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THE CAUCHY PROBLEM FOR THE INFINITESIMAL MODEL IN THE REGIME OF SMALL VARIANCE

FLORIAN PATOUT

We study the asymptotic behavior of solutions of the Cauchy problem associated to a quantitative genetics model with a sexual mode of reproduction. It combines trait-dependent mortality and a nonlinear integral reproduction operator, *the infinitesimal model*. A parameter describes the standard deviation between the offspring and the mean parental traits. We show that under mild assumptions upon the mortality rate m, when the deviations are small, the solutions stay close to a Gaussian profile with small variance, uniformly in time. Moreover, we characterize accurately the dynamics of the mean trait in the population. Our study extends previous results on the existence and uniqueness of stationary solutions for the model. It relies on perturbative analysis techniques with a sharp description of the correction from the Gaussian profile.

A list of symbols can be found on page 1348.

1. Introduction

We investigate solutions $f_\varepsilon \in L^1(\mathbb{R}_+ \times \mathbb{R})$ of the Cauchy problem

$$\begin{cases} \varepsilon^2 \partial_t f_\varepsilon(t,z) + m(z) f_\varepsilon(t,z) = \mathcal{B}_\varepsilon(f_\varepsilon)(t,z), & t>0,\ z \in \mathbb{R}, \\ f_\varepsilon(0,z) = f_\varepsilon^0(z), \end{cases} \tag{$P_t f_\varepsilon$}$$

where $\mathcal{B}_\varepsilon(f)$ is the following nonlinear, homogeneous mixing operator associated with the infinitesimal model of [Fisher 1918], see also [Barton et al. 2017] for a modern perspective:

$$\mathcal{B}_\varepsilon(f)(z) := \frac{1}{\varepsilon\sqrt{\pi}} \iint_{\mathbb{R}^2} \exp\left[-\frac{1}{\varepsilon^2}\left(z - \frac{z_1+z_2}{2}\right)^2\right] f(z_1) \frac{f(z_2)}{\int_{\mathbb{R}} f(z_2')\,dz_2'}\,dz_1\,dz_2. \tag{1-1}$$

This problem originates from quantitative genetics in the context of evolutionary biology. The variable z denotes a phenotypic trait, f_ε is the distribution of the population with respect to z and m is the trait-dependent mortality rate.

The mixing operator $\mathcal{B}_\varepsilon$ models the inheritance of quantitative traits in the population, under the assumption of a sexual mode of reproduction. As formulated in (1-1), it is assumed that offspring traits are distributed normally around the mean of the parental traits $\frac{1}{2}(z_1+z_2)$, with a constant variance, here $\frac{1}{2}\varepsilon^2$. We are interested in the evolutionary dynamics resulting in the selection of well-fitted (low mortality) individuals, i.e., the concentration of the distribution around some dominant traits with standing variance.

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Keywords: infinitesimal model, asymptotic analysis, quantitative genetics, sexual reproduction, perturbative analysis.

In theoretical evolutionary biology, a broad literature deals with this model to describe sexual reproduction; see e.g., [Barfield et al. 2011; Barton et al. 2017; Bulmer 1980; Cotto and Ronce 2014; Huisman and Tufto 2012; Roughgarden 1972; Slatkin 1970; Slatkin and Lande 1976; Tufto 2000; Turelli 2017; Turelli and Barton 1994].

We are interested in the asymptotic behavior of the trait distribution f_ε as ε^2 vanishes. It is expected that the profile concentrates around some particular traits under the influence of selection.

The asymptotic description of concentration around some particular trait(s) has been extensively investigated for various linear operators $\mathcal{B}_\varepsilon$ associated with asexual reproduction such as, for instance, the diffusion operator $f_\varepsilon(t,z)+\varepsilon^2\Delta f_\varepsilon(t,z)$, or the convolution operator $(1/\varepsilon)K(z/\varepsilon)*f_\varepsilon(t,z)$ where K is a probability kernel with unit variance; see [Barles and Perthame 2007; Barles et al. 2009; Diekmann et al. 2005; Lorz et al. 2011; Perthame 2007] for the earliest investigations and [Bouin et al. 2018; Méléard and Mirrahimi 2015; Mirrahimi 2020] for the case of fat-tailed kernels K. In those linear cases, the asymptotic analysis usually leads to a Hamilton–Jacobi equation after performing the Hopf–Cole transform $u_\varepsilon = -\varepsilon\log f_\varepsilon$. Those problems require a careful well-posedness analysis for uniqueness and convergence as $\varepsilon\to 0$; see [Barles et al. 2009; Calvez and Lam 2020; Mirrahimi and Roquejoffre 2016].

Much less is known about the operator $\mathcal{B}_\varepsilon$ defined by (1-1). From a mathematical viewpoint, in the field of probability theory, [Barton et al. 2017] derived the model from a microscopic framework. In [Mirrahimi and Raoul 2013; Raoul 2017], the authors deal with a different scaling than the current small variance assumption $\varepsilon^2\ll 1$ and add a spatial structure in order to derive the celebrated Kirkpatrick and Barton system [1997].

Gaussian distributions will play a pivotal role in our analysis as they are left-invariant by the infinitesimal operator $\mathcal{B}_\varepsilon$; see [Mirrahimi and Raoul 2013; Turelli and Barton 1994]. In [Calvez et al. 2019], the authors studied special *stationary* solutions, having the form

$$\exp\left[\frac{\lambda_\varepsilon t}{\varepsilon^2}\right]F_\varepsilon(z),\quad \text{with } F_\varepsilon(z)=\frac{1}{\varepsilon\sqrt{2\pi}}\exp\left[-\frac{(z-z_*)^2}{2\varepsilon^2}-U_\varepsilon^s(z)\right].$$

In this paper we tackle the Cauchy problem $(P_t f_\varepsilon)$, and we hereby look for solutions that are close to Gaussian distributions uniformly in time of the form

$$f_\varepsilon(t,z)=\frac{1}{\varepsilon\sqrt{2\pi}}\exp\left[\frac{\lambda(t)}{\varepsilon^2}-\frac{(z-z_*(t))^2}{2\varepsilon^2}-U_\varepsilon(t,z)\right]. \tag{1-2}$$

The scalar function λ measures the growth (or decay according to its sign) of the population. The mean of the Gaussian density, z_*, is also the trait at which the population concentrates when $\varepsilon\to 0$. The pair (λ, z_*) will be determined by the analysis at all times. It is somehow related to invariant properties of the operator $\mathcal{B}_\varepsilon$. The function U_ε measures the deviation from the Gaussian profile induced by the selection function m. It is a cornerstone of our analysis that U_ε is Lipschitz continuous with respect to z, uniformly in t and ε. Plugging the transformation (1-2) into $(P_t f_\varepsilon)$ yields the following equivalent problem:

$$-\varepsilon^2\partial_t U_\varepsilon(t,z)+\dot\lambda(t)+(z-z_*(t))\dot z_*(t)+m(z) \\ = I_\varepsilon(U_\varepsilon)(t,z)\exp[U_\varepsilon(t,z)-2U_\varepsilon(t,\bar z(t))+U_\varepsilon(t,z_*(t))], \quad (P_tU_\varepsilon)$$

where $\bar{z}(t)$ is the midpoint between z and $z_*(t)$:

$$\bar{z}(t) = \tfrac{1}{2}(z + z_*(t)),$$

and the functional I_ε is defined by

$$I_\varepsilon(U_\varepsilon)(t,z) = \frac{\iint_{\mathbb{R}^2} \exp\big[-\frac{1}{2}y_1y_2 - \frac{3}{4}(y_1^2+y_2^2) + 2U_\varepsilon(t,\bar{z}) - U_\varepsilon(t,\bar{z}+\varepsilon y_1) - U_\varepsilon(t,\bar{z}+\varepsilon y_2)\big]\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}} \exp\big[-\frac{1}{2}y^2 + U_\varepsilon(t,z_*) - U_\varepsilon(t,z_*+\varepsilon y)\big]\,dy}. \tag{1-3}$$

This functional is the residual shape of the infinitesimal operator (1-1) after suitable transformations. It was first introduced in the formal analysis of [Garnier et al. 2022] and in the study of the corresponding stationary problem in [Calvez et al. 2019]. The Lipschitz continuity of U_ε is pivotal here as it ensures that $I_\varepsilon(U_\varepsilon) \to 1$ when $\varepsilon \to 0$. Thus for small ε, we expect that $(P_t f_\varepsilon)$ is well approximated by the following problem:

$$\dot{\lambda}(t) + (z - z_*(t))\dot{z}_*(t) + m(z) = \exp[U_0(t,z) - 2U_0(t,\bar{z}(t)) + U_0(t,z_*(t))]. \tag{1-4}$$

Interestingly, this characterizes the dynamics of $(\lambda(t), z_*(t))$. By differentiating (1-4) and evaluating at the point $z = z_*(t)$, then simply evaluating (1-4) at $z = z_*(t)$, we find the following pair of relationships:

$$\dot{z}_*(t) + m'(z_*(t)) = 0, \tag{1-5}$$

$$\dot{\lambda}(t) + m(z_*(t)) = 1. \tag{1-6}$$

Then, a more compact way to write the limit problem for $\varepsilon = 0$ is

$$M(t,z) = \exp[U_0(t,z) - 2U_0(t,\bar{z}(t)) + U_0(t,z_*(t))], \tag{P_tU_0}$$

with the notation

$$M(t,z) := 1 + m(z) - m(z_*(t)) - m'(z_*(t))(z - z_*(t)). \tag{1-7}$$

It follows from (1-6) and (1-5) that

$$M(t,z_*(t)) = 1, \quad \partial_z M(t,z_*(t)) = 0. \tag{1-8}$$

An explicit solution of (P_tU_0) exists under the form of an infinite series:

$$V^*(t,z) := \sum_{k\ge 0} 2^k \log(M(t, z_*(t) + 2^{-k}(z - z_*(t)))). \tag{1-9}$$

This formula is obtained by noticing a recursive relation on the first derivative of $\partial_z U_0$, as in Section 2.2 of [Calvez et al. 2019]. The same recursive argument is used here in Section 7G. Interestingly, this series is convergent thanks to the relationships of (1-8). The function V^* is a solution of (P_tU_0), but not the only one. There are two degrees of freedom when solving (P_tU_0), since adding any affine function to U_0 leaves the right-hand side unchanged. Therefore, a general expression of solutions is the following, where the scalar functions p_0 and q_0 are arbitrary:

$$U_0(t,z) = p_0(t) + q_0(t)(z - z_*(t)) + V^*(t,z). \tag{1-10}$$

We have foreseen that the Lipschitz regularity of U_ε was the way to guarantee that $I_\varepsilon(U_\varepsilon) \to 1$ as $\varepsilon \to 0$. As a matter of fact, an important part of [Calvez et al. 2019] is dedicated to proving such regularity for U_ε^s, the solution of the stationary problem

$$\lambda_\varepsilon^s + m(z) = I_\varepsilon(U_\varepsilon^s)(z) \exp\bigl[U_\varepsilon^s(z) - 2U_\varepsilon^s\bigl(\tfrac{1}{2}(z + z_*^s)\bigr) + U_\varepsilon^s(z_*^s)\bigr], \quad z \in \mathbb{R}. \tag{PU_ε stat}$$

The authors introduced an appropriate functional space controlling Lipschitz bound. They were then able to show the existence of U_ε^s and its (local) uniqueness in that space. They also proved that U_ε^s was converging when $\varepsilon \to 0$ towards solutions of $(P_t U_0)$; see Figure 1 for a schematic comparison of the scope of the present article compared to previous work.

Here, to tackle the nonstationary problem $(P_t U_\varepsilon)$, we make the following assumptions of asymptotic growth on the selection function m, when $|z| \to \infty$.

Assumption 1.1. We suppose that the function m is a $\mathcal{C}^5(\mathbb{R})$ function, bounded below. We define the scalar function z_* as the gradient flow

$$\dot{z}_*(t) = -m'(z_*(t)), \quad t > 0, \tag{1-11}$$

associated to a prescribed initial data $z_*(0)$. Next, we make the following assumptions:

- We suppose that $z_*(0)$ lies next to a nondegenerate *local* minimum of m, denoted by z_*^s, such that

$$z_*(t) \xrightarrow[t\to\infty]{} z_*^s. \tag{1-12}$$

- We also require that there exists a uniform positive lower bound on M:

$$\inf_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} M(t,z) > 0. \tag{1-13}$$

- We make growth assumptions on M in the following way:

$$\text{for } k = 1, 2, 3, 4, 5, \quad (1 + |z - z_*|)^\alpha \frac{\partial_z^k M(t,z)}{M(t,z)} \in L^\infty(\mathbb{R}_+ \times \mathbb{R}), \tag{1-14}$$

for some $0 < \alpha < 1$, the same as in Definition 1.2.

- We make a final assumption upon the behavior of m at infinity, roughly that it has superlinear growth, uniformly in time:

$$\limsup_{z\to\infty} \left| \frac{M(t,\bar{z})}{M(t,z)} \right| := a < \frac{1}{2}, \quad \limsup_{z\to\infty} \left| \frac{\partial_z M(t,\bar{z})}{\partial_z M(t,z)} \right| < \infty. \tag{1-15}$$

The first assumption on m and z_* guarantees the following local convexity property, at least for times t large enough:

$$\exists\, \mu_0 > 0, \;\; \exists\, t_0 > 0, \;\; \text{such that } \forall t \ge t_0, \;\; m''(z_*(t)) \ge \mu_0. \tag{1-16}$$

Any convex function m with at least quadratic growth at infinity will satisfy Assumption 1.1, without restriction on the initial data. This type of fitness landscape is fairly standard in the asexual models, e.g., the Fisher geometrical model [Fisher 1999; Martin and Roques 2016] assumes a quadratic fitness function. However, our analysis also encompasses different scenarios with possibly multiple optima, the limiting condition is the positivity of M. This corresponds to a global assumption on the behavior of z_* and m, that reduces the choice of $z_*(0)$.[1] The relationship (1-14) corresponds to algebraic decay assumptions for M, and accordingly, it holds true if m behaves like any (at least quadratic) polynomial function as $|z| \to +\infty$ (as well as (1-15)). The shape of the selection function, even far from the optimum, changes the qualitative behavior of a population; see [Osmond and Klausmeier 2017]. A detailed discussion on the behavior of the solution whether our assumptions are satisfied or not is carried out in Section 9 with some numerical simulations displayed.

The purpose of this work is to rigorously prove the convergence of the solution of $(P_t U_\varepsilon)$ towards a particular solution of $(P_t U_0)$. Given the general shape of U_0, see (1-10), it is natural to decompose U_ε by separating the affine part from the rest:

$$U_\varepsilon(t, z) = p_\varepsilon(t) + q_\varepsilon(t)(z - z_*(t)) + V_\varepsilon(t, z). \tag{1-17}$$

We require accordingly that at all times $t > 0$,

$$V_\varepsilon(t, z_*) = \partial_z V_\varepsilon(t, z_*) = 0,$$

which is another way of saying that the pair $(p_\varepsilon, q_\varepsilon)$ tunes the affine part of U_ε. The pair $(q_\varepsilon, V_\varepsilon)$ is the main unknown of this problem. It is expected that V_ε converges to V^* when $\varepsilon \to 0$. Our analysis will be able to determine the limit of q_ε even if it cannot be identified by the problem at $\varepsilon = 0$. Indeed, in $(P_t U_0)$, the linear part q_0 can be any constant. Our limit candidate for q_ε is q^*, that we define as the solution of the differential equation

$$\dot{q}^*(t) = -m''(z_*(t))q^*(t) + \tfrac{1}{2} m^{(3)}(z_*(t)) - 2m''(z_*(t))m'(z_*(t)), \tag{1-18}$$

corresponding to an initial value of $q^*(0)$. Moreover we define p^* as the function which satisfies, for a given $p^*(0)$,

$$\dot{p}^*(t) = -m'(z_*(t))q^*(t) + m''(z_*(t)). \tag{1-19}$$

These expressions for p^* and q^* are obtained formally by canceling same order (in ε) terms when differentiating $(P_t U_\varepsilon)$ and looking at the main terms when ε is very small. More precisely, we must also evaluate the differentiated problem at $z = z^*$. Thus, those expressions are somehow linked to the formulas for λ and z_* in (1-5) and (1-6). Note that differentiating and evaluating at $z = z_*$ the problem for $\varepsilon > 0$ will be our strategy of proof to tackle the convergence of p_ε and q_ε, in Sections 5A and 5B. Before detailing these technical points, let us note that the function

$$U^*(t, z) := p^*(t) + q^*(t)(z - z_*(t)) + V^*(t, z) \tag{1-20}$$

will be our candidate for the limit of U_ε when $\varepsilon \to 0$. The problem for V_ε equivalent to the problem $(P_t U_\varepsilon)$, using (1-17), is

$$\begin{aligned} M(t, z) - \varepsilon^2(\dot{p}_\varepsilon(t) + \dot{q}_\varepsilon(t)(z - z_*(t)) + m'(z_*(t))q_\varepsilon(t)) - \varepsilon^2 \partial_t V_\varepsilon(t, z) \\ = \mathcal{I}_\varepsilon(q_\varepsilon, V_\varepsilon)(t, z) \exp[V_\varepsilon(t, z) - 2V_\varepsilon(t, \bar{z}(t)) + V_\varepsilon(t, z_*(t))]. \end{aligned} \tag{$P_t V_\varepsilon$}$$

[1] M is structurally positive, based on the formulation of $(P_t U_0)$. The uniform lower bound in (1-13) is mainly technical.

One can notice that thanks to cancellations the functional $I_\varepsilon(U_\varepsilon)$ does not depend on p_ε, which explains for the most part why we focus upon $(q_\varepsilon, V_\varepsilon)$. We choose to write $\mathcal{I}_\varepsilon(q_\varepsilon, V_\varepsilon)(t,z) := I_\varepsilon(U_\varepsilon)(t,z)$ as a functional of both unknowns because we will study variations in both directions. One of the main difficulties to prove the link between $(P_t V_\varepsilon)$ and $(P_t U_0)$ is that, formally, the terms with the time derivatives in q_ε and V_ε vanish when $\varepsilon \to 0$. This makes our study belong to the class of singular limit problems.

Before stating our main result we need to define appropriate functional spaces. We first define a reference space $\mathcal{E}$, similar to the one introduced in [Calvez et al. 2019] for the study of the stationary problem $(PU_\varepsilon \text{ stat})$. However, compared to that case we will need more precise controls, which is why we introduce a subspace $\mathcal{F}$ with more stringent conditions.

Definition 1.2 (functional spaces). We define $\alpha < 2 - \ln 3/\ln 2$ such that $\alpha \in (0,1)$ along with the weight function φ_α:

$$\varphi_\alpha(t,z) := (1 + |z - z_*(t)|)^\alpha.$$

The corresponding functional space $\mathcal{E}$ is given by

$$\begin{aligned}\mathcal{E} = \big\{v \in \mathcal{C}^3(\mathbb{R}_+ \times \mathbb{R}) \;\big|\; \forall t > 0,\ v(t, z_*(t)) = \partial_z v(t, z_*(t)) = 0\big\} \\ \cap \big\{v \in \mathcal{C}^3(\mathbb{R}_+ \times \mathbb{R}) \;\big|\; |\partial_z v(t,z)|,\ \varphi_\alpha(t,z)|\partial_z^2 v(t,z)|,\ \varphi_\alpha(t,z)|\partial_z^3 v(t,z)| \in L^\infty(\mathbb{R}_+ \times \mathbb{R})\big\}\end{aligned}$$

equipped with the norm

$$\|v\|_{\mathcal{E}} = \max\Big\{\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} |\partial_z v(t,z)|,\ \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} (\varphi_\alpha(t,z)|\partial_z^2 v(t,z)|),\ \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} (\varphi_\alpha(t,z)|\partial_z^3 v(t,z)|)\Big\}.$$

We also define the subspace $\mathcal{F}$:

$$\mathcal{F} := \mathcal{E} \cap \big\{v \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}) \;\big|\; |2v(t, \bar{z}(t)) - v(t,z)|,\ \varphi_\alpha(t,z)|\partial_z v(t,\bar{z}(t)) - \partial_z v(t,z)| \in L^\infty(\mathbb{R}_+ \times \mathbb{R})\big\},$$

and we associate to it the corresponding norm

$$\|v\|_{\mathcal{F}} = \max\Big\{\|v\|_{\mathcal{E}},\ \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} (|2v(t,\bar{z}(t)) - v(t,z)|),\ \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} (\varphi_\alpha(t,z)|\partial_z v(t,\bar{z}(t)) - \partial_z v(t,z)|)\Big\}.$$

The condition on α exists for computational reasons, highlighted at the end of the discussion of Proposition 7.7. The threshold coincides with that of the stationary case; see [Calvez et al. 2019, (5.11)]. The weight function φ_α is another similar feature. Its role is mainly to have a uniform bound on the first derivative using previous estimates on further derivatives, for which we need α to be bounded. We refer to Section 7G for comments on the tuning of this parameter.

Since most of this paper is focused around the pair $(q_\varepsilon, V_\varepsilon) \in \mathbb{R} \times \mathcal{F}$, we will use the convenient notation $\|(q,V)\| := \max(|q|, \|V\|_{\mathcal{F}})$. Our main theorem is the following convergence result:

Theorem 1.3 (convergence). *There exist K_0, K_0' and $\varepsilon_0 > 0$ such that if we make the following assumptions on the initial condition, for all $\varepsilon \le \varepsilon_0$:*

$$\|V_\varepsilon(0,\cdot) - V^*(0,\cdot)\|_{\mathcal{F}} \le \varepsilon^2 K_0, \quad |q_\varepsilon(0) - q^*(0)| \le \varepsilon^2 K_0 \quad \textit{and} \quad |p_\varepsilon(0) - p^*(0)| \le \varepsilon^2 K_0,$$

then we have uniform estimates of the solutions of the Cauchy problem:

$$\sup_{t>0} \| V_\varepsilon - V^* \|_{\mathcal{F}} \le \varepsilon^2 K_0', \quad \sup_{t>0} |q_\varepsilon(t) - q^*(t)| \le \varepsilon^2 K_0' \quad \textit{and} \quad \sup_{t>0} |p_\varepsilon(t) - p^*(t)| \le \varepsilon^2 K_0',$$

where q^* *is the solution of* (1-18) *associated to* $q^*(0)$ *and* p^* *is the solution of* (1-19) *associated to* $p^*(0)$. *The function* V^* *is defined in* (1-9).

Therefore, as predicted, the limit of U_ε when $\varepsilon \to 0$ is the function

$$p^*(t) + q^*(t)(z - z_*(t)) + V^*(t, z).$$

Theorem 1.3 establishes the stability, with respect to ε and uniformly in time, of Gaussian distributions around the dynamics of the dominant trait driven by a gradient flow differential equation.

In [Calvez et al. 2019], a fixed-point argument was used to build solutions of the stationary problem (PU_ε stat) when $\varepsilon \ll 1$. Estimates were uniform in ε, in order to pass to the limit $\varepsilon \to 0$. As a matter of fact, their limit problem when $\varepsilon = 0$ [Calvez et al. 2019, Problem PU_0] is consistent with (1-4), without time dependency. However, their method of proof can no longer be applied in our case because the (singular) derivative in time of the Cauchy problem ($P_t f_\varepsilon$) breaks the structure that made the stationary problem equivalent to a fixed-point mapping. In fact, in the present article, ($P_t U_0$) and ($P_t U_\varepsilon$) are different in nature due to the fast time relaxation dynamics. This is one of the main difficulties of this work compared to [Calvez et al. 2019]. For this reason, we replace the fixed-point argument by a perturbative analysis. This program is schematized in Figure 1. We introduce the corrector terms κ_ε and W_ε, our aim is to bound them uniformly:

$$V_\varepsilon(t, z) = V^*(t, z) + \varepsilon^2 W_\varepsilon(t, z), \tag{1-21}$$

$$q_\varepsilon(t) = q^*(t) + \varepsilon^2 \kappa_\varepsilon(t). \tag{1-22}$$

The scalar q^*, perturbed by $\varepsilon^2 \kappa_\varepsilon$, will tune further the affine part of the solution. The function W_ε measures the error made when approximating ($P_t U_\varepsilon$) by ($P_t U_0$). We choose not to perturb p_ε because we will realize in Section 5B that it can be straightforwardly deduced from the analysis.

This decomposition highlights a crucial part of our analysis, coming back to the initial ($P_t f_\varepsilon$). The main part (in ε) of the solution f_ε is quadratic (up to the transform (1-2)). This means that it does not belong to the space of the corrective term V_ε. After this main (quadratic) part of f_ε, of order $1/\varepsilon^2$, the corrective terms are much more precise for small ε: V^* is of order 1, while $\varepsilon^2 W_\varepsilon$ is of order ε^2. The objective of this article is to show that κ_ε and W_ε are uniformly bounded with respect to time and ε.

2. Heuristics and method of proof

For this section only, we focus on the function U_ε instead of V_ε to get a heuristic argument in favor of the decomposition (1-17) and some elements supporting Theorem 1.3. We will denote by R_ε the perturbation such that we look for solutions of ($P_t U_\varepsilon$) of the form

$$U_\varepsilon(t, z) = U^*(t, z) + \varepsilon^2 R_\varepsilon(t, z).$$

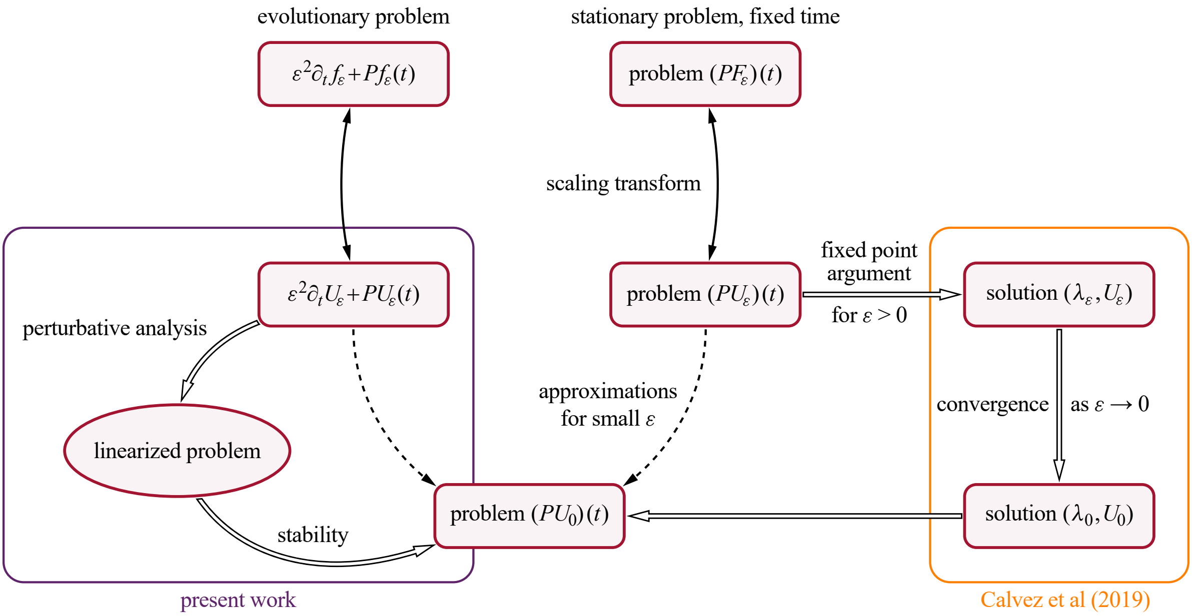

Figure 1. Scope of our paper compared to preceding work.

The function U^*, defined in (1-20), also solves $(P_t U_0)$. Plugging this perturbation into $(P_t U_\varepsilon)$ yields the following perturbed equation for R_ε:

$$M(t,z) - \varepsilon^2 \partial_t U^*(t,z) - \varepsilon^4 \partial_t R_\varepsilon(t,z)$$
$$= I_\varepsilon(U^* + \varepsilon^2 R_\varepsilon)(t,z)$$
$$\times \exp[U^*(t,z) - 2U^*(t,\bar{z}(t)) + U^*(t,z_*(t))] \exp[\varepsilon^2(R_\varepsilon(t,z) - 2R_\varepsilon(t,\bar{z}(t)) + R_\varepsilon(t,z_*(t)))].$$

By using $(P_t U_0)$, one gets that R_ε solves

$$M(t,z) - \varepsilon^2 \partial_t U^*(t,z) - \varepsilon^4 \partial_t R_\varepsilon(t,z)$$
$$= I_\varepsilon(U^* + \varepsilon^2 R_\varepsilon)(t,z) M(t,z) \exp[\varepsilon^2(R_\varepsilon(t,z) - 2R_\varepsilon(t,\bar{z}(t)) + R_\varepsilon(t,z_*(t)))].$$

To prove the boundedness of R_ε, a solution to this nonlinear equation, we shall linearize it and show a stability result on the linearized problem; see Theorem 7.1. We explain here the heuristics about the linearization. We have already said that I_ε is expected to converge to 1. Therefore by linearizing the exponential, a natural linearized equation when ε is small appears to be

$$\varepsilon^2 \partial_t \tilde{R}_\varepsilon(t,z) = M(t,z)(-\tilde{R}_\varepsilon(t,z) + 2\tilde{R}_\varepsilon(t,\bar{z}(t)) - \tilde{R}_\varepsilon(t,z_*(t))), \tag{2-1}$$

For clarity we denote by T the linear operator

$$T(R)(t,z) := M(t,z)(2R(t,\bar{z}(t)) - R(t,z) + R(t,z_*(t))).$$

We know precisely the eigenelements of this linear operator. The eigenvalue 0 has multiplicity two, the eigenspace consisting of affine functions. More generally one can get every eigenvalue by differentiating iteratively the operator T and evaluating at $z = z_*$. This corresponds to the following table:

Eigenvalue:	0	0	$-\frac{1}{2}$	$-\frac{3}{4}$	$\cdots$
Dual eigenvector:	$\delta_{z_*(t)}$	$\delta'_{z_*(t)}$	$\delta''_{z_*(t)}$	$\delta^{(3)}_{z_*(t)}$	$\cdots$

This explains why R_ε should be decomposed between affine parts and the rest, and, as a consequence, why this is also the case for the solution U_ε we are investigating. The scalars p_ε and q_ε of the decomposition (1-17) correspond to the projection of U_ε upon the eigenspace associated to the (double) eigenvalue 0. On the other hand, the rest is expected to remain uniformly bounded since the corresponding eigenvalues are negative, below $-\frac{1}{2}$.

Beyond the heuristics about the stability, this linear analysis also illustrates the discrepancy between V_ε and q_ε in Theorem 1.3. While V_ε is expected to relax to an explicit bounded value arbitrary quickly as $\varepsilon \to 0$ (fast dynamics), this is not true for q_ε, and its limit q^* solves a differential equation (slow dynamics):

$$\dot{q}^*(t) = -m''(z_*(t))q^*(t) + \tfrac{1}{2}m^{(3)}(z_*(t)) - 2m''(z_*(t))m'(z_*(t)).$$

One interpretation of this formula is that, for $\varepsilon > 0$, the second eigenvalue, which corresponds to the affine part, is not 0 as in the table above. Our intuition, given the equation above, is that it is of order $-\varepsilon^2 m''(z_*(t))$. We can guess that this explains why, in Section 8, we obtain directly with contraction arguments that the perturbation of V_ε is bounded (fast dynamics), while to show that the perturbation of q_ε is uniformly bounded, we must deal with an ODE that it solves. This "vanishing" but negative second eigenvalue could also explain why our analysis needs a uniform contraction argument for the affine part while it can be chosen freely at $\varepsilon = 0$; see $(P_t U_0)$.

The technique we will use in the following sections to bound W_ε in $\mathcal{F}$ will seem more natural in light of this formal analysis. The first step will be to work around z_*, the base point of the dual eigenelements in the table above. We derive uniform bounds up to the third derivative to estimate W_ε; see Theorem 7.1.

By plugging the expansions of (1-21) and (1-22) associated to the decomposition (1-17) and the logarithmic transform (1-2) into our original model $(P_t f_\varepsilon)$, we obtain the following main reference equation that we will study in the rest of this article:

$$\begin{aligned} M(t,z) - \varepsilon^2(\dot{p}_\varepsilon(t) + \dot{q}^*(t)(z-z_*) + m'(z_*)q^*(t) + \partial_t V^*(t,z)) - \varepsilon^4(\dot{\kappa}_\varepsilon(t)(z-z_*) + m'(z_*)\kappa_\varepsilon(t) + \partial_t W_\varepsilon(t,z)) \\ = M(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)\exp[\varepsilon^2(W_\varepsilon(t,z) - 2W_\varepsilon(t,\bar{z}(t)) + W_\varepsilon(t,z_*(t)))]. \end{aligned} \tag{2-2}$$

Our main objective will be to linearize (2-2), in order to deduce the boundedness of the unknowns, $(\kappa_\varepsilon W_\varepsilon)$, by working on the linear part of the equations. We will need to investigate different scales (in ε) to capture the different behaviors of each contribution.

We will pay attention to the remaining terms. We will use the classical notation $O(1)$ and $O(\varepsilon)$, and we will write $\|(\kappa_\varepsilon, W_\varepsilon)\| O(\varepsilon)$ to illustrate when the constants of $O(\varepsilon)$ depend on $(\kappa_\varepsilon, W_\varepsilon)$. We also define a refinement of the classical notation $O(\varepsilon)$:

Definition 2.1 ($O^*(\varepsilon^\beta)$). For $\beta \in \mathbb{N}$, we say that a function $g(\varepsilon, t, z)$ is such that $g(\varepsilon, t, z) = O^*(\varepsilon^\beta)$ if there exists ε^* such that for all $\varepsilon \leq \varepsilon^*$,

$$|g(\varepsilon, t, z)| \leq C^* \varepsilon^\beta,$$

and the constants ε^* and C^* depend only on the pair (q^*, V^*).

More generally, when we write $O(\varepsilon)$, the constants involved may a priori depend upon the pair $(\kappa_\varepsilon, W_\varepsilon)$. Our intent is to make the dependency of the constants clear when we linearize. This will prove to be a crucial point when we go back to the nonlinear problem (2-2). We will see that all the terms that do not have a sufficient order in ε, to be negligible, will be $O^*(1)$, and therefore uniformly bounded independently of $(\kappa_\varepsilon, W_\varepsilon)$. A key point of our analysis is to segregate those terms when doing the linearization.

The rest of the paper is organized as follows:

- First we prove some properties upon the reference pair (q^*, V^*) around which all the terms of (2-2) are linearized.
- A key part of our perturbative analysis is to be able to linearize $\mathcal{I}_\varepsilon$, which we do in Section 4 thanks to careful estimates upon the directional derivatives.
- We derive an equation on κ_ε in Section 5A, and later a linear equation for the approximation W_ε, as well as derivations for all of its derivatives in Section 6, while controlling precisely the error terms.
- We next show the boundedness of the solutions of the linear problem in the space $\mathcal{F}$, see Section 7, mainly through maximum principles and a dyadic division of the space to take into account the nonlocal behavior of the infinitesimal operator. This is the content of Theorem 7.1.
- Finally, we tackle the proof of Theorem 1.3 in Section 8, using contraction arguments deduced from the previous section.
- To conclude, in Section 9 we discuss some of our assumptions made in Assumption 1.1, illustrated by some numerical simulations.

3. Preliminary results: estimates of $\mathcal{I}_\varepsilon^*$ and V^*

3A. ***Control of*** (q^*, V^*)**.** Before tackling the main difficulties of this article, we first state some controls on the function V^*, the solution of $(P_t U_0)$. Most of them use the explicit expression of (1-9) and were proved in [Calvez et al. 2019]. To be able to measure this function we introduce another functional space, with more constraints.

Definition 3.1 (subspace $\mathcal{E}^*$). We define $\mathcal{E}^*$ as the following subspace of $\mathcal{E}$:

$$\mathcal{E}^* := \mathcal{E} \cap \{v \in \mathcal{C}^5(\mathbb{R}_+ \times \mathbb{R}) \mid \varphi_\alpha(t, z)|\partial_z^4 v(t, z)|,\ \varphi_\alpha(t, z)|\partial_z^5 v(t, z)| \in L^\infty(\mathbb{R}_+ \times \mathbb{R})\},$$

and we equip it with the norm $\|\cdot\|_*$:

$$\|v\|_* = \max\Big(\|v\|_{\mathcal{E}}, \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \varphi_\alpha(t, z)|\partial_z^4 v(t, z)|, \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \varphi_\alpha(t, z)|\partial_z^5 v(t, z)|\Big).$$

Our intention with the successive definitions of the functional spaces is to be able to measure each term of the decomposition made in (1-21) as follows:

$$V_\varepsilon = V^* + \varepsilon^2 W_\varepsilon, \quad \text{with } V_\varepsilon \in \mathcal{E}, \quad V^* \in \mathcal{E}^* \quad \text{and} \quad W_\varepsilon \in \mathcal{F}.$$

The fact that $V^* \in \mathcal{E}^*$ is part of the claim of the following lemma:

Lemma 3.2 (properties of V^*). *The function V^* belongs to the space $\mathcal{E}^*$. Moreover,*

$$\partial_z^2 V^*(t, z_*) = 2m''(z_*) \quad \textit{and} \quad \partial_z^3 V^*(t, z_*) = \tfrac{4}{3} m^{(3)}(z_*). \tag{3-1}$$

Proof. Precise estimates of the summation operator that defines V^* in (1-9) are studied in [Calvez et al. 2019]. They can be applied there thanks to the decay assumptions about M; see (1-14). The only difference here is that a uniform bound for the fourth and fifth derivative are required. The proofs of those bounds rely solely upon the assumption made in (1-14), for the fourth and fifth derivative of M. This shows that $V^* \in \mathcal{E}^*$. Explicit computations based on (1-9) prove the relationships (3-1). □

A consequence of Lemma 3.2 is that since $m''(z_*(t)) > 0$ for $t > t_0$, thanks to (1-16), we have that V^* is locally convex around $z_*(t)$. However, we need more information about V^* than the space it belongs to. We will bound (q^*, V^*) independently of time. This is the content of the following result:

Proposition 3.3 (uniform bound on (q^*, V^*)). *There exists a constant K^* such that for $j = 0, 1, 2, 3$, we have*

$$\max(\|V^*\|_*, \|q^*\|_{L^\infty(\mathbb{R}_+)}, \|\partial_t \partial_z^j V^*\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})}) \le K^*.$$

Proof. For the estimates upon V^* and $\partial_t V^*$, it is a direct consequence of the definition of $\mathcal{E}^*$ and the explicit formula (1-9). The technique to bound the sums is to distinguish between the small and large indices, as was detailed in [Calvez et al. 2019].

For q^*, one must look to (1-18). The boundedness of q^* is a straightforward consequence of the convexity of m at $z_*(t)$ for large times; see (1-16) and the convergence of z_* to bound the other terms. □

3B. *Estimates of $\mathcal{I}_\varepsilon^*$ and its derivatives.* We next define a notational shortcut for the functional $\mathcal{I}_\varepsilon$ introduced in (1-3), when it is evaluated at the reference pair (q^*, V^*):

$$\mathcal{I}_\varepsilon^* := \mathcal{I}_\varepsilon(q^*, V^*).$$

This section is devoted to getting precise estimates of this function. This will be crucial for the linearization of $\mathcal{I}_\varepsilon(q^* + \varepsilon^2 \kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)$ as can be seen on the full equation (2-2).

Proposition 3.4 (estimation of $\mathcal{I}_\varepsilon^*$). *We have that*

$$\mathcal{I}_\varepsilon^*(t, z) = 1 + O^*(\varepsilon^2),$$

where the constants of $O^*(\varepsilon^2)$ depend only on K^*, as introduced in Proposition 3.3 and as defined by Definition 2.1.

The proof involves exact Taylor expansions in ε. Very similar expansions were performed in Lemma 3.1 of [Calvez et al. 2019]. We adapt the method of proof here, since it will be used extensively throughout this article.

Proof of Proposition 3.4. We recall that by Proposition 3.3, $\max(|q^*|, \|V^*\|_*) \le K^*$, and, by definition,

$$\begin{aligned}\mathcal{I}_\varepsilon^*(t,z) &= \frac{\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} \exp[-\varepsilon q^*(t)(y_1+y_2) + 2V^*(t,\bar z) - V^*(t,\bar z+\varepsilon y_1) - V^*(t,\bar z+\varepsilon y_2)]\, dy_1\, dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon q^*(t) y + V^*(t,z_*) - V^*(t,z_*+\varepsilon y)]\, dy} \\ &:= \frac{N(t,z)}{D(t)},\end{aligned}$$

where Q is the quadratic form appearing after the rescaling of the infinitesimal operator in (1-3):

$$Q(y_1,y_2) := \tfrac{1}{2} y_1 y_2 + \tfrac{3}{4}(y_1^2+y_2^2).$$

This quadratic form will appear very frequently in what follows, mostly, as here, through the bivariate Gaussian distribution it defines. Once and for all, we state that a correct normalization of this Gaussian distribution is

$$\frac{1}{\sqrt{2}\pi}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}\, dy_1\, dy_2 = 1.$$

We start the estimates with the more complicated term, the numerator N. With an exact Taylor expansion inside the exponential, there exists generic $0<\xi_i<1$, for $i=1,2$, such that

$$\begin{aligned}N(t,z) = \frac{1}{\sqrt{2}\pi}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} \exp\Big[&-\varepsilon q^*(t)(y_1+y_2) - \varepsilon(y_1+y_2)\partial_z V^*(t,\bar z) \\ &- \tfrac{1}{2}\varepsilon^2\big(y_1^2\partial_z^2 V^*(t,\bar z+\varepsilon\xi_1 y_1) + y_2^2\partial_z^2 V^*(t,\bar z+\varepsilon\xi_2 y_2)\big)\Big]\, dy_1\, dy_2.\end{aligned}$$

Moreover, we can write, for some $\theta = \theta(y_1,y_2)\in(0,1)$,

$$\begin{aligned}&\exp[-\varepsilon P] = 1-\varepsilon P + \tfrac{1}{2}\varepsilon^2 P^2 \exp[-\theta\varepsilon P], \quad \text{with} \\ &\qquad P := (y_1+y_2)(q^*(t)+\partial_z V^*(t,\bar z)) + \tfrac{1}{2}\varepsilon\big(y_1^2\partial_z^2 V^*(t,\bar z+\varepsilon\xi_1 y_1) + y_2^2\partial_z^2 V^*(t,\bar z+\varepsilon\xi_2 y_2)\big),\end{aligned}$$

such that

$$|P| \le K^*\big(|y_1|+|y_2| + \tfrac{1}{2}\varepsilon(y_1^2+y_2^2)\big). \tag{3-2}$$

Combining the expansions, we find that

$$\begin{aligned}N(t,z) &= \frac{1}{\sqrt{2}\pi}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}\big(1-\varepsilon P + \tfrac{1}{2}\varepsilon^2 P^2\exp[-\theta\varepsilon P]\big)\, dy_1\, dy_2 \\ &= 1 - \varepsilon\frac{1}{\sqrt{2}\pi}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} P\, dy_1\, dy_2 + \frac{\varepsilon^2}{2\sqrt{2}\pi}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} P^2 \exp[-\theta\varepsilon P]\, dy_1\, dy_2.\end{aligned} \tag{3-3}$$

The key part is the cancellation of the terms $O(\varepsilon)$ due to the symmetry of Q:

$$\frac{1}{\sqrt{2}\pi}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}(y_1+y_2)\, dy_1\, dy_2 = 0.$$

Therefore,

$$\frac{\varepsilon}{\sqrt{2\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} P\,dy_1\,dy_2 = \frac{\varepsilon^2}{2\sqrt{2\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}(y_1^2\partial_z^2 V^*(t,\bar{z}+\varepsilon\xi_1 y_1)+y_2^2\partial_z^2 V^*(t,\bar{z})+\varepsilon\xi_2 y_2)\,dy_1\,dy_2,$$

and we get the estimate

$$\left|\frac{\varepsilon}{\sqrt{\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} P\,dy_1\,dy_2\right| \le \frac{\varepsilon^2}{2\sqrt{2\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}(y_1^2+y_2^2)K^*\,dy_1\,dy_2 \le O^*(\varepsilon^2).$$

Thanks to (3-2) it is easy to verify that the last term of (3-3) behaves similarly:

$$\frac{\varepsilon^2}{2\sqrt{2\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} P^2\exp[-\theta\varepsilon P]\,dy_1\,dy_2 = O^*(\varepsilon^2).$$

Indeed, it states that the term P is at most quadratic with respect to y_i, so $Q+\theta\varepsilon P$ is uniformly bounded below by a positive quadratic form for ε small enough. This shows that

$$N(t,z) = 1+O^*(\varepsilon^2).$$

The denominator is easier. With the same arguments, using the Gaussian density, we find that

$$D(t) = 1+O^*(\varepsilon^2).$$

Combining the estimates of N and D, we get the desired result. □

There exists a link between q^* and $\partial_z\mathcal{I}_\varepsilon^*(t,z_*)$, which is in fact the motivation behind the choice of q^*.

Proposition 3.5 (link between q^* and $\partial_z\mathcal{I}_\varepsilon^*(t,z_*)$).

$$\partial_z\mathcal{I}_\varepsilon^*(t,z_*(t)) = \varepsilon^2\big(m''(z_*(t))q^*(t)-\tfrac{1}{2}m^{(3)}(z_*(t))\big)+O^*(\varepsilon^4),$$

where the constants of $O^*(\varepsilon^4)$ only depend on K^*.

The proof of this result was the content of [Calvez et al. 2019, Lemma 3.1] and only requires that the pair (q^*,V^*) is uniformly bounded, as stated in Proposition 3.3. Its proof follows the same procedure of exact Taylor expansions as that of Proposition 3.4.

It will be useful to dispose of estimates of $\partial_z\mathcal{I}_\varepsilon^*$ not only at the point z_*. They are less precise, as stated in the following proposition:

Proposition 3.6 (estimates of the decay of the derivatives of $\mathcal{I}_\varepsilon^*$). *There exists a constant ε_* that depends only on K^* such that for all $\varepsilon\le\varepsilon*$, for $j=1,2,3$,*

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \varphi_\alpha(t,z)|\partial_z^{(j)}\mathcal{I}_\varepsilon^*(t,z)| = O^*(\varepsilon^2).$$

To simplify notations, we introduce the following difference operator that appears in the integral $\mathcal{I}_\varepsilon$; see (1-3):

$$\begin{aligned}\mathcal{D}_\varepsilon(V)(Y,t,z) &:= V(t,\bar{z}) - \tfrac{1}{2}V(t,\bar{z}+\varepsilon y_1) - \tfrac{1}{2}V(t,\bar{z}+\varepsilon y_2), \quad \text{with } Y=(y_1,y_2),\\ \mathcal{D}^*_\varepsilon(V)(y,t) &:= V(t,z_*) - V(t,z_*+\varepsilon y).\end{aligned} \tag{3-4}$$

We will use the following technical lemma giving an estimate of the weight function against the derivatives of a given function.

Lemma 3.7 (influence of the weight function). *There exists a constant C such that for each ball B of $\mathcal{E}^*$ or $\mathcal{F}$, there exists ε_B such that for every $W \in B$, for every $y \in \mathbb{R}$ and $\varepsilon \le \varepsilon_B$, for $j = 2, 3, 4, 5$,*

$$\varphi_\alpha(t,z)|\partial_z^{(j)} W(t,\bar{z}(t)+\varepsilon y)| \le \begin{cases} C\|W\| & \text{if } |y| \le |z - z_*(t)|,\\ (1+|y|^\alpha)\|W\| & \text{otherwise},\end{cases}$$

with $\|W\| = \|W\|_*$ or $\|W\|_{\mathcal{F}}$ depending on the case.

Proposition 3.6 is a prototypical result. It will be followed by a series of similar statements. Therefore, we propose two different proofs. In the first one, we write exact Taylor expansions. However, the formalism is heavy, which is why we propose next a formal argument, where the Taylor expansions are written without the exact remainder for the sake of clarity.

In the rest of this paper more complicated estimates will be proved, in the spirit of Proposition 3.6; see Proposition 4.1 and Lemma 4.8 for instance. The notations and formulas will be very long, so we shall only write the *formal* parts of the argument. However, it can all be made rigorous, as below.

Proof of Proposition 3.6. First, write the expression for the derivative, using our notation $\mathcal{D}_\varepsilon$ introduced in (3-4):

$$\begin{aligned}\partial_z \mathcal{I}^*_\varepsilon(t,z) &= \frac{\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} \exp[-\varepsilon q^*(y_1+y_2) + 2\mathcal{D}_\varepsilon(V^*)(Y,t,z)]\mathcal{D}_\varepsilon(\partial_z V^*)(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon q^* y + \mathcal{D}^*_\varepsilon(V^*)(y,t)]\,dy}\\ &:= \frac{N(t,z)}{D(t)}.\end{aligned} \tag{3-5}$$

We only focus on the numerator. The denominator D can be handled similarly as in the proof of Proposition 3.4, where we show that it is essentially $1 + O^*(\varepsilon^2)$. We perform two Taylor expansions in the numerator N, namely,

$$\begin{aligned}2\mathcal{D}_\varepsilon(V^*)(Y,t,\bar{z}) &= -\varepsilon(y_1+y_2)\partial_z V^*(t,\bar{z}) - \tfrac{1}{2}\varepsilon^2(y_1^2\partial_z^2 V^*(t,\bar{z}+\varepsilon\xi_1 y_1) + y_2^2\partial_z^2 V^*(t,\bar{z}+\varepsilon\xi_2 y_2)),\\ \mathcal{D}_\varepsilon(\partial_z V^*)(Y,t,\bar{z}) &= -\tfrac{1}{2}\varepsilon(y_1+y_2)\partial_z^2 V^*(t,\bar{z}) - \tfrac{1}{4}\varepsilon^2(y_1^2\partial_z^3 V^*(t,\bar{z}+\varepsilon\xi_1 y_1) + y_2^2\partial_z^3 V^*(t,\bar{z}+\varepsilon\xi_2 y_2)),\end{aligned} \tag{3-6}$$

where the ξ_i denote some generic number such that $0 < \xi_i < 1$ for $i = 1, 2$. Moreover, we can write

$$\begin{aligned}&\exp[-\varepsilon P] = 1 - \varepsilon P \exp[-\theta\varepsilon P], \quad \text{with}\\ &P := (y_1+y_2)(\partial_z V^*(t,\bar{z}) + q^*) + \tfrac{1}{2}(y_1^2\partial_z^2 V^*(t,\bar{z}+\varepsilon\xi_1 y_1) + y_2^2\partial_z^2 V^*(t,\bar{z}+\varepsilon\xi_2 y_2)),\end{aligned} \tag{3-7}$$

for some $\theta = \theta(y_1, y_2) \in (0, 1)$. Combining the expansions, we find that

$$\begin{aligned}
&\varphi_\alpha(t,z)\partial_z \mathcal{I}_\varepsilon^*(t,z) \\
&= \frac{\varphi_\alpha(t,z)}{\sqrt{2}\pi} \iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}(1-\varepsilon P \exp[-\theta\varepsilon P]) \\
&\qquad \times \left(-\tfrac{1}{2}\varepsilon(y_1+y_2)\partial_z^2 V^*(t,\bar z) - \tfrac{1}{4}\varepsilon^2(y_1^2\partial_z^3 V^*(t,\bar z+\varepsilon\xi_1 y_1) + y_2^2\partial_z^3 V^*(t,\bar z+\varepsilon\xi_2 y_2))\right) dy_1\,dy_2.
\end{aligned}$$

Crucially, the $O(\varepsilon)$ contribution cancels due to the symmetry of Q, as already observed above:

$$\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}(y_1+y_2)\,dy_1\,dy_2 = 0.$$

So, it remains that

$$\begin{aligned}
&\varphi_\alpha(t,z)N(t,z) \\
&\quad = -\varepsilon^2\frac{\varphi_\alpha(t,z)}{4\sqrt{2}\pi} \iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}(y_1^2\partial_z^3 V^*(t,\bar z+\varepsilon\xi_1 y_1) + y_2^2\partial_z^3 V^*(t,\bar z+\varepsilon\xi_2 y_2))\,dy_1\,dy_2 \\
&\qquad + \varepsilon^2\frac{\varphi_\alpha(t,z)}{2\sqrt{2}\pi} \iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} P\exp[-\theta\varepsilon P](y_1+y_2)\partial_z^2 V^*(t,\bar z)\,dy_1\,dy_2 \\
&\qquad + \varepsilon^3\frac{\varphi_\alpha(t,z)}{4\sqrt{2}\pi} \iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} P\exp[-\theta\varepsilon P](y_1^2\partial_z^3 V^*(t,\bar z+\varepsilon\xi_1 y_1) + y_2^2\partial_z^3 V^*(t,\bar z+\varepsilon\xi_2 y_2))\,dy_1\,dy_2.
\end{aligned}$$

If we forget about the weight in front of each term, clearly the last two contributions are uniform $O^*(\varepsilon)$ since $\varepsilon \le \varepsilon_*$ is small enough and V^* and q^* are uniformly bounded by K^*; see Proposition 3.3. The term P is at most quadratic with respect to y_i, see (3-7), so $Q+\theta\varepsilon P$ is uniformly bounded below by a positive quadratic form for ε small enough.

A difficulty is to add the weight to those estimates. To do so, we use Lemma 3.7, for each integral term appearing in the previous formula, because each time a term of the following form appears:

$$\varphi_\alpha(t,z)\partial_z^{(j)} V^*(t,\bar z+\varepsilon\xi_i y_i). \tag{3-8}$$

Since every ξ_i satisfies $0<\xi_i<1$, the bounds given by Lemma 3.7 ensure that each integral remains bounded by moments of the bivariate Gaussian defined by Q, as if there were no weight function. This concludes the proof of the first estimate of Proposition 3.6.

Bounding the quantity $\varphi_\alpha(t,z)|\partial_z^{(j)}\mathcal{I}_\varepsilon^*(t,z)|$ for $j=2,3$ follows the same steps, as seen here:

$$\begin{aligned}
&\partial_z^2\mathcal{I}_\varepsilon^*(t,z) \\
&\quad = \frac{\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V^*)(Y,t,z)]\left(\mathcal{D}_\varepsilon(\partial_z V^*)^2+\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V^*)\right)(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon q^* y+\mathcal{D}_\varepsilon^*(V^*)(y,t)]\,dy},
\end{aligned}$$

$$\partial_z^3\mathcal{I}_\varepsilon^*(t,z) = \frac{\begin{aligned}&\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V^*)(Y,t,z)] \\ &\qquad\times\left(\mathcal{D}_\varepsilon(\partial_z V^*)^3+\tfrac{3}{2}\mathcal{D}_\varepsilon(\partial_z V^*)\mathcal{D}_\varepsilon(\partial_z^2 V^*)+\tfrac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V^*)\right)(Y,t,z)\,dy_1\,dy_2\end{aligned}}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon q^* y+\mathcal{D}_\varepsilon^*(V^*)(y,t)]\,dy}.$$

The motivation behind going up to the fifth derivative of V^* in Definition 3.1 lies in the terms

$$\tfrac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V^*) \quad \text{and} \quad \tfrac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V^*). \tag{3-9}$$

To gain an order ε^2 as needed in Proposition 3.6 for the estimates, one needs to go up by two orders in the Taylor expansions, which involve fourth and fifth derivatives. The importance of the order ε^2 will later appear in Proposition 4.2 and Section 7. □

We now propose a formal argument which is much simpler to read.

Formal proof of Proposition 3.6. We tackle the first derivative. We use the same notations as previously, see (3-5), and again focus on the numerator N. Formally,

$$\begin{aligned}N(t,z) = \iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} \exp[-\varepsilon(y_1+y_2)(q^* + \partial_z V^*(t,\bar z)) + (y_1^2+y_2^2)O^*(\varepsilon^2)] \\ \times (-\varepsilon(y_1+y_2)\partial_z^2 V^*(t,\bar z) + (y_1^2+y_2^2)O^*(\varepsilon^2))\,dy_1\,dy_2.\end{aligned}$$

Thanks to the linear approximation of the exponential, we find that

$$\begin{aligned}N(t,z) = \iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} (1-\varepsilon(y_1+y_2)(q^* + \partial_z V^*(t,\bar z)) + (y_1^2+y_2^2)O^*(\varepsilon^2)) \\ \times (-\varepsilon(y_1+y_2)\partial_z^2 V^*(t,\bar z) + (y_1^2+y_2^2)O^*(\varepsilon^2))\,dy_1\,dy_2.\end{aligned} \tag{3-10}$$

By sorting out the orders in ε, this can be rewritten as

$$N(t,z) = \varepsilon N_1 + O^*(\varepsilon^2).$$

By symmetry,

$$N_1 := -\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}(\varepsilon(y_1+y_2)\partial_z^2 V^*(t,\bar z))\,dy_1\,dy_2 = 0.$$

To conclude, we notice that we can add the weight function to those estimates and make the same arguments as in the previous proof. □

Proof of Lemma 3.7. If $|z-z_*| \le 1$, then $1+|z-z_*| \le 2$, and the result is immediate by Definitions 3.1 and 1.2 of the adequate functional spaces. Therefore, one can suppose that $|z-z_*| > 1$. We first look at the regime $|y| \le |z-z_*|$. Then, by definition of the norms,

$$\begin{aligned}\varphi_\alpha(t,z)|\partial_z^{(j)} W(t,\bar z+\varepsilon y)| &\le 2\frac{|z-z_*|^\alpha}{|\bar z+\varepsilon y - z_*|^\alpha}|\bar z+\varepsilon y-z_*|^\alpha |\partial_z^{(j)} W(t,\bar z+\varepsilon y)| \\ &\le 2\frac{|z-z_*|^\alpha}{|\bar z+\varepsilon y - z_*|^\alpha}\|W\|.\end{aligned} \tag{3-11}$$

To bound the last quotient, we use the following inequality, that holds true because we are in the regime $|y| \le |z-z_*|$:

$$|\bar z+\varepsilon y-z_*| \ge -|\varepsilon y| + |\bar z - z_*| \ge \tfrac{1}{2}|z-z_*| - \varepsilon|z-z_*|.$$

This yields

$$2\frac{|z-z_*|}{|\bar z+\varepsilon y-z_*|} \le \frac{2}{1/2-\varepsilon}. \tag{3-12}$$

Bringing together (3-11) and (3-12), one gets Lemma 3.7 in the regime $|y| \le |z - z_*|$, on the condition that $\varepsilon < \frac{1}{2}$.

On the contrary, when $|z - z_*| \le |y|$, we have immediately that

$$(1 + |z - z_*|^\alpha)|\partial_z^{(j)} W(t, \bar{z} + \varepsilon y)| \le (1 + |y|^\alpha)\|W\|. \qquad \square$$

4. Linearization of $\mathcal{I}_\varepsilon$ and its derivatives

The first step to obtain a linearized equation on W_ε is to study the nonlinear terms of (2-2). A key point is the study of the functional $\mathcal{I}_\varepsilon$ defined in (1-3), which plays a major role in our study. We will show that it converges uniformly to 1, as we claimed in Section 1, and that its derivatives are uniformly small, with some decay for large z, similarly to what we proved for the function $\mathcal{I}_\varepsilon^*$ in the previous section. This will enable us to linearize $\mathcal{I}_\varepsilon$ and its derivatives in Propositions 4.2 and 4.5.

4A. *Linearization of $\mathcal{I}_\varepsilon$.* We first bound uniformly all the terms that appear during the linearization of $\mathcal{I}_\varepsilon$ by Taylor expansions. One starts by measuring the first order directional derivatives.

Proposition 4.1 (bounds on the directional derivatives of $\mathcal{I}_\varepsilon$). *For any ball B of $\mathbb{R} \times \mathcal{E}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$ we have, for all $(g, V) \in B$ and $H \in \mathcal{E}$:*

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} |\partial_g \mathcal{I}_\varepsilon(g, V)(t, z)| \le \|(g, V)\| O(\varepsilon^2) \tag{4-1}$$

and

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} |\partial_V \mathcal{I}_\varepsilon(g, V) \cdot H(t, z)| \le \|(g, V)\| \, \|H\|_{\mathcal{E}} O(\varepsilon^2). \tag{4-2}$$

Proof. As in the estimates of $\mathcal{I}_\varepsilon^*$ and its derivatives in the previous section, the argument to obtain the result will be to perform exact Taylor expansions. As explained before we will not pay attention to the exact remainders that can be handled exactly as before, and we refer to the proofs of Propositions 3.4 and 3.6 for the details. However, our computations will make clear the order ε^2 of (4-1) and (4-2). First, thanks to the derivation with respect to g, an order of ε is gained straightforwardly:

$$\begin{aligned}
&\partial_g \mathcal{I}_\varepsilon(g, V)(t, z) \\
&= -\varepsilon \Bigg(\frac{\iint_{\mathbb{R}^2} \exp[-Q(y_1, y_2) - \varepsilon g(y_1 + y_2) + 2\mathcal{D}_\varepsilon(V)(Y, t, z)](y_1 + y_2)\, dy_1\, dy_2}{\sqrt{\pi} \int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y, t)]\, dy} \\
&\qquad\qquad - \mathcal{I}_\varepsilon(g, W)(t, z) \frac{\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y, t)] y\, dy}{\sqrt{\pi} \int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y, t)]\, dy} \Bigg).
\end{aligned} \tag{4-3}$$

The common denominator is bounded:

$$\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y, t)]\, dy \ge \int_{\mathbb{R}} \exp\big[-\tfrac{1}{2}|y|^2 - 2\varepsilon |y| \|(g, V)\|\big]\, dy.$$

For the numerators, a supplementary order in ε is gained by symmetry of Q, as in other estimates; see Proposition 3.6 for instance. For the single integral we write

$$\begin{aligned}\int_{\mathbb{R}} e^{-|y|^2/2} y \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\,dy &\le \int_{\mathbb{R}} e^{-|y|^2/2} y \exp[-\varepsilon g y + 2\varepsilon |y| \|(g,V)\|]\,dy \\ &\le \int_{\mathbb{R}} e^{-|y|^2/2} y (1 - \varepsilon g y + O(\varepsilon)|y| \|(g,V)\|)\,dy.\end{aligned}$$

Finally,

$$\int_{\mathbb{R}} e^{-|y|^2/2} y \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\,dy \le \|(g,V)\| O(\varepsilon). \tag{4-4}$$

For the first numerator of (4-3), the computations work in the same way:

$$\begin{aligned}&\iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(Y,t,z)](y_1+y_2)\,dy_1\,dy_2 \\ &\qquad \le \iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2) + O(\varepsilon)(y_1+y_2)\|(g,V)\|](y_1+y_2)\,dy_1\,dy_2 \\ &\qquad \le \iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2)](1 + O(\varepsilon)(y_1+y_2)\|(g,V)\|)(y_1+y_2)\,dy_1\,dy_2 \le \|(g,V)\| O(\varepsilon).\end{aligned} \tag{4-5}$$

Therefore, combining (4-3)–(4-5) we have proven the bound upon the first derivative of $\mathcal{I}_\varepsilon$ in (4-1).

Concerning (4-2), one starts by writing the following formula for the Fréchet derivative:

$$\begin{aligned}&\partial_V \mathcal{I}_\varepsilon(g,V)\cdot H(t,z) \\ &= \frac{\iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(Y,t,z)] 2\mathcal{D}_\varepsilon(H)(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\,dy} \\ &\qquad - \mathcal{I}_\varepsilon(g,V)(t,z) \frac{\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)] \mathcal{D}_\varepsilon^*(H)(y,t)\,dy}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}.\end{aligned} \tag{4-6}$$

The claimed order ε^2 holds true, by similar symmetry arguments. For instance, when we do the Taylor expansions on the numerator of the first term of (4-6), we find that

$$\begin{aligned}&\iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(Y,t,z)] 2\mathcal{D}_\varepsilon(H)(Y,t,z)\,dy_1\,dy_2 \\ &= 2\iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2)](1 - \varepsilon(y_1+y_2)(g + \partial_z V(t,\bar z)) + O(\varepsilon^2)\|V\|_{\mathcal{E}}) \\ &\qquad\qquad \times (-\varepsilon(y_1+y_2)\partial_z H(t,\bar z) + O(\varepsilon^2)(y_1^2+y_2^2)\|H\|_{\mathcal{E}})\,dy_1\,dy_2 \\ &= -2\varepsilon \partial_z H(t,\bar z) \iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2)](y_1+y_2)\,dy_1\,dy_2 \\ &\qquad + \varepsilon^2 \partial_z H(t,\bar z)(g + \partial_z V(t,\bar z)) \iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2)](y_1+y_2)^2\,dy_1\,dy_2 + O(\varepsilon^2)\|H\|_{\mathcal{E}}\|(g,V)\| \\ &\le \|(g,V)\|\|H\|_{\mathcal{E}} O(\varepsilon^2).\end{aligned} \tag{4-7}$$

For the second term of (4-6), we also gain an order ε^2 when making Taylor expansions of $\mathcal{D}^*_\varepsilon(V)$, since $\partial_z H(t, z_*) = 0$:

$$\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}^*_\varepsilon(V)(y,t)]\mathcal{D}^*_\varepsilon(H)(y,t)\,dy$$
$$= -\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + 2\varepsilon|y|\|(g, V)\|]y^2 O(\varepsilon^2)\|H\|_{\mathcal{E}}\,dy$$
$$= -\int_{\mathbb{R}} e^{-|y|^2/2}(1-\varepsilon g y + 2\varepsilon|y|\|(g, V)\|)y^2 O(\varepsilon^2)\|H\|_{\mathcal{E}}\,dy \le \|(g, V)\|\|H\|_{\mathcal{E}} O(\varepsilon^2). \tag{4-8}$$

As before, the denominator of (4-6) has a uniform lower bound, therefore combining (4-6)–(4-8) concludes the proof. □

We have proven all the tools to linearize $\mathcal{I}_\varepsilon$ as follows, thanks to the previous estimates on the directional derivatives of $\mathcal{I}_\varepsilon$.

Proposition 4.2 (linearization of $\mathcal{I}_\varepsilon$). *For any ball B of $\mathbb{R}\times\mathcal{E}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$ we have, for all $(g, W) \in B$,*

$$\mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t, z) = \mathcal{I}^*_\varepsilon(t, z) + O(\varepsilon^3)\|(g, W)\| \tag{4-9}$$
$$= 1 + O^*(\varepsilon^2) + O(\varepsilon^3)\|(g, W)\|, \tag{4-10}$$

where $O(\varepsilon^3)$ only depends on the ball B.

Proof. We write an exact Taylor expansion

$$\mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W) = \mathcal{I}^*_\varepsilon + \varepsilon^2(\partial_g \mathcal{I}_\varepsilon(q^* + \varepsilon^2\xi g, V^* + \varepsilon^2\xi W) + \partial_V \mathcal{I}_\varepsilon(q^* + \varepsilon^2\xi g, V^* + \varepsilon^2\xi W)\cdot W),$$

for some $0 < \xi < 1$. Therefore (4-9) is a direct application of Proposition 4.1 to $g' = q^* + \varepsilon^2\xi g$, $V = V^* + \varepsilon^2\xi W$ and $H = W$. One deduces the estimation of (4-10) from Proposition 3.4. □

As a matter of fact, in (4-10), we have even shown an estimate $1 + O^*(\varepsilon^2) + O(\varepsilon^4)\|(g, W)\|$. However, we choose to reduce arbitrarily the order in ε for consistency reasons with further estimates of this article. It suffices for our purposes.

4B. ***Linearization of $\partial_z\mathcal{I}_\varepsilon$ and decay estimates.*** In order to prove Theorem 1.3, we need to uniformly bound $\|W_\varepsilon\|_{\mathcal{F}}$, and this implies L^∞ bounds of the derivatives of W_ε. To obtain those, our method is to work on the linearized equations they satisfy. Therefore, linearizing $\mathcal{I}_\varepsilon$ is not enough, we need to linearize $\partial_z^{(j)}\mathcal{I}_\varepsilon$ as well, for $j = 1, 2, 3$. For that purpose we need more details than previously about the nature of the negligible terms. More precisely, we need to know how it behaves relatively to the weight function of the space $\mathcal{E}$ and $\mathcal{F}$, that acts by definition upon the second and third derivatives. The objective of this section is to linearize $\partial_z^{(j)}\mathcal{I}_\varepsilon$ to obtain similar results to Proposition 4.2. We first prove the following estimates on the derivatives of $\mathcal{I}_\varepsilon$:

Proposition 4.3 (decay estimate of $\partial_z \mathcal{I}_\varepsilon$). *For any ball B of $\mathbb{R}\times\mathcal{E}$, there exists a constant ε_B that depends only on B such that for any pair (g, V) in B, for all $\varepsilon \le \varepsilon_B$:*

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \varphi_\alpha(t,z)|\partial_z \mathcal{I}_\varepsilon(g,V)(t,z)| \le \|(g,V)\| O(\varepsilon),$$
$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \varphi_\alpha(t,z)|\partial_z^2 \mathcal{I}_\varepsilon(g,V)(t,z)| \le \|(g,V)\| O(\varepsilon),$$
$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \varphi_\alpha(t,z)|\partial_z^3 \mathcal{I}_\varepsilon(g,V)(t,z)| \le \|(g,V)\| O(\varepsilon^\alpha) + \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^3 V\|_\infty,$$

where all the $O(\varepsilon)$ depend only on the ball B, and $\|\varphi_\alpha \partial_z^3 V\|_\infty = \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \varphi_\alpha(t,z)|\partial_z^3 V(t,z)|$.

We are not able to propagate an order ε for all derivatives. There is a factor of order 0 in ε in the third one: $\|\varphi_\alpha \partial_z^3 V\|_\infty / 2^{1-\alpha}$. It will be dealt with using a contraction argument, since $2^{\alpha-1} < k(\alpha) < 1$; and $k(\alpha)$ plays the same role as in Theorem 7.1. This has to be put in parallel with [Calvez et al. 2019, Proposition 4.6].

Proof. We focus on the first derivative, the proof for the second derivative is straightforward to adapt:

$$\varphi_\alpha(t,z)\partial_z \mathcal{I}_\varepsilon(g,V)(t,z) = \varphi_\alpha(t,z)\frac{\iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(Y,t,z)]\mathcal{D}_\varepsilon(\partial_z V)(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}. \tag{4-11}$$

As before, the following formal Taylor expansions hold true for the numerator, ignoring the weight in the first step:

$$\begin{aligned}&\iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(Y,t,z)]\mathcal{D}_\varepsilon(\partial_z V)(Y,t,z)\,dy_1\,dy_2 \\ &\quad= \iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2)](1 - O(\varepsilon)(y_1+y_2)\|(g,V)\|)(-O(\varepsilon)(y_1+y_2)\|(g,V)\|)\,dy_1\,dy_2, \\ &\quad\le O(\varepsilon)\|(g,V)\|.\end{aligned} \tag{4-12}$$

Meanwhile the denominator has a uniform lower bound:

$$\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\,dy \ge \int_{\mathbb{R}} \exp\big[-\tfrac{1}{2}|y|^2 - 2\varepsilon|y|\|(g,V)\|\big]\,dy.$$

The estimate of (4-12) can be made rigorous as in the proof of Proposition 3.6, for instance. Moreover, one can add the weight to bound (4-11) thanks to Lemma 3.7, as explained in the proof of Proposition 3.6. Therefore, the proof of the first estimate of Proposition 4.3 is achieved.

For the second term of Proposition 4.3, involving the second derivative, the arguments and decomposition of the space are the same. We follow the same steps, arriving at the formula

$$\partial_z^2 \mathcal{I}_\varepsilon(g,V)(t,z) = \frac{\iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(Y,t,z)]\big(\mathcal{D}_\varepsilon(\partial_z V)^2 + \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V)\big)(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}.$$

Things are a little bit different for the third derivative, as can be seen in the following explicit formula:

$$\partial_z^3 \mathcal{I}_\varepsilon(t,z) = \frac{\begin{aligned}\iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(Y,t,z)] \\ \times \big(\mathcal{D}_\varepsilon(\partial_z V)^3 + \tfrac{3}{2}\mathcal{D}_\varepsilon(\partial_z V)\mathcal{D}_\varepsilon(\partial_z^2 V) + \tfrac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V)\big)(Y,t,z)\,dy_1\,dy_2\end{aligned}}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y + \mathcal{D}^*_\varepsilon(V)(y,t)]\,dy}. \tag{4-13}$$

All terms in this formula will provide an order ε exactly as before, except for the linear contribution of $\mathcal{D}_\varepsilon(\partial_z^3 V)$ since we lack a priori controls of the fourth derivative of V in $\mathcal{F}$. Therefore, for this term we proceed as follows:

$$\begin{aligned}\varphi_\alpha(t,z)|\mathcal{D}_\varepsilon(\partial_z^3 V)(Y,t,z)| \\ &= (1+|z-z_*|)^\alpha \big|\partial_z^3 V(t,\bar z) - \tfrac{1}{2}\partial_z^3 V(t,\bar z+\varepsilon y_1) - \tfrac{1}{2}\partial_z^3 V(t,\bar z+\varepsilon y_2)\big| \\ &\le (1+|z-z_*|)^\alpha \big(|\partial_z^3 V(t,\bar z)| + \tfrac{1}{2}|\partial_z^3 V(t,\bar z+\varepsilon y_1)| + \tfrac{1}{2}|\partial_z^3 V(t,\bar z+\varepsilon y_2)|\big) \\ &\le 2^{\alpha+1}\|\varphi_\alpha \partial_z^3 V\|_\infty (1+\varepsilon^\alpha(|y_1|^\alpha + |y_2|^\alpha)).\end{aligned} \tag{4-14}$$

For this computation, we used the following property of the weight function, which was also of crucial importance in [Calvez et al. 2019, Lemma 4.5]:

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \frac{\varphi_\alpha(t,z)}{\varphi_\alpha(t,\bar z)} \le 2^\alpha.$$

As a consequence, take $i=1$ or 2. Then

$$\begin{aligned}\varphi_\alpha(t,z)|\partial_z^3 V(\bar z+\varepsilon y_i)| &\le \frac{2^\alpha \varphi_\alpha(t,\bar z)}{(1+|\bar z+\varepsilon y_i - z_*|)^\alpha}\|\varphi_\alpha \partial_z^3 V\|_\infty \\ &\le 2^\alpha\left(1+\frac{|\varepsilon y_i|}{1+|\bar z+\varepsilon y_i - z_*|}\right)^\alpha \|\varphi_\alpha \partial_z^3 V\|_\infty \\ &\le 2^\alpha(1+\varepsilon^\alpha|y_i|^\alpha)\|\varphi_\alpha\partial_z^3 V\|_\infty.\end{aligned}$$

We deduce that

$$\begin{aligned}\varphi_\alpha(t,z)\frac{\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V)(Y,t,z)]\big(\tfrac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V)(Y,t,z)\big)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y + \mathcal{D}^*_\varepsilon(V)(y,t)]\,dy} \\ \le \frac{1}{2^{1-\alpha}}\|\varphi_\alpha\partial_z^3 V\|_\infty + O(\varepsilon^\alpha)\|(g,V)\|,\end{aligned}$$

by subadditivity of $|\cdot|^\alpha$. This justifies (4-14). Once added to other estimates of the terms of (4-13), obtained by Taylor expansions of $\mathcal{D}_\varepsilon$ as before, we get the desired estimate. □

One can notice in the proof that the order $O(\varepsilon)$ is not the sharpest one can possibly get for the first derivative; see (4-12). However, it is sufficient for our purposes. We now detail the control upon the directional derivatives of $\mathcal{I}_\varepsilon$.

Proposition 4.4 (bound of the directional derivatives of $\mathcal{I}_\varepsilon$). *For any ball B of $\mathbb{R}\times\mathcal{E}$, there exists a constant ε_B that depends only on B such that for any pair (g, V) in B and any function $H\in\mathcal{E}$, for every $\varepsilon\le\varepsilon_B$*:

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} (\varphi_\alpha(t,z)|\partial_g\partial_z^{(j)}\mathcal{I}_\varepsilon(g,V)(t,z)|)\le O(\varepsilon)\|(g,V)\|_{\mathcal{E}},\qquad j=1,2,3,\tag{4-15}$$

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} (\varphi_\alpha(t,z)|\partial_V\partial_z^{(j)}\mathcal{I}_\varepsilon(g,V)\cdot H(t,z)|)\le O(\varepsilon)\|H\|_{\mathcal{E}},\qquad j=1,2,\tag{4-16}$$

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} (\varphi_\alpha(t,z)|\partial_V\partial_z^{3}\mathcal{I}_\varepsilon(g,V)\cdot H(t,z)|)\le O(\varepsilon^\alpha)\|H\|_{\mathcal{E}}+\frac{1}{2^{1-\alpha}}\|\varphi_\alpha\partial_z^3 H\|_\infty,\tag{4-17}$$

where the $O(\varepsilon)$ depend only on the ball B.

As for Proposition 4.3, in those estimates, the order of precision $O(\varepsilon)$ is not optimal and we could improve it without it being useful. We will not give the full proof for each estimate of this Proposition. However, we see that it follows the same pattern than in Proposition 4.3, and we will even use those results for the proof. In particular for the third derivative, it is not possible to completely recover an order ε, hence the term

$$\|\varphi_\alpha\partial_z^3 H\|_\infty/2^{1-\alpha}.$$

It comes from the *linear* part $\mathcal{D}_\varepsilon(\partial_z^3 V)$ that appears in $\partial_z^3\mathcal{I}_\varepsilon$, see (4-13). However, it does not prevent us from carrying our analysis since the factor $1/2^{1-\alpha}$ will be absorbed by a contraction argument; see Section 8.

Proof of Proposition 4.4. We first detail the proof of (4-15), because derivatives in g are somehow easier to estimate. The formula for the first derivative is:

$$\begin{aligned}\partial_g\partial_z\mathcal{I}_\varepsilon&(g,V)(t,z)\\ =-\varepsilon\Bigg(&\frac{\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V)(Y,t,z)](y_1+y_2)\mathcal{D}_\varepsilon(\partial_z V)(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}\\ &-\partial_z\mathcal{I}_\varepsilon(g,V)(t,z)\frac{\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]y\,dy}{\sqrt{\pi}\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}\Bigg).\end{aligned}\tag{4-18}$$

The first term of this formula closely resembles the one for $\partial_z I_\varepsilon(g,V)$, with an additional factor $\varepsilon(y_1+y_2)$. We do not detail how to bound it, as it follows the same steps; see the work done following (4-11). For the second term we first use the following bound:

$$\frac{\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]y\,dy}{\sqrt{\pi}\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}\le\frac{\int_{\mathbb{R}}\exp\big[-\frac12|y|^2+2\varepsilon|y|\|(g,V)\|\big]y\,dy}{\int_{\mathbb{R}}\exp\big[-\frac12|y|^2-2\varepsilon|y|\|(g,V)\|\big]\,dy}.\tag{4-19}$$

For ε sufficiently small that depends only on $\|(g,V)\|$ we deduce a uniform bound with moments of the Gaussian distribution. We then use the estimate from Proposition 4.3 on $\partial_z I_\varepsilon(g,V)$, which takes the weight into account, to conclude.

Every other estimate of Proposition 4.4 works along the same lines. We illustrate this with the second derivative in g and z:

$$\begin{aligned}\partial_g\partial_z^2\mathcal{I}_\varepsilon(g,V)(t,z)\\ =-\varepsilon\Biggl(&\frac{\iint_{\mathbb{R}^2}e^{-Q(y_1,y_2)}\exp[-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V)(Y,t,z)](y_1+y_2)\bigl(\mathcal{D}_\varepsilon(\partial_zV)^2+\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2V)\bigr)(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}\\ &-\partial_z^2\mathcal{I}_\varepsilon(g,V)\frac{\int_{\mathbb{R}}e^{-|y|^2/2}y\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}{\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}\Biggr).\end{aligned}\tag{4-20}$$

This is very close to $\partial_z^2 I_\varepsilon$ that has already been estimated in Proposition 4.3, and therefore the same arguments as before hold.

The structure is different for the derivatives in V, as can be seen for $\partial_V\partial_z\mathcal{I}_\varepsilon(g,V)\cdot H$:

$$\begin{aligned}\partial_V\partial_z\mathcal{I}_\varepsilon(g,V)\cdot H(t,z)\\ =&\frac{\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V)(Y,t,z)](2\mathcal{D}_\varepsilon(\partial_zV)\mathcal{D}_\varepsilon(H)+\mathcal{D}_\varepsilon(\partial_zH))(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}\\ &-\partial_z\mathcal{I}_\varepsilon(g,V)(t,z)\frac{\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]\mathcal{D}_\varepsilon^*(H)(y,t)\,dy}{\sqrt{\pi}\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}.\end{aligned}\tag{4-21}$$

The second term can still be bounded using Proposition 4.3 and estimate (4-19) along with the following immediate result:

$$|\mathcal{D}_\varepsilon^*(V)(y,t)|\le\varepsilon|y|\|V\|_{\mathcal{E}}.$$

For the first term, we must do Taylor expansions of $2\mathcal{D}_\varepsilon(\partial_zV)\mathcal{D}_\varepsilon(H)+\mathcal{D}_\varepsilon(\partial_zH)$ to control them with the weight. One ends up with moments of the multidimensional Gaussian distribution just as in all the previous proofs. For instance,

$$\begin{aligned}2\varphi_\alpha(t,z)|\mathcal{D}_\varepsilon(\partial_zV)\mathcal{D}_\varepsilon(H)|(t,z)&\le\varphi_\alpha(t,z)|\mathcal{D}_\varepsilon(\partial_zV)(t,z)|O(\varepsilon)(|y_1|+|y_2|)\|H\|_{\mathcal{E}}\\ &\le O(\varepsilon)(|y_1|+|y_2|+|y_1|^{\alpha+1}+|y_2|^{1+\alpha})(|y_1|+|y_2|)\|H\|_{\mathcal{E}}\|V\|_{\mathcal{E}}.\end{aligned}$$

The same method holds for the second derivative in V and z.

The estimate of the third derivative in g and z is similar to the previous computations with the following formula:

$$\begin{aligned}\partial_g\partial_z^3\mathcal{I}_\varepsilon(t,z)\\ =&\frac{\begin{aligned}-\varepsilon\iint_{\mathbb{R}^2}&\exp[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V)(Y,t,z)]\\ &\times(y_1+y_2)\bigl(\mathcal{D}_\varepsilon(\partial_zV)^3+\tfrac{3}{2}\mathcal{D}_\varepsilon(\partial_zV)\mathcal{D}_\varepsilon(\partial_z^2V)+\tfrac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3V)\bigr)(Y,t,z)\,dy_1\,dy_2\end{aligned}}{\sqrt{\pi}\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}\\ &-\varepsilon\partial_z^3\mathcal{I}_\varepsilon(t,z)\frac{\int_{\mathbb{R}}e^{-|y|^2/2}y\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}{\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}.\end{aligned}\tag{4-22}$$

However, to get the bound (4-17), things are a little bit different, because of the linear term of higher order, $\mathcal{D}_\varepsilon(\partial_z^3 H)$:

$$\partial_V \partial_z^3 \mathcal{I}_\varepsilon(g, V) \cdot H(t,z)$$
$$= \frac{\begin{aligned}&\iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(Y,t,z)] \\ &\qquad \times \Big(\mathcal{D}_\varepsilon(H)\big(2\mathcal{D}_\varepsilon(\partial_z V)^3 + 3\mathcal{D}_\varepsilon(\partial_z V)\mathcal{D}_\varepsilon(\partial_z^2 V) + \tfrac{1}{2}\mathcal{D}_\varepsilon(\partial_z^3 V)\big) + 3\mathcal{D}_\varepsilon(\partial_z H)\mathcal{D}_\varepsilon(\partial_z V)^2 \\ &\qquad\qquad + \tfrac{3}{2}(\mathcal{D}_\varepsilon(\partial_z V)\mathcal{D}_\varepsilon(\partial_z^2 H) + \mathcal{D}_\varepsilon(\partial_z H)\mathcal{D}_\varepsilon(\partial_z^2 V)) + \tfrac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 H)\Big)(Y,t,z)\,dy_1\,dy_2\end{aligned}}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}$$
$$+ \partial_z^3 \mathcal{I}_\varepsilon(t,z) \frac{\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\mathcal{D}_\varepsilon^*(H)(y,t)\,dy}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}.$$

We do not get an order ε from the linear part $\mathcal{D}_\varepsilon(\partial_z^3 H)$, since we do not control the fourth derivative in $\mathcal{E}$. We then proceed with arguments following (4-13) in the proof of Proposition 4.3. $\square$

Thanks to those estimates we are able to write our main result for this part, which is a precise control of the linearization of the derivatives of $\mathcal{I}_\varepsilon$:

Proposition 4.5 (linearization with weight). *For any ball B of $\mathbb{R} \times \mathcal{E}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$ we have, for all $(g, W) \in B$:*

$$\partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t,z) = \partial_z \mathcal{I}_\varepsilon^*(t,z) + \frac{\|(g,W)\|}{\varphi_\alpha(t,z)} O(\varepsilon^3), \tag{4-23}$$
$$\partial_z^2 \mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t,z) = \partial_z^2 \mathcal{I}_\varepsilon^*(t,z) + \frac{\|(g,W)\|}{\varphi_\alpha(t,z)} O(\varepsilon^3), \tag{4-24}$$
$$\partial_z^3 \mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t,z) = \partial_z^3 \mathcal{I}_\varepsilon^*(t,z) + \frac{\varepsilon^2 \|\varphi_\alpha \partial_z^3 W\|_\infty}{2^{1-\alpha}\varphi_\alpha(t,z)} + \frac{\|(g,W)\|}{\varphi_\alpha(t,z)} O(\varepsilon^{2+\alpha}), \tag{4-25}$$

where the $O(\varepsilon^3)$ only depend on the ball B.

Proof. The methodology for (4-23)–(4-25) is the same. We detail for instance how to prove (4-23). One begins by writing the following exact Taylor expansion up to the second order:

$$\begin{aligned}&\partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t,z) \\ &\quad = \partial_z \mathcal{I}_\varepsilon^*(t,z) + \varepsilon^2(\partial_g \partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2 \xi g, V^* + \varepsilon^2 \xi W)(t,z) + \partial_V \partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2 \xi g, V^* + \varepsilon^2 \xi W) \cdot W(t,z)),\end{aligned}$$

with $0 < \xi < 1$. The result for (4-23) is then given by the directional decay estimates of Proposition 4.4 applied to $g' = q^* + \varepsilon^2 \xi g,\ V = V^* + \varepsilon^2 \xi W$ and $H = W$. $\square$

Together with Proposition 3.6, we know exactly how $\partial_z^j I_\varepsilon$ behaves when ε is small:

$$\partial_z^{(j)} \mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t,z) = O^*(\varepsilon^2) + \frac{\|(g,W)\|}{\varphi_\alpha(t,z)} O(\varepsilon^3),$$

where $j = 1, 2$, and the behavior is only slightly different for $j = 3$.

4C. *Refined estimates of $\mathcal{I}_\varepsilon^*$ at $z = z_*$.* To conclude this section dedicated to estimates of $\mathcal{I}_\varepsilon$, we now show that our estimates above can be made much more precise when looking at the particular case of the function $\mathcal{I}_\varepsilon^*$ evaluated at the point z_*. In particular, we will gain information about the sign of the derivatives, that will prove crucial regarding the stability of κ_ε. This additional precision is similar to what was needed in the stationary case [Calvez et al. 2019, Lemma 3.1] where detailed expansions of $\mathcal{I}_\varepsilon$ were needed for the study of the affine part, there named γ_ε. We will find it convenient to use the following notations, as in [Calvez et al. 2019]:

Definition 4.6 (measure notations). We introduce the following measures:

$$dG_\varepsilon^*(Y,z,t) := \frac{G_\varepsilon^*(Y,t,z)}{\iint_{\mathbb{R}^2} G_\varepsilon^*(Y,t,z)\,dy_1\,dy_2} = \frac{\exp[-Q(y_1,y_2)-\varepsilon q^*(y_1+y_2)+2\mathcal{D}_\varepsilon(V^*)(Y,t,z)]}{\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon q^*(y_1+y_2)+2\mathcal{D}_\varepsilon(V^*)(Y,t,z)]\,dy_1\,dy_2}, \tag{4-26}$$

with $Y = (y_1, y_2)$, and

$$dN_\varepsilon^*(y,t) := \frac{N_\varepsilon^*(y,t)}{\int_{\mathbb{R}} N_\varepsilon^*(\cdot\,,t)} := \frac{\exp\bigl[-\frac{1}{2}|y|^2-\varepsilon q^* y+\mathcal{D}_\varepsilon^*(V^*)(y,t)\bigr]}{\int_{\mathbb{R}}\exp\bigl[-\frac{1}{2}|y|^2-\varepsilon q^* y+\mathcal{D}_\varepsilon^*(V^*)(y,t)\bigr]\,dy}. \tag{4-27}$$

Proposition 4.7 (uniform control of the directional derivatives of $\partial_z\mathcal{I}_\varepsilon^*$). *There exist a function of time R_ε^* such that for any ball B of $\mathcal{E}$, there exists a constant ε^* that depends only on K^*, that satisfies for all $\varepsilon \le \varepsilon^*$, for all $H \in \mathcal{E}$:*

$$\partial_g\partial_z\mathcal{I}_\varepsilon^*(t,z_*) = \varepsilon^2 R_\varepsilon^*(t) + O^*(\varepsilon^3) \quad \textit{and} \quad \partial_V\partial_z\mathcal{I}_\varepsilon^*\cdot H(t,z_*) = O^*(\varepsilon^2)\|H\|_{\mathcal{E}}, \tag{4-28}$$

where all the $O^*(\varepsilon^j)$ depend only on K^* defined in Proposition 3.3 and R_ε^* is given by the formula

$$R_\varepsilon^*(t) := m''(t,z_*)\iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)(y_1+y_2)^2\,dy_1\,dy_2. \tag{4-29}$$

So R_ε^* is uniformly bounded and there exists a constant R_0 and time t_0 such that $R_\varepsilon^* \ge R_0 > 0$ for all $t \ge t_0$.

The sign of R_ε^* is directly connected to the behavior of z_* we assumed in the introduction; see (1-16). The derivative in V admits a lower order in ε as in previous estimates; see (4-25) and (4-17) for instance. This lower order term will be absorbed by a contraction argument, see Section 8, once we have a definitive estimate of $\|W_\varepsilon\|_{\mathcal{F}}$; see estimate (8-2).

Proof of Proposition 4.7. First we focus on the bound of the first equation in (4-28). Similarly to (4-18), the explicit formula for the derivative is

$$\begin{aligned}
&\partial_g\partial_z\mathcal{I}_\varepsilon^*(t,z_*)\\
&:= -\varepsilon(I_1+I_2)\\
&= -\varepsilon\Biggl(\frac{\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon q^*(y_1+y_2)+2\mathcal{D}_\varepsilon(V^*)(Y,t,z_*)](y_1+y_2)\mathcal{D}_\varepsilon(\partial_z V^*)(Y,t,z_*)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon q^* y+\mathcal{D}_\varepsilon^*(V^*)(y,t)]\,dy}\\
&\qquad\qquad - \partial_z\mathcal{I}_\varepsilon^*(t,z_*)\frac{\int_{\mathbb{R}} e^{-|y|^2/2} y\exp[-\varepsilon q^* y+\mathcal{D}_\varepsilon^*(V^*)(y,t)]\,dy}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon q^* y+\mathcal{D}_\varepsilon^*(V^*)(y,t)]\,dy}\Biggr).
\end{aligned} \tag{4-30}$$

Thanks to Proposition 4.4, we already know that $|\partial_z \mathcal{I}_\varepsilon^*(t,z_*)| = O^*(\varepsilon^2)$. Moreover, we bound uniformly the second term as follows:

$$\left| \frac{\int_{\mathbb{R}} e^{-|y|^2/2} y \exp[-\varepsilon q^* y + \mathcal{D}_\varepsilon^*(V^*)(y,t)]\,dy}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon q^* y + \mathcal{D}_\varepsilon^*(V^*)(y,t)]\,dy} \right| \le \frac{\int_{\mathbb{R}} \exp\bigl[-\frac{1}{2}|y|^2 + 2\varepsilon K^*|y|\bigr]|y|\,dy}{\sqrt{\pi}\int_{\mathbb{R}} \exp\bigl[-\frac{1}{2}|y|^2 - 2\varepsilon K^*|y|\bigr]\,dy} \le O^*(1),$$

where K^* was defined in Proposition 3.3. This shows that $I_2 = O^*(\varepsilon^2)$. Therefore one can focus on I_1. In order to gather information about the sign of this quantity and not only get a bound in absolute value, we perform exact Taylor expansions of $\mathcal{D}_\varepsilon(\partial_z V^*)$. We divide I_1 by $\mathcal{I}_\varepsilon^*(t,z_*)$, and thanks to the definitions of (4-26) and (4-27) we get

$$\frac{I_1}{\mathcal{I}_\varepsilon^*(t,z_*)} = \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)(y_1+y_2)\mathcal{D}_\varepsilon(\partial_z V^*)(Y,t,z_*)\,dy_1\,dy_2.$$

As usual, we make Taylor expansions: there exists $0 < \xi_1, \xi_2 < 1$ such that

$$\begin{aligned}\frac{I_1}{\mathcal{I}_\varepsilon^*(t,z_*)} = \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)\Bigl(-\varepsilon\tfrac{1}{2}(y_1+y_2)^2\partial_z^2 V^*(t,z_*) - \tfrac{1}{4}\varepsilon^2 y_1^2(y_1+y_2)\partial_z^3 V^*(t,z_*+\varepsilon\xi_1 y_1)\\ - \tfrac{1}{4}\varepsilon^2 y_2^2(y_1+y_2)\partial_z^3 V^*(t,z_*+\varepsilon\xi_2 y_2)\Bigr)\,dy_1\,dy_2. \qquad (4\text{-}31)\end{aligned}$$

We next define R_ε^* as

$$\varepsilon\partial_z^2 V^*(t,z_*)\iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)\tfrac{1}{2}(y_1+y_2)^2\,dy_1\,dy_2 =: \varepsilon R_\varepsilon^*(t),$$

with the following uniform bounds, that come from bounding by moments of a Gaussian distribution:

$$0 < R_0 \le R_\varepsilon^*(t), \quad \forall t \ge t_0.$$

Moreover, it is easy to see that R_ε^* is uniformly bounded. The next terms of (4-31) are of order superior to ε^2 and can be bounded uniformly by

$$\frac{1}{4}\varepsilon^2\left|\iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)(y_1^2(y_1+y_2) + y_2^2(y_1+y_2))K^*\,dy_1\,dy_2\right| \le O^*(\varepsilon^2).$$

Therefore one can rewrite (4-31) as

$$\frac{I_1}{\mathcal{I}_\varepsilon^*(t,z_*)} = -\varepsilon R_\varepsilon^*(t) + O^*(\varepsilon^2).$$

Thanks to Proposition 3.4, we recover a similar estimate for I_1:

$$I_1 = -\varepsilon R_\varepsilon^*(t) + O^*(\varepsilon^2).$$

Finally coming back to (4-30), we have shown that

$$\partial_g\partial_z \mathcal{I}_\varepsilon^*(t,z_*) = \varepsilon^2 R_\varepsilon^*(t) + O^*(\varepsilon^3).$$

This concludes the proof of the first estimate in (4-28). Next, we tackle the proof of the estimate upon the Fréchet derivative in (4-28), where, again, we first divide by $\mathcal{I}_\varepsilon^*(t,z_*)$:

$$\frac{\partial_V \partial_z \mathcal{I}_\varepsilon^* \cdot H(t,z_*)}{\mathcal{I}_\varepsilon^*(t,z_*)} = \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)(\mathcal{D}_\varepsilon(\partial_z V^*)2\mathcal{D}_\varepsilon(H) + \mathcal{D}_\varepsilon(\partial_z H))(Y,t,z_*)\,dy_1\,dy_2 \\ - \frac{\partial_z \mathcal{I}_\varepsilon^*(t,z_*)}{\mathcal{I}_\varepsilon^*(t,z_*)} \int_{\mathbb{R}} dN_\varepsilon^*(y,t)\mathcal{D}_\varepsilon^*(H)(y,t)\,dy. \tag{4-32}$$

Thanks to Propositions 3.6 and 3.4 and a uniform bound on $\mathcal{D}_\varepsilon^*(W)$, we have

$$\left| \frac{\partial_z \mathcal{I}_\varepsilon^*(t,z_*)}{\mathcal{I}_\varepsilon^*(t,z_*)} \int_{\mathbb{R}} dN_\varepsilon^*(y,t)\mathcal{D}_\varepsilon^*(H)(y,t)\,dy \right| \le O^*(\varepsilon^3)\|H\|_{\mathcal{E}}. \tag{4-33}$$

For the first term of (4-32), we first make a bound based on Taylor expansions of $\mathcal{D}_\varepsilon(H)$:

$$|\mathcal{D}_\varepsilon(H)(Y,t,z_*)| \le \tfrac{1}{2}\varepsilon^2(|y_1|^2 + |y_2|^2)\|H\|_{\mathcal{E}}.$$

The key element here is that since $\mathcal{D}_\varepsilon$ is evaluated at z_*, one gains an order in ε because $\partial_z H(t,z_*) = 0$, by definition of $\mathcal{E}$. Therefore, one gets

$$\left| \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)(\mathcal{D}_\varepsilon(\partial_z V^*)2\mathcal{D}_\varepsilon(H))(Y,t,z_*)\,dy_1\,dy_2 \right| \le O^*(\varepsilon^3)\|H\|_{\mathcal{E}}, \tag{4-34}$$

where the additional order in ε is gained through a Taylor expansion of $\mathcal{D}_\varepsilon(\partial_z V^*)$. We finally tackle the last term of (4-32) we did not yet estimate, involving $\mathcal{D}_\varepsilon(\partial_z H)$. Based only on Taylor expansions in $\mathcal{E}$, we do not gain an order ε^3 as in the previous terms, which explains our estimate of order ε^2 in (4-32). Rather, we obtain, for some $0 < \xi < 1$,

$$\iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)\mathcal{D}_\varepsilon(\partial_z H)(Y,t,z_*)\,dy_1\,dy_2 \\ = \varepsilon \frac{\partial_z^2 H(t,z_*)}{2} \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)(y_1+y_2)\,dy_1\,dy_2 \\ + \frac{\varepsilon^2}{4} \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)(y_1^2\partial_z^3 H(t,z_*+\varepsilon\xi y_1) + y_2^2\partial_z^3 H(t,z_*+\varepsilon\xi y_2))\,dy_1\,dy_2. \tag{4-35}$$

It is straightforward, based on multiple similar computations, to deduce that the first moment of dG_ε^* is zero at the leading order. Therefore,

$$\varepsilon \frac{\partial_z^2 H(t,z_*)}{2} \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)(y_1+y_2)\,dy_1\,dy_2 = \varepsilon \frac{\partial_z^2 H(t,z_*)}{2} O^*(\varepsilon) = O^*(\varepsilon^2)\|H\|_{\mathcal{E}}. \tag{4-36}$$

See for instance the proof of Proposition 3.4 for similar computations. In the second term of (4-35), we also cannot do better than an order in ε^2:

$$\frac{\varepsilon^2}{4} \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)(y_1^2\partial_z^3 H(t,z_*+\varepsilon\xi y_1) + y_2^2\partial_z^3 H(t,z_*+\varepsilon\xi y_2))\,dy_1\,dy_2 \\ \le \frac{\varepsilon^2\|H\|_{\mathcal{E}}}{4} \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)(y_1^2+y_2^2)\,dy_1\,dy_2 = O^*(\varepsilon^2)\|H\|_{\mathcal{E}}.$$

Finally, by putting together (4-33)–(4-35) and finally (4-36), the second estimate of (4-28) is proven. □

The order ε^3 of the second equation in (4-28) will be crucial in our analysis around κ_ε, the perturbation of the linear part q_ε defined in (1-22). Next, we provide an accurate linearization of $\partial_z \mathcal{I}_\varepsilon$ compared to the one provided before in Proposition 4.5 and (4-23). This is possible thanks to an evaluation at $z = z_*$, and it will prove useful when tackling the perturbation of the linear part κ_ε. This is the content of the following lemma.

Lemma 4.8 (uniform control of the second Fréchet derivative of $\partial_z \mathcal{I}_\varepsilon$). *For any ball B of $\mathbb{R} \times \mathcal{E}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$ we have, for all $(g, W) \in B$, that*

$$\partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t, z_*) \\ = \partial_z \mathcal{I}^*_\varepsilon(t, z_*) + \varepsilon^2 (\partial_g \partial_z \mathcal{I}^*_\varepsilon(t, z_*) g + (\partial_V \partial_z \mathcal{I}^*_\varepsilon \cdot W)(t, z_*)) + O(\varepsilon^5) \|(g, W)\|. \tag{4-37}$$

Proof. We write $f(p) := \partial_z \mathcal{I}_\varepsilon(q^* + pg, V^* + pW)(t, z)$. We recognize in formula (4-37) a Taylor expansion of f. Then, to prove the estimate of (4-37) it is sufficient to bound $f''(\varepsilon^2)$ uniformly:

$$f''(\varepsilon^2) \le O(\varepsilon) \|(g, W)\|.$$

The formula for f'' is very long, so for clarity we will denote by $A_\varepsilon(p)$ the numerator and by $B_\varepsilon(p)$ the denominator of $f(p)$, respectively, so that when we differentiate we have the structure

$$f''(p) = \frac{A''_\varepsilon(p)}{B_\varepsilon(p)} - 2\frac{A'_\varepsilon(p) B'_\varepsilon(p)}{B_\varepsilon(p)^2} - \frac{A_\varepsilon(p) B''_\varepsilon(p)}{B_\varepsilon(p)^2} + 2\frac{A_\varepsilon(p) B'_\varepsilon(p)^2}{B_\varepsilon(p)^3}. \tag{4-38}$$

The numerator is defined as

$$A_\varepsilon(p) := \iint_{\mathbb{R}^2} \exp[-Q(y_1, y_2) + 2\mathcal{D}_\varepsilon(V^* + pW)(Y, t, z_*) - \varepsilon(q^* + pg)(y_1 + y_2)] \\ \times \mathcal{D}_\varepsilon(\partial_z V^* + pW)(Y, t, z_*)\, dy_1\, dy_2,$$

while the denominator reads

$$B_\varepsilon(p) := \int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon(q^* + pg)y + \mathcal{D}^*_\varepsilon(V^* + pW)(y, t)]\, dy.$$

Therefore we will divide each term by $\mathcal{I}^*_\varepsilon$ to simplify the notations. This will make the measures dG^*_ε, dN^*_ε, introduced in (4-26) and (4-27), appear. For instance,

$$\frac{A_\varepsilon(p)}{\mathcal{I}^*_\varepsilon(t, z_*) B_\varepsilon(p)} \\ := \frac{\iint_{\mathbb{R}^2} dG^*_\varepsilon(Y, t, z_*) \exp[-\varepsilon p g(y_1 + y_2) + 2p\mathcal{D}_\varepsilon(W)(Y, t, z_*)](\mathcal{D}_\varepsilon(\partial_z V^* + p\partial_z V^*)(Y, t, z_*))\, dy_1\, dy_2}{\int_{\mathbb{R}} dN^*_\varepsilon(y, t) \exp[p\mathcal{D}^*_\varepsilon(W)(y, t) - \varepsilon p g y]\, dy}.$$

We notice that any factor of the sum in (4-38) (divided by $\mathcal{I}^*_\varepsilon$) is a sum (and a product) of terms of the form

$$\frac{A^{(j)}_\varepsilon(p) B^{(k)}_\varepsilon(p)}{B_\varepsilon(p) \mathcal{I}^*_\varepsilon(t, z_*)} = \frac{A^{(j)}_\varepsilon(p)}{\mathcal{I}^*_\varepsilon(t, z_*) B_\varepsilon(p)} \frac{B^{(k)}_\varepsilon(p)}{B_\varepsilon(p)},$$

with $j = 0, 1, 2,\ k = 1, 2$ and the constraint $j + k = 2$. It is rather convenient to bound separately each of those terms. For instance, we deal with the second one:

$$\frac{A'_\varepsilon(p)B'_\varepsilon(p)}{B_\varepsilon(p)^2\mathcal{I}^*_\varepsilon(t,z_*)} = \frac{A'_\varepsilon(p)}{\mathcal{I}^*_\varepsilon(t,z_*)B_\varepsilon(p)}\frac{B'_\varepsilon(p)}{B_\varepsilon(p)}. \tag{4-39}$$

The first term of this product is

$$\begin{aligned}
&\frac{A'_\varepsilon(p)}{\mathcal{I}^*_\varepsilon(t,z_*)B_\varepsilon(p)}\\
&:= \frac{\iint_{\mathbb{R}^2} dG^*_\varepsilon(Y,t,z_*)\exp[2p\mathcal{D}_\varepsilon(W)-\varepsilon gp(y_1+y_2)]\mathcal{D}_\varepsilon(\partial_z W)\,dy_1\,dy_2}{\int_{\mathbb{R}} dN^*_\varepsilon(y,t)\exp[2\mathcal{D}^*_\varepsilon(W)(y,t)-\varepsilon gy]\,dy}\\
&\quad+\frac{\iint_{\mathbb{R}^2} dG^*_\varepsilon(Y,t,z_*)\exp[2p\mathcal{D}_\varepsilon(W)-\varepsilon gp(y_1+y_2)]2\mathcal{D}_\varepsilon(\partial_z V^*+p\partial_z W)(\mathcal{D}_\varepsilon(W)-\varepsilon g(y_1+y_2))\,dy_1\,dy_2}{\int_{\mathbb{R}} dN^*_\varepsilon(y,t)\exp[2\mathcal{D}^*_\varepsilon(W)(y,t)-\varepsilon gy]\,dy}.
\end{aligned}$$

The numerator and denominator can be bounded by estimating naively $\mathcal{D}_\varepsilon$:

$$\begin{aligned}
&\left|\frac{A'_\varepsilon(p)}{B_\varepsilon(p)\mathcal{I}^*_\varepsilon(t,z)}\right|\\
&\le \frac{\iint_{\mathbb{R}^2} dG^*_\varepsilon(Y,t,z_*)\exp[3\varepsilon\|(g,W)\|(|y_1|+|y_2|)]\varepsilon(|y_1|+|y_2|)\|(g,W)\|\,dy_1\,dy_2}{\int_{\mathbb{R}} dN^*_\varepsilon(y,t)\exp[-3\varepsilon\|(g,W)\||y|]\,dy}\\
&\quad+\frac{\iint_{\mathbb{R}^2} dG^*_\varepsilon(Y,t,z_*)\exp[3\varepsilon\|(g,W)\|(|y_1|+|y_2|)]\varepsilon^2(|y_1|+|y_2|)^2(3\|(g,W)\|+2K^*)3\|(g,W)\|\,dy_1\,dy_2}{\int_{\mathbb{R}} dN^*_\varepsilon(y,t)\exp[-3\varepsilon\|(g,W)\||y|]\,dy}.
\end{aligned}$$

Therefore, we only get moments of a Gaussian distribution, so the previous bound is in fact

$$\left|\frac{A'_\varepsilon(p)}{B_\varepsilon(p)\mathcal{I}^*_\varepsilon(t,z)}\right| \le O(\varepsilon)\|(g,W)\|. \tag{4-40}$$

With the exact same arguments but more convoluted formulas, one shows that

$$\left|\frac{A''_\varepsilon(p)}{B_\varepsilon(p)\mathcal{I}^*_\varepsilon(t,z)}\right| \le O(\varepsilon)\|(g,W)\|. \tag{4-41}$$

For the quotients of B in (4-38), we lose the structure of the measures dG^*_ε and dN^*_ε, but they are replaced by an actual Gaussian measure $\exp[-y^2/2]$. Therefore, with the same arguments as before, we bound the quotient by the moments of a Gaussian distribution. For instance,

$$\begin{aligned}
\left|\frac{B'_\varepsilon(p)}{B_\varepsilon(p)}\right| &= \left|\frac{\int_{\mathbb{R}} e^{-|y|^2/2}\exp[2\mathcal{D}^*_\varepsilon(V^*+pW)-\varepsilon(q^*+gp)y](2\mathcal{D}^*_\varepsilon(W)-\varepsilon gy)\,dy}{\int_{\mathbb{R}} e^{-|y|^2/2}\exp[2\mathcal{D}^*_\varepsilon(V^*+pW)-\varepsilon(q^*+gp)y]\,dy}\right|\\
&\le \frac{\int_{\mathbb{R}} e^{-|y|^2/2}\exp[3\varepsilon|y|K^*+3\varepsilon\|(g,W)\||y|](3\varepsilon\|(g,W)\||y|)\,dy}{\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-3\varepsilon K^*|y|-3\varepsilon\|(g,W)\||y|]\,dy}\\
&\le O(\varepsilon)\|(g,W)\|.
\end{aligned} \tag{4-42}$$

Multiplying each term of (4-41) by (4-42) and then combining them yields the desired estimate result, given the separation of terms made in (4-38):

$$\left|\frac{f''(p)}{\mathcal{I}_\varepsilon^*(t,z)}\right| \le O(\varepsilon)\|(g,W)\|.$$

Thanks to Proposition 3.4, Lemma 4.8 is proven. □

5. Linearized equation for κ_ε, convergence of p_ε

5A. ***Uniform boundedness of κ_ε.*** Thanks to the estimates of the previous sections, all the useful tools to look at the perturbation κ_ε are made available. We recall that our final goal is to show that κ_ε is bounded as it is the perturbation from q^*; see (1-22). We show in this section that one gets an approximated ODE on κ_ε with good properties when linearizing; see Proposition 5.1. It is obtained by differentiating (2-2) and evaluating at $z = z_*$. This is exactly what suggested the spectral analysis of the formal linearized operator T in the table on page 1297. Now, thanks to our previous set of estimates from Section 4, we are able to carefully justify our linearization. Finally, the limit ODE we introduced for q^* in (1-18) will appear clearly when we do our analysis to balance contributions of smaller order.

To simplify expressions, we introduce the following alternative notations for all $t, z \in \mathbb{R}_+ \times \mathbb{R}$:

$$\Xi_\varepsilon(t,z) := W_\varepsilon(t,z) - 2W_\varepsilon(t,\bar{z}(t)). \tag{5-1}$$

Compared to previous sections, and for the rest of this article, we will work in the space $\mathcal{F}$ that is well suited to measure W_ε and build the linearization results, here for κ_ε. All our previous estimates that were established in $\mathcal{E}$ remain true in $\mathcal{F}$.

Proposition 5.1 (equation on κ_ε). *For any ball B of $\mathbb{R}\times\mathcal{F}$ there exists a constant ε_B that depends only on B such that if $(\kappa_\varepsilon, W_\varepsilon) \in B$ is a solution of (2-2), then for all $\varepsilon \le \varepsilon_B$, we have that κ_ε is a solution of the following ODE*:

$$-\dot{\kappa}_\varepsilon(t) = R_\varepsilon^*(t)\kappa_\varepsilon + O^*(1)\|W_\varepsilon\|_{\mathcal{F}} + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \tag{5-2}$$

where the $O(\varepsilon)$ depend only on B, and the R_ε^* are defined in Proposition 4.7.

Proof. As announced above, one starts by differentiating (2-2). This yields, with the notation Ξ_ε introduced in (5-1),

$$\begin{aligned}
&\partial_z M(t,z) - \varepsilon^2\dot{q}^*(t) - \varepsilon^2\partial_z\partial_t V^*(t,z) - \varepsilon^4\dot{\kappa}_\varepsilon(t) - \varepsilon^4\partial_z\partial_t W_\varepsilon(t,z)\\
&\quad = M(t,z)\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\\
&\qquad\qquad + \partial_z M(t,z)\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\\
&\qquad\qquad\qquad + \varepsilon^2 M(t,z)\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\partial_z\Xi_\varepsilon(t,z).
\end{aligned}$$

When we evaluate the expression at $z = z_*$, the last two terms vanish, since $\partial_z M(t,z_*) = \partial_z\Xi_\varepsilon(t,z_*) = 0$. Therefore, the equation becomes, since $\Xi_\varepsilon(t,z_*) = 0$ and $M(t,z_*) = 1$,

$$-\varepsilon^2\dot{q}^*(t) - \varepsilon^2\partial_z\partial_t V^*(t,z_*) - \varepsilon^4\dot{\kappa}_\varepsilon(t) - \varepsilon^4\partial_z\partial_t W_\varepsilon(t,z_*) = \partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z_*). \tag{5-3}$$

We then use directly the linearization result of Lemma 4.8 that we prepared for that purpose:

$$\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2W_\varepsilon)(t,z_*) \\ = \partial_z\mathcal{I}^*_\varepsilon(t,z_*)+\varepsilon^2(\partial_g\partial_z\mathcal{I}^*_\varepsilon(t,z_*)\kappa_\varepsilon+(\partial_V\partial_z\mathcal{I}^*_\varepsilon\cdot W_\varepsilon)(t,z_*))+O(\varepsilon^5)\|(\kappa_\varepsilon,W_\varepsilon)\|. \quad (5\text{-}4)$$

We see that for most of the terms, we previously provided a careful estimate in Section 4. First, by Proposition 3.5,

$$\partial_z\mathcal{I}^*_\varepsilon(t,z_*)=\varepsilon^2\bigl(m''(z_*)q^*(t)-\tfrac{1}{2}m^{(3)}(z_*)\bigr)+O^*(\varepsilon^4).$$

Plugging this into the asymptotic development of (5-4), we get the following:

$$\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2W_\varepsilon)(t,z_*) \\ =\varepsilon^2\bigl(m''(z_*(t))q^*(t)-\tfrac{1}{2}m^{(3)}(z_*(t))\bigr) \\ +\varepsilon^2(\partial_g\partial_z\mathcal{I}^*_\varepsilon(t,z_*)\kappa_\varepsilon+\partial_V\partial_z\mathcal{I}^*_\varepsilon\cdot W_\varepsilon(t,z_*))+O^*(\varepsilon^4)+O(\varepsilon^5)\|(\kappa_\varepsilon,W_\varepsilon)\|.$$

Combining this with Proposition 4.7 where we got precise estimates at the point z_*, we complete the expansion of $\partial_z\mathcal{I}_\varepsilon$:

$$\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2W_\varepsilon)(t,z_*) \\ =\varepsilon^2\bigl(m''(z_*)q^*(t)-\tfrac{1}{2}m^{(3)}(z_*)\bigr)+\varepsilon^4R^*_\varepsilon(t)\kappa_\varepsilon+O^*(\varepsilon^4)\|W_\varepsilon\|_{\mathcal{F}}+O^*(\varepsilon^4)+O(\varepsilon^5)\|(\kappa_\varepsilon,W_\varepsilon)\|.$$

When we turn back to (5-3), we have shown at this point the following relationship:

$$-\varepsilon^2\dot q^*(t)-\varepsilon^2\partial_z\partial_tV^*(t,z_*)-\varepsilon^4\dot\kappa_\varepsilon(t)-\varepsilon^4\partial_z\partial_tW_\varepsilon(t,z_*) \\ =\varepsilon^2\bigl(m''(z_*)q^*(t)-\tfrac{1}{2}m^{(3)}(z_*)\bigr)+\varepsilon^4R^*_\varepsilon(t)\kappa_\varepsilon+O^*(\varepsilon^4)\|W_\varepsilon\|_{\mathcal{F}}+O^*(\varepsilon^4)+O(\varepsilon^5)\|(\kappa_\varepsilon,W_\varepsilon)\|. \quad (5\text{-}5)$$

To get a stable equation on κ_ε, the terms of order ε^2 must cancel out. This is precisely the role played by the dynamics of q^* defined in (1-18). To see it, we just rewrite a term of (5-5), using $\partial_zV^*(t,z_*)=0$ and Lemma 3.2:

$$\partial_z\partial_tV^*(t,z_*)=m'(z_*)\partial_z^2V^*(t,z_*)=2m'(z_*)m''(z_*).$$

Therefore, we recognize that by the definition of q^* in (1-18), the following terms cancel:

$$\varepsilon^2\bigl(\dot q^*(t)+m''(z_*)q^*(t)-\tfrac{1}{2}m^{(3)}(z_*)+2m''(z_*)m'(z_*)\bigr)=0.$$

We then rewrite the second term of (5-5) of order ε^4:

$$\partial_z\partial_tW_\varepsilon(t,z_*)=m'(z_*)\partial_z^2W_\varepsilon(t,z_*)=O^*(1)\|W_\varepsilon\|_{\mathcal{F}}.$$

Finally, we deduce from (5-5) the following relationship:

$$-\dot\kappa_\varepsilon(t)=R^*_\varepsilon(t)\kappa_\varepsilon+O^*(1)\|W_\varepsilon\|_{\mathcal{F}}+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|.$$

We have proven (5-2). ☐

In this ODE solved by κ_ε, each term plays a separate part. First the function R_ε^* is what guarantees the stability of κ_ε because it is positive for large times. The other terms come from our perturbative analysis methodology. The term $O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|$ measures the error made when linearizing to obtain the ODE, and it ensures that it is of superior order in ε except for the part that comes from the reference point of our linearization: $O^*(1)$. Interestingly, there is also an error term that is not of superior order when linearizing, $O^*(1)\|W_\varepsilon\|_{\mathcal{F}}$, but what saves our contraction argument of Section 8 is that this term only involves W_ε, which we can bound independently, see Section 7.

5B. ***Equation on p_ε.*** We did not perturb the number p_ε as we did for $(q_\varepsilon, V_\varepsilon)$ since it can be straightforwardly computed from our reference equation (2-2). Given the spectral decomposition in the table on page 1297 in the heuristics of Section 2, it is consistent to evaluate (2-2) at $z=z_*$ to gain the necessary information about p_ε. This yields

$$1-\varepsilon^2(\dot{p}_\varepsilon(t)+m'(z_*)q^*(t))-\varepsilon^4 m'(z_*)\kappa_\varepsilon(t)=\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z_*). \tag{5-6}$$

Thanks to Propositions 3.3 and 4.2, and as long as κ_ε is bounded (which we will show in Section 8),

$$\varepsilon^2(\dot{p}_\varepsilon(t)+m'(z_*(t))q^*(t))=O(\varepsilon^2).$$

In this last equation, the order of precision is not enough to recover the equation on p^* when $\varepsilon\to 0$. The problem is that the linearization of $\mathcal{I}_\varepsilon$ made in (4-10) is a little too rough. Coming back to Proposition 3.4, we make the more precise estimate

$$\mathcal{I}_\varepsilon^*(t,z_*)=1-\tfrac{1}{2}\varepsilon^2\partial_z^2 V^*(t,z_*)+O^*(\varepsilon^4). \tag{5-7}$$

The proof of this result is a direct adaptation of that of Proposition 3.4, by making Taylor expansions up to the fourth derivative of V^*, as made possible by the introduction of $\mathcal{E}^*$; see Definition 3.1. This involves computing the moments of the Gaussian distribution $\exp[-Q]$:

$$\frac{1}{\sqrt{2}\pi}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}(y_1^2+y_2^2)\,dy_1\,dy_2=\frac{1}{2}. \tag{5-8}$$

By plugging (5-7) into (5-6), and using (4-9), we find that

$$\dot{p}_\varepsilon(t)+m'(z_*)q^*(t)=\tfrac{1}{2}\partial_z^2 V^*(t,z_*)+O(\varepsilon^2)=m''(z_*)+O(\varepsilon^2). \tag{5-9}$$

We used (3-1) for the last equality. From (5-9), the convergence of p_ε towards p^* defined by (1-19), stated in Theorem 1.3, is straightforward.

6. Linearization results

We finally tackle the complete linearization of (2-2). A preview was given when we studied the equation on κ_ε, however it was local since we had beforehand evaluated at $z_*(t)$. Here, we will provide global (in space) results.

6A. ***Linearization of W_ε.*** A first step is to control the function Ξ_ε, which, we recall, is a byproduct of W_ε, introduced in (5-1).

Lemma 6.1 (control of Ξ_ε). *For any ball B of $\mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$, if $W_\varepsilon \in B$, then Ξ_ε defined in (5-1) satisfies*

$$\exp[\varepsilon^2 \Xi_\varepsilon(t,z)] = 1 + \varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|W_\varepsilon\|_{\mathcal{F}},$$

where $O(\varepsilon^4)$ depends only on the ball B.

Proof. By the choice of the norm in $\mathcal{F}$ and in the setting of $W_\varepsilon \in B$ we have the following uniform control for all t, z:

$$|\Xi_\varepsilon(t,z)| \le \|W_\varepsilon\|_{\mathcal{F}}.$$

Then, by performing an exact Taylor expansion, there exists $0 < \xi < 1$ such that

$$\exp[\varepsilon^2 \Xi_\varepsilon(t,z)] = 1 + \varepsilon^2 \Xi_\varepsilon(t,z) + \tfrac{1}{2}\varepsilon^4 \Xi_\varepsilon(t,z)^2 \exp[\varepsilon^2 \xi\, \Xi_\varepsilon(t,z)].$$

To conclude we uniformly bound the rest for $\varepsilon^2 \le 1/\|W_\varepsilon\|_{\mathcal{F}}$:

$$\left|\tfrac{1}{2}\varepsilon^4 \Xi_\varepsilon(t,z)^2 \exp[\varepsilon^2 \xi\, \Xi_\varepsilon(t,z)]\right| \le \tfrac{1}{2}\varepsilon^4 \|W_\varepsilon\|_{\mathcal{F}}^2. \qquad \square$$

This first result is prototypical of the tools we will employ to linearize the problem (2-2) solved by $(\kappa_\varepsilon, W_\varepsilon)$. We now write the linearized problem satisfied by W_ε.

Proposition 6.2 (linearization for W_ε). *For any ball B of $\mathbb{R} \times \mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$, any pair $(\kappa_\varepsilon, W_\varepsilon) \in B$, a solution of (2-2), satisfies the estimate*

$$-\varepsilon^2 \partial_t W_\varepsilon(t,z) = M(t,z)(\Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|), \tag{6-1}$$

where $O(\varepsilon)$ depends only on B.

Proof. One starts from (2-2):

$$\begin{aligned} M(t,z) - \varepsilon^2(\dot{p}_\varepsilon(t) + m'(z_*)q^*(t) + \dot{q}^*(t)(z-z_*) + \partial_t V^*(t,z)) - \varepsilon^4(\dot{\kappa}_\varepsilon(t)(z-z_*) + m'(z_*)\kappa_\varepsilon(t) + \partial_t W_\varepsilon(t,z)) \\ = M(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2 \Xi_\varepsilon(t,z)]. \end{aligned} \tag{6-2}$$

Thanks to Lemma 6.1 and Proposition 4.2, where we linearized $\mathcal{I}_\varepsilon$, and the term in Ξ_ε, one can expand the right-hand side as follows:

$$\begin{aligned} &M(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2 \Xi_\varepsilon(t,z)] \\ &\quad = M(t,z)(1 + O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|)(1 + \varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &\quad = M(t,z) + \varepsilon^2 M(t,z)\Xi_\varepsilon(t,z) + M(t,z)(O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|). \end{aligned} \tag{6-3}$$

The left-hand side of (6-2) is a little bit more involved. We will use our previous work on $(p_\varepsilon, \kappa_\varepsilon)$. First, thanks to (5-6) which states the relationship satisfied by p_ε, we have

$$-\varepsilon^2(\dot{p}_\varepsilon(t) + m'(z_*)q^*(t)) - \varepsilon^4 \kappa_\varepsilon m'(z_*) = 1 - \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z_*).$$

We then use Proposition 4.2 involving the linearization of $\mathcal{I}_\varepsilon$ to get that

$$-\varepsilon^2(\dot{p}_\varepsilon(t)+m'(z_*)q^*(t))-\varepsilon^4\kappa_\varepsilon m'(z_*)=O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon,W_\varepsilon)\|. \tag{6-4}$$

From Proposition 3.3, we have the following uniform bound:

$$|\partial_t V^*(t,z)|\le K^*. \tag{6-5}$$

Thanks to our preliminary work on κ_ε, and more precisely (5-5), we know that

$$\dot{q}^*(t)+\varepsilon^2\dot{\kappa}_\varepsilon(t)=O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|.$$

Therefore, the affine terms are comparable to M, since M is a superlinear function that admits a uniform lower bound by hypothesis; see (1-13):

$$\left|\frac{(\dot{q}^*(t)+\varepsilon^2\dot{\kappa}_\varepsilon(t))(z-z_*)}{M(t,z)}\right|=O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|. \tag{6-6}$$

When adding up the estimates of (6-5) and (6-6), we have shown that

$$\begin{aligned}-\varepsilon^2(\dot{p}_\varepsilon(t)+m'(z_*)q^*(t)+\dot{q}^*(t)(z-z_*)+\partial_t V^*(t,z))-\varepsilon^4(\dot{\kappa}_\varepsilon(t)(z-z_*)+m'(z_*(t))\kappa_\varepsilon(t)+\partial_t W_\varepsilon(t,z))\\ =M(t,z)(O^*(\varepsilon^2)+O(\varepsilon^3))\|(\kappa_\varepsilon,W_\varepsilon)\|-\varepsilon^4\partial_t W_\varepsilon(t,z).\end{aligned} \tag{6-7}$$

We have divided by M the relationships (6-4) and (6-5), which is possible thanks to the uniform lower bound of M.

Finally, when putting together (6-6) and (6-3) in (6-2), the terms M cancel each other, and we find (6-1) by factoring out ε^2. □

One can notice the similarity between what we just proved rigorously and the heuristics made in (2-1). From this result one can straightforwardly deduce an approximated linear equation satisfied by Ξ_ε.

Corollary 6.3 (linearization in $\Xi_\varepsilon(t,z)$). *For any ball B of $\mathbb{R}\times\mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon\le\varepsilon_B$, any pair $(\kappa_\varepsilon,W_\varepsilon)\in B$ satisfies the estimate*

$$\varepsilon^2\partial_t\Xi_\varepsilon(t,z)=M(t,z)\left(2\frac{M(t,\bar{z}(t))}{M(t,z)}\Xi_\varepsilon(t,\bar{z})-\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|\right), \tag{6-8}$$

where $O(\varepsilon)$ depends only on B.

Remark 6.4. • The reader may notice that the computation of $\partial_t\Xi_\varepsilon$ yields a parasite term $\varepsilon^2\dot{z}_*\partial_z\Xi_\varepsilon(t,\bar{z})$ not dealt with by (6-1). However, this is a lower order term since it satisfies

$$\varepsilon^2\dot{z}_*(t)\partial_z\Xi_\varepsilon(t,z)=O(\varepsilon^2)\|(\kappa_\varepsilon,W_\varepsilon)\|. \tag{6-9}$$

• Under the same assumption as Corollary 6.3, W_ε also satisfies the following linear equation:

$$-\varepsilon^2\partial_t W_\varepsilon(t,z)=M(t,z)(\Xi_\varepsilon(t,z)+O(1)).$$

However, in Section 7, we will study the stability of the solution of the linear problem. We will see that one needs precise estimates about the structure of the nonlinear negligible terms, which explains the more detailed estimate (6-1) and is the purpose of all our previous sections.

6B. *Linearization of $\partial_z W_\varepsilon$.* The computations for $\partial_z W_\varepsilon$ are slightly more complex because of the differentiation of the triple product in the right-hand side (2-2). However, the key point is that when we linearize $\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2\kappa_\varepsilon)$ the derivatives of $\mathcal{I}_\varepsilon$ are negligible in ε. Therefore the intuitive linearized problem for $\partial_z W_\varepsilon$, given by the derivation of the linearized equation for W_ε, actually holds true. This is the content of the following proposition:

Proposition 6.5 (linearization in $\partial_z W_\varepsilon$). *For any ball B of $\mathbb{R} \times \mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$, any pair $(\kappa_\varepsilon, W_\varepsilon) \in B$ solution of (2-2) satisfies the following estimate*:

$$-\varepsilon^2\partial_t\partial_z W_\varepsilon(t,z) = M(t,z)\left(\partial_z\Xi_\varepsilon(t,z) + \frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right) + \partial_z M(t,z)(\Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|), \tag{6-10}$$

where $O(\varepsilon)$ depends only on B.

Proof. One starts by differentiating (2-2) as in the proof of Proposition 5.1 to highlight κ_ε. This yields

$$\begin{aligned}
&\partial_z M(t,z) - \varepsilon^2\dot{q}^*(t) - \varepsilon^2\partial_z\partial_t V^*(t,z) - \varepsilon^4\dot{\kappa}_\varepsilon(t) - \varepsilon^4\partial_z\partial_t W_\varepsilon(t,z)\\
&\quad= M(t,z)\partial_z\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\\
&\qquad\qquad + \partial_z M(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\\
&\qquad\qquad\qquad + \varepsilon^2 M(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\partial_z\Xi_\varepsilon(t,z).
\end{aligned}$$

However, contrary to the case where we were studying $\dot{\kappa}_\varepsilon$, we will not evaluate at z_*. We introduce the notations R_i corresponding to each of the three terms of the right-hand side of the previous equation. We will linearize each R_i starting with R_1, which we estimate thanks to Proposition 4.5 and Lemma 6.1, paired with the estimate of Proposition 3.6:

$$\begin{aligned}
R_1 &:= \partial_z\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)M(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\\
&= M(t,z)\left(\partial_z\mathcal{I}_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1 + \varepsilon^2\Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\
&= M(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1 + \varepsilon^2\Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|).
\end{aligned}$$

Therefore, the final contribution of R_1 is

$$R_1 = M(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right). \tag{6-11}$$

Next, one looks at R_2. Thanks to Proposition 4.2,

$$\begin{aligned}
R_2 &:= \partial_z M(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\\
&= \partial_z M(t,z)(1 + O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|)(1 + \varepsilon^2\Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\
&= \partial_z M(t,z) + \varepsilon^2\partial_z M(t,z)\Xi_\varepsilon(t,z) + \partial_z M(t,z)(O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|).
\end{aligned} \tag{6-12}$$

We finally tackle R_3 with the same techniques, using Proposition 4.2 and Lemma 6.1:

$$\begin{aligned}R_3 &:= \varepsilon^2 M(t,z)\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\partial_z\Xi_\varepsilon(t,z)\\ &= \varepsilon^2 M(t,z)\partial_z\Xi_\varepsilon(t,z)(1+O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon,W_\varepsilon)\|)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon,W_\varepsilon)\|)\\ &= \varepsilon^2 M(t,z)+M(t,z)\frac{O(\varepsilon^4)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}. \end{aligned}\tag{6-13}$$

In that last estimate, we chose to write $O^*(\varepsilon^4)$ as a regular $O(\varepsilon^4)$. Coming back to our initial problem, when we assemble (6-11)–(6-13), we obtain

$$\begin{aligned}&\partial_z M(t,z)-\varepsilon^2\dot q^*(t)-\varepsilon^2\partial_z\partial_t V^*(t,z)-\varepsilon^4\dot\kappa_\varepsilon(t)-\varepsilon^4\partial_z\partial_t W_\varepsilon(t,z)\\ &\quad=\partial_z M(t,z)+\varepsilon^2\partial_z M(t,z)(\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|)\\ &\qquad\qquad+\varepsilon^2 M(t,z)\left(\partial_z\Xi_\varepsilon(t,z)+\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right).\end{aligned}\tag{6-14}$$

We now deal with the left-hand side of (6-14). First, the terms $\partial_z M(t,z)$ on each side cancel. Next, using the ODE that defines q^* in (1-18), our linearized equation on $\dot\kappa_\varepsilon$ stated in (5-2) and finally our bound of $\partial_t V^*$ made in Proposition 3.3, we find that

$$-\varepsilon^2(\dot q^*(t)+\partial_z\partial_t V^*(t,z)+\varepsilon^2\dot\kappa_\varepsilon(t))=O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon,W_\varepsilon)\|.\tag{6-15}$$

Finally, if we divide by M, the following estimate holds true since $\alpha<1$:

$$\left|\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon,W_\varepsilon)\|}{M(t,z)}\right|\le\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}.$$

Plugging this into (6-14), and dividing each side by ε^2, we therefore recover the relationship we wanted to prove:

$$\begin{aligned}-\varepsilon^2\partial_t\partial_z W_\varepsilon(t,z)&=M(t,z)\left(\partial_z\Xi_\varepsilon(t,z)+\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)\\ &\qquad+\partial_z M(t,z)(\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|).\end{aligned}$$ □

We deduce straightforwardly a linearization result upon the quantity $\partial_z\Xi_\varepsilon$.

Corollary 6.6 (linearization for $\partial_z\Xi_\varepsilon(t,z)$). *For any ball B of $\mathbb{R}\times\mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon\le\varepsilon_B$, any pair $(\kappa_\varepsilon,W_\varepsilon)\in B$, a solution of* (2-2), *satisfies the following estimate*:

$$\begin{aligned}\varepsilon^2\partial_t\partial_z\Xi_\varepsilon(t,z)&=M(t,z)\left(\frac{M(t,\bar z)}{M(t,z)}\partial_z\Xi_\varepsilon(t,\bar z)-\partial_z\Xi_\varepsilon(t,z)+\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)\\ &\qquad+\partial_z M(t,z)\left(\frac{\partial_z M(t,\bar z)}{\partial_z M(t,z)}\Xi_\varepsilon(t,\bar z)-\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|\right),\end{aligned}$$

where the $O(\varepsilon)$ depends only on B.

6C. ***Linearization of $\partial_z^2 W_\varepsilon(t,z)$.*** We now tackle the linearized equation for $\partial_z^2 W_\varepsilon$.

Proposition 6.7 (linearization for $\partial_z^2 W_\varepsilon$). *For any ball B of $\mathbb{R}\times\mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$, any pair $(\kappa_\varepsilon, W_\varepsilon) \in B$, a solution of (2-2), satisfies the following estimate*:

$$\begin{aligned}
&-\varepsilon^2\partial_z^2\partial_t W_\varepsilon(t,t)\\
&=\partial_z^2 M(t,z)(\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|)+2\partial_z M(t,z)\left(\partial_z\Xi_\varepsilon(t,z)+\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)\\
&\qquad\qquad+M(t,z)\left(\partial_z^2\Xi_\varepsilon(t,z)+\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right),
\end{aligned}\tag{6-16}$$

where the $O(\varepsilon)$ depend only on B.

In the next sections, we choose to write the second derivative $\partial_z^2\Xi_\varepsilon(t,z)$ in full,

$$\partial_z^2 W_\varepsilon(t,z)-\tfrac12\partial_z^2 W_\varepsilon(t,\bar z),$$

as the factor $\frac12$ will be the key to ensure the uniform boundedness of $\partial_z^2 W_\varepsilon$; see Section 7.

Proof. We start by differentiating (2-2) twice. This yields

$$\partial_z^2 M(t,z)-\varepsilon^2\partial_z^2\partial_t V^*(t,z)-\varepsilon^4\partial_z^2\partial_t W_\varepsilon(t,z)=R_1+R_2+R_3+R_4+R_5+R_6,$$

with the following notations:

$$\begin{aligned}
R_1 &:= \partial_z^2\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2W_\varepsilon)(t,z)M(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)],\\
R_2 &:= 2\partial_z M(t,z)\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)],\\
R_3 &:= 2M(t,z)\varepsilon^2\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\partial_z\Xi_\varepsilon(t,z),\\
R_4 &:= \mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2W_\varepsilon)(t,z)\partial_z^2 M(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)],\\
R_5 &:= 2\varepsilon^2\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2W_\varepsilon)(t,z)\partial_z M(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\partial_z\Xi_\varepsilon(t,z),
\end{aligned}$$

and finally,

$$R_6 := \varepsilon^2 M(t,z)\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)](\varepsilon^2\partial_z\Xi_\varepsilon(t,z)^2+\partial_z^2\Xi_\varepsilon(t,z)).$$

We will estimate each term separately, starting with R_1, for which we apply Proposition 4.5, Lemma 6.1 and Proposition 3.6:

$$\begin{aligned}
R_1 &= M(t,z)\left(\partial_z^2\mathcal{I}_\varepsilon^*(t,z)+\frac{O(\varepsilon^3)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon,W_\varepsilon)\|)\\
&= M(t,z)\left(\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon,W_\varepsilon)\|).
\end{aligned}$$

Therefore, the final estimate of R_1 is

$$R_1 = M(t,z)\left(\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right).\tag{6-17}$$

Next, for the term R_2 we use Propositions 4.5 and 3.6 and find that

$$\begin{aligned}R_2 &= 2\left(\partial_z \mathcal{I}_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)\partial_z M(t,z)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\ &= 2\partial_z M(t,z)\left(\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|).\end{aligned}$$

We can simplify this expression as

$$R_2 = \partial_z M(t,z)\left(\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right). \tag{6-18}$$

The term R_3 will not contribute at the order ε^2, because of Proposition 3.6, and $|\partial_z \Xi_\varepsilon(t,z)| \le \|W_\varepsilon\|_{\mathcal{F}}$:

$$\begin{aligned}R_3 &= 2\varepsilon^2 M(t,z)\partial_z \Xi_\varepsilon(t,z)\left(\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\ &= \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}M(t,z).\end{aligned} \tag{6-19}$$

For R_4, the zeroth order terms are more entangled. With Proposition 4.2 and Lemma 6.1 we have

$$\begin{aligned}R_4 &= \partial_z^2 M(t,z)(1+O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\ &= \partial_z^2 M(t,z)+\varepsilon^2\partial_z^2 M(t,z)(\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|).\end{aligned} \tag{6-20}$$

We see in R_4 the appearance of the term $\varepsilon^2\partial_z^2 M(t,z)\Xi_\varepsilon(t,z)$ which is also in (6-16), and so it is a good opportunity to do at first a summary of the computations when adding (6-17)–(6-20):

$$\begin{aligned}&R_1+R_2+R_3+R_4\\ &\quad= \partial_z^2 M(t,z)+\varepsilon^2\partial_z^2 M(t,z)(\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\ &\qquad\qquad +\varepsilon^2 M(t,z)\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}+\varepsilon^2\partial_z M(t,z)\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}.\end{aligned} \tag{6-21}$$

We continue the estimations by looking at R_5, and thanks to Proposition 4.2 we have

$$\begin{aligned}R_5 &= 2\varepsilon^2\partial_z M(t,z)\partial_z\Xi_\varepsilon(t,z)(1+O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\ &= 2\varepsilon^2\partial_z M(t,z)\partial_z\Xi_\varepsilon(t,z)+\varepsilon^2\partial_z M(t,z)\frac{O^*(\varepsilon)+O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}.\end{aligned} \tag{6-22}$$

Finally, we tackle the last term, R_6, with Proposition 4.2:

$$\begin{aligned}R_6 &= \varepsilon^2 M(t,z)(1+O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\ &\qquad\qquad\times\left(\frac{O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}+\partial_z^2\Xi_\varepsilon(t,z)\right)\\ &= \varepsilon^2 M(t,z)\partial_z^2\Xi_\varepsilon(t,z)+\varepsilon^2 M(t,z)\frac{O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}.\end{aligned} \tag{6-23}$$

Thanks to those last two estimates, (6-22) and (6-23), that we add with the previous result of (6-21), we obtain for the full equation

$$\begin{aligned}
&\partial_z^2 M(t,z) - \varepsilon^2\partial_z^2\partial_t V^*(t,z) - \varepsilon^4\partial_z^2\partial_t W_\varepsilon(t,z)\\
&\quad= \partial_z^2 M(t,z) + \varepsilon^2\partial_z^2 M(t,z)(\Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\
&\qquad\qquad + 2\varepsilon^2\partial_z M(t,z)\left(\partial_z\Xi_\varepsilon(t,z) + \frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)\\
&\qquad\qquad\qquad + \varepsilon^2 M(t,z)\left(\partial_z^2\Xi_\varepsilon(t,z) + \frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right).
\end{aligned}$$

Thanks to Proposition 3.3 we know that $\|\varepsilon^2\partial_z^2\partial_t V^*(t,z)\|_\infty \le O^*(\varepsilon^2)$. Then,

$$\begin{aligned}
&-\varepsilon^4\partial_z^2\partial_t W_\varepsilon(t,t)\\
&\quad= \varepsilon^2\partial_z^2 M(t,z)(\Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\
&\qquad\qquad + 2\varepsilon^2\partial_z M(t,z)\left(\partial_z\Xi_\varepsilon(t,z) + \frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)\\
&\qquad\qquad\qquad + \varepsilon^2 M(t,z)\left(\partial_z^2\Xi_\varepsilon(t,z) + \frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right),
\end{aligned}$$

which proves (6-16) after dividing by ε^2. □

6D. *Linearization of $\partial_z^3 W_\varepsilon(t,z)$.* Our last linearized equation is the one for $\partial_z^3 W_\varepsilon$, and we proceed with the same technique, with slightly more complex formulas.

Proposition 6.8 (linearization in $\partial_z^3 W_\varepsilon$). *For any ball B of $\mathbb{R}\times\mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$, any pair $(\kappa_\varepsilon, W_\varepsilon) \in B$, a solution of (2-2), satisfies the following estimate*:

$$\begin{aligned}
-\varepsilon^2\partial_t\partial_z^3 W_\varepsilon(t,z) &= \partial_z^3 M(t,z)(\Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\
&\quad + 3\partial_z^2 M(t,z)\left(\partial_z\Xi_\varepsilon(t,z) + \frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)\\
&\quad + 3\partial_z M(t,z)\left(\partial_z^2\Xi_\varepsilon(t,z) + \frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)\\
&\quad + M(t,z)\left(\partial_z^3\Xi_\varepsilon(t,z) + \frac{\|\varphi_\alpha\partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}\varphi_\alpha(t,z)} + \frac{O^*(1)+O(\varepsilon^\alpha)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right), \qquad (6\text{-}24)
\end{aligned}$$

where the $O(\varepsilon)$ depend only on B.

Proof of Proposition 6.7. We start, as ever, by differentiating (2-2), but now three times. This yields, for the right-hand side, ten terms:

$$\partial_z^3 M(t,z) - \varepsilon^2\partial_z^3\partial_t V^*(t,z) - \varepsilon^4\partial_z^3\partial_t W_\varepsilon(t,t) = R_1+R_2+R_3+R_4+R_5+R_6+R_7+R_8+R_9+R_{10}, \quad (6\text{-}25)$$

with the following notations:

$$R_1 := \partial_z^3 \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z) M(t,z) \exp[\varepsilon^2 \Xi_\varepsilon(t,z)],$$

$$R_2 := 3\partial_z^2 \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z) \partial_z M(t,z) \exp[\varepsilon^2 \Xi_\varepsilon(t,z)],$$

$$R_3 := 3\varepsilon^2\partial_z^2 \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z) M(t,z) \exp[\varepsilon^2 \Xi_\varepsilon(t,z)]\partial_z \Xi_\varepsilon(t,z),$$

$$R_4 := 6\varepsilon^2\partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z) \partial_z M(t,z) \exp[\varepsilon^2 \Xi_\varepsilon(t,z)]\partial_z \Xi_\varepsilon(t,z),$$

$$R_5 := 3\partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z) \partial_z^2 M(t,z) \exp[\varepsilon^2 \Xi_\varepsilon(t,z)],$$

and, moreover,

$$R_6 := 3\varepsilon^2\partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z) M(t,z) \exp[\varepsilon^2 \Xi_\varepsilon(t,z)](\varepsilon^2\partial_z \Xi_\varepsilon(t,z)^2 + \partial_z^2 \Xi_\varepsilon(t,z)),$$

$$R_7 := 3\varepsilon^2 \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z) \partial_z M(t,z) \exp[\varepsilon^2 \Xi_\varepsilon(t,z)](\varepsilon^2\partial_z \Xi_\varepsilon(t,z)^2 + \partial_z^2 \Xi_\varepsilon(t,z)),$$

$$R_8 := 3\varepsilon^2 \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z) \partial_z^2 M(t,z) \exp[\varepsilon^2 \Xi_\varepsilon(t,z)]\partial_z \Xi_\varepsilon(t,z),$$

$$R_9 := \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z) \partial_z^3 M(t,z) \exp[\varepsilon^2 \Xi_\varepsilon(t,z)].$$

The last term corresponds to the third derivative of the exponential term $\exp[\varepsilon^2 \Xi_\varepsilon]$:

$$R_{10} := \varepsilon^2 \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z) M(t,z) \exp[\varepsilon^2 \Xi_\varepsilon(t,z)] \\ \times (\varepsilon^4 \partial_z \Xi_\varepsilon(t,z)^3 + 3\varepsilon^2 \partial_z \Xi_\varepsilon(t,z)\partial_z^2 \Xi_\varepsilon(t,z) + \partial_z^3 \Xi_\varepsilon(t,z)).$$

We first tackle R_1. We use the linearization of the third derivative of $\mathcal{I}_\varepsilon$ in Proposition 4.5 to find that

$$\begin{aligned} R_1 &= M(t,z)\left(\partial_z^3 \mathcal{I}_\varepsilon^*(t,z) + \frac{\varepsilon^2 \|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}\varphi_\alpha(t,z)} + \frac{O(\varepsilon^{2+\alpha})\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &= \varepsilon^2 M(t,z)\left(\frac{\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}\varphi_\alpha(t,z)} + \frac{O^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|). \end{aligned}$$

We end up with the estimate

$$R_1 = \varepsilon^2 M(t,z)\left(\frac{\|\partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}\varphi_\alpha(t,z)} + \frac{O^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right). \tag{6-26}$$

For R_2, with Proposition 4.5 we have

$$\begin{aligned} R_2 &= 3\partial_z M(t,z)\left(\partial_z^2 \mathcal{I}_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &= 3\partial_z M(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|). \end{aligned}$$

We can simplify this expression to

$$R_2 = \varepsilon^2 \partial_z M(t,z)\left(\frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right). \tag{6-27}$$

For R_3, we get

$$\begin{aligned} R_3 &= 3\varepsilon^2 M(t,z)\partial_z \Xi_\varepsilon(t,z)\left(\partial_z^2 \mathcal{I}_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &= 3\varepsilon^2 M(t,z)\partial_z \Xi_\varepsilon(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|). \end{aligned}$$

We can simplify roughly this expression to

$$R_3 = \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)} M(t,z). \tag{6-28}$$

For R_4 one has very similarly

$$\begin{aligned} R_4 &= 6\varepsilon^2 \partial_z M(t,z)\partial_z \Xi_\varepsilon(t,z)\left(\partial_z \mathcal{I}_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &= 6\varepsilon^2 \partial_z M(t,z)\partial_z \Xi_\varepsilon(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|). \end{aligned}$$

We can simplify this expression to

$$R_4 = \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)} \partial_z M(t,z). \tag{6-29}$$

The expression for R_5 still follows the same road:

$$\begin{aligned} R_5 &= 3\partial_z^2 M(t,z)\left(\partial_z \mathcal{I}_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &= 3\partial_z^2 M(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|). \end{aligned}$$

This last expression can be shortened to

$$R_5 = 3\varepsilon^2 \partial_z^2 M(t,z) \frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}. \tag{6-30}$$

For R_6, the expression is a little more involved due to the second derivative of the exponential:

$$\begin{aligned} R_6 = \varepsilon^2 M(t,z)&\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &\times \left(\frac{O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)} + \partial_z^2 \Xi_\varepsilon(t,z)\right). \end{aligned}$$

We eventually shorten R_6 to

$$R_6 = 3M(t,z) \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}. \tag{6-31}$$

If we bring together all of our previous estimates in (6-26)–(6-31), we obtain that

$$\begin{aligned} R_1+R_2+R_3+R_4+R_5+R_6 \\ = \varepsilon^2 M(t,z)\left(\frac{O^*(1)+O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right) + \varepsilon^2 \partial_z M(t,z)\left(\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right) \\ + \varepsilon^2 \partial_z^2 M(t,z)\left(\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right) + \frac{\varepsilon^2 \|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}\varphi_\alpha(t,z)} M(t,z). \quad (6\text{-}32) \end{aligned}$$

In that first round of estimates, we have shown that all the contributions of the terms with the derivatives of $\mathcal{I}_\varepsilon$ do not appear when linearizing because they are of high order in ε. Therefore, the most meaningful contribution will now appear, because $\mathcal{I}_\varepsilon$ now contributes mainly as 1 and no longer vanishes.

We start with R_7:

$$\begin{aligned} R_7 = 3\varepsilon^2 \partial_z M(t,z)(1+O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|)(1+\varepsilon^2 \Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ \times \left(\frac{O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)} + \partial_z^2 \Xi_\varepsilon(t,z)\right), \end{aligned}$$

which can be rewritten as

$$R_7 = 3\varepsilon^2 \partial_z M(t,z)(1+O^*(\varepsilon)+O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|)\left(\partial_z^2 \Xi_\varepsilon(t,z) + \frac{O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right).$$

Finally, for R_7,

$$R_7 = 3\varepsilon^2 \partial_z M(t,z)\partial_z^2 \Xi_\varepsilon(t,z) + \partial_z M(t,z)\left(\frac{O^*(\varepsilon^3)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right). \quad (6\text{-}33)$$

For R_8, the following estimates hold true:

$$R_8 = 3\varepsilon^2 \partial_z^2 M(t,z)\partial_z \Xi_\varepsilon(t,z)(1+O^*(\varepsilon)+O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|)(1+\varepsilon^2 \Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|).$$

Therefore,

$$R_8 = 3\varepsilon^2 \partial_z^2 M(t,z)\partial_z \Xi_\varepsilon(t,z) + \partial_z^2 M(t,z)\left(\frac{O^*(\varepsilon^3)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right). \quad (6\text{-}34)$$

For the last two terms, the derivatives up to the third order appear. The simplest is given by R_9:

$$\begin{aligned} R_9 &= \partial_z^3 M(t,z)(1+O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|)(1+\varepsilon^2 \Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &= \partial_z^3 M(t,z) + \varepsilon^2 \partial_z^3 M(t,z)(\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|). \quad (6\text{-}35) \end{aligned}$$

At last, for the term R_{10}, we have

$$\begin{aligned} R_{10} = \varepsilon^2 M(t,z)(1+O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|)(1+\varepsilon^2 \Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ \times \left(\frac{O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)} + \partial_z^3 \Xi_\varepsilon(t,z)\right). \quad (6\text{-}36) \end{aligned}$$

This is shortened to

$$R_{10} = \varepsilon^2 M(t,z)\partial_z^3 \Xi_\varepsilon(t,z) + \varepsilon^2 M(t,z)\frac{O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}. \quad (6\text{-}37)$$

We now add every estimate, starting from (6-32) and including (6-33)–(6-37) to obtain

$$\begin{aligned}\sum_{j=1}^{10} R_j = \partial_z^3 M(t,z) &+ \varepsilon^2 \partial_z^3 M(t,z)(\Xi_\varepsilon(t,z) + O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\ &+ 3\varepsilon^2 \partial_z^2 M(t,z)\left(\partial_z \Xi_\varepsilon(t,z) + \frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)\\ &+ 3\varepsilon^2 \partial_z M(t,z)\left(\partial_z^2 \Xi_\varepsilon(t,z) + \frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)\\ &+ \varepsilon^2 M(t,z)\left(\partial_z^3 \Xi_\varepsilon(t,z) + \frac{\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}\varphi_\alpha(t,z)} + \frac{O^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right). \end{aligned} \tag{6-38}$$

To conclude the proof, we deal with the left-hand side of (6-25) as in the linearization of the second derivative, noticing that the terms $\partial_z^3 M$ cancel on each side. $\square$

7. Stability of the linearized equations

Building upon the series of linear approximations, we can study the stability of W_ε in the space $\mathcal{F}$. The first result is to control the different terms of $\mathcal{F}$ in the norm $\|\cdot\|_{\mathcal{F}}$; see Definition 1.2. The weight function introduced in the definition of $\mathcal{E}$ is meant to control the behavior at infinity.

Theorem 7.1 (stability analysis). *For any ball B of $\mathbb{R} \times \mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$, any pair $(\kappa_\varepsilon, W_\varepsilon) \in B$, a solution of (2-2), satisfies the following bounds*:

$$\begin{aligned}\|\Xi_\varepsilon\|_\infty &\le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,\\ \|\partial_z W_\varepsilon\|_\infty &\le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,\\ \|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty &\le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,\\ \|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty &\le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,\\ \|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty &\le O_0^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\| + k(\alpha)\|W_\varepsilon\|_{\mathcal{F}},\end{aligned}$$

where $O_0^*(1) = \max(O^*(1), O(1)\|W_\varepsilon(0,\cdot)\|_{\mathcal{F}})$ and $k(\alpha) < 1$ is a uniform constant.

The proof of this theorem is quite intricate and will be divided into several subsections. The plan is as follows:

- First, we focus on a fixed ball around $z_*(t)$. The first step is to get bounds only on a small time interval on this ball, and the second step is to propagate this bound uniformly in time, locally in space.

- Next, we propagate this bound on the whole space by successively dividing it into growing balls B_n and dyadic rings D_n centered around z_*; see the definitions in (7-2) and (7-3).

The main arguments are the maximum principle coupled with a suitable division of the space that accounts for the nonlocal nature of the infinitesimal operator. The purpose of this dyadic decomposition in rings is to obtain a decay of the norm with respect to the radius of the ring.

7A. ***Division of the space into a ball surrounded by growing balls and dyadic rings.*** Let us first consider a time T_*. Then for all times t and s such that $0 \le t, s \le T_*$, the inequality

$$|z_*(t) - z_*(s)| \le \sup_{s \ge 0} |m'(z_*(s))| T_* := r_*$$

holds true, and the supremum is finite because z_* lives in a bounded domain uniquely determined by m and $z_*(0)$; see (1-5).

We slightly expand this ball by a constant r_0 to be defined later, and define the ball

$$B_0 := \{z : |z - z_*(0)| \le r_0 + r_*\}.$$

Our intention behind this choice is that the ball B_0 satisfies the following property:

$$\forall t \le T_*, \quad \forall z \in B_0, \quad |z - \bar{z}(t)| = \tfrac{1}{2}|z - z_*(t)| = \tfrac{1}{2}|z - z_*(0) + z_*(0) - z_*(t)| \le \tfrac{1}{2} r_0 + r_*. \tag{7-1}$$

We recall that $\bar{z}(t) := \frac{1}{2}(z + z_*(t))$. We will split the rest of the space around B_0 into successive balls. The first ball is defined as $B_1 = \{z : |z - z_*(0)| \le 2r_0 + r_*\}$. It contains B_0, and more importantly, it satisfies for every $t \le T_*$ the following identity on the middle point:

$$|\bar{z}(t) - z_*(0)| = \left|\tfrac{1}{2}(z + z_*(t)) - z_*(0)\right| \le \left|\tfrac{1}{2}(z - z_*(0))\right| + \left|\tfrac{1}{2}(z_*(0) - z_*(t))\right| \le r_0 + r_*.$$

This shows that for any $z \in B_1$ and time $t \le T_*$, the corresponding middle point $\bar{z}(t)$ lies in B_0. More generally, the following lemma holds true if we define, for $n \ge 2$,

$$B_n := \{z : |z - z_*(0)| \le 2^n r_0 + r_*\}. \tag{7-2}$$

Lemma 7.2 (middle point property). *For every time* $0 \le t \le T_*$,

$$\forall n \ge 1, \quad \forall z \in B_n, \quad \bar{z}(t) \in B_{n-1}.$$

The proof will also feature prominently the dyadic rings D_n, defined as

$$D_n := \{2^{n-1} r_0 + r_* \le |z - z_*(0)| \le 2^n r_0 + r_*\}, \tag{7-3}$$

with the convention that $D_0 = B_0$. Note that D_n (a subset of B_n) is the set such that $B_{n-1} \cup D_n = B_n$; see Figure 2. On the rings, we will need the following notations:

$$a_n := \sup_{(t,z) \in \mathbb{R}_+ \times D_n} \left| \frac{M(t, \bar{z})}{M(t, z)} \right|, \quad b_n := \sup_{(t,z) \in \mathbb{R}_+ \times D_n} \left| \frac{\partial_z M(t, \bar{z})}{\partial_z M(t, z)} \right|. \tag{7-4}$$

From the asymptotic hypothesis made in (1-15) on the quotient of M, the sequence a_n is bounded and satisfies $a_n \to a < \frac{1}{2}$ as $n \to \infty$. The sequence b_n is uniformly bounded.

Notations for this section. We will denote by $\|\cdot\|_\infty^n$ the L^∞ norm on $\mathbb{R}_+ \times B_n$:

$$\text{for } n \ge 0, \quad \|\cdot\|_\infty^n := \sup_{(t,z) \in \mathbb{R}_+ \times B_n} |\cdot|. \tag{7-5}$$

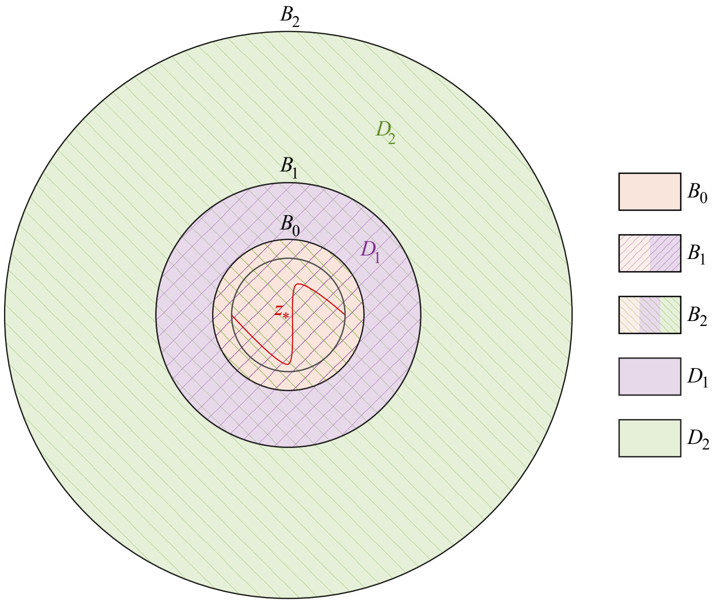

Figure 2. The division of the space into successive dyadic balls and rings.

7B. ***Local bounds on B_0.*** The first step of the proof of Theorem 7.1 consists in getting uniform bounds (in time) on the ball B_0. The estimates on the third derivative are dealt with slightly differently, and are thus delayed to Section 7F.

Proposition 7.3 (local bounds). *For a convenient choice of T^* and r_0 introduced above, and made explicit in (7-7), there exists a constant ε_B that depends only on B, such that with the conditions of Theorem 7.1, W_ε satisfies, for $\varepsilon \leq \varepsilon_B$*:

$$\begin{aligned}
\|\Xi_\varepsilon\|_\infty^0 &\leq O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \\
\|\partial_z W_\varepsilon\|_\infty^0 &\leq O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \\
\|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^0 &\leq O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \\
\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^0 &\leq O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,
\end{aligned}$$

where $O_0^*(1) = \max(O^*(1), O^*(1)\|W_\varepsilon(0,\cdot)\|_{\mathcal{F}})$.

To prove this *local* bound, i.e., in the ball B_0, one must start with the higher order derivative to build a contraction argument. Estimates of the lower order derivatives are then successively deduced by integration. Clearly, our argument for the third derivative is more technical because it involves a lot of terms through the linearized approximation made in Proposition 6.8. Therefore, for reasons of clarity, third derivatives are left out of Proposition 7.3, we will deal with them, locally and on the balls, in Proposition 7.7. We present here our argument on the simpler derivatives up to order two, and we refer to Section 7F for the generalization of the method to the third derivative. Interestingly, to prove the nonlocal estimates on the balls, we will proceed in the reverse way by first dealing with the lower order derivatives.

Proof of Proposition 7.3. By the derivation of the linearized equation in Proposition 6.7, W_ε satisfies the following, see (5-1):

$$\begin{aligned}\varepsilon^2\partial_t\partial_z^2 W_\varepsilon(t,z) = -\partial_z^2 M(t,z)(W_\varepsilon(t,z) - 2W_\varepsilon(t,\bar z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\ - 2\partial_z M(t,z)(\partial_z W_\varepsilon(t,z) - \partial_z W_\varepsilon(t,\bar z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\ + M(t,z)\big(\tfrac{1}{2}\partial_z^2 W_\varepsilon(t,\bar z) - \partial_z^2 W_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\big).\end{aligned}$$

We will use the maximum principle on the ball B_0. The key point is that on this ball, all other factors are controlled by $\|\partial_z^2 W_\varepsilon\|_\infty$. To compare all those terms with $\partial_z^2 W_\varepsilon$, we perform Taylor expansions with respect to the space variable. First, thanks to (7-1), for any $z \in B_0$ we write

$$\partial_z W_\varepsilon(t,\bar z) - \partial_z W_\varepsilon(t,z) \le \left(\frac{r_0}{2} + r_*\right)\|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)}.$$

Similarly, there exists $\xi \in (z, \bar z)$ and $\xi' \in (z_*, \bar z)$ such that

$$\begin{aligned}\Xi_\varepsilon(t,z) &= W_\varepsilon(t,z) - 2W_\varepsilon(t,\bar z) + W_\varepsilon(t,z_*)\\ &= \left(\frac{z-z_*}{2}\right)\partial_z W(t,\bar z) + \frac{1}{2}\frac{(z-z_*)^2}{4}\partial_z^2 W(t,\xi) - \left(\frac{z-z_*}{2}\right)\partial_z W(\bar z) + \frac{1}{2}\frac{(z-z_*)^2}{4}\partial_z^2 W(\xi')\\ &\le \frac{1}{4}\left(\frac{r_0}{2} + r_*\right)^2 \|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)}.\end{aligned} \tag{7-6}$$

Moreover, by the hypothesis made in (1-14) on M, for $j = 1, 2$,

$$\sup_{(t,z)\in\mathbb{R}_+\times B_0}\left|\frac{\partial_z^{(j)} M(t,z)}{M(t,z)}\right| \le O^*(1).$$

Thanks to those a priori bounds, when we evaluate (6-16) at the maximum point of $\partial_z^2 W_\varepsilon$ on B_0, we get

$$\begin{aligned}&\varepsilon^2\partial_t[\|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)}]\\ &\quad\le M(t,z)\Big(\tfrac{1}{2}\|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)} - \|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)}\\ &\qquad + O^*(1)\big(\tfrac{1}{4}\big(\tfrac{1}{2}r_0 + r_*\big)^2 + \tfrac{1}{2}r_0 + r_*\big)\|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)} + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\Big).\end{aligned}$$

The *crucial* step is that we choose T^* and r^* so small, so that

$$O^*(1)\big(\tfrac{1}{4}\big(\tfrac{1}{2}r_0 + r_*\big)^2 + \tfrac{1}{2}r_0 + r_*\big) \le \tfrac{1}{4}. \tag{7-7}$$

The consequence is that

$$\varepsilon^2\partial_t[\|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)}] \le M(t,z)\big(-\tfrac{1}{4}\|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)} + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\big).$$

The function $M(t,z)$ admits a lower bound. Therefore, we can apply the maximum principle, on the ball B_0, and get

$$\|\partial_z^2 W_\varepsilon\|_{L^\infty([0,T^*]\times B_0)} \le \max(O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \|\partial_z^2 W_\varepsilon(0,\cdot)\|_{L^\infty(B_0)}).$$

We now detail how to propagate this bound uniformly in time. One can renew every previous estimate on each interval $I_k := [kT_*, (k+1)T_*]$. By going over the same steps, we notice that the only argument that changes for different k is the center of the ball B_0 around z_*, but interestingly not its radius; see (7-7). Every other estimate is the same and is independent of k. Therefore, since the condition (7-7) is uniform in time ($O^*(1)$ does not depend on time), once the radius is chosen small enough depending only on K^*, see (7-7), we can repeat recursively the estimates on each interval I_k. Considering all $k \in \mathbb{N}$, we have therefore proven that

$$\|\partial_z^2 W_\varepsilon\|_\infty^0 \le \max(O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \|\partial_z^2 W_\varepsilon(0,\cdot)\|_{L^\infty(B_0)}) \le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|. \quad (7\text{-}8)$$

We will use this estimate as the starting point in order to prove the rest of Proposition 7.3. First, notice that adding the weight function φ_α is straightforward, since it is uniformly bounded on B_0:

$$\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^0 \le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|.$$

Next, taking advantage of the fact that both W_ε and $\partial_z W_\varepsilon$ vanish at z^*, we write

$$|\partial_z W_\varepsilon(t,z)| = \left| \int_{z_*(t)}^{z} \partial_z^2 W_\varepsilon(t,z')\,dz' \right| \le (r_0 + 2r_*)\|\partial_z^2 W_\varepsilon\|_\infty^0.$$

As a consequence, using again the expansion of (7-6),

$$|\Xi_\varepsilon(t,z)| = |2W_\varepsilon(t,\bar z(t)) - W_\varepsilon(t,z)| \le \tfrac{1}{4}\left(\tfrac{1}{2}r_0 + r_*\right)^2 \|\partial_z^2 W_\varepsilon\|_\infty^0.$$

Similarly, we get a uniform bound on $\partial_z \Xi_\varepsilon$. Combining those estimates with the first estimate in (7-8), which comes from the maximum principle, the proof of Proposition 7.3 is concluded. □

7C. *Bound on the balls*: Ξ_ε. We will now propagate those bounds beyond the small ball. It is very important to keep the level of precision of $O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|$, to which we will add some decay property due to the specific shape of the rings D_n.

Proposition 7.4 (in the balls, Ξ_ε). *There exists a constant ε_B that depends only on B such that with the conditions of Theorem 7.1, W_ε satisfies, for $\varepsilon \le \varepsilon_B$,*

$$\|\Xi_\varepsilon\|_\infty^n \le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \quad (7\text{-}9)$$

for all $n \ge 1$.

Proof. The starting point of the analysis is the linearized equation given by Corollary 6.3. For $t \in \mathbb{R}_+$, $n \ge 1$, take z in the ball B_n defined previously. We know that

$$\varepsilon^2 \partial_t \Xi_\varepsilon(t,z) = M(t,z)\left(2\frac{M(t,\bar z)}{M(t,z)}\Xi_\varepsilon(t,\bar z) - \Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\right).$$

One can multiply by $\operatorname{sign}(\Xi_\varepsilon)$ this equality to bound the absolute value; it is important to keep the minus sign on the right-hand side. We get

$$\varepsilon^2 \partial_t |\Xi_\varepsilon(t,z)| \le M(t,z)\left(2\frac{M(t,\bar z)}{M(t,z)}|\Xi_\varepsilon(t,\bar z)| - |\Xi_\varepsilon(t,z)| + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\right).$$

Then, from Lemma 7.2, we know that the middle point $\bar{z}$ is in the smaller ball B_{n-1}, and so we have the following estimate:

$$\varepsilon^2 \partial_t |\Xi_\varepsilon(t,z)| \le M(t,z)\left(2\frac{M(t,\bar{z})}{M(t,z)}\|\Xi_\varepsilon\|_\infty^{n-1} - \Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\right).$$

To obtain precise bounds from this inequality, we shall discuss whether the maximum point of Ξ_ε on the ball D_n is reached inside the ring D_{n-1}, defined in (7-3), or not. If it is the case, we obtain a sharper estimate than if it is not the case.

- Suppose that the maximum point that reaches $\|\Xi_\varepsilon\|_\infty^n$ belongs to the ring D_n. We can then control the quotient of M by the sequence a_n defined in (7-4). Moreover, M admits a uniform lower bound by (1-13), thus, we can apply the maximum principle to get

$$\|\Xi_\varepsilon\|_\infty^n \le \max(2a_n\|\Xi_\varepsilon\|_\infty^{n-1} + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \|\Xi_\varepsilon(0,\cdot)\|_{L^\infty(B_n)}). \tag{7-10}$$

We first notice that for all $n \in \mathbb{N}$,

$$\|\Xi_\varepsilon(0,\cdot)\|_{L^\infty(B_n)} \le O_0^*(1).$$

Therefore, from (7-10),

$$\|\Xi_\varepsilon\|_\infty^n \le 2a_n\|\Xi_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|. \tag{7-11}$$

Here lies the motivation behind the introduction of the notation $O_0^*(1)$. It allows us to take into account the initial data and to make recursive estimates that were a priori not possible with (7-10).

- Before going further, we now assume that the maximum point that reaches $\|\Xi_\varepsilon\|_\infty^n$ is outside the ring D_n, in $B_n \setminus D_n = B_{n-1}$. In that case, the estimate of (7-4) is not helpful, as we would need to define $\tilde{a}_n$ to be the supremum over B_n, but then this sequence would not give a contraction factor as in (7-10). Therefore, we simply write for this case

$$\|\Xi_\varepsilon\|_\infty^n \le \|\Xi_\varepsilon\|_\infty^{n-1}. \tag{7-12}$$

- The combination of (7-11) and (7-12) yields

$$\|\Xi_\varepsilon\|_\infty^n \le \max(2a_n\|\Xi_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \ \|\Xi_\varepsilon\|_\infty^{n-1}). \tag{7-13}$$

This inequality guarantees that the sequence $(\|\Xi_\varepsilon\|_\infty^n)_n$ is uniformly bounded. Heuristically, on the right-hand side of (7-13), the geometric part of the maximum satisfies $2a_n \to 2a < 1$ when $n \to \infty$, thanks to (1-15), therefore it ensures a contraction, while the other part of the maximum yields at worst a bound by the term $n = 0$.

We detail more rigorously the steps as it will serve as a model for future proofs. We assume without loss of generality that $2a_n < \theta$ for all $n \in \mathbb{N}$, with, for instance the factor $\theta := a + \frac{1}{2}$, such that $2a < \theta < 1$. We know this is true, but for a finite number of terms, by (7-4). For this handful of terms, we do not need a contraction argument, since the bound (7-9) follows from a finite number of iterations of (7-13). Let f_n be the function $f_n(\xi) = \max(\xi, 2a_n\xi + C)$, and a sequence ξ_n be such that $\xi_{n+1} \le f_n(\xi_n)$. We will then show that for all $n \in \mathbb{N}$,

$$\xi_n \le \max\left(\xi_0, \frac{C}{1-\theta}\right). \tag{7-14}$$

The proof is done by induction, the initial step is obvious. If we now assume that the inequality holds true for a certain $n \in \mathbb{N}$, we get

$$\xi_{n+1} \le f_n(\xi_n) \le \max(\xi_n, 2a_n\xi_n + C).$$

If the previous max is ξ_n, then we immediately deduce by the induction hypothesis the following:

$$\xi_{n+1} \le \xi_n \le \max\left(\xi_0, \frac{C}{1-\theta}\right).$$

Otherwise,

$$\xi_{n+1} \le 2a_n \max\left(\xi_0, \frac{C}{1-\theta}\right) + C.$$

We once again discuss where the maximum point is reached. If it is ξ_0, then we end up with

$$\xi_{n+1} \le 2a_n\xi_0 + C \le (2a_n - \theta)\xi_0 + \xi_0 \le \xi_0.$$

Similarly, if it is not ξ_0,

$$\xi_{n+1} \le \frac{2a_n C}{1-\theta} + C = \frac{(2a_n - \theta)C + C}{1-\theta} \le \frac{C}{1-\theta}.$$

Therefore, we have shown that in all cases,

$$\xi_{n+1} \le \max\left(\xi_0, \frac{C}{1-\theta}\right),$$

which proves (7-14). We conclude, given the bound (7-13), that

$$\|\Xi_\varepsilon\|_\infty^n \le \max\left(\frac{O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{1-\theta}, \|\Xi_\varepsilon\|_\infty^0\right) \le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|. \qquad \square$$

All the remaining proofs of this section will follow this blueprint.

7D. *Bound on the balls*: $\partial_z \Xi_\varepsilon$. We now state a similar result for $\partial_z \Xi_\varepsilon$. We will see the appearance of the weight function φ_α in the estimates. It slightly worsens the expressions but the strategy deployed to prove Proposition 7.4 will still works.

Proposition 7.5 (in the balls, $\partial_z \Xi_\varepsilon$). *There exists a constant ε_B that depends only on B such that upon the condition of Theorem 7.1, W_ε satisfies for $\varepsilon \le \varepsilon_B$*

$$\|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^n \le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,$$

for $n \ge 1$.

Proof. The proof is similar to the bound on Ξ_ε, but we have to take the weight function into account. We start with the linear equation satisfied by $\partial_t\partial_z\Xi_\varepsilon$ in Corollary 6.6. It yields, for $z \in B_n$ and $t \in \mathbb{R}_+$,

$$\begin{aligned}\varepsilon^2\partial_t|\partial_z\Xi_\varepsilon(t,z)| \\ \le M(t,z)\biggl(&\frac{M(t,\bar z)}{M(t,z)}|\partial_z\Xi_\varepsilon(t,\bar z)| - |\partial_z\Xi_\varepsilon(t,z)| + \frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)} \\ &+ \frac{O^*(1)}{\varphi_\alpha(t,z)}\biggl(\frac{\partial_z M(t,\bar z)}{\partial_z M(t,z)}\|\Xi_\varepsilon\|_\infty^{n-1} + \|\Xi_\varepsilon\|_\infty^n + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|\biggr)\biggr). \qquad (7\text{-}15)\end{aligned}$$

In the second factor, thanks to (1-14), we used that

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}}\left(\varphi_\alpha(t,z)\left|\frac{\partial_z M(t,z)}{M(t,z)}\right|\right) \le O^*(1).$$

To take into account the weight function, we make the following computation:

$$\partial_t[\varphi_\alpha\partial_z\Xi_\varepsilon](t,z) = \varphi_\alpha(t,z)\partial_t\partial_z\Xi_\varepsilon(t,z) + \partial_z\Xi_\varepsilon(t,z)\partial_t\varphi_\alpha(t,z).$$

First,

$$\partial_z\Xi_\varepsilon(t,z)\partial_t\varphi_\alpha(t,z) = \alpha\partial_z\Xi_\varepsilon(t,z)\frac{m'(z_*)\,\mathrm{sign}(z-z_*)}{(1+|z-z_*|)^{1-\alpha}} = O^*(1)\partial_z\Xi_\varepsilon(t,z), \qquad (7\text{-}16)$$

and therefore,

$$\varepsilon^2\partial_z\Xi_\varepsilon(t,z)\partial_t\varphi_\alpha(t,z) = O^*(\varepsilon^2)\|(\kappa_\varepsilon,W_\varepsilon)\|. \qquad (7\text{-}17)$$

By multiplying (7-15) by φ_α and taking into account (7-17), we deduce that

$$\begin{aligned}\varepsilon^2\partial_t[\varphi_\alpha|\partial_z\Xi_\varepsilon|](t,z) \\ \le M(t,z)\biggl(&-\varphi_\alpha(t,z)|\partial_z\Xi_\varepsilon(t,z)| + \frac{M(t,\bar z)}{M(t,z)}\varphi_\alpha(t,z)|\partial_z\Xi_\varepsilon(t,\bar z)| + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\| \\ &+ O^*(1)\biggl(\frac{\partial_z M(t,\bar z)}{\partial_z M(t,z)}\|\Xi_\varepsilon\|_\infty^{n-1} + \|\Xi_\varepsilon\|_\infty^n + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|\biggr) + \frac{O^*(\varepsilon^2)\|(\kappa_\varepsilon,W_\varepsilon)\|}{M(t,z)}\biggr),\end{aligned}$$

As in the previous proof, to obtain sharp bounds from the maximum principle, we discuss whether the maximum point of $\varphi_\alpha\partial_z\Xi_\varepsilon$ on B_n is reached on the subset D_n or not. *We now assume that it is the case.*

We can then use the sequences a_n and b_n defined in (7-4) to control the right-hand side of (7-19). Moreover, with Proposition 7.4 we can estimate the terms involving Ξ_ε on the balls. We then find that

$$\begin{aligned}\varepsilon^2\partial_t[\varphi_\alpha|\partial_z\Xi_\varepsilon|](t,z) \\ \le M(t,z)\biggl(&-\varphi_\alpha(t,z)|\partial_z\Xi_\varepsilon(t,z)| + a_n\left|\frac{\varphi_\alpha(t,z)}{\varphi_\alpha(t,\bar z)}\right|\|\varphi_\alpha\partial_z\Xi_\varepsilon\|_\infty^{n-1} + b_n(O_0^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|) \\ &+ O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\| + \frac{O^*(\varepsilon^2)\|(\kappa_\varepsilon,W_\varepsilon)\|}{M(t,z)}\biggr).\end{aligned}$$

The weight function was chosen precisely to satisfy the scaling estimate

$$\sup_{\mathbb{R}_+\times\mathbb{R}} \left| \frac{\varphi_\alpha(t,z)}{\varphi_\alpha(t,\bar{z})} \right| \leq 2^\alpha. \tag{7-18}$$

Since the function $1/M$ has a uniform upper bound, and the sequence b_n is uniformly bounded, we finally conclude that

$$\begin{aligned} &\varepsilon^2 \partial_t[\varphi_\alpha|\partial_z \Xi_\varepsilon|](t,z) \\ &\qquad \leq M(t,z)(-\varphi_\alpha(t,z)|\partial_z \Xi_\varepsilon(t,z)| + 2^\alpha a_n \|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|). \end{aligned} \tag{7-19}$$

The maximum principle applied to (7-19) gives

$$\|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^n \leq \max(2^\alpha a_n \|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \|\varphi_\alpha \partial_z \Xi_\varepsilon(0,\cdot)\|_\infty^n).$$

Notice that for all $n \in \mathbb{N}$,

$$\|\varphi_\alpha \partial_z \Xi_\varepsilon(0,\cdot)\|_{L^\infty(B_n)} \leq \|W_\varepsilon(0,\cdot)\|_{\mathcal{F}} \leq O_0^*(1).$$

Therefore, we obtain finally, in the case where the maximum point of $\varphi_\alpha \partial_z \Xi_\varepsilon$ on B_n is reached on the subset D_n,

$$\|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^n \leq 2^\alpha a_n \|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|. \tag{7-20}$$

When this is not the case, we will only state that

$$\|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^n \leq \|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^{n-1}. \tag{7-21}$$

Combining (7-20) and (7-21), we eventually conclude that

$$\|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^n \leq \max(2^\alpha a_n \|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^{n-1}). \tag{7-22}$$

This implies that the sequence $(\|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^n)_n$ is a contraction, using the same recursive arguments as in the previous proof. Indeed, by hypothesis, $2^\alpha a_n \leq 2^\alpha a < 1$, but for a finite number of terms, which gives for instance a contraction factor $\theta := a + \frac{1}{2}$, such that $2a < \theta < 1$. The second part of the maximum in (7-22) does not perturb the contraction part, and we deduce that

$$\|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^n \leq \max\left(\frac{O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{1-\theta}, \|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^0 \right) \leq O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|.$$

The second inequality uses the local bounds of the ball B_0 made in Proposition 7.3. □

7E. *Bound on the balls*: $\partial_z^2 W_\varepsilon$. We now make a similar statement about the second derivative.

Proposition 7.6 (in the balls, $\partial_z^2 W_\varepsilon$). *There exists a constant ε_B that depends only on B such that with the condition of Theorem 7.1, W_ε satisfies, for $\varepsilon \leq \varepsilon_B$,*

$$\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^n \leq O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,$$

for $n \geq 1$.

Proof. We proceed as in the proof of Proposition 7.5. We already know a linearized approximation for $\partial_z^2 W_\varepsilon$, thanks to (6-16) in Proposition 6.7. Taking the weight into account, one finds that $\varphi_\alpha \partial_z^2 W_\varepsilon$ satisfies the following:

$$\begin{aligned}
&\varepsilon^2 \partial_t[\varphi_\alpha(t,z)\partial_z^2 W_\varepsilon](t,z)\\
&= -\partial_z^2 M(t,z)\varphi_\alpha(t,z)(\Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\
&\qquad - 2\partial_z M(t,z)(\varphi_\alpha(t,z)\partial_z \Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\
&\qquad\quad + M(t,z)\big(\tfrac{1}{2}\varphi_\alpha(t,z)\partial_z^2 W_\varepsilon(t,\bar{z}) - \varphi_\alpha(t,z)\partial_z^2 W_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\big)\\
&\qquad\qquad + O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|. \qquad (7\text{-}23)
\end{aligned}$$

The last term comes from the same computation of $\partial_t \varphi_\alpha$ as that made in (7-17). To continue, we first use the following uniform controls of (1-14):

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \left(\varphi_\alpha(t,z)\left|\frac{\partial_z^2 M(t,z)}{M(t,z)}\right|\right) \le O^*(1) \quad \text{and} \quad \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \left|\frac{\partial_z M(t,z)}{M(t,z)}\right| \le O^*(1).$$

We also need the scaling estimate of the weight function, stated in (7-18). Then, we can bound the right-hand side of (7-23) after factorizing by M, for $t \in \mathbb{R}_+$ and $z \in B_n$:

$$\begin{aligned}
&\varepsilon^2 \partial_t[\varphi_\alpha(t,z)|\partial_z^2 W_\varepsilon|](t,z)\\
&\le M(t,z)\bigg(-\varphi_\alpha(t,z)|\partial_z^2 W_\varepsilon(t,z)| + \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^{n-1} + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\\
&\qquad + O^*(1)(\|\Xi_\varepsilon\|_\infty^n + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\
&\qquad\qquad + O^*(1)(\|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^n + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\bigg).
\end{aligned}$$

Plugging in the controls on the balls of Ξ_ε and $\partial_z \Xi_\varepsilon$ from Propositions 7.4 and 7.5, we have

$$\begin{aligned}
&\varepsilon^2 \partial_t[\varphi_\alpha(t,z)|\partial_z^2 W_\varepsilon|](t,z)\\
&\qquad \le M(t,z)\Big(-\varphi_\alpha(t,z)|\partial_z^2 W_\varepsilon(t,z)| + \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\Big).
\end{aligned}$$

The function $M(t,z)$ admits a positive lower bound by (1-13). We apply the maximum principle to get

$$\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^n \le \max\Big(\frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,\ \|\varphi_\alpha \partial_z^2 W_\varepsilon(0,\cdot)\|_\infty^n\Big).$$

As earlier, we can get rid of the initial data in the maximum by stating that for all $n \in \mathbb{N}$,

$$\|\varphi_\alpha \partial_z^2 W_\varepsilon(0,\cdot)\|_\infty^n \le \|W_\varepsilon(0,\cdot)\|_{\mathcal{F}} \le O_0^*(1). \qquad (7\text{-}24)$$

Then, the recursive arguments are somehow a little easier in that case than in the proofs of Propositions 7.4 and 7.5 since the geometric term, $2^{\alpha-1}$, does not depend on n. Hence we have

$$\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^n \le \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|.$$

Because $2^{\alpha-1} < 1$, we immediately get that

$$\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^n \le \max\left(\frac{O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{1-2^{\alpha-1}}, \|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^0\right) \le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|. \qquad \square$$

7F. ***Local and on-the-balls bound for*** $\boldsymbol{\partial_z^3 W_\varepsilon}$**.** We dedicate this section to the study of $\partial_z^3 W_\varepsilon$ since it does not exactly fit the mold of the previous estimates due to the additional factor $\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty / 2^{1-\alpha}$ in the linearized equation in Proposition 6.8.

- We highlight the difference by first proving the initial bound on the local ball B_0. We write the linear equation solved by $\varphi_\alpha \partial_z^3 W_\varepsilon$:

$$\begin{aligned}
-\varepsilon^2 \partial_t [\varphi_\alpha \partial_z^3 W_\varepsilon](t,z) \\
= \varphi_\alpha(t,z) \partial_z^3 M(t,z)(\Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\
&+ 3\partial_z^2 M(t,z)(\varphi_\alpha(t,z)\partial_z \Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\
&+ 3\partial_z M(t,z)(\varphi_\alpha(t,z)\partial_z^2 \Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\
&+ M(t,z)\left(\varphi_\alpha(t,z)\partial_z^3 \Xi_\varepsilon(t,z) + \frac{\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}} + O^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|\right) \\
&- \varepsilon^2 \partial_z^3 W_\varepsilon(t,z)\partial_t \varphi_\alpha(t,z).
\end{aligned} \tag{7-25}$$

First, one finds that

$$\varepsilon^2 \partial_z^3 \Xi_\varepsilon(t,z)\partial_t \varphi_\alpha(t,z) = O^*(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|.$$

We recall that Ξ_ε, $\partial_z \Xi_\varepsilon$ and $\partial_z^2 \Xi_\varepsilon$ were all uniformly bounded on B_0, with the weight, in Proposition 7.3. Moreover, from (1-13), for $j = 1, 2$,

$$\begin{aligned}
\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \left|\frac{\partial_z^{(j)} M(t,z)}{M(t,z)}\right| \le O^*(1), \\
\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \left(\varphi_\alpha(t,z)\left|\frac{\partial_z^3 M(t,z)}{M(t,z)}\right|\right) \le O^*(1).
\end{aligned} \tag{7-26}$$

Finally,

$$\varphi_\alpha(t,z)|\partial_z^3 W_\varepsilon(t,\bar z)| \le \frac{2^\alpha}{4}|\varphi_\alpha(t,\bar z)\partial_z^3 W_\varepsilon(t,\bar z)|.$$

When plugging all of this into (7-25), we obtain, by evaluating at the point of maximum on B_0,

$$\begin{aligned}
&\varepsilon^2 \partial_t \|\varphi_\alpha(t,\cdot)\partial_z^3 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)} \\
&\le M(t,z)\bigg(-\|\varphi_\alpha(t,\cdot)\partial_z^3 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)} + \frac{1}{2^{2-\alpha}}\|\varphi_\alpha(t,\cdot)\partial_z^3 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)} + \frac{\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}} \\
&\qquad + O_0^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|\bigg).
\end{aligned}$$

Since there is a positive lower bound of M, we recognize a contraction argument on the ball B_0, and for bounded times $0 < t \le T^*$,

$$\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_{L^\infty([0,T^*]\times B_0)} \le \max\Big(\Big(\frac{1}{1-2^{\alpha-2}}\Big)\Big(O_0^*(1)+O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\| + \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty\Big), \|\varphi_\alpha(0,\cdot)\partial_z^3 W_\varepsilon(0,\cdot)\|_{L^\infty(B_0)}\Big).$$

Therefore, since the initial data is controlled by $O_0^*(1)$, we may write

$$\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_{L^\infty([0,T^*]\times B_0)} \le O_0^*(1)+O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\| + \frac{2^{\alpha-1}}{1-2^{\alpha-2}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty.$$

As explained in Section 7A, we can now repeat the procedure on each interval of time

$$I_k := [kT_*, (k+1)T_*],$$

and end up with a bound uniform in time on the ball B_0:

$$\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^0 \le O_0^*(1)+O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\| + \frac{2^{\alpha-1}}{1-2^{\alpha-2}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty. \tag{7-27}$$

• We now proceed to propagate this bound on the balls, starting again from (7-25) and using the maximum principle. For any $t \in \mathbb{R}_+$ and $z \in B_n$, we have

$$\begin{aligned}
&\varepsilon^2 \partial_t[\varphi_\alpha|\partial_z^3 W_\varepsilon|](t,z) \\
&\le M(t,z)\Bigg(-\varphi_\alpha(t,z)|\partial_z^3 W_\varepsilon(t,z)| + \frac{1}{2^{2-\alpha}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^{n-1} + \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty \\
&\qquad + O_0^*(1)+O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\| + \left|\varphi_\alpha(t,z)\frac{\partial_z^3 M(t,z)}{M(t,z)}\right|(\|\Xi_\varepsilon\|_\infty^n + O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\
&\qquad + 3\left|\frac{\partial_z^2 M(t,z)}{M(t,z)}\right|(\|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^n + O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\
&\qquad + 3\left|\frac{\partial_z M(t,z)}{M(t,z)}\right|(\|\varphi_\alpha \partial_z^2 \Xi_\varepsilon\|_\infty^n + O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\Bigg).
\end{aligned}$$

We will use once more our hypothesis (1-14), in the form stated in (7-26). We also need all our previous estimates on the balls: Propositions 7.4, 7.5 and 7.6. We then obtain

$$\begin{aligned}
&\varepsilon^2 \partial_t[\varphi_\alpha|\partial_z^3 W_\varepsilon|](t,z) \\
&\le M(t,z)\Big(-\varphi_\alpha(t,z)|\partial_z^3 W_\varepsilon(t,z)| + \frac{1}{2^{2-\alpha}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^{n-1} + \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty + O_0^*(1)+O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|\Big).
\end{aligned}$$

We recall that the term $\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty$ is a global control on the whole space $\mathbb{R}$ and not localized on the balls. By applying the maximum principle, one gets

$$\begin{aligned}
&\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^n \\
&\le \max\Big(\frac{1}{2^{2-\alpha}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^{n-1} + \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty + O_0^*(1)+O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|,\ \|\varphi_\alpha(0,\cdot)\partial_z^3 W_\varepsilon(0,\cdot)\|_\infty^n\Big).
\end{aligned}$$

We can absorb the initial data in the term $O_0^*(1)$ to deduce that

$$\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^n \le \frac{1}{2^{2-\alpha}} \|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^{n-1} + \frac{1}{2^{1-\alpha}} \|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty + O_0^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|.$$

This sequence is bounded, because its ratio satisfies $2^{\alpha-2} < 1$. Hence,

$$\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^n \le \max\left(\frac{O_0^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|}{1-2^{\alpha-2}} + \frac{2^{\alpha-1}}{1-2^{\alpha-2}} \|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty, \|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^0\right). \tag{7-28}$$

We define $k(\alpha)$ as follows:

$$k(\alpha) := \frac{2^{\alpha-1}}{1-2^{\alpha-2}},$$

and from (7-28) we finally conclude, taking the local bound (7-27) into account, that

$$\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^n \le O_0^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\| + k(\alpha)\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty.$$

We have therefore proven the following proposition:

Proposition 7.7 (in the rings, $\partial_z^3 W_\varepsilon$). *There exists a constant ε_B that depends only on B such that with the condition of Theorem 7.1, W_ε satisfies, for $\varepsilon \le \varepsilon_B$,*

$$\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^n \le O_0^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\| + k(\alpha)\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty,$$

for $n \ge 1$, with

$$0 < k(\alpha) := \frac{2^{\alpha-1}}{1-2^{\alpha-2}} < 1. \tag{7-29}$$

The scalar $k(\alpha)$ is a contraction factor, only upon the condition that

$$\alpha < 2 - \frac{\ln 3}{\ln 2} \approx 0.415. \tag{7-30}$$

We made that assumption prospectively when we introduced $\mathcal{E}$ in Definition 1.2. Beside (7-30), another reason for which α cannot be taken too large is that it worsens the contraction estimate $\varphi_\alpha(t,z) \le 2^\alpha \varphi_\alpha(t,\bar z)$; see (7-18).

7G. *Conclusion*: *proof of Theorem 7.1*. All our previous estimates from Propositions 7.4, 7.5, 7.6 and 7.7 are uniform in n, and therefore apply to the whole space. Therefore, so far, every bound of Theorem 7.1 has been proved except for the one upon $\partial_z W_\varepsilon$. A proof by recursion on the balls could be adapted from that of Proposition 7.5, starting from the linearized equation of Proposition 6.5. We propose here a more concise argument on the whole space, that uses instead the result of Proposition 7.5.

For all times $t > 0$ and $z \in \mathbb{R}$,

$$\begin{aligned} |\partial_z \Xi_\varepsilon(t,z)| &= |\partial_z W_\varepsilon(t,z) - \partial_z W_\varepsilon(t,\bar z)| \\ &\le \frac{O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}. \end{aligned} \tag{7-31}$$

This estimate is helpful for large z, given the weight function. To control uniformly on the whole space, let $h \in \mathbb{R}$ and define N_h^0 as the lowest index such that $z_*(t) + 2^{-k}|h| \in B_0$ for all $k > N_h^0$. From Section 7A we know that $N_h^0 \geq \lceil \ln(|h|/r_0)/\ln(2) \rceil$. Then, by iterating (7-31), we get

$$|\partial_z W_\varepsilon(t, z_* + h)| \leq |\partial_z W_\varepsilon(t, z_* + 2^{-(N_0^h+1)} h)| + (O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \sum_{k=0}^{N_h^0} \frac{1}{\varphi_\alpha(t, z_* + 2^{-k} h)}.$$

Given the control of $\partial_z W_\varepsilon$ on B_0 from Proposition 7.3 and the explicit form of the weight φ_α, we deduce that

$$\partial_z W_\varepsilon(t, z_* + h) \leq O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\| + (O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \sum_{k=0}^{N_h^0} \frac{2^{\alpha k}}{|h|^\alpha}. \tag{7-32}$$

This series has a finite number of terms (roughly $\log(|h|)$), by definition of N_0^h, and otherwise it would not be converging. Indeed, since $2^{N_0^h}/|h| \leq 1/r_0$, this sum is uniformly bounded:

$$\sum_{k=0}^{N_h^0} \frac{2^{\alpha k}}{|h|^\alpha} \leq O^*(1) \frac{2^{\alpha N_h^0}}{|h|^\alpha} \leq O^*(1).$$

Plugging this into (7-32), we have shown, as stated in Theorem 7.1, that

$$\|\partial_z W_\varepsilon\|_\infty \leq O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|. \qquad \square$$

One should note that the weight function φ_α plays a crucial role here, by "compensating" for the diverging terms. Namely, if $\alpha = 0$, the previous argument crumbles starting as early as estimate (7-31). This shows that the weight φ_α is necessary to ensure uniform Lipschitz bounds of W_ε.

8. Proof of Theorem 1.3

We now prove the main result of this paper, that is the boundedness of $(\kappa_\varepsilon, W_\varepsilon)$ in $\mathbb{R} \times \mathcal{F}$. We first suppose that there exists K_0 such that

$$|\kappa_\varepsilon(0)| \leq K_0 \quad \text{and} \quad \|W_\varepsilon(0, \cdot)\|_{\mathcal{F}} \leq K_0, \tag{8-1}$$

and we look to prove that

$$|\kappa_\varepsilon| \leq K_0' \quad \text{and} \quad \|W_\varepsilon\|_{\mathcal{F}} \leq K_0',$$

with K_0' to be determined by the analysis.

Theorem 7.1 grants precise bounds of W_ε, as long as there exists K such that $\|(\kappa_\varepsilon, W_\varepsilon)\| \leq K$. More precisely, there exists a constant C_0^* that depends only on K_0 and K^* and a constant C_K' that depends only K such that

$$\|W_\varepsilon\|_{\mathcal{F}} \leq C_0^* + C_K' \varepsilon^\alpha K + k(\alpha)\|W_\varepsilon\|_{\mathcal{F}}.$$

Therefore, up to renaming the constants,

$$\|W_\varepsilon\|_{\mathcal{F}} \leq \frac{C_0^* + C_K' \varepsilon^\alpha K}{1 - k(\alpha)} \leq C_0^* + C_K' \varepsilon^\alpha K. \tag{8-2}$$

Clearly, a crucial part of this contraction was to ensure that $k(\alpha) < 1$. We can now work on κ_ε. We go back to Proposition 5.1, from which we learned that κ_ε solves

$$-\dot{\kappa}_\varepsilon(t) = R_\varepsilon^*(t)\kappa_\varepsilon + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\| + O^*(1)\|W_\varepsilon\|_{\mathcal{F}}. \tag{8-3}$$

Thanks to our previous contraction argument, we have an estimate of $\|W_\varepsilon\|_{\mathcal{F}}$, and we can get rid of this last term of order 0. We can plug the estimate (8-2) into (8-3) to finally conclude the argument on κ_ε.

Since R_ε^* is a positive function that admits, for $t \geq t_0$, a uniform lower bound R_0, per Proposition 4.7, it is straightforward from (8-2) and (8-3), and our subsequent bounds, that there exists C_0^* and C_K' such that, for all time t,

$$|\kappa_\varepsilon(t)| \leq C_0^* + C_K' \varepsilon^\alpha K. \tag{8-4}$$

Coupled with (8-2), those are the stability results we needed on the pair $(\kappa_\varepsilon, W_\varepsilon)$. We now set a scalar K_0' such that

$$K_0' \geq 2C_0^*. \tag{8-5}$$

Then, choose ε_0 in the following way:

$$\varepsilon_0 := \left(\frac{1}{2C_0'}\right)^{1/\alpha},$$

where C_0' is the constant in (8-2) and (8-4) corresponding to the choice made in (8-5) of the size of the ball K_0'. Then for $\varepsilon \leq \varepsilon_0$, starting from an initial data that satisfies (8-1), the bound is propagated in time, and furthermore

$$\|W_\varepsilon\|_{\mathcal{F}} \leq K_0', \quad |\kappa_\varepsilon| \leq K_0'.$$

Since $V_\varepsilon = V^* + \varepsilon^2 W_\varepsilon$ and $q_\varepsilon = q^* + \varepsilon^2 \kappa_\varepsilon$, Theorem 1.3 is proven. □

9. Numerical simulations and discussion

In this section we display some numerical simulations showing the behavior of the solution of the Cauchy problem for positive ε, and we will provide insight into the structural assumption we made in (1-13).

***Influence of the condition* (1-13).** A first example for our study is to consider a quadratic selection function, as depicted in Figure 3. In that case, according to Theorem 1.3, starting from any initial data $z_*(0)$, the solution f_ε stays close to a Gaussian density with variance ε^2. In addition, its mean z_* converges to the unique minimum of m when the time is large.

Our framework encompasses more general selection functions with multiple local minima, such as the one depicted in Figure 4. The condition in (1-13) restricts somehow the position of those minima. If one assumes that z_* starts from a local minimum, that is $m'(z_*(0)) = 0$, then this condition implies that the selective difference between minima must be inferior to 1: $m(z_*(0)) - m(z_{\text{opt}}) < 1$. We recover the structural condition under which the analysis for the stationary case was performed; see [Calvez et al. 2019].

The selection function depicted in Figure 4, coupled with $z_*(0)$, satisfies the condition (1-13). Then as stated by Theorem 1.3 the population density f_ε concentrates around the local minimum, according to the gradient flow dynamics of Assumption 1.1.

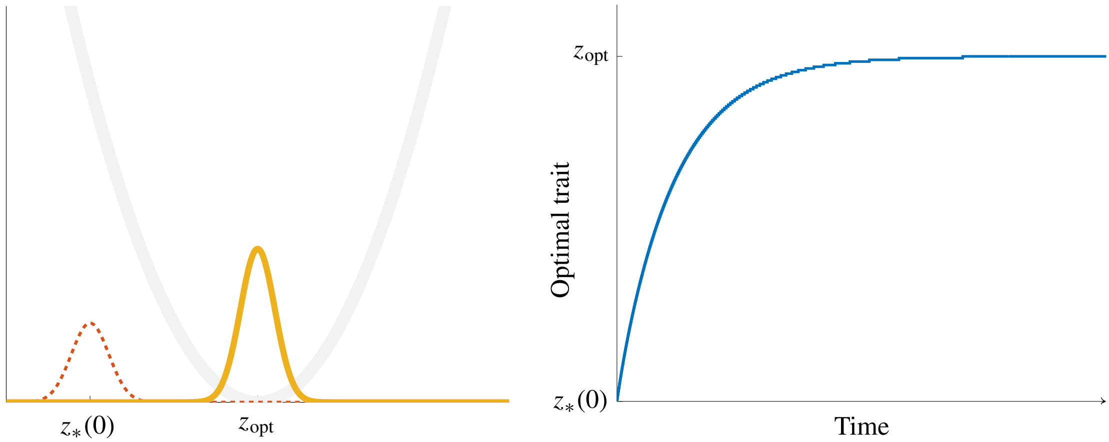

Figure 3. On the left, in dotted red, the initial data $f_\varepsilon(0,\cdot)$, and in orange, the distribution f_ε after a long time. In the background the selection function m with a global optimum z_{opt}. On the right, the trajectory of the dominant trait z_*.

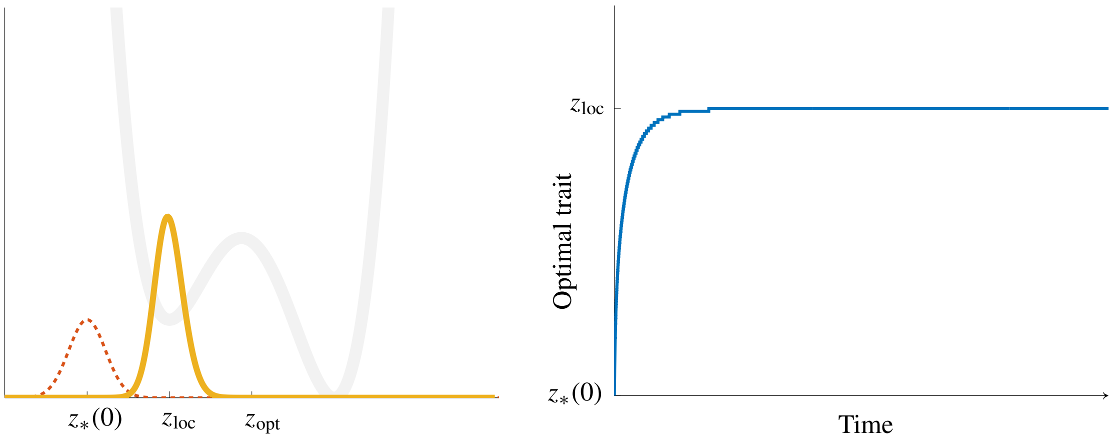

Figure 4. On the left, in dotted red, the initial data $f_\varepsilon(0,\cdot)$, and in orange, the distribution f_ε after a long time. In the background the selection function m with a global optimum z_{opt} and a local optimum z_{loc}. On the right, the trajectory of the dominant trait z_*. The function M is uniformly positive.

A case not taken into account by our methodology is when (1-13) is not satisfied at all times. This is the case if the slopes of the lines between local and global minima are too sharp. For instance, this is true in the case of Figure 5. Interestingly, what is observed is a critical behavior. The solution will first concentrate around the first local minimum before jumping sharply in the attraction basin of the global minimum; see the right-hand picture of Figure 5.

Under this model it would seem that the population will concentrate around the global minimum of selection if it is much better than the other selective optima. Interestingly, the value of the local maximum in between the two minima, that could act as an obstacle between the two convex selection valleys, do not

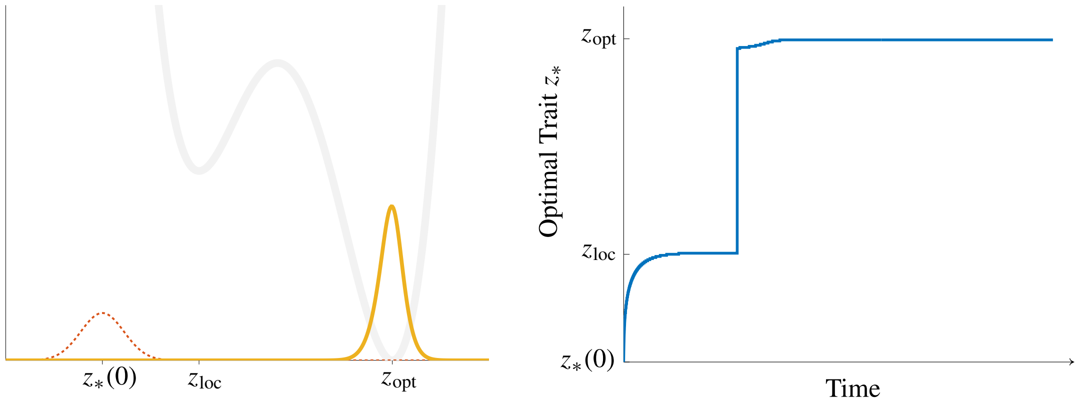

Figure 5. On the left, in dotted red, the initial data $f_\varepsilon(0,\cdot)$, and in orange, the distribution f_ε after a long time. In the background the selection function m with a global optimum z_{opt} and a local optimum z_{loc}. On the right, the trajectory of the dominant trait z_*. The function M is *not* uniformly positive.

appear to play a role. On the other hand, if the global minimum is not much better than a local minima, in the sense that each of them falls under the regime of (1-13), the population can concentrate around this local minimum.

Influence of the sign of q_ε. We introduced the scalar q_ε in (1-17) as part of the decomposition of U_ε between the affine parts and the rest of the function, which we later justified by heuristics on the linearized problem; see the table on page 1297. We can propose a different interpretation of this scalar, related to the Gaussian distribution.

The logarithmic transform (1-2) coupled with the decomposition (1-17) can be rewritten as the following transform on the solution of $(P_t f_\varepsilon)$:

$$f_\varepsilon(t,z)=\frac{1}{\varepsilon\sqrt{2\pi}}\exp\left[\frac{\lambda(t)-\varepsilon^2 p_\varepsilon(t)+\varepsilon^4 q_\varepsilon(t)^2}{\varepsilon^2}-\frac{(z-(z_*(t)-\varepsilon^2 q_\varepsilon(t)))^2}{2\varepsilon^2}-V_\varepsilon(t,z)\right]. \tag{9-1}$$

Therefore one can see that q_ε is the correction to the mean of the Gaussian distribution at the next order in ε. Its sign corresponds to the sign of the error made on the mean of the Gaussian distribution. If q_ε is positive, the correction of z_* lies on its left. This is consistent with the following reasoning on the limit value $q^*=\lim_{\varepsilon\to 0} q_\varepsilon$, defined in (1-18). For clarity, suppose that z_* does not depend on time, that is the regime of the stationary case. Then from (1-18), we find an explicit value for q^*, which coincides with [Calvez et al. 2019, (3.2)]:

$$q^*=\frac{m^{(3)}(z_*)}{2m''(z_*)}.$$

By local convexity of m around z_*, see (1-12), the sign of q^* is the same as the sign of $m^{(3)}(z_*)$. Therefore, if this scalar is positive, selection leans the profile towards the left, which has better selective values than the right, since it is flatter. Therefore, we recover what we deduced from (9-1); the sign of q_ε is linked to the skewness of the selection function m around z_*.

List of symbols

U_ε	Perturbation to solve the Cauchy problem $(P_t f_\varepsilon)$	(1-2)
m	Selection function	Assumption 1.1
I_ε	Residual shape of the infinitesimal operator	(1-3)
$z_*(t)$	Dominant trait in the population	(1-11)
$\bar{z}(t)$	$\frac{1}{2}(z+z_*(t))$	$(P_t U_\varepsilon)$
M	$1+m(z)-m(z_*(t))-m'(z_*(t))(z-z_*(t))$	$(P_t U_0)$
$p_\varepsilon, q_\varepsilon, V_\varepsilon$	$U_\varepsilon(t,z)=p_\varepsilon(t)+q_\varepsilon(t)(z-z_*(t))+V_\varepsilon(t,z)$	(1-17)
$\mathcal{I}_\varepsilon$	I_ε, but as a function of *two* variables: $\mathcal{I}_\varepsilon(q_\varepsilon, V_\varepsilon)=I_\varepsilon(U_\varepsilon)$	$(P_t V_\varepsilon)$
p^*, q^*, V^*	Limits of our unknowns when $\varepsilon\to 0$	(1-19), (1-18), (1-9)
U^*	Limit of U_ε, $U^*(t,z)=p^*(t)+q^*(t)(z-z_*(t))+V^*(t,z)$	(1-20)
R_ε	Decomposition $U_\varepsilon=U^*+\varepsilon^2 R_\varepsilon$	Section 2
$(\kappa_\varepsilon, W_\varepsilon)$	$q_\varepsilon=q^*+\varepsilon^2\kappa_\varepsilon$, $V_\varepsilon=V^*+\varepsilon^2 W_\varepsilon$ perturbations	(1-21), (1-22)
$\mathcal{E}$, $\\|\cdot\\|_{\mathcal{E}}$	Functional space to measure V_ε, and its norm	Definition 1.2
$\mathcal{F}$, $\\|\cdot\\|_{\mathcal{F}}$	Functional space to measure W_ε, and its norm	Definition 1.2
φ_α	Weight function $\varphi_\alpha(t,z)=(1+\lvert z-z_*(t)\rvert)^\alpha$	Definition 1.2
$\\|(g,W)\\|$	$\max(\lvert g\rvert, \\|W\\|_{\mathcal{F}})$ or $\max(\lvert g\rvert, \\|W\\|_{\mathcal{E}})$	Definition 2.1
$\mathcal{E}^*$, $\\|\cdot\\|_*$	Functional space to measure V^*, and its norm	Definition 3.1
K^*	Uniform bound of V^* in $\mathcal{E}^*$	Proposition 3.3
$O^*(\varepsilon)$	Special negligible term $O(\varepsilon)$	Definition 2.1
$\mathcal{I}^*_\varepsilon$	$\mathcal{I}_\varepsilon(q^*, V^*)$	Proposition 3.4
$\partial_g\mathcal{I}_\varepsilon$, $\partial_V\mathcal{I}_\varepsilon$	Fréchet derivatives w.r.t. the first and second variables	Section 4
$Q(y_1,y_2)$	Quadratic form: $\frac{1}{2}y_1y_2+\frac{3}{4}(y_1^2+y_2^2)$	(1-3)
Y	Refers to the pair of traits (y_1,y_2)	(3-4)
$\mathcal{D}_\varepsilon(V)(Y,t,z)$	$V(t,\bar{z})-\frac{1}{2}V(t,\bar{z}+\varepsilon y_1)-\frac{1}{2}V(t,\bar{z}+\varepsilon y_2)$	(3-4)
$\mathcal{D}^*_\varepsilon(V)(y,t)$	$V(t,z_*)-V(t,z_*+\varepsilon y)$	(3-4)
$\\|W\\|_\infty$	$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}}\lvert W(t,z)\rvert$	
dG^*_ε, dN^*_ε	Probability densities	Section 4C, (4-26), (4-27)
$\Xi_\varepsilon(t,z)$	Notation shortcut for $W_\varepsilon(t,z)-2W_\varepsilon(t,\bar{z})$	(5-1)
$O^*_0(1)$	Dependency upon the initial data at $t=0$	Theorem 7.1
B_0	"Small" ball around z_*	Figure 2, Section 7A, Proposition 7.3
r_*, T_*	Parameters in the definition of B_0	Section 7A, (7-7)
B_n, D_n	Successive balls and dyadic rings	Section 7: (7-2), (7-3), Figure 2
$(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$	Controls of M on the rings D_n	(7-4)
$\\|W\\|^n_\infty$	L^∞ norm on $\mathbb{R}_+\times B_n$, for $n\geq 0$	(7-5)

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THE “GOOD” BOUSSINESQ EQUATION: LONG-TIME ASYMPTOTICS

CHRISTOPHE CHARLIER, JONATAN LENELLS AND DENG-SHAN WANG

We consider the initial-value problem for the “good” Boussinesq equation on the line. Using inverse scattering techniques, the solution can be expressed in terms of the solution of a 3×3-matrix Riemann–Hilbert problem. We establish formulas for the long-time asymptotics of the solution by performing a Deift–Zhou steepest descent analysis of a regularized version of this Riemann–Hilbert problem. Our results are valid for generic solitonless Schwartz class solutions whose space-average remains bounded as $t \to \infty$.

1. Introduction

When investigating the bidirectional propagation of small amplitude and long wavelength capillary-gravity waves on the surface of shallow water, J. Boussinesq [1872] derived the classical Boussinesq equation

$$\eta_{tt} - gh_0\eta_{xx} = gh_0\left(\frac{3}{2}\frac{\eta^2}{h_0} + \frac{h_0^2}{3}\eta_{xx}\right)_{xx}, \tag{1-1}$$

where $\eta(x,t)$ is the perturbation-free surface, h_0 is the mean depth, and g is the gravitational constant. This equation was later rediscovered by Keulegan and Patterson [1940]. In nondimensional units, (1-1) can be written as

$$u_{tt} - u_{xx} - (u^2)_{xx} - u_{xxxx} = 0, \tag{1-2}$$

where $u(x,t)$ is a real-valued function and subscripts denote partial derivatives. Equation (1-2) is often referred to as the “bad” Boussinesq equation in contrast to the so-called “good” Boussinesq equation

$$u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx} = 0, \tag{1-3}$$

in which the u_{tt} and u_{xxxx} terms have the same sign, thus making the equation linearly well-posed; see, e.g., [Bona and Sachs 1988; Compaan and Tzirakis 2017; Farah 2009; Himonas and Mantzavinos 2015; Linares 1993] for well-posedness results for (1-3). Equation (1-3) governs small nonlinear oscillations in an elastic beam and is also known as the “nonlinear string equation” [Falkovich et al. 1983].

Deift and Zhou [1993] proposed a steepest descent method for the asymptotic analysis of Riemann–Hilbert (RH) problems. The Deift–Zhou approach has been successfully utilized to determine long-time asymptotics for a large number of integrable equations such as the modified KdV equation [Deift and Zhou 1993], the KdV equation [Deift et al. 1994], the nonlinear Schrödinger equation [Jenkins and McLaughlin 2014; Tovbis et al. 2004], the sine-Gordon equation [Cheng et al. 1999], the Camassa–Holm equation [Boutet de Monvel et al. 2009], the Degasperis–Procesi equation [Boutet de Monvel et al. 2019],

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Keywords: asymptotics, Boussinesq equation, Riemann–Hilbert problem, inverse scattering transform, initial value problem.

and the Toda lattice [Deift et al. 1996]. At his 60th birthday conference in 2005, P. Deift [2008] presented a list of sixteen open problems, among which he pointed out that “The long-time behavior of the solutions of the Boussinesq equation with general initial data is a very interesting problem with many challenges.” The purpose of this paper is to take a first step towards the solution of this problem.

As in [McKean 1981; Deift et al. 1982], we consider the following version of the “good” Boussinesq equation:

$$u_{tt} + \tfrac{4}{3}(u^2)_{xx} + \tfrac{1}{3}u_{xxxx} = 0, \tag{1-4}$$

which can be obtained from (1-3) by a simple shift $u \to u + \frac{1}{2}$ followed by a trivial rescaling. Our main result provides explicit formulas for the long-time asymptotics of the solution $u(x,t)$ of (1-4) in a sector in the right half-plane $\{x > 0,\ t > 0\}$ under the assumption that the initial data lie in the Schwartz class and satisfy the physically natural assumption that $u_t(x,0)$ has zero mean. The proof is based on a Deift–Zhou steepest descent analysis of a 3×3-matrix RH problem, which is parametrized by x and t. This RH problem was derived in [Charlier and Lenells 2022] by performing a spectral analysis of a Lax pair associated to (1-4); it is formulated in the complex plane of the spectral parameter k and has a jump contour consisting of the three lines $\mathbb{R}\cup\omega\mathbb{R}\cup\omega^2\mathbb{R}$, where $\omega = e^{2\pi i/3}$. The steepest descent analysis of this RH problem is severely complicated by the fact that the associated spectral problem is third-order. In fact, even though a version of the inverse scattering formalism was developed for the Boussinesq equation already in [Deift et al. 1982], the results presented here are, to the best of our knowledge, the first asymptotic results for any of the equations (1-2)–(1-4) obtained via steepest descent techniques (there exists a substantial amount of work on the long-time asymptotics for Boussinesq equations based on functional analytic approaches, see, e.g., [Farah 2008; Liu 1997; Linares and Scialom 1995; Wang 2009], but these approaches yield asymptotic information of a much less precise type). In addition to the third-order spectral problem, another complication in the analysis of (1-4) stems from the fact that the associated RH problem is singular at the origin. Therefore, instead of performing the steepest descent analysis of this RH problem directly, we will analyze a regularized version of the RH problem and then transfer the results to the singular problem.

The paper is organized as follows. The main result is stated in Section 2. An overview of the rather involved proof, which also contains a statement of the relevant RH problem, is presented in Section 3. The steepest descent analysis begins in Section 4, where several transformations of the RH problem are implemented. Local parametrices at the three critical points are constructed in Section 5 and the resulting small-norm RH problem is estimated in Section 6. Finally, the asymptotic behavior of $u(x,t)$ is obtained in Section 7.

2. Main result

Equation (1-4) can be rewritten as the system [Zakharov 1974]

$$\begin{cases} w_t + \frac{1}{3}u_{xxx} + \frac{4}{3}(u^2)_x = 0, \\ u_t = w_x, \end{cases} \tag{2-1}$$

which is equivalent to (1-4) provided that $u_1(x) := u_t(x,0)$ satisfies

$$\int_{\mathbb{R}} u_1(x)\,dx = 0. \tag{2-2}$$

The assumption (2-2) ensures that the integral $\int_{\mathbb{R}} u\,dx$ does not grow linearly but is conserved in time. Indeed, letting $u_0(x) := u(x,0)$ and assuming that u has sufficient smoothness and decay, (1-4) implies

$$\frac{d^2}{dt^2}\int_{\mathbb{R}} u\,dx = 0, \quad \text{i.e.,} \quad \int_{\mathbb{R}} u\,dx = \Big(\int_{\mathbb{R}} u_1\,dx\Big)t + \int_{\mathbb{R}} u_0\,dx.$$

Therefore, instead of analyzing (1-4) with initial data $u(x,0)$ and $u_t(x,0)$ directly, we will consider the system (2-1) with initial data $u_0(x) = u(x,0)$ and $w_0(x) = w(x,0)$.

2A. ***Definition of $s(k)$ and $s^A(k)$.*** The formulation of our main result involves two spectral functions $s(k)$ and $s^A(k)$ which are defined as follows (see [Charlier and Lenells 2022] for details). Suppose $u_0(x)$ and $w_0(x)$ are real-valued functions in $\mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz class of rapidly decaying functions on the real line. Let $\omega := e^{2\pi i/3}$ and let, for $j = 1, 2, 3$, $l_j(k) = \omega^j k$. Define $\mathsf{U}(x,k)$ by

$$\mathsf{U}(x,k) = P(k)^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -w_0(x) - u_{0x} & -2u_0(x) & 0 \end{pmatrix} P(k), \tag{2-3}$$

where

$$P(k) = \begin{pmatrix} \omega & \omega^2 & 1 \\ \omega^2 k & \omega k & k \\ k^2 & k^2 & k^2 \end{pmatrix}. \tag{2-4}$$

Let $X(x,k)$ and $X^A(x,k)$ be the 3×3-matrix-valued eigenfunctions defined by the linear Volterra integral equations

$$X(x,k) = I - \int_x^{\infty} e^{(x-x')\widehat{\mathcal{L}(k)}}(\mathsf{U}X)(x',k)\,dx', \tag{2-5a}$$

$$X^A(x,k) = I + \int_x^{\infty} e^{-(x-x')\widehat{\mathcal{L}(k)}}(\mathsf{U}^T X^A)(x',k)\,dx', \tag{2-5b}$$

where $\mathcal{L} = \operatorname{diag}(l_1, l_2, l_3)$, $\widehat{\mathcal{L}}$ denotes the operator which acts on a 3×3 matrix A by $\widehat{\mathcal{L}}A = [\mathcal{L}, A]$ (i.e., $e^{\widehat{\mathcal{L}}}A = e^{\mathcal{L}}Ae^{-\mathcal{L}}$), and U^T denotes the transpose of U. The 3×3-matrix-valued functions $s(k)$ and $s^A(k)$ are defined by

$$s(k) = I - \int_{\mathbb{R}} e^{-x\widehat{\mathcal{L}(k)}}(\mathsf{U}X)(x,k)\,dx, \tag{2-6}$$

$$s^A(k) = I + \int_{\mathbb{R}} e^{x\widehat{\mathcal{L}(k)}}(\mathsf{U}^T X^A)(x,k)\,dx. \tag{2-7}$$

2B. ***Statement of the main result.*** We first state our main result for the system (2-1); the formulation for (1-4) is given as a corollary. For simplicity, we only consider solutions in the Schwartz class, but it will be clear from the text that our result and its proof only require a finite degree of smoothness and decay.

Definition 2.1. We call $\{u(x,t), w(x,t)\}$ a *Schwartz class solution of* (2-1) *with initial data* $u_0, w_0 \in \mathcal{S}(\mathbb{R})$ if

(i) u, w are smooth real-valued functions of $(x,t) \in \mathbb{R}\times[0,\infty)$,

(ii) u, w satisfy (2-1) for $(x,t) \in \mathbb{R}\times[0,\infty)$ and

$$u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \quad x \in \mathbb{R}.$$

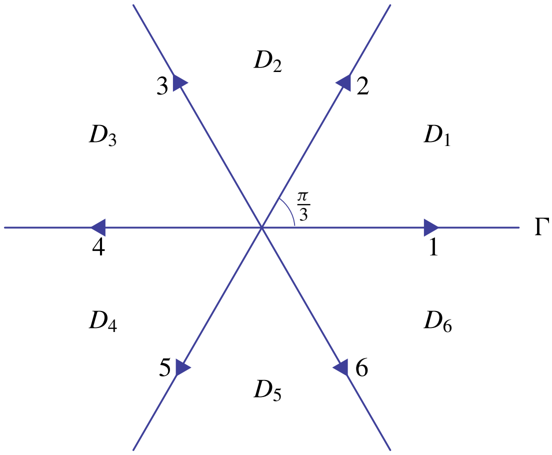

Figure 1. The contour Γ and the open sets D_n, $n = 1, \ldots, 6$, which decompose the complex k-plane.

(iii) u, w have rapid decay as $|x| \to \infty$ in the sense that, for each integer $N \geq 1$ and each $T > 0$,

$$\sup_{\substack{x\in\mathbb{R}\\ t\in[0,T)}} \sum_{i=0}^{N} (1+|x|)^N (|\partial_x^i u| + |\partial_x^i w|) < \infty.$$

Let $\{D_n\}_{n=1}^6$ denote the sectors shown in Figure 1. We make the following two assumptions.

Assumption 2.2 (absence of solitons). Assume that $(s(k))_{11}$ and $(s^A(k))_{11}$ are nonzero for $k \in \bar{D}_1 \setminus \{0\}$ and $k \in \bar{D}_4 \setminus \{0\}$, respectively.

Assumption 2.3 (generic behavior at $k = 0$). Assume that

$$\lim_{k\to 0} k^2 (s(k))_{11} \neq 0, \qquad \lim_{k\to 0} k^2 (s^A(k))_{11} \neq 0.$$

Assumption 2.2 ensures that no solitons are present (the case when s_{11} and s_{11}^A have a finite number of simple poles off the contour can be treated by standard methods; see, e.g., [Fokas and Its 1996], or [Lenells 2012] for a 3×3 matrix case). Assumption 2.3 ensures that s_{11} and s_{11}^A have double poles at $k = 0$, which is the case for generic initial data [Charlier and Lenells 2022].

Define the reflection coefficient $r_1(k)$ by

$$r_1(k) = \frac{(s(k))_{12}}{(s(k))_{11}}, \qquad k \in (0, \infty). \tag{2-8}$$

If $u_0, w_0 \in \mathcal{S}(\mathbb{R})$ are such that Assumptions 2.2 and 2.3 hold, then $r_1(k)$ extends to a smooth function of $k \in [0, \infty)$ with rapid decay as $k \to \infty$ which satisfies $r_1(0) = \omega$ and $|r_1(k)| < 1$ for $k > 0$; see [Charlier and Lenells 2022].

We can now state our main result, which establishes the long-time behavior of $u(x, t)$ in the asymptotic sector $x/t > 0$; see Figures 2 and 3.

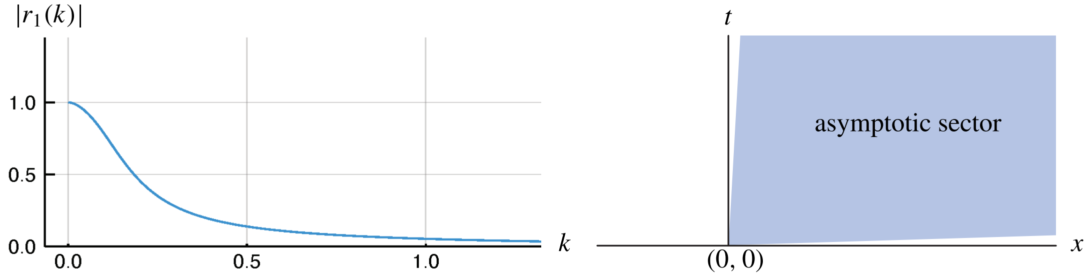

Figure 2. Numerical example showing $|r_1(k)|$ as a function of $k \geq 0$ (left) and an example of an asymptotic sector in the (x, t)-plane where the formula of Theorem 2.4 applies (right).

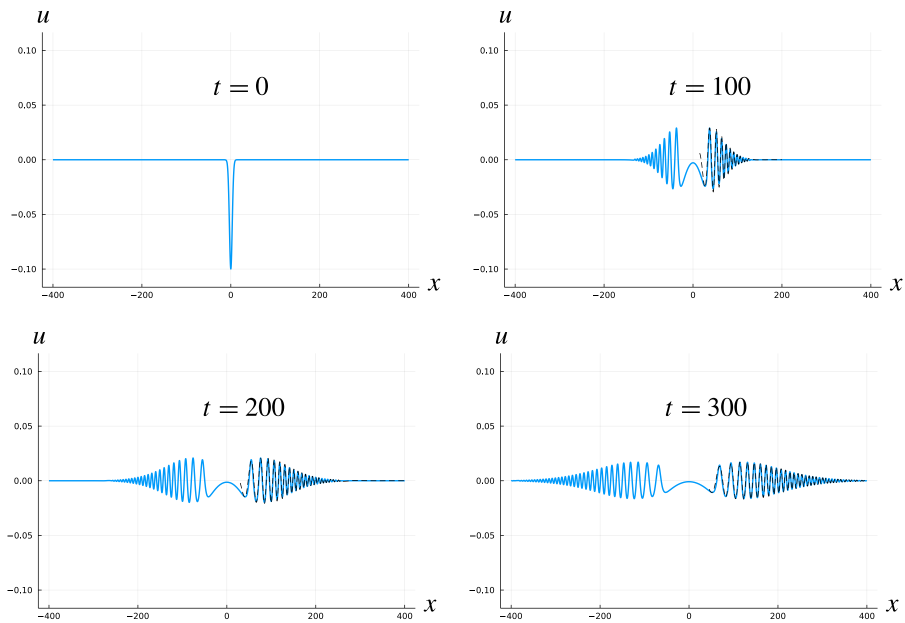

Figure 3. Numerical simulation of the solution $u(x, t)$ of (1-4) with initial data $u(x, 0) = -\frac{1}{10}e^{-x^2/20}$ and $u_t(x, 0) = 0$ (solid blue) together with the asymptotic approximation in (2-9) (dashed black) at the times $t = 0,\ t = 100,\ t = 200$, and $t = 300$. As expected, the asymptotic formula provides a better and better approximation as t increases. The convergence is the slowest for small values of x, which is consistent with the fact that the asymptotic estimate (2-9) is not uniform near $x = 0$.

Theorem 2.4 (long-time asymptotics for (2-1)). *Suppose $\{u(x, t), w(x, t)\}$ is a Schwartz class solution of (2-1) with initial data $u_0, w_0 \in \mathcal{S}(\mathbb{R})$ such that Assumptions 2.2 and 2.3 hold. Then the following asymptotic formula holds uniformly for $\zeta = x/t$ in compact subsets of* $(0, \infty)$ *as $t \to \infty$*:

$$u(x, t) = -\frac{3^{5/4}k_0\sqrt{\nu}}{\sqrt{2t}} \sin\biggl(\frac{19\pi}{12} + \nu \ln(6\sqrt{3}tk_0^2) - \sqrt{3}k_0^2 t - \arg r_1(k_0) - \arg\Gamma(i\nu) \\ + \frac{1}{\pi}\int_{k_0}^{\infty} \ln\left|\frac{s - k_0}{s - \omega k_0}\right| d\ln(1 - |r_1(s)|^2)\biggr) + O\left(\frac{\ln t}{t}\right), \tag{2-9}$$

where Γ *denotes the Gamma function*, $k_0 \equiv k_0(\zeta) = \zeta/2$, *and* $\nu \equiv \nu(\zeta) \geq 0$ *is defined by*

$$\nu = -\frac{1}{2\pi} \ln(1 - |r_1(k_0)|^2).$$

The proof of Theorem 2.4 is presented in Sections 3–7; Section 3 contains an overview of the proof.

As a corollary, we obtain asymptotics of the solution of (1-4) with initial data $u_0(x) = u(x, 0)$ and $u_1(x) = u_t(x, 0)$.

Corollary 2.5 (long-time asymptotics for (1-4)). *Suppose* $u(x, t)$ *is a Schwartz class solution of the "good" Boussinesq equation* (1-4) *with initial data* $u_0, u_1 \in \mathcal{S}(\mathbb{R})$ *such that* $\int_{\mathbb{R}} u_1\, dx = 0$. *Let* $w_0(x) = \int_{-\infty}^{x} u_1(x')\, dx'$ *and define* $r_1 : (0, \infty) \to \mathbb{C}$ *by* (2-8). *Suppose Assumptions* 2.2 *and* 2.3 *hold. Then* u *obeys the asymptotic formula* (2-9) *as* $t \to \infty$ *uniformly for* $\zeta = x/t$ *in compact subsets of* $(0, \infty)$.

Remark 2.6 (asymptotics in the left half-plane). In Theorem 2.4, we have, for conciseness, only presented asymptotics of $u(x, t)$ in a subsector of the right-half plane $x > 0$. A similar formula can be derived by the same methods for a subsector of the left half-plane, except that the formulas there involve $r_2 := s_{12}^A / s_{11}^A$ instead of r_1. Alternatively, asymptotics in the left half-plane can be obtained directly from Theorem 2.4 and the invariance of the Boussinesq equation under space inversion.

Remark 2.7 (asymptotics of w). Theorem 2.4 provides a formula for the asymptotics of u. Our methods can be used to derive an analogous asymptotic formula for w, but since this requires somewhat lengthy estimates of t-derivatives (see (3-6)), we have decided to not include this.

Remark 2.8 (regularity and decay assumptions). The Schwartz class assumption in Theorem 2.4 can be relaxed significantly. In fact, even our current proofs only require a finite degree of smoothness and decay. In light of the developments for integrable equations with second-order spectral problems, we expect that significant further improvements can be obtained by considering solutions in weighted Sobolev spaces. Consider for example the nonlinear Schrödinger equation: In [Deift and Zhou 2003], asymptotic formulas for the solution of the Cauchy problem were established under essentially minimal assumptions on the initial data, and more recently, the error terms in these formulas have been sharpened to become in a certain sense optimal by using the $\bar{\partial}$ generalization of the nonlinear steepest descent method [Borghese et al. 2018; Dieng et al. 2019]. It is an interesting research direction to investigate the regularity and decay requirements necessary for the derivation of asymptotic formulas for integrable equations with higher-order spectral problems. It seems clear that the $\bar{\partial}$ steepest descent method can be effectively applied also in this context. However, the construction of the direct and inverse scattering transforms in the framework of weighted Sobolev spaces is likely to be more involved for spectral problems of at least third order than for second-order problems.

2C. *Notation.* We summarize some notation that will be used throughout the paper. In what follows, $\gamma \subset \mathbb{C}$ denotes an oriented (piecewise smooth) contour.

- If A is an $n \times m$ matrix, then $|A| \geq 0$ is defined by $|A|^2 = \sum_{i,j} |A_{ij}|^2$. Note that $|A + B| \leq |A| + |B|$ and $|AB| \leq |A||B|$.
- c and C will denote generic positive constants which may change within a computation.

- We write $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_- = (-\infty, 0)$.

- For $1 \le p \le \infty$, we write $A \in L^p(\gamma)$ if $|A|$ belongs to $L^p(\gamma)$. Then $A \in L^p(\gamma)$ if and only if each entry A_{ij} belongs to $L^p(\gamma)$. We write $\|A\|_{L^p(\gamma)} := \||A|\|_{L^p(\gamma)}$.

- We define $\dot{L}^3(\gamma)$ as the space of all functions $f : \gamma \to \mathbb{C}$ such that $(1+|k|)^{1/3} f(k) \in L^3(\gamma)$. If γ is bounded, $\dot{L}^3(\gamma) = L^3(\gamma)$, but in general it only holds that $\dot{L}^3(\gamma) \subset L^3(\gamma)$. We turn $\dot{L}^3(\gamma)$ into a Banach space with the norm $\|f\|_{\dot{L}^3(\gamma)} := \|(1+|k|)^{1/3} f\|_{L^3(\gamma)}$.

- We let $\dot{E}^3(\mathbb{C} \setminus \gamma)$ denote the space of all analytic functions $f : \mathbb{C} \setminus \gamma \to \mathbb{C}$ with the property that for each component D of $\mathbb{C} \setminus \gamma$ there exist curves $\{C_n\}_1^\infty$ in D such that the C_n eventually surround each compact subset of D and

$$\sup_{n \ge 1} \int_{C_n} (1+|k|)\, |f(k)|^3\, |dk| < \infty.$$

- For a function f defined in $\mathbb{C} \setminus \gamma$, we let $f_\pm$ denote the nontangential boundary values of f from the left and right sides of γ, respectively, whenever they exist. If $f \in \dot{E}^3(\mathbb{C} \setminus \gamma)$, then $f_\pm$ exist a.e. on γ and $f_\pm \in \dot{L}^3(\gamma)$; see [Lenells 2018, Theorem 4.1].

3. Overview of the proof

The proof of Theorem 2.4 consists of a Deift–Zhou steepest descent analysis of a 3×3 matrix RH problem. The jump contour Γ of this RH problem consists of the three lines $\mathbb{R} \cup \omega\mathbb{R} \cup \omega^2\mathbb{R}$, see Figure 1, and the jump matrix v is given explicitly in terms of $r_1(k)$ defined in (2-8) and the function $r_2(k)$ defined by

$$r_2(k) = \frac{(s^A(k))_{12}}{(s^A(k))_{11}}, \quad k \in (-\infty, 0). \tag{3-1}$$

More precisely, v is defined as follows. Define $\{l_j(k), z_j(k)\}_{j=1}^3$ by

$$l_j(k) = \omega^j k, \quad z_j(k) = \omega^{2j} k^2, \quad k \in \mathbb{C}, \tag{3-2}$$

and define the complex-valued functions $\Phi_{ij}(\zeta, k)$ for $1 \le i \ne j \le 3$ by

$$\Phi_{ij}(\zeta, k) = (l_i - l_j)\zeta + (z_i - z_j),$$

where $\zeta := x/t$. By symmetry, it is enough to consider Φ_{21}, Φ_{31}, and Φ_{32}, which are explicitly given by

$$\begin{aligned} \Phi_{21}(\zeta, k) &= \omega(\omega - 1)k(\zeta - k), \\ \Phi_{31}(\zeta, k) &= (1 - \omega)k(\zeta - \omega^2 k), \\ \Phi_{32}(\zeta, k) &= (1 - \omega^2)k(\zeta - \omega k). \end{aligned}$$

Given a function $f(k)$ of $k \in \mathbb{C}$, we let f^* denote the Schwartz conjugate of f, i.e.,

$$f^*(k) = \overline{f(\bar{k})}.$$

The jump matrix $v(x,t,k)$ is defined for $k \in \Gamma$ by

$$\begin{aligned}
v_1 &= \begin{pmatrix} 1 & -r_1(k)e^{-t\Phi_{21}} & 0 \\ r_1^*(k)e^{t\Phi_{21}} & 1-|r_1(k)|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, &
v_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-|r_2(\omega k)|^2 & -r_2^*(\omega k)e^{-t\Phi_{32}} \\ 0 & r_2(\omega k)e^{t\Phi_{32}} & 1 \end{pmatrix}, \\
v_3 &= \begin{pmatrix} 1-|r_1(\omega^2 k)|^2 & 0 & r_1^*(\omega^2 k)e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ -r_1(\omega^2 k)e^{t\Phi_{31}} & 0 & 1 \end{pmatrix}, &
v_4 &= \begin{pmatrix} 1-|r_2(k)|^2 & -r_2^*(k)e^{-t\Phi_{21}} & 0 \\ r_2(k)e^{t\Phi_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
v_5 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -r_1(\omega k)e^{-t\Phi_{32}} \\ 0 & r_1^*(\omega k)e^{t\Phi_{32}} & 1-|r_1(\omega k)|^2 \end{pmatrix}, &
v_6 &= \begin{pmatrix} 1 & 0 & r_2(\omega^2 k)e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ -r_2^*(\omega^2 k)e^{t\Phi_{31}} & 0 & 1-|r_2(\omega^2 k)|^2 \end{pmatrix},
\end{aligned} \tag{3-3}$$

where v_j denotes the restriction of v to the subcontour of Γ labeled by j in Figure 1. We consider the following RH problem, which is formulated in the L^3-setting to ensure uniqueness (the solution of an $n \times n$-matrix L^p-RH problem is unique whenever it exists provided that $1 \le n \le p$; see [Lenells 2018, Theorem 5.6]).

RH problem 3.1 (L^3-RH problem for m). *Find a 3×3-matrix-valued function $m(x,t,\cdot) \in I + \dot{E}^3(\mathbb{C} \setminus \Gamma)$ such that $m_+(x,t,k) = m_-(x,t,k)v(x,t,k)$ for a.e. $k \in \Gamma$.*

By introducing the row-vector-valued function n by

$$n(x,t,k) = (\omega \;\; \omega^2 \;\; 1)m(x,t,k), \tag{3-4}$$

we can transform the RH problem for m into the following vector RH problem for n.

RH problem 3.2 (L^3-RH problem for n). *Find a 1×3-row-vector-valued function $n(x,t,\cdot) \in (\omega, \omega^2, 1) + \dot{E}^3(\mathbb{C} \setminus \Gamma)$ such that $n_+(x,t,k) = n_-(x,t,k)v(x,t,k)$ for a.e. $k \in \Gamma$.*

For technical reasons, we also need the classical version of this RH problem.

RH problem 3.3 (classical RH problem for n). *Find a 1×3-row-vector-valued function $n(x,t,k)$ with the following properties*:

(i) *$n(x,t,\cdot): \mathbb{C} \setminus \Gamma \to \mathbb{C}^{1\times 3}$ is analytic.*

(ii) *The limits of $n(x,t,k)$ as k approaches $\Gamma \setminus \{0\}$ from the left and right exist, are continuous on $\Gamma \setminus \{0\}$, and are denoted by n_+ and n_-, respectively. Furthermore, they are related by*

$$n_+(x,t,k) = n_-(x,t,k)v(x,t,k), \quad k \in \Gamma \setminus \{0\}. \tag{3-5}$$

(iii) *$n(x,t,k) = (\omega, \omega^2, 1) + O(k^{-1})$ as $k \to \infty$.*

(iv) *$n(x,t,k) = O(1)$ as $k \to 0$.*

The following result was proved in [Charlier and Lenells 2022].

Proposition 3.4. *Suppose the assumptions of Theorem 2.4 hold. Let U be an open subset of $\mathbb{R} \times [0,\infty)$ and suppose for each $(x,t) \in U$ that the solution of the classical RH problem 3.3 for n is unique whenever* *it exists. Then RH problem 3.3 has a unique solution $n(x,t,k)$ for each $(x,t) \in U$ and the solution $\{u(x,t), w(x,t)\}$ of (2-1) can be expressed in terms of $n = (n_1, n_2, n_3)$ by*

$$\begin{cases} u(x,t) = -\dfrac{3}{2}\dfrac{\partial}{\partial x} \lim_{k\to\infty} k(n_3(x,t,k) - 1), \\ w(x,t) = -\dfrac{3}{2}\dfrac{\partial}{\partial t} \lim_{k\to\infty} k(n_3(x,t,k) - 1), \end{cases} \qquad (x,t) \in U. \tag{3-6}$$

To use Proposition 3.4, we need the following lemma.

Lemma 3.5. *Suppose RH problem 3.1 has a solution $m(x,t,\cdot)$ at some point $(x,t) \in \mathbb{R} \times [0,\infty)$. Then $n = (\omega, \omega^2, 1)m$ is the unique solution of RH problem 3.2 at (x,t). Moreover, if the solution of RH problem 3.3 exists, then it is unique and is given by $n = (\omega, \omega^2, 1)m$.*

Proof. The assertion for $n = (\omega, \omega^2, 1)m$ follows as in [Boutet de Monvel et al. 2019, Lemma A.5]. The last claim follows because every solution of RH problem 3.3 is also a solution of RH problem 3.2. ☐

It will follow from the steepest descent analysis (see Lemma 6.3) that there exists a $T > 0$ such that RH problem 3.1 has a unique solution m for $t \geq T$ and x/t in a compact subset of $(0,\infty)$. Thus Proposition 3.4 and Lemma 3.5 imply that the formulas (3-6) for u, w are valid for all $t \geq T$ and x/t in compact subsets of $(0,\infty)$ if n is defined by $n = (\omega, \omega^2, 1)m$. Therefore it is enough to determine the large t asymptotics of m.

3A. ***Steepest descent analysis.*** The large t behavior of m can be obtained by performing a Deift–Zhou steepest descent analysis of RH problem 3.1. The first step in this analysis is to define analytic approximations of the functions r_1 and r_2 appearing in the jump matrix v, as well as of the combination $r_1/(1-|r_1|^2)$. Once these approximations are in place, we can deform the contour in such a way that the new jump is close to the identity matrix everywhere except near three critical points (see Section 4). The critical points are the solutions of the stationary phase equations $\partial\Phi_{21}/\partial k = 0$, $\partial\Phi_{31}/\partial k = 0$, and $\partial\Phi_{32}/\partial k = 0$. For each choice of $1 \leq j < i \leq 3$, $\partial\Phi_{ij}/\partial k = 0$ has a single zero k_{ij} given by

$$k_{21} = \frac{\zeta}{2}, \quad k_{31} = \frac{\omega\zeta}{2}, \quad k_{32} = \frac{\omega^2\zeta}{2}.$$

Writing $k_0 \equiv k_{21}$, these three critical points can be expressed as k_0, ωk_0, and $\omega^2 k_0$; see Figure 4. The signature tables for Φ_{21}, Φ_{31}, and Φ_{32} are shown in Figures 5–7.

Near each of the three critical points, the RH problem can be approximated by a local parametrix which is constructed in Section 5. In fact, since the jump matrix v obeys the symmetries

$$v(x,t,k) = \mathcal{A}v(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}\overline{v(x,t,\bar{k})}^{-1}\mathcal{B}, \quad k \in \Gamma, \tag{3-7}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3-8}$$

the solution m obeys the symmetries

$$m(x,t,k) = \mathcal{A}m(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}\overline{m(x,t,\bar{k})}\mathcal{B}, \quad k \in \mathbb{C} \setminus \Gamma. \tag{3-9}$$

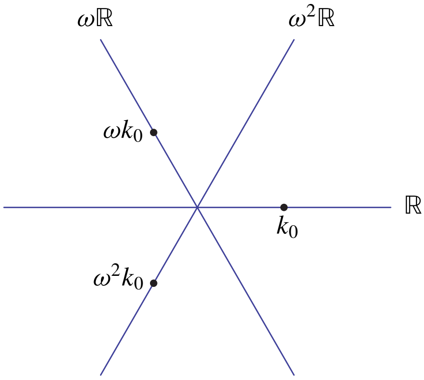

Figure 4. The three critical points k_0, ωk_0, $\omega^2 k_0$ in the complex k-plane for $\zeta > 0$.

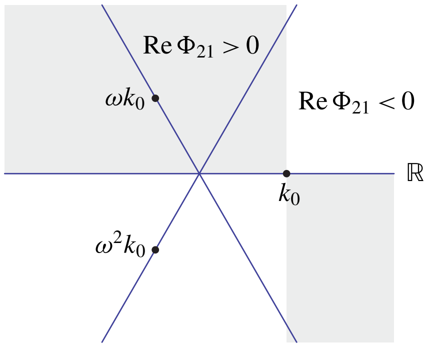

Figure 5. The regions where $\operatorname{Re}\Phi_{21} > 0$ (shaded) and $\operatorname{Re}\Phi_{21} < 0$ (white).

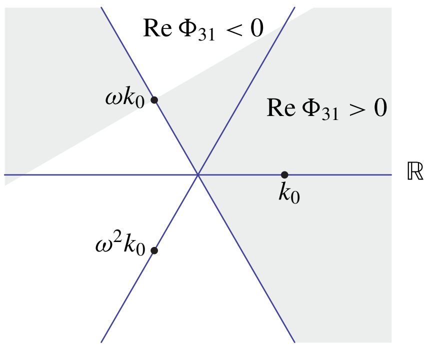

Figure 6. The regions where $\operatorname{Re}\Phi_{31} > 0$ (shaded) and $\operatorname{Re}\Phi_{31} < 0$ (white).

It is therefore sufficient to construct the local parametrix m^{k_0} at k_0, because then the local parametrices at ωk_0 and $\omega^2 k_0$ can be obtained by symmetry. In the end, we arrive at a small-norm RH problem whose solution is estimated in Section 6. Finally, the asymptotics of $u(x,t)$ is obtained in Section 7.

3B. ***Assumptions for the remainder of the paper.*** From here on, we assume that $\{u(x,t), w(x,t)\}$ is a Schwartz class solution of (2-1) with initial data $u_0, w_0 \in \mathcal{S}(\mathbb{R})$ such that Assumptions 2.2 and 2.3 hold.

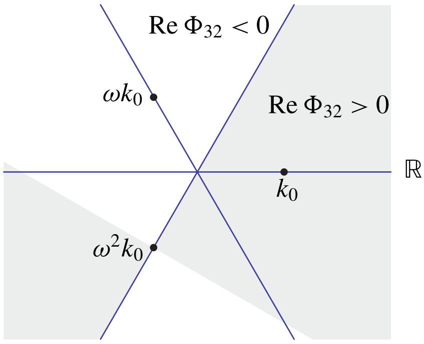

Figure 7. The regions where Re $\Phi_{32} > 0$ (shaded) and Re $\Phi_{32} < 0$ (white).

We also assume that $r_1(k)$ and $r_2(k)$ are defined by (2-8) and (3-1). We let $\mathcal{I}$ denote a fixed compact subset of $(0, \infty)$.

4. Transformations of the RH problem

By performing a number of transformations, we can bring the RH problem 3.1 to a form suitable for determining the long-time asymptotics. More precisely, starting with m, we will define functions $m^{(j)}(x, t, k)$, $j = 1, 2, 3$, such that the RH problem satisfied by $m^{(j)}$ is equivalent to the original RH problem 3.1. The RH problem for $m^{(j)}$ can be formulated as follows, where the contours $\Gamma^{(j)}$ and the jump matrices $v^{(j)}$ are specified below.

RH problem 4.1 (RH problem for $m^{(j)}$). *Find a* 3×3*-matrix-valued function* $m^{(j)}(x, t, \cdot) \in I + \dot{E}^3(\mathbb{C} \setminus \Gamma^{(j)})$ *such that* $m_+^{(j)}(x, t, k) = m_-^{(j)}(x, t, k) v^{(j)}(x, t, k)$ *for a.e.* $k \in \Gamma^{(j)}$.

The jump matrix $v^{(3)}$ obtained after the third transformation has the property that it approaches the identity matrix as $t \to \infty$ everywhere on the contour except near the three critical points k_0, ωk_0, $\omega^2 k_0$. This means that we can find the long-time asymptotics of $m^{(3)}$ by computing the contribution from three small crosses centered at these points.

The symmetries (3-7) and (3-9) will be preserved at each stage of the transformations, so that, for $j = 1, 2, 3$,

$$v^{(j)}(x, t, k) = \mathcal{A} v^{(j)}(x, t, \omega k) \mathcal{A}^{-1} = \mathcal{B} \overline{v^{(j)}(x, t, \bar{k})}^{-1} \mathcal{B}, \quad k \in \Gamma^{(j)}, \tag{4-1}$$

$$m^{(j)}(x, t, k) = \mathcal{A} m^{(j)}(x, t, \omega k) \mathcal{A}^{-1} = \mathcal{B} \overline{m^{(j)}(x, t, \bar{k})} \mathcal{B}, \quad k \in \mathbb{C} \setminus \Gamma^{(j)}. \tag{4-2}$$

4A. ***First transformation.*** The purpose of the first transformation is to remove (except for a small remainder) the jumps across the subcontours $e^{\pi i/3}\mathbb{R}_+$, $\mathbb{R}_-$, and $e^{-\pi i/3}\mathbb{R}_+$ of Γ. To implement this transformation, we need analytic approximations of the functions r_2^*, r_1, and $\hat{r}_1^*$, where $\hat{r}_1(k)$ is defined by

$$\hat{r}_1(k) = \frac{r_1(k)}{1 - r_1(k) r_1^*(k)}.$$

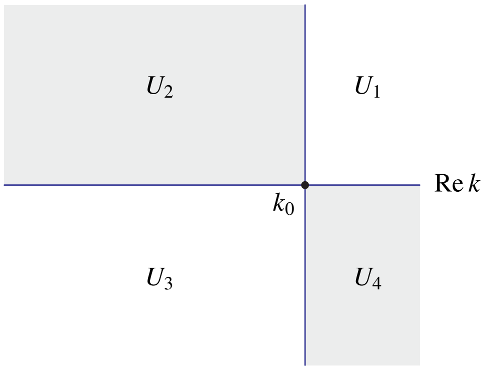

Figure 8. The open sets $\{U_j\}_1^4$ in the complex k-plane.

We introduce open sets $U_j = U_j(\zeta) \subset \mathbb{C}$, $j = 1, \dots, 4$, as in Figure 8, such that

$$U_1 \cup U_3 = \{k \mid \mathrm{Re}\, \Phi_{21}(\zeta, k) < 0\}, \quad U_2 \cup U_4 = \{k \mid \mathrm{Re}\, \Phi_{21}(\zeta, k) > 0\}.$$

Lemma 4.2. *There exist decompositions*

$$\begin{aligned} r_2^*(k) &= r_{2,a}^*(x,t,k) + r_{2,r}^*(x,t,k), && k \in (-\infty, 0], \\ r_1(k) &= r_{1,a}(x,t,k) + r_{1,r}(x,t,k), && k \in [0, k_0], \\ \hat{r}_1^*(k) &= \hat{r}_{1,a}^*(x,t,k) + \hat{r}_{1,r}^*(x,t,k), && k \in [k_0, \infty), \end{aligned} \tag{4-3}$$

where the functions $r_{2,a}^*, r_{2,r}^*, r_{1,a}, r_{1,r}, \hat{r}_{1,a}^*, \hat{r}_{1,r}^*$ have the following properties:

(a) *For each $\zeta \in \mathcal{I}$ and each $t > 0$, $r_{2,a}^*(x,t,k)$ and $r_{1,a}(x,t,k)$ are defined and continuous for $k \in \bar{U}_2$ and analytic for $k \in U_2$, and $\hat{r}_{1,a}^*(x,t,k)$ is defined and continuous for $k \in \bar{U}_1$ and analytic for $k \in U_1$.*

(b) *For each $\zeta \in \mathcal{I}$ and $t > 0$, the functions $r_{2,a}^*$, $r_{1,a}$, and $\hat{r}_{1,a}^*$ satisfy*

$$|r_{2,a}^*(x,t,k)| \le \frac{C|k - \omega k_0|}{1+|k|^2} e^{(t/4)|\mathrm{Re}\, \Phi_{21}(\zeta,k)|}, \quad k \in \bar{U}_2, \tag{4-4a}$$

$$|\partial_x^l (r_{2,a}^*(x,t,k) - r_2^*(0))| \le C|k| e^{(t/4)|\mathrm{Re}\, \Phi_{21}(\zeta,k)|}, \quad k \in \bar{U}_2, \tag{4-4b}$$

$$|\partial_x^l (r_{1,a}(x,t,k) - r_1(0))| \le C|k| e^{(t/4)|\mathrm{Re}\, \Phi_{21}(\zeta,k)|}, \quad k \in \bar{U}_2, \tag{4-4c}$$

$$|\partial_x^l (r_{1,a}(x,t,k) - r_1(k_0))| \le C|k - k_0| e^{(t/4)|\mathrm{Re}\, \Phi_{21}(\zeta,k)|}, \quad k \in \bar{U}_2, \tag{4-4d}$$

$$|\partial_x^l (\hat{r}_{1,a}^*(x,t,k) - \hat{r}_1^*(k_0))| \le C|k - k_0| e^{(t/4)|\mathrm{Re}\, \Phi_{21}(\zeta,k)|}, \quad k \in \bar{U}_1, \tag{4-4e}$$

$$|\partial_x^l \hat{r}_{1,a}^*(x,t,k)| \le \frac{C}{1+|k|} e^{(t/4)|\mathrm{Re}\, \Phi_{21}(\zeta,k)|}, \quad k \in \bar{U}_1, \tag{4-4f}$$

where $l = 0, 1$ and the constant C is independent of ζ, t, k.

(c) *For each $1 \le p \le \infty$ and $l = 0, 1$,*

$$\text{the } L^p\text{-norm of } (1+|\cdot|)\partial_x^l r_{2,r}^*(x,t,\cdot) \text{ on } (-\infty, 0) \text{ is } O(t^{-3/2}), \tag{4-5}$$

$$\text{the } L^p\text{-norms of } \partial_x^l r_{1,r}(x,t,\cdot) \text{ and } \frac{r_{1,r}(x,t,\cdot)}{\cdot - k_0} \text{ on } (0,k_0) \text{ are } O(t^{-3/2}), \tag{4-6}$$

$$\text{the } L^p\text{-norms of } (1+|\cdot|)\partial_x^l \hat{r}_{1,r}^*(x,t,\cdot) \text{ and } \frac{\hat{r}_{1,r}^*(x,t,\cdot)}{\cdot - k_0} \text{ on } (k_0,\infty) \text{ are } O(t^{-3/2}), \tag{4-7}$$

uniformly for $\zeta \in \mathcal{I}$ *as* $t \to \infty$.

Proof. The proof uses the techniques of [Deift and Zhou 1993]. Since these techniques are rather standard by now, we omit the details; see [Lenells 2017, Lemma 4.8] for a proof of a similar lemma. □

In the sequel, we often write $r_{j,a}(k)$ and $r_{j,r}(k)$ instead of $r_{j,a}(x,t,k)$ and $r_{j,r}(x,t,k)$, respectively, for notational convenience.

Recalling that $r_2 = r_{2,a} + r_{2,r}$, we can factorize v_2, v_4, v_6 as

$$v_2 = v_{2,a}^U v_{2,r} v_{2,a}^L, \qquad v_4 = v_{4,a}^U v_{4,r} v_{4,a}^L, \qquad v_6 = v_{6,a}^L v_{6,r} v_{6,a}^U,$$

where the analytic factors are given by

$$v_{2,a}^U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -r_{2,a}^*(\omega k)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \quad v_{2,a}^L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r_{2,a}(\omega k)e^{t\Phi_{32}} & 1 \end{pmatrix},$$

$$v_{4,a}^U = \begin{pmatrix} 1 & -r_{2,a}^*(k)e^{-t\Phi_{21}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v_{4,a}^L = \begin{pmatrix} 1 & 0 & 0 \\ r_{2,a}(k)e^{t\Phi_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{6,a}^L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -r_{2,a}^*(\omega^2 k)e^{t\Phi_{31}} & 0 & 1 \end{pmatrix}, \quad v_{6,a}^U = \begin{pmatrix} 1 & 0 & r_{2,a}(\omega^2 k)e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the small remainders $v_{j,r}$, $j = 2, 4, 6$, are given by the expressions obtained by replacing r_j with $r_{j,r}$ in the definition (3-3) of v_j, i.e.,

$$v_{2,r} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - r_{2,r}(\omega k)r_{2,r}^*(\omega k) & -r_{2,r}^*(\omega k)e^{-t\Phi_{32}} \\ 0 & r_{2,r}(\omega k)e^{t\Phi_{32}} & 1 \end{pmatrix},$$

$$v_{4,r} = \begin{pmatrix} 1 - |r_{2,r}(k)|^2 & -r_{2,r}^*(k)e^{-t\Phi_{21}} & 0 \\ r_{2,r}(k)e^{t\Phi_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{6,r} = \begin{pmatrix} 1 & 0 & r_{2,r}(\omega^2 k)e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ -r_{2,r}^*(\omega^2 k)e^{t\Phi_{31}} & 0 & 1 - r_{2,r}(\omega^2 k)r_{2,r}^*(\omega^2 k) \end{pmatrix}.$$

Define the sectionally analytic function $m^{(1)}$ by

$$m^{(1)}(x,t,k) = m(x,t,k)G(x,t,k),$$

where G is defined by

$$G(x,t,k) = \begin{cases} v_{2,a}^{U}, & k \in D_1, \\ (v_{2,a}^{L})^{-1}, & k \in D_2, \\ v_{4,a}^{U}, & k \in D_3, \\ (v_{4,a}^{L})^{-1}, & k \in D_4, \\ v_{6,a}^{L}, & k \in D_5, \\ (v_{6,a}^{U})^{-1}, & k \in D_6. \end{cases} \tag{4-8}$$

Lemma 4.3. *$G(x,t,k)$ and $G(x,t,k)^{-1}$ are uniformly bounded for $k \in \mathbb{C}\setminus\Gamma$, $t>0$, and $\zeta \in \mathcal{I}$. Moreover, $G = I + O(k^{-1})$ as $k \to \infty$.*

Proof. We have $\operatorname{Re}\Phi_{21}(\zeta,k) > 0$ for $k \in D_3$ (see Figure 5). Therefore, by virtue of (4-4a),

$$|v_{4,a}^{U}(x,t,k) - I| \le \frac{C}{1+|k|} e^{-ct|\operatorname{Re}\Phi_{21}(\zeta,k)|}, \quad k \in D_3,$$

uniformly for $\zeta \in \mathcal{I}$. Since $\operatorname{Re}\Phi_{21}(\zeta,k) < 0$ for $\zeta \in D_4$ (see Figure 5 again), we deduce similarly that

$$|r_{2,a}(x,t,k)| \le \frac{C}{1+|k|} e^{(t/4)|\operatorname{Re}\Phi_{21}(\zeta,k)|}, \quad k \in \bar{U}_3,$$

and hence

$$|(v_{4,a}^{L})^{-1}(x,t,k) - I| \le \frac{C}{1+|k|} e^{-ct|\operatorname{Re}\Phi_{21}(\zeta,k)|}, \quad k \in D_4.$$

We appeal to the $\mathcal{A}$-symmetry of (4-1) to extend these bounds to the other sectors. □

It follows from Lemma 4.3 that m satisfies RH problem 3.1 if and only if $m^{(1)}$ satisfies RH problem 4.1 with $j=1$, where $\Gamma^{(1)} = \Gamma$ and the jump matrix $v^{(1)}$ is given on $\Gamma_1 \cup \Gamma_3 \cup \Gamma_5$ by

$$v_1^{(1)} = v_{6,a}^{U} v_1 v_{2,a}^{U}, \quad v_3^{(1)} = v_{2,a}^{L} v_3 v_{4,a}^{U}, \quad v_5^{(1)} = v_{4,a}^{L} v_5 v_{6,a}^{L},$$

and the small jumps remaining on $\Gamma_2 \cup \Gamma_4 \cup \Gamma_6$ are given by

$$v_j^{(1)} = v_{j,r}, \quad j = 2,4,6.$$

Here Γ_j denotes the subcontour of Γ labeled by j in Figure 1. More explicitly, the jump matrices $v_j^{(1)}$, $j = 1,3,5$, can be expressed as

$$v_1^{(1)} = \begin{pmatrix} 1 & -r_1(k)e^{-t\Phi_{21}} & \beta(k)e^{-t\Phi_{31}} \\ r_1^*(k)e^{t\Phi_{21}} & 1-r_1(k)r_1^*(k) & \alpha(k)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_3^{(1)} = \begin{pmatrix} 1-r_1(\omega^2k)r_1^*(\omega^2k) & \alpha(\omega^2k)e^{-t\Phi_{21}} & r_1^*(\omega^2k)e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ -r_1(\omega^2k)e^{t\Phi_{31}} & \beta(\omega^2k)e^{t\Phi_{32}} & 1 \end{pmatrix},$$

$$v_5^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ \beta(\omega k)e^{t\Phi_{21}} & 1 & -r_1(\omega k)e^{-t\Phi_{32}} \\ \alpha(\omega k)e^{t\Phi_{31}} & r_1^*(\omega k)e^{t\Phi_{32}} & 1-r_1(\omega k)r_1^*(\omega k) \end{pmatrix},$$

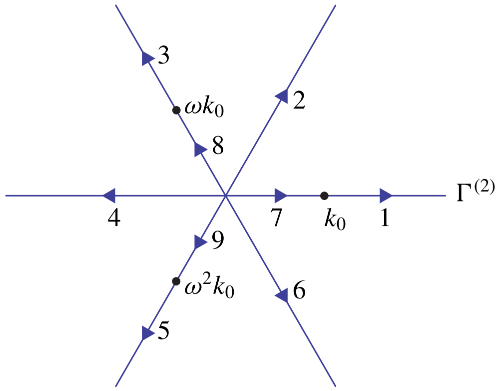

Figure 9. The contour $\Gamma^{(2)}$ in the complex k-plane.

where the functions $\alpha(k) \equiv \alpha(x, t, k)$ and $\beta(k) \equiv \beta(x, t, k)$ are defined by

$$\alpha(k) = -r_{2,a}^*(\omega k)(1 - r_1(k) r_1^*(k)), \quad k \in \mathbb{R}_+,$$
$$\beta(k) = r_{2,a}(\omega^2 k) + r_1(k) r_{2,a}^*(\omega k), \quad k \in \mathbb{R}_+.$$

4B. ***Second transformation.*** Let $\Gamma^{(2)} = \bigcup_{j=1}^{9} \Gamma_j^{(2)}$ denote the contour displayed in Figure 9, where $\Gamma_1^{(2)} = [k_0, \infty)$ etc. For each $\zeta \in \mathcal{I}$, we choose $\delta_1(\zeta, k)$ such that δ_1 is analytic except for the jump across $\Gamma_1^{(2)}$,

$$\delta_{1+}(\zeta, k) = \delta_{1-}(\zeta, k)(1 - |r_1(k)|^2), \quad k \in \Gamma_1^{(2)},$$

and such that

$$\delta_1(\zeta, k) = 1 + O(k^{-1}), \quad k \to \infty. \tag{4-9}$$

The relation $k_0 = \zeta/2$ implies that there exists an $\epsilon > 0$ such that

$$|r(k_0)| \le 1 - \epsilon \quad \text{for all } k \in [k_0, \infty) \text{ and all } \zeta \in \mathcal{I}. \tag{4-10}$$

Hence, by the Plemelj formulas, we find

$$\delta_1(\zeta, k) = \exp\left\{ \frac{1}{2\pi i} \int_{[k_0, \infty)} \frac{\ln(1 - |r_1(s)|^2)}{s - k} \, ds \right\}, \quad k \in \mathbb{C} \setminus \Gamma_1^{(2)}. \tag{4-11}$$

Let $\ln_0(k)$ denote the logarithm of k with branch cut along $\arg k = 0$, i.e., $\ln_0(k) = \ln|k| + i \arg_0 k$ with $\arg_0 k \in (0, 2\pi)$.

Lemma 4.4. *The function $\delta_1(\zeta, k)$ has the following properties*:

(a) *δ_1 can be written as*

$$\delta_1(\zeta, k) = e^{-i\nu \ln_0(k - k_0)} e^{-\chi_1(\zeta, k)}, \tag{4-12}$$

where $\nu \equiv \nu(\zeta) \ge 0$ is defined by

$$\nu = -\frac{1}{2\pi} \ln(1 - |r_1(k_0)|^2), \quad \zeta \in \mathcal{I},$$

and

$$\chi_1(\zeta,k)=\frac{1}{2\pi i}\int_{k_0}^{\infty}\ln_0(k-s)\,d\ln(1-|r_1(s)|^2). \tag{4-13}$$

(b) *For each* $\zeta\in\mathcal{I}$, $\delta_1(\zeta,k)$ *and* $\delta_1(\zeta,k)^{-1}$ *are analytic functions of* $k\in\mathbb{C}\setminus\Gamma_1^{(2)}$ *with continuous boundary values on* $\Gamma_1^{(2)}\setminus\{k_0\}$. *Moreover,*

$$\sup_{\zeta\in\mathcal{I}}\sup_{k\in\mathbb{C}\setminus\Gamma_1^{(2)}}|\delta_1(\zeta,k)^{\pm1}|<\infty. \tag{4-14}$$

(c) δ_1 *obeys the symmetry*

$$\delta_1(\zeta,k)=\overline{\delta_1(\zeta,\bar{k})}^{-1},\quad \zeta\in\mathcal{I},\ k\in\mathbb{C}\setminus\Gamma_1^{(2)}. \tag{4-15}$$

(d) *As* $k\to k_0$ *along a path which is nontangential to* (k_0,∞), *we have*

$$|\chi_1(\zeta,k)-\chi_1(\zeta,k_0)|\le C|k-k_0|(1+|\ln|k-k_0||), \tag{4-16}$$

$$|\partial_x(\chi_1(\zeta,k)-\chi_1(\zeta,k_0))|\le\frac{C}{t}(1+|\ln|k-k_0||), \tag{4-17}$$

where C *is independent of* $\zeta\in\mathcal{I}$. *Furthermore,*

$$|\partial_x\chi_1(\zeta,k_0)|=\frac{1}{t}\left|\partial_u\chi_1(u,v)|_{(u,v)=(\zeta,k_0)}+\frac{1}{2}\partial_v\chi_1(u,v)|_{(u,v)=(\zeta,k_0)}\right|\le\frac{C}{t} \tag{4-18}$$

and

$$\partial_x(\delta_1(\zeta,k)^{\pm1})=\frac{\pm i\nu}{2t(k-k_0)}\delta_1(\zeta,k)^{\pm1}. \tag{4-19}$$

Proof. The lemma follows from (4-11) and relatively straightforward estimates. □

The functions δ_3 and δ_5 defined by

$$\delta_3(\zeta,k)=\delta_1(\zeta,\omega^2k),\quad k\in\mathbb{C}\setminus\Gamma_3^{(2)},$$

$$\delta_5(\zeta,k)=\delta_1(\zeta,\omega k),\quad k\in\mathbb{C}\setminus\Gamma_5^{(2)},$$

satisfy the jump relations

$$\delta_{3+}(\zeta,k)=\delta_{3-}(\zeta,k)(1-|r_1(\omega^2k)|^2),\quad k\in\Gamma_3^{(2)},$$

$$\delta_{5+}(\zeta,k)=\delta_{5-}(\zeta,k)(1-|r_1(\omega k)|^2),\quad k\in\Gamma_5^{(2)}.$$

The jump matrix $v^{(1)}$ cannot be appropriately factorized on the subcontour $\Gamma_1^{(2)}\cup\Gamma_3^{(2)}\cup\Gamma_5^{(2)}$ of $\Gamma^{(2)}$. Hence we introduce $m^{(2)}$ by

$$m^{(2)}(x,t,k)=m^{(1)}(x,t,k)\Delta(\zeta,k),$$

where the 3×3-matrix-valued function $\Delta(\zeta,k)$ is defined by

$$\Delta(\zeta,k)=\begin{pmatrix}\frac{\delta_1(\zeta,k)}{\delta_3(\zeta,k)} & 0 & 0\\ 0 & \frac{\delta_5(\zeta,k)}{\delta_1(\zeta,k)} & 0\\ 0 & 0 & \frac{\delta_3(\zeta,k)}{\delta_5(\zeta,k)}\end{pmatrix}. \tag{4-20}$$

From (4-14) and (4-9), we infer that Δ and Δ^{-1} are uniformly bounded for $\zeta \in \mathcal{I}$ and $k \in \mathbb{C} \setminus (\Gamma_1^{(2)} \cup \Gamma_3^{(2)} \cup \Gamma_5^{(2)})$ and that

$$\Delta(\zeta, k) = I + O(k^{-1}) \quad \text{as } k \to \infty. \tag{4-21}$$

It follows that m satisfies RH problem 3.1 if and only if $m^{(2)}$ satisfies RH problem 4.1 with $j = 2$, where the jump matrix $v^{(2)}$ is given by $v^{(2)} = \Delta_-^{-1} v^{(1)} \Delta_+$. A computation gives

$$v_1^{(2)} = \begin{pmatrix} \frac{\delta_{1+}}{\delta_{1-}} & -\frac{\delta_3\delta_5}{\delta_{1-}\delta_{1+}} r_1(k) e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_{1-}\delta_5}\beta(k)e^{-t\Phi_{31}} \\ \frac{\delta_{1-}\delta_{1+}}{\delta_3\delta_5} r_1^*(k)e^{t\Phi_{21}} & \frac{\delta_{1-}}{\delta_{1+}}(1-r_1(k)r_1^*(k)) & \frac{\delta_{1-}\delta_3}{\delta_5^2}\alpha(k)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-r_1(k)r_1^*(k) & -\frac{\delta_3\delta_5}{\delta_{1-}^2}\frac{r_1(k)}{1-r_1(k)r_1^*(k)}e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_{1-}\delta_5}\beta(k)e^{-t\Phi_{31}} \\ \frac{\delta_{1+}^2}{\delta_3\delta_5}\frac{r_1^*(k)}{1-r_1(k)r_1^*(k)}e^{t\Phi_{21}} & 1 & -r_{2,a}^*(\omega k)\frac{\delta_{1+}\delta_3}{\delta_5^2}e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_2^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-r_{2,r}(\omega k)r_{2,r}^*(\omega k) & -\frac{\delta_1\delta_3}{\delta_5^2}r_{2,r}^*(\omega k)e^{-t\Phi_{32}} \\ 0 & \frac{\delta_5^2}{\delta_1\delta_3}r_{2,r}(\omega k)e^{t\Phi_{32}} & 1 \end{pmatrix},$$

$$v_7^{(2)} = \begin{pmatrix} 1 & -\frac{\delta_3\delta_5}{\delta_1^2}r_1(k)e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5}\beta(k)e^{-t\Phi_{31}} \\ \frac{\delta_1^2}{\delta_3\delta_5}r_1^*(k)e^{t\Phi_{21}} & 1-r_1(k)r_1^*(k) & \frac{\delta_1\delta_3}{\delta_5^2}\alpha(k)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix}.$$

The remaining jumps $v_j^{(2)}$ can be obtained from these matrices together with the $\mathbb{Z}_3$ symmetry (4-1) and are given by

$$v_3^{(2)} = \begin{pmatrix} 1 & -\frac{\delta_{3+}\delta_5}{\delta_1^2}r_{2,a}^*(k)e^{-t\Phi_{21}} & \frac{\delta_{3+}^2}{\delta_1\delta_5}\frac{r_1^*(\omega^2 k)e^{-t\Phi_{31}}}{1-r_1(\omega^2 k)r_1^*(\omega^2 k)} \\ 0 & 1 & 0 \\ -\frac{\delta_1\delta_5}{\delta_{3-}^2}\frac{r_1(\omega^2 k)e^{t\Phi_{31}}}{1-r_1(\omega^2 k)r_1^*(\omega^2 k)} & \frac{\delta_5^2}{\delta_1\delta_{3-}}\beta(\omega^2 k)e^{t\Phi_{32}} & 1-r_1(\omega^2 k)r_1^*(\omega^2 k) \end{pmatrix},$$

$$v_4^{(2)} = \begin{pmatrix} 1-r_{2,r}(k)r_{2,r}^*(k) & -\frac{\delta_3\delta_5}{\delta_1^2}r_{2,r}^*(k)e^{-t\Phi_{21}} & 0 \\ \frac{\delta_1^2}{\delta_3\delta_5}r_{2,r}(k)e^{t\Phi_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_5^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta_1^2}{\delta_3\delta_{5-}}\beta(\omega k)e^{t\Phi_{21}} & 1-r_1(\omega k)r_1^*(\omega k) & -\frac{\delta_3\delta_1}{\delta_{5-}^2}\frac{r_1(\omega k)e^{-t\Phi_{32}}}{1-r_1(\omega k)r_1^*(\omega k)} \\ -\frac{\delta_1\delta_{5+}}{\delta_3^2}r_{2,a}^*(\omega^2 k)e^{t\Phi_{31}} & \frac{\delta_{5+}^2}{\delta_1\delta_3}\frac{r_1^*(\omega k)e^{t\Phi_{32}}}{1-r_1(\omega k)r_1^*(\omega k)} & 1 \end{pmatrix},$$

$$v_6^{(2)} = \begin{pmatrix} 1 & 0 & \frac{\delta_3^2}{\delta_1\delta_5} r_{2,r}(\omega^2 k) e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ -\frac{\delta_1\delta_5}{\delta_3^2} r_{2,r}^*(\omega^2 k) e^{t\Phi_{31}} & 0 & 1 - r_{2,r}(\omega^2 k) r_{2,r}^*(\omega^2 k) \end{pmatrix},$$

$$v_8^{(2)} = \begin{pmatrix} 1 - r_1(\omega^2 k) r_1^*(\omega^2 k) & \frac{\delta_3\delta_5}{\delta_1^2} \alpha(\omega^2 k) e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5} r_1^*(\omega^2 k) e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ -\frac{\delta_1\delta_5}{\delta_3^2} r_1(\omega^2 k) e^{t\Phi_{31}} & \frac{\delta_5^2}{\delta_1\delta_3} \beta(\omega^2 k) e^{t\Phi_{32}} & 1 \end{pmatrix},$$

$$v_9^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta_1^2}{\delta_3\delta_5} \beta(\omega k) e^{t\Phi_{21}} & 1 & -\frac{\delta_1\delta_3}{\delta_5^2} r_1(\omega k) e^{-t\Phi_{32}} \\ \frac{\delta_1\delta_5}{\delta_3^2} \alpha(\omega k) e^{t\Phi_{31}} & \frac{\delta_5^2}{\delta_1\delta_3} r_1^*(\omega k) e^{t\Phi_{32}} & 1 - r_1(\omega k) r_1^*(\omega k) \end{pmatrix}.$$

4C. ***Third transformation.*** The (11)-entry of $v_1^{(2)}$ can be rewritten as

$$(v_1^{(2)})_{11} = 1 - r_1(k) r_1^*(k) = 1 - \frac{\delta_{1+}}{\delta_{1-}} \hat{r}_1(k) \frac{\delta_{1+}}{\delta_{1-}} \hat{r}_1^*(k).$$

Therefore, using the general identity

$$\begin{pmatrix} 1+f_1 f_3 & f_1 & f_2 \\ f_3 & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & f_{1,a} & f_2 - f_1 f_4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1+f_{1,r} f_{3,r} & f_{1,r} & 0 \\ f_{3,r} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ f_{3,a} & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix},$$

where $f_j = f_{j,a} + f_{j,r}$, as well as the relation

$$\beta(k) - r_{2,a}^*(\omega k) r_1(k) = r_{2,a}(\omega^2 k), \quad k \in \mathbb{R}_+,$$

we can factorize $v_1^{(2)}$ for $k \in \Gamma_1^{(2)}$ as

$$v_1^{(2)} = \begin{pmatrix} 1 - \frac{\delta_{1+}^2}{\delta_{1-}^2} \hat{r}_1(k) \hat{r}_1^*(k) & -\frac{\delta_3\delta_5}{\delta_{1-}^2} \hat{r}_1(k) e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_{1-}\delta_5} \beta(k) e^{-t\Phi_{31}} \\ \frac{\delta_{1+}^2}{\delta_3\delta_5} \hat{r}_1^*(k) e^{t\Phi_{21}} & 1 & -r_2^*(\omega k) \frac{\delta_{1+}\delta_3}{\delta_5^2} e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix} = v_1^{(2)A} v_{1,r}^{(2)} v_1^{(2)B}, \tag{4-22}$$

where

$$v_1^{(2)A} = \begin{pmatrix} 1 & -\frac{\delta_3\delta_5}{\delta_{1-}^2} \hat{r}_{1,a}(k) e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_{1-}\delta_5} r_{2,a}(\omega^2 k) e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{1,r}^{(2)} = \begin{pmatrix} 1 - \frac{\delta_{1+}^2}{\delta_{1-}^2} \hat{r}_{1,r}^*(k) \hat{r}_{1,r}(k) & -\frac{\delta_3\delta_5}{\delta_{1-}^2} \hat{r}_{1,r}(k) e^{-t\Phi_{21}} & 0 \\ \frac{\delta_{1+}^2}{\delta_3\delta_5} \hat{r}_{1,r}^*(k) e^{t\Phi_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_1^{(2)B} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta_{1+}^2}{\delta_3\delta_5}\hat{r}_{1,a}^*(k)e^{t\Phi_{21}} & 1 & -\frac{\delta_{1+}\delta_3}{\delta_5^2}r_{2,a}^*(\omega k)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, using the general identity

$$\begin{aligned} &\begin{pmatrix} 1 & f_1 & f_2 \\ f_3 & 1+f_1f_3 & f_4 \\ 0 & 0 & 1 \end{pmatrix} \\ &\quad = \begin{pmatrix} 1 & 0 & 0 \\ f_{3,a} & 1 & f_{4,a}-f_{2,a}f_{3,a} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & f_{1,r} & f_{2,r} \\ f_{3,r} & 1+f_{1,r}f_{3,r} & f_{4,r}-f_{2,a}f_{3,r}-f_{2,r}f_{3,a} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & f_{1,a} & f_{2,a} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

where $f_j = f_{j,a} + f_{j,r}$, as well as the relation

$$\alpha(k) - r_1^*(k)\beta(k) = -r_{2,a}^*(\omega k) - r_1^*(k)r_{2,a}(\omega^2 k), \quad k \in \mathbb{R}_+,$$

we can factorize $v_7^{(2)}$ for $k \in \Gamma_7^{(2)}$ as

$$v_7^{(2)} = \begin{pmatrix} 1 & -\frac{\delta_3\delta_5}{\delta_1^2}r_1(k)e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5}\beta(k)e^{-t\Phi_{31}} \\ \frac{\delta_1^2}{\delta_3\delta_5}r_1^*(k)e^{t\Phi_{21}} & 1-r_1(k)r_1^*(k) & \frac{\delta_1\delta_3}{\delta_5^2}\alpha(k)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix} = v_7^{(2)A}v_{7,r}^{(2)}v_7^{(2)B},$$

where

$$v_7^{(2)A} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta_1^2}{\delta_3\delta_5}r_{1,a}^*(k)e^{t\Phi_{21}} & 1 & -\frac{\delta_1\delta_3}{\delta_5^2}\big(r_{2,a}^*(\omega k)+r_{1,a}^*(k)r_{2,a}(\omega^2 k)\big)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{7,r}^{(2)} = \begin{pmatrix} 1 & -\frac{\delta_3\delta_5}{\delta_1^2}r_{1,r}(k)e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5}\beta_r(k)e^{-t\Phi_{31}} \\ \frac{\delta_1^2}{\delta_3\delta_5}r_{1,r}^*(k)e^{t\Phi_{21}} & 1-|r_{1,r}(k)|^2 & \frac{\delta_1\delta_3}{\delta_5^2}r_{1,r}^*(k)\big(r_{1,r}(k)r_{2,a}^*(\omega k)-r_{2,a}(\omega^2 k)\big)e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_7^{(2)B} = \begin{pmatrix} 1 & -\frac{\delta_3\delta_5}{\delta_1^2}r_{1,a}(k)e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5}\beta_a(k)e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\beta_r(k) := r_{1,r}(k)r_{2,a}^*(\omega k), \quad \beta_a(k) := r_{2,a}(\omega^2 k) + r_{1,a}(k)r_{2,a}^*(\omega k).$$

Let $V_j \equiv V_j(\zeta) \subset \mathbb{C}$, $j = 1, \dots, 4$, denote the open subsets of the complex k-plane displayed in Figure 10. Define the sectionally analytic function $m^{(3)}$ by

$$m^{(3)}(x,t,k) = m^{(2)}(x,t,k)H(x,t,k),$$

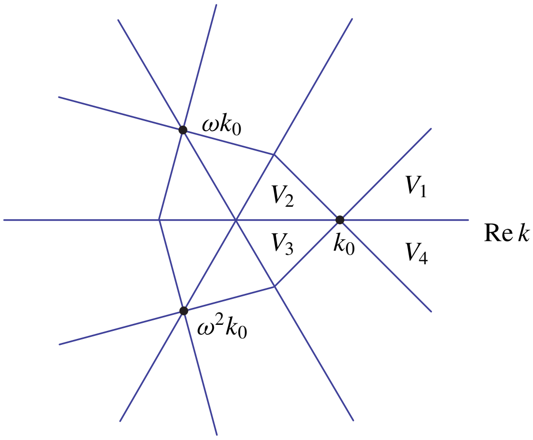

Figure 10. The open sets $\{V_j\}_1^4$ in the complex k-plane.

where H is defined for $k \in D_1 \cup D_6$ by

$$H(x,t,k) = \begin{cases} (v_1^{(2)B})^{-1}, & k \in V_1, \\ (v_7^{(2)B})^{-1}, & k \in V_2, \\ v_7^{(2)A}, & k \in V_3, \\ v_1^{(2)A}, & k \in V_4, \\ I, & \text{elsewhere in } D_1 \cup D_6, \end{cases} \tag{4-23}$$

and extended to all of $\mathbb{C} \setminus \Gamma$ by means of the symmetry $H(x,t,k) = \mathcal{A} H(x,t,\omega k)\mathcal{A}^{-1}$. Let $\Gamma^{(3)}$ be the contour displayed in Figure 11.

Lemma 4.5. *$H(x,t,k)$ is uniformly bounded for $k \in \mathbb{C} \setminus \Gamma^{(3)}$, $t > 0$, and $\zeta \in \mathcal{I}$. Moreover, $H = I + O(k^{-1})$ as $k \to \infty$.*

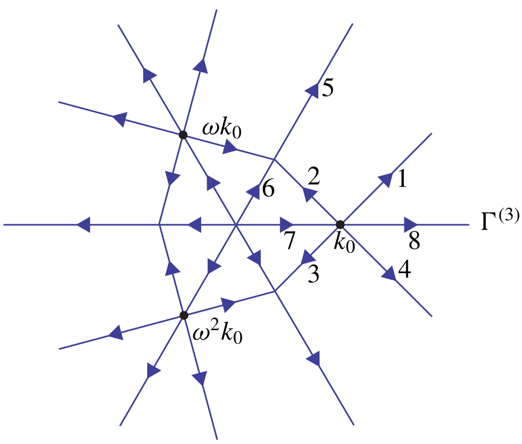

Figure 11. The contour $\Gamma^{(3)}$ in the complex k-plane.

Proof. We present the proof for $k \in V_1 \cup V_2$; the proof for $k \in V_3 \cup V_4$ is similar. Note that $V_j \subset U_j$, $j = 1, \dots, 4$ (see Figures 8 and 10). Note also the identities

$$\Phi_{21}(\zeta, \omega k) = \Phi_{32}(\zeta, k), \quad \Phi_{21}(\zeta, \omega^2 k) = -\Phi_{31}(\zeta, k), \quad \Phi_{21} + \Phi_{32} = \Phi_{31}. \tag{4-24}$$

If $k \in \overline{V}_1$, then $\omega k \in \overline{U}_2$ and (see Figures 5 and 7)

$$\operatorname{Re} \Phi_{21}(\zeta, k) \le 0, \quad \operatorname{Re} \Phi_{32}(\zeta, k) \ge 0.$$

Therefore, using (4-24), (4-4a), (4-4f), and (4-14), we find

$$\begin{aligned}
|((v_1^{(2)B})^{-1})_{21}| &= \left| \frac{\delta_1^2}{\delta_3\delta_5} \hat{r}^*_{1,a}(k) e^{t\Phi_{21}} \right| \le \frac{C}{1+|k|} e^{-ct|\operatorname{Re}\Phi_{21}|}, && k \in V_1,\\
|((v_1^{(2)B})^{-1})_{23}| &= \left| \frac{\delta_1\delta_3}{\delta_5^2} r^*_{2,a}(\omega k) e^{-t\Phi_{32}} \right| \le \frac{C}{1+|k|} e^{-ct|\operatorname{Re}\Phi_{32}|}, && k \in V_1.
\end{aligned}$$

This proves the claim for $k \in V_1$. All entries of $(v_7^{(2)B})^{-1}$ are continuous functions on $\overline{V}_2$. Since $\overline{V}_2$ is compact, the claim follows also for $k \in V_2$. ☐

It follows from Lemma 4.5 that m satisfies RH problem 3.1 if and only if $m^{(3)}$ satisfies RH problem 4.1 with $j = 3$, where $\Gamma^{(3)}$ is the contour displayed in Figure 11 and the jump matrix $v^{(3)}$ is given for $-\frac{\pi}{3} < \arg k \le \frac{\pi}{3}$ by

$$v_1^{(3)} = v_1^{(2)B} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta_1^2}{\delta_3\delta_5} \hat{r}^*_{1,a}(k) e^{t\Phi_{21}} & 1 & -\frac{\delta_1\delta_3}{\delta_5^2} r^*_{2,a}(\omega k) e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_2^{(3)} = (v_7^{(2)B})^{-1} = \begin{pmatrix} 1 & \frac{\delta_3\delta_5}{\delta_1^2} r_{1,a}(k) e^{-t\Phi_{21}} & -\frac{\delta_3^2}{\delta_1\delta_5}\big(r_{2,a}(\omega^2 k) + r_{1,a}(k) r^*_{2,a}(\omega k)\big) e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_3^{(3)} = (v_7^{(2)A})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{\delta_1^2}{\delta_3\delta_5} r^*_{1,a}(k) e^{t\Phi_{21}} & 1 & \frac{\delta_1\delta_3}{\delta_5^2}\big(r^*_{2,a}(\omega k) + r^*_{1,a}(k) r_{2,a}(\omega^2 k)\big) e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_4^{(3)} = v_1^{(2)A} = \begin{pmatrix} 1 & -\frac{\delta_3\delta_5}{\delta_1^2} \hat{r}_{1,a}(k) e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5} r_{2,a}(\omega^2 k) e^{-t\Phi_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_5^{(3)} = v_2^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - r_{2,r}(\omega k) r^*_{2,r}(\omega k) & -\frac{\delta_1\delta_3}{\delta_5^2} r^*_{2,r}(\omega k) e^{-t\Phi_{32}} \\ 0 & \frac{\delta_5^2}{\delta_1\delta_3} r_{2,r}(\omega k) e^{t\Phi_{32}} & 1 \end{pmatrix},$$

$$v_6^{(3)} = v_7^{(2)B} v_2^{(2)} \mathcal{A}^{-1} v_7^{(2)A}(x,t,\omega^2 k)\mathcal{A} = \begin{pmatrix} 1 & \frac{\delta_3\delta_5}{\delta_1^2} g(k) e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5} f(k) e^{-t\Phi_{31}} \\ 0 & 1 - r_{2,r}(\omega k) r_{2,r}^*(\omega k) & -\frac{\delta_1\delta_3}{\delta_5^2} r_{2,r}^*(\omega k) e^{-t\Phi_{32}} \\ 0 & \frac{\delta_5^2}{\delta_1\delta_3} r_{2,r}(\omega k) e^{t\Phi_{32}} & 1 \end{pmatrix},$$

$$v_7^{(3)} = v_{7,r}^{(2)} = \begin{pmatrix} 1 & -\frac{\delta_3\delta_5}{\delta_1^2} r_{1,r}(k) e^{-t\Phi_{21}} & \frac{\delta_3^2}{\delta_1\delta_5} r_{1,r}(k) r_{2,a}^*(\omega k) e^{-t\Phi_{31}} \\ \frac{\delta_1^2}{\delta_3\delta_5} r_{1,r}^*(k) e^{t\Phi_{21}} & 1 - r_{1,r}(k) r_{1,r}^*(k) & \frac{\delta_1\delta_3}{\delta_5^2} r_{1,r}^*(k) h(k) e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_8^{(3)} = v_{1,r}^{(2)} = \begin{pmatrix} 1 - \frac{\delta_{1+}^2}{\delta_{1-}^2} \hat{r}_{1,r}(k) \hat{r}_{1,r}^*(k) & -\frac{\delta_3\delta_5}{\delta_{1-}^2} \hat{r}_{1,r}(k) e^{-t\Phi_{21}} & 0 \\ \frac{\delta_{1+}^2}{\delta_3\delta_5} \hat{r}_{1,r}^*(k) e^{t\Phi_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and extended to the remainder of $\Gamma^{(3)}$ by means of the first symmetry in (4-1). Here the functions $f(k) \equiv f(x,t,k)$, $g(k) \equiv g(x,t,k)$, and $h(k) \equiv h(x,t,k)$ are defined by

$$\begin{aligned} &f(k) = r_{2,a}(\omega^2 k) + r_{1,a}(k) r_2^*(\omega k) + r_{1,a}^*(\omega^2 k), \\ &g(k) = r_{2,r}(\omega k)\big(r_{2,a}(\omega^2 k) + r_{1,a}(k) r_{2,a}^*(\omega k)\big) - r_{1,a}(k)\big(1 - r_{2,r}(\omega k) r_{2,r}^*(\omega k)\big) - r_{2,a}^*(k) - r_{1,a}^*(\omega^2 k) r_{2,a}(\omega k), \\ &h(k) = r_{1,r}(k) r_{2,a}^*(\omega k) - r_{2,a}(\omega^2 k). \end{aligned}$$

The next lemma establishes bounds on f and g and their x-derivatives.

Lemma 4.6. *For $k \in \Gamma_3$, and $l = 0, 1$, we have*

$$|\partial_x^l f(k)| \le C|k| e^{(t/4)|\operatorname{Re}\Phi_{21}(\zeta,k)|}, \quad |\partial_x^l g(k)| \le C|k| e^{(t/4)|\operatorname{Re}\Phi_{21}(\zeta,k)|}. \tag{4-25}$$

Proof. By (4-4b) and (4-4c), we have $r_{2,r}^*(0) = 0$, $r_{2,a}^*(0) = r_2^*(0)$, $r_{1,r}(0) = 0$, and $r_{1,a}(0) = r_1(0)$. Since $r_1(0) = \omega$ and $r_2(0) = 1$ (see [Charlier and Lenells 2022]), we deduce that

$$r_{2,a}^*(0) = r_2^*(0) = 1 \quad \text{and} \quad r_{1,a}(0) = r_1(0) = \omega.$$

In particular,

$$r_{1,a}(0) r_2^*(0) + r_{1,a}^*(0) + r_{2,a}(0) = \omega + \bar{\omega} + 1 = 0. \tag{4-26}$$

To derive the estimate for f, we write

$$\begin{aligned} f(k) &= r_{1,a}^*(\omega^2 k) + r_{2,a}(\omega^2 k) + r_{1,a}(k) r_2^*(\omega k) \\ &= r_{1,a}^*(\omega^2 k) - r_{1,a}^*(0) + r_{2,a}(\omega^2 k) - r_{2,a}(0) + (r_{1,a}(k) - r_{1,a}(0)) r_2^*(\omega k) \\ &\qquad + r_{1,a}(0)(r_2^*(\omega k) - r_2^*(0)) + r_{1,a}(0) r_2^*(0) + r_{1,a}^*(0) + r_{2,a}(0). \end{aligned}$$

Using (4-26) and the fact that $\Gamma_3^{(6)} \subset (\omega^2\mathbb{R}_- \cap U_2)$, the inequalities (4-4b) and (4-4c) imply

$$\begin{aligned}
|f(k)| &\le |r_{1,a}^*(\omega^2 k) - r_{1,a}^*(0)| + |r_{2,a}(\omega^2 k) - r_{2,a}(0)| + C|r_{1,a}(k) - r_{1,a}(0)| + C|k| \\
&\le C|k|(e^{(t/4)|\operatorname{Re}\Phi_{21}(\zeta,\omega\bar{k})|} + e^{(t/4)|\operatorname{Re}\Phi_{21}(\zeta,\omega\bar{k})|} + e^{(t/4)|\operatorname{Re}\Phi_{21}(\zeta,k)|}) \\
&\le C|k|e^{(t/4)|\operatorname{Re}\Phi_{21}(\zeta,k)|}, \qquad k \in \Gamma_3^{(6)}.
\end{aligned}$$

The estimate for $\partial_x f$ is derived in a similar way. Writing

$$g(k) = r_{2,r}(\omega k)\big(r_{2,a}(\omega^2 k) + r_{1,a}(k) r_{2,a}^*(\omega k)\big) + r_{1,a}(k) r_{2,r}(\omega k) r_{2,r}^*(\omega k) - f^*(\omega^2 k),$$

and using that $r_{2,r}$ and $\partial_x r_{2,r}$ vanish at $k = 0$, the estimates for g and $\partial_x g$ follow from the estimates for f and $\partial_x f$. ☐

Lemma 4.7. *The jump matrix $v^{(3)}$ (resp. $\partial_x v^{(3)}$) converges to the identity matrix I (resp. to the zero matrix 0) as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}$ and $k \in \Gamma^{(3)}$ except near the three critical points $k_0, \omega k_0, \omega^2 k_0$. Moreover, the jump matrices $v_j^{(3)}$, $j = 5, 6, 7, 8$, satisfy*

$$\|(1+|\cdot|)\partial_x^l(v^{(3)} - I)\|_{(L^1\cap L^\infty)(\Gamma_5^{(3)})} \le Ct^{-3/2}, \tag{4-27a}$$

$$\|(1+|\cdot|)\partial_x^l(v^{(3)} - I)\|_{L^1(\Gamma_6^{(3)})} \le Ct^{-3/2}, \tag{4-27b}$$

$$\|(1+|\cdot|)\partial_x^l(v^{(3)} - I)\|_{L^\infty(\Gamma_6^{(3)})} \le Ct^{-1}, \tag{4-27c}$$

$$\|(1+|\cdot|)\partial_x^l(v^{(3)} - I)\|_{(L^1\cap L^\infty)(\Gamma_7^{(3)}\cup\Gamma_8^{(3)})} \le Ct^{-3/2}, \tag{4-27d}$$

uniformly for $\zeta \in \mathcal{I}$ and $l = 0, 1$.

Proof. Consider first the jump matrix $v_1^{(3)}$. Since $\operatorname{Re}\Phi_{32} \ge c > 0$ and $\operatorname{Re}\Phi_{21} \le 0$ for $k \in \Gamma_1^{(3)}$, $v_1^{(3)}$ (resp. $\partial_x v_1^{(3)}$) converges to I (resp. to the zero matrix) as $t \to \infty$ by (4-24), (4-4), and (4-14). Note however that the convergence to 0 of the (21) entry is not uniform for k near k_0, because $\operatorname{Re}\Phi_{21}(\zeta, k_0) = 0$. Analogous statements for $v_2^{(3)}$, $v_3^{(3)}$, and $v_4^{(3)}$ can be proved in a similar way.

Since $\operatorname{Re}\Phi_{32} = 0$ for $k \in \Gamma_5^{(3)}$, (4-27a) follows from (4-5), and (4-14).

We next show (4-27b) and (4-27c). We parametrize $\Gamma_6^{(3)}$ by $ue^{\pi i/3}$, $0 \le u \le 2k_0/(1+\sqrt{3})$, and note that

$$\operatorname{Re}\Phi_{31}(\zeta, ue^{\pi i/3}) = \operatorname{Re}\Phi_{21}(\zeta, ue^{\pi i/3}) = \tfrac{3}{2}u(2k_0 - u), \qquad u \in \mathbb{R}.$$

It follows that

$$\begin{cases} \operatorname{Re}\Phi_{31}(\zeta, k) \ge \frac{4}{3}k_0|k|, \\ \operatorname{Re}\Phi_{21}(\zeta, k) \ge \frac{4}{3}k_0|k|, \end{cases} \qquad k \in \Gamma_6^{(3)}.$$

Using (4-25), (4-14), (4-19), and the fact that $\partial_x(t\Phi_{31}) = (1-\omega)k$, we thus find

$$\begin{aligned}
|(v_6^{(3)} - I)_{13}| &\le C|f(k)|e^{-t\operatorname{Re}\Phi_{31}} \le C|k|e^{-tk_0|k|}, && k \in \Gamma_6^{(3)}, \\
|\partial_x(v_6^{(3)})_{13}| &\le C|k|e^{-tk_0|k|}, && k \in \Gamma_6^{(3)}.
\end{aligned}$$

Hence, for $l = 0, 1$, we have

$$\|(1+|\cdot|)\partial_x^l(v_6^{(3)} - I)_{13}\|_{L^1(\Gamma_6^{(3)})} \le \frac{C}{(k_0 t)^2}, \quad \|(1+|\cdot|)\partial_x^l(v_6^{(3)} - I)_{13}\|_{L^\infty(\Gamma_6^{(3)})} \le \frac{C}{k_0 t},$$

and similar estimates apply to the (12)-entry. On the other hand, $\mathrm{Re}\,\Phi_{32} = 0$ for $k \in \Gamma_6^{(3)}$, and hence we can estimate the (23)-entry using (4-14) as

$$|(v_6^{(3)} - I)_{23}| = \left|\frac{\delta_1\delta_3}{\delta_5^2} r_{2,r}^*(\omega k)\right| \le C|r_{2,r}^*(\omega k)|, \quad k \in \Gamma_6^{(3)}.$$

By (4-5), this implies that the L^1 and L^∞ norms of $(1+|\cdot|)(v^{(3)} - I)_{23}$ on $\Gamma_6^{(3)}$ are $O(t^{-3/2})$ as $t \to \infty$. Using also (4-19) and (4-5), we conclude similarly that the L^1 and L^∞ norms of $(1+|\cdot|)\partial_x v_{23}^{(3)}$ on $\Gamma_6^{(3)}$ are $O(t^{-3/2})$ as $t \to \infty$. A similar estimate applies to the (32)-entry and its x-derivative. The (22)-entry is even smaller. This proves (4-27b) and (4-27c).

We finally show (4-27d). Note that $\mathrm{Re}\,\Phi_{32} > 0$ and $\mathrm{Re}\,\Phi_{31} > 0$ for $k \in \mathbb{R}_+$. We conclude from (4-4a) that $|(v_7^{(3)} - I)_{23}|$ and $|(v_7^{(3)} - I)_{13}|$ decay to zero as $t \to \infty$ faster than $|(v_7^{(3)} - I)_{12}|$ and $|(v_7^{(3)} - I)_{21}|$. Moreover, since $\mathrm{Re}\,\Phi_{21} = 0$ for $k \in \mathbb{R}_+$, (4-6) and (4-14) imply

$$|(v_7^{(3)} - I)_{21}| = \left|\frac{\delta_1^2}{\delta_3\delta_5} r_{1,r}^*\right| \le Ct^{-3/2}, \qquad |(v_7^{(3)} - I)_{12}| = \left|\frac{\delta_3\delta_5}{\delta_1^2} r_{1,r}\right| \le Ct^{-3/2},$$

while $|(v_7^{(3)} - I)_{22}|$ is even smaller. Thus,

$$\|v^{(3)} - I\|_{(L^1\cap L^\infty)(\Gamma_7^{(3)})} \le Ct^{-3/2}.$$

To estimate $\partial_x(v_7^{(3)})_{21}$, we use (4-6) and (4-19). This gives

$$|\partial_x(v_7^{(3)})_{21}| \le \left|\partial_x\left(\frac{\delta_3\delta_5}{\delta_1^2}\right) r_{1,r}\right| + \left|\frac{\delta_3\delta_5}{\delta_1^2}\partial_x r_{1,r}\right| \le Ct^{-3/2}.$$

The entries $\partial_x(v_7^{(3)})_{12}$ and $\partial_x(v_7^{(3)})_{22}$ are estimated in a similar way.

The matrix $v_8^{(3)}$ can be estimated in the same way as $v_7^{(3)}$, except that now we need to use (4-7) and to note that $\mathrm{Re}\,\Phi_{21} = 0$ for $k \in (k_0, \infty)$. This proves (4-27d). □

5. Local parametrix at k_0

In Section 4C, we arrived at an RH problem for $m^{(3)}$ with the property that the matrix $v^{(3)} - I$ decays to zero as $t \to \infty$ everywhere except near the three critical points k_0, ωk_0, $\omega^2 k_0$. This means that we only have to consider neighborhoods of these three points when computing the long-time asymptotics of $m^{(3)}$. In this section, we find a local solution m^{k_0} which approximates $m^{(3)}$ near k_0. The basic idea is that in the large t limit, the RH problem for $m^{(3)}$ near k_0 reduces to an RH problem on a cross which can be solved exactly in terms of parabolic cylinder functions [Its 1981; Deift and Zhou 1993].

Let $\epsilon \equiv \epsilon(\zeta) = k_0/2$. Let $D_\epsilon(k_0)$ denote an open disk of radius ϵ centered at k_0. Let $\mathcal{D} = D_\epsilon(k_0) \cup \omega D_\epsilon(k_0) \cup \omega^2 D_\epsilon(k_0)$. Let $\mathcal{X} = k_0 + X$, where X is the contour defined in (A-1). We will also use the notation $\mathcal{X}^\epsilon = \mathcal{X} \cap D_\epsilon(k_0)$ and $\mathcal{X}_j^\epsilon = (k_0 + X_j) \cap D_\epsilon(k_0)$, $j = 1, \dots, 4$, where X_j is defined in (A-1).

In order to relate $m^{(3)}$ to the solution m^X of Lemma A.2, we make a local change of variables for k near k_0 and introduce the new variable $z \equiv z(\zeta, k)$ by

$$z = 3^{1/4}\sqrt{2t}(k - k_0). \tag{5-1}$$

For each $\zeta \in \mathcal{I}$, the map $k \mapsto z$ is a biholomorphism from $D_\epsilon(k_0)$ onto the open disk of radius $3^{1/4}\sqrt{2t}\epsilon$ centered at the origin. Using that

$$\Phi_{21}(\zeta, k) = \Phi_{21}(\zeta, k_0) + i\sqrt{3}(k-k_0)^2,$$

where $\Phi_{21}(\zeta, k_0) = -i\sqrt{3}k_0^2$, we see that

$$t(\Phi_{21}(\zeta, k) - \Phi_{21}(\zeta, k_0)) = \frac{i}{2}z^2.$$

Equations (4-12) and (5-1) imply that, for $\zeta \in \mathcal{I}$ and $k \in D_\epsilon(k_0) \setminus [k_0, \infty)$,

$$\frac{\delta_3\delta_5}{\delta_1^2} = e^{2i\nu \ln_0(z)}(2\sqrt{3}t)^{-i\nu}e^{2\chi_1(\zeta,k)}\delta_3\delta_5 = e^{2i\nu \ln_0(z)}d_0(\zeta, t)d_1(\zeta, k),$$

where the functions $d_0(\zeta, t)$ and $d_1(\zeta, k)$ are defined for $\zeta \in \mathcal{I}$ and $k \in D_\epsilon(k_0) \setminus [k_0, \infty)$ by

$$d_0(\zeta, t) = (2\sqrt{3}t)^{-i\nu}e^{2\chi_1(\zeta,k_0)}\delta_3(\zeta, k_0)\delta_5(\zeta, k_0), \tag{5-2}$$

$$d_1(\zeta, k) = e^{2\chi_1(\zeta,k)-2\chi_1(\zeta,k_0)}\frac{\delta_3(\zeta, k)\delta_5(\zeta, k)}{\delta_3(\zeta, k_0)\delta_5(\zeta, k_0)}. \tag{5-3}$$

Defining $\tilde{m}$ for k near k_0 by

$$\tilde{m}(x, t, k) = m^{(3)}(x, t, k)Y(\zeta, t), \quad k \in D_\epsilon(k_0),$$

where

$$Y(\zeta, t) = \begin{pmatrix} d_0^{1/2}(\zeta, t)e^{-(t/2)\Phi_{21}(\zeta,k_0)} & 0 & 0 \\ 0 & d_0^{-1/2}(\zeta, t)e^{(t/2)\Phi_{21}(\zeta,k_0)} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we find that the jump $\tilde{v}(x, t, k)$ of $\tilde{m}$ across $\mathcal{X}^\epsilon$ is given by

$$\tilde{v}_1 = \begin{pmatrix} 1 & 0 & 0 \\ e^{-2i\nu \ln_0(z)}d_1^{-1}\hat{r}_{1,a}^*(k)e^{iz^2/2} & 1 & -\frac{\delta_1\delta_3}{\delta_5^2}d_0^{1/2}r_{2,a}^*(\omega k)e^{-t\Phi_{32}}e^{-(t/2)\Phi_{21}(\zeta,k_0)} \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_2 = \begin{pmatrix} 1 & e^{2i\nu \ln_0(z)}d_1 r_{1,a}(k)e^{-iz^2/2} & -\frac{\delta_3^2}{\delta_1\delta_5}d_0^{-1/2}\Omega_1(k)e^{-t\Phi_{31}}e^{(t/2)\Phi_{21}(\zeta,k_0)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_3 = \begin{pmatrix} 1 & 0 & 0 \\ -e^{-2i\nu \ln_0(z)}d_1^{-1}r_{1,a}^*(k)e^{iz^2/2} & 1 & \frac{\delta_1\delta_3}{\delta_5^2}d_0^{1/2}\Omega_2(k)e^{-t\Phi_{32}}e^{-(t/2)\Phi_{21}(\zeta,k_0)} \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_4 = \begin{pmatrix} 1 & -e^{2i\nu \ln_0(z)}d_1\hat{r}_{1,a}(k)e^{-iz^2/2} & \frac{\delta_3^2}{\delta_1\delta_5}d_0^{-1/2}r_{2,a}(\omega^2 k)e^{-t\Phi_{31}}e^{(t/2)\Phi_{21}(\zeta,k_0)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\tilde{v}_j$ denotes the restriction of $\tilde{v}$ to $\mathcal{X}_j^\epsilon$, $j=1,2,3,4$, and $\Omega_1(k)\equiv\Omega_1(x,t,k)$ and $\Omega_2(k)\equiv\Omega_2(x,t,k)$ are given by

$$\Omega_1(k)=r_{2,a}(\omega^2 k)+r_{1,a}(k)r_{2,a}^*(\omega k),\quad \Omega_2(k)=r_{2,a}^*(\omega k)+r_{1,a}^*(k)r_{2,a}(\omega^2 k).$$

Define $q\equiv q(\zeta)$ by

$$q=r_1(k_0).$$

For a fixed z, $r_{1,a}(k)\to q$, $\hat{r}_{1,a}^*(k)\to\bar{q}/(1-|q|^2)$, and $d_1(\zeta,k)\to 1$ as $t\to\infty$. This suggests that $\tilde{v}(x,t,k)$ tends to the jump matrix $v^X(x,t,z)$ defined in (A-2) for large t. In other words, the jumps of $m^{(3)}$ for k near k_0 approach those of the function $m^X Y^{-1}$ as $t\to\infty$. This suggests that we approximate $m^{(3)}$ in the neighborhood $D_\epsilon(k_0)$ of k_0 by the 3×3-matrix-valued function m^{k_0} defined by

$$m^{k_0}(x,t,k)=Y(\zeta,t)m^X(q(\zeta),z(\zeta,k))Y(\zeta,t)^{-1},\quad k\in D_\epsilon(k_0).\tag{5-4}$$

The prefactor $Y(\zeta,t)$ on the right-hand side of (5-4) is included so that $m^{k_0}\to I$ on $\partial D_\epsilon(k_0)$ as $t\to\infty$; this ensures that m^{k_0} is a good approximation of $m^{(3)}$ in $D_\epsilon(k_0)$ for large t.

Lemma 5.1. *The function $Y(\zeta,t)$ is uniformly bounded*:

$$\sup_{\zeta\in\mathcal{I}}\sup_{t\geq 2}|\partial_x^l Y(\zeta,t)^{\pm 1}|<C,\quad l=0,1.\tag{5-5}$$

Moreover, the functions $d_0(\zeta,t)$ and $d_1(\zeta,k)$ satisfy

$$|d_0(\zeta,t)|=e^{2\pi\nu},\qquad \zeta\in\mathcal{I},\ t\geq 2,\tag{5-6a}$$

$$|\partial_x d_0(\zeta,t)|\leq C\frac{\ln t}{t},\quad \zeta\in\mathcal{I},\ t\geq 2,\tag{5-6b}$$

and

$$|d_1(\zeta,k)-1|\leq C|k-k_0|(1+|\ln|k-k_0||),\quad \zeta\in\mathcal{I},\ k\in\mathcal{X}^\epsilon,\tag{5-7a}$$

$$|\partial_x d_1(\zeta,k)|\leq\frac{C}{t}|\ln|k-k_0||,\qquad \zeta\in\mathcal{I},\ k\in\mathcal{X}^\epsilon.\tag{5-7b}$$

Proof. The symmetry (4-15) implies

$$|\delta_3(\zeta,k_0)\delta_5(\zeta,k_0)|=|\delta_1(\zeta,\omega^2k_0)\delta_1(\zeta,\omega k_0)|=1,$$

and hence (5-6a) follows because

$$\operatorname{Re}\chi_1(\zeta,k_0)=\frac{1}{2\pi}\int_{k_0}^\infty \pi\, d\ln(1-|r_1(s)|^2)=-\frac{1}{2}\ln(1-|r_1(k_0)|^2)=\pi\nu.$$

Using (5-6a), we obtain

$$\begin{aligned}|\partial_x d_0(\zeta,t)|&=|d_0(\zeta,t)\partial_x\ln d_0(\zeta,t)|=e^{2\pi\nu}|\partial_x\ln d_0(\zeta,t)|\\&\leq C\big(|\ln t\,\partial_x\nu|+|\partial_x\chi_1(\zeta,k_0)|+|\partial_x\ln(\delta_3(\zeta,k_0)\delta_5(\zeta,k_0))|\big),\end{aligned}$$

and thus (5-6b) follows from (4-18) and the fact that $\partial_x=(1/t)\partial_\zeta$. Observing that δ_3 and δ_5 are analytic for $k\in\mathcal{X}^\epsilon$, (5-7a) follows from (4-16). Finally, we have

$$\partial_x d_1(\zeta,k)=d_1(\zeta,k)\,\partial_x\log d_1(\zeta,k).$$

Since

$$\partial_\zeta \log \frac{\delta_3(\zeta,k)\delta_5(\zeta,k)}{\delta_3(\zeta,k_0)\delta_5(\zeta,k_0)}$$

is analytic and $|d_1(\zeta,k)| \le C$ for $k \in \mathcal{X}^\epsilon$, it follows that

$$|\partial_x d_1(\zeta,k)| \le C\left(|\partial_x(\chi_1(\zeta,k)-\chi_1(\zeta,k_0))| + \frac{1}{t}\left|\partial_\zeta \log \frac{\delta_3(\zeta,k)\delta_5(\zeta,k)}{\delta_3(\zeta,k_0)\delta_5(\zeta,k_0)}\right|\right),$$

and so (5-7b) follows from (4-17). □

Lemma 5.2. *For each (x,t), the function $m^{k_0}(x,t,k)$ defined in (5-4) is an analytic and bounded function of $k \in D_\epsilon(k_0) \setminus \mathcal{X}^\epsilon$. Across $\mathcal{X}^\epsilon$, m^{k_0} obeys the jump condition $m_+^{k_0} = m_-^{k_0} v^{k_0}$, where the jump matrix v^{k_0} satisfies*

$$\begin{cases} \|\partial_x^l(v^{(3)}-v^{k_0})\|_{L^1(\mathcal{X}^\epsilon)} \le Ct^{-1}\ln t, \\ \|\partial_x^l(v^{(3)}-v^{k_0})\|_{L^\infty(\mathcal{X}^\epsilon)} \le Ct^{-1/2}\ln t, \end{cases} \qquad \zeta \in \mathcal{I},\ t \ge 2,\ l = 0, 1. \tag{5-8}$$

Furthermore, as $t \to \infty$,

$$\|\partial_x^l(m^{k_0}(x,t,\cdot)^{-1} - I)\|_{L^\infty(\partial D_\epsilon(k_0))} = O(t^{-1/2}), \quad l = 0, 1, \tag{5-9}$$

$$\frac{1}{2\pi i}\int_{\partial D_\epsilon(k_0)} (m^{k_0}(x,t,k)^{-1} - I)\,dk = -\frac{Y(\zeta,t)m_1^X(q(\zeta))Y(\zeta,t)^{-1}}{3^{1/4}\sqrt{2}\sqrt{t}} + O(t^{-1}), \tag{5-10}$$

uniformly for $\zeta \in \mathcal{I}$, and (5-10) can be differentiated with respect to x without increasing the error term.

Proof. We have

$$v^{(3)} - v^{k_0} = Y(\zeta,t)(\tilde{v} - v^X)Y(\zeta,t)^{-1}.$$

Thus, recalling (5-5), the bounds (5-8) follow if we can show that

$$\|\partial_x^l[\tilde{v}(x,t,\cdot) - v^X(x,t,z(\zeta,\cdot))]\|_{L^1(\mathcal{X}_j^\epsilon)} \le Ct^{-1}\ln t, \tag{5-11a}$$

$$\|\partial_x^l[\tilde{v}(x,t,\cdot) - v^X(x,t,z(\zeta,\cdot))]\|_{L^\infty(\mathcal{X}_j^\epsilon)} \le Ct^{-1/2}\ln t \tag{5-11b}$$

for $j = 1,\dots,4$ and $l = 0, 1$. We give the proof of (5-11) for $j = 1$; similar arguments apply when $j = 2, 3, 4$.

For $k \in \mathcal{X}_1^\epsilon$, only the (21) and (23) elements of the matrix $\tilde{v} - v^X$ are nonzero. Using (4-4a), (4-14), (5-6a), and the facts that $\Phi_{21}(\zeta,\omega k) = \Phi_{32}(\zeta,k)$ and $v_{23}^X(q(\zeta),z(\zeta,k)) = 0$ for $k \in \mathcal{X}_1^\epsilon$, $|(\tilde{v}-v^X)_{23}|$ can be estimated as

$$\begin{aligned} |(\tilde{v}-v^X)_{23}| &= \left|\frac{\delta_1\delta_3}{\delta_5^2} d_0^{1/2} r_{2,a}^*(\omega k)e^{-t\Phi_{32}}e^{-(t/2)\Phi_{21}(\zeta,k_0)}\right| \le |r_{2,a}^*(\omega k)|e^{-t\,\mathrm{Re}\,\Phi_{32}} \\ &\le Ce^{(t/4)|\mathrm{Re}\,\Phi_{21}(\zeta,\omega k)|}e^{-t\,\mathrm{Re}\,\Phi_{32}} = Ce^{-(3t/4)|\mathrm{Re}\,\Phi_{32}(\zeta,k)|}, \quad k \in \mathcal{X}_1^\epsilon. \end{aligned}$$

For $k = k_0 + ue^{\pi i/4}$ and $u \ge 0$, we have

$$\mathrm{Re}\,\Phi_{32}(\zeta, k_0 + ue^{\pi i/4}) = \tfrac{1}{2}(9k_0^2 + 6\sqrt{2}k_0 u + \sqrt{3}u^2) \ge c(k_0+u)^2. \tag{5-12}$$

Hence

$$\|(\tilde{v}-v^X)_{23}\|_{L^1(\mathcal{X}_1^\epsilon)} \le C\int_0^{k_0/2} e^{-ct(k_0+u)^2}\,du = C\int_{k_0}^{3k_0/2} e^{-ctv^2}\,dv \le Ce^{-ctk_0^2}$$

and

$$\|(\tilde{v}-v^X)_{23}\|_{L^\infty(\mathcal{X}_1^\epsilon)} \le C\sup_{u\ge 0} e^{-ct(k_0+u)^2} \le Ce^{-ctk_0^2}.$$

To estimate $\partial_x(\tilde{v}-v^X)_{23}$, we first note that

$$\partial_x(\tilde{v}-v^X)_{23} = a_1+a_2+a_3+a_4+a_5,$$

where

$$\begin{aligned}
a_1 &= -\partial_x\left(\frac{\delta_1\delta_3}{\delta_5^2}\right)d_0^{1/2} r_{2,a}^*(\omega k)e^{-t\Phi_{32}}e^{-(t/2)\Phi_{21}(\zeta,k_0)},\\
a_2 &= -\frac{\delta_1\delta_3}{\delta_5^2}\partial_x(d_0^{1/2}) r_{2,a}^*(\omega k)e^{-t\Phi_{32}}e^{-(t/2)\Phi_{21}(\zeta,k_0)},\\
a_3 &= -\frac{\delta_1\delta_3}{\delta_5^2}d_0^{1/2}\partial_x(r_{2,a}^*(\omega k))e^{-t\Phi_{32}}e^{-(t/2)\Phi_{21}(\zeta,k_0)},\\
a_4 &= -\frac{\delta_1\delta_3}{\delta_5^2}d_0^{1/2} r_{2,a}^*(\omega k)\partial_x(e^{-t\Phi_{32}})e^{-(t/2)\Phi_{21}(\zeta,k_0)},\\
a_5 &= -\frac{\delta_1\delta_3}{\delta_5^2}d_0^{1/2} r_{2,a}^*(\omega k)e^{-t\Phi_{32}}\partial_x(e^{-(t/2)\Phi_{21}(\zeta,k_0)}).
\end{aligned}$$

We claim that $\|a_j\|_{(L^1\cap L^\infty)(\mathcal{X}_1^\epsilon)} \le Ce^{-ctk_0^2}$ for $j=1,\dots,5$. These bounds follow from arguments which are similar to those given for $(\tilde{v}-v^X)_{23}$, but more estimates are required. For a_1, we note that $\partial_x(\delta_1\delta_3/\delta_5^2)$ has a pole at $k=k_0$ (see (4-19)) which is canceled by the zero of $r_{2,a}^*(\omega k)$ (see (4-4a)). For a_2 and a_3, we use (5-6b) and (4-4b), respectively. For a_4, we note that $\partial_x(t\Phi_{32}) = \partial_\zeta(\Phi_{32}) = (1-\omega)k$, and for a_5, we observe that $\partial_x(t\Phi_{21}(\zeta,k_0)) = \frac{1}{2}\partial_{k_0}\Phi_{21}(\zeta,k_0) = \omega(\omega-1)k_0$. Therefore, we arrive at

$$\|\partial_x(\tilde{v}-v^X)_{23}\|_{(L^1\cap L^\infty)(\mathcal{X}_1^\epsilon)} \le Ce^{-ctk_0^2}.$$

We next consider the (21)-entry of $\tilde{v}-v^X$. Since $q=r_1(k_0)$, from (4-4e) it follows $\hat{r}_{1,a}^*(k_0)=\hat{r}_1^*(k_0)=\bar{q}/(1-|q|^2)$. Furthermore,

$$e^{(t/4)|\mathrm{Re}\,\Phi_{21}(\zeta,k)|} = e^{(t/4)|\mathrm{Re}(\Phi_{21}(\zeta,k)-\Phi_{21}(\zeta,k_0))|} = e^{(1/4)|\mathrm{Re}(iz^2/2)|} \le e^{|z|^2/8}.$$

Thus $|(\tilde{v}-v^X)_{21}|$ can be estimated as

$$\begin{aligned}
|(\tilde{v}-v^X)_{21}| &= |e^{-2i\nu\ln_0(z)}d_1^{-1}\hat{r}_{1,a}^*(k)e^{iz^2/2} - \hat{r}_{1,a}^*(k_0)e^{-2i\nu\ln_0(z)}e^{iz^2/2}|\\
&= |e^{-2i\nu\ln_0(z)}|\,|(d_1^{-1}-1)\hat{r}_{1,a}^*(k) + (\hat{r}_{1,a}^*(k)-\hat{r}_{1,a}^*(k_0))|\,|e^{iz^2/2}|\\
&\le C(|d_1^{-1}-1|\,|\hat{r}_{1,a}^*(k)| + |\hat{r}_{1,a}^*(k)-\hat{r}_1^*(k_0)|)e^{-|z|^2/2}.\\
&\le C(|d_1^{-1}-1|+|k-k_0|)e^{(t/4)|\mathrm{Re}\,\Phi_{21}(\zeta,k)|}e^{-|z|^2/2}.\\
&\le C(|d_1^{-1}-1|+|k-k_0|)e^{-ct|k-k_0|^2}, \quad k\in\mathcal{X}_1^\epsilon,
\end{aligned}$$

where we have used (4-4e) and (4-4f). Utilizing (5-7a), this gives

$$|(\tilde{v}-v^X)_{21}| \le C|k-k_0|(1+|\ln|k-k_0||)e^{-ct|k-k_0|^2}, \quad k \in \mathcal{X}_1^\epsilon.$$

Hence

$$\|(\tilde{v}-v^X)_{21}\|_{L^1(\mathcal{X}_1^\epsilon)} \le C\int_0^\infty u(1+|\ln u|)e^{-ctu^2}\,du \le Ct^{-1}\ln t$$

and

$$\|(\tilde{v}-v^X)_{21}\|_{L^\infty(\mathcal{X}_1^\epsilon)} \le C\sup_{u\ge 0} u(1+|\ln u|)e^{-ctu^2} \le Ct^{-1/2}\ln t.$$

To analyze $\partial_x(\tilde{v}-v^X)_{21}$, we split it into three parts as follows:

$$\partial_x(\tilde{v}-v^X)_{21} = b_1+b_2+b_3,$$

where

$$\begin{aligned}
b_1 &= \partial_x(e^{-2i\nu\ln_0(z)})\big((d_1^{-1}-1)\hat{r}_{1,a}^*(k)+(\hat{r}_{1,a}^*(k)-\hat{r}_{1,a}^*(k_0))\big)e^{iz^2/2},\\
b_2 &= e^{-2i\nu\ln_0(z)}\partial_x\big((d_1^{-1}-1)\hat{r}_{1,a}^*(k)+(\hat{r}_{1,a}^*(k)-\hat{r}_{1,a}^*(k_0))\big)e^{iz^2/2},\\
b_3 &= e^{-2i\nu\ln_0(z)}\big((d_1^{-1}-1)\hat{r}_{1,a}^*(k)+(\hat{r}_{1,a}^*(k)-\hat{r}_{1,a}^*(k_0))\big)\partial_x e^{iz^2/2}.
\end{aligned}$$

For b_1, we use that $|\partial_x(e^{-2i\nu\ln_0(z)})| \le C/(t(k-k_0))$ for $k\in\mathcal{X}_1^\epsilon$, and thus, by (4-4),

$$\begin{aligned}
\|b_1\|_{L^1(\mathcal{X}_1^\epsilon)} &\le Ct^{-1}\int_0^\infty (1+\ln u)e^{-ctu^2}\,du \le Ct^{-3/2}\ln t,\\
\|b_1\|_{L^\infty(\mathcal{X}_1^\epsilon)} &\le Ct^{-1}\sup_{u\ge 0}(1+\ln u)e^{-ctu^2} \le Ct^{-1}\ln t.
\end{aligned}$$

The norms of b_2 and b_3 are estimated in a similar way. This completes the proof of (5-8).

The variable z goes to infinity as $t\to\infty$ if $k\in\partial D_\epsilon(k_0)$, because

$$|z| = 3^{1/4}\sqrt{2t}|k-k_0|.$$

Thus (A-3) yields

$$m^X(q(\zeta), z(\zeta,k)) = I + \frac{m_1^X(q(\zeta))}{3^{1/4}\sqrt{2t}(k-k_0)} + O(t^{-1}), \quad t\to\infty,$$

uniformly with respect to $k\in\partial D_\epsilon(k_0)$ and $\zeta\in\mathcal{I}$, and this asymptotic formula can be differentiated with respect to x without increasing the error term. Recalling the definition (5-4) of m^{k_0}, this gives

$$(m^{k_0})^{-1} - I = -\frac{Y(\zeta,t)m_1^X(q(\zeta))Y(\zeta,t)^{-1}}{3^{1/4}\sqrt{2t}(k-k_0)} + O(t^{-1}), \quad t\to\infty, \tag{5-13}$$

uniformly for $k\in\partial D_\epsilon(k_0)$ and $\zeta\in\mathcal{I}$. In view of (5-5), the asymptotics (5-13) can be differentiated with respect to x. This proves (5-9). Equation (5-10) follows from (5-13) and Cauchy’s formula. □

6. A small-norm RH problem

We use the symmetry

$$m^{k_0}(x,t,k) = \mathcal{A}m^{k_0}(x,t,\omega k)\mathcal{A}^{-1}$$

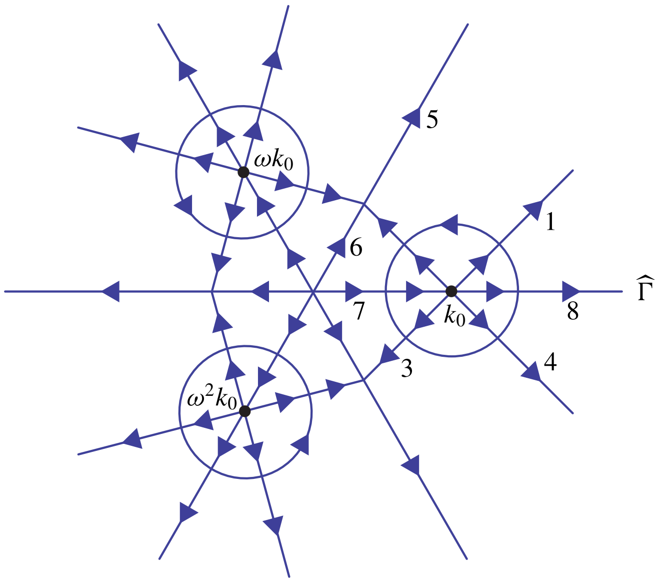

Figure 12. The contour $\widehat{\Gamma} = \Gamma^{(3)} \cup \partial\mathcal{D}$ in the complex k-plane.

to extend the domain of definition of m^{k_0} from $D_\epsilon(k_0)$ to $\mathcal{D}$, where we recall that $\mathcal{D} = D_\epsilon(k_0) \cup \omega D_\epsilon(k_0) \cup \omega^2 D_\epsilon(k_0)$. We will show that the solution $\hat{m}(x,t,k)$ defined by

$$\hat{m} = \begin{cases} m^{(3)}(m^{k_0})^{-1}, & k \in \mathcal{D}, \\ m^{(3)}, & \text{elsewhere}, \end{cases}$$

is small for large t. Let $\widehat{\Gamma} = \Gamma^{(3)} \cup \partial\mathcal{D}$ be the contour displayed in Figure 12 and define the jump matrix $\hat{v}$ by

$$\hat{v} = \begin{cases} v^{(3)}, & k \in \widehat{\Gamma} \setminus \overline{\mathcal{D}}, \\ (m^{k_0})^{-1}, & k \in \partial\mathcal{D}, \\ m_-^{k_0} v^{(3)} (m_+^{k_0})^{-1}, & k \in \widehat{\Gamma} \cap \mathcal{D}. \end{cases}$$

The function $\hat{m}$ satisfies the following RH problem.

RH problem 6.1 (RH problem for $\hat{m}$). *Find a 3×3-matrix-valued function $\hat{m}(x,t,\cdot) \in I + \dot{E}^3(\mathbb{C} \setminus \widehat{\Gamma})$ such that $\hat{m}_+(x,t,k) = \hat{m}_-(x,t,k)\hat{v}(x,t,k)$ for a.e. $k \in \widehat{\Gamma}$.*

Let $\widehat{\mathcal{X}}^\epsilon$ denote the union of the cross $\mathcal{X}^\epsilon$ and its images under the maps $k \mapsto \omega k$ and $k \mapsto \omega^2 k$, i.e., $\widehat{\mathcal{X}}^\epsilon = \mathcal{X}^\epsilon \cup \omega\mathcal{X}^\epsilon \cup \omega^2\mathcal{X}^\epsilon$. Define the contour Γ' by

$$\Gamma' = \widehat{\Gamma} \setminus (\Gamma \cup \widehat{\mathcal{X}}^\epsilon \cup \partial\mathcal{D}).$$

Lemma 6.2. *Let $\hat{w} = \hat{v} - I$. The following estimates hold uniformly for $t \geq 2$ and $\zeta \in \mathcal{I}$:*

$$\|(1+|\cdot|)\partial_x^l \hat{w}\|_{(L^1 \cap L^\infty)(\Gamma)} \leq \frac{C}{k_0 t}, \tag{6-1a}$$

$$\|(1+|\cdot|)\partial_x^l \hat{w}\|_{(L^1 \cap L^\infty)(\Gamma')} \leq C e^{-ct}, \tag{6-1b}$$

$$\|\partial_x^l \hat{w}\|_{(L^1\cap L^\infty)(\partial\mathcal{D})} \le Ct^{-1/2}, \tag{6-1c}$$

$$\|\partial_x^l \hat{w}\|_{L^1(\widehat{\mathcal{X}}^\epsilon)} \le Ct^{-1}\ln t, \tag{6-1d}$$

$$\|\partial_x^l \hat{w}\|_{L^\infty(\widehat{\mathcal{X}}^\epsilon)} \le Ct^{-1/2}\ln t, \tag{6-1e}$$

with $l = 0, 1$.

Proof. Using that $\partial_\zeta^l m_\pm^{k_0}$ and its inverse are uniformly bounded for $k \in \widehat{\Gamma} \cap \mathcal{D}$ and $l = 0, 1$, the estimate (6-1a) follows from Lemma 4.7.

The contour Γ' consists of the set $\left(\bigcup_{j=1}^4 \Gamma_j^{(3)}\right) \setminus \bar{\mathcal{D}}$ and the images of this set under the rotations $k \mapsto \omega k$ and $k \mapsto \omega^2 k$. We estimate the L^1 and L^∞ norms of $(1+|\cdot|)\partial_x^l \hat{w}$ on $\Gamma_j^{(3)} \setminus \bar{\mathcal{D}}$ for $j = 1$; similar arguments apply when $j = 2, 3, 4$, and (6-1b) then follows by symmetry. We parametrize $\Gamma_1^{(3)} \setminus \bar{\mathcal{D}}$ by $k = k_0 + ue^{\pi i/4}$, $u > k_0/2$. Only the (21) and (23) elements of $\hat{w} = v_1^{(3)} - I$ are nonzero. Using (4-4a), (4-14), and (5-12), the (23)-entry can be estimated as

$$|\hat{w}_{23}(x, t, k_0 + ue^{\pi i/4})| \le C|r_{2,a}^*(\omega k)|e^{-t\Phi_{32}} \le Ce^{-(3t/4)\Phi_{32}} \le Ce^{-ct(k_0+u)^2}.$$

The analysis of $|\partial_x \hat{w}_{23}|$ is similar. Using (4-4e), (4-14), and the identity

$$\operatorname{Re}\Phi_{21}(\zeta, k_0 + ue^{\pi i/4}) = -\sqrt{3}u^2, \quad u \ge 0, \tag{6-2}$$

the (21)-entry can be estimated as

$$|\hat{w}_{21}(x, t, k_0 + ue^{\pi i/4})| \le C|\hat{r}_{1,a}^*|e^{t\Phi_{21}} \le Ce^{ct\Phi_{21}} \le Ce^{-ctu^2}, \quad u \ge 0.$$

Using in addition (4-4f) and (4-19), we conclude that $|\partial_x \hat{w}_{21}(x, t, k_0 + ue^{\pi i/4})| \le Ce^{-ctu^2}$. Hence

$$|\partial_x^l \hat{w}(x, t, k_0 + ue^{\pi i/4})| \le Ce^{-ctu^2}, \quad u > k_0/2,\ l = 0, 1.$$

It follows that the L^1 and L^∞ norms of $(1+|\cdot|)\partial_x^l \hat{w},\ l = 0, 1$, are $O(e^{-ct})$ as $t \to \infty$ on $\Gamma_1^{(3)} \setminus \bar{\mathcal{D}}$. This proves (6-1b).

The estimates in (6-1c) are immediate from (5-9).

For $k \in \mathcal{X}^\epsilon$, we have $\hat{w} = m_-^{k_0}(v^{(3)} - v^{k_0})(m_+^{k_0})^{-1}$, so (6-1d) and (6-1e) follow from (5-8) combined with the fact that $\partial_\zeta^l m_\pm^{k_0}$ and its inverse are uniformly bounded for $k \in \widehat{\Gamma} \cap \mathcal{D}$ and $l = 0, 1$. □

For a function h defined on $\widehat{\Gamma}$, the Cauchy transform $\widehat{\mathcal{C}}h$ is defined by

$$(\widehat{\mathcal{C}}h)(z) = \frac{1}{2\pi i}\int_{\widehat{\Gamma}} \frac{h(z')dz'}{z' - z}, \quad z \in \mathbb{C} \setminus \widehat{\Gamma}.$$

If $h \in \dot{L}^3(\widehat{\Gamma})$, then $\widehat{\mathcal{C}}h \in \dot{E}^3(\mathbb{C} \setminus \widehat{\Gamma})$, and the left and right nontangential boundary values of $\widehat{\mathcal{C}}h$, which we denote by $\widehat{\mathcal{C}}_+h$ and $\widehat{\mathcal{C}}_-h$ respectively, exist a.e. on $\widehat{\Gamma}$ and belong to $\dot{L}^3(\widehat{\Gamma})$; furthermore, $\widehat{\mathcal{C}}_\pm \in \mathcal{B}(\dot{L}^3(\widehat{\Gamma}))$ and $\widehat{\mathcal{C}}_+ - \widehat{\mathcal{C}}_- = I$, where $\mathcal{B}(\dot{L}^3(\widehat{\Gamma}))$ denotes the space of bounded linear operators on $\dot{L}^3(\widehat{\Gamma})$; see [Lenells 2018, Theorems 4.1 and 4.2].

The estimates in Lemma 6.2 show that

$$\begin{cases} \|(1+|\cdot|)\partial_x^l \hat{w}\|_{L^1(\widehat{\Gamma})} \le Ct^{-1/2}, \\ \|(1+|\cdot|)\partial_x^l \hat{w}\|_{L^\infty(\widehat{\Gamma})} \le Ct^{-1/2}\ln t, \end{cases} \quad t \ge 2,\ \zeta \in \mathcal{I},\ l = 0, 1, \tag{6-3}$$

and hence, employing the general identity $\|f\|_{L^p} \le \|f\|_{L^1}^{1/p}\|f\|_{L^\infty}^{(p-1)/p}$,

$$\|(1+|\cdot|)\partial_x^l \hat{w}\|_{L^p(\widehat{\Gamma})} \le Ct^{-1/2}(\ln t)^{(p-1)/p}, \quad t \ge 2,\ \zeta \in \mathcal{I},\ l = 0, 1, \tag{6-4}$$

for each $1 \le p \le \infty$. The estimates (6-4) imply that $\hat{w} \in \dot{L}^3(\widehat{\Gamma}) \cap L^\infty(\widehat{\Gamma})$. We define $\widehat{\mathcal{C}}_{\hat{w}} = \widehat{\mathcal{C}}_{\hat{w}(x,t,\cdot)} : \dot{L}^3(\widehat{\Gamma}) + L^\infty(\widehat{\Gamma}) \to \dot{L}^3(\widehat{\Gamma})$ by $\widehat{\mathcal{C}}_{\hat{w}}h := \widehat{\mathcal{C}}_-(h\hat{w})$.

Lemma 6.3. *There exists a $T > 0$ such that $I - \widehat{\mathcal{C}}_{\hat{w}(x,t,\cdot)} \in \mathcal{B}(\dot{L}^3(\widehat{\Gamma}))$ is invertible whenever $t \ge T$ and $\zeta \in \mathcal{I}$.*

Proof. Let $K := \|\widehat{\mathcal{C}}_-\|_{\mathcal{B}(\dot{L}^3(\widehat{\Gamma}))}$. For each $h \in \dot{L}^3(\widehat{\Gamma})$, we have $\|\widehat{\mathcal{C}}_{\hat{w}}h\|_{\dot{L}^3(\widehat{\Gamma})} \le K\|\hat{w}\|_{L^\infty(\widehat{\Gamma})}\|h\|_{\dot{L}^3(\widehat{\Gamma})}$, and thus $\|\widehat{\mathcal{C}}_{\hat{w}}\|_{\mathcal{B}(\dot{L}^3(\widehat{\Gamma}))} \le K\|\hat{w}\|_{L^\infty(\widehat{\Gamma})}$. By (6-3), there exists a $T > 0$ such that $\|\hat{w}\|_{L^\infty(\widehat{\Gamma})} < K^{-1}$ for $t \ge T$. □

In view of Lemma 6.3, we may define $\hat{\mu}(x, t, k)$ for $k \in \widehat{\Gamma}$, $t \ge T$, and $\zeta = x/t \in \mathcal{I}$ by

$$\hat{\mu} = I + (I - \widehat{\mathcal{C}}_{\hat{w}})^{-1}\widehat{\mathcal{C}}_{\hat{w}}I \in I + \dot{L}^3(\widehat{\Gamma}). \tag{6-5}$$

Lemma 6.4. *For $t \ge T$ and $\zeta \in \mathcal{I}$, there exists a unique solution $\hat{m} \in I + \dot{E}^3(\mathbb{C} \setminus \widehat{\Gamma})$ of RH problem 3.1. This solution is given by*

$$\hat{m}(x, t, k) = I + \widehat{\mathcal{C}}(\hat{\mu}\hat{w}) = I + \frac{1}{2\pi i}\int_{\widehat{\Gamma}} \hat{\mu}(x, t, s)\hat{w}(x, t, s)\,\frac{ds}{s-k}. \tag{6-6}$$

Proof. Since $\hat{w} \in \dot{L}^3(\widehat{\Gamma}) \cap L^\infty(\widehat{\Gamma})$, this follows from [Lenells 2018, Proposition 5.8]. □

Lemma 6.5. *Let $1 < p < \infty$. For all sufficiently large t, we have*

$$\|\partial_x^l(\hat{\mu} - I)\|_{L^p(\widehat{\Gamma})} \le Ct^{-1/2}(\ln t)^{(p-1)/p}, \quad l = 0, 1,\ \zeta \in \mathcal{I}.$$

Proof. Let $K_p := \|\widehat{\mathcal{C}}_-\|_{\mathcal{B}(L^p(\widehat{\Gamma}))} < \infty$ and assume t is so large that $\|\hat{w}\|_{L^\infty(\widehat{\Gamma})} < K_p^{-1}$. Standard estimates using the Neumann series show that

$$\|\hat{\mu} - I\|_{L^p(\widehat{\Gamma})} \le \sum_{j=1}^\infty \|\widehat{\mathcal{C}}_{\hat{w}}\|_{\mathcal{B}(L^p(\widehat{\Gamma}))}^{j-1}\|\widehat{\mathcal{C}}_{\hat{w}}I\|_{L^p(\widehat{\Gamma})} \le \sum_{j=1}^\infty K_p^j\|\hat{w}\|_{L^\infty(\widehat{\Gamma})}^{j-1}\|\hat{w}\|_{L^p(\widehat{\Gamma})} = \frac{K_p\|\hat{w}\|_{L^p(\widehat{\Gamma})}}{1 - K_p\|\hat{w}\|_{L^\infty(\widehat{\Gamma})}}.$$

The claim for $l = 0$ now follows from (6-3) and (6-4). Using that

$$\partial_x(\hat{\mu} - I) = \partial_x\sum_{j=1}^\infty(\widehat{\mathcal{C}}_{\hat{w}})^j I = \sum_{j=1}^\infty[(\partial_x\widehat{\mathcal{C}}_{\hat{w}})\widehat{\mathcal{C}}_{\hat{w}}\cdots\widehat{\mathcal{C}}_{\hat{w}} + \cdots + \widehat{\mathcal{C}}_{\hat{w}}\cdots\widehat{\mathcal{C}}_{\hat{w}}(\partial_x\widehat{\mathcal{C}}_{\hat{w}})]I,$$

we find

$$\begin{aligned}\|\partial_x(\hat{\mu}-I)\|_{L^p(\widehat{\Gamma})} &\le \sum_{j=2}^\infty(j-1)\|\widehat{\mathcal{C}}_{\hat{w}}\|_{\mathcal{B}(L^p(\widehat{\Gamma}))}^{j-2}\|\partial_x\widehat{\mathcal{C}}_{\hat{w}}\|_{\mathcal{B}(L^p(\widehat{\Gamma}))}\|\widehat{\mathcal{C}}_{\hat{w}}I\|_{L^p(\widehat{\Gamma})} + \sum_{j=1}^\infty\|\widehat{\mathcal{C}}_{\hat{w}}\|_{\mathcal{B}(L^p(\widehat{\Gamma}))}^{j-1}\|\partial_x\widehat{\mathcal{C}}_{\hat{w}}I\|_{L^p(\widehat{\Gamma})}\\ &\le C\sum_{j=2}^\infty jK_p^{j-2}\|\hat{w}\|_{L^\infty(\widehat{\Gamma})}^{j-2}\|\partial_x\hat{w}\|_{L^\infty(\widehat{\Gamma})}\|\hat{w}\|_{L^p(\widehat{\Gamma})} + \sum_{j=1}^\infty K_p^j\|\hat{w}\|_{L^\infty(\widehat{\Gamma})}^{j-1}\|\partial_x\hat{w}\|_{L^p(\widehat{\Gamma})}\\ &\le C\frac{\|\partial_x\hat{w}\|_{L^\infty(\widehat{\Gamma})}\|\hat{w}\|_{L^p(\widehat{\Gamma})} + \|\partial_x\hat{w}\|_{L^p(\widehat{\Gamma})}}{1 - K_p\|\hat{w}\|_{L^\infty(\widehat{\Gamma})}}\end{aligned}$$

and the claim for $l = 1$ follows from another application of (6-3) and (6-4). □

6A. ***Asymptotics of $\hat{m}$.*** The following nontangential limit exists as $k \to \infty$:

$$L(x,t) := \angle\lim_{k\to\infty} k(\hat{m}(x,t,k) - I) = -\frac{1}{2\pi i}\int_{\widehat{\Gamma}} \hat{\mu}(x,t,k)\hat{w}(x,t,k)\,dk.$$

Lemma 6.6. *As $t \to \infty$,*

$$L(x,t) = -\frac{1}{2\pi i}\int_{\partial\mathcal{D}} \hat{w}(x,t,k)\,dk + O(t^{-1}\ln t) \tag{6-7}$$

and (6-7) can be differentiated termwise with respect to x without increasing the error term.

Proof. Since

$$L(x,t) = -\frac{1}{2\pi i}\int_{\partial\mathcal{D}} \hat{w}(x,t,k)\,dk + L_1(x,t) + L_2(x,t),$$

where

$$L_1(x,t) = -\frac{1}{2\pi i}\int_{\widehat{\Gamma}\setminus\partial\mathcal{D}} \hat{w}(x,t,k)\,dk, \quad L_2(x,t) = -\frac{1}{2\pi i}\int_{\widehat{\Gamma}} (\hat{\mu}(x,t,k) - I)\hat{w}(x,t,k)\,dk,$$

the lemma follows from Lemmas 6.2 and 6.5 and straightforward estimates. □

We infer from (5-10) that the function F defined by

$$F(\zeta,t) = -\frac{1}{2\pi i}\int_{\partial D_\epsilon(k_0)} \hat{w}(x,t,k)\,dk = -\frac{1}{2\pi i}\int_{\partial D_\epsilon(k_0)} ((m^{k_0})^{-1} - I)\,dk$$

satisfies

$$F(\zeta,t) = \frac{Y(\zeta,t)m_1^X(q(\zeta))Y(\zeta,t)^{-1}}{3^{1/4}\sqrt{2}\sqrt{t}} + O(t^{-1}\ln t) \quad \text{as } t\to\infty.$$

The symmetry properties of $\hat{v}$ imply that both $\mathcal{A}\hat{m}(x,t,\omega k)\mathcal{A}^{-1}$ and $\hat{m}(x,t,k)$ satisfy RH problem 6.1; by uniqueness they must be equal, i.e.,

$$\hat{m}(x,t,k) = \mathcal{A}\hat{m}(x,t,\omega k)\mathcal{A}^{-1}, \quad k \in \mathbb{C}\setminus\widehat{\Gamma}.$$

It follows that $\hat{\mu}$ and $\hat{w}$ also obey this symmetry. Using this in (6-7), we find that the leading contribution from $\partial\mathcal{D}$ to the right-hand side of (6-7) is

$$\begin{aligned}-\frac{1}{2\pi i}\int_{\partial\mathcal{D}}\hat{w}(x,t,k)\,dk &= -\frac{1}{2\pi i}\left(\int_{\partial D_\epsilon(k_0)} + \int_{\omega\partial D_\epsilon(k_0)} + \int_{\omega^2\partial D_\epsilon(k_0)}\right)\hat{w}(x,t,k)\,dk \\ &= F(\zeta,t) + \omega\mathcal{A}^{-1}F(\zeta,t)\mathcal{A} + \omega^2\mathcal{A}^{-2}F(\zeta,t)\mathcal{A}^2.\end{aligned}$$

Therefore, (6-7) implies that

$$\begin{aligned}&\partial_x^l \lim_{k\to\infty} k(\hat{m}(x,t,k) - I) \\ &\quad = \partial_x^l\left(\frac{\sum_{j=0}^2 \omega^j\mathcal{A}^{-j}Y(\zeta,t)m_1^X(q(\zeta))Y(\zeta,t)^{-1}\mathcal{A}^j}{3^{1/4}\sqrt{2}\sqrt{t}}\right) + O(t^{-1}\ln t), \quad t\to\infty,\ l = 0,1,\end{aligned} \tag{6-8}$$

uniformly for $\zeta \in \mathcal{I}$.

7. Asymptotics of $u(x, t)$

Recall from the discussion in Section 3 (see Proposition 3.4 and Lemma 3.5) that

$$u(x,t) = -\frac{3}{2}\frac{\partial}{\partial x}\big(\lim_{k\to\infty} k(n_3(x,t,k)-1)\big),$$

where $n = (\omega, \omega^2, 1)m$. Taking the transformations of Section 4 into account, we can write

$$m = \hat{m} H^{-1}\Delta^{-1}G^{-1}$$

for all $k \in \mathbb{C}\setminus\overline{\mathcal{D}}$, where G, Δ, H are defined in (4-8), (4-20), and (4-23), respectively. It follows that

$$u(x,t) = -\frac{3}{2}\frac{\partial}{\partial x}\overset{\angle}{\lim_{k\to\infty}} k(\hat{n}_3(x,t,k)-1) - \frac{3}{2}\frac{d}{dx}\overset{\angle}{\lim_{k\to\infty}} k\left(\frac{\delta_5(\zeta,k)}{\delta_3(\zeta,k)}-1\right), \tag{7-1}$$

where $\hat{n} = (\omega, \omega^2, 1)\hat{m}$. Thus, utilizing (6-8) and the fact that $\overline{\Gamma(i\nu)} = \Gamma(-i\nu)$,

$$\begin{aligned}
u(x,t) &= -\frac{3}{2}\frac{d}{dx}\left((\omega\ \ \omega^2\ \ 1)\frac{\sum_{j=0}^{2}\omega^j\mathcal{A}^{-j}Y(\zeta,t)m_1^X(q(\zeta))Y(\zeta,t)^{-1}\mathcal{A}^j}{3^{1/4}\sqrt{2}\sqrt{t}}\right)_3 + O(t^{-1}\ln t) \\
&= -\frac{3}{2}\frac{d}{dx}\left(\frac{\omega^2 d_0^{-1}e^{t\Phi_{21}(\zeta,k_0)}\beta_{21} + \omega d_0 e^{-t\Phi_{21}(\zeta,k_0)}\beta_{12}}{3^{1/4}\sqrt{2}\sqrt{t}}\right) + O(t^{-1}\ln t) \\
&= -\frac{3\times 2}{2\times 3^{1/4}\sqrt{2t}}\frac{d}{dx}\operatorname{Re}(\omega^2 d_0^{-1}e^{t\Phi_{21}(\zeta,k_0)}\beta_{21}) + O(t^{-1}\ln t), \quad t\to\infty.
\end{aligned}$$

Using the identities

$$\begin{aligned}
|\Gamma(i\nu)| &= \frac{\sqrt{2\pi}}{\sqrt{\nu}\sqrt{e^{\pi\nu}-e^{-\pi\nu}}} = \frac{\sqrt{2\pi}}{\sqrt{\nu}e^{\pi\nu/2}|q|}, \\
\delta_3^{-1}(\zeta,k_0)\delta_5^{-1}(\zeta,k_0) &= \exp\left[i\nu\log(3k_0^2) + \frac{1}{\pi i}\int_{k_0}^{\infty}\log|\omega k_0 - s|\,d\ln(1-|r_1(s)|^2)\right],
\end{aligned}$$

we conclude that, as $t\to\infty$,

$$\begin{aligned}
&u(x,t) \\
&= -\frac{3^{3/4}}{\sqrt{2t}}\frac{d}{dx}\operatorname{Re}\Bigg\{\sqrt{\nu}\exp\Bigg[\frac{4\pi i}{3}+i\nu\ln(6\sqrt{3}tk_0^2)-i\sqrt{3}k_0^2t \\
&\qquad -\frac{1}{\pi i}\int_{k_0}^{\infty}\ln\frac{|s-k_0|}{|s-\omega k_0|}\,d\ln(1-|r_1(s)|^2)+i\left(\frac{\pi}{4}-\arg q-\arg\Gamma(i\nu)\right)\Bigg]\Bigg\}+O(t^{-1}\ln t) \\
&= -\frac{3^{3/4}\sqrt{\nu}}{\sqrt{2t}}\frac{d}{dx}\cos\Bigg(\frac{19\pi}{12}+\nu\ln(6\sqrt{3}tk_0^2)-\sqrt{3}k_0^2t-\arg q \\
&\qquad -\arg\Gamma(i\nu)+\frac{1}{\pi}\int_{k_0}^{\infty}\ln\frac{|s-k_0|}{|s-\omega k_0|}\,d\ln(1-|r_1(s)|^2)\Bigg)+O(t^{-1}\ln t) \\
&= -\frac{3^{5/4}k_0\sqrt{\nu}}{\sqrt{2t}}\sin\Bigg(\frac{19\pi}{12}+\nu\ln(6\sqrt{3}tk_0^2)-\sqrt{3}k_0^2t-\arg q \\
&\qquad -\arg\Gamma(i\nu)+\frac{1}{\pi}\int_{k_0}^{\infty}\ln\frac{|s-k_0|}{|s-\omega k_0|}\,d\ln(1-|r_1(s)|^2)\Bigg)+O(t^{-1}\ln t)
\end{aligned}$$

uniformly for $\zeta\in\mathcal{I}$. This proves (2-9) and completes the proof of Theorem 2.4.

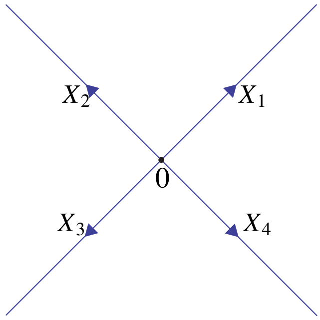

Figure 13. The contour $X = X_1 \cup X_2 \cup X_3 \cup X_4$ defined in (A-1).

Appendix: Exact solution on a cross

Let $X = X_1 \cup \cdots \cup X_4 \subset \mathbb{C}$ be the cross defined by

$$\begin{aligned} X_1 &= \{se^{i\pi/4} \mid 0 \le s < \infty\}, & X_2 &= \{se^{3i\pi/4} \mid 0 \le s < \infty\}, \\ X_3 &= \{se^{-3i\pi/4} \mid 0 \le s < \infty\}, & X_4 &= \{se^{-i\pi/4} \mid 0 \le s < \infty\}, \end{aligned} \tag{A-1}$$

and oriented away from the origin; see Figure 13. Let $\mathbb{D} \subset \mathbb{C}$ denote the open unit disk and define the function $\nu : \mathbb{D} \to (0, \infty)$ by $\nu(q) = -\frac{1}{2\pi} \ln(1 - |q|^2)$. We consider the following family of RH problems parametrized by $q \in \mathbb{D}$.

RH problem A.1 (RH problem for m^X). *Find a 3×3-matrix-valued function $m^X(q, z)$ with the following properties*:

(a) $m^X(q, \cdot) : \mathbb{C} \setminus X \to \mathbb{C}^{3\times 3}$ *is analytic.*

(b) *The limits of $m^X(q, z)$ as z approaches $X \setminus \{0\}$ from the left and right exist, are continuous on $X \setminus \{0\}$, and are related by*

$$m^X_+(q, z) = m^X_-(q, z) v^X(q, z), \quad k \in X \setminus \{0\},$$

where the jump matrix $v^X(q, z)$ is defined by

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 \\ \frac{\bar{q}}{1-|q|^2} z^{-2i\nu(q)} e^{iz^2/2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } z \in X_1, && \begin{pmatrix} 1 & q z^{2i\nu(q)} e^{-iz^2/2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } z \in X_2, \\ &\begin{pmatrix} 1 & 0 & 0 \\ -\bar{q} z^{-2i\nu(q)} e^{iz^2/2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } z \in X_3, && \begin{pmatrix} 1 & \frac{-q}{1-|q|^2} z^{2i\nu(q)} e^{-iz^2/2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } z \in X_4, \end{aligned} \tag{A-2}$$

with the branch cut running along the positive real axis, i.e., $z^{2i\nu(q)} = e^{2i\nu(q) \ln_0(z)}$.

(c) $m^X(q, z) = I + O(z^{-1})$ *as* $z \to \infty$.

(d) $m^X(q, z) = O(1)$ *as* $z \to 0$.

The proof of the following lemma is standard and relies on deriving an explicit formula for the solution m^X in terms of parabolic cylinder functions [Its 1981].

Lemma A.2 (the solution m^X). *The RH problem A.1 has a unique solution $m^X(q,z)$ for each $q \in \mathbb{D}$. This solution satisfies*

$$m^X(q,z) = I + \frac{m_1^X(q)}{z} + O\left(\frac{1}{z^2}\right), \quad z \to \infty,\ q \in \mathbb{D}, \tag{A-3}$$

where the error term is uniform with respect to $\arg z \in [0, 2\pi]$ *and q in compact subsets of $\mathbb{D}$, and the function $m_1^X(q)$ is defined by*

$$m_1^X(q) = \begin{pmatrix} 0 & \beta_{12} & 0 \\ \beta_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad q \in \mathbb{D}, \tag{A-4}$$

where β_{12} and β_{21} are defined by

$$\beta_{12} = \frac{\sqrt{2\pi}\, e^{-\pi i/4} e^{-5\pi\nu/2}}{\bar{q}\,\Gamma(-i\nu)}, \quad \beta_{21} = \frac{\sqrt{2\pi}\, e^{\pi i/4} e^{3\pi\nu/2}}{q\,\Gamma(i\nu)}, \quad q \in \mathbb{D}.$$

Moreover, for each compact subset K of $\mathbb{D}$,

$$\sup_{q\in K} \sup_{z \in \mathbb{C}\setminus X} |\partial_q^l m^X(q,z)| < \infty, \quad l = 0, 1.$$

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QUANTITATIVE OBATA'S THEOREM

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We prove a quantitative version of Obata's theorem involving the shape of functions with null mean value when compared with the cosine of distance functions from single points. The deficit between the diameters of the manifold and of the corresponding sphere is bounded likewise. These results are obtained in the general framework of (possibly nonsmooth) metric measure spaces with curvature-dimension conditions through a quantitative analysis of the transport-ray decompositions obtained by the localization method.

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1. Introduction

One of the core topics in geometric analysis is the deep connection between the geometry of a domain (in a possibly curved space) and spectral properties of the Laplacian defined on it.

The present paper focuses on the first eigenvalue λ_1 of the Laplacian (with Neumann boundary conditions, in case the domain has nonempty boundary). Since the Poincaré(–Wirtinger) inequality plays an important role in analysis and since a lower bound of the first eigenvalue gives an upper bound of the constant in the Poincaré(–Wirtinger) inequality, it is extremely useful to have a good lower estimate of λ_1.

For domains in the Euclidean space, classical estimates of the first eigenvalue of the Laplacian (under Dirichlet or Neumann boundary conditions) date back to Lord Rayleigh [1877], Faber [1923], Krahn [1925], Pólya and Szegő [1951], and Payne and Weinberger [1960], among others. For curved spaces, two major results are due to Lichnerowicz [1958] and Obata [1962]:

Theorem 1.1. *Let (M, g) be an N-dimensional Riemannian manifold with $\mathrm{Ric}_g \geq (N-1)g$. Then $\lambda_1 \geq N$ (Lichnerowicz spectral gap [1958]).*

Moreover, $\lambda_1 = N$ if and only if (M, g) is isometric to the unit sphere $\mathbb{S}^N$ (Obata's theorem [1962]).

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Remark 1.2. On $\mathbb{S}^N$, the first eigenvalue $\lambda_1 = N$ has multiplicity $N+1$. The corresponding eigenspace is spanned by the restriction to $\mathbb{S}^N$ of affine functions of $\mathbb{R}^{N+1}$ (i.e., an L^2-orthogonal basis is composed of the standard coordinate functions $\{x^1, x^2, \dots, x^{N+1}\}$ of $\mathbb{R}^{N+1}$). Equivalently, a function $u : \mathbb{S}^N \to \mathbb{R}$ is a first eigenfunction normalized as $\|u\|_{L^2(\mathbb{S}^N)} = 1$ if and only if there exists $P \in \mathbb{S}^N$ such that $u = \sqrt{N+1}\cos \mathsf{d}_P$, where we denote by d_P the Riemannian distance from the point P.

Our main result is a *quantitative spectral gap* involving the shape of the eigenfunctions (or, more generally, of functions with almost optimal Rayleigh quotient), when compared with the eigenfunctions of the model space $\mathbb{S}^N$ (as in Remark 1.2). In detail, we show that if $\mathrm{Ric}_g \geq (N-1)g$ and $u : M \to \mathbb{R}$ is a first eigenfunction with $\|u\|_{L^2(M)} = 1$, then there exists $P \in M$ such that

$$\|u - \sqrt{N+1}\cos \mathsf{d}_P\|_{L^2(M)} \leq C(N)(\lambda_1 - N)^{\mathrm{O}(1/N)}. \tag{1-1}$$

More generally, the same conclusion holds for every Lipschitz function $u : M \to \mathbb{R}$ with null mean value and $\|u\|_{L^2(M)} = 1$, provided λ_1 on the right-hand-side is replaced by the Dirichlet energy $\int_M |\nabla u|^2 \, d\,\mathrm{vol}_g$.

We will prove (1-1) with tools of optimal transport tailored to study (possibly nonsmooth) metric measure spaces satisfying Ricci curvature lower bounds and dimensional upper bounds in the synthetic sense, the so-called $\mathrm{CD}(K, N)$ spaces introduced in [Sturm 2006a; 2006b; Lott and Villani 2009]. For the sake of this introduction, a metric measure space (m.m.s. for short) is a triple $(X, \mathsf{d}, \mathfrak{m})$, where (X, d) is a compact metric space and $\mathfrak{m}$ is a Borel probability measure, playing the role of reference volume measure. A $\mathrm{CD}(K, N)$ space should be roughly thought of as a possibly nonsmooth metric measure space having Ricci curvature bounded below by $K \in \mathbb{R}$ and dimension bounded above by $N \in (1, \infty)$ in the synthetic sense. The basic idea of the synthetic approach of Lott, Sturm and Villani is to analyze weighted convexity properties of suitable entropy functionals along geodesics in the space of probability measures endowed with the quadratic transportation (also known as Kantorovich–Wasserstein) distance. An important technical assumption throughout the paper is the essentially nonbranching ("e.n.b." for short) property [Rajala and Sturm 2014], which roughly corresponds to requiring that the L^2-optimal transport between two absolutely continuous (with respect to the reference volume measure $\mathfrak{m}$) probability measures is performed along geodesics which do not branch (for the precise definitions see Sections 2A and 2B). Notable examples of spaces satisfying e.n.b. $\mathrm{CD}(K, N)$ include (geodesically convex domains in) smooth Riemannian manifolds with Ricci bounded below by K and dimension bounded above by N, their measured Gromov–Hausdorff limits (i.e., the so-called "Ricci limits") and more generally $\mathrm{RCD}(K, N)$ spaces (i.e., $\mathrm{CD}(K, N)$ spaces with linear Laplacian; see Remark 2.4 for more details), finite-dimensional Alexandrov spaces with curvature bounded below, and Finsler manifolds endowed with a strongly convex norm. A standard example of a space failing to satisfy the essentially nonbranching property is $\mathbb{R}^2$ endowed with the L^∞ norm. Later in the introduction, when discussing the main steps of the proof, we will mention how the essentially nonbranching assumption is used in our arguments.

We will establish our results directly on the more general class of e.n.b. $\mathrm{CD}(N-1, N)$ metric measure spaces. For an m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ we define the nonnegative real number $\lambda^{1,2}_{(X,\mathsf{d},\mathfrak{m})}$ as

$$\lambda^{1,2}_{(X,\mathsf{d},\mathfrak{m})} := \inf\left\{ \frac{\int_X |\nabla u|^2 \, \mathfrak{m}}{\int_X |u|^2 \, \mathfrak{m}} : u \in \mathrm{Lip}(X) \cap L^2(X, \mathfrak{m}),\ u \neq 0,\ \int_X u \, \mathfrak{m} = 0 \right\}, \tag{1-2}$$

where $|\nabla u|$ is the slope (also called local Lipschitz constant) of the Lipschitz function u given by

$$|\nabla u|(x) = \begin{cases} \limsup_{y\to x} |u(x)-u(y)|/\mathsf{d}(x,y) & \text{if } x \text{ is not isolated,} \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that, in case $(X, \mathsf{d}, \mathfrak{m})$ is the m.m.s. corresponding to a smooth compact Riemannian manifold (possibly with boundary) $\lambda^{1,2}_{(X,\mathsf{d},\mathfrak{m})}$ coincides with the first eigenvalue of the problem $-\Delta u = \lambda u$ with Neumann boundary conditions.

Considering the extension of (1-1) to e.n.b. $\mathrm{CD}(N-1, N)$ spaces is natural: indeed a sequence (M_j, g_j) of Riemannian N-manifolds with $\mathrm{Ric}_{g_j} \geq (N-1)g_j$ where the right-hand side of (1-1) converges to zero as $j \to \infty$ may develop singularities and admits a limit (up to subsequences) in the measured Gromov–Hausdorff sense to a possibly nonsmooth e.n.b. $\mathrm{CD}(N-1, N)$ space (actually the limit is, more strongly, $\mathrm{RCD}(N-1, N)$).

In the enlarged class of e.n.b. $\mathrm{CD}(N-1, N)$ spaces (actually already for $\mathrm{RCD}(N-1, N)$ spaces), Obata's rigidity theorem must be modified:

- First of all, $N \in (1, \infty)$ is a (possibly noninteger) real number.
- Even in the case when N is an integer, the round sphere $\mathbb{S}^N$ is not anymore the only case of equality in the Lichnerowicz spectral gap as the spherical suspensions achieve equality as well [Ketterer 2015].

A key geometric property of the spherical suspensions is that they have diameter π, thus saturating Bonnet–Myers diameter upper bound. The first part of our main result is a quantitative control of how close to π the diameter must be, in terms of the spectral gap deficit. The second part of the statement is an L^2-quantitative control of the shape of functions with almost optimal Rayleigh quotient. We can now state our main theorem.

Theorem 1.3 (quantitative Obata's theorem for e.n.b. $\mathrm{CD}(N-1, N)$-spaces). *For every real number $N > 1$ there exists a real constant $C(N) > 0$ with the following properties: If $(X, \mathsf{d}, \mathfrak{m})$ is an essentially nonbranching metric measure space satisfying the* $\mathrm{CD}(N-1, N)$ *condition and* $\mathfrak{m}(X) = 1$ *with* $\mathrm{supp}(\mathfrak{m}) = X$, *then*

$$\pi - \mathrm{diam}(X) \leq C(N)(\lambda^{1,2}_{(X,\mathsf{d},\mathfrak{m})} - N)^{1/N}. \tag{1-3}$$

Moreover, for any Lipschitz function $u : X \to \mathbb{R}$ with $\int_X u\,\mathfrak{m} = 0$ and $\int_X u^2\,\mathfrak{m} = 1$, there exists a distinguished point $P \in X$ such that

$$\|u - \sqrt{N+1}\cos \mathsf{d}_P\|_{L^2(X,\mathfrak{m})} \leq C(N)\left(\int_X |\nabla u|^2\,\mathfrak{m} - N\right)^{\eta}, \quad \eta = \frac{1}{6N+4}. \tag{1-4}$$

Remark 1.4. Although Theorem 1.3 is formulated for e.n.b. $\mathrm{CD}(N-1, N)$ spaces, a statement for e.n.b. $\mathrm{CD}(K, N)$ spaces with $K > 0$ is easily obtained by scaling. Indeed, $(X, \mathsf{d}, \mathfrak{m})$ satisfies $\mathrm{CD}(K, N)$ if and only if, for any $\alpha, \beta \in (0, \infty)$, the scaled metric measure space $(X, \alpha\mathsf{d}, \beta\mathfrak{m})$ satisfies $\mathrm{CD}(\alpha^{-2}K, N)$; see [Sturm 2006b, Proposition 1.4].

Let us compare Theorem 1.3 with related results in the literature. Under the standing assumption that (M, g) is a *smooth* Riemannian N-manifold *without boundary* and with $\mathrm{Ric}_g \geq (N-1)g$:

(1) It follows from Cheng's comparison theorem [1975] that if $\lambda^{1,2}_{(M,g)}$ is close to N then the diameter of M must be close to π. Conversely, Croke [1982] proved that if the diameter is close to π then $\lambda^{1,2}_{(M,g)}$ must be close to N. Bérard, Besson and Gallot [Bérard et al. 1985] sharpened the diameter estimate of Cheng by proving an estimate very similar to (1-3).

(2) Bertrand [2007] established the following stability result for eigenfunctions (see also [Petersen 1999]): for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\lambda_1 \le N + \delta$ and u is an eigenfunction relative to λ_1 normalized so that $\int_M u^2 \, d\,\mathrm{vol}_g = \mathrm{vol}_g(M)$, then there exists a point $P \in M$ such that $\|u - \sqrt{N+1}\cos \mathsf{d}_P\|_{L^\infty(X,\mathfrak{m})} \le \epsilon$.

Theorem 1.3 sharpens and extends the above results in various ways:

- The estimate (1-3) extends [Bérard et al. 1985] to e.n.b. $\mathrm{CD}(N-1, N)$ spaces. These spaces are nonsmooth a priori and may have (convex) boundary. Actually, as the reader will realize, the claim (1-3) will be proved in Section 4 along the way to proving the much harder (1-4), to which the entire Section 5 is devoted.

- The estimate (1-4) extends Bertrand's stability [2007] to the more general class of e.n.b. $\mathrm{CD}(N-1, N)$ spaces and to arbitrary functions (a priori not eigenfunctions) with Rayleigh quotient close to N. The fact that u is an eigenfunction was key in [Bertrand 2007] in order to apply maximum principle and gradient estimates in the spirit of [Li and Yau 1980]. Let us stress that our methods are completely different and work for an arbitrary Lipschitz function satisfying a *small energy* condition but *no PDE* a priori.

Inequality (1-4) naturally fits in the framework of quantitative functional/geometric inequalities. A basic result in this context is the quantitative Euclidean isoperimetric inequality proved by Fusco, Maggi and Pratelli [Fusco et al. 2008] (see also [Figalli et al. 2010; Cicalese and Leonardi 2012] for different proofs) stating that for every Borel set $E \subset \mathbb{R}^n$ of positive and finite volume there exists $\bar{x} \in \mathbb{R}^n$ such that

$$\frac{|E \Delta B_{r_E}(\bar{x})|}{|E|} \le C(N)\left(\frac{\mathsf{P}(E)}{\mathsf{P}(B_{r_E}(\bar{x}))} - 1\right)^{1/2}, \tag{1-5}$$

where r_E is such that $|B_{r_E}(\bar{x})| = |E|$. Quantitative results involving the spectrum of the Laplacian have been proved for domains in $\mathbb{R}^n$, among others, by Hansen and Nadirashvili [1994] in dimension 2, by Melas [1992] for convex bodies, by Fusco, Maggi and Pratelli [Fusco et al. 2009] and Brasco, De Philippis and Velichkov [Brasco et al. 2015] regarding quantitative forms of the Faber–Krahn inequality and by Brasco and Pratelli [2012] regarding quantitative versions of the Krahn–Szegő and Szegő–Weinberger inequalities. More recently, a quantitative version of the Lévy–Gromov isoperimetric inequality was proved for essentially nonbranching $\mathrm{CD}(N-1, N)$ metric measure spaces in [Cavalletti et al. 2019], and a quantitative isoperimetric inequality in the setting of smooth Riemannian manifolds was considered in [Chodosh et al. 2023].

Taking variations in the broad context of metric measure spaces makes the prediction on the sharp exponent η in (1-4) a hard task. Even formulating a conjecture is a challenging question and it could actually be that $\eta = \mathrm{O}(1/N)$ as $N \to \infty$ is already sharp. In the direction of this guess, we notice that the exponent $1/N$ in (1-3) is indeed optimal in the class of metric measure spaces, as a direct computation on the model one-dimensional space $([0, D], |\cdot|, \sin^{N-1}(\cdot)\mathcal{L}^1)$ shows.

Before discussing the main steps in the proof of Theorem 1.3, it is worth recalling remarkable examples of spaces fitting in the assumptions of the result. Let us stress that our main theorem seems new in all of them. The class of essentially nonbranching $\mathrm{CD}(N-1, N)$ spaces includes many notable families of spaces, among them:

- *Geodesically convex domains* in (resp. weighted) Riemannian N-manifolds satisfying $\mathrm{Ric}_g \geq (N-1)g$ (resp. N-Bakry–Émery Ricci curvature bounded below by $N-1$).

- *Measured Gromov Hausdorff limits of Riemannian N-manifolds satisfying* $\mathrm{Ric}_g \geq (N-1)g$ (so-called "Ricci limits") and more generally the class of $\mathrm{RCD}(N-1, N)$ spaces. Indeed Ricci limits are examples of $\mathrm{RCD}(N-1, N)$ spaces (see for instance [Gigli et al. 2015]) and $\mathrm{RCD}(N-1, N)$ spaces are essentially nonbranching $\mathrm{CD}(N-1, N)$ (see [Rajala and Sturm 2014]).

- *Alexandrov spaces with curvature* ≥ 1. Petrunin [2011] proved that the synthetic curvature lower bound in the sense of comparison triangles is compatible with the optimal transport lower bound on the Ricci curvature of Lott, Sturm and Villani (see also [Zhang and Zhu 2010]). Moreover geodesics in Alexandrov spaces with curvature bounded below do not branch. It follows that Alexandrov spaces with curvature bounded from below by 1 and Hausdorff dimension at most N are nonbranching $\mathrm{CD}(N-1, N)$ spaces.

- *Finsler manifolds with strongly convex norm, and satisfying Ricci curvature lower bounds.* More precisely we consider a C^∞-manifold M, endowed with a function $F : TM \to [0, \infty]$ such that $F|_{TM\setminus\{0\}}$ is C^∞ and for each $x \in M$ it holds that $F_x := T_xM \to [0, \infty]$ is a strongly convex norm, i.e.,

$$g^x_{ij}(v) := \frac{\partial^2(F_x^2)}{\partial v^i \partial v^j}(v) \quad \text{is a positive definite matrix at every } v \in T_xM \setminus \{0\}.$$

Under these conditions, it is known that one can write the geodesic equations and the geodesics do not branch: in other words these spaces are nonbranching. We also assume (M, F) to be geodesically complete and endowed with a C^∞ probability measure $\mathfrak{m}$ in such a way that the associated m.m.s. $(X, F, \mathfrak{m})$ satisfies the $\mathrm{CD}(N-1, N)$ condition. This class of spaces has been investigated by Ohta [2009], who established the equivalence between the curvature dimension condition and a Finsler version of the Bakry–Émery N-Ricci tensor bounded from below.

An overview of the proof. The starting point of the proof of Theorem 1.3 is the metric-measure version of the classical *localization technique*. First introduced by Payne and Weinberger [1960] for establishing a sharp Poincaré–Wirtinger inequality for convex domains in $\mathbb{R}^n$, the localization technique has been developed into a general dimension-reduction tool for geometric inequalities in symmetric spaces by Gromov and Milman [1987], Lovász and Simonovits [1993] and Kannan, Lovász and Simonovits [Kannan et al. 1995]. More recently, Klartag [2017] used optimal transportation tools in order to extend the range of applicability of the technique to general Riemannian manifolds. The extension to the metric setting was finally obtained in [Cavalletti and Mondino 2017b]; see Section 2D.

Given a function $u \in L^1(X, \mathfrak{m})$ with $\int_X u\,\mathfrak{m} = 0$, the localization theorem (Theorem 2.10) gives a decomposition of X into a family of one-dimensional sets $\{X_q\}_{q\in Q}$ formed by the transport rays of a Kantorovich potential associated to the optimal transport from the positive part of u (i.e., $\mu_0 := \max\{u, 0\}\,\mathfrak{m}$)

to the negative part of u (i.e., $\mu_1 := \max\{-u, 0\}\,\mathfrak{m}$); each X_q is in particular isometric to a real interval. A first key property of such a decomposition is that each ray X_q carries a natural measure $\mathfrak{m}_q$ (given by the disintegration theorem) in such a way that

$$(X_q, \mathsf{d}, \mathfrak{m}_q) \text{ is a } \mathrm{CD}(N-1, N) \text{ space and } \int_{X_q} u\, \mathfrak{m}_q = 0, \tag{1-6}$$

so that both the geometry of the space and the null mean value constraint are *localized* into a family of one-dimensional spaces. An important ingredient used in the proof of such a decomposition is the essentially nonbranching property which, coupled with $\mathrm{CD}(N-1, N)$ (actually the weaker measure contraction would suffice here), guarantees that the rays form a partition of X (up to an $\mathfrak{m}$-negligible set).

In order to exploit (1-6), as a first step, in Section 3 we prove the one-dimensional counterparts of Theorem 1.3. More precisely, given a one-dimensional $\mathrm{CD}(N-1, N)$ space $(I = [0, D], |\cdot|, \mathfrak{m})$ we show that (Proposition 3.3)

$$\pi - D \le C(N)(\lambda^{1,2}_{(I,|\cdot|,\mathfrak{m})} - N)^{1/N}, \tag{1-7}$$

and that, if $u \in \mathrm{Lip}(I)$ satisfies $\int u\, \mathfrak{m} = 0$ and $\int u^2\, \mathfrak{m} = 1$, then (Theorem 3.11)

$$\min\{\|u - \sqrt{N+1}\cos(\cdot)\|_{L^2(\mathfrak{m})}, \|u + \sqrt{N+1}\cos(\cdot)\|_{L^2(\mathfrak{m})}\} \le C\left(\int |u'|^2\, \mathfrak{m} - N\right)^{\min\{1/2,1/N\}}. \tag{1-8}$$

Combining (1-6) and (1-7), it is not hard to prove (see Theorem 4.3) the first claim (1-3) of Theorem 1.3. Actually, calling Q_ℓ (for "Q long") the set of indices for which $|X_q| \simeq \pi$, we aim to show that $\mathfrak{q}(Q_\ell) \simeq 1$ (i.e., "most rays are long"). As we will discuss in a few lines, this is far from being trivial (in particular, it needs new ideas when compared with [Cavalletti et al. 2019]).

A second crucial property of the decomposition $\{X_q\}_{q\in Q}$, inherited by the variational nature of the construction, is the so-called cyclical monotonicity. This was key in [Cavalletti et al. 2019] for showing that, for $q \in Q_\ell$, the transport ray X_q has its starting point close to a fixed "south pole" P_S, and ends up near a fixed "north pole" P_N (in particular, the distance between P_S and P_N is close to π) (Proposition 5.1).

Then we observe that (1-8) forces, for $q \in Q_\ell$, the fiber $u_q := u\llcorner_{X_q}$ (that is the restriction of u to the corresponding one-dimensional element of the partition) to be L^2 close to a multiple of the cosine of the arclength parametrization along the ray X_q, i.e.,

$$u_q(\cdot) \simeq c_q\sqrt{N+1}\cos(\cdot) \quad \text{along } X_q, \text{ where } c_q = \|u_q\|_{L^2(\mathfrak{m}_q)} \text{ for } q \in Q_\ell \text{ (see (5-13))}. \tag{1-9}$$

The difficulties in order to conclude the proof are mainly two, and are strictly linked:

(1) Show that $Q_\ell \ni q \mapsto c_q$ is almost constant.

(2) Show that $\mathfrak{q}(Q_\ell) \simeq 1$.

Let us stress that at this stage the only given information is that $\int_{Q_\ell} c_q^2\, \mathfrak{q} \simeq 1$. The intuition why (1) and (2) should hold is that an oscillation of c_q would correspond to an oscillation of u "orthogonal to the transport rays", which would be expensive in terms of Dirichlet energy of u. The proofs of the two claims are the most technical part of the work and correspond respectively to Propositions 5.2 and 5.3.

Let us mention that the two difficulties (1) and (2) were not present in the proof of the quantitative Lévy–Gromov inequality in [Cavalletti et al. 2019], where it was sufficient to work with characteristic functions (which have a fixed scale, i.e., they are either 0 or 1).

2. Background material

The goal of this section is to fix the notation and to recall the basic notions/constructions used throughout the paper: in Section 2A we review geodesics in the Wasserstein distance, in Section 2B curvature-dimension conditions, in Section 2C some basics of $\mathrm{CD}(K, N)$ densities on segments of the real line, and in Section 2D the decomposition of the space into transport rays (localization).

2A. *Geodesics in the L^2-Kantorovich–Wasserstein distance.* Let (X, d) be a compact metric space and $\mathfrak{m}$ a Borel probability measure over X. The triple $(X, \mathsf{d}, \mathfrak{m})$ is called metric measure space, m.m.s. for short.

The space of all Borel probability measures over X will be denoted by $\mathcal{P}(X)$. We define the L^2-Kantorovich–Wasserstein distance W_2 between two measures $\mu_0, \mu_1 \in \mathcal{P}(X)$ as

$$W_2(\mu_0, \mu_1)^2 = \inf_{\pi} \int_{X\times X} \mathsf{d}^2(x, y)\, \pi(dx\, dy), \tag{2-1}$$

where the infimum is taken over all $\pi \in \mathcal{P}(X \times X)$ with μ_0 and μ_1 as the first and the second marginal, i.e., $(P_1)_\sharp \pi = \mu_0$, $(P_2)_\sharp \pi = \mu_1$. Of course P_i, $i = 1, 2$, denotes the projection on the first and second factors, respectively, and $(P_i)_\sharp$ is the corresponding push-forward map on measures. As (X, d) is complete, $(\mathcal{P}(X), W_2)$ is also complete.

The space of geodesics of (X, d) is denoted by

$$\mathrm{Geo}(X) := \{\gamma \in C([0, 1], X) : \mathsf{d}(\gamma_s, \gamma_t) = |s - t|\, \mathsf{d}(\gamma_0, \gamma_1) \text{ for every } s, t \in [0, 1]\}.$$

A metric space (X, d) is said to be a *geodesic space* if and only if for each $x, y \in X$ there exists $\gamma \in \mathrm{Geo}(X)$ such that $\gamma_0 = x$, $\gamma_1 = y$. A basic fact of W_2 geometry is that if (X, d) is geodesic then $(\mathcal{P}(X), W_2)$ is geodesic as well. For any $t \in [0, 1]$, let e_t denote the evaluation map:

$$\mathrm{e}_t : \mathrm{Geo}(X) \to X, \quad \mathrm{e}_t(\gamma) := \gamma_t.$$

Any geodesic $(\mu_t)_{t\in[0,1]}$ in $(\mathcal{P}(X), W_2)$ can be lifted to a measure $\nu \in \mathcal{P}(\mathrm{Geo}(X))$, called a *dynamical optimal plan*, such that $(\mathrm{e}_t)_\sharp\, \nu = \mu_t$ for all $t \in [0, 1]$. Given $\mu_0, \mu_1 \in \mathcal{P}(X)$, we denote by $\mathrm{OptGeo}(\mu_0, \mu_1)$ the space of all $\nu \in \mathcal{P}(\mathrm{Geo}(X))$ for which $(\mathrm{e}_0, \mathrm{e}_1)_\sharp\, \nu$ realizes the minimum in (2-1). Here as usual $\sharp$ indicates the push-forward operation. If (X, d) is geodesic, then the set $\mathrm{OptGeo}(\mu_0, \mu_1)$ is nonempty for any $\mu_0, \mu_1 \in \mathcal{P}(X)$.

A set $F \subset \mathrm{Geo}(X)$ is a *set of nonbranching geodesics* if and only if for any $\gamma^1, \gamma^2 \in F$, it holds

$$\text{there exists } \bar{t} \in (0, 1) \text{ such that, for all } t \in [0, \bar{t}\,],\ \gamma^1_t = \gamma^2_t \quad \Longrightarrow \quad \gamma^1_s = \gamma^2_s \quad \text{for all } s \in [0, 1].$$

A measure μ on a measurable space $(\Omega, \mathcal{F})$ is said to be *concentrated* on $F \subset \Omega$ if there exists $E \subset F$ with $E \in \mathcal{F}$ so that $\mu(\Omega \setminus E) = 0$. With this terminology, we next recall the definition of *essentially nonbranching space* from [Rajala and Sturm 2014].

Definition 2.1. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is *essentially nonbranching* if and only if for any $\mu_0, \mu_1 \in \mathcal{P}(X)$, with μ_0, μ_1 absolutely continuous with respect to $\mathfrak{m}$, any element of $\mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of nonbranching geodesics.

2B. ***Curvature-dimension conditions for metric measure spaces.*** The L^2-transport structure described in Section 2A allows us to formulate a generalized notion of Ricci curvature lower bound coupled with a dimension upper bound in the context of possibly nonsmooth metric measure spaces. This corresponds to the $\mathrm{CD}(K, N)$ condition introduced in the seminal works of Sturm [2006a; 2006b] and Lott and Villani [2009], which here is reviewed only for a compact m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ with $\mathfrak{m} \in \mathcal{P}(X)$ and in the case $K > 0$, $1 < N < \infty$ (the basic setting of the present paper).

For $N \in (1, \infty)$, the *N-Rényi relative-entropy functional* $\mathcal{E}_N : \mathcal{P}(X) \to [0, 1]$ is defined as

$$\mathcal{E}_N(\mu) := \int \rho^{1-1/N}\, d\mathfrak{m},$$

where $\mu = \rho\mathfrak{m} + \mu^{\mathrm{sing}}$ is the Lebesgue decomposition of μ with $\mu^{\mathrm{sing}} \perp \mathfrak{m}$.

Given $K \in (0, \infty)$, $N \in (1, \infty)$, and $t \in [0, 1]$, define $\sigma^{(t)}_{K,N} : [0, \infty) \to [0, \infty]$ as

$$\begin{cases} \sigma^{(t)}_{K,N}(0) := t, & \\ \sigma^{(t)}_{K,N}(\theta) := \sin(t\theta\sqrt{K/N})/\sin(\theta\sqrt{K/N}) & \text{if } 0 < \theta < \pi/\sqrt{K/N}, \\ \sigma^{(t)}_{K,N}(\theta) := +\infty & \text{otherwise.} \end{cases} \tag{2-2}$$

Set also

$$\tau^{(t)}_{K,N}(\theta) := t^{1/N}\sigma^{(t)}_{K,N-1}(\theta)^{1-1/N}. \tag{2-3}$$

Definition 2.2 ($\mathrm{CD}(K, N)$). An m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ is said to satisfy $\mathrm{CD}(K, N)$ if for all $\mu_0, \mu_1 \in \mathcal{P}(X)$ absolutely continuous with respect to $\mathfrak{m}$ there exists $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ so that for all $t \in [0, 1]$ it holds $\mu_t := (\mathrm{e}_t)_\#\nu \ll \mathfrak{m}$ and

$$\mathcal{E}_{N'}(\mu_t) \geq \int_{X\times X} \big(\tau^{(1-t)}_{K,N'}(\mathsf{d}(x_0, x_1))\rho_0^{-1/N'}(x_0) + \tau^{(t)}_{K,N'}(\mathsf{d}(x_0, x_1))\rho_1^{-1/N'}(x_1)\big)\, \pi(dx_0, dx_1) \tag{2-4}$$

for all $N' \geq N$, where $\pi = (\mathrm{e}_0, \mathrm{e}_1)_\sharp(\nu)$ and $\mu_i = \rho_i\mathfrak{m}$, $i = 0, 1$.

If $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathrm{CD}(K, N)$ condition then the same is valid for $(\mathrm{supp}(\mathfrak{m}), \mathsf{d}, \mathfrak{m})$; hence we directly assume $X = \mathrm{supp}(\mathfrak{m})$.

For the general definition of $\mathrm{CD}(K, N)$ see [Lott and Villani 2009; Sturm 2006a; 2006b].

Remark 2.3 (case of a smooth Riemannian manifold). It is worth recalling that if (M, g) is a Riemannian manifold of dimension n and $h \in C^2(M)$ with $h > 0$ then, denoting by d_g and vol_g the Riemannian distance and volume measure, the m.m.s. $(M, \mathsf{d}_g, h\,\mathrm{vol}_g)$ satisfies $\mathrm{CD}(K, N)$ with $N \geq n$ if and only if (see [Sturm 2006b, Theorem 1.7])

$$\mathrm{Ric}_{g,h,N} \geq Kg, \quad \mathrm{Ric}_{g,h,N} := \mathrm{Ric}_g - (N-n)\frac{\nabla^2_g h^{1/(N-n)}}{h^{1/(N-n)}},$$

in other words if and only if the weighted Riemannian manifold $(M, g, h\,\mathrm{vol}_g)$ has N-Bakry–Émery Ricci tensor bounded below by K. Note that if $N = n$, the Bakry–Émery Ricci tensor $\mathrm{Ric}_{g,h,N} = \mathrm{Ric}_g$ makes sense only if h is constant. □

Remark 2.4 ($\mathsf{CD}^*(K,N)$, $\mathsf{RCD}^*(K,N)$ and $\mathsf{RCD}(K,N)$). The lack of the local-to-global property of the $\mathsf{CD}(K,N)$ condition (for $K/N \neq 0$) led Bacher and Sturm [2010] to introduce the reduced curvature-dimension condition, denoted by $\mathsf{CD}^*(K,N)$. The $\mathsf{CD}^*(K,N)$ condition asks for the same inequality (2-4) of $\mathsf{CD}(K,N)$ to hold but the coefficients $\tau^{(s)}_{K,N}(\mathsf{d}(\gamma_0,\gamma_1))$ are replaced by the slightly smaller $\sigma^{(s)}_{K,N}(\mathsf{d}(\gamma_0,\gamma_1))$. Let us explicitly notice that, in general, $\mathsf{CD}^*(K,N)$ is weaker than $\mathsf{CD}(K,N)$. A subsequent breakthrough in the theory was obtained with the introduction of the Riemannian curvature dimension condition $\mathsf{RCD}(K,N)$: in the infinite-dimensional case $N=\infty$ this condition was introduced in [Ambrosio et al. 2014] (for finite measures $\mathfrak{m}$, and in [Ambrosio et al. 2015] for σ-finite ones). The finite-dimensional refinements $\mathsf{RCD}(K,N)/\mathsf{RCD}^*(K,N)$ with $N<\infty$ were subsequently studied in [Gigli 2015; Erbar et al. 2015; Ambrosio et al. 2019]. We refer to these articles as well as to the survey papers [Ambrosio 2018; Villani 2019] for a general account on the synthetic formulation of Ricci curvature lower bounds, in particular of the latter Riemannian-type. Here we only briefly recall that it is a stable [Gigli et al. 2015] strengthening of the (resp. reduced) curvature-dimension condition: an m.m.s. satisfies $\mathsf{RCD}(K,N)$ (resp. $\mathsf{RCD}^*(K,N)$) if and only if it satisfies $\mathsf{CD}(K,N)$ (resp. $\mathsf{CD}^*(K,N)$) and the Sobolev space $W^{1,2}(X,\mathfrak{m})$ is a Hilbert space (with the Hilbert structure induced by the Cheeger energy).

To conclude we recall also that recently, the first author together with E. Milman [Cavalletti and Milman 2021] proved the equivalence of $\mathsf{CD}(K,N)$ and $\mathsf{CD}^*(K,N)$, together with the local-to-global property for $\mathsf{CD}(K,N)$, in the framework of essentially nonbranching m.m.s. having $\mathfrak{m}(X)<\infty$. As we will always assume the aforementioned properties to be satisfied by our ambient m.m.s. $(X,\mathsf{d},\mathfrak{m})$, we will use both formulations with no distinction. It is worth also mentioning that an m.m.s. satisfying $\mathsf{RCD}^*(K,N)$ is essentially nonbranching (see [Rajala and Sturm 2014, Corollary 1.2]), implying also the equivalence of $\mathsf{RCD}^*(K,N)$ and $\mathsf{RCD}(K,N)$ (see [Cavalletti and Milman 2021] for details). □

We shall always assume that the m.m.s. $(X,\mathsf{d},\mathfrak{m})$ is essentially nonbranching and satisfies $\mathsf{CD}(K,N)$ for some $K>0$, $N\in(1,\infty)$ with $\mathrm{supp}(\mathfrak{m})=X$. It follows that (X,d) is a geodesic and compact metric space. More precisely: note we assumed from the beginning (X,d) to be compact for the sake of simplicity; however, such an assumption could have been replaced by completeness and separability throughout Sections 2A and 2B, but compactness would have been now a consequence of $\mathsf{CD}(K,N)$ for some $K>0$, $N\in(1,\infty)$.

A useful property of essentially nonbranching $\mathsf{CD}(K,N)$ spaces is the validity of a weak local Poincaré inequality.

Proposition 2.5 (weak local Poincaré inequality). *Let $(X,\mathsf{d},\mathfrak{m})$ be an essentially nonbranching $\mathsf{CD}(K,N)$ space for some $K\geq 0$, $N>1$. For every $u\in \mathrm{Lip}(X)$ it holds*

$$\fint_{B_r(x)}\left|u-\fint_{B_r(x)}u\right|\mathfrak{m}\leq 2^{N+2}r\fint_{B_{2r}(x)}|\nabla u|\mathfrak{m}. \tag{2-5}$$

More generally, for every $p\geq 1$ there exists $C_{p,N}$ such that

$$\fint_{B_r(x)}\left|u-\fint_{B_r(x)}u\right|^p\mathfrak{m}\leq C_{p,N}\,r^p\fint_{B_{10r}(x)}|\nabla u|^p\mathfrak{m}. \tag{2-6}$$

Proof. It is well known that, in essentially nonbranching $\mathrm{CD}(K, N)$ spaces, the W_2 geodesic connecting two absolutely continuous probability measures is unique (indeed, it holds more generally for essentially nonbranching $\mathrm{MCP}(K, N)$ spaces [Cavalletti and Mondino 2017a, Theorem 1.1]). Thus, $(X, \mathsf{d}, \mathfrak{m})$ as in the assumptions enters the framework of [Rajala 2012, Corollary 1] and (2-5) follows.

Recalling that by the Bishop–Gromov inequality [Sturm 2006b, Theorem 2.3] it holds

$$\frac{\mathfrak{m}(B_\rho(x_0))}{\mathfrak{m}(B_1(x_0))} \geq C_N \rho^N$$

for every $\rho \in [0, 1]$, $x_0 \in X$, the second claim (2-6) is a consequence of (2-5) and [Hajłasz and Koskela 2000, Theorem 5.1]. □

2C. CD(*K*, *N*) *densities on segments of the real line.* We will use several times the following terminology: recalling the coefficients σ from (2-2), a nonnegative function h defined on an interval $I \subset \mathbb{R}$ is called *a* $\mathrm{CD}(K, N)$ *density on* I, for $K \in \mathbb{R}$ and $N \in (1, \infty)$, if for all $x_0, x_1 \in I$ and $t \in [0, 1]$

$$h(tx_1 + (1-t)x_0)^{1/(N-1)} \geq \sigma^{(t)}_{K,N-1}(|x_1 - x_0|)h(x_1)^{1/(N-1)} + \sigma^{(1-t)}_{K,N-1}(|x_1 - x_0|)h(x_0)^{1/(N-1)}. \tag{2-7}$$

The link with the definition of $\mathrm{CD}(K, N)$ for an m.m.s. can be summarized as follows (see for instance [Cavalletti and Milman 2021, Theorem A.2]): if h is a $\mathrm{CD}(K, N)$ density on an interval $I \subset \mathbb{R}$ then the m.m.s. $(I, |\cdot|, h(t)\,dt)$ satisfies $\mathrm{CD}(K, N)$; conversely, if the m.m.s. $(\mathbb{R}, |\cdot|, \mu)$ satisfies $\mathrm{CD}(K, N)$ and $I = \mathrm{supp}(\mu)$ is not a point, then $\mu \ll \mathcal{L}^1$ and there exists a representative of the density $h = d\mu/d\mathcal{L}^1$ which is a $\mathrm{CD}(K, N)$ density on I.

A $\mathrm{CD}(K, N)$ density h defined on an interval $I \subset \mathbb{R}$ satisfies the following properties:

- h is lower semicontinuous on I and locally Lipschitz continuous in its interior (this is easily reduced to the corresponding statement for concave functions on I).
- h is strictly positive in the interior of I whenever it does not vanish identically (this follows directly from the definition (2-7)).
- h is locally semiconcave in the interior of I, i.e., for all x_0 in the interior of I, there exists $C_{x_0} \in \mathbb{R}$ so that $x \mapsto h(x) - C_{x_0}(x - x_0)^2$ is concave in a neighborhood of x_0. In particular, h is twice differentiable in I with at most countably many exceptions.

As proven in [Cavalletti and Milman 2021, Lemma A.5], if h is a $\mathrm{CD}(K, N)$ density on an interval I then at any point x in the interior where it is twice differentiable (thus up to at most countably many exceptions) it holds

$$(\log h)''(x) + \frac{1}{N-1}((\log h)'(x))^2 = (N-1)\frac{(h^{1/(N-1)})''(x)}{h^{1/(N-1)}(x)} \leq -K. \tag{2-8}$$

Also the converse implication holds; see [Cavalletti and Milman 2021, Lemma A.6] for the proof and the precise statement.

We next recall some estimates on $\mathrm{CD}(N-1, N)$ densities, which will turn out to be useful in the paper. Let h_N be the model density for the $\mathrm{CD}(N-1, N)$ condition given by

$$h_N(t) := \frac{1}{\omega_N} \sin^{N-1}(t) \quad \text{for } t \in [0, \pi], \tag{2-9}$$

where $\omega_N := \int_0^\pi \sin^{N-1}(t)\, dt$. Let $\epsilon := \pi - D$ and $\lambda_D := \int_0^D h_N(t)\, dt$ for any $D \in [0, \pi]$.

For a proof of the next proposition see for instance [Cavalletti et al. 2019, Proposition A.3].

Proposition 2.6. *Let $h : [0, D] \to [0, +\infty)$ be a $\mathrm{CD}(N-1, N)$ density which integrates to 1 on $[0, D]$. Then, for any $t \in (0, D)$, it holds*

$$\left(\frac{\omega_N}{\omega_N \lambda_D + \epsilon}\right) \min\{h_N(t), h_N(t+\epsilon)\} \le h(t) \le \left(\frac{\omega_N}{\omega_N - \epsilon}\right) \max\{h_N(t), h_N(t+\epsilon)\}. \tag{2-10}$$

Corollary 2.7. *Under the assumptions of Proposition 2.6, there exist a constant $C = C(N) > 0$ and $\epsilon_0 > 0$ with the following property: if $\epsilon \in [0, \epsilon_0]$ then for any $t \in (0, D)$ it holds*

$$|h(t) - h_N(t)| \le C\epsilon. \tag{2-11}$$

Moreover, for $r \in \left(0, \frac{1}{10}\right)$ and $\epsilon \in \left(0, \frac{1}{10} r\right)$ the following improved estimate holds:

$$|h(t) - h_N(t)| \le C r^{N-2} \epsilon \quad \textit{for all } t \in ([0, r] \cup [\pi - r, D]). \tag{2-12}$$

Proof. The validity of (2-11) follows from (2-10) taking into account the Lipschitz continuity of h_N and the asymptotic expansions of

$$\frac{\omega_N}{\omega_N \lambda_D + \epsilon} \quad \text{and} \quad \frac{\omega_N}{\omega_N - \epsilon},$$

as $\epsilon \to 0$. The improved estimate (2-12) on $([0, r] \cup [\pi - r, D])$ follows analogously from (2-10) and the mean value theorem. □

Armed with Corollary 2.7 we can prove that, if $D \in (0, \pi)$ is close to π, then the integrals of the functions sin and cos (and of any bounded function, more in general) with respect to a $\mathrm{CD}(N-1, N)$ density h defined on $[0, D]$ do not differ much from the value of the corresponding integrals computed with respect to the model density h_N.

Corollary 2.8. *Let $f : [0, \pi] \to [-1, 1]$ be Borel measurable. Define $\mathfrak{m}(dt) := h(t)\mathcal{L}^1(dt)$ and $\mathfrak{m}_N(dt) := h_N(t)\mathcal{L}^1(dt)$. Under the assumptions of Proposition 2.6, there exist a constant $C = C(N) > 0$ and $\epsilon_0 > 0$ with the following property: if $\epsilon \in [0, \epsilon_0]$ then*

$$\left| \int_0^D f(t)\, \mathfrak{m}(dt) - \int_0^\pi f(t)\, \mathfrak{m}_N(dt) \right| \le C\epsilon. \tag{2-13}$$

Moreover, for any $r \in \left(0, \frac{1}{10}\right)$ and $\epsilon \in \left(0, \frac{1}{10} r\right)$ the following improved estimate holds

$$\left| \int_0^r f(t)\, \mathfrak{m}(dt) - \int_0^r f(t)\, \mathfrak{m}_N(dt) \right| + \left| \int_{\pi - r}^D f(t)\, \mathfrak{m}(dt) - \int_{\pi - r}^\pi f(t)\, \mathfrak{m}_N(dt) \right| \le C\epsilon r^{N-1}. \tag{2-14}$$

Proof. The conclusion follows from Corollary 2.7 just by integrating on $[0, D]$ and taking into account that $|\int_D^\pi f\,\mathfrak{m}_N| \le C\epsilon^N$. □

2D. ***Localization and L^1-optimal transportation.*** The localization technique has its roots in a work of Payne and Weinberger [1960] and has been developed by Gromov and Milman [1987], Lovász and Simonovits [1993] and Kannan, Lovász and Simonovits [Kannan et al. 1995] in the setting of Euclidean spaces, spheres and Hilbert spaces. The basic idea is to reduce an n-dimensional problem, via tools of convex geometry, to lower-dimensional problems which are easier to handle. In the aforementioned papers, the symmetries of the spaces were heavily used to obtain such a dimensional reduction, typically via iterative bisections. Recently Klartag [2017] found a bridge between L^1-optimal transportation problems and the localization technique yielding the localization theorem in the framework of smooth Riemannian manifolds. Inspired by this approach, the first and the second author in [Cavalletti and Mondino 2017b] proved a localization theorem for essentially nonbranching metric measure spaces satisfying the $\mathrm{CD}(K, N)$ condition. Before stating the result it is worth recalling some basics about the disintegration of a measure associated to a partition (for a comprehensive treatment see the monograph [Fremlin 2006]; for a discussion closer to the spirit of this paper see [Bianchini and Caravenna 2009]; for a one-page summary see [Cavalletti et al. 2019, Appendix B]).

Given a measure space $(X, \mathscr{X}, \mathfrak{m})$, suppose a *partition* of X into *disjoint* sets is given by $\{X_q\}_{q\in Q}$ so that $X = \bigcup_{q\in Q} X_q$. Here Q is the set of indices and $\mathfrak{Q} : X \to Q$ is the quotient map, i.e.,

$$q = \mathfrak{Q}(x) \iff x \in X_q.$$

We endow Q with the *push forward σ-algebra* $\mathscr{Q}$ of $\mathscr{X}$:

$$C \in \mathscr{Q} \iff \mathfrak{Q}^{-1}(C) \in \mathscr{X},$$

i.e., the biggest σ-algebra on Q such that $\mathfrak{Q}$ is measurable. Moreover, the push forward measure $\mathfrak{q} := \mathfrak{Q}_\sharp\,\mathfrak{m}$ defines a natural measure $\mathfrak{q}$ on $(Q, \mathscr{Q})$. The triple $(Q, \mathscr{Q}, \mathfrak{q})$ is called *the quotient measure space*.

Definition 2.9 (consistent and strongly consistent disintegration). A *disintegration* of $\mathfrak{m}$ *consistent with the partition* is a map

$$Q \ni q \mapsto \mathfrak{m}_q \in \mathcal{P}(X, \mathscr{X})$$

such that the following requirements hold:

(1) For all $B \in \mathscr{X}$, the map $q \mapsto \mathfrak{m}_q(B)$ is $\mathfrak{q}$-measurable.

(2) For all $B \in \mathscr{X}$ and $C \in \mathscr{Q}$, the following consistency condition holds:

$$\mathfrak{m}(B \cap \mathfrak{Q}^{-1}(C)) = \int_C \mathfrak{m}_q(B)\,\mathfrak{q}(dq).$$

A disintegration of $\mathfrak{m}$ is called *strongly consistent* if in addition:

(3) For $\mathfrak{q}$-a.e. $q \in Q$, $\mathfrak{m}_q$ is concentrated on $X_q = \mathfrak{Q}^{-1}(q)$.

In the next theorem, for $\mathfrak{q}$-a.e. $q \in Q$, the equivalence class X_q is a geodesic in X. With a slight abuse of notation X_q denotes also the arc-length parametrization on a real interval of the corresponding geodesic; i.e., it is a map from a real interval with image X_q. We will use the following terminology: $q \mapsto \mathfrak{m}_q$ is a $\mathrm{CD}(K, N)$ disintegration if, for $\mathfrak{q}$-a.e. $q \in Q$, $\mathfrak{m}_q = h_q \mathcal{H}^1 \llcorner_{X_q}$, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure and $h_q \circ X_q$ is a $\mathrm{CD}(K, N)$ density, in the sense of (2-7).

Theorem 2.10 [Cavalletti and Mondino 2017b]. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially nonbranching metric measure space satisfying the $\mathrm{CD}(K, N)$ condition for some $K \in \mathbb{R}$ and $N \in [1, \infty)$. Let $f : X \to \mathbb{R}$ be $\mathfrak{m}$-integrable such that $\int_X f\, \mathfrak{m} = 0$, and assume the existence of $x_0 \in X$ such that $\int_X |f(x)|\, \mathsf{d}(x, x_0)\, \mathfrak{m}(dx) < \infty$.*

Then the space X admits a partition $\{X_q\}_{q \in Q}$ and a corresponding (strongly consistent) disintegration of $\mathfrak{m}$, $\{\mathfrak{m}_q\}_{q \in Q}$, such that:

- *For any $\mathfrak{m}$-measurable set $B \subset \mathcal{T}$ it holds*

$$\mathfrak{m}(B) = \int_Q \mathfrak{m}_q(B)\, \mathfrak{q}(dq),$$

where $\mathfrak{q}$ is a probability measure over Q defined on the quotient σ-algebra $\mathcal{Q}$.

- *For $\mathfrak{q}$-almost every $q \in Q$, the set X_q is a geodesic (possibly of zero length) and $\mathfrak{m}_q$ is supported on it. Moreover $q \mapsto \mathfrak{m}_q$ is a $\mathrm{CD}(K, N)$ disintegration.*

- *For $\mathfrak{q}$-almost every $q \in Q$, it holds $\int_{X_q} f\, \mathfrak{m}_q = 0$.*

In Theorem 2.10 we can also distinguish the set of X_α having positive length, whose union forms the so-called *transport set* denoted by $\mathcal{T}$, from the ones having zero length, i.e., points, whose union we usually denote by Z, so to have a decomposition of X into $\mathcal{T}$ and Z. The last point of Theorem 2.10 implies then that $\mathfrak{m}$-a.e. $f \equiv 0$ on Z.

Following the approach of [Klartag 2017], Theorem 2.10 was proven in [Cavalletti and Mondino 2017b] studying the following optimal transportation problem. Let $\mu_0 := f^+\mathfrak{m}$ and $\mu_1 := f^-\mathfrak{m}$, where $f^\pm$ denote the positive and the negative parts of f respectively, and study the L^1-optimal transport problem associated with it:

$$\inf\left\{ \int_{X \times X} \mathsf{d}(x, y)\, \pi(dx\, dy) : \pi \in \mathcal{P}(X \times X),\ (P_1)_\sharp \pi = \mu_0, (P_2)_\sharp \pi = \mu_1 \right\}. \tag{2-15}$$

Then the relevant object to study is given by the dual formulation of the previous minimization problem. By the summability properties of f (see the hypotheses of Theorem 2.10), there exists a 1-Lipschitz function $\phi : X \to \mathbb{R}$ such that π is a minimizer in (2-15) if and only if $\pi(\Gamma) = 1$, where

$$\Gamma := \{(x, y) \in X \times X : \phi(x) - \phi(y) = \mathsf{d}(x, y)\}$$

is the naturally associated d-cyclically monotone set; i.e., for any $(x_1, y_1), \ldots, (x_n, y_n) \in \Gamma$ it holds

$$\sum_{i=1}^n \mathsf{d}(x_i, y_i) \le \sum_{i=1}^n \mathsf{d}(x_i, y_{i+1}), \qquad y_{n+1} = y_1,$$

for any $n \in \mathbb{N}$. The set Γ induces a partial order relation whose maximal chains produce a partition (up to an $\mathfrak{m}$-negligible subset) of the set $\mathcal{T} \subset X$ appearing in the statement of Theorem 2.10, made of one-dimensional subsets. For a summary of the constructions see [Cavalletti et al. 2019, Section 2.5]; for more details see [Cavalletti and Mondino 2017b; Cavalletti and Milman 2021].

3. One-dimensional estimates

The goal of this section is to give a self-contained presentation of the one-dimensional estimates we will use throughout the paper.

3A. ***Bérard–Besson–Gallot explicit lower bound on the model isoperimetric profile.*** For $N > 1$, let

$$\omega_N := \int_0^\pi (\sin t)^{N-1}\, dt \quad \text{and} \quad \mathfrak{m}_N := \frac{1}{\omega_N} (\sin t)^{N-1} \mathcal{L}^1(dt) \llcorner_{[0,\pi]}. \tag{3-1}$$

From now on fix $D \in (0, \pi)$. For $b \in [0, \pi - D]$ and $v \in [0, 1]$, let $R(b, v) \in [b, \pi]$ be uniquely defined by the equation

$$\int_b^{R(b,v)} (\sin t)^{N-1}\, dt = v \int_b^{b+D} (\sin t)^{N-1}\, dt. \tag{3-2}$$

Set

$$\mathcal{I}_{N,D}(v) := \inf\{g(b, v) : b \in [0, \pi - D]\}, \tag{3-3}$$

where

$$g(b, v) := \frac{[\sin(R(b, v))]^{N-1}}{\int_b^{b+D} (\sin t)^{N-1}\, dt}. \tag{3-4}$$

To keep notation short, we also set $\mathcal{I}_N := \mathcal{I}_{N,\pi}$. Notice that $\mathcal{I}_N$ is the isoperimetric profile of $\mathbb{S}^N$ for an integer N. We refer to Section 4 for a brief discussion about the isoperimetric profile; note also that $\mathcal{I}_{N,D}$ is the model isoperimetric profile in the Lévy–Gromov isoperimetric comparison theorem for spaces with Ricci $\geq N - 1$, dimension $\leq N$ and diameter $\leq D$; see [Gromov 1999, Appendix C; Bérard et al. 1985; Milman 2015; Cavalletti and Mondino 2017b].

The proof of the next lemma is inspired by, but somewhat different from, [Bérard et al. 1985, Appendix 1] and the statement generalizes to arbitrary real $N > 1$ the result stated in the reference for an integer $N \geq 2$.

Lemma 3.1 (Bérard–Besson–Gallot explicit isoperimetric lower bound). *Fix $N > 1$ and $D \in [0, \pi]$, and let $\mathcal{I}_{N,D} : [0, 1] \to [0, \infty)$ be defined in (3-3). Then*

$$\frac{\mathcal{I}_{N,D}(v)}{\mathcal{I}_N(v)} \geq \left(\frac{\int_0^{\pi/2} (\cos t)^{N-1}\, dt}{\int_0^{D/2} (\cos t)^{N-1}\, dt} \right)^{1/N} =: C_{N,D} \geq 1 \quad \textit{for all } v \in (0, 1). \tag{3-5}$$

Proof. Let $v' \in (0, 1)$ and $f : [0, \pi - D] \times (0, 1) \to [0, +\infty)$ be defined by

$$v' := \frac{1}{\omega_N} \int_0^{R(b,v)} (\sin t)^{N-1}\, dt \quad \text{and} \quad f(b, v) := \frac{g(b, v)}{\mathcal{I}_N(v)}. \tag{3-6}$$

In particular

$$\mathcal{I}_N(v') = \frac{1}{\omega_N}[\sin R(b,v)]^{N-1}, \tag{3-7}$$

and therefore

$$f(b,v) = \omega_N \left(\int_b^{b+D} (\sin t)^{N-1}\, dt \right)^{-1} \frac{\mathcal{I}_N(v')}{\mathcal{I}_N(v)}. \tag{3-8}$$

Thanks to the explicit expression of the isoperimetric profile $\mathcal{I}_N$ it is possible to compute

$$(\mathcal{I}_N^{N/(N-1)})'' \mathcal{I}_N^{(N-2)/(N-1)} = -N. \tag{3-9}$$

In particular it follows from (3-9) that $\mathcal{I}_N^{N/(N-1)}$ is concave on $(0,1)$.

We now distinguish two cases: $v' \le v$ and $v' > v$.

<u>Case 1</u>: $v' \le v$. First observe that

$$\omega_N v' = \int_0^{R(b,v)} (\sin t)^{N-1}\, dt \ge \int_b^{R(b,v)} (\sin t)^{N-1}\, dt = v \int_b^{b+D} (\sin t)^{N-1}\, dt. \tag{3-10}$$

The concavity observed above, together with (3-10), gives that

$$\frac{\mathcal{I}_N(v')}{\mathcal{I}_N(v)} \ge \left(\frac{v'}{v}\right)^{1-1/N} \ge \left(\omega_N^{-1} \int_b^{b+D} (\sin t)^{N-1}\, dt\right)^{1-1/N}.$$

Hence, taking into account (3-8), we obtain

$$f(b,v) \ge \omega_N^{1/N} \left(\int_b^{b+D} (\sin t)^{N-1}\, dt \right)^{-1/N}. \tag{3-11}$$

<u>Case 2</u>: $v' > v$. A change of variables in the definition of R easily yields

$$R(\pi - b - D, 1 - v) = \pi - R(b,v)$$

and therefore

$$f(b,v) = f(\pi - b - D, 1 - v). \tag{3-12}$$

Moreover

$$\int_0^{R(\pi-b-D,1-v)} (\sin t)^{N-1}\, dt = \int_{R(b,v)}^{\pi} (\sin t)^{N-1}\, dt = (1-v')\omega_N;$$

hence

$$f(\pi - b - D, 1 - v) = \omega_N \left(\int_b^{b+D} (\sin t)^{N-1} \right)^{-1} \frac{\mathcal{I}_N(1-v')}{\mathcal{I}_N(1-v)}. \tag{3-13}$$

Next we observe that, as in the previous case, the concavity of $\mathcal{I}_N^{N/(N-1)}$ yields

$$\frac{\mathcal{I}_N(1-v')}{\mathcal{I}_N(1-v)} \ge \left(\frac{1-v'}{1-v}\right)^{1-1/N}. \tag{3-14}$$

Moreover, it holds

$$\omega_N(1-v') = \int_{R(b,v)}^{\pi} (\sin t)^{N-1}\, dt \ge \int_{R(b,v)}^{b+D} (\sin t)^{N-1}\, dt = (1-v) \int_b^{b+D} (\sin t)^{N-1}\, dt. \tag{3-15}$$

Combining (3-13), (3-14) and (3-15) and taking into account (3-12), we get

$$f(b,v) \geq \omega_N^{1/N}\left(\int_b^{b+D}(\sin t)^{N-1}\,dt\right)^{-1/N}. \tag{3-16}$$

It is now sufficient to observe that the function $x \mapsto \int_x^{x+D}(\sin t)^{N-1}\,dt$ attains its maximum at $x = \pi/2 - D/2$ in order to obtain from (3-11), (3-16), (3-6) and (3-3) that

$$\frac{\mathcal{I}_{N,D}(v)}{\mathcal{I}_N(v)} \geq \left(\frac{\omega_N}{\int_{\pi/2-D/2}^{\pi/2+D/2}(\sin t)^{N-1}\,dt}\right)^{1/N} = C_{N,D} \quad \text{for all } v \in (0,1).$$

Above, the last identity follows from the expression for $C_{N,D}$ introduced in (3-5) thanks to the identity $\cos(\pi/2 - x) = \sin(x)$ and a change of variables. □

Let us study the behavior of $C_{N,D}$ in the asymptotic $D \to \pi$.

Lemma 3.2. *It holds that*

$$\lim_{D\to\pi}\frac{(\pi-D)^N}{C_{N,D}^2 - 1} = 2^{N-1}N^2\int_0^{\pi/2}(\cos t)^{N-1}\,dt. \tag{3-17}$$

Hence there exist $\bar{C} = \bar{C}(N) > 0$ and $\bar{D} = \bar{D}(N) < \pi$ such that

$$C_{N,D}^2 - 1 \geq \bar{C}(\pi - D)^N \quad \textit{for all } D \in [\bar{D}, \pi]. \tag{3-18}$$

Proof. Recalling the expression of $C_{N,D}$ from (3-5), we have

$$C_{N,D}^2 - 1 = \left(\frac{\int_0^{\pi/2}(\cos t)^{N-1}\,dt}{\int_0^{D/2}(\cos t)^{N-1}\,dt}\right)^{2/N} - 1 = \left(1 + \frac{\int_{D/2}^{\pi/2}(\cos t)^{N-1}\,dt}{\int_0^{D/2}(\cos t)^{N-1}\,dt}\right)^{2/N} - 1.$$

Now, as $D \to \pi$, we have the expansion

$$\int_{D/2}^{\pi/2}(\cos t)^{N-1}\,dt = \int_0^{\pi/2-D/2}(\sin t)^{N-1}\,dt \sim \int_0^{\pi/2-D/2}s^{N-1}\,ds \sim \frac{1}{N}\left(\frac{\pi}{2} - \frac{D}{2}\right)^N.$$

Taking into account the asymptotic $(1+x)^\beta - 1 \sim \beta x$, we obtain (3-17).

The second conclusion in the statement easily follows from the first one. □

3B. ***Spectral gap and diameter.*** Building on top of the lower bound of the isoperimetric profile obtained in Lemma 3.1, we next obtain a quantitative spectral gap inequality for Neumann boundary conditions in terms diameters.

The analogous result in the case of smooth Riemannian manifolds was established in [Croke 1982, Theorem B] building upon a quantitative improvement of the Lévy–Gromov inequality and on [Bérard and Meyer 1982] (see also [Bérard et al. 1985, Corollary 17]). The usual strategy to show the improved Neumann spectral gap inequality is based on the observation that a Neumann first eigenfunction of the Laplacian f is a Dirichlet first eigenfunction of the Laplacian on the domains $\{f > 0\}$ and $\{f < 0\}$ (see, for instance, [Matei 2000, Lemma 3.2]). The improved Dirichlet spectral gap inequality is then obtained by rearrangement starting from the isoperimetric inequality.

Proposition 3.3 (1-dimensional quantitative Obata's theorem on the diameter). *Let* $(I, \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$ *be a one-dimensional* $\mathrm{CD}(N-1, N)$ *m.m.s. with* $\mathrm{diam}(I) \le D$. *Then*

$$\frac{\lambda^{1,2}_{(I,\mathsf{d}_{\mathrm{eucl}},\mathfrak{m})}}{N} \ge C^2_{N,D} = \left(\frac{\int_0^{\pi/2} (\cos t)^{N-1}\, dt}{\int_0^{D/2} (\cos t)^{N-1}\, dt} \right)^{2/N}, \tag{3-19}$$

where $C_{N,D}$ *was defined in* (3-5).

In particular, there exists a constant $C_N > 0$ (*more precisely one can choose* $C_N = \bar{C} N$, *where* $\bar{C}$ *was defined in* Lemma 3.2) *such that*

$$C_N(\pi - \mathrm{diam}(I))^N \le \lambda^{1,2}_{(I,\mathsf{d}_{\mathrm{eucl}},\mathfrak{m})} - N. \tag{3-20}$$

Proof. From [Bakry and Qian 2000] (see also [Cavalletti and Mondino 2017c, Section 4.1] for the regularization procedure) we know that $\lambda^{1,2}_{(I,\mathsf{d}_{\mathrm{eucl}},\mathfrak{m})} \ge \lambda^{1,2}_{N,D}$ where $\lambda^{1,2}_{N,D}$ is the first solution $\lambda > 0$ of the eigenvalue problem

$$\ddot{w} + (N-1)\tan(t)\dot{w} + \lambda w = 0, \tag{3-21}$$

on $[-D/2, D/2]$ with Neumann boundary conditions. The eigenfunction associated to the first eigenvalue in (3-21) is unique, up to a multiplicative constant. Therefore, denoting it by $w_{N,D} : [-D/2, D/2] \to (-\infty, +\infty)$, it holds $w_{N,D}(-x) = -w_{N,D}(x)$ for any $x \in [-D/2, D/2]$ as a consequence of the symmetry of (3-21). In particular $w_{N,D}(0) = 0$. Let

$$\mathfrak{m}_{N,D} := \Lambda_{N,D} (\cos t)^{N-1} \mathcal{L}^1 \llcorner_{[-D/2,D/2]},$$

with $\Lambda_{N,D}$ such that $\mathfrak{m}_{N,D}$ is a probability measure. Note that $([-D/2, D/2], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_{N,D})$ is a $\mathrm{CD}(N-1, N)$ m.m.s. with diameter equal to D and $\mathfrak{m}_{N,D}([-D/2, 0]) = \mathfrak{m}_{N,D}([0, D/2]) = \frac{1}{2}$. Hence

$$\lambda^{1,2}_{N,D} = \frac{\int_{-D/2}^{D/2} |w'_{N,D}|^2\, \mathfrak{m}_{N,D}}{\int_{-D/2}^{D/2} |w_{N,D}|^2\, \mathfrak{m}_{N,D}} = \frac{\int_0^{D/2} |w'_{N,D}|^2\, \mathfrak{m}_{N,D}}{\int_0^{D/2} |w_{N,D}|^2\, \mathfrak{m}_{N,D}} \ge \lambda^{1,2,\mathcal{D}}_{N,D}\big(\tfrac{1}{2}\big),$$

where $\lambda^{1,2,\mathcal{D}}_{N,D}\big(\frac{1}{2}\big)$ is the least first eigenvalue of the Laplacian with Dirichlet boundary conditions on one extremum for intervals of volume $\frac{1}{2}$ in $([-D/2, D/2], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_{N,D})$.

Moreover a coarea argument (see for instance [Bérard et al. 1985, Corollary 17; Mondino and Semola 2020, Proposition 3.13]) using Lemma 3.1 gives

$$\lambda^{1,2,\mathcal{D}}_{N,D}\big(\tfrac{1}{2}\big) \ge C^2_{N,D}\, \lambda^{1,2,\mathcal{D}}_{N,\pi}\big(\tfrac{1}{2}\big).$$

Recalling that $\lambda^{1,2,\mathcal{D}}_{N,\pi}\big(\frac{1}{2}\big) = \lambda^{1,2}_{N,\pi} = N$ (see for instance [Bakry and Qian 2000]), we conclude that

$$\lambda^{1,2}_{(I,\mathsf{d}_{\mathrm{eucl}},\mathfrak{m})} \ge \lambda^{1,2}_{N,D} \ge \lambda^{1,2,\mathcal{D}}_{N,D}\big(\tfrac{1}{2}\big) \ge N\, C^2_{N,D}. \tag{3-22}$$

The second part of the statement follows by choosing $D = \mathrm{diam}(I)$ and applying Lemma 3.2. □

A converse of the inequality proved in Proposition 3.3 can be obtained as follows.

Lemma 3.4. *For any $N > 1$ there exists $C = C(N) > 0$ such that if $([0, D], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$ is a one-dimensional $\mathrm{CD}(N-1, N)$ m.m.s. with $D \geq \pi - \epsilon$ then*

$$|\lambda^{1,2}_{([0,D],\mathsf{d}_{\mathrm{eucl}},\mathfrak{m})} - N| \leq C\epsilon.$$

Proof. By the Lichnerowicz spectral gap we already know that $\lambda^{1,2}_{([0,D],\mathsf{d}_{\mathrm{eucl}},\mathfrak{m})} \geq N$. It is therefore enough to prove the existence of $u \in \mathrm{Lip}([0, D])$ such that

$$\|u\|_{L^2([0,D],\mathfrak{m})} = 1, \qquad \int_{[0,D]} u\,\mathfrak{m} = 0, \qquad \int_{[0,D]} |u'|^2\,\mathfrak{m} \leq N + C_N\epsilon. \tag{3-23}$$

Setting $u^*_N(t) := \sqrt{N+1}\cos(t)$ and using Corollary 2.8 we get

$$\left|\int_{[0,D]} u^*_N\,\mathfrak{m}\right| \leq C_N\epsilon, \qquad \left|1 - \int_{[0,D]} |u^*_N|^2\,\mathfrak{m}\right| \leq C_N\epsilon, \qquad \int_{[0,D]} |(u^*_N)'|^2\,\mathfrak{m} \leq N + C_N\epsilon. \tag{3-24}$$

Let $v = u^*_N - \fint_{[0,D]} u^*_N\,\mathfrak{m}$ and $c_v := \|v\|_{L^2([0,D],\mathfrak{m})}$. Using the estimates (3-24), it is straightforward to check that $u = (1/c_v)v$ satisfies (3-23). □

3C. *Spectral gap and shape of eigenfunctions.* Next we establish some basic estimates on eigenfunctions which will be useful later.

Given a one-dimensional $\mathrm{CD}(K, N)$ space $(I, \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$, we know that we can write $\mathfrak{m}(dt) = h\mathcal{L}^1(dt)$ for some $\mathrm{CD}(K, N)$ density h. We start by recalling the definition and basic properties of the Laplace operator Δ. A function $u \in W^{1,2}(I, \mathfrak{m})$ is said to be in the domain of Δ, and we write $u \in \mathrm{Dom}(\Delta)$ if for every $\phi \in C^\infty_c(I)$ it holds

$$\left|\int_I u'\phi'\,\mathfrak{m}\right| \leq C_u\|\phi\|_{L^2(I,\mathfrak{m})}$$

for some $C_u \geq 0$ depending on u. In this case, by the Riesz theorem, there exists a function $\Delta u \in L^2(I, \mathfrak{m})$ such that

$$-\int_I u'\phi'\,\mathfrak{m} = \int_I \Delta u\,\phi\,\mathfrak{m}.$$

It is readily seen that the operator $\mathrm{Dom}(\Delta) \ni u \mapsto \Delta u \in L^2(I, \mathfrak{m})$ is linear.

Moreover, using the properties of $\mathrm{CD}(K, N)$ densities recalled at the beginning of the section, it holds that every $u \in \mathrm{Dom}(\Delta)$ is twice differentiable $\mathcal{L}^1$-a.e. on I and

$$\Delta u = u'' + (\log h)'u', \qquad \mathcal{L}^1\text{-a.e. on } I, \text{ for all } u \in \mathrm{Dom}(\Delta). \tag{3-25}$$

Proposition 3.5. *Let $(I, \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$ be a one-dimensional $\mathrm{CD}(N-1, N)$ m.m.s. Then there exists a constant $C = C(N) > 0$ such that, if u is an eigenfunction of the Laplacian on $(I, \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$ associated to an eigenvalue $\lambda \in [N, 2N]$ and with $\|u\|_2 = 1$, then $u \in W^{2,2}_{\mathrm{loc}}(I, \mathsf{d}_{\mathrm{eucl}}, \mathcal{L}^1)$ and*

$$\|u'' + u\|_{L^2(\mathfrak{m})} \leq C(\lambda - N)^{1/2}. \tag{3-26}$$

Proof. Step 1: We claim that it holds

$$\int_I \Big(u'' - \frac{1}{N}\Delta u\Big)^2\,\mathfrak{m} \leq \int_I \Big(\frac{N-1}{N}(\Delta u)^2 - (N-1)(u')^2\Big)\,\mathfrak{m}. \tag{3-27}$$

Since by assumption $u \in W^{1,2}(I, \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$ is an eigenfunction we have $-\Delta u \in W^{1,2}(I, \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$ as well. Thus we can define the Γ_2 operator as

$$\Gamma_2(u;\phi) := \int_I \left(\tfrac{1}{2}(u')^2 \Delta\phi - (\Delta u)' \, u'\phi\right) \mathfrak{m} \tag{3-28}$$

for all $\phi \in L^\infty(I, \mathfrak{m})$ with $\Delta\phi \in L^\infty(I, \mathfrak{m})$. Using that h satisfies (2-8), a manipulation via integration by parts gives that for all $\phi \geq 0$ as above it holds:

$$\Gamma_2(u;\phi) \geq \int_I \left[(u'')^2 + (N-1)(u')^2 + \frac{1}{N-1}(\Delta u - u'')^2\right]\phi \, \mathfrak{m}. \tag{3-29}$$

By direct computations, one can check that

$$\begin{aligned}(u'')^2 + (N-1)(u')^2 &+ \frac{1}{N-1}(\Delta u - u'')^2 \\ &= (N-1)(u')^2 + \left(u'' - \frac{1}{N}\Delta u\right)^2 + \frac{1}{N}(\Delta u)^2 + \frac{1}{N-1}\left(u'' - \frac{1}{N}\Delta u\right)^2 \quad \mathfrak{m}\text{-a.e.}\end{aligned} \tag{3-30}$$

Plugging (3-30) into (3-29) gives

$$\Gamma_2(u;\phi) \geq \int_I \left[(N-1)(u')^2 + \left(u'' - \frac{1}{N}\Delta u\right)^2 + \frac{1}{N}(\Delta u)^2\right]\phi \, \mathfrak{m}.$$

Choosing $\phi \equiv 1$ yields (3-27).

<u>Step 2</u>: Inserting the eigenvalue relation $\lambda u = -\Delta u$ into (3-27), we obtain

$$\int_I \left(u'' + \frac{\lambda}{N}u\right)^2 \mathfrak{m} \leq \int_I \left(\frac{N-1}{N}(\lambda u)^2 - (N-1)(u')^2\right) \mathfrak{m} = \frac{N-1}{N}\lambda(\lambda - N)\int_I u^2 \, \mathfrak{m}. \tag{3-31}$$

Eventually,

$$\begin{aligned}\int_I (u'' + u)^2 \, \mathfrak{m} &\leq 2\int_I \left|u'' + \frac{\lambda}{N}u\right|^2 \mathfrak{m} + 2\int_I \left|\frac{\lambda - N}{N}u\right|^2 \mathfrak{m} \\ &\leq 2\left(\frac{N-1}{N}\lambda(\lambda - N) + \frac{(\lambda - N)^2}{N^2}\right)\int_I u^2 \, \mathfrak{m} \leq C(N)(\lambda - N)\int_I u^2 \, \mathfrak{m},\end{aligned}$$

where, in the last estimate, we used the assumption $\lambda \leq 2N$. □

The aim of the remaining part of this section is to prove Theorem 3.11 stating roughly that, on any one-dimensional $\mathrm{CD}(N-1, N)$ m.m.s. $(I, \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$, a function $u : I \to \mathbb{R}$ whose 2-Rayleigh quotient is close to N (the optimal one on the model $(N-1, N)$-space) and with L^2-norm equal to 1, is $W^{1,2}$-close to the (normalized) cosine of the distance from one of the extrema of the interval, in quantitative terms.

The conclusion of Theorem 3.11 will be achieved through some intermediate steps. First we estimate the $W^{1,2}$-closeness of a first eigenfunction u^* for $(I, \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$ with the cosine of the distance from one of the extremes of the segment, see Proposition 3.6. Then, we bound the $W^{1,2}$-closeness of the function u from u^* (or $-u^*$), see Proposition 3.10.

Let us observe that

$$\|\cos(\,\cdot\,)\|_{L^2(\mathfrak{m}_N)} = \frac{1}{\sqrt{N+1}}, \tag{3-32}$$

and, by symmetry,

$$\int_{[0,\pi]} \cos(t)\, \mathfrak{m}_N(dt) = 0. \tag{3-33}$$

Proposition 3.6. *For every $N > 1$ there exist constants $C = C(N) > 0$ and $\epsilon_0 = \epsilon_0(N) > 0$ such that for every one dimensional* $\mathrm{CD}(N-1, N)$ *m.m.s.* $([0, D], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$ *and every Neumann eigenfunction u^*, with $\|u^*\|_{L^2(\mathfrak{m})} = 1$, of eigenvalue $\lambda \in [N, 2N]$ it holds*

$$\min\{\|u^* - \sqrt{N+1}\cos(\cdot\,)\|_{L^2(\mathfrak{m})}, \|u^* + \sqrt{N+1}\cos(\cdot\,)\|_{L^2(\mathfrak{m})}\} \le C\delta^{\min\{1/2, 1/N\}}, \tag{3-34}$$

where $\delta := \int |\nabla u^*|^2\, \mathfrak{m} - N < \epsilon_0$. Furthermore the conclusion can be improved to $W^{1,2}$-closeness:

$$\min\{\|(u^* - \sqrt{N+1}\cos(\cdot\,))'\|_{L^2(\mathfrak{m})}, \|(u^* + \sqrt{N+1}\cos(\cdot\,))'\|_{L^2(\mathfrak{m})}\} \le C\delta^{\min\{1/2, 1/N\}}. \tag{3-35}$$

Proof. Let $h : [0, D] \to [0, +\infty)$ be the density of $\mathfrak{m}$ with respect to $\mathcal{L}^1$ and let $x_0 \in (0, D)$ be a maximum point of h. In [Cavalletti et al. 2019, Lemma A.4] it is proved that such a maximum point is unique and that h is strictly increasing on $[0, x_0]$ and strictly decreasing on $[x_0, D]$.

Step 1: In this first step we prove that, given $z \in L^2([0, D], \mathfrak{m})$, any solution of $v'' + v = z$ can be written as

$$v(t) = \int_{x_0}^{t} \sin(t-s) z(s)\, ds + \alpha \sin(t) + \beta \cos(t) \tag{3-36}$$

for some $\alpha, \beta \in \mathbb{R}$. To this aim, it suffices to prove that

$$v_0(t) := \int_{x_0}^{t} \sin(t-s) z(s)\, ds \tag{3-37}$$

solves $v'' + v = z$. First we observe that v_0 is well-defined, since the assumption $z \in L^2((0, D), \mathfrak{m})$ guarantees that $z \in L^1_{\mathrm{loc}}((0, D), \mathcal{L}^1)$ (due to the fact that h is locally bounded from below by a strictly positive constant in the interior of $[0, D]$). The fact that it satisfies $v_0'' + v_0 = z$ follows from an elementary computation.

Step 2: Next, we prove that the function v_0 defined in (3-37) satisfies

$$\|v_0\|_{L^2(\mathfrak{m})} \le \pi \|z\|_{L^2(\mathfrak{m})}. \tag{3-38}$$

Indeed, taking into account that $|\sin| \le 1$, applying the Cauchy–Schwarz inequality, Fubini's theorem and recalling that h is increasing on $[0, x_0]$ and decreasing on $[x_0, D]$, we can compute

$$\begin{aligned}
\|v_0\|^2_{L^2(\mathfrak{m})} &= \int_0^D \left(\int_{x_0}^{t} \sin(t-s) z(s)\, ds \right)^2 h(t)\, dt \le \pi \int_0^D h(t) \left| \int_{x_0}^{t} z^2(s)\, ds \right| dt \\
&= \pi \left(\int_0^{x_0} z^2(s) \int_0^{s} h(t)\, dt ds + \int_{x_0}^{D} z^2(s) \int_s^D h(t)\, dt\, ds \right) \\
&\le \pi^2 \left(\int_0^{x_0} z^2(s) h(s)\, ds + \int_{x_0}^{D} z^2(s) h(s)\, ds \right) = \pi^2 \|z\|^2_{L^2(\mathfrak{m})}.
\end{aligned}$$

Let us remark that from (3-38) it follows applying Cauchy–Schwarz inequality that $\|v_0\|_{L^1(\mathfrak{m})} \le \pi \|z\|_{L^2(\mathfrak{m})}$.

Step 3: Recall from Proposition 3.3 the bound $\pi - D \le C\delta^{1/N}$. Furthermore we know from (3-26) that if u^* is as in the assumptions of the statement, then $(u^*)'' + u^* = z$ on $[0, D]$ for some function z such that

$\|z\|_{L^2(\mathfrak{m})} \leq C\delta^{1/2}$. Hence, as proved in Step 1, u^* can be written as

$$u^*(t) = \int_{x_0}^{t} \sin(t-s)z(s)\,ds + \alpha\sin(t) + \beta\cos(t) \tag{3-39}$$

for some $\alpha, \beta \in \mathbb{R}$. We want to show that there exists $C = C(N) > 0$ such that $|\alpha| + |\beta| \leq C(N)$.

Set $u_0(t) := \int_{x_0}^{t} \sin(t-s)z(s)\,ds$ and recall that, from Step 2, it holds $\|u_0\|_{L^2(\mathfrak{m})} \leq C\delta^{1/2}$. Since by assumption u^* has null mean value, integrating (3-39) over $[0, D]$ with respect to $\mathfrak{m}$ gives

$$0 = \alpha\int_{[0,D]} \sin(t)\mathfrak{m}(dt) + \beta\int_{[0,D]} \cos(t)\mathfrak{m}(dt) + \int_{[0,D]} u_0(t)\mathfrak{m}(dt). \tag{3-40}$$

From the last remark in Step 2 and Corollary 2.8, it follows that

$$\left(\int_{[0,\pi]} \sin^N(t)\,dt + O(\delta^{1/N})\right)\alpha + O(\delta^{1/N})\beta + O(\delta^{1/2}) = 0,$$

giving that

$$\alpha = O(\delta^{1/N})\beta + O(\delta^{1/2}). \tag{3-41}$$

In order to estimate β, we compute the $L^2(\mathfrak{m})$-norm squared of both the left- and right-hand sides of (3-39) to obtain

$$\begin{aligned} 1 = \|u_0\|^2_{L^2(\mathfrak{m})} &+ \alpha^2\|\sin(\cdot)\|^2_{L^2(\mathfrak{m})} + \beta^2\|\cos(\cdot)\|^2_{L^2(\mathfrak{m})} \\ &+ 2\alpha\int u_0(t)\sin(t)\,\mathfrak{m}(dt) + 2\beta\int u_0(t)\cos(t)\,\mathfrak{m}(dt) + 2\alpha\beta\int \sin(t)\cos(t)\,\mathfrak{m}(dt). \end{aligned} \tag{3-42}$$

Plugging (3-41) into (3-42), gives

$$(1 + O(\delta)) + O(\delta^{1/N+1/2})\,\beta + \left(\int_{[0,\pi]} \cos^2(t)\sin^{N-1}(t)\,dt + O(\delta^{1/N})\right)\beta^2 = 0, \tag{3-43}$$

yielding $|\beta| \leq C(N)$ and thus, by (3-41), also $|\alpha| \leq C(N)$.

<u>Step 4</u>: Conclusion. In order to get (3-34), we have to bound $|\alpha|$ and $\min\{|\sqrt{N+1} - \beta|, |\sqrt{N+1} + \beta|\}$ in terms of δ.

From (3-40), Step 3, the last remark in Step 2 and Corollary 2.8 it follows that

$$|\alpha| \leq C(\delta^{1/2} + \delta^{1/N}) \leq C\delta^{\min\{1/N,1/2\}}, \tag{3-44}$$

up to increasing the value of the constant C in the second inequality. Plugging (3-44) into (3-42) gives

$$1 = O(\delta) + O(\delta^{\min\{1,2/N\}}) + O(\delta^{1/2}) + O(\delta^{\min\{1/2,1/N\}}) + \beta^2/(N+1)$$

and therefore

$$\left|1 - \frac{\beta^2}{N+1}\right| = O(\delta^{\min\{1/2,1/N\}}). \tag{3-45}$$

From (3-45) we easily obtain that

$$\min\{|\sqrt{N+1} - \beta|, |\sqrt{N+1} + \beta|\} \leq C\delta^{\min\{1/4,1/(2N)\}}. \tag{3-46}$$

In the case

$$|\sqrt{N+1} - \beta| = \min\{|\sqrt{N+1} - \beta|, |\sqrt{N+1} + \beta|\} \leq C\delta^{\min\{1/4,1/(2N)\}}$$

(respectively $|\sqrt{N+1}+\beta| = \min\{|\sqrt{N+1}-\beta|, |\sqrt{N+1}+\beta|\} \le C\delta^{\min\{1/4,1/(2N)\}}$), it follows that

$$|\sqrt{N+1}+\beta| \ge 2\sqrt{N+1} - C\delta^{\min\{1/4,1/(2N)\}} \ge \sqrt{N+1} \quad \text{for } \delta \le \delta_0(N). \tag{3-47}$$

(resp. $|\sqrt{N+1}-\beta| \ge \sqrt{N+1}$). Plugging (3-47) back into (3-45) gives $|\sqrt{N+1}-\beta| \le C\delta^{\min\{1/2,1/N\}}$ (resp. $|\sqrt{N+1}+\beta| \le C\delta^{\min\{1/2,1/N\}}$). In conclusion, (3-45) and (3-46) can be bootstrapped to give

$$\min\{|\sqrt{N+1}-\beta|, |\sqrt{N+1}+\beta|\} \le C\delta^{\min\{1/2,1/N\}}. \tag{3-48}$$

Combining all these ingredients we can eventually estimate the $L^2(\mathfrak{m})$-distance between the first Neumann eigenfunction and the normalized cosine. Indeed, assuming without loss of generality that $|\sqrt{N+1}-\beta| \le |\sqrt{N+1}+\beta|$ and taking into account (3-44), (3-48), we obtain

$$\begin{aligned}\|u^* - \sqrt{N+1}\cos(\,\cdot\,)\|_{L^2(\mathfrak{m})} &= \|u_0 + \alpha\sin(\,\cdot\,) + \beta\cos(\,\cdot\,) - \sqrt{N+1}\cos(\,\cdot\,)\|_{L^2(\mathfrak{m})} \\ &\le |\alpha|\|\sin(\,\cdot\,)\|_{L^2(\mathfrak{m})} + \|u_0\|_{L^2(\mathfrak{m})} + |\beta - \sqrt{N+1}|\|\cos(\,\cdot\,)\|_{L^2(\mathfrak{m})} \\ &\le C\delta^{\min\{1/2,1/N\}}.\end{aligned}$$

Finally, we improve the $L^2(\mathfrak{m})$-closeness to $W^{1,2}(\mathfrak{m})$-closeness. To this aim, differentiate (3-39) to obtain

$$(u^*)'(t) = \int_{x_0}^t \cos(t-s)z(s)\,ds + \alpha\cos(t) - \beta\sin(t). \tag{3-49}$$

With computations analogous to the ones used to obtain the bound $\|v_0\|_2 \le \pi\|z\|_2$ in Step 2, one can prove that, letting $w_0(t) := \int_{x_0}^t \cos(t-s)\,ds$, it holds $\|w_0\|_2 \le \pi\|z\|_2$. The sought estimate for

$$\min\{\|(u^* - \sqrt{N+1}\cos(\,\cdot\,))'\|_{L^2(\mathfrak{m})}, \|(u^* + \sqrt{N+1}\cos(\,\cdot\,))'\|_{L^2(\mathfrak{m})}\}$$

follows taking into account (3-44) and (3-46). □

We isolate the following corollary, which will be useful later in the paper.

Corollary 3.7. *Under the assumptions of Proposition 3.6, setting $r = \delta^{\gamma/N}$ for some $\gamma \in (0,1)$, it holds*

$$\begin{aligned}\min\{\|u^* - \sqrt{N+1}\cos(\,\cdot\,)\|_{W^{1,2}([0,r],\mathfrak{m})}, \|u^* + \sqrt{N+1}\cos(\,\cdot\,)\|_{W^{1,2}([0,r],\mathfrak{m})}\}& \\ \le C(N)(\delta^{1/2} + r^{N/2}\delta^{\min\{1/2,1/N\}}).&\end{aligned} \tag{3-50}$$

Moreover, for $\eta \in (0, \frac{1}{10}r)$,

$$\begin{aligned}\min\{\|u^* - \sqrt{N+1}\cos(\,\cdot\,)\|_{W^{1,2}([r-\eta,r+\eta],\mathfrak{m})}, \|u^* + \sqrt{N+1}\cos(\,\cdot\,)\|_{W^{1,2}([r-\eta,r+\eta],\mathfrak{m})}\}& \\ \le C(N)(\delta^{1/2} + (r^{N-1}\eta)^{1/2}\delta^{\min\{1/2,1/N\}}).&\end{aligned} \tag{3-51}$$

Proof. It is enough to improve the final estimates in Step 4 of the proof of Proposition 3.6 by using (2-14):

$$\begin{aligned}\|u^* - \sqrt{N+1}\cos(\,\cdot\,)\|_{L^2([0,r],\mathfrak{m})} &= \|u_0 + \alpha\sin(\,\cdot\,) + \beta\cos(\,\cdot\,) - \sqrt{N+1}\cos(\,\cdot\,)\|_{L^2([0,r]\mathfrak{m})} \\ &\le \|u_0\|_{L^2([0,r],\mathfrak{m})} + |\alpha|\|\sin(\,\cdot\,)\|_{L^2([0,r],\mathfrak{m})} + |\beta - \sqrt{N+1}|\|\cos(\,\cdot\,)\|_{L^2([0,r]\mathfrak{m})} \\ &\le C(\delta^{1/2} + \delta^{\min\{1/2,1/N\}}(\|\cos(\,\cdot\,)\|_{L^2([0,r]\mathfrak{m}_N)} + C\delta^{1/N}r^{N-1}\})) \\ &\le C(\delta^{1/2} + r^{N/2}\delta^{\min\{1/2,1/N\}}).\end{aligned}$$

The improved estimate for the first derivative and for the domain $[r-\eta, r+\eta]$ is analogous. □

Lemma 3.8. *For any $N > 1$ there exist $\bar{D} = \bar{D}(N) < \pi$ and $\alpha = \alpha(N) > 0$ such that the following holds. Let $([0, D], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$ be a one-dimensional $\mathrm{CD}(N-1, N)$ m.m.s. with $D \geq \bar{D}$ and u^* any first Neumann eigenfunction, with $\|u^*\|_{L^2(\mathfrak{m})} = 1$.*

Then for any $v \in L^2([0, D], \mathfrak{m})$ with $\|v\|_{L^2(\mathfrak{m})} = 1$ such that $\left|\int vu^*\, \mathfrak{m}\right| \leq \frac{1}{2}$ we have

$$N + \alpha \leq \int_{[0,D]} |v'|^2\, \mathfrak{m}.$$

Proof. We argue by contradiction.

Suppose there is a sequence of $\mathrm{CD}(N-1, N)$ measures $\mathfrak{m}_n = h_n \mathcal{L}^1$ with $\operatorname{supp} h_n = [0, D_n]$ and $D_n \uparrow \pi$ satisfying the following: for every n there exists $v_n \in W^{1,2}([0, D_n], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_n)$ with $\|v_n\|_{L^2(\mathfrak{m}_n)} = 1$ such that

$$\int_{[0,D_n]} |v_n'|^2\, \mathfrak{m}_n \to N \quad \text{as } n \to \infty, \qquad \text{and} \qquad \left|\int v_n\, u_n^*\, \mathfrak{m}_n\right| \leq \frac{1}{2}, \tag{3-52}$$

where u_n^* is a first Neumann eigenfunction on $([0, D_n], \mathsf{d}_{\mathrm{eucl}}, h_n \mathcal{L}^1)$, i.e.,

$$\int_{[0,D_n]} |u_n^*|^2\, \mathfrak{m}_n = 1, \qquad \int_{[0,D_n]} |(u_n^*)'|^2\, \mathfrak{m}_n = \lambda_n \to N, \tag{3-53}$$

where in the last identity we used (3-25), and the convergence of λ_n to N follows from Lemma 3.4.

From Corollary 2.7, the fact that $\operatorname{supp} h_n = [0, D_n]$ with $D_n \uparrow \pi$ implies that (h_n) (extended to the constant $h(D_n)$ on $[D_n, \pi]$) converges uniformly to the model one-dimensional $\mathrm{CD}(N-1, N)$-density $h_N = (1/c_N') \sin^{N-1}$ on $[0, \pi]$. In particular, for every $\eta \in (0, \pi/2)$ the densities h_n restricted to $[\eta, 1-\eta]$ are bounded above and below by strictly positive constants.

The bounds (3-53) then imply that u_n^* (resp. v_n) are uniformly $\frac{1}{2}$-Hölder continuous on $[\eta, \pi - \eta]$ for every $\eta \in (0, \pi/2)$.

Thus, by the Arzelà–Ascoli theorem combined with a standard diagonal argument, there exists $u^* : [0, \pi] \to \mathbb{R}$ (resp. $v : [0, \pi] \to \mathbb{R}$) and a (nonrelabeled for simplicity) subsequence such that $u_n^* \to u^*$ (resp. $v_n \to v$) uniformly on $[\eta, \pi - \eta]$ for every $\eta \in (0, \pi/2)$. It is also easy to check that

$$\int_{[0,\pi]} u_n^* h_n\, \phi\, \mathcal{L}^1 \to \int_{[0,\pi]} u^* h_N\, \phi\, \mathcal{L}^1, \quad \int_{[0,\pi]} v_n h_n\, \phi\, \mathcal{L}^1 \to \int_{[0,\pi]} v h_N\, \phi\, \mathcal{L}^1 \quad \text{for all } \phi \in C([0, \pi]).$$

Combining the last weak convergence statement with the bounds (3-52), (3-53) and with [Gigli et al. 2015, Theorem 6.3] gives

$$\|u^*\|_{L^2([0,\pi],\mathfrak{m}_N)} = \|v\|_{L^2([0,\pi],\mathfrak{m}_N)} = 1, \qquad \left|\int_{[0,\pi]} u^* v\, \mathfrak{m}_N\right| \leq \frac{1}{2},$$

$$\int_{[0,\pi]} |(u^*)'|^2\, \mathfrak{m}_N \leq N, \qquad \int_{[0,\pi]} |v'|^2\, \mathfrak{m}_N \leq N.$$

Therefore, both u^* and v are first Neumann eigenfunctions on the model space $([0, \pi], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_N)$. However the first eigenfunction is unique up to a sign, thus it must hold

$$\left|\int_{[0,\pi]} u^* v\, \mathfrak{m}_N\right| = 1,$$

a contradiction. $\square$

Corollary 3.9. *For every $N > 1$ there exists $\beta = \beta(N) > 0$ with the following property. Let $(I, \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$ be a one-dimensional* $\mathrm{CD}(N-1, N)$ *m.m.s. with $\mathfrak{m}(I) = 1$ and satisfying*

$$\lambda^{1,2}_{(I,\mathsf{d}_{\mathrm{eucl}},\mathfrak{m})} - N < \beta.$$

Then, for any $u \in W^{1,2}(I, \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$ with $\|u\|_{L^2(\mathfrak{m})} = 1$ and $\left|\int_I uu^* \,\mathfrak{m}\right| \le \frac{1}{2}$, where u^* is a first Neumann eigenfunction with $\|u^*\|_{L^2(\mathfrak{m})} = 1$, it holds

$$\lambda^{1,2}_{(I,\mathsf{d}_{\mathrm{eucl}},\mathfrak{m})} + \beta < \int |u'|^2 \,\mathfrak{m}.$$

Proof. First choose $\beta > 0$ sufficiently small so that, by Proposition 3.3, the diameter of $(I, \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$ is bigger than $\bar{D}$. Then conclude by Lemma 3.8 (and decrease the constant $\beta > 0$ if necessary). □

Proposition 3.10. *For every $N > 1$ there exists $\beta = \beta(N) > 0$ with the following property. Let $(I, \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$ be a one-dimensional* $\mathrm{CD}(N-1, N)$ *m.m.s. with $\mathfrak{m}(I) = 1$. Assume there exists $v \in W^{1,2}(I, \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$ with $\|v\|_{L^2(\mathfrak{m})} = 1$ satisfying*

$$\int_I |v'|^2 \,\mathfrak{m} - N < \beta. \tag{3-54}$$

Then it holds

$$\min\{\|v - u^*\|^2_{W^{1,2}(\mathfrak{m})}, \|v + u^*\|^2_{W^{1,2}(\mathfrak{m})}\} \le C\left(\int |v'|^2 \,\mathfrak{m} - \int |(u^*)'|^2 \,\mathfrak{m}\right), \tag{3-55}$$

where u^* is a first Neumann eigenfunction with $\|u^*\|_{L^2(\mathfrak{m})} = 1$.

Proof. We begin by rewriting

$$\begin{aligned}
\int |v'|^2 \,\mathfrak{m} - \int |(u^*)'|^2 \,\mathfrak{m} &= \int |(v - u^*)'|^2 \,\mathfrak{m} + 2 \int (v - u^*)' \,(u^*)' \,\mathfrak{m} \\
&= \int |(v - u^*)'|^2 \,\mathfrak{m} - 2\lambda^{1,2}_{(I,\mathsf{d}_{\mathrm{eucl}},\mathfrak{m})}\left(1 - \int v\, u^* \,\mathfrak{m}\right) \\
&= \int |(v - u^*)'|^2 \,\mathfrak{m} - \lambda^{1,2}_{(I,\mathsf{d}_{\mathrm{eucl}},\mathfrak{m})} \int (v - u^*)^2 \,\mathfrak{m}.
\end{aligned} \tag{3-56}$$

Now (3-54) implies that $\left|\int vu^* \,\mathfrak{m}\right| > \frac{1}{2}$ by Corollary 3.9. Hence, assuming without loss of generality that $\int u^* v \,\mathfrak{m} > \frac{1}{2}$, we get $\left|\int u^*(u^* - v) \,\mathfrak{m}\right| < \frac{1}{2}$. Therefore, Corollary 3.9 yields

$$\int |(v - u^*)'|^2 \,\mathfrak{m} \ge (\lambda^{1,2}_{(I,\mathsf{d}_{\mathrm{eucl}},\mathfrak{m})} + \beta)\|v - u^*\|_2^2.$$

The combination of the last estimate with (3-56) gives

$$\|v - u^*\|_2^2 \le C\left(\int |v'|^2 \,\mathfrak{m} - \int |(u^*)'|^2 \,\mathfrak{m}\right), \tag{3-57}$$

with $C := 1/\beta$. We now improve (3-57) to $W^{1,2}$-closeness, namely (3-55). In order to do so, it suffices to observe that the estimates we obtained above yield

$$\begin{aligned}
\int |(v - u^*)'|^2 \,\mathfrak{m} &\le \lambda^{1,2}_{(I,\mathsf{d}_{\mathrm{eucl}},\mathfrak{m})}\|v - u^*\|_2^2 + \int |v'|^2 \,\mathfrak{m} - \int |(u^*)'|^2 \,\mathfrak{m} \\
&\le C(1 + \lambda^{1,2}_{(I,\mathsf{d}_{\mathrm{eucl}},\mathfrak{m})})\left(\int |v'|^2 \,\mathfrak{m} - \int |(u^*)'|^2 \,\mathfrak{m}\right).
\end{aligned}$$

□

Theorem 3.11 (one-dimensional quantitative Obata's theorem on the function). *For every $N > 1$ there exist constants $C = C(N) > 0$ and $\delta_0 = \delta_0(N) > 0$ with the following property. Let $([0, D], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$ be a one-dimensional $\mathrm{CD}(N-1, N)$ m.m.s. and let $u \in \mathrm{Lip}(I)$ satisfy $\int u\, \mathfrak{m} = 0$ and $\int u^2\, \mathfrak{m} = 1$. If*

$$\delta := \int |u'|^2\, \mathfrak{m} - N \le \delta_0,$$

then

$$\min\{\|u - \sqrt{N+1}\cos(\,\cdot\,)\|_{W^{1,2}(\mathfrak{m})}, \|u + \sqrt{N+1}\cos(\,\cdot\,)\|_{W^{1,2}(\mathfrak{m})}\} \le C\delta^{\min\{1/2, 1/N\}}. \tag{3-58}$$

Moreover, setting $r = \delta^{\gamma/N}$ for some $\gamma \in (0, 1)$, for any $\eta \in \left(0, \frac{1}{10} r\right)$ it holds

$$\begin{aligned}\min\{\|u - \sqrt{N+1}\cos(\,\cdot\,)\|_{W^{1,2}([0,r],\mathfrak{m})}, \|u + \sqrt{N+1}\cos(\,\cdot\,)\|_{W^{1,2}([0,r],\mathfrak{m})}\}&\\ \le C(\delta^{1/2} + r^{N/2}\delta^{\min\{1/2,1/N\}}),&\end{aligned} \tag{3-59}$$

$$\begin{aligned}\min\{\|u^* - \sqrt{N+1}\cos(\,\cdot\,)\|_{W^{1,2}([r-\eta,r+\eta],\mathfrak{m})}, \|u^* + \sqrt{N+1}\cos(\,\cdot\,)\|_{W^{1,2}([r-\eta,r+\eta],\mathfrak{m})}\}&\\ \le C(N)(\delta^{1/2} + (r^{N-1}\eta)^{1/2}\delta^{\min\{1/2,1/N\}}).&\end{aligned} \tag{3-60}$$

Proof. First apply Proposition 3.10 to bound the $W^{1,2}(\mathfrak{m})$-distance between u and a first eigenfunction of the Neumann Laplacian on $([0, D], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m})$, then apply Proposition 3.6 (respectively Corollary 3.7) to bound the $W^{1,2}(\mathfrak{m})$-distance (respectively the $W^{1,2}([0, r], \mathfrak{m})$ or $W^{1,2}([r-\eta, r+\eta], \mathfrak{m})$ distance) between the first eigenfunction and the normalized cosine. The sought estimate follows by the triangle inequality. □

4. Quantitative Obata's theorem on the diameter

Building on top of the one-dimensional results obtained in Section 3, we will derive several quantitative estimates for a general essentially nonbranching m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ satisfying $\mathrm{CD}(K, N)$.

Given an m.m.s. $(X, \mathsf{d}, \mathfrak{m})$, the perimeter $\mathsf{P}(E)$ of a Borel subset $E \subset X$ is defined as

$$\mathsf{P}(E) := \inf\left\{\liminf_{n\to\infty} \int_X |\nabla u_n|\, \mathfrak{m} : u_n \in \mathrm{Lip}(X),\ u_n \to \chi_E \text{ in } L^1_{\mathrm{loc}}(X)\right\}, \tag{4-1}$$

where χ_E is the characteristic function of E. Accordingly $E \subset X$ has finite perimeter in $(X, \mathsf{d}, \mathfrak{m})$ if and only if $\mathsf{P}(E) < \infty$.

The isoperimetric profile $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})} : [0, 1] \to [0, \infty)$ is given by

$$\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) := \inf\{\mathsf{P}(E) : E \subset X,\ \mathfrak{m}(E) = v\}. \tag{4-2}$$

Given a smooth Riemannian manifold (M, g) with finite Riemannian volume $\mathrm{vol}_g(M) < \infty$, let us denote by

$$\mathfrak{m}_g := \frac{1}{\mathrm{vol}_g(M)}\,\mathrm{vol}_g$$

the normalized Riemannian volume measure.

We next recall the improved Lévy–Gromov inequality obtained by Bérard, Besson and Gallot [Bérard et al. 1985, Remark 3.1] for smooth Riemannian N-manifolds with Ricci $\ge N-1$ and with upper bound on the diameter (see also [Milman 2015]).

Theorem 4.1. *Let $(M, \mathsf{d}, \mathfrak{m}_g)$ be the metric measure space associated to a Riemannian manifold (M, g) with dimension $N \in \mathbb{N}$, $N \geq 2$, Ricci bounded from below by $N-1$ and diameter D (recall that, by the Bonnet–Myers theorem, $D \leq \pi$). Then, for any $v \in (0, 1)$, it holds*

$$\frac{\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v)}{\mathcal{I}_N(v)} \geq \left(\frac{\int_0^{\pi/2} (\cos t)^{N-1}\, dt}{\int_0^{D/2} (\cos t)^{N-1}\, dt} \right)^{1/N} =: C_{N,D} \geq 1, \tag{4-3}$$

where $\mathcal{I}_N$, defined in (3-3), for $N \geq 2$, $N \in \mathbb{N}$, is the isoperimetric profile of the normalized round sphere of constant sectional curvature 1 $(\mathbb{S}^N, \mathsf{d}_{\mathbb{S}^N}, \mathfrak{m}_{g_{\mathbb{S}^N}})$.

We extend Theorem 4.1 to the class of essentially nonbranching $\mathrm{CD}(N-1, N)$ metric measure spaces, $N > 1$ any real parameter. In view of [Cavalletti and Mondino 2017b; 2018] the result follows from the one-dimensional improved Lévy–Gromov inequality proved in Lemma 3.1.

Theorem 4.2 (Bérard–Besson–Gallot improved Lévy–Gromov for $\mathrm{CD}(N-1, N)$ e.n.b. spaces). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially nonbranching $\mathrm{CD}(N-1, N)$ m.m.s. with $\mathrm{diam}(X) \leq D$ for some $N > 1$, $D \in (0, \pi]$. Then, for any $v \in (0, 1)$, it holds*

$$\frac{\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v)}{\mathcal{I}_N(v)} \geq \left(\frac{\int_0^{\pi/2} (\cos t)^{N-1}\, dt}{\int_0^{D/2} (\cos t)^{N-1}\, dt} \right)^{1/N} =: C_{N,D} \geq 1, \tag{4-4}$$

where $\mathcal{I}_N$ was defined in (3-3).

Proof. One of the main results in [Cavalletti and Mondino 2017b; 2018] is that for $(X, \mathsf{d}, \mathfrak{m})$ as in the assumptions of the theorem it holds

$$\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) \geq \mathcal{I}_{N,D}(v), \tag{4-5}$$

where $\mathcal{I}_{N,D}$ stands for the model isoperimetric profile defined in (3-3).

The claimed (4-4) follows by combining (4-5) with Lemma 3.1. □

It is also possible to obtain a quantitative spectral gap inequality for Neumann boundary conditions. The analogous result in the case of smooth Riemannian manifolds was established in [Croke 1982, Theorem B] building upon a quantitative improvement of the Lévy–Gromov inequality and on [Bérard and Meyer 1982] (see also [Bérard et al. 1985, Corollary 17]).

Theorem 4.3 (improved spectral gap and quantitative Obata's theorem for $\mathrm{CD}(N-1, N)$ e.n.b. spaces). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially nonbranching $\mathrm{CD}(N-1, N)$ m.m.s. with $\mathrm{diam}(X) \leq D$ for some $N > 1$, $D \in (0, \pi]$. Then*

$$\lambda^{1,2}_{(X,\mathsf{d},\mathfrak{m})} \geq N C^2_{N,D}, \tag{4-6}$$

where $C_{N,D}$ is given in (4-4). Moreover, there exists $C = C_N > 0$ (more precisely one can choose $C_N = \bar{C} N$ where $\bar{C}$ was defined in Lemma 3.2) such that

$$C_N (\pi - \mathrm{diam}(X))^N \leq \lambda^{1,2}_{(X,\mathsf{d},\mathfrak{m})} - N.$$

Proof. Thanks to [Cavalletti and Mondino 2017c, Theorem 4.4] (see also Proposition 3.3) we know that $\lambda^{1,2}_{(X,\mathsf{d},\mathfrak{m})} \geq \lambda^{1,2}_{N,D}$, where $\lambda^{1,2}_{N,D}$ was defined in (3-21).

Let us briefly outline the argument since it will be relevant for addressing the quantitative inequality for the first eigenfunction later in the note. By the very definition of $\lambda^{1,2}_{(X,\mathsf{d},\mathfrak{m})}$ it suffices to prove that, for any $u \in \mathrm{Lip}(X)$ with $\int u\,\mathfrak{m} = 0$ and $\int u^2\,\mathfrak{m} = 1$, it holds

$$\delta(u) := \int_X |\nabla u|^2\,\mathfrak{m} - N \geq C_N(\pi - \mathrm{diam}(X))^N.$$

To this aim, we perform the one-dimensional localization associated to the function u which by assumption has null mean value (this is analogous to the proof of [Cavalletti and Mondino 2017c, Theorem 4.4]; see Section 2D for some basics about one-dimensional localization). We obtain

$$\begin{aligned}\int_X |\nabla u|^2\mathfrak{m} - N\int_X u^2\,\mathfrak{m} &\geq \int_Q \left(\int_{X_q} |u_q'|^2\,\mathfrak{m}_q - N\int_{X_q} u_q^2\,\mathfrak{m}_q\right)\mathfrak{q}(dq)\\ &\geq \int_Q \left(\lambda^{1,2}_{N,\mathrm{diam}(X_q)}\int_{X_q} u_q^2\,\mathfrak{m}_q - N\int_{X_q} u_q^2\,\mathfrak{m}_q\right)\mathfrak{q}(dq)\\ &\geq \int_Q (\lambda^{1,2}_{N,\mathrm{diam}(X)} - N)\int_{X_q} u_q^2\,\mathfrak{m}_q\,\mathfrak{q}(dq) = \lambda^{1,2}_{N,\mathrm{diam}(X)} - N.\end{aligned}$$

Taking into account Proposition 3.3, we conclude that

$$\delta(u) \geq \lambda^{1,2}_{\mathrm{diam}(X),N} - N \geq C_N(\pi - \mathrm{diam}(X))^N$$

and (4-6) can be obtained in an analogous way. □

Remark 4.4. In [Jiang and Zhang 2016] the authors obtained a quantitative version of the estimate for the gap of the diameters in terms of the deficit in the spectral gap for RCD spaces (see Remark 1.3 therein). Their estimate reads as follows: if $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(N-1,N)$ space of diameter $D \leq \pi$, then

$$\lambda^{1,2}_{(X,\mathsf{d},\mathfrak{m})} \geq \frac{N}{1-\cos^N(D/2)}.$$

Theorem 4.3 extends such quantitative control to essentially nonbranching $\mathrm{CD}(N-1,N)$ spaces whose Sobolev space $W^{1,2}$ is a priori non-Hilbert (but just Banach, as for instance on Finsler manifolds).

4A. ***Volume control.*** The aim of this brief subsection is to prove that for a $\mathrm{CD}(N-1,N)$ m.m.s. with diameter close to π we have a quantitative volume control for balls centered at extrema of long rays. The proof is inspired by [Ohta 2007, Lemma 5.1], where the case of maximal diameter π is treated (see also [Cavalletti et al. 2019, Proposition 5.1]).

Proposition 4.5. *Let $(X,\mathsf{d},\mathfrak{m})$ be an m.m.s. satisfying $\mathrm{CD}(N-1,N)$ (actually $\mathrm{MCP}(N-1,N)$ is enough). Let $P_N, P_S \in X$ be such that $\mathsf{d}(P_N,P_S) = \pi - \delta$ for some $\delta \geq 0$. Then, for any $0 < r < \pi - \delta$, it holds*

$$\mathfrak{m}_N([0,r]) \leq \mathfrak{m}(B_r(P_N)) \leq \mathfrak{m}_N([0,r]) + \mathfrak{m}_N([r,r+\delta]), \tag{4-7}$$

where we recall that $\mathfrak{m}_N = (1/\omega_N)(\sin t)^{N-1}dt$ is the model measure on the interval $[0,\pi]$.

Proof. First of all, since $\mathsf{d}(P_N, P_S) = \pi - \delta$, it holds $B_r(P_N) \cap B_{\pi-r-\delta}(P_S) = \varnothing$.

By the Bishop–Gromov inequality implied by the $\mathrm{CD}(N-1, N)$ condition (actually $\mathrm{MCP}(N-1, N)$ is enough), and using that $\mathfrak{m}(X) = 1$, we have

$$\mathfrak{m}(B_r(P_N)) \geq \mathfrak{m}_N([0, r]), \quad \mathfrak{m}(B_{\pi-r-\delta}(P_S)) \geq \mathfrak{m}_N([0, \pi - r - \delta]) = \mathfrak{m}_N([r+\delta, \pi]), \tag{4-8}$$

where the last equality follows from the symmetries of the density $\sin^{N-1}(\,\cdot\,)$. Hence we can compute

$$\begin{aligned}\mathfrak{m}(B_r(P_N)) &\leq 1 - \mathfrak{m}(B_{\pi-r-\delta}(P_S)) \leq 1 - \mathfrak{m}_N([0, \pi - r - \delta]) \\ &= \mathfrak{m}_N([0, r]) + \mathfrak{m}_N([r, r+\delta]).\end{aligned}$$

The claimed conclusion (4-7) follows. □

5. Quantitative Obata's theorem on almost optimal functions

Consider $u \in \mathrm{Lip}(X)$ such that

$$\int_X u\,\mathfrak{m} = 0, \qquad \int_X u^2\,\mathfrak{m} = 1;$$

denote its spectral gap deficit by

$$\delta(u) := \int_X |\nabla u|^2\,\mathfrak{m} - N. \tag{5-1}$$

Since we are interested in quantitative estimates when the spectral gap deficit is small, it is enough to consider the case $\delta(u) \leq 1$. Recall that N is the first eigenvalue for the Neumann Laplacian for the one-dimensional metric measure space $([0, \pi], |\cdot|, \mathfrak{m}_N)$, where $\mathfrak{m}_N := \sin^{N-1}(t)\,dt/\omega_N$ and ω_N is the normalizing constant. In particular

$$N = (N+1) \int_{(0,\pi)} \sin^2(t)\,\mathfrak{m}_N(dt),$$

since, as we already observed, $\int_{(0,\pi)} \cos^2(t)\,\mathfrak{m}_N(dt) = 1/(N+1)$.

Consider the localization associated to the zero-mean function u (see Section 2D for the background and for the relevant bibliography):

$$\mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_q\,\mathfrak{q}(dq),$$

where $\mathcal{T}$ is the transport set associated to the L^1-optimal transport problem between $u^+\mathfrak{m}$ and $u^-\mathfrak{m}$, the positive and the negative parts of u, respectively. It follows that

$$\int_Q \int_{X_q} |u|^2\,\mathfrak{m}_q\,\mathfrak{q}(dq) = \int_{\mathcal{T}} |u|^2\,\mathfrak{m} = \int_X |u|^2\,\mathfrak{m} = 1, \qquad \int_{X\setminus\mathcal{T}} |\nabla u|^2\,\mathfrak{m} = 0. \tag{5-2}$$

Setting $u_q := u|_{X_q}$ and $|c_q| := \big(\int_{X_q} |u_q|^2\,\mathfrak{m}_q\big)^{1/2}$ (for the sign of c_q, see before (5-13)), observe that (5-2) gives

$$\int_Q c_q^2\,\mathfrak{q}(dq) = 1. \tag{5-3}$$

Moreover, the integral constraint $\int_X u\,\mathfrak{m} = 0$ localizes to almost every ray:

$$\int_{X_q} u_q\,\mathfrak{m}_q = 0. \tag{5-4}$$

Since almost each ray $(X_q, \mathsf{d}|_{X_q}, \mathfrak{m}_q)$ is a one-dimensional $\mathrm{CD}(N-1, N)$ space, the Lichnerowicz spectral gap gives

$$\int_{X_q} |u_q'|^2\,\mathfrak{m}_q \geq N c_q^2, \tag{5-5}$$

where $|u_q'|(x)$ denotes the local Lipschitz constant of $u_q : (X_q, \mathsf{d}|_{X_q}) \to \mathbb{R}$ at $x \in X_q$. It is clear that, for each $x \in X_q \subset X$, $|u_q'|(x)$ is bounded by the local Lipschitz constant $|\nabla u|(x)$ of $u : (X, \mathsf{d}) \to \mathbb{R}$:

$$|u_q'|(x) \leq |\nabla u|(x) \quad \text{for all } x \in X_q,\ \mathfrak{q}\text{-a.e. } q \in Q. \tag{5-6}$$

With a slight abuse of notation, in order to keep the formulas short, in the following we will often identify $\mathfrak{q}$ and $\mathfrak{q}\llcorner_{\{q\in Q:\, c_q>0\}}$. Localizing the spectral gap deficit using (5-6) gives

$$\begin{aligned}
\delta(u) = \int_X |\nabla u|^2\,\mathfrak{m} - N &\geq \int_Q \biggl(\int_{X_q} \frac{|u_q'|^2}{c_q^2}\,\mathfrak{m}_q\biggr) c_q^2\,\mathfrak{q}(dq) - N \\
&= \int_Q \biggl[\int_{X_q} \biggl(\frac{|u_q'|^2}{c_q^2} - N\biggr)\,\mathfrak{m}_q\biggr] c_q^2\,\mathfrak{q}(dq) \qquad (5\text{-}7)\\
&= \int_Q \delta(u_q) c_q^2\,\mathfrak{q}(dq), \qquad (5\text{-}8)
\end{aligned}$$

where we set

$$\delta(u_q) := \int_{X_q} \biggl(\frac{|u_q'|^2}{c_q^2} - N\biggr)\,\mathfrak{m}_q,$$

the one-dimensional spectral gap deficit of u_q. From now on, in order to keep notation short, we will write δ for $\delta(u)$. Let $\beta \in (0, 1)$ be a real parameter to be optimized later in the proof and denote the set of "long rays" by

$$Q_\ell := \{q \in Q : \delta(u_q) \leq \delta^\beta \text{ and } c_q > 0\}.$$

It follows from (5-8), Chebyshev's inequality and (5-3) that

$$\int_{Q\setminus Q_\ell} c_q^2\,\mathfrak{q}(dq) \leq \delta^{1-\beta}, \qquad \int_{Q_\ell} c_q^2\,\mathfrak{q}(dq) \geq 1 - \delta^{1-\beta}. \tag{5-9}$$

Hence we can use Proposition 3.3 to deduce that, for all $q \in Q_\ell$,

$$(\pi - |X_q|)^N \leq C_N \delta^\beta, \tag{5-10}$$

where $|X_q|$ denotes the length of the ray X_q. Being the preimage of a measurable function, Q_ℓ is a measurable subset of Q. Adopting the notation $R(E) := \bigcup_{q\in E} X_q$, so that $R(E)$ is the span of the rays corresponding to equivalence classes in E, we claim that

$$\int_{X\setminus R(Q_\ell)} |\nabla u|^2\,\mathfrak{m} \leq (N+1)\delta^{1-\beta}. \tag{5-11}$$

Indeed (5-6), (5-5) and (5-9) yield

$$\int_{R(Q_\ell)} |\nabla u|^2 \, \mathfrak{m} \geq \int_{Q_\ell} \int_{X_q} |u_q'|^2 \, \mathfrak{m}_q \, \mathfrak{q}(dq) \geq N \int_{Q_\ell} c_q^2 \, \mathfrak{q}(dq) \geq N(1-\delta^{1-\beta}).$$

The claim (5-11) follows by combining the last estimate with

$$\int_{X \setminus R(Q_\ell)} |\nabla u|^2 \, \mathfrak{m} + \int_{R(Q_\ell)} |\nabla u|^2 \, \mathfrak{m} = \int_X |\nabla u|^2 \, \mathfrak{m} \leq N + \delta.$$

For each $q \in Q$, we denote by $a(X_q)$ (resp. $b(X_q)$) the initial (resp. final) point of the ray X_q.

Throughout this last section we will often make the identification between the ray X_q and the interval $(0, |X_q|)$.

Proposition 5.1. *There exists a distinguished $\bar{q} \in Q_\ell$ having initial point P_N and final point P_S such that*

$$\mathsf{d}(P_N, a(X_q)) \leq C(N)\delta^{\beta/N}, \quad \mathsf{d}(P_S, b(X_q)) \leq C(N)\delta^{\beta/N} \quad \textit{for all } q \in Q_\ell. \tag{5-12}$$

Proof. Fix any $\bar{q} \in Q_\ell$ and set $P_N := a(X_{\bar{q}})$, $P_S := b(X_{\bar{q}})$. By d-cyclical monotonicity of the transport set $\mathcal{T}$, for any other $q \in Q_\ell$ it holds

$$2\pi - \mathsf{d}(a(X_q), b(X_q)) - \mathsf{d}(P_N, P_S) \geq 2\pi - \mathsf{d}(a(X_q), P_S) - \mathsf{d}(b(X_q), P_N),$$

which we rewrite as

$$\pi - |X_q| + \pi - |X_{\bar{q}}| \geq \pi - \mathsf{d}(a(X_q), P_S) + \pi - \mathsf{d}(b(X_q), P_N).$$

Combining the last estimate with (5-10) gives

$$2C_N \delta^{\beta/N} \geq \pi - \mathsf{d}(a(X_q), P_S) + \pi - \mathsf{d}(b(X_q), P_N).$$

Finally by [Cavalletti et al. 2019, Proposition 5.1] we deduce the existence of a constant, depending only on the dimension N, such that

$$\mathsf{d}(a(X_q), P_N) \leq C(N)\delta^{\beta/N}, \quad \mathsf{d}(b(X_q), P_S) \leq C(N)\delta^{\beta/N},$$

and the claim follows. □

From now on, for every $q \in Q_\ell$ choose the sign of c_q so that

$$\left\| \frac{u_q}{c_q} - \sqrt{N+1}\cos(\cdot) \right\|_{L^2(X_q, \mathfrak{m}_q)} = \min\left\{ \left\| \frac{u_q}{|c_q|} + \sqrt{N+1}\cos(\cdot) \right\|_{L^2(X_q, \mathfrak{m}_q)}, \left\| \frac{u_q}{|c_q|} - \sqrt{N+1}\cos(\cdot) \right\|_{L^2(X_q, \mathfrak{m}_q)} \right\}.$$

From Theorem 3.11 we obtain that for all $q \in Q_\ell$ it holds

$$\left\| \frac{u_q}{c_q} - \sqrt{N+1}\cos(\cdot) \right\|_{L^2(X_q, \mathfrak{m}_q)} \leq C(N)\delta^{\beta \min\{1/2, 1/N\}}. \tag{5-13}$$

The goal of the next section is to globalize estimate (5-13) to the whole space X.

The sought bound will be obtained through two intermediate steps: Firstly, in Proposition 5.2, we control the variance of the map $q \mapsto c_q$ with respect to the measure $\mathfrak{q}$ on the set of long rays Q_ℓ. Then, in Proposition 5.3, we estimate $(1 - \mathfrak{q}(Q_\ell))$ in terms of a power of the deficit.

Below we briefly present the strategy of the proof. In order to fix the ideas, we discuss the heuristics in the rigid case of zero deficit. Actually in the case of zero deficit there is a more streamlined argument (the assumption that u is Lipschitz, combined with the fourth bullet below, gives immediately that $q \mapsto c_q$ is constant); however, the point here is to present a strategy which generalizes to the nonrigid case of nonzero deficit.

In the case where $\delta(u) = 0$, the results of the previous sections give the following conclusions:

- Almost all the transport rays have length π. Moreover, they start from a common point P_N, with $u(P_N) > 0$, and end in a common point P_S, with $u(P_S) < 0$.
- $\mathfrak{m}(B_r(P_N)) = \mathfrak{m}_N([0, r])$ for any $r \in [0, \pi]$.
- For $\mathfrak{q}$-a.e. $q \in Q$, it holds that $\mathfrak{m}_q = \mathfrak{m}_N$ is the model measure for the $\mathrm{CD}(N-1, N)$ condition.
- For $\mathfrak{q}$-a.e. $q \in Q$, it holds that $u_q(\,\cdot\,) = c_q \cos(\mathsf{d}(P_N, \cdot\,))$.

Our aim is to prove that $\mathfrak{q}(Q) = 1$ and that $c_q = 1$ for $\mathfrak{q}$-a.e. $q \in Q$. The basic idea is to apply the Poincaré inequality to balls centered at P_N and having radii converging to 0.

Observe that we can compute

$$\fint_{B_r(P_N)} u\,\mathfrak{m} = \frac{1}{\mathfrak{m}_N([0, r])} \int_Q \int_0^r c_q \cos(t)\mathfrak{m}_N(dt) = \left(\int_Q c_q\,\mathfrak{q}\right) \fint_0^r \cos(t)\,\mathfrak{m}_N(dt). \tag{5-14}$$

Moreover, recalling that $u = 0$ $\mathfrak{m}$-a.e. outside of the transport set, we have

$$\begin{aligned}
&\fint_{B_r(P_N)} \left| u - \fint_{B_r(P_N)} u\,\mathfrak{m} \right|^2 \mathfrak{m} \\
&\quad\overset{(5\text{-}14)}{=} (1-\mathfrak{q}(Q)) \left(\fint_{B_r(P_N)} u\,\mathfrak{m} \right)^2 + \int_Q \fint_0^r \left| c_q \cos(t) - \int_Q c_q\,\mathfrak{q}(dq) \fint_0^r \cos(t)\,\mathfrak{m}_N(dt) \right|^2 \mathfrak{m}_N(dt)\,\mathfrak{q}(dq) \\
&\quad\sim (1-\mathfrak{q}(Q)) \left(\int_Q c_q\,\mathfrak{q}(dq) \right)^2 + \int_Q \left| c_q - \int_Q c_q\,\mathfrak{q}(dq) \right|^2 \mathfrak{q}(dq) \quad \text{as } r \to 0,
\end{aligned} \tag{5-15}$$

where in the last step we relied on the asymptotic $\cos(t) = 1 + o(t)$ as $t \to 0$. Eventually we can compute

$$\fint_{B_{2r}(P_N)} |\nabla u|^2 \mathfrak{m} = \int_Q c_q^2\,\mathfrak{q}(dq) \fint_0^{2r} \sin^2(t)\,\mathfrak{m}_N(dt) = \fint_0^{2r} \sin^2(t)\,\mathfrak{m}_N(dt) \sim r^2 \quad \text{as } r \to 0,$$

where in the last step we relied on the asymptotic $\sin(t) = t + o(t)$ as $t \to 0$.

An application of the Poincaré inequality, in the asymptotic regime $r \downarrow 0$, yields that

$$\int_Q \left| c_q - \int_Q c_q\,\mathfrak{q}(dq) \right|^2 \mathfrak{q}(dq) = 0, \tag{5-16}$$

which implies both the conclusions $\mathfrak{q}(Q) = 1$ and $q \mapsto c_q$ constant $\mathfrak{q}$-a.e. Due to the constraint $\int_Q c_q^2 \mathfrak{q}(dq) = 1$ and the fact that $u(P_N) > 0$, we also get that $c_q = 1$ $\mathfrak{q}$-a.e., as we claimed.

A second heuristic motivation of the fact that the oscillation of the map $q \mapsto c_q$ is controlled by (a power of) the deficit is that "the gradient of u is almost aligned along the rays" in a quantitative L^2-sense, suggesting that u "should not oscillate much in the direction orthogonal to the rays". Note that in the current framework of $\mathrm{CD}(K,N)$ spaces there is no scalar product and the set Q is far from regular, this is the reason why we cannot directly implement this heuristic strategy. However, let us make precise the fact that "the gradient of u is almost aligned along the rays" in a quantitative L^2-sense, since this will be used in the arguments below:

$$\begin{aligned} 0 &\overset{(5\text{-}6)}{\leq} \int_Q \left(\int_{X_q} |\nabla u|^2 - |u_q'|^2 \,\mathfrak{m}_q \right) \mathfrak{q}(dq) = \int_X |\nabla u|^2 \,\mathfrak{m} - \int_Q \left(\int_{X_q} |u_q'|^2 \,\mathfrak{m}_q \right) \mathfrak{q}(dq) \quad \text{(by (5-1),(5-5))} \\ &\leq N + \delta - N \int_Q c_q^2 \,\mathfrak{q}(dq) \overset{(5\text{-}3)}{=} \delta. \end{aligned} \tag{5-17}$$

The proofs of Propositions 5.2 and 5.3 below are based on the idea we just presented, although they are quite technical since one has to handle all the various error terms occurring in the nonrigid case $\delta(u) > 0$.

5A. *Control on the variance.*

Proposition 5.2. *The following estimate holds*:

$$\int_{Q_\ell} \left| c_q - \fint_{Q_\ell} c_q \,\mathfrak{q}(dq) \right|^2 \mathfrak{q}(dq) \leq C(N)(\delta^{4\gamma/N} + \delta^{1-\beta-\gamma+(2\gamma/N)} + \delta^{(\beta-\gamma)\min\{2/N,1\}}) \tag{5-18}$$

for any $0<\beta<1$ *and for any* $0<\gamma<\min\{\beta, 1-\beta\}$.

Proof. In order to bound the variance of $q \mapsto c_q$ on Q_ℓ we wish to prove that it can be controlled by an integral depending on the variation of the function u on a small ball $B_r(P_N)$. Next we will appeal to the fact that in the rigid case the L^2-norm squared of the gradient of u on $B_r(P_N)$ is comparable with r^{N+2} and, at least heuristically, this has to be the case also when dealing with almost rigidity. Some intermediate steps are devoted to reducing to the case where the function u coincides with $c_q \cos(\cdot)$ when restricted to any long ray X_q.

In order to slightly shorten the notation, we will write C in place of $C(N)$ to denote a dimensional constant.

Step 1: We will set $r = \delta^{\gamma/N}$ for a suitable $\gamma \in (0,\beta)$. First of all, notice that the triangle inequality and (5-12) yield

$$[0, r - C\delta^{\beta/N}] \subset X_q \cap B_r(P_N) \subset [0, r + C\delta^{\beta/N}] \tag{5-19}$$

for any $q \in Q_\ell$, where we have identified $[0, r \pm C\delta^{\beta/N}]$ with the set

$$\{z \in X_q : \mathsf{d}(z, a(X_q)) \leq r \pm C\delta^{\beta/N}\}.$$

The minimality of the mean combined with the inclusion (5-19) and with the weak local 2-2 Poincaré inequality (2-6) gives

$$\int_{Q_\ell \times [0, r-C\delta^{\beta/N}]} \left| u - \fint_{Q_\ell \times [0, r-C\delta^{\beta/N}]} u \,\mathfrak{m} \right|^2 \mathfrak{m} \leq \int_{B_r(P_N)} \left| u - \fint_{B_r(P_N)} u \right|^2 \mathfrak{m} \leq C r^2 \int_{B_{10r}(P_N)} |\nabla u|^2 \,\mathfrak{m}. \tag{5-20}$$

Step 2: Next we will obtain a more explicit expression of $\fint_{Q \times [0, r-C\delta^{\beta/N}]} u \,\mathfrak{m}$.

Recall that we will often tacitly identify the ray X_q with the interval $(0, |X_q|)$.

Using Theorem 3.11, Corollary 2.7 and that $\delta_q \le \delta^\beta$ for $q \in Q_\ell$, we estimate

$$\begin{aligned}\Big|\int_{Q_\ell}\int_{[0,r]} u\, \mathfrak{m}_q\, \mathfrak{q}(dq) - \sqrt{N+1}\int_{Q_\ell}\int_{[0,r]} c_q \cos(\,\cdot\,)\mathfrak{m}_q\, \mathfrak{q}(dq)\Big| \\ &\le \int_{Q_\ell} |c_q| \int_{[0,r]} \Big|\frac{u}{c_q} - \sqrt{N+1}\cos(\,\cdot\,)\Big| \mathfrak{m}_q\, \mathfrak{q}(dq) \\ &\le \int_{Q_\ell} |c_q| \sqrt{\mathfrak{m}_q([0,r])}\,\Big\|\frac{u}{c_q} - \sqrt{N+1}\cos(\,\cdot\,)\Big\|_{L^2([0,r],\mathfrak{m}_q)} \mathfrak{q}(dq) \\ &\le C\, r^{N/2}(r^{N/2}\delta^{\beta\min\{1/2,1/N\}} + \delta^{\beta/2}) \int_{Q_\ell} |c_q|\, \mathfrak{q}(dq). \end{aligned} \tag{5-21}$$

Also, using Corollary 2.8, it holds

$$\Big|\int_{Q_\ell}\int_{[0,r]} c_q \cos(\,\cdot\,)\mathfrak{m}_q\, \mathfrak{q}(dq) - \int_{Q_\ell}\int_{[0,r]} c_q \cos(\,\cdot\,)\mathfrak{m}_N\, \mathfrak{q}(dq)\Big| \le C\delta^{\beta/N} r^{N-1} \int_{Q_\ell} |c_q|\, \mathfrak{q}(dq). \tag{5-22}$$

With an analogous estimate involving Corollary 2.8, we also obtain

$$|\mathfrak{m}(Q_\ell \times [0,r]) - \mathfrak{q}(Q_\ell)\mathfrak{m}_N([0,r])| \le C\mathfrak{q}(Q_\ell) r^{N-1}\delta^{\beta/N}. \tag{5-23}$$

The combination of (5-21), (5-22) and (5-23), setting $\bar r := r - C\delta^{\beta/N}$, yields

$$\begin{aligned}\Big| \fint_{Q_\ell\times[0,\bar r]} u\, \mathfrak{m} - \frac{\sqrt{N+1}\int_{Q_\ell\times[0,\bar r]} c_q \cos(\,\cdot\,)\mathfrak{m}_N\, \mathfrak{q}(dq)}{\mathfrak{q}(Q_\ell)(\mathfrak{m}_N([0,\bar r]) - C r^{N-1}\delta^{\beta/N})}\Big| \\ &\le \frac{C\big(\int_{Q_\ell}|c_q|\,\mathfrak{q}(dq)\big)(r^N\delta^{\beta\min\{1/2,1/N\}} + r^{N/2}\delta^{\beta/2} + r^{N-1}\delta^{\beta/N})}{\mathfrak{q}(Q_\ell)(\mathfrak{m}_N([0,\bar r]) - C r^{N-1}\delta^{\beta/N})}. \end{aligned} \tag{5-24}$$

<u>Step 3</u>: In this step we estimate the order in δ of the right-hand side of (5-24) and choose r as

$$r = \delta^{\gamma/N}, \qquad \text{with } \gamma \in (0,\beta). \tag{5-25}$$

Approximating the cosine with its first-order Taylor expansion near to the origin in (5-24), we have

$$\fint_{Q_\ell\times[0,\bar r]} u\, \mathfrak{m} = \frac{\int_{Q_\ell} c_q\, \mathfrak{q}(dq) + \big(\int_{Q_\ell}|c_q|\,\mathfrak{q}(dq)\big)\, O(\delta^{(\beta-\gamma)\min\{1/2,1/N\}})}{\mathfrak{q}(Q_\ell)/\sqrt{N+1}}.$$

Since by Cauchy–Schwarz inequality and (5-3) it holds

$$\Big(\fint_{Q_\ell} c_q\, \mathfrak{q}(dq)\Big)^2 \le \fint_{Q_\ell} c_q^2\, \mathfrak{q}(dq) \le \frac{1}{\mathfrak{q}(Q_\ell)},$$

the last estimate can be rewritten as

$$\Big|\fint_{Q_\ell\times[0,\bar r]} u\, \mathfrak{m} - \sqrt{N+1}\fint_{Q_\ell} c_q\, \mathfrak{q}(dq)\Big|^2 \le \frac{C}{\mathfrak{q}(Q_\ell)}\delta^{(\beta-\gamma)\min\{1,2/N\}}. \tag{5-26}$$

Step 4: The aim of this step is to eventually gain (5-18). We first need the following intermediate inequality, where we assume that $r \gg \delta^{\beta/N}$ is a free parameter, that we will set later:

$$\begin{aligned}
&\int_{Q_\ell}\int_{[0,r]} |u-\sqrt{N+1}c_q|^2 \mathfrak{m}_q\, \mathfrak{q}(dq) \\
&\quad\le 2\int_{Q_\ell}\int_{[0,r]} |u-\sqrt{N+1}c_q\cos(\cdot)|^2\,\mathfrak{m}_q\,\mathfrak{q}(dq) + 2\int_{Q_\ell}\int_{[0,r]} (\sqrt{N+1}|c_q||\cos(\cdot)-1|)^2\,\mathfrak{m}_q\,\mathfrak{q}(dq) \\
&\quad\le C\delta^{\beta\min\{1,2/N\}} r^N \int_{Q_\ell} c_q^2\,\mathfrak{q}(dq) + C\delta^\beta + Cr^4\int_{Q_\ell} c_q^2\,\mathfrak{m}_q([0,r])\,\mathfrak{q}(dq) \quad \text{(by (3-59))} \\
&\quad\le C\delta^{\beta\min\{1,2/N\}} r^N + C\delta^\beta + Cr^4\int_{Q_\ell} c_q^2\,(\mathfrak{m}_N([0,r]) + Cr^{N-1}\delta^{\beta/N})\,\mathfrak{q}(dq) \quad \text{(by (5-10)+(2-12))} \\
&\quad\le C\delta^{\beta\min\{1,2/N\}} r^N + Cr^4\mathfrak{m}_N([0,r])\int_{Q_\ell} c_q^2\,\mathfrak{q}(dq) + C\delta^\beta \le Cr^N(\delta^{\beta\min\{1,2/N\}} + r^4) + C\delta^\beta. \qquad (5\text{-}27)
\end{aligned}$$

In particular, the previous inequality holds true substituting $\bar r := r - C\delta^{\beta/N}$ in place of r, and $r = \delta^{\gamma/N}$ is as in the previous Step 3. We deduce

$$\begin{aligned}
&\mathfrak{m}_N([0,\bar r])(N+1)\int_{Q_\ell}\left|c_q - \fint_{Q_\ell} c_q\,\mathfrak{q}(dq)\right|^2 \mathfrak{q}(dq) \\
&\le (N+1)\int_{Q_\ell}\left|c_q - \fint_{Q_\ell} c_q\,\mathfrak{q}(dq)\right|^2 (\mathfrak{m}_q([0,\bar r]) + Cr^{N-1}\delta^{\beta/N})\,\mathfrak{q}(dq) \\
&\le C\delta^{\beta/N} r^{N-1} + (N+1)\int_{Q_\ell}\left|c_q - \fint_{Q_\ell} c_q\,\mathfrak{q}(dq)\right|^2 \mathfrak{m}_q([0,\bar r])\,\mathfrak{q}(dq) \\
&\le C\delta^{\beta/N} r^{N-1} + 2\int_{Q_\ell}\int_{[0,\bar r]} |u-\sqrt{N+1}c_q|^2\,\mathfrak{m}_q\,\mathfrak{q}(dq) + 2\int_{Q_\ell}\int_{[0,\bar r]}\left|u - \fint_{Q_\ell}\sqrt{N+1}c_q\,\mathfrak{q}(dq)\right|^2 \mathfrak{m}_q\,\mathfrak{q}(dq) \\
&\le C\delta^{\beta/N} r^{N-1} + 2\int_{Q_\ell}\int_{[0,\bar r]} |u-\sqrt{N+1}c_q|^2\,\mathfrak{m}_q\,\mathfrak{q}(dq) + 4\int_{Q_\ell\times[0,\bar r]}\left|u - \fint_{Q_\ell\times[0,\bar r]} u\,\mathfrak{m}\right|^2 \mathfrak{m} \\
&\qquad + 4\int_{Q_\ell}\int_{[0,\bar r]}\left|\fint_{Q_\ell\times[0,\bar r]} u\,\mathfrak{m} - \sqrt{N+1}\fint_{Q_\ell} c_q\,\mathfrak{q}(dq)\right|^2 \mathfrak{m}_q\,\mathfrak{q}(dq).
\end{aligned}$$

Now use (5-20), (5-26), (5-25), (5-27) to continue the chain of inequalities

$$\le C\delta^\gamma(\delta^{(\beta-\gamma)\min\{1,2/N\}} + \delta^{4\gamma/N}) + Cr^2\int_{B_{10r}(P_N)} |\nabla u|^2\,\mathfrak{m}. \qquad (5\text{-}28)$$

Next we wish to bound the term $\int_{B_{10r}(P_N)} |\nabla u|^2\,\mathfrak{m}$. To this aim we observe that

$$\begin{aligned}
&\int_{B_{10r}(P_N)} |\nabla u|^2\,\mathfrak{m} \\
&\le \int_{X\setminus R(Q_\ell)} |\nabla u|^2\,\mathfrak{m} + \int_{Q_\ell}\int_0^{10r+C\delta^{\beta/N}} |u_q'|^2\,\mathfrak{m}_q\,\mathfrak{q}(d\mathfrak{q}) + \delta \quad \text{(by (5-17))} \\
&\le C(\delta^{1-\beta} + \delta^\beta + r^N\delta^{\beta\min\{2/N,1\}} + \delta^{\beta/N}r^{N-1}r^2) + C\int_0^{10r+C\delta^{\beta/N}} \sin(\cdot)^2\,\mathfrak{m}_N \quad \text{(by (5-11), (3-59), (2-12))} \\
&\le C(\delta^{1-\beta} + \delta^\beta + r^N(\delta^{\beta\min\{2/N,1\}} + r^2)). \qquad (5\text{-}29)
\end{aligned}$$

Combine now (5-28) with (5-29) and recall that $r=\delta^{\gamma/N}$ for $0<\gamma<\min\{\beta,1-\beta\}$ to get

$$\delta^{\gamma}\int_{Q_\ell}\Big|c_q-\fint_{Q_\ell}c_q\,\mathfrak{q}(dq)\Big|^2\,\mathfrak{q}(dq)\le C\delta^{\gamma}(\delta^{4\gamma/N}+\delta^{1-\beta-\gamma+(2\gamma/N)}+\delta^{(\beta-\gamma)\min\{1,2/N\}}),$$

which gives the desired estimate (5-18). □

5B. *Control of the measure of long rays.* Following Proposition 5.2, we set

$$\bar{c}:=\fint_{Q_\ell}c_q\,\mathfrak{q}(dq). \tag{5-30}$$

Next we proceed proving that $\mathfrak{q}(Q_\ell)$ is quantitatively close to 1 up to an error of the order of a suitable power of the deficit.

Proposition 5.3. *The following estimate holds*:

$$(1-\mathfrak{q}(Q_\ell))^2\le C(N)(\delta^{4\gamma/N}+\delta^{(\beta-\gamma)/N}+\delta^{1-\beta-\gamma}) \tag{5-31}$$

for any $0<\beta<1$ *and for any* $0<\gamma<\min\{\beta,1-\beta\}$.

Proof. In order to slightly shorten the notation, we will write C in place of $C(N)$ to denote constants depending only on N. Moreover, we will continue to tacitly identify the ray X_q with the interval $(0,|X_q|)$. We achieve (5-31) through three intermediate steps.

<u>Step 1</u>: Aim of this first step is to prove that, for $r=\delta^{\gamma/N}$, $\gamma\in(0,\min\{\beta,1-\beta\})$, letting $\bar{r}:=r-C\delta^{\beta/N}$, it holds

$$(N+1)\int_{Q_\ell}\int_{[0,\bar r]}\Big|c_q\cos(\cdot)-\bar c\,\mathfrak{q}(Q_\ell)\fint_{[0,\bar r]}\cos(\cdot)\mathfrak{m}_N\Big|^2\,\mathfrak{m}_N\,\mathfrak{q}(dq)$$
$$\le\int_{B_r(P_N)}\Big|u-\fint_{B_r(P_N)}u\,\mathfrak{m}\Big|^2\,\mathfrak{m}+C(\delta^{\gamma+(\beta-\gamma)/N}+\delta^{1-\beta}). \tag{5-32}$$

Arguing as in the first steps of the proof of Proposition 5.2, we estimate

$$\begin{aligned}
&\int_{Q_\ell}\int_0^{\bar r}\Big|\sqrt{N+1}c_q\cos(\cdot)-\fint_{B_r(P_N)}u\,\mathfrak{m}\Big|^2\,\mathfrak{m}_N\,\mathfrak{q}(dq)\\
&\le\int_{Q_\ell}\int_0^{\bar r}\Big|\sqrt{N+1}c_q\cos(\cdot)-\fint_{B_r(P_N)}u\,\mathfrak{m}\Big|^2\,\mathfrak{m}_q\,\mathfrak{q}(dq)+C\delta^{\beta/N}r^{N-1}\quad\text{(by (2-12), (5-10))}\\
&\le 2\int_{Q_\ell}\int_0^{\bar r}|\sqrt{N+1}c_q\cos(\cdot)-u|^2\,\mathfrak{m}_q\,\mathfrak{q}(dq)+C\delta^{\beta/N}r^{N-1}+2\int_{Q_\ell}\int_0^{\bar r}\Big|u-\fint_{B_r(P_N)}u\,\mathfrak{m}\Big|^2\,\mathfrak{m}_q\,\mathfrak{q}(dq)\\
&\le 2\int_{Q_\ell}c_q^2\Big\|\frac{u}{c_q}-\sqrt{N+1}\cos(\cdot)\Big\|^2_{L^2([0,\bar r],\mathfrak{m}_q)}\mathfrak{q}(dq)+C\delta^{\beta/N}r^{N-1}+2\int_{Q_\ell}\int_0^{\bar r}\Big|u-\fint_{B_r(P_N)}u\,\mathfrak{m}\Big|^2\,\mathfrak{m}_q\,\mathfrak{q}(dq)\mathfrak{m}\\
&\le 2\int_{B_r(P_N)\cap R(Q_\ell)}\Big|u-\fint_{B_r(P_N)}u\,\mathfrak{m}\Big|^2\,\mathfrak{m}+C\delta^{\beta/N}r^{N-1}\quad\text{(by (3-59), (5-19))}.
\end{aligned}\tag{5-33}$$

In order to achieve (5-32), having in mind to argue by triangle inequality, we are left to bound

$$\mathfrak{m}_N([0,\bar r])\,\mathfrak{q}(Q_\ell)\Big|\fint_{B_r(P_N)}u\,\mathfrak{m}-\sqrt{N+1}\bar c\,\mathfrak{q}(Q_\ell)\fint_{[0,\bar r]}\cos(\cdot)\,\mathfrak{m}_N\Big|^2. \tag{5-34}$$

We start by observing that

$$\left|\int_{B_r(P_N)} u\,\mathfrak{m} - \sqrt{N+1}\,\bar{c}\,\mathfrak{q}(Q_\ell)\int_0^r \cos(\,\cdot\,)\,\mathfrak{m}_N\right| \le \left|\int_{B_r(P_N)\cap R(Q_\ell)} u\,\mathfrak{m} - \sqrt{N+1}\,\bar{c}\,\mathfrak{q}(Q_\ell)\int_0^r \cos(\,\cdot\,)\,\mathfrak{m}_N\right| + \left|\int_{B_r(P_N)\setminus R(Q_\ell)} u\,\mathfrak{m}\right|. \tag{5-35}$$

We first treat the second term of the right-hand side.

From (5-9) we know that $\int_{X\setminus R(Q_\ell)} u^2\,\mathfrak{m} \le \delta^{1-\beta}$; an application of Hölder's inequality and (2-12) yields

$$\int_{B_r(P_N)\setminus R(Q_\ell)} |u|\,\mathfrak{m} \le \delta^{(1-\beta)/2}\sqrt{\mathfrak{m}(B_r(P_N)\setminus R(Q_\ell))} \le C\delta^{(1-\beta)/2} r^{N/2}. \tag{5-36}$$

We estimate the first term in the right-hand side of (5-35) by reducing to (5-21) in the second step of the proof of Proposition 5.2:

$$\begin{aligned}\left|\int_{B_r(P_N)\cap R(Q_\ell)} u\,\mathfrak{m} - \sqrt{N+1}\bar{c}\mathfrak{q}(Q_\ell)\int_0^r \cos(\,\cdot\,)\,\mathfrak{m}_N\right| &\le \left|\int_{B_r(P_N)\cap R(Q_\ell)} u\,\mathfrak{m} - \int_{Q_\ell}\int_{[0,r]} u\mathfrak{m}_q\,\mathfrak{q}(dq)\right| \\ &\quad + \left|\int_{Q_\ell}\int_{[0,r]} u\,\mathfrak{m}_q\,\mathfrak{q}(dq) - \sqrt{N+1}\int_{Q_\ell}\int_{[0,r]} c_q\cos(\,\cdot\,)\,\mathfrak{m}_q\,\mathfrak{q}(dq)\right| \\ &\quad + \left|\int_{Q_\ell}\int_{[0,r]} \sqrt{N+1}c_q\cos(\,\cdot\,)\,\mathfrak{m}_q\,\mathfrak{q}(dq) - \sqrt{N+1}\bar{c}\mathfrak{q}(Q_\ell)\int_0^r \cos(\,\cdot\,)\,\mathfrak{m}_N\right|.\end{aligned}$$

Using (2-12), (3-59), (5-10), (5-19), (5-21), we continue as follows:

$$\le \int_{Q_\ell}\int_{r-C\delta^{\beta/N}}^{r+C\delta^{\beta/N}} |u|\,\mathfrak{m}_q\,\mathfrak{q}(dq) + Cr^{N/2}(\delta^{\beta\min\{1/2,1/N\}}r^{N/2} + \delta^{\beta/2} + r^{(N/2)-1}\delta^{\beta/N})\int_{Q_\ell}|c_q|\,\mathfrak{q}(dq). \tag{5-37}$$

Arguing by triangle inequality bounding first the distance from the normalized cosine (with (3-60)) and then replacing the measures $\mathfrak{m}_q$ with the model measure $\mathfrak{m}_N$ (with (2-12)), we estimate the first summand in the right-hand side of (5-37) as

$$\int_{Q_\ell}\int_{r-C\delta^{\beta/N}}^{r+C\delta^{\beta/N}} |u|\,\mathfrak{m}_q\,\mathfrak{q}(dq) \le C(r^{N-1}\delta^{\beta/N} + r^{(N-1)/2}\delta^{\beta(1/2+1/(2N))})\int_{Q_\ell}|c_q|\,\mathfrak{q}(dq). \tag{5-38}$$

Combining (5-37), (5-38), and choosing $r = \delta^{\gamma/N}$ with $\gamma \in (0, \min\{\beta, 1-\beta\})$ yields

$$\left|\int_{B_r(P_N)\cap R(Q_\ell)} u\,\mathfrak{m} - \bar{c}\sqrt{N+1}\mathfrak{q}(Q_\ell)\int_0^{\bar{r}} \cos(\cdot)\,\mathfrak{m}_N\right| \le C(r^{N-1}\delta^{\beta/N} + r^N\delta^{\beta\min\{1/2,1/N\}} + r^{N/2}\delta^{\beta/2}). \tag{5-39}$$

The combination of (5-35) (5-36) and (5-39) gives

$$\left|\int_{B_r(P_N)} u\,\mathfrak{m} - \bar{c}\sqrt{N+1}\mathfrak{q}(Q_\ell)\int_0^{\bar{r}} \cos(\,\cdot\,)\,\mathfrak{m}_N\right| \le C(r^{N-1}\delta^{\beta/N} + r^N\delta^{\beta\min\{1/2,1/N\}} + r^{N/2}\delta^{\beta/2} + \delta^{(1-\beta)/2}r^{N/2}). \tag{5-40}$$

To bound (5-34), approximating the measure of the ball $B_r(P_N)$ and then the function u with the respective model behaviors, we now estimate

$$
\begin{aligned}
&\mathfrak{m}_N([0,\bar r])\mathfrak{q}(Q_\ell)\left|\fint_{B_r(P_N)} u\,\mathfrak{m}-\bar c\sqrt{N+1}\,\mathfrak{q}(Q_\ell)\fint_0^{\bar r}\cos(\,\cdot\,)\,\mathfrak{m}_N\right|^2\\
&\quad\le 2\mathfrak{m}_N([0,\bar r])\mathfrak{q}(Q_\ell)\left|\fint_{B_r(P_N)} u\,\mathfrak{m}-\frac{1}{\mathfrak{m}_N([0,\bar r])}\int_{B_r(P_N)} u\,\mathfrak{m}\right|^2\\
&\qquad\qquad+2\mathfrak{m}_N([0,\bar r])\mathfrak{q}(Q_\ell)\left|\frac{1}{\mathfrak{m}_N([0,\bar r])}\int_{B_r(P_N)} u\,\mathfrak{m}-\bar c\,\mathfrak{q}(Q_\ell)\sqrt{N+1}\fint_0^{\bar r}\cos(\,\cdot\,)\,\mathfrak{m}_N\right|^2\\
&\quad\le 2\mathfrak{m}_N([0,\bar r])\mathfrak{q}(Q_\ell)\left(\int_{B_r(P_N)} u\,\mathfrak{m}\right)^2\left(\frac{1}{\mathfrak{m}(B_r(P_N))}-\frac{1}{\mathfrak{m}_N([0,\bar r])}\right)^2\\
&\qquad\qquad+2\frac{1}{\mathfrak{m}_N([0,\bar r])}\mathfrak{q}(Q_\ell)\left|\int_{B_r(P_N)} u\,\mathfrak{m}-\bar c\mathfrak{q}(Q_\ell)\sqrt{N+1}\int_0^{\bar r}\cos(\,\cdot\,)\,\mathfrak{m}_N\right|^2. \qquad (5\text{-}41)
\end{aligned}
$$

Estimate the first term by Cauchy–Schwarz and the second term by (5-40):

$$
\le 2\mathfrak{q}(Q_\ell)\left[\frac{(\mathfrak{m}(B_r(P_N))-\mathfrak{m}_N([0,\bar r]))^2}{\mathfrak{m}(B_r(P_N))\mathfrak{m}_N([0,\bar r])}\right]\int_{B_r(P_N)} u^2\,\mathfrak{m}+C\mathfrak{q}(Q_\ell)[\delta^{\gamma+2(\beta-\gamma)/N}+\delta^{\gamma+\beta\min\{1,2/N\}}+\delta^\beta+\delta^{1-\beta}].
$$

Now use Proposition 4.5 and choose $r=\delta^{\gamma/N}$, $\gamma\in(0,\min\{\beta,1-\beta\})$:

$$
\begin{aligned}
&\le 2\left(\int_{B_r(P_N)} u^2\,\mathfrak{m}\right)\left(\frac{\mathfrak{m}_N([\bar r,r+C\delta^{\beta/N}])}{\mathfrak{m}_N([0,\bar r])}\right)^2+C(\delta^{\gamma+2(\beta-\gamma)/N}+\delta^{\gamma+\beta\min\{1,2/N\}}+\delta^\beta+\delta^{1-\beta})\\
&\le C(\delta^{\gamma+2(\beta-\gamma)/N}+\delta^{\gamma+\beta\min\{1,2/N\}}+\delta^\beta+\delta^{1-\beta}), \qquad (5\text{-}42)
\end{aligned}
$$

where the second inequality is obtained by observing that

$$
\begin{aligned}
\int_{B_r(P_N)} u^2\,\mathfrak{m}&=\int_{B_r(P_N)\setminus Q_\ell} u^2\,\mathfrak{m}+\int_{B_r(P_N)\cap Q_\ell} u^2\,\mathfrak{m}\\
&\le \delta^{1-\beta}+2\int_{Q_\ell}\int_{[0,r+C\delta^{\beta/N}]}(u-c_q\sqrt{N+1}\cos(\,\cdot\,))^2\mathfrak{m}_q\,\mathfrak{q}(dq)\\
&\qquad\qquad+2\int_{Q_\ell}\int_{[0,r+C\delta^{\beta/N}]}c_q^2(N+1)\cos^2(\,\cdot\,)\,\mathfrak{m}_q\mathfrak{q}(dq)\\
&\le C(\delta^{1-\beta}+\delta^\beta+\delta^{\gamma+\beta\min\{1,2/N\}}+\delta^\gamma).
\end{aligned}
$$

The claimed estimate (5-32) is eventually obtained via triangle inequality from (5-33) and (5-42)

Step 2: Building upon Proposition 5.2, we shall obtain the bound

$$
\begin{aligned}
&\int_{Q_\ell}\int_{[0,\bar r]}(N+1)\left|\bar c\cos(\,\cdot\,)-\bar c\,\mathfrak{q}(Q_\ell)\fint_0^{\bar r}\cos(\,\cdot\,)\,\mathfrak{m}_N\right|^2\mathfrak{m}_N\,\mathfrak{q}(dq)\\
&\qquad\le 2\int_{B_r(P_N)}\left|u-\fint_{B_r(P_N)} u\,\mathfrak{m}\right|^2\mathfrak{m}+C\delta^\gamma(\delta^{4\gamma/N}+\delta^{(\beta-\gamma)/N})+C\delta^{1-\beta}. \qquad (5\text{-}43)
\end{aligned}
$$

Thanks to the triangle inequality, the error we introduce by replacing $c_q \cos(\cdot)$ with $\bar{c}\cos(\cdot)$ can be controlled by

$$\int_{Q_\ell}\int_{[0,\bar{r}]} |c_q-\bar{c}|^2\cos^2(t)\,\mathfrak{m}_N(dt)\,\mathfrak{q}(dq) \le \mathfrak{m}_N([0,\bar{r}])\int_{Q_\ell}|c_q-\bar{c}|^2\,\mathfrak{q}(dq) \le C\delta^{\gamma}(\delta^{4\gamma/N}+\delta^{(\beta-\gamma)\min\{1,2/N\}})+C\delta^{1-\beta+2\gamma/N}, \tag{5-44}$$

where the last inequality is a consequence of (5-18) and the fact that $\bar{r}\le r=\delta^{\gamma/N}$, $\gamma\in(0,\min\{\beta,1-\beta\})$. The claimed (5-43) follows from (5-44) and (5-32) via triangle inequality.

Step 3: Using the Taylor expansion $\cos(t)=1+O(t^2)$ in the left-hand side of (5-43), we obtain

$$\int_{Q_\ell}\int_0^{\bar{r}}(N+1)|\bar{c}-\bar{c}\mathfrak{q}(Q_\ell)|^2\mathfrak{m}_N\,\mathfrak{q}(dq)\le\int_{B_r(P_N)}\biggl|u-⨍_{B_r(P_N)}u\,\mathfrak{m}\biggr|^2\mathfrak{m}+C\delta^{\gamma}(\delta^{4\gamma/N}+\delta^{(\beta-\gamma)/N})+C\delta^{1-\beta},$$

giving

$$\mathfrak{m}_N([0,\bar{r}])(N+1)\bar{c}^2(1-\mathfrak{q}(Q_\ell))^2\mathfrak{q}(Q_\ell)\le 2\int_{B_r(P_N)}\biggl|u-⨍_{B_r(P_N)}u\,\mathfrak{m}\biggr|^2\mathfrak{m}+C\delta^{\gamma}(\delta^{4\gamma/N}+\delta^{(\beta-\gamma)/N})+C\delta^{1-\beta}.$$

Using the 2-2 Poincaré inequality (2-6) (combined with Bishop–Gromov volume comparison), we obtain

$$\mathfrak{m}_N([0,\bar{r}])(N+1)\bar{c}^2(1-\mathfrak{q}(Q_\ell))^2\mathfrak{q}(Q_\ell)\le Cr^2\int_{B_{10r}(P_N)}|\nabla u|^2\,\mathfrak{m}+C\delta^{\gamma}(\delta^{4\gamma/N}+\delta^{(\beta-\gamma)/N})+C\delta^{1-\beta} \le C\delta^{\gamma}(\delta^{4\gamma/N}+\delta^{(\beta-\gamma)/N})+C\delta^{1-\beta}, \tag{5-45}$$

where in the last estimate we used (5-29) (recall that $r=\delta^{\gamma/N}$).

Using again that $\int_{Q_\ell}|c_q-\bar{c}|^2\,\mathfrak{q}(dq)\le C\delta^{\alpha(N)}$ from (5-18) for some $\alpha(N)>0$, observing that

$$\int_{Q_\ell}(c_q^2-\bar{c}^2)\,\mathfrak{q}(dq)=\int_{Q_\ell}|c_q-\bar{c}|^2\,\mathfrak{q}(dq), \tag{5-46}$$

and recalling (5-9), we get

$$\bar{c}^2\mathfrak{q}(Q_\ell)=\int_{Q_\ell}c_q^2\,\mathfrak{q}(dq)+\int_{Q_\ell}(\bar{c}^2-c_q^2)\,\mathfrak{q}(dq)\ge 1-\delta^{1-\beta}-\int_{Q_\ell}|c_q-\bar{c}|^2\,\mathfrak{q}(dq) \ge 1-\delta^{1-\beta}-C\delta^{\alpha(N)}>\frac{1}{C(N)}>0. \tag{5-47}$$

Plugging (5-47) into (5-45) yields

$$(1-\mathfrak{q}(Q_\ell))^2\le C(\delta^{4\gamma/N}+\delta^{(\beta-\gamma)/N}+\delta^{1-\beta-\gamma}), \tag{5-48}$$

completing the proof. □

Remark 5.4. Observe that a direct consequence of Proposition 5.3 above is an estimate of the measure of the region of the space which is not covered by transport rays, that is, $\{u=0\}$.

Indeed (5-31) implies in particular that

$$\mathfrak{m}(X\setminus\mathcal{T})\le 1-\mathfrak{q}(Q_\ell)\le C(N)(\delta^{2\gamma/N}+\delta^{(\beta-\gamma)/2N}+\delta^{(1-\beta-\gamma)/2}). \tag{5-49}$$

5C. *Proof of the main theorem.* We are now ready to prove the main result putting together the estimates we proved so far. First we reduce to the set spanned by long rays using Proposition 5.3; then, building upon Proposition 5.2 and on Theorem 3.11, we prove that on the set of long rays the function is close to a fixed multiple of the cosine of the distance from the endpoint. Eventually we change the distance from the endpoint of the ray into the distance from a pole thanks to (5-12).

Theorem 5.5. *For any $N \in (1, \infty)$ there exist $C(N) > 0$ and $\delta_0 = \delta_0(N) > 0$ with the following properties. Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially nonbranching $\mathrm{CD}(N-1, N)$ m.m.s. Then, for any $u \in \mathrm{Lip}(X)$ with $\int_X u\, \mathfrak{m} = 0$, $\int_X u^2\, \mathfrak{m} = 1$ and*

$$\delta := \int_X |\nabla u|^2\, \mathfrak{m} - N \leq \delta_0, \tag{5-50}$$

there exists a distinguished point $P \in X$ such that

$$\|u - \sqrt{N+1} \cos(\mathsf{d}(P, \cdot))\|_{L^2(\mathfrak{m})} \leq C(N)\delta^{1/(6N+4)}. \tag{5-51}$$

Proof. Step 1: Let us begin observing that Proposition 5.2 combined with (5-30) and (5-46) gives

$$\left| \int_{Q_\ell} c_q^2\, \mathfrak{q}(dq) - \bar{c}^2\, \mathfrak{q}(Q_\ell) \right| \leq C(N)(\delta^{4\gamma/N} + \delta^{1-\beta-\gamma+(2\gamma/N)} + \delta^{(\beta-\gamma)\min\{2/N,1\}}). \tag{5-52}$$

Since from (5-9) we know that

$$1 - \delta^{1-\beta} \leq \int_{Q_\ell} c_q^2\, \mathfrak{q}(dq) \leq 1,$$

and in Proposition 5.3 we proved that

$$\mathfrak{q}(Q_\ell) \geq 1 - C(N)(\delta^{2\gamma/N} + \delta^{(\beta-\gamma)/2N} + \delta^{(1-\beta-\gamma)/2}), \tag{5-53}$$

from (5-52) we infer that

$$|1 - \bar{c}^2| \leq C(N)(\delta^{2\gamma/N} + \delta^{(1-\beta-\gamma)/2} + \delta^{(\beta-\gamma)/2N}). \tag{5-54}$$

Notice that (5-54) implies (see for instance the proof of (3-48))

$$\min\{|1 - \bar{c}|, |1 + \bar{c}|\} \leq C(N)(\delta^{2\gamma/N} + \delta^{(1-\beta-\gamma)/2} + \delta^{(\beta-\gamma)/2N}). \tag{5-55}$$

Without loss of generality (up to switching the sign of u) we can assume that

$$|1 - \bar{c}| = \min\{|1 - \bar{c}|, |1 + \bar{c}|\}.$$

The combination of Proposition 5.2 and (5-55) gives

$$\begin{aligned} \int_{Q_\ell} |c_q - 1|^2\, \mathfrak{q}(dq) &\leq 2 \int_{Q_\ell} |c_q - \bar{c}|^2\, \mathfrak{q}(dq) + 2|\bar{c} - 1|^2\, \mathfrak{q}(Q_\ell) \\ &\leq C(N)(\delta^{4\gamma/N} + \delta^{1-\beta-\gamma} + \delta^{(\beta-\gamma)/N}). \end{aligned} \tag{5-56}$$

Step 2: Next we let P be equal to P_N given in Proposition 5.1. We get

$$\begin{aligned}
&\|u-\sqrt{N+1}\cos(\mathsf{d}(P,\cdot))\|^2_{L^2(\mathfrak{m})}\\
&\quad=\int_Q\int_{X_q}|u-\sqrt{N+1}\cos(\mathsf{d}(P,\cdot))|^2\,\mathfrak{m}_q\,\mathfrak{q}(dq)+\int_{X\setminus\mathcal{T}}(N+1)\cos(\mathsf{d}(P,\cdot))^2\,\mathfrak{m}\\
&\quad\le\int_{Q_\ell}\int_{X_q}|u-\sqrt{N+1}\cos(\mathsf{d}(P,\cdot))|^2\,\mathfrak{m}_q\,\mathfrak{q}(dq)\\
&\qquad\qquad+2\int_{X\setminus R(Q_\ell)}u^2\,\mathfrak{m}+2(N+1)\,\mathfrak{q}(Q\setminus Q_\ell)+(N+1)\,\mathfrak{m}(X\setminus\mathcal{T}).
\end{aligned}$$

Using (5-9), (5-53) and Remark 5.4, recalling that we are tacitly identifying the ray X_q with the interval $(0,|X_q|)$,

$$\begin{aligned}
&\le\int_{Q_\ell}\int_{X_q}|u-\sqrt{N+1}\cos(\mathsf{d}(P,\cdot))|^2\,\mathfrak{m}_q\,\mathfrak{q}(dq)+C(N)(\delta^{2\gamma/N}+\delta^{(\beta-\gamma)/2N}+\delta^{(1-\beta-\gamma)/2})\\
&\le 2\int_{Q_\ell}\int_{X_q}|u-\sqrt{N+1}\cos(\,\cdot\,)|^2\,\mathfrak{m}_q\,\mathfrak{q}(dq)\\
&\qquad+2\int_{Q_\ell}\int_{X_q}|\cos(\,\cdot\,)-\cos(\mathsf{d}(P,\cdot))|^2\,\mathfrak{m}_q\,\mathfrak{q}(dq)+C(N)(\delta^{2\gamma/N}+\delta^{(\beta-\gamma)/2N}+\delta^{(1-\beta-\gamma)/2}).
\end{aligned}$$

Using triangle inequality to estimate the first term and (5-12) for the second,

$$\begin{aligned}
&\le 4\int_{Q_\ell}\int_{X_q}|u-\sqrt{N+1}c_q\cos(\,\cdot\,)|^2\,\mathfrak{m}_q\,\mathfrak{q}(dq)+C(N)\int_{Q_\ell}|c_q-1|^2\,\mathfrak{q}(dq)\\
&\qquad+C(N)(\delta^{2\gamma/N}+\delta^{(\beta-\gamma)/2N}+\delta^{(1-\beta-\gamma)/2}).
\end{aligned}$$

By (5-13) and (5-56),

$$\begin{aligned}
&\le C(N)\delta^{\beta\min\{1,2/N\}}\int_{Q_\ell}c_q^2\,\mathfrak{q}(dq)+C(N)(\delta^{2\gamma/N}+\delta^{(\beta-\gamma)/2N}+\delta^{(1-\beta-\gamma)/2})\\
&\le C(N)(\delta^{2\gamma/N}+\delta^{(\beta-\gamma)/2N}+\delta^{(1-\beta-\gamma)/2}).
\end{aligned}\tag{5-57}$$

The optimal choice of parameters in (5-57) is $\beta=5N/(6N+4)$ and $\gamma=N/(6N+4)$ giving the claim (5-51). □

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COUNIVERSALITY AND CONTROLLED MAPS ON PRODUCT SYSTEMS OVER RIGHT LCM SEMIGROUPS

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We study the structure of C^*-algebras associated with compactly aligned product systems over group embeddable right LCM semigroups. Towards this end we employ controlled maps and a controlled elimination method that associates the original cores to those of the controlling pair, and we combine these with applications of the C^*-envelope theory for cosystems of nonselfadjoint operator algebras recently produced. We derive several applications of these methods that generalize results on single C^*-correspondences.

First we show that if the controlling group is exact then the couniversal C^*-algebra of the product system coincides with the quotient of the Fock C^*-algebra by the ideal of strong covariance relations. We show that if the controlling group is amenable then the product system is amenable. In particular if the controlling group is abelian then the couniversal C^*-algebra is the C^*-envelope of the tensor algebra.

Secondly we give necessary and sufficient conditions for the Fock C^*-algebra to be nuclear and exact. When the controlling group is amenable we completely characterize nuclearity and exactness of any equivariant injective Nica-covariant representation of the product system.

Thirdly we consider controlled maps that enjoy a saturation property. In this case we induce a compactly aligned product system over the controlling pair that shares the same Fock representation, and preserves injectivity. By using couniversality, we show that they share the same reduced covariance algebras. If in addition the controlling pair is a total order then the fixed point algebra of the controlling group induces a super product system that has the same reduced covariance algebra and is moreover reversible.

1. Introduction

Framework. In the present project we study further the effect of nonselfadjoint operator algebras and boundary theory of group coactions on the theory of C^*-algebras recently initiated by the authors and Dor-On in [Dor-On et al. 2022]. We work in the class of algebras of a compactly aligned product system X over a right LCM semigroup P in a group G with coefficients in a C^*-algebra A (for brevity we will say that such a pair (G, P) is a *weak right* LCM *inclusion*). Continuous product systems of Hilbert spaces were coined by Arveson [1989] for $\mathbb{R}^+$, and their discrete counterparts were studied by Dinh [1991]. Motivated by Pimsner's seminal work [1997], Fowler [1999] studied product systems of correspondences over quasilattices. Since then discrete product systems have been studied by many authors (far too many to list here) and constitute an active area of research in their own right. Recently there has been a growing interest in passing from quasilattices to right LCM semigroups. Kwaśniewski and Larsen [2019]

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studied the Toeplitz–Nica–Pimsner C^*-algebra $\mathcal{NT}(X)$ for right LCM semigroups proving Toeplitz–Cuntz–Krieger-type uniqueness theorems. Here we turn our focus to equivariant quotients with an eye towards Cuntz-type covariant realizations.

One of the main questions in this direction has been to identify the appropriate quotient of $\mathcal{NT}(X)$ so that faithful representations of A lift to faithful representations of the quotient. This cannot be expected to hold unconditionally. The next best hope is thus to locate the quotient of $\mathcal{NT}(X)$ so that faithful representations of A lift to faithful representations of its fixed point algebra. Sehnem [2019] has provided a full answer by introducing the strongly covariant representations. This generalizes the study of Cuntz–Nica–Pimsner relations, initiated by Sims and Yeend [2010], and later continued by Carlsen, Larsen, Sims and Vittadello [Carlsen et al. 2011]. A second aim of [Carlsen et al. 2011] was to use these relations and provide a couniversal object by passing to an appropriate reduced quotient. This was achieved under extra conditions on the product system (such as injectivity or directness of the quasilattice).

Couniversality and boundary representations arise naturally in the context of nonselfadjoint operator algebras and their C^*-envelope in the sense of Arveson. With Dor-On, in [Dor-On et al. 2022] we introduced a coaction variant of the C^*-envelope and used it to fully answer the problem of Carlsen, Larsen, Sims and Vittadello [Carlsen et al. 2011] without any assumptions on the product system X. Even more, the results of [Dor-On et al. 2022] apply to weak right LCM inclusions (G, P) rather than just quasilattices; more specifically, the C^*-envelope $C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G)$ of the Fock tensor algebra $\mathcal{T}_\lambda(X)^+$ with its normal coaction is couniversal for equivariant injective Nica-covariant representations of X. Seeing Sehnem's covariance algebra $A \times_X P$ as the universal C^*-algebra of an induced Fell bundle we further showed that $C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta})$ coincides with the reduced C^*-algebra of this Fell bundle, here denoted by $A \times_{X,\lambda} P$.

The algebraic structure of $C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G)$ was studied in [Dor-On et al. 2022]. Pivotal in this endeavor was the remark that the strong covariance relations of Sehnem are actually filtered through the Fock representation. Following Sehnem [2019], we will denote by $A \times_X P$ the universal C^*-algebra with respect to the strongly covariant representations of X. We further consider the induced quotient $q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$ of $\mathcal{T}_\lambda(X)$ by the strong covariance relations. In [Dor-On et al. 2022] it is shown that the canonical map

$$q_{\mathrm{sc}}(\mathcal{T}_\lambda(X)) \to C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}) \simeq A \times_{X,\lambda} P \tag{1-1}$$

is faithful if and only if the normal coaction of $\mathcal{T}_\lambda(X)$ descends to a normal coaction on $q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$, e.g., when G is exact.

The motivation for the present work is two-fold. On one hand we wish to explore further general settings that entail normality of the coaction of $q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$ and thus identify the algebraic structure of the couniversal object. Our main theorem here is that this happens when (G, P) is controlled by another weak right LCM inclusion $(\mathcal{G}, \mathcal{P})$ with $\mathcal{G}$ exact. When $\mathcal{G}$ is abelian we can further induce dual actions on the C^*-algebras. This has the remarkable consequence that the canonical $*$-epimorphism

$$C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}) \to C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+) \tag{1-2}$$

is faithful. On the other hand we wish to use the couniversal property in such a context and apply it in the identification of C^*-algebras. The quotient by the strong covariance relations is used as a model in several constructions, and this line of reasoning allows to show functoriality without checking a long list of C^*-properties. This is quite pleasing in particular because reduced C^*-algebras do not enjoy a priori universal properties. In fact we follow the reverse route of using the identification of reduced objects and then lift them to $*$-isomorphisms of the universal ones.

Main results. Controlled maps $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ between quasilattice ordered groups were introduced by Laca and Raeburn [1996] with the purpose of extending the range of application of the faithfulness and uniqueness theorems for Toeplitz algebras of quasilattice ordered groups. The key idea is that (G, P) is amenable in the sense of Nica [1992], provided that $\mathcal{G}$ is an amenable group. A similar notion of controlled maps was formulated simultaneously and independently by Crisp [1999] to prove that some Artin monoids inject in their groups. The combination of these two sets of ideas led to the amenability and nonamenability results for Artin monoids by Crisp and the third author [Crisp and Laca 2002]. Similar results can be derived for the Fock algebra $\mathcal{T}_\lambda(X)$ of a product system over P, as it has a $\mathcal{P}$-core that can be expressed as a direct sum of matrix algebras (see for example the proof of Theorem 6.4). As a consequence one obtains for example that compactly aligned product systems over the free semigroup $\mathbb{F}^n_+$ are amenable, although the group $\mathbb{F}^n$ is not, the reason being that the pair $(\mathbb{F}^n, \mathbb{F}^n_+)$ is controlled by its abelianization or by its length map on $(\mathbb{Z}, \mathbb{Z}_+)$.

However this type of argument is no longer valid for equivariant quotients as these relations live in the diagonal of the P-core (and thus in the $\mathcal{P}$-core). An elimination method was recently developed in [Kakariadis 2020] when $(\mathcal{G}, \mathcal{P}) = (\mathbb{Z}^n, \mathbb{Z}^n_+)$ with the purpose of studying nuclearity and exactness properties. By building further on these techniques, in the subsection on page 1458 we give a controlled elimination method for passing from the $\mathcal{P}$-cores to the P-cores of injective Nica-covariant representations. Essentially the method asserts that any relation in a $\mathcal{P}$-core must live at the diagonal and thus in a P-core. We then use this to lift all properties from the realm of the P-fixed point algebras to the $\mathcal{P}$-fixed point algebras. For example this applies to the fixed point algebra property of Sehnem's algebra [2019]; see Corollary 5.8. In particular exactness of $\mathcal{G}$ impacts on the maps appearing in (1-1).

Theorem A (Theorem 6.1). *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right* LCM *inclusions and let X be a compactly aligned product system over P with coefficients in A. Consider the canonical $*$-epimorphisms*

$$q_{\mathrm{sc}}(\mathcal{T}_\lambda(X)) \to A \times_{X,\lambda} P \simeq C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G) \to C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+). \tag{1-3}$$

If $\mathcal{G}$ is exact then the left map is faithful. If in addition $\mathcal{G}$ is abelian then the right map is also faithful.

Theorem A implies that the coaction on $q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$ is normal when $\mathcal{G}$ is exact. As pointed out in [Dor-On et al. 2022], this implies that the reduced Hao–Ng problem over discrete group actions has a positive answer (Remark 6.3). A similar method applies whenever the C^*-envelope functor is stable under crossed products, e.g., for dynamics over abelian locally compact groups or when the tensor algebra is hyperrigid [Katsoulis 2020; Katsoulis and Ramsey 2019], and we leave this to the interested reader. A further consequence of Theorem A is that amenability of $\mathcal{G}$ implies amenability of the product system and thus universality of the reduced constructions (Theorem 6.4). The case of abelian $\mathcal{G}$ directly generalizes the results of [Dor-On and Katsoulis 2020]. There is further potential for Takai duality results even when (G, P) does not admit a dual. A further consequence of Theorem A provides a generalization of the extension theorem of [Katsoulis and Ramsey 2019], which recognizes a Fock tensor algebra by the presence of a coaction (Corollary 6.7).

Another application of the controlled elimination method concerns nuclearity and exactness results. It has been observed by Katsura [2004] that nuclearity of a Cuntz–Pimsner algebra is equivalent to the coefficient algebra being nuclearly embedded in the fixed point algebra. Kakariadis [2020] produced similar results for $\mathbb{Z}_+^n$. In Theorem 6.11 we first give an equivalent characterization for nuclearity of $\mathcal{T}_\lambda(X)$ for right LCM semigroups. Although our original goal was to exploit $A \times_X P$, we tackle any equivariant quotient of $\mathcal{NT}(X)$ that is injective on A.

Theorem B (Theorem 6.12, Theorem 6.13). *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right* LCM *inclusions with $\mathcal{G}$ amenable and let X be a compactly aligned product system over P with coefficients in A. Let (π, t) be an equivariant injective Nica-covariant representation of X. Then*:

(i) *A is exact if and only if $C^*(\pi, t)$ is exact.*

(ii) *$A \hookrightarrow C^*(\pi, t)$ is nuclear if and only if $C^*(\pi, t)$ is nuclear.*

We emphasize that the controlled elimination process occurs at the level of representations. One might be intrigued to introduce a product system Y over $\mathcal{P}$ that would share the same algebras with X over P. However it is not clear that such a procedure gives a compactly aligned product system. For this reason we introduce the notion of saturation for controlled maps, which preserves inclusions of ideals in the semigroups. Under this condition we do get a super product system on the *same* coefficient algebra that does the job.

Theorem C (Theorem 7.7). *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a saturated controlled map between weak right* LCM *inclusions. Let X be an* (*injective*) *compactly aligned product system over P with coefficients in A and let*

$$Y_h := \sum_{p \in \vartheta^{-1}(h)}^{\oplus} X_p \quad \textit{for } h \in \mathcal{P}.$$

Then the collection $Y = \{Y_h\}_{h \in \mathcal{P}}$ is an (*injective*) *compactly aligned product system over $\mathcal{P}$ with coefficients in A such that $\mathcal{T}_\lambda(X)^+ \simeq \mathcal{T}_\lambda(Y)^+$ with*

$$\mathcal{T}_\lambda(X) \simeq \mathcal{T}_\lambda(Y) \quad \textit{and} \quad A \times_{X,\lambda} P \simeq A \times_{Y,\lambda} \mathcal{P},$$

by $*$-isomorphisms that preserve the inclusions $X_p \mapsto Y_{\vartheta(p)}$ for all $p \in P$. These $*$-isomorphisms further lift to $*$-isomorphisms

$$\mathcal{NT}(X) \simeq \mathcal{NT}(Y) \quad \textit{and} \quad A \times_X P \simeq A \times_Y \mathcal{P}$$

that preserve the inclusions $X_p \hookrightarrow Y_{\vartheta(p)}$ for all $p \in P$.

Our method here is to show that the $*$-isomorphism $\mathcal{T}_\lambda(X) \simeq \mathcal{T}_\lambda(Y)$ is canonical on the tensor algebras and then apply the C^*-envelope machinery to induce the $*$-isomorphism $A \times_{X,\lambda} P \simeq A \times_{Y,\lambda} \mathcal{P}$. The saturation property can be induced by free products of abelian total orders, and is preserved by semidirect products. As a notable application of this method we deduce that Sehnem's covariance algebra of a product system over $\mathbb{F}_n^+$ is nothing more than the Cuntz–Pimsner algebra of a single C^*-correspondence, in a similar way that the Nica–Cuntz–Pimsner algebra of $\mathbb{F}_n^+$ coincides with $\mathcal{O}_n$ (Corollary 7.8).

We then take a closer look at total orders. To further motivate these results, recall that the Cuntz algebra $\mathcal{O}_n$ may be viewed as the Cuntz–Pimsner algebra of a Hilbert bimodule over the n^∞-hyperfinite C^*-algebra. In spite of the coefficient algebra of the latter being much larger, Hilbert bimodules are better behaved than other types of C^*-correspondences and they allow for a rich theory, including versions of Takai duality. Here we will show that the situation with $\mathcal{O}_n$ generalizes to product systems that are controlled by exact total orders. Towards this end we consider *reversible* product systems for which the image of every fiber in $A \times_{X,\lambda} P$ is a Hilbert bimodule. We then show that reversible product systems produce all possible covariance algebras for weak right LCM inclusions that are controlled by total orders in a saturated way. The construction relies on using the fixed point algebra and generalizes results of Pimsner [1997], Abadie, Eilers and Exel [Abadie et al. 1998], Schweizer [2001], Kakariadis and Katsoulis [2012], and Meyer and Sehnem [2019]. However our proof uses the C^*-envelope machinery and thus avoids categorical arguments.

Theorem D (Theorem 7.15). *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a saturated controlled map between weak right* LCM *inclusions and suppose that $(\mathcal{G}, \mathcal{P})$ is a total order. Let X be an (injective) product system over P with coefficients in A. Then there exists an (injective) reversible product system Z over $\mathcal{P}$ with coefficients in a C^*-algebra B such that*

$$A \subseteq B \quad \text{and} \quad X_p \subseteq Z_{\vartheta(p)} \quad \text{for all } p \in P \tag{1-4}$$

that satisfies

$$A \times_X P \simeq B \times_Z \mathcal{P} \quad \text{and} \quad A \times_{X,\lambda} P \simeq B \times_{Z,\lambda} \mathcal{P}, \tag{1-5}$$

by $*$-isomorphisms that preserve the inclusions $X_p \hookrightarrow Z_{\vartheta(p)}$ for all $p \in P$.

Semigroup C^*-algebras have been an important source of inspiration for this study. Our results have a direct application to C^*-algebras of right LCM semigroups where $X_p = \mathbb{C}$ for every $p \in P$. In this case the Nica–Toeplitz C^*-algebra is denoted by $C_s^*(P)$ for the Nica-covariant representations of P and Theorem A (and in particular Theorem 6.4) is a direct generalization of [Crisp and Laca 2007, Theorem 4.7]. Faithfulness of the maps of Theorem A has been further investigated in [Kakariadis et al. 2022] for (not-necessarily right LCM) semigroups that embed in exact groups. Theorem B asserts that every quotient of $C_s^*(P)$ is nuclear and aligns with [Li 2013, Corollary 8.3] for quasilattices. Under the saturation property, Theorem C asserts that the operator algebras of P coincide with those of a product system Y over $\mathcal{P}$ with $Y_h = \mathbb{C}^{|\vartheta^{-1}(h)|}$ for $h \in \mathcal{P}$. This follows a recurring idea of obtaining realizations of the same C^*-algebra in different classes. It has been shown in [Li 2017] that $\mathbb{C} \times_{\mathbb{C},\lambda} P$ can be realized as the partial crossed product of the smallest $\mathcal{G}$-invariant subspace of the fixed point algebra of $C_s^*(\mathcal{P})$ by G. Theorem D provides a similar (augmented) realization when ϑ is saturated and $(\mathcal{G}, \mathcal{P})$ is a total order.

Let us close with a remark on controlled maps. It has been known that controlled maps cannot handle HNN extensions of quasilattices as the height map does not have a trivial kernel on the semigroup. In order to resolve this, recently an Huef, Nucinkis, Sehnem and Yang [an Huef et al. 2021] introduced a more general definition of controlled maps for weak quasilattices that allows infinite descending chains and thus produces direct limits of matrix algebras. The controlled elimination arguments we provide here should be compatible with this general definition, as they refer to ideals of representations, which are compatible with direct limits.

Structure of the paper. In Section 2 we review the boundary theory and the theory of the cosystems from [Dor-On et al. 2022]. In Sections 3 and 4 we review the main elements of the product systems theory, and we see how they are enriched under the presence of a controlled map. We have included more details from the aforementioned paper in order to set the ground for the next sections, and also prove additional results that are not covered there. In Section 5 we present the controlled elimination method. Section 6 contains the applications to Sehnem's covariance algebra, the structure of the couniversal C^*-algebra, amenable product systems, nuclearity and exactness, and the reduced Hao–Ng problem. In Section 7 we give the product system reparametrizations under the saturation property with applications to reversible product systems.

2. Operator algebras and their coactions

Operator algebras. The reader may refer to [Blecher and Le Merdy 2004; Paulsen 2002] for the general theory of nonselfadjoint operator algebras and dilations of their representations.

Let $\mathcal{A}$ be an operator algebra, which in this paper means a subalgebra of $\mathcal{B}(H)$ for a Hilbert space H. We say that (C, ι) is a C^*-*cover* of $\mathcal{A}$ if $\iota : \mathcal{A} \to C$ is a completely isometric representation with $C = C^*(\iota(\mathcal{A}))$. The C^*-*envelope* $C^*_{\text{env}}(\mathcal{A})$ of $\mathcal{A}$ is a C^*-cover $(C^*_{\text{env}}(\mathcal{A}), \iota)$ with the following couniversal property: if (C', ι') is a C^*-cover of $\mathcal{A}$ then there exists a (necessarily unique) $*$-epimorphism $\Phi : C' \to C^*_{\text{env}}(\mathcal{A})$ such that $\Phi(\iota'(a)) = \iota(a)$ for all $a \in \mathcal{A}$. Arveson [1969] defined the C^*-envelope in and computed it for a variety of operator algebras, predicting its existence in general. Ten years later Hamana [1979] confirmed Arveson's prediction by proving the existence of injective envelopes for the unital case. The C^*-envelope is the C^*-algebra generated in the injective envelope of $\mathcal{A}$ once this is endowed with the Choi–Effros C^*-structure.

Dritschel and McCullough [2005] provided an alternative proof based on maximal dilations for the unital case. A *dilation* of a representation $\phi : \mathcal{A} \to \mathcal{B}(H)$ is a representation $\phi' : \mathcal{A} \to \mathcal{B}(H')$ such that $H \subseteq H'$ and $\phi(a) = P_H \phi'(a)|_H$ for all $a \in \mathcal{A}$. A completely contractive map $\phi : \mathcal{A} \to \mathcal{B}(H)$ is called *maximal* if every dilation $\phi' : \mathcal{A} \to \mathcal{B}(H')$ is trivial, i.e., $P_H \phi'(a) = \phi(a) = \phi'(a)|_H$ for all $a \in \mathcal{A}$. It follows that the C^*-envelope is the C^*-algebra generated by a maximal completely isometric representation.

It does not hold in general that if $\pi : C^*_{\text{env}}(\mathcal{A}) \to \mathcal{B}(H)$ is a $*$-representation then it is the unique contractive completely positive extension of $\pi|_{\mathcal{A}}$. The algebra $\mathcal{A}$ is called *hyperrigid* if this is the case for any representation π of $C^*_{\text{env}}(\mathcal{A})$. An operator algebra $\mathcal{A}$ is said to be *Dirichlet* if

$$C^*_{\text{env}}(\mathcal{A}) = \overline{\mathcal{A} + \mathcal{A}^*}.$$

Equivalently, $\mathcal{A}$ is Dirichlet if there exists a C^*-cover (C, ι) of $\mathcal{A}$ such that $C = \overline{\iota(\mathcal{A}) + \iota(\mathcal{A})^*}$, in which case $C = C^*_{\text{env}}(\mathcal{A})$. It follows that Dirichlet algebras are automatically hyperrigid.

Coactions on operator algebras. If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of some $\mathcal{B}(H)$ then we write

$$[\mathcal{X}\mathcal{Y}] := \overline{\text{span}}\{xy \mid x \in \mathcal{X},\, y \in \mathcal{Y}\}.$$

All groups and semigroups we consider are discrete and unital. Further, we denote the spatial tensor product by $\otimes$.

For a discrete group G we write u_g for the unitary generator associated with $g \in G$ in the full group C^*-algebra $C^*(G)$. We write λ_g for the generators of the left regular representation $C^*_\lambda(G)$. We write $\lambda : C^*(G) \to C^*_\lambda(G)$ for the canonical $*$-epimorphism. Recall that $C^*(G)$ admits a faithful $*$-homomorphism

$$\Delta : C^*(G) \to C^*(G) \otimes C^*(G) \quad \text{such that } \Delta(u_g) = u_g \otimes u_g,$$

given by the universal property of $C^*(G)$, and with left inverse given by $\text{id} \otimes \chi$ for the character χ of $C^*(G)$. We will require some preliminaries from [Dor-On et al. 2022] on coactions on operator algebras.

Definition 2.1 [Dor-On et al. 2022, Definition 3.1]. Let $\mathcal{A}$ be an operator algebra. A *coaction of G on $\mathcal{A}$* is a completely isometric representation $\delta : \mathcal{A} \to \mathcal{A} \otimes C^*(G)$ such that the linear span of the induced subspaces

$$\mathcal{A}_g := \{a \in \mathcal{A} \mid \delta(a) = a \otimes u_g\}$$

is norm-dense in $\mathcal{A}$, in which case δ satisfies the coaction identity

$$(\delta \otimes \text{id}_{C^*(G)})\delta = (\text{id}_{\mathcal{A}} \otimes \Delta)\delta.$$

If, in addition, the map $(\text{id} \otimes \lambda)\delta$ is injective, then the coaction δ is called *normal*.

If $\mathcal{A}$ is an operator algebra and $\delta : \mathcal{A} \to \mathcal{A} \otimes C^*(G)$ is a coaction on $\mathcal{A}$, then we will refer to the triple $(\mathcal{A}, G, \delta)$ as a *cosystem*. A map $\phi : \mathcal{A} \to \mathcal{A}'$ between two cosystems $(\mathcal{A}, G, \delta)$ and $(\mathcal{A}', G, \delta')$ is said to be *G-equivariant*, or simply equivariant, if $\delta'\phi = (\phi \otimes \text{id})\delta$.

If $(\mathcal{A}, G, \delta)$ is a cosystem then $\mathcal{A}_r \cdot \mathcal{A}_s \subseteq \mathcal{A}_{rs}$ for all $r, s \in G$, since δ is a homomorphism.

Remark 2.2 [Dor-On et al. 2022]. Suppose that $(\mathcal{A}, G, \delta)$ is a cosystem and that δ extends to a $*$-homomorphism $\delta : C^*(\mathcal{A}) \to C^*(\mathcal{A}) \otimes C^*(G)$ that satisfies the coaction identity

$$(\delta \otimes \text{id})\delta(c) = (\text{id} \otimes \Delta)\delta(c) \quad \text{for all } c \in C^*(\mathcal{A}).$$

Then δ is automatically nondegenerate on $C^*(\mathcal{A})$ in the sense that

$$[\delta(C^*(\mathcal{A}))C^*(\mathcal{A}) \otimes C^*(G)] = C^*(\mathcal{A}) \otimes C^*(G).$$

Moreover, Definition 2.1 covers that of full coactions of Quigg [1996] when $\mathcal{A}$ is a C^*-algebra. In this case δ is a faithful $*$-homomorphism and we have that

$$(\mathcal{A}_g)^* = \{a^* \in \mathcal{A} \mid \delta(a^*) = a^* \otimes u_{g^{-1}}\} = \mathcal{A}_{g^{-1}}.$$

Due to the Fell absorption principle, the existence of a "reduced" coaction implies that of a normal coaction.

Proposition 2.3 [Dor-On et al. 2022, Proposition 3.4]. *Let $\mathcal{A}$ be an operator algebra. Suppose there is a group G that induces a grading on $\mathcal{A}$, i.e., there are subspaces $\{\mathcal{A}_g\}_{g\in G}$ such that $\sum_{g\in G} \mathcal{A}_g$ is norm-dense in $\mathcal{A}$, and a completely isometric homomorphism*

$$\delta_\lambda : \mathcal{A} \to \mathcal{A} \otimes C^*_\lambda(G)$$

such that

$$\delta_\lambda(a_g) = a_g \otimes \lambda_g \quad \textit{for all } a_g \in \mathcal{A}_g, \quad \textit{for all } g \in G.$$

Then $\mathcal{A}$ admits a normal coaction δ of G such that $\delta_\lambda = (\mathrm{id} \otimes \lambda)\delta$.

Example 2.4. The reduced group C^*-algebra $C^*_\lambda(G)$ admits a faithful $*$-homomorphism

$$\Delta_\lambda : C^*_\lambda(G) \to C^*_\lambda(G) \otimes C^*_\lambda(G) \quad \text{such that } \Delta_\lambda(\lambda_g) = \lambda_g \otimes \lambda_g.$$

Thus $C^*_\lambda(G)$ admits a normal coaction δ of G such that $\Delta_\lambda = (\mathrm{id} \otimes \lambda)\delta$.

Definition 2.5 [Dor-On et al. 2022, Definition 3.6]. Let $(\mathcal{A}, G, \delta)$ be a cosystem. A triple (C, ι, δ_C) is called a *C^*-cover* for $(\mathcal{A}, G, \delta)$ if (C, ι) is a C^*-cover of $\mathcal{A}$ and $\delta_C : C \to C \otimes C^*(G)$ is a coaction on C such that the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\iota"] \arrow[d, "\delta"] & C \arrow[d, "\delta_C"] \\ \mathcal{A} \otimes C^*(G) \arrow[r, "\iota\otimes\mathrm{id}"] & C \otimes C^*(G) \end{tikzcd}$$

commutes. When the coaction is understood we will say that C is a C^*-cover for $\mathcal{A}$ over G.

Definition 2.6 [Dor-On et al. 2022, Definition 3.7]. Let $(\mathcal{A}, G, \delta)$ be a cosystem. The *C^*-envelope* of $(\mathcal{A}, G, \delta)$ is a C^*-cover $(C^*_{\mathrm{env}}(\mathcal{A}, G, \delta), \iota, \delta_{\mathrm{env}})$ such that: for every C^*-cover (C', ι', δ') of $(\mathcal{A}, G, \delta)$ there exists a $*$-epimorphism $\Phi : C' \to C^*_{\mathrm{env}}(\mathcal{A}, G, \delta)$ that fixes $\mathcal{A}$ and intertwines the coactions, i.e., the diagram

$$\begin{tikzcd} \iota'(\mathcal{A}) \arrow[r, "\delta'"] \arrow[d, "\Phi"] & C' \otimes C^*(G) \arrow[d, "\Phi\otimes\mathrm{id}"] \\ \iota(\mathcal{A}) \arrow[r, "\delta_{\mathrm{env}}"] & C^*_{\mathrm{env}}(\mathcal{A}, G, \delta) \otimes C^*(G) \end{tikzcd}$$

is commutative on $\mathcal{A}$, and thus is commutative on C'.

The existence of the C^*-envelope of a cosystem was proved in [Dor-On et al. 2022] by a direct computation that uses the C^*-envelope of the ambient operator algebra. In order to state the result explicitly we need to make some preliminary remarks and establish the notation. Suppose $(\mathcal{A}, G, \delta)$

is a cosystem, let $i : \mathcal{A} \to C^*_{\text{env}}(\mathcal{A})$ be the C^*-envelope of $\mathcal{A}$, and recall that the spatial tensor product of completely isometric maps is completely isometric. Then the representation of $\mathcal{A}$ obtained via the composition

$$\mathcal{A} \xrightarrow{\delta} \mathcal{A} \otimes C^*(G) \xrightarrow{i \otimes \text{id}} C^*_{\text{env}}(\mathcal{A}) \otimes C^*(G)$$

is completely isometric, and the C^*-algebra

$$C^*((i \otimes \text{id})\delta(\mathcal{A})) := C^*(i(a_g) \otimes u_g \mid g \in G)$$

becomes a C^*-cover of $\mathcal{A}$. This C^*-cover is special because it admits a coaction $\text{id} \otimes \Delta$, such that the triple

$$(C^*(i(a_g) \otimes u_g \mid g \in G),\ (i \otimes \text{id})\delta,\ \text{id} \otimes \Delta)$$

is a C^*-cover for $(\mathcal{A}, G, \delta)$. The following theorem summarizes fundamental results about existence and representations of C^*-envelopes for cosystems.

Theorem 2.7 [Dor-On et al. 2022, Theorem 3.8, Corollary 3.9 and Corollary 3.10]. *Let $(\mathcal{A}, G, \delta)$ be a cosystem and let $i : \mathcal{A} \to C^*_{\text{env}}(\mathcal{A})$ be the inclusion map. Then*

$$(C^*_{\text{env}}(\mathcal{A}, G, \delta),\ \iota,\ \delta_{\text{env}}) \simeq (C^*(i(a_g) \otimes u_g \mid g \in G),\ (i \otimes \text{id})\delta,\ \text{id} \otimes \Delta).$$

If in addition δ is normal on $\mathcal{A}$, then δ_{env} is normal on $C^*_{\text{env}}(\mathcal{A}, G, \delta)$.

Moreover, if $\Phi : C^*_{\text{env}}(\mathcal{A}, G, \delta) \to B$ is a $*$-homomorphism that is completely isometric on $\mathcal{A}$, then it is faithful on the fixed point algebra of $C^*_{\text{env}}(\mathcal{A}, G, \delta)$.

Remark 2.8. A coaction of an abelian group G is equivalent to point-norm continuous actions $\{\beta_\gamma\}_{\gamma \in \hat{G}}$ of the dual group $\hat{G}$. Since every β_γ is a completely isometric automorphism it extends to the C^*-envelope. Hence the C^*-envelope of a cosystem coincides with the usual C^*-envelope of the ambient operator algebra when G is abelian. Equivalently, every coaction of an abelian group on an operator algebra lifts to a coaction on its C^*-envelope. As pointed out in [Dor-On et al. 2022], it is unknown if this is the case for general amenable groups.

Group homomorphisms implement coactions. Note that the following proposition for $\mathcal{G} = \{e_{\mathcal{G}}\}$ says nothing more than that every C^*-cover of a cosystem is a C^*-cover of the ambient operator algebra.

Proposition 2.9. *Let $(\mathcal{A}, G, \delta_G)$ be a (normal) cosystem and let $\vartheta : G \to \mathcal{G}$ be a group homomorphism. Then $\mathcal{G}$ induces a (normal) coaction $\delta_{\mathcal{G}}$ on $\mathcal{A}$. Thus every C^*-cover of $\mathcal{A}$ over G is also a C^*-cover of $\mathcal{A}$ over $\mathcal{G}$.*

Proof. By the universal property of $C^*(G)$ we have a $*$-homomorphism

$$\tilde{\vartheta} : C^*(G) \to C^*(\mathcal{G}), \quad u_g \mapsto u_{\vartheta(g)}.$$

We then have the canonical completely contractive homomorphism

$$\delta_{\mathcal{G}} : \mathcal{A} \xrightarrow{\delta_G} \mathcal{A} \otimes C^*(G) \xrightarrow{\text{id} \otimes \tilde{\vartheta}} \mathcal{A} \otimes C^*(\mathcal{G}),$$

which has $\mathrm{id} \otimes \chi$ as a completely contractive left inverse. By definition we have that

$$\mathcal{A}_h := \{a \in \mathcal{A} \mid \delta_{\mathcal{G}}(a) = a \otimes u_h\} \supseteq \{a \in \mathcal{A}_g \mid \vartheta(g) = h\}$$

and thus

$$\mathcal{A} = \overline{\sum_{g \in G} \mathcal{A}_g} \subseteq \overline{\sum_{h \in \mathcal{G}} \mathcal{A}_h} \subseteq \mathcal{A}.$$

Hence $(\mathrm{id} \otimes \tilde{\vartheta})\delta_G$ defines a coaction of $\mathcal{G}$ on $\mathcal{A}$.

Next suppose that δ_G is normal and let $\delta_{G,\lambda} = (\mathrm{id} \otimes \lambda)\delta_G$. Let $\delta_{\mathcal{G}}$ be the coaction induced by δ_G. By Fell's absorption principle we have that the map $\lambda_g \mapsto \lambda_g \otimes \lambda_{\vartheta(g)}$ gives a faithful $*$-homomorphism of $C^*_\lambda(G)$ and thus we get the induced completely isometric representation

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\delta_{G,\lambda}"] \arrow[d, dotted, "\delta_{\mathcal{G},\lambda}"'] & & \overline{\mathrm{alg}}\{a_g \otimes \lambda_g \mid g \in G\} \arrow[d] \\
\overline{\mathrm{alg}}\{a_g \otimes \lambda_{\vartheta(g)} \mid g \in \mathcal{G}\} & & \overline{\mathrm{alg}}\{a_g \otimes \lambda_g \otimes \lambda_{\vartheta(g)} \mid g \in G\} \arrow[ll, "\delta_{G,\lambda}^{-1} \otimes \mathrm{id}"']
\end{tikzcd}$$

which induces a faithful $*$-homomorphism $\delta_{\mathcal{G},\lambda}$. It follows that $\delta_{\mathcal{G},\lambda} = (\mathrm{id} \otimes \lambda)\delta_{\mathcal{G}}$ and thus $\delta_{\mathcal{G}}$ is a normal coaction of $\mathcal{G}$ on $\mathcal{A}$. $\square$

Let us close this section with some remarks on topological gradings from [Exel 1997; 2017]. Recall that a *topological grading* $\{\mathcal{B}_g\}_{g \in G}$ of a C^*-algebra $\mathcal{B}$ consists of linearly independent subspaces that span a dense subspace of $\mathcal{B}$ and are compatible with the group G, i.e., $\mathcal{B}_g^* = \mathcal{B}_{g^{-1}}$ and $\mathcal{B}_g \cdot \mathcal{B}_h \subseteq \mathcal{B}_{gh}$. By [Exel 1997, Theorem 3.3] the linear independence condition can be substituted by the existence of a conditional expectation on $\mathcal{B}_e$. The maximal C^*-algebra $C^*(\mathcal{B})$ of $\mathcal{B}$ is defined as universal with respect to the representations of $\mathcal{B}$. The reduced C^*-algebra $C^*_\lambda(\mathcal{B})$ of $\mathcal{B}$ is defined by the left regular representation of $\mathcal{B}$ on $\ell^2(\mathcal{B})$.

Definition 2.10. Let $\mathcal{B} = \{\mathcal{B}_g\}_{g \in G}$ be a topological grading over a group G in a C^*-algebra $C^*(\mathcal{B})$ that it generates, with completely contractive Fourier maps $E_g : C^*(\mathcal{B}) \to \mathcal{B}_g$, i.e.,

$$E_g(b) = \delta_{g,h} b \qquad \text{for all } b \in \mathcal{B}_h \text{ and } g, h \in G.$$

An ideal $\mathcal{I} \lhd C^*(\mathcal{B})$ is called *induced* if $\mathcal{I} = \langle \mathcal{I} \cap \mathcal{B}_e \rangle$. An ideal $\mathcal{I} \lhd C^*(\mathcal{B})$ is called *Fourier* if $E_g(f) \subseteq \mathcal{I}$ for every $f \in \mathcal{I}$.

Remark 2.11. It follows that an ideal $\mathcal{I} \lhd C^*(\mathcal{B})$ is Fourier if and only if $E_e(f^* f) \in \mathcal{I}$ for all $f \in \mathcal{I}$. Every induced ideal is a Fourier ideal. The converse holds if G is exact and E_e is a faithful conditional expectation. These can be found at [Exel 2017, Proposition 23.9].

A topological grading defines a *Fell bundle* and once a representation of a Fell bundle is established the two notions are the same. In a loose sense a Fell bundle $\mathcal{B}$ over a discrete group G is a collection of Banach spaces $\{\mathcal{B}_g\}_{g \in G}$, often called *the fibers of* $\mathcal{B}$, that satisfy canonical algebraic properties and the C^*-norm properties; see [Exel 2017, Definition 16.1]. So we will alternate freely between Fell bundles and topologically graded C^*-algebras. Spectral subspaces of coactions on C^*-algebras are an important source of topological gradings.

Definition 2.12. Let δ be a coaction of G on a C^*-algebra C and let $\mathcal{I} \lhd C$ be an ideal of C. We say that the quotient map is *G-equivariant*, or that the quotient $C/\mathcal{I}$ is *G-equivariant* if δ descends to a coaction of G on $C/\mathcal{I}$.

Remark 2.13. If $\delta : C \to C \otimes C^*(G)$ is a coaction and $\mathcal{I} \lhd C$ is an induced ideal then δ descends to a faithful coaction of G on $C/\mathcal{I}$; see for example [Carlsen et al. 2011, Proposition A.1]. The same holds for the normal actions when G is exact; see for example [Carlsen et al. 2011, Proposition A.5].

3. Operator algebras of product systems

C^*-correspondences. A *C^*-correspondence* X over A is a right Hilbert module over A with a left action given by a $*$-homomorphism $\varphi_X : A \to \mathcal{L}X$. We write $\mathcal{L}X$ and $\mathcal{K}X$ for the adjointable operators and the compact operators of X, respectively. For two C^*-correspondences X, Y over the same A we write $X \otimes_A Y$ for the balanced tensor product over A. We say that X is unitarily equivalent to Y (and write $X \simeq Y$) if there is a surjective adjointable operator $U \in \mathcal{L}(X, Y)$ such that $\langle U\xi, U\eta\rangle = \langle \xi, \eta\rangle$ and $U(a\xi b) = aU(\xi)b$ for all $\xi, \eta \in X$ and $a, b \in A$. A C^*-correspondence is called *injective* if the left action is injective.

A *representation* (π, t) of a C^*-correspondence is a left module map that preserves the inner product. Then (π, t) is automatically a bimodule map. Moreover there exists a $*$-homomorphism ψ on $\mathcal{K}X$ such that $\psi(\theta_{\xi,\eta}) = t(\xi)t(\eta)^*$ for all $\theta_{\xi,\eta} \in \mathcal{K}X$. When π is injective, then both t and ψ are isometric. A representation (π, t) is called covariant if it satisfies $\pi(a) = \psi(\varphi_X(a))$ for all a in Katsura's ideal $J_X := \ker\varphi_X^{\perp} \cap \varphi_X^{-1}(\mathcal{K}X)$.

Toeplitz algebras. Let P be a unital subsemigroup of a group G. We will write P^* for the set of elements in P that are invertible in P. A *product system X over P* is a family $\{X_p \mid p \in P\}$ of C^*-correspondences over the same C^*-algebra A such that:

(i) $X_e = A$.

(ii) There are multiplication rules $X_p \otimes_A X_q \simeq_{u_{p,q}} X_{pq}$ for every $p, q \in P \setminus \{e\}$.

(iii) There are multiplication rules $A \otimes_A X_p \simeq_{u_{e,p}} [A \cdot X_p]$ and $X_p \otimes_A A \simeq_{u_{p,e}} [X_p \cdot A] = X_p$ for all $p \in P$.

(iv) The multiplication rules are associative in the sense that
$$u_{pq,r}(u_{p,q} \otimes \mathrm{id}_{X_r}) = u_{p,qr}(\mathrm{id}_{X_p} \otimes u_{q,r}) \quad \text{for all } p, q, r \in P.$$

We say that X is *injective* if every X_p is injective. If $x \in P^*$ then the multiplication rules impose that
$$X_x \otimes_A X_{x^{-1}} \simeq A \simeq X_{x^{-1}} \otimes_A X_x.$$
In particular every such X_x is nondegenerate since
$$A \otimes_A X_x \simeq X_x \otimes_A X_{x^{-1}} \otimes_A X_x \simeq X_x \otimes_A A = X_x.$$

Throughout this work we will be assuming that all left actions are nondegenerate. We do this in order to be able to use freely the results from [Dor-On et al. 2022; Sehnem 2019]. Nevertheless it is possible that this assumption can be removed.

Henceforth we will suppress the use of symbols for the multiplication rules. Thus we write $\xi_p \xi_q$ for the image of $\xi_p \otimes \xi_q$ under $u_{p,q}$, and so

$$\varphi_{pq}(a)(\xi_p \xi_q) = (\varphi_p(a)\xi_p)\xi_q \qquad \text{for all } a \in A \quad \text{and} \quad \xi_p \in X_p, \;\; \xi_q \in X_q.$$

The product system structure gives maps

$$i_p^{pq} : \mathcal{L}X_p \to \mathcal{L}X_{pq} \quad \text{such that } i_p^{pq}(S)(\xi_p \xi_q) = (S\xi_p)\xi_q.$$

If $x \in P^*$ then $i_r^{rx} : \mathcal{L}X_r \to \mathcal{L}X_{rx}$ is a $*$-isomorphism with inverse $i_{rx}^{rxx^{-1}} : \mathcal{L}X_{rx} \to \mathcal{L}X_r$.

Definition 3.1. Let P be a unital subsemigroup of a group G and X be a product system over P with coefficients in A. A *Toeplitz representation* (π, t) *of* X consists of a family of representations (π, t_p) of X_p over A such that

$$t_p(\xi_p)t_q(\xi_q) = t_{pq}(\xi_p \xi_q) \quad \text{for all } \xi_p \in X_p, \;\; \xi_q \in X_q.$$

The *Toeplitz algebra* $\mathcal{T}(X)$ *of* X is the universal C^*-algebra generated by A and X with respect to the representations of X. The *Toeplitz tensor algebra* $\mathcal{T}(X)^+$ *of* X is the subalgebra of $\mathcal{T}(X)$ generated by A and X.

If (π, t) is a Toeplitz representation then we write ψ_p for the induced representation on $\mathcal{K}X_p$. We obtain a bimodule triple $(\psi_r, \psi_{r,s}, \psi_s)$ on the bimodule $(\mathcal{K}X_r, \mathcal{K}(X_s, X_r), \mathcal{K}X_s)$ so that $\psi_{r,s}(\theta_{\xi_r,\xi_s}) = t_r(\xi_r)t_s(\xi_s)^*$. We will often interpret π as t_e or ψ_e to simplify our notation henceforth.

Proposition 3.2 [Dor-On et al. 2022, Proposition 2.4]. *Let X be a product system over P with coefficients in A. Let (π, t) be a Toeplitz representation of X. If $x \in P^*$ then*

$$t_x(X_x)^* = t_{x^{-1}}(X_{x^{-1}}).$$

If $w \in P$ and $x \in P^*$ then

$$i_w^{wx}(k_w) \in \mathcal{K}X_{wx} \quad \textit{and} \quad \psi_{wx}(i_w^{wx}(k_w)) = \psi_w(k_w) \qquad \textit{for all } k_w \in \mathcal{K}X_w.$$

Suppose that $\mathcal{T}(X)$ is faithfully represented by $(\tilde{\pi}, \tilde{t})$. By the universal property of $\mathcal{T}(X)$ there is a canonical $*$-homomorphism

$$\tilde{\delta} : \mathcal{T}(X) \to \mathcal{T}(X) \otimes C^*(G), \quad \tilde{t}(\xi_p) \mapsto \tilde{t}(\xi_p) \otimes u_p.$$

Sehnem [2019, Lemma 2.2] has shown that $\tilde{\delta}$ is a nondegenerate and faithful coaction of $\mathcal{T}(X)$ when X is nondegenerate, with each spectral space $\mathcal{T}(X)_g$, with $g \in G$, given by the products

$$\tilde{t}_{p_1}(\xi_{p_1})\tilde{t}_{p_2}(\xi_{p_2})^* \cdots \tilde{t}_{p_n}(\xi_{p_n})^* \quad \text{for } p_1 p_2^{-1} \cdots p_n^{-1} = g.$$

We will do a little bit more for semigroup homomorphisms.

Definition 3.3. Let P and $\mathcal{P}$ be unital subsemigroups of the groups G and $\mathcal{G}$, respectively. If $\vartheta : G \to \mathcal{G}$ is a group homomorphism such that $\vartheta(P) \subseteq \mathcal{P}$, we write $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ and say that ϑ is a *semigroup preserving homomorphism*.

Proposition 3.4. *Let P be a unital subsemigroup of a group G and X be a product system over P with coefficients in A. Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a semigroup preserving homomorphism and suppose that $(\tilde{\pi}, \tilde{t})$ is a faithful representation of $\mathcal{T}(X)$. Then there is a coaction of $\mathcal{G}$ on $\mathcal{T}(X)$ such that*

$$\tilde{\delta} : \mathcal{T}(X) \to \mathcal{T}(X) \otimes C^*(\mathcal{G}), \quad \tilde{t}(\xi_p) \mapsto \tilde{t}(\xi_p) \otimes u_{\vartheta(p)}.$$

Moreover, each spectral space $\mathcal{T}(X)_h$ with $h \in \mathcal{G}$ is given by the products of the form

$$\tilde{t}_{p_1}(\xi_{p_1})\tilde{t}_{p_2}(\xi_{p_2})^* \cdots \tilde{t}_{p_n}(\xi_{p_n})^* \quad \text{for } \vartheta(p_1)\vartheta(p_2)^{-1} \cdots \vartheta(p_n)^{-1} = h,$$

where we impose that $\tilde{t}_{p_i}(\xi_{p_i}) = I$ when $p_i = e_P$ and $h \neq e_{\mathcal{P}}$.

Proof. The universal property induces a $*$-homomorphism $\tilde{\delta} : \mathcal{T}(X) \to \mathcal{T}(X) \otimes C^*(G)$. Moreover $\tilde{\delta}$ is injective with left inverse given by $\mathrm{id} \otimes \chi$. By construction the fibers $[\mathcal{T}(X)]_g$ contain the generators of $\mathcal{T}(X)$. By Remark 2.2 and the definition of $\mathcal{T}(X)^+$, this gives the coaction of G. Proposition 2.9 provides the coaction of $\mathcal{G}$. □

Remark 3.5. The Fock space representation of Fowler [2002] ensures that A, and thus X, embeds isometrically in $\mathcal{T}(X)$. In short, let $\mathcal{F}(X) = \sum_{q \in P}^{\oplus} X_q$ and for $a \in A$ and $\xi_p \in X_p$ define $(\bar{\pi}, \bar{t}_p)$ by

$$\bar{\pi}(a)\xi_q = \varphi_q(a)\xi_q \quad \text{and} \quad \bar{t}_p(\xi_p)\xi_q = \xi_p\xi_q \qquad \text{for all } \xi_q \in X_q.$$

Then every $(\bar{\pi}, \bar{t}_p)$ defines a representation of X_p and hence it induces a representation of $\mathcal{T}(X)$. By taking the compression at the (e, e)-entry we see that $\bar{\pi}$, and thus $\bar{t}_p$, is injective.

Definition 3.6. Let P be a unital subsemigroup of a group G and X be a product system over P with coefficients in A. The *Fock algebra* $\mathcal{T}_\lambda(X)$ is the C^*-algebra generated by the Fock representation $(\bar{\pi}, \bar{t})$. The *Fock tensor algebra* $\mathcal{T}_\lambda(X)^+$ *of* X is the subalgebra of $\mathcal{T}_\lambda(X)$ generated by A and X.

It is shown in [Dor-On et al. 2022, Proposition 4.1] that the Fock algebra admits an analogous normal coaction. Proposition 2.9 yields the next proposition.

Proposition 3.7. *Let P be a unital subsemigroup of a group G and X be a product system over P with coefficients in A, and let $\mathcal{T}_\lambda(X) = C^*(\bar{\pi}, \bar{t})$ be its associated Fock algebra. If $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ is a semigroup preserving homomorphism then there is a normal coaction of $\mathcal{G}$ on $\mathcal{T}_\lambda(X)$ such that*

$$\bar{\delta}_{\mathcal{G}} : \mathcal{T}_\lambda(X) \to \mathcal{T}_\lambda(X) \otimes C^*(\mathcal{G}), \quad \bar{t}(\xi_p) \mapsto \bar{t}(\xi_p) \otimes u_{\vartheta(p)}.$$

Moreover for each $h \in \mathcal{G}$ the spectral space $\mathcal{T}_\lambda(X)_h$ is the closed linear span of the products of the form

$$\bar{t}_{p_1}(\xi_{p_1})\bar{t}_{p_2}(\xi_{p_2})^* \cdots \bar{t}_{p_{n-1}}(\xi_{p_{n-1}})\bar{t}_{p_n}(\xi_{p_n})^* \quad \text{for } \vartheta(p_1)\vartheta(p_2)^{-1} \cdots \vartheta(p_n)^{-1} = h,$$

where we impose that $\bar{t}_{p_i}(\xi_{p_i}) = I$ when $p_i = e_P$ and $h \neq e_{\mathcal{P}}$.

In turn the coaction of $\mathcal{G}$ induces a faithful conditional expectation of the following form.

Proposition 3.8. *Let P be a unital subsemigroup of a group G and X be a product system over P with coefficients in A. Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a semigroup preserving homomorphism. Then $\mathcal{T}_\lambda(X)$ admits a faithful conditional expectation $\bar{E}_{\mathcal{P}}$ such that*

$$\bar{E}_{\mathcal{P}}(\bar{\psi}_{r,s}(k_{r,s})) = \delta_{\vartheta(r),\vartheta(s)} \bar{\psi}_{r,s}(k_{r,s}) \quad \textit{for all } k_{r,s} \in \mathcal{K}(X_s, X_r).$$

Proof. Let $\bar{\delta}_{\mathcal{G}} : \mathcal{T}_\lambda(X) \to \mathcal{T}_\lambda(X) \otimes C^*(\mathcal{G})$ be the normal coaction and let $\omega_{e,e}$ be the faithful conditional expectation on $C^*_\lambda(\mathcal{G})$. Then $\mathcal{T}_\lambda(X)$ admits the faithful conditional expectation

$$\bar{E}_{\mathcal{P}} := (\mathrm{id} \otimes \omega_{e,e})(\mathrm{id} \otimes \lambda)\bar{\delta}_G.$$

On the other hand for $h \in \mathcal{P}$ let $Y_h := \sum^{\oplus}_{\vartheta(p)=h} X_h$ and consider the projections $Q_h : \mathcal{F}(X) \to Y_h$. We will show that

$$\bar{E}_{\mathcal{P}}(\cdot) = \sum_{h \in \mathcal{P}} Q_h \cdot Q_h.$$

It suffices to check on the spanning elements of the form

$$f := \bar{t}_{p_1}(\xi_{p_1})\bar{t}_{p_2}(\xi_{p_2})^* \cdots \bar{t}_{p_{n-1}}(\xi_{p_{n-1}})\bar{t}_{p_n}(\xi_{p_n})^*,$$

where we impose that $\bar{t}_{p_i}(\xi_{p_i}) = I$ when $p_i = e_P$. For $p \in P$ we directly compute

$$\bar{E}_{\mathcal{P}}(f) = \begin{cases} f\xi_p & \text{if } \vartheta(p_1^{-1}p_2 \cdots p_{n-1}^{-1}p_n) = e_{\mathcal{G}}, \\ 0 & \text{otherwise.} \end{cases}$$

If $f\xi_p \neq 0$ then it is in some X_r with $r = p_1^{-1}p_2 \cdots p_{n-1}^{-1}p_n p$ which gives $\vartheta(r) = \vartheta(p)$. On the other hand we have that

$$\Big(\sum_{h \in \mathcal{P}} Q_h f Q_h\Big)\xi_p = \begin{cases} f\xi_p & \text{if } \vartheta(p_1^{-1}p_2 \cdots p_{n-1}^{-1}p_n p) = \vartheta(p), \\ 0 & \text{otherwise.} \end{cases}$$

We have that $\vartheta(p_1^{-1}p_2 \cdots p_{n-1}^{-1}p_n p) = \vartheta(p)$ if and only if $\vartheta(p_1^{-1}p_2 \cdots p_{n-1}^{-1}p_n) = e_{\mathcal{G}}$ and so

$$\bar{E}_{\mathcal{P}}(f) = \sum_{h \in \mathcal{P}} Q_h f Q_h.$$

For the second part let $r, s \in P$ and $\xi_p \in Y_h$ so that $\vartheta(p) = h$. Then we directly compute

$$\begin{aligned}\bar{E}_{\mathcal{P}}(\bar{\psi}_{r,s}(k_{r,s}))\xi_p = Q_h \bar{\psi}_{r,s}(k_{r,s})\xi_p &= \begin{cases} \bar{\psi}_{r,s}(k_{r,s})\xi_p & \text{if } p = ss',\ \vartheta(p) = \vartheta(rs'), \\ 0 & \text{otherwise,} \end{cases} \\ &= \delta_{\vartheta(r),\vartheta(s)} \bar{\psi}_{r,s}(k_{r,s})\xi_p,\end{aligned}$$

where we used that ϑ is a group homomorphism and so $\vartheta(s)\vartheta(s') = \vartheta(p) = \vartheta(r)\vartheta(s')$. As $p \in P$ is arbitrary the proof is complete. ∎

Covariance algebras and Cuntz–Nica–Pimsner algebras. Let us review Sehnem's strong covariance relations [2019]. We will be using a description presented in [Dor-On et al. 2022]. Let P be a unital subsemigroup of a group G. For a finite set $F \subseteq G$ let

$$K_F := \bigcap_{g \in F} gP.$$

For $r \in P$ and $g \in F$ define the ideal of A given by

$$I_{r^{-1}K_{\{r,g\}}} := \begin{cases} \bigcap_{t \in K_{\{r,g\}}} \ker \varphi_{r^{-1}t} & \text{if } K_{\{r,g\}} \neq \emptyset \text{ and } r \notin K_{\{r,g\}}, \\ A & \text{otherwise.} \end{cases}$$

Then let

$$I_{r^{-1}(r \vee F)} := \bigcap_{g \in F} I_{r^{-1}K_{\{r,g\}}},$$

and define the C^*-correspondences

$$X_F := \bigoplus_{r \in P} X_r I_{r^{-1}(r \vee F)} \quad \text{and} \quad X_F^+ := \bigoplus_{g \in G} X_{gF}.$$

For every $p \in P$ define the representation $(\pi_F, t_{F,p})$ to X_F^+ given by

$$t_{F,p}(\xi_p)(\eta_r) = u_{p,r}(\xi_p \otimes \eta_r) \in X_{pr} I_{(pr)^{-1}(pr \vee pF)}, \quad \text{for all } \eta_r \in X_r I_{r^{-1}(r \vee F)}.$$

It is well defined as $I_{r^{-1}(r \vee F)} = I_{(pr)^{-1}(pr \vee pF)}$ for all $r \in P$, and $I_{r^{-1}(r \vee F)} = I_{(s^{-1}r)^{-1}(s^{-1}r \vee s^{-1}F)}$ for all $r \in sP$. This provides a representation (π_F, t_F) of X on $\mathcal{L}(X_F^+)$ that integrates to a representation

$$\Phi_F : \mathcal{T}(X) \to \mathcal{L}(X_F^+).$$

Now consider the projections $Q_{g,F} : X_F^+ \to X_{gF}$ and define

$$\|f\|_F := \|Q_{e,F} \Phi_F(f) Q_{e,F}\| \quad \text{for all } f \in [\mathcal{T}(X)]_e.$$

In particular we have that

$$t_{F,p}(\xi_p) Q_{g,F} = Q_{pg,F} t_{F,p}(\xi_p) \quad \text{and} \quad t_{F,p}(\xi_p)^* Q_{g,F} = Q_{p^{-1}g,F} t_{F,p}(\xi_p)^*,$$

and so $Q_{e,F}$ is reducing for the fixed point algebra $[\mathcal{T}(X)]_e$ under Φ_F.

Definition 3.9 [Sehnem 2019, Definition 3.2]. A Toeplitz representation is called *strongly covariant* if it vanishes on the ideal $\mathcal{I}_e \lhd [\mathcal{T}(X)]_e$ given by

$$\mathcal{I}_e := \left\{ f \in [\mathcal{T}(X)]_e \;\middle|\; \lim_F \|f\|_F = 0 \right\},$$

where the limit is taken with respect to the partial order induced by inclusion on finite sets of P. The universal C^*-algebra with respect to the strongly covariant representations of X is denoted by $A \times_X P$.

That is, $A \times_X P$ is the quotient $\mathcal{T}(X)/\mathcal{I}_\infty$ for the ideal $\mathcal{I}_\infty \lhd \mathcal{T}(X)$ of *strong covariance relations* generated by $\mathcal{I}_e$. One of the important points of Sehnem's theory is that the inclusion $A \hookrightarrow A \times_X P$ is faithful. As a quotient by an induced ideal of $\mathcal{T}(X)$, the C^*-algebra $A \times_X P$ inherits the coaction of G. The following is the main theorem of [Sehnem 2019].

Theorem 3.10 [Sehnem 2019, Theorem 3.10]. *Let P be a unital subsemigroup of a group G and X be a product system over P with coefficients in A. Then a $*$-homomorphism of $A \times_X P$ is faithful on A if and only if it is faithful on the fixed point algebra $[A \times_X P]_e$.*

Due to the grading $A \times_X P$ is the maximal C^*-algebra of a Fell bundle over G. We consider two reduced versions.

Definition 3.11. Let P be a unital subsemigroup of a group G and X be a product system over P with coefficients in A. We write $A \times_{X,\lambda} P$ for the reduced C^*-algebra of the Fell bundle in $A \times_X P$. If $q : \mathcal{T}(X) \to \mathcal{T}_\lambda(X)$ is the canonical $*$-epimorphism, then we write $q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$ for the quotient of $\mathcal{T}_\lambda(X)$ by the ideal $q(\mathcal{I}_\infty)$.

Remark 3.12. The notation $\mathcal{SC}X$ is used in [Dor-On et al. 2022] to denote the G-Fell bundle inside $A \times_X^G P$. Therefore we have two ways of writing the related C^*-algebras in the sense that

$$A \times_X P = C^*(\mathcal{SC}X) \quad \text{and} \quad A \times_{X,\lambda} P = C^*_\lambda(\mathcal{SC}X).$$

Sehnem [2019, Lemma 3.9] shows that the strong covariance relations do not depend on the group embedding in the following sense. Suppose that P admits two group embeddings $i_G : P \to G$ and $i_H : P \to H$ and write $C^*_{\max}(\mathcal{SC}_G X) = C^*(\pi^G, t^G)$ and $C^*_{\max}(\mathcal{SC}_H X) = C^*(\pi^H, t^H)$. Then there exists a $*$-isomorphism

$$C^*_{\max}(\mathcal{SC}_G X) \to C^*_{\max}(\mathcal{SC}_H X), \quad t^G_{i_G(p)}(\xi_{i_G(p)}) \mapsto t^H_{i_H(p)}(\xi_{i_H(p)}).$$

The $*$-isomorphism between $C^*_{\max}(\mathcal{SC}_G X)$ and $C^*_{\max}(\mathcal{SC}_H X)$ descends to a $*$-isomorphism that fixes X at the reduced level, as well, and thus $A \times_{X,\lambda} P$ does not depend on the group embedding either. Indeed suppose that G is the enveloping group of P and thus there exists a group homomorphism $\gamma : G \to H$ that is injective on P. We then have that there is a $*$-homomorphism between the maximal C^*-algebras induced by the G-Fell bundle and the H-Fell bundle on Sehnem's covariance algebra. Sehnem's result [2019, Lemma 3.9] is that this $*$-homomorphism is faithful. By Fell bundle theory we then get a canonical $*$-epimorphism

$$C^*_\lambda(\mathcal{SC}_G X) \to C^*_\lambda(\mathcal{SC}_H X)$$

that fixes X. Hence by construction it intertwines the normal faithful conditional expectations. Their fixed point algebras are $*$-isomorphic to the fixed point algebras in the maximal C^*-algebras and these are $*$-isomorphic by [Sehnem 2019, Lemma 3.9]. Thus the $*$-epimorphism on the reduced models is faithful.

We see that the representations Φ_F used to define the strong covariance relations are subrepresentations of $\bar{\delta}_{G,\lambda} : \mathcal{T}_\lambda(X) \to \mathcal{T}_\lambda(X) \otimes C^*_\lambda(G)$ for $\bar{\delta}_{G,\lambda} = (\mathrm{id} \otimes \lambda)\bar{\delta}_G$, where $\bar{\delta}_G$ is the normal coaction on the Fock representation. Indeed we can identify

$$X_F^+ = \bigoplus_{\substack{g \in G \\ r \in P}} X_r I_{r^{-1}(r \vee gF)}$$

with a submodule of $\mathcal{F}X \otimes \ell^2(G)$ through the isometry given by

$$X_r I_{r^{-1}(r \vee gF)} \ni \eta_r \mapsto \eta_r \otimes \delta_g \in X_r \otimes \ell^2(G).$$

Recall here that $\mathcal{F}X \otimes \ell^2(G)$ is the exterior tensor product of two modules (seeing $\ell^2(G)$ as a module over $\mathbb{C}$), and there is a faithful $*$-homomorphism

$$\mathcal{T}_\lambda(X) \otimes C^*_\lambda(G) \subseteq \mathcal{L}(\mathcal{F}X) \otimes \mathcal{B}(\ell^2(G)) \hookrightarrow \mathcal{L}(\mathcal{F}X \otimes \ell^2(G)).$$

We then see that

$$t_{F,p}(\xi_p) = (\bar{t}_p(\xi_p) \otimes \lambda_p)|_{X_F^+} = \bar{\delta}_{G,\lambda}(\bar{t}_p(\xi_p))|_{X_F^+} \quad \text{for all } \; p \in P,$$

and likewise for their adjoints. Thus X_F^+ is reducing under $\bar{\delta}_{G,\lambda}(\mathcal{T}_\lambda(X))$. Recall also that X_F is reducing for $[\mathcal{T}(X)]_e$ as the range of the projection $Q_{e,F}$ and so we obtain the representation

$$\bigoplus_{\text{fin } F \subseteq G} \Phi_F(\,\cdot\,)|_{X_F} : [\mathcal{T}(X)]_e \to [\mathcal{T}_\lambda(X)]_e \to \prod_{\text{fin } F \subseteq G} \mathcal{L}(X_F).$$

In particular, by definition we have for an $f \in \mathcal{T}(X)$ that

$$f \in \mathcal{I}_e \quad \iff \quad \bigoplus_{\text{fin } F \subseteq G} \Phi_F(f)|_{X_F} \in c_0(\mathcal{L}(X_F) \mid \text{fin } F \subseteq G).$$

By definition we then get that the diagram

$$\begin{tikzcd}
{[\mathcal{T}(X)]_e} \arrow[r] \arrow[d] & {[\mathcal{T}_\lambda(X)]_e} \arrow[r] \arrow[d, "q_{\mathrm{sc}}"] & \prod_{\text{fin } F \subseteq G} \mathcal{L}(X_F) \arrow[d] \\
{[A \times_X P]_e} \arrow[r] & {[q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))]_e} \arrow[r] & \big(\prod_{\text{fin } F \subseteq G} \mathcal{L}(X_F)\big)/(c_0(\mathcal{L}(X_F) \mid \text{fin } F \subseteq G))
\end{tikzcd}$$

is commutative. Consequently the e-graded $*$-algebraic relations in $\mathcal{T}_\lambda(X)$ and $A \times_X P$ induce relations in $q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$. In particular, since by [Sehnem 2019, Proposition 3.5] A is represented faithfully in the bottom right corner of the above diagram, we obtain the following corollary.

Corollary 3.13 [Dor-On et al. 2022, Corollary 5.5]. *Let P be a unital subsemigroup of a group G and X be a product system over P with coefficients in A. Then we have $A \hookrightarrow q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$. Moreover, a $*$-homomorphism of $q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$ is faithful on A if and only if it is faithful on $[q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))]_e$. Likewise for the reduced C^*-algebra $A \times_{X,\lambda} P$.*

4. Compactly aligned product systems over weak right LCM inclusions

Weak right* LCM *inclusions. A semigroup P is said to be a *right* LCM *semigroup* if it is left-cancellative and satisfies *Clifford's condition* [Lawson 2012; Norling 2014]:

$$\text{for every } \; p, q \in P \; \text{ with } \; pP \cap qP \neq \varnothing \; \text{ there exists a } \; w \in P \; \text{ such that } \; pP \cap qP = wP.$$

In other words, if $p, q \in P$ have a right common multiple then they have a right least common multiple. As we always see a semigroup P inside a group G, it follows that P is by default cancellative, and we will refer to (G, P) simply as a *weak right* LCM *inclusion*. We use the adjective "weak" here to emphasize that we do not assume that the least common multiple property holds for *all* elements in G.

It is clear that w is a right LCM for p, q if and only if wx is a right LCM of p, q for every $x \in P^*$. A *weak quasilattice* (G, P) is a weak right LCM inclusion with $P \cap P^{-1} = \{e\}$, i.e., when least common multiples are unique (whenever they exist).

Definition 4.1. Let (G, P) be a right weak LCM inclusion. A finite set F is said to be $\vee$*-closed* if for any $p, q \in F$ with $pP \cap qP \neq \varnothing$ there exists a unique $w \in F$ such that $pP \cap qP = wP$.

Equivalently, a finite $F \subseteq P$ is $\vee$-closed if and only if the familiar relation

$$p \leq q \quad \Longleftrightarrow \quad q^{-1}p \in P$$

defines a partial order on F. In particular, if F is $\vee$-closed, then $pP \neq qP$ for any $p, q \in F$ with $p \neq q$. Furthermore, any $\vee$-closed set admits maximal and minimal elements. Our terminology here regarding $\vee$-closed sets extends the familiar one from the case where (G, P) is a weak quasilattice order. There is an alternative way for describing $\vee$-closed sets in the context of weak right LCM inclusions. Given a finite subset $F \subseteq P$ we write

$$\mathcal{I}(F) := \{pP \mid p \in F\}$$

for the set of principal ideals defined by F. It then follows that F is $\vee$-closed if and only if $\mathcal{I}(F)$ is closed under intersections and the partial order defined on $\mathcal{I}(F)$ by set theoretic inclusion lifts to a partial order on F.

Let $F \subseteq P$ be a finite set so that $\mathcal{I}(F)$ is closed under intersections. From such a set F we can produce a $\vee$-closed subset $F^\vee$ such that $\mathcal{I}(F) = \mathcal{I}(F^\vee)$ by choosing a minimal set of distinct representatives for the principal ideals. This process does not produce a unique $F^\vee$ in general.

Nica-covariant representations. Following Fowler's work [2002], Brownlowe, Larsen and Stammeier [Brownlowe et al. 2018] and Kwaśniewski and Larsen [2019] considered product systems of right LCM semigroups.

Definition 4.2. A product system X over a weak right LCM semigroup P with coefficients in A is called *compactly aligned* if for $p, q \in P$ with $pP \cap qP = wP$ we have that

$$i_p^w(S) i_q^w(T) \in \mathcal{K}X_w \quad \text{whenever } S \in \mathcal{K}X_p, \quad T \in \mathcal{K}X_q.$$

A note is in order for clarifying that this is independent of the choice of w. Recall that if w' is a right LCM of p, q then $w' = wx$ for some $x \in P^*$. Since $\mathcal{L}X_w \simeq \mathcal{L}X_{wx}$ we have that $i_p^w(S) i_q^w(T) \in \mathcal{K}X_w$ if and only if $i_p^{wx}(S) i_q^{wx}(T) = i_w^{wx}(i_p^w(S) i_q^w(T)) \in \mathcal{K}X_{wx}$ for all $x \in P^*$.

Definition 4.3. Let X be a compactly aligned product system over a right LCM semigroup P with coefficients in A. A *Nica-covariant representation* (π, t) is a Toeplitz representation of A that in addition satisfies the *Nica-covariance* condition: for $S \in \mathcal{K}X_p$ and $T \in \mathcal{K}X_q$ we have that

$$\psi_p(S)\psi_q(T) = \begin{cases} \psi_w(i_p^w(S) i_q^w(T)) & \text{if } pP \cap qP = wP, \\ 0 & \text{otherwise.} \end{cases}$$

The *Toeplitz–Nica–Pimsner algebra* $\mathcal{NT}(X)$ *of* X is the universal C^*-algebra generated by A and X with respect to the representations of X. The *Toeplitz–Nica–Pimsner tensor algebra* $\mathcal{NT}(X)^+$ *of* X is the subalgebra of $\mathcal{T}(X)$ generated by A and X.

Remark 4.4. As noted in [Dor-On et al. 2022], the definition of Nica-covariance requires that the right-hand side is independent of the choice of the least common multiple, i.e., if $pP \cap qP = wP$ and $x \in P^*$ then

$$\psi_w(i_p^w(S)i_q^w(T)) = \psi_{wx}(i_p^{wx}(S)i_q^{wx}(T)) \quad \text{for all } S \in \mathcal{K}X_p, \ \ T \in \mathcal{K}X_q.$$

This is verified in [Dor-On et al. 2022, Proposition 2.4] (see Proposition 3.2 herein) and completes the definition of Nica-covariance in [Kwaśniewski and Larsen 2019].

Remark 4.5. By definition $\mathcal{NT}(X)$ is a quotient of $\mathcal{T}(X)$ by an ideal generated by a subspace of $[\mathcal{T}(X)]_e$. Even though $\mathcal{NT}(X) = \mathcal{T}(X)$ when $P = \mathbb{Z}_+$, this is not the case even when $P = \mathbb{Z}_+^n$. Dor-On and Katsoulis provide a counterexample to this effect in [Dor-On and Katsoulis 2020, Example 5.2]. The same example further shows that $\mathcal{T}(X)^+$ is not completely isometric to $\mathcal{NT}(X)^+$.

Under the assumption of compact alignment, one can check that the Fock representation is automatically Nica-covariant. Thus $\mathcal{NT}(X)$ is nontrivial. As $\mathcal{NT}(X)$ is a quotient of $\mathcal{T}(X)$ by an induced ideal, by [Carlsen et al. 2011, Proposition A.1] the nondegenerate and faithful coaction of $\mathcal{T}(X)$ descends canonically to one on $\mathcal{NT}(X)$. Alternatively one may use the arguments of the proof of Proposition 3.4 for the Toeplitz–Nica–Pimsner tensor algebra to deduce the following.

Proposition 4.6. *Let* (G, P) *be a weak right* LCM *inclusion and* X *be a compactly aligned product system over* P *with coefficients in* A. *Suppose that* $(\hat{\pi}, \hat{t})$ *is a faithful representation of* $\mathcal{NT}(X)$. *Then the canonical* $*$*-homomorphism*

$$\hat{\delta} : \mathcal{NT}(X) \to \mathcal{NT}(X) \otimes C^*(G), \quad \hat{t}(\xi_p) \mapsto \hat{t}(\xi_p) \otimes u_p$$

defines a coaction of G *on* $\mathcal{NT}(X)$.

We have refrained from describing the spectral spaces for the coaction on $\mathcal{NT}(X)$ because of the following additional property of Nica-covariant representations. Let (π, t) be a Nica-covariant representation of X. We compute

$$t_p(X_p)^* t_p(X_p) \cdot t_p(\xi_p)^* t_q(\xi_q) \cdot t_q(X_q)^* t_q(X_q) \subseteq [t_p(X_p)^* \psi_p(\mathcal{K}X_p)\psi_q(\mathcal{K}X_q) t_q(X_q)].$$

Next take a limit by contractive approximate identities in $[t_p(X_p)^* t_p(X_p)]$ and in $[t_q(X_q)^* t_q(X_q)]$, and derive that

$$t_p(\xi_p)^* t_q(\xi_q) \in [t_{p'}(X_{p'}) t_{q'}(X_{q'})^*] \quad \text{for } wP = pP \cap qP, \ \ p' = p^{-1}w, \ \ q' = q^{-1}w,$$

and

$$t_p(\xi_p)^* t_q(\xi_q) = 0 \quad \text{for } pP \cap qP = \varnothing.$$

Hence the C^*-algebra $C^*(\pi, t)$ generated by $\pi(A)$ and $t_p(X_p)$ admits a *Wick ordering* in the sense that

$$C^*(\pi, t) = \overline{\operatorname{span}}\{t_p(\xi_p) t_q(\xi_q)^* \mid \xi_p \in X_p, \ \xi_q \in X_q \text{ and } p, q \in P\}.$$

In particular if $\mathcal{NT}(X) = C^*(\hat{\pi}, \hat{t})$ then the spectral spaces that only matter are of the form

$$\mathcal{NT}(X)_{pq^{-1}} = \overline{\operatorname{span}}\{\hat{t}_p(\xi_p)\hat{t}_q(\xi_q)^* \mid \xi_p \in X_p,\ \xi_q \in X_q\},$$

that is, only for $g \in G$ of the form $g = pq^{-1}$ for some $p, q \in P$.

The following proposition gives a direct criterion to check compact alignment.

Proposition 4.7. *Let (G, P) be a weak right* LCM *inclusion and let $X = \{X_p\}_{p\in P}$ be a product system over $X_e = A$. Let (π, t) be an injective representation of X. Then X is compactly aligned, if and only if for all $p, q \in P$ we have that*

$$t_p(X_p)^* t_q(X_q) \subseteq [t_{p^{-1}w}(X_{p^{-1}w}) t_{q^{-1}w}(X_{q^{-1}w})^*] \quad \text{for } wP = pP \cap qP,$$

if and only if for all $p, q \in P$ we have that

$$t_p(X_p)t_p(X_p)^* t_q(X_q)t_q(X_q)^* \subseteq [t_w(X_w)t_w(X_w)^*] = \psi_w(\mathcal{K}X_w) \quad \text{for } wp = pP \cap qP,$$

with the understanding that the left-hand sides are the zero space when p and q have no right common multiple.

Proof. The first equivalence follows in the same way as [Katsoulis 2020, Proposition 3.2] and it is omitted. By using that $X_p X_p^* X_p$ is dense in X_p for every $p \in P$, we get the second equivalence. □

Let us now pass to the analysis of the cores of a Nica-covariant representation (π, t) of X. For a finite $F \subseteq P$ that is $\vee$-closed we write

$$B_F := \operatorname{span}\{\psi_p(k_p) \mid k_p \in \mathcal{K}X_p,\ p \in F\}.$$

Since F is $\vee$-closed, Nica-covariance implies that B_F is a $*$-subalgebra of $C^*(\pi, t)$. In [Dor-On et al. 2022, Proposition 2.10] we show that every B_F is actually a C^*-subalgebra. Moreover for such an F we write

$$B_{F\cdot P} := \overline{\operatorname{span}}\{\psi_q(k_q) \mid k_q \in \mathcal{K}X_q,\ q \in F \cdot P\}.$$

Likewise this is also a (closed) $*$-subalgebra. Finally we write

$$B_{P\setminus\{e\}} := \overline{\operatorname{span}}\{\psi_p(k_p) \mid k_p \in \mathcal{K}X_p,\ e \neq p \in P\} \quad \text{and} \quad B_P := \pi(A) + B_{P\setminus\{e\}}.$$

We see that $B_{P\setminus\{e\}}$ is an ideal in B_P and thus the sum $\pi(A) + B_{P\setminus\{e\}}$ is indeed closed. We refer to these sets as the *cores* of the representation (π, t). In [Dor-On et al. 2022, Proposition 2.11] we showed that we can exhaust the cores by using finite $\vee$-closed sets, in the sense that

$$B_P = \overline{\bigcup\{B_F \mid F \subseteq P \text{ finite and } \vee\text{-closed}\}}.$$

We denote by $\bar{B}_F$ the cores of $\mathcal{T}_\lambda(X) = C^*(\bar{\pi}, \bar{t})$. Recall that $\mathcal{T}_\lambda(X)$ admits the faithful conditional expectation

$$\bar{E}_P : \mathcal{T}_\lambda(X) \to \bar{B}_P, \quad \bar{t}_p(\xi_p)\bar{t}_q(\xi_q)^* = \delta_{p,q}\bar{t}_p(\xi_p)\bar{t}_q(\xi_q)$$

given by the sum of compressions to the (r, r)-entries in $\mathcal{L}(\mathcal{F}X)$ (see Proposition 3.8).

The Toeplitz–Nica–Pimsner algebra is modeled after the Fock algebra in this context. A compactly aligned product system X over P with coefficients in A is called *amenable* if the Fock representation is faithful on $\mathcal{NT}(X)$. Let us give some equivalent conditions for this to happen.

Theorem 4.8. *Let (G, P) be a weak right* LCM *inclusion and X be a compactly aligned product system over P with coefficients in A. The following are equivalent*:

(i) *The coaction of G on $\mathcal{NT}(X)$ is normal.*

(ii) *The conditional expectation on $\mathcal{NT}(X)$ is faithful.*

(iii) *The Fock representation is faithful on $\mathcal{NT}(X)$.*

(iv) *The representation*

$$\mathcal{NT}(X) \to C^*(\pi, t) \otimes C^*_\lambda(P), \quad \tilde{t}_p(\xi_p) \mapsto t_p(\xi_p) \otimes V_p$$

is faithful for any injective Nica-covariant pair (π, t).

Proof. By the universal property there exists a canonical $*$-representation

$$\mathcal{NT}(X) \to \mathcal{NT}(X) \otimes C^*_\lambda(G)$$

that intertwines the conditional expectations. Thus items (i) and (ii) are equivalent. For the same reason items (ii) and (iii) are equivalent.

Assuming item (iii) we have to show that the representation

$$\mathcal{NT}(X) \to C^*(\pi, t) \otimes C^*_\lambda(P)$$

is faithful on the fixed point algebra. It suffices to show injectivity on the F-boxes for arbitrary $\vee$-closed $F \subseteq P$. To this end suppose that

$$\sum_{p \in F} \psi_p(k_p) = 0$$

for some $k_p \in \mathcal{K}X_p$ and let p_0 be minimal so that $k_{p_0} \neq 0$. Injectivity of π then implies that $\psi_{p_0}(k_{p_0}) \neq 0$ as well. However, if $Q_{p_0} : \ell^2(P) \to \mathbb{C}e_{p_0}$ is the canonical projection, minimality of p_0 yields

$$\psi_{p_0}(k_{p_0}) = I \otimes Q_{p_0} \Bigg(\sum_{p \in F} \psi_p(k_p) \Bigg) I \otimes Q_{p_0} = 0,$$

which is a contradiction. This shows that item (iii) implies item (iv).

Since the $*$-representation

$$\mathcal{NT}(X) \to C^*(\pi, t) \otimes C^*_\lambda(P)$$

intertwines the conditional expectations, we finally have that item (iv) implies item (i). □

On the other hand strongly covariant representations are Nica-covariant (which is expected as Nica-covariance is an e-graded relation in $[\mathcal{T}_\lambda(X)]_e$). It is proven by Sehnem [2019, Proposition 4.2] for quasilattices, but the same proof passes to right LCM semigroups as well [Dor-On et al. 2022, Proposition 5.4]. Hence $A \times_X P$ is a quotient of $\mathcal{NT}(X)$.

Proposition 4.9 [Dor-On et al. 2022, Proposition 5.4; Sehnem 2019, Proposition 4.2]. *Let X be a compactly aligned product system over a right* LCM *semigroup P with coefficients in A. Let*

$$\psi_{F,p} : \mathcal{K}X_p \to \mathcal{L}(X_F^+)$$

be the induced representations from $(\pi_F, t_{F,p})$. A representation (π, t) of X is strongly covariant if and only if it is Nica-covariant and it satisfies

$$\sum_{p \in F} \psi_{F,p}(k_p)|_{X_F} = 0 \quad \Longrightarrow \quad \sum_{p \in F} \psi_p(k_p) = 0$$

for any finite $F \subseteq P$ and $k_p \in \mathcal{K}X_p$.

Carlsen, Larsen, Sims and Vittadello [Carlsen et al. 2011] explored the idea of finding the couniversal C^*-algebra with respect to injective equivariant Nica-covariant representations of X. By using the C^*-envelope machinery we can prove that this object always exists, thus completing the couniversal aspect of their program in the more general context of right weak LCM inclusions.

Definition 4.10. Let (G, P) be a weak right LCM inclusion and X be a compactly aligned product system over P with coefficients in A. We say that a representation (π, t) of X is *couniversal for $\mathcal{NT}(X)$* if

(i) π is faithful,

(ii) $C^*(\pi, t)$ is an equivariant quotient of $\mathcal{NT}(X)$,

(iii) (π, t) factors through any other equivariant quotient of $\mathcal{NT}(X)$ that is injective on A.

Of course the C^*-algebras of couniversal representations are automatically $*$-isomorphic by an equivariant homomorphism. In [Dor-On et al. 2022] we proved that the equivariant representation

$$\mathcal{NT}(X) \to C^*_{\text{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G)$$

that is given by the diagram

$$\begin{tikzcd} \mathcal{NT}(X) \arrow[rr, dotted] \arrow[dr] & & C^*_{\text{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G) \\ & \mathcal{T}_\lambda(X) \arrow[ur] & \end{tikzcd}$$

is couniversal. Let us review the main arguments and see what more we can obtain.

Proposition 4.11 [Dor-On et al. 2022, Proposition 4.4]. *Let (G, P) be a weak right* LCM *inclusion and X be a compactly aligned product system over P with coefficients in A. Let $\Phi : \mathcal{T}_\lambda(X) \to B$ be a $*$-representation such that $\Phi|_{\bar{\pi}(A)}$ is faithful. Then there exists a faithful $*$-homomorphism*

$$\mathcal{T}_\lambda(X) \to B \otimes C^*_\lambda(P), \quad \bar{t}_p(\xi_p) \mapsto \Phi\bar{t}_p(\xi_p) \otimes V_p.$$

As a consequence the injective equivariant representations on product systems generate C^*-covers for the cosystem $(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G)$.

Proposition 4.12 [Dor-On et al. 2022, Proposition 4.5]. *Let (G, P) be a weak right* LCM *inclusion and X be a compactly aligned product system over P with coefficients in A. Let $\Phi : \mathcal{T}_\lambda(X) \to B$ be an equivariant $*$-epimorphism such that $\Phi|_{\overline{\pi}(A)}$ is faithful. Then B is a C^*-cover for the cosystem $(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G)$.*

Another consequence of Proposition 4.11 provides a generalization of the extension theorem of [Katsoulis and Ramsey 2019]. It essentially allows us to recognize a Fock tensor algebra by the presence of a coaction.

Theorem 4.13 (extension theorem). *Let (G, P) be a weak right* LCM *inclusion and X be a compactly aligned product system over P with coefficients in A. Let $\Phi : \mathcal{T}_\lambda(X) \to B$ be a representation of X and set*

$$\mathcal{A} := \overline{\mathrm{alg}}\{\Phi\overline{\pi}(A), \Phi\bar{t}_p(X_p) \mid p \in P\}.$$

Then the following are equivalent:

(i) *$\Phi|_{\mathcal{T}_\lambda(X)^+}$ is completely isometric.*

(ii) *There exists a completely contractive map*

$$\mathcal{A} \to B \otimes C^*(G), \quad \Phi\bar{t}_p(\xi_p) \mapsto \Phi\bar{t}_p(\xi_p) \otimes u_p.$$

(iii) *There exists a completely contractive map*

$$\mathcal{A} \to B \otimes C^*_\lambda(G), \quad \Phi\bar{t}_p(\xi_p) \mapsto \Phi\bar{t}_p(\xi_p) \otimes \lambda_p.$$

(iv) *There exists a completely contractive map*

$$\mathcal{A} \to B \otimes C^*_\lambda(P), \quad \Phi\bar{t}_p(\xi_p) \mapsto \Phi\bar{t}_p(\xi_p) \otimes V_p.$$

Proof. Below we have a diagram of completely contractive representations induced by Propositions 4.11 and 4.12, which are completely positive maps fixing the nonselfadjoint part. If any of the items holds then it makes the representation of $\mathcal{T}_\lambda(X)^+$ to $\mathcal{A}$ completely isometric and the proof is complete.

$$\begin{tikzcd}
\mathcal{T}_\lambda(X)^+ \arrow[d, "\simeq"] \arrow[dr] & \\
\overline{\mathrm{alg}}\{\Phi\bar{t}_p(\xi_p) \otimes u_p \mid \xi_p \in X_p,\, p \in P\} \arrow[d] & \mathcal{A} \\
\overline{\mathrm{alg}}\{\Phi\bar{t}_p(\xi_p) \otimes \lambda_p \mid \xi_p \in X_p,\, p \in P\} \arrow[d] & \\
\overline{\mathrm{alg}}\{\Phi\bar{t}_p(\xi_p) \otimes V_p \mid \xi_p \in X_p,\, p \in P\} \arrow[d, "\simeq"] & \\
\mathcal{T}_\lambda(X)^+ &
\end{tikzcd}$$

$\square$

We now come to the last part of [Dor-On et al. 2022] that connects reduced C^*-algebras with the C^*-envelope. By Corollary 3.13 and Proposition 4.12 we get a canonical $*$-epimorphism

$$q_{\mathrm{sc}}(\mathcal{T}_\lambda(X)) \to A \times_{X,\lambda} P \simeq C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G).$$

We remind the reader of the notation used here and in [Dor-On et al. 2022] as explained in Remark 3.12. The same remark asserts that the C^*-envelope of the cosystem is independent of the group embedding *in this setting*.

Theorem 4.14 [Dor-On et al. 2022, Theorem 4.9, Theorem 5.3 and Corollary 5.6]. *Let (G, P) be a weak right* LCM *inclusion and X be a compactly aligned product system over P with coefficients in A. Then the equivariant $*$-epimorphism*

$$\mathcal{NT}(X) \to C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G)$$

is couniversal. Moreover we have an equivariant $*$-isomorphism

$$C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G) \simeq A \times_{X,\lambda} P.$$

The equivariant $*$-epimorphism

$$q_{\mathrm{sc}}(\mathcal{T}_\lambda(X)) \to A \times_{X,\lambda} P \simeq C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G)$$

is faithful if and only if the coaction of G on $q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$ is normal.

5. Controlled maps

Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a semigroup preserving homomorphism between weak right LCM inclusions and let X be a compactly aligned product system over P with coefficients in A. By Proposition 2.9 the Toeplitz algebra admits a $\mathcal{G}$-grading that contains the G-grading, and the same is true for the fixed point algebras. Of course this may be useless; for example the ϑ-fixed point algebra for the map $\vartheta : G \to \{e\}$ is the entire C^*-algebra. Nevertheless more can be obtained for weak right LCM inclusions as long as we impose axioms that control the map. The following extends the controlled maps on quasilattice ordered groups from [Laca and Raeburn 1996], see also [Fowler 2002] and [Crisp and Laca 2007], to the context of weak right LCM inclusions.

Definition 5.1. A *controlled map* $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ between weak right LCM inclusions is a semigroup preserving homomorphism such that:

(A1) If $pP \cap qP \neq \varnothing$, then $\vartheta(p)\mathcal{P} \cap \vartheta(q)\mathcal{P} = \vartheta(pP \cap qP)\mathcal{P}$.

(A2) If $pP \cap qP \neq \varnothing$ and $\vartheta(p) = \vartheta(q)$, then $p = q$.

It is worth pointing out that in the case where $P = G$ then there is only one right ideal (generated by the identity). Therefore a controlled map in this case is simply an injective group homomorphism due to (A2).

Remark 5.2. It is clear that having $\vartheta(p)\mathcal{P} \cap \vartheta(q)\mathcal{P} = \vartheta(w)\mathcal{P}$ whenever $pP \cap qP = wP$ is equivalent to (A1). Moreover, because of (A2) we have that $\vartheta^{-1}(e_{\mathcal{G}}) \cap P = \{e_G\}$. Indeed as ϑ is a group homomorphism we have that $\vartheta(e_G) = e_{\mathcal{G}}$. Now if $\vartheta(p) = e_{\mathcal{G}}$ for some $p \in P$, then since $pP \cap e_G P = pP \neq \varnothing$ we get by (A1) that $p = e_G$. This extra generality is crucial when we wish to consider the generalized length function given by abelianization on the free monoid $\mathbb{F}_n^+$ [Laca and Raeburn 1996], and, more generally, the Artin monoids of rectangular type [Crisp and Laca 2002].

Remark 5.3. Similar types of maps appear in [Crisp 1999] and [Brownlowe et al. 2018, Section 3]. However the maps therein satisfy the stronger requirement that $\vartheta(p)\mathcal{P} \cap \vartheta(q)\mathcal{P} = \vartheta(pP \cap qP)\mathcal{P}$ for all $p, q \in P$. This means that $p, q \in P$ have a right LCM if and only if $\vartheta(p), \vartheta(q) \in \mathcal{P}$ also do. In our Definition 5.1, the condition (A1) allows the possibility that $\vartheta(p), \vartheta(q)$ have a right LCM in $\mathcal{P}$ even when $pP \cap qP = \varnothing$.

We will investigate the impact of the existence of a controlled map on Nica-covariant representations. Henceforth fix a controlled map $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ between two weak right LCM inclusions. Suppose that (π, t) is a Nica-covariant representation of a compactly aligned product system X over P with coefficients in A. If $p, q \in P$ with $\vartheta(p) = \vartheta(q)$ then by (A2) either $p = q$ or $pP \cap qP = \varnothing$; thus Nica-covariance yields the orthogonality

$$t_p(\xi_p)^* t_q(\xi_q) = \delta_{p,q} \pi(\langle \xi_p, \xi_q \rangle).$$

Hence the C^*-algebra

$$B_{\vartheta^{-1}(h)} := \overline{\operatorname{span}}\{\psi_{p,q}(k_{p,q}) \mid k_{p,q} \in \mathcal{K}(X_q, X_p),\ \vartheta(p) = h = \vartheta(q)\}$$

is a matrix C^*-algebra. For a $\vee$-closed $\mathcal{F} \subseteq \mathcal{P}$ we define

$$B_{\vartheta^{-1}(\mathcal{F})} := \overline{\operatorname{span}}\{B_{\vartheta^{-1}(h)} \mid h \in \mathcal{F}\}.$$

By conditions (A1) and (A2) of Definition 5.1 we get that $\vartheta^{-1}(\mathcal{F})$ is also $\vee$-closed (and thus the above space is a C^*-algebra). Therefore every $B_{\vartheta^{-1}(\mathcal{F})}$ is the inductive limit of the matrix C^*-subalgebras

$$\operatorname{span}\{\psi_{p,q}(k_{p,q}) \mid k_{p,q} \in \mathcal{K}(X_q, X_p),\ p, q \in F,\ \vartheta(p) = \vartheta(q)\} \quad \text{for finite } \vee\text{-closed } F \subseteq \vartheta^{-1}(\mathcal{F}).$$

Taking the closure of the union we obtain the ϑ-fixed point algebra

$$B_{\vartheta^{-1}(\mathcal{P})} := \overline{\operatorname{span}}\{\psi_{p,q}(k_{p,q}) \mid k_{p,q} \in \mathcal{K}(X_q, X_p),\ \vartheta(p) = \vartheta(q)\}.$$

It follows that

$$B_P = \overline{\operatorname{span}}\{\psi_p(\mathcal{K}X_p) \mid p \in P\} \subseteq B_{\vartheta^{-1}(\mathcal{P})}.$$

It is clear that the faithful conditional expectation $\bar{E}_{\mathcal{P}}$ on $C^*(\bar{\pi}, \bar{t}) = \mathcal{T}_\lambda(X)$ described in Proposition 3.8 is onto $\bar{B}_{\vartheta^{-1}(\mathcal{P})}$. We already commented on the effect of semigroup preserving homomorphisms of the form $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ on $\mathcal{T}(X)$ and $\mathcal{T}_\lambda(X)$. We give some basic facts about the effect of controlled maps on $\mathcal{T}_\lambda(X)$.

Proposition 5.4. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right* LCM *inclusions and let X be a compactly aligned product system over P with coefficients in A. Let $\Phi : \mathcal{T}_\lambda(X) \to B$ be a $*$-representation such that $\Phi|_{\overline{\pi}(A)}$ is faithful. Then there exists a faithful $*$-homomorphism*

$$\mathcal{T}_\lambda(X) \to B \otimes C^*_\lambda(\mathcal{P}), \quad \overline{t}_p(\xi_p) \mapsto \Phi\overline{t}_p(\xi_p) \otimes V_{\vartheta(p)}.$$

Proof. The proof follows the same lines as Proposition 4.11 with the observation that $\overline{B}_{\vartheta^{-1}(h)}$ for $h \in \mathcal{G}$ is a matrix algebra. □

As an immediate consequence we have the following corollary which extends Theorem 4.13 to the controlled setting.

Corollary 5.5. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right* LCM *inclusions and let X be a compactly aligned product system over P with coefficients in A. Let $\Phi : \mathcal{T}_\lambda(X) \to B$ be a $*$-representation and set*

$$\mathcal{A} := \overline{\mathrm{alg}}\{\Phi\overline{\pi}(A), \Phi\overline{t}_p(X_p) \mid p \in P\}.$$

Then the following are equivalent:

(i) $\Phi|_{\mathcal{T}_\lambda(X)^+}$ *is completely isometric.*

(ii) *There exists a completely contractive map*

$$\mathcal{A} \to B \otimes C^*(\mathcal{G}), \quad \Phi\overline{t}_p(\xi_p) \mapsto \Phi\overline{t}_p(\xi_p) \otimes u_{\vartheta(p)}.$$

(iii) *There exists a completely contractive map*

$$\mathcal{A} \to B \otimes C^*_\lambda(\mathcal{G}), \quad \Phi\overline{t}_p(\xi_p) \mapsto \Phi\overline{t}_p(\xi_p) \otimes \lambda_{\vartheta(p)}.$$

(iv) *There exists a completely contractive map*

$$\mathcal{A} \to B \otimes C^*_\lambda(\mathcal{P}), \quad \Phi\overline{t}_p(\xi_p) \mapsto \Phi\overline{t}_p(\xi_p) \otimes V_{\vartheta(p)}.$$

Proof. The proof follows as in Theorem 4.13, modulo Proposition 3.7 and Proposition 5.4. □

Controlled elimination. We will require the following lemma for solving polynomial equations in the ϑ-fixed point algebra.

Lemma 5.6. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right* LCM *inclusions and let X be a compactly aligned product system over P with coefficients in A. Let (π, t) be an injective Nica-covariant representation of X.*

(i) *Let p, q be distinct in $\vartheta^{-1}(h)$. For $r, s \in \vartheta^{-1}(h)$ with $(r, s) \neq (p, q)$, we get*

$$t_p(X_p)^* \psi_{r,s}(k_{r,s}) t_q(X_q) = (0) \quad \textit{for all } k_{r,s} \in \mathcal{K}(X_s, X_r).$$

(ii) *Let $\mathcal{F} \subseteq \mathcal{P}$ be $\vee$-closed and $F \subseteq \vartheta^{-1}(\mathcal{F})$ be finite and $\vee$-closed. Let $(r, s) \in F \times F$ with $\vartheta(r) = \vartheta(s)$ and $k_{r,s} \in \mathcal{K}(X_s, X_r)$ such that*

$$\sum_{\substack{r,s \in F \\ \vartheta(r)=\vartheta(s)}} \psi_{r,s}(k_{r,s}) = 0,$$

and suppose that $h \neq e_{\mathcal{G}}$ is minimal in $\mathcal{F}$ so that $k_{p,q} \neq 0$ for distinct $p, q \in \vartheta^{-1}(h)$. Then there exists a $\vee$-closed $\mathcal{F}' \subseteq \mathcal{P}$ and a finite $\vee$-closed $F' \subseteq \vartheta^{-1}(\mathcal{F}')$ with $e_{\mathcal{G}} \notin \mathcal{F}'$ and $|\mathcal{F}'| \leq |\mathcal{F}| - 1$ such that

$$t_p(X_p)^* \psi_{p,q}(k_{p,q}) t_q(X_q) \subseteq B_{F'}.$$

Proof. (i) First we note that condition (A2) of Definition 5.1 yields $pP \cap qP = \varnothing$. By Nica-covariance we have that $t_p(X_p)^* \psi_{r,s}(k_{r,s}) t_q(X_q) = (0)$, unless

$$\exists w, z, v \in P \qquad \text{such that} \quad pP \cap rP = wP, \;\; qP \cap sP = zP \quad \text{and} \quad r^{-1}wP \cap s^{-1}zP = vP. \tag{5-1}$$

If $(r, s) \neq (p, q)$ and $\vartheta(r) = h = \vartheta(s)$, then condition (A2) of Definition 5.1 implies that $pP \cap rP = \varnothing$ or $qP \cap sP = \varnothing$, in which case

$$t_p(X_p)^* \psi_{r,s}(k_{r,s}) t_q(X_q) = (0).$$

(ii) Minimality of h in $\mathcal{F}$ forces minimality of p, q in F. If (5-1) holds, then Nica-covariance yields

$$t_p(\xi_p)^* \psi_{r,s}(k_{r,s}) t_q(\xi_q) \in \psi_{p^{-1}rv, q^{-1}sv}(\mathcal{K}(X_{q^{-1}sv}, X_{p^{-1}rv})),$$

otherwise the product is zero. If $r = s$ and v exists then there are $p', q', x, x', y, y' \in P$ such that

$$pp' = rx, \quad qq' = ry \quad \text{and} \quad xx' = yy'.$$

But then

$$pp'x' = rxx' = ryy' = qq'y,$$

giving the contradiction that $pP \cap qP \neq \varnothing$. Hence in this case the product is zero. We will show that the product is zero also when $\vartheta(p^{-1}rv) = e_{\mathcal{G}} = \vartheta(q^{-1}sv)$ for $r \neq s$ unless $(r, s) = (p, q)$. If $\vartheta(p^{-1}rv) = e_{\mathcal{G}}$ then condition (A2) of Definition 5.1 yields $p \in rP$. Likewise $q \in sP$. Minimality of p, q in F forces either $(p, q) = (r, s)$ or $k_{r,s} = 0$. Set

$$\mathcal{F}' := \{h^{-1}g \mid g \in \mathcal{F}, g > h\} \quad \text{and} \quad F' := \{u^{-1}v \mid u, v \in F, \vartheta(u) = h, u > v\} \subseteq \vartheta^{-1}(\mathcal{F}').$$

We see that $\mathcal{F}'$ is $\vee$-closed with $|\mathcal{F}'| \leq |\mathcal{F}| - 1$ and so F' is $\vee$-closed with

$$|F'| \leq |F \setminus \{p, q\}| = |F| - 2.$$

Moreover we see that $p^{-1}rv, q^{-1}sv \in F'$ whenever v exists. Hence for every $\xi_p \in X_p$ and $\xi_q \in X_q$ there are suitable $k'_{r',s'}$ with nontrivial $r', s' \in F'$ so that

$$0 = \sum_{r,s} t_p(\xi_p)^* \psi_{r,s}(k_{r,s}) t_q(\xi_q) = t_p(\xi_p)^* \psi_{p,q}(k_{p,q}) t_q(\xi_q) + \sum_{r',s'} \psi_{r',s'}(k'_{r',s'}), \qquad \square$$

In the next proposition we show that we can eliminate elements of the form $\psi_{r,s}(k_{r,s})$ for $r \neq s$ with $\vartheta(r) = \vartheta(s)$, from a polynomial equation in the ϑ-fixed point algebra. Such arguments for the left-regular representation appear in [Dinh 1991, Proposition 2.10] and [Laca and Raeburn 1996, Lemma 4.1] for semigroups over quasilattices, i.e., when $X_p = \mathbb{C}$ for every $p \in P$ and (G, P) is a quasilattice. Here we need to move in three directions: (a) beyond one-dimensional fibers, (b) beyond quasilattices, and (c) beyond just the left-regular representation. A step towards this direction is done in [Kakariadis 2020] for quasilattices that are controlled by $(\mathbb{Z}^n, \mathbb{Z}^n_+)$, and here we expand further on this approach.

Proposition 5.7. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right* LCM *inclusions and let X be a compactly aligned product system over P with coefficients in A. Let (π, t) and (π', t') be injective Nica-covariant representations such that there exists a canonical $*$-epimorphism*

$$\Phi : C^*(\pi', t') \to C^*(\pi, t), \qquad \textit{with} \quad \Phi(\pi'(a)) = \pi(a), \;\; \Phi(t'_p(\xi_p)) = t_p(\xi_p).$$

Then Φ is injective on B'_P if and only if it is injective on $B'_{\vartheta^{-1}(\mathcal{P})}$.

Proof. As $B'_P \subseteq B'_{\vartheta^{-1}(\mathcal{P})}$ we need to show just one direction. To this end suppose that Φ is injective on the C^*-subalgebras of the form

$$B'_F = \overline{\operatorname{span}}\{\psi'_p(\mathcal{K}X_p) \mid p \in F\}$$

for every finite $\vee$-closed $F \subseteq P$. We will show that Φ is injective on every

$$B'_{\vartheta^{-1}(\mathcal{F})} = \overline{\operatorname{span}}\{\psi'_{r,s}(k_{r,s}) \mid \vartheta(r) = \vartheta(s) \in \mathcal{F}\}$$

for all $\vee$-closed $\mathcal{F} \subseteq \mathcal{P}$. Our strategy is to show the implication

$$\sum_{\substack{r,s \in F \\ \vartheta(r) = \vartheta(s) \in \mathcal{F}}} \psi'_{r,s}(k_{r,s}) \in \ker \Phi \qquad \Longrightarrow \qquad k_{r,s} = 0 \quad \text{whenever } r \neq s$$

for every finite $\vee$-closed $F \subseteq \vartheta^{-1}(\mathcal{F})$. Then injectivity of Φ in the smaller cores yields

$$\sum_{r \in F} \psi_r(k_r) = \sum_{\substack{r,s \in F \\ \vartheta(r) = \vartheta(s) \in \mathcal{F}}} \psi_{r,s}(k_{r,s}) = 0 \quad \Longrightarrow \quad \sum_{r \in F} \psi'_r(k_r) = 0,$$

and so

$$\sum_{\substack{r,s \in F \\ \vartheta(r) = \vartheta(s) \in \mathcal{F}}} \psi'_{r,s}(k_{r,s}) = \sum_{r \in F} \psi'_r(k_r) = 0.$$

Since F is arbitrary this proves injectivity of Φ on $B'_{\vartheta^{-1}(\mathcal{F})}$. We proceed by induction on the size of $\mathcal{F}$.

Case 1. Assume that $\mathcal{F} = \{h\}$ and let F be a finite $\vee$-closed subset of $\vartheta^{-1}(\mathcal{F})$. Suppose that

$$\sum_{\substack{r,s \in F \\ \vartheta(r) = \vartheta(s) = h}} \psi_{r,s}(k_{r,s}) = 0,$$

and fix $p, q \in \vartheta^{-1}(h)$. Then condition (A2) of Definition 5.1 implies that

$$\psi_p(\mathcal{K}X_p)\psi_{p,q}(k_{p,q})\psi_q(\mathcal{K}X_q) = \psi_p(\mathcal{K}X_p)\Bigg(\sum_{\substack{r,s \in F \\ \vartheta(r) = \vartheta(s) = h}} \psi_{r,s}(k_{r,s})\Bigg)\psi_q(\mathcal{K}X_q) = (0).$$

Using an approximate identity on both sides gives that $\psi_{p,q}(k_{p,q}) = 0$, and the injectivity of ψ implies that $k_{p,q} = 0$. As (p, q) was arbitrary we have that $k_{r,s} = 0$ for all $r, s \in \vartheta^{-1}(h)$ and so

$$\sum_{\substack{r,s \in F \\ \vartheta(r) = \vartheta(s) = h}} \psi'_{r,s}(k_{r,s}) = 0.$$

Hence Φ is injective on $B'_{\vartheta^{-1}(\mathcal{F})}$ whenever $|\mathcal{F}| = 1$.

Case 2. Assume that $\mathcal{F} = \{e_{\mathcal{G}}, h\}$ and let F be a finite $\vee$-closed subset of $\vartheta^{-1}(\mathcal{F})$. Suppose that

$$\sum_{\substack{r,s \in F \\ \vartheta(r)=\vartheta(s)}} \psi_{r,s}(k_{r,s}) = 0.$$

By condition (A1) of Definition 5.1 we have that if $p \neq q$ with $\vartheta(p) = \vartheta(q) \in \mathcal{F}$ then $p, q \in \vartheta^{-1}(h)$. As before and by using item (i) of Lemma 5.6 on p, q we get that

$$\psi_p(\mathcal{K}X_p)\psi_{p,q}(k_{p,q})\psi_q(\mathcal{K}X_q) = \psi_p(\mathcal{K}X_p)\Big(\sum_{\substack{r,s \in F \\ \vartheta(r)=\vartheta(s)}} \psi_{r,s}(k_{r,s})\Big)\psi_q(\mathcal{K}X_q) = (0).$$

Using an approximate identity eventually gives that $k_{p,q} = 0$ whenever $p \neq q$. Hence $k_{r,s} = 0$ whenever $r \neq s$ in F and injectivity of Φ on B'_F gives that

$$\sum_{\substack{r,s \in F \\ \vartheta(r)=\vartheta(s)}} \psi'_{r,s}(k_{r,s}) = 0.$$

Hence Φ is injective on $B'_{\vartheta^{-1}(\mathcal{F})}$ whenever $\mathcal{F} = \{e, h\}$.

Case 3. Assume that $\mathcal{F} = \{h_1, h_2\}$ and let F be a finite $\vee$-closed subset of $\vartheta^{-1}(\mathcal{F})$. Suppose that

$$\sum_{\substack{r,s \in F \\ \vartheta(r)=\vartheta(s)\in\{h_1,h_2\}}} \psi'_{r,s}(k_{r,s}) \in \ker \Phi.$$

Without loss of generality assume that it is written with the understanding that for every $\psi'_{r,s}(k_{r,s})$ we have that either $\psi'_{r,s}(k_{r,s}) = 0$ or that

$$\psi'_{r,s}(k_{r,s}) \notin B'_{\vartheta^{-1}(\vartheta(r)\mathcal{P})}.$$

Choose $h \in \mathcal{F}$ to be minimal such that $\psi'_{p,q}(k_{p,q}) \neq 0$ for distinct $p, q \in \vartheta^{-1}(h)$. Hence $k_{p,q} \neq 0$ and so

$$0 \neq \psi'_{p,q}(k_{p,q}) \notin B'_{\vartheta^{-1}(h\mathcal{P})}.$$

By using Lemma 5.6 item (ii) we have that

$$t_p(X_p)^*\psi_{p,q}(k_{p,q})t_q(X_q) \subseteq B_{\vartheta^{-1}(\mathcal{F}')} \quad \text{for } |\mathcal{F}'| \leq 1,$$

with $e_{\mathcal{G}} \notin \mathcal{F}'$. By using injectivity of Case 2 we then derive that

$$\psi'_p(\mathcal{K}X_p)\psi'_{p,q}(k_{p,q})\psi'_q(\mathcal{K}X_q) \subseteq B'_{\vartheta^{-1}(h\mathcal{P})}.$$

By using approximate identities on both sides we get the contradiction

$$\psi'_{p,q}(k_{p,q}) \in B'_{\vartheta^{-1}(h\mathcal{P})}.$$

Hence Φ is injective on $B'_{\vartheta^{-1}(\mathcal{F})}$ whenever $|\mathcal{F}| \leq 2$.

Case 4. Let $\mathcal{F} \subseteq \mathcal{P}$ be $\vee$-closed with $|\mathcal{F}| = n+1$ and assume that Φ is injective on $B'_{\vartheta^{-1}(\mathcal{F}')}$ for all $\mathcal{F}' \subseteq \mathcal{P}$ with $|\mathcal{F}'| \leq n$. We will show that Φ is injective on $B'_{\vartheta^{-1}(\mathcal{F})}$. To this end let F be a finite $\vee$-closed subset of $\vartheta^{-1}(\mathcal{F})$ and suppose that

$$\sum_{\substack{r,s \in F \\ \vartheta(r)=\vartheta(s) \in \mathcal{F}}} \psi'_{r,s}(k_{r,s}) \in \ker \Phi,$$

with the understanding that for every $\psi'_{r,s}(k_{r,s})$ we have that either $\psi'_{r,s}(k_{r,s}) = 0$ or that

$$\psi'_{r,s}(k_{r,s}) \notin B'_{\vartheta^{-1}(\vartheta(r)\mathcal{P})}.$$

Choose $h \in \mathcal{F}$ to be minimal such that $\psi'_{p,q}(k_{p,q}) \neq 0$ for distinct $p, q \in \vartheta^{-1}(h)$. Hence $k_{p,q} \neq 0$ and so

$$0 \neq \psi'_{p,q}(k_{p,q}) \notin B'_{\vartheta^{-1}(h\mathcal{P})}.$$

By using Lemma 5.6 item (ii) we then have that

$$t_p(X_p)^* \psi_{p,q}(k_{p,q}) t_q(X_q) \subseteq B_{\vartheta^{-1}(\mathcal{F}')} \quad \text{for } |\mathcal{F}'| \leq |\mathcal{F}| - 1 = n.$$

Using the induction hypothesis we then derive that

$$\psi'_p(\mathcal{K}X_p) \psi'_{p,q}(k_{p,q}) \psi'_q(\mathcal{K}X_q) \subseteq B_{\vartheta^{-1}(\mathcal{F}')} \subseteq B'_{\vartheta^{-1}(h\mathcal{P})}.$$

By using approximate identities on both sides we have the contradiction

$$\psi'_{p,q}(k_{p,q}) \in B'_{\vartheta^{-1}(h\mathcal{P})}.$$

This concludes the proof of the proposition. $\square$

Combining with [Sehnem 2019, Theorem 3.10] we get the following corollary.

Corollary 5.8. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right* LCM *inclusions and let X be a compactly aligned product system over P with coefficients in A. Then the following are equivalent for a strongly covariant representation (π, t) of $A \times_X P$:*

(i) *The $*$-representation π is faithful on A.*

(ii) *The induced $*$-representation is faithful on the fixed point algebra B_P of $A \times_X P$.*

(iii) *The induced $*$-representation is faithful on the ϑ-fixed point algebra $B_{\vartheta^{-1}(\mathcal{P})}$ of $A \times_X P$.*

In particular this holds for the $*$-representations of $q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$ and $A \times_{X,\lambda} P$.

A second application of the controlled elimination allows us to pass in between the C^*-envelopes induced by G and $\mathcal{G}$.

Proposition 5.9. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right* LCM *inclusions, and let X be a compactly aligned product system over P with coefficients in A. Let $\bar{\delta}_{\mathcal{G}}$ be the induced coaction of $\mathcal{G}$ on $\mathcal{T}_\lambda(X)$ and $\mathcal{T}_\lambda(X)^+$. Then $C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G)$ inherits a normal coaction of $\mathcal{G}$ and there exists a $\mathcal{G}$-equivariant $*$-isomorphism*

$$C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G) \simeq C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, \mathcal{G}, \bar{\delta}_{\mathcal{G}})$$

that fixes $\mathcal{T}_\lambda(X)^+$.

Proof. By Theorem 2.7 and Propositions 2.9 and 3.7, we get that $C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G)$ admits a normal coaction of $\mathcal{G}$ and therefore there exists a $\mathcal{G}$-equivariant $*$-epimorphism

$$\Phi : C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G) \to C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, \mathcal{G}, \bar{\delta}_{\mathcal{G}})$$

that fixes $\mathcal{T}_\lambda(X)^+$. By construction Φ is $\mathcal{G}$-equivariant, and so it intertwines the faithful conditional expectations induced by $\mathcal{G}$. On the other hand, by Theorem 2.7 the map Φ is faithful on the G-fixed point algebra of $C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G)$. By Proposition 5.7 the map Φ is faithful on the $\mathcal{G}$-fixed point algebra of $C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G)$. Consequently Φ is injective. $\square$

6. Applications

Couniversality of Sehnem's covariance algebra. We will consider weak right LCM inclusions that are controlled by exact groups. In this case we get normality of the coaction of G on $q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$, and thus the latter coincides with $A \times_{X,\lambda} P$ and, by [Dor-On et al. 2022, Theorem 5.3], with $C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G)$. This provides another algebraic description of $C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G)$ by the strong covariance relations in the Fock space representation.

Theorem 6.1. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right* LCM *inclusions and let X be a compactly aligned product system over P with coefficients in A. Consider the canonical $*$-epimorphisms*

$$q_{\mathrm{sc}}(\mathcal{T}_\lambda(X)) \to A \times_{X,\lambda} P \simeq C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G) \to C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+).$$

If $\mathcal{G}$ is exact then the left map is faithful. If in addition $\mathcal{G}$ is abelian then the right map is also faithful.

Proof. First we show that the ideal of the strong covariance relations is $\mathcal{G}$-induced. Let $\mathcal{I}_\lambda$ be the image of the strong covariance relations in $\mathcal{T}_\lambda(X)$ so that $\mathcal{T}_\lambda(X)/\mathcal{I}_\lambda = q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$. Let us denote by $\bar{B}_F$ the cores of the Fock representation $(\bar{\pi}, \bar{t})$ and let $q_{\mathcal{I}_\lambda} : \mathcal{T}_\lambda(X) \to q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$ be the canonical $*$-epimorphism. Proposition 5.7 implies that

$$\mathcal{I}_\lambda \cap \bar{B}_{\vartheta^{-1}(\mathcal{P})} = \overline{\bigcup_{\substack{\text{finite } \vee\text{-closed} \\ \mathcal{F}\subseteq\mathcal{P}}} \ker q_{\mathcal{I}_\lambda} \cap \bar{B}_{\vartheta^{-1}(\mathcal{F})}} = \overline{\bigcup_{\substack{\text{finite } \vee\text{-closed} \\ F\subseteq P}} \ker q_{\mathcal{I}_\lambda} \cap \bar{B}_F} = \mathcal{I}_\lambda \cap \bar{B}_P.$$

Therefore we get that

$$\mathcal{I}_\lambda = \langle \mathcal{I}_\lambda \cap \bar{B}_P \rangle = \langle \mathcal{I}_\lambda \cap \bar{B}_{\vartheta^{-1}(\mathcal{P})} \rangle,$$

showing that $\mathcal{I}_\lambda$ is indeed $\mathcal{G}$-induced.

Consequently, by exactness of $\mathcal{G}$ we derive that the normal coaction of $\mathcal{G}$ on $\mathcal{T}_\lambda(X)$ descends to a normal coaction on the quotient $q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$. Thus by Proposition 4.12 we have that $q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$ is a C^*-cover for $(\mathcal{T}_\lambda(X)^+, \mathcal{G}, \bar{\delta}_{\mathcal{G}})$. Therefore there exists a $\mathcal{G}$-equivariant $*$-epimorphism

$$\Phi : q_{\mathrm{sc}}(\mathcal{T}_\lambda(X)) \to C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, \mathcal{G}, \bar{\delta}_{\mathcal{G}})$$

that fixes $\mathcal{T}_\lambda(X)^+$. The $*$-epimorphism Φ intertwines the coactions (and thus the faithful conditional expectations implemented by normality and exactness of $\mathcal{G}$), and it is faithful on the G-fixed point algebra

$[q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))]_e$ by Corollary 3.13. Hence we derive that Φ is faithful by Corollary 5.8. By Proposition 5.9 we conclude that

$$q_{\mathrm{sc}}(\mathcal{T}_\lambda(X)) \simeq C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, \mathcal{G}, \bar{\delta}_{\mathcal{G}}) \simeq C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G).$$

Now if in addition $\mathcal{G}$ is abelian then $C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+)$ inherits the coaction of $\mathcal{G}$ by the dual gauge action $\hat{\mathcal{G}}$. Due to couniversality we thus derive

$$C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G) \simeq C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, \mathcal{G}, \bar{\delta}_{\mathcal{G}}) \simeq C^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+). \qquad \square$$

Combining with Proposition 2.9 and Proposition 4.12 we get the following corollary.

Corollary 6.2. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right* LCM *inclusions such that $\mathcal{G}$ is exact, and let X be a compactly aligned product system over P with coefficients in A. Then $q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$ is couniversal with respect to both G-equivariant and $\mathcal{G}$-equivariant quotients of $\mathcal{T}_\lambda(X)$ that are faithful on A.*

Remark 6.3. As an immediate consequence of Theorem 6.1 we get that the coaction of G on $q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$ is normal. Therefore one can use the results of [Dor-On et al. 2022] to derive that the reduced Hao–Ng problem for discrete group actions on $A \times_{X,\lambda} P$ has a positive answer when (G, P) is controlled by $(\mathcal{G}, \mathcal{P})$ with $\mathcal{G}$ exact. A similar method applies whenever the C^*-envelope functor is stable under crossed products, e.g., for dynamics over abelian locally compact groups or when the tensor algebra is hyperrigid [Katsoulis 2020; Katsoulis and Ramsey 2019], and we leave this to the interested reader.

Next we consider *amenably controlled* weak right LCM inclusions, i.e., the range of the controlled map is inside an amenable group. In this case the reduced C^*-algebras become universal with respect to classes of representations. First we consider $\mathcal{NT}(X)$. (A variant of) the following has been obtained by Fowler [2002] for nondegenerate product systems over quasilattices. Here we extend it to the weak right LCM inclusions framework with a different approach that does not require nondegeneracy of X.

Theorem 6.4. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right* LCM *inclusions with $\mathcal{G}$ amenable and let X be a compactly aligned product system over P with coefficients in A. Then the Fock representation is faithful on $\mathcal{NT}(X)$.*

Conversely, suppose that (π, t) is an injective $\mathcal{G}$-equivariant Nica-covariant representation of X and for every $\vee$-closed $\mathcal{F} \subseteq \mathcal{P}$ we have linear independence in the ϑ-cores in the sense that

$$B_{\vartheta^{-1}(\mathcal{F})} = \sum_{h \in \mathcal{F}}^{\oplus} B_{\vartheta^{-1}(h)}.$$

Then (π, t) integrates to a faithful representation of $\mathcal{NT}(X)$.

In particular, a Nica-covariant pair (π, t) defines a faithful representation of $\mathcal{NT}(X)$ if and only if the associated representation is $\mathcal{G}$-equivariant and satisfies the condition

$$\sum_{p \in F} \psi_p(k_p) = 0 \quad \iff \quad k_p = 0 \qquad \text{for all } \; p \in F,$$

for every $\vee$-closed $\mathcal{F} \subseteq \mathcal{P}$ and every finite $\vee$-closed $F \subseteq \vartheta^{-1}(\mathcal{F})$.

Proof. Let $(\hat{\pi}, \hat{t})$ be a faithful representation of $\mathcal{NT}(X)$ and consider the canonical $*$-epimorphism

$$\Phi : \mathcal{NT}(X) = C^*(\hat{\pi}, \hat{t}) \to \mathcal{T}_\lambda(X) = C^*(\overline{\pi}, \overline{t}).$$

Let $\overline{E}_{\mathcal{G}}$ be the faithful conditional expectation induced by Proposition 3.8 on $\mathcal{T}_\lambda(X)$. Let $\hat{E}_{\mathcal{G}}$ be the faithful conditional expectation on $\mathcal{NT}(X)$ induced by the amenable $\mathcal{G}$. Since $\Phi \hat{E}_{\mathcal{G}} = \overline{E}_{\mathcal{G}} \Phi$, it suffices to show injectivity of Φ on $\hat{B}_{\mathcal{F}}$ for every $\vee$-closed $\mathcal{F} \subseteq \mathcal{P}$. To this end fix a finite $\vee$-closed $F \subseteq \vartheta^{-1}(\mathcal{F})$ and suppose that

$$f := \sum \{\hat{\psi}_{r_1,r_2}(k_{r_1,r_2}) \mid k_{r_1,r_2} \in \mathcal{K}(X_{r_2}, X_{r_1}), r_1, r_2 \in F, \vartheta(r_1) = \vartheta(r_2)\} \in \ker \Phi.$$

Let h be minimal in $\mathcal{F}$ such that $k_{q_1,q_2} \neq 0$ with $\vartheta(q_1) = \vartheta(q_2) = h$. Using condition (A2) of Definition 5.1 and the Fock space representation we have that

$$k_{q_1,q_2} = Q_{q_1} \Phi(f) Q_{q_2} = 0$$

for the projections $Q_p : \mathcal{F}X \to X_p$, which gives the required contradiction. Thus the Fock representation is injective and also we have linear independence of the cores. The converse follows with a similar proof.

For the last part it is clear that the condition with $F = \{e_G\}$ and $\mathcal{F} = \{e_{\mathcal{G}}\}$ implies that π is injective. Moreover the condition shows that the canonical $*$-epimorphism Φ is injective on the C^*-subalgebras

$$\hat{B}_F = \operatorname{span}\{\hat{\psi}_r(k_r) \mid r \in F\}$$

for every finite $\vee$-closed $F \subseteq P$, and so Φ is injective on $\hat{B}_P$. Thus by Proposition 5.7 the map Φ is injective on $\hat{B}_{\vartheta^{-1}(\mathcal{P})}$ and hence on $\mathcal{NT}(X)$. □

Next we consider the universal covariance algebra $A \times_X P$.

Theorem 6.5. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right* LCM *inclusions with $\mathcal{G}$ amenable and let X be a compactly aligned product system over P with coefficients in A. Then a strongly covariant representation of X integrates to a faithful representation of $A \times_X P$, if and only if it is injective and G-equivariant, if and only if it is injective and $\mathcal{G}$-equivariant.*

Proof. By Theorem 6.4 we have that $A \times_X P$ coincides with $q_{\mathrm{sc}}(\mathcal{T}_\lambda(X))$ and $A \times_{X,\lambda} P$. Thus the result follows from Corollary 6.2. □

Remark 6.6. When (G, P) is amenably controlled then we have a wider selection for a coaction that implements the extension theorem. The diagram in the proof of Corollary 6.7 on page 1466 depicts those. We denote restrictions of $*$-homomorphisms by solid arrows, and we have used Proposition 4.11 for the upper and lower completely isometric maps. Recall that if $\mathcal{G}$ is amenable then $C^*(\mathcal{G}) \simeq C^*_\lambda(\mathcal{G})$ is nuclear, and by [Li 2013] $C^*_\lambda(\mathcal{P})$ is also nuclear.

Corollary 6.7. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right* LCM *inclusions with $\mathcal{G}$ amenable. Suppose that $A, X_p \subseteq B(\mathcal{H})$ for $p \in P$ define a compactly aligned product system $X = \{X_p\}_{p \in P}$ and set*

$$\mathcal{A} := \overline{\operatorname{alg}}\{A, X_p \mid p \in P\}.$$

Then the following are equivalent:

(i) *There is a completely isometric isomorphism*

$$\mathcal{A} \to \mathcal{T}_\lambda(X)^+, \quad \xi_p \mapsto \bar{t}(\xi_p).$$

(ii) *There is a completely contractive map*

$$\mathcal{A} \to \mathcal{T}_\lambda(X)^+ \otimes C^*(G), \quad \xi_p \mapsto \bar{t}(\xi_p) \otimes u_p.$$

(iii) *There is a completely contractive map*

$$\mathcal{A} \to \mathcal{T}_\lambda(X)^+ \otimes C^*_\lambda(G), \quad \xi_p \mapsto \bar{t}(\xi_p) \otimes \lambda_p.$$

(iv) *There is a completely contractive map*

$$\mathcal{A} \to \mathcal{T}_\lambda(X)^+ \otimes C^*_\lambda(P), \quad \xi_p \mapsto \bar{t}(\xi_p) \otimes V_p.$$

(v) *There is a completely contractive map*

$$\mathcal{A} \to \mathcal{T}_\lambda(X)^+ \otimes C^*(\mathcal{G}), \quad \xi_p \mapsto \bar{t}(\xi_p) \otimes u_{\vartheta(p)}.$$

(vi) *There is a completely contractive map*

$$\mathcal{A} \to \mathcal{T}_\lambda(X)^+ \otimes C^*_\lambda(\mathcal{P}), \quad \xi_p \mapsto \bar{t}(\xi_p) \otimes V_{\vartheta(p)}.$$

Proof. The proof follows by the system of completely positive maps fixing the nonselfadjoint part below, where the solid arrows denote the maps that arise from restrictions of $*$-homomorphisms from the appropriate C^*-algebras to the required subalgebras.

$$\begin{tikzcd}
\mathcal{N}\mathcal{T}(X)^+ \arrow[d, "\simeq"] \arrow[dr] & \\
\overline{\mathrm{alg}\{\bar{t}_p(\xi_p) \otimes u_p \mid \xi_p \in X_p,\, p \in P\}}^{\mathcal{N}\mathcal{T}(X)\otimes C^*(G)} \arrow[d] & \mathcal{A} \\
\overline{\mathrm{alg}\{t_p(\xi_p) \otimes u_p \mid \xi_p \in X_p,\, p \in P\}}^{C^*(\pi,t)\otimes C^*(G)} \arrow[d] \arrow[dr] & \\
\overline{\mathrm{alg}\{t_p(\xi_p) \otimes \lambda_p \mid \xi_p \in X_p,\, p \in P\}}^{C^*(\pi,t)\otimes C^*_\lambda(G)} \arrow[d, dotted] & \overline{\mathrm{alg}\{t_p(\xi_p) \otimes \lambda_{\vartheta(p)} \mid \xi_p \in X_p,\, p \in P\}}^{C^*(\pi,t)\otimes C^*(\mathcal{G})} \arrow[d, dotted] \\
\overline{\mathrm{alg}\{t_p(\xi_p) \otimes V_p \mid \xi_p \in X_p,\, p \in P\}}^{C^*(\pi,t)\otimes C^*_\lambda(P)} \arrow[d, "\simeq"] & \overline{\mathrm{alg}\{t_p(\xi_p) \otimes V_{\vartheta(p)} \mid \xi_p \in X_p,\, p \in P\}}^{C^*(\pi,t)\otimes C^*_\lambda(\mathcal{P})} \arrow[dl, "\simeq"] \\
\mathcal{N}\mathcal{T}(X)^+ &
\end{tikzcd}$$

$\square$

Exactness and nuclearity. We will require some results about nuclearity which we record here for convenience.

Lemma 6.8 [Katsura 2004, Proposition B.8]. *Let (π, t) be a representation of a C^*-correspondence X over A such that $\pi(A) \subseteq B$ and $t(X) \subseteq Y$ for a second C^*-correspondence Y over B. If $\pi : A \to B$ is nuclear then the induced map $\psi : \mathcal{K}X \to \mathcal{K}Y$ is nuclear.*

Lemma 6.9 [Kakariadis 2020, Proposition 3.1]. *Let A, A' be C^*-algebras and consider the ideals $I \lhd A$ and $I' \lhd A'$. Suppose we have the commutative diagram of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \varphi_0} & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \tilde{\varphi}} & & \\ 0 & \longrightarrow & I' & \longrightarrow & A' & \longrightarrow & A'/I' & \longrightarrow & 0 \end{array}$$

where $\varphi : A \to A'$ is an injective $*$-homomorphism that satisfies $\varphi(I) \subseteq I'$, $\tilde{\varphi} : A/I \to A'/I'$ is the induced map and $\varphi_0 := \varphi|_I$. If $\varphi : A \to A'$ is nuclear, then φ_0 and $\tilde{\varphi}$ are both nuclear.

Lemma 6.10 [Kakariadis 2020, Proposition 3.3]. *Let A, A' be C^*-algebras and consider the ideals $I \lhd A$ and $I' \lhd A'$. Suppose we have the commutative diagram of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \varphi_0} & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \tilde{\varphi}} & & \\ 0 & \longrightarrow & I' & \longrightarrow & A' & \longrightarrow & A'/I' & \longrightarrow & 0 \end{array}$$

where $\varphi : A \to A'$ is an injective $*$-homomorphism that satisfies $\varphi(I) \subseteq I'$, $\tilde{\varphi} : A/I \to A'/I'$ is the induced map and $\varphi_0 := \varphi|_I$. Suppose further that there exists a contractive approximate identity (e_i) of I' such that $\varphi(a)e_i \in \varphi_0(I)$ for all $a \in A$. If φ_0 and $\tilde{\varphi}$ are nuclear, then so is φ.

First we provide a nuclearity and exactness result for $\mathcal{T}_\lambda(X)$.

Theorem 6.11. *Let (G, P) be a weak right* LCM *inclusion and X be a compactly aligned product system over P with coefficients in A. Let $\overline{E}_P : \mathcal{T}_\lambda(X) \to \overline{B}_P$ be the faithful conditional expectation that arises by compressing to the diagonal. Then the following are equivalent*:

(i) *A is nuclear (resp. exact) and $\overline{E}_P \otimes_{\max} \mathrm{id}_D$ is a faithful conditional expectation on $\mathcal{T}_\lambda(X) \otimes_{\max} D$ for all C^*-algebras D.*

(ii) *$\mathcal{T}_\lambda(X)$ is nuclear (resp. exact).*

Proof. We will show nuclearity; exactness follows in the same way. Notice that for any C^*-algebra D we have the commutative diagram

$$\begin{array}{ccc} C^*(\overline{\pi}, \overline{t}) \otimes_{\max} D & \longrightarrow & C^*(\overline{\pi}, \overline{t}) \otimes D \\ \Big\downarrow{\scriptstyle \overline{E}_P \otimes_{\max} \mathrm{id}} & & \Big\downarrow{\scriptstyle \overline{E}_P \otimes \mathrm{id}} \\ \overline{B}_P \otimes_{\max} D & \longrightarrow & \overline{B}_P \otimes D \end{array}$$

and we recall that $\overline{E}_P \otimes \mathrm{id}$ is faithful on $C^*(\overline{\pi}, \overline{t}) \otimes D$.

Suppose first that $C^*(\bar{\pi}, \bar{t})$ is nuclear. Then trivially $\bar{E}_P \otimes_{\max} \mathrm{id}$ is faithful on $C^*(\bar{\pi}, \bar{t}) \otimes_{\max} D$. Since A is the corner of $C^*(\bar{\pi}, \bar{t})$ at the (e, e)-place we have that A is nuclear, as the compression of a nuclear C^*-algebra.

For the converse, the diagram above implies that it suffices to show that $\bar{B}_P$ is nuclear. Equivalently it suffices to show that $\bar{B}_F$ is nuclear for every finite $\vee$-closed $F \subseteq P$. To this end let $F = \{p_1, \dots, p_n\}$. We choose the enumeration so that it covers the partial order in F in the sense that if $p_m > p_{m'}$ then $m < m'$. We will use induction on n.

For the first step we have that $\bar{\psi}_{p_1}(\mathcal{K}X_{p_1})$ is nuclear as A is nuclear by [Katsura 2004, Proposition B.7]. For the inductive step suppose that $\bar{B}_{F_k}$ is nuclear for $F_k = \{p_1, \dots, p_k\}$ (which is $\vee$-closed by the choice of the enumeration). We will show that so is $\bar{B}_{F_{k+1}}$ for $F_{k+1} = \{p_1, \dots, p_k, p_{k+1}\}$. The enumeration shows that p_{k+1} is minimal in F_{k+1} and hence

$$\bar{B}_{F_{k+1}} = \bar{B}_{F_k} \oplus \bar{\psi}_{p_{k+1}}(\mathcal{K}X_{p_{k+1}}).$$

Indeed let $k_{p_i} \in \mathcal{K}X_{p_i}$ such that

$$\sum_{i=1}^{k+1} \bar{\psi}_{p_i}(k_{p_i}) = 0.$$

Due to minimality of p_{k+1} in F_{k+1} we have that

$$k_{p_{k+1}} = Q_{p_{k+1}} \left(\sum_{i=1}^{k+1} \bar{\psi}_{p_i}(k_{p_i}) \right) Q_{p_{k+1}} = 0,$$

for the projection $Q_{p_{k+1}} : \mathcal{F}X \to X_{p_{k+1}}$. Minimality of p_{k+1} also gives that $\bar{B}_{F_k}$ is an ideal in $\bar{B}_{F_{k+1}}$, and we thus derive the following short exact sequence

$$0 \longrightarrow \bar{B}_{F_k} \longrightarrow \bar{B}_{F_{k+1}} \longrightarrow \bar{\psi}_{p_{k+1}}(\mathcal{K}X_{p_{k+1}}) \longrightarrow 0.$$

Since $\bar{B}_{F_k}$ is nuclear by the inductive hypothesis and $\bar{\psi}_{p_{k+1}}(\mathcal{K}X_{p_{k+1}})$ is nuclear by the base case we have that $\bar{B}_{F_{k+1}}$ is nuclear. Inducing on k gives that $\bar{B}_F = \bar{B}_{F_n}$ is nuclear. □

In the amenably controlled case, and by combining with Theorem 6.4, we can deduce nuclearity and exactness of $\mathcal{NT}(X)$ from nuclearity and exactness of A, and conversely. The exactness equivalence passes to $A \times_X P$, however this fails for nuclearity even for $P = \mathbb{Z}_+$ due to a counterexample of Ozawa in [Katsura 2004]. In [Kakariadis 2020] it is shown that $A \times_X P$ is nuclear if and only if the embedding $A \hookrightarrow A \times_X P$ is nuclear when (G, P) is a quasilattice controlled by $(\mathbb{Z}^n, \mathbb{Z}_+^n)$ that satisfies a minimality condition. In fact this holds for any quotient in between the Toeplitz–Nica–Pimsner and the covariance algebra. Here we generalize to controlled maps by amenable weak right LCM inclusions. Recall that in the amenably controlled case the reduced C^*-algebras are universal.

Theorem 6.12. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right* LCM *inclusions with $\mathcal{G}$ amenable and let X be a compactly aligned product system over P with coefficients in A. Let (π, t) be an equivariant injective Nica-covariant representation of X. Then A is exact if and only if $C^*(\pi, t)$ is exact.*

Proof. We are going to introduce new product systems from X. Therefore in order to make a distinction we will write $\overline{E}_P^X$ for the faithful conditional expectation on the Fock C^*-algebra $\mathcal{T}_\lambda(X)$ of X.

If $C^*(\pi, t)$ is exact then so is A, since exactness passes to C^*-subalgebras. For the converse, by Theorem 6.4 we have that X is amenable and thus $C^*(\pi, t)$ is a quotient of $\mathcal{T}_\lambda(X)$. Hence it suffices to show that $\mathcal{T}_\lambda(X)$ is exact. In view of Theorem 6.11 it suffices to show that $\overline{E}_P^X \otimes_{\max} \mathrm{id}_D$ is faithful on $\mathcal{T}_\lambda(X) \otimes_{\max} D$ for all C^*-algebras D.

Towards this end let the product system $Y = \{Y_p\}_{p \in P}$ be defined by

$$Y_p := \overline{\overline{t}_p(X_p) \odot D}^{\otimes_{\max}} \subseteq \mathcal{T}_\lambda(X) \otimes_{\max} D.$$

That Y is a product system follows as X is a product system. Since X is compactly aligned we have that

$$Y_p Y_p^* Y_q Y_q^* \subseteq \overline{\bar{\psi}_p(\mathcal{K}X_p) \bar{\psi}_q(\mathcal{K}X_q) \odot D}^{\otimes_{\max}} = \overline{\bar{\psi}_w(\mathcal{K}X_w) \odot D}^{\otimes_{\max}} = [Y_w Y_w^*]$$

for $wP = pP \cap qP$, with the understanding that $Y_p Y_p^* Y_q Y_q^* = (0)$ when p and q have no common right common multiple. Thus by Proposition 4.7 we get that Y is a compactly aligned product system over P with coefficients in A.

Again by Theorem 6.4 we have that Y is amenable. Our goal is to show that the identity representation on Y is faithful on $\mathcal{NT}(Y) \simeq \mathcal{T}_\lambda(Y)$, and thus we have that

$$\mathcal{NT}(Y) \simeq \mathcal{T}_\lambda(Y) \simeq \mathcal{T}_\lambda(X) \otimes_{\max} D.$$

We then derive that the faithful conditional expectation $\overline{E}_P^Y$ on $\mathcal{T}_\lambda(Y)$ coincides with $\overline{E}_P^X \otimes_{\max} \mathrm{id}_D$ and the proof will be completed. We will invoke Theorem 6.4.

First we see that the identity representation is $\mathcal{G}$-equivariant. Indeed we have that $(\overline{\pi}, \overline{t})$ admits a coaction $\bar{\delta}_{\mathcal{G}}$ of $\mathcal{G}$ and thus we have an equivariant $*$-homomorphism

$$\bar{\delta}_{\mathcal{G}} \otimes_{\max} \mathrm{id}_D : \mathcal{T}_\lambda(X) \otimes_{\max} D \to (\mathcal{T}_\lambda(X) \otimes C^*(\mathcal{G})) \otimes_{\max} D.$$

By amenability of $\mathcal{G}$ and associativity of the maximal tensor product we get that

$$\begin{aligned}(\mathcal{T}_\lambda(X) \otimes C^*(\mathcal{G})) \otimes_{\max} D &\simeq \mathcal{T}_\lambda(X) \otimes_{\max} C^*(\mathcal{G}) \otimes_{\max} D \\ &\simeq (\mathcal{T}_\lambda(X) \otimes_{\max} D) \otimes_{\max} C^*(\mathcal{G}) \simeq (\mathcal{T}_\lambda(X) \otimes_{\max} D) \otimes C^*(\mathcal{G}),\end{aligned}$$

and thus we deduce that $\bar{\delta}_{\mathcal{G}} \otimes_{\max} \mathrm{id}_D$ is a coaction of $\mathcal{G}$ on $\mathcal{T}_\lambda(X) \otimes_{\max} D$. By construction $\bar{\delta}_{\mathcal{G}} \otimes_{\max} \mathrm{id}_D$ satisfies the coaction identity with aligned fibers in the sense that

$$[\mathcal{T}_\lambda(X) \otimes_{\max} D]_g = \overline{[\mathcal{T}_\lambda(X)]_g \odot D}^{\otimes_{\max}}.$$

Secondly let $F \subseteq P$ be a $\vee$-closed finite set and let $k'_p \in \mathcal{K}Y_p$ such that $\sum_{p \in F} \mathrm{id}(k'_p) = 0$. For every state $\phi \in \mathcal{S}(D)$ we have the completely contractive map

$$\mathrm{id} \otimes_{\max} \phi : [Y_p Y_p^*] \otimes_{\max} D \to \bar{\psi}_p(\mathcal{K}X_p), \quad \bar{\psi}_p(k_p) \otimes d \mapsto \phi(d) \bar{\psi}_p(k_p).$$

Therefore we derive

$$\sum_{p \in F} (\mathrm{id} \otimes_{\max} \phi)(k'_p) = (\mathrm{id} \otimes_{\max} \phi)\left(\sum_{p \in F} k'_p\right) = 0.$$

Note here that this is a relation in $\mathcal{T}_\lambda(X)$ with every $(\mathrm{id} \otimes_{\max} \phi)(k'_p) \in \bar{\psi}_p(\mathcal{K}X_p)$. Thus if p_0 is a minimal element in F such that $(\mathrm{id} \otimes_{\max} \phi)(k'_{p_0}) \neq 0$, then we get

$$P_{p_0}(\mathrm{id} \otimes_{\max} \phi)(k'_{p_0}) P_{p_0} = P_{p_0}(\mathrm{id} \otimes_{\max} \phi)\Big(\sum_{p \in F} k'_p\Big) P_{p_0} = 0,$$

where $P_{p_0} : \mathcal{F}X \to X_{p_0}$ is the canonical projection. However the compression to X_{p_0} is a faithful $*$-representation on $\bar{\psi}_{p_0}(\mathcal{K}X_{p_0})$, and thus we get the contradiction that $(\mathrm{id} \otimes_{\max} \phi)(k'_{p_0}) = 0$. Continuing inductively we deduce that $(\mathrm{id} \otimes_{\max} \phi)(k'_p) = 0$ for all $p \in F$ (one by one for fixed ϕ). As this holds for all ϕ and the family $\{\mathrm{id} \otimes_{\max} \phi\}_{\phi \in \mathcal{S}(D)}$ separates $[Y_p Y_p^*] \otimes_{\max} D$ we get that $k'_p = 0$ for all $p \in F$. Hence the assumptions of Theorem 6.4 hold for Y and the proof is complete. $\square$

Theorem 6.13. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right* LCM *inclusions with $\mathcal{G}$ amenable and let X be a compactly aligned product system over P with coefficients in A. Let (π, t) be an equivariant injective Nica-covariant representation of X. Then $A \hookrightarrow C^*(\pi, t)$ is nuclear if and only if $C^*(\pi, t)$ is nuclear.*

Proof. It is clear that if $C^*(\pi, t)$ is nuclear then $A \hookrightarrow C^*(\pi, t)$ is nuclear. Let us prove the converse. By Theorem 6.4 we have that $\mathcal{T}_\lambda(X) \simeq \mathcal{NT}(X)$ and so (π, t) promotes to a $*$-representation of $\mathcal{T}_\lambda(X)$. Due to amenability, $C^*(\mathcal{G}) = C^*_\lambda(\mathcal{G})$ is nuclear (and so the minimal and the maximal tensor product coincide). Let $\delta : C^*(\pi, t) \to C^*(\pi, t) \otimes C^*_\lambda(\mathcal{G})$ be the coaction of $\mathcal{G}$ and let $E = (\mathrm{id} \otimes E_{\mathcal{G}})\delta$ be the faithful conditional expectation induced on $C^*(\pi, t)$ by the faithful conditional expectation $E_{\mathcal{G}}$ of $C^*_\lambda(\mathcal{G})$. Let D be any C^*-algebra. Associativity of $\otimes_{\max}$ and nuclearity of $C^*_\lambda(\mathcal{G})$ yields

$$D \otimes_{\max} C^*(\pi, t) \otimes_{\max} C^*_\lambda(\mathcal{G}) \simeq (D \otimes_{\max} C^*(\pi, t)) \otimes C^*_\lambda(\mathcal{G}),$$

and so $\mathrm{id}_D \otimes_{\max} \mathrm{id} \otimes_{\max} E_{\mathcal{G}} = (\mathrm{id}_D \otimes_{\max} \mathrm{id}) \otimes E_{\mathcal{G}}$ is faithful on $D \otimes_{\max} C^*(\pi, t) \otimes_{\max} C^*_\lambda(\mathcal{G})$. Hence

$$\mathrm{id}_D \otimes_{\max} E := (\mathrm{id}_D \otimes_{\max} \mathrm{id} \otimes_{\max} E_{\mathcal{G}})(\mathrm{id}_D \otimes_{\max} \delta)$$

is a faithful conditional expectation of $D \otimes_{\max} C^*(\pi, t)$ on $D \otimes_{\max} B_{\mathcal{P}}$. Therefore we have the commutative diagram

$$\begin{tikzcd} C^*(\pi, t) \otimes_{\max} D \arrow[r] \arrow[d] & C^*(\pi, t) \otimes D \arrow[d] \\ B_{\mathcal{P}} \otimes_{\max} D \arrow[r] & B_{\mathcal{P}} \otimes D \end{tikzcd}$$

where the vertical arrows are faithful conditional expectations. Hence it suffices to show that if $\pi : A \to B_{\mathcal{P}}$ is nuclear then the fixed point algebra $B_{\mathcal{P}}$ is nuclear. As the latter is an inductive limit, it suffices to show that nuclearity of π in $B_{\mathcal{P}}$ induces nuclearity of the embedding $B_{\vartheta^{-1}(F)} \hookrightarrow B_{\vartheta^{-1}(\mathcal{P})}$ for every finite $\vee$-closed $\mathcal{F} \subseteq \mathcal{P}$. We will actually show nuclearity of the embedding

$$B_{\vartheta^{-1}(\mathcal{F})} \hookrightarrow B_{\vartheta^{-1}(\mathcal{F} \cdot \mathcal{P})} \subseteq B_{\vartheta^{-1}(\mathcal{P})},$$

where we write

$$\vartheta^{-1}(\mathcal{F} \cdot \mathcal{P}) = \{pP \mid \vartheta(p) \in \mathcal{F}\}.$$

First we remark that $B_{\mathcal{F}}$ contains a contractive approximate identity for $B_{\mathcal{F}\cdot\mathcal{P}}$. Indeed let (e_i) be a contractive approximate identity for $B_{\mathcal{F}}$ so that $\lim_i e_i \psi_p(k_p) = \psi_p(k_p)$ for every $p \in \vartheta^{-1}(\mathcal{F})$. Consequently $\lim_i e_i t_p(\xi_p) = t_p(\xi_p)$ for every $p \in \vartheta^{-1}(\mathcal{F})$ and thus

$$\lim_i e_i t_p(\xi_p) t_r(\xi_r) t_s(\xi_s)^* = t_p(\xi_p) t_r(\xi_r) t_s(\xi_s)^* \quad \text{for all } r, s \in P.$$

Thus $\lim_i e_i \psi_{p,q}(k_{p,q}) = \psi_{p,q}(k_{p,q})$ for every $p, q \in \vartheta^{-1}(\mathcal{F} \cdot P)$.

Now fix a finite $\vee$-closed $\mathcal{F}$. By using maximal elements we can write $\mathcal{F}$ in levels, i.e.,

$$\mathcal{F} = \{h_{11}, \ldots, h_{1n_1}, h_{21}, \ldots, h_{2n_2}, \ldots, h_{m1}, \ldots, h_{mn_m}\},$$

such that every

$$\mathcal{F}_i := \{h_{i1}, \ldots, h_{in_i}\}, \quad \text{with } i \in \{1, \ldots m\},$$

consists of the maximal elements of $\mathcal{F} \setminus \cup_{j=1}^{i-1} \mathcal{F}_j$ and $\mathcal{F}_1$ consists of the maximal elements of $\mathcal{F}$.

We now proceed by induction. For the base case let $h \in \mathcal{P}$ and consider the space

$$Y_h := \sum_{p \in \vartheta^{-1}(h)} t_p(X_p).$$

By using condition (A2) of Definition 5.1 we can equip Y_h with the A-valued bilinear map defined by

$$\langle y_h, y_h' \rangle := y_h^* y_h' \in \pi(A) \quad \text{for all } y_h, y_h' \in Y_h.$$

Then each Y_h becomes a C^*-correspondence over A, since π is faithful. The embedding $Y_h \hookrightarrow [Y_h B_{\vartheta^{-1}(\mathcal{P})}]$ and nuclearity of $\pi(A) \hookrightarrow B_{\vartheta^{-1}(\mathcal{P})}$ imply nuclearity of the embedding

$$B_{\vartheta^{-1}(h)} = [Y_h Y_h^*] \hookrightarrow [Y_h B_{\vartheta^{-1}(\mathcal{P})} Y_h^*] = B_{\vartheta^{-1}(h\mathcal{P})} \quad \text{for all } h \in \mathcal{P},$$

by [Katsura 2004, Proposition B.8]. Maximality of the h_{1j} in $\mathcal{F}$ yields that the $h_{1j}\mathcal{P}$ are minimal in $\{h\mathcal{P} \mid h \in \mathcal{F}\}$ with respect to inclusions. As $\mathcal{F}$ is $\vee$-closed we have that $h_{1j}\mathcal{P} \cap h_{1j'}\mathcal{P} = \varnothing$ for $j \neq j'$. Hence the C^*-algebras $B_{\vartheta^{-1}(h_{1j})}$ are orthogonal and thus the embedding

$$B_{\mathcal{F}_1} = \sum_{j=1}^{n_1}{}^{\oplus} B_{\vartheta^{-1}(h_{1j})} \hookrightarrow \sum_{j=1}^{n_1} B_{\vartheta^{-1}(h_{1j}\mathcal{P})} \subseteq B_{\vartheta^{-1}(\mathcal{F}_1 \cdot \mathcal{P})}$$

is nuclear. For the inductive hypothesis suppose we have shown that the embedding $B_{\vartheta^{-1}(\mathcal{F}')} \hookrightarrow B_{\vartheta^{-1}(\mathcal{F}'\cdot\mathcal{P})}$ is nuclear for

$$\mathcal{F}' = \{h_{11}, \ldots, h_{1n_1}, \ldots h_{i1}, \ldots, h_{ij}\}$$

for some $j \in \{1, \ldots, n_i\}$. If $j < n_i$ then set $h := h_{i(j+1)}$; if $j = n_i$ then set $h = h_{(i+1)1}$. We will show that the embedding

$$B_{\vartheta^{-1}(\mathcal{F}'')} \hookrightarrow B_{\vartheta^{-1}(\mathcal{F}''\cdot\mathcal{P})} \quad \text{for } \mathcal{F}'' := \mathcal{F}' \cup \{h\}$$

is nuclear. By construction $B_{\vartheta^{-1}(\mathcal{F}')}$ is an ideal in $B_{\vartheta^{-1}(\mathcal{F}'')}$ and $B_{\vartheta^{-1}(F'')} = B_{\vartheta^{-1}(h)} + B_{\vartheta^{-1}(\mathcal{F}')}$; thus

$$B_{\vartheta^{-1}(\mathcal{F}'')} / B_{\vartheta^{-1}(\mathcal{F}')} \simeq B_{\vartheta^{-1}(h)} / (B_{\vartheta^{-1}(h)} \cap B_{\vartheta^{-1}(\mathcal{F}')}).$$

Likewise $B_{\vartheta^{-1}(\mathcal{F}'\cdot\mathcal{P})}$ is an ideal of $B_{\vartheta^{-1}(\mathcal{F}''\cdot\mathcal{P})}$. From the base case we have nuclearity of the map

$$B_{\vartheta^{-1}(h)} \hookrightarrow B_{\vartheta^{-1}(h\mathcal{P})} \subseteq B_{\vartheta^{-1}(\mathcal{F}''\cdot\mathcal{P})}.$$

By applying Lemma 6.9 on the commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & B_{\vartheta^{-1}(h)}/B_{\vartheta^{-1}(\mathcal{F}')} \arrow[r] \arrow[d] & B_{\vartheta^{-1}(h)} \arrow[r] \arrow[d] & B_{\vartheta^{-1}(h)}/(B_{\vartheta^{-1}(h)} \cap B_{\vartheta^{-1}(\mathcal{F}')}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B_{\vartheta^{-1}(\mathcal{F}'\cdot P)} \arrow[r] & B_{\vartheta^{-1}(\mathcal{F}''\cdot\mathcal{P})} \arrow[r] & B_{\vartheta^{-1}(\mathcal{F}''\cdot\mathcal{P})}/B_{\vartheta^{-1}(\mathcal{F}'\cdot\mathcal{P})} \arrow[r] & 0 \end{tikzcd}$$

we get that the right vertical arrow is nuclear, i.e., the map

$$B_{\vartheta^{-1}(\mathcal{F}'')}/B_{\vartheta^{-1}(\mathcal{F}')} \simeq B_{\vartheta^{-1}(h)}/(B_{\vartheta^{-1}(h)} \cap B_{\vartheta^{-1}(\mathcal{F}')}) \to B_{\vartheta^{-1}(\mathcal{F}''\cdot\mathcal{P})}/B_{\vartheta^{-1}(\mathcal{F}'\cdot\mathcal{P})}$$

is nuclear. Let $(e_i) \subseteq B_{\vartheta^{-1}(\mathcal{F}')}$ be a contractive approximate identity for $B_{\vartheta^{-1}(\mathcal{F}'\cdot\mathcal{P})}$, and note that

$$B_{\vartheta^{-1}(\mathcal{F}'')} \cdot e_i \subseteq B_{\vartheta^{-1}(\mathcal{F}'')} \cdot B_{\vartheta^{-1}(\mathcal{F}')} = B_{\vartheta^{-1}(\mathcal{F}')}.$$

Using the inductive hypothesis and Lemma 6.10 on the commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & B_{\vartheta^{-1}(\mathcal{F}')} \arrow[r] \arrow[d] & B_{\vartheta^{-1}(\mathcal{F}'')} \arrow[r] \arrow[d] & B_{\vartheta^{-1}(\mathcal{F}'')}/B_{\vartheta^{-1}(\mathcal{F}')} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B_{\vartheta^{-1}(\mathcal{F}'\cdot\mathcal{P})} \arrow[r] & B_{\vartheta^{-1}(\mathcal{F}''\cdot\mathcal{P})} \arrow[r] & B_{\vartheta^{-1}(\mathcal{F}''\cdot\mathcal{P})}/B_{\vartheta^{-1}(\mathcal{F}'\cdot\mathcal{P})} \arrow[r] & 0 \end{tikzcd}$$

we derive that the middle vertical arrow is nuclear, as required. This concludes the inductive step. Now by using induction we derive that $B_{\vartheta^{-1}(\mathcal{F})} \hookrightarrow B_{\vartheta^{-1}(\mathcal{F}\cdot\mathcal{P})}$ is nuclear, and the proof is complete. ☐

7. Saturated controlled maps

A product system reparametrization. Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map between weak right LCM inclusions and let X be a compactly aligned product system over P with coefficients in A. We can then define the C^*-correspondence

$$Y_h := \sum_{p\in\vartheta^{-1}(h)}^{\oplus} X_p \quad \text{for all } h \in \mathcal{P}.$$

One is tempted to consider the family $Y = \{Y_h\}_{h\in\mathcal{P}}$ and associate its C^*-algebras with those of X. However it is not clear that Y is in general a product system (let alone compactly aligned). Nevertheless this happens for controlled maps that satisfy one extra condition.

Definition 7.1. Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a controlled map of weak right LCM inclusions. We say that ϑ is *saturated* if for any $h \in \mathcal{P}$ and $t \in \vartheta^{-1}(h\mathcal{P})$ there exists an $s \in P$ with $\vartheta(s)\mathcal{P} = h\mathcal{P}$ and $t \in sP$.

Remark 7.2. In particular, saturated maps satisfy the following property:

(A3) If $z \in \mathcal{P}^*$ then there exists an $x \in P^*$ such that $\vartheta(x) = z$.

Indeed, we apply the saturation property for $z \in \mathcal{P}^*$ and $t = e_G \in \vartheta^{-1}(z\mathcal{P})$ to obtain an $x \in P$ with $e_G \in xP$. Hence we get that $P = xP$ giving that $x \in P^*$.

The following provides a good supply of saturated controlled maps. Recall that a pair (G, P) is a *total order* if $G = P^{-1} \cup P$ and $P^{-1} \cap P = \{e_G\}$. It is clear that total orders, being lattices, form weak right LCM inclusions.

Proposition 7.3. *Let (G, P) be an abelian total order. For $n \in \mathbb{N} \cup \{\infty\}$ consider the free product $\big(\ast_{i=1}^n G, \ast_{i=1}^n P\big)$ of n copies of (G, P). Then the map*

$$\vartheta : \big(\ast_{i=1}^n G, \ast_{i=1}^n P\big) \to (G, P), \quad (g_1)_{i_1}(g_2)_{i_2} \cdots (g_k)_{i_k} \mapsto g_1 + g_2 + \cdots + g_k$$

is a saturated controlled map.

Proof. For condition (A1) of Definition 5.1, if $\bar{p}, \bar{q} \in \ast_{i=1}^n P$ with

$$\bar{p}\big(\ast_{i=1}^n P\big) \bigcap \bar{q}\big(\ast_{i=1}^n P\big) \neq \varnothing,$$

then the freeness construction implies that either $\bar{p} \leq \bar{q}$ or $\bar{q} \leq \bar{p}$.

For condition (A2) of Definition 5.1 suppose that $\bar{p}, \bar{q}$ have a right LCM and they satisfy $\vartheta(\bar{p}) = \vartheta(\bar{q})$. Without loss of generality assume that $\bar{r} = \bar{p}^{-1}\bar{q} \in \ast_{i=1}^n P$. Then $\vartheta(\bar{r}) = 0$. If $\bar{r} = (r_1)_{i_1} \cdots (r_k)_{i_k}$ then $r_1 + \cdots + r_k = 0$ giving that $r_k \in -P \cap P = \{0\}$. Inductively we get that $r_1 = \cdots = r_k = 0$ and so $\bar{p} = \bar{q}$.

Next we verify that ϑ is saturated. To this end let

$$\bar{p} = (p_1)_{i_1}(p_2)_{i_2} \cdots (p_k)_{i_k},$$

and let $h \in \mathcal{P}$ with

$$h \leq \vartheta(\bar{p}) = p_1 + p_2 + \cdots + p_k.$$

Let $\ell \in \{1, \ldots, k\}$ be the smallest index such that $h \leq p_1 + p_2 + \cdots + p_\ell$. Set

$$h' = \begin{cases} h & \text{if } \ell = 1, \\ h - (p_1 + p_2 + \cdots + p_{\ell-1}) & \text{otherwise,} \end{cases}$$

and notice that $h' \in P$ with $h' \leq p_\ell$. Let

$$\bar{q} = \begin{cases} (h')_{i_1} & \text{if } \ell = 1, \\ (p_1)_{i_1}(p_2)_{i_2} \cdots (p_{\ell-1})_{i_{\ell-1}}(h')_{i_\ell} & \text{otherwise.} \end{cases}$$

Then $\bar{q} \leq \bar{p}$ and $\vartheta(\bar{q}) = h$, as desired. □

Example 7.4. A second example comes from types of semidirect products. Let (G, P) and (H, S) be quasilattice ordered groups and consider an action $\alpha : H \to \mathrm{Aut}(G)$ such that $\alpha|_S : S \to \mathrm{Aut}(P)$ restricts to automorphisms of P. Then we can form the semidirect products $G \rtimes_\alpha H$ and $P \rtimes_\alpha S$ with respect to the relations $\alpha_h(g)h = hg$. The condition on α makes $P \cdot S$ a subsemigroup of the semidirect product, and in [Kakariadis 2020] it is shown that the pair $(G \rtimes_\alpha H, P \rtimes_\alpha S)$ is quasilattice ordered. Now suppose that (G, P) admits an abelian controlled map ϑ_1 in $(\mathcal{G}_1, \mathcal{P}_1)$ and (H, S) admits an abelian controlled map ϑ_2 in $(\mathcal{G}_2, \mathcal{P}_2)$. In order for the semidirect product to inherit the obvious controlled map on $(\mathcal{G}_1 \oplus \mathcal{G}_2, \mathcal{P}_1 \oplus \mathcal{P}_2)$, it is necessary that α is ϑ_1-invariant in the sense that $\vartheta_1\alpha_h = \vartheta_1$ for all $h \in H$. We can then define the homomorphism

$$\vartheta : (G \rtimes_\alpha H, P \rtimes_\alpha S) \to (\mathcal{G}_1 \oplus \mathcal{G}_2, \mathcal{P}_1 \oplus \mathcal{P}_2) \quad \text{such that } \vartheta(gh) = (\vartheta_1(g), \vartheta_2(h)).$$

We claim that if ϑ_1 and ϑ_2 are saturated, then so is ϑ. Suppose that $\vartheta(gh) = (\vartheta_1(g), \vartheta_2(h)) \geq (m, \ell)$. Then there are $s_1, r_1 \in G$ and $s_2, r_2 \in H$ such that

$$g = s_1 r_1, \quad \vartheta_1(s_1) = m \qquad \text{and} \qquad h = s_2 r_2, \quad \vartheta(s_2) = \ell.$$

It follows that $\vartheta(s_1 s_2) = (m, \ell)$ and $gh = s_1 s_2 \alpha_{s_2}^{-1}(r_1) r_2$.

The following examples show that surjectivity is not enough to render a controlled map saturated.

Example 7.5. Take the free quasilattice on two symbols a, b and take ϑ to be its abelianization map. Then for ab and $(0, 1) \in \mathbb{Z}^2$ we have that $\vartheta(ab) = (1, 1) \geq (0, 1)$. However $\{b\} = \vartheta^{-1}((0, 1))$ and $ab \not\geq b$. (Although, Proposition 7.3 induces a saturated map on free quasilattices.)

Example 7.6. Consider the Baumslag–Solitar group $B(3, 3) = \langle a, b \mid a^3 b = b a^3 \rangle$. Recall that every element $x \in B(3, 3)$ admits a unique normal form

$$x = a^{p_1} b^{\varepsilon_1} a^{p_2} \cdots a^{p_k} b^{\varepsilon_k} a^{p_{k+1}} \quad \text{with } p_1, \ldots, p_k \in \{0, 1, 2\}, \quad p_{k+1} \in \mathbb{Z}, \quad k \in \mathbb{Z}_+.$$

Let $B_+(3, 3)$ be its subsemigroup generated by a, b. If x is in its normal form as above, it follows that

$$x = a^{p_1} b^{\varepsilon_1} a^{p_2} \cdots a^{p_k} b^{\varepsilon_k} a^{p_{k+1}} \in B_+(3, 3) \iff \varepsilon_1, \ldots, \varepsilon_k = 1, \quad p_{k+1} \geq 0.$$

By [Spielberg 2012, Theorem 2.11] we have that the pair $(B(3, 3), B_+(3, 3))$ is a quasilattice ordered group. In [Kakariadis 2020] it is shown that the abelianization gives a surjective controlled map

$$\vartheta : (B(3, 3), B_+(3, 3)) \to (\mathbb{Z}^2, \mathbb{Z}^2_+), \quad a^{p_1} b a^{p_2} b \cdots a^{p_k} b a^{p_{k+1}} \mapsto (p_1 + \cdots + p_{k+1}, k).$$

However this map is not saturated. Take $t = a^2 b$ and $h = (1, 1)$ so that

$$\vartheta(t) = (2, 1) \in (1, 1) + \mathbb{Z}^2_+.$$

We have that $\vartheta^{-1}(1, 1) = \{ab, ba\}$ and thus these are the only choices for a possible s with $\vartheta(s) = (1, 1)$ and $s \leq t$. However we see that

$$(ab)^{-1} t = b^{-1} a b \notin B_+(3, 3) \quad \text{and} \quad (ba)^{-1} t = a^{-1} b^{-1} a^2 b \notin B_+(3, 3).$$

Theorem 7.7. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a saturated controlled map between weak right* LCM *inclusions. Let X be an (injective) compactly aligned product system over P with coefficients in A and let*

$$Y_h := \sum_{p \in \vartheta^{-1}(h)}^{\oplus} X_p \quad \text{for } h \in \mathcal{P}.$$

Then the collection $Y = \{Y_h\}_{h \in \mathcal{P}}$ is an (injective) compactly aligned product system over $\mathcal{P}$ with coefficients in A such that $\mathcal{T}_\lambda(X)^+ \simeq \mathcal{T}_\lambda(Y)^+$ with

$$\mathcal{T}_\lambda(X) \simeq \mathcal{T}_\lambda(Y) \quad \text{and} \quad A \times_{X,\lambda} P \simeq A \times_{Y,\lambda} \mathcal{P},$$

by $*$-homomorphisms that preserve the inclusions $X_p \mapsto Y_{\vartheta(p)}$ for all $p \in P$. These $*$-isomorphisms further lift to $*$-isomorphisms

$$\mathcal{NT}(X) \simeq \mathcal{NT}(Y) \quad \text{and} \quad A \times_X P \simeq A \times_Y \mathcal{P}$$

that preserve the inclusions $X_p \hookrightarrow Y_{\vartheta(p)}$ for all $p \in P$.

Proof. Let A act on both the left and right of each Y_h with $h \in \mathcal{P}$ via the usual multiplication of operators. By using condition (A2) of Definition 5.1 we can equip Y_h with the A-valued bilinear map defined by

$$\langle y_h, y_h' \rangle := y_h^* y_h' \in A \subseteq \mathcal{T}_\lambda(X) \quad \text{for all } \; y_h, y_h' \in Y_h.$$

Then each Y_h becomes a C^*-correspondence over A. Since $\ker \varphi_{Y_h} = \bigcap_{p \in \vartheta^{-1}(h)} \ker \varphi_{X_p}$, we have that every Y_h is injective when every X_p is so.

We now show that $Y := \{Y_h\}_{h \in \mathcal{P}}$ is a product system. Since $[Y_h Y_g] \simeq Y_h \otimes_A Y_g$ we have to show that

$$[Y_h Y_g] = Y_{hg} \quad \text{for all } \; h, g \in \mathcal{P}.$$

As $(\overline{\pi}, \overline{t})$ is a Toeplitz representation we have that $Y_h Y_g \subseteq Y_{hg}$ for all $h, g \in \mathcal{P}$. For the reverse inclusion, let $p \in P$ with $\vartheta(p) = hg$, and we will show that $\overline{t}_p(X_p) \in [Y_h Y_g]$. Since ϑ is saturated there are $q, q' \in P$ such that

$$p = qq' \quad \text{and} \quad \vartheta(q)\mathcal{P} = h\mathcal{P}.$$

We can write $\vartheta(q) = hz$ for some $z \in \mathcal{P}^*$ and let $w \in P^*$ with $\vartheta(w) = z$ by condition (A3) of the saturation property. Since $\vartheta(q)\vartheta(q') = \vartheta(p) = hg$ it follows that $\vartheta(q') = z^{-1}g$. We thus conclude that

$$\vartheta(qw^{-1}) = h \quad \text{and} \quad \vartheta(wq') = g.$$

Recall that X_w satisfies $[\overline{t}_{w^{-1}}(X_{w^{-1}})\overline{t}_w(X_w)] = \overline{\pi}(A)$. By taking elementary vectors we get the required

$$\begin{aligned} \overline{t}_p(X_p) &= [\overline{t}_q(X_q)\overline{t}_{q'}(X_{q'})] = [\overline{t}_q(X_q)\overline{\pi}(A)\overline{t}_{q'}(X_{q'})] \\ &= [\overline{t}_q(X_q)\overline{t}_{w^{-1}}(X_{w^{-1}})\overline{t}_w(X_w)\overline{t}_{q'}(X_{q'})] \\ &\subseteq [\overline{t}_{qw^{-1}}(X_{qw^{-1}})\overline{t}_{wq'}(X_{wq'})] \subseteq [Y_{\vartheta(q)\vartheta(w^{-1})} Y_{\vartheta(w)\vartheta(q')}] = [Y_h Y_g]. \end{aligned}$$

Next we show that Y is compactly aligned. Let $h, h' \in \mathcal{P}$ and take $p \in \vartheta^{-1}(h)$ and $q \in \vartheta^{-1}(h')$. If $h \vee h' = \infty$ then $p \vee q = \infty$ as well for all $p \in \vartheta^{-1}(h)$ and $q \in \vartheta^{-1}(h')$, and so

$$Y_h^* Y_{h'} = \sum_{\substack{p \in \vartheta^{-1}(h) \\ q \in \vartheta^{-1}(h')}} \overline{t}_p(X_p)^* \overline{t}_q(X_q) = (0).$$

On the other hand, if $h \vee h' < \infty$ and $p \vee q < \infty$ for $p \in \vartheta^{-1}(h)$ and $q \in \vartheta^{-1}(h')$, then

$$\overline{t}_p(X_p)^* \overline{t}_q(X_q) \subseteq [\overline{t}_{p^{-1}w}(X_{p^{-1}w}) \overline{t}_{q^{-1}w}(X_{q^{-1}w})^*].$$

Since $w = px = qy$ we have that $\vartheta(p^{-1}w) = h^{-1}\vartheta(w) = (h')^{-1}\vartheta(w) = \vartheta(q^{-1}w)$ and also $\vartheta(w) = h \vee h'$. Hence

$$Y_h^* Y_{h'} = \sum_{\substack{p \in \vartheta^{-1}(h) \\ q \in \vartheta^{-1}(h') \\ p \vee q < \infty}} \overline{t}_p(X_p)^* \overline{t}_q(X_q) \subseteq [Y_{h^{-1}(h \vee h')} Y_{h'(h \vee h')}].$$

Thus Proposition 4.7 gives that Y is compactly aligned.

By definition we have that $\mathcal{F}X \simeq \mathcal{F}Y$ (by grouping together summands with the same ϑ-image), and therefore we have that $\mathcal{T}_\lambda(X) \simeq \mathcal{T}_\lambda(Y)$ and that $\mathcal{T}_\lambda(X)^+ \simeq \mathcal{T}_\lambda(Y)^+$. Notice that these identifications are $\mathcal{G}$-compatible. By applying Proposition 5.9 and Theorem 4.14 we then get

$$A \times_{X,\lambda} P \simeq C^*_{\text{env}}(\mathcal{T}_\lambda(X)^+, G, \bar{\delta}_G) \simeq C^*_{\text{env}}(\mathcal{T}_\lambda(X)^+, \mathcal{G}, \bar{\delta}_{\mathcal{G}}) \simeq C^*_{\text{env}}(\mathcal{T}_\lambda(Y)^+, \mathcal{G}, \bar{\delta}_{\mathcal{G}}) \simeq A \times_{Y,\lambda} \mathcal{P}.$$

The second part of the proof is treated likewise. First note that any representation of X lifts to a representation of Y in a unique way, as every fiber of Y is spanned independently by the corresponding fibers of X. Applying similar arguments as above for a representation (π, t) in the place of the Fock representation we see that this correspondence preserves Nica-covariant representations. Hence we get that $\mathcal{NT}(X) \simeq \mathcal{NT}(Y)$.

Finally the $*$-isomorphisms $A \times_{X,\lambda} P \simeq A \times_{Y,\lambda} \mathcal{P}$ gives an injective map

$$[A \times_X P]_{pq^{-1}} \simeq [A \times_{X,\lambda} P]_{pq^{-1}} \hookrightarrow [A \times_{Y,\lambda} \mathcal{P}]_{\vartheta(pq^{-1})} \simeq [A \times_Y \mathcal{P}]_{\vartheta(pq^{-1})}.$$

Therefore we get a commutative diagram

$$\begin{tikzcd} \mathcal{NT}(X) \arrow[r, "\Phi"] \arrow[d, "q_X"] & \mathcal{NT}(Y) \arrow[d, "q_Y"] \\ A \times_X P \arrow[r, "\Psi"] & A \times_Y \mathcal{P} \end{tikzcd}$$

where the upper horizontal arrow is a $*$-isomorphism. Since the ideals of strong covariance relations are induced, it suffices to show that

$$\ker \Phi q_Y \cap [\mathcal{NT}(X)]_{e_{\mathcal{G}}} \subseteq \ker q_X.$$

Equivalently it suffices to show that Ψ is faithful on the $\mathcal{G}$-fixed point algebra defined on $A \times_X P$. However this follows by Corollary 5.8 as Ψ is by definition faithful on A. $\square$

Theorem 7.7 gives a very clear picture for the covariance algebras of a product system over a free product order of the form $\big(*_{i=1}^n G, *_{i=1}^n P\big)$ for an abelian total order (G, P). It is well known that the Cuntz C^*-algebra $\mathcal{O}_n$ for $n \in \mathbb{N}$ can be viewed as either the Nica–Cuntz–Pimsner C^*-algebra of the trivial product system over the free semigroup on n generators or as the Cuntz–Pimsner C^*-algebra of the C^*-correspondence $(\mathbb{C}^n, \mathbb{C})$. Our next result generalizes this fact to arbitrary product systems over the free semigroup.

Corollary 7.8. *Let X be a compactly aligned product system over the free semigroup $\mathbb{F}_n^+ = \langle i_1, \dots, i_n\rangle$. Then $A \times_X \mathbb{F}_n^+ \simeq \mathcal{O}_Y$ for the C^*-correspondence $Y = \sum_{j=1,\dots,n}^{\oplus} X_{i_j}$.*

Reversible product systems and total orders. An application of the theorem of Burns and Hale [1972] asserts that G admits a total order if and only if for every nontrivial finitely generated subgroup H of G there exists a totally ordered L and a nontrivial homomorphism $H \to L$. If $L = \mathbb{Z}$ then the group is called *left indicable*. There are plenty of abelian total orders. Examples include $\mathbb{R}^2$ with the lexicographical order and $\mathbb{Z}^2$ with the semigroup given by the half-plane defined by any line through the origin with irrational slope. Conrad's theorem [1959] asserts that if (G, P) is a total order and G is Archimedean then G embeds in $\mathbb{R}$ so that P embeds in $\mathbb{R}^+$. Here we say that G is *Archimedean* if whenever $e_G < x < y$, there exists an $n \in \mathbb{N}$ such that $y < x^n$. We refer the reader to [Clay and Rolfsen 2016] for an exposition of these results.

There are not many ways for a total order to be controlled by an *abelian* total order.

Proposition 7.9. *Let (G, P) be a total order and let $\vartheta_{\mathrm{ab}} : (G, P) \to (G_{\mathrm{ab}}, P_{\mathrm{ab}})$ be the abelianization map. Then the following are equivalent*:

(i) *There is a controlled map $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ where $(\mathcal{G}, \mathcal{P})$ is an abelian total order.*

(ii) $\vartheta_{\mathrm{ab}}^{-1}(0) \cap P = \{e_G\}$.

(iii) *The abelianization map is a controlled map and $(G_{\mathrm{ab}}, P_{\mathrm{ab}})$ is a total order.*

If any (and thus all) of the above hold then the abelianization is a saturated controlled map.

Proof. If item (i) holds then ϑ factors through the abelianization. Since $\vartheta^{-1}(e_{\mathcal{G}}) \cap P = \{e_G\}$ is a controlled map, we have that $\vartheta_{\mathrm{ab}}^{-1}(0) \cap P = \{e_G\}$ as well.

Assume that item (ii) holds, and we will show that $(G_{\mathrm{ab}}, P_{\mathrm{ab}})$ is a total order. First we clearly have that

$$-P_{\mathrm{ab}} \cup P_{\mathrm{ab}} = \vartheta_{\mathrm{ab}}(P^{-1} \cup P) = G_{\mathrm{ab}}.$$

Next suppose that $-P_{\mathrm{ab}} \cap P_{\mathrm{ab}} \neq \{0\}$ so that there are $h, g \in P_{\mathrm{ab}}$ with $h + g = 0$. As the abelianization map is surjective there are $p, q \in P$ with $pq = e_G$ with $\vartheta(p) = h$ and $\vartheta(q) = g$. As (G, P) is a total order we derive that $p = q = e_G$ and thus $h = g = 0$. Next we show that ϑ_{ab} satisfies conditions (A1) and (A2) of Definition 5.1. Let $p, q \in P$. Then either $p \leq q$ or $q \leq p$ and condition (A1) follows. For condition (A2) suppose without loss of generality that $p \leq q$ with $\vartheta_{\mathrm{ab}}(p) = \vartheta_{\mathrm{ab}}(q)$. Then $q = ps$ for $s \in P \cap \vartheta_{\mathrm{ab}}^{-1}(0)$. Then $s = e_G$ and so $p = q$.

If item (iii) holds then clearly item (i) holds, concluding the equivalences between all items.

For the saturation property let a $t \in P$ and an $h \in P_{\mathrm{ab}}$ such that $\vartheta_{\mathrm{ab}}(t) = h + h'$. Take an $s \in \vartheta_{\mathrm{ab}}^{-1}(h)$ since the abelianization map is surjective. Then either $s \leq t$ or $s > t$. But if $s > t$ then $h = \vartheta_{\mathrm{ab}}(s) > \vartheta_{\mathrm{ab}}(t)$ which is a contradiction. Thus we must have that $s \leq t$. □

Remark 7.10. There are exact total orders for which the abelianization map is not controlled. An example is given by the Klein bottle group

$$\mathbb{K} := \langle x, y \mid x^{-1}yx = y^{-1} \rangle = \langle x, y \mid x = yxy \rangle$$

with the total order induced by the semigroup $\mathbb{K}^+$ generated by x, y in $\mathbb{K}$. It is not hard to see that $\mathbb{K}^+$ induces a total order on $\mathbb{K}$, being left indicable (or since $\mathbb{K}$ is the extension $\mathbb{Z} \rtimes \mathbb{Z}$). Alternatively one can see that every element in $\mathbb{K}$ is written (uniquely) in the form $x^m y^n$ for $m, n \in \mathbb{Z}$ and we take cases: if $m, n \geq 0$ then $x^m y^n \in \mathbb{K}$; if $m \geq 1$ and $n \leq 0$ then we have that $x^m y^n = x^{m-1} y^{-n} x \in \mathbb{K}^+$; if $m = 0$ and $n \leq 0$ then $x^m y^n = y^n \in (\mathbb{K}^+)^{-1}$. By symmetry these cover all cases. We see that $\vartheta_{\mathrm{ab}}(yxy) = \vartheta_{\mathrm{ab}}(x)$ and so $e_{\mathbb{K}} \neq y^2 \in \mathbb{K}^+ \cap \vartheta_{\mathrm{ab}}^{-1}(0)$. In fact we have that $\mathbb{K}_{\mathrm{ab}} = \mathbb{Z} \times \mathbb{Z}_2$ and $\mathbb{K}_{\mathrm{ab}}^+ = \mathbb{Z}^+ \times \mathbb{Z}_2$ and thus it does not define a total order as $-\mathbb{K}_{\mathrm{ab}}^+ \cap \mathbb{K}_{\mathrm{ab}}^+ = \mathbb{Z}_2$.

Definition 7.11. Let (G, P) be a total order and let X be a product system over P with coefficients in A. We say that X is a *reversible* product system if every X_p is a Hilbert bimodule in $A \times_{X,\lambda} P$, i.e., if $A \times_{X,\lambda} P = C^*(\pi, t)$ then $t_p(X_p)t_p(X_p)^* \subseteq A$ for all $p \in P$.

It follows that reversible product systems consist of Hilbert bimodules. The converse holds also for injective product systems, as in this case every strongly covariant representation is Katsura-covariant fiberwise.

Proposition 7.12. *Let (G, P) be a total order and let X be a product system over P with coefficients in A. Suppose that every X_p is injective. If (π, t) is a strongly covariant representation of X then (π, t_p) is a covariant representation of X_p, in the sense of Katsura, for every $p \in P$.*

Therefore an injective product system X is reversible if and only if every X_p is a Hilbert bimodule.

Proof. Fix $p \in P$ and $a \in A$ such that $\varphi_p(a) = k_p \in \mathcal{K}X_p$. In view of strong covariance of Proposition 4.9 and Katsura covariance we have to show that

$$[\pi_F(a) + \psi_{p,F}(k_p)]_{X_F} = 0 \quad \text{for} \;\; F = \{e, p\},$$

where

$$X_F = \bigoplus_{r \in P} X_r I_{r^{-1}(r \vee F)}.$$

Let $r \in P$ with $r = ps$ for some $s \in P$. Then for every $\xi_r = \xi_p \xi_s \in X_r$ and $b \in I_{r^{-1}(r \vee F)}$ we have that

$$\pi_F(a)\xi_r b = (\varphi_p(a)\xi_p)\xi_s b = (k_p\xi_p)\xi_s b = \psi_{F,p}(k_p)\xi_r b.$$

Now suppose that $r < p$. Then by construction $\psi_{F,p}(k_p)\xi_r b = 0$ and we have to show that $\pi_F(a)\xi_r b = 0$ as well. To this end it suffices to show that

$$I_{r^{-1}(r \vee F)} := I_{r^{-1}K_{\{r,e\}}} \bigcap I_{r^{-1}K_{\{r,p\}}} = (0).$$

Since $r < p$ we have that $r \notin K_{\{r,p\}} \subseteq pP$ while $p \in K_{\{r,p\}}$. Therefore $r^{-1}p \neq e_G$ and so

$$I_{r^{-1}K_{\{r,p\}}} = \bigcap_{t \in K_{\{r,p\}}} \ker \varphi_{r^{-1}t} \subseteq \ker \varphi_{r^{-1}p} = (0),$$

and the proof is complete. □

In the case of $(G, P) = (\mathbb{Z}, \mathbb{Z}_+)$, the following result was established in [Kakariadis 2013].

Proposition 7.13. *Let (G, P) be a total order and let X be a product system over P with coefficients in A. Then X is a reversible product system if and only if the tensor algebra $\mathcal{T}_\lambda(X)^+$ is Dirichlet in $A \times_{X,\lambda} P$.*

Proof. Let (π, t) be a faithful representation of $A \times_{X,\lambda} P$. Suppose first that X is a reversible product system so that $t_p(X_p)t_p(X_p)^* \subseteq \pi(A)$ for all $p \in P$. We will show that

$$A \times_{X,\lambda} P = \overline{\operatorname{span}}\{t_s(X_s) + t_r(X_r)^* \mid s, r \in P\}.$$

Let $s, r \in P$. If $rs^{-1} \in P$ then we have that

$$t_s(X_s)t_r(X_r)^* \subseteq [t_s(X_s)t_s(X_s)^* t_{rs^{-1}}(X_{rs^{-1}})^*] \subseteq [\pi(A)t_{rs^{-1}}(X_{rs^{-1}})^*] = t_{rs^{-1}}(X_{rs^{-1}})^*.$$

If $sr^{-1} \in P$ then we have that

$$t_s(X_s)t_r(X_r)^* \subseteq [t_{sr^{-1}}(X_{sr^{-1}})t_r(X_r)t_r(X_r)^*] \subseteq [t_{sr^{-1}}(X_{sr^{-1}})\pi(A)] = t_{sr^{-1}}(X_{sr^{-1}}).$$

Hence

$$A \times_{X,\lambda} P = \overline{\operatorname{span}}\{t_s(X_s)t_r(X_r)^* \mid s, r \in P\} \subseteq \overline{\operatorname{span}}\{t_s(X_s) + t_r(X_r)^* \mid s, r \in P\} \subseteq A \times_{X,\lambda} P,$$

and so $\mathcal{T}_\lambda(X)^+$ is Dirichlet in $A \times_{X,\lambda} P$.

Conversely, assume that $\mathcal{T}_\lambda(X)^+$ is Dirichlet in $A \times_{X,\lambda} P$ and let E be the conditional expectation induced by the coaction of G on $A \times_{X,\lambda} P$. Then $E(\mathcal{T}_\lambda(X)^+) = \pi(A)$ and

$$E(A \times_{X,\lambda} P) = E(\overline{\mathcal{T}_\lambda(X)^+ + (\mathcal{T}_\lambda(X)^+)^*}) = \pi(A).$$

Thus for each $p \in P$ we have that $t_p(X_p)t_p(X_p)^* \subseteq E(A \times_{X,\lambda} P) = \pi(A)$ as desired. □

The next corollary squares with the fact that Popescu's noncommutative disc algebra is not Dirichlet. Recall that for abelian coactions the C^*-envelope of a cosystem coincides with the usual C^*-envelope of the ambient operator algebra.

Corollary 7.14. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a saturated controlled map between weak right* LCM *inclusions and suppose that $(\mathcal{G}, \mathcal{P})$ is an abelian total order. Let X be an injective product system over P with coefficients in A. Then $\mathcal{T}_\lambda(X)^+$ is Dirichlet if and only if every strongly covariant representation (π, t) of X satisfies $t_p(X_p)t_q(X_q)^* \subseteq A$ whenever $\vartheta(p) = \vartheta(q)$.*

Proof. By Theorem 6.1, and since the controlling pair is abelian, the C^*-envelope of $\mathcal{T}_\lambda(X)^+$ is $A \times_X P$. For the injective X, let Y be the injective product system over $\mathcal{P}$ with coefficients in A constructed in Theorem 7.7. By construction we see that Y_h with $h \in \mathcal{P}$ is a Hilbert bimodule if and only if $t_p(X_p)t_q(X_q)^* \subseteq A$ for all $p, q \in \vartheta^{-1}(h)$. By applying Remark 2.8, Theorems 6.1 and 7.7, and Propositions 7.12 and 7.13 we have that the Fock tensor algebra $\mathcal{T}_\lambda(X)^+$ is Dirichlet in $A \times_X P$, if and only if $\mathcal{T}_\lambda(Y)^+$ is Dirichlet in $A \times_Y \mathcal{P}$, if and only if every Y_h with $h \in \mathcal{P}$ is a Hilbert bimodule, if and only if $t_p(X_p)t_q(X_q)^* \subseteq \pi(A)$ whenever $\vartheta(p) = \vartheta(q) = h$ for all $h \in \mathcal{P}$. □

The next theorem shows that, for weak right LCM inclusions that are controlled by total orders in a saturated way, reversible product systems produce all possible covariance algebras.

Theorem 7.15. *Let $\vartheta : (G, P) \to (\mathcal{G}, \mathcal{P})$ be a saturated controlled map between weak right* LCM *inclusions and suppose that $(\mathcal{G}, \mathcal{P})$ is a total order. Let X be an* (*injective*) *product system over P with coefficients in A. Then there exists an* (*injective*) *reversible product system Z over $\mathcal{P}$ with coefficients in a C^*-algebra B such that*

$$A \subseteq B \qquad \textit{and} \qquad X_p \subseteq Z_{\vartheta(p)} \quad \textit{for all} \;\; p \in P \tag{7-1}$$

that satisfies

$$A \times_X P \simeq B \times_Z \mathcal{P} \quad \textit{and} \quad A \times_{X,\lambda} P \simeq B \times_{Z,\lambda} \mathcal{P}, \tag{7-2}$$

by $*$-homomorphisms that preserve the inclusions $X_p \hookrightarrow Z_{\vartheta(p)}$ for all $p \in P$.

Proof. By Theorem 7.7 we can assume that $(G, P) = (\mathcal{G}, \mathcal{P})$. Let (π, t) be a faithful representation of $A \times_{X,\lambda} P$, and let

$$B := B_P = C^*(\{t_s(X_s)t_s(X_s)^* \mid s \in P\} \qquad \text{and} \qquad Z_p := [t_p(X_p)B] \quad \text{for all} \;\; p \in P \backslash \{e\}.$$

The trivial C^*-correspondence structure on $A \times_{X,\lambda} P$ descends to a C^*-correspondence structure on each Z_p over B. Notice here that since (G, P) is totally ordered we automatically have that the product system $Z = \{Z_p\}_{p \in P}$ is compactly aligned. Also $C^*(B, Z) = C^*(\pi, t)$ admits a coaction of G from $A \times_{X,\lambda} P$. Hence by Theorem 4.13 we have that

$$\overline{\operatorname{alg}}\{B, Z_p \mid p \in P\} \simeq \mathcal{T}_\lambda(Z)^+.$$

By construction

$$A \times_{X,\lambda} P = \overline{\mathcal{T}_\lambda(Z)^+ + (\mathcal{T}_\lambda(Z)^+)^*},$$

thus the cosystem of $\mathcal{T}_\lambda(Z)^+$ over G is Dirichlet in a C^*-cover. This gives at the same time that this C^*-cover $A \times_{X,\lambda} P$ is the C^*-envelope of the cosystem $\mathcal{T}_\lambda(Z)^+$ over G, and that Z is reversible by Proposition 7.13. Theorem 4.14 then concludes that

$$B \times_{Z,\lambda} P \simeq C^*_{\text{env}}(\mathcal{T}_\lambda(Z)^+, G, \bar{\delta}_G) \simeq A \times_{X,\lambda} P.$$

For the case of the universal C^*-algebras we proceed as in Theorem 7.7. That is, first we notice that the $*$-isomorphism between the reduced C^*-algebras implies an embedding of the Fell bundles

$$[A \times_X P]_{pq^{-1}} \simeq [A \times_{X,\lambda} P]_{pq^{-1}} \hookrightarrow [B \times_{Z,\lambda} \mathcal{P}]_{pq^{-1}} \simeq [B \times_Z P]_{pq^{-1}}$$

which lifts to a $*$-epimorphism $\Psi : A \times_X P \to B \times_Z P$. Since $X \subseteq Z$ we also have a $*$-epimorphism at the level of the Nica–Toeplitz–Pimsner algebras and thus the diagram

$$\begin{array}{ccc} \mathcal{NT}(X) & \xrightarrow{\Phi} & \mathcal{NT}(Z) \\ \downarrow q_X & & \downarrow q_Z \\ A \times_X P & \xrightarrow{\Psi} & B \times_Z P \end{array}$$

is commutative, and fixes X. Since the ideals of strong covariance relations are induced, it suffices to show that

$$\ker \Phi q_Z \cap [\mathcal{NT}(X)]_e \subseteq \ker q_X.$$

Equivalently it suffices to show that Ψ is faithful on the G-fixed point algebra defined on $A \times_X P$, which by definition is B. However this follows by the property of $A \times_X P$ as $\Psi|_A$ is by construction faithful.

It is left to show that injectivity of X implies injectivity of Z. By Theorem 7.7 we can still assume that $(G, P) = (\mathcal{G}, \mathcal{P})$. To this end let $p \in P$ and $f \in \ker \varphi_p^Z$. We need to show that $f = 0$.

As B_{pP} is an ideal in B we have that $B = B_{\{s<p\}} + B_{pP}$, and let $f_1 \in B_{\{s<p\}}$ and $f_2 \in B_{pP}$ be such that $f = f_1 + f_2$. Let (e_i) be a contractive approximate identity of $\psi_p(\mathcal{K}X_p)$ so that

$$0 = f e_i = f_1 e_i + f_2 e_i.$$

However (e_i) is also a contractive approximate identity for B_{pP} and so

$$\lim_i f_1 e_i = -\lim_\lambda f_2 e_i = f_2.$$

By Nica-covariance $f_1 e_i \in B_p$ for all i, and so we have that $f_2 \in B_p$. Thus we can assume without loss of generality that $f \in B_{\{s \leq p\}}$. As $B_{\{s\leq p\}}$ is the inductive limit of B_F for $F = \{p_1 < p_2 < \cdots < p_n = p\}$ we may assume that

$$f = \sum_{i=1}^n \psi_{p_i}(k_{p_i}) \quad \text{with} \quad k_{p_i} \in \mathcal{K}X_{p_i} \quad \text{and} \quad p_1 < p_2 < \cdots < p_n = p.$$

Recall the representation (π_F, t_F) on

$$X_F = \bigoplus_{r\in P} X_r I^X_{r^{-1}(r\vee F)},$$

and we will show that

$$\sum_{i=1}^n \psi_{F,p_i}(k_{p_i})|_{X_F} = 0.$$

As (π, t) is strongly covariant this will give that $f = 0$ by Proposition 4.9. For $r \geq p$ we have that $f \in \ker \varphi^Z_p \subseteq \ker \varphi^Z_r$, and for every $\eta_r \in X_r I^X_{r^{-1}(r\vee F)}$ we have that $t_r(\eta_r) \in t_r(X_r) \subseteq Z_r$. Hence

$$t_r\left(\sum_{i=1}^n i^r_{p_i}(k_{p_i})(\eta_r)\right) = \sum_{i=1}^n \psi_{p_i}(k_{p_i}) t_r(\eta_r) = f t_r(\eta_r) = 0.$$

As t is isometric we obtain

$$\sum_{i=1}^n \psi_{F,p_i}(k_{p_i})|_{X_r I^X_{r^{-1}(r\vee F)}} = \sum_{i=1}^n i^r_{p_i}(k_{p_i}) = 0, \quad \text{for } r \geq p. \tag{7-3}$$

On the other hand for $r < p$ we have that $r^{-1}p \neq e_G$, and so

$$I^X_{r^{-1}(r\vee F)} \subseteq I^X_{r^{-1}K_{\{r,p\}}} \subseteq \ker \varphi^X_{r^{-1}p} = (0).$$

Hence trivially

$$\sum_{i=1}^n \psi_{F,p_i}(k_{p_i})|_{X_r I^X_{r^{-1}(r\vee F)}} = 0, \quad \text{for } r < p. \tag{7-4}$$

By (7-3) and (7-4) we have that $\sum_{i=1}^n \psi_{p_i,F}(k_{p_i})|_{X_F} = 0$, and the proof is complete. □

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