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SHARP POINTWISE AND UNIFORM ESTIMATES FOR $\bar\partial$

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We use weighted L^2-methods to obtain sharp pointwise estimates for the canonical solution to the equation $\bar\partial u = f$ on smoothly bounded strictly convex domains and the Cartan classical domains when f is bounded in the Bergman metric g. We provide examples to show our pointwise estimates are sharp. In particular, we show that on the Cartan classical domains Ω of rank 2 the maximum blow-up order is greater than $-\log\delta_\Omega(z)$, which was obtained for the unit ball case by Berndtsson. For example, for Ω of type IV(n) with $n \geq 3$, the maximum blow-up order is $\delta(z)^{1-n/2}$ because of the contribution of the Bergman kernel. Additionally, we obtain uniform estimates for the canonical solutions on the polydiscs, strictly pseudoconvex domains and the Cartan classical domains under stronger conditions on f.

1. Introduction

The existence and regularity of solutions to the Cauchy–Riemann equation $\bar\partial u = f$ on a bounded pseudoconvex domain Ω in $\mathbb{C}^n$ is a fundamental topic in several complex variables. Since the kernel of $\bar\partial$ is the set of holomorphic functions, a solution to the Cauchy–Riemann equation is not unique if it exists. However, let $A^2(\Omega) := L^2(\Omega) \cap \ker(\bar\partial)$ denote the Bergman space over Ω. Then the solution to $\bar\partial u = f$ with $u \perp A^2(\Omega)$ is unique, and it is called the canonical solution or L^2-minimal solution because it has minimal L^2-norm among all solutions. Hörmander [1965] showed that if Ω is bounded and pseudoconvex and $f \in L^2_{(0,1)}(\Omega)$ is $\bar\partial$-closed, then there exists a solution u that satisfies the estimate $\|u\|_{L^2} \leq C\|f\|_{L^2}$ for some constant C depending only on the diameter of Ω. In view of Hörmander's result, a natural question arises: does there exist a constant C depending only on Ω such that for any $\bar\partial$-closed $f \in L^\infty_{(0,1)}(\Omega)$ there exists a solution to $\bar\partial u = f$ with $\|u\|_\infty \leq C\|f\|_\infty$? If the answer is yes, we say the $\bar\partial$-equation can be solved with uniform estimates on Ω. An important method for solving the $\bar\partial$-equation is the integral representation for solutions. In this method, one constructs a differential form $B(z, w)$ on $\Omega \times \Omega$ which is an $(n, n-1)$ form in w such that solutions to $\bar\partial u = f$ can be written as

$$u(z) = \int_\Omega B(z, w) \wedge f(w). \tag{1-1}$$

The method of integral representation of solutions was initiated by Cauchy, Leray, Fantappié, etc. On a smoothly bounded strictly pseudoconvex domain Ω in $\mathbb{C}^n$, Henkin [1970] and Grauert and Lieb [1970] constructed integral kernels $B(z, w)$ such that u given by (1-1) satisfies $\|u\|_\infty \leq C\|f\|_\infty$. Kerzman [1971] improved the estimate by showing that $\|u\|_{C^\alpha(\bar\Omega)} \leq C_\alpha\|f\|_\infty$ for any $0 < \alpha < \frac{1}{2}$. Henkin and Romanov [1971] obtained the sharp estimate $\|u\|_{C^{1/2}(\bar\Omega)} \leq C\|f\|_\infty$. For more results on strictly pseudoconvex

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domains, the reader may consult the papers [Krantz 1976; Range and Siu 1972; 1973] and the books [Chen and Shaw 2001; Fornæss and Stensønes 1987; Range 1986].

When the class of domains under consideration is changed from strictly pseudoconvex to weakly pseudoconvex, it is no longer possible to conclude in generality the existence of uniform estimates for $\bar\partial$. Berndtsson [1993], Fornæss [1986] and Sibony [1980] constructed examples of weakly pseudoconvex domains in $\mathbb{C}^2$ and $\mathbb{C}^3$ where uniform estimates for $\bar\partial$ fail. More strikingly, Fornæss and Sibony [1993] constructed a smoothly bounded pseudoconvex domain $\Omega \subset \mathbb{C}^2$ such that $\partial\Omega$ is strictly pseudoconvex except at one point, but any solution to $\bar\partial u = f$ for some given $\bar\partial$-closed $f \in L^\infty_{(0,1)}(\Omega)$ does not belong to $L^p(\Omega)$ for any $2 < p \le \infty$. Range [1978] gave uniform estimates on bounded convex domains in $\mathbb{C}^2$ with real analytic boundaries, and in [Range 1990] gave Hölder estimates on pseudoconvex domains of finite type in $\mathbb{C}^2$. See [Laurent-Thiébaut and Leiterer 1993; Michel and Shaw 1999] for related results. Of particular interest is the unit polydisc $\mathbb{D}^n := \mathbb{D}(0, 1)^n \subset \mathbb{C}^n$, which is pseudoconvex with nonsmooth boundary. When $n = 2$, Henkin [1971] showed that there exists a constant C such that $\|u\|_\infty \le C\|f\|_\infty$ for any $f \in C^1_{(0,1)}(\bar{\mathbb{D}}^2)$. Landucci [1975] obtained the same uniform estimate for the canonical solution on $\mathbb{D}^2$. Chen and McNeal [2020] and Fassina and Pan [2019] generalized Henkin's result to higher dimensions when additional regularity assumptions on f are imposed. It remains open whether uniform estimates hold on $\mathbb{D}^n$ with $n \ge 2$ when f is only assumed to be bounded. See [Dong et al. 2020; Fornæss et al. 2011; Grundmeier et al. 2022] for related results.

A class of pseudoconvex domains in $\mathbb{C}^n$ including $\mathbb{D}^n$ and the unit ball $\mathbb{B}^n$ are the so-called bounded symmetric domains, which up to biholomorphism are Cartesian product(s) of the Cartan classical domains of types I to IV and two domains of exceptional types. In [Henkin and Leiterer 1983, p. 200], the authors asked whether there exists a uniform estimate for the $\bar\partial$-equation on the Cartan classical domains of rank at least 2. Additionally, [Sergeev 1994] conjectured that the $\bar\partial$-equation cannot be solved with uniform estimates on the Cartan classical domains of type IV of dimension $n \ge 3$.

Let $g = (g_{j\bar k})^n_{j,k=1}$ be the Bergman metric on a domain Ω. For a $(0, 1)$-form $f = \sum_{j=1}^n f_j \, d\bar z_j$, one defines

$$\|f\|^2_{g,\infty} := \operatorname{ess\,sup}\left\{ \sum_{j,k=1}^n g^{j\bar k}(z) f_k(z) \overline{f_j(z)} : z \in \Omega \right\},$$

where $(g^{j\bar k})^\tau = (g_{j\bar k})^{-1}$; see (3-1) for details. Berndtsson used weighted L^2 estimates of Donnelly–Fefferman type to prove the following pointwise and uniform estimates.

Theorem 1.1 [Berndtsson 1997, 2001]. *There is a numerical constant C such that for any $\bar\partial$-closed $(0, 1)$-form f on $\mathbb{B}^n$, the canonical solution to $\bar\partial u = f$ satisfies*

$$|u(z)| \le C\|f\|_{g,\infty} \log \frac{2}{1-|z|}, \tag{1-2}$$

and for any $\epsilon > 0$,

$$\|u\|_\infty \le \frac{C}{\epsilon} \|(1 - |z|^2)^{-\epsilon} f\|_{g,\infty}. \tag{1-3}$$

The estimate (1-2) is sharp. If $f(z) := \sum_{k=1}^n z_k(|z|^2 - 1)^{-1} \, d\bar z_k$ then f is $\bar\partial$-closed, $\|f\|_{g,\infty} = 1$ and the canonical solution to $\bar\partial u = f$ is $u = \log(1 - |z|^2) - C_n$, which shows the sharpness of (1-2).

Berndtsson [2001] also pointed out that his proof should generalize to other domains when enough information about the Bergman kernel is known. This result [Berndtsson 1997, 2001] was improved in [Schuster and Varolin 2014] via the "double twisting" method.

Motivated by Berndtsson's results (1-2) and (1-3) and the problems raised by Henkin and Leiterer [1983] and Sergeev [1994], we study sharp pointwise estimates for $\bar\partial u = f$ for any $\bar\partial$-closed (0, 1)-form f with $\|f\|_{g,\infty} < \infty$ and uniform estimates under stronger conditions on f. We generalize Berndtsson's results from $\mathbb{B}^n$ to smoothly bounded strictly pseudoconvex domains and the Cartan classical domains. Our main theorem, Theorem 1.2 (see also Theorem 3.3), for pointwise estimates is stated as follows.

Theorem 1.2. *Let Ω be a smoothly bounded strictly convex domain, a Cartan classical domain or the polydisc, whose Bergman kernel and metric are denoted by K and g, respectively. Then there is a constant C such that for any $\bar\partial$-closed* (0, 1)*-form f with $\|f\|_{g,\infty} < \infty$, the canonical solution to $\bar\partial u = f$ satisfies*

$$|u(z)| \le C\|f\|_{g,\infty} \int_\Omega |K(z, w)|\, dv_w, \quad z \in \Omega. \tag{1-4}$$

Remarks. (i) When Ω is a smoothly bounded strictly pseudoconvex domain, by Fefferman's asymptotic expansion for the Bergman kernel,

$$\int_\Omega |K(z, w)|\, dv_w \approx C \log \frac{1}{\delta_\Omega(z)} \approx \log K(z, z), \quad z \to \partial\Omega.$$

In this case, the estimate (1-4) is sharp. Take for example $\Omega = \mathbb{B}^n$ and $u(z) = \log K(z, z) - c$, where c is chosen so that $P[u] = 0$.

(ii) We will show in Section 3, Theorem 3.4, that if Ω is a smoothly bounded strictly pseudoconvex domain, then (1-4) holds for a solution u which may not be canonical.

(iii) When Ω is the unit polydisc $\mathbb{D}^n$, one has

$$\int_\Omega |K(z, w)|\, dv_w \approx \prod_{j=1}^n \log \frac{2}{1-|z_j|}, \quad z \to \partial\Omega.$$

(iv) When Ω is a Cartan classical domain of rank greater than or equal to 2, the blow-up order of $\int_\Omega |K(z, w)|\, dv_w$ depends on the direction in which z approaches $\partial\Omega$ and it may be larger than $-\log \delta_\Omega(z)$. For example, if $z = tI_2 \in \Omega := \mathrm{II}(2)$, then $\int_\Omega |K(z, w)|\, dv_w \approx \delta_\Omega(z)^{-1/2}$ as $t \to 1^-$. Moreover, if $z = te_1 \oplus te_2 \in \mathrm{IV}(n)$ with $e_j \in \mathcal{U}$ and $n \ge 3$, then $\int_\Omega |K(z, w)|\, dv_w \approx \delta_\Omega(z)^{1-(n/2)}$ as $t \to 1^-$. Here $\mathcal{U}$ denotes the characteristic boundary of Ω.

(v) In Section 6B, we show the estimate (1-4) is sharp on the Cartan classical domains.

Our main theorem for uniform estimates is stated as follows, as a combination of Theorems 4.1 and 4.2.

Theorem 1.3. *Let Ω be either the unit polydisc or a smoothly bounded strictly convex domain, whose Bergman kernel and metric are denoted by K and g, respectively. Then for any $p \in (1, \infty)$, there is a constant C such that for any $\bar\partial$-closed* (0, 1)*-form f, the canonical solution to $\bar\partial u = f$ satisfies*

$$\|u\|_\infty \le C \left\| f(\cdot)\left(\int_\Omega |K(\cdot, w)|\, dv_w\right)^p \right\|_{g,\infty}.$$

For Cartan classical domains, we give a uniform estimate under condition (5-2) in Theorem 5.4.

This paper is organized as follows: In Section 2, we recall and prove some properties of the Bergman kernel and metric which will be used later. In Section 3, we use L^2-methods to establish pointwise estimates on strictly pseudoconvex domains and the Cartan classical domains. In Sections 4 and 5, we obtain uniform estimates on the polydiscs, strictly pseudoconvex domains and the Cartan classical domains under various conditions on f. In Section 6, we verify the sharpness of our pointwise estimates on the Cartan classical domains; in particular, on $\mathrm{IV}(n)$ with $n \geq 3$ we show the estimate has maximum blow-up order $\delta^{1-(n/2)}(z)$.

2. Bergman kernel and metric

The Bergman space $A^2(\Omega)$ on a domain $\Omega \subset \mathbb{C}^n$ is the closed holomorphic subspace of $L^2(\Omega)$. The Bergman projection is the orthogonal projection $P_\Omega : L^2(\Omega) \to A^2(\Omega)$ given by

$$P_\Omega[f](z) = \int_\Omega K(z, w) f(w)\, dv(w),$$

where $K(z, w)$ is the Bergman kernel on Ω and dv is the Lebesgue $\mathbb{R}^{2n}$ measure. We will write $K(z)$ to denote the on-diagonal Bergman kernel $K(z, z)$. When Ω is bounded, the complex Hessian of $\log K(z)$ induces the Bergman metric $B_\Omega(z; X)$ defined by

$$B_\Omega(z; X) := \left(\sum_{j,k=1}^{n} g_{j\bar{k}} X_j \bar{X}_k \right)^{1/2}, \quad g_{j\bar{k}}(z) := \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K(z), \quad \text{for } z \in \Omega, \quad X \in \mathbb{C}^n.$$

The Bergman distance between $z, w \in \Omega$ is

$$\beta_\Omega(z, w) := \inf \left\{ \int_0^1 B_\Omega(\gamma(t); \gamma'(t))\, dt \right\},$$

where the infimum is taken over all piecewise C^1-curves $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = z$ and $\gamma(1) = w$. Throughout the paper,

$$B_a(\epsilon) := \{ z \in \Omega : \beta_\Omega(z, a) \leq \epsilon \} \tag{2-1}$$

will denote the hyperbolic ball in the Bergman metric centered at $a \in \Omega$ of radius ϵ. Additionally, $K(z, w)$, P_Ω and g will always denote the Bergman kernel, the Bergman projection on Ω and the Bergman metric, respectively.

Consider a convex domain Ω that contains no complex lines and $a \in \Omega$. Choose any $a^1 \in \partial\Omega$ such that $\tau_1(a) := |a - a^1| = \operatorname{dist}(a, \partial\Omega)$ and define $V_1 = a + \operatorname{span}(a^1 - a)^\perp$. Let $\Omega_1 = \Omega \cap V_1$ and choose any $a^2 \in \partial\Omega_1$ such that $\tau_2(a) := \|a - a^2\| = \operatorname{dist}(a, \partial\Omega_1)$. Let $V_2 = a + \operatorname{span}(a^1 - a, a^2 - a)^\perp$ and $\Omega_2 = \Omega \cap V_2$. Repeat this process to obtain $a^1, \ldots, a^n$ and $w_k = (a^k - a)/\|a^k - a\|$, $1 \leq k \leq n$. Define

$$D(a; w, r) = \{ z \in \mathbb{C}^n : |\langle z - a, w_k \rangle| < r_k,\ 1 \leq k \leq n \} \tag{2-2}$$

and

$$D(a, r) = \{ z \in \mathbb{C}^n : |z_k - a_k| < r_k,\ 1 \leq k \leq n \}.$$

By [Nikolov and Pflug 2003, Theorem 2], for convex domains that contain no complex lines, the Kobayashi metric and the Bergman metric are comparable. It follows by [Nikolov and Trybuła 2015, Corollary 2] that if Ω is a convex domain with no complex lines, then for every $\epsilon > 0$ there exists constants C_1 and C_2 such that for any a,

$$D(a; w, C_1\tau(a)) \subset B_a(\epsilon) \subset D(a; w, C_2\tau(a)). \tag{2-3}$$

By [Nikolov and Pflug 2003, Theorem 1],

$$\frac{1}{4^n} \le K(a)\prod_{j=1}^{n} \pi\tau_j^2(a) \le \frac{(2n)!}{2^n},$$

which implies that

$$\left(\frac{C_1}{2}\right)^{2n} \le K(a)v(B_a(\epsilon)) \le (2n)!\left(\frac{C_2^2}{2}\right)^{n}.$$

For any open subset A of Ω, we define

$$\|\bar\partial u\|_{g,\infty,A} = \||\bar\partial u(z)|_g\|_{L^\infty(A)}.$$

In the proofs of this paper, C will denote a numerical constant which may be different at each appearance. The Cauchy–Pompeiu formula gives the following useful proposition.

Proposition 2.1. *Let Ω be a bounded convex domain. For any $\epsilon > 0$ sufficiently small, there exists a constant C such that for any complex-valued C^1 function u on Ω*

$$|u(a)| \le C \fint_{B_a(\epsilon)} |u(z)|\,dv_z + C\|\bar\partial u\|_{g,\infty,B_a(\epsilon)}.$$

Proof. After a complex rotation, without loss of generality, we may assume the standard basis for $\mathbb{C}^n$ is $(w_k)_{k=1}^n$, using the notation of (2-2). Let $r_k(a) = C_1\tau_k(a)$, where C_1 is the same constant as in (2-3). Define the metric

$$M_A(z; X) = \left(\sum_{k=1}^{n} \frac{|X_k|^2}{\tau_k(z)^2}\right)^{1/2}, \quad X \in \mathbb{C}^n.$$

It was proved in [McNeal 2001] (see also [McNeal 1994; Nikolov and Pflug 2003]) that

$$M_A(z; X) \approx B_\Omega(z; X), \quad X \in \mathbb{C}^n,$$

where $\approx$ is independent of z and X. So we can choose holomorphic coordinates such that

$$\frac{1}{C}D\left[\frac{1}{\tau_1^2}, \dots, \frac{1}{\tau_n^2}\right] \le [g_{i\bar j}] \le CD\left[\frac{1}{\tau_1^2}, \dots, \frac{1}{\tau_n^2}\right],$$

where $D[a_1, \dots, a_n]$ is a diagonal matrix with diagonal entries $a_1, \dots, a_n$. Therefore

$$\frac{1}{C}D[\tau_1^2, \dots, \tau_n^2] \le [g_{i\bar j}]^{-1} \le CD[\tau_1^2, \dots, \tau_n^2].$$

Additionally, by (2-3) and the definition of the hyperbolic ball (2-1),

$$\tau_k(a) \le C\tau_k(z), \quad z \in D(a; w, C_1\tau(a)),$$

and the constant C is independent of a. Therefore, for $a = (a_1, \dots, a_n) \in \Omega$,

$$
\begin{aligned}
r_1(a)\|\bar\partial_1 u(\,\cdot\,, a_2, \dots, a_n)\|_{L^\infty(D(a_1,r_1))} &\le C \sup_{z\in D(a;w,r)} \sum_{k=1}^{n} r_k(z)\Big|\frac{\partial u}{\partial \overline{w}_k}(z)\Big| \\
&\le C \sup_{z\in D(a;w,r)} \bigg(\sum_{k=1}^{n} \tau_k^2(z)\Big|\frac{\partial u}{\partial \overline{w}_k}(z)\Big|^2\bigg)^{1/2} \\
&\le C\|\bar\partial u\|_{g,\infty,D(a;w,r)} \le C\|\bar\partial u\|_{g,\infty,B_a(\epsilon)}.
\end{aligned}
$$

By Stokes' theorem, for $0 < s_k < r_k$,

$$
u(a) = \frac{1}{2\pi i}\int_{|w_1-a_1|=s_1} \frac{u(w_1, a_2, \dots, a_n)}{w_1 - a_1}\, dw_1 + \frac{1}{2\pi i}\int_{|w_1-a_1|<s_1} \frac{\partial u}{\partial \overline{w}_1}\frac{1}{w_1-a_1}\, dw_1 \wedge d\overline{w}_1.
$$

By polar coordinates and (2-3),

$$
\begin{aligned}
|u(a)| &\le \frac{1}{\pi r_1^2}\int_{|z_1-a_1|<r_1} |u(w_1, a_2, \dots, a_n)|\, dv_{z_1} + \frac{2r_1}{3}\|\bar\partial_1 u(\,\cdot\,, a_2, \dots, a_n)\|_{L^\infty(D(a_1,r_1))} \\
&\le \frac{1}{\pi r_1^2}\int_{|w_1-a_1|<r_1} |u(w_1, a_2, \dots, a_n)|\, dv_{w_1} + C\|\bar\partial u\|_{g,\infty,B_a(\epsilon)}.
\end{aligned}
$$

Using the same estimate on the disc $|w_k - a_k| < s_k$ for $2 \le k \le n$,

$$
\begin{aligned}
|u(a)| &\le \oint_{D(a,C_1\tau(a))} |u(w_1, \dots, w_n)|\, dv_w + C\|\bar\partial u\|_{g,\infty,B_a(\epsilon)} \\
&\le C^{2n}\oint_{B_a(\epsilon)} |u(w)|\, dv_w + C\|\bar\partial u\|_{g,\infty,B_a(\epsilon)}.
\end{aligned}
$$

□

We remark that Proposition 2.1 also holds for smoothly bounded strictly pseudoconvex domains.

For positive real-valued functions f and g on Ω, we say $f \approx g$ for $z \in B_a(\epsilon)$ if for every $\epsilon > 0$ sufficiently small, there exists a constant $C = C(\epsilon, \Omega)$ such that

$$
C^{-1} \le f(z)g(z)^{-1} \le C, \quad z \in B_a(\epsilon)
$$

for all $a \in \Omega$. A similar definition holds for $f \approx g$ for $z \in \Omega$.

A domain Ω is homogeneous if it has a transitive (holomorphic) automorphism group. For convex homogeneous domains, the following results are known; see [Ishi and Yamaji 2011].

Proposition 2.2. *Let Ω be a bounded homogeneous convex domain. Then*

$$
|K(z,a)| \approx K(a) \approx \frac{1}{v(B_a(\epsilon))}, \quad z \in B_a(\epsilon),
$$

and for any $\epsilon > 0$, there is a C_ϵ such that for any $a \in \Omega$

$$
\max_{w\in B_a(\epsilon)} \Big|\frac{K(z,w)}{K(z,a)}\Big| \le C_\epsilon, \quad z \in \Omega.
$$

Let Ω be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and let $-r \in C^\infty(\bar\Omega)$ be a strictly plurisubharmonic defining function for Ω. Define

$$X(z,w) = r(w) + \sum_{j=1}^n \frac{\partial r(w)}{\partial z_j}(z_j - w_j) + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 r(w)}{\partial z_i \partial z_j}(z_i - w_i)(z_j - w_j).$$

It was proved by Fefferman [1974] that there is a $\delta > 0$ such that

$$K(z,w) = \frac{F(z,w)}{X(z,w)^{n+1}} + G(z,w)\log X(z,w)$$

for all $(z,w) \in R_\delta(\Omega) = \{(z,w) \in \Omega \times \Omega : r(z) + r(w) + |z-w|^2 < \delta\}$, where $F, G \in C^\infty(\bar\Omega \times \bar\Omega)$ and $F(z,z) > 0$ on $\partial\Omega$.

When Ω is a smoothly bounded strictly convex domain, the definition of $X(z,w)$ can be simplified. In fact, one can take

$$X(z,a) = h_a(z) = r(a) + \sum_{j=1}^n \frac{\partial r}{\partial z_j}(a)(z_j - a_j).$$

We can take $-r(z)$ to be strictly convex. Then by Taylor's theorem, one can easily see that

$$\operatorname{Re} h_a(z) \approx \operatorname{Re} X(z,w).$$

Moreover, for any $a, z \in \Omega$, we will write $\tilde z = (x_j)_{j=1}^{2n}$ and $\tilde a = (a_j)_{j=1}^{2n}$ if $z = (x_{2j-1} + i x_{2j})_{j=1}^n$ and $a = (a_{2j-1} + i a_{2j})_{j=1}^n$. If we apply Taylor's theorem on the line segment between $\tilde a$ and $\tilde z$, then there is a $\theta \in (0,1)$ such that

$$\operatorname{Re} h_a(z) = r(z) - \sum_{i,j=1}^{2n} \frac{\partial^2 r(\tilde a + \theta(\tilde z - \tilde a))}{\partial \tilde z_i \partial \tilde z_j}(\tilde z_i - \tilde a_i)(\tilde z_j - \tilde a_j) \approx r(z) + |z-a|^2.$$

Therefore, for any $a \in \Omega$,

$$|h_a(z)| \approx \frac{r(z)+r(a)}{2} + |z-a|^2 + \left|\operatorname{Im} \sum_{j=1}^n \frac{\partial r(a)}{\partial z_j}(z_j - a_j)\right|, \quad z \in \Omega.$$

In particular, this implies $h_a(z) \neq 0$. Therefore, by Fefferman's asymptotic expansion [1974] on strictly pseudoconvex domains mentioned above, we know the following.

Lemma 2.3 [Fefferman 1974]. *Let Ω be a smoothly bounded strictly convex domain. Then,*

$$|h_a(z)|^{-n-1} \approx K(a) \approx \frac{1}{v(B_a(\epsilon))}, \quad z \in B_a(\epsilon),$$

and there is a constant C such that

$$\int_\Omega |h_a(z)|^{-n-1}\, dv_z \approx \int_\Omega |K(z,a)|\, dv_z \approx \log \frac{C}{\delta_\Omega(a)}, \quad a \in \Omega,$$

where $\delta_\Omega(\cdot)$ is the Euclidean distance function to $\partial\Omega$. Moreover, for any $\epsilon > 0$, there is a constant C_ϵ such that for any $a \in \Omega$

$$\max_{w \in B_a(\epsilon)} |K(z,w) h_a(z)^{n+1}| \le C_\epsilon, \quad z \in \Omega.$$

Note. We provide some insight into the integration of $|K(z,w)|$ as a Forelli–Rudin-type integral. Roughly, one can view $\partial\Omega$ as a space of homogeneous type with Borel measure dX and quasidistance $|X(z,w)|$. Write

$$|K(z,w)| \approx (\delta(z)+t+|X(\pi(z),\pi(w))|)^{-n-1},$$

where $\pi(z)$ and $\pi(w)$ are the projections of z and w on $\partial\Omega$ along the outer normal direction and $z, w \in R_\delta$. It follows that

$$\int_{\partial\Omega} (\delta(z)+t+|X|)^{-n-1}\,dX \approx (\delta+t)^{-1}.$$

Consequently,

$$\int_\Omega |K(z,w)|\,dv(w) \approx \int_0^C (\delta(z)+t)^{-1}\,dt \approx \log\frac{1}{\delta(z)}.$$

For more information, one can consult the paper of Beatrous and the second author [Beatrous and Li 1993] and the papers of Krantz and the second author [Krantz and Li 2001a; 2001b].

Lemma 2.4. *Let Ω be either a smoothly bounded strictly pseudoconvex domain or a Cartan classical domain. Let $\phi(z) := \gamma\log K(z)$ with $\gamma > 0$. Then, for γ sufficiently small,*

$$\int_\Omega e^{\phi(z)}\,dv_z < \infty \quad \text{and} \quad \|\bar\partial\phi\|^2_{i\partial\bar\partial\phi} \le \tfrac{1}{2}.$$

Proof. When Ω is a smoothly bounded strictly pseudoconvex domain, from Fefferman's asymptotic expansion for the Bergman kernel, one has

$$\phi(z) \approx \gamma\log\frac{1}{\delta(z)} \quad \text{and} \quad \int_\Omega e^{\phi(z)}\,dv(z) \approx \int_\Omega \Big(\frac{1}{\delta(z)}\Big)^\gamma dv \approx \int_0^1 t^{-\gamma}\,dt < \infty.$$

Notice that

$$\|\bar\partial\phi\|^2_{i\partial\bar\partial\phi} = \gamma\|\bar\partial\log K\|^2_g,$$

where g is the Bergman metric. From Fefferman's asymptotic expansion formula (see also [Donnelly 1994]), one gets the boundedness of $\|\bar\partial\log K\|^2_g$. Choose $\gamma > 0$ small enough so that $\|\bar\partial\phi\|^2_g < \frac{1}{2}$. For the Cartan classical domains, the first inequality follows from explicit formulas of the Bergman kernel [Hua 1963], and we compute the precise value of $\|\partial\phi\|_{i\partial\bar\partial\phi}$ in Section 6A. □

3. Pointwise estimates

An upper semicontinuous function ϕ defined on a domain $\Omega \subset \mathbb{C}^n$ with values in $\mathbb{R}\cup\{-\infty\}$ is called plurisubharmonic if its restriction to every complex line is subharmonic. Let $L^2(\Omega,\phi)$ denote the set of measurable functions h satisfying $\int_\Omega |h(z)|^2 e^{-\phi(z)}\,dv_z < \infty$. A C^2 function ϕ is called strongly plurisubharmonic if $i\partial\bar\partial\phi$ is strictly positive definite. Now, let Ω be a bounded pseudoconvex domain and ϕ be strongly plurisubharmonic on Ω. Then, for any $(0,1)$-form $f = \sum_{k=1}^n f_k(z)\,d\bar z_k$, define the norm of f induced by $i\partial\bar\partial\phi$ as (see also [Błocki 2005])

$$|f|^2_{i\partial\bar\partial\phi}(z) := \sum_{j,k=1}^n \phi^{j\bar k}(z)\,\overline{f_j(z)}\,f_k(z), \tag{3-1}$$

where $(\phi^{j\bar{k}})^\tau$ equals the inverse of the complex Hessian matrix $H(\phi)$. Demailly's reformulation [1982; 2012] of Hörmander's theorem [1965] says that *for any $\bar\partial$-closed* (0, 1)*-form f, the $L^2(\Omega, \phi)$ minimal solution to $\bar\partial u = f$ satisfies*

$$\int_\Omega |u|^2 e^{-\phi}\, dv \le \int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\phi}\, dv. \tag{3-2}$$

From this we see that when the (0, 1)-form f is bounded in the Bergman metric g, the canonical solution u to $\bar\partial u = f$ exists and the right-hand side of the estimate (3-2) is finite because it is dominated by a constant times a positive power of the Euclidean volume.

Donnelly and Fefferman [1983] (see also [Berndtsson 1993; 1996; 1997; McNeal 1996; Siu 1996]) modified Hörmander's theorem further as follows.

Theorem 3.1 (Donnelly–Fefferman-type estimate). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, and let ψ and ϕ be plurisubharmonic functions on Ω such that $i\partial\bar\partial\phi > 0$ and $|\partial\phi|^2_{i\partial\bar\partial\phi} \le \frac{1}{2}$. Then the $L^2\big(\Omega, \psi + \frac{1}{2}\phi\big)$ minimal solution u_0 to $\bar\partial u = f$ satisfies*

$$\int_\Omega |u_0|^2 e^{-\psi}\, dv \le 4\int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\psi}\, dv. \tag{3-3}$$

Next we prove the following lemma, using the estimates (3-2) and (3-3).

Lemma 3.2. *Let Ω be a bounded pseudoconvex domain and f be a $\bar\partial$-closed* (0, 1)*-form on Ω. Let ψ and ϕ be plurisubharmonic on Ω and u_0 and u_1 be the L^2-minimal solutions to $\bar\partial u = f$ in $L^2\big(\Omega, \psi + \frac{1}{2}\phi\big)$ and $L^2(\Omega, \phi)$, respectively. Suppose B is a compact subset of Ω and $h \in L^\infty(\Omega)$ with support in B.*

(i) *If $i\partial\bar\partial\phi > 0$ and $\|\partial\phi\|^2_{i\partial\bar\partial\phi} \le \frac{1}{2}$ on Ω, then*

$$\int_B |u_0|\, dv \le 2\left(\int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\psi}\, dv\right)^{1/2}\left(\int_B e^{\psi}\, dv\right)^{1/2} \tag{3-4}$$

and

$$\left|\int_\Omega u_0 \overline{P(h)}\, dv\right| \le 2v(B)\|h\|_\infty \left(\int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\psi}\, dv\right)^{1/2}\left(\int_\Omega \max_{w\in B} |K(z,w)|^2 e^{\psi(z)}\, dv_z\right)^{1/2}. \tag{3-5}$$

(ii) *Additionally,*

$$\int_B |u_1|\, dv \le 2\left(\int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\phi}\, dv\right)^{1/2}\left(\int_B e^{\phi}\, dv\right)^{1/2}$$

and

$$\left|\int_\Omega u_1 \overline{P(h)}\, dv\right| \le 2v(B)\|h\|_\infty \left(\int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\phi}\, dv\right)^{1/2}\left(\int_\Omega \max_{w\in B} |K(z,w)|^2 e^{\phi(z)}\, dv_z\right)^{1/2}.$$

Proof. Let χ_B denote the characteristic function on B, and let $\beta := \chi_B(u_0(z)/|u_0(z)|)$. By (3-3),

$$\left(\int_B |u_0|\, dv\right)^2 = \left|\int_\Omega u_0\bar\beta\, dv\right|^2 \le \int_\Omega |u_0|^2 e^{-\psi}\, dv \int_B |\beta|^2 e^{\psi}\, dv \le 4\int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\psi}\, dv \int_B e^{\psi}\, dv,$$

which proves (3-4). Notice that

$$\left|\int_\Omega u_0\overline{P(h)}\,dv\right|^2 \le \int_\Omega |u_0|^2 e^{-\psi}\,dv \int_\Omega |P(h)|^2 e^{\psi}\,dv$$
$$\le 4\int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\psi}\,dv\cdot v^2(B)\|h\|^2_\infty \int_\Omega \max_{w\in B}|K(z,w)|^2 e^{\psi(z)}\,dv_z,$$

which proves (3-5). Part (ii) can be proved similarly using Hörmander's estimate (3-2) in place of Donnelly–Fefferman's estimate (3-3). □

Theorem 3.3 (key estimate). *Let Ω be a Cartan classical domain or a smoothly bounded strictly convex domain. Then there is a constant C such that for any $\bar\partial$-closed $(0,1)$-form f on Ω with $\|f\|_{g,\infty}<\infty$, the canonical solution to $\bar\partial u = f$ satisfies*

$$|u(z)| \le C\|f\|_{g,\infty}\int_\Omega |K(z,w)|\,dv_w, \quad z\in\Omega.$$

Proof. From the discussion after (3-2) we see that the canonical solution to $\bar\partial u = f$ exists. Suppose first that Ω is a Cartan classical domain. For an arbitrary $a\in\Omega$ and any sufficiently small $\epsilon>0$, let $\beta := \chi_{B_a(\epsilon)}(u(z)/|u(z)|)$, where $\chi_{B_a(\epsilon)}$ is the characteristic function of the hyperbolic ball $B_a(\epsilon)$. Let $\phi := \gamma\log K(z)$ be a plurisubharmonic function on Ω for some chosen γ that satisfies the condition in Lemma 2.4. Define $\psi(z) =: \psi_a(z) := -\log|K(z,a)|$. Then ψ_a is pluriharmonic and bounded on Ω. Also define the function

$$\phi_0 := \psi_a + \tfrac{1}{2}\phi,$$

and let u_0 be the $L^2(\Omega,\phi_0)$ minimal solution to the equation $\bar\partial v = f$. Then by Theorem 3.1,

$$\int_\Omega |u_0|^2 e^{-\psi}\,dv \le 4\gamma^{-1}\int_\Omega |f|^2_g e^{-\psi}\,dv \le 4\gamma^{-1}\|f\|^2_{g,\infty}\int_\Omega e^{-\psi}\,dv < \infty,$$

which implies that $u_0\in L^2(\Omega)$. So $u-u_0\in A^2(\Omega)$ and

$$\int_{B_a(\epsilon)}|u|\,dv = \int_\Omega u\bar\beta\,dv = \int_\Omega u(\overline{\beta - P(\beta)})\,dv = \int_\Omega u_0(\overline{\beta-P(\beta)})\,dv = \int_\Omega u_0\bar\beta\,dv - \int_\Omega u_0\overline{P(\beta)}\,dv.$$

By Lemma 2.4 and (3-4) in Lemma 3.2,

$$\left|\int_\Omega u_0\bar\beta\,dv\right|^2 \le 4\int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\psi_a}\,dv \int_{B_a(\epsilon)} e^{\psi_a}\,dv$$
$$\le C\|f\|^2_{g,\infty}\int_\Omega |K(z,a)|\,dv_z \int_{B_a(\epsilon)} |K(z,a)|^{-1}\,dv_z$$
$$\le C_\epsilon\|f\|^2_{g,\infty}\int_\Omega |K(z,a)|\,dv_z\cdot v(B_a(\epsilon))K(a)^{-1}$$
$$\le C_\epsilon\|f\|^2_{g,\infty} v^2(B_a(\epsilon))\int_\Omega |K(z,a)|\,dv_z,$$

where the last two inequalities hold due to Proposition 2.2 and C_ϵ is a constant depending on ϵ. On the other hand, by (3-5) in Lemma 3.2 and Proposition 2.2 again,

$$\begin{aligned}\left|\int_\Omega u_0\overline{P(\beta)}\,dv\right|^2 &\le Cv^2(B_a(\epsilon))\int_\Omega |f|^2_{i\partial\bar\partial\phi}e^{-\psi_a}\,dv\int_\Omega \max_{w\in\overline{B_a(\epsilon)}}|K(z,w)|^2e^{\psi_a(z)}\,dv_z\\ &\le C_\epsilon v^2(B_a(\epsilon))\int_\Omega |f|^2_{i\partial\bar\partial\phi}(z)|K(z,a)|\,dv_z\int_\Omega |K(z,a)|^{2-1}\,dv_z\\ &\le C_\epsilon\|f\|^2_{g,\infty}v^2(B_a(\epsilon))\left(\int_\Omega |K(z,a)|\,dv_z\right)^2.\end{aligned}$$

Combining the above estimates, one easily sees that

$$\frac{1}{v(B_a(\epsilon))}\int_{B_a(\epsilon)}|u|\,dv\le C_\epsilon\|f\|_{g,\infty}\int_\Omega |K(z,a)|\,dv_z.$$

Fix $\epsilon>0$. By Proposition 2.1, there exists a constant C depending only on Ω such that

$$|u(a)|\le C\|f\|_{g,\infty}\int_\Omega |K(z,a)|\,dv_z.$$

If Ω is instead a smoothly bounded strictly convex domain, then let $\psi_a(z)=(n+1)\log|h_a(z)|$, repeat the argument for the Cartan classical domains and use Lemma 2.3. □

In Section 6, Proposition 6.1, we show that the estimate in Theorem 3.3 is sharp for the Cartan classical domains. When Ω is the unit ball $\mathbb{B}^n$, the *key estimate* reduces to Berndtsson's result (1-2). Now we generalize (1-2) and (1-4) to smoothly bounded strictly pseudoconvex domains.

Theorem 3.4. *Let Ω be a smoothly bounded strictly pseudoconvex domain. Then there is a constant C such that for any $\bar\partial$-closed $(0,1)$-form f on Ω with $\|f\|_{g,\infty}<\infty$, there is a solution u to $\bar\partial u=f$ such that*

$$|u(z)|\le C\|f\|_{g,\infty}\log(1+K(z)),\quad z\in\Omega.$$

Proof. Let $r(z)$ be a strongly plurisubharmonic defining function for Ω such that $r(z)\in C^\infty(\bar\Omega)$ and $r>0$ in Ω. Consider the polynomial

$$X(z,w):=r(w)+\sum_{j=1}^n\left(\frac{\partial r}{\partial w_j}\Big|_w(z_j-w_j)\right)+\frac12\sum_{j,k=1}^n\left(\frac{\partial^2 r}{\partial w_j\partial w_k}\Big|_w(z_j-w_j)(z_k-w_k)\right).$$

Define the region $R_\epsilon=\{(z,w):z,w\in\Omega,\ r(z)+r(w)+|z-w|^2<\epsilon\}$. For $(z,w)\in R_\epsilon$, Fefferman [1974] showed the Bergman kernel on Ω can be expressed as

$$K(z,w)=\frac{F(z,w)}{X(z,w)^{n+1}}+G(z,w)\log X(z,w),\tag{3-6}$$

where $G,F\in C^\infty(\bar\Omega\times\bar\Omega)$, $F(z,z)>0$ on $(\bar\Omega\times\bar\Omega)\cap R_\epsilon$ and "log" denotes the principal branch of the logarithm defined on $\mathbb{C}\setminus(-\infty,0]$. The asymptotic expansion (3-6) implies

$$\int_\Omega |K(z,w)|\,dv_w\le C(1+\log K(z)),\quad z\in\Omega.\tag{3-7}$$

Since the boundary $\partial\Omega$ is compact, for any $\delta > 0$ there are finitely many $b^j \in \partial\Omega$, $j = 1, \dots, m$, such that $\partial\Omega \subset \cup_{j=1}^m \mathbb{B}(b^j, \delta)$. Choose smoothly bounded strictly pseudoconvex domains Ω^j, $j = 1, \dots, m$, such that

$$\mathbb{B}(b^j, 3\delta) \cap \Omega \subset \Omega^j \subset \Omega \cap \mathbb{B}(b^j, 4\delta),$$

where δ is chosen small enough such that for each j, after a polynomial change of variables, each Ω^j is a strictly convex domain. Let $\{\Omega^j\}_{j=m+1}^{m+k}$ be a finite open cover of $\Omega \setminus \cup_{j=1}^m (\mathbb{B}(b^j, \delta) \cap \Omega)$ consisting of balls contained in Ω. In the argument of Theorem 3.3 by letting $\phi_0 = \gamma \log K_\Omega(z)$ (instead of $\gamma \log K_{\Omega^j}(z)$) and using (3-7), we can solve the equation $\bar\partial u^j = f$ on Ω^j with minimal solution u^j satisfying

$$|u^j(z)| \le C \|f\|_{g,\infty} \log(1 + K(z)). \tag{3-8}$$

Let $\{\eta_j\}_{j=1}^{m+k}$ be a partition of unity of $\bar\Omega$ subordinate to the cover $\{B(b^j, \delta)\}_{j=1}^m \cup \{\Omega^j\}_{j=m+1}^{m+k}$, and let $v(z) := \sum_{j=1}^{m+k} \eta_j(z) u^j(z)$. Then $\bar\partial v = f + h$, where $h := \sum_{j=1}^{m+k} u^j \bar\partial \eta_j$ is a $\bar\partial$-closed (0, 1)-form on Ω. By the integral formula in [Grauert and Lieb 1970; Henkin 1970], there is a bounded solution v_0 to the equation $\bar\partial v_0 = h$. Let $u = v - v_0$. Then $\bar\partial u = f$ and by (3-8),

$$|u(z)| \le \sum_{j=1}^k \eta_j(z) C \|f\|_{g,\infty} \log(1 + K(z)) + C \le C \|f\|_{g,\infty} \log(1 + K(z)). \qquad \square$$

Remark. For a smoothly bounded strictly pseudoconvex domain Ω, if the canonical solution is u_0, then for $h \in L^\infty(\Omega)$ with $\|h\|_\infty \le 1$,

$$|P[h(\cdot) \log K(\cdot)](z)| \le C(1 + \log K(z))^2.$$

In fact, letting $\omega_t = \{z \in \Omega : \delta(z) > t\}$, by Fefferman's expansion theorem on the Bergman kernel [Fefferman 1974], we know that

$$\begin{aligned}
|P[h(\cdot) \log K(\cdot)](z)| &\le \int_\Omega |h(w)| \log K(w) |K(z, w)|\, dv(w) \le \|h\|_\infty \int_\Omega \log K(w) |K(z, w)|\, dv(w) \\
&\approx \|h\|_\infty \int_0^c \int_{\partial\Omega_t} \log K(w) |K(z, w)|\, d\sigma(w)\, dt \\
&\le C \|h\|_\infty \int_0^c (-\log t) \frac{1}{\delta(z) + t}\, dt \\
&\le C \|h\|_\infty \left(\frac{1}{\delta(z)} \int_0^{\delta(z)} (-\log t\, dt) + \int_{\delta(z)}^c -\log t \frac{1}{\delta(z) + t}\, dt \right) \\
&\le C \|h\|_\infty \left(\log \frac{C}{\delta(z)} + \int_{\delta(z)}^c -\log(\delta(z) + t) \frac{1}{\delta(z) + t}\, dt \right) \le C \|h\|_\infty \left(\log \frac{C}{\delta(z)} \right)^2.
\end{aligned}$$

Combining this with Theorem 3.4 one gets

$$|u_0(z)| \le C(1 + \log K(z))^2.$$

4. Uniform estimates

In this section, we obtain uniform estimates for the equation $\bar\partial u = f$ on the unit polydisc $\mathbb{D}^n$ and strictly pseudoconvex domains by imposing conditions on f stronger than $\|f\|_{g,\infty} < \infty$.

Theorem 4.1. *For any $p \in (1, \infty)$, there is a constant C such that for any $\bar\partial$-closed $(0, 1)$-form f on $\mathbb{D}^n$, the canonical solution u to $\bar\partial u = f$ satisfies*

$$\|u\|_\infty \le C \left\| f \prod_{j=1}^n \Big(\log\Big(\frac{2}{1-|z_j|^2}\Big)\Big)^p \right\|_{g,\infty}.$$

Proof. Let $A_0 = 2p + \log v(\mathbb{D}^n)$. Choose $0 < \gamma < \frac{1}{2}$ such that $\phi(z) := \gamma \log K(z)$ satisfies Lemma 2.4. Choose $\alpha > 1$ such that $\alpha - \gamma = 1$. Let $K_j(z) := \pi^{-1}(1 - |z_j|^2)^{-2}$, and let

$$\phi_0(z) := \phi(z) - \sum_{j=1}^n p \log(A_0 + \gamma \log K_j(z)) - \alpha \log|K(z, a)|.$$

Since on $\mathbb{D}^n$, $\log|K(z, a)|$ is pluriharmonic and

$$A_0 + \gamma \log K_j = 2p + n \log \pi - \gamma \log \pi - 2\gamma \log(1 - |z_j|^2) \ge 2p,$$

we know that

$$i\partial\bar\partial\phi = \sum_{j=1}^n \left[i\partial\bar\partial(\gamma \log K_j)\Big(1 - \frac{p}{A_0 + \gamma \log K_j}\Big) + \pi \frac{\partial(\gamma \log K_j) \wedge \bar\partial(\gamma \log K_j)}{(A_0 + \gamma \log K_j)^2} \right] \ge \frac{1}{2} \sum_{j=1}^n i\partial\bar\partial(\gamma \log K_j).$$

Thus $|f|^2_{i\partial\bar\partial\phi_0} \le 2|f|^2_{i\partial\bar\partial\phi} = 2|f|^2_g/\gamma$. Let

$$A_p(f) = \left\| f \Big(\prod_{j=1}^n (A_0 - \gamma \log \pi - 2\gamma \log(1 - |z_j|^2)) \Big)^p \right\|_{g,\infty}.$$

As in Theorem 3.3, let u_0 be the $L^2(\mathbb{D}^n, \phi_0)$ minimal solution to $\bar\partial u_0 = f$ and $\beta := e^{i\theta(z)} \chi_{B_a(\epsilon)}$, where $u(z) = |u(z)| e^{i\theta(z)}$. Then

$$\begin{aligned}
\int_{B_a(\epsilon)} |u|\, dv &\le C v(B_a(\epsilon)) \Big(\int_{D(0,1)^n} |f|^2_g e^{-\phi_0}\, dv \Big)^{1/2} \Big(\int_{D(0,1)^n} |K(z, a)|^2 e^{\phi_0(z)}\, dv_z \Big)^{1/2} \\
&\le C A_p(f) v(B_a(\epsilon)) \Big(\int_{D(0,1)^n} \frac{e^{-\phi_0}\, dv_z}{\prod_{j=1}^n (A_0 + \gamma \log K_j(z))^{2p}} \int_{D(0,1)^n} |K(z, a)|^2 e^{\phi_0}\, dv_z \Big)^{1/2} \\
&= C A_p(f) v(B_a(\epsilon)) \prod_{j=1}^n \Big(\int_{D(0,1)} \frac{|K_j(z, a)|^\alpha K_j(z)^{-\gamma}\, dv_z}{(A_0 + \gamma \log K_j(z))^p} \int_{D(0,1)} \frac{|K_j(z, a)|^{2-\alpha} K_j(z)^\gamma\, dv_z}{(A_0 + \gamma \log K_j(z))^p} \Big)^{1/2} \\
&\le C A_p(f) v(B_a(\epsilon)).
\end{aligned}$$

Fix $\epsilon > 0$ sufficiently small. By Proposition 2.1,

$$\begin{aligned}|u(a)| &\le C A_p(f) + C\|f\|_{g,\infty}\\ &\le C\left\| f\prod_{j=1}^{n}(2(n+p)-\log(1-|z_j|^2))^p\right\|_{g,\infty}.\end{aligned}$$

Notice that

$$2(n+p)-\log(1-|z_j|^2) \le 5(n+p)\log\frac{2}{1-|z_j|^2},$$

which completes the proof of the theorem. □

In fact, the finiteness of the right-hand side of the estimate in Theorem 4.1 is a stronger condition than f being L^∞ on $\mathbb{D}^n$. Recently, in [Dong et al. 2020], the first author, Pan and Zhang obtained uniform estimates for the canonical solution to $\bar\partial u = f$ when f is continuous up to the boundary of $\mathbb{D}^n$, and more generally the Cartesian product of smoothly bounded planar domains.

However, the situation for strictly pseudoconvex domains is quite different. The finiteness of the right-hand side of the estimate in either Theorem 4.2 or Theorem 4.3 is a much weaker condition than f being L^∞ on each smooth domain considered. In fact, we allowed f to blow-up on $\partial\Omega$ to order less than $\frac{1}{2}$.

Theorem 4.2. *Let Ω be a smoothly bounded strictly convex domain. For any $p \in (1,\infty)$ and sufficiently small $\gamma > 0$, there exists a constant C such that for any $\bar\partial$-closed $(0,1)$-form f, the canonical solution u to $\bar\partial u = f$ satisfies*

$$\|u\|_\infty \le C\|(1+\log v(\Omega)+\gamma\log K(z))^p f\|_{g,\infty}.$$

Proof. Choose $0 < \gamma < 1/(n+2)$ such that $\phi(z) := \gamma\log K(z)$ satisfies Lemma 2.4, and let $\alpha = \gamma+1$. Let

$$A_0 := 2p + \gamma\log v(\Omega),$$

and let

$$\phi_0(z) = \phi(z) - (n+1)\alpha\log|h_a(z)| - p\log(A_0+\gamma\log K(z)).$$

Notice that

$$i\partial\bar\partial\phi_0 \ge \left(1-\frac{p}{A_0+\phi}\right)i\partial\bar\partial\phi \ge \frac{i\partial\bar\partial\phi}{2}$$

and

$$\gamma K(z) > \gamma v(\Omega)^{-1}.$$

Therefore

$$|f|^2_{i\partial\bar\partial\phi_0} \le \frac{2}{\gamma}|f|^2_g.$$

Define

$$A_p(f) := \|(A_0+\gamma\log K(z))^p f\|_{g,\infty}.$$

Using arguments similar to those in Theorems 3.3 and 4.1 and $\alpha-\gamma=1$,

$$\begin{aligned}
&\int_{B_a(\epsilon)} |u|\,dv \\
&\quad\le Cv(B_a(\epsilon))\Big(\int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\phi_0}\,dv\Big)^{1/2}\Big(\int_\Omega |K(z,a)|^2 e^{\phi_0(z)}\,dv_z\Big)^{1/2}\\
&\quad\le CA_p(f)v(B_a(\epsilon))\Big(\int_\Omega \frac{e^{-\phi_0}}{(A_0+\gamma\log K(z))^{2p}}\,dv_z \int_\Omega |K(z,a)|^2 e^{\phi_0}\,dv_z\Big)^{1/2}\\
&\quad= CA_p(f)v(B_a(\epsilon))\Big(\int_\Omega \frac{K(z)^{-\gamma}|h_a(z)|^{\alpha(n+1)}}{(A_0+\gamma\log K(z))^{p}}\,dv_z \int_\Omega \frac{|K(z,a)|^2 K(z)^{\gamma}|h_a(z)|^{-\alpha(n+1)}}{(A_0+\gamma\log K(z))^{p}}\,dv_z\Big)^{1/2}\\
&\quad\le CA_p(f)v(B_a(\epsilon))\Big(\int_\Omega \frac{|K(z,a)|\,dv_z}{(A_0+\gamma\log K(z))^{p}} \int_\Omega \frac{|K(z,a)|^{2-\alpha} K(z)^{\gamma}}{(A_0+\gamma\log K(z))^{p}}\,dv_z\Big)^{1/2} \le CA_p(f)v(B_a(\epsilon)),
\end{aligned}$$

where the last inequality follows from Fefferman's asymptotic expansion. In fact, since $n/(n+1)<2-\alpha$, if $\Omega_t=\{z:r(z)>t\}$ where $r(z)$ is a defining function satisfying the definition of $h_a(z)$, then

$$\begin{aligned}
\int_\Omega \frac{|K(z,a)|^{2-\alpha}K(z)^\gamma}{(A_0+\gamma\log K(z))^p}\,dv_z &\le C\Big(1+\int_0^\epsilon\int_{\partial\Omega_t}\frac{|K(z,a)|^{2-\alpha}K(z)^\gamma}{(A_0+\gamma\log K(z))^p}\,d\sigma_t(z)\,dt\Big)\\
&\le C\Big(1+\int_0^\epsilon \frac{t^{-(2-\alpha)(n+1)+n}t^{-\gamma(n+1)}}{(A_0-\gamma(n+1)\log t)^p}\,dt\Big)\\
&\le C\Big(1+\int_0^\epsilon \frac{1}{t(\log t)^p}\,dt\Big)\le C\Big(1-\frac{\log\epsilon}{p-1}\Big).
\end{aligned}$$

By Proposition 2.1, for a fixed $\epsilon>0$ sufficiently small, $|u(a)|\le CA_p(f)+\|f\|_{g,\infty}\le CA_p(f)$. □

Using an argument similar to the proof of Theorem 3.4 we get the following generalization of Theorem 4.2 to smoothly bounded strictly pseudoconvex domains.

Theorem 4.3. *Let Ω be a smoothly bounded strictly pseudoconvex domain. Then, for any $p\in(1,\infty)$, there exists a constant C such that for any $\bar\partial$-closed $(0,1)$-form f, there is a solution u to $\bar\partial u=f$ that satisfies*

$$\|u\|_\infty\le C\|(\log K(\cdot))^p f(\cdot)\|_{g,\infty}.$$

Remark. Let $f\in L^\infty_{(0,1)}(\Omega)$ be a $\bar\partial$-closed form on a smoothly bounded strictly pseudoconvex domain Ω. Henkin and Romanov's theorem [1971] states that there exists a solution $u\in C^{1/2}(\Omega)$. Theorem 3.4 implies that one can find a bounded solution when $(\log(1/\delta(z)))^p\delta(z)|f(z)|^2$ is bounded. Moreover, [Lieb and Range 1986, Theorem 2 (i)] shows that uniform estimates hold for the canonical solutions to the $\bar\partial$-equations on Ω.

5. Additional estimates for Cartan classical domains

A domain Ω is symmetric if, for all $a\in\Omega$, there is an involutive automorphism G such that a is isolated in the set of fixed points of G. All bounded symmetric domains are convex and homogeneous. E. Cartan

classical domain	rank r	multiplicity $\boldsymbol{a}$	genus p	dimension N	index k
$\mathrm{I}(m,n),\ m \le n$	m	2	$m+n$	mn	1
$\mathrm{II}(n)$	n	1	$n+1$	$\frac{1}{2}n(n+1)$	1
$\mathrm{III}(2n+\epsilon),\ \epsilon=0$ or 1	n	4	$2(2n+\epsilon-1)$	$n(2n+2\epsilon-1)$	$\frac{1}{2}$
$\mathrm{IV}(n)$	2	$n-2$	n	n	1

Table 1. Characteristics of classical domains.

proved that all bounded symmetric domains in $\mathbb{C}^N$ up to biholomorphism are the Cartesian product(s) of the following four types of Cartan classical domains and two domains of exceptional types.

Definition 5.1. A Cartan classical domain is a domain of one of the following types:

(i) $\mathrm{I}(m,n) := \{z \in M_{(m,n)}(\mathbb{C}) : I_m - zz^* > 0\},\ m \le n.$

(ii) $\mathrm{II}(n) := \{z \in \mathrm{I}(n,n) : z^\tau = z\}.$

(iii) $\mathrm{III}(n) := \{z \in \mathrm{I}(n,n) : z^\tau = -z\}.$

(iv) $\mathrm{IV}(n) := \{z \in \mathbb{C}^n : 1-2|z|^2+|s(z)|^2 > 0 \text{ and } |s(z)| < 1\}$, where $s(z) := \sum_{j=1}^n z_j^2$ and $n > 2$.

Here $z^* := \bar{z}^\tau$ is the conjugate transpose of z.

Let Ω be a Cartan classical domain. Denote the rank, characteristic multiplicity, genus, complex dimension and kernel index of Ω by r, $\boldsymbol{a}$, p, N and k, respectively. Their values are given in Table 1.

Hua [1963] obtained explicit formulas for the Bergman kernels on the Cartan classical domains. For a domain Ω of type I, II or III,

$$K(z,w) = C_\Omega[\det(I - zw^*)]^{-pk},$$

and for a domain of type IV,

$$K(z,w) = C_n\left[1 - 2\sum_{j=1}^n z_j\bar{w}_j + s(z)\overline{s(w)}\right]^{-n}.$$

For any $z \in \Omega$,

$$\delta_\Omega(z) \le K(z)^{-1/(rpk)}.$$

Let $\lambda = pk$. By [Faraut and Korányi 1990, Theorem 3.8], one can write the Bergman kernel on a Cartan classical domain Ω as

$$K(z,w) = h(z,w)^{-\lambda} = \sum_{\boldsymbol{m}\ge 0} (\lambda)_{\boldsymbol{m}} K_{\boldsymbol{m}}(z,w),$$

where

$$\boldsymbol{m} = (m_1,\dots,m_r) \text{ and } \boldsymbol{m} \ge 0 \quad\Longleftrightarrow\quad m_1 \ge m_2 \ge \cdots \ge m_r \ge 0,$$

and

$$(\lambda)_{\boldsymbol{m}} = \frac{\Gamma_\Omega(\boldsymbol{\lambda}+\boldsymbol{m})}{\Gamma_\Omega(\boldsymbol{\lambda})},\quad \Gamma_\Omega(\boldsymbol{s}) = c_\Omega \prod_{j=1}^r \Gamma\Big(s_j - (j-1)\frac{a}{2}\Big),\quad \boldsymbol{\lambda} = (\lambda,\dots,\lambda).$$

Here, $K_{\boldsymbol{m}}$ is the Bergman kernel for homogeneous polynomials in $\mathbb{C}^r$ of degree $|\boldsymbol{m}| = m_1 + \cdots + m_r$. For each Cartan domain Ω, there is a subgroup $\mathcal{K}(\Omega)$ of the unitary group such that for each $z \in \Omega$ there is $k \in \mathcal{K}(\Omega)$ such that $z = k\tilde{z}$ where $\tilde{z} \in \mathbb{C}^r \times \prod_{j=r+1}^{N}\{0\}$ and $K_{\boldsymbol{m}}(z, z) =: K_{\boldsymbol{m}}(\tilde{z}, \tilde{z})$.

The following Forelli–Rudin-type integral was studied in [Faraut and Korányi 1990]:

$$J_{\beta,c}(z) := \int_\Omega K(w)^\beta |K(w, z)|^{1+c-\beta}\, dv_w.$$

By the proof of Theorem 4.1 in [Faraut and Korányi 1990], one has

$$J_{\beta,c}(z) = \sum_{\boldsymbol{m}\ge 0} \frac{|(\mu)_{\boldsymbol{m}}|^2}{((1-k\beta)p)_{\boldsymbol{m}}} K_{\boldsymbol{m}}(z, z), \qquad \mu = \tfrac{1}{2}kp(1+c-\beta).$$

Using Stirling's formula, one can show (see [Faraut and Korányi 1990, (4.3)] or [Engliš and Zhang 2004, (2.9)]) that as m varies,

$$\frac{\left|\left(\frac{1}{2}pk(1-\beta)\right)_{\boldsymbol{m}}\right|^2}{((1-k\beta)p)_{\boldsymbol{m}}} \approx \frac{\left(\frac{1}{2}kp\right)_{\boldsymbol{m}}^2}{(p)_{\boldsymbol{m}}},$$

which implies that

$$J_{\beta,0}(z) \approx J_{0,0}(z) = \int_\Omega |K(z, w)|\, dv(w), \qquad \beta < \frac{1}{pk}. \tag{5-1}$$

Further computations were carried out by Faraut and Koranyi [1990].

Theorem 5.2 [Faraut and Korányi 1990]. *For any $\beta < 1/(pk)$,*

(i) *$J_{\beta,c}(z)$ is bounded for all $z \in \Omega$ if and only if $c < -(r-1)\boldsymbol{a}/(2p)$,*

(ii) *$J_{\beta,c}(z) \approx K(z)^c$ if $c > (r-1)\boldsymbol{a}/(2p)$.*

When $|c| \le (r-1)\boldsymbol{a}/(2p)$, it is difficult to compute $J_{\beta,c}(z)$; see [Korányi 1991; Yan 1992]. Theorem 1 of [Engliš and Zhang 2004], whose parameters are chosen as $\frac{1}{2}p(1+c-\beta)$, $\frac{1}{2}p(1+c-\beta)$ and $p(1-\beta)$, is stated as follows.

Theorem 5.3. *Let Ω be a Cartan classical domain of rank 2 with characteristic boundary $\mathcal{U}$. Then for any $z = te_1 + Te_2$ with $0 \le t \le T < 1$ and $e_1, e_2 \in \mathcal{U}$ the following statements hold:*

(i) *If $2pc = \boldsymbol{a}$, then $J_{\beta,c}(z) \approx (1-t)^{-\boldsymbol{a}/2}(1-T)^{-\boldsymbol{a}/2}[1-\log(1-t)]$.*

(ii) *If $0 < 2pc < \boldsymbol{a}$, then $J_{\beta,c} \approx (1-t)^{-\boldsymbol{a}/2}(1-T)^{-pc}$.*

(iii) *If $c = 0$, then $J_{\beta,c}(z) \approx (1-t)^{-\boldsymbol{a}/2}[1+\log[(1-t)/(1-T)]]$.*

(iv) *If $-\boldsymbol{a} < 2pc < 0$, then $J_{\beta,c}(z) \approx (1-t)^{-pc-\boldsymbol{a}/2}$.*

(v) *If $2pc = -\boldsymbol{a}$, then $J_{\beta,c}(z) \approx 1-\log(1-t)$.*

As a consequence, when Ω is a Cartan classical domain of rank 2 and $z = te_1 + te_2$ with $0 \le t < 1$ and $e_i \in \mathcal{U}$, one has

$$\int_\Omega |K(z, w)|\, dv_w \approx (1-t)^{-\boldsymbol{a}/2} \approx \delta_\Omega(z)^{-\boldsymbol{a}/2}.$$

On the Cartan classical domains, we impose a stronger assumption on f to get bounded solutions to $\bar\partial u = f$. The following result provides a partial answer to the problems raised by Henkin and Leiterer [1983] and Sergeev [1994].

Theorem 5.4. *Let Ω be a Cartan classical domain and $\alpha > 1 + (r-1)\boldsymbol{a}/(2p)$. Then there exists a constant C such that for any $\bar\partial$-closed $(0,1)$-form f, the canonical solution u to $\bar\partial u = f$ satisfies*

$$\|u\|_\infty \le C \left\| \int_\Omega |f|_g^2(z) |K(z,\cdot)|^\alpha \, dv_z \right\|_\infty^{1/2} + C\|f\|_{g,\infty}. \tag{5-2}$$

Proof. As in the proof of Theorem 3.3, for any $a \in \Omega$, let $\beta := \chi_{B_a(\epsilon)} u(z)/|u(z)|$, $\phi := \gamma \log K(z)$ and $\psi_a(z) := -\alpha \log|K(z,a)|$. Then

$$\int_{B_a(\epsilon)} |u| \, dv \le \int_\Omega |u_0 \bar\beta| \, dv + \int_\Omega |u_0 \overline{P(\beta)}| \, dv.$$

By Lemma 2.4 and (3-4),

$$\begin{aligned}
\left| \int_\Omega u_0 \bar\beta \, dv \right| &\le C \left(\int_\Omega |f|_g^2(z) |K(z,a)|^\alpha \, dv_z \right)^{1/2} \left(\int_{B_a(\epsilon)} |K(z,a)|^{-\alpha} \, dv_z \right)^{1/2} \\
&\le C \left(\int_\Omega |f|_g^2(z) |K(z,a)|^\alpha \, dv_z \right)^{1/2} (v(B_a(\epsilon)) K(a)^{-\alpha})^{1/2} \\
&\le C v(B_a(\epsilon))^{(1+\alpha)/2} \left(\int_\Omega |f|_g^2(z) |K(z,a)|^\alpha \, dv_z \right)^{1/2}.
\end{aligned}$$

On the other hand, by (3-5),

$$\begin{aligned}
\left| \int_\Omega u_0 \overline{P(\beta)} \, dv \right| &\le C \left(\int_\Omega |f|_g^2(z) |K(z,a)|^\alpha \, dv_z \right)^{1/2} v(B_a(\epsilon)) \left(\int_\Omega \max_{w \in \overline{B_a(\epsilon)}} |K(z,w)|^2 e^{\psi_a} \, dv_z \right)^{1/2} \\
&\le C \left(\int_\Omega |f|_g^2(z) |K(z,a)|^\alpha \, dv_z \right)^{1/2} v(B_a(\epsilon)) \left(\int_\Omega |K(z,a)|^{2-\alpha} \, dv_z \right)^{1/2}.
\end{aligned}$$

If $\alpha > 1 + (r-1)\boldsymbol{a}/(2p) \ge 1$, then $|K(z,a)|^{2-\alpha}$ is integrable on Ω by Theorem 5.2. Therefore, for any $a \in \Omega$,

$$\frac{1}{v(B_a(\epsilon))} \int_{B_a(\epsilon)} |u| \, dv \le C \left(\int_\Omega |f|_g^2(z) |K(z,a)|^\alpha \, dv_z \right)^{1/2}.$$

Coupling this estimate with Proposition 2.1, we have proved u is bounded. □

6. Sharpness of the pointwise estimates

For the Cartan classical domains, we show that the logarithm of the Bergman kernel has a bounded gradient with respect to the Bergman metric and also verify that Theorem 3.3 is sharp.

6A. *Solutions with logarithmic growth.*

Example 1. Let Ω be a Cartan classical domain and $u(z)=\log K(z)$. Then $P[u](z)$ is a constant function on Ω and there exists a constant c such that $|\bar\partial u|_g^2 = c\,\mathrm{Tr}(zz^*)$.

Proof. Notice that for all $z\in\Omega$,

$$\begin{aligned}P[u](z) &= \int_\Omega u(w)K(z,w)\,dv_w\\ &= \int_\Omega \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta}w)K(z,e^{i\theta}w)\,d\theta\,dv_w\\ &= \int_\Omega \frac{1}{2\pi}\int_0^{2\pi} u(w)K(z,e^{i\theta}w)\,d\theta\,dv_w\\ &= \int_\Omega u(w)K(z,0)\,dv_w = \frac{1}{v(\Omega)}\int_\Omega u(w)\,dv_w,\end{aligned}$$

where the third equality follows by the transformation rule of the Bergman kernel, and the fourth equality follows by the mean value property of (anti)holomorphic functions.

Now we show the second part of the example. For $z\in M_{(m,n)}(\mathbb{C})$, define $V(z):=I_m - zz^*$ and let V_{uv} denote the (u,v) entry of V. Then by [Hua 1963; Lu 1997] (see also [Chen and Li 2019, Proposition 2.1]), for domains of type I, II and III,

$$g^{j\alpha,\overline{k\beta}}(z)=\begin{cases} V_{jk}(\delta_{\alpha\beta}-\sum_{l=1}^m z_{l\alpha}\bar z_{l\beta}), & z\in \mathrm{I}(m,n),\\ V_{jk}\dfrac{V_{\alpha\beta}}{(2-\delta_{j\alpha})(2-\delta_{k\beta})}, & z\in \mathrm{II}(n),\\ \frac14 V_{jk}V_{\alpha\beta}(1-\delta_{j\alpha})(1-\delta_{k\beta}), & z\in\mathrm{III}(n).\end{cases}$$

For matrices $E_{j\alpha}:=(\delta_{ju}\delta_{\alpha v})_{u,v}$, $A:=(a_{uv})_{u,v}\in M_{(n,m)}(\mathbb{C})$,

$$E_{j\alpha}A=(\delta_{ju}a_{\alpha v})_{u,v}\quad\text{and}\quad \frac{\partial V}{\partial z_{j\alpha}}=-E_{j\alpha}z^*.$$

Then for $z\in\mathrm{I}(m,n)$,

$$\begin{aligned}\frac{\partial\log\det V(z)}{\partial z_{j\alpha}} &= \mathrm{Tr}\Big(V^{-1}(z)\frac{\partial V(z)}{\partial z_{j\alpha}}\Big) = -\,\mathrm{Tr}(V^{-1}(z)E_{j\alpha}z^*) = -\,\mathrm{Tr}(E_{j\alpha}z^*V^{-1}(z))\\ &= -\sum_u \delta_{ju}[z^*V^{-1}]_{\alpha u} = -[z^*V^{-1}]_{\alpha j}.\end{aligned}$$

Since $u(z)=\log(\det(V(z)))^{-(m+n)}-\log v(\mathrm{I}(m,n))$ is real-valued,

$$\begin{aligned}|\bar\partial u|_g^2(z) = \sum_{j,\beta,k,\alpha} g^{j\alpha,\overline{k\beta}}\frac{\partial u}{\partial z_{j\alpha}}\overline{\frac{\partial u}{\partial z_{k\beta}}} &= (m+n)^2\sum_{j,\beta,k,\alpha} V_{jk}[I-z^\tau\bar z]_{\alpha\beta}[z^*V^{-1}(z)]_{\alpha j}\overline{[z^*V^{-1}]}_{\beta k}\\ &= (m+n)^2\sum_{\alpha,k}[z^*]_{\alpha k}[(I-z^\tau\bar z)z^\tau\overline{V^{-1}}]_{\alpha k}\\ &= (m+n)^2\sum_{\alpha,k}[z^*]_{\alpha k}[z^\tau]_{\alpha k} = (m+n)^2\,\mathrm{Tr}(zz^*).\end{aligned}$$

For $z \in \mathrm{II}(n)$, using the symmetry of z, we know

$$\frac{\partial V(z)}{\partial z_{j\alpha}} = -\big(1 - \tfrac{1}{2}\delta_{j\alpha}\big)(E_{j\alpha} + E_{\alpha j})z^* \quad \text{and} \quad z^* V^{-1}(z) = (z^* V^{-1}(z))^\tau.$$

Hence,

$$\frac{\partial \log\det V(z)}{\partial z_{j\alpha}} = \mathrm{Tr}\Big(V^{-1}(z)\frac{\partial V}{\partial z_{j\alpha}}(z)\Big) = -\big(1 - \tfrac{1}{2}\delta_{j\alpha}\big)\,\mathrm{Tr}(E_{j\alpha}z^* V^{-1}(z) + z^* V^{-1}(z)E_{\alpha j})$$
$$= -(2-\delta_{j\alpha})\,\mathrm{Tr}(E_{j\alpha}z^* V^{-1}(z)) = -(2-\delta_{j\alpha})[z^* V^{-1}(z)]_{j\alpha}.$$

Since $u(z) = \log(\det(V(z)))^{-(n+1)} - \log v(\mathrm{II}(n))$ and for z symmetric $\overline{z^* V(z)^{-1}} = V(z)^{-1}z$,

$$|\bar\partial u|_g^2(z) = \sum_{j,\beta,k,\alpha} g^{j\alpha,\overline{k\beta}}\,\frac{\partial u}{\partial z_{j\alpha}}\,\overline{\frac{\partial u}{\partial z_{k\beta}}} = (n+1)^2 \sum_{\alpha,\beta,k,j} V_{jk}V_{\alpha\beta}[z^* V^{-1}(z)]_{j\alpha}[\overline{z^* V^{-1}(z)}]_{k\beta}$$
$$= (n+1)^2 \sum_{j,\beta}[z^*]_{j\beta}[V(z)V^{-1}(z)z]_{j\beta} = (n+1)^2\,\mathrm{Tr}(zz^*).$$

The proof for skew-symmetric $z \in \mathrm{III}(n)$ is similar to the preceding proofs.

For a Cartan classical domain $\mathrm{IV}(n)$, let $s(z) := \sum z_j^2$ and $r(z) := 1 - 2|z|^2 + |s(z)|^2$ for $z \in \mathbb{C}^n$. By [Hua 1963], the Bergman kernel $K(z, z)$ equals $cr(z)^{-n}$. Also,

$$g^{j,\bar k}(z) = r(z)(\delta_{jk} - 2z_j\bar z_k) + 2(\bar z_j - \overline{s(z)}z_j)(z_k - s(z)\bar z_k).$$

Notice that

$$(\log(r(z)^{-n}))_{z_j}(\log(r(z)^{-n}))_{z_{\bar k}} = \frac{4n^2}{r(z)^2}[z_j s(\bar z) - \bar z_j][\bar z_k s(z) - z_k]$$

and

$$|\bar\partial u|_g^2(z) = 4n^2 \sum_{j,k=1}^n [r(z)(\delta_{jk} - 2z_j\bar z_k) + 2(\bar z_j - s(\bar z)z_j)(z_k - s(z)\bar z_k)]\frac{(\bar z_j - s(\bar z)z_j)(z_k - s(z)\bar z_k)}{r(z)^2}$$
$$= \sum_{j,k=1}^n \frac{4n^2}{r(z)}(\delta_{jk} - 2z_j\bar z_k)[\bar z_j - \overline{s(z)}z_j][z_k - s(z)\bar z_k] + \sum_{j,k=1}^n \frac{8n^2(\bar z_j - \overline{s(z)}z_j)^2(z_k - s(z)\bar z_k)^2}{r(z)^2}$$
$$=: F(z) + G(z).$$

Thus,

$$F(z)\frac{r}{4n^2} = \sum_{j=1}^n |z_j|^2 - s\bar z_j^2 - z_j^2\bar s + |s|^2|z_j|^2 - 2\sum_{j,k=1}^n |z_j|^2|z_k|^2 - s|z_j|^2\bar z_k^2 - z_j^2|z_k|^2\bar s + |s|^2 z_j^2\bar z_k^2$$
$$= |z|^2 - 2|s|^2 + |s|^2|z|^2 - 2(|z|^4 - s|z|^2\bar s - s|z|^2\bar s + |s|^2 s\bar s)$$
$$= -2|z|^4 + 5|s|^2|z|^2 - 2|s|^2 + |z|^2 - 2|s|^4$$

and

$$G(z) = \frac{8n^2}{r^2}\Big|\sum_{j=1}^n (z_j - s\bar z_j)^2\Big|^2 = \frac{8n^2}{r^2}\big|s - 2s|z|^2 + s^2\bar s\big|^2 = 8n^2|s|^2.$$

Therefore

$$
\begin{aligned}
|\bar\partial u|_g^2 &= \frac{4n^2}{r}[-2|z|^4+5|s|^2|z|^2-2|s|^2+|z|^2-2|s|^4]+\frac{4n^2}{r}2|s(z)|^2 r(z)\\
&= \frac{4n^2}{r}[-2|z|^4+|z|^2|s|^2+|z|^2]\\
&= 4n^2|z|^2 = 4n^2\,\mathrm{Tr}(zz^*). \qquad \square
\end{aligned}
$$

Example 2 shows that the canonical solution to the equation $\bar\partial u = f := \bar\partial \log K(z)$ (here $\|f\|_{g,\infty}<\infty$) given by $\log K(z) - C_\Omega$ is unbounded with logarithmic growth near the boundary of the polydisc.

Example 2. Consider $f(z) := -\sum_{j=1}^n z_j(1-|z_j|^2)^{-1}\,d\bar z_j$ defined on $\mathbb{D}^n$. Then f is $\bar\partial$-closed, $\|f\|_{g,\infty}\le \frac{1}{2}$ and the canonical solution to $\bar\partial u = f$ on $\mathbb{D}^n$ is

$$
u(z) := \sum_{j=1}^n \log(1-|z_j|^2) - n\int_0^1 \log(1-r)\,dr. \tag{6-1}
$$

Proof. We compute directly that u given by (6-1) satisfies $\bar\partial u = f$, and that

$$
|f(z)|_g^2 = \frac{1}{2}\sum_{j=1}^n \frac{(1-|z_j|^2)^2}{(1-|z_j|^2)^2}|z_j|^2 = \frac{|z|^2}{2}.
$$

To verify that u is canonical, notice that

$$
\begin{aligned}
P_{\mathbb{D}^n}\Big[\sum_{j=1}^n \log(1-|w_j|^2)\Big](z) &= \frac{1}{\pi^n}\int_{\mathbb{D}^n}\prod_{j=1}^n \frac{1}{(1-\langle z_j, w_j\rangle)^2}\sum_{k=1}^n \log(1-|w_k|^2)\,dv_{w_1}\cdots dv_{w_n}\\
&= \sum_{k=1}^n \frac{1}{\pi}\int_{\mathbb{D}^n}\frac{\log(1-|w_k|^2)}{(1-\langle z_k, w_k\rangle)^2}\,dv_{w_k}\\
&= \sum_{k=1}^n 2\int_0^1 \log(1-r_k^2)r_k\,dr_k = n\int_0^1 \log(1-r)\,dr. \qquad \square
\end{aligned}
$$

6B. ***A sharp example.*** The maximum blow-up order for a solution to $\bar\partial u = f$ with $\|f\|_{g,\infty}<\infty$ is $\int_\Omega |K(\cdot, w)|\,dv_w$. Here we provide an example to show that Theorem 3.3 is sharp on the Cartan classical domains.

Proposition 6.1. *Let Ω be a Cartan classical domain. Then there is a constant c such that for each $z\in\Omega$, there is a $\bar\partial$-closed $(0,1)$-form f_z on Ω with $\|f_z\|_{g,\infty}=1$ and the canonical solution to $\bar\partial u = f_z$ satisfies*

$$
|u(z)| \ge c\int_\Omega |K(z,w)|\,dv_w.
$$

Proof. For any point $z\in\Omega$, consider the functions $U_z(\cdot) := K(\cdot)^{-1}K(\cdot, z)$ and

$$
f_z(\cdot) := \bar\partial U_z(\cdot) = K(\cdot, z)\bar\partial(K(\cdot)^{-1}).
$$

Then, by Example 1,

$$\begin{aligned}\|f_z\|_{g,\infty} &= \|K(\cdot\,,z)K(\cdot)^{-2}\bar\partial(K(\cdot))\|_{g,\infty} = \|K(\cdot\,,z)K(\cdot)^{-1}\bar\partial(\log K(\cdot))\|_{g,\infty}\\ &\le \|K(\cdot\,,z)K(\cdot)^{-1}\|_\infty\|\bar\partial(\log K(\cdot))\|_{g,\infty} \le C.\end{aligned}$$

The Bergman projection of U_z is

$$P[U_z](\cdot) = \int_\Omega U_z(w)K(\cdot\,,w)\,dv_w = \int_\Omega K(w)^{-1}K(w,z)K(\cdot\,,w)\,dv_w.$$

In particular, by (5-1) with $\beta=-1$,

$$P[U_z](z) = \int_\Omega K(w)^{-1}K(w,z)K(z,w)\,dv_w = \int_\Omega K(w)^{-1}|K(w,z)|^2\,dv_w \approx \int_\Omega |K(z,w)|\,dv_w.$$

The canonical solution to $\bar\partial u = f$ is $u_z := U_z - P[U_z]$ and

$$|u_z(z)| = |1 - J_{-1,0}(z)| \ge c\int_\Omega |K(z,w)|\,dv_w - 1$$

for a uniform constant $c>0$, independent of z. □

6C. ***Blow-up order greater than*** **log.** With the previous example and Theorem 5.3 we will provide the maximum blow-up order when Ω is a Cartan classical domain of rank 2. By Theorem 5.3, for $z=te_1+te_2$ where $e_1,e_2\in\mathcal{U}$,

$$\int_\Omega |K(z,w)|\,dv_w \approx (1-t)^{-a/2} \approx \delta_\Omega(z)^{-a/2} \quad \text{as } t\to 1^-.$$

When Ω is IV(n) with $n\ge 3$,

$$\int_\Omega |K(z,w)|\,dv_w \approx \delta_\Omega(z)^{-n/2+1}.$$

When Ω is III(4) or III(5),

$$\int_\Omega |K(z,w)|\,dv_w \approx \delta_\Omega(z)^{-2}.$$

When Ω is I($2,n$) with $n\ge 2$,

$$\int_\Omega |K(z,w)|\,dv_w \approx \delta_\Omega(z)^{-1}.$$

When Ω is II(2),

$$\int_\Omega |K(z,w)|\,dv_w \approx \delta_\Omega(z)^{-1/2}.$$

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