Analysis & PDE

Volume 16 No. 9 2023

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ANALYSIS AND PDE
Vol. 16 (2023), No. 9, pp. 2079–2107
DOI: 10.2140/apde.2023.16.2079

GLOBAL STABILITY OF SPACETIMES WITH SUPERSYMMETRIC COMPACTIFICATIONS

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This paper proves the stability, with respect to the evolution determined by the vacuum Einstein equations, of the Cartesian product of higher-dimensional Minkowski space with a compact, Ricci-flat Riemannian manifold that admits a spin structure and a nonzero parallel spinor. Such a product includes the example of Calabi–Yau and other special holonomy compactifications, which play a central role in supergravity and string theory. The stability result proved in this paper shows that Penrose's instability argument [2003] does not apply to localised perturbations.

1. Introduction

Let $(\mathbb{R}^{1+n}, \eta_{\mathbb{R}^{1+n}})$ be the $(1+n)$-dimensional Minkowski spacetime, and let (K, k) be a compact, Ricci-flat Riemannian manifold that has a cover that admits a spin structure and a nonzero parallel spinor. The spacetime $\mathcal{M} = \mathbb{R}^{1+n} \times K$ with metric

$$\hat{g} = \eta_{\mathbb{R}^{1+n}} + k \tag{1}$$

is globally hyperbolic and Ricci flat, i.e, it is a solution to the $(1+n+d)$-dimensional vacuum Einstein equations. Such spacetimes play an essential role in supergravity and string theory [Candelas et al. 1985]. In this paper we refer to $(\mathcal{M}, \hat{g})$ as a spacetime with a supersymmetric (SUSY) compactification and (K, k) as the internal manifold.

The simplest spacetime with a supersymmetric compactification, which has been studied since the 1920s, is the Kaluza–Klein spacetime $(\mathbb{R}^{1+3} \times \mathbb{S}^1_\theta, \eta_{\mathbb{R}^{1+3}} + \mathrm{d}\theta^2)$ [Kaluza 1921; Klein 1926]. As shown by Witten in an influential paper [1982], this spacetime is unstable at the semiclassical level. Nonetheless in the same work Witten argued that the spacetime should be classically linearly stable.

By contrast, Penrose has sketched an argument intended to show that spacetimes with supersymmetric compactifications are generically classically unstable, for every dimension n and all internal manifolds, except possibly when the internal manifold is a flat d-dimensional torus [Penrose 2003; 2005]. There are theorems motivated by these considerations that generalise the classical singularity theorems to trapped surfaces of arbitrary codimension [Cipriani and Senovilla 2019; Galloway and Senovilla 2010]. However, the results of the present paper show that for spacetimes with supersymmetric compactifications the instability argued by Penrose does not hold for $n \geq 9$, and we conjecture here that in fact stability holds for $n \geq 3$. The nonnegativity of the spectrum of the Lichnerowicz Laplacian on symmetric 2-tensors, which holds for the internal spaces by the result of Dai, Wang, and Wei [Dai et al. 2005], plays a crucial role

MSC2020: 35L15, 35Q76, 53C25, 83E30.
Keywords: Einstein equation, stability, initial-value problem, SUSY.

in our stability proof. In fact, this nonnegativity, which is conjectured to hold for all compact Ricci-flat manifolds, is sufficient for our result. See Section 2A for details.

In order to state our main theorem, we need to introduce some notation. For the product spacetime $\mathbb{R}^{1+n} \times K$ we denote spacetime indices by $\alpha, \mu, \nu, \dots$, Minkowski indices by $i, j, k, \dots$, and internal indices by $A, B, C, \dots$. For a general pseudo-Riemannian metric g, let $\nabla[g]$ denote its Levi-Civita connection, $\mathrm{Riem}[g]$ its Riemann curvature tensor, $\mathrm{Ric}[g]$ its Ricci curvature, and $\mathrm{d}\mu_g$ its volume form. Define the contraction

$$(R[g] \circ u)_{\mu\nu} = R_{\mu\rho\nu\lambda}[g] u^{\rho\lambda}, \tag{2}$$

which acts on symmetric (0, 2)-tensors $u_{\mu\nu}$. Given the supersymmetric spacetime metric $\hat{g}$ on $\mathbb{R}^{1+n} \times K$, let

$$(g_E)_{\mu\nu} = \hat{g}_{\mu\nu} + 2(\mathrm{d}t)_\mu (\mathrm{d}t)_\nu, \tag{3}$$

where $\mathrm{d}t$ is with respect to the standard Cartesian coordinates on $\mathbb{R}^{1+n}$. On K and $\mathbb{R}^{1+n} \times K$, define the inner products on (0, 2) tensors, respectively, as

$$\langle u, v\rangle_k = k^{AC} k^{BD} u_{AB} v_{CD} \quad \text{and} \quad \langle u, v\rangle_E = g_E^{\mu\nu} g_E^{\rho\sigma} u_{\mu\rho} v_{\nu\sigma}. \tag{4}$$

Define $|u|_k = (\langle u, u\rangle_k)^{1/2}$, and similarly for $|u|_E$.

The following is our main result. The details of some of the concepts appearing in the statement of the theorem appear in Definitions 2.10, 2.11, 2.12, 2.14 and Theorem 2.15.

Theorem 1.1. *Let $n, d \in \mathbb{Z}^+$ be such that $n \geq 9$, and let $N \in \mathbb{Z}^+$ be sufficiently large. Consider a spacetime $(\mathbb{R}^{1+n} \times K, \hat{g} = \eta_{\mathbb{R}^{1+n}} + k)$ with a supersymmetric compactification. Let g_S denote the Schwarzschild metric in the $\eta_{\mathbb{R}^{1+n}}$-wave gauge with mass parameter $C_S \geq 0$.*

There is an $\epsilon > 0$ such that if $(\mathbb{R}^n \times K, \gamma, \kappa)$ is an initial data set satisfying that outside the unit ball the initial data coincides with the product of Schwarzschild initial data with the unperturbed internal metric (i.e., $\gamma = g_S + k$ and $\kappa = 0$ where $|x| \geq 1$) and satisfying

$$\sum_{|I|\leq N} \|\nabla[\gamma]^I (\gamma - \hat{g}|_{t=0})\|^2_{L^2(\mathbb{R}^n\times K)} + \sum_{|I|\leq N-1} \|\nabla[\gamma]^I \kappa\|^2_{L^2(\mathbb{R}^n\times K)} + C_S^2 \leq \epsilon, \tag{5}$$

then there is a solution g of the vacuum Einstein equations on $\mathbb{R}^{1+n} \times K$ with initial data $(\mathbb{R}^n \times K, \gamma, \kappa)$ and satisfying the $\hat{g}$-wave gauge. There is the bound

$$\sup_{(t,x^i,\omega)\in\Sigma_s\times K} t^{2\delta(n)} |g(t, x^i, \omega) - \hat{g}(t, x^i, \omega)|^2_E \lesssim \epsilon, \tag{6}$$

where the decay rate is given by

$$\delta(n) = \tfrac{1}{4}(n-2). \tag{7}$$

Finally $(\mathbb{R}^{1+n} \times K, g)$ is globally hyperbolic and causally geodesically complete.

The stability result obtained in Theorem 1.1 covers a large class of product spacetimes, including many special holonomy compactifications relevant in supergravity and string theory. Although this paper succeeds in its goal of providing a counterexample to the dimension-independent argument in

[Penrose 2003], from a PDE perspective, Theorem 1.1 should be seen as a preliminary result, and we expect that the assumptions that $n \geq 9$ and that the Cauchy data is Schwarzschild near infinity can be relaxed. In fact we make the following conjecture.

Conjecture 1.2. *Spacetimes with a supersymmetric compactification*[1] *and $n = 3$ are nonlinearly stable.*

As explained below, this paper uses a relatively simple vector field argument, while, for example, the proof of global stability for the coupled Einstein–Klein–Gordon system in (1+3)-dimensions [LeFloch and Ma 2016] has required combining vector field arguments with estimates arising from control on the fundamental solution for the wave equation. Such detailed analysis is beyond the scope of this paper, but we intend to explore this in future work. Note that our current method can be easily used to show linear stability as far as $n = 3$.

The decay rate of $|h| \lesssim t^{-\delta(n)}$ arises essentially as a linear estimate. The linearisation of the Einstein equation is

$$(\Box_\eta + \Delta_k + 2R[\hat{g}] \circ) h_{\mu\nu} = 0. \tag{8}$$

To study conservation properties of the linear equations we introduce a novel stress-energy tensor

$$T[h]^\mu{}_\nu = \hat{g}^{\mu\alpha} \langle \nabla[\hat{g}]_\alpha h, \nabla[\hat{g}]_\nu h \rangle_E - \tfrac{1}{2} \hat{g}^{\alpha\beta} \langle \nabla[\hat{g}]_\beta h, \nabla[\hat{g}]_\alpha h \rangle_E \delta^\mu_\nu + \langle R[\hat{g}] \circ h, h \rangle_E \delta^\mu_\nu, \tag{9}$$

which is specifically adapted to the tensorial operator appearing in (8). The conditions on (K, k) imply (see Section 2A) that the energy integral derived from (9) is nonnegative.

The conditions on (K, k) imply that the operator $-(\Delta_k + 2R\circ)$ has a nonnegative discrete spectrum, so a spectral decomposition can be applied to solutions h of the linearised Einstein equation (8). The spectral component corresponding to the zero eigenvalue satisfies an effective wave equation $\Box_\eta (h^0)_{\mu\nu} = 0$, and the components corresponding to positive eigenvalues λ satisfy effective Klein–Gordon equations $(\Box_\eta - \lambda)(h^\lambda)_{\mu\nu} = 0$. A decomposition of this type has been used in the analysis of wave guides, where K is replaced by a compact subset of $\mathbb{R}^d$ with Neumann boundary conditions; see e.g., [Metcalfe and Stewart 2008; Metcalfe et al. 2005]. When applying the vector field method to the wave and Klein–Gordon equations, there is a unified approach using a basic energy of the form $\int \sum_{i=0}^n |\partial_i h|^2 + \lambda |h|^2 \, \mathrm{d}\mu$ that can be strengthened by commuting the equation with Γ, the set of generators of translations, rotations, and boosts. The use of this set of vector fields in the vector field method, with particular application to Klein–Gordon equations, goes back to [Klainerman 1985].

This unified approach then bifurcates: the Klein–Gordon equation does not admit any further commuting first-order operators but the energy has a nonvanishing lower-order term $\lambda |h|^2$; in contrast, the wave equation allows for commutation with the generator of dilations, $S = t\partial_t + r\partial_r$, but the lower-order term in the energy vanishes. For the quasilinear Einstein equation, we refrain from performing a spectral decomposition into wave and Klein–Gordon components. Thus, we use only the unified part of the approach (following especially the treatment of quasilinear Klein–Gordon equations in [Hörmander 1997]),

[1] Recall that the definition of a spacetime with a supersymmetric compactification, as introduced in the opening paragraph of this paper, includes the assumption that the spacetime is a fibre bundle with base space $(\mathbb{R}^{1+n}, \eta_{\mathbb{R}^{1+n}})$. Stability for a certain class of cosmological spacetimes as base spaces is proved in [Branding et al. 2019].

leaving us with a decay rate that is far from the sharp decay rates of the wave and Klein–Gordon equations. In particular, the vector field method can be used to prove decay rates, for the wave and Klein–Gordon equations, of $t^{-(n-1)/2}$ and $t^{-n/2}$, respectively.

In light of this, it seems likely that some novel refinement should allow for a significantly better decay rate than $t^{-\delta(n)}$ with $\delta(n) = \frac{1}{4}(n-2)$. This paper contains two types of refinement. First, the decay rate is shown to be $s^{-2\delta(n)}$, where $s^2 = t^2 - x^2$ inside light cones. The exponent $2\delta(n) = \frac{1}{2}(n-2)$ is much closer to the decay rate for the wave and Klein–Gordon equations. Second, the same decay rates are proved for $\Gamma^I h$ as for h, but, since the Γ contain t- and x-dependent weights, with respect to a translation invariant basis in Minkowski space, derivatives decay faster than the field h itself.

Having obtained a linear estimate that improves with increasing n, we take n large enough that $2\delta(n) - 2 > 1$, so that the nonlinear terms decay sufficiently fast for the linear estimates to remain valid. In particular, we take n large enough that we can ignore all nonlinear structure in the Einstein equation. It is well known that global existence results for semilinear equations in (1+3)-dimensions depend delicately on the nonlinearities, for example the null condition [Klainerman 1986]. Christodoulou and Klainerman [1993] used the vector field method to prove the stability of Minkowski spacetime. One of the major advances in the simplified vector field argument in [Lindblad and Rodnianski 2003; 2005; 2010] was the introduction of the weak null condition and the observation that the Einstein equations in the harmonic gauge satisfy this condition. LeFloch and Ma [2016] identified the relevant nonlinear structures for Klein–Gordon equations coupled to the (1+3)-dimensional Einstein equation.

The dimension of the compact manifold only appears in the required regularity of the initial data, which is given explicitly in Theorem 5.1. The restriction to initial data which is exactly Schwarzschild outside of a compact set mirrors the proof of Minkowski stability in (1+3)-dimensions by Lindblad and Rodnianski [2005].

Background and previous work. Theories of higher-dimensional gravity are of great interest in supergravity and string theory as possible models of quantum gravity. Many of these theories are built around the spacetimes with supersymmetric compactifications discussed above.

The background spacetimes considered in this paper are of the form $\mathbb{R}^{n+1} \times K$, with K compact and Ricci flat, and are hence anisotropic. Among the first stability results for anisotropic spacetimes of a related form was the proof of future stability of flat cosmological spacetimes of the form $M^3 \times S^1$, where M^3 is a flat (2+1)-dimensional Milne spacetime with metric $-dt^2 + t^2 H^2$ and H^2 is a hyperbolic surface, was considered by Choquet-Bruhat and Moncrief [2001]. See also [Andersson 2014; Reiris 2010].

Until now, the only nonlinear stability results for spacetimes with supersymmetric compactification have concerned the simplest Kaluza–Klein case when the internal space is the circle $\mathbb{S}^1$, or in slightly more generality, the flat d-dimensional torus. It was shown by one of the authors [Wyatt 2018] that this spacetime is classically stable to toroidal-independent perturbations. A model problem to remove this restriction with toroidal internal space has recently appeared [Huneau and Stingo 2021]. We remark that in the physics literature, these are known as zero-mode perturbations. An analogous result for cosmological Kaluza–Klein spacetimes, where the Minkowski spacetime is replaced by the four-dimensional Milne spacetime, has also recently been shown [Branding et al. 2019].

The spacetimes of importance in supergravity and string theory involve a nontrivial (i.e., nontoroidal) internal manifold with parallel spinors, such as a Calabi–Yau, G_2 or Spin(7) manifold. Note that a solution of the 10- or 11-dimensional *vacuum* Einstein equations can be considered as a particular solution of the supergravity equations. Local-in-time existence results are known for both the vacuum Einstein equations [Choquet-Bruhat 1952; Choquet-Bruhat and Geroch 1969] and for the supergravity equations [Choquet-Bruhat 1985]. Furthermore, global-in-time existence and decay results for a nonlinear wave equation for 3-form fields, on a fixed background spacetime with compact internal dimensions have been shown in [Ettinger 2015]. The field equation studied in that paper is modelled on the supergravity equations with the gravitational interaction turned off. In our present work, we consider the stability of spacetimes with supersymmetric compactifications as solutions to the vacuum Einstein equations. In future work we intend to study their stability under the supergravity equations.

In addition to determining the dynamics, the Einstein equations also imply that any initial data set must satisfy the constraint equations, which are themselves an important topic of study and have important consequences. A positive mass theorem holds for initial data (Σ, γ, κ) provided that $\Sigma \backslash \Sigma_0$ for some compact Σ_0 is topologically $(\mathbb{R}^n \backslash B) \times K$ for some ball B, that the dimension of the base space is at least $n \geq 3$, that the initial data (Σ, γ, κ) is asymptotically flat in the sense that the metric (including its derivatives) converges to $\delta + k$ sufficiently fast and that κ converges to zero sufficiently rapidly, that the background internal space (K, k) is a simply connected Calabi–Yau manifold, and that the scalar curvature is nonnegative [Dai 2004]. Recent work has shown the existence of such solutions in the case $(K, k) = (\mathbb{T}^d, \delta)$ [Huneau and Vâlcu 2021].

L^2 stability and L^∞ instability. Several people have suggested that the instability argument of Penrose [2003; 2005] should be interpreted as a statement with respect to perturbations that are not localised.[2] This unlocalised interpretation could be stated as saying that SUSY compactifications are unstable against perturbations of the initial data that depend only upon the position in the internal space K but are independent of $x \in \mathbb{R}^n$. Considering the behaviour of the initial data in $x \in \mathbb{R}^n$, this distinction can be interpreted as a being between unlocalised perturbations that merely have a small supremum (for the metric and a suitable number of derivatives) and localised perturbations that have finite and small norms based on the square integral of the perturbation (again including a suitable number of derivatives), such as we use in (5) of Theorem 1.1. We view this as a distinction between, on the one hand, instability in L^∞-based Sobolev spaces and, on the other, stability in L^2-based Sobolev spaces.

Although it is true that SUSY compactifications are unstable against perturbations in L^∞-based Sobolev spaces, this instability does not arise from the presence of the internal space but is already present in Minkowski space for $n \geq 3$. In particular, there is the explicit Kasner solution

$$g = -\,\mathrm{d}t^2 + (1+\epsilon t)^{4/3}\,\mathrm{d}(x^1)^2 + (1+\epsilon t)^{4/3}\,\mathrm{d}(x^2)^2 + (1+\epsilon t)^{-2/3}\,\mathrm{d}(x^3)^2.$$

This is typically considered with (x^1, x^2, x^3) being taken as coordinates on the torus $\mathbb{T}^3$, but it applies equally well on $\mathbb{R}^3$. By taking a tensor product with $(\mathbb{R}^{n-3}, \delta_{\mathbb{R}^{n-3}})$ or $(\mathbb{R}^{n-3} \times K, \delta_{\mathbb{R}^{n-3}} + k)$ one can

[2] We thank the first reviewer for emphasising this perspective.

extend this example to show L^∞ instability also for higher-dimensional Minkowski space and for SUSY compactifications.

The L^∞ instability of Minkowski space and SUSY compactifications can be viewed as part of a broader set of instability phenomena. The L^∞ instability of Minkowski space can be viewed as essentially equivalent to the instability of $(\mathbb{R} \times \mathbb{T}^n, -\mathrm{d}t^2 + \delta_{\mathbb{T}^n})$. Bartnik [1988] has conjectured that a globally hyperbolic spacetime with compact Cauchy surface and satisfying the strong energy condition is either causally incomplete or split as a metric product (and hence flat in the (3+1)-dimensional case). See also [Galloway 2019]. One heuristic justification for this conjecture follows a contradiction argument, which begins by considering what would happen if there were not some major divergence from the original solution. In this case, the metric perturbations would satisfy something close to energy conservation, would exhibit something close to Poincaré recurrence, and would eventually be found in any configuration compatible with the bound on the initial energy. However, just as it is possible to imagine black holes of arbitrarily small mass, it is possible to form trapped surfaces with arbitrarily small energy. Thus, the Poincaré recurrence would imply the eventual formation of trapped surfaces and hence of singularities. This would imply instability, which concludes the contradiction argument. There is a further extension of this belief that if a spacetime with a compact hypersurface does not expand sufficiently rapidly, then metric perturbations will not decay sufficiently rapidly and singularities will form. It is essential to make the distinction between L^∞ and L^2 perturbations when making PDE estimates.

Outline of paper. In Section 2 we introduce: the Lichnerowicz Laplacian, the foliation by hyperboloids, the gauge condition, and the higher-dimensional Schwarzschild-product spacetime. In Section 3 we prove a Sobolev estimate on hyperboloids with respect to wave-like energies. In Section 4 we define an energy functional adapted to the internal manifold and to hyperboloids. Finally in Section 5 we prove the main theorem.

There are four key elements that we add to the standard energy-estimates framework to prove the stability of SUSY compactifications. First, we observe that we can obtain arbitrarily rapid decay by going to sufficiently high dimension and that this decay allows us to control nonlinear terms. Second, the new Sobolev estimates in Section 3 give decay estimates that do not require decomposing metric perturbations into massive and massless parts. Following an argument of Hörmander, the Sobolev estimate in Lemma 3.2 holds on hyperboloids to exploit the fact that the initial data is essentially trivial outside the unit ball. Third, it is possible to introduce an energy that simultaneously enjoys several desirable properties. Namely, the energy introduced in Definition 4.1 is not merely the energy constructed from the energy-momentum tensor (9) for the linearised Einstein equation (8), but we show it is positive using known results on Ricci-flat compact manifolds with special holonomy which we review in Section 2A, and it is the basis for the Sobolev norms in Section 3. Fourth, in defining pointwise norms of derivatives (e.g., Definition 2.4), we commute the equation with the second-order Δ_k rather than just first-order vector fields, which are sufficient in Minkowski space. The higher-order Sobolev estimate in Corollary 4.7 has to use separate indices to count the Minkowski and internal derivatives, because our L^∞-norms use only an even number of derivatives in internal directions, while our L^2-norms use integer number of derivatives. Once we have used these four elements, it is possible to control the nonlinear (including quasilinear) terms in the Einstein equation using standard energy-estimate techniques.

2. Preliminaries

2A. ***Parallel spinors and the Lichnerowicz Laplacian.*** Our main theorem has been stated for an internal manifold that has a cover that admits a spin structure and a nonzero parallel spinor. In this subsection we detail how this condition relates to a linear stability condition involving the eigenvalues of an operator closely related to the Lichnerowicz Laplacian.

Definition 2.1 (Riemannian linear stability). Define $\Delta_k = k^{AB}\nabla[k]_A\nabla[k]_B$ to be the standard Laplacian on (K, k). Let u_{AB} be a symmetric $(0, 2)$ tensor defined on K. Define $\mathcal{L}$ to act on such tensors by

$$(\mathcal{L}u)_{AB} = -\Delta_k u_{AB} - 2(R[k] \circ u)_{AB}. \tag{10}$$

We define a Ricci-flat manifold (K, k) to be Riemannian linearly stable if and only if

$$\int_K \langle \mathcal{L}u, u\rangle_k \, \mathrm{d}\mu_k \geq 0, \tag{11}$$

for all symmetric $(0, 2)$-tensors u_{AB}.

The operator $\mathcal{L}$ is closely related to the Lichnerowicz Laplacian Δ_L, which acts on symmetric tensors by

$$(\Delta_L u)_{AB} = (\mathcal{L}u)_{AB} + \mathrm{Ric}[k]_{AC}u^C{}_B + \mathrm{Ric}[k]^C{}_B u_{AC}. \tag{12}$$

Clearly on a Ricci-flat space these operators are equivalent. The operator $\mathcal{L}$ is self-adjoint and elliptic, and consequently by the compactness of K and spectral theory, it has a discrete set of eigenvalues of finite multiplicity. Hence definition (11) amounts to a condition $\lambda_{\min} \geq 0$ on the lowest eigenvalue $\lambda_{\min}$ of $\mathcal{L}$. For further details see, e.g., [Besse 1987].

Our main Theorem 1.1 in fact applies more generally to internal manifolds which are Riemannian linearly stable. For the purposes of this paper, the crucial relation between spacetimes with a supersymmetric compactification and with an internal space that is Riemannian linearly stable is the following.

Theorem 2.2 [Dai et al. 2005, Theorem 1.1]. *If a compact, Ricci-flat Riemannian manifold (K, k) has a cover which is spin and admits a nonzero parallel spinor then it is Riemannian linearly stable.*

Note that some of the ideas established in [Dai et al. 2005] date back to work of Wang [1991] on the deformation theory of parallel and Killing spinors. A spin manifold (K, k) with a nonzero parallel spinor is Ricci flat and has special holonomy; see [Wang 1989] for a classification. It is not known if any hypotheses on the internal space beyond Ricci flatness are necessary for stability to hold, as all known examples of compact Ricci-flat manifolds admit a spin cover with nonzero parallel spinors. The problem of constructing Ricci-flat manifolds including ones with nonspecial holonomy has been widely studied. A few relevant references on the topic are [Biquard 2013; Brendle and Kapouleas 2017; Tian and Yau 1990; 1991].

The spatial equivalent of the $\hat{g}$-wave gauge was used in the proof of Milne stability [Andersson and Moncrief 2011]. This led to terms involving $\mathcal{L}$ appearing in their PDEs, which were treated using Riemannian linear stability properties specific to the Milne spacetime. Further results on Riemannian linear stability for Einstein manifolds can be found in [Kröncke 2015].

2B. *Cartesian, hyperbolic, and hyperbolic polar coordinates.*

Definition 2.3 (Minkowski space). Let $n \geq 1$ be an integer, let $(x^0, x^1, \dots, x^n) = (t, x^1, \dots, x^n) = (t, \vec{x})$ be Cartesian coordinates parametrising $\mathbb{R}^{1+n}$, and define

$$\eta_{\mathbb{R}^{1+n}} = -\,\mathrm{d}t^2 + \sum_{i=1}^{n} (\mathrm{d}x^i)^2. \tag{13}$$

Define, for $i \in \{1, \dots, n\}$, the translation vector fields T and X_i so that, in the Cartesian coordinates, they are given by

$$X_i = \partial_{x^i}, \quad T = X_0 = \partial_t. \tag{14}$$

Define, for $i, j \in \{0, \dots, n\}$, the vector fields Z_{ij} so that, in the Cartesian coordinates, they are given by

$$Z_{ij} = (\eta_{\mathbb{R}^{1+n}})_{jk} x^k \partial_i - (\eta_{\mathbb{R}^{1+n}})_{ik} x^k \partial_j. \tag{15}$$

Define the collection of Lorentz generators by

$$Z = \{Z_{ij}, T, X_i\}. \tag{16}$$

Define $|x|^2 = \sum_{i=1}^n (x^i)^2$ and define, in the region $t \geq |x|$, the hyperboloidal coordinates to be

$$s = (t^2 - |x|^2)^{1/2}, \quad y = x. \tag{17}$$

Define, for $i \in \{1, \dots, n\}$, the vector fields Y_i so that, in the hyperboloidal coordinates, they are given by

$$Y_i = \partial_{y^i}. \tag{18}$$

For $s_0 \geq 0$, define the spacelike hyperboloidal hypersurface

$$\Sigma_{s_0} = \{(t, x) \in \mathbb{R}^{1+n} : t > 0,\ s = s_0\}. \tag{19}$$

Note that, because $(\eta_{\mathbb{R}^{1+n}})_{00} = -1$, we have $Z_{0i} = t\partial_{x^i} + x_i \partial_t$. Furthermore the collection Z is closed under commutation and forms a basis for the Poincaré Lie algebra.

Definition 2.4 (pointwise derivative norms based on commuting operators). On $\mathbb{R}^{1+n} \times K$, define, for $i \in \{0, \dots, n\}$, X_i, Y_i, and Z_{ij} to be as in $\mathbb{R}^{1+n}$. Let primed roman letters denote spatial indices $i', j' \in \{1, \dots, n+d+1\}$. Define the following collection of vector fields

$$\Gamma = Z \cup \{\Delta_k\}. \tag{20}$$

Note that $[Z, \Delta_k] = 0$. Define $\mathbb{N} = \{0, 1, 2, \dots\}$. We will now define $\{Z_i\}_{i=1}^{(n+1)(n+2)/2}$ to be a reindexing of $\{X_i\}_{i=0}^n \cup \{Z_{ij}\}_{0 \leq i < j \leq n}$, define a multi-index to be an ordered list of arbitrary length of elements from $\left\{1, \dots, \frac{1}{2}(n+1)(n+2)\right\}$, and for a multi-index $I = (i_1, \dots, i_k)$ define the length $|I| = k$ and the differential operator $Z^I = Z_{i_k} \circ \cdots \circ Z_{i_1}$. For $I \in \mathbb{N}$ and $u_{\mu\nu}$ a tensor defined on $\mathbb{R}^{1+n} \times K$, define the generalised multi-index notation

$$|\Gamma^I u|_E^2 = \sum_{I_1 : |I_1| + 2j = |I|} |Z^{I_1} \Delta_k^j u|_E^2, \tag{21}$$

where the sum is taken over all multi-indices I_1 of length $|I_1| = k$ and integers j such that $k + 2j = |I|$.

Definition 2.5 (Sobolev norms). Let $u_{\mu\nu}$ be a tensor defined on $\mathbb{R}^{1+n} \times K$ and let $j \in \mathbb{N}$. Define

$$|\nabla[k]^j u|_E^2 = k^{A_1 B_1} \cdots k^{A_j B_j} g_E^{\mu\nu} g_E^{\rho\sigma} (\nabla[k]_{A_j} \cdots \nabla[k]_{A_1} u_{\mu\rho})(\nabla[k]_{B_j} \cdots \nabla[k]_{B_1} u_{\nu\sigma}). \tag{22}$$

For $\ell \in \mathbb{N}$, define the norms

$$\|u(\,\cdot\,,\,\cdot\,,\omega)\|_{H^\ell(K)} = \left(\int_K \sum_{0 \le j \le \ell} |\nabla[k]^j u(\,\cdot\,,\,\cdot\,,\omega)|_E^2 \,\mathrm{d}\mu_k \right)^{1/2}, \tag{23}$$

$$\|u(t,x,\omega)\|_{L^2(\Sigma_s \times K)} = \left(\int_{\Sigma_s \times K} |u(t,x,\omega)|_E^2 \,\mathrm{d}x \,\mathrm{d}\mu_k \right)^{1/2}, \tag{24}$$

where $\mathrm{d}x = \mathrm{d}x^1 \cdots \mathrm{d}x^n$ is defined to be the flat Euclidean volume form.

Lemma 2.6. $\quad Y_i = X_i + (x_i/t)T, \quad Z_{0i} = tY_i, \quad Z_{ij} = y_i Y_j - y_j Y_i.$

Proof. Since $t = \sqrt{s^2 + y^2}$, by the chain rule, for $j \in \{1, \ldots, n\}$,

$$\frac{\partial}{\partial y^j} = \frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^j} + \frac{\partial t}{\partial y^j}\frac{\partial}{\partial t} = \frac{\partial}{\partial x^j} + \frac{y_j}{t}\frac{\partial}{\partial t},$$

which gives the first result. The second follows from multiplying both sides of the first by t. The third follows from

$$Z_{ij} = x_i X_j - x_j X_i = x_i(X_j + x_j t^{-1} T) - x_j (X_i + x_i t^{-1} T). \qquad \square$$

The following two lemmas relate the t coordinate to the s coordinate.

Lemma 2.7. *Let $s \ge 1$. Suppose $(t_0, x_0) \in \Sigma_s$ and $(t, x) \in \Sigma_s$ with $|x - x_0| \le \frac{1}{2}t_0$. In this case, $\frac{1}{2}t_0 \le t \le 2t_0$.*

Proof. For the graph $t = \sqrt{s^2 + |x|^2}$, the gradient

$$\left| \frac{\partial t}{\partial x} \right| = \left| \frac{x}{\sqrt{s^2 + |x|^2}} \right| \le 1, \tag{25}$$

so the change from t to t_0 is less than the change from $|x|$ to $|x_0|$. $\square$

Lemma 2.8. *There is a constant $C > 0$ such that for all $s > 1$, in the portion of Σ_s where $|x| \le t - 1$, one has $2t - 1 \le s^2 \le t^2$.*

Proof. Observe that $t^2 = s^2 + |x|^2 \ge s^2$. Since $|x|^2 \le t^2 - 2t + 1$, one has $s^2 = t^2 - |x|^2 \ge 2t - 1$. $\square$

The following are standard elliptic estimates; see for example [Besse 1987, Appendix H].

Lemma 2.9 (elliptic estimates on (K, k)). *For $\ell \in \mathbb{N}$ and $u_{\mu\nu}$ a sufficiently regular tensor defined on $\mathbb{R}^{1+n} \times K$, there exist constants $c_1, c_2, c_3 > 0$ such that*

$$\|u\|_{H^{2\ell}(K)} \le c_1 \|(\Delta_k)^\ell u\|_{L^2(K)} + c_2 \|u\|_{L^2(K)} \le c_3 \|u\|_{H^{2\ell}(K)}. \tag{26}$$

In Lemma 2.9, if u is orthogonal to the kernel of Δ_k, then there is a c_1 such that the first estimate holds with $c_2 = 0$.

2C. ***The Einstein equations.*** The theory of the Einstein equations is well known. In this section, we review this theory, for the sake of providing a self-contained presentation in this paper, and in particular to provide a self-contained statement of our main Theorem 1.1.

Definition 2.10 (geometric initial data set). Let $m \in \mathbb{N}^+$. An m-dimensional initial data set is defined to be a triple (Σ, γ, κ) such that Σ is an m-dimensional manifold, $\gamma_{i'j'}$ is a Riemannian metric on Σ, $\kappa_{i'j'}$ is a symmetric 2-tensor on Σ, and the following equations (the constraint equations) are satisfied:

$$R[\gamma] - |\kappa|^2 + (\operatorname{tr}(\kappa))^2 = 0, \quad \nabla[\gamma]_{i'} \operatorname{tr}(\kappa) - \nabla[\gamma]^{j'}(\kappa)_{i'j'} = 0, \tag{27}$$

where $\operatorname{tr}(\kappa) = \gamma^{i'j'}\kappa_{i'j'}$.

Definition 2.11 (solution of the Einstein equations with specified initial data). Let $\mathcal{M}$ be a manifold. A Lorentzian metric g on $\mathcal{M}$ is defined to be a solution of the vacuum Einstein equations if and only if its Ricci curvature vanishes,

$$\operatorname{Ric}[g]_{\mu\nu} = 0. \tag{28}$$

Let (Σ, γ, κ) be a geometric initial data set. A solution to the (geometric) Einstein equations with initial data (Σ, γ, κ) is defined to be a Lorentzian metric g on $I \times \Sigma$ for some interval I where one has: $0 \in I$, g is a solution of the Einstein equations (28), $\{0\} \times \Sigma$ and g restricted to vectors in $T(\{0\} \times \Sigma)$ are isometric in the category of Riemannian manifolds to (Σ, γ), and, with the identification given by this isometry, the second fundamental form of the embedding of $\{0\} \times \Sigma$ into $I \times \Sigma$ is κ.

As is well known, Definition 2.11 is stated in a more restrictive form than necessary. In Definition 2.11, for convenience, we have required that the initial data be specified at $t = 0$. This may initially appear more restrictive than definitions that are stated in other sources. By a translation in the t variable, Definition 2.11 could be stated on any level set of t. Furthermore, because of the freedom to introduce new coordinate systems on the manifold $I \times \Sigma$, Definition 2.11 is actually equivalent to definitions that allow initial data to specified on more general spacelike hypersurfaces.

2D. ***The reduced Einstein equations.*** To obtain a well-posed evolution problem for the Einstein equations we choose a gauge with respect to a fixed Lorentzian metric $\hat{e}_{\mu\nu}$ defined on $\mathcal{M}$.

Definition 2.12 ($\hat{e}$-wave gauge). For Lorentzian metrics g and $\hat{e}$ defined on some manifold $\mathcal{M}$, let $\nabla[g]$ and $\nabla[\hat{e}]$ be the Levi-Civita connections with corresponding Christoffel symbols $\Gamma[g]$ and $\Gamma[\hat{e}]$ in local coordinates. Define the vector field V^γ in local coordinates by

$$V^\gamma = g^{\alpha\beta}(\Gamma^\gamma_{\alpha\beta}[g] - \Gamma^\gamma_{\alpha\beta}[\hat{e}]). \tag{29}$$

Define also $V_\lambda = g_{\lambda\beta}V^\beta$. The $\hat{e}$-wave gauge condition is given by

$$V^\gamma = 0. \tag{30}$$

Recall that the difference of two Christoffel symbols is a tensor, and so V^γ is in fact a well-defined vector field on $\mathcal{M}$.

Definition 2.13 (reduced Einstein equations). Let $\mathcal{M}$ be a manifold with Lorentzian metric $\hat{e}$. A Lorentzian metric g on $\mathcal{M}$ is defined to be a solution of the reduced Einstein equations if and only if

$$g^{\alpha\beta}\nabla[\hat{e}]_\alpha\nabla[\hat{e}]_\beta g_{\mu\nu} - g^{\gamma\delta}(g_{\mu\lambda}\hat{e}^{\lambda\rho}\operatorname{Riem}[\hat{e}]_{\rho\gamma\nu\delta} + g_{\nu\lambda}\hat{e}^{\lambda\rho}\operatorname{Riem}[\hat{e}]_{\rho\gamma\mu\delta}) = Q_{\mu\nu}[g](\nabla[\hat{e}]g, \nabla[\hat{e}]g), \tag{31a}$$

where we have defined

$$\begin{aligned} Q_{\mu\nu}[g](\nabla[\hat{e}]g, \nabla[\hat{e}]g) = g^{\gamma\delta}g^{\alpha\beta}\Big(&\nabla[\hat{e}]_\nu g_{\delta\beta}\nabla[\hat{e}]_\alpha g_{\mu\gamma} + \nabla[\hat{e}]_\mu g_{\gamma\alpha}\nabla[\hat{e}]_\beta g_{\nu\delta} - \tfrac{1}{2}\nabla[\hat{e}]_\nu g_{\delta\beta}\nabla[\hat{e}]_\mu g_{\gamma\alpha} \\ &+ \nabla[\hat{e}]_\gamma g_{\mu\alpha}\nabla[\hat{e}]_\delta g_{\nu\beta} - \nabla[\hat{e}]_\gamma g_{\mu\alpha}\nabla[\hat{e}]_\beta g_{\nu\delta}\Big). \end{aligned} \tag{31b}$$

2E. *The higher-dimensional Schwarzschild spacetime.* In this subsection, the higher-dimensional Schwarzschild solution is considered and its relationship to the initial data for the Einstein equations (28) and the reduced Einstein equations (31) is discussed. The form of the metric follows.

Definition 2.14. Let $n \in \mathbb{Z}$ be such that $n \geq 5$, and let $C_S \in [0, \infty)$. In Schwarzschild coordinates, the Schwarzschild metric is defined, for $(t, \bar{r}, \omega) \in \mathbb{R} \times (C_S^{1/(n-2)}, \infty) \times S^{n-1}$, to be

$$g_S = -\left(1 - \frac{C_S}{\bar{r}^{n-2}}\right)\mathrm{d}t^2 + \left(1 - \frac{C_S}{\bar{r}^{n-2}}\right)^{-1}\mathrm{d}\bar{r}^2 + \bar{r}^2\sigma_{S^{n-1}}. \tag{32}$$

The above metric can also be written in the wave gauge. For $n = 3$, it is sufficient to replace

$$(t, \bar{r}, \omega) \in \mathbb{R} \times (C_S^{1/(n-2)}, \infty) \times S^{n-1}$$

by $(t, x) = (t, r\omega)$ with $r = \bar{r} - M$; the resulting explicit metric can be found in [LeFloch and Ma 2016; Lindblad and Rodnianski 2005]. Although the case $n = 4$ leads to complicated terms involving logarithms, for $n \geq 5$, there is the following theorem.

Theorem 2.15 [Choquet-Bruhat et al. 2006, Section 5.2]. *Let $n \in \mathbb{Z}$ be such that $n \geq 5$, and let $C_S \in [0, \infty)$. There are coordinates (t, x) related to those in Definition 2.14 by $(x^i)_{i=0}^n = (t, r(\bar{r})\omega)$ with*

$$r(\bar{r}) = \bar{r} - \frac{C_S}{2\bar{r}^{n-3}} + O(\bar{r}^{5-2n}),$$

such that the $(x^i)_{i=0}^n$ satisfy the harmonic gauge, that is, the $\eta_{\mathbb{R}^{1+n}}$-wave gauge. Furthermore, there exist functions $h_{00}(R)$, $h(R)$, and $\hat{h}(R)$, defined on an interval around $R = 0$, that are analytic and bounded by a multiple of C_S near $R = 0$, and such that

$$g_S = -\left(1 - \frac{h_{00}(r^{-1})}{r^{n-2}}\right)(\mathrm{d}x^0)^2 + \sum_{i,j=1}^n \left[\left(1 + \frac{h(r^{-1})}{r^{n-2}}\right)\delta^{ij} + \frac{\hat{h}(r^{-1})}{r^{n-2}}\frac{x^i x^j}{r^2}\right]\mathrm{d}x^i\,\mathrm{d}x^j. \tag{33}$$

In particular, the difference between the components of g_S with respect to the harmonic coordinates and the corresponding components of the Minkowski metric are such that any ∂^I derivative decays at least as fast as $C_S r^{-(n-2)-|I|}$.

Note a result in [Dai 2004] ensures that $C_S \geq 0$ for the spacetimes of interest in our main Theorem 1.1.

3. Sobolev estimates on hyperboloids

We begin in Lemma 3.1 by recalling Hörmander's proof of a Sobolev estimate on hyperboloids. This allows us to introduce some of the key ideas that appear in our proof of the main result of this section, Lemma 3.2. The use of the vector field method to prove Sobolev estimates on hyperboloids originates in [Klainerman 1985].

Lemma 3.1 (Sobolev estimate for compactly supported functions on hyperboloids in Minkowski space [Hörmander 1997, Lemma 7.6.1]). *Let ν be the smallest integer greater than $\frac{1}{2}n$, and let $v \in C^{\nu}(\mathbb{R}^{1+n})$ have support in $|x| < t-1$. There is a constant C such that*

$$\sup_{\Sigma_s} t^n |v(t,x)|^2 \le C \sum_{|I|\le\nu} \int_{\Sigma_s} |Z^I v|^2 \,\mathrm{d}x. \tag{34}$$

Proof. Consider a point $(t_0, x_0) \in \Sigma_s$ with $|x_0|^2 \le t_0^2 - 1$. Set $r_0 = \frac{1}{2}t_0$ and $y_0 = x_0$. Set Σ to be the portion of Σ_s on which $|x - x_0| \le r_0$. Let $(t,x) \in \Sigma$. This implies $|t - t_0| \le r_0$, which implies $\frac{1}{2}t \le t_0 \le 2t$. Thus,

$$\sum_{|I|\le\nu} \int_{\Sigma_s} |Z^I v(t,x)|^2 \,\mathrm{d}x \ge C \sum_{|I|\le\nu} \int_{\Sigma_s} |t_0^{|I|} Y^I v(t,y)|^2 \,\mathrm{d}y.$$

The right side can be rewritten, by introducing rescaled coordinates

$$\tilde{y} = 2t_0^{-1}(y - y_0) \quad \text{and} \quad \tilde{v}(\tilde{y}) = v(t,y).$$

One can now decompose the portion of Σ_s where $|x| \le t-1$ into many subregions where t does not vary by more than a factor of 2. Let $\chi(\tilde{y})$ be a smooth cut-off such that χ is 1 on a neighbourhood of 0 and is 0 for $|\tilde{y}| \ge \frac{1}{2}$, it can further be bounded from below. A Sobolev estimate can then be applied to give a further lower bound on v. Combining these yields

$$\begin{aligned}
\sum_{|I|\le\nu} \int_{\Sigma_s} |t_0^{|I|} Y^I v(t,x)|^2 \,\mathrm{d}y &= \sum_{|I|\le\nu} \int_{|\tilde{y}|\le 1} |\partial_{\tilde{y}}^I \tilde{v}(\tilde{y})|^2 t_0^n \,\mathrm{d}\tilde{y} \\
&\ge C t_0^n \sum_{|I|\le\nu} \int_{|\tilde{y}|\le 1} |\partial_{\tilde{y}}^I ((\chi\tilde{v})(\tilde{y}))|^2 \,\mathrm{d}\tilde{y} \\
&\ge C t_0^n |\tilde{v}(0)|^2 \\
&= C t_0^n |v(t_0, x_0)|^2,
\end{aligned}$$

which completes the proof. □

In the following lemma we obtain a Sobolev estimate for functions supported on product spacetimes with specified properties outside a compact set. In particular we obtain a pointwise estimate (36) in terms of the hyperboloidal time s, as well as a t-weighted pointwise estimate on a fixed hyperboloid (37).

Lemma 3.2 (Sobolev estimate for eventually prescribed functions on hyperboloids foliating product spacetimes). *Let $n \ge 4$, let $\tilde{d}$ be the smallest even integer larger than $\frac{1}{2}d$, and let $\tilde{\nu}$ be the smallest integer greater than $\frac{1}{2}n + \tilde{d}$. Let $u_{\mu\nu}$ and $f_{\mu\nu}$ be tensors on $\mathbb{R}^{1+n} \times K$ with f depending only on the Minkowski* *coordinates x^i. Let $u \in C^{\tilde{\nu}}(\mathbb{R}^{1+n} \times K)$ satisfy $u = f$ for $|x| \geq t-1$. Let $f \in C^{\infty}(\mathbb{R}^{1+n} \times K)$ be smooth and be such that, for all $I \in \mathbb{N}$, there is a C_I such that*[3]

$$|\nabla[\hat{g}]^I f|_E \leq C_{|I|}|x|^{-(n-1)/2-|I|}. \tag{35}$$

Let $\delta(n) = \frac{1}{4}(n-2)$. There is a constant C such that

$$\sup_{(t,x^i,\omega)\in\Sigma_s\times K} s^{4\delta(n)}|u(t,x^i,\omega)|_E^2 \leq C \sum_{|I|\leq\tilde{\nu}} \sum_{i=1}^{n} \int_{\substack{\Sigma_s\times K\\ |x|\leq t-1}} |Y_i Z^I u|_E^2 \,\mathrm{d}x\,\mathrm{d}\mu_k + C \sum_{|I|\leq\tilde{\nu}-1} C_I^2. \tag{36}$$

Furthermore there is a constant C such that

$$\sup_{(t,x^i,\omega)\in\Sigma_s\times K} t^{2\delta(n)}|u(t,x^i,\omega)|_E^2 \leq C \sum_{|I|\leq\tilde{\nu}} \sum_{i=1}^{n} \int_{\substack{\Sigma_s\times K\\ |x|\leq t-1}} |Y_i Z^I u|_E^2 \,\mathrm{d}x\,\mathrm{d}\mu_k + C \sum_{|I|\leq\tilde{\nu}-1} C_I^2. \tag{37}$$

Proof. Lemma 2.9 and the standard Sobolev estimate imply

$$\sup_{\omega\in K} |u(\cdot\,,\,\cdot\,,\omega)|_E \leq \|u\|_{H^{\tilde{d}}(K)} \leq \|(\Delta_k)^{\tilde{d}/2}u\|_{L^2(K)} + \|u\|_{L^2(K)},$$

for $\tilde{d}$ the smallest even integer greater than $\frac{1}{2}d$. This choice of $\tilde{d}$ being even is simply to make the elliptic estimate cleaner. Note the trivial estimate

$$\sum_{|I|\leq\tilde{\nu}-\tilde{d}} (|Y_i Z^I(\Delta_k)^{\tilde{d}/2}u|_E^2 + |Y_i Z^I u|_E^2) \leq \sum_{|I|+2j\leq\tilde{\nu}} |Y_i Z^I(\Delta_k)^j u|_E^2.$$

It is thus sufficient to prove in Minkowski space that

$$\sup_{\Sigma_s} s^{n-2}|u(t,x)|_E^2 \leq C \sum_{|I|\leq\tilde{\nu}-\tilde{d}} \sum_{i=1}^{n} \int_{\Sigma_s} |Y_i Z^I u|_E^2 \,\mathrm{d}x + C \sum_{|I|\leq\tilde{\nu}-1} C_I^2, \tag{38}$$

since this would then imply

$$\begin{aligned}\sup_{\Sigma_s\times K} s^{n-2}|u(t,x^i,\omega)|_E^2 &\lesssim \sum_{|I|\leq\tilde{\nu}-\tilde{d}} \sum_{i=1}^{n} \|\sup_K(Y_i Z^I u)\|_{L_x^2}^2 + C \sum_{|I|\leq\tilde{\nu}-1} C_I^2 \\ &\lesssim \sum_{|I|\leq\tilde{\nu}-\tilde{d}} \sum_{i=1}^{n} \|Y_i Z^I(\Delta_k)^{\tilde{d}/2}u\|_{L_x^2 L_K^2}^2 + C \sum_{|I|\leq\tilde{\nu}-1} C_I^2.\end{aligned}$$

For $|x| \geq t-1$ and $(t,x) \in \Sigma_s$, one has $t \sim |x|$, and so

$$s^{n-2}|u(t,x)|_E^2 \leq t^{n-2}|u(t,x)|_E^2 \leq C|x|^{n-2}|u(t,x)|_E^2 \leq C|x|^{n-2}|f(x)|_E^2 \leq CC_0^2.$$

Thus, it remains to prove (38) for $|x| \leq t-1$.

[3]The exponent on f is set to match that corresponding to the exponent arising from the pointwise estimate (36) on u in the region $|t-r| \leq C$. The limiting factor on the exponent in (36) arises from estimates on the hyperboloid, not from the decay of the prescribed function f. If a faster decay rate $t^{-\beta}$ could be proved (using similar methods) on hyperboloids for compact data, then a similar $t^{-\beta}$ decay could be proved for prescribed functions satisfying $f \leq r^{-\beta}$.

Consider the region $|x| \le t-1$. Set $t_{\max} = \frac{1}{2}(s^2+1)$, which is the value of t at which Σ_s intersects $|x| = t-1$ and which satisfies $t \le t_{\max} \le \frac{1}{2}(t^2+1)$ on the portion of Σ_s where $|x| \le t-1$ by Lemma 2.8. Let $\chi : \mathbb{R} \to [0,1]$ be a smooth cut-off function such that $\chi(\alpha) = 1$ for $\alpha < 1$ and $\chi(\alpha) = 0$ for $\alpha > 2$, and define the $(0,2)$ tensor $v_{\mu\nu}(t,x) = \chi(|x|/t_{\max}) u_{\mu\nu}(t,x)$. Observe that $u_{\mu\nu} = v_{\mu\nu}$ in the region $|x| \le t-1$.

Hormander's proof of Lemma 3.1 relies on a carefully chosen rescaling of a portion of the hyperboloid, and the rest of this proof follows the same idea, although the scaling is chosen differently. Recall both the Cartesian (t,x) and hyperboloidal (s,y) coordinates in Minkowski space, which are related via $(s,y) = (\sqrt{t^2-|x|^2}, x)$. Given a choice of s, define $\tilde{y} = s^{-1}y$ and set $\tilde{v}(\tilde{y})$ to be the value of v at hyperboloidal coordinates $(s, s\tilde{y})$. With this, $\mathrm{d}^n\tilde{y} = s^{-n}\,\mathrm{d}y$ and $\partial_{\tilde{y}^i} = s\partial_{y^i} = sY_i$. Recall that $Z_i = tY_i$. Thus, by a Sobolev estimate that exploits the fact that $1 < \frac{1}{2}n < \frac{1}{2}n+1$,

$$\sup_{\Sigma_s} |v(t,x)|_E^2 = \sup |\tilde{v}(\tilde{y})|_E^2 \lesssim \sum_{1\le|J|\le\frac{n}{2}+1} \int |\partial_{\tilde{y}}^J \tilde{v}|_E^2 \,\mathrm{d}^n\tilde{y}.$$

From rescaling and the facts that $s \le t$ and that $Z_{0i} = tY_i$, it follows that

$$\begin{aligned}\sup_{\Sigma_s} |v(t,x)|_E^2 &\lesssim s^{-n} \sum_{1\le|J|\le\frac{n}{2}+1} \int |(sY)^J v|_E^2 \,\mathrm{d}^n y \lesssim s^{-n+2} \sum_{0\le|J|\le\frac{n}{2}} \sum_i \int s^{2|J|} |Y^J Y_i v|_E^2 \,\mathrm{d}^n y \\ &\lesssim s^{-n+2} \sum_{0\le|J|\le\frac{n}{2}} \sum_i \int t^{2|J|} |Y^J Y_i v|_E^2 \,\mathrm{d}^n y \lesssim s^{-n+2} \sum_{0\le|J|\le\frac{n}{2}} \sum_i \int |Y_i Z^J v|_E^2 \,\mathrm{d}^n y.\end{aligned}$$

The last integral can be decomposed into the regions where $|x| \le t-1$ and $|x| > t-1$. Where $|x| \le t-1$, the integral can be bounded by the integral term on the right-hand side of (38) since $\tilde{\nu} - \tilde{d} > \frac{1}{2}n$. Now consider the region $|x| > t-1$. Because of the support of χ, it is sufficient to consider the region $t_{\max} - 1 \le |x| \le 2(t_{\max}-1)$. In this region, $v = \chi f$. When a derivative is applied to v, it is applied to either χ or to f, in which case one obtains an additional factor of $t_{\max}^{-1}$ or $|x|^{-1}$, from the properties of χ and f, respectively. Since $|x|/t_{\max} \in [1,2]$ in the support of $\partial\chi$, effectively one obtains an extra factor of $|x|^{-1}$ in all cases, so $|Y_i Z^J v|_E \le C C_{|J|+1} |x|^{-(n-1)/2-1}$, and

$$\int_{|x|\ge t_{\max}-1} |Y_i Z^J u|_E^2 \,\mathrm{d}x \le C C_{|J|+1}^2 \int_{\mathbb{S}^{n-1}} \int_{t_{\max}-1}^{2(t_{\max}-1)} (|r|^{-(n-1)/2-1})^2 |r|^{n-1} \,\mathrm{d}r\,\mathrm{d}^{n-1}\omega_{\mathbb{S}^{n-1}} \le C C_{|J|+1}^2.$$

Observing that $s \ge Ct^{1/2}$ in the region $|x| \le t-1$ allows us to obtain

$$\begin{aligned}\sup_{\Sigma_s\times K} t^{2\delta(n)} |u|_E^2 &\le \sup_{\Sigma_s\times K\cap\{|x|\le t-1\}} t^{2\delta(n)} |u|_E^2 + \sup_{\Sigma_s\times K\cap\{|x|>t-1\}} t^{2\delta(n)} |u|_E^2 \\ &\lesssim \sup_{\Sigma_s\times K\cap\{|x|\le t-1\}} s^{4\delta(n)} |u|_E^2 + \sup_{\Sigma_s\times K\cap\{|x|>t-1\}} r^{2\delta(n)} |f|_E^2 \\ &\lesssim \sum_{|I|\le\tilde{\nu}} \sum_{i=1}^n \int_{\substack{\Sigma_s\times K \\ |x|\le t-1}} |Y_i Z^I u|_E^2 \,\mathrm{d}x\,\mathrm{d}\mu_k + \sum_{|I|\le\tilde{\nu}-1} C_I^2 + C_0 \sup_{\Sigma_s\times K\cap\{|x|>t-1\}} r^{(n-2)/2} r^{-(n-1)/2}.\end{aligned}$$

In the final line we applied estimate (36) to the first term and assumption (35) to the second term. $\square$

4. Energy integrals and inequalities

4A. ***Basic properties of the energy.*** The energy introduced in the following definition is related to the standard energy used to study quasilinear hyperbolic PDEs, albeit with additional terms included in order to be compatible with the linearised equations (8).

Definition 4.1 (Lichnerowicz-type energy on hyperboloids). Let $n \in \mathbb{Z}^+$ and let $U^{\mu\nu}$ and $u_{\mu\nu}$ be tensors defined on $\mathbb{R}^{1+n} \times K$. For $u, U \in C^1(\mathbb{R}^{1+n} \times K)$ and $s \geq 2$, define

$$\begin{aligned}\mathcal{E}[U;u;s] = \int_{\Sigma_s \times K} \Bigg((s/t)^2 |\partial_t u|_E^2 + \sum_{i=1}^n |Y_i u|_E^2 + \langle \nabla[k]^A u, \nabla[k]_A u\rangle_E - 2\langle R[\hat{g}] \circ u, u\rangle_E \\ - 2U^{\alpha\beta} \langle \nabla[\hat{g}]_\beta u, \partial_t u\rangle_E n_\alpha + U^{\alpha\beta}\langle \nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\beta u\rangle_E \Bigg) \,\mathrm{d}x \,\mathrm{d}\mu_k, \end{aligned} \tag{39}$$

where $n_0 = 1$ and $n_i = -x_i/t$ for $i \in \{1, \ldots, n\}$ and $n_A = 0$, and $\mathrm{d}x$ is the flat Euclidean volume form.

The final terms on the first line could equally well be written as $\langle \nabla[k]^A u, \nabla[k]_A u\rangle_E - 2\langle R[k] \circ u, u\rangle_E$, since the covariant derivative with respect to $\hat{g}$ in directions tangent to K are given by the covariant derivative with respect to k, and similarly for the curvature.

The terms on the second line of (39) are chosen so that, for solutions to the wave equation (42), the change in energy $\mathcal{E}[U;u;s_2] - \mathcal{E}[U;u;s_a]$ is given in (43) by an integral which has an integrand with no terms involving $(\nabla[\hat{g}]u)(\nabla[\hat{g}]\nabla[\hat{g}]u)$. The relevant cancellations to eliminate such terms follow from the properties of $T[U;u]^\mu{}_\nu$ introduced in the proof of Lemma 4.2.

Note that, following [Hörmander 1997; LeFloch and Ma 2016], we have defined $\mathcal{E}[U;u;s]$ so that it is not the naturally induced energy associated with the metric $\hat{g} + U$. This is because we have endowed Σ_s with the flat Euclidean volume form $\mathrm{d}x$, instead of the induced Riemannian volume form $(s/t)\,\mathrm{d}x$.

The following lemma provides us with an energy functional which allows us to measure the perturbation of the spacetime. Note that in (40) we require some weighted t-decay on hyperboloids which we recover from (37) in Lemma 3.2.

Lemma 4.2 (basic properties of the energy). *Take the conditions of Definition 4.1.*

(i) *There is an $\epsilon_n > 0$ such that if*

$$\sup_{\Sigma_s \times K} t|U|_E \leq C\epsilon_n, \tag{40}$$

then for $s \geq 2$,

$$\tfrac{1}{2}\mathcal{E}[U;u;s] \leq \mathcal{E}[0;u;s] \leq 2\mathcal{E}[U;u;s]. \tag{41}$$

(ii) *If $u_{\mu\nu}$ is a solution of*

$$(\hat{g} + U)^{\alpha\beta} \nabla[\hat{g}]_\alpha \nabla[\hat{g}]_\beta u_{\mu\nu} + 2(R[\hat{g}] \circ u)_{\mu\nu} = F_{\mu\nu}, \tag{42}$$

then

$$\begin{aligned}\mathcal{E}[U;u;s_1] = \mathcal{E}[U;u;s_2] + \int_{s_1}^{s_2} \int_{\Sigma_s \times K} \langle F, \partial_t u\rangle_E (s/t) \,\mathrm{d}y \,\mathrm{d}\mu_k \,\mathrm{d}s \\ + \int_{s_1}^{s_2} \int_{\Sigma_s \times K} (-2(\nabla[\hat{g}]_\alpha U^{\alpha\beta})\langle \nabla[\hat{g}]_\beta u, \partial_t u\rangle_E + (\partial_t U^{\alpha\beta})\langle \nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\beta u\rangle_E)(s/t) \,\mathrm{d}y \,\mathrm{d}\mu_k \,\mathrm{d}s. \end{aligned} \tag{43}$$

Proof. We first derive the energy $\mathcal{E}[U; u; s]$ by considering the following nonlinear version of the stress energy tensor (9)

$$T[U; u]^{\mu}{}_{\nu} = (\hat{g}+U)^{\mu\alpha}\langle\nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\nu u\rangle_E - \tfrac{1}{2}(\hat{g}+U)^{\alpha\beta}\langle\nabla[\hat{g}]_\beta u, \nabla[\hat{g}]_\alpha u\rangle_E \delta^\mu_\nu + \langle R[\hat{g}]\circ u, u\rangle_E \delta^\mu_\nu. \quad (44)$$

We calculate

$$\begin{aligned}\nabla[\hat{g}]_\mu T[U; u]^{\mu}{}_{\nu} = {}& \langle(\hat{g}+U)^{\alpha\beta}\nabla[\hat{g}]_\alpha\nabla[\hat{g}]_\beta u, \nabla[\hat{g}]_\nu u\rangle_E + (\hat{g}+U)^{\mu\alpha}\langle\nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\mu\nabla[\hat{g}]_\nu u\rangle_E \\ & - (\hat{g}+U)^{\alpha\beta}\langle\nabla[\hat{g}]_\nu\nabla[\hat{g}]_\beta u, \nabla[\hat{g}]_\alpha u\rangle_E + \nabla[\hat{g}]_\nu\langle R[\hat{g}]\circ u, u\rangle_E \\ & + (\nabla[\hat{g}]_\mu U^{\mu\alpha})\langle\nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\nu u\rangle_E - \tfrac{1}{2}(\nabla[\hat{g}]_\nu U^{\alpha\beta})\langle\nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\beta u\rangle_E. \quad (45)\end{aligned}$$

Let X^μ be a vector field on $\mathbb{R}^{1+n}\times K$ tangent to $\mathbb{R}^{1+n}$. We have

$$\nabla[\hat{g}]_\alpha\nabla[\hat{g}]_\beta u_{\gamma\delta} = \nabla[\hat{g}]_\beta\nabla[\hat{g}]_\alpha u_{\gamma\delta} + \mathrm{Riem}[\hat{g}]_{\alpha\beta\gamma}{}^{\rho}u_{\rho\delta} + \mathrm{Riem}[\hat{g}]_{\alpha\beta\delta}{}^{\rho}u_{\rho\gamma}.$$

Since $(\mathbb{R}^{1+n}, \eta_{\mathbb{R}^{1+n}})$ has zero Riemann curvature, and since the Riemann curvature for a product manifold is given by $\mathrm{Riem}[\hat{g}] = \mathrm{Riem}[\eta_{\mathbb{R}_{1+n}}] + \mathrm{Riem}[k]$, it follows that all components of the Riemann curvature $\mathrm{Riem}[\hat{g}]_{\alpha\beta\gamma}{}^{\delta}$ vanish unless all the indices $\alpha, \beta, \gamma, \delta$ correspond to internal directions tangent to K. Thus, the contraction with a vector tangent to $\mathbb{R}^{1+n}$ vanishes, and, in particular,

$$\mathrm{Riem}[\hat{g}]_{\alpha\beta\gamma\delta}X^\delta = 0. \quad (46)$$

Consequently

$$\langle\nabla[\hat{g}]_\alpha\nabla[\hat{g}]_\beta u, \nabla[\hat{g}]_\nu u\rangle_E X^\alpha = \langle\nabla[\hat{g}]_\beta\nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\nu u\rangle_E X^\alpha.$$

and also

$$\nabla[\hat{g}]_\nu\langle R[\hat{g}]\circ u, u\rangle_E X^\nu = 2\langle R[\hat{g}]\circ u, X^\nu\nabla[\hat{g}]_\nu u\rangle_E.$$

This allows us to calculate

$$\begin{aligned}\nabla[\hat{g}]_\mu(T[U; u]^{\mu}{}_{\nu}X^\nu) = {}& T^{\mu}{}_{\nu}[U]\nabla[\hat{g}]_\mu X^\nu + \langle F, X^\nu\nabla[\hat{g}]_\nu u\rangle_E + (\nabla[\hat{g}]_\mu U^{\mu\alpha})\langle\nabla[\hat{g}]_\alpha u, X^\nu\nabla[\hat{g}]_\nu u\rangle_E \\ & - \tfrac{1}{2}(X^\nu\nabla[\hat{g}]_\nu U^{\alpha\beta})\langle\nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\beta u\rangle_E.\end{aligned}$$

Consider the hyperboloidal energy

$$\begin{aligned}\mathcal{E}[U; u; s] &= \int_{\Sigma_s\times K} -2T[U; u]^{\mu}{}_{\nu}(\partial_t)^\nu n_\mu \,\mathrm{d}x\,\mathrm{d}\mu_k \\ &= \int_{\Sigma_s\times K}\biggl(|\partial_t u|_E^2 + \sum_{i=1}^n|\partial_i u|_E^2 + \sum_{i=1}^n 2(x^i/t)\langle\partial_t u, \partial_i u\rangle_E + k^{AB}\langle\nabla[\hat{g}]_A u, \nabla[\hat{g}]_B u\rangle_E \\ &\qquad - 2\langle R[\hat{g}]\circ u, u\rangle_E - 2U^{\mu\rho}\langle\nabla[\hat{g}]_\rho u, \partial_t u\rangle_E n_\mu + U^{\rho\lambda}\langle\nabla[\hat{g}]_\rho u, \nabla[\hat{g}]_\lambda u\rangle_E\biggr)\,\mathrm{d}x\,\mathrm{d}\mu_k,\end{aligned}$$

where $n_0 = 1$, $n_i = -\eta_{ij}x^j/t$ for $i\in\{1, \dots, n\}$ and $n_A = 0$. Note that

$$\begin{aligned}\mathcal{E}[0; u; s] = \int_{\Sigma_s\times K}\biggl(&|\partial_t u|_E^2 + \sum_{i=1}^n|\partial_i u|_E^2 + 2(x^i/t)\langle\partial_t u, \partial_i u\rangle_E \\ & + \langle\nabla[\hat{g}]^A u, \nabla[\hat{g}]_A u\rangle_E - 2\langle R[\hat{g}]\circ u, u\rangle_E\biggr)\,\mathrm{d}x\,\mathrm{d}\mu_k, \quad (47)\end{aligned}$$

which alternatively can be written in hyperboloidal coordinates as

$$\mathcal{E}[0; u; s] = \int_{\Sigma_s \times K} \left((s/t)^2 |\partial_t u|_E^2 + \sum_{i=1}^{n} |Y_i u|_E^2 + \langle \nabla[\hat{g}]^A u, \nabla[\hat{g}]_A u \rangle_E - 2\langle R[\hat{g}] \circ u, u \rangle_E \right) \mathrm{d}x \, \mathrm{d}\mu_k. \quad (48)$$

Since the contraction of $R[\hat{g}]$ with any direction tangent to $\mathbb{R}^{1+n}$ vanishes, and since $|w|_E \geq |w|_k$ for any tensor field w, it follows from the definition of $\mathcal{L}$ that

$$\begin{aligned}\int_K (\langle \nabla[\hat{g}]^A u, \nabla[\hat{g}]_A u \rangle_E - 2\langle R[\hat{g}] \circ u, u \rangle_E) \, \mathrm{d}\mu_k &\geq \int_K (\langle \nabla[\hat{g}]^A u, \nabla[\hat{g}]_A u \rangle_k - 2\langle R[\hat{g}] \circ u, u \rangle_k) \, \mathrm{d}\mu_k \\ &= \int_K \langle \mathcal{L}u, u \rangle_k \, \mathrm{d}\mu_k.\end{aligned}$$

Thus, from Theorem 2.2 and the condition of Riemannian linear stability (11), it follows that

$$\int_K (\langle \nabla[\hat{g}]^A u, \nabla[\hat{g}]_A u \rangle_E - 2\langle R[\hat{g}] \circ u, u \rangle_E) \, \mathrm{d}\mu_k \geq 0. \quad (49)$$

Thus, $\mathcal{E}[0, u, s] \geq 0$.

Using our previously calculated expression for the divergence of $T[U; u]^\mu{}_\nu X^\nu$, we obtain

$$\begin{aligned}\mathcal{E}[U; u; s_1] = \mathcal{E}[U; u; s_2] &+ \int_{s_1}^{s_2} \int_{\Sigma_s \times K} \langle -2F, \partial_t u \rangle_E (s/t) \, \mathrm{d}y \, \mathrm{d}\mu_k \, \mathrm{d}s \\ &+ \int_{s_1}^{s_2} \int_{\Sigma_s \times K} \big(-2(\nabla[\hat{g}]_\alpha U^{\alpha\beta}) \langle \nabla[\hat{g}]_\beta u, \partial_t u \rangle_E \\ &\qquad + (\partial_t U^{\alpha\beta}) \langle \nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\beta u \rangle_E \big) (s/t) \, \mathrm{d}y \, \mathrm{d}\mu_k \, \mathrm{d}s\end{aligned}$$

via Stoke's theorem. This proves equality (43).

Condition (40) combined with $s \geq Ct^{1/2}$ implies $\sup_{\Sigma_s \times K} |U|_E (t/s)^2 \leq C\varepsilon_n$. For simplicity denote $k^{AB} \langle \nabla[\hat{g}]_A u, \nabla[\hat{g}]_B u \rangle_E$ by $|\partial_A u|_E^2$, then

$$\begin{aligned}\frac{s^2}{2t^2} \left(|\partial_t u|_E^2 \sum_i + |\partial_i u|_E^2 + |\nabla[k] u|_E^2 \right) &\leq \left(|\partial_t u|^2 + \sum_i |\partial_i u|^2 + |\nabla[k] u|_E^2 \right) (1 - |x|/t) \\ &\leq |\partial_t u|_E^2 + |\partial_i u|_E^2 + 2(x^i/t) \langle \partial_t u, \partial_i u \rangle_E + |\nabla[k] u|_E^2. \quad (50)\end{aligned}$$

Using this and Young's inequality we find

$$\begin{aligned}|\mathcal{E}[U; u; s] - \mathcal{E}[0; u; s]| &= \left| \int_{\Sigma_s \times K} (2U^{\alpha\beta} \langle \nabla[\hat{g}]_\alpha u, \partial_t u \rangle_E n_\beta - U^{\alpha\beta} \langle \nabla[\hat{g}]_\alpha u, \nabla[\hat{g}]_\beta u \rangle_E) \, \mathrm{d}x \, \mathrm{d}\mu_k \right| \\ &\leq C\varepsilon_n \mathcal{E}[0; u; s],\end{aligned}$$

and thus the energies are equivalent for sufficiently small ε_n. This proves estimate (41) and the lemma. □

Having defined the energy involving first-order derivatives, we now introduce higher-order energies.

Definition 4.3 (symmetry boosted energy). Let $(\mathbb{R}^{1+n} \times K, \hat{g})$ be a spacetime with a supersymmetric compactification and $N \in \mathbb{N}$. For $k \leq N$, define the energy of a symmetric tensor field g to be

$$\mathcal{E}_{k+1}(s) = \sum_{|I| \leq k} \mathcal{E}[g^{-1} - \hat{g}^{-1}; \Gamma^I g; s]. \quad (51)$$

We end this section with the following Hardy estimate on hyperboloids. The proof is standard; see for example [LeFloch and Ma 2016, Lemma 2.4].

Lemma 4.4 (Hardy estimate on hyperboloids). *Let $u_{\mu\nu}$ be a tensor defined on $\mathbb{R}^{1+n}$. Then one has*

$$\|r^{-1}u\|_{L^2(\Sigma_s)} \lesssim \sum_{i=1}^{n} \|Y_i u\|_{L^2(\Sigma_s)}. \tag{52}$$

4B. ***Preliminary L^2 and L^∞ estimates.*** In our nonlinear estimates we will estimate terms of the form

$$Z^I(\Delta_k)^j(uv) = \sum_{\substack{|I_1|+|I_2|=|I|\\ |J_1|+|J_2|=2j}} Z^{I_1}\nabla[k]^{J_1}u \cdot Z^{I_2}\nabla[k]^{J_2}v. \tag{53}$$

In the following lemma we estimate terms which appear as factors in the right-hand side of (53) in L^2 by using the elliptic estimates of Lemma 2.9 and the Hardy estimate of Lemma 4.4. Note the use of elliptic estimates allows us to avoid commuting derivatives, such as $[\nabla[k], \Delta_k]$, which shortens the argument.

Lemma 4.5 (L^2 estimate for distributed derivatives). *Let $u_{\mu\nu}$ be a tensor defined on $\mathbb{R}^{1+n} \times K$. Suppose N is even, $\ell \in \mathbb{N}$, and $\ell \leq N+1$, then*

$$\sum_{|I|+|J|\leq \ell} \|t^{-1}Z^I\nabla[k]^J u\|_{L^2(\Sigma_s\times K)} \lesssim \mathcal{E}_{N+1}(s)^{1/2}. \tag{54}$$

Proof. We prove the estimate by considering separately the cases of $|I|=0$ and $|I|\neq 0$. Firstly take $|I|\geq 1$, suppose $|J|=2m$ where $m\in\mathbb{N}$, and consider $|I|+|J|=\ell\leq N+1$. Using the elliptic estimates of Lemma 2.9 we find

$$\begin{aligned}\|t^{-1}Z^I\nabla[k]^J u\|_{L^2(\Sigma_s\times K)} &\lesssim \big\| \|t^{-1}Z^I u\|_{H^{2m}(K)}\big\|_{L^2(\Sigma_s)} \lesssim \|t^{-1}Z^I(\Delta_k)^m u\|_{L^2(\Sigma_s\times K)} + \|t^{-1}Z^I u\|_{L^2(\Sigma_s\times K)}\\ &\lesssim \sum_{i=1}^{n}\|Y_i Z^{I-1}(\Delta_k)^m u\|_{L^2(\Sigma_s\times K)} + \sum_{i=1}^{n}\|Y_i Z^{I-1}u\|_{L^2(\Sigma_s\times K)}\\ &\lesssim \mathcal{E}[0;\, Z^{I-1}(\Delta_k)^m u;\, s]^{1/2} + \mathcal{E}[0;\, Z^{I-1}u;\, s]^{1/2} \lesssim \mathcal{E}_\ell(s)^{1/2}.\end{aligned}$$

Next take $|I|\geq 1$ and suppose $|J|=2m+1$ where $m\in\mathbb{N}$. For $|I|+|J|=\ell\leq N+1$, again using Lemma 2.9, we have

$$\begin{aligned}&\|t^{-1}Z^I\nabla[k]^J u\|_{L^2(\Sigma_s\times K)}\\ &\qquad\lesssim \big\| \|t^{-1}Z^I u\|_{H^{2m+1}(K)}\big\|_{L^2(\Sigma_s)}\\ &\qquad\lesssim \sum_{i=1}^{n}\|Y_i Z^{I-1}u\|_{L^2(\Sigma_s\times K)} + \sum_{i=1}^{n}\|Y_i Z^{I-1}(\Delta_k)^m u\|_{L^2(\Sigma_s\times K)} + \|\nabla[k](Z^I(\Delta_k)^m u)\|_{L^2(\Sigma_s\times K)}\\ &\qquad\lesssim \mathcal{E}[0;\, Z^{I-1}u;\, s]^{1/2} + \mathcal{E}[0;\, Z^{I-1}(\Delta_k)^m u;\, s]^{1/2} + \mathcal{E}[0;\, Z^I(\Delta_k)^m u;\, s]^{1/2} \lesssim \mathcal{E}_\ell(s)^{1/2}.\end{aligned}$$

We now turn to the case $|I|=0$. Again we split into the cases of $|J|$ being even and odd. Start with $|J|=2m$ for $m\in\mathbb{N}$. Note that N is chosen to be even so that we have the strict inequality $2m<N+1$.

Applying the Hardy estimate from Lemma 4.4, and recalling that $t \geq r$ on the hyperboloid, yields

$$\begin{aligned}\|t^{-1}\nabla[k]^J u\|_{L^2(\Sigma_s\times K)} &\lesssim \big\| \|r^{-1}u\|_{H^{2m}(K)}\big\|_{L^2(\Sigma_s)} \lesssim \|r^{-1}(\Delta_k)^m u\|_{L^2(\Sigma_s\times K)} + \|r^{-1}u\|_{L^2(\Sigma_s\times K)}\\ &\lesssim \sum_{i=1}^n \|Y_i(\Delta_k)^m u\|_{L^2(\Sigma_s\times K)} + \sum_{i=1}^n \|Y_i u\|_{L^2(\Sigma_s\times K)}\\ &\lesssim \mathcal{E}[0,(\Delta_k)^m u; s]^{1/2} + \mathcal{E}[0,u;s]^{1/2} \lesssim \mathcal{E}_{N+1}(s)^{1/2}.\end{aligned}$$

Finally we have the case $|I| = 0$ and $|J| = 2m+1 \leq N+1$ for $m \in \mathbb{N}$. Again using Lemma 4.4 we obtain

$$\begin{aligned}\|t^{-1}\nabla[k]^J u\|_{L^2(\Sigma_s\times K)} &\lesssim \big\| \|r^{-1}u\|_{H^{2m+1}(K)}\big\|_{L^2(\Sigma_s)} \lesssim \|r^{-1}\nabla[k](\Delta_k)^m u\|_{L^2(\Sigma_s\times K)} + \|r^{-1}u\|_{L^2(\Sigma_s\times K)}\\ &\lesssim \|\nabla[k](\Delta_k)^m u\|_{L^2(\Sigma_s\times K)} + \sum_{i=1}^n \|Y_i u\|_{L^2(\Sigma_s\times K)}\\ &\lesssim \mathcal{E}[0,(\Delta_k)^m u; s]^{1/2} + \mathcal{E}[0,u;s]^{1/2} \lesssim \mathcal{E}_{|J|}(s)^{1/2}.\end{aligned}$$

Adding together the above estimates over all appropriate multi-indices gives the required result. □

Corollary 4.6 (L^2 estimate for eventually prescribed functions on hyperboloids foliating product spacetimes). *Let $n \geq 4$. Let $u_{\mu\nu}$ and $f_{\mu\nu}$ be tensors defined on $\mathbb{R}^{1+n} \times K$ with f depending only on the Minkowski coordinates. Suppose $u = f$ for $|x| \geq t-1$. Let $f \in C^\infty(\mathbb{R}^{1+n} \times K)$ be smooth and such that, for all $I \in \mathbb{N}$, there is a C_I such that*[4]

$$|\nabla[\hat{g}]^I f|_E \leq C_{|I|}|x|^{-(n+1)/2-|I|}. \tag{55}$$

Suppose N is even, $\ell \in \mathbb{N}$, and $\ell \leq N+1$, then

$$\sum_{|I|+|J|\leq \ell} \|(s/t)Z^I\nabla[k]^J u\|_{L^2(\Sigma_s\times K)} \lesssim s\mathcal{E}_{N+1}(s)^{1/2} + \sum_{|I|+|J|\leq\ell} C_{|I|,|J|}. \tag{56}$$

Proof. We will consider separately the regions $|x| \leq t-1$ and $|x| > t-1$. The estimate in the region $|x| \leq t-1$ follows by applying Lemma 4.5 with an additional factor of s. Next consider the region $|x| > t-1 \geq t_0 - 1$, where we let $t_0 = \frac{1}{2}(s^2+1)$ be the value of t at which Σ_s intersects $|x| = t-1$. Using assumption (55) we find

$$\begin{aligned}&\|(s/t)Z^I\nabla[k]^J u\|^2_{L^2(\Sigma_s\times K\cap\{|x|>t-1\})}\\ &\qquad\leq \int_{\Sigma_s\times K\cap\{|x|>t_0-1\}} |Z^I\nabla[k]^J u|_E^2 \,\mathrm{d}x\,\mathrm{d}\mu_k \leq C\int_{\Sigma_s\cap\{|x|>t_0-1\}} |Z^I\nabla[k]^J f|_E^2\,\mathrm{d}x\\ &\qquad\leq CC^2_{|I|,|J|}\int_{\mathbb{S}^{n-1}}\int_{\Sigma_s\cap\{|x|\geq t_0-1\}} (|r|^{-(n+1)/2})^2|r|^{n-1}\,\mathrm{d}r\,\mathrm{d}\omega_{\mathbb{S}^{n-1}}\\ &\qquad\leq CC^2_{|I|,|J|}\int_{\mathbb{S}^{n-1}}\int_{\Sigma_s\cap\{|x|\geq t_0-1\}} r^{-2}\,\mathrm{d}r\,\mathrm{d}\omega_{\mathbb{S}^{n-1}} \leq CC^2_{|I|,|J|}.\end{aligned}$$

Adding together the above estimates over all appropriate multi-indices yields (56). □

[4] Note that the decay assumption on f is stronger here than the assumption (35) in Lemma 3.2.

We next use Lemma 3.2 to obtain L^∞ estimates for terms which appear as factors in the right-hand side of (53).

Corollary 4.7 (higher-order Sobolev estimates). *Let $n \geq 7$. Let $\tilde{d}$, $\tilde{\nu}$, $u_{\mu\nu}$, and $f_{\mu\nu}$ be as defined in Lemma 3.2. Then for $|I|+|J| = \ell \in \mathbb{N}$ there is a constant C such that*

$$\sup_{\Sigma_s \times K} (s^{4\delta(n)} |Z^I \nabla[k]^J u|_E^2 + s^{4\delta(n)-2} |(t/s) Z^I \nabla[k]^J u|_E^2) \\ \leq C \sum_{|I|+2j \leq \tilde{\nu}+\ell+1} \mathcal{E}[0;\, Z^I (\Delta_k)^j u;\, s] + C \sum_{|I| \leq \tilde{\nu}+\ell-1} C_{|I|}^2. \quad (57)$$

Proof. We consider the left-most term in (57) first. Let $\tilde{j}$ be the smallest even integer such that $\tilde{j} \geq |J|$. In particular this means

$$|I|+|J| \leq |I| + \tilde{j} \leq \ell + 1.$$

Recall that $\tilde{d}$ is the smallest even integer larger than $\frac{1}{2}d$ and $\tilde{\nu}$ is the smallest integer greater than $\frac{1}{2}n + \tilde{d}$. Applying Lemma 2.9 yields

$$\sup_K |\nabla[k]^J u|_E \leq \|u\|_{H^{\tilde{d}+\tilde{j}}(K)} \leq \|(\Delta_k)^{(\tilde{d}+\tilde{j})/2} u\|_{L^2(K)} + \|u\|_{L^2(K)}.$$

Thus, using in particular (38), we have

$$\begin{aligned} \sup_{(t,x,\omega)\in\Sigma_s\times K} & s^{4\delta(n)} |Z^I \nabla[k]^J u(t, x^i, \omega)|_E^2 \\ &\lesssim \sum_{|I_1| \leq \tilde{\nu}-\tilde{d}} \sum_{i=1}^{n} \|\sup_K (Y_i Z^{I_1} Z^I \nabla[k]^J u)\|_{L^2(\Sigma_s)}^2 + \sum_{|I_1| \leq \tilde{\nu}-1} C_{I_1}^2 \\ &\lesssim \sum_{|I_1| \leq \tilde{\nu}-\tilde{d}} \sum_{i=1}^{n} (\|Y_i Z^{I+I_1} u\|_{L^2(\Sigma_s\times K)}^2 + \|Y_i Z^{I+I_1} (\Delta_k)^{(\tilde{d}+\tilde{j})/2} u\|_{L^2(\Sigma_s \times K)}^2) + C \sum_{|I_1| \leq \tilde{\nu}-1} C_{I_1}^2 \\ &\lesssim \sum_{|I|+2j \leq \tilde{\nu}+\ell+1} \mathcal{E}[0;\, Z^I (\Delta_k)^j u;\, s] + C \sum_{|I| \leq \tilde{\nu}-1} C_I^2. \end{aligned}$$

To complete the proof for the second term of (57) we observe that $s \geq C t^{1/2}$ in the region $|x| \leq t-1$ while we only have $s \leq t \leq r$ in the region $|x| > t-1$. Since $n \geq 7$ we have $\delta(n) \geq 1$ and thus

$$\begin{aligned} \sup_{\Sigma_s \times K} & s^{4\delta(n)-2} |(t/s) Z^I \nabla[k]^J u|_E^2 \\ &\lesssim \sup_{\Sigma_s \times K \cap \{|x| \leq t-1\}} (t^2/s^4) s^{4\delta(n)} |Z^I \nabla[k]^J u|_E^2 + \sup_{\Sigma_s \times K \cap \{|x| > t-1\}} s^{4\delta(n)-4} r^2 |Z^I \nabla[k]^J f|_E^2 \\ &\lesssim \sum_{|I|+2j \leq \tilde{\nu}+\ell+1} \mathcal{E}[0;\, Z^I (\Delta_k)^j u;\, s] + \sum_{|I| \leq \tilde{\nu}+\ell-1} C_I^2 + C_I^2 \sup_{\Sigma_s \times K \cap \{|x| > t-1\}} r^{(n-2)-2} r^{-(n-1)}. \end{aligned}$$

Note in the final line we applied (35) and the first estimate of (57). $\square$

5. Proof of stability

5A. ***Stability for the reduced Einstein equations.*** We now restate our main Theorem 1.1 in terms of the reduced Einstein equations. For convenience we translate the initial data of Theorem 1.1 to $\{t=4\}$.

Theorem 5.1 (stability for the reduced Einstein equations). *Let $n, d \in \mathbb{Z}^+$ be such that $n \geq 9$, and let $N \in \mathbb{N}$ be an even integer strictly larger than $\frac{1}{2}(n+d+8)$. Let $(\mathbb{R}^{1+n} \times K, \hat{g} = \eta_{\mathbb{R}^{1+n}} + k)$ be a spacetime with a supersymmetric compactification.*

Let $(\{t=4\} \times \mathbb{R}^n \times K, g_0, g_1)$ be Cauchy data for the reduced Einstein equations (31). *Assume that, for $|x| \geq 1$ with respect to Minkowski coordinates on $\mathbb{R}^{1+n}$, $(g_0, g_1) = (g_S + k, 0)$ where g_S is the Schwarzschild metric in the $\eta_{\mathbb{R}^{1+n}}$-wave gauge with parameter $C_S \in [0, \infty)$.*

There is an $\epsilon > 0$ such that, if the initial data satisfies

$$\sum_{|I| \leq N} \|\nabla[g_0]^I (g_0 - \hat{g}|_{t=4})\|^2_{L^2(\mathbb{R}^n \times K)} + \sum_{|I| \leq N-1} \|\nabla[g_0]^I g_1\|^2_{L^2(\mathbb{R}^n \times K)} + C_S^2 \leq \epsilon, \tag{58}$$

then there is a future global solution $g_{\mu\nu}$ of the reduced Einstein equations (31) *with initial data $(h, \partial_t h)|_{t=4} = (g_0, g_1)$. Furthermore, there is the bound*

$$\sup_{(t,x,\omega) \in \Sigma_s \times K} s^{4\delta(n)} |g(t, x^i, \omega) - \hat{g}(t, x^i, \omega)|^2_E \lesssim \epsilon, \tag{59}$$

where $\delta(n)$ was defined in (7).

Proof. Let the perturbation and inverse perturbation be denoted, respectively, by

$$h_{\mu\nu} = g_{\mu\nu} - \hat{g}_{\mu\nu} \quad \text{and} \quad H^{\mu\nu} = g^{\mu\nu} - \hat{g}^{\mu\nu}.$$

Since g is a solution of the reduced Einstein equation (31), it follows that

$$(\hat{g}^{\alpha\beta} + H^{\alpha\beta}) \nabla[\hat{g}]_\alpha \nabla[\hat{g}]_\beta h_{\mu\nu} + 2(R[\hat{g}] \circ h)_{\mu\nu} = Q_{\mu\nu}[g](\nabla[\hat{g}]h, \nabla[\hat{g}]h) + F_{\mu\nu}(H, h), \tag{60}$$

where $Q_{\mu\nu}$ is defined in (31b) and $F_{\mu\nu}$ is defined by

$$F_{\mu\nu}(H, h) = H^{\alpha\beta}(h_{\alpha\delta} \operatorname{Riem}[\hat{g}]^\delta{}_{\mu\nu\beta} + h_{\alpha\delta} \operatorname{Riem}[\hat{g}]^\delta{}_{\nu\mu\beta}) + H^{\alpha\beta}(h_{\mu\delta} \operatorname{Riem}[\hat{g}]^\delta{}_{\alpha\nu\beta} + h_{\nu\delta} \operatorname{Riem}[\hat{g}]^\delta{}_{\alpha\mu\beta}).$$

By commuting the symmetries $Z^I (\Delta_k)^j$ through the system (60) we obtain

$$(\hat{g}^{\alpha\beta} + H^{\alpha\beta}) \nabla[\hat{g}]_\alpha \nabla[\hat{g}]_\beta (Z^I (\Delta_k)^j h_{\mu\nu}) - 2(R[\hat{g}] \circ Z^I (\Delta_k)^j h)_{\mu\nu} = \sum_{i=1}^{3} F^{i,I,j}_{\mu\nu}, \tag{61}$$

where

$$\begin{aligned} F^{1,I,j}_{\mu\nu} &= Z^I (\Delta_k)^j Q_{\mu\nu}[g](\nabla[\hat{g}]h, \nabla[\hat{g}]h), \\ F^{2,I,j}_{\mu\nu} &= Z^I (\Delta_k)^j F_{\mu\nu}(H, h), \\ F^{3,I,j}_{\mu\nu} &= [Z^I (\Delta_k)^j, H^{\alpha\beta} \nabla[\hat{g}]_\alpha \nabla[\hat{g}]_\beta] h_{\mu\nu}. \end{aligned} \tag{62}$$

The symmetry boosted energy is given by

$$\mathcal{E}_{k+1}(s) = \sum_{|I|+2j \leq k} \mathcal{E}[H; Z^I (\Delta_k)^j g; s]. \tag{63}$$

From Lemma 4.2 and the Cauchy–Schwarz inequality we obtain

$$\mathcal{E}_{N+1}(s')^{1/2} \leq \mathcal{E}_{N+1}(4)^{1/2} + \sum_{|I|+2j\leq N} \int_4^{s'} \left(\int_{\Sigma_s \times K} \left(\sum_{i=1}^{3} |F^{i,I,j}|_E^2 + |G^{I,j}|_E^2 \right) \mathrm{d}y \, \mathrm{d}\mu_k \right)^{1/2} \mathrm{d}s, \tag{64}$$

where the $G^{I,j}$ terms arise from applying $Z^I(\Delta_k)^j$ to the terms involving $\nabla[\hat{g}]\gamma$ or $\partial_t\gamma$ on the right side of the energy equality (43). In particular, these can be bounded by

$$|G^{I,j}|_E^2 \leq C|\nabla[\hat{g}]H|_E^2 |Z^I(\Delta_k)^j \nabla[\hat{g}]h|_E^2. \tag{65}$$

The reduced field equations (60) are a system of quasilinear, quasidiagonal wave equations for the perturbation $h_{\mu\nu}$ of the spacetime metric. The existence of unique local solutions emanating from Cauchy data is standard [Choquet-Bruhat 2009, Theorem 4.6 Appendix III].

The proof then follows a bootstrap argument (or continuous induction): we prove that there exist $C > 0$ and $\epsilon > 0$ such that, if $\mathcal{E}_{N+1}(4) + C_S < \epsilon$ and $\mathcal{E}_{N+1}(s) \leq C\epsilon$ for all s, then $\mathcal{E}_{N+1}(s) \leq \epsilon + C\epsilon^2$ for all s and hence $\mathcal{E}_{N+1}(s) \leq \frac{1}{2}C\epsilon$. We note that there is no loss of generality in placing our initial data at $t = 4$.

We consider the integral term on the right-hand side in (64) as the sum of integrals over $\Sigma_s \cap \{|x| \leq t-1\}$ and over $\Sigma_s \cap \{|x| > t-1\}$. Our approach is that, for sufficiently small C_S, in the latter exterior region the solution is identically the product of Schwarzschild with the internal manifold. Thus in the region $|x| \geq t-1$ the perturbation $h_{\mu\nu}$ is only nonzero on its Minkowski indices and on these indices it is identically Schwarzschild. We note that sufficiently small compactly supported initial data on $\{t = 4\} \cap \{|x| \leq 1\}$ can be extended to compactly supported initial data on Σ_4 [LeFloch and Ma 2014, Chapter 39].

Recall from Section 2E that the difference between components of the Minkowski metric and the Schwarzschild metric in wave coordinates decay as $C_S r^{-n+2}$ and the Christoffel symbols decay as $C_S r^{-n+1}$. Along a geodesic parametrised by λ, one has

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} = \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda}.$$

Since $C_S r^{-n+1}$ is integrable in r, there are geodesics along which t and r grow linearly and the $\mathrm{d}x^j/\mathrm{d}\lambda$ approach constant values, not all of which are vanishing. In particular, $\mathrm{d}r/\mathrm{d}t$ asymptotically approaches a constant, and this constant is 1 for null geodesics. The next-to-leading-order term in the geodesic equation arises from the metric, so it is of the form Cr^{-n+2}, which is again integrable. Furthermore, the smaller the mass C_S the sooner this asymptotic behaviour comes to dominate. In particular, if C_S is sufficiently small, then any causal curve launched from within $\Sigma_4 \cap \{|x| \leq t-2\}$ can never reach the region where $|x| \geq t-1$. Furthermore, by uniqueness of solutions to quasilinear wave equations, since the initial data on Σ_4 is identically Schwarzschild for $|x| > t-2$, the solution is identically Schwarzschild for $|x| > t-1$. In particular, when estimating the components of the solution to (61), we can use the Sobolev Lemma 3.2 and Corollary 4.6 on hyperboloids with eventually prescribed functions. (The conclusion of this paragraph is essentially Proposition 2.3 of [LeFloch and Ma 2016].)

The estimate (40) required by Lemma 4.2 is established by combining (37) with the bootstrap assumptions and noting that since $n \geq 9$ we certainly have $\delta(n) > 1$. Similarly since $n \geq 9$ the decay assumptions (55) in Corollary 4.6 and (35) in Lemma 3.2 are satisfied.

We are now in a position to apply the results from Section 4B to the nonlinearities in (64). In general we will distribute $(s/t)(t/s)=1$ across the terms and estimate high-derivative terms with a factor of (s/t) using Corollary 4.6 and low-derivative terms with a factor of (t/s) using Corollary 4.7. We begin by estimating the term $G^{I,j}$. Using (65) we find

$$\begin{aligned}\sum_{|I|+2j\le N}\|G^{I,j}\|_{L^2(\Sigma_s\times K)} &\lesssim \sum_{|I|+|J|\le N}\biggl(\int_{\Sigma_s\times K}|(t/s)\nabla[\hat g]H|_E^2\,|(s/t)Z^I\nabla[k]^J\nabla[\hat g]h|_E^2\,\mathrm{d}y\,\mathrm{d}\mu_k\biggr)^{1/2}\\ &\le \sup_{\Sigma_s\times K}(|(t/s)\nabla[\hat g]h|_E)\biggl(\int_{\Sigma_s\times K}|(s/t)Z^I\nabla[k]^J\nabla[\hat g]h|_E^2\,\mathrm{d}y\,\mathrm{d}\mu_k\biggr)^{1/2}\\ &\lesssim \frac{1}{s^{2\delta(n)-1}}(\mathcal{E}_{\tilde\nu+3}(s)^{1/2}+C_S)(s\mathcal{E}_{N+1}(s)^{1/2}+C_S).\end{aligned}\tag{66}$$

The term $F^1_{\mu\nu}$ involves the standard quadratic derivative nonlinearities of the Einstein equations. Their weak null structure is of course not relevant here since the Minkowski dimension is taken so high. We first look at what type of terms are contained in $F^1_{\mu\nu}$:

$$\begin{aligned}&\sum_{|I|+2j\le N}\|F^{1,I,j}_{\mu\nu}\|_{L^2(\Sigma_s\times K)}\\ &\lesssim \sum_{|I|+|J|\le N}\biggl(\int_{\Sigma_s\times K}|(\hat g+H)^{-1}|_E^2\,|Z^I\nabla[k]^J(\nabla[\hat g]h\nabla[\hat g]h)|_E^2\,\mathrm{d}y\,\mathrm{d}\mu_k\biggr)^{1/2}\\ &\qquad+\sum_{\substack{|I_i|+|J_i|\le N\\ |I_1|+|J_1|\ge 1}}\biggl(\int_{\Sigma_s\times K}|Z^{I_1}\nabla[k]^{J_1}h|_E^2\,|Z^{I_2}\nabla[k]^{J_2}(\nabla[\hat g]h\nabla[\hat g]h)|_E^2\,\mathrm{d}y\,\mathrm{d}\mu_k\biggr)^{1/2}.\end{aligned}\tag{67}$$

We treat the first term on the right-hand side of (67) since the second term is higher-order and thus easier to estimate. Once again we estimate high-derivative terms with a factor of (s/t) using Corollary 4.6 and low-derivative terms with a factor of (t/s) using Corollary 4.7. This yields

$$\begin{aligned}&\sum_{|I|+|J|\le N}\biggl(\int_{\Sigma_s\times K}|(\hat g+H)^{-1}|_E^2\,|Z^I\nabla[k]^J(\nabla[\hat g]h\nabla[\hat g]h)|_E^2\,\mathrm{d}y\,\mathrm{d}\mu_k\biggr)^{1/2}\\ &\qquad\lesssim\sum_{\substack{|I_i|+|J_i|\le N\\ |I_2|+|J_2|\le N/2+1}}\biggl(\int_{\Sigma_s\times K}C|Z^{I_1}\nabla[k]^{J_1}\nabla[\hat g]h|\,|Z^{I_2}\nabla[k]^{J_2}\nabla[\hat g]h|_E^2\,\mathrm{d}y\,\mathrm{d}\mu_k\biggr)^{1/2},\end{aligned}\tag{68}$$

where by symmetry we can assume $|I_2|+|J_2|\le\frac{1}{2}N+1$. After using $(s/t)(t/s)=1$ we find

$$\begin{aligned}&\sum_{\substack{|I_i|+|J_i|\le N\\ |I_2|+|J_2|\le N/2+1}}\biggl(\int_{\Sigma_s\times K}C|(s/t)Z^{I_1}\nabla[k]^{J_1}\nabla[\hat g]h|\,|(t/s)Z^{I_2}\nabla[k]^{J_2}\nabla[\hat g]h|_E^2\,\mathrm{d}y\,\mathrm{d}\mu_k\biggr)^{1/2}\\ &\quad\lesssim\sup_{\Sigma_s\times K}\biggl(\sum_{|I_2|+|J_2|\le N/2+1}|(t/s)Z^{I_2}\nabla[k]^{J_2}\nabla[\hat g]h|_E\biggr)\\ &\qquad\qquad\times\sum_{|I_1|+|J_1|\le N}\biggl(\int_{\Sigma_s\times K}|(s/t)Z^{I_1}\nabla[k]^{J_1}\nabla[\hat g]h|_E^2\,\mathrm{d}y\,\mathrm{d}\mu_k\biggr)^{1/2}\end{aligned}$$

$$\lesssim \frac{1}{s^{2\delta(n)-1}}\Bigg(\sum_{|I|+2j\le\tilde\nu+N/2+3}\mathcal{E}[0;\,Z^I(\Delta_k)^j u;\,s]^{1/2}+C_S\sum_{|I|\le\tilde\nu+N/2}C_I^2\Bigg)(s\mathcal{E}_{N+1}(s)^{1/2}+C_S)$$
$$\lesssim \frac{1}{s^{2\delta(n)-2}}(\mathcal{E}_{\tilde\nu+N/2+4}(s)^{1/2}+C_S)(\mathcal{E}_{N+1}(s)^{1/2}+C_S). \tag{69}$$

The term $F^2_{\mu\nu}$ involves the new nonlinearities which are only nonzero when both $\mu,\nu\in\{A,\dots,B\}$. This means we can control $F^2_{\mu\nu}$ as follows:

$$\sum_{|I|+2j\le N}\|F^{2,I,j}_{\mu\nu}\|_{L^2(\Sigma_s\times K)}\lesssim\sup_{\Sigma_s\times K}\Bigg(\sum_{|I_0|\le N}|\nabla[k]^{I_0}\,\mathrm{Riem}[k]|\Bigg)$$
$$\times\sum_{|I_i|+|J_i|\le N}\Bigg(\int_{\Sigma_s\times K}|Z^{I_1}\nabla[k]^{J_1}h|_E^2\,|Z^{I_2}\nabla[k]^{J_2}h|_E^2\,\mathrm{d}y\,\mathrm{d}\mu_k\Bigg)^{1/2}. \tag{70}$$

The Riemann curvature components of k are bounded (since K is compact) which allows us to control the first factor in (70). To estimate the second factor in (70) we follow the same procedure as in $F^1_{\mu\nu}$, by controlling high-derivatives with a factor of (s/t) using Corollary 4.6 and low-derivatives with a compensating factor of (t/s) using Corollary 4.7. The result of this procedure leads to a term controlled by (69).

The final term $F^3_{\mu\nu}$ is a commutator involving the quasilinear perturbation of the principal part of the differential operator. Note first the identity

$$\sum_{|I|+2j\le N}|F^{3,I,j}_{\mu\nu}|_E\le C\sum_{\substack{|I_i|+|J_i|\le N\\|I_2|+|J_2|\le N-1}}|Z^{I_1}\nabla[k]^{J_1}H|_E\,|Z^{I_1}\nabla[k]^{J_1}\nabla[\hat g]\nabla[\hat g]h|_E. \tag{71}$$

Once again we distribute the product $(s/t)(t/s)=1$ across the two terms appearing here depending on where the derivatives land. The term with high-derivatives gains a factor of (s/t) and is controlled using Corollary 4.6 while the term with low-derivatives absorbs a compensating factor of (t/s) and is estimated using Corollary 4.7. Note that when the term $Z^{I_2}\nabla[k]^{J_2}(\nabla[\hat g]\nabla[\hat g]h)$ is estimated in L^∞, the Sobolev inequality will lead to a symmetry boosted energy at order $\tilde\nu+\frac12 N+5$. We eventually obtain

$$\sum_{|I|+2j\le N}\|F^{3,I,j}_{\mu\nu}\|_{L^2(\Sigma_s\times K)}\lesssim\frac{1}{s^{2\delta(n)-2}}(\mathcal{E}_{\tilde\nu+N/2+5}(s)^{1/2}+C_S)(\mathcal{E}_{N+1}(s)^{1/2}+C_S). \tag{72}$$

Putting these all together, inserting the bootstrap assumptions, and using also $C_S^2<\epsilon$, we find

$$\sum_{|I|+2j\le N}\int_4^{s'}\Bigg(\int_{\Sigma_s\times K}\Bigg(\sum_{i=1}^3|F^{i,I,j}|_E^2+|G^{I,j}|_E^2\Bigg)\mathrm{d}y\,\mathrm{d}\mu_k\Bigg)^{1/2}\mathrm{d}s\lesssim\epsilon\int_4^{s'}\frac{1}{s^{2\delta(n)-2}}\,\mathrm{d}s. \tag{73}$$

For integrability we require $2\delta(n)-2>1$, which is equivalent to each of the following:

$$\delta(n)>\tfrac32\quad\text{and}\quad n>8. \tag{74}$$

This implies $n\ge 9$. For the Sobolev estimates we require

$$\tilde\nu+\tfrac12 N+4\le N. \tag{75}$$

Recalling the definition of $\tilde\nu$ given in Lemma 3.2, this holds provided $N>\frac12(n+d+8)$ and N is even.

Consequently for sufficiently small ϵ and by Grönwall's inequality applied to the energy estimate (64) we find $\mathcal{E}_{\nu+1}(s) \le \frac{1}{2}C_1\epsilon$. We have thus obtained a future global solution $h_{\mu\nu} = g_{\mu\nu} - \hat{g}_{\mu\nu}$ to the reduced Einstein equations which clearly satisfies the decay bounds given in Theorem 5.1. □

Remark 5.2. The system (60) contains quadratic nonlinearities F_{AB} and F_{iA} that are new compared to the weak null terms identified in the proof of Minkowski stability in [Lindblad and Rodnianski 2003; 2010] and the proof of zero-mode Kaluza–Klein stability in [Wyatt 2018].

5B. ***Proof of Theorem 1.1.*** We are now in a position to use the results from Theorem 5.1 in order to prove our main result. Take an initial data set $(\mathbb{R}^n \times K, \gamma, \kappa)$ as specified in Theorem 1.1 with smallness conditions (5). We now transform this data into the form required by Theorem 5.1, which is a standard procedure; see for example [Lindblad and Rodnianski 2005]. We first set $((g_0)_{i'j'}, (g_1)_{i'j'}) = (\gamma_{i'j'}, \kappa_{i'j'})$. Diffeomorphism invariance allows us the freedom to choose the lapse and shift. We set the shift to be zero: $X_{i'} = 0$. We choose the lapse to be a smooth function satisfying

$$\begin{aligned} N(r) &= 1, && r \le \tfrac{1}{2}, \\ |N-1| &\lesssim C_S, && \tfrac{1}{2} \le r \le 1, \\ N(r) &= \left(1 - \frac{h_{00}(r^{-1})}{r^{n-2}}\right)^{1/2}, && r \ge 1. \end{aligned}$$

We relate the lapse and shift with the Cauchy data for the reduced equations in Theorem 5.1 by setting $(g_0)_{00} = -N^2$ and $(g_0)_{0i'} = X_{i'}$. The initial data for $(\partial_t N, \partial_t X_{i'}) = ((g_1)_{00}, (g_1)_{0i'})$ is chosen by satisfying $V^\gamma = 0$. This amounts to solving the following equations on $\mathbb{R}^n \times K$:

$$\begin{aligned} N^{-3}((g_1)_{00} + N^2\gamma^{i'j'}\kappa_{i'j'}) &= {g_0}^{i'j'}\Gamma[\hat{e}]^0_{i'j'}, \\ -N^{-2}\gamma^{i'j'}(g_1)_{0j'} - N^{-1}\gamma^{i'j'}\partial_{j'}N + \gamma^{j'k'}\Gamma^{i'}_{j'k'}[\gamma] &= {g_0}^{j'k'}\Gamma[\hat{e}]^{i'}_{j'k'}. \end{aligned} \tag{76}$$

We have now brought the initial data of Theorem 1.1 into the form of Theorem 5.1. It remains to check that our assumptions on the lapse and shift are compatible with smallness conditions (58). To do this, recall the final sentence of Theorem 2.15. This implies that

$$\begin{aligned} \int_{\{r\ge 1\}\cap\mathbb{R}^n} |\nabla[g_0]^I(-N^2 - \eta_{00})|^2 \, dx &\le \int_{\{r\ge 1\}\cap\mathbb{R}^n} C_S^2 (r^{-(n-2)-|I|})^2 r^{n-1} \, dr \, d^{n-1}\omega_{\mathbb{S}^{n-1}} \\ &\le C_S^2 \int_{\{r\ge 1\}\cap\mathbb{R}^n} r^{-(n-3)-2|I|} \, dr \, d^{n-1}\omega_{\mathbb{S}^{n-1}} \le C C_S^2. \end{aligned}$$

By inverting the expressions (76) for $(\partial_t N, \partial_t X_{i'})$ it is clear that the smallness conditions (58) are satisfied. Furthermore it is a standard result, see for example [Choquet-Bruhat 2009, Theorem 8.3], that the future global solution constructed in Theorem 5.1 is in fact also a solution to the full Einstein equations.

Finally, note that the solution found in Theorem 5.1 is only defined to the future $t \ge 4$. Nonetheless, by time translation, we can treat the initial data as being on $\{t = 0\}$ instead of $\{t = 4\}$, so that Theorem 5.1 ensures the existence of a solution for $t \ge 0$. By time reversibility for the Einstein equation (and the reduced Einstein equation), we similarly obtain a solution for $t \le 0$. Thus, we can construct the global solution required in Theorem 1.1.

It now remains to prove the causal geodesic completeness of $(\mathbb{R}^{1+n} \times K, g)$.

Globally, the metrics g and $\hat{g}$ are very close, in the sense that, with respect to a basis constructed from the X_i and an orthonormal basis on K, their components vanish to order ϵ globally. Denote from now onwards $T = \mathrm{d}t$. This is a globally timelike one-form such that $|g(T, T) - 1| \lesssim \epsilon$. Thus, $g - 2TT$ defines a Riemannian metric. (Note that in the introduction, we used the slightly different Euclidean metric $\hat{g} - 2TT$.) Within this proof, we define, for a vector u, the Euclidean length to be

$$|u|^2 = u^\alpha u^\beta (g_{\alpha\beta} + 2T_\alpha T_\beta). \tag{77}$$

Note that the fact that g and $\hat{g}$ are very close implies the equivalence $|u|_E \sim |u|$.

Consider a causal geodesic γ that is affinely parametrised by λ. For the remainder of this paragraph, let $t = t(\lambda)$ denote the value of the Cartesian coordinate t at the point $\gamma(\lambda)$. By rescaling, we may assume that $\mathrm{d}t/\mathrm{d}\lambda = 1$ at $t = 0$. Let v be the (artificial, Euclidean) speed defined by $v \geq 0$ and

$$v^2 = \left| \frac{\mathrm{d}\gamma^\alpha}{\mathrm{d}\lambda} \right|^2. \tag{78}$$

Since g and $\hat{g}$ are very close, the rate of change in the t direction cannot be (much) greater than the Euclidean speed, i.e.,

$$\left| \frac{\mathrm{d}t}{\mathrm{d}\lambda} \right| = \left| \frac{\mathrm{d}\gamma^0}{\mathrm{d}\lambda} \right| \lesssim v.$$

On the other hand, since γ is causal, the component of $\mathrm{d}\gamma/\mathrm{d}\lambda$ in the T direction cannot vanish faster than the length of the component in the orthogonal spatial directions, and the square of the Euclidean velocity is the sum of the squares of the lengths of the T components and the orthogonal spatial component (up to order ϵ multiplicative errors); thus

$$\left| \frac{\mathrm{d}t}{\mathrm{d}\lambda} \right| = \left| \frac{\mathrm{d}\gamma^0}{\mathrm{d}\lambda} \right| \gtrsim v.$$

In particular, there is the equivalence $|\mathrm{d}t/\mathrm{d}\lambda| \sim v$.

Since $\nabla[g]g = 0$ and $\nabla[g]_{\mathrm{d}\gamma/\mathrm{d}\lambda} \mathrm{d}\gamma/\mathrm{d}\lambda = 0$, the rate of change of the velocity is given by

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} v^2 = 4\left(\frac{\mathrm{d}\gamma^\alpha}{\mathrm{d}\lambda} T_\alpha \right)\left(\frac{\mathrm{d}\gamma^\beta}{\mathrm{d}\lambda} \nabla[g]_{\mathrm{d}\gamma/\mathrm{d}\lambda} T_\beta \right). \tag{79}$$

Since the absolute value of $(\mathrm{d}\gamma^\alpha/\mathrm{d}\lambda)T_\alpha = \mathrm{d}t/\mathrm{d}\lambda$ and the Euclidean length of $\mathrm{d}\gamma/\mathrm{d}\lambda$ are dominated by v,

$$\frac{\mathrm{d}v}{\mathrm{d}\lambda} \lesssim |\nabla[g]_{\mathrm{d}\gamma/\mathrm{d}\lambda} T| v. \tag{80}$$

The $\nabla[g]T$ can be expanded in terms of g and $\nabla[\hat{g}]g$. Both of these have norms that decay as $t^{-\delta(n)}$ due to (74). Thus,

$$\frac{\mathrm{d}v}{\mathrm{d}\lambda} \lesssim \epsilon t^{-\delta(n)} v^2. \tag{81}$$

Thus, for ϵ sufficiently small, a simple bootstrap argument shows that $v \sim 1$ along all of γ, and hence $\mathrm{d}t/\mathrm{d}\lambda \sim 1$. In particular, t is monotone along γ.

Let t_{sup} be the supremum of the t values that are achieved along γ. For contradiction, suppose $t_{\mathrm{sup}} < \infty$. Since the length of the spatial component of $\mathrm{d}\gamma/\mathrm{d}\lambda$ is also uniformly equivalent to v, and hence to $\mathrm{d}t/\mathrm{d}\lambda$, it follows that, as $t \nearrow t_{\mathrm{sup}}$, the curve γ has a limit in $\mathbb{R}^{1+n} \times K$. Because of the global bounds on g and its derivatives, by the standard Picard–Lindelöf theorem for ODEs, the curve γ must smoothly extend through this limiting point, contradicting the definition of t_{sup}. Thus, $t_{\mathrm{sup}} = \infty$. The only other way in which γ can be future incomplete is if t diverges to ∞ in a finite λ interval, but this is also impossible, since $\mathrm{d}t/\mathrm{d}\lambda \sim 1$. By time symmetry, the same argument holds in the past. Thus, any causal geodesic is complete.

The previous construction shows that every causal geodesic goes through each level set of t. Thus, the level sets of t are Cauchy surfaces, and $(\mathbb{R}^{1+n} \times K, g)$ is globally hyperbolic.

Acknowledgements

Andersson thanks the Mathematics Department at Harvard University for hospitality and support during the spring term of 2011, when the initial investigation of this topic was carried out. Andersson is also grateful to the YMSC at Tsinghua University, Beijing, for support and hospitality during part of the work of this paper, and thanks Pin Yu and Po-Ning Chen for helpful discussions. The authors Andersson, Blue, and Wyatt thank the hospitality of the Institut Mittag-Leffler, Djursholm, Sweden during the winter semester of 2019. Wyatt is supported by MIGSAA, a Centre for Doctoral Training funded by EPSRC, the Scottish Funding Council, Heriot–Watt University, and the University of Edinburgh. Yau is supported by the Black Hole Initiative, award 61497, at Harvard University and by NSF grant DMS-1607871 "Analysis, Geometry and Mathematical Physics". We thank the journal reviewers for their thoughtful comments, which helped us improve the paper.

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Received 30 Nov 2020. Revised 25 Feb 2022. Accepted 25 Mar 2022.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 16 No. 9 2023