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DIMENSION-FREE HARNACK INEQUALITIES FOR CONJUGATE HEAT EQUATIONS AND THEIR APPLICATIONS TO GEOMETRIC FLOWS

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Let M be a differentiable manifold endowed with a family of complete Riemannian metrics $g(t)$ evolving under a geometric flow over the time interval $[0, T[$. We give a probabilistic representation for the derivative of the corresponding conjugate semigroup on M which is generated by a Schrödinger-type operator. With the help of this derivative formula, we derive fundamental Harnack-type inequalities in the setting of evolving Riemannian manifolds. In particular, we establish a dimension-free Harnack inequality and show how it can be used to achieve heat kernel upper bounds in the setting of moving metrics. Moreover, by means of the supercontractivity of the conjugate semigroup, we obtain a family of canonical log-Sobolev inequalities. We discuss and apply these results both in the case of the so-called modified Ricci flow and in the case of general geometric flows.

1. Introduction

Let M be a differentiable manifold endowed with a C^1 family of complete Riemannian metrics $g(t)$ indexed by the real interval $[0, T[$, where $T \in]0, \infty]$. The family $g(t)$ describes the evolution of the manifold M under a geometric flow where T is the first time where possibly a blow-up of the curvature occurs. This type of singularity is not excluded in our setting.

More specifically, we consider geometric flows of the type

$$\partial_t g(t) = -2\,\mathrm{Sc}(t) \quad \text{on } M \times [0, T[,$$

where $\mathrm{Sc}(t) = (S_{ij})$ is a general time-dependent symmetric (0, 2)-tensor. For fixed t, with respect to the metric $g(t)$, let $S = g^{ij} S_{ij}$ be the metric trace of the tensor $S(t)$ and Δ_t the Laplace–Beltrami operator acting on functions on M. In practice, the geometric flow deforms the geometry of M and smooths out irregularities in the metric to give it a nicer and more symmetric form, which provides geometric and topological information on the manifold.

Consider operators of the form $L_t = \Delta_t - \nabla^t \phi_t$, where ϕ_t is a time-dependent function on M. We also use the notation $g_t = g(t)$, $S_t = S(t, \cdot)$ and $\mathrm{Sc}_t = \mathrm{Sc}(t)$. In this paper we study the (minimal) fundamental solution to heat equations of the type

$$(L_t - \partial_t)u(t, x) = 0, \quad \text{resp. } (L_t + \partial_t - \varrho_t)u(t, x) = 0,$$

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where $\varrho_t = \partial_t\phi_t + S_t$. The first equation is the classical heat equation, the second one appears naturally as conjugate heat equation. More precisely, we have the following relationship.

Remark 1.1. Set $d\mu_t = \mathrm{e}^{-\phi_t} d\,\mathrm{vol}_t$, where vol_t denotes the Riemannian volume to the metric $g(t)$. Let $\Box = L_t - \partial_t$ be the standard heat operator and $\Box^*$ its formal adjoint with respect to the measure $\mu_t \otimes dt$. Thus,

$$\int_0^T \int_M V \,\Box\, U \, d\mu_t \, dt = \int_0^T \int_M U \,\Box^*\, V \, d\mu_t \, dt$$

for functions $U, V \in C_c^{2,1}(M \times [0, T[)$. From this relation it is immediate that $\Box^* = L_t + \partial_t - \varrho_t$.

Example 1.2. A typical situation covered by this setting is solving a geometric flow equation (e.g., Ricci flow) forward in time and the associated conjugate heat equation backward in time. In the case of the Ricci flow $\partial_t g(t) = -2\,\mathrm{Ric}_t$ and $L_t = \Delta_t$, the conjugate heat equation reads as $\Box^* u = (\Delta_t + \partial_t - R)u = 0$, where $R = \mathrm{tr}\,\mathrm{Ric}$ denotes the (time-dependent) scalar curvature.

It should be mentioned that an important ingredient in the proof of the Poincaré conjecture by Perelman [2002] is a differential Harnack inequality which is related to a gradient estimate for solutions to the conjugate heat equation under forward Ricci flow on a compact manifold. This relation has been one of our motivations to investigate solutions to conjugate heat equations and their associated semigroups also by methods of stochastic analysis.

Let X_t be the diffusion process generated by $L_t = \Delta_t - \nabla^t\phi_t$ (called L_t-diffusion process) which we assume to be nonexplosive up to time T. We consider the two-parameter semigroup associated to L_t,

$$P_{s,t} f(x) := \mathbb{E}[f(X_t) \mid X_s = x], \quad s \le t,$$

which satisfies the heat equation

$$\begin{cases} \dfrac{\partial}{\partial s} P_{s,t} f = -L_s P_{s,t} f, \\ \lim_{s\to t} P_{s,t} f = f. \end{cases}$$

In previous work, we already studied properties of heat equations under geometric flows, like properties of the semigroup $P_{s,t}$ generated by L_t, by adapting probabilistic methods. In [Cheng 2017], for instance, the first author gave functional inequalities equivalent to a lower bound of $\mathrm{Ric}_t + \mathrm{Sc}_t$. In [Cheng and Thalmaier 2018a; 2018c] we established characterizations of upper and lower bounds for $\mathrm{Ric}_t + \mathrm{Sc}_t$ in terms of functional inequalities on the path space over M.

On a more probabilistic side, in [Cheng and Thalmaier 2018b] the authors studied existence and uniqueness of so-called evolution systems of measures related to the semigroup $P_{s,t}$. Using such systems as reference measures, contractivity of the semigroup, as well as log-Sobolev inequality, have been investigated.

Although the evolution system of measures is helpful to shed light on properties of solutions to the heat equation, it is still difficult to obtain a full picture of this measure system, like its relation to the system of volume measures. It has been observed that if one uses volume measures as reference measures, many questions will be related to the conjugate heat equation and not the usual heat equation; see e.g., [Abolarinwa 2015; Cao et al. 2015; Kuang and Zhang 2008].

Recall that $\mu_t(dx) = \mathrm{e}^{-\phi_t(x)}\, d\,\mathrm{vol}_t$, where vol_t is the volume measure with respect to the metric $g(t)$. Let

$$P^{\varrho}_{s,t} f(x) = \mathbb{E}\left[\exp\left(-\int_s^t \varrho(r, X_r)\, dr\right) f(X_t) \,\middle|\, X_s = x\right],$$

where $\varrho(t, x) = \varrho_t(x)$ is given by $\varrho_t := \partial_t \phi_t + S_t$ and

$$\frac{\partial}{\partial t}\mu_t(dx) = -\varrho(t, x)\, \mu_t(dx).$$

According to the Feynman–Kac formula, $P^{\varrho}_{s,t} f$ represents the solution to the equation

$$\frac{\partial}{\partial s}\varphi_s = -(L_s - \varrho_s)\varphi_s, \quad \varphi_t = f,$$

on $[0, t] \times M$ where $t < T$. We note that this equation is conjugate to the heat equation

$$\frac{\partial}{\partial s}u(s, x) = L_s u(s, \cdot)(x).$$

In this paper, we first give probabilistic formulas and estimates for $dP^{\varrho}_{s,t} f$ from which we then derive a dimension-free Harnack inequality. It is interesting to note that by combining the dimension-free Harnack inequalities for $P_{s,t}$ and $P^{\varrho}_{s,t}$, one can obtain new on-diagonal and Gaussian upper bounds for the heat kernel to L_t with respect to μ_t; see Sections 5 and 6. We apply these results then to the following modified geometric flow for g_t combined with the conjugate heat equation for ϕ_t (see, e.g., Corollary 5.3 below), i.e.,

$$\begin{cases} \partial_t g_t = -2(\mathrm{Sc} + \mathrm{Hess}(\phi))_t, \\ \partial_t \phi_t = -(\Delta\phi - S)(t, x). \end{cases} \tag{1-1}$$

As is well known, for $\mathrm{Sc} = \mathrm{Ric}$, this flow represents the gradient flow to Perelman's famous entropy functional, also known as Perelman's $\mathcal{F}$-functional [2002].

Before we give other applications of these Harnack inequalities, let us first compare our results on heat kernel estimates with the known results in this direction. In [Coulibaly-Pasquier 2019], the author used a horizontal coupling of curves to obtain a dimension-free Harnack inequality for $P_{s,t}$ generated by Δ_t, and applied it then to on-diagonal heat kernel estimates by following Grigoryan's argument [1997]. The first difference to our approach is that we use the dimension-free Harnack inequality for the conjugate heat semigroup instead of comparing $P_{s,t}$ to $P^{\varrho}_{s,t}$ by controlling the absolute value of the potential $|\varrho|$. The second difference is that in [Coulibaly-Pasquier 2019], the author used the midpoint $(t+s)/2$ as reference time, so that lower bounds for both $\mathrm{Ric}_t + \mathrm{Sc}_t$ and $\mathrm{Ric}_t - \mathrm{Sc}_t$ for $t \in [0, T]$ are required. Here, in our approach, we adopt the method of [Grigoryan 1997] as well, but the reference time r has greater flexibility. For instance, the reference time r can be chosen close to s so that if one knows that there exist constants κ and K such that $|d\varrho_s| \le \kappa$ and $\mathrm{Ric}_s + \mathrm{Sc}_s + \mathrm{Hess}_s(\phi_s) \ge K$ at the initial time s, then by choosing r close to s, an on-diagonal heat kernel estimate can still be derived under the assumption that $\mathrm{Ric}_t - \mathrm{Sc}_t + \mathrm{Hess}_t(\phi_t) \ge K_1(t)$ for $t \in [0, T]$. For instance, if the geometric flow is a Ricci flow and $\phi = 0$, then $K_1 \equiv 0$ and solely information from the initial manifold is basically enough to derive upper heat kernel bounds. With respect to this point of view, the necessary conditions in our results could be weakened significantly.

Recently, Buzano and Yudowitz [2020] established Gaussian-type heat kernel estimates under a general geometric flow by adapting the methods from [Davies 1987]. In their paper, in the case of a general geometric flow, they assume for vector fields X on M the tensorial inequality

$$0 \le D(\mathrm{Sc}, X) := \frac{\partial}{\partial t} S - \Delta S - 2|S_{ij}|^2 + 4(\nabla_i S_{ij}) X_j - 2(\nabla_j S) X_j + 2R_{ij} X_i X_j - 2S_{ij} X_i X_j. \tag{1-2}$$

It should be remarked that apart from some classical geometric flows where condition (1-2) is easily checked, this condition in general is difficult to verify. Generally speaking, it is an advantage for applications to rely on information about Ric and Sc directly. From this point of view, our conditions are convenient to apply.

Next, we observe that

$$\mu_s(P^{\varrho}_{s,t} f) = \mu_t(f), \quad s \le t, \tag{1-3}$$

which means that the family of measures μ_s plays a role for the semigroup $P^{\varrho}_{s,t}$ similar to that of the invariant measure in the static Riemannian case for the one-parameter semigroup P_t. Log-Sobolev inequalities with respect to the invariant measure are well-established under certain curvature conditions on Riemannian manifolds. They have many applications and are related to other functional inequalities for P_t; see for instance [Bakry 1997; Gross 1975; Wang 2001; 2009]. This leads to the natural question of whether it is possible to prove log-Sobolev inequalities with respect to μ_t in a similar way through functional inequalities for $P^{\varrho}_{s,t}$.

In Section 7 we discuss the relation between contraction properties of the semigroup and log-Sobolev inequalities with respect to μ_t. Using the dimension-free Harnack inequality for $P^{\varrho}_{s,t}$, we give a sufficient condition for supercontractivity of $P^{\varrho}_{s,t}$ which we then use to prove existence of the (defective) log-Sobolev inequality for μ_s. It is well known that the log-Sobolev inequality and Sobolev inequality are important tools to establish upper bounds for the heat kernel; see [Abolarinwa 2016; Băileşteanu 2012; Zhang 2006; Buzano and Yudowitz 2020]. Note that in [Buzano and Yudowitz 2020] the condition for the log-Sobolev inequality is $D(\mathrm{Sc}, X) \ge 0$, which implies in particular $(\partial_t - \Delta_t) S_t \ge 0$. Here besides the curvature condition for the gradient estimate, we add the condition $(\partial_t - L_t)\varrho_t \ge 0$ to derive a super log-Sobolev inequality. The results are then applied to the system (1-1). More specifically, denote by $\rho_t \equiv \rho_t(o, \cdot)$ the distance function to a given base point o in M with respect to the metric g_t, and suppose that the geometric flow g_t and the function ϕ_t satisfy (1-1). Assuming that $\mathrm{Ric}_t + \mathrm{Hess}_t(\phi_t) + \mathrm{Sc}_t \ge K(t)$ and $\mu_t(\exp(\lambda \rho_t^2)) < \infty$ for all $\lambda > 0$ and $t \in [0, T[$, there exists a function β such that

$$\mu_s(f^2 \log f^2) \le r \mu_s\big(|\nabla^s f|_s^2 + \tfrac{1}{4}\varrho_s f^2\big) + \beta_s(r), \quad r > 0,$$

for $f \in C_0^\infty([0, T[\times M)$ and $\mu_s(f^2) = 1$.

The paper is organized as follows. In Section 3 a probabilistic formula for the derivative of the conjugate heat semigroup is given and used to establish a gradient estimate for $P^{\varrho}_{s,t}$ under suitable curvature conditions. In Section 4 we derive two versions of dimension-free Harnack inequalities from the mentioned gradient inequality for $P^{\varrho}_{s,t}$, which are then applied in Section 5 to on-diagonal heat kernel estimates and in Section 6 to Gaussian-type heat kernel estimates via Grigoryan's argument.

These Harnack inequalities are further used in Section 7 to gain sufficient and necessary conditions for supercontractivity of $P^{\varrho}_{s,t}$. In Section 7 we also clarify the relation between supercontractivity of $P^{\varrho}_{s,t}$ and the log-Sobolev inequality with respect to μ_t. These results are then applied to system (1-1) of the modified geometric flow under conjugate heat equation.

2. Brownian motion with respect to evolving manifolds

Let $(M, g_t)_{t\in I}$ be an evolving manifold indexed by $I = [0, T[$. Let ∇^t be the Levi-Civita connection with respect to g_t. Denote by $\mathbb{M} := M \times I$ space-time and consider the bundle

$$TM \xrightarrow{\pi} \mathbb{M},$$

where π is the projection. As observed by Hamilton [1993] there exists a natural space-time connection ∇ on TM considered as bundle over space-time $\mathbb{M}$ such that

$$\begin{cases} \nabla_v X = \nabla^t_v X, \\ \nabla_{\partial_t} X = \partial_t X + \frac{1}{2}(\partial_t g_t)(X, \cdot)^{\sharp g_t} \end{cases}$$

for all $v \in (T_x M, g_t)$ and all time-dependent vector fields X on M. This connection is compatible with the metric, i.e.,

$$\frac{d}{dt}|X|^2_{g_t} = 2\langle X, \nabla_{\partial_t} X\rangle_{g_t}.$$

Remark 2.1. Let $G = \mathrm{O}(n)$, where $n = \dim M$ and consider the G-principal bundle $\mathcal{F} \xrightarrow{\pi} \mathbb{M}$ of orthonormal frames with fibers

$$\mathcal{F}_{(x,t)} = \{u : \mathbb{R}^n \to (T_x M, g_t) \mid u \text{ isometry}\}.$$

As usual, $a \in G$ acts on $\mathcal{F}$ from the right via composition. The connection ∇ gives rise to a G-invariant splitting of the sequence

$$0 \longrightarrow \ker d\pi \longrightarrow T\mathcal{F} \underset{\underset{h}{\curvearrowleft}}{\xrightarrow{d\pi}} \pi^* T\mathbb{M} \longrightarrow 0,$$

which induces a decomposition of $T\mathcal{F}$ as $T\mathcal{F} = V \oplus H := \ker d\pi \oplus h(\pi^* T\mathbb{M})$. For $u \in \mathcal{F}$, the space H_u is the *horizontal space at u* and $V_u = \{v \in T_u\mathcal{F} :\ (d\pi)v = 0\}$ the *vertical space at u*. The bundle isomorphism

$$h : \pi^* T\mathbb{M} \xrightarrow{\sim} H \hookrightarrow T\mathcal{F}, \quad h_u : T_{\pi(u)}\mathbb{M} \xrightarrow{\sim} H_u, \quad u \in \mathcal{F}, \tag{2-1}$$

is the *horizontal lift* of the G-connection.

Corollary 2.2. *To each $X + b\partial_t \in T_{(x,t)}\mathbb{M}$ and each frame $u \in \mathcal{F}_{(x,t)}$, there exists a unique "horizontal lift" $X^* + bD_t \in H_u$ of $X + b\partial_t$ such that*

$$\pi_*(X^* + bD_t) = X + b\partial_t.$$

In explicit terms, X^* is the horizontal lift of X with respect to the metric g_t, and $D_t = (d/ds)|_{s=0} u_s$, where u_s is the horizontal lift based at u of the curve $s \mapsto (x, t + s)$.

We consider curves in $\mathbb{M}$ of the form

$$\gamma_t = (x_t, \ell_t), \quad t \in [0, T[,$$

where ℓ_t is a monotone differentiable transformation on $[0, T[$. The horizontal lift of such a curve γ_t in $\mathbb{M}$ is a curve u_t in $\mathcal{F}$ such that $\pi u_t = \gamma_t$ and $\nabla_{\dot\gamma}(u_t e) = 0$ for each $e \in \mathbb{R}^n$. Then

$$//^{\gamma}_{r,s} := u_s u_r^{-1} : (T_{x_r} M, g_{\ell_r}) \to (T_{x_s} M, g_{\ell_s}), \quad 0 \le r \le s < T,$$

gives parallel transport along γ_t. In the following we consider the special case $\ell_t = t$.

Remark 2.3. Vector fields and differential forms on $\mathbb{M}$ can be seen as time-dependent vector fields and differential forms on M. It is convenient to write objects on $\mathbb{M}$ as G-equivariant functions on $\mathcal{F}$. In particular, then

(1) functions $f \in C^\infty(\mathbb{M})$ read as $\tilde f \in C^\infty(\mathcal{F})$ via $\tilde f := f \circ \pi$,
(2) time-dependent vector fields Y on M read as $\widetilde{Y} : \mathcal{F} \to \mathbb{R}^n$ via $\widetilde{Y}(u) := u^{-1} Y_{\pi(u)}$,
(3) time-dependent differential forms α on M read as $\tilde\alpha : \mathcal{F} \to (\mathbb{R}^n)^*$ via $\tilde\alpha(u) = \alpha_{\pi(u)}(u\,\cdot\,)$.

The following formulas hold:

$$\widetilde{Xf} = X^* \tilde f, \quad \widetilde{\partial_t f} = D_t \tilde f, \quad \widetilde{\nabla_X Y} = X^* \widetilde{Y}, \quad \widetilde{\nabla_{\partial_t} Y} = D_t \widetilde{Y}, \quad \widetilde{\nabla_X \alpha} = X^* \tilde\alpha, \quad \widetilde{\nabla_{\partial_t} \alpha} = D_t \tilde\alpha. \tag{2-2}$$

The proofs are straightforward. For instance, to check the last equality, let u_t be a horizontal curve such that $\pi u_t = \gamma_t = (x, t)$, where $x \in M$ is fixed. Then

$$\begin{aligned}(D_t \tilde\alpha)(u_s) &= \frac{d}{dt}\Big|_{t=s} \tilde\alpha(u_t) = \frac{d}{dt}\Big|_{t=s} \alpha_{\pi(u_t)}(u_t\,\cdot\,) \\ &= \frac{d}{dr}\Big|_{r=0} \alpha_{(x,s+r)}(//_{s,s+r} u_s, \cdot\,) = (\nabla_{\partial_t}\alpha)_{(x,s)}(u_s\,\cdot\,) = (\widetilde{\nabla_{\partial_t}\alpha})(u_s).\end{aligned}$$

Remark 2.4. The vector fields

$$H_i \in \Gamma(T\mathcal{F}), \quad H_i(u) = (ue_i)^* \equiv h_u(ue_i), \quad i = 1, \dots, n,$$

where $(e_1, \dots, e_n)$ denotes the standard basis of $\mathbb{R}^n$, are the standard-horizontal vector fields on $\mathcal{F}$. The operator

$$\Delta_{\text{hor}} = \sum_{i=1}^n H_i^2$$

is called Bochner's horizontal Laplacian on $\mathcal{F}$. Note that

$$\widetilde{\Delta f} = \Delta_{\text{hor}} \tilde f \quad \text{and} \quad \widetilde{\Delta_{\text{rough}} \alpha} = \Delta_{\text{hor}} \tilde\alpha, \tag{2-3}$$

where $\Delta_{\text{rough}} = \operatorname{tr}(\nabla^t)^2$ is the rough Laplacian on differential one-forms. Recall that, for fixed $t \in I$,

$$d\Delta_t f = \operatorname{tr}(\nabla^t)^2 df - \operatorname{Ric}_t(df, \cdot\,) \tag{2-4}$$

by the Weitzenböck formula.

Let $\pi : \mathcal{F} \to \mathbb{M}$ denote the canonical projection. For $\phi \in C^{1,2}(\mathbb{M})$, we define a vector field on $\mathcal{F}$ by

$$H^\phi \in \Gamma(T\mathcal{F}), \quad H^\phi(u) = h_u(\nabla^t \phi(t, \cdot)_x), \quad \pi(u) = (x, t).$$

Consider the following Stratonovich SDE on $\mathcal{F}$:

$$\begin{cases} dU = D_t(U)\,dt + \sum_{i=1}^n H_i(U) \circ dB^i - H^\phi(U)\,dt, \\ U_s = u_s, \quad \pi(u_s) = (x, s),\ s \in [0, T[. \end{cases} \tag{2-5}$$

Here B denotes standard Brownian motion on $\mathbb{R}^n$ (sped up by the factor 2, i.e., $dB^i\,dB^j = 2\delta_{ij}\,dt$) with generator $\Delta_{\mathbb{R}^n}$. Equation (2-5) has a unique solution up to its lifetime $\zeta := \lim_{k\to\infty} \zeta_k$, where

$$\zeta_k := \inf\{t \in [s, T[\, : \rho_t(\pi(U_s), \pi(U_t)) \geq k\}, \quad n \geq 1,\ \inf \varnothing := T, \tag{2-6}$$

and where ρ_t stands for the Riemannian distance induced by the metric $g(t)$. We note that if U solves (2-5) then

$$\pi(U_t) = (X_t, t),$$

where X is a diffusion process on M generated by $L_t = \Delta_t - \nabla^t \phi_t$. In case of $\phi = 0$ we call X a (g_t)-Brownian motion on $\mathbb{M}$.

More precisely, we have the following result.

Proposition 2.5. *Let U be a solution to the SDE* (2-5). *Then*

(1) *for any C^2-function $F : \mathcal{F} \to \mathbb{R}$,*

$$d(F(U)) = (D_t F)(U)\,dt + \sum_{i=1}^n (H_i F)(U)\,dB^i + (\Delta_{\text{hor}} F)(U)\,dt - (H^\phi F)(U)\,dt,$$

(2) *for any C^2-function $f : \mathbb{M} \to \mathbb{R}$, we have*

$$d(f(X)) = (\partial_t f)(X)\,dt + \sum_{i=1}^n (Ue_i f)\,dB^i + (L_t f)(X)\,dt.$$

Let U be a solution to the SDE (2-5) and $\pi(U_t) = (X_t, t)$. Furthermore let

$$//_{r,t} := U_t U_r^{-1} : (T_{x_r} M, g_r) \to (T_{x_t} M, g_t), \quad s \leq r \leq t < T,$$

be the induced parallel transport along X_t (which by construction consists of isometries). We use the notation

$$X_t = X_t^{(s,x)}, \quad t \geq s,$$

if $X_s = x$. Note that $X_t = X_t^{(s,x)}$ solves the equation

$$dX_t^{(s,x)} = U_t \circ dB_t - \nabla^t \phi_t(X_t^{(s,x)})\,dt, \quad X_s^{(s,x)} = x.$$

For any $f \in C_0^2(M)$,

$$f(X_t^{(s,x)}) - f(x) - \int_s^t L_r f(X_r^{(s,x)})\,dr = \int_s^t \langle //_{s,r}^{-1} \nabla^r f(X_r^{(s,x)}), U_s\,dB_r\rangle_s, \quad t \in [s, T[,$$

is a martingale up to the lifetime ζ. In the case $s = 0$, we write again X_t^x instead of $X_t^{(0,x)}$.

3. Derivative formula

Let $L_t = \Delta_t - \nabla^t \phi_t$, where ϕ is $C^{1,2}([0, T[\times M)$. Throughout this section, we assume the diffusion (X_t) generated by L_t is nonexplosive up to time T. Recall that $\mu_t(dx) = \mathrm{e}^{-\phi_t(x)} d\,\mathrm{vol}_t$ and

$$\frac{\partial}{\partial t}\mu_t(dx) = -(\partial_t \phi + S)(t, x)\,\mu_t(dx) = -\varrho(t, x)\,\mu_t(dx),$$

with

$$\varrho(t, x) \equiv \varrho_t(x) = \partial_t \phi(t, x) + S(t, x).$$

For each t, we assume that ϱ_t is bounded from below by a constant depending on t.

For $f \in C_b(M)$ let

$$P^{\varrho}_{s,t} f(x) = \mathbb{E}\left[\exp\left(-\int_s^t \varrho(r, X_r)\,dr\right) f(X_t) \,\Big|\, X_s = x\right]. \tag{3-1}$$

Then

$$\frac{\partial}{\partial s} P^{\varrho}_{s,t} f = -(L_s - \varrho_s) P^{\varrho}_{s,t} f, \quad P^{\varrho}_{t,t} = f. \tag{3-2}$$

That $u(s, x) := P^{\varrho}_{s,t} f(x)$ represents the solution to (3-2) is easily seen from the fact that

$$\exp\left(-\int_s^r \varrho(a, X_a)\,da\right) P^{\varrho}_{r,t} f(X_r^{(s,x)}), \quad r \in [s, t],$$

is a martingale under given assumptions.

Our first step is to establish a derivative formula for $P^{\varrho}_{s,t}$. When the metric is static, the derivative formula for $P_t f$ is known as the Bismut formula [Bismut 1984; Elworthy and Li 1994]. The more general versions in [Thalmaier 1997] have been adapted to Feynman–Kac semigroups in [Thompson 2019].

We fix $s \in [0, T[$ and consider the random family $Q_{s,t} \in \mathrm{Aut}(T_{X_s} M)$, $0 \le s \le t < T$, given as a solution to the ordinary differential equation

$$\frac{dQ_{s,t}}{dt} = -(\mathrm{Ric} + \mathrm{Sc} + \mathrm{Hess}(\phi))_{//_{s,t}} Q_{s,t}, \quad Q_{s,s} = \mathrm{id}, \tag{3-3}$$

where

$$(\mathrm{Ric} + \mathrm{Sc} + \mathrm{Hess}(\phi))_{//_{s,t}} := //_{s,t}^{-1} \circ (\mathrm{Ric}_t + \mathrm{Sc}_t + \mathrm{Hess}_t(\phi_t))(X_t) \circ //_{s,t}.$$

Theorem 3.1. *Let $L_t = \Delta_t - \nabla^t \phi_t$ as above. For each t, assume that both ϱ_t and*

$$(\mathrm{Ric} + \mathrm{Hess}(\phi) + \mathrm{Sc})(t, \cdot\,)$$

are bounded below and that $|d\varrho_t|$ is bounded. Then, for $v \in T_x M$ and $f \in C_b^1(M)$,

$$(dP^{\varrho}_{s,t} f)(v) = \mathbb{E}^{(s,x)}\left[\exp\left(-\int_s^t \varrho_r(X_r)\,dr\right)\left(df(//_{s,t} Q_{s,t} v)(X_t) - f(X_t)\int_s^t d\varrho_r(//_{s,r} Q_{s,r} v)\,dr\right)\right]. \tag{3-4}$$

Proof. By the definition of Q as the solution to (3-3), we have

$$d(U_s^{-1} Q_{s,r}) = -U_s^{-1}(\mathrm{Ric} + \mathrm{Hess}(\phi))_{//_{s,r}} Q_{s,r}. \tag{3-5}$$

Set

$$N_r(v) = dP^{\varrho}_{r,t} f(//_{s,r} Q_{s,r} v), \quad s \le r \le t.$$

Write $N_r(v) = F(U_r, q_r(v))$, where

$$F(u, w) := (dP^{\varrho}_{r,t} f)_x(uw), \quad \pi(u) = (x, r) \text{ and } w \in \mathbb{R}^n,$$
$$q_r(v) := U_s^{-1} Q_{s,r} v.$$

By means of Proposition 2.5, we have

$$d(F(U_r, w)) \overset{\mathrm{m}}{=} (D_r F)(U_r, w)\, dr + (\Delta_{\mathrm{hor}} F)(U_r, w)\, dr - (H^{\phi} F)(U_r, w)\, dr, \tag{3-6}$$

where

$$(D_r F)(u, w) = \partial_r (dP^{\varrho}_{r,t} f)_x(uw) + \tfrac{1}{2}(\partial_r g_r)((dP^{\varrho}_{r,t} f)^{\sharp_{g_r}}, uw),$$

and where $\overset{\mathrm{m}}{=}$ stands for equality modulo the differential of a local martingale. Using the Weitzenböck formula we observe that

$$\begin{aligned}
\partial_r (dP^{\varrho}_{r,t} f) &= -d(L_r - \varrho_r) P^{\varrho}_{r,t} f \\
&= -d(\Delta_r - \nabla^r \phi_r - \varrho_r) P^{\varrho}_{r,t} f \\
&= -\operatorname{tr}(\nabla^t)^2 dP^{\varrho}_{r,t} f + dP^{\varrho}_{r,t} f((\operatorname{Hess} \phi_r)^{\sharp_{g_r}}) + dP^{\varrho}_{r,t} f(\operatorname{Ric}_r^{\sharp_{g_r}}) + d(\varrho_r\, P^{\varrho}_{r,t} f)_{X_r}.
\end{aligned}$$

Taking (2-2) and (2-3) into account, we have

$$(\Delta_{\mathrm{hor}} F)(U_r, w) = \operatorname{tr}(\nabla^r)^2 dP^{\varrho}_{r,t} f(U_r w)$$

and

$$(H^{\phi} F)(U_r, w) = \operatorname{Hess}(\phi_r)((dP^{\varrho}_{r,t} f)^{\sharp_{g_r}}, U_r w) = dP^{\varrho}_{r,t} f((\operatorname{Hess}(\phi_r))^{\sharp_{g_r}})(U_r w).$$

Thus (3-6) can be rewritten as

$$d(F(U_r, w)) \overset{\mathrm{m}}{=} dP^{\varrho}_{r,t} f\, (\operatorname{Ric}_r^{\sharp_{g_r}}) U_r w + d(\varrho_r\, P^{\varrho}_{r,t} f)_{X_r} U_r w + \tfrac{1}{2}(\partial_r g_r)((dP^{\varrho}_{r,t} f)^{\sharp_{g_r}}, U_r w). \tag{3-7}$$

Combining (3-7) and (3-5) we thus obtain

$$\begin{aligned}
dN_r(v) &= d(F(U_r, \cdot))(q_r(v)) + F(U_r, \partial_r q_r(v))\, dr \\
&\overset{\mathrm{m}}{=} d(\varrho_r\, P^{\varrho}_{r,t} f)_{X_r} //_{s,r} Q_{s,r} v\, dr \\
&= \big(\varrho_r(X_r)\, dP^{\varrho}_{r,t} f(//_{s,r} Q_{s,r} v) + P^{\varrho}_{r,t} f(X_r)\, d\varrho_r(//_{s,r} Q_{s,r} v)\big)\, dr,
\end{aligned}$$

Hence we get

$$d\left(\exp\left(-\int_s^r \varrho_u(X_u)\, du\right) N_r(v)\right) \overset{\mathrm{m}}{=} -\exp\left(-\int_s^r \varrho_u(X_u)\, du\right) P^{\varrho}_{r,t} f(X_r)\, d\varrho_r(//_{s,r} Q_{s,r} v)\, dr.$$

Integrating this equality from s to t and taking the expectation gives (3-4). □

Corollary 3.2. *Suppose that ϱ_t is bounded below for each t, and assume that there are functions $\kappa, K \in C([0, T[)$ such that $|d\varrho_t| \le \kappa(t)$, respectively*

$$\operatorname{Ric}_t + \operatorname{Sc}_t + \operatorname{Hess}_t(\phi_t) \ge K(t).$$

Then, for $f \in C_0^\infty(M)$ and $f \ge 0$,

$$|\nabla^s P^\varrho_{s,t} f|_s \le \exp\left(-\int_s^t K(r)\,dr\right) P^\varrho_{s,t} |\nabla^t f|_t + P^\varrho_{s,t} f \int_s^t \kappa(r) \exp\left(-\int_s^r K(u)\,du\right) dr.$$

Proof. The condition $\mathrm{Ric}_t + \mathrm{Sc}_t + \mathrm{Hess}_t(\phi_t) \ge K(t)$ implies

$$|Q_{s,t}| \le \exp\left(-\int_s^t K(r)\,dr\right).$$

The inequality then follows from the bound $|d\varrho_t| \le \kappa(t)$. □

In particular, if $\phi \equiv 0$, Corollary 3.2 then becomes:

Corollary 3.3. *Suppose that S_t is bounded below for each t and assume that there are functions $S, K \in C([0, T[)$ such that $|dS_t| \le \kappa(t)$, respectively*

$$\mathrm{Ric}_t + \mathrm{Sc}_t \ge K(t).$$

Then, for $f \in C_0^\infty(M)$ and $f \ge 0$,

$$|\nabla^s P^\varrho_{s,t} f|_s \le \exp\left(-\int_s^t K(r)\,dr\right) P^\varrho_{s,t} |\nabla^t f|_t + P^\varrho_{s,t} f \int_s^t \kappa(r) \exp\left(-\int_s^r K(u)\,du\right) dr.$$

4. Dimension-free Harnack inequalities

We first derive two Harnack-type inequalities for $P^o_{s,t}$. Such dimension-free Harnack inequalities were studied first by Wang; see, e.g., [Wang 2014] for more results in this direction. When it comes to the evolving metric case, in [Cheng 2017] the author gave the following Harnack inequality for the 2-parameter semigroup $P_{s,t}$ as follows. We denote by $\mathcal{B}_b(M)$ the space of bounded measurable functions on M.

Theorem 4.1. *Suppose that*

$$\mathrm{Ric}_t + \mathrm{Sc}_t + \mathrm{Hess}_t(\phi_t) \ge K(t).$$

Then, for any nonnegative function $f \in \mathcal{B}_b(M)$ and $0 \le s \le t < T$,

$$(P_{s,t} f)^p(x) \le P_{s,t} f^p(y) \exp\left(\frac{p}{4(p-1)\alpha(s,t)} \rho_s^2(x, y)\right),$$

where

$$\alpha(s,t) := \int_s^t \exp\left(2\int_s^r K(u)\,du\right) dr.$$

We first extend such kind of dimension-free Harnack inequality to that for the conjugate semigroup.

Theorem 4.2. *Suppose that ϱ is bounded below, $|d\varrho_t| \le \kappa(t)$, and*

$$\mathrm{Ric}_t + \mathrm{Sc}_t + \mathrm{Hess}_t(\phi_t) \ge K(t).$$

The following two Harnack-type inequalities hold for any $p > 1$:

(i) *For* $0 \le s \le t < T$ *and any nonnegative function* $f \in \mathcal{B}_b(M)$,

$$(P^{\varrho}_{s,t} f)^p(x) \le (P^{\varrho}_{s,t} f^p)(y) \exp\left((p-1)\int_s^t \sup \varrho_r^- \, dr + \frac{p\rho_s^2(x,y)}{4(p-1)\alpha(s,t)} + \frac{p\eta(s,t)\rho_s(x,y)}{\alpha(s,t)}\right), \quad (4\text{-}1)$$

where

$$\alpha(s,t) := \int_s^t \exp\left(2\int_s^r K(u)\,du\right) dr,$$
$$\eta(s,t) := \int_s^t \int_s^v \kappa(r) \exp\left(2\int_s^v K(u)\,du - \int_s^r K(u)\,du\right) dr\,dv.$$

(ii) *For* $0 \le s \le t < T$ *and any nonnegative function* $f \in \mathcal{B}_b(M)$,

$$(P^{\varrho}_{s,t} f)^p(x) \le (P^{\varrho}_{s,t} f^p)(y)\, \mathbb{E}^y\left[\exp\left(-(p-1)\int_s^t \varrho_r(X_r)\,dr\right)\right] \exp\left(\frac{p\rho_s^2(x,y)}{4(p-1)\alpha(s,t)} + \frac{2p\eta(s,t)}{\alpha(s,t)}\rho_s(x,y)\right).$$

Proof. To facilitate the notion we restrict ourselves to the case $s = 0$. By approximation and the monotone class theorem, we may assume that $f \in C^2(M)$ has compact support and $\inf_M f > 0$. Given $x \ne y$ and $t > 0$, let $\gamma : [0,t] \to M$ be the constant-speed g_0-geodesics from x to y of length $\rho_0(x,y)$. Let $\nu_s = d\gamma_s/ds$. Thus we have $|\nu_s|_0 = \rho_0(x,y)/t$. Let

$$h(s) = t\,\frac{\int_0^s \exp\left(2\int_0^r K(u)\,du\right) dr}{\int_0^t \exp\left(2\int_0^r K(u)\,du\right) dr}.$$

Then $h(0) = 0$ and $h(t) = t$. Let $y_s = \gamma_{h(s)}$ and

$$\begin{aligned}\varphi(s) &= \log \mathbb{E}^{y_s}\left(\exp\left(-\int_0^s \varrho_r(X_r)\,dr\right) P^{\varrho}_{s,t} f(X_s)\right)^p \\ &= \log P^{p\varrho}_{0,s}(P^{\varrho}_{s,t} f)^p(y_s), \quad s \in [0,t].\end{aligned}$$

To determine $\varphi'(s)$, we first note that

$$\begin{aligned}d(P^{\varrho}_{s,t} f(X_s))^p &= dM_s + (L_s + \partial_s)(P^{\varrho}_{s,t} f)^p(X_s)\,ds \\ &= dM_s + p(p-1)(P^{\varrho}_{s,t} f)^{p-2}(X_s)\,|\nabla^s P^{\varrho}_{s,t} f|_s^2(X_s)\,ds + p\varrho_s(X_s)(P^{\varrho}_{s,t} f)^p(X_s)\,ds, \quad s \le \zeta_k,\end{aligned}$$

where M_s is the local martingale part of $(P^{\varrho}_{s,t} f)^p(X_s)$. This implies

$$\begin{aligned}&\mathbb{E}^x\left[\left(\exp\left(-\int_0^{s\wedge\zeta_k} \varrho_r(X_r)\,dr\right) P^{\varrho}_{s\wedge\zeta_k,t} f(X_{s\wedge\zeta_k})\right)^p\right] - (P^{\varrho}_{0,t} f)^p(x) \\ &\qquad = p(p-1)\mathbb{E}^x\left[\int_0^{s\wedge\zeta_k} \exp\left(-p\int_0^u \varrho_r(X_r)\,dr\right)(P^{\varrho}_{u,t} f)^{p-2}(X_u)|\nabla^u P^{\varrho}_{u,t} f|_u^2(X_u)\,du\right].\end{aligned}$$

Since $\inf_M f > 0$, by letting $k \to \infty$, we deduce that

$$\begin{aligned}&\mathbb{E}^x\left[\left(\exp\left(-\int_0^s \varrho_r(X_r)\,dr\right) P^{\varrho}_{s,t} f(X_s)\right)^p\right] - (P^{\varrho}_{0,t} f)^p(x) \\ &\qquad = p(p-1)\int_0^s \mathbb{E}^x\left[\exp\left(-p\int_0^u \varrho_r(X_r)\,dr\right)(P^{\varrho}_{u,t} f)^{p-2}|\nabla^u P^{\varrho}_{u,t} f|_u^2(X_u)\right] du. \quad (4\text{-}2)\end{aligned}$$

Hence, for each $x \in M$,

$$\frac{\partial}{\partial s}\mathbb{E}^x\bigg(\exp\Big(-\int_0^s \varrho_r(X_r)\,dr\Big)P_{s,t}^{\varrho} f(X_s)\bigg)^p$$
$$= p(p-1)\mathbb{E}^x\bigg[\exp\Big(-p\int_0^s \varrho_r(X_r)\,dr\Big)(P_{s,t}^{\varrho} f)^{p-2}|\nabla^s P_{s,t}^{\varrho} f|_s^2(X_s)\bigg].$$

Moreover, by adapting Corollary 3.2 for $P_{s,t}^{p\varrho}$, i.e.,

$$|\nabla^0 P_{0,s}^{p\varrho} f|_0 \le \exp\Big(-\int_0^s K(r)\,dr\Big)P_{0,s}^{p\varrho}|\nabla^s f|_s + pP_{0,s}^{p\varrho} f\int_0^s \kappa(r)\exp\Big(-\int_0^r K(u)\,du\Big)\,dr,$$

we thus obtain, for $s \in [0,t]$,

$$\begin{aligned}
\frac{d\varphi(s)}{ds} &= \bigg(\frac{1}{P_{0,s}^{p\varrho}(P_{s,t}^{\varrho} f)^p}\Big\{p(p-1)P_{0,s}^{p\varrho}((P_{s,t}^{\varrho} f)^p\,|\nabla^s \log P_{s,t}^{\varrho} f|_s^2) + h'(s)\langle\nabla^0 P_{0,s}^{p\varrho}(P_{s,t}^{\varrho} f)^p, \nu_s\rangle_0\Big\}\bigg)(y_s)\\
&\ge \bigg(\frac{p}{P_{0,s}^{p\varrho}(P_{s,t}^{\varrho} f)^p}\Big\{(p-1)P_{0,s}^{p\varrho}((P_{s,t}^{\varrho} f)^{p-2}\,|\nabla^s P_{s,t}^{\varrho} f|_s^2)\\
&\qquad - \frac{\rho_0(x,y)}{t}\exp\Big(-\int_0^s K(u)\,du\Big)h'(s)P_{0,s}^{p\varrho}((P_{s,t}^{\varrho} f)^{p-1}|\nabla^s P_{s,t}^{\varrho} f|_s)\\
&\qquad - \frac{\rho_0(x,y)}{t}h'(s)\,P_{0,s}^{p\varrho}\,(P_{s,t}^{\varrho} f)^p\int_0^s \kappa(r)\exp\Big(-\int_0^r K(u)\,du\Big)\,dr\Big\}\bigg)(y_s)\\
&\ge \bigg(\frac{p}{P_{0,s}^{p\varrho}(P_{s,t}^{\varrho} f)^p}P_{0,s}^{p\varrho}\Big\{(P_{s,t}^{\varrho} f)^p\Big((p-1)|\nabla^s \log P_{s,t}^{\varrho} f|_s^2\\
&\qquad - \frac{\rho_0(x,y)}{t}h'(s)\exp\Big(-\int_0^s K(u)\,du\Big)|\nabla^s \log P_{s,t}^{\varrho} f|_s\\
&\qquad - \frac{\rho_0(x,y)}{t}h'(s)\int_0^s \kappa(r)\exp\Big(-\int_0^r K(u)\,du\Big)\,dr\Big)\Big\}\bigg)(y_s)\\
&\ge \frac{-ph'(s)^2}{4(p-1)t^2}\exp\Big(-2\int_0^s K(u)\,du\Big)\rho_0(x,y)^2 - \frac{p}{t}h'(s)\int_0^s \kappa(r)\exp\Big(-\int_0^r K(u)\,du\Big)\,dr\,\rho_0(x,y),
\end{aligned}$$

where the last inequality follows from the simple fact that

$$aA^2 + bA \ge -\frac{b^2}{4a}, \quad a > 0.$$

Since

$$h'(s) = \frac{t\exp\big(2\int_0^s K(u)\,du\big)}{\int_0^t \exp\big(2\int_0^r K(u)\,du\big)\,dr},$$

we thus arrive at

$$\begin{aligned}
\frac{d\varphi(s)}{ds} &\ge -\frac{p\exp\big(\int_0^s 2K(u)\,du\big)}{4(p-1)\big(\int_0^t \exp\big(2\int_0^r K(u)\,du\big)\,dr\big)^2}\,\rho_0(x,y)^2\\
&\qquad - \frac{p\exp\big(2\int_0^s K(u)\,du\big)\int_0^s \kappa(r)\exp\big(-\int_0^r K(u)\,du\big)\,dr}{\int_0^t \exp\big(2\int_0^r K(u)\,du\big)\,dr}\,\rho_0(x,y), \quad s \in [0,t].
\end{aligned}$$

Integrating over s from 0 and t, we get

$$\begin{aligned}
(P^{\varrho}_{0,t}f)^p(x) &\le \mathbb{E}^y\bigg[\bigg(\exp\bigg(-\int_0^t \varrho_r(X_r)\,dr\bigg)f(X_t)\bigg)^p\bigg] \\
&\qquad \times \exp\bigg(\frac{p\rho_0(x,y)^2}{4(p-1)\int_0^t \exp\big(2\int_0^r K(u)\,du\big)\,dr} \\
&\qquad\qquad + \frac{p\int_0^t\int_0^s \kappa(r)\exp\big(2\int_0^s K(u)\,du - \int_0^r K(u)\,du\big)\,dr\,ds}{\int_0^t \exp\big(2\int_0^r K(u)\,du\big)\,dr}\,\rho_0(x,y)\bigg) \\
&\le \mathbb{E}^y\bigg[\exp\bigg(-\int_0^t \varrho_r(X_r)\,dr\bigg)f(X_t)^p\bigg]\exp\bigg((p-1)\int_0^t \sup \varrho_r^-\,dr\bigg) \\
&\qquad \times \exp\bigg(\frac{p\rho_0(x,y)^2}{4(p-1)\int_0^t \exp\big(2\int_0^r K(u)\,du\big)\,dr} \\
&\qquad\qquad + \frac{p\int_0^t\int_0^s \kappa(r)\exp\big(2\int_0^s K(u)\,du - \int_0^r K(u)\,du\big)\,dr\,ds}{\int_0^t \exp\big(2\int_0^r K(u)\,du\big)\,dr}\,\rho_0(x,y)\bigg).
\end{aligned} \tag{4-3}$$

This proves part (i) of the theorem. In addition, by adopting in (4-3) the estimate

$$\mathbb{E}^y\bigg[\bigg(\exp\bigg(-\int_0^t \varrho_r(X_r)\,dr\bigg)f(X_t)\bigg)^p\bigg] \le (P^{\varrho}_{0,t}f^p)(y)\,\mathbb{E}^y\bigg[\exp\bigg(-(p-1)\int_0^t \varrho_r(X_r)\,dr\bigg)\bigg],$$

part (ii) of the theorem follows as well. □

5. On-diagonal heat kernel estimates

Let p be the fundamental solution of $L_t = \Delta_t - \nabla^t\phi_t$ in the sense that

$$\begin{cases} \partial_t p(t,x;s,y) = (\Delta_t - \nabla^t\phi_t)p(t,\cdot\,;s,y)(x), \\ \lim_{t\to s} p(t,x;s,y) = \delta_y(x), \end{cases}$$

where $t > s$. Let p^* be the conjugate heat kernel of p; it is the density of $P^{\varrho}_{s,t}(x,dy)$ with respect to $\mu_t(dy)$, i.e.,

$$P^{\varrho}_{s,t}f(x) = \int p^*(s,x;t,y)f(y)\,\mu_t(dy) = \int p^*(s,x;t,y)f(y)\mathrm{e}^{-\phi_t(y)}\,\mathrm{vol}_t(dy),$$

where vol_t denotes the volume measure with respect to g_t. In [Cheng 2017] the following Harnack inequality for the 2-parameter semigroup $P_{s,t}$ was derived; it can be seen as a special case of Theorem 4.1.

It is interesting to observe that Theorem 4.1 for $P_{s,t}$, along with the Harnack inequality (4-1) for $P^{\varrho}_{s,t}$, will allow us to attain on-diagonal upper bounds for the heat kernel p:

$$p(t,x;s,x) \le \frac{C}{\sqrt{\mu_t(B_t(x,\sqrt{t-s}))\,\mu_s(B_s(x,\sqrt{t-s}))}}$$

for $0 \le s < t < T$ and $x \in M$.

Theorem 5.1. *Suppose that* $\sup \varrho_u^- < \infty$ *for* $u \in [0,T]$. *Let* $0 \le s < t < T$ *and there exists* $r_0 \in (s,t)$ *such that*, *for* $u \in [r_0,t]$,

$$\mathrm{Ric}_u - \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \ge K_1(u),$$

and, *for* $u \in [s, r_0]$,

$$\mathrm{Ric}_u + \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \geq K_2(u) \quad \textit{and} \quad |d\varrho_u| \leq \kappa(u) < \infty.$$

Then the following heat kernel upper bound holds:

$$p(t, x; s, x) \leq \exp\left(\frac{1}{2}\int_s^t \sup \varrho_u^- \, du + \frac{t-s}{4\alpha_1(r_0, t)} + \frac{t - s + 2\eta_2(s, r_0)\sqrt{t-s}}{4\alpha_2(s, r_0)}\right) \times \frac{1}{\sqrt{\mu_t(B_t(x, \sqrt{t-s}))\mu_s(B_s(x, \sqrt{t-s}))}},$$

where

$$\alpha_1(r_0, t) := \int_{r_0}^t \exp\left(2\int_v^t K_1(u)\, du\right) dv, \tag{5-1}$$

$$\alpha_2(s, r_0) := \int_s^{r_0} \exp\left(2\int_s^v K_2(u)\, du\right) dv, \tag{5-2}$$

$$\eta_2(s, r_0) := \int_s^{r_0}\int_s^v \kappa(t) \exp\left(2\int_s^v K_2(u)\, du - \int_s^t K_2(u)\, du\right) dt\, dv. \tag{5-3}$$

Proof. We first observe that

$$\begin{aligned} p(t, x; s, x) &= \int_M p(t, x; r, z)\, p(r, z; s, x)\, \mu_r(dz) \\ &\leq \left(\int_M p(t, x; r, z)^2 \mu_r(dz)\right)^{1/2} \left(\int_M p(r, z; s, x)^2 \mu_r(dz)\right)^{1/2}. \end{aligned}$$

Hence we are left with the task to estimate the two terms

$$I_1 = \int_M p(t, x; r, z)^2\, \mu_r(dz), \quad I_2 = \int_M p(r, z; s, x)^2\, \mu_r(dz).$$

In order to estimate I_1 we proceed with Theorem 4.1. Let $\bar{p}(s, x; u, y) := p(t-s, x; t-u, y)$ for $0 \leq s \leq u \leq t$. Then

$$\partial_s \bar{p}(\,\cdot\,, x; u, y)(s) = -L_{t-s}\bar{p}(s, \cdot\,; u, y)(x).$$

Write $\bar{P}_{s,u} f = \int \bar{p}(s, x; u, y) f(y)\, \mu_{t-s}(dy)$ for all $f \in \mathcal{B}_b(M)$. As

$$\mathrm{Ric}_u - \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \geq K_1(u), \quad u \in [r_0, t],$$

for some $K_1 \in C([r_0, t])$, we obtain that, for $t > r \geq r_0$,

$$\begin{aligned} (\bar{P}_{0,t-r} f)^2(x)\, \mu_t(B_t(x, \sqrt{t-s}))\, \exp\left(-\frac{t-s}{2\int_r^t \exp\left(2\int_v^t K_1(u)\, du\right) dv}\right) \\ \leq \int_M (\bar{P}_{0,t-r} f)^2(x) \exp\left(-\frac{\rho_t^2(x, y)}{2\int_r^t \exp(2\int_v^t K_1(u)\, du)\, dv}\right) \mu_t(dy) \\ \leq \int_M (\bar{P}_{0,t-r} f^2)(y)\, \mu_t(dy) \\ \leq \exp\left(\int_r^t \sup \varrho^-(u, \cdot\,)\, du\right) \int_M f(y)^2\, \mu_r(dy). \end{aligned}$$

Taking

$$f(y) := (k \wedge p(t,x;r,z)), \quad z \in M,\ k \in \mathbb{N},$$

we obtain

$$\int_M (k \wedge p(t,x;r,z))^2\, \mu_r(dy) \le \exp\biggl(\int_r^t \sup \varrho^-(u,\cdot)\, du + \frac{t-s}{2\alpha_1(r,t)}\biggr) \frac{1}{\mu_t(B_t(x,\sqrt{t-s}))},$$

where

$$\alpha_1(r,t) = \int_r^t \exp\biggl(2\int_v^t K_1(u)\, du\biggr)\, dv.$$

Letting $k \to \infty$, we arrive at

$$I_1 = \int_M p(t,x;r,z)^2\, \mu_r(dz) \le \exp\biggl(\int_r^t \sup \varrho^-(u,\cdot)\, du + \frac{t-s}{2\alpha_1(r,t)}\biggr) \frac{1}{\mu_t(B_t(x,\sqrt{t-s}))}.$$

To estimate the second term I_2, we write

$$\int_M p(r,z;s,x)^2\, \mu_r(dz) = \int_M p^*(s,x;r,z)^2\, \mu_r(dz),$$

where p^* denotes the adjoint heat kernel to p. Recall that

$$\begin{cases} \partial_s p^*(s,x;r,z) = -L_s p^*(s,\cdot\,;r,z)(x) + \varrho_s(x) p^*(s,y;r,z), \\ \lim_{s\to r} p^*(s,x;r,z) = \delta_x(z). \end{cases}$$

Let $\{P^{\varrho}_{s,r}\}_{0\le s\le r\le t}$ be the semigroup generated by the operator $L_t - \varrho_t$. By (4-1), this time relying on the assumption

$$\mathrm{Ric}_u + \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \ge K_2(u), \quad u \in [s, r_0],$$

we have that, for $s < r \le r_0$,

$$(P^{\varrho}_{s,r} f)^2(x) \le (P^{\varrho}_{s,r} f^2)(y) \exp\left(\int_s^r \sup \varrho^-(u,\cdot)\, du + \frac{\rho_s^2(x,y)}{2\alpha_2(s,r)} + \frac{2\eta_2(s,r)\rho_s(x,y)}{\alpha_2(s,r)}\right),$$

where

$$\alpha_2(s,r) = \int_s^r \exp\biggl(2\int_s^v K_2(u)\, du\biggr)\, dv,$$
$$\eta_2(s,r) = \int_s^r\int_s^v \kappa(u) \exp\biggl(2\int_s^v K_2(t)\, dt - \int_s^u K_2(t)\, dt\biggr)\, du\, dv.$$

By means of this formula, we can proceed as above to obtain

$$\begin{aligned} (P^{\varrho}_{s,r} f)^2(x)\, \mu_s(B_s(x,\sqrt{t-s})) &\exp\left(-\int_s^r \sup \varrho^-(u,\cdot)\, du - \frac{t-s}{2\alpha_2(s,r)} - \frac{2\eta_2(s,r)\sqrt{t-s}}{\alpha_2(s,r)}\right) \\ &\le \int_M (P^{\varrho}_{s,r} f)^2(x) \exp\left(-\int_s^r \sup \varrho^-(u,\cdot)\, du - \frac{\rho_s^2(x,y)}{2\alpha_2(s,r)} - \frac{2\eta_2(s,r)\rho_s(x,y)}{\alpha_2(s,r)}\right) \mu_s(dy) \\ &\le \int_M (P^{\varrho}_{s,r} f^2)(y)\, \mu_s(dy) = \int_M f(y)^2\, \mu_r(dy). \end{aligned}$$

Thus taking $f(z) := p^*(s,x;r,z) \wedge k$ and letting $k \to \infty$, we obtain

$$\int_M p^*(s,x;r,z)^2\,\mu_r(dz) \le \exp\biggl(\int_s^r \sup\varrho^-(u,\cdot\,)\,du + \frac{t-s+2\eta_2(s,r)\sqrt{t-s}}{2\alpha_2(s,r)}\biggr)\frac{1}{\mu_s(B_s(x,\sqrt{t-s}))}. \tag{5-4}$$

Finally, combining (5-4) we obtain

$$\begin{aligned} p(t,x;s,x) &\le \sqrt{I_1 I_2} \\ &\le \exp\biggl(\frac{1}{2}\int_s^t \sup\varrho^-(u,\cdot\,)\,du + \frac{t-s}{4\alpha_1(r,t)} + \frac{t-s+2\eta_2(s,r)\sqrt{t-s}}{4\alpha_2(s,r)}\biggr) \\ &\qquad\qquad \times \frac{1}{\sqrt{\mu_s(B_s(x,\sqrt{t-s}))\,\mu_t(B_t(x,\sqrt{t-s}))}}, \end{aligned}$$

where the functions $\alpha_1, \alpha_2, \eta_2$ are defined by (5-1). □

Remark 5.2. (1) In [Coulibaly-Pasquier 2019], the author used a horizontal coupling of curves to derive a dimension-free Harnack inequality for the two-parameter semigroup $P_{s,t}$ generated by Δ_t, a method first used by Wang [2011; 2014], and applied it then to establish an upper bound for the heat kernel. A major difference to our approach when $\phi = 0$ is that we use the Harnack inequality again to deal with the term I_2, while in [Coulibaly-Pasquier 2019] a comparison result for $P_{s,t}$ and $P^{\varrho}_{s,t}$ is used so that the conditions there include both upper and lower bounds on ϱ. On the other hand, in [Coulibaly-Pasquier 2019], the middle time $(t+s)/2$ is used as reference time, so that the conditions require ϱ to be bounded and $\mathrm{Ric}_t + \mathrm{Sc}_t$ to have a lower bound on the whole time interval. However the reference time r can be chosen close to s so that if the manifold M is compact, and if $|d\varrho_s| \le \kappa$ and $\mathrm{Ric}_s + \mathrm{Sc}_s + \mathrm{Hess}_s(\phi_s) \ge K_2$ at the initial time s, then for small $\epsilon > 0$ there exists $\delta > 0$ such that, for $u \in [s, s+\delta]$,

$$\mathrm{Ric}_u + \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \ge K_2 - \epsilon \quad \text{and} \quad |d\varrho_u| \le \kappa + \epsilon.$$

Therefore, the coefficients of the upper bound depend on $g(s)$, the lower bound of $\mathrm{Ric}_u - \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u)$, $u \in\,]s,t]$ and $\sup\varrho_u^- < \infty$, $u \in [s,t]$.

(2) Gaussian upper bounds for the heat kernel on evolving manifolds have recently also been obtained by Buzano and Yudowitz [2020]. In their paper, they assume that for each vector field X on M the following tensor inequality holds true:

$$0 \le D(\mathrm{Sc}, X) := \frac{\partial}{\partial t}S - \Delta S - 2|S_{ij}|^2 + 4(\nabla_i S_{ij})X_j - 2(\nabla_j S)X_j + 2R_{ij}X_iX_j - 2S_{ij}X_iX_j.$$

Our approach via dimension-free Harnack inequalities enables us to relax such type of conditions.

We now exemplify our estimates in some specific situations. First we consider the modified geometric flow for g_t combined with the conjugate heat equation for ϕ, i.e.,

$$\begin{cases} \partial_t g(x,t) = -2(\mathrm{Sc} + \mathrm{Hess}(\phi))(x,t), \\ \partial_t \phi_t(x) = -\Delta_t \phi_t(x) - S(x,t). \end{cases} \tag{5-5}$$

The following result holds for system (5-5).

Corollary 5.3. *Suppose that* (g_t, ϕ_t) *evolve by* (5-5) *and that, for* $0 \le s \le t \le T$, *there exists* $r \in [s, t]$ *such that*

$$\mathrm{Ric}_u - \mathrm{Sc}_u \ge K_1(u), \quad u \in [r, t], \; \mathrm{Ric}_u + \mathrm{Sc}_u + 2\,\mathrm{Hess}_u(\phi_u) \ge K_2(u), \quad u \in [s, r].$$

Then

$$p(t, x; s, x) \le \exp\left(\frac{t-s}{4\alpha_1(r,t)} + \frac{t-s}{4\alpha_2(s,r)}\right) \frac{1}{\sqrt{\mu_s(B_s(x, \sqrt{t-s}))\mu_t(B_t(x, \sqrt{t-s}))}},$$

where

$$\alpha_1(r,t) := \int_r^t \exp\left(2\int_v^t K_1(u)\,du\right) dv,$$
$$\alpha_2(s,r) := \int_s^r \exp\left(2\int_s^v K_2(u)\,du\right) dv.$$

Proof. It is immediate from the definition of ϱ that

$$\varrho_t = \partial_t \phi_t + \mathrm{tr}_{g_t}(\mathrm{Sc}_t + \mathrm{Hess}_t(\phi_t)) = \partial_t\phi_t + \Delta_t\phi_t + S_t = 0.$$

The proof is hence completed by applying Theorem 5.1. □

In particular, we may consider the standard geometric flow for the evolution of the metric g, i.e.,

$$\begin{cases} \partial_t g(t) = -2\,\mathrm{Sc}(t), \\ \phi_t(x) = 0. \end{cases} \tag{5-6}$$

For this geometric flow, we have $\varrho = S$ and thus obtain the following result.

Corollary 5.4. *Suppose that* g_t *evolves by* (5-6) *and* $\sup S_t^- < \infty$ *for each* $t \in [0, T[$. *For* $0 < s < t < T$, *there exists* $r \in (s, t)$ *such that*

$$\mathrm{Ric}_u - \mathrm{Sc}_u \ge K_1(u), \quad u \in [r, t],$$
$$|dS_u| \le \kappa(u) < \infty, \quad \mathrm{Ric}_u + \mathrm{Sc}_u \ge K_2(u), \quad u \in [s, r].$$

Then

$$p(t, x; s, x) \le \exp\left(\frac{1}{2}\int_s^t \sup S_u^-\,du + \frac{t-s}{4\alpha_1(r,t)} + \frac{t-s+2\eta_2(s,r)\sqrt{t-s}}{4\alpha_2(s,r)}\right) \frac{1}{\sqrt{\mu_s(B_s(x, \sqrt{t-s}))\mu_t(B_t(x, \sqrt{t-s}))}},$$

where

$$\alpha_1(r,t) := \int_r^t \exp\left(2\int_r^v K_1(u)\,du\right) dv,$$
$$\alpha_2(s,r) := \int_s^r \exp\left(2\int_s^v K_2(u)\,du\right) dv,$$
$$\eta_2(s,r) := \int_s^r \int_s^v \kappa(t) \exp\left(2\int_s^v K_2(u)\,du - \int_s^t K_2(u)\,du\right) dt\,dv.$$

Proof. The result follows from Theorem 5.1 by taking $\mathrm{Hess}(\phi) \equiv 0$ and $\varrho = S$. □

In particular, if we consider the heat kernel estimate under the compact Ricci flow, i.e., g_t evolving by (5-6) with $\mathrm{Sc} = \mathrm{Ric}$, then

$$K_1(t) = 0, \quad \sup R_t^-(\cdot) \le \sup R_s^-(\cdot) < \infty.$$

Hence if we further know that $\mathrm{Ric}_s \geq K$ and $|dR_s| < \kappa$ at time s, then there exists a constant C depending on K, κ, $\sup R^-$ and s, t such that

$$p(t, x; s, x) \leq \frac{C(K, \kappa, \sup R^-, s, t)}{\sqrt{\mu_s(B_s(x, \sqrt{t-s}))\, \mu_t(B_t(x, \sqrt{t-s}))}}.$$

6. Gaussian-type heat kernel estimates

In this section, we apply the dimension-free Harnack inequality to heat kernel estimates. To this end, we need the following lemma, which has been extended from [Grigoryan 1997]. Compared with the results of Section 5 above, the additional condition "$\mathrm{Sc} \geq 0$" is required.

Lemma 6.1. *For $x \in M,\ T_0 > 0,\ p > 1,\ q = p/(2(p-1))$, let*

$$\eta(s, y) = -\frac{\rho_s(x, y)^2}{T_0 - 2q(t-s)}, \quad y \in M,\ 0 < s < t < \frac{T_0}{2q}.$$

If $\mathrm{Sc}_t \geq 0$, then, for any $f \in \mathcal{B}_b^+(M)$,

$$\int_M (P_{s,t}^{\varrho/p} f)^p(y) \mathrm{e}^{\eta(s,y)}\, \mu_s(dy) \leq \int_M f^p(y) \mathrm{e}^{-\rho_t(x,y)^2/T_0}\, \mu_t(dy), \quad s < t < \frac{T_0}{2q}. \tag{6-1}$$

Proof. Let

$$I(s) = \int_M (P_{s,t}^{\varrho/p} f(y))^p \exp(\eta(s, y))\, \mu_s(dy).$$

We first take the derivative of the function η with respect to s,

$$\begin{aligned} \partial_s \eta(\,\cdot\,, y)(s) &= -\frac{2\rho_s(x, y)\partial_s \rho_s(x, y)}{T_0 - 2q(t-s)} + 2q\frac{\rho_s(x, y)^2}{(T_0 - 2q(t-s))^2} \\ &= \frac{2\rho_s(x, y)\int_0^{\rho_s(x,y)} \mathrm{Sc}_s(\dot\gamma_u, \dot\gamma_u)\, du}{T_0 - 2q(t-s)} + 2q\frac{\rho_s(x, y)^2}{(T_0 - 2q(t-s))^2} \geq 2q\frac{\rho_s(x, y)^2}{(T_0 - 2q(t-s))^2}, \end{aligned}$$

where $\gamma : [0, \rho_s(x, y)] \to M$ is a g_s-geodesic connecting x and y. Then we have

$$\begin{aligned} I'(s) &= \int_M p(P_{s,t}^{\varrho/p} f)^{p-1}(y)\Big(-L_s + \frac{\varrho_s}{p}\Big) P_{s,t}^{\varrho/p} f(y) \exp(\eta(s, y))\, d\mu_s \\ &\qquad + \int_M (P_{s,t}^{\varrho/p} f)^p(y) \exp(\eta(s, y))\partial_s \eta(s, y)\, d\mu_s - \int_M (P_{s,t}^{\varrho/p} f)^p(y) \exp(\eta(s, y))\, \varrho_s(y)\, d\mu_s \\ &= -\int_M p(P_{s,t}^{\varrho/p} f)^{p-1} L_s P_{s,t}^{\varrho/p}(y) f \exp(\eta(s, y))\, d\mu_s + \int_M (P_{s,t}^{\varrho/p} f)^p(y) \exp(\eta(s, y))\, \partial_s \eta(s, y)\, d\mu_s \\ &\geq -\int_M p(P_{s,t}^{\varrho/p} f)^{p-1} L_s P_{s,t}^{\varrho/p} f \exp(\eta(s, y))\, d\mu_s \\ &\qquad\qquad + 2q\int_M (P_{s,t}^{\varrho/p} f)^p(y) \exp(\eta(s, y))\frac{\rho_s(x, y)^2}{(T_0 - 2q(t-s))^2}\, d\mu_s \\ &= \int p(p-1)(P_{s,t}^{\varrho/p} f)^p \mathrm{e}^{\eta(s,\cdot)}\bigg(\frac{|\nabla^s P_{s,t}^{\rho/p} f|_s}{P_{s,t}^{\rho/p} f} - \frac{\rho_s(x, y)}{(p-1)(T_0 - 2q(t-s))}\bigg)^2 d\mu_s \geq 0. \end{aligned}$$

By integrating the function I' from s to t, we prove inequality (6-1). □

Theorem 6.2. *Let $p(t,x;s,y)$ be the minimal fundamental solution of*

$$\begin{cases}\partial_t p(t,x;s,y) = L_t p(t,\cdot;s,y)(x),\\ \lim_{t\downarrow s} p(t,x;s,y) = \delta(y).\end{cases}$$

Assume that $\mathrm{Sc}_t \geq 0$ *and* $\sup \varrho_t^- < \infty$ *for* $t \in [0,T]$. *If there exists* $r \in (s,t)$ *such that*

$$\mathrm{Ric}_u - \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \geq K_1(u), \quad u \in [r,t],$$
$$\mathrm{Ric}_u + \mathrm{Sc}_u + \mathrm{Hess}_t \geq K_2(u) \quad \text{and} \quad |d\varrho_u| \leq \kappa(u), \quad u \in [s,r],$$

for some functions $K_1 \in C([r,t])$ *and* $\kappa, K_2 \in C([s,r])$, *then, for any* $\delta > 2$ *and* $s < r \leq t \leq T$,

$$p(t,x;s,y) \leq \frac{C}{\sqrt{\mu_s(B_s(x,\sqrt{t-s}))}\sqrt{\mu_t(B_t(y,\sqrt{t-s}))}} \exp\left(-\frac{\rho_s(x,y)^2}{2\delta(t-s)}\right),$$

where

$$C = \exp\left\{\frac{1}{2}\int_s^t \sup \varrho^-(u,\cdot)\,du + \frac{p(t-s)}{4(2-p)}\left(\frac{1}{\alpha_1(r,t)} + \frac{1}{\alpha_2(s,r)}\right) + \frac{\eta_2(s,r)}{\alpha_2(s,r)} + \frac{1}{2(\delta-2q)}\right\}$$

for $p \in (1+1/(\delta-1), 2)$ *and* $\alpha_1, \alpha_2, \eta_2$ *as in* (5-1).

Proof. Since $\mathrm{Sc}_t \geq 0$, we have $\rho_s \leq \rho_r$ for $r \geq s$. We observe that

$$\begin{aligned}
&p(t,x;s,y)\mathrm{e}^{\rho_s(x,y)^2/(4T_0)}\\
&\qquad \leq p(t,x;s,y)\mathrm{e}^{\rho_r(x,y)^2/(4T_0)}\\
&\qquad \leq \int_M p(t,x;r,z)p(r,z;s,y)\mathrm{e}^{2(\rho_r(x,z)^2+\rho_r(z,y)^2)/(4T_0)}\,\mu_r(dz)\\
&\qquad \leq \left(\int_M p(r,x;r,z)\mathrm{e}^{\rho_r(x,z)^2/T_0}\,\mu_r(dz)\right)^{1/2}\left(\int_M p(t,z;s,y)^2\mathrm{e}^{\rho_r(z,y)^2/T_0}\,\mu_r(dz)\right)^{1/2}, \qquad (6\text{-}2)
\end{aligned}$$

where $T_0 = \delta(t-s)$. Hence we are left with the task to estimate the two terms

$$I_1 = \int_M p(t,x;r,z)^2\mathrm{e}^{\rho_r(x,z)^2/T_0}\,\mu_r(dz),$$
$$I_2 = \int_M p(r,z;s,y)^2\mathrm{e}^{\rho_r(z,y)^2/T_0}\,\mu_r(dz).$$

In order to estimate I_1 we proceed with Theorem 5.1. Recall that by definition

$$\bar{p}(s,x;u,y) = p(t-s,x;t-u,y) \quad \text{for } 0 \leq s \leq u \leq t < T,$$

and write

$$\bar{P}_{s,u} f = \int \bar{p}(s,x;u,y) f(y)\,\mu_{t-s}(dy) \quad \text{for } f \in \mathcal{B}_b(M).$$

As

$$\mathrm{Ric}_u - \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \geq K_1(u), \quad u \in [r,t],$$

for some $K_1 \in C([r,t])$ and for $p \in (1,2)$ such that $q := p/[2(p-1)] < \delta/2$, we have

$$\begin{aligned}
&(\bar{P}_{0,t-r} f)^2(x)\, \mu_t(B_t(x,\sqrt{t-s})) \exp\biggl(-\frac{p(t-s)}{2(2-p)\int_r^t \exp\bigl(2\int_v^t K_1(u)\,du\bigr)\,dv} - \frac{1}{\delta-2q}\biggr)\\
&\qquad \le \int_M (\bar{P}_{0,t-r} f)^2(x) \exp\biggl(-\frac{p\rho_t^2(x,y)}{2(2-p)\int_r^t \exp\bigl(2\int_v^t K_1(u)\,du\bigr)\,dv} - \frac{\rho_t(x,y)^2}{T_0 - 2q(t-s)}\biggr)\,\mu_t(dy)\\
&\qquad \le \int_M (\bar{P}_{0,t-r} f^{2/p})^p(y) \exp\biggl(-\frac{\rho_t(x,y)^2}{T_0-2q(t-s)}\biggr)\,\mu_t(dy)\\
&\qquad \le \exp\biggl(\int_r^t \sup \varrho^-(u,\cdot\,)\,du\biggr) \int_M f(y)^2 \mathrm{e}^{-\rho_r(x,y)^2/T_0}\,\mu_r(dy),
\end{aligned}$$

where the second inequality comes from the dimension-free Harnack inequality (see Theorem 4.1); the third inequality is a consequence of Lemma 6.1. Taking

$$f(y) := (k \wedge p(t,x;r,y))\mathrm{e}^{(k\wedge\rho_r(x,y)^2)/T_0}, \quad y \in M,\ k \in \mathbb{N},$$

we obtain

$$\begin{aligned}
&\int_M (k \wedge p(t,x;r,y))^2 \mathrm{e}^{k\wedge\rho_r(x,y)^2/T_0}\,\mu_r(dy)\\
&\qquad\qquad \le \exp\biggl(\int_r^t \sup\varrho^-(u,\cdot\,)\,du + \frac{p(t-s)}{2(2-p)\alpha_1(r,t)} + \frac{1}{\delta-2q}\biggr)\frac{1}{\mu_t(B_t(x,\sqrt{t-s}))},
\end{aligned}$$

where

$$\alpha_1(r,t) = \int_r^t \exp\biggl(2\int_v^t K_1(u)\,du\biggr)\,dv.$$

Letting $k \to \infty$, we arrive at

$$\begin{aligned}
I_1 &= \int_M p(t,x;r,z)^2 \mathrm{e}^{\rho_r(x,y)^2/(\delta(t-s))}\,\mu_r(dz)\\
&\le \exp\biggl(\int_r^t \sup\varrho^-(u,\cdot\,)\,du + \frac{p(t-s)}{2(2-p)\alpha_1(r,t)} + \frac{1}{\delta-2q}\biggr)\frac{1}{\mu_t(B_t(x,\sqrt{t-s}))}. \qquad (6\text{-}3)
\end{aligned}$$

To estimate the second term I_2, we write

$$\int_M p(r,z;s,y)^2\,\mu_r(dz) = \int_M p^*(s,y;r,z)^2\,\mu_r(dz),$$

where p^* denotes the adjoint heat kernel to p. Recall that

$$\begin{cases} \partial_s p^*(s,y;r,z) = -L_s p^*(s,\cdot\,;r,z)(y) + \varrho_s(y) p^*(s,y;r,z),\\ \lim_{s\uparrow r} p^*(s,y;r,z) = \delta_y(x). \end{cases}$$

Let $\{P_{s,r}^{\varrho}\}_{0\le s\le r\le t}$ be the semigroup generated by the operator $L_t - \varrho_t$. By Theorem 4.1, using the assumption

$$\mathrm{Ric}_t + \mathrm{Sc}_t + \mathrm{Hess}_t(\phi_t) \ge K_2(t) \quad \text{and} \quad |d\varrho_t| \le \kappa(t)$$

for $t \in (s, r_0)$, we have

$$(P^{\varrho/p}_{s,r} f)^2(x) \\ \leq (P^{\rho/p}_{s,r} f^{2/p})^p(y) \exp\biggl(\int_s^r \frac{(2-p)}{p} \sup \varrho^-(u,\cdot)\,du + \frac{p\rho_s^2(x,y)}{2(2-p)\alpha_2(s,r)} + \frac{2\eta_2(s,r)\rho_s(x,y)}{\alpha_2(s,r)}\biggr),$$

where $s \leq r \leq r_0$ and

$$\alpha_2(s,r) = \int_s^r \exp\biggl(2\int_s^v K_2(u)\,du\biggr)\,dv,$$
$$\eta_2(s,r) = \int_s^r \int_s^v \kappa(u) \exp\biggl(2\int_s^v K_2(t)\,dt - \int_s^u K_2(t)\,dt\biggr)\,du\,dv.$$

By means of this formula, we can proceed as above to obtain

$$(P^{\varrho/p}_{s,r} f)^2(x)\mu_s(B_s(x,\sqrt{t-s}))\exp\biggl(-\int_s^r \sup\varrho^-(u,\cdot)\,du - \frac{p(t-s)}{2(2-p)\alpha_2(s,r)} - \frac{2\eta_2(s,r)\sqrt{t-s}}{\alpha_2(s,r)} - \frac{1}{\delta-2q}\biggr) \\ \leq \int_M (P^{\varrho/p}_{s,r} f^{2/p})^p(x)\exp\biggl(-\frac{\rho_s(x,y)^2}{T_0-2q(t-s)}\biggr)\mu_s(dy) \\ \leq \int_M f(y)^2 \exp\biggl(-\frac{\rho_r(x,y)^2}{T_0}\biggr)\mu_r(dy), \tag{6-4}$$

where $T_0 = \delta(t-s)$. Thus taking $f(z) := p^*(s,y;r,z) \wedge k$ and letting $k \to \infty$, we obtain

$$I_2 = \int_M p^*(s,y;r,z)^2 \mathrm{e}^{\rho_r(x,y)^2/(\delta(r-s))}\,\mu_r(dz) \\ \leq \exp\biggl(\int_s^r \sup\varrho^-(u,\cdot)\,du + \frac{p(t-s)}{2(2-p)\alpha_2(s,r)} + \frac{2\eta_2(s,r)\sqrt{t-s}}{\alpha_2(s,r)} + \frac{1}{\delta-2q}\biggr)\frac{1}{\mu_s(B_s(y,\sqrt{t-s}))}.$$

Finally, combining (6-3) and (6-4) we obtain

$$p(t,x;s,y) \leq \sqrt{I_1 I_2} \\ \leq \exp\biggl(\frac{1}{2}\int_s^t \sup\varrho^-(u,\cdot)\,du + \frac{p(t-s)}{4(2-p)}\biggl(\frac{1}{\alpha_1(r,t)} + \frac{1}{\alpha_2(s,r)}\biggr) + \frac{\eta_2(s,r)\sqrt{t-s}}{\alpha_2(s,r)} + \frac{1}{2(\delta-2q)}\biggr) \\ \times \frac{1}{\sqrt{\mu_s(B_s(x,\sqrt{t-s}))\mu_t(B_t(y,\sqrt{t-s}))}}\exp\biggl\{-\frac{\rho_s(x,y)^2}{\delta(t-s)}\biggr\},$$

where the functions $\alpha_1, \alpha_2, \eta_2$ are defined by (6-2). □

Let M be a compact manifold. Since $\mathrm{Ric} + \mathrm{Sc} + \mathrm{Hess}(\phi)$ and $|d\varrho|$ are continuous in time and space, we may choose r in Theorem 6.2 close to s such that if $\mathrm{Ric} + \mathrm{Sc} + \mathrm{Hess}(\phi)$ has a lower bound and $|d\varrho|$ an upper bound at time s, then these terms will also have a lower, resp. upper, bound close to s. We state a consequence of this observation in the following corollary.

Corollary 6.3. *Let M be a compact manifold. Assume $\mathrm{Sc}_u \geq 0$ and $\sup \varrho_u^- < \infty$ for $u \in [0,T]$, as well as $\mathrm{Ric}_u - \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \geq K_1(u)$ for some continuous function $K_1 \in C([0,T])$. For $s \in [0,T[$, if there* *exist constants K_2 and κ such that*

$$\mathrm{Ric}_s + \mathrm{Sc}_s + \mathrm{Hess}_s(\phi_s) \geq K_2 \quad \text{and} \quad |d\varrho_s| \leq \kappa,$$

then, for any $\delta > 2$ and $s \leq t \leq T$, there exists a constant C such that

$$p(t, x; s, y) \leq C \exp\left\{-\frac{\rho_s(x, y)^2}{2\delta(t-s)}\right\} \frac{1}{\sqrt{\mu_s(B(x, \sqrt{t-s}))}\sqrt{\mu_t(B(y, \sqrt{t-s}))}},$$

where C depends on K_1, K_2, κ, $\sup_{[0,T]\times M} \varrho^-$ and s, t.

Proof. As there exist constants K_2 and κ such that

$$\mathrm{Ric}_s + \mathrm{Sc}_s + \mathrm{Hess}_s(\phi_s) \geq K_2 \quad \text{and} \quad |d\varrho_s| \leq \kappa,$$

we conclude that, for any $\epsilon > 0$, there exists $r_0 > s$ such that for $u \in [s, r_0]$ we still have

$$\mathrm{Ric}_u + \mathrm{Sc}_u + \mathrm{Hess}_u(\phi_u) \geq K_2 - \epsilon \quad \text{and} \quad |d\varrho_u| \leq \kappa + \epsilon.$$

Then by means of Theorem 6.2 we can then complete the proof. □

Remark 6.4. (1) Suppose that the geometric flow g_t is evolving as a compact Ricci flow, i.e., the manifold is compact, $\mathrm{Sc}_t = \mathrm{Ric}_t$ and $\mathrm{Ric}_t \geq 0$ for $t \in [0, T[$. We consider the estimate for the heat kernel $p(t, x; s, y)$ generated by Δ_t. Hence if $\mathrm{Ric}_s \geq K$ and $|dR_s| \geq \kappa$ at the initial time s, then, for $T \geq t \geq s$, there exists a constant C depending on K, κ, $\sup R(s, \cdot)$ and s, t such that

$$p(t, x; s, y) \leq C \exp\left\{-\frac{\rho_s(x, y)^2}{2\delta(t-s)}\right\} \frac{1}{\sqrt{\mu_s(B_s(x, \sqrt{t-s}))}\sqrt{\mu_t(B_t(y, \sqrt{t-s}))}}.$$

(2) Theorem 6.2 and Corollary 6.3 can be applied to the modified geometric flow (5-5) and the standard geometric flow (5-6) as well.

7. Super log-Sobolev inequalities and conjugate semigroup properties

The semigroup $P^{\varrho}_{s,t}$ is called supercontractive if it maps $L^p(M, \mu_t)$ into $L^q(M, \mu_s)$, i.e.,

$$\|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)} < \infty$$

for any $1 < p < q < +\infty$ and $0 \leq s \leq t < T$. In the following section, we investigate the relation between supercontractivity of $P^{\varrho}_{s,t}$ and a log-Sobolev inequality with respect to μ_t, which is viewed as another important application of the derivative formula of the conjugate semigroup.

We state first the main results of this section. Thanks to the gradient estimate for $P^{\varrho}_{s,t}$ and the fact that the family of measures $\{\mu_t\}$ takes over the role of the invariant measure, the results can be proved much as in [Wang 2005, Chapter 5] and [Röckner and Wang 2003]. We include proofs in the Appendix for the reader's convenience.

Theorem 7.1. *Assume that ϱ_t is bounded and $(\partial_t - L_t)\varrho_t \geq 0$,*

$$\mathrm{Ric}_t + \mathrm{Hess}_t(\phi_t) + \mathrm{Sc}_t \geq K(t), \quad |d\varrho_t| \leq \kappa(t) \quad \text{for } t \in [0, T[.$$

If

$$\|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)} < \infty \quad \textit{for } 1 < p < q \textit{ and } 0 \le s \le t < T,$$

then, for every $f \in H^1(M, \mu_t)$ *such that* $\|f\|_{2,t} = 1$, $t \in [0, T[$, *the following super log-Sobolev inequalities hold*:

$$\int f^2 \log f^2 \, d\mu_t \le r \int \big(|\nabla^t f|_t^2 + \tfrac{1}{4}\varrho_t f^2\big)\, d\mu_t + \beta_t(r), \quad r > 0, \tag{7-1}$$

where $\beta_t(r) = \tilde{\beta}_t(\gamma_t^{-1}(r), t)$ *and*

$$\begin{aligned}\tilde{\beta}_t(s,t) &= \frac{pq}{q-p}\log\Big(\|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)}\Big) + 2\int_s^t \bigg(\int_r^t \kappa(u)\exp\Big(-\int_r^u K(v)\,dv\Big)\,du\bigg)^2 dr,\\ \gamma(s,t) &= \frac{4p(q-1)}{q-p}\int_s^t \exp\Big(-2\int_r^t K(u)\,du\Big)\,dr,\\ \gamma_t^{-1}(r) &= \inf\{s \in [0,t] : \gamma(s,t) \le r\}.\end{aligned}$$

Remark 7.2. (i) The log-Sobolev inequality (7-1) has been shown to be equivalent to the Sobolev inequality and can hence be used to obtain an upper bound for the heat kernel; see, e.g., [Zhang 2006] for the situation under Ricci flow, and [Buzano and Yudowitz 2020] for a general geometric flow. Moreover, the log-Sobolev inequality can be used to characterize supercontractivity of the conjugate semigroup with respect to the volume measure. Note that in [Buzano and Yudowitz 2020], the authors start with the condition that $D(\mathrm{Sc}, X) \ge 0$, which implies $(\partial_t - \Delta_t)S_t \ge 0$ and has been used in the proof of the log-Sobolev inequality. We follow a different approach but include the condition $(\partial_t - L_t)\varrho_t \ge 0$ to obtain the log-Sobolev inequality.

(ii) In Theorem 7.1 information about $\mathrm{Ric} + \mathrm{Hess}(\phi) + \mathrm{Sc}$ and $|d\varrho|$ on the time interval $[s, t]$ is used to obtain the log-Sobolev inequality (7-1) with respect to the measure μ_t. By a time reversal as in the proof of Theorem 5.1, the lower bound on $\mathrm{Ric} + \mathrm{Hess}(\phi) - \mathrm{Sc}$ and the bound on $|d\varrho|$ allow us to get the log-Sobolev inequality with a modified β_t.

In addition, we observe that the inequality above has a close relationship with supercontractivity of the conjugate semigroup as follows.

Theorem 7.3. *Suppose that there exists a function* $\beta_t : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\mu_t(f^2 \log f^2) \le r\mu_t\big(|\nabla^t f|_t^2 + \tfrac{1}{4}\varrho_t f^2\big) + \beta_t(r) \quad \textit{for all } t \in [0, T[\textit{ and } \|f\|_{2,t} = 1. \tag{7-2}$$

Then

$$\|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)} < \infty \quad \textit{for } 1 < p < q \textit{ and } 0 \le s \le t < T.$$

Proof. The main idea of proof is from [Röckner and Wang 2003]; we include it in the Appendix for the reader's convenience. □

As an application, we specify the general results above in the case of the modified geometric flow (M, g_t, ϕ_t) evolving by

$$\begin{cases}\partial_t g(x,\cdot)(t) = -2(\mathrm{Sc} + \mathrm{Hess}(\phi))(x,t),\\ \partial_t \phi_t = -\Delta_t \phi_t - S_t.\end{cases} \tag{7-3}$$

For this system, we have $\varrho \equiv 0$. Thus, by Theorem 7.1 with $\varrho \equiv 0$, we obtain the following result.

Corollary 7.4. *Assume that* (g_t, ϕ_t) *follows the evolving equation* (7-3) *and*

$$\mathrm{Ric}_t + \mathrm{Hess}_t(\phi_t) + \mathrm{Sc}_t \geq K(t) \quad \textit{for all } t \in [0, T[.$$

Suppose that

$$\|P_{s,t}\|_{(p,t)\to(q,s)} < \infty \quad \textit{for } 1 < p < q \textit{ and } 0 \leq s \leq t < T.$$

Then the super log-Sobolev inequalities

$$\int f^2 \log f^2 \, d\mu_t \leq r \int |\nabla^t f|_t^2 \, d\mu_t + \beta_t(r), \quad r > 0, \tag{7-4}$$

hold for every $f \in H^1(M, \mu_t)$ *such that* $\|f\|_{2,t} = 1,\ t \in [0, T[$, *where* $\beta_t(r) := \tilde{\beta}(\gamma_t^{-1}(r), t)$ *and*

$$\begin{aligned}
\tilde{\beta}(s,t) &= \frac{pq}{q-p} \log(\|P_{s,t}\|_{(p,t)\to(q,s)}),\\
\gamma(s,t) &= \frac{4p(q-1)}{q-p} \int_s^t \exp\left(-2\int_r^t K(u)\, du\right) dr,\\
\gamma_t^{-1}(r) &= \inf\{s \geq 0 : \gamma(s,t) \leq r\}.
\end{aligned}$$

Finally, we are in position to give a necessary and sufficient condition for supercontractivity of $P_{s,t}^{\varrho}$. By the dimension-free Harnack inequality and by using the method from [Röckner and Wang 2003] (see the proof in the Appendix), we have the following result.

Theorem 7.5. *Suppose that* ϱ_t *is bounded from below and*

$$\mathrm{Ric}_t + \mathrm{Hess}_t(\phi_t) + \mathrm{Sc}_t \geq K(t), \quad |d\varrho_t| \leq \kappa(t) \quad \textit{for } t \in [0, T[.$$

Then the condition

$$\|P_{s,t}^{\varrho}\|_{(p,t)\to(q,s)} < \infty \quad \textit{for all } 1 < p < q < \infty \textit{ and } 0 \leq s \leq t < T$$

holds if and only if

$$\mu_t(\exp(\lambda\rho_t^2)) < \infty \quad \textit{for all } \lambda > 0 \textit{ and } t \in [0, T[.$$

Applied to the modified geometric flow ($\varrho \equiv 0$), we have the following corollary from Theorem 7.5.

Corollary 7.6. *Assume that* (g_t, ϕ_t) *evolves according to* (7-3) *and that*

$$\mathrm{Ric}_t + \mathrm{Hess}_t(\phi_t) + \mathrm{Sc}_t \geq K(t), \quad t \in [0, T[,$$

for some function $K \in C([0, T[)$. *Then*

$$\|P_{s,t}^{\varrho}\|_{(p,t)\to(q,s)} < \infty \quad \textit{for } 1 < p < q < \infty \textit{ and } 0 \leq s \leq t < T$$

if and only if

$$\mu_t(\exp(\lambda\rho_t^2)) < \infty \quad \textit{for } \lambda > 0 \textit{ and } t \in [0, T[.$$

Appendix

To prove Theorem 7.1, we first establish a log-Sobolev inequality with respect to the semigroup $P^{\varrho}_{s,t}$.

Proposition A.1. *Assume that*

$$\mathrm{Ric}_t + \mathrm{Hess}_t(\phi_t) + \mathrm{Sc}_t \geq K(t), \quad \sup|\varrho_t| < \infty \quad \textit{and} \quad |d\varrho_t| \leq \kappa(t) \quad \textit{for } t \in [0, T[.$$

Then, for $0 \leq s \leq t < T$ *and* $f \in C_0^\infty(M)$,

$$P^{\varrho}_{s,t}(f^2 \log f^2) \leq 4\left(\int_s^t \exp\left(-2\int_r^t K(u)\,du\right) dr\right) P^{\varrho}_{s,t}|\nabla^t f|_t^2 + P^{\varrho}_{s,t} f^2 \log P^{\varrho}_{s,t} f^2 + 2\int_s^t \left(\int_r^t \kappa(u) \exp\left(-\int_r^u K(v)\,dv\right) du\right)^2 dr\, P^{\varrho}_{s,t} f^2 + \int_s^t P_{s,r}(\varrho_r P_{r,t} f^2)\,dr.$$

Proof. Without loss of generality, we assume that $f \geq \delta$ for some $\delta > 0$. Otherwise, we may take $f_\delta = (f^2+\delta)^{1/2}$ and pass to the limit $\delta \downarrow 0$ to obtain the conclusion.

Now consider the process

$$r \mapsto (P_{r,t} f^2) \log(P_{r,t} f^2)(X_{r\wedge\tau_k}),$$

where as above

$$\tau_k = \inf\{t \in [s, T[\,: \rho_t(o, X_t) \geq k\}, \quad k \geq 1. \tag{A-1}$$

By means of Itô's formula, we have

$$\begin{aligned} d(P^{\varrho}_{r,t} f^2) \log(P^{\varrho}_{r,t} f^2)(X_r) &= dM_r + (L_r + \partial_r)(P^{\varrho}_{r,t} f^2 \log P^{\varrho}_{r,t} f^2)(X_r)\,dr \\ &= dM_r + \left(\frac{1}{P^{\varrho}_{r,t} f^2} |\nabla^r P^{\varrho}_{r,t} f^2|_r^2 + \varrho_r(1 + \log P^{\varrho}_{r,t} f^2) P^{\varrho}_{r,t} f^2\right)(X_r)\,dr, \quad r \leq \tau_k \wedge t, \end{aligned} \tag{A-2}$$

where M_r is a local martingale. On the other hand, by Corollary 3.2, we have the estimate

$$\begin{aligned} |\nabla^r P^{\varrho}_{r,t} f^2|_r &\leq \exp\left(-\int_r^t K(u)\,du\right) P^{\varrho}_{r,t}|\nabla^t f^2|_t + P^{\varrho}_{r,t} f^2 \int_r^t \kappa(u) \exp\left(-\int_r^u K(v)\,dv\right) du \\ &\leq 2\exp\left(-\int_r^t K(u)\,du\right) P^{\varrho}_{r,t}(f|\nabla^t f|_t) + P^{\varrho}_{r,t} f^2 \int_r^t \kappa(u) \exp\left(-\int_r^u K(v)\,dv\right) du \\ &\leq 2\exp\left(-\int_r^t K(u)\,du\right) \sqrt{P^{\varrho}_{r,t}(f^2) P^{\varrho}_{r,t}(|\nabla^t f|_t^2)} + P^{\varrho}_{r,t} f^2 \int_r^t \kappa(u) \exp\left(-\int_r^u K(v)\,dv\right) du, \end{aligned}$$

which gives

$$|\nabla^r P^{\varrho}_{r,t} f^2|_r^2 \leq 4\exp\left(-2\int_r^t K(u)\,du\right) (P^{\varrho}_{r,t} f^2) P^{\varrho}_{r,t}(|\nabla^t f|_t^2) + 2(P^{\varrho}_{r,t} f^2)^2 \left(\int_r^t \kappa(u) \exp\left(-\int_r^u K(v)\,dv\right) du\right)^2.$$

Substituting back into (A-2), we obtain

$$
\begin{aligned}
d\bigg(\exp\Big(-\int_s^r \varrho_u(X_u)\,du\Big)(P^{\varrho}_{r,t}f^2)\log(P^{\varrho}_{r,t}f^2)(X_r)\bigg)&\\
\le dM_r+4\exp\Big(-2\int_r^t K(u)\,du-\int_s^r \varrho_u(X_u)\,du\Big)P^{\varrho}_{r,t}|\nabla^t f|_t^2(X_r)\,dr&\\
+2\bigg(\int_r^t \kappa(u)\exp\Big(-\int_r^u K(v)\,dv\Big)\,du\bigg)^2\exp\Big(-\int_s^r \varrho_u(X_u)\,du\Big)P^{\varrho}_{r,t}f^2(X_r)\,dr&\\
+\varrho_r(X_r)\exp\Big(-\int_s^r \varrho_u(X_u)\,du\Big)P^{\varrho}_{r,t}f^2(X_r)\,dr,\quad 0\le s\le r\le \tau_k\wedge t.&
\end{aligned}
$$

Integrating both sides from s to $t\wedge\tau_k$, taking the expectation, and letting $k\uparrow+\infty$, we obtain by dominated convergence

$$
\begin{aligned}
P^{\varrho}_{s,t}(f^2\log f^2)-P^{\varrho}_{s,t}f^2\log(P^{\varrho}_{s,t}f^2)&\\
\le 4\int_s^t \exp\Big(-2\int_r^t K(u)\,du\Big)\,dr\;P^{\varrho}_{s,t}|\nabla^t f|_t^2&\\
+\int_s^t\bigg\{2\bigg(\int_r^t \kappa(u)\exp\Big(-\int_r^u K(v)\,dv\Big)\,du\bigg)^2\bigg\}\,dr\;P^{\varrho}_{s,t}f^2+\int_s^t P^{\varrho}_{s,r}(\varrho_r P^{\varrho}_{r,t}f^2)\,dr,&
\end{aligned}
$$

or in other words,

$$
\begin{aligned}
P^{\varrho}_{s,t}(f^2\log f^2)\le 4\bigg(\int_s^t \exp\Big(-2\int_r^t K(u)\,du\Big)\,dr\bigg)P^{\varrho}_{s,t}|\nabla^t f|_t^2+P^{\varrho}_{s,t}f^2\log P^{\varrho}_{s,t}f^2&\\
+\int_s^t\bigg\{2\bigg(\int_r^t \kappa(u)\exp\Big(-\int_r^u K(v)\,dv\Big)\,du\bigg)^2\bigg\}\,dr\;P^{\varrho}_{s,t}f^2+\int_s^t P^{\varrho}_{s,r}(\varrho_r P^{\varrho}_{r,t}f^2)\,dr,\quad \text{(A-3)}&
\end{aligned}
$$

completing the proof. □

Proof of Theorem 7.1. Since

$$
\log^+(P^{\varrho}_{s,t}f^2)\le P^{\varrho}_{s,t}f^2\le \exp\bigg(\int_s^t \sup\varrho_u^-\,du\bigg)\|f\|_\infty^2,
$$

we can integrate both sides of the log-Sobolev inequality (A-3) with respect to μ_s. Taking (1-3) into account, we get

$$
\begin{aligned}
\mu_t(f^2\log f^2)\le 4\bigg(\int_s^t \exp\Big(-2\int_r^t K(u)\,du\Big)\,dr\bigg)\mu_t(|\nabla^t f|_t^2)+\mu_s(P^{\varrho}_{s,t}f^2\log P^{\varrho}_{s,t}f^2)&\\
+\int_s^t\bigg\{2\bigg(\int_r^t \kappa(u)\exp\Big(-\int_r^u K(v)\,dv\Big)\,du\bigg)^2\bigg\}dr\;\mu_t(f^2)+\int_s^t \mu_r(\varrho_r P^{\varrho}_{r,t}f^2)\,dr.\quad \text{(A-4)}&
\end{aligned}
$$

For the last term above, we have

$$
\begin{aligned}
\partial_r(\mu_r(\varrho_r P^{\varrho}_{r,t}f^2))&=\mu_r(-\varrho_r^2 P^{\varrho}_{r,t}f^2)+\mu_r((\partial_r\varrho_r)P^{\varrho}_{r,t}f^2)-\mu_r(\varrho_r(L_r-\varrho_r)P^{\varrho}_{r,t}f^2)\\
&=\mu_r((\partial_r\varrho_r-L_r\varrho_r)P^{\varrho}_{r,t}f^2)\ge 0.
\end{aligned}
$$

It then follows that

$$\mu_r(\varrho_r P^{\varrho}_{r,t} f^2) \le \mu_t(\varrho_t f^2).$$

Moreover,

$$\mu_t(f^2 \log f^2) \le 4\left(\int_s^t \exp\left(-2\int_r^t K(u)\,du\right) \vee 1\,dr\right)\mu_t\left(|\nabla^t f|_t^2 + \tfrac{1}{4}\varrho_t f^2\right) + \mu_s(P^{\varrho}_{s,t} f^2 \log P^{\varrho}_{s,t} f^2)$$
$$+ \int_s^t \left\{2\left(\int_r^t \kappa(u) \exp\left(-\int_r^u K(v)\,dv\right) du\right)^2\right\} dr\, \mu_t(f^2).$$

We deal first with the term $\mu_s(P^{\varrho}_{s,t} f^2 \log P^{\varrho}_{s,t} f^2)$. Let $1 < p < q$. For any $h \in]0, 1 - 1/p[$ let

$$r_h = \frac{ph}{p-1} \in]0, 1[.$$

By the Riesz–Thorin interpolation theorem, we have

$$\|P^{\varrho}_{s,t} f\|_{q_h,s} \le \|P^{\varrho}_{s,t}\|^{r_h}_{(p,t)\to(q,s)} \, \|P^{\varrho}_{s,t}\|^{1-r_h}_{(1,t)\to(1,s)} \, \|f\|_{p_h,t}, \quad f \in L^p(M, \mu_s), \tag{A-5}$$

where

$$\frac{1}{p_h} = \frac{1-r_h}{1} + \frac{r_h}{p} \quad \text{and} \quad \frac{1}{q_h} = \frac{1-r_h}{1} + \frac{r_h}{q}.$$

Thus

$$p_h = \frac{1}{1-h} \quad \text{and} \quad q_h = \left(1 - \frac{p(q-1)}{q(p-1)} h\right)^{-1}.$$

Since $\|P^{\varrho}_{s,t}\|_{(1,t)\to(1,s)} \le 1$, we get from (A-5) that

$$\int (P^{\varrho}_{s,t} |f|^{2(1-h)})^{q_h}\, d\mu_s \le \|P^{\varrho}_{s,t}\|^{r_h q_h}_{(p,t)\to(q,s)} \, \|f\|^{q_h/p_h}_{2,t}.$$

Then, for $f \in C_0^\infty(M)$ satisfying $\|f\|_{2,t} = 1$, we have

$$\frac{1}{h}\left(\int (P^{\varrho}_{s,t} |f|^{2(1-h)})^{q_h}\, d\mu_s - \left(\int P^{\varrho}_{s,t} |f|^2\, d\mu_s\right)^{q_h/p_h}\right) = \frac{1}{h}\left(\int (P^{\varrho}_{s,t} |f|^{2(1-h)})^{q_h}\, d\mu_s - 1\right)$$
$$\le \frac{1}{h}(\|P^{\varrho}_{s,t}\|^{r_h q_h}_{(p,t)\to(q,s)} - 1). \tag{A-6}$$

Taking the limit as $h \to 0$ in (A-6), as

$$\lim_{h\to 0} \frac{1}{h}(\|P^{\varrho}_{s,t}\|^{r_h q_h}_{(p,t)\to(q,s)} - 1) = \frac{p}{p-1} \log \|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)},$$

we obtain by dominated convergence

$$\frac{p(q-1)}{q(p-1)} \int P^{\varrho}_{s,t} f^2 \log P^{\varrho}_{s,t} f^2\, d\mu_s - \int P^{\varrho}_{s,t}(f^2 \log f^2)\, d\mu_s \le \frac{p}{p-1} \log \|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)},$$

or equivalently,

$$\mu_s(P^{\varrho}_{s,t} f^2 \log P^{\varrho}_{s,t} f^2) \le \frac{q(p-1)}{p(q-1)}\, \mu_t(f^2 \log f^2) + \frac{q}{q-1} \log \|P^{\varrho}_{s,t}\|_{(p,t)\to(q,s)}. \tag{A-7}$$

Substituting (A-7) back into (A-4), we arrive at

$$\mu_t(f^2 \log f^2) \le \gamma(s,t)\, \mu_t\left(|\nabla^t f|_t^2 + \tfrac{1}{4}\varrho_t f^2\right) + \tilde{\beta}(s,t) \tag{A-8}$$

for every $f \in C_0^\infty(M)$ satisfying $\|f\|_{2,t} = 1$, where

$$\gamma(s,t) = \frac{4p(q-1)}{q-p} \int_s^t \left[\exp\left(-2\int_r^t K(u)\,du\right) \vee 1\right] dr,$$

$$\tilde{\beta}(s,t) = \frac{pq}{q-p} \log(\|P_{s,t}^{\varrho}\|_{(p,t)\to(q,s)}) + 2\frac{p(q-1)}{q-p} \int_s^t \left(\int_r^t \kappa(u) \exp\left(-\int_r^u K(v)\,dv\right) du\right)^2 dr.$$

We complete the proof by letting

$$\beta_t(r) = \tilde{\beta}(\gamma_t^{-1}(r), t). \qquad \square$$

Proof of Theorem 7.3. Let $0 \le s < t < T$ and $f \in C_0^\infty(M)$ such that $f \ge \delta > 0$. To calculate the derivative of $\mu_s(P_{s,t}^{\varrho} f)^{q(s)}$ with respect to s, we start with some preparatory calculations:

$$\begin{aligned}(L_s + \partial_s)&(P_{s,t}^{\varrho} f)^{q(s)} \\ &= L_s(P_{s,t}^{\varrho} f)^{q(s)} + q(s)\,(P_{s,t}^{\varrho} f)^{q(s)-1}(\partial_s P_{s,t}^{\varrho} f) + q'(s)(P_{s,t}^{\varrho} f)^{q(s)} \log P_{s,t}^{\varrho} f \\ &= q(s)(q(s)-1)(P_{s,t}^{\varrho} f)^{q(s)-2} |\nabla^s P_{s,t}^{\varrho} f|_s^2 + q'(s)(P_{s,t}^{\varrho} f)^{q(s)} \log P_{s,t}^{\varrho} f + q(s)\varrho_s(P_{s,t}^{\varrho} f)^{q(s)}. \end{aligned} \tag{A-9}$$

By Corollary 3.2, there exist positive constants $c_1(s,t)$ and $c_2(s,t)$ such that

$$\||\nabla^s P_{s,t}^{\varrho} f|_s^2\|_\infty \le c_1(s,t)\,\|f\|_\infty^2 + c_2(s,t)\,\||\nabla^t f|_t\|_\infty^2.$$

Moreover, $\|P_{s,t}^{\varrho} f\|_\infty \le (P_{s,t}^{\varrho} 1)\,\|f\|_\infty$ and

$$(P_{s,t}^{\varrho} f)^{q(s)} \log^+(P_{s,t}^{\varrho} f) \le (P_{s,t}^{\varrho} f)^{q(s)+1} \le (P_{s,t}^{\varrho} 1)^{q(s)+1}\,\|f\|_\infty^{q(s)+1}.$$

Combining these estimates, we obtain

$$\|(L_s + \partial_s)(P_{s,t}^{\varrho} f)^{q(s)}\|_\infty < \infty.$$

Now, by Theorem 4.1, we see that

$$\begin{aligned}\frac{d}{ds}\mu_s((P_{s,t}^{\varrho} f)^{q(s)}) &= -\mu_s(\varrho_s(P_{s,t}^{\varrho} f)^{q(s)}) + \mu_s(\partial_s(P_{s,t}^{\varrho} f)^{q(s)}) \\ &= -\mu_s(\varrho_s(P_{s,t}^{\varrho} f)^{q(s)}) + \mu_s((L_s + \partial_s)(P_{s,t}^{\varrho} f)^{q(s)}) \\ &= q(s)(q(s)-1)\,\mu_s(|\nabla^s P_{s,t}^{\varrho} f|_s^2 (P_{s,t}^{\varrho} f)^{q(s)-2}) + q'(s)\,\mu_s((P_{s,t}^{\varrho} f)^{q(s)} \log P_{s,t}^{\varrho} f) \\ &\qquad - (1-q(s))\,\mu_s(\varrho_s(P_{s,t}^{\varrho} f)^{q(s)}).\end{aligned}$$

For $\|P_{s,t} f\|_{q(s),s}$, since $\|P_{s,t}^{\varrho} f\|_{q(s),s}^{1-q(s)} = (\mu_s((P_{s,t}^{\varrho} f)^{q(s)}))^{1/q(s)-1}$, we thus find

$$\begin{aligned}\frac{d}{ds}\|P_{s,t}^{\varrho} f\|_{q(s),s} &= (q(s)-1)\|P_{s,t}^{\varrho} f\|_{q(s),s}^{1-q(s)}\,\mu_s(|\nabla^s P_{s,t}^{\varrho} f|_s^2 (P_{s,t}^{\varrho} f)^{q(s)-2}) \\ &\quad + \frac{q'(s)}{q(s)}\|P_{s,t}^{\varrho} f\|_{q(s),s}^{1-q(s)}\,\mu_s((P_{s,t}^{\varrho} f)^{q(s)} \log P_{s,t}^{\varrho} f) \\ &\quad - \frac{q'(s)}{q(s)}\|P_{s,t}^{\varrho} f\|_{q(s),s} \log \|P_{s,t}^{\varrho} f\|_{q(s),s} \\ &\quad + \frac{q(s)-1}{q(s)}\|P_{s,t}^{\varrho} f\|_{q(s),s}^{1-q(s)}\,\mu_s(\varrho_s(P_{s,t}^{\varrho} f)^{q(s)}).\end{aligned}$$

On the other hand, passing from f to $f^{p/2}/\|f^{p/2}\|_{2,s}$ in the log-Sobolev inequality (7-2), we obtain

$$\int f^p \log\left(\frac{f^p}{\|f^{p/2}\|_{2,s}^2}\right) d\mu_s \le r\frac{p^2}{4}\int f^{p-2}|\nabla^s f|_s^2\, d\mu_s + \frac{r}{4}\int f^p \varrho_s\, d\mu_s + \beta_s(r)\|f^{p/2}\|_{2,s}^2.$$

In this inequality, replacing f and p by $P_{s,t}^{\varrho} f$ and $q(s)$ respectively, we obtain

$$\begin{aligned}\mu_s((P_{s,t}^{\varrho} f)^{q(s)} \log(P_{s,t}^{\varrho} f)) - \|P_{s,t}^{\varrho} f\|_{q(s),s}^{q(s)} \log \|P_{s,t}^{\varrho} f\|_{q(s),s}\\ \le r\frac{q(s)}{4}\int (P_{s,t}^{\varrho} f)^{q(s)-2}|\nabla^s P_{s,t}^{\varrho} f|_s^2\, d\mu_s + \frac{r}{4q(s)}\int f^{q(s)}\varrho_s\, d\mu_s + \frac{\beta_s(r)}{q(s)}\|P_{s,t}^{\varrho} f\|_{q(s),s}^{q(s)}. \quad \text{(A-10)}\end{aligned}$$

Now let

$$q(s) = e^{4r^{-1}(t-s)}(p-1)+1, \quad q(t) = p.$$

Note that q is a decreasing function and $q'(s)r/4 + (q(s)-1) \equiv 0$. Thus, combining (A-10) with (A-10), we arrive at

$$\frac{d}{ds}\|P_{s,t}^{\varrho} f\|_{q(s),s} \ge \frac{\beta_s(r)q'(s)}{q(s)^2}\|P_{s,t}^{\varrho} f\|_{q(s),s}, \quad 0 \le s \le t < T.$$

It follows that

$$\|P_{s,t}^{\varrho} f\|_{q(s),s} \le \exp\left(-\int_s^t \frac{\beta_u(r)q'(u)}{q(u)^2}\, du\right)\|f\|_{p,t}. \quad \text{(A-11)}$$

If we impose that $q(s) = q$, then

$$r = 4(t-s)\left(\log\frac{q-1}{p-1}\right)^{-1}.$$

Substituting the value of r into (A-11) yields

$$\|P_{s,t}^{\varrho} f\|_{q,s} \le \exp\left(-\int_s^t \frac{\beta_u(4(t-s)(\log(q-1)/(p-1))^{-1})\, q'(u)}{q(u)^2}\, du\right)\|f\|_{p,t}. \qquad \square$$

Proof of Theorem 7.5. By means of the Harnack inequality (4-1), the theorem can be proved along the lines of [Wang 2005, Theorem 5.7.3] or [Cheng and Thalmaier 2018b]. For the reader's convenience, we include a proof here. We first prove that if $\mu_s(\exp(\lambda\rho_s^2)) < \infty$ for all $\lambda > 0$, then $P_{s,t}$ is supercontractive, i.e., for any $1 < p < q < \infty$, we have

$$\|P_{s,t}^{\varrho}\|_{(p,t)\to(q,s)} < \infty.$$

Let $p > 1$ and $f \in C_b(M)$. For $0 \le s \le t < T$ it follows from the Harnack inequality (4-1) that

$$|(P_{s,t}^{\varrho} f)^p(x)| \le (P_{s,t}^{\varrho}|f|^p)(y)\exp\left((p-1)\int_s^t \sup\varrho_r^-\, dr + \frac{p\rho_s^2(x,y)}{4(p-1)\alpha(s,t)} + \frac{\eta(s,t)\rho_s(x,y)}{\alpha(s,t)}\right).$$

Thus, if $\mu_t(|f|^p) = 1$, then

$$\begin{aligned}1 &\ge |P_{s,t}^{\varrho} f(x)|^p \int \exp\left((1-p)\int_s^t \sup\varrho_r^-\, dr - \frac{p\rho_s^2(x,y)}{4(p-1)\alpha(s,t)} - \frac{\eta(s,t)\rho_s(x,y)}{\alpha(s,t)}\right)\mu_s(dy)\\ &\ge |P_{s,t}^{\varrho} f(x)|^p\, \mu_s(B_s(o,R))\exp\left((1-p)\int_s^t \sup\varrho_r^-\, dr - \frac{p(\rho_s(x)+R)^2}{4(p-1)\alpha(s,t)} - \frac{\eta(s,t)(\rho_s(x)+R)}{\alpha(s,t)}\right), \quad \text{(A-12)}\end{aligned}$$

where $B_s(o, R) = \{y \in M : \rho_s(y) \le R\}$ denotes the geodesic ball (with respect to the metric $g(s)$) of radius R about $o \in M$ and where $\rho_t(\cdot) = \rho_t(o, \cdot)$. Since $\mu_t(\exp(\lambda\rho_t^2)) < \infty$, the system of measures (μ_s) is compact, i.e., there exists $R = R(s) > 0$, possibly depending on s, such that

$$\mu_s(B_s(o, R(s))) = \mu_s(\{x : \rho_s(x) \le R(s)\}) \ge 1 - \frac{\mu_s(\rho_s^2)}{R(s)^2} \ge 2^{-p}$$

(after normalizing μ_s to a probability measure). Combining the last estimate with (A-12), we arrive at

$$1 \ge |P^{\varrho}_{s,t} f(x)|^p\, 2^{-p} \exp\left((1-p)\int_s^t \sup \varrho_r^-\, dr - \frac{\eta(s,t)(\rho_s(x)+R)}{\alpha(s,t)} - \frac{p(\rho_s(x)+R)^2}{4(p-1)\alpha(s,t)}\right),$$

which further implies

$$|P^{\varrho}_{s,t} f(x)| \le 2\exp\left(\frac{p-1}{p}\int_s^t \sup \varrho_r^-\, dr + \frac{\eta(s,t)(\rho_s(x)+R)}{p\alpha(s,t)} + \frac{(\rho_s(x)+R)^2}{4(p-1)\alpha(s,t)}\right), \quad s < t. \tag{A-13}$$

Therefore, we achieve

$$\|P^{\varrho}_{s,t} f\|_{q,s} \le \big(\mu_s(\exp(q(c_1 + c_2\rho_s^2)))\big)^{1/q}$$

for some positive constants c_1, c_2 depending on s and t. Hence, if $\mu_s(\exp(\lambda\rho_s^2)) < \infty$ for any $\lambda > 0$ and $s \in [0, T[$, then $P_{s,t}$ is supercontractive.

Conversely, if the semigroup $P^{\varrho}_{s,t}$ is supercontractive, by Theorem 7.1 the super log-Sobolev inequalities (7-2) holds. We first prove that $\mu_s(\mathrm{e}^{\lambda\rho_s}) < \infty$ for $s \in [0, T[$ and $\lambda > 0$. To this end, let $\rho_{s,k} = \rho_s \wedge k$ and $h_{s,k}(\lambda) = \mu_s(\exp(\lambda\rho_{s,k}))$. Taking $\exp(\lambda\rho_{s,k}/2)$ in the super log-Sobolev inequality (7-1), we obtain

$$\lambda h'_{s,k}(\lambda) - h_{s,k}(\lambda)\log h_{s,k}(\lambda) \le h_{s,k}(\lambda)\lambda^2\left(\frac{r}{4} + \frac{\beta_s(r)}{\lambda^2}\right).$$

This implies

$$\left(\frac{1}{\lambda}\log h_{s,k}(\lambda)\right)' = \frac{\lambda h'_{s,k}(\lambda) - h_{s,k}(\lambda)\log h_{s,k}(\lambda)}{\lambda^2 h_{s,k}(\lambda)} \le \frac{r}{4} + \frac{\beta_s(r)}{\lambda^2}. \tag{A-14}$$

Integrating both sides of (A-14) from λ to 2λ, we obtain

$$h_{s,k}(2\lambda) \le h^2_{s,k}(\lambda)\exp\left(\frac{r}{2}\lambda^2 + \beta_s(r)\right). \tag{A-15}$$

From this inequality, along with the fact that there exists a constant M_s such that

$$\mu_s(\{\lambda\rho_s \ge M_s\}) \le \frac{1}{4}\exp\left(-\frac{r}{2}\lambda^2 - \beta_s(r)\right),$$

we get

$$\begin{aligned}
h_{s,k}(\lambda) &= \int_{\{\lambda\rho_s \ge M_s\}} \exp(\lambda\rho_{s,k})\, d\mu_s + \int_{\{\lambda\rho_s < M_s\}} \exp(\lambda\rho_{s,k})\, d\mu_s \\
&\le \mu_s(\{\lambda\rho_s \ge M_s\})^{1/2}\, \mu_s(\mathrm{e}^{2\lambda\rho_{s,k}})^{1/2} + \mathrm{e}^{M_s}\mu_s(\{\lambda\rho_s < M_s\}) \\
&\le \left(\frac{1}{4}\exp\left(-\frac{r}{2}\lambda^2 - \beta_s(r)\right)\right)^{1/2} \exp\left(\frac{r}{4}\lambda^2 + \frac{1}{2}\beta_s(r)\right) h_{s,k}(\lambda) + \mathrm{e}^{M_s}\mu_s(\{\lambda\rho_s < M_s\}) \\
&\le \frac{1}{2}h_{s,k}(\lambda) + \mathrm{e}^{M_s}\mu_s(\{\lambda\rho_s < M_s\}),
\end{aligned}$$

which implies $h_{s,k}(\lambda) \le 2e^{M_s}\mu_s(\{\lambda\rho_s < M_s\})$ for $s \in [0, T[$. As M_s is independent of k, letting k tend to infinity, we arrive at

$$\mu_s(e^{\lambda\rho_s}) < \infty \quad \text{for all } s \in [0, T[.$$

To prove that moreover $\mu_s(e^{\lambda\rho_s^2}) < \infty$ for $s \in [0, T[$ and $\lambda > 0$, we can follow the argument in [Cheng and Thalmaier 2018b, pp. 22–23]. □

References

[Abolarinwa 2015] A. Abolarinwa, "Differential Harnack estimates for conjugate heat equation under the Ricci flow", *Asian-Eur. J. Math.* **8**:4 (2015), art. id. 1550063. MR Zbl

[Abolarinwa 2016] A. Abolarinwa, "Sobolev-type inequalities and heat kernel bounds along the geometric flow", *Afr. Mat.* **27**:1-2 (2016), 169–186. MR Zbl

[Bakry 1997] D. Bakry, "On Sobolev and logarithmic Sobolev inequalities for Markov semigroups", pp. 43–75 in *New trends in stochastic analysis* (Charingworth, UK, 1994), edited by K. D. Elworthy et al., World Sci., River Edge, NJ, 1997. MR

[Bismut 1984] J.-M. Bismut, *Large deviations and the Malliavin calculus*, Progr. Math. **45**, Birkhäuser, Boston, 1984. MR Zbl

[Băileşteanu 2012] M. Băileşteanu, "Bounds on the heat kernel under the Ricci flow", *Proc. Amer. Math. Soc.* **140**:2 (2012), 691–700. MR Zbl

[Buzano and Yudowitz 2020] R. Buzano and L. Yudowitz, "Gaussian upper bounds for the heat kernel on evolving manifolds", preprint, 2020. arXiv 2007.07112

[Cao et al. 2015] X. Cao, H. Guo, and H. Tran, "Harnack estimates for conjugate heat kernel on evolving manifolds", *Math. Z.* **281**:1-2 (2015), 201–214. MR Zbl

[Cheng 2017] L.-J. Cheng, "Diffusion semigroup on manifolds with time-dependent metrics", *Forum Math.* **29**:4 (2017), 775–798. MR Zbl

[Cheng and Thalmaier 2018a] L.-J. Cheng and A. Thalmaier, "Characterization of pinched Ricci curvature by functional inequalities", *J. Geom. Anal.* **28**:3 (2018), 2312–2345. MR Zbl

[Cheng and Thalmaier 2018b] L.-J. Cheng and A. Thalmaier, "Evolution systems of measures and semigroup properties on evolving manifolds", *Electron. J. Probab.* **23** (2018), art. id. 20. MR Zbl

[Cheng and Thalmaier 2018c] L.-J. Cheng and A. Thalmaier, "Spectral gap on Riemannian path space over static and evolving manifolds", *J. Funct. Anal.* **274**:4 (2018), 959–984. MR Zbl

[Coulibaly-Pasquier 2019] K. A. Coulibaly-Pasquier, "Heat kernel coupled with geometric flow and Ricci flow", pp. 221–256 in *Séminaire de probabilités L* (Tours, France, 2018), edited by C. Donati-Martin et al., Lecture Notes in Math. **2252**, Springer, 2019. MR Zbl

[Davies 1987] E. B. Davies, "Explicit constants for Gaussian upper bounds on heat kernels", *Amer. J. Math.* **109**:2 (1987), 319–333. MR Zbl

[Elworthy and Li 1994] K. D. Elworthy and X.-M. Li, "Formulae for the derivatives of heat semigroups", *J. Funct. Anal.* **125**:1 (1994), 252–286. MR Zbl

[Grigoryan 1997] A. Grigoryan, "Gaussian upper bounds for the heat kernel on arbitrary manifolds", *J. Differential Geom.* **45**:1 (1997), 33–52. MR Zbl

[Gross 1975] L. Gross, "Logarithmic Sobolev inequalities", *Amer. J. Math.* **97**:4 (1975), 1061–1083. MR Zbl

[Hamilton 1993] R. S. Hamilton, "The Harnack estimate for the Ricci flow", *J. Differential Geom.* **37**:1 (1993), 225–243. MR Zbl

[Kuang and Zhang 2008] S. Kuang and Q. S. Zhang, "A gradient estimate for all positive solutions of the conjugate heat equation under Ricci flow", *J. Funct. Anal.* **255**:4 (2008), 1008–1023. MR Zbl

[Perelman 2002] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications", preprint, 2002. Zbl arXiv math/0211159

[Röckner and Wang 2003] M. Röckner and F.-Y. Wang, "Supercontractivity and ultracontractivity for (non-symmetric) diffusion semigroups on manifolds", *Forum Math.* **15**:6 (2003), 893–921. MR Zbl

[Thalmaier 1997] A. Thalmaier, "On the differentiation of heat semigroups and Poisson integrals", *Stochastics Stochastics Rep.* **61**:3-4 (1997), 297–321. MR Zbl

[Thompson 2019] J. Thompson, "Derivatives of Feynman–Kac semigroups", *J. Theoret. Probab.* **32**:2 (2019), 950–973. MR Zbl

[Wang 2001] F.-Y. Wang, "Logarithmic Sobolev inequalities: conditions and counterexamples", *J. Operator Theory* **46**:1 (2001), 183–197. MR Zbl

[Wang 2005] F.-Y. Wang, *Functional inequalities, Markov semigroups and spectral theory*, Sci. Press, Beijing, 2005.

[Wang 2009] F.-Y. Wang, "Log-Sobolev inequality on non-convex Riemannian manifolds", *Adv. Math.* **222**:5 (2009), 1503–1520. MR Zbl

[Wang 2011] F.-Y. Wang, "Harnack inequality for SDE with multiplicative noise and extension to Neumann semigroup on nonconvex manifolds", *Ann. Probab.* **39**:4 (2011), 1449–1467. MR Zbl

[Wang 2014] F.-Y. Wang, *Analysis for diffusion processes on Riemannian manifolds*, Adv. Ser. Statist. Sci. Appl. Prob. **18**, World Sci., Hackensack, NJ, 2014. MR Zbl

[Zhang 2006] Q. S. Zhang, "Some gradient estimates for the heat equation on domains and for an equation by Perelman", *Int. Math. Res. Not.* **2006** (2006), art. id. 92314. MR Zbl

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