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LONG TIME EXISTENCE OF YAMABE FLOW ON SINGULAR SPACES WITH POSITIVE YAMABE CONSTANT

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We establish long-time existence of the normalized Yamabe flow with positive Yamabe constant on a class of manifolds that includes spaces with incomplete cone-edge singularities. We formulate our results axiomatically so that they extend to general stratified spaces as well, provided certain parabolic Schauder estimates hold. The central analytic tool is a parabolic Moser iteration, which yields uniform upper and lower bounds on both the solution and the scalar curvature.

1. Introduction and statement of the main results

The Yamabe conjecture states that for any compact, smooth Riemannian manifold (M, g_0) without boundary there exists a constant scalar curvature metric conformal to g_0. The first proof of this conjecture was initiated by Yamabe [1960] and continued by Trudinger [1968], Aubin [1976] and Schoen [1984]. The proof is based on the calculus of variations and elliptic partial differential equations. An alternative tool for proving the conjecture is due to Hamilton [1989], who utilized the normalized Yamabe flow of a Riemannian manifold (M, g_0), which is a family $g \equiv g(t),\ t \in [0, T]$ of Riemannian metrics on M such that the following evolution equation holds:

$$\partial_t g = -(S-\rho)g, \quad \rho := \mathrm{Vol}_g(M)^{-1} \int_M S \, d\mathrm{Vol}_g. \tag{1-1}$$

Here S is the scalar curvature of g, $\mathrm{Vol}_g(M)$ is the total volume of M with respect to g and ρ is the average scalar curvature of g. The normalization by ρ ensures that the total volume does not change along the flow. Hamilton [1989] introduced the Yamabe flow and also showed its long-time existence. It preserves the conformal class of g_0 and ideally should converge to a constant scalar curvature metric, thereby establishing the Yamabe conjecture by parabolic methods.

Establishing convergence of the normalized Yamabe flow is intricate already in the setting of smooth, compact manifolds. In the case of scalar negative, scalar flat and locally conformally flat scalar positive metrics, convergence is due to Ye [1994]. The case of a nonconformally flat g_0 with positive scalar curvature is delicate and has been studied first by Schwetlick and Struwe [2003] for large energies and later by Brendle [2005; 2007] for arbitrary energies. More specifically, [Schwetlick and Struwe 2003, Section 5] as well as [Brendle 2005, p. 270; 2007, p. 544] invoke the positive mass theorem, which is where the dimensional restriction in [Schwetlick and Struwe 2003; Brendle 2005] and the spin assumption

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in [Brendle 2007, Theorem 4] come from. Assuming [Schoen and Yau 2022] to be correct, [Schwetlick and Struwe 2003; Brendle 2005; 2007] cover all closed manifolds which are not conformally equivalent to spheres.

In the noncompact setting, our understanding is limited. On complete manifolds, long-time existence has been discussed in various settings by Ma [2016], Ma and An [1999] and Schulz [2020]. On incomplete surfaces, where Ricci and Yamabe flows coincide, Giesen and Topping [2010; 2011] constructed a flow that becomes instantaneously complete.

In this work, we study the Yamabe flow on a general class of spaces that includes incomplete spaces with cone-edge (wedge) singularities or, more generally, stratified spaces with iterated cone-edge singularities. This continues a program initiated in [Bahuaud and Vertman 2014; 2019], where existence and convergence of the Yamabe flow has been established in case of negative Yamabe invariant. Here, we study the positive case and, utilizing methods of Akutagawa, Carron and Mazzeo [Akutagawa et al. 2014], we establish long-time existence of the flow under certain mild geometric assumptions. We don't attempt to prove convergence here, in view of [Schwetlick and Struwe 2003; Brendle 2005; 2007] and the fact that we don't have a substitute for the positive mass theorem in the singular setting.

Our main result (see Theorem 5.1 for the precise statement) is as follows.

Theorem 1.1. *Let (M, g_0) be a Riemannian manifold of dimension $n = \dim M \geq 3$ such that the following four assumptions (to be made precise below) hold:*

(1) *The Yamabe constant $Y(M, g_0)$ defined in (1-6) is positive.*

(2) *(M, g_0) is admissible and satisfies a Sobolev inequality as in Definition 1.2.*

(3) *Parabolic Schauder estimates of Definition 1.4 hold on (M, g_0).*

(4) *The initial scalar curvature $S_0 \in C^{1,\alpha}(M)$ as in Assumption 4. Moreover, we also require that $S_0 \in L^{n^2/(2(n-2))}(M)$ and that its negative part $(S_0)_- \in L^\infty(M)$.*

Under these assumptions, a normalized Yamabe flow of g_0 exists within the space of admissible spaces, with infinite existence time.

Examples, where the assumptions of the theorem are satisfied, include spaces with incomplete wedge singularities. More general stratified spaces with iterated cone-edge metrics are also covered, provided parabolic Schauder estimates continue to hold in that setting.

Let us point out two technical novelties of our work.

(1) We prove uniform bounds on the solution and on the scalar curvature along the normalized Yamabe flow without using the maximum principle. We have not found any such argument in the existing literature.

(2) We derive such uniform bounds starting with low initial Sobolev regularity, $S_0 \in H^1(M)$. This low initial regularity forces us to develop very intricate arguments to deal even with the chain rule. We have not seen any such argument in the existing literature on parabolic equations.

We now proceed with explaining the assumptions in detail.

Normalized Yamabe flow and Yamabe constant. Consider a Riemannian manifold (M, g_0), with g_0 normalized such that the total volume $\mathrm{Vol}_{g_0}(M)$ equals 1. The Yamabe flow (1-1) preserves the conformal class of the initial metric g_0 and, assuming $\dim M = n \geq 3$, we can write $g = u^{4/(n-2)} g_0$ for some function $u > 0$ on $M_T = M \times [0, T]$ for some upper time limit $T > 0$. Then the normalized Yamabe flow equation can be equivalently written as an equation for u:

$$\partial_t (u^{(n+2)/(n-2)}) = \tfrac{1}{4}(n+2)(\rho u^{(n+2)/(n-2)} - L_0(u)), \quad L_0 := S_0 - \frac{4(n-1)}{n-2}\Delta_0, \tag{1-2}$$

where L_0 is the conformal Laplacian of g_0, defined in terms of the scalar curvature S_0 and the Laplace Beltrami operator Δ_0 associated to the initial metric g_0. The scalar curvature of the evolving metric g is given by $S = u^{-(n+2)/(n-2)} L_0(u)$, and the volume form of $g = u^{4/(n-2)} g_0$ is given by $d\mathrm{Vol}_g = u^{2n/(n-2)}\, d\mu$, where we write $d\mu := d\mathrm{Vol}_{g_0}$ for the time-independent initial volume form. One computes that

$$\partial_t\, d\mathrm{Vol}_g = -\tfrac{1}{2} n (S - \rho)\, d\mathrm{Vol}_g. \tag{1-3}$$

Hence, the total volume of (M, g) is constant and thus equal to 1 along the flow. The average scalar curvature then takes the form

$$\rho = \int_M S\, d\mathrm{Vol}_g = \int_M L(u) u^{-(n+2)/(n-2)} u^{2n/(n-2)}\, d\mu = \int_M \frac{4(n-1)}{n-2} |\nabla u|^2 + S_0 u^2\, d\mu. \tag{1-4}$$

Explicit computations lead to the following evolution equation for the average scalar curvature:

$$\partial_t \rho = -\frac{n-2}{2} \int_M (\rho - S)^2 u^{2n/(n-2)}\, d\mu. \tag{1-5}$$

The latter evolution equation in particular implies that $\rho \equiv \rho(t)$ is nonincreasing along the flow. We conclude the exposition by defining the Yamabe constant of g_0, which incidentally provides a lower bound for ρ. Let u be a solution of (1-2). We define the $L^q(M)$ spaces with respect to the integration measure $d\mu$.

We define the first Sobolev space $H^1(M)$ as the space of all $v : M \to \mathbb{R}$ such that the first Sobolev norm, defined with respect to $d\mu$ and the pointwise norm associated to g_0, satisfies

$$\|v\|^2_{H^1(M)} := \int_M v^2\, d\mu + \int_M |\nabla v|^2\, d\mu < \infty.$$

Similarly we define $H^1(M, g)$ by using $d\mathrm{Vol}_g$ instead of $d\mu$ and the pointwise norm associated to g. If u and u^{-1} are both bounded, one easily checks that $H^1(M) = H^1(M, g)$.

We define the Yamabe invariant of g_0 as

$$\begin{aligned} Y(M, g_0) &:= \inf_{v \in H^1(M)\setminus\{0\}} \frac{1}{\|v\|^2_{L^{2n/(n-2)}(M)}} \int_M \frac{4(n-1)}{n-2} |\nabla v|^2 + S_0 v^2\, d\mu \\ &\leq \int_M \frac{4(n-1)}{n-2} |\nabla u|^2 + S_0 u^2\, d\mu = \rho \qquad \text{(by (1-4))}, \end{aligned} \tag{1-6}$$

where in the inequality we have used that for any solution u of (1-2), $\|u\|_{L^{2n/(n-2)}(M)} = d\mathrm{Vol}_g(M) \equiv 1$. How one proceeds will depend heavily on the sign of the Yamabe constant. In this paper we will assume that $Y(M, g_0) > 0$. In particular, the average curvature ρ is then positive and uniformly bounded away from zero along the normalized Yamabe flow.

Assumption 1. The Yamabe constant $Y(M, g_0)$ is positive.

A Sobolev inequality and other admissibility assumptions. The Moser iteration arguments in this paper are strongly motivated by the related work in [Akutagawa et al. 2014] on the Yamabe problem on stratified spaces. Thus, similar to that paper, we impose certain admissibility assumptions, which are naturally satisfied by certain compact stratified spaces with iterated cone-edge metrics.

Definition 1.2. Let (M, g_0) be a smooth Riemannian manifold of dimension n. We call (M, g_0) admissible if it satisfies the following conditions:

- (M, g_0) with volume form $d\mu = d\mathrm{Vol}_{g_0}$ has finite volume: $\mathrm{Vol}_{g_0}(M) < \infty$.
- For any $\varepsilon > 0$, there exist finitely many open balls $B_{2R_i}(x_i) \subset M$ such that

$$\mathrm{Vol}_{g_0}\Big(M \backslash \bigcup_i B_{R_i}(x_i)\Big) \leq \varepsilon. \tag{1-7}$$

- Smooth, compactly supported functions $C_c^\infty(M)$ are dense in $H^1(M)$.[1]
- (M, g_0) admits a Sobolev inequality of the following kind: defining $L^q(M)$ spaces with respect to $d\mu$, there exist $A_0, B_0 > 0$ such that for all $f \in H^1(M)$,

$$\|f\|^2_{L^{2n/(n-2)}(M)} \leq A_0 \|\nabla f\|^2_{L^2(M)} + B_0 \|f\|^2_{L^2(M)}. \tag{1-8}$$

The main examples we have in mind are closed manifolds[2] and regular parts of smoothly stratified spaces endowed with iterated cone-edge metrics. See [Akutagawa et al. 2014, Section 2.1] for a definition of the latter. That the Sobolev inequality holds in this case is shown in Proposition 2.2 of the same paper. Note that the list of admissibility assumptions does not contain compactness. Nor do we specify explicitly how the metric g_0 looks near the singular strata of $\overline{M}$, in the case of stratified spaces. Restrictions on the local behavior of the metric will instead be coded in L^q-data, like requiring the initial scalar curvature S_0 to be in $L^q(M)$ for suitable $q > 0$. These requirements are stated in the theorems below, and will vary from statement to statement.

Assumption 2. (M, g_0) is an admissible Riemannian manifold.

In what follows we want to relate the assumption of the Sobolev inequality (1-8) in Definition 1.2 to positivity of the Yamabe constant $Y(M, g_0)$.

Proposition 1.3. *Assume $S_0 \in L^\infty(M)$ and $Y(M, g_0) > 0$. Then (1-8) holds.*

Proof. Indeed, it follows directly from the definition of $Y(M, g_0)$ in (1-6) that

$$\|f\|^2_{L^{2n/(n-2)}(M)} \leq \frac{1}{Y(M, g_0)} \left(\frac{4(n-1)}{n-2} \|\nabla f\|^2_{L^2(M)} + \|S_0\|_{L^\infty(M)} \|f\|^2_{L^2(M)} \right)$$

for all $f \in H^1(M)$. This is indeed the Sobolev inequality (1-8). □

[1] This can be phrased as $H^1_0(M) = H^1(M)$. Note that this rules out M being the interior of a manifold with a codimension 1 boundary.

[2] This includes finite volume, complete manifolds, since any finite volume, complete manifold satisfying the Sobolev inequality is compact; see [Hebey 1996, Lemma 3.2, pp. 18–19 and Remark 2, pp. 56–57].

Parabolic Schauder estimates and short-time existence. Our proof requires intricate arguments involving the heat operator and its mapping properties, as seen in the previous work by the second author jointly with Bahuaud [Bahuaud and Vertman 2014; 2019] in the setting of spaces with incomplete wedge singularities. Here we axiomatize these arguments into a definition of certain parabolic Schauder estimates, having in mind further generalizations to stratified spaces.

Definition 1.4. (M, g_0) satisfies parabolic Schauder estimates if there is a scale of Banach spaces $\{C^{k,\alpha} \equiv C^{k,\alpha}(M \times [0, T])\}_{k\in\mathbb{N}_0}$ of continuous functions on $M \times [0, T]$ with

$$C^{0,\alpha} \supset C^{1,\alpha} \supset C^{2,\alpha} \supset \cdots$$

for some $\alpha \in (0, 1)$ and any $T > 0$, with the following properties:

(1) Algebraic properties of the Banach spaces:

- For any $k \in \mathbb{N}_0$, the constant function 1 exists in $C^{k,\alpha}(M \times [0, T])$.
- For any $k \in \mathbb{N}_0$ and any $u \in C^{k,\alpha}(M \times [0, T])$ uniformly bounded away from zero, we have that the inverse u^{-1} exists in $C^{k,\alpha}(M \times [0, T])$.
- For any $k \geq 2$ and $\ell \leq k$ we have $C^{k,\alpha} \cdot C^{\ell,\alpha} \subseteq C^{\ell,\alpha}$. Writing $\|\cdot\|_{\ell,\alpha}$ for the norm on $C^{\ell,\alpha}$, we have a uniform constant $C_{\ell,\alpha}$ such that for any $u \in C^{k,\alpha}$ and $v \in C^{\ell,\alpha}$,

$$\|u \cdot v\|_{\ell,\alpha} \leq C_{\ell,\alpha}\|u\|_{k,\alpha}\|v\|_{\ell,\alpha}. \tag{1-9}$$

(2) Regularity properties of the Banach spaces:

- We have the inclusions

$$\begin{aligned} C^{0,\alpha}(M \times [0, T]) &\subseteq C^0([0, T], L^2(M)), \\ C^{1,\alpha}(M \times [0, T]) &\subseteq C^0([0, T], H^1(M)), \\ C^{2,\alpha}(M \times [0, T]) &\subseteq L^\infty(M \times [0, T]). \end{aligned} \tag{1-10}$$

  Moreover, for any $u \in C^{0,\alpha}(M \times [0, T])$ and any fixed $p \in M$, the evaluation $u(p, \cdot)$ still lies in $C^{0,\alpha}$. The map $M \ni p \mapsto \|u(p, \cdot)\|_{0,\alpha}$ is again $L^2(M)$.

- If $C^{k,\alpha}([0, T]) \subset C^{k,\alpha}(M \times [0, T])$ consists of functions that are constant on M, then the spaces $C^{2k,\alpha}([0, T])$ are characterized as

$$C^{2k,\alpha}([0, T]) = \{u \in C^{0,\alpha}([0, T]) \mid \partial_t^k u \in C^{0,\alpha}([0, T])\}. \tag{1-11}$$

- For any $k \in \mathbb{N}_0$, the following maps are bounded:

$$\begin{aligned} \partial_t, \Delta_0 : C^{k+2,\alpha}(M \times [0, T]) &\to C^{k,\alpha}(M \times [0, T]), \\ \nabla : C^{k+1,\alpha}(M \times [0, T]) &\to C^{k,\alpha}(M \times [0, T]). \end{aligned} \tag{1-12}$$

(3) Weak maximum principle for elements of the Banach spaces:

- Any $u \in C^{2,\alpha}(M \times [0, T])$ satisfies a weak maximum principle; that is for any Cauchy sequence $\{q_\ell\}_{\ell\in\mathbb{N}} \subset M$ we have

$$\inf_M u = \lim_{\ell\to\infty} u(q_\ell) \implies \lim_{\ell\to\infty} (\Delta_0 u)(q_\ell) \geq 0. \tag{1-13}$$

In the case that the Cauchy sequence $\{q_\ell\}_{\ell\in\mathbb{N}}$ converges to an interior point $p \in M$, where u attains a global minimum, we have that $\Delta_0 u(p) \geq 0$.

(4) Mapping properties of the heat operator:

- The heat operator $e^{t\Delta_0}$ admits the mapping properties

$$
\begin{aligned}
e^{t\Delta_0} &: C^{k,\alpha}(M\times[0,T]) \to C^{k+2,\alpha}(M\times[0,T]),\\
e^{t\Delta_0} &: C^{k,\alpha}(M\times[0,T]) \to t^{\alpha} C^{k+1,\alpha}(M\times[0,T]),\\
e^{t\Delta_0} &: L^{\infty}(M\times[0,T]) \to C^{1,\alpha}(M\times[0,T]).
\end{aligned}
\tag{1-14}
$$

If $e^{t\Delta_0}$ acts without convolution in time, then we have a bounded map

$$e^{t\Delta_0} : C^{k,\alpha}(M) \to C^{k,\alpha}(M\times[0,T]). \tag{1-15}$$

(5) Mapping properties of other solution operators:

- For any positive $a \in C^{1,\alpha}(M\times[0,T])$ uniformly bounded away from zero, there is a solution operator Q for $(\partial_t - a\cdot\Delta_0)u = f,\ u(0) = 0$, with

$$Q : C^{0,\alpha}(M\times[0,T]) \to C^{2,\alpha}(M\times[0,T]). \tag{1-16}$$

  If $a \in C^{2,\alpha}$, then additionally $Q : C^{1,\alpha} \to C^{3,\alpha}$ is bounded.

- For any positive $a \in C^{1,\alpha}(M\times[0,T])$ uniformly bounded away from zero, there is a solution operator R for $(\partial_t - a\cdot\Delta_0)u = 0,\ u(0) = f$, with

$$R : C^{2,\alpha}(M) \to C^{2,\alpha}(M\times[0,T]), \tag{1-17}$$

  where $C^{k,\alpha}(M)$ denotes the subspace of $C^{k,\alpha}(M\times[0,T])$ consisting of time-independent functions. If $a \in C^{2,\alpha}$, then additionally $R : C^{3,\alpha}(M) \to C^{3,\alpha}(M\times[0,T])$ is bounded.

Let us now discuss where such parabolic Schauder estimates hold.

Examples 1.5. (a) Parabolic Schauder estimates hold on smooth compact Riemannian manifolds without boundary by the classical estimates of [Ladyženskaja et al. 1968].

(b) By [Bahuaud and Vertman 2014; 2019], a manifold with a wedge singularity satisfies the parabolic Schauder estimates,[3] assuming that the wedge metric is feasible in the sense of [Bahuaud and Vertman 2019, Definition 2.2]. The proof is based on the microlocal heat kernel description in [Mazzeo and Vertman 2012]. Note that the choice of Banach spaces is not canonical, and instead one can use, for example, the scale of weighted Hölder spaces as in [Vertman 2021].

(c) In view of the recent work by Albin and Gell-Redman [2017], we expect the same parabolic Schauder estimates to hold on general stratified spaces with iterated cone-wedge metrics.

Assumption 3. (M, g_0) satisfies parabolic Schauder estimates.

[3] In fact, in the mapping properties of solution operators Q and R we require here less than in [Bahuaud and Vertman 2019]: in the case $a \in C^{2,\alpha}$ we only ask for $Q : C^{1,\alpha} \to C^{3,\alpha}$ and $R : C^{3,\alpha}(M) \to C^{3,\alpha}$, while in that paper, these additional mapping properties are proved for one order higher.

Using parabolic Schauder estimates, we can prove short time existence and regularity of the renormalized Yamabe flow, exactly as in [Bahuaud and Vertman 2014, Theorem 1.7 and 4.1] and by a slight adaptation of [Bahuaud and Vertman 2019, Proposition 4.8].

Theorem 1.6. *Let (M, g_0) satisfy parabolic Schauder estimates. Assume, moreover, that the scalar curvature S_0 of g_0 lies in $C^{1,\alpha}(M)$. Then the following hold*:

(1) *The Yamabe flow (1-2) admits, for some $T > 0$ sufficiently small, a solution*

$$u \in C^{2,\alpha}(M \times [0, T]) \subseteq C^0([0, T], H^1(M)) \cap L^\infty(M \times [0, T])$$

which is positive and uniformly bounded away from zero.[4]

(2) *If a solution, $u \in C^{2,\alpha}(M \times [0, T])$, to the Yamabe flow (1-2) exists for a given $T > 0$ and is uniformly bounded away from zero, then in fact $u \in C^{3,\alpha}(M \times [0, T])$. In particular, we obtain*

$$S \in C^{1,\alpha}(M \times [0, T]) \subseteq C^0([0, T], H^1(M)).$$

Proof. We shall only provide a brief proof outline. The first statement is proved by setting up a fixed point argument in the Banach space $C^{2,\alpha}(M \times [0, T])$. If $u = 1 + v \in C^{2,\alpha}(M \times [0, T])$ is a solution to (1-2), then v satisfies the equation

$$\partial_t v - (n-1)\Delta_0 v = -\tfrac{1}{4}(n-2)S_0 + \Phi(v), \tag{1-18}$$

where $\Phi : C^{2,\alpha}(M \times [0, T]) \to C^{0,\alpha}(M \times [0, T])$ is a bounded map, in view of the algebraic and regularity properties (1-12) in Definition 1.4. Moreover, Φ is quadratic in its argument, i.e., writing $\|\cdot\|_{k,\alpha}$ for the norm on $C^{k,\alpha}$ for any $k \in \mathbb{N}$, there exists a uniform $C > 0$, such that by (1-9) (compare [Bahuaud and Vertman 2014, Lemma 5.1]), for all $w, w' \in C^{2,\alpha}$,

$$\|\Phi(w)\|_{0,\alpha} \le C\|w\|_{2,\alpha}^2 \quad \text{and} \quad \|\Phi(w) - \Phi(w')\|_{0,\alpha} \le C(\|w\|_{2,\alpha} + \|w'\|_{2,\alpha})\|w - w'\|_{2,\alpha}. \tag{1-19}$$

Now a solution v of (1-18) (and hence also a solution $u = 1 + v$ of (1-2)) is obtained as a fixed point of the map

$$C^{2,\alpha}(M \times [0, T]) \ni v \mapsto e^{t(n-1)\Delta_0}\big(-\tfrac{1}{4}(n-2)S_0 + \Phi(v)\big) \in C^{2,\alpha}(M \times [0, T]), \tag{1-20}$$

which is a contraction mapping on a subset of $C^{2,\alpha}(M \times [0, T])$ for $T > 0$ sufficiently small,[5] by (1-14) in Definition 1.4. One argues exactly as in [Bahuaud and Vertman 2014, Theorem 4.1]. Note that the regularity of the scalar curvature S along the flow is then $S \in C^{0,\alpha}(M \times [0, T])$.

Note also that the fixed point argument is performed in a small ball around zero in $C^{2,\alpha}(M \times [0, T])$, and thus for $T > 0$ sufficiently small, the norm of v is small. Hence $u = 1 + v$ is positive and bounded away from zero.

The second statement improves the regularity of S. By the regularity properties (1-10) in Definition 1.4, we conclude that $\rho, \partial_t \rho \in C^{0,\alpha}([0, T])$. By (1-11), this implies that $\rho \in C^{2,\alpha}([0, T])$. We can now apply

[4] Later on, we will prove uniform lower bounds on u for any finite $T > 0$.

[5] We need to assume that $T > 0$ is sufficiently small in order to control $e^{t(n-1)\Delta_0}(S_0)$.

the mapping properties (1-16) and (1-17)[6] in Definition 1.4 to obtain a solution $u' \in C^{3,\alpha}(M \times [0, T])$ with

$$\partial_t u' - (n-1)u^{-4/(n-2)} \Delta_0 u' = \tfrac{1}{4}(n-2)(\rho u - S_0 u^{(n-6)/(n-2)}), \quad u'(0) = 1. \tag{1-21}$$

The given solution $u \in C^{2,\alpha}$ satisfies the same equation, and we can prove that $u \equiv u'$ by the weak maximum property (1-13) of elements in $C^{2,\alpha}$. Thus, indeed, $u \in C^{3,\alpha}$ and hence $S \in C^{1,\alpha}$. This is basically the argument used in [Bahuaud and Vertman 2019, Proposition 4.8]. □

Remark 1.7. If we assume $Q : C^{2,\alpha} \to C^{4,\alpha}$ and $R : C^{4,\alpha}(M) \to C^{4,\alpha}$ in Definition 1.4, as has been proved in [Bahuaud and Vertman 2019], then the condition $S_0 \in C^{2,\alpha}(M)$ implies, by similar arguments as in Theorem 1.6, that any solution u in $C^{2,\alpha}$ is actually in $C^{4,\alpha}$. This would lead to $S \in C^{2,\alpha}$, in particular, the scalar curvature would stay bounded along the flow. Here, we decided to require less in Definition 1.4, assume less regularity for S_0 and conclude boundedness of S by Moser iteration methods instead.

Regularity of the initial scalar curvature. In view of Theorem 1.6, we arrive at our final assumption on a regularity of the initial scalar curvature S_0 with respect to the scale of Banach spaces in Definition 1.4.

Assumption 4. Assuming that (M, g_0) satisfies parabolic Schauder estimates, we also ask that the initial scalar curvature S_0 be an element of $C^{1,\alpha}(M)$.

In view of Theorem 1.6, this implies that $S \in H^1(M)$. Moreover, since $u \in C^{2,\alpha}(M \times [0, T]) \subset L^\infty$ for $T > 0$ sufficiently small, norms on the Sobolev space $H^1(M)$ with respect to g_0 and norms on the Sobolev space $H^1(M, g)$ with respect to $g = u^{4/(n-2)} g_0$ are equivalent. Thus $S(t)$ lies in the Sobolev space $H^1(M, g(t))$ for any $t \in [0, T]$, which we abbreviate as

$$S \in H^1(M, g). \tag{1-22}$$

Our arguments below will use (1-22) to show that given $S_0 \in L^q(M)$ for

$$q = \frac{n^2}{2(n-2)} = \frac{n}{2} + \frac{n}{n-2} > \frac{n}{2},$$

we may conclude by Moser iteration that $S \in L^\infty(M)$ for positive times. We close this subsection with the observation that on stratified spaces, $S_0 \in L^q(M)$ for $q > \frac{1}{2}n$ and $S_0 \in L^\infty(M)$ basically carry the same geometric restriction. Indeed, consider a cone $(0, 1) \times N$ over a Riemannian manifold (N, g_0), with metric $g_0 = dx^2 \oplus x^2 g_N + h$, where h is smooth in $x \in [0, 1]$ and $|h|_{\bar g} = O(x)$ as $x \to 0$, and where we write $\bar g := dx^2 \oplus x^2 g_N$. Then

$$S_0 \sim \frac{\operatorname{scal}(g_N) - \dim N(\dim N - 1)}{x^2} + O(x^{-1}) \quad \text{as } x \to 0, \tag{1-23}$$

where the higher order term $O(x^{-1})$ comes from the perturbation h. We see that both of the assumptions $S_0 \in L^\infty(M)$ and $S_0 \in L^q(M)$ for $q > \frac{1}{2}n$ imply that the leading term of the metric g_0 is scalar-flat, i.e., $\operatorname{scal}(g_N) = \dim N(\dim N - 1)$.

[6]Here we use the assumption that u is uniformly bounded away from zero and that $1 \in C^{3,\alpha}$ by the algebraic properties of the Banach spaces.

The overarching strategy. Studies of the Yamabe flow usually follow the very rough pattern that we outline here. One first argues that (1-2) has a short-time solution. This is the step we have been concerned with in this section. This step doesn't invoke the sign of the Yamabe constant.

The next step is to show that the flow can be extended to all times. The way one does this is to assume the flow is defined for $t \in (0, T)$ for some maximal time $T < \infty$ and then derive a priori bounds on the solution u and the scalar curvature S, showing that neither of them develop singularities as $t \to T$. One can thus keep flowing past T, establishing long-time existence. This is the step we address in the rest of the paper.

2. The evolution of the scalar curvature and lower bounds

In this section we derive a lower bound on the scalar curvature S along the normalized Yamabe flow. We present an argument that requires neither the maximum principle nor the full set of assumptions in Theorem 1.1, but rather the following assumptions (provided the flow exists):

$$
\begin{aligned}
&\bullet\ S \in H^1(M, g) \text{ along normalized Yamabe flow,}\\
&\bullet\ H^1(M) \text{ and } H^1(M, g) \text{ have equivalent norms,}\\
&\bullet\ C_c^\infty(M) \text{ is dense in } H^1(M),\\
&\bullet\ Y(M, g_0) > 0.
\end{aligned}
\tag{2-1}
$$

These properties clearly follow from Assumptions 1, 2, 3 and 4.

Lemma 2.1. *Let $g = u^{4/(n-2)} g_0$ be a family of metrics evolving according to the normalized Yamabe flow equation (1-2) satisfying (2-1). Then S evolves according to*

$$\partial_t S - (n-1)\Delta S = S(S - \rho), \tag{2-2}$$

where Δ denotes the Laplacian with respect to the time-evolving metric g. We write $S_+ := \max\{S, 0\}$ and $S_- := -\min\{S, 0\}$. Then $S_\pm$ are elements of $H^1(M, g)$ and satisfy

$$\partial_t S_+ - (n-1)\Delta S_+ \leq S_+(S_+ - \rho), \tag{2-3}$$

$$\partial_t S_- - (n-1)\Delta S_- \leq -S_-(S_- + \rho). \tag{2-4}$$

Remark 2.2. Equation (2-2) is to be understood in the weak sense: for any compactly supported smooth test function $\phi \in C_c^\infty(M)$ we have

$$\int_M \partial_t S \cdot \phi \, d\mathrm{Vol}_g + (n-1) \int_M (\nabla S, \nabla \phi)_g \, d\mathrm{Vol}_g = \int_M S(S - \rho) \cdot \phi \, d\mathrm{Vol}_g.$$

Similarly for the partial differential inequalities (2-3) and (2-4) and $\phi \geq 0$, we have

$$\int_M \partial_t S_\pm \cdot \phi \, d\mathrm{Vol}_g + (n-1) \int_M (\nabla S_\pm, \nabla \phi)_g \, d\mathrm{Vol}_g \leq \pm \int_M S_\pm(S_\pm \mp \rho) \cdot \phi \, d\mathrm{Vol}_g.$$

By (2-1), $C_c^\infty(M)$ is dense in $H^1(M) = H^1(M, g)$. Hence we might as well assume $\phi \in H^1(M, g)$ in the weak formulation above.

Proof. Equation (2-2) is well known and can be deduced as follows. Write

$$L_g := S - 4\frac{n-1}{n-2}\Delta$$

for the conformal Laplacian of the metric g. We write $L_0 \equiv L_{g_0}$. If g and g_0 are related by $g = u^{4/(n-2)} g_0$, then L_g and L_0 are related by

$$L_g(\cdot) = u^{-(n+2)/(n-2)} L_0(u\,\cdot).$$

In particular, $S = L_g(1) = u^{-(n+2)/(n-2)} L_0(u)$. We differentiate this equation weakly in time and use (1-2) to replace $\partial_t u = -\frac{1}{4}(n-2)(S-\rho)u$ and get

$$\partial_t S = \tfrac{1}{4}(n+2)(S-\rho)u^{-(n+2)/(n-2)} L_0(u) - \tfrac{1}{4}(n-2)u^{-(n+2)/(n-2)} L_0((S-\rho)u).$$

Applying the transformation rule for L we may rewrite this as

$$\begin{aligned}\partial_t S &= \tfrac{1}{4}(n+2)(S-\rho)L_g(1) - \tfrac{1}{4}(n-2)L_g(S-\rho)\\ &= \tfrac{1}{4}(n+2)(S-\rho)S + (n-1)\Delta S - \tfrac{1}{4}(n-2)(S-\rho)S.\end{aligned}$$

This proves formula (2-2). In order to derive the differential inequality for S_+, consider any $\varepsilon > 0$ and define

$$\psi_\varepsilon(x) := \begin{cases}\sqrt{x^2+\varepsilon^2} - \varepsilon, & x \geq 0,\\ 0, & x < 0.\end{cases}$$

For $v \in H^1(M, g)$ it is readily checked that $\psi_\varepsilon(v) \in H^1(M, g)$ and $\lim_{\varepsilon\to 0} \psi_\varepsilon(v) = v_+$. Furthermore, in the case $x > 0$, we compute the derivatives

$$\psi_\varepsilon'(x) = \frac{x}{\sqrt{x^2+\varepsilon^2}}, \qquad \psi_\varepsilon''(x) = \frac{\varepsilon^2}{(x^2+\varepsilon^2)^{3/2}}.$$

These are both bounded for a fixed $\varepsilon > 0$, so the chain rule applies. Next up we claim, for any $v \in H^1(M, g)$, in the weak sense

$$\Delta\psi_\varepsilon(v) \geq \frac{v}{\sqrt{v^2+\varepsilon^2}}\,\Delta v \equiv \psi_\varepsilon'(v)\,\Delta v. \tag{2-5}$$

This is seen as follows. Let $0 \leq \xi \in C_c^\infty(M)$ be arbitrary and compute

$$\begin{aligned}\int_M \xi\,\Delta\psi_\varepsilon(v)\,d\mathrm{Vol}_g &:= -\int_M (\nabla\xi, \nabla\psi_\varepsilon(v))_g\,d\mathrm{Vol}_g = -\int_M \frac{v}{\sqrt{v^2+\varepsilon^2}}(\nabla\xi, \nabla v)_g\,d\mathrm{Vol}_g\\ &= -\int_M \left(\nabla v, \nabla\left(\frac{v}{\sqrt{v^2+\varepsilon^2}}\xi\right)\right)_g d\mathrm{Vol}_g + \int_M \frac{\xi\varepsilon^2|\nabla v|_g^2}{(v^2+\varepsilon^2)^{3/2}}\,d\mathrm{Vol}_g\\ &\geq -\int_M \left(\nabla v, \nabla\left(\frac{v}{\sqrt{v^2+\varepsilon^2}}\xi\right)\right)_g d\mathrm{Vol}_g\\ &=: \int_M \xi\frac{v}{\sqrt{v^2+\varepsilon^2}}\Delta v\,d\mathrm{Vol}_g.\end{aligned}$$

This proves (2-5), which allows us to deduce that

$$\begin{aligned}\partial_t\psi_\varepsilon(S)-(n-1)\Delta\psi_\varepsilon(S) &\le \begin{cases}\psi_\varepsilon'(S)(\partial_t S-(n-1)\Delta S), & S\ge 0,\\ 0, & S<0,\end{cases}\\ &= \begin{cases}\psi_\varepsilon'(S)S(S-\rho), & S\ge 0,\\ 0, & S<0,\end{cases} \qquad \text{(by (2-2))}\\ &= \frac{S}{\sqrt{S^2+\varepsilon^2}}S_+(S_+-\rho).\end{aligned}$$

Letting $\varepsilon\to 0$ results in (2-3). To prove (2-4), observe that $S_-=S_+-S$. Hence

$$\begin{aligned}\partial_t S_- -(n-1)\Delta S_- &= \partial_t S_+ -(n-1)\Delta S_+ -(\partial_t S-(n-1)\Delta S)\\ &\le S_+(S-\rho)-S(S-\rho)=S_-(S-\rho),\end{aligned}$$

where we have used (2-2) and (2-3) in the inequality step. The only thing which remains to be observed is that $S_-\cdot S=S_-(S_+-S_-)=-S_-^2$. □

We can now derive lower bounds for S by studying the evolution (in-)equalities above. This is usually done by invoking the weak maximum principle for S, which is not available under the assumptions (2-1). Thus, we provide an alternative novel argument, which does not use a maximum principle and which we could not find elsewhere in the literature.

Proposition 2.3. *Let $g=u^{4/(n-2)}g_0$ be a family of metrics evolving according to the normalized Yamabe flow equation (1-2) satisfying (2-1). Then*

$$\|S_-\|_{L^p(M,g)}(t)\le e^{tn\rho(0)/(2p)}\|(S_0)_-\|_{L^p(M)}$$

for all $2\le p\le\infty$. In particular, if $(S_0)_-\in L^\infty(M)$, then $S_-\in L^\infty$ on $[0,T]$ with uniform bounds depending only on T and S_0. Moreover, if $S_0\ge 0$, then $S\ge 0$ along the normalized Yamabe flow for all time.

Proof. The weak formulation of (2-4) is that for any $0\le\xi\in H^1(M,g)$,

$$\int_M \xi\,\partial_t S_-\,d\mathrm{Vol}_g+(n-1)\int_M(\nabla S_-,\nabla\xi)_g\,d\mathrm{Vol}_g\le-\int\xi S_-(S_-+\rho)\,d\mathrm{Vol}_g. \tag{2-6}$$

A problem when manipulating this is that the chain rule fails to hold in general, so we use the same workaround as [Akutagawa et al. 2014, pp. 10–13] (who in turn are following [Gursky 1993, pp. 349–352]). Let $L>0,\ \beta\ge 1$ and define

$$\phi_{\beta,L}(x):=\begin{cases}x^\beta, & x\le L,\\ \beta L^{\beta-1}(x-L)+L^\beta, & x>L,\end{cases} \tag{2-7}$$

$$G_{\beta,L}(x):=\int_0^x\phi_{\beta,L}'(y)^2\,dy=\begin{cases}\dfrac{\beta^2}{2\beta-1}x^{2\beta-1}, & x\le L,\\ \beta^2L^{2(\beta-1)}x-\dfrac{2\beta^2L^{2\beta-1}(\beta-1)}{2\beta-1}, & x>L.\end{cases} \tag{2-8}$$

Finally, we define $H_{\beta,L}(x) := \int_0^x G_{\beta,L}(y)\,dy$ and conclude that

$$H_{\beta,L}(x) = \begin{cases} \dfrac{\beta x^{2\beta}}{2(2\beta-1)}, & x \le L, \\ \dfrac{\beta^2 L^{2(\beta-1)}}{2}(x^2-L^2) - \dfrac{2\beta^2 L^{2\beta-1}(\beta-1)}{2\beta-1}(x-L) + \dfrac{\beta L^{2\beta}}{2(2\beta-1)}, & x > L. \end{cases}$$

The crucial features of these definitions are

$$\phi_{\beta,L}(x) \xrightarrow{L\to\infty} x^\beta, \quad G_{\beta,L}(x) \xrightarrow{L\to\infty} \frac{\beta^2}{2\beta-1}x^{2\beta-1}, \quad H_{\beta,L}(x) \xrightarrow{L\to\infty} \frac{\beta}{2(2\beta-1)}x^{2\beta}.$$

These functions are also dominated by simpler expressions. For instance, $H_{\beta,L}(x) \le \beta^2 x^{2\beta}$ holds for all $L > 0$ and $\beta \ge 1$ as follows: For $x \le L$, there is nothing to show. For $x > L$, we first observe that

$$H_{\beta,L}(x) = \frac{\beta^2}{2}L^{2(\beta-1)}x^2 - \frac{2\beta^2(\beta-1)}{2\beta-1}L^{2\beta-1}x + \frac{\beta(\beta-1)}{2}L^{2\beta}.$$

Dropping the nonpositive middle term and estimating by $x \ge L$, we find

$$H_{\beta,L}(x) \le \frac{\beta^2}{2}x^{2\beta} + \frac{\beta(\beta-1)}{2}x^{2\beta} < \beta^2 x^{2\beta}.$$

Another important property is that $\phi_{\beta,L} \in C^1(\mathbb{R}_+)$, with $\phi'_{\beta,L} \in L^\infty(\mathbb{R}_+)$ for all $L > 0$, and so we may apply the chain rule to $\phi_{\beta,L}(S_-)$. Finally, since we are assuming a C^1 time-dependence, we have $\partial_t H_{\beta,L}(S_-) = (\partial_t S_-)G_{\beta,L}(S_-)$. We will use $\xi := G_{\beta,L}(S_-)$ as a test function in (2-6). Note that by definition, $G_{\beta,L}(x)$ is linear for $x > L$ and hence $G_{\beta,L}(f) \in H^1(M,g)$ whenever $f \in H^1(M,g)$ (here we are also using that $\mathrm{Vol}(M) < \infty$). Then (2-6) implies

$$\int_M \partial_t H_{\beta,L}(S_-)\,d\mathrm{Vol}_g \le -(n-1)\int_M |\nabla\phi_{\beta,L}(S_-)|_g^2\,d\mathrm{Vol}_g - \int_M G_{\beta,L}(S_-)S_-(S_-+\rho)\,d\mathrm{Vol}_g. \tag{2-9}$$

We then use (1-3) to conclude

$$\begin{aligned} \int_M \partial_t H_{\beta,L}(S_-)\,d\mathrm{Vol}_g &= \partial_t \int_M H_{\beta,L}(S_-)\,d\mathrm{Vol}_g + \frac{n}{2}\int_M H_{\beta,L}(S_-)(S-\rho)\,d\mathrm{Vol}_g \\ &= \partial_t \int_M H_{\beta,L}(S_-)\,d\mathrm{Vol}_g - \frac{n}{2}\int_M H_{\beta,L}(S_-)(S_-+\rho)\,d\mathrm{Vol}_g, \end{aligned} \tag{2-10}$$

where the last step uses

$$SH_{\beta,L}(S_-) \equiv (S_+ - S_-)H_{\beta,L}(S_-) = -S_- H_{\beta,L}(S_-).$$

Finally, we need a Sobolev inequality given to us by the positivity of the Yamabe constant, namely for any $f \in H^1(M,g)$ we have by the definition of $Y(M,g_0)$ (note that $Y(M,g_0) = Y(M,g)$ by conformal invariance) that

$$Y(M,g_0)\|f\|^2_{L^{2n/(n-2)}(M,g)} \le 4\frac{n-1}{n-2}\|\nabla f\|^2_{L^2(M,g)} + \int_M Sf^2\,d\mathrm{Vol}_g. \tag{2-11}$$

We set $f=\phi_{\beta,L}(S_-)$. Observe that $\phi_{\beta,L}(S_-)^2S=-\phi_{\beta,L}(S_-)^2S_-$. Then (2-11) implies

$$\begin{aligned}(n-1)\|\nabla\phi_{\beta,L}(S_-)\|^2_{L^2(M,g)} &\geq \frac{n-2}{4}Y(M,g_0)\|\phi_{\beta,L}(S_-)\|^2_{L^{2n/(n-2)}(M,g)}+\frac{n-2}{4}\int_M \phi_{\beta,L}(S_-)^2S_-\,d\mathrm{Vol}_g\\ &\geq \frac{n-2}{4}\int_M \phi_{\beta,L}(S_-)^2S_-\,d\mathrm{Vol}_g. \end{aligned}\tag{2-12}$$

Combining (2-9), (2-10) and (2-12) yields

$$\begin{aligned}\partial_t\int_M H_{\beta,L}(S_-)\,d\mathrm{Vol}_g \leq \int_M\Big(\frac{n}{2}H_{\beta,L}(S_-)-G_{\beta,L}(S_-)S_- - \frac{n-2}{4}\phi_{\beta,L}(S_-)^2\Big)S_-\,d\mathrm{Vol}_g\\ +\int_M\rho\Big(\frac{n}{2}H_{\beta,L}(S_-)-G_{\beta,L}(S_-)S_-\Big)\,d\mathrm{Vol}_g.\end{aligned}$$

We claim the first group of terms on the right-hand side is nonpositive, which follows by a direct computation. We begin by noting that

$$\begin{aligned}&\tfrac{1}{2}nH_{\beta,L}(x)-xG_{\beta,L}(x)-\tfrac{1}{4}(n-2)\phi_{\beta,L}(x)^2\\ &\quad=\begin{cases}\dfrac{(-1)}{4(2\beta-1)}((4\beta+n)(\beta-1)+2)x^{2\beta}, & x\leq L,\\[2ex] \dfrac{L^{2\beta}}{4}\Big(-2\beta^2\Big(\dfrac{x}{L}\Big)^2-\dfrac{2(n-2)\beta(\beta-1)}{2\beta-1}\Big(\dfrac{x}{L}\Big)+(\beta-1)(n+2(\beta-1))\Big), & x>L.\end{cases}\end{aligned}$$

In both cases one checks that the expressions are nonpositive[7] for $\beta\geq 1$. Hence using that $G_{\beta,L}(S_-)\geq 0$ and ρ is nonincreasing by (1-5), we conclude

$$\partial_t\int_M H_{\beta,L}(S_-)\,d\mathrm{Vol}_g\leq\int_M\frac{n\rho}{2}H_{\beta,L}(S_-)\,d\mathrm{Vol}_g\leq\frac{n\rho(0)}{2}\int_M H_{\beta,L}(S_-)\,d\mathrm{Vol}_g.$$

Integrating shows

$$\int_M H_{\beta,L}(S_-)\,d\mathrm{Vol}_g(t)\leq e^{tn\rho(0)/2}\int_M H_{\beta,L}(S_-)\,d\mathrm{Vol}_g(t=0).$$

The conclusion will follow when we take the limit $L\to\infty$, which we can do for the following reason.[8] On the left-hand side we appeal to Fatou's lemma and the pointwise convergence of $H_{\beta,L}$ to find

$$\liminf_{L\to\infty}\int_M H_{\beta,L}(S_-)\,d\mathrm{Vol}_g\geq\int_M\liminf_{L\to\infty}H_{\beta,L}(S_-)\,d\mathrm{Vol}_g=\frac{\beta}{2(2\beta-1)}\int_M S_-^{2\beta}\,d\mathrm{Vol}_g.$$

The right-hand side we deal with by the dominated convergence theorem. We showed on page 488 that $H_{\beta,L}(x)\leq\beta^2x^{2\beta}$ holds for all $L>0$ and $\beta\geq 1$. Since we are assuming $(S_0)_-\in L^\infty(M)$, we can use $\beta^2((S_0)_-)^{2\beta}$ as a dominating integrable function to deduce

$$\liminf_{L\to\infty}\int_M H_{\beta,L}((S_0)_-)\,d\mu=\lim_{L\to\infty}\int_M H_{\beta,L}((S_0)_-)\,d\mu=\frac{\beta}{2(2\beta-1)}\int_M((S_0)_-)^{2\beta}\,d\mu.$$

[7]For the $x\geq L$ case observe that the polynomial is negative for $x=L$, and the expression for $x>L$ clearly has a negative derivative. So the expression remains negative for $x>L$.

[8]This argument is applied several times, without writing out the details in the latter instances.

Combined we have for $\beta \geq 1$,

$$\int_M S_-^{2\beta} \, d\mathrm{Vol}_g \leq e^{tn\rho(0)/2} \int_M (S_0)_-^{2\beta} \, d\mu.$$

This gives the conclusion when writing $2\beta = p$. □

Remark 2.4. Let us again emphasize the novelty of this argument: it circumvents the maximum principle, and one only needs to know that $S \in H^1(M, g)$, as assumed in (2-1).

For completeness, let us also provide the classical and widely known argument (see [Brendle 2005]), using the weak maximum principle: we assume S satisfies (1-13), which is the case if $S \in C^{2,\alpha}(M \times [0, T])$. See Remark 1.7 for conditions which ensure this regularity of S along the flow.

Proposition 2.5. *Assume that $S \in C^0(M \times [0, T])$ satisfies the weak maximum principle (1-13) and that $Y(M, g_0) > 0$. Then S admits a uniform lower bound*

$$S \geq \min \left\{0, \inf_M S_0\right\}.$$

Proof. By the weak maximum principle, we have, for $S_{\min} := \inf_M S$,

$$\partial_t S_{\min} \geq S_{\min}(S_{\min} - \rho).$$

If $S_{\min}$ is negative for all times, then the right-hand side becomes positive, and we get $S_{\min} \geq \inf_M S_0$. If $S_{\min}$ is positive for all times, we can further estimate the right-hand side using $\rho \leq \rho(0)$; see (1-5). Dividing, we then get

$$\frac{\partial_t S_{\min}}{S_{\min}(\rho(0) - S_{\min})} \geq -1.$$

Integrating this differential inequality, we find

$$S_{\min}(t) \geq \frac{\rho(0)(S_0)_{\min}}{e^{\rho(0)t}(\rho(0) - (S_0)_{\min}) + (S_0)_{\min}} \geq 0.$$

If $S_{\min}$ changes sign along the flow, the statement follows by a combination of both estimates. □

3. Uniform bounds on the solution along the flow

The arguments of this section employ the assumptions

$$\begin{aligned} &\bullet\ (M, g_0) \text{ is an admissible manifold,} \\ &\bullet\ H^1(M) \text{ and } H^1(M, g) \text{ have equivalent norms,} \\ &\bullet\ u \in C^0([0, T], H^1(M)) \text{ and } S \in H^1(M, g), \\ &\bullet\ Y(M, g_0) > 0, \end{aligned} \tag{3-1}$$

provided the flow exists. These properties follow from Assumptions 1, 2, 3 and 4.

We begin with the upper bound on u, which follows easily from the lower bound on the scalar curvature S, obtained in Proposition 2.3.

Proposition 3.1. *Let $g = u^{4/(n-2)} g_0$ be a family of metrics, $u > 0$, such that (3-1) holds and the normalized Yamabe flow equation (1-2) holds weakly, with $u(0) = 1$. Assume furthermore that $(S_0)_- \in L^\infty(M)$, where S_0 is the scalar curvature of g_0. Then there exists some uniform constant $0 < C(T) < \infty$, depending only on $T > 0$ and S_0, such that $u \le C(T)$ for all $t \in [0, T]$ with $T < \infty$.*

Proof. We have by (1-1) and (1-5) that

$$\partial_t u = -\tfrac{1}{4}(n-2)(S-\rho)u \le \tfrac{1}{4}(n-2)(S_- + \rho)u \le \tfrac{1}{4}(n-2)(S_- + \rho(0))u.$$

By Proposition 2.3 we have $\|S_-\|_{L^\infty(M)} \le \|(S_0)_-\|_{L^\infty(M)}$, and hence setting

$$C := \tfrac{1}{4}(n-2)(\|(S_0)_-\|_{L^\infty(M)} + \rho(0)),$$

we conclude

$$\partial_t u \le Cu \implies u \le e^{CT} u_0 = e^{CT}. \qquad \square$$

The lower bound is more intricate and in many ways more interesting. The argument will rely on the upper bound on u and the lower bound on S. The proof will be a mixture and modification of the methods in [Akutagawa et al. 2014, pp. 20–21; Brendle 2005, pp. 221–222].

Theorem 3.2. *Let $g = u^{4/(n-2)} g_0$ be a family of metrics, $u > 0$, such that (3-1) holds and the normalized Yamabe flow equation (1-2) holds weakly, with $u(0) = 1$. Assume furthermore that $(S_0)_- \in L^\infty(M)$ and that $S_0 \in L^q(M)$ for some $q > \frac{1}{2}n$. Then there exists some uniform constant $c(T) > 0$, depending only on $T > 0$ and S_0, such that $c(T) \le u$ for all $t \in [0, T]$.*

Proof. By combining (1-2) and (1-1) we may eliminate the term $\partial_t u$ and get

$$-4\frac{n-1}{n-2}\Delta_0 u = (u^{4/(n-2)} S - S_0)u.$$

Using that $(S_0)_- \in L^\infty(M)$ and $u \in L^\infty(M \times [0, T])$, by Proposition 3.1 we may define

$$P := \frac{n-2}{4(n-1)}(S_0 + \|u\|^{4/(n-2)}_{L^\infty(M_T)} \|(S_0)_-\|_{L^\infty(M)}) \in L^q(M).$$

Note that P only depends on S_0 and T. Furthermore, Proposition 2.3 yields

$$(-\Delta_0 + P)u \ge 0. \tag{3-2}$$

Let us explain the proof idea. Assume we can show that there is some $\delta > 0$ such that $u^{-\delta} \in H^1(M)$ uniformly in $t \in [0, T]$. Then (3-2) implies

$$\begin{aligned}(-\Delta_0 - \delta P)u^{-\delta} &= \delta u^{-1-\delta}\Delta_0 u - \delta(1+\delta)u^{-2-\delta}|\nabla u|^2 - \delta P u^{-\delta} \\ &= -\delta u^{-1-\delta}(-\Delta_0 + P)u - \delta(1+\delta)u^{-2-\delta}|\nabla u|^2 \le 0. \end{aligned} \tag{3-3}$$

This is precisely the setting of [Akutagawa et al. 2014, Proposition 1.8], which then concludes by Moser iteration and Sobolev inequality (1-8) that

$$\|u^{-\delta}\|_{L^\infty(M)} \le C\|u^{-\delta}\|_{H^1(M)},$$

where the constant $C > 0$ depends on δP, hence only on T and S_0, but not on t. Under our temporary assumption (3-2), we thus get a uniform bound on $u^{-\delta}$, which gives a uniform lower bound on u.

Hence we only need to show that $u^{-\delta} \in H^1(M)$ uniformly. Let $\varepsilon, \delta > 0$ and (following [Akutagawa et al. 2014, pp. 20–21]) define the functions $\psi_\varepsilon(u) := (u+\varepsilon)^{-\delta}$ and $\phi_\varepsilon(u) := (u+\varepsilon)^{-1-2\delta}$. These are both in $H^1(M)$ since $u \in H^1(M)$. Using ϕ_ε as a test function in the weak formulation of (3-2), we deduce

$$-\frac{1+2\delta}{\delta^2}\|\nabla\psi_\varepsilon(u)\|^2_{L^2(M)} + \int_M Pu\phi_\varepsilon(u)\,d\mu \geq 0,$$

and, using that $u\phi_\varepsilon(u) \leq \psi_\varepsilon(u)^2$ along with the Hölder inequality, we find

$$\|\nabla\psi_\varepsilon(u)\|^2_{L^2(M)} \leq \frac{\delta^2}{1+2\delta}\|P\|_{L^q(M)}\|\psi_\varepsilon(u)^2\|_{L^{q/(q-1)}(M)}. \tag{3-4}$$

Since $q > \frac{1}{2}n$, we have $q/(q-1) < n/(n-2)$ and thus $\|\psi_\varepsilon(u)^2\|_{L^{q/(q-1)}(M)} \leq \|\psi_\varepsilon(u)\|^2_{L^{2n/(n-2)}(M)}$. By the Sobolev inequality (1-8) we know

$$\|\psi_\varepsilon(u)\|^2_{L^{2n/(n-2)}(M)} \leq A_0\|\nabla\psi_\varepsilon(u)\|^2_{L^2(M)} + B_0\|\psi_\varepsilon(u)\|^2_{L^2(M)}. \tag{3-5}$$

Next we need a Poincaré inequality. Let $B \subset M$ be a ball. Then, exactly as in [Akutagawa et al. 2014, Lemma 1.14], there exists a constant $C_B > 0$ such that

$$\|f\|^2_{L^2(M)} \leq C_B(\|\nabla f\|^2_{L^2(M)} + \|f\|^2_{L^2(B)}) \tag{3-6}$$

holds for all $f \in H^1(M)$. Plugging (3-5) and (3-6) into (3-4) yields

$$\|\nabla\psi_\varepsilon(u)\|^2_{L^2(M)} \leq \frac{\delta^2}{1+2\delta}\|P\|_{L^q(M)}((A_0+B_0C_B)\|\nabla\psi_\varepsilon(u)\|^2_{L^2(M)} + B_0C_B\|\psi_\varepsilon(u)\|^2_{L^2(B)}),$$

which is equivalent to

$$\left(1 - \frac{\delta^2}{1+2\delta}\|P\|_{L^q(M)}(A_0+B_0C_B)\right)\|\nabla\psi_\varepsilon(u)\|^2_{L^2(M)} \leq \frac{\delta^2}{1+2\delta}\|P\|_{L^q(M)}B_0C_B\|\psi_\varepsilon(u)\|^2_{L^2(B)}.$$

Choosing $\delta > 0$ small enough so that the left-hand side becomes positive, we get a uniform (meaning now both t- and ε-independent) bound on $\|\nabla\psi_\varepsilon(u)\|_{L^2(M)}$ if we can get a uniform bound on $\|\psi_\varepsilon(u)\|_{L^2(B)}$. The uniform bound on $\|\psi_\varepsilon(u)\|_{L^2(B)}$ will come from the local theory for elliptic supersolutions. Observe that since u satisfies (3-2), we have $u^{2n/(n-2)}$ satisfies (by the same computation as in (3-3))

$$-\Delta_0 u^{2n/(n-2)} + \frac{2n}{n-2}Pu^{2n/(n-2)} \geq 0.$$

Let $R > 0$ be such that $B_{4R}(x) \subset M$ for some $x \in M$. Then, according to [Gilbarg and Trudinger 1983, Theorem 8.18, p. 194], the following weak Harnack inequality holds on $B_{2R}(x)$; namely there is a constant $C > 0$ independent of u but depending on g_0, R and n such that

$$\mathrm{Vol}_g(B_{2R}(x)) \equiv \|u^{2n/(n-2)}\|_{L^1(B_{2R}(x))} \leq C \inf_{B_R(x)} u^{2n/(n-2)}, \tag{3-7}$$

where in the first identification we recall that $d\mathrm{Vol}_g = u^{2n/(n-2)}d\mu$. By the admissibility of (M, g_0), the assumption (1-7) holds and we may take a collection of balls $B_{4R_i}(x_i) \subset M$, indexed by $i = 1, \dots, N < \infty$, with the property that[9]

$$\Big(1 - \mathrm{Vol}_{g_0}\Big(\bigcup_{i=1}^{N} B_{2R_i}(x_i)\Big)\Big)\|u\|^{2n/(n-2)}_{L^\infty(M_T)} < 1. \tag{3-8}$$

Let C_i be the constant in (3-7) for the ball $B_{2R_i}(x_i)$. By summing all the individual inequalities (3-7) for each $i = 1, \dots, N$, we have

$$\sum_{i=1}^{N} \mathrm{Vol}_g(B_{2R_i}(x_i)) \le \sum_{i=1}^{N} C_i \inf_{B_{R_i}(x_i)} u^{2n/(n-2)} \le NC \max_i\big(\inf_{B_{R_i}(x_i)} u^{2n/(n-2)}\big)$$

with $C := \max_i C_i$. The left-hand side can bounded from below by

$$\begin{aligned}\sum_{i=1}^{N} \mathrm{Vol}_g(B_{2R_i}(x_i)) &\ge \mathrm{Vol}_g\Big(\bigcup_{i=1}^{N} B_{2R_i}(x_i)\Big) = 1 - \mathrm{Vol}_g\Big(M \setminus \bigcup_{i=1}^{N} B_{2R_i}(x_i)\Big)\\ &\ge 1 - \mathrm{Vol}_{g_0}\Big(M \setminus \bigcup_{i=1}^{N} B_{2R_i}(x_i)\Big)\|u\|^{2n/(n-2)}_{L^\infty(M_T)} =: c,\end{aligned}$$

which is positive by choice of the balls subject to (3-8). Thus

$$0 < c \le NC \max_i\big(\inf_{B_{R_i}(x_i)} u^{2n/(n-2)}\big).$$

This shows that there has to be a ball $B_{R_i}(x_i)$ with u uniformly bounded from below by $c(T) > 0$ for $t \in [0, T]$. On this ball we thus get a uniform bound $\psi_\varepsilon(u) \ge c(T)^{-\delta}$, which gives our desired t- and ε-independent bound on $\|\psi_\varepsilon(u)\|^2_{L^2(B)}$, and thereby we have that $u^{-\delta} \in H^1(M)$ uniformly. □

Corollary 3.3. *Under the conditions of Theorem 3.2, one can find uniform constants $0 < A(T), B(T) < \infty$, depending only on $T > 0$ and initial scalar curvature S_0 (but not dependent on t), such that for all $f \in H^1(M, g)$,*

$$\|f\|^2_{L^{2n/(n-2)}(M,g)} \le A(T)\|\nabla f\|^2_{L^2(M,g)} + B(T)\|f\|^2_{L^2(M,g)}, \tag{3-9}$$

i.e., (1-8) *holds for the time-dependent metric but with time-independent constants.*

Proof. Due to (1-8) we have, for all $f \in H^1(M) = H^1(M, g)$,

$$\|f\|^2_{L^{2n/(n-2)}(M,g_0)} \le A_0\|\nabla f\|^2_{L^2(M,g_0)} + B_0\|f\|^2_{L^2(M,g_0)}.$$

Using $g = u^{4/(n-2)}g_0$, we conclude a similar estimate with respect to g:

$$\|f\|^2_{L^{2n/(n-2)}(M,g)} \le A(T)\|\nabla f\|^2_{L^2(M,g)} + B(T)\|f\|^2_{L^2(M,g)}, \tag{3-10}$$

where

$$A(T) := A_0\frac{(\sup_{M_T} u)^2}{(\inf_{M_T} u)^2}, \qquad B(T) := B_0\frac{(\sup_{M_T} u)^2}{(\inf_{M_T} u)^{2n/(n-2)}}.$$

Now the statement follows, since $u, u^{-1} \in L^\infty(M \times [0, T])$ by Proposition 3.1 and Theorem 3.2. □

[9]Note that the volume of (M, g_0) is normalized to 1 and thus (3-8) corresponds to (1-7).

We shall need this Sobolev inequality (3-9) when we tackle the upper bound on the scalar curvature S in Section 4.

4. Upper bound on the scalar curvature along the flow

The arguments of this section employ the assumptions

$$\begin{aligned} &\bullet\ (M, g_0) \text{ is an admissible manifold,}\\ &\bullet\ H^1(M) \text{ and } H^1(M,g) \text{ have equivalent norms,}\\ &\bullet\ C_c^\infty(M) \text{ is dense in } H^1(M),\\ &\bullet\ \text{the Sobolev inequality (3-9) holds,}\\ &\bullet\ S \in H^1(M,g) \text{ and } Y(M,g_0) > 0,\end{aligned} \tag{4-1}$$

provided the flow exists. These properties follow from Assumptions 1, 2, 3 and 4, as in the previous section. The Sobolev inequality (3-9) holds under the same assumptions in view of Corollary 3.3. In this section we use (4-1) to show a uniform upper bound on the scalar curvature. More precisely, we will show the following result.

Theorem 4.1. *Let S evolve according to (2-2) with initial curvature $S_0 \in L^{n^2/(2(n-2))}(M)$ and its negative part $(S_0)_- \in L^\infty(M)$. Then, assuming (4-1) holds, there exists a uniform constant $0 < C(T) < \infty$, depending only on $T > 0$ and S_0, such that*

$$\|S\|_{L^\infty(M\times[T/2,T])} \le C(T).$$

The proof proceeds in two steps. The first step is to prove an $L^{n^2/(2(n-2))}(M,g)$-norm bound on S, uniform in $t \in [0,T]$. That uniform bound rests on a chain of arguments of [Brendle 2005, Lemmas 2.2, 2.3, 2.5] (also to be found in [Schwetlick and Struwe 2003, Lemma 3.3]) that apply in our setting as well. In the second step we perform a Moser iteration argument by following [Ma et al. 2012]. Our proofs are close to those in [Brendle 2005] with some additional arguments due to lower regularity.

Lemma 4.2. *Under the conditions of Theorem 4.1, there exists for any finite $T > 0$ a uniform constant $0 < C(T) < \infty$, depending only on T and S_0, such that for all $t \in [0,T]$ we have the estimate*[10]

$$\int_0^T \left(\int_M S^{n^2/(2(n-2))}\, d\mathrm{Vol}_g \right)^{(n-2)/n} dt \le C(T), \qquad \|S\|_{L^{n/2}(M,g)} \le C, \tag{4-2}$$

where the second constant C only depends on S_0, not on T.

Proof. It suffices to prove the statement for S_+ and S_- individually. By Proposition 2.3, the statement holds for the negative part S_-. Thus we only need to prove the claim for S_+. We may therefore assume without loss of generality that $S \ge 0$, so that $S \equiv S_+$, and use (2-3) as the evolution equation.

The claim will follow from the evolution equation (2-2), but we have to argue a bit differently depending on whether $3 \le n \le 4$ or $n > 4$. The idea is the same in all dimensions $n \ge 3$ however. Let

[10] Below, we will denote all uniform positive constants, depending only on T and S_0, either by $C(T)$ or C_T, unless stated otherwise.

us start with $3 \le n \le 4$. Fix any $\sigma > 0$, and set $\beta = \frac{1}{4}n$. Since $\beta \le 1$, the function $x \mapsto (x+\sigma)^\beta$ is in $C^1[0,\infty)$ with bounded derivative. Thus, we may apply the chain rule to $(S+\sigma)^\beta$ and conclude that $(S+\sigma)^\beta \in C^1([0,T]; H^1(M,g))$. We use $\beta^2/(2\beta-1)(S+\sigma)^{2\beta-1}$ as a test function with $\beta = \frac{1}{4}n$ in the weak formulation of (2-3), which yields the inequality

$$\frac{\beta^2}{2\beta-1}\int_M (S+\sigma)^{2\beta-1}\partial_t(S+\sigma)\,d\mathrm{Vol}_g + (n-1)\int_M |\nabla(S+\sigma)^\beta|^2\,d\mathrm{Vol}_g \\ \le \frac{\beta^2}{2\beta-1}\int_M S(S-\rho)(S+\sigma)^{2\beta-1}\,d\mathrm{Vol}_g.$$

Using (1-3) yields

$$\begin{aligned}\frac{\beta}{2(2\beta-1)}\partial_t\int_M (S+\sigma)^{2\beta}\,d\mathrm{Vol}_g &+ (n-1)\int_M |\nabla(S+\sigma)^\beta|^2\,d\mathrm{Vol}_g \\ &\le \frac{\beta}{2\beta-1}\int_M \beta S(S-\rho)(S+\sigma)^{2\beta-1} - \frac{n}{4}(S-\rho)(S+\sigma)^{2\beta}\,d\mathrm{Vol}_g \\ &= -\frac{\beta^2\sigma}{2\beta-1}\int_M (S-\rho)(S+\sigma)^{2\beta-1}\,d\mathrm{Vol}_g \\ &= -\frac{\beta^2\sigma}{2\beta-1}\int_M (S+\sigma-\rho)(S+\sigma)^{2\beta-1}\,d\mathrm{Vol}_g + \frac{\sigma^2\beta^2}{2\beta-1}\int_M (S+\sigma)^{2\beta-1}\,d\mathrm{Vol}_g \\ &\le \frac{\sigma\beta^2(\sigma+\rho(0))}{2\beta-1}\int_M (S+\sigma)^{2\beta-1}\,d\mathrm{Vol}_g \le \frac{\sigma\beta^2(\sigma+\rho(0))}{2\beta-1}\int_M (S+\sigma)^{2\beta}\,d\mathrm{Vol}_g,\end{aligned}$$

where the first equality is due to $\beta = \frac{1}{4}n$, the penultimate inequality uses $\rho(0) \ge \rho(t)$ and the final inequality is due to Hölder with $p = \beta/(\beta-1)$ and $q = \beta$. We want to integrate this inequality in time. Note that any inequality of the form $\partial_t w(t) + a(t) \le bw(t)$ with $a(t) \ge 0$ yields $\partial_t w \le bw$ and hence $w(t) \le e^{bt}w(0)$. Plugging this estimate into the original differential inequality leads to $\partial_t w + a \le be^{bt}w(0)$. Integrating the latter inequality in time yields $w(t) + \int_0^t a(s)\,ds \le e^{bt}w(0)$. We therefore conclude that

$$\int_M (S+\sigma)^{n/2}\,d\mathrm{Vol}_g(T) + \frac{4(n-2)(n-1)}{n}\int_0^T\int_M |\nabla(S+\sigma)^{n/4}|^2\,d\mathrm{Vol}_g \\ \le e^{\sigma(\sigma+\rho(0))nT/2}\int_M (S_0+\sigma)^{n/2}\,d\mu. \quad (4\text{-}3)$$

This is for any $\sigma > 0$. Sending $\sigma \to 0$ and using Fatou's lemma on the left-hand side and the monotone convergence theorem on the right-hand side yields (on dropping the nonnegative term with ∇S)

$$\int_M S^{n/2}\,d\mathrm{Vol}_g(T) \le \int_M S_0^{n/2}\,d\mu.$$

This yields our uniform $L^{n/2}(M,g)$ bound on S in (4-2). Returning to (4-3), we appeal to the Sobolev inequality (3-9) to deduce

$$\int_0^T \|(S+\sigma)^{n/4}\|^2_{L^{2n/(n-2)}(M,g)}\,dt \le \left(\frac{A(T)n}{4(n-2)(n-1)} + TB(T)\right)e^{\sigma(\sigma+\rho(0))nT/2}\int_M (S_0+\sigma)^{n/2}\,d\mu,$$

hence also

$$\int_0^T \left(\int_M |S|^{n^2/(2(n-2))} \, d\mathrm{Vol}_g \right)^{(n-2)/n} dt \le C(T).$$

This proves the claim for $3 \le n \le 4$.

For $n > 4$ the claim will follow similarly, but the above test function does not have bounded derivative for $n > 4$, and we neither know that it is in H^1 nor do we know that the chain rule applies. We therefore argue similarly to the proof of Proposition 2.3, where we introduced the functions $\phi_{\beta,L}$, $G_{\beta,L}$ and $H_{\beta,L}$. We again set $\beta = \frac{1}{4}n$. Using $G_{\beta,L}(S)$ as a test function in (2-2), we find

$$\int_M G_{\beta,L}(S)(\partial_t S) \, d\mathrm{Vol}_g + (n-1) \int_M |\nabla \phi_{\beta,L}(S)|^2 \, d\mathrm{Vol}_g \le \int_M S(S-\rho) G_{\beta,L}(S) \, d\mathrm{Vol}_g.$$

Using the evolution equation (1-3) for the volume form, we have

$$\partial_t \int_M H_{\beta,L}(S) \, d\mathrm{Vol}_g + (n-1) \int_M |\nabla \phi_{\beta,L}(S)|^2 \, d\mathrm{Vol}_g \le \int_M (S-\rho) \left(S G_{\beta,L}(S) - \frac{n}{2} H_{\beta,L}(S) \right) d\mathrm{Vol}_g. \tag{4-4}$$

One readily checks from the definitions of $G_{\beta,L}$ and $H_{\beta,L}$ in Proposition 2.3 that

$$x\, G_{\beta,L}(x) - \frac{n}{2} H_{\beta,L}(x) = \begin{cases} \dfrac{\beta}{2\beta-1} x^{2\beta} \left(\beta - \dfrac{n}{4} \right), & x \le L, \\ \beta^2 L^{2\beta} \left(\left(1 - \dfrac{n}{4}\right) \left(\dfrac{x}{L}\right)^2 + \dfrac{2(\beta-1)}{2\beta-1} \left(\dfrac{n}{2} - 1 \right) \dfrac{x}{L} - \dfrac{n(\beta-1)}{4\beta} \right), & x > L, \end{cases} \tag{4-5}$$

and from this one sees that $x G_{\beta,L}(x) - \frac{1}{2} n H_{\beta,L}(x) \le 0$ for $\beta = \frac{1}{4}n$ and $n \ge 4$ as follows: For $x \le L$ there is nothing to show. For $x > L$, notice that

$$\beta^2 L^{2\beta} \left(\left(1 - \frac{n}{4}\right) \left(\frac{x}{L}\right)^2 + \frac{2(\beta-1)}{2\beta-1} \left(\frac{n}{2} - 1 \right) \frac{x}{L} - \frac{n(\beta-1)}{4\beta} \right) = -\beta^2 (\beta-1) L^{2\beta} \left(\frac{x}{L} - 1 \right)^2 \le 0,$$

where we have substituted $n = 4\beta$ and recognized a square.[11] Hence, using again that ρ is nonincreasing along the flow, we conclude that the inequality

$$\partial_t \int_M H_{\beta,L}(S) \, d\mathrm{Vol}_g + (n-1) \int_M |\nabla \phi_{\beta,L}(S)|^2 \, d\mathrm{Vol}_g \le 0$$

holds for any $L \ge \rho(0)$. This is a differential inequality of the same kind as in the above $3 \le n \le 4$ case. Integrating it we deduce, for any $t \in [0, T]$,

$$\int_M H_{\beta,L}(S) \, d\mathrm{Vol}_g(T) + (n-1) \int_0^T \int_M |\nabla \phi_{\beta,L}(S)|^2 \, d\mathrm{Vol}_g \, dt \le \int_M H_{\beta,L}(S_0) \, d\mu. \tag{4-6}$$

Using $\beta = \frac{1}{4}n$ and letting $L \to \infty$, this yields, using Fatou's lemma and dominated convergence exactly as in the final step of the proof of Proposition 2.3 (neglecting the positive second summand on the left-hand

[11] This is the point where we need $n \ne 3$, since in this case $\beta - 1 < 0$ and the above expression fails to be negative for $x > L$.

side of (4-6)), the inequality

$$\|S\|_{L^{n/2}(M,g)} = \left(\int_M S^{n/2}\, d\mathrm{Vol}_g\right)^{2/n} \leq \left(\int_M |S_0|^{n/2}\, d\mu\right)^{2/n} \equiv C, \tag{4-7}$$

where the constant $C(T) > 0$ depends only on T and S_0. This yields the second estimate in (4-2) for $n > 4$. For the first estimate in (4-2), note that $\phi_{\beta,L}(S) \in H^1(M, g)$[12] and thus by (3-9) and (4-6) we deduce

$$\begin{aligned}
\int_0^T &\left(\int_M |\phi_{\beta,L}(S)|^{2n/(n-2)}\, d\mathrm{Vol}_g\right)^{(n-2)/n} dt \\
&\leq A(T) \int_0^T \int_M |\nabla \phi_{\beta,L}(S)|^2\, d\mathrm{Vol}_g\, dt + B(T) \int_0^T \int_M |\phi_{\beta,L}(S)|^2\, d\mathrm{Vol}_g\, dt \\
&\leq \frac{A(T)}{n-1}\left(\int_M H_{\beta,L}(S_0)\, d\mu - \int_M H_{\beta,L}(S)\, d\mathrm{Vol}_g(T)\right) + B(T) \int_0^T \int_M |\phi_{\beta,L}(S)|^2\, d\mathrm{Vol}_g\, dt.
\end{aligned}$$

Thus, letting $L \to \infty$ we conclude, using Fatou's lemma and dominated convergence as before, with $\beta = \frac{1}{4}n$, and (4-7), that

$$\int_0^T \left(\int_M S^{n^2/(2(n-2))}\, d\mathrm{Vol}_g\right)^{(n-2)/n} dt \leq \left(\frac{nA(T)}{4(n-1)(n-2)} + B(T)T\right) \int_M |S_0|^{n/2}\, d\mu \equiv C(T), \tag{4-8}$$

where the uniform constant $C(T) > 0$ depends only on T and S_0. This proves the first estimate in (4-2) for $n > 4$. □

Lemma 4.3. *Under the conditions of Theorem 4.1, there exists for any finite $T > 0$ a uniform constant $0 < C(T) < \infty$, depending only on T and S_0, such that for all $t \in [0, T]$ we have the estimate*

$$\int_M |S|^{n^2/(2(n-2))}\, d\mathrm{Vol}_g \leq C(T).$$

Proof. As in the previous lemma we have to split the argument into cases based on the dimension. We first show the statement for $n \geq 4$. We will again use the inequality (4-4). However, while in Lemma 4.2 we set $\beta = \frac{1}{4}n$, here we will use the inequality (4-4) with $\beta = n^2/(4(n-2))$. For this choice of β the expression $xG_{\beta,L}(x) - \frac{1}{2}nH_{\beta,L}(x)$ is no longer necessarily nonpositive, and we estimate it against a new approximation function

$$f_{\beta,L}(x) := \begin{cases} \beta x^{2\beta}, & x \leq L, \\ n\beta^2 L^{2\beta-1}x, & x > L. \end{cases} \tag{4-9}$$

By (4-5) one sees that the inequality $xG_{\beta,L}(x) - \frac{1}{2}nH_{\beta,L}(x) \leq f_{\beta,L}(x)$ holds for all $\beta \geq 1$ and $L > 0$ in the case $n \geq 4$. One important aspect to notice is that $f_{\beta,L}(x)$ is linear in x for $x > L$, as opposed to quadratic in x for $H_{\beta,L}(x)$ and $xG_{\beta,L}(x)$. This will become important below. Returning to (4-4) and

[12] Note that a priori we do not know if $S^{n/4} \in H^1(M; g)$ and thus cannot directly apply the Sobolev inequality (3-9) to $S^{n/4}$. However, we do know that $\phi_{\beta,L}(S) \in H^1(M, g)$, since $\phi_{\beta,L}(x)$ is linear for $x > L$ and $S \in H^1(M, g)$ for each fixed time argument.

applying (3-9) to the term $\|\nabla\phi_{\beta,L}(S)\|^2_{L^2(M,g)}$, after some reshuffling we find

$$\partial_t\|H_{\beta,L}(S)\|_{L^1(M,g)} \le (n-1)\frac{B(T)}{A(T)}\|\phi_{\beta,L}(S)^2\|_{L^1(M,g)} - \frac{(n-1)}{A(T)}\|\phi_{\beta,L}(S)\|^2_{L^{2n/(n-2)}(M,g)} + \rho(0)\big\|SG_{\beta,L}(S)-\tfrac{1}{2}nH_{\beta,L}(S)\big\|_{L^1(M,g)} + \|Sf_{\beta,L}(S)\|_{L^1(M,g)}. \tag{4-10}$$

A straightforward computation shows, for all $\beta \ge 1$ and $L > 0$, that

$$12\beta H_{\beta,L}(x) \ge \phi_{\beta,L}(x)^2 \quad \text{and} \quad 4\beta H_{\beta,L}(x) \ge xG_{\beta,L}, \tag{4-11}$$

hold, and here is a way of seeing this: For $x \le L$ these are both obvious from the definitions, so we look at $x > L$. One first notices that

$$\phi_{\beta,L}(x)^2 = \beta^2L^{2(\beta-1)}x^2 - 2\beta(\beta-1)L^{2\beta-1}x + (\beta-1)^2L^{2\beta} \le \beta^2L^{2(\beta-1)}(x^2+2L^2) \le 3\beta^2L^{2(\beta-1)}x^2,$$

where the first inequality comes from dropping the nonpositive linear term and estimating $1 \le \beta$, and the final inequality is simply $L^2 < x^2$. We similarly estimate $H_{\beta,L}(x)$ from below for $x > L$ and find

$$\begin{aligned} H_{\beta,L}(x) &= \left(\frac{\beta^2}{2}\left(\frac{x}{L}\right)^2 - \frac{2\beta^2(\beta-1)}{2\beta-1}\left(\frac{x}{L}\right) + \frac{\beta(\beta-1)}{2}\right)L^{2\beta} \\ &\ge \left(\frac{\beta^2}{2(2\beta-1)}\left(\frac{x}{L}\right)^2 + \frac{\beta(\beta-1)}{2}\right)L^{2\beta} \ge \frac{\beta^2}{2(2\beta-1)}x^2L^{2(\beta-1)}, \end{aligned} \tag{4-12}$$

where the first inequality uses $-x/L \ge -x^2/L^2$ and the second inequality comes from dropping the nonnegative constant term. Using these two estimates one readily sees that

$$12\beta H_{\beta,L}(x) \ge \frac{2\beta}{2\beta-1}3\beta^2x^2L^{2(\beta-1)} \ge 3\beta^2x^2L^{2(\beta-1)} \ge \phi_{\beta,L}(x)^2,$$

showing half of the claim in (4-11). To see the other half, first observe that (for $x > L$) by dropping the nonpositive term in (2-8) we have $xG_{\beta,L}(x) \le \beta^2x^2L^{2(\beta-1)}$. Using (4-12) again we deduce

$$4\beta H_{\beta,L}(x) \ge \frac{2\beta}{2\beta-1}\beta^2x^2L^{2(\beta-1)} \ge \beta^2x^2L^{2(\beta-1)} \ge xG_{\beta,L}(x).$$

This finishes the proof of (4-11), so we arrive by overestimating the right-hand side of (4-10) at the inequality

$$\partial_t\|H_{\beta,L}(S)\|_{L^1(M,g)} \le C_T\|H_{\beta,L}(S)\|_{L^1(M,g)} + \|Sf_{\beta,L}(S)\|_{L^1(M,g)} - \frac{(n-1)}{A(T)}\|\phi_{\beta,L}(S)\|^2_{L^{2n/(n-2)}(M,g)}, \tag{4-13}$$

where the uniform constant $C_T > 0$ is explicitly given by

$$C_T := 12(n-1)\beta\,\frac{A(T)}{B(T)} + \rho(0)\left(\frac{n}{2}+4\beta\right).$$

Introduce the nonnegative, real function $F_{\beta,L}$ via

$$F_{\beta,L}(x) := (xf_{\beta,L}(x))^{1/(2\beta+1)}.$$

Assume $\beta > \frac{1}{4}n$, which holds, for example, for $\beta = n^2/(4(n-2))$. Set $\alpha := n/(4\beta) < 1$ and choose any $\delta > 0$. Observe that by the Hölder inequality in the first estimate and the Young inequality in the second, we obtain

$$\begin{aligned}\|F_{\beta,L}(S)^{2\beta+1}\|_{L^1(M,g)} &\le \|F_{\beta,L}(S)\|^{2\alpha\beta}_{L^{2n\beta/(n-2)}(M,g)}\|F_{\beta,L}(S)\|^{1+2(1-\alpha)\beta}_{L^{2\beta}(M,g)}\\ &\le \delta\alpha\|F_{\beta,L}(S)^{\beta}\|^2_{L^{2n/(n-2)}(M,g)} + \delta^{-\alpha/(1-\alpha)}(1-\alpha)\|F_{\beta,L}(S)\|^{1/(1-\alpha)+2\beta}_{L^{2\beta}(M,g)}.\end{aligned} \tag{4-14}$$

These norms are finite for finite $L > 0$, as follows: The claim is clear for $S \le L$, and the delicate point is the behavior of the function for S large. For $S > L$, $F_{\beta,L}(S) \sim S^{2/(2\beta+1)}$, and (since $2\beta/(2\beta+1) \le 1$) the terms $\|F_{\beta,L}(S)^{\beta}\|_{L^{2n/(n-2)}(M,g)}$ and $\|F_{\beta,L}(S)\|_{L^{2\beta}(M,g)}$ can be controlled via $\|S\|_{L^{2n/(n-2)}(M,g)}$ and $\|S\|_{L^2(M,g)}$, respectively. These latter norms are bounded[13] because of S existing in $C^0([0,T]; H^1(M,g))$ and (3-9).

We can compare $\|F_{\beta,L}(S)^{\beta}\|^2_{L^{2n/(n-2)}(M,g)}$ and $\|\phi_{\beta,L}(S)\|^2_{L^{2n/(n-2)}(M,g)}$ since we have the following pointwise estimates. Directly from the definition we have

$$F_{\beta,L}(x)^{\beta} = \begin{cases} \beta^{\beta/(2\beta+1)}x^{\beta} \le \beta x^{\beta}, & x \le L,\\ (n\beta^2)^{\beta/(2\beta+1)}L^{\beta}\left(\dfrac{x}{L}\right)^{2\beta/(2\beta+1)} \le n\beta L^{\beta-1}x, & x > L.\end{cases}$$

Similarly, we may estimate $\phi_{\beta,L}$ from below as

$$\phi_{\beta,L}(x) = \begin{cases} x^{\beta} = x^{\beta}, & x \le L,\\ \beta L^{\beta-1}x - (\beta-1)L^{\beta} \ge xL^{\beta-1}, & x > L.\end{cases}$$

Combining these two estimates we find $n\beta\phi_{\beta,L}(x) \ge F_{\beta,L}(x)^{\beta}$. By sufficiently shrinking $\delta > 0$ (choosing $\delta \le 4(n-1)/(n^3\beta A(T))$ to be precise), we can thus ensure for all $L > 0$ that

$$\delta\alpha\|F_{\beta,L}(S)^{\beta}\|^2_{L^{2n/(n-2)}(M,g)} \le \frac{(n-1)}{A(T)}\|\phi_{\beta,L}(S)\|^2_{L^{2n/(n-2)}(M,g)},$$

and therefore deduce from (4-13) and (4-14)

$$\partial_t\|H_{\beta,L}(S)\|_{L^1(M,g)} \le C_T\|H_{\beta,L}(S)\|_{L^1(M,g)} + C_T'\|F_{\beta,L}(S)\|^{1/(1-\alpha)+2\beta}_{L^{2\beta}(M,g)}, \tag{4-15}$$

for uniform constants $C_T, C_T' > 0$, where C_T is given above,

$$C_T' := \delta^{-n/(4\beta-n)}\left(\frac{4\beta-n}{4\beta}\right) \quad \text{and} \quad \delta \le \frac{4(n-1)}{n^3\beta A(T)}.$$

The point is that both constants depend only on $T > 0$ and S_0.

We then compare $F_{\beta,L}(x)^{2\beta}$ to $H_{\beta,L}(x)$ as follows: From the definition of $F_{\beta,L}(x)$ again we find

$$F_{\beta,L}(x)^{2\beta} = \begin{cases} \beta^{2\beta/(2\beta+1)}x^{2\beta} \le \beta x^{2\beta}, & x \le L,\\ (n\beta^2)^{2\beta/(2\beta+1)}L^{2\beta}\left(\dfrac{x}{L}\right)^{4\beta/(2\beta+1)} \le n\beta^2L^{2(\beta-1)}x^2, & x > L.\end{cases}$$

[13] This is where it was necessary to estimate $xG_{\beta,L} - \frac{1}{2}nH_{\beta,L} \le f_{\beta,L}$. Otherwise, defining $F_{\beta,L}$ in terms of $xG_{\beta,L} - \frac{1}{2}nH_{\beta,L}$ would cause $F_{\beta,L}(S)$ to go as $S^{3/(2\beta+1)}$ for large L and we would not be able to guarantee that $\|F_{\beta,L}(S)^{\beta}\|^2_{L^{2n/(n-2)}(M,g)}$ is finite.

Consulting (4-12) we find

$$4n\beta H_{\beta,L}(x) \geq \frac{2\beta}{2\beta-1}\begin{cases} n\beta x^{2\beta}, & x \leq L, \\ n\beta^2 L^{2(\beta-1)}x^2, & x > L. \end{cases}$$

We therefore conclude $4n\beta H_{\beta,L}(x) \geq F_{\beta,L}(x)^{2\beta}$. Defining

$$C''_T := \max\{(4n\beta)^{1+2/(4\beta-n)}C'_T, C_T\},$$

we deduce from (4-15) that

$$\partial_t \|H_{\beta,L}(S)\|_{L^1(M,g)} \leq C''_T(1+\|H_{\beta,L}(S)\|^{2/(4\beta-n)}_{L^1(M,g)})\|H_{\beta,L}(S)\|_{L^1(M,g)}.$$

Setting $\beta = n^2/(4(n-2))$, we can rewrite this differential inequality as

$$\partial_t \log(\|H_{\beta,L}(S)\|_{L^1(M,g)}) \leq C''_T(1+\|H_{\beta,L}(S)\|^{n-2/(n)}_{L^1(M,g)}).$$

Integrating this differential inequality in time, we conclude

$$\log(\|H_{\beta,L}(S(T))\|_{L^1(M,g)}) \leq \log(\|H_{\beta,L}(S_0)\|_{L^1(M,g_0)}) + C''_T T + C''_T \int_0^T \|H_{\beta,L}(S)\|^{(n-2)/n}_{L^1(M,g)}\,dt.$$

Taking the limit $L \to \infty$ (using Fatou's lemma and dominated convergence as before in the final step of the proof of Proposition 2.3) and using Lemma 4.2, we deduce

$$\log\|S(T)^{n^2/(2(n-2))}\|_{L^1(M,g)} \leq \log\|S_0^{n^2/(2(n-2))}\|_{L^1(M,g)} + C''_T T + C''_T C(T),$$

which proves the statement for $n \geq 4$.

The above proof would almost work for $n=3$. The problem is that $xG_{\beta,L} - \frac{1}{2}nH_{\beta,L} \leq f_{\beta,L}$ no longer holds true, and one would have a problem showing that the norms in (4-14) are finite. One solution is to redefine the approximation functions $\phi_{\beta,L}$, $G_{\beta,L}$ and $H_{\beta,L}$ to ensure $xG_{\beta,L}(x) - \frac{1}{2}nH_{\beta,L}(x)$ is dominated by a function $f_{\beta,L}$ which, for large x, behaves like at most x rather than x^2. This is a nontrivial task, because it is also important for the above argument that one can find constants (depending on n and β but not L) such that

$$C_1 H_{\beta,L}(x) \geq \phi_{\beta,L}(x)^2, \quad C_2 H_{\beta,L}(x) \geq xG_{\beta,L}(x), \quad C_3 F_{\beta,L}(x)^\beta \leq \phi_{\beta,L}(x), \quad C_4 H_{\beta,L}(x) \geq F_{\beta,L}(x)^{2\beta},$$

where $F_{\beta,L}(x) = (xf_{\beta,L}(x))^{1/(2\beta+1)}$. Consider the following family of approximation functions with $\nu \leq 1$ and $\nu \notin \{0, \frac{1}{2}\}$:

$$\tilde{\phi}_{\beta,L}(x) := \begin{cases} x^\beta, & x \leq L, \\ \dfrac{\beta}{\nu}L^{\beta-\nu}x^\nu + L^\beta\left(1-\dfrac{\beta}{\nu}\right), & x > L, \end{cases} \tag{4-16}$$

$$\tilde{G}_{\beta,L}(x) := \int_0^x \tilde{\phi}'_{\beta,L}(y)^2\,dy = \begin{cases} \dfrac{\beta^2}{2\beta-1}x^{2\beta-1}, & x \geq L, \\ \dfrac{\beta^2 L^{2(\beta-\nu)}}{2\nu-1}x^{2\nu-1} - \dfrac{2\beta^2 L^{2\beta-1}(\beta-\nu)}{(2\nu-1)(2\beta-1)}, & x > L, \end{cases} \tag{4-17}$$

$$\tilde{H}_{\beta,L}(x) := \int_0^x \tilde{G}_{\beta,L}(y)\,dy = \begin{cases} \dfrac{\beta}{2(2\beta-1)}x^{2\beta}, & x \le L, \\ \dfrac{\beta^2 L^{2(\beta-\nu)}}{2\nu(2\nu-1)}x^{2\nu} - \dfrac{2\beta^2 L^{2\beta-1}(\beta-\nu)}{(2\nu-1)(2\beta-1)}x - C_{\beta,\nu}L^{2\beta}, & x > L, \end{cases}$$

where

$$C_{\beta,\nu} := \frac{\beta(\beta(2\beta-1)+4\nu\beta(\nu-\beta)+\nu(1-2\nu))}{2\nu(2\beta-1)(2\nu-1)}.$$

In the $n \ge 4$ case we considered these functions with $\nu = 1$. These functions have the same qualitative properties as before, namely that $\tilde{\phi}_{\beta,L} \xrightarrow{L\to\infty} x^\beta$ and $\tilde{\phi}_{\beta,L} \in C^1(\mathbb{R}_+)$ with $\tilde{\phi}'_{\beta,L} \in L^\infty(\mathbb{R}_+)$, and similarly for $\tilde{G}_{\beta,L}$ and $\tilde{H}_{\beta,L}$. We can therefore use $\tilde{G}_{\beta,L}(S)$ as a test function in (2-2) and deduce the analogue of (4-4), namely

$$\partial_t \int_M \tilde{H}_{\beta,L}(S)\,d\mathrm{Vol}_g + (n-1)\int_M |\nabla \tilde{\phi}_{\beta,L}(S)|^2\,d\mathrm{Vol}_g \le \int_M (S-\rho)\Big(S\tilde{G}_{\beta,L}(S) - \frac{n}{2}\tilde{H}_{\beta,L}(S)\Big)\,d\mathrm{Vol}_g. \quad (4\text{-}18)$$

Consider the expression $x\tilde{G}_{\beta,L}(x) - \frac{1}{2}n\tilde{H}_{\beta,L}(x)$ for $x > L$:

$$x\tilde{G}_{\beta,L}(x) - \frac{n}{2}\tilde{H}_{\beta,L}(x) = \frac{\beta^2 L^{2(\beta-\nu)}}{2\nu(2\nu-1)}\Big(2\nu - \frac{n}{2}\Big)x^{2\nu} + \frac{2\beta^2(\beta-\nu)L^{2\beta-1}}{(2\beta-1)(2\nu-1)}\Big(\frac{n}{2}-1\Big)x + \frac{n}{2}C_{\beta,\nu}L^{2\beta}.$$

From this one sees that when $0 < \nu \le \frac{1}{4}n$ and $\beta \ge \frac{1}{4}n$, the first two terms become negative. So assume from now on that $0 < \nu \le \frac{1}{4}n$ and later we will make a choice of $\beta \ge \frac{1}{4}n$. Introduce the function

$$\tilde{f}_{\beta,L}(x) := \begin{cases} \beta x^{2\beta}, & x \le L, \\ \frac{1}{2}n|C_{\beta,\nu}|L^{2\beta}, & x > L, \end{cases}$$

which has the property that $x\tilde{G} - \frac{1}{2}nH_{\beta,L} \le \tilde{f}_{\beta,L}(x)$ for all $x \ge 0$ and $L > 0$, as long as $\beta \ge \frac{1}{4}n \ge \nu$. Proceeding exactly as in the $n \ge 4$ case, we deduce

$$\begin{aligned} \partial_t \|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)} \le (n-1)\frac{B(T)}{A(T)}\|\tilde{\phi}_{\beta,L}(S)^2\|_{L^1(M,g)} - \frac{(n-1)}{A(T)}\|\tilde{\phi}_{\beta,L}(S)\|^2_{L^{2n/(n-2)}(M,g)} \\ + \rho(0)\big\|S\tilde{G}_{\beta,L}(S) - \tfrac{1}{2}n\tilde{H}_{\beta,L}(S)\big\|_{L^1(M,g)} + \|S\tilde{f}_{\beta,L}(S)\|_{L^1(M,g)}. \quad (4\text{-}19) \end{aligned}$$

We compare $\tilde{\phi}_{\beta,L}(x)^2$, $x\tilde{G}_{\beta,L}(x)$ and $\tilde{H}_{\beta,L}(x)$ as in (4-11) and conclude by similar arguments, for all $\beta \ge 1$, $L > 0$, and some L-independent constants C_1, C_2, that

$$\begin{aligned} C_1\tilde{H}_{\beta,L}(x) &\ge \tilde{\phi}_{\beta,L}(x)^2, \\ C_2\tilde{H}_{\beta,L}(x) &\ge x\tilde{G}_{\beta,L}(x). \end{aligned} \quad (4\text{-}20)$$

We now proceed as before, getting

$$\begin{aligned} &\partial_t \|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)} \\ &\qquad \le C_T\|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)} + \|S\tilde{f}_{\beta,L}(S)\|_{L^1(M,g)} - \frac{(n-1)}{A(T)}\|\tilde{\phi}_{\beta,L}(S)\|^2_{L^{2n/(n-2)}(M,g)}, \quad (4\text{-}21) \end{aligned}$$

where the uniform constant $C_T > 0$ is explicitly given by

$$C_T := C_1(n-1)\,\frac{A(T)}{B(T)} + \rho(0)\big(\tfrac{1}{2}n + C_2\big).$$

Introduce the nonnegative, real function $\tilde{F}_{\beta,L}$ via

$$\tilde{F}_{\beta,L}(x) := (x\tilde{f}_{\beta,L}(x))^{1/(2\beta+1)}.$$

Assume $\beta > \frac{1}{4}n$, which holds, for example, for $\beta = n^2/(4(n-2))$. Set $\alpha := n/(4\beta) < 1$ and choose any $\delta > 0$. Observe that by the Hölder inequality in the first estimate and the Young inequality in the second, we obtain

$$\begin{aligned}\|\tilde{F}_{\beta,L}(S)^{2\beta+1}\|_{L^1(M,g)} &\le \|\tilde{F}_{\beta,L}(S)\|^{2\alpha\beta}_{L^{2n\beta/(n-2)}(M,g)}\|\tilde{F}_{\beta,L}(S)\|^{1+2(1-\alpha)\beta}_{L^{2\beta}(M,g)} \\ &\le \delta\alpha\|\tilde{F}_{\beta,L}(S)^{\beta}\|^2_{L^{2n/(n-2)}(M,g)} + \delta^{-\alpha/(1-\alpha)}(1-\alpha)\|\tilde{F}_{\beta,L}(S)\|^{1/(1-\alpha)+2\beta}_{L^{2\beta}(M,g)}. \end{aligned} \tag{4-22}$$

These integrals are finite for the same reasons as in the $n \ge 4$ case.

We shall from now on set $\nu = \beta/(2\beta+1)$ and $\beta = n^2/(4(n-2))$, which translates into $\nu = \frac{9}{22}$ for $n = 3$. Notice that this choice satisfies $\nu \le \frac{1}{4}n$, so the manipulations up until now are allowed. The reason for choosing this ν is that then

$$\tilde{F}_{\beta,L}(x)^{\beta} = \begin{cases} \beta^{\nu}x^{\beta}, & x \le L, \\ \big(\tfrac{1}{2}n|C_{\beta,\nu}|\big)^{\nu}L^{\beta-\nu}x^{\nu}, & x > L. \end{cases}$$

This is easily comparable to $\tilde{\phi}_{\beta,L}(x)$. Since

$$\tilde{\phi}_{\beta,L}(x) \ge \frac{\beta}{\nu}L^{\beta-\nu}x^{\nu}$$

for $x > L$, we see that by defining

$$C_3^{-1} := \max\left\{\beta^{\nu}, \frac{\nu}{\beta}\left(\frac{n}{2}|C_{\beta,\nu}|\right)^{\nu}\right\}$$

we achieve[14] $C_3\tilde{F}_{\beta,L}(x)^{\beta} \le \tilde{\phi}_{\beta,L}(x)$. So if we choose

$$\delta \le \frac{(n-1)}{A(T)}\frac{4C_3^2\beta}{n},$$

then the inequality

$$\delta\alpha\|\tilde{F}_{\beta,L}(S)^{\beta}\|^2_{L^{2n/(n-2)}(M,g)} - \frac{(n-1)}{A(T)}\|\tilde{\phi}_{\beta,L}(S)\|^2_{L^{2n/(n-2)}(M,g)} \le 0$$

holds for all $L > 0$, and we deduce from (4-21) and (4-22) that

$$\partial_t\|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)} \le C_T\|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)} + C'_T\|\tilde{F}_{\beta,L}(S)\|^{1/(1-\alpha)+2\beta}_{L^{2\beta}(M,g)} \tag{4-23}$$

[14] This is a somewhat delicate point. If one chooses ν small, it is easy to make $x\tilde{G} - \frac{1}{2}n\tilde{H}$ sublinear, but if ν is too small, $\tilde{F}$ will increase faster than $\tilde{\phi}$, ruining the comparison. On the other hand, if ν is bigger than $\frac{1}{4}n$ we see above that $x\tilde{G} - \frac{1}{2}n\tilde{H}$ becomes too large to guarantee the finiteness of the integrals in (4-22).

for uniform constants $C_T, C'_T > 0$, where C_T is given above,

$$C'_T := \delta^{-n/(4\beta-n)}\left(\frac{4\beta-n}{4\beta}\right) \quad \text{and} \quad \delta \leq \frac{(n-1)}{A(T)}\frac{4C_3^2\beta}{n}.$$

The point is that both constants depend only on $T > 0$ and S_0. The final comparison we need is that $C_4 H_{\beta,L}(x) \geq F_{\beta,L}(x)^{2\beta}$ holds for some $C_4 > 0$ independent of L, and here is a way to see that this is doable: For $x \leq L$ both functions are proportional, so there is nothing to show. Inserting $\nu = \beta/(2\beta+1)$ into the definition of $\tilde{H}_{\beta,L}(x)$ yields (for $x > L$)

$$\tilde{H}_{\beta,L}(x) = L^{2\beta}\left(\frac{4\beta^4}{2\beta-1}\left(\frac{x}{L}\right) - \frac{\beta(2\beta+1)^2}{2}\left(\frac{x}{L}\right)^{2\nu} + \beta^2\right),$$

which shows that $\tilde{H}_{\beta,L}(x)$ is dominated by a positive linear term for $x > L$, which will dominate the sublinear term $x^{2\nu}$ of $\tilde{F}_{\beta,L}(x)^{2\beta}$. Defining

$$C''_T := \max\{C_4^{1+2/(4\beta-n)}C'_T, C_T\},$$

we deduce from (4-23) that

$$\partial_t \|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)} \leq C''_T(1 + \|\tilde{H}_{\beta,L}(S)\|^{2/(4\beta-n)}_{L^1(M,g)})\|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)}.$$

The rest of the proof then follows exactly as in the $n \geq 4$ case, giving us our required bound for $n = 3$. □

This completes the first step on the way to Theorem 4.1, proving a uniform $L^{n^2/(2(n-2))}(M, g)$-norm bound on S. Before we can go on to prove Theorem 4.1 by a Moser iteration argument, we need the following parabolic Sobolev inequality.

Lemma 4.4. *Let $A(T)$ and $B(T)$ denote the constants of the (elliptic) Sobolev inequality (3-9). Then for any $f \in H^1(M, g)$ with uniform norm in $t \in [0, T]$, we have (writing $M_T := M \times [0, T]$)*[15]

$$\|f^2\|_{L^{(n+2)/n}(M_T,g)} \leq \frac{n}{n+2}(A(T)\|\nabla f\|^2_{L^2(M_T,g)} + B(T)\|f\|^2_{L^2(M_T,g)}) + \frac{2}{n+2}\sup_{t\in[0,T]}\|f(t)\|^2_{L^2(M,g)}. \tag{4-24}$$

Proof. The statement and the proof are close to [Ma et al. 2012, Equation 12]. We compute

$$\begin{aligned}\int_0^T\int_M f^{2(n+2)/n}\,d\mathrm{Vol}_g\,dt &= \int_0^T\int_M f^2 f^{4/n}\,d\mathrm{Vol}_g\,dt \\ &\leq \int_0^T (\|f\|^2_{L^{2n/(n-2)}(M,g)}\|f\|^{4/n}_{L^2(M,g)})\,dt \\ &\leq \int_0^T (A(T)\|\nabla f\|^2_{L^2(M,g)} + B(T)\|f\|^2_{L^2(M,g)})(\|f\|^{4/n}_{L^2(M,g)})\,dt \\ &\leq (A(T)\|\nabla f\|^2_{L^2(M_T,g)} + B(T)\|f\|^2_{L^2(M_T,g)})\sup_{t\in[0,T]}(\|f\|^{4/n}_{L^2(M,g)}),\end{aligned}$$

[15] We write $L^p(M_T, g) \equiv L^p(M_T, g \oplus dt^2)$ for any $p \geq 1$.

where in the first estimate we applied the Hölder inequality with $p = \frac{1}{2}n$ and $q = n/(n-2)$ and in the second estimate applied (3-9). Raising both sides of the inequality to the power of $n/(n+2)$ and using Young's inequality $AB \le A^p/p + B^q/q$ with $p = (n+2)/n$ and $q = \frac{1}{2}(n+2)$ we arrive at the estimate as claimed. □

Proof of Theorem 4.1. Since we assume that $(S_0)_- \in L^\infty(M)$, we have uniform bounds on S_- by Proposition 2.3. Thus it suffices to prove the statement for S_+. Therefore we may replace S by S_+, replacing the evolution equation (2-2) for S by the inequality (2-3) for S_+. Hence we continue under the assumption $S \equiv S_+ \ge 0$, subject to (2-3).

Let $\eta \in C^1([0, T], \mathbb{R}_+)$ be nondecreasing with $\eta(0) = 0$ and $\|\eta\|_\infty \le 1$. We would like to use $\beta^2\eta^2 S^{2\beta-1}/(2\beta-1)$ (with $\beta > 1$) as a test function in the weak formulation of (2-3). The problem is of course that the chain rule fails to hold in general, so we use the same workaround as in Proposition 2.3 and Lemma 4.3. Let $L > 0$ and define $\phi_{\beta,L}$, $G_{\beta,L}$ and $H_{\beta,L}$ as before. Using $\eta(s)^2 G_{\beta,L}(S)$ as a test function in (2-3) we get

$$\int_M (\partial_s S)\eta^2 G_{\beta,L}(S)\, d\mathrm{Vol}_g + (n-1)\int_M \eta^2 |\nabla \phi_{\beta,L}(S)|^2\, d\mathrm{Vol}_g \le \int_M S G_{\beta,L}(S)|S-\rho|\, d\mathrm{Vol}_g.$$

On the right-hand side we observe (by a direct computation) that

$$S G_{\beta,L}(S) \le \beta^2/(2\beta-1)\phi_{\beta,L}(S)^2.$$

We integrate this in time for any $t \in [0, T]$ and get

$$\int_0^t\int_M (\partial_s S)\eta^2 G_{\beta,L}(S)\, d\mathrm{Vol}_g\, ds + (n-1)\int_0^t\int_M \eta^2|\nabla\phi_{\beta,L}(S)|^2\, d\mathrm{Vol}_g\, ds \\ \le \frac{\beta^2}{2\beta-1}\int_0^t\int_M \eta^2\phi_{\beta,L}(S)^2|S-\rho|\, d\mathrm{Vol}_g\, ds. \tag{4-25}$$

We rewrite the first term on the left-hand side of (4-25) using (1-3) as

$$\begin{aligned}\int_0^t\int_M \eta^2(\partial_s S)G_{\beta,L}(S)\, d\mathrm{Vol}_g\, ds &\equiv \int_0^t\int_M \eta^2\partial_s H_{\beta,L}(S)\, d\mathrm{Vol}_g\, ds \\ &= \int_M \eta^2 H_{\beta,L}(S)\, d\mathrm{Vol}_g(s=t) - 2\int_0^t\int_M \eta\dot\eta H_{\beta,L}(S)\, d\mathrm{Vol}_g\, ds \\ &\quad + \frac{n}{2}\int_0^t\int_M \eta^2 H_{\beta,L}(S)(S-\rho)\, d\mathrm{Vol}_g\, ds,\end{aligned}$$

where we write $\dot\eta \equiv \partial_s\eta$ and use $\eta(0) = 0$. Plugging this into (4-25), we obtain

$$\int_M \eta^2 H_{\beta,L}(S)\, d\mathrm{Vol}_g(s=t) + (n-1)\int_0^t\int_M \eta^2|\nabla\phi_{\beta,L}(S)|^2\, d\mathrm{Vol}_g\, ds \\ \le \int_0^t\int_M \eta^2\left(\frac{\beta^2}{2\beta-1}\phi_{\beta,L}(S)^2 + \frac{n}{2}H_{\beta_L}(S)\right)|S-\rho|\, d\mathrm{Vol}_g\, ds + 2\int_0^t\int_M \eta\dot\eta H_{\beta,L}(S)\, d\mathrm{Vol}_g\, ds.$$

We now take the supremum over $t \in [0, T]$ and appeal to the parabolic Sobolev inequality (4-24) with $f = \eta\phi_{\beta,L}(S)$. The result is

$$\frac{(n-1)}{nA(T)}\Big((n+2)\|\eta^2\phi_{\beta,L}(S)^2\|_{L^{(n+2)/n}(M_T,g)} - 2\sup_{t\in[0,T]}\|\eta\phi_{\beta,L}(S)\|^2_{L^2(M,g)} - nB(T)\|\eta\phi_{\beta,L}(S)\|^2_{L^2(M_T,g)}\Big)$$
$$+\sup_{t\in[0,T]}\int_M \eta^2 H_{\beta,L}(S)\,d\mathrm{Vol}_g$$
$$\leq \int_0^T\int_M \eta^2\left(\frac{\beta^2}{2\beta-1}\phi_{\beta,L}(S)^2 + \frac{n}{2}H_{\beta_L}(S)\right)|S-\rho|\,d\mathrm{Vol}_g\,dt + 2\int_0^T\int_M \eta\dot\eta H_{\beta,L}(S)\,d\mathrm{Vol}_g\,dt. \quad (4\text{-}26)$$

By increasing $A(T) > 0$ if needed, we may assume (also noting that $H_{\beta,L}$ and $\phi^2_{\beta,L}$ are comparable by (4-11)) that

$$\sup_{t\in[0,T]}\int_M \eta^2 H_{\beta,L}(S)\,d\mathrm{Vol}_g - \frac{2(n-1)}{nA(T)}\sup_{t\in[0,T]}\|\eta\phi_{\beta,L}(S)\|^2_{L^2(M,g)} \geq 0$$

for all $\beta \geq 1$ and $L > 0$. We may therefore drop these terms from (4-26). Taking the limit $L \to \infty$ (using Fatou's lemma and the dominated convergence theorem) we get

$$(n-1)\frac{n+2}{nA(T)}\|\eta^2 S^{2\beta}\|_{L^{(n+2)/n}(M_T)} - \frac{B(T)(n-1)}{A(T)}\|\eta S^\beta\|^2_{L^2(M_T,g)}$$
$$\leq \left(\frac{\beta^2}{2\beta-1} + \frac{\beta}{4n(2\beta-1)}\right)\int_0^T\int_M \eta^2 S^{2\beta}|S-\rho|\,d\mathrm{Vol}_g\,dt + \frac{\beta}{2\beta-1}\int_0^T\int_M \eta\dot\eta S^{2\beta}\,d\mathrm{Vol}_g\,dt.$$

Introducing $C := nA(T)/((n+2)(n-1))$ we get, for any $\beta > 1$, the inequality

$$\|\eta^2 S^{2\beta}\|_{L^{(n+2)/n}(M_T)}$$
$$\leq \frac{nB(T)}{n+2}\|\eta S^\beta\|^2_{L^2(M_T)} + C\int_0^T\int_M \eta\dot\eta S^{2\beta}\,d\mathrm{Vol}_g\,dt + 2C\beta\int_0^T\int_M \eta^2 S^{2\beta}|S-\rho|\,d\mathrm{Vol}_g\,dt.$$

We apply the Hölder inequality with $p = n^2/(2(n-2))$ to the last integral on the right-hand side above. Using Lemma 4.3 to get a bound on the integral of $|S-\rho|^p$, we conclude

$$\|\eta^2 S^{2\beta}\|_{L^{(n+2)/n}(M_T)} \leq \frac{nB(T)}{n+2}\|\eta S^\beta\|^2_{L^2(M_T)} + C\int_0^T\int_M \eta\dot\eta S^{2\beta}\,d\mathrm{Vol}_g\,dt + C(T)\beta\|\eta^2 S^{2\beta}\|_{L^N(M_T)}, \quad (4\text{-}27)$$

with $N := p/(p-1) = n^2/(n^2-2n+4) < (n+2)/n$. This is almost the expression we want to iterate, but the presence of $\dot\eta$ means we have to shrink our time interval in the iteration (as is standard for parabolic Moser iteration). The details (inspired by [Ma et al. 2012, pp. 889–890]) follow.

Consider the sequence $t_k := \left(\frac{1}{2} - \left(\frac{1}{2}\right)^k\right)T$ for integers $k \geq 1$. Let $M_k := M \times [t_k, T]$, $M_1 = M_T$ and $M_\infty = M \times \left[\frac{1}{2}T, T\right]$. Choose nondecreasing test functions $\eta_k \in C^1([0,T], \mathbb{R}_+)$ with $\|\eta_k\|_\infty \leq 1$ such that

$$\eta_k(t) = \begin{cases} 0, & t \leq t_{k-1}, \\ 1, & t \geq t_k. \end{cases}$$

The choice of $\{\eta_k\}_k$ can be made subject to a bound on the derivative $0 \le \dot{\eta}_k \le 2^{k+1}/T$, which we henceforth assume. Using these functions in (4-27), we find

$$\begin{aligned}
\|S^{2\beta}\|_{L^{(n+2)/n}(M_k)} &= \|\eta_k^2 S^{2\beta}\|_{L^{(n+2)/n}(M_k)} \le \|\eta_k^2 S^{2\beta}\|_{L^{(n+2)/n}(M_T)} \\
&\le \frac{nB(T)}{n+2}\|\eta_k S^{\beta}\|^2_{L^2(M_T)} + C\int_0^T\int_M \eta_k\dot{\eta}_k S^{2\beta}\, d\mathrm{Vol}_g\, dt + C(T)\beta\|\eta_k^2 S^{2\beta}\|_{L^N(M_T)} \\
&\le \tilde{C}(T)\beta 2^{k+1}\|S^{2\beta}\|_{L^N(M_{k-1})}, \qquad (4\text{-}28)
\end{aligned}$$

where the second inequality uses (4-27) and last step uses $\dot{\eta} \le 2^{k+1}/T$ together with the Hölder inequality to compare L^1- and L^N-norms. This is the equation we will be iterating. Introduce $\gamma := 2\beta N$ and $\rho := (n+2)/(nN) = (n^3+8)/n^3 > 1$. Then (4-28) reads

$$\|S\|_{L^{\rho\gamma}(M_k)} \le (\tilde{C}(T)\gamma 2^k)^{N/\gamma}\|S\|_{L^{\gamma}(M_{k-1})}.$$

Replacing γ by $\rho^m\gamma$ for $m \ge 0$ results in

$$\|S\|_{L^{\rho^{m+1}\gamma}(M_{k+m})} \le (\tilde{C}(T)\rho^m\gamma 2^{k+m})^{N/(\rho^m\gamma)}\|S\|_{L^{\rho^m\gamma}(M_{k+m-1})},$$

which can be iterated down to

$$\|S\|_{L^{\rho^{m+1}\gamma}(M_{k+m})} \le \prod_{i=0}^{m}(\tilde{C}(T)\rho^i\gamma 2^{k+i})^{N/(\rho^i\gamma)}\|S\|_{L^{\gamma}(M_{k-1})}.$$

The expression $\prod_{i=0}^{m}(\tilde{C}(T)\rho^i\gamma 2^{k+i})^{N/(\rho^i\gamma)}$ converges as $m \to \infty$, as one checks by computing the logarithm

$$\begin{aligned}
\lim_{m\to\infty}\log\prod_{i=0}^{m}(\tilde{C}(T)\rho^i\gamma 2^{k+i})^{N/(\rho^i\gamma)} &= \frac{N}{\gamma}\sum_{i=0}^{\infty}\Big(\log(2^k\tilde{C}(T)\gamma)\frac{1}{\rho^i} + \log(2\rho)\frac{i}{\rho^i}\Big) \\
&= \frac{N}{\gamma}\Big(\frac{\rho}{\rho-1}\log(\tilde{C}(T)\gamma 2^k) + \log(2\rho)\frac{\rho}{(\rho-1)^2}\Big).
\end{aligned}$$

We therefore deduce for some uniform constant $C_T > 0$

$$\|S\|_{L^{\infty}(M\times[T/2,T])} \le \lim_{m\to\infty}\|S\|_{L^{\rho^{m+1}\gamma}(M_{k+m})} \le C_T\|S\|_{L^{\gamma}(M_{k-1})} \le C_T\|S\|_{L^{\gamma}(M_T)}.$$

Choosing

$$\beta = \frac{n^2-2n+4}{4(n-2)} \iff \gamma = \frac{n^2}{2(n-2)},$$

we can estimate the right-hand side using Lemma 4.3 and deduce for some uniform constant $C(T) > 0$

$$\|S\|_{L^{\infty}(M\times[T/2,T])} \le C(T). \qquad \square$$

Remark 4.5. It is worth pointing out that we do not assume $S_0 \in L^{\infty}(M)$, only that $S_0 \in L^{n^2/(2(n-2))}(M)$. The above proof tells us that $S \in L^{\infty}(M)$ for positive times, even if the initial curvature is unbounded. This is analogous to the well-known behavior of the heat equation, where the solutions for positive times are often much more regular than the initial data.

5. Long-time existence of the normalized Yamabe flow

We can now establish our main Theorem 1.1, which explicitly reads as follows.

Theorem 5.1. *Let (M, g_0) be a Riemannian manifold of dimension $n = \dim M \geq 3$ such that the following four assumptions hold*:

(1) *The Yamabe constant $Y(M, g_0)$ is positive, i.e., Assumption 1 holds.*

(2) *(M, g_0) is admissible, i.e., Assumption 2 holds.*

(3) *Parabolic Schauder estimates (as defined in Definition 1.4) hold on (M, g_0), i.e., Assumption 3 holds.*

(4) *$S_0 \in C^{1,\alpha}(M)$, i.e., Assumption 4 holds. Moreover, we require that $S_0 \in L^{n^2/(2(n-2))}(M)$ and that its negative part $(S_0)_- \in L^\infty(M)$.*

Under these assumptions, a normalized Yamabe flow $u^{4/(n-2)} g_0$ exists with $u \in C^{3,\alpha}(M \times [0, \infty))$, with infinite existence time, and with scalar curvature $S(t) \in L^\infty(M)$ for all $t > 0$.

Proof. Short time existence of the flow with $u \in C^{3,\alpha}(M \times [0, T'])$ for some small $T' > 0$ is due to Theorem 1.6. Let $T > 0$ be the maximal existence time, so that $u \in C^{3,\alpha}(M \times [0, T))$ with locally uniform control of the Hölder norms in $[0, T)$, but with no uniform control of the norms up to $t = T$. If $T = \infty$, there is nothing to prove. Otherwise, we proceed as follows.

Proposition 2.3 yields a uniform (i.e., depending only on S_0 and the finite T) lower bound on the scalar curvature S. Proposition 3.1 and Theorem 3.2 yield uniform upper and lower bounds on the solution u, so that $u \in L^\infty(M_T)$. This in turn gives us bounds on the Sobolev constants $A(T)$ and $B(T)$ (Corollary 3.3), so we use Theorem 4.1 to argue that $S \in L^\infty(M_T)$. By the evolution equation

$$\partial_t u = -\frac{4}{n-2}(S - \rho)u,$$

we deduce $\partial_t u \in L^\infty(M_T)$. Then, arguing exactly as in [Bahuaud and Vertman 2019, Proposition 2.8], we may then restart the flow and extend the solution past T. For the purpose of self-containment, we provide the argument here.

Let us consider the linearized equation (1-18) with $u = 1 + v$,

$$\partial_t v - (n-1)\Delta_0 v = -\tfrac{1}{4}(n-2)S_0 + \Phi(v), \quad v(0) = 0, \tag{5-1}$$

where $\Phi(v) \in L^\infty(M_T)$, since $u, \partial_t u, \rho \in L^\infty(M_T)$. By the third mapping property in (1-14), we conclude that $v \in C^{1,\alpha}(M \times [0, T])$.[16] Rewrite flow equation (1-2) using $N = (n+2)/(n-2)$ as

$$\partial_t u - (n-1)u^{1-N}\Delta_0 u = \tfrac{1}{4}(n-2)(\rho\, u - S_0 u^{2-N}). \tag{5-2}$$

We will treat the right-hand side of this equation as a fixed element of $C^{0,\alpha}(M \times [0, T])$. Since $u^{1-N} \in C^{1,\alpha}(M \times [0, T])$ is positive and uniformly bounded away from zero, we may apply (1-16) and (1-17) to obtain a solution $u' \in C^{2,\alpha}(M \times [0, T])$ with initial condition $u'(0) = 1$.

[16] Note that we now have uniform control of the $C^{1,\alpha}$-norm up to $t = T$.

Note that $w := u - u'$ solves $\partial_t w - (n-1)u^{1-N}\Delta_0 w = 0$ with zero initial condition. By the weak maximum principle (1-13), $\partial_t w_{\max} \le 0$ and $\partial_t w_{\min} \ge 0$. Due to the initial condition $w(0) = 0$, we deduce $w \equiv 0$ and hence $u = u' \in C^{2,\alpha}(M \times [0, T])$. Thus $u' \in C^{2,\alpha}(M \times [0, T])$ extends $u(t)$ up to $t = T$, and we conclude

$$u \in C^{2,\alpha}(M \times [0, T]).$$

By the second statement of Theorem 1.6, we even have $u \in C^{3,\alpha}(M \times [0, T])$ and can now restart the flow as follows. Consider $u_0 = u(T) \in C^{3,\alpha}(M)$ as the initial condition for the normalized Yamabe flow. By (1-15), $e^{t\Delta_0}u_0 \in C^{3,\alpha}(M \times [0, T])$, where the heat operator acts without convolution in time.

We write $u = f + e^{t\Delta}u_0$ and plug this into the Yamabe flow equation (1-2) with rescaled time $\tau = (t - T)$. This yields an equation for f,

$$[\partial_t - (n-1)(e^{t\Delta_0}u_0)^{1-N}\Delta_0]f = Q_1(f) + Q_2(f, \partial_t f), \quad u'(0) = 0, \tag{5-3}$$

where Q_1 and Q_2 denote linear and quadratic combinations of the elements in brackets, respectively, with coefficients given by polynomials in $e^{t\Delta_0}u_0$, $\partial_t e^{t\Delta_0}u_0$ and $\Delta_0 e^{t\Delta_0}u_0$. Since these coefficients are of higher Hölder regularity $C^{1,\alpha}(M)$, we may set up a contraction mapping argument in $C^{3,\alpha}$ and thus extend u past the maximal existence time T exactly as in the proof of Theorem 1.6. This proves long-time existence. $\square$

Corollary 5.2. *In the setting of the above theorem, we have*

$$\lim_{t\to\infty} \int_M (S-\rho)^2 \, d\mathrm{Vol}_g = 0,$$

and there exists $u_\infty \in L^2(M)$ *such that*

$$\lim_{t\to\infty} \int_M (u - u_\infty)^2 \, d\mu = 0.$$

Proof. By (1-5) we have

$$\partial_t \rho = -\frac{n-2}{2} \int_M (S-\rho)^2 \, d\mathrm{Vol}_g.$$

This shows that $\rho(t)$ is monotonically decreasing, and we know it's bounded from below by $Y(M, g_0) > 0$, so $\lim_{t\to\infty} \rho(t)$ exists. Thus $\int_0^\infty \partial_t \rho(t)\, dt < \infty$, and thus $\partial_t \rho(t)$ must converge to zero as $t \to \infty$. This gives the conclusion on $\int_M (S-\rho)^2 \, d\mathrm{Vol}_g$. By (1-1) we also conclude that

$$\int_M (\partial_t u^{2n/(n-2)}) \, d\mu = -\frac{n}{2} \int_M (S-\rho) u^{2n/(n-2)} \, d\mu = 0,$$

and using u as a test function in (1-2) leads to

$$\frac{n+2}{2n} \int_M \partial_t u^{2n/(n-2)} \, d\mu + (n-1) \int_M |\nabla u|^2 \, d\mu = \tfrac{1}{4}(n+2)\biggl(\rho(t) - \int_M u^2 S_0 \, d\mu\biggr),$$

so

$$\int_M |\nabla u|^2 \, d\mu \le \tfrac{1}{4}(n+2)(\rho(0) + \|(S_0)_-\|_{L^\infty(M)}),$$

where we have used $\int_M u^{2n/(n-2)}\,d\mu = 1$. This shows that u is uniformly bounded in $H^1(M)$ independent of t for all $t \geq 0$. Since the Sobolev embedding $H^1(M) \hookrightarrow L^q(M)$ is compact for $q < 2n/(n-2)$ (see [Akutagawa et al. 2014, Proposition 1.6]), we in particular get that u has a convergent subsequence in $L^2(M)$ as $t \to \infty$, and we call this limit u_∞. $\square$

Remark 5.3. The above methods would also show that $\partial_t u^{(n+2)/(n-2)} \to 0$ in $L^1(M)$, since we may use (1-1) and the Hölder inequality to write

$$\begin{aligned}\|\partial_t u^{(n+2)/(n-2)}\|_{L^1(M)} &\leq \tfrac{1}{4}(n+2)\|(S-\rho)u^{n/(n-2)}\|_{L^2(M)}\|u^{2/(n-2)}\|_{L^2(M)}\\ &\leq \tfrac{1}{4}(n+2)\|(S-\rho)u^{n/(n-2)}\|_{L^2(M)}.\end{aligned}$$

We then use the first part of the corollary to show that the right-hand side tends to 0.

6. Future research directions and open problems

Long time existence alone does *not* guarantee regularity of the limit solution $u_\infty \in L^2(M)$. Indeed, this has to be obstructed for the following two reasons. In the case of closed manifolds, we know that the Yamabe problem is not uniquely solvable on a round sphere, but so far we have not assumed that (M, g_0) is not a sphere. In the singular setting, the Yamabe problem doesn't always have a solution, as demonstrated by Viaclovsky [2010]. We suspect that demanding

$$Y(M, g_0) < \lim_{R\to 0} Y(B_R(p), g_0),$$

for all $p \in \overline{M}$, is the required condition in our setting. Under this assumption, Akutagawa, Carron and Mazzeo are able to solve the Yamabe problem for smoothly stratified spaces in [Akutagawa et al. 2014]. For closed manifolds, this condition becomes $Y(M, g_0) < Y(\mathbb{S}^n, g_{\mathbb{S}^n})$ with the round metric $g_{\mathbb{S}^n}$, which is the assumption used by Brendle [2005] in his study of the Yamabe flow. Brendle's proof of convergence of the Yamabe flow relies on the positive mass theorem, which is not available in the singular setting.

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