ANALYSIS & PDE

Volume 16 No. 4 2023

ANDREA MERLO

MARSTRAND–MATTILA RECTIFIABILITY CRITERION FOR 1-CODIMENSIONAL MEASURES IN CARNOT GROUPS

msp

ANALYSIS AND PDE
Vol. 16 (2023), No. 4, pp. 927–996
DOI: 10.2140/apde.2023.16.927

MARSTRAND–MATTILA RECTIFIABILITY CRITERION FOR 1-CODIMENSIONAL MEASURES IN CARNOT GROUPS

ANDREA MERLO

In this paper, we show that the flatness of tangents of 1-codimensional measures in Carnot groups implies $C^1_{\mathbb{G}}$-rectifiability. As applications we prove a criterion for intrinsic Lipschitz rectifiability of finite perimeter sets in general Carnot groups and we show that measures with $(2n+1)$-density in the Heisenberg groups $\mathbb{H}^n$ are $C^1_{\mathbb{H}^n}$-rectifiable, providing the first non-Euclidean extension of Preiss's rectifiability theorem.

Introduction

In Euclidean spaces the following rectifiability criterion, known as the Marstrand–Mattila rectifiability theorem, is available. It was first proved by J. M. Marstrand [1961] for $m = 2$ and $n = 3$, later extended by P. Mattila [1975] to every $m \le n$ and eventually strengthened by D. Preiss [1987].

Theorem 1. *Suppose ϕ is a Radon measure on $\mathbb{R}^n$ and let $m \in \{1, \dots, n-1\}$. Then the following are equivalent*:

(i) *ϕ is absolutely continuous with respect to the m-dimensional Hausdorff measure $\mathcal{H}^m$, and ϕ-almost all of $\mathbb{R}^n$ can be covered with countably many m-dimensional Lipschitz surfaces.*

(ii) *ϕ satisfies the following two conditions for ϕ-almost every $x \in \mathbb{R}^n$*:

  (a) $0 < \Theta^m_*(\phi, x) \le \Theta^{m,*}(\phi, x) < \infty$.

  (b) $\mathrm{Tan}_m(\phi, x) \subseteq \{\lambda \mathcal{H}^m \llcorner V : \lambda > 0,\ V \in \mathrm{Gr}(n, m)\}$, *where the set of tangent measures* $\mathrm{Tan}_m(\phi, x)$ *is introduced in Definition 1.24.*

The rectifiability of a measure, namely that (i) of Theorem 1 holds, is a global property and as such it is usually very difficult to verify in applications. Rectifiability criteria serve the purpose of characterizing such global properties with local ones, which are usually conditions on the *density* and on the *tangents* of the measure. Most of the more basic criteria impose condition (iia) and the existence of an affine plane $V(x)$, depending only on the point x, on which at small scales the support of the measure is squeezed on around x. The difference between these various elementary criteria relies on how one defines *squeezed on*; for an example see Theorem 15.19 of [Mattila 1995]. However, the existence of just one plane approximating the measure at small scales may be still too difficult to prove in many applications and this is where Theorem 1 comes into play. The Marstrand–Mattila rectifiability criterion says that even if we allow a priori the approximating plane to rotate at different scales, the density hypothesis (iia) guarantees a posteriori this cannot happen almost everywhere.

MSC2020: primary 28A75, 53C17; secondary 22E25, 49Q15.
Keywords: Marstrand–Mattila rectifiability criterion, Preiss's rectifiability theorem, Carnot groups.

It is well known that if a Carnot group $\mathbb{G}$ has Hausdorff dimension $\mathfrak{Q}$, then it is $(\mathfrak{Q}-1)$-purely unrectifiable in the sense of Federer; see for instance Theorem 1.2 of [Magnani 2004]. Despite this geometric irregularity, in the foundational paper [Franchi et al. 2001], B. Franchi, F. Serra Cassano and R. Serapioni introduced the new notion of $C^1_{\mathbb{G}}$-rectifiability in Carnot groups; see Definition 1.34. This definition allowed them to establish De Giorgi's rectifiability theorem for finite perimeter sets in the Heisenberg groups $\mathbb{H}^n$.

Theorem 2 [Franchi et al. 2001, Corollary 7.6]. *Suppose $\Omega \subseteq \mathbb{H}^n$ is a finite perimeter set. Then its reduced boundary $\partial^*_{\mathbb{H}}\Omega$ is $C^1_{\mathbb{H}^n}$-rectifiable.*

It is not hard to see that an open set with smooth boundary is of finite perimeter in $\mathbb{H}^n$, but there are finite perimeter sets in $\mathbb{H}^1$ whose boundary is a fractal from an Euclidean perspective; see for instance [Kirchheim and Serra Cassano 2004]. This means that the Euclidean and $C^1_{\mathbb{G}}$-rectifiability are not equivalent.

The main goal of this paper is to establish a 1-codimensional analogue of Theorem 1 in Carnot groups.

Theorem 3. *Suppose ϕ is a Radon measure on $\mathbb{G}$. Then the following are equivalent*:

(i) *ϕ is absolutely continuous with respect to the $(\mathfrak{Q}-1)$-dimensional Hausdorff measure $\mathcal{H}^{\mathfrak{Q}-1}$, and ϕ-almost all of $\mathbb{G}$ can be covered by countably many $C^1_{\mathbb{G}}$-surfaces.*

(ii) *ϕ satisfies the following two conditions for ϕ-almost every $x \in \mathbb{G}$*:

(a) $0 < \Theta^{\mathfrak{Q}-1}_*(\phi, x) \le \Theta^{\mathfrak{Q}-1,*}(\phi, x) < \infty$.

(b) $\mathrm{Tan}_{\mathfrak{Q}-1}(\phi, x)$ *is contained in* $\mathfrak{M}$, *the family of Haar measures of the elements of* $\mathrm{Gr}(\mathfrak{Q}-1)$, *the* 1-*codimensional homogeneous subgroups of* $\mathbb{G}$.

While the fact that (i) implies (ii) follows from [Vittone et al. 2022, Lemma 3.4 and Corollary 3.6], for instance, the reverse implication is the subject of this work. Besides the already mentioned importance for the applications, Theorem 1 is also relevant because it establishes that $C^1_{\mathbb{G}}$-rectifiability is characterized in the same way as the Euclidean one, and this is the main motivation behind the definition of $\mathscr{P}$-rectifiable measures, given in Definition 4.5. Our main application of Theorem 3 is the proof of the first extension of Preiss's rectifiability theorem outside the Euclidean spaces, which is obtained by combining Theorem 3 with [Merlo 2022, Theorem 1.2]:

Theorem 4. *Suppose ϕ is a Radon measure on the Heisenberg group $\mathbb{H}^n$ such that for ϕ-almost every $x \in \mathbb{H}^n$, we have*

$$0 < \Theta^{2n+1}(\phi, x) := \lim_{r \to 0} \frac{\phi(B(x,r))}{r^{2n+1}} < \infty,$$

where $B(x,r)$ are the metric balls relative to the Koranyi metric. Then ϕ is absolutely continuous with respect to $\mathcal{H}^{2n+1}$, and ϕ-almost all of $\mathbb{H}^n$ can be covered with countably many $C^1_{\mathbb{H}^n}$-regular surfaces.

Finally, an easy adaptation of the arguments used to prove Theorem 3 also provides the following rectifiability criterion for finite perimeter sets in arbitrary Carnot groups. Theorem 5 asserts that if the tangent measures to the boundary of a finite perimeter set are sufficiently close to vertical hyperplanes, then the boundary can be covered by countably many intrinsic Lipschitz graphs.

Theorem 5. *There exists an $\varepsilon_{\mathbb{G}} > 0$ such that if $\Omega \subseteq \mathbb{G}$ is a finite perimeter set for which*

$$\limsup_{r\to 0} d_{x,r}(|\partial\Omega|_{\mathbb{G}}, \mathfrak{M}) := \limsup_{r\to 0} \inf_{\nu\in\mathfrak{M}} \frac{W_1(|\partial\Omega|_{\mathbb{G}} \llcorner B(x,r), \nu \llcorner B(x,r))}{r^{\mathfrak{Q}}} \le \varepsilon_{\mathbb{G}},$$

for $|\partial\Omega|_{\mathbb{G}}$-almost every $x \in \mathbb{G}$, where W_1 is the 1-Wasserstein distance, then $|\partial\Omega|_{\mathbb{G}}$-almost all of $\mathbb{G}$ can be covered with countably many intrinsic Lipschitz graphs.

Before giving an overview of the strategy of the proof, we briefly compare our setting to the Euclidean one and explain why Theorem 3 is so hard won. For the sake of discussion, let us put ourselves in a simplified situation. Assume E is a compact subset of a Carnot group $\mathbb{G} = (\mathbb{R}^n, *)$ such that

(α) there exists an $\eta_1 \in \mathbb{N}$ such that $\eta_1^{-1} r^{\mathfrak{Q}-1} \le \mathcal{H}^{\mathfrak{Q}-1}(E \cap B(x,r)) \le \eta_1 r^{\mathfrak{Q}-1}$ for any $0 < r < \mathrm{diam}(E)$ and any $x \in E$, and

(β) the functions $x \mapsto d_{x,r}(\mathcal{H}^{\mathfrak{Q}-1} \llcorner E, \mathfrak{M})$ converge uniformly to 0 on E as r goes to 0.

Proving that the set E is $C^1_{\mathbb{G}}$-rectifiable is (roughly) equivalent to constructing a plane $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ and a V-intrinsic Lipschitz graph Γ such that $\mathcal{H}^{\mathfrak{Q}-1}(P_V(E \cap \Gamma)) > 0$, where intrinsic Lipschitz graphs are introduced in Definition 1.36 and P_V is the splitting projection on V introduced in 1.10. With this in mind, it is easy to see that the difficulty one has to face when trying to prove Theorem 3 is twofold. On the one hand intrinsic Lipschitz graphs are not Lipschitz in almost any sense of the word as their natural parametrization is Hölder continuous, both from the Euclidean and the intrinsic perspective. On the other hand, splitting projections P_V are just (intrinsic) Hölder continuous maps. This latter complication means that there is no a priori reason for which measure, or even dimension, should be preserved by the projections or the parametrizations. This is indeed the case already in Heisenberg groups $\mathbb{H}^n$, and for further details we refer the reader to [Balogh et al. 2012; 2013].[1]

Unfortunately, the classical approaches to the proof of Theorem 1 all rely on the ideas H. Federer used to prove his celebrated projection theorem, see for instance [Federer 1969, §3.3], and these arguments all *crucially* exploit the fact that orthogonal projections are *Lipschitz*; see [De Lellis 2008; Mattila 1975; 1995; Preiss 1987]. We remark that even in Carnot groups, in some particular cases and for high codimensions, splitting projections are Lipschitz homomorphisms and thus the classical machinery works, although with some highly nontrivial complications; see [Antonelli and Merlo 2022a; 2022b].

This unavoidable technical obstruction of the Hölderianity of intrinsic Lipschitz graphs and of projections implies that, at low codimension, we need to seek a completely different approach. The first pillar of the alternative approach we pursue is the observation, encapsulated in Proposition 1.18, that despite the lack of *metric regularity*, one can still nicely control the measure of the projection of a 1-codimensional set. The other will be combining the classical ideas from [Mattila 1975] with quantitative techniques of [David and Semmes 1993a]. We present here a survey on the strategy of the proof of our main result, Theorem 3, in the simplified hypotheses (α) and (β) for E, that from now on should be considered standing throughout the section.

[1] One could attempt to use *metric projections* instead, however one quickly realizes that in some simple cases, like the Heisenberg groups $\mathbb{H}^n$, splitting projections and metric projections coincide.

The cryptic condition (β) can be reformulated, thanks to Propositions 2.6 and 2.7 in the following more geometric way. For any $\epsilon > 0$ there is a $\mathfrak{r}(\epsilon) > 0$ such that for $\mathcal{H}^{\mathfrak{Q}-1}$-almost any $x \in E$ and any $0 < \rho < \mathfrak{r}(\epsilon)$ there is a plane $V(x, \rho) \in \mathrm{Gr}(\mathfrak{Q} - 1)$, depending on both the point x and the scale ρ, for which

$$E \cap B(x, \rho) \subseteq \{y \in \mathbb{G} : \mathrm{dist}(y, x * V(x, \rho)) \leq \epsilon\rho\}, \tag{1}$$

$$B(y, \epsilon\rho) \cap E \neq \varnothing \quad \text{for any } y \in B\big(x, \tfrac{1}{2}\rho\big) \cap x * V(x, \rho). \tag{2}$$

In Euclidean spaces if a Borel set E satisfies (α), (1) and (2) it is said to be *weakly linear approximable*.[2] The condition (1) says that at small scales E is squeezed on the plane $x * V(x, \rho)$, while (2) implies that inside $B(x, \rho)$ any point of $x * V(x, \rho)$ is very close to E; see Figure 1 on page 931.

Proposition 1 shows that if at some point x the set E has also big projections on some plane W, i.e., (3) holds, then around x the set E is almost a W-intrinsic Lipschitz surface.

Proposition 1. *Let $k > 10\eta_1^2$ and $\omega > 0$. Suppose further that $x \in E$ and $\rho > 0$ are such that*

(i) $d_{x,k\rho}(\mathcal{H}^{\mathfrak{Q}-1} \llcorner E, \mathfrak{M}) \leq \omega$,

(ii) *there exists a plane $W \in \mathrm{Gr}(\mathfrak{Q} - 1)$ such that*

$$(\rho/k)^{\mathfrak{Q}-1} \leq \mathcal{H}^{\mathfrak{Q}-1} \llcorner W(P_W(B(x, \rho) \cap E)), \tag{3}$$

where P_W is the splitting projection on W; see Definition 1.10.

If k^{-1} and ω are suitably small with respect to η_1, there exists an $\alpha = \alpha(\eta_1, k, \omega) > 0$ with the following property. For any $z \in E \cap B(x, \rho)$ and any $y \in B\big(x, \frac{1}{8}k\rho\big) \cap E$ for which $10\omega\rho \leq d(z, y) \leq \frac{1}{2}k\rho$, we have that y is contained in the cone $zC_W(\alpha)$, which is introduced in Definition 1.13.

We remark that thanks to our assumption (β) on E, hypothesis (i) of the above proposition is satisfied almost everywhere on E whenever $\rho < \tilde{\mathfrak{r}}(\omega)$, where $\tilde{\mathfrak{r}}(\omega)$ is suitably small and depends only on ω. Let us explain some of the ideas of the proof of Proposition 1. If the plane W is almost orthogonal to $V(x, \rho)$ (the element of $\mathrm{Gr}(\mathfrak{Q} - 1)$ for which (1) and (2) are satisfied by E at x at scale ρ), we would have that the projection of E on W would be too small and in contradiction with (3); see Figure 2 on page 931.

If the constants k^{-1} and ω are chosen suitably small with respect to η_1 it is possible to show not only that the planes $V(x, \rho)$ and W are not orthogonal but that they must be at a very small angle indeed. In particular, this means that inside $B(x, \rho)$ the plane $x * V(x, \rho)$ must be very close to $x * W$. So close in fact that it can be proved that $E \cap B(x, \rho)$ is contained in a $2\omega\rho$-neighborhood $W_{2\omega\rho}$ of W. This implies that $z, y \in W_{2\omega\rho}$, and since W and $V(x, \rho)$ are at a small angle, it is possible to show that $\mathrm{dist}(y, zW) \leq 4\omega\rho$. Furthermore, by assumption on y, z we have $d(z, y) > 10\omega\rho$ and thus we infer that $\mathrm{dist}(y, zW) \leq 5d(y, z)$. This implies in particular that $y \in zC_W\big(\frac{2}{5}\big)$.

The second step towards the proof of the main result is to show that at any point x of E and for any $\rho > 0$ sufficiently small there is a plane $W_{x,\rho} \in \mathrm{Gr}(\mathfrak{Q} - 1)$ on which E has big projections.

[2] The reader might notice that our definition of weakly linearly approximable sets does not coincide with that which can be commonly found in the literature; see for instance [Balogh et al. 2012, Definition 5.4], [De Lellis 2008, Section 5] and [Mattila 1995, Definition 15.7]. However, the assumption (α) on the AD-regularity of E makes our definition equivalent to all the others.

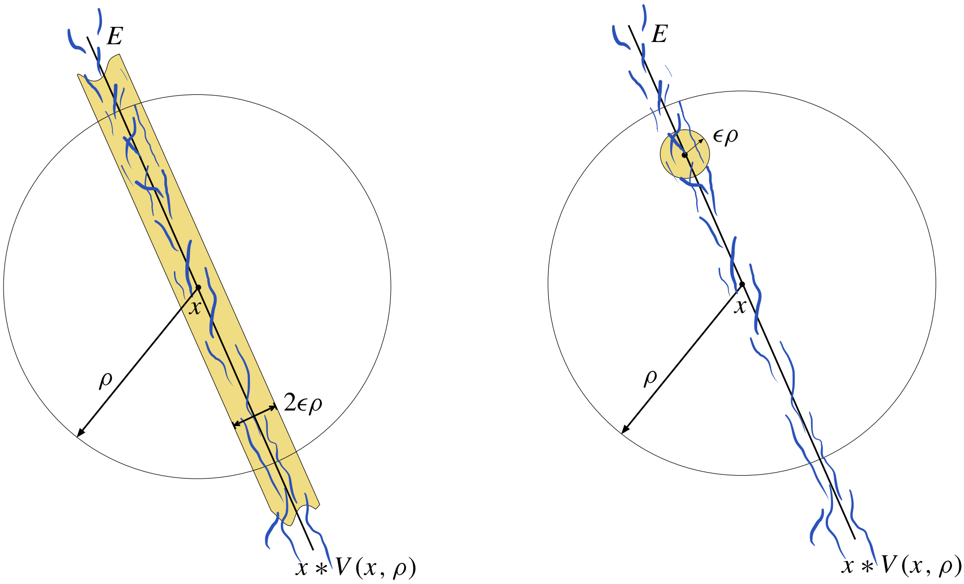

Figure 1. On the left we see that (1) implies that at the scale ρ the set E (collection of blue wavy lines) is contained in a narrow strip of size $2\epsilon\rho$ (shaded yellow) around $x * V(x, \rho)$. On the right we see that (2) implies that any ball centered on the plane $x * V(x, \rho)$ inside $B\big(x, \frac{1}{2}\rho\big)$ and of radius $\epsilon\rho$ (shaded yellow) must meet E.

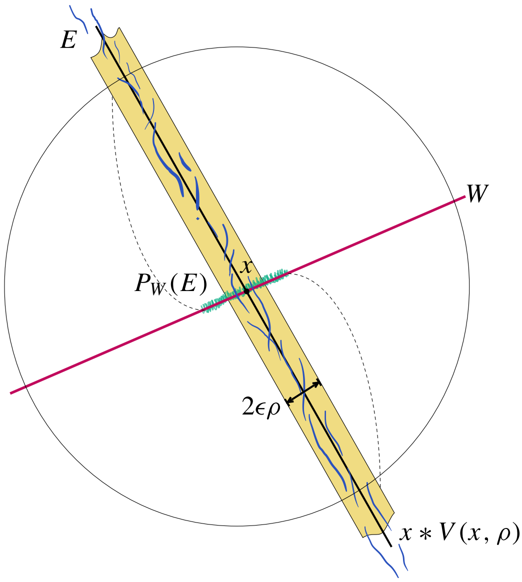

Figure 2. The weak linear approximability of E implies that $E \cap B(x, \rho)$ is contained inside $V_{\omega\rho}$, an $\omega\rho$-neighborhood of the plane $V(x, \rho)$. If $V(x, \rho)$ and W (a red line) are almost orthogonal, i.e., the Euclidean scalar product of their normals is very small, it can be shown that the projection $P_W(E)$ on W of $V_{\omega\rho} \cap B(x, \rho)$ has $\mathcal{H}^{\mathfrak{Q}-1}$-measure smaller than $(\omega\rho)^{\mathfrak{Q}-1}$.

Theorem 6. *There is an $\eta_2 \in \mathbb{N}$ such that for $\mathcal{H}^{\mathfrak{Q}-1}$-almost every $x \in E$ and $\rho > 0$ sufficiently small there is a plane $W_{x,\rho} \in \mathrm{Gr}(\mathfrak{Q}-1)$ for which*

$$\mathcal{H}^{\mathfrak{Q}-1}(P_{W_{x,\rho}}(E \cap B(x,\rho))) \geq \eta_2^{-1}\rho^{\mathfrak{Q}-1}. \tag{4}$$

We now briefly explain the ideas behind the proof of Theorem 6. Fix two parameters $\eta_3 \in \mathbb{N}$ and $\omega > 0$ such that $\omega < 1/\eta_3^{\mathfrak{Q}(\mathfrak{Q}+1)}$ and for which

$$B_+ := B(\delta_{10\eta_3^{-1}}(\mathfrak{n}(W_{x,\rho})), \eta_3^{-1}) \subseteq \{y \in B(0,1) : \langle y, \mathfrak{n}(W_{x,\rho})\rangle > \omega\},$$
$$B_- := B_+ * \delta_{20\eta_3^{-1}}(\mathfrak{n}(W_{x,\rho})^{-1}) \subseteq \{y \in B(0,1) : \langle y, \mathfrak{n}(W_{x,\rho})\rangle < -\omega\},$$

where the δ_λ are the intrinsic dilations introduced in (5) and $\mathfrak{n}(W_{x,\rho}) \in V_1$ is the Euclidean normal of $W_{x,\rho}$. Thanks to assumption (1) on E, for any $0 < \rho < \mathfrak{r}(\omega)$ we have that

$$E \cap B(x,\rho) \subseteq \{y \in B(x,\rho) : \mathrm{dist}(y, x * V(x,\rho)) \leq \omega\rho\}.$$

In particular, thanks to the assumptions on η_3 and ω we infer that $E \cap x\delta_\rho B_+ = \varnothing = E \cap x\delta_\rho B_-$. Let $W_{x,\rho} := V(x,\rho)$, and for any $z \in x\delta_\rho B_+$ define the curve

$$\gamma_z(t) := z\delta_{20\eta_3^{-1}t}(\mathfrak{n}(W_{x,\rho})^{-1}),$$

as t varies in $[0,1]$. The curve γ_z must intersect $W_{x,\rho}$ at the point $P_{W_{x,\rho}}(z)$ since $\gamma_z(1) \in x\delta_\rho B_-$, and as a consequence we have the inclusion $\gamma_z([0,1]) \subseteq P^{-1}_{W_{x,\rho}}(P_{W_{x,\rho}}(z))$. Since conditions (1) and (2) heuristically say that E almost coincides with the plane $x * W_{x,\rho}$ inside $B(x,\rho)$ and it has very few holes, most of the curves γ_z should intersect the set E too.

More precisely, we prove that if some γ_z does not intersect E, there is a small ball U_z centered at some $q \in E$ such that $\gamma_z \cap U_z \neq \varnothing$. It is clear that, defining the set

$$F := E \cup \bigcup_{\substack{z \in x\delta_r B_+ \\ \gamma_z \cap E = \varnothing}} U_z,$$

we have $P_{W_{x,r}}(x\delta_r B_+) \subseteq P_{W_{x,r}}(F)$. So, intuitively speaking adding these balls U_z allows us to close the holes of E. An easy computation proves that $\mathcal{H}^{\mathfrak{Q}-1}(P_{W_{x,r}}(x\delta_r B_+)) \geq r^{\mathfrak{Q}-1}/\eta_3^{\mathfrak{Q}-1}$, and thus in order to be able to conclude the proof of (4) we should have some control over the size of the projection of the balls U_z. This control is achievable thanks to (2) (see Proposition 2.27 and Theorem 2.28), and in particular we are able to show that

$$\mathcal{H}^{\mathfrak{Q}-1}\Bigg(P_{W_{x,r}}\Bigg(\bigcup_{\substack{z \in x\delta_r B_+ \\ \gamma_z \cap E = \varnothing}} U_z\Bigg)\Bigg) \leq \omega r^{\mathfrak{Q}-1}.$$

This implies that E satisfies the big projection properties, i.e., (4) holds with $\eta_2 := 2\eta_3^{\mathfrak{Q}-1}$. This part of the argument is rather delicate and technical. For the details we refer to the proof of Theorem 2.28.

The third step towards the proof of Theorem 3 is achieved in Section 2D, where we prove the following:

Theorem 7. *There exists an intrinsic Lipschitz graph Γ such that $\mathcal{H}^{\mathfrak{Q}-1}(E \cap \Gamma) > 0$.*

The strategy we employ to prove the above theorem is the following. We know that at $\mathcal{H}^{\mathfrak{Q}-1}$-almost every point of $x \in E$ there exists a plane $W_{x,\rho}$ such that $\mathcal{H}^{\mathfrak{Q}-1}(P_{W_{x,\rho}}(E \cap B(x,\rho))) \geq \eta_2^{-1}\rho^{\mathfrak{Q}-1}$. For any $x \in E$ at which the previous inequality holds, we let $\mathscr{B}$ be the points $y \in B(x,\rho)$ for which there is a scale $s \in (0,\rho)$ for which $W_{y,s}$ is almost orthogonal to $W_{x,\rho}$. Choosing the angle between $W_{y,s}$ and $W_{x,\rho}$ sufficiently big it is possible to prove that the projection of $\mathscr{B}$ on $W_{x,\rho}$ is smaller than $\frac{1}{2}\eta_2^{-1}\rho^{\mathfrak{Q}-1}$. This follows from the intuitive idea that if $y \in \mathscr{B}$, the set $E \cap B(y,s)$ is contained in a narrow strip that is almost orthogonal to $W_{x,\rho}$ inside $B(y,s)$ and thus its projection on $W_{x,\rho}$ has very small $\mathcal{H}^{\mathfrak{Q}-1}$-measure. On the other hand, Proposition 1.18 tells us that $\mathcal{S}^{\mathfrak{Q}-1} \llcorner V(P_{W_{x,\rho}}(E \cap B(x,\rho) \setminus \mathscr{B})) \leq 2c(V)\mathcal{S}^{\mathfrak{Q}-1}(E \cap B(x,\rho) \setminus \mathscr{B})$, and this allows us to infer that there are many points $z \in B(x,\rho) \cap E$ for which $W_{z,s}$ is contained in a (potentially large) fixed cone with axis $W_{x,\rho}$ for any $0 < s < \rho$. This uniformity on the scales allows us to infer thanks to Proposition 1 that $E \cap B(x,\rho) \setminus \mathscr{B}$ is an intrinsic Lipschitz graph.

Since the property (β) is stable for the restriction-to-a-subset operation and for the sake of discussion we can assume that (α) is also, Theorem 7 implies by means of a classical argument that $\mathcal{H}^{\mathfrak{Q}-1}$-almost all of E can be covered with intrinsic Lipschitz graphs.

Therefore, we are reduced to seeing how we can improve the regularity of the surfaces Γ_i covering E from intrinsic Lipschitz to $C^1_{\mathbb{G}}$. Since the blowups of $\mathcal{H}^{\mathfrak{Q}-1} \llcorner E$ are almost everywhere flat, the locality of the tangents, i.e., Proposition 1.27, implies that the blowups of the measures $\mathcal{H}^{\mathfrak{Q}-1} \llcorner \Gamma_i$ are flat as well, where we recall that a measure is said to be flat if it is the Haar measure of a 1-codimensional homogeneous subgroup of $\mathbb{G}$. Furthermore, since intrinsic Lipschitz graphs can be extended to boundaries of sets of finite perimeter, see Theorem 1.38, they have an associated normal vector field $\mathfrak{n}_i$. Therefore, for $\mathcal{H}^{\mathfrak{Q}-1}$-almost every $x \in \Gamma_i$, the elements of $\mathrm{Tan}_{\mathfrak{Q}-1}(\mathcal{H}^{\mathfrak{Q}-1} \llcorner \Gamma_i, x)$ are also the perimeter measures of sets with constant horizontal normal $\mathfrak{n}_i(x)$; see Propositions B.12, B.13, and B.16. The above argument shows that on the one hand the $\mathrm{Tan}_{\mathfrak{Q}-1}(\mathcal{H}^{\mathfrak{Q}-1} \llcorner \Gamma_i, x)$ are flat measures and on the other if seen as the boundary of finite perimeter sets, they must have constant horizontal normal coinciding with $\mathfrak{n}_i(x)$ almost everywhere. Therefore, for $\mathcal{H}^{\mathfrak{Q}-1}$-almost every $x \in E \cap \Gamma_i$, the set $\mathrm{Tan}_{\mathfrak{Q}-1}(\mathcal{H}^{\mathfrak{Q}-1} \llcorner \Gamma_i, x)$ must be contained in the family of Haar measures of the 1-codimensional subgroup orthogonal to $\mathfrak{n}_i(x)$. The fact that $E \cap \Gamma_i$ is covered with countably many $C^1_{\mathbb{G}}$-surfaces follows by means of the rigidity of the tangents discussed above and a Whitney-type theorem, which is obtained in Appendix B with an adaptation of the arguments of [Franchi and Serapioni 2016].

Structure of the paper

In Section 1 we recall some well-known facts about Carnot groups and Radon measures. Section 2 is divided in four parts. The main results of Section 2A are Propositions 2.6 and 2.7, which allow us to interpret the flatness of tangents in a more geometric way. Section 2B is devoted to the proof of Proposition 2.11, which is roughly Theorem 6. Section 2C is the technical core of this work and the main result proved in it is Theorem 2.28, which codifies the fact that the flatness of tangents implies big projections on planes. Finally, in Section 2D we put together the results of the previous three subsections to prove Theorem 2.30, which asserts that for any Radon measure satisfying condition (ii) of Theorem 3, there is an intrinsic Lipschitz graph of positive ϕ-measure. In Section 3 we prove that measures with almost-flat tangents and which are asymptotically AD-regular are intrinsic rectifiable, and we will use this in Section 4 to prove Theorem 4.2. In Section 4 we prove Theorem 4.1, which is the main result of the paper, Theorem 4.2 and their consequences. In Appendix A we construct the dyadic cubes that are needed in Section 2 and in Appendix B we recall some well-known facts about finite perimeter sets in Carnot groups and intrinsic Lipschitz graphs whose surface measures have flat tangents.

1. Preliminaries

This preliminary section is divided into four subsections. In Subsections 1A and 1B we introduce the setting, fix notations and prove some basic facts on splitting projections and intrinsic cones. In Section 1C we recall some well-known facts on Radon measures and their blowups and finally in Section 1D we introduce the two main notions of 1-codimensional rectifiable sets available in Carnot groups.

1A. *Carnot groups.* In this subsection we briefly introduce some notations on Carnot groups that we will extensively use throughout the paper. For a detailed account on Carnot groups and sub-Riemannian geometry we refer to [Serra Cassano 2016].

We recall that a *positive grading* of a Lie algebra $\mathfrak{g}$ is a direct-sum decomposition $\mathfrak{g}=V_1\oplus V_2\oplus\cdots\oplus V_s$, for some integer $s\geq 1$, where $V_s\neq 0$ and $[V_1,V_j]\subseteq V_{j+1}$ for all integers $j\in\{1,\dots,s\}$ and where we set $V_{s+1}=0$. A positive grading is said to be a *stratification* if $[V_1,V_j]=V_{j+1}$ for all $j\in\{1,\dots,s\}$. We also recall that the first layer V_1 of a stratification is usually referred to as the *horizontal layer*.

A Carnot group $\mathbb{G}$ of step s is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a stratification $\mathfrak{g}=V_1\oplus V_2\oplus\cdots\oplus V_s$. Throughout the paper we denote by n the topological dimension of $\mathfrak{g}$, by n_j the dimension of V_j and by h_j the number $\sum_{i=1}^j n_i$.

Furthermore, we let $\pi_i:\mathbb{G}\to V_i$ be the projection maps on the i-th layer of the Lie algebra V_i. We shall remark that more often than not, we will shorten the notation to $v_i:=\pi_i v$.

The exponential map $\exp:\mathfrak{g}\to\mathbb{G}$ is a global diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$. Hence, if we choose a basis $\{X_1,\dots,X_n\}$ of $\mathfrak{g}$, any $p\in\mathbb{G}$ can be written in a unique way as $p=\exp(p_1X_1+\cdots+p_nX_n)$. This means that we can identify any $p\in\mathbb{G}$ with the n-tuple $(p_1,\dots,p_n)\in\mathbb{R}^n$ and the group $\mathbb{G}$ itself with $\mathbb{R}^n$ endowed with $*$, the operation determined by the Campbell–Hausdorff formula. From now on, we will always assume that $\mathbb{G}=(\mathbb{R}^n,*)$ and, as a consequence, that the exponential map exp acts as the identity.

The stratification of $\mathfrak{g}$ carries with it a natural family of dilations $\delta_\lambda:\mathfrak{g}\to\mathfrak{g}$, which are Lie algebra automorphisms of $\mathfrak{g}$ and are defined by

$$\delta_\lambda(v_1,\dots,v_s)=(\lambda v_1,\lambda^2 v_2,\dots,\lambda^s v_s),\tag{5}$$

where $v_i\in V_i$. The stratification of the Lie algebra $\mathfrak{g}$ naturally induces a grading on each of its homogeneous Lie subalgebras $\mathfrak{h}$, that is,

$$\mathfrak{h}=V_1\cap\mathfrak{h}\oplus\cdots\oplus V_s\cap\mathfrak{h}.\tag{6}$$

Furthermore, note that since the exponential map acts as the identity, the Lie algebra automorphisms δ_λ are also group automorphisms of $\mathbb{G}$.

Definition 1.1 (homogeneous subgroups). A subgroup V of $\mathbb{G}$ is said to be *homogeneous* if it is a Lie subgroup of $\mathbb{G}$ that is invariant under the dilations δ_λ for any $\lambda > 0$.

Thanks to Lie's theorem and the fact that exp acts as the identity map, homogeneous Lie subgroups of $\mathbb{G}$ are in bijective correspondence through exp with the Lie subalgebras of $\mathfrak{g}$ that are invariant under the dilations δ_λ. Therefore, homogeneous subgroups in $\mathbb{G}$ are identified with the Lie subalgebras of $\mathfrak{g}$ (that in particular are vector subspaces of $\mathbb{R}^n$) that are invariant under the intrinsic dilations δ_λ.

For any nilpotent Lie algebra $\mathfrak{h}$ with stratification $W_1 \oplus \cdots \oplus W_{\bar{s}}$, we define its *homogeneous dimension*

$$\dim_{\mathrm{hom}}(\mathfrak{h}) := \sum_{i=1}^{\bar{s}} i \cdot \dim(W_i).$$

Thanks to (6) we infer that, if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, we have $\dim_{\mathrm{hom}}(\mathfrak{h}) := \sum_{i=1}^{s} i \cdot \dim(\mathfrak{h} \cap V_i)$. It is a classical fact that the Hausdorff dimension[3] with respect to any left-invariant homogeneous metric (see Definition 1.3) of a nilpotent, connected and simply connected Lie group coincides with the homogeneous dimension $\dim_{\mathrm{hom}}(\mathfrak{h})$ of its Lie algebra. Therefore, the above discussion implies that if $\mathfrak{h}$ is a vector subspace of $\mathbb{R}^n$ which is also an α-dimensional homogeneous subgroup of $\mathbb{G}$, we have

$$\alpha = \sum_{i=1}^{s} i \cdot \dim(\mathfrak{h} \cap V_i) = \dim_{\mathrm{hom}}(\mathfrak{h}). \tag{7}$$

Definition 1.2. Let $\mathfrak{Q} := \dim_{\mathrm{hom}}(\mathfrak{g})$, and for any $m \in \{1, \dots, \mathfrak{Q} - 1\}$ we define the m-dimensional Grassmannian of $\mathbb{G}$, denoted by $\mathrm{Gr}(m)$, as the family of all homogeneous subgroups V of $\mathbb{G}$ of Hausdorff dimension m.

Furthermore, thanks to (7) and some easy algebraic considerations that we omit, one deduces that for the elements of $\mathrm{Gr}(\mathfrak{Q} - 1)$ the following identities hold:

$$\dim(V \cap V_1) = n_1 - 1 \qquad \text{and} \qquad \dim(V \cap V_i) = \dim(V_i), \quad \text{for any } i = 2, \dots, s. \tag{8}$$

Thanks to (8), we infer that for any $V \in \mathrm{Gr}(\mathfrak{Q} - 1)$ there exists a $\mathfrak{n}(V) \in V_1$ such that

$$V = \mathscr{V} \oplus V_2 \oplus \cdots \oplus V_s,$$

where $\mathscr{V} := \{w \in V_1 : \langle \mathfrak{n}(V), w \rangle = 0\}$. In the following we will denote by $\mathfrak{N}(V)$ the 1-dimensional homogeneous subgroup generated by the horizontal vector $\mathfrak{n}(V)$. We shall remark that the above discussion implies that the elements of $\mathrm{Gr}(\mathfrak{Q} - 1)$ are hyperplanes in $\mathbb{R}^n$ whose normals lie in V_1. It is not hard to see that the converse holds too and that the elements of $\mathrm{Gr}(\mathfrak{Q} - 1)$ are normal subgroups of $\mathbb{G}$.

For any $p \in \mathbb{G}$, we define the left translation $\tau_p : \mathbb{G} \to \mathbb{G}$ as

$$q \mapsto \tau_p q := p * q.$$

[3]For a definition of Hausdorff dimension, see for instance [Mattila 1995, Definition 4.8].

As already remarked above, we assume without loss of generality that the group operation $*$ is determined by the Campbell–Hausdorff formula, and therefore it has the form

$$p * q = p + q + \mathscr{Q}(p, q) \quad \text{for all } p, q \in \mathbb{R}^n,$$

where $\mathscr{Q} = (\mathscr{Q}_1, \dots, \mathscr{Q}_s) : \mathbb{R}^n \times \mathbb{R}^n \to V_1 \oplus \cdots \oplus V_s$, and the $\mathscr{Q}_i$s have the following properties. For any $i = 1, \dots, s$ and any $p, q \in \mathbb{G}$ we have

(i) $\mathscr{Q}_i(\delta_\lambda p, \delta_\lambda q) = \lambda^i \mathscr{Q}_i(p, q)$,

(ii) $\mathscr{Q}_i(p, q) = -\mathscr{Q}_i(-q, -p)$,

(iii) $\mathscr{Q}_1 = 0$ and $\mathscr{Q}_i(p, q) = \mathscr{Q}_i(p_1, \dots, p_{i-1}, q_1, \dots, q_{i-1})$.

Thus, we can represent the product $*$ more precisely as

$$p * q = (p_1 + q_1, p_2 + q_2 + \mathscr{Q}_2(p_1, q_1), \dots, p_s + q_s + \mathscr{Q}_s(p_1, \dots, p_{s-1}, q_1, \dots, q_{s-1})).$$

Definition 1.3. A metric $d : \mathbb{G} \times \mathbb{G} \to \mathbb{R}$ is said to be homogeneous and left-invariant if for any $x, y \in \mathbb{G}$ we have

(i) $d(\delta_\lambda x, \delta_\lambda y) = \lambda d(x, y)$ for any $\lambda > 0$,

(ii) $d(\tau_z x, \tau_z y) = d(x, y)$ for any $z \in \mathbb{G}$.

Throughout the paper, if not otherwise stated, we will endow the group $\mathbb{G}$ with the following homogeneous and left-invariant metric:

Definition 1.4. For any $g \in \mathbb{G}$, we let

$$\|g\| := \max\{\epsilon_1 |g_1|, \epsilon_2 |g_2|^{1/2}, \dots, \epsilon_s |g_s|^{1/s}\},$$

where $\epsilon_1 = 1$ and $\epsilon_2, \dots, \epsilon_s$ are suitably small parameters depending only on the group $\mathbb{G}$. For the proof that $\|\cdot\|$ is a left-invariant, homogeneous norm on $\mathbb{G}$ for a suitable choice of $\epsilon_2, \dots, \epsilon_s$, we refer to Section 5 of [Franchi et al. 2003]. Furthermore, we define

$$d(x, y) := \|x^{-1} * y\|,$$

and let $B(x, r) := \{z \in \mathbb{G} : d(x, z) < r\}$ be the open metric ball relative to the distance d centered at x at radius $r > 0$.

Remark 1.5. Fix an orthonormal basis $\mathcal{E} := \{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ such that

$$e_j \in V_i, \quad \text{whenever } h_i \le j < h_{i+1}. \tag{9}$$

From the definition of the metric d, it immediately follows that the ball $B(0, r)$ is contained in the box

$$\mathrm{Box}_{\mathcal{E}}(0, r) := \{p \in \mathbb{R}^n : \text{for any } i = 1, \dots, s \text{ whenever } |\langle p, e_j \rangle| \le r^i/\epsilon_i \text{ for any } h_i \le j < h_{j+1}\}.$$

Definition 1.6. For any $0 \le \alpha \le \mathfrak{Q}$, we define the spherical Hausdorff measure to be the Carathéodory completion of the exterior measure that acts on Borel sets $A \subseteq \mathbb{G}$ as

$$\mathcal{S}^\alpha(A) := \sup_{\delta>0} \inf\left\{\sum_{j=1}^{\infty} r_j^\alpha : A \subseteq \bigcup_{j=1}^{\infty} \overline{B(x_j, r_j)},\ r_j \le \delta\right\}.$$

In the following definition, we introduce a family of measures that will be of great relevance throughout the paper.

Definition 1.7 (flat measures). For any $m \in \{1, \dots, \mathfrak{Q}-1\}$ the set of m-dimensional flat measures $\mathfrak{M}(m)$ is defined as

$$\mathfrak{M}(m) := \{\lambda \mathcal{S}^m \llcorner V : \text{for some } \lambda > 0 \text{ and } V \in \mathrm{Gr}(m)\}. \tag{10}$$

In order to simplify notation in the following we let $\mathfrak{M} := \mathfrak{M}(\mathfrak{Q}-1)$.

The following proposition gives a representation of $(\mathfrak{Q}-1)$-flat measures, which will come in handy later on.

Proposition 1.8. *For any $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ we have $\mathcal{S}^{\mathfrak{Q}-1}\llcorner V = \beta^{-1}\mathcal{H}^{n-1}_{\mathrm{eu}}\llcorner V$, where $\beta := \mathcal{H}^{n-1}_{\mathrm{eu}}(B(0,1)\cap V)$ and β does not depend on V.*

Proof. Let $E := \{z \in \mathbb{G} : \langle z_1, \mathfrak{n}(V)\rangle < 0\}$ and let ∂E be the perimeter measure of E; see Definition B.4. Either by direct computation or thanks to identity (2.8) in [Ambrosio et al. 2009], it can be proven that $\partial E = \mathfrak{n}(V)\mathcal{H}^{n-1}_{\mathrm{eu}}\llcorner V$. On the other hand, since the reduced boundary $\partial^* E = V$ of E is a $C^1_{\mathbb{G}}$-surface, see Definition 1.34, thanks to Theorem 4.1 of [Magnani 2017] we conclude that

$$\beta(\|\cdot\|, \mathfrak{n}(V))\mathcal{S}^{\mathfrak{Q}-1}\llcorner V = |\partial E|_{\mathbb{G}} = \mathcal{H}^{n-1}_{\mathrm{eu}}\llcorner V,$$

where $\beta(\|\cdot\|, \mathfrak{n}(V)) := \max_{z\in B(0,1)} \mathcal{H}^{n-1}_{\mathrm{eu}}(B(z,1)\cap V)$. Since $B(0,1)$ is convex as a subset of $\mathbb{R}^n$, [Magnani 2017, Theorem 5.2] implies that

$$\beta(\|\cdot\|, \mathfrak{n}(V)) = \mathcal{H}^{n-1}_{\mathrm{eu}}(B(0,1)\cap V).$$

Finally note that the right-hand side of the above identity does not depend on V since $B(0,1)$ is invariant under rotations of the first layer V_1. □

The above proposition has the following useful consequence:

Proposition 1.9. *A function $\varphi : \mathbb{G} \to \mathbb{R}$ is said to be radially symmetric if there is a profile function $g : [0,\infty) \to \mathbb{R}$ such that $\varphi(x) = g(\|x\|)$. For any $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ and any radially symmetric, positive function φ we have*

$$\int \varphi \, d\mathcal{S}^{\mathfrak{Q}-1}\llcorner V = (\mathfrak{Q}-1)\int s^{\mathfrak{Q}-2} g(s)\, ds.$$

Proof. The thesis of the proposition is trivially satisfied for indicator functions of balls. The general result follows by the monotone convergence theorem. □

1B. *Cones and splitting projections.* For any $V \in \mathrm{Gr}(\mathfrak{Q}-1)$, the group $\mathbb{G}$ can be written as a semidirect product of V and $\mathfrak{N}(V)$, i.e.,

$$\mathbb{G} = V \rtimes \mathfrak{N}(V). \tag{11}$$

In this subsection we adapt some of the results on projections from Subsection 2.2.2 of [Franchi and Serapioni 2016] to the case in which splitting of $\mathbb{G}$ is given by (11).

Definition 1.10 (splitting projections). For any $g \in \mathbb{G}$, there are two unique elements $P_V g \in V$ and $P_{\mathfrak{N}(V)} g \in \mathfrak{N}(V)$ such that

$$g = P_V g * P_{\mathfrak{N}(V)} g.$$

The following result is a particular case of [Franchi and Serapioni 2016, Proposition 2.17].

Proposition 1.11. *For any $V \in \mathrm{Gr}(\mathfrak{Q}-1)$, we let*

$$A_2 g_2 := g_2 - \mathscr{Q}_2(\pi_{\mathscr{V}} g_1, \pi_{\mathfrak{n}(V)} g_1),$$
$$A_i g_i := g_i - \mathscr{Q}_i(\pi_{\mathscr{V}} g_1, A_2 g_2, \dots, A_{i-1} g_{i-1}, \pi_{\mathfrak{n}(V)} g_1, 0, \dots, 0), \quad \textit{whenever } i = 3, \dots, s,$$

where $\pi_{\mathfrak{n}(V)} g_1 := \langle g_1, \mathfrak{n}(V)\rangle \mathfrak{n}(V)$ and $\pi_{\mathscr{V}} g_1 = g_1 - \pi_{\mathfrak{n}(V)} g_1$. With these definitions, the projections P_V and $P_{\mathfrak{N}(V)}$ have the following expressions in coordinates:

$$P_V g = (\pi_{\mathscr{V}} g_1, A_2 g_2, \dots, A_s g_s) \quad \textit{and} \quad P_{\mathfrak{N}(V)} g = (\pi_{\mathfrak{n}(V)} g_1, 0, \dots, 0).$$

Furthermore, for any $x, y \in \mathbb{G}$, the above representations and the fact that V is a normal and homogeneous subgroup of $\mathbb{G}$ imply:

(i) $P_V(x * y) = x * P_V y * P_{\mathfrak{N}(V)} x^{-1}$,

(ii) $P_{\mathfrak{N}(V)}(x * y) = P_{\mathfrak{N}(V)}(x) * P_{\mathfrak{N}(V)}(y) = P_{\mathfrak{N}(V)}(x) + P_{\mathfrak{N}(V)}(y)$,

where here the symbol $+$ has to be interpreted as the sum of vectors.

Remark 1.12. Throughout the paper the reader should always keep in mind that the projections P_V are *not* Lipschitz maps and, as stated in the introduction, this is the major source of the technical problems we have to overcome in order to prove our main result, Theorem 4.1.

The splitting projections allow us to give the following intrinsic notion of cone:

Definition 1.13. For any $\alpha > 0$ and $V \in \mathrm{Gr}(\mathfrak{Q}-1)$, we define the cone $C_V(\alpha)$ as

$$C_V(\alpha) := \{w \in \mathbb{G} : \|P_{\mathfrak{N}(V)}(w)\| \le \alpha \|P_V(w)\|\}.$$

The next proposition is very useful, since one of the major difficulties when dealing with geometric problems in Carnot groups is that $d(x, y) \approx |x - y|^{1/s}$ if x and y are not suitably chosen. However, Proposition 1.14 shows that if $y \notin x C_V(\alpha)$, then $d(x, y)$ is bi-Lipschitz equivalent to the Euclidean distance $|x - y|$.

Proposition 1.14. *For any $x, y \in \mathbb{G}$ for which $x^{-1} y \notin C_V(\alpha)$ for some $\alpha > 0$ and $V \in \mathrm{Gr}(\mathfrak{Q}-1)$, we have*

$$d(x, y) \le \Lambda(\alpha) |\pi_1(x^{-1} y)|, \quad \textit{where } \Lambda(\alpha) := (1 + \alpha^{-1}).$$

Proof. For any $\alpha, \beta > 0$ define

$$C(\alpha) := \bigcup_{V \in \mathrm{Gr}(\mathfrak{Q}-1)} (\mathbb{G} \setminus C_V(\alpha)) \quad \text{and} \quad D(\beta) := \{x \in \mathbb{G} : \|x\| \le \beta |\pi_1(x)|\}.$$

Now let us prove that $C(\alpha) \subseteq D(\Lambda(\alpha))$. For any $w \in C(\alpha)$ there exists a $W \in \mathrm{Gr}(\mathfrak{Q}-1)$ such that $\|P_{\mathfrak{N}(W)}(w)\| > \alpha \|P_W(w)\|$ and, in particular,

$$\|w\| \le \|P_W(w)\| + \|P_{\mathfrak{N}(W)}(w)\| \le (1+\alpha^{-1})\|P_{\mathfrak{N}(W)}(w)\| = (1+\alpha^{-1})|\pi_{\mathfrak{n}(W)}(\pi_1 w)| \le (1+\alpha^{-1})|\pi_1(w)|,$$

where the only identity in the equation above comes from the choice of the metric and Proposition 1.11. This concludes the proof of the inclusion $C(\alpha) \subseteq D(\Lambda(\alpha))$.

Since $x^{-1}y \notin C_V(\alpha)$, then $x^{-1}y \in C(\alpha)$ and hence $d(x, y) = \|x^{-1}y\| \le (1+\alpha^{-1})|\pi_1(x^{-1}y)|$, which concludes the proof of the proposition. □

The following proposition allows us to precisely quantify the distance of a point $g \in \mathbb{G}$ from a plane $V \in \mathrm{Gr}(\mathfrak{Q}-1)$.

Proposition 1.15. *For any $V \in \mathrm{Gr}(\mathcal{Q}-1)$ and any $g \in \mathbb{G}$ we have* $\mathrm{dist}(P_{\mathfrak{N}(V)}g, V) = |\pi_{\mathfrak{n}(V)}g_1|$ *and, in particular*, $\mathrm{dist}(g, V) = |\pi_{\mathfrak{n}(V)}g_1|$. *In addition, for any $g \in \mathbb{G}$ we have*

$$\|P_V(g)\| \le 2\|g\|. \tag{12}$$

Proof. First of all, we note that

$$\mathrm{dist}(P_{\mathfrak{N}(V)}g, V) \le d(P_{\mathfrak{N}(V)}g, 0) = |\pi_{\mathfrak{n}(V)}g_1|, \tag{13}$$

where the last identity above comes from Proposition 1.11 and the definition of the metric. In addition, once again thanks to the definition of the metric, we have

$$\mathrm{dist}(P_{\mathfrak{N}(V)}(g), V) = \inf_{v \in V} \|P_{\mathfrak{N}(V)}(g)^{-1} * v\| \ge \inf_{v \in V} |-\pi_{\mathfrak{n}(V)}g_1 + v_1| = |\pi_{\mathfrak{n}(V)}g_1|. \tag{14}$$

Putting together (13) and (14) we conclude the proof of the identity $\mathrm{dist}(P_{\mathfrak{N}(V)}(g), V) = |\pi_{\mathfrak{n}(V)}g_1|$. Thanks to this, we conclude that

$$\begin{aligned}\mathrm{dist}(g, V) = \inf_{v \in V} d(g, v) &= \inf_{v \in V} d(P_V g * P_{\mathfrak{N}(V)}g, v) \\ &= \inf_{v \in V} d(P_{\mathfrak{N}(V)}g, P_V g^{-1} * v) = \mathrm{dist}(P_{\mathfrak{N}(V)}g, V) = |\pi_{\mathfrak{n}(V)}g_1|,\end{aligned}$$

proving the second claimed identity. In order to conclude the proof of (12) we just note that

$$\|P_V(g)\| = \|g * P_{\mathfrak{N}(V)}g^{-1}\| \le \|g\| + \|P_{\mathfrak{N}(V)}g\| = \|g\| + |\pi_{\mathfrak{n}(V)}g_1| \le \|g\| + |g_1| \le 2\|g\|,$$

where the second identity above comes from the definition of the norm and Proposition 1.11. □

The following result is the analogue of [Franchi and Serapioni 2016, Proposition 2.12] where $\mathbb{M} := V$ and $\mathbb{H} := \mathfrak{N}(V)$.

Proposition 1.16. *For any $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ and any $g \in \mathbb{G}$ we have*

$$\tfrac{1}{3}(\|P_{\mathfrak{N}(V)}g\| + \|P_V g\|) \le \|g\| \le \|P_{\mathfrak{N}(V)}g\| + \|P_V g\|. \tag{15}$$

Proof. The right-hand side of (15) follows directly from the triangular inequality. Furthermore, thanks to Propositions 1.11 and 1.15 we deduce on the one hand that $\|P_{\mathfrak{N}(V)}(g)\| = |\pi_{\mathfrak{n}(V)}(g_1)| \le \|g\|$ and on the other that $\|P_V g\| \le 2\|g\|$. The first inequality in (15) follows from combining these two inequalities. □

The following proposition allows us to estimate the distance of parallel 1-codimensional planes.

Proposition 1.17. *Let $x, y \in \mathbb{G}$ and $V \in \mathrm{Gr}(\mathfrak{Q}-1)$. Defining*

$$\mathrm{dist}(xV, yV) := \max\Big\{\sup_{v\in V}\mathrm{dist}(xv, yV), \sup_{v\in V}\mathrm{dist}(yv, xV)\Big\},$$

we have

(i) $\mathrm{dist}(xV, yV) = \mathrm{dist}(x, yV) = \mathrm{dist}(y, xV) = |\pi_{\mathfrak{n}(V)}(\pi_1(x^{-1}y))|$,

(ii) $\mathrm{dist}(u, xV) \le \mathrm{dist}(u, yV) + \mathrm{dist}(xV, yV)$, *for any $u \in \mathbb{G}$.*

Proof. For any $v \in V$ we have

$$\mathrm{dist}(xv, yV) = \inf_{w\in V}\mathrm{dist}(xv, yw) = \inf_{w\in V} d(x, y(y^{-1}xv^{-1}x^{-1}y)w) = \inf_{w\in V} d(x, yw) = \mathrm{dist}(x, yV),$$

where the second last identity comes from the fact that $v^* := y^{-1}xv^{-1}x^{-1}y \in V$ and from the transitivity of the translation by v^* on V. Therefore, we have $\sup_{v\in V}\mathrm{dist}(xv, yV) = \mathrm{dist}(x, yV)$ and thus by Proposition 1.15 we infer that

$$\begin{aligned}\mathrm{dist}(xV, yV) &= \max\{\mathrm{dist}(x, yV), \mathrm{dist}(y, xV)\} = \max\{|\pi_{\mathfrak{n}(V)}(\pi_1(y^{-1}x))|, |\pi_{\mathfrak{n}(V)}(\pi_1(x^{-1}y))|\}\\ &= |\pi_{\mathfrak{n}(V)}(\pi_1(x^{-1}y))| = \mathrm{dist}(x, yV) = \mathrm{dist}(y, xV),\end{aligned}$$

where the last identity comes from interchanging x and y and exploiting the symmetry of the definition of $\mathrm{dist}(xV, yV)$. In order to prove (ii), let w^* be the element of V for which $\mathrm{dist}(u, yV) = d(u, yw^*)$ and note that

$$\begin{aligned}\mathrm{dist}(u, xV) &= \inf_{v\in V} d(u, xv) \le d(u, yw^*) + \inf_{v\in V} d(yw^*, xv) = \mathrm{dist}(u, yV) + \inf_{v\in V} d(yw^*, xv)\\ &= \mathrm{dist}(u, yV) + \mathrm{dist}(yw^*, xV) \le \mathrm{dist}(u, yV) + \mathrm{dist}(xV, yV).\end{aligned}$$ □

The following result is a direct consequence of [Franchi and Serapioni 2016, Proposition 2.2]. The bound (16) can be obtained with the same argument used by V. Chousionis, K. Fässler and T. Orponen to prove [Chousionis et al. 2019, Lemma 3.6]. In particular, (16) will play the role of a surrogate for the Lipschitzianity of projections. The proof is omitted.

Proposition 1.18. *For any $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ there is a constant $1 \le c(V) \le \mathcal{S}^{\mathfrak{Q}-1}(B(0,2)\cap V) =: C_1$ such that for any $p \in \mathbb{G}$ and any $r > 0$ we have*

$$\mathcal{S}^{\mathfrak{Q}-1}\llcorner V(P_V(B(p,r))) = c(V)r^{\mathfrak{Q}-1}.$$

Furthermore, for any Borel set $A \subseteq \mathbb{G}$ for which $\mathcal{S}^{\mathfrak{Q}-1}(A) < \infty$, we have

$$\mathcal{S}^{\mathfrak{Q}-1}\llcorner V(P_V(A)) \le 2c(V)\mathcal{S}^{\mathfrak{Q}-1}(A). \tag{16}$$

1C. *Densities and tangents of Radon measures.* In this subsection we briefly recall some facts and notations about Radon measures on Carnot groups and their blowups.

Definition 1.19. If ϕ is a Radon measure on $\mathbb{G}$, we define

$$\Theta^m_*(\phi, x) := \liminf_{r\to 0} \frac{\phi(B(x,r))}{r^m} \quad \text{and} \quad \Theta^{m,*}(\phi, x) := \limsup_{r\to 0} \frac{\phi(B(x,r))}{r^m},$$

and say that $\Theta^m_*(\phi, x)$ and $\Theta^{m,*}(\phi, x)$ are the lower and upper m-densities of ϕ at the point $x \in \mathbb{G}$, respectively.

Definition 1.20 (weak convergence of measures). A sequence of Radon measures $\{\mu_i\}_{i\in\mathbb{N}}$ is said to be weakly converging in the sense of measures to some Radon measure ν if, for any continuous functions with compact support $f \in \mathcal{C}_c$, we have

$$\int f\, d\mu_i \to \int f\, d\nu.$$

Throughout the paper, we denote such convergence with the symbol $\mu_i \rightharpoonup \nu$.

Definition 1.21. For any pair of Radon measures ϕ and ψ and any compact set $K \subseteq \mathbb{G}$ we let

$$F_K(\phi, \psi) := \sup\left\{\left|\int f\, d\phi - \int f\, d\psi\right| : f \in \mathrm{Lip}_1^+(K)\right\}, \tag{17}$$

where $\mathrm{Lip}_1^+(K)$ is the set of nonnegative 1-Lipschitz functions whose support is contained in K. Furthermore, if $K = \overline{B(x,r)}$, we shorten the notation to $F_{x,r}(\phi, \psi) := F_{\overline{B(x,r)}}(\phi, \psi)$.

The next lemma is an elementary fact about Radon measures. We omit its proof.

Lemma 1.22. *If ϕ is a Radon measure on $\mathbb{G}$, for any $x \in \mathbb{G}$ there are at most countably many radii $R > 0$ for which $\phi(\partial B(x, R)) > 0$.*

The following proposition allows us to characterize the weak convergence of measures by means of the convergence to 0 of the functionals F_K.

Proposition 1.23. *Assume that $\{\mu_i\}_{i\in\mathbb{N}}$ is a sequence of Radon measures and let μ be a Radon measure on $\mathbb{G}$. Then the following are equivalent*:

(i) $\mu_i \rightharpoonup \mu$.

(ii) $\lim_{i\to\infty} F_K(\mu_i, \mu) = 0$ *for any compact set $K \subseteq \mathbb{G}$.*

Proof. The proof can be achieved with an argument similar to the Euclidean one; see for instance [Preiss 1987, Proposition 1.11]. □

Definition 1.24 (tangent measures). Let ϕ be a Radon measure on $\mathbb{G}$. For any $x \in \mathbb{G}$ and any $r > 0$, we define $T_{x,r}\phi$ to be the Radon measure for which

$$T_{x,r}\phi(B) = \phi(x\delta_r(B)), \quad \text{for any Borel set } B \subseteq \mathbb{G}.$$

For any $m \in \{1, \dots, \mathfrak{Q}\}$ define $\mathrm{Tan}_m(\phi, x)$, *the set of the* m*-dimensional tangent measures to* ϕ *at* x, as the collection of Radon measures ν for which there is an infinitesimal sequence $\{r_i\}_{i\in\mathbb{N}}$ such that $r_i^{-m} T_{x,r}\phi \rightharpoonup \nu$.

Proposition 1.25. *Let* ϕ *be a Radon measure,* $\nu \in \mathrm{Tan}_m(\phi, x)$ *and* $\{r_i\}_{i\in\mathbb{N}}$ *an infinitesimal sequence such that* $r_i^{-m} T_{x,r_i}\phi \rightharpoonup \nu$. *Then, if* $y \in \mathrm{supp}(\nu)$, *there exists a sequence* $\{z_i\}_{i\in\mathbb{N}} \subseteq \mathrm{supp}(\phi)$ *such that* $\delta_{1/r_i}(x^{-1}z_i) \to y$.

Proof. A simple argument by contradiction yields the claim. The proof follows verbatim its Euclidean analogue; see for instance the proof of [De Lellis 2008, Proposition 3.4]. □

Proposition 1.26. *Suppose* ϕ *is a Radon measure on* $\mathbb{G}$ *such that*

$$0 < \Theta^m_*(\phi, x) \le \Theta^{m,*}(\phi, x) < \infty, \quad \text{for } \phi\text{-almost every } x \in \mathbb{G}.$$

Then $\mathrm{Tan}_m(\phi, x) \neq \varnothing$ *for* ϕ*-almost every* $x \in \mathbb{G}$.

Proof. This is an immediate consequence of the local uniform boundedness of the rescaled measures $T_{x,r}\phi$ together with the compactness of measures. See Proposition [Preiss 1987, Proposition 1.12]. □

The following result is the analogue of [De Lellis 2008, Proposition 3.12], which establishes the locality of tangents in the Euclidean space. This proposition is of capital importance since it will ensure that restricting and multiplying a measure with flat tangents by a density will yield a measure still having flat tangents.

Proposition 1.27 (locality of the tangents). *In the hypothesis of Proposition 1.26, for any nonnegative* $\rho \in L^1(\phi)$ *we have* $\mathrm{Tan}_m(\rho\phi, x) = \rho(x)\,\mathrm{Tan}_m(\phi, x)$ *for* ϕ*-almost every* $x \in \mathbb{G}$.

Proof. First of all, let us note that ϕ is locally asymptotically doubling. Indeed,

$$\begin{aligned}\limsup_{r\to 0} \frac{\phi(B(x,2r))}{\phi(B(x,r))} &\le \limsup_{r\to 0} \frac{\phi(B(x,2r))}{(2r)^m} \frac{2^m r^m}{\phi(B(x,r))} \\ &\le \frac{2^m \Theta^{m,*}(\phi, x)}{\Theta^m_*(\phi, x)} < \infty, \qquad \text{for } \phi\text{-almost every } x \in \mathbb{G}. \end{aligned} \tag{18}$$

Thanks to [Heinonen et al. 2015, Theorem 3.4.3], we know that the *Lebesgue differentiation theorem* holds for ϕ; see [Heinonen et al. 2015, page 77]. In particular, the argument that proves the equivalent of this result in Euclidean spaces, see for instance the aforementioned [De Lellis 2008, Proposition 3.12], can be applied verbatim to ϕ. □

Proposition 1.28. *Suppose* ϕ *is a Radon measure supported on a compact set* K *such that for* ϕ*-almost every* $x \in \mathbb{G}$ *we have*

$$0 < \Theta^{\mathfrak{Q}-1}_*(\phi, x) \le \Theta^{\mathfrak{Q}-1,*}(\phi, x) < \infty.$$

Then, for any $\vartheta, \gamma \in \mathbb{N}$, *the set* $E^\phi(\vartheta, \gamma) := \{x \in K : \vartheta^{-1} r^{\mathfrak{Q}-1} \le \phi(B(x,r)) \le \vartheta r^{\mathfrak{Q}-1}$ *for any* $0 < r < 1/\gamma\}$ *is compact.*

Proof. Since K is compact, in order to verify that $E^{\phi}(\vartheta, \gamma)$ is compact, it suffices to prove that it is closed. If $E^{\phi}(\vartheta, \gamma)$ is empty or finite, there is nothing to prove. So, suppose there is a sequence $\{x_i\}_{i\in\mathbb{N}} \subseteq E^{\phi}(\vartheta, \gamma)$ converging to some $x \in K$. Fix an $0 < r < 1/\gamma$ and assume that $\delta > 0$ is so small that $r + \delta < 1/\gamma$. Therefore, if $d(x, x_i) < \delta$ and $r - d(x, x_i) > 0$, we have

$$\begin{aligned}\vartheta^{-1}(r - d(x, x_i))^{\mathfrak{Q}-1} &\leq \phi(B(x_i, r - d(x, x_i))) \leq \phi(B(x, r)) \\ &\leq \phi(B(x_i, r + d(x, x_i))) \leq \vartheta(r + d(x, x_i))^{\mathfrak{Q}-1}.\end{aligned}$$

Taking the limit as i goes to ∞, we see that $x \in E^{\phi}(\vartheta, \gamma)$. □

Proposition 1.29. *With the hypothesis of Proposition 1.28, for any $\vartheta, \gamma, \mu, \nu \in \mathbb{N}$, the set*

$$\mathscr{E}^{\phi}_{\vartheta,\gamma}(\mu, \nu) = \{x \in E^{\phi}(\vartheta, \gamma) : (1 - 1/\mu)\phi(B(x, r)) \leq \phi(B(x, r) \cap E^{\phi}(\vartheta, \gamma)) \text{ for any } 0 < r < 1/\nu\}$$

is compact.

Proof. If $\mathscr{E}^{\phi}_{\vartheta,\gamma}(\mu, \nu)$ is empty or finite, there is nothing to prove. Furthermore, since by Proposition 1.28 we know that the sets $E^{\phi}(\vartheta, \gamma)$ are compact, in order to prove our claim it is sufficient to show that $\mathscr{E}^{\phi}_{\vartheta,\gamma}(\mu, \nu)$ is closed in $E^{\phi}(\vartheta, \gamma)$. Take a sequence $\{y_i\}_{i\in\mathbb{N}} \subseteq \mathscr{E}^{\phi}_{\vartheta,\gamma}(\mu, \nu)$ converging to some $y \in E^{\phi}(\vartheta, \gamma)$. Fix an $0 < r < 1/\nu$ and a $\delta \in \left(0, \frac{1}{4}\right)$ and let $i_0(\delta) \in \mathbb{N}$ be such that for any $i \geq i_0(\delta)$ we have $d(y, y_i) < \delta r$. These choices imply that

$$(1 - 1/\mu)\phi(B(y_i, r - 2d(y, y_i))) \leq \phi(B(y_i, r - 2d(y, y_i)) \cap E^{\phi}(\vartheta, \gamma)) \leq \phi(B(y, r) \cap E^{\phi}(\vartheta, \gamma)).$$

Note that the sequence of functions $f_i(z) := \chi_{B(y_i, r-2d(y, y_i))}(z)$ converges pointwise ϕ-almost everywhere to $\chi_{B(y,r)}(z)$. This is due to the fact that, for any $i \geq i_0(\delta)$, on the one hand we have $\mathrm{supp}(f_i) \Subset B(y, r)$ and on the other the functions f_i are equal to 1 on $B(y, r(1 - 3\delta))$. Thus, the dominated convergence theorem implies that

$$(1 - 1/\mu)\phi(B(y, r)) = \lim_{i\to\infty} (1 - 1/\mu)\phi(B(y_i, r - 2d(y, y_i))) \leq \phi(B(y, r) \cap E^{\phi}(\vartheta, \gamma)).$$

Since r was arbitrarily chosen in $(0, 1/\nu)$, this shows that $y \in \mathscr{E}_{\vartheta,\gamma}(\mu, \nu)$, concluding the proof. □

Proposition 1.30. *With the hypothesis of Proposition 1.28, for any $0 < \epsilon < \frac{1}{10}$ there are $\vartheta_0, \gamma_0 \in \mathbb{N}$ such that for any $\vartheta \geq \vartheta_0$, $\gamma \geq \gamma_0$ and $\mu \in \mathbb{N}$ there is a $\nu = \nu(\vartheta, \gamma, \mu) \in \mathbb{N}$ such that*

$$\phi(K \setminus \mathscr{E}^{\phi}_{\vartheta,\gamma}(\mu, \nu)) \leq \epsilon\phi(K). \tag{19}$$

Proof. The proof is an elementary application of the Lebesgue differentiation theorem that can be found in [Heinonen et al. 2015, page 77]. □

The following result allows us to compare the measure ϕ when restricted to $E^{\phi}(\vartheta, \gamma)$ with the spherical Hausdorff measure. Since the proof is a well-known application of the Lebesgue differentiation theorem that can be found in [Heinonen et al. 2015, page 77], of [Franchi et al. 2015, Theorem 3.1] and the mutual absolute continuity of the spherical and centered Hausdorff measures, see for instance [Franchi et al. 2015], we choose to leave it to the reader.

Proposition 1.31. *Let ϕ be a Radon measure and suppose further that there are $0 < \delta_1 \leq \delta_2 < \infty$ such that*

$$\delta_1 \leq \Theta^m_*(\phi, x) \leq \Theta^{m,*}(\phi, x) \leq \delta_2, \quad \textit{for } \phi\textit{-almost every } x \in E.$$

Then $\delta_1 \mathcal{S}^m \llcorner E \leq \phi \llcorner E \leq \delta_2 2^m \mathcal{S}^m \llcorner E$ and in particular, for any $\vartheta, \gamma \in \mathbb{N}$, we have

$$\vartheta^{-1} \mathcal{S}^{\mathfrak{Q}-1} \llcorner E^\phi(\vartheta, \gamma) \leq \phi \llcorner E^\phi(\vartheta, \gamma) \leq \vartheta 2^{\mathfrak{Q}-1} \mathcal{S}^{\mathfrak{Q}-1} \llcorner E^\phi(\vartheta, \gamma).$$

The following result will be used in the proof of the very important Proposition 2.4. It establishes the natural request that if a sequence of planes V_i in $\mathrm{Gr}(\mathfrak{Q}-1)$ converges in the Grassmannian to some plane $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ (i.e., the normals converge as vectors in V_1), then the surface measures on the V_i converge weakly to the surface measure on V.

Proposition 1.32. *Suppose that $\{V(i)\}_{i \in \mathbb{N}}$ is a sequence of planes in $\mathrm{Gr}(\mathfrak{Q}-1)$ such that $\mathfrak{n}(V(i)) \to \mathfrak{n}$ for some $\mathfrak{n} \in V_1$. Then there exists a $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ such that $\mathfrak{n}(V) = \mathfrak{n}$ and $\mathcal{S}^{\mathfrak{Q}-1} \llcorner V(i) \rightharpoonup \mathcal{S}^{\mathfrak{Q}-1} \llcorner V$.*

Proof. For any continuous function of compact support, $f \in \mathcal{C}_c$, we have thanks to Proposition 1.8 that

$$\lim_{i \to \infty} \int f \, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V(i) - \int f \, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V = \lim_{i \to \infty} \beta^{-1} \left(\int f \, d\mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner V(i) - \int f \, d\mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner V \right) = 0, \quad (20)$$

where the last identity comes from the fact that $\mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner V(i) \rightharpoonup \mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner V$. □

1D. ***Rectifiable sets in Carnot groups.*** In this subsection we recall the two main notions of rectifiability in Carnot groups that will be extensively used throughout the paper. First of all, let us recall the definitions of horizontal vector fields and horizontal distributions.

Definition 1.33. Let $e_1, \dots, e_{n_1}$ be an orthonormal basis of V_1 with respect to the Euclidean scalar product. For any $i = 1, \dots, n_1$ and any $x \in \mathbb{G}$ we let $X_i(x) := \partial_t (x * \delta_t(e_i))|_{t=0}$ and say that the map $X_i : \mathbb{G} \cong \mathbb{R}^n \to \mathbb{R}^n$ so defined is the i-th *horizontal vector field*. Furthermore, we define the *horizontal distribution* of $\mathbb{G}$ to be the following n_1-dimensional distribution of planes in $\mathbb{R}^n$:

$$H\mathbb{G}(x) := \mathrm{span}\{X_1(x), \dots, X_{n_1}(x)\}.$$

Finally, for any open set Ω in $\mathbb{G}$ we denote by $\mathcal{C}^1_0(\Omega, H\mathbb{G})$ the sections of $H\mathbb{G}$ of class $\mathcal{C}^1$ with support contained in Ω.

The definition of regular surfaces we are about to give is reminiscent of the characterization of smooth surfaces in the Euclidean spaces through the local inversion theorem. Heuristically speaking, a $C^1_{\mathbb{G}}$-surface is a set that is transverse to $H\mathbb{G}$ and whose sections with $H\mathbb{G}$ are C^1-surfaces.

Definition 1.34 ($C^1_{\mathbb{G}}$-surfaces). We say that a closed set $C \subseteq \mathbb{G}$ is a $C^1_{\mathbb{G}}$-*surface* if there exists a continuous function $f : \mathbb{G} \to \mathbb{R}$ such that $C = f^{-1}(0)$ and whose horizontal distributional gradient $\nabla_{\mathbb{G}} f := (X_1 f, \dots, X_{n_1} f)$ can be represented by a continuous, never-vanishing section of $H\mathbb{G}$.

Remark 1.35. Thanks to [Serra Cassano 2016, Corollary 4.27], if C is a $C^1_{\mathbb{G}}$-regular surface, then $\mathcal{S}^{\mathfrak{Q}-1} \llcorner C$ is σ-finite.

The second notion of regular surface we give in this subsection is inspired by the characterization of Lipschitz graphs through cones.

Definition 1.36 (intrinsic Lipschitz graphs). Let $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ and E be a Borel subset of V. A function $f : E \to \mathfrak{N}(V)$ is said to be *intrinsic Lipschitz* if there exists an $\alpha > 0$ such that for any $v \in E$ we have

$$\mathrm{gr}(f) := \{wf(w) : w \in E\} \subseteq vf(v)C_V(\alpha).$$

A Borel set $A \subseteq \mathbb{G}$ is said to be a *V-intrinsic Lipschitz graph*, or simply an *intrinsic Lipschitz graph*, if there is an intrinsic Lipschitz function $f : E \subseteq V \to \mathfrak{N}(V)$ such that $A = \mathrm{gr}(f)$.

Proposition 1.37. *Suppose E is a Borel subset of $\mathbb{G}$ and assume there is a plane $W \in \mathrm{Gr}(\mathfrak{Q}-1)$ and an $\alpha > 0$ such that for any $w \in E$ we have $E \subseteq wC_W(\alpha)$. Then E is contained in an intrinsic Lipschitz graph.*

Proof. Thanks to the assumption on E, for any $w_1, w_2 \in E$ we have $w_1^{-1}w_2 \in C_W(\alpha)$. This implies that for any $v \in P_W(E)$ there exists a unique $w \in E$ such that $P_W(w) = v$, otherwise we would have $w_1^{-1}w_2 \in \mathfrak{N}(W)$.

Let $f : P_W(E) \to \mathfrak{N}(V)$ be the map associating every $w \in P_W(E)$ to the only element in its preimage $P_W^{-1}(w)$. With this definition we have that the set $\mathrm{gr}(f) := \{vf(v) : v \in P_W^{-1}(E)\}$ coincides with E and this shows that f is an intrinsic Lipschitz function since $\mathrm{gr}(f) \subseteq vf(v)C_W(\alpha)$ for any $v \in E$. □

The following extension theorem is of capital importance for us:

Theorem 1.38 [Vittone 2012, Theorem 3.4]. *Suppose $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ and let $f : E \to \mathfrak{N}(V)$ be an intrinsic Lipschitz function. Then there is an intrinsic Lipschitz function $\tilde{f} : V \to \mathfrak{N}(V)$ such that $f(v) = \tilde{f}(v)$ for any $v \in E$.*

The following result is an immediate consequence of Theorem 1.38:

Proposition 1.39. *If $f : E \subseteq V \to \mathfrak{N}(V)$ is an intrinsic Lipschitz function, then $\mathcal{S}^{\mathfrak{Q}-1} \llcorner \mathrm{gr}(f)$ is σ-finite.*

Proof. Theorem 1.38 together with [Franchi and Serapioni 2016, Theorem 3.9] immediately implies that $\mathcal{S}^{\mathfrak{Q}-1}(\mathrm{gr}(f) \cap B(0, R)) < \infty$ for any $R > 0$. □

From the notions of $C^1_{\mathbb{G}}$-surfaces and of intrinsic Lipschitz surfaces rise the two following definitions of rectifiability:

Definition 1.40. A Borel set $A \subseteq \mathbb{G}$ of finite $\mathcal{S}^{\mathfrak{Q}-1}$-measure is said to be

(i) *$C^1_{\mathbb{G}}$-rectifiable* if there are countably many $C^1_{\mathbb{G}}$-surfaces Γ_i such that

$$\mathcal{S}^{\mathfrak{Q}-1}\Big(A \setminus \bigcup_{i \in \mathbb{N}} \Gamma_i\Big) = 0,$$

(ii) *intrinsic rectifiable* if there are countably many intrinsic Lipschitz graphs Γ_i such that

$$\mathcal{S}^{\mathfrak{Q}-1}\Big(A \setminus \bigcup_{i \in \mathbb{N}} \Gamma_i\Big) = 0.$$

The following proposition is an adaptation of the well-known fact that Borel sets can be written in an essentially unique way, as the union of a rectifiable and a purely unrectifiable set.

Proposition 1.41 (decomposition theorem). *Suppose $\mathscr{F}$ is a family of Borel sets in $\mathbb{G}$ for which $\mathcal{S}^{\mathfrak{Q}-1}\llcorner C$ is σ-finite for any $C \in \mathscr{F}$. Then, for any Borel set $E \subseteq \mathbb{G}$ such that $\mathcal{S}^{\mathfrak{Q}-1}(E) < \infty$, there are two Borel sets E^u, $E^r \subseteq E$ such that*

(i) $E^u \cup E^r = E$,

(ii) *E^r is contained in a countable union of elements of $\mathscr{F}$,*

(iii) *$\mathcal{S}^{\mathfrak{Q}-1}(E^u \cap C) = 0$ for any $C \in \mathscr{F}$.*

Such a decomposition is unique up to $\mathcal{S}^{\mathfrak{Q}-1}$-null sets, i.e., if F^u and F^r are Borel sets satisfying the three properties listed above, we have $\mathcal{S}^{\mathfrak{Q}-1}(E^r \triangle F^r) = \mathcal{S}^{\mathfrak{Q}-1}(E^u \triangle F^u) = 0$.

Proof. The proof follows verbatim the argument of [De Lellis 2008, Theorem 5.7]. □

Corollary 1.42. *For any Borel set $E \subseteq \mathbb{G}$ such that $\mathcal{S}^{\mathfrak{Q}-1}(E) < \infty$, there are two Borel sets E^u, $E^r \subseteq E$ such that*

(i) $E^u \cup E^r = E$,

(ii) *there are countably many intrinsic Lipschitz functions $f_i : V_i \to \mathfrak{N}(V_i)$, where $V_i \in \mathrm{Gr}(\mathfrak{Q}-1)$, whose graphs cover $\mathcal{S}^{\mathfrak{Q}-1}$-almost all of E^r,*

(iii) *$\mathcal{S}^{\mathfrak{Q}-1}(E^u \cap C) = 0$ for any C-intrinsic Lipschitz graph.*

Proof. Thanks to Proposition 1.39 we know that every intrinsic Lipschitz graph is $\mathcal{S}^{\mathfrak{Q}-1}$-$\sigma$-finite. If we choose $\mathscr{F}$ in the statement of Proposition 1.41 to be the family of all intrinsic Lipschitz graphs of $\mathbb{G}$, we get two sets E^u and E^r whose union is the whole set E, such that E^u has $\mathcal{S}^{\mathcal{Q}-1}$-null intersection with every intrinsic Lipschitz graph and E_r can be covered by countably many graphs of intrinsic Lipschitz functions $f_i : E_i \subseteq V_i \to \mathfrak{N}(V_i)$. The conclusion follows from Theorem 1.38. □

2. The support of 1-codimensional measures with flat tangents is intrinsic rectifiable

Throughout this section *we assume ϕ to be a fixed Radon measure on* $\mathbb{G}$ whose support is a compact set K and such that for ϕ-almost every $x \in \mathbb{G}$ we have

(H1) $0 < \Theta_*^{\mathfrak{Q}-1}(\phi, x) \le \Theta^{\mathfrak{Q}-1,*}(\phi, x) < \infty$,

(H2) $\mathrm{Tan}_{\mathfrak{Q}-1}(\phi, x) \subseteq \mathfrak{M}$, where $\mathfrak{M}$ is the family of 1-codimensional flat measures from Definition 1.7.

The main goal of this section is to prove the following:

Theorem 2.1. *There is an intrinsic Lipschitz graph Γ such that $\phi(\Gamma) > 0$.*

The strategy we employ in order to prove Theorem 2.1 is divided into four parts: First of all in Section 2A we show that hypotheses (H1) and (H2) on ϕ imply that for ϕ-almost any $x \in K$ and $r > 0$ sufficiently small, there is a plane $V_{x,r}$ for which K as a set is very close in the Hausdorff distance to $V_{x,r}$. In Section 2B we prove that if $K \cap B(x, r)$ has a big projection on some plane W, then W is very close to $V_{x,r}$ and there exists an $\alpha > 0$ such that for any $y, z \in B(x, r) \cap K$ for which $d(y, z) \ge \mathrm{dist}(W, V_{x,r})r$, we have $z \in yC_W(\alpha)$. Section 2C is the technical core of this section, and its main result, Theorem 2.28, shows that for ϕ-almost any $x \in K$ we have that the set $B(x, r) \cap K$ has a big projection on $V_{x,r}$. Finally, in Section 2D, making use of the results of the previous subsections, we construct the wanted ϕ-positive intrinsic Lipschitz graph.

2A. ***Geometric implications of flat tangents.*** In this subsection we reformulate the hypothesis (H2) on ϕ in more geometric terms. In order to obtain such a reformulation, we need a way to pass from the purely pointwise information on the flatness of tangents to a more local understanding of the measure ϕ at small scales. In the following Definition 2.2, we introduce two functionals on Radon measures that will be used for this precise objective. These functionals can be considered the Carnot analogue of the functional $d(\cdot\,,\mathfrak{M})$ of Section 2 of [Preiss 1987].

Definition 2.2. For any $x \in \mathbb{G}$ and any $r > 0$ we define the functionals

$$d_{x,r}(\phi,\mathfrak{M}) := \inf_{\substack{\Theta>0\\ V\in\mathrm{Gr}(\mathfrak{Q}-1)}} \frac{F_{x,r}(\phi,\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV)}{r^{\mathfrak{Q}}} \quad\text{and}\quad \tilde{d}_{x,r}(\phi,\mathfrak{M}) := \inf_{\substack{\Theta>0,\ z\in\mathbb{G}\\ V\in\mathrm{Gr}(\mathfrak{Q}-1)}} \frac{F_{x,r}(\phi,\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner zV)}{r^{\mathfrak{Q}}},$$

where $F_{x,r}$ was introduced in (17).

In the following proposition we summarize some useful properties of the functionals $d_{x,r}$ and $\tilde{d}_{x,r}$.

Proposition 2.3. *The functionals $d_{x,r}(\cdot\,,\mathfrak{M})$ and $\tilde{d}_{x,r}(\cdot\,,\mathfrak{M})$ satisfy the following properties*:

(i) *For any $x \in \mathbb{G}$, $k > 0$ and $r > 0$, we have $d_{x,kr}(\phi,\mathfrak{M}) = d_{0,k}(r^{-(\mathfrak{Q}-1)}T_{x,r}\phi,\mathfrak{M})$.*

(ii) *For any $r > 0$, the function $x \mapsto d_{x,r}(\phi,\mathfrak{M})$ is continuous.*

(iii) *For any $x, y \in \mathbb{G}$ and $r, s > 0$ for which $B(y,s) \subseteq B(x,r)$, we have $(s/r)^{\mathfrak{Q}}\tilde{d}_{y,s}(\phi,\mathfrak{M}) \le \tilde{d}_{x,r}(\phi,\mathfrak{M})$.*

(iv) *For any $x \in \mathbb{G}$ and any $s \le r$, we have $(s/r)^{\mathfrak{Q}}d_{x,s}(\phi,\mathfrak{M}) \le d_{x,r}(\phi,\mathfrak{M})$.*

Proof. It is immediate to see that f belongs to $\mathrm{Lip}_1^+(\overline{B(x,kr)})$ if and only if there is a $g \in \mathrm{Lip}_1^+(\overline{B(0,k)})$ such that $f(z) = rg(\delta_{1/r}(x^{-1}z))$. This implies that

$$\begin{aligned}\frac{1}{(kr)^{\mathfrak{Q}}}&\Big(\int f\,d\phi - \Theta\int f\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV\Big)\\ &=\frac{1}{k^{\mathfrak{Q}}r^{\mathfrak{Q}-1}}\Big(\int g(\delta_{1/r}(x^{-1}z))\,d\phi(z) - \Theta\int g(\delta_{1/r}(x^{-1}z))\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV\Big)\\ &=\frac{1}{k^{\mathfrak{Q}}}\Big(\int g(z)\,d\frac{T_{x,r}\phi}{r^{Q-1}}(z) - \Theta\int g(z)\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner V\Big),\end{aligned}$$

and this concludes the proof of (i). To show that the map $x \mapsto d_{x,r}(\phi,\mathfrak{M})$ is continuous, we prove the following stronger fact. There exists a constant $\tilde{C}$ depending only on $\mathbb{G}$ such that for any $x, y \in \mathbb{G}$ with $d(x,y) < 1$ we have

$$|d_{x,r}(\phi,\mathfrak{M}) - d_{y,r}(\phi,\mathfrak{M})| \le \tilde{C}(\mathbb{G})\frac{2(r+2)d(x,y)^{1/s}}{r^{\mathfrak{Q}}}\phi(B(x,r+d(x,y))). \tag{21}$$

In order to prove (21), for any $\epsilon > 0$ we let $\Theta^* > 0$ and $V^* \in \mathrm{Gr}(\mathfrak{Q}-1)$ be such that

$$\left|\int f\,d\frac{T_{y,r}\phi}{r^{\mathfrak{Q}-1}} - \Theta^*\int f\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner V^*\right| \le d_{y,r}(\phi,\mathfrak{M}) + \epsilon, \quad \text{for any } f \in \mathrm{Lip}_1^+(\overline{B(0,1)}).$$

Furthermore, by definition of $d_{y,r}$ we can find an $f^* \in \mathrm{Lip}_1^+(\overline{B(0,1)})$ such that

$$d_{x,r}(\phi, \mathfrak{M}) - \epsilon \leq \left| \int f^* \, d\frac{T_{x,r}\phi}{r^{\mathfrak{Q}-1}} - \Theta^* \int f^* \, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V_* \right|.$$

This choice of f^*, Θ^* and V^* implies that

$$\begin{aligned}
&d_{x,r}(\phi, \mathfrak{M}) - d_{y,r}(\phi, \mathfrak{M}) \\
&\leq \left| \int f^* \, d\frac{T_{x,r}\phi}{r^{\mathfrak{Q}-1}} - \Theta^* \int f^* \, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V^* \right| - \left| \int f^* \, d\frac{T_{y,r}\phi}{r^{\mathfrak{Q}-1}} - \Theta^* \int f^* \, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V^* \right| + 2\epsilon \\
&\leq \left| \int f^* \, d\frac{T_{x,r}\phi}{r^{\mathfrak{Q}-1}} - \int f^* \, d\frac{T_{y,r}\phi}{r^{\mathfrak{Q}-1}} \right| + 2\epsilon \leq r^{-(\mathfrak{Q}-1)} \int |f^*(\delta_{1/r}(x^{-1}w)) - f^*(\delta_{1/r}(y^{-1}w))| \, d\phi(w) + 2\epsilon \\
&\leq r^{-\mathfrak{Q}} \int_{B(x,r+d(x,y))} d(x^{-1}w, y^{-1}w) \, d\phi(x) + 2\epsilon \\
&\leq r^{-\mathfrak{Q}} \big(d(x,y) + C(G) \big(d(x,y)^{1/s} (r + d(x,y))^{(s-1)/s} + d(x,y)^{(s-1)/s} (r + d(x,y))^{1/s} \big) \big) \\
&\qquad\qquad \times \phi(B(x, r + d(x,y))) + 2\epsilon,
\end{aligned}$$

where the last inequality comes from [Franchi and Serapioni 2016, Proposition 2.13] together with the constant $C(\mathbb{G})$. Interchanging x and y, the bound (21) is proved thanks to the arbitrariness of ϵ. Finally, statements (iii) and (iv) follow directly from the definitions. □

The following proposition allows us to rephrase the rather geometric condition on ϕ, the flatness of the tangents, into a more malleable functional-analytic condition that is the ϕ-almost everywhere convergence of the functions $x \mapsto d_{x,kr}(\phi, \mathfrak{M})$ to 0. We omit the straightforward proof.

Proposition 2.4. *Assume μ is a Radon measure on $\mathbb{G}$ such that $0 < \Theta^{\mathfrak{Q}-1}(\mu, x) < \infty$ for μ-almost every $x \in \mathbb{G}$. Then the following are equivalent*:

(i) $\lim_{r\to 0} d_{x,kr}(\mu, \mathfrak{M}) = 0$ *for μ-almost every $x \in \mathbb{G}$ and any $k > 0$.*

(ii) $\mathrm{Tan}_{\mathfrak{Q}-1}(\mu, x) \subseteq \mathfrak{M}$ *for μ-almost every $x \in \mathbb{G}$.*

Notation 2.5. Throughout Section 2 we let $0 < \varepsilon_1 < \frac{1}{10}$ be a fixed constant. Proposition 1.30 yields two natural numbers $\vartheta, \gamma \in \mathbb{N}$, that from now on we consider fixed, such that $\phi(K \setminus E^\phi(\vartheta, \gamma)) \leq \varepsilon_1 \phi(K)$. We can assume without loss of generality, again thanks to Proposition 1.30, that ϑ and γ have the further property that for any $\mu \geq 4\vartheta$ there is a $\nu \in \mathbb{N}$ for which

$$\phi(K \setminus \mathscr{E}^\phi_{\vartheta,\gamma}(\mu, \nu)) \leq \varepsilon_1 \phi(K).$$

For future convenience, we define $\eta := 1/\mathfrak{Q}$ and let

$$\delta_{\mathbb{G}}(\vartheta) := \min\left\{ \frac{1}{2^{4(\mathfrak{Q}+1)}\vartheta}, \frac{\eta^{\mathfrak{Q}+1}(1-\eta)^{\mathfrak{Q}^2-1}}{(32\vartheta)^{\mathfrak{Q}+1}} \right\}.$$

Eventually, if $d_{x,r}(\phi \llcorner E^\phi(\vartheta, \gamma), \mathfrak{M}) + d_{x,r}(\phi, \mathfrak{M}) \leq \delta$ for some $0 < \delta < \delta_{\mathbb{G}}(\vartheta)$, we define $\Pi_\delta(x, r)$ to be the subset of planes $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ for which there exists a $\Theta > 0$ such that

$$F_{x,r}(\phi \llcorner E^\phi(\vartheta, \gamma), \Theta \mathcal{S}^{\mathfrak{Q}-1} \llcorner xV) + F_{x,r}(\phi, \Theta \mathcal{S}^{\mathfrak{Q}-1} \llcorner xV) \leq 2\delta r^{\mathfrak{Q}}. \tag{22}$$

The following two propositions are the main results of this subsection. They are so relevant since they give a more geometric interpretation of the condition we call *flatness of the tangents* and in particular tell us that $E^\phi(\vartheta,\gamma)$ is in essence a weakly linearly approximable set. For a discussion on how this will play a role in the proof of the main result of this work, we refer to the Introduction.

Proposition 2.6. *Let $x\in E^\phi(\vartheta,\gamma)$ be such that $\tilde d_{x,r}(\phi,\mathfrak{M})\leq\delta$ for some $\delta<\delta_{\mathbb{G}}(\vartheta)$ and $0<r<1/\gamma$. Then, for every $V\in\mathrm{Gr}(\mathfrak{Q}-1)$ for which there is a $z\in\mathbb{G}$ and a $\Theta>0$ such that $F_{x,r}(\phi,\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner zV)\leq 2\delta r^{\mathfrak{Q}}$, we have*

$$\sup_{w\in E^\phi(\vartheta,\gamma)\cap B(x,r/4)}\frac{\mathrm{dist}(w,xV)}{r}\leq 2^{2+3/\mathfrak{Q}}\vartheta^{1/\mathfrak{Q}}\delta^{1/\mathfrak{Q}}=:C_2(\vartheta)\delta^{1/\mathfrak{Q}}.$$

Proof. Since $g(w):=\min\{\mathrm{dist}(w,B(x,r)^c),\mathrm{dist}(w,zV)\}$ belongs to $\mathrm{Lip}_1^+(\overline{B(x,r)})$, we deduce that

$$2\delta r^{\mathfrak{Q}}\geq\int g(w)\,d\phi(w)-\Theta\int g(w)\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner zV=\int g(w)\,d\phi(w)\geq\int_{B(x,r/2)}\min\{\tfrac12 r,\mathrm{dist}(w,zV)\}\,d\phi(w).$$

Suppose that y is a point in $\overline{B(x,r/4)}\cap E^\phi(\vartheta,\gamma)$ furthest from zV, and let $D=\mathrm{dist}(y,zV)$. If $D\geq\frac18 r$, this would imply that

$$\begin{aligned}2\delta r^{\mathfrak{Q}}&\geq\int_{B(x,r/2)}\min\{\tfrac12 r,\mathrm{dist}(w,zV)\}\,d\phi(w)\\&\geq\int_{B(y,r/16)}\min\{\tfrac12 r,\mathrm{dist}(w,zV)\}\,d\phi(w)\geq\tfrac1{16}r\phi\big(B\big(y,\tfrac1{16}r\big)\big)\geq\frac{r^{\mathcal{Q}}}{\vartheta 16^{\mathfrak{Q}}},\end{aligned}$$

which is not possible thanks to the choice of δ. This implies that $D<\frac18 r$ and as a consequence, we have

$$\begin{aligned}2\delta r^{\mathfrak{Q}}&\geq\int_{B(x,r/2)}\min\{\tfrac12 r,\mathrm{dist}(w,zV)\}\,d\phi(w)\\&\geq\int_{B(y,D/2)}\min\{12r,\mathrm{dist}(w,zV)\}\,d\phi(w)\geq\tfrac12 D\phi\big(B\big(y,\tfrac12 D\big)\big)\geq\vartheta^{-1}\big(\tfrac12 D\big)^{\mathfrak{Q}},\end{aligned}\tag{23}$$

where the second inequality comes from the fact that $B\big(y,\frac12 D\big)\subseteq B\big(x,\frac12 r\big)$. This implies, thanks to (23), that

$$\sup_{w\in E^\phi(\vartheta,\gamma)\cap B(x,r/4)}\frac{\mathrm{dist}(w,zV)}{r}\leq\frac{D}{r}\leq 2^{1+3/\mathfrak{Q}}\vartheta^{1/\mathfrak{Q}}\delta^{1/\mathfrak{Q}}=\tfrac12 C_2(\vartheta)\delta^{1/\mathfrak{Q}}.$$

Furthermore, since $x\in E^\phi(\vartheta,\gamma)$, we also infer that $\mathrm{dist}(x,zV)/r\leq\frac12 C_2(\vartheta)\delta^{1/\mathfrak{Q}}$. Therefore, thanks to Proposition 1.17, we conclude that

$$\sup_{w\in E^\phi(\vartheta,\gamma)\cap B(x,r/4)}\frac{\mathrm{dist}(w,xV)}{r}\leq\sup_{w\in E^\phi(\vartheta,\gamma)\cap B(x,r/4)}\frac{\mathrm{dist}(w,zV)+\mathrm{dist}(xV,zV)}{r}\leq C_2(\vartheta)\delta^{1/\mathfrak{Q}}.\qquad\square$$

Proposition 2.7. *Let $x\in E^\phi(\vartheta,\gamma)$ and $0<r<1/\gamma$ be such that for some $0<\delta<\delta_{\mathbb{G}}(\vartheta)$ we have*

$$d_{x,r}(\phi,\mathfrak{M})+d_{x,r}(\phi\llcorner E^\phi(\vartheta,\gamma),\mathfrak{M})\leq\delta.\tag{24}$$

Then for any $V\in\Pi_\delta(x,r)$ and any $w\in B\big(x,\frac12 r\big)\cap xV$ we have $E^\phi(\vartheta,\gamma)\cap B(w,\delta^{1/(\mathfrak{Q}+1)}r)\neq\varnothing$.

Proof. By the definition of $\Pi_\delta(x,r)$ (see Notation 2.5), for any $V \in \Pi_\delta(x,r)$, where here we choose $\delta := 2^{-\mathfrak{Q}^2-\mathfrak{Q}}\varepsilon_2$, there exists a $\Theta > 0$ such that

$$\frac{F_{x,r}(\phi, \Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV) + F_{x,r}(\phi\llcorner E^\phi(\vartheta,\gamma), \Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV)}{r^{\mathfrak{Q}}} \leq 2\delta. \tag{25}$$

Therefore, defining $g(x) := \min\{\mathrm{dist}(x, B(0,1)^c), \eta\}$, we infer that

$$\begin{aligned}\vartheta^{-1}(1-\eta)^{\mathfrak{Q}-1}\eta r^{\mathfrak{Q}} - \Theta\eta r^{\mathfrak{Q}} &\leq \eta r\phi(B(x,(1-\eta)r)) - \eta r\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV(B(x,r))\\ &\leq \int rg(\delta_{1/r}(x^{-1}z))\,d\phi(z) - \Theta\int rg(\delta_{1/r}(x^{-1}z))\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV \leq 2\delta r^{\mathfrak{Q}},\end{aligned}$$

where the last inequality above comes from (25) and the fact that $rg(\delta_{1/r}(x^{-1}\cdot)) \in \mathrm{Lip}_1^+(\overline{B(x,r)})$. Simplifying and rearranging the above chain of inequalities we infer that

$$\Theta \geq \vartheta^{-1}(1-\eta)^{\mathfrak{Q}-1} - 2\delta/\eta \geq (2\vartheta)^{-1}(1-\eta)^{\mathfrak{Q}-1} = (2\vartheta)^{-1}(1-1/\mathfrak{Q})^{\mathfrak{Q}-1},$$

where the first inequality comes from the choice of δ and the last equality from that of $\eta = 1/\mathfrak{Q}$; see Notation 2.5. Since the function $\mathfrak{Q} \mapsto (1-1/\mathfrak{Q})^{\mathfrak{Q}-1}$ is decreasing and $\lim_{\mathfrak{Q}\to\infty}(1-1/\mathfrak{Q})^{\mathfrak{Q}-1} = 1/e$, we infer that $\Theta \geq \frac{1}{2}\vartheta e$. Suppose that $\delta^{1/(\mathfrak{Q}+1)} < \lambda < \frac{1}{2}$ and assume that we can find a $w \in xV \cap B\big(x, \frac{1}{2}r\big)$ such that $\phi(B(w,\lambda r) \cap E^\phi(\vartheta,\gamma)) = 0$. This would imply that

$$\begin{aligned}&\Theta\eta(1-\eta)^{\mathfrak{Q}-1}\lambda^{\mathfrak{Q}}r^{\mathfrak{Q}}\\ &\quad= \Theta\eta\lambda r\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV(B(w,(1-\eta)\lambda r))\\ &\quad\leq \Theta\int \lambda rg(\delta_{1/\lambda r}(w^{-1}z))\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV(z)\\ &\quad= \Theta\int \lambda rg(\delta_{1/\lambda r}(w^{-1}z))\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV(z) - \int \lambda rg(\delta_{1/\lambda r}(w^{-1}z))\,d\phi\llcorner E^\phi(\vartheta,\gamma)(z) \leq 2\delta r^{\mathfrak{Q}},\end{aligned} \tag{26}$$

where the inequality on the middle line is a consequence of the fact that, thanks to the precise choice of g, we have $g = \eta$ on $B(0, 1-\eta)$, whereas the last inequality comes from the choice of Θ, V, the fact that $\lambda rg(\delta_{1/\lambda r}(w^{-1}\cdot)) \in \mathrm{Lip}_1^+(\overline{B(x,r)})$ and the constraint on $\phi\llcorner E^\phi(\vartheta,\gamma)$ given by (25). Thanks to (26), the choice of λ and the fact that $\frac{1}{4}e\vartheta < \Theta$, we have that

$$\frac{\delta^{\mathfrak{Q}/(\mathfrak{Q}+1)}}{4e\vartheta}\eta(1-\eta)^{\mathfrak{Q}-1} < \Theta\lambda^{\mathfrak{Q}}\eta(1-\eta)^{\mathfrak{Q}-1} \leq 2\delta.$$

However, a few algebraic computations that we omit show that the above inequality chain is in contradiction with the choice of $\delta < \delta_{\mathbb{G}}(\vartheta)$. □

2B. ***Construction of cones complementing* supp(ϕ) *in case it has big projections on planes.*** This subsection is devoted to the proof of Proposition 2.11, which tells us that if the measure ϕ is well approximated inside a ball $B(x,r)$ by some plane V and if there exists some *other* plane W on which the $\mathcal{S}^{\mathfrak{Q}-1}$-measure of the projection $P_W(\mathrm{supp}(\phi) \cap B(x,r))$ is comparable with $r^{\mathfrak{Q}-1}$, then at scales comparable with r the set $\mathrm{supp}(\phi) \cap B(x,r)$ is a W-intrinsic Lipschitz surface. In other words, we can find an $\alpha > 0$ such that

$$y \in zC_W(\alpha) \quad \text{whenever } y, z \in B(x, r) \cap \operatorname{supp}(\phi) \text{ and } d(z, y) \gtrsim r.$$

Before proceeding with the statement and the proof of Proposition 2.11, we fix some notation that will be extensively used throughout the rest of the paper.

Notation 2.8. Throughout this paragraph we assume that ψ is a Radon measure on $\mathbb{G}$ supported on a compact set K such that $0 < \Theta_*^{\mathfrak{Q}-1}(\psi, x) \le \Theta^{\mathfrak{Q}-1,*}(\psi, x) < \infty$ for ψ-almost every $x \in \mathbb{G}$ and that $\sigma \in \mathbb{N}$ is a fixed positive natural number. First of all, let us define the following two numbers:

$$\zeta(\sigma) := 2^{-50\mathfrak{Q}}\sigma^{-2} \quad \text{and} \quad N(\sigma) := \lfloor -4\log(\zeta(\sigma))\rfloor + 40(\mathfrak{Q}+1).$$

Secondly, we let

$$\begin{aligned} C_3(\sigma) &:= 2^{20}(n_1-1)C_2(\sigma)^2, & C_4(\sigma) &:= 2^{24\mathfrak{Q}}\sigma, \\ C_5(\sigma) &:= C_4(\sigma)(32\zeta(\sigma)^{-2})^{\mathfrak{Q}-1}, & C_6(\sigma) &:= 2^{2\log C_4(\sigma)/(\mathfrak{Q}-1)+N(\sigma)+6}\zeta(\sigma)^{-2}. \end{aligned}$$

Finally, we introduce six further new constants that depend only on σ. Although we could avoid giving an explicit expression for such constants, we choose nonetheless to make them explicit for the following reasons: First of all, having their values helps keep their interactions in proofs under control, getting more precise statements. Secondly, fixing these constants once and for all, we avoid the practice of choosing them *large enough* when necessary. In doing so we hope to help the reader not to get distracted with the problem of whether these choices were legitimate or not.

For the sake of readability, we choose not to make the dependence on σ of the numbers N, ζ and the constants $C_1, \dots, C_6$ explicit in the following. In addition, in the forthcoming definitions, we choose to suppress any dependence on σ in the right-hand side of the expression. We let:

(i) $$A_0(\sigma) := 2\max\left\{\log(C_6) + C_6, \left\lfloor \frac{2\log_2 C_4}{N(\mathfrak{Q}-1)}\right\rfloor + 1, \frac{7\log 2 - 2\log\zeta}{N\log 2 - 2}\right\};$$

(ii) $$k(\sigma) := 80^{N+8}\zeta^{-2}A_0^4(1+e^{8NA_0^2}) \quad \text{and} \quad 0 < R < 2^{-(N+11)}\zeta^2 k;$$

(iii) $$\varepsilon_{\mathbb{G}}(\sigma) := \min\left\{2^{-20}, \frac{2^{2\mathfrak{Q}-n-18}\beta\prod_{j=2}^{s}\epsilon_i^{n_i}}{(A_0k)^{\mathfrak{Q}-1}C_5^2}\right\},$$

where β is the constant introduced in Proposition 1.8, the n_i are the topological dimensions of the i-th layer, V_i, of the Lie algebra $\mathfrak{g}$ and the ϵ_i are the structure constants used to construct the metric; see Definition 1.4;

(iv) $$\varepsilon_2(\sigma) := \min\left\{\frac{\delta_{\mathbb{G}}}{4}, \frac{\varepsilon_{\mathbb{G}}}{(2^{20}C_2^2C_5^2A_0k)(1+36kR^{-1})}, \frac{k-20}{20C_2k}, \frac{1}{2A_0^2C_3 + 2A_0kC_2C_4e^{8NA_0^2}}\right\}^{\mathfrak{Q}^2+\mathfrak{Q}}$$

with $\delta_{\mathbb{G}} = \delta_{\mathbb{G}}(\sigma)$ and $C_2 = C_2(\sigma)$;

(v) $$\varepsilon_3(\sigma) := \frac{1}{2^{2\mathfrak{Q}}C_4^2C_5^2(A_0C_6)^{\mathfrak{Q}-1}}.$$

Since in the rest of Section 2 we make extensive use of the dyadic cubes whose existence is stated in Appendix A, we recall here some of the notation. For any $\xi,\tau\in\mathbb{N}$ for which $\psi(E^\psi(\vartheta,\gamma))>0$, we denote by $\Delta^\psi(\xi,\tau)$ the family of dyadic cubes relative to ψ and to the parameters ξ and τ yielded by Theorem A.2. Furthermore, for any compact subset κ of $E^\psi(\xi,\tau)$ and $l\in\mathbb{N}$ we let

$$\Delta^\psi(\kappa;\xi,\tau,l):=\{Q\in\Delta^\psi(\xi,\tau):Q\cap\kappa\neq\varnothing \text{ and } Q\in\Delta^\psi_j(\xi,\tau) \text{ for some } j\geq l\},$$

where $\Delta^\psi_j(\xi,\tau)$ is the j-th layer of cubes; see Theorem A.2. Finally, for any $Q\in\Delta^\psi(E^\psi(\xi,\tau);\xi,\tau,1)$, we define

$$\alpha(Q):=\tilde d_{\mathfrak{c}(Q),2k\operatorname{diam}Q}(\psi,\mathfrak{M})+\tilde d_{\mathfrak{c}(Q),2k\operatorname{diam}Q}(\psi\llcorner E^\psi(\xi,\tau),\mathfrak{M}),$$

where $\mathfrak{c}(Q)\in Q$ is the center of the cube Q; see Theorem A.2 (v).

Eventually, we recall for the reader's sake some standard nomenclature on dyadic cubes: for any pair of dyadic cubes $Q_1,Q_2\in\Delta^\psi(\xi,\tau)$,

(i) if $Q_1\subseteq Q_2$, then Q_2 is said to be an *ancestor* of Q_1 and Q_1 a *subcube* of Q_2,

(ii) if Q_2 is the smallest cube for which $Q_1\subsetneq Q_2$, then Q_2 is said to be the *parent* of Q_1 and Q_1 the *child* of Q_2.

Notation 2.9. If not otherwise stated, in order to simplify notation throughout Section 2 we will always denote by $\Delta:=\Delta^\phi(\vartheta,\gamma)$ the family of dyadic cubes constructed in Theorem A.2 relative to the measure ϕ, which was fixed at the beginning of this Section, and to the parameters ϑ and γ, fixed in Notation 2.5. Furthermore, we let

$$E(\vartheta,\gamma):=E^\phi(\vartheta,\gamma),\quad \mathscr{E}(\mu,\nu):=\mathscr{E}^\phi_{\vartheta,\gamma}(\mu,\nu)\quad\text{and}\quad \Delta(\kappa,l):=\Delta^\phi(\kappa;\vartheta,\gamma,l).$$

Finally, if the dependence on σ of the constants introduced above is *not* specified, we will always assume that $\sigma=\vartheta$, where once again ϑ is the one natural number fixed in Notation 2.5.

Remark 2.10. For any compact subset κ of $E(\vartheta,\gamma)$, we let $\mathcal{M}(\kappa,l)$ be the set of maximal cubes of $\Delta(\kappa,l)$ ordered by inclusion. The elements of $\mathcal{M}(\kappa,l)$ are pairwise disjoint and enjoy the following properties:

(i) For any $Q\in\Delta(\kappa,l)$ there is a cube $Q_0\in\mathcal{M}(\kappa,l)$ such that $Q\subseteq Q_0$.

(ii) If $Q_0\in\mathcal{M}(\kappa,l)$ and there exists some $Q'\in\Delta(\kappa,l)$ for which $Q_0\subseteq Q'$, then $Q_0=Q'$.

The proof of the following proposition is inspired by the argument employed in proving [David and Semmes 1993b, Lemma 2.19] and its counterpart in the first Heisenberg group $\mathbb{H}^1$ [Chousionis et al. 2019, Lemma 3.8].

Proposition 2.11. *Let $\iota\in\mathbb{N}$ be such that $\iota>N^{-1}(5+\log_2(4k))$, and suppose that Q is a cube in $\Delta(E(\vartheta,\gamma),\iota)$ satisfying the two following conditions*:

(i) $\tilde d_{\mathfrak{c}(Q),4k\operatorname{diam}Q}(\phi\llcorner E(\vartheta,\gamma),\mathfrak{M})+\tilde d_{\mathfrak{c}(Q),4k\operatorname{diam}Q}(\phi,\mathfrak{M})\leq\varepsilon_2$.

(ii) *There exists a plane $W\in\operatorname{Gr}(\mathfrak{Q}-1)$ such that*

$$\frac{\operatorname{diam}Q^{\mathfrak{Q}-1}}{4C_5^2A_0^{\mathfrak{Q}-1}}\leq\mathcal{S}^{\mathfrak{Q}-1}\llcorner W(P_W[\mathfrak{c}(Q)^{-1}(Q\cap E(\vartheta,\gamma))]).\tag{27}$$

Let $x \in E(\vartheta, \gamma) \cap Q$ and $y \in B\big(x, \frac{1}{8}(k-1) \operatorname{diam} Q\big) \cap E(\vartheta, \gamma)$ be two points for which

$$R \operatorname{diam} Q \leq d(x, y) \leq 2^{N+6}\zeta^{-2} R \operatorname{diam} Q. \tag{28}$$

Then, for any $\alpha > (\zeta^2 \varepsilon_{\mathbb{G}}/(6 \cdot 2^{8+N} R^{-1}k))^{-1} =: \alpha_0$, we have $y \in xC_W(\alpha)$.

Remark 2.12. Thanks to the definition of R and k, we have

$$2^{(N+6)}\zeta^{-2}R < 2^{(N+6)}\zeta^{-2} \cdot 2^{-(N+11)}\zeta^2 k = \tfrac{1}{32}k < \tfrac{1}{8}(k-1).$$

This implies that $B(x, 2^{N+6}\zeta^{-2}R \operatorname{diam} Q) \Subset B\big(x, \frac{1}{8}(k-1) \operatorname{diam} Q\big)$, and thus the requested inequality $d(x, y) \geq R \operatorname{diam} Q$ is compatible with the fact that y is chosen in $B\big(x, \frac{1}{8}(k-1) \operatorname{diam} Q\big)$.

Proof of Proposition 2.11. Suppose by contradiction there are two points $x, y \in E(\vartheta, \gamma)$ satisfying the hypothesis of the proposition such that $y \notin xC_W(\alpha)$ for some $\alpha > \alpha_0$. This implies, since the cone $C_W(\alpha)$ is closed by definition, that we have $\pi_1(x^{-1}y) \neq 0$. Furthermore, Proposition 1.14 along with (28) yields

$$\operatorname{diam} Q \leq R^{-1}d(x, y) \leq R^{-1}\Lambda(\alpha)|\pi_1(x^{-1}y)| \leq R^{-1}\Lambda(1)|\pi_1(x^{-1}y)| = 2R^{-1}|\pi_1(x^{-1}y)|, \tag{29}$$

where the last inequality comes from the fact that Λ (the function yielded by Proposition 1.14) is decreasing and from the last identity from the very definition of the function Λ. Let $\rho := \operatorname{diam}(Q)$ and note that Proposition 2.3 (iii) and the fact that $B(x, 4(k-1)\rho) \subseteq B(\mathfrak{c}(Q), 4k\rho)$ imply that

$$\begin{aligned}&\tilde{d}_{x,4(k-1)\rho}(\phi, \mathfrak{M}) + \tilde{d}_{x,4(k-1)\rho}(\phi \llcorner E(\vartheta, \gamma), \mathfrak{M}) \\ &\qquad \leq \left(\frac{k}{(k-1)}\right)^{\mathfrak{Q}} (\tilde{d}_{\mathfrak{c}(Q),4k\rho}(\phi, \mathfrak{M}) + \tilde{d}_{\mathfrak{c}(Q),4k\rho}(\phi \llcorner E(\vartheta, \gamma), \mathfrak{M})) \leq 2^{\mathfrak{Q}}\varepsilon_2.\end{aligned}$$

In addition, we also have that $4(k-1)\rho < 1/\gamma$; indeed,

$$4(k-1)\rho = 4(k-1) \operatorname{diam}(Q) \leq 4(k-1) \cdot 2^{-N\iota+5}/\gamma < 1/\gamma,$$

where the first inequality above comes from Theorem A.2 and the last one from the choice of ι.

Therefore, thanks to Proposition 2.6 and the fact that $2^{\mathfrak{Q}}\varepsilon_2 \leq \delta_{\mathbb{G}}(\vartheta)$, we infer that there exists a plane $V \in \operatorname{Gr}(\mathfrak{Q}-1)$, *that we consider fixed throughout the proof*, such that

$$\sup_{w \in E(\vartheta,\gamma) \cap B(x,(k-1)\rho)} \frac{\operatorname{dist}(w, xV)}{4(k-1)\rho} \leq 2C_2\varepsilon_2^{1/\mathfrak{Q}}. \tag{30}$$

Since $y \in E(\vartheta, \gamma) \cap B(x, (k-1)\rho)$, we deduce from (30) that

$$\operatorname{dist}(y, xV) \leq 8(k-1)C_2\varepsilon_2^{1/\mathfrak{Q}}\rho. \tag{31}$$

In this paragraph we prove that if there exists a point $v \in V$ such that $v_1 \neq 0$ and $|\pi_1(P_W v)| \leq \theta|v_1|$ for some $0 < \theta < 1$, then

$$|\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle| \leq \theta/\sqrt{1-\theta^2}. \tag{32}$$

We note that the assumptions on v_1 imply that

$$|v_1|^2 - \langle \mathfrak{n}(W), v_1\rangle^2 = |v_1 - \langle \mathfrak{n}(W), v_1\rangle \mathfrak{n}(W)|^2 = |\pi_{\mathscr{W}} v_1|^2 = |\pi_1(P_W v)|^2 \leq \theta^2|v_1|^2, \tag{33}$$

where $\pi_{\mathscr{W}}$ is the projection in V_1 onto $W \cap V_1$ that was defined in Proposition 1.11. By means of a few omitted algebraic manipulations of (33), we conclude that $\sqrt{1-\theta^2}|v_1| \le |\langle \mathfrak{n}(W), v_1\rangle|$. Finally, since $\langle \mathfrak{n}(V), v_1\rangle = 0$, thanks to (33) and the Cauchy–Schwartz inequality, we have

$$\begin{aligned}\theta|v_1| &\ge |\langle \pi_{\mathscr{W}} v_1, \mathfrak{n}(V)\rangle| = |\langle v_1 - \langle \mathfrak{n}(W), v_1\rangle \mathfrak{n}(W), \mathfrak{n}(V)\rangle| \\ &= |\langle \mathfrak{n}(W), v_1\rangle\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle| \ge \sqrt{1-\theta^2}|v_1||\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle|. \end{aligned} \tag{34}$$

It is immediate to see that (34) is equivalent to (32), proving the claim.

Given $x, y \in E(\vartheta, \gamma)$ and $V, W \in \mathrm{Gr}(\mathfrak{Q}-1)$ as above, in this paragraph using the counterassumption $x^{-1}y \notin C_W(\alpha)$ we construct a $v \in V$ with $v_1 \neq 0$ that satisfies the bound $|\pi_1(P_W v)| \le \theta|v_1|$ for a suitably small θ. Since $y \notin xC_W(\alpha)$, thanks to Proposition 1.11 we have

$$|\pi_1(P_W(x^{-1}y))| \le \|P_W(x^{-1}y)\| < \alpha^{-1}\|P_{\mathfrak{n}(W)}(x^{-1}y)\| = \alpha^{-1}|\langle \mathfrak{n}(W), \pi_1(x^{-1}y)\rangle| \le \alpha^{-1}|\pi_1(x^{-1}y)|.$$

Defined v to be the point of V for which $d(y, xv) = \mathrm{dist}(y, xV)$, thanks to (31) and the fact that $y \in B\big(x, \tfrac{1}{8}(k-1)\rho\big)$ we have

$$\|v\| \le d(xv, y) + d(y, x) \le \mathrm{dist}(y, xV) + \tfrac{1}{8}((k-1)\rho) \le \big(8C_2\varepsilon_2^{1/\mathfrak{Q}} + \tfrac{1}{8}\big)k\rho < (k-1)\rho,$$

where the last inequality comes from the choice of ε_2. Furthermore, thanks to (29) and (31) we have

$$\begin{aligned}\tfrac{1}{4}R\rho \le \big(\tfrac{1}{2}R - 8C_2k\varepsilon_2^{1/\mathfrak{Q}}\big)\rho &\le |\pi_1(x^{-1}y)| - d(y, xv) \le |\pi_1(x^{-1}y)| - |\pi_1(y^{-1}xv)| \\ &\le |\pi_1(x^{-1}y) - \pi_1(y^{-1}xv)| = |v_1|, \end{aligned} \tag{35}$$

and where the first inequality above, comes from the choice of ε_2. Let us prove that v satisfies the inequality

$$|\pi_1(P_W v)| \le 4R^{-1}k(16C_2\varepsilon_2^{1/\mathfrak{Q}} + 2^{6+N}\zeta^{-2}\alpha^{-1})|v_1|. \tag{36}$$

Since $x^{-1}y \notin C_W(\alpha)$, thanks to Proposition 1.11 we have

$$\begin{aligned}|\pi_1(P_W(v))| &\le |\pi_1(P_W(v)) - \pi_1(P_W(x^{-1}y))| + |\pi_1(P_W(x^{-1}y))| \\ &\le |\pi_1(P_W(y^{-1}xv))| + \|P_W(x^{-1}y)\| \le |\pi_1(P_W(y^{-1}xv))| + \alpha^{-1}\|P_{\mathfrak{N}(W)}(x^{-1}y)\| \\ &\le \|P_W(y^{-1}xv)\| + \alpha^{-1}|\pi_1(x^{-1}y)| \le \|P_W(y^{-1}xv)\| + 2^{6+N}\zeta^{-2}R\alpha^{-1}\rho, \end{aligned} \tag{37}$$

where the last inequality of the last line above comes from (28). Proposition 1.15 together with (31), (35) and (37) implies that

$$\begin{aligned}|\pi_1(P_W(v))| &\overset{(37)}{\le} \|P_W(y^{-1}xv)\| + 2^{6+N}\zeta^{-2}R\alpha^{-1}\rho \le 2\|y^{-1}xv\| + 2^{6+N}\zeta^{-2}R\alpha^{-1}\rho \\ &= 2\,\mathrm{dist}(y, xV) + 2^{6+N}\zeta^{-2}R\alpha^{-1}\rho \overset{(31)}{\le} (16C_2(k-1)\varepsilon_2^{1/\mathfrak{Q}} + 2^{6+N}\zeta^{-2}R\alpha^{-1})\rho \\ &\overset{(35)}{\le} 4R^{-1}k(16C_2\varepsilon_2^{1/\mathfrak{Q}} + 2^{6+N}\zeta^{-2}\alpha^{-1})|v_1| =: \theta(\alpha, \varepsilon_2)|v_1|. \end{aligned} \tag{38}$$

Thanks to the choice of the constants α_0, ε_2, R and k together with some elementary algebraic computations that we omit, it is possible to prove that $\sqrt{1-\theta(\alpha, \varepsilon_2)^2} \ge \frac{1}{2}$. Since $|\pi_1(P_W(v))| \le \theta(\alpha, \varepsilon_2)|\pi_1(v)|$, we deduce thanks to (32) that

$$|\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle| \le \frac{\theta(\alpha, \varepsilon_2)}{\sqrt{1-\theta(\alpha, \varepsilon_2)^2}} \le 2\theta(\alpha, \varepsilon_2). \tag{39}$$

Let us take a step back and let us examine what we have shown so far. Starting from the absurd hypothesis $y^{-1}x \in C_W(\alpha)$ we have shown that there is a nonnull $v \in V$ with $|\pi_1(P_W v)| \le \theta(\alpha, \varepsilon_2)|v|$. This can be alternatively read as the fact that the normals $\mathfrak{n}(V)$ and $\mathfrak{n}(W)$ of V and W respectively are almost orthogonal. However, one should expect this orthogonality to be incompatible with (27).

Let us prove that (39) is in contradiction with (27). Choose some $z \in B\big(x, \frac{1}{8}(k-1)\rho\big) \cap E(\vartheta, \gamma)$ and note that

$$\begin{aligned}
|\langle \mathfrak{n}(V), \pi_1(P_W(x^{-1}z))\rangle| &= |\langle \mathfrak{n}(V), \pi_{\mathscr{W}}(z_1 - x_1)\rangle| \\
&\le |\langle \mathfrak{n}(V), z_1 - x_1\rangle| + |\langle \mathfrak{n}(V), \pi_{\mathfrak{n}(W)}(z_1 - x_1)\rangle| \\
&\le |\langle \mathfrak{n}(V), z_1 - x_1\rangle| + |\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle|\,|\langle z_1 - x_1, \mathfrak{n}(W)\rangle| \\
&\le \|P_{\mathfrak{N}(V)}(x^{-1}z)\| + d(x, z)|\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle| \\
&= \operatorname{dist}(z, xV) + d(x, z)|\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle|,
\end{aligned} \tag{40}$$

where the last identity comes from Proposition 1.15. Inequalities (30), (39), (40) and the choice of z imply that

$$\begin{aligned}
|\langle \mathfrak{n}(V), \pi_1(P_W(x^{-1}z))\rangle| &\overset{(40)}{\le} \operatorname{dist}(z, xV) + d(x, z)|\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle| \\
&\overset{(30),(39)}{\le} 8C_2 k\varepsilon_2^{1/\mathfrak{Q}}\rho + 2\theta(\alpha, \varepsilon_2)d(x, z) \le 8C_2 k\varepsilon_2^{1/\mathfrak{Q}}\rho + 2\theta(\alpha, \varepsilon_2)k\rho.
\end{aligned} \tag{41}$$

Furthermore, defining $\mathfrak{n} := \pi_{\mathscr{W}}(\mathfrak{n}(V))$, it is immediate to see from (39) that $|\mathfrak{n} - \mathfrak{n}(V)| \le 2\theta(\alpha, \varepsilon_2)$, which yields thanks to the triangular inequality and Proposition 1.15 the bound

$$\begin{aligned}
|\langle \mathfrak{n}, \pi_1(P_W(x^{-1}z))\rangle| &\le |\langle \mathfrak{n}(V), \pi_1(P_W(x^{-1}z))\rangle| + |\mathfrak{n} - \mathfrak{n}(V)|\,|\pi_1(P_W(x^{-1}z))| \\
&\le |\langle \mathfrak{n}(V), \pi_1(P_W(x^{-1}z))\rangle| + |\mathfrak{n} - \mathfrak{n}(V)|\,\|P_W(x^{-1}z)\| \\
&\overset{(41)}{\le} (8C_2 k\varepsilon_2^{1/\mathfrak{Q}}\rho + 2\theta(\alpha, \varepsilon_2)k\rho) + 4\theta(\alpha, \varepsilon_2)k\rho \le 8(C_2\varepsilon_2^{1/\mathfrak{Q}} + 3\theta(\alpha, \varepsilon_2))k\rho.
\end{aligned} \tag{42}$$

For the sake of notation, we introduce the set

$$S := \{w \in W : |\langle \mathfrak{n}, w_1\rangle| \le 8(C_2\varepsilon_2^{1/\mathfrak{Q}} + 3\theta(\alpha, \varepsilon_2))k\rho\}.$$

The bound (42) implies that the projection of $x^{-1}E(\vartheta, \gamma) \cap B\big(0, \frac{1}{8}(k-1)\rho\big)$ on W is contained in S, which is a very narrow strip around $V \cap W$ inside W. Furthermore, we recall that from Proposition 1.15 we have

$$P_W\big(B\big(0, \tfrac{1}{8}(k-1)\rho\big)\big) \subseteq B\big(0, 2 \cdot \tfrac{1}{8}(k-1)\rho\big). \tag{43}$$

Finally, putting together (42) and (43), we deduce that

$$\begin{aligned}
P_W\big(x^{-1}E(\vartheta, \gamma) \cap B\big(0, \tfrac{1}{8}(k-1)\rho\big)\big) &\subseteq P_W(x^{-1}E(\vartheta, \gamma)) \cap P_W\big(B\big(0, \tfrac{1}{8}(k-1)\rho\big)\big) \\
&\subseteq S \cap B\big(0, \tfrac{1}{4}(k-1)\rho\big).
\end{aligned} \tag{44}$$

Completing $\{\mathfrak{n}(W), \mathfrak{n}\}$ to an orthonormal basis $\mathcal{E} := \{\mathfrak{n}(W), \mathfrak{n}/|\mathfrak{n}|, e_3, \dots, e_n\}$ of $\mathbb{R}^n$ satisfying (9), thanks to Remark 1.5 we have

$$S \cap B\big(0, 2 \cdot \tfrac{1}{8}(k-1)\rho\big) \subseteq S \cap \mathrm{Box}_{\mathcal{E}}\big(0, \tfrac{1}{4}k\rho\big). \tag{45}$$

The above inclusion together with Tonelli's theorem yields

$$\begin{aligned}
&\mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner W\big(P_W\big(x^{-1}E(\vartheta,\gamma) \cap B\big(0, \tfrac{1}{8}(k-1)\rho\big)\big)\big) \\
&\qquad \le \mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner W\big(S \cap B\big(0, \tfrac{1}{4}(k-1)\rho\big)\big) \le \mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner W\big(S \cap \mathrm{Box}_{\mathcal{E}}\big(0, \tfrac{1}{4}k\rho\big)\big) \\
&\qquad = 16(C_2\varepsilon_2^{1/\mathfrak{Q}} + 3\theta(\alpha,\varepsilon_2))k\rho \cdot 2^{n-2} \prod_{i=2}^{s} \epsilon_i^{-n_i} \big(\tfrac{1}{4}k\rho\big)^{\mathfrak{Q}-2} \\
&\qquad = 2^{n-2\mathfrak{Q}+8} \prod_{i=2}^{s} \epsilon_i^{-n_i} (C_2\varepsilon_2^{1/\mathfrak{Q}} + 3\theta(\alpha,\varepsilon_2))(k\rho)^{\mathfrak{Q}-1}.
\end{aligned} \tag{46}$$

The inclusion (44), the bound (46), Proposition 1.8 and the definition of A_0 finally imply that

$$\begin{aligned}
&\mathcal{S}^{\mathfrak{Q}-1} \llcorner W\big(P_W\big(x^{-1}E(\vartheta,\gamma) \cap B\big(0, \tfrac{1}{8}(k-1)\rho\big)\big)\big) \\
&\qquad \le \mathcal{S}^{\mathfrak{Q}-1} \llcorner W\big(S \cap B\big(0, \tfrac{1}{4}k\rho\big)\big) = \beta^{-1}\mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner W\big(S \cap B\big(0, \tfrac{1}{4}k\rho\big)\big) \\
&\qquad \le \beta^{-1} 2^{n-2\mathfrak{Q}+8} \prod_{i=2}^{s} \epsilon_i^{-n_i} (C_2\varepsilon_2^{1/\mathfrak{Q}} + 3\theta(\alpha,\varepsilon_2))(k\rho)^{\mathfrak{Q}-1} \\
&\qquad = 2^{-10} C_5^{-2} \varepsilon_{\mathbb{G}}^{-1} A_0^{-(\mathfrak{Q}-1)} (C_2\varepsilon_2^{1/\mathfrak{Q}} + 3\theta(\alpha,\varepsilon_2))\rho^{\mathfrak{Q}-1},
\end{aligned} \tag{47}$$

where β is the constant introduced in Proposition 1.8 and where the last identity comes from the definitions of $\varepsilon_{\mathbb{G}}$ and A_0; see Notation 2.8. Furthermore, since $\mathcal{S}^{\mathfrak{Q}-1} \llcorner W(P_W(p * E) = \mathcal{S}^{\mathfrak{Q}-1} \llcorner W(P_W(E))$ for any measurable set E in $\mathbb{G}$, see for instance the proof in [Franchi and Serapioni 2016, Proposition 2.2], we deduce that

$$\begin{aligned}
&\mathcal{S}^{\mathfrak{Q}-1} \llcorner W\big(P_W\big(x^{-1}E(\vartheta,\gamma) \cap B\big(0, \tfrac{1}{8}(k-1)\rho\big)\big)\big) \\
&\qquad = \mathcal{S}^{\mathfrak{Q}-1} \llcorner W\big(P_W\big(\mathfrak{c}(Q)^{-1}E(\vartheta,\gamma) \cap B\big(\mathfrak{c}(Q)^{-1}x, \tfrac{1}{8}(k-1)\rho\big)\big)\big).
\end{aligned}$$

Thanks to the choice of k and the fact that $x \in Q$, we infer that $B(0,\rho) \subseteq B\big(\mathfrak{c}(Q)^{-1}x, \tfrac{1}{8}(k-1)\rho\big)$. Together with (27), this allows us to deduce that

$$\begin{aligned}
&\mathcal{S}^{\mathfrak{Q}-1} \llcorner W\big(P_W\big(x^{-1}E(\vartheta,\gamma) \cap B\big(0, \tfrac{1}{8}(k-1)\rho\big)\big)\big) \\
&\ge \mathcal{S}^{\mathfrak{Q}-1} \llcorner W(P_W(\mathfrak{c}(Q)^{-1}E(\vartheta,\gamma) \cap B(0,\rho))) \ge \mathcal{S}^{\mathfrak{Q}-1} \llcorner W(P_W(\mathfrak{c}(Q)^{-1}(E(\vartheta,\gamma) \cap Q))) \ge \frac{\rho^{\mathfrak{Q}-1}}{4C_5^2 A_0^{\mathfrak{Q}-1}}.
\end{aligned} \tag{48}$$

Putting together (47) and (48) we conclude that

$$2^8 \varepsilon_{\mathbb{G}} \le (C_2\varepsilon_2^{1/\mathfrak{Q}} + 3\theta(\alpha,\varepsilon_2)) = C_2\varepsilon_2^{1/\mathfrak{Q}} + 12R^{-1}k(2C_2\varepsilon_2^{1/\mathfrak{Q}} + 2^{6+N}\zeta^{-2}\alpha^{-1}).$$

The choice of ε_2 and α imply, with some algebraic computations that we omit, that the above inequality is false, showing that the assumption $y \notin xC_W(\alpha)$ is false. We have reached a contradiction, proving the proposition. □

2C. *Flat tangents imply big projections.* We recall that the measure ϕ is supposed to be supported on a compact set K and that for ϕ-almost every $x \in \mathbb{G}$ we assume that $0 < \Theta_*^{\mathfrak{Q}-1}(\phi, x) \le \Theta^{\mathfrak{Q}-1,*}(\phi, x) < \infty$ and $\mathrm{Tan}_{\mathfrak{Q}-1}(\phi, x) \subseteq \mathfrak{M}$. This subsection is devoted to the proof of the following result, which asserts that hypothesis (ii) of Proposition 2.11 is satisfied by the measure $\phi \llcorner E(\vartheta, \gamma)$.

Theorem 2.13. *There exists a compact subset C of $E(\vartheta, \gamma)$ having big measure inside $E(\vartheta, \gamma)$ such that for any cube Q of sufficiently small diameter for which $(1-\varepsilon_3)\phi(Q) \le \phi(Q \cap C)$, there exists a plane $\Pi(Q) \in \mathrm{Gr}(\mathfrak{Q}-1)$ such that*

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q)}(Q \cap C)) \ge \frac{\operatorname{diam} Q^{\mathfrak{Q}-1}}{2A_0^{\mathfrak{Q}-1}}.$$

The compact set C will be constructed in Proposition 2.14 while the scale below which the thesis of Theorem 2.13 is known to hold will be determined in Lemma 2.16 together with the plane $\Pi(Q)$. The reader can find the precise statement of the above result in Theorem 2.28.

In the following it will be useful to reduce to a compact subset C of $E(\vartheta, \gamma)$ where the distance of ϕ from planes is uniformly small below a fixed scale.

Proposition 2.14. *For any $\mu \ge 4\vartheta$, there exists a $\nu \in \mathbb{N}$, a compact subset C of $\mathscr{E}(\mu, \nu)$ and an $\iota_0 \in \mathbb{N}$ such that*

(i) $\phi(K \setminus C) \le 2\varepsilon_1 \phi(K)$,

(ii) $d_{x,4kr}(\phi, \mathfrak{M}) + d_{x,4kr}(\phi \llcorner E(\vartheta, \gamma), \mathfrak{M}) \le 2^{-\mathfrak{Q}^2-\mathfrak{Q}}\varepsilon_2$ *for any $x \in C$ and any $0 < r \le 2^{-\iota_0 N+5}/\gamma$.*

Proof. Since by assumption $\mathrm{Tan}_{\mathfrak{Q}-1}(\phi, x) \subseteq \mathfrak{M}$ for ϕ-almost every $x \in \mathbb{G}$, thanks to Proposition 2.4 we infer that the functions $f_r(x) := d_{x,4kr}(\phi, \mathfrak{M})$ converge ϕ-almost everywhere to 0 on K as r goes to 0. Thanks to Proposition 1.27, the same line of reasoning implies also that $f_r^{\vartheta,\gamma}(x) := d_{x,4kr}(\phi \llcorner E(\vartheta, \gamma), \mathfrak{M})$ converges ϕ-almost everywhere to 0 on $E(\vartheta, \gamma)$. Proposition 2.3 and Severini–Egoroff's theorem yield a compact subset C of $\mathscr{E}(\mu, \nu)$ such that $\phi(E(\vartheta, \gamma) \setminus C) \le \varepsilon_1 \phi(E(\vartheta, \gamma))$ and such that the functions $x \mapsto d_{x,4kr}(\phi, \mathfrak{M}) + d_{x,4kr}(\phi \llcorner E(\vartheta, \gamma), \mathfrak{M})$ converge uniformly to 0 on C as r goes to 0. This directly implies both (i) and (ii) thanks to the choice of ϑ and γ. $\square$

Notation 2.15. From now on *we consider the integer $\mu \ge 4C_4\vartheta$ and the compact set C and the natural numbers ν and ι_0 yielded by Proposition 2.14 to be fixed*. Furthermore, we define $\iota := \max\{\iota_0, \nu\}$.

The following lemma rephrases Propositions 2.6 and 2.7 into the language of dyadic cubes.

Lemma 2.16. *For any cube $Q \in \Delta(C, \iota)$ we have $\alpha(Q) \le \varepsilon_2$. Furthermore, there exists a plane $\Pi(Q) \in \mathrm{Gr}(\mathfrak{Q}-1)$ for which*

(i) $$\sup_{w \in E(\vartheta,\gamma) \cap B(\mathfrak{c}(Q), k \operatorname{diam} Q/2)} \frac{\operatorname{dist}(w, \mathfrak{c}(Q)\Pi(Q))}{2k \operatorname{diam} Q} \le C_2 \varepsilon_2^{1/\mathfrak{Q}}, \quad \textit{and}$$

(ii) *for any $w \in B\big(\mathfrak{c}(Q), \tfrac{1}{2}k \operatorname{diam} Q\big) \cap \mathfrak{c}(Q)\Pi(Q)$ we have $E(\vartheta, \gamma) \cap B(w, 3kC_2\varepsilon_2^{1/(\mathfrak{Q}+1)} \operatorname{diam} Q) \neq \varnothing$.*

Proof. Let $Q \in \Delta(C, \iota)$, fix an $x \in Q \cap C$ and define $\rho := \operatorname{diam} Q$. Thanks to Proposition 2.14 we know that

$$d_{x,4kr}(\phi, \mathfrak{M}) + d_{x,4kr}(\phi \llcorner E(\vartheta,\gamma), \mathfrak{M}) \leq 2^{-\mathfrak{Q}^2-\mathfrak{Q}}\varepsilon_2, \tag{49}$$

for any $r \leq 2^{-\iota N+5}/\gamma$. Thanks to Theorem A.2 (ii) we have that $\rho \leq 2^{-\iota N+5}/\gamma$ and thus by Proposition 2.3 we infer that

$$\begin{aligned} \tilde{d}_{\mathfrak{c}(Q),2k\rho}(\phi, \mathfrak{M}) &\leq 2^{\mathfrak{Q}}\tilde{d}_{x,4k\rho}(\phi, \mathfrak{M}) \leq 2^{-\mathfrak{Q}^2}\varepsilon_2, \\ \tilde{d}_{\mathfrak{c}(Q),2k\rho}(\phi \llcorner E(\vartheta,\gamma), \mathfrak{M}) &\leq 2^{\mathfrak{Q}}\tilde{d}_{x,4k\rho}(\phi \llcorner E(\vartheta,\gamma), \mathfrak{M}) \leq 2^{-\mathfrak{Q}^2}\varepsilon_2. \end{aligned} \tag{50}$$

The bounds in (50) together with Proposition 2.6 imply that $\alpha(Q) \leq 2^{-\mathfrak{Q}^2+1}\varepsilon_2 \leq \varepsilon_2$.

The proof of the second part of the statement is a little more delicate. Since C is a subset of $\mathscr{E}_{\vartheta,\gamma}(\mu,\nu)$, thanks to the choice of μ and ι, by Proposition A.5 we have that $\mathfrak{c}(Q) \in E(\vartheta,\gamma) \cap B(x,\rho)$. Let us choose $\Pi(Q) \in \Pi_\delta(x, 2k\rho)$, where $\delta := 2^{-\mathfrak{Q}^2-\mathfrak{Q}}$, and note that Propositions 1.17 (i), (ii) and 2.6 imply that for any $w \in E(\vartheta,\gamma) \cap B\left(\frac{1}{2}k\rho\right)$ we have

$$\begin{aligned} \operatorname{dist}(w, \mathfrak{c}(Q)V) &\leq \operatorname{dist}(w, xV) + \operatorname{dist}(xV, \mathfrak{c}(Q)V) = \operatorname{dist}(w, xV) + \operatorname{dist}(\mathfrak{c}(Q), xV) \\ &\leq 2 \cdot 2k\rho \cdot C_2(2^{-\mathfrak{Q}^2-\mathfrak{Q}}\varepsilon_2)^{1/\mathfrak{Q}} \leq 2k\rho \cdot 2^{-\mathfrak{Q}} \cdot C_2\varepsilon_2^{1/\mathfrak{Q}}, \end{aligned} \tag{51}$$

where the last inequality comes from (49). This concludes the proof of (i).

Let us move to the proof of (ii). For any $V \in \Pi_\delta(x, 2k\rho)$ and any $w \in B\left(0, \frac{1}{2}k\rho\right) \cap V$, we define

$$w^* := x^{-1}c(Q)wP_{\mathfrak{N}(V)}(\mathfrak{c}(Q)^{-1}x) = P_{\mathfrak{N}(V)}(\mathfrak{c}(Q)^{-1}x)^{-1}P_V(\mathfrak{c}(Q)^{-1}x)^{-1}wP_{\mathfrak{N}(V)}(\mathfrak{c}(Q)^{-1}x) \in V.$$

With a few computations that we omit, it is not difficult to see that

$$d(\mathfrak{c}(Q)w, xw^*) = \|P_{\mathfrak{N}(V)}(\mathfrak{c}(Q)^{-1}x)\| = \operatorname{dist}(\mathfrak{c}(Q), xV) \leq 2^{-(\mathfrak{Q}-2)}k\rho C_2\varepsilon_2^{1/\mathfrak{Q}}, \tag{52}$$

where the second identity follows from Proposition 1.17 and the last inequality from the second last inequality in (51). Thanks to the definition of w^*, the triangle inequality, Proposition 1.15 and the fact that $d(\mathfrak{c}(Q), x) \leq \rho$, the norm of w^* can be estimated as

$$\|w^*\| \leq 2\|P_{\mathfrak{N}(V)}(\mathfrak{c}(Q)^{-1}x)\| + \|P_V(\mathfrak{c}(Q)^{-1}x)\| + \|w\| \leq 2\rho + 2\rho + \tfrac{1}{2}k\rho < k\rho. \tag{53}$$

Thanks to inequalities (49) and (53) and Proposition 2.7, we infer that

$$B(xw^*, 2k\rho \cdot 2^{-\mathfrak{Q}}\varepsilon_2^{1/(\mathfrak{Q}+1)}) \cap E(\vartheta,\gamma) \neq \varnothing.$$

Finally, since $2^{1-\mathfrak{Q}} < C_2$, thanks to (52) we conclude that

$$E(\vartheta,\gamma) \cap B(\mathfrak{c}(Q)w, 3k\rho C_2\varepsilon_2^{1/(\mathfrak{Q}+1)}) \supseteq E(\vartheta,\gamma) \cap B(xw^*, 2k\rho \cdot 2^{-\mathfrak{Q}}\varepsilon_2^{1/(\mathfrak{Q}+1)}) \neq \varnothing. \qquad \square$$

The arguments we will use in the rest of the subsection to prove Proposition 2.18 through Theorem 2.28 follow from an adaptation of the techniques found in Chapter 2, §2 of [David and Semmes 1993a]. The first of such adaptations is the following definition, which is a way of saying that two cubes are close both in metric and in size terms:

Definition 2.17 (neighbor cubes). Let $A := 4A_0^2$ and let $Q_j \in \Delta^{\phi}_{i_j}(\vartheta,\gamma)$ be two cubes with $j = 1, 2$.[4] We say that Q_1 and Q_2 are *neighbors* if

$$\underline{\mathrm{dist}}(Q_1, Q_2) := \inf_{x\in Q_1, y\in Q_2} d(x, y) \underset{\text{(I)}}{\le} A(\operatorname{diam} Q_1 + \operatorname{diam} Q_2) \quad \text{and} \quad |i_1 - i_2| \underset{\text{(II)}}{\le} A.$$

Furthermore, in the following (for the sake of notation), for any $Q \in \Delta(C,\iota)$ we let

$$\mathfrak{n}(Q) := \mathfrak{n}(\Pi(Q)),$$

where $\Pi(Q) \in \mathrm{Gr}(\mathfrak{Q}-1)$ is the plane yielded by Lemma 2.16.

Finally, two planes $V, W \in \mathrm{Gr}(\mathfrak{Q}-1)$ are said to have *compatible orientations* if their normals $\mathfrak{n}(V), \mathfrak{n}(W) \in V_1$ are chosen in such a way that $\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle > 0$. By extension, we will say that two cubes $Q_1, Q_2 \in \Delta(C,\iota)$ have *compatible orientations* themselves if $\Pi(Q_1)$ and $\Pi(Q_2)$ are chosen to have compatible orientations.

Proposition 2.18. *Suppose that $Q_j \in \Delta^{\phi}_{i_j}(\vartheta,\gamma)$ for $j = 1, 2$. Then the following hold*:

(i) *If Q_1 is the parent of Q_2, then Q_1 and Q_2 are neighbors.*

(ii) *If Q_1 and Q_2 are neighbors for any nonnegative integer $k \le \min\{i_1, i_2\}$, then their ancestors $\tilde{Q}_1 \in \Delta^{\phi}_{i_1-k}(\vartheta,\gamma)$ and $\tilde{Q}_2 \in \Delta^{\phi}_{i_2-k}(\vartheta,\gamma)$ are neighbors.*

(iii) *If $Q_1, Q_2 \in \Delta(E(\vartheta,\gamma), 1)$ are neighbors, then $|\log(\operatorname{diam} Q_1/\operatorname{diam} Q_2)| \le 2AN$.*

Proof. Let us prove (i). Since $Q_2 \subseteq Q_1$, we have that (I) of Definition 2.17 follows immediately. On the other hand, since Q_1 is the parent of Q_2, Proposition A.4 implies that

$$|i_1 - i_2| \le \lfloor 2\log_2 C_4/N(\mathfrak{Q}-1)\rfloor + 1 \le 4A_0^2 = A,$$

where the second inequality comes from the choice of A_0 (see Notation 2.8) and this proves (II) of Definition 2.17. In order to prove (ii), we first note that $|(i_1-k)-(i_2-k)| = |i_1-i_2| \le A$ and secondly that

$$\underline{\mathrm{dist}}(\tilde{Q}_1, \tilde{Q}_2) \le \underline{\mathrm{dist}}(Q_1, Q_2) \le A(\operatorname{diam} Q_1 + \operatorname{diam} Q_2) \le A(\operatorname{diam} \tilde{Q}_1 + \operatorname{diam} \tilde{Q}_2).$$

In order to prove (iii), we just need to note that thanks to Theorem A.2 (ii), (v) we infer that

$$\left|\log\frac{\operatorname{diam} Q_1}{\operatorname{diam} Q_2}\right| \le \left|\log\frac{2^{-Ni_1+5}/\gamma}{\zeta^2 2^{-Ni_2-1}/\gamma}\right| = (N|i_2-i_1|+6)\log 2 - 2\log\zeta \le \log(C_6) \le 8A_0^2N = 2AN,$$

where the two last inequalities come from the choice of C_6 and A_0. □

Remark 2.19. If $Q \in \Delta(C,\iota)$ then $\mathfrak{c}(Q) \in E(\vartheta,\gamma)$ thanks to the choices of μ and ι in Notation 2.15 and Proposition A.5.

Remark 2.20. Note that if $Q_1, Q_2 \in \Delta(E(\vartheta,\gamma), 1)$ are neighbors, Proposition 2.18 (iii) implies that

$$e^{-2AN}\operatorname{diam} Q_2 \le \operatorname{diam} Q_1 \le e^{2AN}\operatorname{diam} Q_2.$$

[4]The symbol $\Delta^{\phi}_{i_j}(\vartheta,\gamma)$ denotes the i_j-th layer of dyadic cubes; see Theorem A.2.

Remark 2.20 explicitly tells us that if two cubes $Q_1, Q_2 \in \Delta(C, \iota)$ are neighbors, then they have comparable diameters which are in turn comparable with the distance of their diameters. The information we have on the measure, by means of Lemma 2.16, tells us that ϕ is well approximated by two planes V_1 and V_2 inside the balls $B_1 := B(\mathfrak{c}(Q_1), k \operatorname{diam}(Q_1))$ and $B_2 := B(\mathfrak{c}(Q_2), k \operatorname{diam}(Q_2))$, respectively. However, since we have chosen k in such a way that $k \gg A \approx \operatorname{dist}(\mathfrak{c}(Q_1), \mathfrak{c}(Q_2))/\operatorname{diam}(Q_1)$, the balls B_1 and B_2 have a big overlap while having approximately the same size. Hence, the planes V_1 and V_2 are in essence approximating the same portion of the measure and as a consequence they must be almost the same plane. This heuristic argument is formalized in the following:

Proposition 2.21. *Suppose that $Q_1, Q_2 \in \Delta(C, \iota)$ are two neighbor cubes. Then*

$$(1 - C_3 \varepsilon_2^{2/(\mathfrak{Q}+1)})^{1/2} = (1 - 2^{20}(n_1 - 1) C_2^2 \varepsilon_2^{2/(\mathfrak{Q}+1)})^{1/2} \leq |\langle \mathfrak{n}(Q_1), \mathfrak{n}(Q_2)\rangle|.$$

Proof. Thanks to the definition of k, we have

$$A(\operatorname{diam} Q_1 + \operatorname{diam} Q_2) \leq 2A \max\{\operatorname{diam} Q_1, \operatorname{diam} Q_2\} \leq \tfrac{1}{4} k \max\{\operatorname{diam} Q_1, \operatorname{diam} Q_2\}.$$

Without loss of generality we can assume that $\operatorname{diam} Q_2 \leq \operatorname{diam} Q_1$. Therefore, since the cubes Q_1 and Q_2 are supposed to be neighbors, we deduce that

$$\underline{\operatorname{dist}}(Q_1, Q_2) \leq A(\operatorname{diam} Q_1 + \operatorname{diam} Q_2) \leq \tfrac{1}{4} k \operatorname{diam} Q_1. \tag{54}$$

This implies that for any $z \in Q_1$, we have

$$\begin{aligned} \operatorname{dist}(z, Q_2) &\leq \operatorname{diam} Q_1 + \inf_{y \in Q_1} \operatorname{dist}(y, Q_2) = \operatorname{diam} Q_1 + \underline{\operatorname{dist}}(Q_1, Q_2) \\ &\leq \left(\tfrac{1}{4} k + 1\right) \operatorname{diam} Q_1 < \left(\tfrac{1}{2} k - 1\right) \operatorname{diam} Q_1. \end{aligned} \tag{55}$$

Inequality (55) implies that for any $z \in Q_1$ we have $Q_2 \subseteq B\big(z, \tfrac{1}{2} k \operatorname{diam} Q_1\big)$. This, together with Lemma 2.16 (i), implies that for any $w \in E(\vartheta, \gamma) \cap Q_2$ we have

$$\operatorname{dist}(w, \mathfrak{c}(Q_1)\Pi(Q_1)) \leq 2C_2 \varepsilon_2^{1/\mathfrak{Q}} k \operatorname{diam} Q_1. \tag{56}$$

We now claim that $B_2 := \big\{u \in \mathbb{G} : \operatorname{dist}(u, Q_2) \leq \tfrac{1}{20} k \operatorname{diam} Q_2\big\} \subseteq B\big(\mathfrak{c}(Q_1), \tfrac{1}{2} k \operatorname{diam} Q_1\big)$. In order to prove this inclusion, let $u \in B_2$ and note that

$$\begin{aligned} &\operatorname{dist}(u, \mathfrak{c}(Q_1)) \\ &\quad \leq \inf_{w \in Q_2} (d(u, w) + d(w, \mathfrak{c}(Q_1))) \leq \inf_{w \in Q_2} d(u, w) + \operatorname{diam} Q_1 + \underline{\operatorname{dist}}(Q_1, Q_2) + \operatorname{diam} Q_2 \\ &\quad \underset{u \in B_2}{\leq} \tfrac{1}{20} k \operatorname{diam} Q_2 + \operatorname{diam} Q_1 + \underline{\operatorname{dist}}(Q_1, Q_2) + \operatorname{diam} Q_2 \leq \tfrac{1}{10}(3k + 20) \operatorname{diam} Q_1 < \tfrac{1}{2} k \operatorname{diam} Q_1, \end{aligned} \tag{57}$$

where the second last inequality comes from (54) and the assumption that Q_1 is the cube with the biggest diameter. Inequality (57) concludes the proof of the inclusion $B_2 \subseteq B\big(\mathfrak{c}(Q_1), \tfrac{1}{2} k \operatorname{diam} Q_1\big)$. The inclusion just proved, together with Remark 2.20, the fact that $Q_1, Q_2 \in \Delta(E(\vartheta, \gamma), \iota)$ and Lemma 2.16 (i), implies that for any $u \in E(\vartheta, \gamma) \cap B_2$ we have

$$\operatorname{dist}(u, \mathfrak{c}(Q_1)\Pi(Q_1)) \leq 2C_2 \varepsilon_2^{1/\mathfrak{Q}} k \operatorname{diam} Q_1 \leq 2C_2 e^{2NA} \varepsilon_2^{1/\mathfrak{Q}} k \operatorname{diam} Q_2. \tag{58}$$

Furthermore, thanks to Remark 2.19 we have $\mathfrak{c}(Q_2) \in B_2 \cap E(\vartheta, \gamma)$. Therefore, by Proposition 1.17 for any $u \in B_2 \cap E(\vartheta, \gamma)$ we conclude that

$$\begin{aligned}
&\operatorname{dist}(u, \mathfrak{c}(Q_2)\Pi(Q_1)) \\
&\qquad \leq \operatorname{dist}(u, \mathfrak{c}(Q_1)\Pi(Q_1)) + \operatorname{dist}(\mathfrak{c}(Q_2)\Pi(Q_1), \mathfrak{c}(Q_1)\Pi(Q_1)) \\
&\qquad = \operatorname{dist}(u, \mathfrak{c}(Q_1)\Pi(Q_1)) + \operatorname{dist}(\mathfrak{c}(Q_2), \mathfrak{c}(Q_1)\Pi(Q_1)) \overset{(58)}{\leq} 4C_2 e^{2NA} \varepsilon_2^{1/\mathfrak{Q}} k \operatorname{diam} Q_2. \qquad (59)
\end{aligned}$$

Thanks to Lemma 2.16 (ii), we deduce that for any $y \in B\big(\mathfrak{c}(Q_2), \frac{1}{40}k \operatorname{diam} Q_2\big) \cap \mathfrak{c}(Q_2)\Pi(Q_2)$ there exists some $w(y)$ in $E(\vartheta, \gamma) \cap B(y, 3kC_2\varepsilon_2^{1/(\mathfrak{Q}+1)} \operatorname{diam} Q_2)$. Since by definition $\varepsilon_2 \leq ((k-20)/20C_2k)^{\mathfrak{Q}+1}$, we have

$$\begin{aligned}
\operatorname{dist}(w(y), Q_2) &\leq \inf_{p \in Q_2} d(w(y), y) + d(y, \mathfrak{c}(Q_2)) + d(\mathfrak{c}(Q_2), p) \\
&\leq 3kC_2\varepsilon_2^{1/(\mathfrak{Q}+1)} \operatorname{diam} Q_2 + \tfrac{1}{40}k \operatorname{diam} Q_2 + \operatorname{diam} Q_2 \leq \tfrac{1}{20}k \operatorname{diam} Q_2, \qquad (60)
\end{aligned}$$

where the last inequality comes from the choice of k. Inequality (60) implies that $w(y) \in B_2$, and thanks to (59) we infer that

$$\operatorname{dist}(w(y), \mathfrak{c}(Q_2)\Pi(Q_1)) \leq 4C_2 e^{2NA} \varepsilon_2^{1/\mathfrak{Q}} k \operatorname{diam} Q_2.$$

Summing up, for any $y \in B\big(\mathfrak{c}(Q_2), \frac{1}{40}k \operatorname{diam} Q_2\big) \cap \mathfrak{c}(Q_2)\Pi(Q_2)$, we have

$$\begin{aligned}
\operatorname{dist}(y, \mathfrak{c}(Q_2)\Pi(Q_1)) &\leq d(y, w(y)) + \operatorname{dist}(w(y), \mathfrak{c}(Q_2)\Pi(Q_1)) \\
&\leq 3C_2\varepsilon_2^{1/(\mathfrak{Q}+1)} k \operatorname{diam} Q_2 + 4C_2 e^{2NA}\varepsilon_2^{1/\mathfrak{Q}} k \operatorname{diam} Q_2 \\
&\leq (3C_2 + 4C_2 e^{2NA}\varepsilon_2^{1/(\mathfrak{Q}(\mathfrak{Q}+1))})\varepsilon_2^{1/(\mathfrak{Q}+1)} k \operatorname{diam} Q_2 \leq 6C_2\varepsilon_2^{1/(\mathfrak{Q}+1)} k \operatorname{diam} Q_2, \quad (61)
\end{aligned}$$

where the last inequality comes from the choice of ε_2 and a few elementary algebraic computations that we omit. Furthermore, inequality (61) and Proposition 1.15 imply that

$$|\langle \pi_1(\mathfrak{c}(Q_2)^{-1}y), \mathfrak{n}(Q_1)\rangle| = \|P_{\mathfrak{n}(Q_1)}(\mathfrak{c}(Q_2)^{-1}y)\| = \operatorname{dist}(y, \mathfrak{c}(Q_2)\Pi(Q_1)) \leq 6C_2\varepsilon_2^{1/(\mathfrak{Q}+1)} k \operatorname{diam} Q_2. \quad (62)$$

Suppose $\{v_i\}_{i=1,\dots,n_1-1}$ are the orthonormal vectors of the first layer V_1 spanning the orthogonal complement of $\mathfrak{n}(Q_2)$ inside V_1, and let $y_j := \mathfrak{c}(Q_2)\delta_{k \operatorname{diam} Q_2/80}(v_j)$. Then, from inequality (62), we deduce that

$$\begin{aligned}
1 = |\langle \mathfrak{n}(Q_1), \mathfrak{n}(Q_2)\rangle|^2 + \sum_{j=1}^{n_1-1} |\langle v_j, \mathfrak{n}(Q_1)\rangle|^2 &= |\langle \mathfrak{n}(Q_1), \mathfrak{n}(Q_2)\rangle|^2 + \sum_{j=1}^{n_1-1} \frac{|\langle \pi_1(\mathfrak{c}(Q_2)^{-1}y_j), \mathfrak{n}(Q_1)\rangle|^2}{(k \operatorname{diam} Q_2/80)^2} \\
&\leq |\langle \mathfrak{n}(Q_1), \mathfrak{n}(Q_2)\rangle|^2 + 2^{20}(n_1-1)C_2^2\varepsilon_2^{2/(\mathfrak{Q}+1)}. \qquad \square
\end{aligned}$$

Proposition 2.22. *Let $Q_1, Q_2 \in \Delta(C, \iota)$ be neighbor cubes and suppose that $\Pi(Q_1)$ and $\Pi(Q_2)$, the planes yielded by Lemma 2.16, are chosen with compatible orientations. Then*

$$|\mathfrak{n}(Q_1) - \mathfrak{n}(Q_2)| \leq 2\sqrt{C_3}\varepsilon_2^{1/(\mathfrak{Q}+1)}. \qquad (63)$$

Furthermore, denote by $\tilde{Q}_1$ and $\tilde{Q}_2$ the parent cubes of Q_1 and Q_2, respectively, and assume that the planes $\Pi(\tilde{Q}_1)$ and $\Pi(\tilde{Q}_2)$ have compatible orientations with $\Pi(Q_1)$ and $\Pi(Q_2)$, respectively. Then the $\Pi(Q_i)$ have compatible orientations if and only if the planes $\Pi(\tilde{Q}_i)$ do.

Proof. Since Q_1 and Q_2 are neighbors and have compatible orientations, by definition, $\langle\mathfrak{n}(Q_1),\mathfrak{n}(Q_2)\rangle\geq 0$. Thanks to Proposition 2.21 we infer that

$$|\mathfrak{n}(Q_1)-\mathfrak{n}(Q_2)|^2=2-2\langle\mathfrak{n}(Q_1),\mathfrak{n}(Q_2)\rangle\leq 2-2(1-C_3\varepsilon_2^{2/(\mathfrak{Q}+1)})^{1/2}\leq 2\sqrt{C_3}\varepsilon_2^{1/(\mathfrak{Q}+1)},$$

and (63) is proved. Let us move to the second part of the proposition. Thanks to Proposition 2.18, the pairs $\tilde{Q}_1$ and $\tilde{Q}_2$, Q_1 and $\tilde{Q}_1$, and Q_2 and $\tilde{Q}_2$ are neighbors as well. Therefore Proposition 2.21 implies that

$$\begin{aligned}\langle\mathfrak{n}(\tilde{Q}_1),\mathfrak{n}(\tilde{Q}_2)\rangle&=\langle\mathfrak{n}(Q_1),\mathfrak{n}(Q_2)\rangle+\langle\mathfrak{n}(\tilde{Q}_1)-\mathfrak{n}(Q_1),\mathfrak{n}(Q_2)\rangle+\langle\mathfrak{n}(\tilde{Q}_1),\mathfrak{n}(\tilde{Q}_2)-\mathfrak{n}(Q_2)\rangle\\&\geq(1-C_3\varepsilon_2^{2/(\mathfrak{Q}+1)})^{1/2}-4\sqrt{C_3}\varepsilon_2^{1/(\mathfrak{Q}+1)}\geq\tfrac{1}{10}.\end{aligned}$$

Conversely, if $\Pi(\tilde{Q}_1)$ and $\Pi(\tilde{Q}_2)$ have the same orientation, the same line of reasoning yields that the planes $\Pi(Q_1)$ and $\Pi(Q_2)$ have compatible orientations as well. □

Proposition 2.23. *It is possible to fix an orientation on the planes $\{\Pi(Q):Q\in\Delta(C,\iota)\}$ in such a way that*

$$|\mathfrak{n}(Q_1)-\mathfrak{n}(Q_2)|\leq\tfrac{1}{10},$$

whenever $Q_1,Q_2\in\Delta(C,\iota)$ are neighbors and are contained in the same maximal cube $Q_0\in\mathcal{M}(C,\iota)$, where the set $\mathcal{M}(C,\iota)$ was introduced in Remark 2.10.

Proof. Suppose $Q_i\in\Delta^{\phi}_{j_i}(\vartheta,\gamma)$ for $i=1,2$, and assume without loss of generality that $j_1\leq j_2$. Fix the normal of the plane $\Pi(Q_0)$, and determine the normals of all other planes $\Pi(Q)$ as Q varies in $\Delta(C,\iota)$ by demanding that the orientation of the cube Q is compatible with that of $\tilde{Q}$, its parent cube.

If $Q_1=Q_0$, let us consider the finite sequence $\{\tilde{Q}_i\}_{i=1,\dots,M}$ of ancestors of Q_2 for which $\tilde{Q}_1=Q_2$, $\tilde{Q}_M=Q_0$ and such that $\tilde{Q}_{i+1}$ is the parent of $\tilde{Q}_i$. Then the scalar product between $\mathfrak{n}(Q_0)$ and $\mathfrak{n}(Q_2)$ can be estimated as

$$\langle\mathfrak{n}(Q_0),\mathfrak{n}(Q_2)\rangle\geq\langle\mathfrak{n}(\tilde{Q}_2),\mathfrak{n}(Q_2)\rangle-\sum_{i=2}^{M}|\mathfrak{n}(\tilde{Q}_i)-\mathfrak{n}(\tilde{Q}_{i+1})|\geq(1-C_3\varepsilon_2^{2/(\mathfrak{Q}+1)})-2\sqrt{C_3}M\varepsilon_2^{1/(\mathfrak{Q}+1)},\quad(64)$$

where the last inequality comes from Propositions 2.21 and 2.22 and the fact that the orientation of $\tilde{Q}_i$ and $\tilde{Q}_{i+1}$ were chosen to be compatible. Since Q_0 and Q_2 were assumed to be neighbors, from Definition 2.17 (II) it follows that $M\leq A$ and thus, thanks to (64) and the choice of ε_2, we have

$$\langle\mathfrak{n}(Q_0),\mathfrak{n}(Q_2)\rangle\geq(1-C_3\varepsilon_2^{2/(\mathfrak{Q}+1)})-2\sqrt{C_3}A\varepsilon_2^{1/(\mathfrak{Q}+1)}>0.$$

This proves the statement if one of the cubes is Q_0. The proof of the general case can be obtained with the following argument. Thanks to Proposition 2.22, we know that the orientations of the planes $\Pi(Q_1)$ and $\Pi(Q_2)$ are compatible if and only if the orientations of $\Pi(\tilde{Q}_1)$ and $\Pi(\tilde{Q}_2)$, the planes relative to their parent cubes $\tilde{Q}_1$ and $\tilde{Q}_2$, are compatible.[5] Thus, taking the parents of the parents and so on, one can reduce to the case in which one of the cubes is Q_0. □

[5] Note that this is the case, since by construction we enforced that every element in $\Delta(C,\iota)$ has a compatible orientation with its parent cube.

Definition 2.24. For each cube $Q \in \Delta(C, \iota)$, we let

$$\begin{aligned} G_\pm(Q) &:= \mathfrak{c}(Q)\{u \in B(0, A_0 \operatorname{diam} Q) : \pm\langle \pi_1 u, \mathfrak{n}(Q)\rangle > A_0^{-1} \operatorname{diam} Q\} \\ &= \{u \in B(\mathfrak{c}(Q), A_0 \operatorname{diam} Q) : \pm\langle \pi_1 u - \pi_1(\mathfrak{c}(Q)), \mathfrak{n}(Q)\rangle > A_0^{-1} \operatorname{diam} Q\} \end{aligned}$$

and $G(Q) = G_+(Q) \cup G_-(Q)$. Furthermore, for any $\overline{Q} \in \mathcal{M}(C, \iota)$ we let

$$\mathfrak{G}_\pm(\overline{Q}) := \bigcup_{\substack{Q \in \Delta(C,\iota) \\ Q \subseteq \overline{Q}}} G_\pm(Q) \quad \text{and} \quad \mathfrak{G}(\overline{Q}) := \bigcup_{\substack{Q \in \Delta(C,\iota) \\ Q \subseteq \overline{Q}}} G(Q).$$

For any Q in the set $G(Q)$, there is a ball B with radius comparable with $\operatorname{diam}(Q)$ from which a tubular neighborhood T of the plane $\Pi(Q)$ has been subtracted. The following lemma tells us that our choice of parameters is sufficient to get the inclusion $B \cap E(\vartheta, \gamma) \subseteq T$:

Lemma 2.25. *For any cube Q of $\Delta(C, \iota)$ and any $x \in G(Q)$, we have*

$$\tfrac{1}{2} A_0^{-1} \operatorname{diam} Q \underset{(A)}{\leq} \operatorname{dist}(x, E(\vartheta, \gamma)) \underset{(B)}{\leq} A_0 \operatorname{diam} Q. \tag{65}$$

Proof. Since $A_0 \leq \frac{1}{4}k$, if we let $z \in E(\vartheta, \gamma)$ be the point realizing the minimum distance of x from $E(\vartheta, \gamma)$, we deduce that

$$d(x, z) = \operatorname{dist}(x, E(\vartheta, \gamma)) \leq d(x, \mathfrak{c}(Q)) \leq A_0 \operatorname{diam} Q, \tag{66}$$

where the first inequality above comes from the fact that $\mathfrak{c}(Q) \in E(\vartheta, \gamma)$ (see Remark 2.19) and the last inequality comes from the very definition of $G(Q)$. Note that inequality (66) proves (65) (B). Furthermore, since $1 + A_0 < \frac{1}{2}k$, the bound (66) also implies that $z \in B\big(\mathfrak{c}(Q), \frac{1}{2}k \operatorname{diam} Q\big) \cap E(\vartheta, \gamma)$ and thus, thanks to Lemma 2.16 (i), we deduce that

$$\operatorname{dist}(z, \mathfrak{c}(Q)\Pi(Q)) \leq 2C_2 \varepsilon_2^{1/\mathfrak{Q}} k \operatorname{diam} Q. \tag{67}$$

Let w be an element of $\Pi(Q)$ satisfying the identity $d(z, \mathfrak{c}(Q)w) = \operatorname{dist}(z, \mathfrak{c}(Q)\Pi(Q))$, and note that

$$\begin{aligned} \operatorname{dist}(x, E(\vartheta, \gamma)) = \operatorname{dist}(x, z) &\geq d(x, \mathfrak{c}(Q)w) - d(\mathfrak{c}(Q)w, z) \\ &\geq \operatorname{dist}(\mathfrak{c}(Q)^{-1}x, \Pi(Q)) - \operatorname{dist}(z, \mathfrak{c}(Q)\Pi(Q)) \\ &\underset{(67)}{\geq} |\langle \mathfrak{n}(Q), \pi_1(\mathfrak{c}(Q)^{-1}x)\rangle| - 2C_2 \varepsilon_2^{1/\mathfrak{Q}} k \operatorname{diam} Q \\ &\geq A_0^{-1} \operatorname{diam} Q - 2C_2 \varepsilon_2^{1/\mathfrak{Q}} k \operatorname{diam} Q \geq \tfrac{1}{2} A_0^{-1} \operatorname{diam} Q, \end{aligned} \tag{68}$$

where the second last inequality used the fact that $x \in G(Q)$ and the last inequality used the choice of ε_2 and A_0. □

The following is a disconnection result for $\mathfrak{G}(\overline{Q})$. It tells us that $\mathfrak{G}_+(\overline{Q})$ and $\mathfrak{G}_-(\overline{Q})$ can be regarded as two *sides* of $\mathfrak{G}(\overline{Q})$ in the same way that $G_+(Q)$ and $G_-(Q)$ are the two sides of $G(Q)$. The intuitive idea for which this phenomenon occurs is the following. First, if $Q_1, Q_2 \in \Delta(C, \iota)$ are two cubes contained in Q such that $G_+(Q_1) \cap G_-(Q_2) \neq \varnothing$, then Lemma 2.25 implies that Q_1 and Q_2 must be neighbors.

Since Q_1 and Q_2 are neighbors, the approximating planes $\Pi(Q_1)$ and $\Pi(Q_2)$ are very close thanks to Proposition 2.21. In particular, $G_+(Q_1)$ and $G_-(Q_2)$ are in essence on opposite sides of a plane and thus they cannot intersect, resulting in a contradiction.

Lemma 2.26. *For any $\overline{Q} \in \mathcal{M}(C,\iota)$ we have that the sets $\mathfrak{G}_\pm(\overline{Q})$ are open and $\mathfrak{G}_+(\overline{Q}) \cap \mathfrak{G}_-(\overline{Q}) = \varnothing$.*

Proof. The fact that the $\mathfrak{G}_\pm(\overline{Q})$ are open sets follows immediately from the definitions of the $G_\pm(Q)$. Suppose that $\mathfrak{G}_+(\overline{Q}) \cap \mathfrak{G}_-(\overline{Q}) \neq \varnothing$. Then we can find two cubes $Q_1, Q_2 \in \Delta(C,\iota)$ contained in $\overline{Q}$ such that $G_+(Q_1) \cap G_-(Q_2) \neq \varnothing$ and let x be a point of intersection. In order to fix notations, we also suppose that $Q_i \in \Delta^\phi_{j_i}(\vartheta,\gamma)$ for $i = 1, 2$. Thanks to the definition of $G_\pm(Q)$, we immediately deduce that

$$B(\mathfrak{c}(Q_1), A_0 \operatorname{diam} Q_1) \cap B(\mathfrak{c}(Q_2), A_0 \operatorname{diam} Q_2) \neq \varnothing. \tag{69}$$

This in particular implies that $\underline{\operatorname{dist}}(Q_1, Q_2) \le 2A_0(\operatorname{diam} Q_1 + \operatorname{diam} Q_2)$. Therefore, since $2A_0 \le A$, we have that Q_1 and Q_2 satisfy condition (I) of Definition 2.17. Furthermore, since by construction $x \in G_+(Q_1) \cap G_-(Q_2)$, Lemma 2.25 implies that

$$\frac{\operatorname{diam} Q_1}{2A_0} \le \operatorname{dist}(x, E(\vartheta,\gamma)) \le A_0 \operatorname{diam} Q_1 \quad \text{and} \quad \frac{\operatorname{diam} Q_2}{2A_0} \le \operatorname{dist}(x, E(\vartheta,\gamma)) \le A_0 \operatorname{diam} Q_2. \tag{70}$$

Putting together the bounds in (70), we infer that

$$(2A_0^2)^{-1} \le \frac{\operatorname{diam} Q_1}{\operatorname{diam} Q_2} \le 2A_0^2. \tag{71}$$

Thanks to (71) and Theorem A.2 (ii), (v) we have that

$$(2A_0^2)^{-1} \le \frac{\operatorname{diam} Q_1}{\operatorname{diam} Q_2} \le \frac{2^{-j_1 N+5}/\gamma}{\zeta^2 2^{-j_2 N-1}/\gamma} \quad \text{and} \quad \frac{\zeta^2 2^{-j_1 N-1}/\gamma}{2^{-j_2+5}/\gamma} \le \frac{\operatorname{diam} Q_1}{\operatorname{diam} Q_2} \le 2A_0^2. \tag{72}$$

Finally, thanks to the bounds in (72) together with some computations that we omit, we deduce that

$$|j_2 - j_1| \le \frac{\log(2^7\zeta^{-2}A_0^2)}{N \log 2} \le \log A_0,$$

where the last inequality comes from the choice of A_0. Since $A_0 \ge 2$, we infer that $|j_2 - j_1| \le A$, proving condition (II) of Definition 2.17. This concludes the proof that Q_1 and Q_2 are neighbors.

Now that we know that Q_1 and Q_2 are neighbors, (69) together with Proposition 2.18 (iii) implies that

$$\begin{aligned} d(\mathfrak{c}(Q_1), \mathfrak{c}(Q_2)) &\le d(\mathfrak{c}(Q_1), x) + d(x, \mathfrak{c}(Q_2)) \le A_0(\operatorname{diam} Q_1 + \operatorname{diam} Q_2) \\ &\le A_0(1 + e^{2NA}) \operatorname{diam} Q_2 < \tfrac{1}{2} k \operatorname{diam} Q_2, \end{aligned} \tag{73}$$

where the last inequality comes from the choice of k and of A. Since by (73) and Remark 2.19, we have $\mathfrak{c}(Q_2) \in E(\vartheta,\gamma) \cap B\big(\mathfrak{c}(Q_1), \tfrac{1}{2} k \operatorname{diam} Q_2\big)$, thanks to Lemma 2.16 (i) and Remark 2.20, we deduce that

$$\operatorname{dist}(\mathfrak{c}(Q_2), \mathfrak{c}(Q_1)\Pi(Q_1)) \le 2C_2 k \varepsilon_2^{1/\mathfrak{Q}} \operatorname{diam} Q_1 \le 2C_2 k e^{2NA} \varepsilon_2^{1/\mathfrak{Q}} \operatorname{diam} Q_2. \tag{74}$$

Furthermore, since Q_1 and Q_2 are neighbors, we infer by Proposition 2.22 that

$$|\mathfrak{n}(Q_1) - \mathfrak{n}(Q_2)| \le 2C_3 \varepsilon_2^{1/(\mathfrak{Q}+1)},$$

and this in turn implies that

$$\begin{aligned}
&\langle \pi_1(\mathfrak{c}(Q_1)^{-1}x), \mathfrak{n}(Q_1)\rangle \\
&\qquad = \langle \pi_1(\mathfrak{c}(Q_2)^{-1}x), \mathfrak{n}(Q_2)\rangle + \langle \pi_1(\mathfrak{c}(Q_2)^{-1}x), \mathfrak{n}(Q_1)-\mathfrak{n}(Q_2)\rangle + \langle \pi_1(\mathfrak{c}(Q_1)^{-1}\mathfrak{c}(Q_2)), \mathfrak{n}(Q_1)\rangle \\
&\qquad \le -A_0^{-1}\operatorname{diam} Q_2 + |\pi_1(\mathfrak{c}(Q_2)^{-1}x)|\,|\mathfrak{n}(Q_1)-\mathfrak{n}(Q_2)| + \operatorname{dist}(\mathfrak{c}(Q_2), \mathfrak{c}(Q_1)\Pi(Q_1)) \\
&\qquad \le -A_0^{-1}\operatorname{diam} Q_2 + A_0 \operatorname{diam} Q_2 \cdot 2C_3\varepsilon_2^{1/(\mathfrak{Q}+1)} + 2C_2 k e^{2NA}\varepsilon_2^{1/\mathfrak{Q}} \operatorname{diam} Q_2, \qquad (75)
\end{aligned}$$

where third line above comes from the fact that $x \in G_-(Q_2)$ and the bound on $|\mathfrak{n}(Q_1)-\mathfrak{n}(Q_2)|$ discussed above while the last inequality follows from (74). The chain of inequalities in (75) and the definition of A imply that

$$\langle \pi_1(\mathfrak{c}(Q_1)^{-1}x), \mathfrak{n}(Q_1)\rangle \le (-A_0^{-1} + A_0C_3\varepsilon_2^{1/(\mathfrak{Q}+1)} + C_2 k e^{8NA_0^2}\varepsilon_2^{1/\mathfrak{Q}}) \operatorname{diam} Q_2 \le 0, \qquad (76)$$

where the last inequality comes from the definition of ε_2 and some algebraic computations that we omit. This contradicts the fact that $x \in G_+(Q_1)$, proving that the assumption that $\mathfrak{G}(\overline{Q})_+ \cap \mathfrak{G}_-(\overline{Q}) \neq \varnothing$ was absurd. □

Let us take a step back and explain what the set $\mathfrak{G}(\overline{Q})$ is. Starting from a measure ϕ with flat blowups, in this section we constructed a countable family of pairs $(\mathfrak{c}(Q), \Pi(Q))$, parametrized by the cubes in $\Delta(C,\iota)$ inside $\overline{Q}$, of points of $\operatorname{supp}(\phi)$ and planes that are a good approximation of ϕ around $\mathfrak{c}(Q)$ at the scale $\operatorname{diam} Q$. From this family of pointed planes we built $\mathfrak{G}(\overline{Q})$, which should be imagined as the complement of the union of very thin tubular neighborhoods of the disks $\mathfrak{c}(Q)\Pi(Q) \cap B(\mathfrak{c}(Q), \operatorname{diam} Q)$. So, since the planes $\Pi(Q)$ are very efficiently approximating ϕ one should expect that $\phi(\mathfrak{G}(\overline{Q})) \approx 0$, allowing us to regard $\mathfrak{G}(\overline{Q})^c$ as an extension of $\operatorname{supp}(\phi)$ inside the ball $B(\mathfrak{c}(\overline{Q}), \operatorname{diam} \overline{Q})$. An extension, however, that can ultimately be considered and treated as a countable union of planes. The next proposition shows that $\operatorname{supp}(\phi)$ is quite dense inside $\mathfrak{G}(\overline{Q})^c$.

Proposition 2.27. *Let $\overline{Q} \in \mathcal{M}(C,\iota)$ and define*

$$I(\overline{Q}) := \bigcup_{\substack{Q\in\Delta(C,\iota)\\ Q\subseteq\overline{Q}}} B(\mathfrak{c}(Q), (A_0-2)\operatorname{diam} Q).$$

In addition, for any $x \in I(\overline{Q})$ we let

$$d(x) := \inf_{\substack{Q\in\Delta(C,\iota)\\ Q\subseteq\overline{Q}}} \operatorname{dist}(x, Q) + \operatorname{diam} Q. \qquad (77)$$

Then $\operatorname{dist}(x, E(\vartheta,\gamma)) \le 4A_0^{-1}d(x)$ *whenever $x \in I(\overline{Q}) \setminus \mathfrak{G}(\overline{Q})$.*

Proof. Fix some $x \in I(\overline{Q}) \setminus \mathfrak{G}(\overline{Q})$, and let $Q \subseteq \overline{Q}$ be a cube of $\Delta(C,\iota)$ such that

$$\operatorname{dist}(x, Q) + \operatorname{diam} Q \le \tfrac{4}{3} d(x). \qquad (78)$$

Let Q' be an ancestor of Q in $\Delta(C,\iota)$, possibly Q itself. Since $x \notin \mathfrak{G}(\overline{Q})$, then $x \notin G(Q')$ and, thanks to Proposition 1.15, we have

$$\operatorname{dist}(x, \mathfrak{c}(Q')\Pi(Q')) = |\langle \pi_1(\mathfrak{c}(Q')^{-1}x), \mathfrak{n}(Q')\rangle| \le A_0^{-1}\operatorname{diam} Q', \qquad (79)$$

where the last inequality is true provided that $\operatorname{dist}(x, \mathfrak{c}(Q')) < A_0 \operatorname{diam} Q'$. Since $x \in I(\bar{Q})$, there must exist some $\tilde{Q} \in \Delta(C, \iota)$ such that $\tilde{Q} \subseteq \bar{Q}$ and $x \in B(\mathfrak{c}(\tilde{Q}), (A_0 - 2) \operatorname{diam} \tilde{Q})$. This implies that

$$\operatorname{dist}(x, \mathfrak{c}(\bar{Q})) \leq d(x, \mathfrak{c}(\tilde{Q})) + d(\mathfrak{c}(\tilde{Q}), \mathfrak{c}(\bar{Q})) \leq (A_0 - 2) \operatorname{diam} \tilde{Q} + \operatorname{diam} \bar{Q} < A_0 \operatorname{diam} \bar{Q}. \tag{80}$$

Therefore the inequality $\operatorname{dist}(x, \mathfrak{c}(\bar{Q})) < A_0 \operatorname{diam} \bar{Q}$ is verified and hence (79) holds for $Q' = \bar{Q}$. Let $Q \subseteq Q_0 \subseteq \bar{Q}$ be the smallest cube in $\Delta(C, \iota)$ for which $\operatorname{dist}(x, \mathfrak{c}(Q_0)) < A_0 \operatorname{diam} Q_0$ holds.

Let $w \in \Pi(Q_0)$ be the point for which $d(x, \mathfrak{c}(Q_0)w) = \operatorname{dist}(x, \mathfrak{c}(Q_0)\Pi(Q_0))$, and note that the choice of Q_0 and the bound (79) imply that

$$\begin{aligned}\|w\| = \operatorname{dist}(\mathfrak{c}(Q_0)w, \mathfrak{c}(Q_0)) &\leq d(\mathfrak{c}(Q_0)w, x) + d(x, \mathfrak{c}(Q_0)) \\ &\leq \operatorname{dist}(x, \mathfrak{c}(Q_0)\Pi(Q_0)) + A_0 \operatorname{diam} Q_0 \\ &\leq A_0^{-1} \operatorname{diam} Q_0 + A_0 \operatorname{diam} Q_0 \leq 2A_0 \operatorname{diam} Q_0 < \tfrac{1}{2}k \operatorname{diam} Q_0,\end{aligned} \tag{81}$$

where the last inequality comes from the choice of A_0 and k made in Notation 2.8. Since $Q_0 \in \Delta(C, \iota)$, thanks to inequality (81) we have $\mathfrak{c}(Q_0)w \in B\big(\mathfrak{c}(Q_0), \frac{1}{2}k \operatorname{diam} Q_0\big)$ and thus Lemma 2.16 (ii) implies that $E(\vartheta, \gamma) \cap B(\mathfrak{c}(Q_0)w, 3kC_2\varepsilon_2^{1/(\mathfrak{Q}+1)} \operatorname{diam} Q_0) \neq \varnothing$. Therefore, since by definition of Q_0 the bound (79) holds with $Q' = Q_0$, we have

$$\begin{aligned}\operatorname{dist}(x, E(\vartheta, \gamma)) &\leq d(x, \mathfrak{c}(Q_0)w) + \operatorname{dist}(\mathfrak{c}(Q_0)w, E(\vartheta, \gamma)) \\ &= d(x, \mathfrak{c}(Q_0)\Pi(Q_0)) + \operatorname{dist}(\mathfrak{c}(Q_0)w, E(\vartheta, \gamma)) \\ &\leq A_0^{-1} \operatorname{diam} Q_0 + 3kC_2\varepsilon_2^{1/(\mathfrak{Q}+1)} \operatorname{diam} Q_0 \leq 2A_0^{-1} \operatorname{diam} Q_0,\end{aligned} \tag{82}$$

where the last inequality comes from the choice of ε_2.

If $Q_0 = Q$, then (78) implies that $\operatorname{dist}(x, E(\vartheta, \gamma)) \leq 2A_0^{-1} \operatorname{diam} Q_0 \leq 4A_0^{-1} d(x)$. Otherwise, let Q_1 be the child of Q_0 that contains Q. Thanks to the minimality of Q_0, we have $\operatorname{dist}(x, \mathfrak{c}(Q_1)) \geq A_0 \operatorname{diam} Q_1$, and thus

$$\begin{aligned}\operatorname{dist}(x, Q_1) \geq d(x, \mathfrak{c}(Q_1)) - \operatorname{diam} Q_1 &\geq (A_0 - 1) \operatorname{diam} Q_1 \\ &\geq \frac{A_0 - 1}{C_6} \operatorname{diam} Q_0 \geq \operatorname{diam} Q_0,\end{aligned} \tag{83}$$

where the second last inequality above follows from Proposition A.4 and the fact that Q_0 is the parent of Q_1, whereas the last inequality comes from the choice of A_0. Eventually, thanks to (78), (82), (83) and the fact that $Q \subseteq Q_1$, we deduce that

$$\begin{aligned}\operatorname{dist}(x, E(\vartheta, \gamma)) &\overset{(82)}{\leq} 2A_0^{-1} \operatorname{diam} Q_0 \overset{(83)}{\leq} 2A_0^{-1} \operatorname{dist}(x, Q_1) \\ &\leq 2A_0^{-1} \operatorname{dist}(x, Q) \overset{(78)}{\leq} 4A_0^{-1} d(x).\end{aligned}$$

□

The following is the main result of this subsection. Theorem 2.28 transforms the qualitative information on the relationship between $\mathfrak{G}(Q)^c$ and $\operatorname{supp}(\phi)$ yielded by Proposition 2.27 into a quantitative one, i.e., the bound on projections given in (84). The proof of the theorem reduces to constructing, for any (suitable) cube Q, a family of balls $\{B_i\}_{i \in \mathbb{N}}$ with the two following properties: First, the projection on $\Pi(Q)$ of $\operatorname{supp}(\phi) \cup \bigcup_i B_i$ contains an open set with measure comparable with $\operatorname{diam} Q^{\mathfrak{Q}-1}$. Second, the sum of the radii of the balls B_i is small and in particular the projection on planes of the set $\bigcup_i B_i$ has small measure compared to diam $Q^{\mathfrak{Q}-1}$. The construction of the balls B_i, that the reader may imagine centered at points of $\mathfrak{G}(Q)^c$, relies on the one hand on the previously discussed fact that the set $\mathfrak{G}(\overline{Q})^c$ can be regarded as a countable union of disks and on the other, that the holes of supp(ϕ), seen as a subset of $\mathfrak{G}(\overline{Q})^c$, are really small and patching them does not require too much measure.

Theorem 2.28. *For any cube $Q \in \Delta(C, \iota)$ such that $(1-\varepsilon_3)\phi(Q) \leq \phi(Q \cap C)$, we have*

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q)}(Q \cap C)) \geq \frac{\operatorname{diam} Q^{\mathfrak{Q}-1}}{2A_0^{\mathfrak{Q}-1}}. \tag{84}$$

Proof. Let $Q_0 \in \Delta(C, \iota)$ be such that $(1-\varepsilon_3)\phi(Q_0) \leq \phi(Q_0 \cap C)$, and define

$$F(Q_0) := C \cap Q_0 \cup \bigcup_{Q \in \mathscr{I}(Q_0)} B(\mathfrak{c}(Q), 2C_6 \operatorname{diam} Q),$$

where $\mathscr{I}(Q_0)$ is a family of maximal cubes $Q \in \Delta(E(\vartheta, \gamma), \iota)$ such that $Q \subseteq Q_0$ and $Q \notin \Delta(C, \iota)$. As a first step, we estimate the size of the projection of the balls $\bigcup_{Q \in \mathscr{I}(Q_0)} B(\mathfrak{c}(Q), C_6 \operatorname{diam} Q)$. Thanks to Proposition 1.18 we have

$$\mathcal{S}^{\mathfrak{Q}-1}\biggl(P_{\Pi(Q)}\biggl(\bigcup_{Q \in \mathscr{I}(Q_0)} B(\mathfrak{c}(Q), 2C_6 \operatorname{diam} Q)\biggr)\biggr) \leq 2^{\mathfrak{Q}-1} c(\Pi(Q_0)) C_6^{\mathfrak{Q}-1} \sum_{Q \in \mathscr{I}(Q_0)} \operatorname{diam} Q^{\mathfrak{Q}-1}. \tag{85}$$

We now need to estimate the sum in the right-hand side of (85). Since the cubes in $\mathscr{I}(Q_0)$ are disjoint and they are contained in $\Delta(E(\vartheta, \gamma), \iota)$, thanks to Remark A.3 and the fact that $(1-\varepsilon_3)\phi(Q_0) \leq \phi(Q_0 \cap C)$, we deduce that

$$\begin{aligned} C_5^{-1} \sum_{Q \in \mathscr{I}(Q_0)} \operatorname{diam} Q^{\mathfrak{Q}-1} \leq \sum_{Q \in \mathscr{I}(Q_0)} \phi(Q) = \phi\biggl(\bigcup_{Q \in \mathscr{I}(Q_0)} Q\biggr) \\ \leq \phi(Q_0 \setminus C) \leq \varepsilon_3 \phi(Q_0) \leq \varepsilon_3 C_5 \operatorname{diam} Q_0^{\mathfrak{Q}-1}. \end{aligned} \tag{86}$$

Putting together (85) and (86), we conclude that

$$\begin{aligned} \mathcal{S}^{\mathfrak{Q}-1}\biggl(P_{\Pi(Q)}\biggl(\bigcup_{Q \in \mathscr{I}(Q_0)} B(\mathfrak{c}(Q), 2C_6 \operatorname{diam} Q)\biggr)\biggr) &\leq 2^{\mathfrak{Q}-1} c(\Pi(Q_0)) C_5^2 \varepsilon_3 C_6^{\mathfrak{Q}-1} \operatorname{diam} Q_0^{\mathfrak{Q}-1} \\ &\leq \frac{c(\Pi(Q_0))}{2A_0^{\mathfrak{Q}-1}} \operatorname{diam} Q_0^{\mathfrak{Q}-1}, \end{aligned} \tag{87}$$

where the last inequality comes from the choice of ε_3; see Notation 2.8.

In this first part of the proof of the theorem we have constructed the family of balls B_i mentioned in the introductory paragraph to the statement of the theorem and we have also proved the second necessary property of the B_i, that is the smallness of the measure of their projection. The rest of the proof will be devoted to proving that supp$(\phi) \cup \bigcup_i B_i$ has big projections.

More precisely, the next step in the proof of the theorem is to show that

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(F(Q_0))) \geq \frac{c(\Pi(Q_0)) \operatorname{diam} Q_0^{\mathfrak{Q}-1}}{A_0^{\mathfrak{Q}-1}}. \tag{88}$$

In order to ease notations in the following we let $x = \mathfrak{c}(Q_0)\delta_{10A_0^{-1} \operatorname{diam} Q_0}(\mathfrak{n}(Q_0))$ and define

$$B_+ := B(x, A_0^{-1} \operatorname{diam} Q_0) \quad \text{and} \quad B_- := B(x, A_0^{-1} \operatorname{diam} Q_0)\delta_{20A_0^{-1} \operatorname{diam} Q_0}(\mathfrak{n}(Q_0)^{-1}).$$

Before proceeding further with the proof of (88), we give a brief outline of what we are going to do, hoping to help the reader keep track of the purpose of each computation. As a first step towards the proof of (88), we prove that B_+ and B_- are contained in $G_+(Q_0)$ and $G_-(Q_0)$, respectively. Note that this implies that B_+ and B_- are each on one side of the plane $\Pi(Q_0)$. Let $\overline{Q}$ be the element of $\mathcal{M}(C, \iota)$ containing Q_0 and recall that by Lemma 2.26, $\mathfrak{G}_+(\overline{Q})$ and $\mathfrak{G}_-(\overline{Q})$ are disjoint open sets. This implies in particular that for any horizontal line parallel to the normal of the plane $\Pi(Q_0)$ with starting point in B_+ and end point in B_-, we can find a y in such a segment belonging to the complement of $\mathfrak{G}(Q_0)$. Our final step in the proof of (88) is to show that y belongs to $F(Q_0)$, thus proving the inclusion $P_{\Pi(Q_0)}(B_+) \subseteq P_{\Pi(Q_0)}(F(Q_0))$ and in turn our claim.

Let us proceed with the proof of (88). We will prove that $B_+ \subseteq G_+(Q_0)$ and $B_- \subseteq G_-(Q_0)$ separately, since the computations differ.

Let us begin with the proof of the inclusion $B_+ \subseteq G_+(Q_0)$. For any $\Delta \in \mathbb{G}$ such that $\|\Delta\| \leq A_0^{-1} \operatorname{diam} Q_0$, we have

$$d(\mathfrak{c}(Q_0), x\Delta) = \|\delta_{10A_0^{-1} \operatorname{diam} Q_0}(\mathfrak{n}(Q_0))\Delta\| \leq 11A_0^{-1} \operatorname{diam} Q_0 \leq A_0 \operatorname{diam} Q_0. \tag{89}$$

In addition, the choices of x and Δ imply that

$$\begin{aligned}\langle \pi_1(\mathfrak{c}(Q_0)^{-1}x\Delta), \mathfrak{n}(Q_0)\rangle &= \langle \pi_1(\delta_{10A_0^{-1} \operatorname{diam} Q_0}(\mathfrak{n}(Q_0))\Delta), \mathfrak{n}(Q_0)\rangle \\ &= 10A_0^{-1} \operatorname{diam} Q_0 + \langle \pi_1\Delta, \mathfrak{n}(Q_0)\rangle \geq 9A_0^{-1} \operatorname{diam} Q_0. \end{aligned}\tag{90}$$

The bounds (89) and (90) together with the definitions of B_+ and $G_+(Q_0)$ finally imply that $B_+ \subseteq G_+(Q_0)$.

Let us prove that $B_- \subseteq G_-(Q_0)$. Similar to the previous case, for any $\|\Delta\| \leq A_0^{-1} \operatorname{diam} Q_0$, we have

$$\begin{aligned} d(\mathfrak{c}(Q_0), x\Delta\delta_{20A_0^{-1} \operatorname{diam} Q_0}(\mathfrak{n}(Q_0)^{-1})) &= \|\delta_{10A_0^{-1} \operatorname{diam} Q_0}(\mathfrak{n}(Q_0))\Delta\delta_{20A_0^{-1} \operatorname{diam} Q_0}(\mathfrak{n}(Q_0)^{-1})\| \\ &\leq 31A_0^{-1} \operatorname{diam} Q_0 \leq A_0 \operatorname{diam} Q_0. \end{aligned}\tag{91}$$

Once again, the choices of x and Δ imply that

$$\begin{aligned} &\langle \pi_1(\mathfrak{c}(Q_0)^{-1}x\Delta\delta_{20A_0^{-1} \operatorname{diam} Q_0}(\mathfrak{n}(Q_0)^{-1})), \mathfrak{n}(Q_0)\rangle \\ &\qquad = \langle \pi_1(\delta_{10A_0^{-1} \operatorname{diam} Q_0}(\mathfrak{n}(Q_0))\Delta\delta_{20A_0^{-1} \operatorname{diam} Q_0}(\mathfrak{n}(Q_0)^{-1})), \mathfrak{n}(Q_0)\rangle \\ &\qquad = -10A_0^{-1} \operatorname{diam} Q_0 + \langle \pi_1\Delta, \mathfrak{n}(Q_0)\rangle \leq -9A_0^{-1} \operatorname{diam} Q_0. \end{aligned}\tag{92}$$

The bounds (91) and (92) together with the definitions of B_- and $G_-(Q_0)$ show that $B_- \subseteq G_-(Q_0)$.

Now that we have shown that B_+ and B_- lie on different sides of $\Pi(Q_0)$, we construct horizontal curves parallel to $\mathfrak{n}(Q_0)$ joining B_+ and B_- and we show that each one of these lines intersect $F(Q_0)$.

First of all, let $\overline{Q}$ be the unique cube in $\mathcal{M}(C,\iota)$ containing Q_0. Thanks to Lemma 2.26 we know that the sets $\mathfrak{G}_+(\overline{Q})$ and $\mathfrak{G}_-(\overline{Q})$ are disconnected. With this in mind, for any $a\in B_+$ we define the curve $\gamma_a:[0,1]\to\mathbb{G}$ as

$$\gamma_a(t):=a\delta_{20A_0^{-1}\operatorname{diam} Q_0 t}(\mathfrak{n}(Q_0)^{-1}).$$

By the definition of B_-, it is immediate to see that $\gamma_a(1)\in B_-$. On the other hand, since $\gamma_a(0)\in B_+$ and the image of γ_a is connected, we infer that γ_a must meet the complement of $\mathfrak{G}(\overline{Q})$ at $y=\gamma_a(s)$ for some $s\in(0,1)$.

We now prove that $y\in F(Q_0)$. First, we estimate the distance of y from $\mathfrak{c}(Q_0)$ as

$$\begin{aligned}
d(y,\mathfrak{c}(Q_0))&\leq d(a\delta_{20A_0^{-1}\operatorname{diam} Q_0 s}(\mathfrak{n}(Q_0)^{-1}),\mathfrak{c}(Q_0))\leq d(a,\mathfrak{c}(Q_0))+20A_0^{-1}\operatorname{diam} Q_0 s\\
&\leq d(x,\mathfrak{c}(Q_0))+d(x,a)+20A_0^{-1}\operatorname{diam} Q_0 s\\
&\leq 10A_0^{-1}\operatorname{diam} Q_0+A_0^{-1}\operatorname{diam} Q_0+20A_0^{-1}\operatorname{diam} Q_0 s\\
&\leq 40A_0^{-1}\operatorname{diam} Q_0<(A_0-2)\operatorname{diam} Q_0,
\end{aligned}\tag{93}$$

where the inequality in the third line comes from the definition of x and the fact that $a\in B(x,A_0^{-1}\operatorname{diam} Q_0)$. The above computation together with the fact that $\overline{Q}$ is an ancestor of Q_0 shows that $y\in I(\overline{Q})$. In addition, we have that

$$\begin{aligned}
\operatorname{dist}(y,E(\vartheta,\gamma)\setminus Q_0)&\geq \operatorname{dist}(\mathfrak{c}(Q_0),E(\vartheta,\gamma)\setminus Q_0)-d(y,\mathfrak{c}(Q_0))\\
&\geq 64^{-1}\zeta^2\operatorname{diam} Q_0-40A_0^{-1}\operatorname{diam} Q_0\geq 100A_0^{-1}\operatorname{diam} Q_0,
\end{aligned}\tag{94}$$

where the first inequality in the last line above comes from the second last inequality of (93), Remark 2.19 and Theorem A.2 (v), while the last inequality follows from the choice of A_0. From (93) and (94) we deduce that

$$\operatorname{dist}(y,E(\vartheta,\gamma)\setminus Q_0)\overset{(94)}{\geq}100A_0^{-1}\operatorname{diam} Q_0\overset{(93)}{>}d(y,\mathfrak{c}(Q_0))\geq\operatorname{dist}(y,Q_0\cap E(\vartheta,\gamma)),\tag{95}$$

where the last inequality comes from the fact that $\mathfrak{c}(Q_0)$ belongs to $E(\vartheta,\gamma)$; see Remark 2.19. Therefore, if $z\in E(\vartheta,\gamma)$ is the point of minimal distance of y from $E(\vartheta,\gamma)$, (95) implies that $z\in Q_0\cap E(\vartheta,\gamma)$. Furthermore, since by assumption $y\notin\mathfrak{G}(\overline{Q})$ and by (93) we have $y\in I(\overline{Q})$, Proposition 2.27 implies that

$$d(z,y)=\operatorname{dist}(y,E(\vartheta,\gamma))\leq 4A_0^{-1}d(y)<\tfrac{1}{10}d(y),\tag{96}$$

where the last inequality can be strict only if $d(y)>0$. The definition of the function d, see (77), implies further by (96) that

$$d(z)\geq d(y)-d(z,y)>\tfrac{9}{10}d(y),\tag{97}$$

where last inequality is strict only if $d(y)>0$. Summing up what we know so far about z is that it must be contained in $Q_0\cap E(\vartheta,\gamma)$, however (97) implies that z cannot be contained in a cube $Q\in\Delta(C,\iota)$ with $\operatorname{diam} Q\leq\frac{9}{10}d(y)$.

On the one hand, if $d(y)=0$, the bound (96) implies that $d(y,z)=0$ and thus since $E(\vartheta,\gamma)$ is compact we have $y=z\in E(\vartheta,\gamma)$. This implies in particular that

$$y\in E(\vartheta,\gamma)\cap Q_0\subseteq C\cap Q_0\cup\bigcup_{Q\in\mathscr{I}(Q_0)}Q\subseteq C\cap Q_0\cup\bigcup_{Q\in\mathscr{I}(Q_0)}B(\mathfrak{c}(Q),2C_6\operatorname{diam}Q)=F(Q_0).$$

If, on the other hand, $d(y)>0$, we will now show that $y\in F(Q_0)$. We claim that there is a cube $Q_1\in\Delta(C,\iota)$, contained in Q_0 and possibly coinciding with Q_0 itself, such that

(a) $z\in Q_1$ and for any cube $Q\in\Delta(C,\iota)$ contained in Q_1 we have $z\notin Q$,

(b) $\operatorname{diam}Q_1\geq\frac{9}{10}d(y)$,

(c) there exists a $\tilde{Q}\in\mathscr{I}(Q_0)$, that is a child of Q_1 for which $z\in\tilde{Q}$.

Let us verify that such a cube Q_1 exists. Since $z\in Q_0$, for any cube $Q\in\Delta(C,\iota)$ such that $Q\subseteq Q_0$ and $z\in Q$ we have

$$\tfrac{9}{10}d(y)\leq d(z)\leq\operatorname{dist}(z,Q)+\operatorname{diam}Q=\operatorname{diam}Q,\tag{98}$$

where the first inequality above comes from (97) and the second from the definition of d. Let Q_1 be the smallest cube of $\Delta(C,\iota)$ containing z, and note that for any cube $Q\subseteq Q_1$ belonging to $\Delta(C,\iota)$ we have that $z\notin Q$. This proves (a) and (b). In order to prove (c), we note that any ancestor of Q_1 in $\Delta(E(\vartheta,\gamma),\iota)$ must be contained in $\Delta(C,\iota)$. Furthermore, since the condition $\operatorname{diam}Q_1\geq\frac{9}{10}d(y)$ implies that $z\in E(\vartheta,\gamma)\setminus C$, we infer that there must exist a cube $\tilde{Q}$ in $\mathscr{I}(Q_0)$ for which $z\in\tilde{Q}$. Such a cube must be a child of Q_1 otherwise the maximality of $\tilde{Q}$ would be contradicted.

Let us use (a), (b) and (c) to conclude the proof of the theorem. Items (a), (b) and inequality (96) imply that

$$\operatorname{dist}(y,Q_1)\overset{(a)}{\leq}d(y,z)\overset{(96)}{\leq}\tfrac{1}{10}d(y)\overset{(b)}{\leq}\tfrac{1}{9}\operatorname{diam}Q_1.\tag{99}$$

Therefore, Proposition A.4 together with (c) and (99) implies that

$$\begin{aligned}d(\mathfrak{c}(\tilde{Q}),y)\leq d(\mathfrak{c}(\tilde{Q}),z)+d(z,y)&\leq\operatorname{diam}\tilde{Q}+\tfrac{1}{9}\operatorname{diam}Q_1\\&\leq\operatorname{diam}\tilde{Q}+\tfrac{1}{9}C_6\operatorname{diam}\tilde{Q}<2C_6\operatorname{diam}\tilde{Q}.\end{aligned}\tag{100}$$

The bound (100) finally proves that $y\in F(Q_0)$ thanks to the fact that $\tilde{Q}\in\mathscr{I}(Q_0)$ by (c). Summing up, this shows that for any $a\in B_+$, the curve γ_a meets the set $F(Q_0)$ somewhere.

In turn, this shows that $F(Q_0)$ has big projections. Indeed,

$$\begin{aligned}\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(F(Q_0)))&\geq\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(B(x,A_0^{-1}\operatorname{diam}Q_0)))\\&=c(\Pi(Q_0))A_0^{-(\mathfrak{Q}-1)}\operatorname{diam}Q_0^{\mathfrak{Q}-1},\end{aligned}\tag{101}$$

where the first inequality comes from the fact that the images of the curves γ_a are contained in $P^{-1}_{\Pi(Q_0)}(a)$ for any $a\in B_+$ and the last identity comes from Proposition 1.18. This concludes the proof of the main step of the proof, which was to verify the validity of (88).

In order to conclude the proof of the theorem we just need to put (87) together with (101) to get

$$\begin{aligned}\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(Q_0\cap C)) &\geq \mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(F(Q_0)))-\mathcal{S}^{\mathfrak{Q}-1}\Bigg(P_{\Pi(Q_0)}\Bigg(\bigcup_{Q\in\mathscr{I}(Q_0)}B(\mathfrak{c}(Q),C_6\operatorname{diam}Q_0)\Bigg)\Bigg)\\ &\overset{(87),(101)}{\geq} \frac{c(\Pi(Q_0))}{2A_0^{\mathfrak{Q}-1}}\operatorname{diam}Q_0^{\mathfrak{Q}-1}\geq\frac{\operatorname{diam}Q_0^{\mathfrak{Q}-1}}{2A_0^{\mathfrak{Q}-1}},\end{aligned}$$

where the last inequality comes from the fact that $c(\Pi(Q_0))\geq 1$; see Proposition 1.18. □

2D. *Construction of the ϕ-positive intrinsic Lipschitz graph.* This subsection is devoted to the proof of the main result of Section 2, Theorem 2.1, which we restate here for the reader's convenience:

Theorem 2.1. *There is an intrinsic Lipschitz graph Γ such that $\phi(\Gamma)>0$.*

We outline the proof of Theorem 2.1 here: For a fixed cube $Q\in\mathcal{M}(C,\iota)$, we prove that the family $\mathcal{B}(Q)$ of the maximal subcubes of Q having small projection on $\Pi(Q)$, thanks to Theorem 2.28, is small in measure. Therefore, we can find a cube $Q'\in\Delta(C,\iota)\setminus\mathcal{B}(Q)$ that is contained in Q and for which any subcube $\tilde{Q}$ of Q' has big projections on $\Pi(Q)$. This independence on the scales, thanks to Proposition 2.11, implies that $C\cap Q$ is a $\Pi(Q)$-intrinsic Lipschitz graph.

Proposition 2.29. *Define $\varepsilon_4:=\min\{\varepsilon_1,(32\vartheta C_1C_5A_0^{\mathfrak{Q}-1})^{-1}\}$. There exists a compact set $C_1\subseteq C$ and an $\iota_1\in\mathbb{N}$ such that*

(i) $\phi(C\setminus C_1)\leq\varepsilon_4\phi(C)$,

(ii) *whenever $Q\in\Delta(C_1,\iota_1)$ we have $\big(1-\frac{1}{32}\varepsilon_3\big)\phi(Q)\leq\phi(Q\cap C)$.*

Proof. First of all, we prove that the set $\Delta(C,\iota)$ is a $\phi\llcorner C$ Vitali relation. It is immediate to see that the family $\Delta(C,\iota)$ is a fine covering of C. Furthermore, let E be a Borel set contained in C and suppose $\mathcal{A}\subseteq\Delta(C,\iota)$ is a fine covering of E. Defining $\mathcal{A}^*:=\{Q\in\mathcal{A}: Q \text{ is maximal}\}$, the identity $\bigcup_{Q\in\mathcal{A}}Q=\bigcup_{Q\in\mathcal{A}^*}Q$ is trivially satisfied and thus the family $\mathcal{A}^*$ is still a covering of E. The maximality of the elements of $\mathcal{A}^*$ implies that they are pairwise disjoint and thus $\Delta(C,\iota)$ is a ϕ-Vitali relation in the sense of [Federer 1969, §2.8.16]. Therefore, thanks to [Federer 1969, Theorem 2.9.11], we deduce that

$$\lim_{Q\to x}\frac{\phi(C\cap Q)}{\phi(Q)}=1, \tag{102}$$

for ϕ-almost every $x\in C$. For any $j\in\mathbb{N}$, define the functions $f_j(x):=\phi(C\cap Q_j(x))/\phi(Q_j(x))$, where $Q_j(x)$ is the unique cube of the generation $\Delta_j^\phi(\vartheta,\gamma)$ containing x. Identity (102) implies that $\lim_{j\to\infty}f_j(x)=1$ for ϕ-almost every $x\in C$ and thus, the Severini–Egoroff theorem concludes that we can find a compact subset C_1 of C such that $\phi(C\setminus C_1)\leq\varepsilon_4\phi(C)$ and $f_j(x)$ converges uniformly to 1 on C_1. This proves (i) and (ii) at once. □

Theorem 2.30. *Let C_1 be the compact set from Proposition 2.29. Then there exists a cube $Q'\in\Delta(C_1,2\iota_1)$ such that $Q'\cap C_1$ is an intrinsic Lipschitz graph of positive ϕ-measure.*

Proof. For any $Q_0 \in \mathcal{M}(C_1, 2\iota_1)$, Theorem 2.28 and Proposition 2.29 imply that

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(Q_0 \cap C)) \geq \frac{\operatorname{diam} Q_0^{\mathfrak{Q}-1}}{2A_0^{\mathfrak{Q}-1}}. \tag{103}$$

Therefore, for any $Q_0 \in \mathcal{M}(C_1, 2\iota_1)$ we let $\mathcal{B}(Q_0)$ be the family of the maximal cubes $Q \in \Delta(C_1, 2\iota_1)$ contained in Q_0 for which

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(E(\vartheta, \gamma) \cap Q)) < \frac{\operatorname{diam} Q^{\mathfrak{Q}-1}}{4C_5^2 A_0^{\mathfrak{Q}-1}}, \tag{104}$$

and we define $\mathscr{B}(Q_0) := \bigcup_{Q \in \mathcal{B}(Q_0)} Q$.

The first step of the proof of the theorem is to show that the projection of $\mathscr{B}(Q_0)$ has small measure, or more precisely, that

$$\phi(C \cap [Q_0 \setminus \mathscr{B}(Q_0)]) > \frac{\phi(Q_0)}{8\vartheta C_1 C_5 A_0^{\mathfrak{Q}-1}}, \quad \text{for any } Q_0 \in \mathcal{M}(C_1, 2\iota_1). \tag{105}$$

Throughout this paragraph *we shall assume that $Q_0 \in \mathcal{M}(C_1, 2\iota_1)$ is fixed*. The maximality of the elements of $\mathcal{B}(Q_0)$ implies that they are pairwise disjoint and since by definition we have $Q \cap E(\vartheta, \gamma) \neq \varnothing$, for any $Q \in \mathcal{B}(Q_0)$ Remark A.3 yields

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(E(\vartheta, \gamma) \cap Q)) < \frac{\operatorname{diam} Q^{\mathfrak{Q}-1}}{4C_5^2 A_0^{\mathfrak{Q}-1}} \leq \frac{\phi(Q)}{4C_5 A_0^{\mathfrak{Q}-1}}. \tag{106}$$

Thanks to the fact that $C \subseteq E(\vartheta, \gamma)$, Propositions 1.18 and 1.31 allow us to infer that

$$\phi(C \cap [Q_0 \setminus \mathscr{B}(Q_0)]) \geq \frac{\mathcal{S}^{\mathfrak{Q}-1}(C \cap [Q_0 \setminus \mathscr{B}(Q_0)])}{\vartheta} \geq \frac{\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(C \cap [Q_0 \setminus \mathscr{B}(Q_0)]))}{2c(\Pi(Q_0))\vartheta}. \tag{107}$$

On the other hand, thanks to (103) we conclude that

$$\begin{aligned}\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(C \cap [Q_0 \setminus \mathscr{B}(Q_0)])) &\geq \mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(C \cap Q_0)) - \mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(E(\vartheta, \gamma) \cap \mathscr{B}(Q_0))) \\ &\geq \frac{\operatorname{diam} Q_0^{\mathfrak{Q}-1}}{2A_0^{\mathfrak{Q}-1}} - \sum_{Q \in \mathcal{B}(Q_0)} \mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(E(\vartheta, \gamma) \cap Q))). \end{aligned} \tag{108}$$

Since by definition $Q_0 \cap E(\vartheta, \gamma) \neq \varnothing$, Remark A.3, (106), (108) and the fact that the cubes in $\mathcal{B}(Q_0)$ are disjoint imply that

$$\begin{aligned}\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(C \cap [Q_0 \setminus \mathscr{B}(Q_0)])) &\geq \frac{\phi(Q_0)}{2C_5 A_0^{\mathfrak{Q}-1}} - \frac{1}{4C_5 A_0^{\mathfrak{Q}-1}} \sum_{Q \in \mathcal{B}(Q_0)} \phi(Q) \\ &= \frac{\phi(Q_0)}{2C_5 A_0^{\mathfrak{Q}-1}} - \frac{1}{4C_5 A_0^{\mathfrak{Q}-1}} \phi(\mathscr{B}(Q_0)). \end{aligned} \tag{109}$$

Putting together (107) and (109), we eventually deduce that

$$
\begin{aligned}
2c(\Pi(Q_0))\vartheta\phi(C\cap[Q_0\setminus\mathscr{B}(Q_0)]) &\geq \frac{\phi(Q_0)}{2C_5A_0^{\mathfrak{Q}-1}}-\frac{1}{4C_5A_0^{\mathfrak{Q}-1}}\phi(\mathscr{B}(Q_0))\\
&=\frac{\phi(Q_0)}{4C_5A_0^{\mathfrak{Q}-1}}+\frac{1}{4C_5A_0^{\mathfrak{Q}-1}}\phi(Q_0\setminus\mathscr{B}(Q_0)),
\end{aligned}
\tag{110}
$$

where the last equality above follows from the inclusion $\mathcal{B}(Q_0)\subseteq Q_0$. Inequality (110) together with the fact that $c(\Pi(Q_0))\leq C_1$, see Proposition 1.18, immediately implies (105).

Now that (105) is proved, the second step in the proof is to construct a cube $Q'\in\Delta(C_1,2\iota_1)$ disjoint from $\bigcup_{Q_0\in\mathcal{M}(C_1,2\iota_1)}\mathscr{B}(Q_0)$ such that $\phi(C_1\cap Q')>0$. Every subcube of Q' contained in $\Delta(C_1,2\iota_1)$ thus enjoys a big projections property, and this is what in the end allows us to prove that $C_1\cap Q'$ is contained in an intrinsic Lipschitz graph. Since the elements of $\mathcal{M}(C_1,2\iota_1)$ are pairwise disjoint and their union covers C_1, we infer that

$$
\begin{aligned}
&\phi\Big(C_1\setminus\bigcup_{Q_0\in\mathcal{M}(C_1,2\iota_2)}\mathscr{B}(Q_0)\Big)\\
&\qquad=\phi\Big(\bigcup_{Q_0\in\mathcal{M}(C_1,2\iota_1)}C_1\cap[Q_0\setminus\mathscr{B}(Q_0)]\Big)=\sum_{Q_0\in\mathcal{M}(C_1,2\iota_1)}\phi(C_1\cap[Q_0\setminus\mathscr{B}(Q_0)])\\
&\qquad\geq\sum_{Q_0\in\mathcal{M}(C_1,2\iota_1)}\phi(C\cap[Q_0\setminus\mathscr{B}(Q_0)])-\phi((C\setminus C_1)\cap Q_0)\\
&\qquad\overset{(106)}{\geq}\Big(\sum_{Q_0\in\mathcal{M}(C_1,2\iota_1)}\frac{\phi(Q_0)}{8\vartheta C_1C_5A_0^{\mathfrak{Q}-1}}\Big)-\phi(C\setminus C_1)\geq\frac{\phi(C_1)}{8\vartheta C_1C_5A_0^{\mathfrak{Q}-1}}-\varepsilon_4\phi(C).
\end{aligned}
\tag{111}
$$

Therefore, the choice of ε_4, Proposition 2.29 and (111) imply that

$$
\phi\Big(C_1\setminus\bigcup_{Q_0\in\mathcal{M}(C_1,2\iota_2)}\mathscr{B}(Q_0)\Big)\geq\frac{1-\varepsilon_4}{8\vartheta C_1C_5A_0^{\mathfrak{Q}-1}}\phi(C)-\varepsilon_4\phi(C)\geq\frac{\phi(C)}{16\vartheta C_1C_5A_0^{\mathfrak{Q}-1}}.
\tag{112}
$$

Inequality (112) implies that there must exist a cube $Q_0'\in\mathcal{M}(C_1,2\iota_1)$ such that $\phi\big(C_1\setminus\bigcup_{Q\in\mathcal{B}(Q_0')}Q\big)>0$. Defining $\mathscr{G}$ to be the set of maximal cubes in $\Delta(C_1,2\iota_1)\setminus\mathcal{B}(Q_0')$ contained in Q_0', we can find at least a cube $Q'\in\mathscr{G}$ for which $\phi(C_1\cap Q')>0$. Furthermore, thanks to the maximality of the elements in $\mathcal{B}(Q_0')$ and the fact that $Q'\cap\mathscr{B}(Q_0')=\varnothing$, we also deduce that any subcube of Q' cannot satisfy (104).

In the final step of the proof we show that $C_1\cap Q'$ is contained in an intrinsic Lipschitz graph. Indeed, we claim that

$$
x_1^{-1}x_2\in C_{\Pi(Q_0')}(2\alpha_0),\qquad\text{for any }\ x_1,x_2\in C_1\cap Q',
\tag{113}
$$

where α_0 was defined in Proposition 2.11. Fix $x_1,x_2\in C_1\cap Q'$, and note that there exists a unique $j\in\mathbb{N}$ such that

$$
R\gamma^{-1}2^{-jN+5}\leq d(x_1,x_2)\leq R\gamma^{-1}2^{-(j-1)N+5}.
\tag{114}
$$

For $i=1,2$ we let Q_{x_i} be the unique cubes in the j-th layer of cubes $\Delta_j^{\phi}(\vartheta,\gamma)$ for which $x_i \in Q_{x_i}$. Suppose $Q' \in \Delta_{\bar{j}}^{\phi}(\vartheta,\gamma)$ and note that Theorem A.2 (iv) and (114) imply that

$$R\gamma^{-1}2^{-jN+5} \leq d(x_1,x_2) \leq \operatorname{diam} Q' \leq \gamma^{-1}2^{-\bar{j}N+5}. \tag{115}$$

The chain of inequalities (115) implies that $\bar{j} \leq j$ and thus by Theorem A.2 (i) we infer that $Q_{x_i} \subseteq Q'$ for $i=1,2$. Furthermore, thanks to Theorem A.2 (ii) and (v), for $i=1,2$ we have

$$\begin{aligned} R \operatorname{diam} Q_{x_i} \leq R\gamma^{-1}2^{-jN+5} \leq d(x_1,x_2) &\leq R\gamma^{-1}2^{-(j-1)N+5} \\ &= 2^{N+6}\gamma^{-1}R2^{-jN-1} \leq 2^{N+6}\zeta^{-2}R \operatorname{diam} Q_{x_i}, \end{aligned} \tag{116}$$

since by construction $Q_{x_1} \in \Delta(\vartheta,\gamma)$. In addition to this, since as already remarked $Q_{x_i} \in \Delta(C_1, 2\iota_1)$, Lemma 2.16 implies that $\alpha(Q_{x_i}) \leq \varepsilon_2$ for $i=1,2$. Furthermore, the construction of Q' ensures that for any cube $Q \in \Delta(C_1, 2\iota_1)$ contained in Q', we have

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0')}(E(\vartheta,\gamma)\cap Q)) \geq \frac{\operatorname{diam} Q^{\mathfrak{Q}-1}}{4C_5^2 A_0^{\mathfrak{Q}-1}}. \tag{117}$$

This proves that the hypotheses of Proposition 2.11 are satisfied and thus $x_1 \in x_2 C_{\Pi(Q_0')}(2\alpha_0)$. Finally, $C_1 \cap Q'$ is proved to be contained in an intrinsic Lipschitz graph by means of Proposition 1.37. ☐

Remark 2.31. Note that the proof of Theorem 2.1 is an immediate consequence of Theorem 2.30.

3. The support of 1-codimensional measures with almost-flat tangents is intrinsic rectifiable

A careful examination of the arguments of Section 2 shows that in order to prove Theorem 2.1, we never fully exploited the fact that ϕ-almost everywhere we have $\operatorname{Tan}_{\mathfrak{Q}-1}(\phi,x) \subseteq \mathfrak{M}$. Indeed, we used the flatness of tangents just to show that there exists a set C with large ϕ-measure on which the 1-Wasserstein distance between ϕ and some flat measure — below a certain (uniform on C) scale — is smaller than some fixed constant, which in the specific case of Section 2 is in essence ε_2. See for instance Proposition 2.14 and Lemma 2.16. This quantified closeness to flat measures is sufficient to construct the cones that yield the intrinsic rectifiability property of the set C. This is a typical phenomena occurring even in Euclidean spaces that has been observed explicitly in [David and Semmes 1993a, §II.2.1 Remark 2.5] and less explicitly in [Preiss 1987, Lemma 5.2].

In this section we aim to show how to modify the arguments of Section 2 in order to prove the intrinsic rectifiability of asymptotically AD-regular measures with almost flat tangents.

Throughout this section *we let $\delta \in \mathbb{N}$ be a fixed natural number and ψ be a fixed Radon measure on $\mathbb{G}$* whose support is a compact set K and such that for ψ-almost every $x \in \mathbb{G}$ we have

(H1′) $\delta^{-1} \leq \Theta_*^{\mathfrak{Q}-1}(\psi,x) \leq \Theta^{\mathfrak{Q}-1,*}(\psi,x) \leq \delta$,

(H2′) $\limsup_{r\to 0} d_{x,r}(\psi,\mathfrak{M}) < 4^{-(\mathfrak{Q}+1)^2}\varepsilon_2(2\delta)$.

In the following *we will make extensive use of constants, parameters and sets introduced in Notation 2.8 specializing them for the measure ψ*. For clarity, we stress if not explicitly mentioned throughout this section *we will always assume that $\sigma := 2\delta$*.

The first step in the understanding of the structure of ψ is to show that for any $k > 0$ the limit $\limsup_{r\to 0} d_{x,kr}(\psi, \mathfrak{M})$ can be read as the maximum distance from flat measures among all the elements of $\mathrm{Tan}_{\mathfrak{Q}-1}(\psi, x)$ inside $B(0, k)$:

Proposition 3.1. *For ψ-almost all $x \in \mathbb{G}$ and any $k > 0$ we have*

$$\limsup_{r\to 0} d_{x,kr}(\psi, \mathfrak{M}) = \sup\{d_{0,k}(\nu, \mathfrak{M}) : \nu \in \mathrm{Tan}_{\mathfrak{Q}-1}(\psi, x)\}.$$

Proof. Fix a point $x \in K$ where $\mathrm{Tan}_{\mathfrak{Q}-1}(\psi, x) \neq \varnothing$ and where assumptions (H1′) and (H2′) hold. Recall that this choice of x can be made without loss of generality thanks to Proposition 1.26. Suppose $\{r_i\}_{i\in\mathbb{N}}$ is an infinitesimal sequence such that $\lim_{i\to\infty} d_{x,kr_i}(\psi, x) = \limsup_{r\to 0} d_{x,kr}(\psi, x)$ and assume up to nonrelabeled subsequences that there exists a $\nu \in \mathrm{Tan}_{\mathfrak{Q}-1}(\psi, x)$ such that

$$r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi \rightharpoonup \nu.$$

As a first step let us prove that $\limsup_{r\to 0} d_{x,kr}(\psi, \mathfrak{M}) \le d_{0,k}(\nu, \mathfrak{M})$. For any $0 < \eta < 1$ we let $\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner V$ be an element of $\mathfrak{M}$ such that $F_{0,k}(\nu, \Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner V)/k^{\mathfrak{Q}} \le d_{0,k}(\nu, \mathfrak{M}) + \eta$. With this choice, thanks to the triangle inequality, we infer that

$$\begin{aligned}\limsup_{i\to\infty} d_{0,k}(r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi, \mathfrak{M}) &\le \limsup_{i\to\infty} \frac{F_{0,k}(r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi, \Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner V)}{k^{\mathfrak{Q}}}\\ &\le \limsup_{i\to\infty} \frac{F_{0,k}(r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi, \nu) + F_{0,k}(\nu, \Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner V)}{k^{\mathfrak{Q}}}\\ &\le d_{0,k}(\nu, \mathfrak{M}) + \eta, \end{aligned}\tag{118}$$

where the last inequality comes from the choice of Θ and V and Proposition 1.23. The arbitrariness of η concludes the proof of the first claim.

As a second and final step of the proof, fix a $\mu \in \mathrm{Tan}_{\mathfrak{Q}-1}(\psi, x)$ and show that $\limsup_{r\to 0} d_{x,kr}(\psi, \mathfrak{M}) \ge d_{0,k}(\mu, \mathfrak{M})$. Since $\mu \in \mathrm{Tan}_{\mathfrak{Q}-1}(\psi, x)$, we can find an infinitesimal sequence $\{r_i\}_{i\in\mathbb{N}}$ such that

$$r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi \rightharpoonup \mu.$$

Furthermore, for any $0 < \eta < 2^{-(\mathfrak{Q}+1)}(\delta^{-1} - 2^{-\mathfrak{Q}}\varepsilon_2(2\delta))$ and any $i \in \mathbb{N}$ there exists a $\Theta_i > 0$ and a $V_i \in \mathrm{Gr}(\mathfrak{Q} - 1)$ such that

$$\frac{F_{0,k}(r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi, \Theta_i\mathcal{S}^{\mathfrak{Q}-1}\llcorner V_i)}{k^{\mathfrak{Q}}} \le d_{0,k}(r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi, \mathfrak{M}) + \eta = d_{x,kr_i}(\psi, \mathfrak{M}) + \eta,$$

where the last identity above comes from Proposition 2.3 (i).

Our next task is to show that there exists a compact subinterval I of $(0, \infty)$ such that $\{\Theta_i\}_{i\in\mathbb{N}} \subseteq I$. Thanks to assumption (H2′) on ψ, there exists an $i_0 \in \mathbb{N}$ such that we have $d_{x,kr_i}(\psi, \mathfrak{M}) \le 4^{-(\mathfrak{Q}+1)^2}\varepsilon_2(2\delta)$, for any $i \geq i_0$. This implies for any $i \geq i_0$ that

$$\left|\int g(w)\, d\frac{T_{x,r_i}\psi(w)}{r_i^{\mathfrak{Q}-1}} - \Theta_i \int g(w)\, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(w)\right| \leq F_{0,k}(r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi, \Theta_i \mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i)$$
$$\leq 4^{-(\mathfrak{Q}+1)^2} \varepsilon_2(2\delta) k^{\mathfrak{Q}} + \eta k^{\mathfrak{Q}}, \tag{119}$$

where $g(x) := \max\{k - d(0, x), 0\}$. Thanks to the definition of g and to (119) we infer that

$$\begin{aligned}
&\Theta_i 2^{-\mathfrak{Q}} k^{\mathfrak{Q}} - \frac{k\psi(B(x, kr_i))}{r_i^{\mathfrak{Q}-1}} \\
&\qquad \leq \Theta_i \int_{B(0,k/2)} g(w)\, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(w) - \int_{B(0,k)} k\, \frac{dT_{x,r_i}\psi(w)}{r_i^{\mathfrak{Q}-1}} \\
&\qquad \leq \left|\Theta_i \int g(w)\, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(w) - \int g(w) \frac{dT_{x,r_i}\psi(w)}{r_i^{\mathfrak{Q}-1}}\right| \leq 4^{-(\mathfrak{Q}+1)^2} \varepsilon_2(2\delta) k^{\mathfrak{Q}} + \eta k^{\mathfrak{Q}}.
\end{aligned} \tag{120}$$

On the other hand, a similar argument shows that

$$\begin{aligned}
&\frac{k}{2}\frac{\psi(B(x, kr_i/2))}{r_i^{\mathfrak{Q}-1}} - \Theta_i k^{\mathfrak{Q}} \\
&\qquad = \int_{B(0,k/2)} \frac{k}{2} \frac{dT_{x,r_i}\psi(y)}{r_i^{\mathfrak{Q}-1}} - \Theta_i \int g(y)\, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(y) \\
&\qquad \leq \left|\int g(y) \frac{dT_{x,r_i}\psi(y)}{r_i^{\mathfrak{Q}-1}} - \Theta_i \int g(y)\, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(y)\right| \leq 4^{-(\mathfrak{Q}+1)^2} \varepsilon_2(2\delta) k^{\mathfrak{Q}} + \eta k^{\mathfrak{Q}}.
\end{aligned} \tag{121}$$

Rearranging inequality (120) and dividing both sides by $\left(\frac{1}{2}k\right)^{\mathfrak{Q}}$, thanks to the choice of x and to the arbitrariness of i, we have

$$\limsup_{i\to\infty} \Theta_i \leq 2^{\mathfrak{Q}} \limsup_{i\to\infty} \frac{\psi(B(x, kr_i))}{(kr_i)^{\mathfrak{Q}-1}} + 2^{-(\mathfrak{Q}+1)} \varepsilon_2(2\delta) + 2^{\mathfrak{Q}}\eta \leq 2^{\mathfrak{Q}}(\delta + 1) + 2^{-(\mathfrak{Q}+1)} \varepsilon_2(2\delta), \tag{122}$$

where the second last inequality comes from the fact that (H1′) is satisfied at x and the last inequality from the fact that $\eta < 1$.

Similarly, rearranging inequality (121) and dividing both sides by $k^{\mathfrak{Q}}$, thanks to the arbitrariness of i, we infer that

$$2^{-\mathfrak{Q}}\delta^{-1} \leq \frac{\Theta_*^{\mathfrak{Q}-1}(\psi, x)}{2^{\mathfrak{Q}}} \leq \frac{1}{2^{\mathfrak{Q}}} \liminf_{i\to\infty} \frac{\psi(B(x, kr_i/2))}{(kr_i/2)^{\mathfrak{Q}-1}} \leq \liminf_{i\to\infty} \Theta_i + 4^{-(\mathfrak{Q}+1)^2} \varepsilon_2(2\delta) + \eta. \tag{123}$$

On the other hand, (123) and the choice of η imply that

$$0 < 2^{-(\mathfrak{Q}+1)}(\delta^{-1} - 2^{-\mathfrak{Q}} \varepsilon_2(2\delta)) \leq 2^{-\mathfrak{Q}}(\delta^{-1} - 2^{-\mathfrak{Q}} \varepsilon_2(2\delta)) - \eta \leq \liminf_{i\to\infty} \Theta_i, \tag{124}$$

where the first inequality comes from the choice of $\varepsilon_2(2\delta)$ and the second inequality from that of η. The bounds (122) and (124) together imply that up to taking a nonrelabeled subsequence of $\{\Theta_i\}_{i\in\mathbb{N}}$ we can assume that the Θ_i converge to some $\Theta \in [2^{-(\mathfrak{Q}+1)}(\delta^{-1} - 2^{-\mathfrak{Q}} \varepsilon_2(2\delta)), 2^{\mathfrak{Q}}(\delta + 1) + 2^{-(\mathfrak{Q}+1)} \varepsilon_2(2\delta)]$.

Without loss of generality, we can assume that there exists a $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ such that $\mathfrak{n}(V_i) \to \mathfrak{n}(V)$. Since under such an assumption Proposition 1.32 implies that $\Theta_i \mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i \rightharpoonup \Theta \mathcal{S}^{\mathfrak{Q}-1} \llcorner V$, the triangle inequality implies for any $i \in \mathbb{N}$ that

$$
\begin{aligned}
&d_{0,k}(\mu, \mathfrak{M}) \\
&\quad\le \frac{F_{0,k}(\mu, r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi) + F_{0,k}(r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi, \Theta_i \mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i) + F_{0,k}(\Theta_i \mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i, \Theta \mathcal{S}^{\mathfrak{Q}-1} \llcorner V)}{k^{\mathfrak{Q}}} \\
&\quad\le \frac{F_{0,k}(\mu, r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi)}{k^{\mathfrak{Q}}} + d_{0,k}(r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi, \mathfrak{M}) + \eta + \frac{F_{0,k}(\Theta_i \mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i, \Theta \mathcal{S}^{\mathfrak{Q}-1} \llcorner V)}{k^{\mathfrak{Q}}}.
\end{aligned}
$$

Finally, thanks to the arbitrariness of i and of η and to Proposition 1.23, we infer that

$$d_{0,k}(\mu, \mathfrak{M}) \le \limsup_{i\to\infty} d_{0,k}(r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi, \mathfrak{M}). \qquad \square$$

The following result is the analogue of Proposition 2.14 for ψ as it serves the same purpose, i.e., find a compact subset $\tilde{C}$ of K in such a way that $\psi \llcorner \tilde{C}$ is essentially an AD-regular measure and the functions $x \mapsto d_{x,4kr}(\psi, \mathfrak{M})$ have small supremum norms on $\tilde{C}$ provided r is small enough.

Proposition 3.2. *There exist an $\tilde{\iota}_0 \in \mathbb{N}$ and a $\tilde{\gamma} \in \mathbb{N}$ such that for any $\mu \ge 8C_4(2\delta)\delta$ we can find a $\nu \in \mathbb{N}$ and a compact set $\tilde{C} \subseteq \mathscr{E}^{\psi}_{2\delta,\tilde{\gamma}}(\mu, \nu)$ such that*

(i) $\psi(K \setminus \tilde{C}) \le 2\varepsilon_1 \psi(K)$,

(ii) $d_{x,4k(2\delta)r}(\psi, \mathfrak{M}) + d_{x,4k(2\delta)r}(\psi \llcorner E^{\psi}(2\delta, \tilde{\gamma}), \mathfrak{M}) \le 4^{-\mathfrak{Q}(\mathfrak{Q}+1)} \varepsilon_2(2\delta)$ *for any* $0 < r < 2^{-\tilde{\iota}_0 N(2\delta)+5}/\tilde{\gamma}$ *and any* $x \in \tilde{C}$,

where $\varepsilon_2(2\delta)$ is the constant introduced in Notation 2.8 and ε_1 is chosen in the same way as it was in Notation 2.5.[6]

Proof. First of all, thanks to Propositions 1.28 and 1.30 we can find a $\tilde{\gamma} \in \mathbb{N}$ and a $\nu \in \mathbb{N}$ such that $\psi(K \setminus \mathscr{E}^{\psi}_{2\delta,\tilde{\gamma}}(\mu, \nu)) \le \varepsilon_1 \psi(K)$. Let us now prove that

$$\limsup_{r\to 0} d_{x,4k(2\delta)r}(\psi \llcorner E^{\psi}(2\delta, \tilde{\gamma}), \mathfrak{M}) \le 4^{-(\mathfrak{Q}+1)^2} \varepsilon_2(2\delta), \quad \text{for } \psi\text{-almost every } x \in E^{\psi}(2\delta, \tilde{\gamma}).$$

Recall that for ψ-almost every $x \in E^{\psi}(2\delta, \tilde{\gamma})$, we have that $\mathrm{Tan}_{\mathfrak{Q}-1}(\psi \llcorner E^{\psi}(2\delta, \tilde{\gamma}), x) = \mathrm{Tan}_{\mathfrak{Q}-1}(\psi, x)$. Thanks to this, Proposition 3.1 yields

$$
\begin{aligned}
\limsup_{r\to 0} d_{x,4k(2\delta)r}(\psi \llcorner E^{\psi}(2\delta, \tilde{\gamma}), \mathfrak{M}) &\le \limsup_{r\to 0} \frac{F_{x,4k(2\delta)r}(\psi \llcorner E^{\psi}(2\delta, \tilde{\gamma}), \psi)}{(4k(2\delta)r)^{\mathfrak{Q}}} + d_{x,4k(2\delta)r}(\psi, \mathfrak{M}) \\
&= \limsup_{r\to 0} d_{x,4k(2\delta)r}(\psi, \mathfrak{M}) \le 4^{-(\mathfrak{Q}+1)^2} \varepsilon_2(2\delta),
\end{aligned}
$$

for ψ-almost every $x \in E^{\psi}(2\delta, \tilde{\gamma})$, where the identity in the last line comes from hypothesis (H1′) and the Lebesgue differentiation theorem of [Heinonen et al. 2015, page 77]. Therefore, for ψ-almost every

[6] The reader should notice that the objects and symbols introduced in Notation 2.5 were specific to the measure ϕ. However, ε_1 was just required to be a positive real number smaller than $1/10$.

$x \in E^\psi(2\delta, \tilde\gamma)$ there exists an $r(x) > 0$ such that for every $0 < r < r(x)$,

$$d_{x,4k(2\delta)r}(\psi, \mathfrak{M}) + d_{x,4k(2\delta)r}(\psi \llcorner E^\psi(2\delta, \tilde\gamma), \mathfrak{M}) \leq 4^{-\mathfrak{Q}(\mathfrak{Q}+1)}\varepsilon_2(2\delta).$$

For any $j \in \mathbb{N}$, let us define $E_j := \{x \in \mathscr{E}^\psi_{2\delta,\tilde\gamma}(\mu, \nu) : r(x) > 1/j\}$ and show that the E_j are Borel sets. Thanks to Proposition 2.3 (ii), the map $x \mapsto d_{x,r}(\psi, \mathfrak{M}) + d_{x,r}(\psi \llcorner E^\psi(2\delta, \tilde\gamma), \mathfrak{M})$ is continuous and thus for any $r > 0$ the set $\Omega_r := \{y \in \mathscr{E}^\psi_{2\delta,\tilde\gamma}(\mu, \nu) : d_{y,r}(\psi, \mathfrak{M}) + d_{y,r}(\psi \llcorner E^\psi(2\delta, \tilde\gamma), \mathfrak{M}) < 4^{-\mathfrak{Q}(\mathfrak{Q}+1)}\varepsilon_2(2\delta)\}$ is relatively open in $\mathscr{E}^\psi_{2\delta,\tilde\gamma}(\mu, \nu)$. In particular, if $x \in \Omega_r$ for any $r \in (0, 1/j) \cap \mathbb{Q}$ we have $r(x) > 1/j$ thanks to Proposition 2.3 (iv) and hence $x \in E_j$. On the other hand, if $x \in E_j$ then obviously $x \in \Omega_r$ for any $0 < r < 1/j$. Since $\mathscr{E}^\psi_{2\delta,\tilde\gamma}(\mu, \nu)$ is compact, this shows that the sets E_j are G_δ and thus Borel. Let us note that since ψ-almost every $x \in \mathscr{E}^\psi_{2\delta,\tilde\gamma}(\mu, \nu)$ is contained in some E_j, thanks to the existence of $r(x)$, we infer that

$$\psi\Big(\mathscr{E}^\psi_{2\delta,\tilde\gamma}(\mu, \nu) \setminus \bigcup_{j\in\mathbb{N}} E_j\Big) = 0. \tag{125}$$

Finally, (125) together with the measurability of the nested sets E_j implies that we can find a $j \in \mathbb{N}$ big enough and a compact set $\tilde C$ contained in E_j satisfying items (i) and (ii). □

As in the case of Proposition 2.14, one can impose slightly different conditions on the measure and obtain a family of cubes satisfying the same thesis as Lemma 2.16. From here on we will employ all the notations introduced in Notation 2.8.

Proposition 3.3. *Fixing $\mu \geq 8C_4(2\delta)\delta$ if $\tilde\gamma, \tilde\iota_0, \nu \in \mathbb{N}$ and $\tilde C \Subset \mathscr{E}^\psi_{2\delta,\tilde\gamma}(\mu, \nu)$ are the natural numbers and the compact set yielded by Proposition 3.2, respectively, and defining $\tilde\iota := \max\{\tilde\iota_0, \nu\}$ for any cube $Q \in \Delta^\psi(\tilde C; 2\delta, \tilde\gamma, \tilde\iota)$, we have that $\alpha(Q) \leq \varepsilon_2(2\delta)$ and for any such cube Q there is a plane $\Pi(Q) \in \mathrm{Gr}(\mathfrak{Q} - 1)$ for which*

(i) $$\sup_{w\in E^\psi(2\delta,\tilde\gamma)\cap B(\mathfrak{c}(Q),k(2\delta)\operatorname{diam} Q/2)} \frac{\operatorname{dist}(w, \mathfrak{c}(Q)\Pi(Q))}{2k(2\delta)\operatorname{diam} Q} \leq C_2(2\delta)\varepsilon_2(2\delta)^{1/\mathfrak{Q}}, \quad \textit{and}$$

(ii) *for any $w \in B\big(\mathfrak{c}(Q), \frac{1}{2}k(2\delta)\operatorname{diam} Q\big) \cap \mathfrak{c}(Q)\Pi(Q)$ we have*

$$E^\psi(2\delta, \tilde\gamma) \cap B(w, 3k(2\delta)C_2(2\delta)\varepsilon_2(2\delta)^{1/(\mathfrak{Q}+1)}\operatorname{diam} Q) \neq \varnothing.$$

Proof. Thanks to Proposition 3.2, we can find a $\tilde\gamma \in \mathbb{N}$ and a compact set $\tilde C$ contained in $E^\psi(2\delta, \gamma)$ such that

(i) $\psi(K \setminus \tilde C) \leq 2\varepsilon_1\psi(K)$, where ε_1 was introduced in Notation 2.5,

(ii) $d_{x,4k(2\delta)r}(\psi, \mathfrak{M}) + d_{x,4k(2\delta)r}(\psi \llcorner E^\psi(2\delta, \gamma), \mathfrak{M}) \leq 4^{-\mathfrak{Q}(\mathfrak{Q}+1)}\varepsilon_2(2\delta)$ for any $0 < r < 2^{-\tilde\iota_0 N(2\delta)+5}/\gamma$ and any $x \in \tilde C$.

Thus, if $\Delta^\psi(2\delta, \tilde\gamma)$ is the family of dyadic cubes relative to the parameters $2\delta, \tilde\gamma$ and the measure ψ yielded by Theorem A.2, one can prove that the cubes of $\Delta^\psi(\tilde C; 2\delta, \tilde\gamma, \tilde\iota)$ satisfy (i) and (ii) by using verbatim the argument we employed in the proof of Lemma 2.16. □

As remarked at the beginning of this section, the arguments we used to prove Propositions 2.21 and 2.27, Lemmas 2.25 and 2.26 and Theorem 2.28 just relied on the possibility of proving Lemma 2.16 for the measure ϕ. Proposition 3.3 is the counterpart of Lemma 2.16 for the measure ψ where ϑ has been substituted by 2δ, γ by $\tilde{\gamma}$, and so on. Therefore, repeating the proofs of Section 2C for ψ and its associated parameters and compact set $\tilde{C}$, one can show the following:

Theorem 3.4. *For any cube $Q \in \Delta^{\psi}(\tilde{C}; 2\delta, \tilde{\gamma}, \tilde{\iota})$ such that $(1-\varepsilon_3(2\delta))\phi(Q) \leq \phi(Q \cap \tilde{C})$, we have*

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q)}(Q \cap \tilde{C})) \geq \frac{\operatorname{diam} Q^{\mathfrak{Q}-1}}{2A_0^{\mathfrak{Q}-1}}.$$

Remark 3.5. Similar to what we did in Proposition 2.29, we can construct a compact subset $\tilde{C}_1$ of $\tilde{C}$ and an $\tilde{\iota}_1 \in \mathbb{N}$ satisfying (i) and (ii) of Proposition 2.29, provided ε_3 is substituted with $\varepsilon_3(2\delta)$, ε_4 with $\varepsilon_4(2\delta) := \min\{\varepsilon_1, (64\delta C_1 C_5(2\delta) A_0^{\mathfrak{Q}-1}(2\delta))^{-1}\}$ and $\Delta(C, \iota)$ with $\Delta^{\psi}(\tilde{C}; 2\delta, \gamma, \tilde{\iota}_1)$.

The above remark allows us to construct the ψ-positive intrinsic Lipschitz graph that will be used to prove Theorem 4.2 in Section 4.

Theorem 3.6. *Let $\tilde{C}_1$ be as in Remark 3.5. Then there exists a cube $Q' \in \Delta^{\psi}(\tilde{C}_1; 2\delta, \gamma, 2\tilde{\iota}_1)$ such that $Q' \cap \tilde{C}_1$ is an intrinsic Lipschitz graph of positive ψ-measure.*

Proof. Thanks to Propositions 3.2, 3.3, Remark 3.5 and Theorem 3.4, the argument we used to prove Theorem 2.30 can be applied here verbatim. □

4. Conclusions and discussion of the results

In this section we use the main result of Section 2, i.e., Theorem 2.1, to deduce a number of consequences. First of all we prove the main result of the paper, Theorem 4.1, which is a 1-codimensional extension of the Marstrand–Mattila rectifiability criterion to general Carnot groups. Secondly, we provide in Corollary 4.3 a rigidity result for finite perimeter sets in Carnot groups: we are able to show that if locally a finite perimeter set is not too far from its natural tangent plane, then its boundary is an *intrinsic rectifiable set*; see Definition 1.40. Eventually, we use Theorem 4.1 to prove a 1-codimensional version of Preiss's rectifiability theorem in the Heisenberg groups $\mathbb{H}^n$.

4A. *Main results.* In this subsection we finally conclude the proof of the main results of this work.

Theorem 4.1. *Suppose ϕ is a Radon measure on $\mathbb{G}$ and let $\tilde{d}(\cdot,\cdot)$ be a left-invariant, homogeneous distance on $\mathbb{G}$. Assume further that for ϕ-almost all $x \in \mathbb{G}$ we have*

(i) $$0 < \liminf_{r\to 0} \frac{\phi(\tilde{B}(x,r))}{r^{\mathfrak{Q}-1}} \leq \limsup_{r\to 0} \frac{\phi(\tilde{B}(x,r))}{r^{\mathfrak{Q}-1}} < \infty,$$

where $\tilde{B}(x,r)$ is the ball relative to the metric $\tilde{d}$ centered at x of radius $r > 0$,

(ii) $\operatorname{Tan}_{\mathfrak{Q}-1}(\phi, x) \subseteq \mathfrak{M}$, *where $\mathfrak{M}$ is the family of 1-codimensional flat measures from Definition 1.7.*

Then ϕ is absolutely continuous with respect to $\mathcal{S}^{\mathfrak{Q}-1}$ and ϕ-almost all of $\mathbb{G}$ can be covered with countably many $C^1_{\mathbb{G}}$-hypersurfaces.

Proof. Since $\tilde{d}$ is bi-Lipschitz equivalent to d, see for instance Corollary 5.15 in [Bonfiglioli et al. 2007], hypothesis (i) implies that

$$0 < \Theta_*^{\mathfrak{Q}-1}(\phi, x) \le \Theta^{\mathfrak{Q}-1,*}(\phi, x) < \infty, \tag{126}$$

for ϕ-almost every $x \in \mathbb{G}$. For any $\vartheta, \gamma, R \in \mathbb{N}$ we define

$$E(\vartheta, \gamma, R) := \{x \in \overline{B(0, R)} : \vartheta^{-1} r^{\mathfrak{Q}-1} \le \phi(B(x, r)) \le \vartheta r^{\mathfrak{Q}-1} \text{ for any } 0 < r < 1/\gamma\}.$$

It is possible to prove, with the same arguments used in the proof of Proposition 1.28, that the $E(\vartheta, \gamma, R)$ are compact sets and

$$\phi\Big(\mathbb{G} \setminus \bigcup_{\vartheta,\gamma,R} E(\vartheta, \gamma, R)\Big) = 0. \tag{127}$$

Thus, if A is an $\mathcal{S}^{\mathfrak{Q}-1}$-null Borel set, Proposition 1.31 yields

$$\phi(A) \le \sum_{\vartheta,\gamma,R\in\mathbb{N}} \phi(A \cap E(\vartheta, \gamma, R)) \le \sum_{\vartheta,\gamma,R\in\mathbb{N}} \vartheta 2^{\mathfrak{Q}-1} \mathcal{S}^{\mathfrak{Q}-1}(A \cap E(\vartheta, \gamma, R)) = 0.$$

The above computation proves that ϕ is absolutely continuous with respect to $\mathcal{S}^{\mathfrak{Q}-1}$ and just to fix notations we let $\rho \in L^1(\mathcal{S}^{\mathfrak{Q}-1})$ be such that $\phi = \rho\mathcal{S}^{\mathfrak{Q}-1}$.

As a second step, we show that ϕ-almost all of $\mathbb{G}$ can be covered with countably many intrinsic Lipschitz graphs. Assume by contradiction there are $\vartheta, \gamma, R \in \mathbb{N}$ for which we can find a subset of $E(\vartheta, \gamma, R)$ of positive ϕ-measure that we denote by $E(\vartheta, \gamma, R)^u$ (following the notations of Corollary 1.42) and that has $\mathcal{S}^{\mathfrak{Q}-1}$-null intersection with any intrinsic Lipschitz graph. Thanks to Corollary 2.9.11 of [Federer 1969] it is immediate to see that

$$\vartheta^{-1} \le \Theta_*^{\mathfrak{Q}-1}(\phi \llcorner E(\vartheta, \gamma, R)^u, x) \le \Theta^{\mathfrak{Q}-1,*}(\phi \llcorner E(\vartheta, \gamma, R)^u, x) \le \vartheta,$$

for ϕ-almost every $x \in E(\vartheta, \gamma, R)^u$. Further, from Proposition 1.27, for ϕ-almost every $x \in E(\vartheta, \gamma, R)^u$, we infer that $\mathrm{Tan}_{\mathfrak{Q}-1}(\phi \llcorner E(\vartheta, \gamma, R)^u, x) \subseteq \mathfrak{M}$. And since its hypothesis is satisfied, Theorem 2.1 implies that there exists an intrinsic Lipschitz graph Γ such that $\phi(\Gamma \cap E(\vartheta, \gamma, R)^u) > 0$. However, this is not possible since Proposition 1.31 would yield

$$0 < \phi(\Gamma \cap E(\vartheta, \gamma, R)^u) \le \vartheta 2^{\mathfrak{Q}-1} \mathcal{S}^{\mathfrak{Q}-1}(E(\vartheta, \gamma, R)^u \cap \Gamma),$$

and this contradicts the fact that $E(\vartheta, \gamma, R)$ intersects in a $\mathcal{S}^{\mathfrak{Q}-1}$-null set every intrinsic Lipschitz graph.

Up to this point we have shown that, for any choice of ϑ, γ, R, we have that $\mathcal{S}^{\mathfrak{Q}-1}$-almost all of the sets $E(\vartheta, \gamma, R)$ are covered by countably many intrinsic Lipschitz graphs. Furthermore, since $\phi \ll \mathcal{S}^{\mathfrak{Q}-1}$, thanks to (127) we conclude that ϕ-almost all of $\mathbb{G}$ can be covered by countably many intrinsic Lipschitz graphs. This concludes the first part of the proof of the theorem.

So far we have shown that we can find countably many intrinsic Lipschitz graphs that cover ϕ-almost all of $\mathbb{G}$. Since by Remark B.7 we know that intrinsic Lipschitz graphs are boundaries of finite perimeter sets, if $\mathbb{G}$ is a group where boundaries of finite perimeter sets are $C^1_{\mathbb{G}}$-rectifiable, the proof of the proposition would be completed here. In the moment of writing some broad families of Carnot groups where

De Giorgi's rectifiability theorem is known to hold include step 2 groups (see [Franchi et al. 2003]), groups of type * (see [Marchi 2014]) and groups of diamond type (see [Le Donne and Moisala 2021]).

In this paragraph, we assume that ϑ, γ, $R \in \mathbb{N}$ are fixed. Thanks to Proposition 1.31 we infer that $\mathcal{S}^{\mathfrak{Q}-1} \llcorner E(\vartheta, \gamma, R)$ is mutually absolutely continuous with respect to $\phi \llcorner E(\vartheta, \gamma)$ and in particular that

$$\vartheta^{-1} \leq \rho(x) \leq \vartheta 2^{\mathfrak{Q}-1} \quad \text{for } \mathcal{S}^{\mathfrak{Q}-1}\text{-almost every } x \in E(\vartheta, \gamma, R).$$

Let $\{\gamma_i\}_{i\in\mathbb{N}}$ be the sequence of intrinsic Lipschitz functions $\gamma_i : W_i \to \mathfrak{N}(W_i)$ for which

$$\phi\Big(E(\vartheta, \gamma, R) \setminus \bigcup_{i\in\mathbb{N}} \mathrm{gr}(\gamma_i)\Big) = 0,$$

and let $E_i := \mathrm{epi}(\gamma_i)$ be the epigraph of the function γ_i which is defined in (142). Since $\mathcal{S}^{\mathfrak{Q}-1} \llcorner \mathrm{gr}(\gamma_i)$ and $|\partial E_i|_{\mathbb{G}}$[7] are asymptotically doubling measures by [Franchi and Serapioni 2016, Theorem 3.9] and Theorems B.6 and B.8, respectively, we deduce thanks to Proposition 1.27 that for ϕ-almost every $x \in E(\vartheta, \gamma, R) \cap \mathrm{gr}(\gamma_i)$ we have

$$\begin{aligned} \mathfrak{M} \supseteq \mathrm{Tan}_{\mathfrak{Q}-1}(\phi \llcorner E(\vartheta, \gamma, R) \cap \mathrm{gr}(\gamma_i), x) &= \rho(x)\, \mathrm{Tan}_{\mathfrak{Q}-1}(\mathcal{S}^{\mathfrak{Q}-1} \llcorner \mathrm{gr}(\gamma_i), x) \\ &= \rho(x)\mathfrak{d}(x)\, \mathrm{Tan}_{\mathfrak{Q}-1}(|\partial E_i|_{\mathbb{G}}, x), \end{aligned} \tag{128}$$

where $\mathfrak{d}$ is the density yielded by Remark B.7. Finally, Proposition B.16 implies that

$$\mathrm{Tan}_{\mathfrak{Q}-1}(\phi \llcorner E(\vartheta, \gamma, R) \cap \mathrm{gr}(\gamma_i), x) \subseteq \rho(x)\mathfrak{d}(x)\{\lambda \mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(x) : \lambda \in [L_{\mathbb{G}}^{-1}, l_{\mathbb{G}}^{-1}]\}, \tag{129}$$

for ϕ-almost every $x \in E(\vartheta, \gamma, R) \cap \mathrm{gr}(\gamma_i)$, where $V_i(x) \in \mathrm{Gr}(\mathfrak{Q}-1)$ is the plane orthogonal to $\mathfrak{n}_{E_i}(x)$, the generalized inward normal introduced in Definition B.4, and the constants $l_{\mathbb{G}}$ and $L_{\mathbb{G}}$ are those yielded by Theorem B.6. We now prove that (129) implies that for $\mathcal{S}^{\mathfrak{Q}-1}$-almost every $x \in \mathrm{gr}(\gamma_i) \cap E(\vartheta, \gamma, R)$ and every $\alpha > 0$ we have

$$\lim_{r\to 0} \frac{\mathcal{S}^{\mathfrak{Q}-1}(\mathrm{gr}(\gamma_i) \cap E(\vartheta, \gamma, R) \cap B(x, r) \setminus x X_{V_i(x)}(\alpha))}{r^{\mathfrak{Q}-1}} = 0, \tag{130}$$

where $X_{V_i(x)}(\alpha) := \{w \in \mathbb{G} : \mathrm{dist}(w, V_i(x)) \leq \alpha \|w\|\}$. Thanks to (128) and (129), for $\mathcal{S}^{\mathfrak{Q}-1}$-almost every $x \in \mathrm{gr}(\gamma_i) \cap E(\vartheta, \gamma, R)$ and any sequence $r_j \to 0$, there exists a $\lambda > 0$ for which

$$\frac{T_{x,r} \mathcal{S}^{\mathfrak{Q}-1} \llcorner E(\vartheta, \gamma, R) \cap \mathrm{gr}(\gamma_i)}{r_j^{\mathfrak{Q}-1}} \rightharpoonup \lambda \mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(x). \tag{131}$$

The convergence in (131) implies that

$$\begin{aligned} \lim_{i\to\infty} & \frac{\mathcal{S}^{\mathfrak{Q}-1} \llcorner \mathrm{gr}(\gamma_i) \cap E(\vartheta, \gamma, R)(B(x, r_j) \setminus x X_{V_i(x)}(\alpha))}{r_j^{\mathfrak{Q}-1}} \\ &= \lim_{i\to\infty} \frac{T_{x,r_j}(\mathcal{S}^{\mathfrak{Q}-1} \llcorner \mathrm{gr}(\gamma_i) \cap E(\vartheta, \gamma, R))(B(0, 1) \setminus X_{V_i(x)}(\alpha))}{r_j^{\mathfrak{Q}-1}} \\ &= \lambda(\mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(x))(B(0, 1) \setminus X_{V_i(x)}(\alpha)) = 0, \end{aligned} \tag{132}$$

[7] With $|\partial E_i|_{\mathbb{G}}$ we denote as usual the perimeter measure associated to E_i.

where the second last identity above comes from the fact that $\mathcal{S}^{\mathfrak{Q}-1}(V_i(x)\cap\partial B(0,1)\setminus X_{V_i(x)}(\alpha))=0$ and [De Lellis 2008, Proposition 2.7].

Proposition B.17 and (130) together imply that each one of the intrinsic Lipschitz graphs $\mathrm{gr}(\gamma_i)\cap E(\vartheta,\gamma,R)$ can be covered $\mathcal{S}^{\mathfrak{Q}-1}$-almost all with $C^1_{\mathbb{G}}$-surfaces. In particular this shows that for any ϑ,γ,R the set $E(\vartheta,\gamma,R)$ can be covered $\mathcal{S}^{\mathfrak{Q}-1}$-almost all, and thus ϕ-almost all, by countably many $C^1_{\mathbb{G}}$-surfaces. This, the arbitrariness of $\vartheta,\ \gamma,\ R\in\mathbb{N}$ and (127) conclude the proof of the theorem. □

The following theorem trades off the regularity of tangents, which are assumed only to be close enough to flat measures, with a strengthened hypothesis on the $(\mathfrak{Q}-1)$-density of ϕ.

Theorem 4.2. *Suppose ϕ is a Radon measure on $\mathbb{G}$ and let $\tilde{d}(\cdot,\cdot)$ be a left-invariant, homogeneous distance on $\mathbb{G}$. If there exists a $\delta\in\mathbb{N}$ such that*

$$\delta^{-1}<\liminf_{r\to0}\frac{\phi(\tilde{B}(x,r))}{r^{\mathfrak{Q}-1}}\leq\limsup_{r\to0}\frac{\phi(\tilde{B}(x,r))}{r^{\mathfrak{Q}-1}}<\delta\quad\textit{for }\phi\textit{-almost every }x\in\mathbb{G},\tag{133}$$

where $\tilde{B}(x,r)$ is the ball relative to the metric $\tilde{d}$ centered at x of radius $r>0$, then we can find an $\varepsilon(\delta,\tilde{d})>0$ such that, if

$$\limsup_{r\to0}d_{x,r}(\phi,\mathfrak{M})\leq\varepsilon(\delta,\tilde{d})\quad\textit{for }\phi\textit{-almost every }x\in\mathbb{G},$$

then ϕ is absolutely continuous with respect to $\mathcal{S}^{\mathfrak{Q}-1}$, and ϕ-almost all of $\mathbb{G}$ can be covered with countably many intrinsic Lipschitz surfaces.

Proof. The first step in the proof is to note that since the metric $\tilde{d}$ and d are bi-Lipschitz equivalent, there exists a constant $\mathfrak{c}>1$, which we can assume without loss of generality to be a natural number, such that

$$(\mathfrak{c}\delta)^{-1}<\liminf_{r\to0}\frac{\phi(B(x,r))}{r^{\mathfrak{Q}-1}}\leq\limsup_{r\to0}\frac{\phi(B(x,r))}{r^{\mathfrak{Q}-1}}<\mathfrak{c}\delta\quad\text{for }\phi\text{-almost every }x\in\mathbb{G}.$$

If we let $\varepsilon(\delta,\tilde{d}):=4^{-\mathfrak{Q}(\mathfrak{Q}+1)}\varepsilon_2(\mathfrak{c}\delta)$ then the verbatim repetition of the first part of the argument used to prove Theorem 4.1, where instead of Theorem 2.1 we make use of Theorem 3.6, proves the claim. □

An immediate consequence of Theorem 4.2 is the following:

Corollary 4.3. *Let $\vartheta_{\mathbb{G}}:=\max\{l_{\mathbb{G}}^{-1},L_{\mathbb{G}}\}$, where $l_{\mathbb{G}}$ and $L_{\mathbb{G}}$ are the constants yielded by Theorem B.6, and suppose $\Omega\subseteq\mathbb{G}$ is a finite perimeter set such that*

$$\limsup_{r\to0}d_{x,r}(|\partial\Omega|_{\mathbb{G}},\mathfrak{M})\leq\varepsilon(\vartheta_{\mathbb{G}},d)\quad\textit{for }|\partial\Omega|_{\mathbb{G}}\textit{-almost every }x\in\mathbb{G},$$

where $\varepsilon(\vartheta_G,d)$ is the constant yielded by Theorem 4.2 and d is the metric introduced in Definition 1.4. Then $|\partial\Omega|_{\mathbb{G}}$-almost all of $\mathbb{G}$ can be covered with countably many intrinsic Lipschitz surfaces.

Proof. Theorem B.6 implies that $l_{\mathbb{G}}<\Theta^{\mathfrak{Q}-1}_*(|\partial\Omega|_{\mathbb{G}},x)\leq\Theta^{\mathfrak{Q}-1,*}(|\partial\Omega|_{\mathbb{G}},x)<L_{\mathbb{G}}$ for ϕ-almost every $x\in\mathbb{G}$. Theorem 4.2 directly implies the statement. □

As mentioned at the beginning of this section, the main application of Theorem 4.1 is an extension of Preiss's rectifiability theorem to 1-codimensional measures in $\mathbb{H}^n$.

Theorem 4.4. *Suppose d is the Koranyi metric in $\mathbb{H}^n$ and ϕ is a Radon measure on $\mathbb{H}^n$ such that*

$$0 < \Theta^{2n+1}(\phi, x) := \lim_{r\to 0} \frac{\phi(B(x,r))}{r^{2n+1}} < \infty, \quad \textit{for } \phi\textit{-almost every } x \in \mathbb{H}^n. \tag{134}$$

Then ϕ is absolutely continuous with respect to $\mathcal{S}^{2n+1}$, and ϕ-almost all of $\mathbb{H}^n$ can be covered with $C^1_{\mathbb{H}^n}$-surfaces.

Proof. Thanks to Theorem 1.2 of [Merlo 2022], the almost sure existence of the limit in (134) implies that $\mathrm{Tan}(\phi, x) \subseteq \mathfrak{M}$, for ϕ-almost every $x \in \mathbb{G}$. Thanks to Theorem 4.1, this proves the claim. □

4B. *Discussion of the results.* Theorem 4.1 shows that $C^1_{\mathbb{G}}$-rectifiability in Carnot groups can be characterized by the same conditions on the densities and on the tangents as the Lipschitz rectifiability in Euclidean spaces. With this in mind we introduce the following two definitions:

Definition 4.5 ($\mathscr{P}$-rectifiable measures). Suppose that ϕ is a Radon measure on some Carnot group $\mathbb{G}$ endowed with a left-invariant and homogeneous metric d, and let m be a positive integer. We say that ϕ is $\mathscr{P}_m$-rectifiable if

(i) $0 < \Theta^m_*(\phi, x) \leq \Theta^{m,*}(\phi, x) < \infty$, for ϕ-almost every $x \in \mathbb{G}$,

(ii) $\mathrm{Tan}_m(\phi, x) \subseteq \{\lambda\mu_x : \lambda > 0\}$, for ϕ-almost every $x \in \mathbb{G}$, where μ_x is some Radon measure on $\mathbb{G}$.

Remark 4.6. It was already remarked by P. Mattila [2005] that Definition 4.5 may be considered the correct notion of rectifiability in $\mathbb{H}^1$; see the last paragraph of that work.

Remark 4.7. Instead of condition (ii) of Definition 4.5, we can assume without loss of generality that $\mu_x = \mathcal{H}^m \llcorner V(x)$ for some $V(x) \in \mathrm{Gr}(m)$, where $\mathrm{Gr}(m)$ is the family of m-dimensional homogeneous subgroups of $\mathbb{G}$ introduced in Definition 1.7. This is due to Theorem 3.2 of [Mattila 2005] and Theorem 3.6 of [Onishchik 1993]: the former result tells us that μ_x must be the Haar measure of a closed, dilation-invariant subgroup of $\mathbb{G}$ and the latter that such subgroup is actually a Lie subgroup.

Definition 4.8 ($\mathscr{P}^*$-rectifiable measures). Suppose that ϕ is a Radon measure on some Carnot group $\mathbb{G}$ endowed with a left-invariant and homogeneous metric d, and let m be a positive integer. We say that ϕ is $\mathscr{P}^*_m$-rectifiable if

(i) $0 < \Theta^m_*(\phi, x) \leq \Theta^{m,*}(\phi, x) < \infty$, for ϕ-almost every $x \in \mathbb{G}$,

(ii) $\mathrm{Tan}_m(\phi, x) \subseteq \mathfrak{M}(m)$, for ϕ-almost every $x \in \mathbb{G}$.

The difference between Definitions 4.5 and 4.8 is that in the former the tangent to ϕ is the same plane at every scale, while in the latter the tangents are planes that may vary at different scales. Although there is no a priori reason for which these definition should be equivalent in general, we see that our main result, Theorem 4.1, may be rewritten as follows:

Theorem 4.9. *Suppose ϕ is a Radon measure on $\mathbb{G}$. Then the following are equivalent*:

(i) *ϕ is $\mathscr{P}_{\mathfrak{Q}-1}$-rectifiable.*

(ii) *ϕ is $\mathscr{P}^*_{\mathfrak{Q}-1}$-rectifiable.*

(iii) *ϕ is absolutely continuous with respect to $\mathcal{H}^{\mathfrak{Q}-1}$, and ϕ-almost all of $\mathbb{G}$ can be covered with countably many $C^1_{\mathbb{G}}$-hypersurfaces.*

The notion of $\mathscr{P}$-rectifiable measures is also relevant since in different contexts it appears to imply the right notion of rectifiability. This is summarized in the following theorem, which is an immediate consequence of the Euclidean Marstrand–Mattila rectifiability criterion and Theorem 4.1:

Theorem 4.10. *The following two statements hold*:

(i) *A Radon measure ϕ on $\mathbb{R}^n$ is $\mathscr{P}_m$-rectifiable if and only if it is Euclidean m-rectifiable*;

(ii) *A Radon measure ϕ on $\mathbb{G}$ is $\mathscr{P}_{\mathfrak{Q}-1}$-rectifiable if and only if it is a 1-codimensional $C^1_{\mathbb{G}}$-rectifiable measure.*

In [Mattila et al. 2010], P. Mattila, F. Serra Cassano and R. Serapioni proved in Theorems 3.14 and 3.15 that whenever a good notion of regular surface is available in the Heisenberg group, provided the tangents are selected carefully (see Definition 2.16 of the aforementioned work), a $\mathscr{P}_m$-rectifiable measure is also rectifiable with respect to the family of regular surfaces of the right dimension. However, because of the algebraic structure of the group $\mathbb{H}^n$, there is not an a priori (known) good notion of regular surface that includes the vertical line $\mathcal{V} := \{(0, 0, t) : t \in \mathbb{R}\}$. For this reason the uniform measure $\mathcal{S}^2 \llcorner \mathcal{V}$ is considered to be *nonrectifiable* from the standpoint of [Mattila et al. 2010]. Up to this point Haar measures of not complemented homogeneous subgroups (like the vertical line $\mathcal{V}$ in $\mathbb{H}^1$) were considered nonrectifiable and thus prevented a possible extension of Preiss's theorem to low dimension even in $\mathbb{H}^1$. This was already remarked in [Chousionis and Tyson 2015]. On the other hand, we have the following theorem:

Theorem 4.11. *Let ϕ be a Radon measure on $\mathbb{H}^1$ such that for ϕ-almost every $x \in \mathbb{H}^1$ we have*

$$0 < \Theta^2(\phi, x) := \lim_{r\to 0} \frac{\phi(B(x, r))}{r^2} < \infty,$$

where $B(x, r)$ are the metric balls with respect to the Koranyi metric. Then ϕ is $\mathscr{P}_2$-rectifiable.

Proof. This follows from Proposition 2.2 of [Merlo 2022] and Theorem 1.4 of [Chousionis et al. 2020]. □

As remarked in the previous paragraph, to our knowledge, there is not a good candidate of rectifiability in Carnot groups in the literature for which the density problem may have a positive answer. On the other hand, Theorems 4.4, 4.10 and 4.11 encourage us to state the density problem in Carnot groups in the following way:

Density Problem. *Suppose ϕ is a Radon measure on the Carnot group $\mathbb{G}$. Then there exists a left-invariant distance d on $\mathbb{G}$ such that the following are equivalent*:

(i) *There exists an $\alpha > 0$ such that for ϕ-almost every $x \in \mathbb{G}$ we have*

$$0 < \Theta^\alpha(\phi, x) := \lim_{r\to 0} \frac{\phi(B(x, r))}{r^\alpha} < \infty.$$

(ii) *$\alpha \in \{0, \dots, \mathfrak{Q}\}$, and ϕ is $\mathscr{P}_\alpha$- rectifiable.*

Neither one of the implications of the formulation of the density problem has an easy solution. In [Antonelli and Merlo 2022a], the current author and G. Antonelli proved the implication (ii) $\Rightarrow$ (i) of the Density Problem when the tangent measures to ϕ are supported on complemented subgroups.

Furthermore, as already observed in [Merlo 2022], if d is a left-invariant distance coming from a polynomial norm on $\mathbb{G}$ with the same argument used in [Kirchheim and Preiss 2002] and later on in [Chousionis and Tyson 2015], it is possible to show that if (i) in the Density Problem holds, then $\alpha \in \mathbb{N}$. In $\mathbb{R}^n$ this implies, thanks to Theorem 3.1 of [Ahmadi et al. 2019], that there is an open and dense set Ω in the space of norms (with the distance induced by the Hausdorff distance of the unit balls) for which, for any $\|\cdot\| \in \Omega$, Marstrand's theorem holds.

Appendix A. Dyadic cubes

Throughout this section *we assume ϕ to be a fixed Radon measure on the Carnot group* $\mathbb{G}$, supported on a compact set K, and such that

$$0 < \liminf_{r\to 0} \frac{\phi(B(x,r))}{r^{\mathfrak{Q}-1}} \le \limsup_{r\to 0} \frac{\phi(B(x,r))}{r^{\mathfrak{Q}-1}} < \infty, \quad \text{for } \phi\text{-almost every } x \in \mathbb{G}. \tag{135}$$

There are many constructions in the literature of such dyadic cubes for Radon measures both in Euclidean and in (rather general) metric spaces; see for instance [Christ 1990]. In this section we state the existence of a family of dyadic cubes for ϕ, we list their properties and we prove a number of consequences.

Throughout this Appendix, *we will always assume that ξ and τ are two fixed natural numbers* such that $\phi(E^\phi(\xi,\tau)) > 0$, where the set $E^\phi(\xi,\tau)$ was defined in Proposition 1.28.

Definition A.1. For any subset A of $\mathbb{G}$ and any $\delta > 0$, we let

$$\partial(A,\delta) := \{u \in A : \operatorname{dist}(u, K \setminus A) \le \delta\} \cup \{u \in K \setminus A : \operatorname{dist}(u, A) \le \delta\},$$

where we recall that K is the compact set supporting the measure ϕ.

For the rest of this subsection, we simplify the expressions of the constants introduced in Notation 2.8 to

$$N := N(\xi), \quad \zeta := \zeta(\xi), \quad C_4 := C_4(\xi), \quad C_5 := C_5(\xi), \quad C_6 := C_6(\xi).$$

The construction of the dyadic cubes for the measure ϕ under the hypothesis (135) can be performed with a very similar approach to that employed for AD-regular measures in [David 1991, Appendix 1]. However, since (135) is a weaker condition than the AD-regularity, the construction needs some tweaks. For the sake of completeness we recall that a dyadic lattice for general Radon measures in the Euclidean spaces was constructed in [David and Mattila 2000, Section 3] and that such proof still follows pretty closely the argument of [David 1991, Appendix 1].

In order to adapt the construction in [David 1991], one reduces to discussing the properties of those cubes that intersect the set $E^\phi(\xi,\tau)$, where the measure ϕ behaves locally as an AD-regular measure; see items (iii) and (v) of Theorem A.2 where a uniform bound on the lower density of the measure is crucially exploited. Items (i) and (ii) hold by construction while (iv) can be seen as a fancy way of saying that since ϕ is a radon measure, almost every sphere has null measure. For a complete construction of these cubes we refer to the version of this paper that can be found in the arXiv [Merlo 2020, Subsection A.3].

Theorem A.2. *There are disjoint partitions* $\{\Delta_j^\phi(\xi,\tau)\}_{j\in\mathbb{N}}$, *usually called* ***layers***, *of* K *having the following properties*:

(i) *If* $j \le j'$, $Q\in\Delta_j^\phi(\xi,\tau)$ *and* $Q'\in\Delta_{j'}^\phi(\xi,\tau)$, *then either* Q *contains* Q' *or* $Q\cap Q'=\varnothing$.

(ii) *If* $Q\in\Delta_j^\phi(\xi,\tau)$, *we have* $\operatorname{diam}(Q)\le 2^{-Nj+5}/\tau$.

(iii) *If* $Q\in\Delta_j^\phi(\xi,\tau)$ *and* $Q\cap E^\phi(\xi,\tau)\neq\varnothing$, *then* $C_4^{-1}(2^{-Nj}/\tau)^{\mathfrak{Q}-1}\le\phi(Q)\le C_4(2^{-Nj}/\tau)^{\mathfrak{Q}-1}$.

(iv) *If* $Q\in\Delta_j^\phi(\xi,\tau)$, *we have* $\phi(\partial(Q,\zeta^2 2^{-Nj}/\tau))\le C_4\zeta(2^{-Nj}/)^{\mathfrak{Q}-1}$.

(v) *If* $Q\in\Delta_j^\phi(\xi,\tau)$ *and* $Q\cap E^\phi(\xi,\tau)\neq\varnothing$, *there exists a* $\mathfrak{c}(Q)\in Q$ *such that* $B(\mathfrak{c}(Q),\zeta^2 2^{-Nj-1}/\tau)\subseteq Q$.

We define $\Delta^\phi(\xi,\tau):=\bigcup\{Q: Q\in\Delta_j^\phi(\xi,\tau)$ *for some* $j\in\mathbb{N}\}$ *and call it the family of all dyadic cubes.*

Remark A.3. Part (iii) of Theorem A.2 can be rephrased in the following useful way. Recalling that $C_5(\xi)=C_4(32\zeta^{-2})^{\mathfrak{Q}-1}$ and putting together Theorem A.2 (ii), (iii) and (v) we infer that

(iii)′ if $Q\cap E^\phi(\xi,\tau)\neq\varnothing$, then $C_5^{-1}\operatorname{diam} Q^{\mathfrak{Q}-1}\le\phi(Q)\le C_5\operatorname{diam} Q^{\mathfrak{Q}-1}$.

The families of cubes yielded by Theorem A.2 may have the annoying property that for a fixed cube $Q\in\Delta_j^\phi(\xi,\tau)$, the only subcube of Q in the layer $\Delta_{j+1}^\phi(\xi,\tau)$ contained in Q is just Q itself. The following proposition shows that this is not much of a problem for the cubes intersecting $E^\phi(\xi,\tau)$.

Proposition A.4. *Recall that given two cubes* $Q_1, Q_2\in\Delta^\phi(\xi,\tau)$, *if* Q_2 *is the smallest cube for which* $Q_1\subsetneq Q_2$, *then* Q_2 *is said to be the* ***parent*** *of* Q_1.

Suppose $Q^*\in\Delta_j^\phi(\xi,\tau)$ *is the parent of some cube* $Q\in\Delta_{j+\kappa}^\phi(\xi,\tau)$ *such that* $Q\cap E^\phi(\xi,\tau)\neq\varnothing$. *Then*

$$\kappa<\lfloor 2\log_2 C_4/N(\mathfrak{Q}-1)\rfloor+1 \quad \textit{and} \quad \frac{\operatorname{diam} Q^*}{\operatorname{diam} Q}\le C_6.$$

Proof. Suppose $\tilde Q$ is the ancestor of the cube Q contained in the layer $\Delta_{j'}^\phi(\xi,\tau)$ for some j' for which $j'-j\ge\lfloor 2\log_2 C_4/N(\mathfrak{Q}-1)\rfloor+1$. Then $\tilde Q\cap E^\phi(\xi,\tau)\neq\varnothing$, and thanks to Theorem A.2 (i) and (iii), we infer that

$$\begin{aligned}\phi(\tilde Q\setminus Q)=\phi(\tilde Q)-\phi(Q)&\ge C_4^{-1}\left(\frac{2^{-jN}}{\tau}\right)^{\mathfrak{Q}-1}-C_4\left(\frac{2^{-j'N}}{\tau}\right)^{\mathfrak{Q}-1}\\ &=C_4^{-2}\left(\frac{2^{-jN}}{\tau}\right)^{\mathfrak{Q}-1}(1-C_4^2 2^{-(j'-j)N(\mathfrak{Q}-1)})>0,\end{aligned}\tag{136}$$

where the last inequality above comes from the choice of $j'-j$. It is immediate to see that inequality (136) implies that Q is strictly contained in $\tilde Q$. Therefore, the parent cube of Q must be contained in some $\Delta_{j'-\kappa}^\phi(\xi,\tau)$ with $0\le\kappa<\lfloor 2\log C_4/N(\mathfrak{Q}-1)\rfloor+1$. Hence, thanks to Theorem A.2 (v), we infer that

$$\begin{aligned}\operatorname{diam} Q^*\le 2^{-Nj+5}/\tau=2^{N\kappa+6}\zeta^{-2}\cdot\zeta^2 2^{-N(j+\kappa)-1}/\tau&\le 2^{N\kappa+6}\zeta^{-2}\operatorname{diam} Q\\ &\le 2^{2\log C_4/(\mathfrak{Q}-1)+N+6}\zeta^{-2}\operatorname{diam} Q=C_6\operatorname{diam} Q.\end{aligned}$$

□

The following result tells us that item (v) of Theorem A.2 in some cases can be strengthened to assuming that the center of the cube $\mathfrak{c}(Q)$ is contained in $E^\phi(\xi,\tau)$.

Proposition A.5. *Assume $\mu\in\mathbb{N}$ is such that $\mu\geq 4C_4\xi$. Then, for any cube $Q\in\Delta^\phi(\mathscr{E}^\phi_{\xi,\tau}(\mu,\nu);\xi,\tau,\nu)$, we can find a $\mathfrak{c}(Q)\in E^\phi(\xi,\tau)\cap Q$ such that*

$$B\big(\mathfrak{c}(Q),\tfrac{1}{64}\zeta^2\operatorname{diam} Q\big)\cap K\subseteq Q.$$

Remark A.6. Recall that the set $\mathscr{E}^\phi_{\xi,\tau}(\mu,\nu)$ was introduced in Proposition 1.29 and $\Delta^\phi(\kappa;\xi,\tau,\nu)$ in Notation 2.8.

Proof. In order to prove the proposition it suffices to show that

$$E^\phi(\xi,\tau)\cap Q\setminus\partial\big(Q,\tfrac{1}{32}\zeta^2\operatorname{diam} Q\big)\neq\varnothing. \tag{137}$$

In order to fix ideas, we let $j\geq\nu$ be such that $Q\in\Delta^\phi_j(\xi,\tau)$ and note that since $Q\cap E^\phi(\xi,\tau)\neq\varnothing$, thanks to Theorem A.2 (ii), (iii) and (iv), we have

$$\begin{aligned}
&\phi\big(E^\phi(\xi,\tau)\cap Q\setminus\partial\big(Q,\tfrac{1}{32}\zeta^2\operatorname{diam} Q\big)\big)\\
&\qquad\geq\ \phi(E^\phi(\xi,\tau)\cap Q)-\phi\big(\partial\big(Q,\tfrac{1}{32}\zeta^2\operatorname{diam} Q\big)\big)\overset{\text{A.2 (ii)}}{\geq}\phi(E^\phi(\xi,\tau)\cap Q)-\phi(\partial(Q,\zeta^2 2^{-jN}/\tau))\\
&\qquad\overset{\text{A.2 (iv)}}{\geq}\phi(E^\phi(\xi,\tau)\cap Q)-C_4\zeta(2^{-jN}/\tau)^{\mathfrak{Q}-1}=\phi(Q)-\phi(Q\setminus E^\phi(\xi,\tau))-C_4\zeta(2^{-jN}/\tau)^{\mathfrak{Q}-1}\\
&\qquad\overset{\text{A.2 (iii)}}{\geq}\phi(Q)-\phi(Q\setminus E^\phi(\xi,\tau))-C_4^2\zeta\phi(Q).
\end{aligned}\tag{138}$$

Since $Q\in\Delta^\phi(\mathscr{E}^\phi_{\xi,\tau}(\mu,\nu);\xi,\tau,\nu)$, we have $\operatorname{diam} Q\leq 2^{-N\nu+5}/\tau$ and there exists a $w\in\mathscr{E}^\phi_{\xi,\tau}(\mu,\nu)\cap Q$. Therefore, the definition of $\mathscr{E}^\phi_{\xi,\tau}(\mu,\nu)$ and Theorem A.2 (iii) imply that

$$\begin{aligned}
\phi(Q\setminus E^\phi(\xi,\tau))\leq\phi(B(w,2^{-jN+5}/\tau)\setminus E^\phi(\xi,\tau))&\leq\mu^{-1}\phi(B(w,2^{-jN+5}/\tau))\\
&\leq\mu^{-1}\xi(2^{-jN+5}/\tau)^{\mathfrak{Q}-1}\leq C_4\mu^{-1}\xi\phi(Q).
\end{aligned}\tag{139}$$

Putting together (138) and (139), we conclude that

$$\phi(E^\phi(\xi,\tau)\cap Q\setminus\partial(Q,\zeta^2\operatorname{diam} Q))\geq(1-C_4\mu^{-1}\xi-C_4^2\zeta)\phi(Q)\geq\tfrac14\phi(Q),$$

where the last inequality follows from the fact that $C_4^2\zeta=2^{48\mathfrak{Q}}\xi^2\cdot 2^{-50\mathfrak{Q}}\xi^{-2}\leq\frac12$ and $C_4\mu^{-1}\xi\leq\frac14$. This proves (137) and in turn the proposition. □

Appendix B. Finite perimeter sets in Carnot groups

Throughout this second appendix if not otherwise stated, *we will always endow $\mathbb{G}$ with the box metric d introduced in Definition 1.4.*

Finite perimeter sets and their blow ups. In this subsection we recall the definitions of functions of bounded variation and finite perimeter sets, and we collect from various papers some results that will be useful throughout the paper.

Definition B.1. We say that a function $f:\mathbb{G}\to\mathbb{R}$ is of *local bounded variation* if $f\in L^1_{\mathrm{loc}}(\mathbb{G})$ and

$$\|\nabla_{\mathbb{G}} f\|(\Omega):=\sup\left\{\int_\Omega f(x)\operatorname{div}_{\mathbb{G}}\varphi(x)\,dx:\varphi\in\mathcal{C}^1_0(\Omega,H\mathbb{G}),\ |\varphi(x)|\le 1\right\}<\infty,$$

for any bounded open set $\Omega\subseteq\mathbb{G}$, where $\operatorname{div}_{\mathbb{G}}\varphi:=\sum_{i=1}^{n_1}X_i\varphi_i$ and where $X_1,\dots,X_{n_1}$ are the vector fields introduced in Definition 1.33. We denote by $\mathrm{BV}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$ the set of all functions of locally bounded variation. As usual a Borel set $E\subseteq\mathbb{G}$ is said to be of *finite perimeter* if χ_E is of bounded variation.

The following result is a classical application of Riesz's representation theorem:

Theorem B.2. *If f is a function of bounded variation, then $\|\nabla_{\mathbb{G}} f\|$ is a Radon measure on $\mathbb{G}$. Moreover, there exists a $\|\nabla_{\mathbb{G}} f\|$-measurable horizontal section $\sigma_f:\mathbb{G}\to H\mathbb{G}$ such that $|\sigma_f(x)|=1$ for $\|\nabla_{\mathbb{G}} f\|$-almost every $x\in\mathbb{G}$ and for any open set Ω we have*

$$\int_\Omega f(x)\operatorname{div}_{\mathbb{G}}\varphi(x)\,dx=\int_\Omega\langle\varphi,\sigma_f\rangle\,d\|\nabla_{\mathbb{G}} f\|,\quad \text{for every } \varphi\in\mathcal{C}^1_0(\Omega,H\mathbb{G}).$$

As in the Euclidean spaces functions of bounded variation are compactly embedded in L^1.

Theorem B.3 [Franchi et al. 2003, Theorem 2.16]. *The set* $\mathrm{BV}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$ *is compactly embedded in* $L^1_{\mathrm{loc}}(\mathbb{G})$.

Definition B.4. If $E\subseteq\mathbb{G}$ is a Borel set of locally finite perimeter, we let $|\partial E|_{\mathbb{G}}:=\|\nabla_{\mathbb{G}}\chi_E\|$. Furthermore, we call the horizontal vector $\mathfrak{n}_E(x):=\sigma_{\chi_E}(x)$ the *generalized horizontal inward $\mathbb{G}$-normal to ∂E*. Finally, we define the *reduced boundary* $\partial^*_{\mathbb{G}}E$ to be the set of those $x\in\mathbb{G}$ for which

(i) $|\partial E|_{\mathbb{G}}(B(x,r))>0$ for any $r>0$,

(ii) $\lim_{r\to0}\fint_{B(x,r)}\mathfrak{n}_E\,d|\partial E|_{\mathbb{G}}$ exists,

(iii) $\lim_{r\to0}\left|\fint_{B(x,r)}\mathfrak{n}_E\,d|\partial E|_{\mathbb{G}}\right|_{\mathbb{R}^{n_1}}=1$.

The following lemma on the scaling of the perimeter will come in handy later on.

Lemma B.5. *Assume E is a set of finite perimeter in $\mathbb{G}$ and let $x\in\mathbb{G}$ and $r>0$. Then*

$$|\partial(\delta_{1/r}(x^{-1}E))|_{\mathbb{G}}=r^{-(\mathfrak{Q}-1)}T_{x,r}|\partial E|_{\mathbb{G}}.$$

Proof. For any $\varphi\in C^1_0(\mathbb{G},H\mathbb{G})$, any $x\in\mathbb{G}$ and any $r>0$, defining $\tilde\varphi(z):=\varphi(\delta_{1/r}(x^{-1}z))$, we have the identity

$$\operatorname{div}_{\mathbb{G}}\tilde\varphi(z)=r^{-1}\operatorname{div}_{\mathbb{G}}\varphi(\delta_{1/r}(x^{-1}z)).\tag{140}$$

This, indeed, is due to the fact that

$$\begin{aligned}X_j\tilde\varphi_j(z)&:=\lim_{h\to0}\frac{\tilde\varphi_j(z\delta_h(e_j))-\tilde\varphi_j(z)}{h}\\&=\lim_{h\to0}\frac{\varphi_j(\delta_{1/r}(x^{-1}z\delta_h(e_j)))-\varphi_j(\delta_{1/r}(x^{-1}z))}{h}=r^{-1}X_j\varphi_j(\delta_{1/r}(x^{-1}z)).\end{aligned}$$

Thanks to identity (140) and the fact that the Lebesgue measure is a Haar measure for $\mathbb{G}$, we infer that

$$\int\chi_{\delta_{1/r}(x^{-1}E)}(y)\operatorname{div}_{\mathbb{G}}\varphi(y)\,dy=r^{-\mathfrak{Q}}\int\chi_E\operatorname{div}_{\mathbb{G}}\varphi(\delta_{1/r}(x^{-1}y))\,dy=r^{-(\mathfrak{Q}-1)}\int\chi_E(y)\operatorname{div}_{\mathbb{G}}\tilde\varphi(y)\,dy.$$

It is not hard to see that $\varphi \in C^1_0(\Omega, H\mathbb{G})$ if and only if $\tilde\varphi \in C^1_0(x\delta_r\Omega, H\mathbb{G})$, and thus for any open set Ω we have

$$|\partial(\delta_{1/r}(x^{-1}E))|_{\mathbb{G}}(\Omega) = r^{-(\mathfrak{Q}-1)}|\partial E|_{\mathbb{G}}(x\delta_r\Omega) = r^{-(\mathfrak{Q}-1)}T_{x,r}|\partial E|_{\mathbb{G}}(\Omega). \qquad \square$$

Theorem B.6 [Ambrosio et al. 2009, Theorem 4.16]. *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. Then $|\partial E|_{\mathbb{G}}$ is asymptotically doubling, and more precisely the following holds. For $|\partial E|_{\mathbb{G}}$-almost every $x \in \mathbb{G}$ there exists an $\bar r(x) > 0$ such that*

$$l_{\mathbb{G}} r^{\mathfrak{Q}-1} \leq |\partial E|_{\mathbb{G}}(B(x,r)) \leq L_{\mathbb{G}} 2^{-(\mathfrak{Q}-1)} r^{\mathfrak{Q}-1}, \quad \text{for any } r \in (0, \bar r(x)), \tag{141}$$

where the constants $l_{\mathbb{G}}$ and $L_{\mathbb{G}}$ depend only on $\mathbb{G}$ and the metric d and $|\partial E|_{\mathbb{G}}$ is concentrated on $\partial^*_{\mathbb{G}}E$, i.e., $|\partial E|_{\mathbb{G}}(\mathbb{G} \setminus \partial^*_{\mathbb{G}}E) = 0$.

Remark B.7. Proposition 1.31 and Theorem B.6 imply that $l_{\mathbb{G}}\mathcal{S}^{\mathfrak{Q}-1}\llcorner\partial^*_{\mathbb{G}}E \leq |\partial E|_{\mathbb{G}} \leq L_{\mathbb{G}}\mathcal{S}^{\mathfrak{Q}-1}\llcorner\partial^*_{\mathbb{G}}E$. Therefore, the measures $\mathcal{S}^{\mathfrak{Q}-1}\llcorner\partial^*_{\mathbb{G}}E$ and $|\partial E|_{\mathbb{G}}$ are mutually absolutely continuous. In particular there exists a $\mathfrak{d} \in L^1(|\partial E|_{\mathbb{G}})$ such that

$$\mathcal{S}^{\mathfrak{Q}-1}\llcorner\partial^*_{\mathbb{G}}E = \mathfrak{d}|\partial E|_{\mathbb{G}},$$

and for $|\partial E|_{\mathbb{G}}$-almost every $x \in \mathbb{G}$ we have $L^{-1}_{\mathbb{G}} \leq \mathfrak{d}(x) \leq l^{-1}_{\mathbb{G}}$.

Theorem B.8 [Franchi and Serapioni 2016, Theorem 3.9]. *If $f : V \to \mathfrak{N}(V)$ is an intrinsic Lipschitz map, the epigraph of f,*

$$\operatorname{epi}(f) := \{v * \delta_t(\mathfrak{n}(V)) : t < \langle \pi_1 f(v), \mathfrak{n}(V)\rangle\}, \tag{142}$$

is a set with locally finite $\mathbb{G}$-perimeter.

Since the topological boundary of $\operatorname{epi}(f)$ coincides with $\operatorname{gr}(f)$, thanks to [Franchi and Serapioni 2016, Theorem 3.9], we infer that $|\partial \operatorname{epi}(f)|_{\mathbb{G}}(\mathbb{G}\setminus\partial^*_{\mathbb{G}} \operatorname{epi}(f)) = |\partial \operatorname{epi}(f)|_{\mathbb{G}}(\operatorname{gr}(f)\setminus\partial^*_{\mathbb{G}} \operatorname{epi}(f)) = 0$. In particular, thanks to Remark B.7, we deduce the following proposition:

Proposition B.9. $\mathcal{S}^{\mathfrak{Q}-1}(\operatorname{gr}(f) \setminus \partial^*_{\mathbb{G}} \operatorname{epi}(f)) = 0$.

It is convenient to associate a normal vector field to the graph of every intrinsic Lipschitz function $f : V \to \mathfrak{N}(V)$.

Definition B.10. For any intrinsic Lipschitz function $f : V \to \mathfrak{N}(V)$, we denote by $\mathfrak{n}_f : \partial^*_{\mathbb{G}} \operatorname{epi}(f) \to H\mathbb{G}$ the inward inner $\mathbb{G}$-normal of $\operatorname{epi}(f)$.

Tangents measures versus tangent sets to finite perimeter sets. In this subsection we connect the notion of tangent sets to finite perimeter sets, which is extensively used in the theory of finite perimeter sets, to the notion of tangent measures. This will help us to prove that if the perimeter measure associated to the boundary of a finite perimeter set has flat tangents, then it has a unique tangent that coincides with the plane in $\operatorname{Gr}(\mathfrak{Q}-1)$ orthogonal to the normal.

Definition B.11 (tangent sets). Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter and assume $x \in \partial^*_{\mathbb{G}}E$. We denote by $\operatorname{Tan}(E, x)$ the limit points in the topology of the local convergence in measure of the sets $\{\delta_{1/r}(x^{-1}E)\}_{r>0}$ as $r \to 0$.

For a proof of the following proposition, we refer to [Ambrosio et al. 2009] and in particular to Proposition 5.3.

Proposition B.12. *If E is a set of finite perimeter, for $\mathcal{S}^{\mathfrak{Q}-1}$-almost every $x \in \partial^*_{\mathbb{G}} E$ we have*

(i) $\operatorname{Tan}(E, x) \neq \varnothing$,

(ii) *the elements of* $\operatorname{Tan}(E, x)$ *are sets of locally finite perimeter sets*,

(iii) *for any $F \in \operatorname{Tan}(E, x)$, that $\mathfrak{n}_F(y) = \mathfrak{n}_E(x)$ for $|\partial F|_{\mathbb{G}}$-almost every $y \in \mathbb{G}$.*

The following proposition is a characterization of the tangent measures of perimeter measures.

Proposition B.13. *If E is a set of locally finite perimeter, for $|\partial E|_{\mathbb{G}}$-almost every $x \in \partial^*_{\mathbb{G}} E$ we have the following*:

(i) *If $\{r_i\}_{i\in\mathbb{N}}$ is an infinitesimal sequence such that $\delta_{1/r_i}(x^{-1}E)$ converges locally in measure to some Borel set L, then L is a finite perimeter set and $r_i^{-(\mathfrak{Q}-1)} T_{x,r_i} |\partial E|_{\mathbb{G}} \rightharpoonup |\partial L|_{\mathbb{G}}$. In particular, if $L \in \operatorname{Tan}(E, x)$, then $|\partial L|_{\mathbb{G}} \in \operatorname{Tan}_{\mathfrak{Q}-1}(|\partial E|_{\mathbb{G}}, x)$.*

(ii) *If $\nu \in \operatorname{Tan}_{\mathfrak{Q}-1}(|\partial E|_{\mathbb{G}}, x)$, then there is an $L \in \operatorname{Tan}(E, x)$ such that $\nu = |\partial L|_{\mathbb{G}}$.*

Proof. Let us first prove (i). From now on, thanks to Proposition B.12, we can assume without loss of generality that x is a fixed point where properties (i), (ii) and (iii) of Proposition B.12 hold. Fix now an open and bounded set Ω of $\mathbb{G}$ and note that, defining $E_i := \delta_{1/r_i}(x^{-1}E)$, we have

$$\|\chi_{E_i}\|_{L^1(\Omega)} + \|\nabla_{\mathbb{G}} \chi_{E_i}\|(\Omega) \le \mathcal{L}^n(\Omega) + r_i^{-(\mathfrak{Q}-1)} |\partial E|_{\mathbb{G}}(x\delta_{r_i}\Omega). \tag{143}$$

The above bound implies that χ_{E_i} is a compact sequence in $L^1(\Omega)$ thanks to Theorems B.3 and B.6 and thus the sets E_i converge in $L^1(\Omega)$ to some locally finite perimeter set E which must coincide $\mathcal{L}^n$-almost everywhere with L inside Ω, by the uniqueness of the limit in measure. This implies in particular that for any $\varphi \in \mathcal{C}^1_0(\Omega, H\mathbb{G})$ we have

$$\begin{aligned}\lim_{i\to 0} \int_\Omega \langle \varphi, \mathfrak{n}_{E_i}\rangle \, d|\partial E_i|_{\mathbb{G}} &= \lim_{i\to 0} \int \chi_{E_i\cap\Omega}(y) \operatorname{div}_{\mathbb{G}} \varphi(y)\, dy \\ &= \int \chi_{L\cap\Omega}(y) \operatorname{div}_{\mathbb{G}} \varphi(y)\, dy = \int_\Omega \langle \varphi, \mathfrak{n}_L\rangle \, d|\partial L|_{\mathbb{G}}.\end{aligned} \tag{144}$$

The above identity (144) implies in particular that $\mathfrak{n}_{E_i}|\partial E_i|_{\mathbb{G}} \llcorner \Omega \rightharpoonup \mathfrak{n}_L|\partial L|_{\mathbb{G}} \llcorner \Omega$. However, the arbitrariness of Ω and the well-known fact that the weak convergence implies the convergence of the total variations implies that $|\partial E_i|_{\mathbb{G}} \rightharpoonup |\partial L|_{\mathbb{G}}$. The second part of the statement of (i) follows immediately from Lemma B.5.

We now prove (ii). We can assume without loss of generality that $x = 0$ satisfy the thesis of Theorem B.6 and that $\{r_i\}$ is an infinitesimal sequence such that

$$r_i^{-(\mathfrak{Q}-1)} T_{x,r_i} |\partial E|_{\mathbb{G}} \rightharpoonup \nu \in \operatorname{Tan}_{\mathfrak{Q}-1}(|\partial E|_{\mathbb{G}}, x).$$

Now let $E_i := \delta_{1/r_i}(E)$, so that $|\partial E_i|_{\mathbb{G}} = r_i^{\mathfrak{Q}-1} T_{0,r_i} |\partial E|_{\mathbb{G}}$. For any open and bounded set Ω we can find an $R > 0$ such that $\Omega \subseteq B(0, R)$. Therefore, thanks to Theorem B.3, we have

$$|\partial(\delta_{1/r_i}(x^{-1}E))|_{\mathbb{G}}(\Omega) \le |\partial(\delta_{1/r_i}(x^{-1}E))|_{\mathbb{G}}(B(0,R)) = r_i^{-(\mathfrak{Q}-1)} T_{x,r_i} |\partial E|_{\mathbb{G}}(B(0,R)) = \frac{|\partial E|_{\mathbb{G}}(B(x, Rr_i))}{r_i^{\mathfrak{Q}-1}}.$$

Since we assumed that Theorem B.6 holds at x, we have

$$\limsup_{i\to\infty}|\partial(\delta_{1/r}(x^{-1}E))|_{\mathbb{G}}(\Omega)\le\limsup_{i\to\infty}\frac{|\partial E|_{\mathbb{G}}(B(x,Rr_i))}{r_i^{\mathfrak{Q}-1}}\le L_{\mathbb{G}}R^{\mathfrak{Q}-1}.$$

Thus, thanks to Theorem B.3, the sequence $\{\delta_{1/r_i}(x^{-1}E)\}_{i\in\mathbb{N}}$ is precompact in $L^1_{\mathrm{loc}}(\mathbb{G})$ and since we assumed $\delta_{1/r_i}(x^{-1}E)$ converges locally in measure to L, we have that $\delta_{1/r_i}(x^{-1}E)$ converges in $L^1_{\mathrm{loc}}(\mathbb{G})$ to L. In particular, thanks to Theorem 2.17 of [Franchi et al. 2003], we infer that L is of local finite perimeter. Thus, by definition of the tangent sets, we have $L\in\mathrm{Tan}(E,0)$, and thanks to item (i), we conclude that $r_i^{-(\mathfrak{Q}-1)}T_{0,r_i}|\partial E|_{\mathbb{G}}\rightharpoonup|\partial L|_{\mathbb{G}}$. Thanks to the uniqueness of the limit we conclude that $|\partial L|_{\mathbb{G}}=\nu$. □

Proposition B.14. *If E is an open set of finite perimeter in $\mathbb{G}$, for $\mathcal{S}^{\mathfrak{Q}-1}$-almost any $x\in\partial E$ and any $L\in\mathrm{Tan}(E,x)$ we have $\mathcal{L}^n(L\setminus\mathrm{int}(L))=0$. In particular, the measures $|\partial L|_{\mathbb{G}}$ and $|\partial(\mathrm{int}(L))|_{\mathbb{G}}$ coincide on Borel sets.*

Proof. This proposition follows for instance from Proposition B.12 and [Bellettini and Le Donne 2021, Theorem 1.1]. □

Remark B.15. Let $V_\pm:=\{w\in\mathbb{G}:\pm\langle\mathfrak{n}(V),w\rangle>0\}$. Thanks to (2.8) in [Ambrosio et al. 2009], it is immediate to see that $V_\pm$ are open sets of locally finite perimeter in $\mathbb{G}$ and that $\partial V_\pm=\mp\mathfrak{n}(V)\mathcal{H}^{n-1}_{\mathrm{eu}}\llcorner V$. This implies that the horizontal normal of each of the half spaces determined by V coincides, up to a sign, $|\partial V_\pm|_{\mathbb{G}}$-almost everywhere with $\mathfrak{n}(V)$.

Proposition B.16. *Let $V\in\mathrm{Gr}(\mathfrak{Q}-1)$ and $f:V\to\mathfrak{N}(V)$ be an intrinsic Lipschitz function. Suppose that E is a compact subset of $\mathrm{gr}(\gamma)$ such that*

$$\mathrm{Tan}_{\mathfrak{Q}-1}(|\partial\,\mathrm{epi}(f)|_{\mathbb{G}},x)\subseteq\mathfrak{M},\quad \textit{for } |\partial\,\mathrm{epi}(f)|_{\mathbb{G}}\textit{-almost every } x\in E.$$

Then for $|\partial\,\mathrm{epi}(f)|_{\mathbb{G}}$-almost every $x\in E$, we have

$$\mathrm{Tan}_{\mathfrak{Q}-1}(|\partial\,\mathrm{epi}(f)|_{\mathbb{G}},x)\subseteq\{\lambda\mathcal{S}^{\mathfrak{Q}-1}\llcorner V(x):\lambda\in[L_{\mathbb{G}}^{-1},l_{\mathbb{G}}^{-1}]\},$$

where $V(x)\in\mathrm{Gr}(\mathfrak{Q}-1)$ is the plane orthogonal to $\mathfrak{n}_f(x)$, which is the normal to $\mathrm{gr}(f)$ introduced in Definition B.10, and where the constants $l_{\mathbb{G}}$ and $L_{\mathbb{G}}$ were introduced in Theorem B.6.

Proof. Proposition B.13, the asymptotic AD-regularity of the perimeter and Lebesgue's differentiation theorem at [Heinonen et al. 2015, page 77] imply that for $\mathcal{S}^{\mathfrak{Q}-1}$-almost every $x\in\partial^*_{\mathbb{G}}\,\mathrm{epi}(f)\cap E$ and for every $L\in\mathrm{Tan}(\mathrm{epi}(f),x)$ we have

$$|\partial L|_{\mathbb{G}}=\lambda\mathcal{S}^{\mathfrak{Q}-1}\llcorner V_{L,x}\quad\text{for some }\ V_{L,x}\in\mathrm{Gr}(\mathfrak{Q}-1)\ \text{ and }\ \lambda>0.\tag{145}$$

Furthermore Remark B.7, Proposition 1.8 and a simple computation that we omit, imply that $\lambda\in[l_{\mathbb{G}},L_{\mathbb{G}}]$.

Fix now an $x\in\partial^*\,\mathrm{epi}(f)\cap E$ at which (145) holds and that satisfies the thesis of Proposition B.12, and let $L\in\mathrm{Tan}(\mathrm{epi}(f),x)$. Thanks to these choices, L is a finite perimeter set with constant horizontal normal and Proposition B.9 and (145) tell us that its topological boundary must coincide up to $\mathcal{S}^{\mathfrak{Q}-1}$-null sets with the plane $V_{L,x}$. Therefore, since by Proposition B.14 we can assume without loss of generality that L is an open set, we conclude that L must coincide with one of the two half-spaces determined by $V_{L,x}$. This implies however, thanks to Remark B.15, that

$$\mathfrak{n}(V_{L,x})=\mathfrak{n}_L(y) \quad \text{for } \mathcal{S}^{\mathfrak{Q}-1}\text{-almost every } y\in\partial L. \tag{146}$$

Furthermore, Proposition B.12 (iii) and (146) imply that $\mathfrak{n}(V_{L,x})=\mathfrak{n}_L(y)=\mathfrak{n}_f(x)$ for $\mathcal{S}^{\mathfrak{Q}-1}$-almost all $y\in\partial L$. This shows however that for $\mathcal{S}^{\mathfrak{Q}-1}$-almost all $x\in \mathrm{gr}(f)\cap E$, every element of $\mathrm{Tan}(\mathrm{epi}(f),x)$ is a half-space whose boundary is the plane orthogonal to $\mathfrak{n}_f(x)$ and Proposition B.13 concludes the proof. □

Proposition B.17. *Suppose E is a compact subset of V and let $\gamma: E\subseteq V\to\mathfrak{N}(V)$ be an intrinsic Lipschitz function such that for $\mathcal{S}^{\mathfrak{Q}-1}$-almost every $x\in E$ there exists a plane $V_\gamma(x)\in \mathrm{Gr}(\mathfrak{Q}-1)$ for which*

$$\lim_{r\to 0}\frac{\mathcal{S}^{\mathfrak{Q}-1}(\mathrm{gr}(\gamma)\cap B(x\gamma(x),r)\setminus x\gamma(x)X_{V_\gamma(x\gamma(x))}(\alpha))}{r^{\mathfrak{Q}-1}}=0 \tag{147}$$

whenever $\alpha>0$, and where $X_{V_\gamma(x\gamma(x))}(\alpha):=\{w\in\mathbb{G}:\mathrm{dist}(w,V(x\gamma(x)))\le\alpha\|w\|\}$. Then $\mathrm{gr}(\gamma)$ can be covered with countably many $C^1_{\mathbb{G}}$-surfaces.

Proof. Since the graph map $x\mapsto x*\gamma(x)$ is continuous, let us notice that the set $\mathrm{gr}(\gamma)$ is compact and for any $i\in\mathbb{N}$ let us define the sets

$$A_i:=\{x\in\mathrm{gr}(\gamma): \text{(147) holds at } x \text{ and } \mathcal{S}^{\mathcal{Q}-1}(B(x,r)\cap\mathrm{gr}(\gamma))\ge 2^{-1}L_{\mathbb{G}}^{-1}l_{\mathbb{G}}r^{\mathfrak{Q}-1} \text{ for any } 0<r<1/i\}.$$

As a first step in the proof, we show that the A_i are $\mathcal{S}^{\mathcal{Q}-1}\llcorner\mathrm{gr}(\gamma)$-measurable. It is immediate to see that if we show that the set

$$\tilde{A}_i:=\{x\in\mathrm{gr}(\gamma):\mathcal{S}^{\mathcal{Q}-1}(B(x,r)\cap\mathrm{gr}(\gamma))\ge 2^{-1}L_{\mathbb{G}}^{-1}l_{\mathbb{G}}r^{\mathfrak{Q}-1} \text{ for any } 0<r<1/i\}$$

is closed, the measurability of A_i immediately follows since (147) holds on a set of full $\mathcal{S}^{\mathcal{Q}-1}\llcorner\mathrm{gr}(\gamma)$-measure. Since $\mathrm{gr}(\gamma)$ is closed, to prove the closedness of $\tilde{A}_i$ it is sufficient to show that if a sequence $\{x_j\}_{j\in\mathbb{N}}\subseteq\tilde{A}_i$ converges to some $x\in\mathrm{gr}(\gamma)$, then $x\in\tilde{A}_i$. So, let $0<r<1/i$ and note that if $d(x,x_j)<r$ we have

$$2^{-1}L_{\mathbb{G}}^{-1}l_{\mathbb{G}}(r-d(x,x_j))^{\mathfrak{Q}-1}\le\mathcal{S}^{\mathcal{Q}-1}\llcorner\mathrm{gr}(\gamma)(B(x_j,r-d(x,x_j)))\le\mathcal{S}^{\mathcal{Q}-1}\llcorner\mathrm{gr}(\gamma)(B(x,r)).$$

The arbitrariness of j implies that for any $0<r<1/i$ we have $\mathcal{S}^{\mathcal{Q}-1}\llcorner\mathrm{gr}(\gamma)(B(x,r))\ge 2^{-1}L_{\mathbb{G}}^{-1}l_{\mathbb{G}}r^{\mathfrak{Q}-1}$, proving that $x\in\tilde{A}_i$.

We now prove that the sets A_i cover $\mathcal{S}^{\mathcal{Q}-1}$-almost all $\mathrm{gr}(\gamma)$. Thanks to Theorem 1.38 we can extend γ to an intrinsic Lipschitz function $\tilde{\gamma}: V\to\mathfrak{N}(V)$. Recall now that $\mathrm{gr}(\tilde{\gamma})$ is the boundary of the set of locally finite perimeter $\mathrm{epi}(\tilde{\gamma})$. Thanks to Theorem B.6, this implies that for $|\partial\,\mathrm{epi}(\tilde{\gamma})|_{\mathbb{G}}$-almost every $x\in\mathbb{G}$ there exists a $\bar{r}(x)>0$ such that for any $0<r<\bar{r}(x)$ we have

$$L_{\mathbb{G}}\mathcal{S}^{\mathcal{Q}-1}\llcorner\mathrm{gr}(\tilde{\gamma})(B(x,r))\ge|\partial\,\mathrm{epi}(\tilde{\gamma})|_{\mathbb{G}}(B(x,r))\ge l_{\mathbb{G}}r^{\mathfrak{Q}-1},$$

where the first inequality above comes from Remark B.7. In addition, thanks to [Franchi and Serapioni 2016, Theorem 3.9], [Heinonen et al. 2015, Theorem 3.4.3] and to the Lebesgue differentiation theorem that can be found in [Heinonen et al. 2015, page 77], we deduce that

$$\Theta_*^{\mathfrak{Q}-1}(\mathcal{S}^{\mathfrak{Q}-1}\llcorner \mathrm{gr}(\gamma), x) = \Theta_*^{\mathfrak{Q}-1}(\chi_{\mathrm{gr}(\gamma)}\mathcal{S}^{\mathfrak{Q}-1}\llcorner \mathrm{gr}(\tilde{\gamma}), x) = \Theta_*^{\mathfrak{Q}-1}(\mathcal{S}^{\mathfrak{Q}-1}\llcorner \mathrm{gr}(\tilde{\gamma}), x) \geq L_{\mathbb{G}}^{-1} l_{\mathbb{G}}, \tag{148}$$

for $\mathcal{S}^{\mathfrak{Q}-1}\llcorner \mathrm{gr}(\gamma)$-almost every $x \in \mathbb{G}$. From (148), we infer that for $\mathcal{S}^{\mathfrak{Q}-1}\llcorner \mathrm{gr}(\gamma)$-almost every $x \in \mathbb{G}$ there exists an $r(x) > 0$ such that $\mathcal{S}^{\mathfrak{Q}-1}(B(x,r) \cap \mathrm{gr}(\gamma)) \geq 2^{-1} L_{\mathbb{G}}^{-1} l_{\mathbb{G}} r^{\mathfrak{Q}-1}$ for any $0 < r < r(x)$. Therefore, if $r(x) > 1/i$ and (147) holds at x, then $x \in A_i$ and this concludes the proof of the fact that $\mathcal{S}^{\mathfrak{Q}-1}\big(\mathrm{gr}(\gamma) \setminus \bigcup_{i\in\mathbb{N}} A_i\big) = 0$.

For any $i, j \in \mathbb{N}$ and any $x \in A_i$, we let

$$\rho_{i,j}(x) := \sup\left\{ \frac{|\langle \mathfrak{n}_\gamma(x), \pi_1(x^{-1}y)\rangle|}{d(x,y)} : y \in A_i \text{ and } 0 < d(x,y) < 1/j \right\}.$$

We remark that the functions $\rho_{i,j}$ are measurable for any $i, j \in \mathbb{N}$. Indeed, on the one hand the function $(x,y) \mapsto |\langle \mathfrak{n}_\gamma(x), \pi_1(x^{-1}y)\rangle|/d(x,y)$ is $\mathcal{S}^{\mathfrak{Q}-1}\llcorner \mathrm{gr}(\gamma)$-measurable since it is the quotient of two $\mathcal{S}^{\mathfrak{Q}-1}\llcorner \mathrm{gr}(\gamma)$-measurable functions. On the other, since $\mathbb{G}$ is separable, it is immediate to see that $\rho_{i,j}$ can be rewritten as the supremum on y over a countable subset of $B(x,\delta) \cap A_i$ showing that $\rho_{i,j}$ is indeed measurable. We want to prove that for any $i \in \mathbb{N}$ and any $x \in A_i$ we have

$$\lim_{j\to\infty} \rho_{i,j}(x) = 0. \tag{149}$$

Assume by contradiction this is not the case and that there exists an $i \in \mathbb{N}$ and a $z \in A_i$ for which (149) fails. Then there is a $0 < \mathfrak{c} \leq 1$ and an increasing sequence of natural numbers $\{j_k\}_{k\in\mathbb{N}}$ such that for any $k \in \mathbb{N}$ there is a $y_k \in A_i$ for which $y_k \in B(z, 1/j_k)$ and $|\langle \mathfrak{n}_\gamma(z), \pi_1(z^{-1}y_k)\rangle| > \mathfrak{c}d(z, y_k)$. Thanks to Proposition 1.15, we infer that $y_i \notin zX_{V_\gamma(z)}\big(\frac{1}{2}\mathfrak{c}\big)$; indeed,

$$\mathrm{dist}(V_\gamma(z), z^{-1}y_k) = |\langle \mathfrak{n}_\gamma(z), \pi_1(z^{-1}y_k)\rangle| > \mathfrak{c}d(z, y_k). \tag{150}$$

We now claim that for any $k \in \mathbb{N}$ we have

$$B\big(y_k, \tfrac{1}{4}\mathfrak{c}d(z, y_k)\big) \subseteq B(z, 2d(z, y_k)) \setminus zX_{V_\gamma(z)}\big(\tfrac{1}{4}\mathfrak{c}\big). \tag{151}$$

In order to prove the inclusion (151) we fix a $k \in \mathbb{N}$ and let $w := y_k v$ for some $v \in B\big(y_k, \frac{1}{8}\mathfrak{c}d(z, y_k)\big)$. With these choices Proposition 1.15 and the triangle inequality imply that

$$\begin{aligned}\mathrm{dist}(V_\gamma(z), z^{-1}w) = |\langle \mathfrak{n}_\gamma(z), \pi_1(z^{-1}w)\rangle| &\geq |\langle \mathfrak{n}_\gamma(z), \pi_1(z^{-1}y_k)\rangle| - |\langle \mathfrak{n}_\gamma(z), \pi_1(y_k^{-1}w)\rangle| \\ &\geq \mathfrak{c}d(z, y_k) - d(y_k, w) \geq \mathfrak{c}d(z, w) - (1+\mathfrak{c})d(y_k, w).\end{aligned} \tag{152}$$

Furthermore, thanks to the choice of w we have

$$d(y_k, w) = \|v\| \leq \tfrac{1}{4}\mathfrak{c}d(z, y_k) \leq \tfrac{1}{4}\mathfrak{c}d(z, w) + \tfrac{1}{4}\mathfrak{c}d(y_k, w), \tag{153}$$

$$d(z, w) \leq d(z, y_k) + d(y_k, w) \leq d(z, y_k) + \|v\| \leq \big(1 + \tfrac{1}{8}\mathfrak{c}\big)d(z, y_k) \leq 2d(z, y_k). \tag{154}$$

From (152) we infer in particular that $(4/\mathfrak{c} - 1)d(y_k, w) \leq d(z, w)$. This implies in particular that

$$\mathrm{dist}(V_\gamma(z), z^{-1}w) \overset{(152)}{\geq} \mathfrak{c}d(z, w) - (1+\mathfrak{c})d(y_k, w) \geq \mathfrak{c}d(z, w) - \frac{1+\mathfrak{c}}{4/\mathfrak{c} - 1}d(z, w) \geq \tfrac{1}{4}\mathfrak{c}d(z, w), \tag{155}$$

where the last inequality comes from the fact that $\mathfrak{c} \leq 1$. The inclusion (151) follows immediately from the above bound and (154). Therefore, (151) implies that

$$\begin{aligned}\limsup_{r\to 0} &\frac{\mathcal{S}^{\mathfrak{Q}-1}(\mathrm{gr}(\gamma)\cap B(z,r)\setminus zX_{V_\gamma(z)}(\mathfrak{c}/8))}{r^{\mathfrak{Q}-1}}\\ &\geq \lim_{k\to\infty} \frac{\mathcal{S}^{\mathfrak{Q}-1}(\mathrm{gr}(\gamma)\cap B(z,2d(z,y_k))\setminus zX_{V_\gamma(z)}(\mathfrak{c}/8))}{(2d(z,y_k))^{\mathfrak{Q}-1}}\\ &\geq \lim_{k\to\infty} \frac{\mathcal{S}^{\mathfrak{Q}-1}(\mathrm{gr}(\gamma)\cap B(y_k,\mathfrak{c}d(z,y_k)/8))}{2^{\mathfrak{Q}-1}d(z,y_k)^{\mathfrak{Q}-1}} \geq \lim_{k\to\infty} \frac{L_{\mathbb{G}}^{-1} l_{\mathbb{G}}(\mathfrak{c}d(z,y_k)/8)^{\mathfrak{Q}-1}}{2^{\mathfrak{Q}}d(z,y_k)} = \frac{l_{\mathbb{G}}}{2L_{\mathbb{G}}}\left(\frac{\mathfrak{c}}{16}\right)^{\mathfrak{Q}-1}, \end{aligned} \tag{156}$$

where the second last inequality comes from the fact that $y_k \in A_i$ for any k and that $\frac{1}{8}\mathfrak{c}d(z,y_k) < 1/i$ definitely. However, since by construction (147) holds at any point of A_i, (156) is in contradiction with (147) and thus (149) must hold at any $x \in A_i$. Define f_i to be the function identically 0 on A_i and for any $\iota \in \mathbb{N}$ we let $K_i(\iota)$ be a compact subset of A_i for which

(i) $\mathcal{S}^{\mathfrak{Q}-1}(A_i \setminus K_i(\iota)) \leq 1/\iota$,

(ii) $\mathfrak{n}_\gamma$ is continuous on $K_i(\iota)$,

(iii) $\rho_{i,j}$ converges uniformly to 0 on $K_i(\iota)$.

The existence of $K_i(\iota)$ is implied by Lusin's theorem and Severini–Egoroff's theorem. Thanks to Whitney's extension theorem, see for instance Theorem 5.2 in [Franchi et al. 2003], we infer that we can find a $C^1_{\mathbb{G}}$-function such that $f_{i,\iota}|_K = 0$ and $\nabla_{\mathbb{H}} f_{i,\iota}(x) = \mathfrak{n}_\gamma(x)$ for any $x \in K_i(\iota)$. This implies that A_i, and thus $\mathrm{gr}(\gamma)$, can be covered $\mathcal{S}^{\mathfrak{Q}-1}$-almost all with $C^1_{\mathbb{G}}$-surfaces. □

Acknowledgements

I am deeply grateful to Roberto Monti, without whose guidance and constant encouragement I would have never been able to tackle the density problem in the Carnot setting. I would like to also thank Joan Verdera, who invited me to Barcelona during February of 2019 allowing me to have many fruitful conversations with him and with Xavier Tolsa. I would also like to thank Gioacchino Antonelli, Giacomo Del Nin, Carmelo Puliatti and Davide Vittone for many valuable suggestions and corrections to the early versions of this work. Finally, I am deeply grateful to the anonymous referees whose careful reading found a number of typos and inaccuracies and helped me to significantly improve the exposition.

References

[Ahmadi et al. 2019] A. A. Ahmadi, E. de Klerk, and G. Hall, "Polynomial norms", *SIAM J. Optim.* **29**:1 (2019), 399–422. MR Zbl

[Ambrosio et al. 2009] L. Ambrosio, B. Kleiner, and E. Le Donne, "Rectifiability of sets of finite perimeter in Carnot groups: existence of a tangent hyperplane", *J. Geom. Anal.* **19**:3 (2009), 509–540. MR Zbl

[Antonelli and Merlo 2022a] G. Antonelli and A. Merlo, "On rectifiable measures in Carnot groups: existence of density", *J. Geom. Anal.* **32**:9 (2022), art. id. 239. MR Zbl

[Antonelli and Merlo 2022b] G. Antonelli and A. Merlo, "On rectifiable measures in Carnot groups: Marstrand–Mattila rectifiability criterion", *J. Funct. Anal.* **283**:1 (2022), art. id. 109495. MR Zbl

[Balogh et al. 2012] Z. M. Balogh, K. Fässler, P. Mattila, and J. T. Tyson, "Projection and slicing theorems in Heisenberg groups", *Adv. Math.* **231**:2 (2012), 569–604. MR Zbl

[Balogh et al. 2013] Z. M. Balogh, E. Durand-Cartagena, K. Fässler, P. Mattila, and J. T. Tyson, "The effect of projections on dimension in the Heisenberg group", *Rev. Mat. Iberoam.* **29**:2 (2013), 381–432. MR Zbl

[Bellettini and Le Donne 2021] C. Bellettini and E. Le Donne, "Sets with constant normal in Carnot groups: properties and examples", *Comment. Math. Helv.* **96**:1 (2021), 149–198. MR Zbl

[Bonfiglioli et al. 2007] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer, 2007. MR Zbl

[Chousionis and Tyson 2015] V. Chousionis and J. T. Tyson, "Marstrand's density theorem in the Heisenberg group", *Bull. Lond. Math. Soc.* **47**:5 (2015), 771–788. MR Zbl

[Chousionis et al. 2019] V. Chousionis, K. Fässler, and T. Orponen, "Intrinsic Lipschitz graphs and vertical β-numbers in the Heisenberg group", *Amer. J. Math.* **141**:4 (2019), 1087–1147. MR Zbl

[Chousionis et al. 2020] V. Chousionis, V. Magnani, and J. T. Tyson, "On uniform measures in the Heisenberg group", *Adv. Math.* **363** (2020), art. id. 106980. MR Zbl

[Christ 1990] M. Christ, "A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral", *Colloq. Math.* **60**:2 (1990), 601–628. MR Zbl

[David 1991] G. David, *Wavelets and singular integrals on curves and surfaces*, Lecture Notes in Math. **1465**, Springer, 1991. MR Zbl

[David and Mattila 2000] G. David and P. Mattila, "Removable sets for Lipschitz harmonic functions in the plane", *Rev. Mat. Iberoam.* **16**:1 (2000), 137–215. MR Zbl

[David and Semmes 1993a] G. David and S. Semmes, *Analysis of and on uniformly rectifiable sets*, Math. Surv. Monogr. **38**, Amer. Math. Soc., Providence, RI, 1993. MR Zbl

[David and Semmes 1993b] G. David and S. Semmes, "Quantitative rectifiability and Lipschitz mappings", *Trans. Amer. Math. Soc.* **337**:2 (1993), 855–889. MR Zbl

[De Lellis 2008] C. De Lellis, *Rectifiable sets, densities and tangent measures*, Eur. Math. Soc., Zürich, 2008. MR Zbl

[Federer 1969] H. Federer, *Geometric measure theory*, Grundl. Math. Wissen. **153**, Springer, 1969. MR Zbl

[Franchi and Serapioni 2016] B. Franchi and R. P. Serapioni, "Intrinsic Lipschitz graphs within Carnot groups", *J. Geom. Anal.* **26**:3 (2016), 1946–1994. MR Zbl

[Franchi et al. 2001] B. Franchi, R. Serapioni, and F. Serra Cassano, "Rectifiability and perimeter in the Heisenberg group", *Math. Ann.* **321**:3 (2001), 479–531. MR Zbl

[Franchi et al. 2003] B. Franchi, R. Serapioni, and F. Serra Cassano, "On the structure of finite perimeter sets in step 2 Carnot groups", *J. Geom. Anal.* **13**:3 (2003), 421–466. MR Zbl

[Franchi et al. 2015] B. Franchi, R. P. Serapioni, and F. Serra Cassano, "Area formula for centered Hausdorff measures in metric spaces", *Nonlinear Anal.* **126** (2015), 218–233. MR Zbl

[Heinonen et al. 2015] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. T. Tyson, *Sobolev spaces on metric measure spaces: an approach based on upper gradients*, New Math. Monogr. **27**, Cambridge Univ. Press, 2015. MR Zbl

[Kirchheim and Preiss 2002] B. Kirchheim and D. Preiss, "Uniformly distributed measures in Euclidean spaces", *Math. Scand.* **90**:1 (2002), 152–160. MR Zbl

[Kirchheim and Serra Cassano 2004] B. Kirchheim and F. Serra Cassano, "Rectifiability and parameterization of intrinsic regular surfaces in the Heisenberg group", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **3**:4 (2004), 871–896. MR Zbl

[Le Donne and Moisala 2021] E. Le Donne and T. Moisala, "Semigenerated Carnot algebras and applications to sub-Riemannian perimeter", *Math. Z.* **299**:3-4 (2021), 2257–2285. MR Zbl

[Magnani 2004] V. Magnani, "Unrectifiability and rigidity in stratified groups", *Arch. Math.* (*Basel*) **83**:6 (2004), 568–576. MR Zbl

[Magnani 2017] V. Magnani, "A new differentiation, shape of the unit ball, and perimeter measure", *Indiana Univ. Math. J.* **66**:1 (2017), 183–204. MR Zbl

[Marchi 2014] M. Marchi, "Regularity of sets with constant intrinsic normal in a class of Carnot groups", *Ann. Inst. Fourier (Grenoble)* **64**:2 (2014), 429–455. MR Zbl

[Marstrand 1961] J. M. Marstrand, "Hausdorff two-dimensional measure in 3-space", *Proc. Lond. Math. Soc.* (3) **11** (1961), 91–108. MR Zbl

[Mattila 1975] P. Mattila, "Hausdorff m regular and rectifiable sets in n-space", *Trans. Amer. Math. Soc.* **205** (1975), 263–274. MR Zbl

[Mattila 1995] P. Mattila, *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*, Cambridge Stud. Adv. Math. **44**, Cambridge Univ. Press, 1995. MR Zbl

[Mattila 2005] P. Mattila, "Measures with unique tangent measures in metric groups", *Math. Scand.* **97**:2 (2005), 298–308. MR Zbl

[Mattila et al. 2010] P. Mattila, R. Serapioni, and F. Serra Cassano, "Characterizations of intrinsic rectifiability in Heisenberg groups", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **9**:4 (2010), 687–723. MR Zbl

[Merlo 2020] A. Merlo, "Marstrand–Mattila rectifiability criterion for 1-codimensional measures in Carnot groups", preprint, 2020. Zbl arXiv 2007.03236

[Merlo 2022] A. Merlo, "Geometry of 1-codimensional measures in Heisenberg groups", *Invent. Math.* **227**:1 (2022), 27–148. MR Zbl

[Onishchik 1993] A. L. Onishchik (editor), *Lie groups and Lie algebras, I*, Encycl. Math. Sci. **20**, Springer, 1993. MR Zbl

[Preiss 1987] D. Preiss, "Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities", *Ann. of Math.* (2) **125**:3 (1987), 537–643. MR Zbl

[Serra Cassano 2016] F. Serra Cassano, "Some topics of geometric measure theory in Carnot groups", pp. 1–121 in *Geometry, analysis and dynamics on sub-Riemannian manifolds*, *I* (Marseille, 2014), edited by D. Barilari et al., Eur. Math. Soc., Zürich, 2016. MR Zbl

[Vittone 2012] D. Vittone, "Lipschitz surfaces, perimeter and trace theorems for BV functions in Carnot–Carathéodory spaces", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **11**:4 (2012), 939–998. MR Zbl

[Vittone et al. 2022] A. Julia, S. Nicolussi Golo, and D. Vittone, "Area of intrinsic graphs and coarea formula in Carnot groups", *Math. Z.* **301**:2 (2022), 1369–1406. MR Zbl

Received 16 Dec 2020. Revised 14 Sep 2021. Accepted 28 Oct 2021.

ANDREA MERLO: andrea.merlo@ehu.eus
Departamento de Matemáticas, Universidad del País Vasco, Leioa, Spain

Analysis & PDE
msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2023 is US $405/year for the electronic version, and $630/year (+$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 16 No. 4 2023