

ANALYSIS & PDE

Volume 16

No. 7

2023

GABRIEL S. KOCH

**PARTIAL REGULARITY FOR NAVIER-STOKES AND
LIQUID CRYSTALS INEQUALITIES WITHOUT MAXIMUM
PRINCIPLE**

PARTIAL REGULARITY FOR NAVIER–STOKES AND LIQUID CRYSTALS INEQUALITIES WITHOUT MAXIMUM PRINCIPLE

GABRIEL S. KOCH

In 1985, V. Scheffer discussed partial regularity results for what he called solutions to the *Navier–Stokes inequality*. These maps essentially satisfy the incompressibility condition as well as the local and global energy inequalities and the pressure equation which may be derived formally from the Navier–Stokes system of equations, but they are not required to satisfy the Navier–Stokes system itself.

We extend this notion to a system considered by Fang-Hua Lin and Chun Liu in the mid 1990s related to models of the flow of nematic liquid crystals, which include the Navier–Stokes system when the director field d is taken to be zero. In addition to an extended Navier–Stokes system, the Lin–Liu model includes a further parabolic system which implies an a priori maximum principle for d which they use to establish partial regularity (specifically, $\mathcal{P}^1(\mathcal{S}) = 0$) of solutions.

For the analogous *inequality* one loses this maximum principle, but here we nonetheless establish the partial regularity result $\mathcal{P}^{9/2+\delta}(\mathcal{S}) = 0$, so that in particular the putative singular set \mathcal{S} has space–time Lebesgue measure zero. Under an additional assumption on d for any fixed value of a certain parameter $\sigma \in (5, 6)$ — which for $\sigma = 6$ reduces precisely to the boundedness of d used by Lin and Liu — we obtain the same partial regularity ($\mathcal{P}^1(\mathcal{S}) = 0$) as do Lin and Liu. In particular, we recover the partial regularity result ($\mathcal{P}^1(\mathcal{S}) = 0$) of Caffarelli–Kohn–Nirenberg [1982] for suitable weak solutions of the Navier–Stokes system, and we verify Scheffer’s assertion that the same holds for solutions of the weaker *inequality* as well.

We remark that the proofs of partial regularity both here and in the work of Lin and Liu largely follow the proof in Caffarelli–Kohn–Nirenberg, which in turn used many ideas from an earlier work of Scheffer [1975].

1. Introduction

Fang-Hua Lin and Chun Liu consider the following system in [Lin and Liu 1995; 1996], which reduces to the classical Navier–Stokes system in the case $d \equiv 0$ (here we have set various parameters equal to one for simplicity):

$$\left. \begin{aligned} u_t - \Delta u + \nabla^T \cdot [u \otimes u + \nabla d \odot \nabla d] + \nabla p &= 0 \\ \nabla \cdot u &= 0 \\ d_t - \Delta d + (u \cdot \nabla)d + f(d) &= 0 \end{aligned} \right\} \quad (1-1)$$

with $f = \nabla F$ for a scalar field F given by

$$F(x) := (|x|^2 - 1)^2,$$

so that

$$f(x) = 4(|x|^2 - 1)x$$

MSC2020: 35A21, 35B65, 35D30, 35G50, 35Q35.

Keywords: Navier–Stokes, liquid crystals, partial regularity.

(and in particular $f(0) = 0$). We take the spatial dimension to be three, so that for some $\Omega \subseteq \mathbb{R}^3$ and $T > 0$, we are considering maps of the form

$$u, d : \Omega \times (0, T) \rightarrow \mathbb{R}^3, \quad p : \Omega \times (0, T) \rightarrow \mathbb{R},$$

and here

$$F : \mathbb{R}^3 \rightarrow \mathbb{R}, \quad f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

are fixed as above. As usual, u represents the velocity vector field of a fluid, p is the scalar pressure in the fluid, and, as in nematic liquid crystals models, d corresponds roughly¹ to the *director field* representing the local orientation of rod-like molecules, with u also giving the velocities of the centers of mass of those anisotropic molecules.

In (1-1), for vector fields v and w , the matrix fields $v \otimes w$ and $\nabla v \odot \nabla w$ are defined to be those with entries

$$(v \otimes w)_{ij} = v_i w_j \quad \text{and} \quad (\nabla v \odot \nabla w)_{ij} = v_{,i} \cdot w_{,j} := \frac{\partial v_k}{\partial x_i} \frac{\partial w_k}{\partial x_j}$$

(summing over the repeated index k as per the Einstein convention), and for a matrix field $J = (J_{ij})$, we define the vector field $\nabla^T \cdot J$ by²

$$(\nabla^T \cdot J)_i := J_{ij,j} := \frac{\partial J_{ij}}{\partial x_j}$$

(summing again over j). We think formally of ∇ (as well as any vector field) as a column vector and ∇^T as a row vector, so that each entry of (the column vector) $\nabla^T \cdot J$ is the divergence of the corresponding *row* of J . In what follows, for a vector field v we similarly denote by $\nabla^T v$ the matrix field with i -th row given by $\nabla^T v_i := (\nabla v_i)^T$, i.e.,

$$(\nabla^T v)_{ij} = v_{i,j} := \frac{\partial v_i}{\partial x_j},$$

so that for smooth vector fields v and w we always have

$$\nabla^T \cdot (v \otimes w) = (\nabla^T v)w + v(\nabla \cdot w) = (w \cdot \nabla)v + v(\nabla \cdot w). \quad (1-2)$$

For a scalar field ϕ we set $\nabla^2 \phi := \nabla^T(\nabla \phi)$, and for matrix fields $J = (J_{ij})$ and $K = (K_{ij})$, we let $J : K := J_{ij} K_{ij}$ (summing over repeated indices) denote the (real) Frobenius inner product of the matrices; that is, $J : K = \text{tr}(J^T K)$. We set $|J| := \sqrt{J : J}$ and $|v| := \sqrt{v \cdot v}$, and to minimize cumbersome notation will often abbreviate by writing $\nabla v := \nabla^T v$ for a vector field v where the precise structure of the *matrix* field $\nabla^T v$ is not crucial; for example, $|\nabla v| := |\nabla^T v|$.

We note that by formally taking the divergence $\nabla \cdot$ of the first line in (1-1) we obtain the usual *pressure equation*

$$-\Delta p = \nabla \cdot (\nabla^T \cdot [u \otimes u + \nabla d \odot \nabla d]). \quad (1-3)$$

¹In principle, for d to only represent a *direction* one should have $|d| \equiv 1$. As proposed in [Lin and Liu 1995], $F(d)$ is used to model a Ginzburg–Landau type of relaxation of the pointwise constraint $|d| \equiv 1$. For further discussions on the modeling assumptions leading to systems such as the one above, see e.g., [Lin and Wang 2014] or the appendix of [Lin and Liu 1995].

²Many authors simply write $\nabla \cdot J$, which is perhaps more standard.

As in the Navier–Stokes ($d \equiv 0$) setting, one may formally deduce (see Section 2) from (1-1) the following global and local energy inequalities which one may expect solutions of (1-1) (with appropriate boundary conditions) to satisfy:³

$$\frac{d}{dt} \int_{\Omega} \left[\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + F(d) \right] dx + \int_{\Omega} [|\nabla u|^2 + |\Delta d - f(d)|^2] dx \leq 0 \tag{1-4}$$

for each $t \in (0, T)$, as well as a localized version⁴

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left[\left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} \right) \phi \right] dx + \int_{\Omega} (|\nabla u|^2 + |\nabla^2 d|^2) \phi dx \\ & \leq \int_{\Omega} \left[\left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} \right) (\phi_t + \Delta \phi) + \left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + p \right) u \cdot \nabla \phi \right. \\ & \quad \left. + u \otimes \nabla \phi : \nabla d \odot \nabla d - \underbrace{\phi \nabla^T [f(d)] : \nabla^T d}_{=: \mathcal{R}_f(d, \phi)} \right] dx \tag{1-5} \end{aligned}$$

for $t \in (0, T)$ and each smooth, compactly supported in Ω and nonnegative scalar field $\phi \geq 0$. (For Navier–Stokes, i.e., when $d \equiv 0$, one may omit all terms involving d , even though $0 \neq F(0) \notin L^1(\mathbb{R}^3)$.)

In [Lin and Liu 1995], for smooth and bounded Ω , the global energy inequality (1-4) is used to construct global weak solutions to (1-1) for initial velocity in $L^2(\Omega)$, along with a similarly appropriate condition on the initial value of d which allows (1-4) to be integrated over $0 < t < T$. This is consistent with the pioneering result of J. Leray [1934] for Navier–Stokes (treated later by many other authors using various methods, but always relying on the natural energy as in [Leray 1934]).

In [Lin and Liu 1996], the authors establish a partial regularity result for weak solutions to (1-1) belonging to the natural energy spaces which moreover satisfy the local energy inequality (1-5). The result is of the same type as known partial regularity results for a class of solutions known as *suitable weak solutions* to the Navier–Stokes equations. The program for such partial regularity results for Navier–Stokes was initiated in a series of papers by V. Scheffer in the 1970s and 1980s (see, for example, [Scheffer 1977; 1980] and other works mentioned in [Caffarelli et al. 1982]), and subsequently improved by L. Caffarelli, R. Kohn and L. Nirenberg in [Caffarelli et al. 1982].⁵ They show (as do Lin and Liu [1996]) that the one-dimensional parabolic Hausdorff measure of the (potentially empty) singular set S is zero ($\mathcal{P}^1(S) = 0$, see Definition 2 below), implying that singularities (if they exist) cannot for example contain any smooth one-parameter curve in space-time. The method of proof in [Lin and Liu 1996] largely follows the method of [Caffarelli et al. 1982].

Of course the general system (1-1) is (when $d \not\equiv 0$) substantially more complex than the Navier–Stokes system, and one therefore could not expect a stronger result than the type in [Caffarelli et al. 1982], i.e., $\mathcal{P}^1(S) = 0$; in fact, it is surprising that such a result still holds even when $d \not\equiv 0$. The explanation

³For sufficiently regular solutions one can show that equality holds.

⁴Note that in [Lin and Liu 1996], the term “ $-\mathcal{R}_f(d, \phi)$ ” in (1-5) actually appears incorrectly as “ $+\mathcal{R}_f(d, \phi)$ ”; see Section 2.

⁵Alternative proofs of slight variations of the main results in [Caffarelli et al. 1982] were given in later works such as [Lin 1998; Ladyzhenskaya and Seregin 1999; Vasseur 2007; Kukavica 2009].

for this seems to be that although (1-1) is more complex than Navier–Stokes in view of the additional d components, one can derive an a priori maximum principle for d because of the third equation in (1-1) which substantially offsets this complexity from the viewpoint of regularity. Therefore, under suitable boundary and initial conditions on d , one may assume that d is in fact bounded, a fact which is significantly exploited in [Lin and Liu 1996]. More recently, the authors of [Du et al. 2020] established the same type of result for a related but more complex Q -tensor system; however there, as well, one may obtain a maximum principle which is of crucial importance for proving partial regularity. One is therefore led to the following natural question, which we will address below:

Can one deduce any partial regularity for systems similar in structure to (1-1) but which lack any maximum principle?

In the Navier–Stokes setting, it was asserted by Scheffer [1985] that in fact the proof of the partial regularity result in [Caffarelli et al. 1982] does not require the full set of equations in (1-1). He mentions that the key ingredients are membership of the global energy spaces, the local energy inequality (1-5), the divergence-free condition $\nabla \cdot u = 0$ and the *pressure* equation (1-3) (with $d \equiv 0$ throughout). Scheffer called vector fields satisfying these four requirements solutions to the *Navier–Stokes inequality*, equivalent to solutions to the Navier–Stokes equations with a forcing f which satisfies $f \cdot u \leq 0$ everywhere. In contrast, the results in [Lin and Liu 1996] do very strongly use the third equation in (1-1) in that it implies a maximum principle for d .

In this paper, we explore what happens if one considers the analog of Scheffer’s *Navier–Stokes inequality* for the system (1-1) when $d \not\equiv 0$. That is, we consider triples (u, d, p) with global regularities implied — at least when Ω is bounded and under suitable assumptions on the initial data — by (1-4) which satisfy (1-3) and $\nabla \cdot u = 0$ weakly as well as (a formal consequence of) (1-5), but are *not* necessarily weak solutions of the first and third equations (i.e., the two vector equations) in (1-1). In particular, we will *not* assume that $d \in L^\infty(\Omega \times (0, T))$, which would have been reasonable in view of the third equation in (1-1). We see that without further assumptions, the result is substantially weaker than the $\mathcal{P}^1(S) = 0$ result for Navier–Stokes: following the methods of [Caffarelli et al. 1982; Lin and Liu 1996] we obtain (see Theorem 1) $\mathcal{P}^{9/2+\delta}(S) = 0$ for any $\delta > 0$. This reinforces our intuition that the situation here is substantially more complex than that of Navier–Stokes. On the other hand, we show that under a suitable uniform local decay condition on $|d|^\sigma (|u|^3 + |\nabla d|^3)^{1-\sigma/6}$ with $\sigma \in (5, 6)$ — see (1-14) below, which in particular holds when $d \equiv 0$ as in [Caffarelli et al. 1982] — one in fact obtains $\mathcal{P}^1(S) = 0$ as in [Caffarelli et al. 1982; Lin and Liu 1996]. In particular, we verify the above-mentioned assertion made by Scheffer [1985] regarding partial regularity for the Navier–Stokes inequality.

Our key observation which allows us to work without any maximum principle is that, in view of the global energy (1-4) and the particular forms of F and f , it is reasonable (see Section 2) to assume (1-9); this implies that $d \in L^\infty(0, T; L^6(\Omega))$ which is sufficient for our purposes.

As alluded to above, for our purposes we actually do not require all of the information which appears in (1-5). In view of the fact that

$$|\mathcal{R}_f(d, \phi)| = |\phi \nabla^T [f(d)] : \nabla^T d| \leq 12|d|^2 |\nabla d|^2 \phi + 8 \left(\frac{|\nabla d|^2}{2} \phi \right) \quad (1-6)$$

(see (2-21) below), a consequence of (1-5) is that

$$\mathcal{A}'(t) + \mathcal{B}(t) \leq 8\mathcal{A}(t) + \mathcal{C}(t) \quad \text{for } 0 < t < T, \tag{1-7}$$

with $\mathcal{A}, \mathcal{B}, \mathcal{C} \geq 0$ defined as

$$\mathcal{A}(t) := \int_{\Omega \times \{t\}} \left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} \right) \phi, \quad \mathcal{B}(t) := \int_{\Omega \times \{t\}} (|\nabla u|^2 + |\nabla^2 d|^2) \phi$$

and

$$\begin{aligned} \mathcal{C}(t) := \int_{\Omega \times \{t\}} & \left[\left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} \right) |\phi_t + \Delta \phi| + 12|d|^2 |\nabla d|^2 \phi \right] \\ & + \left| \int_{\Omega \times \{t\}} \left[\left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + p \right) u \cdot \nabla \phi + u \otimes \nabla \phi : \nabla d \odot \nabla d \right] \right|, \end{aligned}$$

where $\int_{\Omega \times \{t\}} g := \int_{\Omega} g(\cdot, t) dx$. Equation (1-7) is nearly sufficient, with the term $\mathcal{A}(t)$ on the right-hand side — in fact, even with u omitted, which cannot be avoided as “ $\mathcal{R}_f(d, \phi)$ ” appears on the right-hand side of (1-5) with a minus sign (see footnote 4) — actually being, for technical reasons, the only⁶ troublesome term. (Note that if $\mathcal{R}_f(d, \phi)$ had appeared with a plus sign in (1-5), one could have simply dropped the troublesome $\phi |\nabla d|^2$ term in (2-21) as a nonpositive quantity.) We therefore use a Grönwall-type argument to hide this term on the left-hand side of (1-7) so that (if $\phi|_{t=0} \equiv 0$)

$$\mathcal{A}'(t) + \mathcal{B}(t) \leq \mathcal{C}(t) + 8e^{8T} \int_0^t \mathcal{C}(\tau) d\tau \quad \text{for } 0 < t < T. \tag{1-8}$$

The (formally derived) local energy inequality (1-8) implies (1-13) below (for an appropriate constant $\bar{C} \sim 8Te^{8T} + 1$), which is sufficient for our purposes. (In fact, for all elements of the proof other than Proposition 8, a weaker form as in (3-5) is sufficient.)

Our main result is the following.

Theorem 1. *Fix any open set $\Omega \subset \mathbb{R}^3$ and any $T, \bar{C} \in (0, \infty)$. Set $\Omega_T := \Omega \times (0, T)$ and suppose $u, d : \Omega_T \rightarrow \mathbb{R}^3$ and $p : \Omega_T \rightarrow \mathbb{R}$ satisfy the following four assumptions:*

(1) *u, d and p belong to the following spaces:⁷*

$$u, d, \nabla d \in L^\infty(0, T; L^2(\Omega)), \quad \nabla u, \nabla d, \nabla^2 d \in L^2(\Omega_T) \tag{1-9}$$

and

$$p \in L^{3/2}(\Omega_T); \tag{1-10}$$

⁶In fact, the appearance of $|d|^2$ on the right-hand side of (1-6), and hence of (1-7) as well, is handled precisely by the assumption that $d \in L^\infty(0, T; L^6(\Omega))$, and is the reason for the slightly weaker results compared to the Navier–Stokes setting (i.e., when $d \equiv 0$).

⁷For a vector field f or matrix field J and scalar function space X , by $f \in X$ or $J \in X$ we mean that all components or entries of f or J belong to X ; by $\nabla^2 f \in X$ we mean all second partial derivatives of all components of f belong to X ; etc.

(2) u is weakly divergence-free:⁸

$$\nabla \cdot u = 0 \quad \text{in } \mathcal{D}'(\Omega_T); \tag{1-11}$$

(3) The following pressure equation holds weakly:⁹

$$-\Delta p = \nabla \cdot [\nabla^T \cdot (u \otimes u + \nabla d \odot \nabla d)] \quad \text{in } \mathcal{D}'(\Omega_T); \tag{1-12}$$

(4) The following local energy inequality holds:¹⁰

$$\begin{aligned} & \int_{\Omega \times \{t\}} (|u|^2 + |\nabla d|^2) \phi \, dx + \int_0^t \int_{\Omega} (|\nabla u|^2 + |\nabla^2 d|^2) \phi \, dx \, d\tau \\ & \leq \bar{C} \int_0^t \left\{ \int_{\Omega \times \{\tau\}} [(|u|^2 + |\nabla d|^2) |\phi_t + \Delta \phi| + |d|^2 |\nabla d|^2 \phi] \, dx \right. \\ & \quad \left. + \left| \int_{\Omega \times \{\tau\}} \left[\left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + p \right) u \cdot \nabla \phi + u \otimes \nabla \phi : \nabla d \odot \nabla d \right] \, dx \right| \right\} \, d\tau \\ & \quad \text{for a.e. } t \in (0, T) \quad \text{and for all } \phi \in C_0^\infty(\Omega \times (0, \infty)) \text{ such that } \phi \geq 0. \end{aligned} \tag{1-13}$$

Let $\mathcal{S} \subset \Omega_T$ be the (potentially empty) set of singular points where $|u| + |\nabla d|$ is not essentially bounded in any neighborhood of each $z \in \mathcal{S}$, and let \mathcal{P}^k be the k -dimensional parabolic Hausdorff outer measure (see Definition 2). The following are then true:

(i) $\mathcal{P}^{9/2+\delta}(\mathcal{S}) = 0$, for any $\delta > 0$ arbitrarily small.

(ii) If¹¹

$$g_\sigma := \sup_{z_0 \in \Omega_T} \left(\limsup_{r \searrow 0} \frac{1}{r^{2+\sigma/2}} \iint_{Q_r(z_0)} |d|^\sigma (|u|^3 + |\nabla d|^3)^{1-\sigma/6} \, dz \right) < \infty \tag{1-14}$$

for some $\sigma \in (5, 6)$, then $\mathcal{P}^1(\mathcal{S}) = 0$.

Note that in the case $d \equiv 0$, we regain the classical result of $\mathcal{P}^1(\mathcal{S}) = 0$ for Navier–Stokes as obtained in, for example, [Caffarelli et al. 1982], and more specifically for the (weaker) Navier–Stokes inequality mentioned in [Scheffer 1985].

We recall that the definition of the outer parabolic Hausdorff measure \mathcal{P}^k is given as follows, see [Caffarelli et al. 1982, pp. 783–784]:

Definition 2 (parabolic Hausdorff measure). For any $\mathcal{S} \subset \mathbb{R}^3 \times \mathbb{R}$ and $k \geq 0$, define

$$\mathcal{P}^k(\mathcal{S}) := \lim_{\delta \searrow 0} \mathcal{P}_\delta^k(\mathcal{S}),$$

⁸Locally integrable functions will always be associated to the standard distribution whose action is integration against a suitable test function so that, e.g., $[\nabla \cdot u](\psi) = -[u](\nabla \psi) := -\int u \cdot \nabla \psi$ for $\psi \in \mathcal{D}(\Omega_T)$.

⁹Note that $u \otimes u + \nabla d \odot \nabla d \in L^{5/3}(\Omega_T) \subset L^1_{\text{loc}}(\Omega_T)$, see (2-18)–(2-19).

¹⁰For brevity, for $\omega \subset \mathbb{R}^3$, we set $\int_{\omega \times \{t\}} g \, dx := \int_\omega g(x, t) \, dx$.

¹¹In general we set $z = (x, t) \in \Omega_T$, $dz := dx \, dt$ and recall from Definition 2 that $Q_r(x_0, t_0) := B_r(x_0) \times (t_0 - r^2, t_0)$.

where

$$\mathcal{P}_\delta^k(\mathcal{S}) := \inf \left\{ \sum_{j=1}^\infty r_j^k \mid \mathcal{S} \subset \bigcup_{j=1}^\infty Q_{r_j}, r_j < \delta, \forall j \in \mathbb{N} \right\}$$

and Q_r is any parabolic cylinder of radius $r > 0$, i.e.,

$$Q_r = Q_r(x, t) := B_r(x) \times (t - r^2, t) \subset \mathbb{R}^3 \times \mathbb{R}$$

for some $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$. We note that \mathcal{P}^k is an outer measure, and all Borel sets are \mathcal{P}^k -measurable.

Remark 3. In the case $\Omega = \mathbb{R}^3$, the condition (1-10) on the pressure follows (locally, at least) from (1-9) and (1-12) if p is taken to be the potential-theoretic solution to (1-12), since (1-9) implies that $u, \nabla d \in L^{10/3}(\Omega_T)$ by interpolation (see (2-18) below) and Sobolev embeddings, and then (1-12) gives $p \in L^{5/3}(\Omega_T) \subset L_{\text{loc}}^{3/2}(\Omega_T)$ by Calderón–Zygmund estimates. For a more general Ω , the existence of such a p can be derived from the motivating equation (1-1) (e.g., by estimates for the Stokes operator); see [Lin and Liu 1996]. Here, however, we will not refer to (1-1) at all and simply *assume* p satisfies (1-10) and address the partial regularity of such a hypothetical set of functions satisfying (1-9)–(1-13).

We note that Theorem 1 does not immediately recover the result of [Lin and Liu 1996] (which would correspond to $\sigma = 6$ in (1-14), which holds when $d \in L^\infty$ as assumed in that paper). Heuristically, however, one can argue as follows:¹²

If d were bounded, then taking for example $D := 24\|d\|_{L^\infty(\Omega_T)}^2 + 8 < \infty$ one would be able to deduce from (1-6) that

$$|\mathcal{R}_f(d, \phi)| \leq D \left(\frac{|\nabla d|^2}{2} \right) \phi.$$

Adjusting the Grönwall-type argument leading to (1-8), one could then deduce from (1-5) that (if $\mathcal{A}(0) = 0$)

$$\mathcal{A}'(t) + \mathcal{B}(t) \leq \tilde{\mathcal{C}}(t) + D e^{DT} \int_0^T \tilde{\mathcal{C}}(\tau) d\tau \quad \text{for } 0 < t < T,$$

where

$$\tilde{\mathcal{C}}(t) := \int_{\Omega \times \{t\}} \left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} \right) |\phi_t + \Delta \phi| + \left| \int_{\Omega \times \{t\}} \left[\left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + p \right) u \cdot \nabla \phi + u \otimes \nabla \phi : \nabla d \odot \nabla d \right] \right|.$$

Using such an energy inequality, one would not need to include the $|d|^6$ term in $E_{3,6}$ (see (3-6)) as one would not need to consider the term coming from $\mathcal{R}_f(d, \phi)$ at all in Proposition 6, and— noting that the L^∞ norm is invariant under the rescaling on d in (3-25)— one could then adjust Lemmas 4 and 7 appropriately to recover the result in [Lin and Liu 1996] using the proof of Theorem 1 below.

Finally, we remark that the majority of the arguments in the proofs given below are not new, with many essentially appearing in [Lin and Liu 1996] or [Caffarelli et al. 1982]. However we feel that our presentation is particularly transparent and may be a helpful addition to the literature, and we include all details so that our results are easily verifiable.

¹²We assume this is roughly the argument in [Lin and Liu 1996], although the details are not explicitly given; see, in particular, [Lin and Liu 1996, (2.45)] which appears without the *remainder* term denoted in [Lin and Liu 1996] by $\mathbf{R}(f, \phi)$, and here by $\mathcal{R}_f(d, \phi)$.

2. Motivation

We will show in this section that the assumptions in Theorem 1 are at least formally satisfied by smooth solutions to the system (1-1).

2.1. Energy identities. As in [Lin and Liu 1996], let us assume that we have smooth solutions to (1-1) which vanish or decay sufficiently at $\partial\Omega$ (assumed smooth, if nonempty) and at spatial infinity as appropriate so that all boundary terms vanish in the following integrations by parts, and proceed to establish smooth versions of (1-4) and (1-5). First, noting the simple identities

$$\nabla^T \cdot (\nabla d \odot \nabla d) = \nabla \left(\frac{|\nabla d|^2}{2} \right) + (\nabla^T d)^T \Delta d \quad (2-1)$$

and

$$[(\nabla^T d)^T \Delta d] \cdot u = [(\nabla^T d)u] \cdot \Delta d = [(u \cdot \nabla)d] \cdot \Delta d, \quad (2-2)$$

at a fixed t one may perform various integrations by parts — keeping in mind that $\nabla \cdot u = 0$ — to see that

$$\begin{aligned} 0 &= \int_{\Omega} [u_t - \Delta u + \nabla^T \cdot (u \otimes u) + \nabla p + \nabla^T \cdot (\nabla d \odot \nabla d)] \cdot u \, dx \\ &= \int_{\Omega} \left[\frac{\partial}{\partial t} \left(\frac{|u|^2}{2} \right) + |\nabla u|^2 + [(u \cdot \nabla)d] \cdot \Delta d \right] dx \end{aligned} \quad (2-3)$$

and — recalling that $f = \nabla F$ so that $[d_t + (u \cdot \nabla)d] \cdot f(d) = \left(\frac{\partial}{\partial t} + u \cdot \nabla \right) [F(d)]$ — that

$$\begin{aligned} 0 &= - \int_{\Omega} [d_t + (u \cdot \nabla)d - (\Delta d - f(d))] \cdot (\Delta d - f(d)) \, dx \\ &= - \int_{\Omega} \left[- \frac{\partial}{\partial t} \left(\frac{|\nabla d|^2}{2} + F(d) \right) + [(u \cdot \nabla)d] \cdot \Delta d - |\Delta d - f(d)|^2 \right] dx. \end{aligned} \quad (2-4)$$

Adding the two gives the **global energy identity for (1-1)**:

$$\frac{d}{dt} \int_{\Omega} \left[\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + F(d) \right] dx + \int_{\Omega} [|\nabla u|^2 + |\Delta d - f(d)|^2] dx = 0 \quad (2-5)$$

in view of the cancellation of the terms in bold in (2-3) and (2-4).

It is not quite straightforward to localize the calculations in (2-3) and (2-4), for example replacing the (global) multiplicative factor $(\Delta d - f(d))$ by $(\Delta d - f(d))\phi$ for a smooth and compactly supported ϕ . Arguing as in [Lin and Liu 1996], one can deduce a local energy identity by instead replacing $(\Delta d - f(d))$ by only a part of its localized version in divergence-form, namely by $\nabla^T \cdot (\phi \nabla^T d)$, at the expense of the appearance of $|\Delta d - f(d)|^2$ anywhere in the local energy.

Recalling (2-1) and (2-2) and noting further that

$$\begin{aligned} [(u \cdot \nabla)d] \cdot [\nabla^T \cdot (\phi \nabla^T d)] &= [(u \cdot \nabla)d] \cdot [\phi \Delta d] + [(u \cdot \nabla)d] \cdot [(\nabla \phi \cdot \nabla)d] \\ &= [(u \cdot \nabla)d] \cdot [\phi \Delta d] + u \otimes \nabla \phi : \nabla d \odot \nabla d \end{aligned}$$

and

$$[\Delta(\nabla^T d)] : \nabla^T d = \Delta \left(\frac{|\nabla d|^2}{2} \right) - |\nabla^2 d|^2,$$

one may perform various integrations by parts to deduce (as $\nabla \cdot u = 0$) that

$$\begin{aligned} 0 &= \int_{\Omega} [u_t - \Delta u + \nabla^T \cdot (u \otimes u) + \nabla p + \nabla^T \cdot (\nabla d \odot \nabla d)] \cdot u \phi \, dx \\ &= \int_{\Omega} \left[\frac{\partial}{\partial t} \left(\frac{|u|^2}{2} \phi \right) + |\nabla u|^2 \phi - \frac{|u|^2}{2} (\phi_t + \Delta \phi) - \left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + p \right) u \cdot \nabla \phi + [(u \cdot \nabla) d] \cdot (\Delta d) \phi \right] dx \end{aligned}$$

and

$$\begin{aligned} 0 &= - \int_{\Omega} [d_t + (u \cdot \nabla) d - (\Delta d - f(d))] \cdot [\nabla^T \cdot (\phi \nabla^T d)] \, dx \\ &= - \int_{\Omega} \left[- \frac{\partial}{\partial t} \left(\frac{|\nabla d|^2}{2} \phi \right) - |\nabla^2 d|^2 \phi + \frac{|\nabla d|^2}{2} (\phi_t + \Delta \phi) \right. \\ &\quad \left. - \nabla^T [f(d)] : \phi \nabla^T d + [(u \cdot \nabla) d] \cdot (\Delta d) \phi + u \otimes \nabla \phi : \nabla d \odot \nabla d \right] dx \end{aligned}$$

for smooth and compactly supported ϕ . Upon adding the two equations above and noting again the cancellation of the terms in bold, we obtain the **local energy identity for (1-1)**:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \left[\left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} \right) \phi \right] dx + \int_{\Omega} (|\nabla u|^2 + |\nabla^2 d|^2) \phi \, dx \\ = \int_{\Omega} \left[\left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} \right) (\phi_t + \Delta \phi) + \left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + p \right) u \cdot \nabla \phi \right. \\ \left. + u \otimes \nabla \phi : \nabla d \odot \nabla d - \underbrace{\phi \nabla^T [f(d)] : \nabla^T d}_{=: \mathcal{R}_f(d, \phi)} \right] dx. \end{aligned} \tag{2-6}$$

Note that the term

$$u \otimes \nabla \phi : \nabla d \odot \nabla d = [(\nabla d \odot \nabla d) \nabla \phi] \cdot u = [(u \cdot \nabla) d] \cdot [(\nabla \phi \cdot \nabla) d]$$

in (2-6) is a more accurate version of what is described in [Lin and Liu 1996] as “ $((u \cdot \nabla) d \odot \nabla d) \cdot \nabla \phi$ ”, and that the term “ $-\mathcal{R}_f(d, \phi)$ ” in (2-6) appears incorrectly in that paper as “ $+\mathcal{R}_f(d, \phi)$ ”.

2.2. Global energy regularity heuristics. Let us first see where the *global* energy identity (2-5) leads us to expect weak solutions to (1-1) to live (and hence why we assume (1-9) in Theorem 1).

To ease notation, in what follows let us fix $\Omega \subset \mathbb{R}^3$, and for $T \in (0, \infty]$ let us set $\Omega_T := \Omega \times (0, T)$ and

$$L_t^r L_x^q(T) := L^r(0, T; L^q(\Omega)).$$

According to (2-5), we expect, so long as

$$M_0 := \frac{1}{2} \|u(\cdot, 0)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla d(\cdot, 0)\|_{L^2(\Omega)}^2 + \|F(d(\cdot, 0))\|_{L^1(\Omega)} < \infty,$$

(which we would assume as a requirement on the initial data), to construct solutions with u in the usual Navier–Stokes spaces:

$$u \in L_t^\infty L_x^2(\infty) \quad \text{and} \quad \nabla u \in L_t^2 L_x^2(\infty). \tag{2-7}$$

As for d we expect as well in view of (2-5) that

$$\nabla d \in L_t^\infty L_x^2(\infty), \quad F(d) \in L_t^\infty L_x^1(\infty) \quad \text{and} \quad [\Delta d - f(d)] \in L_t^2 L_x^2(\infty). \quad (2-8)$$

The norms of all quantities in the spaces given in (2-7) and (2-8) are controlled by either M_0 (the $F(d)$ term) or $(M_0)^{1/2}$ (all other terms), by integrating (2-5) over $t \in (0, \infty)$. Recalling that

$$F(d) := (|d|^2 - 1)^2 \quad \text{and} \quad f(d) := 4(|d|^2 - 1)d, \quad (2-9)$$

one sees that $|f(d)|^2 = 16F(d)|d|^2$, and one can easily confirm the following simple estimates:

$$\|d\|_{L_t^\infty L_x^4(\infty)}^2 \leq \|F(d)\|_{L_t^\infty L_x^1(\infty)}^{1/2} + \|1\|_{L_t^\infty L_x^2(\infty)}, \quad (2-10)$$

$$\|F(d)\|_{L_t^\infty L_x^{3/2}(\infty)}^{1/2} \leq \|d\|_{L_t^\infty L_x^6(\infty)}^2 + \|1\|_{L_t^\infty L_x^3(\infty)}, \quad (2-11)$$

$$\|f(d)\|_{L_t^\infty L_x^2(\infty)}^2 \leq 16\|F(d)\|_{L_t^\infty L_x^{3/2}(\infty)} \|d\|_{L_t^\infty L_x^6(\infty)}^2, \quad (2-12)$$

$$\|\Delta d\|_{L^2(\Omega_T)} \leq \|\Delta d - f(d)\|_{L^2(\Omega_T)} + T^{1/2}\|f(d)\|_{L_t^\infty L_x^2(\infty)}. \quad (2-13)$$

Therefore, if we assume that

$$|\Omega| < \infty, \quad (2-14)$$

and hence

$$1 \in L^\infty(0, \infty; L^2(\Omega)) \cap L^\infty(0, \infty; L^3(\Omega)),$$

(2-8) along with (2-10) and (2-14) implies that

$$d \in L^\infty(0, \infty; L^4(\Omega)) \subset L^\infty(0, \infty; L^2(\Omega)). \quad (2-15)$$

This, along with (2-8), then implies that

$$d \in L^\infty(0, \infty; H^1(\Omega)) \hookrightarrow L^\infty(0, \infty; L^6(\Omega)) \quad (2-16)$$

by the Sobolev embedding, from which (2-11) implies that

$$F(d) \in L_t^\infty L_x^{3/2}(\infty)$$

which, along with (2-12) and (2-16), implies that

$$f(d) \in L_t^\infty L_x^2(\infty)$$

which, finally, in view of (2-13) and the last inclusion in (2-8), implies that

$$\Delta d \in L^2(\Omega_T) \quad \text{for any } T < \infty, \quad (2-17)$$

with the explicit estimate (2-13) which can then further be controlled by M_0 via (2-8) and (2-10)–(2-12).

We therefore see that it is reasonable (in view of the usual elliptic regularity theory) to expect that weak solutions to (1-1) should have the regularities in (1-9) of Theorem 1.

Note further that various interpolations of Lebesgue spaces imply, for example, that for any interval $I \subset \mathbb{R}$ one has

$$L^\infty(I; L^2(\Omega)) \cap L^2(I; L^6(\Omega)) \subset L^{2/\alpha}(I; L^{6/(3-2\alpha)}(\Omega)) \quad \text{for any } \alpha \in [0, 1]; \quad (2-18)$$

for example, one may take $\alpha = \frac{3}{5}$ so that $2/\alpha = 6/(3-2\alpha) = \frac{10}{3}$. Using this along with the Sobolev embedding we expect (as mentioned in Remark 3) that

$$(u \text{ and } \nabla d \in L^{2/\alpha}(0, T; L^{6/(3-2\alpha)}(\Omega))) \quad \text{for any } \alpha \in [0, 1], \quad T < \infty \quad (2-19)$$

with the explicit estimate¹³

$$\|\nabla d\|_{L_t^{2/\alpha} L_x^{6/(3-2\alpha)}(T)}^{2/\alpha} \lesssim T \|\nabla d\|_{L_t^\infty L_x^2(\infty)}^{2/\alpha} + \|\nabla d\|_{L_t^\infty L_x^2(\infty)}^{2/\alpha-2} \|\nabla^2 d\|_{L^2(\Omega_T)}^2.$$

Note that (2-19) along with (2-16), (2-14) and the Sobolev embedding implies that $d \in L^s(0, T; L^\infty(\Omega))$ as well for any $T < \infty$ and $s \in [2, 4)$.

2.3. Local energy regularity heuristics. Here, we will justify the well-posedness of the terms appearing in the *local* energy equality (2-6), based on the expected *global* regularity discussed in the previous section. In fact, all but the final term in (2-6) (where one can furthermore take the essential supremum over $t \in (0, T)$) can be seen to be well defined by (2-19) under the assumptions in (1-9) and (1-10).

The $\mathcal{R}_f(d, \phi)$ term of (2-6) requires some further consideration: in view of (2-9) we see that

$$\frac{1}{4} \nabla^T [f(d)] = \nabla^T [(|d|^2 - 1)d] = 2d \otimes [d \cdot (\nabla^T d)] + (|d|^2 - 1) \nabla^T d. \quad (2-20)$$

Recalling that

$$\mathcal{R}_f(d, \phi) := \phi \nabla^T [f(d)] : \nabla^T d,$$

we therefore have

$$\frac{1}{4} \mathcal{R}_f(d, \phi) = \phi (2d \otimes [d \cdot (\nabla^T d)] : \nabla^T d + |d|^2 |\nabla d|^2) - \phi |\nabla d|^2, \quad (2-21)$$

where we have to be careful how we handle the appearance of, essentially, $|d|^2$ in the first term (the second term is integrable in view of (2-8)). We have, for example, that

$$\|\phi |d|^2 |\nabla d|^2\|_{L^1(\Omega_T)} \leq \|\phi\|_{L^\infty(\Omega_T)} \|d\|_{L^6(\Omega_T)}^2 \|\nabla d\|_{L^3(\Omega_T)}^2$$

and that

$$\|d\|_{L^6(\Omega_T)} < \infty \quad \text{for any } T \in (0, \infty) \quad (2-22)$$

by (2-16), and either

$$\|\phi |\nabla d|^2\|_{L^1(\Omega_T)} \leq \|\phi\|_{L^\infty(\Omega_T)} \|\nabla d\|_{L^2(\Omega_T)}^2 \quad \text{or} \quad \|\phi |\nabla d|^2\|_{L^1(\Omega_T)} \leq \|\phi\|_{L^3(\Omega_T)} \|\nabla d\|_{L^3(\Omega_T)}^2,$$

(recall that ϕ is assumed to have compact support) and, for example, that

$$\|\nabla d\|_{L^{10/3}(\Omega_T)} < \infty \quad \text{for any } T \in (0, \infty) \quad (2-23)$$

by (2-19).

¹³ $A \lesssim B$ means that $A \leq CB$ for some suitably universal constant $C > 0$.

3. Proof of Theorem 1

The first part of Theorem 1 will be a consequence of a certain local L^3 ϵ -regularity criterion (Lemma 4), while the second part will be a consequence of a certain local \dot{H}^1 ϵ -regularity criterion (Lemma 7, which is itself a consequence of Lemma 4). In the remainder of the paper, for a given $z_0 = (x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}$ and $r > 0$, as in [Caffarelli et al. 1982] we will adopt the following notation for the standard parabolic cylinder $Q_r(z_0)$ with time interval $I_r(t_0)$, along with their *centered* versions (indicated with a star):

$$\begin{aligned} I_r(t_0) &:= (t_0 - r^2, t_0), & I_r^*(t_0) &:= (t_0 - \frac{7}{8}r^2, t_0 + \frac{1}{8}r^2), \\ Q_r(z_0) &:= B_r(x_0) \times I_r(t_0), & Q_r^*(z_0) &:= B_r(x_0) \times I_r^*(t_0). \end{aligned} \tag{3-1}$$

These are defined in such a way that $Q_r^*(x_0, t_0) = Q_r(x_0, t_0 + \frac{1}{8}r^2)$, and subsequently that

$$Q_{r/2}(x_0, t_0 + \frac{1}{8}r^2) = B_{r/2}(x_0) \times (t_0 - \frac{1}{8}r^2, t_0 + \frac{1}{8}r^2)$$

is a *centered* cylinder with center (x_0, t_0) .

Lemma 4 (L^3 ϵ -regularity; cf. [Lin and Liu 1996, Theorem 2.6; Caffarelli et al. 1982, Proposition 1]). *Fix any $\bar{C} \in (0, \infty)$. For each $q \in (5, 6]$, there exists a small¹⁴ constant $\bar{\epsilon}_q = \bar{\epsilon}_q(\bar{C}) \in (0, 1)$ such that for any $\bar{z} = (\bar{x}, \bar{t}) \in \mathbb{R}^3 \times \mathbb{R}$ and $\bar{\rho} \in (0, 1]$, the following holds:*

Suppose (see (3-1)) $u, d : Q_1(\bar{z}) \rightarrow \mathbb{R}^3$ and $p : Q_1(\bar{z}) \rightarrow \mathbb{R}$ with

$$\begin{aligned} u, d, \nabla d &\in L^\infty(I_1(\bar{t}); L^2(B_1(\bar{x}))), & \nabla u, \nabla d, \nabla^2 d &\in L^2(Q_1(\bar{z})), \\ p &\in L^{3/2}(Q_1(\bar{z})) \end{aligned} \tag{3-2}$$

satisfy

$$\nabla \cdot u = 0 \quad \text{in } \mathcal{D}'(Q_1(\bar{z})), \tag{3-3}$$

$$-\Delta p = \nabla \cdot (\nabla^T \cdot [u \otimes u + \nabla d \odot \nabla d]) \quad \text{in } \mathcal{D}'(Q_1(\bar{z})), \tag{3-4}$$

*and the following local energy inequality holds:*¹⁵

$$\begin{aligned} &\int_{B_1(\bar{x}) \times \{t\}} (|u|^2 + |\nabla d|^2) \phi \, dx + \int_{\bar{t}-1}^t \int_{B_1(\bar{x})} (|\nabla u|^2 + |\nabla^2 d|^2) \phi \, dx \, d\tau \\ &\leq \bar{C} \int_{\bar{t}-1}^t \left\{ \int_{B_1(\bar{x}) \times \{\tau\}} [(|u|^2 + |\nabla d|^2) |\phi_t + \Delta \phi| + (|u|^3 + |\nabla d|^3) |\nabla \phi| + \bar{\rho} |d|^2 |\nabla d|^2 \phi] \, dx \right. \\ &\quad \left. + \left| \int_{B_1(\bar{x}) \times \{\tau\}} p u \cdot \nabla \phi \, dx \right| \right\} d\tau \\ &\quad \text{for a.e. } t \in I_1(\bar{t}) \text{ and for all } \phi \in C_0^\infty(B_1(\bar{x}) \times (\bar{t} - 1, \infty)) \text{ such that } \phi \geq 0. \end{aligned} \tag{3-5}$$

¹⁴Roughly speaking, $\bar{\epsilon}_q \lesssim (\bar{C})^{-9} (2^{\alpha_q} - 1)^9$ with $\alpha_q := 2(q - 5)/(q - 2)$; in particular, $\bar{\epsilon}_q \rightarrow 0$ as $q \searrow 5$.

¹⁵Since $|(\frac{1}{2}|u|^2 + \frac{1}{2}|\nabla d|^2)u \cdot \nabla \phi + u \otimes \nabla \phi : \nabla d \odot \nabla d| \leq (\frac{1}{2}|u|^3 + \frac{3}{2}|u||\nabla d|^2)|\nabla \phi| \leq (|u|^3 + |\nabla d|^3)|\nabla \phi|$, we note that (1-13) implies (3-5) with $\bar{\rho} = 1$ if $Q_1(\bar{z}) \subseteq \Omega_T$. See also footnote 10.

Set¹⁶

$$E_{3,q} := \iint_{Q_1(\bar{z})} (|u|^3 + |\nabla d|^3 + |p|^{3/2} + |d|^q |\nabla d|^{3(1-q/6)}) dz. \tag{3-6}$$

If $E_{3,q} \leq \bar{\epsilon}_q$, then $u, \nabla d \in L^\infty(Q_{1/2}(\bar{z}))$ with

$$\|u\|_{L^\infty(Q_{1/2}(\bar{z}))}, \|\nabla d\|_{L^\infty(Q_{1/2}(\bar{z}))} \leq \bar{\epsilon}_q^{2/9}.$$

In order to prove Lemma 4, we will require the following two technical propositions. In order to state them, let us fix (recalling (3-1)) for a given $z_0 = (x_0, t_0)$ — to be clear by the context — the abbreviated notations

$$\begin{aligned} r_k &:= 2^{-k}, & B^k &:= B_{r_k}(x_0), \\ I^k &:= I_{r_k}(t_0), & Q^k &:= B^k \times I^k \end{aligned} \tag{3-7}$$

(so that $Q^k = Q_{2^{-k}}(z_0)$) and, for each $k \in \mathbb{N}$, we define the quantities

$$L_k = L_k(z_0) \quad \text{and} \quad R_k = R_k(z_0)$$

(again, the dependence on $z_0 = (x_0, t_0)$ will be clear by context) by¹⁷

$$L_k := \operatorname{ess\,sup}_{t \in I^k} \int_{B^k} (|u(t)|^2 + |\nabla d(t)|^2) dx + \int_{I^k} \int_{B^k} (|\nabla u|^2 + |\nabla^2 d|^2) dx dt \tag{3-8}$$

and

$$R_k := \iint_{Q^k} (|u|^3 + |\nabla d|^3) dz + r_k^{1/3} \iint_{Q^k} |u| |p - \bar{p}_k| dz, \tag{3-9}$$

where

$$\bar{p}_k(t) := \int_{B^k} p(x, t) dx.$$

The terms L_k and R_k correspond roughly to the left- and right-hand sides of the local energy inequality (3-5). We now state the technical propositions, whose proofs we will give in Section 4.

Proposition 5 (cf. [Lin and Liu 1996, Lemma 2.7]). *There exists a large universal constant $C_A > 0$ such that the following holds:*

Fix any $\bar{z} = (\bar{x}, \bar{t}) \in \mathbb{R}^3 \times \mathbb{R}$, and suppose u, d and p satisfy (3-2) and (3-4). Then for any $z_0 \in Q_{1/2}(\bar{z})$ we have (see (3-7)–(3-9))

$$R_{n+1}(z_0) \leq C_A \left(\max_{1 \leq k \leq n} L_k^{3/2}(z_0) + \underbrace{\|p\|_{L^{3/2}(Q_{1/2}(z_0))}^{3/2}}_{\leq E_{3,q} \forall q \geq 0, \text{ cf. (3-6)}} \right) \quad \text{for all } n \geq 2. \tag{3-10}$$

The proof of Proposition 5 uses only the Hölder and Poincaré inequalities, Sobolev embedding and Calderón–Zygmund estimates along with a local decomposition of the pressure (see (4-20)) using the pressure equation (3-4).

¹⁶Note that $E_{3,q} < \infty$ by (3-2) and standard embeddings; see Section 2 along with (3-22) with $\sigma = 6$.

¹⁷We use the standard notation for averages, e.g., $\int_B f(x) dx := \frac{1}{|B|} \int_B f(x) dx$.

Proposition 6 (cf. [Lin and Liu 1996, Lemma 2.8]). *There exists a large universal constant $C_B > 0$ such that the following holds:*

Fix any $\bar{z} = (\bar{x}, \bar{t}) \in \mathbb{R}^3 \times \mathbb{R}$, suppose u, d and p satisfy (3-2), (3-3) and (3-5), and set $E_{3,q}$ as in (3-6). Then for any $z_0 \in Q_{1/2}(\bar{z})$ and any $q \in (5, 6]$, we have (see (3-7)–(3-9))

$$L_n(z_0) \leq \bar{C} \cdot C_B \left(\frac{1}{2^{\alpha_q} - 1} \cdot \max_{k_0 \leq k \leq n} R_k(z_0) + E_{3,q}^{2/3} + (1 + k_0 2^{5k_0}) E_{3,q} \right) \quad \text{for all } n \geq 2 \tag{3-11}$$

for any $k_0 \in \{1, \dots, n - 1\}$, where \bar{C} is the constant from (3-5) and

$$\alpha_q := \frac{2(q - 5)}{q - 2} > 0.$$

The proof of Proposition 6 uses only the local energy inequality (3-5), the divergence-free condition (3-3) on u and elementary estimates. The quantities on either side of (3-11) do not scale (in the sense of (3-25)) the same way (as do those in (3-10)), which is why the energy inequality is necessary.

Let us now prove Lemma 4 using Propositions 5 and 6.

Proof of Lemma 4. Let us fix some $q \in (5, 6]$ and $\bar{C} \in (0, \infty)$. We first note that for any $\phi \geq 0$ as in (3-5) we have¹⁸ (recalling that $\bar{\rho} \leq 1$)

$$\bar{\rho} \iint_{Q^1} |d|^2 |\nabla d|^2 \phi \leq \frac{2}{q} \iint_{Q^1} |d|^q |\nabla d|^{3(1-q/6)} + \left(1 - \frac{2}{q}\right) \iint_{Q^1} |\nabla d|^3 \phi^{(5-\alpha_q)/3},$$

with $\alpha_q := 2(q - 5)/(q - 2) \in (0, \frac{1}{2}]$. Taking ϕ in particular such that $\phi \equiv 1$ on $Q^1 = Q_{1/2}(z_0)$, we see easily from this along with (3-5) that

$$\frac{L_1}{\bar{C}} \lesssim E_{3,q} + E_{3,q}^{2/3} \quad \text{for all } z_0 \in Q_{1/2}(\bar{z}). \tag{3-12}$$

It is also easy to see that

$$L_{n+1} \leq 8L_n \quad \text{for any } n \in \mathbb{N}. \tag{3-13}$$

Hence we may pick $C_0 = C_0(q, \bar{C}) \gg 1$ such that for any $z_0 \in Q_{1/2}(\bar{z})$ — and suppressing the dependence on z_0 in what follows — we have

$$\begin{aligned} L_1, L_2, L_3 &\leq \frac{1}{2} (C_0)^{2/3} (E_{3,q} + E_{3,q}^{2/3}) \quad \text{(by (3-12), (3-13)),} \\ C_A &\leq \frac{1}{2} C_0 \quad \text{and} \quad ((2^{\alpha_q} - 1)^{-1} + 2 + 3 \cdot 2^{15}) \bar{C} \cdot C_B \leq (C_0)^{2/3} \end{aligned} \tag{3-14}$$

for C_A and C_B as in Propositions 5 and 6. Having fixed C_0 — uniformly over $z_0 \in Q_{1/2}(\bar{z})$ — we then choose $\bar{\epsilon}_q \in (0, 1)$ so small that

$$\bar{\epsilon}_q < \frac{1}{(C_0)^6} \iff C_0^2 \bar{\epsilon}_q < \bar{\epsilon}_q^{2/3}.$$

Noting first that $\bar{\epsilon}_q \leq (\bar{\epsilon}_q)^{2/3}$, under the assumption $E_{3,q} \leq \bar{\epsilon}_q$ we in particular see from (3-14) that

$$L_1, L_2, L_3 \leq (C_0 \bar{\epsilon}_q)^{2/3}.$$

¹⁸The inequality in fact holds for any $q \in (2, 6]$.

Then, by Proposition 5 with $n \in \{2, 3\}$ we have

$$\begin{aligned} R_3, R_4 &\leq \frac{1}{2}C_0(\max\{L_1^{3/2}, L_2^{3/2}, L_3^{3/2}\} + \bar{\epsilon}_q) \quad (\text{by (3-10)}) \\ &\leq \frac{1}{2}C_0(C_0 + 1)\bar{\epsilon}_q \leq C_0^2\bar{\epsilon}_q < \bar{\epsilon}_q^{2/3} \end{aligned}$$

which implies due to Proposition 6 with $n = 4$ and $k_0 = 3$ that

$$\begin{aligned} L_4 &\leq C_B((2^{\alpha_q} - 1)^{-1} \max\{R_3, R_4\} + E_{3,q}^{2/3} + (1 + 3 \cdot 2^{15})E_{3,q}) \quad (\text{by (3-11)}) \\ &\leq (C_0\bar{\epsilon}_q)^{2/3}. \end{aligned}$$

Then in turn, Proposition 5 with $n = 4$ gives

$$L_1, L_2, L_3, L_4 \leq (C_0\bar{\epsilon}_q)^{2/3} \implies R_5 < \bar{\epsilon}_q^{2/3} \quad (\text{by (3-10)}),$$

from which Proposition 6 with $n = 5$ and, again, $k_0 = 3$ gives

$$R_3, R_4, R_5 < \bar{\epsilon}_q^{2/3} \implies L_5 \leq (C_0\bar{\epsilon}_q)^{2/3} \quad (\text{by (3-11)}),$$

and continuing we see by induction that Proposition 5 and Proposition 6 (with $k_0 = 3$ fixed throughout) imply that

$$R_n(z_0) < \bar{\epsilon}_q^{2/3} \quad \text{and} \quad L_n(z_0) \leq (C_0\bar{\epsilon}_q)^{2/3} \quad \text{for all } n \geq 3.$$

This, in turn, implies (for example) that (see, e.g., [Wheeden and Zygmund 1977, Theorem 7.16])

$$|u(z_0)|^3 + |\nabla d(z_0)|^3 \leq \bar{\epsilon}_q^{2/3}$$

for all Lebesgue points $z_0 \in Q_{1/2}(\bar{z})$ of $|u|^3 + |\nabla d|^3$ which implies the L^∞ statement, and Lemma 4 is proved. \square

Lemma 4 will be used to prove the first assertion in Theorem 1 as well as the next lemma, which in turn will be used to prove the second assertion in Theorem 1.

Lemma 7 (\dot{H}^1 ϵ -regularity; cf. [Lin and Liu 1996, Theorem 3.1; Caffarelli et al. 1982, Proposition 2]). *Fix any $\bar{C} \in (0, \infty)$ and $\bar{g} \in [1, \infty)$. For each $\sigma \in (5, 6)$, there exists a small constant $\epsilon_\sigma = \epsilon_\sigma(\bar{C}, \bar{g}) > 0$ such that the following holds. Fix $\Omega_T := \Omega \times (0, T)$ as in Theorem 1, and suppose u, d and p satisfy assumptions (1-9)–(1-13). If (recall (3-1))*

$$\limsup_{r \searrow 0} \frac{1}{r^{2+\sigma/2}} \iint_{Q_r^*(z_0)} |d|^\sigma (|u|^3 + |\nabla d|^3)^{1-\sigma/6} dz \leq \bar{g} \tag{3-15}$$

and

$$\limsup_{r \searrow 0} \frac{1}{r} \iint_{Q_r^*(z_0)} (|\nabla u|^2 + |\nabla^2 d|^2) dz \leq \epsilon_\sigma, \tag{3-16}$$

for some $z_0 \in \Omega_T$, then z_0 is a regular point, i.e., $|u|$ and $|\nabla d|$ are essentially bounded in some neighborhood of z_0 .

For the proof of Lemma 7, for $z_0 = (x_0, t_0) \in \Omega_T$ and for $r > 0$ sufficiently small, we define $A_{z_0}, B_{z_0}, C_{z_0}, D_{z_0}, E_{z_0}, F_{z_0}$ (cf. [Lin and Liu 1996, (3.3)]) and G_{q,z_0} using the cylinders $Q_r^*(z_0)$ — whose centers z_0 are in the interior, see (3-1) — by

$$\begin{aligned} A_{z_0}(r) &:= \frac{1}{r} \operatorname{ess\,sup}_{t \in I_r^*(t_0)} \int_{B_r(x_0)} (|u(t)|^2 + |\nabla d(t)|^2) \, dx, & B_{z_0}(r) &:= \frac{1}{r} \iint_{Q_r^*(z_0)} (|\nabla u|^2 + |\nabla^2 d|^2) \, dz, \\ C_{z_0}(r) &:= \frac{1}{r^2} \iint_{Q_r^*(z_0)} (|u|^3 + |\nabla d|^3) \, dz, & D_{z_0}(r) &:= \frac{1}{r^2} \iint_{Q_r^*(z_0)} |p|^{3/2} \, dz, \\ E_{z_0}(r) &:= \frac{1}{r^2} \iint_{Q_r^*(z_0)} |u| \{ | |u|^2 - \overline{|u|^2} | + | |\nabla d|^2 - \overline{|\nabla d|^2} | \} \, dz \end{aligned} \tag{3-17}$$

$$\text{where } \bar{g}^r(t) := \int_{B_r(x_0)} g(y, t) \, dy,$$

$$F_{z_0}(r) := \frac{1}{r^2} \iint_{Q_r^*(z_0)} |u| |p| \, dz, \quad G_{q,z_0}(r) := \frac{1}{r^{2+q/2}} \iint_{Q_r^*(z_0)} |d|^q (|u|^3 + |\nabla d|^3)^{1-q/6} \, dz$$

(note that $G_{0,z_0} \equiv C_{z_0}$) and, for $q \in [0, 6)$, define

$$M_{q,z_0}(r) := \frac{1}{2} [C_{z_0}(r) + G_{q,z_0}^{6/(6-q)}(r)] + D_{z_0}^2(r) + E_{z_0}^{3/2}(r) + F_{z_0}^{3/2}(r). \tag{3-18}$$

The statement in Lemma 7 will follow from Lemma 4 along with the following technical *decay estimate* which will be proved in Section 4.

Proposition 8 (decay estimate; cf. [Lin and Liu 1996, Lemma 3.1; Caffarelli et al. 1982, Proposition 3]). *Fix any $\bar{C} \in (0, \infty)$. There exists some constant $\bar{c} = \bar{c}(\bar{C}) > 0$ such that the following holds. Fix any $q, \sigma \in \mathbb{R}$ with $2 \leq q < \sigma < 6$, and define*

$$\alpha_{\sigma,q} := \frac{6}{\sigma} \cdot \frac{\sigma - q}{6 - q} \in (0, 1). \tag{3-19}$$

If u, d and p satisfy (1-9)–(1-13) for Ω_T as in Theorem 1, and $z_0 \in \Omega_T$ and $\rho_0 \in (0, 1]$ are such that $Q_{\rho_0}^(z_0) \subseteq \Omega_T$ and furthermore*

$$\sup_{\rho \in (0, \rho_0]} B_{z_0}(\rho) \leq 1 \quad \text{and} \quad \sup_{\rho \in (0, \rho_0]} G_{\sigma,z_0}(\rho) \leq \bar{g} \tag{3-20}$$

for some finite $\bar{g} \in [1, \infty)$, then for any $\rho \in (0, \rho_0]$ and $\gamma \in (0, \frac{1}{4}]$ we have

$$M_{q,z_0}(\gamma\rho) \leq \bar{c} \cdot \bar{g}^{6/(6-\sigma)} \left[\gamma^{\alpha_{\sigma,q}/8} (M_{q,z_0} + M_{q,z_0}^{\alpha_{\sigma,q}}) + \gamma^{-15} B_{z_0}^{3\alpha_{\sigma,q}/4} \sum_{k=0}^2 (M_{q,z_0}^{1/2^k} + M_{q,z_0}^{\alpha_{\sigma,q}/2^k}) \right] (\rho). \tag{3-21}$$

(In fact, in the sum over k in (3-21), one can omit the term with $\alpha_{\sigma,q}$ when $k = 0$.)

The key new element in our statement and proof of Proposition 8 (and hence in achieving Lemma 7) is the fact that, for certain $q > 0$ (so that $G_{q,z_0} \neq C_{z_0}$ and hence M_{q,z_0} is notably different from the quantity found in the standard literature, namely M_{0,z_0}), we can still derive an estimate for M_{q,z_0} of the form (3-21), with a constant depending only on \bar{C}, σ and \bar{g} (and not on q). This is made possible (see Claim 4 and its

applications in Section 4.4) by the following interpolation-type estimate for the range of the quantities G_{q,z_0} (including $G_{0,z_0} = C_{z_0}$), a simple consequence of Hölder's inequality:

$$0 \leq q \leq \sigma \leq 6 \implies G_{q,z_0}(r) \leq G_{\sigma,z_0}^{q/\sigma}(r) C_{z_0}^{1-q/\sigma}(r) \quad \text{for all } r > 0. \quad (3-22)$$

The estimate (3-22) follows by writing

$$|d|^q (|u|^3 + |\nabla d|^3)^{1-q/6} = [|d|^\sigma (|u|^3 + |\nabla d|^3)^{1-\sigma/6}]^{q/\sigma} \cdot (|u|^3 + |\nabla d|^3)^{(\sigma-q)/\sigma}$$

and applying Hölder's inequality with

$$1 = \frac{q}{\sigma} + \frac{\sigma - q}{\sigma}$$

to G_{q,z_0} , and noting that $r^{2+q/2} = [r^{2+\sigma/2}]^{q/\sigma} \cdot [r^2]^{1-q/\sigma}$. In particular, if $0 \leq q \leq \sigma < 6$, setting

$$\alpha_{\sigma,q} := \left(1 - \frac{q}{\sigma}\right) \cdot \frac{6}{6-q} \quad \text{and} \quad \beta_{\sigma,q} := \frac{q}{\sigma} \cdot \frac{6}{6-q}$$

and noting that

$$\beta_{\sigma,q} = \frac{6}{6-\sigma} \cdot (1 - \alpha_{\sigma,q}) \leq \frac{6}{6-\sigma},$$

we see that

$$G_{q,z_0}^{6/(6-q)}(r) \stackrel{(3-22)}{\leq} G_{\sigma,z_0}^{\beta_{\sigma,q}}(r) C_{z_0}^{\alpha_{\sigma,q}}(r) \stackrel{(3-15)}{\leq} \bar{g}^{6/(6-\sigma)} \cdot [2M_{q,z_0}^{\alpha_{\sigma,q}}(r)] \quad \text{for all } r > 0 \quad (3-23)$$

as long as $\bar{g} \geq 1$; this leads to the constants appearing in (3-21).

Let us now use Proposition 8 and Lemma 4 to prove Lemma 7.

Proof of Lemma 7. Fix any $\bar{C} \in (0, \infty)$, $\sigma \in (5, 6)$ and $\bar{g} \in [1, \infty)$, and fix¹⁹ any $q = q(\sigma) \in (5, \min\{\sigma, \frac{11}{2}\})$, noting that $6/(6-q) < 12$ and $2(6-q) > 1$; for the chosen q , let $\bar{\epsilon}_q = \bar{\epsilon}_q(\bar{C}) \in (0, 1)$ be the corresponding small constant from Lemma 4.

Let us first note the following important consequence of Lemma 4. Fix Ω_T as in Lemma 4 and $z_0 := (x_0, t_0) \in \Omega_T$, and suppose that

$$M_{q,z_0}(r) \leq \frac{1}{2} \left(\frac{\bar{\epsilon}_q}{3} \right)^{12} \quad (3-24)$$

for some $r \in (0, 1]$ such that $Q_r^*(z_0) \subseteq \Omega_T$. Setting

$$\begin{aligned} u_{z_0,r}(x, t) &:= ru(x_0 + rx, t_0 + r^2t), \\ p_{z_0,r}(x, t) &:= r^2p(x_0 + rx, t_0 + r^2t), \\ d_{z_0,r}(x, t) &:= d(x_0 + rx, t_0 + r^2t), \end{aligned} \quad (3-25)$$

a change of variables from $z = (x, t)$ to

$$(y, s) := (x_0 + rx, t_0 + r^2t) \quad (3-26)$$

¹⁹In the requirement that $q \in (5, \min\{\sigma, \bar{q}\})$, the choice of $\bar{q} := \frac{11}{2}$ is somewhat arbitrary and taken only for concreteness; one could similarly choose any $\bar{q} \in (5, 6)$ and adjust the subsequent constants accordingly.

implies that

$$\int_{Q_1^*(0,0)} (|u_{z_0,r}|^3 + |\nabla d_{z_0,r}|^3 + |p_{z_0,r}|^{3/2} + |d_{z_0,r}|^q (|u_{z_0,r}|^3 + |\nabla d_{z_0,r}|^3)^{1-q/6}) dz = C_{z_0}(r) + D_{z_0}(r) + G_{q,z_0}(r) \leq \left(\frac{\bar{\epsilon}_q}{3}\right)^{12} + \left(\frac{\bar{\epsilon}_q}{3}\right)^6 + \left(\frac{\bar{\epsilon}_q}{3}\right)^{2(6-q)} < \bar{\epsilon}_q.$$

Since $Q_1^*(0, 0) = Q_1(0, \frac{1}{8})$, it follows²⁰ from assumptions (1-9)–(1-13) that $u_{z_0,r}$, $d_{z_0,r}$ and $p_{z_0,r}$ satisfy the assumptions of Lemma 4 with $\bar{z} = (\bar{x}, \bar{t}) := (0, \frac{1}{8})$ and $\bar{\rho} := r^2 \in (0, 1]$, with the same constant \bar{C} (see footnote 15). Since we have just seen that

$$E_{3,q} = E_{3,q}(u_{z_0,r}, d_{z_0,r}, p_{z_0,r}, \bar{z}) < \bar{\epsilon}_q,$$

we therefore conclude by Lemma 4 that

$$|u_{z_0,r}(z)|, |\nabla d_{z_0,r}(z)| \leq \bar{\epsilon}_q^{2/9} \quad \text{for a.e. } z \in Q_{1/2}(0, \frac{1}{8}) = B_{1/2}(0) \times (-\frac{1}{8}, \frac{1}{8})$$

and hence

$$|u(y, s)|, |\nabla d(y, s)| \leq \frac{\bar{\epsilon}_q^{2/9}}{r} \quad \text{for a.e. } (y, s) \in B_{r/2}(x_0) \times (t_0 - \frac{1}{8}r^2, t_0 + \frac{1}{8}r^2).$$

In particular, by definition, $z_0 = (x_0, t_0)$ is a *regular* point, i.e., $|u|$ and $|\nabla d|$ are essentially bounded in a neighborhood of z_0 , so long as (3-24) holds for some sufficiently small $r > 0$.

In view of this fact, setting

$$\delta_\sigma := \frac{1}{2} \left(\frac{\bar{\epsilon}_q(\sigma)}{3}\right)^{12} \quad \text{and} \quad \bar{c}_\sigma := \bar{c} \cdot \bar{g}^{6/(6-\sigma)},$$

we choose $\gamma_\sigma \in (0, \frac{1}{4}]$ so small that

$$\bar{c}_\sigma \gamma_\sigma^{\alpha_{\sigma,q}/8} \leq \frac{1}{4} \left(\frac{\delta_\sigma^{[1-\alpha_{\sigma,q}]}}{2}\right), \tag{3-27}$$

where $\bar{c} = \bar{c}(\bar{C})$ is the constant from Proposition 8 and $\alpha_{\sigma,q}$ is defined as in (3-19); finally, we choose $\epsilon_\sigma \in (0, 1]$ so small that

$$\bar{c}_\sigma \gamma_\sigma^{-15} \epsilon_\sigma^{3\alpha_{\sigma,q}/4} \leq \frac{1}{4} \left(\frac{\delta_\sigma^{[1-\alpha_{\sigma,q}/4]}}{6}\right). \tag{3-28}$$

If $z_0 \in \Omega_T$ is such that (3-15) and (3-16) hold, it implies in particular that there exists some $\rho_0 \in (0, 1]$ such that $Q_{\rho_0}^*(z_0) \subseteq \Omega_T$ and, furthermore,

$$\sup_{\rho \in (0, \rho_0]} G_{\sigma,z_0}(\rho) \leq \bar{g} \tag{3-29}$$

²⁰For example, if one fixes an arbitrary $\phi \in C_0^\infty(Q_1^*(0, 0))$ and sets $\phi^{z_0,r}(x, \tau) := \phi((x - x_0)/r, (\tau - t_0)/r^2)$, then $\phi^{z_0,r} \in C_0^\infty(Q_r^*(z_0)) \subset C_0^\infty(\Omega_T)$. One can therefore use the test function $\phi^{z_0,r}$ in (1-13), make the change of variables $(\xi, s) := ((x - x_0)/r, (\tau - t_0)/r^2)$, so $(x, \tau) = (x_0 + r\xi, t_0 + r^2s)$, and divide both sides of the result by r to obtain the local energy inequality (3-5) for the rescaled functions with $\bar{\rho} = r^2$ (as all terms scale the same way except for $|d|^2|\nabla d|^2\phi^{z_0,r}$ and $\bar{z} = (0, \frac{1}{8})$). The other assumptions are straightforward.

and

$$\sup_{\rho \in (0, \rho_0]} B_{z_0}(\rho) < \epsilon_\sigma. \tag{3-30}$$

It then follows from (3-27), (3-28) and (3-30) — and the facts that $\alpha_{\sigma,q}, \delta_\sigma \leq 1$ — that

$$\bar{c}_\sigma \gamma_\sigma^{\alpha_{\sigma,q}/8} \stackrel{(3-27)}{\leq} \frac{1}{4} \left(\frac{\delta_\sigma^{\lceil 1-\alpha_{\sigma,q} \rceil}}{2} \right) = \frac{1}{4} \left(\frac{\min\{1, \delta_\sigma^{\lceil 1-\alpha_{\sigma,q} \rceil}\}}{2} \right),$$

and that

$$\begin{aligned} \bar{c}_\sigma \gamma_\sigma^{-15} B_{z_0}^{3\alpha_{\sigma,q}/4}(\rho) &\stackrel{(3-30)}{\leq} \bar{c}_\sigma \gamma_\sigma^{-15} \epsilon_\sigma^{3\alpha_{\sigma,q}/4} \stackrel{(3-28)}{\leq} \frac{1}{4} \left(\frac{\delta_\sigma^{\lceil 1-\alpha_{\sigma,q}/4 \rceil}}{6} \right) \\ &= \frac{1}{4} \left(\frac{\min_{k \in \{0,2\}} \{\min\{\delta_\sigma^{\lceil 1-1/2^k \rceil}, \delta_\sigma^{\lceil 1-\alpha_{\sigma,q}/2^k \rceil}\}}}{6} \right) \end{aligned}$$

for all $\rho \leq \rho_0$. Suppose now that z_0 is not a regular point. Then we must have

$$\delta_\sigma < M_{q,z_0}(\rho) \quad \text{for all } \rho \in (0, \rho_0], \tag{3-31}$$

or else (3-24) would hold for some $r \in (0, \rho_0]$ which would imply that z_0 is a regular point as we established above using Lemma 4.

In view of (3-29) and (3-30) — so that in particular (3-20) holds, as we chose $\epsilon_\sigma \leq 1$ — we conclude by the estimate (3-21) of Proposition 8 (along with (3-27), (3-28), (3-30), (3-31) and our calculations above) that

$$M_{q,z_0}(\gamma_\sigma \rho) \leq \frac{1}{2} M_{q,z_0}(\rho) \quad \text{for all } \rho \in (0, \rho_0]$$

for any z_0 which is not a regular point. However, since $\gamma_\sigma^k \rho_0 \in (0, \rho_0]$ for any $k \in \mathbb{N}$, by iterating the estimate above we would conclude for such z_0 that

$$M_{q,z_0}(\gamma_\sigma^n \rho_0) \leq \frac{1}{2} M_{q,z_0}(\gamma_\sigma^{n-1} \rho_0) \leq \frac{1}{2^2} M_{q,z_0}(\gamma_\sigma^{n-2} \rho_0) \leq \dots \leq \frac{1}{2^n} M_{q,z_0}(\rho_0) < \delta_\sigma$$

for a sufficiently large $n \in \mathbb{N}$ which contradicts (3-31) (with $\rho = \gamma_\sigma^n \rho_0$), and hence contradicts our assumption that z_0 is not a regular point. Therefore z_0 must indeed be regular whenever (3-29) and (3-30) hold for our choice of ϵ_σ , which proves Lemma 7. □

In order to prove Theorem 1, we now prove the following general lemma, from which Lemma 4 and Lemma 7 will have various consequences (including Theorem 1 as well as various other historical results, which we point out for the reader’s interest). As a motivation, note first that, for $r > 0$ and $z_1 := (x_1, t_1) \in \mathbb{R}^3 \times \mathbb{R}$, according to the notation in (3-25) a change of variables gives

$$\begin{aligned} \int_{Q_1^*(0,0)} |u_{z_1,r}|^q + |p_{z_1,r}|^{q/2} &= \frac{1}{r^{5-q}} \int_{Q_r^*(x_1,t_1)} |u|^q + |p|^{q/2}, \\ \int_{Q_1^*(0,0)} |\nabla u_{z_1,r}|^q &= \frac{1}{r^{5-2q}} \int_{Q_r^*(x_1,t_1)} |\nabla u|^q \end{aligned}$$

and

$$\int_{Q_1^*(0,0)} |d_{z_1,r}|^q |\nabla d_{z_1,r}|^{3(1-q/6)} = \frac{1}{r^{2+q/2}} \int_{Q_r^*(x_1,t_1)} |d|^q |\nabla d|^{3(1-q/6)} \tag{3-32}$$

for any $q \in [1, \infty)$.

Lemma 9. Fix any open and bounded $\Omega \Subset \mathbb{R}^3$, $T \in (0, \infty)$, $k \geq 0$ and $C_k > 0$, and suppose further that $\mathcal{S} \subseteq \Omega_T := \Omega \times (0, T)$ and that $U : \Omega_T \rightarrow [0, \infty]$ is a nonnegative Lebesgue-measurable function such that the following property holds in general:

$$(x_0, t_0) \in \mathcal{S} \implies \limsup_{r \searrow 0} \frac{1}{r^k} \int_{Q_r^*(x_0,t_0)} U \, dz \geq C_k. \tag{3-33}$$

If, furthermore,

$$U \in L^1(\Omega_T), \tag{3-34}$$

then (recall Definition 2) $\mathcal{P}^k(\mathcal{S}) < \infty$ (and hence the parabolic Hausdorff dimension of \mathcal{S} is at most k) with the explicit estimate

$$\mathcal{P}^k(\mathcal{S}) \leq \frac{5^5}{C_k} \int_{\Omega_T} U \, dz; \tag{3-35}$$

moreover, if $k = 5$, then

$$\mu(\mathcal{S}) \leq \frac{4\pi}{3} \mathcal{P}^5(\mathcal{S}) \leq \frac{5^5 \cdot 4\pi}{3C_5} \int_{\Omega_T} U \, dz \tag{3-36}$$

where μ is the Lebesgue outer measure, and if $k < 5$, then in fact $\mathcal{P}^k(\mathcal{S}) = \mu(\mathcal{S}) = 0$.

Before proving Lemma 9, let us first use it along with Lemma 4 and Lemma 7 to prove Theorem 1.

Proof of Theorem 1. First note that for any $r > 0$ and $z_1 := (x_1, t_1) \in \mathbb{R}^3 \times \mathbb{R}$ such that $Q_r(z_1) \subseteq \Omega_T$, it follows (as in the proof of Lemma 7) that, according to the definitions in (3-25), the rescaled triple $(u_{z_1,r}, d_{z_1,r}, p_{z_1,r})$ satisfies the conditions of Lemma 4 with $\bar{z} := (0, 0)$ and $\bar{\rho} := r^2$. Therefore if $q \in (5, 6]$ and

$$\begin{aligned} & \frac{1}{r^2} \int_{Q_r(x_1,t_1)} |u|^3 + |\nabla d|^3 + |p|^{3/2} + \frac{1}{r^{2+q/2}} \int_{Q_r(x_1,t_1)} |d|^q |\nabla d|^{3(1-q/6)} \\ &= \int_{Q_1(0,0)} |u_{z_1,r}|^3 + |\nabla d_{z_1,r}|^3 + |p_{z_1,r}|^{3/2} + |d_{z_1,r}|^q |\nabla d_{z_1,r}|^{3(1-q/6)} < \bar{\epsilon}_q \end{aligned} \tag{3-37}$$

(with $\bar{\epsilon}_q = \epsilon_q(\bar{C})$ as in Lemma 4), it follows that $|u_{z_1,r}|, |\nabla d_{z_1,r}| \leq C$ on $Q_{1/2}(0, 0)$ for some $C > 0$, and hence $|u|, |\nabla d| \leq C/r$ on $Q_{r/2}(x_1, t_1)$; in particular, every interior point of $Q_{r/2}(x_1, t_1)$ is a regular point, assuming (3-37) holds. Therefore, taking $z_0 := (x_0, t_0)$ such that

$$Q_{r/2}(x_1, t_1) = Q_{r/2}^*(x_0, t_0),$$

(so $x_0 = x_1$ and t_0 is slightly lower than t_1 so that (x_0, t_0) is in the interior of the cylinder $Q_{r/2}(x_1, t_1)$) and letting $\mathcal{S} \subset \Omega_T$ be the singular set of the solution (u, d, p) , we see (in particular) that, since $r^{2+q/2} < r^2$ for $r < 1$,

$$\left. \begin{array}{l} (x_0, t_0) \in \mathcal{S} \\ q \in (5, 6] \end{array} \right\} \implies \limsup_{r \searrow 0} \frac{1}{r^{2+q/2}} \int_{Q_r^*(x_0,t_0)} |u|^3 + |\nabla d|^3 + |p|^{3/2} + |d|^q |\nabla d|^{3(1-q/6)} \geq \bar{\epsilon}_q \tag{3-38}$$

(in fact, (3-38) must hold with \liminf instead of \limsup). Therefore, since (1-9) and (1-10) imply that

$$|u|^3 + |\nabla d|^3 + |p|^{3/2} + |d|^q |\nabla d|^{3(1-q/6)} \in L^1(\Omega_T) \tag{3-39}$$

(for $T < \infty$), we may apply Lemma 9 — it is not hard to see, by using a suitable covering argument, that without loss of generality we can assume Ω is bounded — with $U := |u|^3 + |\nabla d|^3 + |p|^{3/2} + |d|^q |\nabla d|^{3(1-q/6)}$, $k = 2 + \frac{1}{2}q$ and $C_k := \bar{\epsilon}_q$ to see (setting $\delta := \frac{1}{2}(q - 5) \in (0, \frac{1}{2}) \iff 5 < q < 6$ with $2 + \frac{1}{2}q = \frac{9}{2} + \delta$) that

$$\mathcal{P}^{9/2+\delta}(\mathcal{S}) = 0 \quad \text{for any } \delta \in (0, \frac{1}{2}).$$

Before continuing with the proof of Theorem 1, we describe some intermediate results (using only Lemma 4), with historical relevance, for the interest of the reader:

Suppose that (1-14) holds for some $\sigma \in (5, 6)$ which we now fix. We further fix any $q \in (5, \sigma)$, and choose $\gamma_{\sigma,q} > 0$ small enough that

$$\gamma_{\sigma,q}^{1-q/\sigma} (\gamma_{\sigma,q}^{q/\sigma} + (g_\sigma)^{q/\sigma}) < \bar{\epsilon}_q.$$

As in the proof of (3-22), Hölder’s inequality (along with (3-32)) implies that

$$\int_{Q_1(0,0)} |d_{z_1,r}|^q |\nabla d_{z_1,r}|^{3(1-q/6)} \leq (g_\sigma)^{q/\sigma} \left(\int_{Q_1(0,0)} |\nabla d_{z_1,r}|^3 \right)^{1-q/\sigma},$$

so that if

$$\frac{1}{r^2} \int_{Q_r(x_1,t_1)} |u|^3 + |\nabla d|^3 + |p|^{3/2} = \int_{Q_1(0,0)} |u_{z_1,r}|^3 + |\nabla d_{z_1,r}|^3 + |p_{z_1,r}|^{3/2} < \gamma_{\sigma,q}, \tag{3-40}$$

it follows that

$$\int_{Q_1(0,0)} |u_{z_1,r}|^3 + |\nabla d_{z_1,r}|^3 + |p_{z_1,r}|^{3/2} + |d_{z_1,r}|^q |\nabla d_{z_1,r}|^{3(1-q/6)} < \bar{\epsilon}_q$$

and hence $(x_0, t_0) \notin \mathcal{S}$ for (x_0, t_0) as above.

Therefore under the general assumption (1-14) with $\sigma \in (5, 6)$, there exists $\gamma_\sigma > 0$ (e.g., $\gamma_\sigma := \gamma_{\sigma,(5+\sigma)/2}$) such that

$$(x_0, t_0) \in \mathcal{S} \implies \limsup_{r \searrow 0} \frac{1}{r^2} \int_{Q_r^*(x_0,t_0)} |u|^3 + |\nabla d|^3 + |p|^{3/2} \geq \gamma_\sigma. \tag{3-41}$$

Therefore, as long as

$$(u, \nabla d, p) \in L^3(\Omega_T) \times L^3(\Omega_T) \times L^{3/2}(\Omega_T), \tag{3-42}$$

we may apply Lemma 9 with $U := |u|^3 + |\nabla d|^3 + |p|^{3/2}$, $k = 2$ and $C_k := \gamma_\sigma$ to see (similar to Scheffer’s result [1977]) that

$$\mathcal{P}^2(\mathcal{S}) = 0.$$

On the other hand, we know slightly more than (3-42). The assumptions on u and d in (1-9) imply (for example, by (2-18) with $\alpha = \frac{3}{5}$, along with Sobolev embedding) that $u, \nabla d \in L^{10/3}(\Omega_T)$. Suppose we also knew (as in the case when $\Omega = \mathbb{R}^3$) that $p \in L^{5/3}(\Omega_T)$ — which essentially follows from (1-9)

and (1-12), see [Lin and Liu 1996, Theorem 2.5]. Then (3-34) holds with $U := |u|^{10/3} + |\nabla d|^{10/3} + |p|^{5/3}$, and moreover Hölder’s inequality implies that

$$\left(\frac{1}{r^2} \int_{Q_r^*(z_0)} |u|^3 + |\nabla d|^3 + |p|^{3/2}\right)^{10/9} \leq 2^{10/9} |Q_1|^{1/9} \left(\frac{1}{r^{5/3}} \int_{Q_r^*(z_0)} |u|^{10/3} + |\nabla d|^{10/3} + |p|^{5/3}\right)$$

($|Q_1|$ is the Lebesgue measure of the unit parabolic cylinder). In view of (3-41), one could therefore apply Lemma 9 with

$$U := |u|^{10/3} + |\nabla d|^{10/3} + |p|^{5/3}, \quad k = \frac{5}{3} \quad \text{and} \quad C_k = \frac{\gamma_\sigma^{10/9}}{2^{10/9} |Q_1|^{1/9}}$$

to deduce (similar to Scheffer’s result [1980]) that

$$\mathcal{P}^{5/3}(\mathcal{S}) = 0.$$

All of the above follows from Lemma 4 alone. We will now show that Lemma 7 allows one — under assumption (1-14) for some $\sigma \in (5, 6)$, and even if $p \notin L^{5/3}(\Omega_T)$ — to further decrease the dimension of the parabolic Hausdorff measure, with respect to which the singular set has measure zero, from $\frac{5}{3}$ to 1. This was essentially the most significant contribution of [Caffarelli et al. 1982] in the Navier–Stokes setting $d \equiv 0$.

Let us now proceed with the proof of the second assertion in Theorem 1. Suppose d satisfies (1-14) for some $\sigma \in (5, 6)$. Taking $\epsilon_\sigma = \epsilon_\sigma(\bar{C}, g_\sigma) > 0$ as in Lemma 7 with $\bar{g} := g_\sigma$, we see from (3-16) that

$$(x_0, t_0) \in \mathcal{S} \implies \limsup_{r \searrow 0} \frac{1}{r} \int_{Q_r^*(x_0, t_0)} (|\nabla u|^2 + |\nabla^2 d|^2) \geq \epsilon_\sigma,$$

so that (3-33) holds with $U := |\nabla u|^2 + |\nabla^2 d|^2$ and $k = 1$. The second assumption in (1-9) implies that (3-34) holds as well with $U := |\nabla u|^2 + |\nabla^2 d|^2$. Therefore Lemma 9 with $U := |\nabla u|^2 + |\nabla^2 d|^2$, $k = 1$ and $C_k = \epsilon_\sigma$ implies that

$$\mathcal{P}^1(\mathcal{S}) = 0.$$

This completes the proof of Theorem 1 (assuming Lemma 9). □

Proof of Lemma 9. Fix any $\delta > 0$, and any open set V such that

$$\mathcal{S} \subseteq V \subseteq \Omega \times (0, T). \tag{3-43}$$

For each $z := (x, t) \in \mathcal{S}$, by (3-33) we can choose $r_z \in (0, \delta)$ sufficiently small so that $Q_{r_z}^*(z) \subset V$ and

$$\frac{1}{r_z^k} \int_{Q_{r_z}^*(z)} U \geq C_k. \tag{3-44}$$

By a Vitalli covering argument, see [Caffarelli et al. 1982, Lemma 6.1], there exists a sequence $(z_j)_{j=1}^\infty \subseteq \mathcal{S}$ such that

$$\mathcal{S} \subseteq \bigcup_{j=1}^\infty Q_{5r_{z_j}}^*(z_j) \tag{3-45}$$

and such that the set of cylinders $\{Q_{r_{z_j}}^*(z_j)\}_j$ are pairwise disjoint. We therefore see from (3-44) that

$$\sum_{j=1}^{\infty} r_{z_j}^k \leq \frac{1}{C_k} \sum_{j=1}^{\infty} \int_{Q_{r_{z_j}}^*(z_j)} U \leq \frac{1}{C_k} \int_V U \leq \frac{1}{C_k} \int_{\Omega_T} U \tag{3-46}$$

which is finite (and uniformly bounded in δ) by (3-34). Note that according to Definition 2 of the parabolic Hausdorff measure \mathcal{P}^k , (3-46) implies

$$\mathcal{P}^k(S) \leq \frac{5^k}{C_k} \int_V U \leq \frac{5^k}{C_k} \int_{\Omega_T} U \tag{3-47}$$

which establishes (3-35).

Let us now assume that $k \leq 5$. Letting μ be the Lebesgue (outer) measure, note that

$$\mu(Q_{5r_{z_j}}^*) \leq |B_1|(5r_{z_j})^5$$

so that

$$\mu(S) \stackrel{(3-45)}{\leq} |B_1| \sum_{j=1}^{\infty} (5r_{z_j})^5 \leq 5^5 |B_1| \delta^{5-k} \sum_{j=1}^{\infty} r_{z_j}^k \stackrel{(3-46)}{\leq} \delta^{5-k} \frac{5^5 |B_1|}{C_k} \int_{\Omega_T} U, \tag{3-48}$$

since we have chosen $r_z < \delta$ for all $z \in S$. If $k = 5$, (3-48) along with Definition 2 gives the explicit estimate (3-36) on $\mu(S)$. If $k < 5$, since $\delta > 0$ was arbitrary, sending $\delta \rightarrow 0$ we conclude (by (3-34)) that $\mu(S) = 0$ and hence S is Lebesgue measurable with Lebesgue measure zero. We may therefore take V to be an open set such that $\mu(V)$ is arbitrarily small but so that (3-43) still holds, and deduce that $\mathcal{P}^k(S) = 0$ by (3-34) and (3-47). □

4. Proofs of technical propositions

In order to prove Proposition 5 as well as Proposition 8, we will require certain local decompositions of the pressure (cf. [Caffarelli et al. 1982, (2.15)]) as follows:

4.1. Localization of the pressure.

Claim 1. Fix open sets $\Omega_1 \Subset \Omega_2 \Subset \Omega \subset \mathbb{R}^3$ and $\psi \in C_0^\infty(\Omega_2; \mathbb{R})$ with $\psi \equiv 1$ on Ω_1 . Let

$$G^x(y) := \frac{1}{4\pi} \frac{1}{|x - y|} \tag{4-1}$$

be the fundamental solution of $-\Delta$ in \mathbb{R}^3 so that, in particular,

$$\nabla G^x \in L^q(\Omega_2) \quad \text{for any } q \in [1, \frac{3}{2})$$

for any fixed $x \in \mathbb{R}^3$, and set

$$\begin{aligned} G_{\psi,1}^x &:= -G^x \nabla \psi, \\ G_{\psi,2}^x &:= 2\nabla G^x \cdot \nabla \psi + G^x \Delta \psi, \\ G_{\psi,3}^x &:= \nabla G^x \otimes \nabla \psi + \nabla \psi \otimes \nabla G^x + G^x \nabla^2 \psi, \end{aligned}$$

so that

$$G_{\psi,1}^x, G_{\psi,2}^x, G_{\psi,3}^x \in C_0^\infty(\Omega_2) \quad \text{for any fixed } x \in \Omega_1.$$

Suppose $\Pi \in C^2(\Omega; \mathbb{R})$, $v \in C^1(\Omega; \mathbb{R}^3)$ and $K \in C^2(\Omega; \mathbb{R}^{3 \times 3})$. If

$$-\Delta \Pi = \nabla \cdot v \quad \text{in } \Omega, \tag{4-2}$$

then for any $x \in \Omega_1$,

$$\Pi(x) = - \int \nabla G^x \cdot v \psi + \int G_{\psi,1}^x \cdot v + \int G_{\psi,2}^x \Pi. \tag{4-3}$$

Similarly, if

$$-\Delta \Pi = \nabla \cdot (\nabla^T \cdot K) \quad \text{in } \Omega, \tag{4-4}$$

then for any $x \in \Omega_1$,

$$\Pi(x) = S[\psi K](x) + \int G_{\psi,3}^x : K + \int G_{\psi,2}^x \Pi, \tag{4-5}$$

where

$$S[\tilde{K}](x) := \nabla_x \cdot \left(\nabla_x^T \cdot \int G^x \tilde{K} \right) = \int G^x \nabla \cdot (\nabla^T \cdot \tilde{K}) \quad \text{for all } \tilde{K} \in C_0^2(\Omega_2; \mathbb{R}^{3 \times 3});$$

in particular (noting $\nabla^2 G^x \notin L_{loc}^1$), $S : [L^q(\Omega_2)]^{3 \times 3} \rightarrow L^q(\Omega_2)$ for any $q \in (1, \infty)$ is a bounded, linear Calderón–Zygmund operator.

Remark 10. We note, therefore, that under the assumptions (1-9), (1-10) and (1-12), by suitable regularizations one can see that for almost every fixed $t \in (0, T)$, (4-3) and (4-5) hold for a.e. $x \in \Omega_1$ with $\Pi := p(\cdot, t)$, $K := J(\cdot, t)$ and $v := \nabla^T \cdot J(\cdot, t)$, where

$$J := u \otimes u + \nabla d \odot \nabla d.$$

Indeed, under the assumptions (1-9), we have $u, \nabla d \in L^{10/3}(\Omega_T)$ so that (omitting the x -dependence)

$$J(t) \in L^{5/3}(\Omega) \quad \text{for a.e. } t \in (0, T). \tag{4-6}$$

Moreover, since $u, \nabla d \in L^\infty(0, T; L^2(\Omega)) \cap L^{10/3}(\Omega_T)$ and $\nabla u, \nabla^2 d \in L^2(\Omega_T)$, we have

$$\nabla^T \cdot J \in L^2(0, T; L^1(\Omega)) \cap L^{5/4}(\Omega_T)$$

so that

$$\nabla^T \cdot J(t) \in L^1(\Omega) \cap L^{5/4}(\Omega) \quad \text{for a.e. } t \in (0, T). \tag{4-7}$$

Finally, (1-10) implies that

$$p(t) \in L^{3/2}(\Omega) \quad \text{for a.e. } t \in (0, T). \tag{4-8}$$

Fix now any $t \in (0, T)$ such that the inclusions in (4-6)–(4-8) hold. Since $G_{\psi,j}^x \in C_0^\infty$ for $x \in \Omega_1$, the terms in (4-3) and (4-5) containing $G_{\psi,j}^x$ are all well defined for every $x \in \Omega_1$ since $J(t), \nabla^T \cdot J(t), p(t) \in L_{loc}^1(\Omega)$. The term in (4-3) containing ∇G^x is in $L_x^r(\Omega_2)$ for any $r \in [1, \frac{15}{7})$ by Young’s convolution inequality (since Ω_2 is bounded), so that term is well defined for a.e. $x \in \Omega_2$. Indeed, for $R > 0$ such that $\Omega_2 \subseteq B_{R/2}(x_0)$

for some $x_0 \in \mathbb{R}^3$, we have $x - y \in B_R := B_R(0)$ for all $x, y \in \Omega_2$. Setting $G(y) := G^0(y)$ and letting χ_{B_R} be the indicator function of B_R , since ψ is supported in Ω_2 we therefore have

$$-\int \nabla G^x \cdot v \psi = [([\nabla G]\chi_{B_R}) * (v\psi)](x)$$

for all $x \in \Omega_2$. Therefore

$$\begin{aligned} \left\| \int \nabla G^x \cdot v \psi \right\|_{L^r_x(\Omega_2)} &\leq \|([\nabla G]\chi_{B_R}) * v\psi\|_{L^r(\mathbb{R}^3)} \\ &\leq \|[\nabla G]\chi_{B_R}\|_{L^q(\mathbb{R}^3)} \|v\psi\|_{L^s(\mathbb{R}^3)} = \|\nabla G\|_{L^q(B_R)} \|v\psi\|_{L^s(\Omega_2)} < \infty \end{aligned}$$

by Young’s inequality for any $q \in [1, \frac{3}{2})$, $s \in [1, \frac{5}{4})$ and r such that $1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{s}$ (note that $\frac{2}{3} + \frac{4}{5} - 1 = \frac{7}{15}$). Finally, $S[\psi J(t)] \in L^{5/3}(\Omega_2)$ by the Calderón–Zygmund estimates (as $1 < \frac{5}{3} < \infty$) so again that term is defined for a.e. $x \in \Omega_2$.

Regularizing the linear equation (1-12) using a standard spatial mollifier at any $t \in (0, T)$ where (1-12) holds in $\mathcal{D}'(\Omega)$ and where the inclusions in (4-6)–(4-8) hold, applying Claim 1 and passing to limits gives the almost-everywhere convergence (after passing to a suitable subsequence) due, in particular, to the boundedness of the linear operator S on $L^{5/3}(\Omega_2)$.

Proof of Claim 1. Since (extending Π by zero outside of Ω) $\psi \Pi \in C^2_0(\mathbb{R}^3)$, by the classical representation formula, see, e.g., [Gilbarg and Trudinger 1983, (2.17)], for any $x \in \mathbb{R}^3$ we have

$$\psi(x)\Pi(x) = -\int G^x \Delta(\psi \Pi) = -\int G^x (\psi \Delta \Pi + 2\nabla \psi \cdot \nabla \Pi + \Pi \Delta \psi). \tag{4-9}$$

In particular, for a fixed $x \in \Omega_1$ where $\psi \equiv 1$, we have $G^x \nabla \psi \in C^\infty_0(\mathbb{R}^3)$ so that integrating by parts in (4-9) we see that

$$\Pi(x) = \int G^x \psi (-\Delta \Pi) + \int G^x_{\psi,2} \Pi. \tag{4-10}$$

If (4-2) holds, then by (4-10) we have

$$\Pi(x) = \int G^x \psi \nabla \cdot v + \int G^x_{\psi,2} \Pi \tag{4-11}$$

for any $x \in \Omega_1$. One can then carefully integrate by parts once in the first term of (4-11) as follows: for a small $\epsilon > 0$,

$$\int_{|y-x|>\epsilon} G^x \psi \nabla \cdot v \, dy = -\int_{|y-x|>\epsilon} [\nabla(G^x \psi)] \cdot v \, dy + \frac{1}{4\pi\epsilon} \underbrace{\int_{|y-x|=\epsilon} \psi v \cdot \nu_y \, dS_y}_{=O(\epsilon^2)}$$

and since the second term vanishes as $\epsilon \rightarrow 0$ due to the fact that $|\partial B_\epsilon(x)| \lesssim \epsilon^2$, we conclude (since $\nabla G^x \in L^1_{loc}$) that

$$\int G^x \psi \nabla \cdot v = -\int [\nabla(G^x \psi)] \cdot v = -\int \nabla G^x \cdot v \psi + \int G^x_{\psi,1} \cdot v$$

which, along with (4-11), implies (4-3) for any $x \in \Omega_1$.

On the other hand, if (4-4) holds, then by (4-10) we have

$$\Pi(x) = \int G^x \psi \nabla \cdot (\nabla^T \cdot K) + \int G_{\psi,2}^x \Pi \tag{4-12}$$

and one can write

$$\nabla \cdot (\nabla^T \cdot (\psi K)) = [\nabla^2 \psi]^T : K + \nabla^T \psi \cdot [\nabla \cdot K] + \nabla \psi \cdot [\nabla^T \cdot K] + \psi \nabla \cdot (\nabla^T \cdot K)$$

so that (as $\nabla^2 \psi = \nabla^T (\nabla \psi) = \nabla (\nabla^T \psi) = [\nabla^2 \psi]^T$ since $\psi \in \mathcal{C}^2$)

$$\begin{aligned} & \int G^x [\psi \nabla \cdot (\nabla^T \cdot K)] \\ &= \int G^x [\nabla \cdot (\nabla^T \cdot (\psi K))] - \int G^x [\nabla^2 \psi : K] - \int ([G^x \nabla^T \psi] \cdot [\nabla \cdot K] + [G^x \nabla \psi] \cdot [\nabla^T \cdot K]). \end{aligned}$$

Since $G^x \nabla \psi \in \mathcal{C}_0^\infty$ for $x \in \Omega_1$, one can again integrate by parts in the final term to obtain

$$\Pi(x) = \int G^x [\nabla \cdot (\nabla^T \cdot (\psi K))] + \int G_{\psi,3}^x : K + \int G_{\psi,2}^x \Pi$$

for $x \in \Omega_1$ in view of (4-12). Moreover, since $\psi K \in \mathcal{C}_0^2$ and $G^x \in L_{loc}^1$, as usual for convolutions one can change variables to obtain

$$\int G^x \nabla \cdot (\nabla^T \cdot (\psi K)) = \left[\nabla_x \cdot \left(\nabla_x^T \cdot \int G^x \psi K \right) \right] (x) =: S[\psi K](x)$$

which gives us (4-5) for any $x \in \Omega_1$, where S (see, e.g., [Gilbarg and Trudinger 1983, Theorem 9.9]) is a singular integral operator as claimed. (Note that $\nabla^2 G^x \notin L_{loc}^1$ so that one cannot simply integrate by parts twice in this term putting all derivatives on G^x , but $\int G^x \psi K$ is the Newtonian potential of ψK which can be twice differentiated in various senses depending on the regularity of K .) \square

4.2. Proof of Proposition 5. In what follows, for $\mathcal{O} \subseteq \mathbb{R}^3$ and $I \subseteq \mathbb{R}$, we will use the notation

$$\|\cdot\|_{q;\mathcal{O}} := \|\cdot\|_{L^q(\mathcal{O})}, \quad \|\cdot\|_{s;I} := \|\cdot\|_{L^s(I)}, \quad \|\cdot\|_{q,s;\mathcal{O} \times I} := \|\cdot\|_{L^s(I;L^q(\mathcal{O}))} = \left\| \|\cdot\|_{L^q(\mathcal{O})} \right\|_{L^s(I)}$$

and we will abbreviate by writing

$$\|\cdot\|_{q;\mathcal{O} \times I} := \|\cdot\|_{q,q;\mathcal{O} \times I} = \|\cdot\|_{L^q(\mathcal{O} \times I)}.$$

We first note some simple inequalities. Letting $B_r \subset \mathbb{R}^3$ be a ball of radius $r > 0$, from the embedding $W^{1,2}(B_1) \hookrightarrow L^6(B_1)$ applied to functions of the form $g_r(x) = g(rx)$ — or suitably shifted, if the ball is not centered as zero — we obtain

$$\|g_r\|_{6;B_1} \lesssim \|g_r\|_{2;B_1} + \|\nabla g_r\|_{2;B_1} = \|g_r\|_{2;B_1} + r \|(\nabla g)_r\|_{2;B_1}$$

whereupon — noting by a simple change of variables that

$$\|g_r\|_{q;B_1} = r^{-3/q} \|g\|_{q;B_r}$$

for any $q \in [1, \infty)$ — we obtain for any ball B_r of radius $r > 0$ and any g that

$$\|g\|_{6;B_r} \lesssim \frac{1}{r} \|g\|_{2;B_r} + \|\nabla g\|_{2;B_r} \tag{4-13}$$

where the constant is independent of r as well as the center of B_r . Next, for any $v(x, t)$, using Hölder’s inequality to interpolate between L^2 and L^6 we have

$$\|v(t)\|_{3;B_r} \leq \|v(t)\|_{2;B_r}^{1/2} \|v(t)\|_{6;B_r}^{1/2} \stackrel{(4-13)}{\lesssim} r^{-1/2} \|v(t)\|_{2;B_r} + \|v(t)\|_{2;B_r}^{1/2} \|\nabla v(t)\|_{2;B_r}^{1/2}. \tag{4-14}$$

Then for $I_r \subset \mathbb{R}$ with $|I_r| = r^2$ and $Q_r := B_r \times I_r$, Hölder’s inequality in the t variable gives

$$\|v\|_{3;Q_r} \lesssim r^{-1/2} |I_r|^{1/3} \|v\|_{2,\infty;Q_r} + \|v\|_{2,\infty;Q_r}^{1/2} (|I_r|^{1/6} \|\nabla v\|_{2;Q_r})^{1/2}$$

so that

$$r^{-1/6} \|v\|_{3;Q_r} \lesssim \|v\|_{2,\infty;Q_r} + \|v\|_{2,\infty;Q_r}^{1/2} \|\nabla v\|_{2;Q_r}^{1/2} \lesssim \|v\|_{2,\infty;Q_r} + \|\nabla v\|_{2;Q_r}$$

(the first of which is sometimes called the *multiplicative inequality*) with a constant independent of r . From these, noting that $|B_r| \sim r^3$ and $|Q_r| \sim r^5$, it follows easily that, for example,

$$\iint_{Q^n} |v|^3 dz \lesssim \left(\operatorname{ess\,sup}_{t \in I^n} \int_{B^n} |v(t)|^2 dx \right)^{3/2} + \left(\int_{I^n} \int_{B^n} |\nabla v|^2 dx dt \right)^{3/2}. \tag{4-15}$$

Note also that a similar scaling argument applied to Poincaré’s inequality gives the estimate

$$\|g - \overline{g_{B_r}}\|_{q;B_r} \lesssim r \|\nabla g\|_{q;B_r} \sim |B_r|^{1/3} \|\nabla g\|_{q;B_r} \tag{4-16}$$

for any $r > 0$ and $q \in [1, \infty]$, where $\overline{g_{\mathcal{O}}}$ is the average of g in \mathcal{O} for any $\mathcal{O} \subset \mathbb{R}^3$ with $|\mathcal{O}| < \infty$. Note finally that a simple application of Hölder’s inequality gives

$$\|\overline{g_{\mathcal{O}}}\|_{q;\mathcal{O}} \leq \|g\|_{q;\mathcal{O}}. \tag{4-17}$$

Proceeding now with the proof, fix some $\tilde{\phi} \in C_0^\infty(\mathbb{R}^3)$ such that

$$\tilde{\phi} \equiv 1 \quad \text{in } B_{r_2}(0) = B_{1/4}(0) \quad \text{and} \quad \operatorname{supp}(\tilde{\phi}) \subseteq B_{r_1}(0) = B_{1/2}(0).$$

Now fix $\bar{z} = (\bar{x}, \bar{t}) \in \mathbb{R}^3 \times \mathbb{R}$ and $z_0 = (x_0, t_0) \in Q_{1/2}(\bar{z})$, define B^k, I^k and Q^k by (3-7) for this z_0 and define ϕ by $\phi(x) := \tilde{\phi}(x - x_0)$. So

$$\phi \equiv 1 \quad \text{in } B^2 = B_{1/4}(x_0) \quad \text{and} \quad \operatorname{supp}(\phi) \subseteq B^1 = B_{1/2}(x_0) \subset B_1(\bar{x}),$$

since $x_0 \in B_{1/2}(\bar{x})$. The following estimates will clearly depend only on $\tilde{\phi}$, i.e., constants will be uniform for all $z_0 \in Q_{1/2}(\bar{z})$.

First, applying (4-15) to $v \in \{u, \nabla d\}$ and recalling (3-8) we see that

$$\frac{1}{r_n^5} (\|u\|_{3;Q^n}^3 + \|\nabla d\|_{3;Q^n}^3) \lesssim \iint_{Q^n} (|u|^3 + |\nabla d|^3) dz \stackrel{(4-15)}{\lesssim} L_n^{3/2} \tag{4-18}$$

for any n , with a constant independent of n . In particular, for any n we have the estimate

$$\|u\|_{3;Q^n} + \|\nabla d\|_{3;Q^n} \lesssim r_n^{5/3} L_n^{1/2}. \tag{4-19}$$

Next, by Claim 1 and Remark 10 with $\psi := \phi$, $\Omega_2 := B^1$ and $\Omega_1 := B^2$, noting that $p \equiv \phi p$ on $Q^2 = Q_{1/4}(z_0) = B_{1/4}(x_0) \times (t_0 - (\frac{1}{4})^2, t_0)$, as in (4-5) we have

$$p(x, t) = S[\phi J(t)](x) + \int_{B^1 \setminus B^2} (2\nabla G^x \otimes_{\sigma} \nabla \phi + G^x \nabla^2 \phi) : J(t) \, dy + \int_{B^1 \setminus B^2} (2\nabla G^x \cdot \nabla \phi + G^x \Delta \phi) p(t) \, dy, \tag{4-20}$$

at almost every $(x, t) \in Q^2$, where

$$J := u \otimes u + \nabla d \odot \nabla d, \tag{4-21}$$

$2a \otimes_{\sigma} b := a \otimes b + b \otimes a$ and the operator S consisting of second derivatives of the Newtonian potential given by

$$S[\tilde{K}](x) := \nabla_x \cdot \left(\nabla_x^T \cdot \int_{B^1} G^x \tilde{K} \right)$$

for $\tilde{K} \in L^q(B^1)$ is a bounded linear Calderón–Zygmund operator on $L^q(B^1)$ for $1 < q < \infty$. Hence for any $n \in \mathbb{N}$, denoting by χ_n the indicator function for the set $B^n = B_{2^{-n}}(x_0)$ and splitting $\phi = \chi_n \phi + (1 - \chi_n)\phi$ in the first term of (4-20), we can write

$$p = p^{1,n} + p^{2,n} + p^{3,n} \equiv p^{1,n} + p^{2,n} + p^3,$$

where, for almost every $(x, t) \in Q^2$,

$$p(x, t) = \underbrace{S[\chi_n \phi J(t)](x)}_{=: p^{1,n}(x,t)} + \underbrace{S[(1 - \chi_n)\phi J(t)](x)}_{=: p^{2,n}(x,t)} + \underbrace{\int_{B^1 \setminus B^2} (2\nabla G^x \otimes_{\sigma} \nabla \phi + G^x \nabla^2 \phi) : J(t) \, dy + \int_{B^1 \setminus B^2} (2\nabla G^x \cdot \nabla \phi + G^x \Delta \phi) p(t) \, dy}_{=: p^{3,n}(x,t) \equiv p^3(x,t)}$$

(where the last term is clearly independent of n , but we keep the notation $p^{3,n}$ for convenience).

Note first that, by the classical Calderón–Zygmund estimates, there is a universal constant $C_{cz} > 0$ such that, for all $n \in \mathbb{N}$, we have

$$\|p^{1,n}(t)\|_{3/2; B^{n+1}} \leq C_{cz} \|\chi_n \phi J(t)\|_{3/2; \mathbb{R}^3} \leq C_{cz} \|\tilde{\phi}\|_{\infty; \mathbb{R}^3} \|J(t)\|_{3/2; B^n}. \tag{4-22}$$

Next, since the appearance of $\nabla \phi$ in p^3 exactly cuts off a neighborhood of the singularity of G^x (see (4-1)) uniformly for all $x \in B_{1/8}(x_0)$ — as we integrate over $|x_0 - y| \geq \frac{1}{4}$, hence $|x - y| \geq \frac{1}{8}$ — we see that $p^{3,n}(\cdot, t) \in C^\infty(B_{1/8}(x_0))$ for $t \in I_{1/8}(t_0)$ with, in particular,

$$\|\nabla_x p^{3,n}(t)\|_{\infty; B^{n+1}} \stackrel{(n \geq 2)}{\leq} \|\nabla_x p^{3,n}(t)\|_{\infty; B_{1/8}(x_0)} \leq c(\tilde{\phi})(\|J(t)\|_{1; B^1} + \|p(t)\|_{1; B^1}). \tag{4-23}$$

In the term $p^{2,n}$, the singularity coming from G^x is also isolated due to the appearance of χ_n , but it is no longer uniform in n so we must be more careful. As we are integrating over a region which avoids a neighborhood of the singularity at $y = x$ of G^x , we can pass the derivatives in S under the integral sign to write

$$\nabla_x p^{2,n}(x, t) = \int_{B^1 \setminus B^n} \nabla_x [(\nabla_x^2 G^x)^T : \phi J(t)] \, dy = \sum_{k=1}^{n-1} \int_{B^k \setminus B^{k+1}} \nabla_x [(\nabla_x^2 G^x)^T : \phi J(t)] \, dy$$

and note, in view of (4-1), that

$$|\nabla_x^3 G^x(y)| \lesssim \frac{1}{|x-y|^4} \leq (2^{k+2})^4 \lesssim \frac{2^k}{|B^k|} \quad \text{for all } x \in B^{k+2}, y \in (B^{k+1})^c.$$

Therefore, since

$$B^{n+1} = B^{(n-1)+2} \subseteq B^{k+2} \quad \text{for } 1 \leq k \leq n-1,$$

we see that

$$\|\nabla_x p^{2,n}(\cdot, t)\|_{\infty, B^{n+1}} \lesssim c(\tilde{\phi}) \sum_{k=1}^{n-1} 2^k \int_{B^k} |J(y, t)| dy \tag{4-24}$$

for all $t \in I_{1/8}(t_0)$.

Now, recalling the notation

$$\bar{f}_k(t) := \int_{B^k} f(x, t) dx$$

for a function $f(x, t)$ and $k \in \mathbb{N}$, for any $t \in I^2 = (t_0 - (\frac{1}{4})^2, t_0)$ and $n \geq 2$, we estimate

$$\begin{aligned} & \int_{B^{n+1}} |u(x, t)| |p(x, t) - \bar{p}_{n+1}(t)| dx \\ & \leq \sum_{j=1}^3 \int_{B^{n+1}} |u(x, t)| |p^{j,n}(x, t) - \bar{p}_{n+1}^{j,n}(t)| dx \\ & \leq \|u(\cdot, t)\|_{3; B^{n+1}} \sum_{j=1}^3 \|p^{j,n}(\cdot, t) - \bar{p}_{n+1}^{j,n}(t)\|_{3/2; B^{n+1}} \\ & \lesssim \|u(t)\|_{3; B^{n+1}} \left(\|p^{1,n}(t)\|_{3/2; B^{n+1}} + |B^{n+1}| \sum_{j=2}^3 \|\nabla p^{j,n}(t)\|_{\infty; B^{n+1}} \right) \quad (\text{by (4-16), (4-17) and Hölder}) \\ & \lesssim \|u(t)\|_{3; B^{n+1}} \left(\|J(t)\|_{3/2; B^n} + r_{n+1}^3 \left\{ \left(\sum_{k=1}^{n-1} 2^k \int_{B^k} |J(t)| dy \right) + \|J(t)\|_{3/2; B^1} + \|p(t)\|_{3/2; B^1} \right\} \right), \tag{4-25} \end{aligned}$$

where the last inequality follows from (4-22)–(4-24) and Hölder’s inequality. Note further that, setting

$$\mathbb{L}_{J,k} := \left\| \int_{B^k} |J(t)| dy \right\|_{L_t^\infty(I^k)}, \tag{4-26}$$

we have

$$\begin{aligned} \left\| \sum_{k=1}^{n-1} 2^k \int_{B^k} |J(t)| dy \right\|_{L_t^{3/2}(I^{n+1})} & \leq |I^{n+1}|^{2/3} \left(\max_{1 \leq k \leq n-1} \mathbb{L}_{J,k} \right) \sum_{k=1}^{n-1} 2^k \\ & \leq r_{n+1}^{1/3} \max_{1 \leq k \leq n-1} \mathbb{L}_{J,k}, \end{aligned}$$

since $|I^{n+1}| = r_{n+1}^2$ and

$$\sum_{k=1}^{n-1} 2^k = \frac{2^n - 2}{2 - 1} < 2^n = r_n^{-1}.$$

Integrating over $t \in I^{n+1}$ in (4-25), applying Hölder’s inequality in the variable t and recalling by (4-19) that $\|u\|_{3;Q^{n+1}} \lesssim r_{n+1}^{5/3} L_{n+1}^{1/2}$, we obtain

$$\iint_{Q^{n+1}} |u| |p - \bar{p}_{n+1}| dz \lesssim r_{n+1}^{5/3} L_{n+1}^{1/2} \{ \|J\|_{3/2;Q^n} + r_{n+1}^{10/3} \max_{1 \leq k \leq n-1} \mathbb{L}_{J,k} + r_{n+1}^3 (\|J\|_{3/2;Q^1} + \|p\|_{3/2;Q^1}) \}. \tag{4-27}$$

It follows now from (4-21) that

$$\|J\|_{3/2;Q^k} \leq \|u\|_{3;Q^k}^2 + \|\nabla d\|_{3;Q^k}^2 \stackrel{(4-19)}{\lesssim} (r_k^{5/3} L_k^{1/2})^2 = r_k^{10/3} L_k \tag{4-28}$$

and

$$\mathbb{L}_{J,k} \stackrel{(4-26)}{\leq} \left\| \int_{B^k} (|u(\cdot)|^2 + |\nabla d(\cdot)|^2) dy \right\|_{\infty;I^k} \leq L_k. \tag{4-29}$$

Now from (4-21), (4-27)–(4-29) and the simple fact that $\frac{1}{2}r_n = r_{n+1} \leq 1$ we obtain

$$\begin{aligned} r_{n+1}^{1/3} \iint_{Q^{n+1}} |u| |p - \bar{p}_{n+1}| dz &\lesssim L_{n+1}^{1/2} \{ r_{n+1}^{1/3} L_n + r_{n+1}^{1/3} \max_{1 \leq k \leq n-1} L_k + r_1^{10/3} L_1 + \|p\|_{3/2;Q^1} \} \\ &\lesssim L_{n+1}^{1/2} \{ \max_{1 \leq k \leq n} L_k + \|p\|_{3/2;Q^1} \}. \end{aligned}$$

Since

$$\iint_{Q^{n+1}} (|u|^3 + |\nabla d|^3) dz \stackrel{(4-18)}{\lesssim} L_{n+1}^{3/2},$$

adding the previous estimates and recalling (3-8) and (3-9) we have

$$R_{n+1} \lesssim L_{n+1}^{3/2} + L_{n+1}^{1/2} \left(\max_{1 \leq k \leq n} L_k + \|p\|_{3/2;Q^1} \right)$$

(where the constant is universal). This along with (3-13) implies (3-10) and proves Proposition 5. \square

4.3. Proof of Proposition 6. For simplicity, take $\bar{z} = z_0 = (0, 0)$, so that (recall (3-7)) $Q^k = Q^k(0, 0)$, etc., as the rest can be obtained by appropriate shifts.

We want to take the test function ϕ in (3-5) such that $\phi = \phi^n := \chi \psi^n$, where (recall that here $Q^1 = Q^1(0, 0) = B_{1/2}(0) \times (-\frac{1}{4}, 0)$) so χ will be zero in a neighborhood of the parabolic boundary of Q^1)

$$\chi \in C_0^\infty(B_{1/2}(0) \times (-\frac{1}{4}, \infty)), \quad \chi \equiv 1 \text{ in } Q^2, \quad 0 \leq \chi \leq 1 \tag{4-30}$$

and

$$\psi^n(x, t) := \frac{1}{(r_n^2 - t)^{3/2}} e^{-|x|^2/(4(r_n^2 - t))} \quad \text{for } t \leq 0. \tag{4-31}$$

Note that the singularity of ψ^n would naturally be at $(x, t) = (0, r_n^2) \notin Q^1$, so $\psi^n \in C^\infty(\bar{Q}_1)$ and we may extend ψ^n smoothly to $t > 0$ (where its values will actually be irrelevant) for each n so that, in particular,

$\phi^n \in C_0^\infty(B_1(0) \times (-1, \infty))$ as required²¹ in (3-5) with $(\bar{x}, \bar{t}) = (0, 0)$. Furthermore, we have

$$\nabla \psi^n(x, t) = -\frac{x}{2(r_n^2 - t)} \psi^n(x, t) \quad \text{and} \quad \psi_t^n + \Delta \psi^n \equiv 0 \quad \text{in } Q^1. \quad (4-32)$$

Note first that for $(x, t) \in Q^n$ ($n \geq 2$), we have

$$0 \leq |x| \leq r_n \quad \text{and} \quad r_n^2 \leq [r_n^2 - t] \leq 2r_n^2$$

so that

$$r_n^3 = (r_n^2)^{3/2} e^{0/(8r_n^2)} \leq (r_n^2 - t)^{3/2} e^{|x|^2/(4(r_n^2 - t))} \leq (2r_n^2)^{3/2} e^{r_n^2/(4r_n^2)} = 2^{3/2} e^{1/4} r_n^3.$$

Hence

$$\frac{1}{2^{3/2} e^{1/4}} \cdot \frac{1}{r_n^3} \leq \psi^n(x, t) \leq \frac{1}{r_n^3} \quad \text{for all } (x, t) \in Q^n \quad (4-33)$$

and therefore (as $r_n^2 - t > 0$)

$$|\nabla_x \psi^n(x, t)| = \frac{|x|}{2(r_n^2 - t)} |\psi^n(x, t)| \lesssim \frac{r_n}{r_n^2} \cdot \frac{1}{r_n^3} = \frac{1}{r_n^4} \quad \text{for all } (x, t) \in Q^n. \quad (4-34)$$

Next, note similarly that for $2 \leq k \leq n$ and $(x, t) \in Q^{k-1} \setminus Q^k$, we have

$$r_k \leq |x| \leq r_{k-1} = 2r_k \quad \text{and} \quad r_k^2 \leq r_n^2 + r_k^2 \leq [r_n^2 - t] \leq r_n^2 + r_{k-1}^2 \leq 2r_{k-1}^2 = 8r_k^2,$$

so that

$$e^{1/32} r_k^3 = (r_k^2)^{3/2} e^{r_k^2/(32r_k^2)} \leq (r_n^2 - t)^{3/2} e^{|x|^2/(4(r_n^2 - t))} \leq (8r_k^2)^{3/2} e^{(2r_k)^2/(4r_k^2)} = 2^{9/2} e r_k^3.$$

Therefore

$$\frac{1}{2^{9/2} e} \cdot \frac{1}{r_k^3} \leq \psi^n(x, t) \leq \frac{1}{e^{1/32}} \cdot \frac{1}{r_k^3} \quad \text{for all } (x, t) \in Q^{k-1} \setminus Q^k \quad (2 \leq k \leq n) \quad (4-35)$$

and hence, as in (4-34),

$$|\nabla_x \psi^n(x, t)| \lesssim \frac{r_k}{r_k^2} \cdot \frac{1}{r_k^3} = \frac{1}{r_k^4} \quad \text{for all } (x, t) \in Q^{k-1} \setminus Q^k \quad (2 \leq k \leq n). \quad (4-36)$$

We can therefore estimate (for $n \geq 2$ where $\phi^n = \psi^n$ in Q^n):

$$\begin{aligned} & \frac{1}{2^{3/2} e^{1/4}} \cdot \frac{1}{r_n^3} \left[\text{ess sup}_{I^n} \int_{B^n} (|u|^2 + |\nabla d|^2) + \iint_{Q^n} (|\nabla u|^2 + |\nabla^2 d|^2) \right] \\ & \stackrel{(4-33)}{\leq} \text{ess sup}_{I^n} \int_{B^n} (|u|^2 + |\nabla d|^2) \phi^n + \iint_{Q^n} (|\nabla u|^2 + |\nabla^2 d|^2) \phi^n \\ & \leq \bar{C} \left\{ \iint_{Q^1} [(|u|^2 + |\nabla d|^2) |\phi_t^n + \Delta \phi^n| + (|u|^3 + |\nabla d|^3) |\nabla \phi^n| + \bar{\rho} |d|^2 |\nabla d|^2 \phi^n] + \int_{I^1} \left| \int_{B^1} p u \cdot \nabla \phi^n \right| \right\}, \end{aligned}$$

where the last inequality follows from (3-5). Note that

$$\phi_t^n + \Delta \phi^n \stackrel{(4-32)}{=} \psi^n(\chi_t + \Delta \chi) + 2 \nabla \chi \cdot \nabla \psi^n \stackrel{(4-30)}{=} 0 \quad \text{in } Q^2$$

²¹In (3-5) as well, the values of ϕ for $t > \bar{t}$ are actually irrelevant.

and hence, taking $k = 2$ in (4-35) and (4-36), we see that

$$|\phi_t^n + \Delta\phi^n| \lesssim \frac{1}{r_2^3} + \frac{1}{r_2^4} \lesssim 1 \quad \text{on } Q^1, \tag{4-37}$$

so that

$$\iint_{Q^1} (|u|^2 + |\nabla d|^2) |\phi_t^n + \Delta\phi^n| \stackrel{(4-37)}{\lesssim} \iint_{Q^1} (|u|^2 + |\nabla d|^2) \stackrel{(3-6)}{\lesssim} E_{3,q}^{2/3}$$

by Hölder’s inequality. Note similarly that

$$|\nabla\phi^n| = |\chi\nabla\psi^n + \psi^n\nabla\chi| \stackrel{(4-30)}{\lesssim} |\nabla\psi^n| + |\psi^n| \quad \text{on } Q^1$$

so that (since $r_n^4 < r_n^3$) (4-33), (4-34) and (4-35), (4-36), respectively, give

$$|\nabla\phi^n| \lesssim \frac{1}{r_n^4} \quad \text{on } Q^n \quad \text{and} \quad |\nabla\phi^n| \lesssim \frac{1}{r_k^4} \quad \text{on } Q^{k-1} \setminus Q^k \tag{4-38}$$

for any $n \geq 2$ and $2 \leq k \leq n$. Therefore

$$\sum_{k=2}^n \iint_{Q^{k-1} \setminus Q^k} (|u|^3 + |\nabla d|^3) |\nabla\phi^n| \stackrel{(4-38)}{\lesssim} \left[\max_{1 \leq k \leq n-1} (r_k)^{1-\alpha} \iint_{Q^k} (|u|^3 + |\nabla d|^3) \right] \sum_{k=2}^n (r_k)^\alpha$$

and similarly

$$\iint_{Q^n} (|u|^3 + |\nabla d|^3) |\nabla\phi^n| \stackrel{(4-38)}{\lesssim} \left[(r_n)^{1-\alpha} \iint_{Q^n} (|u|^3 + |\nabla d|^3) \right] (r_n)^\alpha$$

for any $\alpha \in (0, 1]$, and we note that

$$\sum_{k=1}^\infty (r_k)^\alpha = \sum_{k=1}^\infty (2^{-\alpha})^k = \frac{1}{2^\alpha - 1} < \infty \quad \text{for any } \alpha > 0. \tag{4-39}$$

Hence in view of the disjoint union

$$Q^1 = \left(\bigcup_{k=2}^n Q^{k-1} \setminus Q^k \right) \cup Q^n \tag{4-40}$$

we have, taking $\alpha = 1$ in (4-39),

$$\iint_{Q^1} (|u|^3 + |\nabla d|^3) |\nabla\phi^n| \lesssim \max_{1 \leq k \leq n} \iint_{Q^k} (|u|^3 + |\nabla d|^3).$$

Similarly, setting

$$\alpha_q := \frac{2(q-5)}{q-2}$$

and noting that $\alpha_q \in (0, \frac{1}{2}]$ for $q \in (5, 6]$, we have

$$\bar{\rho} \iint_{Q^1} |d|^2 |\nabla d|^2 \phi^n \leq \frac{2}{q} \underbrace{\iint_{Q^1} |d|^q |\nabla d|^{3(1-q/6)}}_{\leq E_{3,q}} + \left(1 - \frac{2}{q}\right) \iint_{Q^1} |\nabla d|^3 (\phi^n)^{(5-\alpha_q)/3}$$

uniformly, of course, over $\bar{\rho} \in (0, 1]$. Since

$$\iint_{Q^n} |\nabla d|^3 (\phi^n)^{(5-\alpha_q)/3} \stackrel{(4-33)}{\lesssim} (r_n)^{\alpha_q-5} \iint_{Q^n} |\nabla d|^3 \lesssim (r_n)^{\alpha_q} \iint_{Q^n} |\nabla d|^3$$

(as $\phi^n \lesssim r_n^{-3}$ on Q^n) for $n \geq 2$ and similarly

$$\iint_{Q^k \setminus Q^{k+1}} |\nabla d|^3 (\phi^n)^{(5-\alpha_q)/3} \stackrel{(4-35)}{\lesssim} (r_k)^{\alpha_q-5} \iint_{Q^k} |\nabla d|^3 \lesssim (r_k)^{\alpha_q} \iint_{Q^k} |\nabla d|^3$$

(as $\phi^n \lesssim r_k^{-3}$ on $Q^k \setminus Q^{k+1}$) for $1 \leq k \leq n-1$, we see that (4-39) with $\alpha = \alpha_q$ and (4-40) again give

$$\iint_{Q^1} |\nabla d|^3 (\phi^n)^{(5-\alpha_q)/3} \leq (2^{\alpha_q} - 1)^{-1} \max_{1 \leq k \leq n} \iint_{Q^k} |\nabla d|^3.$$

We therefore see that

$$\bar{\rho} \iint_{Q^1} |d|^2 |\nabla d|^2 \phi^n \lesssim \frac{2}{5} E_{3,q} + \frac{2}{3} (2^{\alpha_q} - 1)^{-1} \max_{1 \leq k \leq n} \iint_{Q^k} |\nabla d|^3 \quad \text{with } \alpha_q := \frac{2(q-5)}{q-2},$$

uniformly for any $\bar{\rho} \in (0, 1]$ and $q \in (5, 6]$.

Putting all of the above together and recalling (3-8), we see that for $n \geq 2$ we have

$$\begin{aligned} \frac{L_n}{\bar{C}} &= \frac{1}{\bar{C}} \left[\text{ess sup}_{I^n} \int_{B^n} (|u|^2 + |\nabla d|^2) + \int_{I^n} \int_{B^n} (|\nabla u|^2 + |\nabla^2 d|^2) \right] \\ &\lesssim E_{3,q} + E_{3,q}^{2/3} + (2^{\alpha_q} - 1)^{-1} \max_{1 \leq k \leq n} \iint_{Q^k} (|u|^3 + |\nabla d|^3) + \int_{I^1} \left| \int_{B^1} pu \cdot \nabla \phi^n \right|. \end{aligned} \quad (4-41)$$

Furthermore, we claim that for $1 \leq k_0 \leq n-1$ we have

$$\int_{I^1} \left| \int_{B^1} pu \cdot \nabla \phi^n \right| \lesssim \max_{k_0 \leq k \leq n} \left(r_k^{1/3} \iint_{Q^k} |p - \bar{p}_k| |u| \right) + k_0 2^{4k_0} \iint_{Q^1} |p| |u|. \quad (4-42)$$

Assuming this for the moment and continuing, for $n \geq 2$, (4-41), (4-42) and Young’s convexity inequality along with the fact that, for any $k_1 \geq 1$, we can estimate

$$\max_{1 \leq k \leq k_1} \iint_{Q^k} (|u|^3 + |\nabla d|^3) \lesssim k_1 2^{5k_1} \iint_{Q^1} (|u|^3 + |\nabla d|^3)$$

imply (recalling (3-9)) that

$$\frac{L_n}{\bar{C}} \lesssim E_{3,q} + E_{3,q}^{2/3} + (2^{\alpha_q} - 1)^{-1} \max_{k_0 \leq k \leq n} R_k + k_0 2^{5k_0} \underbrace{\iint_{Q^1} |u|^3 + |\nabla d|^3 + |p|^{3/2}}_{\leq E_{3,q}}$$

for any $k_0 \in \{1, \dots, n-1\}$, which proves Proposition 6.

To prove (4-42), we consider additional functions χ_k (so that $\chi_k \phi^n = \chi_k \chi \psi^n$) satisfying (recall that $Q^k = Q^k(0, 0) = B_{r_k}(0) \times (-r_k^2, 0)$, so χ_k will be zero in a neighborhood of the parabolic boundary of Q^k)

$$\begin{aligned} \chi_k &\in C_0^\infty(\tilde{Q}_{r_k}) \quad \text{with } \tilde{Q}_r := B_r(0) \times (-r^2, r^2) \quad \text{for } r > 0, \\ \chi_k &\equiv 1 \quad \text{in } \tilde{Q}_{7r_k/8}, \quad 0 \leq \chi_k \leq 1 \quad \text{and} \quad |\nabla \chi_k| \lesssim \frac{1}{r_k} \end{aligned} \quad (4-43)$$

($\chi_k|_{\{t>0\}}$ will again actually be irrelevant) so that in particular (as $\tilde{Q}_{r_{k+2}} \subset \tilde{Q}_{7r_{k+1}/8}$ where $\chi_k \equiv \chi_{k+1} \equiv 1$)

$$\text{supp}(\chi_k - \chi_{k+1}) \subset \tilde{Q}_{r_k} \setminus \tilde{Q}_{r_{k+2}}. \tag{4-44}$$

Then, fixing any $n \geq 2$, writing

$$\chi_0 = \chi_n + \sum_{k=0}^{n-1} (\chi_k - \chi_{k+1})$$

and noting that $\chi_0 \equiv 1$ on $Q^1 = Q_{1/2}(0, 0) \subset Q_{7/8}(0, 0) = Q_{7r_0/8}(0, 0)$, we see that for any fixed $k_0 \in \mathbb{N} \cap [1, n - 1]$ and at each fixed $\tau \in I^1$, we have

$$\begin{aligned} \int_{B^1} pu \cdot \nabla \phi^n &= \int_{B^1} pu \cdot \nabla [\chi_0 \phi^n] && \text{(by (4-43))} \\ &= \int_{B^1} pu \cdot \nabla [\chi_n \phi^n] + \sum_{k=0}^{n-1} \int_{B^1} pu \cdot \nabla [(\chi_k - \chi_{k+1}) \phi^n] \\ &= \int_{B^n} pu \cdot \nabla [\chi_n \phi^n] + \sum_{k=0}^{n-1} \int_{[B^k \setminus B^{k+2}]} pu \cdot \nabla [(\chi_k - \chi_{k+1}) \phi^n] && \text{(by (4-43), (4-44))} \\ &= \int_{B^n} (p - \bar{p}_n) u \cdot \nabla [\chi_n \phi^n] + \sum_{k=0}^{k_0-1} \int_{[B^k \setminus B^{k+2}]} pu \cdot \nabla [(\chi_k - \chi_{k+1}) \phi^n] \\ &\quad + \sum_{k=k_0}^{n-1} \int_{[B^k \setminus B^{k+2}]} (p - \bar{p}_k) u \cdot \nabla [(\chi_k - \chi_{k+1}) \phi^n], \end{aligned} \tag{4-45}$$

where the final equality is due to (3-3), and where

$$\bar{p}_k = \bar{p}_k(\tau) = \int_{B^k} p(x, \tau) dx.$$

Note first that (4-35), (4-36) and (4-44) imply (since $r_{j+1} = 2r_j$ for any j) that

$$\begin{aligned} |\nabla [(\chi_k - \chi_{k+1}) \phi^n]| &\leq |\chi_k - \chi_{k+1}| |\nabla \phi^n| + |\phi^n| |\nabla (\chi_k - \chi_{k+1})| \lesssim r_k^{-4} \\ &\text{on } Q^k \setminus Q^{k+2} = (Q^k \setminus Q^{k+1}) \cup (Q^{k+1} \setminus Q^{k+2}) \end{aligned}$$

for any k , and similarly

$$|\nabla [\chi_n \phi^n]| \leq |\chi_n| |\nabla \phi^n| + |\phi^n| |\nabla \chi_n| \lesssim r_n^{-4} \quad \text{on } Q^n.$$

Therefore we can estimate (recalling again (4-43) and (4-44) when integrating |(4-45)| over $\tau \in I^1$)

$$\int_{\tau \in I^1} \left| \int_{B^1 \times \{\tau\}} pu \cdot \nabla \phi^n \right| \lesssim k_0 2^{4k_0} \iint_{Q^1} |p| |u| + \sum_{k=k_0}^n r_k \iint_{Q^k} |p - \bar{p}_k| |u|$$

which, along with (4-39) with $q = \frac{3}{2}$ implies (4-42) for any $k_0 \in [1, n - 1]$ as desired. □

4.4. Proof of Proposition 8. In this section we prove the technical decay estimate (Proposition 8) used to prove Lemma 7. In all of what follows, recall the definitions in (3-17) and (3-18) of $A_{z_0}, B_{z_0}, C_{z_0}, D_{z_0}, E_{z_0}, F_{z_0}, G_{q,z_0}$ and M_{q,z_0} . We will require the following three claims which essentially appear in [Lin and Liu 1996] and which generalize certain lemmas in [Caffarelli et al. 1982]; however we include full proofs

in order to clarify certain details, and to highlight the role of G_{q,z_0} (not utilized in [Lin and Liu 1996]) in Claim 4 which is therefore²² a slightly refined version of what appears in [Lin and Liu 1996].

Claim 2 (general estimates; cf. [Caffarelli et al. 1982, Lemmas 5.1 and 5.2]). *There exist constants $c_1, c_2 > 0$ such that for any u and d which have the regularities in (1-9) for $\Omega_T := \Omega \times (0, T)$ as in Theorem 1, the estimates*

$$C_{z_0}(\gamma\rho) \leq c_1[\gamma^3 A_{z_0}^{3/2} + \gamma^{-3} A_{z_0}^{3/4} B_{z_0}^{3/4}](\rho) \tag{4-46}$$

and

$$E_{z_0}(\gamma\rho) \leq c_2[C_{z_0}^{1/3} A_{z_0}^{1/2} B_{z_0}^{1/2}](\gamma\rho) \tag{4-47}$$

hold for any $z_0 \in \mathbb{R}^{3+1}$ and $\rho > 0$ such that $Q_\rho^*(z_0) \subseteq \Omega_T$ and any $\gamma \in (0, 1]$.

Claim 3 (estimates requiring the pressure equation; cf. [Caffarelli et al. 1982, Lemmas 5.3 and 5.4]). *There exist constants $c_3, c_4 > 0$ such that for any u, d and p which have the regularities in (1-9) and (1-10) for $\Omega_T := \Omega \times (0, T)$ as in Theorem 1 and which satisfy the pressure equation (1-12), the estimates*

$$D_{z_0}(\gamma\rho) \leq c_3[\gamma(D_{z_0} + A_{z_0}^{3/4} B_{z_0}^{3/4} + C_{z_0}^{1/2}) + \gamma^{-5} A_{z_0}^{3/4} B_{z_0}^{3/2}](\rho) \tag{4-48}$$

and

$$F_{z_0}(\gamma\rho) \leq c_4[\gamma^{1/12}(A_{z_0} + D_{z_0}^{4/3} + C_{z_0}^{2/3}) + \gamma^{-10} A_{z_0}(B_{z_0}^{1/2} + B_{z_0}^2)](\rho). \tag{4-49}$$

hold for any $z_0 \in \mathbb{R}^{3+1}$ and $\rho > 0$ such that $Q_\rho^*(z_0) \subseteq \Omega_T$ and any $\gamma \in (0, \frac{1}{2}]$.

The crucial aspect of the estimates (4-46)–(4-49) — which control $M_{q,z_0}(\gamma\rho)$ — in proving Lemma 7 (through Proposition 8) is that whenever a negative power of γ appears, there is always a factor of B_{z_0} as well, which will be small when proving Lemma 7. Positive powers of γ will similarly be small; in each term evaluated at ρ (see also (4-52) below), we must have either γ^α or $B_{z_0}^\alpha$ for some $\alpha > 0$.

To complete the proof of Proposition 8, we require the following.

Claim 4 (estimate requiring the local energy inequality; cf. [Caffarelli et al. 1982, Lemma 5.5]). *There exists a constant $c_5 > 0$ such that for any u, d and p which have the regularities in (1-9) and (1-10) for $\Omega_T := \Omega \times (0, T)$ as in Theorem 1 and such that u satisfies the weak divergence-free property (1-11) and the local energy inequality (1-13) holds for some constant $\bar{C} \in (0, \infty)$, the estimate*

$$A_{z_0}\left(\frac{\rho}{2}\right) \leq c_5 \cdot \bar{C}[C_{z_0}^{2/3} + E + F_{z_0} + (1 + [\cdot]^2)G_q^{4/(6-q)} + (G_q^{2/(6-q)} + C_{z_0}^{1/3})B_{z_0}^{1/2}](\rho) \tag{4-50}$$

holds for any $q \in [2, 6)$ and any $z_0 \in \mathbb{R}^{3+1}$ and $\rho > 0$ such that $Q_\rho^*(z_0) \subseteq \Omega_T$.

Postponing the proof of the claims, let us use them to prove the proposition. In all of what follows, we note the simple facts that, for any $\rho > 0$ and $\alpha \in (0, 1]$,

$$\begin{aligned} \mathcal{K} \in \{A_{z_0}, B_{z_0}\} &\implies \mathcal{K}(\alpha\rho) \leq \alpha^{-1}\mathcal{K}(\rho), \\ \mathcal{K} \in \{C_{z_0}, D_{z_0}, E_{z_0}, F_{z_0}\} &\implies \mathcal{K}(\alpha\rho) \leq \alpha^{-2}\mathcal{K}(\rho), \\ G_{q,z_0}(\alpha\rho) &\leq \alpha^{-2-q/2}G_{q,z_0}(\rho). \end{aligned} \tag{4-51}$$

²²Note that $G_{z_0}(r) \lesssim \|d\|_\infty$ uniformly in r (and z_0), though in our setting we may have $d \notin L^\infty$.

Proof of Proposition 8. Fixing z_0 and ρ_0 as in Proposition 8, under the assumptions in the proposition we see that estimates (4-46)–(4-50) hold for all $\rho \in (0, \rho_0]$, $\gamma \in (0, \frac{1}{2}]$ and $q \in [2, 6)$ by Claims 2, 3 and 4.

Note first that (4-46), (4-47) and (4-51) imply that

$$E_{z_0}(\gamma\rho) \lesssim [A_{z_0} B_{z_0}^{1/2} + \gamma^{-2} A_{z_0}^{3/4} B_{z_0}^{3/4}](\rho)$$

and hence, for example, there exists some $c_6 > 0$ such that

$$E_{z_0}(\gamma\rho) \leq c_6[\gamma^2 A_{z_0} + \gamma^{-2}(A_{z_0}^{1/2} B_{z_0}^{1/2} + A_{z_0} B_{z_0})](\rho), \tag{4-52}$$

for $\rho \in (0, \rho_0]$ and $\gamma \in (0, \frac{1}{2}]$ (in fact, for $\gamma \in (0, 1]$) and that it follows from (4-50), assumption (3-20) and the assumption that $\rho_0 \leq 1$ that there exists some $c_7 > 0$ such that

$$(\bar{C})^{-1} A_{z_0} \left(\frac{\rho}{2}\right) \leq c_7[C_{z_0}^{2/3} + E_{z_0} + F_{z_0} + G_{q,z_0}^{4/(6-q)} + (G_{q,z_0}^{2/(6-q)} + C_{z_0}^{1/3}) B_{z_0}^{1/2}](\rho),$$

and hence, recalling (3-18), we have that, for some $c_8 > 0$,

$$(\bar{C})^{-3/2} A_{z_0}^{3/2} \left(\frac{\rho}{2}\right) \leq c_8[M_{q,z_0}(\rho) + M_{q,z_0}^{1/2}(\rho) B_{z_0}^{3/4}(\rho)] \tag{4-53}$$

for $\rho \in (0, \rho_0]$. We note as well that, as in (3-23), if $\sigma \in [q, 6)$ and if (3-20) holds for some $\bar{g} \geq 1$, then

$$G_{q,z_0}^{6/(6-q)}(\gamma\rho) \stackrel{(3-22)}{\leq} \bar{g}^{6/(6-\sigma)} \cdot C_{z_0}^{\alpha_{\sigma,q}}(\gamma\rho) \stackrel{(4-46)}{\leq} \bar{g}^{6/(6-\sigma)} \cdot [\gamma^3 A_{z_0}^{3/2} + \gamma^{-3} A_{z_0}^{3/4} B_{z_0}^{3/4}]^{\alpha_{\sigma,q}}(\rho) \tag{4-54}$$

for $\rho \in (0, \rho_0]$. Now, writing $\gamma\rho = 2\gamma \cdot (\frac{1}{2}\rho)$ for $2\gamma \leq \frac{1}{2}$, it follows from (4-46), (4-48), (4-49), (4-52), (4-54) and (3-18) followed by an application of (4-51) (with $\alpha = \frac{1}{2}$) to all terms except for A_{z_0} along with the facts that $\gamma, B_{z_0}(\rho) \leq 1$ (so that you can always estimate positive powers by 1) as well as the fact that $\alpha_{\sigma,q} \in (0, 1)$ that

$$\begin{aligned} M_{q,z_0}(\gamma\rho) &\leq [C_{z_0} + G_{q,z_0}^{6/(6-q)} + D_{z_0}^2 + E_{z_0}^{3/2} + F_{z_0}^{3/2}](\gamma\rho) \\ &\lesssim \left[\gamma^3 A_{z_0}^{3/2} \left(\frac{\rho}{2}\right) + \gamma^{-3} A_{z_0}^{3/4} \left(\frac{\rho}{2}\right) B_{z_0}^{3/4}(\rho)\right] + \bar{g}^{6/(6-\sigma)} \cdot \left[\gamma^3 A_{z_0}^{3/2} \left(\frac{\rho}{2}\right) + \gamma^{-3} A_{z_0}^{3/4} \left(\frac{\rho}{2}\right) B_{z_0}^{3/4}(\rho)\right]^{\alpha_{\sigma,q}} \\ &\quad + \left[\gamma M_{q,z_0}^{1/2}(\rho) + \gamma^{-5} A_{z_0}^{3/4} \left(\frac{\rho}{2}\right) (B_{z_0}^{3/4}(\rho) + B_{z_0}^{3/2}(\rho))\right]^2 \\ &\quad + \left[\gamma^2 A_{z_0} \left(\frac{\rho}{2}\right) + \gamma^{-2} \left(A_{z_0}^{1/2} \left(\frac{\rho}{2}\right) B_{z_0}^{1/2}(\rho) + A_{z_0} \left(\frac{\rho}{2}\right) B_{z_0}(\rho)\right)\right]^{3/2} \\ &\quad + \left[\gamma^{1/12} \left(A_{z_0} \left(\frac{\rho}{2}\right) + M_{q,z_0}^{2/3}(\rho)\right) + \gamma^{-10} A_{z_0} \left(\frac{\rho}{2}\right) (B_{z_0}^{1/2}(\rho) + B_{z_0}^2(\rho))\right]^{3/2} \\ &\lesssim (1 + \bar{g}^{6/(6-\sigma)}) \left[\gamma^{\alpha_{\sigma,q}/8} \left(M_{q,z_0}(\rho) + \left[A_{z_0}^{3/2} \left(\frac{\rho}{2}\right)\right]^{\alpha_{\sigma,q}} + \left[A_{z_0}^{3/2} \left(\frac{\rho}{2}\right)\right]\right) \right. \\ &\quad \left. + \gamma^{-15} \left(\left[A_{z_0}^{3/2} \left(\frac{\rho}{2}\right)\right]^{\alpha_{\sigma,q}/2} + \left[A_{z_0}^{3/2} \left(\frac{\rho}{2}\right)\right]^{1/2} + \left[A_{z_0}^{3/2} \left(\frac{\rho}{2}\right)\right]\right) B_{z_0}^{3\alpha_{\sigma,q}/4}(\rho) \right] \end{aligned}$$

so long as $\gamma \in (0, \frac{1}{4}]$. Noting that $1 \leq \bar{g}^{6/(6-\sigma)}$, the estimate (3-21) for such γ and for $\rho \in (0, \rho_0]$ now follows from the estimate above along with (4-53) as, in particular, (4-53) implies — as $\gamma, B_{z_0}(\rho) \leq 1$

and $\alpha_{\sigma,q} \in (0, 1)$ — that

$$(\bar{C})^{-3/2} A_{z_0}^{3/2} \left(\frac{\rho}{2}\right) \lesssim M_{q,z_0}(\rho) + \gamma^{-15-\alpha_{\sigma,q}/8} M_{q,z_0}^{1/2}(\rho) B_{z_0}^{3\alpha_{\sigma,q}/4}(\rho)$$

which we apply to the terms above with a positive power of γ , and that

$$(\bar{C})^{-3/2} A_{z_0}^{3/2} \left(\frac{\rho}{2}\right) \lesssim M_{q,z_0}(\rho) + M_{q,z_0}^{1/2}(\rho),$$

which we apply to the terms above with a negative power of γ , completing the proof. \square

Let us now prove the claims.

Proof of Claim 2. For simplicity, we will suppress the dependence on $z_0 = (x_0, t_0)$ in what follows.

Let us first prove (4-46). Note that for any $r \leq \rho$, at any fixed $t \in I_r^*$, taking $v \in \{u, \nabla d\}$ we have

$$\begin{aligned} \int_{B_r} |v|^2 dx &\leq \int_{B_\rho} \left| |v|^2 - \overline{|v|^2}^\rho \right| dx + |B_r| \overline{|v|^2}^\rho \\ &\lesssim \rho \int_{B_\rho} |\nabla |v|^2| dx + \left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |v|^2 dx \end{aligned}$$

due to Poincaré’s inequality (4-16). Since $|\nabla |v|^2| \leq |v| |\nabla v|$ almost everywhere, Hölder’s inequality then implies that

$$\|v\|_{2;B_r}^2 \lesssim \rho \|v\|_{2;B_\rho} \|\nabla v\|_{2;B_\rho} + \left(\frac{r}{\rho}\right)^3 \|v\|_{2;B_\rho}^2. \tag{4-55}$$

Therefore

$$\begin{aligned} \|v\|_{3;B_r}^3 &\lesssim \frac{1}{r^{3/2}} (\|v\|_{2;B_r}^2)^{3/2} + \|v\|_{2;B_r}^{3/2} \|\nabla v\|_{2;B_r}^{3/2} && \text{(by (4-14))} \\ &\lesssim \left(1 + \left(\frac{\rho}{r}\right)^{3/2}\right) \|v\|_{2;B_\rho}^{3/2} \|\nabla v\|_{2;B_\rho}^{3/2} + \frac{1}{r^{3/2}} \left(\frac{r}{\rho}\right)^{9/2} \|v\|_{2;B_\rho}^3 && \text{(by (4-55)).} \end{aligned}$$

Summing over $v \in \{u, \nabla d\}$, we see that

$$\begin{aligned} \|u\|_{3;B_r}^3 + \|\nabla d\|_{3;B_r}^3 &\lesssim \left(1 + \left(\frac{\rho}{r}\right)^{3/2}\right) (\|u\|_{2;B_\rho}^2 + \|\nabla d\|_{2;B_\rho}^2)^{3/4} (\|\nabla u\|_{2;B_\rho}^2 + \|\nabla^2 d\|_{2;B_\rho}^2)^{3/4} + \frac{r^3}{\rho^{9/2}} (\|u\|_{2;B_\rho}^2 + \|\nabla d\|_{2;B_\rho}^2)^{3/2}. \end{aligned}$$

Now integrating over $t \in I_r^*$ (where $|I_r^*| = r^2$), Hölder’s inequality implies that

$$\begin{aligned} r^2 C(r) &\lesssim |I_r^*|^{1/4} \left(1 + \left(\frac{\rho}{r}\right)^{3/2}\right) \| \|u\|_{2;B_\rho}^2 + \|\nabla d\|_{2;B_\rho}^2 \|_{\infty;I_r^*}^{3/4} (\|\nabla u\|_{2;Q_\rho^*}^2 + \|\nabla^2 d\|_{2;Q_\rho^*}^2)^{3/4} \\ &\quad + |I_r^*| \frac{r^3}{\rho^{9/2}} \| \|u\|_{2;B_\rho}^2 + \|\nabla d\|_{2;B_\rho}^2 \|_{\infty;I_r^*}^{3/2} \\ &\lesssim r^{1/2} \left(1 + \left(\frac{\rho}{r}\right)^{3/2}\right) (\rho A(\rho))^{3/4} (\rho B(\rho))^{3/4} + \frac{r^5}{\rho^{9/2}} (\rho A(\rho))^{3/2}, \end{aligned}$$

which, upon dividing both sides by r^2 , setting $\gamma := r/\rho$ and noting that $1 \leq \gamma^{-3/2}$, precisely gives (4-46).

Next, to prove (4-47), we use the Poincaré–Sobolev inequality

$$\|g - \bar{g}^r\|_{q^*; B_r} \leq c_q \|\nabla g\|_{q; B_r}$$

(the constant is independent of r due to the relationship between q and q^*) corresponding to the embedding $W^{1,q} \hookrightarrow L^{q^*}$ for $q < 3$ (in \mathbb{R}^3) and $q^* = 3q/(3 - q)$. Taking $q = 1$, at any $t \in I_r^*$ and for $v \in \{u, \nabla d\}$ the Hölder and Poincaré–Sobolev inequalities give us

$$\int_{B_r} |u| | |v|^2 - \overline{|v|^2}^r | dx \leq \|u\|_{3; B_r} \| |v|^2 - \overline{|v|^2}^r \|_{3/2; B_r} \lesssim \|u\|_{3; B_r} \|\nabla(|v|^2)\|_{1; B_r} \lesssim \|u\|_{3; B_r} \|v\|_{2; B_r} \|\nabla v\|_{2; B_r}.$$

Summing this first over $v \in \{u, \nabla d\}$ at a fixed t and then integrating over $t \in I_r^*$, we see that

$$\begin{aligned} r^2 E(r) &\lesssim \int_{I_r^*} \|u\|_{3; B_r} (\|u\|_{2; B_r}^2 + \|\nabla d\|_{2; B_r}^2)^{1/2} (\|\nabla d\|_{2; B_r}^2 + \|\nabla^2 d\|_{2; B_r}^2)^{1/2} dt \\ &\lesssim \|u\|_{3; Q_r^*} (\|u\|_{2; B_r}^2 + \|\nabla d\|_{2; B_r}^2)^{1/2} \|_{6; I_r^*} (\|\nabla u\|_{2; Q_r^*}^2 + \|\nabla^2 d\|_{2; Q_r^*}^2)^{1/2} \\ &\lesssim |I_r^*|^{1/6} (\|u\|_{3; Q_r^*}^3)^{1/3} \| \|u\|_{2; B_r}^2 + \|\nabla d\|_{2; B_r}^2 \|_{\infty; I_r^*}^{1/2} (\|\nabla u\|_{2; Q_r^*}^2 + \|\nabla^2 d\|_{2; Q_r^*}^2)^{1/2} \\ &\lesssim r^{1/3} (r^2 C(r))^{1/3} (r A(r))^{1/2} (r B(r))^{1/2} = r^2 [C^{1/3} A^{1/2} B^{1/2}](r) \end{aligned}$$

which proves (4-47) and completes the proof of Claim 2. □

Proof of Claim 3. As in (4-3) of Claim 1, for any $t \in I_r^*(z_0)$ and almost every $x \in B_{3\rho/4}(x_0)$ (with $r \leq \rho$), using a smooth cut-off function ψ equal to one in $\Omega_1 := B_{3\rho/4}(x_0)$ and supported in $\Omega_2 := B_\rho(x_0)$ so that

$$|\nabla \psi| \lesssim \rho^{-1} \quad \text{and} \quad |\Delta \psi| \lesssim \rho^{-2}, \tag{4-56}$$

we use Remark 10 to write $\Pi := p(\cdot, t)$ as

$$p(x, t) = - \underbrace{\int \nabla G^x \cdot v(t) \psi dy}_{=: p_1(x, t)} + \underbrace{\int G_{\psi, 1}^x \cdot v(t) dy}_{=: p_2(x, t)} + \underbrace{\int G_{\psi, 2}^x p(\cdot, t) dy}_{=: p_3(x, t)}$$

with

$$G_{\psi, 1}^x := -G^x \nabla \psi, \quad G_{\psi, 2}^x := 2\nabla G^x \cdot \nabla \psi + G^x \Delta \psi \quad \text{and} \quad v(t) := [\nabla^T \cdot (u \otimes u + \nabla d \odot \nabla d)](\cdot, t).$$

Our goal is to estimate $p(x, t)$ for $x \in B_{\rho/2}(x_0)$.

Both p_2 and p_3 contain derivatives of ψ in each term so that the integrand can only be nonzero when $|y - x_0| > \frac{3}{4}\rho$, and hence for $x \in B_{\rho/2}(x_0)$ one has

$$|x - y| \geq \frac{1}{4}\rho \implies |G^x(y)| \lesssim \rho^{-1} \quad \text{and} \quad |\nabla G^x(y)| \lesssim \rho^{-2}. \tag{4-57}$$

From (4-56) and (4-57) and the fact that ψ is supported in $B_\rho(x_0)$, we have (omitting the dependence on t , and noting that the constants in the inequalities are independent of t as they come only from G^x and ψ)

$$\begin{aligned} \sup_{x \in B_{\rho/2}(x_0)} |p_2(x)| &\lesssim \rho^{-2} \int_{B_\rho(x_0)} (|u| |\nabla u| + |\nabla d| |\nabla^2 d|) dy \\ &\lesssim \rho^{-2} \left(\int_{B_\rho(x_0)} (|u|^2 + |\nabla d|^2) dy \right)^{1/2} \left(\int_{B_\rho(x_0)} (|\nabla u|^2 + |\nabla^2 d|^2) dy \right)^{1/2} \end{aligned} \tag{4-58}$$

and similarly

$$\sup_{x \in B_{\rho/2}(x_0)} |p_3(x)| \lesssim \rho^{-3} \int_{B_\rho(x_0)} |p| \, dy. \tag{4-59}$$

For p_1 , Young’s inequality for convolutions (setting $R := 2\rho$ as in Remark 10) with $\frac{2}{3} + 1 = \frac{3}{4} + \frac{11}{12}$ gives

$$\begin{aligned} \|p_1\|_{3/2; B_\rho(x_0)} &\lesssim \left\| \frac{1}{|\cdot|^2} \right\|_{4/3; B_{2\rho}(0)} \|(|u| + |\nabla d|)(|\nabla u| + |\nabla^2 d|)\|_{12/11; B_\rho(x_0)} \\ &\lesssim \rho^{1/4} \|(|u| + |\nabla d|)(|\nabla u| + |\nabla^2 d|)\|_{12/11; B_\rho(x_0)} \end{aligned}$$

and then Hölder’s inequality with $\frac{11}{12} = \frac{1}{4} + \frac{1}{6} + \frac{1}{2}$ gives

$$\begin{aligned} \|p_1\|_{3/2; B_\rho(x_0)}^{3/2} &\lesssim (\rho^{1/4} \|(|u| + |\nabla d|)^{1/2}\|_{4; B_\rho(x_0)} \|(|u| + |\nabla d|)^{1/2}\|_{6; B_\rho(x_0)} \| |\nabla u| + |\nabla^2 d| \|_{2; B_\rho(x_0)})^{3/2} \\ &\lesssim \rho^{3/8} (\rho A(\rho))^{3/8} \| |u| + |\nabla d| \|_{3; B_\rho(x_0)}^{3/4} \| |\nabla u| + |\nabla^2 d| \|_{2; B_\rho(x_0)}^{3/2}. \end{aligned} \tag{4-60}$$

For the following, we fix now any $r \in (0, \frac{\rho}{2}]$ and omit the dependence on x_0, t_0 and z_0 in $B_r(x_0), B_\rho(x_0), I^*(t_0), A_{z_0}, B_{z_0}, C_{z_0}$ and D_{z_0} (we will retain z_0 in the notation for F_{z_0} to distinguish it from $F = \nabla f$).

To first prove (4-48), we note that (4-58) implies — since $r \leq \frac{1}{2}\rho$ — that

$$\begin{aligned} \int_{B_r} |p_2|^{3/2} \, dx &\lesssim r^3 \rho^{-3} \left(\int_{B_\rho} (|u|^2 + |\nabla d|^2) \, dy \right)^{3/4} \left(\int_{B_\rho} (|\nabla u|^2 + |\nabla^2 d|^2) \, dy \right)^{3/4} \\ &\leq r^3 \rho^{-3} (\rho A(\rho))^{3/4} \left(\int_{B_\rho} (|\nabla u|^2 + |\nabla^2 d|^2) \, dy \right)^{3/4} \end{aligned}$$

so that, integrating over $t \in I_r^*$ and using Hölder’s inequality, we have

$$r^{-2} \iint_{Q_r^*} |p_2|^{3/2} \, dz \lesssim r^{-2} r^3 \rho^{-9/4} A^{3/4}(\rho) \cdot |I_\rho^*|^{1/4} (\rho B(\rho))^{3/4} = \frac{r}{\rho} \cdot [(AB)^{3/4}](\rho), \tag{4-61}$$

and that (4-59) similarly implies that

$$r^{-2} \iint_{Q_r^*} |p_3|^{3/2} \, dz \lesssim r \rho^{-9/2} \int_{I_r^*} \left(\int_{B_\rho} |p| \, dy \right)^{3/2} \lesssim \frac{r}{\rho} \cdot D(\rho). \tag{4-62}$$

Finally, integrating (4-60) over $t \in I_r^*$, Hölder’s inequality with $1 = \frac{1}{4} + \frac{3}{4}$ gives

$$\begin{aligned} r^{-2} \|p_1\|_{3/2; Q_r^*}^{3/2} &\lesssim r^{-2} \rho^{3/4} A^{3/8}(\rho) \| |u| + |\nabla d| \|_{3; Q_\rho^*}^{3/4} \| |\nabla u| + |\nabla^2 d| \|_{2; Q_\rho^*}^{3/2} \\ &\lesssim r^{-2} \rho^{3/4} A^{3/8}(\rho) (\rho^2 C(\rho))^{1/4} (\rho B(\rho))^{3/4} = (C^{1/4}(\rho)) \cdot \left(\left(\frac{r}{\rho} \right)^{-2} A^{3/8}(\rho) B^{3/4}(\rho) \right). \end{aligned}$$

Multiplying and dividing by $(r/\rho)^{\alpha/2}$ for any $\alpha \in \mathbb{R}$, Cauchy’s inequality gives

$$r^{-2} \|p_1\|_{3/2; Q_r^*}^{3/2} \lesssim \left(\frac{r}{\rho} \right)^\alpha C^{1/2}(\rho) + \left(\frac{r}{\rho} \right)^{-\alpha-4} A^{3/4}(\rho) B^{3/2}(\rho). \tag{4-63}$$

Since we want a positive power of $\gamma = r/\rho$ in the first term and a negative one in the second (because it contains B which will be small), we want to take $\alpha > 0$. Choosing $\alpha = 1$ purely to make the following

expression simpler, since $p = p_3 + p_2 + p_1$, we see from (4-61)–(4-63) that

$$D(r) \lesssim \frac{r}{\rho} \cdot [D + (AB)^{3/4} + C^{1/2}](\rho) + \left(\frac{r}{\rho}\right)^{-5} [A^{3/4} B^{3/2}](\rho)$$

which implies (4-48) for $\gamma := r/\rho \leq \frac{1}{2}$.

To prove (4-49), we note that $F_{z_0}(r) \leq F_1(r) + F_2(r) + F_3(r)$, where we set

$$F_j(r) := \frac{1}{r^2} \iint_{Q_r} |p_j| |u| dz.$$

To estimate F_1 we use Hölder's inequality and (4-60) to see that (in fact, for $r \leq \rho$)

$$\begin{aligned} \int_{B_r} |p_1| |u| dx &\leq \|u\|_{3; B_\rho} \|p_1\|_{3/2; B_\rho} \\ &\lesssim \|u\|_{3; B_\rho} \cdot \rho^{1/4} (\rho A(\rho))^{1/4} \| |u| + |\nabla d| \|_{3; B_\rho}^{1/2} \| |\nabla u| + |\nabla^2 d| \|_{2; B_\rho} \\ &\leq \rho^{1/2} A^{1/4}(\rho) \| |u| + |\nabla d| \|_{3; B_\rho}^{3/2} \| |\nabla u| + |\nabla^2 d| \|_{2; B_\rho} \end{aligned}$$

and hence the Cauchy–Schwarz inequality in time gives

$$\begin{aligned} F_1(r) &\lesssim r^{-2} \rho^{1/2} A^{1/4}(\rho) \| |u| + |\nabla d| \|_{3; Q_\rho^*}^{3/2} \| |\nabla u| + |\nabla^2 d| \|_{2; Q_\rho^*} \\ &\lesssim r^{-2} \rho^{1/2} A^{1/4}(\rho) (\rho^2 C(\rho))^{1/2} (\rho B(\rho))^{1/2} \\ &= \left(\left(\frac{r}{\rho}\right)^\alpha C^{1/2}(\rho)\right) \cdot \left(\left(\frac{r}{\rho}\right)^{-2-\alpha} [A^{1/4} B^{1/2}](\rho)\right) \\ &\lesssim \left(\left(\frac{r}{\rho}\right)^\alpha C^{1/2}(\rho)\right)^{4/3} + \left(\left(\frac{r}{\rho}\right)^{-2-\alpha} [A^{1/4} B^{1/2}](\rho)\right)^4 \end{aligned}$$

for any $\alpha \in \mathbb{R}$. Taking, say, $\alpha = \frac{1}{2}$, we have

$$F_1(r) \lesssim \left(\frac{r}{\rho}\right)^{2/3} C^{2/3}(\rho) + \left(\frac{r}{\rho}\right)^{-10} [AB^2](\rho). \quad (4-64)$$

Now for F_2 note that, using (4-58), we have (since $r \leq \frac{1}{2}\rho$)

$$\begin{aligned} \int_{B_r} |p_2| |u| dx &\lesssim \rho^{-2} \int_{B_\rho} (|u| |\nabla u| + |\nabla d| |\nabla^2 d|) dy \int_{B_r} |u| dx \\ &\lesssim \rho^{-2} \| |u| + |\nabla d| \|_{2; B_\rho} \| |\nabla u| + |\nabla^2 d| \|_{2; B_\rho} (r^3)^{1/2} \|u\|_{2; B_r} \\ &\lesssim \rho^{-2} r^{3/2} (\rho A(\rho)) \| |\nabla u| + |\nabla^2 d| \|_{2; B_\rho} \end{aligned}$$

so that integrating over $t \in I_r^*$ and using Hölder's inequality in time we have

$$F_2(r) \lesssim \frac{1}{r^2} \frac{r^{3/2}}{\rho^2} (\rho A(\rho)) (\rho B(\rho))^{1/2} (r^2)^{1/2} = \left(\frac{r}{\rho}\right)^{1/2} [AB^{1/2}](\rho). \quad (4-65)$$

For F_3 , using (4-59) and Hölder’s inequality, we see that

$$\begin{aligned} \frac{1}{r^2} \int_{B_r} |p_3| |u| \, dx &\leq \frac{1}{r^2 \rho^3} \left(\int_{B_\rho} |p| \, dy \right) \left(\int_{B_r} |u| \, dx \right) \\ &\leq \frac{1}{r^2 \rho^3} \left(\int_{B_\rho} |p|^{3/2} \, dx \right)^{2/3} (\rho^3)^{1/3} \left(\int_{B_r} (|u|^{1/2})^4 \, dx \right)^{1/4} \left(\int_{B_r} (|u|^{1/2})^6 \, dx \right)^{1/6} (r^3)^{7/12} \end{aligned}$$

which gives us (setting $\gamma := r/\rho$)

$$\begin{aligned} F_3(r) &\lesssim \frac{1}{r^{1/4} \rho^2} (rA(r))^{1/4} \left(\iint_{Q_\rho^*} |p|^{3/2} \, dx \right)^{2/3} \left(\iint_{Q_r^*} |u|^3 \, dx \right)^{1/6} (r^2)^{1/6} \\ &\leq \frac{1}{r^{1/4} \rho^2} (rA(r))^{1/4} (\rho^2 D(\rho))^{2/3} (r^2 C(r))^{1/6} (r^2)^{1/6} \\ &\leq \left(\frac{r}{\rho} \right)^{2/3} (\gamma^{-1} A)^{1/4}(\rho) D^{2/3}(\rho) (\gamma^{-2} C)^{1/6}(\rho) = \left(\frac{r}{\rho} \right)^{1/12} A^{1/4}(\rho) D^{2/3}(\rho) C^{1/6}(\rho) \end{aligned}$$

by (4-51). Hence Young’s inequality implies

$$F_3(r) \lesssim \left(\frac{r}{\rho} \right)^{1/12} (A(\rho) + D^{4/3}(\rho) + C^{2/3}(\rho)). \tag{4-66}$$

Adding (4-64)–(4-66) and passing to the smallest powers of $\gamma = r/\rho (< 1)$ we see that

$$F_{z_0}(r) \lesssim \left(\frac{r}{\rho} \right)^{1/12} (A + D^{4/3} + C^{2/3})(\rho) + \left(\frac{r}{\rho} \right)^{-10} [A(B^{1/2} + B^2)](\rho)$$

which implies (4-49), and completes the proof of Claim 3. □

Proof of Claim 4. We will again omit the dependence on z_0 (except in F_{z_0}).

To estimate $A(\rho/2)$, we use the local energy inequality (1-13) with a nonnegative cut-off function $\phi \in C_0^\infty(Q_\rho^*)$ which is equal to 1 in $Q_{\rho/2}^*$, with

$$|\nabla \phi| \lesssim \rho^{-1} \quad \text{and} \quad |\phi_t|, |\nabla^2 \phi| \lesssim \rho^{-2}.$$

We’ll need to estimate terms which control those that appear on the right-hand side of the local energy inequality (1-13), which we’ll call $I-V$ (all of which depend on ρ) as follows.

$$\begin{aligned} I := \iint_{Q_\rho^*} (|u|^2 + |\nabla d|^2) |\phi_t + \Delta \phi| \, dz &\lesssim \rho^{-2} \| |u|^2 + |\nabla d|^2 \|_{3/2; Q_\rho^*} (\rho^5)^{1/3} \\ &\lesssim \rho^{-2} (\rho^2 C(\rho))^{2/3} (\rho^5)^{1/3} = \rho C^{2/3}(\rho). \end{aligned} \tag{4-67}$$

Using the assumption (1-11) that $\nabla \cdot u = 0$ weakly and indicating by \bar{g}^ρ the average of a function g in B_ρ , we have

$$II := \int_{I_\rho^*} \left| \int_{B_\rho} (|u|^2 + |\nabla d|^2) u \cdot \nabla \phi \, dx \right| dt = \int_{I_\rho^*} \left| \int_{B_\rho} [(|u|^2 - \overline{|u|^2}^\rho) + (|\nabla d|^2 - \overline{|\nabla d|^2}^\rho)] u \cdot \nabla \phi \, dx \right| dt,$$

hence

$$II \lesssim \rho^{-1} (\rho^2 E(\rho)) = \rho E(\rho). \tag{4-68}$$

Clearly we have

$$III := \iint_{Q_\rho^*} |pu \cdot \nabla \phi| dz \lesssim \rho^{-1}(\rho^2 F_{z_0}(\rho)) = \rho F_{z_0}(\rho). \tag{4-69}$$

Using the weak divergence-free condition $\nabla \cdot u = 0$ in (1-11) to write (see (1-2))

$$(u \cdot \nabla)d = \nabla^T \cdot (d \otimes u)$$

(at almost every x) and integrating by parts we have

$$\begin{aligned} IV &:= \int_{I_\rho^*} \left| \int_{B_\rho} u \otimes \nabla \phi : \nabla d \odot \nabla d \, dx \right| dt \\ &= \int_{I_\rho^*} \left| \int_{B_\rho} [(u \cdot \nabla)d] \cdot [(\nabla \phi \cdot \nabla)d] \, dx \right| dt \\ &= \int_{I_\rho^*} \left| \int_{B_\rho} [\nabla^T \cdot (d \otimes u)] \cdot [(\nabla \phi \cdot \nabla)d] \, dx \right| dt \\ &= \int_{I_\rho^*} \left| - \int_{B_\rho} d \otimes u : \nabla^T [(\nabla \phi \cdot \nabla)d] \, dx \right| dt, \end{aligned}$$

and clearly

$$|\nabla^T [(\nabla \phi \cdot \nabla)d]| \lesssim |\nabla^2 \phi| |\nabla d| + |\nabla \phi| |\nabla^2 d|.$$

Therefore, for $q \in [2, 6]$ we have²³

$$\begin{aligned} IV &\lesssim \iint_{Q_\rho^*} |d| |u| (\rho^{-2} |\nabla d| + \rho^{-1} |\nabla^2 d|) dz \\ &\leq \| |d| |u| \|_{2; Q_\rho^*} (\rho^{-2} \|\nabla d\|_{2; Q_\rho^*} + \rho^{-1} \|\nabla^2 d\|_{2; Q_\rho^*}) \\ &\lesssim \| |d| |u| \|_{2; Q_\rho^*} (\rho^{-2} \cdot \rho^{5/6} \|\nabla d\|_{3; Q_\rho^*} + \rho^{-1} \|\nabla^2 d\|_{2; Q_\rho^*}) \\ &\leq (\rho^3 G_2(\rho))^{1/2} (\rho^{-2} \cdot \rho^{5/6} (\rho^2 C(\rho))^{1/3} + \rho^{-1} (\rho B(\rho))^{1/2}) \\ &= \rho (G_2(\rho))^{1/2} (C^{1/3}(\rho) + B^{1/2}(\rho)) \\ &\leq \rho (G_q^{2/q}(\rho) C^{1-2/q}(\rho))^{1/2} (C^{1/3}(\rho) + B^{1/2}(\rho)) \tag{by (3-22)}, \end{aligned}$$

so that

$$IV \lesssim \rho [G_q^{1/q} (C^{5/6-1/q} + C^{1/2-1/q} B^{1/2})](\rho). \tag{4-70}$$

Similarly, for $q \in [2, 6]$ we have

$$V := \iint_{Q_\rho^*} |d|^2 |\nabla d|^2 \phi \, dz \lesssim \rho^3 G_2(\rho) \stackrel{(3-22)}{\leq} \rho^3 G_q^{2/q}(\rho) C^{1-2/q}(\rho). \tag{4-71}$$

²³Note that it is only the appearance of $\nabla^2 d$ in the estimate of term IV which forces us to include u in the definition of G_{q,z_0} . Indeed, switching the roles of u (which appears in C_{z_0} along with ∇d) and ∇d (which appears in G_{q,z_0} even with u omitted), one could otherwise control the term IV in precisely the same way. If u is omitted in G_{q,z_0} , one could still obtain the same estimate of IV if one takes $q = 6$, but this would dramatically weaken the statement of Theorem 1. The remainder of the proof of Theorem 1 does not require (but is not harmed by) the inclusion of u in G_{q,z_0} .

Finally, using (4-67)–(4-71), the local energy inequality (1-13) (with constant \bar{C}) gives

$$\begin{aligned} (\bar{C})^{-1} \frac{\rho}{2} A\left(\frac{\rho}{2}\right) &\lesssim I + II + III + IV + V \\ &\lesssim \rho[C^{2/3} + E + F_{z_0} + G_q^{1/q}(C^{5/6-1/q} + C^{1/2-1/q} B^{1/2}) + [\cdot]^2 G_q^{2/q} C^{1-2/q}](\rho) \\ &\lesssim \rho[C^{2/3} + E + F_{z_0} + (1 + [\cdot]^2) G_q^{4/(6-q)} + (G_q^{2/(6-q)} + C^{1/3}) B^{1/2}](\rho) \end{aligned}$$

as long as $2 \leq q < 6$, as in that case we have

$$\begin{aligned} G_q^{1/q} C^{5/6-1/q} &= (G_q^{4/(6-q)})^{(6-q)/(4q)} (C^{2/3})^{(5q-6)/(4q)} \\ &\leq \left(\frac{6-q}{4q}\right) G_q^{4/(6-q)} + \left(\frac{5q-6}{4q}\right) C^{2/3} \leq \frac{3}{4} G_q^{4/(6-q)} + \frac{5}{4} C^{2/3}, \\ G_q^{1/q} C^{1/2-1/q} &= (G_q^{2/(6-q)})^{(6-q)/(2q)} (C^{1/3})^{(3q-6)/(2q)} \\ &\leq \left(\frac{6-q}{2q}\right) G_q^{2/(6-q)} + \left(\frac{3q-6}{2q}\right) C^{1/3} \leq \frac{3}{2} G_q^{2/(6-q)} + \frac{3}{2} C^{1/3}, \\ G_q^{2/q} C^{1-2/q} &= (G_q^{4/(6-q)})^{(6-q)/(2q)} (C^{2/3})^{(3q-6)/(2q)} \\ &\leq \left(\frac{6-q}{2q}\right) G_q^{4/(6-q)} + \left(\frac{3q-6}{2q}\right) C^{2/3} \leq \frac{3}{2} G_q^{4/(6-q)} + \frac{3}{2} C^{2/3}. \end{aligned}$$

This implies (4-50) and proves Claim 4. □

Acknowledgements

The author would like to offer his sincere thanks to Prof. Arghir Zarnescu for many insightful discussions, for introducing him to the field of liquid crystals models, and for suggesting a problem which led to this publication. The author would also like to thank Prof. Camillo De Lellis for introducing him to Scheffer's notion of Navier–Stokes inequalities. Finally, the author would like to thank the anonymous referee for insightful comments about a previous draft of this article.

References

- [Caffarelli et al. 1982] L. Caffarelli, R. Kohn, and L. Nirenberg, “Partial regularity of suitable weak solutions of the Navier–Stokes equations”, *Comm. Pure Appl. Math.* **35**:6 (1982), 771–831. MR Zbl
- [Du et al. 2020] H. Du, X. Hu, and C. Wang, “Suitable weak solutions for the co-rotational Beris–Edwards system in dimension three”, *Arch. Ration. Mech. Anal.* **238**:2 (2020), 749–803. MR Zbl
- [Gilbarg and Trudinger 1983] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundle Math. Wissen. **224**, Springer, 1983. MR Zbl
- [Kukavica 2009] I. Kukavica, “Partial regularity results for solutions of the Navier–Stokes system”, pp. 121–145 in *Partial differential equations and fluid mechanics*, edited by J. C. Robinson and J. L. Rodrigo, London Math. Soc. Lecture Note Ser. **364**, Cambridge Univ. Press, 2009. MR Zbl
- [Ladyzhenskaya and Seregin 1999] O. A. Ladyzhenskaya and G. A. Seregin, “On partial regularity of suitable weak solutions to the three-dimensional Navier–Stokes equations”, *J. Math. Fluid Mech.* **1**:4 (1999), 356–387. MR Zbl
- [Leray 1934] J. Leray, “Sur le mouvement d’un liquide visqueux emplissant l’espace”, *Acta Math.* **63**:1 (1934), 193–248. MR Zbl

- [Lin 1998] F. Lin, “A new proof of the Caffarelli–Kohn–Nirenberg theorem”, *Comm. Pure Appl. Math.* **51**:3 (1998), 241–257. MR Zbl
- [Lin and Liu 1995] F.-H. Lin and C. Liu, “Nonparabolic dissipative systems modeling the flow of liquid crystals”, *Comm. Pure Appl. Math.* **48**:5 (1995), 501–537. MR Zbl
- [Lin and Liu 1996] F.-H. Lin and C. Liu, “Partial regularity of the dynamic system modeling the flow of liquid crystals”, *Discrete Contin. Dynam. Systems* **2**:1 (1996), 1–22. MR Zbl
- [Lin and Wang 2014] F. Lin and C. Wang, “Recent developments of analysis for hydrodynamic flow of nematic liquid crystals”, *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **372**:2029 (2014), art. id. 20130361. MR Zbl
- [Scheffer 1977] V. Scheffer, “Hausdorff measure and the Navier–Stokes equations”, *Comm. Math. Phys.* **55**:2 (1977), 97–112. MR Zbl
- [Scheffer 1980] V. Scheffer, “The Navier–Stokes equations on a bounded domain”, *Comm. Math. Phys.* **73**:1 (1980), 1–42. MR Zbl
- [Scheffer 1985] V. Scheffer, “A solution to the Navier–Stokes inequality with an internal singularity”, *Comm. Math. Phys.* **101**:1 (1985), 47–85. MR Zbl
- [Vasseur 2007] A. F. Vasseur, “A new proof of partial regularity of solutions to Navier–Stokes equations”, *Nonlinear Differential Equations Appl.* **14**:5-6 (2007), 753–785. MR Zbl
- [Wheeden and Zygmund 1977] R. L. Wheeden and A. Zygmund, *Measure and integral: an introduction to real analysis*, Pure Appl. Math. **43**, Dekker, New York, 1977. MR Zbl

Received 28 Oct 2021. Accepted 7 Jan 2022.

GABRIEL S. KOCH: g.koch@sussex.ac.uk
Department of Mathematics, University of Sussex, Brighton, United Kingdom

Analysis & PDE

msp.org/apde

EDITORS-IN-CHIEF

Patrick Gérard Université Paris Sud XI, France
patrick.gerard@universite-paris-saclay.fr

Clément Mouhot Cambridge University, UK
c.mouhot@dpmms.cam.ac.uk

BOARD OF EDITORS

| | | | |
|--------------------|--|-----------------------|--|
| Massimiliano Berti | Scuola Intern. Sup. di Studi Avanzati, Italy berti@sissa.it | William Minicozzi II | Johns Hopkins University, USA minicozz@math.jhu.edu |
| Zbigniew Błocki | Uniwersytet Jagielloński, Poland zbigniew.blocki@uj.edu.pl | Werner Müller | Universität Bonn, Germany mueller@math.uni-bonn.de |
| Charles Fefferman | Princeton University, USA cf@math.princeton.edu | Igor Rodnianski | Princeton University, USA irod@math.princeton.edu |
| David Gérard-Varet | Université de Paris, France david.gerard-varet@imj-prg.fr | Yum-Tong Siu | Harvard University, USA siu@math.harvard.edu |
| Colin Guillarmou | Université Paris-Saclay, France colin.guillarmou@universite-paris-saclay.fr | Terence Tao | University of California, Los Angeles, USA tao@math.ucla.edu |
| Ursula Hamenstaedt | Universität Bonn, Germany ursula@math.uni-bonn.de | Michael E. Taylor | Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu |
| Vadim Kaloshin | University of Maryland, USA vadim.kaloshin@gmail.com | Gunther Uhlmann | University of Washington, USA gunther@math.washington.edu |
| Izabella Laba | University of British Columbia, Canada ilaba@math.ubc.ca | András Vasy | Stanford University, USA andras@math.stanford.edu |
| Anna L. Mazzucato | Penn State University, USA alm24@psu.edu | Dan Virgil Voiculescu | University of California, Berkeley, USA dvv@math.berkeley.edu |
| Richard B. Melrose | Massachusetts Inst. of Tech., USA rbm@math.mit.edu | Jim Wright | University of Edinburgh, UK j.r.wright@ed.ac.uk |
| Frank Merle | Université de Cergy-Pontoise, France merle@ihes.fr | Maciej Zworski | University of California, Berkeley, USA zworski@math.berkeley.edu |

PRODUCTION

production@msp.org
Silvio Levy, Scientific Editor

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2023 is US \$405/year for the electronic version, and \$630/year (+\$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2023 Mathematical Sciences Publishers

ANALYSIS & PDE

Volume 16 No. 7 2023

| | |
|--|------|
| Simplices in thin subsets of Euclidean spaces ALEX IOSEVICH and ÁKOS MAGYAR | 1485 |
| Resonances for Schrödinger operators on infinite cylinders and other products T. J. CHRISTIANSEN | 1497 |
| A structure theorem for elliptic and parabolic operators with applications to homogenization of operators of Kolmogorov type MALTE LITSGÅRD and KAJ NYSTRÖM | 1547 |
| Dimension-free Harnack inequalities for conjugate heat equations and their applications to geometric flows LI-JUAN CHENG and ANTON THALMAIER | 1589 |
| The regularity of the boundary of vortex patches for some nonlinear transport equations JUAN CARLOS CANTERO, JOAN MATEU, JOAN OROBITG and JOAN VERDERA | 1621 |
| Directional square functions NATALIA ACCOMAZZO, FRANCESCO DI PLINIO, PAUL HAGELSTEIN, IOANNIS PARISSIS and LUZ RONCAL | 1651 |
| Partial regularity for Navier–Stokes and liquid crystals inequalities without maximum principle GABRIEL S. KOCH | 1701 |