \Big\|_E^p \bigg)^{1/p} \leqslant \bigg(\mathbb{E} \Big\| \sum_{i=1}^n \delta_i(t) \partial_i f(\varepsilon\xi(t)) \Big\|_E^p \bigg)^{1/p}.$$

We conclude by noting that the couple $(\varepsilon\xi(t), \xi(t))$ has the same law as the couple $(\varepsilon, \xi(t))$. This can be seen as a proxy of the rotational invariance of the Gaussian measure (compare with the proof of Proposition 28 below). □

Theorems 1 and 6 are consequences of the following lemma.

Lemma 19. *Let $(E, \|\cdot\|_E)$ be a Banach space with Rademacher type 2. Then there exists a constant $K = K(E) \in (0,\infty)$ such that for every $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$, every $f : \mathcal{C}_n \to E$ satisfies*

$$\|f - \mathbb{E}_{\sigma_n} f\|^2_{L_2(\sigma_n;E)} \leqslant \frac{K}{\varepsilon}\sum_{i=1}^n \int_0^\infty e^{-\varepsilon t}\|\partial_i P_t f\|^2_{L_2(\sigma_n;E)}\,\frac{\mathrm{d}t}{t^\varepsilon}. \tag{55}$$

Proof. We will apply Proposition 18 to $P_t f$ instead of f. We have that

$$\begin{aligned}\|f - \mathbb{E}_{\sigma_n} f\|_{L_2(\sigma_n;E)} = \left\|\int_0^\infty \Delta P_t f\,\mathrm{d}t\right\|_{L_2(\sigma_n;E)} &= 2\left\|\int_0^\infty \Delta P_{2t} f\,\mathrm{d}t\right\|_{L_2(\sigma_n;E)} \leqslant 2\int_0^\infty \|\Delta P_{2t} f\|_{L_2(\sigma_n;E)}\,\mathrm{d}t\\ &\overset{(53)}{\leqslant} 2\int_0^\infty \left(\mathbb{E}\left\|\sum_{i=1}^n \delta_i(t)\partial_i P_t f(\varepsilon)\right\|_E^2\right)^{1/2}\frac{\mathrm{d}t}{\sqrt{e^{2t}-1}}.\end{aligned} \tag{56}$$

Suppose now that E has Rademacher type 2 with constant T. Then for $\varepsilon \in (0,1)$, by (56) and the Rademacher type condition for centered random variables [Ledoux and Talagrand 1991, Proposition 9.11], we have

$$\begin{aligned}\|f - \mathbb{E}_{\sigma_n} f\|_{L_2(\sigma_n;E)} &\leqslant 4T\int_0^\infty \left(\sum_{i=1}^n \|\partial_i P_t f\|^2_{L_2(\sigma_n;E)}\right)^{1/2}\frac{\mathrm{d}t}{\sqrt{e^{2t}-1}}\\ &\leqslant 4T\left(\int_0^\infty \sum_{i=1}^n \|\partial_i P_t f\|^2_{L_2(\sigma_n;E)}\frac{\mathrm{d}t}{(e^{2t}-1)^\varepsilon}\right)^{1/2}\left(\int_0^\infty \frac{\mathrm{d}t}{(e^{2t}-1)^{1-\varepsilon}}\right)^{1/2},\end{aligned} \tag{57}$$

where in the second line we used the Cauchy–Schwarz inequality. Therefore, since

$$\int_0^\infty \frac{\mathrm{d}t}{(e^{2t}-1)^{1-\varepsilon}} \asymp \frac{1}{\varepsilon} \quad \text{as } \varepsilon \to 0^+,$$

we deduce that there exists a universal constant $C \in (0,\infty)$ with

$$\|f - \mathbb{E}_{\sigma_n} f\|^2_{L_2(\sigma_n;E)} \leqslant \frac{C\cdot T^2}{\varepsilon}\sum_{i=1}^n \int_0^\infty \|\partial_i P_t f\|^2_{L_2(\sigma_n;E)}\,\frac{\mathrm{d}t}{(e^{2t}-1)^\varepsilon},$$

and the conclusion follows readily since $e^{2t} - 1 \geqslant te^t$ for every $t \geqslant 0$. □

Proof of Theorems 1 and 6. By Bonami's hypercontractive inequalities [1970], since the semigroup commutes with partial derivatives, we get that for every $t \geqslant 0$ and $i \in \{1,\dots,n\}$,

$$\|\partial_i P_t f\|_{L_2(\sigma_n;E)} = \|P_t \partial_i f\|_{L_2(\sigma_n;E)} \leqslant \|\partial_i f\|_{L_{1+e^{-2t}}(\sigma_n;E)}. \tag{58}$$

Therefore, the conclusion of Theorem 6 follows by combining Lemma 19, (58) and Lemma 14. Moreover, in view of Lemma 17, Theorem 6 readily implies (8). In order to prove (9), one can just apply (8) for $\varepsilon \asymp \sigma(f)^{-1}$. $\square$

Remark 20. It was pointed out to us by an anonymous referee that plugging in the standard application of Hölder's inequality (46) along with hypercontractivity to bound the middle term of (57) cannot remove the dependence on ε in inequality (8). Indeed, by hypercontractivity and Hölder's inequality, we have

$$\int_0^\infty \Big(\sum_{i=1}^n \|\partial_i P_t f\|^2_{L_2(\sigma_n;E)}\Big)^{1/2} \frac{\mathrm{d}t}{\sqrt{e^{2t}-1}} \leqslant \int_0^1 \Big(\sum_{i=1}^n a_i^{(1-u^2)/(1+u^2)} b_i^{2u^2/(1+u^2)}\Big)^{1/2} \frac{\mathrm{d}u}{\sqrt{1-u^2}},$$

where $a_i = \|\partial_i f\|^2_{L_1(\sigma_n;E)}$ and $b_i = \|\partial_i f\|^2_{L_2(\sigma_n;E)}$. Suppose, for contradiction, that for every $n \geqslant 1$ and every $0 \leqslant a_i \leqslant b_i$ where $i \in \{1, \ldots, n\}$, we have

$$\int_0^1 \Big(\sum_{i=1}^n a_i^{(1-u^2)/(1+u^2)} b_i^{2u^2/(1+u^2)}\Big)^{1/2} \frac{\mathrm{d}u}{\sqrt{1-u^2}} \lesssim \Big(\sum_{i=1}^n \frac{b_i}{1+\log(b_i/a_i)}\Big)^{1/2}.$$

Equivalently, we have

$$\int_0^1 \Big(\sum_{i=1}^n p_i \exp\Big(-\frac{1-u^2}{1+u^2}x_i\Big)\cdot(1+x_i)\Big)^{1/2} \frac{\mathrm{d}u}{\sqrt{1-u^2}} \lesssim 1, \tag{59}$$

where $x_i = \log(b_i/a_i) \geqslant 0$ and $\big(\sum_{k=1}^n b_k/(1+\log(b_k/a_k))\big)p_i = b_i/(1+\log(b_i/a_i))$. The parameters $n \geqslant 1$, $x_i \geqslant 0$ and the weights p_i are all arbitrary, thus we conclude from (59) that for every positive random variable X, the inequality

$$\int_0^1 \sqrt{\mathbb{E}\Big[\exp\Big(-\frac{1-u^2}{1+u^2}X\Big)\cdot(1+X)\Big]} \frac{\mathrm{d}u}{\sqrt{1-u^2}} \lesssim 1 \tag{60}$$

holds true. To reach a contradiction, consider a discrete random variable $X \geqslant 0$ such that

$$\sum_{k\geqslant 0} \sqrt{\mathbb{P}\{1+X \in [2^k, 2^{k+1})\}} = \infty, \tag{61}$$

and notice that

$$\begin{aligned}
\int_0^1 \sqrt{\mathbb{E}\Big[\exp\Big(-\frac{1-u^2}{1+u^2}X\Big)\cdot(1+X)\Big]} \frac{\mathrm{d}u}{\sqrt{1-u^2}} &\geqslant \frac{1}{\sqrt{2}}\int_0^1 \sqrt{\mathbb{E}[\exp(-v(1+X))\cdot(1+X)]}\,\frac{\mathrm{d}v}{\sqrt{v}} \\
&> \frac{1}{\sqrt{2}}\int_0^1 \sqrt{\sum_{k=0}^\infty \exp(-v2^{k+1})\cdot 2^k\cdot\mathbb{P}\{1+X\in[2^k,2^{k+1})\}}\,\frac{\mathrm{d}v}{\sqrt{v}} \\
&\geqslant \frac{1}{2\sqrt{2}}\sum_{\ell=0}^\infty 2^{-\ell/2}\sqrt{\sum_{k=0}^\infty \exp(-2^{k-\ell})\cdot 2^k\cdot\mathbb{P}\{1+X\in[2^k,2^{k+1})\}} \\
&\geqslant \frac{1}{2\sqrt{2e}}\sum_{\ell=0}^\infty \sqrt{\mathbb{P}\{1+X\in[2^\ell,2^{\ell+1})\}} = \infty,
\end{aligned}$$

where in the last inequality we bounded the inner sum by the $k=\ell$ term. This contradicts (60).

Remark 21. A combination of Proposition 18 and Lemma 16 implies a different Talagrand type strengthening of the vector-valued discrete Poincaré inequality (5) for spaces of Rademacher type 2, which is weaker than (7); see also [Chatterjee 2014, Theorem 5.4] for a similar scalar-valued inequality. For a function $f : \mathcal{C}_n \to E$, we will use the notation $\boldsymbol{D} f : \mathcal{C}_n \to E^n$ for the gradient vector

$$\boldsymbol{D} f \stackrel{\text{def}}{=} (\partial_1 f, \dots, \partial_n f).$$

Then, the first inequality in (57) can be rewritten as

$$\|f - \mathbb{E}_{\sigma_n} f\|_{L_2(\sigma_n;E)} \lesssim_E \int_0^\infty \biggl(\sum_{i=1}^n \|\partial_i P_t f\|_{L_2(\sigma_n;E)}^2\biggr)^{1/2} \frac{\mathrm{d}t}{\sqrt{e^{2t}-1}} = \int_0^\infty \|P_t \boldsymbol{D} f\|_{L_2(\sigma_n;\ell_2^n(E))} \frac{\mathrm{d}t}{\sqrt{e^{2t}-1}}.$$

Now, by the hypercontractivity of $\{P_t\}_{t\geqslant 0}$, we have

$$\|P_t \boldsymbol{D} f\|_{L_2(\sigma_n;\ell_2^n(E))} \leqslant \|\boldsymbol{D} f\|_{L_{1+e^{-2t}}(\sigma_n;\ell_2^n(E))}.$$

Therefore, combining the last two inequalities, we get

$$\|f - \mathbb{E}_{\sigma_n} f\|_{L_2(\sigma_n;E)} \lesssim_E \int_0^\infty \|\boldsymbol{D} f\|_{L_{1+e^{-2t}}(\sigma_n;\ell_2^n(E))} \frac{\mathrm{d}t}{\sqrt{e^{2t}-1}} \lesssim \int_0^\infty e^{-t/2} \|\boldsymbol{D} f\|_{L_{1+e^{-2t}}(\sigma_n;\ell_2^n(E))} \frac{\mathrm{d}t}{\sqrt{t}},$$

and Lemma 16 then implies that

$$\|f - \mathbb{E}_{\sigma_n} f\|_{L_2(\sigma_n;E)} \lesssim_E \frac{\|\boldsymbol{D} f\|_{L_2(\sigma_n;\ell_2^n(E))}}{1 + \sqrt{\log\bigl(\|\boldsymbol{D} f\|_{L_2(\sigma_n;\ell_2^n(E))} / \|\boldsymbol{D} f\|_{L_1(\sigma_n;\ell_2^n(E))}\bigr)}}. \tag{62}$$

The argument above shows that spaces of Rademacher type 2 satisfy (62) and the reverse implication is clear by choosing a function of the form $f(\varepsilon) = \sum_{i=1}^n \varepsilon_i x_i$. When $E = \mathbb{C}$, this coincides with (16) where $p = 2$; see also Remark 32 below for comparison with (6).

4. Influence inequalities under martingale type

In this section, we shall present two proofs of Theorems 2 and 7, one probabilistic and one Fourier analytic. As a warmup, we present a simple proof of Talagrand's inequality in Gauss space for functions with values in a space of martingale type 2 using a classical stochastic representation for the variance. The scalar-valued case of this inequality was shown in [Cordero-Erausquin and Ledoux 2012] via semigroup methods which do not seem to be adaptable to the case of vector-valued functions (see Section 4.3 for a harmonic analytic variant). We will denote by γ_n the standard Gaussian measure on $\mathbb{R}^n$, i.e., the measure $\mathrm{d}\gamma_n(x) = \exp(-\|x\|_2^2/2)/(2\pi)^{n/2}\,\mathrm{d}x$, where $\|\cdot\|_2$ denotes the usual Euclidean norm on $\mathbb{R}^n$.

4.1. ***A simple stochastic proof in Gauss space.*** We will denote by $\{U_t\}_{t\geqslant 0}$ the Ornstein–Uhlenbeck semigroup on $\mathbb{R}^n$, whose action on an integrable function $f : \mathbb{R}^n \to E$, where $(E, \|\cdot\|_E)$ is a Banach space, is given by the Mehler formula

$$U_t f(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1-e^{-2t}}\,y)\,\mathrm{d}\gamma_n(y), \quad \text{for all } t \geqslant 0 \text{ and } x \in \mathbb{R}^n. \tag{63}$$

Let $\{X_t\}_{t\geqslant 0}$ be an Ornstein–Uhlenbeck process, i.e., a stochastic process of the form $X_t = e^{-t}X_0 + e^{-t}B_{e^{2t}-1}$, where $\{B_t\}_{t\geqslant 0}$ is a standard Brownian motion and X_0 is a standard Gaussian random vector, independent of $\{B_t\}_{t\geqslant 0}$. We will use the following well-known consequence of the Clark–Ocone formula; see [Capitaine et al. 1997] for a proof and further applications in functional inequalities.

Lemma 22. *Let $(E, \|\cdot\|_E)$ be a Banach space. For every smooth function $f : \mathbb{R}^n \to E$, we have*

$$f(X_s) - U_s f(X_0) = \int_0^s \nabla(U_{s-t} f)(X_t)\cdot \mathrm{d}B_t, \quad \textit{for all } s > 0. \tag{64}$$

We will also need the following one-sided version of the Itô isometry for 2-smooth spaces, which is essentially due to Dettweiler [1991]. We include the crux of the (simple) proof for completeness.

Proposition 23. *Let $(E, \|\cdot\|_E)$ be a Banach space of martingale type 2. Then there exists $M \in (0,\infty)$ such that for every $n \in \mathbb{N}$, if $\{B_t\}_{t\geqslant 0}$ is a standard Brownian motion on $\mathbb{R}^n$ and $\{Y_t\}_{t\geqslant 0}$ is an E^n-valued square integrable stochastic process adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ of $\{B_t\}_{t\geqslant 0}$, then*

$$\mathbb{E}\left\|\int_0^\infty Y_t\cdot \mathrm{d}B_t\right\|_E^2 \leqslant M^2\int_0^\infty \mathbb{E}\left\|\sum_{i=1}^n G(i)Y_t(i)\right\|_E^2 \mathrm{d}t, \tag{65}$$

where $G = (G(1),\dots,G(n))$ is a standard Gaussian random vector on $\mathbb{R}^n$, independent of $\{\mathcal{F}_t\}_{t\geqslant 0}$.

Proof. We shall assume that $\{Y_t\}_{t\geqslant 0}$ is a simple process of the form

$$Y_t(i) = \sum_{k=1}^N \alpha_{t_k}(i)\cdot \mathbb{1}_{(t_k,t_{k+1}]}, \quad \text{for all } i \in \{1,\dots,n\},$$

where $0 = t_1 < t_2 < \cdots < t_{N+1}$ and each $\alpha_{t_k}(i)$ is an $\mathcal{F}_{t_k}$-measurable random variable. The general case will follow by standard approximation arguments. By definition,

$$\int_0^\infty Y_t\cdot \mathrm{d}B_t = \sum_{k=1}^N\sum_{i=1}^n \alpha_{t_k}(i)\cdot (B_{t_{k+1}}(i) - B_{t_k}(i))$$

and $\left\{\sum_{i=1}^n \alpha_{t_k}(i)(B_{t_{k+1}}(i) - B_{t_k}(i))\right\}_{k=1}^N$ is a martingale difference sequence. Therefore, if M is the martingale type 2 constant of E, then

$$\mathbb{E}\left\|\int_0^\infty Y_t\cdot \mathrm{d}B_t\right\|_E^2 \leqslant M^2\sum_{k=1}^N \mathbb{E}\left\|\sum_{i=1}^n \alpha_{t_k}(i)\cdot (B_{t_{k+1}}(i) - B_{t_k}(i))\right\|_E^2. \tag{66}$$

Now, for a fixed k, $(B_{t_{k+1}}(i) - B_{t_k}(i))_{i=1}^n$ conditioned on $\mathcal{F}_{t_k}$ is equidistributed to a Gaussian random vector with covariance matrix $(t_{k+1}-t_k)\cdot \mathrm{Id}_n$. Therefore,

$$\mathbb{E}\left[\left\|\sum_{i=1}^n \alpha_{t_k}(i)\cdot (B_{t_{k+1}}(i) - B_{t_k}(i))\right\|_E^2 \,\Big|\, \mathcal{F}_{t_k}\right] = (t_{k+1}-t_k)\mathbb{E}\left[\left\|\sum_{i=1}^n G(i)\alpha_{t_k}(i)\right\|_E^2 \,\Big|\, \mathcal{F}_{t_k}\right], \tag{67}$$

where $G = (G(1), \dots, G(n))$ is a standard Gaussian random vector, independent of $\{\mathcal{F}_t\}_{t\geqslant 0}$. Hence, after taking expectation in (67) and summing over k, (66) becomes

$$\mathbb{E}\left\| \int_0^\infty Y_t \cdot \mathrm{d}B_t \right\|_E^2 \leqslant M^2 \sum_{k=1}^N (t_{k+1} - t_k) \mathbb{E}\left\| \sum_{i=1}^n G(i)\alpha_{t_k}(i) \right\|_E^2 = M^2 \int_0^\infty \mathbb{E}\left\| \sum_{i=1}^n G(i) Y_t(i) \right\|_E^2 \mathrm{d}t. \qquad \square$$

We are now well equipped to prove the following result.

Theorem 24. *Let $(E, \|\cdot\|_E)$ be a Banach space with martingale type 2. Then there exists $C = C(E) \in (0, \infty)$ such that for every $n \in \mathbb{N}$, every smooth function $f : \mathbb{R}^n \to E$ satisfies*

$$\|f - \mathbb{E}_{\gamma_n} f\|_{L_2(\gamma_n;E)}^2 \leqslant C \sum_{i=1}^n \|\partial_i f\|_{L_2(\log L)^{-1}(\gamma_n;E)}^2. \tag{68}$$

Proof. If E has martingale type 2 with constant M, then Lemma 22 and Proposition 23 imply that

$$\mathbb{E}[\|f(X_s) - U_s f(X_0)\|_E^2 \mid X_0] \leqslant M^2 \int_0^s \mathbb{E}\left[\left\| \sum_{i=1}^n G(i) \partial_i U_{s-t} f(X_t) \right\|_E^2 \,\middle|\, X_0 \right] \mathrm{d}t, \quad \text{for all } s > 0.$$

Thus, applying the Rademacher type 2 condition for Gaussian variables, we deduce that

$$\mathbb{E}[\|f(X_s) - U_s f(X_0)\|_E^2 \mid X_0] \leqslant M^2 T^2 \int_0^s \sum_{i=1}^n \mathbb{E}\left[\left\| \partial_i U_{s-t} f(X_t) \right\|_E^2 \,\middle|\, X_0 \right] \mathrm{d}t, \quad \text{for all } s > 0, \tag{69}$$

where T is the Rademacher type 2 constant of E. Now, integrating (69) with respect to the standard Gaussian random vector X_0 and using the stationarity of the Ornstein–Uhlenbeck process $\{X_t\}_{t\geqslant 0}$ along with Nelson's hypercontractive inequalities [1966; 1973], we derive, for all $s > 0$,

$$\begin{aligned} \mathbb{E}\|f(X_s) - U_s f(X_0)\|_E^2 &\leqslant M^2 T^2 \sum_{i=1}^n \int_0^s \|\partial_i U_{s-t} f\|_{L_2(\gamma_n;E)}^2 \,\mathrm{d}t \\ &= M^2 T^2 \sum_{i=1}^n \int_0^s e^{-2(s-t)} \|U_{s-t} \partial_i f\|_{L_2(\gamma_n;E)}^2 \,\mathrm{d}t \leqslant M^2 T^2 \sum_{i=1}^n \int_0^s e^{-2t} \|\partial_i f\|_{L_{1+e^{-2t}}(\gamma_n;E)}^2 \,\mathrm{d}t, \end{aligned} \tag{70}$$

where the equality follows from the standard commutation relation $\partial_i U_{s-t} f = e^{-(s-t)} U_{s-t} \partial_i f$. Since for every $i \in \{1, \dots, n\}$ the correlation $\mathbb{E} X_0(i) X_s(i)$ equals e^{-s}, taking $s \to \infty$ in (70) we get

$$\|f - \mathbb{E}_{\gamma_n} f\|_{L_2(\gamma_n;E)}^2 \leqslant M^2 T^2 \int_0^\infty e^{-2t} \|f\|_{L_{1+e^{-2t}}(\gamma_n;E)}^2 \,\mathrm{d}t,$$

and the conclusion follows by Lemma 14. $\square$

4.2. ***A proof of Theorems 2 and 7 via the Eldan–Gross process.*** Eldan and Gross [2022] constructed a clever stochastic process on the cube which resembles the behavior of Brownian motion on $\mathbb{R}^n$ and used it to prove several important inequalities relating the variance and influences of Boolean functions. We shall briefly describe their construction.

Let $\{B_t\}_{t\geqslant 0}=\{(B_t(1),\dots,B_t(n))\}_{t\geqslant 0}$ be a standard Brownian motion on $\mathbb{R}^n$. For every $i\in\{1,\dots,n\}$ and $t\geqslant 0$, consider the stopping time $\tau_t(i)$ given by

$$\tau_t(i)\stackrel{\text{def}}{=}\inf\{s\geqslant 0:|B_s(i)|>t\},$$

and then let $X_t(i)\stackrel{\text{def}}{=}B_{\tau_t(i)}(i)$. Then the jump process $\{X_t\}_{t\geqslant 0}\stackrel{\text{def}}{=}\{(X_t(1),\dots,X_t(n))\}_{t\geqslant 0}$ satisfies the following properties (see [Eldan and Gross 2022, Section 3] for detailed proofs):

(1) For every $t\geqslant 0$ and $i\in\{1,\dots,n\}$, $|X_t(i)|=t$ almost surely, and in fact $X_t\sim\mathrm{Unif}\{-t,t\}^n$.

(2) The process $\{X_t\}_{t\geqslant 0}$ is a martingale.

(3) For every coordinate $i\in\{1,\dots,n\}$, the jump probabilities of $\{X_t(i)\}_{t\geqslant 0}$ are

$$\mathbb{P}\{\operatorname{sign}X_{t+h}(i)\neq\operatorname{sign}X_t(i)\}=\frac{h}{2(t+h)},\quad\text{for all }\ t,h>0.\tag{71}$$

Proof of Theorems 2 and 7. Fix a function $f:\mathcal{C}_n\to E$ and recall (see, e.g., [O'Donnell 2014]) that there exists a unique multilinear polynomial on $\mathbb{R}^n$ which coincides with f on $\mathcal{C}_n$, i.e., we can write

$$f(\varepsilon)=\sum_{A\subseteq\{1,\dots,n\}}\hat{f}(A)\prod_{i\in A}\varepsilon_i,\quad\text{for all }\ \varepsilon\in\mathcal{C}_n,\tag{72}$$

for some coefficients $\hat{f}(A)\in E$. By abuse of notation, we will also denote by f that unique multilinear extension on $\mathbb{R}^n$. Since f is a multilinear polynomial and $\{X_t\}_{t\geqslant 0}$ is a martingale with independent coordinates, it follows that the process $\{f(X_t)\}_{t\geqslant 0}$ is itself a martingale.

Fix some large $N\in\mathbb{N}$ and for $k\in\{0,1,\dots,N\}$, let $t_k=k/N$ and $M_k=f(X_{t_k})$. Since E has martingale type 2, there exists $M=M(E)\in(0,\infty)$ such that

$$\|f-\mathbb{E}_{\sigma_n}f\|^2_{L_2(\sigma_n;E)}=\mathbb{E}\|M_N-M_0\|_E^2\leqslant M^2\sum_{k=1}^N\mathbb{E}\|M_k-M_{k-1}\|_E^2.\tag{73}$$

Now, for a fixed $k\in\{1,\dots,N\}$, since $M_k-M_{k-1}=f(X_{t_k})-f(X_{t_{k-1}})$, Taylor's formula gives

$$M_k-M_{k-1}=\sum_{i=1}^n(X_{t_k}(i)-X_{t_{k-1}}(i))\cdot\frac{\partial f}{\partial x_i}(X_{t_{k-1}})+R_k(f),\tag{74}$$

where $\partial f/\partial x_i$ are the usual partial derivatives of f on $\mathbb{R}^n$ and the remainder $R_k(f)$ satisfies

$$\|R_k(f)\|_E\leqslant\frac12\sum_{i,j=1}^n\left\|\frac{\partial^2 f}{\partial x_i\partial x_j}\right\|_{L_\infty([-1,1]^n;E)}|X_{t_k}(i)-X_{t_{k-1}}(i)|\cdot|X_{t_k}(j)-X_{t_{k-1}}(j)|.\tag{75}$$

However, since f is a multilinear polynomial, all second derivatives of the form $\partial^2 f/\partial x_i^2$ vanish and (75) implies that

$$\|R_k(f)\|_E\leqslant K(f)\cdot\sum_{\substack{i,j=1\\ i\neq j}}^n|X_{t_k}(i)-X_{t_{k-1}}(i)|\cdot|X_{t_k}(j)-X_{t_{k-1}}(j)|,\tag{76}$$

for some $K(f) \in (0, \infty)$, so that

$$\mathbb{E}\|R_k(f)\|_E^2 \leqslant n^2 K(f)^2 \cdot \sum_{\substack{i,j=1\\ i\neq j}}^{n} \mathbb{E}|X_{t_k}(i) - X_{t_{k-1}}(i)|^2 \cdot \mathbb{E}|X_{t_k}(j) - X_{t_{k-1}}(j)|^2. \tag{77}$$

The fact that only $i \neq j$ enters the sum will be crucial below to ensure that the error tends to zero as $N \to +\infty$ after summing over k. Now, by (71), we have

$$\operatorname{sign}(X_{t_{k-1}}(i)) \cdot (X_{t_k}(i) - X_{t_{k-1}}(i)) = \begin{cases} -\frac{2k-1}{N} & \text{with probability } \frac{1}{2k}, \\ \frac{1}{N} & \text{with probability } \frac{2k-1}{2k}, \end{cases}$$

so the conditional second moment of the increments is

$$\mathbb{E}[|X_{t_k}(i) - X_{t_{k-1}}(i)|^2 \mid X_{t_{k-1}}(i)] = \frac{1}{2k}\left(\frac{2k-1}{N}\right)^2 + \frac{2k-1}{2k}\frac{1}{N^2} = \frac{2k-1}{N^2}. \tag{78}$$

By the tower property of conditional expectation, the estimate (77) can finally be written as

$$\mathbb{E}\|R_k(f)\|_E^2 \lesssim \frac{k^2 n^4 K(f)^2}{N^4}, \tag{79}$$

and thus (74) implies that

$$\mathbb{E}\|M_k - M_{k-1}\|_E^2 \lesssim \mathbb{E}\left\|\sum_{i=1}^{n}(X_{t_k}(i) - X_{t_{k-1}}(i)) \cdot \frac{\partial f}{\partial x_i}(X_{t_{k-1}})\right\|_E^2 + \frac{k^2 n^4 K(f)^2}{N^4}. \tag{80}$$

Since $\{X_t\}_{t\geqslant 0}$ is a martingale, the sequence $(X_{t_k}(i) - X_{t_{k-1}}(i))_{i=1}^n$ is a sequence of independent entered random variables, when conditioned on $\{X_s\}_{s\leqslant t_{k-1}}$. Therefore, applying the Rademacher type condition for centered random variables [Ledoux and Talagrand 1991, Proposition 9.11] and (78), we deduce that

$$\mathbb{E}\left[\left\|\sum_{i=1}^{n}(X_{t_k}(i) - X_{t_{k-1}}(i)) \cdot \frac{\partial f}{\partial x_i}(X_{t_{k-1}})\right\|_E^2 \,\middle|\, \{X_s\}_{s\leqslant t_{k-1}}\right] \lesssim \frac{kT^2}{N^2}\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}(X_{t_{k-1}})\right\|_E^2, \tag{81}$$

where T is the type 2 constant of E. By the tower property of conditional expectation, (80) combined with (81) gives

$$\mathbb{E}\|M_k - M_{k-1}\|_E^2 \lesssim \frac{kT^2}{N^2}\sum_{i=1}^{n}\mathbb{E}\left\|\frac{\partial f}{\partial x_i}(X_{t_{k-1}})\right\|_E^2 + \frac{k^2 n^4 K(f)^2}{N^4}. \tag{82}$$

Now, summing over $k \in \{1, \dots, N\}$ and using (73), we get

$$\|f - \mathbb{E}_{\sigma_n} f\|_{L_2(\sigma_n;E)}^2 \lesssim M^2 T^2 \sum_{i=1}^{n}\frac{1}{N}\sum_{k=1}^{N}\frac{k}{N}\mathbb{E}\left\|\frac{\partial f}{\partial x_i}(X_{t_{k-1}})\right\|_E^2 + \frac{n^4 K(f)^2 M^2}{N}, \tag{83}$$

which as $N \to \infty$ becomes

$$\|f - \mathbb{E}_{\sigma_n} f\|_{L_2(\sigma_n;E)}^2 \lesssim M^2 T^2 \sum_{i=1}^{n}\int_0^1 t\,\mathbb{E}\left\|\frac{\partial f}{\partial x_i}(X_t)\right\|_E^2 \mathrm{d}t. \tag{84}$$

Since X_t is uniformly distributed on $\{-t,t\}^n$, the random variable $\partial f/\partial x_i(X_t)$ satisfies

$$\frac{\partial f}{\partial x_i}(X_t)=\sum_{\substack{A\subseteq\{1,\dots,n\}\\ i\in A}}\hat{f}(A)\prod_{j\in A\setminus\{i\}}X_t(j)\sim\sum_{\substack{A\subseteq\{1,\dots,n\}\\ i\in A}}t^{|A|-1}\hat{f}(A)\prod_{j\in A\setminus\{i\}}\varepsilon_j=P_{\log(1/t)}\frac{\partial f}{\partial x_i}(\varepsilon), \tag{85}$$

where $\sim$ denotes equality in distribution, ε is uniformly distributed on $\mathcal{C}_n$ and the last equality follows, e.g., by [O'Donnell 2014, Proposition 2.47]. Therefore, by (85) and the change of variables $u=\log(1/t)$, we can rewrite (84) as

$$\|f-\mathbb{E}_{\sigma_n}f\|^2_{L_2(\sigma_n;E)}\lesssim M^2T^2\sum_{i=1}^n\int_0^\infty e^{-2u}\Big\|P_u\frac{\partial f}{\partial x_i}\Big\|^2_{L_2(\sigma_n;E)}\,\mathrm{d}u. \tag{86}$$

In the scalar-valued case, formula (84) is then an equality with $M^2T^2=1$ and appears in [Eldan and Gross 2022]. However, in this case, its equivalent form (86) can also be proved by elementary semigroup arguments as in [Cordero-Erausquin and Ledoux 2012] which we can follow to conclude the proof. Using hypercontractivity [Bonami 1970] and (86), we get

$$\|f-\mathbb{E}_{\sigma_n}f\|^2_{L_2(\sigma_n;E)}\lesssim M^2T^2\sum_{i=1}^n\int_0^\infty e^{-2u}\Big\|\frac{\partial f}{\partial x_i}\Big\|^2_{L_{1+e^{-2u}}(\sigma_n;E)}\,\mathrm{d}u.$$

The conclusions of Theorems 2 and 7 now follow from (86) combined with Lemmas 14 and 17 since for every $i\in\{1,\dots,n\}$, we have $\partial f/\partial x_i(\varepsilon)=\varepsilon_i\partial_i f(\varepsilon)$ for every $\varepsilon\in\mathcal{C}_n$. □

4.3. ***A proof of Theorems 2 and 7 by Littlewood–Paley–Stein theory.*** We shall now present a second more analytic proof of Theorems 2 and 7. The main tool for this proof is a deep vector-valued Littlewood–Paley–Stein inequality (see [Stein 1970]) due to Xu [2020], which is the culmination of the series of works [Xu 1998; Martínez et al. 2006] (see also [Hytönen 2007] for some similar inequalities for UMD targets). We will need the following statement which is a special case of [Xu 2020, Theorem 2].

Theorem 25 (Xu). *Let $(E,\|\cdot\|_E)$ be a Banach space of martingale type 2. Then there exists a constant $C=C(E)\in(0,\infty)$ such that for a symmetric diffusion semigroup $\{T_t\}_{t>0}$ on a probability space (Ω,μ), every function $f:\Omega\to E$ satisfies*

$$\|f-\mathbb{E}_\mu f\|^2_{L_2(\mu;E)}\leqslant C^2\int_0^\infty\|t\partial_tT_tf\|^2_{L_2(\mu;E)}\frac{\mathrm{d}t}{t}. \tag{87}$$

Second proof of Theorems 2 and 7. Since E has martingale type 2, there exists $T\in(0,\infty)$ such that E also has Rademacher type 2 with constant T. Then, applying Proposition 18 to P_tf and using the Rademacher type condition for centered random variables [Ledoux and Talagrand 1991, Proposition 9.11], we deduce that

$$\|\Delta P_{2t}f\|^2_{L_2(\sigma_n;E)}\leqslant\frac{1}{e^{2t}-1}\mathbb{E}\Big\|\sum_{i=1}^n\delta_i(t)\partial_iP_tf(\varepsilon)\Big\|^2_E\leqslant\frac{4T^2}{e^{2t}-1}\sum_{i=1}^n\|\partial_iP_tf\|^2_{L_2(\sigma_n;E)},\quad\text{for all } t\geqslant0. \tag{88}$$

Plugging (88) into (87) for $\{T_t\}_{t\geqslant 0}=\{P_t\}_{t\geqslant 0}$ and doing a change of variables, we get

$$\begin{aligned}\|f-\mathbb{E}_{\sigma_n}f\|^2_{L_2(\sigma_n;E)}\leqslant 4C^2\int_0^\infty \|t\Delta P_{2t}f\|^2_{L_2(\sigma_n;E)}\frac{\mathrm{d}t}{t}\leqslant 8C^2T^2\int_0^\infty \frac{2t}{e^{2t}-1}\sum_{i=1}^n\|\partial_i P_t f\|^2_{L_2(\sigma_n;E)}\,\mathrm{d}t\\ \leqslant 8C^2T^2\sum_{i=1}^n\int_0^\infty e^{-t}\|\partial_i P_t f\|^2_{L_2(\sigma_n;E)}\,\mathrm{d}t.\end{aligned}\tag{89}$$

As before, the conclusion now follows from hypercontractivity [Bonami 1970] along with Lemmas 14 and 17. □

Remark 26. A careful inspection of the proof of [Xu 1998, Theorem 3.1] shows that if we denote by $X_2(E)$ the least constant C in Xu's inequality (87), then $X_2(E)\gtrsim M_2(E)$, where $M_2(E)$ is the martingale type 2 constant of E. On the other hand, in [Xu 2020] it is shown that

$$X_2(E)\lesssim \sup_{t\geqslant 0}\|t\partial_t T_t\|_{L_2(\mu;E)\to L_2(\mu;E)}M_2(E),\tag{90}$$

and

$$\sup_{t\geqslant 0}\|t\partial_t T_t\|_{L_2(\mu;E)\to L_2(\mu;E)}<\infty$$

is proven as a consequence of the uniform convexity of E^*. Specifically for the case of the heat semigroup $\{P_t\}_{t\geqslant 0}$ on $\mathcal{C}_n$, a different proof of this statement which only relies on Pisier's K-convexity theorem [1982] is presented in [Eskenazis and Ivanisvili 2020, Lemma 37]. In the particular case of $E=\ell_p$, where $p\geqslant 2$, an optimization of the argument of that result using the recent proof of Weissler's conjecture on the domain of contractivity of the complex heat flow by Ivanisvili and Nazarov [2022] reveals that

$$\sup_{t\geqslant 0}\|t\Delta P_t\|_{L_2(\sigma_n;\ell_p)\to L_2(\sigma_n;\ell_p)}\lesssim\sqrt{p},\quad \text{for all } n\in\mathbb{N}.\tag{91}$$

Therefore, since the Rademacher and martingale type 2 constants of ℓ_p are both of the order of $\sqrt{p}$, the probabilistic proof of Theorem 2 presented in Section 4.2 shows that for every $n\in\mathbb{N}$, every function $f:\mathcal{C}_n\to\ell_p$, where $p\geqslant 2$, satisfies

$$\|f-\mathbb{E}_{\sigma_n}f\|^2_{L_2(\sigma_n;\ell_p)}\lesssim p^2\sum_{i=1}^n\frac{\|\partial_i f\|^2_{L_2(\sigma_n;\ell_p)}}{1+\log(\|\partial_i f\|_{L_2(\sigma_n;\ell_p)}/\|\partial_i f\|_{L_1(\sigma_n;\ell_p)})},$$

whereas the proof via Xu's inequality (87) implies a weaker $O(p^3)$ bound because of the current best known bounds (90) and (91). We refer to [Xu 2021; 2022] for recent updates on the optimal order of the constant $X_2(E)$.

5. Vector-valued L_1-L_p inequalities

In this section, we will prove Theorems 3 and 4. We start by presenting a joint strengthening of the two results for functions from the Gauss space instead of the discrete hypercube.

5.1. ***A stronger theorem in Gauss space.*** For a smooth function $f:\mathbb{R}^n\to E$, where $(E,\|\cdot\|_E)$ is a Banach space, and $p\in[1,\infty)$, we will use the shorthand notation

$$\|\nabla f\|_{L_p(\gamma_n;E)} \stackrel{\text{def}}{=} \bigg(\int_{\mathbb{R}^n}\Big\|\sum_{i=1}^n y_i\partial_i f\Big\|^p_{L_p(\gamma_n;E)}\,\mathrm{d}\gamma_n(y)\bigg)^{1/p}.$$

In [Pisier 1986, Corollary 2.4], the author presented an argument of Maurey showing that for every Banach space $(E,\|\cdot\|_E)$, $p\in[1,\infty)$ and $n\in\mathbb{N}$, every smooth function $f:\mathbb{R}^n\to E$ satisfies

$$\|f-\mathbb{E}_{\gamma_n}f\|_{L_p(\gamma_n;E)} \leqslant \frac{\pi}{2}\|\nabla f\|_{L_p(\gamma_n;E)}. \tag{92}$$

In this section, we will prove the following Talagrand type strengthening of (92).

Theorem 27. *For every $p\in(1,\infty)$, there exists $C_p\in(0,\infty)$ such that the following holds. For every Banach space $(E,\|\cdot\|_E)$ and $n\in\mathbb{N}$, every smooth function $f:\mathbb{R}^n\to E$ satisfies*

$$\|f-\mathbb{E}_{\gamma_n}f\|_{L_p(\gamma_n;E)} \leqslant C_p\frac{\|\nabla f\|_{L_p(\gamma_n;E)}}{1+\sqrt{\log(\|\nabla f\|_{L_p(\gamma_n;E)}/\|\nabla f\|_{L_1(\gamma_n;E)})}}. \tag{93}$$

We will denote by $\mathcal{L}$ the (negative) generator of the Ornstein–Uhlenbeck semigroup $\{U_t\}_{t\geqslant 0}$, whose action on a smooth function $f:\mathbb{R}^n\to E$ is given by

$$\mathcal{L}f(x)=\Delta f(x)-\sum_{i=1}^n x_i\partial_i f(x),\quad \text{for all } x\in\mathbb{R}^n.$$

We will need the following (classical) Gaussian analogue of Proposition 18.

Proposition 28. *Let $(E,\|\cdot\|_E)$ be a Banach space and $p\in[1,\infty)$. Then for every $n\in\mathbb{N}$, every smooth function $f:\mathbb{R}^n\to E$ satisfies*

$$\Big\|\frac{\partial}{\partial t}U_tf\Big\|_{L_p(\gamma_n;E)} \leqslant \frac{1}{\sqrt{e^{2t}-1}}\|\nabla f\|_{L_p(\gamma_n;E)},\quad \textit{for all } t\geqslant 0. \tag{94}$$

Proof. Here we can follow Maurey's trick [Pisier 1986], setting

$$X_t=e^{-t}X+\sqrt{1-e^{-2t}}\,Y$$

and

$$Y_t=-\sqrt{1-e^{-2t}}\,X+e^{-t}Y=\sqrt{e^{2t}-1}\cdot\frac{\partial}{\partial t}X_t$$

for given independent standard Gaussian vectors $X,Y\in\mathbb{R}^n$. Then, we have

$$\frac{\partial}{\partial t}U_tf(X)=\frac{\partial}{\partial t}\mathbb{E}_Yf(X_t)=\frac{1}{\sqrt{e^{2t}-1}}\mathbb{E}_Y\sum_{i=1}^n\partial_if(X_t)Y_t(i),$$

and we conclude the proof using Jensen's inequality together with the fact that (X_t,Y_t) has the same distribution as (X,Y) for every $t\geqslant 0$. □

Proof of Theorem 27. Arguing as in (56) and using (94) for $U_t f$ instead of f, we can write

$$\begin{aligned}\|f-\mathbb{E}_{\gamma_n} f\|_{L_p(\gamma_n;E)} &\leqslant 2\int_0^\infty \|\mathcal{L}U_{2t}f\|_{L_p(\gamma_n;E)}\,\mathrm{d}t \overset{(94)}{\leqslant} 2\int_0^\infty \|\nabla U_t f\|_{L_p(\gamma_n;E)}\frac{\mathrm{d}t}{\sqrt{e^{2t}-1}}\\ &= 2\int_0^\infty e^{-t}\|U_t\nabla f\|_{L_p(\gamma_n;E)}\frac{\mathrm{d}t}{\sqrt{e^{2t}-1}} \lesssim \int_0^\infty e^{-t}\|U_t\nabla f\|_{L_p(\gamma_n;E)}\frac{\mathrm{d}t}{\sqrt{t}}. \end{aligned}\tag{95}$$

Now, by Nelson's hypercontractive inequalities [1966; 1973] and Kahane's inequality [1964] for Gaussian variables, we have

$$\begin{aligned}\|U_t\nabla f\|_{L_p(\gamma_n;E)} &= \bigg(\int_{\mathbb{R}^n}\Big\|\sum_{i=1}^n y_i U_t\partial_i f\Big\|_{L_p(\gamma_n;E)}^p \mathrm{d}\gamma_n(y)\bigg)^{1/p}\\ &\leqslant \bigg(\int_{\mathbb{R}^n}\Big\|\sum_{i=1}^n y_i \partial_i f\Big\|_{L_{1+(p-1)e^{-2t}}(\gamma_n;E)}^p \mathrm{d}\gamma_n(y)\bigg)^{1/p} \lesssim_p \|\nabla f\|_{L_{1+(p-1)e^{-2t}}(\gamma_n;E)}, \end{aligned}\tag{96}$$

and the conclusion follows from (95), (96) and Lemma 16. □

5.2. ***Proof of Theorem 3.*** Recall that a Banach space $(E,\|\cdot\|_E)$ has cotype $q\in[2,\infty)$ with constant $C\in(0,\infty)$ if for every $n\in\mathbb{N}$ and $x_1,\dots,x_n\in E$,

$$\sum_{i=1}^n \|x_i\|_E^q \leqslant C^q\int_{\mathcal{C}_n}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|_E \mathrm{d}\sigma_n(\varepsilon).\tag{97}$$

The discrete vector-valued L_1-L_p inequality of Theorem 3 can be proven along the same lines as Theorem 27 using Proposition 18 instead of Proposition 28.

Proof of Theorem 3. Suppose that E has cotype $q\in[2,\infty)$. It has been observed in the proof of [Ivanisvili et al. 2020, Proposition 4.2] that [Pisier 1986, Proposition 3.2] implies the estimate

$$\bigg(\mathbb{E}\Big\|\sum_{i=1}^n \delta_i(t)\partial_i f(\varepsilon)\Big\|_E^p\bigg)^{1/p} \leqslant \frac{B_p}{(1-e^{-2t})^{1/2-1/\max\{p,q\}}}\bigg(\mathbb{E}\Big\|\sum_{i=1}^n \delta_i\partial_i f(\varepsilon)\Big\|_E^p\bigg)^{1/p},\quad \text{for all } t\geqslant 0,\tag{98}$$

for some $B_p=B_p(E)\in(0,\infty)$, where $\delta=(\delta_1,\dots,\delta_n)$ is a random vector, uniformly distributed on $\mathcal{C}_n$, which is independent of ε. Therefore, combining (52), (98) and integrating, we deduce that

$$\begin{aligned}\|f-\mathbb{E}_{\sigma_n} f\|_{L_p(\sigma_n;E)} &= 2\Big\|\int_0^\infty \Delta P_{2t}f\,\mathrm{d}t\Big\|_{L_p(\sigma_n;E)} \leqslant 2\int_0^\infty \|\Delta P_{2t} f\|_{L_p(\sigma_n;E)}\,\mathrm{d}t\\ &\overset{(52)\wedge(98)}{\leqslant} 2B_p\int_0^\infty e^{-t}\bigg(\mathbb{E}\Big\|\sum_{i=1}^n \delta_i\partial_i P_t f(\varepsilon)\Big\|_E^p\bigg)^{1/p}\frac{\mathrm{d}t}{(1-e^{-2t})^{1-1/\max\{p,q\}}}\\ &\lesssim B_p\int_0^\infty e^{-t/2}\bigg(\mathbb{E}\Big\|\sum_{i=1}^n \delta_i\partial_i P_t f(\varepsilon)\Big\|_E^p\bigg)^{1/p}\frac{\mathrm{d}t}{t^{1-1/\max\{p,q\}}}. \end{aligned}\tag{99}$$

Arguing as in (96) by using the hypercontractivity of $\{P_t\}_{t\geqslant 0}$ and Kahane's inequality, we get

$$\left(\mathbb{E}\Big\|\sum_{i=1}^n \delta_i\partial_i P_t f(\varepsilon)\Big\|_E^p\right)^{1/p} \lesssim_p \left(\mathbb{E}\Big\|\sum_{i=1}^n \delta_i\partial_i f(\varepsilon)\Big\|_E^{p(t)}\right)^{1/p(t)}, \tag{100}$$

where $p(t) = 1+(p-1)e^{-2t}$ and (15) follows from (99), (100) and (44) with $\alpha_p(E) = 1/\max\{p, q\}$. □

An inspection of the above proofs shows that one can also get the following Orlicz space strengthenings of Theorems 3 and 27 using Lemma 15 instead of Lemma 16.

Theorem 29. *Let $(E, \|\cdot\|_E)$ be a Banach space of cotype q and $p \in [1,\infty)$. Then there exists $C_p = C_p(E) \in (0,\infty)$ such that for every $\theta \in (0,1)$ and $n \in \mathbb{N}$, every $f : \mathcal{C}_n \to E$ satisfies*

$$\|f - \mathbb{E}_{\sigma_n} f\|_{L_p(\sigma_n;E)} \leqslant \frac{C_p}{\theta^{(p-1)/p}} \cdot \|\nabla f\|_{L_p(\log L)^{-p/\max\{p,q\}+\theta}(\sigma_n;E)} \tag{101}$$

Theorem 30. *For every $p \in [1,\infty)$, there exists $C_p \in (0,\infty)$ such that the following holds. For every Banach space $(E, \|\cdot\|_E)$, $\theta \in (0,1)$ and $n \in \mathbb{N}$, every smooth function $f : \mathbb{R}^n \to E$ satisfies*

$$\|f - \mathbb{E}_{\gamma_n} f\|_{L_p(\gamma_n;E)} \leqslant \frac{C_p}{\theta^{(p-1)/p}} \cdot \|\nabla f\|_{L_p(\log L)^{-p/2+\theta}(\gamma_n;E)} \tag{102}$$

5.3. ***Proof of Theorem 4.*** Since $E = \mathbb{C}$ has cotype 2, the proof of Theorem 3 implies that in the scalar-valued case, (15) holds with an exponent $\alpha_p(\mathbb{C}) = 1/\max\{p, 2\}$ for every $p \in (1,\infty)$. In order to boost this exponent to $\frac{1}{2}$ we shall use the following deep result of Lust-Piquard [1998]; see also [Ben Efraim and Lust-Piquard 2008] for a slightly neater argument with better dependence on p and further applications.

Theorem 31 (Lust-Piquard). *For every $p \in (1,\infty)$, there exists $\beta_p \in (0,\infty)$ such that for every $n \in \mathbb{N}$, every function $f : \mathcal{C}_n \to \mathbb{C}$ satisfies*

$$\beta_p \|\Delta^{1/2} f\|_{L_p(\sigma_n)} \leqslant \|\nabla f\|_{L_p(\sigma_n)}. \tag{103}$$

Proof of Theorem 4. By Khintchine's inequality [1923], for every function $f : \mathcal{C}_n \to \mathbb{C}$, we have

$$\left(\sum_{i=1}^n (\partial_i f(\varepsilon))^2\right)^{1/2} \asymp_p \left(\mathbb{E}\Big|\sum_{i=1}^n \delta_i\partial_i f(\varepsilon)\Big|^p\right)^{1/p}, \quad \text{for all } \varepsilon \in \mathcal{C}_n, \tag{104}$$

where the expectation is with respect to $\delta = (\delta_1,\ldots,\delta_n)$ uniformly distributed on $\mathcal{C}_n$. Therefore, if $F : \mathcal{C}_n \to L_p(\sigma_n)$ is given by $[F(\varepsilon)](\delta) = \sum_{i=1}^n \delta_i\partial_i f(\varepsilon)$, then (103), (104) and Theorem 5 imply that

$$\begin{aligned}\|f - \mathbb{E}_{\sigma_n} f\|_{L_p(\sigma_n)} \overset{(103)}{\lesssim_p} \|\nabla\Delta^{-1/2} f\|_{L_p(\sigma_n)} &\overset{(104)}{\asymp_p} \|\Delta^{-1/2} F\|_{L_p(\sigma_n;L_p(\sigma_n))} \\ &\overset{(20)}{\lesssim_p} \frac{\|F\|_{L_p(\sigma_n;L_p(\sigma_n))}}{1+\sqrt{\log(\|F\|_{L_p(\sigma_n;L_p(\sigma_n))}/\|F\|_{L_1(\sigma_n;L_p(\sigma_n))})}}.\end{aligned}$$

The conclusion now follows since, again by Khintchine's inequality (104), the function F satisfies $\|F\|_{L_p(\sigma_n;L_p(\sigma_n))} \asymp_p \|\nabla f\|_{L_p(\sigma_n)}$ and $\|F\|_{L_1(\sigma_n;L_p(\sigma_n))} \asymp_p \|\nabla f\|_{L_1(\sigma_n)}$. □

Remark 32. We note in passing that for $p=2$, (16) is a consequence of Talagrand's influence inequality (6). To see this, note that it has been observed in [Chatterjee 2014, Theorem 5.4] that Talagrand's inequality (6) along with an application of Jensen's inequality imply that for every $n \in \mathbb{N}$, every $f : \mathcal{C}_n \to \mathbb{C}$ satisfies

$$\mathrm{Var}_{\sigma_n}(f) \leqslant C \frac{\|\nabla f\|_{L_2(\sigma_n)}^2}{1+\log(u(f))},$$

where $u(f) = \big(\sum_{i=1}^n \|\partial_i f\|_{L_2(\sigma_n)}^2\big)/\big(\sum_{i=1}^n \|\partial_i f\|_{L_1(\sigma_n)}^2\big)$ and $C \in (0,\infty)$ is a universal constant. Then, (16) for $p=2$ follows by Minkowski's integral inequality, since

$$u(f) = \frac{\sum_{i=1}^n \|\partial_i f\|_{L_2(\sigma_n)}^2}{\sum_{i=1}^n \|\partial_i f\|_{L_1(\sigma_n)}^2} \geqslant \frac{\|\nabla f\|_{L_2(\sigma_n)}^2}{\|\nabla f\|_{L_1(\sigma_n)}^2}.$$

Using the vector-valued Bakry–Meyer inequality of Theorem 8 instead of Theorem 5, one obtains the following Orlicz space strengthening of Theorem 4.

Theorem 33. *For every $p \in (1,\infty)$, there exists $C_p \in (0,\infty)$ such that for every $n \in \mathbb{N}$, every $f : \mathcal{C}_n \to \mathbb{C}$ satisfies*

$$\|f - \mathbb{E}_{\sigma_n} f\|_{L_p(\sigma_n)} \leqslant C_p \|\nabla f\|_{L_p(\log L)^{-p/2}(\sigma_n)}. \tag{105}$$

6. Holomorphic multipliers and the vector-valued Bakry–Meyer theorem

In this section, we will present the proofs of Theorems 5, 8 and 9. In the proof of Theorem 9, we will need some preliminary terminology from discrete Fourier analysis. Recall that for every Banach space $(E, \|\cdot\|_E)$ and every $n \in \mathbb{N}$, all functions $f : \mathcal{C}_n \to E$ admit a unique expansion of the form

$$f(\varepsilon) = \sum_{A \subseteq \{1,\ldots,n\}} \hat{f}(A) w_A(\varepsilon), \quad \text{for all } \varepsilon \in \mathcal{C}_n,$$

where the Walsh function $w_A : \mathcal{C}_n \to \{-1,1\}$ is given by $w_A(\varepsilon) = \prod_{i \in A} \varepsilon_i$ for $\varepsilon \in \mathcal{C}_n$. In this basis, the action of the hypercube Laplacian on f can be written as

$$\Delta f = \sum_{A \subseteq \{1,\ldots,n\}} |A| \hat{f}(A) w_A.$$

Suppose now that $r \in (0,\infty)$ and that $h : (0,r) \to \mathbb{C}$ is a function. Then, for every $\alpha \in (0,\infty)$, the operator $h(\Delta^{-\alpha})$ is defined spectrally by

$$h(\Delta^{-\alpha}) \stackrel{\mathrm{def}}{=} \sum_{\substack{A \subseteq \{1,\ldots,n\} \\ |A| > r^{-1/\alpha}}} h(|A|^{-\alpha}) \hat{f}(A) w_A. \tag{106}$$

Finally, for a function $f : \mathcal{C}_n \to E$ and $k \in \{0,1,\ldots,n\}$ we will define the k-th level Rademacher projection of f to be the function with Walsh expansion

$$\mathrm{Rad}_k f \stackrel{\mathrm{def}}{=} \sum_{\substack{A \subseteq \{1,\ldots,n\} \\ |A| = k}} \hat{f}(A) w_A.$$

Pisier's deep K-convexity theorem [1982] asserts that a Banach space $(E, \|\cdot\|_E)$ has nontrivial Rademacher type if and only if for every $p \in (1, \infty)$, there exist $M_p = M_p(E) \in (0, \infty)$ such that for every $n \in \mathbb{N}$ and $k \in \{1, \dots, n\}$, every $f : \mathcal{C}_n \to E$ satisfies $\|\mathrm{Rad}_k f\|_{L_p(\sigma_n;E)} \leqslant M_p^k \|f\|_{L_p(\sigma_n;E)}$.

6.1. *Proof of Theorem 5.* Although Theorem 5 is a formal consequence of Theorem 8 and Lemma 17, we present a short self-contained proof.

Proof of Theorem 5. Since $P_t = e^{-t\Delta}$, we can express the action of $\Delta^{-\alpha}$ on functions with expectation equal to 0 as

$$\Delta^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty P_t \frac{\mathrm{d}t}{t^{1-\alpha}}. \tag{107}$$

Hence, every function $f : \mathcal{C}_n \to E$ with $\mathbb{E}_{\sigma_n} f = 0$ satisfies

$$\|\Delta^{-\alpha} f\|_{L_p(\sigma_n;E)} \leqslant \frac{1}{\Gamma(\alpha)} \int_0^\infty \|P_t f\|_{L_p(\sigma_n;E)} \frac{\mathrm{d}t}{t^{1-\alpha}}. \tag{108}$$

If E has nontrivial type, it is a standard consequence of Pisier's K-convexity theorem [1982] that there exist $K_p = K_p(E) \in (0, \infty)$ and $\eta_p = \eta_p \in \big(0, \frac{1}{2}\big]$, independent of n and f, such that

$$\mathbb{E}_{\sigma_n} f = 0 \quad \implies \quad \|P_t f\|_{L_p(\sigma_n;E)} \leqslant K_p e^{-2\eta_p t} \|f\|_{L_p(\sigma_n;E)}, \qquad \text{for all } t \geqslant 0. \tag{109}$$

Combining (108) and (109), we deduce that

$$\|\Delta^{-\alpha} f\|_{L_p(\sigma_n;E)} \leqslant \frac{K_p}{\Gamma(\alpha)} \int_0^\infty e^{-\eta_p t} \|P_{t/2} f\|_{L_p(\sigma_n;E)} \frac{\mathrm{d}t}{t^{1-\alpha}},$$

and the conclusion follows by hypercontractivity [Bonami 1970] and Lemma 16. □

6.2. *Proof of Theorem 9.* The proof of Theorem 9 relies on the following result of Mendel and Naor [2014]; see also [Eskenazis and Ivanisvili 2020] for a different proof and further results in this direction.

Theorem 34 (Mendel–Naor). *Let $(E, \|\cdot\|_E)$ be a Banach space of nontrivial type and $p \in (1, \infty)$. Then there exist $c_p = c_p(E)$, $C_p = C_p(E) \in (0, \infty)$ and $A_p = A_p(E) \in [1, \infty)$ such that for every $n \in \mathbb{N}$ and $d \in \{1, \dots, n\}$, the following holds. Every function $f : \mathcal{C}_n \to E$ whose Fourier coefficients $\hat{f}(A)$ vanish for all subsets $A \subseteq \{1, \dots, n\}$ with $|A| < d$ satisfies*

$$\|P_t f\|_{L_p(\sigma_n;E)} \leqslant C_p e^{-c_p d \min\{t, t^{A_p}\}} \|f\|_{L_p(\sigma_n;E)}. \tag{110}$$

Using identity (107) and (110), we see that every such function $f : \mathcal{C}_n \to E$ satisfies

$$\frac{\|\Delta^{-\alpha} f\|_{L_p(\sigma_n;E)}}{\|f\|_{L_p(\sigma_n;E)}} \leqslant \frac{C_p}{\Gamma(\alpha)} \int_0^1 e^{-c_p d t^{A_p}} \frac{\mathrm{d}t}{t^{1-\alpha}} + \frac{C_p}{\Gamma(\alpha)} \int_1^\infty e^{-c_p d t} \frac{\mathrm{d}t}{t^{1-\alpha}} \leqslant \frac{K_p(\alpha)}{d^{\alpha/A_p}}, \tag{111}$$

for some $K_p(\alpha) = K_p(\alpha, E) \in (0, \infty)$.

Proof of Theorem 9. Let $d_p(\alpha) = \lceil (2K_p(\alpha)/r)^{A_p/\alpha} \rceil$, where $K_p(\alpha)$ is the same as in (111), so that every function $f : \mathcal{C}_n \to E$ whose Fourier coefficients $\hat{f}(A)$ vanish for all subsets $A \subseteq \{1, \dots, n\}$ with

$|A| < d_p(\alpha)$ satisfies $\|\Delta^{-\alpha} f\|_{L_p(\sigma_n;E)} \leqslant \frac{1}{2} r \|f\|_{L_p(\sigma_n;E)}$. Iterating this inequality, we get

$$\|\Delta^{-\alpha\ell} f\|_{L_p(\sigma_n;E)} \leqslant \left(\frac{r}{2}\right)^{\ell} \|f\|_{L_p(\sigma_n;E)}, \quad \text{for all } \ell \geqslant 1, \tag{112}$$

for every such function f.

Now, let $f : \mathcal{C}_n \to E$ be an arbitrary function and write

$$f(\varepsilon) = \underbrace{\sum_{k=0}^{d_p(\alpha)-1} \mathrm{Rad}_k\, f(\varepsilon)}_{f_1(\varepsilon)} + \underbrace{\sum_{k=d_p(\alpha)}^{n} \mathrm{Rad}_k\, f(\varepsilon)}_{f_2(\varepsilon)}, \quad \text{for all } \varepsilon \in \mathcal{C}_n.$$

By Pisier’s K-convexity theorem [1982], we have

$$\|h(\Delta^{-\alpha}) f_1\|_{L_p(\sigma_n;E)} \leqslant \sum_{k=\lfloor r^{-1/\alpha}\rfloor+1}^{d_p(\alpha)-1} |h(k^{-\alpha})|\, \|\mathrm{Rad}_k\, f\|_{L_p(\sigma_n;E)} \leqslant \left(\sum_{k=\lfloor r^{-1/\alpha}\rfloor+1}^{d_p(\alpha)-1} |h(k^{-\alpha})| M_p^k\right) \|f\|_{L_p(\sigma_n;E)}, \tag{113}$$

for some $M_p = M_p(E) \in (0,\infty)$. To bound the action of $h(\Delta^{-\alpha})$ on f_2, consider the power series expansion $h(z) = \sum_{\ell\geqslant 0} c_\ell z^\ell$ of h around 0, which converges absolutely and uniformly on $\overline{\mathbb{D}}_{r/2}$. Then, the triangle inequality implies that

$$\|h(\Delta^{-\alpha}) f_2\|_{L_p(\sigma_n;E)} \leqslant \sum_{\ell\geqslant 0} |c_\ell|\, \|\Delta^{-\alpha\ell} f_2\|_{L_p(\sigma_n;E)} \overset{(112)}{\leqslant} \left(\sum_{\ell\geqslant 0} |c_\ell| \left(\frac{r}{2}\right)^{\ell}\right) \|f_2\|_{L_p(\sigma_n;E)}. \tag{114}$$

Finally, observe that, again by Pisier’s K-convexity theorem,

$$\begin{aligned} \|f_2\|_{L_p(\sigma_; E)} = \|f - f_1\|_{L_p(\sigma_n;E)} &\leqslant \|f\|_{L_p(\sigma_n;E)} + \sum_{k=0}^{d_p(\alpha)-1} \|\mathrm{Rad}_k\, f\|_{L_p(\sigma_n;E)} \\ &\leqslant \left(1 + \sum_{k=0}^{d_p(\alpha)-1} M_p^k\right) \|f\|_{L_p(\sigma_n;E)}, \end{aligned} \tag{115}$$

for some $M_p = M_p(E) \in (0,\infty)$. The conclusion follows readily from (113), (114) and (115). □

6.3. *Proof of Theorem 8.* Equipped with Theorem 9, we can now deduce Theorem 8 from (26). We will also need the following simple lemma.

Lemma 35. *For every Banach space $(E, \|\cdot\|_E)$, every function $f : \mathcal{C}_n \to E$ and every $\alpha \in (0,\infty)$,*

$$\|(\Delta+1)^{-\alpha} f(\varepsilon)\|_E \leqslant [(\Delta+1)^{-\alpha} \|f\|_E](\varepsilon), \quad \textit{for all } \varepsilon \in \mathcal{C}_n. \tag{116}$$

Proof. A change of variables shows that

$$(\Delta+1)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} P_t \frac{\mathrm{d}t}{t^{1-\alpha}}, \tag{117}$$

so that for every $\varepsilon \in \mathcal{C}_n$, we have

$$\begin{aligned}\|(\Delta+1)^{-\alpha} f(\varepsilon)\|_E &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} \|P_t f(\varepsilon)\|_E \frac{\mathrm{d}t}{t^{1-\alpha}} \\ &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} [P_t \|f\|_E](\varepsilon) \frac{\mathrm{d}t}{t^{1-\alpha}} = [(\Delta+1)^{-\alpha} \|f\|_E](\varepsilon),\end{aligned}$$

where the second inequality follows from Jensen's inequality because P_t is an averaging operator. □

Proof of Theorems 5 and 8. Let $\phi, \psi : \mathbb{D}_1 \to \mathbb{C}$ be two holomorphic branches of

$$\phi(z) = (1+z)^{\alpha} \quad \text{and} \quad \psi(z) = (1+z)^{-\alpha}, \qquad \text{for all } z \in \mathbb{D}_1,$$

on $\mathbb{D}_1$. By Theorem 9, the operators $\phi(\Delta^{-1})$ and $\psi(\Delta^{-1})$ are bounded on $L_p(\sigma_n; E)$, where $p \in (1, \infty)$, with operator norms independent of n. In other words, there exist constants $\lambda_p(\alpha, E), \Lambda_p(\alpha, E) \in (0, \infty)$ such that for every $n \in \mathbb{N}$, every function $f : \mathcal{C}_n \to E$ satisfies

$$\lambda_p(\alpha, E) \|\Delta^{-\alpha} f\|_{L_p(\sigma_n; E)} \leqslant \|(\Delta+1)^{-\alpha} f\|_{L_p(\sigma_n; E)} \leqslant \Lambda_p(\alpha, E) \|\Delta^{-\alpha} f\|_{L_p(\sigma_n; E)}. \tag{118}$$

Combining (118) with Lemma 35 and inequality (26) of Bakry and Meyer [1982a; 1982b], we get

$$\begin{aligned}\|\Delta^{-\alpha} f\|_{L_p(\sigma_n; E)} &\overset{(118)}{\leqslant} \lambda_p(\alpha, E)^{-1} \|(\Delta+1)^{-\alpha} f\|_{L_p(\sigma_n; E)} \overset{(116)}{\leqslant} \lambda_p(\alpha, E)^{-1} \big\|(\Delta+1)^{-\alpha} \|f\|_E\big\|_{L_p(\sigma_n)} \\ &\overset{(118)}{\leqslant} \frac{\Lambda_p(\alpha, \mathbb{C})}{\lambda_p(\alpha, E)} \big\|\Delta^{-\alpha} \|f\|_E\big\|_{L_p(\sigma_n)} \overset{(26)}{\leqslant} \frac{\Lambda_p(\alpha, \mathbb{C}) K_p(\alpha)}{\lambda_p(\alpha, E)} \|f\|_{L_p(\log L)^{-p\alpha}(\sigma_n; E)},\end{aligned}$$

for some $K_p(\alpha) \in (0, \infty)$, and the conclusion of Theorem 8 follows. □

7. Influence inequalities in nonpositive curvature

Theorems 11 and 12 will be proven by combining Theorem 6 with results from geometry and Banach space theory.

Proof of Theorem 11. It immediately follows from definition (29) that if a metric space $\mathcal{M}$ has Talagrand type (p, ψ) with constant $\tau \in (0, \infty)$ and another metric space $\mathcal{N}$ embeds bi-Lipschitzly in $\mathcal{M}$ with distortion $D \in [1, \infty)$, then $\mathcal{N}$ has Talagrand type (p, ψ) with constant τD. Let G be a Gromov hyperbolic group equipped with the shortest path metric d_{G} associated to the Cayley graph of any (finite) generating set S. Then, by a theorem of Ostrovskii [2014], $(\mathsf{G}, d_{\mathsf{G}})$ admits a bi-Lipschitz embedding of bounded distortion into any nonsuperreflexive Banach space. In particular, $(\mathsf{G}, d_{\mathsf{G}})$ embeds bi-Lipschitzly in the classical exotic Banach space $(\mathbb{J}, \|\cdot\|_{\mathbb{J}})$ of James [1978], which has Rademacher type 2 yet is not superreflexive. By Theorem 6, there exists a universal constant $C \in (0, \infty)$ such that for every $\varepsilon \in (0, 1)$, $(\mathbb{J}, \|\cdot\|_{\mathbb{J}})$ has Talagrand type $(2, \psi_{2,1-\varepsilon})$ with constant $C/\sqrt{\varepsilon}$, and thus the same holds true for the group $(\mathsf{G}, d_{\mathsf{G}})$. □

The binary $\mathbb{R}$-tree of depth d is the geodesic metric space which is obtained by replacing every edge of the combinatorial binary tree of depth d by the interval $[0, 1]$. In order to prove Theorem 12, we will need the following structural result for Riemannian manifolds of pinched negative curvature which is essentially due to Naor, Peres, Schramm and Sheffield [Naor et al. 2006].

Theorem 36. *Fix $n \in \mathbb{N}$ and $r, R \in (0,\infty)$ with $r < R$. Then there exists $N \in \mathbb{N}$ and $D \in (0,\infty)$ such that any n-dimensional complete, simply connected Riemannian manifold (M, g) with sectional curvature in $[-R,-r]$ embeds bi-Lipschitzly with distortion at most D in a product of N binary $\mathbb{R}$-trees of infinite depth.*

In [Naor et al. 2006, Corollary 6.5], the authors proved an analogue of Theorem 36, in which binary $\mathbb{R}$-trees are replaced by $\mathbb{R}$-trees of infinite degree. In order to prove the (stronger) theorem presented here, one needs to repeat the argument of that paper verbatim, replacing the use of [Buyalo and Schroeder 2005] with a more recent result of Dranishnikov and Schroeder [2005], who showed that the hyperbolic space $\mathbb{H}^m$ admits a quasi-isometric embedding in a finite product of binary $\mathbb{R}$-trees of infinite depth.

We shall also need the following slight refinement of a result of Bourgain [1986].

Proposition 37. *Let $(E, \|\cdot\|_E)$ be a nonsuperreflexive Banach space. For every $d \in \mathbb{N}$, the binary $\mathbb{R}$-tree of depth d embeds in E with distortion at most* 4.

Proof. Fix $d \in \mathbb{N}$, let $\mathbb{B}_d$ be the combinatorial binary tree of depth d and denote its root by r. There exists a natural enumeration $\sigma : \mathbb{B}_d \to \{1, \dots, 2^{d+1}-1\}$ of the vertices of $\mathbb{B}_d$ with the following property: if x, y are two leaves of the tree whose least common ancestor is z, then $\sigma((z,x])$ and $\sigma((z,y])$ are two disjoint subsets of $\{1,\dots,2^{d+1}-1\}$ such that one of the inequalities

$$\max \sigma((z,x]) < \min \sigma((z,y]) \quad \text{or} \quad \max \sigma((z,y]) < \min \sigma((z,x]) \tag{119}$$

holds true. To see this, one can "draw" the binary tree and label the vertices from top to bottom along an arbitrary path. After reaching a leaf, one should return to the nearest ancestor with an unlabeled child and continue labeling along an arbitrary downwards path starting at this child. This process should continue until the whole tree has been labeled.

Since E is nonsuperreflexive, by a classical theorem of Pták [Pisier 2016, Theorem 11.10] (which is often attributed to James), there exists vectors $\{x_k\}_{k=1}^{2^{d+1}-1}$ such that for every scalar $a_1, \dots, a_{2^{d+1}-1}$,

$$\frac{1}{4} \sup_{j \in \{1,\dots,2^{d+1}-1\}} \left\{ \left| \sum_{i<j} a_i \right| + \left| \sum_{i \geqslant j} a_i \right| \right\} \leqslant \left\| \sum_{i=1}^{2^{d+1}-1} a_i x_i \right\|_E \leqslant \sum_{i=1}^{2^{d+1}-1} |a_i|. \tag{120}$$

Let $\overline{\mathbb{B}}_d$ be the binary $\mathbb{R}$-tree of depth d. For a point $a \in \overline{\mathbb{B}}_d$ suppose that a belongs in the edge $\{v, w\}$ of $\mathbb{B}_d$ and that v is closer to the root than w. Consider the embedding $\psi : \overline{\mathbb{B}}_d \to E$ given by

$$\psi(a) \stackrel{\text{def}}{=} \sum_{u \in [r,a] \cap \mathbb{B}_d} x_{\sigma(u)} + d_{\overline{\mathbb{B}}_d}(v, a) \cdot x_{\sigma(w)}.$$

Let $a, b \in \overline{\mathbb{B}}_d$ and suppose that c is their least common ancestor. Then, there are downward paths $\{s_1, \dots, s_{j+1}\}$ and $\{t_1, \dots, t_{k+1}\}$ in $\mathbb{B}_d$ such that $a \in [s_j, s_{j+1})$, $b \in [t_k, t_{k+1})$ and s_1 and t_1 are the two distinct children of c. In this notation, the embedding ψ satisfies

$$\psi(a)-\psi(b)=\sum_{i=1}^{j} x_{\sigma(s_i)}+\delta x_{\sigma(s_{j+1})}-\sum_{i=1}^{k} x_{\sigma(t_i)}-\varepsilon x_{\sigma(t_{k+1})}, \tag{121}$$

where $\delta=d_{\overline{\mathbb{B}}_d}(s_j,a)$ and $\varepsilon=d_{\overline{\mathbb{B}}_d}(t_k,b)$. Since $\|x_i\|_X\leqslant 1$, it is clear that

$$\|\psi(a)-\psi(b)\|_E\leqslant j+\delta+k+\varepsilon=d_{\overline{\mathbb{B}}_d}(a,b).$$

On the other hand, by the property (119) of σ, we can assume without loss of generality that

$$\max\{\sigma(s_1),\dots,\sigma(s_{j+1})\}<\min\{\sigma(t_1),\dots,\sigma(t_{k+1})\}.$$

Then (121) and (120) imply that

$$\|\psi(a)-\psi(b)\|_E\geqslant \tfrac{1}{4}(j+\delta+k+\varepsilon)=\tfrac{1}{4}d_{\overline{\mathbb{B}}_d}(a,b).$$

Therefore ψ is the desired bi-Lipschitz embedding. □

Proof of Theorem 12. It follows from definition (29) that if a metric space $\mathcal{M}$ has Talagrand type (p,ψ) with constant $\tau\in(0,\infty)$ and another metric space $\mathcal{N}$ is such that every finite subset of $\mathcal{N}$ embeds bi-Lipscitzly in $\mathcal{M}$ with distortion at most $K\in[1,\infty)$, then $\mathcal{N}$ has Talagrand type (p,ψ) with constant τK. Let (M,g) be a Riemannian manifold of pinched negative curvature equipped with its Riemannian distance d_{M}. Then, by Theorem 36, there exists $N\in\mathbb{N}$ and $D\in(0,\infty)$ such that $(\mathsf{M},d_{\mathsf{M}})$ embeds with distortion at most D in a product of N binary $\mathbb{R}$-trees of infinite depth. In particular, every finite subset X of M embeds with distortion at most D in a product of N binary $\mathbb{R}$-trees of depth d, for some d depending on the cardinality of X. Therefore, by Proposition 37 (see also the discussion following Theorem 2.1 in [Ostrovskii 2014]), X embeds with distortion at most $K=K(N,D)\in(0,\infty)$ in every nonsuperreflexive Banach space. In particular, X embeds with distortion at most K in the classical exotic Banach space $(\mathbb{J},\|\cdot\|_{\mathbb{J}})$ of James [1978], which has Rademacher type 2 yet is not superreflexive. By Theorem 6, there exists a universal constant $C\in(0,\infty)$ such that for every $\varepsilon\in(0,1)$, $(\mathbb{J},\|\cdot\|_{\mathbb{J}})$ has Talagrand type $(2,\psi_{2,1-\varepsilon})$ with constant $C/\sqrt{\varepsilon}$ and thus the same holds for the Riemannian manifold $(\mathsf{M},d_{\mathsf{M}})$. □

8. Embeddings of nonlinear quotients of the cube and Talagrand type

We will now prove that Talagrand type is an obstruction to embeddings of quotients of $\mathscr{C}_n$.

Proof of Theorem 13. Suppose that $(\mathcal{M},d_{\mathcal{M}})$ has Talagrand type (p,ψ) with constant τ and let $\mathscr{R}\subseteq\mathscr{C}_n\times\mathscr{C}_n$ be an equivalence relation. Let $f:\mathscr{C}_n/\mathscr{R}\to\mathcal{M}$ be a map satisfying

$$s\rho_{\mathscr{C}_n/\mathscr{R}}([\zeta],[\eta])\leqslant d_{\mathcal{M}}(f([\zeta]),f([\eta]))\leqslant sD\rho_{\mathscr{C}_n/\mathscr{R}}([\zeta],[\eta]),\quad \text{for all } [\zeta],[\eta]\in\mathscr{C}_n/\mathscr{R}, \tag{122}$$

where $s\in(0,\infty)$ and $D\geqslant 1$. Consider the lifting $F:\mathscr{C}_n\to\mathcal{M}$ given by $F(\varepsilon)=f([\varepsilon])$, where $\varepsilon\in\mathscr{C}_n$. Then, since $\mathcal{M}$ has Talagrand type (p,ψ) with constant τ, we have

$$\int_{\mathscr{C}_n\times\mathscr{C}_n} d_{\mathcal{M}}(F(\varepsilon),F(\delta))^p\,\mathrm{d}\sigma_{2n}(\varepsilon,\delta)\leqslant\tau^p\sum_{i=1}^{n}\|\partial_i F\|^p_{L_{\psi}(\sigma_n)}. \tag{123}$$

The bi-Lipschitz condition (122) and the definition of F imply that

$$d_{\mathcal{M}}(F(\varepsilon), F(\delta)) = d_{\mathcal{M}}(f([\varepsilon]), f([\delta])) \overset{(122)}{\geqslant} s\rho_{\mathcal{C}_n/\mathcal{R}}([\varepsilon], [\delta]), \quad \text{for all } \varepsilon, \delta \in \mathcal{C}_n. \tag{124}$$

On the other hand, for every $\varepsilon \in \mathcal{C}_n$,

$$\begin{aligned}\mathfrak{d}_i F(\varepsilon) &= \tfrac{1}{2} d_{\mathcal{M}}(F(\varepsilon), F(\varepsilon_1, \dots, \varepsilon_{i-1}, -\varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_n)) \\ &\overset{(122)}{\leqslant} \tfrac{1}{2} sD\rho_{\mathcal{C}_n/\mathcal{R}}([\varepsilon], [(\varepsilon_1, \dots, \varepsilon_{i-1}, -\varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_n)]) = \tfrac{1}{2} sD \mathbb{1}_{\partial_i \mathcal{R}}(\varepsilon),\end{aligned}$$

since $\rho_{\mathcal{C}_n/\mathcal{R}}([\varepsilon], [(\varepsilon_1, \dots, \varepsilon_{i-1}, -\varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_n)]) \in \{0, 1\}$ for every $\varepsilon \in \mathcal{C}_n$ and it vanishes if and only if $(\varepsilon, (\varepsilon_1, \dots, \varepsilon_{i-1}, -\varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_n)) \in \mathcal{R}$. Therefore,

$$\|\mathfrak{d}_i F\|_{L_\psi(\sigma_n)} \leqslant \tfrac{1}{2} sD \|\mathbb{1}_{\partial_i \mathcal{R}}\|_{L_\psi(\sigma_n)} = \frac{sD}{2\psi^{-1}(\sigma_n(\partial_i \mathcal{R})^{-1})}. \tag{125}$$

Combining (123), (124) and (125), we deduce that

$$\frac{s^p D^p \tau^p}{2^p} \sum_{i=1}^n \psi^{-1}(\sigma_n(\partial_i \mathcal{R})^{-1})^{-p} \geqslant s^p \mathsf{a}_p(\mathcal{R})^p,$$

and the conclusion follows. □

Remark 38. A metric space $(\mathcal{M}, d_{\mathcal{M}})$ is said to have Enflo type $p \in (0, \infty)$ with constant $T \in (0, \infty)$ if for every $n \in \mathbb{N}$, every function $f : \mathcal{C}_n \to \mathcal{M}$ satisfies

$$\int_{\mathcal{C}_n} d_{\mathcal{M}}(f(\varepsilon), f(-\varepsilon))^p \, \mathrm{d}\sigma_n(\varepsilon) \leqslant T^p \sum_{i=1}^n \|\mathfrak{d}_i F\|^p_{L_p(\sigma_n)}. \tag{126}$$

While Talagrand type is meant to be a refinement of Enflo type (where the Young function is $\psi(t) = t^p$), the attentive reader will notice that the left-hand sides of the two inequalities are different. This difference is mainly superficial (and originates from Enflo's original definition of "roundedness" of a metric space [1969]) and all interesting geometric applications of Enflo type could be recovered with either definition. Since we discuss the bi-Lipschitz geometry of quotients of $(\mathcal{C}_n, \rho)$, it is more natural to define Talagrand type by (29) in order to be able to get distortion lower bounds for quotients $\mathcal{C}_n/\mathcal{R}$ satisfying $(\varepsilon, -\varepsilon) \in \mathcal{R}$ for every $\varepsilon \in \mathcal{C}_n$.

Remark 39. Theorem 13 provides distortion lower bounds for the embedding of quotients of $(\mathcal{C}_n, \rho)$ by an arbitrary equivalence relation $\mathcal{R}$ into spaces with prescribed Talagrand type. While we are not aware of any such bounds in the literature (except perhaps the bound (38) which one can deduce from Enflo type p), it is worth mentioning that there exist L_p-nonembeddability results for more structured quotients of $\mathcal{C}_n$. In particular, we refer the reader to [Khot and Naor 2006], where the authors provide lower bounds for the L_1-distortion of quotients of $\mathcal{C}_n$ by linear codes and by the action of transitive subgroups of the symmetric group S_n. As the proofs of that paper rely on delicate properties of both these structured quotients and L_p spaces, it seems improbable that they can be easily modified to give nonembeddability results into spaces with given Talagrand type.

9. Concluding remarks and open problems

In this final section, we shall present a few remarks regarding the preceding results and indicate some potentially interesting directions of future research.

9.1. ***Talagrand type and linear type.*** In order to highlight the relation of our results with Talagrand's original inequality (6), we decided to state Theorems 1, 2, 6 and 7 only for spaces of Rademacher or martingale type 2. In the terminology of Definition 10, one has the following more general results for spaces of Rademacher or martingale type s. Here and throughout, we will denote by $\psi_{s,\delta} : [0,\infty) \to [0,\infty)$ a Young function with $\psi_{s,\delta}(t) = t^s \log^{-\delta}(e+t)$ for large enough $t > 0$.

Theorem 40 (Rademacher type and Talagrand type). *Fix $s \in (1,2]$. If a Banach space $(E, \|\cdot\|_E)$ has Rademacher type s, then for every $\varepsilon \in (0, s/2)$, E has Talagrand type $(s, \psi_{s,s/2-\varepsilon})$.*

Theorem 41 (martingale type and Talagrand type). *Fix $s \in (1,2]$. If a Banach space $(E, \|\cdot\|_E)$ has martingale type s, then E also has Talagrand type $(s, \psi_{s,s/2})$.*

Since for every $s \in (1,2]$ there exist spaces of Rademacher type s which do not have martingale type s (see [James 1978; Pisier and Xu 1987]), the following natural question poses itself.

Question 1. Does every Banach space of Rademacher type s also have Talagrand type $(s, \psi_{s,s/2})$?

9.2. ***Talagrand type of $L_1(\mu)$.*** It is worth emphasizing that the proofs of both Theorems 40 and 41 crucially rely on the fact that $s > 1$ due to the use of Bonami's hypercontractive inequalities [1970]. In the following theorem, we establish the Talagrand type of L_1. It is worth emphasizing the somewhat surprising fact that Theorem 42 below shows that a stronger property than the trivial "Enflo type 1" inequality holds true in L_1.

Theorem 42. *For every measure μ, the Banach space $L_1(\mu)$ has Talagrand type $(1, \psi_{1,1})$.*

Proof. Since Talagrand type is a local invariant, it clearly suffices to consider the case that μ is the counting measure on $\mathbb{N}$ and thus $L_1(\mu)$ is isometric to ℓ_1. We will employ a classical result of Schoenberg [1938], according to which there exists a function $\mathfrak{s} : \mathbb{R} \to \ell_2$ such that $\mathfrak{s}(0) = 0$ and

$$\|\mathfrak{s}(a) - \mathfrak{s}(b)\|_{\ell_2}^2 = |a-b|, \quad \text{for all } a, b \in \mathbb{R}.$$

Consider the mapping $\bar{\mathfrak{s}} : \ell_1 \to \ell_2(\ell_2)$, given by

$$\bar{\mathfrak{s}}(a_1, a_2, \dots) = (\mathfrak{s}(a_1), \mathfrak{s}(a_2), \dots),$$

and observe that for $a = (a_1, a_2, \dots)$, $b = (b_1, b_2, \dots) \in \ell_1$,

$$\|\bar{\mathfrak{s}}(a) - \bar{\mathfrak{s}}(b)\|_{\ell_2(\ell_2)}^2 = \sum_{i=1}^{\infty} \|\mathfrak{s}(a_i) - \mathfrak{s}(b_i)\|_{\ell_2}^2 = \sum_{i=1}^{\infty} |a_i - b_i| = \|a-b\|_{\ell_1}.$$

Fix $n \in \mathbb{N}$ and a function $f : \mathcal{C}_n \to \ell_1$. Consider the composition $g : \mathcal{C}_n \to \ell_2(\ell_2)$ given by $g = \bar{\mathfrak{s}} \circ f$. Then, we have

$$\mathbb{E}_{\sigma_n \times \sigma_n} \|f(\varepsilon) - f(\delta)\|_{\ell_1} = \mathbb{E}_{\sigma_n \times \sigma_n} \|g(\varepsilon) - g(\delta)\|^2_{\ell_2(\ell_2)} = \mathbb{E}_{\sigma_n} \|g(\varepsilon) - \mathbb{E}_{\sigma_n} g\|^2_{\ell_2(\ell_2)} \lesssim \sum_{i=1}^n \|\partial_i g\|^2_{L_2(\log L)^{-1}(\sigma_n;\ell_2(\ell_2))},$$

where the last inequality follows from Theorem 7. Combining this with the pointwise identity

$$\|\partial_i g(\varepsilon)\|_{\ell_2(\ell_2)} = \frac{1}{2}\|g(\varepsilon) - g(\varepsilon_1, \dots, \varepsilon_{i-1}, -\varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_n)\|_{\ell_2(\ell_2)} = \frac{1}{\sqrt{2}}\|\partial_i f(\varepsilon)\|^{1/2}_{\ell_1}$$

and the fact that for every $h : \{-1, 1\}^n \to \mathbb{R}_+$,

$$\|\sqrt{h}\|^2_{L_2(\log L)^{-1}(\sigma_n)} \asymp \|h\|_{L_1(\log L)^{-1}(\sigma_n)},$$

we deduce that

$$\mathbb{E}_{\sigma_n \times \sigma_n} \|f(\varepsilon) - f(\delta)\|_{\ell_1} \lesssim \sum_{i=1}^n \|\partial_i f\|_{L_1(\log L)^{-1}(\sigma_n;\ell_1)}. \qquad \square$$

The argument used in the proof of Theorem 42 to derive the Talagrand type of ℓ_1 from the Talagrand type of ℓ_2 is very specifically tailored to $L_1(\mu)$ spaces. It remains an interesting open problem to investigate the Talagrand type of noncommutative L_1-spaces.

Question 2. Does the Schatten trace class $(\mathsf{S}_1, \|\cdot\|_{\mathsf{S}_1})$ have Talagrand type $(1, \psi_{1,1})$?

9.3. ***Vector-valued Riesz transforms.*** The optimal $L_1 - L_p$ inequality for scalar-valued functions (see Theorem 33) was derived by combining the vector-valued Bakry–Meyer inequality of Theorem 8 and Lust-Piquard's Theorem 31. In fact, the same argument gives the following implication.

Theorem 43. *Let $(E, \|\cdot\|_E)$ be a K-convex Banach space such that for some $\alpha \in \left(0, \frac{1}{2}\right]$, $p \in (1, \infty)$ and $K \in (0, \infty)$, the following property holds. For every $n \in \mathbb{N}$, every $f : \mathcal{C}_n \to E$ satisfies*

$$\|\Delta^\alpha f\|_{L_p(\sigma_n;E)} \leqslant K\|\nabla f\|_{L_p(\sigma_n;E)}. \tag{127}$$

Then there exists $C = C(\alpha, p, K) \in (0, \infty)$ such that for every $f : \mathcal{C}_n \to E$,

$$\|f - \mathbb{E}_{\sigma_n} f\|_{L_p(\sigma_n;E)} \leqslant C\|\nabla f\|_{L_p(\log L)^{-\alpha p}(\sigma_n;E)}.$$

Therefore, the following question seems natural.

Question 3. Fix $\alpha \in \left(0, \frac{1}{2}\right]$ and $p \in [1, \infty)$. Which target spaces $(E, \|\cdot\|_E)$ satisfy (127) with a constant K independent of n?

In the case of Gauss space, it has been shown by Pisier [1988] that dimension-free Riesz transform inequalities hold true provided that the target space E has the UMD property. In particular, this means that in the case of UMD spaces, Theorem 30 can be improved as follows.

Theorem 44. *Let $(E, \|\cdot\|_E)$ be a UMD Banach space. Then for every $p \in (1, \infty)$, there exists $C_p = C_p(E) \in (0, \infty)$ such that for $n \in \mathbb{N}$, every smooth function $f : \mathbb{R}^n \to E$ satisfies*

$$\|f - \mathbb{E}_{\gamma_n} f\|_{L_p(\gamma_n;E)} \leqslant C_p \cdot \|\nabla f\|_{L_p(\log L)^{-p/2}(\gamma_n;E)}$$

9.4. ***Talagrand type of nonpositively curved spaces.*** A geodesic metric space $(\mathcal{M}, d_{\mathcal{M}})$ is an Alexandrov space of (global) nonpositive curvature (or simply a CAT(0) space) if for every quadruple of points $x, y, z, m \in \mathcal{M}$ such that m is a metric midpoint of x and y, i.e., a point for which $d_{\mathcal{M}}(x, m) = d_{\mathcal{M}}(y, m) = \frac{1}{2} d_{\mathcal{M}}(x, y)$, we have

$$d_{\mathcal{M}}(z, m)^2 \leqslant \tfrac{1}{2} d_{\mathcal{M}}(z, x)^2 + \tfrac{1}{2} d_{\mathcal{M}}(z, y)^2 - \tfrac{1}{4} d_{\mathcal{M}}(x, y)^2.$$

Complete Riemannian manifolds of nonpositive sectional curvature are examples of CAT(0) spaces. Let $\psi_{2,\delta} : [0, \infty) \to [0, \infty)$ be a Young function with $\psi_{2,\delta}(t) = t^2 \log^{-\delta}(e + t)$ for large enough $t > 0$. In Theorems 11 and 12, we showed that Gromov hyperbolic groups and complete Riemannian manifolds of pinched negative curvature have Talagrand type $(2, \psi_{1-\varepsilon})$ for every $\varepsilon \in (0, 1)$. On the other hand, a classical inductive argument essentially going back to Enflo [1969] shows that all Alexandrov spaces of nonpositive curvature have Enflo type 2, which is closely related to Talagrand type $(2, \psi_{2,0})$. We believe that the following question deserves further investigation.

Question 4. Does there exist some $\delta \in (0, 1]$ such that every Alexandrov space of nonpositive curvature has Talagrand type $(2, \psi_{2,\delta})$? More ambitiously, does every Alexandrov space of nonpositive curvature have Talagrand type $(2, \psi_{2,1})$?

9.5. CAT(0) ***spaces as test spaces for superreflexivity.*** In Proposition 37, we showed that all binary $\mathbb{R}$-trees of finite depth embed with uniformly bounded distortion into any nonsuperreflexive Banach space. It was communicated to us by Florent Baudier that using this proposition and the barycentric gluing technique (see [Baudier 2007] and the survey [Baudier 2022]), one can in fact prove that the binary $\mathbb{R}$-tree of *infinite* depth admits a bi-Lipschitz embedding into any nonsuperreflexive Banach space. Then, an inductive argument (see, e.g., [Ostrovskii 2014, Remark 2.2]) shows that any finite product of binary $\mathbb{R}$-trees also embeds bi-Lipschitzly into any nonsuperreflexive Banach space. Therefore, one deduces from Theorem 36 that every finite-dimensional complete, simply connected Riemannian manifold of pinched negative curvature embeds bi-Lipschitzly into any nonsuperreflexive Banach space. Conversely, since all binary trees embed in the hyperbolic plane $\mathbb{H}^2$, if a Banach space E bi-Lipschitzly contains $\mathbb{H}^2$, then E cannot be superreflexive by Bourgain's theorem [1986]. In conclusion, we deduce the following characterization.

Theorem 45. *A Banach space $(E, \|\cdot\|_E)$ is nonsuperreflexive if and only if for every $n \in \mathbb{N}$, every n-dimensional complete, simply connected Riemannian manifold (M, g) of pinched negative curvature equipped with the Riemannian distance d_{M} admits a bi-Lipschitz embedding in E.*

In recent years, there have been plenty of such characterizations in the literature, although one can argue that this is not a particularly novel one due to its close relation to Bourgain's characterization in terms of trees. We believe the following stronger question deserves further investigation.

Question 5. Which Alexandrov spaces of nonpositive curvature admit a bi-Lipschitz embedding into every nonsuperreflexive Banach space?

There are plenty of CAT(0) spaces which do not embed into finite products of binary $\mathbb{R}$-trees and in order to prove that they embed into all nonsuperreflexive Banach spaces, one may need to employ interesting structural properties of such spaces. On the other hand, there exist CAT(0) spaces which do not embed into L_1, which is of course nonsuperreflexive. Indeed, if every CAT(0) space admitted a bi-Lipschitz embedding into L_1, then every classical expander (which is also an expander with respect to L_1 by Matoušek's extrapolation lemma for Poincaré inequalities [1997]), would be an expander with respect to all CAT(0) spaces and this is known to be false by important work of Mendel and Naor [2015].

9.6. ***General hypercontractive semigroups.*** In [Cordero-Erausquin and Ledoux 2012], the authors established versions of Talagrand's (scalar-valued) inequality (6) in the setting of hypercontractive Markov semigroups satisfying some minimal assumptions. At first glance, the arguments which we use in the present paper to obtain vector-valued extensions of (6) seem to rely more heavily on specific properties of the Hamming cube, such as identity (54) from [Ivanisvili et al. 2020] or the Eldan–Gross process [Eldan and Gross 2022]. Nevertheless, we strongly believe that there are versions of our results for other hypercontractive Markov semigroups satisfying some fairly general assumptions.

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