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# IDA AND HANKEL OPERATORS ON FOCK SPACES

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We introduce a new space IDA of locally integrable functions whose integral distance to holomorphic functions is finite, and use it to completely characterize boundedness and compactness of Hankel operators on weighted Fock spaces. As an application, for bounded symbols, we show that the Hankel operator $H_f$ is compact if and only if $H_{\bar{f}}$ is compact, which complements the classical compactness result of Berger and Coburn. Motivated by recent work of Bauer, Coburn, and Hagger, we also apply our results to the Berezin–Toeplitz quantization.

## 1. Introduction

Denote by $L^2$ the Hilbert space of all Gaussian square-integrable functions $f$ on $\mathbb{C}^n$, that is,

$$\int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2}\, dv(z) < \infty,$$

where $v$ is the standard Lebesgue measure on $\mathbb{C}^n$. The Fock space $F^2$ (aka Segal–Bargmann space) consists of all holomorphic functions in $L^2$. The orthogonal projection of $L^2$ onto $F^2$ is denoted by $P$ and called the Bergman projection. For a suitable function $f : \mathbb{C}^n \to \mathbb{C}$, the Hankel operator $H_f$ and the Toeplitz operator $T_f$ are defined on $F^2$ by

$$H_f = (I - P)M_f \quad \text{and} \quad T_f = PM_f.$$

The function $f$ is referred to as the symbol of $H_f$ and $T_f$. Since $P$ is a bounded operator, it follows that both $H_f$ and $T_f$ are well-defined and bounded on $F^2$ if $f$ is a bounded function. For unbounded symbols, despite considerable efforts, see, e.g., [Bauer 2005; Berger and Coburn 1994; Coburn et al. 2021; Hu and Wang 2018], characterization of boundedness or compactness of these operators has remained an open problem for more than 20 years.

In this paper, as a natural evolution from BMO (see [John and Nirenberg 1961; Zhu 2012]), we introduce a notion of integral distance to holomorphic (aka analytic) functions IDA and use it to completely characterize boundedness and compactness of Hankel operators on Fock spaces. Recently, in [Hu and Virtanen 2022], which continues our present work, we used IDA in the Hilbert space setting to characterize the Schatten class properties of Hankel operators. Indeed, the space IDA is broad in scope, and should have more applications, which we hope to demonstrate in future work in connection with Toeplitz operators.

All our results are proved for weighted Fock spaces $F^p(\varphi)$ consisting of holomorphic functions for which

$$\int_{\mathbb{C}^n} |f(z)|^p e^{-p\varphi(z)}\, dv(z) < \infty,$$

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where $0 < p < \infty$ and $\varphi$ is a suitable weight function (see Section 2 for further details). Obviously, with $p = 2$ and $\varphi(z) = (\alpha/2)|z|^2$, we obtain the weighted Fock space $F^2_\alpha$. The study of $L^p$-type Fock spaces was initiated in [Janson et al. 1987] and has since grown considerably, as seen in [Zhu 2012].

We also revisit and complement a surprising result due to [Berger and Coburn 1987], which states that for bounded symbols

$$H_f : F^2 \to L^2 \text{ is compact if and only if } H_{\bar{f}} \text{ is compact.}$$

In particular, we give a new proof and show that this phenomenon remains true for Hankel operators from $F^p(\varphi)$ to $L^q(\varphi)$ for general weights. What also makes this result striking is that it is not true for Hankel operators acting on other important function spaces, such as Hardy or Bergman spaces.

As an application, we will apply our results to the Berezin–Toeplitz quantization, which complements the results in [Bauer et al. 2018].

**1A. *Main results.*** We introduce the following new function spaces to characterize bounded and compact Hankel operators. Let $0 < s \le \infty$ and $0 < q < \infty$. For $f \in L^q_{\text{loc}}$, set

$$(G_{q,r}(f)(z))^q = \inf_{h \in H(B(z,r))} \frac{1}{|B(z,r)|} \int_{B(z,r)} |f - h|^q \, dv, \quad z \in \mathbb{C}^n,$$

where $H(B(z,r))$ stands for the set of holomorphic functions in the ball $B(z,r)$. We say that $f \in L^q_{\text{loc}}$ is in $\text{IDA}^{s,q}$ if

$$\|f\|_{\text{IDA}^{s,q}} = \|G_{q,1}(f)\|_{L^s} < \infty.$$

We further write $\text{BDA}^q$ for $\text{IDA}^{\infty,q}$ and say that $f \in \text{VDA}^q$ if

$$\lim_{z \to \infty} G_{q,1}(f)(z) = 0.$$

The properties of these spaces will be studied in Section 3.

We denote by $\mathcal{S}$ the set of all measurable functions $f$ that satisfy the condition in (2-7), which ensures that the Hankel operator $H_f$ is densely defined on $F^p(\varphi)$ provided that $0 < p < \infty$ and $\varphi$ is a suitable weight. Notice that the symbol class $\mathcal{S}$ contains all bounded functions. Further, we write $\text{Hess}_{\mathbb{R}}\,\varphi$ for the Hessian of $\varphi$ and E for the $2n \times 2n$ identity matrix — these concepts will be discussed in more detail in Section 2. It is important to notice that the condition $\text{Hess}_{\mathbb{R}}\,\varphi \simeq \text{E}$ in the following theorems is satisfied by the classical Fock space $F^2$, the Fock spaces $F^2_\alpha$ generated by *standard weights* $\varphi(z) = (\alpha/2)|z|^2$, $\alpha > 0$, Fock–Sobolev spaces, and a large class of nonradial weights.

**Theorem 1.1.** *Let $f \in \mathcal{S}$ and suppose that* $\text{Hess}_{\mathbb{R}}\,\varphi \simeq \text{E}$ *as in* (2-1).

(a) *For $0 < p \le q < \infty$ and $q \ge 1$, $H_f : F^p(\varphi) \to L^q(\varphi)$ is bounded if and only if $f \in \text{BDA}^q$, and $H_f$ is compact if and only if $f \in \text{VDA}^q$. For the operator norm of $H_f$, we have the estimate*

$$\|H_f\| \simeq \|f\|_{\text{BDA}^q}. \tag{1-1}$$

(b) *For* $1 \le q < p < \infty$, $H_f : F^p(\varphi) \to L^q(\varphi)$ *is bounded if and only if it is compact, which is equivalent to* $f \in \mathrm{IDA}^{s,q}$, *where* $s = pq/(p-q)$, *and*

$$\|H_f\| \simeq \|f\|_{\mathrm{IDA}^{s,q}}. \tag{1-2}$$

(c) *For* $0 < p \le q \le 1$ *and* $f \in L^\infty$, $H_f : F^p(\varphi) \to L^q(\varphi)$ *is bounded with*

$$\|H_f\| \le C\|f\|_{L^\infty} \tag{1-3}$$

*and compact if and only if* $f \in \mathrm{VDA}^q$.

We first note that Theorem 1.1 is new even for Hankel operators acting from $F^2$ to $L^2$. Previously only characterizations for $H_f$ and $H_{\bar f}$ to be *simultaneously* bounded (or *simultaneously* compact) were known. These were given in terms of the bounded (or vanishing) mean oscillation of $f$ in [Bauer 2005] for $F^2$ and in [Hu and Wang 2018] for Hankel operators from $F^p_\alpha$ to $L^q_\alpha$. In Theorem 7.1 of Section 7, we obtain these results as a simple consequence of Theorem 1.1. We also mention our recent work [Hu and Virtanen 2022], which gives a complete characterization of Schatten class Hankel operators.

Theorem 1.1 should also be compared with the results for Hankel operators on Bergman spaces $A^p$. Indeed, characterizations for boundedness and compactness can be found in [Axler 1986] for antianalytic symbols, in [Hagger and Virtanen 2021] for bounded symbols, and in [Hu and Lu 2019; Li 1994; Luecking 1992; Pau et al. 2016] for unbounded symbols. These two cases are different to study because of properties such as $F^p \subset F^q$ for $p \le q$ (as opposed to $A^q \subset A^p$) and certain nice geometry on the boundary of these bounded domains, which in turn helps with the treatment of the $\bar\partial$-problem.

What is very different about the results on Hankel operators acting on these two types of spaces is that our next result is only true in Fock spaces (see [Hagger and Virtanen 2021] for an interesting counterexample for the Bergman space).

**Theorem 1.2.** *Let* $f \in L^\infty$ *and suppose that* $\mathrm{Hess}_{\mathbb{R}}\, \varphi \simeq \mathrm{E}$ *as in* (2-1). *If* $0 < p \le q < \infty$ *or* $1 \le q < p < \infty$, *then* $H_f : F^p(\varphi) \to L^q(\varphi)$ *is compact if and only if* $H_{\bar f}$ *is compact.*

For Hankel operators on the Fock space $F^2$, Theorem 1.2 was proved in [Berger and Coburn 1987] using $C^*$-algebra and Hilbert space techniques and in [Stroethoff 1992] using elementary methods. More recently in [Hagger and Virtanen 2021], limit operator techniques were used to treat the reflexive Fock spaces $F^p_\alpha$. However, our result is new even in the Hilbert space case because of the more general weights that we consider. As a natural continuation of our present work, in [Hu and Virtanen 2022], we prove that, for $f \in L^\infty$, the Hankel operator $H_f$ is in the Schatten class $S_p$ if and only if $H_{\bar f}$ is in the Schatten class $S_p$ provided that $1 < p < \infty$.

As an application and further generalization of our results, in Section 6, we provide a complete characterization of those $f \in L^\infty$ for which

$$\lim_{t \to 0} \|T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}\|_t = 0 \tag{1-4}$$

for all $g \in L^\infty$, where $T_f^{(t)} = P^{(t)} M_f : F_t^2(\varphi) \to F_t^2(\varphi)$ and $P^{(t)}$ is the orthogonal projection of $L_t^2(\varphi)$ onto $F_t^2(\varphi)$. Here $L_t^2 = L^2(\mathbb{C}^n, d\mu_t)$ and

$$d\mu_t(z) = \frac{1}{t^n} \exp\left\{-2\varphi\left(\frac{z}{\sqrt{t}}\right)\right\} dv(z).$$

The importance of the semiclassical limit in (1-4) stems from the fact that it is one of the essential ingredients of the deformation quantization of [Rieffel 1989; 1990] in mathematical physics. Our conclusion related to (1-4) extends and complements the main result in [Bauer et al. 2018].

**1B.** ***Approach.*** A careful inspection shows that the methods and techniques used in [Berger and Coburn 1986; 1987; Hagger and Virtanen 2021; Perälä et al. 2014; Stroethoff 1992] depend heavily upon the following three aspects. First, the explicit representation of the Bergman kernel $K(z, w)$ for standard weights $\varphi(z) = (\alpha/2)|z|^2$ has the property that

$$K(z, w)e^{-(\alpha/2)|z|^2-(\alpha/2)|w|^2} = e^{(\alpha/2)|z-w|^2}. \tag{1-5}$$

However, for the class of weights we consider, this quadratic decay is known not to hold (even in dimension $n = 1$) and is expected to be very rare [Christ 1991]. The second aspect involves the Weyl unitary operator $W_a$ defined as

$$W_a f = f \circ \tau_a k_a,$$

where $\tau_a$ is the translation by $a$ and $k_a$ is the normalized reproducing kernel. As a unitary operator on $F^p_\alpha$ (or on $L^p_\alpha$), $W_a$ plays a very important role in the theory of the Fock spaces $F^p_\alpha$ (see [Zhu 2012]). Unfortunately, no analogue of Weyl operators is currently available for $F^p(\varphi)$ when $\varphi \neq (\alpha/2)|w|^2$. The third aspect we mention is Banach (or Hilbert) space techniques, such as the adjoint (for example, $H_f^*$) and the duality. However, when $0 < p < 1$, $F^p(\varphi)$ is only an $F$-space (in the sense of [Rudin 1973]) and the usual Banach space techniques can no longer be applied.

To overcome the three difficulties mentioned above, we introduce function spaces IDA, BDA and VDA, and develop their theory, which we use to characterize those symbols $f$ such that $H_f$ are bounded (or compact) from $F^p(\varphi)$ to $L^q(\varphi)$. Our characterization of the boundedness of $H_f$ extends the main results of [Bauer 2005; Hu and Wang 2018; Perälä et al. 2014]. It is also worth noting that as a natural generalization of BMO, the space IDA will have its own interest and will likely be useful to study other (related) operators (such as Toeplitz operators).

In our analysis, we appeal to the $\bar\partial$-techniques several times. As the canonical solution to $\bar\partial u = g\partial f$, $H_f g$ is naturally connected with the $\bar\partial$-theory. Hörmander's theory provides us with the $L^2$-estimate, but less is known about $L^p$-estimates on $\mathbb{C}^n$ when $p \neq 2$. With the help of a certain auxiliary integral operator, we obtain $L^p$-estimates of the Berndtsson–Anderson solution [1982] to the $\bar\partial$-equation. Our approach to handling weights whose curvature is uniformly comparable to the Euclidean metric form is similar to the treatment in [Schuster and Varolin 2012] which was initiated in [Berndtsson and Ortega Cerdà 1995], and a number of the techniques we use here were inspired by this approach. Although the work in [Berndtsson and Ortega Cerdà 1995] is restricted to $n = 1$, some of the results were extended to higher dimensions in [Lindholm 2001], and the others are easy to modify.

The outline of the paper is as follows. In Section 2 we study preliminary results on the Bergman kernel which are needed throughout the paper, and we also establish estimates for the $\bar\partial$-solution developed in [Berndtsson and Andersson 1982]. In Section 3, a notion of function spaces $\mathrm{IDA}^{s,q}$ is introduced. We obtain a useful decomposition for functions in $\mathrm{IDA}^{s,q}$ (compare with the decompositions of BMO and VMO). Using this decomposition, we obtain the completeness of $\mathrm{IDA}^{s,q}/H(\mathbb{C}^n)$ in $\|\cdot\|_{\mathrm{IDA}^{s,q}}$. In Sections 4 and 5 we prove Theorems 1.1 and 1.2, respectively. For the latter theorem, we also appeal to the Calderón–Zygmund theory of singular integrals, and in particular employ the Ahlfors–Beurling operator to obtain certain estimates on $\partial$- and $\bar\partial$-derivatives. In Section 6, we present an application of our results to quantization. In the last section, we give further remarks together with two conjectures.

Throughout the paper, $C$ stands for positive constants which may change from line to line, but does not depend on functions being considered. Two quantities $A$ and $B$ are called equivalent, denoted by $A \simeq B$, if there exists some $C$ such that $C^{-1}A \le B \le CA$.

## 2. Preliminaries

Let $\mathbb{C}^n = \mathbb{R}^{2n}$ be the $n$-dimensional complex Euclidean space and denote by $v$ the Lebesgue measure on $\mathbb{C}^n$. For $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in $\mathbb{C}^n$, we write $z \cdot \bar w = z_1\bar w_1 + \cdots + z_n \bar w_n$ and $|z| = \sqrt{z\cdot \bar z}$. Let $H(\mathbb{C}^n)$ be the family of all holomorphic functions on $\mathbb{C}^n$. Given a domain $\Omega$ in $\mathbb{C}^n$ and a positive Borel measure $\mu$ on $\Omega$, we denote by $L^p(\Omega, d\mu)$ the space of all Lebesgue measurable functions $f$ on $\Omega$ for which

$$\|f\|_{L^p(\Omega,d\mu)} = \left\{\int_\Omega |f|^p \, d\mu\right\}^{1/p} < \infty \quad \text{for } 0 < p < \infty$$

and $\|f\|_{L^\infty(\Omega,dv)} = \operatorname{ess\,sup}_{z\in\Omega} |f(z)| < \infty$ for $p = \infty$. For ease of notation, we simply write $L^p$ for the space $L^p(\mathbb{C}^n, dv)$.

**2A. *Weighted Fock spaces.*** For a real-valued weight $\varphi \in C^2(\mathbb{C}^n)$ and $0 < p < \infty$, denote by $L^p(\varphi)$ the space $L^p(\mathbb{C}^n, e^{-p\varphi}dv)$ with norm $\|\cdot\|_{p,\varphi} = \|\cdot\|_{L^p(\mathbb{C}^n, e^{-p\varphi}dv)}$. Then the Fock space $F^p(\varphi)$ is defined as

$$F^p(\varphi) = L^p(\varphi) \cap H(\mathbb{C}^n),$$
$$F^\infty(\varphi) = \left\{f \in H(\mathbb{C}^n) : \|f\|_{\infty,\varphi} = \sup\nolimits_{z\in\mathbb{C}^n} |f(z)|e^{-\varphi(z)} < \infty\right\}.$$

For $1 \le p \le \infty$, $F^p(\varphi)$ is a Banach space in the norm $\|\cdot\|_{p,\varphi}$ and $F^2(\varphi)$ is a Hilbert space. For $0 < p < 1$, $F^p(\varphi)$ is an $F$-space with metric given by $d(f, g) = \|f - g\|_{p,\varphi}^p$.

Other related and widely studied holomorphic function spaces include the Bergman spaces $A^p_\alpha(\mathbb{B}^n)$ of the unit ball $\mathbb{B}^n$ consisting of all holomorphic functions $f$ in $L^p(\mathbb{B}^n, dv_\alpha)$, where $0 < p < \infty$, $dv_\alpha(z) = (1 - |z|^2)^\alpha \, dv(z)$ and $\alpha > -1$.

In this paper we are interested in Fock spaces $F^p(\varphi)$ with certain uniformly convex weights $\varphi$. More precisely, suppose $\varphi = \varphi(x_1, x_2, \dots, x_{2n}) \in C^2(\mathbb{R}^{2n})$ is real-valued, and there are positive constants $m$ and $M$ such that $\operatorname{Hess}_{\mathbb{R}} \varphi$, the real Hessian, satisfies

$$m\mathrm{E} \le \operatorname{Hess}_{\mathbb{R}} \varphi(x) = \left(\frac{\partial^2\varphi(x)}{\partial x_j \partial x_k}\right)_{j,k=1}^{2n} \le M\mathrm{E}, \tag{2-1}$$

where E is the $2n \times 2n$ identity matrix; above, for symmetric matrices $A$ and $B$, we used the convention that $A \le B$ if $B - A$ is positive semidefinite. When (2-1) is satisfied, we write $\operatorname{Hess}_{\mathbb{R}} \varphi \simeq \mathrm{E}$. A typical model of such weights is given by $\varphi(z) = (\alpha/2)|z|^2$ for $z = (z_1, z_2, \dots, z_n)$ with $z_j = x_{2j-1} + \mathrm{i}x_{2j}$, which induces the weighted Fock space $F^p_\alpha$ studied by many authors (see, e.g., [Zhu 2012]). Another popular example is $\varphi(z) = |z|^2 - \frac{1}{2}\log(1+|z|^2)$, which gives the so-called Fock–Sobolev spaces studied for example in [Cho and Zhu 2012]. Notice that the weights $\varphi$ satisfying (2-1) are not only radial functions, as the example $\varphi(z) = |z|^2 + \sin[(z_1 + \bar{z}_1)/2]$ clearly shows.

For $x = (x_1, x_2, \dots, x_{2n})$, $t = (t_1, t_2, \dots, t_{2n}) \in \mathbb{R}^{2n}$, write $z_j = x_{2j-1} + \mathrm{i}x_{2j}$, $\xi_j = t_{2j-1} + \mathrm{i}t_{2j}$ and $\xi = (\xi_1, \xi_2, \dots, \xi_n)$. An elementary calculation similar to that on page 125 of [Krantz 1992] shows

$$\operatorname{Re} \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial z_k}(z)\xi_j\xi_k + \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(z)\xi_j\bar{\xi}_k = \frac{1}{2}\sum_{j,k=1}^{2n} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(x)t_j t_k \geq \frac{1}{2}m|\xi|^2.$$

Replacing $\xi$ with $\mathrm{i}\xi$ in the above inequality gives

$$-\operatorname{Re} \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial z_k}(z)\xi_j\xi_k + \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(z)\xi_j\bar{\xi}_k \geq \frac{1}{2}m|\xi|^2.$$

Thus,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(z)\xi_j\bar{\xi}_k \geq \frac{1}{2}m|\xi|^2.$$

Similarly, we have an upper bound for the complex Hessian of $\varphi$. Therefore, $m\omega_0 \leq dd^c\varphi \leq M\omega_0$, where $\omega_0 = dd^c|z|^2$ is the Euclidean Kähler form on $\mathbb{C}^n$ and $d^c = \frac{1}{4}\sqrt{-1}(\bar{\partial} - \partial)$. This implies that the theory in [Schuster and Varolin 2012; Hu and Lv 2014] is applicable in the present setting.

For $z \in \mathbb{C}^n$ and $r > 0$, let $B(z, r) = \{w \in \mathbb{C}^n : |w - z| < r\}$ be the ball with center at $z$ with radius $r$. For the proof of the following weighted Bergman inequality, we refer to Proposition 2.3 of [Schuster and Varolin 2012].

**Lemma 2.1.** *Suppose $0 < p \leq \infty$. For each $r > 0$ there is some $C > 0$ such that if $f \in F^p(\varphi)$ then*

$$|f(z)e^{-\varphi(z)}|^p \leq C\int_{B(z,r)} |f(\xi)e^{-\varphi(\xi)}|^p \, dv(\xi).$$

It follows from the preceding lemma that $\|f\|_{q,\varphi} \leq C\|f\|_{p,\varphi}$ and

$$F^p(\varphi) \subseteq F^q(\varphi) \quad \text{for } 0 < p \leq q \leq \infty. \tag{2-2}$$

This inclusion is completely different from that of the Bergman spaces.

**Lemma 2.2.** *There exist positive constants $\theta$ and $C_1$, depending only on $n$, $m$ and $M$, such that*

$$|K(z, w)| \leq C_1 e^{\varphi(z)+\varphi(w)} e^{-\theta|z-w|} \quad \textit{for all } z, w \in \mathbb{C}^n, \tag{2-3}$$

*and there exist positive constants $C_2$ and $r_0$ such that*

$$|K(z, w)| \geq C_2 e^{\varphi(z)+\varphi(w)} \tag{2-4}$$

*for $z \in \mathbb{C}^n$ and $w \in B(z, r_0)$.*

The estimate (2-3) appeared in [Christ 1991] for $n = 1$ and in [Delin 1998] for $n \geq 2$, while the inequality (2-4) can be found in [Schuster and Varolin 2012].

For $z \in \mathbb{C}^n$, write

$$k_z(\cdot) = \frac{K(\cdot, z)}{\sqrt{K(z, z)}}$$

for the normalized Bergman kernel. Then Lemma 2.2 implies

$$\frac{1}{C}e^{\varphi(z)} \le \|K(\,\cdot\,,z)\|_{p,\varphi} \le Ce^{\varphi(z)} \quad \text{and} \quad \frac{1}{C} \le \|k_z\|_{p,\varphi} \le C \quad \text{for } z \in \mathbb{C}^n, \tag{2-5}$$

and $\lim_{|z|\to\infty} k_z(\xi) = 0$ uniformly in $\xi$ on compact subsets of $\mathbb{C}^n$.

**2B. *The Bergman projection.*** For Fock spaces, we denote by $P$ the orthogonal projection of $L^2(\varphi)$ onto $F^2(\varphi)$, and refer to it as the Bergman projection. It is well known that $P$ can be represented as an integral operator

$$Pf(z) = \int_{\mathbb{C}^n} K(z,w) f(w) e^{-2\varphi(w)}\, dv(w) \tag{2-6}$$

for $z \in \mathbb{C}^n$, where $K(\,\cdot\,,\cdot\,)$ is the Bergman (reproducing) kernel of $F^2(\varphi)$.

As a consequence of Lemma 2.2, it follows that the Bergman projection $P$ is bounded on $L^p(\varphi)$ for $1 \le p \le \infty$, and $P|_{F^p(\varphi)} = \mathrm{I}$ for $0 < p \le \infty$; for further details, see Proposition 3.4 and Corollary 3.7 of [Schuster and Varolin 2012].

**2C. *Hankel operators.*** To define Hankel operators with unbounded symbols, consider

$$\Gamma = \left\{ \textstyle\sum_{j=1}^N a_j K(\,\cdot\,, z_j) : N \in \mathbb{N},\ a_j \in \mathbb{C},\ z_j \in \mathbb{C}^n \text{ for } 1 \le j \le N \right\},$$

and the symbol class

$$\mathcal{S} = \{ f \text{ measurable on } \mathbb{C}^n : fg \in L^1(\varphi) \text{ for } g \in \Gamma \}. \tag{2-7}$$

Given $f \in \mathcal{S}$, the Hankel operator $H_f = (\mathrm{I} - P)M_f$ with symbol $f$ is well-defined on $\Gamma$. According to Proposition 2.5 of [Hu and Virtanen 2020], for $0 < p < \infty$, the set $\Gamma$ is dense in $F^p(\varphi)$, and hence the Hankel operator $H_f$ is densely defined on $F^p(\varphi)$.

**2D. *Lattices in* $\mathbb{C}^n$.** Given $r > 0$, a sequence $\{a_k\}_{k=1}^\infty$ in $\mathbb{C}^n$ is called an $r$-lattice if the balls $\{B(a_k, r)\}_{k=1}^\infty$ cover $\mathbb{C}^n$ and $\{B(a_k, r/(2\sqrt{n}))\}_{k=1}^\infty$ are pairwise disjoint. A typical model of an $r$-lattice is the sequence

$$\left\{ \frac{r}{\sqrt{n}}(m_1 + k_1\mathrm{i}, m_2 + k_2\mathrm{i}, \ldots, m_n + k_n\mathrm{i}) \in \mathbb{C}^n : m_j, k_j \in \mathbb{Z},\ j = 1, 2, \ldots, n \right\}. \tag{2-8}$$

Notice that there exists an integer $N$ depending only on the dimension of $\mathbb{C}^n$ such that, for any $r$-lattice $\{a_k\}_{k=1}^\infty$,

$$1 \le \sum_{k=1}^\infty \chi_{B(a_k, 2r)}(z) \le N \tag{2-9}$$

for $z \in \mathbb{C}^n$, where $\chi_E$ is the characteristic function of $E \subset \mathbb{C}^n$. These well-known facts are explained in [Zhu 2012] when $n = 1$ and they can be easily generalized to any $n \in \mathbb{N}$.

**2E. *Fock Carleson measures.*** In the theory of Bergman spaces, Carleson measures provide an essential tool for treating various problems, especially in connection with bounded operators, functions of bounded mean oscillation, and their applications; see, e.g., [Zhu 2005]. In Fock spaces, Carleson measures play a similar role; see [Zhu 2012] for the Fock spaces $F^p_\alpha$. Carleson measures for Fock–Sobolev spaces were described in [Cho and Zhu 2012]. In [Schuster and Varolin 2012], Carleson measures for generalized Fock spaces (which include the weights considered in the present work) were used to study bounded and compact Toeplitz operators. Finally, their generalization to $(p, q)$-Fock Carleson measures was carried out in [Hu and Lv 2014], which is indispensable to the study of operators between distinct Banach spaces and will be applied to analyze Hankel operators acting from $F^p(\varphi)$ to $L^q(\varphi)$ in our work.

We recall the basic theory of these measures. Let $0 < p, q < \infty$ and let $\mu \geq 0$ be a positive Borel measure on $\mathbb{C}^n$. We call $\mu$ a $(p, q)$-*Fock Carleson measure* if the embedding $\mathrm{I} : F^p(\varphi) \to L^q(\mathbb{C}^n, e^{-q\varphi} d\mu)$ is bounded. Further, the measure $\mu$ is referred to as a vanishing $(p, q)$-Fock Carleson measure if in addition

$$\lim_{j\to\infty} \int_{\mathbb{C}^n} |f_j(z)e^{-\varphi(z)}|^q \, d\mu(z) = 0$$

whenever $\{f_j\}_{j=1}^{\infty}$ is bounded in $F^p(\varphi)$ and converges to 0 uniformly on any compact subset of $\mathbb{C}^n$ as $j \to \infty$. Fock Carleson measures were completely characterized in [Hu and Lv 2014] and we only add the following simple result, which is trivial for Banach spaces and can be easily proved in the other cases.

**Proposition 2.3.** *Let* $0 < p, q < \infty$ *and* $\mu$ *be a positive Borel measure on* $\mathbb{C}^n$*. Then* $\mu$ *is a vanishing* $(p, q)$*-Fock Carleson measure if and only if the inclusion map* I *is compact from* $F^p(\varphi) \to L^q(\mathbb{C}^n, d\mu)$.

*Proof.* It is not difficult to show that the image of the unit ball of $F^p(\varphi)$ under the inclusion is relatively compact in $L^q(\mathbb{C}^n, e^{q\varphi} d\mu)$. We leave out the details. □

**2F. *Differential forms and an auxiliary integral operator.*** As in [Krantz 1992], given two nonnegative integers $s, t \leq n$, we write

$$\omega = \sum_{|\alpha|=s,|\beta|=t} \omega_{\alpha,\beta} \, dz^\alpha \wedge d\bar{z}^\beta \tag{2-10}$$

for a differential form of type $(s, t)$. We denote by $L_{s,t}$ the family of all $(s, t)$-forms $\omega$ as in (2-10) with coefficients $\omega_{\alpha,\beta}$ measurable on $\mathbb{C}^n$ and set

$$|\omega| = \sum_{|\alpha|=s,|\beta|=t} |\omega_{\alpha,\beta}| \quad \text{and} \quad \|\omega\|_{p,\varphi} = \||\omega|\|_{p,\varphi}. \tag{2-11}$$

Given a weight function $\varphi$ satisfying (2-1), we define an integral operator $A_\varphi$ as

$$A_\varphi(\omega)(z) = \int_{\mathbb{C}^n} e^{\langle 2\partial\varphi, z-\xi\rangle} \sum_{j<n} \omega(\xi) \wedge \frac{\partial|\xi - z|^2 \wedge (2\bar{\partial}\partial\varphi(\xi))^j \wedge (\bar{\partial}\partial|\xi - z|^2)^{n-1-j}}{j! \, |\xi - z|^{2n-2j}} \tag{2-12}$$

for $\omega \in L_{0,1}$, where

$$\langle \partial\varphi(\xi), z - \xi\rangle = \sum_{j=1}^{n} \frac{\partial\varphi}{\partial\xi_j}(\xi)(z_j - \xi_j)$$

as denoted on page 92 in [Berndtsson and Andersson 1982].

For an $(s_1, t_1)$-form $\omega_A$ and an $(s_2, t_2)$-form $\omega_B$ with $s_1 + s_2 \leq n$, $t_1 + t_2 \leq n$, it is easy to verify that $|\omega_A \wedge \omega_B| \leq |\omega_A| |\omega_B|$. Therefore, for the $(n, n)$-form inside the integral of the right-hand side of (2-12), we obtain

$$\left|\omega(\xi)\wedge\frac{\partial|\xi-z|^2\wedge(2\bar\partial\partial\varphi)^j\wedge(\bar\partial\partial|\xi-z|^2)^{n-1-j}}{j!\,|\xi-z|^{2n-2j}}\right|\le C\frac{|\omega(\xi)|}{|\xi-z|^{2n-2j-1}}$$

because $\mathrm{i}\partial\bar\partial\,\varphi(\xi)\simeq\mathrm{i}\partial\bar\partial\,|\xi|^2$.

Recall that

$$\Gamma=\left\{\sum_{j=1}^N a_jK_{z_j}:N\in\mathbb{N},\ a_j\in\mathbb{C},\ z_j\in\mathbb{C}^n\text{ for }1\le j\le N\right\}$$

is dense in $F^p(\varphi)$ for all $0<p<\infty$.

**Lemma 2.4.** *Suppose* $1\le p\le\infty$.

(I) *There is a constant C such that* $\|A_\varphi(\omega)\|_{p,\varphi}\le C\|\omega\|_{p,\varphi}$ *for* $\omega\in L_{0,1}$.

(II) *For* $g\in\Gamma$ *and* $f\in C^2(\mathbb{C}^n)$ *satisfying* $|\bar\partial f|\in L^p$, *it holds that* $\bar\partial A_\varphi(g\bar\partial f)=g\bar\partial f$.

*Proof.* Let $z\in\mathbb{C}^n$. By (2-1), using Taylor expansion of $\varphi$ at $\xi$, we get

$$\varphi(z)-\varphi(\xi)\ge 2\,\mathrm{Re}\sum_{j=1}^n\frac{\partial\varphi(\xi)}{\partial\xi_j}(z_j-\xi_j)+m|z-\xi|^2.$$

Then (2-12) gives

$$|A_\varphi(\omega)(z)e^{-\varphi(z)}|\le C\int_{\mathbb{C}^n}|\omega(\xi)|e^{-\varphi(\xi)}\left\{\frac{1}{|\xi-z|}+\frac{1}{|\xi-z|^{2n-1}}\right\}e^{-m|\xi-z|^2}\,dv(\xi).\tag{2-13}$$

For $l<2n$ fixed, define another integral operator $\mathcal{A}_l$ as

$$\mathcal{A}_l:h\mapsto\int_{\mathbb{C}^n}h(\xi)\frac{e^{-m|\xi-z|^2}}{|\xi-z|^l}\,dv(\xi).$$

It is easy to verify, by interpolation, that $\mathcal{A}_l$ is bounded on $L^p$ for $1\le p\le\infty$. Therefore,

$$\begin{aligned}\|A_\varphi(\omega)\|_{p,\varphi}&\le C\|(\mathcal{A}_1+\mathcal{A}_{2n-1})(|\omega|e^{-\varphi})\|_{L^p}\\&\le C(\|\mathcal{A}_1\|_{L^p\to L^p}+\mathcal{A}_{2n-1}\|_{L^p\to L^p})\|\omega\|_{p,\varphi},\end{aligned}$$

which completes the proof of part (A).

Notice that the convexity assumption in (2-1) yields $dd^c\varphi\simeq\omega_0$, which in turn means that $|\partial\bar\partial\varphi(\xi)|\simeq 1$. We use $p'$ to denote the conjugate of $p$, $1/p+1/p'=1$. Now, for $f\in C^2(\mathbb{C}^n)$ satisfying $|\bar\partial f|\in L^p$, and $z,z_0\in\mathbb{C}^n$, we have

$$\begin{aligned}&\int_{\mathbb{C}^n}|K(\xi,z_0)\,\bar\partial f(\xi)|\sum_{j=0}^{n-1}\frac{e^{-\varphi(\xi)}|\bar\partial\partial\varphi(\xi)|^j}{|\xi-z|^{2n-2j-1}}\,dv(\xi)\\&\qquad\le C\left\{\sup_{\xi\in B(z,1)}|K(\xi,z_0)\,\bar\partial f(\xi)e^{-\varphi(\xi)}|+\int_{\mathbb{C}^n\setminus B(z,1)}|K(\xi,z_0)\,\bar\partial f(\xi)|e^{-\varphi(\xi)}\,dv(\xi)\right\}\\&\qquad\le Ce^{\varphi(z_0)}\Big\{\sup_{\xi\in B(z,1)}|\bar\partial f(\xi)|+\|\bar\partial f\|_{L^p}\|K(\,\cdot\,,z_0)\|_{p',\varphi}\Big\}<\infty.\end{aligned}$$

Hence, for $g \in \Gamma$ and $z \in \mathbb{C}^n$, it holds that

$$\int_{\mathbb{C}^n} |g(\xi)\bar{\partial} f(\xi)| \sum_{j=0}^{n-1} \frac{e^{-\varphi(\xi)}|\bar{\partial}\partial\varphi(\xi)|^j}{|\xi - z|^{2n-2j-1}}\, dv(\xi) < \infty.$$

From Proposition 10 of [Berndtsson and Andersson 1982], we get (B) (pay attention to the mistake in the last line of that result where $f$ is left out on the right-hand side). □

**Corollary 2.5.** *Suppose $f \in \mathcal{S} \cap C^1(\mathbb{C}^n)$ and $|\bar{\partial} f| \in L^s$ with some $1 \le s \le \infty$. For $g \in \Gamma$, it holds that*

$$H_f(g) = A_\varphi(g\bar{\partial} f) - P(A_\varphi(g\bar{\partial} f)). \tag{2-14}$$

*Proof.* Given $f \in \mathcal{S} \cap C^1(\mathbb{C}^n)$ with $|\bar{\partial} f| \in L^s$ and $g \in \Gamma$, we have $\|g\bar{\partial} f\|_{1,\varphi} \le \|g\|_{s',\varphi}\|\bar{\partial} f\|_{L^s} < \infty$, where $s'$ is the conjugate of $s$. Lemma 2.4 implies that $u = A_\varphi(g\bar{\partial} f) \in L^1(\varphi)$ and $\bar{\partial} u = g\bar{\partial} f$. Then $fg - u \in L^1(\varphi)$. Notice that $\bar{\partial}(fg - u) = g\bar{\partial} f - \bar{\partial} u = 0$, and so $fg - u \in F^1(\varphi)$. Since $P|_{F^1_\varphi} = \mathrm{I}$, we have

$$fg - u = P(fg - u) = P(fg) - P(u).$$

This shows that $H_f(g) = u - P(u)$. □

## 3. The space IDA

We now introduce a new space to characterize boundedness and compactness of Hankel operators. The space IDA is related to the space of bounded mean oscillation BMO (see, e.g., [John and Nirenberg 1961; Zhu 2012]), which has played an important role in many branches of analysis and their applications for decades. We find that IDA is also broad in scope and should have more applications in operator theory and related areas.

**3A.** ***Definitions and preliminary lemmas.*** Let $0 < q < \infty$ and $r > 0$. For $f \in L^q_{\mathrm{loc}}$ (the collection of $q$-th locally Lebesgue integrable functions on $\mathbb{C}^n$), following [Luecking 1992], we define $G_{q,r}(f)$ as

$$G_{q,r}(f)(z) = \inf\left\{\left(\frac{1}{|B(z,r)|}\int_{B(z,r)} |f - h|^q\, dv\right)^{1/q} : h \in H(B(z,r))\right\} \tag{3-1}$$

for $z \in \mathbb{C}^n$.

**Definition 3.1.** Suppose $0 < s \le \infty$ and $0 < q < \infty$. The space $\mathrm{IDA}^{s,q}$ (integral distance to holomorphic functions) consists of all $f \in L^q_{\mathrm{loc}}$ such that

$$\|f\|_{\mathrm{IDA}^{s,q}} = \|G_{q,1}(f)\|_{L^s} < \infty.$$

The space $\mathrm{IDA}^{\infty,q}$ is also denoted by $\mathrm{BDA}^q$. The space $\mathrm{VDA}^q$ consists of all $f \in \mathrm{BDA}^q$ such that

$$\lim_{z\to\infty} G_{q,1}(f)(z) = 0.$$

We will see in Section 6 that $\mathrm{IDA}^{s,q}$ is an extension of the space $\mathrm{IMO}^{s,q}$ introduced in [Hu and Wang 2018].

Notice that the space $\mathrm{BDA}^2$ was first introduced in the context of the Bergman spaces of the unit disk in [Luecking 1992], where it is called the space of functions with bounded distance to *analytic* functions (BDA).

**Remark 3.2.** As is the case with the classical $\mathrm{BMO}^q$ and $\mathrm{VMO}^q$ spaces, we have

$$\mathrm{BDA}^{q_2} \subset \mathrm{BDA}^{q_1} \quad \text{and} \quad \mathrm{VDA}^{q_2} \subset \mathrm{VDA}^{q_1}$$

properly for $0 < q_1 < q_2 < \infty$.

Let $0 < q < \infty$. For $z \in \mathbb{C}^n$, $f \in L^q(B(z,r), dv)$ and $r > 0$, we define the $q$-th mean of $|f|$ over $B(z,r)$ by setting

$$M_{q,r}(f)(z) = \left( \frac{1}{|B(z,r)|} \int_{B(z,r)} |f|^q \, dv \right)^{1/q}.$$

For $\omega \in L_{0,1}$, we set $M_{q,r}(\omega)(z) = M_{q,r}(|\omega|)(z)$.

**Lemma 3.3.** *Suppose $0 < q < \infty$. Then for $f \in L^q_{\mathrm{loc}}$, $z \in \mathbb{C}^n$ and $r > 0$, there is some $h \in H(B(z,r))$ such that*

$$M_{q,r}(f-h)(z) = G_{q,r}(f)(z) \tag{3-2}$$

*and*

$$\sup_{w \in B(z,r/2)} |h(w)| \le C \|f\|_{L^q(B(z,r),dv)}, \tag{3-3}$$

*where the constant $C$ is independent of $f$ and $r$.*

*Proof.* Let $f \in L^q_{\mathrm{loc}}$, $z \in \mathbb{C}^n$ and $r > 0$. Taking $h = 0$ in the integrand of (3-1), we get

$$G_{q,r}(f)(z) \le M_{q,r}(f)(z) < \infty.$$

Then for $j = 1, 2, \ldots$, we can pick $h_j \in H(B(z,r))$ such that

$$M_{q,r}(f - h_j)(z) \to G_{q,r}(f)(z) \tag{3-4}$$

as $j \to \infty$. Hence, for $j$ sufficiently large,

$$M_{q,r}(h_j)(z) \le C\{M_{q,r}(f-h_j)(z) + M_{q,r}(f)(z)\} \le C M_{q,r}(f)(z). \tag{3-5}$$

This shows that $\{h_j\}_{j=1}^{\infty}$ is a normal family. Thus, we can find a subsequence $\{h_{j_k}\}_{k=1}^{\infty}$ and a function $h \in H(B(z,r))$ so that $\lim_{k\to\infty} h_{j_k}(w) \to h(w)$ for $w \in B(z,r)$. By (3-4), applying Fatou's lemma, we have

$$G_{q,r}(f)(z) \le M_{q,r}(f-h)(z) \le \liminf_{k\to\infty} M_{q,r}(f - h_{j_k})(z) = G_{q,r}(f)(z),$$

which proves (3-2). It remains to note that, with the plurisubharmonicity of $|h|^q$, for $w \in B(z, r/2)$, we have

$$|h(w)| \le M_{q,r/2}(h)(w) \le C M_{q,r}(h)(z) \le C M_{q,r}(f)(z),$$

which completes the proof. □

**Corollary 3.4.** *For $0 < s < r$, there is a constant $C > 0$ such that for $f \in L^q_{\mathrm{loc}}$ and $w \in B(z, r-s)$, it holds that*

$$G_{q,s}(f)(w) \le M_{q,s}(f-h)(w) \le C G_{q,r}(f)(z), \tag{3-6}$$

*where $h$ is as in Lemma 3.3.*

*Proof.* For $0 < s < r$ and $w \in B(z, r-s)$, we have $B(w, s) \subset B(z, r)$. Then, the first estimate in (3-6) comes from the definition of $G_{q,s}(f)$, while (3-2) yields

$$M_{q,s}(f-h)(w) \le C M_{q,r}(f-h)(z) = C G_{q,r}(f)(z),$$

which completes the proof. □

For $z \in \mathbb{C}^n$ and $r > 0$, let

$$A^q(B(z,r), dv) = L^q(B(z,r), dv) \cap H(B(z,r))$$

be the $q$-th Bergman space over $B(z, r)$. Denote by $P_{z,r}$ the corresponding Bergman projection induced by the Bergman kernel for $A^2(B(z,r), dv)$. It is well known that $P_{z,r}(f)$ is well-defined for $f \in L^1(B(z,r), dv)$.

**Lemma 3.5.** *Suppose* $1 \le q < \infty$ *and* $0 < s < r$. *There is a constant* $C > 0$ *such that, for* $f \in L^q_{\mathrm{loc}}$ *and* $w \in B(z, r - s/(2))$,

$$G_{q,s}(f)(w) \le M_{q,s}(f - P_{z,r}(f))(w) \le C G_{q,r}(f)(z) \quad \text{for } z \in \mathbb{C}^n. \tag{3-7}$$

*Proof.* We only need to prove the second inequality. Suppose $1 < q < \infty$. Notice that $P_{0,1}$ is the standard Bergman projection on the unit ball of $\mathbb{C}^n$. Theorem 2.11 of [Zhu 2005] implies that

$$\|P_{0,1}\|_{L^q(B(0,1),dv) \to A^q(B(0,1),dv)} < \infty.$$

Now for $r > 0$ fixed and $f \in L^q((B(0,r), dv)$, set $f_r(w) = f(rw)$. Then

$$\|f_r\|_{L^q(B(0,1),dv)} = r^{-2n/q} \|f\|_{L^q(B(0,1),dv)}.$$

Furthermore, it is easy to verify that the operator $f \mapsto P_{0,1}(f_r)(\cdot/r)$ is self-adjoint and idempotent, and it maps $L^2((B(0,r), dv)$ onto $A^2((B(0,r), dv)$. Therefore,

$$P_{0,r}(f)(z) = P_{0,1}(f_r)\left(\frac{z}{r}\right) \quad \text{for } f \in L^q(B(0,r), dv),$$

and hence

$$\|P_{0,r}\|_{L^q(B(0,r),dv) \to A^q(B(0,r),dv)} = \|P_{0,1}\|_{L^q(B(0,1),dv) \to A^q(B(0,1),dv)}.$$

Now for $z \in \mathbb{C}^n$ and $r > 0$, using a suitable dilation, it follows that

$$\|P_{z,r}\|_{L^q(B(z,r),dv) \to A^q(B(z,r),dv)} = \|P_{0,1}\|_{L^q(B(0,1),dv) \to A^q(B(0,1),dv)} < \infty. \tag{3-8}$$

Unfortunately, $P_{z,r}$ is not bounded on $L^1(B(z,r), dv)$, but with the same approach as above, by Fubini's theorem and Theorem 1.12 of [Zhu 2005], we have

$$\|P_{z,r}\|_{L^1(B(z,r),dv) \to A^1(B(z,r),(r^2 - |\cdot - z|^2)dv)} \le C \tag{3-9}$$

for $z \in \mathbb{C}^n$ and $r > 0$.

Choose $h$ as in Lemma 3.3. Then $h \in A^q(B(z,r), dv)$ because $f \in L^q_{\mathrm{loc}}$. Thus, $P_{z,r}(h) = h$. Now for $w \in B(z, (r-s)/2)$ and $1 \le q < \infty$,

$$
\begin{aligned}
&\left\{\int_{B(w,s)} |f - P_{z,r}(f)|^q \, dv\right\}^{1/q} \\
&\qquad \le C\left\{\int_{B(z,(r+s)/2)} |f - P_{z,r}(f)|^q \, dv\right\}^{1/q} \\
&\qquad \le C\left\{\int_{B(z,r)} |f(\xi) - P_{z,r}(f)(\xi)|^q (r^2 - |\xi - z|^2)\, dv(\xi)\right\}^{1/q} \\
&\qquad \le C\left\{\left[\int_{B(z,r)} |f-h|^q \, dv\right]^{1/q} + \left[\int_{B(z,r)} |P_{z,r}(f-h)(\xi)|^q (r^2 - |\xi - z|^2)\, dv(\xi)\right]^{1/q}\right\} \\
&\qquad \le C\left\{\int_{B(z,r)} |f-h|^q \, dv\right\}^{1/q}. \qquad (3\text{-}10)
\end{aligned}
$$

From this and Lemma 3.3, (3-7) follows. □

Given $t > 0$, let $\{a_j\}_{j=1}^\infty$ be a $(t/2)$-lattice, set $J_z = \{j : z \in B(a_j, t)\}$ and denote by $|J_z|$ the cardinal number of $J_z$. By (2-9), $|J_z| = \sum_{j=1}^\infty \chi_{B(a_j,t)}(z) \le N$. Choose a partition of unity $\{\psi_j\}_{j=1}^\infty$, $\psi_j \in C^\infty(\mathbb{C}^n)$, subordinate to $\{B(a_j, t/2)\}$ such that

$$
\begin{gathered}
\operatorname{supp} \psi_j \subset B(a_j, t/2), \quad \psi_j(z) \ge 0, \quad \sum_{j=1}^\infty \psi_j(z) = 1, \\
|\bar\partial \psi_j(z)| \le C t^{-1}, \quad \sum_{j=1}^\infty \bar\partial \psi_j(z) = 0.
\end{gathered} \qquad (3\text{-}11)
$$

Given $f \in L^q_{\mathrm{loc}}$, for $j = 1, 2, \ldots$, pick $h_j \in H(B(a_j, t))$ as in Lemma 3.3 so that

$$M_{q,t}(f - h_j)(a_j) = G_{q,t}(f)(a_j).$$

Define

$$f_1 = \sum_{j=1}^\infty h_j \psi_j \quad \text{and} \quad f_2 = f - f_1. \qquad (3\text{-}12)$$

Notice that $f_1(z)$ is a finite sum for every $z \in \mathbb{C}^n$ and hence well-defined because we have $\operatorname{supp} \psi_j \subset B(a_j, t/2) \subset B(a_j, t)$.

Inspired by a similar treatment on pages 254–255 of [Luecking 1992], using the partition of unity, we can prove the following estimate.

**Lemma 3.6.** *Suppose* $0 < q < \infty$. *For* $f \in L^q_{\mathrm{loc}}$ *and* $t > 0$, *decomposing* $f = f_1 + f_2$ *as in* (3-12), *we have* $f_1 \in C^2(\mathbb{C}^n)$ *and*

$$|\bar\partial f_1(z)| + M_{q,t/2}(\bar\partial f_1)(z) + M_{q,t/2}(f_2)(z) \le C G_{q,2t}(f)(z) \qquad (3\text{-}13)$$

*for* $z \in \mathbb{C}^n$, *where the constant* $C$ *is independent of* $f$.

*Proof.* Observe first that $f_1 \in C^2(\mathbb{C}^n)$ follows directly from the properties of the functions $h_j$ and $\psi_j$. For $z \in \mathbb{C}^n$, we may assume $z \in B(a_1, t/2)$ without loss of generality. Then for those $j$ that satisfy $\bar\partial \psi_j(z) \ne 0$,

$|h_j - h_1|^q$ is plurisubharmonic on $B(z, t/2) \subset B(a_j, t)$. Hence, by Corollary 3.4,

$$\begin{aligned}
|\bar\partial f_1(z)| = \left|\sum_{j=1}^{\infty}(h_j(z)-h_1(z))\bar\partial\psi_j(z)\right| &\le \sum_{j=1}^{\infty}|h_j(z)-h_1(w)||\bar\partial\psi_j(z)| \\
&\le C\sum_{\{j:\,|a_j-z|<t/2\}} M_{q,t/4}(h_j-h_1)(z) \\
&\le C\sum_{\{j:\,|a_j-z|<t/2\}} [M_{q,t/4}(f-h_j)(z)+M_{q,t/4}(f-h_1)(w)] \\
&\le C\sum_{\{j:\,|a_j-z|<t/2\}} G_{q,t}(f)(a_j).
\end{aligned}$$

Thus, using Corollary 3.4 again, we get

$$|\bar\partial f_1(z)| \le C G_{q,3t/2}(f)(z) \quad \text{for } z \in \mathbb{C}^n,$$

and so,

$$M_{q,t/2}(\bar\partial f_1)(z)^q \le C\frac{1}{|B(z,t/2)|}\int_{B(z,t/2)} G_{q,3t/2}(f)(w)^q\,dw \le C G_{q,2t}(f)(z)^q.$$

Similarly, we have $|f_2(\xi)|^q \le C\sum_{j=1}^{\infty}|f(\xi)-h_j(\xi)|^q\psi_j(\xi)^q$, and so

$$M_{q,t/2}(f_2)(z)^q \le C\sum_{j=1}^{\infty}\frac{1}{|B(z,t/2)|}\int_{B(z,t/2)}|f-h_j|^q\psi_j^q\,dv \le C\sum_{\{j:\,|a_j-z|<t/2\}} G_{q,t}(f)(a_j)^q.$$

Therefore,

$$M_{q,t/2}(f_2)(z) \le C G_{q,3t/2}(f)(z).$$

Combining this and the other two estimates above gives (3-13). □

Given $\{\psi_j\}$ as in (3-11), we have another decomposition $f = \mathfrak{F}_1 + \mathfrak{F}_2$, where

$$\mathfrak{F}_1 = \sum_{j=1}^{\infty} P_{a_j,t}(f)\psi_j \quad \text{and} \quad \mathfrak{F}_2 = f - \mathfrak{F}_1. \tag{3-14}$$

When $q = 2$, the two decompositions coincide.

**Corollary 3.7.** *Suppose $1 \le q < \infty$. For $f \in L^q_{\mathrm{loc}}$ and $t > 0$, we have $\mathfrak{F}_1 \in C^2(\mathbb{C}^n)$ and*

$$|\bar\partial\mathfrak{F}_1(z)| + M_{q,t/2}(\bar\partial\mathfrak{F}_1)(z) + M_{q,t/2}(\mathfrak{F}_2)(z) \le C G_{q,2t}(f)(z) \tag{3-15}$$

*for $z \in \mathbb{C}^n$, where the constant $C$ is independent of $f$.*

*Proof.* The proof can be carried out as that of Lemma 3.6 using (3-7) instead of (3-6). We omit the details. □

**3B.** ***The decomposition.*** In our analysis, we will appeal to $\bar\partial$-techniques several times. Let $\Omega \subset \mathbb{C}^n$ be strongly pseudoconvex with $C^4$ boundary, and let $S$ be a $\bar\partial$-closed (0, 1) form on $\Omega$ with $L^p$ coefficients,

$1 \le p \le \infty$. As in [Krantz 1992], we denote by $\mathrm{H}_\Omega(S)$ the Henkin solution of $\bar\partial$-equation $\bar\partial u = S$ on $\Omega$. We observe that Theorem 10.3.9 of that work implies that, for $1 \le q < \infty$,

$$\|\mathrm{H}_\Omega(S)\|_{L^q(\Omega,dv)} \le C\|S\|_{L^q(\Omega,dv)}, \tag{3-16}$$

where the constant $C$ is independent of $S$ and of "small" perturbations of the boundary. (We note that the second item in Theorem 10.3.9 of [Krantz 1992] is stated incorrectly and should read $\|u\|_{L^q} \le C_p\|f\|_p$ instead.) Indeed, to deduce (3-16), we consider three cases. First, for $1 \le q < (2n+2)/(2n+1)$,

$$\|\mathrm{H}_\Omega(S)\|_{L^q(\Omega,dv)} \le C\|S\|_{L^1(\Omega,dv)} \le C\|S\|_{L^q(\Omega,dv)}.$$

For $q = (2n+2)/(2n+1)$, take $1 < p = q < 2n+2$ and $q_1 = (2n+2)/(2n) > q$. Then $1/q_1 = 1/p - 1/(2n+2)$, and by the second item in Theorem 10.3.9 of [Krantz 1992], we have

$$\|\mathrm{H}_\Omega(S)\|_{L^q(\Omega,dv)} \le C\|\mathrm{H}_\Omega(S)\|_{L^{q_1}(\Omega,dv)} \le C\|S\|_{L^p(\Omega,dv)}.$$

Finally, for $q > (2n+2)/(2n+1)$, choose $p$ so that $1/q = 1/p - 1/(2n+2)$. Then $1 < p < 2n+2$ and $p < q$. Now Theorem 10.3.9 of [Krantz 1992] implies

$$\|\mathrm{H}_\Omega(S)\|_{L^q(\Omega,dv)} \le C\|S\|_{L^p(\Omega,dv)} \le C\|S\|_{L^q(\Omega,dv)}.$$

**Theorem 3.8.** *Suppose* $1 \le q < \infty$, $0 < s < \infty$, *and* $f \in L^q_{\mathrm{loc}}$. *Then* $f \in \mathrm{IDA}^{s,q}$ *if and only if* $f$ *admits a decomposition* $f = f_1 + f_2$ *such that*

$$f_1 \in C^2(\mathbb{C}^n), \quad M_{q,r}(\bar\partial f_1) + M_{q,r}(f_2) \in L^s \tag{3-17}$$

*for some* (*or any*) $r > 0$. *Furthermore, for fixed* $\tau, r > 0$, *it holds that*

$$\|f\|_{\mathrm{IDA}^{s,q}} \simeq \|G_{q,\tau}(f)\|_{L^s} \simeq \inf\{\|M_{q,r}(\bar\partial f_1)\|_{L^s} + \|M_{q,r}(f_2)\|_{L^s}\}, \tag{3-18}$$

*where the infimum is taken over all possible decompositions* $f = f_1 + f_2$ *that satisfy* (3-17) *with a fixed* $r$.

*Proof.* First, given $0 < r < R < \infty$, we have some $\boldsymbol{a}_1, \boldsymbol{a}_1, \dots, \boldsymbol{a}_m \in \mathbb{C}^n$ so that $B(0, R) \subset \bigcup_{j=1}^m B(\boldsymbol{a}_j, r)$. Then, for $g \in L^q_{\mathrm{loc}}$,

$$M_{q,R}(g)(z)^s \le C\sum_{j=1}^m M_{q,r}(g)(z+\boldsymbol{a}_j)^s, \quad z \in \mathbb{C}^n,$$

and

$$\int_{\mathbb{C}^n} M_{q,R}(g)(z)^s\,dv(z) \le C\sum_{j=1}^m \int_{\mathbb{C}^n} M_{q,r}(g)(z+\boldsymbol{a}_j)^s\,dv(z) \le C\int_{\mathbb{C}^n} M_{q,r}(g)(z)^s\,dv(z). \tag{3-19}$$

This implies that (3-17) holds for some $r$ if and only if it holds for any $r$.

Suppose that $f \in L^q_{\mathrm{loc}}$ with $\|G_{q,\tau}(f)\|_{L^s} < \infty$ for some $\tau > 0$ and decompose $f = f_1 + f_2$ as in Lemma 3.6 with $t = \tau/2$. Then $f_1 \in C^2(\mathbb{C}^n)$ and

$$|\bar\partial f_1(z)| + M_{q,\tau/4}(\bar\partial f_1)(z) + M_{q,\tau/4}(f_2)(z) \le CG_{q,\tau}(f)(z).$$

Now for any $r > 0$, we have

$$\|M_{q,r}(\bar{\partial} f_1)\|_{L^s} + \|M_{q,r}(f_2)\|_{L^s} \le C\|G_{q,\tau}(f)\|_{L^s}. \tag{3-20}$$

This implies that, $f = f_1 + f_2$ satisfies (3-17).

Conversely, suppose $f = f_1 + f_2$ with $f_1 \in C^2(\mathbb{C}^n)$ and $M_{q,r}(\bar{\partial} f_1) + M_{q,r}(f_2) \in L^s$ for some $r > 0$ as in Theorem 3.8. Then, for any $\tau > 0$,

$$\|G_{q,\tau}(f_2)\|_{L^s} \le C\|M_{q,\tau}(f_2)\|_{L^s} \le C\|M_{q,r}(f_2)\|_{L^s}. \tag{3-21}$$

So $f_2 \in \mathrm{IDA}^{s,q}$. To consider $f_1$, we write $u = \mathrm{H}_{B(z,2\tau)}(\bar{\partial} f_1)$ for the Henkin solution of the equation $\bar{\partial} u = \bar{\partial} f_1$ on $B(z, 2\tau)$. From (3-16) and (3-17), $u$ satisfies

$$M_{q,2\tau}(u)(z) \le C M_{q,2\tau}(\bar{\partial} f_1)(z) \quad \text{for } z \in \mathbb{C}^n, \tag{3-22}$$

which implies that $u \in L^q(B(z, 2\tau), dv)$. Similarly to (3-10),

$$M_{q,\tau}(\mathrm{P}_{z,2\tau}(u))(z) \le C M_{q,2\tau}(u)(z).$$

Thus,

$$\begin{aligned} M_{q,\tau}(u - \mathrm{P}_{z,2\tau}(u))(z) &\le M_{q,\tau}(u)(z) + M_{q,\tau}(\mathrm{P}_{z,2\tau}(u))(z) \\ &\le C M_{q,2\tau}(u)(z). \end{aligned} \tag{3-23}$$

Since

$$f_1 - u \in L^q(B(z, 2\tau), dv) \quad \text{and} \quad \bar{\partial}(f_1 - u) = 0,$$

we have

$$f_1 - u \in A^q(B(z, 2\tau), dv).$$

Notice that $P_{z,2\tau}|_{A^q(B(z,2\tau),dv)} = \mathrm{I}$, and so

$$f_1(\xi) - \mathrm{P}_{z,2\tau}(f_1)(\xi) = u(\xi) - \mathrm{P}_{z,2\tau}(u)(\xi) \quad \text{for } \xi \in B(z, 2\tau). \tag{3-24}$$

Combining (3-22), (3-23) and (3-24), we get

$$\begin{aligned} M_{q,\tau}(f_1 - \mathrm{P}_{z,2\tau}(f_1))(z) &= M_{q,\tau}(u - \mathrm{P}_{z,2\tau}(u))(z) \\ &\le M_{q,2\tau}(u)(z) \le C M_{q,2\tau}(\bar{\partial} f_1)(z). \end{aligned}$$

Therefore, by (3-19),

$$\begin{aligned} \|G_{q,\tau}(f_1)\|_{L^s} &\le \|M_{q,r}(f_1 - \mathrm{P}_{z,2\tau}(f_1))\|_{L^s} \\ &\le C\|M_{q,2\tau}(\bar{\partial} f_1)\|_{L^s} \le C\|M_{q,r}(\bar{\partial} f_1)\|_{L^s}. \end{aligned}$$

This and (3-21) yield

$$\|G_{q,\tau}(f)\|_{L^s} \le C\{\|M_{q,r}(\bar{\partial} f_1)\|_{L^s} + \|M_{q,r}(f_2)\|_{L^s}\}. \tag{3-25}$$

Thus, $f = f_1 + f_2 \in \mathrm{IDA}^{s,q}$.

It remains to note that the norm equivalence (3-18) follows from (3-20) and (3-25). □

With a similar proof we have the following corollary.

**Corollary 3.9.** *Suppose* $1 \le q < \infty$, *and* $f \in L^q_{\text{loc}}$. *Then* $f \in \mathrm{BDA}^q$ (*or* $\mathrm{VDA}^q$) *if and only if* $f = f_1 + f_2$, *where*

$$f_1 \in C^2(\mathbb{C}^n), \quad \bar\partial f_1 \in L^\infty_{0,1} \quad (\textit{or} \lim_{z\to\infty} |\bar\partial f_1| = 0) \tag{3-26}$$

*and*

$$M_{q,r}(f_2) \in L^\infty \quad (\textit{or} \lim_{z\to\infty} M_{q,r}(f_2) = 0) \tag{3-27}$$

*for some* (*or any*) $r > 0$. *Furthermore*,

$$\|f\|_{\mathrm{BDA}^q} \simeq \inf\{\|\bar\partial f_1\|_{L^\infty_{0,1}} + \|M_{q,r}(f_2)\|_{L^\infty}\},$$

*where the infimum is taken over all possible decompositions* $f = f_1 + f_2$, *with* $f_1$ *and* $f_2$ *satisfying the conditions in* (3-26) *and* (3-27).

**Corollary 3.10.** *Suppose* $1 \le q < \infty$. *Different values of* $r$ *give equivalent seminorms* $\|G_{q,r}(\,\cdot\,)\|_{L^s}$ *on* $\mathrm{IDA}^{s,q}$ *when* $0 < s < \infty$ *and on both* $\mathrm{BDA}^q$ *and* $\mathrm{VDA}^q$ *when* $s = \infty$.

**Remark 3.11.** Recall that each $f$ in $\mathrm{BMO}^q$ can be decomposed as $f = f_1 + f_2$, where $f_1$ is of bounded oscillation BO and $f_2$ has a bounded average $\mathrm{BA}^q$ (see [Zhu 2012] for the one-dimensional case and [Lv 2019] for the general case). Furthermore, we may choose $f_1$ to be a Lipschitz function in $C^2(\mathbb{C}^n)$ (see Corollary 3.37 of [Zhu 2012]); that is, $f \in \mathrm{BMO}^q$ if and only if $f = f_1 + f_2$ with all $\partial f_1/\partial x_j \in L^\infty$ for $j = 1, 2, \ldots, 2n$ and $f_2 \in \mathrm{BA}^q$, or in the language of complex analysis both $\bar\partial f_1$ and $\bar\partial \bar f_1$ are bounded. Therefore, $f \in \mathrm{BMO}^q$ if and only if $f, \bar f \in \mathrm{BDA}^q$. For a similar relationship between $\mathrm{IMO}^q$ and the IDA spaces, see Lemma 6.1 of [Hu and Virtanen 2022] and Theorem 7.1 below.

**3C.** ***IDA as a Banach space.*** We next prove that $\mathrm{IDA}^{s,q}/H(\mathbb{C}^n)$ with $1 \le s, q < \infty$ is a Banach space when equipped with the induced norm

$$\|f + H(\mathbb{C}^n)\| = \|f\|_{\mathrm{IDA}^{s,q}} \tag{3-28}$$

for $f \in \mathrm{IDA}^{s,q}$.

**Theorem 3.12.** *For* $1 \le s, q < \infty$, *the quotient space* $\mathrm{IDA}^{s,q}/H(\mathbb{C}^n)$ *is a Banach space with the norm induced by* $\|\cdot\|_{\mathrm{IDA}^{s,q}}$.

*Proof.* Obviously $H(\mathbb{C}^n) \subset \mathrm{IDA}^{s,q}$. Now given $f \in \mathrm{IDA}^{s,q}$ and $h \in H(\mathbb{C}^n)$, we have $G_{q,r}(f) = G_{q,r}(f+h)$. This means that the norm in (3-28) is well-defined on $\mathrm{IDA}^{s,q}/H(\mathbb{C}^n)$. If $\|f\|_{\mathrm{IDA}^{s,q}} = 0$, then $G_{q,r}(f)(z) = 0$ in $\mathbb{C}^n$. By Lemma 3.3, $f \in H(B(z,r))$ and hence $f \in H(\mathbb{C}^n)$.

Let $f_1, f_2 \in \mathrm{IDA}^{s,q}$ and $z \in \mathbb{C}^n$. According to Lemma 3.3, there are functions $h_j$ holomorphic in $B(z,r)$ such that

$$M_{q,r}(f_j - h_j)(z) = G_{q,r}(f_j)(z) \quad \text{for } j = 1, 2.$$

Then, since

$$M_{q,r}((f_1+f_2)-(h_1+h_2))(z) \le M_{q,r}(f_1-h_1)(z) + M_{q,r}(f_2-h_2)(z),$$

we have

$$G_{q,r}(f_1+f_2)(z) \le G_{q,r}(f_1)(z) + G_{q,r}(f_2)(z) \quad \text{for } z \in \mathbb{C}^n.$$

Hence, $\|f_1+f_2\|_{\mathrm{IDA}^{s,q}} \le \|f_1\|_{\mathrm{IDA}^{s,q}}+\|f_2\|_{\mathrm{IDA}^{s,q}}$. In addition, $\|f\|_{\mathrm{IDA}^{s,q}} \ge 0$ and $\|af\|_{\mathrm{IDA}^{s,q}} = |a|\|f\|_{\mathrm{IDA}^{s,q}}$ for $a \in \mathbb{C}$. Therefore, $\|\cdot\|_{\mathrm{IDA}^{s,q}}$ induces a norm on $\mathrm{IDA}^{s,q}/H(\mathbb{C}^n)$.

It remains to prove that the norm is complete. Suppose that $\{f_m\}_{m=1}^\infty$ is a Cauchy sequence in

$$\|\cdot\|_{\mathrm{IDA}^{s,q}} = \|G_{q,1}(\cdot)\|_{L^s}.$$

According to Corollary 3.10, we may assume that $\{f_m\}_{m=1}^\infty$ is a Cauchy sequence in $\|G_{q,r}(\cdot)\|_{L^s}$ with $r>0$ fixed. We now embark on proving that, for some $f \in \mathrm{IDA}^{s,q}$, $\lim_{m\to\infty}\|G_{q,r/2}(f_m-f)\|_{L^s}=0$, which implies $\{f_m\}_{m=1}^\infty$ converges to some $f \in \mathrm{IDA}^{s,q}$ in the $\|\cdot\|_{\mathrm{IDA}^{s,q}}$-topology. For this purpose, let $\{a_j\}_{j=1}^\infty$ be some $t=(r/4)$-lattice. We decompose each $f_m$ similarly to (3-14) as

$$f_{m,1} = \sum_{j=1}^\infty P_{a_j,r}(f_m)\psi_j \quad \text{and} \quad f_{m,2} = f_m - f_{m,1},$$

where $\{\psi_j\}_{j=1}^\infty$ is the partition of unity subordinate to $\{B(a_j,r/4)\}_{j=1}^\infty$ as in (3-11). It follows from Corollary 3.7 that

$$\begin{aligned} M_{q,r/8}(f_{m,2}-f_{k,2})(z)^s &= M_{q,r/8}\biggl((f_m-f_k) - \sum_{j=1}^\infty P_{a_j,t}(f_m-f_k)\psi_j\biggr)(z)^s \\ &\le CG_{q,r/2}(f_m-f_k)(z)^s \\ &\le C\int_{B(z,r/2)} G_{q,r}(f_m-f_k)(\xi)^s\,dv(\xi). \end{aligned}$$

This implies that $\{f_{m,2}\}_{j=1}^\infty$ converges to some function $f_2$ in the $L^q_{\mathrm{loc}}$-topology. In addition, by Lemma 3.5, we have

$$M_{q,r/2}(f_{m,2}-f_{k,2}-\mathrm{P}_{z,r}(f_{m,2}-f_{k,2}))(z) \le CG_{q,r}(f_{m,2}-f_{k,2})(z).$$

Letting $k\to\infty$ and applying Fatou's lemma, we get

$$\begin{aligned} G_{q,r/2}(f_{m,2}-f_2)(z)^s &\le M_{q,r/2}(f_{m,2}-f_2-\mathrm{P}_{z,r}(f_{m,2}-f_2))(z)^s \\ &\le C\liminf_{k\to\infty} G_{q,r}(f_{m,2}-f_{k,2})(z)^s. \end{aligned}$$

Integrate both sides over $\mathbb{C}^n$ and apply Fatou's lemma again to obtain the estimate

$$\int_{\mathbb{C}^n} G_{q,r/2}(f_{m,2}-f_2)^s\,dv \le C\liminf_{k\to\infty}\|f_{m,2}-f_{k,2}\|_{\mathrm{IDA}^{s,q}}.$$

Therefore,

$$\lim_{m\to\infty}\|f_{m,2}-f_2\|_{\mathrm{IDA}^{s,q}} = 0. \tag{3-29}$$

Next we consider $\{f_{m,1}\}_{m=1}^\infty$. Applying the estimate (3-15) to $f_m-f_k$,

$$|\bar\partial(f_{m,1}-f_{k,1})(z)| \le CG_{q,r/2}(f_m-f_k)(z). \tag{3-30}$$

Hence, $\{\bar\partial f_{m,1}\}_{m=1}^\infty$ is a Cauchy sequence in $L^s_{0,1}$ (see (2-11)). We may assume $\bar\partial f_{m,1} \to S = \sum_{j=1}^n S_j\,d\bar z_j$ under the $L^s_{0,1}$-norm. Since $\bar\partial^2=0$, $\bar\partial f_{m,1}$ is trivially $\bar\partial$-closed, and so, as the $L^s_{0,1}$ limit of $\{\bar\partial f_{m,1}\}_{m=1}^\infty$, $S$ is also $\bar\partial$-closed weakly. Let $\phi(z)=\frac{1}{2}|z|^2$ and $g=1\in\Gamma$, and define

$$f_1(z) = A_\phi(S) \quad \text{and} \quad f^*_{m,1} = A_\phi(\bar\partial f_{m,1}).$$

Then, by Lemma 2.4,

$$f_1,\ f^*_{m,1} \in L^s(\phi) \subset L^s_{\mathrm{loc}}, \quad \bar\partial f^*_{m,1} = \bar\partial f_{m,1},$$

and $\{f^*_{m,1}\}_{m=1}^\infty$ converges to $f_1$ in $L^s(\phi)$. Therefore, for $\psi \in C_c^\infty(\mathbb{C}^n)$ (the family of all $C^\infty$ functions with compact support) and $j = 1, 2, \dots, n$, it holds that

$$-\left\langle f_1, \frac{\partial\psi}{\partial z_j}\right\rangle_{L^2} = -\lim_{m\to\infty}\left\langle f^*_{m,1}, \frac{\partial\psi}{\partial z_j}\right\rangle_{L^2} = \lim_{m\to\infty}\left\langle \frac{\partial f^*_{m,1}}{\partial \bar z_j}, \psi\right\rangle_{L^2} = \lim_{m\to\infty}\left\langle \frac{\partial f_{m,1}}{\partial \bar z_j}, \psi\right\rangle_{L^2} = \langle S_j, \psi\rangle_{L^2}.$$

Hence, $\bar\partial f_1 = S$ weakly. Then for $\mathrm{H}_{B(z,r)}(\bar\partial f_{m,1} - S)$, the Henkin solution to the equation $\bar\partial u = \bar\partial f_{m,1} - S$ on $B(z,r)$, (3-16) gives

$$\|\mathrm{H}_{B(z,r)}(\bar\partial f_{m,1} - S)\|_{L^q(B(z,r),dv)} \le C\|\bar\partial f_{m,1} - S\|_{L^q(B(z,r),dv)}. \tag{3-31}$$

In addition, according to (3-24), it holds that

$$(f_{m,1} - f_1) - \mathrm{P}_{z,r}(f_{m,1} - f_1) = \mathrm{H}_{B(z,r)}(\bar\partial f_{m,1} - S) - \mathrm{P}_{z,r}(\mathrm{H}_{B(z,r)}(\bar\partial f_{m,1} - S))$$

on $B(z,r)$. Therefore, by (3-8), (3-9), and (3-31) we have

$$\begin{aligned}
&\|(f_{m,1} - f_1) - \mathrm{P}_{z,r}(f_{m,1} - f_1)\|^q_{L^q(B(z,r/2),\,dv)}\\
&\qquad = \|\mathrm{H}_{B(z,r)}(\bar\partial f_{m,1} - S) - \mathrm{P}_{z,r}(\mathrm{H}_{B(z,r)}(\bar\partial f_{m,1} - S))\|^q_{L^q(B(z,r/2),\,dv)}\\
&\qquad \le C\|\mathrm{H}_{B(z,r)}(\bar\partial f_{m,1} - S)\|^q_{L^q(B(z,r),\,dv)}\\
&\qquad \le C\|\bar\partial f_{m,1} - S\|^q_{L^q(B(z,r),dv)}.
\end{aligned} \tag{3-32}$$

Since $S = \lim_{k\to\infty} \bar\partial f_{k,1}$ in $L^s_{0,1}$, by Fatou's lemma,

$$\begin{aligned}
\|\bar\partial f_{m,1} - S\|^q_{L^q(B(z,r),dv)} &\le C \liminf_{k\to\infty} \|\bar\partial(f_{m,1} - f_{k,1})\|^q_{L^q(B(z,r),dv)}\\
&\le C \liminf_{k\to\infty} G_{q,2r}(f_{m,1} - f_{k,1})(z)^q,
\end{aligned} \tag{3-33}$$

where the last inequality follows from (3-30). We combine (3-32) and (3-33) to get

$$\|(f_{m,1} - f_1) - \mathrm{P}_{z,r}(f_{m,1} - f_1)\|^s_{L^q(B(z,r/2),dv)} \le C \liminf_{k\to\infty} G_{q,2r}(f_{m,1} - f_{k,1})(z)^s.$$

Integrating both sides over $\mathbb{C}^n$ with respect to $dv$ and applying Fatou's lemma once more gives the estimates

$$\begin{aligned}
\|f_{m,1} - f_1\|^s_{\mathrm{IDA}^{s,q}} &\le C\int_{\mathbb{C}^n} \|(f_{m,1} - f_1) - \mathrm{P}_{z,r}(f_{m,1} - f_1)\|^s_{L^q(B(z,r/2))}\,dv\\
&\le C\int_{\mathbb{C}^n} \liminf_{k\to\infty} G_{q,2r}(f_{m,1} - f_{k,1})^s\,dv\\
&\le C \liminf_{k\to\infty} \|f_{m,1} - f_{k,1}\|^s_{\mathrm{IDA}^{s,q}}.
\end{aligned} \tag{3-34}$$

Therefore, $\lim_{m\to\infty} \|f_{m,1} - f_1\|_{\mathrm{IDA}^{s,q}} = 0$. Set $f = f_1 + f_2 \in L^q_{\mathrm{loc}}$. From (3-29) and (3-34) it follows that

$$\lim_{m\to\infty} \|f_m - f\|_{\mathrm{IDA}^{s,q}} \le \lim_{m\to\infty} (\|f_{m,1} - f_1\|_{\mathrm{IDA}^{s,q}} + \|f_{m,2} - f_2\|_{\mathrm{IDA}^{s,q}}) = 0,$$

which completes the proof of the completeness and of the theorem. □

**Corollary 3.13.** *Let $1 \le q < \infty$. With the norm induced by $\|\cdot\|_{\mathrm{BDA}^q}$, the quotient space $\mathrm{BDA}^q/H(\mathbb{C}^n)$ is a Banach space and $\mathrm{VDA}^q$ is a closed subspace of $\mathrm{BDA}^q$.*

*Proof.* The proof of Theorem 3.12 works for $s = \infty$, so $\mathrm{BDA}^q/H(\mathbb{C}^n)$ is a Banach space in $\|\cdot\|_{\mathrm{BDA}^q}$. That $\mathrm{VDA}^q$ is a closed subspace of $\mathrm{BDA}^q$ can be proved in a standard way. □

## 4. Proof of Theorem 1.1

Given two $F$-spaces X and Y, we write $B(\mathrm{X})$ for the unit ball of X. A linear operator T from X to Y is bounded (or compact) if $\mathrm{T}(B(\mathrm{X}))$ is bounded (or relatively compact) in Y. The collection of all bounded (and compact) operators from X to Y is denoted by $\mathcal{B}(\mathrm{X}, \mathrm{Y})$ (and by $\mathcal{K}(\mathrm{X}, \mathrm{Y})$ respectively). We use $\|\mathrm{T}\|_{\mathrm{X}\to\mathrm{Y}}$ to denote the corresponding operator norm. In particular, we recall that when $0 < p < 1$, the Fock space $F^p(\varphi)$ with the metric given by $d(f, g) = \|f - g\|_{p,\varphi}^p$ is an $F$-space.

To deal with the boundedness and compactness of Hankel operators, we need an additional result involving positive measures and their averages. More precisely, given a positive Borel measure $\mu$ on $\mathbb{C}^n$ and $r > 0$, we write $\hat{\mu}_r(z) = \mu(B(z, r))$. Notice, in particular, $\hat{\mu}_r$ is a constant multiple of the averaging function induced by the measure $\mu$.

**Lemma 4.1.** *Suppose $0 < p \le 1$ and $r > 0$. There is a constant $C$ such that, for $\mu$ a positive Borel measure on $\mathbb{C}^n$, $\Omega$ a domain in $\mathbb{C}^n$, and $g \in H(\mathbb{C}^n)$, it holds that*

$$\left(\int_\Omega |g(\xi)e^{-\varphi(\xi)}|\,d\mu(\xi)\right)^p \le C\int_{\Omega_r^+} |g(\xi)e^{-\varphi(\xi)}|^p \hat{\mu}_r(\xi)^p\,dv(\xi),$$

*where $\Omega_r^+ = \bigcup_{\{z\in\Omega\}} B(z, r)$.*

*Proof.* Let $\{a_j\}_{j=1}^\infty$ be an $(r/4)$-lattice. Notice that

$$\hat{\mu}_{r/4}(a_j) \le C \inf_{w\in B(a_j,r/2)} \hat{\mu}_r(w)$$

for all $j \in \mathbb{N}$ and $(a+b)^p \le a^p + b^p$ for $a, b \ge 0$. Then

$$\begin{aligned}
\left(\int_\Omega |g(\xi)e^{-\varphi(\xi)}|\,d\mu(\xi)\right)^p &\le \sum_{j=1}^\infty \left(\int_{B(a_j,r/4)\cap\Omega} |g(\xi)e^{-\varphi(\xi)}|\,d\mu(\xi)\right)^p \\
&\le C \sum_{\{j:\, B(a_j,r/4)\cap\Omega\neq\varnothing\}} \sup_{\xi\in B(a_j,r/4)\cap\Omega} |g(\xi)e^{-\varphi(\xi)}|^p \hat{\mu}_{r/4}(a_j)^p \\
&\le C \sum_{\{j:\, B(a_j,r/4)\cap\Omega\neq\varnothing\}} \hat{\mu}_{r/4}(a_j)^p \int_{B(a_j,r/2)} |g(\xi)e^{-\varphi(\xi)}|^p\,dv(\xi) \\
&\le C \sum_{\{j:\, B(a_j,r/4)\cap\Omega\neq\varnothing\}} \int_{B(a_j,r/2)} |g(\xi)e^{-\varphi(\xi)}|^p \hat{\mu}_r(\xi)^p\,dv(\xi) \\
&\le C\int_{\Omega_r^+} |g(\xi)e^{-\varphi(\xi)}|^p \hat{\mu}_r(\xi)^p\,dv(\xi),
\end{aligned}$$

which completes the proof. □

**Remark 4.2.** To prove compactness of Hankel operators on spaces that are not necessarily Banach spaces, we use the following result. For $0 < p, q < \infty$, $H_f : F^p(\varphi) \to L^q(\varphi)$ is compact if and only if

$$\lim_{m\to\infty} \|H_f(g_m)\|_{q,\varphi} = 0$$

for any sequences $\{g_m\}_{m=1}^\infty$ in $B(F^p(\varphi))$ satisfying

$$\lim_{m\to\infty} \sup_{w\in E} |g_m(w)| = 0$$

for compact subsets $E$ in $\mathbb{C}^n$.

Necessity is trivial. To prove sufficiency, we notice that $B(F^p(\varphi))$ is a normal family, so for any sequence $\{g_m\}_{m=1}^\infty \subset B(F^p(\varphi))$, there exist a holomorphic function $g_0$ on $\mathbb{C}^n$ and a subsequence $\{g_{m_j}\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty} \sup_{w\in E} |g_{m_j}(w) - g_0(w)| = 0.$$

This and Fatou's lemma imply that $g_0 \in B(F^p(\varphi))$, and hence by the hypothesis, we get

$$\lim_{j\to\infty} \|H_f(g_{m_j}) - H_f(g_0)\|_{q,\varphi} = \lim_{j\to\infty} \|H_f(g_{m_j} - g_0)\|_{q,\varphi} = 0.$$

Thus, $H_f(B(F^p(\varphi)))$ is sequentially compact in $L^q(\varphi)$, that is, the Hankel operator $H_f : F^p(\varphi) \to L^q(\varphi)$ is compact.

## 4A. *The case* $0 < p \le q < \infty$ *and* $q \ge 1$.

*Proof of Theorem 1.1*(a). By (2-3)–(2-5),

$$\|k_z\|_{p,\varphi} \le C, \qquad \sup_{\xi\in B(z,r_0)} |k_z(\xi)|e^{-\varphi(\xi)} \ge C \quad \text{and} \quad \lim_{z\to\infty} \sup_{w\in E} |k_z(w)| = 0 \tag{4-1}$$

for any compact subset $E \subset \mathbb{C}^n$. As in the proof of Theorem 4.2 of [Hu and Lu 2019], there is an $r_0$ such that, for all $z \in \mathbb{C}^n$, we have

$$\begin{aligned}\|H_f(k_z)\|_{q,\varphi}^q &\ge \int_{B(z,r_0)} |fk_z - P(fk_z)|^q e^{-q\varphi}\, dv \\ &\ge C\frac{1}{|B(z,r_0)|} \int_{B(z,r_0)} \left| f - \frac{1}{k_z} P(fk_z)\right|^q dv \ge CG^q_{q,r_0}(f)(z).\end{aligned} \tag{4-2}$$

If $H_f \in \mathcal{B}(F^p(\varphi), L^q(\varphi))$,

$$\|f\|_{\mathrm{BDA}^q} \le C\|H_f\|_{F^p(\varphi)\to L^q(\varphi)} < \infty; \tag{4-3}$$

if $H_f \in \mathcal{K}(F^p(\varphi), L^q(\varphi))$, then $f \in \mathrm{VDA}^q$ because

$$\lim_{z\to\infty} G^q_{q,r_0}(f)(z) \le C \lim_{z\to\infty} \|H_f(k_z)\|_{q,\varphi} = 0. \tag{4-4}$$

Next we prove sufficiency. Suppose that $f \in \mathrm{BDA}^q$ and decompose $f = f_1 + f_2$ as in (3-12). Write $d\mu = |f_2|^q dv$ and $d\nu = |\bar\partial f_1|^q dv$. According to Theorem 2.6 of [Hu and Lv 2014] and Corollary 3.9, both $d\mu$ and $d\nu$ are $(p,q)$-Fock Carleson measures. We claim that both $f_1, f_2 \in \mathcal{S}$. Indeed, since $q \geq 1$, we can use Lemma 4.1 with $\Omega = \mathbb{C}^n$ and the measure $|f_2| d\nu$ to get

$$\begin{aligned}\int_{\mathbb{C}^n} |f_2(\xi)K(\xi,z)|e^{-\varphi(\xi)}\,dv(\xi) &\leq C\int_{\mathbb{C}^n} M_{1,r}(f_2)(\zeta)|K(\zeta,z)|e^{-\varphi(\zeta)}\,dv(\zeta)\\ &\leq C\int_{\mathbb{C}^n} M_{q,r}(f_2)(\zeta)|K(\zeta,z)|e^{-\varphi(\zeta)}\,dv(\zeta).\end{aligned} \tag{4-5}$$

Since $f \in \mathrm{BDA}^q$, Lemma 3.6 implies

$$\int_{\mathbb{C}^n} |f_2(\xi)K(\xi,z)|e^{-\varphi(\xi)}\,dv(\xi) \leq C\|f\|_{BDA^q}\int_{\mathbb{C}^n} |K(\xi,z)|e^{-\varphi(\xi)}\,dv(\xi) < \infty$$

for $z \in \mathbb{C}^n$. Hence, $f_2 \in \mathcal{S}$, and so also $f_1 = f - f_2 \in \mathcal{S}$ because $f \in \mathcal{S}$ by the hypothesis. Since the Bergman projection $P$ is bounded on $L^q(\varphi)$ when $q \geq 1$, we have, for $g \in \Gamma$,

$$\begin{aligned}\|H_{f_2}(g)\|_{q,\varphi} &\leq (1 + \|P\|_{L^q(\varphi)\to F^q(\varphi)})\|f_2 g\|_{q,\varphi}\\ &\leq C\|M_{q,r}(f_2)\|_{L^\infty}\|g\|_{q,\varphi} \leq C\|M_{q,r}(f_2)\|_{L^\infty}\|g\|_{p,\varphi},\end{aligned}$$

where the second inequality follows from Lemma 4.1. For $H_{f_1}(g)$ with $g \in \Gamma$, Corollary 2.5 shows that $H_{f_1}(g) = A_\varphi(g\bar\partial f_1) - P(A_\varphi(g\bar\partial f_1))$. Lemma 2.4 implies

$$\|H_{f_1}(g)\|_{q,\varphi} \leq C\|g\,|\bar\partial f_1|\|_{q,\varphi} \leq C\|\bar\partial f_1\|_{L^\infty}\|g\|_{q,\varphi} \leq C\|\bar\partial f_1\|_{L^\infty}\|g\|_{p,\varphi}. \tag{4-6}$$

From the above estimates and the fact that $\Gamma$ is dense in $F^p(\varphi)$, it follows that, for $0 < p \leq q < \infty$, we have

$$\|H_f\|_{F^p(\varphi)\to L^q(\varphi)} \leq C\{\|\bar\partial f_1\|_{L^\infty} + \|M_{q,r}(f_2)\|_{L^\infty}\} \leq C\|f\|_{\mathrm{BDA}^q}, \tag{4-7}$$

where the latter inequality follows from Lemma 3.6.

For compactness, suppose $f \in \mathrm{VDA}^q$ so that $f = f_1 + f_2$ is as (3-12). Notice that both $d\mu = |f_2|^q\,dv$ and $d\nu = |\bar\partial f_1|^q\,dv$ are vanishing $(p,q)$-Fock Carleson measures. Let $\{g_m\}$ be a bounded sequence in $F^p(\varphi)$ converging to zero uniformly on compact subsets of $\mathbb{C}^n$. Then

$$\|H_{f_2}(g_m)\|_{L^q(\varphi)} \leq \|g_m f_2\|_{q,\varphi} + \|P(g_m f_2)\|_{q,\varphi} \leq C\bigg(\int_{\mathbb{C}} |g_m e^{-\varphi}|^q\,d\mu\bigg)^{1/q} \to 0$$

as $m \to \infty$. To prove $\lim_{m\to\infty}\|H_{f_1}(g_m)\|_{L^q(\varphi)} = 0$, for each $m$ we pick some $g_m^* \in \Gamma$ so that $\|g_m - g_m^*\|_{p,\varphi} < 1/m$. Clearly, $\{g_m^*\}_{m=1}^\infty$ is bounded in $F^p(\varphi)$, and $\lim_{z\to\infty}\sup_{w\in E}|g_m^*(w)| = 0$ for any compact subset $E$. Again by Corollary 2.5,

$$\|H_{f_1}(g_m^*)\|_{L^q(\varphi)} \leq C\|g_m^*\bar\partial f_1\|_{L^q(\varphi)} \leq C\|g_m^*\|_{L^q(\mathbb{C}^n,d\nu)} \to 0 \quad \text{as } m \to \infty.$$

Thus, since Lemma 3.6 guarantees $H_{f_1} \in \mathcal{B}(F^p(\varphi), L^q(\varphi))$, it follows that $\lim_{m\to\infty}\|H_{f_1}(g_m)\|_{L^q(\varphi)} = 0$, and so

$$H_f = H_{f_1} + H_{f_2} \in \mathcal{K}(F^p(\varphi), L^q(\varphi)).$$

Finally, it remains to notice that the norm equivalence (1-1) follows from (4-3) and (4-7). □

**4B. *The case* $1 \le q < p < \infty$.** We can now prove the case $q < p$ under the assumption that $q \ge 1$.

*Proof of Theorem 1.1*(b). Suppose that $H_f \in \mathcal{B}(F^p(\varphi), L^q(\varphi))$. Because the proof of sufficiency is similar to the implication $(A) \Rightarrow (C)$ of Theorem 4.4 in [Hu and Lu 2019], we only give the sketch here.

Indeed, take $r_0$ as in (4-1), and set $t = r_0/4$. Let $\{a_j\}_{j=1}^\infty$ be a $(t/2)$-lattice. By Lemma 2.4 of [Hu and Lv 2014], $\big\|\sum_{j=1}^\infty \lambda_j k_{a_j}\big\|_{p,\varphi} \le C\|\{\lambda_j\}\|_{l^p}$ for all $\{\lambda_j\}_{j=1}^\infty \in l^p$, where the constant $C$ is independent of $\{\lambda_j\}_{j=1}^\infty$. Let $\{\phi_j\}_{j=1}^\infty$ be the sequence of Rademacher functions on the interval [0, 1]. Using the boundedness of $H_f$, we get

$$\left\| H_f\left(\sum_{j=1}^\infty \lambda_j \phi_j(s) k_{a_j}(\,\cdot\,)\right)\right\|_{q,\varphi} \le C\|H_f\|_{F^p(\varphi)\to L^q(\varphi)} \|\{|\lambda_j|^q\}\|_{l^{p/q}}^{1/q} \tag{4-8}$$

for $s \in [0, 1]$. On the other hand,

$$\int_{B(a_j,t)} |H_f(k_z)(\xi) e^{-\varphi(\xi)}|^q \, dv(\xi) \ge C G_{q,t}(f)(a_j)^q. \tag{4-9}$$

This and Khintchine's inequality yield

$$\int_0^1 \left\| H_f\left(\sum_{j=1}^\infty \lambda_j \phi_j(s) k_{a_j}(\,\cdot\,)\right)\right\|_{q,\varphi}^q dt \ge C \sum_{j=1}^\infty |\lambda_j|^q G_{q,t}(f)(a_j)^q.$$

Combining this with (4-8) gives

$$\sum_{j=1}^\infty |\lambda_j|^q G_{q,t}(f)(a_j)^q \le C\|H_f\|_{F^p(\varphi)\to L^q(\varphi)}^q \|\{|\lambda_j|^q\}\|_{l^{p/q}}$$

for all $\{|\lambda_j|^q\}_{j=1}^\infty \in l^{p/q}$. By duality with the exponentials $p/q$ and its conjugate,

$$\sum_{j=1}^\infty G_{q,t}(f)(a_j)^{pq/(p-q)} \le C\|H_f\|_{F^p(\varphi)\to L^q(\varphi)}^{pq/(p-q)}.$$

Therefore, by (3-7),

$$\begin{aligned}\int_{\mathbb{C}^n} G_{q,t/2}(f)(z)^{pq/(p-q)} \, dv(z) &\le \sum_{j=1}^\infty \int_{B(a_j,t/2)} G_{q,t/2}(f)(z)^{pq/(p-q)} \, dv(z) \\ &\le C\|H_f\|_{F^p(\varphi)\to L^q(\varphi)}^{pq/(p-q)},\end{aligned} \tag{4-10}$$

which means that $f \in \mathrm{IDA}^{s,q}$ with the estimate $\|f\|_{\mathrm{IDA}^{s,q}} \le C\|H_f\|$.

It should be pointed out that the right-hand side of the estimate (4.24) (the analogue of (4-10) above) in [Hu and Lu 2019] should read $C\|H_f\|_{A^p_\omega \to L^q_\omega}^{pq/(p-q)}$, and not $C\|H_f\|_{A^p_\omega \to L^q_\omega}$ as stated there.

Conversely, suppose $f \in \mathrm{IDA}^{s,q}$. As before, decompose $f = f_1 + f_2$ as in (3-12). From Lemma 3.6 we know that $\|M_{q,r}(f_2)\|_{pq/(p-q)} \le C\|f\|_{\mathrm{IDA}^{s,q}}$. Applying Hölder's inequality to the right-hand side integral in (4-5) with exponent $pq/(p-q)$ and its conjugate exponent $t$, since we have $\|K(\,\cdot\,, z)\|_{t,\varphi} < \infty$, it follows that

$$\int_{\mathbb{C}^n} |f_2(\xi) K_z(\xi)| e^{-\varphi(\xi)} \, dv(\xi) \le C\|M_{q,r}(f_2)\|_{pq/(p-q)} \cdot \|K_z\|_{t,\varphi} < \infty.$$

This implies $f_2 \in \mathcal{S}$, and so also $f_1 \in \mathcal{S}$.

Now for $d\nu = |\bar\partial f_1|^q dv$, applying Hölder's inequality again with $p/(p-q)$ and its conjugate exponent $p/q$, we get

$$\begin{aligned}\|\hat\nu_r\|^{p/(p-q)}_{L^{p/(p-q)}} &= C\int_{\mathbb{C}^n}\left\{\int_{B(\xi,r)}|\bar\partial f_1(\zeta)|^q\,dv(\zeta)\right\}^{p/(p-q)}dv(\xi)\\ &\le C\int_{\mathbb{C}^n}dv(\xi)\int_{B(\xi,r)}|\bar\partial f_1(\zeta)|^{pq/(p-q)}\,dv(\zeta)\\ &\simeq C\int_{\mathbb{C}^n}|\bar\partial f_1(\zeta)|^{pq/(p-q)}\,dv(\zeta)<\infty. \end{aligned}\tag{4-11}$$

Theorem 2.8 of [Hu and Lv 2014] shows that $\nu$ is a vanishing $(p,q)$-Fock Carleson measure; that is, the multiplier $M_{f_1}: g \mapsto g|\bar\partial f_1|$ is compact from $F^p(\varphi)$ to $L^q(\varphi)$ (see Proposition 2.3). Therefore, by Lemma 2.4(A), $A_\varphi(\,\cdot\,\bar\partial f_1)$ is compact from $F^p(\varphi)$ to $L^q(\varphi)$. Moreover, $\Gamma$ is dense in $F^p(\varphi)$ and, by Corollary 2.5, $H_{f_1}(g) = A_\varphi(g\,\bar\partial f_1) - P\circ A_\varphi(g\,\bar\partial f_1)$ for $g\in\Gamma$. Hence, $H_{f_1}: F^p(\varphi)\to L^q(\varphi)$ is compact and we obtain the norm estimate

$$\|H_{f_1}\|_{F^p(\varphi)\to L^q(\varphi)} \le C \sup_{\{g\in F^p(\varphi):\,\|g\|_{p,\varphi}\le 1\}}\|A_\varphi(g\bar\partial f_1)\|_{q,\varphi} \le C\|\bar\partial f_1\|_{pq/(p-q)}. \tag{4-12}$$

Similarly to (4-11), using Lemma 3.6, for $d\mu = |f_2|^q dv$, we get

$$\begin{aligned}\|\hat\mu_r\|^{p/(p-q)}_{L^{p/(p-q)}} &= C\int_{\mathbb{C}^n}\left\{\int_{B(\xi,r)}|f_2(\zeta)|^q\,dv(\zeta)\right\}^{p/(p-q)}dv(\xi)\\ &= C\|M_{q,r}(f_2)\|^{pq/(p-q)}_{pq/(p-q)} \le C\|f\|^s_{\mathrm{IDA}^{s,q}}<\infty.\end{aligned}$$

Hence, $d\mu = |f_2|^q\,dv$ is a vanishing $(p,q)$-Fock Carleson measure. It follows from Proposition 2.3 that the identity operator

$$\mathrm{I}: F^p(\varphi)\to L^q(\mathbb{C}^n, e^{-q\varphi}\,d\mu)$$

is compact. Using the inequality

$$\|H_{f_2}(g)\|_{q,\varphi} \le C\|f_2 g\|_{q,\varphi} = C\|\mathrm{I}(g)\|_{L^q(\mathbb{C},\,e^{-q\varphi}d\mu)}, \tag{4-13}$$

we see that $H_{f_2}$ is compact from $F^p(\varphi)$ to $L^q(\varphi)$.

It remains to notice that the norm equivalence in (1-2) follows from combining the estimates in (4-10), (4-12), and (4-13). □

**Remark 4.3.** In [Stroethoff 1992], it was proved that for bounded symbols $f$, the Hankel operator $H_f: F^2\to L^2$ is compact if and only if

$$\|(I-P)(f\circ\phi_\lambda)\|\to 0 \tag{4-14}$$

as $|\lambda|\to\infty$, where $\phi_\lambda(z) = z+\lambda$. This characterization was recently generalized to $F^p_\alpha$ with $1<p<\infty$ in [Hagger and Virtanen 2021]. Here we note that, using a generalization of Lemma 8.2 of [Zhu 2012] to the setting of $\mathbb{C}^n$, one can prove that Stroethoff's result remains true for Hankel operators acting from $F^p_\alpha$ to $L^q_\alpha$ whenever $1\le p,q<\infty$ even for unbounded symbols.

**4C.** ***The case* $0 < p \le q \le 1$ *with bounded symbols.*** We start with the following preliminary lemma whose proof can be completed with a standard $\varepsilon$ argument.

**Lemma 4.4.** *Suppose that* $0 < p < \infty$, $h \in L^\infty$ *and* $\lim_{z\to\infty} h(z) = 0$. *Then for any bounded sequence* $\{g_j\}_{j=1}^\infty$ *in* $L^p_\varphi$ *satisfying* $\lim_{j\to\infty} g_j(z) = 0$ *uniformly on compact subsets of* $\mathbb{C}^n$, *it holds that* $\lim_{j\to\infty} \|g_j h\|_{p,\varphi} = 0$.

*Proof.* If $R$ is sufficiently large, there is a $C > 0$ such that

$$\begin{aligned}\|g_j h\|^p_{p,\varphi} &= \biggl(\int_{B(0,R)} + \int_{\mathbb{C}^n\setminus B(0,R)}\biggr) |g_j(\xi)h(\xi)e^{-\varphi(\xi)}|^p \, dv(\xi) \\ &\le \|h\|^p_{L^\infty} \sup_{|\xi|\le R} |g_j(\xi)e^{-\varphi(\xi)}|^p + C\|g_j\|^p_{p,\varphi} \to 0\end{aligned}$$

as $j \to \infty$. □

*Proof of Theorem 1.1*(c). Suppose that $f \in \mathcal{S}$. Then $f \in L^q_{\mathrm{loc}}$ for $0 < q \le 1$, and we may decompose $f = f_1 + f_2$ as in (3-12) with $t = r/2$. We claim that, for $g \in \Gamma$,

$$\|H_{f_1}(g)\|^q_{q,\varphi} \le C \int_{\mathbb{C}^n} |g(\xi)e^{-\varphi(\xi)}|^q \, \|\bar\partial f_1\|^q_{L^\infty(B(\xi,r),dv)} \, dv(\xi), \tag{4-15}$$

$$\|H_{f_2}(g)\|^q_{q,\varphi} \le C \int_{\mathbb{C}^n} |g(\xi)e^{-\varphi(\xi)}|^q M_{1,r}(f_2)(\xi)^q \, dv(\xi). \tag{4-16}$$

To estimate $\|H_{f_1}(g)\|_{q,\varphi}$, we use the representation

$$H_{f_1}(g) = A_\varphi(g\bar\partial f_1) - P(A_\varphi(g\bar\partial f_1))$$

(see (2-14)), which suggests that we define a measure $d\mu_z$ as

$$d\mu_z(\xi) = |\bar\partial f_1(\xi)| \biggl\{\frac{1}{|\xi - z|} + \frac{1}{|\xi - z|^{2n-1}}\biggr\} e^{-m|\xi - z|} \, dv(\xi).$$

Then there is a constant $C$ such that, for $w \in \mathbb{C}^n$,

$$\int_{B(w,r)} |\bar\partial f_1(\xi)| \biggl\{\frac{1}{|\xi - z|} + \frac{1}{|\xi - z|^{2n-1}}\biggr\} e^{-m|\xi - z|^2} \, dv(\xi) \le C \int_{B(w,r)} d\mu_z(\xi).$$

Also, it is easy to verify that

$$\widehat{(\mu_z)}_r(w) \le C \sup_{\eta\in B(w,r)} |\bar\partial f_1(\eta)| e^{-m|w-z|},$$

where the constant $C$ is independent of $z, w \in \mathbb{C}^n$. Recall that

$$A_\varphi(g\bar\partial f_1)(z) = \int_{\mathbb{C}^n} e^{\langle 2\partial\varphi, z-\xi\rangle} \sum_{j<n} g(\xi)\bar\partial f_1(\xi) \wedge \frac{\partial|\xi - z|^2 \wedge (2\bar\partial\partial\varphi(\xi))^j \wedge (\bar\partial\partial|\xi - z|^2)^{n-1-j}}{j!\,|\xi - z|^{2n-2j}}.$$

Therefore, using (2-13) and Lemma 4.1, we get

$$\begin{aligned}|A_\varphi(g\bar\partial f_1)(z)e^{-\varphi(z)}|^q &\le C\biggl(\int_{\mathbb{C}^n} |g(\xi)e^{-\varphi(\xi)}| \, d\mu_z(\xi)\biggr)^q \\ &\le C \int_{\mathbb{C}^n} |g(\xi)e^{-\varphi(\xi)}|^q \, \|\bar\partial f_1\|^q_{L^\infty(B(\xi,r),dv)} e^{-qm|\xi - z|} \, dv(\xi).\end{aligned} \tag{4-17}$$

Fubini's theorem yields

$$\begin{aligned}\|A_\varphi(g\bar\partial f_1)\|_{q,\varphi}^q &\le C\int_{\mathbb{C}^n} dv(z)\int_{\mathbb{C}^n}|g(\xi)e^{-\varphi(\xi)}|^q\,\|\bar\partial f_1\|^q_{L^\infty(B(\xi,r),dv)}e^{-qm|\xi-z|}\,dv(\xi)\\ &\le C\int_{\mathbb{C}^n}|g(\xi)e^{-\varphi(\xi)}|^q\,\|\bar\partial f_1\|^q_{L^\infty(B(\xi,r),dv)}\,dv(\xi).\end{aligned}\tag{4-18}$$

To deal with $P(A_\varphi(g\bar\partial f_1))$, we use Lemma 2.2 to obtain positive constants $\theta$ and $C$ so that, for $z\in\mathbb{C}^n$, we have

$$\begin{aligned}\int_{\mathbb{C}^n}|K(w,z)|e^{-m|\xi-z|}e^{-\varphi(z)}\,dv(z) &\le Ce^{\varphi(w)}\int_{\mathbb{C}^n}e^{-m|\xi-z|}e^{-\theta|w-z|}\,dv(z)\\ &= Ce^{\varphi(w)}\biggl(\int_{\{z:|z-\xi|\ge|z-w|\}}+\int_{\{z:|z-\xi|<|z-w|\}}\biggr)e^{-m|w-z|}e^{-\theta|\xi-z|}\,dv(z)\\ &\le Ce^{\varphi(w)}e^{-\tau|\xi-w|},\end{aligned}$$

where $\tau=\min\{\theta,m\}$. Therefore, (4-17) and Fubini's theorem yield

$$\begin{aligned}|P(A_\varphi(g\bar\partial f_1))(w)| &\le C\int_{\mathbb{C}^n}|g(\xi)e^{-\varphi(\xi)}|\,\|\bar\partial f_1\|_{L^\infty(B(\xi,r/2),dv)}\,dv(\xi)\int_{\mathbb{C}^n}|K(w,z)|e^{-\theta|\xi-z|}e^{-\varphi(z)}\,dv(z)\\ &\le Ce^{\varphi(w)}\int_{\mathbb{C}^n}|g(\xi)e^{-\varphi(\xi)}|\,\|\bar\partial f_1\|_{L^\infty(B(\xi,r/2),dv)}e^{-\tau|\xi-w|}\,dv(\xi).\end{aligned}$$

Lemma 4.1 again gives

$$\|P(A_\varphi(g\bar\partial f_1))(w)\|_{q,\varphi}^q\le C\int_{\mathbb{C}^n}|g(\xi)e^{-\varphi(\xi)}|^q\,\|\bar\partial f_1\|^q_{L^\infty(B(\xi,r),dv)}\,dv(\xi).$$

Combining this and (4-18), we get (4-15).

For (4-16), notice first that

$$\|f_2g\|_{q,\varphi}^q\le C\int_{\mathbb{C}^n}|g(\xi)e^{-\varphi(\xi)}|^q M_{q,r}^q(f_2)(\xi)\,dv(\xi),\tag{4-19}$$

and, by Lemma 4.1 with the measure $M_{1,r/2}(f_2)\,dv$, we have

$$\begin{aligned}|P(f_2g)(z)|^q &\le C\biggl(\int_{\mathbb{C}^n}|g(\xi)K(z,\xi)e^{-2\varphi(\xi)}|M_{1,r/2}(f_2)(\xi)\,dv(\xi)\biggr)^q\\ &\le C\int_{\mathbb{C}^n}|g(\xi)K(z,\xi)e^{-2\varphi(\xi)}|^q M_{1,r}(f_2)(\xi)^q\,dv(\xi).\end{aligned}\tag{4-20}$$

Integrating both sides of (4-20) against $e^{-q\varphi}\,dv$ over $\mathbb{C}^n$ and using (2-5), we get

$$\|P(f_2g)\|_{q,\varphi}^q\le C\int_{\mathbb{C}^n}|g(\xi)e^{-\varphi(\xi)}|^q M_{1,r}(f_2)(\xi)^q\,dv(\xi).\tag{4-21}$$

This and (4-19) imply (4-16).

Now we suppose that $f\in L^\infty$ and $0<p\le q<1$. For $g\in H(\mathbb{C}^n)$, similarly to the proof of (4-16), we have

$$\|H_f(g)\|_{q,\varphi}\le C\biggl(\int_{\mathbb{C}^n}|g(\xi)e^{-\varphi(\xi)}|^q M_{1,r}(f)(\xi)^q\,dv(\xi)\biggr)^{1/q}\le C\|f\|_{L^\infty}\|g\|_{p,\varphi}.$$

This implies boundedness of $H_f$ with the norm estimate (1-3).

For the second assertion, suppose first that $\lim_{|z|\to\infty} G_{q,r}(f)(z)=0$ for some $r>0$ and write $f=f_1+f_2$ as above. Since the unit ball $B(F^p(\varphi))$ of $F^p(\varphi)$ is a normal family, to show that $H_f$ is compact from $F^p(\varphi)$ to $L^q(\varphi)$, it suffices to prove that, for $k=1,2$,

$$\lim_{j\to\infty}\|H_{f_k}(g_j)\|_{q,\varphi}=\lim_{j\to\infty}\|f_kg_j-P(f_kg_j)\|_{q,\varphi}=0$$

for any bounded sequence $\{g_j\}_{j=1}^{\infty}$ in $F^p(\varphi)$ with the property that

$$\lim_{j\to\infty}\sup_{w\in E}|g_j(w)|=0$$

for $E$ compact in $\mathbb{C}^n$. From the assumption that $\lim_{z\to\infty} M_{q,r}(f_2)(z)=0$, it follows that $d\mu=|f_2|^q\,dv$ is a vanishing $(p,q)$-Fock Carleson measure (see Theorem 2.7 of [Hu and Lv 2014] and Proposition 2.3). Therefore, we get

$$\|f_2g_j\|_{q,\varphi}=\|g_j\|_{L^q(\mathbb{C}^n,|f_2|^q dv)}\to 0\quad\text{as } j\to\infty.$$

Notice also that $\|g\|_{q,\varphi}\le C\|g\|_{p,\varphi}$ for $g\in F^q(\varphi)$ and $p\le q$. Further, by (4-16), we obtain

$$M_{1,r}(f_2)(\xi)\le\|f_2\|_{L^\infty}^{1-q}M_{q,r}(f_2)(\xi)^q,$$

and applying Lemma 4.4 to $h=M_{q,r}(f_2)^{q^2}$, we get

$$\begin{aligned}\|H_{f_2}g_j\|_{q,\varphi}^q&\le C\int_{\mathbb{C}^n}|g_j(\xi)e^{-\varphi(\xi)}|^q M_{1,r}(f_2)(\xi)^q\,dv(\xi)\\&\le C\|f_2\|_{L^\infty}^{(1-q)q}\int_{\mathbb{C}^n}|g_j(\xi)e^{-\varphi(\xi)}|^q M_{q,r}(f_2)(\xi)^{q^2}\,dv(\xi)\to 0\end{aligned}$$

as $j\to\infty$. So $H_{f_2}\in\mathcal{K}(F^p(\varphi),L^q(\varphi))$. As for $H_{f_1}$, it follows from Lemma 3.6 that

$$\|\bar\partial f_1\|_{L^\infty(B(\xi,r),dv)}\le CG_{q,r}(f)(\xi)\to 0\quad\text{when } \xi\to\infty.$$

Therefore, by (4-15),

$$\|H_{f_1}(g_j)\|_{q,\varphi}^q\le C\int_{\mathbb{C}^n}|g_j(\xi)e^{-\varphi(\xi)}|^q\,\|\bar\partial f_1\|_{L^\infty(B(\xi,r),dv)}^q\,dv(\xi)\to 0$$

as $j\to\infty$, and hence we have $H_{f_1}\in\mathcal{K}(F^p(\varphi),L^q(\varphi))$.

Conversely, suppose that $H_f$ is compact from $F^p(\varphi)$ to $L^q(\varphi)$. Then, as in (4-4), we have

$$\lim_{z\to\infty}G_{q,r}(f)(z)\le C\lim_{z\to\infty}\|H_f(k_z)\|_{q,\varphi}=0 \tag{4-22}$$

for $r\in(0,r_0]$ fixed. We claim that (4-22) is valid for any $r>0$. To see this, we consider the Hankel operator $H_f$ on the Fock space $F_\alpha^p$. From (4-22), using the sufficiency part, it follows that $H_f$ is compact from $F_\alpha^p$ to $L^q(\mathbb{C}^n,e^{-(q\alpha/2)|z|^2}\,dv)$. Notice that the equality (1-5) yields

$$\inf_{w\in B(z,r)}|K(w,z)|\ge C>0$$

for any $r>0$ fixed, where the constant $C$ is independent of $z\in\mathbb{C}^n$. As in (4-2), we have

$$\lim_{z\to\infty}G_{q,r}(f)(z)\le C\lim_{z\to\infty}\|H_f(k_z)\|_{L^q(\mathbb{C}^n,e^{-(q\alpha/2)|z|^2}dv)}=0.$$

Thus, $f\in\mathrm{VDA}^q$. $\square$

The following Corollary 4.5 is a direct consequence of the proof of Theorem 1.1(c) which we use to complement and extend the classical result of Berger and Coburn in the next section.

**Corollary 4.5.** *Suppose that* $0 < q < 1$ *and* $f \in L^\infty$. *Then the limit* $\lim_{z\to\infty} G_{q,r}(f)(z) = 0$ *is independent of* $r > 0$.

## 5. Proof of Theorem 1.2

*Proof of the case* $0 < p \le q < \infty$. For $R > 0$, let $\{a_k\}_{k=1}^\infty$ be the $(R/2)$-lattice

$$\left\{ \frac{R}{2\sqrt{n}}(m_1 + k_1 \mathrm{i}, m_2 + k_2 \mathrm{i}, \dots, m_n + k_n \mathrm{i}) \in \mathbb{C}^n : m_j, k_j \in \mathbb{Z},\ j = 1, 2, \dots, n \right\}.$$

Choose $\rho \in C^\infty(\mathbb{C}^n)$ such that

$$0 \le \rho \le 1, \quad \rho|_{B(0,1/2)} \equiv 1, \quad \operatorname{supp} \rho \subseteq B\big(0, \tfrac{3}{4}\big).$$

Then $\|\nabla \rho\|_{L^\infty} < \infty$ and

$$0 < \sum_{k=1}^\infty \rho((z - a_k)/R) \le C$$

for $z \in \mathbb{C}^n$. Define $\psi_{j,R} \in C^\infty(\mathbb{C}^n)$ by

$$\psi_{j,R}(z) = \frac{\rho((z - a_j)/R)}{\sum_{k=1}^\infty \rho((z - a_k)/R)}.$$

Then $\{\psi_{j,R}\}_{j=1}^\infty$ is a partition of unity subordinate to $\{B(a_j, R)\}_{j=1}^\infty$ and

$$R\|\nabla \psi_{j,R}(\,\cdot\,)\|_{L^\infty} \le C, \tag{5-1}$$

where the constant $C$ is independent of $j$ and $R$.

Now we suppose that $f \in L^\infty$ and $H_f \in \mathcal{K}(F^p(\varphi), L^q(\varphi))$. Theorem 1.1 and Corollary 4.5 imply that

$$\lim_{z\to\infty} G_{q,2R}(f)(z) = 0 \tag{5-2}$$

for $R > 0$ fixed. As in (3-2), pick $h_{j,R} \in H(B(a_j, 2R))$ so that

$$\frac{1}{|B(a_j, 2R)|} \int_{B(a_j,2R)} |f - h_{j,R}|^q \, dv = G_{q,2R}(f)(a_j)^q. \tag{5-3}$$

By (3-3),

$$\sup_{z\in B(a_j,R)} |h_{j,R}(z)| \le C\|f\|_{L^\infty}.$$

Set

$$f_{1,R} = \sum_{j=1}^\infty \psi_{j,R}\, h_{j,R} \quad \text{and} \quad f_{2,R} = f - f_{1,R}.$$

From estimates (2-9) and (3-3), it follows that there is a positive constant $C$ such that

$$\|f_{1,R}\|_{L^\infty} + \|f_{2,R}\|_{L^\infty} \le C\|f\|_{L^\infty} \tag{5-4}$$

for $R > 0$. Lemma 3.6 and (5-2) imply that

$$\lim_{z\to\infty} M_{q,R}(\bar{f}_{2,R})(z) = \lim_{z\to\infty} M_{q,R}(f_{2,R})(z) = 0,$$

and so

$$H_{\bar{f}_{2,R}} \in \mathcal{K}(F^p(\varphi), L^q(\varphi)). \tag{5-5}$$

Recall that $P_{z,R}$ is the standard Bergman projection from $L^2(B(z,R), dv)$ to $A^2(B(z,R), dv)$. Since $h_{j,R}$ is bounded on $B(a_j, R)$, we have $h_{j,R} = P_{a_j,R}(h_{j,R})$, that is,

$$\overline{h_{j,R}(z)} = \frac{1}{\pi}\int_{B(a_j,R)} \frac{R^2\overline{h_{j,R}(\xi)}\,dv(\xi)}{(R^2-(\xi-a_j)\cdot\overline{(z-a_j)})^{n+1}}, \quad z\in B(a_j,R).$$

Hence,

$$|\bar\partial\overline{h_{j,R}(z)}| \le C\frac{\|h_{j,R}\|_{L^\infty(B(z,R),dv)}}{R} \quad \text{for } z\in\overline{B(a_j,3R/4)}. \tag{5-6}$$

Notice that $\operatorname{supp}\psi_{j,R}\,h_{j,R}\subseteq\overline{B(a_j,3R/4)}$, and the estimates (5-1) and (5-6) imply that

$$|\bar\partial\,\bar{f}_{1,R}| \le \sum_{j=1}^{\infty}|(\bar\partial\psi_{j,R})\bar{h}_{j,R}| + \sum_{j=1}^{\infty}\psi_{j,R}|\bar\partial(\bar{h}_{j,R})| \le C\frac{\|f\|_{L^\infty}}{R}.$$

Therefore, using (4-6) (when $q\ge 1$) and (4-15) (when $q<1$), we have

$$\|H_{\bar{f}_{1,R}}\|^p_{F^p(\varphi)\to L^q(\varphi)} \le C\|\bar\partial\,\bar{f}_{1,R}\|_{L^\infty} \le C\frac{\|f\|_{L^\infty}}{R}.$$

The constants $C$ above are all independent of $f$ and $R$. Therefore,

$$\|H_{\bar f} - H_{\bar{f}_{2,R}}\|_{F^p(\varphi)\to L^q(\varphi)} = \|H_{\bar{f}_{1,R}}\|_{F^p(\varphi)\to L^q(\varphi)} \le C\frac{\|f\|_{L^\infty}}{R} \to 0$$

as $R\to\infty$. Finally, using (5-5) and the fact that $\mathcal{K}(F^p(\varphi), L^q(\varphi))$ is closed under the operator norm, we see that $H_{\bar f}\in\mathcal{K}(F^p(\varphi), L^q(\varphi))$, which completes the proof. $\square$

To deal with the case $1\le q<p<\infty$, we use the Ahlfors–Beurling operator, which is a very well-known Calderón–Zygmund operator on $L^p(\mathbb{C})$, $1<p<\infty$, defined as

$$\mathfrak{T}(f)(z) = \text{p.v.} -\frac{1}{\pi}\int_{\mathbb{C}}\frac{f(\xi)}{(\xi-z)^2}\,dv(\xi),$$

where p.v. means the Cauchy principal value. The Ahlfors–Beurling operator connects harmonic analysis and complex analysis, and it is of fundamental importance in several areas of mathematics including PDE and quasiconformal mappings. See [Ahlfors 2006; Astala et al. 2009] for further details and examples.

**Lemma 5.1.** *Suppose $1<s<\infty$. Then there is some constant $C$, depending only on $s$, such that, for $f\in C^2(\mathbb{C}^n)\cap L^\infty$ and $j=1,2,\dots,n$,*

$$\left\|\frac{\partial f}{\partial z_j}\right\|_{L^s} \le C\left\|\frac{\partial f}{\partial\bar z_j}\right\|_{L^s}. \tag{5-7}$$

*Proof.* We consider the case $n = 1$ first. Let $f \in C^2(\mathbb{C}) \cap L^\infty$. If $\|\partial f/\partial\bar z\|_{L^s} = 0$, then $f \in H(\mathbb{C}) \cap L^\infty$, which implies that the function $f$ is constant and the estimate (5-7) follows. Next we suppose that $\|\partial f/\partial\bar z\|_{L^s} > 0$. Take $\psi(r) \in C^\infty(\mathbb{R})$ to be decreasing such that $\psi(x) = 1$ for $x \le 0$, $\psi(x) = 0$ for $x \ge 1$, and $0 \le -\psi'(x) \le 2$ for $x \in \mathbb{R}$. For $R > 0$ fixed, we set $\psi_R(x) = \psi(x - R)$ for $x \in \mathbb{R}$ and define $f_R(z) = f(z)\psi_R(|z|)$ for $z \in \mathbb{C}$. Since $f \in C^2(\mathbb{C}) \cap L^\infty$, it is obvious that $f_R(z) \in C_c^2(\mathbb{C})$, the set of $C^2$ functions on $\mathbb{R}^2$ with compact support. From Theorem 2.1.1 of [Chen and Shaw 2001], it follows that

$$f_R(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C}} \frac{\partial f_R/\partial\bar z}{\xi - z}\, d\xi \wedge d\bar\xi.$$

Notice that $\partial f_R/\partial\bar z = \psi_R(\partial f/\partial\bar z) + f(\partial\psi_R/\partial\bar z)$. By Lemma 2 on page 52 of [Ahlfors 2006], we get

$$\frac{\partial f_R}{\partial z}(z) = \mathfrak{T}\Big(\frac{\partial f_R}{\partial\bar z}\Big)(z) = \mathfrak{T}\Big(\psi_R\frac{\partial f}{\partial\bar z}\Big)(z) + \mathfrak{T}\Big(f\frac{\partial\psi_R}{\partial\bar z}\Big)(z). \tag{5-8}$$

Now for $r > 0$ and $|z| < r$, when $R$ is sufficiently large, it holds that

$$\Big|\mathfrak{T}\Big(f\frac{\partial\psi_R}{\partial\bar z}\Big)\Big|(z) \le \frac{\|f\|_{L^\infty}}{\pi(R-r)^2}\int_{R\le|\xi|\le R+1} dv(\xi) \le \frac{3R\|f\|_{L^\infty}}{(R-r)^2},$$

and hence

$$\Big\|\mathfrak{T}\Big(f\frac{\partial\psi_R}{\partial\bar z}\Big)\Big\|_{L^s(D(0,r),dv)} \le \Big\|\frac{\partial f}{\partial\bar z}\Big\|_{L^s}, \tag{5-9}$$

where $D(0, r) = \{z \in \mathbb{C} : |z| < r\}$. In addition, by the boundedness of $\mathfrak{T}$ on $L^s$ (see, for example, the estimate (11) on page 53 in [Ahlfors 2006]), we get

$$\Big\|\mathfrak{T}\Big(\psi_R\frac{\partial f}{\partial\bar z}\Big)\Big\|_{L^s} \le C\Big\|\psi_R\frac{\partial f}{\partial\bar z}\Big\|_{L^s} \le C\Big\|\frac{\partial f}{\partial\bar z}\Big\|_{L^s}. \tag{5-10}$$

For $R$ sufficiently large, from (5-8), (5-9) and (5-10) it follows that

$$\Big\|\frac{\partial f}{\partial z}\Big\|_{L^s(D(0,r),dv)} = \Big\|\frac{\partial f_R}{\partial z}\Big\|_{L^s(D(0,r),dv)} \le C\Big\|\frac{\partial f}{\partial\bar z}\Big\|_{L^s}.$$

Therefore,

$$\Big\|\frac{\partial f}{\partial z}\Big\|_{L^s} \le C\Big\|\frac{\partial f}{\partial\bar z}\Big\|_{L^s}. \tag{5-11}$$

Now for $n \ge 2$ and $f \in L^\infty \cap C^2(\mathbb{C}^n)$, by (5-11), we have

$$\begin{aligned}\int_{\mathbb{C}^n}\Big|\frac{\partial f}{\partial z_1}(\xi)\Big|^s dv(\xi) &= \int_{\mathbb{C}^{n-1}} dv(\xi')\int_{\mathbb{C}}\Big|\frac{\partial f}{\partial z_1}(\xi_1,\xi')\Big|^s dv(\xi_1)\\ &\le C\int_{\mathbb{C}^{n-1}} dv(\xi')\int_{\mathbb{C}}\Big|\frac{\partial f}{\partial \bar z_1}(\xi_1,\xi')\Big|^s dv(\xi_1).\end{aligned}$$

This implies (5-7) for $j = 1$. Similarly, (5-7) holds for $j = 2, \dots, n$, and the proof is complete. □

*Proof of the case* $1 \le q < p < \infty$. Notice first that if $H_f \in \mathcal{K}(F^p(\varphi), L^q(\varphi)$, then by Theorem 1.1, we have $f \in \mathrm{IDA}^{s,q}$ with $s = pq/(p-q) > 1$. We use a decomposition $f = f_1 + f_2$ as in (3-17) with $r = 1$.

Furthermore, by (5-4), we may assume that $\|f_1\|_{L^\infty} \le C\|f\|_{L^\infty}$. Then, from Lemma 5.1 it follows that

$$\|\bar\partial\,\bar f_1\|_{L^s} \le C\sum_{j=1}^n \left\|\frac{\partial\bar f}{\partial\bar z_j}\right\|_{L^s} = C\sum_{j=1}^n \left\|\frac{\partial f}{\partial z_j}\right\|_{L^s} \le C\sum_{j=1}^n \left\|\frac{\partial f}{\partial\bar z_j}\right\|_{L^s} \le C\|\bar\partial f_1\|_{L^s}.$$

We also observe that $\|M_{q,r}(\bar f_2)\|_{L^s} = \|M_{q,r}(f_2)\|_{L^s} < \infty$. Now Theorem 3.8 implies that $\bar f = \bar f_1 + \bar f_2 \in \mathrm{IDA}^{s,q}$, and hence, by Theorem 1.1, we get $H_{\bar f} \in \mathcal{K}(F^p(\varphi), L^q(\varphi))$. □

**Remark 5.2.** Notice that it follows from the preceding proof that

$$\|H_{\bar f}\|_{F^p(\varphi)\to L^q(\varphi)} \le C\|H_f\|_{F^p(\varphi)\to L^q(\varphi)}.$$

## 6. Application to Berezin–Toeplitz quantization

As an application and further generalization of our results, we consider deformation quantization in the sense of [Rieffel 1989; 1990] and focus on one of its essential ingredients in the noncompact setting of $\mathbb{C}^n$ that involves the limit condition

$$\lim_{t\to 0}\|T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}\|_{F_t^2(\varphi)\to F_t^2(\varphi)} = 0.$$

Recently this and related questions were studied in [Bauer and Coburn 2016; Bauer et al. 2018; Fulsche 2020], which also provide further physical background and references for this type of quantization.

Recall that $\varphi \in C^2(\mathbb{C}^n)$ is real-valued and $\mathrm{Hess}_{\mathbb{R}}\,\varphi \simeq \mathrm{E}$, where E is the $2n\times 2n$-unit matrix. For $t>0$, we set

$$d\mu_t(z) = \frac{1}{t^n}\exp\left\{-2\varphi\left(\frac{z}{\sqrt t}\right)\right\}dv(z)$$

and denote by $L_t^2(\varphi)$ the space of all Lebesgue measurable functions $f$ in $\mathbb{C}^n$ such that

$$\|f\|_t = \left\{\int_{\mathbb{C}^n}|f|^2\,d\mu_t(z)\right\}^{1/2}.$$

Further, we let $F_t^2(\varphi) = L_t^2(\varphi)\cap H(\mathbb{C}^n)$. Then clearly $F_1^2(\varphi) = F^2(\varphi)$ and $L_1^2(\varphi) = L^2(\varphi)$ in terms of the spaces that were considered in the previous sections. Given $f\in L^\infty$, we use the orthogonal projection $P^{(t)}$ from $L_t^2(\varphi)$ onto $F_t^2(\varphi)$ to define the Toeplitz operator $T_f^{(t)}$ and the Hankel operator $H_f^{(t)}$, respectively, by

$$T_f^{(t)} = P^{(t)}M_f \quad\text{and}\quad H_f^{(t)} = (\mathrm{I}-P^{(t)})M_f.$$

Let $U_t$ be the dilation acting on measurable functions in $\mathbb{C}^n$ as

$$U_t : f \mapsto f(\,\cdot\sqrt t).$$

It is easy to verify that $U_t$ is a unitary operator from $L_t^2(\varphi)$ to $L^2(\varphi)$ (as well as a unitary operator from $F_t^2(\varphi)$ to $F^2(\varphi)$). Further, we have $U_tP^{(t)}U_t^{-1} = P^{(1)}$, which implies that

$$U_tT_f^{(t)}U_t^{-1} = T_{f(\,\cdot\sqrt t)},\quad U_tH_f^{(t)}U_t^{-1} = H_{f(\,\cdot\sqrt t)}. \tag{6-1}$$

Therefore,

$$\|T_f^{(t)}\|_{F_t^2(\varphi)\to F_t^2(\varphi)} = \|T_{f(\,\cdot\sqrt t)}\|_{F^2(\varphi)\to F^2(\varphi)} \tag{6-2}$$

and

$$\|H_f^{(t)}\|_{F_t^2(\varphi)\to L_t^2(\varphi)} = \|H_{f(\cdot\sqrt{t})}\|_{F^2(\varphi)\to L^2(\varphi)}. \tag{6-3}$$

Given $f \in L^2_{\mathrm{loc}}$, for $z \in \mathbb{C}^n$ and $r > 0$ set

$$MO_{2,r}(f)(z) = \left\{\frac{1}{|B(z,r)|}\int_{B(z,r)} |f - f_{B(z,r)}|^2\,dv\right\}^{1/2}$$

where $f_S = (1/|S|)\int_S f\,dv$ for $S \subset \mathbb{C}^n$ measurable.

The following definitions of BMO and VMO are analogous to the classical definition introduced in [John and Nirenberg 1961], but they differ from those widely used in the study of Bergman and Fock spaces.

**Definition 6.1.** We denote by BMO the set of all $f \in L^2_{\mathrm{loc}}$ such that

$$\|f\|_* = \sup_{z\in\mathbb{C}^n,\, r>0} MO_{2,r}(f)(z) < \infty$$

and by VMO the set of all $f \in \mathrm{BMO}$ such that

$$\lim_{r\to 0}\sup_{z\in\mathbb{C}^n} MO_{2,r}(f)(z) = 0.$$

**Definition 6.2.** We define $\mathrm{BDA}_*$ to be the family of all $f \in L^2_{\mathrm{loc}}$ such that

$$\|f\|_{\mathrm{BDA}_*} = \sup_{z\in\mathbb{C}^n, r>0} G_{2,r}(f)(z) < \infty$$

and $\mathrm{VDA}_*$ to be the subspace of all $f \in \mathrm{BDA}_*$ such that

$$\lim_{r\to 0}\sup_{z\in\mathbb{C}^n} G_{2,r}(f)(z) = 0.$$

Given a family $X$ of functions on $\mathbb{C}^n$, we set $\overline{X} = \{\bar f : f \in X\}$.

**Proposition 6.3.** *It holds that*

$$\mathrm{BMO} = \mathrm{BDA}_* \cap \overline{\mathrm{BDA}_*} \quad \textit{and} \quad \mathrm{VMO} = \mathrm{VDA}_* \cap \overline{\mathrm{VDA}_*}.$$

*Furthermore, we have*

$$\|f\|_{\mathrm{BMO}_*} \simeq \|f\|_{\mathrm{BDA}_*} + \|\bar f\|_{\mathrm{BDA}_*} \tag{6-4}$$

*for* $f \in L^2_{\mathrm{loc}}$.

*Proof.* From a careful inspection of the proof of Proposition 2.5 in [Hu and Wang 2018], it follows that there is a constant $C > 0$ such that, for $f \in L^2_{\mathrm{loc}}$ and $z \in \mathbb{C}^n$, $r > 0$, there is a constant $c(z)$ for which

$$\left\{\frac{1}{|B(z,r)|}\int_{B(z,r)} |f - c(z)|^2\,dv\right\}^{1/2} \le C\{G_{2,r}(f)(z) + G_{2,r}(\bar f)(z)\}.$$

It is easy to verify that

$$MO_{2,r}(f)(z) \le \left\{\frac{1}{|B(z,r)|}\int_{B(z,r)} |f - c(z)|^2\,dv\right\}^{1/2},$$

and hence

$$MO_{2,r}(f)(z) \le C\{G_{2,r}(f)(z) + G_{2,r}(\bar{f})(z)\}.$$

On the other hand, by definition, we have

$$G_{2,r}(f)(z) \le MO_{2,r}(f)(z).$$

Thus, we have $C_1$ and $C_2$, independent of $f$, $r$ and $z$, such that

$$\begin{aligned} C_1\{G_{2,r}(f)(z) + G_{2,r}(\bar{f})(z)\} &\le MO_{2,r}(f)(z) \\ &\le C_2\{G_{2,r}(f)(z) + G_{2,r}(\bar{f})(z)\}. \end{aligned} \tag{6-5}$$

Therefore, $f \in \mathrm{BMO}$ (or $f \in \mathrm{VMO}$) if and only if $f \in \mathrm{BDA}_* \cap \overline{\mathrm{BDA}_*}$ (or $f \in \mathrm{VDA}_* \cap \overline{\mathrm{VDA}_*}$). The estimate in (6-4) follows from (6-5). □

**Theorem 6.4.** *Suppose $f \in L^\infty$. Then for all $g \in L^\infty$, it holds that*

$$\lim_{t\to 0} \|T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}\|_{F_t^2(\varphi)\to F_t^2(\varphi)} = 0 \tag{6-6}$$

*if and only if $f \in \overline{\mathrm{VDA}_*}$.*

*Proof.* Given $f \in L^\infty$, it follows from (6-3) that

$$\|(H_{\bar{f}}^{(t)})^*\|_{L_t^2(\varphi)\to F_t^2(\varphi)} = \|H_{\bar{f}}^{(t)}\|_{F_t^2(\varphi)\to L_t^2(\varphi)} = \|H_{f(\cdot\sqrt{t})}\|_{F^2(\varphi)\to L^2(\varphi)}.$$

This and Theorem 1.1 imply

$$\frac{1}{C}\|G_{2,1}(f(\cdot\sqrt{t}))\|_{L^\infty} \le \|(H_{\bar{f}}^{(t)})^*\|_{L_t^2(\varphi)\to F_t^2(\varphi)} \le C\|G_{2,1}(f(\cdot\sqrt{t}))\|_{L^\infty}, \tag{6-7}$$

where the constant $C$ is independent of $f$ and $t$.

Suppose $f \in \overline{\mathrm{VDA}_*}$. Then, by definition, we have

$$\lim_{r\to 0} \sup_{z\in\mathbb{C}^n} G_{2,r}(\bar{f})(z) = 0.$$

It is easy to verify that

$$G_{2,1}(f(\cdot\sqrt{t}))(z) = G_{2,\sqrt{t}}(f)(z\sqrt{t}).$$

Now by (6-7), we get

$$\lim_{t\to 0} \|(H_{\bar{f}}^{(t)})^*\|_{L_t^2(\varphi)\to F_t^2(\varphi)} \le C \lim_{t\to 0} \|G_{2,\sqrt{t}}(\bar{f})\|_{L^\infty} = 0. \tag{6-8}$$

In addition, for $f, g \in L^\infty$, it is easy to verify that

$$T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)} = -(H_{\bar{f}}^{(t)})^* H_g^{(t)}. \tag{6-9}$$

Therefore, for all $g \in L^\infty$,

$$\lim_{t\to 0} \|T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}\|_{F_t^2(\varphi)\to F_t^2(\varphi)} \le \|g\|_{L^\infty} \lim_{t\to 0} \|(H_{\bar{f}}^{(t)})^*\|_{L_t^2(\varphi)\to F_t^2(\varphi)} = 0,$$

which gives (6-6).

Conversely, suppose that (6-6) holds for every $g \in L^\infty$. Let $g = \bar{f} \in L^\infty$. Then it follows from (6-9) that

$$\begin{aligned}\lim_{t\to 0} \|H_{\bar{f}}^{(t)}\|^2_{F_t^2(\varphi)\to L_t^2(\varphi)} &= \lim_{t\to 0} \|(H_{\bar{f}}^{(t)})^* H_{\bar{f}}^{(t)}\|_{F_t^2(\varphi)\to F_t^2(\varphi)} \\ &= \lim_{t\to 0} \|T_f^{(t)} T_{\bar{f}}^{(t)} - T_{|f|^2}^{(t)}\|_{F_t^2(\varphi)\to F_t^2(\varphi)} = 0.\end{aligned}$$

This and (6-7) imply that $f \in \overline{\mathrm{VDA}_*}$. □

Combining Proposition 6.3 with Theorem 6.4, we obtain the following corollary, which is the main result of [Bauer et al. 2018] when $\varphi(z) = \frac{1}{8}|z|^2$.

**Corollary 6.5.** *Suppose $f \in L^\infty$. Then for all $g \in L^\infty$, it holds that*

$$\lim_{t\to 0} \|T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}\| = 0 \quad \textit{and} \quad \lim_{t\to 0} \|T_g^{(t)} T_f^{(t)} - T_{fg}^{(t)}\| = 0 \tag{6-10}$$

*if and only if $g \in \mathrm{VMO}$. Here $\|\cdot\| = \|\cdot\|_{F_t^2(\varphi)\to F_t^2(\varphi)}$.*

## 7. Further remarks

For $1 \le p, q < \infty$, we have characterized those $f \in \mathcal{S}$ for which $H_f : F^p(\varphi) \to L^q(\varphi)$ is bounded (or compact). For small exponents $0 < p < q < 1$, we have proved that this characterization remains true for compactness when $f \in L^\infty$. We also note that when $p \le q$ and $q \ge 1$, boundedness and compactness of Hankel operators $H_f : F^p(\varphi) \to L^p(\varphi)$ depend on $q$ (see Remark 3.2 and Theorem 1.1), while for $p > q$ we cannot say the same — we note that we have no statement analogous to Remark 3.2 for $\mathrm{IDA}^{s,q}$.

Moreover, for harmonic symbols $f \in \mathcal{S}$ and $0 < p, q < \infty$, using the Hardy–Littlewood theorem on the submean value (see Lemma 2.1 of [Hu et al. 2007], for example), we are able to characterize boundedness of $H_f : F^p(\varphi) \to L^q(\varphi)$ with the space $\mathrm{IDA}^{s,q}$. We will return to this topic in a future publication.

We also note that the space $F^\infty(\varphi)$ does not appear in our results because $\Gamma$ is not dense in it. Instead, it may be possible to consider the space

$$f^\infty(\varphi) = \{f \in F^\infty(\varphi) : fe^{-\varphi} \in C_0(\mathbb{C}^n)\},$$

which can be viewed as the closure of $\Gamma$ in $F^\infty(\varphi)$, and extend our results to this setting.

Regarding weights, the Fock spaces studied in this paper are defined with weights $\varphi \in C(\mathbb{C}^n)$ satisfying $\mathrm{Hess}_{\mathbb{R}}\, \varphi \simeq \mathrm{E}$. As stated in Section 2A, these weights are contained in the class considered in [Schuster and Varolin 2012]. Now, we note that for the weights $\varphi$ in that work, $i\partial\bar{\partial}\varphi \simeq \omega_0$, and from Hörmander's theorem on the canonical solution to the $\bar{\partial}$-equation it follows that

$$\|H_f g\|^2_{2,\varphi} \le \int_{\mathbb{C}^n} |g\bar{\partial} f|^2_{i\partial\bar{\partial}} e^{-2\varphi}\, dv \le C \|g|\bar{\partial} f|\|^2_{2,\varphi},$$

and hence we know that the conclusions of Theorem 1.1 remain true when $q = 2$ (see Theorem 4.3 of [Hu and Virtanen 2022]). Upon these observations, we raise the following conjecture.

**Conjecture 1.** Suppose $\varphi \in C^2(\mathbb{C}^n)$ satisfying $i\partial\bar{\partial}\varphi \simeq \omega_0$. Then for $f \in \mathcal{S}$ and $0 < p, q < \infty$, $H_f \in \mathcal{B}(F^p(\varphi), L^q(\varphi))$ if and only if $f \in \mathrm{IDA}^{s,q}$, where $s = pq/(p-q)$ if $p > q$ and $s = \infty$ if $p \le q$.

In the literature, there are a number of interesting results on the simultaneous boundedness (and compactness) of Hankel operators $H_f$ and $H_{\bar{f}}$. These types of characterizations often involve the function spaces $\mathrm{BMO}^q$ and $\mathrm{IMO}^{s,q}$ in their conditions; see, e.g., [Hu and Wang 2018; Zhu 2012]. For $1 \le q < \infty$ and $1 \le s \le \infty$, set $\overline{\mathrm{IDA}^{s,q}} = \{\bar{f} : f \in \mathrm{IDA}^{s,q}\}$. Then Proposition 2.5 of [Hu and Wang 2018] shows that $\mathrm{IDA}^{s,q} \cap \overline{\mathrm{IDA}^{s,q}} = \mathrm{IMO}^{s,q}$ and the results of Section 4 provide a description of the simultaneous boundedness (or compactness) of $H_f$ and $H_{\bar{f}}$ as seen in the following theorem, where as before, we set $s = pq/(p-q)$ if $p > q$ and $s = \infty$ if $p \le q$.

**Theorem 7.1.** *Let $\varphi \in C^2(\mathbb{C}^n)$ be real-valued*, $\mathrm{Hess}_{\mathbb{R}}\, \varphi \simeq \mathrm{E}$, *and let $f \in \mathcal{S}$. For $1 \le p, q < \infty$, Hankel operators $H_f$ and $H_{\bar{f}}$ are simultaneously bounded from $F^p(\varphi)$ to $L^q(\varphi))$ if and only if $f \in \mathrm{IMO}^{s,q}$.*

We state one more conjecture related to Theorem 1.2, in which we proved that for $f \in L^\infty$ and $0 < p < \infty$, $H_f$ is compact on $F^p(\varphi)$ if and only if $H_{\bar{f}}$ in compact on $F^p(\varphi)$. Recall that this phenomenon does not occur for Hankel operators on the Bergman space or on the Hardy space. As predicted in [Zhu 2012], and verified for Hankel operators on the weighted Fock spaces $F^p(\alpha)$ with $1 < p < \infty$ in [Hagger and Virtanen 2021], a partial explanation for this difference is the lack of bounded holomorphic or harmonic functions on the entire complex plane. From this point of view it is natural to suggest that a similar result should remain true for Hankel operators mapping from $F^p(\varphi)$ to $L^q(\varphi)$.

**Conjecture 2.** Suppose that $\varphi \in C^2(\mathbb{C}^n)$ satisfies $i\partial\bar{\partial}\varphi \simeq \omega_0$ and $0 < p, q < \infty$. Then for $f \in L^\infty$, $H_f \in \mathcal{K}(F^p(\varphi), L^q(\varphi))$ if and only if $H_{\bar{f}} \in \mathcal{K}(F^p(\varphi), L^q(\varphi))$.

Notice that $\mathrm{IDA}^{s,q} \cap L^\infty$ is a Banach algebra under the norm $\|\cdot\|_{\mathrm{IDA}^{s,q}} + \|\cdot\|_\infty$. We can also express Conjecture 2 in algebraic terms; that is, we conjecture that $\mathrm{IDA}^{s,q} \cap L^\infty$ on $\mathbb{C}^n$ is closed under the conjugate operation $f \mapsto \bar{f}$, where $1 < s \le \infty$ and $0 < q < \infty$.

Related to our work on quantization and Theorem 6.4 in particular, we conclude this section with the following problem: characterize those $f \in L^\infty$ for which it holds that

$$\lim_{t \to 0} \|T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}\|_{S_2} = 0$$

for all $g \in L^\infty$, where $\|\cdot\|_{S_2}$ stands for the Hilbert–Schmidt norm. It would also be important to consider this question for other Schatten classes $S_p$.


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