ANALYSIS & PDE

Volume 16 No. 7 2023

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ANALYSIS AND PDE
Vol. 16 (2023), No. 7, pp. 1485–1496
DOI: 10.2140/apde.2023.16.1485

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SIMPLICES IN THIN SUBSETS OF EUCLIDEAN SPACES

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Let Δ be a nondegenerate simplex on k vertices. We prove that there exists a threshold $s_k < k$ such that any set $A \subseteq \mathbb{R}^k$ of Hausdorff dimension $\dim A \geq s_k$ necessarily contains a similar copy of the simplex Δ.

1. Introduction

A classical problem of geometric Ramsey theory is to show that sufficiently large sets contain a given geometric configuration. The underlying settings can be Euclidean space, the integer lattice or vector spaces over finite fields. By a geometric configuration, we mean the collection of finite point sets obtained from a given finite set $F \subseteq \mathbb{R}^k$ via translations, rotations and dilations.

If the size is measured in terms of the positivity of the Lebesgue density, then it is known that large sets in $\mathbb{R}^k$ contain a translated and rotated copy of all sufficiently large dilates of any nondegenerate simplex Δ with k vertices [Bourgain 1986]. However, on the scale of the Hausdorff dimension $s < k$ this question is not very well understood. The only affirmative result in this direction was obtained by Iosevich and Liu [2019].

In the other direction, a construction due to Keleti [2008] shows that there exists a set $A \subseteq \mathbb{R}$ of full Hausdorff dimension which does not contain any nontrivial 3-term arithmetic progression. In two dimensions an example due to Falconer [2013] and Maga [2010] shows that there exists a set $A \subseteq \mathbb{R}^2$ of Hausdorff dimension 2 which does not contain the vertices of an equilateral triangle, or more generally a nontrivial similar copy of a given nondegenerate triangle. It seems plausible that examples of such sets exist in all dimensions, but this is not currently known. See [Fraser and Pramanik 2018] for related results.

The purpose of this paper is to show that measurable sets $A \subseteq \mathbb{R}^k$ of sufficiently large Hausdorff dimension $s < k$ contain a similar copy of any given nondegenerate k-simplex with bounded eccentricity. Our arguments make use of and have some similarity to those of Lyall and Magyar [2020]. We also extend our results to bounded degree distance graphs. For the special cases of a path (or chain) and, more generally, a tree, similar but somewhat stronger results were obtained in [Bennett et al. 2016] and [Iosevich and Taylor 2019].

2. Main results

Let $V = \{v_1, \ldots, v_k\} \subset \mathbb{R}^k$ be a nondegenerate k-simplex, a set of k vertices which are in *general position* spanning a $(k-1)$-dimensional affine subspace. For $1 \leq j \leq k$, let $r_j(V)$ be the distance of the vertex v_j

The research of Iosevich was partially supported by the National Science Foundation grant no. HDR TRIPODS - 1934962. The research of Magyar was partially supported by the National Science Foundation grant no. NSF-DMS 1600840.
MSC2020: 28A75, 42B15.
Keywords: simplexes, Hausdorff dimension, graphs.

to the affine subspace spanned by the remaining vertices v_i, $i \neq j$, and define $r(V) := \min_{1\leq j\leq k} r_j(V)$. Let $d(V)$ denote the diameter of the simplex, which is also the maximum distance between two vertices. Then the quantity $\delta(V) := r(V)/d(V)$, which is positive if and only if V is nondegenerate, measures how close the simplex V is to being degenerate.

We say that a simplex V' is *similar* to V, if $V' = x + \lambda \cdot U(V)$ for some $x \in \mathbb{R}^k$, $\lambda > 0$ and $U \in \mathrm{SO}(k)$; that is if V' is obtained from V by a translation, dilation and rotation.

Theorem 1. *Let $k \in \mathbb{N}$ and $\delta > 0$. There exists $s_0 = s_0(k, \delta) < k$ such that if E is a compact subset of $\mathbb{R}^k$ of Hausdorff dimension* $\dim E \geq s_0$, *then E contains the vertices of a simplex V' similar to V, for any nondegenerate k-simplex V with $\delta(V) \geq \delta$.*

Remarks. (1) Note that the dimension condition is sharp for $k = 2$, as a construction due to Maga [2010] shows the existence of a set $E \subseteq \mathbb{R}^2$ with $\dim(E) = 2$ that does not contain any equilateral triangle or more generally a similar copy of any given triangle.

While we do not currently have an example showing that the dimension condition is sharp when $k > 2$, we have some indications that this should be the case. In the finite field setting, one can show that $\mathbb{F}_q^d$ (the d-dimensional vector space over the field with q elements) contains a d-dimensional equilateral simplex if and only if $(d+1)/2^d$ is a square in $\mathbb{F}_q$; see the appendix in [Bennett et al. 2014]. This allows one to construct an $\mathbb{F}_q^d$ that does not contain a d-dimensional equilateral simplex under a suitable arithmetic assumption on $\mathbb{F}_q$. While such an assumption is not meaningful in $\mathbb{R}^d$, the Fourier analytic methods used in this paper would likely to extend to the finite field setting. At the very least, this says that if the dimensional assumption in Theorem 1 is not sharp, a very different approach would be required to establish a positive result.

(2) It is also interesting to note that the proof of Theorem 1 above proves much more than just the existence of vertices of V' similar to V inside E. The proof proceeds by constructing a natural measure on the set of simplexes and proving an upper and a lower bound on this measure. This argument shows that an infinite "statistically" correct "amount" of simplexes V' exist that satisfy the conclusion of the theorem, shedding considerable light on the structure of sets of positive upper Lebesgue density.

(3) Theorem 1 establishes a nontrivial exponent $s_0 < k$, but the proof yields s_0 very close to k and not explicitly computable. The analogous results in the finite field setting (see e.g., [Hart and Iosevich 2008], [Iosevich and Parshall 2019]) suggest that it may be possible to obtain explicit exponents, but this would require a fundamentally different approach to certain lower bounds obtained in the proof of Theorem 1.

A distance graph is a connected finite graph embedded in Euclidean space, with a set of vertices $V = \{v_0, v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$ and a set of edges $E \subseteq \{(i, j) : 0 \leq i < j \leq n\}$. We say that a graph $\Gamma = (V, E)$ has degree at most k if $|V_j| \leq k$ for all $1 \leq j \leq n$, where $V_j = \{v_i : (i, j) \in E\}$. The graph Γ is called *proper* if the sets $V_j \cup \{v_j\}$ for all j are in general position, in the sense that $V_j \cup \{v_j\}$ is not contained in a subspace of dimension smaller than $|V_j| - 1$. Let $r(\Gamma)$ be the minimum of the distances from the vertices v_j to the corresponding affine subspace spanned by the sets V_j, and note that $r(\Gamma) > 0$ if Γ is proper. Let $d(\Gamma)$ denote the length of the longest edge of Γ, and let $\delta(\Gamma) := r(\Gamma)/d(\Gamma)$.

We say that a distance graph $\Gamma' = (V', E)$ is *isometric* to Γ and write $\Gamma' \simeq \Gamma$, if there is a one-to-one and onto mapping $\phi : V \to V'$ so that $|\phi(v_i) - \phi(v_j)| = |v_i - v_j|$ for all $(i, j) \in E$. One may picture Γ' obtained from Γ by a translation followed by rotating the edges around the vertices, if possible. By $\lambda \cdot \Gamma$ we mean the dilate of the distance graph Γ by a factor $\lambda > 0$, and we say that Γ' is *similar* to Γ if Γ' is isometric to $\lambda \cdot \Gamma$.

Theorem 2. *Let $\delta > 0$, $n \geq 1$, $1 \leq k < d$, and let E be a compact subset of $\mathbb{R}^k$ of Hausdorff dimension $s < d$. There exists $s_0 = s_0(n, d, \delta) < d$ such that if $s \geq s_0$, then E contains a distance graph Γ' similar to Γ, for any proper distance graph $\Gamma = (V, E)$ of degree at most k, with $V \subseteq \mathbb{R}^d$, $|V| = n$ and $\delta(\Gamma) \geq \delta$.*

Note that Theorem 2 implies Theorem 1, as a nondegenerate simplex is a proper distance graph of degree $k - 1$.

3. Proof of Theorem 1

Let $E \subseteq B(0, 1)$ be a compact subset of the unit ball $B(0, 1)$ in $\mathbb{R}^k$ of Hausdorff dimension $s < k$. It is well known that there is a probability measure μ supported on E such that $\mu(B(x, r)) \leq C_\mu r^s$ for all balls $B(x, r)$. The following observation shows that we may take $C_\mu = 4$ for our purposes.[1]

Lemma 1. *There exists a set $E' \subseteq B(0, 1)$ of the form $E' = \rho^{-1}(F - u)$ for some $\rho > 0$, $u \in \mathbb{R}^k$ and $F \subseteq E$, and a probability measure μ' supported on E' which satisfies*

$$\mu'(B(x, r)) \leq 4r^s, \quad \textit{for all } x \in \mathbb{R}^k,\ r > 0. \tag{3-1}$$

Proof. Let $K := \inf(S)$, where

$$S := \{C \in \mathbb{R} : \mu(B(x, r)) \leq Cr^s,\ \forall B(x, r)\}.$$

By Frostman's lemma [Mattila 1995], we have that $S \neq \varnothing$ and $K > 0$, moreover,

$$\mu(B(x, r)) \leq 2Kr^s,$$

for all balls $B(x, r)$. There exists a ball $Q = B(v, \rho)$ of radius ρ such that $\mu(Q) \geq \frac{1}{2} K\rho^s$. We translate E so Q is centered at the origin, set $F = E \cap Q$ and denote by μ_F the induced probability measure on F:

$$\mu_F(A) = \frac{\mu(A \cap F)}{\mu(F)}.$$

Note that for all balls $B = B(x, r)$,

$$\mu_F(B) \leq \frac{2Kr^s}{(1/2)K\rho^s} = 4\left(\frac{r}{\rho}\right)^s.$$

Finally, we define the probability measure μ' as $\mu'(A) := \mu_F(\rho A)$. It is supported on $E' = \rho^{-1} F \subseteq B(0, 1)$ and satisfies

$$\mu'(B(x, r)) = \mu_F(B(\rho x, \rho r)) \leq 4r^s. \qquad \square$$

[1] We would like to thank Giorgis Petridis for bringing this observation to our attention.

Clearly E contains a similar copy of V if the same holds for E', thus one can pass from E to E' in proving our main results, assuming that (3-1) holds. Given $\varepsilon > 0$, let $\psi_\varepsilon(x) = \varepsilon^{-k}\psi(x/\varepsilon) \ge 0$, where $\psi \ge 0$ is a Schwarz function whose Fourier transform, $\hat{\psi}$, is a compactly supported smooth function satisfying $\hat{\psi}(0) = 1$ and $0 \le \hat{\psi} \le 1$.

We define $\mu_\varepsilon := \mu * \psi_\varepsilon$. Note that μ_ε is a continuous function satisfying $\|\mu_\varepsilon\|_\infty \le C\varepsilon^{s-k}$ with an absolute constant $C = C_\psi > 0$, by Lemma 1.

Let $V = \{v_0 = 0, \dots, v_{k-1}\}$ be a given nondegenerate simplex and note that in proving Theorem 1 we may assume that $d(V) = 1$, and hence $\delta(V) = r(V)$. A simplex $V' = \{x_0 = 0, x_1, \dots, x_{k-1}\}$ is isometric to V if for every $1 \le j \le k$ one has that $x_j \in S_{x_1,\dots,x_{j-1}}$, where

$$S_{x_1,\dots,x_{j-1}} = \{y \in \mathbb{R}^k : |y - x_i| = |v_j - v_i|,\ 0 \le i < j\}$$

is a sphere of dimension $k - j$ and of radius $r_j = r_j(V) \ge r(V) > 0$. Let $\sigma_{x_1,\dots,x_{j-1}}$ denote its normalized surface area measure.

Given $0 < \lambda$ and $\varepsilon \le 1$, define the multilinear expression

$$T_{\lambda V}(\mu_\varepsilon) := \int \mu_\varepsilon(x)\mu_\varepsilon(x - \lambda x_1)\cdots\mu_\varepsilon(x - \lambda x_{k-1})\, d\sigma(x_1)\, d\sigma_{x_1}(x_2)\cdots d\sigma_{x_1,\dots,x_{k-2}}(x_{k-1})\, dx, \quad (3\text{-}2)$$

which may be viewed as a weighted count of the isometric copies of $\lambda\Delta$.

3.1. ***Upper bounds.*** A crucial part of our approach is to show that the averages $T_{\lambda V}(\mu_\varepsilon)$ have a limit as $\varepsilon \to 0$, for which one needs the following upper bound.

Lemma 2. *There exists a constant $C_k > 0$, depending only on k, such that*

$$|T_{\lambda V}(\mu_{2\varepsilon}) - T_{\lambda V}(\mu_\varepsilon)| \le C_k r(V)^{-1/2}\lambda^{-1/2}\varepsilon^{(k-1/2)(s-k)+1/4}. \quad (3\text{-}3)$$

As an immediate corollary we have the following:

Lemma 3. *Let $k - \frac{1}{4k} \le s < k$. There exists*

$$T_{\lambda V}(\mu) := \lim_{\varepsilon\to 0} T_{\lambda V}(\mu_\varepsilon), \quad (3\text{-}4)$$

and moreover,

$$|T_{\lambda V}(\mu) - T_{\lambda V}(\mu_\varepsilon)| \le C_k r(V)^{-1/2}\lambda^{-1/2}\varepsilon^{(k-1/2)(s-k)+1/4}. \quad (3\text{-}5)$$

Indeed, the left side of (3-5) can be written as a telescopic sum:

$$\sum_{j\ge 0} T_{\lambda V}(\mu_{2\varepsilon_j}) - T_{\lambda V}(\mu_{\varepsilon_j}), \quad \text{with } \varepsilon_j = 2^{-j}\varepsilon.$$

Proof of Lemma 2. Write $\Delta\mu_\varepsilon := \mu_{2\varepsilon} - \mu_\varepsilon$. Then

$$\prod_{j=1}^{k-1}\mu_{2\varepsilon}(x - \lambda x_j) - \prod_{j=1}^{k-1}\mu_\varepsilon(x - \lambda x_j) = \sum_{j=1}^{k}\Delta_j(\mu_\varepsilon),$$

where

$$\Delta_j(\mu_\varepsilon) = \prod_{i\ne j}\mu_{\varepsilon_{ij}}(x - \lambda x_i)\Delta\mu_\varepsilon(x - \lambda x_j), \quad (3\text{-}6)$$

and where $\varepsilon_{ij} = 2\varepsilon$ for $i < j$ and $\varepsilon_{ij} = \varepsilon$ for $i > j$. Since the arguments below are the same for all $1 \le j \le k-1$, assume $j = k-1$ for simplicity of notations. Writing $f *_\lambda g(x) := \int f(x - \lambda y) g(y)\, dy$, and using $\|\mu_\varepsilon\|_\infty \le C\varepsilon^{s-k}$, we have for $\Delta T(\mu_\varepsilon) := T_{\lambda V}(\mu_\varepsilon) - T_{\lambda V}(\mu_{2\varepsilon})$ that

$$|\Delta T(\mu_\varepsilon)| \lesssim \varepsilon^{(k-2)(s-d)} \int \left| \int \mu_\varepsilon(x) \Delta\mu_\varepsilon *_\lambda \sigma_{x_1,\dots,x_{k-2}}(x)\, dx \right| d\omega(x_1, \dots, x_{k-2}), \tag{3-7}$$

where $d\omega(x_1, \dots, x_{k-2}) = d\sigma(x_1) \cdots d\sigma_{x_1,\dots,x_{k-3}}(x_{k-2})$ for $k > 3$, and where for $k = 3$ we have that $d\omega(x_1) = d\sigma(x_1)$, which is the normalized surface area measure on the sphere $S = \{y : |y| = |v_1|\}$.

The inner integral is of the form

$$|\langle \mu_\varepsilon, \Delta\mu_\varepsilon *_\lambda \sigma_{x_1,\dots,x_{k-2}} \rangle| \lesssim \varepsilon^{s-d} \|\Delta\mu_\varepsilon *_\lambda \sigma_{x_1,\dots,x_{k-2}}\|_2.$$

Thus by Cauchy–Schwarz and Plancherel's identity,

$$|\Delta_{k-1} T(\mu_\varepsilon)|^2 \lesssim \varepsilon^{2(k-1)(s-d)} \int |\widehat{\Delta\mu_\varepsilon}(\xi)|^2 I_\lambda(\xi)\, d\xi,$$

where

$$I_\lambda(\xi) = \int |\hat{\sigma}_{x_1,\dots,x_{k-2}}(\lambda\xi)|^2\, d\omega(x_1, \dots, x_{k-2}).$$

Since $S_{x_1,\dots,x_{k-2}}$ is a one-dimensional circle of radius $r_{k-1} \ge r(V) > 0$ contained in an affine subspace orthogonal to $M_{x_1,\dots,x_{k-2}} = \operatorname{span}\{x_1, \dots, x_{k-2}\}$, we have that

$$|\hat{\sigma}_{x_1,\dots,x_{k-2}}(\lambda\xi)|^2 \lesssim (1 + r(V)\lambda \operatorname{dist}(\xi, M_{x_1,\dots,x_{k-2}}))^{-1}.$$

Since the measure $\omega(x_1, \dots, x_{k-2})$ is invariant with respect to the change of variables $(x_1, \dots, x_{k-2}) \to (Ux_1, \dots, Ux_{k-2})$ for any rotation $U \in \mathrm{SO}(k)$, one estimates

$$\begin{aligned} I_\lambda(\xi) &\lesssim \iint (1 + r(V)\lambda \operatorname{dist}(\xi, M_{Ux_1,\dots,Ux_{k-2}}))^{-1}\, d\omega(x_1, \dots, x_{k-2})\, dU \\ &= \iint (1 + r(V)\lambda \operatorname{dist}(U\xi, M_{x_1,\dots,x_{k-2}}))^{-1}\, d\omega(x_1, \dots, x_{k-2})\, dU \\ &= \iint (1 + r(V)\lambda|\xi| \operatorname{dist}(\eta, M_{x_1,\dots,x_{k-2}}))^{-1}\, d\omega(x_1, \dots, x_{k-2})\, d\sigma_{k-2}(\eta) \lesssim (1 + r(V)\lambda|\xi|)^{-1}, \end{aligned}$$

where we have written $\eta := |\xi|^{-1} U\xi$ and σ_{k-1} denotes the surface area measure on the unit sphere $S^{k-1} \subseteq \mathbb{R}^k$.

Note that $\widehat{\Delta\mu_\varepsilon}(\xi) = \hat{\mu}(\xi)(\hat{\psi}(2\varepsilon\xi) - \hat{\psi}(\varepsilon\xi))$, which is supported on $|\xi| \lesssim \varepsilon^{-1}$ and is essentially supported on $|\xi| \approx \varepsilon^{-1}$. Indeed, writing

$$J := \int |\widehat{\Delta\mu_\varepsilon}(\xi)|^2 I_\lambda(\xi)\, d\xi = \int_{|\xi| \le \varepsilon^{-1/2}} |\widehat{\Delta\mu_\varepsilon}(\xi)|^2 I_\lambda(\xi)\, d\xi + \int_{\varepsilon^{-1/2} \le |\xi| \lesssim \varepsilon^{-1}} |\widehat{\Delta\mu_\varepsilon}(\xi)|^2 I_\lambda(\xi)\, d\xi =: J_1 + J_2$$

and using $|\hat{\psi}(2\varepsilon\xi) - \hat{\psi}(\varepsilon\xi)| \lesssim \varepsilon^{1/2}$ for $|\xi| \le \varepsilon^{-1/2}$, we estimate

$$J_1 \lesssim \varepsilon^1/2 \int |\hat{\mu}(\xi)|^2 (\hat{\psi}(2\varepsilon\xi) + \hat{\psi}(\varepsilon\xi))\, d\xi \lesssim \varepsilon^{1/2+s-k},$$

as

$$\int |\hat{\mu}(\xi)|^2 \hat{\psi}(\varepsilon\xi)\, d\xi = \int \mu_\varepsilon(x)\, d\mu(x) \lesssim \varepsilon^{s-k}.$$

On the other hand, as $I_\lambda(\xi) \lesssim \varepsilon^{1/2} r(V)^{-1}\lambda^{-1}$ for $|\xi| \geq \varepsilon^{-1/2}$, we have

$$J_2 \lesssim \varepsilon^{1/2} r(V)^{-1}\lambda^{-1} \int |\hat{\mu}(\xi)|^2 \hat{\phi}(\varepsilon\xi)\, d\xi \lesssim r(V)^{-1}\lambda^{-1}\varepsilon^{1/2+s-k},$$

where we have written $\hat{\phi}(\xi) = (\hat{\psi}(2\xi) - \hat{\psi}(\xi))^2$. Plugging these estimates into (3-7), we obtain

$$|\Delta T(\mu_\varepsilon)|^2 \lesssim r(V)^{-1}\lambda^{-1}\varepsilon^{1/2+(2k-1)(s-d)},$$

and (3-5) follows. □

The support of μ_ε is not compact, however, as it is a rapidly decreasing function, it can be made to be supported in a small neighborhood of the support of μ without changing our main estimates. Let $\phi_\varepsilon(x) := \phi(c\varepsilon^{-1/2}x)$ with some small absolute constant $c > 0$, where $0 \leq \phi(x) \leq 1$ is a smooth cut-off, which equals one for $|x| \leq \frac{1}{2}$ and is zero for $|x| \geq 2$. Define $\tilde{\psi}_\varepsilon = \psi_\varepsilon\phi_\varepsilon$ and $\tilde{\mu}_\varepsilon = \mu * \tilde{\psi}_\varepsilon$. It is easy to see that $\tilde{\mu}_\varepsilon \leq \mu_\varepsilon$ and $\int \tilde{\mu}_\varepsilon \geq \frac{1}{2}$, if $c > 0$ is chosen sufficiently small. Using the trivial upper bound, for $k - 1/(4k) \leq s < k$ we have

$$|T_{\lambda V}(\mu_\varepsilon) - T_{\lambda V}(\tilde{\mu}_\varepsilon)| \leq C_k \|\mu_\varepsilon\|_\infty^{k-1} \|\mu_\varepsilon - \tilde{\mu}_\varepsilon\|_\infty \leq C_k \varepsilon^{1/2},$$

and it follows that estimate (3-5) remains true with μ_ε replaced with $\tilde{\mu}_\varepsilon$.

3.2. *Lower bounds.* Let $f_\varepsilon := c\varepsilon^{k-s}\tilde{\mu}_\varepsilon$, where $c = c_\psi > 0$ is a constant such that $0 \leq f_\varepsilon \leq 1$ and $\int f_\varepsilon\, dx = c'\varepsilon^{k-s}$. Let $\alpha := c'\varepsilon^{k-s}$ and note that the set $A_\varepsilon := \left\{x : f_\varepsilon(x) \geq \frac{1}{2}\alpha\right\}$ has measure $|A_\varepsilon| \geq \frac{1}{2}\alpha$. If one defines the averages

$$T_{\lambda V}(A_\varepsilon) = \int \mathbf{1}_{A_\varepsilon}(x)\mathbf{1}_{A_\varepsilon}(x - \lambda x_1) \cdots \mathbf{1}_{A_\varepsilon}(x - \lambda x_{k-1})\, d\sigma(x_1) \cdots d\sigma_{x_1,\ldots,x_{k-2}}(x_{k-1})\, dx,$$

then clearly

$$T_{\lambda V}(\tilde{\mu}_\varepsilon) \geq c\alpha^k T_{\lambda V}(A_\varepsilon).$$

The averages $T_{\lambda V}(A_\varepsilon)$ represent the density of isometric copies of the simplex $\lambda\Delta$ in a set A_ε of measure $|A_\varepsilon| \geq \frac{\alpha}{2} > 0$, which was studied in [Lyall and Magyar 2020] in the more general context of k-degenerate distance graphs. We recall one of the main results of the aforementioned paper; see Theorem 2 (ii) together with Estimate (18):

Theorem 3 [Lyall and Magyar 2020]. *Let $A \subseteq [0, 1]^k$ and $|A| \geq \alpha > 0$. Then there exists an interval I of length $|I| \geq \exp(-C\alpha^{-C_k})$, such that for all $\lambda \in I$, one has*

$$|T_{\lambda V}(A))| \geq c\alpha^k.$$

Thus for all $\lambda \in I$,

$$T_{\lambda V}(\tilde{\mu}_\varepsilon) \geq c > 0 \tag{3-8}$$

for a constant $c = c(k, \psi, r(V)) > 0$. Now, let

$$T_V(\tilde{\mu}_\varepsilon) := \int_0^1 \lambda^{1/2} T_{\lambda V}(\tilde{\mu}_\varepsilon)\, d\lambda.$$

For $k - \frac{1}{4k} \le s < k$, by (3-5) we have that

$$|T_{\lambda V}(\mu) - T_{\lambda V}(\tilde{\mu}_\varepsilon)| \le C_k r(V)^{-1/2} \lambda^{-1/2} \varepsilon^{1/8},$$

it follows that

$$\int_0^1 \lambda^{1/2} |T_{\lambda V}(\mu) - T_{\lambda V}(\tilde{\mu}_\varepsilon)|\, d\lambda \le C_k r(V)^{-1/2} \varepsilon^{1/8}, \tag{3-9}$$

and in particular $\int_0^1 \lambda^{1/2} T_{\lambda V}(\mu)\, d\lambda < \infty$. On the other hand, by (3-8), one has that

$$\int_0^1 \lambda^{1/2} T_{\lambda V}(\tilde{\mu}_\varepsilon)\, d\lambda \ge \exp(-\varepsilon^{-C_k(k-s)}). \tag{3-10}$$

Assume that $r(V) \ge \delta$, fix a small $\varepsilon = \varepsilon_{k,\delta} > 0$ and then choose $s = s(\varepsilon, \delta) < k$ such that

$$C_k \delta^{-1/2} \varepsilon^{1/8} < \tfrac{1}{2} \exp(-\varepsilon^{-C_k(k-s)}),$$

which ensures that

$$\int_0^1 \lambda^{1/2} T_{\lambda V}(\mu)\, d\lambda > 0.$$

Thus there exists $\lambda > 0$ such that $T_{\lambda V}(\mu) > 0$. Fix such a λ, and assume indirectly that $E^k = E \times \cdots \times E$ does not contain any simplex isometric to λV, i.e., any point of the compact configuration space $S_{\lambda V} \subseteq \mathbb{R}^{k^2}$ of such simplexes. By compactness, this implies that there is some $\eta > 0$ such that the η-neighborhood of E^k also does not contain any simplex isometric to λV. Since the support of $\tilde{\mu}_\varepsilon$ is contained in the $C_k \varepsilon^{1/2}$-neighborhood of E, as $E = \operatorname{supp} \mu$, it follows that $T_{\lambda V}(\tilde{\mu}_\varepsilon) = 0$ for all $\varepsilon < c_k \eta^2$ and hence $T_{\lambda V}(\mu) = 0$, contradicting our choice of λ. This proves Theorem 1.

4. The configuration space of isometric distance graphs

Let $\Gamma_0 = (V_0, E)$ be a fixed proper distance graph, with vertex set $V_0 = \{v_0 = 0, v_1, \dots, v_n\} \subseteq \mathbb{R}^d$ of degree $k < d$. Let $t_{ij} = |v_i - v_j|^2$ for $(i, j) \in E$. A distance graph $\Gamma = (V, E)$ with $V = \{x_0 = 0, x_1, \dots, x_n\}$ is isometric to Γ_0 if and only if $\boldsymbol{x} = (x_1, \dots, x_n) \in S_{\Gamma_0}$, where

$$S_{\Gamma_0} = \{(x_1, \dots, x_n) \in \mathbb{R}^{dn} : |x_i - x_j|^2 = t_{ij},\ \forall 0 \le i < j \le n,\ (i, j) \in E\}.$$

We call the algebraic set S_{Γ_0} the *configuration space* of isometric copies of Γ_0. Note that S_{Γ_0} is the zero set of the family $\mathcal{F} = \{f_{ij} : (i, j) \in E\}$ with $f_{ij}(\boldsymbol{x}) = |x_i - x_j|^2 - t_{ij}$, thus it is a special case of the general situation described in Section 5.

If $\Gamma \simeq \Gamma_0$ with vertex set $V = \{x_0 = 0, x_1, \dots, x_n\}$ is proper, then $\boldsymbol{x} = (x_1, \dots, x_n)$ is a nonsingular point of S_{Γ_0}. Indeed, for a fixed $1 \le j \le n$, let Γ_j be the distance graph obtained from Γ by removing the vertex x_j together with all edges emanating from it. By induction we may assume that $\boldsymbol{x}' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ is a nonsingular point, i.e., the gradient vectors $\nabla_{\boldsymbol{x}'} f_{ik}(\boldsymbol{x})$, $(i, k) \in E$, $i \ne j$, $k \ne j$, are linearly independent. Since Γ is proper, the gradient vectors $\nabla_{x_j} f_{ij}(\boldsymbol{x}) = 2(x_i - x_j)$, $(i, j) \in E$, are also linearly independent, hence $\boldsymbol{x}$ is a nonsingular point. In fact we have shown that the partition of coordinates $\boldsymbol{x} = (y, z)$ with $y = x_j$ and $z = \boldsymbol{x}'$ is admissible and hence (6-4) holds.

Let $r_0 = r(\Gamma_0) > 0$. It is clear that if $\Gamma \simeq \Gamma_0$ and $|x_j - v_j| \leq \eta_0$ for all $1 \leq j \leq n$, for a sufficiently small $\eta_0 = \eta(r_0) > 0$, then Γ is proper and $r(\Gamma) \geq \frac{1}{2} r_0$. For a given $1 \leq j \leq n$, let $X_j := \{x_i \in V : (i, j) \in E\}$ and define

$$S_{X_j} := \{x \in \mathbb{R}^d : |x - x_i|^2 = t_{ij},\ \forall x_i \in X_j\}.$$

As explained in Section 6, S_{X_j} is a sphere of dimension $d - |X_j| \geq 1$ with radius $r(X_j) \geq \frac{1}{2} r_0$. Let σ_{X_j} denote the surface area measure on S_{X_j} and write $\nu_{X_j} := \phi_j \sigma_{X_j}$, where ϕ_j is a smooth cut-off function supported in an η-neighborhood of v_j with $\phi_j(v_j) = 1$.

Write $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\phi(\boldsymbol{x}) := \prod_{j=1}^n \phi_j(x_j)$. Then by (6-4) and (6-5) one has

$$\int g(\boldsymbol{x})\phi(\boldsymbol{x})\, d\omega_{\mathcal{F}}(\boldsymbol{x}) = c_j(\Gamma_0) \iint g(\boldsymbol{x})\phi(\boldsymbol{x}')\, d\nu_{X_j}(x_j)\, d\omega_{\mathcal{F}_j}(\boldsymbol{x}'), \tag{4-1}$$

where $\boldsymbol{x}' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ and $\mathcal{F}_j = \{f_{il} : (i, l) \in E,\ l \neq j\}$. The constant $c_j(\Gamma_0) > 0$ is the reciprocal of the volume of the parallelotope with sides $x_j - x_i$, $(i, j) \in E$, which is easily shown to be at least $c_k r_0^k$, as the distance of each vertex to the opposite face is at least $\frac{1}{2} r_0$ on the support of ϕ.

5. Proof of Theorem 2

Let $d > k$ and again, without loss of generality, assume that $d(\Gamma) = 1$ and hence $\delta(\Gamma) = r(\Gamma)$. Given $\lambda, \varepsilon > 0$, define the multilinear expression

$$T_{\lambda\Gamma_0}(\mu_\varepsilon) := \int \cdots \int \mu_\varepsilon(x)\mu_\varepsilon(x - \lambda x_1) \cdots \mu_\varepsilon(x - \lambda x_n)\phi(x_1, \ldots, x_n)\, d\omega_{\mathcal{F}}(x_1, \ldots, x_n)\, dx. \tag{5-1}$$

Given a proper distance graph $\Gamma_0 = (V, E)$ on $|V| = n$ vertices of degree $k < n$, one has the following upper bound.

Lemma 4. *There exists a constant $C = C_{n,d,k}(r_0) > 0$ such that*

$$|T_{\lambda\Gamma_0}(\mu_{2\varepsilon}) - T_{\lambda\Gamma_0}(\mu_\varepsilon)| \leq C\lambda^{-1/2}\varepsilon^{(n+1/2)(s-d)+1/4}. \tag{5-2}$$

This implies again that in dimensions $d - 1/(4n+2) \leq s \leq d$, the limit $T_{\lambda\Gamma_0}(\mu) := \lim_{\varepsilon\to 0} T_{\lambda\Gamma_0}(\mu_\varepsilon)$ exists. Also, the lower bound (3-8) holds for distance graphs of degree k, as was shown for a large class of graphs, the so-called k-degenerate distance graphs; see [Lyall and Magyar 2020]. Thus one may argue exactly as in Section 3 to prove that there exists a $\lambda > 0$ for which

$$T_{\lambda\Gamma_0}(\mu) > 0, \tag{5-3}$$

and Theorem 2 follows from the compactness of the configuration space $S_{\lambda\Gamma_0} \subseteq \mathbb{R}^{dn}$. It remains to prove Lemma 4.

Proof of Lemma 4. Write $\Delta T(\mu_\varepsilon) := T_{\lambda\Gamma_0}(\mu_\varepsilon) - T_{\lambda\Gamma_0}(\mu_{2\varepsilon})$. Then we have $\Delta T(\mu_\varepsilon) = \sum_{j=1} \Delta_j T(\mu_\varepsilon)$, where $\Delta_j T(\mu_\varepsilon)$ is given by (5-1) with $\mu_\varepsilon(x - \lambda x_j)$ replaced by $\Delta\mu_\varepsilon(x - \lambda x_j)$ given in (3-6), and $\mu_\varepsilon(x - \lambda x_i)$ by $\mu_{2\varepsilon}(x - \lambda x_j)$ for $i > j$. Then by (4-1) we have the analogue of estimate (3-7):

$$|\Delta T(\mu_\varepsilon)| \lesssim \varepsilon^{(n-1)(s-d)} \int \left| \iint \mu_\varepsilon(x)\Delta\mu_\varepsilon *_\lambda \nu_{X_j}(x)\, dx \right| \phi(\boldsymbol{x}')\, d\omega_{\mathcal{F}_j}(\boldsymbol{x}'), \tag{5-4}$$

where $\phi(\boldsymbol{x}') = \prod_{i \neq j} \phi(x_j)$. Thus by Cauchy–Schwarz and Plancherel's identity,

$$|\Delta_j T^{\varepsilon}(\mu)|^2 \lesssim \varepsilon^{2n(s-d)} \int |\widehat{\Delta_\varepsilon \mu}(\xi)|^2 I_\lambda^j(\xi)\, d\xi,$$

where

$$I_\lambda^j(\xi) = \int |\hat{\nu}_{X_j}(\lambda\xi)|^2 \phi(\boldsymbol{x}')\, d\omega_{\mathcal{F}_j}(\boldsymbol{x}').$$

Recall that on the support of $\phi(\boldsymbol{x}')$ we have that S_{X_j} is a sphere of dimension at least 1 and of radius $r \geq \frac{1}{2} r_0 > 0$, contained in an affine subspace orthogonal to span X_j. Thus,

$$|\hat{\nu}_{X_j}(\lambda\xi)|^2 \lesssim (1 + r_0 \lambda \operatorname{dist}(\xi, \operatorname{span} X_j))^{-1}.$$

Let $U : \mathbb{R}^d \to \mathbb{R}^d$ be a rotation, and for $\boldsymbol{x}' = (x_i)_{i \neq j}$ write $U\boldsymbol{x}' = (Ux_i)_{i \neq j}$. As explained in Section 6, the measure $\omega_{\mathcal{F}_j}$ is invariant under the transformation $\boldsymbol{x}' \to U\boldsymbol{x}'$, hence

$$\begin{aligned} I_\lambda(\xi) &\lesssim \iint (1 + r_0 \lambda \operatorname{dist}(\xi, \operatorname{span} UX_j))^{-1}\, d\omega_{\mathcal{F}_j}(\boldsymbol{x}')\, dU \\ &= \iint (1 + r_0 \lambda |\xi| \operatorname{dist}(\eta, \operatorname{span} X_j))^{-1}\, d\sigma_{d-1}(\eta)\, d\omega\mathcal{F}_j(\boldsymbol{x}') \lesssim (1 + r_0 \lambda |\xi|)^{-1}, \end{aligned}$$

where we have written again $\eta := |\xi|^{-1} U\xi \in S^{d-1}$.

Then we argue as in Lemma 2, noting that as $\widehat{\Delta\mu_\varepsilon}(\xi)$ is essentially supported on $|\xi| \approx \varepsilon^{-1}$, we have that

$$|\Delta T(\mu_\varepsilon)|^2 \lesssim r_0^{-1} \lambda^{-1} \varepsilon^{2n(s-d)+1/2} \int |\hat{\mu}(\xi)|^2 \hat{\phi}(\varepsilon\xi)\, d\xi \lesssim r_0^{-1} \lambda^{-1} \varepsilon^{(2n+1)(s-d)+1/2},$$

with $\tilde{\mu}_\varepsilon = \mu_\varepsilon$ or $\tilde{\mu}_\varepsilon = \mu * \phi_\varepsilon$. This proves Lemma 4. $\square$

6. Measures on real algebraic sets

Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a family of polynomials $f_i : \mathbb{R}^d \to \mathbb{R}$. We will describe certain measures supported on the algebraic set

$$S_{\mathcal{F}} := \{x \in \mathbb{R}^d : f_1(x) = \cdots = f_n(x) = 0\}. \tag{6-1}$$

A point $x \in S_{\mathcal{F}}$ is called *nonsingular* if the gradient vectors

$$\nabla f_1(x), \ldots, \nabla f_n(x)$$

are linearly independent. Let $S_{\mathcal{F}}^0$ denote the set of nonsingular points. It is well known that if $S_{\mathcal{F}}^0 \neq \varnothing$, then it is a relative open, dense subset of $S_{\mathcal{F}}$, and moreover it is an $(d-n)$-dimensional submanifold of $\mathbb{R}^d$. If $x \in S_{\mathcal{F}}^0$, then there exists a set of coordinates $J = \{j_1, \ldots, j_n\}$, with $1 \leq j_1 < \cdots < j_n \leq d$, such that

$$j_{\mathcal{F},J}(x) := \det\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{1 \leq i \leq n,\, j \in J} \neq 0. \tag{6-2}$$

Accordingly, we will call a set of coordinates J *admissible* if (6-2) holds for at least one point $x \in S^0_{\mathcal{F}}$ and will denote by $S_{\mathcal{F},J}$ the set of such points. For a given set of coordinates x_J let $\nabla_{x_J} f(x) := (\partial_{x_j} f(x))_{j\in J}$ and note that J is admissible if and only if the gradient vectors

$$\nabla_{x_J} f_1(x), \dots, \nabla_{x_J} f_n(x)$$

are linearly independent for at least one point $x \in S_{\mathcal{F}}$. It is clear that, unless $S_{\mathcal{F},J} = \varnothing$, it is a relative open and dense subset of $S_{\mathcal{F}}$ and is also a $(d-n)$-dimensional submanifold, moreover $S^0_{\mathcal{F}}$ is the union of the sets $S_{\mathcal{F},J}$ for all admissible J.

We define a measure, near a point $x_0 \in S_{\mathcal{F},J}$, as follows. For simplicity of notation assume that $J = \{1, \dots, n\}$ and let

$$\Phi(x) := (f_1, \dots, f_n, x_{n+1}, \dots, x_d).$$

Then $\Phi : U \to V$ is a diffeomorphism on some open set $x_0 \in U \subseteq \mathbb{R}^d$ to its image $V = \Phi(U)$, moreover $S_{\mathcal{F}} = \Phi^{-1}(V \cap \mathbb{R}^{d-n})$. Indeed, $x \in S_{\mathcal{F}} \cap U$ if and only if $\Phi(x) = (0, \dots, 0, x_{n+1}, \dots, x_d) \in V$. Let $I = \{n+1, \dots, d\}$ and write $x_I := (x_{n+1}, \dots, x_d)$. Let $\Psi(x_I) = \Phi^{-1}(0, x_I)$ and in local coordinates let x_I define the measure $\omega_{\mathcal{F}}$ via

$$\int g \, d\omega_{\mathcal{F}} := \int g(\Psi(x_I)) \operatorname{Jac}_{\Phi}^{-1}(\Psi(x_I)) \, dx_I, \tag{6-3}$$

for a continuous function g supported on U. Note that $\operatorname{Jac}_{\Phi}(x) = j_{\mathcal{F},J}(x)$, i.e., the Jacobian of the mapping Φ at $x \in U$ is equal to the expression given in (6-2), and that the measure $d\omega_{\mathcal{F}}$ is supported on $S_{\mathcal{F}}$. Define the local coordinates $y_j = f_j(x)$ for $1 \le j \le n$ and $y_j = x_j$ for $n < j \le d$. Then

$$dy_1 \wedge \cdots \wedge dy_d = df_1 \wedge \cdots \wedge df_n \wedge dx_{n+1} \wedge \cdots \wedge dx_d = \operatorname{Jac}_{\Phi}(x)\, dx_1 \wedge \cdots \wedge dx_d,$$

and thus

$$dx_1 \wedge \cdots \wedge dx_d = \operatorname{Jac}_{\Phi}(x)^{-1} df_1 \wedge \cdots \wedge df_n \wedge dx_{n+1} \wedge \cdots \wedge dx_d = df_1 \wedge \cdots \wedge df_n \wedge d\omega_{\mathcal{F}}.$$

This shows that the measure $d\omega_{\mathcal{F}}$ (given as a differential $(d-n)$-form on $S_{\mathcal{F}} \cap U$) is independent of the choice of local coordinates x_I. Then $\omega_{\mathcal{F}}$ is defined on $S^0_{\mathcal{F}}$ and moreover the set $S^0_{\mathcal{F}} \backslash S_{\mathcal{F},J}$ is of measure zero with respect to ω_F, as it is a proper analytic subset on $\mathbb{R}^{d-n}$ in any other admissible local coordinates.

Let $x = (z, y)$ be a partition of coordinates in $\mathbb{R}^d$, with $y = x_{J_2}$, $z = X_{J_1}$, and assume that for $i = 1, \dots, m$ the functions f_i depend only on the z-variables. We say that the partition of coordinates is *admissible* if there is a point $x = (z, y) \in S_{\mathcal{F}}$ such that both the gradient vectors $\nabla_z f_1(x), \dots, \nabla_z f_m(x)$ and the vectors $\nabla_y f_{m+1}(x), \dots, \nabla_y f_n(x)$ form a linearly independent system. Partition the system $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ with $\mathcal{F}_1 = \{f_1, \dots, f_m\}$ and $\mathcal{F}_2 = \{f_{m+1}, \dots, f_n\}$. Then there is a set $J_1' \subseteq J_1$ for which

$$j_{\mathcal{F}_1, J_1'}(z) := \det\left(\frac{\partial f_i}{\partial x_j}(z)\right)_{1 \le i \le m,\, j \in J_1'} \neq 0,$$

and also a set $J_2' \subseteq J_2$ such that

$$j_{\mathcal{F}_2, J_2'}(z, y) := \det\left(\frac{\partial f_i}{\partial x_j}(z, y)\right)_{m+1 \le i \le n,\, j \in J_2'} \neq 0.$$

Since $\nabla_y f_i \equiv 0$ for $1 \le i \le m$, it follows that the set of coordinates $J' = J_1' \cup J_2'$ is admissible, moreover,

$$j_{\mathcal{F},J'}(y,z) = j_{\mathcal{F}_1,J_1'}(z)\, j_{\mathcal{F}_2,J_2'}(y,z).$$

For fixed z, let $f_{i,z}(y) := f_i(z,y)$ and let $\mathcal{F}_{2,z} = \{f_{m+1,z}, \dots, f_{n,z}\}$. Then clearly $j_{\mathcal{F}_2,J_2'}(y,z) = j_{\mathcal{F}_{2,z},J_2'}(y)$ as it only involves partial derivatives with respect to the y-variable. Thus we have an analogue of Fubini's theorem, namely,

$$\int g(x)\, d\omega_{\mathcal{F}}(x) = \iint g(z,y)\, d\omega_{\mathcal{F}_{2,z}}(y)\, d\omega_{\mathcal{F}_1}(z). \tag{6-4}$$

Consider now algebraic sets given as the intersection of spheres. Let $x_1, \dots, x_m \in \mathbb{R}^d$, $t_1, \dots, t_m > 0$ and $\mathcal{F} = \{f_1, \dots, f_m\}$, where $f_i(x) = |x - x_i|^2 - t_i$ for $i = 1, \dots, m$. Then $S_{\mathcal{F}}$ is the intersection of spheres centered at the points x_i of radius $r_i = t_i^{1/2}$. If the set of points $X = \{x_1, \dots, x_m\}$ is in general position (i.e., they span an $(m-1)$-dimensional affine subspace), then a point $x \in S_{\mathcal{F}}$ is nonsingular if $x \notin \operatorname{span} X$, i.e., if x cannot be written as linear combination of $x_1, \dots, x_m$. Indeed, since $\nabla f_i(x) = 2(x - x_i)$, we have that

$$\sum_{i=1}^m a_i \nabla f_i(x) = 0 \quad \Longleftrightarrow \quad \sum_{i=1}^m a_i x = \sum_{i=1}^m a_i x_i,$$

which implies that $\sum_{i=1}^m a_i = 0$ and $\sum_{i=1}^m a_i x_i = 0$. By replacing the equations $|x - x_i|^2 = t_i$ with $|x - x_1|^2 - |x - x_i|^2 = t_1 - t_i$, which is of the form $x \cdot (x_1 - x_i) = c_i$, for $i = 2, \dots, m$, it follows that $S_{\mathcal{F}}$ is the intersection of the sphere with an $(n-1)$-codimensional affine subspace Y, perpendicular to the affine subspace spanned by the points x_i. Thus $S_{\mathcal{F}}$ is an m-codimensional sphere of $\mathbb{R}^d$ if $S_{\mathcal{F}}$ has one point $x \notin \operatorname{span}\{x_1, \dots, x_m\}$ and all of its points are nonsingular. Let x' be the orthogonal projection of x to $\operatorname{span} X$. If $y \in Y$ is a point with $|y - x'| = |x - x'|$ then by the Pythagorean theorem we have that $|y - x_i| = |x - x_i|$ and hence $y \in S_{\mathcal{F}}$. It follows that $S_{\mathcal{F}}$ is a sphere centered at x' and contained in Y.

Let $T = T_X$ be the inner product matrix with entries $t_{ij} := (x - x_i) \cdot (x - x_j)$ for $x \in S_{\mathcal{F}}$. Since

$$(x - x_i) \cdot (x - x_j) = \tfrac{1}{2}(t_i + t_j - |x_i - x_j|^2),$$

the matrix T is independent of x. We will show that $d\omega_{\mathcal{F}} = c_T\, d\sigma_{S_{\mathcal{F}}}$, where $d\sigma_{S_{\mathcal{F}}}$ denotes the surface area measure on the sphere $S_{\mathcal{F}}$ and $c_T = 2^{-m} \det(T)^{-1/2} > 0$, i.e., for a function $g \in C_0(\mathbb{R}^d)$,

$$\int_{S_{\mathcal{F}}} g(x)\, d\omega_{\mathcal{F}}(x) = c_T \int_{S_{\mathcal{F}}} g(x)\, d\sigma_{S_{\mathcal{F}}}(x). \tag{6-5}$$

Let $x \in S_{\mathcal{F}}$ be fixed and let $e_1, \dots, e_d$ be an orthonormal basis so that the tangent space $T_x S_{\mathcal{F}}$ equals $\operatorname{span}\{e_{m+1}, \dots, e_d\}$, and moreover we have that $\operatorname{span}\{\nabla f_1, \dots, \nabla f_m\} = \operatorname{span}\{e_1, \dots, e_m\}$. Let $x_1, \dots, x_n$ be the corresponding coordinates on $\mathbb{R}^d$ and note that in these coordinates the surface area measure, as a $(d-m)$-form at x, is

$$d\sigma_{S_{\mathcal{F}}}(x) = dx_{m+1} \wedge \cdots \wedge dx_d.$$

On the other hand, in local coordinates $x_I = (x_{m+1}, \ldots, x_d)$, it is easy to see from (6-2)–(6-3) that $j_{\mathcal{F},J}(x) = 2^m \operatorname{vol}(x - x_1, \ldots, x - x_m)$, and hence

$$d\omega_{\mathcal{F}}(x) = 2^{-m} \operatorname{vol}(x - x_1, \ldots, x - x_m)^{-1}\, dx_{m+1} \wedge \cdots \wedge dx_d,$$

where $\operatorname{vol}(x - x_1, \ldots, x - x_m)$ is the volume of the parallelotope with side vectors $x - x_j$. Finally, it is a well-known fact from linear algebra that

$$\operatorname{vol}(x - x_1, \ldots, x - x_m)^2 = \det(T),$$

i.e., the volume of a parallelotope is the square root of the Gram matrix formed by the inner products of its side vectors.

References

[Bennett et al. 2014] M. Bennett, A. Iosevich, and J. Pakianathan, "Three-point configurations determined by subsets of $\mathbb{F}_q^2$ via the Elekes–Sharir paradigm", *Combinatorica* **34**:6 (2014), 689–706. MR Zbl

[Bennett et al. 2016] M. Bennett, A. Iosevich, and K. Taylor, "Finite chains inside thin subsets of $\mathbb{R}^d$", *Anal. PDE* **9**:3 (2016), 597–614. MR Zbl

[Bourgain 1986] J. Bourgain, "A Szemerédi type theorem for sets of positive density in $\boldsymbol{R}^k$", *Israel J. Math.* **54**:3 (1986), 307–316. MR Zbl

[Falconer 2013] K. Falconer, "Some problems in measure combinatorial geometry", lecture slides, Erdős Centennial, Budapest, Hungary, 2013, available at http://www.renyi.hu/conferences/erdos100/slides/falconer.pdf.

[Fraser and Pramanik 2018] R. Fraser and M. Pramanik, "Large sets avoiding patterns", *Anal. PDE* **11**:5 (2018), 1083–1111. MR Zbl

[Hart and Iosevich 2008] D. Hart and A. Iosevich, "Ubiquity of simplices in subsets of vector spaces over finite fields", *Anal. Math.* **34**:1 (2008), 29–38. MR Zbl

[Iosevich and Liu 2019] A. Iosevich and B. Liu, "Equilateral triangles in subsets of $\mathbb{R}^d$ of large Hausdorff dimension", *Israel J. Math.* **231**:1 (2019), 123–137. MR Zbl

[Iosevich and Parshall 2019] A. Iosevich and H. Parshall, "Embedding distance graphs in finite field vector spaces", *J. Korean Math. Soc.* **56**:6 (2019), 1515–1528. MR Zbl

[Iosevich and Taylor 2019] A. Iosevich and K. Taylor, "Finite trees inside thin subsets of $\mathbb{R}^d$", pp. 51–56 in *Modern methods in operator theory and harmonic analysis*, Springer Proc. Math. Stat. **291**, Springer, 2019. MR

[Keleti 2008] T. Keleti, "Construction of one-dimensional subsets of the reals not containing similar copies of given patterns", *Anal. PDE* **1**:1 (2008), 29–33. MR Zbl

[Lyall and Magyar 2020] N. Lyall and Á. Magyar, "Distance graphs and sets of positive upper density in $\mathbb{R}^d$", *Anal. PDE* **13**:3 (2020), 685–700. MR Zbl

[Maga 2010] P. Maga, "Full dimensional sets without given patterns", *Real Anal. Exchange* **36**:1 (2010), 79–90. MR Zbl

[Mattila 1995] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics **44**, Cambridge Univ. Press, 1995. MR Zbl

Received 7 Sep 2020. Revised 28 Sep 2021. Accepted 19 Nov 2021.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 16 No. 7 2023