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THE PRESCRIBED CURVATURE PROBLEM FOR ENTIRE HYPERSURFACES IN MINKOWSKI SPACE

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We prove three results in this paper: First, we prove, for a wide class of functions $\varphi \in C^2(\mathbb{S}^{n-1})$ and $\psi(X, \nu) \in C^2(\mathbb{R}^{n+1} \times \mathbb{H}^n)$, there exists a unique, entire, strictly convex, spacelike hypersurface $\mathcal{M}_u$ satisfying $\sigma_k(\kappa[\mathcal{M}_u]) = \psi(X, \nu)$ and $u(x) \to |x| + \varphi(x/|x|)$ as $|x| \to \infty$. Second, when $k = n-1, n-2$, we show the existence and uniqueness of an entire, k-convex, spacelike hypersurface $\mathcal{M}_u$ satisfying $\sigma_k(\kappa[\mathcal{M}_u]) = \psi(x, u(x))$ and $u(x) \to |x| + \varphi(x/|x|)$ as $|x| \to \infty$. Last, we obtain the existence and uniqueness of entire, strictly convex, downward translating solitons $\mathcal{M}_u$ with prescribed asymptotic behavior at infinity for σ_k curvature flow equations. Moreover, we prove that the downward translating solitons $\mathcal{M}_u$ have bounded principal curvatures.

1. Introduction

Let $\mathbb{R}^{n,1}$ be the Minkowski space with the Lorentzian metric

$$ds^2 = \sum_{i=1}^{n} dx_i^2 - dx_{n+1}^2.$$

In this paper, we will devote ourselves to the study of spacelike hypersurfaces with prescribed σ_k curvature in Minkowski space $\mathbb{R}^{n,1}$. Here, σ_k is the k-th elementary symmetric polynomial, i.e.,

$$\sigma_k(\kappa) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \kappa_{i_1} \cdots \kappa_{i_k}.$$

Any such hypersurface $\mathcal{M}$ can be written locally as a graph of a function $x_{n+1} = u(x),\ x \in \mathbb{R}^n$, satisfying the spacelike condition

$$|Du| < 1. \tag{1-1}$$

More precisely, we focus on the equation

$$\sigma_k(\kappa[\mathcal{M}_u]) = \psi(X, \nu), \tag{1-2}$$

where $X = (x, u(x))$ is the position vector of $\mathcal{M}_u = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$, $\nu = (Du, 1)/\sqrt{1 - |Du|^2}$ is the future-directed unit normal lying on the hyperboloid $\mathbb{H}^n$, and $\kappa[\mathcal{M}_u] = (\kappa_1, \ldots, \kappa_n)$ is the set of principal curvatures of $\mathcal{M}_u$. Thus (1-2) can be rewritten as

$$\sigma_k(\kappa[\mathcal{M}_u]) = \psi(x, u(x), Du). \tag{1-3}$$

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Notice that the functions ψ in the right-hand sides of (1-2) and (1-3) are different. Slightly extending the notation, we use the same symbol here.

The classical Minkowski problem asks for the construction of a strictly convex compact surface Σ whose Gaussian curvature is a given positive function $f(\nu(X))$, where $\nu(X)$ denotes the normal to Σ at X. This problem has been discussed by Nirenberg [1953], Pogorelov [1978], and Cheng and Yau [1976]. The general problem of finding strictly convex hypersurfaces with prescribed surface area measures is called the Christoffel–Minkowski problem. This type of problem can be reduced to a fully nonlinear equation of the form (1-2). It may be traced back to Aleksandrov [1942], who established the problem of prescribing zeroth curvature measure. The prescribed curvature measure problem in convex geometry has been extensively studied by Aleksandrov [1956], Pogorelov [1953], Guan, Lin, and Ma [Guan et al. 2009], and Guan, Li, and Li [Guan et al. 2012]. A more general form of the prescribed curvature measure problem can be expressed as (1-3). In particular, Guan, Ren, and Wang [Guan et al. 2015] solved this problem in Euclidean space for convex hypersurfaces. Other related studies and references about the Minkowski problem may be found in [Bakelman and Kantor 1974; Caffarelli et al. 1986; 1988; Guan and Guan 2002; Oliker 1984; Treibergs and Wei 1983].

In Minkowski space, there have been fruitful results on the prescribed curvature problem for spacelike entire hypersurfaces. In [Treibergs 1982] and [Choi and Treibergs 1990], the authors obtained the existence of entire hypersurfaces with constant mean curvature. Li [1995] then extended [Treibergs 1982] and proved the existence of constant Gauss curvature hypersurfaces with Gauss image a unit ball. The existence of constant Gauss curvature hypersurfaces with Gauss image the convex hull in B_1 of an arbitrary closed set $\mathcal{F} \subset \mathbb{S}^{n-1}$ was proved by Guan, Jian, and Schoen [Guan et al. 2006a] and Bayard and Schnürer [2009]. Later, [Bayard 2006] and [Bayard and Delanoë 2009] considered the prescribed scalar curvature problem for entire, spacelike hypersurfaces under different settings. More recently, the second and third authors showed the existence of entire, spacelike, constant σ_k curvature hypersurfaces in [Wang and Xiao 2022].

Our goal here is to construct entire, spacelike hypersurfaces satisfying (1-2) in Minkowski space. The main results of this paper follow.

The first result is to construct entire, strictly convex, spacelike hypersurfaces satisfying (1-2).

Theorem 1. *Suppose φ is a C^2 function defined on $\mathbb{S}^{n-1}$, i.e., $\varphi \in C^2(\mathbb{S}^{n-1})$, $\psi(X,\nu) \in C^2(\mathbb{R}^{n+1} \times \mathbb{H}^n)$ is a positive function, and $c_1 \geqslant \psi(X,\nu) \geqslant c_2$ for some positive constants c_1, c_2. We further assume that $\psi_{x_{n+1}} \geqslant 0$ (or $\psi_u \geqslant 0$). If either $\psi^{-1/k}(X,\nu)$ is locally strictly convex with respect to X for any ν or ψ only depends on ν, then there exists a unique, entire, strictly convex, spacelike hypersurface $\mathcal{M}_u = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$ satisfying (1-2). Moreover, as $|x| \to \infty$,*

$$u(x) \to |x| + \varphi\left(\frac{x}{|x|}\right). \tag{1-4}$$

Remark 2. Indeed, from the proof of the C^2 global estimate Lemma 10, we can see that the assumption that $\psi(X,\nu)$ does not depend on X can be replaced by a weaker assumption; that is, $\psi^{-1/k}(X,\nu)$ is convex with respect to X, and the corresponding form $\psi(x,u,Du)$ does not depend on $|x|$.

Remark 3. In the proof, we only can see that the hypersurface $\mathcal{M}_u$ we constructed is convex. In order to say it's strictly convex, we need to apply the constant rank theorem (see [Guan et al. 2006b, Theorem 1.2; Wang and Xiao 2022, Theorem 27]) and the splitting theorem (see [Wang and Xiao 2022, Theorem 28]) to obtain that, if $\mathcal{M}_u$ has a degenerate point in the interior, then $\mathcal{M}_u = \mathcal{M}^l \times \mathbb{R}^{n-l}$, where $\mathcal{M}^l \subset \mathbb{R}^{l,1}$ is a strictly convex, spacelike hypersurface. This contradicts (1-4).

Before stating our second result, we need the following definition.

Definition 4. A C^2 regular hypersurface $\mathcal{M} \subset \mathbb{R}^{n,1}$ is k-convex if the principal curvatures of $\mathcal{M}$ at $X \in \mathcal{M}$ satisfy $\kappa[X] \in \Gamma_k$ for all $X \in \mathcal{M}$, where Γ_k is the Gårding cone

$$\Gamma_k = \{\kappa \in \mathbb{R}^n \mid \sigma_m(\kappa) > 0,\ m = 1, \dots, k\}.$$

Using the newly developed methods in [Ren and Wang 2019; 2023], we are able to generalize results in [Bayard 2006] to prove the following.

Theorem 5. *Suppose φ is some C^2 function defined on $\mathbb{S}^{n-1}$ and $\psi(x, u(x)) \in C^2(\mathbb{R}^{n+1})$ is a positive function satisfying $c_1 \geqslant \psi(x, u(x)) \geqslant c_2$ for $c_1, c_2 > 0$. We further assume that $k = n-1, n-2$ and $\psi_u \geqslant 0$. Then there exists a unique, k-convex, spacelike hypersurface $\mathcal{M}_u = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$ satisfying*

$$\sigma_k(\kappa[\mathcal{M}_u]) = \psi(x, u(x)). \tag{1-5}$$

Moreover, as $|x| \to \infty$,

$$u(x) \to |x| + \varphi\left(\frac{x}{|x|}\right). \tag{1-6}$$

Remark 6. Notice that unlike in the strictly convex case (Theorem 1), in this theorem, we only prove the existence result for the case when ψ depends on x and $u(x)$ (ψ is independent of Du). This is because the proofs of Lemma 12 (C^2 boundary estimates for k-convex hypersurfaces) and Lemma 15 (C^1 local estimates for k-convex hypersurfaces) crucially rely on the fact that ψ is independent of Du.

Now, let's consider the σ_k curvature flow with a forcing term in Minkowski space:

$$\frac{dX}{dt} = -\left(\mathcal{C} - \frac{\sigma_k^{1/k}(\kappa[\mathcal{M}_u])}{\binom{n}{k}^{1/k}}\right)\nu, \tag{1-7}$$

where $\kappa[\mathcal{M}_u] \in \Gamma_k$. This can be rewritten as the equation for the height function u:

$$\frac{u_t}{\sqrt{1-|Du|^2}} = \frac{\sigma_k^{1/k}(\kappa[\mathcal{M}_u])}{\binom{n}{k}^{1/k}} - \mathcal{C}. \tag{1-8}$$

The downward translating soliton to (1-8) is of the form

$$u(x,t) = u(x) - t, \tag{1-9}$$

where $u(x)$ satisfies

$$\left(\frac{\sigma_k}{\binom{n}{k}}\right)^{1/k}(\kappa[\mathcal{M}_u]) = \mathcal{C} - \frac{1}{\sqrt{1-|Du|^2}}. \tag{1-10}$$

Equation (1-10) can be viewed as the "degenerate" type of (1-2). In this case, we prove the following.

Theorem 7. *Suppose φ is a C^2 function defined on $\mathbb{S}^{n-1}_{\tilde{\mathcal{C}}} := \{x \in \mathbb{R}^n \mid |x| = \tilde{\mathcal{C}}\}$, where $\tilde{\mathcal{C}} = \sqrt{1-(1/\mathcal{C})^2}$ and $\mathcal{C} > 1$ is a constant. There exists a unique, strictly convex solution $u : \mathbb{R}^n \to \mathbb{R}$ of* (1-10) *such that, as $|x| \to \infty$,*

$$u(x) \to \tilde{\mathcal{C}}|x| - \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}\log|x| + \varphi\left(\tilde{\mathcal{C}}\frac{x}{|x|}\right). \tag{1-11}$$

Moreover, $\mathcal{M}_u = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$ has bounded principal curvatures.

When $k = 1$, (1-10) has been studied in [Ju et al. 2010; Spruck and Xiao 2016]; when $k = 2$, (1-10) has been studied in [Bayard 2023].

Remark 8. Under our assumptions on ψ, we can see that the linearized operators of (1-2), (1-5), and (1-10) satisfy the maximum principle. Therefore, the uniqueness properties in Theorem 1, 5, and 7 follow from the maximum principle directly.

The rest of this paper is organized as follows. In Section 2, we introduce some basic formulas and notation. The solvability of (1-2) and (1-5) on a bounded domain (Dirichlet problem) is discussed in Section 3. We prove the local C^1 and C^2 estimates for solutions of (1-2) and (1-5) in Section 4. This leads to the completion of the proof of our first two main results, Theorems 1 and 5, in Section 5. Section 6 and Section 7 are devoted to Theorem 7. In particular, in Section 6, we study the radially symmetric solution to (1-10), this solution will be used to construct barrier functions in Section 7. We finish the proof of Theorem 7 in Section 7.

2. Preliminaries

In this paper, we will follow notation in [Wang and Xiao 2022]. For the readers convenience, we will include some basic notation and formulas in this section. For more details, one can refer to [Choi and Treibergs 1990; Li 1995]. Readers who are already familiar with calculations in Minkowski space can skip this section.

We first recall that the Minkowski space $\mathbb{R}^{n,1}$ is $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric

$$ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2.$$

Throughout this paper, $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^{n,1}$.

2.1. ***Vertical graphs in*** $\mathbb{R}^{n,1}$**.** A spacelike hypersurface $\mathcal{M}$ in $\mathbb{R}^{n,1}$ is a codimension 1 submanifold whose induced metric is Riemannian. Locally, $\mathcal{M}$ can be written as the graph of a function, i.e.,

$$\mathcal{M}_u = \{X = (x, u(x)) \mid x \in \mathbb{R}^n\},$$

satisfying the spacelike condition (1-1). We let $E = (0, \dots, 0, 1)$. Then the height function of $\mathcal{M}$ is $u(x) = -\langle X, E\rangle$. It's easy to see that the induced metric and second fundamental form of $\mathcal{M}$ are given by

$$g_{ij} = \delta_{ij} - D_{x_i}uD_{x_j}u, \quad 1 \leqslant i, j \leqslant n,$$

and

$$h_{ij} = \frac{u_{x_ix_j}}{\sqrt{1-|Du|^2}},$$

respectively, while the timelike unit normal vector field to $\mathcal{M}$ is

$$\nu = \frac{(Du, 1)}{\sqrt{1-|Du|^2}},$$

where $Du = (u_{x_1}, \dots, u_{x_n})$ and $D^2u = (u_{x_i x_j})$ denote the ordinary gradient and Hessian, respectively, of u. By a straightforward calculation, we have that the principle curvatures of $\mathcal{M}$ are eigenvalues of the symmetric matrix $A = (a_{ij})$ given by

$$a_{ij} = \frac{1}{w}\gamma^{ik} u_{kl} \gamma^{lj},$$

where $\gamma^{ik} = \delta_{ik} + u_i u_k/(w(1+w))$ and $w = \sqrt{1-|Du|^2}$. Note that (γ^{ij}) is invertible with inverse $(\gamma_{ij}) = \delta_{ij} - u_i u_j/(1+w)$, which is the square root of (g_{ij}).

Let $\mathcal{S}$ be the vector of $n \times n$ symmetric matrices and

$$\mathcal{S}_k = \{A \in \mathcal{S} \mid \lambda(A) \in \Gamma_k\},$$

where $\lambda(A) = (\lambda_1, \dots, \lambda_n)$ is the set of eigenvalues of A. Define a function F by

$$F(A) = \sigma_k(\lambda(A)), \quad A \in \mathcal{S}_k.$$

Then (1-3) can be written as

$$F\Big(\frac{1}{w}\gamma^{ik} u_{kl} \gamma^{lj}\Big) = \psi(x, u(x), Du). \tag{2-1}$$

Throughout this paper, we write

$$F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A) \quad \text{and} \quad F^{ij,kl} = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}.$$

Now, let $\{\tau_1, \tau_2, \dots, \tau_n\}$ be a local orthonormal frame on $T\mathcal{M}$. We will use ∇ to denote the induced Levi-Civita connection on $\mathcal{M}$. For a function v on $\mathcal{M}$, we write $v_i = \nabla_{\tau_i} v$, $v_{ij} = \nabla_{\tau_i}\nabla_{\tau_j} v$, etc. In particular, we have

$$|\nabla u| = \sqrt{g^{ij} u_{x_i} u_{x_j}} = \frac{|Du|}{\sqrt{1-|Du|^2}}.$$

Using normal coordinates, we also need the following well-known fundamental equations for a hypersurface $\mathcal{M}$ in $\mathbb{R}^{n,1}$:

$$\begin{aligned} X_{ij} &= h_{ij}\nu && \text{(Gauss formula)},\\ (\nu)_i &= h_{ij}\tau_j && \text{(Weigarten formula)},\\ h_{ijk} &= h_{ikj} && \text{(Codazzi equation)},\\ R_{ijkl} &= -(h_{ik}h_{jl} - h_{il}h_{jk}) && \text{(Gauss equation)}, \end{aligned} \tag{2-2}$$

and the Ricci identity

$$h_{ijkl} = h_{ijlk} + h_{mj}R_{imlk} + h_{im}R_{jmlk} = h_{klij} - (h_{mj}h_{il} - h_{ml}h_{ij})h_{mk} - (h_{mj}h_{kl} - h_{ml}h_{kj})h_{mi}. \tag{2-3}$$

2.2. ***The Gauss map.*** Let $\mathcal{M}$ be an entire, strictly convex, spacelike hypersurface, and let $\nu(X)$ be the timelike unit normal vector to $\mathcal{M}$ at X. It's well known that the hyperbolic space $\mathbb{H}^n(-1)$ is canonically embedded in $\mathbb{R}^{n,1}$ as the hypersurface

$$\langle X, X\rangle = -1, \quad x_{n+1} > 0.$$

By translation parallel to the origin, we can regard $\nu(X)$ as a point in $\mathbb{H}^n(-1)$. In this way, we define the Gauss map

$$G : \mathcal{M} \to \mathbb{H}^n(-1), \quad X \mapsto \nu(X).$$

Next, let's consider the support function of $\mathcal{M}$. We write

$$v := \langle X, \nu\rangle = \frac{1}{\sqrt{1-|Du|^2}}\Big(\sum_i x_i \frac{\partial u}{\partial x_i} - u\Big).$$

Let $\{e_1, \dots, e_n\}$ be an orthonormal frame on $\mathbb{H}^n$. We will also write $\{e_1^*, \dots, e_n^*\}$ for the pull-back of e_i by the Gauss map G. Similarly to the convex geometry case, we write

$$\Lambda_{ij} = v_{ij} - v\delta_{ij},$$

which is the hyperbolic Hessian. Here the v_{ij} denote the covariant derivatives with respect to the hyperbolic metric.

Let $\overline{\nabla}$ be the connection of the ambient space. Then we have

$$X = \sum_i v_i e_i - v\nu$$

and

$$\overline{\nabla}_{e_j^*} X = \sum_k (e_j(v_k)e_k + v_k \overline{\nabla}_{e_j} e_k) - v_j \nu - v\overline{\nabla}_{e_j}\nu = \sum_k \Lambda_{kj} e_k.$$

Note also that

$$g_{ij} = \langle \overline{\nabla}_{e_i^*} X, \overline{\nabla}_{e_j^*} X\rangle = \sum_k \Lambda_{ik}\Lambda_{kj} \tag{2-4}$$

and

$$h_{ij} = \langle \overline{\nabla}_{e_i^*} X, \overline{\nabla}_{e_j}\nu\rangle = \Lambda_{ij}. \tag{2-5}$$

This implies that the eigenvalues of the hyperbolic Hessian are equal to the curvature radius of $\mathcal{M}$. Therefore, (1-2) can be written as

$$F(v_{ij} - v\delta_{ij}) = \frac{1}{\psi(X, \nu)}, \tag{2-6}$$

where $F(A) = (\sigma_n/\sigma_{n-k})(\lambda(A))$. Moreover, it is clear that

$$(\overline{\nabla}_{e_j}\overline{\nabla}_{e_i}\nu)^{\perp} = \delta_{ij}\nu, \tag{2-7}$$

which yields, for $k = 1, 2, \dots, n+1$,

$$\nabla_{e_j}\nabla_{e_i} x_k = x_k \delta_{ij}, \tag{2-8}$$

where x_k is the coordinate function.

2.3. ***Legendre transform.*** Suppose $\mathcal{M}$ is an entire, strictly convex, spacelike hypersurface. Then $\mathcal{M}$ is the graph of a convex function

$$x_{n+1} = -\langle X, E\rangle = u(x_1, \dots, x_n),$$

where $E = (0, \dots, 0, 1)$. We introduce the Legendre transform

$$\xi_i = \frac{\partial u}{\partial x_i}, \quad u^* = \sum x_i \xi_i - u.$$

Next, we calculate the first and second fundamental forms in terms of ξ_i. Since it is well known that

$$\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = \left(\frac{\partial^2 u^*}{\partial \xi_i \partial \xi_j}\right)^{-1},$$

we have that the first and the second fundamental forms can be rewritten as

$$g_{ij} = \delta_{ij} - \xi_i \xi_j \quad \text{and} \quad h_{ij} = \frac{u^{*ij}}{\sqrt{1-|\xi|^2}},$$

where (u^{*ij}) denotes the inverse matrix of (u^*_{ij}) and $|\xi|^2 = \sum_i \xi_i^2$. Now, let W be the Weingarten matrix of $\mathcal{M}$. Then

$$(W^{-1})_{ij} = \sqrt{1-|\xi|^2}\, g_{ik} u^*_{kj}.$$

From the discussion above, we can see that if $\mathcal{M}_u = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$ is an entire, strictly convex, spacelike hypersurface satisfying $\sigma_k(\kappa[\mathcal{M}]) = \psi$, then the Legendre transform of u, denoted by u^*, satisfies

$$F(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = \frac{\sigma_n}{\sigma_{n-k}}(\kappa^*[w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}]) = \frac{1}{\psi}. \tag{2-9}$$

Here, $w^* = \sqrt{1-|\xi|^2}$, and $(\gamma^*_{ij}) = \delta_{ij} - \xi_i\xi_j/(1+w^*)$ is the square root of the matrix (g_{ij}).

3. The Dirichlet problem

We will divide this section into two subsections. In the first subsection, we only consider the convex solution to (1-2). In the second subsection, we restrict ourselves to the cases when $k = n-1$ ($n \geqslant 3$), $n-2$ ($n \geqslant 5$), and we will consider the k-convex, spacelike solution to (1-5). When $k = 2$, this problem has been studied in [Bayard 2003; Urbas 2003].

3.1. ***Dirichlet problem for $1 \leqslant k \leqslant n$.*** Recall that in [Wang and Xiao 2022] we proved the following:

Lemma 9. *Let $\mathcal{F} \subset \mathbb{S}^{n-1}$, $\widetilde{F} = \mathrm{Conv}(\mathcal{F})$, and u^* be a solution of*

$$\begin{cases} \hat{F}(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = \binom{n}{k}^{-1/k} & \text{in } \widetilde{F}, \\ u^* = \varphi & \text{on } \partial\widetilde{F}, \end{cases} \tag{3-1}$$

where $\hat{F}(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = (\sigma_n/\sigma_{n-k})^{1/k}(\kappa^*[w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}])$. Then the Legendre transform of u^*, denoted by u, satisfies, when $x/|x| \in \mathcal{F}$,

$$u(x) - |x| \to -\varphi\left(\frac{x}{|x|}\right) \quad \textit{uniformly as } |x| \to \infty. \tag{3-2}$$

Notice that the proof of the above lemma is independent of the equation that the function u^* satisfies. Therefore, adapting the above lemma to the settings in this paper, this lemma tells us that if a strictly convex function $u^*: B_1 \to \mathbb{R}$ satisfies $u^*(\xi) = -\varphi(\xi)$ for $\xi \in \partial B_1$, then the Legendre transform of u^*, denoted by u, satisfies $u(x) \to |x| + \varphi(x/|x|)$ as $|x| \to \infty$. Moreover, by [Wang and Xiao 2022, Theorem 4], there exist two solutions $\underline{u}$ and $\bar{u}$ such that

$$\sigma_k(\kappa[\mathcal{M}_{\underline{u}}]) = c_1, \quad \sigma_k(\kappa[\mathcal{M}_{\bar{u}}]) = c_2,$$

and, as $|x| \to \infty$,

$$\underline{u}(x) - |x|,\ \bar{u}(x) - |x| \to \varphi\left(\frac{x}{|x|}\right).$$

Here, the constants c_1, c_2 are the same as those in Theorem 1. Throughout this paper, we will denote the Legendre transforms of $\underline{u}$ and $\bar{u}$ by $\underline{u}^*$ and $\bar{u}^*$, respectively. It's easy to see that $\underline{u}^*$ and $\bar{u}^*$ are the super- and subsolutions of (2-9).

Combining the discussions above with Section 2, we conclude that in order to find an entire, strictly convex solution u of (1-3), we only need to solve the equation

$$\begin{cases} F(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = \psi^* & \text{in } B_1, \\ u^* = -\varphi & \text{on } \partial B_1, \end{cases} \tag{3-3}$$

where

$$\psi^*(\xi, u^*, Du^*) = \frac{1}{\psi(x, u, Du)} = \frac{1}{\psi(Du^*, \xi\cdot Du^* - u^*, \xi)}$$

and

$$F(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = \frac{\sigma_n}{\sigma_{n-k}}(\kappa^*[w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}]).$$

Note that, by our assumption in Theorem 1, we have

$$\psi^*_{u^*} = \frac{\psi_u}{\psi^2} \geqslant 0. \tag{3-4}$$

Thus, (3-3) possesses the maximum principle.

Notice that (3-3) is degenerate on ∂B_1. Therefore, we will consider the approximate equation

$$\begin{cases} F(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = \psi^* & \text{in } B_r, \\ u^* = \underline{u}^* & \text{on } \partial B_r, \end{cases} \tag{3-5}$$

where $0 < r < 1$.

By the continuity method, we know that, if we can obtain a prior estimates up to the second order, then we can show (3-5) has a unique, strictly convex solution u^{r*}. In view of the super- and subsolutions $\underline{u}^*$ and $\bar{u}^*$, the C^0 estimates are easy to obtain. The C^1 estimates can be derived by following the argument in Section 9.2 of [Ren et al. 2020]. The C^2 estimate on the boundary can be derived from Lemma 27 in [Ren et al. 2020] and the argument of Bo Guan [Guan 1999]. In the following, we only need to consider the global C^2 estimate.

Let $\mathcal{M}_u = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$ be a strictly convex, spacelike hypersurface, $v = \langle X, \nu\rangle$ be the support function of $\mathcal{M}_u$, and u^* be the Legendre transform of u. From Sections 2.2 and 2.3, we know that $\lambda[v_{ij} - v\delta_{ij}] = \kappa^*[w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}]$. Therefore, studying the global C^2 estimate of (3-5) is equivalent to studying the global C^2 estimate of (2-6).

For our convenience, we will consider the equation

$$\hat{F}(\Lambda) = \left(\frac{\sigma_n}{\sigma_{n-k}}\right)^{1/k}(\Lambda) = \tilde{\psi}, \tag{3-6}$$

where $\Lambda = (\Lambda_{ij}) = (v_{ij} - v\delta_{ij})$, $\tilde{\psi} = \psi^{-1/k}(X, \nu)$, and the v_{ij} are the covariant derivatives with respect to the hyperbolic metric.

We will write $\lambda[\Lambda] = (\lambda_1, \lambda_2, \dots, \lambda_n)$ for the set of eigenvalues of the matrix Λ. We define the Riemann curvature tensor

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal frame on $\mathbb{H}^n$; we use the notation

$$R_{ijkl} = R(e_i, e_j)e_k \cdot e_l \quad \text{and} \quad R^l_{ijk} = g^{lp} R_{ijkp}.$$

Then the commutation formulas are

$$v_{ijk} - v_{ikj} = R^l_{jki} v_l \quad \text{and} \quad v_{ijkl} - v_{ijlk} = R^m_{kli} v_{jm} + R^m_{klj} v_{im}.$$

Note that, in hyperbolic space, we have

$$R_{ijkl} = g_{ik} g_{jl} - g_{il} g_{jk}.$$

Therefore, given an orthonormal frame on $\mathbb{H}^n$, we obtain the geometric formulas

$$\Lambda_{ijk} = \Lambda_{ikj} \quad \text{and} \quad \Lambda_{lkji} - \Lambda_{lkij} = v_{lkji} - v_{lkij} = -v_{lj}\delta_{ik} + v_{li}\delta_{jk} - v_{jk}\delta_{il} + v_{ik}\delta_{jl}. \tag{3-7}$$

Lemma 10. *Let v be the solution of* (3-6) *in a bounded domain $U \subset \mathbb{H}^n$. Denote the set of eigenvalues of $(v_{ij} - v\delta_{ij})$ by $\lambda[v_{ij} - v\delta_{ij}] = (\lambda_1, \dots, \lambda_n)$. Then*

$$\lambda_{\max} \leqslant \max\{C, \lambda|_{\partial U}\},$$

where $\lambda_{\max} = \max\{\lambda_1, \dots, \lambda_n\}$ and C is a positive constant only depending on U and $\tilde{\psi}$.

Proof. Set

$$M = \max_{P \in \bar{U}} \max_{\substack{|\xi|=1 \\ \xi \in T_P \mathbb{H}^n}} (\log \Lambda_{\xi\xi} + N x_{n+1}),$$

where x_{n+1} is the coordinate function. Without loss of generality, we assume M is achieved at an interior point $P_0 \in U$ for some direction ξ_0. Chose an orthonormal frame $\{e_1, \dots, e_n\}$ around P_0 such that $e_1(P_0) = \xi_0$ and $\Lambda_{ij}(P_0) = \lambda_i \delta_{ij}$.

Now, let's consider the test function

$$\phi = \log \Lambda_{11} + N x_{n+1}.$$

At its maximum point P_0, we have

$$0 = \phi_i = \frac{\Lambda_{11i}}{\Lambda_{11}} + N(x_{n+1})_i, \tag{3-8}$$

$$0 \geqslant \phi_{ii} = \frac{\Lambda_{11ii}}{\Lambda_{11}} - \frac{\Lambda^2_{11i}}{\Lambda^2_{11}} + N(x_{n+1})_{ii}. \tag{3-9}$$

Note that $(x_{n+1})_{ij} = x_{n+1}\delta_{ij}$; thus

$$\hat{F}^{ii}\phi_{ii} = \frac{\hat{F}^{ii}\Lambda_{11ii}}{\Lambda_{11}} - \frac{\hat{F}^{ii}\Lambda_{11i}^2}{\Lambda_{11}^2} + Nx_{n+1}\sum_i \hat{F}^{ii}. \tag{3-10}$$

In view of (3-7),

$$\Lambda_{11ii} = \Lambda_{i11i} = \Lambda_{i1i1} + v_{ii} - v_{11} = \Lambda_{ii11} + \Lambda_{ii} - \Lambda_{11}.$$

This yields

$$\hat{F}^{ii}\Lambda_{11ii} = \hat{F}^{ii}\Lambda_{ii11} + \hat{F}^{ii}\Lambda_{ii} - \Lambda_{11}\sum_i \hat{F}^{ii}. \tag{3-11}$$

Differentiating (3-6) twice, we obtain

$$\hat{F}^{ii}\Lambda_{ii11} = -\hat{F}^{pq,rs}\Lambda_{pq1}\Lambda_{rs1} + \tilde{\psi}_{11} = -\hat{F}^{pp,qq}\Lambda_{pp1}\Lambda_{qq1} - \sum_{p\neq q}\frac{\hat{F}^{pp} - \hat{F}^{qq}}{\lambda_p - \lambda_q}\Lambda_{pq1}^2 + \tilde{\psi}_{11}. \tag{3-12}$$

By the concavity of $(\sigma_n/\sigma_{n-k})^{1/k}$, we can see that the first term on the right-hand side is nonnegative. Combining (3-10)–(3-12), we have

$$\begin{aligned}\hat{F}^{ii}\phi_{ii} &\geqslant \frac{\tilde{\psi}_{11}}{\Lambda_{11}} - \frac{1}{\Lambda_{11}}\sum_{p\neq q}\frac{\hat{F}^{pp} - \hat{F}^{qq}}{\lambda_p - \lambda_q}\Lambda_{pq1}^2 - \frac{\hat{F}^{ii}\Lambda_{11i}^2}{\Lambda_{11}^2} + (Nx_{n+1} - 1)\sum_i \hat{F}^{ii} \\ &\geqslant \frac{\tilde{\psi}_{11}}{\Lambda_{11}} + \frac{1}{\Lambda_{11}}\sum_{i\neq 1}\frac{\hat{F}^{ii} - \hat{F}^{11}}{\lambda_1 - \lambda_i}\Lambda_{11i}^2 - \frac{\hat{F}^{ii}\Lambda_{11i}^2}{\Lambda_{11}^2} + (Nx_{n+1} - 1)\sum_i \hat{F}^{ii}.\end{aligned} \tag{3-13}$$

We need an explicit expression of $\hat{F}^{ii}$. A straightforward calculation gives

$$k\hat{F}^{k-1}\hat{F}^{ii} = \frac{\sigma_n^{ii}\sigma_{n-k} - \sigma_n\sigma_{n-k}^{ii}}{\sigma_{n-k}^2}, \tag{3-14}$$

where $\sigma_l^{ii} = \partial\sigma_l/\partial\lambda_i$ for $1 \leqslant l \leqslant n$. We find that

$$\begin{aligned}\sigma_n^{ii}\sigma_{n-k} - \sigma_n\sigma_{n-k}^{ii} &= \sigma_{n-1}(\lambda|i)(\lambda_i\sigma_{n-k-1}(\lambda|i) + \sigma_{n-k}(\lambda|i)) - \lambda_i\sigma_{n-1}(\lambda|i)\sigma_{n-k-1}(\lambda|i) \\ &= \sigma_{n-1}(\lambda|i)\sigma_{n-k}(\lambda|i).\end{aligned}$$

Here and in the following, $\sigma_l(\lambda|a)$ and $\sigma_l(\lambda|ab)$ are the l-th elementary symmetric polynomials of $\lambda_1, \dots, \lambda_n$ with $\lambda_a = 0$ and $\lambda_a = \lambda_b = 0$, respectively. It follows that

$$k\hat{F}^{k-1}\hat{F}^{ii} = \frac{\sigma_{n-1}(\lambda|i)\sigma_{n-k}(\lambda|i)}{\sigma_{n-k}^2}. \tag{3-15}$$

Therefore, we get

$$\begin{aligned}k\hat{F}^{k-1}(\hat{F}^{ii} - \hat{F}^{11}) &= \frac{1}{\sigma_{n-k}^2}[\sigma_{n-1}(\lambda|i)\sigma_{n-k}(\lambda|i) - \sigma_{n-1}(\lambda|1)\sigma_{n-k}(\lambda|1)] \\ &= \frac{\sigma_{n-2}(\lambda|1i)}{\sigma_{n-k}^2}[\lambda_1\sigma_{n-k}(\lambda|i) - \lambda_i\sigma_{n-k}(\lambda|1)] \\ &= \frac{\sigma_{n-2}(\lambda|1i)(\lambda_1 - \lambda_i)}{\sigma_{n-k}^2}[(\lambda_1 + \lambda_i)\sigma_{n-k-1}(\lambda|1i) + \sigma_{n-k}(\lambda|1i)].\end{aligned} \tag{3-16}$$

When $i \geqslant 2$, we can see that

$$k\hat{F}^{k-1}\left(\frac{\hat{F}^{ii}-\hat{F}^{11}}{\lambda_1-\lambda_i}-\frac{\hat{F}^{ii}}{\lambda_1}\right)=\frac{\sigma_{n-2}(\lambda\,|\,1i)}{\sigma_{n-k}^2}[(\lambda_1+\lambda_i)\sigma_{n-k-1}(\lambda\,|\,1i)+\sigma_{n-k}(\lambda\,|\,1i)-\sigma_{n-k}(\lambda\,|\,i)]$$
$$=\frac{\sigma_{n-2}(\lambda\,|\,1i)}{\sigma_{n-k}^2}\lambda_i\sigma_{n-k-1}(\lambda\,|\,1i)=\frac{\sigma_{n-1}(\lambda\,|\,1)}{\sigma_{n-k}^2}\sigma_{n-k-1}(\lambda\,|\,1i)>0. \tag{3-17}$$

Plugging (3-17) into (3-13), we obtain

$$\hat{F}^{ii}\phi_{ii}\geqslant\frac{\tilde{\psi}_{11}}{\Lambda_{11}}-\hat{F}^{11}\frac{\Lambda_{11i}^2}{\Lambda_{11}^2}+(Nx_{n+1}-1)\sum_i\hat{F}^{ii}=\frac{\tilde{\psi}_{11}}{\Lambda_{11}}-\hat{F}^{11}N^2(y_{n+1})_1^2+(Nx_{n+1}-1)\sum_i\hat{F}^{ii}. \tag{3-18}$$

Here, in the last equality, we have used (3-8).

Now, let's calculate $\tilde{\psi}_{11}$. We denote by $\overline{\nabla}$ the connection of the ambient space and by $\{e_1^*, e_2^*, \dots, e_n^*\}$ the pull back of $\{e_1, e_2, \dots, e_n\}$ via the Gauss map. Differentiating $\tilde{\psi}$ with respect to e_1 twice, we get

$$\tilde{\psi}_1=d_X\psi^{-1/k}(\overline{\nabla}_{e_1^*}X)+d_\nu\psi^{-1/k}(e_1) \tag{3-19}$$

and

$$\begin{aligned}\tilde{\psi}_{11}&=d_Xd_X\psi^{-1/k}(\overline{\nabla}_{e_1^*}X,\overline{\nabla}_{e_1^*}X)+d_X\psi^{-1/k}(\overline{\nabla}_{e_1}\overline{\nabla}_{e_1^*}X)\\&\qquad+2d_Xd_\nu\psi^{-1/k}(e_1,\overline{\nabla}_{e_1^*}X)+d_\nu d_\nu\psi^{-1/k}(e_1,e_1)+d_\nu\psi^{-1/k}(\overline{\nabla}_{e_1}e_1)\\&\geqslant c_0\Lambda_{11}^2+d_X\psi^{-1/k}\left(\overline{\nabla}_{e_1}\sum_k\Lambda_{k1}e_k\right)+2d_Xd_\nu\psi^{-1/k}\left(e_1,\sum_l\Lambda_{l1}e_l\right)\\&\qquad+d_\nu d_\nu\psi^{-1/k}(e_1,e_1)+d_\nu\psi^{-1/k}(\nu)\\&\geqslant c_0\Lambda_{11}^2+\sum_k d_X\psi^{-1/k}(\Lambda_{k11}e_k+\Lambda_{k1}\delta_{k1}\nu)-C\lambda_1-C\\&\geqslant c_0\Lambda_{11}^2+\sum_k\Lambda_{11k}d_X\psi^{-1/k}(e_k)-C\lambda_1-C,\end{aligned} \tag{3-20}$$

where the first inequality comes from the locally strict convexity assumption on $\psi^{-1/k}$, i.e., for any spacelike vector $\xi\in\mathbb{R}^{n,1}$,

$$d_Xd_X\psi^{-1/k}(\xi,\xi)\geqslant c_0|\xi|_E^2\geqslant c_0|\xi|_M^2.$$

Here $c_0>0$ is some constant depending on the defining domain, and $|\cdot|_E$ and $|\cdot|_M$ are the Euclidean norm and Minkowski norm, respectively. At the point P_0, in view of (3-8) and the assumption that $\psi_{x_{n+1}}\geqslant 0$, we derive

$$\begin{aligned}\frac{\tilde{\psi}_{11}}{\Lambda_{11}}&\geqslant c_0\lambda_1-N\sum_k(x_{n+1})_kd_X\psi^{-1/k}(e_k)-C-\frac{C}{\lambda_1}\\&=c_0\lambda_1+\frac{N}{k}\psi^{-1/k-1}d_X\psi(\nabla x_{n+1})-C-\frac{C}{\lambda_1}\\&=c_0\lambda_1+\frac{N}{k}\psi^{-1/k-1}d_X\psi\left(-\frac{\partial}{\partial x_{n+1}}+x_{n+1}\nu\right)-C-\frac{C}{\lambda_1}\end{aligned}$$

$$
\begin{aligned}
&= c_0\lambda_1 + \frac{N}{k}\psi^{-1/k-1} d_X\psi\left(|x|^2\frac{\partial}{\partial x_{n+1}} + x_{n+1}\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}\right) - C - \frac{C}{\lambda_1}\\
&= c_0\lambda_1 + \frac{N|x|^2}{k}\psi^{-1/k-1}\frac{\partial\psi}{\partial x_{n+1}} + \frac{N}{k}\psi^{-1/k-1}x_{n+1}\sum_{i=1}^n x_i\frac{\partial\psi}{\partial x_i} - C - \frac{C}{\lambda_1}\\
&\geqslant c_0\lambda_1 + \frac{N}{k}\psi^{-1/k-1}x_{n+1}\sum_{i=1}^n x_i\frac{\partial\psi}{\partial x_i} - C - \frac{C}{\lambda_1} \geqslant -C - \frac{C}{\lambda_1}.
\end{aligned}
\tag{3-21}
$$

Here, in the last inequality, we have assumed $\lambda_1 = \lambda_1(|\psi|_{C^2}) > 0$ is large at P_0. On the other hand, note that the functional $\hat{F}$ is concave and homogenous of degree 1. Therefore,

$$
\sum_i \hat{F}^{ii} = \hat{F}(\lambda) + \sum_i \hat{F}^{ii}(1-\lambda_i) \geqslant \hat{F}(1) = \binom{n}{k}^{-1/k}. \tag{3-22}
$$

Combining (3-18)–(3-22), we obtain

$$
0 \geqslant \hat{F}^{ii}\phi_{ii} \geqslant -C - \frac{C}{\lambda_1} - \frac{C}{\lambda_1}N^2(x_{n+1})_1^2 + (Nx_{n+1}-1)\binom{n}{k}^{-1/k}.
$$

Letting N and λ_1 be sufficiently large, we obtain a contradiction. This completes the proof of Lemma 10.

Notice that this is the only place we need the locally strict convexity assumption of $\psi^{-1/k}$ in Theorem 1. It's also clear that the above proof can be easily modified to the case when $\psi^{-1/k}$ is convex with respect to X and the corresponding $\psi(x, u(x), Du)$ does not depend on $|x|$ (see the second inequality in (3-21)), as stated in the Remark 2. Therefore, (3-5) is solvable when either $\psi^{-1/k}$ is locally strictly convex with respect to X or $\psi^{-1/k}$ is convex with respect to X and $\psi(x, u(x), Du(x))$ does not depend on $|x|$. □

3.2. *Dirichilet problem for $k = n-1, n-2$.* Let $n \in \mathbb{N}$ and $\Omega_n := \{x \in \mathbb{R}^n \mid \underline{u}(x) = n\}$. We will consider the Dirichlet problem

$$
\begin{cases} \sigma_k(\kappa[\mathcal{M}_u]) = \psi(x, u(x)) & \text{in } \Omega_n,\\ u = n & \text{on } \partial\Omega_n. \end{cases} \tag{3-23}
$$

Note that since $\underline{u}$ is strictly convex, Ω_n is strictly convex. It's easy to see that if u is a solution of (3-23), then $\underline{u} \leqslant u \leqslant \bar{u}$. Therefore, in order to find a k-convex solution u for (3-23), we only need to study the C^1 and C^2 estimates of u.

3.2.1. *C^1 estimate for* (3-23).

Lemma 11. *Let u be a solution of* (3-23), *then $|Du| < C < 1$. Here C is a constant depending on $|D\underline{u}|_{\overline{\Omega}_n}$ and ψ.*

Proof. Let $V = -\langle \nu, E\rangle = 1/\sqrt{1-|Du|^2}$, and consider the test function $\phi = \ln V + Ku$, where $K > 0$ is to be determined. If ϕ achieves its maximum at an interior point $P_0 \in \mathcal{M}_u$, then at this point, we may choose a normal coordinate $\{\tau_1, \dots, \tau_n\}$ such that $h_{ij} = \kappa_i\delta_{ij}$. Since at P_0 we have

$$
\phi_i = \frac{V_i}{V} + Ku_i = 0 \quad \text{and} \quad 0 \geqslant \phi_{ii} = \frac{V_{ii}}{V} - \frac{V_i^2}{V^2} + Ku_{ii},
$$

a straightforward calculation yields

$$0 \geqslant -\frac{\langle \nabla \sigma_k, E\rangle}{V} - \frac{\sigma_k^{ii}\kappa_i^2 u_i^2}{V^2} + Kk\psi V + \sigma_k^{ii}\kappa_i^2.$$

Note that $|\langle \nabla \sigma_k, E\rangle| \leqslant CV^2$, where C only depends on $|\psi|_{C^1}$. Choosing $K > C+1$, we have

$$-\frac{\langle \nabla \sigma_k, E\rangle}{V} - \frac{\sigma_k^{ii}\kappa_i^2 u_i^2}{V^2} + Kk\psi V + \sigma_k^{ii}\kappa_i^2 > 0.$$

This leads to a contradiction. □

3.2.2. C^2 *boundary estimates for* (3-23). Now, we will establish the C^2 boundary estimate. For our convenience, we will consider the solvability of the Dirichlet problem

$$\begin{cases} G(Du, D^2u) = \sigma_k\big(\frac{1}{w}\gamma^{ik}u_{kl}\gamma^{lj}\big) = \psi(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3-24}$$

where Ω is strictly convex. We will follow the idea of [Caffarelli et al. 1988].

Infinitesimal stretching. If u is a solution of (3-24), let $v(x) = t^{-1}u(tx)$, where $t > 0$. Then the principal curvatures of $\mathcal{M}_v$ satisfy $\kappa[\mathcal{M}_v(x)] = t\kappa[\mathcal{M}_u(tx)]$. Therefore,

$$G(Dv, D^2v) = t^k\psi(tx, u(tx)) = t^k\psi(tx, tv(x)). \tag{3-25}$$

We write $\dot{v} = (d/dt)v = -t^{-2}u(tx) + x \cdot Du(tx)$; when $t = 1$,

$$\dot{v} = x \cdot Du(x) - u(x).$$

Differentiating (3-25) with respect to t then evaluating at $t = 1$, we obtain

$$G^{ij}\partial_{ij}\dot{v} + G^s\partial_s\dot{v} = k\psi + \psi_z(v + \dot{v}) + x\psi_x.$$

Writing $L := G^{ij}\partial_{ij} + G^s\partial_s$, we have

$$L(x \cdot Du - u) = k\psi + \psi_z(u + x \cdot Du - u) + x\psi_x = k\psi + x\psi_x + \psi_z x \cdot Du. \tag{3-26}$$

Infinitesimal rotation in Minkowski space. It is well known that Lorentz boosts are isometries of $\mathbb{R}^{n,1}$. Keeping the coordinates $x' = (x_1, \dots, x_{n-1})$ fixed, we rotate in the (x_n, u) variables:

$$\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}\begin{bmatrix} x_n \\ u \end{bmatrix} = \begin{bmatrix} \cosh\theta x_n + \sinh\theta u \\ \cosh\theta u + \sinh\theta x_n \end{bmatrix}.$$

To the first order in θ, the image of $(x, u(x))$ under such a rotation is

$$(x', x_n + u(x)\theta, u(x) + x_n\theta).$$

Therefore, to the first order in θ, the image of

$$(x', x_n - u(x)\theta, u(x', x_n - u(x)\theta))$$

is $(x', x_n, u(x', x_n - u(x)\theta) + x_n\theta)$. Considering this image as the graph of the function

$$v(x) = u(x', x_n - u(x)\theta) + x_n\theta + \text{higher order in } \theta,$$

we have

$$\begin{aligned} G(Dv, D^2v) &= \psi(x', x_n - u(x)\theta, u(x', x_n - u(x)\theta)) + \text{higher order in } \theta \\ &= \psi(x', x_n - u(x)\theta, v(x) - x_n\theta) + \text{higher order in } \theta. \end{aligned}$$

Notice that $(dv/d\theta)|_{\theta=0} = x_n - u_n u$, so we obtain

$$G^{ij}\partial_{ij}(x_n - u_n u) + G^s\partial_s(x_n - u_n u) = \psi_n(-u(x)) + \psi_z(x_n - u_n u - x_n). \tag{3-27}$$

Thus, we conclude that

$$L(x_n - uu_n) = -u\psi_n - u_n u\psi_z. \tag{3-28}$$

Lemma 12. *Let u be a solution of* (3-24), *then $|D^2u| < C$ on $\partial\Omega$. Here C is a constant depending on Ω and ψ.*

Proof. For any $p \in \partial\Omega$, we suppose p is the origin and that the x_n-axis is the interior normal of $\partial\Omega$ at p. We may also assume the boundary near the origin p is represented by

$$x_n = \frac{1}{2}\sum_{\alpha=1}^{n-1}\lambda_\alpha x_\alpha^2 + O(|x'|^3), \quad x' = (x_1, \dots, x_{n-1}),$$

where $\lambda_\alpha > 0$, $1 \leqslant \alpha \leqslant n-1$, are the principal curvatures of $\partial\Omega$ at the origin. Let $T_\alpha = \partial_\alpha + \lambda_\alpha(x_\alpha\partial_n - x_n\partial_\alpha)$. Note that $G^{ij}u_{ij\alpha} + G^s u_{s\alpha} = \psi_\alpha + \psi_z u_\alpha$. In view of the fact that (3-23) is invariant under rotation (see (3.1) in [Caffarelli et al. 1988]), we get

$$|LT_\alpha u| \leqslant C. \tag{3-29}$$

Moreover, it's easy to see we have $|T_\alpha u| \leqslant C|x'|^2$ on $\partial\Omega$ near the origin. In the following, we write $\Omega_\beta := \Omega \cap \{x_n < \beta\}$. Set

$$h = (x \cdot Du - u) - \frac{\delta}{\beta}(x_n - uu_n).$$

On $\partial\Omega \cap \partial\Omega_\beta$, note that $u = 0$, so we have $x \cdot Du \leqslant C_1|x'|^2$. This implies, on $\partial\Omega \cap \partial\Omega_\beta$,

$$h = x \cdot Du - \frac{\delta}{\beta}x_n \leqslant \left(C_1 - \frac{\delta}{\beta}a\right)|x'|^2, \tag{3-30}$$

where $a > 0$ depends on the principal curvatures of $\partial\Omega$. Notice that u is a spacelike function, so we suppose $|Du| \leqslant \theta_0$ in $\overline{\Omega}$ for some $\theta_0 \in (0, 1)$. Then we have $0 \leqslant -u \leqslant \theta_0\beta$ in Ω_β. Therefore, on $\{x_n = \beta\}$,

$$h = \beta u_n + \sum_{\alpha=1}^{n-1}x_\alpha u_\alpha - u + \frac{\delta}{\beta}uu_n - \delta \leqslant \beta\theta_0 + C\beta^{1/2} + \theta_0\beta + \theta_0^2\delta - \delta \leqslant C\beta^{1/2} + \delta(\theta_0 - 1) \tag{3-31}$$

with C being independent of β and δ. Moreover,

$$Lh = k\psi + x\psi_x + \psi_z x \cdot Du - \frac{\delta}{\beta}(-u\psi_n - u_n u\psi_z) \geqslant k\psi - C\beta^{1/2} - C\delta \geqslant \frac{k}{2}\psi, \tag{3-32}$$

where δ and β are small positive constants.

Now choose $A = A(\delta) > 0$ large enough that

$$Ah \leqslant -|T_\alpha u| \quad \text{on } \partial\Omega_\beta \qquad \text{and} \qquad LAh > |LT_\alpha u| \quad \text{in } \Omega_\beta.$$

By the maximum principle, we conclude that

$$Ah \pm T_\alpha u \leqslant 0 \quad \text{in } \bar{\Omega}_\beta.$$

On the other hand, we have $h(0) = T_\alpha u(0) = 0$. Therefore,

$$|\partial_n T_\alpha u(0)| \leqslant -Ah_n(0) \leqslant \frac{A\delta}{\beta},$$

which yields

$$|u_{n\alpha}(0)| \leqslant C. \tag{3-33}$$

Next, following the notation in Section 2.1, we write $a_{ij} = \frac{1}{w}\gamma^{ik}u_{kl}\gamma^{lj}$, where $w = \sqrt{1-|Du|^2}$ and $\gamma^{ik} = \delta_{ik} + u_i u_k/(w(1+w))$. A straightforward calculation yields, at the origin,

$$\begin{aligned} a_{\alpha\alpha} &= \frac{u_{\alpha\alpha}}{w} = -\frac{u_n\lambda_\alpha}{w}, \quad a_{\alpha n} = \frac{u_{\alpha n}}{w^2} && \text{for } 1 \leqslant \alpha \leqslant n-1, \\ a_{nn} &= \frac{u_{nn}}{w^3}, \qquad\qquad a_{ij} = 0 && \text{for all other } 1 \leqslant i, \quad j \leqslant n. \end{aligned} \tag{3-34}$$

Since $\partial\Omega$ is smooth, we know there exists $r_0 > 0$ and $z_p = (0, \dots, 0, r_0)$ such that $B_{r_0}(z_p) \subset \Omega$ and $\bar{B}_{r_0}(z_p) \cap \partial\Omega = p$. Here $B_{r_0}(z_p)$ is a ball of radius r_0 centered at z_p. Let

$$\bar{u} = -\sqrt{R^2 + r_0^2} + \sqrt{R^2 + |x - z_p|^2},$$

where $x = (x_1, \dots, x_n)$ and $R > 0$ is a constant to be determined. A straightforward calculation yields

$$\sigma_k\Big(\frac{1}{w}\gamma^{ik}\bar{u}_{kl}\gamma^{lj}\Big) = \binom{n}{k}\frac{1}{R} < c_2$$

when $R = R(c_2) > 0$ is sufficiently large. Here c_2 is the lower bound for ψ defined in Theorem 5. Therefore, $\bar{u}$ is a supersolution of (3-24). By the strong maximum principal, we have $u < \bar{u}$ in $B_{r_0}(z_p)$. Applying the Hopf lemma, we obtain

$$\frac{r_0}{\sqrt{R^2 + r_0^2}} = -\bar{u}_n(p) < -u_n(p).$$

In view of (3-34) and [Trudinger 1995, (2.5)], (3-24) can be written as

$$\frac{1}{w^k}\bigg[\frac{1}{w^2}(-u_n)^{k-1}\sigma_{k-1}(\lambda)u_{nn} + P\bigg] = \psi,$$

where P depends on w, $u_{\alpha\beta}$, and $u_{\alpha n}$, which are bounded by some uniform constants depending on n, k, $\partial\Omega$, $\|u\|_{C^1(\bar{\Omega})}$, and $\lambda = (\lambda_1, \dots, \lambda_{n-1})$. Moreover, by our assumption that ψ is bounded, we obtain an upper bound for $u_{nn}(0)$. The lower bound for $u_{nn}(0)$ comes from the fact that $\mathcal{M}_u$ is k-convex, which implies $\sum_{i=1}^n a_{ii} > 0$.

Finally, since $p \in \partial\Omega$ is arbitrary, we get

$$|D^2u(x)| \leqslant C \quad \text{for any } x \in \partial\Omega. \qquad \square$$

3.2.3. *C^2 global estimate for* (3-23). Finally, we will prove the C^2 global estimate. In this subsubsection, for greater generality, we will assume $\psi = \psi(X, \nu)$.

Lemma 13. *Let u be a solution of* (3-24) *with $\psi = \psi(X, \nu)$, then*

$$|D^2u| < \max\{C, \max_{\partial\Omega} |D^2u|\}$$

on Ω. Here C is a constant depending on $|Du|_\Omega$ and ψ.

Proof. We consider the following test function whose form first appeared in [Guan et al. 2015]:

$$\phi = \log\log P - N\langle\nu, E\rangle.$$

Here, $P := \sum_l e^{\kappa_l}$, and N is a sufficiently large constant to be determined later.

We may assume that the maximum of ϕ is achieved at some point $P_0 \in \mathcal{M}_u$, where u is the solution of (3-24). Suppose $\{\tau_1, \tau_2, \dots, \tau_n\}$ is a normal coordinate near P_0 such that, at P_0,

$$h_{ij} = \kappa_i\delta_{ij} \quad \text{and} \quad \kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_n.$$

Differentiating the function ϕ twice at P_0, we have

$$\phi_i = \frac{P_i}{P\log P} + Nh_{ii}u_i = 0, \tag{3-35}$$

and

$$\begin{aligned}\phi_{ii} &= \frac{P_{ii}}{P\log P} - \frac{P_i^2}{P^2\log P} - \frac{P_i^2}{(P\log P)^2} - Nh_{ii}^2\langle\nu, E\rangle + \sum_s Nu_sh_{isi}\\ &= \frac{1}{P\log P}\bigg[\sum_l e^{\kappa_l}h_{llii} + \sum_l e^{\kappa_l}h_{lli}^2 + \sum_{p\neq q}\frac{e^{\kappa_p}-e^{\kappa_q}}{\kappa_p-\kappa_q}h_{pqi}^2 - \Big(\frac{1}{P} + \frac{1}{P\log P}\Big)P_i^2\bigg]\\ &\qquad - Nh_{ii}^2\langle\nu, E\rangle + \sum_s Nu_sh_{iis}.\end{aligned}$$

Contracting with σ_k^{ii}, we get

$$\begin{aligned}\sigma_k^{ii}\phi_{ii} = \frac{\sigma_k^{ii}}{P\log P}\bigg[\sum_l e^{\kappa_l}h_{llii} + \sum_l e^{\kappa_l}h_{lli}^2 + \sum_{p\neq q}\frac{e^{\kappa_p}-e^{\kappa_q}}{\kappa_p-\kappa_q}h_{pqi}^2 - \Big(\frac{1}{P} + \frac{1}{P\log P}\Big)P_i^2\bigg]\\ - N\sigma_k^{ii}\kappa_i^2\langle\nu, E\rangle + \sum_s Nu_s\sigma_k^{ii}h_{iis}.\end{aligned} \tag{3-36}$$

At P_0, differentiating (1-2) twice yields

$$\sigma_k^{ii}h_{iil} = d_X\psi(\tau_l) + \kappa_l d_\nu\psi(\tau_l) \tag{3-37}$$

and

$$\sigma_k^{ii}h_{iill} + \sigma_k^{pq,rs}h_{pql}h_{rsl} \geqslant -C - Ch_{11}^2 + \sum_s h_{sll}d_\nu\psi(\tau_s), \tag{3-38}$$

where C is some uniform constant only depending on ψ. Note that

$$h_{llii} = h_{iill} - h_{ii}h_{ll}^2 + h_{ii}^2h_{ll}. \tag{3-39}$$

Inserting (3-38) and (3-39) into (3-36), we obtain

$$\begin{aligned}\sigma_k^{ii}\phi_{ii} \geqslant \frac{1}{P\log P}\biggl[\sum_l e^{\kappa_l}\Bigl(-C - C\kappa_1^2 - \sigma_k^{pq,rs}h_{pql}h_{rsl} + \sum_s h_{sll}d_\nu\psi(\tau_s)\Bigr)\\ + \sum_l \sigma_k^{ii}e^{\kappa_l}h_{lli}^2 + \sigma_k^{ii}\sum_{p\neq q}\frac{e^{\kappa_p}-e^{\kappa_q}}{\kappa_p-\kappa_q}h_{pqi}^2 - \Bigl(\frac{1}{P}+\frac{1}{P\log P}\Bigr)\sigma_k^{ii}P_i^2\biggr]\\ -N\sigma_k^{ii}\kappa_i^2\langle\nu, E\rangle + \sum_s Nu_s\sigma_k^{ii}h_{sii} - \sigma_k^{ii}\kappa_i^2. \quad (3\text{-}40)\end{aligned}$$

By (3-35) and (3-37), we have

$$\frac{1}{P\log P}\sum_s\sum_l e^{\kappa_l}h_{sll}d_\nu\psi(\tau_s) + \sum_s Nu_s\sigma_k^{ii}h_{sii} \geqslant -C.$$

Now, for any constant $K > 1$, we write

$$\begin{aligned}A_i &= e^{\kappa_i}\biggl[K(\sigma_k)_i^2 - \sum_{p\neq q}\sigma_k^{pp,qq}h_{ppi}h_{qqi}\biggr],\\ B_i &= 2\sum_{l\neq i}\sigma_k^{ii,ll}e^{\kappa_l}h_{lli}^2, \quad C_i = \sigma_k^{ii}\sum_l e^{\kappa_l}h_{lli}^2,\\ D_i &= 2\sum_{l\neq i}\sigma_k^{ll}\frac{e^{\kappa_l}-e^{\kappa_i}}{\kappa_l-\kappa_i}h_{lli}^2, \quad E_i = \frac{1+\log P}{P\log P}\sigma_k^{ii}P_i^2.\end{aligned}$$

Combining

$$-\sum_l \sigma_k^{pq,rs}h_{pql}h_{rsl} = \sum_{p\neq q}\sigma_k^{pp,qq}h_{pql}^2 - \sum_{p\neq q}\sigma_k^{pp,qq}h_{ppl}h_{qql}$$

with (3-40), we get

$$\sigma_k^{ii}\phi_{ii} \geqslant \frac{1}{P\log P}\sum_i(A_i+B_i+C_i+D_i-E_i) + (-N\langle\nu, E\rangle - 1)\sigma_k^{ii}\kappa_i^2 - C\kappa_1. \quad (3\text{-}41)$$

Claim 1. *For any given $0<\varepsilon<\frac{1}{2}$, we let $\alpha = (1-2\varepsilon)/(1+\varepsilon)$. There exists a positive constant $\delta < \frac{1}{2}$ such that, for any $|\kappa_i| \leqslant \delta\kappa_1,\ 1\leqslant i\leqslant n$, if the constant K and the maximum principal curvature κ_1 are both sufficiently large, we have*

$$A_i+B_i+C_i+D_i-E_i-\frac{\alpha}{P\log P}\sigma_k^{ii}P_i^2 \geqslant 0.$$

Applying Lemma 6 in [Ren and Wang 2019], we can see that when K is chosen to be sufficiently large, we have $A_i \geqslant 0$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}P_i^2 &= e^{2\kappa_i}h_{iii}^2 + 2\sum_{l\neq i}e^{\kappa_i+\kappa_l}h_{iii}h_{lli} + \Bigl(\sum_{l\neq i}e^{\kappa_l}h_{lli}\Bigr)^2\\ &\leqslant e^{2\kappa_i}h_{iii}^2 + 2\sum_{l\neq i}e^{\kappa_i+\kappa_l}h_{iii}h_{lli} + (P-e^{\kappa_i})\sum_{l\neq i}e^{\kappa_l}h_{lli}^2. \quad (3\text{-}42)\end{aligned}$$

Thus,

$$
\begin{aligned}
B_i + C_i + D_i - E_i - \frac{\alpha}{P\log P}\sigma_k^{ii}P_i^2 \\
\geqslant 2\sum_{l\neq i} e^{\kappa_l}\sigma_k^{ll,ii}h_{lli}^2 + 2\sum_{l\neq i}\frac{e^{\kappa_l}-e^{\kappa_i}}{\kappa_l-\kappa_i}\sigma_k^{ll}h_{lli}^2 - \frac{1+\alpha}{\log P}\sum_{l\neq i} e^{\kappa_l}\sigma_k^{ii}h_{lli}^2 + \frac{1+\alpha+\log P}{P\log P}\sum_{l\neq i} e^{\kappa_l+\kappa_i}\sigma_k^{ii}h_{lli}^2 \\
+ e^{\kappa_i}\sigma_k^{ii}h_{iii}^2 - \frac{1+\alpha+\log P}{P\log P}e^{2\kappa_i}\sigma_k^{ii}h_{iii}^2 - 2\frac{1+\alpha+\log P}{P\log P}\sum_{l\neq i} e^{\kappa_i+\kappa_l}\sigma_k^{ii}h_{iii}h_{lli}.
\end{aligned}
\tag{3-43}
$$

Let ε be equal to the ε_T in Lemma 12 of [Ren and Wang 2019]. Then we know there exists a positive constant $\delta < \varepsilon$ such that, when $|\kappa_i| < \delta\kappa_1$,

$$
(2-\varepsilon)\sum_{l\neq i} e^{\kappa_l}\sigma_k^{ll,ii}h_{lli}^2 + (2-\varepsilon)\sum_{l\neq i}\frac{e^{\kappa_l}-e^{\kappa_i}}{\kappa_l-\kappa_i}\sigma_k^{ll}h_{lli}^2 - \frac{1+\alpha}{\log P}\sum_{l\neq i} e^{\kappa_l}\sigma_k^{ii}h_{lli}^2 \geqslant 0. \tag{3-44}
$$

On the other hand, we have

$$
\sum_{l\neq i,1} e^{\kappa_l+\kappa_i}\sigma_k^{ii}h_{lli}^2 - 2\sum_{l\neq i,1} e^{\kappa_i+\kappa_l}\sigma_k^{ii}h_{iii}h_{lli} \geqslant -\sum_{l\neq i,1} e^{\kappa_l+\kappa_i}\sigma_k^{ii}h_{iii}^2. \tag{3-45}
$$

It follows that

$$
\begin{aligned}
B_i + C_i + D_i - E_i - \frac{\alpha}{P\log P}\sigma_k^{ii}P_i^2 \\
\geqslant \frac{1+\alpha+\log P}{P\log P}e^{\kappa_1+\kappa_i}\sigma_k^{ii}h_{11i}^2 + e^{\kappa_i}\sigma_k^{ii}h_{iii}^2 - \frac{1+\alpha+\log P}{P\log P}\sum_{l\neq 1} e^{\kappa_l+\kappa_i}\sigma_k^{ii}h_{iii}^2 \\
-2\frac{1+\alpha+\log P}{P\log P}e^{\kappa_i+\kappa_1}\sigma_k^{ii}h_{iii}h_{11i} + \varepsilon e^{\kappa_1}\sigma_k^{11,ii}h_{11i}^2 + \varepsilon\frac{e^{\kappa_1}-e^{\kappa_i}}{\kappa_1-\kappa_i}\sigma_k^{11}h_{11i}^2.
\end{aligned}
\tag{3-46}
$$

A straightforward calculation shows that, when κ_1 is very large, the following inequalities hold:

$$
e^{\kappa_i}\sigma_k^{ii}h_{iii}^2 - \frac{1+\alpha+\log P}{P\log P}\sum_{l\neq 1} e^{\kappa_l+\kappa_i}\sigma_k^{ii}h_{iii}^2 \geqslant \left(\frac{e^{\kappa_1}}{P} - \frac{1+\alpha}{\log P}\right)e^{\kappa_i}\sigma_k^{ii}h_{iii}^2 \geqslant \frac{1}{n+1}e^{\kappa_i}\sigma_k^{ii}h_{iii}^2,
$$

and

$$
-2\frac{1+\alpha+\log P}{P\log P}e^{\kappa_i+\kappa_1}\sigma_k^{ii}|h_{iii}h_{11i}| \geqslant -\frac{3}{P}e^{\kappa_i+\kappa_1}\sigma_k^{ii}|h_{iii}h_{11i}| \geqslant -3e^{\kappa_i}\sigma_k^{ii}|h_{iii}h_{11i}|.
$$

Moreover, it is easy to see that

$$
e^{\kappa_1}\sigma_k^{11,ii}h_{11i}^2 + \frac{e^{\kappa_1}-e^{\kappa_i}}{\kappa_1-\kappa_i}\sigma_k^{11}h_{11i}^2 = e^{\kappa_i}\sigma_k^{11,ii}h_{11i}^2 + \frac{e^{\kappa_1}-e^{\kappa_i}}{\kappa_1-\kappa_i}\sigma_k^{ii}h_{11i}^2. \tag{3-47}
$$

By the Taylor expansion, we have

$$
\frac{e^{\kappa_1}-e^{\kappa_i}}{\kappa_1-\kappa_i}\sigma_k^{ii}h_{11i}^2 = e^{\kappa_i}\sum_{m\geqslant 1}\frac{(\kappa_1-\kappa_i)^{m-1}}{m!}\sigma_k^{ii}h_{11i}^2. \tag{3-48}
$$

Combining the previous four formulas with (3-46), when κ_1 is sufficiently large and $|\kappa_i| < \delta\kappa_1$, we obtain

$$B_i + C_i + D_i - E_i - \frac{\alpha}{P\log P}\sigma_k^{ii}P_i^2 \geqslant e^{\kappa_i}\sigma_k^{ii}\left[\frac{1}{n+1}h_{iii}^2 - 3|h_{iii}h_{11i}| + \varepsilon\sum_{m\geqslant 1}\frac{(\kappa_1-\kappa_i)^{m-1}}{m!}h_{11i}^2\right] \geqslant 0.$$

Therefore, Claim 1 is proved.

Recalling Section 4 of [Ren and Wang 2019] and the proof of Theorem 14 in [Ren and Wang 2023], we know the following claim is true.

Claim 2. *Suppose $k = n-1$ ($n \geqslant 3$) or $k = n-2$ ($n \geqslant 5$). For any index $1 \leqslant i \leqslant n$, if the positive constant K and the maximum principal curvature κ_1 are both sufficiently large, we have*

$$A_i + B_i + C_i + D_i - E_i \geqslant 0.$$

By Claims 1 and 2, (3-41) becomes

$$0 \geqslant \sum_{|\kappa_i|<\delta\kappa_1}\frac{\alpha}{(P\log P)^2}\sigma_k^{ii}P_i^2 + (-N\langle\nu, E\rangle - 1)\sigma_k^{ii}\kappa_i^2 - C\kappa_1. \tag{3-49}$$

Here, the constant δ is the constant chosen in Claim 1. Choosing $N > 0$ such that

$$\sigma_k^{11}\kappa_1^2(-N\langle\nu, E\rangle - 1) - C\kappa_1 > 0,$$

we get a contradiction. Therefore, our desired estimate follows immediately. □

By Lemmas 11, 12, and 13, we conclude that, when $k = n-1,\ n-2$, the Dirichlet problem (3-23) admits a k-convex solution.

4. The local estimates

We will devote this section to establishing the local C^1 and C^2 estimates for the solution u of (1-3).

4.1. ***Local C^1 estimates.*** In this subsection, we will prove the local C^1 estimate. We will split it into two cases. In the first case, we will assume u is a convex solution of (1-2); in the second case, we will assume u is a k-convex solution of (1-5). Note that in both cases our results hold for $1 \leqslant k \leqslant n$.

For strictly convex, spacelike hypersurfaces, [Bayard and Schnürer 2009] proved the following local gradient estimate lemma.

Lemma 14 [Bayard and Schnürer 2009, Lemma 5.1]. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $u, \bar{u}, \Psi : \Omega \to \mathbb{R}^n$ be strictly spacelike. Assume that u is strictly convex and $u < \bar{u}$ in Ω. Also assume that, near $\partial\Omega$, we have $\Psi > \bar{u}$. Consider the set with $u > \Psi$. For every x in this set, we have the following gradient estimate for u:*

$$\frac{1}{\sqrt{1-|Du|^2}} \leqslant \frac{1}{u(x)-\Psi(x)}\cdot\sup_{\{u>\Psi\}}\frac{\bar{u}-\Psi}{\sqrt{1-|D\Psi|^2}}.$$

For k-convex, spacelike hypersurfaces, [Bayard 2006] proved a similar result when $k = 2$. In the following, we will extend it to all k. Our argument is a modification of that in [Bayard 2006]. We would also like to mention that the basic idea of this argument appeared in [Chou and Wang 2001].

Lemma 15. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $u, \bar{u}, \Psi : \Omega \to \mathbb{R}^n$ be strictly spacelike. Assume that $\mathcal{M}_u = \{(x, u(x)) \mid x \in \Omega\}$ is a k-convex hypersurface satisfying*

$$\sigma_k(\kappa[\mathcal{M}_u]) = \psi(x, u(x))$$

and $u \leqslant \bar{u}$ in Ω. Also assume that, near $\partial\Omega$, we have $\Psi > \bar{u}$. Consider the set with $u > \Psi$. For every x in this set, we have the following gradient estimate for u:

$$\frac{1}{\sqrt{1-|Du|^2}} \leqslant \left[\frac{1}{u(x)-\Psi(x)} \cdot \sup_{\{u>\Psi\}} (\bar{u}-\Psi)\right]^N C.$$

Here, $N = N(n, k)$ is a uniform constant only depending on n and k, and $C = C(\bar{u}-\Psi, |\Psi|_{C^2}, |\psi|_{C^1})$ is a uniform constant depending on the upper bound of $\bar{u}-\Psi$, $1/\sqrt{1-|D\Psi|^2}$, $D^2\Psi$, and $|\psi|_{C^1}$.

Proof. Consider the test function

$$\phi = (u-\Psi)^N(-\langle \nu, E\rangle),$$

where N is a large undetermined constant. Assume the function ϕ achieves its maximum at P. We may choose a local normal coordinate $\{\tau_1, \dots, \tau_n\}$ such that, at P, we have $h_{ij} = \kappa_i\delta_{ij}$. Differentiating ϕ twice at P, we have

$$\begin{aligned} 0 &= \frac{\phi_i}{\phi} = N\frac{u_i-\Psi_i}{u-\Psi} + \frac{h_{im}u_m}{-\langle\nu, E\rangle}, \\ 0 &\geqslant \frac{\phi_{ii}}{\phi} - \frac{\phi_i^2}{\phi^2} = N\frac{u_{ii}-\Psi_{ii}}{u-\Psi} - N\frac{(u_i-\Psi_i)^2}{(u-\Psi)^2} + \frac{\sum_m h_{im}^2(-\langle\nu, E\rangle) + \sum_m h_{imi}u_m}{-\langle\nu, E\rangle} - \frac{\left(\sum_m h_{im}u_m\right)^2}{(-\langle\nu, E\rangle)^2}. \end{aligned} \tag{4-1}$$

Contracting with σ_k^{ii}, we get

$$0 \geqslant \frac{\sigma_k^{ii}\phi_{ii}}{\phi} = N\frac{\sigma_k^{ii}u_{ii} - \sigma_k^{ii}\Psi_{ii}}{u-\Psi} - N\frac{\sigma_k^{ii}(u_i-\Psi_i)^2}{(u-\Psi)^2} + \sigma_k^{ii}\kappa_i^2 + \frac{\sigma_k^{ii}\sum_m h_{iim}u_m}{-\langle\nu, E\rangle} - \frac{\sigma_k^{ii}\kappa_i^2u_i^2}{(-\langle\nu, E\rangle)^2}. \tag{4-2}$$

Without loss of generality, we may assume that, at P,

$$u_1^2 \geqslant \frac{|\nabla u|^2}{n},$$

where ∇ is the Levi-Civita connection on $\mathcal{M}_u$. By (4-1), we have

$$\kappa_1 = \frac{N\langle\nu, E\rangle}{u-\Psi}\left(1 - \frac{\Psi_1}{u_1}\right).$$

We may also assume $|\nabla u(P)|$ is sufficiently large that $|\Psi_1/u_1| < \frac{1}{2}$. Then, at P, we can see

$$\kappa_1 < \frac{N}{2}\frac{\langle\nu, E\rangle}{u-\Psi}. \tag{4-3}$$

Thus, if N is sufficiently large, κ_1 is negative and its norm is large. Using inequality (26) in [Lin and Trudinger 1994], we obtain

$$\sum_{i\geqslant 2}\sigma_k^{ii}\kappa_i^2 \geqslant \eta\sigma_k^{11}\kappa_1^2,$$

where η is a uniform constant only depending on n and k. Therefore,

$$\sigma_k^{ii}\kappa_i^2 - \frac{\sigma_k^{ii}\kappa_i^2 u_i^2}{(-\langle \nu, E\rangle)^2} \geqslant \sum_{i\geqslant 2}\sigma_k^{ii}\kappa_i^2 - \Big(1-\frac{1}{n}\Big)\sum_{i\geqslant 2}\sigma_k^{ii}\kappa_i^2 \geqslant \frac{\eta}{n}\sigma_k^{11}\kappa_1^2 := \eta_0\sigma_k^{11}\kappa_1^2.$$

By (4-3), we get

$$\sigma_k^{ii}\kappa_i^2 - \frac{\sigma_k^{ii}\kappa_i^2 u_i^2}{(-\langle \nu, E\rangle)^2} \geqslant \frac{\eta_0 N^2}{4}\sigma_k^{11}\frac{(-\langle \nu, E\rangle)^2}{(u-\Psi)^2}. \tag{4-4}$$

Inserting (1-2) and (4-4) into (4-2) yields

$$0 \geqslant N(u-\Psi)[\sigma_k^{ii}\kappa_i(-\langle \nu, E\rangle) - \sigma_k^{ii}\Psi_{ii}] - N\sigma_k^{ii}(u_i-\Psi_i)^2 + (u-\Psi)^2\frac{\sum_m \psi_m u_m}{-\langle \nu, E\rangle} + \frac{\eta_0 N^2}{4}\sigma_k^{11}(-\langle \nu, E\rangle)^2. \tag{4-5}$$

Noticing that

$$\psi_m = \sum_{l=1}^n \psi_{x_l}\Big\langle \tau_m, \frac{\partial}{\partial x_l}\Big\rangle + \psi_u\langle -\tau_m, E\rangle,$$

we calculate

$$\frac{\sum_m \psi_m u_m}{-\langle \nu, E\rangle} \geqslant -C(1+\langle -\nu, E\rangle). \tag{4-6}$$

Combining (4-5) with (4-6), we get

$$0 \geqslant -(n-k+1)N(\bar u-\Psi)\sigma_{k-1}|\nabla^2\Psi| - 2(n-k+1)N\sigma_{k-1}(|\nabla u|^2+|\nabla\Psi|^2) - C(\bar u-\Psi)^2(1+\langle -\nu, E\rangle) + \frac{\eta_0 N^2}{4}\sigma_k^{11}(-\langle \nu, E\rangle)^2. \tag{4-7}$$

Notice that, when $\kappa_1 < 0$, we have

$$\sigma_{k-1} = \kappa_1\sigma_{k-2}(\kappa\,|\,1) + \sigma_{k-1}(\kappa\,|\,1) \leqslant \sigma_k^{11}.$$

Moreover, $-\langle \nu, E\rangle = \sqrt{1+|\nabla u|^2}$. With N sufficiently large in (4-7), we obtain the desired estimate. □

4.2. ***The Pogorelov-type local C^2 estimates.*** Recall that in [Wang and Xiao 2022] (see Lemma 24) we proved the Pogorelov-type local C^2 estimate for strictly convex, spacelike, constant σ_k curvature hypersurfaces. With small modifications, we can show the following.

Lemma 16. *Let u^{r*} be the solution of* (3-5) *and u^r be the Legendre transform of u^{r*}. For any given $s > 2C_0+1$, where $C_0 > \min \bar u$ is an arbitrary constant, let $r_s > 0$ be a positive number such that, when $r > r_s$, we have $u^r|_{\partial\Omega_r} > s$, where $\Omega_r = Du^{r*}(B_r)$. Let $\kappa_{\max}(x)$ be the largest principal curvature of $\mathcal{M}_{u^r}$ at x, where $\mathcal{M}_{u^r} = \{(x, u^r(x)) \mid x \in \Omega_r\}$. Then, for $r > r_s$, we have*

$$\max_{\mathcal{M}_{u^r}}(s-u^r)\kappa_{\max} \leqslant C. \tag{4-8}$$

Here, C depends on the local C^1 estimates of u^r and s.

In the rest of this subsection, we will establish the Pogorelov-type local C^2 estimates for the k-convex solution of (1-2), where $k = n-1$ $(n \geqslant 3)$, $n-2$ $(n \geqslant 5)$.

Lemma 17. *Let u^n be the k-convex solution of* (3-23) *with $\psi = \psi(X, \nu)$, where $k = n-1$ ($n \geqslant 3$), $n-2$ ($n \geqslant 5$). For any given $s > 1$, let $m > s$. Then $u^m|_{\partial\Omega_m} = m > s$. Let $\kappa_{\max}(x)$ be the largest principal curvature of $\mathcal{M}_{u^m}$ at x, where $\mathcal{M}_{u^m} = \{(x, u^m(x)) \mid x \in \Omega_m\}$. Then, for $m > s$, we have*

$$\max_{\mathcal{M}_{u^m}} (s - u^m)\kappa_{\max} \leqslant C.$$

Here, C depends on the local C^1 estimates of u^m and s.

Proof. In this proof, for our convenience when there is no confusion, we will drop the superscript on u^m. Now, on Ω_m, we consider the following test function whose form first appeared in [Guan et al. 2015]:

$$\phi = \beta \log(s - u) + \log\log P - N\langle \nu, E\rangle.$$

Here the function P is defined by

$$P = \sum_l e^{\kappa_l},$$

and β and N are constants to be determined later.

Letting $U_s = \{x \in \mathbb{R}^n \mid u(x) < s\}$, we may assume that the maximum of ϕ is achieved at $P_0 \in U_s$. Choose a local normal coordinate $\{\tau_1, \tau_2, \ldots, \tau_n\}$ such that $h_{ij} = \kappa_i \delta_{ij}$ and $\kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_n$ at P_0.

Differentiating the function ϕ twice at P_0, we get

$$\phi_i = -\frac{\beta u_i}{s - u} + \frac{P_i}{P \log P} + N h_{ii} u_i = 0 \tag{4-9}$$

and

$$\begin{aligned} 0 \geqslant \phi_{ii} &= \frac{P_{ii}}{P \log P} - \frac{P_i^2}{P^2 \log P} - \frac{P_i^2}{(P \log P)^2} + \frac{\beta h_{ii}\langle \nu, E\rangle}{s - u} - \frac{\beta u_i^2}{(s-u)^2} - N h_{ii}^2 \langle \nu, E\rangle + \sum_s N u_s h_{isi} \\ &= \frac{1}{P \log P}\left[\sum_l e^{\kappa_l} h_{llii} + \sum_l e^{\kappa_l} h_{lli}^2 + \sum_{p \neq q} \frac{e^{\kappa_p} - e^{\kappa_q}}{\kappa_p - \kappa_q} h_{pqi}^2 - \left(\frac{1}{P} + \frac{1}{P \log P}\right) P_i^2\right] \\ &\qquad + \frac{\beta h_{ii}\langle \nu, E\rangle}{s - u} - \frac{\beta u_i^2}{(s-u)^2} - N h_{ii}^2 \langle \nu, E\rangle + \sum_s N u_s h_{iis}. \end{aligned}$$

Contracting with σ_k^{ii}, we have

$$\begin{aligned} \sigma_k^{ii}\phi_{ii} &= \frac{\sigma_k^{ii}}{P \log P}\left[\sum_l e^{\kappa_l} h_{llii} + \sum_l e^{\kappa_l} h_{lli}^2 + \sum_{p \neq q} \frac{e^{\kappa_p} - e^{\kappa_q}}{\kappa_p - \kappa_q} h_{pqi}^2 - \left(\frac{1}{P} + \frac{1}{P \log P}\right) P_i^2\right] \\ &\qquad + \frac{\beta \sigma_k^{ii} \kappa_i \langle \nu, E\rangle}{s - u} - \frac{\beta \sigma_k^{ii} u_i^2}{(s-u)^2} - N \sigma_k^{ii} \kappa_i^2 \langle \nu, E\rangle + \sum_s N u_s \sigma_k^{ii} h_{iis}. \end{aligned} \tag{4-10}$$

At P_0, differentiating (1-2) twice yields,

$$\sigma_k^{ii} h_{iil} = d_X \psi(\tau_l) + \kappa_l d_\nu \psi(\tau_l) \tag{4-11}$$

and

$$\sigma_k^{ii} h_{iill} + \sigma_k^{pq,rs} h_{pql} h_{rsl} \geqslant -C - C h_{11}^2 + \sum_s h_{sll} d_\nu \psi(\tau_s), \tag{4-12}$$

where C is some uniform constant. Note that

$$h_{llii} = h_{iill} - h_{ii}h_{ll}^2 + h_{ii}^2 h_{ll}. \tag{4-13}$$

Inserting (4-12) and (4-13) into (4-10), we obtain

$$\begin{aligned}\sigma_k^{ii}\phi_{ii} \geqslant \frac{1}{P\log P}\bigg[\sum_l e^{\kappa_l}\Big(-C - C\kappa_1^2 - \sigma_k^{pq,rs}h_{pql}h_{rsl} + \sum_s h_{sll}d_\nu\psi(\partial_s)\Big) \\ + \sum_l \sigma_k^{ii}e^{\kappa_l}h_{lli}^2 + \sigma_k^{ii}\sum_{p\neq q}\frac{e^{\kappa_p}-e^{\kappa_q}}{\kappa_p-\kappa_q}h_{pqi}^2 - \Big(\frac{1}{P}+\frac{1}{P\log P}\Big)\sigma_k^{ii}P_i^2\bigg] \\ + \frac{\beta k\sigma_k\langle\nu, E\rangle}{s-u} - \frac{\beta\sigma_k^{ii}u_i^2}{(s-u)^2} - N\sigma_k^{ii}\kappa_i^2\langle\nu, E\rangle + \sum_s Nu_s\sigma_k^{ii}h_{sii} - \sigma_k^{ii}\kappa_i^2.\end{aligned} \tag{4-14}$$

From (4-9) and (4-11), we deduce

$$\frac{1}{P\log P}\sum_j\sum_l e^{\kappa_l}h_{jll}d_\nu\psi(\tau_j) + \sum_j Nu_j\sigma_k^{ii}h_{sii} \geqslant \sum_l d_\nu\psi(\tau_l)\frac{\beta u_l}{s-u} - C.$$

For any constant $K > 1$, write

$$A_i = e^{\kappa_i}\bigg[K(\sigma_k)_i^2 - \sum_{p\neq q}\sigma_k^{pp,qq}h_{ppi}h_{qqi}\bigg], \quad B_i = 2\sum_{l\neq i}\sigma_k^{ii,ll}e^{\kappa_l}h_{lli}^2,$$

$$C_i = \sigma_k^{ii}\sum_l e^{\kappa_l}h_{lli}^2, \quad D_i = 2\sum_{l\neq i}\sigma_k^{ll}\frac{e^{\kappa_l}-e^{\kappa_i}}{\kappa_l-\kappa_i}h_{lli}^2, \quad E_i = \frac{1+\log P}{P\log P}\sigma_k^{ii}P_i^2.$$

Note that

$$-\sum_l \sigma_k^{pq,rs}h_{pql}h_{rsl} = \sum_{p\neq q}\sigma_k^{pp,qq}h_{pql}^2 - \sum_{p\neq q}\sigma_k^{pp,qq}h_{ppl}h_{qql}.$$

Therefore, (4-14) becomes

$$\begin{aligned}\sigma_k^{ii}\phi_{ii} \geqslant \frac{1}{P\log P}\sum_i(A_i+B_i+C_i+D_i-E_i) + \frac{\beta k\sigma_k\langle\nu, E\rangle}{s-u} - \frac{\beta\sigma_k^{ii}u_i^2}{(s-u)^2} \\ + (-N\langle\nu, E\rangle - 1)\sigma_k^{ii}\kappa_i^2 + \sum_l d_\nu\psi(\tau_l)\frac{\beta u_l}{s-u} - C\kappa_1.\end{aligned} \tag{4-15}$$

Following the same argument as that in the proof of Lemma 13, from (4-15) we obtain

$$\begin{aligned}0 \geqslant \sum_{|\kappa_i|<\delta\kappa_1}\frac{\alpha}{(P\log P)^2}\sigma_k^{ii}P_i^2 + \frac{\beta k\sigma_k\langle\nu, E\rangle}{s-u} - \frac{\beta\sigma_k^{ii}u_i^2}{(s-u)^2} \\ + (-N\langle\nu, E\rangle - 1)\sigma_k^{ii}\kappa_i^2 + \sum_l d_\nu\psi(\tau_l)\frac{\beta u_l}{s-u} - C\kappa_1.\end{aligned} \tag{4-16}$$

Here, the constant δ is the same constant as the one chosen in Claim 1 of Lemma 13. Moreover, by (4-9),

$$-\frac{\beta\sigma_k^{ii}u_i^2}{(s-u)^2} \geqslant -\frac{\sigma_k^{ii}}{\beta}\bigg[2\Big(\frac{P_i}{P\log P}\Big)^2 + 2N^2u_i^2\kappa_i^2\bigg].$$

Choosing $\beta > 0$ such that $\alpha\beta > 2$, (4-16) implies

$$0 \geqslant \frac{\beta k \sigma_k \langle \nu, E\rangle}{s-u} - \sum_{|\kappa_i| \geqslant \delta\kappa_1} \frac{\beta \sigma_k^{ii} u_i^2}{(s-u)^2} + (-N\langle \nu, E\rangle - 1)\sigma_k^{ii}\kappa_i^2 + \sum_l d_\nu \psi(\tau_l) \frac{\beta u_l}{s-u} - C\kappa_1 - \sum_{|\kappa_i| < \delta\kappa_1} \frac{\sigma_k^{ii}}{\beta} 2N^2 u_i^2 \kappa_i^2. \tag{4-17}$$

Now, first choose $N > 0$ such that

$$\frac{1}{2} \sum_{|\kappa_i| \geqslant \delta\kappa_1} \sigma_k^{ii} \kappa_i^2 (-N\langle \nu, E\rangle - 1) - C\kappa_1 \geqslant 0.$$

Then choose $\beta = \beta(N)$ sufficiently large such that

$$\sum_{|\kappa_i| < \delta\kappa_1} \left(\sigma_k^{ii} \kappa_i^2 (-N\langle \nu, E\rangle - 1) - \frac{\sigma_k^{ii}}{\beta} 2N^2 u_i^2 \kappa_i^2 \right) \geqslant 0.$$

We deduce

$$\frac{\beta C}{s-u} + \sum_{|\kappa_i| \geqslant \delta\kappa_1} \frac{2\beta \sigma_k^{ii} u_i^2}{(s-u)^2} \geqslant \sum_{|\kappa_i| \geqslant \delta\kappa_1} \sigma_k^{ii} \kappa_i^2 (-N\langle \nu, E\rangle - 1). \tag{4-18}$$

If

$$\frac{C}{s-u} \geqslant \sum_{|\kappa_i| \geqslant \delta\kappa_1} \frac{2\beta \sigma_k^{ii} u_i^2}{(s-u)^2},$$

we get

$$\frac{2C\beta}{s-u} \geqslant \sigma_k^{11} \kappa_1^2 (-N\langle \nu, E\rangle - 1) \geqslant c_0 (N-1)\kappa_1,$$

which implies the desired estimate. If

$$\frac{C}{s-u} \leqslant \sum_{|\kappa_i| \geqslant \delta\kappa_1} \frac{2\beta \sigma_k^{ii} u_i^2}{(s-u)^2},$$

we let i_0 denote the index of the maximum value element of the set

$$\left\{ \frac{2\beta \sigma_k^{ii} u_i^2}{(s-u)^2} \,\middle|\, |\kappa_i| \geqslant \delta\kappa_1 \right\}.$$

Then, we obtain the following, which implies our desired estimate:

$$4n \frac{\beta \sigma_k^{i_0 i_0} u_{i_0}^2}{(s-u)^2} \geqslant \sigma_k^{i_0 i_0} \kappa_{i_0}^2 (-N\langle \nu, E\rangle - 1) \geqslant C(N-1)\sigma_k^{i_0 i_0} \delta^2 \kappa_1^2.$$

□

5. The prescribed curvature problem

We will prove Theorem 1 and 5 in this section.

Let's consider the proof of Theorem 1 first. Recall that in Section 3.1, we have solved the approximate Dirichlet problem (3-5) on B_r for $r < 1$. We will denote the strictly convex solution of (3-5) by u^{r*}. We further denote the Legendre transform of (B_r, u^{r*}) by (Ω_r, u^r), where $\Omega_r = Du^{r*}(B_r)$ is the domain of u^r. By Lemmas 19 and 20 in [Wang and Xiao 2022], we have

$$\underline{u} \leqslant u^r \leqslant \bar{u} \quad \text{in } \Omega_r. \tag{5-1}$$

In the following, we will write $\widetilde{\Omega}_r = D\underline{u}^*(B_r)$ for the domain of $\underline{u}_r := \underline{u}|_{\widetilde{\Omega}_r}$. It is not difficult to see that these domains are increasing, namely,

$$\widetilde{\Omega}_r \subset \widetilde{\Omega}_s \quad \text{for } r < s.$$

Moreover, by the choice of $\underline{u}$ in Section 3.1, we have

$$\underline{u}|_{\partial\widetilde{\Omega}_r} \to +\infty \quad \text{as } r \to 1.$$

Thus, by the comparison principle, we have

$$u_r|_{\partial\Omega_r} = [\xi \cdot Du_r^*(\xi) - u_r^*(\xi)]|_{\partial B_r} \geqslant [\xi \cdot D\underline{u}^*(\xi) - \underline{u}^*(\xi)]|_{\partial B_r} = \underline{u}|_{\partial\widetilde{\Omega}_r}. \tag{5-2}$$

From this we can see that, as $r \to 1$, $u_r|_{\partial\Omega_r} \to +\infty$. This in turn implies, for any compact set $\mathcal{K} \subset \mathbb{R}^n$, there exists a constant $c_{\mathcal{K}} = c(\mathcal{K}) < 1$ such that, when $r > c_{\mathcal{K}}$, $\Omega_r \supset \mathcal{K}$. Therefore, for any compact set $\mathcal{K} \subset \mathbb{R}^n$, we can apply Lemmas 14 and 16 to obtain uniform C^1 and C^2 bounds for u^r in $\mathcal{K}$.

More precisely, in order to obtain the local C^1 estimate, we introduce a new subsolution $\underline{u}_1$ of (1-2), where $\underline{u}_1$ satisfies

$$\sigma_k(\kappa_1, \dots, \kappa_n) = c_1 + 100$$

and, as $|x| \to \infty$,

$$\underline{u}_1 \to |x| + \varphi\left(\frac{x}{|x|}\right).$$

By the strong maximum principle, we have, when $x \in \mathbb{R}^n$,

$$\underline{u}_1(x) < \underline{u}(x).$$

Thus, for any compact convex domain $\mathcal{K}$, let

$$2\delta = \min_{\mathcal{K}}(\underline{u} - \underline{u}_1).$$

We define a strict spacelike function $\Psi = \underline{u}_1 + \delta$. Set $\mathcal{K}' = \{x \in \mathbb{R}^n \mid \Psi \leqslant \bar{u}\}$. Since, as $|x| \to \infty$, we have $\underline{u}_1 - \bar{u} \to 0$, we know that $\mathcal{K}'$ is a compact set only depending on $\mathcal{K}$. Applying Lemma 14, for any (Ω_r, u^r), if $\mathcal{K}' \subset \Omega_r$, we have the gradient estimate

$$\sup_{\mathcal{K}} \frac{1}{\sqrt{1 - |Du^r|^2}} \leqslant \frac{1}{\delta} \sup_{\mathcal{K}'} \frac{\bar{u} - \Psi}{\sqrt{1 - |D\Psi|^2}}.$$

Next, we want to show that, for any given compact set $\mathcal{K} \subset \mathbb{R}^n$, the set $\{|D^2 u^r|\}$ is uniformly bounded in $\mathcal{K}$. Without loss of generality, let's consider any $B_R \subset \mathbb{R}^n$. Let $C_0 = \max_{B_R} \bar{u}$ and $s = 2C_0 + 1$ in Lemma 16. Set $U_s = \{x \in \mathbb{R}^n \mid \underline{u}(x) < s\}$. Then by our earlier discussion, it's easy to see that there exists $r_s > 0$ such that, when $r > r_s$, we have $\Omega_r \supset U_s$. Applying Lemma 16, we obtain, when $r > r_s$,

$$\sup_{B_R} \kappa_{\max}(M_{u^r}) \leqslant C.$$

Here C depends on the upper bound of $1/\sqrt{1 - |Du^r|^2}$ on $\bar{U}_s$, which is independent of r. Using the classical regularity theorem and convergence theorem, we conclude that (Ω_r, u^r) converges locally smoothly to an entire, smooth convex function u satisfying (1-2). In view of (5-1) and the asymptotic behavior of $\underline{u}$ and $\bar{u}$, we know that, as $|x| \to \infty$, we have $u \to |x| + \varphi(x/|x|)$. Moreover, by Remark 2, we also know that u is strictly convex. Therefore, its Gauss map image is B_1, i.e., $Du(\mathbb{R}^n) = B_1$.

Theorem 5 follows by replacing Lemmas 14 and 16 in the proof of Theorem 1 with Lemmas 15 and 17.

6. The radial downward translating soliton

We will now study the radially symmetric downward translating soliton. Recall that we say $\mathcal{M}_u$ is a downward translating soliton when its principal curvatures satisfy

$$\sigma_k(\kappa[\mathcal{M}_u]) = \binom{n}{k}\left(\mathcal{C} - \frac{1}{\sqrt{1-|Du|^2}}\right)^k, \tag{6-1}$$

where $\mathcal{C} > 1$ is a constant. We want to point out that in this section and the next, $\mathcal{C}$ is the fixed constant in (6-1). We also write

$$\tilde{\mathcal{C}} = \sqrt{1 - \frac{1}{\mathcal{C}^2}}$$

as in Theorem 7. The following theorem is a generalization of Theorem 1 in [Bayard 2023].

Theorem 18. *Let $\mathcal{C} > 1$ be a positive constant. Then there exists a strictly convex radial solution $u : \mathbb{R}^n \to \mathbb{R}$ of* (6-1) *satisfying*

$$|Du| \to \tilde{\mathcal{C}} \quad \text{as } |x| \to +\infty.$$

Moreover, $u(x)$ has the following asymptotic expansion as $|x| \to \infty$:

$$u(x) = \tilde{\mathcal{C}}|x| - \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}\log|x| + c_0 + o(1) \tag{6-2}$$

for some constant $c_0 \in \mathbb{R}$. In particular, the radial solution u is unique up to the addition of a constant.

For radial solutions, we will reduce (6-1) to an ODE. Let $u = u(r)$ and $y = \partial u/\partial r$. Then a straightforward calculation yields

$$D_i u = y\frac{x_i}{|x|} \quad \text{and} \quad D^2_{ij}u = \frac{y}{|x|}\left(\delta_{ij} - \frac{x_i x_j}{|x|^2}\right) + y'\frac{x_i x_j}{|x|^2}.$$

Therefore,

$$\kappa[\mathcal{M}_u] = \frac{1}{\sqrt{1-y^2}}\left(\frac{y'}{1-y^2}, \frac{y}{r}, \dots, \frac{y}{r}\right),$$

and (6-1) becomes

$$\frac{1}{(1-y^2)^{k/2}}\frac{y^{k-1}}{r^{k-1}}\left(\frac{k}{n}\frac{y'}{1-y^2} + \frac{n-k}{n}\frac{y}{r}\right) = \left(\mathcal{C} - \frac{1}{\sqrt{1-y^2}}\right)^k. \tag{6-3}$$

By a small modification of the proof of Proposition 2.1 in [Bayard 2023], we obtain the following.

Proposition 19. *Under the hypotheses of Theorem 18, there exists a solution y of* (6-3), *which is defined on* $[0, +\infty)$ *and smooth on* $(0, +\infty)$, *such that*

$$y(0) = 0, \quad 0 \leqslant y < \tilde{\mathcal{C}}, \quad \lim_{r\to+\infty} y(r) = \tilde{\mathcal{C}}, \quad y'(0) = \mathcal{C} - 1, \quad \text{and} \quad y' > 0 \quad \text{on } [0, +\infty).$$

Moreover, as $r \to 0+$, we have

$$\kappa[\mathcal{M}_u(r)] \to (\mathcal{C} - 1)(1, 1, \dots, 1).$$

Since the proof is a small modification of the proof of Proposition 2.1 in [Bayard 2023], we skip it here. Now, let's study the asymptotic behavior of y.

Proposition 20. *Let y be the solution of (6-3). Then y has the following asymptotic expansion as $r \to \infty$:*

$$y(r) = \tilde{\mathcal{C}} - \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}\frac{1}{r} + O\left(\frac{1}{r^2}\right).$$

Proof. By Proposition 19, we may assume

$$y(r) = \tilde{\mathcal{C}} - \frac{z}{r}. \tag{6-4}$$

Then we have

$$\sqrt{1-y^2} - \frac{1}{\mathcal{C}} = \frac{1 - 1/\mathcal{C}^2 - y^2}{\sqrt{1-y^2} + 1/\mathcal{C}} = \frac{z}{r}A(r), \quad \text{where } A(r) = \frac{\sqrt{1-1/\mathcal{C}^2} + y}{\sqrt{1-y^2} + 1/\mathcal{C}}. \tag{6-5}$$

Differentiating (6-4) then substituting it into (6-3), we get

$$\frac{k}{n}\frac{y^{k-1}}{1-y^2}\left(-\frac{z'}{r^k} + \frac{z}{r^{k+1}}\right) + \frac{n-k}{n}\frac{y^k}{r^k} = \mathcal{C}^k\left(\sqrt{1-y^2} - \frac{1}{\mathcal{C}}\right)^k. \tag{6-6}$$

By (6-5), (6-6) can be simplified as

$$\frac{k}{n}\frac{y^{k-1}}{1-y^2}\left(-z' + \frac{z}{r}\right) + \frac{n-k}{n}y^k = \mathcal{C}^k z^k A^k(r).$$

Thus, we obtain

$$z' = -B(r)z^k + C(r), \tag{6-7}$$

where

$$B(r) = \mathcal{C}^k \frac{n}{k}\frac{1-y^2}{y^{k-1}}A^k(r) \quad \text{and} \quad C(r) = \frac{z}{r} + \frac{n-k}{k}y(1-y^2). \tag{6-8}$$

Applying Proposition 19, we can see that

$$\lim_{r\to+\infty} B(r) = \frac{n}{k}\mathcal{C}^{2k-2}\tilde{\mathcal{C}} \quad \text{and} \quad \lim_{r\to+\infty} C(r) = \frac{n-k}{k}\frac{1}{\mathcal{C}^2}\tilde{\mathcal{C}}.$$

Here, we have used $\lim_{r\to\infty}(z/r) = 0$, which is a direct consequence of Proposition 19. The next lemma is a generalization of Proposition A.2 in [Bayard 2023].

Lemma 21. *Assume $z : (0, +\infty) \to \mathbb{R}$ is a positive solution of the equation*

$$z' = -A(r)z^k + B(r),$$

where $A, B : (0, \infty) \to \mathbb{R}$ are continuous functions such that

$$\lim_{r\to+\infty} A(r) = A_0 > 0 \quad \textit{and} \quad \lim_{r\to+\infty} B(r) = B_0 > 0.$$

Then

$$\lim_{r\to+\infty} z(r) = \sqrt[k]{\frac{B_0}{A_0}}.$$

Proof. In order to prove this lemma, we only need to prove the following claim.

Claim 3. *Assume $z : (0, +\infty) \to \mathbb{R}$ is a positive solution of the equation*

$$z' = A_0 z^k + B_0,$$

with $A_0 < 0$ and $B_0 > 0$ constants. Then

$$\lim_{r\to\infty} z(r) = \left(-\frac{B_0}{A_0}\right)^{1/k}.$$

If this claim is true, following the same argument as Proposition A.2 in [Bayard 2023], we can prove Lemma 21. We will prove this claim below.

Without loss of generality, let's consider the positive solution of the equation

$$z' = B - z^k \tag{6-9}$$

instead. We will show that

$$\lim_{r\to\infty} z(r) = B^{1/k}. \tag{6-10}$$

First, since z is a positive solution of (6-9), let's assume $0 < z(r_0) = z_0 < B^{1/k}$. Then we have $z_0 < z(r) < B^{1/k}$ on (r_0, ∞). Writing $z_1 = B^{1/k}$, we get

$$z^k - B = (z - z_1)(z^{k-1} + z^{k-2} z_1 + \cdots + z_1^{k-1}).$$

Therefore, (6-9) can be written as

$$-dr = \left[\frac{A_1}{z - z_1} + \frac{Q_{k-2}(z)}{z^{k-1} + z^{k-2} z_1 + \cdots + z_1^{k-1}}\right] dz, \tag{6-11}$$

where $A_1 = z_1^{1-k}/k$ and $Q_{k-2}(z)$ is a polynomial of degree $k-2$. It's easy to see that

$$Q_{k-2}(z) = -A_1 z^{k-2} + Q(k-3)(z)$$

and $Q_{k-3}(z)$ is a polynomial of degree $k-3$. Integrating (6-11) from r_0 to r yields

$$-r + r_0 = A_1 \ln\left|\frac{z(r) - z_1}{z_0 - z_1}\right| - \int_{z_0}^{z(r)} \frac{A_1 z^{k-2}}{z^{k-1} + z^{k-2} z_1 + \cdots + z_1^{k-1}}\, dz + \int_{z_0}^{z(r)} \frac{Q_{k-3}(z)}{z^{k-1} + z^{k-2} z_1 + \cdots + z_1^{k-1}}\, dz. \tag{6-12}$$

Notice that, as $r \to \infty$, the left-hand side of (6-12) goes to $-\infty$, while

$$-\int_{z_0}^{z(r)} \frac{A_1 z^{k-2}}{z^{k-1} + z^{k-2} z_1 + \cdots + z_1^{k-1}}\, dz \geqslant -A_1 \ln\left|\frac{z_1}{z_0}\right|$$

and

$$\left|\int_{z_0}^{z(r)} \frac{Q_{k-3}(z)}{z^{k-1} + z^{k-2} z_1 + \cdots + z_1^{k-1}}\, dz\right|$$

is bounded. Therefore, $\lim_{r\to\infty} z(r) = z_1 = B^{1/k}$. We similarly prove the case when $z(r_0) = z_0 > z_1$. □

From Lemma 21 and (6-7), we conclude

$$\lim_{r\to+\infty} z(r) = \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}.$$

We further assume

$$z(r) = \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}} + \frac{w(r)}{r}.$$

Inserting it into (6-7), we get

$$w' = -D(r)w + F(r),$$

where

$$D(r) = B(r)\sum_{i=1}^{k}\binom{k}{i}\left(\frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}\right)^{k-i}\left(\frac{w}{r}\right)^{i-1}$$

and

$$F(r) = r\left(C(r) - \frac{B(r)}{\mathcal{C}^{2k}}\frac{n-k}{n}\right) + \frac{w}{r}.$$

Notice that $\lim_{r\to+\infty}(w/r) = 0$ and $D(r)$ has a uniform positive lower bound. In the following, we want to find a positive upper bound for $F(r)$. Using the expressions (6-8) for $B(r)$ and $C(r)$, we obtain

$$\begin{aligned} F(r) &= \frac{w}{r} + z + \frac{n-k}{k}\frac{1-y^2}{y^{k-1}} r\left[y^k - \left(\frac{A(r)}{\mathcal{C}}\right)^k\right] \\ &= \frac{w}{r} + z + \frac{n-k}{k}\frac{1-y^2}{y^{k-1}} r\left(y - \frac{A(r)}{\mathcal{C}}\right)\sum_{i=1}^{k} y^{k-i}\left(\frac{A(r)}{\mathcal{C}}\right)^{i-1}. \end{aligned} \tag{6-13}$$

Therefore, we only need to show $r(y - A(r)/\mathcal{C})$ is bounded as $r\to\infty$. By (6-5), we have

$$\begin{aligned} r\left(y - \frac{A(r)}{\mathcal{C}}\right) &= r\left(y - \frac{1}{\mathcal{C}}\frac{\sqrt{1-1/\mathcal{C}^2}+y}{\sqrt{1-y^2}+1/\mathcal{C}}\right) \\ &= \frac{r(y\sqrt{1-y^2} - (1/\mathcal{C})\sqrt{1-1/\mathcal{C}^2})}{\sqrt{1-y^2}+1/\mathcal{C}}. \end{aligned} \tag{6-14}$$

Combining (6-14) with the expression for y and (6-5), we can derive

$$\begin{aligned} y\sqrt{1-y^2} - \frac{1}{\mathcal{C}}\sqrt{1-\frac{1}{\mathcal{C}^2}} &= \left(\sqrt{1-\frac{1}{\mathcal{C}^2}} - \frac{z}{r}\right)\left(\frac{1}{\mathcal{C}} + \frac{zA(r)}{r}\right) - \frac{1}{\mathcal{C}}\sqrt{1-\frac{1}{\mathcal{C}^2}} \\ &= \frac{z}{r}\left(-\frac{1}{\mathcal{C}} + A(r)\sqrt{1-\frac{1}{\mathcal{C}^2}}\right) - \frac{z^2A(r)}{r^2}. \end{aligned} \tag{6-15}$$

From (6-14), (6-15), and Lemma 21, we conclude that $r(y - A(r)/\mathcal{C})$ is uniformly bounded from above. Thus, $F(r)$ has an uniform upper bound. Applying Proposition A.3 in [Bayard 2023], we obtain a uniform upper bound for w. □

It's not hard to see that Theorem 18 follows from Propositions 19 and 20.

7. The existence results

In this section we will prove Theorem 7. First, we want to prove the following existence theorem.

Proposition 22. *Suppose φ is a C^2 function defined on $\mathbb{S}^{n-1}_{\tilde{C}} := \{x \in \mathbb{R}^n \mid |x| = \tilde{C}\}$, where $\tilde{C} = \sqrt{1-(1/C)^2}$. There exists a unique, strictly convex solution $u : \mathbb{R}^n \to \mathbb{R}$ of* (1-10) *such that, as $|x| \to \infty$,*

$$u(x) \to \tilde{C}|x| - \frac{1}{C^2}\sqrt[k]{\frac{n-k}{n}}\log|x| + \varphi\left(\tilde{C}\frac{x}{|x|}\right). \tag{7-1}$$

7.1. ***Constructing barriers.*** We first construct the barrier functions of (1-10). Following the ideas of [Spruck and Xiao 2016; Treibergs 1982], we denote the radial solution of (1-10) by $z_0^k(|x|)$, whose asymptotic expansion satisfies (6-2) with $c_0 = 0$. Let

$$p_i(\tilde{C}y) = D\varphi(\tilde{C}y) + (-1)^{i+1}2M\tilde{C}y, \quad i = 1, 2,$$

for any $y \in \mathbb{S}^{n-1}$. Set

$$z_i^k(x, y) = \varphi(\tilde{C}y) - p_i(\tilde{C}y)\cdot\tilde{C}y + z_0^k(|x + p_i(\tilde{C}y)|) \quad \text{for all } x \in \mathbb{R}^n, \quad y \in \mathbb{S}^{n-1}.$$

Then

$$q_1^k(x) = \sup_{y\in\mathbb{S}^{n-1}} z_1^k(x, y)$$

is a subsolution of (1-10) and

$$q_2^k = \inf_{y\in\mathbb{S}^{n-1}} z_2^k(x, y)$$

is a supersolution of (1-10). Moreover, $q_1^k(x) \leqslant q_2^k(x)$, and, when $|x| \to +\infty$, we have

$$q_i^k(x) \to \tilde{C}|x| - \frac{1}{C^2}\sqrt[k]{\frac{n-k}{n}}\log|x| + \varphi\left(\tilde{C}\frac{x}{|x|}\right), \quad i = 1, 2.$$

7.2. ***The Dirichlet problem.*** First, let's solve (1-10) for the case $k = n$. For any $t > \min_{\mathbb{R}^n} q_2^n$, we let

$$\partial\Omega_t = \{x \in \mathbb{R}^n \mid q_1^n(x) < t < q_2^n(x)\}$$

and Ω_t be a smooth, strictly convex domain in $\mathbb{R}^n$. Consider the Dirichlet problem

$$\begin{cases} \sigma_n^{1/n}(\kappa(\mathcal{M}_{u_t})) = C + \langle \nu, E\rangle & \text{in } \Omega_t, \\ u_t = t & \text{on } \partial\Omega_t. \end{cases} \tag{7-2}$$

By a small modification of [Delanoë 1990], we know that there exists a unique solution u_t of (7-2). Then, applying the local C^1 and C^2 estimates obtained in [Bayard and Schnürer 2009], we conclude that there exists a subsequence $\{u_{t_i}\}_{i=1}^\infty$ ($t_i \to \infty$ as $i \to \infty$) that converges to an entire, strictly convex solution u of (1-10) for $k = n$. Moreover, it's easy to see that $u(x)$ satisfies the desired asymptotic behavior as $|x| \to \infty$. From now on, we will denote this solution by u^n. We will also denote the Legendre transform of u^n by u^{n*}.

Next, we consider the case when $k < n$. We denote the Legendre transform of z_0^k by $(z_0^k)^*$; that is,

$$(z_0^k)^*(\tau) = r\cdot\frac{\partial z_0^k}{\partial r} - z_0^k(r), \quad \text{where } \tau = \frac{\partial z_0^k}{\partial r}.$$

Using the asymptotic expansion of z_0 derived in Section 6, we know

$$(z_0^k)^*(\tau) = \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}(\log r - 1) + O\Big(\frac{1}{r}\Big).$$

Writing its principal part as

$$(\tilde{z}_0^k)^*(\tau) = \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}(\log r(\tau) - 1),$$

it is clear that $(\tilde{z}_0^k)^*$ is unbounded in $B_{\tilde{\mathcal{C}}}$.

To make sure our solution is convex, we consider the dual Dirichlet problem on B_τ for any $\tau < \tilde{\mathcal{C}}$:

$$\begin{cases} \hat{F}(w^*\gamma_{ik}^*u_{kl}^*\gamma_{lj}^*) = \dfrac{\binom{n}{k}^{-1/k}}{\mathcal{C} - 1/\sqrt{1-|\xi|^2}} & \text{in } B_\tau, \\ u^* = u^{n*} + (z_0^k)^* - (z_0^n)^* & \text{on } \partial B_\tau. \end{cases} \tag{7-3}$$

Here, we have

$$w^* = \sqrt{1-|\xi|^2}, \quad \gamma_{ij}^* = \delta_{ij} - \frac{\xi_i\xi_j}{1+w^*}, \quad u_{kl}^* = \frac{\partial^2 u}{\partial\xi_k\partial\xi_l}, \quad \hat{F}(w^*\gamma_{ik}^*u_{kl}^*\gamma_{lj}^*) = \Big(\frac{\sigma_n}{\sigma_{n-k}}(\kappa^*[w^*\gamma_{ik}^*u_{kl}^*\gamma_{lj}^*])\Big)^{1/k},$$

and $\kappa^*[w^*\gamma_{ik}^*u_{kl}^*\gamma_{lj}^*] = (\kappa_1^*, \dots, \kappa_n^*)$ is the set of eigenvalues of the matrix $(w^*\gamma_{ik}^*u_{kl}^*\gamma_{lj}^*)$. The solvability of (7-3) has been established in Section 3. Therefore, by standard PDE theorems, in order to prove Proposition 22, we only need to obtain local C^1 and C^2 estimates for the translating soliton equation (1-10). In order to do so, we will need the following lemma.

Lemma 23. *Let $u^{\tau*}$ be a solution to* (7-3) *and u^τ be the Legendre transform of $u^{\tau*}$. Then, for any $x \in Du^{\tau*}(B_\tau)$, we have $q_1^k(x) \leqslant u^\tau(x) \leqslant q_2^k(x)$.*

Proof. Without causing confusion we shall drop the superscript τ in the proof. We only need to prove that

$$z_1^k(x, y) \leqslant u(x) \leqslant z_2^k(x, y)$$

for any $x \in Du^{\tau*}(B_\tau)$ and $y \in \mathbb{S}^{n-1}$. This is equivalent to proving

$$(z_2^k)^*(\xi, y) \leqslant u^*(\xi) \leqslant (z_1^k)^*(\xi, y)$$

for any $\xi \in B_\tau$ and $y \in \mathbb{S}^{n-1}$. Since we have

$$\begin{aligned}(z_i^k)^*(\xi, y) &= (z_0^k)^*(|\xi|) - p_i(\tilde{\mathcal{C}}y)\cdot\xi - \varphi(\tilde{\mathcal{C}}y) + p_i(\tilde{\mathcal{C}}y)\cdot\tilde{\mathcal{C}}y \\ &= (z_0^k)^*(|\xi|) - (z_0^n)^*(|\xi|) + (z_i^n)^*(\xi, y)\end{aligned} \tag{7-4}$$

and

$$(z_2^n)^*(\xi, y) < u^{n*}(\xi) < (z_1^n)^*(\xi, y),$$

we obtain, on ∂B_τ,

$$(z_2^k)^*(\xi, y) \leqslant u^*(\xi) \leqslant (z_1^k)^*(\xi, y).$$

By the comparison principle, we finish the proof. □

7.3. *Local C^1 and C^2 estimates.* Similar to Lemma 14, we have the following local C^1 estimate lemma for translating solitons.

Lemma 24. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $u, \bar{u}, \Psi : \Omega \to \mathbb{R}^n$ be strictly $\mathcal{C}$-spacelike, i.e.,*

$$|Du|,\ |D\bar{u}|,\ |D\Psi| < \tilde{\mathcal{C}}.$$

Assume that u is strictly convex and $u \leqslant \bar{u}$ in Ω. Also assume that, near $\partial\Omega$, we have $\Psi > \bar{u}$. Consider the set with $u > \Psi$. For every x in that set, we have the following gradient estimate for u:

$$\frac{1}{\sqrt{\tilde{\mathcal{C}}^2 - |Du|^2}} \leqslant \frac{1}{u(x) - \Psi(x)} \cdot \sup_{\{u > \Psi\}} \frac{\bar{u} - \Psi}{\sqrt{\tilde{\mathcal{C}}^2 - |D\psi|^2}}.$$

Since the proof is the same as the proof of Lemma 5.1 in [Bayard and Schnürer 2009], we skip it here.

We now construct Ψ. Following the argument in Section 4 of [Bayard 2023], let

$$\Psi(x) = -A_0 + \tilde{\mathcal{C}}\sqrt{1 + |x|^2}.$$

It is clear that, when $|x|$ is sufficiently large, we have $\Psi(x) > q_2(x)$. On the other hand, for any compact set $\mathcal{K} \subset \mathbb{R}^n$, we can always choose A_0 large enough that $\Psi(x) < q_1(x)$ in $\mathcal{K}$. Applying Lemma 24 we obtain that, for any $\mathcal{K} \subset \mathbb{R}^n$ and any strictly convex function $q_1(x) < u(x) < q_2(x)$ satisfying (1-10), whose domain of definition contains $\mathcal{K}$, there exists a local C^1 bound $C_{\mathcal{K}}$ for $u(x)$ in $\mathcal{K}$ that only depends on $\mathcal{K}$.

Using the idea of [Wang and Xiao 2022], we can prove the following Pogorelov-type local C^2 estimate for translating solitons.

Lemma 25. *Let u be the solution of* (1-10) *defined on Ω. For any given $s > \min_{\mathbb{R}^n} u(x) + 1$, suppose $u|_{\partial\Omega} > s$. Let $\kappa_{\max}(x)$ be the largest principal curvature of $\mathcal{M}_u = \{(x, u(x)) \mid x \in \Omega\}$ at x. Then we have*

$$\max_{\mathcal{M}_u} (s - u)\kappa_{\max} \leqslant C_1.$$

Here, C_1 only depends on the local C^1 estimate of u. More specifically, C_1 depends on the lower bound of $\mathcal{C} + \langle \nu, E \rangle$.

Following the argument in Section 5, we complete the proof of Proposition 22.

7.4. *Proof of Theorem 7.* In this subsection, we will prove that the hypersurface $\mathcal{M}_u$ constructed in Proposition 22 has bounded principal curvatures. This completes the proof of Theorem 7. For our convenience, in the following, we will drop the superscript k, and the updated configuration z_0^k now becomes z_0.

Suppose u is a strictly convex solution of (1-10) and u^* is the Legendre transform of u. Then u^* satisfies

$$\hat{F}(w^* \gamma^*_{ik} u^*_{kl} \gamma^*_{lj}) = \frac{\binom{n}{k}^{-1/k}}{\mathcal{C} - 1/\sqrt{1 - |\xi|^2}} \quad \text{in } B_{\tilde{\mathcal{C}}}. \tag{7-5}$$

We also denote the Legendre transform of z_0 by z_0^*; that is,

$$z_0^*(\tau) = r \cdot \frac{\partial z_0}{\partial r} - z_0(r), \quad \text{where } \tau = \frac{\partial z_0}{\partial r}.$$

Using the asymptotic expansion of z_0 derived in Section 6, we know

$$z_0^*(\tau) = \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}(\log r - 1) + O\Big(\frac{1}{r}\Big).$$

Writing its principal part as

$$\tilde{z}_0^*(\tau) = \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}(\log r(\tau) - 1),$$

it is clear that $\tilde{z}_0^*(\tau)$ is unbounded in $B_{\tilde{\mathcal{C}}}$.

Lemma 26. *Let u^* and $\tilde{z}_0^*$ be defined as above. Then we have*

$$\lim_{\xi\to\xi_0}(u^*(\xi) - \tilde{z}_0^*(|\xi|)) = -\varphi(\xi_0) \quad \textit{for any } \xi_0 \in \partial B_{\tilde{\mathcal{C}}}, \ \ \xi \in B_{\tilde{\mathcal{C}}}. \tag{7-6}$$

Proof. We use the auxiliary functions $z_i(x, y)$, $i = 1, 2$, constructed in Section 7.1. It's easy to see that

$$z_1(x, y) < u(x) < z_2(x, y) \quad \text{for any } x \in \mathbb{R}^n, \ \ y \in \mathbb{S}^{n-1}.$$

By the strict convexity of $z_i(x, y)$, we have

$$z_2^*(\xi, y) < u^*(\xi) < z_1^*(\xi, y) \quad \text{for any } \xi \in B_{\tilde{\mathcal{C}}}, \ \ y \in \mathbb{S}^{n-1}. \tag{7-7}$$

Notice that

$$z_i^*(\xi, y) = z_0^*(|\xi|) - p_i(\tilde{\mathcal{C}}y)\cdot\xi - \varphi(\tilde{\mathcal{C}}y) + p_i(\tilde{\mathcal{C}}y)\cdot\tilde{\mathcal{C}}y.$$

Therefore, letting $\tilde{\mathcal{C}}y = \xi_0$ and $\xi \to \xi_0$, we get

$$z_i(\xi, \tilde{\mathcal{C}}^{-1}\xi_0) - z_0^*(|\xi|) \to -\varphi(\xi_0).$$

This together with (7-7) yields (7-6). □

Now we let

$$\partial = \xi_i\frac{\partial}{\partial\xi_j} - \xi_j\frac{\partial}{\partial\xi_i}$$

be the angular derivative. Similar to Section 10 in [Ren et al. 2020], we obtain following lemmas.

Lemma 27. *Let u^* be the solution of* (7-5). *Then $|\partial u^*|$ is bounded above by a constant depending on $|\varphi|_{C^1}$, and $\partial^2 u^*$ is bounded above by a constant depending on $|\varphi|_{C^2}$.*

Proof. Noticing that $\partial|\xi|^2 = 0$, we have that the angular derivative of the right-hand side of (7-5) is zero. Therefore, following the proof of Lemmas 29 and 30 in [Ren et al. 2020], we have

$$F^{ij}w^*\gamma_{ik}^*(\partial(u^* - \tilde{z}_0^*))_{kl}\gamma_{lj}^* = 0 \quad \text{and} \quad F^{ij}w^*\gamma_{ik}^*(\partial^2(u^* - \tilde{z}_0^*))_{kl}\gamma_{lj}^* \geqslant 0.$$

In view of (7-6) and the maximum principle, we obtain the desired estimates. □

Lemma 28. *Let u^* be the solution of* (7-5). *There is a positive constant b such that*

$$\sqrt{\tilde{\mathcal{C}}^2 - |\xi|^2}\,|\partial^2 u^*| < b.$$

Proof. We consider $u^* - \tilde{z}_0^*$, which has C^0 bound on $B_{\tilde{\mathcal{C}}}$. Since $\partial^2 u^* = \partial^2(u^* - \tilde{z}_0^*)$, the rest of the proof is the same as that of Lemma 5.3 in [Li 1995]. □

Lemma 29. *Suppose $a_0 < r < \tilde{\mathcal{C}}$ for some $a_0 \in (0, \tilde{\mathcal{C}})$ and $\mathbb{S}^{n-1}(r) = \{\xi \in \mathbb{R}^n \mid \sum \xi_i^2 = r^2\}$. For any point $\hat{\xi} \in \mathbb{S}^{n-1}(r)$, there is a function*

$$\bar{u}_0^* = z_0^* + b_1\xi_1 + \cdots + b_n\xi_n + b$$

such that

$$\bar{u}_0^*(\hat{\xi}) = u^*(\hat{\xi})$$

and

$$\bar{u}_0^*(\hat{\xi}) > u^*(\xi) \quad \text{for any } \xi \in \mathbb{S}^{n-1}(r) \setminus \{\hat{\xi}\}.$$

Here, $b_1, \dots, b_n$ are constants depending on $\hat{\xi}$, and b is a positive constant independent of $\hat{\xi}$ and r.

Proof. The proof is almost the same as the proof of Lemma 5.4 in [Li 1995]. We only need to replace u, $\bar{u}$, and $-\bar{k}\sqrt{1-|x|^2}$ by $u^* - \tilde{z}_0^*$, $\bar{u}_0^* - \tilde{z}_0^*$, and $z_0^* - \tilde{z}_0^*$, respectively, in Li's proof. □

Similarly, we can prove the following lemma analogous to Lemma 5.5 in [Li 1995].

Lemma 30. *Suppose $a_0 < r < \tilde{\mathcal{C}}$ for some $a_0 \in (0, \tilde{\mathcal{C}})$ and $\mathbb{S}^{n-1}(r) = \{\xi \in \mathbb{R}^n \mid \sum \xi_i^2 = r^2\}$. For any point $\hat{\xi} \in \mathbb{S}^{n-1}(r)$, there is a function*

$$\underline{u}_0^* = z_0^* + a_1\xi_1 + \cdots + a_n\xi_n - a$$

such that

$$\underline{u}_0^*(\hat{\xi}) = u^*(\hat{\xi})$$

and

$$\underline{u}_0^*(\hat{\xi}) < u^*(\xi) \quad \text{for any } \xi \in \mathbb{S}^{n-1}(r) \setminus \{\hat{\xi}\}.$$

Here, $a_1, \dots, a_n$ and a are constants depending on $\hat{\xi}$, $a > 0$, and $a\sqrt{\tilde{\mathcal{C}}^2 - |\hat{\xi}|^2} < C_1$, where C_1 is a positive constant only depending on $|\varphi|_{C^2}$.

Using Lemmas 29 and 30 we can show the following.

Lemma 31. *Let u be the solution of* (1-10) *and u^* be the Legendre transform of u. There are positive constants $d_2 > d_1$ such that*

$$0 < d_1 \leqslant u(\tilde{\mathcal{C}}^2 - |Du|^2) \leqslant d_2. \tag{7-8}$$

Here, d_2 depends on $|u|_{C^0(\Omega)}$, and $\Omega = \{x \in \mathbb{R}^n \mid |Du| \leqslant a_0\}$.

Proof. We modify the proof of Li [1995]. We first consider the lower bound. For any $\hat{\xi} \in \mathbb{S}^{n-1}(r)$, using Lemma 29, we have

$$u^*(\hat{\xi}) = \bar{u}_0^*(\hat{\xi}) \qquad \text{and} \qquad u^*(\xi) < \bar{u}_0^*(\xi) \quad \text{for } \xi \in \mathbb{S}^{n-1}(r) \setminus \{\hat{\xi}\}.$$

Thus, using that $\bar{u}_0^*$ is a supersolution, we get $u^*(\xi) < \bar{u}_0^*(\xi)$ in B_r. Therefore, at $\hat{\xi}$, we get

$$u(\hat{x}) = \hat{\xi} \cdot Du^* - u^* > \hat{\xi} \cdot D\bar{u}_0^* - \bar{u}_0^* = z_0(\hat{r}) - b,$$

where we assume $\hat{x} = Du^*(\hat{\xi})$ and $z_0'(\hat{r}) := \partial z_0/\partial r(\hat{r}) = |\hat{\xi}|$. Thus, at $\hat{x}$, we have

$$u(\tilde{\mathcal{C}}^2 - |Du|^2) > z_0(\hat{r})(\tilde{\mathcal{C}}^2 - |z_0'(\hat{r})|^2) - b(\tilde{\mathcal{C}}^2 - |\hat{\xi}|^2). \tag{7-9}$$

Using the asymptotic behavior of z_0, we have

$$z_0(\tilde{\mathcal{C}}^2-|z_0'|^2)=\left[\tilde{\mathcal{C}}r-\frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}\log r+O\left(\frac{1}{r}\right)\right]\left[\tilde{\mathcal{C}}^2-\left(\tilde{\mathcal{C}}-\frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}\frac{1}{r}+O\left(\frac{1}{r^2}\right)\right)^2\right]=2\frac{\tilde{\mathcal{C}}^2}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}+o(1)$$

We write

$$2c_0=2\frac{\tilde{\mathcal{C}}^2}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}.$$

Therefore, by (7-9), we obtain

$$u(\tilde{\mathcal{C}}^2-|Du|^2)>\tfrac{1}{2}c_0$$

for r sufficiently close to $\tilde{\mathcal{C}}$. We further assume $r>a_0$, since for $r<a_0$, without loss of generality, we can assume $u\geqslant 1$. Therefore,

$$u(\tilde{\mathcal{C}}^2-|\hat{\xi}|^2)\geqslant\tilde{\mathcal{C}}^2-a_0^2.$$

Thus, we obtain the uniform lower bound. For the upper bound, we apply a similar argument. For r sufficiently close to $\tilde{\mathcal{C}}$ and still assuming $r\geqslant a_0$, we have

$$u(\tilde{\mathcal{C}}^2-|Du|^2)<z_0(\hat{r})(\tilde{\mathcal{C}}^2-|z_0'(\hat{r})|^2)+a(\tilde{\mathcal{C}}^2-|\hat{\xi}|^2)\leqslant 3c_0+C_1\tilde{\mathcal{C}}.$$

We have obtained a uniform upper bound. □

Finally, we are ready to adapt the ideas in [Li 1995; Ren et al. 2020] to estimate the principal curvatures of $\mathcal{M}_u$.

Proposition 32. *Let u be the solution of* (1-10). *Then the hypersurface $\mathcal{M}_u=\{(x,u(x))\mid x\in\mathbb{R}^n\}$ has bounded principal curvatures.*

Proof. We will establish a Pogorelov-type interior estimate. For any $s>0$, consider

$$\phi=e^{-s/(s-u)}[u(\mathcal{C}+\langle\nu,E\rangle)]^{-N}P_m^{1/m},$$

where $P_m=\sum_j\kappa_j^m$ and $m,N>0$ are constants to be determined later. Without loss of generality, we also assume $u\geqslant 1$ in $\mathbb{R}^n$. It's easy to see that ϕ achieves its local maximum at an interior point of $U_s=\{x\in\mathbb{R}^n\mid u(x)<s\}$; we will assume this point is x_0. We can choose a local normal coordinate $\{\tau_1,\dots,\tau_n\}$ such that, at x_0, we have $h_{ij}=\kappa_i\delta_{ij}$ and $\kappa_1\geqslant\kappa_2\geqslant\cdots\geqslant\kappa_n$.

Differentiating $\log\phi$ at x_0, we get

$$\frac{\phi_i}{\phi}=\frac{\sum_j\kappa_j^{m-1}h_{jji}}{P_m}-N\frac{h_{ii}\langle\tau_i,E\rangle}{\mathcal{C}+\langle\nu,E\rangle}-N\frac{u_i}{u}-\frac{su_i}{(s-u)^2}=0 \tag{7-10}$$

and

$$\begin{aligned}\frac{\phi_{ii}}{\phi}-\frac{\phi_i^2}{\phi^2}=&\frac{1}{P_m}\left[\sum_j\kappa_j^{m-1}h_{jjii}+(m-1)\sum_j\kappa_j^{m-2}h_{jji}^2+\sum_{p\neq q}\frac{\kappa_p^{m-1}-\kappa_q^{m-1}}{\kappa_p-\kappa_q}h_{pqi}^2\right]\\&-\frac{m}{P_m^2}\left(\sum_j\kappa_j^{m-1}h_{jji}\right)^2-N\sum_l h_{ili}\frac{\langle\tau_l,E\rangle}{\mathcal{C}+\langle\nu,E\rangle}+Nh_{ii}^2\frac{-\langle\nu,E\rangle}{\mathcal{C}+\langle\nu,E\rangle}\\&+Nh_{ii}^2\frac{u_i^2}{(\mathcal{C}+\langle\nu,E\rangle)^2}+N\frac{h_{ii}\langle\nu,E\rangle}{u}+N\frac{u_i^2}{u^2}+s\frac{h_{ii}\langle\nu,E\rangle}{(s-u)^2}-2s\frac{u_i^2}{(s-u)^3}\leqslant 0.\end{aligned} \tag{7-11}$$

By (1-10), we derive

$$\sigma_k^{ii}h_{iij}=\binom{n}{k}k(\mathcal{C}+\langle\nu,E\rangle)^{k-1}(-h_{jj}u_j)$$

and

$$\begin{aligned}\sigma_k^{ii}h_{iijj}&=-\sigma_k^{pq,rs}h_{pqj}h_{rsj}+\binom{n}{k}k(k-1)(\mathcal{C}+\langle\nu,E\rangle)^{k-2}h_{jj}^2u_j^2\\&\qquad+\binom{n}{k}k(\mathcal{C}+\langle\nu,E\rangle)^{k-1}\Big(-\sum_l h_{jjl}u_l+h_{jj}^2\langle\nu,E\rangle\Big)\\&\geqslant-\sigma_k^{pq,rs}h_{pqj}h_{rsj}+\binom{n}{k}k(\mathcal{C}+\langle\nu,E\rangle)^{k-1}\Big(-\sum_l h_{jjl}u_l\Big)-K_0(\mathcal{C}+\langle\nu,E\rangle)^{k-1}\kappa_1^2,\end{aligned}\tag{7-12}$$

where $K_0=K_0(n,k,\mathcal{C})>0$ is a constant depending on $n,\ k$, and $\mathcal{C}$. Recall that, in Minkowski space,

$$h_{jjii}=h_{iijj}+h_{ii}^2h_{jj}-h_{ii}h_{jj}^2.$$

Thus,

$$\sigma_k^{ii}h_{jjii}=\sigma_k^{ii}h_{iijj}+\sigma_k^{ii}h_{ii}^2h_{jj}-\sigma_k^{ii}h_{ii}h_{jj}^2\geqslant\sigma_k^{ii}h_{iijj}-k\binom{n}{k}(\mathcal{C}+\langle\nu,E\rangle)^kh_{jj}^2.\tag{7-13}$$

Combining (7-13) with (7-11), we obtain

$$\begin{aligned}0&\geqslant\sigma_k^{ii}\frac{\phi_{ii}}{\phi}=\frac{\sigma_k^{ii}}{P_m}\Big[\sum_j\kappa_j^{m-1}h_{jjii}+(m-1)\sum_j\kappa_j^{m-2}h_{jji}^2+\sum_{p\neq q}\frac{\kappa_p^{m-1}-\kappa_q^{m-1}}{\kappa_p-\kappa_q}h_{pqi}^2\Big]\\&\quad-\frac{m\sigma_k^{ii}}{P_m^2}\Big(\sum_j\kappa_j^{m-1}h_{jji}\Big)^2-N\sigma_k^{ii}\sum_l h_{ili}\frac{\langle\tau_l,E\rangle}{(\mathcal{C}+\langle\nu,E\rangle)}+N\sigma_k^{ii}h_{ii}^2\frac{-\langle\nu,E\rangle}{\mathcal{C}+\langle\nu,E\rangle}\\&\quad+N\sigma_k^{ii}h_{ii}^2\frac{u_i^2}{(\mathcal{C}+\langle\nu,E\rangle)^2}+N\sigma_k^{ii}\frac{h_{ii}\langle\nu,E\rangle}{u}+N\sigma_k^{ii}\frac{u_i^2}{u^2}+s\frac{\sigma_k^{ii}h_{ii}\langle\nu,E\rangle}{(s-u)^2}-2s\frac{\sigma_k^{ii}u_i^2}{(s-u)^3}\\&\geqslant-K_0(\mathcal{C}+\langle\nu,E\rangle)^{k-1}\kappa_1+\sum_i(A_i+B_i+C_i+D_i-E_i)+\binom{n}{k}k(\mathcal{C}+\langle\nu,E\rangle)^{k-1}\frac{-\sum_{j,l}h_{jjl}\kappa_j^{m-1}u_l}{P_m}\\&\quad-Nk\binom{n}{k}(\mathcal{C}+\langle\nu,E\rangle)^{k-2}\sum_l\kappa_lu_l^2+N\sigma_k^{ii}\kappa_i^2\frac{-\langle\nu,E\rangle}{\mathcal{C}+\langle\nu,E\rangle}+N\sigma_k^{ii}h_{ii}^2\frac{u_i^2}{(\mathcal{C}+\langle\nu,E\rangle)^2}\\&\quad+N\sigma_k^{ii}\frac{h_{ii}\langle\nu,E\rangle}{u}+N\sigma_k^{ii}\frac{u_i^2}{u^2}+s\frac{\sigma_k^{ii}h_{ii}\langle\nu,E\rangle}{(s-u)^2}-2s\frac{\sigma_k^{ii}u_i^2}{(s-u)^3}.\end{aligned}\tag{7-14}$$

Here,

$$A_i=\frac{\kappa_i^{m-1}}{P_m}\Big[K(\sigma_k)_i^2-\sum_{p,q}\sigma_k^{pp,qq}h_{ppi}h_{qqi}\Big]\quad\text{for some constant }\ K>1,$$

$$B_i=\frac{2\kappa_j^{m-1}}{P_m}\sum_j\sigma_k^{jj,ii}h_{jji}^2,\qquad C_i=\frac{m-1}{P_m}\sigma_k^{ii}\sum_j\kappa_j^{m-2}h_{jji}^2,$$

$$D_i=\frac{2\sigma_k^{jj}}{P_m}\sum_{j\neq i}\frac{\kappa_j^{m-1}-\kappa_i^{m-1}}{\kappa_j-\kappa_i}h_{jji}^2,\qquad E_i=\frac{m\sigma_k^{ii}}{P_m^2}\Big(\sum_j\kappa_j^{m-1}h_{jji}\Big)^2.$$

By Lemmas 8 and 9 and Corollary 10 in [Li et al. 2016], we can assume the following claim holds.

Claim 4. *There exist two small positive constants δ and $\eta < 1$. If $\kappa_k \leqslant \delta\kappa_1$, we have*

$$\sum_i A_i + B_i + C_i + D_i - \left(1 + \frac{\eta}{m}\right) E_i \geqslant 0, \tag{7-15}$$

where $m > 0$ is sufficiently large.

If (7-15) doesn't hold, we would have $\kappa_k > \delta\kappa_1$. Since $\sigma_k \leqslant \binom{n}{k}\mathcal{C}^k$, we get

$$\delta^{k-1}\kappa_1^k \leqslant \kappa_1\kappa_2\cdots\kappa_k \leqslant \sigma_k \leqslant \binom{n}{k}\mathcal{C}^k.$$

Since this gives an upper bound for κ_1 at x_0 directly, we would be done. Therefore, we assume (7-15) holds. Plugging (7-15) into (7-14) yields

$$\begin{aligned} 0 \geqslant{}& -K_0(\mathcal{C} + \langle \nu, E\rangle)^{k-1}\kappa_1 + \eta \frac{\sigma_k^{ii}}{P_m^2}\Big(\sum_j \kappa_j^{m-1} h_{jji}\Big)^2 - k\binom{n}{k}(\mathcal{C} + \langle \nu, E\rangle)^{k-1}|\nabla u|^2\Big(\frac{N}{u} + \frac{s}{(s-u)^2}\Big) \\ &+ N\sigma_k^{ii}\kappa_i^2 \frac{-\langle \nu, E\rangle}{\mathcal{C} + \langle \nu, E\rangle} + N\sigma_k^{ii} h_{ii}^2 \frac{u_i^2}{(\mathcal{C} + \langle \nu, E\rangle)^2} + N\sigma_k^{ii}\frac{h_{ii}\langle \nu, E\rangle}{u} \\ &+ N\sigma_k^{ii}\frac{u_i^2}{u^2} + s\frac{\sigma_k^{ii} h_{ii}\langle \nu, E\rangle}{(s-u)^2} - 2s\frac{\sigma_k^{ii} u_i^2}{(s-u)^3}. \end{aligned} \tag{7-16}$$

From (7-10), we obtain

$$\begin{aligned} \left(\frac{\sum_j \kappa_j^{m-1} h_{jji}}{P_m}\right)^2 ={}& N^2 \frac{\kappa_i^2 u_i^2}{(\mathcal{C} + \langle \nu, E\rangle)^2} + N^2\frac{u_i^2}{u^2} + \frac{s^2 u_i^2}{(s-u)^4} - 2N^2 \frac{\kappa_i u_i^2}{u(\mathcal{C} + \langle \nu, E\rangle)} \\ &- 2Ns\frac{\kappa_i u_i^2}{(\mathcal{C} + \langle \nu, E\rangle)(s-u)^2} + 2Ns\frac{u_i^2}{u(s-u)^2}. \end{aligned} \tag{7-17}$$

Inserting (7-17) into (7-16), we derive

$$\begin{aligned} 0 \geqslant{}& -K_0(\mathcal{C}+\langle \nu, E\rangle)^{k-1}\kappa_1 + \eta\frac{s^2\sigma_k^{ii}u_i^2}{(s-u)^4} + N(N\eta+1)\sigma_k^{ii}\kappa_i^2\frac{u_i^2}{(\mathcal{C}+\langle \nu, E\rangle)^2} - 2N^2\eta\frac{\sigma_k^{ii}\kappa_i u_i^2}{u(\mathcal{C}+\langle \nu, E\rangle)} \\ &- 2Ns\eta\frac{\sigma_k^{ii}\kappa_i u_i^2}{(\mathcal{C}+\langle \nu, E\rangle)(s-u)^2} + 2Ns\eta\frac{\sigma_k^{ii}u_i^2}{u(s-u)^2} + N\sigma_k^{ii}\frac{h_{ii}\langle \nu, E\rangle}{u} + N(\eta N+1)\sigma_k^{ii}\frac{u_i^2}{u^2} + s\frac{\sigma_k^{ii}h_{ii}\langle \nu, E\rangle}{(s-u)^2} \\ &- 2s\frac{\sigma_k^{ii}u_i^2}{(s-u)^3} - k\binom{n}{k}(\mathcal{C}+\langle \nu, E\rangle)^{k-1}|\nabla u|^2\Big(\frac{N}{u} + \frac{s}{(s-u)^2}\Big) + N\sigma_k^{ii}\kappa_i^2\frac{-\langle \nu, E\rangle}{\mathcal{C}+\langle \nu, E\rangle}. \end{aligned} \tag{7-18}$$

It's clear that

$$|\nabla u| = \frac{|Du|}{\sqrt{1 - |Du|^2}} < -\langle \nu, E\rangle \leqslant \mathcal{C}. \tag{7-19}$$

We also notice that, for any $1 \leqslant i \leqslant n$, we have $\sigma_k^{ii}\kappa_i \leqslant \binom{n}{k}\mathcal{C}^k$ (no summation). By a simple calculation, we get, when $N > 1/\eta^2$,

$$\eta\frac{s^2\sigma_k^{ii}u_i^2}{(s-u)^4} + 2Ns\eta\frac{\sigma_k^{ii}u_i^2}{u(s-u)^2} - 2s\frac{\sigma_k^{ii}u_i^2}{(s-u)^3} \geqslant 0. \tag{7-20}$$

Moreover, applying Lemma 31, we know there exist two positive constants $\tilde{d}_2 > \tilde{d}_1 > 0$ such that

$$\tilde{d}_1 \leqslant u(\mathcal{C} + \langle \nu, E\rangle) \leqslant \tilde{d}_2. \tag{7-21}$$

Therefore, for $N > 1/\eta^2$ sufficiently large, combining (7-19)–(7-21) with (7-18) yields

$$\begin{aligned} 0 \geqslant & -K_0(\mathcal{C} + \langle \nu, E\rangle)^{k-1}\kappa_1 - \frac{2N^2}{\tilde{d}_1}|\nabla u|^2\sigma_k^{ii}\kappa_i - 2Ns\frac{|\nabla u|^2\sigma_k^{ii}\kappa_i}{(\mathcal{C} + \langle \nu, E\rangle)(s-u)^2} \\ & - N\mathcal{C}\sigma_k^{ii}\kappa_i - \mathcal{C}\sigma_k^{ii}\kappa_i\frac{s}{(s-u)^2} - k\mathcal{C}^2\binom{n}{k}(\mathcal{C} + \langle \nu, E\rangle)^{k-1}\frac{s}{(s-u)^2} \\ & - k\binom{n}{k}\mathcal{C}^2(\mathcal{C} + \langle \nu, E\rangle)^{k-1}N + N\frac{c_0\sigma_k\kappa_1}{\mathcal{C} + \langle \nu, E\rangle}. \end{aligned}$$

It's easy to see that the above inequality yields, at x_0,

$$\kappa_1 \leqslant K(N, \mathcal{C}, \tilde{d}_1)\frac{s^2}{(s-u)^2}.$$

Therefore, in U_s, by (7-21), we have

$$\phi \leqslant K(N, \mathcal{C}, \tilde{d}_1)e^{-s/(s-u)}\frac{s^2}{(s-u)^2}.$$

Note that, for any $t \in [0, s]$,

$$\varphi(t) = e^{-s/(s-t)}\frac{s^2}{(s-t)^2} \leqslant 4e^{-2}.$$

We obtain, at any point $x \in U_s$,

$$\phi \leqslant K(N, \mathcal{C}, \tilde{d}_1). \tag{7-22}$$

Now, for any $x \in \mathbb{R}^n$, we can choose $s > 0$ large enough that $x \in U_{s/2}$. Then, by (7-22) and (7-21), we conclude that

$$\kappa_1(x) \leqslant K(N, \mathcal{C}, \tilde{d}_1, \tilde{d}_2).$$

Since x is arbitrary, we have finished proving Proposition 32. □

Theorem 7 follows from Propositions 22 and 32 immediately.

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Volume 17 No. 1 2024

The prescribed curvature problem for entire hypersurfaces in Minkowski space 1
Changyu Ren, Zhizhang Wang and Ling Xiao

Anisotropic micropolar fluids subject to a uniform microtorque: the stable case 41
Antoine Remond-Tiedrez and Ian Tice

Strong ill-posedness for SQG in critical Sobolev spaces 133
In-Jee Jeong and Junha Kim

Large-scale regularity for the stationary Navier–Stokes equations over non-Lipschitz boundaries 171
Mitsuo Higaki, Christophe Prange and Jinping Zhuge

On a family of fully nonlinear integrodifferential operators: from fractional Laplacian to nonlocal Monge–Ampère 243
Luis A. Caffarelli and María Soria-Carro

Propagation of singularities for gravity-capillary water waves 281
Hui Zhu

Shift equivalences through the lens of Cuntz–Krieger algebras 345
Toke Meier Carlsen, Adam Dor-On and Søren Eilers