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GLOBAL REGULARITY FOR THE NONLINEAR WAVE EQUATION WITH SLIGHTLY SUPERCRITICAL POWER

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We consider the defocusing nonlinear wave equation $\Box u = |u|^{p-1}u$ in $\mathbb{R}^3 \times [0,\infty)$. We prove that for any initial datum with a scaling-subcritical norm bounded by M_0 the equation is globally well-posed for $p = 5+\delta$, where $\delta \in (0, \delta_0(M_0))$.

1. Introduction

We consider the Cauchy problem for the nonlinear defocusing wave equation on $\mathbb{R}^3$, that is,

$$\begin{cases} \Box u = |u|^{p-1}u, \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) \in (\dot{H}^1 \cap \dot{H}^2) \times H^1, \end{cases} \tag{1}$$

where $u : \mathbb{R}^3 \times I \to \mathbb{R}$, $p > 1$ and $\Box = -\partial_{tt} + \Delta$ is the d'Alembertian. For sufficiently regular solutions of (1) the energy

$$E(u)(t) := \int \frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla u|^2 + \frac{|u|^{p+1}}{p+1}\,\mathrm{d}x$$

is conserved, i.e., $E(t) = E$. Moreover, there is a natural scaling associated to (1): for $\lambda > 0$ the map

$$u \mapsto u_\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda t)$$

preserves solutions of (1). Correspondingly, the energy rescales like $E(u_\lambda)(t) = \lambda^{(5-p)/(p-1)}E(u)(t)$ and hence the equation is energy-supercritical for $p > 5$. Our goal is to show that given any (possibly large) initial data (u_0, u_1), the supercritical nonlinear defocusing wave equation (1) is globally well-posed at least for an open interval of exponents $p \in [5, 5+\delta_0)$.

Theorem 1.1. *Let $\|(u_0,u_1)\|_{\dot{H}^1\cap\dot{H}^2\times H^1} \le M_0$. Then there exists $\delta_0 = \delta_0(M_0) > 0$ such that for any $\delta \in (0,\delta_0)$ there exists a global solution u of* (1) *with $p = 5+\delta$ from the initial data (u_0,u_1). Moreover, there exists a universal constant $C > 1$ such that for any time t*

$$\|(u,\partial_t u)(t)\|_{\dot{H}^1\cap\dot{H}^2\times H^1} \le \|(u_0,u_1)\|_{\dot{H}^1\cap\dot{H}^2\times H^1} e^{C(1+(CE(u))^{CE(u)^{352}})} \tag{2}$$

and we have the global spacetime bound

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times\mathbb{R})} \le C(1+(CE(u))^{CE(u)^{352}}).$$

In particular, the solution scatters as $t \to \pm\infty$.

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Global regularity and scattering for the energy-critical regime was established in [Struwe 1988; Grillakis 1990]. The classical results in the critical case were recently improved to obtain explicit double exponential bounds [Tao 2006b] and to allow a critical nonlinearity with an extra logarithmic factor $f(u) = u^5 \log(2+u^2)$ in the case of spherical symmetric data [Tao 2007]. Exploiting the method introduced in [Tao 2006b; Roy 2009] could remove the assumption of spherical symmetry for slightly log log-supercritical growth. In two-dimensions, global regularity has also been established for the slightly supercritical nonlinearity $f(u) = ue^{u^2}$ in [Struwe 2011]. For the classical supercritical nonlinearity $f(u) = |u|^{p-1}u$ with $p > 5$, global existence and scattering of solutions still holds for small data in scaling-invariant spaces, for instance in $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$, where

$$s_p := 1 + \frac{\delta}{2(p-1)}$$

is the critical Sobolev exponent. For general large data, however, the problem of global regularity and scattering is still open: apart from conditional regularity results in terms of the critical Sobolev regularity [Kenig and Merle 2011; Killip and Visan 2011], global solutions have been built only from particular classes of initial data [Krieger and Schlag 2017; Beceanu and Soffer 2018] or for a nonlinearity satisfying the null condition as in [Wang and Yu 2016; Miao et al. 2019].

Our result should be seen in line with [Tao 2006b; Roy 2009], pushing global regularity in a slightly supercritical regime. Although the nonlinearity considered in those papers has a logarithmically supercritical growth at infinity, it still comes, up to lower-order terms, with the scaling associated to the critical case $p = 5$. Correspondingly, both the scaling-invariant quantities of the critical regime, as well as some logarithmically higher integrability, are controlled by the energy. Instead, we consider the supercritical nonlinearity (1) and achieve global existence and scattering by paying the price of working on bounded sets of initial data, as previously done for other equations, such as SQG [Coti Zelati and Vicol 2016] and Navier–Stokes [Colombo and Haffter 2021]. As in [Roy 2009; Coti Zelati and Vicol 2016; Colombo and Haffter 2021], the crucial ingredient of the proof of Theorem 1.1 is a (quantitative) long-time estimate. In the spherically symmetric case, the classical Morawetz inequality gives an a priori spacetime bound as long as the solution exists. The following result replaces this long-time estimate in the absence of symmetry assumptions.

Theorem 1.2 (a priori spacetime bound). *There exists a universal constant $C \geq 1$ such that, for any solution $(u, \partial_t u) \in L^\infty(J, (\dot{H}^1 \cap \dot{H}^2 \times H^1)(\mathbb{R}^3))$ of (1) with $p = 5+\delta$, $\delta \in (0,1)$, defining $M := \|u\|_{L^\infty(\mathbb{R}^3 \times J)}$, $E := E(u)$ and $L := \|(u, \partial_t u)\|_{L^\infty(J,(\dot{H}^{s_p} \times \dot{H}^{s_p-1})(\mathbb{R}^3))}$ the following hold:*

- *If $\min\{EM^{\delta/2}, L\} < c_0$, then $\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J)} \leq 1$.*
- *If $\min\{EM^{\delta/2}, L\} \geq c_0$ and $(CEM^{\delta/2}L)^{C(EM^{\delta/2}L)^{176}} \leq 2^{1/\delta}$, then*

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J)} \leq (CEM^{\frac{\delta}{2}}L)^{C(EM^{\delta/2}L)^{176}}. \tag{3}$$

Corollary 1.3. *There exists a universal constant $C \geq 1$ such that the following holds. Let $M_0 > 0$ be given. Then there exists $\delta_0 = \delta_0(M_0) > 0$ such that, for any solution $(u, \partial_t u) \in L^\infty(J, (\dot{H}^1 \cap \dot{H}^2 \times H^1)(\mathbb{R}^3))$*

of (1) *with* $p = 5 + \delta$ *for* $\delta \in (0, \delta_0]$ *and with* $\|(u, \partial_t u)\|_{L^\infty(J,(\dot{H}^1 \cap \dot{H}^2 \times H^1)(\mathbb{R}^3))} \le M_0$, *we have the a priori spacetime bound*

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J)} \le \max\{1, (CE(u)M_0^{\frac{\delta}{2}})^{C(E(u)M_0^{\delta/2})^{352}}\}. \tag{4}$$

Remark 1.4. From the proof, we observe that δ_0 has the following dependence as $M_0 \to \infty$: there exists $C' \ge 1$ such that

$$\delta_0 := \min\left\{1, \frac{\ln 2}{\ln M_0}, \frac{\ln 2}{\ln(C'E)(C'E)^{352}}\right\}.$$

Theorem 1.1 follows from Corollary 1.3 and a continuity argument, taking advantage of the fact that, if the estimate (4) involves in the right-hand side higher-order norms of the solution itself, which we a priori don't control for large times, on the other side they appear only to the power δ and hence can be kept under control for δ small.

The proof of Theorem 1.2 follows instead the scheme introduced in [Tao 2006b] to obtain double exponential bounds on critical Strichartz norms based on Bourgain's "induction on energy" method [1999]. In [Roy 2009], the scheme has been successfully applied to a log-supercritical equation assuming a (subcritical) a priori bound M on $\|u\|_{L^\infty(\mathbb{R}^3 \times J)}$: indeed, it was noticed that the induction on the energy, which does not allow the inclusion of the a priori bound M, can actually be bypassed by a simpler ad-hoc argument. We will use the latter strategy also in our case. Rather than controlling an L^4L^{12} norm as performed in the mentioned papers, we estimate an $L^{2(p-1)}$ norm, which is scaling-critical for every p. To follow their line of proof, we need to overcome some issues related to the supercritical nature of our equation: for instance, a fundamental use of the equation in all critical global regularity results is the localized energy equality and the subsequent potential energy decay, first used in [Struwe 1988; Grillakis 1990; Shatah and Struwe 1993]. In the supercritical regime, the localized energy inequality becomes less powerful, since the nonlinear term is estimated this time in terms of a power of the length of the time interval besides the energy itself (see Lemma 4.5). To be able to still take advantage of this localized energy inequality, we need a control on the length of the so-called unexceptional intervals, which was not derived before in [Tao 2006b; Roy 2009] and seems to work in the supercritical case only. To achieve this control, we introduce another scaling-invariant norm of u accounting for more differentiability, namely $L^\infty \dot{H}^{s_p}$. This quantity, which appears in the final estimate (3), was not needed in [Tao 2006b; Roy 2009]. It turns out to be fundamental to bound the length of unexceptional intervals by performing a mass concentration in $\dot{H}^{s_p}$, rather than in $\dot{H}^1$ (see Lemma 6.2), and thereby obtaining an upper bound on the mass concentration radius.

The strategy of proof of Theorem 1.1 is very flexible and we plan to apply it in a future work to the radial supercritical Schrödinger equation. For instance, as regards the initial data, the statement of Theorem 1.1 is written with $(u_0, u_1) \in \dot{H}^1 \cap \dot{H}^2 \times H^1$ and in the proof we take advantage of the embedding of $H^{3/2+\epsilon}$ in L^∞. However, we will investigate whether a similar result holds just above the critical threshold, namely for $(u_0, u_1) \in \dot{H}^1 \cap \dot{H}^{1+\epsilon} \times H^\epsilon$ for some $\epsilon > 0$, with δ_0 depending on ϵ.

2. Preliminaries

2A. *Energy-flux equality.* With the notation of [Shatah and Struwe 1998], we introduce the forward-in-time wave cone, the truncated cone and their boundaries centered at $z_0 = (x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}$ defined by

$$\begin{aligned} K(z_0) &:= \{z = (x,t) \in \mathbb{R}^4 : |x - x_0| \le t - t_0\}, \\ K_s^t(z_0) &:= K(z_0) \cap (\mathbb{R}^3 \times [s,t]), \\ M_s^t(z_0) &:= \{z = (x,r) \in \mathbb{R}^3 \times (s,t) : |x - x_0| = r - t_0\}, \\ D(t; z_0) &:= K(z_0) \cap (\mathbb{R}^3 \times t). \end{aligned}$$

Correspondingly, we introduce the localized energy as well as the energy flux

$$\begin{aligned} E(u; D(t;z_0)) &:= \int_{D(t;z_0)} \frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla u|^2 + \frac{|u|^{p+1}}{p+1}\,\mathrm{d}x, \\ \mathrm{Flux}(u; M_s^t(z_0)) &:= \int_{M_s^t(z_0)} \frac{1}{2}\left|\nabla u - \frac{x - x_0}{|x - x_0|}\partial_t u\right|^2 + \frac{|u|^{p+1}}{p+1}\,\frac{\mathrm{d}\sigma}{\sqrt{2}}. \end{aligned}$$

Let us recall that for any sufficiently regular solution we have the energy-flux identity

$$E(u; D(t;z_0)) + \mathrm{Flux}(u; M_s^t(z_0)) = E(u; D(s;z_0)) \tag{5}$$

for any $0 < s < t$. Indeed, (5) is obtained by integrating $(\Box u - |u|^{p-1}u)\,\partial_t u$ on $K_s^t(z_0)$; see for instance [Shatah and Struwe 1998]. Whenever $z_0 = (0,0)$, we will not write the dependence on z_0; we will write $\Gamma_+(I)$ for the forward wave cone centered at 0 and truncated by I,

$$\Gamma_+(I) := \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} : |x| < t,\ t \in I\},$$

and we define $e(t) := E(u; D(t))$. We can then rewrite (5) for any $0 < s < t$ as

$$e(t) - e(s) = \int_{M_s^t} \frac{1}{2}\left|\nabla u - \frac{x}{t}\,\partial_t u\right|^2 + \frac{|u|^{p+1}}{p+1}\,\frac{\mathrm{d}\sigma}{2}.$$

2B. *Strichartz estimates.* Let $u : \mathbb{R}^3 \times I \to \mathbb{R}$ solve the linear wave equation $\Box u = F$. Let $m \in \left[1, \frac{3}{2}\right)$. Then for any $(q,r) \in (2,\infty] \times [1,\infty)$ wave-m-admissible and for any conjugate pair $(\tilde{q}, \tilde{r}) \in [1,+\infty] \times [1,+\infty]$ with

$$\frac{1}{\tilde{q}} + \frac{3}{\tilde{r}} - 2 = \frac{1}{q} + \frac{3}{r} = \frac{3}{2} - m, \tag{6}$$

we have

$$\|u\|_{L^q(I,L^r)} + \|(u, \partial_t u)\|_{L^\infty(I,\dot{H}^m \times \dot{H}^{m-1})} \le C(\|(u,\partial_t u)(t_0)\|_{\dot{H}_x^m \times \dot{H}_x^{m-1}} + \|F\|_{L^{\tilde{q}}(I,L^{\tilde{r}})}), \tag{7}$$

where $t_0 \in I$ is a generic time. The above Strichartz estimates are classical and we refer for instance to [Ginibre and Velo 1995; Keel and Tao 1998; Lindblad and Sogge 1995; Sogge 1995]. Notice that $(q,r) = (2(p-1), 2(p-1))$ is wave-s_p-admissible and all (q,r) wave-s_p-admissible are scaling-critical. Moreover, the constant C can be taken independent of $m \in \left[1, \frac{5}{4}\right]$.

2C. ***Localized Strichartz estimates.*** By the finite speed of propagation, we can localize the above Strichartz estimates on wave cones. Let $I=[a,b]$ and $m\in\left[1,\frac{3}{2}\right)$. For any solution $u:\mathbb{R}^3\times I\to\mathbb{R}$ of a linear wave equation $\Box u=F$, we have for any (q,r) wave-m-admissible and any conjugate pair $(\tilde{q},\tilde{r})$ satisfying (6) the localized estimate

$$\|u\|_{L^qL^r(\Gamma_+(I))}\lesssim\|(u,\partial_t u)(b)\|_{(\dot{H}^m\times\dot{H}^{m-1})(\mathbb{R}^3)}+\|F\|_{L^{\tilde{q}}L^{\tilde{r}}(\Gamma_+(I))}. \tag{8}$$

As a consequence, if $I=[a,b]=J_1\cup J_2$, we have

$$\|u\|_{L^qL^r(\Gamma_+(J_1))}\lesssim\|(u,\partial_t u)(b)\|_{(\dot{H}^m\times\dot{H}^{m-1})(\mathbb{R}^3)}+\|F\|_{L^{\tilde{q}}L^{\tilde{r}}(\Gamma_+(J_1\cup J_2))}.$$

2D. ***Littlewood–Paley projection.*** We follow the presentation of [Tao 2006a]. Fix $\phi\in C_c^\infty(\mathbb{R}^d)$ radially symmetric, $0\le\phi\le 1$ such that $\operatorname{supp}\phi\subseteq B_2(0)$ and $\phi\equiv 1$ on $B_1(0)$. For $N\in 2^{\mathbb{Z}}$, introduce the Fourier multipliers

$$\begin{aligned}\widehat{P_{\le N}f}(\xi)&:=\phi(\xi/N)\hat{f}(\xi),\\ \widehat{P_{>N}f}(\xi)&:=(1-\phi(\xi/N))\hat{f}(\xi),\\ \widehat{P_N f}(\xi)&:=(\phi(\xi/N)-\phi(2\xi/N))\hat{f}(\xi).\end{aligned}$$

The above projections can equivalently be written as convolution operators and the Young inequality shows that the Littlewood–Paley projections are bounded on L^p for any $1\le p\le+\infty$. Moreover, we have the Bernstein inequalities

$$\|P_{\le N}f\|_{L^q_x(\mathbb{R}^d)}\lesssim_{p,q}N^{d\left(\frac{1}{p}-\frac{1}{q}\right)}\|P_{\le N}f\|_{L^p_x(\mathbb{R}^d)} \tag{9}$$

for $1\le p\le q\le+\infty$ and the same holds with $P_N f$ in place of $P_{\le N}f$. Moreover, for $1<p<+\infty$ we also recall the fundamental Littlewood–Paley inequality

$$\|f\|_{L^p(\mathbb{R}^d)}\sim\left\|\left(\sum_{N\in 2^{\mathbb{Z}}}|P_N f|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^d)}. \tag{10}$$

2E. ***Dependence of constants.*** In the rest of the paper, all constants will be independent of the choice of $\delta\in[0,1)$. We keep the estimates in scaling-invariant form (for instance, in all the statements of the lemmas in Sections 3–6). We write the terms in the estimate in terms of simpler scaling-invariant quantities, such as $E\|u\|_{L^\infty}^{\delta/2}$, $\|u\|_{L^{2(p-1)}}$, $\|u\|_{L^\infty\dot{H}^{s_p}}$, $ET^{-\delta/(p-1)}$ (see for instance (16)).

3. Spacetime norm bound under a scaling-invariant smallness assumption

In this section, we recall that the Strichartz estimates give a universal control on the critical $L^{2(p-1)}$ spacetime norm, which is in particular independent of the length of the time interval of existence, provided that the solution satisfies a suitable scaling-invariant smallness assumption. In our context, we formulate the smallness assumption in terms of the critical $\dot{H}^{s_p}$ norm as well as a scaling-invariant combination of the energy and the L^∞ norm.

Lemma 3.1. *Let $p = 5 + \delta$ for $\delta \in (0, 1)$ and consider a solution $(u, \partial_t u) \in L^\infty(I, \dot H^1 \cap \dot H^2 \times H^1)$ to* (1). *Assume additionally that $\|u\|_{L^\infty(\mathbb{R}^3\times I)} \le M$. There exists a universal $0 < c_0 < 1$ such that if*

$$EM^{\frac{\delta}{2}} \le c_0 \quad \text{or} \quad \|(u, \partial_t u)\|_{L^\infty(I,(\dot H^{s_p}\times \dot H^{s_p-1})(\mathbb{R}^3))} \le c_0,$$

then

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times I)} \le 1. \tag{11}$$

Proof. Let us first assume that $EM^{\delta/2} \le c_0$ for a $c_0 < 1$ yet to be chosen. By interpolation

$$\|u\|_{L^{2(p-1)}} \le \|u\|_{L^\infty}^{\frac{\delta}{p-1}} \|u\|_{L^8}^{\frac{4}{p-1}}.$$

We notice that $(8, 8)$ is wave-1-admissible. By the Strichartz estimate (7) (with $m = 1$ and $(\tilde q, \tilde r) = (2, \frac{3}{2})$), Hölder and the Sobolev embedding $\dot H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ we have

$$\|u\|_{L^8_{t,x}} \lesssim E^{\frac12} + \||u|^{p-1}u\|_{L^2L^{3/2}} \lesssim E^{\frac12} + \||u|^{p-1}\|_{L^2_{t,x}}\|u\|_{L^\infty L^6} \lesssim E^{\frac12}(1 + \|u\|^{p-1}_{L^{2(p-1)}}).$$

Summarizing, we have obtained that for a $C \ge 1$

$$\|u\|_{L^{2(p-1)}} \le C(M^{\frac{\delta}{2}}E)^{\frac{2}{p-1}}(1 + \|u\|^4_{L^{2(p-1)}}),$$

from which (11) follows setting $c_0 := (4C)^{-(p-1)/2} < 1$.

Let us now assume $\|(u, \partial_t u)\|_{L^\infty(\dot H^{s_p}\times \dot H^{s_p-1})} \le c_0'$ for a $0 < c_0' < 1$. Observing that $(2(p-1), 2(p-1))$ is wave-s_p-admissible, by the Strichartz estimate (7) (with $m = s_p$ and $(\tilde q, \tilde r) = (2, 6(p-1)/(3p+1))$), Hölder and the Sobolev embedding $\dot H^{s_p}(\mathbb{R}^3) \hookrightarrow L^{3(p-1)/2}(\mathbb{R}^3)$, we have

$$\begin{aligned}\|u\|_{L^{2(p-1)}} &\lesssim \|(u, \partial_t u)\|_{L^\infty(\dot H^{s_p}\times \dot H^{s_p-1})} + \||u|^{p-1}u\|_{L^2L^{6(p-1)/(3p+1)}}\\ &\lesssim \|(u, \partial_t u)\|_{L^\infty(\dot H^{s_p}\times \dot H^{s_p-1})} + \||u|^{p-1}\|_{L^2_{t,x}}\|u\|_{L^\infty L^{3(p-1)/2}}\\ &\lesssim \|(u, \partial_t u)\|_{L^\infty(\dot H^{s_p}\times \dot H^{s_p-1})}(1 + \|u\|^{p-1}_{L^{2(p-1)}}).\end{aligned}$$

Calling C' the constant in the above inequality, (11) follows by setting $c_0' := (4C')^{-1}$. □

4. Spacetime norm decay in forward wave cones

The goal of this section is to prove the following proposition, which identifies a subinterval J (of quantified length) with small $L^{2(p-1)}$ norm of u in any sufficiently large given interval $I = [T_1, T_2]$. The main difference to the energy-critical case $p = 5$ [Tao 2006b, Corollary 4.11] lies in the fact that the largeness requirement on I can no longer be reached by simply choosing T_2 big enough (see Remark 4.3).

Proposition 4.1 (spacetime-norm decay). *Let $p = 5 + \delta$ with $\delta \in (0, 1)$, $I = [T_1, T_2] \subset (0, \infty)$ and consider a solution $(u, \partial_t u) \in L^\infty(I, \dot H^1 \cap \dot H^2 \times H^1)$ to* (1). *Assume that $\|u\|_{L^\infty(\mathbb{R}^3\times I)} \le M$. There exists a universal constant $0 < C_2 < 1$ such that if $0 < \eta < 1$ is such that*

$$\eta < C_2(EM^{\frac{\delta}{2}})^{\frac{7}{6(p-1)}} \tag{12}$$

then the following holds for any A satisfying

$$A > (C_2\eta^{-1})^{\frac{12(p-1)}{5}}(EM^{\frac{\delta}{2}})^{\frac{14}{5}} : \tag{13}$$

if T_1 and T_2 are such that

$$\frac{T_2}{T_1} \geq A^{3(C_2\eta^{-1})^{6(p-1)(p+1)/5}(EM^{\delta/2})^{(9p+19)/10}\max\{(C_2\eta^{-1})^{-6(p-1)^2/5}(EM^{\delta/2})^{9(p-1)/10},(M^{(p-1)/2}T_2)^{\delta/2}\}}, \tag{14}$$

then there exists a subinterval $J = [t', At'] \subseteq I$ with

$$\|u\|_{L^{2(p-1)}(\Gamma_+(J))} \leq \eta.$$

Remark 4.2 (simplified assumptions in the large energy regime). In the large energy regime $EM^{\delta/2} \geq c_0$, with c_0 defined through Lemma 3.1, the hypothesis (12) can be simplified to

$$\eta < C_2 c_0^{\frac{7}{6(p-1)}} := c_0',$$

where we observe that $0 < c_0' \leq 1$. Moreover, the assumption (14) can be replaced by the stronger condition

$$\frac{T_2}{T_1} \geq A^{3(C_2\eta^{-1})^{6(p-1)(p+1)/5}(EM^{\delta/2})^{9p+19/10}\max\{c_0^{(p-1)/2},(M^{(p-1)/2}T_2)^{\delta/2}\}}. \tag{15}$$

Remark 4.3. The assumptions of Proposition 4.1 comprise an upper bound on T_1 for any fixed η satisfying (12), A satisfying (13) and T_2 satisfying (14). However, this will not be the spirit of the application of this proposition: we will rather fix T_1 and consider (14) as a condition on T_2 and δ. This condition may sound strange since, when all other parameters are fixed, (14) is not verified for large T_2. On the other hand, we will instead fix

$$T_2 := T_1 A^{3(C_2\eta^{-1})^{6(p-1)(p+1)/5}(EM^{\delta/2})^{(9p+19)/10}}$$

and notice that (14) is verified for δ sufficiently small.

As a first step to the proof of Proposition 4.1, we show that if the $L^{2(p-1)}$ norm of u in a strip is bounded from below, the Strichartz estimates imply a lower bound on the $L^\infty L^{p+1}$ norm in the same interval.

Lemma 4.4 (lower bound on global and local potential energy). *Let $p = 5+\delta$ with $\delta \in (0,1)$ and $\eta \in (0,1]$. Consider a solution $(u, \partial_t u) \in L^\infty(I, \dot{H}^1 \cap \dot{H}^2 \times H^1)$ to (1). Assume that $\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times I)} \geq \eta$ and $\|u\|_{L^\infty(\mathbb{R}^3\times I)} \leq M$. Then there exists $0 < C_1 \leq 1$ universal such that*

$$\|u\|^{p+1}_{L^\infty(I,L^{p+1})} \geq C_1 \eta^{\frac{12}{5}(p-1)}(M^{\frac{\delta}{2}}E)^{-\frac{9}{5}}M^{-\frac{\delta}{2}}. \tag{16}$$

Moreover, by finite speed of propagation the same estimate can be obtained by replacing $\mathbb{R}^3 \times I$ by any truncated forward wave cone $\Gamma_+(I)$.

Proof. Let $0 < \eta \le 1$. By shrinking I, we can assume without loss of generality that $\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times I)} = \eta$. We observe that we control all wave-1-admissible spacetime norms with the energy. Indeed, fix (q, r) wave-1-admissible. By the Strichartz estimate (7) with $m = 1$ and Hölder

$$\|u\|_{L^q L^r} \lesssim E^{\frac{1}{2}} + \||u|^{p-1}u\|_{L^2 L^{3/2}} \lesssim E^{\frac{1}{2}} + \|u\|_{L^\infty L^6}\||u|^{p-1}\|_{L^2_{t,x}} \lesssim E^{\frac{1}{2}} + E^{\frac{1}{2}}\eta^{p-1} \lesssim E^{\frac{1}{2}}. \quad (17)$$

We observe that the pair $(3, 18)$ is wave-1-admissible and that $(3, 18)$ and $(\infty, p+1)$ interpolate to $\big(\big(\frac{5}{6}(p+1)+3, \frac{5}{6}(p+1)+3\big) = \big(8+\frac{5}{6}\delta, 8+\frac{5}{6}\delta\big)$. By interpolation and (17), we thus have

$$\|u\|^{2(p-1)}_{L^{2(p-1)}} \le \|u\|^{\frac{7}{6}\delta}_{L^\infty_{t,x}} \|u\|^{8+\frac{5}{6}\delta}_{L^{8+(5/6)\delta}_{t,x}} \le M^{\frac{7}{6}\delta}\|u\|^{\frac{5}{6}(p+1)}_{L^\infty L^{p+1}}\|u\|^3_{L^3 L^{18}} \lesssim (M^{\frac{\delta}{2}}E)^{\frac{3}{2}} M^{\frac{5}{12}\delta}\|u\|^{\frac{5}{6}(p+1)}_{L^\infty L^{p+1}}. \qquad \square$$

We now come to a localized energy inequality of Morawetz-type which, in the critical case $p = 5$, implies the potential energy decay and hence it is crucial for the global regularity in the critical case [Grillakis 1990; Struwe 1988]. In the supercritical case, the former localized energy inequality degenerates and will only lead to some decay estimate on bounded intervals: indeed the presence of the extra term $b^{\delta/(p+1)}$ in the right-hand side of (18) below makes the inequality interesting only when an estimate on the length of the interval is at hand.

Lemma 4.5. *Let $\delta \in [0, 1)$ and $p = 5 + \delta$. For any $0 < a < b$ and any weak finite energy solution $(u, \partial_t u) \in C([a, b], \dot{H}^1 \cap L^{p+1}) \cap L^p([a, b], L^{2p}) \times C([a, b], L^2)$ of (1), we have*

$$\int_{|x|\le b} |u(x, b)|^{p+1}\,\mathrm{d}x \lesssim \frac{a}{b}E + e(b) - e(a) + b^{\frac{\delta}{p+1}}(e(b) - e(a))^{\frac{2}{p+1}}. \quad (18)$$

Proof. Let us first assume that $u \in C^2(\mathbb{R}^3 \times [a, b])$ is a classical solution of (1). We follow the notation of [Shatah and Struwe 1993; Bahouri and Shatah 1998] and introduce the quantities

$$\begin{aligned}
Q_0 &:= \frac{1}{2}((\partial_t u)^2 + |\nabla u|^2) + \frac{|u|^{p+1}}{p+1} + \partial_t u\Big(\frac{x}{t}\cdot\nabla u\Big),\\
P_0 &:= \frac{x}{t}\Big(\frac{(\partial_t u)^2}{2} - \frac{|\nabla u|^2}{2} - \frac{|u|^{p+1}}{p+1}\Big) + \nabla u\Big(\partial_t u + \frac{x}{t}\cdot\nabla u + \frac{u}{t}\Big),\\
R_0 &:= \Big(1 - \frac{4}{p+1}\Big)|u|^{p+1}.
\end{aligned}$$

Observe $R_0 \ge 0$. Multiplying (1) by $(t\,\partial_t u + x\cdot\nabla u + u)$ one obtains $\partial_t(t\,Q_0 + \partial_t u\,u) - \operatorname{div}(tP_0) + R_0 = 0$; see [Shatah and Struwe 1998, Chapter 2.3]. Integrating on K_a^b (recall the definitions in Section 2), we obtain

$$\begin{aligned}
b\int_{D(b)} Q_0\,\mathrm{d}x - a\int_{D(a)} Q_0\,\mathrm{d}x + \int_{K_a^b} R_0\,\mathrm{d}x\,\mathrm{d}t&\\
= -\int_{D(b)} \partial_t u\,u\,\mathrm{d}x + \int_{D(a)} \partial_t u\,u\,\mathrm{d}x &+ \int_{M_a^b}\Big(t\,Q_0 + \partial_t u\,u + t\,P_0\cdot\frac{x}{|x|}\Big)\frac{\mathrm{d}\sigma}{\sqrt{2}}\\
= \int_{M_a^b} t\Big(\partial_t u + \frac{x}{t}\cdot\nabla u + \frac{u}{t}\Big)^2\frac{\mathrm{d}\sigma}{\sqrt{2}}&, \quad (19)
\end{aligned}$$

where in the second equality we used the computations of [Bahouri and Shatah 1998, Section 2] for $p=5$ to rewrite the last addend on the right-hand side. Indeed, on M_a^b the integrand

$$t\,Q_0 + \partial_t u + P_0 \cdot \frac{x}{|x|} = t(\partial_t u)^2 + 2\partial_t u\, x\cdot\nabla u + \partial_t u\, u$$

is now independent of p. Proceeding as in [Bahouri and Gérard 1999], we estimate on K_a^b

$$\partial_t u\, \frac{x}{t}\cdot\nabla u \le \frac{(\partial_t u)^2}{2} + \frac{1}{2}\left|\frac{x}{t}\cdot\nabla u\right|^2 \le \frac{(\partial_t u)^2}{2} + \frac{1}{2}|\nabla u|^2. \tag{20}$$

We infer from (19)–(20), the positivity of R_0 and the conservation of the energy that

$$\begin{aligned}\int_{D(b)} \frac{|u|^{p+1}}{p+1}\,\mathrm{d}x &\le \frac{a}{b}\int_{D(a)} Q_0\,\mathrm{d}x + \frac{1}{b}\int_{M_a^b} t\left(\partial_t u + \frac{x}{t}\cdot\nabla u + \frac{u}{t}\right)^2 \frac{\mathrm{d}\sigma}{\sqrt{2}}\\ &\le \frac{a}{b}\int_{D(a)}\left(\frac{|u|^{p+1}}{p+1} + (\partial_t u)^2 + |\nabla u|^2\right)\mathrm{d}x + \frac{1}{b}\int_{M_a^b} t\left(\partial_t u + \frac{x}{t}\cdot\nabla u + \frac{u}{t}\right)^2 \frac{\mathrm{d}\sigma}{\sqrt{2}}\\ &\le \frac{a}{b}E + \frac{1}{b}\int_{M_a^b} t\left(\partial_t u + \frac{x}{t}\cdot\nabla u + \frac{u}{t}\right)^2 \frac{\mathrm{d}\sigma}{\sqrt{2}}.\end{aligned}$$

We estimate the last term on the right-hand side as in [Bahouri and Gérard 1999]: we use (5) to bound

$$\frac{1}{b}\int_{M_a^b} t\left(\partial_t u + \frac{x}{t}\cdot\nabla u + \frac{u}{t}\right)^2 \frac{\mathrm{d}\sigma}{\sqrt{2}} \le 2(e(b)-e(a)) + 2\int_{M_a^b}\frac{u^2}{t^2}\frac{\mathrm{d}\sigma}{\sqrt{2}}.$$

The main difference with respect to the energy-critical regime is the estimate of the second addend which now deteriorates with b. Indeed, we estimate by Hölder

$$\int_{M_a^b}\frac{u^2}{t^2}\frac{\mathrm{d}\sigma}{\sqrt{2}} \le b^{\frac{\delta}{p+1}}\left(\int_{M_a^b}\frac{|u|^{p+1}}{p+1}\frac{\mathrm{d}\sigma}{\sqrt{2}}\right)^{\frac{2}{p+1}} \lesssim b^{\frac{\delta}{p+1}}(e(b)-e(a))^{\frac{2}{p+1}}.$$

Collecting terms, we have obtained (18) for classical solutions $u \in C^2(\mathbb{R}^3\times[a,b])$.

If u is a weak finite-energy solution of (1) as in the statement, we proceed as in [Bahouri and Gérard 1999]: we fix a family of mollifiers $\{\rho_\epsilon\}_{\epsilon>0}$ in space and define $u_\epsilon := u*\rho_\epsilon$. Then, setting

$$f_\epsilon = -|u_\epsilon|^{p-1}u_\epsilon + (|u|^{p-1}u)*\rho_\epsilon,$$

$u_\epsilon \in C^2(\mathbb{R}^3\times[a,b])$ is a classical solution of

$$\Box u_\epsilon = |u_\epsilon|^{p-1}u_\epsilon + f_\epsilon. \tag{21}$$

By assumption, $f_\epsilon \in L^1([a,b],L^2)$ can be treated as a source term. We then deduce (18) by proving the analogous local energy inequality for a nonlinear wave equation with right-hand side (21) and pass to the limit $\epsilon\to 0$. We refer to [Bahouri and Gérard 1999, Lemma 2.3] for details. □

Lemma 4.5 can be viewed as decay estimate for the potential energy. Again, when compared to the critical case [Tao 2006b, Corollary 4.10], the supercriticality of the equation weakens the decay by introducing a new dependence on T_2, the endpoint of the interval to which the decay estimate is applied, which deteriorates as $T_2 \to +\infty$.

Proposition 4.6 (potential energy decay in forward wave cones). *Let $I = [T_1, T_2] \subset (0, +\infty)$ and consider a solution $(u, \partial_t u) \in L^\infty(I, \dot H^1 \cap \dot H^2 \times H^1)$ to* (1) *with $p = 5 + \delta$ for some $\delta \in (0, 1)$. Let $0 < \theta$ such that*

$$ET_2^{-\frac{\delta}{p-1}}\theta^{-(p+1)} > 1. \tag{22}$$

Let $A > 0$ be such that

$$A \ge ET_2^{-\frac{\delta}{p-1}}\theta^{-(p+1)} \quad \textit{and} \quad A^{3ET_2^{-\delta/(p-1)}\theta^{-(p+1)}\max\{1,\theta^{-(p+1)(p-1)/2}\}}T_1 \le T_2. \tag{23}$$

Then there exists a subinterval of the form $J = [t', At']$ such that

$$\|u\|_{L^\infty L^{p+1}(\Gamma_+(J))} \lesssim T_2^{\frac{\delta}{(p-1)(p+1)}}\theta.$$

Notice that θ in the previous statement is not dimensional.

Proof. Let $\theta > 0$ be as in (22) and fix $A \ge ET_2^{-\delta/(p-1)}\theta^{-(p+1)}$. Let N to be chosen later be such that $A^{2N}T_1 \le T_2$, namely

$$\bigcup_{i=1}^{N}[A^{2(n-1)}T_1, A^{2n}T_1] \subseteq I.$$

Since e is nondecreasing in time (see (5)), we have $e(A^{2n}t) - e(A^{2(n-1)}t) \ge 0$ for all n and

$$0 \le \sum_{n=1}^{N} e(A^{2n}T_1) - e(A^{2(n-1)}T_1) = e(A^{2N}T_1) - e(T_1) \le E.$$

Hence there exists $n_0 \in \{1, \dots, N\}$ such that $e(A^{2n_0}T_1) - e(A^{2(n_0-1)}T_1) \le EN^{-1}$. Splitting the interval as

$$[A^{2(n_0-1)}T_1, A^{2n_0}T_1] = [A^{2(n_0-1)}T_1, A^{2n_0-1}T_1] \cup [A^{2n_0-1}T_1, A^{2n_0}T_1],$$

we have, applying Lemma 4.5 with $a := A^{2(n_0-1)}T_1$ and varying $b \in [A^{2n_0-1}T_1, A^{2n_0}T_1]$, that

$$\begin{aligned}\|u\|^{p+1}_{L^\infty L^{p+1}(\Gamma_+([A^{2n_0-1}T_1, A^{2n_0}T_1])} &\lesssim \frac{1}{A}E + EN^{-1} + (A^{2n_0}T_1)^{\frac{\delta}{p+1}}(EN^{-1})^{\frac{2}{p+1}} \\ &\lesssim T_2^{\frac{\delta}{p-1}}\theta^{p+1} + EN^{-1} + T_2^{\frac{\delta}{p+1}}(EN^{-1})^{\frac{2}{p+1}} \lesssim T_2^{\frac{\delta}{p-1}}\theta^{p+1},\end{aligned}$$

provided

$$(EN^{-1})^{\frac{2}{p+1}} \le T_2^{\frac{2\delta}{(p-1)(p+1)}}\theta^{p+1} \quad \text{and} \quad EN^{-1} \le T_2^{\frac{\delta}{p-1}}\theta^{p+1},$$

or equivalently,

$$ET_2^{-\frac{\delta}{p-1}}\theta^{-(p+1)}\max\{1, \theta^{-\frac{(p+1)(p-1)}{2}}\} \le N.$$

For the latter, we have to ask that $[T_1, A^{2N}T_1] \subseteq [T_1, T_2]$, which is enforced by the second requirement in (23). □

Proof of Proposition 4.1. Fix $0 < \theta$ yet to be determined such that $ET_2^{-\delta/(p-1)}\theta^{-(p+1)} > 1$. Fix $A \geq ET_2^{-\delta/(p-1)}\theta^{-(p+1)}$ and assume that (23) holds. By Proposition 4.6, there exists a subinterval J of the form $J := [t', At']$ and $C' \geq 1$ such that

$$\|u\|_{L^\infty L^{p+1}(\Gamma_+(J))} \leq C' T_2^{\frac{\delta}{(p-1)(p+1)}} \theta. \tag{24}$$

We claim that if we choose θ appropriately, we have $\|u\|_{L^{2(p-1)}(\Gamma_+(J))} \leq \eta$. Indeed, assume by contradiction that $\|u\|_{L^{2(p-1)}(\Gamma_+(J))} \geq \eta$. Then we have from Lemma 4.4

$$\|u\|_{L^\infty L^{p+1}(\Gamma_+(J))} \geq C_1 \eta^{\frac{12(p-1)}{5(p+1)}} (M^{\frac{\delta}{2}} E)^{-\frac{9}{5(p+1)}} M^{-\frac{\delta}{2(p+1)}}.$$

Choosing θ to be

$$\theta := \frac{C_1}{2C'} \eta^{\frac{12(p-1)}{5(p+1)}} (M^{\frac{\delta}{2}} E)^{-\frac{9}{5(p+1)}} M^{-\frac{\delta}{2(p+1)}} T_2^{-\frac{\delta}{(p+1)(p-1)}},$$

we reach a contradiction with (24). Let us now verify the hypothesis on θ: We observe that

$$ET_2^{-\frac{\delta}{p-1}} \theta^{-(p+1)} = (C_1(2C')^{-1})^{-(p+1)} \eta^{-\frac{12(p-1)}{5}} (EM^{\frac{\delta}{2}})^{\frac{14}{5}}$$

such that hypothesis (22) is enforced if

$$0 < \eta < (C_1^{-1} 2C')^{\frac{5(p+1)}{12(p-1)}} (EM^{\frac{\delta}{2}})^{\frac{7}{6(p-1)}}.$$

This explains the hypotheses (12) and (13) with the choice

$$C_2 := (C_1^{-1} 2C')^{\frac{5(p+1)}{12(p-1)}}.$$

We also rewrite the largeness hypothesis on I, namely the second formula in (23), in terms of η,

$$\begin{aligned} \theta^{-\frac{(p+1)(p-1)}{2}} &= (C_1(2C')^{-1})^{-\frac{(p+1)(p-1)}{2}} \eta^{-\frac{6(p-1)^2}{5}} (EM^{\frac{\delta}{2}})^{\frac{9(p-1)}{10}} M^{\frac{\delta(p-1)}{4}} T_2^{\frac{\delta}{2}} \\ &= (C_2\eta^{-1})^{\frac{6(p-1)^2}{5}} (M^{\frac{p-1}{2}} T_2)^{\frac{\delta}{2}} (EM^{\frac{\delta}{2}})^{\frac{9(p-1)}{10}}, \end{aligned}$$

so that

$$\begin{aligned} &\max\{1, \theta^{-\frac{(p+1)(p-1)}{2}}\} \\ &\quad = (C_2\eta^{-1})^{\frac{6(p-1)^2}{5}} (EM^{\frac{\delta}{2}})^{\frac{9(p-1)}{10}} \max\{(C_2\eta^{-1})^{\frac{-6(p-1)^2}{5}} (EM^{\frac{\delta}{2}})^{\frac{-9(p-1)}{10}}, (M^{\frac{p-1}{2}} T_2)^{\frac{\delta}{2}}\}. \end{aligned}$$

This shows that (14) implies the second inequality in (23). □

5. Asymptotic stability

Let $u : \mathbb{R}^3 \times I \to \mathbb{R}$ solve an inhomogeneous wave equation $\Box u = F$. We now introduce the free evolution u_{l,t_0} from time t_0, that is, the unique solution of the free wave equation $\Box u_{l,t_0} = 0$ which agrees with u at time t_0, i.e., $(u_{l,t_0}, \partial_t u_{l,t_0})(t_0) = (u, \partial_t u)(t_0)$. We recall that, from solving the linear wave equation in Fourier space, we have the representation formula

$$u_{l,t_0}(t) = \cos(t\sqrt{-\Delta}) u(t_0) + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} \partial_t u(t_0),$$

where we use Fourier multiplier notation; see for instance [Sogge 1995]. From this representation as well as the Strichartz estimate (7), it follows that for any $m \in \left[1, \frac{3}{2}\right)$ and any (p, q) satisfying (6) we have the estimate

$$\|(u_{l,t_0}, \partial_t u_{l,t_0})\|_{L^\infty(I,\dot H^m\times \dot H^{m-1})} + \|u_{l,t_0}\|_{L^{2(p-1)}(\mathbb{R}^3\times I)} \lesssim \|(u, \partial_t u)(t_0)\|_{\dot H^m\times \dot H^{m-1}}. \tag{25}$$

From Duhamel's principle it follows that we can write for $t \in I$

$$u(t) = u_{l,t_0}(t) + \int_{t_0}^{t} \frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} F(t')\,\mathrm{d}t'. \tag{26}$$

We recall from [Shatah and Struwe 1998, Chapter 4] that for $t \neq t'$ we have the explicit expression

$$\frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} F(t') = \frac{1}{4\pi(t-t')} \int_{|x-x'|=|t-t'|} F(t',x')\,\mathrm{d}\mathcal{H}^2(x').$$

We recall that the linear evolution enjoys asymptotic stability in the following sense.

Lemma 5.1 (asymptotic stability for the linear evolution). *Let $p = 5+\delta$ with $\delta \in (0,1)$. Let u be a solution to* (1) *on $\mathbb{R}^3 \times I'$ with $\|u\|_{L^\infty(\mathbb{R}^3\times I')} \le M$. Then for any $I = [t_1,t_2] \subseteq I'$ and any $t \in I' \setminus I$ we have*

$$\|u_{l,t_2}(t) - u_{l,t_1}(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim (EM^{\frac{\delta}{2}})^{\frac{2p}{3(p-1)}} \operatorname{dist}(t,I)^{-\frac{2}{p-1}}.$$

Proof. From (5) we deduce that

$$\partial_t e(t) \ge \int_{|x|=t} \frac{|u(y,t)|^{p+1}}{p+1}\,\mathrm{d}\mathcal{H}^2(y).$$

Integrating in time, by translation invariance and time reversibility, we have

$$\int_I \int_{|x'-x|=|t'-t|} |u(x',t')|^{p+1}\,\mathrm{d}\mathcal{H}^2(x')\,\mathrm{d}t' \lesssim E$$

for any $(x,t) \in \mathbb{R}^3 \times I'$. Using (26), we write for $t \in I' \setminus I$

$$u_{l,t_2}(t) - u_{l,t_1}(t) = -\frac{1}{4\pi} \int_{t_1}^{t_2} \frac{1}{|t-t'|} \int_{|x-x'|=|t-t'|} |u(x',t')|^p\,\mathrm{d}\mathcal{H}^2(x')\,\mathrm{d}t'.$$

We apply Hölder with

$$\left(\frac{3(p-1)}{2p}, \frac{3(p-1)}{p-3}\right) = \left(\frac{p+1+\frac{\delta}{2}}{p}, \frac{p+1+\frac{\delta}{2}}{1+\frac{\delta}{2}}\right)$$

to estimate for any $x \in \mathbb{R}^3$

$$\begin{aligned}
&|u_{l,t_2}(x,t) - u_{l,t_1}(x,t)| \\
&\qquad\lesssim \int_{t_1}^{t_2} \frac{1}{|t-t'|} \int_{|x-x'|=|t-t'|} |u(x',t')|^p\,\mathrm{d}\mathcal{H}^2(x')\,\mathrm{d}t' \\
&\qquad\lesssim \left(\int_{t_1}^{t_2}\int_{|x-x'|=|t-t'|} |u|^{p+1+\frac{\delta}{2}}(x',t')\,\mathrm{d}\mathcal{H}^2(x')\,\mathrm{d}t'\right)^{\frac{2p}{3(p-1)}} \left(\int_{t_1}^{t_2} \frac{\mathrm{d}t'}{|t-t'|^{\frac{3(p-1)}{p-3}-2}}\right)^{\frac{p-3}{3(p-1)}} \\
&\qquad\lesssim \left(\|u\|^{\frac{\delta}{2}}_{L^\infty(\mathbb{R}^3\times I)} \int_{t_1}^{t_2}\int_{|x-x'|=|t-t'|} |u|^{p+1}(x',t')\mathrm{d}\mathcal{H}^2(x')\,\mathrm{d}t'\right)^{\frac{2p}{3(p-1)}} \operatorname{dist}(t,I)^{-\frac{2}{p-1}} \\
&\qquad\lesssim (M^{\frac{\delta}{2}}E)^{\frac{2p}{3(p-1)}} \operatorname{dist}(t,I)^{-\frac{2}{p-1}}.
\end{aligned}$$

□

The importance of the above asymptotic stability lies in the following corollary.

Corollary 5.2. *Let $p = 5 + \delta$ with $\delta \in (0,1)$ and $I = [t_-, t_+]$. Consider a solution $(u, \partial_t u) \in L^\infty(I, \dot{H}^1 \cap \dot{H}^2 \times H^1)$ to* (1) *and assume that $\|u\|_{L^\infty(\mathbb{R}^3\times I)} \le M$. Consider $I_1 = [t_1, t_2]$ and $I_2 = [t_2, t_3]$ for any $t_- \le t_1 < t_2 < t_3 \le t_+$. Then*

$$\|u_{l,t_3} - u_{l,t_+}\|_{L^{2(p-1)}(\Gamma_+(I_1))} \lesssim \frac{|I_1|^{\frac{1}{2(p-1)}}}{|I_2|^{\frac{1}{2(p-1)}}} (EM^{\frac{\delta}{2}})^{\frac{p}{6(p-1)}} \|u\|^{\frac{3}{4}}_{L^\infty(I,(\dot{H}^{s_p}\times\dot{H}^{s_p-1}))}.$$

Proof. We observe that the pair $\big(\infty, \frac{3}{2}(p-1)\big)$ is wave-s_p-admissible, where we recall that $s_p := 1 + \delta/(2(p-1))$ is the critical Sobolev regularity of (1). We estimate by Hölder

$$\|u_{l,t_3} - u_{l,t_+}\|_{L^{2(p-1)}(\Gamma_+(I_1))} \lesssim |I_1|^{\frac{1}{2(p-1)}} \|u_{l,t_2} - u_{l,t_3}\|^{\frac{1}{4}}_{L^\infty(\mathbb{R}^3\times I_1)} \|u_{l,t_3} - u_{l,t_+}\|^{\frac{3}{4}}_{L^\infty L^{3(p-1)/2}(\Gamma_+(I_1))}.$$

Observe that $v := u_{l,t_3} - u_{l,t_+}$ solves $\Box v = 0$ with $v(t_3) = u(t_3) - u_{l,t_+}(t_3)$. Hence by the Strichartz estimate (7) and (25) we have

$$\begin{aligned}
\|v\|_{L^\infty L^{3(p-1)/2}(\Gamma_+(I_1))} &\lesssim \|(v, \partial_t v)(t_3)\|_{(\dot{H}^{s_p}\times\dot{H}^{s_p-1})(\mathbb{R}^3)} \\
&\lesssim \|(u, \partial_t u)(t_3)\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}} + \|(u_{l,t_+}, \partial_t u_{l,t_+})(t_3)\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}} \\
&\lesssim \|(u, \partial_t u)(t_3)\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}} + \|(u, \partial_t u)(t_+)\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}} \\
&\lesssim \|(u, \partial_t u)\|_{L^\infty(I,(\dot{H}^{s_p}\times\dot{H}^{s_p-1}))}.
\end{aligned}$$

□

6. A reverse Sobolev inequality and mass concentration

The section is devoted to proving that, if u solves (1), then there exists a suitable ball with controlled size which contains an amount of L^2 norm, quantified in terms of $\|u\|_{L^{2(p-1)}}$ and $\|u\|_{H^s}$. A key ingredient in the proof is the reverse Sobolev inequality of Tao, generalized for any $s \in \big(0, \frac{3}{2}\big)$. We present the proof for completeness, since the original argument used the fact that p was integer.

Proposition 6.1. *Let $0 < s < \frac{3}{2}$ and $\frac{1}{q} := \frac{1}{2} - \frac{s}{3}$. Let $f \in \dot{H}^s(\mathbb{R}^3)$. Then there exists $x \in \mathbb{R}^3$ and $0 < r \le \frac{2}{N}$ such that*

$$\left(\frac{1}{r^{2s}} \int_{B(x,r)} f^2(y)\,\mathrm{d}y\right)^{\frac{1}{2}} \gtrsim \|P_{\ge N} f\|^{\left(\frac{3}{2s}\right)^2}_{L^q(\mathbb{R}^3)} \|f\|^{1-\left(\frac{3}{2s}\right)^2}_{\dot{H}^s}. \tag{27}$$

Proof. By replacing f with $\tilde{f}(x) := (1/\|f\|_{\dot{H}^s}) f(x)$ we can assume without loss of generality that $\|f\|_{\dot{H}^s} = 1$.

Step 1: *Let $g \in \dot{H}^s$ with $\|g\|_{\dot{H}^s} \le 1$. Then there exists $\overline{N} \in 2^{\mathbb{Z}}$ such that*

$$\|g\|^{\frac{3}{2s}}_{L^q} \lesssim \|P_{\overline{N}} g\|_{L^q}, \tag{28}$$

and as a consequence

$$\|g\|^{\left(\frac{3}{2s}\right)^2}_{L^q} \overline{N}^{\frac{3}{q}} \lesssim \|P_{\overline{N}} g\|_{L^\infty}. \tag{29}$$

From (10), Plancherel's theorem and the hypothesis $\|g\|_{\dot H^s} \le 1$, we infer that

$$\sum_{N\in 2^{\mathbb{Z}}} N^{2s}\|P_N g\|_{L^2}^2 \lesssim 1. \tag{30}$$

By interpolation, (30) and the definition of q we see that (29) is a consequence of (28); indeed

$$\|P_{\overline{N}} g\|_{L^q} \le \|P_{\overline{N}} g\|_{L^2}^{\frac{2}{q}} \|P_{\overline{N}} g\|_{L^\infty}^{1-\frac{2}{q}} = \overline{N}^{-\frac{2s}{q}} (\overline{N}^{2s}\|P_{\overline{N}} g\|_{L^2}^2)^{\frac{1}{q}} \|P_{\overline{N}} g\|_{L^\infty}^{1-\frac{2}{q}} \lesssim \overline{N}^{-\frac{2s}{q}} \|P_{\overline{N}} g\|_{L^\infty}^{\frac{2s}{3}}.$$

We are left to prove (28). Let us fix $M\in\mathbb{N}$ big enough such that $\frac{q}{2}\in(M-1,M]$. With this choice of M, we ensure the subadditivity of the map $x\mapsto x^{q/(2M)}$. We then write, using the hypothesis, (10), the aforementioned subadditivity, a reordering and Hölder,

$$\begin{aligned}
\|g\|_{L^q}^q &\lesssim \int\Big(\sum_{M\in 2^{\mathbb{Z}}} |P_M g(x)|^2\Big)^{\frac{q}{2}}\,\mathrm{d}x = \int\prod_{i=1}^M\Big(\sum_{N_i\in 2^{\mathbb{Z}}}|P_{N_i} g(x)|^2\Big)^{\frac{q}{2M}}\,\mathrm{d}x\\
&\le \int\prod_{i=1}^M\sum_{N_i\in 2^{\mathbb{Z}}}|P_{N_i} g(x)|^{\frac{q}{M}}\,\mathrm{d}x \lesssim \sum_{N_1\le\cdots\le N_M}\int\prod_{i=1}^M |P_{N_i} g(x)|^{\frac{q}{M}}\,\mathrm{d}x\\
&\lesssim \Big(\sup_{N\in 2^{\mathbb{Z}}}\|P_N g\|_{L^q}\Big)^{\frac{q(M-2)}{M}} \sum_{N_1\le\cdots\le N_M}\Big(\int |P_{N_1} g(x)|^{\frac{q}{2}}|P_{N_M} g(x)|^{\frac{q}{2}}\,\mathrm{d}x\Big)^{\frac{2}{M}}.
\end{aligned}$$

In all sums on $N_1\le\cdots\le N_M$, we intend that each N_i belongs to $2^{\mathbb{Z}}$. We claim that the second factor is bounded by a constant. Indeed, we estimate the last integral for fixed N_1 and N_M using Hölder by

$$\begin{aligned}
&\Big(\int |P_{N_1} g(x)|^{\frac{q}{2}}|P_{N_M} g(x)|^{\frac{q}{2}}\,\mathrm{d}x\Big)^{\frac{2}{M}}\\
&\qquad\le \Big(\|P_{N_1} g\|_{L^\infty}^{\frac{M}{2}} \int |P_{N_1} g(x)|^{\frac{q-M}{2}} |P_{N_M} g(x)|^{\frac{q-M}{2}} |P_{N_M} g(x)|^{\frac{M}{2}}\,\mathrm{d}x\Big)^{\frac{2}{M}}\\
&\qquad\le \|P_{N_1} g\|_{L^\infty} \|P_{N_1} g\|_{L^q}^{\frac{q-M}{M}} \|P_{N_M} g\|_{L^q}^{\frac{q-M}{M}} \|P_{N_M} g\|_{L^{q/2}}.
\end{aligned}$$

By Bernstein's inequality (9) and the definition of q, we have

$$\|P_{N_1} g\|_{L^\infty}\|P_{N_M} g\|_{L^{\frac{q}{2}}} \lesssim N_1^{\frac{3}{2}} N_M^{\frac{3}{2}-\frac{6}{q}} \|P_{N_1} g\|_{L^2}\|P_{N_M} g\|_{L^2} = N_1^{\frac{3}{2}} N_M^{2s-\frac{3}{2}} \|P_{N_1} g\|_{L^2}\|P_{N_M} g\|_{L^2}.$$

Combining the three estimates, we deduce that

$$\begin{aligned}
\|g\|_{L^q}^q &\lesssim \Big(\sup_{N\in 2^{\mathbb{Z}}}\|P_N g\|_{L^q}\Big)^{q-2} \sum_{N_1\le\cdots\le N_M} \|P_{N_1} g\|_{L^\infty}\|P_{N_M} g\|_{L^{q/2}}\\
&\lesssim \Big(\sup_{N\in 2^{\mathbb{Z}}}\|P_N g\|_{L^q}\Big)^{q-2} \sum_{N_1\le\cdots\le N_M} N_1^{\frac{3}{2}-s} N_M^{s-\frac{3}{2}} (N_1^{2s}\|P_{N_1} g\|_{L^2}^2 + N_M^{2s}\|P_{N_M} g\|_{L^2}^2).
\end{aligned}$$

Let us consider the first addend on the right-hand side (the second is handled analogously):

$$\begin{aligned}
\sum_{N_1\le\cdots\le N_M} N_1^{\frac{3}{2}-s} N_M^{s-\frac{3}{2}} N_1^{2s}\|P_{N_1} g\|_{L^2}^2 &\le \sum_{n_1\in\mathbb{Z}} 2^{2n_1 s}\|P_{2^{n_1}} g\|_{L^2}^2 \sum_{n_M=n_1}^{\infty} (n_M-n_1)^{M-2} 2^{-(\frac{3}{2}-s)(n_M-n_1)}\\
&\lesssim \sum_{n_1\in\mathbb{Z}} 2^{2n_1 s}\|P_{2^{n_1}} g\|_{L^2}^2 \lesssim 1,
\end{aligned}$$

where we used that for fixed $s \in \left(0, \frac{3}{2}\right)$ the series $\| P_{2^{n_1}} g\|_{L^2}^2 \sum_{n=0}^{\infty} n^{M-2} 2^{-(3/2-s)n}$ converges for every $M \in \mathbb{N}$ as well as (30). We conclude from (31) that

$$\|g\|_{L^q}^{\frac{3}{2s}} = \|g\|_{L^q}^{\frac{q}{q-2}} \lesssim \sup_{N \in 2^{\mathbb{Z}}} \|P_N\|_{L^q},$$

which implies (28).

Step 2: *Let* $\overline{N}, N \in 2^{\mathbb{Z}}$ *and define* $\psi_{\overline{N}} := \overline{N}^3 \psi(\overline{N} x)$, *where* ψ *is a bump function supported in* $B_1(0)$ *whose Fourier transform has magnitude* ~ 1 *on* $B_{100}(0)$. *Then we can rewrite*

$$P_{\overline{N}} P_{\geq N} f = \widetilde{P}_{\overline{N}}(f * \psi_{\overline{N}}),$$

where $\widetilde{P}_{\overline{N}}$ *is a Fourier multiplier which is bounded on* L^∞.

The claimed identity of Fourier multipliers follows by setting $\mathcal{F}(\widetilde{P}_{\overline{N}})(\xi) := \Psi(\xi/\overline{N})$, where

$$\Psi(\xi) := (\varphi(\xi) - \varphi(2\xi))(1 - \varphi(\xi \overline{N}/N))\hat{\psi}(\xi)^{-1}.$$

To verify that $\widetilde{P}_{\overline{N}}$ is bounded on L^∞, for $g \in L^\infty$ we estimate by Young and a change of variables

$$\|\widetilde{P}_{\overline{N}} g\|_{L^\infty} \lesssim \|\mathcal{F}^{-1}(\Psi(\xi/\overline{N}))\|_{L^1} \|g\|_{L^\infty} = \|\mathcal{F}^{-1}(\Psi)\|_{L^1} \|g\|_{L^\infty}.$$

Observe that $\Psi \in C_c^\infty(\mathbb{R}^3) \subseteq \mathcal{S}(\mathbb{R}^3)$, so that $\|\mathcal{F}^{-1}(\Psi)\|_{L^1} < +\infty$.

Step 3: *Conclusion of the proof.*

We apply Step 1 to $g = P_{\geq N} f$ to deduce that there exist $\overline{N} \in 2^{\mathbb{Z}}$ such that

$$\|P_{\geq N} f\|_{L^q}^{\left(\frac{3}{2s}\right)^2} \overline{N}^{\frac{3}{q}} \lesssim \|P_{\overline{N}} P_{\geq N} f\|_{L^\infty}.$$

We observe that $\overline{N} \geq \frac{N}{2}$ because otherwise $P_{\overline{N}} P_{\geq N} f = 0$. By Step 2, we deduce that there exists $x \in \mathbb{R}^3$ such that

$$\|P_{\geq N} f\|_{L^q}^{\left(\frac{3}{2s}\right)^2} \overline{N}^{\frac{3}{q}} \lesssim |\psi_{\overline{N}} * f(x)| \leq \overline{N}^{\frac{3}{2}} \left(\int_{B(x, \frac{1}{\overline{N}})} f^2(y) \, dy \right)^{\frac{1}{2}} \|\psi\|_{L^2}.$$

Combining the two inequalities, we obtain the claimed inequality (27) with $r := \frac{1}{\overline{N}} \in \left(0, \frac{2}{N}\right]$. □

The proposition above will be applied with $s = s_p$; the choice of $s \neq 1$ is in turn fundamental in the main theorem, since it allows us to give an upper bound on the r_0 given by the mass concentration only in terms of E, M, $\|u\|_{L^\infty \dot{H}^{s_p}}$.

Lemma 6.2 (mass concentration). *Let* $p = 5 + \delta$ *for* $\delta \in (0, 1)$ *and let* $0 < \eta \leq 1$. *Assume*

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times I)} \geq \eta \quad \text{and} \quad \|u\|_{L^\infty(\mathbb{R}^3 \times I)} \leq M.$$

Then, for any $1 \leq s \leq s_p := 1 + \delta/(2(p-1))$ *there exists* $(x, t) \in \mathbb{R}^3 \times I$ *and* $r > 0$ *such that*

$$\frac{1}{r^{2s}} \int_{B(x,r)} u^2(y,t) \, dy \gtrsim \|u\|_{L^\infty(I, \dot{H}^{s_p}(\mathbb{R}^3))}^{-\alpha_0} (M^{\frac{\delta}{2}} E)^{-\alpha_1} M^{-(s_p - s)(p-1)} \eta^{\alpha_2}, \tag{31}$$

where $\alpha_i = \alpha_i(s) \geq 0$ are defined as

$$\alpha_0 := (\gamma-2)\frac{s-1}{s_p-1}, \quad \alpha_1 := \frac{3}{10}\gamma(3-2s)+\frac{\gamma-2}{2}\frac{s_p-s}{s_p-1} \quad \textit{and} \quad \alpha_2 := \frac{3-2s}{5}2(p-1)\gamma \quad \textit{for } \gamma := \frac{9}{2s^2}.$$

Moreover,

$$|I| \gtrsim \eta^{2(p-1)}\|u\|^{-\alpha_0'}_{L^\infty(I,\dot H^{s_p}(\mathbb{R}^3))}(EM^{\frac{\delta}{2}})^{-\alpha_1'}M^{\frac{(s-1)(p-1)}{2}}r^s, \tag{32}$$

where $\alpha_i'(s) \geq 0$ are defined as

$$\alpha_0' := 2(p-1)-\frac{(s-1)(p-1)(p+1)}{\delta} \quad \textit{and} \quad \alpha_1' := \frac{(s-1)(p-1)}{\delta}.$$

Proof. Fix $1 \leq s \leq s_p = 1+\delta/(2(p-1))$ and set

$$\frac{1}{q} := \frac{1}{2}-\frac{s}{3},$$

the conjugate Sobolev exponent. By shrinking I, we can always assume that $\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times I)} = \eta$. Recalling the proof of Lemma 4.4, we have that for any wave-1-admissible (q,r)

$$\|u\|_{L^qL^r} \lesssim E^{\frac{1}{2}}. \tag{33}$$

Step 1: *We find a frequency scale $N \in 2^{\mathbb{Z}}$, where $\|P_{\geq N}f\|_{L^{2(p-1)}(\mathbb{R}^3\times I)} \gtrsim \eta$.*

By Hölder and Bernstein (9) with exponents $2(p-1)$ and $6(p-1)/(s+3) \in [6,q^*]$ we estimate

$$\|P_{<N}u\|_{L^{2(p-1)}} \lesssim |I|^{\frac{1}{2(p-1)}}\|P_{<N}u\|_{L^\infty L^{2(p-1)}} \lesssim |I|^{\frac{1}{2(p-1)}}N^{\frac{s}{2(p-1)}}\|u\|_{L^\infty L^{6(p-1)/(s+3)}}.$$

We observe that by interpolation and the Sobolev embedding of $\dot H^{s_p} \hookrightarrow L^{3(p-1)/2}$

$$\|u\|_{L^\infty L^{6(p-1)/(s+3)}} \leq \|u\|^{1-\frac{(s-1)(p+1)}{2\delta}}_{L^\infty \frac{3(p-1)}{2}}\|u\|^{\frac{(s-1)(p+1)}{2\delta}}_{L^\infty L^{p+1}} \lesssim \|u\|^{1-\frac{(s-1)(p+1)}{2\delta}}_{L^\infty \dot H^{s_p}}(EM^{\frac{\delta}{2}})^{\frac{(s-1)}{2\delta}}M^{-\frac{(s-1)}{4}}.$$

Thus if we choose the frequency scale $N \in 2^{\mathbb{Z}}$ such that

$$|I|^{\frac{1}{2(p-1)}}N^{\frac{s}{2(p-1)}}\|u\|^{1-\frac{(s-1)(p+1)}{2\delta}}_{L^\infty \dot H^{s_p}}(EM^{\frac{\delta}{2}})^{\frac{(s-1)}{2\delta}}M^{-\frac{(s-1)}{4}} = c\eta \tag{34}$$

for a universal small constant $0 < c \ll 1$, we can ensure that $\|P_{\geq N}u\|_{L^{2(p-1)}(\mathbb{R}^3\times I)} \gtrsim \eta$.

Step 2: *We deduce a lower bound of $\|P_{\geq N}u\|_{L^\infty(I,L^q(\mathbb{R}^3))}$ in terms of η, E, M.*

Observe that the pair $(3,18)$ is wave-1-admissible and $(3,18)$ and (∞,q) interpolate to $\left(\frac{5}{6}q+3,\frac{5}{6}q+3\right)$. Using (33) and (34), we have by Hölder

$$\begin{aligned}\eta^{2(p-1)} &\lesssim \|P_{\geq N}u\|^{2(p-1)}_{L^{2(p-1)}_{t,x}} \lesssim \|P_{\geq N}u\|^{2(p-1)-(\frac{5}{6}q+3)}_{L^\infty_{t,x}}\|P_{\geq N}u\|^{\frac{5}{6}q+3}_{L^{(5/6)q+3}_{t,x}}\\ &\lesssim M^{5+2\delta-\frac{5}{6}q}\|P_{\geq N}u\|^3_{L^3L^{18}}\|P_{\geq N}u\|^{\frac{5}{6}q}_{L^\infty L^q}\\ &\lesssim M^{\frac{5}{6}q(\frac{6}{q}+\frac{3}{2q}\delta-1)}(M^{\frac{\delta}{2}}E)^{\frac{3}{2}}\|P_{\geq N}u\|^{\frac{5}{6}q}_{L^\infty L^q};\end{aligned}$$

hence after some easy algebraic manipulations

$$\begin{aligned}\|P_{\geq N}u\|_{L^\infty L^q} &\gtrsim \eta^{\frac{12}{5q}(p-1)}(M^{\frac{\delta}{2}}E)^{-\frac{9}{5q}}M^{-(\frac{6}{q}+\frac{3}{2q}\delta-1)}\\ &= \eta^{\frac{(3-2s)}{5}2(p-1)}(M^{\frac{\delta}{2}}E)^{-\frac{3}{10}(3-2s)}M^{-\frac{1}{2}(s_p-s)(p-1)}.\end{aligned}$$

Step 3: We apply the reverse Sobolev of Proposition 6.1 to conclude that there exists $(x,t)\in\mathbb{R}^3\times I$ and $0<r\leq\frac{2}{N}$ such that

$$\frac{1}{r^{2s}}\int_{B(x,r)}u^2(y,t)\,\mathrm{d}y \gtrsim \|u\|^{2-\gamma}_{L^\infty(I,\dot H^s(\mathbb{R}^3))}(\eta^{\frac{(3-2s)}{5}2(p-1)}(M^{\frac{\delta}{2}}E)^{-\frac{3}{10}(3-2s)}M^{-\frac{1}{2}(s_p-s)(p-1)})^\gamma, \quad (35)$$

where $\gamma:=9/(2s^2)$. Moreover from (34) we get

$$\begin{aligned}|I| &= \frac{(c\eta)^{2(p-1)}M^{\frac{(s-1)(p-1)}{2}}}{\|u\|^{2(p-1)-\frac{(s-1)(p-1)(p+1)}{\delta}}_{L^\infty\dot H^{s_p}}(EM^{\frac{\delta}{2}})^{\frac{(s-1)(p-1)}{\delta}}N^s}\\ &\gtrsim \eta^{2(p-1)}\frac{M^{\frac{(s-1)(p-1)}{2}}}{\|u\|^{2(p-1)-\frac{(s-1)(p-1)(p+1)}{\delta}}_{L^\infty\dot H^{s_p}}(EM^{\frac{\delta}{2}})^{\frac{(s-1)(p-1)}{\delta}}}r^s.\end{aligned}$$

We now rewrite (35): by interpolation and energy conservation,

$$\|u\|_{L^\infty\dot H^s}\leq E^{\frac{(s_p-s)(p-1)}{\delta}}\|u\|^{\frac{2(s-1)(p-1)}{\delta}}_{L^\infty\dot H^{s_p}}.$$

Observe that $\gamma\geq 2$ for $s\in\left(0,\frac{3}{2}\right)$. Thus we have

$$\|u\|^{2-\gamma}_{L^\infty\dot H^s}\gtrsim (M^{\frac{\delta}{2}}E)^{\frac{(s_p-s)(p-1)(2-\gamma)}{\delta}}\|u\|^{\frac{2(s-1)(p-1)(2-\gamma)}{\delta}}_{L^\infty\dot H^{s_p}}M^{\frac{(s_p-s)(p-1)(\gamma-2)}{2}},$$

so that

$$\begin{aligned}&\frac{1}{r^{2s}}\int_{B(x,r)}u^2(y,t)\,\mathrm{d}y\\ &\qquad\gtrsim \|u\|^{-(\gamma-2)\frac{s-1}{s_p-1}}_{L^\infty\dot H^{s_p}}(M^{\frac{\delta}{2}}E)^{-[\frac{3}{10}\gamma(3-2s)+\frac{\gamma-2}{2}\frac{s_p-s}{s_p-1}]}M^{-(s_p-s)(p-1)}\eta^{\frac{3-2s}{5}2(p-1)\gamma}. \quad\square\end{aligned}$$

Remark 6.3 (optimization of exponents on η, $\|u\|_{L^\infty\dot H^{s_p}}$ and $EM^{\delta/2}$). Whilst the free powers of M in (31) and (32) are fixed by scaling, the other powers come from interpolation and can be optimized. Since we are not aiming at an optimal double exponential bound, we can take in Step 2 of the proof of Lemma 6.2 any Strichartz-1-pair (q',r') (here $(3,18)$) such that (∞,q) and (q',r') interpolate to $(\tilde r,\tilde r)$ with $\tilde r\leq 2(p-1)$. Alternatively, to optimize the exponents α_1 and α_2, we first suppose that the endpoint $(2,\infty)$ was Strichartz-1-admissible, interpolate in Step 2 between $(2,\infty)$ and (∞,q) and conclude in Step 3 as before. We then approximate $(2,\infty)$ by wave-1-admissible pairs $(2+\epsilon,6(2+\epsilon)/\epsilon)$. Letting $\epsilon\to 0$,

$$\frac{3-2s}{6}(\gamma)+\frac{\gamma-2}{2}\frac{s_p-s}{s_p-1}$$

and $\alpha_2(s)$ approaches

$$\frac{3-2s}{3}(p-1).$$

In the very same way, the free exponents in Lemma 4.4 can be optimized. Proceeding in this way, we would obtain the lower bound, for any $\omega > 0$ (and an implicit constant depending on ω),

$$\|u\|_{L^\infty L^{p+1}}^{p+1} \gtrsim \eta^{2(p-1)+\omega}(EM^{\frac{\delta}{2}})^{-(1+\omega)}M^{-\frac{\delta}{2}}.$$

7. Proof of Theorem 1.2 and Corollary 1.3

We have now assembled all necessary tools to prove Theorem 1.2. We outline now its main steps which are based on the scheme of [Tao 2006b] and its adaptation in [Roy 2009].

Let $(u, \partial_t u) \in L^\infty(J, \dot{H}^1 \cap \dot{H}^2 \times H^1)$ solve (1). Whenever the scaling-invariant smallness assumption of the first item of Theorem 1.2 holds, then Lemma 3.1 gives the desired spacetime bound. Otherwise, we split J into subintervals J_i such that on each subinterval the $L^{2(p-1)}$ spacetime is completely under control and substantial, i.e., $\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J_i)} = 1$ for all but eventually the last subinterval. The estimate (3) is then equivalent to estimating the number of such subintervals and relies on the following key arguments:

(i) Using Lemma 4.4, we deduce that on each J_i also the potential energy $L^\infty(J_i, L^{p+1})$ is substantial: it has a quantitative lower bound in terms of E, M and L.

(ii) Lemma 6.2 allows us to identify a ball $B = B(x_i, r_i)$ such that mass concentrates on B at time $t = t_i \in J_i$. The mass concentration can be extended to a neighborhood of t_i using that the local mass is Lipschitz in time. At the same time, the size of intervals J_i where such concentration happens is bounded from below in terms of E, M, L and r_i.

(iii) In the scheme of [Tao 2006b], the previous observation together with the finite speed of propagation is used to remove a cone in spacetime, containing the mass-concentration “bubble”, and to construct a new solution $\tilde{u}$ with smaller energy than u which coincides with u outside the cone. This allows us to perform an induction on the level sets of the energy since for sufficiently small energy the claimed estimate holds by Lemma 3.1. In our setting, such an induction argument seems not applicable, since the solution $\tilde{u}$ does not need to obey the same a priori bounds on the $L^\infty(J, \dot{H}^1 \cap \dot{H}^2 \times H^1)$ norm as u.

(iv) As in [Roy 2009] we bypass the induction on the energy by an ad-hoc argument. By time reversal and translation symmetry and the lower bound on the length of (ii), it is enough to estimate the length of $K_+ = J \cap [t_0, +\infty)$, where $(x_0 = 0, t_0)$ is the point where mass concentration occurs at the minimal mass concentration radius (among those individuated before). As in (ii), the mass concentration at minimal radius extends to a neighborhood $\tilde{J}_0$ of t_0. We then look at the truncated-in-time cone $\Gamma_+(K_+)$ and we define a new splitting of K_+ in subintervals $\tilde{J}_i$ such that the $L^{2(p-1)}$ norm on every truncated-in-time cone $\Gamma_+(\tilde{J}_i)$ is substantial and such that $\tilde{J}_1 \subset \tilde{J}_0$. The asymptotic stability of Section 5 controls inductively the size $|\tilde{J}_{j+1}| \lesssim |\tilde{J}_j|$. Moreover, the size of $\tilde{J}_0$ is controlled from below by the mass concentration argument in (ii) and from above by an upper bound on the mass concentration radius (which was not needed in [Roy 2009]). If $|K_+|$ was too large, then by the decay of the potential energy Proposition 4.6 there must be a subinterval such that on the truncated-in-time cone the $L^{2(p-1)}$ spacetime norm is small. By construction, such subinterval cannot be covered by many $\tilde{J}_i$, which means that one of them has to be sufficiently large, contradicting the upper bound on $|\tilde{J}_0|$.

Proof of Theorem 1.2. Let $p = 5 + \delta$ with $\delta \in (0, 1)$, $J = [t_-, t_+]$ and consider a solution $(u, \partial_t u) \in L^\infty(J, ((\dot{H}^1 \cap \dot{H}^2) \times H^1)(\mathbb{R}^3))$ to (1) as in the statement. If either $EM^{\delta/2} < c_0$ or $L < c_0$, then we conclude by Lemma 3.1 that $\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J)} \leq 1$. For the rest of the argument, we thus may assume the lower bound

$$\min\{EM^{\frac{\delta}{2}}, L\} \geq c_0,$$

where $c_0 > 0$ is the universal constant given by Lemma 3.1.

Let $C > 2c_0^{-2}$ be a universal constant that will be fixed at the end of the proof. The inequality imposed on C guarantees that $CLEM^{\delta/2} > 2$.

Moreover, we may assume without loss of generality that $\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J)} \geq 1$. We then split J into subintervals $J_1, \ldots, J_l$ such that

- $\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J_i)} = 1$ for $i = 1, \ldots, l-1$,
- $\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J_l)} \leq 1$.

We call J_i exceptional if

$$\|u_{l,t_+}\|_{L^{2(p-1)}(\mathbb{R}^3 \times J_i)} + \|u_{l,t_-}\|_{L^{2(p-1)}(\mathbb{R}^3 \times J_i)} \geq B_{\text{exc}}^{-1}$$

for some $B_{\text{exc}} \geq 1$ yet to be defined. We have by Strichartz estimates (7) that

$$\|u_{l,t_+}\|_{L^{2(p-1)}(\mathbb{R}^3 \times J)}, \ \|u_{l,t_-}\|_{L^{2(p-1)}(\mathbb{R}^3 \times J)} \lesssim L.$$

In particular, J cannot consist of too many exceptional intervals. More precisely, calling the number of exceptional intervals $N_{\text{exc}} := |\{i \in \{1, \ldots, l\} : J_i \text{ exceptional}\}|$, we have the bound

$$N_{\text{exc}} \lesssim LB_{\text{exc}}.$$

Between two exceptional intervals there can lie a chain $K = J_{i_0} \cup \cdots \cup J_{i_1}$ of unexceptional intervals. However, since a chain K of unexceptional intervals has to be confined between two exceptional intervals (or one of its endpoints is t_- or t_+), the number of chains of unexceptional intervals N_{chain} is comparable to N_{exc}, that is,

$$N_{\text{chain}} \lesssim N_{\text{exc}}.$$

For a chain $K = J_{i_0} \cup \cdots \cup J_{i_1}$ of unexceptional intervals, we define $N(K) := i_1 + 1 - i_0$ to be the number of intervals it is made of. Summarizing, we have

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J)}^{2(p-1)} \leq N_{\text{exc}} + N_{\text{chain}} \sup_K N(K) \lesssim LB_{\text{exc}}(1 + \sup_K N(K)).$$

The proof is thus concluded with the following lemma and with the choice of B_{exc} in (36) below. □

Lemma 7.1. *There exists a universal constant $C \geq 1$ such that the following holds: Consider a solution $(u, \partial_t u) \in L^\infty(J, (\dot{H}^1 \cap \dot{H}^2 \times H^1)(\mathbb{R}^3))$ of* (1) *with $p = 5 + \delta$, $\delta \in (0, 1)$. Define $M := \|u\|_{L^\infty(\mathbb{R}^3 \times J)}$, $E := E(u)$ and $L := \|(u, \partial_t u)\|_{L^\infty(J, (\dot{H}^{s_p} \times \dot{H}^{s_p - 1})(\mathbb{R}^3))}$ on $J = [t_-, t_+]$ and set*

$$B_{\text{exc}} := (CEM^{\frac{\delta}{2}}L)^{C(EM^{\delta/2}L)^{176}}. \tag{36}$$

Assume that $B_{\mathrm{exc}}^{\delta/2} \le 2$ and that

$$\min\{EM^{\frac{\delta}{2}}, L\} \ge c_0. \tag{37}$$

Then for any chain of unexceptional intervals, that is, for any $K = J_{i_0} \cup \cdots \cup J_{i_1} \subseteq J$ with

$$\begin{aligned} \|u\|_{L^{2(p-1)}(\mathbb{R}^3\times J_i)} &= 1, \\ \|u_{l,t_+}\|_{L^{2(p-1)}(\mathbb{R}^3\times J_i)} + \|u_{l,t_-}\|_{L^{2(p-1)}(\mathbb{R}^3\times J_i)} &\le B_{\mathrm{exc}}^{-1}, \end{aligned} \tag{38}$$

for all $i \in \{i_0, \dots, i_1\}$, we have the estimate

$$N(K) \lesssim B_{\mathrm{exc}}.$$

Proof of Lemma 7.1. Step 0: *Let α_0, α_0', α_1 and α_1' be defined through Lemma 6.2 for $s = s_p$, that is, for $\gamma := 2(3/(2s_p))^2 \in \left[\frac{7}{2}, \frac{9}{2}\right]$,*

$$\alpha_0 = \gamma - 2 \in \left[\tfrac{3}{2}, \tfrac{5}{2}\right], \quad \alpha_1 = \frac{6\gamma}{5(p-1)} \in \left[\tfrac{3}{4}, \tfrac{3}{2}\right], \quad \alpha_0' = 5 + \tfrac{3}{2}\delta \in \left[5, \tfrac{13}{2}\right] \quad \textit{and} \quad \alpha_1' = \tfrac{1}{2}. \tag{39}$$

We prove that there exists $(t_0, x_0, r_0) \in K \times \mathbb{R}^3 \times (0, +\infty)$ such that

(i) *mass concentrates in $B(x_0, r_0)$ at time t_0, i.e.,*

$$\frac{1}{r_0^{2s_p}} \int_{B(x_0,r_0)} u^2(y, t_0)\, \mathrm{d}y \ge C_6 L^{-\alpha_0} (EM^{\frac{\delta}{2}})^{-\alpha_1}, \tag{40}$$

(ii) *the length of the J_i is uniformly bounded from below in terms of r_0, i.e., for all $i = i_0, \dots, i_1$*

$$|J_i| \ge C_7 L^{-\alpha_0'} (EM^{\frac{\delta}{2}})^{-\alpha_1'} M^{\frac{\delta}{4}} r_0^{s_p}. \tag{41}$$

From (i), *we immediately also deduce the lower bound on the mass concentration radius*

$$r_0 \gtrsim (L^{-\alpha_0} (EM^{\frac{\delta}{2}})^{-\alpha_1})^{\frac{p-1}{4}} M^{-\frac{p-1}{2}}. \tag{42}$$

By (38), we can apply the mass concentration Lemma 6.2 with $\eta = 1$ and $s = s_p$ to find that for any $i \in \{i_0, \dots, i_1\}$ there exists $(t_i, x_i, r_i) \in J_i \times \mathbb{R}^3 \times (0, +\infty)$ such that

$$\begin{aligned} \frac{1}{r_i^{2s_p}} \int_{B(x_i,r_i)} u^2(y, t_i)\, \mathrm{d}y &\ge C_6 L^{-\alpha_0} (EM^{\frac{\delta}{2}})^{-\alpha_1}, \\ |J_i| &\ge C_7 L^{-\alpha_0'} (EM^{\frac{\delta}{2}})^{-\alpha_1'} M^{\frac{\delta}{4}} r_i^{s_p}. \end{aligned}$$

Defining the minimal mass concentration radius $r_0 := \min_{i\in\{i_0,\dots,i_1\}} r_i$ and calling the associated point in spacetime (x_0, t_0), we reach (i) and (ii). The lower bound on the mass concentration radius (42) is a consequence of the simple observation that the left-hand side of (40) can be bounded from above, up to constants, by $r_0^{3-2s_p} M^2 = r_0^{4/(p-1)} M^2$. By time and space translation symmetry, we can assume without loss of generality that $x_0 = 0$ and that $t_0 = r_0$ such that $B(x_0, r_0) \times \{t_0\}$ lies in the forward wave cone centered in $(0, 0)$. In view of (ii) it is enough to prove

$$|K| \lesssim L^{-\alpha_0'} (EM^{\frac{\delta}{2}})^{-\alpha_1'} M^{\frac{\delta}{4}} B_{\mathrm{exc}} r_0^{s_p}.$$

Moreover, by time reversal symmetry, it is enough to estimate $K_+ := K \cap [t_0, +\infty)$, i.e., to show

$$|K_+| \lesssim L^{-\alpha_0'}(EM^{\frac{\delta}{2}})^{-\alpha_1'} M^{\frac{\delta}{4}} B_{\mathrm{exc}} r_0^{s_p}. \tag{43}$$

Step 1: *We find a cylinder $B(x_0, r_0) \times \tilde{J}_0 \subseteq \Gamma_+(K_+)$ in spacetime such that*:

(i) *Mass still concentrates in $B(x_0, r_0)$ for any $t \in \tilde{J}_0$, i.e., for $t \in \tilde{J}_0$ it holds*

$$\frac{1}{r_0^{2s_p}} \int_{B(x_0,r_0)} u^2(y,t)\,\mathrm{d}y \geq \frac{C_6}{2} L^{-\alpha_0}(M^{\frac{\delta}{2}} E)^{-\alpha_1}. \tag{44}$$

(ii) *$\tilde{J}_0$ has controlled length, i.e.*,

$$L^{-\frac{\alpha_0}{2}}(M^{\frac{\delta}{2}} E)^{-\frac{\alpha_1+1}{2}} M^{\frac{\delta}{4}} r_0^{s_p} \lesssim |\tilde{J}_0| \leq M^{\frac{\delta}{4}} r_0^{s_p}.$$

(iii) *$\tilde{J}_0$ does not carry too much of the spacetime norm. More precisely*,

$$\|u\|^{2(p-1)}_{L^{2(p-1)}(\mathbb{R}^3 \times \tilde{J}_0)} \lesssim L^{\alpha_0' - \frac{\alpha_0}{2}}. \tag{45}$$

The local mass is Lipschitz in time with Lipschitz constant at most $\|\partial_t u\|_{L^\infty(J, L^2(\mathbb{R}^3))} \lesssim E^{1/2}$. More precisely, we have

$$\left| \left(\int_{B(x_0,r_0)} u^2(y,t)\,\mathrm{d}y \right)^{\frac{1}{2}} - \left(\int_{B(x_0,r_0)} u^2(y,t_0)\,\mathrm{d}y \right)^{\frac{1}{2}} \right| \lesssim E^{\frac{1}{2}} |t - t_0|.$$

In particular, if

$$E^{\frac{1}{2}} |t - t_0| \leq c_1 L^{-\frac{\alpha_0}{2}} (M^{\frac{\delta}{2}} E)^{-\frac{\alpha_1}{2}} r_0^{s_p}$$

for a universal $0 < c_1 \ll 1$ yet to be chosen sufficiently small, then we still have the mass concentration on the bubble $B(x_0, r_0) \times \tilde{J}_0$, where

$$\tilde{J}_0 := [t_0, t_0 + c_1 L^{-\frac{\alpha_0}{2}} (M^{\frac{\delta}{2}} E)^{-\frac{\alpha_1+1}{2}} M^{\frac{\delta}{4}} r_0^{s_p}].$$

More precisely, for any $t \in \tilde{J}_0$, (44) holds. We observe that

$$|\tilde{J}_0| = c_1 M^{\frac{\delta}{4}} L^{-\frac{\alpha_0}{2}} (EM^{\frac{\delta}{2}})^{-\frac{1}{2}(\alpha_1+1)} r_0^{s_p} \leq c_1 c_0^{-\frac{1}{2}(\alpha_0+\alpha_1+1)} M^{\frac{\delta}{4}} r_0^{s_p} \tag{46}$$

such that we can choose $c_1 < c_0^{5/2}$ to ensure (ii). Finally, if $K_+ \subset \tilde{J}_0$ is a strict subset, then $|K_+| \leq |\tilde{J}_0|$ and (43) holds (for big enough constants in the definition of B_{exc}). Thus we can assume that $\tilde{J}_0 \subseteq K_+$ and hence $B(x_0, r_0) \times \tilde{J}_0 \subseteq \Gamma_+(K_+)$. Finally, let us argue that $\tilde{J}_0$ cannot be covered by too many unexceptional intervals and thus cannot carry too much spacetime norm. Indeed, from (41), (46) and (37) we deduce that $\tilde{J}_0$ can be covered by at most

$$\frac{c_1 L^{-\frac{\alpha_0}{2}} (EM^{\frac{\delta}{2}})^{-\frac{1}{2}(\alpha_1+1)} M^{\frac{\delta}{4}} r_0^{s_p}}{C_7 L^{-\alpha_0'} (EM^{\frac{\delta}{2}})^{-\alpha_1'} M^{\frac{\delta}{4}} r_0^{s_p}} \lesssim L^{\alpha_0' - \frac{\alpha_0}{2}}$$

many intervals of the family $\{J_i\}_{i=i_0}^{i_1}$. Hence by (38) we deduce (45).

Step 2: *Let*

$$\tilde{\eta} := c_2(LEM^{\frac{\delta}{2}})^{-\frac{3}{2}} \in (0, c_0'), \tag{47}$$

with c_0' defined through Remark 4.2 (so that $\tilde{\eta}$ is admissible for the spacetime norm decay on large intervals). For a suitable choice of the universal constant c_2, we truncate $\Gamma_+(K_+)$ into wave cones $\{\Gamma_+(\tilde{J}_i)\}_{i=1}^k$ such that:

(i) *Each of them carries substantial spacetime norm $\tilde{\eta}$, i.e., $\|u\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_i))} = \tilde{\eta}$ for $i = 1, \dots, k-1$ and $\|u\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_k))} \leq \tilde{\eta}$.*

(ii) *The first interval is not too long, that is, $\tilde{J}_1 \subseteq \tilde{J}_0$.*

For an $\tilde{\eta}$ yet to be chosen, we will truncate $\Gamma_+(K_+)$ into wave cones $\{\Gamma_+(\tilde{J}_i)\}_{i=1}^k$ such that

$$\|u\|_{L^{2(p-1}(\Gamma_+(\tilde{J}_i))} = \tilde{\eta} \quad \text{for } i = 1, \dots, k-1 \qquad \text{and} \qquad \|u\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_k))} \leq \tilde{\eta}.$$

We come to the choice of $\tilde{\eta}$. Let us estimate the spacetime norm on the mass concentration cylinder from above

$$\int_{\tilde{J}_0}\int_{B(x_0,r_0)} u^2(y,t)\,\mathrm{d}y\,\mathrm{d}t \lesssim \left(\int_{\Gamma_+(\tilde{J}_0)} |u|^{2(p-1)}(y,t)\,\mathrm{d}y\,\mathrm{d}t\right)^{\frac{1}{p-1}} |\tilde{J}_0|^{\frac{p-2}{p-1}} r_0^{\frac{3(p-2)}{p-1}}$$

and from below, using (44),

$$\int_{\tilde{J}_0}\int_{B(x_0,r_0)} u^2(y,t)\,\mathrm{d}y\,\mathrm{d}t \gtrsim |\tilde{J}_0| L^{-\alpha_0}(M^{\frac{\delta}{2}}E)^{-\alpha_1} r_0^{2s_p}.$$

We have obtained, using the definition of $\tilde{J}_0$ from Step 1, that

$$\|u\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_0))} \gtrsim (L^{-\alpha_0}(EM^{\frac{\delta}{2}})^{-\alpha_1})^{\frac{2p-1}{4(p-1)}} (E^{-1} r_0^{\frac{\delta}{p-1}})^{\frac{1}{4(p-1)}}.$$

Using (45), we obtain an upper bound on r_0, that is,

$$\begin{aligned} r_0^{\delta} &\lesssim (L^{\alpha_0}(EM^{\frac{\delta}{2}})^{\alpha_1})^{(2p-1)(p-1)} E^{p-1} \|u\|^{4(p-1)^2}_{L^{2(p-1)}(\Gamma_+(\tilde{J}_0))} \\ &\lesssim (L^{\alpha_0}(EM^{\frac{\delta}{2}})^{\alpha_1})^{(2p-1)(p-1)} E^{p-1} L^{(\alpha_0' - \frac{\alpha_0}{2})2(p-1)} \\ &= M^{-\frac{\delta(p-1)}{2}} L^{2(p-1)(\alpha_0(p-1)+\alpha_0')} (EM^{\frac{\delta}{2}})^{(p-1)(\alpha_1(2p-1)+1)}. \end{aligned} \tag{48}$$

On the other hand, using the lower bound on r_0 given by (42), we can estimate furthermore, recalling (37) and (39), that

$$\begin{aligned} \|u\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_0))} &\gtrsim (L^{-\alpha_0}(EM^{\frac{\delta}{2}})^{-\alpha_1})^{\frac{2p-1}{4(p-1)} + \frac{\delta}{16(p-1)}} (EM^{\frac{\delta}{2}})^{-\frac{1}{4(p-1)}} \\ &= L^{-\frac{9}{16}\alpha_0} (EM^{\frac{\delta}{2}})^{-(\frac{9}{16}\alpha_1 + \frac{1}{4(p-1)})} \gtrsim (LEM^{\frac{\delta}{2}})^{-\frac{3}{2}}. \end{aligned}$$

Thus choosing $\tilde{\eta} := c_2(LEM^{\delta/2})^{-3/2}$, for a small universal constant $0 < c_2 < 1$, we ensure that $\tilde{J}_1 \subseteq \tilde{J}_0$. Choosing c_2 even smaller, namely $c_2 \leq c_0' c_0^3$, we ensure that $\tilde{\eta} \in (0, c_0')$, with c_0' given by Remark 4.2.

Step 3: *We prove the following dichotomy* (*analogous to* [Tao 2006b, Lemma 5.2]). *Let* $j \in \{1, \dots, k-1\}$. *Then*, *for some universal constants* $C_8 > 8$ *and* $C_9 < 1$, *either*

$$|\tilde{J}_{j+1}| \leq C_8 \tilde{\eta}^{-15} |\tilde{J}_j| \quad or \quad |\tilde{J}_j| \geq C_9 \tilde{\eta}^5 M^{\frac{\delta}{4}} B_{\text{exc}} r_0^{s_p}.$$

Consider two subsequent intervals $\tilde{J}_j = [t_{j-1}, t_j]$ and $\tilde{J}_{j+1} = [t_j, t_{j+1}]$ for some $j \in \{1, \dots, k-1\}$. We have by the localized Strichartz estimates (8) (with $(\tilde{q}, \tilde{r}) = (2, 6(p-1)/(3p+1))$ and $v := u - u_{l,t_{j+1}}$ solving $\Box v = |u|^{p-1}u$ with initial datum $(v, \partial_t v)(t_{j+1}) = (0, 0)$) and Hölder that

$$\begin{aligned}\|u - u_{l,t_{j+1}}\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} &\lesssim \||u|^{p-1}u\|_{L^{\tilde{q}}L^{\tilde{r}}(\Gamma_+(\tilde{J}_j \cup \tilde{J}_{j+1}))} \\ &\lesssim \|u\|_{L^\infty L^{3(p-1)/2}(\Gamma_+(\tilde{J}_j \cup \tilde{J}_{j+1}))} \|u\|^{p-1}_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j \cup \tilde{J}_{j+1}))} \\ &\lesssim \|u\|^{\frac{\delta}{3(p-1)}}_{L^\infty(\mathbb{R}^3 \times J)} \|u\|^{\frac{2(p+1)}{3(p-1)}}_{L^\infty L^{p+1}(\mathbb{R}^3 \times J)} \tilde{\eta}^{p-1} \lesssim (EM^{\frac{\delta}{2}})^{\frac{2}{3(p-1)}} \tilde{\eta}^{p-1}.\end{aligned}$$

Using (37) and (47), we have

$$\tilde{\eta}^{p-2}(EM^{\frac{\delta}{2}})^{\frac{2}{3(p-1)}} \leq c_2^{\frac{4}{9(p-1)}} L^{-\frac{4}{9(p-1)}} \tilde{\eta}^{p-2-\frac{4}{9(p-1)}} \leq (c_2 c_0^{-1})^{\frac{4}{9(p-1)}} \leq (c_0')^{\frac{4}{9(p-1)}} c_0^{\frac{8}{9(p-1)}} \leq c_0^{\frac{8}{9(p-1)}},$$

where we recall that from the choice of c_0 in Lemma 3.1, it is clear that it beats also the constant arising from Strichartz estimates. We infer $\|u - u_{l,t_{j+1}}\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} \leq \tilde{\eta}$. Since $\|u\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} = \tilde{\eta}$ by construction, the triangular inequality implies

$$\|u_{l,t_{j+1}}\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} \gtrsim \tilde{\eta}.$$

This now gives rise to a dichotomy: either $\|u_{l,t_{j+1}} - u_{l,t_+}\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} \gtrsim \tilde{\eta}$ or the scattering solution u_{l,t_+} is nonnegligible $\|u_{l,t_+}\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} \gtrsim \tilde{\eta}$.

Case 1: Assume $\|u_{l,t_{j+1}} - u_{l,t_+}\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} \gtrsim \tilde{\eta}$. Then in view of Corollary 5.2, we have

$$|\tilde{J}_{j+1}| \lesssim \tilde{\eta}^{-2(p-1)} (EM^{\frac{\delta}{2}})^{\frac{p}{3}} L^{\frac{3(p-1)}{2}} |\tilde{J}_j| \lesssim \tilde{\eta}^{-2(p-1)} (EM^{\frac{\delta}{2}} L)^{\frac{15}{2}} |\tilde{J}_j| \lesssim \tilde{\eta}^{-15} |\tilde{J}_j|,$$

where in the second inequality we used (37) and in the last the definition (47).

Case 2: Assume $\|u_{l,t_+}\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} \gtrsim \tilde{\eta}$. Recall that K_+ consists of unexceptional intervals. Hence we need at least $\tilde{\eta} B_{\text{exc}}$ many of them to cover $\tilde{J}_j$. Recalling the lower bound on the length of unexceptional intervals, the definition of $\tilde{\eta}$, (37) and that $\alpha_0' > \alpha_1'$ from (39), we have

$$\begin{aligned}|\tilde{J}_j| &\geq C_7 \tilde{\eta} L^{-\alpha_0'} (EM^{\frac{\delta}{2}})^{-\alpha_1'} M^{\frac{\delta}{4}} B_{\text{exc}} r_0^{s_p} = C_7 \tilde{\eta} (EM^{\frac{\delta}{2}} L)^{-\alpha_0'} (EM^{\frac{\delta}{2}})^{\alpha_0' - \alpha_1'} M^{\frac{\delta}{4}} B_{\text{exc}} r_0^{s_p} \\ &\geq C_7 \tilde{\eta}^{1 + \frac{2}{3}\alpha_0'} c_2^{-\frac{2\alpha_0'}{3}} c_0^{\alpha_0' - \alpha_1'} M^{\frac{\delta}{4}} B_{\text{exc}} r_0^{s_p} \geq C_9 \tilde{\eta}^{\frac{11}{2}} M^{\frac{\delta}{4}} B_{\text{exc}} r_0^{s_p},\end{aligned}$$

where in the last inequality we introduced a universal constant $C_9 \leq C_7 c_2^{-2\alpha_0'/3} c_0^{\alpha_0' - \alpha_1'}$.

Step 4: *We show that*

$$|K_+| \leq C_9 \tilde{\eta}^{\frac{11}{2}} M^{\frac{\delta}{4}} B_{\text{exc}} r_0^{s_p}.$$

Since $0 < \tilde{\eta} \leq 1$, *this implies in particular that* $|K_+| \leq C_9 M^{\delta/4} B_{\text{exc}} r_0^{s_p}$ *and we achieved* (43), *thereby concluding the proof.*

Let us therefore assume by contradiction that $|K_+| > C_9\tilde{\eta}^{11/2} M^{\delta/4} B_{\rm exc} r_0^{s_p}$. We call $\tilde{J}_{j_1}$ the first interval for which $|\tilde{J}_1 \cup \dots \cup \tilde{J}_{j_1}| > C_9\tilde{\eta}^{11/2} M^{\delta/4} B_{\rm exc} r_0^{s_p}$. We observe that up to choosing the constant C in the definition of $B_{\rm exc}$ big enough, we may assume that

$$\tilde{\eta}^{\frac{11}{2}} B_{\rm exc} > \max\left\{ \frac{2}{C_9}, 1 \right\}. \tag{49}$$

By the definition of j_1, we then have:

(i) $j_1 \neq 1$. Indeed, by Steps 1 and 2, $|\tilde{J}_1| \le |\tilde{J}_0| \le M^{\delta/4} r_0^{s_p}$.

(ii) For every $j \in \{1, \dots, j_1 - 1\}$ we have $|\tilde{J}_{j+1}| \le C_8 \tilde{\eta}^{-15} |\tilde{J}_j|$. This follows from Step 3 since the second option in the dichotomy is ruled out.

Let us call $[T_1, T_2] := \tilde{J}_2 \cup \dots \tilde{J}_{j_1-1}$. We want to apply the spacetime norm decay result of Proposition 4.1 on $I = [T_1, T_2]$ with $\eta = \frac{\tilde{\eta}}{4}$. Recall that by choice of $\tilde{\eta}$ in Step 2, we have that $\frac{\tilde{\eta}}{4} \in (0, c_0')$ is admissible for the spacetime norm decay. We need thus a lower bound on the length of I. By construction, Step 2 and (ii)

$$C_9 \tilde{\eta}^{\frac{11}{2}} M^{\frac{\delta}{4}} B_{\rm exc} r_0^{s_p} \le |\tilde{J}_1| + \dots + |\tilde{J}_{j_1}| \le M^{\frac{\delta}{4}} r_0^{s_p} + (T_2 - T_1) + C_8 \tilde{\eta}^{-15} (T_2 - T_1),$$

so that

$$T_2 - T_1 \ge \frac{1}{2C_8} \tilde{\eta}^{\frac{41}{2}} M^{\frac{\delta}{4}} B_{\rm exc} r_0^{s_p}. \tag{50}$$

On the other hand, we have from Step 2 and the lower bound on r_0 (42)

$$\begin{aligned} T_1 &\le r_0 + M^{\frac{\delta}{4}} r_0^{s_p} = M^{\frac{\delta}{4}} r_0^{s_p} (1 + r_0^{1-s_p} M^{\frac{-\delta}{4}}) \lesssim M^{\frac{\delta}{4}} r_0^{s_p} (1 + (L^{\alpha_0} (E M^{\frac{\delta}{2}})^{\alpha_1})^{\frac{2\delta}{(p-1)^2}}) \\ &\lesssim M^{\frac{\delta}{4}} r_0^{s_p} \tilde{\eta}^{-\frac{2(\alpha_0+\alpha_1)\delta}{\gamma(p-1)^2}} \lesssim \tilde{\eta}^{-\frac{1}{4}} M^{\frac{\delta}{4}} r_0^{s_p}. \end{aligned}$$

Summarizing, we have obtained

$$\frac{T_2}{T_1} \ge \frac{T_2 - T_1}{T_1} \ge C_{10} \tilde{\eta}^{21} B_{\rm exc}. \tag{51}$$

We now claim that to reach a contradiction, it is enough to find A and a constant $C \ge 1$ such that we can verify the following three requirements:

(R1) A satisfies the hypothesis (13) of Proposition 4.1, that is,

$$A > (4C_2\tilde{\eta}^{-1})^{\frac{12(p-1)}{5}} (E M^{\frac{\delta}{2}})^{\frac{14}{5}}.$$

(R2) The interval I is sufficiently large to apply Proposition 4.1, i.e., (14) is satisfied. In view of (51), we can enforce (15) if

$$\begin{aligned} B_{\rm exc} &= (CEM^{\frac{\delta}{2}} L)^{C(EM^{\delta/2}L)^{176}} \\ &\ge C_{10}^{-1} \tilde{\eta}^{-21} A^{3(4C_2\tilde{\eta}^{-1})^{6(p-1)(p+1)/5} (EM^{\delta/2})^{(9p+19)/10} \max\{c_0^{(p-1)/2}, (M^{(p-1)/2} T_2)^{\delta/2}\}}. \end{aligned}$$

(R3) Moreover $\sqrt{A} > 2C_8 \tilde{\eta}^{-15}$.

Observe that (R3) ensures in particular that $A > 4$. If (R1)–(R3) hold, we are in the position to conclude the proof following [Roy 2009]. The difficulty in the supercritical case instead lies in verifying the requirements (R1)–(R3). Indeed, if (R1)–(R3) hold, we infer from Proposition 4.1 that there exists $[t_1', At_1'] \subseteq \tilde{J}_2 \cup \dots \tilde{J}_{j_1-1}$ such that

$$\|u\|_{L^{2(p-1)}(\Gamma_+([t_1',At_1']))} \le \frac{\tilde{\eta}}{4}.$$

In particular, $[t_1', At_1']$ is covered by at most two consecutive intervals of the family $\{J_j\}_{j=2}^{j_1-1}$. We claim that then there exists $j \in \{2, \dots, j_1 - 1\}$ such that

$$|\tilde{J}_j| \ge \frac{\sqrt{A}}{2} |\tilde{J}_{j-1}|. \tag{52}$$

Notice that in view of (R3), the claim contradicts (ii) such that we reached a contradiction. Indeed, assume first, that $[t_1', At_1']$ is covered by one interval $\tilde{J}_j$ for some $j \in \{2, \dots, j_1 - 1\}$. Then, recalling that $A > 4$, we have

$$|\tilde{J}_j| \ge t_1'(A-1) \ge \frac{A}{2} t_1' \ge \frac{A}{2} |\tilde{J}_{j-1}| \ge \frac{\sqrt{A}}{2} |\tilde{J}_{j-1}|.$$

Assume now that $[t_1', At_1']$ is covered by two intervals $\tilde{J}_j = [a_j, b_j]$ and $\tilde{J}_{j+1} = [a_{j+1}, b_{j+1}]$ for some $j \in \{2, \dots, j_1 - 2\}$. We consider two cases. First, if $b_j \le \sqrt{A} t_1'$, then $|\tilde{J}_{j+1}| \ge t_1'(A - \sqrt{A})$ and $|\tilde{J}_j| \le \sqrt{A} t_1'$ such that

$$|\tilde{J}_{j+1}| \ge (\sqrt{A} - 1) |\tilde{J}_j| \ge \frac{\sqrt{A}}{2} |\tilde{J}_j|.$$

Second, if $b_j > \sqrt{A} t_1'$, then $|\tilde{J}_j| \ge (\sqrt{A} - 1) t_1'$ and $|\tilde{J}_{j-1}| \le t_1'$ such that

$$|\tilde{J}_j| \ge (\sqrt{A} - 1) |\tilde{J}_{j-1}| \ge \frac{\sqrt{A}}{2} |\tilde{J}_{j-1}|.$$

This proves (52).

To conclude the proof, we are left to verify the requirements (R1)-(R3) by choosing A and C. We observe that the right-hand side of (R1) can be bounded from above using (47) and (37) by

$$(4C_2 \tilde{\eta}^{-1})^{\frac{12(p-1)}{5}} (EM^{\frac{\delta}{2}})^{\frac{14}{5}} \le C_{11} \tilde{\eta}^{-14}$$

such that (R1) and (R3) are enforced if we set

$$A := C_{12} \tilde{\eta}^{-30}$$

for $C_{12} := \max\{3C_8, C_{11}\}^2$. We are left to verify (R2). We observe that from (49)

$$T_2 = T_1 + (T_2 - T_1) \lesssim \tilde{\eta}^{-1} M^{\frac{\delta}{4}} r_0^{s_p} + \tilde{\eta}^{\frac{11}{2}} M^{\frac{\delta}{4}} B_{\text{exc}} r_0^{s_p} \lesssim M^{\frac{\delta}{4}} B_{\text{exc}} r_0^{s_p}.$$

Combining this with the upper bound on r_0 in (48) and using (39), we obtain

$$\begin{aligned}(M^{\frac{p-1}{2}} T_2)^{\frac{\delta}{2}} &\lesssim (M^{\frac{8+3\delta}{4}} B_{\text{exc}} r_0^{s_p})^{\frac{\delta}{2}} \lesssim B_{\text{exc}}^{\frac{\delta}{2}} L^{s_p(p-1)(\alpha_0(p-1)+\alpha_0')} (EM^{\frac{\delta}{2}})^{\frac{s_p}{2}(p-1)(\alpha_1(2p-1)+1)} \\ &\lesssim B_{\text{exc}}^{\frac{\delta}{2}} (EM^{\frac{\delta}{2}} L)^{105} \le C_{13} B_{\text{exc}}^{\frac{\delta}{2}} \tilde{\eta}^{-70}.\end{aligned}$$

We now bound the right-hand side of (R2) from above using again (47) and (37) by

$$
\begin{aligned}
C_{10}^{-1}\tilde{\eta}^{-21}(C_{12}\tilde{\eta}^{-30})^{3(4C_2\tilde{\eta}^{-1})^{42}(EM^{\delta/2})^{(9p+19)/10}\max\{c_0^{(p-1)/2},(M^{\delta(p-1)/2}T_2)^{\delta/2}\}} \\
&\le C_{10}^{-1}\tilde{\eta}^{-21}(C_{12}\tilde{\eta}^{-30})^{3C_{13}(4C_2\tilde{\eta}^{-1})^{42}(c_2c_0^{-1}\tilde{\eta}^{-1})^{(9p+19)/15}\tilde{\eta}^{-70}B_{\text{exc}}^{\delta/2}} \\
&\le (C'EM^{\frac{\delta}{2}}L)^{C'\tilde{\eta}^{-117}B_{\text{exc}}^{\delta/2}} \le (CEM^{\frac{\delta}{2}}L)^{(C/2)(EM^{\delta/2}L)^{176}B_{\text{exc}}^{\delta/2}}
\end{aligned}
$$

for a big enough constant $C, C' \ge 1$. We now define B_{exc} to be

$$B_{\text{exc}} := (CEM^{\frac{\delta}{2}}L)^{C(EM^{\delta/2}L)^{176}}$$

for the same constant C. With this definition, (R2) is enforced since we assumed $B_{\text{exc}}^{\delta/2} \le 2$. □

Proof of Corollary 1.3. Consider a solution $(u, \partial_t u) \in L^\infty(J, (\dot{H}^1 \cap \dot{H}^2 \times H^1)(\mathbb{R}^3))$ of (1) with $p = 5 + \delta$ for $\delta \in [0, 1)$ and with

$$\|(u, \partial_t u)\|_{L^\infty(J,(\dot{H}^1\cap\dot{H}^2\times H^1)(\mathbb{R}^3))} \le M_0.$$

By interpolation, conservation of the energy and the Sobolev embeddings $(\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3) \hookrightarrow W^{1,6}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, we observe

$$
\begin{aligned}
L &:= \|(u, \partial_t u)\|_{L^\infty(J,\dot{H}^{s_p}\times\dot{H}^{s_p-1})} \le E^{1-\frac{\delta}{2(p-1)}} M_0^{\frac{\delta}{2(p-1)}}, \\
M &:= \|u\|_{L^\infty(\mathbb{R}^3\times J)} \le C_S M_0.
\end{aligned}
$$

By Theorem 1.2, if $\min\{EM^{\delta/2}, L\} < c_0$, then $\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times J)} \le 1$. Otherwise, we may assume $\min\{EM^{\delta/2}, L\} \ge c_0$ and we fix $0 \le \delta \le \min\{1, (\ln 2)/(\ln M_0)\}$. We estimate as above

$$EM^{\frac{\delta}{2}}L \le C_S^{\frac{\delta}{2}(1+\frac{\delta}{2(p-1)})} c_0^{-\frac{\delta}{2(p-1)}} E^2 M_0^{\delta(1-\frac{p+1}{4(p-1)})} \le 2C_S c_0^{-1} E^2 =: (C'E)^2$$

for $C' := (2C_S c_0^{-1})^1/2$. Thus the corollary follows, if we can meet the smallness requirement of Theorem 1.2 which now reads, setting $\overline{C} := \sqrt{C}C'$,

$$((\overline{C}E)^{2C(C'E)^{352}})^\delta \le 2.$$

The latter holds defining

$$\delta_0 := \min\left\{1, \frac{\ln 2}{\ln M_0}, \frac{\ln 2}{\ln(\overline{C}E)2C(\overline{C}E)^{352}}\right\}.$$

Observe that δ_0 depends on M_0 only, since $E = E(u_0, u_1)$ depends on the initial data only. □

8. Proof of Theorem 1.1

By time reversibility, it is enough to consider forward-in-time solutions. Thanks to classical local well-posedness and existence theory [Sogge 1995], the proof of Theorem 1.1 consists in establishing an a priori bound on $\|(u, \partial_t u)\|_{L^\infty([0,T],\dot{H}^1\cap\dot{H}^2\times H^1)}$ which is uniform in T.

Lemma 8.1 (local boundedness). *Let $\delta \in (0,1)$, $p = 5+\delta$ and consider a solution*

$$(u, \partial_t u) \in L^\infty(I, \dot{H}^1 \cap \dot{H}^2 \times H^1)$$

to (1) *on $I = [t_0, t_1]$. Then there exists a universal constant $C_l \geq 1$ such that if*

$$\|u\|^{p-1}_{L^{2(p-1)}(\mathbb{R}^3 \times I)} < C_l^{-1}, \tag{53}$$

then

$$\|(u, \partial_t u)\|_{L^\infty(I, \dot{H}^1 \cap \dot{H}^2 \times H^1)} \leq C_l \|(u, \partial_t u)(t_0)\|_{H^1 \cap \dot{H}^2 \times H^1}.$$

Proof. For $t \in I$, define $Z(t) := \|(u, \partial_t u)(t)\|_{H^1 \cap \dot{H}^2 \times H^1}$. By Strichartz estimate (7), Hölder and the Sobolev embedding of $\dot{H}^1 \hookrightarrow L^6$ we have

$$\begin{aligned} Z(t) &\lesssim Z(t_0) + \||u|^{p-1} u\|_{L^2([t_0,t], L^{3/2})} + \|\nabla(|u|^{p-1} u)\|_{L^2([t_0,t], L^{3/2})} \\ &\lesssim Z(t_0) + \||u|^{p-1}\|_{L^2(\mathbb{R}^3 \times [t_0,t])} (\|u\|_{L^\infty([t_0,t], L^6)} + \|\nabla u\|_{L^\infty([t_0,t], L^6)}) \\ &\lesssim Z(t_0) + \|u\|^{p-1}_{L^{2(p-1)}(\mathbb{R}^3 \times [t_0,t])} \sup_{t' \in [t_0,t]} Z(t'). \end{aligned}$$

We set $Y(t) := \sup_{t' \in [t_0,t]} Z(t')$. Observe that Y is nondecreasing, continuous, $Y(t_0) = Z(t_0)$ and

$$Y(t) \leq C(Z(t_0) + \|u\|^{p-1}_{L^{2(p-1)}(\mathbb{R}^3 \times I)} Y(t)) \tag{54}$$

for any $t \in I$. Setting $C_l := 2C$, we have by monotonicity that $Y(t) \leq C_l Z(t_0)$ for all $t \in [t_0, \bar{t}]$, where $\bar{t} := \sup\{t \in [t_0, t_1] : Y(t) \leq C_l Z(t_0)\}$. We claim that if $\|u\|^{p-1}_{L^{2(p-1)}(\mathbb{R}^3 \times I)} \leq C_l^{-1}$, then $\bar{t} = t_1$. Assume by contradiction that $\bar{t} < t_1$. By continuity $Y(\bar{t}) = C_l Z(t_0)$ and by the validity of (54) at $\bar{t}$, we obtain

$$C_l Z(t_0) = Y(\bar{t}) \leq C Z(t_0) + C \|u\|^{p-1}_{L^{2(p-1)}(\mathbb{R}^3 \times I)} Y(\bar{t}) < 2C Z(t_0) = C_l Z(t_0),$$

which is a contradiction. □

We achieve an a priori bound on $(u, \partial_t u)$ in $L^\infty([0,T], \dot{H}^1 \cap \dot{H}^2 \times H^1)$, uniform in T, by iterating Lemma 8.1 on a partition $\{I_n\}_{n=1}^N$ of $[0,T]$, where the smallness assumption (53)

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times I_n)} < C_l^{-\frac{1}{p-1}}$$

is satisfied by construction. Corollary 1.3 is crucial to control N, independent of T, in terms of a double exponential in E and $\|(u, \partial_t u)\|^\delta_{L^\infty \dot{H}^1 \cap H^2 \times H^1}$. The crucial observation is that in the limit as $\delta \to 0$, N is a double exponential of the energy which in turn is controlled by the initial data only. This will allow us to iterate the local bound obtained in Lemma 8.1 on bounded sets of initial data for δ small enough.

Proof of Theorem 1.1. Fix $(u_0, u_1) \in \dot{H}^1 \cap \dot{H}^2 \times H^1$. Consider $(u, \partial_t u)$ a solution to (1) with $p = 5+\delta$ for $\delta \in (0,1)$. We introduce the set

$$\mathcal{F} := \{T \in [0, +\infty) : \|(u, \partial_t u)\|_{L^\infty([0,T], \dot{H}^1 \cap \dot{H}^2 \times H^1)} \leq M_0\}$$

for some $M_0 = M_0(\|(u_0, u_1)\|_{\dot{H}^1 \cap \dot{H}^2 \times H^1})$ yet to be chosen large enough. We claim that $\mathcal{F} = [0, +\infty)$. For $M_0 \geq \|(u_0, u_1)\|_{\dot{H}^1 \cap \dot{H}^2 \times H^1}$, it is clear that $0 \in \mathcal{F}$ and, by continuity, that $\mathcal{F}$ is a closed set. We show openness. Let $T \in \mathcal{F}$. By continuity, there exists $\epsilon > 0$ such that for all $T' \in [0, T+\epsilon)$ we have

$$\|(u, \partial_t u)\|_{L^\infty([0,T'],\dot{H}^1\cap\dot{H}^2\times H^1)} \leq 2M_0.$$

Fix such a T' and let us show that $T' \in \mathcal{F}$. If $\delta \leq \delta_0(2M_0)$, with δ_0 given through Corollary 1.3, then

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times[0,T'])} \leq \max\{1, (CE(2M_0)^{\frac{\delta}{2}})^{C(E(2M_0)^{\delta/2})^{352}}\}. \tag{55}$$

We can split $[0, T']$ into subintervals $\{J_i\}_{i=1}^N$ such that

- $\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times J_i)} = \frac{1}{2}C_l^{-1/(p-1)}$ for $i = 1, \dots, N-1$,
- $\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times J_N)} \leq \frac{1}{2}C_l^{-1/(p-1)}$,

and we deduce by iterating Lemma 8.1 that

$$\|(u, \partial_t u)\|_{L^\infty([0,T'],\dot{H}^1\cap\dot{H}^2\times H^1)} \leq C_l^N \|(u_0, u_1)\|_{H^1\cap\dot{H}^2\times H^1}. \tag{56}$$

Moreover, from (55) we have the upper bound

$$N \leq 2C_l^{\frac{1}{p-1}} \max\{1, (CE(2M_0)^{\frac{\delta}{2}})^{C(E(2M_0)^{\delta/2})^{352}}\}. \tag{57}$$

We want to show that with an appropriate choice of $M_0 = M_0(\|(u_0, u_1)\|_{H^1\cap\dot{H}^2\times H^1})$ and of $\delta = \delta(\|(u_0, u_1)\|_{H^1\cap\dot{H}^2\times H^1})$, we have

$$N \leq (\ln C_l)^{-1} \ln(M_0/\|(u_0, u_1)\|_{H^1\cap\dot{H}^2\times H^1}), \tag{58}$$

which in view of (56) implies $\|(u, \partial_t u)\|_{L^\infty([0,T'],\dot{H}^1\cap\dot{H}^2\times H^1)} \leq M_0$ concluding the proof. Observe that for M_0 fixed, we have that the right-hand side of (57) as $\delta \to 0$ converges, more precisely

$$\lim_{\delta\to 0} 2C_l^{\frac{1}{p-1}} \max\{1, (CE(2M_0)^{\frac{\delta}{2}})^{C(E(2M_0)^{\delta/2})^{352}}\} = 2C_l^{\frac{1}{4}} \max\{1, (CE)^{CE^{352}}\}. \tag{59}$$

We now choose M_0 such that the right-hand side of (58) exceeds (59) by a factor 2; that is, we choose $M_0(E, \|(u_0, u_1)\|_{H^1\cap\dot{H}^2\times H^1})$ such that

$$(\ln C_l)^{-1} \ln(M_0/\|(u_0, u_1)\|_{H^1\cap\dot{H}^2\times H^1}) \geq 4C_l^{\frac{1}{4}} \max\{1, (CE)^{CE^{352}}\}$$

or, equivalently,

$$M_0 \geq \|(u_0, u_1)\|_{H^1\cap\dot{H}^2\times H^1} e^{4C_l^{1/4} \ln C_l \max\{1,(CE)^{CE^{352}}\}}.$$

Finally, by (57) we can choose $\bar{\delta}_0 = \bar{\delta}_0(M_0) < \delta_0(2M_0)$ even smaller such that for all $\delta \in (0, \bar{\delta}_0)$ we have

$$N \leq 4C_l^{\frac{1}{4}} \max\{1, (CE)^{CE^{352}}\}. \tag{60}$$

This finishes the proof that $F = [0, +\infty)$ and in particular the solution $(u, \partial_t u)$ cannot blow up. Recalling the choice of M_0, we then obtain (2). As a byproduct of the upper bound (60) on N, independent of the size of the interval, we also obtain

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times[0,+\infty)} \leq \frac{1}{2}C_l^{-\frac{1}{p-1}} 4C_l^{\frac{1}{4}} \max\{1, (CE)^{CE^{352}}\} \leq 2\max\{1, (CE)^{CE^{352}}\},$$

where we used that $C_l \geq 1$. □

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