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OVERDETERMINED BOUNDARY PROBLEMS WITH NONCONSTANT DIRICHLET AND NEUMANN DATA

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We consider the overdetermined boundary problem for a general second-order semilinear elliptic equation on bounded domains of $\mathbb{R}^n$, where one prescribes both the Dirichlet and Neumann data of the solution. We are interested in the case where the data are not necessarily constant and where the coefficients of the equation can depend on the position, so that the overdetermined problem does not generally admit a radial solution. Our main result is that, nevertheless, under minor technical hypotheses nontrivial solutions to the overdetermined boundary problem always exist.

1. Introduction

The study of overdetermined boundary problems, that is, problems where one prescribes both Dirichlet and Neumann data, has grown into a major field of research in the theory of elliptic PDEs since its appearance in Lord Rayleigh's classic treatise [1877]. An outburst of activity started with the groundbreaking paper [Serrin 1971], where he combined an adaptation of Alexandrov's moving planes method with a subtle refinement of the maximum principle to prove a symmetry result for an overdetermined problem. More precisely, Serrin proved that, under mild technical hypotheses, positive solutions to elliptic equations of the form

$$\Delta u + F(u) = 0$$

inside a bounded domain $\Omega \subset \mathbb{R}^n$ satisfying the boundary conditions

$$u = 0 \quad \text{and} \quad \partial_\nu u = -c \quad \text{on } \partial\Omega, \tag{1-1}$$

where c is an unspecified constant that can be picked freely, only exist if Ω is a ball, in which case u is radial. The result remains true if F also depends on the norm of the gradient of u and if we replace the Laplacian by other position-independent operators of variational form [Cianchi and Salani 2009].

The influence of Serrin's result is such that the very considerable body of literature devoted to overdetermined boundary problems is mostly limited to proofs that solutions need to be radial in cases that can be handled using the method of moving planes. Without attempting to be comprehensive, some remarkable results about overdetermined boundary value problems include alternative approaches to radial symmetry results using P-functions [Garofalo and Lewis 1989; Kawohl 1998] or Pohozaev-type integral identities [Brandolini et al. 2008; Magnanini and Poggesi 2020a; 2020b], extensions of the moving plane method to the hyperbolic space and the hemisphere [Kumaresan and Prajapat 1998], to degenerate

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elliptic equations such as the p-Laplace equation [Damascelli et al. 1999], and to exterior [Aftalion and Busca 1998; Garofalo and Sartori 1999], unbounded [Farina and Valdinoci 2010] or nonsmooth domains [Prajapat 1998], and stability of symmetry [Aftalion et al. 1999]. Another direction of research that has attracted considerable recent attention is the study of connections with the theory of constant mean curvature surfaces and the construction of nontrivial solutions to Serrin-type problems in exterior domains [Traizet 2014; del Pino et al. 2015; Ros et al. 2020]. Nontrivial solutions for partially overdetermined problems or with degenerate ellipticity are also known to exist [Alessandrini and Garofalo 1989; Fragalà and Gazzola 2008; Fragalà et al. 2006; Farina and Valdinoci 2013].

In two surprising papers, Pacard and Sicbaldi [2009] and Delay and Sicbaldi [2015] proved the existence of extremal domains with small volume for the first eigenvalue of the Laplacian in any compact Riemannian manifold, that is, domains for which the overdetermined problem for the linear elliptic equation

$$\Delta_g u + \lambda u = 0$$

has a positive solution with zero Dirichlet data and constant Neumann data. Here Δ_g is the Laplacian operator associated with a Riemannian metric g on a compact manifold and the constant λ (which one eventually chooses as the first Dirichlet eigenvalue of the domain Ω) is not specified a priori. Very recently we managed to show the existence of nontrivial solutions, with the same overdetermined Dirichlet and Neumann conditions, for fairly general semilinear elliptic equations of second order with possibly nonconstant coefficients [Domínguez-Vázquez et al. 2019].

In all these results, the fact that one is imposing precisely the standard overdetermined boundary conditions (1-1) plays a crucial role. Roughly speaking, this is because one can relate the existence of overdetermined solutions with the critical points of certain functional via a variational argument. Therefore, the gist of the argument in these papers is that the overdetermined condition with constant data is connected with the local extrema for a natural energy functional, restricted to a specific class of functions labeled by points in the physical space. This ultimately permits one to derive the existence of solutions from the fact that a continuous function attains its maximum on a compact manifold. However, this strategy is successful only for constant boundary data. To our best knowledge, the only result in the literature which considers nonconstant (albeit special) Neumann data in relation to overdetermined boundary problems is [Bianchini et al. 2014].

In the recent paper [Domínguez-Vázquez et al. 2021], we have constructed new families of compactly supported stationary solutions to the three-dimensional Euler equation by proving that there are solutions to an associated overdetermined problem in two dimensions where one prescribes (modulo constants that can be picked freely) zero Dirichlet data and nonconstant Neumann data. The proof uses crucially that the space is two-dimensional, which ensures that the kernel and cokernel of a certain operator are one-dimensional, and does not work in higher dimensions.

Our objective in this paper is to prove the existence of solutions to overdetermined problems where one prescribes general Dirichlet and Neumann data (just as before, up to unspecified constants). For concreteness, we consider the model semilinear equation

$$Lu + \lambda F(x, u) = 0 \tag{1-2}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, with Dirichlet and Neumann boundary conditions

$$u = f_0(x), \quad \nu \cdot A(x)\nabla u = -c f_1(x) \quad \text{on } \partial\Omega. \tag{1-3}$$

Here f_0, f_1 are functions on $\mathbb{R}^n$, F is a function on $\mathbb{R}^n \times \mathbb{R}$, λ, c are unspecified positive constants, ν is the outer unit normal on $\partial\Omega$ and L is the second-order operator

$$Lu := a_{ij}(x)\, \partial_{ij} u + b_i(x)\, \partial_i u,$$

where $A(x) = (a_{ij}(x))$ is a (symmetric) matrix-valued function on $\mathbb{R}^n$ satisfying the (possibly nonuniform) ellipticity condition

$$\min_{|\xi|=1} \xi \cdot A(x)\xi > 0 \quad \text{for all } x \in \mathbb{R}^n.$$

Theorem 1.1. *Given any noninteger $s > 2$, let us take any functions F, f_0, f_1, b of class C^s and A of class C^{s+2}. Assume that the functions $F(\cdot\,, f_0(\cdot))$ and f_1 are positive and that the function f_0 has a nondegenerate critical point. Then there is a family of domains $\Omega_{\varepsilon,\bar\lambda}$ for which the overdetermined problem (1-2)-(1-3) admits a solution.*

More precisely, let $p \in \mathbb{R}^n$ be a nondegenerate critical point of f_0. Then, for any $\varepsilon \neq 0$ small enough and $\bar\lambda > 0$, the following statements hold:

(i) *The domain $\Omega_{\varepsilon,\bar\lambda}$ is a small deformation of the ball of radius ε centered at p, characterized by an equation of the form $|x - p|^2 < \varepsilon^2 + O(\varepsilon^3)$.*

(ii) *The dependence of λ and c on the parameter ε is of the form*

$$\lambda = \varepsilon^{-2}\bar\lambda, \quad c = \varepsilon^{-1}\bar c,$$

where $\bar c = \bar c(\varepsilon, \bar\lambda)$ is a positive constant of order 1.

Remark 1.2. In the case of the torsion problem, i.e., $\Delta u + \lambda = 0$ (i.e., $F(x, u) = 1$ in the previous notation), the condition that f_0 has a critical point can be relaxed: it is enough that the function $G_\kappa := f_0 + \kappa \log f_1$ has at least one nondegenerate critical point for some constant $\kappa > 0$. The statement then applies if p is a nondegenerate critical point of G_κ and taking $\bar\lambda := n\kappa > 0$ (not necessarily small).

Also, it is easy to obtain different variations on our main theorem following the same method of proof. In fact, one obtains new results even for the linear equation $\Delta u + b(x) \cdot \nabla u + \lambda f(x) = 0$ with standard overdetermined boundary data $f_0 := 0$, $f_1 := 1$; specifically, if p is a nondegenerate zero of the vector field $n\nabla f - f b$, then the statement still holds taking any $\bar\lambda > 0$. This does not follow from [Domínguez-Vázquez et al. 2019]. However, we shall not pursue these generalizations here.

Compared with [Domínguez-Vázquez et al. 2019], a major difference is that the theorem does not only ensure the existence of domains where the overdetermined problem under consideration admits a nontrivial solution, but also specifies the points around which those domains are located. This immediately permits one to translate this existence result to problems that are only defined in a subset of $\mathbb{R}^n$ or on a differentiable manifold. In view of the heuristic but fruitful connection between overdetermined boundary problems and the study of CMC hypersurfaces, a result that is somehow akin to our existence results for overdetermined boundary problems for semilinear equations is Ye's classical theorem [1991] on foliations by small CMC spheres on n-dimensional Riemannian manifolds.

The paper is organized as follows. We will start by setting up the problem in Section 2. For clarity of exposition, in Sections 2 to 4 we have chosen to assume that the matrix $A(x)$ is the identity and carry out the proof in this context. An essential ingredient of the proof is the computation of asymptotic expansions for the solution to the Dirichlet problem in small perturbations of a ball of radius $\varepsilon \ll 1$, when the constants λ and c scale with the radius as in Theorem 1.1. This computation is carried out in Section 3. These asymptotic estimates are put to use in Section 4, where we prove Theorem 1.1 in the particular case when $A(x) = I$. To obtain the general result, in Section 5 we show that the case of a general matrix-valued function $A(x)$ reduces to the study of the easiest case $A(x) = I$ subject to an inessential perturbation of order ε^2. Making this precise, however, involves using a heavier notation and geodesic-type normal coordinates adapted to the matrix $A(x)$ that might unnecessarily obscure the simple ideas the proof is based on. As a side remark, let us point out that the reason we ask for more regularity of the matrix A (which is of class C^{s+2} in contrast with the C^s regularity of the other functions) is precisely due to our use of geodesic coordinates.

2. Setting up the problem

For clarity of exposition, until Section 5 we will assume that $A(x) = I$. This assumption will enable us to obtain more compact expressions for the various quantities that appear in the problem and it will make it easier to point out the salient features of the proof.

Let us fix a point $p \in \mathbb{R}^n$ and introduce rescaled coordinates $z \in \mathbb{R}^n$ centered at p as

$$z := \frac{x-p}{\varepsilon},$$

where ε is a suitably small nonzero constant. We now consider spherical coordinates $(r, \omega) \in \mathbb{R}^+ \times \mathbb{S}$ for z, defined as

$$r := |z| = \left|\frac{x-p}{\varepsilon}\right|, \quad \omega := \frac{z}{|z|} = \frac{x-p}{|x-p|}.$$

Here and in what follows,

$$\mathbb{S} := \{\omega \in \mathbb{R}^n : |\omega| = 1\}$$

denotes the unit sphere of dimension $n-1$. For simplicity of notation, we will notationally omit the dependence on the point p. Also, with some abuse of notation, we will denote the expression of the function $u(x)$ in these coordinates simply by $u(r, \omega)$.

Let us now consider a C^{s+1} function $B : \mathbb{S} \to \mathbb{R}$ and, for suitably small ε, let us describe the domain in terms of the above coordinates as

$$\Omega_{p,\varepsilon B} := \{r < 1 + \varepsilon B(\omega)\}. \tag{2-1}$$

We now consider (1-2) in the domain $\Omega_{p,\varepsilon B}$ and choose the constants λ, c as

$$\lambda =: \varepsilon^{-2}\bar{\lambda}, \quad c =: \varepsilon^{-1}\bar{c},$$

where we think of ε as a small constant and of $\bar{\lambda}$, $\bar{c}$ as positive constants of order 1. Equation (1-2) can then be rewritten in the rescaled coordinates as

$$\tilde{L}u + \bar{\lambda}\widetilde{F}(z, u) = 0, \tag{2-2}$$

where

$$\widetilde{F}(z, u) := F(p + \varepsilon z, u)$$

and $\tilde{L}$ is the differential operator

$$\tilde{L}u = \Delta u + \varepsilon \tilde{b}(z) \cdot \nabla u,$$

with $\tilde{b}_i(z) := b_i(p + \varepsilon z)$. We also denote the functions f_0 and f_1 in these coordinates as

$$\tilde{f}_0(z) := f_0(p + \varepsilon z), \qquad \tilde{f}_1(z) := f_1(p + \varepsilon z).$$

Here and in what follows, Δ and ∇ denote the Laplacian and gradient operators in the rescaled coordinates z.

The Dirichlet boundary condition on $\partial\Omega_{p,\varepsilon B}$ can be simply written in rescaled hyperspherical coordinates as

$$u(1 + \varepsilon B(\omega), \omega) = \tilde{f}_0(1 + \varepsilon B(\omega), \omega) =: \tilde{f}_0(\varepsilon, \omega). \tag{2-3}$$

We notice that $\tilde{f}_0(0, \omega) = f_0(p)$. Analogously, the Neumann boundary condition reads as

$$\partial_\nu u(1 + \varepsilon B(\omega), \omega) = -\bar{c}\tilde{f}_1(1 + \varepsilon B(\omega), \omega),$$

where ν is the outwards normal unit vector on $\partial\Omega_{p,\varepsilon B}$.

We denote by $C^s_{\mathrm{Dir}}(\mathbb{B})$ the space of C^s functions on the unit n-dimensional ball $\mathbb{B} := \{|z| < 1\}$ with zero trace to the boundary. Also, $\mathcal{K} \subset C^\infty(\mathbb{S})$ denotes the restriction to the unit sphere of the space of linear functions on $\mathbb{R}^n$,

$$\mathcal{K} := \{V \cdot z : |z| = 1,\ V \in \mathbb{R}^n\}.$$

Equivalently, $\mathcal{K}$ is the eigenspace of the Laplacian $\Delta_{\mathbb{S}}$ of the unit sphere corresponding to the second eigenvalue, $n - 1$. Also, in what follows we will denote the partial derivatives of F (or $\widetilde{F}$) as

$$F'(x, u) := \partial_u F(x, u), \qquad \nabla F(x, u) := \nabla_x F(x, u), \qquad \partial_j F(x, u) := \partial_{x_j} F(x, u).$$

The following lemma is a reformulation of [Domínguez-Vázquez et al. 2019, Theorem 2.3 and Proposition 2.4]. Here $s > 2$ is assumed to be a noninteger real.

Lemma 2.1. *For each $p \in \mathbb{R}^n$, there is some $\bar{\lambda}_p > 0$ such that the following statements hold for all $\bar{\lambda} \in (0, \bar{\lambda}_p)$:*

(i) *There is a unique function $\phi_{p,\bar{\lambda}}(r)$ of class C^{s+2} satisfying the ODE*

$$\phi_{p,\bar{\lambda}}''(r) + \frac{n-1}{r}\phi_{p,\bar{\lambda}}'(r) + \bar{\lambda}F(p, f_0(p) + \phi_{p,\bar{\lambda}}(r)) = 0$$

and the boundary condition $\phi_{p,\bar{\lambda}}(1) = 0$ which is regular at $r = 0$. The function $\phi_{p,\bar{\lambda}}$ is well-defined for $r \in [0, 1 + \delta_p]$, with $\delta_p > 0$. Furthermore, $\phi_{p,\bar{\lambda}}(r) > 0$ for $r < 1$ and $\phi_{p,\bar{\lambda}}'(1) < 0$.

(ii) *The operator*

$$T_{p,\bar\lambda} v := \Delta v + \bar\lambda\, F'(p,\, f_0(p) + \phi_{p,\bar\lambda}(|z|))v$$

defines an invertible map $T_{p,\bar\lambda} : C^{s+1}_{\mathrm{Dir}}(\mathbb{B}) \to C^{s-1}(\mathbb{B})$.

(iii) *Consider the map* $H_{p,\bar\lambda}$ *defined for each function* ψ *on the boundary of the ball as*

$$H_{p,\bar\lambda}\psi := -{\phi_{p,\bar\lambda}}'(1)\, \partial_\nu v_\psi + {\phi_{p,\bar\lambda}}''(1) v_\psi,$$

where v_ψ *is the only solution to the problem* $T_{p,\bar\lambda} v_\psi = 0$ *on* $\mathbb{B}$, $v_\psi|_{\partial\mathbb{B}} = \psi$. *Then* $H_{p,\bar\lambda}$ *maps* $C^{s+1}(\mathbb{S}) \to C^s(\mathbb{S})$, *its kernel is* $\mathcal{K}$, *and its range is the set* $C^s(\mathbb{B}) \cap \mathcal{K}^\perp$ *of* C^s *functions orthogonal to* $\mathcal{K}$. *Furthermore,*

$$\|\psi\|_{C^{s+1}} \leqslant C_{p,\bar\lambda} \|H_{p,\bar\lambda}\psi\|_{C^s} \tag{2-4}$$

for all $\psi \in C^{s+1} \cap \mathcal{K}^\perp$.

(iv) *The function* $\phi_{p,\bar\lambda}$ *satisfies* $\|{\phi_{p,\bar\lambda}}'\|_{C^s((0,1+\delta_p))} \leqslant C\bar\lambda$ *and is of class* C^s *in* p *and* $\bar\lambda$.

Remark 2.2. When the equation is linear (that is, $F(x,u) = f(x)$), one can take $\bar\lambda_p$ arbitrarily large and

$$\phi_{p,\bar\lambda}(r) = -\frac{\bar\lambda}{2n} f(p)\,(r^2-1).$$

The operator $H_{p,\bar\lambda}$ is then

$$H_{p,\bar\lambda}\psi = \frac{\bar\lambda}{n} f(p)\,(\Lambda_0\psi - \psi),$$

where $\Lambda_0 := [(n/2-1)^2 - \Delta_{\mathbb{S}}]^{1/2} - n/2 + 1$ is the Dirichlet–Neumann map of the ball.

In what follows we shall always assume that $\bar\lambda < \bar\lambda_p$.

Proposition 2.3. *For any* ε *small enough and any function* $B \in C^{s+1}(\mathbb{S})$ *with* $\|B\|_{C^{s+1}} < 1$, *there is a unique function* $u = u_{p,\varepsilon,\bar\lambda,B}$ *in a small neighborhood of* $f_0(p) + \phi_{p,\bar\lambda}$ *in* $C^{s+1}(\Omega_{p,\varepsilon B})$ *that satisfies* (2-2) *and the Dirichlet boundary condition* (2-3).

Proof. Let $\chi_{p,\varepsilon B} : \mathbb{B} \to \Omega_{p,\varepsilon B}$ be the diffeomorphism defined in spherical coordinates as

$$(\rho,\omega) \mapsto ([1 + \varepsilon\chi(\rho)\, B(\omega)]\,\rho, \omega),$$

where $\chi(\rho)$ is a smooth cutoff function that is zero for $\rho < \frac14$ and 1 for $\rho > \frac12$. Then one can define a map

$$\mathcal{H}_{p,\bar\lambda,B} : (-\varepsilon_p, \varepsilon_p) \times C^{s+1}_{\mathrm{Dir}}(\mathbb{B}) \to C^{s-1}(\mathbb{B})$$

as

$$\mathcal{H}_{p,\bar\lambda,B}(\varepsilon,\phi) := [\tilde L(\phi\circ\chi^{-1}_{p,\varepsilon B})]\circ\chi_{p,\varepsilon B} + E\circ\chi_{p,\varepsilon B} + \bar\lambda[\widetilde F(\,\cdot\,, \tilde f_0 + \phi\circ\chi^{-1}_{p,\varepsilon B})]\circ\chi_{p,\varepsilon B},$$

with the function E defined as

$$E := \tilde L \tilde f_0. \tag{2-5}$$

Note that $\|E\|_{C^{s-1}(\Omega_{p,\varepsilon B})} \leqslant C\varepsilon^2$ because $\tilde f_0(z) := f_0(p+\varepsilon z)$. Clearly, $\mathcal{H}_{p,\bar\lambda,B}(\varepsilon,\phi) = 0$ if and only if $u := \tilde f_0 + \phi\circ\chi^{-1}_{p,\varepsilon B}$ solves the Dirichlet problem (2-2)-(2-3) in $\Omega_{p,\varepsilon B}$.

Note that, by definition and using (2-5), $\mathcal{H}_{p,\bar{\lambda},B}(0,\phi_{p,\bar{\lambda}})=0$. Also, a short computation shows that the derivative of $\mathcal{H}_{p,\bar{\lambda},B}(\varepsilon,\phi)$ with respect to ϕ satisfies

$$D_\phi \mathcal{H}_{p,\bar{\lambda},B}(0,\phi_{p,\bar{\lambda}})=T_{p,\bar{\lambda}},$$

so it is an invertible map $C^{s+1}_{\mathrm{Dir}}(\mathbb{B})\to C^{s-1}(\mathbb{B})$; see Lemma 2.1. The implicit function theorem in Banach spaces then ensures that, for any ε close enough to 0, there is a unique function ϕ^ε in a small neighborhood of $\phi_{p,\bar{\lambda}}$ in $C^{s+1}_{\mathrm{Dir}}(\mathbb{B})$ satisfying

$$\mathcal{H}_{p,\bar{\lambda},B}(\varepsilon,\phi^\varepsilon)=0.$$

Then $u_{p,\varepsilon,\bar{\lambda},B}:=\tilde{f}_0+\phi^\varepsilon\circ\chi^{-1}_{p,\varepsilon B}$ is the desired solution to the Dirichlet problem in $\Omega_{p,\varepsilon B}$. □

We will henceforth denote by

$$\mathbb{P}_{p,\bar{\lambda},\varepsilon B}: C^{s+1}(\mathbb{S})\to C^{s+1}(\Omega_{p,\varepsilon B})$$

the map $\psi\mapsto v_\psi$, where v_ψ is the only solution to the problem

$$T_{p,\bar{\lambda}}v_\psi=0 \quad \text{in } \Omega_{p,\varepsilon B},$$

with the boundary condition

$$v_\psi(1+\varepsilon B(\omega),\omega)=\psi(\omega).$$

Note that the existence and uniqueness of v_ψ is an easy consequence of Lemma 2.1.

For future reference, let us record here the definition of the associated Dirichlet–Neumann operator $\Lambda_{p,\bar{\lambda},\varepsilon B}: C^{s+1}(\mathbb{S})\to C^s(\mathbb{S})$,

$$\Lambda_{p,\bar{\lambda},\varepsilon B}\psi(\omega):=\nu\cdot A\nabla\mathbb{P}_{p,\bar{\lambda},\varepsilon B}\psi(1+\varepsilon B(\omega),\omega).$$

As $\Lambda_{p,\bar{\lambda},\varepsilon B}$ reduces to the standard Dirichlet–Neumann map Λ_0 when $\varepsilon=\bar{\lambda}=0$, it is standard that

$$\|\Lambda_{p,\bar{\lambda},\varepsilon B}-\Lambda_{p,\bar{\lambda},0}\|_{C^{s+1}(\mathbb{S})\to C^s(\mathbb{S})}\leqslant C|\varepsilon|, \tag{2-6}$$

$$\|\Lambda_{p,\bar{\lambda},\varepsilon B}-\Lambda_0\|_{C^{s+1}(\mathbb{S})\to C^s(\mathbb{S})}\leqslant C(|\varepsilon|+\bar{\lambda}). \tag{2-7}$$

3. Asymptotic expansions

In this section we compute asymptotic formulas for the solution to the Dirichlet problem in the domain (2-1) obtained in Proposition 2.3, valid for $|\varepsilon|\ll 1$. Let us begin with the estimates for the solutions to the Dirichlet problem:

Proposition 3.1. *The function $u_{p,\varepsilon,\bar{\lambda},B}$ is of the form*

$$u_{p,\varepsilon,\bar{\lambda},B}=f_0(p)+\phi_{p,\bar{\lambda}}(r)+\varepsilon\{W_{p,\bar{\lambda}}(r)\cdot z+\mathbb{P}_{p,\bar{\lambda},\varepsilon B}[\nabla f_0(p)\cdot\omega-\phi_{p,\bar{\lambda}}{}'(1)\,B]\}+O(\varepsilon^2),$$

where $W_{p,\bar{\lambda}}:[0,1+\delta_p]\to\mathbb{R}^n$ is a function with $\|W_{p,\bar{\lambda}}\|_{C^{s+1}}\leqslant C\bar{\lambda}$.

Remark 3.2. In the case when $F(x,u)=f(x)$, the formula is slightly more explicit:

$$u_{p,\varepsilon,\bar\lambda,B} = f_0(p) - \frac{\bar\lambda}{2n} f(p)\,(r^2-1) \\ + \varepsilon\left\{\left[\nabla f_0(p) - \frac{\bar\lambda(r^2-1)}{2n+4}\left(\nabla f(p) - \frac{f(p)b(p)}{n}\right)\right]\cdot z + \frac{\bar\lambda\, f(p)}{n}\mathbb{P}_{\varepsilon B}B\right\} + O(\varepsilon^2).$$

Here we are using the notation $\mathbb{P}_{\varepsilon B}\equiv \mathbb{P}_{p,0,\varepsilon B}$, which does not depend on p because $F'=0$.

Proof. Note that $u_0 := f_0(p)+\phi_{p,\bar\lambda}(r)$ satisfies the equation

$$\Delta u_0 + \bar\lambda F(p,u_0) = 0, \quad u_0|_{r=1} = f_0(p).$$

Let us write $u_1 := (u_{p,\varepsilon,\bar\lambda,B} - u_0)/\varepsilon$ and observe that

$$\widetilde{F}(z,u_{p,\varepsilon,\bar\lambda,B}) = F(p+\varepsilon z, u_0+\varepsilon u_1) = F(p,u_0) + \varepsilon[\nabla F(p,u_0)\cdot z + F'(p,u_0)u_1] + O(\varepsilon^2).$$

As $\tilde{L}u_{p,\varepsilon,\bar\lambda,B} + \bar\lambda\widetilde{F}(z,u_{p,\varepsilon,\bar\lambda,B}) = 0$ with the boundary condition

$$u_{p,\varepsilon,\bar\lambda,B}(1+\varepsilon B(\omega),\omega) = \tilde{f}_0(1+\varepsilon B(\omega),\omega) = f_0(p) + \varepsilon\nabla f_0(p)\cdot\omega + O(\varepsilon^2),$$

this ensures that u_1 satisfies an equation of the form

$$T_{p,\bar\lambda}u_1 + \bar\lambda\nabla F(p,u_0)\cdot z + b(p)\cdot\frac{z}{r}\,\phi_{p,\bar\lambda}{}'(r) + O(\varepsilon) = 0$$

in $\Omega_{p,\varepsilon B}$ and the boundary condition

$$u_1(1+\varepsilon B(\omega),\omega) = \nabla f_0(p)\cdot\omega - \phi_{p,\bar\lambda}{}'(1)\,B(\omega) + O(\varepsilon).$$

To analyze u_1, we start by noting that

$$u_1^* := \mathbb{P}_{p,\bar\lambda,\varepsilon B}[\nabla f_0(p)\cdot\omega - \phi_{p,\bar\lambda}{}'(1)\,B(\omega)]$$

satisfies the equation $T_{p,\bar\lambda}u_1^* = 0$ in $\Omega_{p,\varepsilon B}$ and the boundary condition

$$u_1^*(1+\varepsilon B(\omega),\omega) = \nabla f_0(p)\cdot\omega - \phi_{p,\bar\lambda}{}'(1)\,B(\omega).$$

It is an easy consequence of Lemma 2.1 that the equation

$$T_{p,\bar\lambda}w + \bar\lambda\nabla F(p,u_0(|z|))\cdot z + b(p)\cdot\frac{z}{r}\,u_0'(|z|) = 0 \quad \text{in } \mathbb{B}, \quad w|_{\partial\mathbb{B}} = 0,$$

has a unique solution w, which is then of the form $w = W_{p,\bar\lambda}(|z|)\cdot z$ for some $\mathbb{R}^n$-valued function $W_{p,\bar\lambda}$. Specifically, its j-th component $W_j(r) := W_{p,\bar\lambda}(r)\cdot e_j$ satisfies the ODE

$$W_j''(r) + \frac{n+1}{r}W_j'(r) + \bar\lambda F'(p,u_0(r))W_j(r) + \bar\lambda\,\partial_j F(p,u_0(r)) + b_j(p)\,\frac{u_0'(r)}{r} = 0,$$

with the boundary condition $W_j(1)=0$ and the requirement that W_j must be regular at 0. As $u_0(r)$ is well-defined up to $r=1+\delta_p$, so is $W_j(r)$. The function $W_{p,\bar\lambda}$ is obviously bounded as

$$\|W_{p,\bar\lambda}\|_{C^{s+1}((0,1+\delta_p))} \leqslant C\bar\lambda\|\partial_j F(p,u_0)\|_{C^{s-1}((0,1+\delta_p))} + C\left\|\frac{u_0'}{r}\right\|_{C^{s-1}((0,1+\delta_p))}.$$

Since $\|u_0'\|_{C^s((0,1+\delta_p))} \leqslant C\bar\lambda$ by Lemma 2.1, we infer that $\|W_{p,\bar\lambda}\|_{C^{s+1}} = O(\bar\lambda)$ as well.

By construction, we immediately obtain that $u_1 = u_1^* + w + O(\varepsilon)$, so the proposition follows. The expression of Remark 3.2 follows from the same argument taking into account the formula for $\phi_{p,\bar\lambda}$ provided in Remark 2.2. □

Next we obtain asymptotic formulas for the normal derivative of u:

Proposition 3.3. *The normal derivative of the function $u_{p,\varepsilon,\bar\lambda,B}$ satisfies*

$$\partial_\nu u_{p,\varepsilon,\bar\lambda,B} = {\phi_{p,\bar\lambda}}'(1) + \varepsilon\{H_{p,\bar\lambda}B + [\nabla f_0(p) + V_{p,\bar\lambda}]\cdot\omega\} + O(\varepsilon^2),$$

where the constant vector $V_{p,\bar\lambda} \in \mathbb{R}^n$ satisfies $|V_{p,\bar\lambda}| \leqslant C\bar\lambda$.

Remark 3.4. When $F(x,u) = f(x)$, one can obtain a more compact formula:

$$\partial_\nu u_{p,\varepsilon,\bar\lambda,B} = -\frac{\bar\lambda}{n} f(p) + \varepsilon\left\{-\frac{\bar\lambda}{n} f(p)\,(B - \Lambda_0 B) + \nabla f_0(p)\cdot\omega - \frac{\bar\lambda}{n+2}\left(\nabla f(p) - \frac{f(p)b(p)}{n}\right)\cdot\omega\right\} + O(\varepsilon^2). \tag{3-1}$$

Proof. Since the boundary of $\Omega_{p,\varepsilon B}$ is the zero set of the function $r - \varepsilon B(\omega) - 1$, it is clear that its unit normal vector at the point $(1 + \varepsilon B(\omega), \omega)$ is

$$\nu = \left(\omega - \frac{\varepsilon}{1+\varepsilon B(\omega)}\nabla_{\mathbb{S}} B(\omega)\right)\left(1 + \frac{\varepsilon^2}{(1+\varepsilon B(\omega))^2}|\nabla_{\mathbb{S}} B(\omega)|^2\right)^{-1/2} = \omega - \varepsilon\nabla_{\mathbb{S}} B(\omega) + O(\varepsilon^2),$$

where $\nabla_{\mathbb{S}}$ denotes covariant differentiation on the unit sphere.

Using this formula, it follows from Proposition 3.1 that

$$\begin{aligned}\partial_\nu u_{p,\varepsilon,\bar\lambda,B} &= \nu\cdot\nabla u_{p,\varepsilon,\bar\lambda,B}(1+\varepsilon B(\omega),\omega)\\ &= {\phi_{p,\bar\lambda}}'(1+\varepsilon B(\omega)) + \varepsilon\{(rW_{p,\bar\lambda})'(1)\cdot\omega + \nu\cdot\nabla\mathbb{P}_{p,\bar\lambda,\varepsilon B}[\nabla f_0(p)\cdot\omega - {\phi_{p,\bar\lambda}}'(1)\,B]\} + O(\varepsilon^2).\end{aligned}$$

Since $\phi_{p,\bar\lambda}(r)$ is C^{s+1}-smooth for $r < 1 + \delta_p$, let us now expand ${\phi_{p,\bar\lambda}}'$ and use the definition of the operator $\Lambda_{p,\bar\lambda,\varepsilon B}$ to write

$$\partial_\nu u_{p,\varepsilon,\bar\lambda,B} = {\phi_{p,\bar\lambda}}'(1) + \varepsilon\{{\phi_{p,\bar\lambda}}''(1)B - {\phi_{p,\bar\lambda}}'(1)\,\Lambda_{p,\bar\lambda,\varepsilon B}B + \Lambda_{p,\bar\lambda,\varepsilon B}(\nabla f_0(p)\cdot\omega) + W'_{p,\bar\lambda}(1)\cdot\omega\} + O(\varepsilon^2).$$

Let us now recall that $H_{p,\bar\lambda}B := {\phi_{p,\bar\lambda}}''(1)B - {\phi_{p,\bar\lambda}}'(1)\,\Lambda_{p,\bar\lambda,0}B$ (see Lemma 2.1) and that the usual Dirichlet–Neumann map of the ball satisfies $\Lambda_0(V\cdot\omega) = V\cdot\omega$ for all $V \in \mathbb{R}^n$. Therefore, we can use the bounds (2-6)-(2-7) and the estimate $|V_{p,\bar\lambda}| \leqslant C\bar\lambda$ with

$$V_{p,\bar\lambda} := W'_{p,\bar\lambda}(1),$$

proven in Proposition 3.1, to obtain the formula of the statement. The expression of Remark 3.4 follows from the above argument after taking into account the expression for $u_{p,\varepsilon,\bar\lambda,B}$ given in Remark 3.2. □

4. Proof of Theorem 1.1 when $A(x) = I$

For any given point $p \in \mathbb{R}^n$, let us now define a map

$$\mathcal{F}_{p,\bar\lambda} : (-\varepsilon_p, \varepsilon_p) \times X^1_{s+1} \to C^s(\mathbb{S}),$$

with $X_s^1 := \{b \in C^s(\mathbb{S}) : \|b\|_{C^s} < 1\}$, as

$$\mathcal{F}_{p,\bar\lambda}(\varepsilon, B) := \partial_\nu u_{p,\varepsilon,\bar\lambda,B} - \frac{\phi_{p,\bar\lambda}{}'(1)}{f_1(p)} \tilde f_1.$$

Roughly speaking, this map measures how far the Dirichlet solution $u_{p,\varepsilon,\bar\lambda,B}$ is from satisfying the Neumann condition in the domain $\Omega_{p,\varepsilon B}$ with a constant

$$\bar c := -\frac{\phi_{p,\bar\lambda}{}'(1)}{f_1(p)} > 0.$$

An immediate consequence of the asymptotic formulas for $\partial_\nu u_{p,\varepsilon,\bar\lambda,B}$ proved in Proposition 3.3 and the fact that

$$\tilde f_1(1+\varepsilon B(\omega), \omega) = f_1(p) + \varepsilon \nabla f_1(p)\cdot\omega + O(\varepsilon^2),$$

is the following:

Proposition 4.1. *For any fixed $p \in \mathbb{R}^n$, any $B \in X_{s+1}^1(\mathbb{S})$ and any $|\varepsilon| < \varepsilon_p$,*

$$\mathcal{F}_{p,\bar\lambda}(\varepsilon, B) = \varepsilon\left\{ H_{p,\bar\lambda} B + \left[\nabla f_0(p) - \frac{\phi_{p,\bar\lambda}{}'(1)}{f_1(p)}\nabla f_1(p) + V_{p,\bar\lambda}\right]\cdot\omega\right\} + O(\varepsilon^2).$$

Remark 4.2. When $F(x,u) = f(x)$, one can obtain a slightly more explicit formula:

$$\mathcal{F}_{p,\bar\lambda}(\varepsilon, B) = \varepsilon\left\{-\frac{\bar\lambda}{n} f(p)\,(B - \Lambda_0 B) + \left[\nabla f_0(p) + \frac{\bar\lambda\, f(p)}{n\, f_1(p)}\nabla f_1(p)\right]\cdot\omega - \frac{\bar\lambda}{n+2}\left[\nabla f(p) - \frac{f(p) b(p)}{n}\right]\cdot\omega\right\} + O(\varepsilon^2). \quad (4\text{-}1)$$

It then follows that the function $\mathcal{F}_{p,\bar\lambda}(\varepsilon, B)/\varepsilon$ can be defined at $\varepsilon = 0$ by continuity. Furthermore, its derivative with respect to B involves the operator $H_{p,\bar\lambda}$, whose kernel was shown to be the space $\mathcal{K}$ in Lemma 2.1. Consequently, let us define the spaces

$$\mathcal{X}_s := \{b \in C^s(\mathbb{S}) : \mathcal{P}_{\mathcal{K}} b = 0\}, \quad \mathcal{X}_s^1 := \{b \in \mathcal{X}_s : \|b\|_{C^s} < 1\},$$

with $\mathcal{P}_{\mathcal{K}}$ being the orthogonal projector onto the subspace $\mathcal{K}$. We also define the operator

$$\mathcal{P} b := b - \mathcal{P}_{\mathcal{K}} b.$$

It is clear from these expressions that $\mathcal{P}$ maps each space $C^s(\mathbb{S})$ into itself and $\mathcal{X}_s^1 \subset X_s^1$.

By Proposition 4.1, we can now define a map

$$\mathcal{G}_{p,\bar\lambda} : (-\varepsilon_p, \varepsilon_p) \times \mathcal{X}_{s+1}^1 \to \mathcal{X}_s$$

as

$$\mathcal{G}_{p,\bar\lambda}(\varepsilon, B) := \frac{\mathcal{P}\mathcal{F}_{p,\bar\lambda}(\varepsilon, B)}{\varepsilon}.$$

Lemma 4.3. *Let $U \subset \mathbb{R}^n$ be any bounded domain. For any $\bar\lambda \in (0, \bar\lambda_U)$, with*

$$\bar\lambda_U := \inf_{p\in U} \bar\lambda_p > 0,$$

there exist some $\varepsilon_{U,\bar\lambda} > 0$ and a C^s function $Y_{\varepsilon,\bar\lambda} : U \to \mathbb{R}^n$ such that

$$\partial_\nu u_{p,\varepsilon,\bar\lambda,B_{\varepsilon,p,\bar\lambda}} - \frac{\phi_{p,\bar\lambda}{}'(1)}{f_1(p)} \tilde f_1 = Y_{\varepsilon,\bar\lambda}(p)\cdot\omega$$

for all $p \in U$ and all $|\varepsilon| < \varepsilon_{U,\bar\lambda}$. Here $Y_{\varepsilon,\bar\lambda}(p) := Y(\varepsilon, p, \bar\lambda)$ is of class C^s in all its arguments, and can be interpreted as a family of parametrized vector fields on U, and $B_{\varepsilon,p,\bar\lambda}$ is a certain function in $\mathcal{X}^1_{s+1}$.

Proof. Let us begin by showing that the Fréchet derivative $D_B\mathcal{G}_{p,\bar\lambda}(0,0) : \mathcal{X}_{s+1} \to \mathcal{X}_s$ is one-to-one. To see this, note that Proposition 4.1 and the fact that $\mathcal{P}(A\cdot\omega) = 0$ for any $A \in \mathbb{R}^n$ imply that the derivative of $\mathcal{G}_{p,\bar\lambda}$ with respect to B is of the form

$$D_B\mathcal{G}_{p,\bar\lambda}(\varepsilon, 0) = H_{p,\bar\lambda} + \mathcal{E},$$

with $\|\mathcal{E}\|_{\mathcal{X}_{s+1}\to\mathcal{X}_s} \leqslant C|\varepsilon|$. Here we have used that, by Lemma 2.1, $\mathcal{P}H_{p,\bar\lambda} = H_{p,\bar\lambda}$ because the range of the elliptic first-order operator $H_{p,\bar\lambda}$ is contained in $\mathcal{K}^\perp$. The estimate (2-4) then ensures that $D_B\mathcal{G}_{p,\bar\lambda}(\varepsilon, 0)$ is an invertible map $\mathcal{X}_{s+1} \to \mathcal{X}_s$ provided that ε is small enough.

As $\mathcal{G}_{p,\bar\lambda}(0,0) = 0$, the invertibility of $D_B\mathcal{G}_{p,\bar\lambda}(\varepsilon, 0)$ implies, via the implicit function theorem, that for any ε small enough, there is a unique function $B_{\varepsilon,p,\bar\lambda}$ in a small neighborhood of 0 such that

$$\mathcal{G}_{p,\bar\lambda}(\varepsilon, B_{\varepsilon,p,\bar\lambda}) = 0.$$

By the definition of $\mathcal{F}_{p,\bar\lambda}$ and the fact that $\mathcal{K} = \{Y\cdot\omega : Y \in \mathbb{R}^n\}$, this implies that there is some $Y(\varepsilon, p, \bar\lambda) \in \mathbb{R}^n$ such that

$$\partial_\nu u_{p,\varepsilon,\bar\lambda,B_{\varepsilon,p,\bar\lambda}} - \frac{\phi_{p,\bar\lambda}{}'(1)}{f_1(p)} \tilde f_1 = Y(\varepsilon, p, \bar\lambda)\cdot\omega.$$

Furthermore, $Y(\varepsilon, p, \bar\lambda)$ is a C^s-smooth function of its arguments because so is the left-hand side of this identity. □

Let us now note that the asymptotic expression of the vector field $Y_{\varepsilon,\bar\lambda}(p)$ can be read off Proposition 4.1:

Lemma 4.4. *The vector field $Y_{\varepsilon,\bar\lambda}$ is of the form*

$$Y_{\varepsilon,\bar\lambda}(p) = \varepsilon\left[\nabla f_0(p) - \frac{\phi_{p,\bar\lambda}{}'(1)}{f_1(p)}\nabla f_1(p) + V_{p,\bar\lambda}\right] + O(\varepsilon^2).$$

When $F(x,u) = f(x)$, one can write down the more precise expression

$$Y_{\varepsilon,\bar\lambda}(p) = \varepsilon\left\{\nabla f_0(p) + \frac{\bar\lambda\, f(p)}{n\, f_1(p)}\nabla f_1(p) - \frac{\bar\lambda}{n+2}\left[\nabla f(p) - \frac{f(p)b(p)}{n}\right]\right\} + O(\varepsilon^2).$$

Proof of Theorem 1.1 when $A(x)=I$ and of Remark 1.2. Let us suppose that p^* is a nondegenerate critical point of the function f_0. As $\phi_{p,\bar\lambda}{}'(1)=O(\bar\lambda)$ by Lemma 2.1, Lemma 4.4 implies that

$$\frac{Y_{\varepsilon,\bar\lambda}(p)}{\varepsilon}=\nabla f_0(p)+\mathcal{E},$$

with an error bounded as $\|\mathcal{E}\|_{C^1(U)}\leqslant C_U|\bar\lambda|+C_U|\varepsilon|$ for any bounded domain $U\ni p^*$. If $|\bar\lambda|$ and $|\varepsilon|$ are small enough, it is then standard that there is a unique point $p_{\varepsilon,\bar\lambda}$ in a small neighborhood of p^* such that

$$Y_{\varepsilon,\bar\lambda}(p_{\varepsilon,\bar\lambda})=0.$$

By Lemma 4.3, and setting $\bar c:=-\phi'_{p_{\varepsilon,\bar\lambda},\bar\lambda}(1)/f_1(p_{\varepsilon,\bar\lambda})$, this ensures that

$$\partial_\nu u_{\varepsilon,p_{\varepsilon,\bar\lambda},\bar\lambda,B_{\varepsilon,p,\bar\lambda}}+\bar c\tilde f_1\equiv 0,$$

which implies the claim of the theorem with the domain $\Omega_{p_{\varepsilon,\bar\lambda},\varepsilon B_{\varepsilon,p,\bar\lambda}}$.

To prove Remark 1.2 on overdetermined solutions for the torsion problem, let us assume that $F(x,u)=f(x)=1$ and that p^* is a nondegenerate critical point of the function $f_0+\kappa\log f_1$ for some constant $\kappa>0$. In this case, since $f(x)=1$ and $b(x)=0$, Lemma 4.4 implies that

$$\frac{Y_{\varepsilon,\bar\lambda}(p)}{\varepsilon}=\nabla f_0(p)+\frac{\bar\lambda}{n}\nabla\log f_1(p)+\mathcal{E}',$$

with $\|\mathcal{E}'\|_{C^1(U)}\leqslant C_U\varepsilon$. As one can pick any positive value of $\bar\lambda$ by Remark 2.2, let us fix $\bar\lambda=\bar\lambda^*:=n\kappa>0$. The previous argument then allows us to conclude that, for any ε small enough, there exists some point p_ε close to p^* for which $Y_{\varepsilon,\bar\lambda}(p_\varepsilon)=0$. Note that the condition that p^* is a nondegenerate critical point of $f_0+\kappa\log f_1$ is crucially used to solve

$$\nabla f_0(p_\varepsilon)+\kappa\nabla\log f_1(p_\varepsilon)=-\mathcal{E}'$$

for small $\varepsilon>0$ via the inverse function theorem. As above, this implies the existence of solutions to the overdetermined torsion problem. The case of $f_0=0$, $f_1=1$ and $F(x,u)=f(x)$ is handled similarly, so Remark 1.2 then follows. □

5. Introduction of a nonconstant matrix $A(x)$ and conclusion of the proof

In this section we will show why the proof of Theorem 1.1 carried out for the case when $A(x)=I$ remains valid, with only minor variations, in the case of a general matrix $A(x)$.

The key idea is that we are constructing domains that are small deformations of the ball of radius ε, with $\varepsilon\ll 1$. Over scales of order ε, the function $A(x)$ is essentially constant, so it stands to reason that one might be able to compensate for the effect of having a nonconstant matrix $A(x)$ (at least, to some orders when considering an asymptotic expansion in ε) by deforming the balls accordingly. More visually, this would correspond essentially to picking an ellipsoidal domain at each point x with axes determined by the matrix $A(x)$.

The way to implement this idea is through (a rescaling of) the normal coordinates associated with the matrix-valued function A, which we now regard as a Riemannian metric on $\mathbb{R}^n$ of class C^{s+2}. These are defined through the exponential map at a point $p \in \mathbb{R}^n$,

$$\exp_p^A : U_p \to \mathbb{R}^n,$$

which maps a certain domain $U_p \subset \mathbb{R}^n$ diffeomorphically onto its image. It is standard [DeTurck and Kazdan 1981] that $\exp_p^A(Z)$ is a C^{s+1} function of $Z \in U_p$ and of $p \in \mathbb{R}^n$. The normal coordinates at p are just the Cartesian coordinates $Z = (Z_1, \dots, Z_n)$ on $U_p \subset \mathbb{R}^n$. In these coordinates, the metric reads as $\hat{A}(Z) = I + O(|Z|^2)$. More precisely, $\hat{A}(Z) = (\hat{a}_{ij}(Z))$ is given by the pullback by the exponential map of the metric tensor, which is well known to be of the form

$$(\exp_p^A)^*[a_{ij}(x)\, dx_i\, dx_j] =: \hat{a}_{ij}(Z)\, dZ_i\, dZ_j,$$

with functions $\hat{a}_{ij}$ of class $C^s(U_p)$ such that

$$\hat{a}_{ij}(0) = \delta_{ij}, \quad \partial_{Z_k}\hat{a}_{ij}(0) = 0.$$

Therefore, normal coordinates enable us to write the matrix as the identity plus a C^s-smooth quadratic error. Incidentally, it is well known that the leading-order contribution of the error is determined by the curvature of the metric A at the point p.

We are now ready to reformulate the overdetermined problem with a general function A as a small perturbation of the case $A(x) = I$. For each function $B \in C^{s+1}(\mathbb{S})$ with $\|B\|_{C^{s+1}} < 1$ and each ε small enough, one can then define the domain $\Omega_{p,\varepsilon B} \subset \mathbb{R}^n$ (which will play the same role as (2-1)) as

$$\Omega_{p,\varepsilon B} := \{\exp_p^A(\varepsilon z) : |z| < 1 + \varepsilon B(z/|z|)\}.$$

Note that, in terms of the spherical coordinates associated with a point z,

$$r := |z| \in (0, \infty), \quad \omega := \frac{z}{|z|} \in \mathbb{S},$$

the above condition reads simply as $r < 1 + \varepsilon B(\omega)$. In the domain $\Omega_{p,\varepsilon B}$, (1-2) reads in the rescaled normal coordinates z at p as

$$\hat{L}u + \bar{\lambda}\widehat{F}(z, u) = 0,$$

where $\widehat{F}(z, u) := F(\exp_p^A(\varepsilon z), u)$ and now the linear operator $\hat{L}$ is of the form

$$\hat{L}u := \hat{a}_{ij}(\varepsilon z)\, \partial_{z_i z_j} u + \varepsilon \hat{b}_i(\varepsilon z)\, \partial_{z_i} u,$$

with $\hat{a}_{ij}(Z)$ as above and some functions $\hat{b}_i(Z)$ of class C^s.

Therefore,

$$\hat{L}u = \Delta u + \varepsilon \hat{b}_i(\varepsilon z)\, \partial_{z_i} u + \mathcal{E}u,$$

where the error term is bounded as $\|\mathcal{E}u\|_{C^{s-1}} \leqslant C\varepsilon^2 \|u\|_{C^{s+1}}$ and $\hat{L}u - \mathcal{E}u$ is just like the operator $\tilde{L}u$ introduced below (2-2). One can now go over the proof of Theorem 1.1 and readily see that all the arguments remain valid when one introduces an error of this form in the expressions. This is not surprising, as the proof only uses the formulas for the terms in the equations that are of zeroth and first order in ε. Since the nondegenerate critical points of f_0 do not depend on the coordinate system, Theorem 1.1 is then proven for a general matrix-valued function A.

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References

[Aftalion and Busca 1998] A. Aftalion and J. Busca, "Radial symmetry of overdetermined boundary-value problems in exterior domains", *Arch. Ration. Mech. Anal.* **143**:2 (1998), 195–206. MR Zbl

[Aftalion et al. 1999] A. Aftalion, J. Busca, and W. Reichel, "Approximate radial symmetry for overdetermined boundary value problems", *Adv. Differential Equations* **4**:6 (1999), 907–932. MR Zbl

[Alessandrini and Garofalo 1989] G. Alessandrini and N. Garofalo, "Symmetry for degenerate parabolic equations", *Arch. Ration. Mech. Anal.* **108**:2 (1989), 161–174. MR Zbl

[Bianchini et al. 2014] C. Bianchini, A. Henrot, and P. Salani, "An overdetermined problem with non-constant boundary condition", *Interfaces Free Bound.* **16**:2 (2014), 215–241. MR Zbl

[Brandolini et al. 2008] B. Brandolini, C. Nitsch, P. Salani, and C. Trombetti, "Serrin-type overdetermined problems: an alternative proof", *Arch. Ration. Mech. Anal.* **190**:2 (2008), 267–280. MR Zbl

[Cianchi and Salani 2009] A. Cianchi and P. Salani, "Overdetermined anisotropic elliptic problems", *Math. Ann.* **345**:4 (2009), 859–881. MR Zbl

[Damascelli et al. 1999] L. Damascelli, F. Pacella, and M. Ramaswamy, "Symmetry of ground states of p-Laplace equations via the moving plane method", *Arch. Ration. Mech. Anal.* **148**:4 (1999), 291–308. MR Zbl

[Delay and Sicbaldi 2015] E. Delay and P. Sicbaldi, "Extremal domains for the first eigenvalue in a general compact Riemannian manifold", *Discrete Contin. Dyn. Syst.* **35**:12 (2015), 5799–5825. MR Zbl

[DeTurck and Kazdan 1981] D. M. DeTurck and J. L. Kazdan, "Some regularity theorems in Riemannian geometry", *Ann. Sci. École Norm. Sup.* (4) **14**:3 (1981), 249–260. MR Zbl

[Domínguez-Vázquez et al. 2019] M. Domínguez-Vázquez, A. Enciso, and D. Peralta-Salas, "Solutions to the overdetermined boundary problem for semilinear equations with position-dependent nonlinearities", *Adv. Math.* **351** (2019), 718–760. MR Zbl

[Domínguez-Vázquez et al. 2021] M. Domínguez-Vázquez, A. Enciso, and D. Peralta-Salas, "Piecewise smooth stationary Euler flows with compact support via overdetermined boundary problems", *Arch. Ration. Mech. Anal.* **239**:3 (2021), 1327–1347. MR Zbl

[Farina and Valdinoci 2010] A. Farina and E. Valdinoci, "Flattening results for elliptic PDEs in unbounded domains with applications to overdetermined problems", *Arch. Ration. Mech. Anal.* **195**:3 (2010), 1025–1058. MR Zbl

[Farina and Valdinoci 2013] A. Farina and E. Valdinoci, "On partially and globally overdetermined problems of elliptic type", *Amer. J. Math.* **135**:6 (2013), 1699–1726. MR Zbl

[Fragalà and Gazzola 2008] I. Fragalà and F. Gazzola, "Partially overdetermined elliptic boundary value problems", *J. Differential Equations* **245**:5 (2008), 1299–1322. MR Zbl

[Fragalà et al. 2006] I. Fragalà, F. Gazzola, and B. Kawohl, "Overdetermined problems with possibly degenerate ellipticity: a geometric approach", *Math. Z.* **254**:1 (2006), 117–132. MR Zbl

[Garofalo and Lewis 1989] N. Garofalo and J. L. Lewis, "A symmetry result related to some overdetermined boundary value problems", *Amer. J. Math.* **111**:1 (1989), 9–33. MR Zbl

[Garofalo and Sartori 1999] N. Garofalo and E. Sartori, "Symmetry in exterior boundary value problems for quasilinear elliptic equations via blow-up and a priori estimates", *Adv. Differential Equations* **4**:2 (1999), 137–161. MR Zbl

[Kawohl 1998] B. Kawohl, "Symmetry or not?", *Math. Intelligencer* **20**:2 (1998), 16–22. MR Zbl

[Kumaresan and Prajapat 1998] S. Kumaresan and J. Prajapat, "Serrin's result for hyperbolic space and sphere", *Duke Math. J.* **91**:1 (1998), 17–28. MR Zbl

[Magnanini and Poggesi 2020a] R. Magnanini and G. Poggesi, "Nearly optimal stability for Serrin's problem and the soap bubble theorem", *Calc. Var. Partial Differential Equations* **59**:1 (2020), art. id. 35. MR Zbl

[Magnanini and Poggesi 2020b] R. Magnanini and G. Poggesi, "Serrin's problem and Alexandrov's soap bubble theorem: enhanced stability via integral identities", *Indiana Univ. Math. J.* **69**:4 (2020), 1181–1205. MR Zbl

[Pacard and Sicbaldi 2009] F. Pacard and P. Sicbaldi, "Extremal domains for the first eigenvalue of the Laplace–Beltrami operator", *Ann. Inst. Fourier* (*Grenoble*) **59**:2 (2009), 515–542. MR Zbl

[del Pino et al. 2015] M. del Pino, F. Pacard, and J. Wei, "Serrin's overdetermined problem and constant mean curvature surfaces", *Duke Math. J.* **164**:14 (2015), 2643–2722. MR Zbl

[Prajapat 1998] J. Prajapat, "Serrin's result for domains with a corner or cusp", *Duke Math. J.* **91**:1 (1998), 29–31. MR Zbl

[Rayleigh 1877] J. W. Strutt, 3rd baron Rayleigh, *The theory of sound, I*, Macmillan, London, 1877.

[Ros et al. 2020] A. Ros, D. Ruiz, and P. Sicbaldi, "Solutions to overdetermined elliptic problems in nontrivial exterior domains", *J. Eur. Math. Soc.* **22**:1 (2020), 253–281. MR Zbl

[Serrin 1971] J. Serrin, "A symmetry problem in potential theory", *Arch. Ration. Mech. Anal.* **43** (1971), 304–318. MR Zbl

[Traizet 2014] M. Traizet, "Classification of the solutions to an overdetermined elliptic problem in the plane", *Geom. Funct. Anal.* **24**:2 (2014), 690–720. MR Zbl

[Ye 1991] R. Ye, "Foliation by constant mean curvature spheres", *Pacific J. Math.* **147**:2 (1991), 381–396. MR Zbl

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