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SIMPLICES IN THIN SUBSETS OF EUCLIDEAN SPACES

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Let Δ be a nondegenerate simplex on k vertices. We prove that there exists a threshold $s_k < k$ such that any set $A \subseteq \mathbb{R}^k$ of Hausdorff dimension $\dim A \geq s_k$ necessarily contains a similar copy of the simplex Δ.

1. Introduction

A classical problem of geometric Ramsey theory is to show that sufficiently large sets contain a given geometric configuration. The underlying settings can be Euclidean space, the integer lattice or vector spaces over finite fields. By a geometric configuration, we mean the collection of finite point sets obtained from a given finite set $F \subseteq \mathbb{R}^k$ via translations, rotations and dilations.

If the size is measured in terms of the positivity of the Lebesgue density, then it is known that large sets in $\mathbb{R}^k$ contain a translated and rotated copy of all sufficiently large dilates of any nondegenerate simplex Δ with k vertices [Bourgain 1986]. However, on the scale of the Hausdorff dimension $s < k$ this question is not very well understood. The only affirmative result in this direction was obtained by Iosevich and Liu [2019].

In the other direction, a construction due to Keleti [2008] shows that there exists a set $A \subseteq \mathbb{R}$ of full Hausdorff dimension which does not contain any nontrivial 3-term arithmetic progression. In two dimensions an example due to Falconer [2013] and Maga [2010] shows that there exists a set $A \subseteq \mathbb{R}^2$ of Hausdorff dimension 2 which does not contain the vertices of an equilateral triangle, or more generally a nontrivial similar copy of a given nondegenerate triangle. It seems plausible that examples of such sets exist in all dimensions, but this is not currently known. See [Fraser and Pramanik 2018] for related results.

The purpose of this paper is to show that measurable sets $A \subseteq \mathbb{R}^k$ of sufficiently large Hausdorff dimension $s < k$ contain a similar copy of any given nondegenerate k-simplex with bounded eccentricity. Our arguments make use of and have some similarity to those of Lyall and Magyar [2020]. We also extend our results to bounded degree distance graphs. For the special cases of a path (or chain) and, more generally, a tree, similar but somewhat stronger results were obtained in [Bennett et al. 2016] and [Iosevich and Taylor 2019].

2. Main results

Let $V = \{v_1, \ldots, v_k\} \subset \mathbb{R}^k$ be a nondegenerate k-simplex, a set of k vertices which are in *general position* spanning a $(k-1)$-dimensional affine subspace. For $1 \leq j \leq k$, let $r_j(V)$ be the distance of the vertex v_j

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to the affine subspace spanned by the remaining vertices $v_i,\ i \neq j$, and define $r(V) := \min_{1 \leq j \leq k} r_j(V)$. Let $d(V)$ denote the diameter of the simplex, which is also the maximum distance between two vertices. Then the quantity $\delta(V) := r(V)/d(V)$, which is positive if and only if V is nondegenerate, measures how close the simplex V is to being degenerate.

We say that a simplex V' is *similar* to V, if $V' = x + \lambda \cdot U(V)$ for some $x \in \mathbb{R}^k$, $\lambda > 0$ and $U \in \mathrm{SO}(k)$; that is if V' is obtained from V by a translation, dilation and rotation.

Theorem 1. *Let $k \in \mathbb{N}$ and $\delta > 0$. There exists $s_0 = s_0(k, \delta) < k$ such that if E is a compact subset of $\mathbb{R}^k$ of Hausdorff dimension* $\dim E \geq s_0$, *then E contains the vertices of a simplex V' similar to V, for any nondegenerate k-simplex V with $\delta(V) \geq \delta$.*

Remarks. (1) Note that the dimension condition is sharp for $k = 2$, as a construction due to Maga [2010] shows the existence of a set $E \subseteq \mathbb{R}^2$ with $\dim(E) = 2$ that does not contain any equilateral triangle or more generally a similar copy of any given triangle.

While we do not currently have an example showing that the dimension condition is sharp when $k > 2$, we have some indications that this should be the case. In the finite field setting, one can show that $\mathbb{F}_q^d$ (the d-dimensional vector space over the field with q elements) contains a d-dimensional equilateral simplex if and only if $(d+1)/2^d$ is a square in $\mathbb{F}_q$; see the appendix in [Bennett et al. 2014]. This allows one to construct an $\mathbb{F}_q^d$ that does not contain a d-dimensional equilateral simplex under a suitable arithmetic assumption on $\mathbb{F}_q$. While such an assumption is not meaningful in $\mathbb{R}^d$, the Fourier analytic methods used in this paper would likely to extend to the finite field setting. At the very least, this says that if the dimensional assumption in Theorem 1 is not sharp, a very different approach would be required to establish a positive result.

(2) It is also interesting to note that the proof of Theorem 1 above proves much more than just the existence of vertices of V' similar to V inside E. The proof proceeds by constructing a natural measure on the set of simplexes and proving an upper and a lower bound on this measure. This argument shows that an infinite "statistically" correct "amount" of simplexes V' exist that satisfy the conclusion of the theorem, shedding considerable light on the structure of sets of positive upper Lebesgue density.

(3) Theorem 1 establishes a nontrivial exponent $s_0 < k$, but the proof yields s_0 very close to k and not explicitly computable. The analogous results in the finite field setting (see e.g., [Hart and Iosevich 2008], [Iosevich and Parshall 2019]) suggest that it may be possible to obtain explicit exponents, but this would require a fundamentally different approach to certain lower bounds obtained in the proof of Theorem 1.

A distance graph is a connected finite graph embedded in Euclidean space, with a set of vertices $V = \{v_0, v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$ and a set of edges $E \subseteq \{(i, j) : 0 \leq i < j \leq n\}$. We say that a graph $\Gamma = (V, E)$ has degree at most k if $|V_j| \leq k$ for all $1 \leq j \leq n$, where $V_j = \{v_i : (i, j) \in E\}$. The graph Γ is called *proper* if the sets $V_j \cup \{v_j\}$ for all j are in general position, in the sense that $V_j \cup \{v_j\}$ is not contained in a subspace of dimension smaller than $|V_j| - 1$. Let $r(\Gamma)$ be the minimum of the distances from the vertices v_j to the corresponding affine subspace spanned by the sets V_j, and note that $r(\Gamma) > 0$ if Γ is proper. Let $d(\Gamma)$ denote the length of the longest edge of Γ, and let $\delta(\Gamma) := r(\Gamma)/d(\Gamma)$.

We say that a distance graph $\Gamma' = (V', E)$ is *isometric* to Γ and write $\Gamma' \simeq \Gamma$, if there is a one-to-one and onto mapping $\phi : V \to V'$ so that $|\phi(v_i) - \phi(v_j)| = |v_i - v_j|$ for all $(i, j) \in E$. One may picture Γ' obtained from Γ by a translation followed by rotating the edges around the vertices, if possible. By $\lambda \cdot \Gamma$ we mean the dilate of the distance graph Γ by a factor $\lambda > 0$, and we say that Γ' is *similar* to Γ if Γ' is isometric to $\lambda \cdot \Gamma$.

Theorem 2. *Let $\delta > 0$, $n \geq 1$, $1 \leq k < d$, and let E be a compact subset of $\mathbb{R}^k$ of Hausdorff dimension $s < d$. There exists $s_0 = s_0(n, d, \delta) < d$ such that if $s \geq s_0$, then E contains a distance graph Γ' similar to Γ, for any proper distance graph $\Gamma = (V, E)$ of degree at most k, with $V \subseteq \mathbb{R}^d$, $|V| = n$ and $\delta(\Gamma) \geq \delta$.*

Note that Theorem 2 implies Theorem 1, as a nondegenerate simplex is a proper distance graph of degree $k - 1$.

3. Proof of Theorem 1

Let $E \subseteq B(0, 1)$ be a compact subset of the unit ball $B(0, 1)$ in $\mathbb{R}^k$ of Hausdorff dimension $s < k$. It is well known that there is a probability measure μ supported on E such that $\mu(B(x, r)) \leq C_\mu r^s$ for all balls $B(x, r)$. The following observation shows that we may take $C_\mu = 4$ for our purposes.[1]

Lemma 1. *There exists a set $E' \subseteq B(0, 1)$ of the form $E' = \rho^{-1}(F - u)$ for some $\rho > 0$, $u \in \mathbb{R}^k$ and $F \subseteq E$, and a probability measure μ' supported on E' which satisfies*

$$\mu'(B(x, r)) \leq 4r^s, \quad \textit{for all } x \in \mathbb{R}^k,\ r > 0. \tag{3-1}$$

Proof. Let $K := \inf(S)$, where

$$S := \{C \in \mathbb{R} : \mu(B(x, r)) \leq Cr^s,\ \forall B(x, r)\}.$$

By Frostman's lemma [Mattila 1995], we have that $S \neq \varnothing$ and $K > 0$, moreover,

$$\mu(B(x, r)) \leq 2Kr^s,$$

for all balls $B(x, r)$. There exists a ball $Q = B(v, \rho)$ of radius ρ such that $\mu(Q) \geq \frac{1}{2} K \rho^s$. We translate E so Q is centered at the origin, set $F = E \cap Q$ and denote by μ_F the induced probability measure on F:

$$\mu_F(A) = \frac{\mu(A \cap F)}{\mu(F)}.$$

Note that for all balls $B = B(x, r)$,

$$\mu_F(B) \leq \frac{2Kr^s}{(1/2)K\rho^s} = 4\Big(\frac{r}{\rho}\Big)^s.$$

Finally, we define the probability measure μ' as $\mu'(A) := \mu_F(\rho A)$. It is supported on $E' = \rho^{-1} F \subseteq B(0, 1)$ and satisfies

$$\mu'(B(x, r)) = \mu_F(B(\rho x, \rho r)) \leq 4r^s. \qquad \square$$

[1] We would like to thank Giorgis Petridis for bringing this observation to our attention.

Clearly E contains a similar copy of V if the same holds for E', thus one can pass from E to E' in proving our main results, assuming that (3-1) holds. Given $\varepsilon > 0$, let $\psi_\varepsilon(x) = \varepsilon^{-k}\psi(x/\varepsilon) \geq 0$, where $\psi \geq 0$ is a Schwarz function whose Fourier transform, $\hat{\psi}$, is a compactly supported smooth function satisfying $\hat{\psi}(0) = 1$ and $0 \leq \hat{\psi} \leq 1$.

We define $\mu_\varepsilon := \mu * \psi_\varepsilon$. Note that μ_ε is a continuous function satisfying $\|\mu_\varepsilon\|_\infty \leq C\varepsilon^{s-k}$ with an absolute constant $C = C_\psi > 0$, by Lemma 1.

Let $V = \{v_0 = 0, \ldots, v_{k-1}\}$ be a given nondegenerate simplex and note that in proving Theorem 1 we may assume that $d(V) = 1$, and hence $\delta(V) = r(V)$. A simplex $V' = \{x_0 = 0, x_1, \ldots, x_{k-1}\}$ is isometric to V if for every $1 \leq j \leq k$ one has that $x_j \in S_{x_1,\ldots,x_{j-1}}$, where

$$S_{x_1,\ldots,x_{j-1}} = \{y \in \mathbb{R}^k : |y - x_i| = |v_j - v_i|,\ 0 \leq i < j\}$$

is a sphere of dimension $k - j$ and of radius $r_j = r_j(V) \geq r(V) > 0$. Let $\sigma_{x_1,\ldots,x_{j-1}}$ denote its normalized surface area measure.

Given $0 < \lambda$ and $\varepsilon \leq 1$, define the multilinear expression

$$T_{\lambda V}(\mu_\varepsilon) := \int \mu_\varepsilon(x)\mu_\varepsilon(x - \lambda x_1) \cdots \mu_\varepsilon(x - \lambda x_{k-1})\, d\sigma(x_1)\, d\sigma_{x_1}(x_2) \cdots d\sigma_{x_1,\ldots,x_{k-2}}(x_{k-1})\, dx, \tag{3-2}$$

which may be viewed as a weighted count of the isometric copies of $\lambda\Delta$.

3.1. ***Upper bounds.*** A crucial part of our approach is to show that the averages $T_{\lambda V}(\mu_\varepsilon)$ have a limit as $\varepsilon \to 0$, for which one needs the following upper bound.

Lemma 2. *There exists a constant $C_k > 0$, depending only on k, such that*

$$|T_{\lambda V}(\mu_{2\varepsilon}) - T_{\lambda V}(\mu_\varepsilon)| \leq C_k r(V)^{-1/2}\lambda^{-1/2}\varepsilon^{(k-1/2)(s-k)+1/4}. \tag{3-3}$$

As an immediate corollary we have the following:

Lemma 3. *Let $k - \frac{1}{4k} \leq s < k$. There exists*

$$T_{\lambda V}(\mu) := \lim_{\varepsilon \to 0} T_{\lambda V}(\mu_\varepsilon), \tag{3-4}$$

and moreover,

$$|T_{\lambda V}(\mu) - T_{\lambda V}(\mu_\varepsilon)| \leq C_k r(V)^{-1/2}\lambda^{-1/2}\varepsilon^{(k-1/2)(s-k)+1/4}. \tag{3-5}$$

Indeed, the left side of (3-5) can be written as a telescopic sum:

$$\sum_{j \geq 0} T_{\lambda V}(\mu_{2\varepsilon_j}) - T_{\lambda V}(\mu_{\varepsilon_j}), \quad \text{with } \varepsilon_j = 2^{-j}\varepsilon.$$

Proof of Lemma 2. Write $\Delta\mu_\varepsilon := \mu_{2\varepsilon} - \mu_\varepsilon$. Then

$$\prod_{j=1}^{k-1} \mu_{2\varepsilon}(x - \lambda x_j) - \prod_{j=1}^{k-1} \mu_\varepsilon(x - \lambda x_j) = \sum_{j=1}^{k} \Delta_j(\mu_\varepsilon),$$

where

$$\Delta_j(\mu_\varepsilon) = \prod_{i \neq j} \mu_{\varepsilon_{ij}}(x - \lambda x_i)\Delta\mu_\varepsilon(x - \lambda x_j), \tag{3-6}$$

and where $\varepsilon_{ij} = 2\varepsilon$ for $i < j$ and $\varepsilon_{ij} = \varepsilon$ for $i > j$. Since the arguments below are the same for all $1 \le j \le k-1$, assume $j = k-1$ for simplicity of notations. Writing $f *_\lambda g(x) := \int f(x-\lambda y)g(y)\,dy$, and using $\|\mu_\varepsilon\|_\infty \le C\varepsilon^{s-k}$, we have for $\Delta T(\mu_\varepsilon) := T_{\lambda V}(\mu_\varepsilon) - T_{\lambda V}(\mu_{2\varepsilon})$ that

$$|\Delta T(\mu_\varepsilon)| \lesssim \varepsilon^{(k-2)(s-d)} \int \left| \int \mu_\varepsilon(x)\Delta\mu_\varepsilon *_\lambda \sigma_{x_1,\dots,x_{k-2}}(x)\,dx \right| d\omega(x_1,\dots,x_{k-2}), \tag{3-7}$$

where $d\omega(x_1,\dots,x_{k-2}) = d\sigma(x_1)\cdots d\sigma_{x_1,\dots,x_{k-3}}(x_{k-2})$ for $k > 3$, and where for $k = 3$ we have that $d\omega(x_1) = d\sigma(x_1)$, which is the normalized surface area measure on the sphere $S = \{y : |y| = |v_1|\}$.

The inner integral is of the form

$$|\langle \mu_\varepsilon, \Delta\mu_\varepsilon *_\lambda \sigma_{x_1,\dots,x_{k-2}}\rangle| \lesssim \varepsilon^{s-d}\|\Delta\mu_\varepsilon *_\lambda \sigma_{x_1,\dots,x_{k-2}}\|_2.$$

Thus by Cauchy–Schwarz and Plancherel's identity,

$$|\Delta_{k-1}T(\mu_\varepsilon)|^2 \lesssim \varepsilon^{2(k-1)(s-d)} \int |\widehat{\Delta\mu_\varepsilon}(\xi)|^2 I_\lambda(\xi)\,d\xi,$$

where

$$I_\lambda(\xi) = \int |\hat\sigma_{x_1,\dots,x_{k-2}}(\lambda\xi)|^2\,d\omega(x_1,\dots,x_{k-2}).$$

Since $S_{x_1,\dots,x_{k-2}}$ is a one-dimensional circle of radius $r_{k-1} \ge r(V) > 0$ contained in an affine subspace orthogonal to $M_{x_1,\dots,x_{k-2}} = \operatorname{span}\{x_1,\dots,x_{k-2}\}$, we have that

$$|\hat\sigma_{x_1,\dots,x_{k-2}}(\lambda\xi)|^2 \lesssim (1 + r(V)\lambda\operatorname{dist}(\xi, M_{x_1,\dots,x_{k-2}}))^{-1}.$$

Since the measure $\omega(x_1,\dots,x_{k-2})$ is invariant with respect to the change of variables $(x_1,\dots,x_{k-2}) \to (Ux_1,\dots,Ux_{k-2})$ for any rotation $U \in \mathrm{SO}(k)$, one estimates

$$\begin{aligned}
I_\lambda(\xi) &\lesssim \iint (1 + r(V)\lambda\operatorname{dist}(\xi, M_{Ux_1,\dots,Ux_{k-2}}))^{-1}\,d\omega(x_1,\dots,x_{k-2})\,dU \\
&= \iint (1 + r(V)\lambda\operatorname{dist}(U\xi, M_{x_1,\dots,x_{k-2}}))^{-1}\,d\omega(x_1,\dots,x_{k-2})\,dU \\
&= \iint (1 + r(V)\lambda|\xi|\operatorname{dist}(\eta, M_{x_1,\dots,x_{k-2}}))^{-1}\,d\omega(x_1,\dots,x_{k-2})\,d\sigma_{k-2}(\eta) \lesssim (1 + r(V)\lambda|\xi|)^{-1},
\end{aligned}$$

where we have written $\eta := |\xi|^{-1}U\xi$ and σ_{k-1} denotes the surface area measure on the unit sphere $S^{k-1} \subseteq \mathbb{R}^k$.

Note that $\widehat{\Delta\mu_\varepsilon}(\xi) = \hat\mu(\xi)(\hat\psi(2\varepsilon\xi) - \hat\psi(\varepsilon\xi))$, which is supported on $|\xi| \lesssim \varepsilon^{-1}$ and is essentially supported on $|\xi| \approx \varepsilon^{-1}$. Indeed, writing

$$J := \int |\widehat{\Delta\mu_\varepsilon}(\xi)|^2 I_\lambda(\xi)\,d\xi = \int_{|\xi|\le\varepsilon^{-1/2}} |\widehat{\Delta\mu_\varepsilon}(\xi)|^2 I_\lambda(\xi)\,d\xi + \int_{\varepsilon^{-1/2}\le|\xi|\lesssim\varepsilon^{-1}} |\widehat{\Delta\mu_\varepsilon}(\xi)|^2 I_\lambda(\xi)\,d\xi =: J_1 + J_2$$

and using $|\hat\psi(2\varepsilon\xi) - \hat\psi(\varepsilon\xi)| \lesssim \varepsilon^{1/2}$ for $|\xi| \le \varepsilon^{-1/2}$, we estimate

$$J_1 \lesssim \varepsilon^1/2 \int |\hat\mu(\xi)|^2(\hat\psi(2\varepsilon\xi) + \hat\psi(\varepsilon\xi))\,d\xi \lesssim \varepsilon^{1/2+s-k},$$

as

$$\int |\hat\mu(\xi)|^2\hat\psi(\varepsilon\xi)\,d\xi = \int \mu_\varepsilon(x)\,d\mu(x) \lesssim \varepsilon^{s-k}.$$

On the other hand, as $I_\lambda(\xi) \lesssim \varepsilon^{1/2} r(V)^{-1}\lambda^{-1}$ for $|\xi| \geq \varepsilon^{-1/2}$, we have

$$J_2 \lesssim \varepsilon^{1/2} r(V)^{-1}\lambda^{-1} \int |\hat{\mu}(\xi)|^2 \hat{\phi}(\varepsilon\xi)\, d\xi \lesssim r(V)^{-1}\lambda^{-1}\varepsilon^{1/2+s-k},$$

where we have written $\hat{\phi}(\xi) = (\hat{\psi}(2\xi) - \hat{\psi}(\xi))^2$. Plugging these estimates into (3-7), we obtain

$$|\Delta T(\mu_\varepsilon)|^2 \lesssim r(V)^{-1}\lambda^{-1}\varepsilon^{1/2+(2k-1)(s-d)},$$

and (3-5) follows. □

The support of μ_ε is not compact, however, as it is a rapidly decreasing function, it can be made to be supported in a small neighborhood of the support of μ without changing our main estimates. Let $\phi_\varepsilon(x) := \phi(c\varepsilon^{-1/2}x)$ with some small absolute constant $c > 0$, where $0 \leq \phi(x) \leq 1$ is a smooth cut-off, which equals one for $|x| \leq \frac{1}{2}$ and is zero for $|x| \geq 2$. Define $\tilde{\psi}_\varepsilon = \psi_\varepsilon \phi_\varepsilon$ and $\tilde{\mu}_\varepsilon = \mu * \tilde{\psi}_\varepsilon$. It is easy to see that $\tilde{\mu}_\varepsilon \leq \mu_\varepsilon$ and $\int \tilde{\mu}_\varepsilon \geq \frac{1}{2}$, if $c > 0$ is chosen sufficiently small. Using the trivial upper bound, for $k - 1/(4k) \leq s < k$ we have

$$|T_{\lambda V}(\mu_\varepsilon) - T_{\lambda V}(\tilde{\mu}_\varepsilon)| \leq C_k \|\mu_\varepsilon\|_\infty^{k-1} \|\mu_\varepsilon - \tilde{\mu}_\varepsilon\|_\infty \leq C_k \varepsilon^{1/2},$$

and it follows that estimate (3-5) remains true with μ_ε replaced with $\tilde{\mu}_\varepsilon$.

3.2. *Lower bounds.* Let $f_\varepsilon := c\varepsilon^{k-s}\tilde{\mu}_\varepsilon$, where $c = c_\psi > 0$ is a constant such that $0 \leq f_\varepsilon \leq 1$ and $\int f_\varepsilon\, dx = c'\varepsilon^{k-s}$. Let $\alpha := c'\varepsilon^{k-s}$ and note that the set $A_\varepsilon := \{x : f_\varepsilon(x) \geq \frac{1}{2}\alpha\}$ has measure $|A_\varepsilon| \geq \frac{1}{2}\alpha$. If one defines the averages

$$T_{\lambda V}(A_\varepsilon) = \int \mathbf{1}_{A_\varepsilon}(x)\mathbf{1}_{A_\varepsilon}(x - \lambda x_1) \cdots \mathbf{1}_{A_\varepsilon}(x - \lambda x_{k-1})\, d\sigma(x_1) \cdots d\sigma_{x_1,\ldots,x_{k-2}}(x_{k-1})\, dx,$$

then clearly

$$T_{\lambda V}(\tilde{\mu}_\varepsilon) \geq c\alpha^k T_{\lambda V}(A_\varepsilon).$$

The averages $T_{\lambda V}(A_\varepsilon)$ represent the density of isometric copies of the simplex $\lambda\Delta$ in a set A_ε of measure $|A_\varepsilon| \geq \frac{\alpha}{2} > 0$, which was studied in [Lyall and Magyar 2020] in the more general context of k-degenerate distance graphs. We recall one of the main results of the aforementioned paper; see Theorem 2 (ii) together with Estimate (18):

Theorem 3 [Lyall and Magyar 2020]. *Let $A \subseteq [0, 1]^k$ and $|A| \geq \alpha > 0$. Then there exists an interval I of length $|I| \geq \exp(-C\alpha^{-C_k})$, such that for all $\lambda \in I$, one has*

$$|T_{\lambda V}(A))| \geq c\alpha^k.$$

Thus for all $\lambda \in I$,

$$T_{\lambda V}(\tilde{\mu}_\varepsilon) \geq c > 0 \tag{3-8}$$

for a constant $c = c(k, \psi, r(V)) > 0$. Now, let

$$T_V(\tilde{\mu}_\varepsilon) := \int_0^1 \lambda^{1/2} T_{\lambda V}(\tilde{\mu}_\varepsilon)\, d\lambda.$$

For $k - \frac{1}{4k} \le s < k$, by (3-5) we have that

$$|T_{\lambda V}(\mu) - T_{\lambda V}(\tilde{\mu}_\varepsilon)| \le C_k r(V)^{-1/2} \lambda^{-1/2} \varepsilon^{1/8},$$

it follows that

$$\int_0^1 \lambda^{1/2} |T_{\lambda V}(\mu) - T_{\lambda V}(\tilde{\mu}_\varepsilon)| \, d\lambda \le C_k r(V)^{-1/2} \varepsilon^{1/8}, \tag{3-9}$$

and in particular $\int_0^1 \lambda^{1/2} T_{\lambda V}(\mu) \, d\lambda < \infty$. On the other hand, by (3-8), one has that

$$\int_0^1 \lambda^{1/2} T_{\lambda V}(\tilde{\mu}_\varepsilon) \, d\lambda \ge \exp(-\varepsilon^{-C_k(k-s)}). \tag{3-10}$$

Assume that $r(V) \ge \delta$, fix a small $\varepsilon = \varepsilon_{k,\delta} > 0$ and then choose $s = s(\varepsilon, \delta) < k$ such that

$$C_k \delta^{-1/2} \varepsilon^{1/8} < \tfrac{1}{2} \exp(-\varepsilon^{-C_k(k-s)}),$$

which ensures that

$$\int_0^1 \lambda^{1/2} T_{\lambda V}(\mu) \, d\lambda > 0.$$

Thus there exists $\lambda > 0$ such that $T_{\lambda V}(\mu) > 0$. Fix such a λ, and assume indirectly that $E^k = E \times \cdots \times E$ does not contain any simplex isometric to λV, i.e., any point of the compact configuration space $S_{\lambda V} \subseteq \mathbb{R}^{k^2}$ of such simplexes. By compactness, this implies that there is some $\eta > 0$ such that the η-neighborhood of E^k also does not contain any simplex isometric to λV. Since the support of $\tilde{\mu}_\varepsilon$ is contained in the $C_k \varepsilon^{1/2}$-neighborhood of E, as $E = \operatorname{supp} \mu$, it follows that $T_{\lambda V}(\tilde{\mu}_\varepsilon) = 0$ for all $\varepsilon < c_k \eta^2$ and hence $T_{\lambda V}(\mu) = 0$, contradicting our choice of λ. This proves Theorem 1.

4. The configuration space of isometric distance graphs

Let $\Gamma_0 = (V_0, E)$ be a fixed proper distance graph, with vertex set $V_0 = \{v_0 = 0, v_1, \dots, v_n\} \subseteq \mathbb{R}^d$ of degree $k < d$. Let $t_{ij} = |v_i - v_j|^2$ for $(i, j) \in E$. A distance graph $\Gamma = (V, E)$ with $V = \{x_0 = 0, x_1, \dots, x_n\}$ is isometric to Γ_0 if and only if $\boldsymbol{x} = (x_1, \dots, x_n) \in S_{\Gamma_0}$, where

$$S_{\Gamma_0} = \{(x_1, \dots, x_n) \in \mathbb{R}^{dn} : |x_i - x_j|^2 = t_{ij}, \ \forall 0 \le i < j \le n, \ (i, j) \in E\}.$$

We call the algebraic set S_{Γ_0} the *configuration space* of isometric copies of Γ_0. Note that S_{Γ_0} is the zero set of the family $\mathcal{F} = \{f_{ij} : (i, j) \in E\}$ with $f_{ij}(\boldsymbol{x}) = |x_i - x_j|^2 - t_{ij}$, thus it is a special case of the general situation described in Section 5.

If $\Gamma \simeq \Gamma_0$ with vertex set $V = \{x_0 = 0, x_1, \dots, x_n\}$ is proper, then $\boldsymbol{x} = (x_1, \dots, x_n)$ is a nonsingular point of S_{Γ_0}. Indeed, for a fixed $1 \le j \le n$, let Γ_j be the distance graph obtained from Γ by removing the vertex x_j together with all edges emanating from it. By induction we may assume that $\boldsymbol{x}' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ is a nonsingular point, i.e., the gradient vectors $\nabla_{\boldsymbol{x}'} f_{ik}(\boldsymbol{x}), \ (i, k) \in E$, $i \neq j, \ k \neq j$, are linearly independent. Since Γ is proper, the gradient vectors $\nabla_{x_j} f_{ij}(\boldsymbol{x}) = 2(x_i - x_j)$, $(i, j) \in E$, are also linearly independent, hence $\boldsymbol{x}$ is a nonsingular point. In fact we have shown that the partition of coordinates $\boldsymbol{x} = (y, z)$ with $y = x_j$ and $z = \boldsymbol{x}'$ is admissible and hence (6-4) holds.

Let $r_0 = r(\Gamma_0) > 0$. It is clear that if $\Gamma \simeq \Gamma_0$ and $|x_j - v_j| \le \eta_0$ for all $1 \le j \le n$, for a sufficiently small $\eta_0 = \eta(r_0) > 0$, then Γ is proper and $r(\Gamma) \ge \frac{1}{2} r_0$. For a given $1 \le j \le n$, let $X_j := \{x_i \in V : (i, j) \in E\}$ and define

$$S_{X_j} := \{x \in \mathbb{R}^d : |x - x_i|^2 = t_{ij},\ \forall x_i \in X_j\}.$$

As explained in Section 6, S_{X_j} is a sphere of dimension $d - |X_j| \ge 1$ with radius $r(X_j) \ge \frac{1}{2} r_0$. Let σ_{X_j} denote the surface area measure on S_{X_j} and write $\nu_{X_j} := \phi_j \sigma_{X_j}$, where ϕ_j is a smooth cut-off function supported in an η-neighborhood of v_j with $\phi_j(v_j) = 1$.

Write $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\phi(\boldsymbol{x}) := \prod_{j=1}^n \phi_j(x_j)$. Then by (6-4) and (6-5) one has

$$\int g(\boldsymbol{x})\phi(\boldsymbol{x})\, d\omega_{\mathcal{F}}(\boldsymbol{x}) = c_j(\Gamma_0) \iint g(\boldsymbol{x})\phi(\boldsymbol{x}')\, d\nu_{X_j}(x_j)\, d\omega_{\mathcal{F}_j}(\boldsymbol{x}'), \tag{4-1}$$

where $\boldsymbol{x}' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ and $\mathcal{F}_j = \{f_{il} : (i, l) \in E,\ l \ne j\}$. The constant $c_j(\Gamma_0) > 0$ is the reciprocal of the volume of the parallelotope with sides $x_j - x_i,\ (i, j) \in E$, which is easily shown to be at least $c_k r_0^k$, as the distance of each vertex to the opposite face is at least $\frac{1}{2} r_0$ on the support of ϕ.

5. Proof of Theorem 2

Let $d > k$ and again, without loss of generality, assume that $d(\Gamma) = 1$ and hence $\delta(\Gamma) = r(\Gamma)$. Given $\lambda, \varepsilon > 0$, define the multilinear expression

$$T_{\lambda\Gamma_0}(\mu_\varepsilon) := \int \cdots \int \mu_\varepsilon(x)\mu_\varepsilon(x - \lambda x_1) \cdots \mu_\varepsilon(x - \lambda x_n)\phi(x_1, \dots, x_n)\, d\omega_{\mathcal{F}}(x_1, \dots, x_n)\, dx. \tag{5-1}$$

Given a proper distance graph $\Gamma_0 = (V, E)$ on $|V| = n$ vertices of degree $k < n$, one has the following upper bound.

Lemma 4. *There exists a constant $C = C_{n,d,k}(r_0) > 0$ such that*

$$|T_{\lambda\Gamma_0}(\mu_{2\varepsilon}) - T_{\lambda\Gamma_0}(\mu_\varepsilon)| \le C\lambda^{-1/2}\varepsilon^{(n+1/2)(s-d)+1/4}. \tag{5-2}$$

This implies again that in dimensions $d - 1/(4n+2) \le s \le d$, the limit $T_{\lambda\Gamma_0}(\mu) := \lim_{\varepsilon \to 0} T_{\lambda\Gamma_0}(\mu_\varepsilon)$ exists. Also, the lower bound (3-8) holds for distance graphs of degree k, as was shown for a large class of graphs, the so-called k-degenerate distance graphs; see [Lyall and Magyar 2020]. Thus one may argue exactly as in Section 3 to prove that there exists a $\lambda > 0$ for which

$$T_{\lambda\Gamma_0}(\mu) > 0, \tag{5-3}$$

and Theorem 2 follows from the compactness of the configuration space $S_{\lambda\Gamma_0} \subseteq \mathbb{R}^{dn}$. It remains to prove Lemma 4.

Proof of Lemma 4. Write $\Delta T(\mu_\varepsilon) := T_{\lambda\Gamma_0}(\mu_\varepsilon) - T_{\lambda\Gamma_0}(\mu_{2\varepsilon})$. Then we have $\Delta T(\mu_\varepsilon) = \sum_{j=1} \Delta_j T(\mu_\varepsilon)$, where $\Delta_j T(\mu_\varepsilon)$ is given by (5-1) with $\mu_\varepsilon(x - \lambda x_j)$ replaced by $\Delta\mu_\varepsilon(x - \lambda x_j)$ given in (3-6), and $\mu_\varepsilon(x - \lambda x_i)$ by $\mu_{2\varepsilon}(x - \lambda x_j)$ for $i > j$. Then by (4-1) we have the analogue of estimate (3-7):

$$|\Delta T(\mu_\varepsilon)| \lesssim \varepsilon^{(n-1)(s-d)} \int \left| \iint \mu_\varepsilon(x)\Delta\mu_\varepsilon *_\lambda \nu_{X_j}(x)\, dx \right| \phi(\boldsymbol{x}')\, d\omega_{\mathcal{F}_j}(\boldsymbol{x}'), \tag{5-4}$$

where $\phi(\boldsymbol{x}') = \prod_{i \neq j} \phi(x_j)$. Thus by Cauchy–Schwarz and Plancherel's identity,

$$|\Delta_j T^{\varepsilon}(\mu)|^2 \lesssim \varepsilon^{2n(s-d)} \int |\widehat{\Delta_{\varepsilon}\mu}(\xi)|^2 I_{\lambda}^{j}(\xi)\, d\xi,$$

where

$$I_{\lambda}^{j}(\xi) = \int |\hat{\nu}_{X_j}(\lambda\xi)|^2 \phi(\boldsymbol{x}')\, d\omega_{\mathcal{F}_j}(\boldsymbol{x}').$$

Recall that on the support of $\phi(\boldsymbol{x}')$ we have that S_{X_j} is a sphere of dimension at least 1 and of radius $r \geq \frac{1}{2} r_0 > 0$, contained in an affine subspace orthogonal to span X_j. Thus,

$$|\hat{\nu}_{X_j}(\lambda\xi)|^2 \lesssim (1 + r_0\lambda \operatorname{dist}(\xi, \operatorname{span} X_j))^{-1}.$$

Let $U : \mathbb{R}^d \to \mathbb{R}^d$ be a rotation, and for $\boldsymbol{x}' = (x_i)_{i \neq j}$ write $U\boldsymbol{x}' = (Ux_i)_{i \neq j}$. As explained in Section 6, the measure $\omega_{\mathcal{F}_j}$ is invariant under the transformation $\boldsymbol{x}' \to U\boldsymbol{x}'$, hence

$$\begin{aligned} I_{\lambda}(\xi) &\lesssim \iint (1 + r_0\lambda \operatorname{dist}(\xi, \operatorname{span} UX_j))^{-1}\, d\omega_{\mathcal{F}_j}(\boldsymbol{x}')\, dU \\ &= \iint (1 + r_0\lambda|\xi| \operatorname{dist}(\eta, \operatorname{span} X_j))^{-1}\, d\sigma_{d-1}(\eta)\, d\omega\mathcal{F}_j(\boldsymbol{x}') \lesssim (1 + r_0\lambda|\xi|)^{-1}, \end{aligned}$$

where we have written again $\eta := |\xi|^{-1} U\xi \in S^{d-1}$.

Then we argue as in Lemma 2, noting that as $\widehat{\Delta\mu_{\varepsilon}}(\xi)$ is essentially supported on $|\xi| \approx \varepsilon^{-1}$, we have that

$$|\Delta T(\mu_{\varepsilon})|^2 \lesssim r_0^{-1}\lambda^{-1}\varepsilon^{2n(s-d)+1/2} \int |\hat{\mu}(\xi)|^2 \hat{\phi}(\varepsilon\xi)\, d\xi \lesssim r_0^{-1}\lambda^{-1}\varepsilon^{(2n+1)(s-d)+1/2},$$

with $\tilde{\mu}_{\varepsilon} = \mu_{\varepsilon}$ or $\tilde{\mu}_{\varepsilon} = \mu * \phi_{\varepsilon}$. This proves Lemma 4. □

6. Measures on real algebraic sets

Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a family of polynomials $f_i : \mathbb{R}^d \to \mathbb{R}$. We will describe certain measures supported on the algebraic set

$$S_{\mathcal{F}} := \{x \in \mathbb{R}^d : f_1(x) = \cdots = f_n(x) = 0\}. \tag{6-1}$$

A point $x \in S_{\mathcal{F}}$ is called *nonsingular* if the gradient vectors

$$\nabla f_1(x), \ldots, \nabla f_n(x)$$

are linearly independent. Let $S_{\mathcal{F}}^0$ denote the set of nonsingular points. It is well known that if $S_{\mathcal{F}}^0 \neq \varnothing$, then it is a relative open, dense subset of $S_{\mathcal{F}}$, and moreover it is an $(d-n)$-dimensional submanifold of $\mathbb{R}^d$. If $x \in S_{\mathcal{F}}^0$, then there exists a set of coordinates $J = \{j_1, \ldots, j_n\}$, with $1 \leq j_1 < \cdots < j_n \leq d$, such that

$$j_{\mathcal{F},J}(x) := \det\Big(\frac{\partial f_i}{\partial x_j}(x)\Big)_{1 \leq i \leq n,\, j \in J} \neq 0. \tag{6-2}$$

Accordingly, we will call a set of coordinates J *admissible* if (6-2) holds for at least one point $x \in S^0_{\mathcal{F}}$ and will denote by $S_{\mathcal{F},J}$ the set of such points. For a given set of coordinates x_J let $\nabla_{x_J} f(x) := (\partial_{x_j} f(x))_{j\in J}$ and note that J is admissible if and only if the gradient vectors

$$\nabla_{x_J} f_1(x), \dots, \nabla_{x_J} f_n(x)$$

are linearly independent for at least one point $x \in S_{\mathcal{F}}$. It is clear that, unless $S_{\mathcal{F},J} = \varnothing$, it is a relative open and dense subset of $S_{\mathcal{F}}$ and is also a $(d-n)$-dimensional submanifold, moreover $S^0_{\mathcal{F}}$ is the union of the sets $S_{\mathcal{F},J}$ for all admissible J.

We define a measure, near a point $x_0 \in S_{\mathcal{F},J}$, as follows. For simplicity of notation assume that $J = \{1, \dots, n\}$ and let

$$\Phi(x) := (f_1, \dots, f_n, x_{n+1}, \dots, x_d).$$

Then $\Phi : U \to V$ is a diffeomorphism on some open set $x_0 \in U \subseteq \mathbb{R}^d$ to its image $V = \Phi(U)$, moreover $S_{\mathcal{F}} = \Phi^{-1}(V \cap \mathbb{R}^{d-n})$. Indeed, $x \in S_{\mathcal{F}} \cap U$ if and only if $\Phi(x) = (0, \dots, 0, x_{n+1}, \dots, x_d) \in V$. Let $I = \{n+1, \dots, d\}$ and write $x_I := (x_{n+1}, \dots, x_d)$. Let $\Psi(x_I) = \Phi^{-1}(0, x_I)$ and in local coordinates let x_I define the measure $\omega_{\mathcal{F}}$ via

$$\int g\, d\omega_{\mathcal{F}} := \int g(\Psi(x_I))\, \mathrm{Jac}_{\Phi}^{-1}(\Psi(x_I))\, dx_I, \tag{6-3}$$

for a continuous function g supported on U. Note that $\mathrm{Jac}_\Phi(x) = j_{\mathcal{F},J}(x)$, i.e., the Jacobian of the mapping Φ at $x \in U$ is equal to the expression given in (6-2), and that the measure $d\omega_{\mathcal{F}}$ is supported on $S_{\mathcal{F}}$. Define the local coordinates $y_j = f_j(x)$ for $1 \le j \le n$ and $y_j = x_j$ for $n < j \le d$. Then

$$dy_1 \wedge \cdots \wedge dy_d = df_1 \wedge \cdots \wedge df_n \wedge dx_{n+1} \wedge \cdots \wedge dx_d = \mathrm{Jac}_\Phi(x)\, dx_1 \wedge \cdots \wedge dx_d,$$

and thus

$$dx_1 \wedge \cdots \wedge dx_d = \mathrm{Jac}_\Phi(x)^{-1} df_1 \wedge \cdots \wedge df_n \wedge dx_{n+1} \wedge \cdots \wedge dx_d = df_1 \wedge \cdots \wedge df_n \wedge d\omega_{\mathcal{F}}.$$

This shows that the measure $d\omega_{\mathcal{F}}$ (given as a differential $(d-n)$-form on $S_{\mathcal{F}} \cap U$) is independent of the choice of local coordinates x_I. Then $\omega_{\mathcal{F}}$ is defined on $S^0_{\mathcal{F}}$ and moreover the set $S^0_{\mathcal{F}} \backslash S_{\mathcal{F},J}$ is of measure zero with respect to ω_F, as it is a proper analytic subset on $\mathbb{R}^{d-n}$ in any other admissible local coordinates.

Let $x = (z, y)$ be a partition of coordinates in $\mathbb{R}^d$, with $y = x_{J_2}$, $z = X_{J_1}$, and assume that for $i = 1, \dots, m$ the functions f_i depend only on the z-variables. We say that the partition of coordinates is *admissible* if there is a point $x = (z, y) \in S_{\mathcal{F}}$ such that both the gradient vectors $\nabla_z f_1(x), \dots, \nabla_z f_m(x)$ and the vectors $\nabla_y f_{m+1}(x), \dots, \nabla_y f_n(x)$ form a linearly independent system. Partition the system $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ with $\mathcal{F}_1 = \{f_1, \dots, f_m\}$ and $\mathcal{F}_2 = \{f_{m+1}, \dots, f_n\}$. Then there is a set $J_1' \subseteq J_1$ for which

$$j_{\mathcal{F}_1, J_1'}(z) := \det\left(\frac{\partial f_i}{\partial x_j}(z)\right)_{1\le i\le m,\, j\in J_1'} \neq 0,$$

and also a set $J_2' \subseteq J_2$ such that

$$j_{\mathcal{F}_2, J_2'}(z, y) := \det\left(\frac{\partial f_i}{\partial x_j}(z, y)\right)_{m+1\le i\le n,\, j\in J_2'} \neq 0.$$

Since $\nabla_y f_i \equiv 0$ for $1 \le i \le m$, it follows that the set of coordinates $J' = J_1' \cup J_2'$ is admissible, moreover,

$$j_{\mathcal{F},J'}(y,z) = j_{\mathcal{F}_1,J_1'}(z)\, j_{\mathcal{F}_2,J_2'}(y,z).$$

For fixed z, let $f_{i,z}(y) := f_i(z,y)$ and let $\mathcal{F}_{2,z} = \{f_{m+1,z},\dots,f_{n,z}\}$. Then clearly $j_{\mathcal{F}_2,J_2'}(y,z) = j_{\mathcal{F}_{2,z},J_2'}(y)$ as it only involves partial derivatives with respect to the y-variable. Thus we have an analogue of Fubini's theorem, namely,

$$\int g(x)\, d\omega_{\mathcal{F}}(x) = \iint g(z,y)\, d\omega_{\mathcal{F}_{2,z}}(y)\, d\omega_{\mathcal{F}_1}(z). \tag{6-4}$$

Consider now algebraic sets given as the intersection of spheres. Let $x_1,\dots,x_m \in \mathbb{R}^d$, $t_1,\dots,t_m > 0$ and $\mathcal{F} = \{f_1,\dots,f_m\}$, where $f_i(x) = |x-x_i|^2 - t_i$ for $i = 1,\dots,m$. Then $S_{\mathcal{F}}$ is the intersection of spheres centered at the points x_i of radius $r_i = t_i^{1/2}$. If the set of points $X = \{x_1,\dots,x_m\}$ is in general position (i.e., they span an $(m-1)$-dimensional affine subspace), then a point $x \in S_{\mathcal{F}}$ is nonsingular if $x \notin \operatorname{span} X$, i.e., if x cannot be written as linear combination of $x_1,\dots,x_m$. Indeed, since $\nabla f_i(x) = 2(x-x_i)$, we have that

$$\sum_{i=1}^m a_i \nabla f_i(x) = 0 \quad \Longleftrightarrow \quad \sum_{i=1}^m a_i x = \sum_{i=1}^m a_i x_i,$$

which implies that $\sum_{i=1}^m a_i = 0$ and $\sum_{i=1}^m a_i x_i = 0$. By replacing the equations $|x-x_i|^2 = t_i$ with $|x-x_1|^2 - |x-x_i|^2 = t_1 - t_i$, which is of the form $x \cdot (x_1 - x_i) = c_i$, for $i = 2,\dots,m$, it follows that $S_{\mathcal{F}}$ is the intersection of the sphere with an $(n-1)$-codimensional affine subspace Y, perpendicular to the affine subspace spanned by the points x_i. Thus $S_{\mathcal{F}}$ is an m-codimensional sphere of $\mathbb{R}^d$ if $S_{\mathcal{F}}$ has one point $x \notin \operatorname{span}\{x_1,\dots,x_m\}$ and all of its points are nonsingular. Let x' be the orthogonal projection of x to $\operatorname{span} X$. If $y \in Y$ is a point with $|y-x'| = |x-x'|$ then by the Pythagorean theorem we have that $|y-x_i| = |x-x_i|$ and hence $y \in S_{\mathcal{F}}$. It follows that $S_{\mathcal{F}}$ is a sphere centered at x' and contained in Y.

Let $T = T_X$ be the inner product matrix with entries $t_{ij} := (x-x_i)\cdot(x-x_j)$ for $x \in S_{\mathcal{F}}$. Since

$$(x-x_i)\cdot(x-x_j) = \tfrac{1}{2}(t_i + t_j - |x_i - x_j|^2),$$

the matrix T is independent of x. We will show that $d\omega_{\mathcal{F}} = c_T\, d\sigma_{S_{\mathcal{F}}}$, where $d\sigma_{S_{\mathcal{F}}}$ denotes the surface area measure on the sphere $S_{\mathcal{F}}$ and $c_T = 2^{-m} \det(T)^{-1/2} > 0$, i.e., for a function $g \in C_0(\mathbb{R}^d)$,

$$\int_{S_{\mathcal{F}}} g(x)\, d\omega_{\mathcal{F}}(x) = c_T \int_{S_{\mathcal{F}}} g(x)\, d\sigma_{S_{\mathcal{F}}}(x). \tag{6-5}$$

Let $x \in S_{\mathcal{F}}$ be fixed and let $e_1,\dots,e_d$ be an orthonormal basis so that the tangent space $T_x S_{\mathcal{F}}$ equals $\operatorname{span}\{e_{m+1},\dots,e_d\}$, and moreover we have that $\operatorname{span}\{\nabla f_1,\dots,\nabla f_m\} = \operatorname{span}\{e_1,\dots,e_m\}$. Let $x_1,\dots,x_n$ be the corresponding coordinates on $\mathbb{R}^d$ and note that in these coordinates the surface area measure, as a $(d-m)$-form at x, is

$$d\sigma_{S_{\mathcal{F}}}(x) = dx_{m+1} \wedge \cdots \wedge dx_d.$$

On the other hand, in local coordinates $x_I = (x_{m+1}, \dots, x_d)$, it is easy to see from (6-2)–(6-3) that $j_{\mathcal{F},J}(x) = 2^m \operatorname{vol}(x - x_1, \dots, x - x_m)$, and hence

$$d\omega_{\mathcal{F}}(x) = 2^{-m} \operatorname{vol}(x - x_1, \dots, x - x_m)^{-1}\, dx_{m+1} \wedge \cdots \wedge dx_d,$$

where $\operatorname{vol}(x - x_1, \dots, x - x_m)$ is the volume of the parallelotope with side vectors $x - x_j$. Finally, it is a well-known fact from linear algebra that

$$\operatorname{vol}(x - x_1, \dots, x - x_m)^2 = \det(T),$$

i.e., the volume of a parallelotope is the square root of the Gram matrix formed by the inner products of its side vectors.

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RESONANCES FOR SCHRÖDINGER OPERATORS ON INFINITE CYLINDERS AND OTHER PRODUCTS

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We study the resonances of Schrödinger operators on the infinite product $X = \mathbb{R}^d \times \mathbb{S}^1$, where d is odd, $\mathbb{S}^1$ is the unit circle, and the potential V lies in $L^\infty_c(X)$. This paper shows that at high energy, resonances of the Schrödinger operator $-\Delta + V$ on $X = \mathbb{R}^d \times \mathbb{S}^1$ which are near the continuous spectrum are approximated by the resonances of $-\Delta + V_0$ on X, where the potential V_0 is given by averaging V over the unit circle. These resonances are, in turn, given in terms of the resonances of a Schrödinger operator on $\mathbb{R}^d$ which lie in a bounded set. If the potential is smooth, we obtain improved localization of the resonances, particularly in the case of simple, rank 1 poles of the corresponding scattering resolvent on $\mathbb{R}^d$. In that case, we obtain the leading order correction for the location of the corresponding high-energy resonances. In addition to direct results about the location of resonances, we show that at high energies away from the resonances, the resolvent of the model operator $-\Delta + V_0$ on X approximates that of $-\Delta + V$ on X. If $d = 1$, in certain cases this implies the existence of an asymptotic expansion of solutions of the wave equation. Again for the special case of $d = 1$, we obtain a resonant rigidity type result for the zero potential among all real-valued smooth potentials.

1. Introduction

We study the Schrödinger operator $-\Delta + V$ on the manifold $X = \mathbb{R}^d \times \mathbb{S}^1$ with the product metric, where d is odd, $\mathbb{S}^1$ is the unit circle, and $V \in L^\infty_c(X)$. In the special case $d = 1$, X is the infinite cylinder $\mathbb{R} \times \mathbb{S}^1$. We show that in the large energy limit, resonances near the continuous spectrum are well approximated by those of $-\Delta + V_0$, where V_0 is the average of V over $\mathbb{S}^1$: $V_0(x) = \frac{1}{2\pi} \int_0^{2\pi} V(x, \theta)\, d\theta$. By a separation of variables argument, these, in turn, are determined by the low energy resonances of the Schrödinger operator $-\sum_{j=1}^d \partial^2/\partial x_j^2 + V_0$ on $\mathbb{R}^d$. In the case of smooth potentials V, for simple rank 1 poles of the (scattering) resolvent of $-\sum_{j=0}^d \partial^2/\partial x_j^2 + V_0$, we find the leading-order corrections to the location of the corresponding poles of the resolvent of $-\Delta + V$ on X. Among other things, this allows us to prove that no other smooth real-valued potential on $\mathbb{R} \times \mathbb{S}^1$ has the same resonances as the zero potential. For potentials with $V_0 \equiv 0$, we show the existence of large resonance-free regions. When $d = 1$ and $V \in C^\infty_c(X; \mathbb{R})$, under certain hypotheses on the potential V_0 we are able to give an asymptotic expansion of solutions of the wave equation. For the case of $d = 1$ we study a simple example of a nontrivial potential V with $V_0 \equiv 0$ and locate some of the corresponding resonances. Some of these results are reminiscent of Drouot's results [2018] for rapidly oscillating potentials on $\mathbb{R}^d$.

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Let $\Delta \leq 0$ denote the Laplacian on $X = \mathbb{R}^d \times \mathbb{S}^1$ with the product metric. For $V \in L^\infty_c(X)$ the Schrödinger operator $-\Delta + V$ has continuous spectrum $[0, \infty)$, with multiplicity which increases at each *threshold* j^2, for $j \in \mathbb{N}_0$. For $\operatorname{Im} \zeta > 0$, set $R_V(\zeta) = (-\Delta + V - \zeta^2)^{-1}$. This (scattering) resolvent has a meromorphic continuation to $\hat{Z}$, the minimal Riemann surface for which $\tau_l(\zeta) \stackrel{\text{def}}{=} (\zeta^2 - l^2)^{1/2}$ is a single-valued analytic function for each $l \in \mathbb{N}_0$. The resonances are poles of the resolvent $R_V(\zeta)$. We refer to the portion of $\hat{Z}$ for which $\operatorname{Im} \tau_l(\zeta) > 0$ for all $l \in \mathbb{N}_0$ as the *physical space*. In this set R_V is a bounded operator on $L^2(X)$, away from a discrete set of points which correspond to (square roots of) eigenvalues. For $l \in \mathbb{N}_0$ and $\rho > 0$, denote by $B_l(\rho)$ the connected component of $\{\zeta \in \hat{Z} : |\tau_l(\zeta)| < \rho\}$ which nontrivially intersects both the physical space and the set $\{\zeta \in \hat{Z} : \operatorname{Re} \tau_0(\zeta) > 0\}$. Using as the coordinate $\tau_l(\zeta)$, $B_l(\rho)$ is identified with the disk of radius ρ in the complex plane, centered at the origin, and this identification is compatible with the complex structure of $\hat{Z}\restriction_{B_l(\rho)}$ if $\rho < \sqrt{2l-1}$. The point $\tau_l(\zeta) = 0$ in $B_l(\rho)$ corresponds to the l-th threshold. We study the resonances of $-\Delta + V$ in $B_l(\rho)$, or $B_l(\alpha \log l)$, as $l \to \infty$. Results of Section 6 show that these are the high-energy resonances "near" the continuous spectrum which have $\operatorname{Re} \tau_0 > 0$.

For a function $V \in L^\infty_c(X)$ and $m \in \mathbb{Z}$ define

$$V_m(x) = \frac{1}{2\pi} \int_0^{2\pi} V(x, \theta) e^{-im\theta} \, d\theta,$$

so that $V(x, \theta) = \sum_{m=-\infty}^{\infty} V_m(x) e^{im\theta}$. The minimal assumption on a potential V in most of this paper will be that

$$V \in L^\infty_c(X) \quad \text{and} \quad \|V_m\|_{L^\infty} = O(|m|^{-\delta}) \quad \text{for some } \delta \text{ with } 0 < \delta \leq \tfrac{1}{2}. \tag{1-1}$$

Note that this imposes an assumption on δ as well, which we shall include when we invoke hypothesis (1-1). We use the notation $\Delta_0 = \sum_{j=1}^d \partial^2/\partial x_j^2$ for the Laplacian on $\mathbb{R}^d$,

$$R_{V_0,0}(\lambda) = (-\Delta_0 + V_0 - \lambda^2)^{-1}, \quad \text{if } \operatorname{Im} \lambda > 0 \tag{1-2}$$

with the same notation for its meromorphic continuation to the complex plane — see Section 3A. The poles of $R_{V_0,0}$ in $\mathbb{C}$ are the resonances of $-\Delta_0 + V_0$. The multiplicity $m_{V_0,0}(\lambda_0)$ of a resonance of $-\Delta_0 + V_0$ at λ_0 is given by the dimension of the range of the singular part of the resolvent at λ_0; this is discussed further in Section 4.

Theorem 1.1. *Let $X = \mathbb{R}^d \times \mathbb{S}^1$, d odd, and let $V \in L^\infty_c(X)$ satisfy $\|V_m\|_{L^\infty} = O(|m|^{-\delta})$ for some δ with $0 < \delta \leq \frac{1}{2}$. Suppose $\lambda_0 \in \mathbb{C}$, $\lambda_0 \neq 0$, is a resonance of $-\Delta_0 + V_0$ on $\mathbb{R}^d$, of multiplicity $m_{V_0,0}(\lambda_0)$. Let $\rho \in \mathbb{R}$, $\rho > |\lambda_0|$. Then there are $C_0 > 0$, $L > 0$ so that for $l > L$, $l \in \mathbb{N}$ there are exactly $2m_{V_0,0}(\lambda_0)$ resonances, when counted with multiplicity, of $-\Delta + V$ in the set*

$$\{\zeta \in B_l(\rho) : |\tau_l(\zeta) - \lambda_0| < C_0 l^{-\delta/(m_{V_0,0}(\lambda_0))}\}.$$

Here, and elsewhere in the paper, the apparent "doubling" of the number of poles (when counted with multiplicity) on X as compared with those on $\mathbb{R}^d$ is due to the fact that for $l \in \mathbb{N}$, l^2 is an eigenvalue of $-d^2/d\theta^2$ on $\mathbb{S}^1$ of multiplicity two. This can be seen immediately in the simplest case, $V \equiv V_0$, by separating variables.

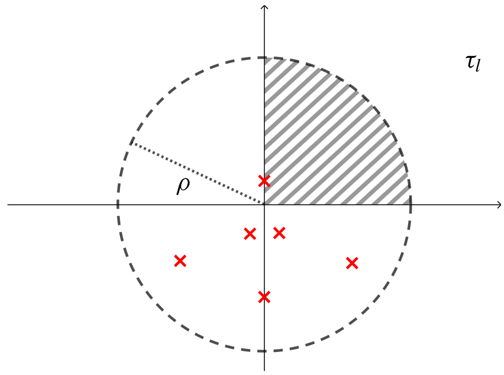

Figure 1. A schematic showing resonances of $-\Delta + V$ in $B_l(\rho)$, pictured in the τ_l-coordinate. Each red × indicates a single resonance of even multiplicity or a cluster of resonances. The hatched region indicates the portion of $B_l(\rho)$ which lies in the physical space. By comparing Figure 2, Section 3B one can see how this fits in the larger picture.

In this paper we refer to any pole of the resolvent as a resonance, including those which correspond to eigenvalues. The second part of Theorem 1.2, for which V is assumed to be smooth, implies an improved localization of the resonances for smooth potentials.

The next theorem shows that, other than possible poles near the threshold, the poles as described above are all the poles in $B_l(\rho)$ for sufficiently large l.

Theorem 1.2. *Let $X = \mathbb{R}^d \times \mathbb{S}^1$, d odd, and suppose V satisfies the hypothesis* (1-1). *Choose $\rho > 0$ so that if λ_j is a pole of $R_{V_0,0}(\lambda)$, then $|\lambda_j| \neq \rho$. Set*

$$\Lambda_\rho = \{\lambda_j \in \mathbb{C} : |\lambda_j| < \rho \text{ and } \lambda_j \text{ is a pole of } R_{V_0,0}(\lambda)\}.$$

Let $\epsilon' > 0$ be so that $\epsilon' < \min\{|\lambda_j| : \lambda_j \in \Lambda_\rho,\ \lambda_j \neq 0\}$. Then there are $\widetilde{C},\ L > 0$ so that for $l > L$, $l \in \mathbb{N}$, there are no resonances of $-\Delta + V$ in

$$\{\zeta \in B_l(\rho) : |\tau_l(\zeta)| > \epsilon' \text{ and } |\tau_l(\zeta) - \lambda_j| > \widetilde{C} l^{-\delta/m_{V_0,0}(\lambda_j)} \text{ for all } \lambda_j \in \Lambda_\rho\}.$$

Moreover, if V is smooth for perhaps larger L and $\widetilde{C}$, for $l > L$ there are no resonances in

$$\{\zeta \in B_l(\rho) : |\tau_l(\zeta)| > \epsilon' \text{ and } |\tau_l(\zeta) - \lambda_j| > \widetilde{C} l^{-2/(m_{V_0,0}(\lambda_j))} \text{ for all } \lambda_j \in \Lambda_\rho\}.$$

In addition, if $R_{V_0,0}(\lambda)$ is analytic in a neighborhood of the origin, then there are no poles in $B_l(\epsilon')$ for l sufficiently large.

We comment that smoothness of the potential V is more than is needed for the second part of Theorem 1.2. It would suffice to have $V \in C^k(X)$, for some k sufficiently large. In order to simplify the proofs, we have not tracked the value of k which is needed.

To help visualize these theorems, we include Figure 1, which is a schematic showing the resonances of $-\Delta + V$ in $B_l(\rho)$ for large l, using the τ_l-coordinate. This schematic is familiar from odd-dimensional scattering theory; that this should be so is a consequence of Theorems 1.1–1.3. One difference is that in this diagram, the only portion of $B_l(\rho)$ which lies in the physical space is the portion which is in the first quadrant, indicated by hatching. Another is that each x indicates either a single resonance of even multiplicity, or a cluster of resonances. See Figure 2 to see how B_l fits in a larger context.

For Schrödinger operators on $\mathbb{R}^d$, the behavior of the singularities of the resolvent at the origin is delicate. For example, notions of multiplicity of a resonance which agree at points away from the origin may differ at the origin; see [Dyatlov and Zworski 2019, Theorem 2.8]. These same sorts of issues arise at thresholds in the case under study here, and accounts for the fact that this next theorem, which concerns resonances very near the thresholds, is weaker than the previous ones.

Theorem 1.3. *Let V satisfy* (1-1) *and suppose the resolvent of $-\Delta_0 + V_0$ on $\mathbb{R}^d$ has a pole at* 0 *of order $r > 0$, and multiplicity $m_{V_0,0}(0)$ Then there are $C,\ L > 0$ so that $-\Delta + V$ on X has at least $2m_{V_0,0}(0)$ resonances, when counted with multiplicity, in $B_l(Cl^{-\delta/r})$ when $l > L$, $l \in \mathbb{N}$. Moreover, there is an $\epsilon > 0$ so that $-\Delta + V$ has no poles in $B_l(\epsilon) \setminus B_l(Cl^{-\delta/r})$ when $l > L$. If $V \in C_c^\infty(X)$, then this can be improved to show that there is a $C_1 > 0$ so that $-\Delta + V$ has no poles in $B_l(\epsilon) \setminus B_l(C_1 l^{-2/r})$ when $l > L$. Moreover, under the hypothesis* (1-1), *if $r = 1$ there are exactly $2m_{V_0,0}(0)$ resonances of $-\Delta + V$ in $B_l(Cl^{-\delta})$ for $l > L$.*

Suppose for the moment that V_0 is real-valued. In this case, it is well known that if $d = 1$ the order of the pole of the resolvent of $-d^2/dx^2 + V_0$ at 0 cannot exceed 1, and if it is 1, then $m_{V_0,0}(0) = 1$ [Dyatlov and Zworski 2019, Theorem 2.7]. If $d \geq 3$ is odd, then the order of the pole of the resolvent of $-\Delta_0 + V_0$ at 0 cannot exceed 2 [Dyatlov and Zworski 2019, Lemma 3.16]. For general V and r, the order of the pole at 0 can be bounded from above in terms of $m_{V_0,0}(0)$, and in the case $d = 1$, $m_{V_0,0}(0)$ can be bounded above by r.

It is of particular interest to understand poles of the resolvent R_V near the physical region. In Section 6 we show that there are large regions near the physical region that contain no resonances. A consequence of those results is that large energy resonances near the continuous spectrum and having $\operatorname{Re} \tau_0(\zeta) > 0$ are contained in regions of the form $B_l(\rho)$, where ρ depends on how near the continuous spectrum we wish to look. In Section 6 we further justify our focus on the resonances in sets $B_l(\rho)$.

Theorems 1.1–1.3 combined with results of Section 6 yield the following corollary. Here $d_{\hat{Z}}$ is a distance on $\hat{Z}$, defined in Section 6. The boundary of the physical region corresponds to the continuous spectrum. In the corollary, we use $\{\zeta_j^\flat\}$ to denote a sequence of points in $\hat{Z}$, to distinguish them from ζ_l which is used elsewhere to denote a particular mapping from an open subset of the complex plane into $\hat{Z}$.

Corollary 1.4. *Let $V \in L_c^\infty(X; \mathbb{R})$ satisfy* (1-1). *Then $R_V(\zeta)$ has a sequence $\{\zeta_j^\flat\}_{j=1}^\infty$ of poles satisfying both $|\tau_0(\zeta_j^\flat)| \to \infty$ as $j \to \infty$ and $d_{\hat{Z}}(\zeta_j^\flat$, physical region$) \to 0$ as $j \to \infty$ if and only if $R_{V_0,0}(\lambda)$ has at least one pole in $i[0, \infty)$.*

In particular, if $d = 1$, by [Reed and Simon 1978, Theorem XIII.110] if $\int_X V \leq 0$ then $R_V(\zeta)$ has such a sequence of poles approaching the physical space. In contrast, if $V_0(x) \geq 0$ for all x and V_0 is nontrivial, $R_V(\zeta)$ does not have such a sequence of poles. Note that for any fixed $k_0 \in \mathbb{N}$, we have $|\tau_0(\zeta_j^\flat)| \to \infty$ as $j \to \infty$ if and only if $|\tau_{k_0}(\zeta_j^\flat)| \to \infty$ as $j \to \infty$. We remark that we could prove an analog of Corollary 1.4 for complex-valued potentials as well.

If we enlarge the region centered at the threshold l^2 with increasing l, we have less fine localization of the resonances, see Theorem 7.1. However, when V_0, the average of the potential, is identically zero, we can get a larger resonance-free region. The difference in the next result for $d = 1$ and $d \geq 3$ is due to the fact that the resolvent of $-d^2/dx^2$ on $\mathbb{R}$ has a pole at the origin, but that of $-\Delta_0$ on $\mathbb{R}^d$ for $d \geq 3$ odd does not.

Theorem 1.5. *Let $V \in L^\infty_c(X)$ satisfy* (1-1), *and suppose $V_0 \equiv 0$. If $d = 1$ there are $\alpha,\ c_0 > 0$ so that for $l \in \mathbb{N}$ sufficiently large there are no resonances of $-\Delta + V$ in the set $\{\zeta \in B_l(\alpha \log l) : |\tau_l(\zeta)| > c_0/l^\delta\}$. If $d \geq 3$ is odd, there is an $\alpha > 0$ so that for l sufficiently large there are no resonances of $-\Delta + V$ in the set $B_l(\alpha \log l)$.*

There is a sense in which this theorem is sharp; see Proposition 12.6 for a computation for the case $d = 1$ with the potential $V(x, \theta) = 2\chi_{I_0}(x) \cos\theta$, where χ_{I_0} is the characteristic function of the interval $[-1, 1]$.

We can find the leading correction term for high-energy resonances of $-\Delta + V$ which correspond to simple resonances of $-\Delta_0 + V_0$. In the next theorem, ∇_0 is the gradient on $\mathbb{R}^d$, so that

$$\nabla_0 f = \left(\frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f, \dots, \frac{\partial}{\partial x_d} f\right).$$

Theorem 1.6. *Let $X = \mathbb{R}^d \times \mathbb{S}^1$, d odd, $V \in C^\infty_c(X)$, and suppose $\lambda_0 \in \mathbb{C}$ is a simple pole of the scattering resolvent $R_{V_0,0}$ of $-\Delta_0 + V_0$ on $\mathbb{R}^d$, and that the residue of $R_{V_0,0}$ at λ_0 has rank* 1. *Suppose for any $\chi \in C^\infty_c(\mathbb{R}^d)$,*

$$\chi\left(R_{V_0,0}(\lambda) - \frac{i}{\lambda - \lambda_0} u \otimes u\right)\chi \tag{1-3}$$

is analytic near $\lambda = \lambda_0$. Let $\rho > |\lambda_0|$. Then there are $\epsilon,\ L > 0$ so that for $l > L$ there are exactly two poles of $R_V(\zeta)$, when counted with multiplicity, in $\{\zeta \in B_l(|\lambda_0| + 1) : |\tau_l(\zeta) - \lambda_0| < \epsilon\}$, and each pole of $R_V(\zeta)$ in this set satisfies

$$\tau_l(\zeta) = \lambda_0 - \frac{i}{4l^2} \sum_{k \neq 0} \frac{1}{k^2} \int_{\mathbb{R}} (k^2 V_{-k} V_k u^2 + (\nabla_0 V_{-k} \cdot \nabla_0 V_k) u^2)(x)\, dx + O(l^{-3}).$$

We note that the normalization of the singularity in (1-3) is chosen so that if V is real-valued and $\lambda_0 \in i[0, \infty)$, then u is real-valued. There is some further discussion of u at the beginning of Section 10. Proposition 12.3 shows that the leading correction may be rather different for a nonsmooth potential by considering the special case of the potential on $\mathbb{R} \times \mathbb{S}^1$ given by $V(x, \theta) = 2\cos\theta\, \chi_{I_0}(x)$, where χ_{I_0} is the characteristic function of the interval $[-1, 1]$. As for Theorem 1.2, the proof of Theorem 1.6 only needs V to be C^k for some k sufficiently large. Since Theorem 1.8 requires smoothness of the potential only for an application of Theorem 1.6, the same is true for it. Again, we have chosen not to track this value in the interest of simplifying proofs.

If $V_0 \in L^\infty_c(\mathbb{R}^d; \mathbb{R})$ and the operator $-\Delta_0 + V_0$ on $L^2(\mathbb{R}^d)$ has a simple negative eigenvalue $-\beta^2$, then this negative eigenvalue corresponds to a simple pole of $R_{V_0,0}$ on the positive imaginary axis at $i|\beta|$, and the residue has rank 1. By Theorem 1.1 (or Corollary 1.4), in this case R_V has a sequence of poles approaching the physical space. If $V \in C^\infty_c(X; \mathbb{R})$, the poles approach the physical space very rapidly.

Theorem 1.7. *Suppose $V \in C_c^\infty(X;\mathbb{R})$ and $\lambda_0 \in \mathbb{C}$ is a simple pole of $R_{V_0}(\lambda)$ with $\operatorname{Re}\lambda_0 = 0$, with residue of R_{V_0} at λ_0 having rank 1. Then there is an $\epsilon > 0$ so that if $\{\zeta_l^\flat\}_{l=L}^\infty \subset \hat{Z}$ is a sequence of poles of R_V with $\zeta_l^\flat \in B_l(|\lambda_0|+1)$ and $|\tau_l(\zeta_l^\flat) - \lambda_0| < \epsilon$, then $\operatorname{Re}\tau_l(\zeta_l^\flat) = O(l^{-N})$ for any N. In particular, this implies that if $\operatorname{Im}\lambda_0 > 0$, then $d_{\hat{Z}}(\zeta_l^\flat, \text{physical region}) = O(l^{-N})$.*

Proposition 12.3 demonstrates the necessity of assuming some regularity of the potential, at least for $d = 1$, by studying the resonances very near the l-th threshold for a certain real-valued potential with a jump singularity. These resonances in $B_l(1)$ arise from the pole of $R_{0,0}(\lambda)$ at $\lambda_0 = 0$. They have

$$|\tau_l(\zeta_l^\flat)| = O(l^{-3/2})$$

and, for a subsequence of l's tending to infinity,

$$|\operatorname{Im}(\tau_l(\zeta_l^\flat))| > \tfrac{1}{10} l^{-3/2}.$$

This paper was initially motivated by the case $d = 1$, as $\mathbb{R} \times \mathbb{S}^1$ provides a particularly simple example of a manifold with infinite cylindrical ends and as such provides a testing ground for studying resonances for Schrödinger operators on such manifolds. Most of the proofs of the preceding theorems are essentially identical for any odd dimension of the factor $\mathbb{R}^d$, so we have included the more general results. However, Theorems 1.8 and 1.9 are particular to the $d = 1$ case.

As a corollary of Theorems 1.1, 1.3, and 1.6, we get in the case $d = 1$ a uniqueness-type result for the zero potential among smooth real-valued potentials.

Theorem 1.8. *Let $V \in C_c^\infty(\mathbb{R}\times\mathbb{S}^1;\mathbb{R})$. Suppose for each $\rho > 0$ there is a sequence*

$$\{l_j\}_{j=1}^\infty = \{l_j(\rho)\}_{j=1}^\infty \subset \mathbb{N}$$

with $l_j \to \infty$ when $j \to \infty$ so that in $B_{l_j}(\rho)$ the resonances of $-\Delta + V$ and $-\Delta$ on $X = \mathbb{R}\times\mathbb{S}^1$ are the same. Then $V \equiv 0$.

This result is false if we omit the hypothesis that V is real-valued. For example, for $V_1 \in C_c^\infty(\mathbb{R})$ set $V(x,\theta) = V_1(x)e^{i\theta}$. Then the operators $-\Delta + V$ and $-\Delta$ have the same resonances; see [Autin 2011] or [Christiansen 2004, Section 4]. This example can be easily generalized.

As part of our study of the distribution of resonances, we prove that, in a suitable sense, near the physical region of $\hat{Z}$, R_V is well approximated by R_{V_0} away from the poles of R_{V_0}; see Proposition 5.4 and Lemma 5.5. In the case $d = 1$, this and results of [Christiansen and Datchev 2022] give a wave expansion; see Theorem 1.9.

Let $X = \mathbb{R}\times\mathbb{S}^1$, $V \in C_c^\infty(X;\mathbb{R})$, and suppose $-\Delta + V$ has finitely many eigenvalues $\mu_1, \mu_2, \ldots, \mu_J$, repeated with multiplicity, with associated orthonormal eigenfunctions $\{\eta_j\}$, so that $(-\Delta + V)\eta_j = \mu_j\eta_j$. Let u satisfy

$$\frac{\partial^2}{\partial t^2}u - \Delta u + Vu = 0, \tag{1-4}$$
$$(u, u_t)\restriction_{t=0} = (f_1, f_2) \in C_c^\infty(X) \times C_c^\infty(X).$$

Theorem 1.9. *Let $X = \mathbb{R} \times \mathbb{S}^1$ and $V,\ f_1,\ f_2 \in C_c^\infty(X)$, with V real-valued, and suppose $-d^2/dx^2 + V_0$ on $\mathbb{R}$ has no negative eigenvalues and no resonance at* 0*. Let u be the solution of* (1-4) *on* $[0, \infty) \times X$*. Then for each $k_0 \in \mathbb{N}$ we can write $u(t) = u_e(t) + u_{\mathrm{thr},k_0}(t) + u_{r,k_0}(t)$, where*

$$u_e(t, x, \theta) = \sum_{\substack{\mu_j \in \sigma_p(-\Delta+V) \\ \mu_j \neq 0}} \eta_j(x, \theta)\left(\cos((\mu_j)^{1/2} t)\langle f_1, \eta_j\rangle + \frac{\sin((\mu_j)^{1/2} t)}{(\mu_j)^{1/2}}\langle f_2, \eta_j\rangle\right) + \sum_{\substack{\mu_j \in \sigma_p(-\Delta+V) \\ \mu_j = 0}} \eta_j(x, \theta)(\langle f_1, \eta_j\rangle + t\langle f_2, \eta_j\rangle) \quad (1\text{-}5)$$

and

$$u_{\mathrm{thr},k_0}(t, x, \theta) = b_{0,0,+}(x, \theta) + \sum_{k=0}^{k_0-1} t^{-1/2-k} \sum_{j=1}^{\infty} (e^{itj} b_{j,k,+}(x, \theta) + e^{-itj} b_{j,k,-}(x, \theta))$$

for some $b_{j,k,\pm} \in \langle x\rangle^{1/2+2k+\epsilon} L^2(X)$. For any $\chi \in C_c^\infty(X)$ there is a constant C so that

$$\sum_{j=1}^{\infty} \|\chi b_{j,k,\pm}\|_{L^2(X)} < C, \quad k = 0, 1, 2, \dots, k_0 - 1$$

and

$$\|\chi u_{r,k_0}(t)\|_{L^2(X)} \le C t^{-k_0} \quad \textit{for } t \textit{ sufficiently large.}$$

The assumption that $-d^2/dx^2 + V_0$ on $\mathbb{R}$ has no negative eigenvalues and no resonance at 0 means, by Theorem 1.2, that R_V has at most finitely many poles on the boundary of the physical space. In particular, this means at most finitely many eigenvalues of $-\Delta + V$, so that the sum in u_e is finite. Further, there are at most finitely many poles at thresholds, and this implies via results of [Christiansen and Datchev 2022] that at most finitely many of the $b_{j,0,\pm}$ are nonzero.

If $-d^2/dx^2 + V_0$ on $\mathbb{R}$ has one or more negative eigenvalues, it seems plausible that there is an asymptotic expansion of solutions of the wave equation on compact sets. Since in this case by Theorem 1.7 the resolvent R_V may have a sequence of poles rapidly approaching, but not lying in, the continuous spectrum, such an expansion would need to take these into account and is more complicated — see for example [Tang and Zworski 2000] for an expansion in a Euclidean scattering setting with resonances approaching the continuous spectrum. In our setting proving the existence of such an expansion may use techniques similar to those of [Christiansen and Datchev 2022] but does not follow directly from the results of that work. Proving this is outside the scope of this paper.

In this paper we have, for simplicity, limited ourselves to the case of Schrödinger operators on $\mathbb{R}^d \times \mathbb{S}$. However, many of our results for L^∞ potentials hold as well for Schrödinger operators with Dirichlet or Neumann boundary conditions on $\mathbb{R}^{d-1} \times (0, \infty) \times \mathbb{S}$ or on $\mathbb{R}^d \times (0, \pi)$.

1A. ***Relation to previous work.*** This paper was inspired in part by two different sets of papers. The first are papers which study eigenvalues and resonances of Schrödinger operators on $\mathbb{R}^d$ with rapidly oscillating potentials, and includes [Borisov 2006; Borisov and Gadylshin 2006; Duchêne and Weinstein 2011; Duchêne et al. 2014; 2015; Dimassi 2016; Drouot 2018]. Of these the most closely related to this paper is that of Drouot [2018], which studies the distribution of resonances of Schrödinger operators $-\Delta_0 + V_\epsilon$ on $\mathbb{R}^d$ with d odd. Here

$$V_\epsilon(x) = V_0(x) + \sum_{k\in\mathbb{Z}^d, k\neq 0} V_k(x)e^{ik\cdot x/\epsilon}, \quad x \in \mathbb{R}^d.$$

Drouot shows in quantitative ways that in the limit $\epsilon \downarrow 0$, resonances of $-\Delta_0 + V_\epsilon$ near the continuous spectrum are well approximated by those of $-\Delta_0 + V_0$. In addition, he proves some refinements related, for example, to the leading order correction of the positions of the resonances. Theorems 1.1, 1.2, 1.3, 1.5, and 1.6, as well as some computations in Section 12, are inspired by results in [Drouot 2018]. However, the proofs are quite different. In part, this is because the different setting requires different techniques. Additionally, Drouot's results come mainly from studying regularized determinants. While this has the potential of producing in some instances more refined results than we obtain here, it requires a substantial amount of technical work. We have chosen instead to mostly avoid determinants, or to work only with determinants of operators of the type $I + F$, where F is finite rank. Instead, we use an operator Rouché theorem of Gohberg and Sigal [1971]. In some places this may allow for sharper results than could be obtained by using a regularized determinant. We note in addition that in the setting of [Drouot 2018], the resonances lie on the complex plane, while for us, the resonances lie on a Riemann surface which is a countable but infinite cover of the complex plane, with infinitely many branch points. This means that some of the techniques used in [Drouot 2018] cannot be applied here.

A less direct source of inspiration is work done on the distribution of eigenvalues of the Schrödinger operator $-\Delta_{\mathbb{S}^n} + W$ on the sphere $\mathbb{S}^n$ (and certain other compact manifolds), $n \geq 2$; see for example [Weinstein 1977; Widom 1979]. In this setting, eigenvalues of the Schrödinger operator occur in bands. Roughly speaking, these authors show that a suitable average of the potential W can be used to obtain information about the distribution of high-energy eigenvalues of the Schrödinger operator within these bands. This average is over closed geodesics, rather than over all of $\mathbb{S}^n$. Of course, our function $V_0(x)$ is the average of the potential V over the cross section of $\mathbb{S}^1$, the unique closed geodesic on $\mathbb{S}^1$.

This paper was originally motivated by the $d = 1$ case, which gives $X = \mathbb{R} \times \mathbb{S}^1$, a manifold with an infinite cylindrical end. The spectral and scattering theory of manifolds with infinite cylindrical ends has been studied in, for example, [Goldstein 1974; Guillopé 1989; Melrose 1993]. There is a large literature studying the existence of eigenvalues and, in certain settings, the locations of resonances for such manifolds and the related problems of waveguides which have a "one-dimensional infinity" as our $d = 1$ case does; see, e.g., [Levitin and Marletta 2008] or the monograph [Exner and Kovařík 2015]. This monograph also includes some results for manifolds with "higher-dimensional infinity". Many of these results focus on low-energy eigenvalues or resonances. We mention the papers [Christiansen 2002; 2004; Christiansen and Datchev 2021; Christiansen and Zworski 1995; Parnovski 1995; Edward 2002] which are more directly connected with high-energy behavior.

1B. ***Comments regarding other product manifolds.*** This paper studies only Schrödinger operators on $\mathbb{R}^d \times \mathbb{S}^1$, where d is odd. Here we comment on why we require that d be odd and on the choice of $\mathbb{S}^1$ for the second factor.

For Euclidean scattering, e.g., for the Schrödinger operator $-\Delta_{\mathbb{R}^d} + V_{\mathbb{R}^d}$ on $\mathbb{R}^d$ with $V_{\mathbb{R}^d} \in L^\infty_c(\mathbb{R}^d)$, the space to which the resolvent continues is determined by the dimension: for odd d the meromorphic continuation is to the complex plane, and for even d the meromorphic continuation is to Λ, the logarithmic cover of $\mathbb{C} \setminus \{0\}$. This means that certain questions related to the distribution of resonances are more difficult in even dimensional Euclidean scattering than in odd dimensional Euclidean scattering. For the problem we consider here, the Riemann surface on which the resonances live is a bit involved to describe when d is odd; see Section 3B. The Riemann surface when d is even is much more complicated, requiring as its building block Λ rather than $\mathbb{C}$. It is, however, clear that some of our results, appropriately interpreted, hold if d is even as long as we stay away from thresholds. In the interest of clarity we do not pursue this here.

Next we turn to the choice of the factor $\mathbb{S}^1$. There are three things that make this an especially nice choice:

(1) The spacing between the distinct eigenvalues grows as the eigenvalues grow.

(2) Upon averaging in $\mathbb{S}^1$, we get a model operator that we understand fairly well.

(3) There is a choice of eigenfunctions of the Laplacian on $\mathbb{S}^1$ so that a product of two eigenfunctions is again an eigenfunction: $e^{ij\theta}e^{ik\theta} = e^{i(j+k)\theta}$.

Not all of our results require this last property. In view of [Weinstein 1977; Widom 1979], it would be natural to think of replacing $\mathbb{S}^1$ with $\mathbb{S}^m$. Of course, the spacing of distinct eigenvalues of the Laplacian on $\mathbb{S}^m$ is similar to that for $\mathbb{S}^1$. However, when using a factor $\mathbb{S}^m$ with $m > 1$, obtaining a model operator is much more complicated, and it seems any results for general potentials would likely be substantially weaker.

1C. ***Ideas from the proofs.*** Our starting point for the study of resonances of $-\Delta + V$ is an identification of the resonances with the points ζ for which the operator $I + (V - V_0)R_{V_0}(\zeta)\chi$ has nontrivial null space. Here $R_W(\zeta)$ is the meromorphic continuation of the resolvent of $-\Delta + W$, and $\chi \in L^\infty_c(X)$ satisfies $\chi V = V$ and is, for convenience, chosen independent of θ. By separating variables, we can understand R_{V_0} in terms of the resolvent of $-\sum_{j=1}^d \partial^2/\partial x_j^2 + V_0(x)$ on $\mathbb{R}^d$.

We use two well-known and related properties of the resolvent of $-\sum_{j=1}^d \partial^2/\partial x_j^2 + V_0(x)$ on $\mathbb{R}^d$. One is the estimate

$$\left\| \tilde{\chi} \left(-\sum_{j=1}^d \partial^2/\partial x_j^2 + V_0 - (\lambda + i0)^2 \right)^{-1} \tilde{\chi} \right\| = O(|\lambda|^{-1})$$

as $\lambda \to \infty$ for $\lambda \in \mathbb{R}$ and $\tilde{\chi} \in L^\infty_c(\mathbb{R}^d)$. The second is the existence of a logarithmic resonance-free neighborhood of the real axis.

An immediate consequence of this second fact and the fact that the distance between thresholds of our operator $-\Delta + V$ on X increases at high energy is that if $V = V_0$, at high energy near the thresholds the resonances of $-\Delta + V_0$ on X are determined by low-energy resonances of $-\sum_{j=1}^d \partial^2/\partial x_j^2 + V_0$ on $\mathbb{R}^d$. Moreover, using these facts and an operator Rouché theorem of Gohberg and Sigal [1971], we are able to show that at high energy near the thresholds the zeros of $I + (V - V_0)R_{V_0}\chi$ are approximated by the poles of $\chi R_{V_0}\chi$. These ideas underlie the proofs of the L^∞ results of Theorems 1.1–1.3 and 1.5. They are also central to the proofs of the smooth versions of these results and of Theorem 1.6, although these proofs require some additional study of the resolvent of $-\Delta + V_0$ when V_0 is smooth.

1D. ***Organization.*** In Section 3 we recall some results from Euclidean scattering and show that the resolvent of $-\Delta + V$ on X has a meromorphic continuation to $\hat{Z}$. (We note that this latter is known; see Section 3 for references.) We define the multiplicity of a pole of the resolvent, and give several useful identities involving it in Section 4. In addition, this section introduces some notation and results related to the operator Rouché theorem of Gohberg and Sigal [1971]. With these preliminaries we prove Theorems 1.1 and 1.2 in the case of an L^∞ potential V, using results from [Gohberg and Sigal 1971]. Section 6 contains more discussion of the Riemann surface $\hat{Z}$ and shows the existence of resonance-free regions which are, at high energy, near the physical region and away from thresholds. This provides the missing pieces of the proof of Corollary 1.4. Combining these with the resolvent estimates of Section 5 and results of [Christiansen and Datchev 2022] proves Theorem 1.9.

Section 8 contains preliminary computations which are needed to refine our results for smooth potentials. The smooth case of Theorem 1.2 is proved with techniques similar to that of the L^∞ result, but using in addition results of Section 8.

In Section 10 we prove Theorems 1.6 and 1.7. We do this using Fredholm determinants, but determinants of the form $\det(I + F)$, where F is a finite-rank operator. Theorem 1.8 follows rather directly from the earlier results. Finally, in Section 12, in the case $d = 1$ we give approximations of some of the high-energy resonances for a particularly simple potential which has $V_0 \equiv 0$ and which is not smooth.

2. Notation and conventions

On $X = \mathbb{R}^d \times \mathbb{S}^1$ we use the coordinates (x, θ) or (x', θ'), with $x,\ x' \in \mathbb{R}^d$ and $\theta,\ \theta' \in [0, 2\pi)$.

Throughout the paper, $V \in L^\infty_c(X)$ and $l \in \mathbb{N}_0$, and the dimension d of $\mathbb{R}^d$ is odd. We use C to stand for a positive constant, the value of which may change without comment.

Suppose A and B are linear operators on domains in $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{S}^1)$, respectively, and are given by

$$(Af)(x) = \int_{\mathbb{R}^d} A(x, x') f(x')\, dx' \quad \text{and} \quad (Bg)(\theta) = \int_0^{2\pi} B(\theta, \theta') g(\theta')\, d\theta.$$

Then A and B give rise to linear operators on domains in $L^2(X)$, which we again denote by A and B, and which are given by

$$(Ah)(x, \theta) = \int_{\mathbb{R}^d} A(x, x') h(x', \theta)\, dx' \quad \text{and} \quad (Bh)(x, \theta) = \int_{\mathbb{R}^d} B(\theta, \theta') h(x, \theta')\, d\theta'.$$

For $f,\ g \in L^2(\mathbb{R}^d)$, the operator $f \otimes g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is defined via

$$((f \otimes g)h)(x) = f(x) \int_{\mathbb{R}^d} g(x') h(x')\, dx'.$$

If $f,\ g \in L^2(X)$, the operator $f \otimes g$ on $L^2(X)$ is defined analogously.

We list some repeatedly used notation for the convenience of the reader:

- The Laplacians on $\mathbb{R}^d$ and X are given, respectively, by

$$\Delta_0 = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \quad \text{and} \quad \Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial \theta^2}.$$

- $V_m(x) = \frac{1}{2\pi} \int_0^{2\pi} V(x,\theta) e^{-im\theta}\, d\theta$ for $m \in \mathbb{Z}$.
- $V^\# = V^\#(x,\theta) = V(x,\theta) - V_0(x)$.
- $B_l(\rho)$ and $D_l(\lambda_0, \rho)$ are open sets in $\hat{Z}$, defined in Sections 1 and 5, respectively.
- R_V is the (scattering) resolvent of $-\Delta + V$ on X; see Section 3B.
- $R_{V_0,0}$ is the (scattering) resolvent of $-\Delta_0 + V_0$ on $\mathbb{R}^d$; see Section 3.
- $m_V(\zeta_0)$ is the multiplicity of $\zeta_0 \in \hat{Z}$ as a pole of R_V; see (4-1).
- $m_{V_0,0}(\lambda_0)$ is the multiplicity of $\lambda_0 \in \mathbb{C}$ as a pole of $R_{V_0,0}$; see (4-2).
- $\zeta_l : \{z \in \mathbb{C} : |z| < \sqrt{2l-1}\} \to B_l(\sqrt{2l-1}) \subset \hat{Z}$ is the (local) inverse of

$$B_l(\sqrt{2l-1}) \ni \zeta \mapsto \tau_l(\zeta) \in \{z \in \mathbb{C} : |z| < \sqrt{2l-1}\} \subset \mathbb{C}.$$

3. Odd-dimensional Euclidean scattering and continuation of the resolvent

We begin by fixing notation and recalling some well-known facts from Euclidean scattering theory. We then use these to give a self-contained proof that the resolvent of $-\Delta + V$ on X has a meromorphic continuation to $\hat{Z}$.

3A. ***The Euclidean resolvent.*** Let $V_0 \in L^\infty_c(\mathbb{R}^d)$, d odd, and set

$$R_{V_0,0}(\lambda) = (-\Delta_0 + V_0 - \lambda^2)^{-1} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

when $\operatorname{Im} \lambda > 0$. The 0 in the second place in the subscript is to help us think of this as a model operator, as we shall see. We shall later use the explicit expression for the resolvent as an integral when $d = 1$, $f \in L^2(\mathbb{R})$, and $\operatorname{Im} \lambda > 0$:

$$(R_{0,0}(\lambda) f)(x) = \frac{i}{2\lambda} \int e^{i\lambda|x-x'|} f(x')\, dx' \quad \text{for } d = 1. \tag{3-1}$$

From this we can see immediately that if $\chi \in L^\infty_c(\mathbb{R})$, then $\chi R_{0,0}(\lambda)\chi$ has a meromorphic continuation to $\mathbb{C} \setminus \{0\}$. The same is true when $d \geq 3$ is odd: if $\chi \in L^\infty_c(\mathbb{R}^d)$, then $\chi R_{0,0}(\lambda)\chi$ has an analytic continuation to the complex plane, see [Dyatlov and Zworski 2019, Theorem 3.3]. In higher dimensions, the Schwartz kernel is given in terms of a Hankel function. It is well known, see [Dyatlov and Zworski 2019, Theorem 3.8], that if $V_0,\ \chi \in L^\infty_c(\mathbb{R}^d)$, then $\chi R_{V_0,0}(\lambda)\chi$ has a meromorphic continuation to the complex plane. Alternatively, restricting the domain and enlarging the range, $R_{V_0,0}(\lambda) : L^2_c(\mathbb{R}^d) \to H^2_{\text{loc}}(\mathbb{R}^d)$ has a meromorphic continuation to $\mathbb{C}$.

The following lemma is well known, but we include it for completeness, as it is crucial for our arguments.

Lemma 3.1. *Let* $V_0,\ \chi \in L^\infty_c(\mathbb{R}^d)$. *Then there are constants* $C_0,\ C_1 > 0$ *so that* $\chi R_{V_0,0}(\lambda)\chi$ *is analytic in* $\{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| > C_0,\ \operatorname{Im}\lambda > -C_1 \log(1 + |\operatorname{Re}\lambda|)\}$. *Moreover, in this region* $\|\chi R_{V_0,0}(\lambda)\chi\| = O(|\lambda|^{-1})$.

Proof. Without loss of generality, we may assume $\chi V_0 = V_0$. Then

$$\chi R_{V_0,0}(\lambda)\chi = \chi R_{0,0}(\lambda)\chi(I + V_0 R_{0,0}(\lambda)\chi)^{-1}.$$

Since from (3-1) when $d = 1$ or [Dyatlov and Zworski 2019, Theorem 3.1] when $d \geq 3$, there is a $C > 0$ so that

$$\|V R_{0,0}(\lambda)\chi\| \leq C e^{C(\operatorname{Im}\lambda)_-}/|\lambda|,$$

where $(\operatorname{Im}\lambda)_- = \max(0, -\operatorname{Im}\lambda)$; the lemma follows immediately. □

3B. ***The resolvent of $-\Delta + V$ on X and the Riemann surface $\hat{Z}$.*** Recall that when $d = 1$, X is a manifold with infinite cylindrical ends. For a manifold with infinite cylindrical ends, the space to which the resolvent of a Schrödinger operator continues is determined by the distinct eigenvalues of the Laplacian on the cross-section of the end(s). Here that means $\{j^2\}_{j\in\mathbb{N}_0}$, since this is the set of (distinct) eigenvalues of $-d^2/d\theta^2$ on $\mathbb{S}^1$. As we show below, the resolvent for $-\Delta + V$ on $\mathbb{R}^d \times \mathbb{S}^1$ has a meromorphic continuation to the same space as that of the resolvent of $-\Delta + V$ on $\mathbb{R} \times \mathbb{S}^1$, provided d is odd.

For $j \in \mathbb{N}_0$ and $\zeta \in \mathbb{C}$, $\operatorname{Im}\zeta > 0$, set

$$\tau_j(\zeta) \stackrel{\text{def}}{=} (\zeta^2 - j^2)^{1/2}$$

with $\operatorname{Im}\tau_j(\zeta) > 0$. Set $\tau_{-j}(\zeta) = \tau_j(\zeta)$ if $j \in \mathbb{N}$.

The Riemann surface $\hat{Z}$ is defined to be the minimal Riemann surface on which, for each $j \in \mathbb{N}_0$, τ_j is a single-valued analytic function on $\hat{Z}$. We briefly describe its construction and some of its properties. Note that $\tau_0(\zeta) = \zeta$ for ζ in the upper half-plane, and this has, of course, an analytic continuation to $\mathbb{C}$. Now $\tau_1(\zeta) = \tau_{-1}(\zeta)$ is an analytic function of $\zeta \in \mathbb{C} \setminus ((-\infty, 1] \cup [1, \infty))$, and there is a minimal Riemann surface $\hat{Z}_1$ so that τ_1 extends analytically to $\hat{Z}_1$. This is a double cover of $\mathbb{C}$, ramified at the points ± 1. This process can be repeated for each $j \in \mathbb{N}$, resulting in a minimal Riemann surface $\hat{Z}$ on which τ_j is analytic for each $j \in \mathbb{N}_0$. We define a projection $p : \hat{Z} \to \mathbb{C}$ as follows. For ζ in the physical space, identified with the upper half-plane, $p(\zeta) = \zeta$, and p is in general the analytic continuation of this function. Then $\hat{Z}$ has infinitely many ramification points which project under p to $j \in \mathbb{Z} \setminus \{0\}$. We call the set $\{\zeta \in \hat{Z} : \operatorname{Im}\tau_j(\zeta) > 0 \text{ for all } j \in \mathbb{N}_0\}$ the *physical space*, or *physical region*. For further discussion of this Riemann surface; see [Melrose 1993, Section 6.6].

We shall say that a point $\zeta_0 \in \hat{Z}$ *corresponds to a threshold* if $\tau_0(\zeta_0) \in \mathbb{Z}$. Note that with this definition, all the ramification points of $\hat{Z}$ correspond to thresholds. In addition, the set of points corresponding to thresholds includes those points projecting to 0. These might naturally also be considered ramification points of $\hat{Z}$, as in some sense by choosing ζ^2 to originally be our spectral parameter we have already made the cuts corresponding to the zero threshold.

In order to separate variables below, we introduce the orthogonal projections $\mathcal{P}_k : L^2(X) \to L^2(X)$ defined for $k \in \mathbb{Z}$ by

$$(\mathcal{P}_k f)(x,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(x,\theta')(e^{ik(\theta-\theta')} + e^{-ik(\theta-\theta')})\, d\theta' \quad \text{if } k \in \mathbb{N},$$
$$(\mathcal{P}_0 f)(x,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(x,\theta')\, d\theta'.$$

We shall use these throughout the paper.

Let $V \in L^\infty_c(X)$. For $\zeta \in \mathbb{C}$ with $\operatorname{Im}\zeta > 0$, set $R_V(\zeta) = (-\Delta + V - \zeta^2)^{-1}$. Consider first the special case where $V \in L^\infty_c(X)$ is independent of θ. Then $V = V_0$, and we can think of V_0 as an element of $L^\infty_c(X)$ or of $L^\infty_c(\mathbb{R}^d)$. In this special case we can separate variables to obtain

$$R_{V_0}(\zeta) = \sum_{k=0}^{\infty} R_{V_0,0}(\tau_k(\zeta))\mathcal{P}_k. \tag{3-2}$$

The explicit expression (3-2) for R_{V_0} using separation of variables shows that if $\chi \in L^\infty_c(X)$, then $\chi R_{V_0}\chi$ and $R_{V_0} : L^2_c(X) \to H^2_{\mathrm{loc}}(X)$ have meromorphic continuations to $\hat{Z}$. In fact, the same is true for $\chi R_V \chi$ and R_V for general $V \in L^\infty_c(X)$. This is well known, at least when $d = 1$, see [Goldstein 1974; Guillopé 1989; Melrose 1993], though we sketch a proof below, valid for all odd d.

If $\zeta \in \mathbb{C}$, $\operatorname{Im}\zeta > 0$, then

$$(-\Delta + V - \zeta^2)R_0(\zeta) = I + VR_0(\zeta).$$

Multiplying by a function $\chi \in L^\infty_c(X)$ with $\chi V = V$,

$$(-\Delta + V - \zeta^2)R_0(\zeta)\chi = \chi(I + VR_0(\zeta)\chi),$$

implying that

$$\chi R_0(\zeta)\chi = \chi R_V(\zeta)\chi(I + VR_0(\zeta)\chi) \tag{3-3}$$

or

$$\chi R_V(\zeta)\chi = \chi R_0(\zeta)\chi(I + VR_0(\zeta)\chi)^{-1}. \tag{3-4}$$

Using $I - VR_0(\zeta)\chi(I + VR_0(\zeta)\chi)^{-1} = (I + VR_0(\zeta)\chi)^{-1}$ and (3-4) yields

$$(I + VR_0(\zeta)\chi)^{-1} = I - VR_V(\zeta)\chi; \tag{3-5}$$

compare [Dyatlov and Zworski 2019, (2.2.15)–(2.2.16)]. Likewise, writing

$$V^\# \stackrel{\text{def}}{=} V - V_0, \tag{3-6}$$

we find, making the additional hypothesis that $\chi V^\# = V^\#$,

$$\chi R_{V_0}(\zeta)\chi = \chi R_V(\zeta)\chi(I + V^\# R_{V_0}(\zeta)\chi) \quad \text{and} \quad (I + V^\# R_{V_0}(\zeta)\chi)^{-1} = I - V^\# R_V(\zeta)\chi. \tag{3-7}$$

Each of these is helpful. Since $VR_0(\zeta)\chi : L^2(X) \to L^2(X)$ is compact and has a meromorphic extension to $\hat{Z}$, and $I + VR_0(\zeta)\chi$ is invertible for ζ in the physical space with $\operatorname{Im}\zeta$ sufficiently large, meromorphic Fredholm theory ensures that $(I + VR_0(\zeta)\chi)^{-1}$ is a meromorphic operator-valued function on $\hat{Z}$, and each

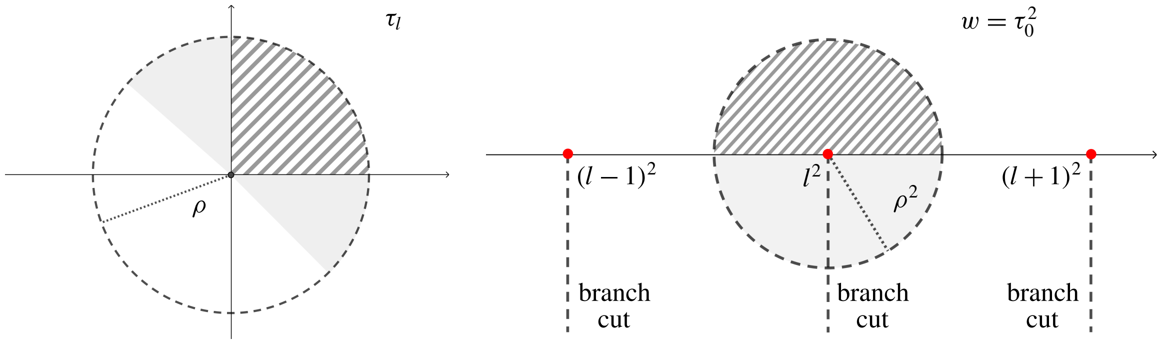

Figure 2. On the left, $B_l(\rho)$ in the τ_l-coordinate; on the right, a *portion* of $B_l(\rho)$ in the $w = (\tau_0(\zeta))^2$-coordinate for larger context. In the w diagram, $(-\Delta + V - w)^{-1}$ is bounded in the upper half-plane and the red dots on the horizontal axis indicate thresholds. The hatching denotes the portion of $B_l(\rho)$ in the physical region; the shaded region indicates the rest which is visible in the w plane diagram.

of (3-3)–(3-5) and (3-7) holds on all of $\hat{Z}$. Moreover, writing $I + VR_0 = (I + VR_0(I-\chi))(I + VR_0\chi)$ and noting that $(I + VR_0(I-\chi))^{-1} = I - VR_0(I-\chi)$, this shows that

$$R_V(\zeta) = R_0(\zeta)(I + VR_0(\zeta)\chi))^{-1}(I - VR_0(\zeta)(I-\chi)) : L^2_c(X) \to H^2_{\mathrm{loc}}(X)$$

has a meromorphic continuation to $\hat{Z}$.

We note from (3-2) that R_{V_0} is bounded on $L^2(X)$ when ζ is in the physical space and is away from a discrete set of poles (corresponding to eigenvalues). The same is true of R_V.

Throughout this paper we shall mainly work with subsets of $B_l(\sqrt{2l-1}) \subset \hat{Z}$, for $l \in \mathbb{N}$. We recall $B_l(\rho)$ is defined to be the connected component of $\{\zeta \in \hat{Z} : |\tau_l(\zeta)| < \rho\}$ which has nonempty intersection with both the physical space and the portion of $\hat{Z}$ with $\operatorname{Re}\tau_0(\zeta) > 0$. The choice of $\sqrt{2l-1}$ in $B_l(\sqrt{2l-1})$ is made because then (for $l \geq 1$) $B_l(\sqrt{2l-1})$ contains only a single point of $\hat{Z}$ corresponding to a threshold, the one associated with the eigenvalue l^2 of $-d^2/d\theta^2$ on $\mathbb{S}^1$. If $\epsilon > 0$, then $z = \tau_l(\zeta)$ gives the complex structure of $\hat{Z}\restriction_{B_l(\sqrt{2l-1}-\epsilon)}$, and $B_l(\sqrt{2l-1}-\epsilon)$ is naturally identified with a disk $B_{\mathbb{C}}(\sqrt{2l-1}-\epsilon)$ of radius $\sqrt{2l-1}-\epsilon$ in $\mathbb{C}$, centered at the origin. In this coordinate z, we have that $z = 0$ corresponds to the threshold l^2 and the intersection of $B_{\mathbb{C}}(\sqrt{2l-1}-\epsilon)$ with the first quadrant corresponds to a region in physical space, and so has $\operatorname{Im}\tau_k > 0$ for all $k \in \mathbb{N}_0$. If z lies in the intersection of $B_{\mathbb{C}}(\sqrt{2l-1}-\epsilon)$ with the fourth quadrant, then $\operatorname{Im}\tau_k(\zeta(z)) < 0$ for $0 \leq k \leq l$ and $\operatorname{Im}\tau_k(\zeta(z)) > 0$ for $k > l$. On the other hand, if z lies in the intersection of $B_{\mathbb{C}}(\sqrt{2l-1}-\epsilon)$ with the second quadrant, then $\operatorname{Im}\tau_k(\zeta(z)) < 0$ for $0 \leq k \leq l-1$ and $\operatorname{Im}\tau_k(\zeta(z)) > 0$ for $k \geq l$. Figure 2 shows a schematic of $B_l(\rho)$ and, for context, the portion of $B_l(\rho)$ which is visible in the $w = (\tau_0(\zeta))^2$ plane. We note that while we have used the spectral parameter ζ^2 in the definition of $R_V(\zeta)$ to be consistent with the usual odd-dimensional Euclidean scattering resolvent, the diagram on the right in Figure 2 uses as spectral parameter $w = (\tau_0(\zeta))^2$ to make a more easily digested diagram. To put the diagram in context, think of $(-\Delta + V - w)^{-1}$ as having meromorphic continuation from the upper half-plane to $\{w \in \mathbb{C} \setminus (\bigcup_{j=0}^{\infty}(j^2 + i(-\infty, 0]))\}$ (which can, of course, be identified with a subset of $\hat{Z}$).

On the open set $B_l(\sqrt{2l-1}-\epsilon)$, $z=\tau_l(\zeta)$ is a coordinate compatible with the complex structure of $\hat{Z}$. Thus it is natural to use τ_l as a local coordinate. We write

$$\zeta_l : \{z \in \mathbb{C} : |z| < \sqrt{2l-1}-\epsilon\} \to B_l(\sqrt{2l-1}-\epsilon) \subset \hat{Z}$$

as the function satisfying

$$\zeta_l(\tau_l(\zeta)) = \zeta \quad \text{for all } \zeta \in B_l(\sqrt{2l-1}-\epsilon).$$

We note that if $\zeta \in B_l(\sqrt{2l-1}-\epsilon)$, then $\operatorname{Re}\tau_j(\zeta) > 0$ if $0 \le j < l$, and $\operatorname{Im}\tau_j(\zeta) > 0$ if $j > l$.

The next lemma follows easily from (3-2) and Lemma 3.1, but is fundamental to many of the results of this paper.

Lemma 3.2. *Let $V_0 \in L^\infty_c(\mathbb{R})$, $\alpha > 0$, and $\chi \in L^\infty_c(X)$. Then for l sufficiently large, uniformly for $\zeta \in B_l(\alpha \log l)$, we have $\|\chi(I-\mathcal{P}_l)R_{V_0}(\zeta)\chi\| = O(l^{-1/2})$.*

Proof. Set $\tau_l = z$ and $|z| < \alpha \log l$. Then using the identity

$$\tau_k^2 = \tau_l^2 + l^2 - k^2,$$

for l sufficiently large, $|\tau_k(\zeta_l(z))| > \sqrt{l}$ for $k \in \mathbb{N}_0$, $k \neq l$. Moreover, $\operatorname{Im}\tau_k(\zeta_l(z)) > 0$ if $k > l$, and $|\operatorname{Im}\tau_k(\zeta_l(z))| = O(1)$ if $k < l$. Then the lemma follows from Lemma 3.1 and the representation of $R_{V_0,0}$ given by (3-2). □

4. Multiplicities of poles and results of [Gohberg and Sigal 1971]

For an operator A depending meromorphically on $\zeta \in \mathbb{C}$ or $\zeta \in \hat{Z}$, let $\Xi(A, \zeta_0)$ denote the principal part of the Laurent expansion of A at ζ_0. For $V \in L^\infty_c(X)$ and $\zeta_0 \in \hat{Z}$, define

$$m_V(\zeta_0) \overset{\text{def}}{=} \operatorname{rank} \Xi(R_V, \zeta_0)(L^2_c(X)). \tag{4-1}$$

Suppose $\chi \in L^\infty_c(X)$ satisfies $\chi V = V$ (and, if $V \equiv 0$, χ is nontrivial). Then it follows from an expansion of R_V at its singularities as in [Dyatlov and Zworski 2019, Theorems 2.5, 2.7, 3.9, 3.17] and a unique continuation result, e.g., [Jerison and Kenig 1985, Remark 6.7], that $m_V(\zeta_0) = \operatorname{rank} \Xi(\chi R_V \chi, \zeta_0)$. Note that if R_V is analytic at ζ_0, then $m_V(\zeta_0) = 0$.

If $V_0 \in L^\infty_c(\mathbb{R}^d)$ and $\lambda_0 \in \mathbb{C}$ we define

$$m_{V_0,0}(\lambda_0) \overset{\text{def}}{=} \operatorname{rank} \Xi(R_{V_0,0}, \lambda_0)(L^2_c(\mathbb{R}^d)). \tag{4-2}$$

Again, the second 0 in the subscript is meant to help us think of this as corresponding to a model. As for m_V, if $\chi \in L^\infty_c(\mathbb{R})$ satisfies $\chi V = V$ (and χ is nontrivial if $V_0 \equiv 0$), then $m_{V_0,0}(\lambda_0) = \operatorname{rank} \Xi(\chi R_{V_0,0}\chi, \lambda_0)$.

We recall some definitions and results of [Gohberg and Sigal 1971], adapted to our setting.

Let A be a bounded linear operator on a complex Hilbert space $\mathcal{H}$, depending meromorphically on $z \in \Omega \subset \mathbb{C}$, where Ω is a domain. Near a point $z_0 \in \Omega$, we have $A(z) = \sum_{j=-n}^{\infty}(z-z_0)^j A_j$. If the operators $A_{-1}, \dots, A_{-n}$ are finite rank, then we say A is *finitely meromorphic* at z_0. If A is finitely meromorphic at each $z_0 \in \Omega$, then A is *finitely meromorphic on* Ω. Suppose that A is a compact operator on $\mathcal{H}$, A is finitely meromorphic on Ω, and $I + A(z_1)$ is invertible for some $z_1 \in \Omega$. Then by the meromorphic Fredholm theorem, $(I + A(z))^{-1}$ is finitely meromorphic on Ω.

Suppose A is a finitely meromorphic operator on a domain Ω, with $(I+A)^{-1}$ also finitely meromorphic on Ω. Below we denote the derivative of A with respect to z by $\dot{A}$. Then for $z_0 \in \Omega$, define

$$M(I + A, z_0) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \operatorname{tr} \int_{\gamma_{z_0}} \dot{A}(z)(I + A(z))^{-1}\, dz,$$

where γ_{z_0} is a positively oriented circle, centered at z_0 with radius ϵ. Here we choose ϵ small enough that $\{|z - z_0| \le \epsilon\} \subset \Omega$ and neither A nor $(I + A)^{-1}$ has poles in the set $\{z : 0 < |z - z_0| \le \epsilon\}$.

Our definition of finitely meromorphic is local, so it makes sense on domains in $\hat{Z}$ as well, using a local coordinate compatible with the complex structure of $\hat{Z}$. Likewise, we can define $M(I + A, \zeta_0)$ for such operators. (This requires the choice of a circle small enough that it has in its interior at most one ramification point of $\hat{Z}$.)

We will say the linear operator A on the Hilbert space $\mathcal{H}$ satisfies hypotheses (H1) on a domain $\Omega \subset \mathbb{C}$ if A is a finitely meromorphic, compact operator defined on Ω, and $I + A$ is invertible for at least one point in Ω and hence has a finitely meromorphic inverse in Ω.

The following lemma is a direct consequence of [Gohberg and Sigal 1971, Proposition 5].

Lemma 4.1 [Gohberg and Sigal 1971, Proposition 5]. *Suppose $A,\ B : \mathcal{H} \to \mathcal{H}$ satisfy hypotheses* (H1), *and suppose B and $(I + B)^{-1}$ are analytic on Ω. Then for $z_0 \in \Omega$,*

$$M(I + A, z_0) = M((I + A)(I + B), z_0).$$

Let $T : L^2(X) \to L^2(X)$ be a bounded linear operator. We shall repeatedly make use of the straightforward identities

$$I + T\mathcal{P}_l = (I + \mathcal{P}_l T\mathcal{P}_l)(I + (I - \mathcal{P}_l)T\mathcal{P}_l) \quad \text{and} \quad (I + (I - \mathcal{P}_l)T\mathcal{P}_l)^{-1} = I - (I - \mathcal{P}_l)T\mathcal{P}_l. \tag{4-3}$$

Lemma 4.2. *Let $A : L^2(X) \to L^2(X)$ satisfy hypotheses* (H1) *on a domain Ω. Then for $z_0 \in \Omega$,*

$$M(I + A\mathcal{P}_l, z_0) = M(I + \mathcal{P}_l A\mathcal{P}_l, z_0).$$

Proof. Using (4-3) implies that

$$M(I+A\mathcal{P}_l, z_0) = \frac{1}{2\pi i} \operatorname{tr} \int_{\gamma_{z_0}} \dot{A}(z)\mathcal{P}_l(I+A(z)\mathcal{P}_l)^{-1}\, dz = \frac{1}{2\pi i} \operatorname{tr} \int_{\gamma_{z_0}} \dot{A}(z)\mathcal{P}_l(I+\mathcal{P}_l A(z)\mathcal{P}_l)^{-1}\, dz, \tag{4-4}$$

where γ_{z_0} is a small circle centered at z_0 as in the definition of $M(I + A, z_0)$.

Because $\mathcal{P}_l$ is a projection, using the cyclicity of the trace, $\operatorname{tr}(B\mathcal{P}_l) = \operatorname{tr}(\mathcal{P}_l B\mathcal{P}_l)$ for a trace class operator $B : L^2(X) \to L^2(X)$. Using this in (4-4) gives

$$M(I + A\mathcal{P}_l, z_0) = \frac{1}{2\pi i} \operatorname{tr} \int_{\gamma_{z_0}} \mathcal{P}_l \dot{A}(z)\mathcal{P}_l(I + \mathcal{P}_l A(z)\mathcal{P}_l)^{-1}\, dz = M(I + \mathcal{P}_l A\mathcal{P}_l, z_0). \qquad \square$$

The following proposition is a variant of a well-known result in the study of resonances of Schrödinger operators on $\mathbb{R}^d$; compare, e.g., [Dyatlov and Zworski 2019, Theorem 3.15].

Proposition 4.3. *Suppose $V \in L^\infty_c(X)$ is nontrivial, and let $\chi \in L^\infty_c(X)$ satisfy $\chi V = V$. Then the operator $R_V(\zeta)$ has a pole at $\zeta_0 \in \hat{Z}$ if and only if the operator $I + VR_0(\zeta)\chi$ has nontrivial null space at ζ_0. Moreover, if ζ_0 does not correspond to a threshold, then*

$$m_V(\zeta_0) = M(I + VR_0\chi, \zeta_0).$$

Proof. A proof follows by essentially the same method as [Dyatlov and Zworski 2019, Theorem 3.15]. □

We recall the notation $V^\# = V - V_0$. Another useful identity is the following.

Lemma 4.4. *Let $\chi \in L^\infty_c(X)$ satisfy $\chi V = V$ and $\chi V_0 = V_0$. Then for $\zeta_0 \in \hat{Z}$ so that ζ_0 does not correspond to a threshold, we have*

$$m_V(\zeta_0) = M(I + V^\# R_{V_0}\chi, \zeta_0) + m_{V_0}(\zeta_0).$$

Proof. We first note that

$$I + VR_0\chi = (I + V^\# R_0\chi(I + V_0R_0\chi)^{-1})(I + V_0R_0\chi) = (I + V^\# R_{V_0}\chi)(I + V_0R_0\chi). \tag{4-5}$$

Thus using Proposition 4.3 and [Gohberg and Sigal 1971, Theorem 5.2] gives

$$\begin{aligned} m_V(\zeta_0) = M(I + VR_0\chi, \zeta_0) &= M(I + V^\# R_{V_0}\chi, \zeta_0) + M(I + V_0R_0\chi, \zeta_0) \\ &= M(I + V^\# R_{V_0}\chi, \zeta_0) + m_{V_0}(\zeta_0). \end{aligned}$$ □

Lemma 4.5. *Suppose $V, \chi \in L^\infty_c(X)$, with $\chi V = V$, and χ is independent of θ. Let $\alpha > 0$. Then there is an $L > 0$ so that for $l > L$*

$$M(I + VR_0\chi, \zeta_0) = M(I + \mathcal{P}_l(I + VR_0(I - \mathcal{P}_l)\chi)^{-1}VR_0\mathcal{P}_l\chi, \zeta_0)$$

for any $\zeta_0 \in B_l(\alpha \log l)$.

Proof. We begin by writing

$$I + VR_0\chi = (I + VR_0(I - \mathcal{P}_l)\chi)(I + (I + VR_0(I - \mathcal{P}_l)\chi)^{-1}VR_0\mathcal{P}_l\chi)$$

and noting that since by Lemma 3.2 $\|VR_0(I - \mathcal{P}_l)\chi\| = O(l^{-1/2})$ uniformly on $B_l(\alpha \log l)$ there is an $L > 0$ so that for $l > L$, $(I + VR_0(I - \mathcal{P}_l)\chi)^{-1}$ is analytic on $B_l(\alpha \log l)$. Thus for these l by Lemma 4.1 $M(I + VR_0\chi, \zeta_0) = M(I + (I + VR_0(I - \mathcal{P}_l)\chi)^{-1}VR_0\mathcal{P}_l\chi, \zeta_0)$ for any $\zeta_0 \in B_l(\alpha \log l)$. An application of Lemma 4.2 completes the proof. □

Lemma 4.6. *Let $V, \chi \in L^\infty_c(X)$, with V satisfying* (1-1), *$\chi V = V$, and χ independent of θ. Set $A_{l,V} = (I + VR_0(I - \mathcal{P}_l)\chi)^{-1}$ and $B_{l,V} = VR_0\mathcal{P}_l\chi$. Let $K \subset \mathbb{C}$ be a compact set such that $R_{V_0,0}$ is analytic on K, and suppose $0 \notin K$ if $d = 1$. Choose $\rho > 0$ so that $K \subset \{\lambda \in \mathbb{C} : |\lambda| < \rho\}$, and set $K_l = \{\zeta \in B_l(\rho) : \tau_l(\zeta) \in K\}$. Then for sufficiently large l,*

$$\|\mathcal{P}_l(A_{l,V}B_{l,V} - A_{l,V_0}B_{l,V_0})\| = O(l^{-\delta}) \tag{4-6}$$

and

$$\|(I + \mathcal{P}_lA_{l,V_0}B_{l,V_0})^{-1}\mathcal{P}_l(A_{l,V}B_{l,V} - A_{l,V_0}B_{l,V_0})\| = O(l^{-\delta}) \tag{4-7}$$

uniformly for $\zeta \in K_l$.

Proof. We write

$$\mathcal{P}_l(A_{l,V}B_{l,V} - A_{l,V_0}B_{l,V_0}) = \mathcal{P}_l(A_{l,V} - A_{l,V_0})B_{l,V} + \mathcal{P}_l A_{l,V_0}(B_{l,V} - B_{l,V_0}). \tag{4-8}$$

By Lemma 3.2, $\|A_{l,V} - I\| = O(l^{-1/2})$ and $\|A_{l,V_0} - I\| = O(l^{-1/2})$ uniformly on $B_l(\rho)$, so that the first term on the left-hand side is $O(l^{-1/2})$. Moreover,

$$\mathcal{P}_l A_{l,V_0}(B_{l,V} - B_{l,V_0}) = A_{l,V_0}\mathcal{P}_l(B_{l,V} - B_{l,V_0}) = A_{l,V_0}\mathcal{P}_l V^{\#} R_0 \mathcal{P}_l,$$

and $\|\mathcal{P}_l V^{\#}\mathcal{P}_l\| = O(l^{-\delta})$ by our assumption on V. Hence the norm of the second term on the right-hand side of (4-8) is $O(l^{-\delta})$. This proves (4-6).

On K_l,

$$I + \mathcal{P}_l A_{l,V_0}B_{l,V_0} = I + \mathcal{P}_l B_{l,V_0} + O(l^{-1/2}) = I + \mathcal{P}_l V_0 R_0 \chi + O(l^{-1/2}). \tag{4-9}$$

But

$$(I + \mathcal{P}_l V_0 R_0 \chi)^{-1} = I - \mathcal{P}_l + (I - V_0 R_{V_0,0}(\tau_l)\chi)\mathcal{P}_l = I - \mathcal{P}_l + T\mathcal{P}_l,$$

where T is given by $T = (I + V_0 R_{0,0}(\tau_l)\chi)^{-1} = I - V_0 R_{V_0,0}(\tau_l)\chi$. By our choice of K, we have that T is uniformly bounded for $\tau_l \in K$ or for $\zeta \in K_l$, and hence $(I + \mathcal{P}_l V_0 R_0 \chi)^{-1}$ is bounded on K_l. Using (4-9), this shows $(I + \mathcal{P}_l A_{l,V_0}B_{l,V_0})^{-1}$ is bounded on K_l, and thus, by (4-6), we get (4-7). □

5. A resolvent estimate and localizing the resonances in the L^∞ case: Proofs of Theorems 1.1, 1.2, and 1.3

In this section we prove Theorems 1.1–1.3 in the case of an L^∞ potential V, providing a high-energy localization of the resonances in sets $B_l(\rho)$. We also prove Proposition 5.4 and Lemma 5.5, which show that the resolvent for the potential V_0 is, at high energies, a good approximation of the resolvent for the potential V away from poles.

We shall use notation for a disk in the τ_l-coordinate in $B_l(\rho)$. For $\lambda_0 \in \mathbb{C}$ and $r_0 > 0$, set $\rho = |\lambda_0| + r_0 + 1$, and define, for $2l > \rho^2 + 1$, $D_l(\lambda_0, r_0) \subset B_l(\rho) \subset \hat{Z}$ by

$$D_l(\lambda_0, r_0) \overset{\text{def}}{=} \{\zeta \in B_l(\rho) : |\tau_l(\zeta) - \lambda_0| < r_0\}.$$

A preliminary step in the proof of Theorem 1.1 is the following proposition, which provides an initial localization of the resonances.

Proposition 5.1. *Let $V \in L^\infty_c(X)$ satisfy* (1-1). *Suppose $\lambda_0 \in \mathbb{C}$, $\lambda_0 \neq 0$ is a resonance of $-\Delta_0 + V_0$ on $\mathbb{R}^d$, of multiplicity $m_{V_0,0}(\lambda_0)$. Then there are L, $\epsilon > 0$ so that*

$$\sum_{\substack{\zeta \in D_l(\lambda_0,\epsilon) \\ m_V(\zeta) > 0}} m_V(\zeta) = 2m_{V_0,0}(\lambda_0)$$

when $l > L$.

Proof. Choose $\epsilon > 0$ so that $R_{V_0,0}(\lambda)$ is analytic on $0 < |\lambda - \lambda_0| \le \epsilon$ and $\epsilon < |\lambda_0|$. By our expression (3-2) for R_{V_0}, using separation of variables and Lemmas 3.1 and 3.2, $m_{V_0}(\zeta_l(\lambda_0)) = 2m_{V_0,0}(\lambda_0)$ for l sufficiently large. The factor of 2 on the right comes from the fact that the range of $\mathcal{P}_l$ (as an operator on $L^2(\mathbb{S}^1)$) has rank 2 for $l > 0$. Choose $\chi \in L^\infty_c(X)$ independent of θ so that $\chi V = V$. From Proposition 4.3 and our choice of ϵ, for l sufficiently large,

$$m_{V_0}(\zeta_l(\lambda_0)) = M(I + V_0R_0\chi, \zeta_l(\lambda_0)) = \sum_{\substack{\zeta\in D_l(\lambda_0,\epsilon)\\ M(I+V_0R_0\chi,\zeta)\neq 0}} M(I + V_0R_0\chi, \zeta).$$

Lemma 4.5 implies that if $W = V_0$ or $W = V$,

$$M(I + WR_0\chi, \zeta') = M(I + \mathcal{P}_l(I + WR_0(I - \mathcal{P}_l)\chi)^{-1}WR_0\mathcal{P}_l\chi, \zeta') \quad \text{for } \zeta' \in D_l(\lambda_0, \epsilon) \tag{5-1}$$

if l is sufficiently large.

By Lemma 4.6 and an operator Rouché theorem [Gohberg and Sigal 1971, Theorem 2.2], for l sufficiently large,

$$\begin{aligned}\sum_{\substack{\zeta\in D_l(\lambda_0,\epsilon)\\ M(I+\mathcal{P}_l(I+VR_0(I-\mathcal{P}_l)\chi)^{-1}VR_0\mathcal{P}_l\chi,\zeta)\neq 0}} & M(I + \mathcal{P}_l(I + VR_0(I - \mathcal{P}_l)\chi)^{-1}VR_0\mathcal{P}_l\chi, \zeta)\\ = \sum_{\substack{\zeta\in D_l(\lambda_0,\epsilon)\\ M(I+\mathcal{P}_l(I+V_0R_0(I-\mathcal{P}_l)\chi)^{-1}V_0R_0\mathcal{P}_l\chi,\zeta)\neq 0}} & M(I + \mathcal{P}_l(I + V_0R_0(I - \mathcal{P}_l)\chi)^{-1}V_0R_0\mathcal{P}_l\chi, \zeta).\end{aligned} \tag{5-2}$$

Combining (5-1) (with $W = V$ and with $W = V_0$), (5-2), and another application of Proposition 4.3, this time with V, proves the proposition. □

5A. *Proofs of Theorems 1.1 and 1.2 for $V \in L^\infty_c(X)$.* Theorem 1.1 follows from combining the result of Theorem 1.2 for L^∞ potentials and Proposition 5.1. In this section we prove Theorem 1.2 for L^∞ potentials V.

Recall by the definition of $\Xi(R_{V_0,0}, \lambda_0)$, if $\lambda_0 \in \mathbb{C}$ is a pole of $R_{V_0,0}$, then $R_{V_0,0}(\lambda) - \Xi(R_{V_0,0}(\lambda), \lambda_0)$ is analytic at λ_0. Define

$$R^{\text{reg}}_{V_0}(\zeta; \lambda_0, l) \overset{\text{def}}{=} R_{V_0}(\zeta) - \Xi(R_{V_0}, \zeta_l(\lambda_0)). \tag{5-3}$$

For l sufficiently large, by (3-2) and Lemma 3.2

$$R^{\text{reg}}_{V_0}(\zeta; \lambda_0, l) = R_{V_0}(\zeta) - \Xi(R_{V_0,0}(\lambda), \lambda_0)|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l. \tag{5-4}$$

Note that if R_{V_0} is analytic at $\zeta_l(\lambda_0)$, then $R^{\text{reg}}_{V_0}(\zeta; \lambda_0, l) = R_{V_0}(\zeta)$.

Lemma 5.2. *Suppose V, $\chi \in L^\infty_c(X)$ and V satisfies* (1-1). *Let $\lambda_0 \in \mathbb{C}$ and $R^{\text{reg}}_{V_0} = R^{\text{reg}}_{V_0}(\zeta; \lambda_0, l)$ be the operator defined in* (5-3). *If $R_{V_0,0}(\lambda)$ is analytic for $0 < |\lambda - \lambda_0| \le \epsilon$, then for l sufficiently large,*

$$V^\# R^{\text{reg}}_{V_0}(\zeta)\chi = V^\# R^{\text{reg}}_{V_0}(\zeta; \lambda_0, l)\chi$$

is analytic on $\overline{D}_l(\lambda_0, \epsilon)$, and as $l \to \infty$ the estimate $\|\chi R^{\text{reg}}_{V_0}(\zeta)V^\# R^{\text{reg}}_{V_0}(\zeta)\chi\| = O(l^{-\delta})$ holds uniformly for $\zeta \in \overline{D}_l(\lambda_0, \epsilon)$.

Proof. Without loss of generality we can assume χ is independent of θ and $\chi V = V$. By (3-2) and Lemma 3.2, for l sufficiently large, $R^{\mathrm{reg}}_{V_0}(\zeta)$ is analytic and bounded in $\overline{D}_l(\lambda_0,\epsilon)$. We write

$$\chi R^{\mathrm{reg}}_{V_0}(\zeta)V^{\#}R^{\mathrm{reg}}_{V_0}(\zeta)\chi = \chi R^{\mathrm{reg}}_{V_0}\chi(I-\mathcal{P}_l)V^{\#}R^{\mathrm{reg}}_{V_0}\chi + \chi R^{\mathrm{reg}}_{V_0}\chi\mathcal{P}_l V^{\#}R^{\mathrm{reg}}_{V_0}\chi(I-\mathcal{P}_l) + \chi R^{\mathrm{reg}}_{V_0}\chi\mathcal{P}_l V^{\#}R^{\mathrm{reg}}_{V_0}\chi\mathcal{P}_l. \quad (5\text{-}5)$$

Now for $\zeta \in \overline{D}_l(\lambda_0,\epsilon)$ and l sufficiently large, $\|\chi R^{\mathrm{reg}}_{V_0}\chi(I-\mathcal{P}_l)\| = O(l^{-1/2})$ uniformly in $\overline{D}_l(\lambda_0,\epsilon)$. Since $\|V_m\| = O(|m|^{-\delta})$ we have $\|\mathcal{P}_l V^{\#}\mathcal{P}_l\| = O(l^{-\delta})$, and so

$$\|\mathcal{P}_l V^{\#}R^{\mathrm{reg}}_{V_0}\chi\mathcal{P}_l\| = \|\mathcal{P}_l V^{\#}\mathcal{P}_l R^{\mathrm{reg}}_{V_0}\chi\mathcal{P}_l\| = O(l^{-\delta}). \qquad \square$$

A related lemma which we also need is the following.

Lemma 5.3. *Let $V,\ \chi \in L^{\infty}_c(X)$ with V satisfying* (1-1). *Let $K \subset \mathbb{C}$ be a compact set on which $R_{V_0,0}$ is analytic and suppose $K \subset \{\lambda \in \mathbb{C} : |\lambda| < \rho\}$. Set $K_l \overset{\mathrm{def}}{=} \{\zeta \in B_l(\rho) : \tau_l(\zeta) \in K\} \subset \hat{Z}$. Then for l sufficiently large, $\|\chi R_{V_0}V^{\#}R_{V_0}\chi\| = O(l^{-\delta})$ uniformly on K_l.*

Proof. This lemma can be proved by mimicking the proof of Lemma 5.2. Alternatively, it can be proved by covering K_l with a finite number of neighborhoods on which Lemma 5.2 holds. $\square$

Proof of Theorem 1.2 for $V \in L^{\infty}_c(X)$. We shall use the identities (3-7). Thus poles of R_V in $B_l(\rho)$ are the values of $\zeta \in B_l(\rho)$ such that $I + V^{\#}R_{V_0}(\zeta)\chi$ is not invertible. Here $\chi \in C^{\infty}_c(X)$ satisfies $\chi V = V$ and is independent of θ.

(1) For each $\lambda_j \in \Lambda_\rho,\ \lambda_j \neq 0$, let $\epsilon_j > 0$ be as guaranteed by Proposition 5.1, so that there are exactly $2m_{V_0,0}(\lambda_0)$ resonances (counted with multiplicity) of $-\Delta + V$ in $D_l(\lambda_j,\epsilon_j)$ for l sufficiently large. Set

$$K = \{\lambda \in \mathbb{C} : \epsilon' \le |\lambda| \le \rho \text{ and } |\lambda - \lambda_j| \ge \epsilon_j \text{ for all } \lambda_j \in \Lambda_\rho\},$$

$$K_l = \{\zeta \in B_l(\rho+1) : \tau_l(\zeta) \in K\} = \overline{B}_l(\rho) \setminus \Bigl(D_l(0,\epsilon') \bigcup_{\lambda_j \in \Lambda_\rho} D_l(\lambda_j,\epsilon_j)\Bigr).$$

By an application of Lemma 5.3, for l sufficiently large, $I + V^{\#}R_{V_0}(\zeta)\chi$ is invertible by its Neumann series on K_l. Thus by (3-7) R_V has no poles on K_l for l sufficiently large.

(2) Now we work on $D_l(\lambda_j,\epsilon_j)$ and set $R^{\mathrm{reg}}_{V_0}(\zeta) = R^{\mathrm{reg}}_{V_0}(\zeta; l, \lambda_j)$, so that

$$R^{\mathrm{reg}}_{V_0}(\zeta) = R_{V_0}(\zeta) - \Xi(R_{V_0,0}(\lambda),\lambda_j)|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l$$

for l sufficiently large. By our choice of ϵ_j this is analytic on $\overline{D}_l(\lambda_j,\epsilon_j)$ for large enough l. Then by Lemma 5.2 $I + V^{\#}R^{\mathrm{reg}}_{V_0}(\zeta)\chi$ is invertible in $D_l(\lambda_j,\epsilon_j)$, with

$$(I + V^{\#}R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^{-1} = I - V^{\#}R^{\mathrm{reg}}_{V_0}(\zeta)\chi + O_{L^2(X)\to L^2(X)}(l^{-\delta})$$

for $\zeta \in \overline{D}_l(\lambda_j,\epsilon_j)$. Thus on $\overline{D}_l(\lambda_j,\epsilon_j)$,

$$I + V^{\#}R_{V_0}\chi = (I + V^{\#}R^{\mathrm{reg}}_{V_0}(\zeta)\chi)(I + (I + V^{\#}R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^{-1}V^{\#}\Xi(R_{V_0,0}(\lambda),\lambda_0)|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l\chi). \quad (5\text{-}6)$$

By (5-6) and (4-3), $I+V^\# R_{V_0}\chi$ is invertible at a point $\zeta\in D_l(\lambda_j,\epsilon_j)$ if and only if

$$I+\mathcal{P}_l(I+V^\# R_{V_0}^{\mathrm{reg}}(\zeta)\chi)^{-1}V^\#\Xi(R_{V_0,0}(\lambda),\lambda_0)|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l\chi$$

is invertible at ζ. There is a C_j so that $\|\chi\,\Xi(R_{V_0,0},\lambda_j)\chi\|\le C_j|\lambda-\lambda_j|^{-m_{V_0,0}(\lambda_j)}$ on $\{\lambda\in\mathbb{C}:|\lambda-\lambda_j|\le\epsilon_j\}$; see [Dyatlov and Zworski 2019, Theorems 2.5 and 3.9]. Thus on $D_l(\lambda_j,\epsilon_j)$, using Lemma 5.2,

$$\begin{aligned}\|\mathcal{P}_l(I&+V^\# R_{V_0}^{\mathrm{reg}}(\zeta)\chi)^{-1}V^\#\Xi(R_{V_0,0}(\lambda,\lambda_0))|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l\chi\|\\&=\Big\|\sum_{j=0}^{\infty}\mathcal{P}_l(-V^\# R_{V_0}^{\mathrm{reg}}(\zeta))^jV^\#\Xi(R_{V_0,0}(\lambda,\lambda_0))|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l\chi\Big\|\\&\le\|\mathcal{P}_l(I-V^\# R_{V_0}^{\mathrm{reg}}(\zeta))V^\#\Xi(R_{V_0,0}(\lambda,\lambda_0))|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l\chi\|+C_j'l^{-\delta}|\tau_l(\zeta)-\lambda_j|^{-m_{V_0,0}(\lambda_j)}.\end{aligned}$$

Now we use Lemma 3.2, $\|V_m\|_{L^\infty}=O(|m|^{-\delta})$, and the fact that $\mathcal{P}_l$ commutes with $R_{V_0,0}$ so that

$$\|\mathcal{P}_l(I-V^\# R_{V_0}^{\mathrm{reg}}(\zeta))V^\#\mathcal{P}_l\|=O(l^{-\delta})$$

on $\overline{D}_l(\lambda_j,\epsilon_j)$. Thus there is a (new) C_j' so that

$$\|\mathcal{P}_l(I+V^\# R_{V_0}^{\mathrm{reg}}(\zeta))^{-1}V^\#\Xi(R_{V_0,0}(\lambda,\lambda_0))|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l\chi\|\le C_j'l^{-\delta}|\tau_l(\zeta)-\lambda_j|^{-m_{V_0,0}(\lambda_j)}$$

on $\overline{D}_l(\lambda_j,\epsilon_j)$. Therefore

$$I+\mathcal{P}_l(I+R_{V_0}^{\mathrm{reg}}(\zeta))^{-1}\Xi(R_{V_0,0}(\lambda,\lambda_0))|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l\chi$$

is invertible in this region if $|\tau_l(\zeta)-\lambda_j|\ge C_jl^{-\delta/m_{V_0,0}(\lambda_j)}$, where we can take $C_j=(2C_j')^{1/m_{V_0,0}(\lambda_j)}$. Taking $\widetilde{C}=\max_{\lambda_j\in\Lambda_\rho}C_j$ finishes the proof of Theorem 1.2 away from $\tau_l=0$.

(3) If $R_{V_0,0}(\lambda)$ does not have a pole at the origin, then there is a $\delta>0$ so that for l sufficiently large, $R_{V_0}(\zeta)$ is analytic in $\overline{B}_l(\delta)$. Thus by Lemma 5.3, for l sufficiently large, $R_V(\zeta)$ is analytic in $\overline{B}_l(\delta)$. ☐

5B. ***Approximating the resolvent R_V.*** In a sense made precise below in Proposition 5.4 and Lemma 5.5, at high energies R_{V_0} approximates R_V well away from resonances. The first result is useful for neighborhoods of thresholds.

Proposition 5.4. *Let $V,\ \chi\in L^\infty_c(X)$, with V satisfying* (1-1). *Let $K\subset\mathbb{C}$ be a compact set on which $R_{V_0,0}$ is analytic and suppose $K\subset\{\lambda\in\mathbb{C}:|\lambda|<\rho\}$. Define $K_l\overset{\mathrm{def}}{=}\{\zeta\in B_l(\rho):\tau_l(\zeta)\in K\}\subset\hat{Z}$. Then for l sufficiently large, R_V is analytic on K_l. Moreover, if $\chi\in L^\infty_c(X)$, then $\|\chi(R_V(\zeta)-R_{V_0}(\zeta))\chi\|=O(l^{-\delta})$ uniformly for $\zeta\in K_l$.*

Proof. Without loss of generality we may assume χ is independent of θ and satisfies $\chi V=V$. Then $\chi R_{V_0}\chi=\chi R_V\chi(I+V^\# R_{V_0}\chi)$. Since by Lemma 5.3 $\|(V^\# R_{V_0}\chi)^2\|\le\frac12$ on K_l for l sufficiently large, $I+V^\# R_{V_0}\chi$ is invertible as $(I+V^\# R_{V_0}\chi)^{-1}=\sum_{j=0}^\infty(-V^\# R_{V_0}\chi)^j$, and thus R_V is analytic on K_l. Moreover,

$$\chi(R_V(\zeta)-R_{V_0}(\zeta))\chi=\chi\sum_{j=1}^{\infty}R_{V_0}(\zeta)(-V^\# R_{V_0}(\zeta)\chi)^j.$$

By applying Lemma 5.3 twice, this becomes

$$\chi(R_V(\zeta)-R_{V_0}(\zeta))\chi=-\chi R_{V_0}(\zeta)V^\# R_{V_0}(\zeta)\chi+O_{L^2\to L^2}(l^{-\delta})=O_{L^2\to L^2}(l^{-\delta}).$$ ☐

A similar result with a similar proof is the following lemma. The points $\zeta \in \hat{Z}$ considered in this lemma lie on the boundary of the physical space, but are away from the thresholds.

Lemma 5.5. *Let* $V, \chi \in L^\infty_c(X)$, *with* V *satisfying* (1-1). *Then there are constants* $M, L > 0$ *so that*

$$\text{if } l > L,\ \zeta \in B_l(\sqrt{2l-1}),\ \tau_l(\zeta) \in i[0,\infty),\ \text{and } M < \frac{\tau_l(\zeta)}{i} < \sqrt{2l-1} - \frac{M}{\sqrt{l}},$$

$$\text{then } \|\chi(R_V(\zeta) - R_{V_0}(\zeta))\chi\| = O(l^{-\delta}). \tag{5-7}$$

Likewise, there are constants $M_1, L_1 > 0$ *so that*

$$\text{if } l > L_1,\ \zeta \in B_l(\sqrt{2l-1}),\ \tau_l(\zeta) \in [0,\infty),\ \text{and } M_1 < \tau_l(\zeta) < \sqrt{2l-1} - \frac{M_1}{\sqrt{l}},$$

$$\text{then } \|\chi(R_V(\zeta) - R_{V_0}(\zeta))\chi\| = O(l^{-\delta}). \tag{5-8}$$

Proof. This proof is very similar to the proof of Proposition 5.4. We outline the proof of the first statement only, as the proof of the second is analogous.

Without loss of generality, we may assume χ is independent of θ and satisfies $\chi V = V$.

We next note that if $\zeta \in B_l(\sqrt{2l-1})$, then for $l > 3$ either $|\tau_l(\zeta)| > \frac{1}{4}\sqrt{2l-1}$ or $|\tau_{l-1}(\zeta)| > \frac{1}{4}\sqrt{2l-1}$ or both are true. In either case, if $\tau_l(\zeta) \in i[0,\infty)$, then there is a $c_0 > 0$ so that $|\tau_j(\zeta)| > c_0 l^{1/2}$ for $j \neq l, l-1$. Moreover, again with $\tau_l(\zeta) \in i[0,\infty)$, $\operatorname{Im}\tau_j(\zeta) > 0$ if $j > l$ and $\operatorname{Im}\tau_j(\zeta) = 0$ if $0 \le j < l$.

Suppose $\zeta \in B_l(\sqrt{2l-1})$, $\tau_l(\zeta) \in i[0,\infty)$, and $|\tau_l(\zeta)| > \frac{1}{4}\sqrt{2l-1}$. Then using Lemma 3.1 and (3-2) we see that

$$\|\chi R_{V_0}(\zeta)\chi(I - \mathcal{P}_{l-1})\| = O(l^{-1/2}).$$

By Lemma 3.1 there is a $C > 0$ so that if $\lambda \in \mathbb{R}$, $|\lambda| > C$, then

$$\|V^\#\|_{L^\infty}\|\chi R_{V_0,0}(\lambda)\chi\| \le \tfrac{1}{2}.$$

Choose $M > C + 1$; then if $\tau_l(\zeta) \in i[0,\infty)$ with

$$\frac{\tau_l(\zeta)}{i} < \sqrt{2l-1} - \frac{M}{\sqrt{l}},$$

for l sufficiently large $|\tau_{l-1}(\zeta)| > C$. Now we restrict ourselves to $\tau_l(\zeta) \in i[0,\infty)$ with

$$\frac{1}{4}\sqrt{2l-1} < \frac{\tau_l(\zeta)}{i} < \sqrt{2l-1} - \frac{M}{\sqrt{l}}.$$

Since $\|\mathcal{P}_{l-1}V^\#\mathcal{P}_{l-1}\| = O(l^{-\delta})$ by our assumption on $\|V_m\|_{L^\infty}$,

$$\|\chi R_{V_0}(\zeta)\mathcal{P}_{l-1}V^\# R_{V_0}(\zeta)\mathcal{P}_{l-1}\chi\| = O(l^{-\delta}),$$

and we can follow the proof of Lemma 5.2 to show that $\|\chi R_{V_0}(\zeta)V^\# R_{V_0}(\zeta)\chi\| = O(l^{-\delta})$. Then

$$\begin{aligned}\|\chi(R_V(\zeta) - R_{V_0}(\zeta))\chi\| &= \|\chi R_{V_0}(\zeta)\chi((I + V^\# R_{V_0}(\zeta)\chi)^{-1} - I)\| \\ &= \|\chi R_{V_0}(\zeta)V^\# R_{V_0}(\zeta)\chi\| + O(l^{-\delta}) = O(l^{-\delta}),\end{aligned}$$

proving the lemma when $\tau_l(\zeta) \in i[0,\infty)$ with $\frac{1}{4}\sqrt{2l-1} < \frac{1}{i}\tau_l(\zeta) < \sqrt{2l-1} - M/\sqrt{l}$. A similar argument, singling out $\mathcal{P}_l$ rather than $\mathcal{P}_{l-1}$, handles the case when $\tau_l(\zeta) \in i[0,\infty)$ with $\frac{1}{4}\sqrt{2l-1} < |\tau_{l-1}(\zeta)|$. □

5C. *Proof of Theorem 1.3.* Theorem 1.3 concerns poles of R_V arising as perturbations of threshold poles of $R_{V_0}(\zeta)$. Using separation of variables as in (3-2), these threshold poles, in turn, correspond to a pole of $R_{V_0,0}(\lambda)$ at $\lambda = 0$.

We begin with a lemma about poles of $R_{V_0}(\lambda)$ at the origin. This result is well known if V_0 is real-valued.

Lemma 5.6. *Suppose $V_0 \in L^\infty_c(\mathbb{R}^d)$, and near $\lambda = 0$*

$$R_{V_0,0}(\lambda) = \sum_{k=1}^{k_0} \frac{1}{\lambda^k} A_k + A(\lambda), \tag{5-9}$$

where A is analytic in a neighborhood of the origin. Then $m_{V_0,0}(0) = \max_{0\le t\le 1} \operatorname{rank}(A_1 + tA_2)$.

Since A_1, A_2 are finite-rank, the rank of $A_1 + tA_2$ is equal to its maximum for all but a finite number of values of t in [0, 1].

Proof. Using the expansion (5-9) and the identity $(-\Delta_0 + V_0 - \lambda^2)R_{V_0,0}(\lambda) = I$ shows that for $k > 0$, $(-\Delta_0+V_0)A_k = A_{k+2}$, where we use the convention $A_{k+2} = 0$ if $k+2 > k_0$. Just as in [Dyatlov and Zworski 2019, Theorem 2.5], one can use this and the fact that $-\Delta_0+V_0$ commutes with $R_{V_0,0}$ to show that for $j \in \mathbb{N}$, $\operatorname{Ran}(A_{2j}) \subset \operatorname{Ran}(A_2)$ and $\operatorname{Ran}(A_{2j+1}) \subset \operatorname{Ran}(A_1)$. Here $\operatorname{Ran}(A_k)$ denotes the range of the operator A_k on $L^2_c(\mathbb{R}^d)$. Since $m_{V_0,0}(0) = \dim\big(\bigcup_{k=1}^{k_0} \operatorname{Ran}(A_k)\big)$, this shows $m_{V_0,0}(0) = \dim(\operatorname{Ran} A_1 \cup \operatorname{Ran} A_2)$. But

$$\dim(\operatorname{Ran} A_1 \cup \operatorname{Ran} A_2) = \max_{t\in[0,1]} \dim \operatorname{Ran}(A_1 + tA_2) = \max_{t\in[0,1]} \operatorname{rank}(A_1 + tA_2),$$

proving the lemma. □

Lemma 5.7. *Let $V \in L^\infty_c(X)$ satisfy* (1-1). *Let $\epsilon > 0$ be chosen so that $R_{V_0,0}(\lambda)$ has no poles in $\{\lambda \in \mathbb{C} : 0 < |\lambda| < 2\epsilon\}$, and let $\gamma_l \subset B_l(2\epsilon) \subset \hat{Z}$ be the curve $\{|\tau_l| = \epsilon\}$ with positive orientation. Then for $t \in [0, 1]$ and l sufficiently large*,

$$\operatorname{rank} \int_{\gamma_l} (1 + t\tau_l(\zeta)) R_V(\zeta)\, d\tau_l(\zeta) \ge \operatorname{rank} \int_{\gamma_l} (1 + t\tau_l(\zeta)) R_{V_0}(\zeta)\, d\tau_l(\zeta).$$

Proof. We assume $V^\#$ is nontrivial, since otherwise there is nothing to prove.

We first point out that if $R_{V_0,0}(\lambda) = \sum_{k=1}^{k_0} \lambda^{-k} A_k + A(\lambda)$, with $A(\lambda)$ analytic near $\lambda = 0$, then for l sufficiently large

$$\int_{\gamma_l} (1 + t\tau_l(\zeta)) R_{V_0}(\zeta)\, d\tau_l(\zeta) = \int_{\gamma_l} (1 + t\tau_l(\zeta)) R_{V_0,0}(\tau_l(\zeta)) \mathcal{P}_l\, d\tau_l(\zeta) = 2\pi i (A_1 + tA_2)\mathcal{P}_l.$$

Let $\chi \in L^\infty_c(X)$ satisfy $\chi V = V$, with χ independent of θ. Using Proposition 5.4, for l sufficiently large,

$$\left\| \int_{\gamma_l} (1 + t\tau_l(\zeta)) \chi (R_V(\zeta) - R_{V_0}(\zeta)) \chi\, d\tau_l(\zeta) \right\| = O(l^{-\delta}).$$

Thus

$$\left\| \int_{\gamma_l} (1 + t\tau_l(\zeta)) \chi R_V(\zeta) \chi\, d\tau_l(\zeta) - 2\pi i \chi (A_1 + tA_2)\mathcal{P}_l \chi \right\| = O(l^{-\delta}),$$

and this implies that for l sufficiently large,

$$\operatorname{rank} \int_{\gamma_l} (1 + t\tau_l(\zeta))\chi R_V(\zeta)\chi \, d\tau_l(\zeta) \geq 2 \operatorname{rank}(\chi(A_1 + tA_2)\chi). \tag{5-10}$$

But since $(-\Delta_0 + V_0)^{k_0} A_j = 0$ for $j = 1, 2$, a unique continuation theorem, e.g., [Jerison and Kenig 1985], ensures that $\operatorname{rank}(A_1 + tA_2) = \operatorname{rank}(\chi(A_1 + tA_2)\chi)$, and similarly

$$\operatorname{rank} \int_{\gamma_l} (1 + t\tau_l(\zeta))\chi R_V(\zeta)\chi \, d\tau_l(\zeta) = \operatorname{rank} \int_{\gamma_l} (1 + t\tau_l(\zeta)) R_V(\zeta) \, d\tau_l(\zeta). \qquad \square$$

Lemma 5.8. *Let* $V_0,\ \chi \in L^\infty_c(\mathbb{R}^d)$, *with* $\chi V_0 = V_0$. *Suppose* $R_{V_0}(\lambda)$ *has a pole of order* 1 *at the origin. Then for* l *sufficiently large*, $2(m_{V_0,0}(0) - m_{0,0}(0)) = M(I + V_0 R_0 \chi, \zeta_l(0))$.

Proof. We note here that the requirement that l is sufficiently large is to ensure that, using (3-2), any poles of R_{V_0} at $\zeta_l(0)$ arise from poles of R_{V_0} at the origin. Then via separation of variables it suffices to show that

$$m_{V_0,0}(0) - m_{0,0}(0) = M(I + V_0 R_{0,0}(\lambda)\chi, 0).$$

For $d = 1$, then $m_{V_0,0}(0) = 1$ and if V_0 is real-valued, this follows immediately from [Dyatlov and Zworski 2019, (2.2.31)]. For complex-valued V_0, the proof is similar, if one uses the assumption that R_{V_0} has a simple pole at the origin. When $d \geq 3$ is odd, the lemma follows as in the proof of [Dyatlov and Zworski 2019, Theorem 3.15]. In each case, the assumption that the pole is of order 1 is important. $\square$

Lemma 5.9. *Let* $V \in L^\infty_c(X)$ *satisfy* (1-1). *Let* $\epsilon > 0$ *be chosen so that* $R_{V_0,0}(\lambda)$ *has no poles in* $\{\lambda \in \mathbb{C} : 0 < |\lambda| < 2\epsilon\}$. *Suppose* $R_{V_0,0}(\lambda)$ *has a pole of order* 1 *at the origin*, *with residue of rank* $m_{V_0,0}(0)$. *Then for* l *sufficiently large*,

$$\sum_{\substack{\zeta \in D_l(\epsilon) \\ m_V(\zeta) \neq 0}} m_V(\zeta) \leq 2 m_{V_0,0}(0).$$

Proof. Let $\chi \in L^\infty_c(X)$ be independent of θ and satisfy $\chi V = V$. We first claim that for any $\zeta_0 \in \hat{Z}$,

$$m_V(\zeta_0) \leq M(I + V R_0 \chi, \zeta_0) + m_0(\zeta_0). \tag{5-11}$$

If ζ_0 does not correspond to a threshold, then $m_0(\zeta_0) = 0$ and this follows from the stronger Proposition 4.3. If ζ_0 does correspond to a threshold, this follows from a simplified adaptation of the proof of [Dyatlov and Zworski 2019, Theorem 3.15].

Arguing as in the proof of Proposition 5.1, using Lemmas 4.5 and 4.6 and an operator Rouché theorem [Gohberg and Sigal 1971, Theorem 2.2], for l sufficiently large,

$$\sum_{\substack{\zeta \in B_l(\epsilon) \\ M(I + V R_0 \chi, \zeta) \neq 0}} M(I + V R_0 \chi, \zeta) = \sum_{\substack{\zeta \in B_l(\epsilon) \\ M(I + V_0 R_0 \chi, \zeta) \neq 0}} M(I + V_0 R_0 \chi, \zeta) = M(I + V_0 R_0 \chi, \zeta_l(0)). \tag{5-12}$$

But by our assumptions and Lemma 5.8, $M(I + V_0 R_0 \chi, \zeta_l(0)) = 2(m_{V_0,0}(0) - m_{0,0}(0))$ for l sufficiently large. Using this, (5-12), and applying (5-11) completes the proof. $\square$

Proof of Theorem 1.3 under the assumption $\|V_m\|_{L^\infty} = O(|m|^{-\delta})$. Let $\epsilon > 0$ be as in the statement of Lemma 5.7. By applying Lemmas 5.6 and 5.7, we see that for l sufficiently large,

$$\sum_{\substack{\zeta \in B_l(\epsilon) \\ m_V(\zeta) \neq 0}} m_V(\zeta) \geq \sum_{\substack{\zeta \in B_l(\epsilon) \\ m_{V_0}(\zeta) \neq 0}} m_{V_0}(\zeta) = 2m_{V_0,0}(0).$$

Thus for l sufficiently large R_V has at least $2m_{V_0,0}(0)$ poles in $B_l(\epsilon)$. If $R_{V_0,0}(\lambda)$ has a simple pole at the origin, then applying in addition Lemma 5.9 we see that R_V has at exactly $2m_{V_0,0}(0)$ poles in $B_l(\epsilon)$.

To finish the proof of the theorem for the L^∞ case we need to refine the estimate on the location of the resonances in $B_l(\epsilon)$. We do this by showing that there is a $C > 0$ so that there are no resonances in $B_l(\epsilon) \setminus B_l(Cl^{-\delta/r})$ for l sufficiently large. This follows almost exactly the proof of Theorem 1.2, point 2, with λ_j replaced by 0. The difference here is that the bound on the singular part of $\chi R_{V_0} \chi$ at the origin is given by $\|\chi\, \Xi(R_{V_0}, 0)\chi\| \leq C|\lambda|^{-r}$; that is, $m_{V_0,0}(\lambda_j)$ is replaced by r rather than $m_{V_0,0}(0)$. Having made this minor adaptation, the remainder of the proof follows without change. □

6. Resonance-free regions, poles of R_V and $R_{\overline{V}}$, and the proofs of Corollary 1.4 and Theorem 1.9

Thus far we have focused on resonances in the sets $B_l(\rho)$, for l large. In this section we justify this by showing that the high-energy resonances near the physical space which also have $\operatorname{Re} \tau_0(\zeta) > 0$ lie in $B_l(\rho)$, for ρ sufficiently large. We do this by showing the existence of large resonance-free regions in $B_l(\sqrt{2l-1})$. We discuss $\hat{Z}$ further, focusing on the region near the physical space. We describe the relationship between the resolvents R_V and $R_{\overline{V}}$, where $\overline{V}$ is the complex conjugate of V; see Lemma 6.2. This lemma shows that we can understand the poles of R_V which are near the physical space and have $\operatorname{Re} \tau_0(\zeta) < 0$ by understanding the poles of $R_{\overline{V}}$ which are near the physical space and have $\operatorname{Re} \tau_0(\zeta) > 0$.

Lemma 6.1. *Let $V \in L^\infty_c(X)$. Then for any $0 < \gamma < 1$ there are $M_+,\ c_+ > 0$ so that the region*

$$U_l^+ \overset{\text{def}}{=} \{\zeta \in B_l(\sqrt{2l-1}) : M_+ < \operatorname{Re}(\tau_l(\zeta)) < \gamma\sqrt{2l},\ \operatorname{Im} \tau_l(\zeta) > -c_+ \log \operatorname{Re}(\tau_l(\zeta))\}$$

contains no poles of R_V for l sufficiently large. Likewise, for any $\alpha > 0$ and $0 < \gamma < 1$, there is a constant $M_- > 0$ so that

$$U_l^- \overset{\text{def}}{=} \{\zeta \in B_l(\sqrt{2l-1}) : M_- < \operatorname{Im}(\tau_l(\zeta)) < \gamma\sqrt{2l},\ \operatorname{Re} \tau_l(\zeta) > -\alpha\}$$

contains no poles of R_V for l sufficiently large.

The region U_l^+ is reminiscent of the logarithmic resonance-free regions familiar from potential scattering on $\mathbb{R}^d$. We note that there is substantial overlap between U_l^+ and U_{l+1}^-.

Proof. Let $\chi \in L^\infty_c(X)$ be independent of θ and satisfy $\chi V = V$ and $0 \leq \chi \leq 1$. To prove the lemma, we use $\chi R_V(\zeta)\chi = \chi R_0(\zeta)(I + V R_0(\zeta)\chi)^{-1}$ and the representation (3-2) via separation of variables.

From (3-2) and the estimate $\|\chi R_{0,0}(\lambda)\chi\| \leq Ce^{(C \operatorname{Im} \lambda)_-}/|\lambda|$, there are constants $C_1,\ C_2$ so that

$$\|V R_0(\zeta)\chi\| \leq \sup_{j \in \mathbb{N}_0} \left(\frac{C_1 e^{C_2 (\operatorname{Im} \tau_j(\zeta))_-}}{|\tau_j(\zeta)|} \right).$$

First consider U_l^+. Set $c_+ = 1/C_2 - \delta_+$, where $\delta_+ > 0,\ \delta_+ < 1/C_2$, and take $M_+ > (2C_1)^{1/(\delta_+ C_2)}$. Then if $\zeta \in U_l^+$,

$$\frac{C_1 e^{C_2(\operatorname{Im}\tau_l(\zeta))_-}}{|\tau_l(\zeta)|} < \tfrac{1}{2}.$$

If $j < l$ and $\zeta \in U_l^+$, then $|\tau_j(\zeta)| \geq |\tau_l(\zeta)|$ and a computation shows

$$\frac{e^{C_2(\operatorname{Im}\tau_j(\zeta))_-}}{|\tau_j(\zeta)|} < \frac{e^{C_2(\operatorname{Im}\tau_l(\zeta))_-}}{|\tau_l(\zeta)|}.$$

On the other hand, for $j > l$, if $\zeta \in U_l^+$, then

$$\operatorname{Re}(\tau_j(\zeta))^2 \leq \operatorname{Re}(\tau_{l+1}(\zeta))^2 = (\operatorname{Re}\tau_l(\zeta))^2 - 2l - (\operatorname{Im}\tau_l(\zeta))^2 - 1 \leq -2l(1-\gamma^2).$$

Since $\operatorname{Im}\tau_j(\zeta) > 0$ for $j > l$ and $\zeta \in B_l(\sqrt{2l-1})$, this is enough to show that

$$\frac{C_1 e^{C_2(\operatorname{Im}\tau_j(\zeta))_-}}{|\tau_j(\zeta)|} < \tfrac{1}{2}$$

for $\zeta \in U_l^+$ and l sufficiently large. Then $\|VR_0(\zeta)\chi\| < \frac{1}{2}$, and $I + VR_0(\zeta)\chi$ is invertible.

For U_l^-, choose $M_- > 0$ so that $16\|V\|_{L^\infty} < M_-^2$. Then using (3-2) and $\|R_{0,0}(\lambda)\| \leq 1/(\operatorname{dist}(\lambda^2, [0,\infty)))$ for $\operatorname{Im}\lambda > 0$, for $\zeta \in U_l^-$,

$$\Big\| VR_0(\zeta)\chi \sum_{j\geq l} \mathcal{P}_j \Big\| \leq \|V\|_{L^\infty} \sup_{j\geq l} \frac{1}{(\operatorname{dist}\tau_j^2, [0,\infty))} \leq \frac{8\|V\|_{L^\infty}}{M_-^2} \leq \tfrac{1}{2}.$$

Next we show that

$$\Big\| VR_0(\zeta) \sum_{0\leq j<l} \mathcal{P}_j \chi \Big\| \leq \tfrac{1}{2}$$

in U_l^- for sufficiently large l. Using the orthogonality of the projections $\sum_{j\geq l}\mathcal{P}_j$ and $\sum_{0\leq j<l}\mathcal{P}_j$ this will complete our proof that $I + VR_0\chi$ is invertible. Note that

$$\tau_{l-1}^2 = 2l - (\operatorname{Im}\tau_l)^2 + (\operatorname{Re}\tau_l)^2 - 1 + 2i\operatorname{Re}(\tau_l)\operatorname{Im}(\tau_l).$$

Thus $|\tau_{l-1}| \geq \sqrt{(1-\gamma^2)2l} + O(1)$ and $-\operatorname{Im}(\tau_{l-1}) \leq 2\alpha/\sqrt{1-\gamma^2} + O(l^{-1/2})$, so for l sufficiently large,

$$\frac{C_1 e^{C_2(\operatorname{Im}\tau_{l-1}(\zeta))_-}}{|\tau_{l-1}(\zeta)|} < \tfrac{1}{2}$$

for $\zeta \in U_l^-$. But if $0 \leq j < l-1$ and $\zeta \in U_l^-$,

$$\frac{C_1 e^{C_2(\operatorname{Im}\tau_j(\zeta))_-}}{|\tau_j(\zeta)|} < \frac{C_1 e^{C_2(\operatorname{Im}\tau_{l-1}(\zeta))_-}}{|\tau_{l-1}(\zeta)|}.$$

This ensures that

$$\Big\| VR_0(\zeta) \sum_{0\leq j<l} \mathcal{P}_l \chi \Big\| < \tfrac{1}{2}$$

so that $I + VR_0(\zeta)\chi$ is invertible on U_l^- for l sufficiently large. □

We remark that we have not made an effort to optimize the results of Lemma 6.1, as in this paper we are concentrating instead on regions near the thresholds, where, as we have seen, resonances can occur.

Before proving Corollary 1.4, we discuss $\hat{Z}$ and the boundary of the physical space a bit more. To motivate the discussion, consider the simpler case of the Schrödinger operator $-\Delta_0 + V_0$ on $\mathbb{R}^d$, where we use λ^2 as the spectral parameter in defining the (scattering) resolvent. Thus, given a value $E > 0$, there are two points, $\pm\sqrt{E}$ corresponding to the spectral parameter E on the boundary of the physical space, with $R_{V_0,0}(\pm\sqrt{E}) = (-\Delta_0 + V_0 - (\sqrt{E} \pm i0))^{-1}$.

There is a similar phenomena in the case of $-\Delta + V$ on $\mathbb{R}^d \times \mathbb{S}^1$, but it is notationally harder to describe. Given $E > 0$, let $\sqrt{E} \pm i0 \in \hat{Z}$ be the points on $\hat{Z}$ with $R_V(\sqrt{E} \pm i0) = (-\Delta + V - E \mp i0)^{-1}$. Equivalently, we could define $\sqrt{E} \pm i0$ to be the point in $\hat{Z}$ with $\tau_j(\sqrt{E} \pm i0) = \pm\sqrt{E - j^2}$ if $j^2 \le E$, and $\tau_j(\sqrt{E} \pm i0) = i\sqrt{j^2 - E}$ if $j^2 > E$. By our definition of $B_l(\rho)$, if $l_E = \lfloor\sqrt{E}\rfloor$ and $l_E > 0$, then $\sqrt{E} + i0 \in B_{l_E}(\sqrt{2l_E - 1})$, but $\sqrt{E} - i0 \notin B_{l_E}(\sqrt{2l_E - 1})$. Thus there is some sense in which we have been studying only "half" of the boundary of the physical space. However, we shall see in Lemma 6.2 that this suffices for understanding the behavior of the resolvent, if we consider both the resolvent of $-\Delta + V$ and that of $-\Delta + \overline{V}$.

Thus, to fully cover points on the boundary of the physical space, we need to define another type of open set in $\hat{Z}$, analogous to $B_l(\rho)$. For $l \in \mathbb{N}$ and $\rho > 0$, denote by $B_l^{\pm}(\rho)$ the connected component of $\{\zeta \in \hat{Z} : |\tau_l(\zeta)| < \rho\}$ which intersects the physical space and includes a region with $\pm\operatorname{Re}\tau_0(\zeta) > 0$. With the $+$ sign, we get the set $B_l(\rho)$ defined in the introduction: $B_l^+(\rho) = B_l(\rho)$. If $l_E = \lfloor\sqrt{E}\rfloor$ and $\sqrt{E} - l_E < \rho$, then the point $\sqrt{E} - i0$ corresponding to E on the boundary of the physical space as defined above has $\sqrt{E} - i0 \in B_{l_E}^-(\rho)$. Hence any point on the boundary of the physical space lies in

$$B_0^+(1) \cup \left(\bigcup_{l=1}^{\infty} B_l^+(\sqrt{2l-1})\right) \cup \left(\bigcup_{l=1}^{\infty} B_l^-(\sqrt{2l-1})\right).$$

As before, we make the choice of $\sqrt{2l-1}$ for ρ as that is the largest value of ρ for which $B_l^{\pm}(\rho)$ contains only a single point corresponding to a threshold. For certain combinations of l and ρ it can happen that $B_l^+(\rho) = B_l^-(\rho)$.

Consider a Schrödinger operator on d-dimensional Euclidean space with potential $V_0 \in L_c^{\infty}(\mathbb{R}^d)$ and scattering resolvent $R_{V_0,0}(\lambda)$. When $\operatorname{Im}\lambda > 0$, that is λ is in the physical space,

$$R_{V_0,0}(\lambda) = (-\Delta_0 + V_0 - \lambda^2)^{-1} = ((-\Delta_0 + \overline{V}_0 - \bar{\lambda}^2)^{-1})^* = (R_{\overline{V}_0 0}(-\bar{\lambda}))^*.$$

Here $\overline{V}_0$ and $\bar{\lambda}$ denote the usual complex conjugates. For odd d the identity $R_{V_0,0}(\lambda) = (R_{\overline{V}_0 0}(-\bar{\lambda}))^*$ then holds by meromorphic continuation for all $\lambda \in \mathbb{C}$. In particular, this implies λ_0 is a pole of $R_{V_0,0}(\lambda)$ if and only if $-\bar{\lambda}_0$ is a pole of $R_{\overline{V}_0 0}(\lambda)$. For real-valued V, this is the well-known symmetry of resonances for symmetric Schrödinger operators in odd dimensions.

We turn to the analog of this result for R_V, which is shown in a similar way. Suppose ζ is in the physical space, here identified with the upper half-plane, so that $R_V(\zeta) = (-\Delta + V - \zeta^2)^{-1}$. Thus $(R_{\overline{V}}(-\bar{\zeta}))^* = R_V(\zeta)$. For general $\zeta \in \hat{Z}$, we define $-\zeta^{\dagger} \in \hat{Z}$ to be the point in $\hat{Z}$ which satisfies $\tau_j(-\zeta^{\dagger}) = -\overline{\tau_j}(\zeta)$ for all j. This is an antiholomorphic mapping, and if ζ is in the physical space, identified with the upper half-plane, the mapping $\zeta \mapsto -\zeta^\dagger$ agrees with the mapping $\zeta \mapsto -\bar{\zeta}$. Then the identity

$$(R_{\overline{V}}(-\zeta^\dagger))^* = R_V(\zeta), \quad \text{where } \tau_j(-\zeta^\dagger) = -\overline{\tau_j}(\zeta), \text{ for all } j \in \mathbb{N}_0 \tag{6-1}$$

holds for all $\zeta \in \hat{Z}$ by meromorphic continuation. In particular, this means that $\zeta_0 \in \hat{Z}$ is a pole of $R_V(\zeta)$ if and only if $-\zeta_0^\dagger$ is a pole of $R_{\overline{V}}(\zeta)$. Note that if $\zeta \in B_l^+(\rho) = B_l(\rho)$, then $-\zeta^\dagger \in B_l^-(\rho)$. Thus to study the poles of $R_V(\zeta)$ in $B_l^-(\rho)$ it suffices to study the poles of $R_{\overline{V}}(\zeta)$ in $B_l^+(\rho) = B_l(\rho)$. Likewise, an estimate on $R_{\overline{V}}$ in $B_l^+(\sqrt{2l-1})$ implies an estimate on R_V in $B_l^-(\sqrt{2l-1})$.

We summarize these results in the following lemma.

Lemma 6.2. *If $V_0 \in L_c^\infty(\mathbb{R}^d)$, then λ_0 is a pole of $R_{V_0,0}(\lambda)$ if and only if $-\bar{\lambda}_0$ is a pole of $R_{\overline{V}_0 0}(\lambda)$. Let $V \in L_c^\infty(X)$. Then $\zeta_0 \in \hat{Z}$ is a pole of $R_V(\zeta)$ if and only if $-\zeta_0^\dagger$ is a pole of $R_{\overline{V}}(\zeta)$. Here $\bar{\lambda}_0$, $\overline{V}$, and $\overline{V}_0$ are the complex conjugates of λ_0, V, and V_0, respectively, and $-\zeta^\dagger$ is as defined in* (6-1).

We define a distance on $\hat{Z}$ as follows: for $\zeta, \zeta' \in \hat{Z}$,

$$d_{\hat{Z}}(\zeta, \zeta') \overset{\text{def}}{=} \sup_j |\tau_j(\zeta) - \tau_j(\zeta')|. \tag{6-2}$$

That this is well defined and a metric is shown in [Christiansen and Datchev 2021, Section 5.1]. Note that if $\zeta, \zeta' \in \hat{Z}$ satisfy $\tau_j(\zeta) \neq -\tau_j(\zeta')$, then since $\tau_j(\zeta)^2 - \tau_j(\zeta')^2 = \tau_l(\zeta)^2 - \tau_l(\zeta')^2$,

$$|\tau_j(\zeta) - \tau_j(\zeta')| = |\tau_l(\zeta) - \tau_l(\zeta')| \left| \frac{\tau_l(\zeta) + \tau_l(\zeta')}{\tau_j(\zeta) + \tau_j(\zeta')} \right|.$$

In particular, this implies that for any $\rho > 0$ there is an $L = L(\rho)$ so that if $l \geq L$ and $\zeta, \zeta' \in B_l(\rho)$ then

$$d_{\hat{Z}}(\zeta, \zeta') = |\tau_l(\zeta) - \tau_l(\zeta')|.$$

Proof of Corollary 1.4. Recall our hypotheses include that V is real-valued, ensuring that V_0 is real-valued as well.

The operator-valued function $R_V(\zeta)$ has a sequence $\{\zeta_j^\flat\}$ of poles satisfying $|\tau_0(\zeta_j^\flat)| \to \infty$ as $j \to \infty$ and $d_{\hat{Z}}(\zeta_j^\flat, \text{physical space}) \to 0$ only if $R_V(\zeta)$ has infinitely many poles in $\bigcup_{l=1}^\infty B_l(\sqrt{2l-1})$ or infinitely many poles in $\bigcup_{l=1}^\infty B_l^-(\sqrt{2l-1})$ (or both). If $R_V(\zeta)$ has infinitely many poles in $\bigcup_{l=1}^\infty B_l^-(\sqrt{2l-1})$, then by Lemma 6.2, $R_{\overline{V}}(\zeta) = R_V(\zeta)$ has infinitely many poles in $\bigcup_{l=1}^\infty B_l(\sqrt{2l-1})$. Thus it suffices to study sequences of poles in $\bigcup_{l=1}^\infty B_l(\sqrt{2l-1})$.

Note that while $B_l(\sqrt{2l-1})$ contains only a single threshold, $B_l(\sqrt{2l-1})$ and $B_{l+1}(\sqrt{2l+1})$ are not disjoint and in fact have substantial overlap which contains an interval of the continuous spectrum. Moreover, for l sufficiently large the sets U_l^+ and U_{l+1}^- of Lemma 6.1 have nontrivial intersection. Applying Lemma 6.1 we see that in order to have a sequence of resonances contained in $\bigcup_{l=1}^\infty B_l(\sqrt{2l-1})$ and approaching the continuous spectrum (and with $|\tau_0| \to \infty$), the resonances must lie in $\bigcup_{l=1}^\infty B_l(M)$ for some M. But then the corollary follows from an application of Theorems 1.1–1.3. ☐

We now have the ingredients we need to prove Theorem 1.9.

Proof of Theorem 1.9. The hypotheses on $-d^2/dx^2 + V_0$ and the expression (3-2) mean that the resolvent $R_{V_0}(\zeta)$ has no poles on the boundary of the physical space. Moreover, since for any $\tilde{\chi} \in C_c^\infty(\mathbb{R})$ there is a constant C so that $\|\tilde{\chi} R_{V_0,0}(\lambda)\tilde{\chi}\| \le C$ for all $\lambda \in \mathbb{R} \cup i[0,\infty)$, for any $\chi \in C_c^\infty(X)$ there is a $C_1 > 0$ so that $\|\chi R_{V_0}(\zeta)\chi\| \le C_1$ for all ζ in the boundary of the physical space.

Corollary 1.4 shows that there are no poles of the resolvent R_V in the continuous spectrum at high energy. Proposition 5.4 and Lemma 5.5 show that when ζ is in the boundary of the physical space and $\zeta \in B_l(\sqrt{2l-1})$, the cut-off resolvent of $-\Delta + V$ satisfies $\|\chi R_V(\zeta)\chi - \chi R_{V_0}(\zeta)\chi\| = O(l^{-1/2})$. Thus $\|\chi R_{V_0}(\zeta)\chi\|$ is uniformly bounded on the boundary of the physical space when $|\tau_0(\zeta)|$ is sufficiently large. Hence by [Christiansen and Datchev 2021, Theorem 5.6] the hypotheses of [Christiansen and Datchev 2022, Theorem 4.1] hold. Theorem 1.9 then follows directly. □

7. Larger neighborhoods of the threshold l^2

In this section we consider poles of $R_V(\zeta)$ in neighborhoods $B_l(\alpha \log l)$ and $B_l(\alpha(\log l)^{1-\epsilon})$ of the l-th threshold. We prove Theorem 1.5 for potentials with $V_0 \equiv 0$ and the related, but weaker, Theorem 7.1 which holds for a general potential $V \in L_c^\infty(X)$.

The proof of Theorem 1.5 is similar to that of the proof of Theorem 1.2 for L^∞ potentials.

Proof of Theorem 1.5. Choose $\chi \in L_c^\infty(X)$, $\chi V = V$, and χ independent of θ. We write

$$\chi R_0 V R_0 \chi$$
$$= \chi R_0 \mathcal{P}_l V R_0 \mathcal{P}_l \chi + \chi R_0(1-\mathcal{P}_l) V R_0 \mathcal{P}_l \chi + \chi \mathcal{P}_l R_0 V R_0 (1-\mathcal{P}_l)\chi + \chi R_0(1-\mathcal{P}_l) V R_0 (1-\mathcal{P}_l)\chi. \quad (7\text{-}1)$$

Let $\alpha' > 0$, and let $\zeta \in B_l(\alpha'|\log l|)$, where l is large enough that $B_l(\alpha'|\log l|)$ contains only a single point of $\hat{Z}$ which corresponds to a threshold. Let $\zeta \in B_l(\alpha' \log l)$ satisfy $|\tau_l(\zeta)| \ge 1$. Then by Lemma 3.2,

$$\|\chi R_0(\zeta)(1-\mathcal{P}_l)\chi\| = O(l^{-1/2}),$$

and by (3-1) and [Dyatlov and Zworski 2019, Theorem 3.1],

$$\|\chi R_0(\zeta)\mathcal{P}_l \chi\| = O\left(\frac{e^{C(\operatorname{Im}\tau_l(\zeta))_-}}{|\tau_l(\zeta)|}\right)$$

for some $C > 0$. Using this estimate and $\mathcal{P}_l V \mathcal{P}_l = O(l^{-\delta})$ in (7-1) shows

$$\|\chi R_0(\zeta) V R_0(\zeta)\chi\| = O(l^{-\delta} e^{2C(\operatorname{Im}\tau_l(\zeta))_-}).$$

Thus from (7-1) there is a $C_1 > 0$ so that $I + V R_0(\zeta)\chi$ is invertible if l is sufficiently large, $\zeta \in B_l(\alpha' \log l)$, $|\tau_l(\zeta)| \ge 1$, and $e^{2C(\operatorname{Im}\tau_l(\zeta))_-} \le C_1 l^\delta$. This last item may be ensured by requiring $|\tau_l| \le \alpha \log l$, for suitably chosen $\alpha > 0$, $\alpha \le \alpha'$, and taking l sufficiently large. Recall that $-\Delta + V$ has no resonances in regions where $I + V R_0 \chi$ is invertible, see Proposition 4.3.

Applying Theorems 1.2 and 1.3 shows that if $d = 1$ there is a $c_0 > 0$ so that when l is sufficiently large the region $\{\zeta \in B_l(\alpha \log l) : 1 \ge |\tau_l(\zeta)| > c_0 l^{-\delta}\}$ contains no resonances, and if $d > 1$ there are no resonances in $B_l(1)$ for l sufficiently large. □

A similar proof gives the next theorem.

Theorem 7.1. *Let $V \in L^\infty_c(X)$ satisfy* (1-1), *and let $\epsilon > 0$. Then there is a $c_0 = c_0(\epsilon, V) > 0$ so that for l sufficiently large, the region*

$$\{\zeta \in B_l(c_0(\log l)^{1/(d+\epsilon)}) : |\tau_l(\zeta) - \lambda'| \geq (1 + |\lambda'|^2)^{-(d+\epsilon)/2} \text{ for every } \lambda' \in \mathbb{C} : m_{V_0,0}(\lambda') > 0\}$$

contains no poles of $R_V(\zeta)$.

Proof. We assume $V^\# = V - V_0 \not\equiv 0$, since otherwise there is nothing to prove.

Choose $\chi \in L^\infty_c(X)$ so that $\chi V = V$ and χ is independent of θ. We may think of χ as an element of $L^\infty_c(\mathbb{R}^d)$ as well.

Set

$$A_\epsilon \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} : |\lambda - \lambda'| \geq (1 + |\lambda'|^2)^{-(d+\epsilon)/2} \text{ for every } \lambda' \in \mathbb{C} : m_{V_0,0}(\lambda') > 0\}.$$

We shall use, from the proof of [Dyatlov and Zworski 2019, Theorem 3.54], that there is a $C > 0$ so that

$$\|(I + V_0 R_{0,0}(\lambda))^{-1}\| \leq C \exp(C|\lambda|^{d+\epsilon}) \quad \text{if } \lambda \in A_\epsilon. \tag{7-2}$$

Choose $\alpha' > 0$. If $\zeta \in B_l(\alpha' \log l)$,

$$\begin{aligned} \chi R_{V_0}(\zeta)\mathcal{P}_l\chi &= \chi R_{V_0,0}(\tau_l(\zeta))\mathcal{P}_l\chi \\ &= \chi R_{0,0}(\tau_l(\zeta))\chi(I + V_0 R_{0,0}(\tau_l(\zeta))\chi)^{-1}\mathcal{P}_l. \end{aligned}$$

Thus, if $\zeta \in B_l(\alpha' \log l)$ with $\tau_l \in A_\epsilon$ and $|\tau_l(\zeta)| \geq 1$, then

$$\begin{aligned} \|\chi R_{V_0}(\zeta)\mathcal{P}_l\chi\| &\leq C \exp(C(\operatorname{Im} \tau_l(\zeta)_-)) \exp(C|\tau_l(\zeta)|^{d+\epsilon}) \\ &\leq C \exp(C|\tau_l(\zeta)|^{d+\epsilon}). \end{aligned} \tag{7-3}$$

Here and below we allow the constant C to change from line to line, and note that it depends on V, ϵ, and χ, but not l.

Let $\zeta \in B_l(\alpha' \log l)$ with $\tau_l \in A_\epsilon$ and $|\tau_l(\zeta)| \geq 1$. Writing $\chi R_{V_0}\chi$ as in (7-1) and applying Lemma 3.2 and (7-3), we find that for these ζ, if l is sufficiently large,

$$\|\chi R_{V_0}(\zeta)V^\# R_{V_0}(\zeta)\chi\| \leq C_1 l^{-\delta} \exp(C_1|\tau_l(\zeta)|^{d+\epsilon}) \tag{7-4}$$

for some C_1. Now we can choose $c_0 > 0$ sufficiently small and $L > 0$ sufficiently large so that

$$\text{if } |\tau_l(\zeta)| \leq c_0(\log l)^{1/(d+\epsilon)} \text{ and } l > L \quad \text{then } C_1 l^{-\delta} \exp(C_1|\tau_l(\zeta)|^{d+\epsilon}) \leq \tfrac{1}{2}$$

ensuring that $I + V^\# R_{V_0}(\zeta)\chi$ is invertible.

Recalling that with $V^\#$ nontrivial if $I + V^\# R_{V_0}(\zeta)\chi$ is invertible then ζ is not a resonance of $-\Delta + V$ proves the theorem. ☐

8. Expansion of $\mathcal{P}_l(I + V^\# R^{\text{reg}}_{V_0}\chi)^{-1}V^\#\mathcal{P}_l$ for smooth V

This section contains preliminary computations which allow us to refine some of our results when V is smooth. We begin with a straightforward lemma about Schrödinger operators on $\mathbb{R}^d$.

Lemma 8.1. *Let $V_0,\ \chi \in C_c^\infty(\mathbb{R}^d)$ and $J \in \mathbb{N}$. Then as an operator from $H^s(\mathbb{R}^d)$ to $H^{s-2J}(\mathbb{R}^d)$,*

$$\chi R_{V_0,0}(\lambda)\chi = -\sum_{j=1}^{J} \frac{1}{\lambda^{2j}} \chi(-\Delta_0 + V_0)^{j-1}\chi + \frac{1}{\lambda^{2J}} \chi R_{V_0,0}(\lambda)(-\Delta_0 + V_0)^J \chi. \tag{8-1}$$

Proof. First assume λ is in the physical region, that is, $\operatorname{Im}\lambda > 0$. Then the $J = 1$ case follows from rearranging the equality

$$(-\Delta_0 + V_0 - \lambda^2)R_{V_0,0}(\lambda) = R_{V_0,0}(\lambda)(-\Delta_0 + V_0 - \lambda^2) = I$$

to get

$$R_{V_0,0}(\lambda) = \frac{1}{\lambda^2}(-I + R_{V_0,0}(\lambda)(-\Delta_0 + V_0)).$$

The general case follows by induction.

Since both sides of (8-1) have meromorphic continuations to the complex plane, the equality holds for all λ. □

We shall use the following Hilbert spaces: for $n \in \mathbb{N}_0$,

$$H_{(0,n)}(X) \stackrel{\text{def}}{=} \left\{ u \in L^2(X) : \frac{\partial^\alpha}{\partial x^\alpha} u \in L^2(X) \text{ if } |\alpha| \le n \right\} \quad \text{with } \|u\|^2_{H_{(0,n)}} = \sum_{|\alpha| \le n} \left\| \frac{\partial^\alpha}{\partial x^\alpha} u \right\|^2_{L^2(X)}.$$

Here we use the usual multi-index notation for $\alpha = (\alpha_1, \dots, \alpha_d)$. This allows us to indicate mapping properties of operators which act differently in the x and θ variables.

One of the main results of this section is the following proposition. Recall that $R^{\text{reg}}_{V_0}(\zeta) = R^{\text{reg}}_{V_0}(\zeta; \lambda_0, l)$ is defined in (5-3).

Proposition 8.2. *Let $V,\ \chi \in C_c^\infty(X)$ satisfy $\chi V = V$. In addition, suppose χ is independent of θ. Let $\lambda_0 \in \mathbb{C}$, and suppose $R_{V_0,0}(\lambda)$ is analytic on $0 < |\lambda - \lambda_0| \le \epsilon$. Then, for $R^{\text{reg}}_{V_0}(\zeta) = R^{\text{reg}}_{V_0}(\zeta; \lambda_0, l)$ and $\zeta \in D_l(\lambda_0, \epsilon)$,*

$$\left\| \mathcal{P}_l (I + V^\# R^{\text{reg}}_{V_0}(\zeta)\chi)^{-1} V^\# \mathcal{P}_l + \frac{1}{l^2} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \left(\frac{\tau_l^2 - k^2}{4k^2} V_{-k} V_k - \frac{V_{-k}}{4k^2}(-\Delta_0 + V_0) V_k \right) \mathcal{P}_l \right\|_{H_{(0,8)}(X) \to L^2(X)} = O(l^{-3}),$$

where the error is uniform on $D_l(\lambda_0, \epsilon)$ for l sufficiently large.

To prove this proposition we use Lemmas 8.3–8.6. In each of these, $V,\ \lambda_0,\ R^{\text{reg}}_{V_0}(\zeta)$, and ϵ are as in Proposition 8.2. Some of these computations rely on the identity $e^{\pm ik\theta}e^{\pm il\theta} = e^{\pm i(k+l)\theta}$ and hence use the structure of the eigenfunctions of the Laplacian on $\mathbb{S}^1$ in an essential way.

For $l \in \mathbb{N}$, let $\mathcal{P}_{l\pm} : L^2(X) \to L^2(X)$ denote orthogonal projection onto $L^2(\mathbb{R}^d_x)e^{\pm il\theta}$, so that

$$(\mathcal{P}_{l\pm} f)(x, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(x, \theta') e^{\pm il(\theta - \theta')}\, d\theta',$$

for $l > 0$ and $\mathcal{P}_l = \mathcal{P}_{l+} + \mathcal{P}_{l-}$.

Lemma 8.3. *Under the hypotheses of Proposition 8.2,*

$$\left\| \mathcal{P}_l V^\# R^{\mathrm{reg}}_{V_0}(\zeta) V^\# \mathcal{P}_l - \frac{1}{l^2} \sum_{k\in\mathbb{Z},\, k\neq 0} \left(\frac{\tau_l^2 - k^2}{4k^2} V_{-k} V_k - \frac{V_{-k}}{4k^2}(-\Delta_0 + V_0) V_k \right) \mathcal{P}_l \right\|_{H_{(0,n+6)}\to H_{(0,n)}} = O(l^{-3})$$

uniformly for $\zeta \in D_l(\lambda_0, \epsilon)$ when l is sufficiently large.

Proof. Since $V \in C_c^\infty(X)$, we have $\|V_m\|_{L^\infty} = O(|m|^{-N})$ for any N, so $\|\mathcal{P}_l V^\# \mathcal{P}_l\| = O(l^{-N})$. Thus, choosing l sufficiently large so that (5-4) holds, it suffices to consider

$$\mathcal{P}_l V^\# R^{\mathrm{reg}}_{V_0}(\zeta)(I - \mathcal{P}_l) V^\# \mathcal{P}_l = \mathcal{P}_l V^\# R_{V_0}(\zeta)(I - \mathcal{P}_l) V^\# \mathcal{P}_l.$$

Then

$$\begin{aligned} \mathcal{P}_l V^\# R^{\mathrm{reg}}_{V_0}(I - \mathcal{P}_l) V^\# \mathcal{P}_l &= \sum_{\pm} \sum_{\substack{k\in\mathbb{Z} \\ 0<|k|,\, k\neq -l}} V_{\mp k} R_{V_0,0}(\tau_{l+k}) V_{\pm k} \mathcal{P}_{l\pm} \\ &= \sum_{\pm} \sum_{\substack{k\in\mathbb{Z} \\ 0<|k|<l^{1/2}}} V_{\mp k} R_{V_0,0}(\tau_{l+k}) V_{\pm k} \mathcal{P}_{l\pm} + O_{L^2\to L^2}(l^{-N}). \end{aligned}$$

Here we use the rapid decay of $\|V_m\|$ to bound the error obtained when we restrict the values of k in the sum. Using Lemma 8.1 with $J = 3$ gives

$$\mathcal{P}_l V^\# R^{\mathrm{reg}}_{V_0}(I - \mathcal{P}_l) V^\# \mathcal{P}_l = \sum_{\pm} \sum_{\substack{k\in\mathbb{Z} \\ 0<|k|<l^{1/2}}} V_{\mp k} \left(\frac{-1}{\tau_{l+k}^2} - \frac{1}{\tau_{l+k}^4}(-\Delta_0 + V_0) \right) V_{\pm k} \mathcal{P}_{l\pm} + O(l^{-3}), \tag{8-2}$$

where the error is as an operator from $H_{(0,n+6)}(X)$ to $H_{(0,n)}(X)$ and is uniform in $D_l(\lambda_0, \epsilon)$. Since we have restricted $|k|$ to be relatively small compared with l, we can expand $\tau_{l\pm k}$ asymptotically in l. Thus, with each sum over $k \in \mathbb{Z}$ with $0 < |k| < l^{1/2}$, using $\tau_{l\pm k}^2 = \tau_l^2 \mp 2lk - k^2$ gives

$$\begin{aligned} \sum_{0<|k|<l^{1/2}} \frac{1}{\tau_{l+k}^2} V_{-k} V_k = \frac{1}{2} \sum_{0<|k|<l^{1/2}} \left(\frac{1}{\tau_{l+k}^2} + \frac{1}{\tau_{l-k}^2} \right) V_{-k} V_k &= \sum_{0<|k|<l^{1/2}} \frac{\tau_l^2 - k^2}{(\tau_l^2 - k^2)^2 - 4k^2 l^2} V_{-k} V_k \\ &= \frac{-1}{4l^2} \sum_{0<|k|<l^{1/2}} \left(\frac{\tau_l^2 - k^2}{k^2} \right) V_{-k} V_k + O(l^{-4}). \end{aligned} \tag{8-3}$$

Here and below the error is uniform in $D_l(\lambda_0, \epsilon)$ when l is sufficiently large.

For the second term in (8-2), we write

$$\begin{aligned} \sum_{0<|k|<l^{1/2}} \frac{1}{\tau_{l+k}^4} V_{\mp k}(-\Delta_0 + V_0) V_{\pm k} &= \sum_{0<|k|<l^{1/2}} \frac{1}{(\tau_l^2 - 2lk - k^2)^2} V_{\mp k}(-\Delta_0 + V_0) V_{\pm k} \\ &= \frac{1}{4l^2} \sum_{0<|k|<l^{1/2}} \frac{1}{k^2} V_{\mp k}(-\Delta_0 + V_0) V_{\pm k} + O(l^{-3}). \end{aligned}$$

Note that

$$\sum_{0<|k|<l^{1/2}} \frac{1}{k^2} V_{\mp k}(-\Delta_0 + V_0) V_{\pm k} = \sum_{0<|k|<l^{1/2}} \frac{1}{k^2} V_{-k}(-\Delta_0 + V_0) V_k, \tag{8-4}$$

since the sum is over $k \in \mathbb{Z}$, with $0 < |k| < l^{1/2}$. The rapid decay in m of $\|V_m\|_{C^p}$ means we can replace the sums in (8-3) and (8-4) over $0 < |k| < l^{1/2}$ by sums over all nonzero $k \in \mathbb{Z}$, with an error which is $O(l^{-N})$. □

The next lemma is an algebraic identity.

Lemma 8.4. *For any $V \in C_c^\infty(X)$*

$$\sum_{\substack{m,j\in\mathbb{Z}\\ m,j\neq 0,\, m\neq -j}} \frac{1}{j(j+m)} V_m V_j V_{-m-j} = 0.$$

We give two different proofs.

Proof. For this proof, we show that for each $j_0 \neq 0$, $m_0 \neq 0$ the coefficient of $V_{j_0} V_{m_0} V_{-j_0-m_0}$ in the sum is zero. This proof is purely algebraic in nature.

If $m_0 \neq \pm j_0$, then there are six possibilities for the pair (j, m) which will give a term containing $V_{m_0} V_{j_0} V_{-m_0-j_0}$: (j_0, m_0), (m_0, j_0), $(-m_0-j_0, m_0)$, $(m_0, -j_0-m_0)$, $(j_0, -m_0-j_0)$, $(-m_0-j_0, j_0)$. Thus the sum of the coefficients of $V_{m_0} V_{j_0} V_{-m_0-j_0}$ is

$$\frac{1}{j_0(j_0+m_0)} + \frac{1}{m_0(j_0+m_0)} + \frac{1}{j_0(j_0+m_0)} - \frac{1}{j_0 m_0} - \frac{1}{j_0 m_0} + \frac{1}{m_0(j_0+m_0)} = 0.$$

A similar argument when $j_0 = m_0$ shows the coefficient of $V_{j_0}^2 V_{-2j_0}$ is zero as well. □

Alternate proof of Lemma 8.4. For this proof, we use that V_j is the j-th Fourier coefficient of V. Though in our applications V_j depends on x, that dependence is not important here so we will suppress it.

Set

$$W(\theta) = \sum_{j\neq 0} \frac{1}{j} V_j e^{ij\theta}$$

and note $d/d\theta\, W(\theta) = V(\theta) - V_0$. Then

$$\int_0^{2\pi} (V(\theta) - V_0)(W(\theta))^2 \, d\theta = \frac{1}{3} (W(\theta))^3 \Big|_0^{2\pi} = 0 \tag{8-5}$$

by the fundamental theorem of calculus. But

$$\begin{aligned}\sum_{\substack{m,j\in\mathbb{Z}\\ m,j\neq 0\\ m\neq -j}} \frac{1}{j(j+m)} V_m V_j V_{-m-j} &= \sum_{\substack{m,j\in\mathbb{Z}\\ m,j\neq 0\\ m\neq -j}} \frac{-1}{jm} V_m V_j V_{-m-j}\\ &= -\int_0^{2\pi} (V(\theta) - V_0)(W(\theta))^2 \, d\theta,\end{aligned} \tag{8-6}$$

where the last equality uses $e^{ij\theta} e^{im\theta} = e^{i(j+m)\theta}$ and the fact that the integral of a function over a circle is its zeroth Fourier coefficient. Combining (8-5) and (8-6) proves the lemma. □

Lemma 8.5. *Under the hypotheses of Proposition 8.2, if l is sufficiently large*

$$\|\mathcal{P}_l (V^\# R_{V_0}^{\mathrm{reg}})^2 V^\# \mathcal{P}_l\|_{H_{(0,n+6)}(X) \to H_{(0,n)}(X)} = O(l^{-3}) \quad \textit{uniformly for } \zeta \in D_l(\lambda_0, \epsilon).$$

Proof. Again we use that $\|\mathcal{P}_l V^\# \mathcal{P}_l\| = O(l^{-N})$ for any N. This implies

$$\mathcal{P}_l (V^\# R^{\mathrm{reg}}_{V_0})^2 V^\# \mathcal{P}_l = \mathcal{P}_l (V^\# R^{\mathrm{reg}}_{V_0}(I-\mathcal{P}_l))^2 V^\# \mathcal{P}_l + O_{L^2\to L^2}(l^{-N}).$$

Note that for $\zeta \in D_l(\lambda_0,\epsilon)$ and l sufficiently large, $R^{\mathrm{reg}}_{V_0}(\zeta)(I-\mathcal{P}_l) = R_{V_0}(\zeta)(I-\mathcal{P}_l)$. Then

$$\begin{aligned}\mathcal{P}_l&(V^\# R_{V_0}(I-\mathcal{P}_l))^2 V^\# \mathcal{P}_l\\ &= \mathcal{P}_l \sum_{\pm} e^{\pm i(j+k+m)\theta} \sum_{\substack{k,m,j\in\mathbb{Z}\\ k,m+k\neq 0,-2l\\ m,j\neq 0}} V_{\pm j} R_{V_0,0}(\tau_{l+k+m}) V_{\pm m} R_{V_0,0}(\tau_{l+k}) V_{\pm k}\mathcal{P}_{l\pm}\\ &= \sum_{\pm} \sum_{\substack{k,m+k\neq 0,-2l\\ m\neq 0,\, k,m\in\mathbb{Z}}} V_{\mp(k+m)} R_{V_0,0}(\tau_{l+k+m}) V_{\pm m} R_{V_0,0}(\tau_{l+k}) V_{\pm k}\mathcal{P}_{l\pm} + O(l^{-N}).\end{aligned} \tag{8-7}$$

By Lemma 8.1, for $k, m+k \neq 0, -2l$,

$$\Big\| \chi R_{V_0,0}(\tau_{l+k+m}) V_{\pm m} R_{V_0,0}(\tau_{l+k}) V_{\pm k} - \frac{1}{\tau^2_{l+k+m}\tau^2_{l+k}} \chi V_{\pm m} V_{\pm k}\Big\|_{H^{n+6}(\mathbb{R}^d)\to H^n(\mathbb{R}^d)} = O(l^{-3}\|V_{\pm k}\|_{C^{6+n}}\|V_{\pm m}\|_{C^{6+n}}).$$

This implies (with sums still over $\mathbb{Z}$), using $\|V_m\|_{C^p} = O(|m|^{-N})$, that

$$\begin{aligned}\mathcal{P}_l (V^\# R_{V_0}(I-\mathcal{P}_l))^2 V^\# \mathcal{P}_l &= \sum_{\pm} \sum_{k,m,k+m\neq 0,-2l} \frac{1}{\tau^2_{l+k+m}\tau^2_{l+k}} V_{\mp(k+m)} V_{\pm m} V_{\pm k}\mathcal{P}_{l\pm} + O(l^{-3})\\ &= \sum_{\pm} \sum_{0<|k|,\,|k+m|<l^{1/2},\, m\neq 0} \frac{1}{\tau^2_{l+k+m}\tau^2_{l+k}} V_{\mp(k+m)} V_{\pm m} V_{\pm k}\mathcal{P}_{l\pm} + O(l^{-3})\\ &= \sum_{\pm} \sum_{0<|k|,\,|k+m|<l^{1/2},\, m\neq 0} \frac{1}{4l^2 k(k+m)} V_{\mp(k+m)} V_{\pm m} V_{\pm k}\mathcal{P}_{l\pm} + O(l^{-3})\\ &= \sum_{\pm} \sum_{0\neq k,k+m,m} \frac{1}{4l^2 k(k+m)} V_{\mp(k+m)} V_{\pm m} V_{\pm k}\mathcal{P}_{l\pm} + O(l^{-3}).\end{aligned} \tag{8-8}$$

Here errors are as operators from $H_{(0,n+6)}(X)$ to $H_{(0,n)}(X)$, and are uniform in $D_l(\lambda_0,\epsilon)$ when l is sufficiently large. But the final sum in (8-8) is zero by Lemma 8.4. □

Lemma 8.6. *Under the hypotheses of Proposition 8.2 for $j\geq 3$, $j\in\mathbb{N}$, and l sufficiently large,*

$$\|(V^\# R^{\mathrm{reg}}_{V_0}(\zeta))^j V^\#\mathcal{P}_l\|_{H_{(0,8)}(X)\to L^2(X)} = O(l^{-3})$$

uniformly for $\zeta\in D_l(\lambda_0,\epsilon)$.

Proof. By Lemma 8.5,

$$\|\mathcal{P}_l (V^\# R^{\mathrm{reg}}_{V_0})^2 V^\#\mathcal{P}_l\|_{H_{(0,n+6)}\to H_{(0,n)}} = O(l^{-3}).$$

This gives

$$\begin{aligned}(V^\# R^{\mathrm{reg}}_{V_0})^3 V^\#\mathcal{P}_l &= V^\# R^{\mathrm{reg}}_{V_0}(I-\mathcal{P}_l)(V^\# R^{\mathrm{reg}}_{V_0})^2 V^\#\mathcal{P}_l + V^\# R^{\mathrm{reg}}_{V_0}\mathcal{P}_l (V^\# R^{\mathrm{reg}}_{V_0})^2 V^\#\mathcal{P}_l\\ &= V^\# R^{\mathrm{reg}}_{V_0}(I-\mathcal{P}_l)(V^\# R^{\mathrm{reg}}_{V_0})^2 V^\#\mathcal{P}_l + O(l^{-3})\end{aligned} \tag{8-9}$$

as an operator from $H_{(0,n+6)}(X)$ to $H_{(0,n)}(X)$. Using that $\mathcal{P}_l$ commutes with R_{V_0} and $\|\mathcal{P}_l V^\# \mathcal{P}_l\| = O(l^{-N})$ for any N gives

$$(V^\# R_{V_0}^{\text{reg}})^2 V^\# \mathcal{P}_l = (V^\# R_{V_0}(I-\mathcal{P}_l))^2 V^\# \mathcal{P}_l + V^\# R_{V_0}^{\text{reg}} \mathcal{P}_l V^\# R_{V_0}(I-\mathcal{P}_l) V^\# \mathcal{P}_l + O_{L^2\to L^2}(l^{-N}).$$

Using this in (8-9) yields

$$(V^\# R_{V_0}^{\text{reg}})^3 V^\# \mathcal{P}_l = (V^\# R_{V_0}(I-\mathcal{P}_l))^3 V^\# \mathcal{P}_l \\ + V^\# R_{V_0}(I-\mathcal{P}_l) V^\# R_{V_0}^{\text{reg}} \mathcal{P}_l V^\# R_{V_0}(I-\mathcal{P}_l) V^\# \mathcal{P}_l + O_{H_{(0,n+6)}\to H_{(0,n)}}(l^{-3}). \quad (8\text{-}10)$$

For large l, Lemma 8.1 applied with $J=1$ shows that

$$\|(V^\# R_{V_0}(I-\mathcal{P}_l))^3 V^\# \mathcal{P}_l\|_{H_{(0,6)}(X)\to L^2(X)} = O(l^{-3}).$$

Choose $\chi \in C_c^\infty(X)$ independent of θ so that $V\chi = V$. We write the second term on the right in (8-10) as the composition of three operators, with the grouping indicated below by the brackets:

$$V^\# R_{V_0}(I-\mathcal{P}_l) V^\# R_{V_0}^{\text{reg}} \mathcal{P}_l V^\# R_{V_0}(I-\mathcal{P}_l) V^\# \mathcal{P}_l \\ = \big[V^\# R_{V_0}(I-\mathcal{P}_l) V^\#\big]\big[\chi R_{V_0}^{\text{reg}} \mathcal{P}_l \chi\big]\big[\mathcal{P}_l V^\# R_{V_0}(I-\mathcal{P}_l) V^\# \mathcal{P}_l\big]. \quad (8\text{-}11)$$

By Lemma 8.1,

$$\|V^\# R_{V_0}(I-\mathcal{P}_l) V^\#\|_{H_{(0,n+2)}\to H_{(0,n)}} = O(l^{-1}).$$

The second operator, $\chi R_{V_0}^{\text{reg}} \mathcal{P}_l \chi$, is bounded. By Lemma 8.3, the third is $O(l^{-2})$ as an operator from $H_{(0,n+6)}$ to $H_{(0,n)}$. Thus we have proved the lemma when $j=3$.

The case of $j>3$ follows from the $j=3$ case. □

We now can prove Proposition 8.2.

Proof of Proposition 8.2. For l sufficiently large, on $D_l(\lambda_0, \epsilon)$,

$$\mathcal{P}_l (I + V^\# R_{V_0}^{\text{reg}}(\zeta)\chi)^{-1} V^\# \mathcal{P}_l = \mathcal{P}_l \sum_{j=0}^{\infty} (-V^\# R_{V_0}^{\text{reg}}(\zeta)\chi)^j V^\# \mathcal{P}_l.$$

The proposition then follows from an application of Lemmas 8.3, 8.5, and 8.6, and recalling that $\|\mathcal{P}_l V^\# \mathcal{P}_l\| = O(l^{-N})$. □

The proof of Theorem 1.6 uses the next lemma, which computes an expression related to the leading term of

$$\mathcal{P}_l (I + V^\# R_{V_0}^{\text{reg}}(\zeta_l(z)))^{-1} V^\# \mathcal{P}_l.$$

Lemma 8.7. *Suppose $V \in C_c^\infty(X)$ and $u \in H^2(\mathbb{R}^d)$ satisfies $(-\Delta_0 + V_0 - \lambda_0^2)u = 0$. Then*

$$-\int_{\mathbb{R}^d} u((z^2-k^2)V_{-k}V_k u - V_{-k}(-\Delta_0+V_0)(V_k u))\,dx = \int_{\mathbb{R}^d} ((k^2+\lambda_0^2-z^2)u^2 V_{-k}V_k + u^2 \nabla_0 V_{-k}\cdot\nabla_0 V_k)\,dx.$$

Proof. We first compute $\int_{\mathbb{R}} u V_{-k}(-\Delta_0 + V_0)(V_k u)\,dx$. Expanding and then integrating by parts yields

$$\begin{aligned}\int_{\mathbb{R}^d} & u V_{-k}(-\Delta_0+V_0)(V_k u)\,dx \\ &= -\int_{\mathbb{R}^d}(u^2 V_{-k}\Delta_0 V_k + 2V_{-k}u\nabla_0 V_k\cdot\nabla_0 u)\,dx + \int_{\mathbb{R}^d} uV_{-k}V_k(-\Delta_0+V_0)u\,dx \\ &= -\int_{\mathbb{R}^d} u^2 V_{-k}\Delta_0 V_k\,dx + \int_{\mathbb{R}^d} u^2\sum_{j=1}^d \frac{\partial}{\partial x_j}\Big(V_{-k}\frac{\partial}{\partial x_j}V_k\Big)\,dx + \lambda_0^2\int_{\mathbb{R}^d} u^2 V_{-k}V_k\,dx \\ &= \int_{\mathbb{R}^d} u^2\nabla_0 V_{-k}\cdot\nabla_0 V_k\,dx + \lambda_0^2\int_{\mathbb{R}^d} u^2 V_{-k}V_k\,dx. \end{aligned} \tag{8-12}$$

Using this, we find

$$\int_{\mathbb{R}^d}((z^2-k^2)V_{-k}V_k u^2 - uV_{-k}(-\Delta_0+V_0)(V_k u))\,dx = -\int_{\mathbb{R}^d}(((k^2+\lambda_0^2-z^2)V_{-k}V_k + \nabla_0 V_{-k}\cdot\nabla_0 V_k)u^2)\,dx,$$

completing the proof. □

The proof of the next lemma uses some of the same ideas as that of Proposition 8.2. This result will be used in the proof of Theorem 1.7.

Lemma 8.8. *Suppose $V \in C_c^\infty(X;\mathbb{R})$. Let $\lambda_0 \in i\mathbb{R}$ be a simple pole of $R_{V_0,0}(\lambda)$ with residue of rank* 1. *Let $M > |\lambda_0|$ and $N \in \mathbb{N}$, and suppose $R_{V_0,0}(\lambda) - \Xi(R_{V_0,0}(\lambda),\lambda_0)$ is analytic for $|\lambda-\lambda_0| \le \epsilon$. Then if $\chi \in C_c^\infty(X;\mathbb{R})$ is independent of θ and satisfies $V\chi = V$, there exists an $s = s(N) \in \mathbb{N}$ and an $A_N = A_N(\tau_l, l) : H_{(0,s)}(X) \to L^2(X)$ so that for l sufficiently large,*

$$\|\mathcal{P}_l(I + V^\# R^{\mathrm{reg}}_{V_0,0}(\zeta)\chi)^{-1}V^\#\mathcal{P}_l - A_N(\tau_l(\zeta), l)\|_{H_{(0,s)}(X)\to L^2(X)} = O(l^{-N}) \tag{8-13}$$

uniformly for $\zeta \in \overline{D}_l(\lambda_0,\epsilon)$. Moreover, $A_N(z,l)$ depends analytically on z in the set $\{z\in\mathbb{C} : |z-\lambda_0|\le\epsilon\}$ and if $z \in i\mathbb{R}$, then $A_N(z,l)$ is symmetric on $C_c^\infty(X)\subset L^2(X)$. Furthermore,

$$\|\mathcal{P}_{l\pm}A_N\mathcal{P}_{l\mp}\|_{H_{(0,s)}(X)\to L^2(X)} = O(l^{-N})$$

for any N.

Proof. By Lemma 5.2, if $j > 2N$, then on $\overline{D}_l(\lambda_0,\epsilon)$ we have

$$\|(V^\# R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^j\|_{L^2(X)\to L^2(X)} = O(l^{-N}).$$

Thus

$$\Big\|(I+V^\# R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^{-1} - \sum_{j=0}^{2N}(-V^\# R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^j\Big\|_{L^2(X)\to L^2(X)} = O(l^{-N}). \tag{8-14}$$

Now we write, for l sufficiently large,

$$R^{\mathrm{reg}}_{V_0} = R^{\mathrm{reg}}_{V_0}\mathcal{P}_l + R_{V_0}(I-\mathcal{P}_l). \tag{8-15}$$

From our assumptions on V_0 and the pole of $R_{V_0,0}$ at λ_0, there is a $u \in C^\infty(\mathbb{R}^d;\mathbb{R})$ so that for $|\lambda-\lambda_0| \le \epsilon$, $R_{V_0,0}(\lambda) - i/(\lambda-\lambda_0) u\otimes u$ is analytic. Then for l sufficiently large

$$R^{\mathrm{reg}}_{V_0}(\zeta)\mathcal{P}_l = R^{\mathrm{reg}}_{V_0}(\zeta;\lambda_0,l)\mathcal{P}_l = R_{V_0,0}(\tau_l(\zeta))\mathcal{P}_l - \frac{i}{\tau_l(\zeta)-\lambda_0}(u\otimes u)\mathcal{P}_l.$$

If $\tau_l = \tau_l(\zeta) \in i\mathbb{R}$ and $\zeta \in \overline{D}_l(\lambda_0,\epsilon)$, the operator $\chi R^{\mathrm{reg}}_{V_0}(\zeta)\mathcal{P}_l\chi$ is symmetric on $C^\infty_c(X)$. On the other hand, for $k\neq l$, writing τ_k for $\tau_k(\zeta)$ and using Lemma 8.1 yields

$$\begin{aligned}\chi R_{V_0}\mathcal{P}_k\chi &= \chi R_{V_0,0}(\tau_k)\mathcal{P}_k\chi\\ &= -\chi\sum_{j=1}^{N}\frac{1}{(\tau_l^2+l^2-k^2)^j}(-\Delta_0+V_0)^{j-1}\mathcal{P}_k\chi\\ &\qquad\qquad + \chi\frac{1}{(\tau_l^2+l^2-k^2)^N}R_{V_0}(\tau_k)(-\Delta_0+V_0)^N\mathcal{P}_k\chi.\end{aligned} \tag{8-16}$$

If $\tau_l^2\in\mathbb{R}$, then

$$\chi\frac{1}{(\tau_l^2+l^2-k^2)^j}(-\Delta_0+V_0)^{j-1}\mathcal{P}_k\chi$$

is symmetric on $C^\infty_c(X)$. Set

$$T_N = T_N(\tau_l,l) = R^{\mathrm{reg}}_{V_0,0}(\tau_l)\mathcal{P}_l - \sum_{k\neq l}\sum_{j=1}^{N}\frac{1}{(\tau_l^2+l^2-k^2)^j}(-\Delta_0+V_0)^{j-1}\mathcal{P}_k. \tag{8-17}$$

Note that T_N is an analytic operator-valued function of τ_l for $\zeta\in\overline{D}_l(\lambda_0,\epsilon)$, where $|\tau_l-\lambda_0|\le\epsilon$. Using (8-16),

$$\|\chi(R^{\mathrm{reg}}_{V_0} - T_N)\chi\|_{H_{(0,2N+t)}(X)\to H_{(0,t)}(X)} = O(l^{-N}),$$

if $|\tau_l-\lambda_0|\le\epsilon$, and $\chi T_N(\tau_l,l)\chi$ is symmetric on $C^\infty_c(X)$ if $\tau_l\in i\mathbb{R}$. Moreover, by (8-14),

$$\left\|(I+V^\# R^{\mathrm{reg}}_{V_0}(\zeta_l(\tau_l))\chi)^{-1} - \sum_{j=0}^{2N}(-V^\# T_N(\tau_l,l))^j\chi\right\|_{H_{(0,s(N))}\to L^2} = O(l^{-N})$$

if $s(N)\ge 4N^2$. Thus if we define

$$A_N = A_N(\tau_l,l) = \mathcal{P}_l\sum_{j=0}^{2N}(-V^\# T_N)^j V^\#\mathcal{P}_l \tag{8-18}$$

then A_N satisfies (8-13), A_N is an analytic function of τ_l if $|\tau_l-\lambda_0|\le\epsilon$, and $A_N(\tau_l,l)$ is symmetric on $C^\infty_c(X)$ if $\tau_l\in i\mathbb{R}$.

To show that $\|\mathcal{P}_{l\pm}A_N\mathcal{P}_{l\mp}\|_{H_{(0,s)}\to L^2} = O(l^{-N})$, consider a term $\mathcal{P}_{l+}(V^\# T_N)^j V^\#\mathcal{P}_{l-}$. We write

$$\mathcal{P}_{l+}(V^\# T_N)^j V^\#\mathcal{P}_{l-} = \sum_{\substack{m_1+m_2+\cdots+m_{j+1}=2l\\ m_k\neq 0}} V_{m_1}e^{im_1\theta}T_N V_{m_2}e^{im_2\theta}T_N\cdots V_{m_j}e^{im_j\theta}T_N V_{m_{j+1}}e^{im_{j+1}\theta}\mathcal{P}_{l-}.$$

Thus we see that at least one m_n must have absolute value at least $2l/(j+1)$. Since $\|V_m\|_{C^r} = O(|m|^{-p})$ for any fixed r, any p, we obtain

$$\|\mathcal{P}_{l+}(V^\# T_N)^j V^\# \mathcal{P}_{l-}\|_{H_{(0,s)}\to L^2} = O(l^{-N})$$

for some sufficiently large s. Thus the result for $\mathcal{P}_{l+} A_N \mathcal{P}_{l-}$ follows from our expression (8-18) for A_N. The result for $\mathcal{P}_{l-} A_N \mathcal{P}_{l+}$ follows similarly. □

9. Proofs of the smooth case of Theorem 1.2 and Theorem 1.3

The first application of our results in the previous section is to improve the localization of the resonances when $V \in C_c^\infty(X)$.

Proof of Theorem 1.2 for $V \in C_c^\infty(X)$. Let $\lambda_j \in \Lambda_\rho$ and choose $\epsilon > 0$ so that there are no poles of $R_{V_0,0}(\lambda)$ in $0 < |\lambda - \lambda_j| \le \epsilon$. We will show that there is a $C_j > 0$ so that there are no poles of $R_V(\zeta)$ in $\zeta \in D_l(\lambda_j, \epsilon)$ with $|\tau_l(\zeta) - \lambda_j| > C_j l^{-2/(m_{V_0,0}(\lambda_j))}$ when l is sufficiently large.

Choose $\chi \in C_c^\infty(X)$ so that $\chi V = V$ and χ is independent of θ. As previously, if l is sufficiently large,

$$R_{V_0}^{\mathrm{reg}}(\zeta) = R_{V_0}^{\mathrm{reg}}(\zeta; \lambda_j, l) = R_{V_0}(\zeta) - \Xi(R_{V_0,0}, \lambda_j)|_{\lambda=\tau_l(\zeta)} \mathcal{P}_l$$

and note that $R_{V_0}^{\mathrm{reg}}(\zeta; \lambda_j, l)$ is analytic on $\overline{D}_l(\lambda_j, \epsilon)$. By (3-7), any poles of $R_V(\zeta)$ in $D_l(\lambda_j, \epsilon)$ are points at which $I + \mathcal{P}_l(I + V^\# R_{V_0}^{\mathrm{reg}}(\zeta)\chi)^{-1} V^\# \Xi(R_{V_0,0}, \lambda_j)\chi \mathcal{P}_l$ has nontrivial null space.

Using the smoothness of V, for any fixed $s \in \mathbb{N}$ there is a constant $C > 0$ (depending on s, V_0, λ_j) with

$$\|V^\# \Xi(R_{V_0,0}, \lambda_j)|_{\lambda=\tau_l(\zeta)} \chi \mathcal{P}_l\|_{L^2(X)\to H_{(0,s)}(X)} \le \frac{C}{|\tau_l(\zeta) - \lambda_j|^{m_{V_0,0}(\lambda_j)}}, \tag{9-1}$$

[Dyatlov and Zworski 2019, Theorems 2.5, 2.7, 3.9, and 3.17]. Thus on $D_l(\lambda_j, \epsilon)$, for l sufficiently large by Proposition 8.2,

$$\|\mathcal{P}_l(I + V^\# R_{V_0}^{\mathrm{reg}}(\zeta)\chi)^{-1} V^\# \Xi(R_{V_0,0}, \lambda_j)|_{\lambda=\tau_l(\zeta)} \chi \mathcal{P}_l\|_{L^2(X)\to L^2(X)} \le \frac{C}{l^2 |\tau_l(\zeta) - \lambda_j|^{m_{V_0,0}(\lambda_j)}},$$

for some C. Thus there is a $C_j > 0$ so that if $\zeta \in D_l(\lambda_j, \epsilon)$ and $|\tau_l(\zeta) - \lambda_j| > C_j l^{-2/m_{V_0,0}(\lambda_j)}$, then $I + \mathcal{P}_l(I + V^\# R_{V_0}^{\mathrm{reg}}(\zeta)\chi)^{-1} V^\# \Xi(R_{V_0,0}, \lambda_j)\mathcal{P}_l$ is invertible, and ζ is not a resonance.

Since $\lambda_j \in \Lambda_\rho$ is arbitrary, Λ_ρ contains only finitely many elements and we have already proved the theorem for the case of an L^∞ potential V, this suffices to prove the smooth version of the theorem. □

The proof of the smooth case of Theorem 1.3 is almost identical, given our earlier results.

Proof of Theorem 1.3 for $V \in C_c^\infty(X)$. Recall that we have already proved the L^∞ case of this theorem. Thus, the proof follows just as in the proof of the smooth case of Theorem 1.2, except that estimate (9-1) is replaced by

$$\|V^\# \Xi(R_{V_0,0}, 0)|_{\lambda=\tau_l(\zeta)} \chi \mathcal{P}_l\|_{L^2(X)\to H_{(0,s)}(X)} \le \frac{C}{|\tau_l(\zeta)|^r}.$$ □

10. Proofs of Theorems 1.6 and 1.7

We prove Theorems 1.6 and 1.7 in this section, using results of Section 8.

Before turning to the proofs of the theorems, we say something more about the function u of (1-3). The mapping properties of the resolvent mean that for any $\epsilon > 0$ away from its poles, we have the map

$$R_{V_0,0}(\lambda) : e^{-(\epsilon+\max(0,-\operatorname{Im}\lambda))|x|}L^2(\mathbb{R}^d) \to e^{(\epsilon+\max(0,-\operatorname{Im}\lambda))|x|}L^2(\mathbb{R}^d).$$

With $R_{V_0,0}(\lambda)^t$ denoting the transpose, we have the symmetry $R_{V_0,0}(\lambda)^t = R_{V_0,0}(\lambda)$, checked first for $\operatorname{Im}\lambda > 0$ and then holding by analytic continuation for all λ. This implies that if $R_{V_0,0}(\lambda)$ has a simple pole of rank 1 at λ_0, then there is a $u \in e^{(\epsilon+\max(0,-\operatorname{Im}\lambda))|x|}L^2(\mathbb{R}^d)$ so that (1-3) holds, where the operator $u \otimes u$ is understood as an operator between weighted L^2 spaces.

Now we turn more directly to the proofs, beginning with a preliminary lemma.

Lemma 10.1. *Let λ_0 be a pole of $R_{V_0,0}$ and set $R^{\mathrm{reg}}_{V_0}(\zeta) = R^{\mathrm{reg}}_{V_0}(\zeta;\lambda_0,l)$. Let $\chi \in C_c^\infty(X)$ be independent of θ and satisfy $\chi V = V$, with χ nontrivial. Suppose $R_{V_0,0}(\lambda)$ is analytic for $0 < |\lambda-\lambda_0| \le \epsilon$. Then there is an $L > 0$ so that for $l > L$, if $\zeta_0 \in D_l(\lambda_0,\epsilon)$, then*

$$M(I+V^\# R_{V_0}(\zeta)\chi,\zeta_0) = M(I+(I+V^\# R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^{-1}V^\#\Xi(R_{V_0,0}(\lambda),\lambda_0)\restriction_{\lambda=\tau_l(\zeta)}\mathcal{P}_l,\zeta_0).$$

Proof. By Lemma 5.2, there is an $L > 0$ so that $I + V^\# R^{\mathrm{reg}}_{V_0}(\zeta)\chi$ is invertible on $D_l(\lambda_0,\epsilon)$ for $l > L$. Then if $l > L$ and $\zeta_0 \in D_l(\lambda_0,\epsilon)$,

$$\begin{aligned} M(I+V^\# R_{V_0}\chi,\zeta_0) &= M((I+V^\# R^{\mathrm{reg}}_{V_0}(\zeta)\chi)(I+(I+V^\# R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^{-1}V^\#\Xi(R_{V_0,0}(\lambda),\lambda_0)\restriction_{\lambda=\tau_l(\zeta)}\mathcal{P}_l),\zeta_0) \\ &= M(I+(I+V^\# R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^{-1}V^\#\Xi(R_{V_0,0}(\lambda),\lambda_0)\restriction_{\lambda=\tau_l(\zeta)}\mathcal{P}_l,\zeta_0), \end{aligned}$$

where the second equality uses Lemma 4.1. □

Given $f \in C_c^\infty(\mathbb{R}^d)$, define $h_{\pm l} \in C_c^\infty(X)$ by $h_{\pm l}(x,\theta) = f(x)e^{\pm il\theta}/\sqrt{2\pi}$. For $z_0 \in \mathbb{C}$ and an operator $A : H_{(0,s)}(X) \to L^2(X)$ set

$$\mathcal{D}_A(z) = \det\Bigl(I + \frac{i}{z-z_0}(Ah_l \otimes h_{-l} + Ah_{-l}\otimes h_l)\Bigr). \tag{10-1}$$

Here "det" is the Fredholm determinant. In this special case it is easily calculated to be

$$\mathcal{D}_A(z) = \frac{1}{(z-z_0)^2}\Bigl\{\Bigl(z-z_0+i\int_X h_{-l}(Ah_l)\Bigr)\Bigl(z-z_0+i\int_X h_l(Ah_{-l})\Bigr)+\int_X h_{-l}(Ah_{-l})\int_X h_l(Ah_l)\Bigr\}. \tag{10-2}$$

Proposition 10.2. *Let $z_0 \in \mathbb{C}$, $\epsilon > 0$, and set $U_\epsilon = \{z \in \mathbb{C} : |z-z_0| < \epsilon\}$. Suppose there are L_1, $m_0 \ge \frac{1}{2}$ and $s \in \mathbb{N}$ so that for $l > L_1$, $l \in \mathbb{N}$ and $z \in U_\epsilon$ there are linear operators $S_l = S_l(z)$ and $T_l = T_l(z)$ mapping $H_{(0,s)}(X)$ to $L^2(X)$ which are operator-valued functions analytic on U_ϵ satisfying*:

- $\sup_{z\in U_\epsilon} \|\mathcal{P}_l S_l(z)\mathcal{P}_l - T_l(z)\mathcal{P}_l\|_{H_{(0,s)}(X)\to L^2(X)} = O(l^{-m_0})$,
- $T_l(z)\mathcal{P}_l = \mathcal{P}_{l+}T_l(z)\mathcal{P}_{l+} + \mathcal{P}_{l-}T_l(z)\mathcal{P}_{l-}$ *and* $\sup_{z\in U_\epsilon}\|T_l(z)\|_{H_{(0,s)}(X)\to L^2(X)} = O(l^{-1/2})$.

Then given $f \in C_c^\infty(\mathbb{R}^d)$, for l sufficiently large the functions $(z-z_0)^2\mathcal{D}_{S_l}(z)$ and $(z-z_0)^2\mathcal{D}_{T_l}(z)$ have exactly two zeros, counted with multiplicity, in U_ϵ, and they lie in $U_{\epsilon/2}$. Moreover, there is a labeling of these two sets of zeros as $z_{S_l\pm}$ and $z_{T_l\pm}$, so that $|z_{S_l\pm} - z_{T_l\pm}| = O(l^{-m_0})$.

Proof. By translating if necessary, we may assume $z_0 = 0$.

Our assumptions on T_l imply that $F_\pm(z) = F_\pm(z;l) \stackrel{\text{def}}{=} z + i\int_X h_{\mp l}(T_l(z)h_{\pm l})$ is analytic on U_ϵ and satisfies $F_\pm(z) = z + O(l^{-1/2})$ uniformly on U_ϵ. Applying Rouché's theorem to the pair $F_\pm(z)$ and the function z, we see that $F_\pm$ has, for l sufficiently large, exactly one zero in the set $U_{\epsilon/4}$ and no zeros in $U_\epsilon \setminus U_{\epsilon/4}$. We label this zero as $z_{T_l\pm}$. Since $\int_X h_{\pm l}(T_l h_{\pm l}) = 0$, we have that $z^2\mathcal{D}_{T_l}(z) = F_+(z)F_-(z)$ and $z_{T_l\pm}$ are the zeros of $z^2\mathcal{D}_{T_l}$.

We write

$$F_\pm(z;l) = z + i\int_X h_{\mp l}(T_l(z)h_{\pm l}) = (z - z_{T_l\pm})\varphi_\pm(z;l), \tag{10-3}$$

with $\varphi_\pm$ analytic on U_ϵ for l sufficiently large. An application of the maximum principle shows that there is a $C > 0$ independent of l so that for l sufficiently large,

$$\frac{1}{C} \le |\varphi_\pm(z;l)| \le C \quad \text{for all } z \in U_{3\epsilon/4}. \tag{10-4}$$

Next consider the intermediary

$$G_\pm(z) = G_\pm(z;l) \stackrel{\text{def}}{=} z + i\int_X h_{\mp l}(S_l(z)h_{\pm l}) = z + i\int_X h_{\mp l}(T_l(z)h_{\pm l}) + O(l^{-m_0}).$$

Our estimate $G_\pm - F_\pm = O(l^{-m_0})$, (10-3), and (10-4) allow an application of Rouché's theorem to the pair $F_\pm$, $G_\pm$ on a disk with center $z_{T_l\pm}$ and radius $c_0 l^{-m_0}$ for an appropriate choice of $c_0 > 0$ and for l sufficiently large. This shows that for l sufficiently large, $G_\pm$ has exactly one zero (counting multiplicity) in $U_{\epsilon/3}$. We label this zero $z_{I,l,\pm}$ (the "I" here stands for intermediate, as this is an intermediate step). We have shown $|z_{I,l,\pm} - z_{T_l,\pm}| = O(l^{-m_0})$. As before, by the maximum principle we may write

$$G_\pm(z;l) = (z - z_{I,l,\pm})\varphi_{I\pm}(z;l), \quad \text{with } \frac{1}{C} \le |\varphi_{I\pm}(z;l)| \le C, \quad \text{for all } z \in U_{3\epsilon/4} \tag{10-5}$$

for some constant C independent of l, and for l sufficiently large.

Now consider $z^2\mathcal{D}_{S_l}(z)$. By our assumptions on S_l and T_l,

$$z^2\mathcal{D}_{S_l}(z) = G_+(z)G_-(z) + O(l^{-2m_0}) = (z - z_{I,l,+})(z - z_{I,l,-})\varphi_{I+}(z)\varphi_{I-}(z) + O(l^{-2m_0}).$$

Thus we can apply Rouché's theorem again, this time to the pair $z^2\mathcal{D}_{S_l}(z)$ and $G_+(z;l)G_-(z;l)$ at a distance proportional to l^{-m_0} of $z_{I,l,\pm}$, proving the proposition. □

We apply this proposition in the proof of Theorem 1.6.

Proof of Theorem 1.6. We assume that $V^\# \not\equiv 0$, since otherwise there is nothing to prove. Choose $\chi \in C_c^\infty(X)$ with $\chi V = V$, and χ independent of θ.

Let $R_{V_0}^{\text{reg}}(\zeta) = R_{V_0}^{\text{reg}}(\zeta;\lambda_0,l)$, and let $\epsilon,\ L > 0$ be as in Lemma 10.1. For $l > L$ the function

$$F_l(\zeta) \stackrel{\text{def}}{=} (\tau_l(\zeta) - \lambda_0)^2 \det(I + (I + V^\# R_{V_0}^{\text{reg}}(\zeta)\chi)^{-1} V^\# \Xi(R_{V_0,0},\lambda_0)|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l)$$

is analytic on $D_l(\lambda_0,\epsilon)$. Moreover, the order of vanishing of F_l at $\zeta_0 \in D_l(\lambda_0,\epsilon)$ is given by

$$M(I + (I + V^\# R_{V_0}^{\text{reg}}(\zeta)\chi)^{-1} V^\# \Xi(R_{V_0,0},\lambda_0)|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l,\ \zeta_0) + m_{V_0}(\zeta_0),$$

see [Gohberg and Sigal 1971, Theorem 5.1]. Note that for $\zeta_0 \in D_l(\lambda_0, \epsilon)$ and l sufficiently large, $m_{V_0}(\zeta_0) \neq 0$ if and only if $\tau_l(\zeta_0) = \lambda_0$. For $\lambda_0 \neq 0$, combining this with Lemmas 10.1 and 4.4, we see that the poles of R_V in $D_l(\lambda_0, \epsilon)$ are, for $l > L$, given by the zeros of F_l, and the multiplicities agree. If $\lambda_0 = 0$, the same is true, but as in the proof of Theorem 1.3 we use Lemmas 5.6, 5.7, and 5.9.

To prove the theorem, we will apply Proposition 10.2 with the following choices: $z = \tau_l(\zeta)$, $z_0 = \lambda_0$, $f(x) = \chi(x)u(x)$ so that $h_{\pm l}(x, \theta) = \chi(x)u(x)e^{\pm il\theta}/\sqrt{2\pi}$,

$$S_l = S_l(z) = (I + V^{\#} R_V^{\mathrm{reg}}(\zeta_l(z)))^{-1} V^{\#} \mathcal{P}_l,$$
$$T_l = T_l(z) = \frac{-1}{l^2} \sum_{k \neq 0} \left(\frac{z^2 - k^2}{4k^2} V_{-k} V_k - \frac{1}{4k^2} V_{-k}(-\Delta_0 + V_0) V_k \right) \mathcal{P}_l,$$

and $s = 8$. By Proposition 8.2 we have, in the notation of Proposition 10.2, $m_0 = 3$. Note that using the coordinate $z = \tau_l(\zeta)$, we have $F_l(\zeta_l(z)) = (z - \lambda_0)^2 \mathcal{D}_{S_l}(z)$, where $\mathcal{D}_{S_l}$ is as defined via (10-1).

The function $(z - \lambda_0)^2 \mathcal{D}_{T_l}(z)$ has a single zero of multiplicity 2 in U_ϵ, and by Lemma 8.7 this is the zero of

$$z - \lambda_0 + \frac{i}{4l^2} \sum_{k \neq 0} \int_{\mathbb{R}} \left(\frac{k^2 + \lambda_0^2 - z^2}{k^2} u^2 V_{-k} V_k + \frac{u^2 \nabla_0 V_{-k} \cdot \nabla_0 V_k}{k^2} \right)$$

near $z = \lambda_0$. This zero is given by

$$z_{T_l \pm} = \lambda_0 - \frac{i}{4l^2} \sum_{k \neq 0} \int_{\mathbb{R}} \left(u^2 V_{-k} V_k + \frac{u^2 \nabla_0 V_{-k} \cdot \nabla_0 V_k}{k^2} \right) + O(l^{-4}).$$

By Proposition 10.2, the zeros of $(z - \lambda_0)^2 \mathcal{D}_{S_l}(z)$ in U_ϵ are within $O(l^{-m_0}) = O(l^{-3})$ of the zero (of multiplicity 2) of $(z - \lambda_0)^2 \mathcal{D}_{T_l}(z)$ in U_ϵ, thus completing the proof. □

The proof of Theorem 1.7 is similar.

Proof of Theorem 1.7. We prove the theorem by showing that for any $N \in \mathbb{N}$ there is an $\epsilon > 0$ so that for $l \in \mathbb{N}$ sufficiently large if $\zeta_l^\flat \in D_l(\lambda_0, \epsilon)$ and $\zeta_l^\flat$ is a pole of $R_V(\zeta)$, then $\operatorname{Re} \tau_l(\zeta_l^\flat) = O(l^{-N})$.

Choose $\chi \in C_c^\infty(X; \mathbb{R})$ so that $\chi V = V$ and χ is independent of θ. Choose ϵ, $L > 0$, as in Lemma 10.1. Let $u \in C^\infty(\mathbb{R}^d)$ be such that $R_{V_0,0}(\lambda) - i/(\lambda - \lambda_0) u \otimes u$ is analytic for λ near λ_0. Our assumptions on V and λ_0 imply that u is real-valued. We apply Proposition 10.2 in a way very similar to the proof of Theorem 1.6. We make the following choices: $z = \tau_l(\zeta)$, $z_0 = \lambda_0$, $h_{\pm l}(x, \theta) = \chi(x)u(x)e^{\pm il\theta}/\sqrt{2\pi}$, and $S_l = S_l(z) = (I + V^{\#} R_V^{\mathrm{reg}}(\zeta_l(z))\chi)^{-1} V^{\#} \mathcal{P}_l$, where $R_V^{\mathrm{reg}}(\zeta) = R_V^{\mathrm{reg}}(\zeta; \lambda_0, l)$. For l sufficiently large, S_l is analytic on U_ϵ. Let $A_N = A_N(z, l)$ be the operator from Lemma 8.8, and set

$$T_l = T_l(z; N) = \mathcal{P}_{l+} A_N \mathcal{P}_{l+} + \mathcal{P}_{l-} A_N \mathcal{P}_{l-}.$$

By Lemma 8.8, there is an $s \in \mathbb{N}$ so that

$$\| \mathcal{P}_l S_l(z) \mathcal{P}_l - T_l(z) \|_{H_{(0,s)}(X) \to L^2(X)} = O(l^{-N})$$

uniformly for $z \in U_\epsilon$. Thus for our application of Proposition 10.2 we have $m_0 = N$.

Following the proof of Theorem 1.6, the poles of R_V in $D_l(\lambda_0, \epsilon)$ are determined by the zeros of $(z - \lambda_0)^2 \mathcal{D}_{S_l}(z)$ in U_ϵ, using $U_\epsilon \ni z = \tau_l(\zeta)$. By Proposition 10.2, these zeros are approximated by those of $(z-\lambda_0)^2\mathcal{D}_{T_l}(z)$ in U_ϵ, with an error which is $O(l^{-N})$. We compete the proof by showing that for l sufficiently large the zeros of $\mathcal{D}_{T_l}(z)$ in U_ϵ lie on the imaginary axis.

Set $a_\pm(z;l) \overset{\text{def}}{=} \int_X h_{\mp l}(T_l(z)h_{\pm l}) = \int_X \overline{h_{\pm l}}(T_l(z)h_{\pm l})$. From Lemma 8.8 and the definition of T_l, if $z \in U_\epsilon \cap i\mathbb{R}$, then $T_l(iz)$ is symmetric on $C_c^\infty(X) \subset L^2(X)$. In particular, this implies that if $z \in i\mathbb{R}\cap U_\epsilon$ then $a_\pm(z;l) \in \mathbb{R}$. Since $a_\pm(z;l)$ is analytic for $z \in U_\epsilon$ and is real-valued for $z \in i\mathbb{R}\cap U_\epsilon$, we must have

$$a_\pm(z;l) = \bar{a}_\pm(-\bar{z};l) \quad \text{for } z \in U_\epsilon. \tag{10-6}$$

We remark that since $\lambda_0 \in i\mathbb{R}$, we have $z \in U_\epsilon$ if and only if $-\bar{z} \in U_\epsilon$.

From the proof of Proposition 10.2, the zeros of $(z-\lambda_0)^2\mathcal{D}_{T_l}(z)$ in U_ϵ are given by the zeros of $z-\lambda_0+ia_\pm(z,l)$ in U_ϵ, and there is, for l sufficiently large, exactly one such zero for each choice of $\pm$. We denote these zeros by $z_{T_l\pm}$ and focus on the zero for the "+" sign, z_{T_l+}. Using $\lambda_0 \in i\mathbb{R}$,

$$\begin{aligned} z_{T_l+}-\lambda_0+ia_+(z_{T_l+};l) = 0 = \overline{z_{T_l+}-\lambda_0+ia_+(z_{T_l+};l)} &= -(-\overline{z_{T_l+}}-\lambda_0+i\bar{a}_+(z_{T_l+};l)) \\ &= -(-\overline{z_{T_l+}}-\lambda_0+ia_+(-\overline{z_{T_l+}};l)), \end{aligned}$$

where the last equality uses (10-6). Hence $-\overline{z_{T_l+}}$ is also a zero of $z-\lambda_0+ia_+(z;l)$ in U_ϵ, and since there is exactly one such zero, it must be that $-\overline{z_{T_l+}} = z_{T_l+}$, and thus $z_{T_l+} \in i\mathbb{R}$. The same argument shows $z_{T_l-} \in i\mathbb{R}$. □

11. Proof of Theorem 1.8, the resonant uniqueness of $V \equiv 0$ when $d = 1$

Theorem 1.8, a result on the resonant rigidity of the zero potential on $\mathbb{R}\times\mathbb{S}^1$, follows rather directly from Theorems 1.1, 1.3, and 1.6.

Proof of Theorem 1.8. Suppose $X = \mathbb{R}\times\mathbb{S}^1$ and V is as in Theorem 1.8. Then by Theorems 1.1 and 1.3, the one-dimensional operator $-d^2/dx^2 + V_0$ on $\mathbb{R}$ must have a resonance at the origin and nowhere else, and this resonance must have multiplicity 1. But since $V_0 \in L_c^\infty(\mathbb{R})$, by well-known results for one-dimensional Schrödinger operators, $V_0 \equiv 0$; see for example [Zworski 1987].

The operator $R_{0,0}(\lambda) - i/(2\lambda)1\otimes 1$ is analytic at the origin. Using this in Theorem 1.6 along with the fact that R_V has poles at a sequence of thresholds tending to infinity, we find

$$\sum_{k\neq 0} \frac{1}{k^2}\int_{\mathbb{R}} (k^2 V_k V_{-k} + V_k' V_{-k}')(x)\,dx = 0.$$

But since $V_{-k}(x) = \overline{V}_k(x)$ for a real-valued potential V, this implies $V_k \equiv 0$ for all k, and hence $V \equiv 0$. □

12. The potential $V(x,\theta) = 2\chi_{I_0}(x)\cos\theta$ on $\mathbb{R}\times\mathbb{S}^1$

In this section we investigate the resonances near the l-th threshold of the Schrödinger operator with potential $V(x,\theta) = 2\chi_{I_0}(x)\cos\theta$ on $X = \mathbb{R}\times\mathbb{S}^1$. Here $\chi_{I_0}(x)$ is the characteristic function of the interval $I_0 = [-1,1]$, so $\chi_{I_0}(x) = 1$ if $|x| \le 1$ and $\chi_{I_0}(x) = 0$ if $|x| > 1$. This potential has $V_0 \equiv 0$ so that $V^\# = V$. Proposition 12.3 shows that the resonances nearest the threshold, which correspond to perturbations of the pole at the origin for $R_{0,0}(\lambda)$, are, for this potential, localized in a different way than for smooth potentials;

compare Theorem 1.6. By Proposition 12.6, there is a sense in which Theorem 1.5 is sharp. We remark that some of the computations of this section are reminiscent of those found in [Drouot 2018, Section 2].

In all of this section,

$$V(x,\theta) = 2\chi_{I_0}(x)\cos\theta \quad \text{and} \quad X = \mathbb{R}\times\mathbb{S}^1.$$

We will use this preliminary lemma.

Lemma 12.1. *For* $\lambda,\ \lambda' \in \mathbb{C}$, $\lambda \neq \pm\lambda'$,

$$\chi_{I_0} R_{0,0}(\lambda)\chi_{I_0} R_{0,0}(\lambda')\chi_{I_0} = \frac{1}{(\lambda')^2-\lambda^2}\chi_{I_0}(R_{0,0}(\lambda')-R_{0,0}(\lambda))\chi_{I_0} + \frac{i}{4\lambda\lambda'(\lambda+\lambda')}e^{i(\lambda+\lambda')}(\phi_\lambda\otimes\phi_{\lambda'}+\phi_{-\lambda}\otimes\phi_{-\lambda'}), \tag{12-1}$$

where

$$\phi_{\pm\lambda}(x) = e^{\pm i\lambda x}\chi_{I_0}(x).$$

Moreover, if $\tau\in\mathbb{C}$, $\tau\neq\pm\lambda$, *applying the operator* $\chi_{I_0}R_{0,0}(\tau)$ *to the function* $\chi_{I_0}(x)e^{i\lambda x}$ *yields*

$$(\chi_{I_0}R_{0,0}(\tau)\chi_{I_0}e^{i\lambda\star})(x) = \chi_{I_0}(x)\left(\frac{1}{\lambda^2-\tau^2}e^{i\lambda x} + \frac{1}{2\tau(\lambda-\tau)}e^{-i\lambda}e^{i\tau(1+x)} + \frac{1}{2\tau(\tau+\lambda)}e^{i\lambda}e^{i\tau(1-x)}\right). \tag{12-2}$$

Proof. The first can be seen, for example, by using (3-1), the explicit expression for the Schwartz kernel of $R_{0,0}$, and evaluating

$$\int_{-1}^{1} e^{i\lambda|x-x''|+i\lambda'|x''-x'|}\,dx''$$

for $|x|,\ |x'|\leq 1$. Likewise, (12-2) follows from an explicit computation using (3-1). □

12A. ***Resonances near the threshold*** $\boldsymbol{\tau_l = 0}$ ***for*** $\boldsymbol{V(x,\theta) = 2\chi_{I_0}(x)\cos\theta}$**.** Since in this section we concentrate on the resonance near the threshold, we work on $B_l(1)$. A preliminary step is the following.

Lemma 12.2. *Let* $R_0^{\mathrm{reg}}(\zeta) = R_0^{\mathrm{reg}}(\zeta;0,l)$. *Then for* l *sufficiently large, uniformly on* $B_l(1)$,

$$\|\mathcal{P}_l((I+VR_0^{\mathrm{reg}}(\zeta)\chi_{I_0})^{-1}V + VR_0^{\mathrm{reg}}(\zeta)V + (VR_0^{\mathrm{reg}}(\zeta))^3V)\mathcal{P}_l\| = O(l^{-2}).$$

Proof. Using the Neumann series,

$$(I+VR_0^{\mathrm{reg}}(\zeta)\chi_{I_0})^{-1}V = \sum_{j=0}^{\infty}(-VR_0^{\mathrm{reg}}(\zeta))^jV.$$

By Lemma 5.2, $\|(-VR_0^{\mathrm{reg}}(\zeta))^j\| = O(l^{-2})$ on $B_l(1)$ if $j\geq 4$ and l is sufficiently large. This ensures the Neumann series for $(I+VR_0^{\mathrm{reg}}(\zeta)\chi_{I_0})^{-1}$ converges, and

$$\left\|(I+VR_0^{\mathrm{reg}}(\zeta)\chi_{I_0})^{-1}V - \sum_{j=0}^{3}(-VR_0^{\mathrm{reg}}(\zeta))^jV\right\| = O(l^{-2})$$

on $B_l(1)$.

Now we note that our explicit expression for V means that $\mathcal{P}_lV\mathcal{P}_l = 0$. Likewise, it implies that $\mathcal{P}_l(VR_0^{\mathrm{reg}}(\zeta))^2V\mathcal{P}_l = 0$, completing the proof. □

Proposition 12.3. *For l sufficiently large, the poles of $R_V(\zeta)$ in $B_l(1)$ satisfy*

$$\tau_l(\zeta) = \frac{1}{4l\sqrt{2l}}(-1 - i + e^{i2\sqrt{2l}}) + O(l^{-2}).$$

Proof. We give a proof similar to that of Theorem 1.6 using Proposition 10.2.

Let R_0^{reg} be as in Lemma 12.2, and restrict ζ to $\zeta \in B_l(1)$. Note

$$R_{0,0}(\lambda) - \frac{i}{2\lambda} 1 \otimes 1$$

is regular at $\lambda = 0$. Set $z = \tau_l(\zeta)$,

$$S_l(z) = (I + V R_0^{\text{reg}}(\zeta_l(z))\chi_{I_0})^{-1} V \mathcal{P}_l, \quad \text{and} \quad h_{\pm l}(x,\theta) = \frac{1}{\sqrt{2\pi}} \chi_{I_0}(x) e^{\pm il\theta}$$

We use $\mathcal{D}_{S_l}$ as is defined by (10-1) and U_ϵ as in Proposition 10.2. Then just as in the proof of Theorem 1.6, the poles of R_V in $B_l(1)$ are identified via $z = \tau_l(\zeta)$ with the zeros of $z^2 \mathcal{D}_{S_l}(z)$ in U_1. Set $z_0 = 0$ and $T_l = \mathcal{P}_l(-V R_0^{\text{reg}}(\zeta) V - (V R_0^{\text{reg}}(\zeta))^3 V)\mathcal{P}_l$. Then by Lemma 12.2, in our application of Proposition 10.2 we can take $s = 0$ and $m_0 = 2$. We claim that uniformly for $z \in U_1$,

$$z^2 \mathcal{D}_{T_l}(z) = \left(z + \frac{1}{2(2l)^{3/2}}(1 - e^{2i\sqrt{2l}} + i) + O(l^{-2}) \right)^2. \tag{12-3}$$

Assuming for the moment that (12-3) holds, this shows that the two zeros (when counted with multiplicity) of $z^2 \mathcal{D}_{T_l}(z)$ in U_1 satisfy

$$z = \frac{-1 - i + e^{2i\sqrt{2l}}}{2(2l)^{3/2}} + O(l^{-2}).$$

An application of Proposition 10.2 and Lemma 12.2 then proves the proposition.

We now turn to showing (12-3). We use

$$R_0^{\text{reg}}(\zeta_l(z)) V \mathcal{P}_l = \sum_{\pm} (e^{\pm i\theta} R_{0,0}(\tau_{l+1}) + e^{\mp i\theta} R_{0,0}(\tau_{l-1}))\chi_{I_0} \mathcal{P}_{l\pm}, \tag{12-4}$$

where $\tau_{l\pm 1} = \tau_{l\pm 1}(\zeta_l(z))$, so that

$$\mathcal{P}_l V R_0^{\text{reg}}(\zeta_l(z)) V \mathcal{P}_l = \chi_{I_0}(R_{0,0}(\tau_{l-1}) + R_{0,0}(\tau_{l+1}))\chi_{I_0} \mathcal{P}_l. \tag{12-5}$$

Then using (12-2) gives

$$\int_X h_{\mp l} V R_0^{\text{reg}}(\zeta_l(z)) V h_{\pm l} = \frac{-i}{2(2l)^{3/2}}(1 - e^{2i\sqrt{2l}}) + \frac{1}{2(2l)^{3/2}} + O(l^{-2}) \tag{12-6}$$

uniformly on U_1. Now note

$$\int_X h_{\mp l}(V R_0^{\text{reg}})^3 V h_{\pm l} = \int_X (V R_0^{\text{reg}} V h_{\mp l})(\chi_{I_0}(R_0^{\text{reg}} V)^2 h_{\pm l}). \tag{12-7}$$

By (12-2),

$$\|(V R_0^{\text{reg}} V h_{\mp l})\| = O(l^{-1}) \quad \text{and} \quad \|\chi_{I_0}(R_0^{\text{reg}} V)^2 h_{\pm l}\| = O(l^{-1}).$$

Using the expression for $\mathcal{D}_{T_l}$ as in (10-2) and equations (12-5)–(12-7) completes the proof of (12-3). □

12B. ***Existence of poles of R_V within $\approx \log l$ of the l-th threshold, for $V(x,\theta) = 2\chi_{I_0}(x)\cos\theta$.*** As a point of comparison with Theorem 1.5, for the special case $V(x,\theta) = 2\chi_{I_0}(x)\cos\theta$ on $X = \mathbb{R}\times\mathbb{S}^1$ we consider the existence of poles of $R_V(\zeta)$ in $D_l(\alpha\log l)$ with $|\tau_l(\zeta)| > 1$.

Again, we use the coordinate $z = \tau_l(\zeta)$ on $B_l(\alpha\log l)$, and the functions ϕ_λ are as defined in Lemma 12.1.

Lemma 12.4. *Let $\alpha > 0$ be fixed, and set $z = \tau_l(\zeta)$. For l sufficiently large, uniformly on $B_l(\alpha\log l)\setminus B_l(1)$ we have*

$$\left\| \mathcal{P}_l(I + VR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l))^{-1}VR_0(\zeta)\chi_{I_0}\mathcal{P}_l + (f_+\otimes\phi_z + f_-\otimes\phi_{-z})\mathcal{P}_l - \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\mathcal{P}_l \right\| = O\left(\frac{1}{l^{5/2}}e^{2(\operatorname{Im} z)_-}\right) + O(l^{-3/2}), \quad (12\text{-}8)$$

where

$$f_\pm(x) = f_\pm(x,z,l) = \frac{ie^{iz}}{4z}\chi_{I_0}(x)\left(\frac{e^{i\tau_{l+1}}}{\tau_{l+1}(z+\tau_{l+1})}\phi_{\pm\tau_{l+1}} + \frac{e^{i\tau_{l-1}}}{\tau_{l-1}(z+\tau_{l-1})}\phi_{\pm\tau_{l-1}}\right).$$

For notational simplicity, we have written $\tau_{l\pm1}$ for $\tau_{l\pm1}(\zeta_l(z))$.

Proof. We use

$$(I + VR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l))^{-1} = \sum_{j=0}^{\infty}(-VR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l))^j$$

since $\|VR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l)\| = O(l^{-1/2})$. This estimate, along with others in this proof, are uniform for $\zeta\in B_l(\alpha\log l)\setminus B_l(1)$. By Lemma 12.1, (3-1), and the explicit expression for V, we see that

$$\|\chi_{I_0}R_0(\zeta)(I-\mathcal{P}_l)VR_0(\zeta)\chi_{I_0}\mathcal{P}_l\| = O(e^{2(\operatorname{Im} z)_-}/(l|z|)) \quad \text{for } \zeta\in B_l(\alpha\log l)$$

for l sufficiently large. Moreover, this same lemma implies that if $|j-l|\le 2$, then

$$\|\chi_{I_0}(VR_0(\zeta)(I-\mathcal{P}_l))^2\chi_{I_0}\mathcal{P}_j\| = O(l^{-3/2})$$

uniformly on $B_l(\alpha\log l)$. This ensures that

$$\left\| \left((I + VR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l))^{-1} - \sum_{j=0}^{2}(-VR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l))^j\right)VR_0(\zeta)\chi_{I_0}\mathcal{P}_l \right\| = O\left(\frac{1}{l^{5/2}|z|}e^{2(\operatorname{Im} z)_-}\right). \quad (12\text{-}9)$$

Since, as in the proof of Proposition 12.3, $\mathcal{P}_lV\mathcal{P}_l = 0$ and $\mathcal{P}_l(VR_0(I-\mathcal{P}_l))^2V\mathcal{P}_l = 0$, it suffices to use $-\mathcal{P}_lVR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l)VR_0(\zeta)\mathcal{P}_l$ to approximate $\mathcal{P}_l(I + VR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l))^{-1}VR_0(\zeta)\chi_{I_0}\mathcal{P}_l$ with the desired accuracy.

Using Lemma 12.1 and its notation,

$$\begin{aligned}
&\mathcal{P}_l V R_0(\zeta_l(z))(I-\mathcal{P}_l) V R_0(\zeta_l(z))\chi_{I_0}\mathcal{P}_l \\
&\quad = \chi_{I_0}(R_{0,0}(\tau_{l+1})\chi_{I_0} R_{0,0}(z) + R_{0,0}(\tau_{l-1})\chi_{I_0} R_{0,0}(z))\chi_{I_0}\mathcal{P}_l \\
&\quad = \frac{1}{\tau_{l+1}^2 - z^2}\chi_{I_0}(R_{0,0}(\tau_{l+1}) - R_{0,0}(z))\chi_{I_0}\mathcal{P}_l + \frac{i e^{i(z+\tau_{l+1})}}{4z\tau_{l+1}(z+\tau_{l+1})}(\phi_{\tau_{l+1}}\otimes\phi_z + \phi_{-\tau_{l+1}}\otimes\phi_{-z})\mathcal{P}_l \\
&\qquad\quad + \frac{1}{\tau_{l-1}^2 - z^2}\chi_{I_0}(R_{0,0}(\tau_{l-1}) - R_{0,0}(z))\chi_{I_0}\mathcal{P}_l + \frac{i e^{i(z+\tau_{l-1})}}{4z\tau_{l-1}(z+\tau_{l-1})}(\phi_{\tau_{l-1}}\otimes\phi_z + \phi_{-\tau_{l-1}}\otimes\phi_{-z})\mathcal{P}_l.
\end{aligned}$$

Note that

$$\left\| \frac{1}{\tau_{l\pm1}^2 - z^2}\chi_{I_0} R_{0,0}(\tau_{l\pm1})\chi_{I_0}\right\| = O(l^{-3/2})$$

and

$$\left\| \left(\frac{1}{\tau_{l+1}^2 - z^2} + \frac{1}{\tau_{l-1}^2 - z^2}\right)\chi_{I_0} R_{0,0}(z)\chi_{I_0} - \frac{1}{2l^2}\chi_{I_0} R_{0,0}(z)\chi_{I_0}\right\| = O(l^{-4}|z|^{-1}e^{2(\operatorname{Im} z)_-}).$$

This gives

$$\begin{aligned}
&\mathcal{P}_l(V R_0(\zeta_l(z))(I-\mathcal{P}_l) V R_{0,0}(\zeta_l(z))\chi_{I_0}\mathcal{P}_l \\
&= \frac{i e^{i(z+\tau_{l+1})}}{4z\tau_{l+1}(z+\tau_{l+1})}(\phi_{\tau_{l+1}}\otimes\phi_z + \phi_{-\tau_{l+1}}\otimes\phi_{-z})\mathcal{P}_l + \frac{i e^{i(z+\tau_{l-1})}}{4z\tau_{l-1}(z+\tau_{l-1})}(\phi_{\tau_{l-1}}\otimes\phi_z + \phi_{-\tau_{l-1}}\otimes\phi_{-z})\mathcal{P}_l \\
&\qquad - \frac{1}{2l^2}R_{0,0}(z)\mathcal{P}_l + O_{L^2\to L^2}\left(\frac{1}{l^{5/2}|z|}e^{2(\operatorname{Im} z)_-}\right) + O_{L^2\to L^2}(l^{-3/2}), \qquad (12\text{-}10)
\end{aligned}$$

and completes the proof. □

Note that the functions $f_\pm$ and $\phi_\pm$ in Lemma 12.4 depend holomorphically on z in the set

$$\{z \in \mathbb{C} : 1 \le z \le \alpha \log l\}.$$

The function g_l of the next lemma appears in the proof of Proposition 12.6, as its zeros approximate the locations of the poles of $R_V(\zeta)$ away from the threshold in $B_l(\alpha \log l)$, if $\alpha < 1$. A discussion of the Lambert W function can be found, for example, in [Corless et al. 1996]. This next lemma is very similar to [Drouot 2018, Lemma 2.4].

Lemma 12.5. *The zeros of*

$$g_l(z) \overset{\text{def}}{=} \left(1 - \frac{1}{z8l\sqrt{2l}}e^{2i(\sqrt{2l}+z)}\right)^2 - \left(\frac{1}{8lz\sqrt{2l}}(ie^{2iz} + e^{2iz})\right)^2$$

are given by $z_\nu^\pm = z_\nu^\pm(l) = \frac{i}{2}\mathcal{W}_\nu((-ie^{2i\sqrt{2l}} \mp i \pm 1)/(4l\sqrt{2l}))$, *where* $\mathcal{W}_\nu$ *is the* ν*-th branch of the Lambert W function. In particular, we have* $z_1^+ \sim -\frac{3i}{4}\log l$. *Moreover, for* l *sufficiently large there is an* $r_0 > 0$ *independent of* l *so that if* $w \in \mathbb{C}$ *and* $|w| < r_0$, *then*

$$|g_l(z_1^+(l) + w)| \ge \tfrac{2}{3}|w|. \qquad (12\text{-}11)$$

Proof. The zeros of g_l are solutions of

$$1 - \frac{1}{z8l\sqrt{2l}} e^{2i(\sqrt{2l}+z)} = \pm \frac{1}{8lz\sqrt{2l}} (ie^{2iz} + e^{2iz})$$

and so satisfy

$$ze^{-2iz} = \frac{1}{8l\sqrt{2l}} (e^{2i\sqrt{2l}} \pm 1 \pm i).$$

Solutions of this equation are given by

$$z_\nu^{\pm} = \frac{i}{2} \mathcal{W}_\nu \left(\frac{1}{4l\sqrt{2l}} (-ie^{2i\sqrt{2l}} \mp i \pm 1) \right).$$

From [Corless et al. 1996, (4.20)], we have $z_1^+ \sim -\frac{3i}{4} \log l$ as $l \to \infty$.

To finish the proof, we set $\gamma = 1/(8l\sqrt{2l})$ and write

$$g_l(z) = \left(1 + \frac{\gamma}{z} e^{2iz} (-e^{2i\sqrt{2l}} - 1 - i)\right) \left(1 + \frac{\gamma}{z} e^{2iz} (-e^{2i\sqrt{2l}} + 1 + i)\right).$$

Now we evaluate at $z = z_1^+ + w$, with $w \in \mathbb{C}$, $|w|$ small, to find

$$\begin{aligned} g_l(z_1^+ + w) &= \left(1 + \frac{z_1^+ e^{2iw}}{z_1^+ + w} \frac{\gamma}{z_1^+ e^{-2iz_1^+}} (-e^{2i\sqrt{2l}} - 1 - i)\right) \left(1 + \frac{z_1^+ e^{2iw}}{z_1^+ + w} \frac{\gamma}{z_1^+ e^{-2iz_1^+}} (-e^{2i\sqrt{2l}} + 1 + i)\right) \\ &= \left(1 - \frac{z_1^+ e^{2iw}}{z_1^+ + w}\right) \left(1 + \frac{z_1^+ e^{2iw}}{z_1^+ + w} \frac{-e^{2i\sqrt{2l}} + 1 + i}{e^{2i\sqrt{2l}} + 1 + i}\right), \end{aligned}$$

where for the second equality we have used $z_1^+ e^{-2iz_1^+} = \gamma(e^{2i\sqrt{2l}} + 1 + i)$. This gives, then, recalling $|z_1^+| \to \infty$ as $l \to \infty$,

$$g_l(z_1^+ + w) = (-2iw + O(|w|/|z_1^+|) + O(|w|^2)) \left(\frac{2(i+1)}{e^{2i\sqrt{2l}} + 1 + i} + O(|w|) \right)$$

for $|w|$ small. Then there is a $r_0 > 0$ independent of l so that for l sufficiently large and $|w| < r_0$, $|g_l(z_1^+ + w)| > \frac{2}{3}|w|$. □

Proposition 12.6. *For $V(x, \theta) = 2\chi_{I_0}(x) \cos\theta$ and l sufficiently large, $R_V(\zeta)$ has a pole at a point $\zeta_l^+ \in B_l\left(\frac{7}{8}\log l\right)$ with ζ_l^+ satisfying*

$$\tau_l(\zeta_l^+) = \frac{i}{2} \mathcal{W}_1 \left(\frac{1}{4l\sqrt{2l}} (ie^{2i\sqrt{2l}} - i + 1) \right) + O(l^{-1/2+\epsilon})$$

for any $\epsilon > 0$.

Proof. We continue to use $z = \tau_l(\zeta)$ and work in a region with $1 < |z| < \frac{7}{8} \log l$.

Using Lemma 12.4,

$$\mathcal{P}_l (I + VR_0(\zeta)\chi_{I_0}(I - \mathcal{P}_l))^{-1} VR_0(\zeta)\chi_{I_0}\mathcal{P}_l = F\mathcal{P}_l + \frac{1}{2l^2} \chi_{I_0} R_{0,0}(z) \chi_{I_0} \mathcal{P}_l + A,$$

where, with notation from Lemma 12.4,

$$F = F(z, l) = -f_+ \otimes \phi_z - f_- \otimes \phi_{-z}$$

and $\|A\| = O(l^{-5/2}e^{(2\operatorname{Im} z)_-}) + O(l^{-3/2})$ on $B_l\big(\frac{7}{8}\log l\big) \setminus B_1(l)$. We recall that the poles of R_V in $B_l\big(\frac{7}{8}\log l\big) \setminus B_1(l)$ are the zeros of $I + \mathcal{P}_l(I + VR_0(\zeta)\chi_{I_0}(I - \mathcal{P}_l))^{-1}VR_0(\zeta)\chi_{I_0}\mathcal{P}_l$ in $B_l\big(\frac{7}{8}\log l\big) \setminus B_1(l)$. We write

$$I + \mathcal{P}_l(I + VR_0(\zeta)\chi_{I_0}(I - \mathcal{P}_l))^{-1}VR_0(\zeta)\chi_{I_0}\mathcal{P}_l$$
$$= \left(I + \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\mathcal{P}_l\right)\left(I + \left(I + \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\mathcal{P}_l\right)^{-1}(F\mathcal{P}_l + A)\right) \tag{12-12}$$

since

$$I + \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\mathcal{P}_l$$

is invertible here. For notational convenience, set

$$S = S_l = \left(I + \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\mathcal{P}_l\right)^{-1},$$

and note that

$$S = I - \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\mathcal{P}_l + O_{L^2\to L^2}(l^{-4}e^{4(\operatorname{Im} z)_-}).$$

We first consider the poles of $I + SF\mathcal{P}_l$. These poles are given by the zeros of the function

$$\widetilde{\mathcal{D}}_l(z) \stackrel{\text{def}}{=} \det(I + SF\mathcal{P}_{l\pm}) = \left(1 - \int_{\mathbb{R}}(Sf_+)\phi_z\right)\left(1 - \int_{\mathbb{R}}(Sf_-)\phi_{-z}\right) - \left(\int_{\mathbb{R}}(Sf_-)\phi_z\right)\left(\int_{\mathbb{R}}(Sf_+)\phi_{-z}\right)$$

with twice the multiplicity. A computation and use of the approximations $\tau_{l+1} = i\sqrt{2l} + O(l^{-1/2})$ and $\tau_{l-1} = \sqrt{2l} + O(l^{-1/2})$ show that

$$\widetilde{\mathcal{D}}_l(z) = g_l(z) + O(l^{-3/2}) + O(l^{-2}\log l e^{2(\operatorname{Im} z)_-}),$$

where g_l is the function of Lemma 12.5. We note that both g_l and $\widetilde{\mathcal{D}}_l$ are analytic in z if $1 < |z| < \frac{7}{8}\log l$. We use $z_1^+(l)$ as in Lemma 12.5. Recalling that $\operatorname{Im} z_1^+ \sim -\frac{3}{4}\log l$, the estimate (12-11) combined with Rouché's theorem shows that $\widetilde{\mathcal{D}}_l(z)$ has a zero within $O(l^{-1/2+\epsilon})$, for any $\epsilon > 0$, of $z_1^+(l)$. This, in turn, means that

$$(I + SF\mathcal{P}_l)^{-1} = \left(I + \left(I + \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\right)^{-1}F\mathcal{P}_l\right)^{-1}$$

has a single pole of multiplicity two at a point satisfying $z = z_1^+(l) + O(l^{-1/2+\epsilon})$. Moreover, we can find a $c_0 = c_0(\epsilon)$ so that

$$\left\|\left(I + \left(I - \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\right)F\mathcal{P}_l\right)^{-1}\right\| = O(l^{1+\epsilon})$$

when the distance from z to the pole is given by $c_0 l^{-1/2+\epsilon}$.

Now using our estimate on $\|A\|$ we can apply the operator Rouché theorem to the pair $I + SF\mathcal{P}_l$ and $I + SF\mathcal{P}_l + SA$, to find that $I + SF\mathcal{P}_l + SA$ has two poles (when counted with multiplicity) which are, using the z-coordinate, within $O(l^{-1/2+\epsilon})$ of $z_1^+(l)$. □

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A STRUCTURE THEOREM FOR ELLIPTIC AND PARABOLIC OPERATORS WITH APPLICATIONS TO HOMOGENIZATION OF OPERATORS OF KOLMOGOROV TYPE

MALTE LITSGÅRD AND KAJ NYSTRÖM

We consider the operators

$$\nabla_X \cdot (A(X)\nabla_X), \quad \nabla_X \cdot (A(X)\nabla_X) - \partial_t, \quad \nabla_X \cdot (A(X)\nabla_X) + X \cdot \nabla_Y - \partial_t,$$

where $X \in \Omega$, $(X, t) \in \Omega \times \mathbb{R}$ and $(X, Y, t) \in \Omega \times \mathbb{R}^m \times \mathbb{R}$, respectively, and where $\Omega \subset \mathbb{R}^m$ is an (unbounded) Lipschitz domain with defining function $\psi : \mathbb{R}^{m-1} \to \mathbb{R}$ being Lipschitz with constant bounded by M. Assume that the elliptic measure associated to the first of these operators is mutually absolutely continuous with respect to the surface measure $\mathrm{d}\sigma(X)$ and that the corresponding Radon–Nikodym derivative or Poisson kernel satisfies a scale-invariant reverse Hölder inequality in L^p, for some fixed p, $1 < p < \infty$, with constants depending only on the constants of A, m and the Lipschitz constant of ψ, M. Under this assumption we prove that the same conclusions are also true for the parabolic measures associated to the second and third operators with $\mathrm{d}\sigma(X)$ replaced by the surface measures $\mathrm{d}\sigma(X)\,\mathrm{d}t$ and $\mathrm{d}\sigma(X)\,\mathrm{d}Y\,\mathrm{d}t$, respectively. This structural theorem allows us to reprove several results previously established in the literature, as well as to deduce new results in, for example, the context of homogenization for operators of Kolmogorov type. Our proof of the structural theorem is based on recent results established by the authors concerning boundary Harnack inequalities for operators of Kolmogorov type in divergence form with bounded, measurable and uniformly elliptic coefficients.

1. Introduction

Let $\Omega \subset \mathbb{R}^m$, $m \geq 2$, be an (unbounded) Lipschitz domain

$$\Omega = \{X = (x, x_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : x_m > \psi(x)\}, \tag{1-1}$$

where $\psi : \mathbb{R}^{m-1} \to \mathbb{R}$ is Lipschitz with constant bounded by M. Let $A = A(X) = \{a_{i,j}(X)\}$ be a real $m \times m$ matrix-valued, measurable function such that $A(X)$ is symmetric and

$$\kappa^{-1}|\xi|^2 \leq \sum_{i,j=1}^{m} a_{i,j}(X)\xi_i\xi_j \leq \kappa|\xi|^2, \tag{1-2}$$

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for some $1 \leq \kappa < \infty$ and for all $\xi \in \mathbb{R}^m$, $X \in \mathbb{R}^m$. We consider the divergence form operators

$$\begin{aligned}\mathcal{L}_{\mathcal{E}} &:= \nabla_X \cdot (A(X)\nabla_X),\\ \mathcal{L}_{\mathcal{P}} &:= \nabla_X \cdot (A(X)\nabla_X) - \partial_t,\\ \mathcal{L}_{\mathcal{K}} &:= \nabla_X \cdot (A(X)\nabla_X) + X\cdot\nabla_Y - \partial_t,\end{aligned}$$

in $\mathbb{R}^{2m+1}$, $m \geq 1$, equipped with coordinates $(X, Y, t) := (x_1, \dots, x_m, y_1, \dots, y_m, t) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$. Obviously $\mathcal{L}_{\mathcal{E}}$ only makes reference to the X-coordinate, $\mathcal{L}_{\mathcal{P}}$ makes reference to the X- and t-coordinates and $\mathcal{L}_{\mathcal{K}}$ makes reference to all coordinates. The subscripts $\mathcal{E}$, $\mathcal{P}$, $\mathcal{K}$, refer to elliptic, parabolic and Kolmogorov.

$\mathcal{L}_{\mathcal{E}}$ is the standard second-order elliptic PDE with only measurable, bounded and uniformly elliptic coefficients, much-studied ever since the breakthroughs of Moser, Nash, De Giorgi and others. $\mathcal{L}_{\mathcal{P}}$ is the corresponding parabolic version, and $\mathcal{L}_{\mathcal{K}}$ is an operator of Kolmogorov type in divergence form, which up to now has only been modestly studied and understood. Recently, in [Golse et al. 2019] the authors extended the De Giorgi–Nash–Moser (DGNM) theorem, which in its original form only considers elliptic or parabolic equations in divergence form, to (hypoelliptic) equations with rough coefficients including the operator $\mathcal{L}_{\mathcal{K}}$ assuming (1-2). Their result is the correct scale- and translation-invariant estimates for local Hölder continuity and the Harnack inequality for weak solutions.

To give some perspective on the operator $\mathcal{L}_{\mathcal{K}}$, recall that the operator

$$\mathcal{K} := \nabla_X \cdot \nabla_X + X \cdot \nabla_Y - \partial_t$$

was originally introduced and studied by Kolmogorov [1934]. He noted that $\mathcal{K}$ is an example of a degenerate parabolic operator having strong regularity properties, and he proved that $\mathcal{K}$ has a fundamental solution which is smooth off its diagonal. Today, using the terminology introduced by Hörmander [1967], we can conclude that $\mathcal{K}$ is hypoelliptic. Naturally, for the operator $\mathcal{L}_{\mathcal{K}}$, assuming only measurable coefficients and (1-2), the methods of Kolmogorov and Hörmander cannot be directly applied to establish the DGNM theorem and related estimates.

In this paper we are interested in the L^p Dirichlet problem for the operators $\mathcal{L}_{\mathcal{E}}$, $\mathcal{L}_{\mathcal{P}}$, $\mathcal{L}_{\mathcal{K}}$ in the (unbounded) Lipschitz domains Ω, $\Omega \times \mathbb{R}$ and $\Omega \times \mathbb{R}^m \times \mathbb{R}$ respectively, and where $X \in \Omega$, $(X, t) \in \Omega \times \mathbb{R}$ and $(X, Y, t) \in \Omega \times \mathbb{R}^m \times \mathbb{R}$. In particular, we consider the operators $\mathcal{L}_{\mathcal{P}}$ and $\mathcal{L}_{\mathcal{K}}$ in t-independent and (Y, t)-independent domains, respectively. We introduce a (physical) measure $\sigma_{\mathcal{K}}$ on $\partial\Omega \times \mathbb{R}^m \times \mathbb{R}$,

$$d\sigma_{\mathcal{K}}(X, Y, t) := \sqrt{1 + |\nabla_x \psi(x)|^2}\, dx\, dY\, dt, \quad (X, Y, t) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}. \tag{1-3}$$

We refer to $\sigma_{\mathcal{K}}$ as the surface measure on $\partial\Omega \times \mathbb{R}^m \times \mathbb{R}$, where the subscript $\mathcal{K}$ indicates that we consider a setting appropriate for operators of Kolmogorov type. The corresponding measures relevant for $\mathcal{L}_{\mathcal{E}}$ and $\mathcal{L}_{\mathcal{P}}$ are $\sigma_{\mathcal{E}}$ and $\sigma_{\mathcal{P}}$,

$$d\sigma_{\mathcal{E}}(X) := \sqrt{1 + |\nabla_x \psi(x)|^2}\, dx, \quad d\sigma_{\mathcal{P}}(X, t) := d\sigma_{\mathcal{E}}(X)\, dt, \tag{1-4}$$

where $X \in \partial\Omega$ and $(X, t) \in \partial\Omega \times \mathbb{R}$, respectively.

The main results of the paper are Theorems 3.1, 3.2 and 3.3, stated in Section 3 below. Using these theorems we can derive new results concerning the L^p Dirichlet problem for $\mathcal{L}_{\mathcal{K}}$ using results previously only proved for $\mathcal{L}_{\mathcal{E}}$ or $\mathcal{L}_{\mathcal{P}}$, and we can also conclude that some results proved in the literature concerning $\mathcal{L}_{\mathcal{P}}$ are straightforward consequences of the corresponding results for $\mathcal{L}_{\mathcal{E}}$. In particular, the main result of [Fabes and Salsa 1983] concerning parabolic measure is a consequence of the classical result of [Dahlberg 1977] concerning harmonic measure. Our proofs of Theorems 3.1, 3.2 and 3.3 are based on our recent results in [Litsgård and Nyström 2022] concerning boundary Harnack inequalities for operators of Kolmogorov type in divergence form with bounded, measurable and uniformly elliptic coefficients.

Theorem 3.1, 3.2 and 3.3, and their consequences, are deduced under the assumptions:

(A1) $\Omega \subset \mathbb{R}^m$ is a (unbounded) Lipschitz domain with constant M.

(A2) A satisfies (1-2) with constant κ.

(A3) A satisfies the qualitative assumptions stated in (2-16) and (2-17) below.

All quantitative estimates will only depend on m, κ and M, and Theorems 3.1 and 3.2 are by their nature of local character. However, we have chosen to state our results in the unbounded geometric setting $\Omega \times \mathbb{R}^m \times \mathbb{R}$. To avoid being diverted by additional technical issues caused by the unbounded setting, we assume (2-16). Equation (2-17) is only imposed to ensure that all results (e.g., the existence of fundamental solutions) and all estimates used in the paper can be found in the literature. One can dispense of (2-17) at the expense of additional arguments.

We consider the following problems and we refer to the bulk of the paper for all definitions, and in particular for the definition of weak solutions to $\mathcal{L}_{\mathcal{K}} u = 0$ in $\Omega \times \mathbb{R}^m \times \mathbb{R}$.

Definition. Assume that $\Omega \subset \mathbb{R}^m$ is an (unbounded) Lipschitz domain with constant M. Assume that A satisfies (1-2) with constant κ, and (2-16). Given $p \in (1, \infty)$, we say that the Dirichlet problem for $\mathcal{L}_{\mathcal{K}} u = 0$ in $\Omega \times \mathbb{R}^m \times \mathbb{R}$ is solvable in $L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\, d\sigma_{\mathcal{K}})$ if there exists, for every $f \in L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\, d\sigma_{\mathcal{K}})$, a weak solution to the Dirichlet problem

$$\begin{cases} \mathcal{L}_{\mathcal{K}} u = 0 & \text{in } \Omega \times \mathbb{R}^m \times \mathbb{R}, \\ u = f & \text{nontangentially on } \partial\Omega \times \mathbb{R}^m \times \mathbb{R}, \end{cases}$$

and a constant c, depending only on m, κ, M and p, such that

$$\|N(u)\|_{L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\, d\sigma_{\mathcal{K}})} \le c \|f\|_{L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\, d\sigma_{\mathcal{K}})},$$

where $N(u)$ is introduced in Section 2G. For short we say that $D^p_{\mathcal{K}}(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\, d\sigma_{\mathcal{K}})$ is solvable. If the solution is unique then we say that the Dirichlet problem for $\mathcal{L}_{\mathcal{K}} u = 0$ in Ω is uniquely solvable in $L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\, d\sigma_{\mathcal{K}})$. For short we write that $D^p_{\mathcal{K}}(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\, d\sigma_{\mathcal{K}})$ is uniquely solvable. The notions that $D^p_{\mathcal{E}}(\partial\Omega,\, d\sigma_{\mathcal{E}})$ and $D^p_{\mathcal{P}}(\partial\Omega \times \mathbb{R},\, d\sigma_{\mathcal{P}})$ are uniquely solvable are defined analogously.

Using our structural theorems (i.e., combining Theorems 3.1, 3.2 and 3.3) we can conclude that if $D^p_{\mathcal{E}}(\partial\Omega,\, d\sigma_{\mathcal{E}})$ is uniquely solvable for some $p \in (1, \infty)$, then also $D^p_{\mathcal{K}}(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\, d\sigma_{\mathcal{K}})$ is uniquely solvable. We can use this insight to state a number of results concerning the solvability of $D^p_{\mathcal{K}}(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\, d\sigma_{\mathcal{K}})$ and in particular we can conclude the following.

Theorem 1.1. *Assume* (A1)–(A3). *Assume also*

$$A(x, x_m) = A(x), \quad x \in \mathbb{R}^{m-1},\ x_m \in \mathbb{R}, \tag{1-5}$$

i.e., A is independent of x_m. Then there exists $\delta = \delta(m, \kappa, M) \in (0, 1)$ such that if $2 - \delta < p < \infty$, then $D^p_{\mathcal{K}}(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\ \mathrm{d}\sigma_{\mathcal{K}})$ is uniquely solvable.

Theorem 1.2. *Assume* (A1)–(A3). *Assume also*

$$A(x, x_m + 1) = A(x, x_m), \quad x \in \mathbb{R}^{m-1},\ x_m \in \mathbb{R}, \tag{1-6}$$

i.e., A is 1*-periodic in x_m, and that A satisfies a Dini-type condition in the x_m-variable,*

$$\int_0^1 \frac{\theta(\varrho)^2}{\varrho}\,\mathrm{d}\varrho < \infty, \tag{1-7}$$

where $\theta(\varrho) := \sup\{|A(x, \lambda_1) - A(x, \lambda_2)| : x \in \mathbb{R}^{m-1},\ |\lambda_1 - \lambda_2| \le \varrho\}$. Then there exists $\delta = \delta(m, \kappa, M) \in$ $(0, 1)$ such that if $2 - \delta < p < \infty$, then $D^p_{\mathcal{K}}(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\ \mathrm{d}\sigma_{\mathcal{K}})$ is uniquely solvable.

Using our structural theorems it follows that Theorem 1.1 is a consequence of [Jerison and Kenig 1981] and that Theorem 1.2 is a consequence of [Kenig and Shen 2011]. By the same argument we can conclude that the main result in [Fabes and Salsa 1983] is a consequence of [Dahlberg 1977] and that the main result in [Castro and Strömqvist 2018] is a consequence of [Kenig and Shen 2011].

With Theorem 1.2 in place we are also able to analyze a homogenization problem for operators of Kolmogorov type. In this case we assume, in addition to (1-2), that

$$A(X + Z) = A(X) \quad \text{for all } Z \in \mathbb{Z}^m, \tag{1-8}$$

and that

$$\int_0^1 \frac{\Theta(\varrho)^2}{\varrho}\,\mathrm{d}\varrho < \infty, \tag{1-9}$$

where $\Theta(\varrho) := \sup\{|A(X) - A(\widetilde{X})| : X, \widetilde{X} \in \mathbb{R}^m,\ |X - \widetilde{X}| \le \varrho\}$. That is, A is periodic with respect to the lattice $\mathbb{Z}^m$ and A satisfies a Dini condition in all variables.

We consider, for $\epsilon > 0$, the operator $\mathcal{L}^\epsilon_{\mathcal{E}}$,

$$\mathcal{L}^\epsilon_{\mathcal{E}} := \nabla_X \cdot (A^\epsilon(X)\nabla_X), \quad A^\epsilon(X) := A(X/\epsilon). \tag{1-10}$$

Let

$$\bar{\mathcal{L}}_{\mathcal{E}} := \nabla_X \cdot (\bar{A}\nabla_X),$$

where the matrix $\bar{A}$ is determined by

$$\bar{A}\alpha := \int_{(0,1)^m} A(X)\nabla_X w_\alpha(X)\,\mathrm{d}X, \quad \alpha \in \mathbb{R}^m, \tag{1-11}$$

and the auxiliary function w_α solves the problem

$$\begin{cases} \nabla_X \cdot (A(X)\nabla_X w_\alpha(X)) = 0 \quad \text{in } (0, 1)^m, \\ w_\alpha(X) - \alpha X \text{ is 1-periodic (in all variables)}, \\ \int_{(0,1)^m} (w_\alpha(X) - \alpha X)\,\mathrm{d}X = 0. \end{cases}$$

Finally, we also introduce

$$\mathcal{L}_{\mathcal{K}}^{\epsilon} := \mathcal{L}_{\mathcal{E}}^{\epsilon} + X \cdot \nabla_Y - \partial_t, \quad \overline{\mathcal{L}}_{\mathcal{K}} := \overline{\mathcal{L}}_{\mathcal{E}} + X \cdot \nabla_Y - \partial_t. \tag{1-12}$$

We prove the following homogenization result.

Theorem 1.3. *Assume* (A1)–(A3). *Assume also* (1-8) *and* (1-9). *Then there exists* $\delta = \delta(m, \kappa, M) \in (0, 1)$ *such that the following is true. Consider* $\epsilon > 0$. *Given* p, $2-\delta < p < \infty$, *and* $f \in L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R}, \mathrm{d}\sigma_{\mathcal{K}})$, *there exists a unique weak solution* u_ϵ *to the Dirichlet problem*

$$\begin{cases} \mathcal{L}_{\mathcal{K}}^{\epsilon} u_\epsilon = 0 & \text{in } \Omega \times \mathbb{R}^m \times \mathbb{R}, \\ \quad u_\epsilon = f & \text{nontangentially on } \partial\Omega \times \mathbb{R}^m \times \mathbb{R}, \end{cases}$$

and a constant $c = c(m, \kappa, M, p)$, $1 \le c < \infty$, *such that*

$$\|N(u_\epsilon)\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\, \mathrm{d}\sigma_{\mathcal{K}})} \le c\|f\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\, \mathrm{d}\sigma_{\mathcal{K}})}.$$

Moreover, $u_\epsilon \to \bar{u}$ *locally uniformly in* $\Omega \times \mathbb{R}^m \times \mathbb{R}$ *as* $\epsilon \to 0$, *and* $\bar{u}$ *is the unique weak solution to the Dirichlet problem*

$$\begin{cases} \overline{\mathcal{L}}_{\mathcal{K}} \bar{u} = 0 & \text{in } \Omega \times \mathbb{R}^m \times \mathbb{R}, \\ \quad \bar{u} = f & \text{nontangentially on } \partial\Omega \times \mathbb{R}^m \times \mathbb{R}, \end{cases} \tag{1-13}$$

and there exists a constant $c = c(m, \kappa, M, p)$, $1 \le c < \infty$, *such that*

$$\|N(\bar{u})\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\, \mathrm{d}\sigma_{\mathcal{K}})} \le c\|f\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\, \mathrm{d}\sigma_{\mathcal{K}})}.$$

Theorem 1.2 and the first part of Theorem 1.3 were proved in [Kenig and Shen 2011] for $\mathcal{L}_{\mathcal{E}}$. In that work the Neumann and regularity problems are also treated. The theory for the Neumann and regularity problems is based on the use of integral identities to estimate certain nontangential maximal functions. Homogenization of Neumann and regularity problems for $\mathcal{L}_{\mathcal{P}}$ and $\mathcal{L}_{\mathcal{K}}$ remain interesting open problems.

To be clear, the main idea of this paper is that results concerning the L^p Dirichlet problem for the operator $\mathcal{L}_{\mathcal{K}}$ in domains $\Omega \times \mathbb{R}^m \times \mathbb{R}$ (and for the operator $\mathcal{L}_{\mathcal{P}}$ in domains $\Omega \times \mathbb{R}$) can be derived from the corresponding results for the operator $\mathcal{L}_{\mathcal{E}}$ in Ω, using boundary estimates and in particular boundary Harnack inequalities for the operator $\mathcal{L}_{\mathcal{K}}$ ($\mathcal{L}_{\mathcal{P}}$). In the case of $\mathcal{L}_{\mathcal{K}}$ the latter results are established in [Litsgård and Nyström 2022]; however, the relevant results in that work hold for more general operators

$$\nabla_X \cdot (A(X, Y, t)\nabla_X) + X \cdot \nabla_Y - \partial_t,$$

and in the more general class of domains

$$\{(X, Y, t) = (x, x_m, y, y_m, t) \in \mathbb{R}^{2m+1} : x_m > \tilde{\psi}(x, y, t)\}.$$

In particular, in [Litsgård and Nyström 2022] we allow for (Y, t)-dependent coefficients and domains. Therefore, one can repeat the analysis of this paper, taking any result concerning the solvability of the L^p Dirichlet problem for parabolic operators

$$\nabla_X \cdot (A(X, t)\nabla_X) - \partial_t,$$

in $\mathrm{Lip}\big(1, \frac{1}{2}\big)$ domains, as the point of departure. The results are the corresponding results for the operator

$$\nabla_X \cdot (A(X,t)\nabla_X) + X \cdot \nabla_Y - \partial_t$$

in Y-independent Lipschitz-type domains. Similarly, focusing only on $\mathcal{L}_{\mathcal{E}}$ and $\mathcal{L}_{\mathcal{P}}$, one can replace $\Omega \subset \mathbb{R}^m$ by an NTA-domain in the sense of [Jerison and Kenig 1982], having an $(m-1)$-dimensional Ahlfors-regular boundary in the sense of [David and Semmes 1991; 1993]; see also [David and Jerison 1990].

The rest of the paper is organized as follows. In Section 2, which is of more preliminary nature, we introduce notation and state definitions including the notion of weak solutions. In this section we also discuss the Dirichlet problem, see Theorem 2.1, and we point out that in Theorems 1.3 and 1.4 in [Litsgård and Nyström 2021] we simply missed stating the obvious restriction $u \in L^\infty(\Omega \times \mathbb{R}^m \times \mathbb{R})$ under which the proofs there are given. With this clarification, Theorem 2.1 is a special case of Theorem 1.4 in [Litsgård and Nyström 2021]. In Section 3 we state our structural theorems: Theorems 3.1, 3.2 and 3.3. In Section 4 we state a number of lemmas concerning the interior regularity of weak solutions and concerning the boundary behavior of nonnegative solutions to $\mathcal{L}_{\mathcal{K}} u = 0$; the latter were recently established in [Litsgård and Nyström 2022]. In Section 5 we prove Theorems 3.1 and 3.2. In Section 6 we prove Theorem 3.3 and hence, as outlined above and as a consequence, we prove Theorems 1.1 and 1.2. In Section 7 we also give, as we believe that the argument may be of independent interest in the case of operators of Kolmogorov type, a proof of Theorem 1.1 using Rellich-type inequalities along the proof of the corresponding result for the heat equation in [Fabes and Salsa 1983]. In Section 8 we apply our findings to homogenization, giving new results for homogenization of operators of Kolmogorov type, and in particular we prove Theorem 1.3.

2. Preliminaries

2A. ***Group law and metric.*** The natural family of dilations jointly for the operators $\mathcal{L}_{\mathcal{E}}$, $\mathcal{L}_{\mathcal{P}}$, $\mathcal{L}_{\mathcal{K}}$, $(\delta_r)_{r>0}$, on $\mathbb{R}^{N+1}$, $N := 2m$, is defined by

$$\delta_r(X,Y,t) = (rX, r^3 Y, r^2 t) \tag{2-1}$$

for $(X,Y,t) \in \mathbb{R}^{N+1}$, $r > 0$. Furthermore, the classes of operators $\mathcal{L}_{\mathcal{E}}$, $\mathcal{L}_{\mathcal{P}}$, $\mathcal{L}_{\mathcal{K}}$ are closed under the group law

$$(\widetilde{X}, \widetilde{Y}, \tilde{t}) \circ (X,Y,t) = (\widetilde{X} + X, \widetilde{Y} + Y - t\widetilde{X}, \tilde{t} + t), \tag{2-2}$$

where $(X,Y,t), (\widetilde{X}, \widetilde{Y}, \tilde{t}) \in \mathbb{R}^{N+1}$. Note that

$$(X,Y,t)^{-1} = (-X, -Y - tX, -t), \tag{2-3}$$

and hence

$$(\widetilde{X}, \widetilde{Y}, \tilde{t})^{-1} \circ (X,Y,t) = (X - \widetilde{X}, Y - \widetilde{Y} + (t-\tilde{t})\widetilde{X}, t - \tilde{t}), \tag{2-4}$$

whenever $(X,Y,t), (\widetilde{X}, \widetilde{Y}, \tilde{t}) \in \mathbb{R}^{N+1}$.

Given $(X,Y,t) \in \mathbb{R}^{N+1}$ we let

$$\|(X,Y,t)\| := |(X,Y)| + |t|^{1/2}, \quad |(X,Y)| := |X| + |Y|^{1/3}. \tag{2-5}$$

We recall the following pseudotriangular inequalities: there exists a positive constant c such that

$$\|(X,Y,t)^{-1}\| \leq c\|(X,Y,t)\|, \quad \|(X,Y,t)\circ(\widetilde{X},\widetilde{Y},\tilde{t})\| \leq c(\|(X,Y,t)\| + \|(\widetilde{X},\widetilde{Y},\tilde{t})\|), \tag{2-6}$$

whenever $(X,Y,t), (\widetilde{X},\widetilde{Y},\tilde{t}) \in \mathbb{R}^{N+1}$. Using (2-6) it follows immediately that

$$\|(\widetilde{X},\widetilde{Y},\tilde{t})^{-1}\circ(X,Y,t)\| \leq c\,\|(X,Y,t)^{-1}\circ(\widetilde{X},\widetilde{Y},\tilde{t})\|, \tag{2-7}$$

whenever $(X,Y,t), (\widetilde{X},\widetilde{Y},\tilde{t}) \in \mathbb{R}^{N+1}$. Let

$$d((X,Y,t),(\widetilde{X},\widetilde{Y},\tilde{t})) := \tfrac{1}{2}(\|(\widetilde{X},\widetilde{Y},\tilde{t})^{-1}\circ(X,Y,t)\| + \|(X,Y,t)^{-1}\circ(\widetilde{X},\widetilde{Y},\tilde{t})\|). \tag{2-8}$$

Using (2-7) it follows that

$$\|(\widetilde{X},\widetilde{Y},\tilde{t})^{-1}\circ(X,Y,t)\| \approx d((X,Y,t),(\widetilde{X},\widetilde{Y},\tilde{t})) \approx \|(X,Y,t)^{-1}\circ(\widetilde{X},\widetilde{Y},\tilde{t})\|, \tag{2-9}$$

with constants of comparison independent of $(X,Y,t), (\widetilde{X},\widetilde{Y},\tilde{t}) \in \mathbb{R}^{N+1}$. Again using (2-6) we also see that

$$d((X,Y,t),(\widetilde{X},\widetilde{Y},\tilde{t})) \leq c\big(d((X,Y,t),(\widehat{X},\widehat{Y},\hat{t})) + d((\widehat{X},\widehat{Y},\hat{t}),(\widetilde{X},\widetilde{Y},\tilde{t}))\big), \tag{2-10}$$

whenever $(X,Y,t), (\widehat{X},\widehat{Y},\hat{t}), (\widetilde{X},\widetilde{Y},\tilde{t}) \in \mathbb{R}^{N+1}$, and hence d is a symmetric quasidistance. Based on d we introduce the balls

$$\mathcal{B}_r(X,Y,t) := \{(\widetilde{X},\widetilde{Y},\tilde{t}) \in \mathbb{R}^{N+1} : d((\widetilde{X},\widetilde{Y},\tilde{t}),(X,Y,t)) < r\} \tag{2-11}$$

for $(X,Y,t) \in \mathbb{R}^{N+1}$ and $r > 0$. The measure of the ball $\mathcal{B}_r(X,Y,t)$ is $|\mathcal{B}_r(X,Y,t)| = c(m)r^{\boldsymbol{q}}$, where $\boldsymbol{q} := 4m+2$.

2B. ***Surface cubes and reference points.*** Let $\Omega \subset \mathbb{R}^m$, $m \geq 2$, be an (unbounded) Lipschitz domain as defined in (1-1) and with constant M. Let

$$\Sigma := \partial\Omega \times \mathbb{R}^m \times \mathbb{R} = \{(x, x_m, y, y_m, t) \in \mathbb{R}^{N+1} : x_m = \psi(x)\}. \tag{2-12}$$

An observation is that $(\Sigma, d, \mathrm{d}\sigma_{\mathcal{K}})$ is a space of homogeneous type in the sense of [Coifman and Weiss 1971], with homogeneous dimension $\boldsymbol{q}-1$. Furthermore, $(\mathbb{R}^{N+1}, d, \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t)$ is also a space of homogeneous type in the sense of [Coifman and Weiss 1971], but with homogeneous dimension $\boldsymbol{q}$.

Let

$$Q := (-1,1)^m \times (-1,1)^m \times (-1,1)$$

and

$$Q_r = \delta_r Q := \{(rX, r^3Y, r^2t) : (X,Y,t) \in Q\}.$$

Given a point $(X_0, Y_0, t_0) \in \mathbb{R}^{N+1}$ we let

$$Q_r(X_0,Y_0,t_0) := (X_0,Y_0,t_0)\circ Q_r := \{(X_0,Y_0,t_0)\circ(X,Y,t) : (X,Y,t) \in Q_r\}.$$

Furthermore, if $(X_0,Y_0,t_0) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$ then we set

$$\Delta_r(X_0,Y_0,t_0) := (\partial\Omega \times \mathbb{R}^m \times \mathbb{R}) \cap Q_r(X_0,Y_0,t_0).$$

We will frequently, and for brevity, write Q_r and Δ_r for $Q_r(X_0, Y_0, t_0)$ and $\Delta_r(X_0, Y_0, t_0)$ whenever the point (X_0, Y_0, t_0) is clear from the context. At instances we will simply also write Δ for $\Delta_r(X_0, Y_0, t_0)$ whenever the point (X_0, Y_0, t_0) and the scale r do not have to be stated explicitly. Given a positive constant c, $c\Delta := \Delta_{cr}(X_0, Y_0, t_0)$.

Given $\varrho > 0$ and $\Lambda > 0$, we let

$$\begin{aligned} A^+_{\varrho,\Lambda} &:= \big(0, \Lambda\varrho, 0, -\tfrac{2}{3}\Lambda\varrho^3, \varrho^2\big) \in \mathbb{R}^{m-1} \times \mathbb{R} \times \mathbb{R}^{m-1} \times \mathbb{R} \times \mathbb{R}, \\ A^-_{\varrho,\Lambda} &:= \big(0, \Lambda\varrho, 0, \tfrac{2}{3}\Lambda\varrho^3, -\varrho^2\big) \in \mathbb{R}^{m-1} \times \mathbb{R} \times \mathbb{R}^{m-1} \times \mathbb{R} \times \mathbb{R}, \end{aligned} \tag{2-13}$$

and

$$A^\pm_{\varrho,\Lambda}(X_0, Y_0, t_0) := (X_0, Y_0, t_0) \circ A^\pm_{\varrho,\Lambda},$$

whenever $(X_0, Y_0, t_0) \in \mathbb{R}^{N+1}$. Furthermore, given $\Delta := \Delta_r(X_0, Y_0, t_0)$ we let

$$A^\pm_{\Delta,\Lambda} := A^\pm_{r,\Lambda}(X_0, Y_0, t_0).$$

2C. *Qualitative assumptions on the coefficients.* Central to our arguments are the boundary estimates recently proved in [Litsgård and Nyström 2022], where we considered solutions to the equation $\mathcal{L}u = 0$, where $\mathcal{L}$ is the operator

$$\nabla_X \cdot (A(X, Y, t)\nabla_X) + X \cdot \nabla_Y - \partial_t \tag{2-14}$$

in $\mathbb{R}^{N+1}$, $N = 2m$, $m \geq 1$, $(X, Y, t) := (x_1, \dots, x_m, y_1, \dots, y_m, t) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$. We assume that

$$A = A(X, Y, t) = \{a_{i,j}(X, Y, t)\}_{i,j=1}^m$$

is a real-valued, $m \times m$-dimensional, symmetric-matrix-valued function satisfying

$$\kappa^{-1}|\xi|^2 \leq \sum_{i,j=1}^m a_{i,j}(X, Y, t)\xi_i\xi_j, \quad |A(X, Y, t)\xi \cdot \zeta| \leq \kappa|\xi||\zeta|, \tag{2-15}$$

for some $\kappa \in [1, \infty)$, and for all $\xi, \zeta \in \mathbb{R}^m$, $(X, Y, t) \in \mathbb{R}^{N+1}$. Throughout [Litsgård and Nyström 2022] we also assume that

$$A = A(X, Y, t) \equiv I_m \text{ outside some arbitrary but fixed compact subset of } \mathbb{R}^{N+1}, \tag{2-16}$$

and that

$$a_{i,j} \in C^\infty(\mathbb{R}^{N+1}) \tag{2-17}$$

for all $i, j \in \{1, \dots, m\}$. In [Litsgård and Nyström 2022] the assumptions in (2-16) and (2-17) are only used in a qualitative fashion. In particular, from the perspective of the operator, the constants of the quantitative estimates in that work only depend on m and κ. To be consistent with that paper, in (A1)–(A3) we have included the qualitative assumptions stated in (2-16), (2-17).

2D. *Function spaces.* Let $U_X \subset \mathbb{R}^m$, $U_Y \subset \mathbb{R}^m$ be bounded domains, i.e., bounded, open and connected sets in $\mathbb{R}^m$. Let $J \subset \mathbb{R}$ be an open and bounded interval. We denote by $H^1_X(U_X)$ the Sobolev space of functions $g \in L^2(U_X)$ whose distribution gradient in U_X lies in $(L^2(U_X))^m$, i.e.,

$$H^1_X(U_X) := \{g \in L^2_X(U_X) : \nabla_X g \in (L^2(U_X))^m\},$$

and we set

$$\|g\|_{H^1_X(U_X)} := \|g\|_{L^2(U_X)} + \||\nabla_X g\||_{L^2(U_X)}, \quad g \in H^1_X(U_X).$$

We let $H^1_{X,0}(U_X)$ denote the closure of $C_0^\infty(U_X)$ in the norm of $H^1_X(U_X)$. If U_X is a bounded Lipschitz domain, then $C^\infty(\overline{U}_X)$ is dense in $H^1_X(U_X)$. In particular, equivalently we could define $H^1_X(U_X)$ as the closure of $C^\infty(\overline{U}_X)$ in the norm $\|\cdot\|_{H^1_X(U_X)}$. We let $H^{-1}_X(U_X)$ denote the dual to $H^1_X(U_X)$, whose elements act on functions in $H^1_{X,0}(U_X)$ through the duality pairing $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{H^{-1}_X(U_X),H^1_{X,0}(U_X)}$.

In analogy with the definition of $H^1_X(U_X)$, we let $W(U_X\times U_Y\times J)$ be the closure of $C^\infty(\overline{U_X\times U_Y\times J})$ in the norm

$$\begin{aligned}
&\|u\|_{W(U_X\times U_Y\times J)}\\
&:= \|u\|_{L^2_{Y,t}(U_Y\times J,H^1(U_X))} + \|(-X\cdot\nabla_Y+\partial_t)u\|_{L^2_{Y,t}(U_Y\times J,H^{-1}_X(U_X))}\\
&:= \bigg(\iint_{U_Y\times J}\|u(\cdot,Y,t)\|^2_{H^1_X(U_X)}\,\mathrm{d}Y\,\mathrm{d}t\bigg)^{1/2} + \bigg(\iint_{U_Y\times J}\|(-X\cdot\nabla_Y+\partial_t)u(\cdot,Y,t)\|^2_{H^{-1}_X(U_X)}\,\mathrm{d}Y\,\mathrm{d}t\bigg)^{1/2}. \qquad (2\text{-}18)
\end{aligned}$$

In particular, $W(U_X\times U_Y\times J)$ is a Banach space and $u\in W(U_X\times U_Y\times J)$ if and only if

$$u\in L^2_{Y,t}(U_Y\times J, H^1_X(U_X)) \quad\text{and}\quad (-X\cdot\nabla_Y+\partial_t)u\in L^2_{Y,t}(U_Y\times J, H^{-1}_X(U_X)). \qquad (2\text{-}19)$$

Let $\Omega\subset\mathbb{R}^m$, $m\ge 2$, be an (unbounded) Lipschitz domain as defined in (1-1) and with constant M. We say that $u\in W_{\text{loc}}(\Omega\times\mathbb{R}^m\times\mathbb{R})$ if $u\in W(U_X\times U_Y\times J)$ whenever $U_X\subset\mathbb{R}^m$, $U_Y\subset\mathbb{R}^m$ are bounded domains, $J\subset\mathbb{R}$ is an open and bounded interval, and $\overline{U_X\times U_Y\times J}$ is compactly contained in $\Omega\times\mathbb{R}^m\times\mathbb{R}$.

2E. *Weak solutions.* Let U_X, U_Y and J be as introduced in the previous subsection. We say that u is a weak solution to

$$\mathcal{L}_{\mathcal{K}}u = 0 \quad \text{in } U_X\times U_Y\times J \qquad (2\text{-}20)$$

if $u\in W(U_X\times U_Y\times J)$ and if

$$0=\iiint_{U_X\times U_Y\times J} A(X)\nabla_X u\cdot\nabla_X\phi\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t + \iint_{U_Y\times J}\langle(-X\cdot\nabla_Y+\partial_t)u(\cdot,Y,t),\phi(\cdot,Y,t)\rangle\,\mathrm{d}Y\,\mathrm{d}t \qquad (2\text{-}21)$$

for all $\phi\in L^2_{Y,t}(U_Y\times J, H^1_{X,0}(U_X))$. Here, again, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{H^{-1}_X(U_X),H^1_{X,0}(U_X)}$ is the duality pairing between $H^{-1}_X(U_X)$ and $H^1_{X,0}(U_X)$.

Definition. Let $\Omega\subset\mathbb{R}^m$, $m\ge 2$, be an (unbounded) Lipschitz domain as defined in (1-1) and with constant M. We say that u is a weak solution to

$$\mathcal{L}_{\mathcal{K}}u = 0 \quad \text{in } \Omega\times\mathbb{R}^m\times\mathbb{R} \qquad (2\text{-}22)$$

if $u\in W_{\text{loc}}(\Omega\times\mathbb{R}^m\times\mathbb{R})$ and if u satisfies (2-21), whenever $\overline{U_X\times U_Y\times J}$ is compactly contained in $\Omega\times\mathbb{R}^m\times\mathbb{R}$.

Note that if u is a weak solution to the equation $\mathcal{L}_{\mathcal{K}}u=0$ in $\Omega\times\mathbb{R}^m\times\mathbb{R}$, then it is a weak solution in the sense of distributions, i.e.,

$$\iiint\big(A(X)\nabla_X u\cdot\nabla_X\phi - u(-X\cdot\nabla_Y+\partial_t)\phi\big)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t = 0, \qquad (2\text{-}23)$$

whenever $\phi\in C_0^\infty(\Omega\times\mathbb{R}^m\times\mathbb{R})$.

2F. ***The Dirichlet problem and associated boundary measures.*** In [Litsgård and Nyström 2021] we conducted a study of the existence and uniqueness of weak solutions to

$$\nabla_X \cdot (A(X, Y, t)\nabla_X u) + X \cdot \nabla_Y u - \partial_t u = 0,$$

as well as the existence and uniqueness of weak solutions to the Dirichlet problem with continuous boundary data. In [Litsgård and Nyström 2021], Theorems 1.2, 1.3, and 1.4, are particularly relevant to this paper. Theorem 1.2 in [loc. cit.] concerns the existence of weak solutions to (2-24). However, in [loc. cit.] a stronger notion of weak solutions is used, see Definition 2 there, as we there demand certain Sobolev regularity up to the boundary of $\Omega \times \mathbb{R}^m \times \mathbb{R}$. Theorem 1.3 in [loc. cit.] concerns the uniqueness of weak solutions to (2-24) and in Theorem 1.4 in [loc. cit.] we consider the continuous Dirichlet problem and the representation of the solution using associated parabolic measures. We here state the following consequence of these results.

Theorem 2.1. *Assume that A satisfies* (1-2) *and* (2-16). *Let $f \in C_0(\partial\Omega \times \mathbb{R}^m \times \mathbb{R})$. Then there exists $u \in C(\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R})$ such that $u = u_f$ is a weak solution to the Dirichlet problem*

$$\begin{cases} \mathcal{L}_{\mathcal{K}} u = 0 & \text{in } \Omega \times \mathbb{R}^m \times \mathbb{R}, \\ \quad u = f & \text{on } \partial\Omega \times \mathbb{R}^m \times \mathbb{R}, \end{cases} \tag{2-24}$$

in the sense of the Definition on page 1555. If u is bounded, then $u = u_f$ is the unique weak solution to (2-24) *and in this case there exists, for every $(X, Y, t) \in \Omega \times \mathbb{R}^m \times \mathbb{R}$, a unique probability measure $\omega_{\mathcal{K}}(X, Y, t, \cdot)$ on $\partial\Omega \times \mathbb{R}^m \times \mathbb{R}$ such that*

$$u(X, Y, t) = \iiint_{\partial\Omega \times \mathbb{R}^m \times \mathbb{R}} f(\widetilde{X}, \widetilde{Y}, \tilde{t})\, \mathrm{d}\omega_{\mathcal{K}}(X, Y, t, \widetilde{X}, \widetilde{Y}, \tilde{t}). \tag{2-25}$$

Proof. As stated above, the notion of weak solutions introduced in the Definition on page 1555 is weaker than the notion of weak solutions introduced in Definition 2 in [Litsgård and Nyström 2021]. In particular, concerning the existence part of Theorem 2.1, Theorems 1.2–1.4 in that work give a stronger result. Concerning uniqueness and Theorems 1.3 and 1.4 of that work, an important piece of information is neglected in the statements of these two theorems. As can be seen from the proofs of Theorems 1.3 and 1.4 there, this information concerns the fact that in the unbounded setting $\Omega \times \mathbb{R}^m \times \mathbb{R}$ we need a condition at infinity to ensure uniqueness, and what we prove is the uniqueness of bounded weak solutions. In particular, in Theorem 1.3 it should be stated that $g \in W(\mathbb{R}^{N+1}) \cap L^\infty(\mathbb{R}^{N+1})$ and that u is unique if $u \in L^\infty(\Omega \times \mathbb{R}^m \times \mathbb{R})$. Similarly, in Theorem 1.4 it should be stated that u is unique if $u \in L^\infty(\Omega \times \mathbb{R}^m \times \mathbb{R})$. In Theorems 1.3 and 1.4 we simply missed stating the obvious restriction $u \in L^\infty(\Omega \times \mathbb{R}^m \times \mathbb{R})$ under which the proofs in that work are given. With this clarification, Theorem 2.1 is a special case of Theorem 1.4 in [Litsgård and Nyström 2021]. □

The measure $\omega_{\mathcal{K}}(X, Y, t, E)$ introduced in Theorem 2.1 is referred to as the parabolic measure, or Kolmogorov measure to distinguish it from the parabolic measure associated to $\mathcal{L}_{\mathcal{P}}$, associated to $\mathcal{L}_{\mathcal{K}}$ in $\Omega \times \mathbb{R}^m \times \mathbb{R}$, at $(X, Y, t) \in \Omega \times \mathbb{R}^m \times \mathbb{R}$ and of $E \subset \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$. Properties of $\omega_{\mathcal{K}}(X, Y, t, \cdot)$ govern the Dirichlet problem in (2-24). The corresponding elliptic and parabolic measures on $\partial\Omega$ and $\partial\Omega \times \mathbb{R}$, $\omega_{\mathcal{E}}$ and $\omega_{\mathcal{P}}$, are introduced analogously.

2G. ***The nontangential maximal operator.*** Given an (unbounded) Lipschitz domain $\Omega \subset \mathbb{R}^m$ with constant M,

$$(X_0, Y_0, t_0) = ((x_0, \psi(x_0)), Y_0, t_0) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R},$$

and $\eta > 0$, we introduce the (nontangential) cone

$$\Gamma^\eta(X_0, Y_0, t_0) := \{(X, Y, t) \in \Omega \times \mathbb{R}^m \times \mathbb{R} : d((X, Y, t), (X_0, Y_0, t_0)) < \eta|x_m - \psi(x_0)|\}. \tag{2-26}$$

Given a function u defined in $\Omega \times \mathbb{R}^m \times \mathbb{R}$ we consider the nontangential maximal operator

$$N^\eta(u)(X_0, Y_0, t_0) := \sup_{(X,Y,t)\in\Gamma^\eta(X_0,Y_0,t_0)} |u(X, Y, t)|. \tag{2-27}$$

If f is defined on $\partial\Omega \times \mathbb{R}^m \times \mathbb{R}$ and $(X_0, Y_0, t_0) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$, then we say that $u(X_0, Y_0, t_0) = f(X_0, Y_0, t_0)$ nontangentially (n.t.) if

$$\lim_{\substack{(X,Y,t)\in\Gamma^\eta(X_0,Y_0,t_0)\\ (X,Y,t)\to(X_0,Y_0,t_0)}} u(X, Y, t) = f(X_0, Y_0, t_0),$$

where $\eta = \eta(M)$ is chosen so that $(\partial\Omega \times \mathbb{R}^m \times \mathbb{R}) \cap \Gamma^\eta(X_0, Y_0, t_0) = \{(X_0, Y_0, t_0)\}$. With this choice of η we simply write $N(u)$ for $N^\eta(u)$. Furthermore, given $\delta > 0$ we introduce the truncated cone

$$\Gamma^\eta_\delta(X_0, Y_0, t_0) := \Gamma^\eta(X_0, Y_0, t_0) \cap \mathcal{B}_\delta(X_0, Y_0, t_0), \tag{2-28}$$

and the truncated nontangential maximal operator

$$N^\eta_\delta(u)(X_0, Y_0, t_0) := \sup_{(X,Y,t)\in\Gamma^\eta_\delta(X_0,Y_0,t_0)} |u(X, Y, t)|. \tag{2-29}$$

Again with η fixed, we write $N_\delta(u)$ for $N^\eta_\delta(u)$. For more on nontangential maximal functions in the elliptic context we refer to [Kenig 1994].

2H. ***Conventions.*** Throughout the paper we will use following conventions. By c we will, if not otherwise stated, denote a constant satisfying $1 \le c < \infty$. We write $c_1 \lesssim c_2$ if c_1/c_2 is bounded from above by a positive constant depending only on m, κ, and M, if not otherwise stated. We write $c_1 \approx c_2$ if $c_1 \lesssim c_2$ and $c_2 \lesssim c_1$.

Given a point $(X, Y, t) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$, we let $\pi_X(X, Y, t) := X$, $\pi_{X,t}(X, Y, t) := (X, t)$. Similarly, if $\Delta \subset \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$, then we let $\pi_X(\Delta)$ denote the projection of Δ onto the X-coordinate, we let $\pi_{X,t}(\Delta)$ denote the projection of Δ onto the (X, t)-coordinates.

3. Statements of the structural theorems

Our structural theorems concern the quantitative relations between the measures $\omega_{\mathcal{E}}$, $\omega_{\mathcal{P}}$, $\omega_{\mathcal{K}}$ and the (physical) measures $\sigma_{\mathcal{E}}$, $\sigma_{\mathcal{P}}$, $\sigma_{\mathcal{K}}$. We first prove the following relations between the measures.

Theorem 3.1. *Assume* (A1)–(A3). *Let $\omega_{\mathcal{E}}$, $\omega_{\mathcal{P}}$, and $\omega_{\mathcal{K}}$ be the elliptic, parabolic and Kolmogorov measures associated to $\mathcal{L}_{\mathcal{E}}$, $\mathcal{L}_{\mathcal{P}}$, $\mathcal{L}_{\mathcal{K}}$ in Ω, $\Omega \times \mathbb{R}$ and $\Omega \times \mathbb{R}^m \times \mathbb{R}$, respectively. Then there exist*

$\Lambda = \Lambda(m, M)$, $1 \le \Lambda < \infty$ and $c = c(m, \kappa, M)$, $1 \le c < \infty$ such that the following is true. Consider $\Delta := \Delta_r(X_0, Y_0, t_0) \subset \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$. Then

$$\frac{\sigma_{\mathcal{K}}(\Delta)\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \tilde{\Delta})}{\sigma_{\mathcal{K}}(\tilde{\Delta})} \approx \frac{\sigma_{\mathcal{P}}(\pi_{X,t}(\Delta))\omega_{\mathcal{P}}(\pi_{X,t}(A^+_{c\Delta,\Lambda}), \pi_{X,t}(\tilde{\Delta}))}{\sigma_{\mathcal{P}}(\pi_{X,t}(\tilde{\Delta}))} \approx \frac{\sigma_{\mathcal{E}}(\pi_X(\Delta))\omega_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), \pi_X(\tilde{\Delta}))}{\sigma_{\mathcal{E}}(\pi_X(\tilde{\Delta}))},$$

whenever $\tilde{\Delta} \subset \Delta$.

Theorem 3.1 states that the measures $\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \cdot\,)$, $\omega_{\mathcal{P}}(\pi_{X,t}(A^+_{c\Delta,\Lambda}), \cdot\,)$, $\omega_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), \cdot\,)$ are all comparable in the sense stated when evaluated on the surface cube $\tilde{\Delta} \subset \Delta$. As we will prove, if $\tilde{\Delta} = \Delta_{\tilde{r}}$ and if

$$\lim_{\tilde{r}\to 0} \frac{\omega_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), \pi_X(\Delta_{\tilde{r}}))}{\sigma_{\mathcal{E}}(\pi_X(\Delta_{\tilde{r}}))} \tag{3-1}$$

exists, then also the limits

$$\lim_{\tilde{r}\to 0} \frac{\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \Delta_{\tilde{r}})}{\sigma_{\mathcal{K}}(\Delta_{\tilde{r}})} \quad \text{and} \quad \lim_{\tilde{r}\to 0} \frac{\omega_{\mathcal{P}}(\pi_{X,t}(A^+_{c\Delta,\Lambda}), \pi_{X,t}(\Delta_{\tilde{r}}))}{\sigma_{\mathcal{P}}(\pi_{X,t}(\Delta_{\tilde{r}}))} \tag{3-2}$$

exist and all limits are comparable in the sense of Theorem 3.1. Indeed, using (3-1) we will be able to deduce that the Poisson kernels

$$\begin{aligned} K_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), X) &:= \frac{\mathrm{d}\omega_{\mathcal{E}}}{\mathrm{d}\sigma_{\mathcal{E}}}(\pi_X(A^+_{c\Delta,\Lambda}), X), \\ K_{\mathcal{P}}(\pi_{X,t}(A^+_{c\Delta,\Lambda}), X, t) &:= \frac{\mathrm{d}\omega_{\mathcal{P}}}{\mathrm{d}\sigma_{\mathcal{P}}}(\pi_{X,t}(A^+_{c\Delta,\Lambda}), X, t), \\ K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, Y, t) &:= \frac{\mathrm{d}\omega_{\mathcal{K}}}{\mathrm{d}\sigma_{\mathcal{K}}}(A^+_{c\Delta,\Lambda}, X, Y, t) \end{aligned}$$

are all well-defined on Δ and that

$$\begin{aligned} \sigma_{\mathcal{K}}(\Delta)K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, Y, t) &\approx \sigma_{\mathcal{P}}(\pi_{X,t}(\Delta))K_{\mathcal{P}}(\pi_{X,t}(A^+_{c\Delta,\Lambda}), X, t) \\ &\approx \sigma_{\mathcal{E}}(\pi_X(\Delta))K_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), X), \end{aligned}$$

whenever $(X, Y, t) \in \Delta$.

Given q, $1 < q < \infty$, we say that $K_{\mathcal{E}}(X) := K_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), X) \in B_q(\pi_X(\Delta), \mathrm{d}\sigma_{\mathcal{E}})$ with constant Γ, $1 \le \Gamma < \infty$, if

$$\left(\fint_{\pi_X(\tilde{\Delta})} |K_{\mathcal{E}}(X)|^q \, \mathrm{d}\sigma_{\mathcal{E}}(X)\right)^{1/q} \le \Gamma\left(\fint_{\pi_X(\tilde{\Delta})} |K_{\mathcal{E}}(X)| \, \mathrm{d}\sigma_{\mathcal{E}}(X)\right) \tag{3-3}$$

for all $\tilde{\Delta} \subset \Delta$. Analogously, $K_{\mathcal{P}}(X, t) := K_{\mathcal{P}}(\pi_{X,t}(A^+_{c\Delta,\Lambda}), X, t) \in B_q(\pi_{X,t}(\Delta), \mathrm{d}\sigma_{\mathcal{P}})$ and $K_{\mathcal{K}}(X, Y, t) := K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, Y, t) \in B_q(\Delta, \mathrm{d}\sigma_{\mathcal{K}})$, with constant Γ, if

$$\begin{aligned} \left(\fint\!\!\!\fint_{\pi_{X,t}(\tilde{\Delta})} |K_{\mathcal{P}}(X, t)|^q \, \mathrm{d}\sigma_{\mathcal{P}}(X, t)\right)^{1/q} &\le \Gamma\left(\fint\!\!\!\fint_{\pi_{X,t}(\tilde{\Delta})} |K_{\mathcal{P}}(X, t)| \, \mathrm{d}\sigma_{\mathcal{P}}(X, t)\right), \\ \left(\fint\!\!\!\fint\!\!\!\fint_{\tilde{\Delta}} |K_{\mathcal{K}}(X, Y, t)|^q \, \mathrm{d}\sigma_{\mathcal{K}}(X, Y, t)\right)^{1/q} &\le \Gamma\left(\fint\!\!\!\fint\!\!\!\fint_{\tilde{\Delta}} |K_{\mathcal{K}}(X, Y, t)| \, \mathrm{d}\sigma_{\mathcal{K}}(X, Y, t)\right), \end{aligned} \tag{3-4}$$

respectively, for all $\tilde{\Delta} \subset \Delta$.

We can now state our second main result.

Theorem 3.2. *Assume* (A1)–(A3). *Let $\omega_{\mathcal{E}}$, $\omega_{\mathcal{P}}$, and $\omega_{\mathcal{K}}$ be as in the statement of Theorem 3.1. Then there exist $\Lambda = \Lambda(m, M)$, $1 \le \Lambda < \infty$ and $c = c(m, \kappa, M)$, $1 \le c < \infty$, such that the following is true. Consider $\Delta := \Delta_r(X_0, Y_0, t_0) \subset \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$. Assume that $\omega_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), \cdot)$ is mutually absolutely continuous on $\pi_X(\Delta)$ with respect to $\sigma_{\mathcal{E}}$ and that the associated Poisson kernel $K_{\mathcal{E}}(X) := K_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), X)$ satisfies*

$$K_{\mathcal{E}} \in B_q(\pi_X(\Delta),\ \mathrm{d}\sigma_{\mathcal{E}})$$

for some q, $1 < q < \infty$, and with constant Γ, $1 \le \Gamma < \infty$. Then $\omega_{\mathcal{P}}(\pi_{X,t}(A^+_{c\Delta,\Lambda}), \cdot)$ and $\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \cdot)$ are mutually absolutely continuous on $\pi_{X,t}(\Delta)$ and Δ with respect to $\sigma_{\mathcal{P}}$ and $\sigma_{\mathcal{K}}$, respectively, and the associated Poisson kernels $K_{\mathcal{P}}(X, t) := K_{\mathcal{P}}(\pi_{X,t}(A^+_{c\Delta,\Lambda}), X, t)$ and $K_{\mathcal{K}}(X, Y, t) := K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, Y, t)$ satisfy

$$K_{\mathcal{P}} \in B_q(\pi_{X,t}(\Delta),\ \mathrm{d}\sigma_{\mathcal{P}}), \qquad K_{\mathcal{K}} \in B_q(\Delta,\ \mathrm{d}\sigma_{\mathcal{K}}),$$

with constant $\widetilde{\Gamma} = \widetilde{\Gamma}(m, \kappa, M, \Gamma)$.

We also prove the following theorem.

Theorem 3.3. *Assume* (A1)–(A3). *Let $p \in (1, \infty)$ be given and let q denote the index dual to p. Assume that $\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \cdot)$ is mutually absolutely continuous on Δ with respect to $\sigma_{\mathcal{K}}$ for all $\Delta := \Delta_r(X_0, Y_0, t_0) \subset \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$. Then the following statements are equivalent*:

(i) *$K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \cdot, \cdot, \cdot) \in B_q(\Delta,\ \mathrm{d}\sigma_{\mathcal{K}})$ for all $\Delta \subset \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$, with a uniform constant Γ.*

(ii) *$D^p_{\mathcal{K}}(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\ \mathrm{d}\sigma_{\mathcal{K}})$ is solvable.*

Furthermore, if $D^p_{\mathcal{K}}(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\ \mathrm{d}\sigma_{\mathcal{K}})$ is solvable then it is uniquely solvable.

4. Local regularity and boundary estimates

In this section we state a number of the lemmas concerning the interior regularity of weak solution and the boundary behavior of nonnegative solutions. The boundary estimates are proven in [Litsgård and Nyström 2022] for the more general operators stated in (2-14), assuming (2-15), (2-16) and (2-17). Concerning geometry, in that work we considered unbounded domains $\widetilde{\Omega} \subset \mathbb{R}^{N+1}$ of the form

$$\widetilde{\Omega} = \{(X, Y, t) = (x, x_m, y, y_m, t) \in \mathbb{R}^{N+1} : x_m > \tilde{\psi}(x, y, y_m, t)\}, \tag{4-1}$$

imposing restrictions on $\tilde{\psi}$ of Lipschitz character accounting for the underlying non-Euclidean group structure. In particular, we also allowed for (Y, t)-dependent domains. Up to a point, the results in [Litsgård and Nyström 2022] are established allowing $A = A(X, Y, t)$ and $\tilde{\psi} = \tilde{\psi}(x, y, y_m, t)$ to depend on all variables with y_m included. However, the more refined results established are derived assuming in addition that A as well as ψ are independent of the variable y_m. The reason for this is discussed in detail in that work. Obviously, the operators $\mathcal{L}_{\mathcal{K}}$ considered in this paper are, as $A = A(X)$, special cases of the more general operators of Kolmogorov type considered there. Also, the geometric setting of that work is more demanding compared to the domains considered in this paper, as $\Omega \times \mathbb{R}^m \times \mathbb{R}$ is a special case of the domains in (4-1).

Below we formulate the necessary auxiliary and boundary-type estimate results needed in our proofs, and in particular in the proofs of Theorems 3.1, 3.2 and 3.3, in the context of $\mathcal{L}_{\mathcal{K}}$ as these results follow from [Litsgård and Nyström 2022]. For the corresponding results for $\mathcal{L}_{\mathcal{E}}$ and $\mathcal{L}_{\mathcal{P}}$ we refer to [Kenig 1994] and [Fabes and Safonov 1997; Fabes et al. 1986; 1999; Nyström 1997], respectively.

4A. ***Energy estimates and local regularity.*** Consider $(X_0, Y_0, t_0) \subset \mathbb{R}^{N+1}$. In the following we will frequently use the notation $Q_\varrho := Q_\varrho(X_0, Y_0, t_0)$ for $\varrho > 0$.

Lemma 4.1. *Assume that u is a weak solution to $\mathcal{L}_{\mathcal{K}}u = 0$ in $Q_{2r} = Q_{2r}(X_0, Y_0, t_0) \subset \mathbb{R}^{N+1}$. Then*

$$\iiint_{Q_r} |\nabla_X u|^2 \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t \lesssim \frac{1}{r^2} \iiint_{Q_{2r}} |u|^2 \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t.$$

Proof. This is an energy estimate that can be proven using standard arguments. We refer to [Litsgård and Nyström 2022] for further details. □

The following two lemmas are proved in [Golse et al. 2019].

Lemma 4.2. *Assume that u is a weak solution to $\mathcal{L}_{\mathcal{K}}u = 0$ in $Q_{2r}(X_0, Y_0, t_0) \subset \mathbb{R}^{N+1}$. Given $p \in [1, \infty)$ there exists a constant $c = c(m, \kappa, p)$, $1 \le c < \infty$ such that*

$$\sup_{Q_r} |u| \le c \left(⨍\!\!⨍\!\!⨍_{Q_{2r}} |u|^p \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t \right)^{1/p}. \tag{4-2}$$

Lemma 4.3. *Assume that u is a weak solution to $\mathcal{L}_{\mathcal{K}}u = 0$ in $Q_{2r}(X_0, Y_0, t_0) \subset \mathbb{R}^{N+1}$. Then there exists $\alpha = \alpha(m, \kappa) \in (0, 1)$ such that*

$$|u(X, Y, t) - u(\widetilde{X}, \widetilde{Y}, \tilde{t})| \lesssim \left(\frac{d((X, Y, t), (\widetilde{X}, \widetilde{Y}, \tilde{t}))}{r} \right)^{\alpha} \sup_{Q_{2r}} |u|, \tag{4-3}$$

whenever $(X, Y, t), (\widetilde{X}, \widetilde{Y}, \tilde{t}) \in Q_r(X_0, Y_0, t_0)$.

To state the Harnack inequality we introduce

$$Q_r^-(X_0, Y_0, t_0) := Q_r(X_0, Y_0, t_0) \cap \{(X, Y, t) : t_0 - r^2 < t < t_0\}. \tag{4-4}$$

The following Harnack inequality is proved in [Golse et al. 2019].

Lemma 4.4. *There exist constants $c = c(m, \kappa) > 1$ and $\alpha, \beta, \gamma, \theta \in (0, 1)$, with $0 < \alpha < \beta < \gamma < \theta^2$, such that the following is true. Assume that u is a nonnegative weak solution to $\mathcal{L}_{\mathcal{K}}u = 0$ in $Q_r^-(X_0, Y_0, t_0) \subset \mathbb{R}^{N+1}$. Then,*

$$\sup_{\widetilde{Q}_r^-(X_0, Y_0, t_0)} u \le c \inf_{\widetilde{Q}_r^+(X_0, Y_0, t_0)} u,$$

where

$$\widetilde{Q}_r^+(X_0, Y_0, t_0) = \{(X, Y, t) \in Q_{\theta r}^-(X_0, Y_0, t_0) : t_0 - \alpha r^2 \le t \le t_0\},$$
$$\widetilde{Q}_r^-(X_0, Y_0, t_0) = \{(X, Y, t) \in Q_{\theta r}^-(X_0, Y_0, t_0) : t_0 - \gamma r^2 \le t \le t_0 - \beta r^2\}.$$

Remark. Note that the constants $\alpha, \beta, \gamma, \theta$ appearing in Lemma 4.4 cannot be chosen arbitrarily.

4B. ***Estimates for (nonnegative) solutions.*** We refer to [Litsgård and Nyström 2022] for the proofs of the following results.

Lemma 4.5. *Assume* (A1)–(A3). *Let* $(X_0, Y_0, t_0) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$ *and* $r > 0$. *Let* u *be a weak solution of* $\mathcal{L}_{\mathcal{K}} u = 0$ *in* $(\Omega \times \mathbb{R}^m \times \mathbb{R}) \cap Q_{2r}(X_0, Y_0, t_0)$, *vanishing continuously on* $(\partial\Omega \times \mathbb{R}^m \times \mathbb{R}) \cap Q_{2r}(X_0, Y_0, t_0)$. *Then, there exists* $\alpha = \alpha(m, \kappa, M) \in (0, 1)$ *such that*

$$u(X, Y, t) \lesssim \left(\frac{d((X, Y, t), (X_0, Y_0, t_0))}{r}\right)^{\alpha} \sup_{(\Omega \times \mathbb{R}^m \times \mathbb{R}) \cap Q_{2r}(X_0, Y_0, t_0)} u, \tag{4-5}$$

whenever $(X, Y, t) \in (\Omega \times \mathbb{R}^m \times \mathbb{R}) \cap Q_{r/c}(X_0, Y_0, t_0)$.

Lemma 4.6. *Let* Ω *and* A *be as in Lemma 4.5. There exist* $\Lambda = \Lambda(m, M)$, $c = c(m, \kappa, M)$, *and* $\gamma = \gamma(m, \kappa, M)$, $0 < \gamma < \infty$, *such that the following holds. Let* $(X_0, Y_0, t_0) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$ *and* $r > 0$. *Assume that* u *is a nonnegative weak solution to* $\mathcal{L}_{\mathcal{K}} u = 0$ *in* $(\Omega \times \mathbb{R}^m \times \mathbb{R}) \cap Q_{2r}(X_0, Y_0, t_0)$. *Then*

$$\begin{aligned} u(X, Y, t) &\lesssim (\varrho/d)^{\gamma} u(A^{+}_{\varrho,\Lambda}(X_0, Y_0, t_0)), \\ u(X, Y, t) &\gtrsim (d/\varrho)^{\gamma} u(A^{-}_{\varrho,\Lambda}(X_0, Y_0, t_0)), \end{aligned} \tag{4-6}$$

whenever $(X, Y, t) \in (\Omega \times \mathbb{R}^m \times \mathbb{R}) \cap Q_{2\varrho/c}(X_0, Y_0, t_0)$, $0 < \varrho < r/c$, *where* $d := d((X, Y, t), \partial\Omega \times \mathbb{R}^m \times \mathbb{R})$.

Theorem 4.7. *Let* Ω *and* A *be as in Lemma 4.5. Then there exist* $\Lambda = \Lambda(m, M)$ *and* $c = c(m, \kappa, M)$ *such that the following holds. Let* $(X_0, Y_0, t_0) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$ *and* $r > 0$. *Assume that* u *is a nonnegative weak solution to* $\mathcal{L}_{\mathcal{K}} u = 0$ *in* $(\Omega \times \mathbb{R}^m \times \mathbb{R}) \cap Q_{2r}(X_0, Y_0, t_0)$, *vanishing continuously on* $(\partial\Omega \times \mathbb{R}^m \times \mathbb{R}) \cap Q_{2r}(X_0, Y_0, t_0)$. *Then*

$$u(X, Y, t) \lesssim u(A^{+}_{\varrho,\Lambda}(X_0, Y_0, t_0)),$$

whenever $(X, Y, t) \in (\Omega \times \mathbb{R}^m \times \mathbb{R}) \cap Q_{2\varrho/c}(X_0, Y_0, t_0)$, $0 < \varrho < r/c$.

Theorem 4.8. *Let* Ω *and* A *be as in Lemma 4.5. Then there exist* $\Lambda = \Lambda(m, M)$ *and* $c = c(m, \kappa, M)$ *such that the following holds. Let* $(X_0, Y_0, t_0) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$ *and* $r > 0$. *Assume that* u *and* v *are nonnegative weak solutions to* $\mathcal{L}_{\mathcal{K}} u = 0$ *in* $\Omega \times \mathbb{R}^m \times \mathbb{R}$, *vanishing continuously on* $(\partial\Omega \times \mathbb{R}^m \times \mathbb{R}) \cap Q_{2r}(X_0, Y_0, t_0)$. *Let* $\varrho_0 = r/c$,

$$\begin{aligned} m_1^+ &= v(A^{+}_{\varrho_0,\Lambda}(X_0, Y_0, t_0)), \quad & m_1^- &= v(A^{-}_{\varrho_0,\Lambda}(X_0, Y_0, t_0)), \\ m_2^+ &= u(A^{+}_{\varrho_0,\Lambda}(X_0, Y_0, t_0)), \quad & m_2^- &= u(A^{-}_{\varrho_0,\Lambda}(X_0, Y_0, t_0)), \end{aligned} \tag{4-7}$$

and assume $m_1^- > 0$, $m_2^- > 0$. *Then there exist constants* $c_1 = c_1(m, M)$ *and*

$$c_2 = c_2(m, \kappa, M, m_1^+/m_1^-, m_2^+/m_2^-),$$

$1 \le c_1, c_2 < \infty$, *such that if we let* $\varrho_1 = \varrho_0/c_1$, *then*

$$c_2^{-1} \frac{v(A_{\varrho,\Lambda}(\widetilde{X}_0, \widetilde{Y}_0, \tilde{t}_0))}{u(A_{\varrho,\Lambda}(\widetilde{X}_0, \widetilde{Y}_0, \tilde{t}_0))} \le \frac{v(X, Y, t)}{u(X, Y, t)} \le c_2 \frac{v(A_{\varrho,\Lambda}(\widetilde{X}_0, \widetilde{Y}_0, \tilde{t}_0))}{u(A_{\varrho,\Lambda}(\widetilde{X}_0, \widetilde{Y}_0, \tilde{t}_0))},$$

whenever $(X, Y, t) \in (\Omega \times \mathbb{R}^m \times \mathbb{R}) \cap Q_{\varrho/c_1}(\widetilde{X}_0, \widetilde{Y}_0, \tilde{t}_0)$ *for some* $0 < \varrho < \varrho_1$ *and* $(\widetilde{X}_0, \widetilde{Y}_0, \tilde{t}_0) \in (\partial\Omega \times \mathbb{R}^m \times \mathbb{R}) \cap Q_{\varrho_1}(X_0, Y_0, t_0)$.

4C. *Estimates of Green's functions and parabolic measures.* The adjoint operator of $\mathcal{L}_{\mathcal{K}}$ is defined as

$$\mathcal{L}_{\mathcal{K}}^* := \nabla_X \cdot (A(X)\nabla_X) - X \cdot \nabla_Y + \partial_t, \tag{4-8}$$

as A is assumed to be symmetric.

Remark. We remark that for nonnegative weak solutions to the adjoint equation $\mathcal{L}_{\mathcal{K}}^* u = 0$, adjoint versions of Lemma 4.6, Theorem 4.7, and Theorem 4.8 hold. The statements in the adjoint versions are the same, except that the roles of $A^+_{\varrho,\Lambda}(X_0, Y_0, t_0)$ and $A^-_{\varrho,\Lambda}(X_0, Y_0, t_0)$ are reversed.

Definition. A fundamental solution for $\mathcal{L}_{\mathcal{K}}$ is a continuous and positive function $\Gamma_{\mathcal{K}} = \Gamma_{\mathcal{K}}(X, Y, t, \widetilde{X}, \widetilde{Y}, \tilde{t})$, defined for $\tilde{t} < t$ and $(X, Y), (\widetilde{X}, \widetilde{Y}) \in \mathbb{R}^N$, such that

(i) $\Gamma_{\mathcal{K}}(\cdot, \cdot, \cdot, \widetilde{X}, \widetilde{Y}, \tilde{t})$ is a weak solution of $\mathcal{L}_{\mathcal{K}} u = 0$ in $\mathbb{R}^N \times (\tilde{t}, \infty)$ and $\Gamma_{\mathcal{K}}(X, Y, t, \cdot, \cdot, \cdot)$ is a weak solution of $\mathcal{L}_{\mathcal{K}}^* u = 0$ in $\mathbb{R}^N \times (-\infty, t)$,

(ii) for any bounded function $\phi \in C(\mathbb{R}^N)$ and $(X, Y), (\widetilde{X}, \widetilde{Y}) \in \mathbb{R}^N$, we have

$$\lim_{\substack{(X,Y,t)\to(\widetilde{X},\widetilde{Y},\tilde{t})\\ t>\tilde{t}}} u(X, Y, t) = \phi(\widetilde{X}, \widetilde{Y}), \qquad \lim_{\substack{(\widetilde{X},\widetilde{Y},\tilde{t})\to(X,Y,t)\\ t>\tilde{t}}} v(\widetilde{X}, \widetilde{Y}, \tilde{t}) = \phi(X, Y), \tag{4-9}$$

where

$$\begin{aligned} u(X, Y, t) &:= \iint_{\mathbb{R}^N} \Gamma_{\mathcal{K}}(X, Y, t, \widetilde{X}, \widetilde{Y}, \tilde{t})\, \phi(\widetilde{X}, \widetilde{Y})\, \mathrm{d}\widetilde{X}\, \mathrm{d}\widetilde{Y}, \\ v(\widetilde{X}, \widetilde{Y}, \tilde{t}) &:= \iint_{\mathbb{R}^N} \Gamma_{\mathcal{K}}(X, Y, t, \widetilde{X}, \widetilde{Y}, \tilde{t})\, \phi(X, Y)\, \mathrm{d}X\, \mathrm{d}Y. \end{aligned} \tag{4-10}$$

Lemma 4.9. *Assume that A satisfies* (2-17). *Then there exists a fundamental solution to $\mathcal{L}_{\mathcal{K}}$ in the sense of the Definition above. Let $\Gamma_{\mathcal{K}}(X, Y, t, \widetilde{X}, \widetilde{Y}, \tilde{t})$ be the fundamental solution to $\mathcal{L}_{\mathcal{K}}$. Then we have the upper bound*

$$\Gamma_{\mathcal{K}}(X, Y, t, \widetilde{X}, \widetilde{Y}, \tilde{t}) \lesssim \frac{1}{d((X, Y, t), (\widetilde{X}, \widetilde{Y}, \tilde{t}))^{\mathbf{q}-2}} \tag{4-11}$$

for all $(X, Y, t), (\widetilde{X}, \widetilde{Y}, \tilde{t})$ with $t > \tilde{t}$.

Proof. We refer to [Delarue and Menozzi 2010; Di Francesco and Pascucci 2005; Polidoro 1997] for the existence of the fundamental solution for $\mathcal{L}$ under the additional condition that the coefficients are Hölder continuous. See also [Lanconelli et al. 2020]. For the quantitative estimate we refer to Lemma 4.17 in [Litsgård and Nyström 2022] and the subsequent discussion. □

Assume that $\Omega \subset \mathbb{R}^m$ is an (unbounded) Lipschitz domain with constant M. We define the Green's function associated to $\mathcal{L}_{\mathcal{K}}$ for $\Omega \times \mathbb{R}^m \times \mathbb{R}$, with pole at $(\widehat{X}, \widehat{Y}, \hat{t}) \in \Omega \times \mathbb{R}^m \times \mathbb{R}$, as

$$\begin{aligned} G_{\mathcal{K}}(X, Y, t, \widehat{X}, \widehat{Y}, \hat{t}) = {} & \Gamma_{\mathcal{K}}(X, Y, t, \widehat{X}, \widehat{Y}, \hat{t}) \\ & - \iiint_{\partial\Omega\times\mathbb{R}^m\times\mathbb{R}} \Gamma_{\mathcal{K}}(\widetilde{X}, \widetilde{Y}, \tilde{t}, \widehat{X}, \widehat{Y}, \hat{t})\, \mathrm{d}\omega_{\mathcal{K}}(X, Y, t, \widetilde{X}, \widetilde{Y}, \tilde{t}), \end{aligned} \tag{4-12}$$

where $\Gamma_{\mathcal{K}}$ is the fundamental solution to the operator $\mathcal{L}_{\mathcal{K}}$. If we instead consider $(X, Y, t) \in \Omega \times \mathbb{R}^m \times \mathbb{R}$ as fixed, then, for $(\widehat{X}, \widehat{Y}, \hat{t}) \in \Omega \times \mathbb{R}^m \times \mathbb{R}$,

$$G_{\mathcal{K}}(X, Y, t, \widehat{X}, \widehat{Y}, \hat{t}) = \Gamma_{\mathcal{K}}(X, Y, t, \widehat{X}, \widehat{Y}, \hat{t}) - \iiint_{\partial\Omega\times\mathbb{R}^m\times\mathbb{R}} \Gamma_{\mathcal{K}}(X, Y, t, \widetilde{X}, \widetilde{Y}, \tilde{t})\, \mathrm{d}\omega^*_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, \widetilde{X}, \widetilde{Y}, \tilde{t}), \tag{4-13}$$

where $\omega^*_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, \cdot\,)$ is the associated adjoint Kolmogorov measure relative to $(\widehat{X}, \widehat{Y}, \hat{t})$ and $\Omega \times \mathbb{R}^m \times \mathbb{R}$. The corresponding Green's functions associated to $\mathcal{L}_{\mathcal{E}}$ and $\mathcal{L}_{\mathcal{P}}$, for Ω and $\Omega \times \mathbb{R}$, are denoted by $G_{\mathcal{E}}$ and $G_{\mathcal{P}}$, respectively.

Let $\theta \in C_0^\infty(\mathbb{R}^{N+1})$. The following representation formulas are proved in Lemma 8.3 in [Litsgård and Nyström 2022]:

$$\begin{aligned}
\theta(\widehat{X}, \widehat{Y}, \hat{t}) &= \iiint_{\partial\Omega\times\mathbb{R}^m\times\mathbb{R}} \theta(X, Y, t)\, \mathrm{d}\omega_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, X, Y, t) \\
&\quad - \iiint_{\Omega\times\mathbb{R}^m\times\mathbb{R}} A(X)\nabla_X G_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, X, Y, t) \cdot \nabla_X \theta(X, Y, t)\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}t \\
&\quad + \iiint_{\Omega\times\mathbb{R}^m\times\mathbb{R}} G_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, X, Y, t)(X \cdot \nabla_Y - \partial_t)\theta(X, Y, t)\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}t, \\
\theta(\widehat{X}, \widehat{Y}, \hat{t}) &= \iiint_{\partial\Omega\times\mathbb{R}^m\times\mathbb{R}} \theta(X, Y, t)\, \mathrm{d}\omega^*_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, X, Y, t) \\
&\quad - \iiint_{\Omega\times\mathbb{R}^m\times\mathbb{R}} A(X)\nabla_X G_{\mathcal{K}}(X, Y, t, \widehat{X}, \widehat{Y}, \hat{t}) \cdot \nabla_X \theta(X, Y, t)\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}t \\
&\quad + \iiint_{\Omega\times\mathbb{R}^m\times\mathbb{R}} G_{\mathcal{K}}(X, Y, t, \widehat{X}, \widehat{Y}, \hat{t})(-X \cdot \nabla_Y + \partial_t)\theta(X, Y, t)\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}t,
\end{aligned} \tag{4-14}$$

whenever $(\widehat{X}, \widehat{Y}, \hat{t}) \in \Omega \times \mathbb{R}^m \times \mathbb{R}$.

The following lemmas, Lemmas 4.10 and 4.11, are proved in [Litsgård and Nyström 2022]; see in particular Section 8. Theorem 4.12 stated below is one of the main results in that work.

Lemma 4.10. *Let Ω and A be as in Lemma 4.5. Then there exist $\Lambda = \Lambda(m, M)$, $1 \le \Lambda < \infty$, $c = c(m, \kappa, M)$, $1 \le c < \infty$, such that the following is true. Let $(X_0, Y_0, t_0) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$, $0 < \varrho < \infty$. Then*

$$\begin{aligned}
\varrho^{\boldsymbol{q}-2} G_{\mathcal{K}}(X, Y, t, A^+_{\varrho,\Lambda}(X_0, Y_0, t_0)) &\lesssim \omega_{\mathcal{K}}(X, Y, t, \Delta_\varrho(X_0, Y_0, t_0)) \\
&\lesssim \varrho^{\boldsymbol{q}-2} G_{\mathcal{K}}(X, Y, t, A^-_{c\varrho,\Lambda}(X_0, Y_0, t_0)),
\end{aligned}$$

whenever $(X, Y, t) \in \Omega \times \mathbb{R}^m \times \mathbb{R}$, $t \ge t_0 + c\varrho^2$.

Lemma 4.11. *Let Ω and A be as in Lemma 4.5. Then there exist $\Lambda = \Lambda(m, M)$, $1 \le \Lambda < \infty$, $c = c(m, \kappa, M)$, $1 \le c < \infty$, such that the following is true. Let $(X_0, Y_0, t_0) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$, $0 < \varrho < \infty$. Then*

$$G_{\mathcal{K}}(X, Y, t, A^-_{\varrho,\Lambda}(X_0, Y_0, t_0)) \lesssim G_{\mathcal{K}}(X, Y, t, A^+_{\varrho,\Lambda}(X_0, Y_0, t_0)) \lesssim G_{\mathcal{K}}(X, Y, t, A^-_{\varrho,\Lambda}(X_0, Y_0, t_0)),$$

whenever $(X, Y, t) \in \Omega \times \mathbb{R}^m \times \mathbb{R}$, $t \ge t_0 + c\varrho^2$.

Theorem 4.12. *Let Ω and A be as in Lemma 4.5. Then there exist $\Lambda = \Lambda(m, M)$, $1 \le \Lambda < \infty$, $c = c(m, \kappa, M)$, $1 \le c < \infty$, such that the following is true. Let $(X_0, Y_0, t_0) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$, $0 < \varrho_0 < \infty$. Then*

$$\omega_{\mathcal{K}}\big(A^+_{c\varrho_0,\Lambda}(X_0, Y_0, t_0), \Delta_{2\varrho}(\widetilde{X}_0, \widetilde{Y}_0, \tilde{t}_0)\big) \lesssim \omega_{\mathcal{K}}\big(A^+_{c\varrho_0,\Lambda}(X_0, Y_0, t_0), \Delta_{\varrho}(\widetilde{X}_0, \widetilde{Y}_0, \tilde{t}_0)\big)$$

for all $\Delta_\varrho(\widetilde{X}_0, \widetilde{Y}_0, \tilde{t}_0)$, $(\widetilde{X}_0, \widetilde{Y}_0, \tilde{t}_0) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$ such that $\Delta_\varrho(\widetilde{X}_0, \widetilde{Y}_0, \tilde{t}_0) \subset \Delta_{4\varrho_0}(X_0, Y_0, t_0)$.

5. Proof of the structural theorems: Theorems 3.1 and 3.2

The purpose of the section is to prove Theorems 3.1 and 3.2. Throughout the section we assume (A1)–(A3). Let $\omega_{\mathcal{E}}$, $\omega_{\mathcal{P}}$, and $\omega_{\mathcal{K}}$ be as in the statement of Theorem 3.1.

5A. ***Proof of Theorem 3.1.*** To prove Theorem 3.1 we need to prove that there exist $\Lambda = \Lambda(m, M)$, $1 \le \Lambda < \infty$, $c = c(m, \kappa, M)$, $1 \le c < \infty$, such that if $\Delta := \Delta_r(X_0, Y_0, t_0) \subset \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$, then the estimates stated in the theorems hold whenever $\tilde{\Delta} \subset \Delta$. The proof of Theorem 3.1 is based on the relation between $\omega_{\mathcal{E}}$, $\omega_{\mathcal{P}}$, $\omega_{\mathcal{K}}$ and the corresponding Green's functions and boundary Harnack inequalities.

To start the proof we first note that an immediate consequence of Lemma 4.10 is that there exists $c = c(m, \kappa, M)$, $1 \le c < \infty$, such that given $\Delta := \Delta_r(X_0, Y_0, t_0) \subset \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$, we have

$$\tilde{r}^{\mathbf{q}-2} G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^+_{\tilde{\Delta},\Lambda}) \lesssim \omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \tilde{\Delta}) \lesssim \tilde{r}^{\mathbf{q}-2} G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^-_{c\tilde{\Delta},\Lambda}), \tag{5-1}$$

whenever $\tilde{\Delta} = \Delta_{\tilde{r}} \subset \Delta$. Using this, and the corresponding results for $\mathcal{L}_{\mathcal{E}}$ and $\mathcal{L}_{\mathcal{P}}$, see [Kenig 1994] and [Fabes and Safonov 1997; Fabes et al. 1986; 1999; Nyström 1997], we obtain

$$\frac{G_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), \pi_X(A^+_{\tilde{\Delta},\Lambda}))}{G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^-_{c\tilde{\Delta},\Lambda})} \lesssim \frac{\sigma_{\mathcal{K}}(\tilde{\Delta})\omega_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), \pi_X(\tilde{\Delta}))}{\sigma_{\mathcal{E}}(\pi_X(\tilde{\Delta}))\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \tilde{\Delta})} \lesssim \frac{G_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), \pi_X(A^-_{c\tilde{\Delta},\Lambda}))}{G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^+_{\tilde{\Delta},\Lambda})}, \tag{5-2}$$

and

$$\frac{G_{\mathcal{P}}(\pi_{X,t}(A^+_{c\Delta,\Lambda}), \pi_{X,t}(A^+_{\tilde{\Delta},\Lambda}))}{G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^-_{c\tilde{\Delta},\Lambda})} \lesssim \frac{\sigma_{\mathcal{K}}(\tilde{\Delta})\omega_{\mathcal{P}}(\pi_{X,t}(A^+_{c\Delta,\Lambda}), \pi_{X,t}(\tilde{\Delta}))}{\sigma_{\mathcal{P}}(\pi_{X,t}(\tilde{\Delta}))\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \tilde{\Delta})} \lesssim \frac{G_{\mathcal{P}}(\pi_{X,t}(A^+_{c\Delta,\Lambda}), \pi_{X,t}(A^-_{c\tilde{\Delta},\Lambda}))}{G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^+_{\tilde{\Delta},\Lambda})}.$$

To this end we will now prove the theorem only for $\omega_{\mathcal{K}}$, the proof for $\omega_{\mathcal{P}}$ being analogous. We first relate $G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^-_{c\tilde{\Delta},\Lambda})$ and $G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^+_{\tilde{\Delta},\Lambda})$. Using that $G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \cdot, \cdot, \cdot)$ solves the adjoint equation we can apply the adjoint version of Lemma 4.6 to conclude that

$$G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^+_{\tilde{\Delta},\Lambda}) \gtrsim G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^+_{c\tilde{\Delta},\Lambda}),$$
$$G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^-_{c\tilde{\Delta},\Lambda}) \gtrsim G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^-_{\tilde{\Delta},\Lambda}).$$

Hence

$$\frac{G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^+_{c\tilde{\Delta},\Lambda})}{G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^-_{c\tilde{\Delta},\Lambda})} \lesssim \frac{G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^+_{\tilde{\Delta},\Lambda})}{G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^-_{c\tilde{\Delta},\Lambda})} \lesssim \frac{G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^+_{\tilde{\Delta},\Lambda})}{G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^-_{\tilde{\Delta},\Lambda})}. \tag{5-3}$$

Therefore, applying Lemma 4.11 twice,

$$\frac{G_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, A^{+}_{\tilde{\Delta},\Lambda})}{G_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, A^{-}_{c\tilde{\Delta},\Lambda})} \approx 1. \tag{5-4}$$

Furthermore, by the standard elliptic Harnack inequality

$$G_{\mathcal{E}}(\pi_X(A^{+}_{c\Delta,\Lambda}), \pi_X(A^{+}_{\tilde{\Delta},\Lambda})) \approx G_{\mathcal{E}}(\pi_X(A^{+}_{c\Delta,\Lambda}), \pi_X(A^{-}_{c\tilde{\Delta},\Lambda})). \tag{5-5}$$

Putting (5-2)–(5-5) together we can conclude that

$$\frac{\sigma_{\mathcal{K}}(\tilde{\Delta})\omega_{\mathcal{E}}(\pi_X(A^{+}_{c\Delta,\Lambda}), \pi_X(\tilde{\Delta}))}{\sigma_{\mathcal{E}}(\pi_X(\tilde{\Delta}))\omega_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, \tilde{\Delta})} \approx \frac{G_{\mathcal{E}}(\pi_X(A^{+}_{c\Delta,\Lambda}), \pi_X(A^{+}_{\tilde{\Delta},\Lambda}))}{G_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, A^{+}_{\tilde{\Delta},\Lambda})}. \tag{5-6}$$

Next, using Theorem 4.8

$$\frac{G_{\mathcal{E}}(\pi_X(A^{+}_{c\Delta,\Lambda}), \pi_X(A^{+}_{\tilde{\Delta},\Lambda}))}{G_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, A^{+}_{\tilde{\Delta},\Lambda})} \approx \frac{G_{\mathcal{E}}(\pi_X(A^{+}_{c\Delta,\Lambda}), \pi_X(A^{+}_{\Delta,\Lambda}))}{G_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, A^{+}_{\Delta,\Lambda})}.$$

Furthermore, $G_{\mathcal{E}}(\pi_X(A^{+}_{c\Delta,\Lambda}), \pi_X(A^{+}_{\Delta,\Lambda})) \approx r^{-m} \approx (r\sigma_{\mathcal{E}}(\pi_X(\Delta)))^{-1}$ by classical estimates for the fundamental solution second-order elliptic equations in divergence form; see [Kenig 1994]. We claim that

$$G_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, A^{+}_{\Delta,\Lambda}) \approx r^{2-\boldsymbol{q}} \approx (r\sigma_{\mathcal{K}}(\Delta))^{-1}. \tag{5-7}$$

To prove this we first note that the upper bound on $G_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, A^{+}_{\Delta,\Lambda})$ follows from Lemma 4.9. The proof of the lower bound on $G_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, A^{+}_{\Delta,\Lambda})$ is a bit more subtle but can be achieved analogously to the proof of the estimate in display (9.11) in [Litsgård and Nyström 2022]. Using (5-7), we deduce

$$\frac{G_{\mathcal{E}}(\pi_X(A^{+}_{c\Delta,\Lambda}), \pi_X(A^{+}_{\tilde{\Delta},\Lambda}))}{G_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, A^{+}_{\tilde{\Delta},\Lambda})} \approx \frac{G_{\mathcal{E}}(\pi_X(A^{+}_{c\Delta,\Lambda}), \pi_X(A^{+}_{\Delta,\Lambda}))}{G_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, A^{+}_{\Delta,\Lambda})} \approx \frac{\sigma_{\mathcal{K}}(\Delta)}{\sigma_{\mathcal{E}}(\pi_X(\Delta))}.$$

Combing this with (5-6),

$$\frac{\sigma_{\mathcal{E}}(\pi_X(\Delta))\sigma_{\mathcal{K}}(\tilde{\Delta})\omega_{\mathcal{E}}(\pi_X(A^{+}_{c\Delta,\Lambda}), \pi_X(\tilde{\Delta}))}{\sigma_{\mathcal{K}}(\Delta)\sigma_{\mathcal{E}}(\pi_X(\tilde{\Delta}))\omega_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, \tilde{\Delta})} \approx 1.$$

This proves Theorem 3.1.

5B. ***Proof of Theorem 3.2.*** Again we will only prove the theorem for $\omega_{\mathcal{K}}$, the proof for $\omega_{\mathcal{P}}$ being analogous. Assume that $\omega_{\mathcal{E}}(\pi_X(A^{+}_{c\Delta,\Lambda}), \cdot)$ is mutually absolutely continuous on $\pi_X(\Delta)$ with respect to $\sigma_{\mathcal{E}}$ and that the associated Poisson kernel $K_{\mathcal{E}}(X) := K_{\mathcal{E}}(\pi_X(A^{+}_{c\Delta,\Lambda}), X)$ satisfies

$$K_{\mathcal{E}} \in B_q(\pi_X(\Delta),\, \mathrm{d}\sigma_{\mathcal{E}})$$

for some q, $1 < q < \infty$, and with constant Γ, $1 \le \Gamma < \infty$. To prove Theorem 3.2 for $\omega_{\mathcal{K}}$ we have to prove that $\omega_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, \cdot)$ is mutually absolutely continuous on Δ with respect to $\sigma_{\mathcal{K}}$, and that the associated Poisson kernel $K_{\mathcal{K}}(X, Y, t) := K_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, X, Y, t)$ satisfies $K_{\mathcal{K}} \in B_q(\Delta,\, \mathrm{d}\sigma_{\mathcal{K}})$ with a constant $\widetilde{\Gamma} = \widetilde{\Gamma}(m, \kappa, M, \Gamma)$.

Let $\mathrm{d}\mu_{\mathcal{K}} := \omega_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), \pi_X(\cdot))\,\mathrm{d}Y\,\mathrm{d}t$. To prove that $\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \cdot)$ is absolutely continuous on Δ with respect to $\sigma_{\mathcal{K}}$ it suffices to prove that $\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \cdot) \ll \mu_{\mathcal{K}}$ on Δ and that $\mu_{\mathcal{K}} \ll \sigma_{\mathcal{K}}$ on Δ. Recall that $\mathrm{d}\sigma_{\mathcal{K}}(X, Y, t) = \mathrm{d}\sigma_{\mathcal{E}}(X)\,\mathrm{d}Y\,\mathrm{d}t$. However, as $\mu_{\mathcal{K}}$ and $\sigma_{\mathcal{K}}$ are defined through the stated product structure, it follows immediately that $\mu_{\mathcal{K}} \ll \sigma_{\mathcal{K}}$ on Δ as $\omega_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), \cdot) \ll \sigma_{\mathcal{E}}$ on $\pi_X(\Delta)$. In particular, by the assumptions it suffices to prove that $\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \cdot)$ is absolutely continuous on Δ with respect to $\mu_{\mathcal{K}}$ and we will do this by using Theorem 3.1.

Recall that we previously observed that $(\Sigma, d, \mathrm{d}\sigma_{\mathcal{K}})$, where Σ was introduced in (2-12), is a space of homogeneous type in the sense of [Coifman and Weiss 1971]. By the results in [Christ 1990] there exists what we here will refer to as a dyadic grid on Σ having a number of important properties in relation to d. To formulate this we introduce, for any $(X, Y, t) \in \Sigma$ and $E \subset \Sigma$,

$$\operatorname{dist}((X, Y, t), E) := \inf\{d((X, Y, t), (\widetilde{X}, \widetilde{Y}, \tilde{t})) : (\widetilde{X}, \widetilde{Y}, \tilde{t}) \in E\}, \tag{5-8}$$

and we let

$$\operatorname{diam}(E) := \sup\{d((X, Y, t), (\widetilde{X}, \widetilde{Y}, \tilde{t})) : (X, Y, t),\ (\widetilde{X}, \widetilde{Y}, \tilde{t}) \in E\}. \tag{5-9}$$

Using [Christ 1990] we can conclude that there exist constants $\alpha > 0$, $\beta > 0$ and $c_* < \infty$ such that for each $k \in \mathbb{Z}$ there exists a collection of Borel sets, $\mathbb{D}_k$, which we will call cubes, such that

$$\mathbb{D}_k := \{Q^k_j \subset \Sigma : j \in \mathfrak{I}_k\},$$

where $\mathfrak{I}_k$ denotes some index set depending on k, satisfying:

(i) $\Sigma = \bigcup_j Q^k_j$ for each $k \in \mathbb{Z}$.

(ii) If $m \geq k$ then either $Q^m_i \subset Q^k_j$ or $Q^m_i \cap Q^k_j = \varnothing$.

(iii) For each (j, k) and each $m < k$, there is a unique i such that $Q^k_j \subset Q^m_i$.

(iv) $\operatorname{diam}(Q^k_j) \leq c_* 2^{-k}$.

(v) Each Q^k_j contains $\Sigma \cap \mathcal{B}_{\alpha 2^{-k}}(X^k_j, Y^k_j, t^k_j)$ for some $(X^k_j, Y^k_j, t^k_j) \in \Sigma$.

(vi) $\sigma_{\mathcal{K}}(\{(X, Y, t) \in Q^k_j : \operatorname{dist}((X, Y, t), \Sigma \setminus Q^k_j) \leq \varrho\, 2^{-k}\}) \leq c_*\, \varrho^{\beta}\, \sigma_{\mathcal{K}}(Q^k_j)$ for all k, j and for all $\varrho \in (0, \alpha)$.

We shall denote by $\mathbb{D} = \mathbb{D}(\Sigma)$ the collection of all Q^k_j, i.e.,

$$\mathbb{D} := \bigcup_k \mathbb{D}_k.$$

Note that (iv) and (v) above imply that for each cube $Q \in \mathbb{D}_k$ there is a point $(X_Q, Y_Q, t_Q) \in \Sigma$ and a cube $Q_r(X_Q, Y_Q, t_Q)$ such that $r \approx 2^{-k} \approx \operatorname{diam}(Q)$ and

$$\Delta_r(X_Q, Y_Q, t_Q) \subset Q \subset \Delta_{cr}(X_Q, Y_Q, t_Q) \tag{5-10}$$

for some uniform constant c. We let

$$\Delta_Q := \Delta_r(X_Q, Y_Q, t_Q), \tag{5-11}$$

and we shall refer to the point (X_Q, Y_Q, t_Q) as the center of Q. Given a dyadic cube $Q \subset \Sigma$, we define its γ dilate by

$$\gamma Q := \Delta_{\gamma\,\mathrm{diam}(Q)}(X_Q, Y_Q, t_Q). \tag{5-12}$$

For a dyadic cube $Q \in \mathbb{D}_k$, we let $\ell(Q) = 2^{-k}$, and we shall refer to this quantity as the length of Q. Clearly, $\ell(Q) \approx \mathrm{diam}(Q)$.

We now prove that $\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \cdot\,)$ is absolutely continuous on Δ with respect to $\mu_{\mathcal{K}}$ using Theorem 3.1. Indeed, let $E \subset \Delta$ and $\delta > 0$, and let $\{Q_j\}$ be a (finite) dyadic Vitali covering of E such that

$$\mu_{\mathcal{K}}\Big(\bigcup Q_j\Big) < \mu_{\mathcal{K}}(E) + \delta,$$

and such that $\gamma Q_i \cap \gamma Q_j = \varnothing$ for some small $\gamma = \gamma(m, M) > 0$, whenever $i \neq j$. Using Theorem 3.1 and the doubling property of $\omega_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), \cdot\,)$ we see that

$$\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, Q_j) \le \omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \Delta_{cQ_j}) \lesssim \omega_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), \pi_X(\Delta_{cQ_j}))\ell(cQ_j)^{3m+2} \lesssim \mu_{\mathcal{K}}(\gamma Q_j), \tag{5-13}$$

where now the implicit constants may depend on $|\Delta|$, which is fixed. Hence

$$\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, E) \le \sum_j \omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, Q_j) \lesssim \sum_j \mu_{\mathcal{K}}(\gamma Q_j) \lesssim \mu_{\mathcal{K}}\Big(\bigcup Q_j\Big) \lesssim (\mu_{\mathcal{K}}(E) + \delta). \tag{5-14}$$

In particular, given $\epsilon > 0$ there exists $\delta = \delta(m, \kappa, M, \epsilon, |\Delta|) > 0$ such that if $E \subset \Delta$, and if $\mu_{\mathcal{K}}(E) < \delta$, then $\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, E) < \epsilon$, proving that $\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \cdot\,) \ll \mu_K$.

By the above we can conclude that $\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \cdot\,) \ll \sigma_{\mathcal{K}}$ on Δ and that

$$K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, Y, t) := \frac{\mathrm{d}\omega_{\mathcal{K}}}{\mathrm{d}\sigma_{\mathcal{K}}}(A^+_{c\Delta,\Lambda}, X, Y, t) = \lim_{\tilde r \to 0} \frac{\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \Delta_{\tilde r}(X, Y, t))}{\sigma_{\mathcal{K}}(\Delta_{\tilde r}(X, Y, t))}$$

exists and is well-defined for $\sigma_{\mathcal{K}}$-almost every $(X, Y, t) \in \Delta$. Using Theorem 3.1

$$\begin{aligned}\sigma_{\mathcal{K}}(\Delta)K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, Y, t) &\approx \sigma_{\mathcal{P}}(\pi_{X,t}(\Delta))K_{\mathcal{P}}(\pi_{X,t}(A^+_{c\Delta,\Lambda}), X, t)\\ &\approx \sigma_{\mathcal{E}}(\pi_X(\Delta))K_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), X),\end{aligned} \tag{5-15}$$

whenever $(X, Y, t) \in \Delta$. Using the assumption on $K_{\mathcal{E}}(X) = K_{\mathcal{E}}(\pi_X(A^+_{c\Delta,\Lambda}), X)$, and (5-15), it follows that $K_{\mathcal{K}}(X, Y, t) := K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, Y, t)$ satisfies

$$K_{\mathcal{K}} \in B_q(\Delta,\, \mathrm{d}\sigma_{\mathcal{K}}),$$

with a constant $\widetilde{\Gamma} = \widetilde{\Gamma}(m, \kappa, M, \Gamma)$. This completes the proof of Theorem 3.2.

6. The L^p Dirichlet problem for $\mathcal{L}_{\mathcal{K}}$: Theorem 3.3

Recall the notation Σ introduced in (2-12). Given $f \in L^1_{\mathrm{loc}}(\Sigma,\, \mathrm{d}\sigma_{\mathcal{K}})$, we let

$$M(f)(X, Y, t) := \sup_{\Delta_r = \Delta_r(X,Y,t) \subset \Sigma} ⨍\!\!\!\!\!\!\iiint_{\Delta_r} |f|\, \mathrm{d}\sigma_{\mathcal{K}}$$

denote the Hardy–Littlewood maximal function of f, with respect to $\sigma_{\mathcal{K}}$. In the following we assume that $\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda},\cdot)$ is mutually absolutely continuous on Δ with respect to $\sigma_{\mathcal{K}}$ for every $\Delta := \Delta_r(X_0, Y_0, t_0) \subset \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$.

We first prove that (i) implies (ii) and hence we assume, given $\Delta \subset \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$, that $K_{\mathcal{K}}(A^+_{c\Delta,\Lambda},\cdot,\cdot,\cdot) \in B_q(\Delta, \mathrm{d}\sigma_{\mathcal{K}})$. As $\omega_{\mathcal{K}}$ is a doubling measure we can use the classical results of Coifman and Fefferman [1974, Theorem IV] to conclude that $K_{\mathcal{K}}(A^+_{c\Delta,\Lambda},\cdot,\cdot,\cdot) \in B_{\tilde q}(\Delta, \mathrm{d}\sigma_{\mathcal{K}})$ for some $\tilde q > q$ independent of Δ. Let $\tilde p$ be the index dual to $\tilde q$ and note that $\tilde p < p$.

Consider first $f \in C_0(\partial\Omega \times \mathbb{R}^m \times \mathbb{R})$. Let $(X_0, Y_0, t_0) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$, and recall the (nontangential) cone $\Gamma^\eta(X_0, Y_0, t_0)$. Let $(\widehat X, \widehat Y, \hat t) \in \Gamma^\eta(X_0, Y_0, t_0)$ and let $\delta := d((\widehat X, \widehat Y, \hat t), (X_0, Y_0, t_0))$. Then, by Theorem 2.1 we know that there exists a unique bounded weak solution to $\mathcal{L}_{\mathcal{K}} u = 0$ in $\Omega \times \mathbb{R}^m \times \mathbb{R}$, with $u = f$ on $\partial\Omega \times \mathbb{R}^m \times \mathbb{R}$. Furthermore,

$$u(\widehat X, \widehat Y, \hat t) = \iiint_{\partial\Omega} K_{\mathcal{K}}(\widehat X, \widehat Y, \hat t, X, Y, t) f(X, Y, t)\, \mathrm{d}\sigma_{\mathcal{K}}(X, Y, t).$$

We write

$$\begin{aligned} u(\widehat X, \widehat Y, \hat t) &= \iiint_{\Delta_{4\delta}(X_0,Y_0,t_0)} K_{\mathcal{K}}(\widehat X, \widehat Y, \hat t, X, Y, t) f(X, Y, t)\, \mathrm{d}\sigma_{\mathcal{K}}(X, Y, t) \\ &\qquad + \sum_{j=2}^{\infty} \iiint_{R_j(X_0,Y_0,t_0)} K_{\mathcal{K}}(\widehat X, \widehat Y, \hat t, X, Y, t) f(X, Y, t)\, \mathrm{d}\sigma_{\mathcal{K}}(X, Y, t) \\ &=: u_1(\widehat X, \widehat Y, \hat t) + \sum_{j=2}^{\infty} u_j(\widehat X, \widehat Y, \hat t), \end{aligned}$$

where $R_j(X_0, Y_0, t_0) := \Delta_{2^{j+1}\delta}(X_0, Y_0, t_0) \setminus \Delta_{2^j\delta}(X_0, Y_0, t_0)$. Using

$$K_{\mathcal{K}}(\widehat X, \widehat Y, \hat t, X, Y, t) = \frac{\mathrm{d}\omega_{\mathcal{K}}}{\mathrm{d}\sigma_{\mathcal{K}}}(\widehat X, \widehat Y, \hat t, X, Y, t) = \lim_{\tilde r \to 0} \frac{\omega_{\mathcal{K}}(\widehat X, \widehat Y, \hat t, \Delta_{\tilde r}(X, Y, t))}{\sigma_{\mathcal{K}}(\Delta_{\tilde r}(X, Y, t))}, \tag{6-1}$$

in combination with Theorem 4.7, we see that

$$K_{\mathcal{K}}(\widehat X, \widehat Y, \hat t, X, Y, t) \lesssim K_{\mathcal{K}}(A^+_{c\Delta_{4\delta},\Lambda}, X, Y, t),$$

whenever $(X, Y, t) \in \Delta_{4\delta}(X_0, Y_0, t_0)$, and where $\Delta_{4\delta} := \Delta_{4\delta}(X_0, Y_0, t_0)$. Hence, using Cauchy–Schwarz,

$$\begin{aligned} |u_1(\widehat X, \widehat Y, \hat t)| &\le \sigma_{\mathcal{K}}(\Delta_{4\delta}) \biggl(\fint\!\!\fint\!\!\fint_{\Delta_{4\delta}} |K_{\mathcal{K}}(A^+_{c\Delta_{4\delta},\Lambda}, X, Y, t)|^{\tilde q}\, \mathrm{d}\sigma_{\mathcal{K}} \biggr)^{1/\tilde q} (M(|f|^{\tilde p})(X_0, Y_0, t_0))^{1/\tilde p} \\ &\le c\,\omega_{\mathcal{K}}(A^+_{c\Delta_{4\delta},\Lambda}, \Delta_{4\delta}) (M(|f|^{\tilde p})(X_0, Y_0, t_0))^{1/\tilde p} \\ &\le c (M(|f|^{\tilde p})(X_0, Y_0, t_0))^{1/\tilde p} \end{aligned}$$

by (i). Similarly, using also Lemma 4.5 we have

$$K_{\mathcal{K}}(\widehat X, \widehat Y, \hat t, X, Y, t) \lesssim 2^{-\alpha j} K_{\mathcal{K}}(A^+_{c\Delta_{2^j\delta},\Lambda}, X, Y, t),$$

whenever $(X, Y, t) \in R_j(X_0, Y_0, t_0)$. Using this estimate, and essentially just repeating the estimates conducted in the estimate of u_1, we deduce that

$$|u_j(\widehat X, \widehat Y, \hat t)| \le c 2^{-\alpha j} (M(|f|^{\tilde p})(X_0, Y_0, t_0))^{1/\tilde p}.$$

In particular,

$$|u(\widehat{X}, \widehat{Y}, \hat{t})| \le |u_1(\widehat{X}, \widehat{Y}, \hat{t})| + \sum_{j=2}^{\infty} |u_j(\widehat{X}, \widehat{Y}, \hat{t})| \le c(M(|f|^{\tilde{p}})(X_0, Y_0, t_0))^{1/\tilde{p}},$$

and hence

$$N(u)(X_0, Y_0, t_0) \le c(M(|f|^{\tilde{p}})(X_0, Y_0, t_0))^{1/\tilde{p}}.$$

We can conclude that

$$\begin{aligned}\|N(u)\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\, d\sigma_{\mathcal{K}})} &\le c\|(M(|f|^{\tilde{p}}))^{1/\tilde{p}}\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\, d\sigma_{\mathcal{K}})}\\ &\le c\|f\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\, d\sigma_{\mathcal{K}})},\end{aligned} \tag{6-2}$$

by the continuity of the Hardy–Littlewood maximal function and where the constant c depends only on (m, κ, M, p). We now remove the restriction that $f \in C_0(\partial\Omega \times \mathbb{R}^m \times \mathbb{R})$. Indeed, given $f \in L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R}, d\sigma_{\mathcal{K}})$ there exist, by density of $C_0(\partial\Omega \times \mathbb{R}^m \times \mathbb{R})$ in $L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R}, d\sigma_{\mathcal{K}})$, a sequence of functions $\{f_j\}$, $f_j \in C_0(\partial\Omega \times \mathbb{R}^m \times \mathbb{R})$, converging to f in $L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R}, d\sigma_{\mathcal{K}})$. In particular, there exists a sequence of functions $\{u_j\}$, where u_j is the unique bounded weak solution to $\mathcal{L}_{\mathcal{K}} u_j = 0$ in $\Omega \times \mathbb{R}^m \times \mathbb{R}$, with $u_j = f_j$ on $\partial\Omega \times \mathbb{R}^m \times \mathbb{R}$. By (6-2),

$$\|N(u_k - u_l)\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\, d\sigma_{\mathcal{K}})} \le c\|f_k - f_l\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\, d\sigma_{\mathcal{K}})} \to 0 \quad \text{as } k, l \to \infty. \tag{6-3}$$

Consider $U_X \times U_Y \times J \subset \mathbb{R}^{N+1}$, where $U_X \subset \mathbb{R}^m$ and $U_Y \subset \mathbb{R}^m$ are bounded domains and $J = (a, b)$ with $-\infty < a < b < \infty$. Assume that $\overline{U_X \times U_Y \times J}$ is compactly contained in $\Omega \times \mathbb{R}^m \times \mathbb{R}$ and that the distance from $\overline{U_X \times U_Y \times J}$ to $\partial\Omega \times \mathbb{R}^m \times \mathbb{R}$ is $r > 0$. By a covering argument with cubes of size, say, $r/2$, Lemma 4.2, and the finiteness of $N(u_j)$ in $L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R}, d\sigma_{\mathcal{K}})$, it follows that $\{u_j\}$ is uniformly bounded in $L^2(U_X \times U_Y \times J)$, whenever $\overline{U_X \times U_Y \times J}$ is compactly contained in $\Omega \times \mathbb{R}^m \times \mathbb{R}$. Using this, and the energy estimate of Lemma 4.1, we can conclude that

$$\|\nabla_X u_j\|_{L^2(U_X\times U_Y\times J)} \text{ is uniformly bounded.} \tag{6-4}$$

Using (6-4) and the weak formulation of the equation $\mathcal{L}_{\mathcal{K}} u_j = 0$ it follows that $(X \cdot \nabla_Y - \partial_t)u_j$ is uniformly bounded, with respect to j, in $L^2_{Y,t}(U_Y \times J, H^{-1}_X(U_X))$. Let $W(U_X \times U_Y \times J)$ be defined as in (2-18). By the above argument we can conclude, whenever $\overline{U_X \times U_Y \times J}$ is compactly contained in $\Omega \times \mathbb{R}^m \times \mathbb{R}$, that

$$\|u_j\|_{W(U_X\times U_Y\times J)} \text{ is uniformly bounded.} \tag{6-5}$$

Using (6-3), and arguing as in the deductions in (6-4) and (6-5), we can also conclude that

$$\|u_k - u_l\|_{W(U_X\times U_Y\times J)} \to 0 \quad \text{as } k, l \to \infty. \tag{6-6}$$

Using (6-6) it follows that a subsequence $\{u_{j_k}\}$ of $\{u_j\}$ converges to a weak solution u to

$$\mathcal{L}_{\mathcal{K}} u = 0 \quad \text{in } \Omega \times \mathbb{R}^m \times \mathbb{R},$$

and that

$$\|N(u)\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\, d\sigma_{\mathcal{K}})} \le c\|f\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\, d\sigma_{\mathcal{K}})}.$$

Note also, using the notation introduced above, that

$$\|N(u - u_j)\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\, d\sigma_{\mathcal{K}})} \le c\|f - f_j\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\, d\sigma_{\mathcal{K}})} \to 0 \quad \text{as } j \to \infty. \tag{6-7}$$

To complete the proof that (i) implies (ii) we have to prove that $u = f$ n.t. on $\partial\Omega \times \mathbb{R}^m \times \mathbb{R}$. Consider $f \in L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\ \mathrm{d}\sigma_{\mathcal{K}})$ and let $\{f_j\}$, $f_j \in C_0(\partial\Omega \times \mathbb{R}^m \times \mathbb{R})$, be a sequence of functions converging to f in $L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\ \mathrm{d}\sigma_{\mathcal{K}})$. Let $(X_0, Y_0, t_0) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$ be a Lebesgue point of f. Given $\delta > 0$ we have

$$N_\delta(u-f)(X_0,Y_0,t_0) \le N_\delta(u-u_j)(X_0,Y_0,t_0)+N_\delta(u_j-f_j)(X_0,Y_0,t_0)+M(f-f_j)(X_0,Y_0,t_0), \quad (6\text{-}8)$$

where N_δ was introduced in (2-29) and $N_\delta(u-f)(X_0, Y_0, t_0)$ should be interpreted as

$$\sup_{(X,Y,t)\in\Gamma^\eta_\delta(X_0,Y_0,t_0)} |u(X,Y,t) - f(X_0,Y_0,t_0)|.$$

In the following we assume, as we may without loss of generality, that $(0, 0, 0) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$. Given $\epsilon > 0$ small and $R \gg 1$, let

$$S_\epsilon(R,\delta) := \{(X,Y,t) \in \Delta_R(0,0,0) : N_\delta(u-f)(X,Y,t) > \epsilon\}.$$

Using (6-8), weak estimates and (6-7) we deduce

$$\sigma_{\mathcal{K}}(S_\epsilon(R,\delta)) \le c\epsilon^{-p}(\|f-f_j\|^p_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\ \mathrm{d}\sigma_{\mathcal{K}})} + \|N_\delta(u_j-f_j)\|^p_{L^p(\Delta_R(0,0,0),\ \mathrm{d}\sigma_{\mathcal{K}})}). \quad (6\text{-}9)$$

Now letting $\delta \to 0$, $j \to \infty$, $R \to \infty$, in that order, we deduce that the set of points $(X_0, Y_0, t_0) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$ at which

$$\lim_{\substack{(X,Y,t)\in\Gamma^\eta(X_0,Y_0,t_0)\\(X,Y,t)\to(X_0,Y_0,t_0)}} |u(X,Y,t) - f(X_0,Y_0,t_0)| > \epsilon$$

has $\sigma_{\mathcal{K}}$ measure zero. As ϵ is arbitrary we can conclude that $u = f$ n.t. on $\partial\Omega \times \mathbb{R}^m \times \mathbb{R}$.

Next we prove that (ii) implies (i) and hence we assume that $D^p_{\mathcal{K}}(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\ \mathrm{d}\sigma_{\mathcal{K}})$ is solvable. Let $(X_0, Y_0, t_0) \in \partial\Omega$, $\Delta := \Delta_r(X_0, Y_0, t_0) \subset \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$ and $f \in C_0(\Delta)$, $f \ge 0$. Let u be the unique bounded solution to the Dirichlet problem with boundary data f. Then

$$u(A^+_{c\Delta,\Lambda}) = \iiint_\Delta K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, Y, t) f(X,Y,t)\, \mathrm{d}\sigma_{\mathcal{K}}(X,Y,t).$$

Using the estimate in Lemma 4.2, and (ii),

$$\begin{aligned} u(A^+_{c\Delta,\Lambda}) &\lesssim \left(\fint\!\!\!\iint_{Q_{r/c}(A^+_{c\Delta,\Lambda})} |u(X,Y,t)|^p \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t \right)^{1/p} \\ &\lesssim \left(\frac{1}{\sigma_{\mathcal{K}}(\Delta)} \iiint_{4\Delta} |N(u)(X,Y,t)|^p \,\mathrm{d}\sigma_{\mathcal{K}}(X,Y,t) \right)^{1/p} \\ &\lesssim \left(\frac{1}{\sigma_{\mathcal{K}}(\Delta)} \iiint_{\Delta} |f(X,Y,t)|^p \,\mathrm{d}\sigma_{\mathcal{K}}(X,Y,t) \right)^{1/p}. \end{aligned}$$

In particular, for all $f \in C_0(\Delta)$ with $\|f\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\ \mathrm{d}\sigma_{\mathcal{K}})} = 1$, we have

$$\left| \iiint_\Delta K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, Y, t) f(X,Y,t)\, \mathrm{d}\sigma_{\mathcal{K}}(X,Y,t) \right| \le \left(\frac{1}{\sigma_{\mathcal{K}}(\Delta)} \right)^{1/p}.$$

Hence, since $(\Delta, \sigma_{\mathcal{K}})$ is a finite measure space,

$$\biggl(\iiint_{\Delta} |K_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, X, Y, t)|^{q}\, \mathrm{d}\sigma_{\mathcal{K}}(X, Y, t)\biggr)^{1/q} \leq \biggl(\frac{1}{\sigma_{\mathcal{K}}(\Delta)}\biggr)^{1/p}.$$

Furthermore, Lemmas 4.5 and 4.6 imply

$$\iiint_{\Delta} K_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, X, Y, t)\, \mathrm{d}\sigma_{\mathcal{K}}(X, Y, t) = \omega_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, \Delta) \gtrsim 1.$$

Combining the estimates,

$$\biggl(\fint\!\!\fint\!\!\fint_{\Delta} |K_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, X, Y, t)|^{q}\, \mathrm{d}\sigma_{\mathcal{K}}(X, Y, t)\biggr)^{1/q} \lesssim \fint\!\!\fint\!\!\fint_{\Delta} K_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, X, Y, t)\, \mathrm{d}\sigma_{\mathcal{K}}(X, Y, t).$$

Hence $K_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, \cdot, \cdot, \cdot) \in B_q(\Delta, \mathrm{d}\sigma_{\mathcal{K}})$ and the proof that (ii) implies (i) is complete. Put together we have proved that the statements in Theorem 3.3(i) and (ii) are equivalent.

6A. ***Proof of the uniqueness statement in Theorem 3.3.*** Having proved that Theorem 3.3(i) and (ii) are equivalent it remains to prove that if $D^{p}_{\mathcal{K}}(\partial\Omega \times \mathbb{R}^m \times \mathbb{R}, \mathrm{d}\sigma_{\mathcal{K}})$ is solvable, then $D^{p}_{\mathcal{K}}(\partial\Omega \times \mathbb{R}^m \times \mathbb{R}, \mathrm{d}\sigma_{\mathcal{K}})$ is uniquely solvable. That is, we have to prove that if $N(u) \in L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R}, \mathrm{d}\sigma_{\mathcal{K}})$, and if u is a weak solution to the Dirichlet problem

$$\begin{cases} \mathcal{L}_{\mathcal{K}} u = 0 & \text{in } \Omega \times \mathbb{R}^m \times \mathbb{R}, \\ \quad u = 0 & \text{n.t. on } \partial\Omega \times \mathbb{R}^m \times \mathbb{R}, \end{cases}$$

then $u \equiv 0$ in $\Omega \times \mathbb{R}^m \times \mathbb{R}$. Note that the proof of this is considerably more involved compared to the corresponding arguments in the elliptic setting [Kenig 1994; Kenig and Shen 2011]. One reason is, again, the (time-)lag in the Harnack inequality for parabolic equations.

To start the proof we fix $(\widehat{X}, \widehat{Y}, \hat{t}) \in \Omega \times \mathbb{R}^m \times \mathbb{R}$ and we intend to prove that $u(\widehat{X}, \widehat{Y}, \hat{t}) = 0$. Let $\theta \in C_0^{\infty}(\Omega \times \mathbb{R}^m \times \mathbb{R})$ with $\theta = 1$ in a neighborhood of $(\widehat{X}, \widehat{Y}, \hat{t})$. Then, using (4-14),

$$\begin{aligned} u(\widehat{X}, \widehat{Y}, \hat{t}) &= (u\theta)(\widehat{X}, \widehat{Y}, \hat{t}) \\ &= -\iiint A(X)\nabla_X G_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, X, Y, t) \cdot \nabla_X (u\theta)(X, Y, t)\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}t \\ &\qquad + \iiint G_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, X, Y, t)(X \cdot \nabla_Y - \partial_t)(u\theta)(X, Y, t)\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}t. \end{aligned} \tag{6-10}$$

By the results in [Golse et al. 2019], see Lemma 4.3, we know that any weak solution to $\mathcal{L}_{\mathcal{K}} u = 0$ is Hölder continuous. As A is independent of (Y, t), it follows that partial derivatives of u with respect to Y and t are also weak solutions. As a consequence, as A is independent of (Y, t), any weak solution to $\mathcal{L}_{\mathcal{K}} u = 0$ is C^{∞}-smooth as a function of (Y, t). Hence the term $(X \cdot \nabla_Y - \partial_t)(u\theta)$ appearing in the last display is well-defined. Using (6-10), and that $\mathcal{L}_{\mathcal{K}} u = 0$,

$$|u(\widehat{X}, \widehat{Y}, \hat{t})| \lesssim (I + II + III), \tag{6-11}$$

where

$$
\begin{aligned}
I &:= \iiint |G_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, X, Y, t)||\nabla_X u(X, Y, t)||\nabla_X \theta(X, Y, t)| \,\mathrm{d}X \,\mathrm{d}Y \,\mathrm{d}t, \\
II &:= \iiint |\nabla_X G_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, X, Y, t)||u(X, Y, t)||\nabla_X \theta(X, Y, t)| \,\mathrm{d}X \,\mathrm{d}Y \,\mathrm{d}t, \\
III &:= \iiint |G_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, X, Y, t)||u(X, Y, t)||(\partial_t - X \cdot \nabla_Y)\theta(X, Y, t)| \,\mathrm{d}X \,\mathrm{d}Y \,\mathrm{d}t.
\end{aligned}
\tag{6-12}
$$

Recall the notation $Q := (-1, 1)^m \times (-1, 1)^m \times (-1, 1)$. Given $(\widehat{X}, \widehat{Y}, \hat{t}) = (\hat{x}, \hat{x}_m, \widehat{Y}, \hat{t}) \in \Omega \times \mathbb{R}^m \times \mathbb{R}$ fixed, we have

$$((\hat{x}, \psi(\hat{x})), \widehat{Y}, \hat{t}) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$$

fixed. We consider $Q_R((\hat{x}, \psi(\hat{x})), \widehat{Y}, \hat{t}) = ((\hat{x}, \psi(\hat{x})), \widehat{Y}, \hat{t}) \circ Q_R$ and we let ϵ and R satisfy

$$\epsilon < \lambda/8, \quad R > 8\lambda, \quad \text{where } \lambda := \hat{x}_m - \psi(\hat{x}).$$

When taking limits, we will always first let $\epsilon \to 0$ before letting $R \to \infty$.

Let $\varphi_1 = \varphi_1(X, Y, t) \in C_0^\infty(Q_{2R}((\hat{x}, \psi(\hat{x})), \widehat{Y}, \hat{t}))$, $0 \le \varphi_1 \le 1$, be such that $\varphi_1 \equiv 1$ on $Q_R((\hat{x}, \psi(\hat{x})), \widehat{Y}, \hat{t})$. Let $\varphi_2 = \varphi_2(X) = \varphi_2(x, x_m)$ be a smooth function with range $[0, 1]$ such that $\varphi_2(x, x_m) \equiv 1$ on $\{(x, x_m) : x_m \ge \psi(x) + 2\epsilon\}$ and $\varphi_2(x, x_m) \equiv 0$ on $\{(x, x_m) : x_m \le \psi(x) + \epsilon\}$. Note that φ_1 can be constructed so that $\|R\nabla_X \varphi_1\|_{L^\infty} + \|R^2(X \cdot \nabla_Y - \partial_t)\varphi_1\|_{L^\infty} \lesssim 1$. Similarly, φ_2 can be constructed so that $\|\epsilon \nabla_X \varphi_2\|_{L^\infty} \le c$, where c is independent of ϵ. We let

$$\theta = \theta(X, Y, t) = \theta(x, x_m, Y, t) := \varphi_1(X, Y, t)\varphi_2(x, x_m).$$

Then $\theta \in C_0^\infty(Q_{2R}((\hat{x}, \psi(\hat{x})), \widehat{Y}, \hat{t}))$, $0 \le \theta \le 1$, $\theta \equiv 1$ on the set of points $(X, Y, t) = (x, x_m, Y, t) \in Q_R((\hat{x}, \psi(\hat{x})), \widehat{Y}, \hat{t})$ which satisfy $x_m \ge \psi(x) + 2\epsilon$ and $\theta \equiv 0$ on the set of points in $(X, Y, t) = (x, x_m, Y, t) \in Q_R((\hat{x}, \psi(\hat{x})), \widehat{Y}, \hat{t})$ which satisfy $x_m \le \psi(x) + \epsilon$. Let

(i) $D_1 := Q_{2R}((\hat{x}, \psi(\hat{x})), \widehat{Y}, \hat{t}) \cap \{(X, Y, t) : \psi(x) + \epsilon < x_m < \psi(x) + 2\epsilon\}$,

(ii) $D_2 := Q_{2R}((\hat{x}, \psi(\hat{x})), \widehat{Y}, \hat{t}) \cap \{(X, Y, t) : \psi(x) + R < x_m < \psi(x) + 2R\}$,

(iii) $D_3 := D_4 \cap \{(X, Y, t) : \psi(x) + 2\epsilon \le x_m \le \psi(x) + R\}$,

where

$$D_4 := Q_{2R}((\hat{x}, \psi(\hat{x})), \widehat{Y}, \hat{t}) \setminus Q_R((\hat{x}, \psi(\hat{x})), \widehat{Y}, \hat{t}).$$

Using this notation, the domains where the integrands in I, II, and III are nonzero are contained in the union $D_1 \cup D_2 \cup D_3$. By the construction of θ,

(i′) $\|\epsilon \nabla_X \theta\|_{L^\infty(D_1)} + \|R^2(X \cdot \nabla_Y - \partial_t)\theta\|_{L^\infty(D_1)} \le c$,

(ii′) $\|R \nabla_X \theta\|_{L^\infty(D_2)} + \|R^2(X \cdot \nabla_Y - \partial_t)\theta\|_{L^\infty(D_2)} \le c$,

(iii′) $\|R \nabla_X \theta\|_{L^\infty(D_3)} + \|R^2(X \cdot \nabla_Y - \partial_t)\theta\|_{L^\infty(D_3)} \le c$,

where c is a constant which is independent of ϵ and R. Note that if $(X, Y, t) \in D_3$, then $\theta(X, Y, t) = \varphi_1(X, Y, t)$ and this explains (iii′).

Using the sets D_1, D_2, and D_3, and letting

$$G_{\mathcal{K}}(\cdot\,,\cdot\,,\cdot) := G_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, \cdot\,, \cdot, \cdot),$$

we see that

$$I + II + III \lesssim T_1 + T_2 + T_3, \tag{6-13}$$

where

$$\begin{aligned}
T_1 &:= \frac{1}{\epsilon^2}\iiint_{D_1} (\epsilon|G_{\mathcal{K}}||\nabla_X u| + \epsilon|\nabla_X G_{\mathcal{K}}||u| + \epsilon^2 R^{-2}|G_{\mathcal{K}}||u|)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t,\\
T_2 &:= \frac{1}{R^2}\iiint_{D_2} (R|G_{\mathcal{K}}||\nabla_X u| + R|\nabla_X G_{\mathcal{K}}||u| + |G_{\mathcal{K}}||u|)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t,\\
T_3 &:= \frac{1}{R^2}\iiint_{D_3} (R|G_{\mathcal{K}}||\nabla_X u| + R|\nabla_X G_{\mathcal{K}}||u| + |G_{\mathcal{K}}||u|)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t.
\end{aligned}$$

We need to estimate T_1, T_2, and T_3. To improve readability we will in the following use the notation

$$\Delta_\varrho := (\partial\Omega \times \mathbb{R}^m \times \mathbb{R}) \cap Q_\varrho((\hat{x}, \psi(\hat{x})), \widehat{Y}, \hat{t}) \quad \text{for } \varrho > 0.$$

We first consider T_1. We start by estimating the contribution from the term $|G_{\mathcal{K}}||u|$ and in this case we prove a harder estimate than we need. The argument will be used for further reference. Note that

$$\begin{aligned}
&\frac{1}{\epsilon^2}\iiint_{D_1} |G_{\mathcal{K}}||u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\\
&\qquad\lesssim \iiint_{\Delta_{2R}} \widetilde{N}_\epsilon(u)\biggl(\frac{1}{\epsilon}\int_{\psi(x)+\epsilon}^{\psi(x)+2\epsilon} \frac{G_{\mathcal{K}}((x, x_m), Y, t)}{\epsilon}\,\mathrm{d}x_m\biggr)\,\mathrm{d}\sigma_{\mathcal{K}}\\
&\qquad\lesssim \|\widetilde{N}_\epsilon(u)\|_{L^p(\Delta_{2R},\,\mathrm{d}\sigma_{\mathcal{K}})}\biggl(\iiint_{\Delta_{2R}} \biggl(\frac{1}{\epsilon}\int_{\psi(x)+\epsilon}^{\psi(x)+2\epsilon} \frac{G_{\mathcal{K}}((x, x_m), Y, t)}{\epsilon}\,\mathrm{d}x_m\biggr)^q\,\mathrm{d}\sigma_{\mathcal{K}}\biggr)^{1/q},
\end{aligned}$$

where $\widetilde{N}_\epsilon$ is a truncated maximal operator defined as

$$\widetilde{N}_\epsilon(u)(X, Y, t) := \sup_{\psi(x)<x_m<\psi(x)+2\epsilon} |u((x, x_m), Y, t)|.$$

Using Lemma 4.10 and the definition of $K_{\mathcal{K}}$, see (6-1), we have, for every $(X, Y, t) \in \Delta_{2R}$, $1 \le \sigma \le 2$, and denoting by e_m the unit vector in $\mathbb{R}^m$ pointing into Ω in the x_m-direction,

$$\lim_{\epsilon\to 0} \frac{G_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, X + \sigma\epsilon e_m, Y, t)}{\epsilon} \lesssim \lim_{\epsilon\to 0} \frac{\omega_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, \Delta_{c\sigma\epsilon}(X, Y, t))}{\epsilon^{q-1}} \lesssim K_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, X, Y, t).$$

Note that if $\hat{t} \le t$, then this is trivial as the left-hand side is identically zero. If $\hat{t} > t$, then we may apply Lemma 4.10 in the deduction as we are considering the limiting situation $\epsilon \to 0$. Using these estimates, and Lebesgue's theorem on dominated convergence, we obtain

$$\begin{aligned}
&\limsup_{\epsilon\to 0} \frac{1}{\epsilon^2}\iiint_{D_1} |G_{\mathcal{K}}||u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\\
&\qquad\lesssim \bigl(\limsup_{\epsilon\to 0} \|\widetilde{N}_\epsilon(u)\|_{L^p(\Delta_{2R},\,\mathrm{d}\sigma_{\mathcal{K}})}\bigr)\|K_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, \cdot\,, \cdot, \cdot)\|_{L^q(\Delta_{2R},\,\mathrm{d}\sigma_{\mathcal{K}})} = 0,
\end{aligned} \tag{6-14}$$

as u vanishes at the boundary in the nontangential sense. We next consider the term

$$\frac{1}{\epsilon}\iiint_{D_1}|G_{\mathcal{K}}||\nabla_X u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t.$$

In this case, we first note, using Lemma 4.6 and the construction of D_1, that if ϵ is small enough, then

$$G_{\mathcal{K}}(X,Y,t)=G_{\mathcal{K}}(\widehat{X},\widehat{Y},\hat{t},X,Y,t)\lesssim(R/\lambda)^{\gamma}G_{\mathcal{K}}(A^{+}_{c\Delta_R,\Lambda},X,Y,t),\tag{6-15}$$

whenever $(X,Y,t)\in D_1$. Let $\{Q_j\}$ be all Whitney cubes in a Whitney decomposition of $\Omega\times\mathbb{R}^m\times\mathbb{R}$ which intersects D_1. Then $|Q_j|\approx\epsilon^{q}$. Using (6-15) and Hölder's inequality

$$\begin{aligned}\frac{1}{\epsilon}&\iiint_{D_1}|G_{\mathcal{K}}||\nabla_X u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\\ &\lesssim(R/\lambda)^{\gamma}\frac{1}{\epsilon}\sum_j\iiint_{Q_j}G_{\mathcal{K}}(A^{+}_{c\Delta_R,\Lambda},X,Y,t)|\nabla_X u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\\ &\lesssim(R/\lambda)^{\gamma}\frac{1}{\epsilon}\sum_j\biggl(\iiint_{Q_j}|G_{\mathcal{K}}(A^{+}_{c\Delta_R,\Lambda},X,Y,t)|^2\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\biggr)^{1/2}\biggl(\iiint_{Q_j}|\nabla_X u|^2\,\mathrm{d}X\mathrm{d}Y\,\mathrm{d}t\biggr)^{1/2}.\end{aligned}\tag{6-16}$$

Using the adjoint version of Lemmas 4.6, and 4.11, we see that

$$\sup_{4Q_j}G_{\mathcal{K}}(A^{+}_{c\Delta_R,\Lambda},X,Y,t)\lesssim\inf_{4Q_j}G_{\mathcal{K}}(A^{+}_{c\Delta_R,\Lambda},X,Y,t).\tag{6-17}$$

Furthermore, using the energy estimate of Lemma 4.1, assuming that the Whitney decomposition is such that $8Q_j\subset\Omega\times\mathbb{R}^m\times\mathbb{R}$,

$$\iiint_{Q_j}|\nabla_X u|^2\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\lesssim\epsilon^{-2}\iiint_{2Q_j}|u|^2\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\lesssim\epsilon^{-2}|Q_j|(\sup_{2Q_j}|u|)^2.\tag{6-18}$$

Using (6-16)–(6-18) we deduce

$$\frac{1}{\epsilon}\iiint_{D_1}|G_{\mathcal{K}}||\nabla_X u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\lesssim(R/\lambda)^{\gamma}\frac{1}{\epsilon^2}\sum_j|Q_j|\bigl(\inf_{4Q_j}G_{\mathcal{K}}(A^{+}_{c\Delta_R,\Lambda},X,Y,t)\bigr)\bigl(\sup_{2Q_j}|u(X,Y,t)|\bigr).\tag{6-19}$$

Using Lemma 4.2

$$\sup_{2Q_j}|u|\lesssim\biggl(\fint\!\!\fint\!\!\fint_{4Q_j}|u|\,\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\biggr).\tag{6-20}$$

This inequality in combination with (6-19) imply that

$$\frac{1}{\epsilon}\iiint_{D_1}|G_{\mathcal{K}}||\nabla_X u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\lesssim(R/\lambda)^{\gamma}\frac{1}{\epsilon^2}\iiint_{\widetilde{D}_1}G_{\mathcal{K}}(A^{+}_{c\Delta_R,\Lambda},X,Y,t)|u(X,Y,t)|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t,\tag{6-21}$$

where $\widetilde{D}_1$ is the enlargement of D_1 defined as the union of the cubes $\{4Q_j\}$. We can now repeat the argument leading up to (6-14), with $G_{\mathcal{K}}$ replaced by $G_{\mathcal{K}}(A^{+}_{c\Delta_R,\Lambda},\cdot,\cdot,\cdot)$ and with D_1 replaced by $\widetilde{D}_1$, to conclude that

$$\limsup_{\epsilon\to0}\frac{1}{\epsilon}\iiint_{D_1}|G_{\mathcal{K}}||\nabla_X u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t=0.\tag{6-22}$$

The remaining term in T_1 can be handled analogously and hence we can conclude that

$$T_1 \to 0, \quad \text{as } \epsilon \to 0. \tag{6-23}$$

Next we consider T_2 and we first consider the contribution from the term

$$\frac{1}{R^2}\iiint_{D_2} |G_{\mathcal{K}}|\,|u| \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t. \tag{6-24}$$

In this case we first note, using Lemma 4.9, that

$$G_{\mathcal{K}}(X,Y,t) = G_{\mathcal{K}}(\widehat{X},\widehat{Y},\hat{t},X,Y,t) \le \Gamma_{\mathcal{K}}(\widehat{X},\widehat{Y},\hat{t},X,Y,t) \lesssim R^{2-\boldsymbol{q}},$$

whenever $(X,Y,t)\in D_2$. Hence,

$$\begin{aligned}\frac{1}{R^2}\iiint_{D_2} |G_{\mathcal{K}}|\,|u| \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t &\lesssim R^{1-\boldsymbol{q}}\iiint_{\Delta_{2R}} N(u)\,\mathrm{d}\sigma_{\mathcal{K}}\\ &\lesssim R^{1-\boldsymbol{q}}R^{(\boldsymbol{q}-1)(1-1/p)}\|N(u)\|_{L^p(\Delta_{2R},\,\mathrm{d}\sigma_{\mathcal{K}})}\\ &= R^{(1-\boldsymbol{q})/p}\|N(u)\|_{L^p(\Delta_{2R},\,\mathrm{d}\sigma_{\mathcal{K}})} \to 0, \quad \text{as } R\to\infty.\end{aligned}$$

We next consider the contribution from the term

$$\frac{1}{R}\iiint_{D_2} |G_{\mathcal{K}}|\,|\nabla_X u| \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t.$$

Using the energy estimate of Lemma 4.1, as well as Lemma 4.2,

$$\biggl(\iiint_{D_2} |\nabla_X u|^2 \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\biggr)^{1/2} \lesssim R^{-1-\boldsymbol{q}/2}\iiint_{\widetilde{D}_2} |u| \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t,$$

where $\widetilde{D}_2$ is an enlargement of D_2. Using this, and also again using the bound on $G_{\mathcal{K}}$ stated above, we see that

$$\begin{aligned}\frac{1}{R}\iiint_{D_2} |G_{\mathcal{K}}|\,|\nabla_X u| \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t &\lesssim R^{1-\boldsymbol{q}/2}R^{-1-\boldsymbol{q}/2}\iiint_{\widetilde{D}_2} |u| \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\\ &\lesssim R^{1-\boldsymbol{q}}\iiint_{\Delta_{4R}} |N(u)|\,\mathrm{d}\sigma_K\\ &\lesssim R^{1-\boldsymbol{q}}R^{(\boldsymbol{q}-1)/q}\|N(u)\|_{L^p(\Delta_{4R},\,\mathrm{d}\sigma_{\mathcal{K}})}\\ &\lesssim R^{(1-\boldsymbol{q})/p}\|N(u)\|_{L^p(\Delta_{4R},\,\mathrm{d}\sigma_{\mathcal{K}})} \to 0, \quad \text{as } R\to\infty.\end{aligned}$$

The remaining term in T_2 can be handled analogously and hence we can conclude that

$$T_2 \to 0, \quad \text{as } R\to\infty. \tag{6-25}$$

Finally we consider T_3. The term in T_3 containing the integrand $|G_{\mathcal{K}}|\,|u|$ can be handled as we handled the term in (6-24). For the other terms we first recall that by construction $G_{\mathcal{K}}(\widehat{X},\widehat{Y},\hat{t},X,Y,t)\neq 0$ if and only if $t<\hat{t}$. Furthermore, for $(X,Y,t)\in D_3$ fixed, $G_{\mathcal{K}}(\cdot\,,\cdot\,,\cdot, X,Y,t)$ is a nonnegative solution to $\mathcal{L}_{\mathcal{K}}u=0$ in $(\Omega\times\mathbb{R}^m\times\mathbb{R})\cap Q_R((\hat{x},\psi(\hat{x})),\widehat{Y},\hat{t})$. In particular, if R is large enough, then by Theorem 4.7 we have that

$$G_{\mathcal{K}}(\widehat{X},\widehat{Y},\hat{t},X,Y,t) \lesssim G_{\mathcal{K}}(A^{+}_{c^{-1}\Delta_R,\Lambda},X,Y,t), \tag{6-26}$$

whenever $(X,Y,t)\in D_3$ and we can ensure that $A^+_{c^{-1}\Delta_R,\Lambda}\subset Q_{R/2}((\hat{x},\psi(\hat{x})),\widehat{Y},\hat{t})$. To proceed we let $C=C(m)\gg 1$ be a large but fixed constant, and we introduce

$$D_3^*:=D_3\cap\{(X,Y,t):\psi(x)+2\epsilon\le x_m\le\psi(x)+R/C\}.$$

Then the domain of integration in the terms defining T_3 is partitioned into integration over D_3^* and $D_3\setminus D_3^*$. Integration over the latter set can be handled as we handled T_2. Therefore we here only consider the remaining terms in T_3 but with domain of integration defined by D_3^*. We now let $\{Q_j\}$ be all Whitney cubes in a Whitney decomposition of $\Omega\times\mathbb{R}^m\times\mathbb{R}$ which intersects D_3^*. Focusing on the term in T_3 containing the integrand $|G_{\mathcal{K}}||\nabla_X u|$ we see that

$$\begin{aligned}\frac{1}{R}\iiint_{D_3^*}&|G_{\mathcal{K}}||\nabla_X u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\\ &\le\frac{1}{R}\sum_j\iiint_{Q_j\cap D_3^*}|G_{\mathcal{K}}||\nabla_X u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\\ &\lesssim\frac{1}{R}\sum_j|Q_j|^{1/2}l(Q_j)^{-1}\biggl(\iiint_{Q_j\cap D_3^*}|G_{\mathcal{K}}|^2\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\biggr)^{1/2}\biggl(\fint\!\!\fint\!\!\fint_{4Q_j}|u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\biggr)\\ &\lesssim\frac{1}{R}\sum_j|Q_j|l(Q_j)^{-1}\bigl(\sup_{Q_j}G_{\mathcal{K}}(A^+_{c^{-1}\Delta_R,\Lambda},X,Y,t)\bigr)\biggl(\fint\!\!\fint\!\!\fint_{4Q_j}|u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\biggr),\end{aligned}\tag{6-27}$$

where we have used Lemma 4.1, Lemma 4.2 and (6-26). Furthermore, (6-17) remains valid in this context and hence

$$\begin{aligned}\bigl(\sup_{Q_j}G_{\mathcal{K}}(A^+_{c^{-1}\Delta_R,\Lambda},X,Y,t)\bigr)&\biggl(\fint\!\!\fint\!\!\fint_{4Q_j}|u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\biggr)\\ &\lesssim\biggl(\fint\!\!\fint\!\!\fint_{4Q_j}G_{\mathcal{K}}(A^+_{c^{-1}\Delta_R,\Lambda},X,Y,t)|u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\biggr).\end{aligned}\tag{6-28}$$

Combining these insights we see, using the notation $\delta(X):=(x_m-\psi(x))$, that

$$\begin{aligned}\frac{1}{R}\iiint_{D_3^*}|G_{\mathcal{K}}||\nabla_X u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t&\lesssim\frac{1}{R}\sum_j l(Q_j)^{-1}\biggl(\iiint_{4Q_j}G_{\mathcal{K}}(A^+_{c^{-1}\Delta_R,\Lambda},X,Y,t)|u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\biggr)\\ &\lesssim\frac{1}{R}\biggl(\iiint_{\widetilde{D}_3^*}G_{\mathcal{K}}(A^+_{c^{-1}\Delta_R,\Lambda},X,Y,t)|u|\delta(X)^{-1}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\biggr),\end{aligned}\tag{6-29}$$

where $\widetilde{D}_3^*$ is a slight enlargement of D_3^* due to the enlargement from Q_j to $4Q_j$. In particular,

$$\frac{1}{R}\iiint_{D_3^*}|G_{\mathcal{K}}||\nabla_X u|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\lesssim\frac{1}{R}\biggl(\iiint_{D_5}G_{\mathcal{K}}(A^+_{c^{-1}\Delta_R,\Lambda},X,Y,t)|u|\delta(X)^{-1}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\biggr),\tag{6-30}$$

where D_5 is defined as the set

$$(\Omega\times\mathbb{R}^m\times\mathbb{R})\cap\bigl(Q_{cR}(\widehat{X},\widehat{Y},\hat{t})\setminus\{(X,Y,t):(x,\psi(x),Y,t)\in\Delta_{R/c},\ \psi(x)\le x_m<\psi(x)+2cR\}\bigr)$$

for some $c = c(m) \gg 1$. Note that points in D_5 can be represented as

$$(X, Y, t) = ((x, \psi(x)), Y, t) + (0, \delta(X), 0, 0),$$

where $((x, \psi(x)), Y, t) \in \Delta_{cR} \setminus \Delta_{R/c}$. Consider one such point (X, Y, t). We claim that

$$G_{\mathcal{K}}(A^+_{c^{-1}\Delta_R,\Lambda}, X, Y, t)\delta(X)^{-1} \lesssim M\big(K_{\mathcal{K}}(A^+_{c^{-1}\Delta_R,\Lambda}, \cdot)\chi_{\Delta_{2cR}\setminus\Delta_{R/(2c)}}(\cdot)\big)((x, \psi(x)), Y, t), \tag{6-31}$$

where again M denotes the Hardy–Littlewood maximal function on $\partial\Omega \times \mathbb{R}^m \times \mathbb{R}$ with respect to $\sigma_{\mathcal{K}}$, and $\chi_{\Delta_{cR}\setminus\Delta_{R/c}}(\cdot)$ is the indicator function for the set $\Delta_{cR} \setminus \Delta_{R/c}$. To prove (6-31) we simply note, using Lemma 4.10, that

$$G_{\mathcal{K}}(A^+_{c^{-1}\Delta_R,\Lambda}, X, Y, t)\delta(X)^{-1} \lesssim \frac{\omega_{\mathcal{K}}(A^+_{c^{-1}\Delta_R,\Lambda}, \Delta_{cr}((x, \psi(x)), Y, t))}{\sigma_{\mathcal{K}}(\Delta_{cr}((x, \psi(x)), Y, t))},$$

where $r := \delta(X)$, and that $\omega_{\mathcal{K}}(A^+_{c^{-1}\Delta_R,\Lambda}, \Delta_{cr}((x, \psi(x)), Y, t))$ can be expressed as

$$\iiint_{\Delta_{cr}((x,\psi(x)),Y,t)} K_{\mathcal{K}}(A^+_{c^{-1}\Delta_R,\Lambda}, \widetilde{X}, \widetilde{Y}, \tilde{t})\, \mathrm{d}\sigma_{\mathcal{K}}(\widetilde{X}, \widetilde{Y}, \tilde{t})$$
$$= \iiint_{\Delta_{cr}((x,\psi(x)),Y,t)} K_{\mathcal{K}}(A^+_{c^{-1}\Delta_R,\Lambda}, \widetilde{X}, \widetilde{Y}, \tilde{t})\chi_{\Delta_{2cR}\setminus\Delta_{R/(2c)}}(\widetilde{X}, \widetilde{Y}, \tilde{t})\, \mathrm{d}\sigma_{\mathcal{K}}(\widetilde{X}, \widetilde{Y}, \tilde{t}).$$

Using (6-31) we can continue the estimate in (6-30) to conclude that

$$\frac{1}{R}\iiint_{D_3^*} |G_{\mathcal{K}}||\nabla_X u|\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}t \lesssim \iiint_{\Delta_{cR}\setminus\Delta_{R/c}} M\big(K_{\mathcal{K}}(A^+_{c^{-1}\Delta_R,\Lambda}, \cdot)\chi_{\Delta_{2cR}\setminus\Delta_{R/(2c)}}(\cdot)\big)N(u)\, \mathrm{d}\sigma_{\mathcal{K}}.$$

Hence, the term on the left-hand side in the last display can estimated by

$$\bigg(\iiint_{\Delta_{cR}\setminus\Delta_{R/c}} |K_{\mathcal{K}}(A^+_{c^{-1}\Delta_R,\Lambda}, \cdot)|^q\, \mathrm{d}\sigma_{\mathcal{K}}\bigg)^{1/q}\bigg(\iiint_{\Delta_{cR}\setminus\Delta_{R/c}} |N(u)|^p\, \mathrm{d}\sigma_{\mathcal{K}}\bigg)^{1/p}$$
$$\lesssim (\sigma_{\mathcal{K}}(\Delta_{cR}))^{1/q-1}\bigg(\iiint_{\Sigma\setminus\Delta_{R/c}} |N(u)|^p\, \mathrm{d}\sigma_{\mathcal{K}}\bigg)^{1/p} \to 0,$$

as $R \to \infty$. This completes the estimate of the term in T_3 containing the integrand $|G_{\mathcal{K}}||\nabla_X u|$. The term containing the integrand $|\nabla_X G_{\mathcal{K}}||u|$ can be estimated in a similar manner. We omit further details and claim that

$$T_3 \to 0, \quad \text{as } R \to \infty. \tag{6-32}$$

To summarize, we have proved that

$$|u(\widehat{X}, \widehat{Y}, \hat{t})| \lesssim \lim_{R\to\infty} \limsup_{\epsilon\to 0} (I + II + III) \lesssim \lim_{R\to\infty} \limsup_{\epsilon\to 0} (T_1 + T_2 + T_3) = 0; \tag{6-33}$$

i.e., $|u(\widehat{X}, \widehat{Y}, \hat{t})| = 0$, and as $(\widehat{X}, \widehat{Y}, \hat{t})$ is an arbitrary but fixed point in the argument, we can conclude that $u \equiv 0$ in $\Omega \times \mathbb{R}^m \times \mathbb{R}$. This completes the proof of uniqueness and hence the proof of Theorem 3.3.

7. An alternative proof of Theorem 1.1 along the lines of [Fabes and Salsa 1983]

In this section we give, as we believe that the argument may be of independent interest in the case of operators of Kolmogorov type, a proof of the key estimate underlying Theorem 1.1 using Rellich-type inequalities instead of the structural theorem. Hence, the proof is along the lines of the corresponding proof for the heat equation in [Fabes and Salsa 1983]. To avoid formal calculations and manipulations we will, for simplicity, throughout the section assume

$$\text{(A1)–(A3) and that } \partial\Omega \text{ is } C^\infty\text{-smooth.} \tag{7-1}$$

The assumptions in (7-1) will only be used in a qualitative fashion and the constants of our quantitative estimates will only depend on m, κ and M. The general case follows by approximation arguments that we leave to the interested reader.

In addition to (7-1) we also assume (1-5), i.e., that A is independent of x_m. Then the unique bounded solution to the Dirichlet problem $\mathcal{L}_{\mathcal{K}}u = 0$ in $\Omega\times\mathbb{R}^m\times\mathbb{R}$, $u = f \in C_0(\partial\Omega\times\mathbb{R}^m\times\mathbb{R})$, equals

$$u(\widehat{X},\widehat{Y},\hat{t}) = \iiint_{\partial\Omega\times\mathbb{R}^m\times\mathbb{R}} K_{\mathcal{K}}(\widehat{X},\widehat{Y},\hat{t},X,Y,t)\,f(X,Y,t)\,\mathrm{d}\sigma_{\mathcal{K}}(X,Y,t),$$

and due to (7-1),

$$K_{\mathcal{K}}(\widehat{X},\widehat{Y},\hat{t},X,Y,t) := A(x)\nabla_X G_{\mathcal{K}}(\widehat{X},\widehat{Y},\hat{t},X,Y,t)\cdot N(X)$$

for all $(X,Y,t)\in\partial\Omega\times\mathbb{R}^m\times\mathbb{R}$ and where $N(X)$ is the outer unit normal to $\partial\Omega$ at $X\in\partial\Omega$.

We are going to prove that if $\Delta := \Delta_r(X_0,Y_0,t_0)\subset\partial\Omega\times\mathbb{R}^m\times\mathbb{R}$, then

$$\biggl(⨍\!\!\!\!\iiint_{\tilde\Delta} |K_{\mathcal{K}}(A^+_{c\Delta,\Lambda},X,Y,t)|^2\,\mathrm{d}\sigma_{\mathcal{K}}(X,Y,t)\biggr)^{1/2} \lesssim \biggl(⨍\!\!\!\!\iiint_{\tilde\Delta} |K_{\mathcal{K}}(A^+_{c\Delta,\Lambda},X,Y,t)|\,\mathrm{d}\sigma_{\mathcal{K}}(X,Y,t)\biggr) \tag{7-2}$$

for all $\tilde\Delta\subset\Delta$. In fact, we claim that it suffices to prove (7-2) for $\tilde\Delta=\Delta$. To see this, we assume that (7-2) holds for all Δ with $\tilde\Delta$ replaced by Δ, and we start by noting that we have the representations

$$\begin{aligned} K_{\mathcal{K}}(A^+_{c\Delta,\Lambda},X,Y,t) &= A(x)\nabla_X G_{\mathcal{K}}(A^+_{c\Delta,\Lambda},X,Y,t)\cdot N(X)\\ &= \frac{\mathrm{d}\omega_{\mathcal{K}}}{\mathrm{d}\sigma_{\mathcal{K}}}(A^+_{c\Delta,\Lambda},X,Y,t) = \lim_{\hat r\to 0}\frac{\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda},\Delta_{\hat r}(X,Y,t))}{\sigma_{\mathcal{K}}(\Delta_{\hat r}(X,Y,t))}\end{aligned}$$

for $(X,Y,t)\in\Delta$. Consider $(X,Y,t)\in\tilde\Delta$ and $\hat r>0$ small. Writing $\hat\Delta := \Delta_{\hat r}(X,Y,t)$ and

$$\frac{\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda},\hat\Delta)}{\sigma_{\mathcal{K}}(\hat\Delta)} = \frac{\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda},\hat\Delta)\;\omega_{\mathcal{K}}(A^+_{c\tilde\Delta,\Lambda},\hat\Delta)}{\omega_{\mathcal{K}}(A^+_{c\tilde\Delta,\Lambda},\hat\Delta)\;\sigma_{\mathcal{K}}(\hat\Delta)}, \tag{7-3}$$

we first apply Lemma 4.10 to deduce

$$\frac{\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda},\hat\Delta)}{\omega_{\mathcal{K}}(A^+_{c\tilde\Delta,\Lambda},\hat\Delta)} \lesssim \frac{G_{\mathcal{K}}(A^+_{c\Delta,\Lambda},A^-_{c\hat\Delta,\Lambda})}{G_{\mathcal{K}}(A^+_{c\tilde\Delta,\Lambda},A^-_{c\hat\Delta,\Lambda})}. \tag{7-4}$$

Next, applying Theorem 4.8 in (7-4), and passing to the limit by letting $\hat{r} \to 0$ in (7-3),

$$K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, Y, t) \lesssim \frac{G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^-_{4\tilde{\Delta},\Lambda})}{G_{\mathcal{K}}(A^+_{c\tilde{\Delta},\Lambda}, A^-_{4\tilde{\Delta},\Lambda})} K_{\mathcal{K}}(A^+_{c\tilde{\Delta},\Lambda}, X, Y, t).$$

Using this, and (7-2) with Δ replaced by $\tilde{\Delta}$ (which holds by the assumption), we deduce

$$\left(\fint\!\!\fint\!\!\fint_{\tilde{\Delta}} |K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, Y, t)|^2 \, \mathrm{d}\sigma_{\mathcal{K}}(X, Y, t)\right)^{1/2} \lesssim \frac{G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^-_{4\tilde{\Delta},\Lambda})}{G_{\mathcal{K}}(A^+_{c\tilde{\Delta},\Lambda}, A^-_{4\tilde{\Delta},\Lambda})\, \sigma_{\mathcal{K}}(\tilde{\Delta})}. \tag{7-5}$$

However, again using the bound $G_{\mathcal{K}}(A^+_{c\tilde{\Delta},\Lambda}, A^-_{4\tilde{\Delta},\Lambda}) \gtrsim \tilde{r}^{2-\boldsymbol{q}}$, see (5-7), we see that

$$\frac{G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^-_{4\tilde{\Delta},\Lambda})}{G_{\mathcal{K}}(A^+_{c\tilde{\Delta},\Lambda}, A^-_{4\tilde{\Delta},\Lambda})} \frac{1}{\sigma_{\mathcal{K}}(\tilde{\Delta})} \lesssim \tilde{r}^{-1} G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^-_{4\tilde{\Delta},\Lambda}). \tag{7-6}$$

Next, using Lemma 4.11, Lemma 4.10 and Theorem 4.12, in that order, we deduce

$$G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, A^-_{4\tilde{\Delta},\Lambda}) \lesssim \tilde{r}^{2-\boldsymbol{q}} \omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \tilde{\Delta}), \tag{7-7}$$

and hence, by combining the estimates above, see that

$$\left(\fint\!\!\fint\!\!\fint_{\tilde{\Delta}} |K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, Y, t)|^2 \, \mathrm{d}\sigma_{\mathcal{K}}(X, Y, t)\right)^{1/2} \lesssim \frac{\omega_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, \tilde{\Delta})}{\sigma_{\mathcal{K}}(\tilde{\Delta})}, \tag{7-8}$$

which completes the proof of our claim.

Based on the above it remains to prove (7-2) for $\tilde{\Delta} = \Delta$ and the rest of the proof is devoted to this. We note that we can without loss of generality assume that $(X_0, Y_0, t_0) = (0, 0, 0)$. A key observation in the following argument, and this is a consequence of A and Ω being independent of (Y, t), is that

$$K_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, X, Y, t) \text{ depends on } (\widehat{Y}, \hat{t}, Y, t) \text{ only through the differences } (\widehat{Y} - Y), (\hat{t} - t).$$

In particular,

$$K_{\mathcal{K}}(\widehat{X}, \widehat{Y}, \hat{t}, X, Y, t) = K_{\mathcal{K}}(\widehat{X}, \widehat{Y} - Y, \hat{t} - t, X, 0, 0). \tag{7-9}$$

Note that Δ is invariant under the change of coordinates $(X, Y, t) \to (X, -Y, -t)$. Hence,

$$\begin{aligned} I &:= \iiint_{\Delta} |K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, Y, t)|^2 \, \mathrm{d}\sigma_{\mathcal{K}}(X, Y, t) \\ &= (-1)^{m+1} \iiint_{\Delta} |K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, -Y, -t)|^2 \, \mathrm{d}\sigma_{\mathcal{K}}(X, Y, t). \end{aligned}$$

Using (7-9), Harnack's inequality, i.e., Lemma 4.4, and more specifically Lemma 4.6, we see that

$$K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, -Y, -t) \lesssim K_{\mathcal{K}}(A^+_{4c\Delta,\Lambda}, X, Y, t)$$

for all $(X, Y, t) \in \Delta$. Hence,

$$|K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, -Y, -t)|^2 \lesssim K_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, -Y, -t) K_{\mathcal{K}}(A^+_{4c\Delta,\Lambda}, X, Y, t) \tag{7-10}$$

for all $(X, Y, t) \in \Delta$. Let

$$\phi \in C_0^\infty(\mathbb{R}^{N+1} \setminus (\{A^+_{c\Delta,\Lambda}\} \cup \{A^+_{4c\Delta,\Lambda}\}))$$

be such that

$$\phi(X, Y, t) = 1, \tag{7-11}$$

whenever $(X, Y, t) = ((x, x_m), Y, t)$ is such that $(x, Y, t) \in [-r, r]^{m-1} \times [-r^3, r^3]^m \times [-r^2, r^2]$, $x_m \in [\psi(x) - r/16, \psi(x) + r/16]$, and

$$\phi(X, Y, t) = 0, \tag{7-12}$$

whenever $(X, Y, t) = ((x, x_m), Y, t)$ is in the complement of the set defined through the restrictions $(x, Y, t) \in [-2r, 2r]^{m-1} \times [-(2r)^3, (2r)^3]^m \times [-(2r)^2, (2r)^2]$, $x_m \in [\psi(x) - r/8, \psi(x) + r/8]$. Furthermore, we choose ϕ so that

$$|\nabla_X \phi(X, Y, t)| \lesssim r^{-1}, \quad |(X \cdot \nabla_Y - \partial_t)\phi(X, Y, t)| \lesssim r^{-2}, \tag{7-13}$$

whenever $(X, Y, t) \in \mathbb{R}^{N+1}$. We introduce

$$v(X, Y, t) := G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, -Y, -t), \quad \tilde{v}(X, Y, t) := G_{\mathcal{K}}(A^+_{4c\Delta,\Lambda}, X, Y, t), \tag{7-14}$$

and

$$\Psi(X, Y, t) := \phi(X, Y, t)\, \partial_{x_m} v(X, Y, t). \tag{7-15}$$

Recalling that $\mathcal{L}^*_{X,Y,t} = \nabla_X \cdot (A(X)\nabla_X) - X \cdot \nabla_Y + \partial_t$ and using the definition of the Green's function, we see that

$$\begin{aligned} 0 &= \iiint_{\Omega\times\mathbb{R}^m\times\mathbb{R}} \mathcal{L}^* G_{\mathcal{K}}(A^+_{4c\Delta,\Lambda}, X, Y, t)\Psi(X, Y, t)\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}t \\ &= \iiint_{\Omega\times\mathbb{R}^m\times\mathbb{R}} \mathcal{L}^* \tilde{v}(X, Y, t)\Psi(X, Y, t)\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}t. \end{aligned}$$

Hence

$$\begin{aligned} 0 = \iiint_{\Omega\times\mathbb{R}^m\times\mathbb{R}} \big(\mathcal{L}^* \tilde{v}(X, Y, t)\Psi(X, Y, t) - \tilde{v}(X, Y, t)\mathcal{L}\Psi(X, Y, t)\big)\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}t \\ + \iiint_{\Omega\times\mathbb{R}^m\times\mathbb{R}} \tilde{v}(X, Y, t)\mathcal{L}\Psi(X, Y, t)\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}t. \end{aligned}$$

Using this identity, and integrating by parts,

$$\begin{aligned} 0 = \iiint_{\partial\Omega\times\mathbb{R}^m\times\mathbb{R}} K_{\mathcal{K}}(A^+_{4c\Delta,\Lambda}, X, Y, t)\Psi(X, Y, t)\, \mathrm{d}\sigma_{\mathcal{K}}(X, Y, t) \\ + \iiint_{\Omega\times\mathbb{R}^m\times\mathbb{R}} \tilde{v}(X, Y, t)\mathcal{L}\Psi(X, Y, t)\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}t. \end{aligned} \tag{7-16}$$

Note that by construction, $\Psi(X, Y, t) = \partial_{x_m} v(X, Y, t)$ if $(X, Y, t) \in \Delta$. Consider the vector field $A(x)N(X)$. Obviously, $A(x)N(X) \cdot N(X) \le \kappa$ by the boundedness of A and hence we can write

$$e_m = T(X) + c(X)A(x)N(X)$$

for all $(X, Y, t) \in \Delta$ and for some function $c(\cdot)$ such that $c(X) \ge c(m, \kappa, M)$ for all $(X, Y, t) \in \Delta$. Here $T(X)$ denotes a vector tangent to $\partial\Omega$ at X. Using these observations we see that

$$\Psi(X, Y, t) = \partial_{x_m} v(X, Y, t) = c(X)A(x)N(X) \cdot \nabla_X v(X, Y, t),$$

whenever $(X, Y, t) \in \Delta$. In particular, using this and the fact that $K_{\mathcal{K}}(A^+_{c\Delta,\Lambda})$ and Ψ are nonnegative functions,

$$I \lesssim \left| \iiint_{\Omega\times\mathbb{R}^m\times\mathbb{R}} \tilde{v}(X, Y, t)\mathcal{L}\Psi(X, Y, t)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t \right|.$$

We next observe that

$$\begin{aligned}\mathcal{L}\Psi(X, Y, t) &= (\nabla_X(A(x)\nabla_X) + X\cdot\nabla_Y - \partial_t)\Psi \\ &= 2A(X)\nabla_X(\partial_{x_m} v)\nabla_X\phi + \partial_{x_m} v\mathcal{L}\phi + \phi\mathcal{L}(\partial_{x_m} v),\end{aligned}$$

and that

$$\mathcal{L}v(X, Y, t) = \mathcal{L}(G(A_{c\Delta,\Lambda}, X, -Y, -t)) = (\mathcal{L}^* G_{\mathcal{K}})(A_{c\Delta,\Lambda}, X, -Y, -t) = 0.$$

Using this we see that

$$\mathcal{L}(\partial_{x_m} v) = \mathcal{L}(\partial_{x_m} v) - \partial_{x_m}\mathcal{L}(v) = \partial_{y_m} v.$$

In particular,

$$\mathcal{L}\Psi(X, Y, t) = 2A(x)\nabla_X(\partial_{x_m} v)\nabla_X\phi + \partial_{x_m} v\mathcal{L}\phi + \phi\partial_{y_m} v.$$

We note that these calculations essentially only use that A is independent of x_m. Recall that ϕ satisfies (7-11)–(7-13) and let

$$E = (\Omega\times\mathbb{R}^m\times\mathbb{R}) \cap \overline{\{(X, Y, t) : \phi(X, Y, t) \neq 0\}}.$$

Using this notation and elementary manipulations,

$$I \lesssim I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}I_1 &:= r^{-2}\iiint_E |\nabla_X G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, -Y, -t)|G_{\mathcal{K}}(A^+_{4c\Delta,\Lambda}, X, Y, t)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t,\\ I_2 &:= r^{-1}\iiint_E |\nabla_X G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, -Y, -t)||\nabla_X G_{\mathcal{K}}(A^+_{4c\Delta,\Lambda}, X, Y, t)|\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t,\\ I_3 &:= r^{-1}\iiint_E |\nabla_X\partial_{x_m} G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, -Y, -t)|G_{\mathcal{K}}(A^+_{4c\Delta,\Lambda}, X, Y, t)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t,\\ I_4 &:= \iiint_E |\partial_{y_m} G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, -Y, -t)|G_{\mathcal{K}}(A^+_{4c\Delta,\Lambda}, X, Y, t)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t.\end{aligned}$$

Using the energy estimate of Lemma 4.1, and that

$$|G_{\mathcal{K}}(A^+_{c\Delta,\Lambda}, X, -Y, -t)| + |G_{\mathcal{K}}(A^+_{4c\Delta,\Lambda}, X, Y, t)| \lesssim r^{2-\boldsymbol{q}},$$

whenever $(X, Y, t) \in E$, we deduce that

$$I_1 + I_2 \lesssim \sigma_{\mathcal{K}}(\Delta)^{-1}.$$

Similarly, using a slightly more involved argument, a Whitney decomposition, Lemma 4.1 and the fact that A is independent of x_m, we can proceed in a manner similar to the proof of Lemma 2.6 in [Nyström 2017] to also deduce that

$$I_3 + I_4 \lesssim \sigma_{\mathcal{K}}(\Delta)^{-1}.$$

Putting these estimates together we can conclude that

$$\iiint_{\Delta} |K_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, X, Y, t)|^2 \, \mathrm{d}\sigma_{\mathcal{K}}(X, Y, t) = I \lesssim \sigma_{\mathcal{K}}(\Delta)^{-1},$$

whenever $\Delta \subset \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$. Furthermore, as $1 \lesssim \omega_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, \Delta)$, we have

$$\biggl(⨌_{\Delta} |K_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, X, Y, t)|^2 \, \mathrm{d}\sigma_{\mathcal{K}}\biggr)^{1/2} \lesssim \biggl(⨌_{\Delta} |K_{\mathcal{K}}(A^{+}_{c\Delta,\Lambda}, X, Y, t)| \, \mathrm{d}\sigma_{\mathcal{K}}\biggr),$$

which is (7-2) with $\tilde{\Delta} = \Delta$. This completes the proof.

8. Applications to homogenization: Theorem 1.3

By making the change of variables $(X, Y, t) \mapsto (\widetilde{X}, \widetilde{Y}, \tilde{t})$, $(X, Y, t) = (\epsilon\widetilde{X}, \epsilon^3\widetilde{Y}, \epsilon^2\tilde{t})$, the boundary

$$\partial\Omega \times \mathbb{R}^m \times \mathbb{R} = \{(X, Y, t) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} : x_m = \psi(x)\}$$

is transformed into

$$\partial\Omega_\epsilon \times \mathbb{R}^m \times \mathbb{R} := \{(\widetilde{X}, \widetilde{Y}, \tilde{t}) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} : \tilde{x}_m = \psi_\epsilon(\tilde{x})\},$$

where $\psi_\epsilon(x) := \epsilon^{-1}\psi(\epsilon x)$. Note that ψ and ψ_ϵ have the same Lipschitz constant. Let

$$v_\epsilon(\widetilde{X}, \widetilde{Y}, \tilde{t}) := u_\epsilon(X, Y, t), \qquad f_\epsilon(\tilde{x}, \psi_\epsilon(\tilde{x}), \widetilde{Y}, \tilde{t}) := f(x, \psi(x), Y, t).$$

Then,

$$\begin{cases} \mathcal{L}^{\epsilon}_{\mathcal{K}} u_\epsilon = 0 & \text{in } \Omega \times \mathbb{R}^m \times \mathbb{R}, \\ \quad u_\epsilon = f & \text{n.t. on } \partial\Omega \times \mathbb{R}^m \times \mathbb{R}, \end{cases} \tag{8-1}$$

where $\mathcal{L}^{\epsilon}_{\mathcal{K}}$ is as in (1-12), if and only if

$$\begin{cases} \mathcal{L}^{1}_{\mathcal{K}} v_\epsilon = 0 & \text{in } \Omega_\epsilon \times \mathbb{R}^m \times \mathbb{R}, \\ \quad v_\epsilon = f_\epsilon & \text{n.t. on } \partial\Omega_\epsilon \times \mathbb{R}^m \times \mathbb{R}. \end{cases} \tag{8-2}$$

By Theorem 1.2 we see that (8-2) has a unique weak solution which satisfies

$$\|N(v_\epsilon)\|_{L^p(\partial\Omega_\epsilon \times \mathbb{R}^m \times \mathbb{R},\, \mathrm{d}\sigma_{\mathcal{K}})} \lesssim \|f_\epsilon\|_{L^p(\partial\Omega_\epsilon \times \mathbb{R}^m \times \mathbb{R},\, \mathrm{d}\sigma_{\mathcal{K}})}.$$

Changing back to the (X, Y, t)-coordinates, we get that (8-1) has a unique weak solution satisfying the estimate

$$\|N(u_\epsilon)\|_{L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\, \mathrm{d}\sigma_{\mathcal{K}})} \lesssim \|f\|_{L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\, \mathrm{d}\sigma_{\mathcal{K}})}, \tag{8-3}$$

and in the last two displays the implicit constants are also allowed to depend on p, but are independent of ϵ and f. This settles the proof of the first part of Theorem 1.3.

To settle the proof of the second part of Theorem 1.3 we want to let $\epsilon \to 0$ and prove, given $f \in L^p(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\, \mathrm{d}\sigma_{\mathcal{K}})$, that $u_\epsilon \to \bar{u}$ and that $\bar{u}$ is a weak solution to the Dirichlet problem

$$\begin{cases} \bar{\mathcal{L}}_{\mathcal{K}} \bar{u} = 0 & \text{in } \Omega \times \mathbb{R}^m \times \mathbb{R}, \\ \quad \bar{u} = f & \text{n.t. on } \partial\Omega \times \mathbb{R}^m \times \mathbb{R}, \end{cases} \tag{8-4}$$

and that

$$\|N(\bar{u})\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\,\mathrm{d}\sigma_{\mathcal{K}})} \lesssim \|f\|_{L^p(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\,\mathrm{d}\sigma_{\mathcal{K}})}, \tag{8-5}$$

where the implicit constant also is allowed to depend on p. Note that $\bar{A}$ is a constant matrix, and once existence is established, uniqueness for the problem stated follows from the uniqueness part of Theorem 3.3. We also note that in the following it suffices to consider the case $p = 2$, again by the classical arguments in [Coifman and Fefferman 1974].

Consider $U_X \times U_Y \times J \subset \mathbb{R}^{N+1}$, where $U_X \subset \mathbb{R}^m$ and $U_Y \subset \mathbb{R}^m$ are bounded domains and $J = (a, b)$, with $-\infty < a < b < \infty$. Assume that $\overline{U_X \times U_Y \times J}$ is contained in $\Omega \times \mathbb{R}^m \times \mathbb{R}$ and that the distance from $\overline{U_X \times U_Y \times J}$ to $\partial\Omega \times \mathbb{R}^m \times \mathbb{R}$ is $r > 0$. By a covering argument with cubes of size, say, $r/2$, Lemma 4.2, and (8-3), it follows that u_ϵ is uniformly bounded, with respect to ϵ, in $L^2(U_X \times U_Y \times J)$, whenever $\overline{U_X \times U_Y \times J} \subset \Omega \times \mathbb{R}^m \times \mathbb{R}$. Using this, and the energy estimate of Lemma 4.1, we can conclude that

$$\|\nabla_X u_\epsilon\|_{L^2(U_X\times U_Y\times J)} \text{ is uniformly bounded in } \epsilon. \tag{8-6}$$

Next, using (8-6) and the weak formulation of the equation $\mathcal{L}^\epsilon_{\mathcal{K}} u_\epsilon = 0$ it follows that $(X \cdot \nabla_Y - \partial_t)u_\epsilon$ is uniformly bounded, with respect to ϵ, in $L^2_{Y,t}(U_Y \times J, H^{-1}_X(U_X))$. Let $W(U_X \times U_Y \times J)$ be defined as in (2-18). By the above argument we can conclude, whenever $\overline{U_X \times U_Y \times J}$ is compactly contained in $\Omega \times \mathbb{R}^m \times \mathbb{R}$, that

$$\|u_\epsilon\|_{W(U_X\times U_Y\times J)} \text{ is uniformly bounded in } \epsilon, \tag{8-7}$$

and, by ellipticity of A^ϵ, that

$$\|A^\epsilon \nabla_X u_\epsilon\|_{(L^2_{X,Y,t}(U_X\times U_Y\times J))^m} \text{ is uniformly bounded in } \epsilon. \tag{8-8}$$

Using the Sobolev embedding theorem one can prove that there exists a compact injection

$$W(U_X \times U_Y \times J) \to L^2(U_X \times U_Y \times J). \tag{8-9}$$

Using this, (8-7) and (8-8) we see that there exists a subsequence of $\{u_\epsilon\}$, still denoted by $\{u_\epsilon\}$, such that

$$\begin{aligned} u_\epsilon &\to \bar{u} && \text{in } L^2(U_X \times U_Y \times J), \\ A^\epsilon \nabla_X u_\epsilon &\to \xi && \text{weakly in } (L^2(U_X \times U_Y \times J))^m, \\ (X \cdot \nabla_Y - \partial_t)u_\epsilon &\to (X \cdot \nabla_Y - \partial_t)\bar{u} && \text{weakly in } L^2_{Y,t}(U_Y \times J, H^{-1}_X(U_X)). \end{aligned} \tag{8-10}$$

In particular,

$$u_\epsilon \to \bar{u} \quad \text{weakly in } W(U_X \times U_Y \times J).$$

Furthermore, using this and the local regularity estimate in Lemma 4.3 we also have that

$$u_\epsilon \to \bar{u}, \quad \text{locally uniformly in } \Omega \times \mathbb{R}^m \times \mathbb{R} \text{ as } \epsilon \to 0.$$

We now have sufficient information to pass to the limit in the weak formulation of the equation $\mathcal{L}^\epsilon_{\mathcal{K}} u_\epsilon = 0$ and doing so we obtain

$$0 = \iiint_{U_X\times U_Y\times J} \xi \cdot \nabla_X \phi \;\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t + \iint_{U_Y\times J} \langle(-X \cdot \nabla_Y + \partial_t)\bar{u}(\,\cdot\,, Y, t), \phi(\,\cdot\,, Y, t)\rangle \,\mathrm{d}Y\,\mathrm{d}t \tag{8-11}$$

for all $\phi \in L^2_{Y,t}(U_Y \times J, H^1_{X,0}(U_X))$. We need to show that $\xi = \bar{A}\nabla_X \bar{u}$. To this end, we consider the functions

$$w^\epsilon_\alpha(X) := \epsilon w_\alpha(X/\epsilon), \tag{8-12}$$

with w_α defined as in (1-11). Following [Cioranescu and Donato 1999], we see that

$$\begin{aligned} w^\epsilon_\alpha &\to \alpha \cdot X \quad \text{weakly in } H^1_X(U_X),\\ w^\epsilon_\alpha &\to \alpha \cdot X \quad \text{in } L^2(U_X). \end{aligned} \tag{8-13}$$

In particular

$$A^\epsilon \nabla_X w^\epsilon_\alpha \to \bar{A}\alpha \quad \text{weakly in } (L^2(U_X))^m \tag{8-14}$$

and

$$\int A^\epsilon(X)\nabla_X w^\epsilon_\alpha \cdot \nabla_X \phi \,\mathrm{d}X = 0 \tag{8-15}$$

for all $\phi \in C^\infty_0(U_X)$; see [Cioranescu and Donato 1999, Section 8.1].

Pick $\varphi \in C^\infty_0(U_X)$, $\psi \in C^\infty_0(U_Y \times J)$. We choose $\phi = \varphi u_\epsilon \psi$ in (8-15), and integrate with respect to Y and t:

$$0 = \iiint (A^\epsilon(X)\nabla_X w^\epsilon_\alpha \cdot \nabla_X u_\epsilon)\varphi\psi \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t + \iiint (A^\epsilon(X)\nabla_X w^\epsilon_\alpha \cdot \nabla_X \varphi)u_\epsilon\psi \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t. \tag{8-16}$$

Picking $\varphi w^\epsilon_\alpha \psi$ as a test function in the weak formulation of $\mathcal{L}^\epsilon_{\mathcal{K}} u_\epsilon = 0$ yields

$$\begin{aligned} 0 = \iiint ((A^\epsilon(X)\nabla_X u_\epsilon \cdot \nabla_X w^\epsilon_\alpha)\varphi\psi + (A^\epsilon(X)\nabla_X u_\epsilon \cdot \nabla_X \varphi) w^\epsilon_\alpha \psi) \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t&\\ + \iiint (X \cdot \nabla_Y \psi - \partial_t \psi)\varphi w^\epsilon_\alpha u_\epsilon \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t&, \end{aligned}$$

where we have used that φ and w^ϵ_α only depend on X and that ψ only depends on Y and t. Subtracting the expression in the last display from (8-16) yields

$$\begin{aligned} 0 = \iiint ((A^\epsilon(X)\nabla_X w^\epsilon_\alpha \cdot \nabla_X \varphi)u_\epsilon\psi - (A^\epsilon(X)\nabla_X u_\epsilon \cdot \nabla_X \varphi) w^\epsilon_\alpha \psi) \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t&\\ - \iiint (X \cdot \nabla_Y \psi - \partial_t \psi)\varphi w^\epsilon_\alpha u_\epsilon \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t&. \end{aligned} \tag{8-17}$$

Using (8-10), (8-13), and (8-14), we see that

$$\begin{aligned} \iiint ((A^\epsilon(X)\nabla_X w^\epsilon_\alpha \cdot \nabla_X \varphi)u_\epsilon\psi) \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t &\to \iiint ((\bar{A}\alpha \cdot \nabla_X \varphi)\bar{u}\psi) \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t,\\ \iiint (A^\epsilon(X)\nabla_X u_\epsilon \cdot \nabla_X \varphi) w^\epsilon_\alpha \psi \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t &\to \iiint (\xi \cdot \nabla_X \varphi)(\alpha \cdot X)\psi \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t,\\ \iiint (X \cdot \nabla_Y \psi - \partial_t \psi)\varphi w^\epsilon_\alpha u_\epsilon \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t &\to \iiint (X \cdot \nabla_Y \psi - \partial_t \psi)(\alpha \cdot X)\varphi\bar{u} \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t, \end{aligned}$$

as $\epsilon \to 0$; i.e., passing to the limit in (8-17) we obtain

$$\iiint \big((\bar{A}\alpha \cdot \nabla_X \varphi)\bar{u}\psi - (\xi \cdot \nabla_X \varphi)(\alpha \cdot X)\psi - (X \cdot \nabla_Y \psi - \partial_t \psi)(\alpha \cdot X)\varphi\bar{u}\big) \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t = 0.$$

Using that

$$(\nabla_X\varphi)(\alpha\cdot X)\psi=\nabla_X(\varphi(\alpha\cdot X)\psi)-\alpha\varphi\psi,$$

and (8-11), now with $\phi=(\alpha\cdot X)\varphi\psi$ as test function, we get

$$\iiint ((\bar{A}\alpha\cdot\nabla_X\varphi)\bar{u}\psi-(\xi\cdot\alpha)\varphi\psi)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t=0. \tag{8-18}$$

Since $\bar{A}$ is constant, this implies that

$$\xi\cdot\alpha=(\bar{A}\nabla_X\bar{u})\cdot\alpha\quad\text{for all }\alpha\in\mathbb{R}^m,$$

and consequently, $\xi=\bar{A}\nabla_X\bar{u}$. In particular, $\{u_\epsilon\}_{\epsilon>0}$ has a subsequence that converges weakly to $\bar{u}$ and $\bar{u}$ is a weak solution to $\bar{\mathcal{L}}_{\mathcal{K}}\bar{u}=0$ in $\Omega\times\mathbb{R}^m\times\mathbb{R}$.

Next, assume that $f\in C_0(\partial\Omega\times\mathbb{R}^m\times\mathbb{R})$. Then

$$u_\epsilon(X,Y,t)=\iiint K_\epsilon(X,Y,t,\widetilde{X},\widetilde{Y},\tilde{t})f(\widetilde{X},\widetilde{Y},\tilde{t})\,\mathrm{d}\sigma_{\mathcal{K}}(\widetilde{X},\widetilde{Y},\tilde{t}), \tag{8-19}$$

and we need to extract a convergent subsequence from the sequence of kernels $\{K_\epsilon\}$. Using the representation in (8-19) we see that if

$$(X,Y,t)\in U_X\times U_Y\times J\quad\text{and}\quad\operatorname{dist}(U_X,\partial\Omega\times\mathbb{R}^m\times\mathbb{R})\ge 2r, \tag{8-20}$$

then as above, i.e., again using a covering argument, Lemma 4.2 and (8-3), we deduce that

$$\left|\iiint K_\epsilon(X,Y,t,\widetilde{X},\widetilde{Y},\tilde{t})f(\widetilde{X},\widetilde{Y},\tilde{t})\,\mathrm{d}\sigma_{\mathcal{K}}(\widetilde{X},\widetilde{Y},\tilde{t})\right|=|u_\epsilon(X,Y,t)|\le c\|f\|_{L^2(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\,\mathrm{d}\sigma_{\mathcal{K}})}$$

for some positive constant $c<\infty$ independent of ϵ. It thus follows by duality that

$$\|K_\epsilon(X,Y,t,\cdot,\cdot,\cdot)\|_{L^2(\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\,\mathrm{d}\sigma_{\mathcal{K}})}$$

is bounded uniformly in ϵ for (X,Y,t) as in (8-20). This clearly implies that

$$\|K_\epsilon\|_{L^2(U_X\times U_Y\times J\times\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\,\mathrm{d}\sigma_{\mathcal{K}})}$$

is bounded uniformly in ϵ. Thus, for a subsequence,

$$K_\epsilon\to\overline{K},\quad\text{as }\epsilon\to 0,\quad\text{weakly in }L^2(U_X\times U_Y\times J\times\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\ \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\,\mathrm{d}\sigma_{\mathcal{K}}).$$

Suppose now that $\{u_{\epsilon_j}\}$ converges weakly in $W(U_X\times U_Y\times J)$ to $\bar{u}$. Then, by the above argument there exists a subsequence $\{\epsilon_{j'}\}$ of $\{\epsilon_j\}$ such that $K_{\epsilon_{j'}}$ converges weakly to $\overline{K}$ in

$$L^2(U_X\times U_Y\times J\times\partial\Omega\times\mathbb{R}^m\times\mathbb{R},\ \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\,\mathrm{d}\sigma_{\mathcal{K}}).$$

This implies, as $u_\epsilon(X,Y,t)\to\bar{u}(X,Y,t)$, and by continuity for all (X,Y,t) as in (8-20), that

$$\begin{aligned}u_\epsilon(X,Y,t)&=\iiint K_\epsilon(X,Y,t,\widetilde{X},\widetilde{Y},\tilde{t})f(\widetilde{X},\widetilde{Y},\tilde{t})\,\mathrm{d}\sigma_{\mathcal{K}}(\widetilde{X},\widetilde{Y},\tilde{t})\\&\to\iiint \overline{K}(X,Y,t,\widetilde{X},\widetilde{Y},\tilde{t})f(\widetilde{X},\widetilde{Y},\tilde{t})\,\mathrm{d}\sigma_{\mathcal{K}}(\widetilde{X},\widetilde{Y},\tilde{t})=\bar{u}(X,Y,t),\end{aligned}$$

as $\epsilon \to 0$ and for all (X, Y, t) as in (8-20). As $U_X \times U_Y \times J$ is arbitrary in this argument, we conclude that for a certain subsequence of $\{u_\epsilon\}_{\epsilon>0}$,

$$u_\epsilon \to \bar{u} \quad \text{weakly in } W_{\text{loc}}(\Omega \times \mathbb{R}^m \times \mathbb{R}),$$

and

$$K_\epsilon \to \overline{K} \quad \text{weakly in } L^2_{\text{loc}}(\Omega \times \mathbb{R}^m \times \mathbb{R} \times \partial\Omega \times \mathbb{R}^m \times \mathbb{R},\ \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\,\mathrm{d}\sigma_{\mathcal{K}}). \tag{8-21}$$

Furthermore,

$$\overline{\mathcal{L}}_{\mathcal{K}}\bar{u} = 0 \quad \text{in } \Omega \times \mathbb{R}^m \times \mathbb{R},$$

and

$$\bar{u}(X, Y, t) = \iiint \overline{K}(X, Y, t, \widetilde{X}, \widetilde{Y}, \tilde{t}) f(\widetilde{X}, \widetilde{Y}, \tilde{t})\,\mathrm{d}\sigma_{\mathcal{K}}(\widetilde{X}, \widetilde{Y}, \tilde{t}),$$

whenever $(X, Y, t) \in \Omega \times \mathbb{R}^m \times \mathbb{R}$. Note that the space

$$L^2_{\text{loc}}(\Omega \times \mathbb{R}^m \times \mathbb{R} \times \partial\Omega \times \mathbb{R}^m \times \mathbb{R},\ \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\,\mathrm{d}\sigma_{\mathcal{K}})$$

in (8-21) should be interpreted as local only in the first three variables X, Y and t. As $\bar{A}$ is a constant matrix, the Kolmogorov measure $\omega_{\overline{\mathcal{L}}_{\mathcal{K}}}$ is absolutely continuous with respect to $\sigma_{\mathcal{K}}$ and this can be seen as a consequence of Theorem 1.1. In particular, the problem $D^2_{\mathcal{K}}(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\ \mathrm{d}\sigma_{\mathcal{K}})$ is uniquely solvable for the operator $\overline{\mathcal{L}}_{\mathcal{K}}$ and $\overline{K}(X, Y, t, \widetilde{X}, \widetilde{Y}, \tilde{t})$ is the Radon–Nikodym derivative of the Kolmogorov measure $\omega_{\overline{\mathcal{L}}_{\mathcal{K}}}(X, Y, t, \cdot)$ with respect to $\sigma_{\mathcal{K}}$ at $(\widetilde{X}, \widetilde{Y}, \tilde{t}) \in \partial\Omega \times \mathbb{R}^m \times \mathbb{R}$. As a consequence, using Theorem 3.3 we can conclude that for $f \in C_0(\partial\Omega \times \mathbb{R}^m \times \mathbb{R})$ given, $\bar{u}$ is the unique solution to the problem in (8-4) which satisfies (8-5). For $f \in L^2(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\ \mathrm{d}\sigma_{\mathcal{K}})$ the same conclusion follows from the density of $C_0(\partial\Omega \times \mathbb{R}^m \times \mathbb{R})$ in $L^2(\partial\Omega \times \mathbb{R}^m \times \mathbb{R},\ \mathrm{d}\sigma_{\mathcal{K}})$; see the final part in the proof of (i) implies (ii) in Theorem 3.3 for reference. Summing up, the proof of Theorem 1.3 is complete.

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DIMENSION-FREE HARNACK INEQUALITIES FOR CONJUGATE HEAT EQUATIONS AND THEIR APPLICATIONS TO GEOMETRIC FLOWS

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Let M be a differentiable manifold endowed with a family of complete Riemannian metrics $g(t)$ evolving under a geometric flow over the time interval $[0, T[$. We give a probabilistic representation for the derivative of the corresponding conjugate semigroup on M which is generated by a Schrödinger-type operator. With the help of this derivative formula, we derive fundamental Harnack-type inequalities in the setting of evolving Riemannian manifolds. In particular, we establish a dimension-free Harnack inequality and show how it can be used to achieve heat kernel upper bounds in the setting of moving metrics. Moreover, by means of the supercontractivity of the conjugate semigroup, we obtain a family of canonical log-Sobolev inequalities. We discuss and apply these results both in the case of the so-called modified Ricci flow and in the case of general geometric flows.

1. Introduction

Let M be a differentiable manifold endowed with a C^1 family of complete Riemannian metrics $g(t)$ indexed by the real interval $[0, T[$, where $T \in \,]0, \infty]$. The family $g(t)$ describes the evolution of the manifold M under a geometric flow where T is the first time where possibly a blow-up of the curvature occurs. This type of singularity is not excluded in our setting.

More specifically, we consider geometric flows of the type

$$\partial_t g(t) = -2\,\mathrm{Sc}(t) \quad \text{on } M \times [0, T[,$$

where $\mathrm{Sc}(t) = (S_{ij})$ is a general time-dependent symmetric $(0, 2)$-tensor. For fixed t, with respect to the metric $g(t)$, let $S = g^{ij} S_{ij}$ be the metric trace of the tensor $S(t)$ and Δ_t the Laplace–Beltrami operator acting on functions on M. In practice, the geometric flow deforms the geometry of M and smooths out irregularities in the metric to give it a nicer and more symmetric form, which provides geometric and topological information on the manifold.

Consider operators of the form $L_t = \Delta_t - \nabla^t \phi_t$, where ϕ_t is a time-dependent function on M. We also use the notation $g_t = g(t)$, $S_t = S(t, \cdot)$ and $\mathrm{Sc}_t = \mathrm{Sc}(t)$. In this paper we study the (minimal) fundamental solution to heat equations of the type

$$(L_t - \partial_t)u(t, x) = 0, \quad \text{resp. } (L_t + \partial_t - \varrho_t)u(t, x) = 0,$$

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where $\varrho_t = \partial_t \phi_t + S_t$. The first equation is the classical heat equation, the second one appears naturally as conjugate heat equation. More precisely, we have the following relationship.

Remark 1.1. Set $d\mu_t = \mathrm{e}^{-\phi_t} d\,\mathrm{vol}_t$, where vol_t denotes the Riemannian volume to the metric $g(t)$. Let $\Box = L_t - \partial_t$ be the standard heat operator and $\Box^*$ its formal adjoint with respect to the measure $\mu_t \otimes dt$. Thus,

$$\int_0^T \int_M V \,\Box\, U \,d\mu_t\,dt = \int_0^T \int_M U \,\Box^*\, V \,d\mu_t\,dt$$

for functions $U, V \in C_c^{2,1}(M \times [0, T[)$. From this relation it is immediate that $\Box^* = L_t + \partial_t - \varrho_t$.

Example 1.2. A typical situation covered by this setting is solving a geometric flow equation (e.g., Ricci flow) forward in time and the associated conjugate heat equation backward in time. In the case of the Ricci flow $\partial_t g(t) = -2\,\mathrm{Ric}_t$ and $L_t = \Delta_t$, the conjugate heat equation reads as $\Box^* u = (\Delta_t + \partial_t - R)u = 0$, where $R = \mathrm{tr}\,\mathrm{Ric}$ denotes the (time-dependent) scalar curvature.

It should be mentioned that an important ingredient in the proof of the Poincaré conjecture by Perelman [2002] is a differential Harnack inequality which is related to a gradient estimate for solutions to the conjugate heat equation under forward Ricci flow on a compact manifold. This relation has been one of our motivations to investigate solutions to conjugate heat equations and their associated semigroups also by methods of stochastic analysis.

Let X_t be the diffusion process generated by $L_t = \Delta_t - \nabla^t \phi_t$ (called L_t-diffusion process) which we assume to be nonexplosive up to time T. We consider the two-parameter semigroup associated to L_t,

$$P_{s,t} f(x) := \mathbb{E}[f(X_t) \mid X_s = x], \quad s \le t,$$

which satisfies the heat equation

$$\begin{cases} \dfrac{\partial}{\partial s} P_{s,t} f = -L_s P_{s,t} f, \\ \lim_{s \to t} P_{s,t} f = f. \end{cases}$$

In previous work, we already studied properties of heat equations under geometric flows, like properties of the semigroup $P_{s,t}$ generated by L_t, by adapting probabilistic methods. In [Cheng 2017], for instance, the first author gave functional inequalities equivalent to a lower bound of $\mathrm{Ric}_t + \mathrm{Sc}_t$. In [Cheng and Thalmaier 2018a; 2018c] we established characterizations of upper and lower bounds for $\mathrm{Ric}_t + \mathrm{Sc}_t$ in terms of functional inequalities on the path space over M.

On a more probabilistic side, in [Cheng and Thalmaier 2018b] the authors studied existence and uniqueness of so-called evolution systems of measures related to the semigroup $P_{s,t}$. Using such systems as reference measures, contractivity of the semigroup, as well as log-Sobolev inequality, have been investigated.

Although the evolution system of measures is helpful to shed light on properties of solutions to the heat equation, it is still difficult to obtain a full picture of this measure system, like its relation to the system of volume measures. It has been observed that if one uses volume measures as reference measures, many questions will be related to the conjugate heat equation and not the usual heat equation; see e.g., [Abolarinwa 2015; Cao et al. 2015; Kuang and Zhang 2008].

Recall that $\mu_t(dx) = e^{-\phi_t(x)}\, d\,\mathrm{vol}_t$, where vol_t is the volume measure with respect to the metric $g(t)$. Let

$$P^{\varrho}_{s,t} f(x) = \mathbb{E}\left[\exp\left(-\int_s^t \varrho(r, X_r)\, dr\right) f(X_t) \;\middle|\; X_s = x\right],$$

where $\varrho(t,x) = \varrho_t(x)$ is given by $\varrho_t := \partial_t \phi_t + S_t$ and

$$\frac{\partial}{\partial t}\mu_t(dx) = -\varrho(t,x)\,\mu_t(dx).$$

According to the Feynman–Kac formula, $P^{\varrho}_{s,t} f$ represents the solution to the equation

$$\frac{\partial}{\partial s}\varphi_s = -(L_s - \varrho_s)\varphi_s, \quad \varphi_t = f,$$

on $[0,t] \times M$ where $t < T$. We note that this equation is conjugate to the heat equation

$$\frac{\partial}{\partial s} u(s,x) = L_s u(s,\cdot)(x).$$

In this paper, we first give probabilistic formulas and estimates for $dP^{\varrho}_{s,t} f$ from which we then derive a dimension-free Harnack inequality. It is interesting to note that by combining the dimension-free Harnack inequalities for $P_{s,t}$ and $P^{\varrho}_{s,t}$, one can obtain new on-diagonal and Gaussian upper bounds for the heat kernel to L_t with respect to μ_t; see Sections 5 and 6. We apply these results then to the following modified geometric flow for g_t combined with the conjugate heat equation for ϕ_t (see, e.g., Corollary 5.3 below), i.e.,

$$\begin{cases} \partial_t g_t = -2(\mathrm{Sc} + \mathrm{Hess}(\phi))_t, \\ \partial_t \phi_t = -(\Delta\phi - S)(t,x). \end{cases} \tag{1-1}$$

As is well known, for $\mathrm{Sc} = \mathrm{Ric}$, this flow represents the gradient flow to Perelman's famous entropy functional, also known as Perelman's $\mathcal{F}$-functional [2002].

Before we give other applications of these Harnack inequalities, let us first compare our results on heat kernel estimates with the known results in this direction. In [Coulibaly-Pasquier 2019], the author used a horizontal coupling of curves to obtain a dimension-free Harnack inequality for $P_{s,t}$ generated by Δ_t, and applied it then to on-diagonal heat kernel estimates by following Grigoryan's argument [1997]. The first difference to our approach is that we use the dimension-free Harnack inequality for the conjugate heat semigroup instead of comparing $P_{s,t}$ to $P^{\varrho}_{s,t}$ by controlling the absolute value of the potential $|\varrho|$. The second difference is that in [Coulibaly-Pasquier 2019], the author used the midpoint $(t+s)/2$ as reference time, so that lower bounds for both $\mathrm{Ric}_t + \mathrm{Sc}_t$ and $\mathrm{Ric}_t - \mathrm{Sc}_t$ for $t \in [0,T]$ are required. Here, in our approach, we adopt the method of [Grigoryan 1997] as well, but the reference time r has greater flexibility. For instance, the reference time r can be chosen close to s so that if one knows that there exist constants κ and K such that $|d\varrho_s| \le \kappa$ and $\mathrm{Ric}_s + \mathrm{Sc}_s + \mathrm{Hess}_s(\phi_s) \ge K$ at the initial time s, then by choosing r close to s, an on-diagonal heat kernel estimate can still be derived under the assumption that $\mathrm{Ric}_t - \mathrm{Sc}_t + \mathrm{Hess}_t(\phi_t) \ge K_1(t)$ for $t \in [0,T]$. For instance, if the geometric flow is a Ricci flow and $\phi = 0$, then $K_1 \equiv 0$ and solely information from the initial manifold is basically enough to derive upper heat kernel bounds. With respect to this point of view, the necessary conditions in our results could be weakened significantly.

Recently, Buzano and Yudowitz [2020] established Gaussian-type heat kernel estimates under a general geometric flow by adapting the methods from [Davies 1987]. In their paper, in the case of a general geometric flow, they assume for vector fields X on M the tensorial inequality

$$0 \le D(\mathrm{Sc}, X) := \frac{\partial}{\partial t} S - \Delta S - 2|S_{ij}|^2 + 4(\nabla_i S_{ij}) X_j - 2(\nabla_j S) X_j + 2R_{ij} X_i X_j - 2S_{ij} X_i X_j. \tag{1-2}$$

It should be remarked that apart from some classical geometric flows where condition (1-2) is easily checked, this condition in general is difficult to verify. Generally speaking, it is an advantage for applications to rely on information about Ric and Sc directly. From this point of view, our conditions are convenient to apply.

Next, we observe that

$$\mu_s(P^{\varrho}_{s,t} f) = \mu_t(f), \quad s \le t, \tag{1-3}$$

which means that the family of measures μ_s plays a role for the semigroup $P^{\varrho}_{s,t}$ similar to that of the invariant measure in the static Riemannian case for the one-parameter semigroup P_t. Log-Sobolev inequalities with respect to the invariant measure are well-established under certain curvature conditions on Riemannian manifolds. They have many applications and are related to other functional inequalities for P_t; see for instance [Bakry 1997; Gross 1975; Wang 2001; 2009]. This leads to the natural question of whether it is possible to prove log-Sobolev inequalities with respect to μ_t in a similar way through functional inequalities for $P^{\varrho}_{s,t}$.

In Section 7 we discuss the relation between contraction properties of the semigroup and log-Sobolev inequalities with respect to μ_t. Using the dimension-free Harnack inequality for $P^{\varrho}_{s,t}$, we give a sufficient condition for supercontractivity of $P^{\varrho}_{s,t}$ which we then use to prove existence of the (defective) log-Sobolev inequality for μ_s. It is well known that the log-Sobolev inequality and Sobolev inequality are important tools to establish upper bounds for the heat kernel; see [Abolarinwa 2016; Băileşteanu 2012; Zhang 2006; Buzano and Yudowitz 2020]. Note that in [Buzano and Yudowitz 2020] the condition for the log-Sobolev inequality is $D(\mathrm{Sc}, X) \ge 0$, which implies in particular $(\partial_t - \Delta_t) S_t \ge 0$. Here besides the curvature condition for the gradient estimate, we add the condition $(\partial_t - L_t)\varrho_t \ge 0$ to derive a super log-Sobolev inequality. The results are then applied to the system (1-1). More specifically, denote by $\rho_t \equiv \rho_t(o, \cdot)$ the distance function to a given base point o in M with respect to the metric g_t, and suppose that the geometric flow g_t and the function ϕ_t satisfy (1-1). Assuming that $\mathrm{Ric}_t + \mathrm{Hess}_t(\phi_t) + \mathrm{Sc}_t \ge K(t)$ and $\mu_t(\exp(\lambda \rho_t^2)) < \infty$ for all $\lambda > 0$ and $t \in [0, T[$, there exists a function β such that

$$\mu_s(f^2 \log f^2) \le r \mu_s\big(|\nabla^s f|_s^2 + \tfrac{1}{4}\varrho_s f^2\big) + \beta_s(r), \quad r > 0,$$

for $f \in C_0^\infty([0, T[\times M)$ and $\mu_s(f^2) = 1$.

The paper is organized as follows. In Section 3 a probabilistic formula for the derivative of the conjugate heat semigroup is given and used to establish a gradient estimate for $P^{\varrho}_{s,t}$ under suitable curvature conditions. In Section 4 we derive two versions of dimension-free Harnack inequalities from the mentioned gradient inequality for $P^{\varrho}_{s,t}$, which are then applied in Section 5 to on-diagonal heat kernel estimates and in Section 6 to Gaussian-type heat kernel estimates via Grigoryan's argument.

These Harnack inequalities are further used in Section 7 to gain sufficient and necessary conditions for supercontractivity of $P^{\varrho}_{s,t}$. In Section 7 we also clarify the relation between supercontractivity of $P^{\varrho}_{s,t}$ and the log-Sobolev inequality with respect to μ_t. These results are then applied to system (1-1) of the modified geometric flow under conjugate heat equation.

2. Brownian motion with respect to evolving manifolds

Let $(M, g_t)_{t\in I}$ be an evolving manifold indexed by $I = [0, T[$. Let ∇^t be the Levi-Civita connection with respect to g_t. Denote by $\mathbb{M} := M \times I$ space-time and consider the bundle

$$TM \xrightarrow{\pi} \mathbb{M},$$

where π is the projection. As observed by Hamilton [1993] there exists a natural space-time connection ∇ on TM considered as bundle over space-time $\mathbb{M}$ such that

$$\begin{cases} \nabla_v X = \nabla^t_v X, \\ \nabla_{\partial_t} X = \partial_t X + \frac{1}{2}(\partial_t g_t)(X, \cdot)^{\sharp g_t} \end{cases}$$

for all $v \in (T_x M, g_t)$ and all time-dependent vector fields X on M. This connection is compatible with the metric, i.e.,

$$\frac{d}{dt}|X|^2_{g_t} = 2\langle X, \nabla_{\partial_t} X\rangle_{g_t}.$$

Remark 2.1. Let $G = \mathrm{O}(n)$, where $n = \dim M$ and consider the G-principal bundle $\mathcal{F} \xrightarrow{\pi} \mathbb{M}$ of orthonormal frames with fibers

$$\mathcal{F}_{(x,t)} = \{u : \mathbb{R}^n \to (T_x M, g_t) \mid u \text{ isometry}\}.$$

As usual, $a \in G$ acts on $\mathcal{F}$ from the right via composition. The connection ∇ gives rise to a G-invariant splitting of the sequence

$$0 \longrightarrow \ker d\pi \longrightarrow T\mathcal{F} \underset{\underset{h}{\curvearrowleft}}{\xrightarrow{d\pi}} \pi^* T\mathbb{M} \longrightarrow 0,$$

which induces a decomposition of $T\mathcal{F}$ as $T\mathcal{F} = V \oplus H := \ker d\pi \oplus h(\pi^* T\mathbb{M})$. For $u \in \mathcal{F}$, the space H_u is the *horizontal space at u* and $V_u = \{v \in T_u\mathcal{F} : (d\pi)v = 0\}$ the *vertical space at u*. The bundle isomorphism

$$h : \pi^* T\mathbb{M} \xrightarrow{\sim} H \hookrightarrow T\mathcal{F}, \quad h_u : T_{\pi(u)}\mathbb{M} \xrightarrow{\sim} H_u, \quad u \in \mathcal{F}, \tag{2-1}$$

is the *horizontal lift* of the G-connection.

Corollary 2.2. *To each $X + b\partial_t \in T_{(x,t)}\mathbb{M}$ and each frame $u \in \mathcal{F}_{(x,t)}$, there exists a unique "horizontal lift" $X^* + bD_t \in H_u$ of $X + b\partial_t$ such that*

$$\pi_*(X^* + bD_t) = X + b\partial_t.$$

In explicit terms, X^* is the horizontal lift of X with respect to the metric g_t, and $D_t = (d/ds)|_{s=0}u_s$, where u_s is the horizontal lift based at u of the curve $s \mapsto (x, t+s)$.

We consider curves in $\mathbb{M}$ of the form

$$\gamma_t = (x_t, \ell_t), \quad t \in [0, T[,$$

where ℓ_t is a monotone differentiable transformation on $[0, T[$. The horizontal lift of such a curve γ_t in $\mathbb{M}$ is a curve u_t in $\mathcal{F}$ such that $\pi u_t = \gamma_t$ and $\nabla_{\dot\gamma}(u_t e) = 0$ for each $e \in \mathbb{R}^n$. Then

$$//^{\gamma}_{r,s} := u_s u_r^{-1} : (T_{x_r} M, g_{\ell_r}) \to (T_{x_s} M, g_{\ell_s}), \quad 0 \le r \le s < T,$$

gives parallel transport along γ_t. In the following we consider the special case $\ell_t = t$.

Remark 2.3. Vector fields and differential forms on $\mathbb{M}$ can be seen as time-dependent vector fields and differential forms on M. It is convenient to write objects on $\mathbb{M}$ as G-equivariant functions on $\mathcal{F}$. In particular, then

(1) functions $f \in C^\infty(\mathbb{M})$ read as $\tilde f \in C^\infty(\mathcal{F})$ via $\tilde f := f \circ \pi$,

(2) time-dependent vector fields Y on M read as $\widetilde{Y} : \mathcal{F} \to \mathbb{R}^n$ via $\widetilde{Y}(u) := u^{-1} Y_{\pi(u)}$,

(3) time-dependent differential forms α on M read as $\tilde\alpha : \mathcal{F} \to (\mathbb{R}^n)^*$ via $\tilde\alpha(u) = \alpha_{\pi(u)}(u\,\cdot\,)$.

The following formulas hold:

$$\widetilde{Xf} = X^* \tilde f, \quad \widetilde{\partial_t f} = D_t \tilde f, \quad \widetilde{\nabla_X Y} = X^* \widetilde{Y}, \quad \widetilde{\nabla_{\partial_t} Y} = D_t \widetilde{Y}, \quad \widetilde{\nabla_X \alpha} = X^* \tilde\alpha, \quad \widetilde{\nabla_{\partial_t} \alpha} = D_t \tilde\alpha. \tag{2-2}$$

The proofs are straightforward. For instance, to check the last equality, let u_t be a horizontal curve such that $\pi u_t = \gamma_t = (x, t)$, where $x \in M$ is fixed. Then

$$\begin{aligned}(D_t\tilde\alpha)(u_s) &= \frac{d}{dt}\Big|_{t=s} \tilde\alpha(u_t) = \frac{d}{dt}\Big|_{t=s} \alpha_{\pi(u_t)}(u_t\,\cdot\,) \\ &= \frac{d}{dr}\Big|_{r=0} \alpha_{(x,s+r)}(//_{s,s+r} u_s, \cdot\,) = (\nabla_{\partial_t}\alpha)_{(x,s)}(u_s\,\cdot\,) = (\widetilde{\nabla_{\partial_t}\alpha})(u_s).\end{aligned}$$

Remark 2.4. The vector fields

$$H_i \in \Gamma(T\mathcal{F}), \quad H_i(u) = (ue_i)^* \equiv h_u(ue_i), \quad i = 1, \dots, n,$$

where $(e_1, \dots, e_n)$ denotes the standard basis of $\mathbb{R}^n$, are the standard-horizontal vector fields on $\mathcal{F}$. The operator

$$\Delta_{\text{hor}} = \sum_{i=1}^n H_i^2$$

is called Bochner's horizontal Laplacian on $\mathcal{F}$. Note that

$$\widetilde{\Delta f} = \Delta_{\text{hor}} \tilde f \quad \text{and} \quad \widetilde{\Delta_{\text{rough}}\alpha} = \Delta_{\text{hor}} \tilde\alpha, \tag{2-3}$$

where $\Delta_{\text{rough}} = \operatorname{tr}(\nabla^t)^2$ is the rough Laplacian on differential one-forms. Recall that, for fixed $t \in I$,

$$d\Delta_t f = \operatorname{tr}(\nabla^t)^2 df - \operatorname{Ric}_t(df, \cdot\,) \tag{2-4}$$

by the Weitzenböck formula.

Let $\pi : \mathcal{F} \to \mathbb{M}$ denote the canonical projection. For $\phi \in C^{1,2}(\mathbb{M})$, we define a vector field on $\mathcal{F}$ by

$$H^\phi \in \Gamma(T\mathcal{F}), \quad H^\phi(u) = h_u(\nabla^t \phi(t,\cdot)_x), \quad \pi(u) = (x,t).$$

Consider the following Stratonovich SDE on $\mathcal{F}$:

$$\begin{cases} dU = D_t(U)\,dt + \sum_{i=1}^n H_i(U) \circ dB^i - H^\phi(U)\,dt, \\ U_s = u_s, \quad \pi(u_s) = (x,s),\ s \in [0,T[. \end{cases} \tag{2-5}$$

Here B denotes standard Brownian motion on $\mathbb{R}^n$ (sped up by the factor 2, i.e., $dB^i\,dB^j = 2\delta_{ij}\,dt$) with generator $\Delta_{\mathbb{R}^n}$. Equation (2-5) has a unique solution up to its lifetime $\zeta := \lim_{k\to\infty} \zeta_k$, where

$$\zeta_k := \inf\{t \in [s,T[\, : \rho_t(\pi(U_s), \pi(U_t)) \geq k\}, \quad n \geq 1,\ \inf\varnothing := T, \tag{2-6}$$

and where ρ_t stands for the Riemannian distance induced by the metric $g(t)$. We note that if U solves (2-5) then

$$\pi(U_t) = (X_t, t),$$

where X is a diffusion process on M generated by $L_t = \Delta_t - \nabla^t \phi_t$. In case of $\phi = 0$ we call X a (g_t)-Brownian motion on $\mathbb{M}$.

More precisely, we have the following result.

Proposition 2.5. *Let U be a solution to the SDE* (2-5)*. Then*

(1) *for any C^2-function $F : \mathcal{F} \to \mathbb{R}$,*

$$d(F(U)) = (D_t F)(U)\,dt + \sum_{i=1}^n (H_i F)(U)\,dB^i + (\Delta_{\text{hor}} F)(U)\,dt - (H^\phi F)(U)\,dt,$$

(2) *for any C^2-function $f : \mathbb{M} \to \mathbb{R}$, we have*

$$d(f(X)) = (\partial_t f)(X)\,dt + \sum_{i=1}^n (U e_i f)\,dB^i + (L_t f)(X)\,dt.$$

Let U be a solution to the SDE (2-5) and $\pi(U_t) = (X_t, t)$. Furthermore let

$$//_{r,t} := U_t U_r^{-1} : (T_{x_r} M, g_r) \to (T_{x_t} M, g_t), \quad s \leq r \leq t < T,$$

be the induced parallel transport along X_t (which by construction consists of isometries). We use the notation

$$X_t = X_t^{(s,x)}, \quad t \geq s,$$

if $X_s = x$. Note that $X_t = X_t^{(s,x)}$ solves the equation

$$dX_t^{(s,x)} = U_t \circ dB_t - \nabla^t \phi_t(X_t^{(s,x)})\,dt, \quad X_s^{(s,x)} = x.$$

For any $f \in C_0^2(M)$,

$$f(X_t^{(s,x)}) - f(x) - \int_s^t L_r f(X_r^{(s,x)})\,dr = \int_s^t \langle //_{s,r}^{-1} \nabla^r f(X_r^{(s,x)}), U_s\,dB_r \rangle_s, \quad t \in [s,T[,$$

is a martingale up to the lifetime ζ. In the case $s = 0$, we write again X_t^x instead of $X_t^{(0,x)}$.

3. Derivative formula

Let $L_t = \Delta_t - \nabla^t \phi_t$, where ϕ is $C^{1,2}([0, T[\times M)$. Throughout this section, we assume the diffusion (X_t) generated by L_t is nonexplosive up to time T. Recall that $\mu_t(dx) = \mathrm{e}^{-\phi_t(x)} d\,\mathrm{vol}_t$ and

$$\frac{\partial}{\partial t}\mu_t(dx) = -(\partial_t \phi + S)(t, x)\, \mu_t(dx) = -\varrho(t, x)\, \mu_t(dx),$$

with

$$\varrho(t, x) \equiv \varrho_t(x) = \partial_t \phi(t, x) + S(t, x).$$

For each t, we assume that ϱ_t is bounded from below by a constant depending on t.

For $f \in C_b(M)$ let

$$P^{\varrho}_{s,t} f(x) = \mathbb{E}\left[\exp\left(-\int_s^t \varrho(r, X_r)\, dr\right) f(X_t) \,\Big|\, X_s = x\right]. \tag{3-1}$$

Then

$$\frac{\partial}{\partial s} P^{\varrho}_{s,t} f = -(L_s - \varrho_s) P^{\varrho}_{s,t} f, \quad P^{\varrho}_{t,t} = f. \tag{3-2}$$

That $u(s, x) := P^{\varrho}_{s,t} f(x)$ represents the solution to (3-2) is easily seen from the fact that

$$\exp\left(-\int_s^r \varrho(a, X_a)\, da\right) P^{\varrho}_{r,t} f(X^{(s,x)}_r), \quad r \in [s, t],$$

is a martingale under given assumptions.

Our first step is to establish a derivative formula for $P^{\varrho}_{s,t}$. When the metric is static, the derivative formula for $P_t f$ is known as the Bismut formula [Bismut 1984; Elworthy and Li 1994]. The more general versions in [Thalmaier 1997] have been adapted to Feynman–Kac semigroups in [Thompson 2019].

We fix $s \in [0, T[$ and consider the random family $Q_{s,t} \in \mathrm{Aut}(T_{X_s} M)$, $0 \le s \le t < T$, given as a solution to the ordinary differential equation

$$\frac{dQ_{s,t}}{dt} = -(\mathrm{Ric} + \mathrm{Sc} + \mathrm{Hess}(\phi))_{//_{s,t}} Q_{s,t}, \quad Q_{s,s} = \mathrm{id}, \tag{3-3}$$

where

$$(\mathrm{Ric} + \mathrm{Sc} + \mathrm{Hess}(\phi))_{//_{s,t}} := //_{s,t}^{-1} \circ (\mathrm{Ric}_t + \mathrm{Sc}_t + \mathrm{Hess}_t(\phi_t))(X_t) \circ //_{s,t}.$$

Theorem 3.1. *Let $L_t = \Delta_t - \nabla^t \phi_t$ as above. For each t, assume that both ϱ_t and*

$$(\mathrm{Ric} + \mathrm{Hess}(\phi) + \mathrm{Sc})(t, \cdot)$$

are bounded below and that $|d\varrho_t|$ is bounded. Then, for $v \in T_x M$ and $f \in C^1_b(M)$,

$$(dP^{\varrho}_{s,t} f)(v) = \mathbb{E}^{(s,x)}\left[\exp\left(-\int_s^t \varrho_r(X_r)\, dr\right)\left(df(//_{s,t} Q_{s,t} v)(X_t) - f(X_t)\int_s^t d\varrho_r(//_{s,r} Q_{s,r} v)\, dr\right)\right]. \tag{3-4}$$

Proof. By the definition of Q as the solution to (3-3), we have

$$d(U_s^{-1} Q_{s,r}) = -U_s^{-1}(\mathrm{Ric} + \mathrm{Hess}(\phi))_{//_{s,r}} Q_{s,r}. \tag{3-5}$$

Set

$$N_r(v) = dP^{\varrho}_{r,t} f(//_{s,r} Q_{s,r} v), \quad s \le r \le t.$$

Write $N_r(v) = F(U_r, q_r(v))$, where

$$F(u, w) := (dP^{\varrho}_{r,t} f)_x(uw), \quad \pi(u) = (x, r) \text{ and } w \in \mathbb{R}^n,$$
$$q_r(v) := U_s^{-1} Q_{s,r} v.$$

By means of Proposition 2.5, we have

$$d(F(U_r, w)) \overset{\mathrm{m}}{=} (D_r F)(U_r, w)\, dr + (\Delta_{\mathrm{hor}} F)(U_r, w)\, dr - (H^{\phi} F)(U_r, w)\, dr, \tag{3-6}$$

where

$$(D_r F)(u, w) = \partial_r (dP^{\varrho}_{r,t} f)_x(uw) + \tfrac{1}{2}(\partial_r g_r)((dP^{\varrho}_{r,t} f)^{\sharp_{g_r}}, uw),$$

and where $\overset{\mathrm{m}}{=}$ stands for equality modulo the differential of a local martingale. Using the Weitzenböck formula we observe that

$$\begin{aligned}
\partial_r(dP^{\varrho}_{r,t} f) &= -d(L_r - \varrho_r) P^{\varrho}_{r,t} f \\
&= -d(\Delta_r - \nabla^r \phi_r - \varrho_r) P^{\varrho}_{r,t} f \\
&= -\operatorname{tr}(\nabla^t)^2 dP^{\varrho}_{r,t} f + dP^{\varrho}_{r,t} f((\operatorname{Hess} \phi_r)^{\sharp_{g_r}}) + dP^{\varrho}_{r,t} f(\operatorname{Ric}^{\sharp_{g_r}}_r) + d(\varrho_r\, P^{\varrho}_{r,t} f)_{X_r}.
\end{aligned}$$

Taking (2-2) and (2-3) into account, we have

$$(\Delta_{\mathrm{hor}} F)(U_r, w) = \operatorname{tr}(\nabla^r)^2 dP^{\varrho}_{r,t} f(U_r w)$$

and

$$(H^{\phi} F)(U_r, w) = \operatorname{Hess}(\phi_r)((dP^{\varrho}_{r,t} f)^{\sharp_{g_r}}, U_r w) = dP^{\varrho}_{r,t} f((\operatorname{Hess}(\phi_r))^{\sharp_{g_r}})(U_r w).$$

Thus (3-6) can be rewritten as

$$d(F(U_r, w)) \overset{\mathrm{m}}{=} dP^{\varrho}_{r,t} f\, (\operatorname{Ric}^{\sharp_{g_r}}_r) U_r w + d(\varrho_r\, P^{\varrho}_{r,t} f)_{X_r} U_r w + \tfrac{1}{2}(\partial_r g_r)((dP^{\varrho}_{r,t} f)^{\sharp_{g_r}}, U_r w). \tag{3-7}$$

Combining (3-7) and (3-5) we thus obtain

$$\begin{aligned}
dN_r(v) &= d(F(U_r, \cdot))(q_r(v)) + F(U_r, \partial_r q_r(v))\, dr \\
&\overset{\mathrm{m}}{=} d(\varrho_r\, P^{\varrho}_{r,t} f)_{X_r} //_{s,r} Q_{s,r} v\, dr \\
&= \big(\varrho_r(X_r)\, dP^{\varrho}_{r,t} f(//_{s,r} Q_{s,r} v) + P^{\varrho}_{r,t} f(X_r)\, d\varrho_r(//_{s,r} Q_{s,r} v)\big)\, dr,
\end{aligned}$$

Hence we get

$$d\left(\exp\left(-\int_s^r \varrho_u(X_u)\, du\right) N_r(v)\right) \overset{\mathrm{m}}{=} -\exp\left(-\int_s^r \varrho_u(X_u)\, du\right) P^{\varrho}_{r,t} f(X_r)\, d\varrho_r(//_{s,r} Q_{s,r} v)\, dr.$$

Integrating this equality from s to t and taking the expectation gives (3-4). □

Corollary 3.2. *Suppose that ϱ_t is bounded below for each t, and assume that there are functions $\kappa, K \in C([0, T[)$ such that $|d\varrho_t| \le \kappa(t)$, respectively*

$$\operatorname{Ric}_t + \operatorname{Sc}_t + \operatorname{Hess}_t(\phi_t) \ge K(t).$$

Then, *for* $f \in C_0^\infty(M)$ *and* $f \geq 0$,

$$|\nabla^s P^{\varrho}_{s,t} f|_s \leq \exp\left(-\int_s^t K(r)\,dr\right) P^{\varrho}_{s,t} |\nabla^t f|_t + P^{\varrho}_{s,t} f \int_s^t \kappa(r) \exp\left(-\int_s^r K(u)\,du\right) dr.$$

Proof. The condition $\mathrm{Ric}_t + \mathrm{Sc}_t + \mathrm{Hess}_t(\phi_t) \geq K(t)$ implies

$$|Q_{s,t}| \leq \exp\left(-\int_s^t K(r)\,dr\right).$$

The inequality then follows from the bound $|d\varrho_t| \leq \kappa(t)$. □

In particular, if $\phi \equiv 0$, Corollary 3.2 then becomes:

Corollary 3.3. *Suppose that* S_t *is bounded below for each* t *and assume that there are functions* $S, K \in C([0, T[)$ *such that* $|dS_t| \leq \kappa(t)$, *respectively*

$$\mathrm{Ric}_t + \mathrm{Sc}_t \geq K(t).$$

Then, *for* $f \in C_0^\infty(M)$ *and* $f \geq 0$,

$$|\nabla^s P^{\varrho}_{s,t} f|_s \leq \exp\left(-\int_s^t K(r)\,dr\right) P^{\varrho}_{s,t} |\nabla^t f|_t + P^{\varrho}_{s,t} f \int_s^t \kappa(r) \exp\left(-\int_s^r K(u)\,du\right) dr.$$

4. Dimension-free Harnack inequalities

We first derive two Harnack-type inequalities for $P^o_{s,t}$. Such dimension-free Harnack inequalities were studied first by Wang; see, e.g., [Wang 2014] for more results in this direction. When it comes to the evolving metric case, in [Cheng 2017] the author gave the following Harnack inequality for the 2-parameter semigroup $P_{s,t}$ as follows. We denote by $\mathcal{B}_b(M)$ the space of bounded measurable functions on M.

Theorem 4.1. *Suppose that*

$$\mathrm{Ric}_t + \mathrm{Sc}_t + \mathrm{Hess}_t(\phi_t) \geq K(t).$$

Then, *for any nonnegative function* $f \in \mathcal{B}_b(M)$ *and* $0 \leq s \leq t < T$,

$$(P_{s,t} f)^p(x) \leq P_{s,t} f^p(y) \exp\left(\frac{p}{4(p-1)\alpha(s,t)} \rho_s^2(x,y)\right),$$

where

$$\alpha(s,t) := \int_s^t \exp\left(2\int_s^r K(u)\,du\right) dr.$$

We first extend such kind of dimension-free Harnack inequality to that for the conjugate semigroup.

Theorem 4.2. *Suppose that* ϱ *is bounded below*, $|d\varrho_t| \leq \kappa(t)$, *and*

$$\mathrm{Ric}_t + \mathrm{Sc}_t + \mathrm{Hess}_t(\phi_t) \geq K(t).$$

The following two Harnack-type inequalities hold for any $p > 1$:

(i) *For* $0 \le s \le t < T$ *and any nonnegative function* $f \in \mathcal{B}_b(M)$,

$$(P^{\varrho}_{s,t} f)^p(x) \le (P^{\varrho}_{s,t} f^p)(y) \exp\left((p-1)\int_s^t \sup \varrho_r^- \, dr + \frac{p\rho_s^2(x,y)}{4(p-1)\alpha(s,t)} + \frac{p\eta(s,t)\rho_s(x,y)}{\alpha(s,t)}\right), \quad (4\text{-}1)$$

where

$$\alpha(s,t) := \int_s^t \exp\left(2\int_s^r K(u)\,du\right) dr,$$

$$\eta(s,t) := \int_s^t \int_s^v \kappa(r) \exp\left(2\int_s^v K(u)\,du - \int_s^r K(u)\,du\right) dr\,dv.$$

(ii) *For* $0 \le s \le t < T$ *and any nonnegative function* $f \in \mathcal{B}_b(M)$,

$$(P^{\varrho}_{s,t} f)^p(x) \le (P^{\varrho}_{s,t} f^p)(y)\, \mathbb{E}^y\left[\exp\left(-(p-1)\int_s^t \varrho_r(X_r)\,dr\right)\right] \exp\left(\frac{p\rho_s^2(x,y)}{4(p-1)\alpha(s,t)} + \frac{2p\eta(s,t)}{\alpha(s,t)}\rho_s(x,y)\right).$$

Proof. To facilitate the notion we restrict ourselves to the case $s = 0$. By approximation and the monotone class theorem, we may assume that $f \in C^2(M)$ has compact support and $\inf_M f > 0$. Given $x \ne y$ and $t > 0$, let $\gamma : [0,t] \to M$ be the constant-speed g_0-geodesics from x to y of length $\rho_0(x,y)$. Let $\nu_s = d\gamma_s/ds$. Thus we have $|\nu_s|_0 = \rho_0(x,y)/t$. Let

$$h(s) = t\,\frac{\int_0^s \exp\left(2\int_0^r K(u)\,du\right) dr}{\int_0^t \exp\left(2\int_0^r K(u)\,du\right) dr}.$$

Then $h(0) = 0$ and $h(t) = t$. Let $y_s = \gamma_{h(s)}$ and

$$\varphi(s) = \log \mathbb{E}^{y_s}\left(\exp\left(-\int_0^s \varrho_r(X_r)\,dr\right) P^{\varrho}_{s,t} f(X_s)\right)^p$$

$$= \log P^{p\varrho}_{0,s}(P^{\varrho}_{s,t} f)^p(y_s), \quad s \in [0,t].$$

To determine $\varphi'(s)$, we first note that

$$d(P^{\varrho}_{s,t} f(X_s))^p = dM_s + (L_s + \partial_s)(P^{\varrho}_{s,t} f)^p(X_s)\,ds$$

$$= dM_s + p(p-1)(P^{\varrho}_{s,t} f)^{p-2}(X_s)\,|\nabla^s P^{\varrho}_{s,t} f|_s^2(X_s)\,ds + p\varrho_s(X_s)(P^{\varrho}_{s,t} f)^p(X_s)\,ds, \quad s \le \zeta_k,$$

where M_s is the local martingale part of $(P^{\varrho}_{s,t} f)^p(X_s)$. This implies

$$\mathbb{E}^x\left[\left(\exp\left(-\int_0^{s\wedge\zeta_k} \varrho_r(X_r)\,dr\right) P^{\varrho}_{s\wedge\zeta_k,t} f(X_{s\wedge\zeta_k})\right)^p\right] - (P^{\varrho}_{0,t} f)^p(x)$$

$$= p(p-1)\mathbb{E}^x\left[\int_0^{s\wedge\zeta_k} \exp\left(-p\int_0^u \varrho_r(X_r)\,dr\right)(P^{\varrho}_{u,t} f)^{p-2}(X_u)|\nabla^u P^{\varrho}_{u,t} f|_u^2(X_u)\,du\right].$$

Since $\inf_M f > 0$, by letting $k \to \infty$, we deduce that

$$\mathbb{E}^x\left[\left(\exp\left(-\int_0^s \varrho_r(X_r)\,dr\right) P^{\varrho}_{s,t} f(X_s)\right)^p\right] - (P^{\varrho}_{0,t} f)^p(x)$$

$$= p(p-1)\int_0^s \mathbb{E}^x\left[\exp\left(-p\int_0^u \varrho_r(X_r)\,dr\right)(P^{\varrho}_{u,t} f)^{p-2}|\nabla^u P^{\varrho}_{u,t} f|_u^2(X_u)\right] du. \quad (4\text{-}2)$$

Hence, for each $x \in M$,

$$\frac{\partial}{\partial s}\mathbb{E}^x\left(\exp\left(-\int_0^s \varrho_r(X_r)\,dr\right)P^{\varrho}_{s,t}f(X_s)\right)^p = p(p-1)\mathbb{E}^x\left[\exp\left(-p\int_0^s \varrho_r(X_r)\,dr\right)(P^{\varrho}_{s,t}f)^{p-2}|\nabla^s P^{\varrho}_{s,t}f|_s^2(X_s)\right].$$

Moreover, by adapting Corollary 3.2 for $P^{p\varrho}_{s,t}$, i.e.,

$$|\nabla^0 P^{p\varrho}_{0,s}f|_0 \le \exp\left(-\int_0^s K(r)\,dr\right)P^{p\varrho}_{0,s}|\nabla^s f|_s + pP^{p\varrho}_{0,s}f\int_0^s \kappa(r)\exp\left(-\int_0^r K(u)\,du\right)dr,$$

we thus obtain, for $s \in [0,t]$,

$$\begin{aligned}
\frac{d\varphi(s)}{ds} &= \left(\frac{1}{P^{p\varrho}_{0,s}(P^{\varrho}_{s,t}f)^p}\left\{p(p-1)P^{p\varrho}_{0,s}((P^{\varrho}_{s,t}f)^p\,|\nabla^s \log P^{\varrho}_{s,t}f|_s^2) + h'(s)\langle\nabla^0 P^{p\varrho}_{0,s}(P^{\varrho}_{s,t}f)^p, \nu_s\rangle_0\right\}\right)(y_s)\\
&\ge \left(\frac{p}{P^{p\varrho}_{0,s}(P^{\varrho}_{s,t}f)^p}\left\{(p-1)P^{p\varrho}_{0,s}((P^{\varrho}_{s,t}f)^{p-2}\,|\nabla^s P^{\varrho}_{s,t}f|_s^2)\right.\right.\\
&\qquad - \frac{\rho_0(x,y)}{t}\exp\left(-\int_0^s K(u)\,du\right)h'(s)P^{p\varrho}_{0,s}((P^{\varrho}_{s,t}f)^{p-1}|\nabla^s P^{\varrho}_{s,t}f|_s)\\
&\qquad \left.\left. - \frac{\rho_0(x,y)}{t}h'(s)\,P^{p\varrho}_{0,s}\,(P^{\varrho}_{s,t}f)^p\int_0^s \kappa(r)\exp\left(-\int_0^r K(u)\,du\right)dr\right\}\right)(y_s)\\
&\ge \left(\frac{p}{P^{p\varrho}_{0,s}(P^{\varrho}_{s,t}f)^p}P^{p\varrho}_{0,s}\left\{(P^{\varrho}_{s,t}f)^p\left((p-1)|\nabla^s \log P^{\varrho}_{s,t}f|_s^2\right.\right.\right.\\
&\qquad - \frac{\rho_0(x,y)}{t}h'(s)\exp\left(-\int_0^s K(u)\,du\right)|\nabla^s \log P^{\varrho}_{s,t}f|_s\\
&\qquad \left.\left.\left. - \frac{\rho_0(x,y)}{t}h'(s)\int_0^s \kappa(r)\exp\left(-\int_0^r K(u)\,du\right)dr\right)\right\}\right)(y_s)\\
&\ge \frac{-ph'(s)^2}{4(p-1)t^2}\exp\left(-2\int_0^s K(u)\,du\right)\rho_0(x,y)^2 - \frac{p}{t}h'(s)\int_0^s \kappa(r)\exp\left(-\int_0^r K(u)\,du\right)dr\;\rho_0(x,y),
\end{aligned}$$

where the last inequality follows from the simple fact that

$$aA^2 + bA \ge -\frac{b^2}{4a}, \quad a > 0.$$

Since

$$h'(s) = \frac{t\exp\left(2\int_0^s K(u)\,du\right)}{\int_0^t \exp\left(2\int_0^r K(u)\,du\right)dr},$$

we thus arrive at

$$\begin{aligned}
\frac{d\varphi(s)}{ds} \ge &-\frac{p\exp\left(\int_0^s 2K(u)\,du\right)}{4(p-1)\left(\int_0^t \exp\left(2\int_0^r K(u)\,du\right)dr\right)^2}\,\rho_0(x,y)^2\\
&- \frac{p\exp\left(2\int_0^s K(u)\,du\right)\int_0^s \kappa(r)\exp\left(-\int_0^r K(u)\,du\right)dr}{\int_0^t \exp\left(2\int_0^r K(u)\,du\right)dr}\,\rho_0(x,y), \quad s \in [0,t].
\end{aligned}$$

Integrating over s from 0 and t, we get

$$\begin{aligned}(P^{\varrho}_{0,t}f)^p(x) &\le \mathbb{E}^y\biggl[\biggl(\exp\biggl(-\int_0^t \varrho_r(X_r)\,dr\biggr)f(X_t)\biggr)^p\biggr]\\ &\qquad\times\exp\biggl(\frac{p\rho_0(x,y)^2}{4(p-1)\int_0^t\exp\bigl(2\int_0^r K(u)\,du\bigr)\,dr}\\ &\qquad\qquad+\frac{p\int_0^t\int_0^s\kappa(r)\exp\bigl(2\int_0^s K(u)\,du-\int_0^r K(u)\,du\bigr)\,dr\,ds}{\int_0^t\exp\bigl(2\int_0^r K(u)\,du\bigr)\,dr}\,\rho_0(x,y)\biggr)\\ &\le \mathbb{E}^y\biggl[\exp\biggl(-\int_0^t \varrho_r(X_r)\,dr\biggr)f(X_t)^p\biggr]\exp\biggl((p-1)\int_0^t\sup\varrho_r^-\,dr\biggr)\\ &\qquad\times\exp\biggl(\frac{p\rho_0(x,y)^2}{4(p-1)\int_0^t\exp\bigl(2\int_0^r K(u)\,du\bigr)\,dr}\\ &\qquad\qquad+\frac{p\int_0^t\int_0^s\kappa(r)\exp\bigl(2\int_0^s K(u)\,du-\int_0^r K(u)\,du\bigr)\,dr\,ds}{\int_0^t\exp\bigl(2\int_0^r K(u)\,du\bigr)\,dr}\,\rho_0(x,y)\biggr).\end{aligned}\tag{4-3}$$

This proves part (i) of the theorem. In addition, by adopting in (4-3) the estimate

$$\mathbb{E}^y\biggl[\biggl(\exp\biggl(-\int_0^t\varrho_r(X_r)\,dr\biggr)f(X_t)\biggr)^p\biggr]\le(P^{\varrho}_{0,t}f^p)(y)\,\mathbb{E}^y\biggl[\exp\biggl(-(p-1)\int_0^t\varrho_r(X_r)\,dr\biggr)\biggr],$$

part (ii) of the theorem follows as well. □

5. On-diagonal heat kernel estimates

Let p be the fundamental solution of $L_t=\Delta_t-\nabla^t\phi_t$ in the sense that

$$\begin{cases}\partial_t p(t,x;s,y)=(\Delta_t-\nabla^t\phi_t)p(t,\cdot\,;s,y)(x),\\ \lim_{t\to s}p(t,x;s,y)=\delta_y(x),\end{cases}$$

where $t>s$. Let p^* be the conjugate heat kernel of p; it is the density of $P^{\varrho}_{s,t}(x,dy)$ with respect to $\mu_t(dy)$, i.e.,

$$P^{\varrho}_{s,t}f(x)=\int p^*(s,x;t,y)f(y)\,\mu_t(dy)=\int p^*(s,x;t,y)f(y)\mathrm{e}^{-\phi_t(y)}\,\mathrm{vol}_t(dy),$$

where vol_t denotes the volume measure with respect to g_t. In [Cheng 2017] the following Harnack inequality for the 2-parameter semigroup $P_{s,t}$ was derived; it can be seen as a special case of Theorem 4.1.

It is interesting to observe that Theorem 4.1 for $P_{s,t}$, along with the Harnack inequality (4-1) for $P^{\varrho}_{s,t}$, will allow us to attain on-diagonal upper bounds for the heat kernel p:

$$p(t,x;s,x)\le\frac{C}{\sqrt{\mu_t(B_t(x,\sqrt{t-s}))\,\mu_s(B_s(x,\sqrt{t-s}))}}$$

for $0\le s<t<T$ and $x\in M$.

Theorem 5.1. *Suppose that* $\sup\varrho_u^-<\infty$ *for* $u\in[0,T]$. *Let* $0\le s<t<T$ *and there exists* $r_0\in(s,t)$ *such that*, *for* $u\in[r_0,t]$,

$$\mathrm{Ric}_u-\mathrm{Sc}_u+\mathrm{Hess}_u(\phi_u)\ge K_1(u),$$

and, *for* $u \in [s, r_0]$,

$$\mathrm{Ric}_u + \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \geq K_2(u) \quad \textit{and} \quad |d\varrho_u| \leq \kappa(u) < \infty.$$

Then the following heat kernel upper bound holds:

$$p(t, x; s, x) \leq \exp\left(\frac{1}{2}\int_s^t \sup \varrho_u^- \, du + \frac{t-s}{4\alpha_1(r_0, t)} + \frac{t-s+2\eta_2(s, r_0)\sqrt{t-s}}{4\alpha_2(s, r_0)}\right) \times \frac{1}{\sqrt{\mu_t(B_t(x, \sqrt{t-s}))\mu_s(B_s(x, \sqrt{t-s}))}},$$

where

$$\alpha_1(r_0, t) := \int_{r_0}^t \exp\left(2\int_v^t K_1(u)\, du\right) dv, \tag{5-1}$$

$$\alpha_2(s, r_0) := \int_s^{r_0} \exp\left(2\int_s^v K_2(u)\, du\right) dv, \tag{5-2}$$

$$\eta_2(s, r_0) := \int_s^{r_0}\int_s^v \kappa(t) \exp\left(2\int_s^v K_2(u)\, du - \int_s^t K_2(u)\, du\right) dt\, dv. \tag{5-3}$$

Proof. We first observe that

$$\begin{aligned} p(t, x; s, x) &= \int_M p(t, x; r, z)\, p(r, z; s, x)\, \mu_r(dz) \\ &\leq \left(\int_M p(t, x; r, z)^2 \mu_r(dz)\right)^{1/2} \left(\int_M p(r, z; s, x)^2 \mu_r(dz)\right)^{1/2}. \end{aligned}$$

Hence we are left with the task to estimate the two terms

$$I_1 = \int_M p(t, x; r, z)^2\, \mu_r(dz), \quad I_2 = \int_M p(r, z; s, x)^2\, \mu_r(dz).$$

In order to estimate I_1 we proceed with Theorem 4.1. Let $\bar{p}(s, x; u, y) := p(t-s, x; t-u, y)$ for $0 \leq s \leq u \leq t$. Then

$$\partial_s \bar{p}(\,\cdot\,, x; u, y)(s) = -L_{t-s}\bar{p}(s, \cdot\,; u, y)(x).$$

Write $\overline{P}_{s,u} f = \int \bar{p}(s, x; u, y) f(y)\, \mu_{t-s}(dy)$ for all $f \in \mathcal{B}_b(M)$. As

$$\mathrm{Ric}_u - \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \geq K_1(u), \quad u \in [r_0, t],$$

for some $K_1 \in C([r_0, t])$, we obtain that, for $t > r \geq r_0$,

$$\begin{aligned} (\overline{P}_{0,t-r} f)^2(x)\, \mu_t(B_t(x, \sqrt{t-s}))\, \exp\left(-\frac{t-s}{2\int_r^t \exp\left(2\int_v^t K_1(u)\, du\right) dv}\right) \\ &\leq \int_M (\overline{P}_{0,t-r} f)^2(x) \exp\left(-\frac{\rho_t^2(x, y)}{2\int_r^t \exp(2\int_v^t K_1(u)\, du)\, dv}\right) \mu_t(dy) \\ &\leq \int_M (\overline{P}_{0,t-r} f^2)(y)\, \mu_t(dy) \\ &\leq \exp\left(\int_r^t \sup \varrho^-(u, \cdot\,)\, du\right) \int_M f(y)^2\, \mu_r(dy). \end{aligned}$$

Taking

$$f(y) := (k \wedge p(t, x; r, z)), \quad z \in M,\ k \in \mathbb{N},$$

we obtain

$$\int_M (k \wedge p(t, x; r, z))^2 \, \mu_r(dy) \le \exp\biggl(\int_r^t \sup \varrho^-(u, \cdot\,)\, du + \frac{t-s}{2\alpha_1(r,t)}\biggr) \frac{1}{\mu_t(B_t(x, \sqrt{t-s}))},$$

where

$$\alpha_1(r,t) = \int_r^t \exp\biggl(2\int_v^t K_1(u)\, du\biggr)\, dv.$$

Letting $k \to \infty$, we arrive at

$$I_1 = \int_M p(t, x; r, z)^2 \, \mu_r(dz) \le \exp\biggl(\int_r^t \sup \varrho^-(u, \cdot\,)\, du + \frac{t-s}{2\alpha_1(r,t)}\biggr) \frac{1}{\mu_t(B_t(x, \sqrt{t-s}))}.$$

To estimate the second term I_2, we write

$$\int_M p(r, z; s, x)^2 \, \mu_r(dz) = \int_M p^*(s, x; r, z)^2 \, \mu_r(dz),$$

where p^* denotes the adjoint heat kernel to p. Recall that

$$\begin{cases} \partial_s p^*(s, x; r, z) = -L_s p^*(s, \cdot\,; r, z)(x) + \varrho_s(x) p^*(s, y; r, z), \\ \lim_{s \to r} p^*(s, x; r, z) = \delta_x(z). \end{cases}$$

Let $\{P^\varrho_{s,r}\}_{0 \le s \le r \le t}$ be the semigroup generated by the operator $L_t - \varrho_t$. By (4-1), this time relying on the assumption

$$\mathrm{Ric}_u + \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \ge K_2(u), \quad u \in [s, r_0],$$

we have that, for $s < r \le r_0$,

$$(P^\varrho_{s,r} f)^2(x) \le (P^\varrho_{s,r} f^2)(y) \exp\left(\int_s^r \sup \varrho^-(u, \cdot\,)\, du + \frac{\rho_s^2(x,y)}{2\alpha_2(s,r)} + \frac{2\eta_2(s,r)\rho_s(x,y)}{\alpha_2(s,r)}\right),$$

where

$$\alpha_2(s,r) = \int_s^r \exp\biggl(2\int_s^v K_2(u)\, du\biggr)\, dv,$$

$$\eta_2(s,r) = \int_s^r \int_s^v \kappa(u) \exp\biggl(2\int_s^v K_2(t)\, dt - \int_s^u K_2(t)\, dt\biggr)\, du\, dv.$$

By means of this formula, we can proceed as above to obtain

$$\begin{aligned}
(P^\varrho_{s,r} f)^2(x)\, \mu_s(B_s(x, \sqrt{t-s})) \exp\left(-\int_s^r \sup \varrho^-(u, \cdot\,)\, du - \frac{t-s}{2\alpha_2(s,r)} - \frac{2\eta_2(s,r)\sqrt{t-s}}{\alpha_2(s,r)}\right) \\
\le \int_M (P^\varrho_{s,r} f)^2(x) \exp\left(-\int_s^r \sup \varrho^-(u, \cdot\,)\, du - \frac{\rho_s^2(x,y)}{2\alpha_2(s,r)} - \frac{2\eta_2(s,r)\rho_s(x,y)}{\alpha_2(s,r)}\right) \mu_s(dy) \\
\le \int_M (P^\varrho_{s,r} f^2)(y)\, \mu_s(dy) = \int_M f(y)^2 \, \mu_r(dy).
\end{aligned}$$

Thus taking $f(z) := p^*(s,x;r,z) \wedge k$ and letting $k \to \infty$, we obtain

$$\int_M p^*(s,x;r,z)^2\,\mu_r(dz) \le \exp\biggl(\int_s^r \sup \varrho^-(u,\cdot\,)\,du + \frac{t-s+2\eta_2(s,r)\sqrt{t-s}}{2\alpha_2(s,r)}\biggr)\frac{1}{\mu_s(B_s(x,\sqrt{t-s}))}. \tag{5-4}$$

Finally, combining (5-4) we obtain

$$\begin{aligned} p(t,x;s,x) &\le \sqrt{I_1 I_2} \\ &\le \exp\left(\frac{1}{2}\int_s^t \sup \varrho^-(u,\cdot\,)\,du + \frac{t-s}{4\alpha_1(r,t)} + \frac{t-s+2\eta_2(s,r)\sqrt{t-s}}{4\alpha_2(s,r)}\right) \\ &\qquad \times \frac{1}{\sqrt{\mu_s(B_s(x,\sqrt{t-s}))\,\mu_t(B_t(x,\sqrt{t-s}))}}, \end{aligned}$$

where the functions $\alpha_1, \alpha_2, \eta_2$ are defined by (5-1). □

Remark 5.2. (1) In [Coulibaly-Pasquier 2019], the author used a horizontal coupling of curves to derive a dimension-free Harnack inequality for the two-parameter semigroup $P_{s,t}$ generated by Δ_t, a method first used by Wang [2011; 2014], and applied it then to establish an upper bound for the heat kernel. A major difference to our approach when $\phi = 0$ is that we use the Harnack inequality again to deal with the term I_2, while in [Coulibaly-Pasquier 2019] a comparison result for $P_{s,t}$ and $P^{\varrho}_{s,t}$ is used so that the conditions there include both upper and lower bounds on ϱ. On the other hand, in [Coulibaly-Pasquier 2019], the middle time $(t+s)/2$ is used as reference time, so that the conditions require ϱ to be bounded and $\mathrm{Ric}_t + \mathrm{Sc}_t$ to have a lower bound on the whole time interval. However the reference time r can be chosen close to s so that if the manifold M is compact, and if $|d\varrho_s| \le \kappa$ and $\mathrm{Ric}_s + \mathrm{Sc}_s + \mathrm{Hess}_s(\phi_s) \ge K_2$ at the initial time s, then for small $\epsilon > 0$ there exists $\delta > 0$ such that, for $u \in [s, s+\delta]$,

$$\mathrm{Ric}_u + \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \ge K_2 - \epsilon \quad \text{and} \quad |d\varrho_u| \le \kappa + \epsilon.$$

Therefore, the coefficients of the upper bound depend on $g(s)$, the lower bound of $\mathrm{Ric}_u - \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u)$, $u \in \,]s,t]$ and $\sup \varrho_u^- < \infty$, $u \in [s,t]$.

(2) Gaussian upper bounds for the heat kernel on evolving manifolds have recently also been obtained by Buzano and Yudowitz [2020]. In their paper, they assume that for each vector field X on M the following tensor inequality holds true:

$$0 \le D(\mathrm{Sc}, X) := \frac{\partial}{\partial t}S - \Delta S - 2|S_{ij}|^2 + 4(\nabla_i S_{ij})X_j - 2(\nabla_j S)X_j + 2R_{ij}X_iX_j - 2S_{ij}X_iX_j.$$

Our approach via dimension-free Harnack inequalities enables us to relax such type of conditions.

We now exemplify our estimates in some specific situations. First we consider the modified geometric flow for g_t combined with the conjugate heat equation for ϕ, i.e.,

$$\begin{cases} \partial_t g(x,t) = -2(\mathrm{Sc} + \mathrm{Hess}(\phi))(x,t), \\ \partial_t \phi_t(x) = -\Delta_t \phi_t(x) - S(x,t). \end{cases} \tag{5-5}$$

The following result holds for system (5-5).

Corollary 5.3. *Suppose that* (g_t, ϕ_t) *evolve by* (5-5) *and that, for* $0 \le s \le t \le T$, *there exists* $r \in [s, t]$ *such that*

$$\mathrm{Ric}_u - \mathrm{Sc}_u \ge K_1(u), \quad u \in [r, t], \ \mathrm{Ric}_u + \mathrm{Sc}_u + 2\,\mathrm{Hess}_u(\phi_u) \ge K_2(u), \quad u \in [s, r].$$

Then

$$p(t, x; s, x) \le \exp\left(\frac{t-s}{4\alpha_1(r,t)} + \frac{t-s}{4\alpha_2(s,r)}\right) \frac{1}{\sqrt{\mu_s(B_s(x, \sqrt{t-s}))\mu_t(B_t(x, \sqrt{t-s}))}},$$

where

$$\alpha_1(r,t) := \int_r^t \exp\left(2\int_v^t K_1(u)\,du\right) dv,$$
$$\alpha_2(s,r) := \int_s^r \exp\left(2\int_s^v K_2(u)\,du\right) dv.$$

Proof. It is immediate from the definition of ϱ that

$$\varrho_t = \partial_t \phi_t + \mathrm{tr}_{g_t}(\mathrm{Sc}_t + \mathrm{Hess}_t(\phi_t)) = \partial_t \phi_t + \Delta_t \phi_t + S_t = 0.$$

The proof is hence completed by applying Theorem 5.1. □

In particular, we may consider the standard geometric flow for the evolution of the metric g, i.e.,

$$\begin{cases} \partial_t g(t) = -2\,\mathrm{Sc}(t), \\ \phi_t(x) = 0. \end{cases} \tag{5-6}$$

For this geometric flow, we have $\varrho = S$ and thus obtain the following result.

Corollary 5.4. *Suppose that* g_t *evolves by* (5-6) *and* $\sup S_t^- < \infty$ *for each* $t \in [0, T[$. *For* $0 < s < t < T$, *there exists* $r \in (s, t)$ *such that*

$$\mathrm{Ric}_u - \mathrm{Sc}_u \ge K_1(u), \quad u \in [r, t],$$
$$|dS_u| \le \kappa(u) < \infty, \quad \mathrm{Ric}_u + \mathrm{Sc}_u \ge K_2(u), \quad u \in [s, r].$$

Then

$$p(t, x; s, x) \le \exp\left(\frac{1}{2}\int_s^t \sup S_u^-\,du + \frac{t-s}{4\alpha_1(r,t)} + \frac{t-s+2\eta_2(s,r)\sqrt{t-s}}{4\alpha_2(s,r)}\right) \frac{1}{\sqrt{\mu_s(B_s(x, \sqrt{t-s}))\mu_t(B_t(x, \sqrt{t-s}))}},$$

where

$$\alpha_1(r,t) := \int_r^t \exp\left(2\int_r^v K_1(u)\,du\right) dv,$$
$$\alpha_2(s,r) := \int_s^r \exp\left(2\int_s^v K_2(u)\,du\right) dv,$$
$$\eta_2(s,r) := \int_s^r \int_s^v \kappa(t) \exp\left(2\int_s^v K_2(u)\,du - \int_s^t K_2(u)\,du\right) dt\,dv.$$

Proof. The result follows from Theorem 5.1 by taking $\mathrm{Hess}(\phi) \equiv 0$ and $\varrho = S$. □

In particular, if we consider the heat kernel estimate under the compact Ricci flow, i.e., g_t evolving by (5-6) with $\mathrm{Sc} = \mathrm{Ric}$, then

$$K_1(t) = 0, \quad \sup R_t^-(\,\cdot\,) \le \sup R_s^-(\,\cdot\,) < \infty.$$

Hence if we further know that $\mathrm{Ric}_s \geq K$ and $|dR_s| < \kappa$ at time s, then there exists a constant C depending on K, κ, $\sup R^-$ and s, t such that

$$p(t, x; s, x) \leq \frac{C(K, \kappa, \sup R^-, s, t)}{\sqrt{\mu_s(B_s(x, \sqrt{t-s}))\, \mu_t(B_t(x, \sqrt{t-s}))}}.$$

6. Gaussian-type heat kernel estimates

In this section, we apply the dimension-free Harnack inequality to heat kernel estimates. To this end, we need the following lemma, which has been extended from [Grigoryan 1997]. Compared with the results of Section 5 above, the additional condition "$\mathrm{Sc} \geq 0$" is required.

Lemma 6.1. *For $x \in M,\ T_0 > 0,\ p > 1,\ q = p/(2(p-1))$, let*

$$\eta(s, y) = -\frac{\rho_s(x, y)^2}{T_0 - 2q(t-s)}, \quad y \in M,\ 0 < s < t < \frac{T_0}{2q}.$$

If $\mathrm{Sc}_t \geq 0$, then, for any $f \in \mathcal{B}_b^+(M)$,

$$\int_M (P_{s,t}^{\varrho/p} f)^p(y) \mathrm{e}^{\eta(s,y)}\, \mu_s(dy) \leq \int_M f^p(y) \mathrm{e}^{-\rho_t(x,y)^2/T_0}\, \mu_t(dy), \quad s < t < \frac{T_0}{2q}. \tag{6-1}$$

Proof. Let

$$I(s) = \int_M (P_{s,t}^{\varrho/p} f(y))^p \exp(\eta(s, y))\, \mu_s(dy).$$

We first take the derivative of the function η with respect to s,

$$\begin{aligned}\partial_s \eta(\cdot\,, y)(s) &= -\frac{2\rho_s(x, y)\partial_s \rho_s(x, y)}{T_0 - 2q(t-s)} + 2q \frac{\rho_s(x, y)^2}{(T_0 - 2q(t-s))^2} \\ &= \frac{2\rho_s(x, y) \int_0^{\rho_s(x,y)} \mathrm{Sc}_s(\dot\gamma_u, \dot\gamma_u)\, du}{T_0 - 2q(t-s)} + 2q \frac{\rho_s(x, y)^2}{(T_0 - 2q(t-s))^2} \geq 2q \frac{\rho_s(x, y)^2}{(T_0 - 2q(t-s))^2},\end{aligned}$$

where $\gamma : [0, \rho_s(x, y)] \to M$ is a g_s-geodesic connecting x and y. Then we have

$$\begin{aligned} I'(s) &= \int_M p(P_{s,t}^{\varrho/p} f)^{p-1}(y) \Big(-L_s + \frac{\varrho_s}{p}\Big) P_{s,t}^{\varrho/p} f(y) \exp(\eta(s, y))\, d\mu_s \\ &\qquad + \int_M (P_{s,t}^{\varrho/p} f)^p(y) \exp(\eta(s, y)) \partial_s \eta(s, y)\, d\mu_s - \int_M (P_{s,t}^{\varrho/p} f)^p(y) \exp(\eta(s, y))\, \varrho_s(y)\, d\mu_s \\ &= -\int_M p(P_{s,t}^{\varrho/p} f)^{p-1} L_s P_{s,t}^{\varrho/p}(y) f \exp(\eta(s, y))\, d\mu_s + \int_M (P_{s,t}^{\varrho/p} f)^p(y) \exp(\eta(s, y))\, \partial_s \eta(s, y)\, d\mu_s \\ &\geq -\int_M p(P_{s,t}^{\varrho/p} f)^{p-1} L_s P_{s,t}^{\varrho/p} f \exp(\eta(s, y))\, d\mu_s \\ &\qquad\qquad + 2q \int_M (P_{s,t}^{\varrho/p} f)^p(y) \exp(\eta(s, y)) \frac{\rho_s(x, y)^2}{(T_0 - 2q(t-s))^2}\, d\mu_s \\ &= \int p(p-1)(P_{s,t}^{\varrho/p} f)^p \mathrm{e}^{\eta(s,\cdot)} \bigg(\frac{|\nabla^s P_{s,t}^{\rho/p} f|_s}{P_{s,t}^{\rho/p} f} - \frac{\rho_s(x, y)}{(p-1)(T_0 - 2q(t-s))}\bigg)^2 d\mu_s \geq 0. \end{aligned}$$

By integrating the function I' from s to t, we prove inequality (6-1). □

Theorem 6.2. *Let $p(t,x;s,y)$ be the minimal fundamental solution of*

$$\begin{cases}\partial_t p(t,x;s,y) = L_t p(t,\cdot;s,y)(x),\\ \lim_{t\downarrow s} p(t,x;s,y) = \delta(y).\end{cases}$$

Assume that $\mathrm{Sc}_t \geq 0$ *and* $\sup \varrho_t^- < \infty$ *for* $t \in [0,T]$. *If there exists* $r \in (s,t)$ *such that*

$$\begin{aligned}\mathrm{Ric}_u - \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) &\geq K_1(u), \quad u \in [r,t],\\ \mathrm{Ric}_u + \mathrm{Sc}_u + \mathrm{Hess}_t \geq K_2(u) \quad &\textit{and} \quad |d\varrho_u| \leq \kappa(u), \quad u \in [s,r],\end{aligned}$$

for some functions $K_1 \in C([r,t])$ *and* $\kappa, K_2 \in C([s,r])$, *then, for any* $\delta > 2$ *and* $s < r \leq t \leq T$,

$$p(t,x;s,y) \leq \frac{C}{\sqrt{\mu_s(B_s(x,\sqrt{t-s}))}\sqrt{\mu_t(B_t(y,\sqrt{t-s}))}} \exp\left(-\frac{\rho_s(x,y)^2}{2\delta(t-s)}\right),$$

where

$$C = \exp\left\{\frac{1}{2}\int_s^t \sup \varrho^-(u,\cdot)\,du + \frac{p(t-s)}{4(2-p)}\left(\frac{1}{\alpha_1(r,t)} + \frac{1}{\alpha_2(s,r)}\right) + \frac{\eta_2(s,r)}{\alpha_2(s,r)} + \frac{1}{2(\delta-2q)}\right\}$$

for $p \in (1+1/(\delta-1),2)$ *and* $\alpha_1, \alpha_2, \eta_2$ *as in* (5-1).

Proof. Since $\mathrm{Sc}_t \geq 0$, we have $\rho_s \leq \rho_r$ for $r \geq s$. We observe that

$$\begin{aligned}&p(t,x;s,y)\mathrm{e}^{\rho_s(x,y)^2/(4T_0)}\\ &\quad\leq p(t,x;s,y)\mathrm{e}^{\rho_r(x,y)^2/(4T_0)}\\ &\quad\leq \int_M p(t,x;r,z)p(r,z;s,y)\mathrm{e}^{2(\rho_r(x,z)^2+\rho_r(z,y)^2)/(4T_0)}\,\mu_r(dz)\\ &\quad\leq \left(\int_M p(r,x;r,z)\mathrm{e}^{\rho_r(x,z)^2/T_0}\,\mu_r(dz)\right)^{1/2}\left(\int_M p(t,z;s,y)^2\mathrm{e}^{\rho_r(z,y)^2/T_0}\,\mu_r(dz)\right)^{1/2},\end{aligned} \tag{6-2}$$

where $T_0 = \delta(t-s)$. Hence we are left with the task to estimate the two terms

$$\begin{aligned}I_1 &= \int_M p(t,x;r,z)^2\mathrm{e}^{\rho_r(x,z)^2/T_0}\,\mu_r(dz),\\ I_2 &= \int_M p(r,z;s,y)^2\mathrm{e}^{\rho_r(z,y)^2/T_0}\,\mu_r(dz).\end{aligned}$$

In order to estimate I_1 we proceed with Theorem 5.1. Recall that by definition

$$\bar{p}(s,x;u,y) = p(t-s,x;t-u,y) \quad \text{for } 0 \leq s \leq u \leq t < T,$$

and write

$$\bar{P}_{s,u}f = \int \bar{p}(s,x;u,y)f(y)\,\mu_{t-s}(dy) \quad \text{for } f \in \mathcal{B}_b(M).$$

As

$$\mathrm{Ric}_u - \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \geq K_1(u), \quad u \in [r,t],$$

for some $K_1 \in C([r,t])$ and for $p \in (1,2)$ such that $q := p/[2(p-1)] < \delta/2$, we have

$$
\begin{aligned}
(\bar{P}_{0,t-r} f)^2(x)\, \mu_t(B_t(x,\sqrt{t-s})) &\exp\left(-\frac{p(t-s)}{2(2-p)\int_r^t \exp\left(2\int_v^t K_1(u)\,du\right) dv} - \frac{1}{\delta-2q}\right) \\
&\le \int_M (\bar{P}_{0,t-r} f)^2(x) \exp\left(-\frac{p\rho_t^2(x,y)}{2(2-p)\int_r^t \exp\left(2\int_v^t K_1(u)\,du\right) dv} - \frac{\rho_t(x,y)^2}{T_0-2q(t-s)}\right) \mu_t(dy) \\
&\le \int_M (\bar{P}_{0,t-r} f^{2/p})^p(y) \exp\left(-\frac{\rho_t(x,y)^2}{T_0-2q(t-s)}\right) \mu_t(dy) \\
&\le \exp\left(\int_r^t \sup \varrho^-(u,\cdot\,)\,du\right) \int_M f(y)^2 \mathrm{e}^{-\rho_r(x,y)^2/T_0}\, \mu_r(dy),
\end{aligned}
$$

where the second inequality comes from the dimension-free Harnack inequality (see Theorem 4.1); the third inequality is a consequence of Lemma 6.1. Taking

$$
f(y) := (k \wedge p(t,x;r,y))\mathrm{e}^{(k\wedge \rho_r(x,y)^2)/T_0}, \quad y \in M,\ k \in \mathbb{N},
$$

we obtain

$$
\begin{aligned}
\int_M (k \wedge p(t,x;r,y))^2 \mathrm{e}^{k\wedge \rho_r(x,y)^2/T_0}\, \mu_r(dy) \\
\le \exp\left(\int_r^t \sup \varrho^-(u,\cdot\,)\,du + \frac{p(t-s)}{2(2-p)\alpha_1(r,t)} + \frac{1}{\delta-2q}\right) \frac{1}{\mu_t(B_t(x,\sqrt{t-s}))},
\end{aligned}
$$

where

$$
\alpha_1(r,t) = \int_r^t \exp\left(2\int_v^t K_1(u)\,du\right) dv.
$$

Letting $k \to \infty$, we arrive at

$$
\begin{aligned}
I_1 &= \int_M p(t,x;r,z)^2 \mathrm{e}^{\rho_r(x,y)^2/(\delta(t-s))}\, \mu_r(dz) \\
&\le \exp\left(\int_r^t \sup \varrho^-(u,\cdot\,)\,du + \frac{p(t-s)}{2(2-p)\alpha_1(r,t)} + \frac{1}{\delta-2q}\right) \frac{1}{\mu_t(B_t(x,\sqrt{t-s}))}. \qquad (6\text{-}3)
\end{aligned}
$$

To estimate the second term I_2, we write

$$
\int_M p(r,z;s,y)^2\, \mu_r(dz) = \int_M p^*(s,y;r,z)^2\, \mu_r(dz),
$$

where p^* denotes the adjoint heat kernel to p. Recall that

$$
\begin{cases}
\partial_s p^*(s,y;r,z) = -L_s p^*(s,\cdot\,;r,z)(y) + \varrho_s(y) p^*(s,y;r,z), \\
\lim_{s\uparrow r} p^*(s,y;r,z) = \delta_y(x).
\end{cases}
$$

Let $\{P_{s,r}^{\varrho}\}_{0\le s\le r\le t}$ be the semigroup generated by the operator $L_t - \varrho_t$. By Theorem 4.1, using the assumption

$$
\mathrm{Ric}_t + \mathrm{Sc}_t + \mathrm{Hess}_t(\phi_t) \ge K_2(t) \quad \text{and} \quad |d\varrho_t| \le \kappa(t)
$$

for $t \in (s, r_0)$, we have

$$(P^{\varrho/p}_{s,r} f)^2(x) \le (P^{\rho/p}_{s,r} f^{2/p})^p(y) \exp\biggl(\int_s^r \frac{(2-p)}{p} \sup \varrho^-(u,\cdot)\,du + \frac{p\rho_s^2(x,y)}{2(2-p)\alpha_2(s,r)} + \frac{2\eta_2(s,r)\rho_s(x,y)}{\alpha_2(s,r)}\biggr),$$

where $s \le r \le r_0$ and

$$\alpha_2(s,r) = \int_s^r \exp\biggl(2\int_s^v K_2(u)\,du\biggr)\,dv,$$
$$\eta_2(s,r) = \int_s^r \int_s^v \kappa(u) \exp\biggl(2\int_s^v K_2(t)\,dt - \int_s^u K_2(t)\,dt\biggr)\,du\,dv.$$

By means of this formula, we can proceed as above to obtain

$$\begin{aligned}(P^{\varrho/p}_{s,r} f)^2(x)\mu_s(B_s(x,\sqrt{t-s}))\exp\biggl(-\int_s^r \sup\varrho^-(u,\cdot)\,du - \frac{p(t-s)}{2(2-p)\alpha_2(s,r)} - \frac{2\eta_2(s,r)\sqrt{t-s}}{\alpha_2(s,r)} - \frac{1}{\delta-2q}\biggr) \\ \le \int_M (P^{\varrho/p}_{s,r} f^{2/p})^p(x) \exp\biggl(-\frac{\rho_s(x,y)^2}{T_0-2q(t-s)}\biggr)\mu_s(dy) \\ \le \int_M f(y)^2 \exp\biggl(-\frac{\rho_r(x,y)^2}{T_0}\biggr)\mu_r(dy), \qquad (6\text{-}4)\end{aligned}$$

where $T_0 = \delta(t-s)$. Thus taking $f(z) := p^*(s,y;r,z) \wedge k$ and letting $k \to \infty$, we obtain

$$\begin{aligned}I_2 &= \int_M p^*(s,y;r,z)^2 \mathrm{e}^{\rho_r(x,y)^2/(\delta(r-s))}\,\mu_r(dz) \\ &\le \exp\biggl(\int_s^r \sup\varrho^-(u,\cdot)\,du + \frac{p(t-s)}{2(2-p)\alpha_2(s,r)} + \frac{2\eta_2(s,r)\sqrt{t-s}}{\alpha_2(s,r)} + \frac{1}{\delta-2q}\biggr)\frac{1}{\mu_s(B_s(y,\sqrt{t-s}))}.\end{aligned}$$

Finally, combining (6-3) and (6-4) we obtain

$$\begin{aligned}p(t,x;s,y) &\le \sqrt{I_1 I_2} \\ &\le \exp\biggl(\frac{1}{2}\int_s^t \sup\varrho^-(u,\cdot)\,du + \frac{p(t-s)}{4(2-p)}\biggl(\frac{1}{\alpha_1(r,t)} + \frac{1}{\alpha_2(s,r)}\biggr) + \frac{\eta_2(s,r)\sqrt{t-s}}{\alpha_2(s,r)} + \frac{1}{2(\delta-2q)}\biggr) \\ &\qquad \times \frac{1}{\sqrt{\mu_s(B_s(x,\sqrt{t-s}))\mu_t(B_t(y,\sqrt{t-s}))}} \exp\biggl\{-\frac{\rho_s(x,y)^2}{\delta(t-s)}\biggr\},\end{aligned}$$

where the functions $\alpha_1, \alpha_2, \eta_2$ are defined by (6-2). □

Let M be a compact manifold. Since $\mathrm{Ric} + \mathrm{Sc} + \mathrm{Hess}(\phi)$ and $|d\varrho|$ are continuous in time and space, we may choose r in Theorem 6.2 close to s such that if $\mathrm{Ric} + \mathrm{Sc} + \mathrm{Hess}(\phi)$ has a lower bound and $|d\varrho|$ an upper bound at time s, then these terms will also have a lower, resp. upper, bound close to s. We state a consequence of this observation in the following corollary.

Corollary 6.3. *Let M be a compact manifold. Assume $\mathrm{Sc}_u \ge 0$ and $\sup \varrho_u^- < \infty$ for $u \in [0,T]$, as well as $\mathrm{Ric}_u - \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \ge K_1(u)$ for some continuous function $K_1 \in C([0,T])$. For $s \in [0,T[$, if there* *exist constants K_2 and κ such that*

$$\mathrm{Ric}_s + \mathrm{Sc}_s + \mathrm{Hess}_s(\phi_s) \geq K_2 \quad \textit{and} \quad |d\varrho_s| \leq \kappa,$$

then, for any $\delta > 2$ and $s \leq t \leq T$, there exists a constant C such that

$$p(t, x; s, y) \leq C \exp\left\{-\frac{\rho_s(x, y)^2}{2\delta(t-s)}\right\} \frac{1}{\sqrt{\mu_s(B(x, \sqrt{t-s}))}\sqrt{\mu_t(B(y, \sqrt{t-s}))}},$$

where C depends on K_1, K_2, κ, $\sup_{[0,T]\times M} \varrho^-$ and s, t.

Proof. As there exist constants K_2 and κ such that

$$\mathrm{Ric}_s + \mathrm{Sc}_s + \mathrm{Hess}_s(\phi_s) \geq K_2 \quad \text{and} \quad |d\varrho_s| \leq \kappa,$$

we conclude that, for any $\epsilon > 0$, there exists $r_0 > s$ such that for $u \in [s, r_0]$ we still have

$$\mathrm{Ric}_u + \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \geq K_2 - \epsilon \quad \text{and} \quad |d\varrho_u| \leq \kappa + \epsilon.$$

Then by means of Theorem 6.2 we can then complete the proof. □

Remark 6.4. (1) Suppose that the geometric flow g_t is evolving as a compact Ricci flow, i.e., the manifold is compact, $\mathrm{Sc}_t = \mathrm{Ric}_t$ and $\mathrm{Ric}_t \geq 0$ for $t \in [0, T[$. We consider the estimate for the heat kernel $p(t, x; s, y)$ generated by Δ_t. Hence if $\mathrm{Ric}_s \geq K$ and $|dR_s| \geq \kappa$ at the initial time s, then, for $T \geq t \geq s$, there exists a constant C depending on K, κ, $\sup R(s, \cdot)$ and s, t such that

$$p(t, x; s, y) \leq C \exp\left\{-\frac{\rho_s(x, y)^2}{2\delta(t-s)}\right\} \frac{1}{\sqrt{\mu_s(B_s(x, \sqrt{t-s}))}\sqrt{\mu_t(B_t(y, \sqrt{t-s}))}}.$$

(2) Theorem 6.2 and Corollary 6.3 can be applied to the modified geometric flow (5-5) and the standard geometric flow (5-6) as well.

7. Super log-Sobolev inequalities and conjugate semigroup properties

The semigroup $P^{\varrho}_{s,t}$ is called supercontractive if it maps $L^p(M, \mu_t)$ into $L^q(M, \mu_s)$, i.e.,

$$\|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)} < \infty$$

for any $1 < p < q < +\infty$ and $0 \leq s \leq t < T$. In the following section, we investigate the relation between supercontractivity of $P^{\varrho}_{s,t}$ and a log-Sobolev inequality with respect to μ_t, which is viewed as another important application of the derivative formula of the conjugate semigroup.

We state first the main results of this section. Thanks to the gradient estimate for $P^{\varrho}_{s,t}$ and the fact that the family of measures $\{\mu_t\}$ takes over the role of the invariant measure, the results can be proved much as in [Wang 2005, Chapter 5] and [Röckner and Wang 2003]. We include proofs in the Appendix for the reader's convenience.

Theorem 7.1. *Assume that ϱ_t is bounded and $(\partial_t - L_t)\varrho_t \geq 0$,*

$$\mathrm{Ric}_t + \mathrm{Hess}_t(\phi_t) + \mathrm{Sc}_t \geq K(t), \quad |d\varrho_t| \leq \kappa(t) \quad \textit{for } t \in [0, T[.$$

If

$$\|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)} < \infty \quad \textit{for } 1 < p < q \textit{ and } 0 \le s \le t < T,$$

then, for every $f \in H^1(M, \mu_t)$ *such that* $\|f\|_{2,t} = 1$, $t \in [0, T[$, *the following super log-Sobolev inequalities hold*:

$$\int f^2 \log f^2 \, d\mu_t \le r \int \big(|\nabla^t f|_t^2 + \tfrac{1}{4}\varrho_t f^2\big)\, d\mu_t + \beta_t(r), \quad r > 0, \tag{7-1}$$

where $\beta_t(r) = \tilde{\beta}_t(\gamma_t^{-1}(r), t)$ *and*

$$\begin{aligned}
\tilde{\beta}_t(s,t) &= \frac{pq}{q-p}\log\Big(\|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)}\Big) + 2\int_s^t \bigg(\int_r^t \kappa(u)\exp\Big(-\int_r^u K(v)\,dv\Big)\,du\bigg)^2 dr,\\
\gamma(s,t) &= \frac{4p(q-1)}{q-p}\int_s^t \exp\Big(-2\int_r^t K(u)\,du\Big)\,dr,\\
\gamma_t^{-1}(r) &= \inf\{s \in [0,t] : \gamma(s,t) \le r\}.
\end{aligned}$$

Remark 7.2. (i) The log-Sobolev inequality (7-1) has been shown to be equivalent to the Sobolev inequality and can hence be used to obtain an upper bound for the heat kernel; see, e.g., [Zhang 2006] for the situation under Ricci flow, and [Buzano and Yudowitz 2020] for a general geometric flow. Moreover, the log-Sobolev inequality can be used to characterize supercontractivity of the conjugate semigroup with respect to the volume measure. Note that in [Buzano and Yudowitz 2020], the authors start with the condition that $D(\mathrm{Sc}, X) \ge 0$, which implies $(\partial_t - \Delta_t)S_t \ge 0$ and has been used in the proof of the log-Sobolev inequality. We follow a different approach but include the condition $(\partial_t - L_t)\varrho_t \ge 0$ to obtain the log-Sobolev inequality.

(ii) In Theorem 7.1 information about $\mathrm{Ric} + \mathrm{Hess}(\phi) + \mathrm{Sc}$ and $|d\varrho|$ on the time interval $[s,t]$ is used to obtain the log-Sobolev inequality (7-1) with respect to the measure μ_t. By a time reversal as in the proof of Theorem 5.1, the lower bound on $\mathrm{Ric} + \mathrm{Hess}(\phi) - \mathrm{Sc}$ and the bound on $|d\varrho|$ allow us to get the log-Sobolev inequality with a modified β_t.

In addition, we observe that the inequality above has a close relationship with supercontractivity of the conjugate semigroup as follows.

Theorem 7.3. *Suppose that there exists a function* $\beta_t : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\mu_t(f^2 \log f^2) \le r\mu_t\big(|\nabla^t f|_t^2 + \tfrac{1}{4}\varrho_t f^2\big) + \beta_t(r) \quad \textit{for all } t \in [0, T[\textit{ and } \|f\|_{2,t} = 1. \tag{7-2}$$

Then

$$\|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)} < \infty \quad \textit{for } 1 < p < q \textit{ and } 0 \le s \le t < T.$$

Proof. The main idea of proof is from [Röckner and Wang 2003]; we include it in the Appendix for the reader's convenience. □

As an application, we specify the general results above in the case of the modified geometric flow (M, g_t, ϕ_t) evolving by

$$\begin{cases} \partial_t g(x,\cdot)(t) = -2(\mathrm{Sc} + \mathrm{Hess}(\phi))(x,t),\\ \partial_t \phi_t = -\Delta_t \phi_t - S_t. \end{cases} \tag{7-3}$$

For this system, we have $\varrho \equiv 0$. Thus, by Theorem 7.1 with $\varrho \equiv 0$, we obtain the following result.

Corollary 7.4. *Assume that* (g_t, ϕ_t) *follows the evolving equation* (7-3) *and*

$$\mathrm{Ric}_t + \mathrm{Hess}_t(\phi_t) + \mathrm{Sc}_t \geq K(t) \quad \textit{for all } t \in [0, T[.$$

Suppose that

$$\|P_{s,t}\|_{(p,t)\to(q,s)} < \infty \quad \textit{for } 1 < p < q \textit{ and } 0 \leq s \leq t < T.$$

Then the super log-Sobolev inequalities

$$\int f^2 \log f^2 \, d\mu_t \leq r \int |\nabla^t f|_t^2 \, d\mu_t + \beta_t(r), \quad r > 0, \tag{7-4}$$

hold for every $f \in H^1(M, \mu_t)$ *such that* $\|f\|_{2,t} = 1$, $t \in [0, T[$, *where* $\beta_t(r) := \tilde{\beta}(\gamma_t^{-1}(r), t)$ *and*

$$\begin{aligned}
\tilde{\beta}(s,t) &= \frac{pq}{q-p} \log(\|P_{s,t}\|_{(p,t)\to(q,s)}), \\
\gamma(s,t) &= \frac{4p(q-1)}{q-p} \int_s^t \exp\left(-2\int_r^t K(u)\,du\right) dr, \\
\gamma_t^{-1}(r) &= \inf\{s \geq 0 : \gamma(s,t) \leq r\}.
\end{aligned}$$

Finally, we are in position to give a necessary and sufficient condition for supercontractivity of $P^{\varrho}_{s,t}$. By the dimension-free Harnack inequality and by using the method from [Röckner and Wang 2003] (see the proof in the Appendix), we have the following result.

Theorem 7.5. *Suppose that* ϱ_t *is bounded from below and*

$$\mathrm{Ric}_t + \mathrm{Hess}_t(\phi_t) + \mathrm{Sc}_t \geq K(t), \quad |d\varrho_t| \leq \kappa(t) \quad \textit{for } t \in [0, T[.$$

Then the condition

$$\|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)} < \infty \quad \textit{for all } 1 < p < q < \infty \textit{ and } 0 \leq s \leq t < T$$

holds if and only if

$$\mu_t(\exp(\lambda\rho_t^2)) < \infty \quad \textit{for all } \lambda > 0 \textit{ and } t \in [0, T[.$$

Applied to the modified geometric flow ($\varrho \equiv 0$), we have the following corollary from Theorem 7.5.

Corollary 7.6. *Assume that* (g_t, ϕ_t) *evolves according to* (7-3) *and that*

$$\mathrm{Ric}_t + \mathrm{Hess}_t(\phi_t) + \mathrm{Sc}_t \geq K(t), \quad t \in [0, T[,$$

for some function $K \in C([0, T[)$. *Then*

$$\|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)} < \infty \quad \textit{for } 1 < p < q < \infty \textit{ and } 0 \leq s \leq t < T$$

if and only if

$$\mu_t(\exp(\lambda\rho_t^2)) < \infty \quad \textit{for } \lambda > 0 \textit{ and } t \in [0, T[.$$

Appendix

To prove Theorem 7.1, we first establish a log-Sobolev inequality with respect to the semigroup $P^{\varrho}_{s,t}$.

Proposition A.1. *Assume that*

$$\mathrm{Ric}_t + \mathrm{Hess}_t(\phi_t) + \mathrm{Sc}_t \geq K(t), \quad \sup|\varrho_t| < \infty \quad \text{and} \quad |d\varrho_t| \leq \kappa(t) \quad \text{for } t \in [0, T[.$$

Then, for $0 \leq s \leq t < T$ *and* $f \in C_0^\infty(M)$,

$$P^{\varrho}_{s,t}(f^2 \log f^2) \leq 4\biggl(\int_s^t \exp\biggl(-2\int_r^t K(u)\,du\biggr)dr\biggr)P^{\varrho}_{s,t}|\nabla^t f|_t^2 + P^{\varrho}_{s,t} f^2 \log P^{\varrho}_{s,t} f^2$$
$$+2\int_s^t \biggl(\int_r^t \kappa(u)\exp\biggl(-\int_r^u K(v)\,dv\biggr)du\biggr)^2 dr\, P^{\varrho}_{s,t} f^2 + \int_s^t P_{s,r}(\varrho_r P_{r,t} f^2)\,dr.$$

Proof. Without loss of generality, we assume that $f \geq \delta$ for some $\delta > 0$. Otherwise, we may take $f_\delta = (f^2+\delta)^{1/2}$ and pass to the limit $\delta \downarrow 0$ to obtain the conclusion.

Now consider the process

$$r \mapsto (P_{r,t} f^2)\log(P_{r,t} f^2)(X_{r\wedge\tau_k}),$$

where as above

$$\tau_k = \inf\{t \in [s, T[\, : \rho_t(o, X_t) \geq k\}, \quad k \geq 1. \tag{A-1}$$

By means of Itô's formula, we have

$$\begin{aligned} d(P^{\varrho}_{r,t} f^2)\log(P^{\varrho}_{r,t} f^2)(X_r) &= dM_r + (L_r + \partial_r)(P^{\varrho}_{r,t} f^2 \log P^{\varrho}_{r,t} f^2)(X_r)\,dr \\ &= dM_r + \biggl(\frac{1}{P^{\varrho}_{r,t} f^2}|\nabla^r P^{\varrho}_{r,t} f^2|_r^2 + \varrho_r(1+\log P^{\varrho}_{r,t} f^2)P^{\varrho}_{r,t} f^2\biggr)(X_r)\,dr, \quad r \leq \tau_k \wedge t, \end{aligned} \tag{A-2}$$

where M_r is a local martingale. On the other hand, by Corollary 3.2, we have the estimate

$$\begin{aligned} |\nabla^r P^{\varrho}_{r,t} f^2|_r &\leq \exp\biggl(-\int_r^t K(u)\,du\biggr)P^{\varrho}_{r,t}|\nabla^t f^2|_t + P^{\varrho}_{r,t} f^2 \int_r^t \kappa(u)\exp\biggl(-\int_r^u K(v)\,dv\biggr)du \\ &\leq 2\exp\biggl(-\int_r^t K(u)\,du\biggr)P^{\varrho}_{r,t}(f|\nabla^t f|_t) + P^{\varrho}_{r,t} f^2 \int_r^t \kappa(u)\exp\biggl(-\int_r^u K(v)\,dv\biggr)du \\ &\leq 2\exp\biggl(-\int_r^t K(u)\,du\biggr)\sqrt{P^{\varrho}_{r,t}(f^2)P^{\varrho}_{r,t}(|\nabla^t f|_t^2)} + P^{\varrho}_{r,t} f^2 \int_r^t \kappa(u)\exp\biggl(-\int_r^u K(v)\,dv\biggr)du, \end{aligned}$$

which gives

$$\begin{aligned} &|\nabla^r P^{\varrho}_{r,t} f^2|_r^2 \\ &\quad\leq 4\exp\biggl(-2\int_r^t K(u)\,du\biggr)(P^{\varrho}_{r,t} f^2)P^{\varrho}_{r,t}(|\nabla^t f|_t^2) + 2(P^{\varrho}_{r,t} f^2)^2\biggl(\int_r^t \kappa(u)\exp\biggl(-\int_r^u K(v)\,dv\biggr)du\biggr)^2. \end{aligned}$$

Substituting back into (A-2), we obtain

$$
\begin{aligned}
d\bigg(&\exp\Big(-\int_s^r \varrho_u(X_u)\,du\Big)(P^{\varrho}_{r,t}f^2)\log(P^{\varrho}_{r,t}f^2)(X_r)\bigg)\\
&\le dM_r+4\exp\Big(-2\int_r^t K(u)\,du-\int_s^r \varrho_u(X_u)\,du\Big)P^{\varrho}_{r,t}|\nabla^t f|_t^2(X_r)\,dr\\
&\quad+2\bigg(\int_r^t \kappa(u)\exp\Big(-\int_r^u K(v)\,dv\Big)du\bigg)^2\exp\Big(-\int_s^r \varrho_u(X_u)\,du\Big)P^{\varrho}_{r,t}f^2(X_r)\,dr\\
&\qquad+\varrho_r(X_r)\exp\Big(-\int_s^r \varrho_u(X_u)\,du\Big)P^{\varrho}_{r,t}f^2(X_r)\,dr,\quad 0\le s\le r\le \tau_k\wedge t.
\end{aligned}
$$

Integrating both sides from s to $t\wedge\tau_k$, taking the expectation, and letting $k\uparrow+\infty$, we obtain by dominated convergence

$$
\begin{aligned}
P^{\varrho}_{s,t}&(f^2\log f^2)-P^{\varrho}_{s,t}f^2\log(P^{\varrho}_{s,t}f^2)\\
&\le 4\int_s^t \exp\Big(-2\int_r^t K(u)\,du\Big)\,dr\ P^{\varrho}_{s,t}|\nabla^t f|_t^2\\
&\quad+\int_s^t\bigg\{2\bigg(\int_r^t \kappa(u)\exp\Big(-\int_r^u K(v)\,dv\Big)du\bigg)^2\bigg\}\,dr\ P^{\varrho}_{s,t}f^2+\int_s^t P^{\varrho}_{s,r}(\varrho_r P^{\varrho}_{r,t}f^2)\,dr,
\end{aligned}
$$

or in other words,

$$
\begin{aligned}
P^{\varrho}_{s,t}(f^2\log f^2)&\le 4\bigg(\int_s^t \exp\Big(-2\int_r^t K(u)\,du\Big)\,dr\bigg)P^{\varrho}_{s,t}|\nabla^t f|_t^2+P^{\varrho}_{s,t}f^2\log P^{\varrho}_{s,t}f^2\\
&\quad+\int_s^t\bigg\{2\bigg(\int_r^t \kappa(u)\exp\Big(-\int_r^u K(v)\,dv\Big)du\bigg)^2\bigg\}\,dr\ P^{\varrho}_{s,t}f^2+\int_s^t P^{\varrho}_{s,r}(\varrho_r P^{\varrho}_{r,t}f^2)\,dr,
\end{aligned}
\tag{A-3}
$$

completing the proof. □

Proof of Theorem 7.1. Since

$$
\log^+(P^{\varrho}_{s,t}f^2)\le P^{\varrho}_{s,t}f^2\le \exp\bigg(\int_s^t \sup\varrho_u^-\,du\bigg)\|f\|_\infty^2,
$$

we can integrate both sides of the log-Sobolev inequality (A-3) with respect to μ_s. Taking (1-3) into account, we get

$$
\begin{aligned}
\mu_t(f^2\log f^2)&\le 4\bigg(\int_s^t \exp\Big(-2\int_r^t K(u)\,du\Big)\,dr\bigg)\mu_t(|\nabla^t f|_t^2)+\mu_s(P^{\varrho}_{s,t}f^2\log P^{\varrho}_{s,t}f^2)\\
&\quad+\int_s^t\bigg\{2\bigg(\int_r^t \kappa(u)\exp\Big(-\int_r^u K(v)\,dv\Big)du\bigg)^2\bigg\}\,dr\,\mu_t(f^2)+\int_s^t \mu_r(\varrho_r P^{\varrho}_{r,t}f^2)\,dr.
\end{aligned}
\tag{A-4}
$$

For the last term above, we have

$$
\begin{aligned}
\partial_r(\mu_r(\varrho_r P^{\varrho}_{r,t}f^2))&=\mu_r(-\varrho_r^2 P^{\varrho}_{r,t}f^2)+\mu_r((\partial_r\varrho_r)P^{\varrho}_{r,t}f^2)-\mu_r(\varrho_r(L_r-\varrho_r)P^{\varrho}_{r,t}f^2)\\
&=\mu_r((\partial_r\varrho_r-L_r\varrho_r)P^{\varrho}_{r,t}f^2)\ge 0.
\end{aligned}
$$

It then follows that

$$\mu_r(\varrho_r P^{\varrho}_{r,t} f^2) \le \mu_t(\varrho_t f^2).$$

Moreover,

$$\mu_t(f^2\log f^2) \le 4\biggl(\int_s^t \exp\biggl(-2\int_r^t K(u)\,du\biggr)\vee 1\,dr\biggr)\mu_t\bigl(|\nabla^t f|_t^2+\tfrac14\varrho_t f^2\bigr)+\mu_s(P^{\varrho}_{s,t}f^2\log P^{\varrho}_{s,t}f^2)$$
$$+\int_s^t\biggl\{2\biggl(\int_r^t \kappa(u)\exp\biggl(-\int_r^u K(v)\,dv\biggr)du\biggr)^2\biggr\}dr\,\mu_t(f^2).$$

We deal first with the term $\mu_s(P^{\varrho}_{s,t}f^2\log P^{\varrho}_{s,t}f^2)$. Let $1<p<q$. For any $h\in\,]0,1-1/p[$ let

$$r_h=\frac{ph}{p-1}\in\,]0,1[.$$

By the Riesz–Thorin interpolation theorem, we have

$$\|P^{\varrho}_{s,t}f\|_{q_h,s}\le \|P^{\varrho}_{s,t}\|^{r_h}_{(p,t)\to(q,s)}\,\|P^{\varrho}_{s,t}\|^{1-r_h}_{(1,t)\to(1,s)}\,\|f\|_{p_h,t},\qquad f\in L^p(M,\mu_s),\tag{A-5}$$

where

$$\frac{1}{p_h}=\frac{1-r_h}{1}+\frac{r_h}{p}\quad\text{and}\quad \frac{1}{q_h}=\frac{1-r_h}{1}+\frac{r_h}{q}.$$

Thus

$$p_h=\frac{1}{1-h}\quad\text{and}\quad q_h=\biggl(1-\frac{p(q-1)}{q(p-1)}h\biggr)^{-1}.$$

Since $\|P^{\varrho}_{s,t}\|_{(1,t)\to(1,s)}\le 1$, we get from (A-5) that

$$\int (P^{\varrho}_{s,t}|f|^{2(1-h)})^{q_h}\,d\mu_s\le \|P^{\varrho}_{s,t}\|^{r_hq_h}_{(p,t)\to(q,s)}\,\|f\|^{q_h/p_h}_{2,t}.$$

Then, for $f\in C_0^\infty(M)$ satisfying $\|f\|_{2,t}=1$, we have

$$\frac1h\biggl(\int (P^{\varrho}_{s,t}|f|^{2(1-h)})^{q_h}\,d\mu_s-\biggl(\int P^{\varrho}_{s,t}|f|^2\,d\mu_s\biggr)^{q_h/p_h}\biggr)=\frac1h\biggl(\int (P^{\varrho}_{s,t}|f|^{2(1-h)})^{q_h}\,d\mu_s-1\biggr)$$
$$\le\frac1h(\|P^{\varrho}_{s,t}\|^{r_hq_h}_{(p,t)\to(q,s)}-1).\tag{A-6}$$

Taking the limit as $h\to 0$ in (A-6), as

$$\lim_{h\to0}\frac1h(\|P^{\varrho}_{s,t}\|^{r_hq_h}_{(p,t)\to(q,s)}-1)=\frac{p}{p-1}\log\|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)},$$

we obtain by dominated convergence

$$\frac{p(q-1)}{q(p-1)}\int P^{\varrho}_{s,t}f^2\log P^{\varrho}_{s,t}f^2\,d\mu_s-\int P^{\varrho}_{s,t}(f^2\log f^2)\,d\mu_s\le\frac{p}{p-1}\log\|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)},$$

or equivalently,

$$\mu_s(P^{\varrho}_{s,t}f^2\log P^{\varrho}_{s,t}f^2)\le\frac{q(p-1)}{p(q-1)}\,\mu_t(f^2\log f^2)+\frac{q}{q-1}\log\|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)}.\tag{A-7}$$

Substituting (A-7) back into (A-4), we arrive at

$$\mu_t(f^2\log f^2)\le\gamma(s,t)\,\mu_t\bigl(|\nabla^t f|_t^2+\tfrac14\varrho_t f^2\bigr)+\tilde\beta(s,t)\tag{A-8}$$

for every $f \in C_0^\infty(M)$ satisfying $\|f\|_{2,t} = 1$, where

$$\gamma(s,t) = \frac{4p(q-1)}{q-p}\int_s^t \left[\exp\left(-2\int_r^t K(u)\,du\right) \vee 1\right] dr,$$

$$\tilde\beta(s,t) = \frac{pq}{q-p}\log(\|P_{s,t}^\varrho\|_{(p,t)\to(q,s)}) + 2\frac{p(q-1)}{q-p}\int_s^t \left(\int_r^t \kappa(u)\exp\left(-\int_r^u K(v)\,dv\right)du\right)^2 dr.$$

We complete the proof by letting

$$\beta_t(r) = \tilde\beta(\gamma_t^{-1}(r), t). \qquad \square$$

Proof of Theorem 7.3. Let $0 \le s < t < T$ and $f \in C_0^\infty(M)$ such that $f \ge \delta > 0$. To calculate the derivative of $\mu_s(P_{s,t}^\varrho f)^{q(s)}$ with respect to s, we start with some preparatory calculations:

$$\begin{aligned}(L_s + \partial_s)&(P_{s,t}^\varrho f)^{q(s)}\\ &= L_s(P_{s,t}^\varrho f)^{q(s)} + q(s)\,(P_{s,t}^\varrho f)^{q(s)-1}(\partial_s P_{s,t}^\varrho f) + q'(s)(P_{s,t}^\varrho f)^{q(s)}\log P_{s,t}^\varrho f\\ &= q(s)(q(s)-1)(P_{s,t}^\varrho f)^{q(s)-2}|\nabla^s P_{s,t}^\varrho f|_s^2 + q'(s)(P_{s,t}^\varrho f)^{q(s)}\log P_{s,t}^\varrho f + q(s)\varrho_s(P_{s,t}^\varrho f)^{q(s)}. \qquad \text{(A-9)}\end{aligned}$$

By Corollary 3.2, there exist positive constants $c_1(s,t)$ and $c_2(s,t)$ such that

$$\||\nabla^s P_{s,t}^\varrho f|_s^2\|_\infty \le c_1(s,t)\,\|f\|_\infty^2 + c_2(s,t)\,\||\nabla^t f|_t\|_\infty^2.$$

Moreover, $\|P_{s,t}^\varrho f\|_\infty \le (P_{s,t}^\varrho 1)\,\|f\|_\infty$ and

$$(P_{s,t}^\varrho f)^{q(s)}\log^+(P_{s,t}^\varrho f) \le (P_{s,t}^\varrho f)^{q(s)+1} \le (P_{s,t}^\varrho 1)^{q(s)+1}\,\|f\|_\infty^{q(s)+1}.$$

Combining these estimates, we obtain

$$\|(L_s + \partial_s)(P_{s,t}^\varrho f)^{q(s)}\|_\infty < \infty.$$

Now, by Theorem 4.1, we see that

$$\begin{aligned}\frac{d}{ds}\mu_s((P_{s,t}^\varrho f)^{q(s)}) &= -\mu_s(\varrho_s(P_{s,t}^\varrho f)^{q(s)}) + \mu_s(\partial_s(P_{s,t}^\varrho f)^{q(s)})\\ &= -\mu_s(\varrho_s(P_{s,t}^\varrho f)^{q(s)}) + \mu_s((L_s+\partial_s)(P_{s,t}^\varrho f)^{q(s)})\\ &= q(s)(q(s)-1)\,\mu_s(|\nabla^s P_{s,t}^\varrho f|_s^2(P_{s,t}^\varrho f)^{q(s)-2}) + q'(s)\,\mu_s((P_{s,t}^\varrho f)^{q(s)}\log P_{s,t}^\varrho f)\\ &\qquad - (1-q(s))\,\mu_s(\varrho_s(P_{s,t}^\varrho f)^{q(s)}).\end{aligned}$$

For $\|P_{s,t} f\|_{q(s),s}$, since $\|P_{s,t}^\varrho f\|_{q(s),s}^{1-q(s)} = (\mu_s((P_{s,t}^\varrho f)^{q(s)}))^{1/q(s)-1}$, we thus find

$$\begin{aligned}\frac{d}{ds}\|P_{s,t}^\varrho f\|_{q(s),s} &= (q(s)-1)\|P_{s,t}^\varrho f\|_{q(s),s}^{1-q(s)}\mu_s(|\nabla^s P_{s,t}^\varrho f|_s^2(P_{s,t}^\varrho f)^{q(s)-2})\\ &\quad + \frac{q'(s)}{q(s)}\|P_{s,t}^\varrho f\|_{q(s),s}^{1-q(s)}\mu_s((P_{s,t}^\varrho f)^{q(s)}\log P_{s,t}^\varrho f)\\ &\quad - \frac{q'(s)}{q(s)}\|P_{s,t}^\varrho f\|_{q(s),s}\log\|P_{s,t}^\varrho f\|_{q(s),s}\\ &\quad + \frac{q(s)-1}{q(s)}\|P_{s,t}^\varrho f\|_{q(s),s}^{1-q(s)}\mu_s(\varrho_s(P_{s,t}^\varrho f)^{q(s)}).\end{aligned}$$

On the other hand, passing from f to $f^{p/2}/\|f^{p/2}\|_{2,s}$ in the log-Sobolev inequality (7-2), we obtain

$$\int f^p \log\biggl(\frac{f^p}{\|f^{p/2}\|_{2,s}^2}\biggr)\,d\mu_s \le r\frac{p^2}{4}\int f^{p-2}|\nabla^s f|_s^2\,d\mu_s + \frac{r}{4}\int f^p \varrho_s\,d\mu_s + \beta_s(r)\|f^{p/2}\|_{2,s}^2.$$

In this inequality, replacing f and p by $P^{\varrho}_{s,t}f$ and $q(s)$ respectively, we obtain

$$\begin{aligned}\mu_s((P^{\varrho}_{s,t}f)^{q(s)}\log(P^{\varrho}_{s,t}f)) - \|P^{\varrho}_{s,t}f\|_{q(s),s}^{q(s)}\log\|P^{\varrho}_{s,t}f\|_{q(s),s}\\ \le r\frac{q(s)}{4}\int (P^{\varrho}_{s,t}f)^{q(s)-2}|\nabla^s P^{\varrho}_{s,t}f|_s^2\,d\mu_s + \frac{r}{4q(s)}\int f^{q(s)}\varrho_s\,d\mu_s + \frac{\beta_s(r)}{q(s)}\|P^{\varrho}_{s,t}f\|_{q(s),s}^{q(s)}.\end{aligned}\tag{A-10}$$

Now let

$$q(s) = \mathrm{e}^{4r^{-1}(t-s)}(p-1)+1, \quad q(t) = p.$$

Note that q is a decreasing function and $q'(s)r/4 + (q(s)-1) \equiv 0$. Thus, combining (A-10) with (A-10), we arrive at

$$\frac{d}{ds}\|P^{\varrho}_{s,t}f\|_{q(s),s} \ge \frac{\beta_s(r)q'(s)}{q(s)^2}\|P^{\varrho}_{s,t}f\|_{q(s),s}, \quad 0 \le s \le t < T.$$

It follows that

$$\|P^{\varrho}_{s,t}f\|_{q(s),s} \le \exp\biggl(-\int_s^t \frac{\beta_u(r)q'(u)}{q(u)^2}\,du\biggr)\|f\|_{p,t}.\tag{A-11}$$

If we impose that $q(s) = q$, then

$$r = 4(t-s)\biggl(\log\frac{q-1}{p-1}\biggr)^{-1}.$$

Substituting the value of r into (A-11) yields

$$\|P^{\varrho}_{s,t}f\|_{q,s} \le \exp\biggl(-\int_s^t \frac{\beta_u(4(t-s)(\log(q-1)/(p-1))^{-1})\,q'(u)}{q(u)^2}\,du\biggr)\|f\|_{p,t}. \qquad \square$$

Proof of Theorem 7.5. By means of the Harnack inequality (4-1), the theorem can be proved along the lines of [Wang 2005, Theorem 5.7.3] or [Cheng and Thalmaier 2018b]. For the reader's convenience, we include a proof here. We first prove that if $\mu_s(\exp(\lambda\rho_s^2)) < \infty$ for all $\lambda > 0$, then $P_{s,t}$ is supercontractive, i.e., for any $1 < p < q < \infty$, we have

$$\|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)} < \infty.$$

Let $p > 1$ and $f \in C_b(M)$. For $0 \le s \le t < T$ it follows from the Harnack inequality (4-1) that

$$|(P^{\varrho}_{s,t}f)^p(x)| \le (P^{\varrho}_{s,t}|f|^p)(y)\exp\biggl((p-1)\int_s^t \sup\varrho_r^-\,dr + \frac{p\rho_s^2(x,y)}{4(p-1)\alpha(s,t)} + \frac{\eta(s,t)\rho_s(x,y)}{\alpha(s,t)}\biggr).$$

Thus, if $\mu_t(|f|^p) = 1$, then

$$\begin{aligned}1 &\ge |P^{\varrho}_{s,t}f(x)|^p\int \exp\biggl((1-p)\int_s^t \sup\varrho_r^-\,dr - \frac{p\rho_s^2(x,y)}{4(p-1)\alpha(s,t)} - \frac{\eta(s,t)\rho_s(x,y)}{\alpha(s,t)}\biggr)\mu_s(dy)\\ &\ge |P^{\varrho}_{s,t}f(x)|^p\,\mu_s(B_s(o,R))\exp\biggl((1-p)\int_s^t \sup\varrho_r^-\,dr - \frac{p(\rho_s(x)+R)^2}{4(p-1)\alpha(s,t)} - \frac{\eta(s,t)(\rho_s(x)+R)}{\alpha(s,t)}\biggr),\end{aligned}\tag{A-12}$$

where $B_s(o, R) = \{y \in M : \rho_s(y) \le R\}$ denotes the geodesic ball (with respect to the metric $g(s)$) of radius R about $o \in M$ and where $\rho_t(\cdot) = \rho_t(o, \cdot)$. Since $\mu_t(\exp(\lambda\rho_t^2)) < \infty$, the system of measures (μ_s) is compact, i.e., there exists $R = R(s) > 0$, possibly depending on s, such that

$$\mu_s(B_s(o, R(s))) = \mu_s(\{x : \rho_s(x) \le R(s)\}) \ge 1 - \frac{\mu_s(\rho_s^2)}{R(s)^2} \ge 2^{-p}$$

(after normalizing μ_s to a probability measure). Combining the last estimate with (A-12), we arrive at

$$1 \ge |P^{\varrho}_{s,t} f(x)|^p \, 2^{-p} \exp\left((1-p)\int_s^t \sup \varrho_r^- \, dr - \frac{\eta(s,t)(\rho_s(x)+R)}{\alpha(s,t)} - \frac{p(\rho_s(x)+R)^2}{4(p-1)\alpha(s,t)}\right),$$

which further implies

$$|P^{\varrho}_{s,t} f(x)| \le 2\exp\left(\frac{p-1}{p}\int_s^t \sup \varrho_r^- \, dr + \frac{\eta(s,t)(\rho_s(x)+R)}{p\alpha(s,t)} + \frac{(\rho_s(x)+R)^2}{4(p-1)\alpha(s,t)}\right), \quad s < t. \tag{A-13}$$

Therefore, we achieve

$$\|P^{\varrho}_{s,t} f\|_{q,s} \le \big(\mu_s(\exp(q(c_1 + c_2\rho_s^2)))\big)^{1/q}$$

for some positive constants c_1, c_2 depending on s and t. Hence, if $\mu_s(\exp(\lambda\rho_s^2)) < \infty$ for any $\lambda > 0$ and $s \in [0, T[$, then $P_{s,t}$ is supercontractive.

Conversely, if the semigroup $P^{\varrho}_{s,t}$ is supercontractive, by Theorem 7.1 the super log-Sobolev inequalities (7-2) holds. We first prove that $\mu_s(\mathrm{e}^{\lambda\rho_s}) < \infty$ for $s \in [0, T[$ and $\lambda > 0$. To this end, let $\rho_{s,k} = \rho_s \wedge k$ and $h_{s,k}(\lambda) = \mu_s(\exp(\lambda\rho_{s,k}))$. Taking $\exp(\lambda\rho_{s,k}/2)$ in the super log-Sobolev inequality (7-1), we obtain

$$\lambda h'_{s,k}(\lambda) - h_{s,k}(\lambda)\log h_{s,k}(\lambda) \le h_{s,k}(\lambda)\lambda^2\left(\frac{r}{4} + \frac{\beta_s(r)}{\lambda^2}\right).$$

This implies

$$\Big(\frac{1}{\lambda}\log h_{s,k}(\lambda)\Big)' = \frac{\lambda h'_{s,k}(\lambda) - h_{s,k}(\lambda)\log h_{s,k}(\lambda)}{\lambda^2 h_{s,k}(\lambda)} \le \frac{r}{4} + \frac{\beta_s(r)}{\lambda^2}. \tag{A-14}$$

Integrating both sides of (A-14) from λ to 2λ, we obtain

$$h_{s,k}(2\lambda) \le h^2_{s,k}(\lambda)\exp\Big(\frac{r}{2}\lambda^2 + \beta_s(r)\Big). \tag{A-15}$$

From this inequality, along with the fact that there exists a constant M_s such that

$$\mu_s(\{\lambda\rho_s \ge M_s\}) \le \frac{1}{4}\exp\Big(-\frac{r}{2}\lambda^2 - \beta_s(r)\Big),$$

we get

$$\begin{aligned}
h_{s,k}(\lambda) &= \int_{\{\lambda\rho_s \ge M_s\}} \exp(\lambda\rho_{s,k})\, d\mu_s + \int_{\{\lambda\rho_s < M_s\}} \exp(\lambda\rho_{s,k})\, d\mu_s \\
&\le \mu_s(\{\lambda\rho_s \ge M_s\})^{1/2}\, \mu_s(\mathrm{e}^{2\lambda\rho_{s,k}})^{1/2} + \mathrm{e}^{M_s}\mu_s(\{\lambda\rho_s < M_s\}) \\
&\le \Big(\frac{1}{4}\exp\Big(-\frac{r}{2}\lambda^2 - \beta_s(r)\Big)\Big)^{1/2}\exp\Big(\frac{r}{4}\lambda^2 + \frac{1}{2}\beta_s(r)\Big)h_{s,k}(\lambda) + \mathrm{e}^{M_s}\mu_s(\{\lambda\rho_s < M_s\}) \\
&\le \frac{1}{2}h_{s,k}(\lambda) + \mathrm{e}^{M_s}\mu_s(\{\lambda\rho_s < M_s\}),
\end{aligned}$$

which implies $h_{s,k}(\lambda) \le 2e^{M_s}\mu_s(\{\lambda\rho_s < M_s\})$ for $s \in [0, T[$. As M_s is independent of k, letting k tend to infinity, we arrive at

$$\mu_s(e^{\lambda\rho_s}) < \infty \quad \text{for all } s \in [0, T[.$$

To prove that moreover $\mu_s(e^{\lambda\rho_s^2}) < \infty$ for $s \in [0, T[$ and $\lambda > 0$, we can follow the argument in [Cheng and Thalmaier 2018b, pp. 22–23]. □

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THE REGULARITY OF THE BOUNDARY OF VORTEX PATCHES FOR SOME NONLINEAR TRANSPORT EQUATIONS

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We prove the persistence of boundary smoothness of vortex patches for a nonlinear transport equation in $\mathbb{R}^n$ with velocity field given by convolution of the density with an odd kernel, homogeneous of degree $-(n-1)$ and of class $C^2(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$. This allows the velocity field to have nontrivial divergence. The quasigeostrophic equation in $\mathbb{R}^3$ and the Cauchy transport equation in the plane are examples.

1. Introduction

The vorticity form of the Euler equation in the plane is

$$\begin{aligned} \partial_t \omega(x,t) + v(x,t)\cdot\nabla\omega(x,t) &= 0,\\ v(x,t) &= (\nabla^\perp N * \omega(\,\cdot\,,t))(x),\\ \omega(x,0) &= \omega_0(x), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^2,\ t \in \mathbb{R},\ N = \frac{1}{2\pi}\log|x|$ is the fundamental solution of the laplacian in the plane, $\nabla^\perp N$ is a rotation of ∇N of 90° in the counterclockwise direction and ω_0 is the initial vorticity. A deep result of Yudovich [1963] asserts that the vorticity equation is well-posed in L^∞_c, the measurable bounded functions with compact support. A vortex patch is the special weak solution of (1) when the initial condition is the characteristic function of a bounded domain D_0. Since the vorticity equation is a transport equation, vorticity is conserved along trajectories and thus $\omega(x,t) = \chi_{D_t}(x)$ for some domain D_t. A challenging problem, raised in the eighties, was to show that boundary smoothness persists for all times. Specifically, if D_0 has boundary of class $C^{1+\gamma},\ 0<\gamma<1$, then one would like D_t to have boundary of the same class for all times. This was viewed as a 2-dimensional problem which featured some of the main difficulties of the regularity problem for the Euler equation in $\mathbb{R}^3$. It was conjectured, on the basis of numerical simulations, that the boundary of D_t could become of infinite length in finite time [Majda 1986]. Chemin [1993] proved that boundary regularity persists for all times using paradifferential calculus, and Bertozzi and Constantin [1993] found shortly after a minimal beautiful proof based on methods of classical analysis with a geometric flavor.

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The vortex patch problem was considered for the aggregation equation with newtonian kernel in higher dimensions in [Bertozzi et al. 2016]. The equation is

$$\begin{aligned} \partial_t \rho(x,t) + \operatorname{div}(\rho(x,t)v(x,t)) &= 0, \\ v(x,t) &= -(\nabla N * \rho(\,\cdot\,,t))(x), \\ \rho(x,0) &= \rho_0(x), \end{aligned} \tag{2}$$

$x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. In [Bertozzi et al. 2012] a well-posedness theory in L^∞_c was developed, following the path of [Yudovich 1963; 2002, Theorem 8.1]. When the initial condition is the characteristic function of a bounded domain, one calls the unique weak solution a vortex patch, as for the vorticity equation. One proves in [Bertozzi et al. 2016] that if the boundary of D_0 is of class $C^{1+\gamma}$, $0<\gamma<1$, then the solution of (2) with initial condition $\rho_0 = \chi_{D_0}$ is of the form

$$\rho(x,t) = \frac{1}{1-t}\chi_{D_t}(x), \quad x \in \mathbb{R}^n,\ 0 \le t < 1,$$

where D_t is a $C^{1+\gamma}$ domain for all $t<1$. The restriction to times less than 1 obeys a blow-up phenomenon studied in [Bertozzi et al. 2012]. Hence the preceding result is the analog of Chemin's theorem for the aggregation equation. See [Bae and Kelliher 2021] for a more general result concerning striated regularity.

After a change in the time scale the aggregation equation for vortex patches becomes the nonlinear transport equation

$$\begin{aligned} \partial_t \rho(x,t) + v(x,t)\cdot\nabla\rho(x,t) &= 0, \\ v(x,t) &= -(\nabla N * \rho(\,\cdot\,,t))(x), \\ \rho(x,0) &= \chi_{D_0}(x), \end{aligned} \tag{3}$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$, where N is the fundamental solution of the laplacian in $\mathbb{R}^n$ and D_0 is a bounded domain. In this formulation one proves in [Bertozzi et al. 2016] that if D_0 is of class $C^{1+\gamma}$, then there is a solution of (3) of the form $\chi_{D_t}(x)$ with D_t a domain of class $C^{1+\gamma}$. To the best of our knowledge there is no well-posedness theory in L^∞_c for (3), for a general initial condition in L^∞_c. However, if the initial condition is the characteristic function of a domain D_0, not necessarily smooth, one has existence and uniqueness for the transport equation (3). For existence, solve the equation (2) with initial condition $\rho_0(x) = \chi_{D_0}(x)$. Then the unique solution has the form $\rho(x,t) = (1/(1-t))\chi_{D_t}(x)$ and hence, after changing the time scale as in [Bertozzi et al. 2012], one obtains a solution for (3) which is a vortex patch. For uniqueness, we resort to an argument which combines results of [Clop et al. 2016a; 2016b] to prove that each weak solution of (3) in L^∞_c is lagrangian and so a vortex patch. Changing the time scale, one obtains a weak solution of (2), which is unique.

The proof follows the scheme of [Bertozzi and Constantin 1993] and overcomes difficulties related to the fact that the velocity field has a nonzero divergence and to the higher-dimensional context. The reader can consult [Bertozzi et al. 2016] for connections with the existing literature and for references to models leading to various aggregation equations.

This paper originated from an attempt to deeply understand the role of the kernel that gives the velocity field. For the aggregation equation the kernel is $-\nabla N$ and for the vorticity equation in the plane the kernel is a rotation of 90° of ∇N. These are odd kernels, smooth off the origin and homogeneous of degree $-(n-1)$. We wondered what would happen for the Cauchy kernel

$$\frac{1}{2\pi z} = L(\nabla N), \quad \text{with } L(x, y) = (x, -y),\ z = (x, y) \in \mathbb{R}^2 = \mathbb{C}.$$

Although apparently there is no model leading to the nonlinear transport equation given by the Cauchy kernel, from the mathematical perspective the question makes sense. We then embarked in the study of the nonlinear transport equation

$$\begin{aligned} &\partial_t \rho(z, t) + v(z, t) \cdot \nabla \rho(z, t) = 0, \\ &v(z, t) = \Big(\frac{1}{2\pi z} * \rho(\,\cdot\,, t)\Big)(z), \\ &\rho(z, 0) = \chi_{D_0}(z), \end{aligned} \tag{4}$$

where $z = (x, y)$ is the complex variable and D_0 is a bounded domain with $C^{1+\gamma}$ boundary, $0 < \gamma < 1$. A first remark is that apparently there does not exist a well-posedness theory in L^∞_c for the equation above, but this does not prevent the study of smooth vortex patches, as a particular subclass of L^∞_c enjoying a bit of smoothness.

To grasp what could be expected we looked at an initial datum which is the characteristic function of the domain enclosed by an ellipse

$$D_0 = \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\right\}.$$

We proved that there exists a weak solution of (4) of the form $\rho(z, t) = \chi_{D_t}(z)$, with D_t the domain enclosed by an ellipse with semiaxes $a(t)$ and $b(t)$ collapsing to a segment on the horizontal axis as $t \to \infty$.

A key remark is that (4) is not rotation invariant. Fix an angle $0 < \theta < \frac{\pi}{2}$ and consider as initial domain the set enclosed by a tilted ellipse

$$D_0 = e^{i\theta}\left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\right\}.$$

As before we find a weak solution of (4) of the form $\rho(z, t) = \chi_{D_t}(z)$, with D_t the domain enclosed by an ellipse with semiaxes $a(t)$ and $b(t)$ forming an angle $\theta(t)$ with the horizontal axis. The evolution is different according to whether $0 < \theta \le \frac{\pi}{4}$ or $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. Under the assumption that $a_0 > b_0$, in the case $0 < \theta \le \frac{\pi}{4}$ the semiaxis $a(t)$ increases as $t \to \infty$ to a positive number a_∞, $b(t)$ decreases to 0 and $\theta(t)$ decreases to a positive angle θ_∞. Hence D_t collapses into an interval on a line forming a positive angle with the horizontal axis. If $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, then for small times $a(t)$ decreases and $b(t)$ increases, so that the ellipse at time t tends initially to become a circle. This happens until a critical time is reached after which $a(t)$ increases and $b(t)$ decreases. The angle $\theta(t)$ decreases for all positive times and at some point it becomes $\frac{\pi}{4}$; after that one falls into the regime of the first case and the domain D_t collapses as $t \to \infty$, into a segment on a line which forms a positive angle with the horizontal axis. The case $a_0 < b_0$ is similar and can be reduced to the previous situation by conjugation (symmetry with respect to the horizontal axis).

Detailed proofs of the results just described can be found in Section 7. What they show is that the behavior of vortex patches for the Cauchy transport equation can be much more complicated than for the vorticity or aggregation equations. This is also easily understood if one looks at the divergence of the vector field in (4). If ∂ and $\bar\partial$ denote respectively the derivatives with respect to the z- and $\bar z$-variables, then we get

$$2\,\bar\partial v(z, t) = \rho(z, t)$$

and

$$2\,\partial v(z,t) = -\frac{1}{\pi}\,\text{p.v.}\int \frac{1}{(z-w)^2}\rho(w,t)\,dA(w) = \mathrm{B}(\rho(\,\cdot\,,t))(z),$$

where B is the Beurling transform, one of the basic Calderón–Zygmund operators in the plane. Here dA is 2-dimensional Lebesgue measure. The divergence of v is given by

$$\begin{aligned}\operatorname{div} v = \Re(2\,\partial v) &= -\,\text{p.v.}\frac{1}{\pi}\int \Re\left(\frac{1}{(z-w)^2}\right)\rho(w,t)\,dA(w)\\ &= -\,\text{p.v.}\frac{1}{\pi}\left(\frac{x^2-y^2}{|z|^4}\star\rho(\,\cdot\,,t)\right)(z).\end{aligned}$$

The last convolution is a Calderón–Zygmund operator (a second-order Riesz transform) and so it does not map bounded functions into bounded functions. The most one can say a priori on the divergence of the velocity field is that it is a BMO function in the plane, provided the density $\rho(\,\cdot\,,t)$ is a bounded function. It is a well-known fact, already used in [Bertozzi and Constantin 1993; Chemin 1993], that if D is a domain with boundary of class $C^{1+\gamma}$, then an even Calderón–Zygmund operator applied to χ_D is a bounded function. Thus we indeed expect $\operatorname{div} v$ to be bounded. Nevertheless, the expression of the divergence of the field in terms of a Calderón–Zygmund operator applied to the density is potentially difficult to handle.

We have succeeded in proving that there exists a weak solution of (4) of the form χ_{D_t} with D_t a domain with boundary of class $C^{1+\gamma}$ for all times $t\in\mathbb{R}$. This weak solution is unique in the class of characteristic functions of $C^{1+\gamma}$ domains.

The Cauchy kernel belongs to a wider class for which the preceding well-posedness theorem holds. We refer to the class of kernels in $\mathbb{R}^n$ which are odd, homogeneous of degree $-(n-1)$ and of class $C^2(\mathbb{R}^n\setminus\{0\},\mathbb{R}^n)$. Interesting examples of such kernels are those of the form $L(\nabla N)$, where L is a linear mapping from $\mathbb{R}^n$ into itself and N is the fundamental solution of the laplacian in $\mathbb{R}^n$. They are harmonic off the origin. In particular in $\mathbb{R}^3$ one can take $L(x_1,x_2,x_3)=(-x_2,x_1,0)$. The corresponding field is divergence-free and the associated equation is the well-known quasigeostrophic equation. See [García et al. 2022] for recent results on rotating vortex patches for the quasigeostrophic equation.

Our main result is the following.

Theorem. *Let $k:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}^n$ be an odd function, homogeneous of degree $-(n-1)$ and of class $C^2(\mathbb{R}^n\setminus\{0\},\mathbb{R}^n)$. Let D_0 be a bounded domain with boundary of class $C^{1+\gamma}$, $0<\gamma<1$. Then the nonlinear transport equation*

$$\begin{aligned}\partial_t\rho(x,t)+v(x,t)\cdot\nabla\rho(x,t)&=0,\\ v(x,t)&=(k\star\rho(\,\cdot\,,t))(x),\\ \rho(x,0)&=\chi_{D_0}(x),\end{aligned}\tag{5}$$

$x\in\mathbb{R}^n$, $t\in\mathbb{R}$, has a weak solution of the form

$$\rho(x,t)=\chi_{D_t}(x),\quad x\in\mathbb{R}^n,\ t\in\mathbb{R},$$

with D_t a bounded domain with boundary of class $C^{1+\gamma}$.

This solution is unique in the class of characteristic functions of domains with boundary of class $C^{1+\gamma}$.

For the notion of weak solution see [Majda and Bertozzi 2002, Chapter 8].

A remark on the special case in which the kernel k is divergence-free is in order. In this case, in particular for the quasigeostrophic equation, one has well-posedness in L^∞_c. Existence can be proved following closely the argument in [Majda and Bertozzi 2002, Chapter 8] for the vorticity equation (for the smooth case see [Cantero 2021]). For uniqueness one resorts to [Nieto et al. 2001] whenever the kernel has the special form $L(\nabla N)$ with L a linear map from $\mathbb{R}^n$ into itself. Indeed, in that work uniqueness in L^∞_c is proven for the continuity equation in higher dimensions with velocity field given by convolution with $\pm\nabla N$. The changes needed to take care of the case $L(\nabla N)$ are straightforward. If k is divergence-free and satisfies the general hypothesis stated in the theorem, then one appeals to [Crippa and Stefani 2021], where uniqueness is proved for lagrangian solutions, and to [Clop et al. 2016a; 2016b], in which one shows that a weak solution is lagrangian.

The paper is organized as follows. In the next section we present an outline of the proof, in which only a few facts are proven. The other sections are devoted to presenting complete proofs of our results. Section 3 is devoted to an auxiliary result. In Section 4 an appropriate defining function for the patch at time t is constructed. Section 5 deals with the material derivative of the gradient of the defining function and its expression in terms of differences of commutators. In Section 6 we estimate the differences of commutators in the Hölder norm on the boundary via Whitney's extension theorem. Domains enclosed by ellipses as initial patches for the Cauchy transport equation are studied in Section 7 and the unexpected phenomena that turn up along the vortex patch evolution are described in detail. Finally, there is an Appendix on the existence of principal values of singular integrals in a very special context.

Constants will be denoted by C, mostly without an explicit reference to innocuous parameters, and may be different at different occurrences. If D is a domain with smooth boundary $\sigma = \sigma_{\partial D}$ denotes the surface measure on ∂D and when there is no confusion possible we omit the subscript. The exterior unit normal vector to ∂D at the point x is denoted by $\vec{n}(x) = (n_1(x), \dots, n_n(x))$, without explicit reference to the boundary.

2. Outline of the proof

The proof follows the general scheme devised in [Bertozzi and Constantin 1993]. There are serious obstructions caused by the fact that the field is not divergence-free and we will explain below how to confront them. The reader will find useful to consult [Bertozzi and Constantin 1993; Bertozzi et al. 2016].

2.1. ***The contour dynamics equation.*** Assume that one has a weak solution of (5) of the form $\rho(x,t) = \chi_{D_t}(x)$, D_t being a bounded domain of class $C^{1+\gamma}$ for t in some interval $[0,T]$. The field $v(\cdot,t)$ is Lipschitz. This is due to the fact that our kernel has homogeneity $-(n-1)$ and so ∇v is given by a matrix whose entries are even convolution Calderón–Zygmund operators applied to the characteristic function of D_t plus, possibly, a constant multiple of such a characteristic function (coming from a delta function at the origin). Since D_t has boundary of class $C^{1+\gamma}$ all entries of the matrix ∇v are functions in $L^\infty(\mathbb{R}^n)$ [Bertozzi and Constantin 1993]. Thus the equation of particle trajectories (the flow mapping)

$$\frac{dX(\alpha,t)}{dt} = v(X(\alpha,t),t), \qquad X(\alpha,0) = \alpha, \tag{6}$$

has a unique solution and $X(\,\cdot\,,t)$ is a bilipschitz mapping of $\mathbb{R}^n$ into itself, $0 \le t \le T$. Indeed one has the usual estimate

$$\|\nabla X(\,\cdot\,,t)\|_\infty \le \exp \int_0^t \|\nabla v(\,\cdot\,,s)\|_\infty \, ds. \tag{7}$$

Since k is homogeneous of degree $-(n-1)$ and smooth off the origin we have

$$k = \partial_1(x_1 k) + \partial_2(x_2 k) + \cdots + \partial_n(x_n k), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n \setminus \{0\}. \tag{8}$$

This follows straightforwardly from Euler's theorem on homogeneous functions.

Assume that $\rho(x,t) = \chi_{D_t}(x)$ is a weak solution of the general equation (5). The velocity field is

$$\begin{aligned} v(\,\cdot\,,t) = \chi_{D_t} \star k &= \chi_{D_t} \star (\partial_1(x_1 k) + \cdots + \partial_n(x_n k)) \\ &= \partial_1 \chi_{D_t} \star (x_1 k) + \cdots + \partial_n \chi_{D_t} \star (x_n k) \\ &= -n_1 \, d\sigma_{\partial D_t} \star (x_1 k) - \cdots - n_n \, d\sigma_{\partial D_t} \star (x_n k). \end{aligned}$$

Thus

$$\begin{aligned} v(x,t) &= -\sum_{j=1}^n \int_{\partial D_t} (x_j - y_j) k(x-y) n_j(y) \, d\sigma_{\partial D_t}(y) \\ &= -\int_{\partial D_t} k(x-y) \langle x-y, \vec{n}(y) \rangle \, d\sigma_{\partial D_t}(y), \quad x \in \mathbb{R}^n. \end{aligned} \tag{9}$$

The next step is to set $x = X(\alpha,t)$ and to make the change of variables $y = X(\beta,t)$ in the preceding surface integral. To do this conveniently let $T_1(\beta), \ldots, T_{n-1}(\beta)$ be an orthonormal basis of the tangent space to ∂D_0 at the point $\beta \in \partial D_0$ and let $DX(\,\cdot\,,t)$ be the differential of $X(\,\cdot\,,t)$ as a differentiable mapping from ∂D_0 into $\mathbb{R}^n$. The vectors $DX(\beta,t)(T_j(\beta))$ are tangent to ∂D_t at the point $X(\beta,t)$ for $1 \le j \le n-1$. Hence the vector

$$\textstyle\bigwedge_{j=1}^{n-1} DX(\beta,t)(T_j(\beta)) \tag{10}$$

is orthogonal to ∂D_t at the point $X(\beta,t)$ and a different choice of the orthonormal basis $T_j(\beta)$, $1 \le j \le n-1$, has the effect of introducing a $\pm$ sign in front of (10). We may choose the $T_j(\beta)$ so that $\vec{n}(\beta), T_1(\beta), \ldots, T_{n-1}(\beta)$ gives the standard orientation of $\mathbb{R}^n$. Substituting the expression (9) for the velocity field in (6) and making the change of variables $y = X(\beta,t)$ we get

$$\begin{aligned} \frac{d}{dt} X(\alpha,t) &= v(X(\alpha,t),t) \\ &= -\int_{\partial D_0} k(X(\alpha,t) - X(\beta,t)) \Big\langle X(\alpha,t) - X(\beta,t), \textstyle\bigwedge_{j=1}^{n-1} DX(\beta,t)(T_j(\beta)) \Big\rangle \, d\sigma_{\partial D_0}(\beta). \end{aligned}$$

Let $X : \partial D_0 \to \mathbb{R}^n$ be a mapping of class $C^{1+\gamma}$ such that for some constant $\mu > 0$

$$|X(\alpha) - X(\beta)| \ge \frac{1}{\mu} |\alpha - \beta|, \quad \alpha, \beta \in \partial D_0. \tag{11}$$

In other words $X \in C^{1+\gamma}(\partial D_0, \mathbb{R}^n)$, X is bilipschitz onto the image and μ is a Lipschitz constant for the inverse mapping.

Define a mapping $F(X) : \partial D_0 \to \mathbb{R}^n$ by

$$F(X)(\alpha) = -\int_{\partial D_0} k(X(\alpha) - X(\beta)) \Big\langle X(\alpha) - X(\beta), \textstyle\bigwedge_{j=1}^{n-1} DX(\beta)(T_j(\beta)) \Big\rangle \, d\sigma_{\partial D_0}(\beta). \tag{12}$$

The contour dynamics equation (CDE) is

$$\begin{aligned} \frac{dX(\alpha, t)}{dt} &= F(X(\,\cdot\,, t))(\alpha), \quad \alpha \in \partial D_0, \\ X(\,\cdot\,, 0) &= I, \end{aligned}$$

where I denotes the identity mapping on ∂D_0.

We conclude that if there exists a weak solution of the type we are looking for, then the flow restricted to ∂D_0 is a solution of the CDE.

To proceed in the reverse direction, we need some preparation. Let Ω be the open set in the Banach space $C^{1+\gamma}(\partial D_0, \mathbb{R}^n)$ consisting of those $X \in C^{1+\gamma}(\partial D_0, \mathbb{R}^n)$ satisfying (11) for some $\mu > 0$. The set Ω is open in $C^{1+\gamma}(\partial D_0, \mathbb{R}^n)$ and the CDE can be thought of as an ODE in the open set Ω. We want to show that a solution $X(\,\cdot\,, t)$ to the CDE in an interval $(-T, T)$ provides a weak solution of the nonlinear transport equation (5). Clearly $X(\,\cdot\,, t)$ maps ∂D_0 onto a $(n-1)$-dimensional hypersurface S_t. The goal now is to identify an open set D_t with boundary S_t. If we add the hypothesis that ∂D_0 is connected, and hence a connected $(n-1)$-dimensional hypersurface of class $C^{1+\gamma}$, then the analog of the Jordan curve theorem holds [Guillemin and Pollack 1974, p. 89]. Then the complement of ∂D_0 in $\mathbb{R}^n$ has only one bounded connected component which is D_0. In the same vein, the complement of S_t has only one bounded connected component, which we denote by D_t, so that the boundary of D_t is S_t. The definition of D_t is less direct if we drop the assumption that ∂D_0 is connected. We proceed as follows. Let S_t^j, $1 \le j \le m$, be the connected components of S_t. Denote by U_t^j the bounded connected component of the complement of S_t^j in $\mathbb{R}^n$. Among the U_t^j there is one, say U_t^1, that contains all the others. This is so at time $t = 0$ because D_0 is connected and this property is preserved by the flow $X(\,\cdot\,, t)$. We set $D_t = U_t^1 \setminus \big(\bigcup_{j=2}^m \overline{U}_t^j\big)$, so that the boundary of D_t is S_t.

Indeed, as the reader may have noticed, it is not necessary to assume that D_0 is connected in our theorem. It can be any bounded open set with $C^{1+\gamma}$ boundary. Then the argument we have just described is applied to each connected component.

Define a velocity field by

$$v(x, t) = (k \star \chi_{D_t})(x), \quad x \in \mathbb{R}^n, \ t \in (-T, T). \tag{13}$$

Since D_t has boundary of class $C^{1+\gamma}$, the field $v(\,\cdot\,, t)$ is Lipschitz for each $t \in (-T, T)$ and the equation of the flow (6) has a unique solution which is a bilipschitz mapping of $\mathbb{R}^n$ onto itself whose restriction to ∂D_0 is the solution of the CDE we were given. Thus $X(D_0, t) = D_t$ and χ_{D_t} is a weak solution of the nonlinear transport equation (5).

2.2. ***The local theorem.*** As a first step we solve the CDE locally in time. For this we look at the CDE as an ODE in the open set Ω of the Banach space $C^{1+\gamma}(\partial D_0, \mathbb{R}^n)$. To show local existence and uniqueness we apply the Picard theorem. First one has to check that $F(X) \in C^{1+\gamma}(\partial D_0, \mathbb{R}^n)$ for each $X \in \Omega$. After taking a derivative in α in (12) one gets a p.v. integral on ∂D_0, which defines a Calderón–Zygmund operator (not of convolution type) with respect to the underlying measure $d\sigma_{\partial D_0}$, acting on a function satisfying a Hölder condition of order γ. The result is again a Hölder function of the same order, since one shows that Calderón–Zygmund operators of the type one gets preserve Hölder spaces. In a second step one needs to prove that $F(X)$ is locally a Lipschitz function of the variable X or, equivalently, that the differential $DF(X)$ of F at the point $X \in \Omega$ is locally bounded in X. Again one has to estimate operators of Calderón–Zygmund type with respect Hölder spaces of order γ. These estimates, subtle at some points, are proved in full detail in [Bertozzi et al. 2016] for the kernel $k = -\nabla N$. The variations needed to cover the present situation are minor and are left to the reader. It is important that, as in [Bertozzi et al. 2016], the time interval on which the local solution exists depends continuously only on the dimension n, the kernel k, the diameter of D_0, the $(n-1)$-dimensional surface measure of ∂D_0 and the constant $q(D_0)$ determining the $C^{1+\gamma}$ character of ∂D_0, whose definition we discuss below.

Let D be a bounded domain with boundary of class $C^{1+\gamma}$. Then there exists a defining function of class $C^{1+\gamma}$, that is, a function $\varphi \in C^{1+\gamma}(\mathbb{R}^n)$, such that $D = \{x \in \mathbb{R}^n : \varphi(x) < 0\}$ and $\nabla\varphi(x) \neq 0$ if $\varphi(x) = 0$. We set

$$q(D) = \inf\left\{ \frac{\|\nabla\varphi\|_{\gamma,\partial D}}{|\nabla\varphi|_{\inf}} : \varphi \text{ a defining function of } D \text{ of class } C^{1+\gamma} \right\}, \tag{14}$$

where $|\nabla\varphi(x)| = \sqrt{\sum_{j=1}^n \partial_j\varphi(x)^2}$,

$$\|\nabla\varphi\|_{\gamma,\partial D} = \sup\left\{ \frac{|\nabla\varphi(x) - \nabla\varphi(y)|}{|x-y|^\gamma} : x, y \in \partial D, x \neq y \right\},$$
$$|\nabla\varphi|_{\inf} = \inf\{|\nabla\varphi(x)| : \varphi(x) = 0\}.$$

There is here an important variation with respect to [Bertozzi and Constantin 1993; Bertozzi et al. 2016]: the Hölder seminorm of order γ of $\nabla\varphi$ is taken in those papers in the whole of $\mathbb{R}^n$. For reasons that will become clear later on we need to restrict our attention to the boundary of D and this requires finer estimates.

2.3. ***Global existence: a priori estimates.*** Assume that the maximal time of existence for the solution $X(\cdot, t)$ of the CDE is T. By this we mean that $X(\cdot, t)$ is defined for $t \in (-T, T)$ but cannot be extended to a larger interval. We want to prove that $T = \infty$. For that it suffices to prove that for some constant $C = C(T)$ one has

$$\operatorname{diam}(D_t) + \sigma(\partial D_t) + q(D_t) \le C, \quad t \in (-T, T). \tag{15}$$

If the preceding inequality holds, then we take $t_0 < T$ close enough to T so that after the application of the existence and uniqueness theorem for the CDE to the domain D_{t_0} at time t_0 we get an interval of existence for the solution which goes beyond T (the same argument applies to the lower extreme $-T$).

To obtain (15) we look for a priori estimates in terms of $\|\nabla v\|_\infty$. For $\operatorname{diam}(D_t)$ and $\sigma(\partial D_t)$ this is straightforward in view of (7). The core of the paper is the a priori estimate of $q(D_t)$, which we get by constructing an appropriate defining function $\Phi(\cdot\,,t)$ for D_t satisfying

$$|\nabla\Phi(\cdot\,,t)|_{\inf} \geq |\nabla\varphi_0|_{\inf} \exp\biggl(-C_n\int_0^t \|\nabla v(\cdot\,,s)\|_\infty\,ds\biggr), \qquad t>0, \tag{16}$$

$$\|\nabla\Phi(\cdot\,,t)\|_{\gamma,\partial D_t} \leq \|\nabla\varphi_0\|_{\gamma,\partial D_0} \exp\biggl(C_n\int_0^t (1+\|\nabla v(\cdot\,,s)\|_\infty)\,ds\biggr), \quad t>0. \tag{17}$$

As it was pointed out in [Bertozzi et al. 2016] if one transports a defining function φ_0 of D_0 by $\varphi_t=\varphi_0\circ X^{-1}(\cdot\,,t)$, then $\nabla\varphi_t$ may have jumps at the boundary of D_t for $t\neq 0$ and so φ_t is not necessarily differentiable. In [Bertozzi et al. 2016] one shows that

$$\lim_{D_t\ni y\to x}\nabla\varphi(y,t)=\lim_{D_t\ni y\to x}\det\nabla X^{-1}(y,t)\,\frac{|\nabla\varphi_0(X^{-1}(x,t))|}{\det D(x)}\,\vec{n}(x), \qquad x\in\partial D_t, \tag{18}$$

$$\lim_{\mathbb{R}^n\setminus\bar{D}_t\ni y\to x}\nabla\varphi(y,t)=\lim_{\mathbb{R}^n\setminus\bar{D}_t\ni y\to x}\det\nabla X^{-1}(y,t)\,\frac{|\nabla\varphi_0(X^{-1}(x,t))|}{\det D(x)}\,\vec{n}(x), \quad x\in\partial D_t, \tag{19}$$

where $X^{-1}(\cdot\,,t)$ is the inverse mapping of $X(\cdot\,,t)$ and $D(x)$ is the differential at x of the restriction of $X^{-1}(\cdot\,,t)$ to ∂D_t, as a differentiable mapping from ∂D_t onto ∂D_0. Define

$$\Phi(x,t)=\begin{cases}0, & x\in\partial D_t,\\ \det\nabla X(X^{-1}(x,t),t)\varphi(x,t), & x\notin\partial D_t.\end{cases} \tag{20}$$

We show in Section 4 that $\Phi(x,t)$ is a defining function of D_t of class $C^{1+\gamma}$.

The definition of Φ yields a formula for its material derivative $D/(Dt)=\partial_t+v\cdot\nabla$, namely,

$$\frac{D\Phi}{Dt}=\operatorname{div}(v)\,\Phi. \tag{21}$$

Taking gradient in the preceding identity one gets

$$\frac{D(\nabla\Phi)}{Dt}=\nabla(\operatorname{div}(v))\,\Phi+\operatorname{div}(v)\nabla\Phi-(\nabla v)^t(\nabla\Phi), \tag{22}$$

where $(\nabla v)^t$ stands for the transpose of the matrix ∇v. The right-hand side of (22) can be split into two terms which behave differently. The first is $\nabla(\operatorname{div}(v))\,\Phi$ and the second $\operatorname{div}(v)\nabla\Phi-(\nabla v)^t(\nabla\Phi)$. We prove that the second term is a finite sum of differences of commutators, which can be shown, with some effort, to have the right estimates. The first term does not combine with others to yield a commutator and because of that we call it the solitary term. A priori it is the most singular term on the right-hand side of (22), since it contains second-order derivatives of v. We show that the solitary term extends continuously to ∂D_t by 0 and so it can be ignored at the price of working only on the boundary of D_t for all t.

To prove that the solitary term extends continuously to the boundary by 0 we need a recent result of [Vasin 2017] whose statement is as follows. Let T be a convolution homogeneous even Calderón–Zygmund operator of the type

$$T(f)(x)=\text{p.v.}\int_{\mathbb{R}^n} L(x-y)f(y)\,dy=\lim_{\epsilon\to 0}\int_{|y-x|>\epsilon} L(x-y)f(y)\,dy, \tag{23}$$

where L is an even kernel, homogeneous of degree $-n$, satisfying the smoothness condition $L \in C^1(\mathbb{R}^n \setminus \{0\})$ and the cancellation property $\int_{|x|=1} L(x)\,d\sigma(x) = 0$. The function f is in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$, and the principal value integral (23) is defined a.e. on $\mathbb{R}^n$. Vasin's result states that if D is a bounded domain with boundary of class $C^{1+\gamma}$ then

$$|\nabla T(\chi_D)(x)|\ \operatorname{dist}(x, \partial D)^{1-\gamma} \le C, \quad x \in D \cup (\mathbb{R}^n \setminus \bar{D}), \tag{24}$$

where the constant C depends only on n, γ and the constants giving the smoothness of ∂D. We provide a proof of (24) in Section 3 for completeness.

One applies (24) to the second derivatives of the velocity field $v = k \star \chi_{D_t}$ with t fixed. One has in the distributions sense

$$\partial_j k = \text{ p.v. } \partial_j k + \vec{c}_j\, \delta_0, \quad \text{with } \vec{c}_j = \int_{|\xi|=1} k(\xi)\xi_j\, d\sigma(\xi), \tag{25}$$

and so

$$(\partial_j v)(x) = (\text{ p.v. } \partial_j k \star \chi_{D_t})(x) + \vec{c}_j\, \chi_{D_t}(x), \quad x \in D_t \cup (\mathbb{R}^n \setminus \bar{D}_t),$$

and, taking a second derivative,

$$(\partial_l \partial_j v)(x) = \partial_l(\text{ p.v. } \partial_j k \star \chi_{D_t})(x), \quad x \in D_t \cup (\mathbb{R}^n \setminus \bar{D}_t),\ 1 \le l, j \le n. \tag{26}$$

By (24) applied to the operator T associated with the kernel $L = \partial_j k$

$$|(\partial_l \partial_j v)(x)|\operatorname{dist}(x, \partial D_t)^{1-\gamma} \le C(t), \quad x \in D_t \cup (\mathbb{R}^n \setminus \bar{D}_t),\ 1 \le l, j \le n, \tag{27}$$

where $C(t)$ depends on n, γ and the constants related to the smoothness of ∂D_t. This implies that the solitary term has limit 0 at the boundary of ∂D_t, coming from the complement, because $|\Phi(x,t)|$ is comparable to $\operatorname{dist}(x, \partial D_t)$ as x approaches ∂D_t ($\Phi(\cdot, t)$ is continuously differentiable and vanishes on the boundary but the gradient does not).

It is worth remarking that if each component of the kernel k is harmonic off the origin, then (24) can be obtained readily from the fact that $T(\chi_D)$ satisfies a Hölder condition of order γ in D, which is the main lemma of [Mateu et al. 2009].

From (22) at boundary points, and thus without the solitary term, one gets straightforwardly (16). Thus the a priori estimate of $q(D_t)$ is reduced to (17).

We turn now our attention to the second term in (22). We prove that the i-th component of the vector $\operatorname{div}(v)\nabla\Phi - (\nabla v)^t(\nabla\Phi)$ evaluated at the point $x \in \mathbb{R}^n$ is a sum of $n-1$ terms, each of which is a difference of two commutators. In fact, the i-th component is

$$\sum_{j \ne i} \text{p.v.}\int_{D_t} \partial_j k_j(x-y)(\partial_i \Phi(x) - \partial_i \Phi(y))\,dy - \text{ p.v.}\int_{D_t} \partial_i k_j(x-y)(\partial_j \Phi(x) - \partial_j \Phi(y))\,dy. \tag{28}$$

It is crucial here that we obtain differences of commutators, which provides eventually an extra cancellation.

In [Bertozzi and Constantin 1993] it was shown that the Hölder seminorm of order γ of each commutator in (28) can be estimated by $C_n\,(1 + \|\nabla v(\cdot, t)\|_\infty)\|\nabla\Phi(\cdot, t)\|_{\gamma, \mathbb{R}^n}$. This is not enough in our situation, because of the presence of the factor $\|\nabla\Phi(\cdot, t)\|_{\gamma, \mathbb{R}^n}$, which should be replaced by a boundary quantity like $\|\nabla\Phi(\cdot, t)\|_{\gamma, \partial D_t}$.

To obtain the correct estimate we transform the j-the term in (28) into a difference of two boundary commutators:

$$\text{p.v.}\int_{\partial D_t} k_j(x-y)(\partial_i\Phi(x)-\partial_i\Phi(y))n_j(y)\,d\sigma(y) - \text{p.v.}\int_{\partial D_t} k_j(x-y)(\partial_j\Phi(x)-\partial_j\Phi(y))n_i(y)\,d\sigma(y). \quad (29)$$

It is worth emphasizing here that it is not true that the commutator

$$\text{p.v.}\int_{D_t} \partial_j k_j(x-y)(\partial_i\Phi(x)-\partial_i\Phi(y))\,dy$$

equals

$$\text{p.v.}\int_{\partial D_t} k_j(x-y)(\partial_i\Phi(x)-\partial_i\Phi(y))n_j(y)\,d\sigma(y).$$

What is true is that the difference of two commutators in the j-th term of (28) equals the difference of two commutators in (29). There is some magic here in arranging all terms so that certain hidden cancellation takes place. To get the right estimates on the boundary commutators one cannot adapt [Bertozzi and Constantin 1993, Lemma 7.3, p. 26] to the underlying measure $d\sigma$ on ∂D_t, because this would give a constant of the type

$$C_t = \sup_{x\in\partial D_t}\sup_{r>0}\frac{\sigma(B(x,r))}{r^{n-1}},$$

which can be estimated by the Lipschitz constant of $X(\cdot,t)$, namely, $\exp\int_0^t \|\nabla v(\cdot,s)\|_\infty\,ds$. This exponential constant is by far too large.

One needs to replace the standard bound $C_n\,(1+\|\nabla v(\cdot,t)\|_\infty)\|\nabla\Phi(\cdot,t)\|_{\gamma,\mathbb{R}^n}$ for a “solid” commutator of the type (28) by $C_n\,(1+\|\nabla v(\cdot,t)\|_\infty)\|\nabla\Phi(\cdot,t)\|_{\gamma,\partial D_t}$. Here we have used the term solid commutator to indicate that the integration is on D_t with respect to n-dimensional Lebesgue measure as opposed to a boundary commutator in which the integration is on the boundary of ∂D_t with respect to surface measure σ. To get the estimate in terms of $\|\nabla\Phi(\cdot,t)\|_{\gamma,\partial D_t}$ we resort to the difference-of-commutators structure, which allows us to appeal to Whitney’s extension theorem, the reason being that one can switch between a difference of boundary commutators and a difference of solid commutators via the divergence theorem. The final outcome is (17).

Of course for those cases in which the kernel is divergence-free, the quasigeostrophic equation in particular, one does not need the boundary commutators and getting the commutator formula (28) suffices to complete the proof as in [Bertozzi and Constantin 1993]. Indeed in these cases the transported defining function is already a genuine defining function, since the gradient has no jump according (18) and (19), or appealing to a regularization argument, as in [Radu 2022].

To complete the proof from the a priori estimates is a standard reasoning. One needs a logarithmic inequality for $\|\nabla v(\cdot,t)\|_\infty$, which is a consequence of the boundedness of $T(\chi_D)$ for an even smooth convolution Calderón–Zygmund operator T and a domain D with boundary of class $C^{1+\gamma}$, and of the particular form of the constant. One obtains

$$\|\nabla v(\cdot,t)\|_\infty \le \frac{C_n}{\gamma}\left(1+\log^+\left(|D_t|^{1/n}\frac{\|\nabla\Phi\|_{\gamma,\partial D_t}}{|\nabla\Phi|_{\inf}}\right)\right), \quad (30)$$

where C_n is a dimensional constant and $|D|$ stands for the n-dimensional Lebesgue measure of the measurable set D. The novelty in inequality (30) is that $\|\nabla\Phi\|_{\gamma,\partial D_t}$ is now replacing the larger constant $\|\nabla\Phi\|_{\gamma,\mathbb{R}^n}$ which appears in [Bertozzi and Constantin 1993; Bertozzi et al. 2016, Corollary 6.3] in dealing with the corresponding inequality. This follows from a scrutiny of the constants that appear along the proof and an application of the implicit differentiation formula.

Inserting (16) and (17) in (30) one gets, for a dimensional constant C,

$$\|\nabla v(\,\cdot\,,t)\|_\infty \le C + C\int_0^t (1+\|\nabla v(\,\cdot\,,s)\|_\infty)\,ds,$$

which yields, by Gronwall,

$$\|\nabla v(x,t)\|_\infty \le C\,e^{Ct}, \quad -T<t<T,$$

and this completes the proof of (15).

The reader may have observed that it is not strictly necessary for the proof to use the quantity $q(D_t)$, defined in (14). Nevertheless, it is the canonical quantity to take into consideration and helps to make some statements clearer. We will use it again in Section 4.

3. An auxiliary result

The result we are referring to is the following and can be found in [Vasin 2017].

Lemma. *Let $D\subset\mathbb{R}^n$ be a bounded domain with boundary of class $C^{1+\gamma}$, $0<\gamma<1$, and L an even kernel in $C^1(\mathbb{R}^n\setminus 0)$, homogeneous of degree $-n$. Then*

$$|\nabla(L\star\chi_D)(x)|\,\operatorname{dist}(x,\partial D)^{1-\gamma}\le C, \quad x\in\mathbb{R}^n\setminus\partial D,$$

where C is a constant depending only on D.

Proof. Placing the gradient on the characteristic function of D we obtain

$$\nabla(L\star\chi_D) = L\star(-\vec{n}\,d\sigma_{\partial D}).$$

Fix $x\in D$ and set $d=d(x)=\operatorname{dist}(x,\partial D)$. By the divergence theorem

$$(L\star\vec{n}\,d\sigma_{\partial D})(x) = (L\star\vec{n}\,d\sigma_{\partial B(x,d)})(x) - \int_{D\setminus B(x,d)}\nabla L(x-y)\,dy.$$

Now

$$(L\star\vec{n}\,d\sigma_{\partial B(x,d)})(x) = \int_{|y-x|=d} L(x-y)\vec{n}(y)\,d\sigma(y) = \int_{|z|=d} L(z)\vec{n}(z)\,d\sigma(z)$$

and the last integral clearly vanishes, owing to the oddness of $L(z)\vec{n}(z)$. Thus

$$\nabla(L\star\chi_D)(x) = -(L\star\vec{n}\,d\sigma_{\partial D})(x) = \int_{D\setminus B(x,d)}\nabla L(x-y)\,dy,$$

and

$$\operatorname{dist}(x,\partial D)\,|\nabla(L\star\chi_D)(x)| = d\left|\int_{D\setminus B(x,d)}\nabla L(x-y)\,dy\right| \le d\int_{D\setminus B(x,d)}\frac{C}{|y-x|^{n+1}}\,dy \le C.$$

Therefore in proving the lemma one can assume that $d \le \frac{1}{2} r_0$, where $r_0 = r_0(D)$ has the property that, given a point p in the boundary of D, $B(p, 2r_0) \cap D$ is the set of points in $B(p, 2r_0)$ lying below the graph of a $C^{1+\gamma}$ function defined on the tangent hyperplane through p.

We assume, without loss of generality, that 0 is the closest point of ∂D to x and that the tangent hyperplane to ∂D at 0 is $\{x \in \mathbb{R}^n : x_n = 0\}$. We also assume that $D \cap B(0, 2r_0) = \{x \in \mathbb{R}^n : x_n < \varphi(x')\}$, where $x' = (x_1, \dots, x_{n-1})$, $\varphi \in C^{1+\gamma}(B'(0, 2r_0))$, $B'(0, 2r_0) = \{x' \in \mathbb{R}^{n-1} : |x'| < 2r_0\}$. In particular,

$$|\varphi(x')| \le \|\nabla\varphi\|_{\gamma, B'(0,2r_0)} |x'|^{1+\gamma}, \quad x' \in B'(0, 2r_0).$$

We clearly have

$$\int_{D \setminus B(x,d)} \nabla L(x-y)\, dy = \int_{(D \setminus B(x,d)) \cap B(0,r_0)} \nabla L(x-y)\, dy + \int_{D \cap B^c(0,r_0)} \nabla L(x-y)\, dy.$$

The second term above is easy to estimate:

$$\left| \int_{D \cap B^c(0,r_0)} \nabla L(x-y)\, dy \right| \le \int_{B^c(0,r_0)} \frac{dy}{|y|^{n+1}}\, dy \le \frac{C}{r_0}.$$

For the first term one uses the fact that if H is a halfspace then $L \star \chi_H$ vanishes on H. This follows from the fact that the preceding statement is true for balls instead of halfspaces [Mateu et al. 2009] and a straightforward limiting argument. Then one has

$$\begin{aligned}
\int_{(D \setminus B(x,d)) \cap B(0,r_0)} \nabla L(x-y)\, dy &= \int_{(D \setminus B(x,d)) \cap B(0,r_0)} \nabla L(x-y)\, dy - \int_{H_-} \nabla L(x-y)\, dy \\
&= \int_{(D \setminus H_-) \cap B(0,r_0)} \nabla L(x-y)\, dy \\
&\qquad - \int_{(H_- \setminus (D \cup B(x,d))) \cap B(0,r_0)} \nabla L(x-y)\, dy - \int_{H_- \cap B^c(0,r_0)} \nabla L(x-y)\, dy,
\end{aligned}$$

and the last term is estimated as we did above with D in place of H_-. The remaining two terms are tangential and they are treated similarly. For the first we set

$$\int_{(D \setminus H_-) \cap B(0,r_0)} \nabla L(x-y)\, dy = \int_{(D \setminus H_-) \cap B(0,2d)} \nabla L(x-y)\, dy + \int_{(D \setminus H_-) \cap (B(0,r_0) \setminus B(0,2d))} \nabla L(x-y)\, dy.$$

Since for $x \in D \setminus H_-$ one has $|y - x| \ge d$, we get

$$\begin{aligned}
\left| \int_{(D \setminus H_-) \cap B(0,2d)} \nabla L(x-y)\, dy \right| &\le \frac{C}{d^{n+1}} |(D \setminus H_-) \cap B(0, 2d)| \\
&\le \frac{C}{d^{n+1}} \int_0^{2d} \rho^{n-1}\, \sigma\{\theta \in S^{n-1} : \rho\theta \in D \setminus H_-\}\, d\rho \\
&\le \frac{C}{d^{n+1}} \int_0^{2d} \rho^{n-1+\gamma}\, d\rho = C\, d^{\gamma-1}.
\end{aligned}$$

Finally

$$\left|\int_{(D\setminus H_-)\cap(B(0,r_0)\setminus B(0,2d))} \nabla L(x-y)\,dy\right| \le C \int_{(D\setminus H_-)\cap(B(0,r_0)\setminus B(0,2d))} \frac{1}{|y|^{n+1}}\,dy$$
$$\le C \int_0^{2d} \frac{1}{\rho^{n+1}} \rho^{n-1+\gamma}\,d\rho = C\,d^{\gamma-1}. \qquad \square$$

It is an interesting fact that the preceding lemma implies the main lemma in [Mateu et al. 2009], which states that under the hypothesis of Vasin's lemma the function $L \star \chi_D$ satisfies a Hölder condition of order γ on D and on $\mathbb{R}^n \setminus \overline{D}$. Incidentally, it is worth mentioning that this result has been proved independently by various authors at different times and with various degrees of generality. We are grateful to M. Lanza de Cristoforis for bringing to our attention the oldest reference we are aware of, namely, the paper of Carlo Miranda [1965].

We give an account of the proof of this fact only for the statement concerning D. In the exterior of D one applies similar arguments.

Take two points x and y in D. Let $d = \operatorname{dist}(x, \partial D)$ be the distance from x to the boundary. As before, we assume, without loss of generality, that 0 is the closest point of ∂D to x and that the tangent hyperplane to ∂D at 0 is $\{x \in \mathbb{R}^n : x_n = 0\}$. We can also assume that $D \cap B(0, 2r_0) = \{x \in \mathbb{R}^n : x_n < \varphi(x')\}$, where $x' = (x_1, \dots, x_{n-1})$, $\varphi \in C^{1+\gamma}(B'(0, 2r_0))$, $B'(0, 2r_0) = \{x' \in \mathbb{R}^{n-1} : |x'| < 2r_0\}$. Then

$$|\varphi(x')| \le \|\nabla\varphi\|_{\gamma, B'(0,2r_0)} |x'|^{1+\gamma}, \quad x' \in B'(0, 2r_0). \tag{31}$$

As in [Mateu et al. 2009] we can reduce matters to the case in which $d \le \frac{1}{2} r_0$, because otherwise we resort to the smoothness of $L \star \chi_D$ on the domain $\left\{z \in D : \operatorname{dist}(z, \partial D) > \frac{1}{2} r_0\right\}$.

Let K be the closed cone with aperture 45° and axis the negative x_n-axis. That is

$$K = \{x \in \mathbb{R}^n : -\sqrt{2}\, x_n \ge |x|\}.$$

We say that x and y are in nontangential position if $x, y \in K$. Otherwise they are in tangential position.

Assume first that $x, y \in D$ are in nontangential position and distinguish two cases. The first is $y \in B(0, 2d) \setminus B(0, d)$. Apply the mean value theorem on an arc contained in $K \cap (B(0, 2d) \setminus B(0, d))$ of length comparable to $|y - x|$. One gets

$$|f(y) - f(x)| \le C \sup\{\operatorname{dist}(\xi, \partial D)^{\gamma-1} : \xi \in K \cap (B(0, 2d) \setminus B(0, d))\}|y - x|. \tag{32}$$

We claim that there exists an absolute constant c_0 with $0 < c_0 < 1$ satisfying

$$\operatorname{dist}(\xi, \partial D) \ge c_0\, |\xi_n|, \quad \xi \in K \cap B(0, r_0), \tag{33}$$

provided r_0 is small enough. Let $p \in \partial D$ be such that $|\xi - p| = \operatorname{dist}(\xi, \partial D)$. Since $p = (p', p_n)$ is on the graph of φ we have, by (31), $|p_n| \le C\, |p'|^{1+\epsilon} \le C\, r_0^{\epsilon}\, |p'|$. Thus

$$\begin{aligned} |\xi_n| &\le |\xi_n - p_n| + |p_n| \le |\xi - p| + C\, r_0^{\epsilon}\, |p'| \\ &\le |\xi - p| + C\, r_0^{\epsilon}\, (|p' - \xi'| + |\xi'|) \\ &\le |\xi - p|(1 + C\, r_0^{\epsilon}) + C\, r_0^{\epsilon}\, |\xi_n|, \end{aligned}$$

where in the last inequality we used that $|\xi| \le \sqrt{2}|\xi_n|,\ \xi \in K$. Taking r_0 so small that $C\, r_0^{\epsilon} \le \frac{1}{2}$ we obtain

$$|\xi_n| \le 2(1 + C\, r_0^{\epsilon})|\xi - p| = 2(1 + C\, r_0^{\epsilon})\, \mathrm{dist}(\xi, \partial D).$$

Indeed, the constant C is the previous string of inequalities is $\sqrt{2}\, \|\nabla\varphi\|_{\gamma, B'(0,2r_0)}$, which also depends on r_0. But this is not an obstruction because it decreases with r_0.

Therefore, by (32),

$$\begin{aligned}|f(y) - f(x)| &\le C\,(c_0|\xi_n|)^{\gamma-1}|y - x| \le C\, c_0^{\gamma-1} d^{\gamma-1}|y - x|^{1-\gamma}|y - x|^{\gamma}\\ &\le C\, c_0^{\gamma-1} d^{\gamma-1}(3d)^{1-\gamma}|y - x|^{\gamma} = C\,|y - x|^{\gamma}.\end{aligned}$$

Let us turn our attention to the case $y \in K \cap B^c(0, 2d)$. Note that there exists an absolute constant $C_0 > 1$ such that

$$|y - x| \le C_0\, |y_n - x_n|, \quad y \in K \cap B^c(0, 2d).$$

Apply the fundamental theorem of calculus on the interval with endpoints x and y and estimate the gradient of f by a constant times the distance to the boundary raised to the power $\gamma - 1$. By (33) we obtain

$$\begin{aligned}|f(y) - f(x)| &\le C \int_0^1 \mathrm{dist}(x + t(y - x), \partial D)^{\gamma-1}|y - x|\, dt\\ &\le C\, c_0^{\gamma-1} \int_0^1 |x_n + t(y_n - x_n)|^{\gamma-1}|y - x|\, dt\\ &= C\, \frac{|y - x|}{|y_n - x_n|} \int_0^{|y_n - x_n|} (d + \tau)^{\gamma-1}\, d\tau\\ &= C\, C_0\,((d + |y_n - x_n|)^{\gamma} - d^{\gamma}) \le C\,|y_n - x_n|^{\gamma} \le C\,|y - x|^{\gamma},\end{aligned}$$

as desired.

We are left with the case in which x and y are in tangential position, that is, $y \in D \cap (\mathbb{R}^n \setminus K)$. In [Mateu et al. 2009] there is a reduction argument to the nontangential case, which we now reproduce for completeness. Take a point $p \in \partial D$ with $|y - p| = \mathrm{dist}(y, \partial D)$ and let $\vec{N}$ be the exterior unit normal vector to ∂D at p. We will take r_0 so small that $\vec{N}$ is very close to the exterior unit normal vector $\vec{n}$ to ∂D at 0. Then the ray $y - t\vec{N},\ t > 0$, will intersect K at some point y_0 and the pairs x, y_0 and y, y_0 will be in tangential position. Let us seek a condition on t so that $y - t\vec{N} \in K$, that is, so that

$$|y - t\vec{N}| \le \sqrt{2}|\langle y - t\vec{N}, \vec{n}\rangle|. \tag{34}$$

Here $\langle\,\cdot\,,\cdot\,\rangle$ denotes the scalar product in $\mathbb{R}^n$. Since $|\langle y - t\vec{N}, \vec{n}\rangle| \ge t\langle\vec{N}, \vec{n}\rangle - |y|$ and $|y - t\vec{N}| \le |y| + t$, a sufficient condition for (34) is

$$(1 + \sqrt{2})|y| \le t(\sqrt{2}\langle\vec{N}, \vec{n}\rangle - 1).$$

Take r_0 small enough so that $\sqrt{2}\langle\vec{N}, \vec{n}\rangle - 1 \ge (\sqrt{2} - 1)/2$. A simpler sufficient condition for (34) is

$$|y| \le c_0\, t, \quad \text{with } c_0 = \frac{1}{2}\,\frac{\sqrt{2} - 1}{\sqrt{2} + 1}.$$

Define t_0 by $|y| = c_0\, t_0$ and then set $y_0 = y - t_0 \vec{N}$. By construction, $y_0 \in K \cap D$ and the pairs x, y_0 and y, y_0 are in nontangential position. Hence we only have to check that

$$|y - y_0| \le C_0\, |x - y|. \tag{35}$$

We have $c_0\, |y - y_0| = c_0\, t_0 = |y|$. On the other hand, the condition $y \notin K$ is exactly $|y| < \sqrt{2}\, |y'|$, and clearly $|x - y| \ge |y'|$. Therefore (35) holds with an absolute constant C_0.

4. The defining function for D_t

In this section we prove that the function Φ defined by (20) is a defining function of D_t of class $C^{1+\gamma}$. Our assumption now is that the CDE has a solution $X(\,\cdot\,, t)$ for t in an interval $(-T; T)$ and that D_t is the domain with $\partial D_t = X(\partial D_0, t)$ which has been defined in Section 2.1. The field defined by (13) has a flow map (6) whose restriction to ∂D_0 is precisely the solution of the CDE.

Taking the gradient in (20) we get, for $x \notin \partial D_t$,

$$\nabla \Phi(x, t) = \det \nabla X(X^{-1}(x, t), t)\, \nabla \varphi(x, t) + \nabla(\det \nabla X(X^{-1}(x, t), t))\varphi(x, t). \tag{36}$$

In [Bertozzi et al. 2016, Section 8] it was shown that $\nabla X^{-1}(\,\cdot\,, t)$ satisfies a Hölder condition of order γ on the open set $\mathbb{R}^n \setminus \partial D_t$ (but may have jumps at ∂D_t). What remains to be proved is that $\nabla\Phi(\,\cdot\,, t)$ extends continuously to ∂D_t. This is straightforward for the first term in the right-hand side of (36), just by the jump formulas (18) and (19). We have

$$\lim_{\mathbb{R}^n \setminus \partial D_t \ni y \to x} \det \nabla X(X^{-1}(y, t), t)\, \nabla \varphi(y, t) = \frac{|\nabla \varphi_0(X^{-1}(x, t))|}{\det D(x)}\, \vec{n}(x),$$

where $D(x)$ is the differential at $x \in \partial D_t$ of $X^{-1}(\,\cdot\,, t)$ viewed as a differentiable mapping from ∂D_t into ∂D_0.

The second term in the right-hand side of (36) tends to 0 as x approaches a point in ∂D_t. Proving this requires some work. For the sake of simplicity of notation let us consider positive times t less than T. Since $X(\,\cdot\,, t)$ is a continuously differentiable function of t with values in the Banach space $C^{1+\gamma}(\partial D_0, \mathbb{R}^n)$, the constants $q(D_s)$ determining the $C^{1+\gamma}$ smoothness of the boundary of D_s are uniformly bounded for $0 \le s \le t$. Hence

$$\|\nabla v(\,\cdot\,, s)\|_\infty \le C(t), \quad 0 \le s \le t, \tag{37}$$

$$\|\nabla v(\,\cdot\,, s)\|_{\gamma, D_s} + \|\nabla v(\,\cdot\,, s)\|_{\gamma, \mathbb{R}^n \setminus \bar{D}_s} \le C(t), \quad 0 \le s \le t, \tag{38}$$

where $C(t)$ denotes here and in the sequel a positive constant depending on t but not on $s \in [0, t]$. Inequality (37) follows from the fact, already mentioned, that standard even convolution Calderón–Zygmund operators are bounded on characteristic functions of $C^{1+\gamma}$ domains with bounds controlled by the constants giving the smoothness of the domain (see, for instance, (30)). Inequality (38) has appeared in the literature several times with various degrees of generality, as we mentioned in the previous section, where a complete proof was presented. In [Mateu et al. 2009] the reader will find another accessible proof independent of Vasin's lemma. The constants are not logarithmic, but this is not relevant here. The statement is that if T is an even smooth (of class C^1) convolution homogeneous Calderón–Zygmund operator and D is a domain with boundary of class $C^{1+\gamma}$, $0<\gamma<1$, then $T(\chi_D)$ satisfies a Hölder condition of order γ in D and in $\mathbb{R}^n \setminus \overline{D}$.

As we said in Section 2 one applies (24) to the second-order derivatives of the field v to conclude that

$$|\partial_j \partial_k v(x,s)| \le C(t)\ \operatorname{dist}(x, \partial D_s)^{\gamma-1}, \quad x \notin \partial D_s,\ 0 \le s \le t,\ 1 \le j,k \le n. \tag{39}$$

See (26) and (27).

Combining (7), the analogous inequality with $\nabla X(\cdot\,,t)$ replaced by $\nabla X^{-1}(\cdot\,,t)$ and (37) we get

$$C(t)^{-1} \le \|\nabla X(\cdot\,,s)\|_\infty \le C(t), \quad 0 \le s \le t. \tag{40}$$

Therefore $X(\cdot\,,s)$ is a bilipschitz homeomorphism of $\mathbb{R}^n$ and consequently, for all $\alpha \in R^n$,

$$C(t)^{-1} \operatorname{dist}(\alpha, \partial D_0) \le \operatorname{dist}(X(\alpha,s), \partial D_s) \le C(t) \operatorname{dist}(\alpha, \partial D_0), \quad 0 \le s \le t. \tag{41}$$

Now let us turn to the second term in the right-hand side of (36)

$$II(x) = \nabla(\det \nabla X(X^{-1}(x,t),t))\varphi(x,t) = \varphi_0(\alpha)\, \nabla_x J(\alpha,t), \tag{42}$$

where we have set $x = X(\alpha,t)$ and $J(\alpha,t) = \det \nabla X(\alpha,t)$. The jacobian satisfies

$$\frac{d}{dt} J(\alpha,t) = \operatorname{div} v(X(\alpha,t),t)\, J(\alpha,t)$$

and so

$$J(\alpha,t) = \exp \int_0^t \operatorname{div} v(X(\alpha,s),s)\, ds.$$

Hence $\nabla_x J(\alpha,t)$ is

$$\left(\exp \int_0^t \operatorname{div} v(X(\alpha,s),s)\, ds\right) \left(\int_0^t \operatorname{div}((\nabla v)^t (X(\alpha,s),s))\, \nabla X(\alpha,s)\, ds\right) \nabla X^{-1}(x,t), \tag{43}$$

where the divergence of a matrix is the vector with components the divergence of rows. Combining (37), (39), (40), (41), (42) and (43) we get

$$\begin{aligned} |II(x)| &\le C(t)\, |\varphi_0(\alpha)| \int_0^t \operatorname{dist}(X(\alpha,s), \partial D_s)^{\gamma-1}\, ds \\ &\le C(t)\, |\varphi_0(\alpha)|\ \operatorname{dist}(\alpha, \partial D_0)^{\gamma-1} \le C(t)\ \operatorname{dist}(\alpha, \partial D_0)^{\gamma}. \end{aligned}$$

If $\operatorname{dist}(x, \partial D_t) \to 0$ then $\operatorname{dist}(\alpha, \partial D_0) \to 0$ and thus $II(x) \to 0$.

5. The commutators

The material derivative $D/(Dt) = \partial_t + v \cdot \nabla$ of the defining function of the previous section is

$$\frac{D}{Dt}\Phi(x) = \frac{d}{dt}(J(\alpha,t)\, \varphi_0(\alpha)) = \operatorname{div} v(X(\alpha,t),t)\, J(\alpha,t)\, \varphi_0(\alpha) = \operatorname{div} v(x,t)\, \Phi(x,t),$$

which proves (21). Taking derivatives in the equation above and rearranging terms one obtains

$$\frac{D}{Dt}\nabla\Phi = \nabla(\operatorname{div} v)\Phi + (\operatorname{div} v)\nabla\Phi - (\nabla v)^t (\nabla\Phi). \tag{44}$$

The first term tends to 0 at the boundary of D_t, by (39). This section is devoted to proving that the second term in the right-hand side, namely $(\operatorname{div} v)\nabla\Phi - (\nabla v)^t(\nabla\Phi)$, is a sum of $n-1$ terms, each of which is a difference of boundary commutators. It clearly suffices to prove that each coordinate is a sum of $n-1$ differences of boundary commutators. We present the details for the first coordinate, which is

$$\partial_2 v_2\, \partial_1\Phi - \partial_1 v_2\, \partial_2\Phi + \cdots + \partial_n v_n\, \partial_1\Phi - \partial_1 v_n\, \partial_n\Phi. \tag{45}$$

Let us work with the first term $\partial_2 v_2\, \partial_1\Phi - \partial_1 v_2\, \partial_2\Phi$. The others are treated similarly. The preceding expression is evaluated at (x,t) with $x \in \partial D_t$. To lighten the notation we set $D = D_t$, so that t is fixed, and $\chi = \chi_{D_t}$. Recall that $v(\cdot\,,t) = k \star \chi$ and so

$$v_j(\cdot\,,t) = k_j \star \chi, \quad 1 \le j \le n.$$

By (25) we have in the distributions sense

$$\partial_j v_j(\cdot\,,t) = \partial_j k_j \star \chi = \text{p.v. } \partial_j k_j \star \chi + c_j \chi, \quad 1 \le j \le n,$$

where $c_j = \int_{|\xi|=1} k_j(\xi)\xi_j \, d\sigma(\xi)$. Thus

$$\partial_2 v_2(\cdot\,,t)\, \partial_1\Phi(\cdot\,,t) = (\partial_2 k_2 \star \chi)(\cdot)\, \partial_1\Phi(\cdot\,,t) = \text{p.v.}(\partial_2 k_2 \star \chi)\, \partial_1\Phi + c_2\chi\partial_1\Phi$$

and

$$\partial_2 k_2 \star (\chi\partial_1\Phi) = \text{p.v. } \partial_2 k_2 \star (\chi\partial_1\Phi) + c_2\chi\partial_1\Phi,$$

which yields

$$\partial_2 v_2(\cdot\,,t)\, \partial_1\Phi(\cdot\,,t) = \text{p.v.}(\partial_2 k_2 \star \chi)\, \partial_1\Phi - \text{p.v. } \partial_2 k_2 \star (\chi\partial_1\Phi) + \partial_2 k_2 \star (\chi\partial_1\Phi). \tag{46}$$

Similarly

$$\partial_1 v_2(\cdot\,,t)\, \partial_2\Phi(\cdot\,,t) = \text{p.v.}(\partial_1 k_2 \star \chi)\, \partial_2\Phi - \text{p.v. } \partial_1 k_2 \star (\chi\partial_2\Phi) + \partial_1 k_2 \star (\chi\partial_2\Phi). \tag{47}$$

Since $\chi\partial_j\Phi = \partial_j(\chi\Phi)$, $1 \le j \le n$, we have

$$\partial_2 k_2 \star (\chi\partial_1\Phi) = \partial_1 k_2 \star (\chi\partial_2\Phi),$$

and subtracting (47) from (46) yields

$$\begin{aligned}\partial_2 v_2(\cdot\,,t)\, &\partial_1\Phi(\cdot\,,t) - \partial_1 v_2(\cdot\,,t)\, \partial_2\Phi(\cdot\,,t) \\ &= \text{p.v.}(\partial_2 k_2 \star \chi)\, \partial_1\Phi - \text{p.v. } \partial_2 k_2 \star (\chi\partial_1\Phi) - \big(\text{p.v.}(\partial_1 k_2 \star \chi)\, \partial_2\Phi - \text{p.v. } \partial_1 k_2 \star (\chi\partial_2\Phi)\big),\end{aligned} \tag{48}$$

which is the difference of two solid commutators. Here we are using the term "solid" to indicate that the integration is taken with respect to n-dimensional Lebesgue measure. Our next task is to bring the solid commutators to the boundary.

Formula (48) is an identity between distributions and is not a priori obvious that the principal value integrals exist at boundary points. The same can be said about the principal values on the boundary which appear in the calculation below. That they do exist in our context is a routine argument, which we postpone to the Appendix.

Let $x \in \partial D$. Given $\epsilon > 0$ set $D_\epsilon = D \setminus \overline{B(x,\epsilon)}$. By the divergence theorem

$$
\begin{aligned}
(\text{p.v. } \partial_2 k_2 \star (\chi \partial_1 \Phi))(x) &= \lim_{\epsilon \to 0} \int_{D_\epsilon} \partial_2 k_2(x-y) \partial_1 \Phi(y)\, dy \\
&= -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} k_2(x-y) \partial_1 \Phi(y) n_2(y)\, d\sigma(y) + \int_D k_2(x-y) \partial_{21} \Phi(y)\, dy \\
&= -\text{p.v.} \int_{\partial D} k_2(x-y) \partial_1 \Phi(y) n_2(y)\, d\sigma(y) + \int_D k_2(x-y) \partial_{12} \Phi(y)\, dy \\
&\qquad + \lim_{\epsilon \to 0} \int_{\partial B(x,\epsilon) \cap D} k_2(x-y) \partial_1 \Phi(y) n_2(y)\, d\sigma_\epsilon(y),
\end{aligned}
$$

where σ_ϵ is the surface measure on $\partial B(x,\epsilon)$. We do not need to compute explicitly the term

$$
\lim_{\epsilon \to 0} \int_{\partial B(x,\epsilon) \cap D} k_2(x-y) \partial_1 \Phi(y) n_2(y)\, d\sigma_\epsilon(y),
$$

nor to worry about the second-order derivative of Φ which has appeared, because they will eventually cancel out (a routine regularization argument takes care of the actual presence of the second derivatives of Φ).

We turn now to the computation of $(\text{p.v. } \partial_{12} N \star \chi)(x)$. We have

$$
\begin{aligned}
(\text{p.v. } \partial_2 k_2 \star \chi)(x) &= \lim_{\epsilon \to 0} \int_{D_\epsilon} \partial_2 k_2(x-y)\, dy \\
&= -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} k_2(x-y) n_2(y)\, d\sigma(y) \\
&= -\text{p.v.} \int_{\partial D} k_2(x-y) n_2(y)\, d\sigma(y) + \lim_{\epsilon \to 0} \int_{\partial B(x,\epsilon) \cap D} k_2(x-y) n_2(y)\, d\sigma_\epsilon(y).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&(\text{p.v. } \partial_2 k_2 \star (\chi \partial_1 \Phi))(x) - (\text{p.v. } \partial_2 k_2 \star \chi)(x) \partial_1 \Phi(x) \\
&\quad = \text{p.v.} \int_{\partial D} k_2(x-y) n_2(y)\, d\sigma(y)\, \partial_1 \Phi(x) - \text{p.v.} \int_{\partial D} k_2(x-y) \partial_1 \Phi(y) n_2(y)\, d\sigma(y) \\
&\qquad + \int_D k_2(x-y) \partial_{21} \Phi(y)\, dy, \qquad (49)
\end{aligned}
$$

since

$$
\lim_{\epsilon \to 0} \int_{\partial B(x,\epsilon) \cap D} k_2(x-y) (\partial_1 \Phi(y) - \partial_1 \Phi(x))\, d\sigma_\epsilon(y) = 0,
$$

because k_2 is homogeneous of order $-(n-1)$ and $\partial_1 \Phi$ is continuous at x. The conclusion is that the solid commutator in the left-hand side of (49) is a boundary commutator plus and additional term involving second-order derivatives of Φ. This term will disappear soon and in the final formulas no second derivatives of Φ are present, so that the $C^{1+\gamma}$ condition on Φ is enough.

Proceeding in a similar way we find

$$
\begin{aligned}
&(\text{p.v. } \partial_1 k_2 \star (\chi \partial_2 \Phi))(x) - (\text{p.v. } \partial_1 k_2 \star \chi)(x) \partial_2 \Phi(x) \\
&\quad = \text{p.v.} \int_{\partial D} k_2(x-y) n_1(y)\, d\sigma(y)\, \partial_2 \Phi(x) - \text{p.v.} \int_{\partial D} k_2(x-y) \partial_2 \Phi(y) n_1(y)\, d\sigma(y) \\
&\qquad + \int_D k_2(x-y) \partial_{12} \Phi(y)\, dy, \qquad (50)
\end{aligned}
$$

Subtracting (50) from (49) we see that the difference of the two solid commutators in (48) is exactly, on the boundary of D, a difference of boundary commutators. Hence $((\operatorname{div} v)I - (\nabla v)^t)(\nabla\Phi)$ is a sum of $n-1$ terms, each being a difference of two vector-valued boundary commutators.

6. Hölder estimate of differences of boundary commutators

We keep the notation of the previous section $D = D_t$, $\nabla\Phi = \nabla\Phi(\,\cdot\,, t)$, with t fixed. Our goal is to estimate the Hölder seminorm of order γ on ∂D of the difference of two boundary commutators. For instance,

$$\begin{aligned} DB(x) := \text{p.v.}\int_{\partial D} k_2(x-y)\partial_2\Phi(y)n_1(y)\,d\sigma(y) - \text{p.v.}\int_{\partial D} k_2(x-y)n_1(y)\,d\sigma(y)\,\partial_2\Phi(x) \\ - \left(\text{p.v.}\int_{\partial D} k_2(x-y)\partial_1\Phi(y)n_2(y)\,d\sigma(y) - \text{p.v.}\int_{\partial D} k_2(x-y)n_2(y)\,d\sigma(y)\,\partial_1\Phi(x)\right). \end{aligned}$$

The general case follows immediately by the same arguments. The strategy consists in exploiting the fact that $DB(x)$ is also, for $x \in \partial D$, a difference $DS(x)$ of two solid commutators, as we checked in the previous section. That is, $DB(x)$ for $x \in \partial D$ is identical to

$$DS(x) = DS(\Phi)(x) := \big(\text{p.v. } \partial_2 k_2 \star (\chi\partial_1\Phi) - (\text{p.v. } \partial_2 k_2 \star \chi)\partial_1\Phi\big) - \big(\partial_1 k_2 \star (\chi\partial_2\Phi) - (\partial_1 k_2 \star \chi)\partial_2\Phi\big).$$

By [Bertozzi and Constantin 1993, Corollary, p. 24, and Lemma, p. 26], estimating each commutator separately, we have $\|DS\|_{\gamma,\mathbb{R}^n} \le C_n \|\nabla v(\,\cdot\,, t)\|_\infty \|\nabla\Phi\|_{\gamma,\mathbb{R}^n}$, which is not good enough, because we need $\|\nabla\Phi\|_{\gamma,\partial D}$ in place of $\|\nabla\Phi\|_{\gamma,\mathbb{R}^n}$.

We now consider the jet

$$(0, \partial_1\Phi, \dots, \partial_n\Phi)$$

on ∂D. By Whitney's extension theorem [Stein 1970, Chapter VI, p. 177] there exists Ψ of class $C^{1+\gamma}(\mathbb{R}^n)$ such that $\Psi = 0$ and $\nabla\Psi = \nabla\Phi$ on ∂D, satisfying

$$\|\nabla\Psi\|_{\gamma,\mathbb{R}^n} \le C_n \left(\|\nabla\Phi\|_{\gamma,\partial D} + \sup\left\{\frac{|\nabla\Phi(x)\cdot(y-x)|}{|y-x|^{1+\gamma}} : y \ne x,\ y, x \in \partial D\right\}\right). \tag{51}$$

This precise estimate is not stated explicitly in Stein's book but it follows from the proof. We claim that

$$\sup\left\{\frac{|\nabla\Phi(x)\cdot(y-x)|}{|y-x|^{1+\gamma}} : y \ne x,\ y, x \in \partial D\right\} \le 2^{(1+\gamma)/2}\|\nabla\Phi\|_{\gamma,\partial D}. \tag{52}$$

We postpone the proof of the claim and we complete the estimate of $\|DB\|_{\gamma,\partial D}$.

The extension Ψ of the jet $(0, \partial_1\Phi, \dots, \partial_n\Phi)$ on ∂D, given by Whitney's extension theorem, satisfies, in view of (51) and (52) ,

$$\|\nabla\Psi\|_{\gamma,\mathbb{R}^n} \le C_{n,\gamma}\,\|\nabla\Phi\|_{\gamma,\partial D}.$$

Since $\nabla\Psi = \nabla\Phi$ on ∂D, the differences of solid commutators $DS(\Phi)$ and $DS(\Psi)$ are equal on ∂D. Thus

$$\begin{aligned} \|DB\|_{\gamma,\partial D} = \|DS(\Psi)\|_{\gamma,\partial D} \le \|DS(\Psi)\|_{\gamma,\mathbb{R}^n} \\ \le C_n\,(\|\nabla v(\,\cdot\,,t)\|_\infty + 1)\,\|\nabla\Psi\|_{\gamma,\mathbb{R}^n} \le C_n\,(\|\nabla v(\,\cdot\,,t)\|_\infty + 1)\,\|\nabla\Phi\|_{\gamma,\partial D}. \end{aligned}$$

This can be used to prove the a priori estimate (17) as in [Bertozzi and Constantin 1993].

We turn now to the proof of the claim (52). Fix a point $x \in \partial D$. Assume without loss of generality that $x = 0$ and $\nabla\Phi(0) = (0, \dots, 0, \partial_n\Phi(0)),\ \partial_n\Phi(0) > 0$. Define $\delta = \delta(x)$ by

$$\delta^{-\gamma} = 2\,\frac{\|\nabla\Phi\|_{\gamma,\partial D}}{|\nabla\Phi(0)|}.$$

This choice of δ implies that the normal vector $\nabla\Phi(y)$ remains for $y \in B(0,\delta) \cap \partial D$ in the ball $B(\nabla\Phi(0), |\nabla\Phi(0)|/2)$. Indeed

$$|\nabla\Phi(y) - \nabla\Phi(0)| \le \|\nabla\Phi\|_{\gamma,\partial D}\,\delta^{\gamma} = \frac{|\nabla\Phi(0)|}{2}.$$

Then given $y \in B(0,\delta) \cap \partial D$, the tangent hyperplane to ∂D at y forms an angle less than 30° with the horizontal plane and thus ∂D is the graph of a function $y_n = \varphi(y_n')$ which satisfies a Lipschitz condition with constant less than 1. Here we have used the standard notation $y = (y', y_n),\ y' = (y_1, \dots, y_{n-1})$. The function φ is defined in the open set U which is the projection of $B(0,\delta) \cap \partial D$ into $\mathbb{R}^{n-1}$ defined by $y \to y'$. By the implicit function theorem φ is of class $C^{1+\gamma}$ in its domain.

Note that for each $y \in \partial D \cap B(0,\delta)$, the segment $\{t\,y' : 0 \le t \le 1\}$ is contained in U, as an elementary argument shows. The mean value theorem on that segment for the function $t \to \varphi(ty')$ yields

$$\frac{|\nabla\Phi(0)\cdot y|}{|y|^{1+\gamma}} = \frac{|\nabla\Phi(0)|\,|\varphi(y')|}{|y|^{1+\gamma}} \le \frac{|\nabla\Phi(0)|}{|y|^{1+\gamma}} \sup\{|\nabla\varphi(z')| : z' \in U,\ |z'| \le |y'|\}\,|y'|.$$

By implicit differentiation

$$\partial_j\varphi(z') = -\frac{\partial_j\Phi(z', \varphi(z'))}{\partial_n\Phi(z', \varphi(z'))}, \quad 1 \le j \le n-1,$$

and so, recalling that $\partial_j\Phi(0) = 0,\ 1 \le j \le n-1$, and that $z = (z', \varphi(z'))$,

$$|\nabla\varphi(z')| \le \frac{\|\nabla\Phi\|_{\gamma,\partial D}\,|z|^{\gamma}}{|\partial_n\Phi(z)|} \le \frac{2}{|\nabla\Phi(0)|}\,\|\nabla\Phi\|_{\gamma,\partial D}\,2^{\gamma/2}\,|z'|^{\gamma}, \quad |z'| \le |y'|,$$

because

$$\begin{aligned} |\partial_n\Phi(z)| &\ge |\partial_n\Phi(0)| - |\partial_n\Phi(z) - \partial_n\Phi(0)| \\ &\ge |\nabla\Phi(0)| - \|\nabla\Phi\|_{\gamma,\partial D}\,\delta^{\gamma} = \frac{|\nabla\Phi(0)|}{2} \end{aligned}$$

and

$$|z| = (|z'|^2 + \varphi(z')^2)^{1/2} \le \sqrt{2}\,|z'|.$$

Thus

$$\frac{|\nabla\Phi(0)\cdot y|}{|y|^{1+\gamma}} \le 2^{1+\gamma/2}\,\|\nabla\Phi\|_{\gamma,\partial D}, \quad y \in \partial D \cap B(0,\delta).$$

If $y \in \partial D \setminus B(0,\delta)$,

$$\frac{|\nabla\Phi(0)\cdot y|}{|y|^{1+\gamma}} \le \frac{|\nabla\Phi(0)|}{|y|^{\gamma}} \le \frac{|\nabla\Phi(0)|}{\delta^{\gamma}} = 2\,\|\nabla\Phi\|_{\gamma,\partial D},$$

which completes the proof of (52).

7. Domains enclosed by ellipses as Cauchy patches

In this section we consider the transport equation in the plane given by the Cauchy kernel

$$\begin{aligned} &\partial_t \rho(z,t) + v(z,t)\cdot\nabla\rho(z,t) = 0, \\ &v(z,t) = \Big(\frac{1}{\pi z} * \rho(\,\cdot\,,t)\Big)(z), \\ &\rho(z,0) = \chi_{D_0}(z), \end{aligned} \tag{53}$$

$z = x + iy \in \mathbb{C} = \mathbb{R}^2$ and $t \in \mathbb{R}$. Note that we have changed the normalization of the velocity field in (4) by a factor of 2.

We take the initial patch to be the domain enclosed by an ellipse

$$E_0 = \left\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{a_0^2} + \frac{y^2}{b_0^2} < 1\right\}.$$

We will show that the solution provided by the theorem is of the form $\chi_{E_t}(z)$, with

$$E_t = \left\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{a(t)^2} + \frac{y^2}{b(t)^2} < 1\right\}$$

and

$$a(t) = a_0 \frac{(a_0+b_0)\,e^{2t}}{b_0 + a_0\,e^{2t}}, \quad t\in\mathbb{R}, \tag{54}$$

$$b(t) = b_0 \frac{(a_0+b_0)}{b_0 + a_0\,e^{2t}}, \quad t\in\mathbb{R}. \tag{55}$$

As $t \to \infty$, $a(t) \to a_0 + b_0$ and $b(t) \to 0$, so that the ellipse at time t degenerates into the segment $[-(a_0{+}b_0), a_0{+}b_0]$ as $t \to +\infty$ and into the segment $i[-(a_0{+}b_0), a_0{+}b_0]$ on the vertical axis as $t \to -\infty$.

Since (53) is not rotation invariant, one has to consider also the case of an initial patch given by the domain enclosed by a tilted ellipse

$$E(a,b,\theta) = e^{i\theta}\left\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\right\}.$$

In this case the straight line containing the semiaxis of length a makes an angle θ with the horizontal axis and we take $0 < \theta < \frac{\pi}{2}$.

Assume that the initial patch is $E_0 = E(a_0, b_0, \theta_0)$. Then we will show that the solution given by the theorem is χ_{E_t} with $E_t = E(a(t), b(t), \theta(t))$, where $a(t)$, $b(t)$ and $\theta(t)$ are the unique solutions of the system

$$\begin{aligned} a'(t) &= \frac{2}{a_0+b_0}\, a(t)\, b(t)\, \cos(2\theta(t)), \\ b'(t) &= -\frac{2}{a_0+b_0}\, a(t)\, b(t)\, \cos(2\theta(t)), \\ \theta'(t) &= -\frac{2}{a_0+b_0}\, \frac{a(t)\, b(t)}{a(t)-b(t)}\, \sin(2\theta(t)), \end{aligned} \tag{56}$$

with initial conditions $a(0) = a_0,\ b(0) = b_0,\ \theta(0) = \theta_0$.

We start the proof by assuming that the patch D_t of the weak solution provided by the theorem is indeed E_t. Let $z(t)$ be the trajectory of the particle that at time 0 is at $z(0) \in \partial E_0$. Then

$$\frac{dz}{dt} = v(z(t), t), \quad z(0) \in \partial E_0, \tag{57}$$

and v is the velocity field of (53). It is a well-known fact that v can be explicitly computed [Hmidi et al. 2015]. One has

$$v(z,t) = \Big(\frac{1}{\pi z} \star \chi_{E_t}\Big)(z) = \bar{z} - q(t)\, e^{-2\theta(t)} z, \quad z \in E_t,\ q(t) = \frac{a(t) - b(t)}{a(t) + b(t)}. \tag{58}$$

Indeed in [Hmidi et al. 2015] only the case $\theta(t) = 0$ is dealt with, but the general case follows easily from the behavior under rotations of a convolution with the Cauchy kernel.

To lighten the notation we do not stress the dependence on t and write $a = a(t),\ b = b(t),\ \theta = \theta(t)$, $q = q(t),\ z = z(t) = x(t) + iy(t) = x + iy$. The condition $z(t) \in \partial E_t$ is equivalent to $e^{-i\theta(t)} z(t) \in \partial E(a(t), b(t), 0)$, which is

$$\frac{(x\ \cos(\theta) + y\ \sin(\theta))^2}{a^2} + \frac{(-x\ \sin(\theta) + y\ \cos(\theta))^2}{b^2} = 1,$$

and can also be written more concisely as

$$\frac{\langle z, e^{i\theta}\rangle^2}{a^2} + \frac{\langle z, i\, e^{i\theta}\rangle^2}{b^2} = 1. \tag{59}$$

Here we have denoted by $\langle u, v\rangle$ the scalar product of the vectors u and v. Now proceed as follows. Take the derivative in (59) with respect to t and then replace $z'(t)$ by the expression of the field given by (58). We get an equation containing a, b, θ and z, which determines $z(t)$, the solution of the CDE. This equation is

$$0 = \frac{\langle z, e^{i\theta}\rangle}{a^2}\big(\langle z, e^{i\theta}\, i\, \theta' - q\, e^{3i\theta}\rangle + \langle \bar{z}, e^{i\theta}\rangle\big) - \frac{a'}{a^3}\langle z, e^{i\theta}\rangle^2 + \frac{\langle z, i\, e^{i\theta}\rangle}{b^2}\big(\langle z, -e^{i\theta}\, \theta' - i\, q\, e^{3i\theta}\rangle + \langle \bar{z}, i\, e^{i\theta}\rangle\big) - \frac{b'}{b^3}\langle z, i\, e^{i\theta}\rangle^2. \tag{60}$$

Evaluate at $z = z(t) = a(t)e^{i\theta(t)}$ (which is a vertex of the ellipse at time t). One gets the equation

$$a' = 2\frac{a\, b}{a + b}\ \cos(2\theta). \tag{61}$$

Evaluating at the other vertex of the ellipse at time t, that is, at $z = z(t) = b(t)\, i\, e^{i\theta(t)}$, yields

$$b' = -2\frac{a\, b}{a + b}\ \cos(2\theta). \tag{62}$$

Adding (61) and (62) we see that $a + b$ is constant, then equal to $a_0 + b_0$. Thus we have the first two equations in (56).

Before getting the third equation let us solve the case in which the initial ellipse has axes parallel to the coordinate axes ($\theta_0 = 0$). In this case set $\theta(t) = 0,\ t \in \mathbb{R}$. Replacing in (62) a by $a_0 + b_0 - b$ and solving we get (55) and then (54).

Now take the domain $E_t = E(a(t), b(t), 0)$, the vector field

$$v(z,t) = \Big(\frac{1}{\pi z} \star \chi_{E_t}\Big)(z) = \bar{z} - q(t)\, z, \quad z \in E_t,$$

and the flow

$$\frac{dz}{dt} = \bar{z} - q(t)\, z, \quad z(0) \in \partial E_0,$$

The preceding system is

$$\frac{dx(t)}{dt} = \frac{2b(t)}{a_0 + b_0} x(t), \quad \frac{dy(t)}{dt} = -\frac{2a(t)}{a_0 + b_0} y(t).$$

Then the flow map is linear on E_0 and given by a diagonal matrix. Hence the flow preserves the coordinate axes and maps ∂E_0 into an ellipse with axes parallel to the coordinate axes enclosing a domain $\widetilde{E}_t$. But (61) and (62) say exactly that the vertices of ∂E_t belong to $\partial \widetilde{E}_t$. Thus $\widetilde{E}_t = E_t$ and so χ_{E_t} is the unique weak solution of the Cauchy transport equation in the class of characteristic functions of $C^{1+\gamma}$ domains.

Let us now go back to the general case and obtain a third equation involving θ'. Impose that the intersection of the ellipse ∂E_t with the positive real axis belongs to the image of ∂E_0 under the flow. In other words replace $z(t)$ in (60) by

$$\bigg(\Big(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\Big)^{-1/2}, 0\bigg).$$

After a lengthy computation one gets

$$\theta' = -\frac{2}{a_0 + b_0}\, \frac{ab}{a-b}\, \sin(2\theta), \tag{63}$$

provided $a \neq b$.

We know claim that the system (56) has a unique solution defined for all times $t \in \mathbb{R}$ provided $a_0 \neq b_0$. The case $a_0 = b_0$ corresponds to an initial disc and so to the case $\theta_0 = 0$, which has been discussed before. Consider the open set

$$\Omega = \big\{(a, b, \theta) \in \mathbb{R}^3 : a > 0,\ b > 0,\ a \neq b \text{ and } 0 < \theta < \tfrac{\pi}{2}\big\}.$$

Clearly a unique solution of the system exists locally in time for any initial condition $(a_0, b_0, \theta_0) \in \Omega$, because the function giving the system is C^∞ in Ω. We claim that this solution exists for all times. Assume that the maximal interval of existence is $(-T, T)$ for some $0 < T < \infty$. By the first two equations of the system (56) $|a'|$ and $|b'|$ are bounded above by $2(a_0 + b_0)$ and hence the limits $\lim_{t\to T} a(t) = a(T)$ and $\lim_{t\to T} b(t) = b(T)$ exist. We also have

$$\left|\frac{a'}{a}\right| \le 2 \quad \text{and} \quad \left|\frac{b'}{b}\right| \le 2$$

and so

$$0 < a_0 e^{-2T} \le a(T) \le a_0 e^{2T} \quad \text{and} \quad 0 < b_0 e^{-2T} \le b(T) \le b_0 e^{2T}.$$

Note that $\theta'(t)$ cannot vanish. Otherwise, by (63), $\theta(t) = 0$ for some t, and in this case we have already checked that the system can be solved for all times. Hence θ' has constant sign. When $a_0 > b_0$, the function θ decreases, and if $a_0 < b_0$, the function θ increases. In any case we have that there exists

$$\theta(T) = \lim_{t \to T} \theta(t).$$

We cannot have $\theta(T) = 0$ or $\theta(T) = \frac{\pi}{2}$ because we have solved the equation in these cases for all times. For the same reason we cannot have $a(T) = b(T)$. Therefore $(a(T), b(T), \theta(T)) \in \Omega$ and we can solve the system past T, which is a contradiction.

We proceed now to prove that the domain $E_t = E(a(t), b(t), \theta(t))$ enclosed by the ellipse provided by the solution of (56) yields the weak solution χ_{E_t} of the transport equation (53) with initial condition $D_0 = E_0$. We consider the field (58) and the trajectory (57) of a particle initially at the boundary point $z(0) \in \partial E_0$. Since the velocity field is linear in E_t the flow is a linear function of $z(0) \in E_0$. Thus the initial ellipse ∂E_0 is mapped into an ellipse $\partial \widetilde{E}_t$ enclosing $\widetilde{E}_t$, the image of E_0 under the flow map. To show that χ_{E_t} is a weak solution of the Cauchy transport equation we only need to ascertain that $E_t = \widetilde{E}_t$. But the three equations of (56) simply mean that the vertices of E_t and the intersection of E_t with the horizontal axis are in the image of ∂E_0 under the flow map. It is now a simple matter to realize that there is only one ellipse centered at the origin containing those three points.

A surprising result arises when examining the asymptotic behavior as $t \to \infty$ of the weak solution of the Cauchy transport equation (53) when the initial condition is $E(a_0, b_0, \theta_0)$, with $a_0 \neq b_0$ and $\theta_0 > 0$. We know that the solution of the system (56) never leaves the open set Ω. In particular $a(t) - b(t)$ has a definite sign determined by the initial condition. Assume for definiteness that $a_0 - b_0 > 0$, so that $a(t) - b(t) > 0,\ t \in \mathbb{R}$, and hence $\theta(t)$ is a decreasing function. Then the limit $\theta_\infty = \lim_{t\to\infty} \theta(t)$ exists. The system (56) readily yields that the function $(a - b)\sin(2\theta)$ has vanishing derivative, so that

$$(a(t) - b(t))\sin(2\theta(t)) = (a_0 - b_0)\sin(2\theta_0), \quad t \in \mathbb{R}. \tag{64}$$

Thus $(a_0 + b_0)\sin(2\theta(t)) \geq (a(t) - b(t))\sin(2\theta(t)) = (a_0 - b_0)\sin(2\theta_0)$ and taking limits

$$\sin(2\theta_\infty) \geq \frac{a_0 - b_0}{a_0 + b_0}\sin(2\theta_0) > 0,$$

which means that the limit angle θ_∞ is positive. In other words, the axes of the ellipses at time t do not approach the coordinate axes.

Assume that $0 < \theta_0 \leq \frac{\pi}{4}$. Since $\theta(t)$ decreases, $0 < 2\theta(t) < \frac{\pi}{2},\ t > 0$, which implies that $a(t)$ increases and $b(t)$ decreases. By (62)

$$b(t) = b_0 \exp \int_0^t -\frac{2}{a_0 + b_0} a(s)\cos(2\theta(s))\, ds \leq b_0 \exp\left(-\frac{2a_0}{a_0 + b_0}\cos(2\theta_0)\, t\right),$$

and so $b_\infty = 0$, provided $\theta_0 < \frac{\pi}{4}$. If $\theta_0 = \frac{\pi}{4}$ we break the integral above into two pieces, the first between 0 and 1 and the second between 1 and t. We get, for some constant C independent of t,

$$b(t) \leq C\ \exp\left(-\frac{2a_0}{a_0 + b_0}\cos(2\theta(1))\,(t - 1)\right), \quad t > 1,$$

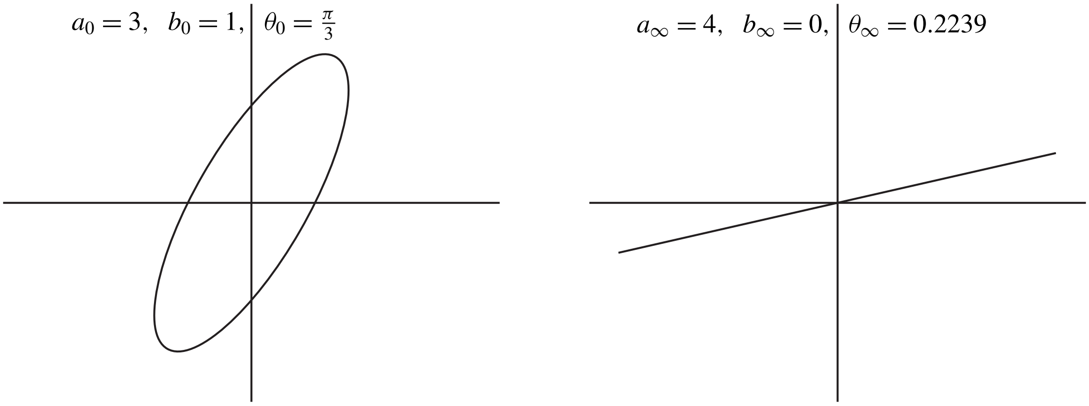

Figure 1. The initial ellipse and the final segment.

which again yields $b_\infty = 0$. By (64)

$$\sin(2\theta_\infty) = \frac{a_0 - b_0}{a_0 + b_0} \sin(2\theta_0), \tag{65}$$

which determines the limit angle in terms of the initial data.

Let us turn now to the case $\frac{\pi}{4} < \theta_0 < \frac{\pi}{2}$. In Figure 1 one can see the initial ellipse and the final segment. In view of the first two equations of the system (56), at least for a short time $a(t)$ decreases and $b(t)$ increases. If one has $\frac{\pi}{4} \le \theta_\infty$, then $\cos(2\theta(t)) < 0$ for $t > 0$ and $a(t)$ decreases and $b(t)$ increases for all times. Integrating the third equation in (56) we obtain

$$\tan(\theta(t)) = \tan(\theta_0) \exp\left(\frac{-4}{a_0 + b_0} \int_0^t \frac{a(s)b(s)}{a(s) - b(s)}\, ds \right) \le \tan(\theta_0) \exp\left(-\frac{-4b_0^2}{a_0^2 - b_0^2}\, t \right).$$

Letting $t \to \infty$ we get $\tan(\theta_\infty) = 0$, which is impossible. Hence $\theta_\infty < \frac{\pi}{4}$. Then for some t_0 we have $\theta(t_0) < \frac{\pi}{4}$, which brings us into the previous case, in particular to the expression (65) for the limiting angle θ_∞.

Arguing similarly with $t \to -\infty$ we get (65) with $\sin(2\theta_\infty)$ replaced by $\sin(2\theta_{-\infty})$, where $\theta_{-\infty} = \lim_{t\to-\infty} \theta(t)$. Thus $\theta_{-\infty} = \frac{\pi}{2} - \theta_\infty$.

The case $a_0 < b_0$ is reduced to $a_0 > b_0$ by taking conjugates (symmetry with respect to the horizontal axis). Indeed, (53) is invariant by taking conjugates, as a simple computation shows. If one has $a_0 < b_0$ and an angle θ_0, the symmetric ellipse has semiaxes $A_0 = b_0$, $B_0 = a_0$ and angle $\theta_0' = \frac{\pi}{2} - \theta_0$.

Appendix: Existence of principal values

The first fact we prove in this section is the following.

Lemma. *Let D be a bounded domain with boundary of class $C^{1+\gamma}$, $0 < \gamma < 1$. Let $L : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be an even kernel, continuous on $\mathbb{R}^n \setminus \{0\}$, homogeneous of degree $-n$, which satisfies cancellation property*

$\int_{|\xi|=1} L(\xi)\, d\sigma(\xi) = 0$. *Then for each* $x \in \partial D$ *the principal value*

$$(\text{p.v. } L \star \chi_D)(x) = \lim_{\epsilon \to 0} \int_{\{y \in D : |y-x| > \epsilon\}} L(x-y)\, dy$$

exists.

Proof. Without loss of generality assume that $x = 0$, that the tangent hyperplane to ∂D at 0 is $\{y \in \mathbb{R}^n : y_n = 0\}$ and that $r_0 > 0$ is so small that there exists a function

$$\varphi \in C^{1+\gamma}(B'(0, 2r_0)), \quad B'(0, 2r_0) = \{y \in \mathbb{R}^n : |y'| < 2r_0\}, \quad y' = (y_1, \dots, y_{n-1}),$$

such that $D \cap B(0, r_0) = \{y \in B(0, r_0) : y_n < \varphi(y')\}$.

For $0 < r$ set

$$S_r = \{y \in \mathbb{R}^n : |y| = r\}, \quad S_r^+ = \{y \in S_r : y_n > 0\} \quad \text{and} \quad S_r^- = \{y \in S_r : y_n < 0\}.$$

Since L is even,

$$0 = \int_{S_r} L(y)\, d\sigma(y) = \int_{S_r^+} L(y)\, d\sigma(y) + \int_{S_r^-} L(y)\, d\sigma(y) = 2 \int_{S_r^-} L(y)\, d\sigma(y).$$

Set $H_- = \{y \in \mathbb{R}^n : y_n < 0\}$. For $0 < \delta < \epsilon < r_0$ we then have

$$-\int_{\{y \in D : |y| > \epsilon\}} L(y)\, dy + \int_{\{y \in D : |y| > \delta\}} L(y)\, dy = \int_{\{y \in \mathbb{R}^n : \delta < |y| < \epsilon\} \cap (D \setminus H_-)} L(y)\, dy - \int_{\{y \in \mathbb{R}^n : \delta < |y| < \epsilon\} \cap (H_- \setminus D)} L(y)\, dy.$$

The tangential domains $(D \setminus H_-) \cap B(0, \epsilon)$ and $(H_- \setminus D) \cap B(0, \epsilon)$ are very small. Indeed,

$$\left| \int_{\{y \in \mathbb{R}^n : \delta < |y| < \epsilon\} \cap (D \setminus H_-)} L(y)\, dy \right| \le \int_\delta^\epsilon \frac{1}{\rho^n} \rho^{n-1} \sigma\{\theta \in S^{n-1} : \rho\theta \in D \setminus H_-\}\, d\rho \le C \int_\delta^\epsilon \rho^{-1+\gamma}\, d\rho \le \frac{C}{\gamma} \epsilon^\gamma.$$

One obtains in the same way

$$\left| \int_{\{y \in \mathbb{R}^n : \delta < |y| < \epsilon\} \cap (H_- \setminus D)} L(y)\, dy \right| \le \frac{C}{\gamma} \epsilon^\gamma,$$

and so the proof is complete. □

The second result is the following.

Lemma. *Let* D *be a bounded domain with boundary of class* $C^{1+\gamma}$, $0 < \gamma < 1$. *Let* $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ *be an odd kernel of class* $C^1(\mathbb{R}^n \setminus \{0\})$, *homogeneous of degree* $-(n-1)$. *Let* φ *be a function defined on* ∂D *satisfying a Hölder condition of some positive order on* ∂D. *Then for each* $x \in \partial D$ *and each* $1 \le j \le n$ *the principal value*

$$(\text{p.v. } K \star \varphi n_j\, d\sigma)(x) = \lim_{\epsilon \to 0} \int_{\{y \in \partial D : |y-x| > \epsilon\}} K(x-y)\, \varphi(y)\, n_j(y)\, d\sigma(y)$$

exists.

Proof. It is easy to get rid of φ. Indeed

$$\begin{aligned}\int_{\{y\in\partial D:\epsilon<|y-x|\}} & K(x-y)\varphi(y)n_j(y)\,d\sigma(y)\\ &= \int_{\{y\in\partial D:\epsilon<|y-x|\}} K(x-y)(\varphi(y)-\varphi(x))n_j(y)\,d\sigma(y)+\varphi(x)\int_{\{y\in\partial D:\epsilon<|y-x|\}} K(x-y)n_j(y)\,d\sigma(y)\end{aligned}$$

and the first integral in the right-hand side tends as $\epsilon\to 0$ to the absolutely convergent integral

$$\int_{\partial D} K(x-y)(\varphi(y)-\varphi(x))n_j(y)\,d\sigma(y).$$

Hence we can assume that φ is identically 1.

We can also assume, as in the proof of the previous lemma, that $x=0$, the tangent hyperplane to ∂D at 0 is $\{y\in\mathbb{R}^n : y_n=0\}$ and the domain D inside $B(0,\epsilon)$ is exactly $\{y\in B(0,\epsilon) : y_n<\varphi(y')\}$. By the divergence theorem

$$\begin{aligned}\int_{\{y\in\partial D:\epsilon<|y|\}} K(-y)n_j(y)\,d\sigma(y) &= -\int_{\{y\in D:\epsilon<|y|\}} \partial_j K(y)\,dy+\int_{\{y\in D:|y|=\epsilon\}} K(y)n_j(y)\,d\sigma(y)\\ &= -I+II,\end{aligned}$$

where in the last identity one is defining I and II.

To apply the previous lemma to I we need to check that $\partial_j K(y)$, which is continuous off the origin, even and homogeneous of degree $-n$, and has vanishing integral on the unit sphere. By the divergence theorem

$$\int_{1<|y|<2} \partial_j K(y)\,dy=\int_{|y|=2} K(y)\,n_j(y)\,d\sigma(y)-\int_{|y|=1} K(y)\,n_j(y)\,d\sigma(y),$$

which is 0, since the two integrals over the spheres are the same by homogeneity. Hence, changing to polar coordinates,

$$0=\int_{1<|y|<2}\partial_j K(y)\,dy=\log 2\int_{|\theta|=1} K(\theta)\,d\sigma(\theta),$$

which takes care of I.

For the term II, set, as before, $H_-=\{y\in\mathbb{R}^n : y_n<0\}$. We then have

$$\begin{aligned}\int_{\{y\in D:|y|=\epsilon\}} K(y)n_j(y)\,d\sigma(y) &= \int_{\{y\in D\setminus H_-:|y|=\epsilon\}} K(y)\,n_j(y)\,d\sigma(y)\\ &\quad+\int_{\{y\in H_-:|y|=\epsilon\}} K(y)\,n_j(y)\,d\sigma(y)-\int_{\{y\in H_-\setminus D:|y|=\epsilon\}} K(y)\,n_j(y)\,d\sigma(y).\end{aligned}$$

The first and third terms tend to 0 with ϵ, because the domains of integration are tangential. Indeed,

$$\sigma(\partial B(0,\epsilon)\cap(D\setminus H_-))+\sigma(\partial B(0,\epsilon)\cap(H_-\setminus D))\le C\,\epsilon^{n-1+\gamma}$$

and so the absolute value of the first and third terms can be estimated by $C\,\epsilon^{\gamma}$.

It only remains to note that the second term is independent of ϵ, by homogeneity. □

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DIRECTIONAL SQUARE FUNCTIONS

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Quantitative formulations of Fefferman's counterexample for the ball multiplier are naturally linked to square function estimates for conical and directional multipliers. We develop a novel framework for these square function estimates, based on a directional embedding theorem for Carleson sequences and multiparameter time-frequency analysis techniques. As applications we prove sharp or quantified bounds for Rubio-de Francia-type square functions of conical multipliers and of multipliers adapted to rectangles pointing along N directions. A suitable combination of these estimates yields a new and currently best-known logarithmic bound for the Fourier restriction to an N-gon, improving on previous results of A. Córdoba. Our directional Carleson embedding extends to the weighted setting, yielding previously unknown weighted estimates for directional maximal functions and singular integrals.

1. Motivation and main results

The celebrated theorem of Charles Fefferman [1971] shows that the ball multiplier is an unbounded operator on $L^p(\mathbb{R}^n)$ for all $p \neq 2$ whenever $n \geq 2$. A well-known argument, originally due to Yves Meyer [de Guzmán 1981], exhibits the intimate relationship of the ball multiplier with vector-valued estimates for directional singular integrals along all possible directions. Fefferman [1971] proved the impossibility of such estimates by testing these vector-valued inequalities on a Kakeya set.

Besicovitch or Kakeya sets are compact sets in the Euclidean space that contain a line segment of unit length in every direction. Sets of this type with zero Lebesgue measure do exist. However, in two dimensions, Kakeya sets are necessarily of full Hausdorff dimension. The question of the Hausdorff

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dimension of Kakeya sets can be then formulated as a question of quantitative boundedness of the Kakeya maximal function, which is a maximal directional average along rectangles of fixed eccentricity and pointing along arbitrary directions.

The importance of the ball multiplier for the summation of higher dimensional Fourier series, as well as its intimate connection to Kakeya sets, have motivated a host of problems in harmonic analysis which have been driving relevant research since the 1970s. Finitary or smooth models of the ball multiplier such as the polygon multiplier and the Bochner–Riesz means quantify the failure of boundedness of the ball multiplier and formalize the close relation of these operators with directional maximal and singular averages.

This paper is dedicated to the study of a variety of operators in the plane that are all connected in one way or another with the ball multiplier. Our point of view is through the analysis of directional operators mapping into $L^p(\mathbb{R}^2;\ell^q)$-spaces where the inner ℓ^q-norm is taken with respect to the set of directions. Different values of q are relevant in our analysis but the cases $q=2$ and $q=\infty$ are of particular interest. On one hand, the case $q=\infty$ arises when considering maximal directional averages and the corresponding differentiation theory along directions; see [Bateman 2013; Christ et al. 1986; Di Plinio and Parissis 2021; Katz 1999] for classical and recent work on the subject. On the other hand, the case $q=2$ is especially relevant for Meyer's argument that bounds the norm of a vector-valued directional Hilbert transform by the norm of the ball multiplier. It also arises when dealing with square functions associated to conical or directional Fourier multipliers of the type

$$f \mapsto \{C_j f : j = 1, \dots, N\},$$

where each C_j is adapted to a different coordinate pair and the C_j have disjoint or well-separated Fourier support. These estimates are directional analogues of the celebrated square function estimate for Fourier restriction to families of disjoint cubes, due to Rubio de Francia [1985], and they appear naturally when seeking quantitative estimates on the N-gon Fourier multiplier.

While such square function estimates have been considered previously in the literature, and usually approached directly via weighted norm inequalities, our treatment is novel and leads to improved and in certain cases sharp estimates in terms of the cardinality of the set of directions. It rests on a new directional Carleson measure condition and corresponding embedding theorem, which is subsequently applied to intrinsic directional square functions of time-frequency nature. The link between the abstract Carleson embedding theorem and the applications is provided by directional, one- and two-parameter time-frequency analysis models. The latter allow us to reduce estimates for directional operators to those of the corresponding intrinsic square functions involving directional wave packet coefficients. We note that in the fixed coordinate system case, related square functions have appeared in [Lacey 2007], while a single-scale directional square function similar to those of Section 4 is present in [Di Plinio et al. 2018] by Guo, Thiele, Zorin-Kranich and the second author.

Having clarified the context of our investigation, we turn to the detailed description of our main results and techniques.

A new approach to directional square functions. While we address several types of square functions associated to directional multipliers, our analysis of each relies on a common first step. This is an

L^4-square function inequality for abstract Carleson measures associated with one- and two-parameter collections of rectangles in $\mathbb{R}^2$, pointing along a finite set of N directions; this setup is presented in Section 2 and the central result is Theorem C. Section 2 builds upon the proof technique first introduced in [Katz 1999] and revisited in [Bateman 2013] in the study of sharp weak L^2-bounds for maximal directional operators. Our main novel contributions are the formulation of an abstract directional Carleson condition which is flexible enough to be applied in the context of time-frequency square functions, and the realization that square functions in L^4 can be treated in a TT^*-like fashion. The advancements over [Bateman 2013; Katz 1999] also include the possibility of handling two-parameter collections of rectangles.

In Section 4, we verify that the Carleson condition, which is a necessary assumption in the directional embedding of Theorem C, is satisfied by the intrinsic directional wave packet coefficients associated with certain time-frequency tile configurations, and Theorem C may be thus applied to obtain sharp estimates for discrete time-frequency models of directional Rubio de Francia square functions (for instance). Establishing the Carleson condition requires a precise control of spatial tails of the wave packets; this control is obtained by a careful use of Journé's product theory lemma.

The estimates obtained for the time-frequency model square functions are then applied to three main families of operators described below. All of them are defined in terms of an underlying set of N directions. As in Fefferman's counterexample for the ball multiplier, the Kakeya set is the main obstruction for obtaining uniform estimates. Depending on the type of operator, the usable estimates will be restricted in the range $2 < p < 4$ for square function estimates or in the range $\frac{3}{4} < p < 4$ for the self-adjoint case of the polygon multiplier. The fact that the estimates should be logarithmic in N in the L^p-ranges above is directed by the Besicovitch construction of the Kakeya set. It is easy to see that for p outside this range the only available estimates are essentially trivial polynomial estimates. Further obstructions deter any estimates for Rubio-de-Francia-type square function in the range $p < 2$ already in the one-directional case.

Sharp Rubio de Francia square function estimates in the directional setting. Section 5 concerns quantitative estimates of Rubio de Francia type for the square function associated with N finitely overlapping cone multipliers, of both rough and smooth type. Beginning with the seminal article of Nagel, Stein and Wainger [Nagel et al. 1978], square functions of this type are crucial in the theory of maximal operators, in particular along lacunary directions; see for instance [Parcet and Rogers 2015; Sjögren and Sjölin 1981]. In the case of N uniformly spaced cones, logarithmic estimates with unspecified dependence were proved by A. Córdoba [1982] using weighted theory.

In order to make the discussion above more precise, and to give a flavor of the results of this paper, we introduce some basic notation. Let $\tau \subset [0, 2\pi)$ be an interval and consider the corresponding smooth restriction to the frequency cone subtended by τ, namely

$$C_\tau^\circ f(x) := \int_0^{2\pi}\int_0^\infty \hat{f}(\varrho e^{i\vartheta})\beta_\tau(\vartheta) e^{ix\cdot\varrho e^{i\vartheta}} \varrho \, d\varrho \, d\vartheta, \quad x \in \mathbb{R}^2,$$

where β_τ is a smooth indicator on τ; namely it is supported in τ and is identically 1 on the middle half of τ.

One of the main results of this paper is a quantitative estimate for a square function associated with the smooth conical multipliers of a finite collection of intervals with bounded overlap. In the statement of the theorem below $\ell^2_{\boldsymbol{\tau}}$ denotes the ℓ^2-norm on the finite set of directions $\boldsymbol{\tau}$.

Theorem A. *Let $\boldsymbol{\tau} = \{\tau\}$ be a finite collection of intervals in $[0, 2\pi)$ with bounded overlap, namely*

$$\Big\|\sum_{\tau\in\boldsymbol{\tau}} \mathbf{1}_\tau\Big\|_\infty \lesssim 1.$$

We then have the square function estimate

$$\|\{C_\tau^\circ f\}\|_{L^p(\mathbb{R}^2;\ell^2_{\boldsymbol{\tau}})} \lesssim_p (\log \#\boldsymbol{\tau})^{\frac{1}{2}-\frac{1}{p}} \|f\|_p$$

for $2 \le p < 4$, as well as the restricted-type analogue valid for all measurable sets E

$$\|\{C_\tau^\circ(f\mathbf{1}_E)\}\|_{L^4(\mathbb{R}^2;\ell^2_{\boldsymbol{\tau}})} \lesssim (\log \#\boldsymbol{\tau})^{\frac{1}{4}} |E|^{\frac{1}{4}} \|f\|_\infty.$$

The dependence on $\#\boldsymbol{\tau}$ in the estimates above is best possible.

The sharp estimate of Theorem A above can be suitably bootstrapped in order to provide an estimate for rough conical frequency projections; the precise statement can be found in Theorem J of Section 5. The sharpness of the estimates in Theorem A above is discussed in Section 8.6.

A similar square function estimate associated with disjoint rectangular directional frequency projections is presented in Section 6. This is a square function that is very close in spirit to the one originally considered in [Rubio de Francia 1985], and especially to the two-parameter version from [Journé 1985] and revisited in [Lacey 2007]. The novel element is the directional aspect which comes from the fact that the frequency rectangles are allowed to point along a set of N different directions. Our method of proof can deal equally well with one-parameter rectangular projections or collections of arbitrary eccentricities. As before we prove a sharp — in terms of the number of directions — estimate for the smooth square function associated with rectangular frequency projections along N directions; this is the content of Theorem K. The main term in the upper bound of Theorem K matches the logarithmic lower bound associated with the Kakeya set.

The polygon multiplier. The square function estimates discussed above may be combined with suitable vector-valued estimates in the directional setting in order to obtain a quantitative estimate for the operator norm of the N-gon multiplier, namely the Fourier restriction to a regular N-gon $\mathcal{P}_N$,

$$T_{\mathcal{P}_N} f(x) := \int_{\mathcal{P}_N} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi, \quad x \in \mathbb{R}^2. \tag{1.1}$$

In Section 7 we give the details and proof of the following quantitative estimate for the polygon multiplier.

Theorem B. *Let $\mathcal{P}_N$ be a regular N-gon in $\mathbb{R}^2$ and $T_{\mathcal{P}_N}$ be the corresponding Fourier restriction operator defined in* (1.1). *We have the estimate*

$$\|T_{\mathcal{P}_N} : L^p(\mathbb{R}^2)\| \lesssim (\log N)^{4\left|\frac{1}{2}-\frac{1}{p}\right|}, \quad \tfrac{4}{3} < p < 4.$$

We limit ourselves to treating the regular N-gon case; however, it will be clear from the proof that this restriction may be significantly weakened by requiring instead a well-distribution-type assumption on the arcs defining the polygon, similar to the one that is implicit in Theorem A.

Precise L^p-bounds for the N-gon multiplier as a function of N quantify Fefferman's counterexample and so the failure of boundedness of the ball multiplier when $p \ne 2$. A logarithmic-type estimate for $T_{\mathcal{P}_N}$

was first obtained in [Córdoba 1977]. While the exact dependence in that work is not explicitly tracked, the upper bound on the operator norm obtained there must be necessarily larger than $O(\log N)^{5/4}$ for p close to the endpoints of the relevant interval; see Remark 7.12 and Section 8.4 for details. While the dependence obtained in Theorem B is a significant improvement over previous results, it does not match the currently best-known lower bound, which is the same as that for the Meyer lemma constant in Lemma 7.21 and Section 8.1.

Remark. Let $\delta > 0$ and T_j be a smooth frequency restriction to one of the $O(\delta^{-1})$ tangential $\delta \times \delta^2$ boxes covering the δ^2 neighborhood of $\mathbb{S}^1$. Unlike the sharp forward square function estimate we prove in this article, the *reverse square function* estimate

$$\|f\|_p \leq C_{p,\delta} \|\{T_j f : 1 \leq j \leq O(1/\delta)\}\|_{L^p(\mathbb{R}^2;\ell^2_j)} \tag{1.2}$$

holds with $C_{4,\delta} = O(1)$ at the endpoint $p = 4$. For the proof of this L^4-decoupling estimate, see [Córdoba 1977; Fefferman 1973]. An extension to the range $2 < p < 4$ is at the moment only possible via vector-valued methods, which introduce the loss $C_{p,\delta} = O(|\log \delta|^{1/2-1/p})$. In fact (1.2) with the loss $C_{p,\delta}$ claimed above follows easily from Lemma 7.18; the details are contained in Remark 7.22.

Reverse square function inequalities of the type (1.2) have been popularized by Wolff in his proof of local smoothing estimates in the large p regime; see also [Garrigós and Seeger 2010; Łaba and Pramanik 2006; Łaba and Wolff 2002; Pramanik and Seeger 2007]. We refer to [Carbery 2015] for a proof that the $p = 2n/(n-1)$ case of the $\mathbb{S}^{n-1}$ reverse square function estimate implies the corresponding $L^n(\mathbb{R}^n)$ Kakeya maximal inequality, as well as the Bochner–Riesz conjecture. In [Carbery 2015], the author also asks whether a δ-free estimate holds in the range $2 < p < 2n/(n-1)$. At the moment this is not known in any dimension.

On a different but related note, weakening (1.2) by replacing the right-hand side with the larger square function of $\|f_j\|_p$ yields a sample (weak) *decoupling* inequality: a full range of sharp decoupling inequalities for hypersurfaces with curvature have been established starting from the recent, seminal paper [Bourgain and Demeter 2015]. In the case of $\mathbb{S}^1$, the weak decoupling inequality holds in the wider range $2 \leq p \leq 6$, with $C_\varepsilon \delta^{-\varepsilon}$-type bounds outside of $[2, 4]$; our methods do not seem to provide insights on the quantitative character of weak decoupling in this wider range.

Weighted estimates for the maximal directional function. The simplest example of an application of the directional Carleson embedding theorem is the adjoint of the directional maximal function; this was already noticed by Bateman [2013], re-elaborating on the approach of [Katz 1999]. By duality, the L^2-directional Carleson embedding theorem of Section 2 yields the sharp bound for the weak-$(2, 2)$-norm of the maximal Hardy–Littlewood maximal function M_N along N *arbitrary directions*

$$\|M_N : L^2(\mathbb{R}^2) \to L^{2,\infty}(\mathbb{R}^2)\| \sim \sqrt{\log N};$$

this result first appeared in the quoted article [Katz 1999].

Theorem C may be extended to the directional weighted setting. We describe this extension in Section 3, see Theorem D, and derive several novel weighted estimates for directional maximal and singular integrals as an application.

More specifically, our weighted Carleson embedding Theorem D yields a Fefferman–Stein-type inequality for the operator M_N with sharp dependence on the number of directions; this result is the content of Theorem E. Specializing to A_1-weights in the directional setting yields the first sharp weighted result for the maximal function along arbitrary directions. Furthermore, Theorem F contains an $L^{2,\infty}(w)$-estimate for the maximal directional singular integrals along N directions, for suitable directional weights w, with a quantified logarithmic dependence in N. This is a weighted counterpart of the results of [Demeter 2010; Demeter and Di Plinio 2014].

2. An L^2-inequality for directional Carleson sequences

In this section we prove an abstract L^2-inequality for certain Carleson sequences adapted to sets of directions: the main result is Theorem C below. The Carleson sequences we will consider are indexed by parallelograms with long side pointing in a given set of directions in $\mathbb{R}^2$, and possessing certain natural properties. The definitions below are motivated by the applications we have in mind, all of them lying in the realm of directional singular and averaging operators.

2.1. ***Parallelograms and sheared grids.*** Fix a coordinate system and the associated horizontal and vertical projections of $A \subset \mathbb{R}^2$:

$$\pi_1(A) := \{x \in \mathbb{R} : \{x\} \times \mathbb{R} \cap A \neq \varnothing\}, \quad \pi_2(A) := \{y \in \mathbb{R} : \mathbb{R} \times \{y\} \cap A \neq \varnothing\}.$$

Fix a finite set of slopes $S \subset [-1, 1]$. Throughout, we indicate by $N = \#S$ the number of elements of S. In general we will deal with sets of directions

$$V := \{(1, s) : s \in S\}, \quad V^{\perp} := \{(-s, 1) : s \in S\}.$$

We will conflate the descriptions of directions in terms of slopes in S and in terms of vectors in V with no particular mention.

For each $s \in S$ let

$$\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

be the corresponding shearing matrix. A *parallelogram along s* is the image $P = A_s(I \times J)$ of the rectangular box $I \times J$ in the fixed coordinate system with $|I| \geq |J|$. We denote the collection of parallelograms along s by $\mathcal{P}_s^2$ and

$$\mathcal{P}_S^2 := \bigcup_{s \in S} \mathcal{P}_s^2.$$

In order to describe the setup for our general result we introduce a collection of directional dyadic grids of parallelograms. In order to define these grids we consider the two-parameter product dyadic grid

$$\mathcal{D}_0^2 := \{R = I \times J : I, J \in \mathcal{D}(\mathbb{R}),\ |I| \geq |J|\}$$

obtained by taking the cartesian product of the standard dyadic grid $\mathcal{D}(\mathbb{R})$ with itself; we note that we only consider the rectangles in $\mathcal{D} \times \mathcal{D}$ whose horizontal side is longer than their vertical one. Define the

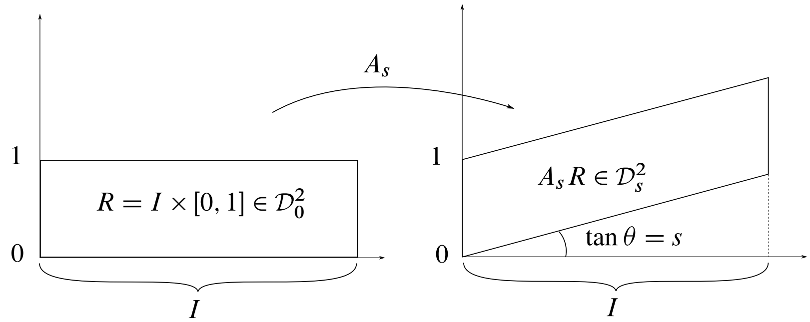

Figure 1. The axis-parallel rectangle $R \in \mathcal{D}^2_0$ is mapped to the slanted parallelogram $A_s R \in \mathcal{D}^2_s$.

sheared grids

$$\mathcal{D}^2_s := \{A_s R : R \in \mathcal{D}^2_0\}, \quad s \in S, \qquad \mathcal{D}^2_S := \bigcup_{s\in S} \mathcal{D}^2_s.$$

We will also use the notation

$$\mathcal{D}^2_{s,k_1,k_2} := \{A_s R : R = I \times J \in \mathcal{D}^2_0,\, |I| = 2^{-k_1},\ |J| = 2^{-k_2}\}, \quad s \in S,\ k_1, k_2 \in \mathbb{Z},\ k_1 \le k_2.$$

Note that $\mathcal{D}^2_s$ is a special subcollection of $\mathcal{P}^2_s$. In particular, $R \in \mathcal{D}^2_s$ is a parallelogram oriented along $v = (1, s)$ with vertical sides parallel to the y-axis and such that $\pi_1(R)$ is a standard dyadic interval. Furthermore our assumptions on S and the definition of $\mathcal{D}^2_0$ imply that the parallelograms in $\mathcal{D}^2_S$ have long side with slope $|s| \le 1$ and a vertical short side. See Figure 1. With a slight abuse of language we will continue referring to the rectangles in $\mathcal{D}^2_S$ as *dyadic*.

Several results in this paper will involve collections of parallelograms $\mathcal{R} \subset \mathcal{D}^2_S$. Writing $\mathcal{R}_s := \mathcal{R} \cap \mathcal{D}^2_s$ we have the natural decomposition of $\mathcal{R}$ into $\#S = N$ subcollections

$$\mathcal{R} = \bigcup_{s\in S} \mathcal{R}_s.$$

In general for any collection $\mathcal{R}$ of parallelograms we will use the notation

$$\mathsf{sh}(\mathcal{R}) := \bigcup_{R\in\mathcal{R}} R$$

for the *shadow* of the collection. Finally, for any collection of parallelograms $\mathcal{R}$ we define the corresponding maximal operator

$$\mathrm{M}_{\mathcal{R}} f(x) := \sup_{R\in\mathcal{R}} \langle |f| \rangle_R \mathbf{1}_R(x), \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^2),\ x \in \mathbb{R}^2. \tag{2.2}$$

We will also use the following notation for directional maximal functions:

$$\mathrm{M}_v f(x) := \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(x+tv)|\,\mathrm{d}t, \quad \mathrm{M}_j f(x) := \mathrm{M}_{e_j} f(x), \quad j \in \{1,2\},\ x \in \mathbb{R}^2. \tag{2.3}$$

If $V \subset \mathbb{R}^2$ is a compact set of directions with $0 \notin V$, we write

$$\mathrm{M}_V f := \sup_{v\in V} \mathrm{M}_v f. \tag{2.4}$$

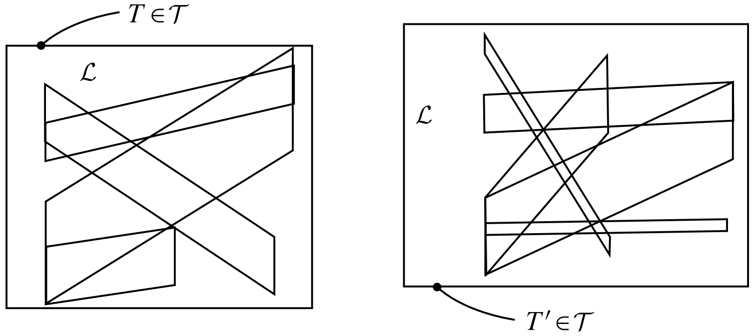

Figure 2. A collection $\mathcal{L}$ subordinate to a collection $\mathcal{T} \subset \mathcal{P}^2_0$.

In the definitions above and throughout the paper we use the notation

$$\langle g \rangle_E = \fint_E g := \frac{1}{|E|} \int_E g(x)\,\mathrm{d}x$$

whenever g is a locally integrable function in $\mathbb{R}^2$ and $E \subset \mathbb{R}^2$ has finite measure.

2.5. ***An embedding theorem for directional Carleson sequences.*** In this section we will be dealing with Carleson-type sequences $a = \{a_R\}_{R \in \mathcal{D}^2_S}$, indexed by dyadic parallelograms. In order to define them precisely we need a preliminary notion.

Definition 2.6. Let $\mathcal{L} \subset \mathcal{P}^2_S$ be a collection of parallelograms and let $s \in S$. We will say that $\mathcal{L}$ is subordinate to a collection $\mathcal{T} \subset \mathcal{P}^2_s$ if for each $L \in \mathcal{L}$ there exists $T \in \mathcal{T}$ such that $L \subset T$; see Figure 2.

It is important to stress that collections $\mathcal{L}$ are subordinate to rectangles $\mathcal{T} \subset \mathcal{P}^2_s$ *having a fixed slope* s. The Carleson sequences $a = \{a_R\}_{R \in \mathcal{R}}$ we will be considering will fall under the scope of the following definition.

Definition 2.7. Let $a = \{a_R\}_{R \in \mathcal{D}^2_S}$ be a sequence of nonnegative numbers. Then a will be called an L^∞*-normalized* Carleson sequence if for every $\mathcal{L} \subset \mathcal{D}^2_S$ which is subordinate to some collection $\mathcal{T} \subset \mathcal{P}^2_\tau$ for some fixed $\tau \in S$, we have

$$\sum_{L \in \mathcal{L}} a_L \leq |\mathsf{sh}(\mathcal{T})|$$

and the quantity

$$\mathsf{mass}_a := \sum_{R \in \mathcal{D}^2_S} a_R$$

is finite. Given a Carleson sequence $a = \{a_R : R \in \mathcal{D}^2_S\}$ and a collection $\mathcal{R} \subset \mathcal{D}^2_S$ we define the corresponding *balayage*

$$T_{\mathcal{R}}(a)(x) := \sum_{R \in \mathcal{R}} a_R \frac{\mathbf{1}_R(x)}{|R|}, \quad x \in \mathbb{R}^2. \tag{2.8}$$

We write $T(a)$ for $T_{\mathcal{R}}(a)$ when $\mathcal{R} = \mathcal{D}^2_S$. For $1 \leq p \leq 2$ we then define the balayage norms

$$\mathsf{mass}_{a,p}(\mathcal{R}) := \|T_{\mathcal{R}}(a)\|_{L^p}.$$

Note that $\mathsf{mass}_{a,1}(\mathcal{R}) = \sum_{R \in \mathcal{R}} a_R \leq \mathsf{mass}_a$.

Remark 2.9 (elementary properties of mass). Let $\mathcal{R} \subset \mathcal{D}^2_\tau$ for some fixed $\tau \in S$. Then $\mathcal{R}$ is subordinate to itself and if a is an L^∞-normalized Carleson sequence we have

$$\mathsf{mass}_{a,1}(\mathcal{R}) = \sum_{R\in\mathcal{R}} a_R \leq |\mathsf{sh}(\mathcal{R})|, \quad \mathcal{R} \subset \mathcal{D}^2_\tau \text{ for some fixed } \tau \in S.$$

Also, the very definition of mass and the log-convexity of the L^p-norm imply

$$\mathsf{mass}_{a,p}(\mathcal{R}) \leq \mathsf{mass}_{a,1}(\mathcal{R})^{1-\frac{2}{p'}}\, \mathsf{mass}_{a,2}(\mathcal{R})^{\frac{2}{p'}} \tag{2.10}$$

for all $1 \leq p \leq 2$, with p' its dual exponent.

We are now ready to state the main result of this section. The result below should be interpreted as a reverse Hölder-type bound for the balayages of directional Carleson sequences.

Theorem C. *Let $S \subset [-1, 1]$ be a finite set of N slopes and $\mathcal{R} \subset \mathcal{D}^2_S$. Suppose that the maximal operators $\{\mathrm{M}_{\mathcal{R}_s} : s \in S\}$ satisfy*

$$\sup_{s\in S} \|\mathrm{M}_{\mathcal{R}_s} : L^p \to L^{p,\infty}\| \lesssim (p')^\gamma, \quad p \to 1^+,$$

for some $\gamma \geq 0$. Then for every L^∞-normalized Carleson sequence $a = \{a_R\}_{R\in\mathcal{D}^2_S}$

$$\mathsf{mass}_{a,2}(\mathcal{R}) \lesssim (\log N)^{\frac{1}{2}} ((1+\gamma)\log\log N)^{\frac{\gamma}{2}}\, \mathsf{mass}_{a,1}(\mathcal{R})^{\frac{1}{2}}.$$

The proof of Theorem C occupies the next subsection. The argument relies on several lemmas, whose proof is postponed to Section 2.23.

Remark 2.11. There are essentially two cases in the assumption of Theorem C above. If for each $s \in S$ the family $\mathcal{R}_s$ happens to be a one-parameter family, then the corresponding maximal operator $\mathrm{M}_{\mathcal{R}_s}$ is of weak-type-(1, 1), whence the assumption holds with $\gamma = 0$. In the generic case that $\mathcal{R} = \mathcal{D}^2_S$, for each s the operator $\mathrm{M}_{\mathcal{R}_s} = \mathrm{M}_{\mathcal{D}^2_S}$ is a skewed copy of the strong maximal operator and the assumption holds with $\gamma = 1$.

2.12. ***Main line of proof of Theorem C.*** Throughout the proof, we use the following partial order between parallelograms $Q, R \in \mathcal{D}^2_S$:

$$Q \leq R \quad \overset{\text{def}}{\iff} \quad Q \cap R \neq \varnothing,\ \ \pi_1(Q) \subseteq \pi_1(R). \tag{2.13}$$

Notice that, since $Q, R \in \mathcal{D}^2_S$, we have that $\pi_1(R), \pi_1(Q)$ belong to the standard dyadic grid $\mathcal{D}$ on $\mathbb{R}$.

It is convenient to encode the main inequality of Theorem C by means of the following dimensionless quantity associated with a collection $\mathcal{R} \subset \mathcal{D}^2_S$ and a Carleson sequence $a = \{a_R\}_{R\in\mathcal{D}^2_S}$:

$$\mathsf{U}_p(\mathcal{R}) := \sup_{\substack{\mathcal{L}\subset\mathcal{R}\\ a=\{a_R\}}} \frac{\mathsf{mass}_{a,p}(\mathcal{L})}{\mathsf{mass}_{a,1}(\mathcal{L})^{\frac{1}{p}}},$$

where the supremum is taken over all finite subcollections $\mathcal{L} \subset \mathcal{R}$ and all L^∞-normalized Carleson sequences $a = \{a_R\}_{R\in\mathcal{D}^2_S}$. There is an easy, albeit lossy, a priori estimate for $\mathsf{U}_p(\mathcal{R})$ for general $\mathcal{R} \subset \mathcal{D}^2_S$.

Lemma 2.14. *Let $S \subset [-1,1]$ be a finite set of N slopes and $a = \{a_R\}_{R\in\mathcal{R}}$ be a normalized Carleson sequence as above. For every $\mathcal{R} \subset \mathcal{D}^2_S$ we have the estimate*

$$\mathsf{U}_p(\mathcal{R}) \lesssim N^{\frac{1}{p'}} \sup_{s\in S} \|\mathrm{M}_{\mathcal{R}_s} : L^{p'} \to L^{p',\infty}\|, \quad 1 < p < \infty.$$

Theorem C is then an easy consequence of the following bootstrap-type estimate. For an arbitrary finite collection of parallelograms $\mathcal{R} \subset \mathcal{D}^2_S$ we will prove the estimate

$$\mathsf{U}_2(\mathcal{R})^2 \lesssim (\log \mathsf{U}_2(\mathcal{R}))^\gamma \log N, \tag{2.15}$$

with absolute implicit constant. Note also that the boundedness assumption on $\mathrm{M}_{\mathcal{R}_s}$ for some $p < 2$ and Lemma 2.14 yield the a priori estimate $\mathsf{U}_2(\mathcal{R}) \lesssim N^{1/2}$. Inserting this a priori estimate into (2.15) and bootstrapping will then complete the proof of Theorem C. It thus suffices to prove (2.15) to obtain Theorem C.

The remainder of the section is dedicated to the proof of (2.15). We begin by expanding the square of the L^2-norm of $T_{\mathcal{R}}(a)$ as follows:

$$\mathsf{mass}_{a,2}(\mathcal{R})^2 = \|T_{\mathcal{R}}(a)\|_2^2 \le 2 \sum_{R\in\mathcal{R}} a_R \frac{1}{|R|} \int_R \sum_{\substack{Q\in\mathcal{R}\\ Q\le R}} a_Q \frac{\mathbf{1}_Q}{|Q|} =: 2 \sum_{R\in\mathcal{R}} a_R B_R^{\mathcal{R}}. \tag{2.16}$$

For any $\mathcal{L} \subset \mathcal{R}$ and $R \in \mathcal{R}$ we have implicitly defined

$$B_R^{\mathcal{L}} := \frac{1}{|R|} \int_R \sum_{\substack{Q\in\mathcal{L}\\ Q\le R}} a_Q \frac{\mathbf{1}_Q}{|Q|}. \tag{2.17}$$

Remark 2.18. Observe that for any $\mathcal{L} \subset \mathcal{R}$ and every fixed $s \in S$ we have

$$\bigcup \{R \in \mathcal{R}_s : B_R^{\mathcal{L}} > \lambda\} \subset \left\{x \in \mathbb{R}^2 : \mathrm{M}_{\mathcal{R}_s}\Big[\sum_{Q\in\mathcal{L}} a_Q \frac{\mathbf{1}_Q}{|Q|}\Big](x) > \lambda\right\},$$

which by our assumption on the weak (p,p) norm of $\mathrm{M}_{\mathcal{R}_s}$ implies

$$\sup_{s\in S} \left|\bigcup \left\{R \in \mathcal{R}_s : B_R^{\mathcal{L}} > \lambda\right\}\right| \lesssim (p')^\gamma \frac{\mathsf{mass}_{a,p}(\mathcal{L})^p}{\lambda^p}, \quad p \to 1^+.$$

For a numerical constant $\lambda \ge 1$, to be chosen at the end of the proof, a nonnegative integer k and $s \in S$ we consider subcollections of $\mathcal{R}_s$ as follows:

$$\mathcal{R}_{s,k} := \{R : R \in \mathcal{R}_s,\ \lambda k \le B_R^{\mathcal{R}} < \lambda(k+1)\}, \quad k \in \mathbb{N},\ s \in S. \tag{2.19}$$

Using (2.16) we have

$$\begin{aligned} \|T_{\mathcal{R}}(a)\|_2^2 &\lesssim \sum_{s\in S} \sum_{k=0}^{N} k\lambda \sum_{R\in\mathcal{R}_{s,k}} a_R + N \sup_{s\in S} \Big[\sum_{k>N} k\lambda \sum_{R\in\mathcal{R}_{s,k}} a_R\Big] \\ &\lesssim \lambda(\log N)\, \mathsf{mass}_{a,1}(\mathcal{R}) + \lambda N \sum_{k>N} k \sup_{s\in S} |\mathsf{sh}(\mathcal{R}_{s,k})|. \end{aligned} \tag{2.20}$$

Here $\lambda > 0$ is the constant used to define the collections $\mathcal{R}_{s,k}$ and in the last lines we used the definition of a Carleson sequence and Remark 2.9.

The following lemma encodes the exponential decay relation between mass and $B_R^{\mathcal{L}}$ and is in fact the main step of the proof of Theorem C.

Lemma 2.21. *Let $a = \{a_R : R \in \mathcal{D}_S^2\}$ be an L^∞-normalized Carleson sequence, $S \subset [-1,1]$, and $\mathcal{L}, \mathcal{R} \subset \mathcal{D}_S^2$ with $\mathcal{L} \subseteq \mathcal{R}$. We assume that for some $p \in [1,2)$*

$$A_p := \sup_{s\in S} \|\mathrm{M}_{\mathcal{R}_s} : L^p \to L^{p,\infty}\| < +\infty.$$

If $\lambda \geq C \max(1, A_p \mathsf{U}_2(\mathcal{L})^{2/p'})$ for a sufficiently large numerical constant $C > 1$ then there exists $\mathcal{L}_1 \subset \mathcal{L}$ such that

(i) $\mathsf{mass}_{a,1}(\mathcal{L}_1) \leq \frac{1}{2}\mathsf{mass}_{a,1}(\mathcal{L})$,

(ii) *fixing $s \in S$ and denoting by $\mathcal{R}_s'$ the collection of rectangles R in $\mathcal{R}_s$ with $B_R^{\mathcal{L}} > \lambda$, see* (2.17), *we have that*

$$B_R^{\mathcal{L}} \leq \lambda + B_R^{\mathcal{L}_1} \quad \textit{for all } R \in \mathcal{R}_s'.$$

The final lemma we make use of in the argument translates the exponential decay of the mass of each $\mathcal{R}_{s,k}$ into exponential decay of the support size, which is what we need in the estimate (2.20).

Lemma 2.22. *Let $S \subset [-1,1]$ and define the collections $\mathcal{R}_{s,k}$ by* (2.19) *with λ defined as in Lemma 2.21 for $\mathcal{L} = \mathcal{R}$*

$$\lambda := C \max(1, A_p \mathsf{U}_2(\mathcal{R})^{\frac{2}{p'}}).$$

We assume that the operators $\{\mathrm{M}_{\mathcal{R}_s} : s \in S\}$ map $L^p(\mathbb{R}^2)$ to $L^{p,\infty}(\mathbb{R}^2)$ uniformly with constant A_p. For $k \geq 1$ we then have the estimate

$$|\mathsf{sh}(\mathcal{R}_{s,k})| \lesssim 2^{-k}\,\mathsf{mass}_{a,1}(\mathcal{R}),$$

with absolute implicit constant.

With these lemmas in hand we now return to the proof of (2.15). Substituting the estimate of Lemma 2.22 into (2.20) yields

$$\|T_{\mathcal{R}}(a)\|_2^2 \lesssim \lambda\,\mathsf{mass}_{a,1}(\mathcal{R})\Big[(\log N) + N \sum_{k \geq \log N} k2^{-k}\Big] \lesssim \lambda\,\mathsf{mass}_{a,1}(\mathcal{R})(\log N).$$

This was proved for an arbitrary collection $\mathcal{R}$ and so also for every $\mathcal{L} \subset \mathcal{R}$. Thus the estimate above and our assumption $A_p \lesssim (p')^\gamma$ imply

$$\mathsf{U}_2(\mathcal{R})^2 \lesssim \lambda(\log N), \quad \lambda \gtrsim \max(1, (p')^\gamma \mathsf{U}_2(\mathcal{R})^{\frac{2}{p'}}).$$

Now observe that we can assume $\mathsf{U}_2(\mathcal{R}) \gtrsim 1$; otherwise there is nothing to prove. In this case we can take

$$\lambda \simeq (p')^\gamma \mathsf{U}_2(\mathcal{R})^{\frac{2}{p'}}$$

for every $p > 1$. The choice $p' := (\log \mathsf{U}_2(\mathcal{R}))$ guarantees that $[\mathsf{U}_2(\mathcal{R})]^{1/p'} \lesssim 1$ and leads to

$$\mathsf{U}_2(\mathcal{R})^2 \lesssim (\log \mathsf{U}_2(\mathcal{R}))^\gamma \log N.$$

This is the desired estimate (2.15) and so the proof of Theorem C is complete.

2.23. *Proofs of the lemmas.*

Proof of Lemma 2.14. We follow the proof of [Lacey 2007, Lemma 3.11]. Take $\mathcal{R}$ to be some finite collection and $\|g\|_{p'} = 1$ such that

$$\Big\| \sum_{R \in \mathcal{R}} a_R \frac{\mathbf{1}_R}{|R|} \Big\|_p = \int \sum_{R \in \mathcal{R}} a_R \frac{\mathbf{1}_R}{|R|} g.$$

Define $\mathcal{R}' := \{R \in \mathcal{R} : \langle g \rangle_R > [cN/\operatorname{mass}_{a,1}(\mathcal{R})]^{1/p'}\}$ for some $c > 1$ and $\mathcal{R}'_s := \mathcal{R}' \cap \mathcal{D}^2_s$ for $s \in S$. Then,

$$\int \sum_{R \in \mathcal{R}} a_R \frac{\mathbf{1}_R}{|R|} g \le \sum_{R \in \mathcal{R} \setminus \mathcal{R}'} a_R \langle g \rangle_R + \Big\| \sum_{R \in \mathcal{R}'} a_R \frac{\mathbf{1}_R}{|R|} \Big\|_p \le (cN)^{\frac{1}{p'}} \Big(\sum_{R \in \mathcal{R}} a_R \Big)^{\frac{1}{p}} + N \sup_{s \in S} \Big\| \sum_{R \in \mathcal{R}'_s} a_R \frac{\mathbf{1}_R}{|R|} \Big\|_p.$$

This means

$$\Big\| \sum_{R \in \mathcal{R}} a_R \frac{\mathbf{1}_R}{|R|} \Big\|_p \lesssim (cN)^{\frac{1}{p'}} \Bigg(1 + \frac{N^{\frac{1}{p}}}{c^{\frac{1}{p'}}} \sup_{s \in S} \frac{\| \sum_{R \in \mathcal{R}'_s} a_R (\mathbf{1}_R/|R|) \|_p}{\big(\sum_{R \in \mathcal{R}'_s} a_R \big)^{\frac{1}{p}}} \frac{\big(\sum_{R \in \mathcal{R}'_s} a_R \big)^{\frac{1}{p}}}{\big(\sum_{R \in \mathcal{R}_s} a_R \big)^{\frac{1}{p}}} \Bigg) \Big(\sum_{R \in \mathcal{R}} a_R \Big)^{\frac{1}{p}}.$$

We have proved that for an arbitrary collection $\mathcal{R}$ we have

$$\mathsf{U}_p(\mathcal{R}) \le (cN)^{\frac{1}{p'}} \Bigg(1 + \frac{N^{\frac{1}{p}}}{c^{\frac{1}{p'}}} \sup_s \mathsf{U}_p(\mathcal{R}'_s) \frac{\operatorname{mass}_{a,1}(\mathcal{R}'_s)^{\frac{1}{p}}}{\operatorname{mass}_{a,1}(\mathcal{R})^{\frac{1}{p}}} \Bigg).$$

We claim that $\sup_{s \in S} \mathsf{U}_p(\mathcal{R}'_s) \lesssim \sup_{s \in S} \|\mathrm{M}_{\mathcal{R}_s} : L^{p'} \to L^{p',\infty}\|$. Assuming this for a moment and using Remark 2.9 we can estimate

$$\begin{aligned} \sum_{R \in \mathcal{R}'_s} a_R \le |\operatorname{sh}(\mathcal{R}'_s)| &\le |\{\mathrm{M}_{\mathcal{R}_s}(g) > (cN/\operatorname{mass}_{a,1}(\mathcal{R}))^{1/p'}\}| \\ &\le \sup_{s \in S} \|\mathrm{M}_{\mathcal{R}_s} : L^{p'} \to L^{p',\infty}\|^{p'} \frac{\operatorname{mass}_{a,1}(\mathcal{R})}{cN}. \end{aligned}$$

This proves the proposition upon choosing $c \gtrsim \sup_{s \in S} \|\mathrm{M}_{\mathcal{R}_s} : L^{p'} \to L^{p',\infty}\|^{p'}$.

We have to prove the claim. Note that since $\mathcal{R}'_s$ is a collection in a fixed direction, the inequality $\mathsf{U}_{\mathcal{R}'_s} \lesssim \sup_{s \in S} \|\mathrm{M}_{\mathcal{R}_s} : L^{p'} \to L^{p',\infty}\|$ follows by the John–Nirenberg inequality in the product setting and Remark 2.9; see [Lacey 2007, Lemma 3.11]. □

Proof of Lemma 2.21. By the invariance under shearing of our statement, we can work in the case $s = 0$. Therefore, $\mathcal{R}'_0$ will stand for the collection of rectangles in $\mathcal{R}_0$ such that $B^{\mathcal{L}}_R > \lambda$, where $\lambda \ge C$ and $C > 1$ will be specified at the end of the proof. We write $R = I_R \times L_R$ for $R \in \mathcal{R}_0$.

Inside-outside splitting. For $I \in \{\pi_1(R) : R \in \mathcal{R}'_0\}$ and any interval K we define

$$\mathcal{L}^{\text{in}}_{I,K} := \{Q \in \mathcal{L} : Q \le I \times K,\ \pi_2(Q) \subset 3K\}, \quad \mathcal{L}^{\text{out}}_{I,K} := \{Q \in \mathcal{L} : Q \le I \times K,\ \pi_2(Q) \not\subseteq 3K\},$$

where we recall that the definition of partial order $Q \le R$ was given in (2.13). Set also

$$B^{\text{in}}_{I,K} := \fint_{I \times K} \sum_{Q \in \mathcal{L}^{\text{in}}_{I,K}} \frac{a_Q}{|Q|} \mathbf{1}_Q, \quad B^{\text{out}}_{I,K} := \fint_{I \times K} \sum_{Q \in \mathcal{L}^{\text{out}}_{I,K}} \frac{a_Q}{|Q|} \mathbf{1}_Q.$$

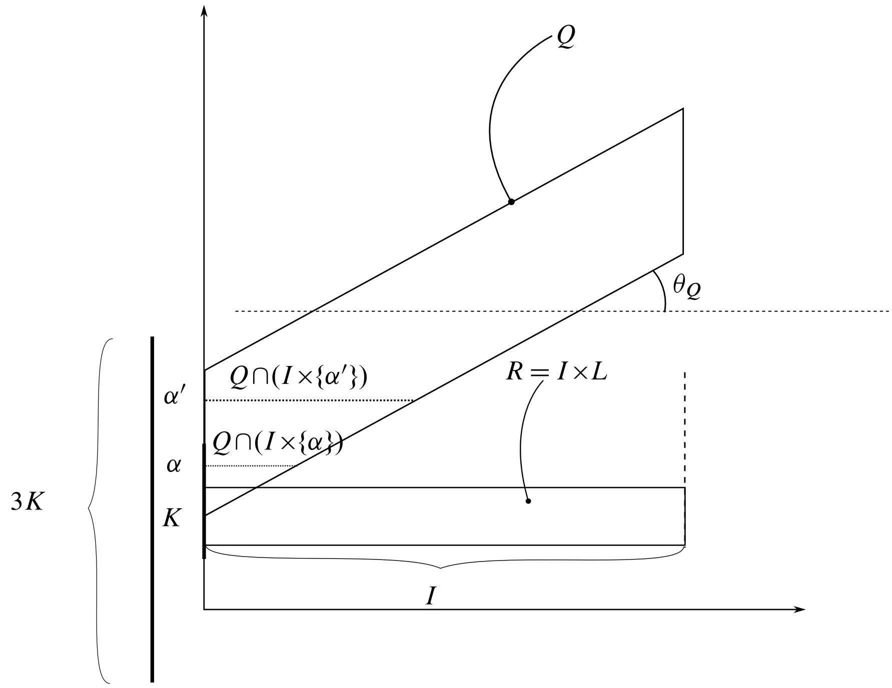

Figure 3. A rectangle Q with angle θ_Q intersecting $R = I \times L \subset I \times K$.

We claim that if $K \subset \mathbb{R}$ is any interval then for all $\alpha \in K$ we have

$$\fint_{I\times\{\alpha\}} \sum_{Q\in\mathcal{L}^{\text{out}}_{I,K}} a_Q \frac{\mathbf{1}_Q}{|Q|} = \sum_{Q\in\mathcal{L}^{\text{out}}_{I,K}} a_Q \frac{|Q\cap(I\times\{\alpha\})|}{|Q|} \lesssim \fint_{I\times 3K} \sum_{Q\in\mathcal{L}^{\text{out}}_{I,K}} a_Q \frac{\mathbf{1}_Q}{|Q|}. \tag{2.24}$$

To see this note that in order for a Q-term appearing in the sum of the left-hand side above to be nonzero we must have

$$\pi_1(Q) \subset I, \quad \pi_2(Q)\cap K \neq \varnothing, \quad \pi_2(Q)\cap\mathbb{R}\setminus 3K \neq \varnothing.$$

Let us write $\theta_Q = \arctan\sigma$ if $Q \in \mathcal{D}^2_\sigma$ for some $\sigma \in S$. A computation then reveals that

$$|Q\cap(I\times\{\alpha\})| = \min(|J_Q|, \operatorname{dist}(\alpha, \mathbb{R}\setminus\pi_2(Q)))\cot\theta_Q.$$

We also observe that $\pi_2(Q)\cap(3K\setminus K)$ contains an interval $A = A(\alpha)$ of length $|K|/3$, whence for all $\alpha' \in A$ we have

$$\operatorname{dist}(\alpha,\mathbb{R}\setminus\pi_2(Q)) \le \operatorname{dist}(\alpha,\alpha') + \operatorname{dist}(\alpha',\mathbb{R}\setminus\pi_2(Q)) \lesssim |K| + \operatorname{dist}(\alpha',\mathbb{R}\setminus\pi_2(Q)) \lesssim \operatorname{dist}(\alpha',\mathbb{R}\setminus\pi_2(Q));$$

see Figure 3. This clearly implies that for every $\alpha \in K$ we have

$$|Q\cap(I\times\{\alpha\})| \lesssim \fint_A |Q\cap(I\times\{\alpha'\})|\,\mathrm{d}\alpha' \lesssim \fint_{3K} |Q\cap(I\times\{\alpha'\})|\,\mathrm{d}\alpha',$$

which proves the claim.

Smallness of the local average. We now use the previously obtained (2.24) to prove (ii). Let $\mathcal{R}_0^\star$ denote the family of parallelograms $R = I_R \times L_R \in \mathcal{R}_0'$ such that $B^{\text{out}}_{I_R,L_R} > \lambda$. For each such R let K_R be the maximal interval $K \in \{L_R, 3L_R, \dots, 3^k L_R, \dots\}$ such that $B^{\text{out}}_{I_R,K} > \lambda$; the existence of the maximal interval K_R is guaranteed for example by the a priori estimate of Lemma 2.14 and the assumption $R \in \mathcal{R}_0^\star$. Obviously $K_R \supseteq L_R$ and $B^{\text{out}}_{I_R,3K_R} \le \lambda$.

We show that for $R \in \mathcal{R}_0^\star$ we have

$$\fint_R \sum_{Q \in \mathcal{L}^{\text{out}}_{I_R,K_R}} a_Q \frac{\mathbf{1}_Q}{|Q|} \le \kappa\lambda \tag{2.25}$$

for some numerical constant $\kappa \ge 1$. Indeed it is a consequence of (2.24) that

$$\begin{aligned}\fint_{I_R\times\{\alpha\}} \sum_{Q \in \mathcal{L}^{\text{out}}_{I_R,K_R}} a_Q \frac{\mathbf{1}_Q}{|Q|} &\lesssim \fint_{I_R\times 3K_R} \sum_{Q \in \mathcal{L}^{\text{out}}_{I_R,K_R}} a_Q \frac{\mathbf{1}_Q}{|Q|} \\ &\le \fint_{I_R\times 3K_R} \sum_{Q \in \mathcal{L}^{\text{out}}_{I_R,3K_R}} a_Q \frac{\mathbf{1}_Q}{|Q|} + \fint_{I_R\times 3K_R} \sum_{Q \in \mathcal{L}^{\text{out}}_{I_R,K_R}\setminus \mathcal{L}^{\text{out}}_{I_R,3K_R}} a_Q \frac{\mathbf{1}_Q}{|Q|}.\end{aligned}$$

The first summand is estimated using the maximality of K_R:

$$\fint_{I_R\times 3K_R} \sum_{Q \in \mathcal{L}^{\text{out}}_{I_R,3K_R}} a_Q \frac{\mathbf{1}_Q}{|Q|} = B^{\text{out}}_{I_R,3K_R} \le \lambda.$$

The second summand can be further analyzed by observing that the cubes Q appearing in the sum above satisfy $\pi_1(Q) \subset I$ and $\pi_2(Q) \subset 9K_R$ since $Q \notin \mathcal{L}^{\text{out}}_{I_R,3K_R}$, that is, $\mathcal{L}^{\text{out}}_{I_R,3K_R} \setminus \mathcal{L}^{\text{out}}_{I_R,K_R}$ is subordinate to the singleton collection $\{I_R \times 9K_R\}$. Applying the Carleson sequence property

$$\fint_{I_R\times 3K_R} \sum_{Q \in \mathcal{L}^{\text{out}}_{I_R,K_R}\setminus \mathcal{L}^{\text{out}}_{I_R,3K_R}} a_Q \frac{\mathbf{1}_Q}{|Q|} \le \sum_{Q \in \mathcal{L}^{\text{out}}_{I_R,K_R}\setminus \mathcal{L}^{\text{out}}_{I_R,3K_R}} a_Q \frac{|Q \cap (I_R \times 3K_R)|}{|Q||I_R \times 3K_R|} \lesssim 1 \le \lambda \tag{2.26}$$

by our assumption on λ. Combining the estimates above shows that

$$\fint_{I_R\times\{\alpha\}} \sum_{Q \in \mathcal{L}^{\text{out}}_{I_R,K_R}} a_Q \frac{\mathbf{1}_Q}{|Q|} \lesssim \lambda$$

for all $\alpha \in K_R$. Since $\pi_2(R) \subset K$ this implies (2.25).

Observe that if $R = I_R \times L_R \in \mathcal{R}_0' \setminus \mathcal{R}_0^\star$ then

$$B^{\text{out}}_{I_R,L_R} = \fint_{I_R\times L_R} \sum_{Q \in \mathcal{L}^{\text{out}}_{I_R,L_R}} a_Q \frac{\mathbf{1}_Q}{|Q|} \le \lambda.$$

Defining the subcollection $\mathcal{L}_1$. We set

$$\mathcal{L}_1' := \bigcup_{R\in\mathcal{R}_0^\star} \mathcal{L}^{\text{in}}_{I_R,K_R}, \quad \mathcal{L}_1'' := \bigcup_{R\in\mathcal{R}_0'\setminus\mathcal{R}_0^\star} \mathcal{L}^{\text{in}}_{I_R,L_R}, \quad \mathcal{L}_1 := \mathcal{L}_1' \cup \mathcal{L}_1''.$$

Now note that for each $R \in \mathcal{R}_0^\star$ and $K = K_R \in \mathcal{K}_{\pi_1(R)}$ we have that

$$B_R^{\mathcal{L}} \leq \fint_R \sum_{Q \in \mathcal{L}^{\mathrm{out}}_{I_R \times K_R}} a_Q \frac{\mathbf{1}_Q}{|Q|} + \fint_R \sum_{Q \in \mathcal{L}^{\mathrm{in}}_{I_R \times K_R}} a_Q \frac{\mathbf{1}_Q}{|Q|} \leq \kappa\lambda + B_R^{\mathcal{L}_1},$$

while for $R \in \mathcal{R}_0' \setminus \mathcal{R}_0^\star$ the same estimate holds using L_R in place of K_R. It remains to show the desired estimate for $\mathsf{mass}_{a,1}(\mathcal{L}_1)$ in (i) of the lemma.

Smallness of $\mathsf{mass}_{a,1}(\mathcal{L}_1)$. By the definition of the collections $\mathcal{L}^{\mathrm{in}}_{I,K}$ we have that

$$\mathsf{sh}(\mathcal{L}_1) \subset \bigcup_{R \in \mathcal{R}_0^\star} I_R \times 3K_R \cup \bigcup_{R \in \mathcal{R}_0' \setminus \mathcal{R}_0^\star} I_R \times 3L_R.$$

If $K = K_R$ for some $R \in \mathcal{R}_0^\star$ we have by definition that $B^{\mathrm{out}}_{I_R,K_R} > \lambda$. On the other hand for $R \in \mathcal{R}_0' \setminus \mathcal{R}_0^\star$ we have that $B_R^{\mathcal{L}} = B^{\mathcal{L}}_{I_R,L_R} > \lambda$.

Define

$$E := \left\{(x,y) \in \mathbb{R}^2 : \mathrm{M}_v\left[\sum_{Q \in \mathcal{L}} a_Q \frac{\mathbf{1}_Q}{|Q|}\right](x,y) \geq \frac{\lambda}{2}\right\},$$

where $\mathrm{M}_v = \mathrm{M}_{(1,s)} = \mathrm{M}_1$ is the directional Hardy–Littlewood maximal operator acting in the direction $v = (1,s) = (1,0)$, see (2.3), since we have assumed $s = 0$. We will show that

$$\bigcup_{R \in \mathcal{R}_0^\star} I_R \times 3K_R \subset \{(x,y) \in \mathbb{R}^2 : \mathrm{M}_2(\mathbf{1}_E)(x,y) \geq C\}$$

for a sufficiently small constant $C > 0$, where M_2 is as in (2.3). To this end let us define

$$\psi(\alpha) := \frac{1}{|I_R|} \int_{I_R \times \{\alpha\}} \sum_{Q \in \mathcal{L}^{\mathrm{out}}_{I_R,K_R}} a_Q \frac{\mathbf{1}_Q}{|Q|}.$$

Note that

$$\lambda < B^{\mathrm{out}}_{I_R,K_R} = \fint_{K_R} \psi(\alpha)\,\mathrm{d}\alpha \leq \frac{1}{|K_R|} \int_{\{K_R : \psi(\alpha) > \lambda/2\}} \psi(\alpha)\,\mathrm{d}\alpha + \frac{\lambda}{2} \leq \frac{c\lambda}{|K_R|}\left|\left\{K_R : \psi(\alpha) > \frac{\lambda}{2}\right\}\right| + \frac{\lambda}{2},$$

which readily yields the existence of $K' \subset K_R$, with

$$|K_R| \lesssim |K'|, \qquad \inf_{x \in I_R} \inf_{y \in K'} \mathrm{M}_v\left[\sum_{Q \in \mathcal{L}^{\mathrm{out}}_{I_R,K_R}} a_Q \frac{\mathbf{1}_Q}{|Q|}\right](x,y) > \frac{\lambda}{2}.$$

This in turn implies that $\mathrm{M}_2(\mathbf{1}_E) \gtrsim 1$ on $I_R \times 3K_R$. Now we can conclude

$$\left|\bigcup_{R \in \mathcal{R}_0^\star} I_R \times 3K_R\right| \leq |\{\mathrm{M}_2(\mathbf{1}_E) \gtrsim 1\}| \lesssim |E| \lesssim \frac{1}{\lambda}\,\mathsf{mass}_{a,1}(\mathcal{L})$$

by the weak-(1, 1) inequality of the directional Hardy–Littlewood maximal operator $\mathrm{M}_{(1,0)}$.

On the other hand we have for the rectangles $R \in \mathcal{R}_0' \setminus \mathcal{R}_0^\star$ that

$$\bigcup_{R \in \mathcal{R}_0' \setminus \mathcal{R}_0^\star} I_R \times 3L_R \subset \left\{\mathrm{M}_{\mathcal{R}_0}\left(\sum_{Q \in \mathcal{L}} a_Q \frac{\mathbf{1}_Q}{|Q|}\right) > \frac{\lambda}{3}\right\}.$$

Thus we get by the weak (p,p) assumption for $\mathrm{M}_{\mathcal{R}_0}$ that

$$\Big|\bigcup_{R\in\mathcal{R}_0'\setminus\mathcal{R}_0^\star} I_R\times 3L_R\Big| \le \Big|\Big\{\mathrm{M}_{\mathcal{R}_0}\Big(\sum_{Q\in\mathcal{L}} a_Q\frac{\mathbf{1}_Q}{|Q|} > \frac{\lambda}{3}\Big)\Big\}\Big| \lesssim \frac{A_p^p}{\lambda^p}\,\mathsf{mass}_{a,p}(\mathcal{L}) \lesssim \frac{A_p^p}{\lambda^p}\,\mathsf{mass}_{a,1}(\mathcal{L})\mathsf{U}_2(\mathcal{L})^{2(p-1)}.$$

By the subordination property of $\mathcal{L}_1$ we get

$$\mathsf{mass}_{a,1}(\mathcal{L}_1) \le \Big|\bigcup_{R\in\mathcal{R}_0^\star} I_R\times 3K_R \cup \bigcup_{R\in\mathcal{R}_0'\setminus\mathcal{R}_0^\star} I_R\times 3L_R\Big| \le \frac{1}{2}\,\mathsf{mass}_{a,1}(\mathcal{L}),$$

upon choosing $\lambda \ge C\max(1, A_p\mathsf{U}_2(\mathcal{L})^{2/p'})$ with sufficiently large $C>1$. □

Proof of Lemma 2.22. Fix $s\in S$ and choose λ in the definition of $\mathcal{R}_{s,k}$ to be the value given by Lemma 2.21 with $\mathcal{L}=\mathcal{R}=\bigcup_{s\in S}\mathcal{R}_s$. Let $j=0$ and $\mathcal{L}_0=\mathcal{L}_j:=\mathcal{R}$. Construct $\mathcal{L}_1=\mathcal{L}_{j+1}\subset\mathcal{R}$ such that $\mathsf{mass}_{a,1}(\mathcal{L}_1)\le\frac{1}{2}\,\mathsf{mass}_{a,1}(\mathcal{L}_0)$. Since $B_R^{\mathcal{L}_0}>k\lambda$ for all $R\in\mathcal{R}_{s,k}$, we have

$$\lambda k < B_R^{\mathcal{L}_0} \le \lambda + B_R^{\mathcal{L}_1} \quad\Longrightarrow\quad B_R^{\mathcal{L}_1} > \lambda(k-1).$$

Repeat the procedure recursively with $j+1$ in place of j. When $j=k-1$, we have reached the collection $\mathcal{L}_{k-1}$ with $\mathsf{mass}_{a,1}(\mathcal{L}_{k-1})\lesssim 2^{-k}\,\mathsf{mass}_{a,1}(\mathcal{L}_0)$ and $B_R^{\mathcal{L}_{k-1}}>\lambda$. This last condition and Remark 2.18 imply that

$$\mathsf{sh}(\mathcal{R}_{s,k}) \subset \Big\{\mathrm{M}_{\mathcal{R}_s}\Big[\sum_{Q\in\mathcal{L}_{k-1}} a_Q\frac{\mathbf{1}_Q}{|Q|}\Big] > \lambda\Big\}$$

and so, using (2.10),

$$\begin{aligned}|\mathsf{sh}(\mathcal{R}_{s,k})| &\le \frac{A_p^p}{\lambda^p}\,\mathsf{mass}_{a,p}(\mathcal{L}_{k-1})^p \le \frac{A_p^p}{\lambda^p}\,\mathsf{mass}_{a,1}(\mathcal{L}_{k-1})^{p-\frac{2p}{p'}}\,\mathsf{mass}_{a,2}(\mathcal{L}_{k-1})^{\frac{2p}{p'}}\\ &\le 2^{-k}\,\mathsf{mass}_{a,1}(\mathcal{L}_0)\frac{CA_p^p}{\lambda^p}\Big(\frac{\mathsf{mass}_{a,2}(\mathcal{L}_0)^2}{\mathsf{mass}_{a,1}(\mathcal{L}_0)}\Big)^{p-1} = 2^{-k}\,\mathsf{mass}_{a,1}(\mathcal{L}_0)\frac{CA_p^p}{\lambda^p}\mathsf{U}_2(\mathcal{L}_0)^{2(p-1)}\end{aligned}$$

and the lemma follows by the definition of λ since $\mathcal{L}_0=\mathcal{R}$. □

3. A weighted Carleson embedding and applications to directional maximal operators

In this section, we provide a weighted version of the directional Carleson embedding theorem. We then derive, as applications, novel weighted norm inequalities for maximal and singular directional operators.

The proof of the weighted Carleson embedding follows the strategy used for Theorem C, with suitable modifications. In order to simplify the presentation, we restrict our scope to collections of parallelograms $\mathcal{R}=\{\bigcup\mathcal{R}_s : s\in S\}$ with the property that the maximal operator $\mathrm{M}_{\mathcal{R}_s}$ associated to each collection $\mathcal{R}_s$ satisfies the appropriate weighted weak-(1,1) inequality. This is the case, for instance, when the collections $\mathcal{R}_s$ are of the form

$$\mathcal{R}_s \subset \mathcal{D}^2_{s,k}, \quad \mathcal{D}^2_{s,k} := \bigcup_{k_1\le k}\mathcal{D}^2_{s,k_1,k} \tag{3.1}$$

for a fixed $k \in \mathbb{Z}$. In other words, the parallelograms in direction s have fixed vertical sidelength and arbitrary eccentricity.

3.2. ***Directional weights.*** Let S be a set of slopes and $w, u \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ be nonnegative functions, which we refer to as *weights* from now on. Our weight classes are related to the maximal operator

$$\mathrm{M}_{S;2} := \mathrm{M}_V \circ \mathrm{M}_{(0,1)},$$

recalling that $M_V = M_{\{(1,s):s\in S\}}$ is the directional maximal operator defined in (2.4). We introduce the two-weight directional constant

$$[w,u]_S := \sup_{x\in\mathbb{R}^2} \frac{\mathrm{M}_{S;2}w(x)}{u(x)}.$$

We pause to point out some relevant examples of pairs w, u with $[w,u]_S < \infty$. Recall that, for $p > 2$, $\|\mathrm{M}_{S;2}\|_{p\to p} \lesssim (\log \#S)^{1/p}$; this is actually a special case of Theorem C and interpolation. Therefore, if $g \geq 0$ belongs to the unit sphere of $L^p(\mathbb{R}^2)$,

$$w := \sum_{\ell=0}^{\infty} \frac{\mathrm{M}^{[\ell]}_{S;2} g}{2^\ell \|\mathrm{M}_{S;2}\|^{\ell}_{p\to p}}$$

satisfies $[w,w]_S \leq 2\|\mathrm{M}_{S;2}\|_{p\to p}$; here $T^{[\ell]}$ denotes ℓ-fold composition of an operator T with itself. We also highlight the relevance of $[w,u]_S$ in Theorem D below by noticing that

$$\sup_{s\in S} \|\mathrm{M}_{\mathcal{D}^2_{s,k}} \colon L^1(u) \to L^{1,\infty}(w)\| \lesssim [w,u]_S,$$

with absolute implicit constant. This result is obtained via the classical Fefferman–Stein inequality in direction s paired with the remark that $\mathrm{M}_{\mathcal{D}^2_{s,k}} w \lesssim \mathrm{M}_{S;2} w \leq [w,u]_S u$.

3.3. ***Weighted Carleson sequences.*** We begin with the weighted analogue of Definition 2.7, which is given with respect to a fixed weight w.

Definition 3.4. Let $a = \{a_R\}_{R\in\mathcal{D}^2_S}$ be a sequence of nonnegative numbers. Then a will be called an L^∞*-normalized* w-Carleson sequence if for every $\mathcal{L} \subset \mathcal{D}^2_S$ which is subordinate to some collection $\mathcal{T} \subset \mathcal{P}^2_\tau$ for some fixed $\tau \in S$, we have

$$\sum_{L\in\mathcal{L}} a_L \leq w(\mathsf{sh}(\mathcal{T})), \quad \mathsf{mass}_a := \sum_{R\in\mathcal{D}^2_S} a_R < \infty.$$

As before, if $\mathcal{R} \subset \mathcal{D}^2_\tau$ for some fixed $\tau \in S$ then $\mathcal{R}$ is subordinate to itself and

$$\mathsf{mass}_{a,1}(\mathcal{R}) = \sum_{R\in\mathcal{R}} a_R \leq w(\mathsf{sh}(\mathcal{R})), \quad \mathcal{R} \subset \mathcal{D}^2_\tau \text{ for some fixed } \tau \in S.$$

Throughout this section all Carleson sequences and related quantities are taken with respect to some fixed weight w which is suppressed from the notation. We can now state our weighted Carleson embedding theorem.

Theorem D. *Let $S \subset [-1,1]$ be a finite set of N slopes and $\mathcal{R} \subset \mathcal{D}^2_S$. Let w, u be weights with $[w,u]_S < \infty$ and such that*

$$\sup_{s\in S} \|\mathrm{M}_{\mathcal{R}_s} : L^1(u) \to L^{1,\infty}(w)\| \lesssim [w,u]_S.$$

Then for every L^∞-normalized w-Carleson sequence $a = \{a_R\}_{R\in\mathcal{D}^2_S}$ we have

$$\left(\int |T_{\mathcal{R}}(a)(x)|^2 \frac{\mathrm{d}x}{\mathrm{M}_{\mathcal{R}}u(x)}\right)^{\frac{1}{2}} \lesssim (\log N)^{\frac{1}{2}} [w,u]_S \,\mathsf{mass}_{a,1}(\mathcal{R})^{\frac{1}{2}}.$$

3.5. ***Proof of Theorem D.*** We follow the proof of Theorem C and only highlight the differences to accommodate the weighted setting. Write $\sigma := [\mathrm{M}_{\mathcal{R}}u]^{-1}$. Expanding the $L^2(\sigma)$-norm we have

$$\|T_{\mathcal{R}}(a)\|^2_{L^2(\sigma)} \le 2 \sum_{R\in\mathcal{R}} a_R \sum_{\substack{Q\in\mathcal{R}\\ Q\le R}} a_Q \frac{\sigma(Q\cap R)}{|Q||R|}.$$

From the definition of σ we have that

$$\sigma(Q\cap R) \le \frac{|Q\cap R|}{\inf_Q \mathrm{M}_{\mathcal{R}}u} \le \frac{|Q|}{u(Q)}|Q\cap R|,$$

whence

$$\|T_{\mathcal{R}}(a)\|^2_{L^2(\sigma)} \le 2 \sum_{R\in\mathcal{R}} a_R \fint_R \sum_{\substack{Q\in\mathcal{R}\\ Q\le R}} a_Q \frac{\mathbf{1}_Q}{u(Q)} := 2\sum_{R\in\mathcal{R}} a_R B^{\mathcal{R}}_R,$$

where now for any $\mathcal{L} \subset \mathcal{R}$ we have defined

$$B^{\mathcal{L}}_R := \fint_R \sum_{\substack{Q\in\mathcal{L}\\ Q\le R}} a_Q \frac{\mathbf{1}_Q}{u(Q)}.$$

Defining the families $\mathcal{R}_{s,k}$ for $s \in S$ and $k \in \mathbb{N}$ as in (2.19) we then have the estimate

$$\|T_{\mathcal{R}}(a)\|^2_{L^2(\sigma)} \le 2\lambda \left[(\log N)\,\mathsf{mass}_{a,1}(\mathcal{R}) + N \sum_{k>\log N} k \sup_{s\in S} w(\mathsf{sh}(\mathcal{R}_{s,k}))\right].$$

Again $\lambda > 0$ is a constant that will be determined later in the proof and in the last line we used the w-Carleson assumption for the sequence $a = \{a_R\}$ for rectangles in a fixed direction.

We need the weighted version of Lemma 2.21, which is given under the standing assumptions of Theorem D.

Lemma 3.6. *Let $a = \{a_R : R \in \mathcal{D}^2_S\}$ be an L^∞-normalized w-Carleson sequence, $s \in S \subset [-1,1]$, and $\mathcal{L}, \mathcal{R} \subset \mathcal{D}^2_S$ with $\mathcal{L} \subseteq \mathcal{R}$. For every $\lambda > C[w,u]_S$, where C is a suitably chosen absolute constant, there exists $\mathcal{L}_1 \subset \mathcal{L}$ such that*

(i) $\mathsf{mass}_{a,1}(\mathcal{L}_1) \le \frac{1}{2}\mathsf{mass}_{a,1}(\mathcal{L})$,

(ii) *denoting by $\mathcal{R}'_s$ the collection of rectangles R in $\mathcal{R}_s$ with $B^{\mathcal{L}}_R > \lambda$ we have that*

$$B^{\mathcal{L}}_R \le \lambda + B^{\mathcal{L}_1}_R \quad \textit{for all } R \in \mathcal{R}'_s.$$

Proof. We can assume that $s = 0$ and let $\mathcal{R}_0'$ be the collection of rectangles in $\mathcal{R}_0$ such that $B_R^{\mathcal{L}} > \lambda$, where λ is as in the statement of the lemma and C will be specified at the end of the proof. For $I \in \{\pi_1(R) : R \in \mathcal{R}_0'\}$ and any interval $K \subset \mathbb{R}$ we define $\mathcal{L}_{I,K}^{\text{in}}$ and $\mathcal{L}_{I,K}^{\text{out}}$ as in the proof of Theorem C, but now we set

$$B_{I,K}^{\text{in}} := \fint_{I\times K} \sum_{Q\in\mathcal{L}_{I,K}^{\text{in}}} \frac{a_Q}{u(Q)}\mathbf{1}_Q, \quad B_{I,K}^{\text{out}} := \fint_{I\times K} \sum_{Q\in\mathcal{L}_{I,K}^{\text{out}}} \frac{a_Q}{u(Q)}\mathbf{1}_Q.$$

We define $\mathcal{R}_0''$ to be the subcollection of those $R = I\times L \in \mathcal{R}_0'$ such that $B_{I,L}^{\text{out}} \le \lambda$. By linearity we get for each $R \in \mathcal{R}_0''$ that $B_R^{\mathcal{L}} \le \lambda + B_{I,L}^{\text{in}} \le \lambda + B_R^{\mathcal{L}_1''}$, where

$$\mathcal{L}_1{}'' := \bigcup_{R=I\times L\in\mathcal{R}_0''} \mathcal{L}_{I,L}^{\text{in}}, \quad \mathsf{sh}(\mathcal{L}_1'') \subset \bigcup_{R=I\times L\in\mathcal{R}_0''} I\times 3L.$$

Since $\mathcal{R}_0'' \subset \mathcal{R}_0'$ we conclude as before that

$$\begin{aligned} w(\mathsf{sh}(\mathcal{L}_1'')) &\le w\Big(\bigcup_{R=I\times L\in\mathcal{R}_0''} I\times 3L\Big) \le w\bigg(\bigg\{\mathrm{M}_{\mathcal{R}_0}\Big(\sum_{Q\in\mathcal{L}} \frac{a_Q\mathbf{1}_Q}{u(Q)}\Big) > \frac{\lambda}{3}\bigg\}\bigg) \\ &\lesssim \frac{[w,u]_S}{\lambda}\int_{\mathbb{R}^2}\sum_{Q\in\mathcal{R}} a_Q\frac{\mathbf{1}_Q}{u(Q)}\,\mathrm{d}u = \frac{[w,u]_S}{\lambda}\,\mathsf{mass}_{a,1}(\mathcal{L}) \end{aligned}$$

by the two-weight weak-type-(1, 1) inequality for $\mathrm{M}_{\mathcal{R}_s} = \mathrm{M}_{\mathcal{R}_0}$. Now $\mathcal{L}_1''$ is subordinate to the collection $\{I\times 3L : I\times L \in \mathcal{R}_0''\}$. Using the definition of a Carleson sequence we have

$$\sum_{Q\in\mathcal{L}_1''} a_Q \le w\Big(\bigcup_{R=I\times L\in\mathcal{R}_0''} I\times 3L\Big) \lesssim \frac{[w,u]_S}{\lambda}\,\mathsf{mass}_{a,1}(\mathcal{L}),$$

and so $\mathsf{mass}_{a,1}(\mathcal{L}_1'') \lesssim [w,u]_S\,\mathsf{mass}_{a,1}(\mathcal{L})/\lambda$.

It remains to deal with parallelograms

$$R = I\times L \in \mathcal{R}_0^\star := \mathcal{R}_0' \setminus \mathcal{R}_0'', \quad B_{I,L}^{\text{out}} > \lambda.$$

We define the maximal K_R such that $B_{I,K_R}^{\text{out}} > \lambda$ as before; the existence of this maximal interval can be guaranteed for example by assuming the collection $\mathcal{R}$ is finite. We have for each $R = I\times L \in \mathcal{R}_0^\star$ that $B_{I,L}^{\text{out}} > \lambda$ so $K_R \supset L$ and $B_{I,3K_R}^{\text{out}} \le \lambda$ by maximality.

Now using (2.24) we get that

$$\Delta := \sum_{Q\in\mathcal{L}_{I,3K_R}^{\text{out}}} a_Q\frac{|Q\cap(I\times\{\alpha\})|}{u(Q)|I|} \lesssim \sum_{Q\in\mathcal{L}_{I,3K_R}^{\text{out}}} a_Q\frac{|Q\cap(I\times 3K_R)|}{u(Q)|3K_R||I|} = \fint_{I\times 3K_R}\sum_{Q\in\mathcal{L}_{I,3K_R}^{\text{out}}} a_Q\frac{\mathbf{1}_Q}{u(Q)} \lesssim \lambda$$

by the maximality of K_R. On the other hand

$$\Xi := \sum_{Q\in\mathcal{L}_{I,K}^{\text{out}}\setminus\mathcal{L}_{I,3K}^{\text{out}}} a_Q\fint_{I\times\{\alpha\}}\frac{\mathbf{1}_Q}{u(Q)} \lesssim \sum_{Q\subset I\times 9K} a_Q\frac{|Q\cap(I\times 3K)|}{|I\times 3K|u(Q)}.$$

Since $\mathrm{M}_{\mathcal{R}_s} w \le \mathrm{M}_V \mathrm{M}_2 w \le [w,u]_S\, u$ uniformly in s we get that for $Q \subset I \times 9K$

$$u(Q) \gtrsim [w,u]_S^{-1} \frac{w(I \times 9K)}{|I \times 9K|}|Q|$$

and by this and the w-Carleson property for all Q subordinate to $I \times 9K$ we get

$$\Xi \lesssim [w,u]_S \le \lambda$$

provided $\lambda \ge [w,u]_S$. We now define

$$\mathcal{L}_1{}' := \bigcup_{R \in \mathcal{R}_0^\star} L^{\mathrm{in}}_{\pi_1(R),K_R}$$

so that

$$\mathsf{sh}(\mathcal{L}_1') \subset \bigcup_{R \in \mathcal{R}_0^\star} \pi_1(R) \times K_R.$$

Arguing as in the unweighted case of Theorem C we can estimate

$$w(\mathsf{sh}(\mathcal{L}_1')) \le w\Big(\bigcup_{R \in \mathcal{R}_0^\star} \pi_1(R) \times K_R\Big) \lesssim w(\{\mathrm{M}_2(\mathbf{1}_E) \gtrsim 1\}),$$

where

$$E := \Big\{(x,y) \in \mathbb{R}^2 : \mathrm{M}_v\Big[\sum_{Q \in \mathcal{L}} a_Q \frac{\mathbf{1}_Q}{u(Q)}\Big](x,y) \ge \frac{\lambda}{2}\Big\}.$$

In the definition of E above we have that $\mathrm{M}_v = \mathrm{M}_{(1,s)} = \mathrm{M}_1$ since we have reduced to the case $v = (1,s) = (1,0)$. Using the subordination property of $\mathcal{L}_1'$ and the Fefferman–Stein inequality once in the direction e_2 for M_2 and once in the direction $v = (1,s) = (1,0)$ for M_v we estimate

$$\mathsf{mass}_{a,1}(\mathcal{L}_1') \le w\Big(\bigcup_{R \in \mathcal{R}_0^\star} \pi_1(R) \times K_R\Big) \lesssim \frac{1}{\lambda} \sum_{Q \in \mathcal{L}} a_Q \frac{\mathrm{M}_V \mathrm{M}_2 w(Q)}{u(Q)} \le \frac{[w,u]_S}{\lambda}\, \mathsf{mass}_{a,1}(\mathcal{L}).$$

We have thus proved the lemma upon setting $\mathcal{L}_1 := \mathcal{L}_1'' \cup \mathcal{L}_1'$ and choosing $\lambda \ge C[w,u]_S$ for a sufficiently large numerical constant $C > 1$. □

Repeating the steps in the proof of Lemma 2.22 for λ as in the statement of Lemma 3.6 we get for the sets $\mathcal{R}_{s,k}$ defined with respect to this λ that

$$w(\mathsf{sh}(\mathcal{R}_{s,k})) \lesssim 2^{-k}\, \mathsf{mass}_{a,1}(\mathcal{R}),$$

and this completes the proof of Theorem D.

3.7. ***Applications of Theorem D.*** The first corollary of Theorem D is a two-weighted estimate for the directional maximal operator M_V from (2.4).

Theorem E. *Let $V \subset \mathbb{S}^1$ be a finite set of N slopes and w be a weight on $\mathbb{R}^2$. Then*

$$\|\mathrm{M}_V : L^2(\widetilde{\mathrm{M}}_V w) \to L^{2,\infty}(w)\| \lesssim \sqrt{\log N}, \quad \widetilde{\mathrm{M}}_V := \mathrm{M}_V \circ \mathrm{M}_V \circ \max\{\mathrm{M}_{(1,0)}, \mathrm{M}_{(0,1)}\}.$$

Remark 3.8. In the proof below, we argue for almost horizontal V, and in place of $\max\{\mathrm{M}_{(1,0)}, \mathrm{M}_{(0,1)}\}$ we use $\mathrm{M}_{(0,1)}$. The usage of $\max\{\mathrm{M}_{(1,0)}, \mathrm{M}_{(0,1)}\}$ enables the statement of the theorem to be invariant under rotation of V.

Proof of Theorem E. By standard limiting arguments, it suffices to prove that for each $k \in \mathbb{Z}$ the estimate

$$\|\mathrm{M}_{\mathcal{R}} : L^2(z) \to L^{2,\infty}(w)\| \lesssim \sqrt{\log N}, \quad z := \mathrm{M}_{\mathcal{R}} \circ \mathrm{M}_V \circ \mathrm{M}_{(0,1)} w, \tag{3.9}$$

when $\mathcal{R}$ is a one-parameter collection as in (3.1), holds uniformly in k.

For a nonnegative function $f \in \mathcal{S}(\mathbb{R}^2)$ let Uf be a linearization of $\mathrm{M}_{\mathcal{R}} f$, namely

$$\mathrm{M}_{\mathcal{R}} f(x) = Uf(x) = \frac{1}{|R(x)|} \int_{R(x)} f(y)\,\mathrm{d}y = \sum_{R \in \mathcal{R}} \langle f \rangle_R \mathbf{1}_{F_R}(x), \quad F_R := \{x \in R : R(x) = R\}.$$

By duality, (3.9) turns into

$$\|U^*(w\mathbf{1}_E)\|_{L^2(z^{-1})} \lesssim \sqrt{\log N}\sqrt{w(E)} \quad \text{for all } E \subset \mathbb{R}^2. \tag{3.10}$$

We can easily calculate

$$U^*(w\mathbf{1}_E) = \sum_{R \in \mathcal{R}} w(E \cap F_R) \frac{\mathbf{1}_R}{|R|}$$

and it is routine to check that $\{w(E \cap F_R)\}_{R \in \mathcal{R}}$ is a w-Carleson sequence according to Definition 3.4. The main point here is that the sets $\{E \cap F_R\}_{R \in \mathcal{R}}$ are by definition pairwise disjoint and $F_R \subseteq R$ for each $R \in \mathcal{R}$.

Setting $u := \mathrm{M}_V \circ \mathrm{M}_{(0,1)} w$, if S are the slopes of V, it is clear that $[w, u]_S \lesssim 1$ and that $z^{-1} = (\mathrm{M}_{\mathcal{R}} u)^{-1}$. Therefore (3.10) follows from an application of Theorem D. □

We may in turn use Theorem E to establish a weighted norm inequality for maximal directional singular integrals with controlled dependence on the cardinality $\#V = N$. Similar considerations may be used to yield weighted bounds for directional singular integrals in $L^p(\mathbb{R}^2)$ for $p > 2$; we do not pursue this issue.

Theorem F. *Let K be a standard Calderón–Zygmund convolution kernel on $\mathbb{R}$ and $V \subset \mathbb{S}^1$ be a finite set of N slopes. For $v \in V$ we define*

$$T_v f(x) = \sup_{\varepsilon > 0} \left| \int_{\varepsilon < t < \frac{1}{\varepsilon}} f(x + tv) K(t)\,\mathrm{d}t \right|, \quad T_V f(x) = \sup_{v \in V} |T_v f(x)|.$$

Let w be a weight on $\mathbb{R}^2$ with $[w]_{A_1^V} := \|\mathrm{M}_V w / w\|_\infty < \infty$. Then

$$\|T_V : L^2(w) \to L^{2,\infty}(w)\| \lesssim (\log N)^{\frac{3}{2}} [w]_{A_1^V}^{\frac{5}{2}}.$$

We sketch the proof, which is a weighted modification of the arguments for [Demeter and Di Plinio 2014, Theorem 1]. Hunt's classical exponential good-λ inequality, see [Demeter and Di Plinio 2014, Proposition 2.2] for a proof, may be upgraded to

$$w(\{x \in \mathbb{R}^2 : T_v f(x) > 2\lambda, \mathrm{M}_v f(x) \le \gamma\lambda\}) \lesssim \exp\left(-\frac{c}{\gamma [w]_{A_1^V}}\right) w(\{x \in \mathbb{R}^2 : T_v f(x) > \lambda\}) \tag{3.11}$$

by using that $[w]_{A_1^V}$ dominates the A_∞ constant of the one-dimensional weight $t \mapsto w(x+tv)$ for all $x \in \mathbb{R}^2$, $v \in V$, together with Fubini's theorem. With (3.11) in hand, Theorem F follows from Theorem E via standard good-λ inequalities, selecting $(\gamma)^{-1} \sim [w]_{A_1^V} \log N$. Note that the right-hand side of the estimate in the conclusion of Theorem E becomes $[w]_{A_1}^{3/2} \sqrt{\log N}$ when the estimate is specified to A_1^V weights as the ones we consider here.

4. Tiles, adapted families, and intrinsic square functions

We define here some general notions of tiles and adapted families of wave-packets: definitions in this spirit have appeared in, among others [Barrionuevo and Lacey 2003; Demeter and Di Plinio 2014; Lacey and Li 2006; 2010; Lacey 2007]. These will be essential for the time-frequency analysis square functions we use in this paper in order to model the main operators of interest. After presenting these abstract definitions we show some general orthogonality estimates for wave packet coefficients. We then detail how these notions are specialized in three particular cases of interest.

4.1. ***Tiles and wavelet coefficients.*** Throughout this section we fix a finite set of slopes $S \subset [-1, 1]$. Remember that alternatively we will refer to the set of vectors $V := \{(1, s) : s \in S\}$. A *tile* is a set $t := R_t \times \Omega_t \subset \mathbb{R}^2 \times \mathbb{R}^2$, where $R_t \in \mathcal{D}_S^2$ and $\Omega_t \subset \mathbb{R}^2$ is a measurable set, and $|R_t||\Omega_t| \gtrsim 1$. We denote by $s(t) \in S$ the slope such that $R_t \in \mathcal{D}_{s(t)}^2$, and then

$$R_t = A_{s(t)}(I_t \times J_t), \quad \text{with } I_t \times J_t \in \mathcal{D}_0^2.$$

We also use the notation $v_t := (1, s(t))$. There are several different collections of tiles used in this paper, they will generically be denoted by $\boldsymbol{T}, \boldsymbol{T}_1, \boldsymbol{T}'$ or similar. Given any collection of tiles $\boldsymbol{T}$ we will often use the notation $\mathcal{R}_{\boldsymbol{T}} := \{R_t : t \in \boldsymbol{T}\}$ to denote the collection of spatial components of the tiles in $\boldsymbol{T}$. The exact geometry of these tiles will be clear from context; however, several estimates hold for generic collections of tiles, as we will see in Section 4.3.

Let $t = R_t \times \Omega_t$ be a tile and $M \geq 2$. We denote by $\mathcal{A}_t^M$ the collection of Schwartz functions ϕ on $\mathbb{R}^2$ such that

(i) $\operatorname{supp}(\hat{\phi}) \subset \Omega_t$,

(ii) there holds

$$\sup_{0 \leq \alpha, \beta \leq M} \sup_{x \in \mathbb{R}^2} |R_t|^{\frac{1}{2}} |I|^\alpha |J|^\beta \left(1 + \frac{|x \cdot v_t|}{|I||v_t|}\right)^M \left(1 + \frac{|x \cdot e_2|}{|J|}\right)^M |\partial_{v_t}^\alpha \partial_{e_2}^\beta \phi(x + c_{R_t})| \leq 1.$$

In the above display c_{R_t} refers to the center of R_t and

$$\partial_{v_t}(\cdot) := \frac{v_t}{|v_t|} \cdot \nabla(\cdot).$$

An immediate consequence of property (ii) is the normalization

$$\sup_{\phi \in \mathcal{A}_t^M} \|\phi\|_2 \lesssim 1.$$

We thus refer to $\mathcal{A}_t^M$ as the collection of L^2*-normalized wave packets adapted to* t *of order* M. For our purposes, it will suffice to work with moderate values of M, say $2^3 \leq M \leq 2^{50}$. In fact, we use

$M = M_0 = 2^{50}$ in the definition of the *intrinsic wavelet coefficient* associated with the tile t and the Schwartz function f:

$$a_t(f) := \sup_{\phi \in \mathcal{A}_t^{M_0}} |\langle f, \phi\rangle|^2, \quad M_0 = 2^{50}. \tag{4.2}$$

This section is dedicated to square functions involving wavelet coefficients associated with particular collections of tiles which formally look like

$$\Delta_{\boldsymbol{T}}(f)^2 := \sum_{t \in \boldsymbol{T}} a_t(f) \frac{\mathbf{1}_{R_t}}{|R_t|}, \quad \boldsymbol{T} \text{ is a collection of tiles.}$$

We begin by proving some general global and local orthogonality estimates for collections of tiles with finitely overlapping frequency components. These estimates will be essential in showing that the sequence $\{a_t(f)\}_{t\in\boldsymbol{T}}$ is Carleson in the sense of Section 2, when $|f| \le \mathbf{1}_E$ for some measurable set $E \subset \mathbb{R}^2$ with $0 < |E| < \infty$. This in turn will allow us to use the directional Carleson embedding of Theorem C in order to conclude corresponding estimates for intrinsic square functions defined on collections of tiles.

4.3. ***Orthogonality estimates for collections of tiles.*** We begin with an easy orthogonality estimate for wave packet coefficients. For completeness we present a sketch of proof which has a TT^* flavor. The argument follows the lines of proof of [Lacey 2007, Proposition 3.3].

Lemma 4.4. *Let $\boldsymbol{T}$ be a set of tiles such that $\sum_{t\in\boldsymbol{T}} \mathbf{1}_{\Omega_t} \lesssim 1$, let $M \ge 2^3$ and $\{\phi_t : t \in \boldsymbol{T}\}$ be such that $\phi_t \in \mathcal{A}_t^M$ for all $t \in \boldsymbol{T}$. We have the estimate*

$$\sum_{t\in\boldsymbol{T}} |\langle f, \phi_t\rangle|^2 \lesssim \|f\|_2^2, \tag{4.5}$$

and as a consequence

$$\sum_{t\in\boldsymbol{T}} a_t(f) \lesssim \|f\|_2^2.$$

Proof. Fix $M \ge 2^3$. It suffices to prove that for $\|f\|_2 = 1$ and an arbitrary adapted family of wave packets $\{\phi_t : \phi_t \in \mathcal{A}_t^M,\ t \in \boldsymbol{T}\}$ there holds

$$B := \sum_{t\in\boldsymbol{T}} |\langle f, \phi_t\rangle|^2 \lesssim 1. \tag{4.6}$$

Let us first fix some $\Omega \in \Omega(\boldsymbol{T}) := \{\Omega_t : t \in \boldsymbol{T}\}$ and consider the family

$$\boldsymbol{T}(\{\Omega\}) := \{t \in \boldsymbol{T} : \Omega_t = \Omega\}.$$

To prove (4.6), we introduce

$$B_\Omega(g) := \sum_{t \in \boldsymbol{T}(\{\Omega\})} |\langle g, \phi_t\rangle|^2, \quad S_\Omega(g) := (\hat{g}\mathbf{1}_\Omega)^\vee.$$

We claim that $B_\Omega(g) \lesssim \|g\|_2^2$ for all g, uniformly in $\Omega \in \Omega(\boldsymbol{T})$. Assuming the claim for a moment and remembering the finite overlap assumption on the frequency components of the tiles we have

$$B = \sum_{\Omega\in\Omega(\boldsymbol{T})} B_\Omega(S_\Omega f) \lesssim \sum_{\Omega\in\Omega(\boldsymbol{T})} \|S_\Omega(f)\|_2^2 \le \Big\| \sum_{\Omega\in\Omega(\boldsymbol{T})} \mathbf{1}_\Omega \Big\|_\infty^2 \|f\|_2^2 \lesssim 1$$

as desired. It thus suffices to prove the claim. To this end let

$$P_\Omega(g) := \sum_{t \in \boldsymbol{T}(\{\Omega\})} \langle g, \phi_t \rangle \phi_t.$$

Then for any g with $\|g\|_2 = 1$ we have that $B_\Omega(g) = \langle P_\Omega(g), g\rangle \leq \|P_\Omega(g)\|_2$ and it suffices to prove that $\|P_\Omega(g)\|_2^2 \lesssim B_\Omega(g)$. A direct computation reveals that

$$\|P_\Omega(g)\|_2^2 \leq B_\Omega(g) \sup_{t' \in \boldsymbol{T}(\{\Omega\})} \sum_{t \in \boldsymbol{T}(\{\Omega\})} |\langle \phi_t, \phi_{t'} \rangle| \lesssim B,$$

where the second inequality in the last display above follows by the polynomial decay of the wave packets $\{\phi_t : \Omega_t = \Omega\}$. This completes the proof of the lemma. □

We present below a localized orthogonality statement which is needed in order to verify that the coefficients $a_t(f)$ form a Carleson sequence in the sense of Section 2. Verifying this Carleson condition relies on a variation of Journé's lemma that can be found in [Cabrelli et al. 2006, Lemma 3.23]; we rephrase it here adjusted to our notation. In the statement of the lemma below we denote by $\mathrm{M}_{\mathcal{P}_s^2}$ the *maximal operator* corresponding to the collection $\mathcal{P}_s^2$, where $s \in S$ is a fixed slope. Note that the proof in [Cabrelli et al. 2006] corresponds to the case of slope $s = 0$ but the general case $s \in S$ follows easily by a change of variables. Remember here that we have $S \subset [-1, 1]$.

In the statement of the lemma below two parallelograms are called *incomparable* if none of them is contained in the other.

Lemma 4.7. *Let $s \in S$ be a slope and $\mathcal{T} \subset \mathcal{D}_s^2$ be a collection of pairwise incomparable parallelograms. Define*

$$\mathsf{sh}^\star(\mathcal{T}) := \{\mathrm{M}_{\mathcal{P}_s^2} \mathbf{1}_{\mathsf{sh}(\mathcal{T})} > 2^{-6}\}$$

and for each $R \in \mathcal{T}$ let u_R be the least integer u such that $2^u R \not\subset \mathsf{sh}^\star(\mathcal{T})$. Then

$$\sum_{\substack{R \in \mathcal{T} \\ u_R = u}} |R| \lesssim 2^u |\mathsf{sh}(\mathcal{T})|.$$

With the suitable analogue of Journé's lemma in hand we are ready to state and prove the localized orthogonality condition for the coefficients $a_t(f)$.

Lemma 4.8. *Let $s \in S$ be a slope, $\mathcal{T} \subset \mathcal{P}_s^2$ be a given collection of parallelograms and $\boldsymbol{T}$ be a collection of tiles such that*

$$\mathcal{R}_{\boldsymbol{T}} := \{R_t : t \in \boldsymbol{T}\}$$

is subordinate to $\mathcal{T}$. Then we have

$$\sum_{t \in \boldsymbol{T}} a_t(f) \lesssim |\mathsf{sh}(\mathcal{T})| \|f\|_\infty^2.$$

Proof. We first make a standard reduction that allows us to pass to a collection of dyadic rectangles. To do this we use that there exist at most 9^2 shifted dyadic grids $\mathcal{D}_{s,j}^2$ such that for each parallelogram $T \in \mathcal{T}$ there exists $\widetilde{T} \in \bigcup_j \mathcal{D}_{s,j}^2$ with $T \subset \widetilde{T}$ and $|T| \leq |\widetilde{T}| \lesssim |T|$; see for example [Hytönen et al. 2013]. Now note that for each $\widetilde{T} \in \widetilde{\mathcal{T}}$ we have

$$\frac{|T \cap \widetilde{T}|}{|\widetilde{T}|} \gtrsim 1, \quad \mathsf{sh}(\widetilde{\mathcal{T}}) \subset \{\mathrm{M}_{\mathcal{P}_S^2}(\mathbf{1}_{\mathsf{sh}(\mathcal{T})}) \gtrsim 1\}$$

and so $|\mathsf{sh}(\widetilde{\mathcal{T}})| \lesssim |\mathsf{sh}(\mathcal{T})|$. Now it is clear that we can replace $\mathcal{T}$ with the dyadic collection $\widetilde{\mathcal{T}}$ in the assumption. Furthermore there is no loss in generality with assuming that $\mathcal{T}$ is a pairwise incomparable collection. We do so in the rest of the proof and continue using the notation $\mathcal{T}$ assuming it is a dyadic collection.

Since $\mathcal{R}_{\boldsymbol{T}}$ is subordinate to $\mathcal{T}$ we have the decomposition

$$\boldsymbol{T} = \bigcup_{T \in \mathcal{T}} \boldsymbol{T}(T), \quad \boldsymbol{T}(T) := \{t \in \boldsymbol{T} : R_t \subset T\}.$$

Now if f is supported on $\mathsf{sh}^\star(\mathcal{T})$ and $\phi_t \in \mathcal{A}_t^{M_0}$ for each $t \in \boldsymbol{T}$ then

$$\sum_{t \in \boldsymbol{T}} |\langle f, \phi_t\rangle|^2 \lesssim \|f\|_2^2 \le |\mathsf{sh}^\star(\mathcal{T})| \|f\|_\infty^2 \lesssim |\mathsf{sh}(\mathcal{T})| \|f\|_\infty^2$$

by Lemma 4.4. We may thus assume that f is supported outside $\mathsf{sh}^\star(\mathcal{T})$. By Lemma 4.7 it then suffices to prove that

$$\sum_{t \in \boldsymbol{T}(T)} |\langle f, \phi_t\rangle|^2 \lesssim 2^{-10u} |T|$$

whenever u is the least integer such that $2^u T \not\subset \mathsf{sh}^\star(\mathcal{T})$ and $\|f\|_\infty = 1$. As f is supported off $\mathsf{sh}^\star(\mathcal{T})$ we have for this choice of u that

$$f = \sum_{n \ge 0} f_n, \quad f_n := f \mathbf{1}_{2^{u+n}T \setminus 2^{u+n-1}T}.$$

Let z_T be the center of T and suppose that $T = A_s(I_T \times J_T)$, with $I_T \times J_T \in \mathcal{D}_0^2$; remember that we write $v_s := (1, s)$. Let

$$\chi_T(x) := \left(1 + \frac{(x - z_T) \cdot v_s}{|I_T||v_s|}\right)^{-20} (1 + |J_T|^{-1}(x - z_T) \cdot e_2)^{-20}.$$

Observe preliminarily that

$$\|f_n \chi_T\|_\infty \lesssim 2^{-20(u+n)}$$

so that for any constant $c > 0$ we have

$$\begin{aligned} \Big(\sum_{t \in \boldsymbol{T}(T)} |\langle f, \phi_t\rangle|^2\Big)^{\frac{1}{2}} &\le \sum_{n \ge 0} \Big(\sum_{t \in \boldsymbol{T}(T)} |\langle f_n, \phi_t\rangle|^2\Big)^{\frac{1}{2}} = \sum_{n \ge 0} \Big(\sum_{t \in \boldsymbol{T}(T)} |\langle f_n c^{-1} \chi_T, c\chi_T^{-1}\phi_t\rangle|^2\Big)^{\frac{1}{2}} \\ &\lesssim \sum_{n \ge 0} \|f_n \chi_T\|_2 \lesssim \sum_{n \ge 0} \|f_n \chi_T\|_\infty |2^{u+n} T|^{\frac{1}{2}} \lesssim 2^{-5u} |T|^{\frac{1}{2}} \end{aligned}$$

as claimed. To pass to the second line we have used estimate (4.5) of Lemma 4.4 together with the easily verifiable fact that for each $t \in \boldsymbol{T}(T)$ the wave-packet $c\chi_T^{-1}\phi_t$ is adapted to t with order $M_0 - 20 \ge 2^3$ provided the absolute constant c is chosen small enough. ☐

4.9. ***The intrinsic square function associated with rough frequency cones.*** Let $s \in S$ be our finite set of slopes. As usual we write $v_s := (1, s)$ for $s \in S$ and $V := \{v_s : s \in S\}$ and switch between the description of directions as slopes or vectors as desired with no particular mention. Now assume we are given a finitely overlapping collection of arcs $\{\omega_s\}_{s \in S}$ with each $\omega_s \subset \mathbb{S}^1$ centered at $(v_s/|v_s|)^\perp$. We will adopt the notation

$$\omega_s := \left(\left(\frac{v_{s^-}}{|v_{s^-}|} \right)^\perp, \left(\frac{v_{s^+}}{|v_{s^+}|} \right)^\perp \right)$$

assuming that the positive direction on the circle is counterclockwise and $s^- < s < s^+$.

For $s \in S$ we define the conical sectors

$$\Omega_{s,k} := \left\{ \xi \in \mathbb{R}^2 : 2^{k-1} < |\xi| < 2^{k+1},\ \frac{\xi}{|\xi|} \in \omega_s \right\}, \quad k \in \mathbb{Z}; \tag{4.10}$$

these are an overlapping cover of the cone

$$C_s := \left\{ \xi \in \mathbb{R}^2 \setminus \{0\} : \frac{\xi}{|\xi|} \in \omega_s \right\},$$

with $k \in \mathbb{Z}$ playing the role of the annular parameter. Each sector $\Omega_{s,k}$ is strictly contained in the cone C_s.

For each $s \in S$ let $\ell_s \in \mathbb{Z}$ be chosen such that $2^{-\ell_s} < |\omega_s| \le 2^{-\ell_s+1}$. We perform a further discretization of each conical sector $\Omega_{s,k}$ by considering Whitney-type decompositions with respect to the distance to the lines determined by the boundary rays r_{s^-} and r_{s^+}; here r_{s^+} denotes the ray emanating from the origin in the direction of $v_{s^+}^\perp$ and similarly for r_{s^-}. For each sector $\Omega_{s,k}$ a central piece which we call $\Omega_{s,k,0}$ is left uncovered by these Whitney decompositions. This is merely a technical issue and we will treat these central pieces separately in what follows.

To make this precise let s, k be fixed and define the regions

$$\begin{aligned} \Omega_{s,k,m} &:= \left\{ \xi \in \Omega_{s,k} : \frac{1}{3} 2^{-|m|-1} \le \frac{\operatorname{dist}(\xi, r_{s^+})}{|\omega_s|} \le \frac{1}{3} 2^{-|m|+1} \right\}, \quad m > 0, \\ \Omega_{s,k,m} &:= \left\{ \xi \in \Omega_{s,k} : \frac{1}{3} 2^{-|m|-1} \le \frac{\operatorname{dist}(\xi, r_{s^-})}{|\omega_s|} \le \frac{1}{3} 2^{-|m|+1} \right\}, \quad m < 0. \end{aligned} \tag{4.11}$$

The central part that was left uncovered corresponds to $m = 0$ and is described as

$$\Omega_{s,k,0} := \left\{ \xi \in \Omega_{s,k} : \min(\operatorname{dist}(\xi, r_{s^-}), \operatorname{dist}(\xi, r_{s^+})) \ge \frac{1}{2} \frac{1}{3} |\omega_s| \right\}. \tag{4.12}$$

Notice that the collection $\{\Omega_{s,k,m}\}_{m \in \mathbb{N}}$ is a finitely overlapping cover of $\Omega_{s,k}$. Furthermore the family $\{\Omega_{s,k,m}\}_{s,k,m}$ has finite overlap as the cones $\{C_s\}_{s \in S}$ have finite overlap and for fixed s the family $\{\Omega_{s,k,m}\}_{k,m}$ is Whitney both in k and m.

These geometric considerations are depicted in Figure 4.

The collection of tiles $\boldsymbol{T}$ corresponding to this decomposition is obtained as

$$\boldsymbol{T} := \bigcup_{s \in S} \boldsymbol{T}_s^- \cup \boldsymbol{T}_s^0 \cup \boldsymbol{T}_s^+ \tag{4.13}$$

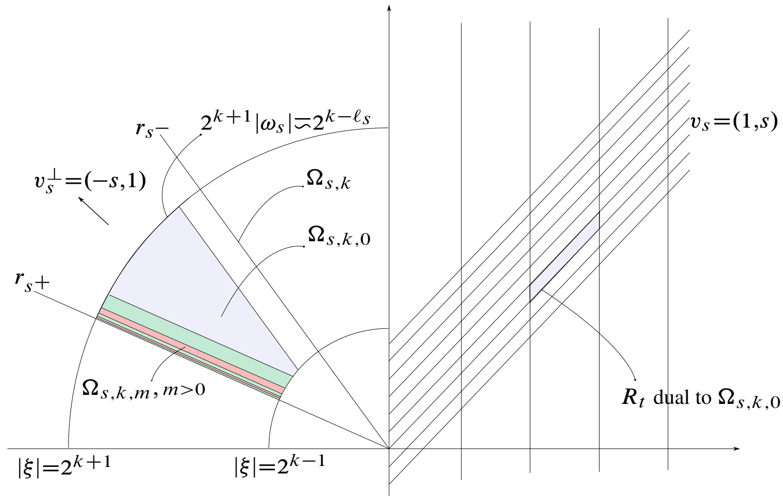

Figure 4. The decomposition of the sector $\Omega_{s,k}$ into Whitney regions, and the spatial grid corresponding to the middle region $\Omega_{s,k,0}$.

where

$$
\begin{aligned}
\boldsymbol{T}_s^- &:= \bigcup_{k\in\mathbb{Z},\, m<0} \boldsymbol{T}_{s^-,k,m}, \quad \boldsymbol{T}_{s^-,\,k,m} := \{t = R_t \times \Omega_{s,k,m} : R_t \in \mathcal{D}_{s^-,k,k-\ell_s+|m|}\}, \quad m<0,\\
\boldsymbol{T}_s^0 &:= \bigcup_{k\in\mathbb{Z}} \boldsymbol{T}_{s,k,0}, \qquad\quad \boldsymbol{T}_{s,k,0} := \{t = R_t \times \Omega_{s,k,0} : R_t \in \mathcal{D}_{s,k,k-\ell_s}\},\\
\boldsymbol{T}_s^+ &:= \bigcup_{k\in\mathbb{Z},\, m>0} \boldsymbol{T}_{s^+,k,m}, \quad \boldsymbol{T}_{s^+,\,k,m} := \{t = R_t \times \Omega_{s,k,m} : R_t \in \mathcal{D}_{s^+,k,k-\ell_s+|m|}\}, \quad m>0.
\end{aligned}
\tag{4.14}
$$

We stress here that for each cone C_s we introduce tiles in three possible directions v_{s^-}, v_s, v_{s^+}. This turns out to be a technical nuisance more than anything else as the total number of directions is still comparable to $\#S$, and our estimates will be uniform over all S with the same cardinality. However in order to avoid confusion we set

$$
S^* := S \cup \{s^- : s \in S\} \cup \{s^+ : s \in S\} =: S^- \cup S \cup S^+. \tag{4.15}
$$

Note also that for fixed s, k, m the choice of scales for R_t yields that the tile $t = R_t \times \Omega_{s,k,m}$ obeys the uncertainty principle in both radial and tangential directions.

We then define the associated intrinsic square function by

$$
\Delta_{\boldsymbol{T}}(f) := \Big(\sum_{t\in\boldsymbol{T}} a_t(f) \frac{\mathbf{1}_{R_t}}{|R_t|}\Big)^{\frac12}, \tag{4.16}
$$

where the set of slopes S are kept implicit in the notation. Here we remember the notation $a_t(f)$ that was introduced in (4.2). Using the orthogonality estimates of Section 4.3 as input for Theorem C, we readily obtain the estimates of the following theorem.

Theorem G. *We have the estimates*

$$\|\Delta_{\boldsymbol{T}} : L^p(\mathbb{R}^2)\| \lesssim_p (\log \#S)^{\frac{1}{2}-\frac{1}{p}} (\log\log \#S)^{\frac{1}{2}-\frac{1}{p}}, \quad 2 \le p < 4, \tag{4.17}$$

$$\sup_{E,f} \frac{\|\Delta_{\boldsymbol{T}}(f\mathbf{1}_E)\|_4}{|E|^{\frac{1}{4}}} \lesssim (\log \#S)^{\frac{1}{4}} (\log\log \#S)^{\frac{1}{4}}, \tag{4.18}$$

where the supremum in the last display is taken over all measurable sets $E \subset \mathbb{R}^2$ of finite positive measure and all Schwartz functions f on $\mathbb{R}^2$ with $\|f\|_\infty \le 1$.

Proof. First of all, observe that the case $p = 2$ of (4.17) is exactly the conclusion of Lemma 4.4. By restricted weak-type interpolation it thus suffices to prove (4.18) to obtain the remaining cases of (4.17); we turn to the former task.

For convenience define $S^* := S \cup \{s^- : s \in S\} \cup \{s^+ : s \in S\} =: S^- \cup S \cup S^+$; note that this is the actual set of slopes of tiles in $\boldsymbol{T}$. Let

$$\mathcal{R}_{\boldsymbol{T}} := \{R_t : t \in \boldsymbol{T}\} \subset \mathcal{D}^2_{S^*}.$$

Observe that we can write

$$\Delta_{\boldsymbol{T}}(f\mathbf{1}_E)^2 = \sum_{R\in\mathcal{R}_{\boldsymbol{T}}} \bigg(\sum_{t\in\boldsymbol{T}:R_t=R} a_t(f\mathbf{1}_E) \bigg) \frac{\mathbf{1}_R}{|R|} =: \sum_{R\in\mathcal{R}_{\boldsymbol{T}}} a_R \frac{\mathbf{1}_R}{|R|},$$

where

$$a := \bigg\{ a_R = \sum_{t\in\boldsymbol{T}:R_t=R} a_t(f\mathbf{1}_E) : R \in \mathcal{R}_{\boldsymbol{T}} \bigg\}.$$

We fix E and f as in the statement and we will obtain (4.18) from an application of Theorem C to the Carleson sequence $a = \{a_R\}_{R\in\mathcal{R}_{\boldsymbol{T}}}$.

First, $\mathsf{mass}_a \lesssim |E|$ as a consequence of Lemma 4.4 since

$$\sum_{R\in\mathcal{R}_{\boldsymbol{T}}} a_R = \sum_{R\in\mathcal{R}_{\boldsymbol{T}}} \sum_{t\in\boldsymbol{T}:R_t=R} a_t(f\mathbf{1}_E) = \sum_{t\in\boldsymbol{T}} a_t(f\mathbf{1}_E) \lesssim \|f\mathbf{1}_E\|_2^2 \lesssim |E|.$$

Further, the fact that a is (a constant multiple of) an L^∞-normalized Carleson sequence is a consequence of the localized estimate of Lemma 4.8. To verify this we need to check the validity of Definition 2.7 for the sequence a above. To that end let $\mathcal{L} \subset \mathcal{D}^2_{S^*}$ be a collection of parallelograms which is subordinate to $\mathcal{T} \subset \mathcal{D}^2_\sigma$ for some fixed $\sigma \in S^*$. Then

$$\sum_{R\in\mathcal{L}} a_R = \sum_{R\in\mathcal{L}} \sum_{t\in\boldsymbol{T}:R_t=R} a_t(f\mathbf{1}_E) = \sum_{t\in\boldsymbol{T}_{\mathcal{L}}} a_t(f\mathbf{1}_E),$$

where $\boldsymbol{T}_{\mathcal{L}} := \{t \in \boldsymbol{T} : R_t \in \mathcal{L}\}$. By Lemma 4.8 the right-hand side of the display above can be estimated by a constant multiple of $|\mathsf{sh}(\mathcal{T})|\|f\mathbf{1}_E\|_\infty^2 \le |\mathsf{sh}(\mathcal{T})|$. This shows the desired property in the definition of a Carleson sequence.

Finally if $\boldsymbol{T}_\sigma := \{t \in \boldsymbol{T} : s(t) = \sigma\}$ for $\sigma \in S^*$, we have that

$$\sup_{\sigma\in S^*} \|\mathrm{M}_{\mathcal{R}_{\boldsymbol{T}_\sigma}} : L^p(\mathbb{R}^2) \to L^{p,\infty}(\mathbb{R}^2)\| \lesssim p', \quad p \to 1^+.$$

Indeed note that for fixed direction $\sigma \in S^*$ each maximal operator appearing in the estimate above is bounded by the strong maximal operator in the coordinates (v, e_2) with $v = (1, \sigma)$.

Now Theorem C applies to the Carleson sequence $a = \{a_R\}_{R \in \mathcal{R}_{\boldsymbol{T}}}$ yielding

$$\|\Delta_{\boldsymbol{T}}(f\mathbf{1}_E)\|_4^4 = \|T_{R_{\boldsymbol{T}}}(a)\|_2^2 \lesssim (\log \#S^*)(\log\log \#S^*)\,\mathsf{mass}_a \lesssim (\log \#S)(\log\log \#S)|E|,$$

which is the claimed estimate (4.18) as $\#S^* \simeq \#S$. The proof of Theorem G is thus complete. □

4.19. ***The intrinsic square function associated with smooth frequency cones.*** The tiles in the previous subsection were used to model rough frequency projections on a collection of essentially disjoint cones. Indeed note that all decompositions were of Whitney type with respect to all the singular sets of the corresponding rough multiplier. In the case of smooth frequency projections on cones we need a simplified collection of tiles that we briefly describe below.

Assuming S is a finite set of slopes and the arcs $\{\omega_s\}_{s\in S}$ on $\mathbb{S}^1$ have finite overlap as before we now define for $s \in S$ and $k \in \mathbb{Z}$ the collections

$$\boldsymbol{T}_{s,k} := \{t = R_t \times \Omega_{s,k} : R_t \in \mathcal{D}_{s,k-\ell_s,k}\}, \quad \boldsymbol{T}_s := \bigcup_{k\in\mathbb{Z}} \boldsymbol{T}_{s,k}, \quad \boldsymbol{T} := \bigcup_{s\in S} \boldsymbol{T}_s, \tag{4.20}$$

with $\Omega_{s,k}$ given by (4.10). Here we also assume that $2^{-\ell_s} \le |\omega_s| \le 2^{-\ell_s+1}$. Notice that each conical sector $\Omega_{s,k}$ now generates exactly one frequency component of possible tiles in contrast with the previous subsection where we need a whole Whitney collection for every s and every k; in fact the tiles $\boldsymbol{T}_{s,k}$ are for all practical purposes the same as the tiles $\boldsymbol{T}_{s,k,0}$ considered in Section 4.9. It is of some importance to note here that for each fixed $s \in S$ the collection $\mathcal{R}_{\boldsymbol{T}} := \{R_t : t \in \boldsymbol{T}\}$ consists of parallelograms of fixed eccentricity 2^{ℓ_s} and thus the corresponding maximal operator $\mathrm{M}_{\mathcal{R}_{\boldsymbol{T}_s}}$ is of weak-type-$(1,1)$ uniformly in $s \in S$:

$$\sup_{s\in S} \|\mathrm{M}_{\mathcal{R}_{\boldsymbol{T}_s}} : L^1(\mathbb{R}^2) \to L^{1,\infty}(\mathbb{R}^2)\| \lesssim 1.$$

The intrinsic square function $\Delta_{\boldsymbol{T}}$ is formally given as in (4.16) but defined with respect to the new collection of tiles defined in (4.20). A repetition of the arguments that led to the proof of Theorem G yields the following.

Theorem H. *For* $\boldsymbol{T}$ *defined by* (4.20) *we have the estimates*

$$\|\Delta_{\boldsymbol{T}} : L^p(\mathbb{R}^2)\| \lesssim_p (\log \#S)^{\frac12 - \frac1p}, \quad 2 \le p < 4,$$
$$\sup_{E,f} \frac{\|\Delta_{\boldsymbol{T}}(f\mathbf{1}_E)\|_4}{|E|^{\frac14}} \lesssim (\log \#S)^{\frac14},$$

where the supremum in the last display is taken over all measurable sets $E \subset \mathbb{R}^2$ *of finite positive measure and all Schwartz functions* f *on* $\mathbb{R}^2$ *with* $\|f\|_\infty \le 1$.

4.21. ***The intrinsic square function associated with rough frequency rectangles.*** The considerations in this subsection aim at providing the appropriate time-frequency analysis in order to deal with a Rubio-de-Francia-type square function, given by frequency projections on disjoint rectangles in finitely many directions. The intrinsic setup is described by considering again a finite set of slopes S and corresponding directions V. Suppose that we are given a finitely overlapping collection of rectangles $\mathcal{F} = \bigcup_{s\in S} \mathcal{F}_s$, consisting of rectangles which are tensor products of intervals in the coordinates $v, v^\perp,\ v = (1, s)$, for some $s \in S$. Namely a rectangle $F \in \mathcal{F}_s$ is a *rotation* by s of an axis-parallel rectangle. We stress that the rectangles in each collection $\mathcal{F}_s$ are generic two-parameter rectangles, namely their sides have independent lengths (there is no restriction on their eccentricity).

We also note that $\mathcal{F}_s$ consists of rectangles rather than parallelograms and this difference is important when one deals with rough frequency projections. Our techniques are sufficient to deal with the case of parallelograms as well but we just choose to detail the setup for the rectangular case. The interested reader will have no trouble adjusting the proof for variations of our main statement below for the case of parallelograms, or for the case that the families $\mathcal{F}_s$ are in fact one-parameter families.

Given $F \in \mathcal{F}_s$ we define a two-parameter Whitney discretization as follows. Let $F = \mathrm{rot}_s(I \times J) + y_F$ for some $y_F \in \mathbb{R}^2$, where rot_s denotes counterclockwise rotation by s about the origin and $I \times J$ is an axis parallel rectangle centered at the origin. Note that $I = (-|I|/2, |I|/2)$ and similarly for J. Then we define for $(k_1, k_2) \in \mathbb{N}^2,\ k_1, k_2 \neq 0$,

$$W_{k_1,k_2}(F) := \left\{ \xi \in I \times J : \frac{1}{3}2^{-k_1-1} \leq \frac{1}{2} - \frac{|\xi_1|}{|I|} \leq \frac{1}{3}2^{-k_1+1},\ \frac{1}{3}2^{-k_2-1} \leq \frac{1}{2} - \frac{|\xi_2|}{|J|} \leq \frac{1}{3}2^{-k_2+1} \right\}.$$

The definition has to be adjusted for $k_1 = 0$ or $k_2 = 0$. For example we define for $k_2 \neq 0$

$$W_{0,k_2}(F) := \left\{ \xi \in I \times J : \frac{1}{2}|I| - |\xi_1| \geq \frac{1}{2}\frac{1}{3}|I|,\ \frac{1}{3}2^{-k_2-1}|J| \leq \frac{1}{2}|J| - |\xi_2| \leq \frac{1}{3}2^{-k_2+1}|J| \right\}$$

and symmetrically for $k_1 \neq 0$ and $k_2 = 0$. Finally

$$W_{0,0}(F) := \left\{ \xi \in I \times J : \frac{1}{2}|I| - |\xi_1| \geq \frac{1}{2}\frac{1}{3}|I|,\ \frac{1}{2}|J| - |\xi_2| \geq \frac{1}{2}\frac{1}{3}|J| \right\}.$$

Then for $k = (k_1, k_2) \in \mathbb{N}^2$ we set $\Omega_{s,k_1,k_2}(F) := \mathrm{rot}_s(W_{k_1,k_2}(F)) + y_F$.

We can define tiles for this system as follows. If $F \in \mathcal{F}_s$ for some $s \in S$ and $F = \mathrm{rot}_s(I \times J) + y_F$ with $I \times J$ as above, then we choose $\ell_I^F, \ell_J^F \in \mathbb{Z}$ such that $2^{\ell_I^F} < |I| \leq 2^{\ell_I^F+1}$ and $2^{\ell_J^F} < |J| \leq 2^{\ell_J^F+1}$. We will have

$$\boldsymbol{T}^{\mathcal{F}} := \bigcup_{s\in S} \boldsymbol{T}_s^{\mathcal{F}}, \quad \boldsymbol{T}_s^{\mathcal{F}} := \bigcup_{F\in\mathcal{F}_s} \boldsymbol{T}_s(F), \quad \boldsymbol{T}_s(F) := \bigcup_{(k_1,k_2)\in\mathbb{N}^2} \boldsymbol{T}_{s,k_1,k_2}(F), \quad F \in \mathcal{F}_s, \tag{4.22}$$

where

$$\boldsymbol{T}_{s,k_1,k_2}(F) := \{t = R_t \times \Omega_{s,k_1,k_2}(F) : R_t \in \mathcal{D}_{s,-k_2+\ell_J^F,-k_1+\ell_I^F}\}, \quad F \in \mathcal{F}_s.$$

Note again that the tiles defined above obey the uncertainty principle in both $v, v^\perp$ for every fixed $v = (1, s)$ with $s \in S$.

The intrinsic square function associated with the collection $\mathcal{F}$ is denoted by $\Delta_{\boldsymbol{T}^{\mathcal{F}}}$ and formally has the same definition as (4.16), where now the $\boldsymbol{T}$ are given by the collection $\boldsymbol{T}^{\mathcal{F}}$ of (4.22). The corresponding theorem is the intrinsic analogue of a multiparameter directional Rubio de Francia square function estimate.

Theorem I. *Let $\mathcal{F}$ be a finitely overlapping collection of two-parameter rectangles in directions given by S*

$$\Big\| \sum_{F\in\mathcal{F}} \mathbf{1}_F \Big\|_\infty \lesssim 1.$$

Consider the collection of tiles $\boldsymbol{T}^{\mathcal{F}}$ defined in (4.22) *and let $\Delta_{\boldsymbol{T}^{\mathcal{F}}}$ be the corresponding intrinsic square function. We have the estimates*

$$\|\Delta_{\boldsymbol{T}^{\mathcal{F}}} : L^p(\mathbb{R}^2)\| \lesssim_p (\log \#S)^{\frac{1}{2}-\frac{1}{p}} (\log\log \#S)^{\frac{1}{2}-\frac{1}{p}}, \quad 2 \le p < 4,$$
$$\sup_{E,f} \frac{\|\Delta_{\boldsymbol{T}^{\mathcal{F}}}(f\mathbf{1}_E)\|_4}{|E|^{\frac{1}{4}}} \lesssim (\log \#S)^{\frac{1}{4}} (\log\log \#S)^{\frac{1}{4}},$$

where the supremum in the last display is taken over all measurable sets $E \subset \mathbb{R}^2$ of finite positive measure and all Schwartz functions f on $\mathbb{R}^2$ with $\|f\|_\infty \le 1$.

Remark 4.23. As before, there is slight improvement in the case of one-parameter spatial components in each direction. More precisely suppose that $\mathcal{F} = \bigcup_{s\in S} \mathcal{F}_s$ is a given collection of disjoint rectangles in directions given by S. If for each $s \in S$ the family $\mathcal{R}_{\mathcal{F}_s} := \{R_t : t \in \boldsymbol{T}_{\mathcal{F}_s}\}$ yields a weak-type-(1, 1) maximal operator then the estimates of Theorem I hold without the log log-terms.

Remark 4.24. Suppose that $\mathcal{R} = \bigcup_{s\in S} \mathcal{R}_s \subset \mathcal{P}^2_S$ is a family of parallelograms in directions given by s; namely we have that if $R \in \mathcal{R}_s$ then $R = A_s(I \times J) + y_R$ for some rectangle $I \times J$ in $\mathbb{R}^2$ with sides parallel to the coordinate axes and centered at 0, and $y_R \in \mathbb{R}^2$. Now there is an obvious way to construct a Whitney partition of each $R \in \mathcal{R}$. Indeed we just define the frequency components

$$\Omega_{s,k_1,k_2}(R) := A_s(W_{k_1,k_2}(I \times J)) + y_R,$$

with $W_{k_1,k_2}(I \times J)$ as constructed before. Then

$$\boldsymbol{T}_{s,k_1,k_2}(R) := \{R_t \times \Omega_{s,k_1,k_2}(R) : R_t \in \mathcal{D}_{s,-k_2+\ell^F_J,-k_1+\ell^F_I}\}, \quad R \in \mathcal{R}_s,$$

and $\boldsymbol{T}$ are given as in (4.22). With this definition there is a corresponding intrinsic square function $\Delta_{\boldsymbol{T}_{\mathcal{R}}}$ which satisfies the bounds of Theorem I. The improvement of Remark 4.23 is also valid if $\mathcal{R} = \bigcup_{s\in S} \mathcal{R}_s$ and each $\mathcal{R}_s$ consists of rectangles of fixed eccentricity.

The proof of Theorem I relies again on the global and local orthogonality estimates of Section 4.3 and a subsequent application of the directional Carleson embedding theorem, Theorem C. We omit the details.

5. Sharp bounds for conical square functions

We begin this section by recalling the definition for the smooth conical frequency projections given in Section 1. Let $\tau \subset [0, 2\pi)$ be an interval and consider the corresponding rough cone multiplier

$$C_\tau f(x) := \int_0^{2\pi}\int_0^\infty \hat{f}(\varrho e^{i\vartheta}) \mathbf{1}_\tau(\vartheta) e^{ix\cdot\varrho e^{i\vartheta}} \varrho \, d\varrho \, d\vartheta, \quad x \in \mathbb{R}^2,$$

and its smooth analogue

$$C_\tau^\circ f(x) := \int_0^{2\pi}\int_0^\infty \hat{f}(\varrho e^{i\vartheta})\beta\left(\frac{\vartheta - c_\tau}{|\tau|/2}\right)e^{ix\cdot\varrho e^{i\vartheta}}\,\varrho\,d\varrho\,d\vartheta, \quad x \in \mathbb{R}^2, \tag{5.1}$$

where β is a smooth function on $\mathbb{R}$ supported on $[-1, 1]$ and equal to 1 on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and c_τ, $|\tau|$ stand respectively for the center and length of τ.

This section is dedicated to the proofs of two related theorems concerning conical square functions. The first is a quantitative estimate for a square function associated with the smooth conical multipliers of a finite collection of intervals with bounded overlap given in Theorem A, namely the estimates

$$\|\{C_\tau^\circ f\}\|_{L^p(\mathbb{R}^2;\ell^2_{\boldsymbol{\tau}})} \lesssim_p (\log \#\boldsymbol{\tau})^{\frac{1}{2}-\frac{1}{p}}\|f\|_p$$

for $2 \le p < 4$, as well as the restricted-type analogue valid for all measurable sets E

$$\|\{C_\tau^\circ (f\mathbf{1}_E)\}\|_{L^4(\mathbb{R}^2;\ell^2_{\boldsymbol{\tau}})} \lesssim (\log \#\boldsymbol{\tau})^{\frac{1}{4}}|E|^{\frac{1}{4}}\|f\|_\infty,$$

under the assumption of finite overlap

$$\Big\|\sum_{\tau\in\boldsymbol{\tau}}\mathbf{1}_\tau\Big\|_\infty \lesssim 1. \tag{5.2}$$

The second theorem concerns an estimate for the rough conical square function for a collection of finitely overlapping cones $\boldsymbol{\tau}$.

Theorem J. *Let $\boldsymbol{\tau}$ be a finite collection intervals in $[0, 2\pi)$ with finite overlap as in (5.2). Then the square function estimate*

$$\|\{C_\tau f\}\|_{L^p(\mathbb{R}^2;\ell^2_{\boldsymbol{\tau}})} \lesssim_p (\log \#\boldsymbol{\tau})^{1-\frac{2}{p}}(\log\log \#\boldsymbol{\tau})^{\frac{1}{2}-\frac{1}{p}}\|f\|_p \tag{5.3}$$

holds for each $2 \le p < 4$.

Theorem A is sharp, in terms of $\log \#\boldsymbol{\omega}$-dependence, for all $2 \le p < 4$ and for $p = 4$ up to the restricted type. Theorem J improves on [Córdoba 1982, Theorem 1], where the dependence on cardinality is unspecified. Examples providing a lower bound of $(\log \#\boldsymbol{\omega})^{1/2-1/p}\|f\|_p$ for the left-hand side of (5.3), and showing the sharpness of Theorem A, are detailed in Section 8.

The remainder of the section is articulated as follows. In the upcoming Section 5.4 we show Theorem A. The subsequent subsection is dedicated to the proof of Theorem J.

5.4. ***Proof of Theorem A.*** We are given a finite collection of intervals $\omega \in \boldsymbol{\omega}$ having bounded overlap as in (5.2). By finite splitting we may reduce to the case of $\omega \in \boldsymbol{\omega}$ being pairwise disjoint; we treat this case throughout.

The first step in the proof of Theorem A is a radial decoupling. Let ψ be a smooth radial function on $\mathbb{R}^2$ with

$$\mathbf{1}_{[1,2]}(|\xi|) \le \psi(\xi) \le \mathbf{1}_{[2^{-1},2^2]}(|\xi|)$$

and define the Littlewood–Paley projection

$$S_k f(x) := \int \psi(2^{-k}\xi)\hat{f}(\xi)\,e^{ix\cdot\xi}\,d\xi, \quad x \in \mathbb{R}^2.$$

The following weighted Littlewood–Paley inequality is contained in [Bennett and Harrison 2012, Proposition 4.1].

Proposition 5.5 [Bennett and Harrison 2012]. *Let w be a nonnegative locally integrable function. Then*

$$\int_{\mathbb{R}^2} |f|^2 w \lesssim \int_{\mathbb{R}^2} \sum_{k\in\mathbb{Z}} |S_k(f)|^2 \mathrm{M}^{[3]} w,$$

with implicit constant independent of w, f, where we recall that $\mathrm{M}^{[3]}$ denotes the three-fold iteration of the Hardy–Littlewood maximal operator M *with itself.*

We may easily deduce the next lemma from the proposition.

Lemma 5.6. *For any $p \geq 2$ we have*

$$\|\{C_\tau^\circ f\}\|_{L^p(\mathbb{R}^2;\ell^2_\tau)} \lesssim \Bigg\| \Bigg(\sum_{k\in\mathbb{Z},\,\tau\in\boldsymbol{\tau}} |C_\tau^\circ S_k(f)|^2 \Bigg)^{\frac12} \Bigg\|_p. \tag{5.7}$$

Proof. The case $p = 2$ is trivial so we assume $p > 2$. Letting $r := \frac{p}{2} > 1$ there exists some $w \in L^{r'}(\mathbb{R}^2)$ with $\|w\|_{r'} = 1$ such that

$$\|\{C_\tau^\circ f\}\|^2_{L^p(\mathbb{R}^2;\ell^2_\tau)} = \sum_{\tau\in\boldsymbol{\tau}} \int_{\mathbb{R}^2} |C_\tau^\circ f|^2 w \lesssim \sum_{k\in\mathbb{Z},\,\tau\in\boldsymbol{\tau}} \int_{\mathbb{R}^2} |C_\tau^\circ S_k(f)|^2 M^{[3]} w$$

and the lemma follows by Hölder's inequality and the boundedness of $M^{[3]}$ on $L^{r'}(\mathbb{R}^2)$. □

The second and final step of the proof of Theorem A is the reduction of the operator appearing in the right-hand side of (5.7) to the model operator of Theorem H.

In order to match the notation of Section 4.9 we write $\{\omega_s\}_{s\in S}$ for the collection of arcs in $\mathbb{S}^1$ corresponding to the collection of intervals $\boldsymbol{\tau}$, namely for $\tau \in \boldsymbol{\tau}$ we implicitly define $s = s_\tau$ by means of $v_s^\perp/|v_s^\perp| := \mathrm{e}^{ic_\tau} = (1,s)/|(1,s)|$. We set $S := \{s_\tau : \tau \in \boldsymbol{\tau}\}$ and define the corresponding arcs in $\mathbb{S}^1$ as

$$\omega_{s_\tau} := \{\mathrm{e}^{i\theta} : \theta \in \tau\}.$$

Now the cone C_τ is the same thing as the cone C_s and $\#S = \#\boldsymbol{\tau}$. Similarly we write $C_\tau^\circ = C_{s_\tau}^\circ$ so the cones can now be indexed by $s \in S$. Define ℓ_s such that $2^{-\ell_s} \leq |\omega_s| \leq 2^{-\ell_s+1}$.

By finite splitting and rotational invariance there is no loss in generality with assuming that $S \subset [-1, 1]$. Notice that the support of the multiplier of $C_s^\circ S_k$ is contained in the frequency sector $\Omega_{s,k}$ defined in (4.10). By standard procedures of time-frequency analysis, as for example in [Demeter and Di Plinio 2014, Section 6], the operator $C_s^\circ S_k$ can be recovered by appropriate averages of operators

$$\mathsf{C}_{s,k} f := \sum_{t\in\boldsymbol{T}_{s,k}} \langle f, \phi_t\rangle \phi_t,$$

where $\phi_t \in \mathcal{A}_t^{8M_0}$ for all $t \in \boldsymbol{T}_{s_\omega,k}$ and $\boldsymbol{T}_{s,k}$ is defined in (4.20). Here $M_0 = 2^{50}$ is as chosen in (4.2). Fixing s, k for the moment we preliminarily observe that for each $\nu \geq 1$ the collection

$\mathcal{R}_{s,k} := \mathcal{R}_{\boldsymbol{T}_{s,k}} = \{R_t : t \in \boldsymbol{T}_{s,k}\}$ can be partitioned into subcollections $\{\mathcal{R}^j_{s,k,\nu} : 1 \leq j \leq 2^{8\nu}\}$ with the property that

$$R_1, R_2 \in \mathcal{R}^j_{s,k,\nu} \quad \Longrightarrow \quad 2^{2\nu+4} R_1 \cap 2^{2\nu+4} R_2 = \varnothing.$$

We will also use below the Schwartz decay of $\phi_t \in \mathcal{A}^{M_0}_t$ in the form

$$\sqrt{|R_t|}|\phi_t| \lesssim \mathbf{1}_{R_t} + \sum_{\nu \geq 0} 2^{-8M_0\nu} \sum_{\substack{\rho \in \mathcal{R}_{s,k} \\ \rho \not\subset 2^\nu R_t,\, \rho \subset 2^{\nu+1} R_t}} \mathbf{1}_\rho.$$

Using Schwartz decay of ϕ_t twice, in particular to bound by an absolute constant the second factor obtained by Cauchy–Schwarz after the first step, we get

$$\begin{aligned}
|\mathsf{C}_{s,k} f|^2 &\lesssim \Bigg(\sum_{t \in \boldsymbol{T}_{s,k}} |\langle f, \phi_t\rangle|^2 \frac{|\phi_t|}{\sqrt{|R_t|}}\Bigg)\Bigg(\sum_{t \in \boldsymbol{T}_{s,k}} \sqrt{|R_t|}|\phi_t|\Bigg) \\
&\lesssim \sum_{t \in \boldsymbol{T}_{s,k}} |\langle f, \phi_t\rangle|^2 \frac{\mathbf{1}_{R_t}}{|R_t|} + \sum_{\nu \geq 0} 2^{-8M_0\nu} \sum_{t \in \boldsymbol{T}_{s,k}} \sum_{\substack{\rho \in \mathcal{R}_{s,k} \\ \rho \not\subset 2^\nu R_t, \rho \subset 2^{\nu+1} R_t}} |\langle f, \phi_t\rangle|^2 \frac{\mathbf{1}_\rho}{|\rho|} \\
&\leq \sum_{t \in \boldsymbol{T}_{s,k}} |\langle f, \phi_t\rangle|^2 \frac{\mathbf{1}_{R_t}}{|R_t|} + \sum_{\nu \geq 0} 2^{-8M_0\nu} \sum_{R \in \mathcal{R}_{s,k}} \sum_{j=1}^{2^{8\nu}} \sum_{\substack{\rho \in \mathcal{R}_{s,k} \\ \rho \not\subset 2^\nu R_t, \rho \subset 2^{\nu+1} R_t}} |\langle f, \phi_t\rangle|^2 \frac{\mathbf{1}_\rho}{|\rho|}.
\end{aligned}$$

Now for fixed ω, k, ν, j and $t \in \boldsymbol{T}_{s,k}$ observe that there is at most one $\rho = \rho^j_{s,k,\nu}(t) \in \mathcal{R}^j_{\omega,k,\nu}$ such that $\rho \not\subset 2^\nu R_t,\ \rho \subset 2^{\nu+1} R_t$. Thus the estimate above can be written in the form

$$|\mathsf{C}_{s,k} f|^2 \lesssim \sum_{t \in \boldsymbol{T}_{s,k}} |\langle f, \phi_t\rangle|^2 \frac{\mathbf{1}_{R_t}}{|R_t|} + \sum_{\nu \geq 0} 2^{-8M_0\nu} \sum_{j=1}^{2^{8\nu}} \sum_{t \in \boldsymbol{T}_{s,k}} |\langle f, \phi_t\rangle|^2 \frac{\mathbf{1}_{\rho^j_{s,k,\nu}(t)}}{|\rho^j_{s,k,\nu}(t)|}.$$

Observe that if $t \in \boldsymbol{T}_{s,k}$,

$$\phi_t \in \mathcal{A}^{8M_0}_t,\ \ \rho \in \mathcal{R}_{s,k},\ \ \rho \subset 2^{\nu+1} R_t \quad \Longrightarrow \quad 2^{-4M\nu}|\langle f, \phi_t\rangle|^2 \leq a_{t_\rho}(f),$$

where $t_\rho = \rho \times \Omega_{s,k} \in \boldsymbol{T}_{s,k}$ is the unique tile with spatial localization given by ρ; this is because $2^{-4M\nu}\phi_t \in \mathcal{A}^{M_0}_{t_\rho}$. We thus conclude that

$$|\mathsf{C}_{s,k} f|^2 \lesssim \sum_{t \in \boldsymbol{T}_{s,k}} a_t(f) \frac{\mathbf{1}_{R_t}}{|R_t|}. \tag{5.8}$$

Comparing with the definition of $\Delta_{\boldsymbol{T}}$ given in (4.16) we may summarize the discussion in the lemma below.

Lemma 5.9. *Let* $1 < p < \infty$. *Then*

$$\sup_{\|f\|_p=1} \Bigg\| \Bigg(\sum_{k \in \mathbb{Z},\, \tau \in \boldsymbol{\tau}} |C^\circ_\tau S_k(f)|^2\Bigg)^{\frac{1}{2}} \Bigg\|_p \lesssim \sup_{\|f\|_p=1} \|\Delta_{\boldsymbol{T}}(f)\|_p,$$

where

$$\boldsymbol{T} := \bigcup_{s\in S}\bigcup_{k\in\mathbb{Z}} \boldsymbol{T}_{s,k}$$

and $\boldsymbol{T}_{s,k}$ *is defined in* (4.20).

The proof of the upper bound in Theorem A is then completed by juxtaposing the estimates of Lemmas 5.6 and 5.9 with Theorem H. For the optimality of the estimate see Section 8.6.

5.10. ***Proof of Theorem J.*** The proof of Theorem J is necessarily more involved than its smooth counterpart Theorem A. In particular we need to decompose each cone not only in the radial direction as before, but also in the directions perpendicular to the singular boundary of each cone. We describe this procedure below.

Consider a collection of intervals $\boldsymbol{\tau} = \{\tau\}$ as in the statement. By the same correspondence as in the proof of Theorem A we pass to a family $\{\omega_s\}_{s\in S}$ consisting of finitely overlapping arcs on $\mathbb{S}^1$ centered at $v_s^\perp/|v_s^\perp|$ and corresponding cones C_s. Note that the sectors $\{\Omega_{s,k}\}_{s\in S,k\in\mathbb{Z}}$ defined in (4.10) form a finitely overlapping cover of $\bigcup_{s\in S} C_s$. We remember here that $v_s = (1,s)$, that the interval ω_s is given by $(v_{s^-}^\perp, v_{s^+}^\perp)$, and that the positive direction is counterclockwise.

Now, for each fixed $s\in S$ the cover $\{\Omega_{s,k,m}\}_{(k,m)\in\mathbb{Z}^2}$ defined in (4.11), (4.12), is a Whitney cover of $\Omega_{s,k}$ in the product sense: for each $\Omega_{s,k,m}$ the distance from the origin is comparable to 2^k and the distance to the boundary is comparable to $2^{-|m||\omega_s|}$.

The radial decomposition in k will be taken care of by the Littlewood–Paley decomposition $\{S_k\}_{k\in\mathbb{Z}}$, defined as in the proof of Theorem J. Now for fixed s,k we consider a smooth partition of unity subordinated to the cover $\{\Omega_{s,k,m}\}_{m\in\mathbb{Z}}$. Note that one can easily achieve that by choosing $\{\varphi_{s,m}\}_{m<0}$ to be a one-sided (contained in C_s) Littlewood–Paley decomposition in the negative direction $v^- = v_{s^-}$, and constant in the direction $(v^-)^\perp$ when $m<0$, and similarly one can define $\varphi_{s,m}$ when $m>0$, with respect to the positive direction v^+. The central piece $\Omega_{s,k,0}$ corresponds to φ_{s_0} defined implicitly as

$$\varphi_{s,0} = \mathbf{1}_{C_s} - \sum_{m\in\mathbb{Z}} \varphi_{s,m}.$$

Now the desired partition of unity is

$$\pi_{s,k,m}(\xi) := \mathbf{1}_{C_s}(\xi)\varphi_{s,m}(\xi)\psi_k(\xi) = \varphi_{s,m}(\xi)\psi_k(\xi),$$

where $\psi_k := \psi(2^{-k}\,\cdot)$, with the ψ constructed in the proof of Theorem A. Remember that $S_k f := (\psi_k \hat{f})^\vee$ and let us define $\Phi_{s,m} f := (\varphi_{s,m}\hat{f})^\vee$.

An important step in the proof is the following square function estimate in $L^p(\mathbb{R}^2)$, with $2\le p<4$, that decouples the Whitney pieces in every cone C_s. It comes at a loss in N, which appears to be inevitable because of the directional nature of the problem.

Lemma 5.11. *Let* $\{C_s\}_{s\in S}$ *be a family of frequency cones, given by a family of finitely overlapping arcs* $\boldsymbol{\omega} := \{\omega_s\}_{s\in S}$ *as above. For* $2\le p<4$ *there holds*

$$\|\{C_s f\}\|_{L^p(\mathbb{R}^2;\ell^2_{\boldsymbol{\omega}})} \lesssim \frac{1}{4-p}(\log \#S)^{\frac12-\frac1p}\|\{S_k\Phi_{s,m} f\}\|_{L^p(\mathbb{R}^2;\ell^2_{\boldsymbol{\omega}\times\mathbb{Z}\times\mathbb{Z}\}})}.$$

Proof. Observe that the desired estimate is trivial for $p=2$ so let us fix some $p\in(2,4)$. There exists some $g\in L^q$ with $q=(p/2)'=p/(p-2)$ such that

$$A^2:=\|\{C_s f\}\|^2_{L^p(\mathbb{R}^2;\ell^2_\omega)}=\int_{\mathbb{R}^2}\sum_{s\in S}|C_s f|^2 g$$

and so by Proposition 5.5 we get

$$A^2\lesssim\sum_{k\in\mathbb{Z}}\sum_{s\in S}\int_{\mathbb{R}^2}|C_s S_k f|^2\mathrm{M}^{[3]}g,$$

where we recall that $\mathrm{M}^{[3]}$ denotes three iterations of the Hardy–Littlewood maximal operator M. Fixing s for a moment we use Proposition 5.5 in the directions v_{s^-}, v_s and v_{s^+} to further estimate

$$\int_{\mathbb{R}^2}|C_s f|^2\mathrm{M}^{[3]}g\lesssim\sum_{m\in\mathbb{Z}}\sum_{\varepsilon\in\{-,0,+\}}\int_{\mathbb{R}^2}|S_k\Phi_{s,m}f|^2\mathrm{M}^{[3]}_{v_{s^\varepsilon}}\mathrm{M}^{[3]}g,$$

where we adopted the convention $s^0:=s$ for brevity, and M_v is given by (2.3). Remember also that $\Phi_{s,m}$ for $m>0$ corresponds to directions s^+, while $\Phi_{s,m}$ corresponds to directions s^- for $m<0$, and to directions $s^0=s$ for $m=0$. Now for any $v\in\mathbb{S}^1$ and $r>1$ we have that

$$\mathrm{M}^{[3]}_v G\lesssim(r')^2[\mathrm{M}_v G^r]^{\frac{1}{r}};$$

see for example [Pérez 1994]. Thus $\mathrm{M}^{[3]}_{v_{s^\varepsilon}}\mathrm{M}^{[3]}g\lesssim(r')^2[\mathrm{M}_{V^*}[\mathrm{M}^{[3]}G]^r]^{1/r}$, where $\mathrm{M}_{V^*}f:=\sup_{v\in V^*}\mathrm{M}_v f$, where here we use $V^*:=\{(1,s):s\in S^*\}$ with S^* as in (4.15), and $\mathrm{M}_{V^*}f:=\sup_{w\in V^*}\mathrm{M}_w(f)$.

It is known [Katz 1999] that M_{V^*} maps $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ with a bound $(\log\#V^*)^{1/p}$ for $p>2$. As $p<4$ there exists a choice of $1<r<p/(2(p-2))$ so that $p/(r(p-2))>2$ and a theorem from [Katz 1999] applies. Using this fact together with Hölder's inequality proves the lemma. □

The proof of Theorem J can now be completed as follows. For each $(s,k,m)\in S\times\mathbb{Z}\times\mathbb{Z}$ the operator $S_k\Phi_{s,m}$ is a smooth frequency projection adapted to the rectangular box $\Omega_{s,k,m}$. Following the same procedure that led to (5.8) in the proof of Theorem A we can approximate each piece $S_k\Phi_{s,m}f$ by an operator of the form

$$\mathsf{C}_{s^\varepsilon,k,m}f:=\sum_{t\in\boldsymbol{T}_{s^\varepsilon,k,m}}\langle f,\phi_t\rangle\phi_t,\quad|\mathsf{C}_{s^\varepsilon,k,m}f|^2\lesssim\sum_{t\in\boldsymbol{T}_{s^\varepsilon,k,m}}a_t(f)\frac{\mathbf{1}_{R_t}}{|R_t|},$$

where s^ε follows the sign of m and coincides with s if $m=0$. The collections of tiles $\boldsymbol{T}_{s^\varepsilon,k,m}$ are the ones given in (4.14). Now Lemma 5.11 and Theorem G are combined to complete the proof of Theorem J.

6. Directional Rubio de Francia square functions

In his seminal paper Rubio de Francia [1985] proved a one-sided Littlewood–Paley inequality for arbitrary intervals on the line. This estimate was later extended by Journé [1985] to the case of rectangles (n-dimensional intervals) in $\mathbb{R}^n$; a proof more akin to the arguments of the present paper appears in [Lacey 2007]. The aim of this subsection is to present a generalization of the one-sided Littlewood–Paley inequality to the case of rectangles in $\mathbb{R}^2$ with sides parallel to a given set of directions. The set of directions is to be finite, necessarily, because of Kakeya counterexamples.

As in the case of cones of Section 5 we will present two versions, one associated with smooth frequency projections and one with rough. To set things up let S be a finite set of slopes and V be the corresponding directions. We consider a family of rotated rectangles $\mathcal{F}$ as in Section 4.21, where $\mathcal{F} = \bigcup_{s\in S} \mathcal{F}_s$. For each $s \in S$ a rectangle $F \in \mathcal{F}_s$ is a rotation by s of an axis parallel rectangle, so that the sides of R are parallel to $(v, v^\perp)$ with $v = (1, s)$. We will write $F = \mathrm{rot}_s(I_F \times J_F) + y_F$ for some $y_F \in \mathbb{R}^2$ in order to identify the axes-parallel rectangle $I_F \times J_F$ producing F by an s-rotation; this writing assumes that $I_F \times J_F$ is centered at the origin.

Now for each $F \in \mathcal{F}$ we consider the rough frequency projection

$$P_F f(x) := \int_{\mathbb{R}^2} \hat{f}(\xi)\mathbf{1}_F(\xi)e^{ix\cdot\xi}\,\mathrm{d}\xi, \quad x \in \mathbb{R}^2,$$

and its smooth analogue

$$P_F^\circ f(x) := \int_{\mathbb{R}^2} \hat{f}(\xi)\gamma_F(\xi)e^{ix\cdot\xi}\,\mathrm{d}\xi, \quad x \in \mathbb{R}^2,$$

where γ_R is a smooth function on $\mathbb{R}^2$, supported in R, and identically 1 on $\mathrm{rot}_s\big(\frac{1}{2}I \times \frac{1}{2}J\big)$.

We first state the smooth square function estimate.

Theorem K. *Let $\mathcal{F}$ be a collection of rectangles in $\mathbb{R}^2$ with sides parallel to $(v, v^\perp)$ for some v in a finite set of directions V. Assume that $\mathcal{F}$ has finite overlap. Then*

$$\|\{P_F^\circ f\}\|_{L^p(\mathbb{R}^2;\ell^2_{\mathcal{F}})} \lesssim_p (\log\#V)^{\frac{1}{2}-\frac{1}{p}}(\log\log\#V)^{\frac{1}{2}-\frac{1}{p}}\|f\|_p$$

for $2 \le p < 4$, as well as the restricted-type analogue valid for all measurable sets E

$$\|\{P_F^\circ(f\mathbf{1}_E)\}\|_{L^4(\mathbb{R}^2;\ell^2_{\mathcal{F}})} \lesssim (\log\#V)^{\frac{1}{4}}(\log\log\#V)^{\frac{1}{4}}|E|^{\frac{1}{4}}\|f\|_\infty.$$

The dependence on $\#V$ in the estimates above is best possible up the doubly logarithmic term.

Remark 6.1. We record a small improvement of the estimates above in some special cases. Suppose that for fixed $s \in S$ all the rectangles $F \in \mathcal{F}_s$ have one side-length fixed, or that they have fixed eccentricity. In both these cases the collections of spatial components of the tiles needed to discretize these operators, $\mathcal{R}_{\boldsymbol{T}_s^{\mathcal{F}}} := \{R_t : t \in \boldsymbol{T}_s^{\mathcal{F}}\}$, with $\boldsymbol{T}_s^{\mathcal{F}}$ as in (4.22), give rise to maximal operators that are of weak-type $(1,1)$. Then Remark 4.23 shows that the estimates of Theorem K hold without the doubly logarithmic terms, and as shown in Section 8.2 this is best possible.

The rough version of this Rubio-de-Francia-type theorem is slightly worse in terms of the dependence on the number of directions. The reason for that is that, as in the case of conical projections, passing from rough to smooth in the directional setting incurs a loss of logarithmic terms, essentially originating in the corresponding maximal function bound.

Theorem L. *Let $\mathcal{F}$ be a collection of rectangles in $\mathbb{R}^2$ with sides parallel to $(v, v^\perp)$ for some v in a finite set of directions V. Assume that $\mathcal{F}$ has finite overlap. Then the following square function estimate holds for $2 \le p < 4$:*

$$\|\{P_F f\}\|_{L^p(\mathbb{R}^2;\ell^2_{\mathcal{F}})} \lesssim_p (\log\#V)^{\frac{3}{2}-\frac{3}{p}}(\log\log\#V)^{\frac{1}{2}-\frac{1}{p}}\|f\|_p.$$

The proofs of these theorems follow the by now familiar path of introducing local Littlewood–Paley decompositions on each multiplier, approximating with time-frequency analysis operators, establishing a directional Carleson condition on the wave-packet coefficients and finally applying Theorem C. We will very briefly comment on the proofs below.

Proof of Theorems L and K. We first sketch the proof of Theorem L, which is slightly more involved. The first step here is a decoupling lemma which is completely analogous to Lemma 5.11 with the difference that now we need to use two directional Littlewood–Paley decompositions, while in the case of cones only one. This explains the extra logarithmic term of the statement.

Remember that $\mathcal{F} = \bigcup_s \mathcal{F}_s$, with $s = (1, v)$ for some $v \in V$; here s gives the directions $(v, v^{\perp})$ of the rectangles in $\mathcal{F}_s$. Using the finitely overlapping Whitney decomposition of Section 4.21 we have for each $F \in \mathcal{F}_s$ a collection of tiles

$$\boldsymbol{T}_s(F) := \bigcup_{(k_1,k_2)\in\mathbb{Z}^2} \boldsymbol{T}_{s,k_1,k_2}(F)$$

as in (4.22). Let us for a moment fix s and $F \in \mathcal{F}_s$. The frequency components of the tiles in $\boldsymbol{T}_s(F)$ form a two-parameter Whitney decomposition of F, so let $\{\phi_{F,k_1,k_2}\}_{(k_1,k_2)\in\mathbb{Z}^2}$ be a smooth partition of unity subordinated to this cover and denote by Φ_{F,k_1,k_2} the Fourier multiplier with symbol ϕ_{F,k_1,k_2}.

The promised analogue of Lemma 5.11 is the following estimate: for $2 \le p < 4$ there holds

$$\|\{P_F f\}\|_{L^p(\mathbb{R}^2;\ell^2_{\mathcal{F}})} \lesssim \frac{1}{(4-p)^2} (\log \# V)^{1-\frac{2}{p}} \|\{\Phi_{s,k_1,k_2} f\}\|_{L^p(\mathbb{R}^2;\ell^2_{\mathcal{F}\times\mathbb{Z}\times\mathbb{Z}\}})}. \tag{6.2}$$

The proof of this estimate is a two-parameter repetition of the proof of Lemma 5.11, where one applies Proposition 5.5 once in the direction of v and once in the direction of $v^{\perp}$. Using the familiar scheme we can approximate each $\Phi_{s,k_1,k_2} f$ by time-frequency analysis operators

$$\mathsf{P}_{F,k_1,k_2} f := \sum_{t\in\boldsymbol{T}_{s,k_1,k_2}(F)} \langle f, \phi_t\rangle \phi_t, \quad |\mathsf{P}_{F,k_1,k_2} f|^2 \lesssim \sum_{t\in\boldsymbol{T}_{s,k_1,k_2}(F)} a_t(f) \frac{\mathbf{1}_{R_t}}{|R_t|}$$

and by (6.2) the proof of Theorem L follows by corresponding bounds for the intrinsic square function of Theorem I, defined with respect to the tiles $\boldsymbol{T}^{\mathcal{F}}$ given by (4.22).

For Theorem K things are a bit simpler as the decoupling step of (6.2) is not needed. Apart from that one needs to consider for each F a new set of tiles which is very easy to define: If $F \in \mathcal{F}_s$ with $F = \mathrm{rot}_s(I_F \times J_F) + y_F$,

$$\boldsymbol{T}'(F) := \{t = R_t \times F : R_t \in \mathcal{D}^2_{s,\ell_J,\ell_I}\},$$

and then $\boldsymbol{T}' := \bigcup_{F\in\mathcal{F}} \boldsymbol{T}'(F)$. One can recover P_F° by operators of the form

$$\mathsf{P}^{\circ}_F f := \sum_{t\in\boldsymbol{T}_s(F)} \langle f, \phi_t\rangle \phi_t, \quad |\mathsf{P}^{\circ}_F f|^2 \lesssim \sum_{t\in\boldsymbol{T}_s(F)} a_t(f) \frac{\mathbf{1}_{R_t}}{|R_t|}$$

as before. Using the orthogonality estimates of Section 4.3 in Theorem C yields the upper bound in Theorem K. The optimality of the estimates in the statement of Theorem K is discussed in Section 8.2. □

7. The multiplier problem for the polygon

Let $\mathcal{P} = \mathcal{P}_N$ be a regular N-gon and $T_{\mathcal{P}_N}$ be the corresponding Fourier restriction operator on $\mathcal{P}$

$$T_{\mathcal{P}} f(x) := \int_{\mathbb{R}^2} \hat{f}(\xi)\mathbf{1}_{\mathcal{P}}(\xi)\mathrm{e}^{ix\cdot\xi}\,\mathrm{d}\xi, \quad x \in \mathbb{R}^2.$$

In this subsection we prove Theorem B, namely we will prove the estimate

$$\|T_{\mathcal{P}_N} : L^p(\mathbb{R}^2)\| \lesssim (\log N)^{4\left|\frac{1}{2}-\frac{1}{p}\right|}, \quad \tfrac{4}{3} < p < 4.$$

The idea is to reduce the multiplier problem for the polygon to the directional square function estimates of Theorem K and combine those with vector-valued inequalities for directional averages and directional Hilbert transforms.

We introduce some notation. The large integer N is fixed throughout and left implicit in the notation. By scaling, it will be enough to consider a regular polygon $\mathcal{P}$ with the following geometric properties: First, $\mathcal{P}$ has vertices

$$\{v_j = \mathrm{e}^{i\vartheta_j} : 1 \le j \le N+1\}, \quad v_j := \exp(2\pi j/N),$$

on the unit circle $\mathbb{S}^1$, with $\vartheta_1 = \vartheta_{N+1} = 0$ and oriented counterclockwise so that $\vartheta_{j+1} - \vartheta_j > 0$. The associated Fourier restriction operator is then defined by

$$T_{\mathcal{P}} f := (\mathbf{1}_{\mathcal{P}}\hat{f})^{\vee}.$$

The proof of the estimate of Theorem B for $T_{\mathcal{P}}$ occupies the remainder of this section; by self-duality of the estimate it will suffice to consider the range $2 \le p < 4$.

7.1. ***A preliminary decomposition.*** Let N be a large positive integer and take κ such that $2^{\kappa-1} < N \le 2^{\kappa}$. For each $-2\kappa \le k \le 0$ consider a smooth radial multiplier m_k which is supported on the annulus

$$A_k := \left\{\xi \in \mathbb{R}^2 : 1 - \frac{2^{-k-1}}{2^{2\kappa}} < |\xi| < 1 - \frac{2^{-k-5}}{2^{2\kappa}}\right\}$$

and is identically 1 on the smaller annulus

$$a_k := \left\{\xi \in \mathbb{R}^2 : 1 - \frac{2^{-k-2}}{2^{2\kappa}} < |\xi| < 1 - \frac{2^{-k-4}}{2^{2\kappa}}\right\}.$$

Now consider the corresponding radial multiplier operators T_k

$$T_k f := (m_k \hat{f})^{\vee}, \quad m_\kappa := \sum_{k=-2\kappa}^{0} m_k.$$

We note that m_κ is supported in the annulus

$$\left\{\xi \in \mathbb{R}^2 : \frac{1}{2} < |\xi| < 1 - \frac{2^{-5}}{2^{2\kappa}}\right\}.$$

With this in mind let us consider radial functions $m_0, m_{\mathcal{P}} \in \mathcal{S}(\mathbb{R}^2)$, with $0 \le m_0, m_{\mathcal{P}} \le 1$, such that

$$(m_0 + m_\kappa + m_{\mathcal{P}})\mathbf{1}_{\mathcal{P}} = \mathbf{1}_{\mathcal{P}}, \tag{7.2}$$

with the additional requirement that

$$\operatorname{supp}(m_{\mathcal{P}}) \subset A_{\mathcal{P}} := \{\xi \in \mathbb{R}^2 : 1 - 2^{-2\kappa-3} \le |\xi| \le 1 + 2^{-2\kappa-3}\}. \tag{7.3}$$

Defining

$$\widehat{T_0 f} := \hat{f} m_0, \quad \widehat{T_\kappa f} := \hat{f} m_\kappa, \quad \widehat{O_{\mathcal{P}} f} := \hat{f} m_{\mathcal{P}} \mathbf{1}_{\mathcal{P}},$$

identity (7.2) implies that $T_{\mathcal{P}} = T_0 + T_\kappa + O_{\mathcal{P}}$. Observing that T_0 is bounded on p for all $1 < p < \infty$ with bounds $O_p(1)$ we have

$$\|T_{\mathcal{P}}\|_{L^p(\mathbb{R}^2)} \lesssim_p 1 + \|T_\kappa\|_{L^p(\mathbb{R}^2)} + \|O_{\mathcal{P}}\|_{L^p(\mathbb{R}^2)}, \quad 1 < p < \infty. \tag{7.4}$$

7.5. ***Estimating*** $\boldsymbol{T_\kappa}$**.** We aim for the estimate

$$\|T_\kappa f\|_p \lesssim \kappa^{4(\frac{1}{2}-\frac{1}{p})} \|f\|_p, \quad 2 \le p < 4. \tag{7.6}$$

The case $p = 2$ is obvious, whence it suffices to prove the restricted-type version at the endpoint $p = 4$

$$\|T_\kappa(f \mathbf{1}_E)\|_4 \lesssim \kappa |E|^{\frac{1}{4}} \|f\|_\infty. \tag{7.7}$$

Now we have that for any g

$$|T_\kappa g| = \left| \sum_{k=-2\kappa}^{0} T_k g \right| \lesssim \left(\sum_{k=-2\kappa}^{0} |T_k g|^4 \right)^{\frac{1}{4}} \kappa^{\frac{3}{4}}$$

and thus

$$\|T_\kappa g\|_4 \lesssim \kappa^{\frac{3}{4}} \left(\sum_{k=-2\kappa}^{0} \|T_k g\|_4^4 \right)^{\frac{1}{4}}. \tag{7.8}$$

Let $\{\omega_j : j \in J\}$ be the collection of intervals on $\mathbb{S}^1$ centered at $v_j := \exp(2\pi i j/N)$ and of length $2^{-\kappa}$. Note that these intervals have finite overlap and their centers v_j form a $\sim 1/N$-net on $\mathbb{S}^1$. Now let $\{\beta_j : j \in J\}$ be a smooth partition of unity subordinated to the finitely overlapping open cover $\{\omega_j : j \in J\}$ so that each β_j is supported in ω_j. We can decompose each T_k as

$$\widehat{(T_k f)}(\xi) = \sum_{j \in J} m_k(|\xi|) \beta_j\left(\frac{\xi}{|\xi|}\right) \hat{f}(\xi) =: \sum_{j \in J} m_{j,k}(\xi) \hat{f}(\xi), =: \sum_{j \in J} \widehat{(T_{j,k} f)}(\xi), \quad \xi \in \mathbb{R}^2.$$

For $s_j \in S$ and $-2\kappa \le k \le 0$ we define the conical sectors

$$\Omega_{j,k} := \{\xi \in \mathbb{R}^2 : \xi \in A_k, \ \xi/|\xi| \in \omega_j\}$$

and note that each one of the multipliers $m_{j,k}$ is supported in $\Omega_{j,k}$. Each $\Omega_{j,k}$ is an annular sector around the circle of radius $1 - 2^{-k}/2^{2\kappa}$ of width $\sim 2^{-k}/2^{2\kappa}$, where $-2\kappa \le k \le 0$. It is a known observation, usually attributed to Córdoba [1977, Theorem 2] or C. Fefferman [1973], that for such parameters we have

$$\sum_{j,j' \in J} \mathbf{1}_{\Omega_{j,k} + \Omega_{j',k}} \lesssim 1. \tag{7.9}$$

This pointwise inequality and Plancherel's theorem allow us to decouple the pieces $T_{j,k}$ in L^4; for each fixed k as above we have

$$\|T_k f\|_4 \lesssim \Big\| \Big(\sum_{j\in J} |T_{j,k} f|^2 \Big)^{\frac{1}{2}} \Big\|_4; \tag{7.10}$$

see also the proof of Lemma 7.18 below for a vector-valued version of this estimate. Combining the last estimate with (7.8) and dominating the ℓ^2-norm by the ℓ^1-norm yields

$$\begin{aligned} \|T_\kappa f\|_4 &\lesssim \kappa^{\frac{3}{4}} \Big(\int_{\mathbb{R}^2} \sum_{k=-2\kappa}^{0} \Big(\sum_{j\in J} |T_{j,k} f|^2 \Big)^2 \Big)^{\frac{1}{4}} \lesssim \kappa^{\frac{3}{4}} \Big(\int_{\mathbb{R}^2} \Big[\sum_{k=-2\kappa}^{0} \Big(\sum_{j\in J} |T_{j,k} f|^2 \Big)^2 \Big]^{\frac{1}{2} 2} \Big)^{\frac{1}{4}} \\ &\le \kappa^{\frac{3}{4}} \Big(\int_{\mathbb{R}^2} \Big[\sum_{k=-2\kappa}^{0} \sum_{j\in J} |T_{j,k} f|^2 \Big]^2 \Big)^{\frac{1}{4}} =: \kappa^{\frac{3}{4}} \|\Delta_{J,\kappa} f\|_4, \end{aligned}$$

with

$$\Delta_{J,\kappa} f := \Big(\sum_{k=-2\kappa}^{0} \sum_{j\in J} |T_{j,k} f|^2 \Big)^{\frac{1}{2}}.$$

But now note that $\{T_{j,k}\}_{j,k}$ is a finitely overlapping family of smooth frequency projections on a family of rectangles in at most $\sim N$ directions. Furthermore all these rectangles have one side of fixed length since $|\omega_j| = 2^{-\kappa}$ for all $j \in J$. So Theorem K with the improvement of Remark 6.1 applies to yield

$$\|\Delta_{J,\kappa} f\|_4 \lesssim (\log \# N)^{\frac{1}{4}} \|f\|_\infty |E|^{\frac{1}{4}} \simeq \kappa^{\frac{1}{4}} \|f\|_\infty |E|^{\frac{1}{4}}. \tag{7.11}$$

The last two displays establish (7.7) and thus (7.6).

Remark 7.12. The term T_κ is also present in the argument of [Córdoba 1977]. Therein, an upper estimate of order $O(\kappa^{5/4})$ for p near 4 is obtained, by using the triangle inequality and the bound $\sup\{\|T_k\|_{L^4(\mathbb{R}^2)} : -2\kappa \le k \le 0\} \sim \kappa^{1/4}$ for the smooth restriction to a single annulus.

7.13. *Estimating $O_{\mathcal{P}}$.* In this subsection we will prove the estimate

$$\|O_{\mathcal{P}} f\|_p \lesssim \kappa^{4(\frac{1}{2}-\frac{1}{p})} \|f\|_p. \tag{7.14}$$

Let Φ be a smooth radial function with support in the annular region $\{\xi \in \mathbb{R}^2 : 1 - c2^{-2\kappa} < |\xi| < 1 + c2^{-2\kappa}\}$, where c is a fixed small constant, and satisfying $0 \le \Phi \le 1$. Let $\{\beta_j : j \in J\}$ be a partition of unity on $\mathbb{S}^1$ relative to intervals ω_j as in Section 7.5. Define the Fourier multiplier operators on $\mathbb{R}^2$

$$\widehat{T_j f}(\xi) := \Phi(\xi) \beta_j \Big(\frac{\xi}{|\xi|} \Big) \hat{f}(\xi), \quad \xi \in \mathbb{R}^2. \tag{7.15}$$

The operators T_j satisfy a square function estimate

$$\begin{aligned} \|\{T_j f\}\|_{L^p(\mathbb{R}^2;\ell^2_J)} &\lesssim \kappa^{\frac{1}{2}-\frac{1}{p}} \|f\|_p, \quad 2 \le p < 4, \\ \|\{T_j (f \mathbf{1}_E)\}\|_{L^4(\mathbb{R}^2;\ell^2_J)} &\lesssim \kappa^{\frac{1}{4}} |E|^{\frac{1}{4}} \|f\|_\infty, \end{aligned} \tag{7.16}$$

which follows in the same way as (7.11), by using Theorem K with the improvement of Remark 6.1. They also obey a vector-valued estimate

$$\begin{aligned} \|\{T_j f_j\}\|_{L^p(\mathbb{R}^2;\ell^2_J)} &\lesssim \kappa^{\frac{1}{2}-\frac{1}{p}} \|\{f_j\}\|_{L^p(\mathbb{R}^2;\ell^2_J)}, \quad 2 \le p < 4, \\ \|\{T_j(f_j \mathbf{1}_F)\}\|_{L^4(\mathbb{R}^2;\ell^2_J)} &\lesssim \kappa^{\frac{1}{4}} |F|^{\frac{1}{4}} \|\{f_j\}\|_{L^\infty(\mathbb{R}^2;\ell^2_J)}. \end{aligned} \tag{7.17}$$

These estimates are easy to prove. Indeed note that it suffices to prove the endpoint-restricted estimate at $p = 4$. Using the Fefferman–Stein inequality for fixed $j \in J$ we can estimate for each function g with $\|g\|_2 = 1$

$$\begin{aligned} \int_{\mathbb{R}^2} \sum_{j\in J} |T_j(f_j \mathbf{1}_F)|^2 g \lesssim \sum_{j \in J} \int_{\mathbb{R}^2} |f_j \mathbf{1}_F|^2 \mathrm{M}_j g &\le \|\{f_j\}\|^2_{L^\infty(\mathbb{R}^2;\ell^2_J)} \int_F \sup_{j\in J} \mathrm{M}_j g \\ &\lesssim |F|^{\frac{1}{2}} \| \sup_{j\in J} \mathrm{M}_j g\|_{L^{2,\infty}(\mathbb{R}^2)}, \end{aligned}$$

where M_j is the Hardy–Littlewood maximal operator with respect to the collection of parallelograms in $\mathcal{D}^2_{s_j,-2\kappa,-\kappa}$ with s_j defined through $(-s_j, 1) := v_j$. Now $\sup_{j\in J} M_j$ is the maximal directional maximal operator and the number of directions involved in its definition is comparable to $N \sim 2^\kappa$. Then the maximal theorem from [Katz 1999] applies to give the estimate

$$\| \sup_{j\in J} \mathrm{M}_j g\|_{L^{2,\infty}(\mathbb{R}^2)} \lesssim \kappa^{\frac{1}{2}}.$$

This proves the second of the estimates (7.17) and thus both of them by interpolation.

In the estimate for $O_{\mathcal{P}}$ we will also need the following decoupling result.

Lemma 7.18. *Let $2 \le p < 4$. Then*

$$\Big\| \sum_j T_j f_j \Big\|_p \lesssim \kappa^{\frac{1}{2}-\frac{1}{p}} \|\{f_j\}\|_{L^p(\mathbb{R}^2;\ell^2_J)}.$$

Proof. Note that the case $p = 2$ of the conclusion is trivial due to the finite overlap of the supports of the multipliers of the operators T_j. Thus by vector-valued restricted-type interpolation of the operator

$$\{f_j\} \mapsto O(\{f_j\}) := \sum_{j\in J} T_j f_j$$

it suffices to prove a restricted type $L^{4,1} \to L^4$ estimate:

$$\|O(\{f_j\})\|_4 \lesssim \kappa^{\frac{1}{4}} |E|^{\frac{1}{4}} \tag{7.19}$$

for functions with $\|\{f_j\}\|_{\ell^2} \le \mathbf{1}_E$. To do so note that the finite overlap of the supports of $\widehat{T_j f_j} * \widehat{T_k f_k}$ over j, k, as in (7.9), gives

$$\|O(\{f_j\})\|_4 \lesssim \|\{T_j f_j\}\|_{L^4(\mathbb{R}^2;\ell^2_J)}$$

and the restricted-type estimate (7.19) follows from (7.17). □

We come to the main argument for $O_{\mathcal{P}}$. Let $m_{\mathcal{P}}$ be as in (7.2)–(7.3) and T_j be the multiplier operators from (7.15) corresponding to the choice $\Phi = m_{\mathcal{P}}$. Then obviously

$$m_{\mathcal{P}} \hat{f} = \sum_{j\in J} \widehat{T_j f}.$$

We may also tweak Φ and the partition of unity on $\mathbb{S}^1$ to obtain further multiplier operators $\widetilde{T}_j$ as in (7.15) and such that the Fourier transform of the symbol of $\widetilde{T}_j$ equals 1 on the support of the symbol of T_j. With these definitions in hand we estimate for $2 < p < 4$

$$\begin{aligned}\|O_{\mathcal{P}} f\|_p = \Big\|\sum_j \widetilde{T}_j(T_j T_{\mathcal{P}} f)\Big\|_p &\lesssim \kappa^{\frac{1}{2}-\frac{1}{p}} \|\{T_{\mathcal{P}}(T_j f)\}\|_{L^p(\mathbb{R}^2;\ell^2_J)} \\ &= \kappa^{\frac{1}{2}-\frac{1}{p}} \|\{H_j H_{j+1}(T_j f)\}\|_{L^p(\mathbb{R}^2;\ell^2_J)}. \end{aligned} \tag{7.20}$$

The first inequality is an application of Lemma 7.18 for $\widetilde{T}_j$. The last equality is obtained by observing that the polygon multiplier $T_{\mathcal{P}}$ on the support of each T_j may be written as a (sum of $O(1)$) directional biparameter multipliers $H_j H_{j+1}$ of iterated Hilbert transform type, where H_j is a Hilbert transform along the direction ν_j, which is the unit vector perpendicular to the j-th side of the polygon, and pointing inside the polygon; these are at most $\sim N$ such directions.

In order to complete our estimate for $O_{\mathcal{P}}$ we need the following Meyer-type lemma for directional Hilbert transforms of the form

$$H_v f(x) := \int_{\mathbb{R}^2} \hat{f}(\xi) \mathbf{1}_{\{\xi\cdot v>0\}} e^{i x\cdot\xi}\, \mathrm{d}\xi, \quad x \in \mathbb{R}^2.$$

Lemma 7.21. *Let $V \subset \mathbb{S}^1$ be a finite set of directions and H_v be the Hilbert transform in the direction v. Then for $\frac{4}{3} < p < 4$ we have*

$$\|\{H_v f_v\}\|_{L^p(\mathbb{R}^2;\ell^2_V)} \lesssim (\log \#V)^{\left|\frac{1}{2}-\frac{1}{p}\right|} \|\{f_v\}\|_{L^p(\mathbb{R}^2;\ell^2_V)}.$$

The dependence on $\#V$ is best possible.

Proof. It suffices to prove the estimate for $2 < p < 4$. The proof is by way of duality and uses the following inequality for the Hilbert transform: for $r > 1$ and w a nonnegative locally integrable function we have

$$\int_{\mathbb{R}^2} |H_v f|^2 w \lesssim \int_{\mathbb{R}^2} |f|^2 (\mathrm{M}_v |w|^r)^{\frac{1}{r}},$$

with M_v given by (2.3). See for example [Pérez 1994]. Using this we have for a suitable $g \in L^{(p/2)'}$ of norm 1 that

$$\begin{aligned}\|\{H_v f_v\}\|^2_{L^p(\mathbb{R}^2;\ell^2_V)} = \int_{\mathbb{R}^2} \sum_{v\in V} |H_v f_v|^2 g &\lesssim \sum_{v\in V} \int_{\mathbb{R}^2} |f_v|^2 (\mathrm{M}_v |g|^r)^{\frac{1}{r}} \\ &\lesssim \|\{f_v\}\|^2_{L^p(\mathbb{R}^2;\ell^2_V)} \|(\mathrm{M}_V |g|^r)^{\frac{1}{r}}\|_{L^{(p/2)'}(\mathbb{R}^2)},\end{aligned}$$

with $\mathrm{M}_V g := \sup_{v\in V} \mathrm{M}_v g$. Now for $2 < p < 4$ there is a choice of $1 < r < p/(2(p-2))$ so that $p/(r(p-2)) > 2$. This means that the maximal theorem from [Katz 1999] applies again to give

$$\|(\mathrm{M}_V |g|^r)^{\frac{1}{r}}\|_{L^{(p/2)'}(\mathbb{R}^2)} \lesssim (\log \#V)^{1-\frac{2}{p}},$$

and so the proof of the upper bound is complete. The optimality is discussed in Section 8.1. □

Let us now go back to the estimate for $O_{\mathcal{P}}$. The left-hand side of (7.20) contains a double Hilbert transform. By an iterated application of Lemma 7.21 we thus have

$$\|\{H_j H_{j+1}(T_j f)\}\|_{L^p(\mathbb{R}^2;\ell^2_J)} \lesssim \kappa^{1-\frac{2}{p}} \|\{T_j f)\}\|_{L^p(\mathbb{R}^2;\ell^2_J)}$$

since the number of directions is $N = 2^{\kappa}$. The final estimate for the right-hand side of the display above is a direct application of (7.16), which together with (7.20) yields the estimate for $\|O_{\mathcal{P}} f\|_p$ claimed in (7.14).

Now the decomposition (7.4), together with the estimate of Section 7.5 for T_κ and the estimate (7.14) for $O_{\mathcal{P}}$, completes the proof of Theorem B.

Remark 7.22. Consider a function f in $\mathbb{R}^2$ such that $\operatorname{supp}(\hat{f}) \subseteq A_\delta$, where A_δ is an annulus of width δ^2 around $\mathbb{S}^1$. Decomposing A_δ into a union of $O(1/\delta)$ finitely overlapping annular boxes of radial width δ^2 and tangential width δ, we can write $f = \sum_{j\in J} T_j f$, where each T_j is a smooth frequency projection onto one of these annular boxes, indexed by j. Then if $\widetilde{T}_j$ is a multiplier operator whose symbol is identically 1 on the frequency support of $T_j f$ and supported on a slightly larger box, we can write $f = \sum_j \widetilde{T}_j T_j f$, as in (7.20) above. Then Lemma 7.18 yields

$$\|f\|_{L^p(\mathbb{R}^2)} \lesssim (\log(1/\delta))^{\frac{1}{2}-\frac{1}{p}} \|\{T_j f\}\|_{L^p(\mathbb{R}^2;\ell^2_J)}.$$

This is the inverse square function estimate claimed in the remark after Theorem B in Section 1.

8. Lower bounds and concluding remarks

8.1. ***Sharpness of Meyer's lemma.*** We briefly sketch the quantitative form of Fefferman's counterexample [1971] proving the sharpness of Lemma 7.21. Let N be a large dyadic integer. Using a standard Besicovitch-type construction we produce rectangles $\{R_j : j = 1, \dots, N\}$ with sidelengths $1 \times 1/N$, so that the long side of R_j is oriented along $v_j := \exp(2\pi i j/N)$. Now we consider the set E to be the union of these rectangles and

$$\left| E := \bigcup_{j=1}^{N} R_j \right| \lesssim \frac{1}{\log N}.$$

Denoting by $\widetilde{R}_j$ the 2-translate of R_j in the direction of v_j we gather that $\{\widetilde{R}_j : j = 1, \dots, N\}$ is a pairwise disjoint collection. Furthermore if H_j is the Hilbert transform in direction v_j, there holds

$$|H_j \mathbf{1}_{R_j}| \geq c \mathbf{1}_{\widetilde{R}_j}.$$

Therefore for all $1 < p < \infty$

$$\left\| \left(\sum_{j=1}^{N} |H_j \mathbf{1}_{R_j}|^2 \right)^{\frac{1}{2}} \right\|_p \geq c \left| \bigcup_{j=1}^{N} \widetilde{R}_j \right|^{\frac{1}{p}} \geq c,$$

while for $p \leq 2$

$$\left\| \left(\sum_{j=1}^{N} |\mathbf{1}_{R_j}|^2 \right)^{\frac{1}{2}} \right\|_p \leq \left(\sum_{j=1}^{N} |R_j| \right)^{\frac{1}{2}} |E|^{\frac{1}{p}-\frac{1}{2}} \lesssim (\log N)^{\frac{1}{2}-\frac{1}{p}}.$$

Self-duality of the square function estimate then gives the optimality of the estimate of Lemma 7.21.

8.2. ***Sharpness of the directional square function bound.*** In this subsection we prove that the bound of Theorem L is best possible, up to the doubly logarithmic terms. In particular we prove that the bound of Remark 6.1 is best possible.

We begin by showing a lower bound for the rough square function estimate

$$\|\{P_F g\}\|_{L^p(\mathbb{R}^2;\ell^2_{\mathcal{F}})} \leq \|\{P_F\} : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2;\ell^2_{\mathcal{F}})\| \|g\|_p, \quad 2 \leq p < 4, \tag{8.3}$$

where the notation is as in Section 6. Now as in [Fefferman 1971] one can easily show that the estimate above implies the vector-valued inequality for directional averages, for directions corresponding to the directions of rectangles in $\mathcal{F}$. For this let $\#V = N$, where V is the set of directions of rectangles in $\mathcal{F}$. Now consider functions $\{g_F\}_{F\in\mathcal{F}}$ with compact Fourier support; by modulating these functions we can assume that $\operatorname{supp}(\hat{g}_F) \subset B(c_F, A)$ for some $A > 1$ and $\{c_F\}_{F\in\mathcal{F}}$ a $100AN$-net in $\mathbb{R}^2$. Then if F is a rectangle centered at c_F with short side 1 parallel to a direction $v_F \in V$ and long side of length N parallel to $v_F^\perp$, then we have that $|P_F g_F| = |A_{v_F} g_F|$, where A_{v_F} is the averaging operator

$$A_{v_F} f(x) := 2N \int_{|t|\leq 1/2} \int_{N|s|<1} f(x - tv_F - sv_F^\perp)\, \mathrm{d}t\, \mathrm{d}s, \quad x \in \mathbb{R}^2.$$

Note that this is a single-scale average with respect to rectangles of dimensions $1 \times 1/N$ in the directions $v_F, v_F^\perp$ respectively. Since the frequency supports of these functions are well-separated we gather that for all choices of signs $\varepsilon_F \in \{-1, 1\}$ we have

$$\sum_{T\in\mathcal{F}} |P_T G|^2 := \sum_{T\in\mathcal{F}} \Big| P_T \Big(\sum_{F\in\mathcal{F}} \varepsilon_F g_F \Big) \Big|^2 = \sum_{T\in\mathcal{F}} |P_T g_T|^2.$$

Thus applying (8.3) with the function G as above and averaging over random signs we get

$$\|\{A_{v_F} g_F\}\|_{L^p(\mathbb{R}^2;\ell^2_{\mathcal{F}})} \leq \|\{P_F\} : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2;\ell^2_{\mathcal{F}})\| \|\{g_F\}\|_{L^p(\mathbb{R}^2;\ell^2_{\mathcal{F}})}, \quad 2 \leq p < 4.$$

Now we just need to note that as in Section 8.1 we have that

$$A_{v_F} \mathbf{1}_{R_F} \gtrsim \mathbf{1}_{\tilde{R}_F},$$

where $\{R_F\}_{F\in\mathcal{F}}$ are the rectangles used in the Besicovitch construction in Section 8.1. As before we get

$$\|\{P_F\} : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2;\ell^2_{\mathcal{F}})\| \gtrsim (\log \#V)^{\frac{1}{2}-\frac{1}{p}}.$$

For $p < 2$ the square function estimate (8.3) is known to fail even in the case of a single directions; see for example the counterexample in [Rubio de Francia 1985, §1.5].

One can use the same argument in order to show a lower bound for the norm of the smooth square function

$$\|\{P_F^\circ g\}\|_{L^p(\mathbb{R}^2;\ell^2_{\mathcal{F}})} \leq \|\{P_F^\circ\} : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2;\ell^2_{\mathcal{F}})\| \|g\|_p, \quad 2 \leq p < 4.$$

Indeed, following the exact same steps we can deduce a vector-valued inequality for smooth averages

$$A_{v_F}^\circ f(x) := \int_{\mathbb{R}} \int_{\mathbb{R}} f(x - tv_F - sv_F^\perp) \gamma_F(t, s)\, \mathrm{d}t\, \mathrm{d}s, \quad x \in \mathbb{R}^2,$$

where γ_F is the smooth product bump function used in the definition of P°_F in Section 6. By a direct computation one easily shows the analogous lower bound $A^\circ_{v_F}\mathbf{1}_{R_F} \gtrsim \mathbf{1}_{\widetilde{R}_F}$ for the rectangles of the Besicovitch construction and this completes the proof of the lower bound for smooth projections as well.

8.4. ***Sharpness of Córdoba's bound for radial multipliers.*** Firstly we remember the definition of each radial multiplier P_δ: Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a smooth function which is supported in $[-1, 1]$ and define

$$P_\delta f(x) := \int_{\mathbb{R}^2} \hat{f}(\xi)\Phi(\delta^{-1}(1-|\xi|))\mathrm{e}^{ix\cdot\xi}\,\mathrm{d}\xi, \quad x \in \mathbb{R}^2.$$

These smooth radial multipliers were used extensively in Section 7. Córdoba [1979] proved the bound

$$\|P_\delta f\|_p \lesssim (\log 1/\delta)^{\left|\frac{1}{2}-\frac{1}{p}\right|}\|f\|_p, \quad \tfrac{3}{4} \le p \le 4.$$

In fact the same bound is implicitly proved in Section 7 in a more refined form, but only in the open range $p \in \left(\frac{3}{4}, 4\right)$ with weak-type analogues at the endpoints. More precisely we have discretized P_δ into a sum of pieces $\{P_{\delta,j}\}_{j\in J}$, where each $P_{\delta,j}$ is a smooth projection onto an annular box of width δ and length $\sqrt{\delta}$, pointing along one of N equispaced directions v_j. Then it follows from the considerations in Section 7 that

$$\begin{aligned} \|\{P_{\delta,j} f\}\|_{L^p(\mathbb{R}^2;\ell^2_j)} &\lesssim \log(1/\delta)^{\frac{1}{2}-\frac{1}{p}}\|f\|_p, \quad 2 < p < 4, \\ \|\{P_{\delta,j} f\mathbf{1}_F\}\|_{L^4(\mathbb{R}^2;\ell^2_j)} &\lesssim \log(1/\delta)^{\frac{1}{4}}\|f\|_\infty|F|^{\frac{1}{4}}. \end{aligned} \tag{8.5}$$

Obviously one gets the same bound by duality for $\frac{4}{3} < p < 2$, while the L^2-bound is trivial. Now these estimates imply Córdoba's estimate for P_δ in the open range $\left(\frac{3}{4}, 4\right)$ by the decoupling inequality (7.10), also due to Córdoba. On the other hand Córdoba's estimate is sharp. Indeed one uses the same rescaling and modulation arguments as in the previous subsection in order to deduce a vector-valued inequality for smooth averages starting by Córdoba's estimate. Testing this vector-valued estimate against the rectangles of the Besicovitch construction proves the familiar lower bound for P_δ and thus also shows the optimality of the estimates in (8.5). We omit the details.

8.6. ***Lower bounds for the conical square function.*** We conclude this section with a simple example that provides a lower bound for the operator norm of the conical square function $\|C_\omega(f) : \ell^2_{\boldsymbol{\omega}}\|$ of Theorem J and the smooth conical square function $\|C^\circ_\omega : \ell^2_{\boldsymbol{\omega}}\|$ of Theorem A. The considerations in this subsection also rely on the Besicovitch construction so we adopt again the notation of Section 8.1 for the rectangles $\{R_j : 1 \le j \le N\}$ and their union E. Let H^+_j denote the frequency projection in the half-space $\{\xi \in \mathbb{R}^2 : \xi \cdot v_j > 0\}$, where $v_j := \exp(2\pi ij/N)$. We begin by observing that

$$H^+_j f - H^+_{j+1} f = C_j P_+ f - C_j P_- f, \tag{8.7}$$

where P_+, P_- denote the rough frequency projections in the upper and lower half-space respectively and C_{v_j} is the multiplier associated with the cone bordered by v_j, v_{j+1}. Since H^+_j is a linear combination of the identity with the usual directional Hilbert transform H_j along v_j we conclude that

$$\left\|\left(\sum_{j=1}^{N} |(H_{j+1}-H_j)f|^2\right)^{\frac{1}{2}}\right\|_p \lesssim \|\{C_j\} : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2;\ell^2_j)\|\|f\|_p, \quad 2 \le p < 4.$$

Now note that for each fixed $1 \le k \le N$ we have

$$\mathbf{1}_{\widetilde{R}_k} \sum_j (H_j - H_{j+1})\mathbf{1}_{R_j} = \mathbf{1}_{\widetilde{R}_k} H_k \mathbf{1}_{R_k} \gtrsim \mathbf{1}_{\widetilde{R}_k} \tag{8.8}$$

if $\widetilde{R}_k$ is a sufficiently large translation of R_k in the positive direction v_k. Thus

$$\left| \int \mathbf{1}_{\bigcup_k \widetilde{R}_k} \sum_{j=1}^{N} (H_{j+1} - H_j)\mathbf{1}_{R_j} \right| \gtrsim \left| \sum_k \int_{\widetilde{R}_k} \mathbf{1}_{\widetilde{R}_k} \right| \simeq 1.$$

On the other hand the left-hand side of the display above is bounded by a constant multiple of

$$\|\{C_j\} : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2; \ell_j^2)\| \left\| \left(\sum_j \mathbf{1}_{R_j}^2 \right)^{\frac{1}{2}} \right\|_{p'} \lesssim \|C_V : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2; \ell^2)\| (\log N)^{\frac{1}{2} - \frac{1}{p'}}$$

for all $2 \le p < 4$. We thus conclude that

$$\|\{C_j\} : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2; \ell^2)\| \gtrsim (\log N)^{\frac{1}{2} - \frac{1}{p}}, \quad 2 \le p < 4.$$

We explain how this counterexample can be modified to get a lower square function estimate for the smooth cone multipliers C_ω° from (5.1) matching the upper bound of Theorem A. For $t \in \mathbb{R}$ write $v_j^t := \exp(2\pi i (j+t)/N)$ and let H_j^t and $H_j^{t,+}$ be the directional Hilbert transform and analytic projection along v_j^t, respectively. Let $\delta > 0$ be a small parameter to be chosen later and for each $1 \le j \le N$ let ω_j be an interval of size δN^{-1} centered around $2\pi j/N$. Arguing as in (8.7),

$$C_{\omega_j}^\circ P_+ f - C_{\omega_j}^\circ P_- f = \int_{N|t|<\delta} \alpha\left(\frac{Nt}{\delta}\right) (H_j^{t,+} f - H_{j+1}^{t,+} f)\, \mathrm{d}t$$

for a suitable nonnegative averaging function α which equals 1 on $\left[-\frac{1}{4}, \frac{1}{4}\right]$. Now, if $\widetilde{R}_k$ is again a sufficiently large translation of R_k in the positive direction v_k and δ is chosen sufficiently small depending only on the translation amount, the analogue of (8.8) is

$$\mathbf{1}_{\widetilde{R}_k} \inf_{N|t|<\delta} \sum_{j=1}^{N} (H_j^t - H_{j+1}^t)\mathbf{1}_{R_j} = \mathbf{1}_{\widetilde{R}_k} \inf_{N|t|<\delta} H_k^t[\mathbf{1}_{R_k}] \gtrsim \mathbf{1}_{\widetilde{R}_k}.$$

The lower bound for $\|\{C_{\omega_j}\} : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2; \ell_j^2)\|$ then follows exactly as in the previous case.

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PARTIAL REGULARITY FOR NAVIER–STOKES AND LIQUID CRYSTALS INEQUALITIES WITHOUT MAXIMUM PRINCIPLE

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In 1985, V. Scheffer discussed partial regularity results for what he called solutions to the *Navier–Stokes inequality*. These maps essentially satisfy the incompressibility condition as well as the local and global energy inequalities and the pressure equation which may be derived formally from the Navier–Stokes system of equations, but they are not required to satisfy the Navier–Stokes system itself.

We extend this notion to a system considered by Fang-Hua Lin and Chun Liu in the mid 1990s related to models of the flow of nematic liquid crystals, which include the Navier–Stokes system when the director field d is taken to be zero. In addition to an extended Navier–Stokes system, the Lin–Liu model includes a further parabolic system which implies an a priori maximum principle for d which they use to establish partial regularity (specifically, $\mathcal{P}^1(\mathcal{S})=0$) of solutions.

For the analogous *inequality* one loses this maximum principle, but here we nonetheless establish the partial regularity result $\mathcal{P}^{9/2+\delta}(\mathcal{S})=0$, so that in particular the putative singular set $\mathcal{S}$ has space- time Lebesgue measure zero. Under an additional assumption on d for any fixed value of a certain parameter $\sigma\in(5,6)$ — which for $\sigma=6$ reduces precisely to the boundedness of d used by Lin and Liu — we obtain the same partial regularity ($\mathcal{P}^1(\mathcal{S})=0$) as do Lin and Liu. In particular, we recover the partial regularity result ($\mathcal{P}^1(\mathcal{S})=0$) of Caffarelli–Kohn–Nirenberg [1982] for suitable weak solutions of the Navier–Stokes system, and we verify Scheffer's assertion that the same holds for solutions of the weaker *inequality* as well.

We remark that the proofs of partial regularity both here and in the work of Lin and Liu largely follow the proof in Caffarelli–Kohn–Nirenberg, which in turn used many ideas from an earlier work of Scheffer [1975].

1. Introduction

Fang-Hua Lin and Chun Liu consider the following system in [Lin and Liu 1995; 1996], which reduces to the classical Navier–Stokes system in the case $d\equiv 0$ (here we have set various parameters equal to one for simplicity):

$$\left.\begin{aligned} u_t-\Delta u+\nabla^T\cdot[u\otimes u+\nabla d\odot\nabla d]+\nabla p=0\\ \nabla\cdot u=0\\ d_t-\Delta d+(u\cdot\nabla)d+f(d)=0\end{aligned}\right\}\tag{1-1}$$

with $f=\nabla F$ for a scalar field F given by

$$F(x):=(|x|^2-1)^2,$$

so that

$$f(x)=4(|x|^2-1)x$$

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(and in particular $f(0)=0$). We take the spatial dimension to be three, so that for some $\Omega\subseteq\mathbb{R}^3$ and $T>0$, we are considering maps of the form

$$u, d:\Omega\times(0,T)\to\mathbb{R}^3,\qquad p:\Omega\times(0,T)\to\mathbb{R},$$

and here

$$F:\mathbb{R}^3\to\mathbb{R},\qquad f:\mathbb{R}^3\to\mathbb{R}^3$$

are fixed as above. As usual, u represents the velocity vector field of a fluid, p is the scalar pressure in the fluid, and, as in nematic liquid crystals models, d corresponds roughly[1] to the *director field* representing the local orientation of rod-like molecules, with u also giving the velocities of the centers of mass of those anisotropic molecules.

In (1-1), for vector fields v and w, the matrix fields $v\otimes w$ and $\nabla v\odot\nabla w$ are defined to be those with entries

$$(v\otimes w)_{ij}=v_iw_j\quad\text{and}\quad(\nabla v\odot\nabla w)_{ij}=v_{,i}\cdot w_{,j}:=\frac{\partial v_k}{\partial x_i}\frac{\partial w_k}{\partial x_j}$$

(summing over the repeated index k as per the Einstein convention), and for a matrix field $J=(J_{ij})$, we define the vector field $\nabla^T\cdot J$ by[2]

$$(\nabla^T\cdot J)_i:=J_{ij,j}:=\frac{\partial J_{ij}}{\partial x_j}$$

(summing again over j). We think formally of ∇ (as well as any vector field) as a column vector and ∇^T as a row vector, so that each entry of (the column vector) $\nabla^T\cdot J$ is the divergence of the corresponding *row* of J. In what follows, for a vector field v we similarly denote by $\nabla^T v$ the matrix field with i-th row given by $\nabla^T v_i:=(\nabla v_i)^T$, i.e.,

$$(\nabla^T v)_{ij}=v_{i,j}:=\frac{\partial v_i}{\partial x_j},$$

so that for smooth vector fields v and w we always have

$$\nabla^T\cdot(v\otimes w)=(\nabla^T v)w+v(\nabla\cdot w)=(w\cdot\nabla)v+v(\nabla\cdot w).\tag{1-2}$$

For a scalar field ϕ we set $\nabla^2\phi:=\nabla^T(\nabla\phi)$, and for matrix fields $J=(J_{ij})$ and $K=(K_{ij})$, we let $J:K:=J_{ij}K_{ij}$ (summing over repeated indices) denote the (real) Frobenius inner product of the matrices; that is, $J:K=\operatorname{tr}(J^TK)$. We set $|J|:=\sqrt{J:J}$ and $|v|:=\sqrt{v\cdot v}$, and to minimize cumbersome notation will often abbreviate by writing $\nabla v:=\nabla^T v$ for a vector field v where the precise structure of the *matrix* field $\nabla^T v$ is not crucial; for example, $|\nabla v|:=|\nabla^T v|$.

We note that by formally taking the divergence $\nabla\cdot$ of the first line in (1-1) we obtain the usual *pressure equation*

$$-\Delta p=\nabla\cdot(\nabla^T\cdot[u\otimes u+\nabla d\odot\nabla d]).\tag{1-3}$$

[1] In principle, for d to only represent a *direction* one should have $|d|\equiv 1$. As proposed in [Lin and Liu 1995], $F(d)$ is used to model a Ginzburg–Landau type of relaxation of the pointwise constraint $|d|\equiv 1$. For further discussions on the modeling assumptions leading to systems such as the one above, see e.g., [Lin and Wang 2014] or the appendix of [Lin and Liu 1995].

[2] Many authors simply write $\nabla\cdot J$, which is perhaps more standard.

As in the Navier–Stokes ($d \equiv 0$) setting, one may formally deduce (see Section 2) from (1-1) the following global and local energy inequalities which one may expect solutions of (1-1) (with appropriate boundary conditions) to satisfy:[3]

$$\frac{d}{dt}\int_\Omega \left[\frac{|u|^2}{2}+\frac{|\nabla d|^2}{2}+F(d)\right]dx+\int_\Omega [|\nabla u|^2+|\Delta d-f(d)|^2]\,dx \le 0 \tag{1-4}$$

for each $t \in (0,T)$, as well as a localized version[4]

$$\begin{aligned}\frac{d}{dt}\int_\Omega\left[\left(\frac{|u|^2}{2}+\frac{|\nabla d|^2}{2}\right)\phi\right]dx+\int_\Omega(|\nabla u|^2+|\nabla^2 d|^2)\phi\,dx \\ \le \int_\Omega\Bigg[\left(\frac{|u|^2}{2}+\frac{|\nabla d|^2}{2}\right)(\phi_t+\Delta\phi)+\left(\frac{|u|^2}{2}+\frac{|\nabla d|^2}{2}+p\right)u\cdot\nabla\phi \\ +u\otimes\nabla\phi:\nabla d\odot\nabla d-\underbrace{\phi\nabla^T[f(d)]:\nabla^T d}_{=:\mathcal{R}_f(d,\phi)}\Bigg]dx\end{aligned} \tag{1-5}$$

for $t \in (0,T)$ and each smooth, compactly supported in Ω and nonnegative scalar field $\phi \ge 0$. (For Navier–Stokes, i.e., when $d \equiv 0$, one may omit all terms involving d, even though $0 \ne F(0) \notin L^1(\mathbb{R}^3)$.)

In [Lin and Liu 1995], for smooth and bounded Ω, the global energy inequality (1-4) is used to construct global weak solutions to (1-1) for initial velocity in $L^2(\Omega)$, along with a similarly appropriate condition on the initial value of d which allows (1-4) to be integrated over $0 < t < T$. This is consistent with the pioneering result of J. Leray [1934] for Navier–Stokes (treated later by many other authors using various methods, but always relying on the natural energy as in [Leray 1934]).

In [Lin and Liu 1996], the authors establish a partial regularity result for weak solutions to (1-1) belonging to the natural energy spaces which moreover satisfy the local energy inequality (1-5). The result is of the same type as known partial regularity results for a class of solutions known as *suitable weak solutions* to the Navier–Stokes equations. The program for such partial regularity results for Navier–Stokes was initiated in a series of papers by V. Scheffer in the 1970s and 1980s (see, for example, [Scheffer 1977; 1980] and other works mentioned in [Caffarelli et al. 1982]), and subsequently improved by L. Caffarelli, R. Kohn and L. Nirenberg in [Caffarelli et al. 1982].[5] They show (as do Lin and Liu [1996]) that the one-dimensional parabolic Hausdorff measure of the (potentially empty) singular set S is zero ($\mathcal{P}^1(S)=0$, see Definition 2 below), implying that singularities (if they exist) cannot for example contain any smooth one-parameter curve in space-time. The method of proof in [Lin and Liu 1996] largely follows the method of [Caffarelli et al. 1982].

Of course the general system (1-1) is (when $d \not\equiv 0$) substantially more complex than the Navier–Stokes system, and one therefore could not expect a stronger result than the type in [Caffarelli et al. 1982], i.e., $\mathcal{P}^1(S)=0$; in fact, it is surprising that such a result still holds even when $d \not\equiv 0$. The explanation

[3]For sufficiently regular solutions one can show that equality holds.

[4]Note that in [Lin and Liu 1996], the term "$-\mathcal{R}_f(d,\phi)$" in (1-5) actually appears incorrectly as "$+\mathcal{R}_f(d,\phi)$"; see Section 2.

[5]Alternative proofs of slight variations of the main results in [Caffarelli et al. 1982] were given in later works such as [Lin 1998; Ladyzhenskaya and Seregin 1999; Vasseur 2007; Kukavica 2009].

for this seems to be that although (1-1) is more complex than Navier–Stokes in view of the additional d components, one can derive an a priori maximum principle for d because of the third equation in (1-1) which substantially offsets this complexity from the viewpoint of regularity. Therefore, under suitable boundary and initial conditions on d, one may assume that d is in fact bounded, a fact which is significantly exploited in [Lin and Liu 1996]. More recently, the authors of [Du et al. 2020] established the same type of result for a related but more complex *Q-tensor* system; however there, as well, one may obtain a maximum principle which is of crucial importance for proving partial regularity. One is therefore led to the following natural question, which we will address below:

***Can one deduce any partial regularity for systems similar in structure to* (1-1) *but which lack any maximum principle*?**

In the Navier–Stokes setting, it was asserted by Scheffer [1985] that in fact the proof of the partial regularity result in [Caffarelli et al. 1982] does not require the full set of equations in (1-1). He mentions that the key ingredients are membership of the global energy spaces, the local energy inequality (1-5), the divergence-free condition $\nabla \cdot u = 0$ and the *pressure* equation (1-3) (with $d \equiv 0$ throughout). Scheffer called vector fields satisfying these four requirements solutions to the *Navier–Stokes inequality*, equivalent to solutions to the Navier–Stokes equations with a forcing f which satisfies $f \cdot u \leq 0$ everywhere. In contrast, the results in [Lin and Liu 1996] do very strongly use the third equation in (1-1) in that it implies a maximum principle for d.

In this paper, we explore what happens if one considers the analog of Scheffer's *Navier–Stokes inequality* for the system (1-1) when $d \not\equiv 0$. That is, we consider triples (u, d, p) with global regularities implied — at least when Ω is bounded and under suitable assumptions on the initial data — by (1-4) which satisfy (1-3) and $\nabla \cdot u = 0$ weakly as well as (a formal consequence of) (1-5), but are *not* necessarily weak solutions of the first and third equations (i.e., the two vector equations) in (1-1). In particular, we will *not* assume that $d \in L^\infty(\Omega \times (0, T))$, which would have been reasonable in view of the third equation in (1-1). We see that without further assumptions, the result is substantially weaker than the $\mathcal{P}^1(S) = 0$ result for Navier–Stokes: following the methods of [Caffarelli et al. 1982; Lin and Liu 1996] we obtain (see Theorem 1) $\mathcal{P}^{9/2+\delta}(S) = 0$ for any $\delta > 0$. This reinforces our intuition that the situation here is substantially more complex than that of Navier–Stokes. On the other hand, we show that under a suitable uniform local decay condition on $|d|^\sigma(|u|^3 + |\nabla d|^3)^{1-\sigma/6}$ with $\sigma \in (5, 6)$ — see (1-14) below, which in particular holds when $d \equiv 0$ as in [Caffarelli et al. 1982] — one in fact obtains $\mathcal{P}^1(S) = 0$ as in [Caffarelli et al. 1982; Lin and Liu 1996]. In particular, we verify the above-mentioned assertion made by Scheffer [1985] regarding partial regularity for the Navier–Stokes inequality.

Our key observation which allows us to work without any maximum principle is that, in view of the global energy (1-4) and the particular forms of F and f, it is reasonable (see Section 2) to assume (1-9); this implies that $d \in L^\infty(0, T; L^6(\Omega))$ which is sufficient for our purposes.

As alluded to above, for our purposes we actually do not require all of the information which appears in (1-5). In view of the fact that

$$|\mathcal{R}_f(d, \phi)| = |\phi \nabla^T[f(d)] : \nabla^T d| \leq 12|d|^2|\nabla d|^2\phi + 8\left(\frac{|\nabla d|^2}{2}\phi\right) \tag{1-6}$$

(see (2-21) below), a consequence of (1-5) is that

$$\mathcal{A}'(t)+\mathcal{B}(t)\le 8\mathcal{A}(t)+\mathcal{C}(t) \quad \text{for } 0<t<T, \tag{1-7}$$

with $\mathcal{A},\mathcal{B},\mathcal{C}\ge 0$ defined as

$$\mathcal{A}(t):=\int_{\Omega\times\{t\}}\Big(\frac{|u|^2}{2}+\frac{|\nabla d|^2}{2}\Big)\phi, \quad \mathcal{B}(t):=\int_{\Omega\times\{t\}}(|\nabla u|^2+|\nabla^2 d|^2)\phi$$

and

$$\mathcal{C}(t):=\int_{\Omega\times\{t\}}\Big[\Big(\frac{|u|^2}{2}+\frac{|\nabla d|^2}{2}\Big)|\phi_t+\Delta\phi|+12|d|^2|\nabla d|^2\phi\Big] \\ +\Big|\int_{\Omega\times\{t\}}\Big[\Big(\frac{|u|^2}{2}+\frac{|\nabla d|^2}{2}+p\Big)u\cdot\nabla\phi+u\otimes\nabla\phi:\nabla d\odot\nabla d\Big]\Big|,$$

where $\int_{\Omega\times\{t\}} g := \int_\Omega g(\,\cdot\,,t)\,dx$. Equation (1-7) is nearly sufficient, with the term $\mathcal{A}(t)$ on the right-hand side — in fact, even with u omitted, which cannot be avoided as "$\mathcal{R}_f(d,\phi)$" appears on the right-hand side of (1-5) with a minus sign (see footnote 4) — actually being, for technical reasons, the only[6] troublesome term. (Note that if $\mathcal{R}_f(d,\phi)$ had appeared with a plus sign in (1-5), one could have simply dropped the troublesome $\phi|\nabla d|^2$ term in (2-21) as a nonpositive quantity.) We therefore use a Grönwall-type argument to hide this term on the left-hand side of (1-7) so that (if $\phi|_{t=0}\equiv 0$)

$$\mathcal{A}'(t)+\mathcal{B}(t)\le\mathcal{C}(t)+8e^{8T}\int_0^t\mathcal{C}(\tau)\,d\tau \quad \text{for } 0<t<T. \tag{1-8}$$

The (formally derived) local energy inequality (1-8) implies (1-13) below (for an appropriate constant $\bar{C}\sim 8Te^{8T}+1$), which is sufficient for our purposes. (In fact, for all elements of the proof other than Proposition 8, a weaker form as in (3-5) is sufficient.)

Our main result is the following.

Theorem 1. *Fix any open set $\Omega\subset\mathbb{R}^3$ and any $T,\bar{C}\in(0,\infty)$. Set $\Omega_T:=\Omega\times(0,T)$ and suppose $u,d:\Omega_T\to\mathbb{R}^3$ and $p:\Omega_T\to\mathbb{R}$ satisfy the following four assumptions*:

(1) *u, d and p belong to the following spaces*:[7]

$$u,\ d,\ \nabla d\in L^\infty(0,T;L^2(\Omega)), \quad \nabla u,\ \nabla d,\ \nabla^2 d\in L^2(\Omega_T) \tag{1-9}$$

and

$$p\in L^{3/2}(\Omega_T); \tag{1-10}$$

[6] In fact, the appearance of $|d|^2$ on the right-hand side of (1-6), and hence of (1-7) as well, is handled precisely by the assumption that $d\in L^\infty(0,T;L^6(\Omega))$, and is the reason for the slightly weaker results compared to the Navier–Stokes setting (i.e., when $d\equiv 0$).

[7] For a vector field f or matrix field J and scalar function space X, by $f\in X$ or $J\in X$ we mean that all components or entries of f or J belong to X; by $\nabla^2 f\in X$ we mean all second partial derivatives of all components of f belong to X; etc.

(2) *u is weakly divergence-free*:[8]

$$\nabla \cdot u = 0 \quad \textit{in } \mathcal{D}'(\Omega_T); \tag{1-11}$$

(3) *The following pressure equation holds weakly*:[9]

$$-\Delta p = \nabla \cdot [\nabla^T \cdot (u \otimes u + \nabla d \odot \nabla d)] \quad \textit{in } \mathcal{D}'(\Omega_T); \tag{1-12}$$

(4) *The following local energy inequality holds*:[10]

$$\begin{aligned}\int_{\Omega\times\{t\}} (|u|^2 + |\nabla d|^2)\phi \, dx + \int_0^t \int_\Omega (|\nabla u|^2 + |\nabla^2 d|^2)\phi \, dx \, d\tau \\ \le \bar{C} \int_0^t \Bigg\{ \int_{\Omega\times\{\tau\}} [(|u|^2 + |\nabla d|^2)|\phi_t + \Delta\phi| + |d|^2|\nabla d|^2\phi] \, dx \\ + \left| \int_{\Omega\times\{\tau\}} \left[\left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + p \right) u \cdot \nabla\phi + u \otimes \nabla\phi : \nabla d \odot \nabla d \right] dx \right| \Bigg\} \, d\tau \\ \textit{for a.e. } t \in (0, T) \quad \textit{and} \quad \textit{for all } \phi \in C_0^\infty(\Omega \times (0, \infty)) \textit{ such that } \phi \ge 0.\end{aligned} \tag{1-13}$$

Let $\mathcal{S} \subset \Omega_T$ be the (potentially empty) set of singular points where $|u| + |\nabla d|$ is not essentially bounded in any neighborhood of each $z \in \mathcal{S}$, and let $\mathcal{P}^k$ be the k-dimensional parabolic Hausdorff outer measure (see Definition 2). The following are then true:

(i) $\mathcal{P}^{9/2+\delta}(\mathcal{S}) = 0$, *for any $\delta > 0$ arbitrarily small.*

(ii) *If*[11]

$$g_\sigma := \sup_{z_0 \in \Omega_T} \left(\limsup_{r \searrow 0} \frac{1}{r^{2+\sigma/2}} \iint_{Q_r(z_0)} |d|^\sigma (|u|^3 + |\nabla d|^3)^{1-\sigma/6} \, dz \right) < \infty \tag{1-14}$$

for some $\sigma \in (5, 6)$, then $\mathcal{P}^1(\mathcal{S}) = 0$.

Note that in the case $d \equiv 0$, we regain the classical result of $\mathcal{P}^1(\mathcal{S}) = 0$ for Navier–Stokes as obtained in, for example, [Caffarelli et al. 1982], and more specifically for the (weaker) Navier–Stokes inequality mentioned in [Scheffer 1985].

We recall that the definition of the outer parabolic Hausdorff measure $\mathcal{P}^k$ is given as follows, see [Caffarelli et al. 1982, pp. 783–784]:

Definition 2 (parabolic Hausdorff measure). For any $\mathcal{S} \subset \mathbb{R}^3 \times \mathbb{R}$ and $k \ge 0$, define

$$\mathcal{P}^k(\mathcal{S}) := \lim_{\delta \searrow 0} \mathcal{P}^k_\delta(\mathcal{S}),$$

---

[8]Locally integrable functions will always be associated to the standard distribution whose action is integration against a suitable test function so that, e.g., $[\nabla \cdot u](\psi) = -[u](\nabla\psi) := -\int u \cdot \nabla\psi$ for $\psi \in \mathcal{D}(\Omega_T)$.

[9]Note that $u \otimes u + \nabla d \odot \nabla d \in L^{5/3}(\Omega_T) \subset L^1_{\mathrm{loc}}(\Omega_T)$, see (2-18)–(2-19).

[10]For brevity, for $\omega \subset \mathbb{R}^3$, we set $\int_{\omega\times\{t\}} g \, dx := \int_\omega g(x, t) \, dx$.

[11]In general we set $z = (x, t) \in \Omega_T$, $dz := dx \, dt$ and recall from Definition 2 that $Q_r(x_0, t_0) := B_r(x_0) \times (t_0 - r^2, t_0)$.

where

$$\mathcal{P}^k_\delta(\mathcal{S}) := \inf\left\{ \sum_{j=1}^{\infty} r_j^k \;\middle|\; \mathcal{S} \subset \bigcup_{j=1}^{\infty} Q_{r_j},\ r_j < \delta,\ \forall j \in \mathbb{N} \right\}$$

and Q_r is any parabolic cylinder of radius $r > 0$, i.e.,

$$Q_r = Q_r(x,t) := B_r(x) \times (t - r^2, t) \subset \mathbb{R}^3 \times \mathbb{R}$$

for some $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$. We note that $\mathcal{P}^k$ is an outer measure, and all Borel sets are $\mathcal{P}^k$-measurable.

Remark 3. In the case $\Omega = \mathbb{R}^3$, the condition (1-10) on the pressure follows (locally, at least) from (1-9) and (1-12) if p is taken to be the potential-theoretic solution to (1-12), since (1-9) implies that $u, \nabla d \in L^{10/3}(\Omega_T)$ by interpolation (see (2-18) below) and Sobolev embeddings, and then (1-12) gives $p \in L^{5/3}(\Omega_T) \subset L^{3/2}_{\mathrm{loc}}(\Omega_T)$ by Calderón–Zygmund estimates. For a more general Ω, the existence of such a p can be derived from the motivating equation (1-1) (e.g., by estimates for the Stokes operator); see [Lin and Liu 1996]. Here, however, we will not refer to (1-1) at all and simply *assume* p satisfies (1-10) and address the partial regularity of such a hypothetical set of functions satisfying (1-9)–(1-13).

We note that Theorem 1 does not immediately recover the result of [Lin and Liu 1996] (which would correspond to $\sigma = 6$ in (1-14), which holds when $d \in L^\infty$ as assumed in that paper). Heuristically, however, one can argue as follows:[12]

If d were bounded, then taking for example $D := 24\|d\|^2_{L^\infty(\Omega_T)} + 8 < \infty$ one would be able to deduce from (1-6) that

$$|\mathcal{R}_f(d,\phi)| \le D\left(\frac{|\nabla d|^2}{2}\right)\phi.$$

Adjusting the Grönwall-type argument leading to (1-8), one could then deduce from (1-5) that (if $\mathcal{A}(0) = 0$)

$$\mathcal{A}'(t) + \mathcal{B}(t) \le \tilde{\mathcal{C}}(t) + De^{DT} \int_0^T \tilde{\mathcal{C}}(\tau)\, d\tau \quad \text{for } 0 < t < T,$$

where

$$\tilde{\mathcal{C}}(t) := \int_{\Omega\times\{t\}} \left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2}\right)|\phi_t + \Delta\phi| + \left| \int_{\Omega\times\{t\}} \left[\left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + p\right) u\cdot\nabla\phi + u\otimes\nabla\phi : \nabla d \odot \nabla d\right]\right|.$$

Using such an energy inequality, one would not need to include the $|d|^6$ term in $E_{3,6}$ (see (3-6)) as one would not need to consider the term coming from $\mathcal{R}_f(d,\phi)$ at all in Proposition 6, and — noting that the L^∞ norm is invariant under the rescaling on d in (3-25) — one could then adjust Lemmas 4 and 7 appropriately to recover the result in [Lin and Liu 1996] using the proof of Theorem 1 below.

Finally, we remark that the majority of the arguments in the proofs given below are not new, with many essentially appearing in [Lin and Liu 1996] or [Caffarelli et al. 1982]. However we feel that our presentation is particularly transparent and may be a helpful addition to the literature, and we include all details so that our results are easily verifiable.

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[12] We assume this is roughly the argument in [Lin and Liu 1996], although the details are not explicitly given; see, in particular, [Lin and Liu 1996, (2.45)] which appears without the *remainder* term denoted in [Lin and Liu 1996] by $\mathbf{R}(f,\phi)$, and here by $\mathcal{R}_f(d,\phi)$.

2. Motivation

We will show in this section that the assumptions in Theorem 1 are at least formally satisfied by smooth solutions to the system (1-1).

2.1. ***Energy identities.*** As in [Lin and Liu 1996], let us assume that we have smooth solutions to (1-1) which vanish or decay sufficiently at $\partial\Omega$ (assumed smooth, if nonempty) and at spatial infinity as appropriate so that all boundary terms vanish in the following integrations by parts, and proceed to establish smooth versions of (1-4) and (1-5). First, noting the simple identities

$$\nabla^T \cdot (\nabla d \odot \nabla d) = \nabla\left(\frac{|\nabla d|^2}{2}\right) + (\nabla^T d)^T \Delta d \tag{2-1}$$

and

$$[(\nabla^T d)^T \Delta d] \cdot u = [(\nabla^T d) u] \cdot \Delta d = [(u \cdot \nabla) d] \cdot \Delta d, \tag{2-2}$$

at a fixed t one may perform various integrations by parts — keeping in mind that $\nabla \cdot u = 0$ — to see that

$$\begin{aligned} 0 &= \int_\Omega [u_t - \Delta u + \nabla^T \cdot (u \otimes u) + \nabla p + \nabla^T \cdot (\nabla d \odot \nabla d)] \cdot u \, dx \\ &= \int_\Omega \left[\frac{\partial}{\partial t}\left(\frac{|u|^2}{2}\right) + |\nabla u|^2 + \mathbf{[(u \cdot \nabla) d] \cdot \Delta d}\right] dx \end{aligned} \tag{2-3}$$

and — recalling that $f = \nabla F$ so that $[d_t + (u \cdot \nabla) d] \cdot f(d) = \left(\frac{\partial}{\partial t} + u \cdot \nabla\right)[F(d)]$ — that

$$\begin{aligned} 0 &= -\int_\Omega [d_t + (u \cdot \nabla) d - (\Delta d - f(d))] \cdot (\Delta d - f(d)) \, dx \\ &= -\int_\Omega \left[-\frac{\partial}{\partial t}\left(\frac{|\nabla d|^2}{2} + F(d)\right) + \mathbf{[(u \cdot \nabla) d] \cdot \Delta d} - |\Delta d - f(d)|^2\right] dx. \end{aligned} \tag{2-4}$$

Adding the two gives the ***global energy identity for* (1-1)**:

$$\frac{d}{dt} \int_\Omega \left[\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + F(d)\right] dx + \int_\Omega [|\nabla u|^2 + |\Delta d - f(d)|^2] \, dx = 0 \tag{2-5}$$

in view of the cancellation of the terms in bold in (2-3) and (2-4).

It is not quite straightforward to localize the calculations in (2-3) and (2-4), for example replacing the (global) multiplicative factor $(\Delta d - f(d))$ by $(\Delta d - f(d))\phi$ for a smooth and compactly supported ϕ. Arguing as in [Lin and Liu 1996], one can deduce a local energy identity by instead replacing $(\Delta d - f(d))$ by only a part of its localized version in divergence-form, namely by $\nabla^T \cdot (\phi \nabla^T d)$, at the expense of the appearance of $|\Delta d - f(d)|^2$ anywhere in the local energy.

Recalling (2-1) and (2-2) and noting further that

$$\begin{aligned} [(u \cdot \nabla) d] \cdot [\nabla^T \cdot (\phi \nabla^T d)] &= [(u \cdot \nabla) d] \cdot [\phi \Delta d] + [(u \cdot \nabla) d] \cdot [(\nabla \phi \cdot \nabla) d] \\ &= [(u \cdot \nabla) d] \cdot [\phi \Delta d] + u \otimes \nabla \phi : \nabla d \odot \nabla d \end{aligned}$$

and

$$[\Delta(\nabla^T d)] : \nabla^T d = \Delta\left(\frac{|\nabla d|^2}{2}\right) - |\nabla^2 d|^2,$$

one may perform various integrations by parts to deduce (as $\nabla \cdot u = 0$) that

$$\begin{aligned}0 &= \int_\Omega [u_t - \Delta u + \nabla^T \cdot (u \otimes u) + \nabla p + \nabla^T \cdot (\nabla d \odot \nabla d)] \cdot u\phi \, dx \\ &= \int_\Omega \left[\frac{\partial}{\partial t}\left(\frac{|u|^2}{2}\phi\right) + |\nabla u|^2 \phi - \frac{|u|^2}{2}(\phi_t + \Delta\phi) - \left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + p\right) u \cdot \nabla\phi + \boldsymbol{[(u \cdot \nabla) d] \cdot (\Delta d) \phi} \right] dx\end{aligned}$$

and

$$\begin{aligned}0 &= -\int_\Omega [d_t + (u \cdot \nabla) d - (\Delta d - f(d))] \cdot [\nabla^T \cdot (\phi \nabla^T d)] \, dx \\ &= -\int_\Omega \Bigg[-\frac{\partial}{\partial t}\left(\frac{|\nabla d|^2}{2}\phi\right) - |\nabla^2 d|^2 \phi + \frac{|\nabla d|^2}{2}(\phi_t + \Delta\phi) \\ &\qquad\qquad - \nabla^T [f(d)] : \phi \nabla^T d + \boldsymbol{[(u \cdot \nabla) d] \cdot (\Delta d) \phi} + u \otimes \nabla\phi : \nabla d \odot \nabla d \Bigg] dx\end{aligned}$$

for smooth and compactly supported ϕ. Upon adding the two equations above and noting again the cancellation of the terms in bold, we obtain the ***local energy identity for* (1-1):**

$$\begin{aligned}&\frac{d}{dt} \int_\Omega \left[\left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} \right) \phi \right] dx + \int_\Omega (|\nabla u|^2 + |\nabla^2 d|^2) \phi \, dx \\ &\qquad = \int_\Omega \Bigg[\left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} \right)(\phi_t + \Delta\phi) + \left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + p \right) u \cdot \nabla\phi \\ &\qquad\qquad + u \otimes \nabla\phi : \nabla d \odot \nabla d - \underbrace{\phi \nabla^T [f(d)] : \nabla^T d}_{=:\mathcal{R}_f(d,\phi)} \Bigg] dx. \end{aligned} \tag{2-6}$$

Note that the term

$$u \otimes \nabla\phi : \nabla d \odot \nabla d = [(\nabla d \odot \nabla d) \nabla\phi] \cdot u = [(u \cdot \nabla) d] \cdot [(\nabla\phi \cdot \nabla) d]$$

in (2-6) is a more accurate version of what is described in [Lin and Liu 1996] as "$((u \cdot \nabla) d \odot \nabla d) \cdot \nabla\phi$", and that the term "$-\mathcal{R}_f(d, \phi)$" in (2-6) appears incorrectly in that paper as "$+\mathcal{R}_f(d, \phi)$".

2.2. *Global energy regularity heuristics.* Let us first see where the *global* energy identity (2-5) leads us to expect weak solutions to (1-1) to live (and hence why we assume (1-9) in Theorem 1).

To ease notation, in what follows let us fix $\Omega \subset \mathbb{R}^3$, and for $T \in (0, \infty]$ let us set $\Omega_T := \Omega \times (0, T)$ and

$$L_t^r L_x^q(T) := L^r(0, T; L^q(\Omega)).$$

According to (2-5), we expect, so long as

$$M_0 := \tfrac{1}{2}\|u(\cdot, 0)\|_{L^2(\Omega)}^2 + \tfrac{1}{2}\|\nabla d(\cdot, 0)\|_{L^2(\Omega)}^2 + \|F(d(\cdot, 0))\|_{L^1(\Omega)} < \infty,$$

(which we would assume as a requirement on the initial data), to construct solutions with u in the usual Navier–Stokes spaces:

$$u \in L_t^\infty L_x^2(\infty) \quad \text{and} \quad \nabla u \in L_t^2 L_x^2(\infty). \tag{2-7}$$

As for d we expect as well in view of (2-5) that

$$\nabla d \in L_t^\infty L_x^2(\infty), \quad F(d) \in L_t^\infty L_x^1(\infty) \quad \text{and} \quad [\Delta d - f(d)] \in L_t^2 L_x^2(\infty). \tag{2-8}$$

The norms of all quantities in the spaces given in (2-7) and (2-8) are controlled by either M_0 (the $F(d)$ term) or $(M_0)^{1/2}$ (all other terms), by integrating (2-5) over $t \in (0, \infty)$. Recalling that

$$F(d) := (|d|^2 - 1)^2 \quad \text{and} \quad f(d) := 4(|d|^2 - 1)d, \tag{2-9}$$

one sees that $|f(d)|^2 = 16F(d)|d|^2$, and one can easily confirm the following simple estimates:

$$\|d\|^2_{L_t^\infty L_x^4(\infty)} \le \|F(d)\|^{1/2}_{L_t^\infty L_x^1(\infty)} + \|1\|_{L_t^\infty L_x^2(\infty)}, \tag{2-10}$$

$$\|F(d)\|^{1/2}_{L_t^\infty L_x^{3/2}(\infty)} \le \|d\|^2_{L_t^\infty L_x^6(\infty)} + \|1\|_{L_t^\infty L_x^3(\infty)}, \tag{2-11}$$

$$\|f(d)\|^2_{L_t^\infty L_x^2(\infty)} \le 16\|F(d)\|_{L_t^\infty L_x^{3/2}(\infty)} \|d\|^2_{L_t^\infty L_x^6(\infty)}, \tag{2-12}$$

$$\|\Delta d\|_{L^2(\Omega_T)} \le \|\Delta d - f(d)\|_{L^2(\Omega_T)} + T^{1/2}\|f(d)\|_{L_t^\infty L_x^2(\infty)}. \tag{2-13}$$

Therefore, if we assume that

$$|\Omega| < \infty, \tag{2-14}$$

and hence

$$1 \in L^\infty(0, \infty; L^2(\Omega)) \cap L^\infty(0, \infty; L^3(\Omega)),$$

(2-8) along with (2-10) and (2-14) implies that

$$d \in L^\infty(0, \infty; L^4(\Omega)) \subset L^\infty(0, \infty; L^2(\Omega)). \tag{2-15}$$

This, along with (2-8), then implies that

$$d \in L^\infty(0, \infty; H^1(\Omega)) \hookrightarrow L^\infty(0, \infty; L^6(\Omega)) \tag{2-16}$$

by the Sobolev embedding, from which (2-11) implies that

$$F(d) \in L_t^\infty L_x^{3/2}(\infty)$$

which, along with (2-12) and (2-16), implies that

$$f(d) \in L_t^\infty L_x^2(\infty)$$

which, finally, in view of (2-13) and the last inclusion in (2-8), implies that

$$\Delta d \in L^2(\Omega_T) \quad \text{for any } T < \infty, \tag{2-17}$$

with the explicit estimate (2-13) which can then further be controlled by M_0 via (2-8) and (2-10)–(2-12).

We therefore see that it is reasonable (in view of the usual elliptic regularity theory) to expect that weak solutions to (1-1) should have the regularities in (1-9) of Theorem 1.

Note further that various interpolations of Lebesgue spaces imply, for example, that for any interval $I \subset \mathbb{R}$ one has

$$L^\infty(I; L^2(\Omega)) \cap L^2(I; L^6(\Omega)) \subset L^{2/\alpha}(I; L^{6/(3-2\alpha)}(\Omega)) \quad \text{for any } \alpha \in [0, 1]; \tag{2-18}$$

for example, one may take $\alpha = \frac{3}{5}$ so that $2/\alpha = 6/(3-2\alpha) = \frac{10}{3}$. Using this along with the Sobolev embedding we expect (as mentioned in Remark 3) that

$$(u \text{ and}) \ \nabla d \in L^{2/\alpha}(0, T; L^{6/(3-2\alpha)}(\Omega)) \quad \text{for any } \alpha \in [0, 1], \ T < \infty \tag{2-19}$$

with the explicit estimate[13]

$$\|\nabla d\|^{2/\alpha}_{L_t^{2/\alpha} L_x^{6/(3-2\alpha)}(T)} \lesssim T \|\nabla d\|^{2/\alpha}_{L_t^\infty L_x^2(\infty)} + \|\nabla d\|^{2/\alpha - 2}_{L_t^\infty L_x^2(\infty)} \|\nabla^2 d\|^2_{L^2(\Omega_T)}.$$

Note that (2-19) along with (2-16), (2-14) and the Sobolev embedding implies that $d \in L^s(0, T; L^\infty(\Omega))$ as well for any $T < \infty$ and $s \in [2, 4)$.

2.3. ***Local energy regularity heuristics.*** Here, we will justify the well-posedness of the terms appearing in the *local* energy equality (2-6), based on the expected *global* regularity discussed in the previous section. In fact, all but the final term in (2-6) (where one can furthermore take the essential supremum over $t \in (0, T)$) can be seen to be well defined by (2-19) under the assumptions in (1-9) and (1-10).

The $\mathcal{R}_f(d, \phi)$ term of (2-6) requires some further consideration: in view of (2-9) we see that

$$\tfrac{1}{4}\nabla^T[f(d)] = \nabla^T[(|d|^2 - 1)d] = 2d \otimes [d \cdot (\nabla^T d)] + (|d|^2 - 1)\nabla^T d. \tag{2-20}$$

Recalling that

$$\mathcal{R}_f(d, \phi) := \phi \nabla^T[f(d)] : \nabla^T d,$$

we therefore have

$$\tfrac{1}{4}\mathcal{R}_f(d, \phi) = \phi(2d \otimes [d \cdot (\nabla^T d)] : \nabla^T d + |d|^2 |\nabla d|^2) - \phi |\nabla d|^2, \tag{2-21}$$

where we have to be careful how we handle the appearance of, essentially, $|d|^2$ in the first term (the second term is integrable in view of (2-8)). We have, for example, that

$$\|\phi |d|^2 |\nabla d|^2\|_{L^1(\Omega_T)} \le \|\phi\|_{L^\infty(\Omega_T)} \|d\|^2_{L^6(\Omega_T)} \|\nabla d\|^2_{L^3(\Omega_T)}$$

and that

$$\|d\|_{L^6(\Omega_T)} < \infty \quad \text{for any } T \in (0, \infty) \tag{2-22}$$

by (2-16), and either

$$\|\phi |\nabla d|^2\|_{L^1(\Omega_T)} \le \|\phi\|_{L^\infty(\Omega_T)} \|\nabla d\|^2_{L^2(\Omega_T)} \quad \text{or} \quad \|\phi |\nabla d|^2\|_{L^1(\Omega_T)} \le \|\phi\|_{L^3(\Omega_T)} \|\nabla d\|^2_{L^3(\Omega_T)},$$

(recall that ϕ is assumed to have compact support) and, for example, that

$$\|\nabla d\|_{L^{10/3}(\Omega_T)} < \infty \quad \text{for any } T \in (0, \infty) \tag{2-23}$$

by (2-19).

[13] $A \lesssim B$ means that $A \le CB$ for some suitably universal constant $C > 0$.

3. Proof of Theorem 1

The first part of Theorem 1 will be a consequence of a certain local L^3 *ϵ-regularity* criterion (Lemma 4), while the second part will be a consequence of a certain local $\dot{H}^1$ *ϵ-regularity* criterion (Lemma 7, which is itself a consequence of Lemma 4). In the remainder of the paper, for a given $z_0 = (x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}$ and $r > 0$, as in [Caffarelli et al. 1982] we will adopt the following notation for the standard parabolic cylinder $Q_r(z_0)$ with time interval $I_r(t_0)$, along with their *centered* versions (indicated with a star):

$$\begin{aligned} I_r(t_0) &:= (t_0 - r^2, t_0), & I_r^*(t_0) &:= \big(t_0 - \tfrac{7}{8}r^2, t_0 + \tfrac{1}{8}r^2\big), \\ Q_r(z_0) &:= B_r(x_0) \times I_r(t_0), & Q_r^*(z_0) &:= B_r(x_0) \times I_r^*(t_0). \end{aligned} \tag{3-1}$$

These are defined in such a way that $Q_r^*(x_0, t_0) = Q_r\big(x_0, t_0 + \frac{1}{8}r^2\big)$, and subsequently that

$$Q_{r/2}\big(x_0, t_0 + \tfrac{1}{8}r^2\big) = B_{r/2}(x_0) \times \big(t_0 - \tfrac{1}{8}r^2, t_0 + \tfrac{1}{8}r^2\big)$$

is a *centered* cylinder with center (x_0, t_0).

Lemma 4 (L^3 ϵ-regularity; cf. [Lin and Liu 1996, Theorem 2.6; Caffarelli et al. 1982, Proposition 1]). *Fix any $\overline{C} \in (0, \infty)$. For each $q \in (5, 6]$, there exists a small*[14] *constant $\bar{\epsilon}_q = \bar{\epsilon}_q(\overline{C}) \in (0, 1)$ such that for any $\bar{z} = (\bar{x}, \bar{t}) \in \mathbb{R}^3 \times \mathbb{R}$ and $\bar{\rho} \in (0, 1]$, the following holds*:

Suppose (*see* (3-1)) $u, d : Q_1(\bar{z}) \to \mathbb{R}^3$ *and* $p : Q_1(\bar{z}) \to \mathbb{R}$ *with*

$$\begin{gathered} u, d, \nabla d \in L^\infty(I_1(\bar{t}); L^2(B_1(\bar{x}))), \quad \nabla u, \nabla d, \nabla^2 d \in L^2(Q_1(\bar{z})), \\ p \in L^{3/2}(Q_1(\bar{z})) \end{gathered} \tag{3-2}$$

satisfy

$$\nabla \cdot u = 0 \quad \textit{in } \mathcal{D}'(Q_1(\bar{z})), \tag{3-3}$$

$$-\Delta p = \nabla \cdot (\nabla^T \cdot [u \otimes u + \nabla d \odot \nabla d]) \quad \textit{in } \mathcal{D}'(Q_1(\bar{z})), \tag{3-4}$$

and the following local energy inequality holds:[15]

$$\begin{aligned} &\int_{B_1(\bar{x}) \times \{t\}} (|u|^2 + |\nabla d|^2)\phi \, dx + \int_{\bar{t}-1}^{t} \int_{B_1(\bar{x})} (|\nabla u|^2 + |\nabla^2 d|^2)\phi \, dx \, d\tau \\ &\quad \le \overline{C} \int_{\bar{t}-1}^{t} \Bigg\{ \int_{B_1(\bar{x}) \times \{\tau\}} [(|u|^2 + |\nabla d|^2)|\phi_t + \Delta\phi| + (|u|^3 + |\nabla d|^3)|\nabla\phi| + \bar{\rho}|d|^2|\nabla d|^2\phi] \, dx \\ &\qquad\qquad + \bigg| \int_{B_1(\bar{x}) \times \{\tau\}} p u \cdot \nabla\phi \, dx \bigg| \Bigg\} d\tau \\ &\qquad\qquad \textit{for a.e. } t \in I_1(\bar{t}) \textit{ and for all } \phi \in C_0^\infty(B_1(\bar{x}) \times (\bar{t} - 1, \infty)) \textit{ such that } \phi \ge 0. \end{aligned} \tag{3-5}$$

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[14]Roughly speaking, $\bar{\epsilon}_q \lesssim (\bar{C})^{-9}(2^{\alpha_q} - 1)^9$ with $\alpha_q := 2(q-5)/(q-2)$; in particular, $\bar{\epsilon}_q \to 0$ as $q \searrow 5$.

[15]Since $\big|\big(\frac{1}{2}|u|^2 + \frac{1}{2}|\nabla d|^2\big)u \cdot \nabla\phi + u \otimes \nabla\phi : \nabla d \odot \nabla d\big| \le \big(\frac{1}{2}|u|^3 + \frac{3}{2}|u||\nabla d|^2\big)|\nabla\phi| \le (|u|^3 + |\nabla d|^3)|\nabla\phi|$, we note that (1-13) implies (3-5) with $\bar{\rho} = 1$ if $Q_1(\bar{z}) \subseteq \Omega_T$. See also footnote 10.

Set[16]

$$E_{3,q} := \iint_{Q_1(\bar z)} (|u|^3 + |\nabla d|^3 + |p|^{3/2} + |d|^q |\nabla d|^{3(1-q/6)})\,dz. \tag{3-6}$$

If $E_{3,q} \le \bar\epsilon_q$, *then* $u, \nabla d \in L^\infty(Q_{1/2}(\bar z))$ *with*

$$\|u\|_{L^\infty(Q_{1/2}(\bar z))},\ \|\nabla d\|_{L^\infty(Q_{1/2}(\bar z))} \le \bar\epsilon_q^{2/9}.$$

In order to prove Lemma 4, we will require the following two technical propositions. In order to state them, let us fix (recalling (3-1)) for a given $z_0 = (x_0, t_0)$ — to be clear by the context — the abbreviated notations

$$\begin{aligned} r_k &:= 2^{-k}, & B^k &:= B_{r_k}(x_0),\\ I^k &:= I_{r_k}(t_0), & Q^k &:= B^k \times I^k \end{aligned} \tag{3-7}$$

(so that $Q^k = Q_{2^{-k}}(z_0)$) and, for each $k \in \mathbb{N}$, we define the quantities

$$L_k = L_k(z_0) \quad \text{and} \quad R_k = R_k(z_0)$$

(again, the dependence on $z_0 = (x_0, t_0)$ will be clear by context) by[17]

$$L_k := \operatorname*{ess\,sup}_{t \in I^k} \fint_{B^k} (|u(t)|^2 + |\nabla d(t)|^2)\,dx + \int_{I^k}\fint_{B^k} (|\nabla u|^2 + |\nabla^2 d|^2)\,dx\,dt \tag{3-8}$$

and

$$R_k := \fint\!\!\fint_{Q^k} (|u|^3 + |\nabla d|^3)\,dz + r_k^{1/3} \fint\!\!\fint_{Q^k} |u|\,|p - \bar p_k|\,dz, \tag{3-9}$$

where

$$\bar p_k(t) := \fint_{B^k} p(x,t)\,dx.$$

The terms L_k and R_k correspond roughly to the left- and right-hand sides of the local energy inequality (3-5). We now state the technical propositions, whose proofs we will give in Section 4.

Proposition 5 (cf. [Lin and Liu 1996, Lemma 2.7]). *There exists a large universal constant* $C_A > 0$ *such that the following holds*:

Fix any $\bar z = (\bar x, \bar t) \in \mathbb{R}^3 \times \mathbb{R}$, *and suppose* u, d *and* p *satisfy* (3-2) *and* (3-4). *Then for any* $z_0 \in Q_{1/2}(\bar z)$ *we have* (*see* (3-7)–(3-9))

$$R_{n+1}(z_0) \le C_A\Big(\max_{1\le k\le n} L_k^{3/2}(z_0) + \underbrace{\|p\|^{3/2}_{L^{3/2}(Q_{1/2}(z_0))}}_{\le E_{3,q}\ \forall q\ge 0,\ \text{cf. (3-6)}}\Big) \quad \textit{for all}\ n \ge 2. \tag{3-10}$$

The proof of Proposition 5 uses only the Hölder and Poincaré inequalities, Sobolev embedding and Calderón–Zygmund estimates along with a local decomposition of the pressure (see (4-20)) using the pressure equation (3-4).

[16]Note that $E_{3,q} < \infty$ by (3-2) and standard embeddings; see Section 2 along with (3-22) with $\sigma = 6$.

[17]We use the standard notation for averages, e.g., $\fint_B f(x)\,dx := \frac{1}{|B|}\int_B f(x)\,dx$.

Proposition 6 (cf. [Lin and Liu 1996, Lemma 2.8]). *There exists a large universal constant $C_B > 0$ such that the following holds*:

Fix any $\bar{z} = (\bar{x}, \bar{t}) \in \mathbb{R}^3 \times \mathbb{R}$, suppose u, d and p satisfy (3-2), (3-3) *and* (3-5), *and set $E_{3,q}$ as in* (3-6). *Then for any $z_0 \in Q_{1/2}(\bar{z})$ and any $q \in (5, 6]$, we have* (*see* (3-7)–(3-9))

$$L_n(z_0) \le \overline{C} \cdot C_B \left(\frac{1}{2^{\alpha_q} - 1} \cdot \max_{k_0 \le k \le n} R_k(z_0) + E_{3,q}^{2/3} + (1 + k_0 2^{5k_0}) E_{3,q} \right) \quad \textit{for all } n \ge 2 \tag{3-11}$$

for any $k_0 \in \{1, \dots, n-1\}$, where $\overline{C}$ is the constant from (3-5) *and*

$$\alpha_q := \frac{2(q-5)}{q-2} > 0.$$

The proof of Proposition 6 uses only the local energy inequality (3-5), the divergence-free condition (3-3) on u and elementary estimates. The quantities on either side of (3-11) do not scale (in the sense of (3-25)) the same way (as do those in (3-10)), which is why the energy inequality is necessary.

Let us now prove Lemma 4 using Propositions 5 and 6.

Proof of Lemma 4. Let us fix some $q \in (5, 6]$ and $\overline{C} \in (0, \infty)$. We first note that for any $\phi \ge 0$ as in (3-5) we have[18] (recalling that $\bar{\rho} \le 1$)

$$\bar{\rho} \iint_{Q^1} |d|^2 |\nabla d|^2 \phi \le \frac{2}{q} \iint_{Q^1} |d|^q |\nabla d|^{3(1-q/6)} + \left(1 - \frac{2}{q}\right) \iint_{Q^1} |\nabla d|^3 \phi^{(5-\alpha_q)/3},$$

with $\alpha_q := 2(q-5)/(q-2) \in \left(0, \frac{1}{2}\right]$. Taking ϕ in particular such that $\phi \equiv 1$ on $Q^1 = Q_{1/2}(z_0)$, we see easily from this along with (3-5) that

$$\frac{L_1}{\overline{C}} \lesssim E_{3,q} + E_{3,q}^{2/3} \quad \text{for all } z_0 \in Q_{1/2}(\bar{z}). \tag{3-12}$$

It is also easy to see that

$$L_{n+1} \le 8 L_n \quad \text{for any } n \in \mathbb{N}. \tag{3-13}$$

Hence we may pick $C_0 = C_0(q, \overline{C}) \gg 1$ such that for any $z_0 \in Q_{1/2}(\bar{z})$ — and suppressing the dependence on z_0 in what follows — we have

$$\begin{gathered} L_1, L_2, L_3 \le \tfrac{1}{2}(C_0)^{2/3}(E_{3,q} + E_{3,q}^{2/3}) \quad \text{(by (3-12), (3-13))}, \\ C_A \le \tfrac{1}{2} C_0 \quad \text{and} \quad ((2^{\alpha_q} - 1)^{-1} + 2 + 3 \cdot 2^{15}) \overline{C} \cdot C_B \le (C_0)^{2/3} \end{gathered} \tag{3-14}$$

for C_A and C_B as in Propositions 5 and 6. Having fixed C_0 — uniformly over $z_0 \in Q_{1/2}(\bar{z})$ — we then choose $\bar{\epsilon}_q \in (0, 1)$ so small that

$$\bar{\epsilon}_q < \frac{1}{(C_0)^6} \quad \Longleftrightarrow \quad C_0^2 \bar{\epsilon}_q < \bar{\epsilon}_q^{2/3}.$$

Noting first that $\bar{\epsilon}_q \le (\bar{\epsilon}_q)^{2/3}$, under the assumption $E_{3,q} \le \bar{\epsilon}_q$ we in particular see from (3-14) that

$$L_1, L_2, L_3 \le (C_0 \bar{\epsilon}_q)^{2/3}.$$

[18]The inequality in fact holds for any $q \in (2, 6]$.

Then, by Proposition 5 with $n \in \{2, 3\}$ we have

$$\begin{aligned} R_3, R_4 &\le \tfrac{1}{2} C_0 (\max\{L_1^{3/2}, L_2^{3/2}, L_3^{3/2}\} + \bar{\epsilon}_q) \quad \text{(by (3-10))} \\ &\le \tfrac{1}{2} C_0 (C_0 + 1) \bar{\epsilon}_q \le C_0^2 \bar{\epsilon}_q < \bar{\epsilon}_q^{2/3} \end{aligned}$$

which implies due to Proposition 6 with $n = 4$ and $k_0 = 3$ that

$$\begin{aligned} L_4 &\le C_B ((2^{\alpha_q} - 1)^{-1} \max\{R_3, R_4\} + E_{3,q}^{2/3} + (1 + 3 \cdot 2^{15}) E_{3,q}) \quad \text{(by (3-11))} \\ &\le (C_0 \bar{\epsilon}_q)^{2/3}. \end{aligned}$$

Then in turn, Proposition 5 with $n = 4$ gives

$$L_1, L_2, L_3, L_4 \le (C_0 \bar{\epsilon}_q)^{2/3} \quad \Longrightarrow \quad R_5 < \bar{\epsilon}_q^{2/3} \quad \text{(by (3-10))},$$

from which Proposition 6 with $n = 5$ and, again, $k_0 = 3$ gives

$$R_3, R_4, R_5 < \bar{\epsilon}_q^{2/3} \quad \Longrightarrow \quad L_5 \le (C_0 \bar{\epsilon}_q)^{2/3} \quad \text{(by (3-11))},$$

and continuing we see by induction that Proposition 5 and Proposition 6 (with $k_0 = 3$ fixed throughout) imply that

$$R_n(z_0) < \bar{\epsilon}_q^{2/3} \quad \text{and} \quad L_n(z_0) \le (C_0 \bar{\epsilon}_q)^{2/3} \qquad \text{for all } n \ge 3.$$

This, in turn, implies (for example) that (see, e.g., [Wheeden and Zygmund 1977, Theorem 7.16])

$$|u(z_0)|^3 + |\nabla d(z_0)|^3 \le \bar{\epsilon}_q^{2/3}$$

for all Lebesgue points $z_0 \in Q_{1/2}(\bar{z})$ of $|u|^3 + |\nabla d|^3$ which implies the L^∞ statement, and Lemma 4 is proved. ☐

Lemma 4 will be used to prove the first assertion in Theorem 1 as well as the next lemma, which in turn will be used to prove the second assertion in Theorem 1.

Lemma 7 ($\dot{H}^1$ ϵ-regularity; cf. [Lin and Liu 1996, Theorem 3.1; Caffarelli et al. 1982, Proposition 2]). *Fix any $\bar{C} \in (0, \infty)$ and $\bar{g} \in [1, \infty)$. For each $\sigma \in (5, 6)$, there exists a small constant $\epsilon_\sigma = \epsilon_\sigma(\bar{C}, \bar{g}) > 0$ such that the following holds. Fix $\Omega_T := \Omega \times (0, T)$ as in Theorem 1, and suppose u, d and p satisfy assumptions* (1-9)–(1-13). *If* (*recall* (3-1))

$$\limsup_{r \searrow 0} \frac{1}{r^{2+\sigma/2}} \iint_{Q_r^*(z_0)} |d|^\sigma (|u|^3 + |\nabla d|^3)^{1-\sigma/6} \, dz \le \bar{g} \tag{3-15}$$

and

$$\limsup_{r \searrow 0} \frac{1}{r} \iint_{Q_r^*(z_0)} (|\nabla u|^2 + |\nabla^2 d|^2) \, dz \le \epsilon_\sigma, \tag{3-16}$$

for some $z_0 \in \Omega_T$, then z_0 is a regular point, i.e., $|u|$ and $|\nabla d|$ are essentially bounded in some neighborhood of z_0.

For the proof of Lemma 7, for $z_0 = (x_0, t_0) \in \Omega_T$ and for $r > 0$ sufficiently small, we define A_{z_0}, B_{z_0}, C_{z_0}, D_{z_0}, E_{z_0}, F_{z_0} (cf. [Lin and Liu 1996, (3.3)]) and G_{q,z_0} using the cylinders $Q^*_r(z_0)$ — whose *centers* z_0 are in the interior, see (3-1) — by

$$A_{z_0}(r) := \frac{1}{r} \operatorname*{ess\,sup}_{t \in I^*_r(t_0)} \int_{B_r(x_0)} (|u(t)|^2 + |\nabla d(t)|^2)\, dx, \quad B_{z_0}(r) := \frac{1}{r} \iint_{Q^*_r(z_0)} (|\nabla u|^2 + |\nabla^2 d|^2)\, dz,$$

$$C_{z_0}(r) := \frac{1}{r^2} \iint_{Q^*_r(z_0)} (|u|^3 + |\nabla d|^3)\, dz, \quad D_{z_0}(r) := \frac{1}{r^2} \iint_{Q^*_r(z_0)} |p|^{3/2}\, dz,$$

$$E_{z_0}(r) := \frac{1}{r^2} \iint_{Q^*_r(z_0)} |u| \big\{ \big| |u|^2 - \overline{|u|^2}^r \big| + \big| |\nabla d|^2 - \overline{|\nabla d|^2}^r \big| \big\}\, dz \tag{3-17}$$

$$\text{where } \bar{g}^r(t) := \fint_{B_r(x_0)} g(y, t)\, dy,$$

$$F_{z_0}(r) := \frac{1}{r^2} \iint_{Q^*_r(z_0)} |u|\, |p|\, dz, \quad G_{q,z_0}(r) := \frac{1}{r^{2+q/2}} \iint_{Q^*_r(z_0)} |d|^q (|u|^3 + |\nabla d|^3)^{1-q/6}\, dz$$

(note that $G_{0,z_0} \equiv C_{z_0}$) and, for $q \in [0, 6)$, define

$$M_{q,z_0}(r) := \tfrac{1}{2}[C_{z_0}(r) + G^{6/(6-q)}_{q,z_0}(r)] + D^2_{z_0}(r) + E^{3/2}_{z_0}(r) + F^{3/2}_{z_0}(r). \tag{3-18}$$

The statement in Lemma 7 will follow from Lemma 4 along with the following technical *decay estimate* which will be proved in Section 4.

Proposition 8 (decay estimate; cf. [Lin and Liu 1996, Lemma 3.1; Caffarelli et al. 1982, Proposition 3]). *Fix any $\overline{C} \in (0, \infty)$. There exists some constant $\bar{c} = \bar{c}(\overline{C}) > 0$ such that the following holds. Fix any $q, \sigma \in \mathbb{R}$ with $2 \le q < \sigma < 6$, and define*

$$\alpha_{\sigma,q} := \frac{6}{\sigma} \cdot \frac{\sigma - q}{6 - q} \in (0, 1). \tag{3-19}$$

If u, d and p satisfy (1-9)–(1-13) *for Ω_T as in Theorem 1, and $z_0 \in \Omega_T$ and $\rho_0 \in (0, 1]$ are such that $Q^*_{\rho_0}(z_0) \subseteq \Omega_T$ and furthermore*

$$\sup_{\rho \in (0, \rho_0]} B_{z_0}(\rho) \le 1 \quad \text{and} \quad \sup_{\rho \in (0, \rho_0]} G_{\sigma, z_0}(\rho) \le \bar{g} \tag{3-20}$$

for some finite $\bar{g} \in [1, \infty)$, then for any $\rho \in (0, \rho_0]$ and $\gamma \in \big(0, \frac{1}{4}\big]$ we have

$$M_{q,z_0}(\gamma\rho) \le \bar{c} \cdot \bar{g}^{6/(6-\sigma)} \bigg[\gamma^{\alpha_{\sigma,q}/8} (M_{q,z_0} + M^{\alpha_{\sigma,q}}_{q,z_0}) + \gamma^{-15} B^{3\alpha_{\sigma,q}/4}_{z_0} \sum_{k=0}^{2} (M^{1/2^k}_{q,z_0} + M^{\alpha_{\sigma,q}/2^k}_{q,z_0}) \bigg](\rho). \tag{3-21}$$

(In fact, in the sum over k in (3-21), one can omit the term with $\alpha_{\sigma,q}$ when $k = 0$.)

The key new element in our statement and proof of Proposition 8 (and hence in achieving Lemma 7) is the fact that, for certain $q > 0$ (so that $G_{q,z_0} \neq C_{z_0}$ and hence M_{q,z_0} is notably different from the quantity found in the standard literature, namely M_{0,z_0}), we can still derive an estimate for M_{q,z_0} of the form (3-21), with a constant depending only on $\overline{C}$, σ and $\bar{g}$ (and not on q). This is made possible (see Claim 4 and its applications in Section 4.4) by the following interpolation-type estimate for the range of the quantities G_{q,z_0} (including $G_{0,z_0} = C_{z_0}$), a simple consequence of Hölder's inequality:

$$0 \le q \le \sigma \le 6 \quad \Longrightarrow \quad G_{q,z_0}(r) \le G^{q/\sigma}_{\sigma,z_0}(r) C^{1-q/\sigma}_{z_0}(r) \quad \text{for all } r > 0. \tag{3-22}$$

The estimate (3-22) follows by writing

$$|d|^q (|u|^3 + |\nabla d|^3)^{1-q/6} = [|d|^\sigma (|u|^3 + |\nabla d|^3)^{1-\sigma/6}]^{q/\sigma} \cdot (|u|^3 + |\nabla d|^3)^{(\sigma-q)/\sigma}$$

and applying Hölder's inequality with

$$1 = \frac{q}{\sigma} + \frac{\sigma - q}{\sigma}$$

to G_{q,z_0}, and noting that $r^{2+q/2} = [r^{2+\sigma/2}]^{q/\sigma} \cdot [r^2]^{1-q/\sigma}$. In particular, if $0 \le q \le \sigma < 6$, setting

$$\alpha_{\sigma,q} := \left(1 - \frac{q}{\sigma}\right) \cdot \frac{6}{6-q} \quad \text{and} \quad \beta_{\sigma,q} := \frac{q}{\sigma} \cdot \frac{6}{6-q}$$

and noting that

$$\beta_{\sigma,q} = \frac{6}{6-\sigma} \cdot (1 - \alpha_{\sigma,q}) \le \frac{6}{6-\sigma},$$

we see that

$$G^{6/(6-q)}_{q,z_0}(r) \overset{(3\text{-}22)}{\le} G^{\beta_{\sigma,q}}_{\sigma,z_0}(r) C^{\alpha_{\sigma,q}}_{z_0}(r) \overset{(3\text{-}15)}{\le} \bar{g}^{6/(6-\sigma)} \cdot [2 M^{\alpha_{\sigma,q}}_{q,z_0}(r)] \quad \text{for all } r > 0 \tag{3-23}$$

as long as $\bar{g} \ge 1$; this leads to the constants appearing in (3-21).

Let us now use Proposition 8 and Lemma 4 to prove Lemma 7.

Proof of Lemma 7. Fix any $\overline{C} \in (0, \infty)$, $\sigma \in (5, 6)$ and $\bar{g} \in [1, \infty)$, and fix[19] any $q = q(\sigma) \in \left(5, \min\{\sigma, \frac{11}{2}\}\right)$, noting that $6/(6-q) < 12$ and $2(6-q) > 1$; for the chosen q, let $\bar{\epsilon}_q = \bar{\epsilon}_q(\overline{C}) \in (0, 1)$ be the corresponding small constant from Lemma 4.

Let us first note the following important consequence of Lemma 4. Fix Ω_T as in Lemma 4 and $z_0 := (x_0, t_0) \in \Omega_T$, and suppose that

$$M_{q,z_0}(r) \le \frac{1}{2}\left(\frac{\bar{\epsilon}_q}{3}\right)^{12} \tag{3-24}$$

for some $r \in (0, 1]$ such that $Q^*_r(z_0) \subseteq \Omega_T$. Setting

$$\begin{aligned} u_{z_0,r}(x, t) &:= r u(x_0 + r x, t_0 + r^2 t), \\ p_{z_0,r}(x, t) &:= r^2 p(x_0 + r x, t_0 + r^2 t), \\ d_{z_0,r}(x, t) &:= d(x_0 + r x, t_0 + r^2 t), \end{aligned} \tag{3-25}$$

a change of variables from $z = (x, t)$ to

$$(y, s) := (x_0 + r x, t_0 + r^2 t) \tag{3-26}$$

[19]In the requirement that $q \in (5, \min\{\sigma, \bar{q}\})$, the choice of $\bar{q} := \frac{11}{2}$ is somewhat arbitrary and taken only for concreteness; one could similarly choose any $\bar{q} \in (5, 6)$ and adjust the subsequent constants accordingly.

implies that

$$\int_{Q_1^*(0,0)} (|u_{z_0,r}|^3 + |\nabla d_{z_0,r}|^3 + |p_{z_0,r}|^{3/2} + |d_{z_0,r}|^q (|u_{z_0,r}|^3 + |\nabla d_{z_0,r}|^3)^{1-q/6})\, dz$$
$$= C_{z_0}(r) + D_{z_0}(r) + G_{q,z_0}(r) \le \left(\frac{\bar\epsilon_q}{3}\right)^{12} + \left(\frac{\bar\epsilon_q}{3}\right)^{6} + \left(\frac{\bar\epsilon_q}{3}\right)^{2(6-q)} < \bar\epsilon_q.$$

Since $Q_1^*(0,0) = Q_1\big(0, \frac{1}{8}\big)$, it follows[20] from assumptions (1-9)–(1-13) that $u_{z_0,r}$, $d_{z_0,r}$ and $p_{z_0,r}$ satisfy the assumptions of Lemma 4 with $\bar z = (\bar x, \bar t) := \big(0, \frac{1}{8}\big)$ and $\bar\rho := r^2 \in (0,1]$, with the same constant $\overline{C}$ (see footnote 15). Since we have just seen that

$$E_{3,q} = E_{3,q}(u_{z_0,r}, d_{z_0,r}, p_{z_0,r}, \bar z) < \bar\epsilon_q,$$

we therefore conclude by Lemma 4 that

$$|u_{z_0,r}(z)|,\ |\nabla d_{z_0,r}(z)| \le \bar\epsilon_q^{2/9} \quad \text{for a.e. } z \in Q_{1/2}\big(0, \tfrac{1}{8}\big) = B_{1/2}(0) \times \big(-\tfrac{1}{8}, \tfrac{1}{8}\big)$$

and hence

$$|u(y,s)|,\ |\nabla d(y,s)| \le \frac{\bar\epsilon_q^{2/9}}{r} \quad \text{for a.e. } (y,s) \in B_{r/2}(x_0) \times \big(t_0 - \tfrac{1}{8}r^2, t_0 + \tfrac{1}{8}r^2\big).$$

In particular, by definition, $z_0 = (x_0, t_0)$ is a *regular* point, i.e., $|u|$ and $|\nabla d|$ are essentially bounded in a neighborhood of z_0, so long as (3-24) holds for some sufficiently small $r > 0$.

In view of this fact, setting

$$\delta_\sigma := \frac{1}{2}\left(\frac{\bar\epsilon_{q(\sigma)}}{3}\right)^{12} \quad \text{and} \quad \bar c_\sigma := \bar c \cdot \bar g^{6/(6-\sigma)},$$

we choose $\gamma_\sigma \in \big(0, \frac{1}{4}\big]$ so small that

$$\bar c_\sigma \gamma_\sigma^{\alpha_{\sigma,q}/8} \le \frac{1}{4}\left(\frac{\delta_\sigma^{[1-\alpha_{\sigma,q}]}}{2}\right), \tag{3-27}$$

where $\bar c = \bar c(\overline{C})$ is the constant from Proposition 8 and $\alpha_{\sigma,q}$ is defined as in (3-19); finally, we choose $\epsilon_\sigma \in (0,1]$ so small that

$$\bar c_\sigma \gamma_\sigma^{-15} \epsilon_\sigma^{3\alpha_{\sigma,q}/4} \le \frac{1}{4}\left(\frac{\delta_\sigma^{[1-\alpha_{\sigma,q}/4]}}{6}\right). \tag{3-28}$$

If $z_0 \in \Omega_T$ is such that (3-15) and (3-16) hold, it implies in particular that there exists some $\rho_0 \in (0,1]$ such that $Q^*_{\rho_0}(z_0) \subseteq \Omega_T$ and, furthermore,

$$\sup_{\rho \in (0,\rho_0]} G_{\sigma,z_0}(\rho) \le \bar g \tag{3-29}$$

[20]For example, if one fixes an arbitrary $\phi \in C_0^\infty(Q_1^*(0,0))$ and sets $\phi^{z_0,r}(x,\tau) := \phi((x-x_0)/r, (\tau - t_0)/r^2)$, then $\phi^{z_0,r} \in C_0^\infty(Q_r^*(z_0)) \subset C_0^\infty(\Omega_T)$. One can therefore use the test function $\phi^{z_0,r}$ in (1-13), make the change of variables $(\xi, s) := ((x-x_0)/r, (\tau-t_0)/r^2)$, so $(x,\tau) = (x_0 + r\xi, t_0 + r^2 s)$, and divide both sides of the result by r to obtain the local energy inequality (3-5) for the rescaled functions with $\bar\rho = r^2$ (as all terms scale the same way except for $|d|^2|\nabla d|^2\phi^{z_0,r}$) and $\bar z = \big(0, \frac{1}{8}\big)$. The other assumptions are straightforward.

and

$$\sup_{\rho\in(0,\rho_0]} B_{z_0}(\rho) < \epsilon_\sigma. \tag{3-30}$$

It then follows from (3-27), (3-28) and (3-30) — and the facts that $\alpha_{\sigma,q}$, $\delta_\sigma \le 1$ — that

$$\bar{c}_\sigma \gamma_\sigma^{\alpha_{\sigma,q}/8} \overset{(3\text{-}27)}{\le} \frac{1}{4}\left(\frac{\delta_\sigma^{[1-\alpha_{\sigma,q}]}}{2}\right) = \frac{1}{4}\left(\frac{\min\{1, \delta_\sigma^{[1-\alpha_{\sigma,q}]}\}}{2}\right),$$

and that

$$\begin{aligned}\bar{c}_\sigma \gamma_\sigma^{-15} B_{z_0}^{3\alpha_{\sigma,q}/4}(\rho) &\overset{(3\text{-}30)}{\le} \bar{c}_\sigma \gamma_\sigma^{-15}\epsilon_\sigma^{3\alpha_{\sigma,q}/4} \overset{(3\text{-}28)}{\le} \frac{1}{4}\left(\frac{\delta_\sigma^{[1-\alpha_{\sigma,q}/4]}}{6}\right)\\ &= \frac{1}{4}\left(\frac{\min_{k\in\{0,2\}}\{\min\{\delta_\sigma^{[1-1/2^k]}, \delta_\sigma^{[1-\alpha_{\sigma,q}/2^k]}\}\}}{6}\right)\end{aligned}$$

for all $\rho \le \rho_0$. Suppose now that z_0 is not a regular point. Then we must have

$$\delta_\sigma < M_{q,z_0}(\rho) \quad \text{for all } \rho \in (0, \rho_0], \tag{3-31}$$

or else (3-24) would hold for some $r \in (0, \rho_0]$ which would imply that z_0 is a regular point as we established above using Lemma 4.

In view of (3-29) and (3-30) — so that in particular (3-20) holds, as we chose $\epsilon_\sigma \le 1$ — we conclude by the estimate (3-21) of Proposition 8 (along with (3-27), (3-28), (3-30), (3-31) and our calculations above) that

$$M_{q,z_0}(\gamma_\sigma \rho) \le \tfrac{1}{2} M_{q,z_0}(\rho) \quad \text{for all } \rho \in (0, \rho_0]$$

for any z_0 which is not a regular point. However, since $\gamma_\sigma^k \rho_0 \in (0, \rho_0]$ for any $k \in \mathbb{N}$, by iterating the estimate above we would conclude for such z_0 that

$$M_{q,z_0}(\gamma_\sigma^n \rho_0) \le \frac{1}{2} M_{q,z_0}(\gamma_\sigma^{n-1}\rho_0) \le \frac{1}{2^2} M_{q,z_0}(\gamma_\sigma^{n-2}\rho_0) \le \cdots \le \frac{1}{2^n} M_{q,z_0}(\rho_0) < \delta_\sigma$$

for a sufficiently large $n \in \mathbb{N}$ which contradicts (3-31) (with $\rho = \gamma_\sigma^n \rho_0$), and hence contradicts our assumption that z_0 is not a regular point. Therefore z_0 must indeed be regular whenever (3-29) and (3-30) hold for our choice of ϵ_σ, which proves Lemma 7. □

In order to prove Theorem 1, we now prove the following general lemma, from which Lemma 4 and Lemma 7 will have various consequences (including Theorem 1 as well as various other historical results, which we point out for the reader's interest). As a motivation, note first that, for $r > 0$ and $z_1 := (x_1, t_1) \in \mathbb{R}^3 \times \mathbb{R}$, according to the notation in (3-25) a change of variables gives

$$\int_{Q_1^*(0,0)} |u_{z_1,r}|^q + |p_{z_1,r}|^{q/2} = \frac{1}{r^{5-q}} \int_{Q_r^*(x_1,t_1)} |u|^q + |p|^{q/2},$$

$$\int_{Q_1^*(0,0)} |\nabla u_{z_1,r}|^q = \frac{1}{r^{5-2q}} \int_{Q_r^*(x_1,t_1)} |\nabla u|^q$$

and

$$\int_{Q_1^*(0,0)} |d_{z_1,r}|^q |\nabla d_{z_1,r}|^{3(1-q/6)} = \frac{1}{r^{2+q/2}} \int_{Q_r^*(x_1,t_1)} |d|^q |\nabla d|^{3(1-q/6)} \tag{3-32}$$

for any $q \in [1, \infty)$.

Lemma 9. *Fix any open and bounded $\Omega \Subset \mathbb{R}^3$, $T \in (0, \infty)$, $k \geq 0$ and $C_k > 0$, and suppose further that $\mathcal{S} \subseteq \Omega_T := \Omega \times (0, T)$ and that $U : \Omega_T \to [0, \infty]$ is a nonnegative Lebesgue-measurable function such that the following property holds in general*:

$$(x_0, t_0) \in \mathcal{S} \quad \Longrightarrow \quad \limsup_{r \searrow 0} \frac{1}{r^k} \int_{Q_r^*(x_0,t_0)} U\, dz \geq C_k. \tag{3-33}$$

If, furthermore,

$$U \in L^1(\Omega_T), \tag{3-34}$$

then (recall Definition 2) $\mathcal{P}^k(\mathcal{S}) < \infty$ (and hence the parabolic Hausdorff dimension of $\mathcal{S}$ is at most k) with the explicit estimate

$$\mathcal{P}^k(\mathcal{S}) \leq \frac{5^5}{C_k} \int_{\Omega_T} U\, dz; \tag{3-35}$$

moreover, if $k = 5$, then

$$\mu(\mathcal{S}) \leq \frac{4\pi}{3} \mathcal{P}^5(\mathcal{S}) \leq \frac{5^5 \cdot 4\pi}{3C_5} \int_{\Omega_T} U\, dz \tag{3-36}$$

where μ is the Lebesgue outer measure, and if $k < 5$, then in fact $\mathcal{P}^k(\mathcal{S}) = \mu(\mathcal{S}) = 0$.

Before proving Lemma 9, let us first use it along with Lemma 4 and Lemma 7 to prove Theorem 1.

Proof of Theorem 1. First note that for any $r > 0$ and $z_1 := (x_1, t_1) \in \mathbb{R}^3 \times \mathbb{R}$ such that $Q_r(z_1) \subseteq \Omega_T$, it follows (as in the proof of Lemma 7) that, according to the definitions in (3-25), the rescaled triple $(u_{z_1,r}, d_{z_1,r}, p_{z_1,r})$ satisfies the conditions of Lemma 4 with $\bar{z} := (0, 0)$ and $\bar{\rho} := r^2$. Therefore if $q \in (5, 6]$ and

$$\begin{aligned}\frac{1}{r^2} \int_{Q_r(x_1,t_1)} |u|^3 + |\nabla d|^3 + |p|^{3/2} + \frac{1}{r^{2+q/2}} \int_{Q_r(x_1,t_1)} |d|^q |\nabla d|^{3(1-q/6)} \\ = \int_{Q_1(0,0)} |u_{z_1,r}|^3 + |\nabla d_{z_1,r}|^3 + |p_{z_1,r}|^{3/2} + |d_{z_1,r}|^q |\nabla d_{z_1,r}|^{3(1-q/6)} < \bar{\epsilon}_q\end{aligned} \tag{3-37}$$

(with $\bar{\epsilon}_q = \epsilon_q(\bar{C})$ as in Lemma 4), it follows that $|u_{z_1,r}|$, $|\nabla d_{z_1,r}| \leq C$ on $Q_{1/2}(0, 0)$ for some $C > 0$, and hence $|u|$, $|\nabla d| \leq C/r$ on $Q_{r/2}(x_1, t_1)$; in particular, every interior point of $Q_{r/2}(x_1, t_1)$ is a regular point, assuming (3-37) holds. Therefore, taking $z_0 := (x_0, t_0)$ such that

$$Q_{r/2}(x_1, t_1) = Q_{r/2}^*(x_0, t_0),$$

(so $x_0 = x_1$ and t_0 is slightly lower than t_1 so that (x_0, t_0) is in the interior of the cylinder $Q_{r/2}(x_1, t_1)$) and letting $\mathcal{S} \subset \Omega_T$ be the singular set of the solution (u, d, p), we see (in particular) that, since $r^{2+q/2} < r^2$ for $r < 1$,

$$\left.\begin{array}{c}(x_0, t_0) \in \mathcal{S} \\ q \in (5, 6]\end{array}\right\} \quad \Longrightarrow \quad \limsup_{r \searrow 0} \frac{1}{r^{2+q/2}} \int_{Q_r^*(x_0,t_0)} |u|^3 + |\nabla d|^3 + |p|^{3/2} + |d|^q |\nabla d|^{3(1-q/6)} \geq \bar{\epsilon}_q \tag{3-38}$$

(in fact, (3-38) must hold with lim inf instead of lim sup). Therefore, since (1-9) and (1-10) imply that

$$|u|^3+|\nabla d|^3+|p|^{3/2}+|d|^q|\nabla d|^{3(1-q/6)}\in L^1(\Omega_T) \tag{3-39}$$

(for $T<\infty$), we may apply Lemma 9 — it is not hard to see, by using a suitable covering argument, that without loss of generality we can assume Ω is bounded — with $U:=|u|^3+|\nabla d|^3+|p|^{3/2}+|d|^q|\nabla d|^{3(1-q/6)}$, $k=2+\frac{1}{2}q$ and $C_k:=\bar{\epsilon}_q$ to see $\big($setting $\delta:=\frac{1}{2}(q-5)\in\big(0,\frac{1}{2}\big)\iff 5<q<6$ with $2+\frac{1}{2}q=\frac{9}{2}+\delta\big)$ that

$$\mathcal{P}^{9/2+\delta}(\mathcal{S})=0 \quad \text{for any } \delta\in\big(0,\tfrac{1}{2}\big).$$

Before continuing with the proof of Theorem 1, we describe some intermediate results (using only Lemma 4), with historical relevance, for the interest of the reader:

Suppose that (1-14) holds for some $\sigma\in(5,6)$ which we now fix. We further fix any $q\in(5,\sigma)$, and choose $\gamma_{\sigma,q}>0$ small enough that

$$\gamma_{\sigma,q}^{1-q/\sigma}(\gamma_{\sigma,q}^{q/\sigma}+(g_\sigma)^{q/\sigma})<\bar{\epsilon}_q.$$

As in the proof of (3-22), Hölder's inequality (along with (3-32)) implies that

$$\int_{Q_1(0,0)}|d_{z_1,r}|^q|\nabla d_{z_1,r}|^{3(1-q/6)}\le(g_\sigma)^{q/\sigma}\bigg(\int_{Q_1(0,0)}|\nabla d_{z_1,r}|^3\bigg)^{1-q/\sigma},$$

so that if

$$\frac{1}{r^2}\int_{Q_r(x_1,t_1)}|u|^3+|\nabla d|^3+|p|^{3/2}=\int_{Q_1(0,0)}|u_{z_1,r}|^3+|\nabla d_{z_1,r}|^3+|p_{z_1,r}|^{3/2}<\gamma_{\sigma,q}, \tag{3-40}$$

it follows that

$$\int_{Q_1(0,0)}|u_{z_1,r}|^3+|\nabla d_{z_1,r}|^3+|p_{z_1,r}|^{3/2}+|d_{z_1,r}|^q|\nabla d_{z_1,r}|^{3(1-q/6)}<\bar{\epsilon}_q$$

and hence $(x_0,t_0)\notin\mathcal{S}$ for (x_0,t_0) as above.

Therefore under the general assumption (1-14) with $\sigma\in(5,6)$, there exists $\gamma_\sigma>0$ (e.g., $\gamma_\sigma:=\gamma_{\sigma,(5+\sigma)/2}$) such that

$$(x_0,t_0)\in\mathcal{S} \quad\Longrightarrow\quad \limsup_{r\searrow 0}\frac{1}{r^2}\int_{Q_r^*(x_0,t_0)}|u|^3+|\nabla d|^3+|p|^{3/2}\ge\gamma_\sigma. \tag{3-41}$$

Therefore, as long as

$$(u,\nabla d,p)\in L^3(\Omega_T)\times L^3(\Omega_T)\times L^{3/2}(\Omega_T), \tag{3-42}$$

we may apply Lemma 9 with $U:=|u|^3+|\nabla d|^3+|p|^{3/2}$, $k=2$ and $C_k:=\gamma_\sigma$ to see (similar to Scheffer's result [1977]) that

$$\mathcal{P}^2(\mathcal{S})=0.$$

On the other hand, we know slightly more than (3-42). The assumptions on u and d in (1-9) imply (for example, by (2-18) with $\alpha=\frac{3}{5}$, along with Sobolev embedding) that $u,\ \nabla d\in L^{10/3}(\Omega_T)$. *Suppose* we also knew (as in the case when $\Omega=\mathbb{R}^3$) that $p\in L^{5/3}(\Omega_T)$ — which essentially follows from (1-9) and (1-12), see [Lin and Liu 1996, Theorem 2.5]. Then (3-34) holds with $U := |u|^{10/3} + |\nabla d|^{10/3} + |p|^{5/3}$, and moreover Hölder's inequality implies that

$$\left(\frac{1}{r^2}\int_{Q_r^*(z_0)} |u|^3 + |\nabla d|^3 + |p|^{3/2}\right)^{10/9} \le 2^{10/9}|Q_1|^{1/9}\left(\frac{1}{r^{5/3}}\int_{Q_r^*(z_0)} |u|^{10/3} + |\nabla d|^{10/3} + |p|^{5/3}\right)$$

($|Q_1|$ is the Lebesgue measure of the unit parabolic cylinder). In view of (3-41), one could therefore apply Lemma 9 with

$$U := |u|^{10/3} + |\nabla d|^{10/3} + |p|^{5/3}, \quad k = \tfrac{5}{3} \quad \text{and} \quad C_k = \frac{\gamma_\sigma{}^{10/9}}{2^{10/9}|Q_1|^{1/9}}$$

to deduce (similar to Scheffer's result [1980]) that

$$\mathcal{P}^{5/3}(\mathcal{S}) = 0.$$

All of the above follows from Lemma 4 alone. We will now show that Lemma 7 allows one — under assumption (1-14) for some $\sigma \in (5, 6)$, and even if $p \notin L^{5/3}(\Omega_T)$ — to further decrease the dimension of the parabolic Hausdorff measure, with respect to which the singular set has measure zero, from $\frac{5}{3}$ to 1. This was essentially the most significant contribution of [Caffarelli et al. 1982] in the Navier–Stokes setting $d \equiv 0$.

Let us now proceed with the proof of the second assertion in Theorem 1. Suppose d satisfies (1-14) for some $\sigma \in (5, 6)$. Taking $\epsilon_\sigma = \epsilon_\sigma(\overline{C}, g_\sigma) > 0$ as in Lemma 7 with $\bar{g} := g_\sigma$, we see from (3-16) that

$$(x_0, t_0) \in \mathcal{S} \quad \Longrightarrow \quad \limsup_{r\searrow 0} \frac{1}{r}\int_{Q_r^*(x_0,t_0)} (|\nabla u|^2 + |\nabla^2 d|^2) \ge \epsilon_\sigma,$$

so that (3-33) holds with $U := |\nabla u|^2 + |\nabla^2 d|^2$ and $k = 1$. The second assumption in (1-9) implies that (3-34) holds as well with $U := |\nabla u|^2 + |\nabla^2 d|^2$. Therefore Lemma 9 with $U := |\nabla u|^2 + |\nabla^2 d|^2$, $k = 1$ and $C_k = \epsilon_\sigma$ implies that

$$\mathcal{P}^1(\mathcal{S}) = 0.$$

This completes the proof of Theorem 1 (assuming Lemma 9). □

Proof of Lemma 9. Fix any $\delta > 0$, and any open set V such that

$$\mathcal{S} \subseteq V \subseteq \Omega \times (0, T). \tag{3-43}$$

For each $z := (x, t) \in \mathcal{S}$, by (3-33) we can choose $r_z \in (0, \delta)$ sufficiently small so that $Q_{r_z}^*(z) \subset V$ and

$$\frac{1}{r_z^k}\int_{Q_{r_z}^*(z)} U \ge C_k. \tag{3-44}$$

By a Vitalli covering argument, see [Caffarelli et al. 1982, Lemma 6.1], there exists a sequence $(z_j)_{j=1}^\infty \subseteq \mathcal{S}$ such that

$$\mathcal{S} \subseteq \bigcup_{j=1}^\infty Q_{5r_{z_j}}^*(z_j) \tag{3-45}$$

and such that the set of cylinders $\{Q^*_{r_{z_j}}(z_j)\}_j$ are pairwise disjoint. We therefore see from (3-44) that

$$\sum_{j=1}^{\infty} r_{z_j}^k \le \frac{1}{C_k}\sum_{j=1}^{\infty}\int_{Q^*_{r_{z_j}}(z_j)} U \le \frac{1}{C_k}\int_V U \le \frac{1}{C_k}\int_{\Omega_T} U \tag{3-46}$$

which is finite (and uniformly bounded in δ) by (3-34). Note that according to Definition 2 of the parabolic Hausdorff measure $\mathcal{P}^k$, (3-46) implies

$$\mathcal{P}^k(\mathcal{S}) \le \frac{5^k}{C_k}\int_V U \le \frac{5^k}{C_k}\int_{\Omega_T} U \tag{3-47}$$

which establishes (3-35).

Let us now assume that $k \le 5$. Letting μ be the Lebesgue (outer) measure, note that

$$\mu(Q^*_{5r_{z_j}}) \le |B_1|(5r_{z_j})^5$$

so that

$$\mu(\mathcal{S}) \overset{(3\text{-}45)}{\le} |B_1|\sum_{j=1}^{\infty}(5r_{z_j})^5 \le 5^5|B_1|\delta^{5-k}\sum_{j=1}^{\infty} r_{z_j}^k \overset{(3\text{-}46)}{\le} \delta^{5-k}\frac{5^5|B_1|}{C_k}\int_{\Omega_T} U, \tag{3-48}$$

since we have chosen $r_z < \delta$ for all $z \in \mathcal{S}$. If $k = 5$, (3-48) along with Definition 2 gives the explicit estimate (3-36) on $\mu(S)$. If $k < 5$, since $\delta > 0$ was arbitrary, sending $\delta \to 0$ we conclude (by (3-34)) that $\mu(\mathcal{S}) = 0$ and hence $\mathcal{S}$ is Lebesgue measurable with Lebesgue measure zero. We may therefore take V to be an open set such that $\mu(V)$ is arbitrarily small but so that (3-43) still holds, and deduce that $\mathcal{P}^k(\mathcal{S}) = 0$ by (3-34) and (3-47). □

4. Proofs of technical propositions

In order to prove Proposition 5 as well as Proposition 8, we will require certain local decompositions of the pressure (cf. [Caffarelli et al. 1982, (2.15)]) as follows:

4.1. *Localization of the pressure.*

Claim 1. *Fix open sets $\Omega_1 \Subset \Omega_2 \Subset \Omega \subset \mathbb{R}^3$ and $\psi \in \mathcal{C}_0^\infty(\Omega_2; \mathbb{R})$ with $\psi \equiv 1$ on Ω_1. Let*

$$G^x(y) := \frac{1}{4\pi}\frac{1}{|x-y|} \tag{4-1}$$

be the fundamental solution of $-\Delta$ in $\mathbb{R}^3$ so that, in particular,

$$\nabla G^x \in L^q(\Omega_2) \quad \textit{for any } q \in \left[1, \tfrac{3}{2}\right)$$

for any fixed $x \in \mathbb{R}^3$, and set

$$\begin{aligned} G^x_{\psi,1} &:= -G^x\nabla\psi, \\ G^x_{\psi,2} &:= 2\nabla G^x\cdot\nabla\psi + G^x\Delta\psi, \\ G^x_{\psi,3} &:= \nabla G^x\otimes\nabla\psi + \nabla\psi\otimes\nabla G^x + G^x\nabla^2\psi, \end{aligned}$$

so that

$$G^x_{\psi,1}, G^x_{\psi,2}, G^x_{\psi,3} \in \mathcal{C}^\infty_0(\Omega_2) \quad \text{for any fixed } x \in \Omega_1.$$

Suppose $\Pi \in \mathcal{C}^2(\Omega;\mathbb{R})$, $v \in \mathcal{C}^1(\Omega;\mathbb{R}^3)$ *and* $K \in \mathcal{C}^2(\Omega;\mathbb{R}^{3\times 3})$. *If*

$$-\Delta\Pi = \nabla\cdot v \quad \text{in } \Omega, \tag{4-2}$$

then for any $x \in \Omega_1$,

$$\Pi(x) = -\int \nabla G^x \cdot v\psi + \int G^x_{\psi,1}\cdot v + \int G^x_{\psi,2}\Pi. \tag{4-3}$$

Similarly, if

$$-\Delta\Pi = \nabla\cdot(\nabla^T\cdot K) \quad \text{in } \Omega, \tag{4-4}$$

then for any $x \in \Omega_1$,

$$\Pi(x) = S[\psi K](x) + \int G^x_{\psi,3} : K + \int G^x_{\psi,2}\Pi, \tag{4-5}$$

where

$$S[\widetilde{K}](x) := \nabla_x\cdot\left(\nabla^T_x\cdot\int G^x\widetilde{K}\right) = \int G^x\nabla\cdot(\nabla^T\cdot\widetilde{K}) \quad \text{for all } \widetilde{K}\in\mathcal{C}^2_0(\Omega_2;\mathbb{R}^{3\times 3});$$

in particular (noting $\nabla^2 G^x \notin L^1_{\text{loc}}$*),* $S : [L^q(\Omega_2)]^{3\times 3} \to L^q(\Omega_2)$ *for any* $q \in (1,\infty)$ *is a bounded, linear Calderón–Zygmund operator.*

Remark 10. We note, therefore, that under the assumptions (1-9), (1-10) and (1-12), by suitable regularizations one can see that for almost every fixed $t \in (0,T)$, (4-3) and (4-5) hold for a.e. $x \in \Omega_1$ with $\Pi := p(\cdot,t)$, $K := J(\cdot,t)$ and $v := \nabla^T\cdot J(\cdot,t)$, where

$$J := u\otimes u + \nabla d\odot\nabla d.$$

Indeed, under the assumptions (1-9), we have $u, \nabla d \in L^{10/3}(\Omega_T)$ so that (omitting the x-dependence)

$$J(t) \in L^{5/3}(\Omega) \quad \text{for a.e. } t\in(0,T). \tag{4-6}$$

Moreover, since $u, \nabla d \in L^\infty(0,T;L^2(\Omega)) \cap L^{10/3}(\Omega_T)$ and $\nabla u, \nabla^2 d \in L^2(\Omega_T)$, we have

$$\nabla^T\cdot J \in L^2(0,T;L^1(\Omega)) \cap L^{5/4}(\Omega_T)$$

so that

$$\nabla^T\cdot J(t) \in L^1(\Omega)\cap L^{5/4}(\Omega) \quad \text{for a.e. } t\in(0,T). \tag{4-7}$$

Finally, (1-10) implies that

$$p(t) \in L^{3/2}(\Omega) \quad \text{for a.e. } t\in(0,T). \tag{4-8}$$

Fix now any $t\in(0,T)$ such that the inclusions in (4-6)–(4-8) hold. Since $G^x_{\psi,j} \in \mathcal{C}^\infty_0$ for $x\in\Omega_1$, the terms in (4-3) and (4-5) containing $G^x_{\psi,j}$ are all well defined for every $x\in\Omega_1$ since $J(t)$, $\nabla^T\cdot J(t)$, $p(t) \in L^1_{\text{loc}}(\Omega)$. The term in (4-3) containing ∇G^x is in $L^r_x(\Omega_2)$ for any $r\in\left[1,\frac{15}{7}\right)$ by Young's convolution inequality (since Ω_2 is bounded), so that term is well defined for a.e. $x\in\Omega_2$. Indeed, for $R>0$ such that $\Omega_2 \subseteq B_{R/2}(x_0)$

for some $x_0 \in \mathbb{R}^3$, we have $x - y \in B_R := B_R(0)$ for all $x, y \in \Omega_2$. Setting $G(y) := G^0(y)$ and letting χ_{B_R} be the indicator function of B_R, since ψ is supported in Ω_2 we therefore have

$$-\int \nabla G^x \cdot v\psi = [([\nabla G]\chi_{B_R}) * (v\psi)](x)$$

for all $x \in \Omega_2$. Therefore

$$\begin{aligned}\left\| \int \nabla G^x \cdot v\psi \right\|_{L^r_x(\Omega_2)} &\le \|([\nabla G]\chi_{B_R}) * v\psi\|_{L^r(\mathbb{R}^3)} \\ &\le \|[\nabla G]\chi_{B_R}\|_{L^q(\mathbb{R}^3)} \|v\psi\|_{L^s(\mathbb{R}^3)} = \|\nabla G\|_{L^q(B_R)} \|v\psi\|_{L^s(\Omega_2)} < \infty\end{aligned}$$

by Young's inequality for any $q \in \left[1, \frac{3}{2}\right)$, $s \in \left[1, \frac{5}{4}\right)$ and r such that $1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{s}$ $\left(\text{note that } \frac{2}{3} + \frac{4}{5} - 1 = \frac{7}{15}\right)$. Finally, $S[\psi J(t)] \in L^{5/3}(\Omega_2)$ by the Calderón–Zygmund estimates $\left(\text{as } 1 < \frac{5}{3} < \infty\right)$ so again that term is defined for a.e. $x \in \Omega_2$.

Regularizing the linear equation (1-12) using a standard spatial mollifier at any $t \in (0, T)$ where (1-12) holds in $\mathcal{D}'(\Omega)$ and where the inclusions in (4-6)–(4-8) hold, applying Claim 1 and passing to limits gives the almost-everywhere convergence (after passing to a suitable subsequence) due, in particular, to the boundedness of the linear operator S on $L^{5/3}(\Omega_2)$.

Proof of Claim 1. Since (extending Π by zero outside of Ω) $\psi\Pi \in \mathcal{C}^2_0(\mathbb{R}^3)$, by the classical representation formula, see, e.g., [Gilbarg and Trudinger 1983, (2.17)], for any $x \in \mathbb{R}^3$ we have

$$\psi(x)\Pi(x) = -\int G^x \Delta(\psi\Pi) = -\int G^x(\psi\Delta\Pi + 2\nabla\psi \cdot \nabla\Pi + \Pi\Delta\psi). \tag{4-9}$$

In particular, for a fixed $x \in \Omega_1$ where $\psi \equiv 1$, we have $G^x\nabla\psi \in \mathcal{C}^\infty_0(\mathbb{R}^3)$ so that integrating by parts in (4-9) we see that

$$\Pi(x) = \int G^x\psi(-\Delta\Pi) + \int G^x_{\psi,2}\Pi. \tag{4-10}$$

If (4-2) holds, then by (4-10) we have

$$\Pi(x) = \int G^x\psi\nabla \cdot v + \int G^x_{\psi,2}\Pi \tag{4-11}$$

for any $x \in \Omega_1$. One can then carefully integrate by parts once in the first term of (4-11) as follows: for a small $\epsilon > 0$,

$$\int_{|y-x|>\epsilon} G^x\psi\nabla \cdot v\,dy = -\int_{|y-x|>\epsilon} [\nabla(G^x\psi)] \cdot v\,dy + \frac{1}{4\pi\epsilon} \underbrace{\int_{|y-x|=\epsilon} \psi v \cdot \nu_y\,dS_y}_{=\mathcal{O}(\epsilon^2)}$$

and since the second term vanishes as $\epsilon \to 0$ due to the fact that $|\partial B_\epsilon(x)| \lesssim \epsilon^2$, we conclude (since $\nabla G^x \in L^1_{\text{loc}}$) that

$$\int G^x\psi\nabla \cdot v = -\int [\nabla(G^x\psi)] \cdot v = -\int \nabla G^x \cdot v\psi + \int G^x_{\psi,1} \cdot v$$

which, along with (4-11), implies (4-3) for any $x \in \Omega_1$.

On the other hand, if (4-4) holds, then by (4-10) we have

$$\Pi(x) = \int G^x \psi \nabla \cdot (\nabla^T \cdot K) + \int G^x_{\psi,2} \Pi \tag{4-12}$$

and one can write

$$\nabla \cdot (\nabla^T \cdot (\psi K)) = [\nabla^2 \psi]^T : K + \nabla^T \psi \cdot [\nabla \cdot K] + \nabla \psi \cdot [\nabla^T \cdot K] + \psi \nabla \cdot (\nabla^T \cdot K)$$

so that (as $\nabla^2 \psi = \nabla^T (\nabla \psi) = \nabla(\nabla^T \psi) = [\nabla^2 \psi]^T$ since $\psi \in \mathcal{C}^2$)

$$\begin{aligned}\int G^x [\psi \nabla \cdot (\nabla^T \cdot K)] \\ = \int G^x [\nabla \cdot (\nabla^T \cdot (\psi K))] - \int G^x [\nabla^2 \psi : K] - \int ([G^x \nabla^T \psi] \cdot [\nabla \cdot K] + [G^x \nabla \psi] \cdot [\nabla^T \cdot K]).\end{aligned}$$

Since $G^x \nabla \psi \in \mathcal{C}^\infty_0$ for $x \in \Omega_1$, one can again integrate by parts in the final term to obtain

$$\Pi(x) = \int G^x [\nabla \cdot (\nabla^T \cdot (\psi K))] + \int G^x_{\psi,3} : K + \int G^x_{\psi,2} \Pi$$

for $x \in \Omega_1$ in view of (4-12). Moreover, since $\psi K \in \mathcal{C}^2_0$ and $G^x \in L^1_{\text{loc}}$, as usual for convolutions one can change variables to obtain

$$\int G^x \nabla \cdot (\nabla^T \cdot (\psi K)) = \left[\nabla_x \cdot \left(\nabla^T_x \cdot \int G^x \psi K\right)\right](x) =: S[\psi K](x)$$

which gives us (4-5) for any $x \in \Omega_1$, where S (see, e.g., [Gilbarg and Trudinger 1983, Theorem 9.9]) is a singular integral operator as claimed. (Note that $\nabla^2 G^x \notin L^1_{\text{loc}}$ so that one cannot simply integrate by parts twice in this term putting all derivatives on G^x, but $\int G^x \psi K$ is the Newtonian potential of ψK which can be twice differentiated in various senses depending on the regularity of K.) □

4.2. ***Proof of Proposition 5.*** In what follows, for $\mathcal{O} \subseteq \mathbb{R}^3$ and $I \subseteq \mathbb{R}$, we will use the notation

$$\|\cdot\|_{q;\mathcal{O}} := \|\cdot\|_{L^q(\mathcal{O})}, \quad \|\cdot\|_{s;I} := \|\cdot\|_{L^s(I)}, \quad \|\cdot\|_{q,s;\mathcal{O}\times I} := \|\cdot\|_{L^s(I;L^q(\mathcal{O}))} = \big\| \|\cdot\|_{L^q(\mathcal{O})} \big\|_{L^s(I)}$$

and we will abbreviate by writing

$$\|\cdot\|_{q;\mathcal{O}\times I} := \|\cdot\|_{q,q;\mathcal{O}\times I} = \|\cdot\|_{L^q(\mathcal{O}\times I)}.$$

We first note some simple inequalities. Letting $B_r \subset \mathbb{R}^3$ be a ball of radius $r > 0$, from the embedding $W^{1,2}(B_1) \hookrightarrow L^6(B_1)$ applied to functions of the form $g_r(x) = g(rx)$ — or suitably shifted, if the ball is not centered as zero — we obtain

$$\|g_r\|_{6;B_1} \lesssim \|g_r\|_{2;B_1} + \|\nabla g_r\|_{2;B_1} = \|g_r\|_{2;B_1} + r\|(\nabla g)_r\|_{2;B_1}$$

whereupon — noting by a simple change of variables that

$$\|g_r\|_{q;B_1} = r^{-3/q} \|g\|_{q;B_r}$$

for any $q \in [1, \infty)$ — we obtain for any ball B_r of radius $r > 0$ and any g that

$$\|g\|_{6;B_r} \lesssim \frac{1}{r}\|g\|_{2;B_r} + \|\nabla g\|_{2;B_r} \tag{4-13}$$

where the constant is independent of r as well as the center of B_r. Next, for any $v(x,t)$, using Hölder's inequality to interpolate between L^2 and L^6 we have

$$\|v(t)\|_{3;B_r} \le \|v(t)\|_{2;B_r}^{1/2}\|v(t)\|_{6;B_r}^{1/2} \overset{(4\text{-}13)}{\lesssim} r^{-1/2}\|v(t)\|_{2;B_r} + \|v(t)\|_{2;B_r}^{1/2}\|\nabla v(t)\|_{2;B_r}^{1/2}. \tag{4-14}$$

Then for $I_r \subset \mathbb{R}$ with $|I_r| = r^2$ and $Q_r := B_r \times I_r$, Hölder's inequality in the t variable gives

$$\|v\|_{3;Q_r} \lesssim r^{-1/2}|I_r|^{1/3}\|v\|_{2,\infty;Q_r} + \|v\|_{2,\infty;Q_r}^{1/2}(|I_r|^{1/6}\|\nabla v\|_{2;Q_r})^{1/2}$$

so that

$$r^{-1/6}\|v\|_{3;Q_r} \lesssim \|v\|_{2,\infty;Q_r} + \|v\|_{2,\infty;Q_r}^{1/2}\|\nabla v\|_{2;Q_r}^{1/2} \lesssim \|v\|_{2,\infty;Q_r} + \|\nabla v\|_{2;Q_r}$$

(the first of which is sometimes called the *multiplicative inequality*) with a constant independent of r. From these, noting that $|B_r| \sim r^3$ and $|Q_r| \sim r^5$, it follows easily that, for example,

$$⨏\!\!⨏_{Q^n} |v|^3\,dz \lesssim \left(\operatorname*{ess\,sup}_{t\in I^n} ⨏_{B^n} |v(t)|^2\,dx\right)^{3/2} + \left(\int_{I^n} ⨏_{B^n} |\nabla v|^2\,dx\,dt\right)^{3/2}. \tag{4-15}$$

Note also that a similar scaling argument applied to Poincaré's inequality gives the estimate

$$\|g - \overline{g_{B_r}}\|_{q;B_r} \lesssim r\|\nabla g\|_{q;B_r} \sim |B_r|^{1/3}\|\nabla g\|_{q;B_r} \tag{4-16}$$

for any $r > 0$ and $q \in [1, \infty]$, where $\overline{g_{\mathcal{O}}}$ is the average of g in $\mathcal{O}$ for any $\mathcal{O} \subset \mathbb{R}^3$ with $|\mathcal{O}| < \infty$. Note finally that a simple application of Hölder's inequality gives

$$\|\overline{g_{\mathcal{O}}}\|_{q;\mathcal{O}} \le \|g\|_{q;\mathcal{O}}. \tag{4-17}$$

Proceeding now with the proof, fix some $\tilde{\phi} \in C_0^\infty(\mathbb{R}^3)$ such that

$$\tilde{\phi} \equiv 1 \quad \text{in } B_{r_2}(0) = B_{1/4}(0) \qquad \text{and} \qquad \operatorname{supp}(\tilde{\phi}) \subseteq B_{r_1}(0) = B_{1/2}(0).$$

Now fix $\bar{z} = (\bar{x}, \bar{t}) \in \mathbb{R}^3 \times \mathbb{R}$ and $z_0 = (x_0, t_0) \in Q_{1/2}(\bar{z})$, define B^k, I^k and Q^k by (3-7) for this z_0 and define ϕ by $\phi(x) := \tilde{\phi}(x - x_0)$. So

$$\phi \equiv 1 \quad \text{in } B^2 = B_{1/4}(x_0) \qquad \text{and} \qquad \operatorname{supp}(\phi) \subseteq B^1 = B_{1/2}(x_0) \subset B_1(\bar{x}),$$

since $x_0 \in B_{1/2}(\bar{x})$. The following estimates will clearly depend only on $\tilde{\phi}$, i.e., constants will be uniform for all $z_0 \in Q_{1/2}(\bar{z})$.

First, applying (4-15) to $v \in \{u, \nabla d\}$ and recalling (3-8) we see that

$$\frac{1}{r_n^5}(\|u\|_{3;Q^n}^3 + \|\nabla d\|_{3;Q^n}^3) \lesssim ⨏\!\!⨏_{Q^n} (|u|^3 + |\nabla d|^3)\,dz \overset{(4\text{-}15)}{\lesssim} L_n^{3/2} \tag{4-18}$$

for any n, with a constant independent of n. In particular, for any n we have the estimate

$$\|u\|_{3;Q^n} + \|\nabla d\|_{3;Q^n} \lesssim r_n^{5/3} L_n^{1/2}. \tag{4-19}$$

Next, by Claim 1 and Remark 10 with $\psi := \phi$, $\Omega_2 := B^1$ and $\Omega_1 := B^2$, noting that $p \equiv \phi p$ on $Q^2 = Q_{1/4}(z_0) = B_{1/4}(x_0) \times \big(t_0 - \big(\tfrac{1}{4}\big)^2, t_0\big)$, as in (4-5) we have

$$p(x,t) = S[\phi J(t)](x) + \int_{B^1\setminus B^2} (2\nabla G^x \otimes_\sigma \nabla\phi + G^x \nabla^2\phi) : J(t)\,dy + \int_{B^1\setminus B^2} (2\nabla G^x \cdot \nabla\phi + G^x \Delta\phi)\,p(t)\,dy, \quad (4\text{-}20)$$

at almost every $(x,t) \in Q^2$, where

$$J := u \otimes u + \nabla d \odot \nabla d, \quad (4\text{-}21)$$

$2a \otimes_\sigma b := a \otimes b + b \otimes a$ and the operator S consisting of second derivatives of the Newtonian potential given by

$$S[\widetilde{K}](x) := \nabla_x \cdot \left(\nabla_x^T \cdot \int_{B^1} G^x \widetilde{K} \right)$$

for $\widetilde{K} \in L^q(B^1)$ is a bounded linear Calderón–Zygmund operator on $L^q(B^1)$ for $1 < q < \infty$. Hence for any $n \in \mathbb{N}$, denoting by χ_n the indicator function for the set $B^n = B_{2^{-n}}(x_0)$ and splitting $\phi = \chi_n\phi + (1-\chi_n)\phi$ in the first term of (4-20), we can write

$$p = p^{1,n} + p^{2,n} + p^{3,n} \equiv p^{1,n} + p^{2,n} + p^3,$$

where, for almost every $(x,t) \in Q^2$,

$$p(x,t) = \underbrace{S[\chi_n \phi J(t)](x)}_{=:p^{1,n}(x,t)} + \underbrace{S[(1-\chi_n)\phi J(t)](x)}_{=:p^{2,n}(x,t)} + \underbrace{\int_{B^1\setminus B^2} (2\nabla G^x \otimes_\sigma \nabla\phi + G^x\nabla^2\phi) : J(t)\,dy + \int_{B^1\setminus B^2} (2\nabla G^x \cdot \nabla\phi + G^x \Delta\phi)\,p(t)\,dy}_{=:p^{3,n}(x,t) \equiv p^3(x,t)}$$

(where the last term is clearly independent of n, but we keep the notation $p^{3,n}$ for convenience).

Note first that, by the classical Calderón–Zygmund estimates, there is a universal constant $C_{cz} > 0$ such that, for all $n \in \mathbb{N}$, we have

$$\|p^{1,n}(t)\|_{3/2;B^{n+1}} \le C_{cz} \|\chi_n \phi J(t)\|_{3/2;\mathbb{R}^3} \le C_{cz} \|\tilde{\phi}\|_{\infty;\mathbb{R}^3} \|J(t)\|_{3/2;B^n}. \quad (4\text{-}22)$$

Next, since the appearance of $\nabla\phi$ in p^3 exactly cuts off a neighborhood of the singularity of G^x (see (4-1)) uniformly for all $x \in B_{1/8}(x_0)$ — as we integrate over $|x_0 - y| \ge \frac{1}{4}$, hence $|x-y| \ge \frac{1}{8}$ — we see that $p^{3,n}(\cdot,t) \in \mathcal{C}^\infty(B_{1/8}(x_0))$ for $t \in I_{1/8}(t_0)$ with, in particular,

$$\|\nabla_x p^{3,n}(t)\|_{\infty;B^{n+1}} \overset{(n\ge 2)}{\le} \|\nabla_x p^{3,n}(t)\|_{\infty;B_{1/8}(x_0)} \le c(\tilde{\phi})(\|J(t)\|_{1;B^1} + \|p(t)\|_{1;B^1}). \quad (4\text{-}23)$$

In the term $p^{2,n}$, the singularity coming from G^x is also isolated due to the appearance of χ_n, but it is no longer uniform in n so we must be more careful. As we are integrating over a region which avoids a neighborhood of the singularity at $y = x$ of G^x, we can pass the derivatives in S under the integral sign to write

$$\nabla_x p^{2,n}(x,t) = \int_{B^1\setminus B^n} \nabla_x[(\nabla_x^2 G^x)^T : \phi J(t)]\,dy = \sum_{k=1}^{n-1} \int_{B^k\setminus B^{k+1}} \nabla_x[(\nabla_x^2 G^x)^T : \phi J(t)]\,dy$$

and note, in view of (4-1), that

$$|\nabla_x^3 G^x(y)| \lesssim \frac{1}{|x-y|^4} \le (2^{k+2})^4 \lesssim \frac{2^k}{|B^k|} \quad \text{for all } x \in B^{k+2},\ y \in (B^{k+1})^c.$$

Therefore, since

$$B^{n+1} = B^{(n-1)+2} \subseteq B^{k+2} \quad \text{for } 1 \le k \le n-1,$$

we see that

$$\|\nabla_x p^{2,n}(\,\cdot\,,t)\|_{\infty,B^{n+1}} \lesssim c(\tilde{\phi}) \sum_{k=1}^{n-1} 2^k \fint_{B^k} |J(y,t)|\,dy \tag{4-24}$$

for all $t \in I_{1/8}(t_0)$.

Now, recalling the notation

$$\bar{f}_k(t) := \fint_{B^k} f(x,t)\,dx$$

for a function $f(x,t)$ and $k \in \mathbb{N}$, for any $t \in I^2 = \left(t_0 - \left(\frac{1}{4}\right)^2, t_0\right)$ and $n \ge 2$, we estimate

$$\begin{aligned}
&\int_{B^{n+1}} |u(x,t)|\,|p(x,t) - \bar{p}_{n+1}(t)|\,dx \\
&\le \sum_{j=1}^{3} \int_{B^{n+1}} |u(x,t)|\,|p^{j,n}(x,t) - \bar{p}^{j,n}_{n+1}(t)|\,dx \\
&\le \|u(\,\cdot\,,t)\|_{3;B^{n+1}} \sum_{j=1}^{3} \|p^{j,n}(\,\cdot\,,t) - \bar{p}^{j,n}_{n+1}(t)\|_{3/2;B^{n+1}} \\
&\lesssim \|u(t)\|_{3;B^{n+1}} \left(\|p^{1,n}(t)\|_{3/2;B^{n+1}} + |B^{n+1}| \sum_{j=2}^{3} \|\nabla p^{j,n}(t)\|_{\infty;B^{n+1}} \right) \qquad \text{(by (4-16), (4-17) and Hölder)} \\
&\lesssim \|u(t)\|_{3;B^{n+1}} \left(\|J(t)\|_{3/2;B^n} + r_{n+1}^3 \left\{ \left(\sum_{k=1}^{n-1} 2^k \fint_{B^k} |J(t)|\,dy \right) + \|J(t)\|_{3/2;B^1} + \|p(t)\|_{3/2;B^1} \right\} \right),
\end{aligned} \tag{4-25}$$

where the last inequality follows from (4-22)–(4-24) and Hölder's inequality. Note further that, setting

$$\mathbb{L}_{J,k} := \left\| \fint_{B^k} |J(t)|\,dy \right\|_{L_t^\infty(I^k)}, \tag{4-26}$$

we have

$$\begin{aligned}
\left\| \sum_{k=1}^{n-1} 2^k \fint_{B^k} |J(t)|\,dy \right\|_{L_t^{3/2}(I^{n+1})} &\le |I^{n+1}|^{2/3} \big(\max_{1\le k\le n-1} \mathbb{L}_{J,k} \big) \sum_{k=1}^{n-1} 2^k \\
&\le r_{n+1}^{1/3} \max_{1\le k\le n-1} \mathbb{L}_{J,k},
\end{aligned}$$

since $|I^{n+1}| = r_{n+1}^2$ and

$$\sum_{k=1}^{n-1} 2^k = \frac{2^n - 2}{2-1} < 2^n = r_n^{-1}.$$

Integrating over $t \in I^{n+1}$ in (4-25), applying Hölder's inequality in the variable t and recalling by (4-19) that $\|u\|_{3;Q^{n+1}} \lesssim r_{n+1}^{5/3} L_{n+1}^{1/2}$, we obtain

$$\iint_{Q^{n+1}} |u|\,|p - \overline{p_{n+1}}|\,dz \\ \lesssim r_{n+1}^{5/3} L_{n+1}^{1/2} \big\{ \|J\|_{3/2;Q^n} + r_{n+1}^{10/3} \max_{1\le k\le n-1} \mathbb{L}_{J,k} + r_{n+1}^3 (\|J\|_{3/2;Q^1} + \|p\|_{3/2;Q^1}) \big\}. \tag{4-27}$$

It follows now from (4-21) that

$$\|J\|_{3/2;Q^k} \le \|u\|_{3;Q^k}^2 + \|\nabla d\|_{3;Q^k}^2 \overset{(4\text{-}19)}{\lesssim} (r_k^{5/3} L_k^{1/2})^2 = r_k^{10/3} L_k \tag{4-28}$$

and

$$\mathbb{L}_{J,k} \overset{(4\text{-}26)}{\le} \left\| \fint_{B^k} (|u(\,\cdot\,)|^2 + |\nabla d(\,\cdot\,)|^2)\,dy \right\|_{\infty;I^k} \le L_k. \tag{4-29}$$

Now from (4-21), (4-27)–(4-29) and the simple fact that $\frac{1}{2} r_n = r_{n+1} \le 1$ we obtain

$$r_{n+1}^{1/3} \fint\!\!\fint_{Q^{n+1}} |u|\,|p - \bar{p}_{n+1}|\,dz \lesssim L_{n+1}^{1/2} \big\{ r_{n+1}^{1/3} L_n + r_{n+1}^{1/3} \max_{1\le k\le n-1} L_k + \underbrace{r_1^{10/3}}_{\le 1} L_1 + \|p\|_{3/2;Q^1} \big\} \\ \lesssim L_{n+1}^{1/2} \big\{ \max_{1\le k\le n} L_k + \|p\|_{3/2;Q^1} \big\}.$$

Since

$$\fint\!\!\fint_{Q^{n+1}} (|u|^3 + |\nabla d|^3)\,dz \overset{(4\text{-}18)}{\lesssim} L_{n+1}^{3/2},$$

adding the previous estimates and recalling (3-8) and (3-9) we have

$$R_{n+1} \lesssim L_{n+1}^{3/2} + L_{n+1}^{1/2} \big(\max_{1\le k\le n} L_k + \|p\|_{3/2;Q^1} \big)$$

(where the constant is universal). This along with (3-13) implies (3-10) and proves Proposition 5. □

4.3. ***Proof of Proposition 6.*** For simplicity, take $\bar{z} = z_0 = (0, 0)$, so that (recall (3-7)) $Q^k = Q^k(0, 0)$, etc., as the rest can be obtained by appropriate shifts.

We want to take the test function ϕ in (3-5) such that $\phi = \phi^n := \chi \psi^n$, where (recall that here $Q^1 = Q^1(0, 0) = B_{1/2}(0) \times \big(-\frac{1}{4}, 0\big)$ so χ will be zero in a neighborhood of the parabolic boundary of Q^1)

$$\chi \in \mathcal{C}_0^\infty\big(B_{1/2}(0) \times \big(-\tfrac{1}{4}, \infty\big)\big), \quad \chi \equiv 1 \text{ in } Q^2, \quad 0 \le \chi \le 1 \tag{4-30}$$

and

$$\psi^n(x, t) := \frac{1}{(r_n^2 - t)^{3/2}} e^{-|x|^2/(4(r_n^2 - t))} \quad \text{for } t \le 0. \tag{4-31}$$

Note that the singularity of ψ^n would naturally be at $(x, t) = (0, r_n^2) \notin Q^1$, so $\psi^n \in \mathcal{C}^\infty(\overline{Q}_1)$ and we may extend ψ^n smoothly to $t > 0$ (where its values will actually be irrelevant) for each n so that, in particular,

$\phi^n \in \mathcal{C}_0^\infty(B_1(0)\times(-1,\infty))$ as required[21] in (3-5) with $(\bar{x},\bar{t})=(0,0)$. Furthermore, we have

$$\nabla\psi^n(x,t) = -\frac{x}{2(r_n^2-t)}\psi^n(x,t) \quad \text{and} \quad \psi_t^n + \Delta\psi^n \equiv 0 \qquad \text{in } Q^1. \tag{4-32}$$

Note first that for $(x,t)\in Q^n$ $(n\ge 2)$, we have

$$0 \le |x| \le r_n \quad \text{and} \quad r_n^2 \le [r_n^2 - t] \le 2r_n^2$$

so that

$$r_n^3 = (r_n^2)^{3/2} e^{0/(8r_n^2)} \le (r_n^2-t)^{3/2} e^{|x|^2/(4(r_n^2-t))} \le (2r_n^2)^{3/2} e^{r_n^2/(4r_n^2)} = 2^{3/2}e^{1/4} r_n^3.$$

Hence

$$\frac{1}{2^{3/2}e^{1/4}}\cdot\frac{1}{r_n^3} \le \psi^n(x,t) \le \frac{1}{r_n^3} \quad \text{for all } (x,t)\in Q^n \tag{4-33}$$

and therefore (as $r_n^2 - t > 0$)

$$|\nabla_x\psi^n(x,t)| = \frac{|x|}{2(r_n^2-t)}|\psi^n(x,t)| \lesssim \frac{r_n}{r_n^2}\cdot\frac{1}{r_n^3} = \frac{1}{r_n^4} \quad \text{for all } (x,t)\in Q^n. \tag{4-34}$$

Next, note similarly that for $2\le k\le n$ and $(x,t)\in Q^{k-1}\setminus Q^k$, we have

$$r_k \le |x| \le r_{k-1} = 2r_k \quad \text{and} \quad r_k^2 \le r_n^2 + r_k^2 \le [r_n^2 - t] \le r_n^2 + r_{k-1}^2 \le 2r_{k-1}^2 = 8r_k^2,$$

so that

$$e^{1/32} r_k^3 = (r_k^2)^{3/2} e^{r_k^2/(32r_k^2)} \le (r_n^2-t)^{3/2} e^{|x|^2/(4(r_n^2-t))} \le (8r_k^2)^{3/2} e^{(2r_k)^2/(4r_k^2)} = 2^{9/2} e r_k^3.$$

Therefore

$$\frac{1}{2^{9/2}e}\cdot\frac{1}{r_k^3} \le \psi^n(x,t) \le \frac{1}{e^{1/32}}\cdot\frac{1}{r_k^3} \quad \text{for all } (x,t)\in Q^{k-1}\backslash Q^k \quad (2\le k\le n) \tag{4-35}$$

and hence, as in (4-34),

$$|\nabla_x\psi^n(x,t)| \lesssim \frac{r_k}{r_k^2}\cdot\frac{1}{r_k^3} = \frac{1}{r_k^4} \quad \text{for all } (x,t)\in Q^{k-1}\backslash Q^k \quad (2\le k\le n). \tag{4-36}$$

We can therefore estimate (for $n\ge 2$ where $\phi^n = \psi^n$ in Q^n):

$$\begin{aligned}
&\frac{1}{2^{3/2}e^{1/4}}\cdot\frac{1}{r_n^3}\left[\operatorname*{ess\,sup}_{I^n}\int_{B^n}(|u|^2+|\nabla d|^2) + \iint_{Q^n}(|\nabla u|^2+|\nabla^2 d|^2)\right]\\
&\overset{(4\text{-}33)}{\le} \operatorname*{ess\,sup}_{I^n}\int_{B^n}(|u|^2+|\nabla d|^2)\phi^n + \iint_{Q^n}(|\nabla u|^2+|\nabla^2 d|^2)\phi^n\\
&\le \bar{C}\left\{\iint_{Q^1}[(|u|^2+|\nabla d|^2)|\phi_t^n+\Delta\phi^n| + (|u|^3+|\nabla d|^3)|\nabla\phi^n| + \bar{\rho}|d|^2|\nabla d|^2\phi^n] + \int_{I^1}\left|\int_{B^1} pu\cdot\nabla\phi^n\right|\right\},
\end{aligned}$$

where the last inequality follows from (3-5). Note that

$$\phi_t^n + \Delta\phi^n \overset{(4\text{-}32)}{=} \psi^n(\chi_t + \Delta\chi) + 2\nabla\chi\cdot\nabla\psi^n \overset{(4\text{-}30)}{\equiv} 0 \quad \text{in } Q^2$$

[21] In (3-5) as well, the values of ϕ for $t > \bar{t}$ are actually irrelevant.

and hence, taking $k = 2$ in (4-35) and (4-36), we see that

$$|\phi_t^n + \Delta\phi^n| \lesssim \frac{1}{r_2^3} + \frac{1}{r_2^4} \lesssim 1 \quad \text{on } Q^1, \tag{4-37}$$

so that

$$\iint_{Q^1} (|u|^2 + |\nabla d|^2)|\phi_t^n + \Delta\phi^n| \overset{(4\text{-}37)}{\lesssim} \iint_{Q^1} (|u|^2 + |\nabla d|^2) \overset{(3\text{-}6)}{\lesssim} E_{3,q}^{2/3}$$

by Hölder's inequality. Note similarly that

$$|\nabla\phi^n| = |\chi\nabla\psi^n + \psi^n\nabla\chi| \overset{(4\text{-}30)}{\lesssim} |\nabla\psi^n| + |\psi^n| \quad \text{on } Q^1$$

so that (since $r_n^4 < r_n^3$) (4-33), (4-34) and (4-35), (4-36), respectively, give

$$|\nabla\phi^n| \lesssim \frac{1}{r_n^4} \quad \text{on } Q^n \qquad \text{and} \qquad |\nabla\phi^n| \lesssim \frac{1}{r_k^4} \quad \text{on } Q^{k-1}\backslash Q^k \tag{4-38}$$

for any $n \geq 2$ and $2 \leq k \leq n$. Therefore

$$\sum_{k=2}^{n} \iint_{Q^{k-1}\backslash Q^k} (|u|^3 + |\nabla d|^3)|\nabla\phi^n| \overset{(4\text{-}38)}{\lesssim} \left[\max_{1\leq k\leq n-1} (r_k)^{1-\alpha} ⨌_{Q^k} (|u|^3 + |\nabla d|^3)\right] \sum_{k=2}^{n} (r_k)^\alpha$$

and similarly

$$\iint_{Q^n} (|u|^3 + |\nabla d|^3)|\nabla\phi^n| \overset{(4\text{-}38)}{\lesssim} \left[(r_n)^{1-\alpha} ⨌_{Q^n} (|u|^3 + |\nabla d|^3)\right](r_n)^\alpha$$

for any $\alpha \in (0, 1]$, and we note that

$$\sum_{k=1}^{\infty} (r_k)^\alpha = \sum_{k=1}^{\infty} (2^{-\alpha})^k = \frac{1}{2^\alpha - 1} < \infty \quad \text{for any } \alpha > 0. \tag{4-39}$$

Hence in view of the disjoint union

$$Q^1 = \left(\bigcup_{k=2}^{n} Q^{k-1}\backslash Q^k\right) \cup Q^n \tag{4-40}$$

we have, taking $\alpha = 1$ in (4-39),

$$\iint_{Q^1} (|u|^3 + |\nabla d|^3)|\nabla\phi^n| \lesssim \max_{1\leq k\leq n} ⨌_{Q^k} (|u|^3 + |\nabla d|^3).$$

Similarly, setting

$$\alpha_q := \frac{2(q-5)}{q-2}$$

and noting that $\alpha_q \in \left(0, \frac{1}{2}\right]$ for $q \in (5, 6]$, we have

$$\bar\rho \iint_{Q^1} |d|^2 |\nabla d|^2 \phi^n \leq \frac{2}{q} \underbrace{\iint_{Q^1} |d|^q |\nabla d|^{3(1-q/6)}}_{\leq E_{3,q}} + \left(1 - \frac{2}{q}\right) \iint_{Q^1} |\nabla d|^3 (\phi^n)^{(5-\alpha_q)/3}$$

uniformly, of course, over $\bar\rho \in (0,1]$. Since

$$\iint_{Q^n} |\nabla d|^3 (\phi^n)^{(5-\alpha_q)/3} \overset{(4\text{-}33)}{\lesssim} (r_n)^{\alpha_q - 5} \iint_{Q^n} |\nabla d|^3 \lesssim (r_n)^{\alpha_q} \fint\!\!\fint_{Q^n} |\nabla d|^3$$

(as $\phi^n \lesssim r_n^{-3}$ on Q^n) for $n \geq 2$ and similarly

$$\iint_{Q^k \setminus Q^{k+1}} |\nabla d|^3 (\phi^n)^{(5-\alpha_q)/3} \overset{(4\text{-}35)}{\lesssim} (r_k)^{\alpha_q - 5} \iint_{Q^k} |\nabla d|^3 \lesssim (r_k)^{\alpha_q} \fint\!\!\fint_{Q^k} |\nabla d|^3$$

(as $\phi^n \lesssim r_k^{-3}$ on $Q^k \setminus Q^{k+1}$) for $1 \leq k \leq n-1$, we see that (4-39) with $\alpha = \alpha_q$ and (4-40) again give

$$\iint_{Q^1} |\nabla d|^3 (\phi^n)^{(5-\alpha_q)/3} \leq (2^{\alpha_q} - 1)^{-1} \max_{1 \leq k \leq n} \fint\!\!\fint_{Q^k} |\nabla d|^3.$$

We therefore see that

$$\bar\rho \iint_{Q^1} |d|^2 |\nabla d|^2 \phi^n \lesssim \frac{2}{5} E_{3,q} + \frac{2}{3} (2^{\alpha_q} - 1)^{-1} \max_{1 \leq k \leq n} \fint\!\!\fint_{Q^k} |\nabla d|^3 \quad \text{with } \alpha_q := \frac{2(q-5)}{q-2},$$

uniformly for any $\bar\rho \in (0,1]$ and $q \in (5,6]$.

Putting all of the above together and recalling (3-8), we see that for $n \geq 2$ we have

$$\begin{aligned} \frac{L_n}{\overline{C}} &= \frac{1}{\overline{C}} \left[\operatorname*{ess\,sup}_{I^n} \fint_{B^n} (|u|^2 + |\nabla d|^2) + \int_{I^n} \fint_{B^n} (|\nabla u|^2 + |\nabla^2 d|^2) \right] \\ &\lesssim E_{3,q} + E_{3,q}^{2/3} + (2^{\alpha_q} - 1)^{-1} \max_{1 \leq k \leq n} \fint\!\!\fint_{Q^k} (|u|^3 + |\nabla d|^3) + \int_{I^1} \left| \int_{B^1} pu \cdot \nabla \phi^n \right|. \end{aligned} \tag{4-41}$$

Furthermore, we claim that for $1 \leq k_0 \leq n-1$ we have

$$\int_{I^1} \left| \int_{B^1} pu \cdot \nabla \phi^n \right| \lesssim \max_{k_0 \leq k \leq n} \left(r_k^{1/3} \fint\!\!\fint_{Q^k} |p - \bar p_k| \, |u| \right) + k_0 2^{4k_0} \iint_{Q^1} |p| \, |u|. \tag{4-42}$$

Assuming this for the moment and continuing, for $n \geq 2$, (4-41), (4-42) and Young's convexity inequality along with the fact that, for any $k_1 \geq 1$, we can estimate

$$\max_{1 \leq k \leq k_1} \fint\!\!\fint_{Q^k} (|u|^3 + |\nabla d|^3) \lesssim k_1 2^{5k_1} \iint_{Q^1} (|u|^3 + |\nabla d|^3)$$

imply (recalling (3-9)) that

$$\frac{L_n}{\overline{C}} \lesssim E_{3,q} + E_{3,q}^{2/3} + (2^{\alpha_q} - 1)^{-1} \max_{k_0 \leq k \leq n} R_k + k_0 2^{5k_0} \underbrace{\iint_{Q^1} |u|^3 + |\nabla d|^3 + |p|^{3/2}}_{\leq E_{3,q}}$$

for any $k_0 \in \{1, \ldots, n-1\}$, which proves Proposition 6.

To prove (4-42), we consider additional functions χ_k (so that $\chi_k \phi^n = \chi_k \chi \psi^n$) satisfying (recall that $Q^k = Q^k(0,0) = B_{r_k}(0) \times (-r_k^2, 0)$, so χ_k will be zero in a neighborhood of the parabolic boundary of Q^k)

$$\begin{gathered} \chi_k \in C_0^\infty(\widetilde{Q}_{r_k}) \quad \text{with } \widetilde{Q}_r := B_r(0) \times (-r^2, r^2) \quad \text{for } r > 0, \\ \chi_k \equiv 1 \quad \text{in } \widetilde{Q}_{7r_k/8}, \quad 0 \leq \chi_k \leq 1 \quad \text{and} \quad |\nabla \chi_k| \lesssim \frac{1}{r_k} \end{gathered} \tag{4-43}$$

($\chi_k|_{\{t>0\}}$ will again actually be irrelevant) so that in particular (as $\widetilde{Q}_{r_{k+2}} \subset \widetilde{Q}_{7r_{k+1}/8}$ where $\chi_k \equiv \chi_{k+1} \equiv 1$)

$$\operatorname{supp}(\chi_k - \chi_{k+1}) \subset \widetilde{Q}_{r_k} \backslash \widetilde{Q}_{r_{k+2}}. \tag{4-44}$$

Then, fixing any $n \geq 2$, writing

$$\chi_0 = \chi_n + \sum_{k=0}^{n-1} (\chi_k - \chi_{k+1})$$

and noting that $\chi_0 \equiv 1$ on $Q^1 = Q_{1/2}(0,0) \subset Q_{7/8}(0,0) = Q_{7r_0/8}(0,0)$, we see that for any fixed $k_0 \in \mathbb{N} \cap [1, n-1]$ and at each fixed $\tau \in I^1$, we have

$$\begin{aligned}
\int_{B^1} pu \cdot \nabla \phi^n &= \int_{B^1} pu \cdot \nabla[\chi_0 \phi^n] && \text{(by (4-43))} \\
&= \int_{B^1} pu \cdot \nabla[\chi_n \phi^n] + \sum_{k=0}^{n-1} \int_{B^1} pu \cdot \nabla[(\chi_k - \chi_{k+1})\phi^n] \\
&= \int_{B^n} pu \cdot \nabla[\chi_n \phi^n] + \sum_{k=0}^{n-1} \int_{[B^k \backslash B^{k+2}]} pu \cdot \nabla[(\chi_k - \chi_{k+1})\phi^n] && \text{(by (4-43), (4-44))} \\
&= \int_{B^n} (p - \bar{p}_n) u \cdot \nabla[\chi_n \phi^n] + \sum_{k=0}^{k_0-1} \int_{[B^k \backslash B^{k+2}]} pu \cdot \nabla[(\chi_k - \chi_{k+1})\phi^n] \\
&\qquad + \sum_{k=k_0}^{n-1} \int_{[B^k \backslash B^{k+2}]} (p - \bar{p}_k) u \cdot \nabla[(\chi_k - \chi_{k+1})\phi^n], && \text{(4-45)}
\end{aligned}$$

where the final equality is due to (3-3), and where

$$\bar{p}_k = \bar{p}_k(\tau) = \fint_{B^k} p(x,\tau)\, dx.$$

Note first that (4-35), (4-36) and (4-44) imply (since $r_{j+1} = 2r_j$ for any j) that

$$\begin{gathered}
|\nabla[(\chi_k - \chi_{k+1})\phi^n]| \leq |\chi_k - \chi_{k+1}|\,|\nabla \phi^n| + |\phi^n|\,|\nabla(\chi_k - \chi_{k+1})| \lesssim r_k^{-4} \\
\text{on } \; Q^k \setminus Q^{k+2} = (Q^k \setminus Q^{k+1}) \cup (Q^{k+1} \setminus Q^{k+2})
\end{gathered}$$

for any k, and similarly

$$|\nabla[\chi_n \phi^n]| \leq |\chi_n|\,|\nabla \phi^n| + |\phi^n|\,|\nabla \chi_n| \lesssim r_n^{-4} \quad \text{on } \; Q^n.$$

Therefore we can estimate (recalling again (4-43) and (4-44) when integrating $|(4\text{-}45)|$ over $\tau \in I^1$)

$$\int_{\tau \in I^1} \left| \int_{B^1 \times \{\tau\}} pu \cdot \nabla \phi^n \right| \lesssim k_0 2^{4k_0} \iint_{Q^1} |p|\,|u| + \sum_{k=k_0}^{n} r_k \fint\!\!\fint_{Q^k} |p - \bar{p}_k|\,|u|$$

which, along with (4-39) with $q = \frac{3}{2}$ implies (4-42) for any $k_0 \in [1, n-1]$ as desired. □

4.4. ***Proof of Proposition 8.*** In this section we prove the technical decay estimate (Proposition 8) used to prove Lemma 7. In all of what follows, recall the definitions in (3-17) and (3-18) of A_{z_0}, B_{z_0}, C_{z_0}, D_{z_0}, E_{z_0}, F_{z_0}, G_{q,z_0} and M_{q,z_0}. We will require the following three claims which essentially appear in [Lin and Liu 1996] and which generalize certain lemmas in [Caffarelli et al. 1982]; however we include full proofs in order to clarify certain details, and to highlight the role of G_{q,z_0} (not utilized in [Lin and Liu 1996]) in Claim 4 which is therefore[22] a slightly refined version of what appears in [Lin and Liu 1996].

Claim 2 (general estimates; cf. [Caffarelli et al. 1982, Lemmas 5.1 and 5.2]). *There exist constants $c_1,\ c_2 > 0$ such that for any u and d which have the regularities in* (1-9) *for $\Omega_T := \Omega \times (0, T)$ as in Theorem 1, the estimates*

$$C_{z_0}(\gamma\rho) \le c_1[\gamma^3 A_{z_0}^{3/2} + \gamma^{-3} A_{z_0}^{3/4} B_{z_0}^{3/4}](\rho) \tag{4-46}$$

and

$$E_{z_0}(\gamma\rho) \le c_2[C_{z_0}^{1/3} A_{z_0}^{1/2} B_{z_0}^{1/2}](\gamma\rho) \tag{4-47}$$

hold for any $z_0 \in \mathbb{R}^{3+1}$ and $\rho > 0$ such that $Q_\rho^*(z_0) \subseteq \Omega_T$ and any $\gamma \in (0, 1]$.

Claim 3 (estimates requiring the pressure equation; cf. [Caffarelli et al. 1982, Lemmas 5.3 and 5.4]). *There exist constants $c_3,\ c_4 > 0$ such that for any $u,\ d$ and p which have the regularities in* (1-9) *and* (1-10) *for $\Omega_T := \Omega \times (0, T)$ as in Theorem 1 and which satisfy the pressure equation* (1-12), *the estimates*

$$D_{z_0}(\gamma\rho) \le c_3[\gamma(D_{z_0} + A_{z_0}^{3/4} B_{z_0}^{3/4} + C_{z_0}^{1/2}) + \gamma^{-5} A_{z_0}^{3/4} B_{z_0}^{3/2}](\rho) \tag{4-48}$$

and

$$F_{z_0}(\gamma\rho) \le c_4[\gamma^{1/12}(A_{z_0} + D_{z_0}^{4/3} + C_{z_0}^{2/3}) + \gamma^{-10} A_{z_0}(B_{z_0}^{1/2} + B_{z_0}^2)](\rho). \tag{4-49}$$

hold for any $z_0 \in \mathbb{R}^{3+1}$ and $\rho > 0$ such that $Q_\rho^*(z_0) \subseteq \Omega_T$ and any $\gamma \in \left(0, \tfrac{1}{2}\right]$.

The crucial aspect of the estimates (4-46)–(4-49) — which control $M_{q,z_0}(\gamma\rho)$ — in proving Lemma 7 (through Proposition 8) is that whenever a negative power of γ appears, there is always a factor of B_{z_0} as well, which will be small when proving Lemma 7. Positive powers of γ will similarly be small; in each term evaluated at ρ (see also (4-52) below), we must have either γ^α or $B_{z_0}^\alpha$ for some $\alpha > 0$.

To complete the proof of Proposition 8, we require the following.

Claim 4 (estimate requiring the local energy inequality; cf. [Caffarelli et al. 1982, Lemma 5.5]). *There exists a constant $c_5 > 0$ such that for any $u,\ d$ and p which have the regularities in* (1-9) *and* (1-10) *for $\Omega_T := \Omega \times (0, T)$ as in Theorem 1 and such that u satisfies the weak divergence-free property* (1-11) *and the local energy inequality* (1-13) *holds for some constant $\bar{C} \in (0, \infty)$, the estimate*

$$A_{z_0}\left(\frac{\rho}{2}\right) \le c_5 \cdot \bar{C}[C^{2/3} + E + F_{z_0} + (1 + [\,\cdot\,]^2) G_q^{4/(6-q)} + (G_q^{2/(6-q)} + C^{1/3}) B^{1/2}](\rho) \tag{4-50}$$

holds for any $q \in [2, 6)$ and any $z_0 \in \mathbb{R}^{3+1}$ and $\rho > 0$ such that $Q_\rho^*(z_0) \subseteq \Omega_T$.

Postponing the proof of the claims, let us use them to prove the proposition. In all of what follows, we note the simple facts that, for any $\rho > 0$ and $\alpha \in (0, 1]$,

$$\begin{aligned} \mathcal{K} \in \{A_{z_0}, B_{z_0}\} \qquad &\Longrightarrow \qquad \mathcal{K}(\alpha\rho) \le \alpha^{-1}\mathcal{K}(\rho), \\ \mathcal{K} \in \{C_{z_0}, D_{z_0}, E_{z_0}, F_{z_0}\} \qquad &\Longrightarrow \qquad \mathcal{K}(\alpha\rho) \le \alpha^{-2}\mathcal{K}(\rho), \\ G_{q,z_0}(\alpha\rho) &\le \alpha^{-2-q/2} G_{q,z_0}(\rho). \end{aligned} \tag{4-51}$$

[22] Note that $G_{z_0}(r) \lesssim \|d\|_\infty$ uniformly in r (and z_0), though in our setting we may have $d \notin L^\infty$.

Proof of Proposition 8. Fixing z_0 and ρ_0 as in Proposition 8, under the assumptions in the proposition we see that estimates (4-46)–(4-50) hold for all $\rho \in (0, \rho_0]$, $\gamma \in \left(0, \frac{1}{2}\right]$ and $q \in [2, 6)$ by Claims 2, 3 and 4.

Note first that (4-46), (4-47) and (4-51) imply that

$$E_{z_0}(\gamma\rho) \lesssim [A_{z_0} B_{z_0}^{1/2} + \gamma^{-2} A_{z_0}^{3/4} B_{z_0}^{3/4}](\rho)$$

and hence, for example, there exists some $c_6 > 0$ such that

$$E_{z_0}(\gamma\rho) \le c_6[\gamma^2 A_{z_0} + \gamma^{-2}(A_{z_0}^{1/2} B_{z_0}^{1/2} + A_{z_0} B_{z_0})](\rho), \tag{4-52}$$

for $\rho \in (0, \rho_0]$ and $\gamma \in \left(0, \frac{1}{2}\right]$ (in fact, for $\gamma \in (0, 1]$) and that it follows from (4-50), assumption (3-20) and the assumption that $\rho_0 \le 1$ that there exists some $c_7 > 0$ such that

$$(\bar{C})^{-1} A_{z_0}\left(\frac{\rho}{2}\right) \le c_7[C_{z_0}^{2/3} + E_{z_0} + F_{z_0} + G_{q,z_0}^{4/(6-q)} + (G_{q,z_0}^{2/(6-q)} + C{z_0}^{1/3}) B_{z_0}^{1/2}](\rho),$$

and hence, recalling (3-18), we have that, for some $c_8 > 0$,

$$(\bar{C})^{-3/2} A_{z_0}^{3/2}\left(\frac{\rho}{2}\right) \le c_8[M_{q,z_0}(\rho) + M_{q,z_0}^{1/2}(\rho) B_{z_0}^{3/4}(\rho)] \tag{4-53}$$

for $\rho \in (0, \rho_0]$. We note as well that, as in (3-23), if $\sigma \in [q, 6)$ and if (3-20) holds for some $\bar{g} \ge 1$, then

$$G_{q,z_0}^{6/(6-q)}(\gamma\rho) \overset{(3\text{-}22)}{\le} \bar{g}^{6/(6-\sigma)} \cdot C_{z_0}^{\alpha_{\sigma,q}}(\gamma\rho) \overset{(4\text{-}46)}{\le} \bar{g}^{6/(6-\sigma)} \cdot [\gamma^3 A_{z_0}^{3/2} + \gamma^{-3} A_{z_0}^{3/4} B_{z_0}^{3/4}]^{\alpha_{\sigma,q}}(\rho) \tag{4-54}$$

for $\rho \in (0, \rho_0]$. Now, writing $\gamma\rho = 2\gamma \cdot \left(\frac{1}{2}\rho\right)$ for $2\gamma \le \frac{1}{2}$, it follows from (4-46), (4-48), (4-49), (4-52), (4-54) and (3-18) followed by an application of (4-51) $\left(\text{with } \alpha = \frac{1}{2}\right)$ to all terms except for A_{z_0} along with the facts that $\gamma,\ B_{z_0}(\rho) \le 1$ (so that you can always estimate positive powers by 1) as well as the fact that $\alpha_{\sigma,q} \in (0, 1)$ that

$$\begin{aligned}
M_{q,z_0}(\gamma\rho) &\le [C_{z_0} + G_{q,z_0}^{6/(6-q)} + D_{z_0}^2 + E_{z_0}^{3/2} + F_{z_0}^{3/2}](\gamma\rho) \\
&\lesssim \left[\gamma^3 A_{z_0}^{3/2}\left(\frac{\rho}{2}\right) + \gamma^{-3} A_{z_0}^{3/4}\left(\frac{\rho}{2}\right) B_{z_0}^{3/4}(\rho)\right] + \bar{g}^{6/(6-\sigma)} \cdot \left[\gamma^3 A_{z_0}^{3/2}\left(\frac{\rho}{2}\right) + \gamma^{-3} A_{z_0}^{3/4}\left(\frac{\rho}{2}\right) B_{z_0}^{3/4}(\rho)\right]^{\alpha_{\sigma,q}} \\
&\quad + \left[\gamma M_{q,z_0}^{1/2}(\rho) + \gamma^{-5} A_{z_0}^{3/4}\left(\frac{\rho}{2}\right)(B_{z_0}^{3/4}(\rho) + B_{z_0}^{3/2}(\rho))\right]^2 \\
&\quad + \left[\gamma^2 A_{z_0}\left(\frac{\rho}{2}\right) + \gamma^{-2}\left(A_{z_0}^{1/2}\left(\frac{\rho}{2}\right) B_{z_0}^{1/2}(\rho) + A_{z_0}\left(\frac{\rho}{2}\right) B_{z_0}(\rho)\right)\right]^{3/2} \\
&\quad + \left[\gamma^{1/12}\left(A_{z_0}\left(\frac{\rho}{2}\right) + M_{q,z_0}^{2/3}(\rho)\right) + \gamma^{-10} A_{z_0}\left(\frac{\rho}{2}\right)(B_{z_0}^{1/2}(\rho) + B_{z_0}^2(\rho))\right]^{3/2} \\
&\lesssim (1 + \bar{g}^{6/(6-\sigma)})\left[\gamma^{\alpha_{\sigma,q}/8}\left(M_{q,z_0}(\rho) + \left[A_{z_0}^{3/2}\left(\frac{\rho}{2}\right)\right]^{\alpha_{\sigma,q}} + \left[A_{z_0}^{3/2}\left(\frac{\rho}{2}\right)\right]\right)\right. \\
&\qquad \left. + \gamma^{-15}\left(\left[A_{z_0}^{3/2}\left(\frac{\rho}{2}\right)\right]^{\alpha_{\sigma,q}/2} + \left[A_{z_0}^{3/2}\left(\frac{\rho}{2}\right)\right]^{1/2} + \left[A_{z_0}^{3/2}\left(\frac{\rho}{2}\right)\right]\right) B_{z_0}^{3\alpha_{\sigma,q}/4}(\rho)\right]
\end{aligned}$$

so long as $\gamma \in \left(0, \frac{1}{4}\right]$. Noting that $1 \le \bar{g}^{6/(6-\sigma)}$, the estimate (3-21) for such γ and for $\rho \in (0, \rho_0]$ now follows from the estimate above along with (4-53) as, in particular, (4-53) implies — as $\gamma,\ B_{z_0}(\rho) \le 1$

and $\alpha_{\sigma,q} \in (0,1)$ — that

$$(\overline{C})^{-3/2} A_{z_0}^{3/2}\Big(\frac{\rho}{2}\Big) \lesssim M_{q,z_0}(\rho) + \gamma^{-15-\alpha_{\sigma,q}/8} M_{q,z_0}^{1/2}(\rho) B_{z_0}^{3\alpha_{\sigma,q}/4}(\rho)$$

which we apply to the terms above with a positive power of γ, and that

$$(\overline{C})^{-3/2} A_{z_0}^{3/2}\Big(\frac{\rho}{2}\Big) \lesssim M_{q,z_0}(\rho) + M_{q,z_0}^{1/2}(\rho),$$

which we apply to the terms above with a negative power of γ, completing the proof. □

Let us now prove the claims.

Proof of Claim 2. For simplicity, we will suppress the dependence on $z_0 = (x_0, t_0)$ in what follows.

Let us first prove (4-46). Note that for any $r \le \rho$, at any fixed $t \in I_r^*$, taking $v \in \{u, \nabla d\}$ we have

$$\begin{aligned}\int_{B_r} |v|^2\,dx &\le \int_{B_\rho} \big||v|^2 - \overline{|v|^2}^{\rho}\big|\,dx + |B_r|\,\overline{|v|^2}^{\rho} \\ &\lesssim \rho \int_{B_\rho} \big|\nabla |v|^2\big|\,dx + \Big(\frac{r}{\rho}\Big)^3 \int_{B_\rho} |v|^2\,dx\end{aligned}$$

due to Poincaré's inequality (4-16). Since $\big|\nabla|v|^2\big| \le |v|\,|\nabla v|$ almost everywhere, Hölder's inequality then implies that

$$\|v\|_{2;B_r}^2 \lesssim \rho \|v\|_{2;B_\rho} \|\nabla v\|_{2;B_\rho} + \Big(\frac{r}{\rho}\Big)^3 \|v\|_{2;B_\rho}^2. \tag{4-55}$$

Therefore

$$\begin{aligned}\|v\|_{3;B_r}^3 &\lesssim \frac{1}{r^{3/2}} (\|v\|_{2;B_r}^2)^{3/2} + \|v\|_{2;B_r}^{3/2} \|\nabla v\|_{2;B_r}^{3/2} && \text{(by (4-14))} \\ &\lesssim \Big(1 + \Big(\frac{\rho}{r}\Big)^{3/2}\Big) \|v\|_{2;B_\rho}^{3/2} \|\nabla v\|_{2;B_\rho}^{3/2} + \frac{1}{r^{3/2}}\Big(\frac{r}{\rho}\Big)^{9/2} \|v\|_{2;B_\rho}^3 && \text{(by (4-55))}.\end{aligned}$$

Summing over $v \in \{u, \nabla d\}$, we see that

$$\begin{aligned}&\|u\|_{3;B_r}^3 + \|\nabla d\|_{3;B_r}^3 \\ &\quad\lesssim \Big(1 + \Big(\frac{\rho}{r}\Big)^{3/2}\Big) (\|u\|_{2;B_\rho}^2 + \|\nabla d\|_{2;B_\rho}^2)^{3/4} (\|\nabla u\|_{2;B_\rho}^2 + \|\nabla^2 d\|_{2;B_\rho}^2)^{3/4} + \frac{r^3}{\rho^{9/2}} (\|u\|_{2;B_\rho}^2 + \|\nabla d\|_{2;B_\rho}^2)^{3/2}.\end{aligned}$$

Now integrating over $t \in I_r^*$ (where $|I_r^*| = r^2$), Hölder's inequality implies that

$$\begin{aligned}r^2 C(r) &\lesssim |I_r^*|^{1/4} \Big(1 + \Big(\frac{\rho}{r}\Big)^{3/2}\Big) \big\| \|u\|_{2;B_\rho}^2 + \|\nabla d\|_{2;B_\rho}^2 \big\|_{\infty;I_r^*}^{3/4} (\|\nabla u\|_{2;Q_\rho^*}^2 + \|\nabla^2 d\|_{2;Q_\rho^*}^2)^{3/4} \\ &\qquad\qquad + |I_r^*| \frac{r^3}{\rho^{9/2}} \big\| \|u\|_{2;B_\rho}^2 + \|\nabla d\|_{2;B_\rho}^2 \big\|_{\infty;I_r^*}^{3/2} \\ &\lesssim r^{1/2} \Big(1 + \Big(\frac{\rho}{r}\Big)^{3/2}\Big) (\rho A(\rho))^{3/4} (\rho B(\rho))^{3/4} + \frac{r^5}{\rho^{9/2}} (\rho A(\rho))^{3/2},\end{aligned}$$

which, upon dividing both sides by r^2, setting $\gamma := r/\rho$ and noting that $1 \le \gamma^{-3/2}$, precisely gives (4-46).

Next, to prove (4-47), we use the Poincaré–Sobolev inequality

$$\|g-\bar{g}^r\|_{q^*;B_r} \le c_q \|\nabla g\|_{q;B_r}$$

(the constant is independent of r due to the relationship between q and q^*) corresponding to the embedding $W^{1,q} \hookrightarrow L^{q^*}$ for $q<3$ (in $\mathbb{R}^3$) and $q^* = 3q/(3-q)$. Taking $q=1$, at any $t \in I_r^*$ and for $v \in \{u, \nabla d\}$ the Hölder and Poincaré–Sobolev inequalities give us

$$\int_{B_r} |u|\big||v|^2 - \overline{|v|^2}^r\big|\,dx \le \|u\|_{3;B_r} \||v|^2-\overline{|v|^2}^r\|_{3/2;B_r} \lesssim \|u\|_{3;B_r}\|\nabla(|v|^2)\|_{1;B_r} \lesssim \|u\|_{3;B_r}\|v\|_{2;B_r}\|\nabla v\|_{2;B_r}.$$

Summing this first over $v \in \{u, \nabla d\}$ at a fixed t and then integrating over $t \in I_r^*$, we see that

$$\begin{aligned} r^2 E(r) &\lesssim \int_{I_r^*} \|u\|_{3;B_r}(\|u\|^2_{2;B_r} + \|\nabla d\|^2_{2;B_r})^{1/2}(\|\nabla d\|^2_{2;B_r} + \|\nabla^2 d\|^2_{2;B_r})^{1/2}\,dt \\ &\lesssim \|u\|_{3;Q_r^*}\big\|(\|u\|^2_{2;B_r} + \|\nabla d\|^2_{2;B_r})^{1/2}\big\|_{6;I_r^*}(\|\nabla u\|^2_{2;Q_r^*} + \|\nabla^2 d\|^2_{2;Q_r^*})^{1/2} \\ &\lesssim |I_r^*|^{1/6}(\|u\|^3_{3;Q_r^*})^{1/3}\big\|\|u\|^2_{2;B_r} + \|\nabla d\|^2_{2;B_r}\big\|^{1/2}_{\infty;I_r^*}(\|\nabla u\|^2_{2;Q_r^*} + \|\nabla^2 d\|^2_{2;Q_r^*})^{1/2} \\ &\lesssim r^{1/3}(r^2C(r))^{1/3}(rA(r))^{1/2}(rB(r))^{1/2} = r^2[C^{1/3}A^{1/2}B^{1/2}](r) \end{aligned}$$

which proves (4-47) and completes the proof of Claim 2. □

Proof of Claim 3. As in (4-3) of Claim 1, for any $t \in I_r^*(z_0)$ and almost every $x \in B_{3\rho/4}(x_0)$ (with $r \le \rho$), using a smooth cut-off function ψ equal to one in $\Omega_1 := B_{3\rho/4}(x_0)$ and supported in $\Omega_2 := B_\rho(x_0)$ so that

$$|\nabla\psi| \lesssim \rho^{-1} \quad \text{and} \quad |\Delta\psi| \lesssim \rho^{-2}, \tag{4-56}$$

we use Remark 10 to write $\Pi := p(\,\cdot\,,t)$ as

$$p(x,t) = \underbrace{-\int \nabla G^x \cdot v(t)\psi\,dy}_{=:p_1(x,t)} + \underbrace{\int G^x_{\psi,1}\cdot v(t)\,dy}_{=:p_2(x,t)} + \underbrace{\int G^x_{\psi,2}\,p(\,\cdot\,,t)\,dy}_{=:p_3(x,t)}$$

with

$$G^x_{\psi,1} := -G^x\nabla\psi, \quad G^x_{\psi,2} := 2\nabla G^x\cdot\nabla\psi + G^x\Delta\psi \quad \text{and} \quad v(t) := [\nabla^T\cdot(u\otimes u + \nabla d\odot\nabla d)](\,\cdot\,,t).$$

Our goal is to estimate $p(x,t)$ for $x \in B_{\rho/2}(x_0)$.

Both p_2 and p_3 contain derivatives of ψ in each term so that the integrand can only be nonzero when $|y-x_0| > \frac{3}{4}\rho$, and hence for $x \in B_{\rho/2}(x_0)$ one has

$$|x-y| \ge \tfrac14\rho \quad \Longrightarrow \quad |G^x(y)| \lesssim \rho^{-1} \text{ and } |\nabla G^x(y)| \lesssim \rho^{-2}. \tag{4-57}$$

From (4-56) and (4-57) and the fact that ψ is supported in $B_\rho(x_0)$, we have (omitting the dependence on t, and noting that the constants in the inequalities are independent of t as they come only from G^x and ψ)

$$\begin{aligned} \sup_{x\in B_{\rho/2}(x_0)} |p_2(x)| &\lesssim \rho^{-2}\int_{B_\rho(x_0)} (|u||\nabla u| + |\nabla d||\nabla^2 d|)\,dy \\ &\lesssim \rho^{-2}\bigg(\int_{B_\rho(x_0)}(|u|^2+|\nabla d|^2)\,dy\bigg)^{1/2}\bigg(\int_{B_\rho(x_0)}(|\nabla u|^2+|\nabla^2 d|^2)\,dy\bigg)^{1/2} \end{aligned} \tag{4-58}$$

and similarly

$$\sup_{x\in B_{\rho/2}(x_0)} |p_3(x)| \lesssim \rho^{-3} \int_{B_\rho(x_0)} |p|\, dy. \tag{4-59}$$

For p_1, Young's inequality for convolutions (setting $R := 2\rho$ as in Remark 10) with $\frac{2}{3}+1=\frac{3}{4}+\frac{11}{12}$ gives

$$\begin{aligned}\|p_1\|_{3/2;B_\rho(x_0)} &\lesssim \left\|\frac{1}{|\cdot|^2}\right\|_{4/3;B_{2\rho}(0)} \|(|u|+|\nabla d|)(|\nabla u|+|\nabla^2 d|)\|_{12/11;B_\rho(x_0)}\\ &\lesssim \rho^{1/4}\|(|u|+|\nabla d|)(|\nabla u|+|\nabla^2 d|)\|_{12/11;B_\rho(x_0)}\end{aligned}$$

and then Hölder's inequality with $\frac{11}{12}=\frac{1}{4}+\frac{1}{6}+\frac{1}{2}$ gives

$$\begin{aligned}\|p_1\|^{3/2}_{3/2;B_\rho(x_0)} &\lesssim (\rho^{1/4}\|(|u|+|\nabla d|)^{1/2}\|_{4;B_\rho(x_0)}\|(|u|+|\nabla d|)^{1/2}\|_{6;B_\rho(x_0)}\||\nabla u|+|\nabla^2 d|\|_{2;B_\rho(x_0)})^{3/2}\\ &\lesssim \rho^{3/8}(\rho A(\rho))^{3/8}\||u|+|\nabla d|\|^{3/4}_{3;B_\rho(x_0)}\||\nabla u|+|\nabla^2 d|\|^{3/2}_{2;B_\rho(x_0)}.\end{aligned} \tag{4-60}$$

For the following, we fix now any $r\in\left(0,\frac{\rho}{2}\right]$ and omit the dependence on x_0, t_0 and z_0 in $B_r(x_0)$, $B_\rho(x_0)$, $I^*(t_0)$, A_{z_0}, B_{z_0}, C_{z_0} and D_{z_0} (we will retain z_0 in the notation for F_{z_0} to distinguish it from $F=\nabla f$).

To first prove (4-48), we note that (4-58) implies — since $r\le\frac{1}{2}\rho$ — that

$$\begin{aligned}\int_{B_r}|p_2|^{3/2}\,dx &\lesssim r^3\rho^{-3}\left(\int_{B_\rho}(|u|^2+|\nabla d|^2)\,dy\right)^{3/4}\left(\int_{B_\rho}(|\nabla u|^2+|\nabla^2 d|^2)\,dy\right)^{3/4}\\ &\le r^3\rho^{-3}(\rho A(\rho))^{3/4}\left(\int_{B_\rho}(|\nabla u|^2+|\nabla^2 d|^2)\,dy\right)^{3/4}\end{aligned}$$

so that, integrating over $t\in I_r^*$ and using Hölder's inequality, we have

$$r^{-2}\iint_{Q_r^*}|p_2|^{3/2}\,dz \lesssim r^{-2}r^3\rho^{-9/4}A^{3/4}(\rho)\cdot|I_\rho^*|^{1/4}(\rho B(\rho))^{3/4} = \frac{r}{\rho}\cdot[(AB)^{3/4}](\rho), \tag{4-61}$$

and that (4-59) similarly implies that

$$r^{-2}\iint_{Q_r^*}|p_3|^{3/2}\,dz \lesssim r\rho^{-9/2}\int_{I_r^*}\left(\int_{B_\rho}|p|\,dy\right)^{3/2} \lesssim \frac{r}{\rho}\cdot D(\rho). \tag{4-62}$$

Finally, integrating (4-60) over $t\in I_r^*$, Hölder's inequality with $1=\frac{1}{4}+\frac{3}{4}$ gives

$$\begin{aligned}r^{-2}\|p_1\|^{3/2}_{3/2;Q_r^*} &\lesssim r^{-2}\rho^{3/4}A^{3/8}(\rho)\||u|+|\nabla d|\|^{3/4}_{3;Q_\rho^*}\||\nabla u|+|\nabla^2 d|\|^{3/2}_{2;Q_\rho^*}\\ &\lesssim r^{-2}\rho^{3/4}A^{3/8}(\rho)(\rho^2 C(\rho))^{1/4}(\rho B(\rho))^{3/4} = (C^{1/4}(\rho))\cdot\left(\left(\frac{r}{\rho}\right)^{-2}A^{3/8}(\rho)B^{3/4}(\rho)\right).\end{aligned}$$

Multiplying and dividing by $(r/\rho)^{\alpha/2}$ for any $\alpha\in\mathbb{R}$, Cauchy's inequality gives

$$r^{-2}\|p_1\|^{3/2}_{3/2;Q_r^*} \lesssim \left(\frac{r}{\rho}\right)^{\alpha}C^{1/2}(\rho)+\left(\frac{r}{\rho}\right)^{-\alpha-4}A^{3/4}(\rho)B^{3/2}(\rho). \tag{4-63}$$

Since we want a positive power of $\gamma=r/\rho$ in the first term and a negative one in the second (because it contains B which will be small), we want to take $\alpha>0$. Choosing $\alpha=1$ purely to make the following expression simpler, since $p = p_3 + p_2 + p_1$, we see from (4-61)–(4-63) that

$$D(r) \lesssim \frac{r}{\rho} \cdot [D + (AB)^{3/4} + C^{1/2}](\rho) + \left(\frac{r}{\rho}\right)^{-5} [A^{3/4} B^{3/2}](\rho)$$

which implies (4-48) for $\gamma := r/\rho \le \frac{1}{2}$.

To prove (4-49), we note that $F_{z_0}(r) \le F_1(r) + F_2(r) + F_3(r)$, where we set

$$F_j(r) := \frac{1}{r^2} \iint_{Q_r} |p_j|\,|u|\,dz.$$

To estimate F_1 we use Hölder's inequality and (4-60) to see that (in fact, for $r \le \rho$)

$$\begin{aligned}
\int_{B_r} |p_1|\,|u|\,dx &\le \|u\|_{3;B_\rho} \|p_1\|_{3/2;B_\rho} \\
&\lesssim \|u\|_{3;B_\rho} \cdot \rho^{1/4} (\rho A(\rho))^{1/4} \||u| + |\nabla d|\|_{3;B_\rho}^{1/2} \||\nabla u| + |\nabla^2 d|\|_{2;B_\rho} \\
&\le \rho^{1/2} A^{1/4}(\rho) \||u| + |\nabla d|\|_{3;B_\rho}^{3/2} \||\nabla u| + |\nabla^2 d|\|_{2;B_\rho}
\end{aligned}$$

and hence the Cauchy–Schwarz inequality in time gives

$$\begin{aligned}
F_1(r) &\lesssim r^{-2} \rho^{1/2} A^{1/4}(\rho) \||u| + |\nabla d|\|_{3;Q_\rho^*}^{3/2} \||\nabla u| + |\nabla^2 d|\|_{2;Q_\rho^*} \\
&\lesssim r^{-2} \rho^{1/2} A^{1/4}(\rho) (\rho^2 C(\rho))^{1/2} (\rho B(\rho))^{1/2} \\
&= \left(\left(\frac{r}{\rho}\right)^{\alpha} C^{1/2}(\rho)\right) \cdot \left(\left(\frac{r}{\rho}\right)^{-2-\alpha} [A^{1/4} B^{1/2}](\rho)\right) \\
&\lesssim \left(\left(\frac{r}{\rho}\right)^{\alpha} C^{1/2}(\rho)\right)^{4/3} + \left(\left(\frac{r}{\rho}\right)^{-2-\alpha} [A^{1/4} B^{1/2}](\rho)\right)^{4}
\end{aligned}$$

for any $\alpha \in \mathbb{R}$. Taking, say, $\alpha = \frac{1}{2}$, we have

$$F_1(r) \lesssim \left(\frac{r}{\rho}\right)^{2/3} C^{2/3}(\rho) + \left(\frac{r}{\rho}\right)^{-10} [AB^2](\rho). \tag{4-64}$$

Now for F_2 note that, using (4-58), we have $\left(\text{since } r \le \frac{1}{2}\rho\right)$

$$\begin{aligned}
\int_{B_r} |p_2|\,|u|\,dx &\lesssim \rho^{-2} \int_{B_\rho} (|u|\,|\nabla u| + |\nabla d|\,|\nabla^2 d|)\,dy \int_{B_r} |u|\,dx \\
&\lesssim \rho^{-2} \||u| + |\nabla d|\|_{2;B_\rho} \||\nabla u| + |\nabla^2 d|\|_{2;B_\rho} (r^3)^{1/2} \|u\|_{2;B_r} \\
&\lesssim \rho^{-2} r^{3/2} (\rho A(\rho)) \||\nabla u| + |\nabla^2 d|\|_{2;B_\rho}
\end{aligned}$$

so that integrating over $t \in I_r^*$ and using Hölder's inequality in time we have

$$F_2(r) \lesssim \frac{1}{r^2} \frac{r^{3/2}}{\rho^2} (\rho A(\rho)) (\rho B(\rho))^{1/2} (r^2)^{1/2} = \left(\frac{r}{\rho}\right)^{1/2} [AB^{1/2}](\rho). \tag{4-65}$$

For F_3, using (4-59) and Hölder's inequality, we see that

$$\frac{1}{r^2}\int_{B_r}|p_3||u|\,dx \le \frac{1}{r^2\rho^3}\left(\int_{B_\rho}|p|\,dy\right)\left(\int_{B_r}|u|\,dx\right)$$
$$\le \frac{1}{r^2\rho^3}\left(\int_{B_\rho}|p|^{3/2}\,dx\right)^{2/3}(\rho^3)^{1/3}\left(\int_{B_r}(|u|^{1/2})^4\,dx\right)^{1/4}\left(\int_{B_r}(|u|^{1/2})^6\,dx\right)^{1/6}(r^3)^{7/12}$$

which gives us (setting $\gamma := r/\rho$)

$$F_3(r) \lesssim \frac{1}{r^{1/4}\rho^2}(rA(r))^{1/4}\left(\iint_{Q_\rho^*}|p|^{3/2}\,dx\right)^{2/3}\left(\iint_{Q_r^*}|u|^3\,dx\right)^{1/6}(r^2)^{1/6}$$
$$\le \frac{1}{r^{1/4}\rho^2}(rA(r))^{1/4}(\rho^2 D(\rho))^{2/3}(r^2C(r))^{1/6}(r^2)^{1/6}$$
$$\le \left(\frac{r}{\rho}\right)^{2/3}(\gamma^{-1}A)^{1/4}(\rho)D^{2/3}(\rho)(\gamma^{-2}C)^{1/6}(\rho) = \left(\frac{r}{\rho}\right)^{1/12}A^{1/4}(\rho)D^{2/3}(\rho)C^{1/6}(\rho)$$

by (4-51). Hence Young's inequality implies

$$F_3(r) \lesssim \left(\frac{r}{\rho}\right)^{1/12}(A(\rho)+D^{4/3}(\rho)+C^{2/3}(\rho)). \tag{4-66}$$

Adding (4-64)–(4-66) and passing to the smallest powers of $\gamma = r/\rho (< 1)$ we see that

$$F_{z_0}(r) \lesssim \left(\frac{r}{\rho}\right)^{1/12}(A+D^{4/3}+C^{2/3})(\rho) + \left(\frac{r}{\rho}\right)^{-10}[A(B^{1/2}+B^2)](\rho)$$

which implies (4-49), and completes the proof of Claim 3. □

Proof of Claim 4. We will again omit the dependence on z_0 (except in F_{z_0}).

To estimate $A(\rho/2)$, we use the local energy inequality (1-13) with a nonnegative cut-off function $\phi \in C_0^\infty(Q_\rho^*)$ which is equal to 1 in $Q_{\rho/2}^*$, with

$$|\nabla\phi| \lesssim \rho^{-1} \quad \text{and} \quad |\phi_t|,\ |\nabla^2\phi| \lesssim \rho^{-2}.$$

We'll need to estimate terms which control those that appear on the right-hand side of the local energy inequality (1-13), which we'll call I–V (all of which depend on ρ) as follows.

$$I := \iint_{Q_\rho^*}(|u|^2+|\nabla d|^2)|\phi_t+\Delta\phi|\,dz \lesssim \rho^{-2}\||u|^2+|\nabla d|^2\|_{3/2;Q_\rho^*}(\rho^5)^{1/3}$$
$$\lesssim \rho^{-2}(\rho^2 C(\rho))^{2/3}(\rho^5)^{1/3} = \rho C^{2/3}(\rho). \tag{4-67}$$

Using the assumption (1-11) that $\nabla\cdot u = 0$ weakly and indicating by $\bar{g}^\rho$ the average of a function g in B_ρ, we have

$$II := \int_{I_\rho^*}\left|\int_{B_\rho}(|u|^2+|\nabla d|^2)u\cdot\nabla\phi\,dx\right|dt = \int_{I_\rho^*}\left|\int_{B_\rho}[(|u|^2-\overline{|u|^2}^\rho)+(|\nabla d|^2-\overline{|\nabla d|^2}^\rho)]u\cdot\nabla\phi\,dx\right|dt,$$

hence

$$II \lesssim \rho^{-1}(\rho^2E(\rho)) = \rho E(\rho). \tag{4-68}$$

Clearly we have

$$III := \iint_{Q_\rho^*} |pu \cdot \nabla\phi|\, dz \lesssim \rho^{-1}(\rho^2 F_{z_0}(\rho)) = \rho F_{z_0}(\rho). \tag{4-69}$$

Using the weak divergence-free condition $\nabla \cdot u = 0$ in (1-11) to write (see (1-2))

$$(u \cdot \nabla)d = \nabla^T \cdot (d \otimes u)$$

(at almost every x) and integrating by parts we have

$$\begin{aligned} IV &:= \int_{I_\rho^*} \left| \int_{B_\rho} u \otimes \nabla\phi : \nabla d \odot \nabla d\, dx \right| dt \\ &= \int_{I_\rho^*} \left| \int_{B_\rho} [(u \cdot \nabla)d] \cdot [(\nabla\phi \cdot \nabla)d]\, dx \right| dt \\ &= \int_{I_\rho^*} \left| \int_{B_\rho} [\nabla^T \cdot (d \otimes u)] \cdot [(\nabla\phi \cdot \nabla)d]\, dx \right| dt \\ &= \int_{I_\rho^*} \left| -\int_{B_\rho} d \otimes u : \nabla^T [(\nabla\phi \cdot \nabla)d]\, dx \right| dt, \end{aligned}$$

and clearly

$$|\nabla^T [(\nabla\phi \cdot \nabla)d]| \lesssim |\nabla^2\phi|\,|\nabla d| + |\nabla\phi|\,|\nabla^2 d|.$$

Therefore, for $q \in [2, 6]$ we have[23]

$$\begin{aligned} IV &\lesssim \iint_{Q_\rho^*} |d|\,|u|(\rho^{-2}|\nabla d| + \rho^{-1}|\nabla^2 d|)\, dz \\ &\le \| |d|\,|u| \|_{2;Q_\rho^*}(\rho^{-2}\|\nabla d\|_{2;Q_\rho^*} + \rho^{-1}\|\nabla^2 d\|_{2;Q_\rho^*}) \\ &\lesssim \| |d|\,|u| \|_{2;Q_\rho^*}(\rho^{-2} \cdot \rho^{5/6}\|\nabla d\|_{3;Q_\rho^*} + \rho^{-1}\|\nabla^2 d\|_{2;Q_\rho^*}) \\ &\le (\rho^3 G_2(\rho))^{1/2}(\rho^{-2} \cdot \rho^{5/6}(\rho^2 C(\rho))^{1/3} + \rho^{-1}(\rho B(\rho))^{1/2}) \\ &= \rho(G_2(\rho))^{1/2}(C^{1/3}(\rho) + B^{1/2}(\rho)) \\ &\le \rho(G_q^{2/q}(\rho) C^{1-2/q}(\rho))^{1/2}(C^{1/3}(\rho) + B^{1/2}(\rho)) \qquad \text{(by (3-22))}, \end{aligned}$$

so that

$$IV \lesssim \rho[G_q^{1/q}(C^{5/6-1/q} + C^{1/2-1/q}B^{1/2})](\rho). \tag{4-70}$$

Similarly, for $q \in [2, 6]$ we have

$$V := \iint_{Q_\rho^*} |d|^2|\nabla d|^2\phi\, dz \lesssim \rho^3 G_2(\rho) \overset{(3\text{-}22)}{\le} \rho^3 G_q^{2/q}(\rho) C^{1-2/q}(\rho). \tag{4-71}$$

[23] Note that it is only the appearance of $\nabla^2 d$ in the estimate of term IV which forces us to include u in the definition of G_{q,z_0}. Indeed, switching the roles of u (which appears in C_{z_0} along with ∇d) and ∇d (which appears in G_{q,z_0} even with u omitted), one could otherwise control the term IV in precisely the same way. If u is omitted in G_{q,z_0}, one could still obtain the same estimate of IV if one takes $q = 6$, but this would dramatically weaken the statement of Theorem 1. The remainder of the proof of Theorem 1 does not require (but is not harmed by) the inclusion of u in G_{q,z_0}.

Finally, using (4-67)–(4-71), the local energy inequality (1-13) (with constant $\bar{C}$) gives

$$
\begin{aligned}
(\bar{C})^{-1}\frac{\rho}{2}A\Big(\frac{\rho}{2}\Big) &\lesssim I+II+III+IV+V\\
&\lesssim \rho[C^{2/3}+E+F_{z_0}+G_q^{1/q}(C^{5/6-1/q}+C^{1/2-1/q}B^{1/2})+[\,\cdot\,]^2G_q^{2/q}C^{1-2/q}](\rho)\\
&\lesssim \rho[C^{2/3}+E+F_{z_0}+(1+[\,\cdot\,]^2)G_q^{4/(6-q)}+(G_q^{2/(6-q)}+C^{1/3})B^{1/2}](\rho)
\end{aligned}
$$

as long as $2\le q<6$, as in that case we have

$$
\begin{aligned}
G_q^{1/q}C^{5/6-1/q} &= (G_q^{4/(6-q)})^{(6-q)/(4q)}(C^{2/3})^{(5q-6)/(4q)}\\
&\le \Big(\frac{6-q}{4q}\Big)G_q^{4/(6-q)}+\Big(\frac{5q-6}{4q}\Big)C^{2/3}\le \frac{3}{4}G_q^{4/(6-q)}+\frac{5}{4}C^{2/3},\\
G_q^{1/q}C^{1/2-1/q} &= (G_q^{2/(6-q)})^{(6-q)/(2q)}(C^{1/3})^{(3q-6)/(2q)}\\
&\le \Big(\frac{6-q}{2q}\Big)G_q^{2/(6-q)}+\Big(\frac{3q-6}{2q}\Big)C^{1/3}\le \frac{3}{2}G_q^{2/(6-q)}+\frac{3}{2}C^{1/3},\\
G_q^{2/q}C^{1-2/q} &= (G_q^{4/(6-q)})^{(6-q)/(2q)}(C^{2/3})^{(3q-6)/(2q)}\\
&\le \Big(\frac{6-q}{2q}\Big)G_q^{4/(6-q)}+\Big(\frac{3q-6}{2q}\Big)C^{2/3}\le \frac{3}{2}G_q^{4/(6-q)}+\frac{3}{2}C^{2/3}.
\end{aligned}
$$

This implies (4-50) and proves Claim 4. □

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