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THE "GOOD" BOUSSINESQ EQUATION: LONG-TIME ASYMPTOTICS

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We consider the initial-value problem for the "good" Boussinesq equation on the line. Using inverse scattering techniques, the solution can be expressed in terms of the solution of a 3×3-matrix Riemann–Hilbert problem. We establish formulas for the long-time asymptotics of the solution by performing a Deift–Zhou steepest descent analysis of a regularized version of this Riemann–Hilbert problem. Our results are valid for generic solitonless Schwartz class solutions whose space-average remains bounded as $t \to \infty$.

1. Introduction

When investigating the bidirectional propagation of small amplitude and long wavelength capillary-gravity waves on the surface of shallow water, J. Boussinesq [1872] derived the classical Boussinesq equation

$$\eta_{tt} - gh_0\eta_{xx} = gh_0\left(\frac{3}{2}\frac{\eta^2}{h_0} + \frac{h_0^2}{3}\eta_{xx}\right)_{xx}, \tag{1-1}$$

where $\eta(x,t)$ is the perturbation-free surface, h_0 is the mean depth, and g is the gravitational constant. This equation was later rediscovered by Keulegan and Patterson [1940]. In nondimensional units, (1-1) can be written as

$$u_{tt} - u_{xx} - (u^2)_{xx} - u_{xxxx} = 0, \tag{1-2}$$

where $u(x,t)$ is a real-valued function and subscripts denote partial derivatives. Equation (1-2) is often referred to as the "bad" Boussinesq equation in contrast to the so-called "good" Boussinesq equation

$$u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx} = 0, \tag{1-3}$$

in which the u_{tt} and u_{xxxx} terms have the same sign, thus making the equation linearly well-posed; see, e.g., [Bona and Sachs 1988; Compaan and Tzirakis 2017; Farah 2009; Himonas and Mantzavinos 2015; Linares 1993] for well-posedness results for (1-3). Equation (1-3) governs small nonlinear oscillations in an elastic beam and is also known as the "nonlinear string equation" [Falkovich et al. 1983].

Deift and Zhou [1993] proposed a steepest descent method for the asymptotic analysis of Riemann–Hilbert (RH) problems. The Deift–Zhou approach has been successfully utilized to determine long-time asymptotics for a large number of integrable equations such as the modified KdV equation [Deift and Zhou 1993], the KdV equation [Deift et al. 1994], the nonlinear Schrödinger equation [Jenkins and McLaughlin 2014; Tovbis et al. 2004], the sine-Gordon equation [Cheng et al. 1999], the Camassa–Holm equation [Boutet de Monvel et al. 2009], the Degasperis–Procesi equation [Boutet de Monvel et al. 2019],

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and the Toda lattice [Deift et al. 1996]. At his 60th birthday conference in 2005, P. Deift [2008] presented a list of sixteen open problems, among which he pointed out that "The long-time behavior of the solutions of the Boussinesq equation with general initial data is a very interesting problem with many challenges." The purpose of this paper is to take a first step towards the solution of this problem.

As in [McKean 1981; Deift et al. 1982], we consider the following version of the "good" Boussinesq equation:

$$u_{tt} + \tfrac{4}{3}(u^2)_{xx} + \tfrac{1}{3}u_{xxxx} = 0, \tag{1-4}$$

which can be obtained from (1-3) by a simple shift $u \to u + \frac{1}{2}$ followed by a trivial rescaling. Our main result provides explicit formulas for the long-time asymptotics of the solution $u(x,t)$ of (1-4) in a sector in the right half-plane $\{x > 0,\ t > 0\}$ under the assumption that the initial data lie in the Schwartz class and satisfy the physically natural assumption that $u_t(x,0)$ has zero mean. The proof is based on a Deift–Zhou steepest descent analysis of a 3×3-matrix RH problem, which is parametrized by x and t. This RH problem was derived in [Charlier and Lenells 2022] by performing a spectral analysis of a Lax pair associated to (1-4); it is formulated in the complex plane of the spectral parameter k and has a jump contour consisting of the three lines $\mathbb{R} \cup \omega\mathbb{R} \cup \omega^2\mathbb{R}$, where $\omega = e^{2\pi i/3}$. The steepest descent analysis of this RH problem is severely complicated by the fact that the associated spectral problem is third-order. In fact, even though a version of the inverse scattering formalism was developed for the Boussinesq equation already in [Deift et al. 1982], the results presented here are, to the best of our knowledge, the first asymptotic results for any of the equations (1-2)–(1-4) obtained via steepest descent techniques (there exists a substantial amount of work on the long-time asymptotics for Boussinesq equations based on functional analytic approaches, see, e.g., [Farah 2008; Liu 1997; Linares and Scialom 1995; Wang 2009], but these approaches yield asymptotic information of a much less precise type). In addition to the third-order spectral problem, another complication in the analysis of (1-4) stems from the fact that the associated RH problem is singular at the origin. Therefore, instead of performing the steepest descent analysis of this RH problem directly, we will analyze a regularized version of the RH problem and then transfer the results to the singular problem.

The paper is organized as follows. The main result is stated in Section 2. An overview of the rather involved proof, which also contains a statement of the relevant RH problem, is presented in Section 3. The steepest descent analysis begins in Section 4, where several transformations of the RH problem are implemented. Local parametrices at the three critical points are constructed in Section 5 and the resulting small-norm RH problem is estimated in Section 6. Finally, the asymptotic behavior of $u(x,t)$ is obtained in Section 7.

2. Main result

Equation (1-4) can be rewritten as the system [Zakharov 1974]

$$\begin{cases} w_t + \tfrac{1}{3}u_{xxx} + \tfrac{4}{3}(u^2)_x = 0, \\ u_t = w_x, \end{cases} \tag{2-1}$$

which is equivalent to (1-4) provided that $u_1(x) := u_t(x,0)$ satisfies

$$\int_{\mathbb{R}} u_1(x)\, dx = 0. \tag{2-2}$$

The assumption (2-2) ensures that the integral $\int_{\mathbb{R}} u\,dx$ does not grow linearly but is conserved in time. Indeed, letting $u_0(x) := u(x,0)$ and assuming that u has sufficient smoothness and decay, (1-4) implies

$$\frac{d^2}{dt^2}\int_{\mathbb{R}} u\,dx = 0, \quad \text{i.e.,} \quad \int_{\mathbb{R}} u\,dx = \Big(\int_{\mathbb{R}} u_1\,dx\Big)t + \int_{\mathbb{R}} u_0\,dx.$$

Therefore, instead of analyzing (1-4) with initial data $u(x,0)$ and $u_t(x,0)$ directly, we will consider the system (2-1) with initial data $u_0(x) = u(x,0)$ and $w_0(x) = w(x,0)$.

2A. ***Definition of $s(k)$ and $s^A(k)$.*** The formulation of our main result involves two spectral functions $s(k)$ and $s^A(k)$ which are defined as follows (see [Charlier and Lenells 2022] for details). Suppose $u_0(x)$ and $w_0(x)$ are real-valued functions in $\mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz class of rapidly decaying functions on the real line. Let $\omega := e^{2\pi i/3}$ and let, for $j = 1,2,3$, $l_j(k) = \omega^j k$. Define $\mathsf{U}(x,k)$ by

$$\mathsf{U}(x,k) = P(k)^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -w_0(x) - u_{0x} & -2u_0(x) & 0 \end{pmatrix} P(k), \tag{2-3}$$

where

$$P(k) = \begin{pmatrix} \omega & \omega^2 & 1 \\ \omega^2 k & \omega k & k \\ k^2 & k^2 & k^2 \end{pmatrix}. \tag{2-4}$$

Let $X(x,k)$ and $X^A(x,k)$ be the 3×3-matrix-valued eigenfunctions defined by the linear Volterra integral equations

$$X(x,k) = I - \int_x^{\infty} e^{(x-x')\widehat{\mathcal{L}(k)}}(\mathsf{U}X)(x',k)\,dx', \tag{2-5a}$$

$$X^A(x,k) = I + \int_x^{\infty} e^{-(x-x')\widehat{\mathcal{L}(k)}}(\mathsf{U}^T X^A)(x',k)\,dx', \tag{2-5b}$$

where $\mathcal{L} = \operatorname{diag}(l_1, l_2, l_3)$, $\widehat{\mathcal{L}}$ denotes the operator which acts on a 3×3 matrix A by $\widehat{\mathcal{L}}A = [\mathcal{L}, A]$ (i.e., $e^{\widehat{\mathcal{L}}}A = e^{\mathcal{L}}Ae^{-\mathcal{L}}$), and U^T denotes the transpose of U. The 3×3-matrix-valued functions $s(k)$ and $s^A(k)$ are defined by

$$s(k) = I - \int_{\mathbb{R}} e^{-x\widehat{\mathcal{L}(k)}}(\mathsf{U}X)(x,k)\,dx, \tag{2-6}$$

$$s^A(k) = I + \int_{\mathbb{R}} e^{x\widehat{\mathcal{L}(k)}}(\mathsf{U}^T X^A)(x,k)\,dx. \tag{2-7}$$

2B. ***Statement of the main result.*** We first state our main result for the system (2-1); the formulation for (1-4) is given as a corollary. For simplicity, we only consider solutions in the Schwartz class, but it will be clear from the text that our result and its proof only require a finite degree of smoothness and decay.

Definition 2.1. We call $\{u(x,t), w(x,t)\}$ a *Schwartz class solution of* (2-1) *with initial data* $u_0, w_0 \in \mathcal{S}(\mathbb{R})$ if

(i) u, w are smooth real-valued functions of $(x,t) \in \mathbb{R}\times[0,\infty)$,

(ii) u, w satisfy (2-1) for $(x,t) \in \mathbb{R}\times[0,\infty)$ and

$$u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \quad x \in \mathbb{R}.$$

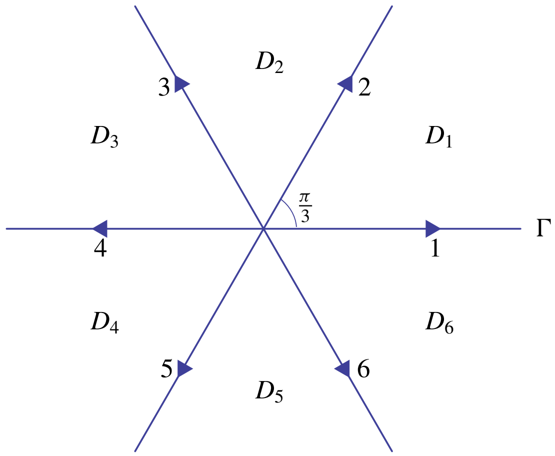

Figure 1. The contour Γ and the open sets D_n, $n = 1, \dots, 6$, which decompose the complex k-plane.

(iii) u, w have rapid decay as $|x| \to \infty$ in the sense that, for each integer $N \geq 1$ and each $T > 0$,

$$\sup_{\substack{x\in\mathbb{R}\\ t\in[0,T)}} \sum_{i=0}^{N} (1+|x|)^N (|\partial_x^i u| + |\partial_x^i w|) < \infty.$$

Let $\{D_n\}_{n=1}^6$ denote the sectors shown in Figure 1. We make the following two assumptions.

Assumption 2.2 (absence of solitons). Assume that $(s(k))_{11}$ and $(s^A(k))_{11}$ are nonzero for $k \in \bar{D}_1 \setminus \{0\}$ and $k \in \bar{D}_4 \setminus \{0\}$, respectively.

Assumption 2.3 (generic behavior at $k = 0$). Assume that

$$\lim_{k\to 0} k^2 (s(k))_{11} \neq 0, \qquad \lim_{k\to 0} k^2 (s^A(k))_{11} \neq 0.$$

Assumption 2.2 ensures that no solitons are present (the case when s_{11} and s_{11}^A have a finite number of simple poles off the contour can be treated by standard methods; see, e.g., [Fokas and Its 1996], or [Lenells 2012] for a 3×3 matrix case). Assumption 2.3 ensures that s_{11} and s_{11}^A have double poles at $k = 0$, which is the case for generic initial data [Charlier and Lenells 2022].

Define the reflection coefficient $r_1(k)$ by

$$r_1(k) = \frac{(s(k))_{12}}{(s(k))_{11}}, \qquad k \in (0, \infty). \tag{2-8}$$

If $u_0, w_0 \in \mathcal{S}(\mathbb{R})$ are such that Assumptions 2.2 and 2.3 hold, then $r_1(k)$ extends to a smooth function of $k \in [0, \infty)$ with rapid decay as $k \to \infty$ which satisfies $r_1(0) = \omega$ and $|r_1(k)| < 1$ for $k > 0$; see [Charlier and Lenells 2022].

We can now state our main result, which establishes the long-time behavior of $u(x, t)$ in the asymptotic sector $x/t > 0$; see Figures 2 and 3.

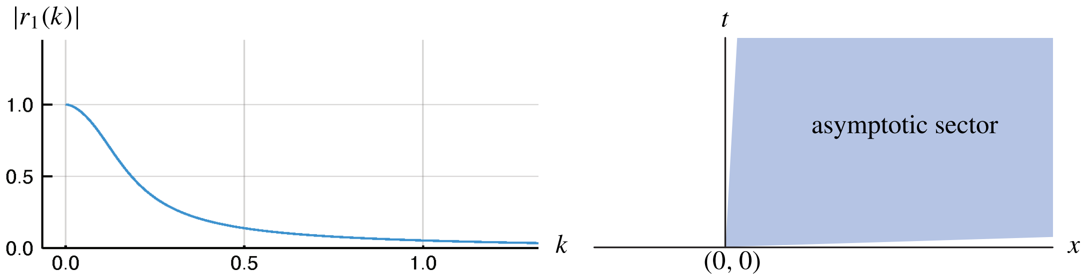

Figure 2. Numerical example showing $|r_1(k)|$ as a function of $k \geq 0$ (left) and an example of an asymptotic sector in the (x, t)-plane where the formula of Theorem 2.4 applies (right).

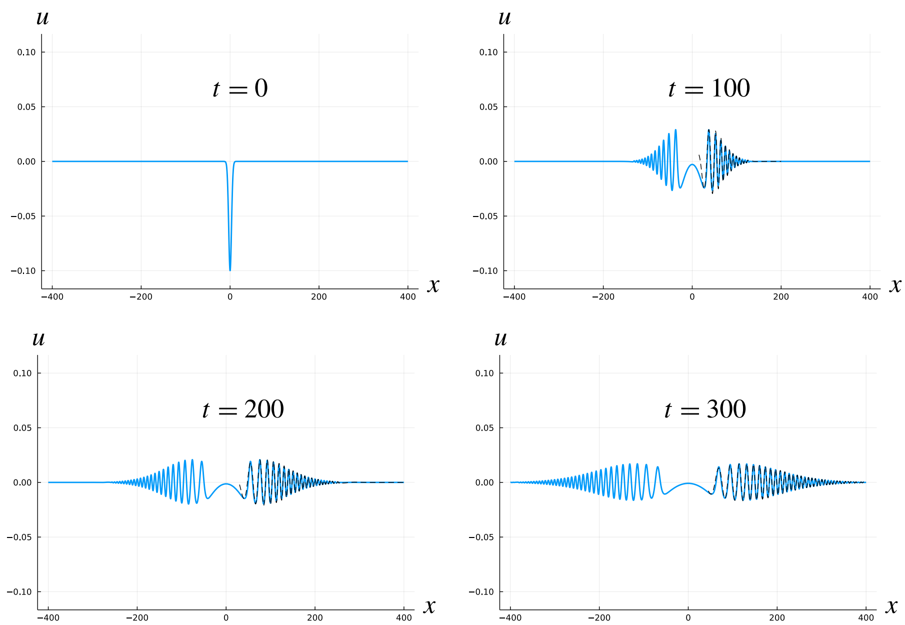

Figure 3. Numerical simulation of the solution $u(x, t)$ of (1-4) with initial data $u(x, 0) = -\frac{1}{10}e^{-x^2/20}$ and $u_t(x, 0) = 0$ (solid blue) together with the asymptotic approximation in (2-9) (dashed black) at the times $t = 0$, $t = 100$, $t = 200$, and $t = 300$. As expected, the asymptotic formula provides a better and better approximation as t increases. The convergence is the slowest for small values of x, which is consistent with the fact that the asymptotic estimate (2-9) is not uniform near $x = 0$.

Theorem 2.4 (long-time asymptotics for (2-1)). *Suppose* $\{u(x, t), w(x, t)\}$ *is a Schwartz class solution of* (2-1) *with initial data* $u_0, w_0 \in \mathcal{S}(\mathbb{R})$ *such that Assumptions 2.2 and 2.3 hold. Then the following asymptotic formula holds uniformly for* $\zeta = x/t$ *in compact subsets of* $(0, \infty)$ *as* $t \to \infty$:

$$u(x,t) = -\frac{3^{5/4}k_0\sqrt{\nu}}{\sqrt{2t}} \sin\biggl(\frac{19\pi}{12} + \nu \ln(6\sqrt{3}tk_0^2) - \sqrt{3}k_0^2 t - \arg r_1(k_0) - \arg \Gamma(i\nu) + \frac{1}{\pi}\int_{k_0}^{\infty} \ln\left|\frac{s-k_0}{s-\omega k_0}\right| d\ln(1-|r_1(s)|^2)\biggr) + O\left(\frac{\ln t}{t}\right), \quad (2\text{-}9)$$

where Γ *denotes the Gamma function*, $k_0 \equiv k_0(\zeta) = \zeta/2$, *and* $\nu \equiv \nu(\zeta) \geq 0$ *is defined by*

$$\nu = -\frac{1}{2\pi} \ln(1 - |r_1(k_0)|^2).$$

The proof of Theorem 2.4 is presented in Sections 3–7; Section 3 contains an overview of the proof.

As a corollary, we obtain asymptotics of the solution of (1-4) with initial data $u_0(x) = u(x, 0)$ and $u_1(x) = u_t(x, 0)$.

Corollary 2.5 (long-time asymptotics for (1-4)). *Suppose* $u(x, t)$ *is a Schwartz class solution of the "good" Boussinesq equation* (1-4) *with initial data* $u_0, u_1 \in \mathcal{S}(\mathbb{R})$ *such that* $\int_{\mathbb{R}} u_1 \, dx = 0$. *Let* $w_0(x) = \int_{-\infty}^{x} u_1(x') \, dx'$ *and define* $r_1 : (0, \infty) \to \mathbb{C}$ *by* (2-8). *Suppose Assumptions 2.2 and 2.3 hold. Then* u *obeys the asymptotic formula* (2-9) *as* $t \to \infty$ *uniformly for* $\zeta = x/t$ *in compact subsets of* $(0, \infty)$.

Remark 2.6 (asymptotics in the left half-plane). In Theorem 2.4, we have, for conciseness, only presented asymptotics of $u(x, t)$ in a subsector of the right-half plane $x > 0$. A similar formula can be derived by the same methods for a subsector of the left half-plane, except that the formulas there involve $r_2 := s_{12}^A / s_{11}^A$ instead of r_1. Alternatively, asymptotics in the left half-plane can be obtained directly from Theorem 2.4 and the invariance of the Boussinesq equation under space inversion.

Remark 2.7 (asymptotics of w). Theorem 2.4 provides a formula for the asymptotics of u. Our methods can be used to derive an analogous asymptotic formula for w, but since this requires somewhat lengthy estimates of t-derivatives (see (3-6)), we have decided to not include this.

Remark 2.8 (regularity and decay assumptions). The Schwartz class assumption in Theorem 2.4 can be relaxed significantly. In fact, even our current proofs only require a finite degree of smoothness and decay. In light of the developments for integrable equations with second-order spectral problems, we expect that significant further improvements can be obtained by considering solutions in weighted Sobolev spaces. Consider for example the nonlinear Schrödinger equation: In [Deift and Zhou 2003], asymptotic formulas for the solution of the Cauchy problem were established under essentially minimal assumptions on the initial data, and more recently, the error terms in these formulas have been sharpened to become in a certain sense optimal by using the $\bar{\partial}$ generalization of the nonlinear steepest descent method [Borghese et al. 2018; Dieng et al. 2019]. It is an interesting research direction to investigate the regularity and decay requirements necessary for the derivation of asymptotic formulas for integrable equations with higher-order spectral problems. It seems clear that the $\bar{\partial}$ steepest descent method can be effectively applied also in this context. However, the construction of the direct and inverse scattering transforms in the framework of weighted Sobolev spaces is likely to be more involved for spectral problems of at least third order than for second-order problems.

2C. ***Notation.*** We summarize some notation that will be used throughout the paper. In what follows, $\gamma \subset \mathbb{C}$ denotes an oriented (piecewise smooth) contour.

- If A is an $n \times m$ matrix, then $|A| \geq 0$ is defined by $|A|^2 = \sum_{i,j} |A_{ij}|^2$. Note that $|A + B| \leq |A| + |B|$ and $|AB| \leq |A||B|$.
- c and C will denote generic positive constants which may change within a computation.

- We write $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_- = (-\infty, 0)$.
- For $1 \le p \le \infty$, we write $A \in L^p(\gamma)$ if $|A|$ belongs to $L^p(\gamma)$. Then $A \in L^p(\gamma)$ if and only if each entry A_{ij} belongs to $L^p(\gamma)$. We write $\|A\|_{L^p(\gamma)} := \| |A| \|_{L^p(\gamma)}$.
- We define $\dot{L}^3(\gamma)$ as the space of all functions $f : \gamma \to \mathbb{C}$ such that $(1+|k|)^{1/3} f(k) \in L^3(\gamma)$. If γ is bounded, $\dot{L}^3(\gamma) = L^3(\gamma)$, but in general it only holds that $\dot{L}^3(\gamma) \subset L^3(\gamma)$. We turn $\dot{L}^3(\gamma)$ into a Banach space with the norm $\|f\|_{\dot{L}^3(\gamma)} := \|(1+|k|)^{1/3} f\|_{L^3(\gamma)}$.
- We let $\dot{E}^3(\mathbb{C} \setminus \gamma)$ denote the space of all analytic functions $f : \mathbb{C} \setminus \gamma \to \mathbb{C}$ with the property that for each component D of $\mathbb{C} \setminus \gamma$ there exist curves $\{C_n\}_1^\infty$ in D such that the C_n eventually surround each compact subset of D and

$$\sup_{n \ge 1} \int_{C_n} (1+|k|)\, |f(k)|^3\, |dk| < \infty.$$

- For a function f defined in $\mathbb{C} \setminus \gamma$, we let $f_\pm$ denote the nontangential boundary values of f from the left and right sides of γ, respectively, whenever they exist. If $f \in \dot{E}^3(\mathbb{C} \setminus \gamma)$, then $f_\pm$ exist a.e. on γ and $f_\pm \in \dot{L}^3(\gamma)$; see [Lenells 2018, Theorem 4.1].

3. Overview of the proof

The proof of Theorem 2.4 consists of a Deift–Zhou steepest descent analysis of a 3×3 matrix RH problem. The jump contour Γ of this RH problem consists of the three lines $\mathbb{R} \cup \omega\mathbb{R} \cup \omega^2\mathbb{R}$, see Figure 1, and the jump matrix v is given explicitly in terms of $r_1(k)$ defined in (2-8) and the function $r_2(k)$ defined by

$$r_2(k) = \frac{(s^A(k))_{12}}{(s^A(k))_{11}}, \quad k \in (-\infty, 0). \tag{3-1}$$

More precisely, v is defined as follows. Define $\{l_j(k), z_j(k)\}_{j=1}^3$ by

$$l_j(k) = \omega^j k, \quad z_j(k) = \omega^{2j} k^2, \quad k \in \mathbb{C}, \tag{3-2}$$

and define the complex-valued functions $\Phi_{ij}(\zeta, k)$ for $1 \le i \ne j \le 3$ by

$$\Phi_{ij}(\zeta, k) = (l_i - l_j)\zeta + (z_i - z_j),$$

where $\zeta := x/t$. By symmetry, it is enough to consider Φ_{21}, Φ_{31}, and Φ_{32}, which are explicitly given by

$$\Phi_{21}(\zeta, k) = \omega(\omega - 1)k(\zeta - k),$$
$$\Phi_{31}(\zeta, k) = (1 - \omega)k(\zeta - \omega^2 k),$$
$$\Phi_{32}(\zeta, k) = (1 - \omega^2)k(\zeta - \omega k).$$

Given a function $f(k)$ of $k \in \mathbb{C}$, we let f^* denote the Schwartz conjugate of f, i.e.,

$$f^*(k) = \overline{f(\bar{k})}.$$

The jump matrix $v(x,t,k)$ is defined for $k\in\Gamma$ by

$$v_1=\begin{pmatrix}1 & -r_1(k)e^{-t\Phi_{21}} & 0\\ r_1^*(k)e^{t\Phi_{21}} & 1-|r_1(k)|^2 & 0\\ 0 & 0 & 1\end{pmatrix},\qquad v_2=\begin{pmatrix}1 & 0 & 0\\ 0 & 1-|r_2(\omega k)|^2 & -r_2^*(\omega k)e^{-t\Phi_{32}}\\ 0 & r_2(\omega k)e^{t\Phi_{32}} & 1\end{pmatrix},$$
$$v_3=\begin{pmatrix}1-|r_1(\omega^2k)|^2 & 0 & r_1^*(\omega^2k)e^{-t\Phi_{31}}\\ 0 & 1 & 0\\ -r_1(\omega^2k)e^{t\Phi_{31}} & 0 & 1\end{pmatrix},\quad v_4=\begin{pmatrix}1-|r_2(k)|^2 & -r_2^*(k)e^{-t\Phi_{21}} & 0\\ r_2(k)e^{t\Phi_{21}} & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\tag{3-3}$$
$$v_5=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -r_1(\omega k)e^{-t\Phi_{32}}\\ 0 & r_1^*(\omega k)e^{t\Phi_{32}} & 1-|r_1(\omega k)|^2\end{pmatrix},\qquad v_6=\begin{pmatrix}1 & 0 & r_2(\omega^2k)e^{-t\Phi_{31}}\\ 0 & 1 & 0\\ -r_2^*(\omega^2k)e^{t\Phi_{31}} & 0 & 1-|r_2(\omega^2k)|^2\end{pmatrix},$$

where v_j denotes the restriction of v to the subcontour of Γ labeled by j in Figure 1. We consider the following RH problem, which is formulated in the L^3-setting to ensure uniqueness (the solution of an $n\times n$-matrix L^p-RH problem is unique whenever it exists provided that $1\le n\le p$; see [Lenells 2018, Theorem 5.6]).

RH problem 3.1 (L^3-RH problem for m). *Find a* 3×3-*matrix-valued function* $m(x,t,\cdot)\in I+\dot{E}^3(\mathbb{C}\setminus\Gamma)$ *such that* $m_+(x,t,k)=m_-(x,t,k)v(x,t,k)$ *for a.e.* $k\in\Gamma$.

By introducing the row-vector-valued function n by

$$n(x,t,k)=(\omega\ \ \omega^2\ \ 1)m(x,t,k),\tag{3-4}$$

we can transform the RH problem for m into the following vector RH problem for n.

RH problem 3.2 (L^3-RH problem for n). *Find a* 1×3-*row-vector-valued function* $n(x,t,\cdot)\in(\omega,\omega^2,1)+\dot{E}^3(\mathbb{C}\setminus\Gamma)$ *such that* $n_+(x,t,k)=n_-(x,t,k)v(x,t,k)$ *for a.e.* $k\in\Gamma$.

For technical reasons, we also need the classical version of this RH problem.

RH problem 3.3 (classical RH problem for n). *Find a* 1×3-*row-vector-valued function* $n(x,t,k)$ *with the following properties*:

(i) $n(x,t,\cdot):\mathbb{C}\setminus\Gamma\to\mathbb{C}^{1\times 3}$ *is analytic.*

(ii) *The limits of* $n(x,t,k)$ *as* k *approaches* $\Gamma\setminus\{0\}$ *from the left and right exist, are continuous on* $\Gamma\setminus\{0\}$, *and are denoted by* n_+ *and* n_-, *respectively. Furthermore, they are related by*

$$n_+(x,t,k)=n_-(x,t,k)v(x,t,k),\qquad k\in\Gamma\setminus\{0\}.\tag{3-5}$$

(iii) $n(x,t,k)=(\omega,\omega^2,1)+O(k^{-1})$ *as* $k\to\infty$.

(iv) $n(x,t,k)=O(1)$ *as* $k\to 0$.

The following result was proved in [Charlier and Lenells 2022].

Proposition 3.4. *Suppose the assumptions of Theorem 2.4 hold. Let* U *be an open subset of* $\mathbb{R}\times[0,\infty)$ *and suppose for each* $(x,t)\in U$ *that the solution of the classical RH problem 3.3 for* n *is unique whenever* *it exists. Then RH problem 3.3 has a unique solution $n(x,t,k)$ for each $(x,t) \in U$ and the solution $\{u(x,t), w(x,t)\}$ of* (2-1) *can be expressed in terms of $n = (n_1, n_2, n_3)$ by*

$$\begin{cases} u(x,t) = -\dfrac{3}{2}\dfrac{\partial}{\partial x} \lim_{k\to\infty} k(n_3(x,t,k)-1), \\ w(x,t) = -\dfrac{3}{2}\dfrac{\partial}{\partial t} \lim_{k\to\infty} k(n_3(x,t,k)-1), \end{cases} \qquad (x,t) \in U. \tag{3-6}$$

To use Proposition 3.4, we need the following lemma.

Lemma 3.5. *Suppose RH problem 3.1 has a solution $m(x,t,\cdot)$ at some point $(x,t) \in \mathbb{R} \times [0,\infty)$. Then $n = (\omega, \omega^2, 1)m$ is the unique solution of RH problem 3.2 at (x,t). Moreover, if the solution of RH problem 3.3 exists, then it is unique and is given by $n = (\omega, \omega^2, 1)m$.*

Proof. The assertion for $n = (\omega, \omega^2, 1)m$ follows as in [Boutet de Monvel et al. 2019, Lemma A.5]. The last claim follows because every solution of RH problem 3.3 is also a solution of RH problem 3.2. □

It will follow from the steepest descent analysis (see Lemma 6.3) that there exists a $T > 0$ such that RH problem 3.1 has a unique solution m for $t \geq T$ and x/t in a compact subset of $(0,\infty)$. Thus Proposition 3.4 and Lemma 3.5 imply that the formulas (3-6) for u, w are valid for all $t \geq T$ and x/t in compact subsets of $(0,\infty)$ if n is defined by $n = (\omega, \omega^2, 1)m$. Therefore it is enough to determine the large t asymptotics of m.

3A. *Steepest descent analysis.* The large t behavior of m can be obtained by performing a Deift–Zhou steepest descent analysis of RH problem 3.1. The first step in this analysis is to define analytic approximations of the functions r_1 and r_2 appearing in the jump matrix v, as well as of the combination $r_1/(1-|r_1|^2)$. Once these approximations are in place, we can deform the contour in such a way that the new jump is close to the identity matrix everywhere except near three critical points (see Section 4). The critical points are the solutions of the stationary phase equations $\partial\Phi_{21}/\partial k = 0$, $\partial\Phi_{31}/\partial k = 0$, and $\partial\Phi_{32}/\partial k = 0$. For each choice of $1 \leq j < i \leq 3$, $\partial\Phi_{ij}/\partial k = 0$ has a single zero k_{ij} given by

$$k_{21} = \frac{\zeta}{2}, \quad k_{31} = \frac{\omega\zeta}{2}, \quad k_{32} = \frac{\omega^2\zeta}{2}.$$

Writing $k_0 \equiv k_{21}$, these three critical points can be expressed as k_0, ωk_0, and $\omega^2 k_0$; see Figure 4. The signature tables for Φ_{21}, Φ_{31}, and Φ_{32} are shown in Figures 5–7.

Near each of the three critical points, the RH problem can be approximated by a local parametrix which is constructed in Section 5. In fact, since the jump matrix v obeys the symmetries

$$v(x,t,k) = \mathcal{A}v(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}\overline{v(x,t,\bar{k})}^{-1}\mathcal{B}, \quad k \in \Gamma, \tag{3-7}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3-8}$$

the solution m obeys the symmetries

$$m(x,t,k) = \mathcal{A}m(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}\overline{m(x,t,\bar{k})}\mathcal{B}, \quad k \in \mathbb{C} \setminus \Gamma. \tag{3-9}$$

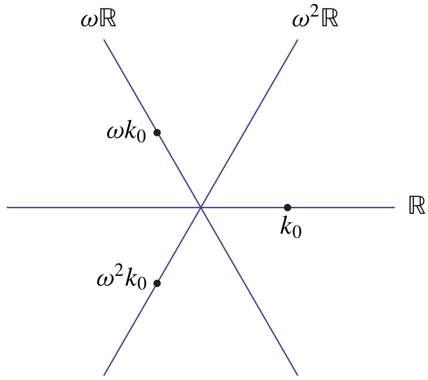

Figure 4. The three critical points k_0, ωk_0, $\omega^2 k_0$ in the complex k-plane for $\zeta > 0$.

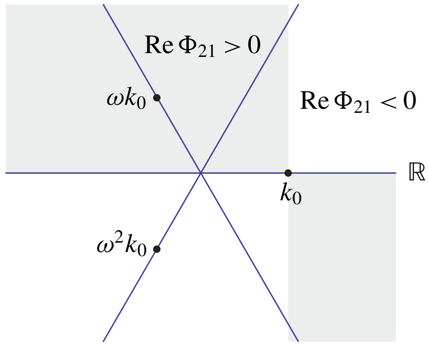

Figure 5. The regions where $\mathrm{Re}\,\Phi_{21} > 0$ (shaded) and $\mathrm{Re}\,\Phi_{21} < 0$ (white).

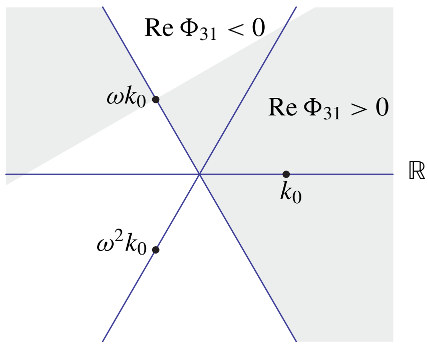

Figure 6. The regions where $\mathrm{Re}\,\Phi_{31} > 0$ (shaded) and $\mathrm{Re}\,\Phi_{31} < 0$ (white).

It is therefore sufficient to construct the local parametrix m^{k_0} at k_0, because then the local parametrices at ωk_0 and $\omega^2 k_0$ can be obtained by symmetry. In the end, we arrive at a small-norm RH problem whose solution is estimated in Section 6. Finally, the asymptotics of $u(x,t)$ is obtained in Section 7.

3B. ***Assumptions for the remainder of the paper.*** From here on, we assume that $\{u(x,t), w(x,t)\}$ is a Schwartz class solution of (2-1) with initial data $u_0, w_0 \in \mathcal{S}(\mathbb{R})$ such that Assumptions 2.2 and 2.3 hold.

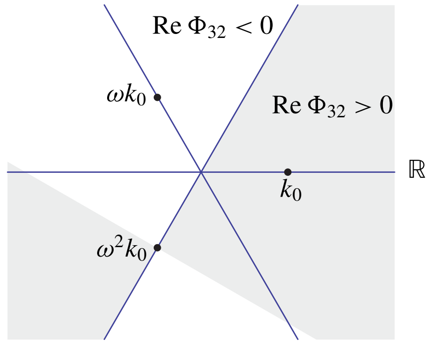

Figure 7. The regions where Re $\Phi_{32} > 0$ (shaded) and Re $\Phi_{32} < 0$ (white).

We also assume that $r_1(k)$ and $r_2(k)$ are defined by (2-8) and (3-1). We let $\mathcal{I}$ denote a fixed compact subset of $(0, \infty)$.

4. Transformations of the RH problem

By performing a number of transformations, we can bring the RH problem 3.1 to a form suitable for determining the long-time asymptotics. More precisely, starting with m, we will define functions $m^{(j)}(x, t, k)$, $j = 1, 2, 3$, such that the RH problem satisfied by $m^{(j)}$ is equivalent to the original RH problem 3.1. The RH problem for $m^{(j)}$ can be formulated as follows, where the contours $\Gamma^{(j)}$ and the jump matrices $v^{(j)}$ are specified below.

RH problem 4.1 (RH problem for $m^{(j)}$). *Find a* 3×3*-matrix-valued function* $m^{(j)}(x, t, \cdot) \in I + \dot{E}^3(\mathbb{C} \setminus \Gamma^{(j)})$ *such that* $m_+^{(j)}(x, t, k) = m_-^{(j)}(x, t, k) v^{(j)}(x, t, k)$ *for a.e.* $k \in \Gamma^{(j)}$.

The jump matrix $v^{(3)}$ obtained after the third transformation has the property that it approaches the identity matrix as $t \to \infty$ everywhere on the contour except near the three critical points k_0, ωk_0, $\omega^2 k_0$. This means that we can find the long-time asymptotics of $m^{(3)}$ by computing the contribution from three small crosses centered at these points.

The symmetries (3-7) and (3-9) will be preserved at each stage of the transformations, so that, for $j = 1, 2, 3$,

$$v^{(j)}(x, t, k) = \mathcal{A} v^{(j)}(x, t, \omega k) \mathcal{A}^{-1} = \mathcal{B} \overline{v^{(j)}(x, t, \bar{k})}^{-1} \mathcal{B}, \quad k \in \Gamma^{(j)}, \tag{4-1}$$

$$m^{(j)}(x, t, k) = \mathcal{A} m^{(j)}(x, t, \omega k) \mathcal{A}^{-1} = \mathcal{B} \overline{m^{(j)}(x, t, \bar{k})} \mathcal{B}, \quad k \in \mathbb{C} \setminus \Gamma^{(j)}. \tag{4-2}$$

4A. ***First transformation.*** The purpose of the first transformation is to remove (except for a small remainder) the jumps across the subcontours $e^{\pi i/3}\mathbb{R}_+$, $\mathbb{R}_-$, and $e^{-\pi i/3}\mathbb{R}_+$ of Γ. To implement this transformation, we need analytic approximations of the functions r_2^*, r_1, and $\hat{r}_1^*$, where $\hat{r}_1(k)$ is defined by

$$\hat{r}_1(k) = \frac{r_1(k)}{1 - r_1(k) r_1^*(k)}.$$

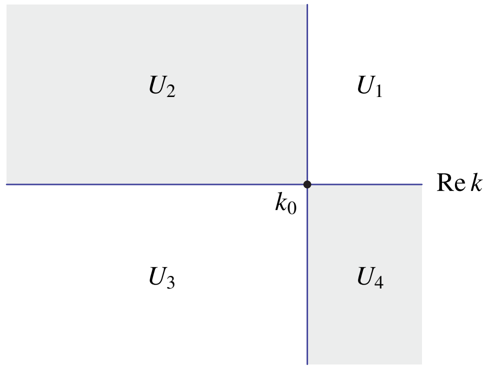

Figure 8. The open sets $\{U_j\}_1^4$ in the complex k-plane.

We introduce open sets $U_j = U_j(\zeta) \subset \mathbb{C}$, $j = 1, \dots, 4$, as in Figure 8, such that

$$U_1 \cup U_3 = \{k \mid \mathrm{Re}\, \Phi_{21}(\zeta, k) < 0\}, \quad U_2 \cup U_4 = \{k \mid \mathrm{Re}\, \Phi_{21}(\zeta, k) > 0\}.$$

Lemma 4.2. *There exist decompositions*

$$\begin{aligned} r_2^*(k) &= r_{2,a}^*(x,t,k) + r_{2,r}^*(x,t,k), \quad k \in (-\infty, 0], \\ r_1(k) &= r_{1,a}(x,t,k) + r_{1,r}(x,t,k), \quad k \in [0, k_0], \\ \hat{r}_1^*(k) &= \hat{r}_{1,a}^*(x,t,k) + \hat{r}_{1,r}^*(x,t,k), \quad k \in [k_0, \infty), \end{aligned} \tag{4-3}$$

where the functions $r_{2,a}^*$, $r_{2,r}^*$, $r_{1,a}$, $r_{1,r}$, $\hat{r}_{1,a}^*$, $\hat{r}_{1,r}^*$ have the following properties:

(a) *For each $\zeta \in \mathcal{I}$ and each $t > 0$, $r_{2,a}^*(x,t,k)$ and $r_{1,a}(x,t,k)$ are defined and continuous for $k \in \bar{U}_2$ and analytic for $k \in U_2$, and $\hat{r}_{1,a}^*(x,t,k)$ is defined and continuous for $k \in \bar{U}_1$ and analytic for $k \in U_1$.*

(b) *For each $\zeta \in \mathcal{I}$ and $t > 0$, the functions $r_{2,a}^*$, $r_{1,a}$, and $\hat{r}_{1,a}^*$ satisfy*

$$|r_{2,a}^*(x,t,k)| \le \frac{C|k - \omega k_0|}{1 + |k|^2} e^{(t/4)|\mathrm{Re}\, \Phi_{21}(\zeta,k)|}, \quad k \in \bar{U}_2, \tag{4-4a}$$

$$|\partial_x^l (r_{2,a}^*(x,t,k) - r_2^*(0))| \le C|k| e^{(t/4)|\mathrm{Re}\, \Phi_{21}(\zeta,k)|}, \quad k \in \bar{U}_2, \tag{4-4b}$$

$$|\partial_x^l (r_{1,a}(x,t,k) - r_1(0))| \le C|k| e^{(t/4)|\mathrm{Re}\, \Phi_{21}(\zeta,k)|}, \quad k \in \bar{U}_2, \tag{4-4c}$$

$$|\partial_x^l (r_{1,a}(x,t,k) - r_1(k_0))| \le C|k - k_0| e^{(t/4)|\mathrm{Re}\, \Phi_{21}(\zeta,k)|}, \quad k \in \bar{U}_2, \tag{4-4d}$$

$$|\partial_x^l (\hat{r}_{1,a}^*(x,t,k) - \hat{r}_1^*(k_0))| \le C|k - k_0| e^{(t/4)|\mathrm{Re}\, \Phi_{21}(\zeta,k)|}, \quad k \in \bar{U}_1, \tag{4-4e}$$

$$|\partial_x^l \hat{r}_{1,a}^*(x,t,k)| \le \frac{C}{1 + |k|} e^{(t/4)|\mathrm{Re}\, \Phi_{21}(\zeta,k)|}, \quad k \in \bar{U}_1, \tag{4-4f}$$

where $l = 0, 1$ and the constant C is independent of ζ, t, k.

(c) *For each $1 \le p \le \infty$ and $l = 0, 1$,*

$$\textit{the } L^p\textit{-norm of } (1 + |\cdot|)\partial_x^l r_{2,r}^*(x,t,\cdot) \textit{ on } (-\infty, 0) \textit{ is } O(t^{-3/2}), \tag{4-5}$$

$$\text{the } L^p\text{-norms of } \partial_x^l r_{1,r}(x,t,\cdot) \text{ and } \frac{r_{1,r}(x,t,\cdot)}{\cdot - k_0} \text{ on } (0,k_0) \text{ are } O(t^{-3/2}), \tag{4-6}$$

$$\text{the } L^p\text{-norms of } (1+|\cdot|)\partial_x^l \hat{r}^*_{1,r}(x,t,\cdot) \text{ and } \frac{\hat{r}^*_{1,r}(x,t,\cdot)}{\cdot - k_0} \text{ on } (k_0,\infty) \text{ are } O(t^{-3/2}), \tag{4-7}$$

uniformly for $\zeta \in \mathcal{I}$ *as* $t \to \infty$.

Proof. The proof uses the techniques of [Deift and Zhou 1993]. Since these techniques are rather standard by now, we omit the details; see [Lenells 2017, Lemma 4.8] for a proof of a similar lemma. □

In the sequel, we often write $r_{j,a}(k)$ and $r_{j,r}(k)$ instead of $r_{j,a}(x,t,k)$ and $r_{j,r}(x,t,k)$, respectively, for notational convenience.

Recalling that $r_2 = r_{2,a} + r_{2,r}$, we can factorize v_2, v_4, v_6 as

$$v_2 = v^U_{2,a} v_{2,r} v^L_{2,a}, \quad v_4 = v^U_{4,a} v_{4,r} v^L_{4,a}, \quad v_6 = v^L_{6,a} v_{6,r} v^U_{6,a},$$

where the analytic factors are given by

$$v^U_{2,a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -r^*_{2,a}(\omega k)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \quad v^L_{2,a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r_{2,a}(\omega k)e^{t\Phi_{32}} & 1 \end{pmatrix},$$

$$v^U_{4,a} = \begin{pmatrix} 1 & -r^*_{2,a}(k)e^{-t\Phi_{21}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v^L_{4,a} = \begin{pmatrix} 1 & 0 & 0 \\ r_{2,a}(k)e^{t\Phi_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v^L_{6,a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -r^*_{2,a}(\omega^2 k)e^{t\Phi_{31}} & 0 & 1 \end{pmatrix}, \quad v^U_{6,a} = \begin{pmatrix} 1 & 0 & r_{2,a}(\omega^2 k)e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the small remainders $v_{j,r}$, $j = 2, 4, 6$, are given by the expressions obtained by replacing r_j with $r_{j,r}$ in the definition (3-3) of v_j, i.e.,

$$v_{2,r} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-r_{2,r}(\omega k)r^*_{2,r}(\omega k) & -r^*_{2,r}(\omega k)e^{-t\Phi_{32}} \\ 0 & r_{2,r}(\omega k)e^{t\Phi_{32}} & 1 \end{pmatrix},$$

$$v_{4,r} = \begin{pmatrix} 1-|r_{2,r}(k)|^2 & -r^*_{2,r}(k)e^{-t\Phi_{21}} & 0 \\ r_{2,r}(k)e^{t\Phi_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{6,r} = \begin{pmatrix} 1 & 0 & r_{2,r}(\omega^2 k)e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ -r^*_{2,r}(\omega^2 k)e^{t\Phi_{31}} & 0 & 1-r_{2,r}(\omega^2 k)r^*_{2,r}(\omega^2 k) \end{pmatrix}.$$

Define the sectionally analytic function $m^{(1)}$ by

$$m^{(1)}(x,t,k) = m(x,t,k)G(x,t,k),$$

where G is defined by

$$G(x,t,k) = \begin{cases} v_{2,a}^U, & k \in D_1, \\ (v_{2,a}^L)^{-1}, & k \in D_2, \\ v_{4,a}^U, & k \in D_3, \\ (v_{4,a}^L)^{-1}, & k \in D_4, \\ v_{6,a}^L, & k \in D_5, \\ (v_{6,a}^U)^{-1}, & k \in D_6. \end{cases} \tag{4-8}$$

Lemma 4.3. *$G(x,t,k)$ and $G(x,t,k)^{-1}$ are uniformly bounded for $k \in \mathbb{C}\setminus\Gamma$, $t > 0$, and $\zeta \in \mathcal{I}$. Moreover, $G = I + O(k^{-1})$ as $k \to \infty$.*

Proof. We have $\operatorname{Re}\Phi_{21}(\zeta,k) > 0$ for $k \in D_3$ (see Figure 5). Therefore, by virtue of (4-4a),

$$|v_{4,a}^U(x,t,k) - I| \le \frac{C}{1+|k|} e^{-ct|\operatorname{Re}\Phi_{21}(\zeta,k)|}, \quad k \in D_3,$$

uniformly for $\zeta \in \mathcal{I}$. Since $\operatorname{Re}\Phi_{21}(\zeta,k) < 0$ for $\zeta \in D_4$ (see Figure 5 again), we deduce similarly that

$$|r_{2,a}(x,t,k)| \le \frac{C}{1+|k|} e^{(t/4)|\operatorname{Re}\Phi_{21}(\zeta,k)|}, \quad k \in \bar{U}_3,$$

and hence

$$|(v_{4,a}^L)^{-1}(x,t,k) - I| \le \frac{C}{1+|k|} e^{-ct|\operatorname{Re}\Phi_{21}(\zeta,k)|}, \quad k \in D_4.$$

We appeal to the $\mathcal{A}$-symmetry of (4-1) to extend these bounds to the other sectors. □

It follows from Lemma 4.3 that m satisfies RH problem 3.1 if and only if $m^{(1)}$ satisfies RH problem 4.1 with $j = 1$, where $\Gamma^{(1)} = \Gamma$ and the jump matrix $v^{(1)}$ is given on $\Gamma_1 \cup \Gamma_3 \cup \Gamma_5$ by

$$v_1^{(1)} = v_{6,a}^U v_1 v_{2,a}^U, \quad v_3^{(1)} = v_{2,a}^L v_3 v_{4,a}^U, \quad v_5^{(1)} = v_{4,a}^L v_5 v_{6,a}^L,$$

and the small jumps remaining on $\Gamma_2 \cup \Gamma_4 \cup \Gamma_6$ are given by

$$v_j^{(1)} = v_{j,r}, \quad j = 2, 4, 6.$$

Here Γ_j denotes the subcontour of Γ labeled by j in Figure 1. More explicitly, the jump matrices $v_j^{(1)}$, $j = 1, 3, 5$, can be expressed as

$$v_1^{(1)} = \begin{pmatrix} 1 & -r_1(k)e^{-t\Phi_{21}} & \beta(k)e^{-t\Phi_{31}} \\ r_1^*(k)e^{t\Phi_{21}} & 1-r_1(k)r_1^*(k) & \alpha(k)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_3^{(1)} = \begin{pmatrix} 1-r_1(\omega^2k)r_1^*(\omega^2k) & \alpha(\omega^2k)e^{-t\Phi_{21}} & r_1^*(\omega^2k)e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ -r_1(\omega^2k)e^{t\Phi_{31}} & \beta(\omega^2k)e^{t\Phi_{32}} & 1 \end{pmatrix},$$

$$v_5^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ \beta(\omega k)e^{t\Phi_{21}} & 1 & -r_1(\omega k)e^{-t\Phi_{32}} \\ \alpha(\omega k)e^{t\Phi_{31}} & r_1^*(\omega k)e^{t\Phi_{32}} & 1-r_1(\omega k)r_1^*(\omega k) \end{pmatrix},$$

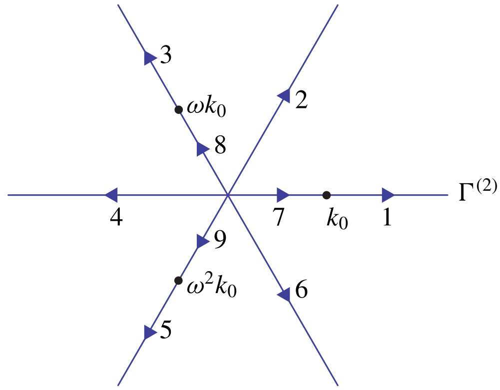

Figure 9. The contour $\Gamma^{(2)}$ in the complex k-plane.

where the functions $\alpha(k) \equiv \alpha(x,t,k)$ and $\beta(k) \equiv \beta(x,t,k)$ are defined by

$$\alpha(k) = -r_{2,a}^*(\omega k)(1 - r_1(k) r_1^*(k)), \quad k \in \mathbb{R}_+,$$
$$\beta(k) = r_{2,a}(\omega^2 k) + r_1(k) r_{2,a}^*(\omega k), \quad k \in \mathbb{R}_+.$$

4B. ***Second transformation.*** Let $\Gamma^{(2)} = \bigcup_{j=1}^9 \Gamma_j^{(2)}$ denote the contour displayed in Figure 9, where $\Gamma_1^{(2)} = [k_0, \infty)$ etc. For each $\zeta \in \mathcal{I}$, we choose $\delta_1(\zeta, k)$ such that δ_1 is analytic except for the jump across $\Gamma_1^{(2)}$,

$$\delta_{1+}(\zeta, k) = \delta_{1-}(\zeta, k)(1 - |r_1(k)|^2), \quad k \in \Gamma_1^{(2)},$$

and such that

$$\delta_1(\zeta, k) = 1 + O(k^{-1}), \quad k \to \infty. \tag{4-9}$$

The relation $k_0 = \zeta/2$ implies that there exists an $\epsilon > 0$ such that

$$|r(k_0)| \le 1 - \epsilon \quad \text{for all } k \in [k_0, \infty) \text{ and all } \zeta \in \mathcal{I}. \tag{4-10}$$

Hence, by the Plemelj formulas, we find

$$\delta_1(\zeta, k) = \exp\left\{ \frac{1}{2\pi i} \int_{[k_0,\infty)} \frac{\ln(1 - |r_1(s)|^2)}{s - k} \, ds \right\}, \quad k \in \mathbb{C} \setminus \Gamma_1^{(2)}. \tag{4-11}$$

Let $\ln_0(k)$ denote the logarithm of k with branch cut along $\arg k = 0$, i.e., $\ln_0(k) = \ln|k| + i \arg_0 k$ with $\arg_0 k \in (0, 2\pi)$.

Lemma 4.4. *The function $\delta_1(\zeta, k)$ has the following properties*:

(a) *δ_1 can be written as*

$$\delta_1(\zeta, k) = e^{-i\nu \ln_0(k - k_0)} e^{-\chi_1(\zeta, k)}, \tag{4-12}$$

where $\nu \equiv \nu(\zeta) \ge 0$ is defined by

$$\nu = -\frac{1}{2\pi} \ln(1 - |r_1(k_0)|^2), \quad \zeta \in \mathcal{I},$$

and

$$\chi_1(\zeta, k) = \frac{1}{2\pi i} \int_{k_0}^{\infty} \ln_0(k-s)\, d\ln(1-|r_1(s)|^2). \tag{4-13}$$

(b) *For each* $\zeta \in \mathcal{I}$, $\delta_1(\zeta, k)$ *and* $\delta_1(\zeta, k)^{-1}$ *are analytic functions of* $k \in \mathbb{C} \setminus \Gamma_1^{(2)}$ *with continuous boundary values on* $\Gamma_1^{(2)} \setminus \{k_0\}$. *Moreover*,

$$\sup_{\zeta \in \mathcal{I}} \sup_{k \in \mathbb{C}\setminus\Gamma_1^{(2)}} |\delta_1(\zeta, k)^{\pm 1}| < \infty. \tag{4-14}$$

(c) δ_1 *obeys the symmetry*

$$\delta_1(\zeta, k) = \overline{\delta_1(\zeta, \bar{k})}^{-1}, \quad \zeta \in \mathcal{I},\ k \in \mathbb{C} \setminus \Gamma_1^{(2)}. \tag{4-15}$$

(d) *As* $k \to k_0$ *along a path which is nontangential to* (k_0, ∞), *we have*

$$|\chi_1(\zeta, k) - \chi_1(\zeta, k_0)| \le C|k-k_0|(1+|\ln|k-k_0||), \tag{4-16}$$

$$|\partial_x(\chi_1(\zeta, k) - \chi_1(\zeta, k_0))| \le \frac{C}{t}(1+|\ln|k-k_0||), \tag{4-17}$$

where C *is independent of* $\zeta \in \mathcal{I}$. *Furthermore*,

$$|\partial_x \chi_1(\zeta, k_0)| = \frac{1}{t}\Big|\partial_u \chi_1(u, v)|_{(u,v)=(\zeta,k_0)} + \frac{1}{2}\partial_v \chi_1(u, v)|_{(u,v)=(\zeta,k_0)}\Big| \le \frac{C}{t} \tag{4-18}$$

and

$$\partial_x(\delta_1(\zeta, k)^{\pm 1}) = \frac{\pm i\nu}{2t(k-k_0)}\delta_1(\zeta, k)^{\pm 1}. \tag{4-19}$$

Proof. The lemma follows from (4-11) and relatively straightforward estimates. □

The functions δ_3 and δ_5 defined by

$$\delta_3(\zeta, k) = \delta_1(\zeta, \omega^2 k), \quad k \in \mathbb{C} \setminus \Gamma_3^{(2)},$$

$$\delta_5(\zeta, k) = \delta_1(\zeta, \omega k), \quad k \in \mathbb{C} \setminus \Gamma_5^{(2)},$$

satisfy the jump relations

$$\delta_{3+}(\zeta, k) = \delta_{3-}(\zeta, k)(1-|r_1(\omega^2 k)|^2), \quad k \in \Gamma_3^{(2)},$$

$$\delta_{5+}(\zeta, k) = \delta_{5-}(\zeta, k)(1-|r_1(\omega k)|^2), \quad k \in \Gamma_5^{(2)}.$$

The jump matrix $v^{(1)}$ cannot be appropriately factorized on the subcontour $\Gamma_1^{(2)} \cup \Gamma_3^{(2)} \cup \Gamma_5^{(2)}$ of $\Gamma^{(2)}$. Hence we introduce $m^{(2)}$ by

$$m^{(2)}(x, t, k) = m^{(1)}(x, t, k)\Delta(\zeta, k),$$

where the 3×3-matrix-valued function $\Delta(\zeta, k)$ is defined by

$$\Delta(\zeta, k) = \begin{pmatrix} \frac{\delta_1(\zeta,k)}{\delta_3(\zeta,k)} & 0 & 0 \\ 0 & \frac{\delta_5(\zeta,k)}{\delta_1(\zeta,k)} & 0 \\ 0 & 0 & \frac{\delta_3(\zeta,k)}{\delta_5(\zeta,k)} \end{pmatrix}. \tag{4-20}$$

From (4-14) and (4-9), we infer that Δ and Δ^{-1} are uniformly bounded for $\zeta \in \mathcal{I}$ and $k \in \mathbb{C} \setminus (\Gamma_1^{(2)} \cup \Gamma_3^{(2)} \cup \Gamma_5^{(2)})$ and that

$$\Delta(\zeta, k) = I + O(k^{-1}) \quad \text{as } k \to \infty. \tag{4-21}$$

It follows that m satisfies RH problem 3.1 if and only if $m^{(2)}$ satisfies RH problem 4.1 with $j = 2$, where the jump matrix $v^{(2)}$ is given by $v^{(2)} = \Delta_-^{-1} v^{(1)} \Delta_+$. A computation gives

$$v_1^{(2)} = \begin{pmatrix} \frac{\delta_{1+}}{\delta_{1-}} & -\frac{\delta_3\delta_5}{\delta_{1-}\delta_{1+}} r_1(k) e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_{1-}\delta_5}\beta(k)e^{-t\Phi_{31}} \\ \frac{\delta_{1-}\delta_{1+}}{\delta_3\delta_5} r_1^*(k) e^{t\Phi_{21}} & \frac{\delta_{1-}}{\delta_{1+}}(1-r_1(k)r_1^*(k)) & \frac{\delta_{1-}\delta_3}{\delta_5^2}\alpha(k)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-r_1(k)r_1^*(k) & -\frac{\delta_3\delta_5}{\delta_{1-}^2}\frac{r_1(k)}{1-r_1(k)r_1^*(k)}e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_{1-}\delta_5}\beta(k)e^{-t\Phi_{31}} \\ \frac{\delta_{1+}^2}{\delta_3\delta_5}\frac{r_1^*(k)}{1-r_1(k)r_1^*(k)}e^{t\Phi_{21}} & 1 & -r_{2,a}^*(\omega k)\frac{\delta_{1+}\delta_3}{\delta_5^2}e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_2^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-r_{2,r}(\omega k)r_{2,r}^*(\omega k) & -\frac{\delta_1\delta_3}{\delta_5^2} r_{2,r}^*(\omega k)e^{-t\Phi_{32}} \\ 0 & \frac{\delta_5^2}{\delta_1\delta_3} r_{2,r}(\omega k)e^{t\Phi_{32}} & 1 \end{pmatrix},$$

$$v_7^{(2)} = \begin{pmatrix} 1 & -\frac{\delta_3\delta_5}{\delta_1^2} r_1(k)e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5}\beta(k)e^{-t\Phi_{31}} \\ \frac{\delta_1^2}{\delta_3\delta_5} r_1^*(k)e^{t\Phi_{21}} & 1-r_1(k)r_1^*(k) & \frac{\delta_1\delta_3}{\delta_5^2}\alpha(k)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix}.$$

The remaining jumps $v_j^{(2)}$ can be obtained from these matrices together with the $\mathbb{Z}_3$ symmetry (4-1) and are given by

$$v_3^{(2)} = \begin{pmatrix} 1 & -\frac{\delta_{3+}\delta_5}{\delta_1^2} r_{2,a}^*(k)e^{-t\Phi_{21}} & \frac{\delta_{3+}^2}{\delta_1\delta_5}\frac{r_1^*(\omega^2 k)e^{-t\Phi_{31}}}{1-r_1(\omega^2 k)r_1^*(\omega^2 k)} \\ 0 & 1 & 0 \\ -\frac{\delta_1\delta_5}{\delta_{3-}^2}\frac{r_1(\omega^2 k)e^{t\Phi_{31}}}{1-r_1(\omega^2 k)r_1^*(\omega^2 k)} & \frac{\delta_5^2}{\delta_1\delta_{3-}}\beta(\omega^2 k)e^{t\Phi_{32}} & 1-r_1(\omega^2 k)r_1^*(\omega^2 k) \end{pmatrix},$$

$$v_4^{(2)} = \begin{pmatrix} 1-r_{2,r}(k)r_{2,r}^*(k) & -\frac{\delta_3\delta_5}{\delta_1^2} r_{2,r}^*(k)e^{-t\Phi_{21}} & 0 \\ \frac{\delta_1^2}{\delta_3\delta_5} r_{2,r}(k)e^{t\Phi_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_5^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta_1^2}{\delta_3\delta_{5-}}\beta(\omega k)e^{t\Phi_{21}} & 1-r_1(\omega k)r_1^*(\omega k) & -\frac{\delta_3\delta_1}{\delta_{5-}^2}\frac{r_1(\omega k)e^{-t\Phi_{32}}}{1-r_1(\omega k)r_1^*(\omega k)} \\ -\frac{\delta_1\delta_{5+}}{\delta_3^2} r_{2,a}^*(\omega^2 k)e^{t\Phi_{31}} & \frac{\delta_{5+}^2}{\delta_1\delta_3}\frac{r_1^*(\omega k)e^{t\Phi_{32}}}{1-r_1(\omega k)r_1^*(\omega k)} & 1 \end{pmatrix},$$

$$v_6^{(2)} = \begin{pmatrix} 1 & 0 & \frac{\delta_3^2}{\delta_1\delta_5} r_{2,r}(\omega^2 k)e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ -\frac{\delta_1\delta_5}{\delta_3^2} r_{2,r}^*(\omega^2 k)e^{t\Phi_{31}} & 0 & 1-r_{2,r}(\omega^2 k)r_{2,r}^*(\omega^2 k) \end{pmatrix},$$

$$v_8^{(2)} = \begin{pmatrix} 1-r_1(\omega^2 k)r_1^*(\omega^2 k) & \frac{\delta_3\delta_5}{\delta_1^2}\alpha(\omega^2 k)e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5} r_1^*(\omega^2 k)e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ -\frac{\delta_1\delta_5}{\delta_3^2} r_1(\omega^2 k)e^{t\Phi_{31}} & \frac{\delta_5^2}{\delta_1\delta_3}\beta(\omega^2 k)e^{t\Phi_{32}} & 1 \end{pmatrix},$$

$$v_9^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta_1^2}{\delta_3\delta_5}\beta(\omega k)e^{t\Phi_{21}} & 1 & -\frac{\delta_1\delta_3}{\delta_5^2} r_1(\omega k)e^{-t\Phi_{32}} \\ \frac{\delta_1\delta_5}{\delta_3^2}\alpha(\omega k)e^{t\Phi_{31}} & \frac{\delta_5^2}{\delta_1\delta_3} r_1^*(\omega k)e^{t\Phi_{32}} & 1-r_1(\omega k)r_1^*(\omega k) \end{pmatrix}.$$

4C. *Third transformation.* The (11)-entry of $v_1^{(2)}$ can be rewritten as

$$(v_1^{(2)})_{11} = 1 - r_1(k)r_1^*(k) = 1 - \frac{\delta_{1+}}{\delta_{1-}}\hat{r}_1(k)\frac{\delta_{1+}}{\delta_{1-}}\hat{r}_1^*(k).$$

Therefore, using the general identity

$$\begin{pmatrix} 1+f_1f_3 & f_1 & f_2 \\ f_3 & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & f_{1,a} & f_2-f_1f_4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1+f_{1,r}f_{3,r} & f_{1,r} & 0 \\ f_{3,r} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ f_{3,a} & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix},$$

where $f_j = f_{j,a} + f_{j,r}$, as well as the relation

$$\beta(k) - r_{2,a}^*(\omega k)r_1(k) = r_{2,a}(\omega^2 k), \quad k \in \mathbb{R}_+,$$

we can factorize $v_1^{(2)}$ for $k \in \Gamma_1^{(2)}$ as

$$v_1^{(2)} = \begin{pmatrix} 1-\frac{\delta_{1+}^2}{\delta_{1-}^2}\hat{r}_1(k)\hat{r}_1^*(k) & -\frac{\delta_3\delta_5}{\delta_{1-}^2}\hat{r}_1(k)e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_{1-}\delta_5}\beta(k)e^{-t\Phi_{31}} \\ \frac{\delta_{1+}^2}{\delta_3\delta_5}\hat{r}_1^*(k)e^{t\Phi_{21}} & 1 & -r_2^*(\omega k)\frac{\delta_{1+}\delta_3}{\delta_5^2}e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix} = v_1^{(2)A} v_{1,r}^{(2)} v_1^{(2)B}, \tag{4-22}$$

where

$$v_1^{(2)A} = \begin{pmatrix} 1 & -\frac{\delta_3\delta_5}{\delta_{1-}^2}\hat{r}_{1,a}(k)e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_{1-}\delta_5} r_{2,a}(\omega^2 k)e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{1,r}^{(2)} = \begin{pmatrix} 1-\frac{\delta_{1+}^2}{\delta_{1-}^2}\hat{r}_{1,r}^*(k)\hat{r}_{1,r}(k) & -\frac{\delta_3\delta_5}{\delta_{1-}^2}\hat{r}_{1,r}(k)e^{-t\Phi_{21}} & 0 \\ \frac{\delta_{1+}^2}{\delta_3\delta_5}\hat{r}_{1,r}^*(k)e^{t\Phi_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_1^{(2)B} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta_{1+}^2}{\delta_3\delta_5}\hat{r}_{1,a}^*(k)e^{t\Phi_{21}} & 1 & -\frac{\delta_{1+}\delta_3}{\delta_5^2}r_{2,a}^*(\omega k)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, using the general identity

$$\begin{aligned}\begin{pmatrix} 1 & f_1 & f_2 \\ f_3 & 1+f_1f_3 & f_4 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ f_{3,a} & 1 & f_{4,a}-f_{2,a}f_{3,a} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & f_{1,r} & f_{2,r} \\ f_{3,r} & 1+f_{1,r}f_{3,r} & f_{4,r}-f_{2,a}f_{3,r}-f_{2,r}f_{3,a} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & f_{1,a} & f_{2,a} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\end{aligned}$$

where $f_j = f_{j,a} + f_{j,r}$, as well as the relation

$$\alpha(k) - r_1^*(k)\beta(k) = -r_{2,a}^*(\omega k) - r_1^*(k)r_{2,a}(\omega^2 k), \quad k \in \mathbb{R}_+,$$

we can factorize $v_7^{(2)}$ for $k \in \Gamma_7^{(2)}$ as

$$v_7^{(2)} = \begin{pmatrix} 1 & -\frac{\delta_3\delta_5}{\delta_1^2}r_1(k)e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5}\beta(k)e^{-t\Phi_{31}} \\ \frac{\delta_1^2}{\delta_3\delta_5}r_1^*(k)e^{t\Phi_{21}} & 1-r_1(k)r_1^*(k) & \frac{\delta_1\delta_3}{\delta_5^2}\alpha(k)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix} = v_7^{(2)A}v_{7,r}^{(2)}v_7^{(2)B},$$

where

$$v_7^{(2)A} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta_1^2}{\delta_3\delta_5}r_{1,a}^*(k)e^{t\Phi_{21}} & 1 & -\frac{\delta_1\delta_3}{\delta_5^2}\big(r_{2,a}^*(\omega k)+r_{1,a}^*(k)r_{2,a}(\omega^2 k)\big)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{7,r}^{(2)} = \begin{pmatrix} 1 & -\frac{\delta_3\delta_5}{\delta_1^2}r_{1,r}(k)e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5}\beta_r(k)e^{-t\Phi_{31}} \\ \frac{\delta_1^2}{\delta_3\delta_5}r_{1,r}^*(k)e^{t\Phi_{21}} & 1-|r_{1,r}(k)|^2 & \frac{\delta_1\delta_3}{\delta_5^2}r_{1,r}^*(k)\big(r_{1,r}(k)r_{2,a}^*(\omega k)-r_{2,a}(\omega^2 k)\big)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_7^{(2)B} = \begin{pmatrix} 1 & -\frac{\delta_3\delta_5}{\delta_1^2}r_{1,a}(k)e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5}\beta_a(k)e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\beta_r(k) := r_{1,r}(k)r_{2,a}^*(\omega k), \quad \beta_a(k) := r_{2,a}(\omega^2 k) + r_{1,a}(k)r_{2,a}^*(\omega k).$$

Let $V_j \equiv V_j(\zeta) \subset \mathbb{C}$, $j = 1, \dots, 4$, denote the open subsets of the complex k-plane displayed in Figure 10. Define the sectionally analytic function $m^{(3)}$ by

$$m^{(3)}(x,t,k) = m^{(2)}(x,t,k)H(x,t,k),$$

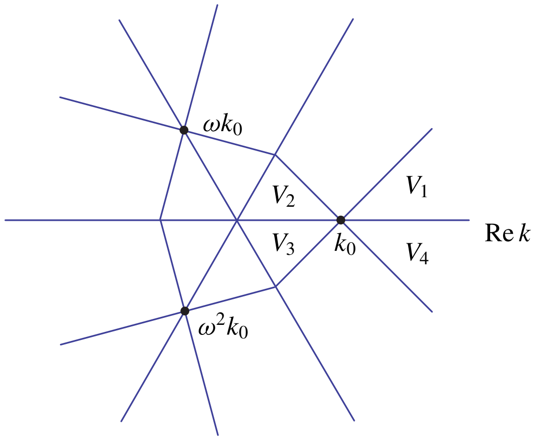

Figure 10. The open sets $\{V_j\}_1^4$ in the complex k-plane.

where H is defined for $k \in D_1 \cup D_6$ by

$$H(x,t,k) = \begin{cases} (v_1^{(2)B})^{-1}, & k \in V_1, \\ (v_7^{(2)B})^{-1}, & k \in V_2, \\ v_7^{(2)A}, & k \in V_3, \\ v_1^{(2)A}, & k \in V_4, \\ I, & \text{elsewhere in } D_1 \cup D_6, \end{cases} \tag{4-23}$$

and extended to all of $\mathbb{C} \setminus \Gamma$ by means of the symmetry $H(x,t,k) = \mathcal{A} H(x,t,\omega k)\mathcal{A}^{-1}$. Let $\Gamma^{(3)}$ be the contour displayed in Figure 11.

Lemma 4.5. *$H(x,t,k)$ is uniformly bounded for $k \in \mathbb{C} \setminus \Gamma^{(3)}$, $t > 0$, and $\zeta \in \mathcal{I}$. Moreover, $H = I + O(k^{-1})$ as $k \to \infty$.*

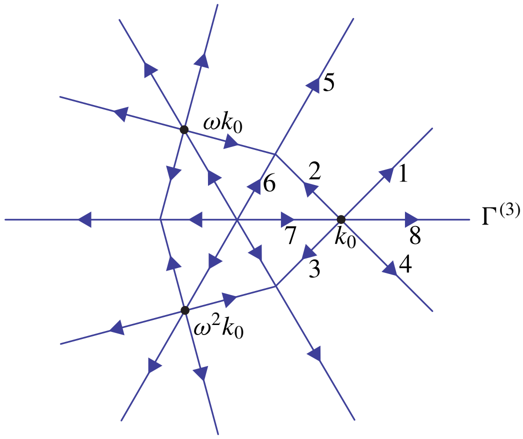

Figure 11. The contour $\Gamma^{(3)}$ in the complex k-plane.

Proof. We present the proof for $k \in V_1 \cup V_2$; the proof for $k \in V_3 \cup V_4$ is similar. Note that $V_j \subset U_j$, $j = 1, \dots, 4$ (see Figures 8 and 10). Note also the identities

$$\Phi_{21}(\zeta, \omega k) = \Phi_{32}(\zeta, k), \quad \Phi_{21}(\zeta, \omega^2 k) = -\Phi_{31}(\zeta, k), \quad \Phi_{21} + \Phi_{32} = \Phi_{31}. \tag{4-24}$$

If $k \in \overline{V}_1$, then $\omega k \in \overline{U}_2$ and (see Figures 5 and 7)

$$\operatorname{Re}\Phi_{21}(\zeta, k) \le 0, \quad \operatorname{Re}\Phi_{32}(\zeta, k) \ge 0.$$

Therefore, using (4-24), (4-4a), (4-4f), and (4-14), we find

$$\begin{aligned} |((v_1^{(2)B})^{-1})_{21}| &= \left| \frac{\delta_1^2}{\delta_3\delta_5} \hat{r}_{1,a}^*(k) e^{t\Phi_{21}} \right| \le \frac{C}{1+|k|} e^{-ct|\operatorname{Re}\Phi_{21}|}, \qquad k \in V_1, \\ |((v_1^{(2)B})^{-1})_{23}| &= \left| \frac{\delta_1\delta_3}{\delta_5^2} r_{2,a}^*(\omega k) e^{-t\Phi_{32}} \right| \le \frac{C}{1+|k|} e^{-ct|\operatorname{Re}\Phi_{32}|}, \quad k \in V_1. \end{aligned}$$

This proves the claim for $k \in V_1$. All entries of $(v_7^{(2)B})^{-1}$ are continuous functions on $\overline{V}_2$. Since $\overline{V}_2$ is compact, the claim follows also for $k \in V_2$. □

It follows from Lemma 4.5 that m satisfies RH problem 3.1 if and only if $m^{(3)}$ satisfies RH problem 4.1 with $j = 3$, where $\Gamma^{(3)}$ is the contour displayed in Figure 11 and the jump matrix $v^{(3)}$ is given for $-\frac{\pi}{3} < \arg k \le \frac{\pi}{3}$ by

$$v_1^{(3)} = v_1^{(2)B} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta_1^2}{\delta_3\delta_5}\hat{r}_{1,a}^*(k)e^{t\Phi_{21}} & 1 & -\frac{\delta_1\delta_3}{\delta_5^2} r_{2,a}^*(\omega k) e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_2^{(3)} = (v_7^{(2)B})^{-1} = \begin{pmatrix} 1 & \frac{\delta_3\delta_5}{\delta_1^2} r_{1,a}(k) e^{-t\Phi_{21}} & -\frac{\delta_3^2}{\delta_1\delta_5}\big(r_{2,a}(\omega^2 k) + r_{1,a}(k) r_{2,a}^*(\omega k)\big)e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_3^{(3)} = (v_7^{(2)A})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{\delta_1^2}{\delta_3\delta_5} r_{1,a}^*(k) e^{t\Phi_{21}} & 1 & \frac{\delta_1\delta_3}{\delta_5^2}\big(r_{2,a}^*(\omega k) + r_{1,a}^*(k) r_{2,a}(\omega^2 k)\big)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_4^{(3)} = v_1^{(2)A} = \begin{pmatrix} 1 & -\frac{\delta_3\delta_5}{\delta_1^2}\hat{r}_{1,a}(k)e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5} r_{2,a}(\omega^2 k) e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_5^{(3)} = v_2^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - r_{2,r}(\omega k) r_{2,r}^*(\omega k) & -\frac{\delta_1\delta_3}{\delta_5^2} r_{2,r}^*(\omega k) e^{-t\Phi_{32}} \\ 0 & \frac{\delta_5^2}{\delta_1\delta_3} r_{2,r}(\omega k) e^{t\Phi_{32}} & 1 \end{pmatrix},$$

$$v_6^{(3)} = v_7^{(2)B} v_2^{(2)} \mathcal{A}^{-1} v_7^{(2)A}(x,t,\omega^2 k)\mathcal{A} = \begin{pmatrix} 1 & \frac{\delta_3\delta_5}{\delta_1^2} g(k) e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5} f(k) e^{-t\Phi_{31}} \\ 0 & 1 - r_{2,r}(\omega k) r_{2,r}^*(\omega k) & -\frac{\delta_1\delta_3}{\delta_5^2} r_{2,r}^*(\omega k) e^{-t\Phi_{32}} \\ 0 & \frac{\delta_5^2}{\delta_1\delta_3} r_{2,r}(\omega k) e^{t\Phi_{32}} & 1 \end{pmatrix},$$

$$v_7^{(3)} = v_{7,r}^{(2)} = \begin{pmatrix} 1 & -\frac{\delta_3\delta_5}{\delta_1^2} r_{1,r}(k) e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5} r_{1,r}(k) r_{2,a}^*(\omega k) e^{-t\Phi_{31}} \\ \frac{\delta_1^2}{\delta_3\delta_5} r_{1,r}^*(k) e^{t\Phi_{21}} & 1 - r_{1,r}(k) r_{1,r}^*(k) & \frac{\delta_1\delta_3}{\delta_5^2} r_{1,r}^*(k) h(k) e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_8^{(3)} = v_{1,r}^{(2)} = \begin{pmatrix} 1 - \frac{\delta_{1+}^2}{\delta_{1-}^2} \hat{r}_{1,r}(k) \hat{r}_{1,r}^*(k) & -\frac{\delta_3\delta_5}{\delta_{1-}^2} \hat{r}_{1,r}(k) e^{-t\Phi_{21}} & 0 \\ \frac{\delta_{1+}^2}{\delta_3\delta_5} \hat{r}_{1,r}^*(k) e^{t\Phi_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and extended to the remainder of $\Gamma^{(3)}$ by means of the first symmetry in (4-1). Here the functions $f(k) \equiv f(x,t,k)$, $g(k) \equiv g(x,t,k)$, and $h(k) \equiv h(x,t,k)$ are defined by

$$\begin{aligned} f(k) &= r_{2,a}(\omega^2 k) + r_{1,a}(k) r_2^*(\omega k) + r_{1,a}^*(\omega^2 k), \\ g(k) &= r_{2,r}(\omega k)\big(r_{2,a}(\omega^2 k) + r_{1,a}(k) r_{2,a}^*(\omega k)\big) - r_{1,a}(k)\big(1 - r_{2,r}(\omega k) r_{2,r}^*(\omega k)\big) - r_{2,a}^*(k) - r_{1,a}^*(\omega^2 k) r_{2,a}(\omega k), \\ h(k) &= r_{1,r}(k) r_{2,a}^*(\omega k) - r_{2,a}(\omega^2 k). \end{aligned}$$

The next lemma establishes bounds on f and g and their x-derivatives.

Lemma 4.6. *For $k \in \Gamma_3$, and $l = 0, 1$, we have*

$$|\partial_x^l f(k)| \le C|k| e^{(t/4)|\mathrm{Re}\,\Phi_{21}(\zeta,k)|}, \quad |\partial_x^l g(k)| \le C|k| e^{(t/4)|\mathrm{Re}\,\Phi_{21}(\zeta,k)|}. \tag{4-25}$$

Proof. By (4-4b) and (4-4c), we have $r_{2,r}^*(0) = 0$, $r_{2,a}^*(0) = r_2^*(0)$, $r_{1,r}(0) = 0$, and $r_{1,a}(0) = r_1(0)$. Since $r_1(0) = \omega$ and $r_2(0) = 1$ (see [Charlier and Lenells 2022]), we deduce that

$$r_{2,a}^*(0) = r_2^*(0) = 1 \quad \text{and} \quad r_{1,a}(0) = r_1(0) = \omega.$$

In particular,

$$r_{1,a}(0) r_2^*(0) + r_{1,a}^*(0) + r_{2,a}(0) = \omega + \bar{\omega} + 1 = 0. \tag{4-26}$$

To derive the estimate for f, we write

$$\begin{aligned} f(k) &= r_{1,a}^*(\omega^2 k) + r_{2,a}(\omega^2 k) + r_{1,a}(k) r_2^*(\omega k) \\ &= r_{1,a}^*(\omega^2 k) - r_{1,a}^*(0) + r_{2,a}(\omega^2 k) - r_{2,a}(0) + (r_{1,a}(k) - r_{1,a}(0)) r_2^*(\omega k) \\ &\qquad + r_{1,a}(0)(r_2^*(\omega k) - r_2^*(0)) + r_{1,a}(0) r_2^*(0) + r_{1,a}^*(0) + r_{2,a}(0). \end{aligned}$$

Using (4-26) and the fact that $\Gamma_3^{(6)} \subset (\omega^2\mathbb{R}_- \cap U_2)$, the inequalities (4-4b) and (4-4c) imply

$$\begin{aligned}|f(k)| &\le |r_{1,a}^*(\omega^2 k) - r_{1,a}^*(0)| + |r_{2,a}(\omega^2 k) - r_{2,a}(0)| + C|r_{1,a}(k) - r_{1,a}(0)| + C|k| \\ &\le C|k|\big(e^{(t/4)|\mathrm{Re}\,\Phi_{21}(\zeta,\omega\bar{k})|} + e^{(t/4)|\mathrm{Re}\,\Phi_{21}(\zeta,\omega\bar{k})|} + e^{(t/4)|\mathrm{Re}\,\Phi_{21}(\zeta,k)|}\big) \\ &\le C|k|e^{(t/4)|\mathrm{Re}\,\Phi_{21}(\zeta,k)|}, \quad k \in \Gamma_3^{(6)}.\end{aligned}$$

The estimate for $\partial_x f$ is derived in a similar way. Writing

$$g(k) = r_{2,r}(\omega k)\big(r_{2,a}(\omega^2 k) + r_{1,a}(k) r_{2,a}^*(\omega k)\big) + r_{1,a}(k) r_{2,r}(\omega k) r_{2,r}^*(\omega k) - f^*(\omega^2 k),$$

and using that $r_{2,r}$ and $\partial_x r_{2,r}$ vanish at $k = 0$, the estimates for g and $\partial_x g$ follow from the estimates for f and $\partial_x f$. □

Lemma 4.7. *The jump matrix* $v^{(3)}$ (*resp.* $\partial_x v^{(3)}$) *converges to the identity matrix* I (*resp. to the zero matrix* 0) *as* $t \to \infty$ *uniformly for* $\zeta \in \mathcal{I}$ *and* $k \in \Gamma^{(3)}$ *except near the three critical points* k_0, ωk_0, $\omega^2 k_0$. *Moreover, the jump matrices* $v_j^{(3)}$, $j = 5, 6, 7, 8$, *satisfy*

$$\|(1+|\cdot|)\partial_x^l (v^{(3)} - I)\|_{(L^1\cap L^\infty)(\Gamma_5^{(3)})} \le Ct^{-3/2}, \tag{4-27a}$$

$$\|(1+|\cdot|)\partial_x^l (v^{(3)} - I)\|_{L^1(\Gamma_6^{(3)})} \le Ct^{-3/2}, \tag{4-27b}$$

$$\|(1+|\cdot|)\partial_x^l (v^{(3)} - I)\|_{L^\infty(\Gamma_6^{(3)})} \le Ct^{-1}, \tag{4-27c}$$

$$\|(1+|\cdot|)\partial_x^l (v^{(3)} - I)\|_{(L^1\cap L^\infty)(\Gamma_7^{(3)}\cup\Gamma_8^{(3)})} \le Ct^{-3/2}, \tag{4-27d}$$

uniformly for $\zeta \in \mathcal{I}$ *and* $l = 0, 1$.

Proof. Consider first the jump matrix $v_1^{(3)}$. Since $\mathrm{Re}\,\Phi_{32} \ge c > 0$ and $\mathrm{Re}\,\Phi_{21} \le 0$ for $k \in \Gamma_1^{(3)}$, $v_1^{(3)}$ (resp. $\partial_x v_1^{(3)}$) converges to I (resp. to the zero matrix) as $t \to \infty$ by (4-24), (4-4), and (4-14). Note however that the convergence to 0 of the (21) entry is not uniform for k near k_0, because $\mathrm{Re}\,\Phi_{21}(\zeta, k_0) = 0$. Analogous statements for $v_2^{(3)}$, $v_3^{(3)}$, and $v_4^{(3)}$ can be proved in a similar way.

Since $\mathrm{Re}\,\Phi_{32} = 0$ for $k \in \Gamma_5^{(3)}$, (4-27a) follows from (4-5), and (4-14).

We next show (4-27b) and (4-27c). We parametrize $\Gamma_6^{(3)}$ by $ue^{\pi i/3}$, $0 \le u \le 2k_0/(1+\sqrt{3})$, and note that

$$\mathrm{Re}\,\Phi_{31}(\zeta, ue^{\pi i/3}) = \mathrm{Re}\,\Phi_{21}(\zeta, ue^{\pi i/3}) = \tfrac{3}{2}u(2k_0 - u), \quad u \in \mathbb{R}.$$

It follows that

$$\begin{cases}\mathrm{Re}\,\Phi_{31}(\zeta,k) \ge \tfrac{4}{3}k_0|k|, \\ \mathrm{Re}\,\Phi_{21}(\zeta,k) \ge \tfrac{4}{3}k_0|k|,\end{cases} \quad k \in \Gamma_6^{(3)}.$$

Using (4-25), (4-14), (4-19), and the fact that $\partial_x(t\Phi_{31}) = (1-\omega)k$, we thus find

$$\begin{aligned}|(v_6^{(3)} - I)_{13}| &\le C|f(k)|e^{-t\,\mathrm{Re}\,\Phi_{31}} \le C|k|e^{-tk_0|k|}, \quad k \in \Gamma_6^{(3)}, \\ |\partial_x (v_6^{(3)})_{13}| &\le C|k|e^{-tk_0|k|}, \quad k \in \Gamma_6^{(3)}.\end{aligned}$$

Hence, for $l = 0, 1$, we have

$$\|(1+|\cdot|)\partial_x^l (v_6^{(3)} - I)_{13}\|_{L^1(\Gamma_6^{(3)})} \le \frac{C}{(k_0 t)^2}, \quad \|(1+|\cdot|)\partial_x^l (v_6^{(3)} - I)_{13}\|_{L^\infty(\Gamma_6^{(3)})} \le \frac{C}{k_0 t},$$

and similar estimates apply to the (12)-entry. On the other hand, $\operatorname{Re}\Phi_{32} = 0$ for $k \in \Gamma_6^{(3)}$, and hence we can estimate the (23)-entry using (4-14) as

$$|(v_6^{(3)} - I)_{23}| = \left|\frac{\delta_1\delta_3}{\delta_5^2} r_{2,r}^*(\omega k)\right| \le C|r_{2,r}^*(\omega k)|, \quad k \in \Gamma_6^{(3)}.$$

By (4-5), this implies that the L^1 and L^∞ norms of $(1+|\cdot|)(v^{(3)} - I)_{23}$ on $\Gamma_6^{(3)}$ are $O(t^{-3/2})$ as $t \to \infty$. Using also (4-19) and (4-5), we conclude similarly that the L^1 and L^∞ norms of $(1+|\cdot|)\partial_x v_{23}^{(3)}$ on $\Gamma_6^{(3)}$ are $O(t^{-3/2})$ as $t \to \infty$. A similar estimate applies to the (32)-entry and its x-derivative. The (22)-entry is even smaller. This proves (4-27b) and (4-27c).

We finally show (4-27d). Note that $\operatorname{Re}\Phi_{32} > 0$ and $\operatorname{Re}\Phi_{31} > 0$ for $k \in \mathbb{R}_+$. We conclude from (4-4a) that $|(v_7^{(3)} - I)_{23}|$ and $|(v_7^{(3)} - I)_{13}|$ decay to zero as $t \to \infty$ faster than $|(v_7^{(3)} - I)_{12}|$ and $|(v_7^{(3)} - I)_{21}|$. Moreover, since $\operatorname{Re}\Phi_{21} = 0$ for $k \in \mathbb{R}_+$, (4-6) and (4-14) imply

$$|(v_7^{(3)} - I)_{21}| = \left|\frac{\delta_1^2}{\delta_3\delta_5} r_{1,r}^*\right| \le Ct^{-3/2}, \quad |(v_7^{(3)} - I)_{12}| = \left|\frac{\delta_3\delta_5}{\delta_1^2} r_{1,r}\right| \le Ct^{-3/2},$$

while $|(v_7^{(3)} - I)_{22}|$ is even smaller. Thus,

$$\|v^{(3)} - I\|_{(L^1\cap L^\infty)(\Gamma_7^{(3)})} \le Ct^{-3/2}.$$

To estimate $\partial_x(v_7^{(3)})_{21}$, we use (4-6) and (4-19). This gives

$$|\partial_x(v_7^{(3)})_{21}| \le \left|\partial_x\left(\frac{\delta_3\delta_5}{\delta_1^2}\right) r_{1,r}\right| + \left|\frac{\delta_3\delta_5}{\delta_1^2}\partial_x r_{1,r}\right| \le Ct^{-3/2}.$$

The entries $\partial_x(v_7^{(3)})_{12}$ and $\partial_x(v_7^{(3)})_{22}$ are estimated in a similar way.

The matrix $v_8^{(3)}$ can be estimated in the same way as $v_7^{(3)}$, except that now we need to use (4-7) and to note that $\operatorname{Re}\Phi_{21} = 0$ for $k \in (k_0, \infty)$. This proves (4-27d). □

5. Local parametrix at k_0

In Section 4C, we arrived at an RH problem for $m^{(3)}$ with the property that the matrix $v^{(3)} - I$ decays to zero as $t \to \infty$ everywhere except near the three critical points k_0, ωk_0, $\omega^2 k_0$. This means that we only have to consider neighborhoods of these three points when computing the long-time asymptotics of $m^{(3)}$. In this section, we find a local solution m^{k_0} which approximates $m^{(3)}$ near k_0. The basic idea is that in the large t limit, the RH problem for $m^{(3)}$ near k_0 reduces to an RH problem on a cross which can be solved exactly in terms of parabolic cylinder functions [Its 1981; Deift and Zhou 1993].

Let $\epsilon \equiv \epsilon(\zeta) = k_0/2$. Let $D_\epsilon(k_0)$ denote an open disk of radius ϵ centered at k_0. Let $\mathcal{D} = D_\epsilon(k_0) \cup \omega D_\epsilon(k_0) \cup \omega^2 D_\epsilon(k_0)$. Let $\mathcal{X} = k_0 + X$, where X is the contour defined in (A-1). We will also use the notation $\mathcal{X}^\epsilon = \mathcal{X} \cap D_\epsilon(k_0)$ and $\mathcal{X}_j^\epsilon = (k_0 + X_j) \cap D_\epsilon(k_0)$, $j = 1, \dots, 4$, where X_j is defined in (A-1).

In order to relate $m^{(3)}$ to the solution m^X of Lemma A.2, we make a local change of variables for k near k_0 and introduce the new variable $z \equiv z(\zeta, k)$ by

$$z = 3^{1/4}\sqrt{2t}(k - k_0). \tag{5-1}$$

For each $\zeta \in \mathcal{I}$, the map $k \mapsto z$ is a biholomorphism from $D_\epsilon(k_0)$ onto the open disk of radius $3^{1/4}\sqrt{2t}\epsilon$ centered at the origin. Using that

$$\Phi_{21}(\zeta, k) = \Phi_{21}(\zeta, k_0) + i\sqrt{3}(k-k_0)^2,$$

where $\Phi_{21}(\zeta, k_0) = -i\sqrt{3}k_0^2$, we see that

$$t(\Phi_{21}(\zeta, k) - \Phi_{21}(\zeta, k_0)) = \frac{i}{2}z^2.$$

Equations (4-12) and (5-1) imply that, for $\zeta \in \mathcal{I}$ and $k \in D_\epsilon(k_0) \setminus [k_0, \infty)$,

$$\frac{\delta_3\delta_5}{\delta_1^2} = e^{2i\nu \ln_0(z)}(2\sqrt{3}t)^{-i\nu}e^{2\chi_1(\zeta,k)}\delta_3\delta_5 = e^{2i\nu\ln_0(z)}d_0(\zeta, t)d_1(\zeta, k),$$

where the functions $d_0(\zeta, t)$ and $d_1(\zeta, k)$ are defined for $\zeta \in \mathcal{I}$ and $k \in D_\epsilon(k_0) \setminus [k_0, \infty)$ by

$$d_0(\zeta, t) = (2\sqrt{3}t)^{-i\nu}e^{2\chi_1(\zeta,k_0)}\delta_3(\zeta, k_0)\delta_5(\zeta, k_0), \tag{5-2}$$

$$d_1(\zeta, k) = e^{2\chi_1(\zeta,k)-2\chi_1(\zeta,k_0)}\frac{\delta_3(\zeta, k)\delta_5(\zeta, k)}{\delta_3(\zeta, k_0)\delta_5(\zeta, k_0)}. \tag{5-3}$$

Defining $\tilde{m}$ for k near k_0 by

$$\tilde{m}(x, t, k) = m^{(3)}(x, t, k)Y(\zeta, t), \quad k \in D_\epsilon(k_0),$$

where

$$Y(\zeta, t) = \begin{pmatrix} d_0^{1/2}(\zeta, t)e^{-(t/2)\Phi_{21}(\zeta,k_0)} & 0 & 0 \\ 0 & d_0^{-1/2}(\zeta, t)e^{(t/2)\Phi_{21}(\zeta,k_0)} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we find that the jump $\tilde{v}(x, t, k)$ of $\tilde{m}$ across $\mathcal{X}^\epsilon$ is given by

$$\tilde{v}_1 = \begin{pmatrix} 1 & 0 & 0 \\ e^{-2i\nu\ln_0(z)}d_1^{-1}\hat{r}_{1,a}^*(k)e^{iz^2/2} & 1 & -\frac{\delta_1\delta_3}{\delta_5^2}d_0^{1/2}r_{2,a}^*(\omega k)e^{-t\Phi_{32}}e^{-(t/2)\Phi_{21}(\zeta,k_0)} \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_2 = \begin{pmatrix} 1 & e^{2i\nu\ln_0(z)}d_1 r_{1,a}(k)e^{-iz^2/2} & -\frac{\delta_3^2}{\delta_1\delta_5}d_0^{-1/2}\Omega_1(k)e^{-t\Phi_{31}}e^{(t/2)\Phi_{21}(\zeta,k_0)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_3 = \begin{pmatrix} 1 & 0 & 0 \\ -e^{-2i\nu\ln_0(z)}d_1^{-1}r_{1,a}^*(k)e^{iz^2/2} & 1 & \frac{\delta_1\delta_3}{\delta_5^2}d_0^{1/2}\Omega_2(k)e^{-t\Phi_{32}}e^{-(t/2)\Phi_{21}(\zeta,k_0)} \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_4 = \begin{pmatrix} 1 & -e^{2i\nu\ln_0(z)}d_1\hat{r}_{1,a}(k)e^{-iz^2/2} & \frac{\delta_3^2}{\delta_1\delta_5}d_0^{-1/2}r_{2,a}(\omega^2 k)e^{-t\Phi_{31}}e^{(t/2)\Phi_{21}(\zeta,k_0)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\tilde{v}_j$ denotes the restriction of $\tilde{v}$ to $\mathcal{X}_j^\epsilon$, $j=1,2,3,4$, and $\Omega_1(k)\equiv\Omega_1(x,t,k)$ and $\Omega_2(k)\equiv\Omega_2(x,t,k)$ are given by

$$\Omega_1(k)=r_{2,a}(\omega^2k)+r_{1,a}(k)r_{2,a}^*(\omega k),\quad \Omega_2(k)=r_{2,a}^*(\omega k)+r_{1,a}^*(k)r_{2,a}(\omega^2k).$$

Define $q\equiv q(\zeta)$ by

$$q=r_1(k_0).$$

For a fixed z, $r_{1,a}(k)\to q$, $\hat{r}_{1,a}^*(k)\to\bar{q}/(1-|q|^2)$, and $d_1(\zeta,k)\to 1$ as $t\to\infty$. This suggests that $\tilde{v}(x,t,k)$ tends to the jump matrix $v^X(x,t,z)$ defined in (A-2) for large t. In other words, the jumps of $m^{(3)}$ for k near k_0 approach those of the function m^XY^{-1} as $t\to\infty$. This suggests that we approximate $m^{(3)}$ in the neighborhood $D_\epsilon(k_0)$ of k_0 by the 3×3-matrix-valued function m^{k_0} defined by

$$m^{k_0}(x,t,k)=Y(\zeta,t)m^X(q(\zeta),z(\zeta,k))Y(\zeta,t)^{-1},\quad k\in D_\epsilon(k_0). \tag{5-4}$$

The prefactor $Y(\zeta,t)$ on the right-hand side of (5-4) is included so that $m^{k_0}\to I$ on $\partial D_\epsilon(k_0)$ as $t\to\infty$; this ensures that m^{k_0} is a good approximation of $m^{(3)}$ in $D_\epsilon(k_0)$ for large t.

Lemma 5.1. *The function $Y(\zeta,t)$ is uniformly bounded*:

$$\sup_{\zeta\in\mathcal{I}}\sup_{t\geq 2}|\partial_x^l Y(\zeta,t)^{\pm1}|<C,\quad l=0,1. \tag{5-5}$$

Moreover, the functions $d_0(\zeta,t)$ and $d_1(\zeta,k)$ satisfy

$$|d_0(\zeta,t)|=e^{2\pi\nu},\qquad \zeta\in\mathcal{I},\ t\geq 2, \tag{5-6a}$$

$$|\partial_x d_0(\zeta,t)|\leq C\frac{\ln t}{t},\quad \zeta\in\mathcal{I},\ t\geq 2, \tag{5-6b}$$

and

$$|d_1(\zeta,k)-1|\leq C|k-k_0|(1+|\ln|k-k_0||),\quad \zeta\in\mathcal{I},\ k\in\mathcal{X}^\epsilon, \tag{5-7a}$$

$$|\partial_x d_1(\zeta,k)|\leq\frac{C}{t}|\ln|k-k_0||,\qquad \zeta\in\mathcal{I},\ k\in\mathcal{X}^\epsilon. \tag{5-7b}$$

Proof. The symmetry (4-15) implies

$$|\delta_3(\zeta,k_0)\delta_5(\zeta,k_0)|=|\delta_1(\zeta,\omega^2k_0)\delta_1(\zeta,\omega k_0)|=1,$$

and hence (5-6a) follows because

$$\operatorname{Re}\chi_1(\zeta,k_0)=\frac{1}{2\pi}\int_{k_0}^\infty \pi\, d\ln(1-|r_1(s)|^2)=-\frac{1}{2}\ln(1-|r_1(k_0)|^2)=\pi\nu.$$

Using (5-6a), we obtain

$$\begin{aligned}|\partial_x d_0(\zeta,t)|&=|d_0(\zeta,t)\partial_x\ln d_0(\zeta,t)|=e^{2\pi\nu}|\partial_x\ln d_0(\zeta,t)|\\&\leq C\big(|\ln t\,\partial_x\nu|+|\partial_x\chi_1(\zeta,k_0)|+|\partial_x\ln(\delta_3(\zeta,k_0)\delta_5(\zeta,k_0))|\big),\end{aligned}$$

and thus (5-6b) follows from (4-18) and the fact that $\partial_x=(1/t)\partial_\zeta$. Observing that δ_3 and δ_5 are analytic for $k\in\mathcal{X}^\epsilon$, (5-7a) follows from (4-16). Finally, we have

$$\partial_x d_1(\zeta,k)=d_1(\zeta,k)\,\partial_x\log d_1(\zeta,k).$$

Since

$$\partial_\zeta \log \frac{\delta_3(\zeta,k)\delta_5(\zeta,k)}{\delta_3(\zeta,k_0)\delta_5(\zeta,k_0)}$$

is analytic and $|d_1(\zeta,k)| \le C$ for $k \in \mathcal{X}^\epsilon$, it follows that

$$|\partial_x d_1(\zeta,k)| \le C\left(|\partial_x(\chi_1(\zeta,k)-\chi_1(\zeta,k_0))| + \frac{1}{t}\left|\partial_\zeta \log \frac{\delta_3(\zeta,k)\delta_5(\zeta,k)}{\delta_3(\zeta,k_0)\delta_5(\zeta,k_0)}\right|\right),$$

and so (5-7b) follows from (4-17). □

Lemma 5.2. *For each* (x,t), *the function* $m^{k_0}(x,t,k)$ *defined in* (5-4) *is an analytic and bounded function of* $k \in D_\epsilon(k_0) \setminus \mathcal{X}^\epsilon$. *Across* $\mathcal{X}^\epsilon$, m^{k_0} *obeys the jump condition* $m_+^{k_0} = m_-^{k_0} v^{k_0}$, *where the jump matrix* v^{k_0} *satisfies*

$$\begin{cases} \|\partial_x^l(v^{(3)}-v^{k_0})\|_{L^1(\mathcal{X}^\epsilon)} \le Ct^{-1}\ln t, \\ \|\partial_x^l(v^{(3)}-v^{k_0})\|_{L^\infty(\mathcal{X}^\epsilon)} \le Ct^{-1/2}\ln t, \end{cases} \qquad \zeta \in \mathcal{I},\ t \ge 2,\ l = 0,1. \tag{5-8}$$

Furthermore, as $t \to \infty$,

$$\|\partial_x^l(m^{k_0}(x,t,\cdot)^{-1} - I)\|_{L^\infty(\partial D_\epsilon(k_0))} = O(t^{-1/2}), \quad l = 0,1, \tag{5-9}$$

$$\frac{1}{2\pi i}\int_{\partial D_\epsilon(k_0)} (m^{k_0}(x,t,k)^{-1} - I)\,dk = -\frac{Y(\zeta,t)m_1^X(q(\zeta))Y(\zeta,t)^{-1}}{3^{1/4}\sqrt{2}\sqrt{t}} + O(t^{-1}), \tag{5-10}$$

uniformly for $\zeta \in \mathcal{I}$, *and* (5-10) *can be differentiated with respect to x without increasing the error term.*

Proof. We have

$$v^{(3)} - v^{k_0} = Y(\zeta,t)(\tilde{v} - v^X)Y(\zeta,t)^{-1}.$$

Thus, recalling (5-5), the bounds (5-8) follow if we can show that

$$\|\partial_x^l[\tilde{v}(x,t,\cdot) - v^X(x,t,z(\zeta,\cdot))]\|_{L^1(\mathcal{X}_j^\epsilon)} \le Ct^{-1}\ln t, \tag{5-11a}$$

$$\|\partial_x^l[\tilde{v}(x,t,\cdot) - v^X(x,t,z(\zeta,\cdot))]\|_{L^\infty(\mathcal{X}_j^\epsilon)} \le Ct^{-1/2}\ln t \tag{5-11b}$$

for $j = 1,\dots,4$ and $l = 0,1$. We give the proof of (5-11) for $j = 1$; similar arguments apply when $j = 2,3,4$.

For $k \in \mathcal{X}_1^\epsilon$, only the (21) and (23) elements of the matrix $\tilde{v} - v^X$ are nonzero. Using (4-4a), (4-14), (5-6a), and the facts that $\Phi_{21}(\zeta,\omega k) = \Phi_{32}(\zeta,k)$ and $v_{23}^X(q(\zeta), z(\zeta,k)) = 0$ for $k \in \mathcal{X}_1^\epsilon$, $|(\tilde{v} - v^X)_{23}|$ can be estimated as

$$\begin{aligned} |(\tilde{v} - v^X)_{23}| &= \left|\frac{\delta_1\delta_3}{\delta_5^2} d_0^{1/2} r_{2,a}^*(\omega k) e^{-t\Phi_{32}} e^{-(t/2)\Phi_{21}(\zeta,k_0)}\right| \le |r_{2,a}^*(\omega k)| e^{-t\operatorname{Re}\Phi_{32}} \\ &\le Ce^{(t/4)|\operatorname{Re}\Phi_{21}(\zeta,\omega k)|} e^{-t\operatorname{Re}\Phi_{32}} = Ce^{-(3t/4)|\operatorname{Re}\Phi_{32}(\zeta,k)|}, \quad k \in \mathcal{X}_1^\epsilon. \end{aligned}$$

For $k = k_0 + ue^{\pi i/4}$ and $u \ge 0$, we have

$$\operatorname{Re}\Phi_{32}(\zeta, k_0 + ue^{\pi i/4}) = \tfrac{1}{2}(9k_0^2 + 6\sqrt{2}k_0 u + \sqrt{3}u^2) \ge c(k_0+u)^2. \tag{5-12}$$

Hence

$$\|(\tilde{v}-v^X)_{23}\|_{L^1(\mathcal{X}_1^\epsilon)} \le C\int_0^{k_0/2} e^{-ct(k_0+u)^2}\,du = C\int_{k_0}^{3k_0/2} e^{-ctv^2}\,dv \le Ce^{-ctk_0^2}$$

and

$$\|(\tilde{v}-v^X)_{23}\|_{L^\infty(\mathcal{X}_1^\epsilon)} \le C\sup_{u\ge 0} e^{-ct(k_0+u)^2} \le Ce^{-ctk_0^2}.$$

To estimate $\partial_x(\tilde{v}-v^X)_{23}$, we first note that

$$\partial_x(\tilde{v}-v^X)_{23} = a_1+a_2+a_3+a_4+a_5,$$

where

$$\begin{aligned}
a_1 &= -\partial_x\left(\frac{\delta_1\delta_3}{\delta_5^2}\right) d_0^{1/2} r_{2,a}^*(\omega k) e^{-t\Phi_{32}} e^{-(t/2)\Phi_{21}(\zeta,k_0)},\\
a_2 &= -\frac{\delta_1\delta_3}{\delta_5^2}\partial_x(d_0^{1/2}) r_{2,a}^*(\omega k) e^{-t\Phi_{32}} e^{-(t/2)\Phi_{21}(\zeta,k_0)},\\
a_3 &= -\frac{\delta_1\delta_3}{\delta_5^2} d_0^{1/2}\partial_x(r_{2,a}^*(\omega k)) e^{-t\Phi_{32}} e^{-(t/2)\Phi_{21}(\zeta,k_0)},\\
a_4 &= -\frac{\delta_1\delta_3}{\delta_5^2} d_0^{1/2} r_{2,a}^*(\omega k)\partial_x(e^{-t\Phi_{32}}) e^{-(t/2)\Phi_{21}(\zeta,k_0)},\\
a_5 &= -\frac{\delta_1\delta_3}{\delta_5^2} d_0^{1/2} r_{2,a}^*(\omega k) e^{-t\Phi_{32}}\partial_x(e^{-(t/2)\Phi_{21}(\zeta,k_0)}).
\end{aligned}$$

We claim that $\|a_j\|_{(L^1\cap L^\infty)(\mathcal{X}_1^\epsilon)} \le Ce^{-ctk_0^2}$ for $j=1,\dots,5$. These bounds follow from arguments which are similar to those given for $(\tilde{v}-v^X)_{23}$, but more estimates are required. For a_1, we note that $\partial_x(\delta_1\delta_3/\delta_5^2)$ has a pole at $k=k_0$ (see (4-19)) which is canceled by the zero of $r_{2,a}^*(\omega k)$ (see (4-4a)). For a_2 and a_3, we use (5-6b) and (4-4b), respectively. For a_4, we note that $\partial_x(t\Phi_{32}) = \partial_\zeta(\Phi_{32}) = (1-\omega)k$, and for a_5, we observe that $\partial_x(t\Phi_{21}(\zeta,k_0)) = \frac{1}{2}\partial_{k_0}\Phi_{21}(\zeta,k_0) = \omega(\omega-1)k_0$. Therefore, we arrive at

$$\|\partial_x(\tilde{v}-v^X)_{23}\|_{(L^1\cap L^\infty)(\mathcal{X}_1^\epsilon)} \le Ce^{-ctk_0^2}.$$

We next consider the (21)-entry of $\tilde{v}-v^X$. Since $q=r_1(k_0)$, from (4-4e) it follows $\hat{r}_{1,a}^*(k_0) = \hat{r}_1^*(k_0) = \bar{q}/(1-|q|^2)$. Furthermore,

$$e^{(t/4)|\mathrm{Re}\,\Phi_{21}(\zeta,k)|} = e^{(t/4)|\mathrm{Re}(\Phi_{21}(\zeta,k)-\Phi_{21}(\zeta,k_0))|} = e^{(1/4)|\mathrm{Re}(iz^2/2)|} \le e^{|z|^2/8}.$$

Thus $|(\tilde{v}-v^X)_{21}|$ can be estimated as

$$\begin{aligned}
|(\tilde{v}-v^X)_{21}| &= |e^{-2i\nu\ln_0(z)} d_1^{-1}\hat{r}_{1,a}^*(k) e^{iz^2/2} - \hat{r}_{1,a}^*(k_0) e^{-2i\nu\ln_0(z)} e^{iz^2/2}|\\
&= |e^{-2i\nu\ln_0(z)}|\,|(d_1^{-1}-1)\hat{r}_{1,a}^*(k) + (\hat{r}_{1,a}^*(k)-\hat{r}_{1,a}^*(k_0))|\,|e^{iz^2/2}|\\
&\le C(|d_1^{-1}-1|\,|\hat{r}_{1,a}^*(k)| + |\hat{r}_{1,a}^*(k)-\hat{r}_1^*(k_0)|)e^{-|z|^2/2}.\\
&\le C(|d_1^{-1}-1| + |k-k_0|)e^{(t/4)|\mathrm{Re}\,\Phi_{21}(\zeta,k)|}e^{-|z|^2/2}.\\
&\le C(|d_1^{-1}-1| + |k-k_0|)e^{-ct|k-k_0|^2}, \quad k\in\mathcal{X}_1^\epsilon,
\end{aligned}$$

where we have used (4-4e) and (4-4f). Utilizing (5-7a), this gives

$$|(\tilde{v}-v^X)_{21}| \le C|k-k_0|(1+|\ln|k-k_0||)e^{-ct|k-k_0|^2}, \quad k \in \mathcal{X}_1^\epsilon.$$

Hence

$$\|(\tilde{v}-v^X)_{21}\|_{L^1(\mathcal{X}_1^\epsilon)} \le C\int_0^\infty u(1+|\ln u|)e^{-ctu^2}\,du \le Ct^{-1}\ln t$$

and

$$\|(\tilde{v}-v^X)_{21}\|_{L^\infty(\mathcal{X}_1^\epsilon)} \le C\sup_{u\ge 0} u(1+|\ln u|)e^{-ctu^2} \le Ct^{-1/2}\ln t.$$

To analyze $\partial_x(\tilde{v}-v^X)_{21}$, we split it into three parts as follows:

$$\partial_x(\tilde{v}-v^X)_{21} = b_1+b_2+b_3,$$

where

$$\begin{aligned}
b_1 &= \partial_x(e^{-2i\nu\ln_0(z)})\big((d_1^{-1}-1)\hat{r}_{1,a}^*(k)+(\hat{r}_{1,a}^*(k)-\hat{r}_{1,a}^*(k_0))\big)e^{iz^2/2},\\
b_2 &= e^{-2i\nu\ln_0(z)}\partial_x\big((d_1^{-1}-1)\hat{r}_{1,a}^*(k)+(\hat{r}_{1,a}^*(k)-\hat{r}_{1,a}^*(k_0))\big)e^{iz^2/2},\\
b_3 &= e^{-2i\nu\ln_0(z)}\big((d_1^{-1}-1)\hat{r}_{1,a}^*(k)+(\hat{r}_{1,a}^*(k)-\hat{r}_{1,a}^*(k_0))\big)\partial_x e^{iz^2/2}.
\end{aligned}$$

For b_1, we use that $|\partial_x(e^{-2i\nu\ln_0(z)})| \le C/(t(k-k_0))$ for $k\in\mathcal{X}_1^\epsilon$, and thus, by (4-4),

$$\begin{aligned}
\|b_1\|_{L^1(\mathcal{X}_1^\epsilon)} &\le Ct^{-1}\int_0^\infty (1+\ln u)e^{-ctu^2}\,du \le Ct^{-3/2}\ln t,\\
\|b_1\|_{L^\infty(\mathcal{X}_1^\epsilon)} &\le Ct^{-1}\sup_{u\ge 0}(1+\ln u)e^{-ctu^2} \le Ct^{-1}\ln t.
\end{aligned}$$

The norms of b_2 and b_3 are estimated in a similar way. This completes the proof of (5-8).

The variable z goes to infinity as $t\to\infty$ if $k\in\partial D_\epsilon(k_0)$, because

$$|z| = 3^{1/4}\sqrt{2t}|k-k_0|.$$

Thus (A-3) yields

$$m^X(q(\zeta), z(\zeta,k)) = I + \frac{m_1^X(q(\zeta))}{3^{1/4}\sqrt{2t}(k-k_0)} + O(t^{-1}), \quad t\to\infty,$$

uniformly with respect to $k\in\partial D_\epsilon(k_0)$ and $\zeta\in\mathcal{I}$, and this asymptotic formula can be differentiated with respect to x without increasing the error term. Recalling the definition (5-4) of m^{k_0}, this gives

$$(m^{k_0})^{-1} - I = -\frac{Y(\zeta,t)m_1^X(q(\zeta))Y(\zeta,t)^{-1}}{3^{1/4}\sqrt{2t}(k-k_0)} + O(t^{-1}), \quad t\to\infty, \tag{5-13}$$

uniformly for $k\in\partial D_\epsilon(k_0)$ and $\zeta\in\mathcal{I}$. In view of (5-5), the asymptotics (5-13) can be differentiated with respect to x. This proves (5-9). Equation (5-10) follows from (5-13) and Cauchy’s formula. ☐

6. A small-norm RH problem

We use the symmetry

$$m^{k_0}(x,t,k) = \mathcal{A}m^{k_0}(x,t,\omega k)\mathcal{A}^{-1}$$

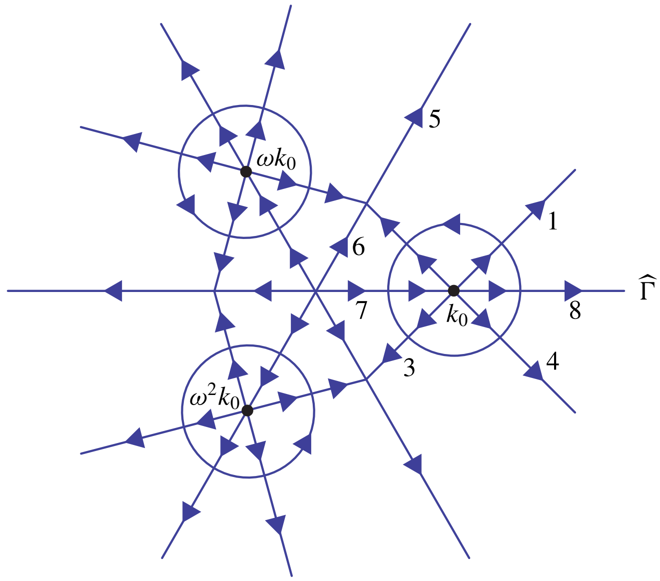

Figure 12. The contour $\widehat{\Gamma} = \Gamma^{(3)} \cup \partial\mathcal{D}$ in the complex k-plane.

to extend the domain of definition of m^{k_0} from $D_\epsilon(k_0)$ to $\mathcal{D}$, where we recall that $\mathcal{D} = D_\epsilon(k_0) \cup \omega D_\epsilon(k_0) \cup \omega^2 D_\epsilon(k_0)$. We will show that the solution $\hat{m}(x,t,k)$ defined by

$$\hat{m} = \begin{cases} m^{(3)}(m^{k_0})^{-1}, & k \in \mathcal{D}, \\ m^{(3)}, & \text{elsewhere}, \end{cases}$$

is small for large t. Let $\widehat{\Gamma} = \Gamma^{(3)} \cup \partial\mathcal{D}$ be the contour displayed in Figure 12 and define the jump matrix $\hat{v}$ by

$$\hat{v} = \begin{cases} v^{(3)}, & k \in \widehat{\Gamma} \setminus \overline{\mathcal{D}}, \\ (m^{k_0})^{-1}, & k \in \partial\mathcal{D}, \\ m_-^{k_0} v^{(3)} (m_+^{k_0})^{-1}, & k \in \widehat{\Gamma} \cap \mathcal{D}. \end{cases}$$

The function $\hat{m}$ satisfies the following RH problem.

RH problem 6.1 (RH problem for $\hat{m}$). *Find a 3×3-matrix-valued function $\hat{m}(x,t,\cdot) \in I + \dot{E}^3(\mathbb{C} \setminus \widehat{\Gamma})$ such that $\hat{m}_+(x,t,k) = \hat{m}_-(x,t,k)\hat{v}(x,t,k)$ for a.e. $k \in \widehat{\Gamma}$.*

Let $\widehat{\mathcal{X}}^\epsilon$ denote the union of the cross $\mathcal{X}^\epsilon$ and its images under the maps $k \mapsto \omega k$ and $k \mapsto \omega^2 k$, i.e., $\widehat{\mathcal{X}}^\epsilon = \mathcal{X}^\epsilon \cup \omega\mathcal{X}^\epsilon \cup \omega^2\mathcal{X}^\epsilon$. Define the contour Γ' by

$$\Gamma' = \widehat{\Gamma} \setminus (\Gamma \cup \widehat{\mathcal{X}}^\epsilon \cup \partial\mathcal{D}).$$

Lemma 6.2. *Let $\hat{w} = \hat{v} - I$. The following estimates hold uniformly for $t \geq 2$ and $\zeta \in \mathcal{I}$:*

$$\|(1+|\cdot|)\partial_x^l \hat{w}\|_{(L^1 \cap L^\infty)(\Gamma)} \leq \frac{C}{k_0 t}, \tag{6-1a}$$

$$\|(1+|\cdot|)\partial_x^l \hat{w}\|_{(L^1 \cap L^\infty)(\Gamma')} \leq C e^{-ct}, \tag{6-1b}$$

$$\|\partial_x^l \hat{w}\|_{(L^1\cap L^\infty)(\partial\mathcal{D})} \le Ct^{-1/2}, \tag{6-1c}$$

$$\|\partial_x^l \hat{w}\|_{L^1(\widehat{\mathcal{X}}^\epsilon)} \le Ct^{-1}\ln t, \tag{6-1d}$$

$$\|\partial_x^l \hat{w}\|_{L^\infty(\widehat{\mathcal{X}}^\epsilon)} \le Ct^{-1/2}\ln t, \tag{6-1e}$$

with $l = 0, 1$.

Proof. Using that $\partial_\zeta^l m_\pm^{k_0}$ and its inverse are uniformly bounded for $k \in \widehat{\Gamma} \cap \mathcal{D}$ and $l = 0, 1$, the estimate (6-1a) follows from Lemma 4.7.

The contour Γ' consists of the set $\left(\bigcup_{j=1}^4 \Gamma_j^{(3)}\right) \setminus \overline{\mathcal{D}}$ and the images of this set under the rotations $k \mapsto \omega k$ and $k \mapsto \omega^2 k$. We estimate the L^1 and L^∞ norms of $(1+|\cdot|)\partial_x^l \hat{w}$ on $\Gamma_j^{(3)} \setminus \overline{\mathcal{D}}$ for $j = 1$; similar arguments apply when $j = 2, 3, 4$, and (6-1b) then follows by symmetry. We parametrize $\Gamma_1^{(3)} \setminus \overline{\mathcal{D}}$ by $k = k_0 + ue^{\pi i/4}$, $u > k_0/2$. Only the (21) and (23) elements of $\hat{w} = v_1^{(3)} - I$ are nonzero. Using (4-4a), (4-14), and (5-12), the (23)-entry can be estimated as

$$|\hat{w}_{23}(x, t, k_0 + ue^{\pi i/4})| \le C|r_{2,a}^*(\omega k)|e^{-t\Phi_{32}} \le Ce^{-(3t/4)\Phi_{32}} \le Ce^{-ct(k_0+u)^2}.$$

The analysis of $|\partial_x \hat{w}_{23}|$ is similar. Using (4-4e), (4-14), and the identity

$$\operatorname{Re} \Phi_{21}(\zeta, k_0 + ue^{\pi i/4}) = -\sqrt{3}u^2, \quad u \ge 0, \tag{6-2}$$

the (21)-entry can be estimated as

$$|\hat{w}_{21}(x, t, k_0 + ue^{\pi i/4})| \le C|\hat{r}_{1,a}^*|e^{t\Phi_{21}} \le Ce^{ct\Phi_{21}} \le Ce^{-ctu^2}, \quad u \ge 0.$$

Using in addition (4-4f) and (4-19), we conclude that $|\partial_x \hat{w}_{21}(x, t, k_0 + ue^{\pi i/4})| \le Ce^{-ctu^2}$. Hence

$$|\partial_x^l \hat{w}(x, t, k_0 + ue^{\pi i/4})| \le Ce^{-ctu^2}, \quad u > k_0/2,\ l = 0, 1.$$

It follows that the L^1 and L^∞ norms of $(1+|\cdot|)\partial_x^l \hat{w}$, $l = 0, 1$, are $O(e^{-ct})$ as $t \to \infty$ on $\Gamma_1^{(3)} \setminus \overline{\mathcal{D}}$. This proves (6-1b).

The estimates in (6-1c) are immediate from (5-9).

For $k \in \mathcal{X}^\epsilon$, we have $\hat{w} = m_-^{k_0}(v^{(3)} - v^{k_0})(m_+^{k_0})^{-1}$, so (6-1d) and (6-1e) follow from (5-8) combined with the fact that $\partial_\zeta^l m_\pm^{k_0}$ and its inverse are uniformly bounded for $k \in \widehat{\Gamma} \cap \mathcal{D}$ and $l = 0, 1$. ☐

For a function h defined on $\widehat{\Gamma}$, the Cauchy transform $\widehat{\mathcal{C}}h$ is defined by

$$(\widehat{\mathcal{C}}h)(z) = \frac{1}{2\pi i}\int_{\widehat{\Gamma}} \frac{h(z')dz'}{z'-z}, \quad z \in \mathbb{C} \setminus \widehat{\Gamma}.$$

If $h \in \dot{L}^3(\widehat{\Gamma})$, then $\widehat{\mathcal{C}}h \in \dot{E}^3(\mathbb{C} \setminus \widehat{\Gamma})$, and the left and right nontangential boundary values of $\widehat{\mathcal{C}}h$, which we denote by $\widehat{\mathcal{C}}_+h$ and $\widehat{\mathcal{C}}_-h$ respectively, exist a.e. on $\widehat{\Gamma}$ and belong to $\dot{L}^3(\widehat{\Gamma})$; furthermore, $\widehat{\mathcal{C}}_\pm \in \mathcal{B}(\dot{L}^3(\widehat{\Gamma}))$ and $\widehat{\mathcal{C}}_+ - \widehat{\mathcal{C}}_- = I$, where $\mathcal{B}(\dot{L}^3(\widehat{\Gamma}))$ denotes the space of bounded linear operators on $\dot{L}^3(\widehat{\Gamma})$; see [Lenells 2018, Theorems 4.1 and 4.2].

The estimates in Lemma 6.2 show that

$$\begin{cases} \|(1+|\cdot|)\partial_x^l \hat{w}\|_{L^1(\widehat{\Gamma})} \le Ct^{-1/2}, \\ \|(1+|\cdot|)\partial_x^l \hat{w}\|_{L^\infty(\widehat{\Gamma})} \le Ct^{-1/2}\ln t, \end{cases} \quad t \ge 2,\ \zeta \in \mathcal{I},\ l = 0, 1, \tag{6-3}$$

and hence, employing the general identity $\|f\|_{L^p} \le \|f\|_{L^1}^{1/p} \|f\|_{L^\infty}^{(p-1)/p}$,

$$\|(1+|\cdot|)\partial_x^l \hat{w}\|_{L^p(\widehat{\Gamma})} \le Ct^{-1/2}(\ln t)^{(p-1)/p}, \quad t \ge 2,\ \zeta \in \mathcal{I},\ l = 0, 1, \tag{6-4}$$

for each $1 \le p \le \infty$. The estimates (6-4) imply that $\hat{w} \in \dot{L}^3(\widehat{\Gamma}) \cap L^\infty(\widehat{\Gamma})$. We define $\widehat{\mathcal{C}}_{\hat{w}} = \widehat{\mathcal{C}}_{\hat{w}(x,t,\cdot)} : \dot{L}^3(\widehat{\Gamma}) + L^\infty(\widehat{\Gamma}) \to \dot{L}^3(\widehat{\Gamma})$ by $\widehat{\mathcal{C}}_{\hat{w}} h := \widehat{\mathcal{C}}_-(h\hat{w})$.

Lemma 6.3. *There exists a $T > 0$ such that $I - \widehat{\mathcal{C}}_{\hat{w}(x,t,\cdot)} \in \mathcal{B}(\dot{L}^3(\widehat{\Gamma}))$ is invertible whenever $t \ge T$ and $\zeta \in \mathcal{I}$.*

Proof. Let $K := \|\widehat{\mathcal{C}}_-\|_{\mathcal{B}(\dot{L}^3(\widehat{\Gamma}))}$. For each $h \in \dot{L}^3(\widehat{\Gamma})$, we have $\|\widehat{\mathcal{C}}_{\hat{w}} h\|_{\dot{L}^3(\widehat{\Gamma})} \le K \|\hat{w}\|_{L^\infty(\widehat{\Gamma})} \|h\|_{\dot{L}^3(\widehat{\Gamma})}$, and thus $\|\widehat{\mathcal{C}}_{\hat{w}}\|_{\mathcal{B}(\dot{L}^3(\widehat{\Gamma}))} \le K \|\hat{w}\|_{L^\infty(\widehat{\Gamma})}$. By (6-3), there exists a $T > 0$ such that $\|\hat{w}\|_{L^\infty(\widehat{\Gamma})} < K^{-1}$ for $t \ge T$. ☐

In view of Lemma 6.3, we may define $\hat{\mu}(x, t, k)$ for $k \in \widehat{\Gamma}$, $t \ge T$, and $\zeta = x/t \in \mathcal{I}$ by

$$\hat{\mu} = I + (I - \widehat{\mathcal{C}}_{\hat{w}})^{-1} \widehat{\mathcal{C}}_{\hat{w}} I \in I + \dot{L}^3(\widehat{\Gamma}). \tag{6-5}$$

Lemma 6.4. *For $t \ge T$ and $\zeta \in \mathcal{I}$, there exists a unique solution $\hat{m} \in I + \dot{E}^3(\mathbb{C} \setminus \widehat{\Gamma})$ of RH problem 3.1. This solution is given by*

$$\hat{m}(x, t, k) = I + \widehat{\mathcal{C}}(\hat{\mu}\hat{w}) = I + \frac{1}{2\pi i} \int_{\widehat{\Gamma}} \hat{\mu}(x, t, s)\hat{w}(x, t, s) \frac{ds}{s - k}. \tag{6-6}$$

Proof. Since $\hat{w} \in \dot{L}^3(\widehat{\Gamma}) \cap L^\infty(\widehat{\Gamma})$, this follows from [Lenells 2018, Proposition 5.8]. ☐

Lemma 6.5. *Let $1 < p < \infty$. For all sufficiently large t, we have*

$$\|\partial_x^l (\hat{\mu} - I)\|_{L^p(\widehat{\Gamma})} \le Ct^{-1/2}(\ln t)^{(p-1)/p}, \quad l = 0, 1,\ \zeta \in \mathcal{I}.$$

Proof. Let $K_p := \|\widehat{\mathcal{C}}_-\|_{\mathcal{B}(L^p(\widehat{\Gamma}))} < \infty$ and assume t is so large that $\|\hat{w}\|_{L^\infty(\widehat{\Gamma})} < K_p^{-1}$. Standard estimates using the Neumann series show that

$$\|\hat{\mu} - I\|_{L^p(\widehat{\Gamma})} \le \sum_{j=1}^{\infty} \|\widehat{\mathcal{C}}_{\hat{w}}\|_{\mathcal{B}(L^p(\widehat{\Gamma}))}^{j-1} \|\widehat{\mathcal{C}}_{\hat{w}} I\|_{L^p(\widehat{\Gamma})} \le \sum_{j=1}^{\infty} K_p^j \|\hat{w}\|_{L^\infty(\widehat{\Gamma})}^{j-1} \|\hat{w}\|_{L^p(\widehat{\Gamma})} = \frac{K_p \|\hat{w}\|_{L^p(\widehat{\Gamma})}}{1 - K_p \|\hat{w}\|_{L^\infty(\widehat{\Gamma})}}.$$

The claim for $l = 0$ now follows from (6-3) and (6-4). Using that

$$\partial_x(\hat{\mu} - I) = \partial_x \sum_{j=1}^{\infty} (\widehat{\mathcal{C}}_{\hat{w}})^j I = \sum_{j=1}^{\infty} [(\partial_x \widehat{\mathcal{C}}_{\hat{w}})\widehat{\mathcal{C}}_{\hat{w}} \cdots \widehat{\mathcal{C}}_{\hat{w}} + \cdots + \widehat{\mathcal{C}}_{\hat{w}} \cdots \widehat{\mathcal{C}}_{\hat{w}} (\partial_x \widehat{\mathcal{C}}_{\hat{w}})] I,$$

we find

$$\begin{aligned}
\|\partial_x(\hat{\mu} - I)\|_{L^p(\widehat{\Gamma})} &\le \sum_{j=2}^{\infty} (j-1) \|\widehat{\mathcal{C}}_{\hat{w}}\|_{\mathcal{B}(L^p(\widehat{\Gamma}))}^{j-2} \|\partial_x \widehat{\mathcal{C}}_{\hat{w}}\|_{\mathcal{B}(L^p(\widehat{\Gamma}))} \|\widehat{\mathcal{C}}_{\hat{w}} I\|_{L^p(\widehat{\Gamma})} + \sum_{j=1}^{\infty} \|\widehat{\mathcal{C}}_{\hat{w}}\|_{\mathcal{B}(L^p(\widehat{\Gamma}))}^{j-1} \|\partial_x \widehat{\mathcal{C}}_{\hat{w}} I\|_{L^p(\widehat{\Gamma})} \\
&\le C \sum_{j=2}^{\infty} j K_p^{j-2} \|\hat{w}\|_{L^\infty(\widehat{\Gamma})}^{j-2} \|\partial_x \hat{w}\|_{L^\infty(\widehat{\Gamma})} \|\hat{w}\|_{L^p(\widehat{\Gamma})} + \sum_{j=1}^{\infty} K_p^j \|\hat{w}\|_{L^\infty(\widehat{\Gamma})}^{j-1} \|\partial_x \hat{w}\|_{L^p(\widehat{\Gamma})} \\
&\le C \frac{\|\partial_x \hat{w}\|_{L^\infty(\widehat{\Gamma})} \|\hat{w}\|_{L^p(\widehat{\Gamma})} + \|\partial_x \hat{w}\|_{L^p(\widehat{\Gamma})}}{1 - K_p \|\hat{w}\|_{L^\infty(\widehat{\Gamma})}}
\end{aligned}$$

and the claim for $l = 1$ follows from another application of (6-3) and (6-4). ☐

6A. ***Asymptotics of $\hat{m}$.*** The following nontangential limit exists as $k \to \infty$:

$$L(x,t) := \angle\lim_{k\to\infty} k(\hat{m}(x,t,k) - I) = -\frac{1}{2\pi i}\int_{\widehat{\Gamma}} \hat{\mu}(x,t,k)\hat{w}(x,t,k)\,dk.$$

Lemma 6.6. *As $t \to \infty$,*

$$L(x,t) = -\frac{1}{2\pi i}\int_{\partial\mathcal{D}} \hat{w}(x,t,k)\,dk + O(t^{-1}\ln t) \tag{6-7}$$

and (6-7) *can be differentiated termwise with respect to x without increasing the error term.*

Proof. Since

$$L(x,t) = -\frac{1}{2\pi i}\int_{\partial\mathcal{D}} \hat{w}(x,t,k)\,dk + L_1(x,t) + L_2(x,t),$$

where

$$L_1(x,t) = -\frac{1}{2\pi i}\int_{\widehat{\Gamma}\setminus\partial\mathcal{D}} \hat{w}(x,t,k)\,dk, \quad L_2(x,t) = -\frac{1}{2\pi i}\int_{\widehat{\Gamma}} (\hat{\mu}(x,t,k) - I)\hat{w}(x,t,k)\,dk,$$

the lemma follows from Lemmas 6.2 and 6.5 and straightforward estimates. □

We infer from (5-10) that the function F defined by

$$F(\zeta,t) = -\frac{1}{2\pi i}\int_{\partial D_\epsilon(k_0)} \hat{w}(x,t,k)\,dk = -\frac{1}{2\pi i}\int_{\partial D_\epsilon(k_0)} ((m^{k_0})^{-1} - I)\,dk$$

satisfies

$$F(\zeta,t) = \frac{Y(\zeta,t)m_1^X(q(\zeta))Y(\zeta,t)^{-1}}{3^{1/4}\sqrt{2}\sqrt{t}} + O(t^{-1}\ln t) \quad \text{as } t \to \infty.$$

The symmetry properties of $\hat{v}$ imply that both $\mathcal{A}\hat{m}(x,t,\omega k)\mathcal{A}^{-1}$ and $\hat{m}(x,t,k)$ satisfy RH problem 6.1; by uniqueness they must be equal, i.e.,

$$\hat{m}(x,t,k) = \mathcal{A}\hat{m}(x,t,\omega k)\mathcal{A}^{-1}, \quad k \in \mathbb{C}\setminus\widehat{\Gamma}.$$

It follows that $\hat{\mu}$ and $\hat{w}$ also obey this symmetry. Using this in (6-7), we find that the leading contribution from $\partial\mathcal{D}$ to the right-hand side of (6-7) is

$$\begin{aligned}-\frac{1}{2\pi i}\int_{\partial\mathcal{D}} \hat{w}(x,t,k)\,dk &= -\frac{1}{2\pi i}\biggl(\int_{\partial D_\epsilon(k_0)} + \int_{\omega\partial D_\epsilon(k_0)} + \int_{\omega^2\partial D_\epsilon(k_0)}\biggr)\hat{w}(x,t,k)\,dk \\ &= F(\zeta,t) + \omega\mathcal{A}^{-1}F(\zeta,t)\mathcal{A} + \omega^2\mathcal{A}^{-2}F(\zeta,t)\mathcal{A}^2.\end{aligned}$$

Therefore, (6-7) implies that

$$\begin{aligned}&\partial_x^l \lim_{k\to\infty} k(\hat{m}(x,t,k) - I) \\ &\quad = \partial_x^l\biggl(\frac{\sum_{j=0}^2 \omega^j\mathcal{A}^{-j}Y(\zeta,t)m_1^X(q(\zeta))Y(\zeta,t)^{-1}\mathcal{A}^j}{3^{1/4}\sqrt{2}\sqrt{t}}\biggr) + O(t^{-1}\ln t), \quad t \to \infty,\ l = 0, 1,\end{aligned} \tag{6-8}$$

uniformly for $\zeta \in \mathcal{I}$.

7. Asymptotics of $u(x,t)$

Recall from the discussion in Section 3 (see Proposition 3.4 and Lemma 3.5) that

$$u(x,t) = -\frac{3}{2}\frac{\partial}{\partial x}\big(\lim_{k\to\infty} k(n_3(x,t,k)-1)\big),$$

where $n = (\omega, \omega^2, 1)m$. Taking the transformations of Section 4 into account, we can write

$$m = \hat{m}H^{-1}\Delta^{-1}G^{-1}$$

for all $k \in \mathbb{C}\setminus\overline{\mathcal{D}}$, where G, Δ, H are defined in (4-8), (4-20), and (4-23), respectively. It follows that

$$u(x,t) = -\frac{3}{2}\frac{\partial}{\partial x}\overset{\angle}{\lim_{k\to\infty}} k(\hat{n}_3(x,t,k)-1) - \frac{3}{2}\frac{d}{dx}\overset{\angle}{\lim_{k\to\infty}} k\left(\frac{\delta_5(\zeta,k)}{\delta_3(\zeta,k)}-1\right), \tag{7-1}$$

where $\hat{n} = (\omega, \omega^2, 1)\hat{m}$. Thus, utilizing (6-8) and the fact that $\overline{\Gamma(i\nu)} = \Gamma(-i\nu)$,

$$\begin{aligned}
u(x,t) &= -\frac{3}{2}\frac{d}{dx}\left((\omega\ \ \omega^2\ \ 1)\frac{\sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}Y(\zeta,t)m_1^X(q(\zeta))Y(\zeta,t)^{-1}\mathcal{A}^j}{3^{1/4}\sqrt{2}\sqrt{t}}\right)_3 + O(t^{-1}\ln t)\\
&= -\frac{3}{2}\frac{d}{dx}\left(\frac{\omega^2 d_0^{-1}e^{t\Phi_{21}(\zeta,k_0)}\beta_{21} + \omega d_0 e^{-t\Phi_{21}(\zeta,k_0)}\beta_{12}}{3^{1/4}\sqrt{2}\sqrt{t}}\right) + O(t^{-1}\ln t)\\
&= -\frac{3\times 2}{2\times 3^{1/4}\sqrt{2t}}\frac{d}{dx}\operatorname{Re}(\omega^2 d_0^{-1}e^{t\Phi_{21}(\zeta,k_0)}\beta_{21}) + O(t^{-1}\ln t), \quad t\to\infty.
\end{aligned}$$

Using the identities

$$\begin{aligned}
|\Gamma(i\nu)| &= \frac{\sqrt{2\pi}}{\sqrt{\nu}\sqrt{e^{\pi\nu}-e^{-\pi\nu}}} = \frac{\sqrt{2\pi}}{\sqrt{\nu}e^{\pi\nu/2}|q|},\\
\delta_3^{-1}(\zeta,k_0)\delta_5^{-1}(\zeta,k_0) &= \exp\left[i\nu\log(3k_0^2) + \frac{1}{\pi i}\int_{k_0}^{\infty}\log|\omega k_0 - s|\,d\ln(1-|r_1(s)|^2)\right],
\end{aligned}$$

we conclude that, as $t\to\infty$,

$$\begin{aligned}
&u(x,t)\\
&= -\frac{3^{3/4}}{\sqrt{2t}}\frac{d}{dx}\operatorname{Re}\left\{\sqrt{\nu}\exp\left[\frac{4\pi i}{3} + i\nu\ln(6\sqrt{3}tk_0^2) - i\sqrt{3}k_0^2 t\right.\right.\\
&\qquad\left.\left. -\frac{1}{\pi i}\int_{k_0}^{\infty}\ln\frac{|s-k_0|}{|s-\omega k_0|}\,d\ln(1-|r_1(s)|^2) + i\left(\frac{\pi}{4} - \arg q - \arg\Gamma(i\nu)\right)\right]\right\} + O(t^{-1}\ln t)\\
&= -\frac{3^{3/4}\sqrt{\nu}}{\sqrt{2t}}\frac{d}{dx}\cos\left(\frac{19\pi}{12} + \nu\ln(6\sqrt{3}tk_0^2) - \sqrt{3}k_0^2 t - \arg q\right.\\
&\qquad\left. - \arg\Gamma(i\nu) + \frac{1}{\pi}\int_{k_0}^{\infty}\ln\frac{|s-k_0|}{|s-\omega k_0|}\,d\ln(1-|r_1(s)|^2)\right) + O(t^{-1}\ln t)\\
&= -\frac{3^{5/4}k_0\sqrt{\nu}}{\sqrt{2t}}\sin\left(\frac{19\pi}{12} + \nu\ln(6\sqrt{3}tk_0^2) - \sqrt{3}k_0^2 t - \arg q\right.\\
&\qquad\left. - \arg\Gamma(i\nu) + \frac{1}{\pi}\int_{k_0}^{\infty}\ln\frac{|s-k_0|}{|s-\omega k_0|}\,d\ln(1-|r_1(s)|^2)\right) + O(t^{-1}\ln t)
\end{aligned}$$

uniformly for $\zeta \in \mathcal{I}$. This proves (2-9) and completes the proof of Theorem 2.4.

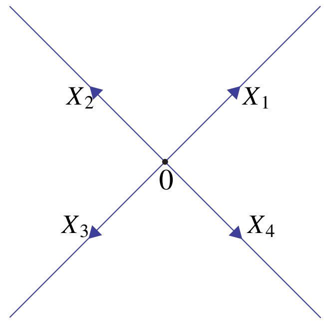

Figure 13. The contour $X = X_1 \cup X_2 \cup X_3 \cup X_4$ defined in (A-1).

Appendix: Exact solution on a cross

Let $X = X_1 \cup \cdots \cup X_4 \subset \mathbb{C}$ be the cross defined by

$$\begin{aligned} X_1 &= \{se^{i\pi/4} \mid 0 \le s < \infty\}, \qquad X_2 = \{se^{3i\pi/4} \mid 0 \le s < \infty\}, \\ X_3 &= \{se^{-3i\pi/4} \mid 0 \le s < \infty\}, \quad X_4 = \{se^{-i\pi/4} \mid 0 \le s < \infty\}, \end{aligned} \tag{A-1}$$

and oriented away from the origin; see Figure 13. Let $\mathbb{D} \subset \mathbb{C}$ denote the open unit disk and define the function $\nu : \mathbb{D} \to (0, \infty)$ by $\nu(q) = -\frac{1}{2\pi} \ln(1 - |q|^2)$. We consider the following family of RH problems parametrized by $q \in \mathbb{D}$.

RH problem A.1 (RH problem for m^X). *Find a* 3×3*-matrix-valued function* $m^X(q, z)$ *with the following properties*:

(a) $m^X(q, \cdot) : \mathbb{C} \setminus X \to \mathbb{C}^{3\times 3}$ *is analytic.*

(b) *The limits of* $m^X(q, z)$ *as* z *approaches* $X \setminus \{0\}$ *from the left and right exist*, *are continuous on* $X \setminus \{0\}$, *and are related by*

$$m^X_+(q, z) = m^X_-(q, z) v^X(q, z), \quad k \in X \setminus \{0\},$$

where the jump matrix $v^X(q, z)$ *is defined by*

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 \\ \frac{\bar{q}}{1-|q|^2} z^{-2i\nu(q)} e^{iz^2/2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } z \in X_1, \qquad \begin{pmatrix} 1 & q z^{2i\nu(q)} e^{-iz^2/2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } z \in X_2, \\ &\begin{pmatrix} 1 & 0 & 0 \\ -\bar{q} z^{-2i\nu(q)} e^{iz^2/2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } z \in X_3, \qquad \begin{pmatrix} 1 & \frac{-q}{1-|q|^2} z^{2i\nu(q)} e^{-iz^2/2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } z \in X_4, \end{aligned} \tag{A-2}$$

with the branch cut running along the positive real axis, i.e., $z^{2i\nu(q)} = e^{2i\nu(q) \ln_0(z)}$.

(c) $m^X(q, z) = I + O(z^{-1})$ *as* $z \to \infty$.

(d) $m^X(q, z) = O(1)$ *as* $z \to 0$.

The proof of the following lemma is standard and relies on deriving an explicit formula for the solution m^X in terms of parabolic cylinder functions [Its 1981].

Lemma A.2 (the solution m^X). *The RH problem A.1 has a unique solution $m^X(q, z)$ for each $q \in \mathbb{D}$. This solution satisfies*

$$m^X(q, z) = I + \frac{m_1^X(q)}{z} + O\left(\frac{1}{z^2}\right), \quad z \to \infty, \ q \in \mathbb{D}, \tag{A-3}$$

where the error term is uniform with respect to $\arg z \in [0, 2\pi]$ *and q in compact subsets of* $\mathbb{D}$, *and the function $m_1^X(q)$ is defined by*

$$m_1^X(q) = \begin{pmatrix} 0 & \beta_{12} & 0 \\ \beta_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad q \in \mathbb{D}, \tag{A-4}$$

where β_{12} and β_{21} are defined by

$$\beta_{12} = \frac{\sqrt{2\pi} e^{-\pi i/4} e^{-5\pi\nu/2}}{\bar{q}\Gamma(-i\nu)}, \quad \beta_{21} = \frac{\sqrt{2\pi} e^{\pi i/4} e^{3\pi\nu/2}}{q\Gamma(i\nu)}, \quad q \in \mathbb{D}.$$

Moreover, *for each compact subset K of* $\mathbb{D}$,

$$\sup_{q \in K} \sup_{z \in \mathbb{C} \setminus X} |\partial_q^l m^X(q, z)| < \infty, \quad l = 0, 1.$$

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