ANALYSIS & PDE

Volume 17 No. 2 2024

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ANALYSIS AND PDE
Vol. 17 (2024), No. 2, pp. 535–585
DOI: 10.2140/apde.2024.17.535

SEMICLASSICAL EIGENVALUE ESTIMATES UNDER MAGNETIC STEPS

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We establish accurate eigenvalue asymptotics and, as a by-product, sharp estimates of the splitting between two consecutive eigenvalues for the Dirichlet magnetic Laplacian with a nonuniform magnetic field having a jump discontinuity along a smooth curve. The asymptotics hold in the semiclassical limit, which also corresponds to a large magnetic field limit and is valid under a geometric assumption on the curvature of the discontinuity curve.

1. Introduction

The paper studies a semiclassical Schrödinger operator with a step magnetic field and Dirichlet boundary conditions, in a smooth bounded domain. The aim is to give accurate estimates of the lower eigenvalues in the semiclassical limit.

Let Ω be an open, bounded, and simply connected subset of $\mathbb{R}^2$ with smooth C^1 boundary. We consider a simple smooth curve $\Gamma \subset \mathbb{R}^2$ that splits $\mathbb{R}^2$ into two disjoint unbounded open sets, P_1 and P_2, and such that Γ is a semistraight line when $|x|$ tends to $+\infty$. We assume that Γ decomposes Ω into two sets Ω_1 and Ω_2 as follows (see Figure 1):

(1) Γ intersects $\partial\Omega$ transversally at two distinct points.

(2) $\Omega_1 := \Omega \cap P_1 \neq \varnothing$ and $\Omega_2 := \Omega \cap P_2 \neq \varnothing$.

Let $h > 0$ and $\boldsymbol{F} = (F_1, F_2) \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ be a magnetic potential whose associated magnetic field is

$$\operatorname{curl} \boldsymbol{F} = a_1 \mathbb{1}_{P_1} + a_2 \mathbb{1}_{P_2}, \quad \boldsymbol{a} := (a_1, a_2) \in \mathbb{R}^2,\ a_1 \neq a_2. \tag{1-1}$$

When restricted to Ω, the vector field $\boldsymbol{F}$ satisfies

$$\operatorname{curl} \boldsymbol{F} = a_1 \mathbb{1}_{\Omega_1} + a_2 \mathbb{1}_{\Omega_2}, \quad \boldsymbol{a} := (a_1, a_2) \in \mathbb{R}^2,\ a_1 \neq a_2 \text{ and } \boldsymbol{F} \in L^4(\Omega). \tag{1-2}$$

Note that the curve Γ separates the two regions Ω_1 and Ω_2 which are assigned with different values of the magnetic field. For this reason, we refer to Γ as the *magnetic edge*. We consider the quadratic form on $H^1_0(\Omega)$

$$u \mapsto \mathcal{Q}_h(u) = \int_\Omega |(h\nabla - i\boldsymbol{F})u|^2\, dx. \tag{1-3}$$

MSC2020: 35P15, 35P20, 81Q20.
Keywords: semiclassical analysis, magnetic Laplacian, magnetic steps.

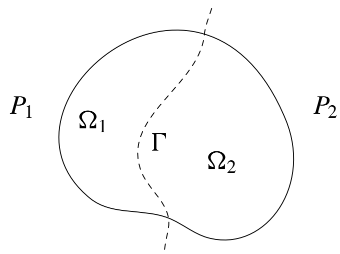

Figure 1. The curve Γ transversally cuts $\partial\Omega$ at two points and splits Ω into two regions, Ω_1 and Ω_2.

This quadratic form is closed on the form domain $H_0^1(\Omega)$. By the Friedrichs extension procedure, we can associate its Dirichlet realization in Ω

$$\mathcal{P}_h := -(h\nabla - i\boldsymbol{F})^2 = -\sum_{j=1}^{2}(h\partial_{x_j} - iF_j)^2, \tag{1-4}$$

whose domain is

$$\mathrm{Dom}(\mathcal{P}_h) = \{u \in L^2(\Omega) : (h\nabla - i\boldsymbol{F})^j u \in L^2(\Omega),\ j \in \{1,2\},\ u|_{\partial\Omega} = 0\}. \tag{1-5}$$

The operator $\mathcal{P}_h$ is self-adjoint, has compact resolvent, and its spectrum is an increasing sequence, $(\lambda_n(h))_{n\in\mathbb{N}}$, of real eigenvalues listed with multiplicities.

In this contribution, we aim at giving the asymptotic expansion of the low-lying eigenvalues of $\mathcal{P}_h$, in the semiclassical limit, i.e., when h tends to 0.

Schrödinger operators with a discontinuous magnetic field, like $\mathcal{P}_h$, appear in many models in nanophysics such as in quantum transport while studying the transport properties of a bidimensional electron gas [Reijniers and Peeters 2000; Peeters and Matulis 1993]. In that context, the magnetic edge is *straight* and bound states interestingly feature currents flowing along the magnetic edge.

The present contribution addresses another appealing question on the influence of the shape of the magnetic edge on the energy of the bound states. We give an affirmative answer by providing sharp semiclassical eigenvalue asymptotics under a single "well" hypothesis on the curvature of the magnetic edge (see Assumption 1.1 and Theorem 1.2 below). Loosely speaking, our hypothesis says that we perform a local deformation of the magnetic edge so that its curvature has a unique nondegenerate maximum.

Another important occurrence of magnetic Laplace operators is in the Ginzburg–Landau model of superconductivity [Saint-James and de Gennes 1963]. In bounded domains, the spectral properties of these operators can describe interesting physical situations. In the context of superconductivity, accurate information about the lowest eigenvalues is important for giving a precise description of the concentration of superconductivity in a type-II superconductor. Moreover, it improves the estimates of the third critical field, H_{C_3}, that marks the onset of superconductivity in the domain. We refer the reader to [Assaad and Kachmar 2022; Assaad 2021] for discontinuous field cases, and to [Fournais and Helffer 2006; Helffer and Pan 2003; Lu and Pan 1999a; 1999b; 2000; Bonnaillie-Noël and Fournais 2007; Bonnaillie-Noël and Dauge 2006; Bernoff and Sternberg 1998; Tilley and Tilley 1990] for a further discussion in smooth fields cases. In the present paper, the Dirichlet realization of $\mathcal{P}_h$ in the bounded domain Ω can physically correspond to a superconductor which is set in the normal (nonsuperconducting) state at its boundary.

Using symmetry and scaling arguments, one can reduce the problem to the study of cases of $\boldsymbol{a} = (a_1, a_2)$, where $a_1 = 1$ and $a_2 = a \in [-1, 1)$. Moreover, we will soon make a more restrictive choice of cases of $\boldsymbol{a}$ (see (1-11) below). Towards justifying the upcoming choice of $\boldsymbol{a}$, we introduce the effective operator $\mathfrak{h}_a[\xi]$ with a discontinuous field, defined on $\mathbb{R}$ and parametrized by $\xi \in \mathbb{R}$:

$$\mathfrak{h}_a[\xi] = -\frac{d^2}{d\tau^2} + (\xi + b_a(\tau)\tau)^2, \tag{1-6}$$

where

$$b_a(\tau) = \mathbf{1}_{\mathbb{R}_+}(\tau) + a\mathbf{1}_{\mathbb{R}_-}(\tau). \tag{1-7}$$

This operator arises from the approximation by the case where $\Omega = \mathbb{R}^2$ and $\Gamma = \{x_2 = 0\}$, τ corresponding to the variable x_2 and ξ being the dual variable of x_1. The known spectral properties of $\mathfrak{h}_a[\xi]$, obtained earlier in [Hislop et al. 2016; Assaad et al. 2019; Assaad and Kachmar 2022], are recalled in Section 2A. Here, we only present some features of this operator that are useful to this introduction. The bottom of the spectrum of $\mathfrak{h}_a[\xi]$, denoted by $\mu_a(\xi)$, is a simple eigenvalue for $a \neq 0$, usually called the *band function* in the literature. Minimizing the band function leads us to introduce

$$\beta_a = \inf_{\xi \in \mathbb{R}} \mu_a(\xi). \tag{1-8}$$

We list the following properties of β_a, depending on the values of a:

<u>Case $a = -1$</u>: In the case where $\Omega = \mathbb{R}^2$ and $\Gamma = \{x_2 = 0\}$, this case is called the "symmetric trapping magnetic steps" and is well-understood in the literature (see, e.g., [Hislop et al. 2016]). In this case, the study of $\mathfrak{h}_a[\xi]$ can be reduced to that of the de Gennes operator (a harmonic oscillator on the half-axis with Neumann condition at the origin). We refer the reader to [Fournais and Helffer 2010] for the spectral properties of this operator. Here,

$$\Theta_0 := \beta_{-1} \approxeq 0.59 \tag{1-9}$$

is attained by $\mu_{-1}(\cdot)$ at a unique and nondegenerate minimum $\xi_0 = -\sqrt{\Theta_0}$. Moreover, $\beta_{-1} = \mu_{-1}(\xi_0)$ is a simple eigenvalue of $\mathfrak{h}_{-1}[\xi_0]$.

<u>Case $-1 < a < 0$</u>: This case is called the "asymmetric trapping magnetic steps" and is studied in many works (see [Assaad and Kachmar 2022; Assaad et al. 2019; Hislop et al. 2016]). We have $|a|\Theta_0 < \beta_a < \min(|a|, \Theta_0)$ and β_a is attained by $\mu_a(\cdot)$ at a unique $\zeta_a < 0$ [Assaad and Kachmar 2022]

$$\mu_a(\zeta_a) = \beta_a. \tag{1-10}$$

Moreover, the minimum is nondegenerate, i.e., $\mu_a''(\zeta_a) > 0$.

<u>Case $a = 0$</u>: This corresponds to the "magnetic wall" case studied for instance in [Reijniers and Peeters 2000; Hislop et al. 2016]. We refer to [Hislop et al. 2016, Section 2] for this case.

For $\xi \leq 0$, we have

$$\sigma(h_a[\xi]) = \sigma_{\mathrm{ess}}(h_a[\xi]) = [\xi^2, +\infty),$$

where σ and σ_{ess} respectively denote the spectrum and essential spectrum.

For $\xi > 0$,

$$\sigma_{\text{ess}}(h_a[\xi]) = [\xi^2, +\infty)$$

and $h_a[\xi]$ may have positive eigenvalues $\lambda < \xi^2$. Consequently, $\beta_0 = \mu_0(0) = \inf \sigma_{\text{ess}} \mathfrak{h}_0[0] = 0$, and β_0 is not an eigenvalue of $\mathfrak{h}_a[\xi]$ for all $\xi \in \mathbb{R}$.

Case $0 < a < 1$: This corresponds with the "nontrapping magnetic steps" case; see [Assaad et al. 2019; Hislop and Soccorsi 2015; Iwatsuka 1985]. Here, $\beta_a = a$ and $\mu_a(\cdot)$ doesn't achieve a minimum; the infimum is attained at $+\infty$.

A key ingredient in establishing the asymptotics of the eigenvalues $\lambda_n(h)$ is that β_a is an eigenvalue of $\mathfrak{h}_a[\xi]$ for some $\xi \in \mathbb{R}$. We will use the corresponding eigenfunction in constructing quasimodes of the operator $\mathcal{P}_h$. The above discussion shows that β_a is an eigenvalue only when $a \in [-1, 0)$. The case $a = -1$ is excluded from our study, despite the fact that β_{-1} is an eigenvalue of $\mathfrak{h}_{-1}[\xi_0]$. Except when Γ is an axis of symmetry of Ω as in [Hislop et al. 2016], the situation is more difficult and the curvature will play a more important role. We hope to treat this case in a future work. This explains our choice to work under the following assumption on $\boldsymbol{a}$ (thus on the magnetic field curl $\boldsymbol{F}$) throughout the paper:

$$\boldsymbol{a} = (1, a), \quad \text{with } -1 < a < 0. \tag{1-11}$$

Under assumption (1-11), we introduce two spectral invariants

$$c_2(a) = \frac{1}{2}\mu_a''(\zeta_a) > 0 \quad \text{and} \quad M_3(a) = \frac{1}{3}\Big(\frac{1}{a} - 1\Big)\zeta_a \phi_a(0)\phi_a'(0) < 0, \tag{1-12}$$

where μ_a and ζ_a are introduced in (1-8) and (1-10), and ϕ_a is the positive L^2-normalized eigenfunction of $\mathfrak{h}_a[\zeta_a]$ corresponding to β_a.

Furthermore, we work under the following assumption:

Assumption 1.1. The curvature $\Gamma \ni s \mapsto k(s)$ at the magnetic edge has a unique maximum

$$k(s) < k(s_0) =: k_{\max} \quad \text{for } s \neq s_0.$$

This maximum is attained in $\Gamma \cap \Omega$ and is nondegenerate:

$$k_2 := k''(s_0) < 0.$$

The goal of this paper is to prove the following theorem:

Theorem 1.2. *Let $n \in \mathbb{N}^*$ and $\boldsymbol{a} = (1, a)$, with $-1 < a < 0$. Under Assumption 1.1, the n-th eigenvalue $\lambda_n(h)$ of $\mathcal{P}_h$, defined in* (1-4), *satisfies, as $h \to 0$,*

$$\lambda_n(h) = h\beta_a + h^{3/2} k_{\max} M_3(a) + h^{7/4}(2n-1)\sqrt{\frac{k_2 M_3(a) c_2(a)}{2}} + \mathcal{O}(h^{15/8}),$$

where β_a, $c_2(a)$ and $M_3(a)$ are the spectral quantities introduced in (1-8) *and* (1-12).

Remark 1.3. This theorem extends [Assaad and Kachmar 2022, Theorem 4.5], where the first two terms in the expansion of the first eigenvalue were determined with a remainder in $\mathcal{O}(h^{5/3})$. The proof of Theorem 1.2 partially relies on decay estimates of the eigenfunctions with the right scale; see Section 6 and [Assaad and Kachmar 2022]. In fact, away from the edge Γ, the eigenfunctions decay exponentially at the scale $h^{-1/2}$ of the distance to Γ, while, along Γ, they decay exponentially with a scale of $h^{-1/8}$ of the tangential distance on Γ to the point with maximum curvature.

Comparison with earlier situations. It is useful to compare the asymptotics of $\lambda_n(h)$ in Theorem 1.2 with those obtained in the literature for regular domains submitted to uniform magnetic fields. In bounded planar domains with smooth boundary, subject to unit magnetic fields and when the *Neumann* boundary condition is imposed, the low-lying eigenvalues of the linear operator, analogous to $\mathcal{P}_h$, admit the following asymptotics as h tends to 0 (see, e.g., [Fournais and Helffer 2006]):

$$\lambda_n(h) = h\Theta_0 - h^{3/2}\tilde{k}_{\max}C_1 + h^{7/4}C_1\Theta_0^{1/4}(2n-1)\sqrt{\tfrac{3}{2}\tilde{k}_2} + \mathcal{O}(h^{15/8}),$$

where Θ_0 is as in (1-9), $C_1 > 0$ is some spectral value, and $\tilde{k}_{\max}$ and $\tilde{k}_2$ are positive constants introduced in what follows. In this uniform field/Neumann condition situation, the eigenstates localize near the boundary of the domain. More precisely, they localize near the point $\tilde{s}$ with maximum curvature $k(\tilde{s})$ of this boundary, assuming the uniqueness and nondegeneracy of this point. We define $\tilde{k}_{\max} = k(\tilde{s})$ and $\tilde{k}_2 = -k''(\tilde{s}) > 0$. In [Fournais and Helffer 2006], the foregoing localization of eigenstates restricted the study to the boundary, involving a family of one-dimensional effective operators which act in the normal direction to the boundary. These are the de Gennes operators

$$\mathfrak{h}^N[\xi] = -\frac{d^2}{d\tau^2} + (\xi+\tau)^2,$$

defined on $\mathbb{R}_+$ with Neumann boundary condition at $\tau = 0$, and parametrized by $\xi \in \mathbb{R}$. We recover the value Θ_0 as an *effective energy* associated to $(\mathfrak{h}^N[\xi])_\xi$,

$$\Theta_0 = \inf_{\xi\in\mathbb{R}} \mu^N(\xi),$$

where $\mu^N(\xi)$ is the bottom of the spectrum $\sigma(\mathfrak{h}^N[\xi])$ of $\mathfrak{h}^N[\xi]$, for $\xi \in \mathbb{R}$.

Back to our discontinuous field case with *Dirichlet* boundary condition, we prove that our eigenstates are localized near the magnetic edge Γ, and more particularly, near the point with maximum curvature of this edge (see Section 6). Analogously to the aforementioned uniform field/Neumann condition situation, our study near Γ involves the family of one-dimensional effective operators $(\mathfrak{h}_a[\xi])_{\xi\in\mathbb{R}}$ which act in the normal direction to the edge Γ, along with the associated effective energy β_a.

At this stage, it is natural to discuss our problem when the Dirichlet boundary conditions are replaced by Neumann boundary ones. In this situation, one can prove the concentration of the eigenstates of the operator $\mathcal{P}_h$ near the points of intersection between the edge Γ and the boundary $\partial\Omega$. This was shown in [Assaad 2021, Theorem 6.1] at least for the lowest eigenstate. In such settings, a geometric condition is usually imposed related to the angles formed at the intersection $\Gamma\cap\partial\Omega$; see [Assaad 2021, Assumption 1.3 and Remark 1.4]. The localization of the eigenstates near $\Gamma\cap\partial\Omega$ will involve effective models that are genuinely two-dimensional, i.e., they cannot be fibered to one-dimensional operators; see [Assaad 2021, Section 3]. Studying this case may show similarity features with the case of piecewise smooth bounded domains with corners submitted to uniform magnetic fields, treated in [Bonnaillie-Noël and Dauge 2006];

see also [Bonnaillie-Noël et al. 2007; Bonnaillie-Noël 2005; Bonnaillie-Noël 2003; 2007] for studies on corner domains. Such similarities were first revealed in [Assaad 2021, Section 1.3]. More precisely, one expects the result in the discontinuous field/Neumann condition situation to be similar to that in [Bonnaillie-Noël and Dauge 2006, Theorem 7.1]. Such a result is worth establishing in a future work.

Theorem 1.2 permits us to deduce the splitting between the ground-state energy (lowest eigenvalue) and the energy of the first excited state of $\mathcal{P}_h$. More precisely, introducing the spectral gap

$$\Delta(h) := \lambda_2(h) - \lambda_1(h),$$

we get by Theorem 1.2:

Corollary 1.4. *Under the conditions in Theorem 1.2, we have as $h \to 0$*

$$\Delta(h) = h^{7/4}\sqrt{2k_2 M_3(a) c_2(a)} + \mathcal{O}(h^{15/8}).$$

Apart from its own interest, estimating the foregoing spectral gap has potential applications in nonlinear bifurcation problems, for instance, in the context of the Ginzburg–Landau model of superconductivity (see [Fournais and Helffer 2010, Section 13.5.1]).

Remark 1.5. Altering the regularity/geometry of the edge Γ may lead to radical changes in Theorem 1.2.

- If Γ is a piecewise smooth curve (a broken edge) then we have to analyze a new model in the full plane (reminiscent of a model in [Assaad 2021]). We expect analogies with domains with corners in a uniform magnetic field [Bonnaillie-Noël 2003].
- If we relax Assumption 1.1 by allowing the curvature k to have two symmetric maxima, then a tunnel effect may occur and the splitting in Theorem 1.2 becomes of exponential order. This was recently analyzed in [Fournais et al. 2022] based on the analysis of this paper and [Bonnaillie-Noël et al. 2022].
- If the curvature along Γ or a part of Γ is constant, then we expect that the magnitude of the splitting in Theorem 1.2 will change too, probably leading to multiple eigenvalues. It would be desirable to get accurate estimates in this setting. We expect analogies with disc domains in a nonuniform magnetic field [Fournais and Persson-Sundqvist 2015].

Heuristics of the proofs. Our proof of Theorem 1.2 is purely variational. The derivation of the eigenvalue upper bound is rather standard. It is obtained by computing the energy of a well-chosen trial state, $v_{h,n}^{\mathrm{app}}$, constructed by expressing the operator in a Frenet frame near the point of maximum curvature and doing WKB like expansions (for the operator and the trial state).

Proving the eigenvalue lower bound is more involved. The idea is to project the actual bound state, $v_{h,n}$, on the trial state $v_{h,n}^{\mathrm{app}}$, and to prove that this provides us with a well-chosen trial state for a one-dimensional effective operator, $H_a^{\mathrm{harm}} = -c_2(a)\partial_\sigma^2 - \frac{1}{2}k_2 M_3(a)\sigma^2$. To validate this method, we need sharp estimates of the tangential derivative of the actual bound state, which we derive via a simple, but lengthy and quite technical method involving Agmon estimates and other implementations from one-dimensional model operators. At this stage, one advantage of our approach seems its applicability with weaker regularity assumptions on the magnetic edge or the magnetic field, which could be useful in other situations as well, like the study of the three-dimensional problem in [Helffer and Morame 2004].

Outline of the paper. The paper is organized as follows. Sections 2 and 3 contain the necessary material on the model one-dimensional problems for flat and curved magnetic edges, respectively. Section 4 is devoted to the eigenvalue upper bounds matching with the asymptotics of Theorem 1.2. Here, we give the construction of the aforementioned trial state $v_{h,n}^{\mathrm{app}}$.

In Sections 5 and 6, we estimate the tangential derivative of the actual bound states, after being truncated and properly expressed in rescaled variables. The tangential derivative estimate of the L^2 norm will follow straightforwardly from the main result of Section 5. However, a higher-regularity estimate will require additional work in Section 6.

In Section 7, using the actual bound states, we construct trial states for the effective one-dimensional operator, and eventually prove the eigenvalue lower bounds of Theorem 1.2. Finally, we give two appendices, Appendix A on the Frenet coordinates near the magnetic edge, and Appendix B on the control of a remainder term that we meet in Section 7.

2. Fiber operators

2A. ***Band functions.*** Let $a \in [-1, 0)$. We first introduce some constants whose definition involves the following family of fiber operators in $L^2(\mathbb{R})$:

$$\mathfrak{h}_a[\xi] = -\frac{d^2}{d\tau^2} + V_a(\xi, \tau), \tag{2-1}$$

where $\xi \in \mathbb{R}$ is a parameter,

$$V_a(\xi, \tau) = (\xi + b_a(\tau)\tau)^2, \quad b_a(\tau) = \mathbf{1}_{\mathbb{R}_+}(\tau) + a\mathbf{1}_{\mathbb{R}_-}(\tau), \tag{2-2}$$

and the domain of $\mathfrak{h}_a[\xi]$ is given by

$$\mathrm{Dom}(\mathfrak{h}_a[\xi]) = B^2(\mathbb{R}).$$

Here the space $B^n(I)$ is defined for a positive integer n and an open interval $I \subset \mathbb{R}$ as

$$B^n(I) = \left\{ u \in L^2(I) : \tau^i \frac{d^j u}{d\tau^j} \in L^2(I) \text{ for all } i, j \in \mathbb{N} \text{ such that } i + j \le n \right\}. \tag{2-3}$$

The operator $\mathfrak{h}_a[\xi]$ is essentially self-adjoint and has compact resolvent. Actually, it can also be defined as the Friedrichs realization starting from the closed quadratic form

$$u \mapsto q_a[\xi](u) = \int_{\mathbb{R}} (|u'(\tau)|^2 + V_a(\xi, \tau)|u(\tau)|^2)\, d\tau \tag{2-4}$$

defined on $B^1(\mathbb{R})$.

For $(a, \xi) \in [-1, 0) \times \mathbb{R}$, the ground-state energy (bottom of the spectrum) $\mu_a(\xi)$ of $\mathfrak{h}_a[\xi]$ can be characterized by

$$\mu_a(\xi) = \inf_{u \in B^1(\mathbb{R}), u \ne 0} \frac{q_a[\xi](u)}{\|u\|^2_{L^2(\mathbb{R})}}, \tag{2-5}$$

and $\xi \mapsto \mu_a(\xi)$ will be called the band function.

We then introduce the *step constant* at a by

$$\beta_a := \inf_{\xi \in \mathbb{R}} \mu_a(\xi). \tag{2-6}$$

For $a = -1$, it is easy to identify by symmetrization $\mu_{-1}(\xi)$ with the ground-state energy of the Neumann realization of $-(d^2/d\tau^2) + (\tau + \xi)^2$ in $\mathbb{R}_+$ and therefore

$$\beta_{-1} = \Theta_0, \tag{2-7}$$

where Θ_0 is the celebrated de Gennes constant.

By the general theory for the Schrödinger operator, $\mu_a(\xi)$ is, for each $\xi \in \mathbb{R}$, a simple eigenvalue, that we associate with a unique *positive* L^2-normalized eigenfunction denoted by $\varphi_{a,\xi}$, i.e., satisfying

$$\varphi_{a,\xi} > 0, \quad (\mathfrak{h}_a[\xi] - \mu_a(\xi))\varphi_{a,\xi} = 0 \quad \text{and} \quad \int_{\mathbb{R}} |\varphi_{a,\xi}(\tau)|^2 \, d\tau = 1. \tag{2-8}$$

By Kato's theory, the band function μ_a is an analytic function on $\mathbb{R}$. Its derivative was computed in [Hislop and Soccorsi 2015] (see also [Assaad et al. 2019, Proposition A.4]),

$$\mu_a'(\xi) = \Bigl(1 - \frac{1}{a}\Bigr)\bigl(\varphi_{a,\xi}'(0)^2 + (\mu_a(\xi) - \xi^2)\varphi_{a,\xi}(0)^2\bigr), \tag{2-9}$$

which results from the following Feynman–Hellmann formula (see [Assaad et al. 2019, equation (A.9); Bolley and Helffer 1993; Dauge and Helffer 1993]):

$$\mu_a'(\xi) = 2 \int_{\mathbb{R}} (\xi + b_a(\tau)\tau) |\varphi_{a,\xi}(\tau)|^2 \, d\tau. \tag{2-10}$$

2B. ***Properties of band functions/states.*** For $a \in (-1, 0)$, the following results were recently established in [Assaad and Kachmar 2022; Assaad et al. 2019; Hislop et al. 2016]:

(1) $|a|\Theta_0 < \beta_a < \min(|a|, \Theta_0)$.

(2) There exists a unique $\zeta_a \in \mathbb{R}$ such that $\beta_a = \mu_a(\zeta_a)$.

(3) $\zeta_a < 0$, $\mu_a''(\zeta_a) > 0$ and the ground state $\phi_a := \varphi_{a,\zeta_a}$ satisfies

$$\phi_a'(0) < 0 \text{ and } \zeta_a = -\sqrt{\beta_a + (\phi_a'^2(0)/\phi_a^2(0))}.$$

In particular, using (2-10) for $\xi = \zeta_a$, we observe that the functions ϕ_a and $(\zeta_a + b_a(\tau)\tau)\phi_a$ are orthogonal

$$\int_{\mathbb{R}} (\zeta_a + b_a(\tau)\tau) |\phi_a(\tau)|^2 \, d\tau = 0. \tag{2-11}$$

Moreover, the ground-state ϕ_a satisfies the following decay estimates:

Proposition 2.1. *Let $a \in [-1, 0)$. For any $\gamma > 0$, there exists a positive constant C_γ such that*

$$\int_{\mathbb{R}} e^{\gamma|\tau|} (|\phi_a(\tau)|^2 + |\phi_a'(\tau)|^2) \, d\tau \le C_\gamma.$$

Consequently, for all $n \in \mathbb{N}^*$ there exists $C_n > 0$ such that

$$\int_{\mathbb{R}} |\tau|^n |\phi_a(\tau)|^2 \, d\tau \le C_n. \tag{2-12}$$

The proof is classical by using Agmon's approach for proving decay estimates. We omit it and refer the reader to [Fournais and Helffer 2010, Theorem 7.2.2] or to the proof of Lemma 2.4 below.

2C. *Moments.* Later in the paper, we will encounter the *moments*

$$M_n(a) = \int_{-\infty}^{+\infty} \frac{1}{b_a(\tau)} (\zeta_a + b_a(\tau)\tau)^n |\phi_a(\tau)|^2 \, d\tau, \tag{2-13}$$

which are finite according to (2-12).

For $n \in \{1, 2, 3\}$, they were computed in [Assaad and Kachmar 2022] and we have

$$M_1(a) = 0, \tag{2-14}$$

$$M_2(a) = -\frac{1}{2}\beta_a \int_{-\infty}^{+\infty} \frac{1}{b_a(\tau)} |\phi_a(\tau)|^2 \, d\tau + \frac{1}{4}\Big(\frac{1}{a} - 1\Big)\zeta_a \phi_a(0)\phi_a'(0), \tag{2-15}$$

$$M_3(a) = \frac{1}{3}\Big(\frac{1}{a} - 1\Big)\zeta_a \phi_a(0)\phi_a'(0). \tag{2-16}$$

Remark 2.2. From the properties of the band function recalled in Section 2B, we get that $M_3(a)$ is negative for $-1 < a < 0$ and vanishes for $a = -1$.

Remark 2.3. The next identities follow in a straightforward manner from the foregoing formulas of the moments:

$$\int_{-\infty}^{+\infty} \tau(\zeta_a + b_a(\tau)\tau)|\phi_a(\tau)|^2 \, d\tau = M_2(a),$$

$$\int_{-\infty}^{+\infty} \tau(\zeta_a + b_a(\tau)\tau)^2|\phi_a(\tau)|^2 \, d\tau = M_3(a) - \zeta_a M_2(a),$$

$$\int_{-\infty}^{+\infty} b_a(\tau)\tau^2(\zeta_a + b_a(\tau)\tau)|\phi_a(\tau)|^2 \, d\tau = M_3(a) - 2\zeta_a M_2(a),$$

$$\int_{-\infty}^{+\infty} \tau|\phi_a(\tau)|^2 \, d\tau = -\zeta_a \int_{-\infty}^{+\infty} \frac{1}{b_a(\tau)} |\phi_a(\tau)|^2 \, d\tau,$$

$$\int_{-\infty}^{+\infty} \tau|\phi_a'(\tau)|^2 \, d\tau = \beta_a\zeta_a \int_{-\infty}^{+\infty} \frac{1}{b_a(\tau)} |\phi_a(\tau)|^2 \, d\tau + 2M_3(a) - 2\zeta_a M_2(a).$$

We will also encounter the moment

$$I_2(a) := \int_{\mathbb{R}} (\zeta_a + b_a(\tau)\tau)\phi_a \mathfrak{R}_a[(\zeta_a + b_a(\tau)\tau)\phi_a] \, d\tau, \tag{2-17}$$

involving the resolvent $\mathfrak{R}_a$, which is an operator defined on $L^2(\mathbb{R})$ by means of the following lemma:

Lemma 2.4. *If $u \in L^2(\mathbb{R})$ is orthogonal to ϕ_a, we define $(\mathfrak{h}_a[\zeta_a] - \beta_a)^{-1}u$ in $L^2(\mathbb{R})$ as the unique solution v orthogonal to ϕ_a to*

$$(\mathfrak{h}_a[\zeta_a] - \beta_a)v = u.$$

We introduce the regularized resolvent $\mathfrak{R}_a$ in $\mathcal{L}(L^2(\mathbb{R}))$ by

$$\mathfrak{R}_a(u)=\begin{cases}0 & \text{if } u \parallel \phi_a,\\ (\mathfrak{h}_a[\zeta_a]-\beta_a)^{-1}u & \text{if } u\perp\phi_a\end{cases}\tag{2-18}$$

(*extended by linearity*). *Then, for any $\gamma\geq 0$, $\mathfrak{R}_a$ and $(d/d\tau)\circ\mathfrak{R}_a$ are two bounded operators on $L^2(\mathbb{R},\exp(\gamma|\tau|)\,d\tau)$.*

Proof. We follow Agmon's approach. Consider $v\in\mathrm{Dom}(\mathfrak{h}_a[\zeta_a])$ and $u\in L^2(\mathbb{R},\exp(\gamma|\tau|)\,d\tau)$ such that

$$(\mathfrak{h}_a[\zeta_a]-\beta_a)v=u.$$

For all $\gamma>0$ and $N>1$, consider the continuous function on $\mathbb{R}$

$$\Phi_{\gamma,N}(\tau)=\min(\gamma|\tau|,N).$$

Observe that $\Phi_{\gamma,N}\in H^1_{\mathrm{loc}}(\mathbb{R})$ and

$$|\Phi'_{\gamma,N}(\tau)|=\begin{cases}\gamma & \text{if } \gamma|\tau|<N,\\ 0 & \text{if } \gamma|\tau|>N.\end{cases}$$

Integration by parts yields

$$\begin{aligned}\langle u,e^{2\Phi_{\gamma,N}}v\rangle&=\langle(\mathfrak{h}_a[\zeta_a]-\beta_a)v,e^{2\Phi_{\gamma,N}}v\rangle\\&=\|(e^{\Phi_{\gamma,N}}v)'\|^2+\int_{\mathbb{R}}((\zeta_a+b\tau)^2-\beta_a)|e^{\Phi_{\gamma,N}}v|^2\,d\tau-\|\Phi'_{\gamma,N}e^{\Phi_{\varepsilon,N}}v\|^2\\&\geq\|(e^{\Phi_{\gamma,N}}v)'\|^2+\int_{\mathbb{R}}((\zeta_a+b\tau)^2-\beta_a-\gamma^2)|e^{\Phi_{\gamma,N}}v|^2\,d\tau.\end{aligned}$$

Choose $A_\gamma>1$ so that, for $|\tau|\geq A_\gamma$, we have $(\zeta_a+b\tau)^2-\beta_a-\gamma^2\geq 1$; consequently, for $N\geq\gamma A_\gamma$,

$$\langle u,e^{2\Phi_{\gamma,N}}v\rangle\geq\|(e^{\Phi_{\gamma,N}}v)'\|^2+\int_{\{|\tau|\geq A_\gamma\}}|e^{\Phi_{\gamma,N}}v|^2\,d\tau-(\beta_a+\gamma^2)e^{2\gamma A_\gamma}\|v\|^2.$$

Using the Cauchy–Schwarz inequality, we get further

$$\|e^{\Phi_{\gamma,N}}u\|\,\|e^{\Phi_{\gamma,N}}v\|\geq\|(e^{\Phi_{\gamma,N}}v)'\|^2+\int_{\{|\tau|\geq A_\gamma\}}|e^{\Phi_{\gamma,N}}v|^2\,d\tau-(\beta_a+\gamma^2)e^{2\gamma A_\gamma}\|v\|^2.$$

Rearranging the terms in (2-19) and using Cauchy's inequality

$$\|e^{\Phi_{\gamma,N}}u\|\,\|e^{\Phi_{\gamma,N}}v\|\leq 2\|e^{\Phi_{\gamma,N}}u\|^2+\tfrac{1}{2}\|e^{\Phi_{\gamma,N}}v\|^2,$$

we get

$$\|(e^{\Phi_{\gamma,N}}v)'\|^2+\frac{1}{2}\int_{\{|\tau|\geq A_\gamma\}}|e^{\Phi_{\gamma,N}}v|^2\,d\tau\leq(\beta_a+\gamma^2+1)e^{2\gamma A_\gamma}\|v\|^2+2\|e^{\Phi_{\gamma,N}}u\|^2.$$

We end up with the estimate

$$\int|e^{\Phi_{\gamma,N}}v'|^2\,d\tau+\int|e^{\Phi_{\gamma,N}}v|^2\,d\tau\leq C_\gamma(\|v\|^2+\|e^{\Phi_\gamma}u\|^2),$$

where we note that the right-hand side is independent of N.

Since $\Phi_{\gamma,N}$ is nonnegative and monotone increasing with respect to N, we get by monotone convergence that $e^{\Phi_\gamma} v$ and $e^{\Phi_\gamma} v'$ belong to $L^2(\mathbb{R})$ and satisfy

$$\int |e^{\Phi_\gamma} v'|^2\, d\tau + \int |e^{\Phi_\gamma} v|^2\, d\tau \le C_\gamma (\|v\|^2 + \|e^{\Phi_\gamma} u\|^2), \tag{2-19}$$

where

$$\Phi_\gamma(\tau) = \lim_{N\to+\infty} \Phi_{\gamma,N}(\tau) = \gamma|\tau|.$$

To finish the proof, we note that, since the regularized resolvent is bounded and $\Phi_\gamma \ge 0$,

$$\|v\|^2 = \|\mathfrak{R}_a u\|^2 \le \|\mathfrak{R}_a\|^2\, \|u\|^2 \le \|\mathfrak{R}_a\|^2\, \|e^{\Phi_\gamma} u\|^2. \qquad \square$$

Proposition 2.5. *For any $a \in (-1, 0)$, it holds*

$$\mu_a''(\zeta_a) = 2(1 - 4I_2(a)) > 0. \tag{2-20}$$

Proof. First we notice $(\zeta_a + b_a(\tau)\tau)\phi_a$ is orthogonal to ϕ_a in $L^2(\mathbb{R})$ (see (2-10)). Thus $\mathfrak{R}_a[(\zeta_a + b_a(\tau)\tau)\phi_a]$ is well-defined as $(\mathfrak{h}_a[\zeta_a] - \beta_a)^{-1}(\zeta_a + b_a(\tau)\tau)\phi_a$. Let $z \in \mathbb{R}$, and $E_a(z)$ be the lowest eigenvalue of the operator $H_a(z)$, defined on $L^2(\mathbb{R})$ as

$$H_a(z) := \mathfrak{h}_a[\zeta_a + z] = -\frac{d^2}{d\tau^2} + (\zeta_a + z + b_a(\tau)\tau)^2.$$

We adopt the same proof of [Fournais and Helffer 2006, Proposition A.3] (replacing P_0 by $H_a(0) - \beta_a$ there) to get the identity in (2-20). Finally, by [Assaad and Kachmar 2022], $\mu''(\zeta_a) > 0$. $\square$

3. One-dimensional model involving the curvature

We consider a new family of fiber operators which are obtained by adding to the fiber operators in Section 2 new terms that will be related to the geometry of the magnetic edge. This family was introduced earlier in [Assaad and Kachmar 2022] and their definition is reminiscent of the weighted operators introduced in the context of the Neumann Laplacian with a uniform magnetic field [Helffer and Morame 2001].

We introduce the parameters

$$a \in (-1, 0), \quad \delta \in \left(0, \tfrac{1}{12}\right), \quad M > 0, \quad h_0 > 0 \quad \text{and} \quad \kappa \in [-M, M],$$

which satisfy

$$M h_0^{1/2-\delta} < \tfrac{1}{3},$$

and will be fixed throughout this section.

Consider on $(-h^{-\delta}, h^{-\delta})$ the positive function $a_{\kappa,h}(\tau) = (1 - \kappa h^{1/2}\tau)$, the Hilbert space $L^2((-h^{-\delta}, h^{-\delta}); a_{\kappa,h}\, d\tau)$ with the inner product

$$\langle u, v\rangle = \int_{-h^{-\delta}}^{h^{-\delta}} u(\tau)\overline{v(\tau)}\,(1 - \kappa h^{1/2}\tau)\, d\tau,$$

and, for $\xi \in \mathbb{R}$, the operator

$$\mathcal{H}_{a,\xi,\kappa,h} = -\frac{d^2}{d\tau^2} + (b_a(\tau)\tau + \xi)^2 + \kappa h^{1/2}(1 - \kappa h^{1/2}\tau)^{-1}\partial_\tau + 2\kappa h^{1/2}\tau\Big(b_a(\tau)\tau + \xi - \kappa h^{1/2} b_a(\tau)\frac{\tau^2}{2}\Big)^2$$
$$- \kappa h^{1/2} b_a(\tau)\tau^2(b_a(\tau)\tau + \xi) + \kappa^2 h b_a(\tau)^2 \frac{\tau^4}{4}, \tag{3-1}$$

where b_a is the function in (2-2) and

$$\mathrm{Dom}(\mathcal{H}_{a,\xi,\kappa,h}) = \{u \in H^2(-h^{-\delta}, h^{-\delta}) : u(\pm h^{-\delta}) = 0\}. \tag{3-2}$$

The operator $\mathcal{H}_{a,\xi,\kappa,h}$ is a self-adjoint operator in $L^2((-h^{-\delta}, h^{-\delta}); a_{\kappa,h}\, d\tau)$ with compact resolvent. We denote by $(\lambda_n(\mathcal{H}_{a,\xi,\kappa,h}))_{n\geq 1}$ its sequence of min-max eigenvalues. The first eigenvalue can be expressed as

$$\lambda_1(\mathcal{H}_{a,\xi,\kappa,h}) = \inf\{q_{a,\xi,\kappa,h}(u) : u \in H_0^1(-h^{-\delta}, h^{-\delta}) \text{ and } \|u\|_{L^2((-h^{-\delta},h^{-\delta});a_{\kappa,h}\,d\tau)} = 1\}, \tag{3-3}$$

where

$$q_{a,\xi,\kappa,h}(u) = \int_{-h^{-\delta}}^{h^{-\delta}} \Big(|u'(\tau)|^2 + (1+2\kappa h^{1/2}\tau)\Big(b_a(\tau)\tau + \xi - \kappa h^{1/2} b_a(\tau)\frac{\tau^2}{2}\Big)^2 u^2(\tau)\Big)(1-\kappa h^{1/2}\tau)\, d\tau. \tag{3-4}$$

By Cauchy's inequality, we write, for any $\varepsilon \in (0,1)$,

$$\Big(b_a(\tau)\tau + \xi - \kappa h^{1/2} b_a(\tau)\frac{\tau^2}{2}\Big)^2 \geq (1-\varepsilon)(b_a(\tau)\tau + \xi)^2 - \varepsilon^{-1}\kappa^2 h b_a(\tau)^2 \frac{\tau^4}{4}.$$

Noticing that $h\tau^4 \leq h^{1-4\delta}$ for $\tau \in (h^{-\delta}, h^{\delta})$ and optimizing with respect to ε, we choose $\varepsilon = h^{1/2-2\delta}$ and get

$$\Big(b_a(\tau)\tau + \xi - \kappa h^{1/2} b_a(\tau)\frac{\tau^2}{2}\Big)^2 \geq (1-h^{1/2-2\delta})(b_a(\tau)\tau + \xi)^2 - \kappa^2 b_a(\tau)^2 h^{1/2-2\delta}. \tag{3-5}$$

We plug (3-5) in (3-4) to get, for some $C_0 > 0$,

$$q_{a,\xi,\kappa,h}(u) \geq (1 - C_0 h^{1/2-2\delta}) q_a[\xi](u) - C_0 h^{1/2-2\delta} \|u\|^2_{L^2(-h^{-\delta},h^{-\delta})}, \tag{3-6}$$

where $q_a[\xi]$ is the quadratic form in (2-4). The min-max principle ensures that

$$q_a[\xi](u) \geq \beta_a \|u\|^2_{L^2(-h^{-\delta},h^{-\delta})} \quad \text{for all } u \in H_0^1(-h^{-\delta}, h^{-\delta}). \tag{3-7}$$

Since $\beta_a > 0$, (3-6) and (3-7) imply

$$q_{a,\xi,\kappa,h}(u) \geq (1 - C h^{1/2-2\delta}) q_a[\xi](u), \tag{3-8}$$

with $C = (1+\beta_a^{-1})C_0$. From (3-8) and the min-max principle we deduce the lower bounds in Lemma 3.1 below (see [Assaad and Kachmar 2022, Section 4.2] for details).

Lemma 3.1. *Given $a \in (-1, 0)$, there exist positive constants $\varepsilon_0(a)$, $\varepsilon_1(a)$, $\varepsilon_2(a)$, $c_0(a)$, $h_0(a)$, $C_0(a)$ such that, for all $h \in (0, h_0(a))$,*

- *For $|\xi - \zeta_a| \geq \varepsilon_0(a)$, we have*
$$\lambda_1(\mathcal{H}_{a,\xi,\kappa,h}) \geq \beta_a + c_0(a).$$
- *For $\varepsilon_2(a)h^{1/4-\delta} \leq |\xi - \zeta_a| \leq \varepsilon_0(a)$, we have*
$$\lambda_1(\mathcal{H}_{a,\xi,\kappa,h}) \geq \beta_a + \varepsilon_1(a)(\xi - \zeta_a)^2.$$
- *For $|\xi - \zeta_a| \leq \varepsilon_2(a)h^{1/4-\delta}$, we have*
$$\lambda_1(\mathcal{H}_{a,\xi,\kappa,h}) \geq \beta_a + c_2(a)|\xi - \zeta_a|^2 + \kappa M_3(a) h^{1/2} - C_0(a)\max(h^{1/2}|\xi - \zeta_a|, |\xi - \zeta_a|^3, h),$$

where

$$c_2(a) = \tfrac{1}{2}\mu_a''(\zeta_a) > 0. \tag{3-9}$$

We can now state the following:

Proposition 3.2. *There exists $\hat{c}_0(a) > 0$ and, for all $\varepsilon \in (0,1)$, there exist $C_\varepsilon, h_\varepsilon > 0$ such that, for all $h \in (0, h_\varepsilon)$ and $\xi \in \mathbb{R}$, the following inequality holds*:

$$\lambda_1(\mathcal{H}_{a,\xi,\kappa,h}) \geq \beta_a + \hat{c}_0(a)\min((\xi-\zeta_a)^2, \varepsilon) + \kappa M_3(a) h^{1/2} - C_\varepsilon h.$$

Proof. In the third item of Lemma 3.1, we estimate the remainder term

$$\max(h^{1/2}|\xi-\zeta_a|, |\xi-\zeta_a|^3, h) \leq (\eta^{-1}+1)h + \eta|\xi-\zeta_a|^2 + |\xi-\zeta_a|^3$$

for all $\eta \in (0,1)$. Choosing $\eta = c_2(a)/(4C_0(a))$, where $C_0(a)$ is the constant in Lemma 3.1, we deduce from Lemma 3.1 the lower bound for the eigenvalue $\lambda_1(\mathcal{H}_{a,\xi,\kappa,h})$, with

$$\hat{c}_0(a) = \frac{1}{2}\min\left(\varepsilon_1(a), \frac{c_0(a)}{\varepsilon_0(a)^2}, c_0(a)\right). \qquad \square$$

4. Upper bound

We establish an upper bound of the n-th eigenvalue $\lambda_n(h)$ of $\mathcal{P}_h$, which was defined in (1-4). This will involve the spectral value β_a introduced in (2-6), the moment $M_3(a) < 0$ introduced in (2-16), and $c_2(a) > 0$ the value defined in (3-9). In this section, we consider two parameters $\eta \in \left(0, \frac{1}{8}\right)$ and $\delta \in \left(0, \frac{1}{2}\right)$.

Theorem 4.1. *Let $n \in \mathbb{N}^*$ and $\boldsymbol{a} = (1, a)$, with $-1 < a < 0$. Under Assumption 1.1, there exist $h_0 > 0$ and $C_0 > 0$ such that, for all $h \in (0, h_0)$, the n-th eigenvalue $\lambda_n(h)$ of the operator $\mathcal{P}_h$ defined in* (1-4) *satisfies*

$$\lambda_n(h) \leq h\beta_a + h^{3/2}k_{\max}M_3(a) + h^{7/4}(2n-1)\sqrt{\frac{k_2 M_3(a) c_2(a)}{2}} + C_0 h^{15/8}, \tag{4-1}$$

where $c_2(a)$ and $M_3(a)$ were introduced in (1-12).

Proof. The approach is similar to the one used in the literature in establishing upper bounds for the low-lying eigenvalues of operators defined on smooth bounded domains, like Schrödinger operators with uniform magnetic fields (and Neumann boundary conditions) or the Laplacian (with Robin boundary conditions). For instance, one can see [Bernoff and Sternberg 1998; Fournais and Helffer 2006; Helffer and Kachmar 2017]. The proof relies on the construction of quasimodes localized near the point of maximal curvature on Γ.

Let $h \in (0,1)$. Working near Γ, we start by expressing the operator $\mathcal{P}_h$ in the adapted (s,t)-coordinates there (see Appendix A):

$$\widetilde{\mathcal{P}}_h = -\mathfrak{a}^{-1}(h\partial_s - i\widetilde{F}_1)\mathfrak{a}^{-1}(h\partial_s - i\widetilde{F}_1) - \mathfrak{a}^{-1}(h\partial_t - i\widetilde{F}_2)\mathfrak{a}(h\partial_t - i\widetilde{F}_2). \tag{4-2}$$

Recall that we assume that the maximum is attained for $s = 0$, hence $k_{\max} = k(0)$, and having Lemma A.1, we perform a global change of gauge ω such that the magnetic potential $\boldsymbol{F}$ satisfies in Ω near the edge Γ, when expressed in the (s,t)-coordinates,

$$\widetilde{\boldsymbol{F}}(s,t)=\begin{pmatrix}-b_a(t)\big(t-\frac{1}{2}t^2k(s)\big)\\ 0\end{pmatrix}, \tag{4-3}$$

where $t\mapsto b_a(t)$ is defined by

$$b_a(t)=\mathbb{1}_{\mathbb{R}_+}(t)+a\mathbb{1}_{\mathbb{R}_-}(t),\quad t\in\mathbb{R}.$$

Performing the change of variables

$$\sigma=h^{-1/8}s\quad\text{and}\quad\tau=h^{-1/2}t,$$

the operator $\widetilde{\mathcal{P}}_h$ becomes in the (σ,τ)-coordinates

$$\check{\mathcal{P}}_h=-\check{\mathfrak{a}}^{-1}(h^{7/8}\partial_\sigma+ih^{1/2}b_a(\tau)\tau\check{\mathfrak{a}}_2)\check{\mathfrak{a}}^{-1}(h^{7/8}\partial_\sigma+ih^{1/2}b_a(\tau)\tau\check{\mathfrak{a}}_2)-h\check{\mathfrak{a}}^{-1}\partial_\tau\check{\mathfrak{a}}\partial_\tau, \tag{4-4}$$

with

$$\check{\mathfrak{a}}(\sigma,\tau;h)=1-h^{1/2}\tau k(h^{1/8}\sigma)\quad\text{and}\quad\check{\mathfrak{a}}_2(\sigma,\tau;h)=1-\tfrac{1}{2}h^{1/2}\tau k(h^{1/8}\sigma). \tag{4-5}$$

It is convenient to introduce the operator

$$\mathcal{P}_h^{\text{new}}=e^{-i\sigma\zeta_a/h^{3/8}}h^{-1}\check{\mathcal{P}}_he^{i\sigma\zeta_a/h^{3/8}}-\beta_a, \tag{4-6}$$

where ζ_a is introduced in Section 2B and we get

$$\begin{aligned}\mathcal{P}_h^{\text{new}}=-\check{\mathfrak{a}}^{-1}\partial_\tau\check{\mathfrak{a}}\partial_\tau-\beta_a-\check{\mathfrak{a}}^{-1}\big(h^{3/8}\partial_\sigma+i(\zeta_a+b_a(\tau)\tau)-ib_a(\tau)\tau(1-\check{\mathfrak{a}}_2)\big)&\\ \times\check{\mathfrak{a}}^{-1}\big(h^{3/8}\partial_\sigma+i(\zeta_a+b_a(\tau)\tau)-ib_a(\tau)\tau(1-\check{\mathfrak{a}}_2)\big)&.\end{aligned}$$

Using the boundedness and the smoothness of k, and the fact that $k'(0)=0$ and $k''(0)<0$, we write

$$\begin{aligned}\check{\mathfrak{a}}(\sigma,\tau;h)&=1-h^{1/2}\tau k(0)-h^{3/4}\tau\sigma^2\frac{k''(0)}{2}+h^{7/8}e_{1,h}(\sigma,\tau),\\ \check{\mathfrak{a}}_2(\sigma,\tau;h)&=1-h^{1/2}\tau\frac{k(0)}{2}-h^{3/4}\tau\sigma^2\frac{k''(0)}{4}+h^{7/8}e_{2,h}(\sigma,\tau),\\ \check{\mathfrak{a}}^{-1}(\sigma,\tau;h)&=1+h^{1/2}\tau k(0)+h^{3/4}\tau\sigma^2\frac{k''(0)}{2}+h^{7/8}e_{3,h}(\sigma,\tau),\\ \check{\mathfrak{a}}^{-2}(\sigma,\tau;h)&=1+2h^{1/2}\tau k(0)+h^{3/4}\tau\sigma^2k''(0)+h^{7/8}e_{4,h}(\sigma,\tau),\end{aligned}$$

where $(e_{i,h})_{i=1,\dots,4}$ are functions of σ and τ having the property that there exist C and h_0 such that,[1] for $h\in(0,h_0)$, $\sigma\in(-h^{-\eta},h^{-\eta})$ and $\tau\in(-h^{-\rho},h^{-\rho})$ we have

$$|e_{1,h}(\sigma,\tau)|+|e_{2,h}(\sigma,\tau)|\le C|\tau\sigma^3|,\quad |e_{3,h}(\tau,\sigma)|+|e_{4,h}(\tau,\sigma)|\le C(\sigma^6+\tau^4+1), \tag{4-7}$$

and

$$\sum_{i=1}^{4}\bigg(\sum_{j=1}^{2}\big(|\partial_\tau^je_{i,h}(\sigma,\tau)|+|\partial_\sigma^je_{i,h}(\sigma,\tau)|\big)+|\partial_{\sigma\tau}^2e_{i,h}(\sigma,\tau)|\bigg)\le C(|\sigma|^5+|\tau|^3+1). \tag{4-8}$$

[1]The following conditions on the length scales of τ and σ (namely that $\sigma\in(-h^{-\delta},h^{-\delta})$ and $\tau\in(-h^{-\rho},h^{-\rho})$), as well as (4-7) and (4-8) below, are set for a later use in the paper.

Hence,

$$\mathcal{P}_h^{\text{new}} = P_0 + h^{3/8} P_1 + h^{1/2} P_2 + h^{3/4} P_3 + h^{7/8} Q_h, \tag{4-9}$$

where

$$\begin{aligned}
P_0 &= -\partial_\tau^2 + (\zeta_a + b_a(\tau)\tau)^2 - \beta_a,\\
P_1 &= -2i(\zeta_a + b_a(\tau)\tau)\partial_\sigma,\\
P_2 &= k(0)[2\tau(\zeta_a + b_a(\tau)\tau)^2 - b_a(\tau)\tau^2(\zeta_a + b_a(\tau)\tau)] + k(0)\partial_\tau,\\
P_3 &= -\partial_\sigma^2 + \frac{k''(0)}{2}\sigma^2[2\tau(\zeta_a + b_a(\tau)\tau)^2 - b_a(\tau)\tau^2(\zeta_a + b_a(\tau)\tau)] + \frac{k''(0)}{2}\sigma^2\partial_\tau,
\end{aligned} \tag{4-10}$$

and

$$Q_h = \mathcal{E}_{1,h}(\sigma,\tau)\partial_\sigma^2 + \mathcal{E}_{2,h}(\sigma,\tau)\partial_\sigma + \mathcal{E}_{3,h}(\sigma,\tau)\partial_\tau + \mathcal{E}_{4,h}(\sigma,\tau). \tag{4-11}$$

Here the terms $(\mathcal{E}_{i,h})_{i=1,\dots,4}$ are functions in σ and τ having the property that there exist C and h_0 such that, for $h \in (0, h_0)$, $\sigma \in (-h^{-\eta}, h^{-\eta})$ and $\tau \in (-h^{-\rho}, h^{-\rho})$, we have

$$|\mathcal{E}_{i,h}(\sigma,\tau)| + |\partial_\sigma \mathcal{E}_{i,h}(\sigma,\tau)| + |\partial_\tau \mathcal{E}_{i,h}\sigma,\tau)| \le C\,(|\sigma|^6 + |\tau|^6 + 1). \tag{4-12}$$

In what follows, we will construct, for each $n \in \mathbb{N}^*$, a trial function $\phi_n \in \operatorname{Dom} \mathcal{P}_h^{\text{new}}$ satisfying

$$\begin{aligned}
\left\| \mathcal{P}_h^{\text{new}}\phi_n - \left(h^{1/2} k_{\max} M_3(a) + h^{3/4}(2n-1)\sqrt{\frac{k_2 M_3(a) c_2(a)}{2}} \right)\phi_n \right\|_{L^2(\mathbb{R}^2, h^{5/8}\tilde{\mathfrak{a}}\,d\sigma\,d\tau)}&\\
= \mathcal{O}(h^{7/8}) \|\phi_n\|_{L^2(\mathbb{R}^2, h^{5/8}\tilde{\mathfrak{a}}\,d\sigma\,d\tau)}&
\end{aligned} \tag{4-13}$$

(recall $k_2 = k''(0)$).

The result in (4-13), once established, will imply by the spectral theorem the existence of an eigenvalue $\lambda_n^{\text{new}}(h)$ of $\mathcal{P}_h^{\text{new}}$ such that

$$\lambda_n^{\text{new}}(h) = h^{1/2} k_{\max} M_3(a) + h^{3/4}(2n-1)\sqrt{\frac{k_2 M_3(a) c_2(a)}{2}} + \mathcal{O}(h^{7/8}). \tag{4-14}$$

Furthermore, by the definition of $\mathcal{P}_h^{\text{new}}$ in (4-6) we have

$$\sigma(\mathcal{P}_h) = h\,\sigma(\mathcal{P}_h^{\text{new}}).$$

Thus, (4-14) will yield the result in (4-1). Hence, the discussion above shows that establishing (4-13) is sufficient to complete the proof of the theorem.

We construct the trial functions in the form

$$\phi_h(\sigma,\tau) = h^{-5/16}\chi(h^\eta\sigma)\chi(h^\rho\tau)g(\sigma,\tau), \tag{4-15}$$

where χ is a smooth cut-off function supported in $(-1, 1)$ and $g = g[h]$ will be determined in $L^2(\mathbb{R}^2)$ with rapid decay at infinity. First we set

$$g[h] = g_0 + h^{3/8} g_1 + h^{1/2} g_2 + h^{3/4} g_3, \tag{4-16}$$

with $g_i \in L^2(\mathbb{R}^2)$ for $i = 0, \dots, 3$, and

$$\mu = \mu(h) = \mu_0 + h^{3/8}\mu_1 + h^{1/2}\mu_2 + h^{3/4}\mu_3, \tag{4-17}$$

with $\mu_i \in \mathbb{R}$ for $i = 0, \dots, 3$. We will search for μ and g satisfying on $\mathbb{R}^2$

$$(\mathcal{P}_h^{\text{new}} - \mu)g = \mathcal{O}(h^{7/8}). \tag{4-18}$$

More precisely, using the expansion of $\mathcal{P}_h^{\text{new}}$ in (4-9), we will search for μ_i and g_i satisfying the system of equations

$$\begin{cases} (e_0)\colon\ (P_0 - \mu_0)g_0 = 0, \\ (e_1)\colon\ (P_0 - \mu_0)g_1 + (P_1 - \mu_1)g_0 = 0, \\ (e_2)\colon\ (P_0 - \mu_0)g_2 + (P_2 - \mu_2)g_0 = 0, \\ (e_3)\colon\ (P_0 - \mu_0)g_3 + (P_1 - \mu_1)g_1 + (P_3 - \mu_3)g_0 = 0. \end{cases}$$

Let $u_0 = \phi_a$ be the positive normalized eigenfunction of the operator $\mathfrak{h}_a[\zeta_a]$ (in (2-1)) corresponding to the lowest eigenvalue β_a.

Obviously, the pair

$$(\mu_0, g_0) = (0, u_0 f) \tag{4-19}$$

is a solution of (e_0) for any $f \in \mathcal{S}(\mathbb{R}_\sigma)$.

We implement this choice of (μ_0, g_0) in (e_1) and write

$$P_0 g_1 = -(P_1 - \mu_1)g_0 = [2i(\zeta_a + b_a(\tau)\tau)\partial_\sigma + \mu_1]u_0 f.$$

Noticing that $(\zeta_a + b_a(\tau)\tau)u_0$ is orthogonal to u_0 in $L^2(\mathbb{R})$, $\mathfrak{R}_a[(\zeta_a + b_a(\tau)\tau)u_0]$ is well-defined with $\mathfrak{R}_a$ in (2-18) (see (2-11) and Remark 2.2), and the pair

$$(\mu_1, g_1) = \big(0, 2i\mathfrak{R}_a[(\zeta_a + b_a(\tau)\tau)u_0]\partial_\sigma f\big) \tag{4-20}$$

is a solution of (e_1).

Similarly,

$$P_0 g_2 = -(P_2 - \mu_2)g_0 = \big[-k_{\max}\big(2\tau(\zeta_a + b_a(\tau)\tau)^2 - b_a(\tau)\tau^2(\zeta_a + b_a(\tau)\tau)\big) + \mu_2\big]u_0 f - k_{\max} f\,\partial_\tau u_0.$$

From Remark 2.3, we observe that $[2\tau(\zeta_a + b_a(\tau)\tau)^2 - b_a(\tau)\tau^2(\zeta_a + b_a(\tau)\tau) - M_3(a)]u_0$ is orthogonal to u_0 in $L^2(\mathbb{R})$. Moreover, the normalization of u_0 in $L^2(\mathbb{R})$ yields $\partial_\tau u_0 \perp u_0$. Hence, the pair

$$\begin{aligned}(\mu_2, g_2) = \big(&k_{\max} M_3(a), \\ &-k_{\max}\mathfrak{R}_a([2\tau(\zeta_a + b_a(\tau)\tau)^2 - b_a(\tau)\tau^2(\zeta_a + b_a(\tau)\tau) - M_3(a)]u_0 + \partial_\tau u_0)f\big)\end{aligned} \tag{4-21}$$

is a solution of equation (e_2).

Finally, we consider equation (e_3):

$$P_0 g_3 = -P_1 g_1 - (P_3 - \mu_3)g_0.$$

We will search for μ_3 and f satisfying

$$\big(P_1 g_1(\sigma, \cdot) + (P_3 - \mu_3)g_0(\sigma, \cdot)\big) \perp u_0(\cdot) \tag{4-22}$$

for every fixed σ. This orthogonality result will allow us to choose

$$g_3(\sigma,\cdot) = -\mathfrak{R}_a[P_1 g_1(\sigma,\cdot) + (P_3-\mu_3)g_0(\sigma,\cdot)] \tag{4-23}$$

in order to satisfy (e_3). To that end, the aforementioned choice of g_0, g_1 and g_2 gives for any fixed σ

$$\begin{aligned}
&\langle P_1 g_1(\sigma,\cdot) + (P_3-\mu_3)g_0(\sigma,\cdot), u_0(\cdot)\rangle_{L^2(\mathbb{R})}\\
&= 4\partial_\sigma^2 f(\sigma)\int_{\mathbb{R}}(\zeta_a + b_a(\tau)\tau)u_0\mathfrak{R}_a[(\zeta_a+b_a(\tau)\tau)u_0]\,d\tau + \frac{k_2}{2}\sigma^2 f(\sigma)\int_{\mathbb{R}} u_0\partial_\tau u_0\,d\tau\\
&\qquad + \int_{\mathbb{R}}\Big(-\partial_\sigma^2 f(\sigma) + \frac{k_2}{2}\sigma^2 f(\sigma)[2\tau(\zeta_a+b_a(\tau)\tau)^2 - b_a(\tau)\tau^2(\zeta_a+b_a(\tau)\tau)] - \mu_3 f(\sigma)\Big)u_0^2\,d\tau\\
&= -(1-4I_2(a))\partial_\sigma^2 f(\sigma) + \frac{k_2 M_3(a)}{2}\sigma^2 f(\sigma) - \mu_3 f(\sigma) \quad (\text{using } \|u_0\|_{L^2(\mathbb{R})}=1)\\
&= -c_2(a)\partial_\sigma^2 f(\sigma) + \frac{k_2 M_3(a)}{2}\sigma^2 f(\sigma) - \mu_3 f(\sigma),
\end{aligned} \tag{4-24}$$

where $I_2(a)$ is introduced in (2-17) and (2-20), and $c_2(a)$ is introduced in (1-12).

We consider the harmonic oscillator on $\mathbb{R}$

$$H_a^{\mathrm{harm}} := -c_2(a)\frac{d^2}{d\sigma^2} + \frac{1}{2}k_2 M_3(a)\sigma^2. \tag{4-25}$$

For each $n\in\mathbb{N}^*$, let $f_n\in\mathcal{S}(\mathbb{R})$ be the n-th normalized eigenfunction of H_a^{harm} corresponding to the eigenvalue $(2n-1)\sqrt{k_2M_3(a)c_2(a)/2}$. The choice

$$f = f_n \quad\text{and}\quad \mu_3 = (2n-1)\sqrt{\frac{k_2 M_3(a)c_2(a)}{2}} \tag{4-26}$$

makes the expression in (4-24) equal to zero, hence realizing the orthogonality result in (4-22).

We can now gather the above results. For each $n\in\mathbb{N}^*$, we choose μ in (4-17) and $g = g_{(n)}$ in (4-16) such that μ_i, g_i and f are as in (4-19)–(4-21), (4-23) and (4-26).

For h sufficiently small, using the properties of Q_h in (4-11) and (4-12), the fact that $f\in\mathcal{S}(\mathbb{R})$, the decay properties of ϕ_a in Proposition 2.1 and those of the resolvent $\mathfrak{R}_a$ in (2-18), the foregoing choice of g and μ implies (4-18).

Now, we consider the trial function (see (4-15)) associated with $g_{(n)}$. Using again the decay properties of u_0 and f, and Lemma 2.4 for getting the same properties for the g_j, one can neglect the effect of the cut-off functions in the computation while concluding from (4-18) the desired result in (4-13). We omit further details of the computation, and refer the reader to [Fournais and Helffer 2006, Sections 2–3]. □

Remark 4.2. The formal construction of the pairs $(\mu_i, g_i)_{i=0,\dots,3}$ in the proof of Theorem 4.1 can be pushed to any order, assuming that the curve Γ is C^∞ smooth. Using the same approach we can construct pairs $(\mu_i, g_i)_{i\in\mathbb{N}^*}$ for defining quasimodes yielding an accurate upper bound of the eigenvalue $\lambda_n(h)$, which is an infinite expansion of powers of $h^{1/8}$. This upper bound will agree with the one in Theorem 4.1 up to the order $h^{7/4}$; see [Bernoff and Sternberg 1998; Fournais and Helffer 2006; Helffer and Kachmar 2017].

Remark 4.3. In the derivation of the lower bound in Section 7, the operator H_a^{harm} introduced in (4-25) plays the role of an effective operator in the tangential variable. In light of (4-16), (4-19), (4-20), (4-21) and (4-26), the quasimode

$$v_{h,n}^{\mathrm{app}}=\phi_a(\tau)f_n(\sigma)+2ih^{3/8}\mathfrak{R}_a\big((\zeta_a+b_a(\tau)\tau)\phi_a(\tau)\big)\partial_\sigma f_n(\sigma)+h^{1/2}g_2(\sigma,\tau)$$

is a candidate for the profile of an actual eigenfunction of the operator $\mathcal{P}_h$, after rescaling and a gauge transformation.

5. Functions localized near the magnetic edge

In this section, we consider functions satisfying the energy bound[2] in (5-1), which are consequently localized near the maximum of the curvature of the magnetic edge Γ. We will be able to estimate the tangential derivative of such functions.

As we shall see in Section 5A, bound states and their first-order tangential derivatives are examples of the functions we discuss in this section.

5A. ***Localization hypotheses.*** We fix $t_0>0$ so that the Frenet coordinates recalled in Appendix A are valid in $\{d(x,\Gamma)<t_0\}$. We recall our assumption that the curvature of Γ attains its maximum at a unique point defined by the tangential coordinate $s=0$.

Let $\theta\in\big(0,\frac{3}{8}\big)$ be a fixed constant. Consider a family of functions $(g_h)_{h\in(0,h_0]}$ in $H^1(\Omega)$ for which there exist positive constants C_1, C_2 such that, for $h\in(0,h_0]$,

$$\mathcal{Q}_h(g_h)\le(h\beta_a+h^{3/2}M_3(a)k_{\max}+C_1h^{7/4})\|g_h\|^2_{L^2(\Omega)}+C_2h^{5/2-\theta},\tag{5-1}$$

where $\mathcal{Q}_h$ is the quadratic form introduced in (1-3).

Suppose also that there exist constants $\alpha, C>0$ and a family $(r_h)_{h\in(0,h_0]}\subset\mathbb{R}_+$ such that

$$\limsup_{h\to0_+} r_h<+\infty,\tag{5-2}$$

and the following two estimates hold:

$$\int_\Omega\big(|g_h|^2+h^{-1}|(h\nabla-i\boldsymbol{F})g_h|^2\big)\exp(\alpha h^{-1/2}d(x,\Gamma))\,dx\le Cr_h,\tag{5-3}$$

$$\int_{d(x,\Gamma)\le t_0}\big(|g_h(x)|^2+h^{-1}|(h\nabla-i\boldsymbol{F})g_h|^2\big)\exp(\alpha h^{-1/8}|s(x)|)\,dx\le Cr_h.\tag{5-4}$$

We can derive from the decay estimates in (5-3) and (5-4) four estimates.

The two first estimates follow from the inequality $e^z\ge z^N/N!$ for $z\ge0$ and read: for $N\ge1$, there exist $C_N, h_N>0$ such that, for all $h\in(0,h_N]$, we have

$$A_N(g_h):=\int_\Omega(d(x,\Gamma))^N\big(|g_h(x)|^2+h^{-1}|(h\nabla-i\boldsymbol{F})g_h(x)|^2\big)\,dx\le C_Nh^{N/2}r_h,\tag{5-5}$$

[2]This is coherent with (4-1) if we consider the function a normalized bound state.

and, for $\rho \in \left(0, \frac{1}{2}\right)$, there exist $C_{N,\rho}, h_{N,\rho} > 0$ such that, for all $h \in (0, h_{N,\rho}]$,

$$B_N(g_h) := \int_{d(x,\Gamma)\le h^\rho} |s(x)|^N \left(|g_h(x)|^2 + h^{-1}|(h\nabla - i\boldsymbol{F})g_h(x)|^2\right) dx \le C_N h^{N/8} r_h. \tag{5-6}$$

The two last estimates imply that, for a fixed $\rho \in \left(0, \frac{1}{2}\right)$, and $N \ge 1$, there exist $C_{N,\rho}, h_{N,\rho} > 0$ such that, for all $h \in (0, h_{N,\rho}]$, we have

$$\int_{d(x,\Gamma)\ge h^\rho} \left(|g_h(x)|^2 + h^{-1}|(h\nabla - i\boldsymbol{F})g_h(x)|^2\right) dx \le C_{N,\rho}\, h^N r_h, \tag{5-7}$$

and for $\eta \in \left(0, \frac{1}{8}\right)$, there exist $C_{N,\rho,\eta}, h_{N,\rho,\eta} > 0$ such that, for all $h \in (0, h_{N,\rho,\eta}]$, we have

$$\int_{\substack{d(x,\Gamma)\le h^\rho \\ |s(x)|\ge h^\eta}} \left(|g_h(x)|^2 + h^{-1}|(h\nabla - i\boldsymbol{F})g_h|^2\right) dx \le C_{N,\rho,\eta}\, h^N r_h. \tag{5-8}$$

In fact, (5-7) and (5-8) follow in a straightforward manner from (5-3) and (5-4) after noticing that

$$\int_{d(x,\Gamma)\ge h^\rho} \left(|g_h(x)|^2 + h^{-1}|(h\nabla - i\boldsymbol{F})g_h|^2\right) dx \le C r_h \exp(-\alpha h^{\rho-1/2}),$$

$$\int_{\substack{d(x,\Gamma)\le h^\rho \\ |s(x)|\ge h^\eta}} \left(|g_h(x)|^2 + h^{-1}|(h\nabla - i\boldsymbol{F})g_h|^2\right) dx \le C r_h \exp(-\alpha h^{\eta-1/8}).$$

5B. ***Rescaled functions and tangential estimates.*** Let $\delta \in \left(0, \frac{1}{12}\right)$ and $\eta \in \left(0, \frac{1}{8}\right)$ be two fixed constants. Consider the function w_h defined as

$$w_h(\sigma, \tau) = h^{5/16}\chi(h^\eta\sigma)\chi(h^\delta\tau)\tilde{g}_h(h^{1/8}\sigma, h^{1/2}\tau), \tag{5-9}$$

where $\tilde{g}_h$ is the function assigned to g_h by the Frenet coordinates as in (A-3), namely

$$\tilde{g}_h(s,t) = g_h(x),$$

and $\chi \in C_c^\infty(\mathbb{R}),\ \operatorname{supp}\chi \subset [-1, 1],\ 0 \le \chi \le 1$ and $\chi = 1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Note that, due to our conditions on δ and η, w_h can be seen as a function on $\mathbb{R}^2$, and its L^2-norm can be estimated by using (A-7) and (5-5) as follows:

$$\|w_h\|^2_{L^2(\mathbb{R}^2)} = (1 + \mathcal{O}(h^{1/2}))\|g_h\|^2_{L^2(\Omega)}. \tag{5-10}$$

Under our hypotheses on the function g_h (particularly (5-1) for $\theta \in \left(0, \frac{3}{8}\right)$ and (5-3)–(5-4)), we can estimate the tangential derivative of the function w_h.

Proposition 5.1. *For all $\theta \in \left(0, \frac{3}{8}\right)$, there exist constants $C_\theta, h_\theta > 0$ such that, if $h \in (0, h_\theta]$, and g_h satisfies* (5-1)$_\theta$, (5-3) *and* (5-4), *then the function w_h introduced in* (5-9) *satisfies the estimate*

$$\|(h^{3/8}\partial_\sigma - i\zeta_a)w_h\|_{L^2(\mathbb{R}^2)} \le C h^{3/8-\theta/2}(\|w_h\|_{L^2(\mathbb{R}^2)} + \sqrt{r_h} + h^{3/8-3\theta/4}). \tag{5-11}$$

Proof. The proof is split into four steps.

Step 1: We localize the integrals defining the L^2-norm and the quadratic form of g_h to the neighborhood, $\mathcal{N}_h = \{x \in \Omega : d(x,\Gamma) \le h^{1/2-\delta},\ |s(x)| \le h^{\eta}\}$, of the point of maximal curvature, $s=0$. In fact, by the decay estimates in (5-7) and (5-8),

$$\|g_h\|^2_{L^2(\Omega)} = \int_{\mathcal{N}_h} |g_h(x)|^2\,dx + \mathcal{O}(h^\infty) \quad\text{and}\quad \mathcal{Q}_h(g_h) = \int_{\mathcal{N}_h} |(h\nabla - i\boldsymbol{F})g_h|^2\,dx + \mathcal{O}(h^\infty).$$

We refine the localization of these integrals by using the decay estimates in (5-5) and (5-6), the change of variable formulas in (A-7) and the expansions

$$k(s) = \kappa + \mathcal{O}(s^2), \quad \mathfrak{a}(s,t) = 1 - t\kappa + \mathcal{O}(s^2 t), \quad \mathfrak{a}^{-2} = 1 + 2t\kappa + \mathcal{O}(s^2 t),$$

where we set $\kappa = k_{\max}$. More precisely,

$$\|g_h\|^2_{L^2(\Omega)} = \int_{\mathbb{R}}\int_{-h^{1/2-\delta}}^{h^{1/2-\delta}} |\tilde g_h|^2 (1 - t\kappa)\,ds\,dt + \int_{\mathbb{R}}\int_{-h^{1/2-\delta}}^{h^{1/2-\delta}} \mathcal{O}(s^2 t)|\tilde g_h|^2\,ds\,dt + \mathcal{O}(h^\infty).$$

To estimate the second term in the right-hand side we use the Cauchy–Schwarz inequality to obtain

$$\int_{\mathbb{R}}\int_{-h^{1/2-\delta}}^{h^{1/2-\delta}} s^2|t|\,|\tilde g_h|^2\,ds\,dt \le \biggl(\int_{\mathbb{R}}\int_{-h^{1/2-\delta}}^{h^{1/2-\delta}} t^2|\tilde g_h|^2\,ds\,dt\biggr)^{1/2}\biggl(\int_{\mathbb{R}}\int_{-h^{1/2-\delta}}^{h^{1/2-\delta}} s^4|\tilde g_h|^2\,ds\,dt\biggr)^{1/2}.$$

Hence by (5-5) (with $N=2$) and (5-6) (with $N=4$) we get

$$\int_{\mathbb{R}}\int_{-h^{1/2-\delta}}^{h^{1/2-\delta}} s^2|t|\,|\tilde g_h(s,t)|^2\,ds\,dt = \mathcal{O}(h^{3/4})r_h.$$

Implementing the above, we have

$$\|g_h\|^2_{L^2(\Omega)} \le \int_{\mathbb{R}}\int_{-h^{-\delta}}^{h^{-\delta}} |w_h|^2(1 - h^{1/2}\tau\kappa)\,d\sigma\,d\tau + \mathcal{O}(h^{3/4})r_h + \mathcal{O}(h^\infty) \tag{5-12}$$

and

$$\mathcal{Q}_h(g_h) = \int_{\mathbb{R}}\int_{-h^{1/2-\delta}}^{h^{1/2-\delta}} \Bigl(|h\partial_t \tilde g_h|^2 + (1+2\kappa t)\Bigl|\Bigl(h\partial_s + ib_a(t)\Bigl(t - \frac{\kappa t^2}{2}\Bigr)\Bigr)\tilde g_h\Bigr|^2\Bigr)(1-\kappa t)\,ds\,dt + \mathcal{O}(h^\infty) + \mathcal{O}(R_h), \tag{5-13}$$

where

$$R_h = \int_{\mathbb{R}^2} s^2|t|\Bigl(|h\partial_t\tilde g_h|^2 + \Bigl|\Bigl(h\partial_s + ib_a(t)\Bigl(t - \frac{k(s)t^2}{2}\Bigr)\Bigr)\tilde g_h\Bigr|^2\Bigr)\,ds\,dt + \int_{\mathbb{R}^2} s^4t^4|\tilde g_h|^2\,ds\,dt + \biggl(\int_{\mathbb{R}^2} s^4t^4|\tilde g_h|^2\,ds\,dt\biggr)^{1/2}\|(h\nabla - i\boldsymbol{F})g_h\|_{L^2(\Omega)}.$$

Proceeding as above for the treatment of $\int_{\mathbb{R}^2} s^4t^4|\tilde g_h|^2\,ds\,dt$, we infer from (5-1), (5-5) and (5-6) that

$$R_h \le C\bigl((A_2(g_h)B_4(g_h))^{1/2}h + (A_8(g_h)B_8(g_h))^{1/2} + (A_8(g_h)B_8(g_h))^{1/4}h^{1/2}\bigr) = \mathcal{O}(h^{7/4}r_h).$$

Now, coming back to (5-1), we get after performing a change of variable and dividing by h that[3]

$$\int_{\mathbb{R}}\int_{-h^{-\delta}}^{h^{-\delta}}\Big(|\partial_\tau w_h|^2+(1+2\kappa h^{1/2}\tau)\Big|\Big(h^{3/8}\partial_\sigma+i\Big(b_a(\tau)\tau-\kappa h^{1/2}b_a(\tau)\frac{\tau^2}{2}\Big)\Big)w_h\Big|^2\Big)(1-\kappa h^{1/2}\tau)\,d\sigma\,d\tau$$
$$\le(\beta_a+h^{1/2}M_3(a)\kappa+\mathcal{O}(h^{3/4}))m_h+\mathcal{O}(h^{3/4}r_h)+\mathcal{O}(h^{3/2-\theta}),\quad(5\text{-}14)$$

where

$$m_h:=\int_{\mathbb{R}}\int_{-h^{-\delta}}^{h^{-\delta}}|w_h|^2(1-\kappa h^{1/2}\tau)\,d\sigma\,d\tau=(1+o(1))\|w_h\|^2_{L^2(\mathbb{R}^2)}.\quad(5\text{-}15)$$

In the sequel, we set

$$M_h=m_h+r_h.\quad(5\text{-}16)$$

Next we perform a Fourier transform with respect to σ and denote the transform of w_h by

$$\hat{w}_h(\xi,t)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}w_h(\sigma,t)e^{-i\sigma\xi}\,d\sigma.$$

Then it is immediate from (5-14) and (5-15) that we have

$$\int_{\mathbb{R}}\int_{-h^{-\delta}}^{h^{-\delta}}\Big(|\partial_\tau\hat{w}_h|^2+(1+2\kappa h^{1/2}\tau)\Big|\Big(h^{3/8}\xi+b_a(\tau)\tau-\kappa h^{1/2}b_a(\tau)\frac{\tau^2}{2}\Big)\hat{w}_h\Big|^2\Big)(1-\kappa h^{1/2}\tau)\,d\xi\,d\tau$$
$$\le(\beta_a+h^{1/2}M_3(a)\kappa)m_h+\mathcal{O}(h^{3/4}M_h)+\mathcal{O}(h^{3/2-\theta}),\quad(5\text{-}17)$$

and m_h introduced in (5-15) now satisfies

$$m_h=\int_{\mathbb{R}}\int_{-h^{-\delta}}^{h^{-\delta}}|\hat{w}_h|^2(1-\kappa h^{1/2}\tau)\,d\xi\,d\tau.\quad(5\text{-}18)$$

Step 2: We introduce

$$f_h(\xi)=q_{a,\zeta,\kappa,h}(\hat{w}_h)\big|_{\zeta=h^{3/8}\xi},\quad(5\text{-}19)$$

where $q_{a,\zeta,\kappa,h}$ is the quadratic form introduced in (3-4). We rewrite (5-17) as

$$\int_{\mathbb{R}}f_h(\xi)\,d\xi\le(\beta_a+h^{1/2}M_3(a)\kappa)m_h+\mathcal{O}(h^{3/4}M_h)+\mathcal{O}(h^{3/2-\theta}).\quad(5\text{-}20)$$

Fix a positive constant $\varepsilon<1$. Then by Proposition 3.2,

$$f_h(\xi)\ge\int_{-h^{-\delta}}^{h^{-\delta}}\big(\beta_a+\hat{c}_0(a)\min((h^{3/8}\xi-\zeta_a)^2,\varepsilon)+h^{1/2}M_3(a)\kappa-C_\varepsilon h\big)|\hat{w}_h|^2(1-h^{1/2}\kappa\tau)\,d\tau.\quad(5\text{-}21)$$

Inserting this into (5-20) we get

$$\int_{\mathbb{R}}\int_{-h^{-\delta}}^{h^{-\delta}}\hat{c}_0(a)\min((h^{3/8}\xi-\zeta_a)^2,\varepsilon)|\hat{w}_h|^2(1-h^{1/2}\kappa\tau)\,d\xi\,d\tau=\mathcal{O}(h^{3/4}M_h)+\mathcal{O}(h^{3/2-\theta}),$$

from which we infer the two estimates

$$\int_{|h^{3/8}\xi-\zeta_a|^2<\varepsilon}\int_{-h^{-\delta}}^{h^{-\delta}}|h^{3/8}\xi-\zeta_a|^2|\hat{w}_h|^2(1-h^{1/2}\kappa\tau)\,d\xi\,d\tau=\mathcal{O}(h^{3/4}M_h)+\mathcal{O}(h^{3/2-\theta}),\quad(5\text{-}22)$$

$$\int_{|h^{3/8}\xi-\zeta_a|^2\ge\varepsilon}\int_{-h^{-\delta}}^{h^{-\delta}}|\hat{w}_h|^2(1-h^{1/2}\kappa\tau)\,d\xi\,d\tau=\mathcal{O}(h^{3/4}M_h)+\mathcal{O}(h^{3/2-\theta}).\quad(5\text{-}23)$$

[3] Replacing the cut-off functions in (5-9) by 1 in the integrals produces $\mathcal{O}(h^\infty)$ errors by (5-7) and (5-8).

Step 3: Noticing the simple decomposition

$$\int_{\mathbb{R}}\int_{-h^{-\delta}}^{h^{-\delta}}|\hat{w}_h|^2(1-h^{1/2}\kappa\tau)\,d\xi\,d\tau$$
$$=\int_{|h^{3/8}\xi-\zeta_a|^2<\varepsilon}\int_{-h^{-\delta}}^{h^{-\delta}}|\hat{w}_h|^2(1-h^{1/2}\kappa\tau)\,d\xi\,d\tau+\int_{|h^{3/8}\xi-\zeta_a|^2\geq\varepsilon}\int_{-h^{-\delta}}^{h^{-\delta}}|\hat{w}_h|^2(1-h^{1/2}\kappa\tau)\,d\xi\,d\tau, \quad (5\text{-}24)$$

we get from (5-23) and (5-18)

$$\int_{|h^{3/8}\xi-\zeta_a|^2<\varepsilon}\int_{-h^{-\delta}}^{h^{-\delta}}|\hat{w}_h|^2(1-h^{1/2}\kappa\tau)\,d\xi\,d\tau=m_h+\mathcal{O}(h^{3/4}M_h)+\mathcal{O}(h^{3/2-\theta}). \quad (5\text{-}25)$$

Similarly, we decompose the integral in (5-20) as

$$\int_{\mathbb{R}}f_h(\xi)\,d\xi=\int_{|h^{3/8}\xi-\zeta_a|^2<\varepsilon}f_h(\xi)\,d\xi+\int_{|h^{3/8}\xi-\zeta_a|^2\geq\varepsilon}f_h(\xi)\,d\xi. \quad (5\text{-}26)$$

We write a lower bound of the integral on $\{|h^{3/8}\xi-\zeta_a|^2\geq\varepsilon\}$ by using (5-21). Noting that $\hat{c}_0(a)>0$, we get, by (5-25),

$$\int_{|h^{3/8}\xi-\zeta_a|^2<\varepsilon}f_h(\xi)\,d\xi\geq\big(\beta_a+h^{1/2}M_3(a)\kappa+\mathcal{O}(h)\big)m_h+\mathcal{O}(h^{3/4}M_h)+\mathcal{O}(h^{3/2-\theta}).$$

Inserting this into (5-26) and using (5-20), we get

$$\int_{|h^{3/8}\xi-\zeta_a|^2\geq\varepsilon}f_h(\xi)\,d\xi=\mathcal{O}(h^{3/4}M_h)+\mathcal{O}(h^{3/2-\theta}). \quad (5\text{-}27)$$

Step 4: We write a lower bound for $f_h(\xi)$ by gathering (5-19) and (3-8), thereby obtaining

$$\int_{|h^{3/8}\xi-\zeta_a|^2\geq\varepsilon}f_h(\xi)\,d\xi\geq(1-Ch^{1/2-2\delta})\int_{|h^{3/8}\xi-\zeta_a|^2\geq\varepsilon}\int_{\mathbb{R}}\big(|\partial_\tau\hat{w}_h|^2+|(b_a(\tau)\tau+h^{3/8}\xi)\hat{w}_h|^2\big)\,d\xi\,d\tau.$$

Using (5-27) and the inequality (note that $|b_a|\leq 1$ since $|a|<1$)

$$(b_a(\tau)\tau+h^{3/8}\xi)^2\geq\tfrac{1}{2}(h^{3/8}\xi)^2-2\tau^2,$$

we get

$$\frac{1}{2}\int_{|h^{3/8}\xi-\zeta_a|^2\geq\varepsilon}\int_{\mathbb{R}}|h^{3/8}\xi\,\hat{w}_h|^2\,d\xi\,d\tau\leq 2\int_{|h^{3/8}\xi-\zeta_a|^2\geq\varepsilon}\int_{\mathbb{R}}\tau^2|\hat{w}_h|^2\,d\xi\,d\tau+\mathcal{O}(h^{3/4}M_h)+\mathcal{O}(h^{3/2-\theta}). \quad (5\text{-}28)$$

Let $p=1/\theta$ and $q=1/(1-\theta)$. By the Hölder inequality, (5-5) and (5-23), we write

$$\begin{aligned}
\int_{|h^{3/8}\xi-\zeta_a|^2\geq\varepsilon}\int_{\mathbb{R}}\underbrace{\tau^2|\hat{w}_h|^2}_{=\tau^2|\hat{w}_h|^{2\theta}|\hat{w}_h|^{2-2\theta}}\,d\xi\,d\tau
&\leq\bigg(\int_{|\xi_h-\zeta_a|^2\geq\varepsilon}\int_{\mathbb{R}}\tau^{2p}|\hat{w}_h|^{2p\theta}\,d\xi\,d\tau\bigg)^{1/p}\bigg(\int_{|\xi_h-\zeta_a|^2\geq\varepsilon}\int_{\mathbb{R}}|\hat{w}_h|^{q(2-2\theta)}\,d\xi\,d\tau\bigg)^{1/q}\\
&\leq\bigg(\int_{\mathbb{R}^2}\tau^{2p}|w_h|^2\,d\tau\,ds\bigg)^{1/p}\bigg(\int_{|\xi_h-\zeta_a|^2\geq\varepsilon}\int_{\mathbb{R}}|\hat{w}_h|^2\,d\xi\,d\tau\bigg)^{1/q}\\
&=\mathcal{O}(h^{3/4(1-\theta)}M_h)+\mathcal{O}(M_h^\theta\,h^{(1-\theta)(3/2-\theta)})\\
&=\mathcal{O}(h^{3/4(1-\theta)}M_h)+\mathcal{O}(h^{3/2-5\theta/2}),
\end{aligned}$$

where, in the last step, we used Young's inequality,

$$\begin{aligned} M_h^\theta h^{(1-\theta)(3/2-\theta)} &= M_h\theta h^{\theta(3/4-\theta)} h^{(1-\theta)(3/2-\theta)-\theta(3/4-\theta)} \\ &\le \theta M_h h^{3/4-\theta} + (1-\theta)h^{3/2-\theta} h^{-(3/4-\theta)\theta/(1-\theta)} \\ &\le \theta M_h h^{3/4-\theta} + (1-\theta)h^{3/2-5\theta/2} \quad \text{for } 0<\theta<\tfrac{3}{8}. \end{aligned}$$

Inserting this estimate into (5-28), we get

$$\int_{|h^{3/8}\xi-\zeta_a|^2\ge\varepsilon}\int_{\mathbb{R}} |h^{3/8}\xi\hat{w}_h|^2\,d\xi\,d\tau = \mathcal{O}(h^{3/4-\theta}M_h)+\mathcal{O}(h^{3/2-5\theta/2}).$$

Collecting the foregoing estimate and those in (5-22) and (5-23), we deduce that

$$\int_{\mathbb{R}^2} |(h^{3/8}\partial_\sigma - i\zeta_a)w_h|^2\,d\sigma\,d\tau = \int_{\mathbb{R}}\int_{-h^{-\delta}}^{h^{-\delta}} |h^{3/8}\xi-\zeta_a|^2|\hat{w}_h|^2\,d\xi\,d\tau = \mathcal{O}(h^{3/4-\theta}M_h)+\mathcal{O}(h^{3/2-5\theta/2}).$$

With (5-15) and (5-16) in mind, this implies (5-11) as stated in the proposition. □

6. Localization of bound states

In this section, we fix a labeling $n\ge 1$ and denote by $\psi_{h,n}$ a normalized eigenfunction of the operator $\mathcal{P}_h$ with eigenvalue $\lambda_n(h)$. By Theorem 4.1, it holds

$$\mathcal{Q}_h(\psi_{h,n}) \le (h\beta_a + h^{3/2}M_3(a)k_{\max} + C_1h^{7/4})\|\psi_{h,n}\|^2_{L^2(\Omega)}, \tag{6-1}$$

where $\mathcal{Q}_h$ is the quadratic form introduced in (1-3).

The decay estimates in Sections 6A and 6B follow by standard semiclassical Agmon estimates. We refer to [Helffer and Morame 2001; Fournais and Helffer 2006] for details in the case of the Laplacian with a smooth magnetic field, and to [Assaad and Kachmar 2022] for adaptations in the piecewise constant field discussed here.

Using the aforementioned decay estimates, the bound state $\psi_{h,n}$ satisfies the hypotheses in Section 5. Namely the estimates in $(5\text{-}1)_\theta$, (5-3) and (5-4) hold with $g_h=\psi_{h,n}$, $r_h=1$ and for any $\theta\in\left(0,\tfrac{3}{8}\right)$. Consequently, we will be able to estimate its tangential derivative (see Proposition 6.2). Estimating the second-order tangential derivative of $\psi_{h,n}$ (as in Proposition 6.3) requires the analysis of the decay of its first-order tangential derivative in order to verify the hypotheses of Section 5.

6A. ***Decay away from the edge.*** The derivation of an Agmon decay estimate relies on the following useful lower bound of the quadratic form [Assaad and Kachmar 2022, Section 4.3]. For every $R_0>1$, there exists a positive constant C_0 and $h_0>0$ such that, for $h\in(0,h_0]$,

$$\mathcal{Q}_h(u) \ge \int_\Omega (U_{h,a}(x) - C_0R_0^{-2}h)|u(x)|^2\,dx \quad (u\in H^1_0(\Omega)), \tag{6-2}$$

where $\mathcal{Q}_h$ is introduced in (1-3) and

$$U_{h,a}(x) = \begin{cases} |a|h & \text{if } \operatorname{dist}(x,\Gamma) > R_0h^{1/2}, \\ \beta_a h & \text{if } \operatorname{dist}(x,\Gamma) < R_0h^{1/2}. \end{cases}$$

Note that the decay property is a consequence of $\beta_a < |a|$. Following [Fournais and Helffer 2010, Theorem 8.2.4], it results from the foregoing lower bound that the eigenfunction $\psi_{h,n}$ decays roughly like $\exp(-\alpha_0 h^{-1/2} d(x,\Gamma))$ for some constant $\alpha_0 > 0$. More precisely, the following holds:

$$\int_\Omega \big(|\psi_{h,n}|^2 + h^{-1}|(h\nabla - i\boldsymbol{F})\psi_{h,n}|^2\big)\exp(2\alpha_0 h^{-1/2}\, d(x,\Gamma))\,dx \le C. \tag{6-3}$$

6B. *Decay along the edge.* Here we discuss tangential estimates along the edge Γ. Recall that $s = 0$ corresponds to the (unique) point of maximal curvature.

The starting point is the following refined lower bound of the quadratic form [Assaad and Kachmar 2022, Section 4.3]:

$$\mathcal{Q}_h(u) \ge \int_\Omega (U^\Gamma_{h,a}(x) - C_0 h^{7/4})|u|^2\,dx \quad (u \in H^1_0(\Omega)), \tag{6-4}$$

where, with $x = \Phi(s,t)$, $\kappa(s) = k_{\max} - \varepsilon_0 s^2$ and ε_0 a positive constant,

$$U^\Gamma_{h,a}(x) = \begin{cases} |a|h & \text{if } \operatorname{dist}(x,\Gamma) \ge 2h^{1/6}, \\ \beta_a h + M_3(a)\kappa(s)h^{3/2} & \text{if } \operatorname{dist}(x,\Gamma) < 2h^{1/6}. \end{cases}$$

Here we recall that $M_3(a)$ is negative so the potential in the second zone is minimal at the point of maximal curvature. The lower bound (6-4) can be derived along the same arguments in [Fournais and Helffer 2010, Proposition 8.3.3, Remark 8.3.6] and by using Proposition 3.2.

The eigenfunction $\psi_{n,h}$ decays exponentially roughly like $\exp(-\alpha_1 h^{-1/8} s(x))$ for some constant $\alpha_1 > 0$. More precisely, picking t_0 sufficiently small so that the Frenet coordinates recalled in Appendix A are valid in $\{d(x,\Gamma) < t_0\}$, we have

$$\int_{d(x,\Gamma)\le t_0} \big(|\psi_{h,n}(x)|^2 + h^{-1}|(h\nabla - i\boldsymbol{F})\psi_{h,n}|^2\big)\exp(2\alpha_1 h^{-1/8}|s(x)|)\,dx \le C. \tag{6-5}$$

Remark 6.1. We observe, by collecting (6-1), (6-3) and (6-5), that the eigenfunction $g_h = \psi_{h,n}$ satisfies the hypotheses of Proposition 5.1, namely

- (5-1) holds for any $\theta \in \big(0, \frac{3}{8}\big)$,
- (5-3) and (5-4) hold with $0 < \alpha \le \min(2\alpha_1, 2\alpha_2)$ and $r_h = 1$.

6C. *Estimating tangential frequency.* The localization of the eigenfunction $\psi_{h,n}$ is to be measured by two parameters $\rho \in \big(0, \frac{1}{2}\big)$ and $\eta \in \big(0, \frac{1}{8}\big)$. We will choose $\rho = \frac{1}{2} - \delta$ with $\delta \in \big(0, \frac{1}{12}\big)$; i.e., we are assuming

$$\tfrac{5}{12} < \rho < \tfrac{1}{2}.$$

We introduce the function

$$u_{h,n}(\sigma,\tau) = h^{5/16}\chi(h^\eta\sigma)\chi(h^\delta\tau)\tilde{\psi}_{h,n}(h^{1/8}\sigma, h^{1/2}\tau), \tag{6-6}$$

where $\tilde{\psi}_{h,n}$ is the function assigned to $\psi_{h,n}$ by the Frenet coordinates as in (A-3), $\chi \in C^\infty_c(\mathbb{R})$, $\operatorname{supp}\chi \subset [-1,1]$, $0 \le \chi \le 1$ and $\chi = 1$ on $\big[-\frac{1}{2}, \frac{1}{2}\big]$. Note that $u_{h,n}$ can be seen as a function on $\mathbb{R}^2$, and by (5-10) (applied with $g_h = \psi_{h,n}$), its L^2-norm satisfies

$$\|u_{h,n}\|^2_{L^2(\mathbb{R}^2)} = \|\psi_{h,n}\|^2_{L^2(\Omega)}(1+\mathcal{O}(h^{1/2})) = 1+\mathcal{O}(h^{1/2}), \tag{6-7}$$

since $\psi_{h,n}$ is normalized in $L^2(\Omega)$.

Using Proposition 5.1, we can estimate the tangential derivative of $u_{h,n}$. More precisely, we apply this proposition with $g_h = \psi_{h,n}$, $r_h = 1$ and *any* $0 < \theta < \frac{3}{8}$ (see Remark 6.1). In this case, the function introduced in (5-9) is given by $w_h = u_{h,n}$.

Proposition 6.2. *For all* $\theta \in \big(0, \frac{3}{8}\big)$, *there exist constants* $C_\theta, h_\theta > 0$ *such that*, *for all* $h \in (0, h_\theta]$,

$$\|(h^{3/8}\partial_\sigma - i\zeta_a)u_{h,n}\|_{L^2(\mathbb{R}^2)} \le C_\theta\, h^{3/8-\theta}.$$

We can estimate higher-order tangential derivatives of $u_{h,n}$.

Proposition 6.3. *For all* $\theta \in \big(0, \frac{3}{4}\big)$, *there exist constants* $C_\theta, h_\theta > 0$ *such that*, *for all* $h \in (0, h_\theta]$,

$$\|(h^{3/8}\partial_\sigma - i\zeta_a)^2 u_{h,n}\|_{L^2(\mathbb{R}^2)} \le C_\theta\, h^{3/4-\theta}, \tag{6-8}$$

where $u_{h,n}$ *is introduced in* (6-6).

Before proceeding with the proof of Proposition 6.3, we introduce the notation, $r_h = \widetilde{\mathcal{O}}(h^\gamma)$ for a positive number γ, to mean

$$\text{for all } \theta \in (0,\gamma), \text{ there exists } C_\theta, h_\theta > 0 \text{ such that, for all } h \in (0,h_\theta),\quad |r_h| \le C_\theta h^{\gamma-\theta}. \tag{6-9}$$

Proof of Proposition 6.3. We will apply Proposition 5.1 with an adequate choice of the function g_h defining the function w_h in (5-9).

We introduce the function φ_h on Ω as

$$\varphi_h(x) = f(x)\psi_{h,n}(x), \tag{6-10}$$

where $f(x) = (1-\chi(\operatorname{dist}(x,\partial\Omega)/t_1))\,\chi(\operatorname{dist}(x,\Gamma)/t_0)$, t_1 and t_0 are constants so that the set $\{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > t_1\}$ contains the point of maximum curvature and the transformation in (A-1) is a diffeomorphism, $\chi \in C_c^\infty(\mathbb{R})$, $\operatorname{supp}\chi \subset [-1,1]$, $0 \le \chi \le 1$ and $\chi = 1$ on $\big[-\frac{1}{2}, \frac{1}{2}\big]$. Then we define

$$\tilde{g}_h(s,t) = (h^{1/2}\partial_s - i\zeta_a)\tilde{\varphi}_h(s,t), \tag{6-11}$$

where $\tilde{\varphi}_h$ is the function assigned to φ_h by (A-3). Notice that, using the notation in (6-9), the conclusion of Proposition 6.2 can be written as

$$\|g_h\|_{L^2(\Omega)} = \widetilde{\mathcal{O}}(h^{3/8}). \tag{6-12}$$

We will show that g_h satisfies $(5\text{-}1)_\theta$ for *any* $\theta \in \big(0, \frac{3}{8}\big)$, and that (5-3) and (5-4) hold with

$$r_h = \|g_h\|^2_{L^2(\Omega)} + h^{3/4}. \tag{6-13}$$

This will be done in several steps outlined below.

- In Step 1, we establish rough decay estimates for g_h in the normal and tangential directions (see (6-20)). These estimates are nevertheless weaker than the estimates in (5-3) and (5-4) that we wish to prove.
- In Step 2, we show that g_h is in the domain of the operator $\mathcal{P}_h$ introduced in (1-4).
- In Step 3, using the rough estimates obtained in Steps 1 and 2, we can verify that (5-1) holds for any $\theta \in \left(0, \frac{3}{8}\right)$.
- In Step 4, using the estimates obtained in Steps 1 and 3, and the Agmon method, we derive the decay estimates for g_h as in (5-3) and (5-4) with r_h given in (6-13).
- In Step 5, we can apply the conclusion of Proposition 5.1 and conclude the proof of Proposition 6.3.

Step 1: We show that the function g_h decays exponentially in the normal and tangential directions. We select the constant t_0 so that the two functions

$$x \mapsto \operatorname{dist}(x, \Gamma) \quad \text{and} \quad x \mapsto s(x)$$

are smooth in the neighborhood, Γ_{2t_0}, of the edge Γ. Consequently, the transformation in (A-1) is valid in Γ_{2t_0}. Since we encounter integrals of the function g_h, which is supported in $\Gamma_{t_0} \cap \Omega$, we select the gauge given in Lemma A.1. In particular, by (A-4), we have

$$|\boldsymbol{F}(x)| = \mathcal{O}(\operatorname{dist}(x, \Gamma)) \quad \text{on } \Omega \cap \Gamma_{t_0}. \tag{6-14}$$

Let $\alpha_2 \in \left(0, \frac{1}{2}\min(\alpha_0, \alpha_1)\right)$, where α_0, α_1 are the positive constants in (6-3) and (6-5). We introduce on Ω the weight functions

$$\Phi_{\mathrm{norm}}(x) = \exp\left(\frac{\alpha_2 \operatorname{dist}(x, \Gamma)}{h^{1/2}}\right) \quad \text{and} \quad \Phi_{\tan}(x) = \exp\left(\frac{\alpha_2\, s(x)}{h^{1/8}}\right). \tag{6-15}$$

By Remark 6.1, we can use (5-5) for $\psi_{h,n}$. It results from (6-5), (6-14), the Hölder inequality, and our choice of α_2, that, for $j \in \{1, 2\}$,

$$\begin{aligned}\int_\Omega |\boldsymbol{F}|^{2j} |\psi_{h,n}|^2 \Phi_{\tan}^2\, dx &= \int_{\Omega\cap\Gamma_{t_0}} |\boldsymbol{F}|^{2j} |\psi_{h,n}|^2 \Phi_{\tan}^2\, dx + \mathcal{O}(h^\infty) \\ &\le A_{4j}(\psi_{h,n})^{1/2} \|\Phi_{\tan}^2 \psi_{h,n}\|_{L^2(\Omega)} + \mathcal{O}(h^\infty) = \mathcal{O}(h^j),\end{aligned} \tag{6-16}$$

where $A_{4j}(\,\cdot\,)$ is defined in (5-5) and

$$\begin{aligned}\int_\Omega |\boldsymbol{F}\cdot(h\nabla - i\boldsymbol{F})\psi_{h,n}|^2 \Phi_{\tan}^2\, dx &= \int_{\Omega\cap\Gamma_{t_0}} |\boldsymbol{F}\cdot(h\nabla - i\boldsymbol{F})\psi_{h,n}|^2 \Phi_{\tan}^2\, dx + \mathcal{O}(h^\infty) \\ &\le A_4(\psi_{h,n})^{1/2} \|\Phi_{\tan}^2 (h\nabla - i\boldsymbol{F})\psi_{h,n}\|_{L^2(\Omega)} + \mathcal{O}(h^\infty) = \mathcal{O}(h^2).\end{aligned}$$

Similarly, we estimate the $L^2(\Omega)$-norms of $\boldsymbol{F}\psi_{h,n}\Phi_{\mathrm{norm}}$, $(\boldsymbol{F}\cdot\boldsymbol{F})\psi_{h,n}\Phi_{\mathrm{norm}}$ and $\Phi_{\mathrm{norm}}\boldsymbol{F}\cdot(h\nabla - i\boldsymbol{F})\psi_{h,n}$ using (6-3). Eventually, we get the estimates

$$\begin{aligned}&\|\boldsymbol{F}\psi_{h,n}\Phi_{\mathrm{norm}}\|_{L^2(\Omega\cap\Gamma_{2t_0};\mathbb{R}^2)} + \|\boldsymbol{F}\psi_{h,n}\Phi_{\tan}\|_{L^2(\Omega\cap\Gamma_{2t_0};\mathbb{R}^2)} \\ &\qquad \le Ch^{1/2}\|\boldsymbol{F}\cdot\nabla(\psi_{h,n}\Phi_{\mathrm{norm}})\|_{L^2(\Omega\cap\Gamma_{2t_0};\mathbb{R}^2)} + \|\boldsymbol{F}\cdot\nabla(\psi_{h,n}\Phi_{\tan})\|_{L^2(\Omega\cap\Gamma_{2t_0})} \le C.\end{aligned} \tag{6-17}$$

Furthermore, the following two estimates hold:

$$
\begin{aligned}
\|\psi_{h,n}\Phi_{\mathrm{norm}}\|_{L^2(\Omega\cap\Gamma_{2t_0})}+\|\psi_{h,n}\Phi_{\tan}\|_{L^2(\Omega\cap\Gamma_{2t_0})}&\le C,\\
\|\psi_{h,n}\Phi_{\mathrm{norm}}\|_{H^1(\Omega\cap\Gamma_{2t_0})}+\|\psi_{h,n}\Phi_{\tan}\|_{H^1(\Omega\cap\Gamma_{2t_0})}&\le Ch^{-1/2}.
\end{aligned}
\tag{6-18}
$$

Notice that for $w_\# := \psi_{h,n}\Phi_\#$, ($\#\in\{\mathrm{norm},\tan\}$), we have, with $\mathcal{P}_h$ the operator introduced in (1-4),

$$
\mathcal{P}_h w_\# = \lambda_n(h)w_\# - 2h\nabla\Phi_\#\cdot(h\nabla - i\boldsymbol{F})\psi_{h,n} - h^2\Delta\Phi_\#\,\psi_{h,n}.
$$

Hence, noting that $\mathcal{P}_h = -h^2\Delta + 2ih\boldsymbol{F}\cdot\nabla + ih\,\mathrm{div}\boldsymbol{F} + |\boldsymbol{F}|^2$, we find by (4-1), (6-16) and (6-17),

$$
\begin{aligned}
h^2\|\Delta w_\#\|_{L^2(\Omega\cap\Gamma_{2t_0})} \le \big(\|\mathcal{P}_h w_\#\|_{L^2(\Omega)} + \|(h\nabla - i\boldsymbol{F})w_\#\|_{L^2(\Omega\cap\Gamma_{2t_0})} + h\|\mathrm{div}\,\boldsymbol{F}\,w_\#\|_{L^2(\Omega\cap\Gamma_{2t_0})}&\\
+2h\|\boldsymbol{F}\cdot\nabla w_\#\|_{L^2(\Omega\cap\Gamma_{t_0})} + \||\boldsymbol{F}|^2 w_\#\|_{L^2(\Omega\cap\Gamma_{2t_0})}\big) &= \mathcal{O}(h).
\end{aligned}
$$

By the L^2-elliptic estimates for the Dirichlet problem in $\Gamma_{2t_0}\cap\Omega$, and noting that $w_\#$ satisfies the Dirichlet condition,

$$
\|w_\#\|_{H^2(\Omega\cap\Gamma_{t_0})} \le C(t_0,\Omega)(\|\Delta w_\#\|_{L^2(\Omega\cap\Gamma_{2t_0})} + \|w_\#\|_{L^2(\Omega\cap\Gamma_{2t_0})}).
$$

Consequently, we get the estimate

$$
\|\psi_{h,n}\Phi_{\mathrm{norm}}\|_{H^2(\Omega\cap\Gamma_{t_0})}+\|\psi_{h,n}\Phi_{\tan}\|_{H^2(\Omega\cap\Gamma_{t_0})}\le Ch^{-1}. \tag{6-19}
$$

Now we can derive decay estimates of the function g_h introduced in (6-11). Controlling the decay of the magnetic gradient of g_h requires a decay estimate of $\psi_{h,n}$ in the H^2 norm. Actually, collecting (6-18) and (6-19), we observe that

$$
\begin{aligned}
\|g_h\Phi_{\mathrm{norm}}\|_{L^2(\Gamma_{t_0})} + h^{-1/2}\|((h\nabla - i\boldsymbol{F})g_h)\Phi_{\mathrm{norm}}\|_{L^2(\Gamma_{t_0};\mathbb{R}^2)} &\le C,\\
\|g_h\Phi_{\tan}\|_{L^2(\Gamma_{t_0})} + h^{-1/2}\|((h\nabla - i\boldsymbol{F})g_h)\Phi_{\tan}\|_{L^2(\Gamma_{t_0};\mathbb{R}^2)} &\le C.
\end{aligned}
\tag{6-20}
$$

<u>Step 2</u>: By the definition of g_h in (6-11), this function is compactly supported in $\Omega\cap\Gamma_{t_0}$. Hence, there exists a regular open set ω such that, for $h\in(0,h_0]$, $\mathrm{supp}\,g_h\subset\omega\subset\bar{\omega}\subset\Omega\cap\Gamma_{2t_0}$. Consequently g_h satisfies the Dirichlet boundary condition on $\partial\omega$. To prove that g_h is in the domain of the operator $\mathcal{P}_h$, it suffices to establish that

$$
\partial_s\tilde{\psi}_{h,n}\in H^2(\Phi^{-1}(\omega)). \tag{6-21}
$$

To that end, we consider the spectral equation satisfied by the eigenfunction $\psi_{h,n}$

$$
-(h\nabla - i\boldsymbol{F})^2\psi_{h,n} = \lambda_n(h)\psi_{h,n}. \tag{6-22}
$$

Using (A-5) with the potential $\widetilde{\boldsymbol{F}}$ in (4-3), (6-22) reads in the (s,t)-coordinates as

$$
-\big(\mathfrak{a}^{-1}(h\partial_s - i\widetilde{F}_1)\mathfrak{a}^{-1}(h\partial_s - i\widetilde{F}_1) + h^2\mathfrak{a}^{-1}\partial_t\mathfrak{a}\partial_t\big)\tilde{\psi}_{h,n} = \lambda_n(h)\tilde{\psi}_{h,n}, \tag{6-23}
$$

that is,

$$
h^2(\mathfrak{a}^{-2}\partial_s^2\tilde{\psi}_{h,n} + \partial_t^2\tilde{\psi}_{h,n}) = f_1(s,t)\partial_s\tilde{\psi}_{h,n} + f_2(s,t)\partial_t\tilde{\psi}_{h,n} + f_3(s,t)\tilde{\psi}_{h,n}, \tag{6-24}
$$

where

$$f_1(s,t) = -h^2\mathfrak{a}^{-3}tk'(s) - 2i\mathfrak{a}^{-2}b_a(t)\Big(t - \frac{t^2}{2}k(s)\Big),$$
$$f_2(s,t) = h^2\mathfrak{a}^{-1}k(s),$$
$$f_3(s,t) = -ih\mathfrak{a}^{-3}tk'(s)b_a(t)\Big(t - \frac{t^2}{2}k(s)\Big) + h\mathfrak{a}^{-2}\frac{t^2}{2}k'(s) + \mathfrak{a}^{-2}b_a^2(t)\Big(t - \frac{t^2}{2}k(s)\Big)^2 - \lambda_n(h).$$

We differentiate with respect to s in (6-24), and get

$$h^2(\mathfrak{a}^{-2}\partial_s^2 + \partial_t^2)(\partial_s\tilde{\psi}_{h,n}) \\ = (f_1 - h^2\partial_s\mathfrak{a}^{-2})\partial_s^2\tilde{\psi}_{h,n} + f_2\,\partial_s\partial_t\tilde{\psi}_{h,n} + (\partial_s f_1 + f_3)\,\partial_s\tilde{\psi}_{h,n} + \partial_s f_2\,\partial_t\tilde{\psi}_{h,n} + \partial_s f_3\,\tilde{\psi}_{h,n}. \tag{6-25}$$

Having $s \mapsto k(s)$ smooth, $\mathfrak{a} = 1 - tk(s)$ for $t \in (-2t_0, 2t_0)$, and $\psi_{n,h} \in \operatorname{Dom}\mathcal{P}_h$ ensures that the function in the right-hand side of (6-25) is in $L^2(\Phi^{-1}(\Omega \cap \Gamma_{2t_0}))$. Hence $\partial_s\tilde{\psi}_{h,n} \in H^1(\Omega \cap \Gamma_{2t_0})$ and satisfies

$$(\mathfrak{a}^{-2}\partial_s^2 + \partial_t^2)\partial_s\tilde{\psi}_{h,n} \in L^2(\Phi^{-1}(\Omega \cap \Gamma_{2t_0})). \tag{6-26}$$

Hence (6-21) follows from (6-26) using the interior elliptic estimates associated with the differential operator $L := (\mathfrak{a}^{-2}\partial_s^2 + \partial_t^2)$.

Step 3: We prove that

$$\mathcal{Q}_h(g_h) = \lambda_n(h)\|g_h\|^2_{L^2(\Omega)} + \widetilde{\mathcal{O}}(h^{5/2}), \tag{6-27}$$

where $\mathcal{Q}_h$ is the quadratic form introduced in (1-3).

With the notation introduced in (6-9), the estimates in (4-1) and (6-27) yield (5-1) for *any* $\theta \in \big(0, \frac{3}{8}\big)$.

We start by noticing that

$$\langle \mathcal{P}_h\varphi_h, G_h\rangle_{L^2(\Omega)} = \lambda_n(h)\langle\varphi_h, G_h\rangle_{L^2(\Omega)} + \langle(\mathcal{P}_h - \lambda_n(h))\varphi_h, G_h\rangle_{L^2(\Omega)}, \tag{6-28}$$

where φ_h is defined in (6-10) and

$$\widetilde{G}_h(s,t) = -(h^{1/2}\partial_s - i\zeta_a)g_h.$$

Recall that φ_h and G_h are compactly supported in $\Omega \cap \Gamma_{t_0}$ so that we can use the Frenet coordinates valid near the edge Γ. By (6-19) we have

$$\|(\mathcal{P}_h - \lambda_n(h))\varphi_h\|_{L^2(\Omega)} = \mathcal{O}(h^\infty) \tag{6-29}$$

and by (6-20)

$$\|G_h\|_{L^2(\Omega)} = \mathcal{O}(1). \tag{6-30}$$

By Hölder's inequality, we infer from (6-29) and (6-30)

$$\langle(\mathcal{P}_h - \lambda_n(h))\varphi_h, G_h\rangle_{L^2(\Omega)} = \mathcal{O}(h^\infty). \tag{6-31}$$

Furthermore, computing the integrals in the Frenet coordinates and integrating by parts, we find

$$\langle\varphi_h, G_h\rangle_{L^2(\Omega)} = \langle\mathfrak{a}(h^{1/2}\partial_s - i\zeta_a)\tilde{\varphi}_h + h^{1/2}(\partial_s\mathfrak{a})\tilde{\varphi}_h, \tilde{g}_h\rangle_{L^2(\mathbb{R}^2)} = \|g_h\|^2_{L^2(\Omega)} + \mathcal{O}(h^{9/8})\|g_h\|_{L^2(\Omega)}. \tag{6-32}$$

Here we get the $\mathcal{O}(h^{9/8})$ remainder by using that $\partial_s \mathfrak{a} = \mathcal{O}(ts)$, the Hölder inequality and Remark 6.1 on the decay estimates in (5-5) and (5-6) for $\psi_{h,n}$ as follows:

$$|\langle \mathfrak{a}(\partial_s \mathfrak{a})\tilde{\varphi}_h, \tilde{g}_h\rangle_{L^2(\mathbb{R}^2)}| \le C(A_4(\psi_{h,n})B_4(\psi_{h,n}))^{1/4}\|g_h\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(h^{5/8})\|g_h\|_{L^2(\mathbb{R}^2)}.$$

By (4-1) and (6-12), we infer from (6-32)

$$\lambda_n(h)\langle \varphi_h, G_h\rangle_{L^2(\Omega)} = \lambda_n(h)\|g_h\|^2_{L^2(\Omega)} + \widetilde{\mathcal{O}}(h^{5/2}). \tag{6-33}$$

Therefore, inserting the estimates in (6-33) and (6-31) into (6-28), we find

$$\langle \mathcal{P}_h\varphi_h, G_h\rangle_{L^2(\Omega)} = \lambda_n(h)\|g_h\|^2_{L^2(\Omega)} + \widetilde{\mathcal{O}}(h^{5/2}). \tag{6-34}$$

Now, by Lemma A.2 (used with $\phi = 0$), we get

$$\operatorname{Re}\langle \mathcal{P}_h\varphi_h, G_h\rangle = \mathcal{Q}_h(g_h) - h^{1/2}\operatorname{Re}\langle R_h, g_h\rangle_{L^2(\Omega)}, \tag{6-35}$$

where the function R_h is defined via (A-3) as

$$\widetilde{R}_h(s,t) = (h\partial_s - i\widetilde{F}_1)\big((\partial_s \mathfrak{a}^{-1} - i\mathfrak{a}^{-1}\partial_s\widetilde{F}_1)(h\partial_s - i\widetilde{F}_1)\tilde{\varphi}_h - i\mathfrak{a}^{-1}(\partial_s\widetilde{F}_1)\tilde{\varphi}_h\big) + h^2\partial_t(\partial_s\mathfrak{a})\partial_t\tilde{\varphi}_h. \tag{6-36}$$

Our choice of gauge in Lemma A.1 ensures that $\widetilde{F}_2 = 0$ and $\widetilde{F}_1 = \mathcal{O}(t)$. By Remark 6.1 and (A-7), we have

$$\int_{\mathbb{R}}\int_{-t_0}^{t_0} |t|^N\big(|\tilde{\varphi}_h|^2 + \mathfrak{a}^{-1}h^{-1}|(h\partial_s - i\widetilde{F}_1)\tilde{\varphi}_h|^2 + h|\partial_t\tilde{\varphi}_h|^2\big)\mathfrak{a}\,ds\,dt = \mathcal{O}(h^{N/2}),$$
$$\int_{\mathbb{R}}\int_{-t_0}^{t_0} |s|^N\big(|\tilde{\varphi}_h|^2 + \mathfrak{a}^{-1}h^{-1}|(h\partial_s - i\widetilde{F}_1)\tilde{\varphi}_h|^2 + h|\partial_t\tilde{\varphi}_h|^2\big)\mathfrak{a}\,ds\,dt = \mathcal{O}(h^{N/8}).$$

Furthermore, by (6-19),

$$\int_{\mathbb{R}}\int_{-t_0}^{t_0} |t|^N(|\partial_s^2\tilde{\varphi}_h|^2 + |\partial_t^2\tilde{\varphi}_h|^2)\,ds\,dt = \mathcal{O}(h^{N/2-2}),$$
$$\int_{\mathbb{R}}\int_{-t_0}^{t_0} |s|^N(|\partial_s^2\tilde{\varphi}_h|^2 + |\partial_t^2\tilde{\varphi}_h|^2)\,ds\,dt = \mathcal{O}(h^{N/8-2}).$$

Now we can estimate $\widetilde{R}_h$ in (6-36), by expressing it as

$$\widetilde{R}_h = m_1(h\partial_s - i\widetilde{F}_1)^2\tilde{\varphi}_h + (m_2 + h\partial_s m_1)(h\partial_s - i\widetilde{F}_1)\tilde{\varphi}_h + h(\partial_s m_2)\tilde{\varphi}_h + h^2 m_3\partial_t^2\tilde{\varphi}_h + h^2(\partial_t m_3)\partial_t\tilde{\varphi}_h,$$

where

$$\begin{aligned} m_1 &= \partial_s\mathfrak{a}^{-1} - i\mathfrak{a}^{-1}\partial_s\widetilde{F}_1 = \mathcal{O}(ts), & \partial_s m_1 &= \mathcal{O}(t),\\ m_2 &= -i\mathfrak{a}^{-1}\partial_s\widetilde{F}_1 = \mathcal{O}(t^2 s), & \partial_s m_2 &= \mathcal{O}(t^3 s^2),\\ m_3 &= \partial_s\mathfrak{a} = \mathcal{O}(ts), & \partial_t m_3 &= \mathcal{O}(s). \end{aligned}$$

We get then that the norm of R_h satisfies

$$\|R_h\|_{L^2(\Omega)} = \mathcal{O}(h^{13/8}). \tag{6-37}$$

By Hölder's inequality, we infer from (6-37) and (6-12) the estimate

$$h^{1/2}|\operatorname{Re}\langle R_h, g_h\rangle_{L^2(\Omega)}| \le h^{1/2}\|R_h\|_{L^2(\Omega)}\|g_h\|_{L^2(\Omega)} = \widetilde{\mathcal{O}}(h^{5/2}).$$

Consequently, (6-34) and (6-35) yield (6-27).

Step 4: We refine the exponential decay of g_h. To that end, consider a fixed constant $0<\alpha<\frac{1}{4}\alpha_2$, where α_2 is the constant in (6-15), and a real-valued Lipschitz function $\phi_{h,\alpha}\ge 0$, which will be either

$$\phi_{h,\alpha}(x)=\phi_{h,\alpha}^{\mathrm{norm}}(x):=\alpha h^{-1/2}\operatorname{dist}(x,\Gamma) \quad \text{or} \quad \phi_{h,\alpha}(x)=\phi_{h,\alpha}^{\tan}(x):=\alpha h^{-1/8}s(x).$$

We introduce the function $G_{h,\alpha}$ defined via (A-3) as

$$\widetilde{G}_{h,\alpha}(s,t)=-(h^{1/2}\partial_s - i\zeta_a)(e^{2\phi_{h,\alpha}}\tilde{g}_h(s,t)).$$

Since $\alpha<\frac{1}{4}\alpha_2$, we infer from (6-18) and (6-20)

$$\begin{aligned}\int_\Omega (\operatorname{dist}(x,\Gamma))^2|e^{\phi_{h,\alpha}}\varphi_h(x)|^2\,dx &= \mathcal{O}(h),\\ \int_\Omega (s(x))^2|e^{\phi_{h,\alpha}}\varphi_h(x)|^2\,dx &= \mathcal{O}(h^{1/4}),\\ \|G_{h,\alpha}\|_{L^2(\Omega)} &= \mathcal{O}(1),\end{aligned}$$

and also

$$\langle \mathcal{P}_h\varphi_h, G_{h,\alpha}\rangle_{L^2(\Omega)} = \lambda_n(h)\|e^{\phi_{h,\alpha}}g_h\|^2_{L^2(\Omega)} + \widetilde{\mathcal{O}}(h^{19/8}),$$

which results similarly to (6-34).

Now, we write by Lemma A.2,

$$\operatorname{Re}\langle \mathcal{P}_h\varphi_h, G_{h,\alpha}\rangle = \mathcal{Q}_h(e^{\phi_{h,\alpha}}g_h) - h^2\||\nabla\phi_{h,\alpha}|e^{\phi_{h,\alpha}}g_h\|^2_{L^2(\Omega)} - h^{1/2}\operatorname{Re}\langle R_h, e^{2\phi_{h,\alpha}}g_h\rangle_{L^2(\Omega)},$$

where R_h is introduced in (6-36). Since $\alpha<\frac{1}{4}\alpha_2$, we get from (6-18) and (6-19),

$$\|e^{\phi_{h,\alpha}}R_h\|_{L^2(\Omega)} = \mathcal{O}(h^{9/8}) \quad \text{and} \quad \langle R_h, e^{2\phi_{h,\alpha}}g_h\rangle_{L^2(\Omega)} = \mathcal{O}(h^{9/8})\|g_h\|_{L^2(\Omega)}.$$

Collecting the foregoing estimates, we get

$$\mathcal{Q}_h(e^{\phi_{h,\alpha}}g_h) = \lambda_n(h)\|e^{\phi_{h,\alpha}}g_h\|^2_{L^2(\Omega)} + \widetilde{\mathcal{O}}(h^{5/2}). \tag{6-38}$$

Now we can select $\alpha>0$ small enough so that the following two estimates hold. The first estimate is

$$\int_\Omega \big(|g_h|^2 + h^{-1}|(h\nabla - i\boldsymbol{F})g_h|^2\big)\exp(\alpha h^{-1/2}d(x,\Gamma))\,dx \le C\|g_h\|^2_{L^2(\Omega)} + \widetilde{\mathcal{O}}(h^{3/2}), \tag{6-39}$$

and it follows after choosing $\phi_{h,\alpha}=\alpha h^{-1/2}\operatorname{dist}(x,\Gamma)$ and using (6-2). The second estimate follows by choosing $\phi_{h,\alpha}=\alpha h^{-1/8}s(x)$ and using (5-4); it reads as

$$\int_\Omega \big(|g_h|^2 + h^{-1}|(h\nabla - i\boldsymbol{F})g_h|^2\big)\exp(\alpha h^{-1/8}s(x))\,dx \le C\|g_h\|^2_{L^2(\Omega)} + \widetilde{\mathcal{O}}(h). \tag{6-40}$$

Step 5: Let $\theta \in \left(0, \frac{3}{8}\right)$. Collecting the estimates in (6-27), (6-39) and (6-40), we observe that the function g_h satisfies $(5\text{-}1)_\theta$, (5-3) and (5-4) with $r_h = \mathcal{O}(h^{3/4-\theta})$. We can then apply Proposition 5.1 and get (recall that $\|w_h\|_{L^2(\Omega)} \sim \|g_h\|_{L^2(\Omega)} \le \sqrt{r_h}$ by (5-10))

$$\|(h^{3/8}\partial_\sigma - i\zeta_a)w_h\|_{L^2(\Omega)} \le C_\theta h^{3/8-\theta/2}\big(\|g_h\|_{L^2(\Omega)} + \sqrt{r_h} + h^{3/8-3\theta/4}\big) = \mathcal{O}(h^{3/4-5\theta/4}).$$

Since this holds for any $\theta \in \left(0, \frac{3}{8}\right)$, we get that $\|(h^{3/8}\partial_\sigma - i\zeta_a)w_h\|_{L^2(\Omega)} = \widetilde{\mathcal{O}}(h^{3/4})$, thereby finishing the proof of Proposition 6.3. □

7. Lower bound

We fix a labeling $n \ge 1$ corresponding to the eigenvalue $\lambda_n(h)$ of the operator $\mathcal{P}_h$ introduced in (1-4). The purpose of this section is to obtain an accurate lower bound for $\lambda_n(h)$. This will be done by doing a spectral reduction via various auxiliary operators.

7A. ***Useful operators.*** We introduce operators, on the real line and in the plane, which will be useful to carry out a spectral reduction for the operator $\mathcal{P}_h$ and deduce the eigenvalue lower bounds that match with the established eigenvalue asymptotics in Theorem 1.2.

These new operators are defined via the spectral characteristics of the model operator introduced in Section 2B, namely, the spectral constants $\beta_a > 0$ and $\zeta_a < 0$ introduced in (1-10) and (1-12), and the positive normalized eigenfunction $\phi_a \in L^2(\mathbb{R})$ corresponding to β_a. We introduce the two operators

$$R_0^- : \psi \in L^2(\mathbb{R}^2) \mapsto \int_{\mathbb{R}} \phi_a(\tau)\psi(\,\cdot\,,\tau)\,d\tau \in L^2(\mathbb{R}), \tag{7-1}$$

$$R_0^+ : f \in L^2(\mathbb{R}) \mapsto f \otimes \phi_a \in L^2(\mathbb{R}^2), \tag{7-2}$$

where $(f \otimes \phi_a)(\sigma,\tau) := f(\sigma)\phi_a(\tau)$.

Note that $R_0^+R_0^-$ is an orthogonal projector on $L^2(\mathbb{R}^2)$ whose image is $L^2(\mathbb{R}) \otimes \operatorname{span}(\phi_a)$. It is easy to check that the operator norms of $R_0^\pm$ are equal to 1; hence, for any $f \in L^2(\mathbb{R})$ and $\psi \in L^2(\mathbb{R}^2)$, we have

$$\|R_0^+ f\|_{L^2(\mathbb{R})} \le \|f\|_{L^2(\mathbb{R})}, \quad \|R_0^-\psi\|_{L^2(\mathbb{R})} \le \|\psi\|_{L^2(\mathbb{R}^2)}, \quad \|R_0^+R_0^-\psi\|_{L^2(\mathbb{R}^2)} \le \|\psi\|_{L^2(\mathbb{R}^2)}. \tag{7-3}$$

If we denote by π_a the projector in $L^2(\mathbb{R}_\tau)$ on the vector space generated by ϕ_a, we notice that

$$\Pi_0 := R_0^+R_0^- = I \otimes \pi_a. \tag{7-4}$$

7B. ***Structure of bound states.*** Our aim is to determine a rough approximation of the bound state $\psi_{h,n}$ of $\mathcal{P}_h$ satisfying

$$\mathcal{P}_h\psi_{h,n} = \lambda_n(h)\psi_{h,n}, \tag{7-5}$$

this approximation being valid near the point of maximum curvature and reading as follows in the Frenet coordinates:

$$\tilde{\psi}_{h,n}(s,t) \approx h^{-5/16}e^{i\zeta_a s/h^{1/2}}\phi_a(h^{-1/2}t).$$

Associated with $\psi_{h,n}$, we introduced in (6-6) the function $u_{h,n}$ which can be seen as a function on $\mathbb{R}^2$ with L^2-norm satisfying (6-7). We recall that

$$u_{h,n}(\sigma,\tau)=h^{5/16}\chi(h^\eta\sigma)\chi(h^\delta\tau)\tilde{\psi}_{h,n}(h^{1/8}\sigma,h^{1/2}\tau),$$

where $\tilde{\psi}_{h,n}$ is the function assigned to $\psi_{h,n}$ by (A-3), $\chi\in C_c^\infty(\mathbb{R})$, $\operatorname{supp}\chi\subset[-1,1]$, $0\le\chi\le 1$ and $\chi=1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$.

We consider the function defined as

$$v_{h,n}(\sigma,\tau)=e^{-i\zeta_a\sigma/h^{3/8}}u_{h,n}(\sigma,\tau). \tag{7-6}$$

Approximating the function $v_{h,n}\sim\chi(h^\eta\sigma)\chi(h^\delta\tau)\phi_a(\tau)$ is the aim of the next proposition, which also yields an approximation of the bound state $\psi_{h,n}$ by the previous considerations.

Proposition 7.1. *Let $\mathcal{P}_h^{\mathrm{new}}$ be the operator in* (4-6). *The following hold*:

(1) $\|\mathcal{P}_h^{\mathrm{new}}v_{h,n}-(h^{-1}\lambda_n(h)-\beta_a)v_{h,n}\|_{L^2(\mathbb{R}^2)}=\mathcal{O}(h^\infty)$.

(2) $\|v_{h,n}\|_{L^2(\mathbb{R}^2)}=1+\mathcal{O}(h^{1/2})$.

(3) $\|v_{h,n}-\Pi_0v_{h,n}\|_{L^2(\mathbb{R}^2)}=\mathcal{O}(h^{1/4})$.

(4) $\|\partial_\tau v_{h,n}-\partial_\tau\Pi_0v_{h,n}\|_{L^2(\mathbb{R}^2)}+\|\tau(v_{h,n}-\Pi_0v_{h,n})\|_{L^2(\mathbb{R}^2)}=\mathcal{O}(h^{1/4})$.

Proof. Proof of item (1). Let z_h be the function supported near Γ and defined in the Frenet coordinates by means of (A-3) as

$$\tilde{z}_h(s,t)=\chi(h^{-1/8+\eta}s)\chi(h^{-1/2+\delta}t). \tag{7-7}$$

We introduce the function involving the commutator of $\mathcal{P}_h$ and z_h acting on $\psi_{h,n}$,

$$f_h=[\mathcal{P}_h,z_h]\psi_{h,n}=(\mathcal{P}_hz_h-z_h\mathcal{P}_h)\psi_{h,n}. \tag{7-8}$$

By Remark 6.1, we may use the localization estimates in (5-7) and (5-8) with $g_h=\psi_{h,n}$ and $r_h=1$. Consequently,

$$\int_{\mathbb{R}^2}|\tilde{f}_h(s,t)|^2\,ds\,dt\le C\int_\Omega|f_h(x)|^2\,dx=\mathcal{O}(h^\infty),$$

where $\tilde{f}_h$ which is assigned to the function f_h in (7-8) is supported in the set

$$\left\{\left\{|s|\ge\tfrac{1}{2}h^{\eta-1/8}\right\}\cup\{|t|\ge\tfrac{1}{2}h^{\delta-1/2}\}\right\}\cap\{\{|s|\le h^{\eta-1/8}\}\cap\{|t|\le h^{\delta-1/2}\}\}.$$

We infer from (7-5), (4-2), (4-4) and (6-6),

$$\check{\mathcal{P}}_hu_{h,n}-\lambda_n(h)u_{h,n}=h^{5/16}\check{f}_h,$$

where

$$\check{f}_h(\sigma,\tau)=\tilde{f}_h(h^{1/8}\sigma,h^{1/2}\tau).$$

Consequently, after performing the change of variable ($\sigma=h^{-1/8}s$, $\tau=h^{-1/2}t$),

$$\|\check{\mathcal{P}}_hu_{h,n}-\lambda_n(h)u_{h,n}\|^2_{L^2(\mathbb{R}^2)}=\|\tilde{f}_h\|^2_{L^2(\mathbb{R}^2)}=\mathcal{O}(h^\infty). \tag{7-9}$$

By (4-6) and (7-6), we observe that

$$\check{\mathcal{P}}_h u_{h,n} = h e^{i\zeta_a\sigma/h^{3/8}}(\mathcal{P}_h^{\mathrm{new}} + \beta_a) v_{h,n},$$

which after being inserted into (7-9), yields the estimate in item (1).

Remark 7.2. By (6-21), $\partial_\sigma v_{h,n} \in H^2(\mathbb{R}^2)$. Furthermore, by (6-19), the function f_h in (7-8) satisfies $\|\partial_\sigma \check{f}_h\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(h^\infty)$. A slight adjustment of the proof of item (1) then yields

$$\|\partial_\sigma \mathcal{P}_h^{\mathrm{new}} v_{h,n} - (h^{-1}\lambda_n(h) - \beta_a)\partial_\sigma v_{h,n}\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(h^\infty).$$

Proof of item (2). By the normalization of $\psi_{h,n}$ and Remark 6.1, we have

$$1 = \int_\Omega |\psi_{h,n}|^2\,dx = \int_{\{|s(x)|<h^{-\eta+1/8},|t(x)|<h^{-\delta+1/2}\}} |\psi_{h,n}|^2\,dx + \mathcal{O}(h^\infty),$$
$$\int_\Omega (1-z_h^2)|\psi_h|^2\,dx = \mathcal{O}(h^\infty),$$
$$\int_\Omega \mathrm{dist}(x,\Gamma)|\psi_{h,n}|^2\,dx = \mathcal{O}(h^{1/2}).$$

We notice that the function z_h introduced above in (7-7) equals 1 in $\{|s(x)|<\frac{1}{2}h^{-\eta+1/8}, |t(x)|<\frac{1}{2}h^{-\delta+1/2}\}$.

Now we infer from (A-7)

$$\int_{\{|s|<h^{-\eta+1/8},\ |t|<h^{-\delta+1/8}\}} |\tilde{\psi}_{h,n}(s,t)|^2|t|\,ds\,dt \le C\int_\Omega \mathrm{dist}(x,\Gamma)|\psi_{h,n}|^2\,dx = \mathcal{O}(h^{1/2})$$

and

$$\begin{aligned}\int_{\{|s|<h^{-\eta+1/8},\ |t|<h^{-\delta+1/8}\}} |\tilde{\psi}_{h,n}(s,t)|^2\,ds\,dt &= \int_{\{|s|<h^{-\eta+1/8},\ |t|<h^{-\delta+1/8}\}} |\tilde{\psi}_{h,n}(s,t)|^2(1-tk(s))\,ds\,dt\\ &\quad+\int_{\{|s|<h^{-\eta+1/8},\ |t|<h^{-\delta+1/8}\}} |\tilde{\psi}_{h,n}(s,t)|^2\,tk(s)\,ds\,dt\\ &= 1+\mathcal{O}(h^{1/2}).\end{aligned}$$

Similarly we get

$$\int_{\{|s|<\frac{1}{2}h^{-\eta+1/8},\ |t|<\frac{1}{2}h^{-\delta+1/8}\}} (1-\tilde{z}_h^2)\,|\tilde{\psi}_{h,n}(s,t)|^2\,ds\,dt = \mathcal{O}(h^{1/2}).$$

Consequently, returning to (7-6), doing a change of variables and noticing that $\tilde{z}_h$ is supported in $\{|s|<h^{-\eta+1/8}, |t|<h^{-\delta+1/8}\}$, we get

$$\begin{aligned}\|v_{h,n}\|^2_{L^2(\mathbb{R}^2)} &= \int_{\{|s|<h^{-\eta+1/8},\ |t|<h^{-\delta+1/8}\}} |\tilde{\psi}_{h,n}|^2\,ds\,dt - \int_{\{|s|<h^{-\eta+1/8},\ |t|<h^{-\delta+1/8}\}} (1-\tilde{z}_h^2)|\tilde{\psi}_h|^2\,ds\,dt\\ &= 1+\mathcal{O}(h^{1/2}).\end{aligned}$$

Proof of items (3) *and* (4).

<u>Step 1</u>: We recall that the $\widetilde{\mathcal{O}}$ notation was introduced in (6-9). Note that Proposition 6.2 yields

$$\|h^{3/8}\partial_\sigma v_{h,n}\|_{L^2(\mathbb{R}^2)} = \widetilde{\mathcal{O}}(h^{3/8}). \tag{7-10}$$

By Remark 6.1, we can use (5-13) and (5-14) with $g_h = \psi_{h,n}$, $r_h = 1$ (and $w_h = \check{u}_{h,n}$). In the same vein, we can use (5-5) and (5-6) too. Since $u_{h,n} = e^{i\zeta_a\sigma/h^{3/8}} v_{h,n}$, we get

$$\int_{\mathbb{R}^2} \big(|\partial_\tau v_{h,n}|^2 + |h^{3/8}\partial_\sigma v_{h,n} + i(b_a(\tau)\tau + \zeta_a)v_{h,n}|^2\big)\, d\tau\, d\sigma \le (\beta_a + \mathcal{O}(h^{1/2}))\|v_{h,n}\|^2_{L^2(\mathbb{R}^2)}. \tag{7-11}$$

By Cauchy's inequality and (7-10), we obtain, for any $\varepsilon > 0$,

$$\begin{aligned}\int_{\mathbb{R}^2} |h^{3/8}\partial_\sigma v_{h,n} + i(b_a(\tau)\tau+\zeta_a)v_{h,n}|^2\, d\sigma\, d\tau &\ge \int_{\mathbb{R}^2} \big((1-\varepsilon)|(b_a(\tau)\tau+\zeta_a)v_{h,n}|^2 - \varepsilon^{-1}|h^{3/8}\partial_\sigma v_{h,n}|^2\big)\, d\sigma\, d\tau \\ &\ge (1-\varepsilon)\int_{\mathbb{R}^2} |(b_a(\tau)\tau+\zeta_a)v_{h,n}|^2\, d\sigma\, d\tau - \widetilde{\mathcal{O}}(\varepsilon^{-1}h^{3/4}).\end{aligned}$$

We choose $\varepsilon = h^{3/8}$ and insert the resulting inequality into (7-11) to get

$$\int_{\mathbb{R}^2} \big(|\partial_\tau v_{h,n}|^2 + |(b_a(\tau)\tau + \zeta_a)v_{h,n}|^2\big)\, d\tau\, d\sigma \le \beta_a + \widetilde{\mathcal{O}}(h^{3/8}). \tag{7-12}$$

<u>Step 2</u>: In light of (7-4), let us introduce

$$r := \Pi_0 v_{h,n} \quad \text{and} \quad r_\perp := (I - \Pi_0)v_{h,n} = (I \otimes (I - \pi_a))v_{h,n}. \tag{7-13}$$

Using the last relation, and since the orthogonal projection π_a commutes with the operator $\mathfrak{h}_a[\zeta_a]$, we have the following two identities for almost every $\sigma \in \mathbb{R}$:

$$\int_{\mathbb{R}} |v_{h,n}(\sigma,\tau)|^2\, d\tau = \int_{\mathbb{R}} |r(\sigma,\tau)|^2\, d\tau + \int_{\mathbb{R}} |r_\perp(\sigma,\tau)|^2\, d\tau$$

and

$$\begin{aligned} q_{\zeta_a}(v_{h,n}(\sigma,\cdot)) &:= \int_{\mathbb{R}} \big(|\partial_\tau v_{h,n}(\sigma,\tau)|^2 + |(b_a(\tau)\tau + \zeta_a)v_{h,n}(\sigma,\tau)|^2\big)\, d\tau \\ &= q_{\zeta_a}(r(\sigma,\cdot)) + q_{\zeta_a}(r_\perp(\sigma,\cdot)) \\ &\ge \beta_a \int_{\mathbb{R}} |r(\sigma,\tau)|^2\, d\tau + \mu_2(\zeta_a)\int_{\mathbb{R}} |r_\perp(\sigma,\tau)|^2\, d\tau, \end{aligned} \tag{7-14}$$

by the min-max principle, where $\mu_2(\zeta_a)$ is the second eigenvalue of the operator $\mathfrak{h}_a[\zeta_a]$, satisfying $\mu_2(\zeta_a) > \beta_a$ (see Section 2A). Integrating with respect to σ, we get

$$\begin{aligned}\int_{\mathbb{R}^2} \big(|\partial_\tau v_{h,n}(\sigma,\tau)|^2 + |(b_a(\tau)\tau + \zeta_a)v_{h,n}(\sigma,\tau)|^2\big)\, d\sigma\, d\tau \\ \ge \beta_a \int_{\mathbb{R}^2} |r(\sigma,\tau)|^2\, d\sigma\, d\tau + \mu_2(\zeta_a)\int_{\mathbb{R}^2} |r_\perp(\sigma,\tau)|^2\, d\sigma\, d\tau.\end{aligned} \tag{7-15}$$

We deduce from (7-12) and the first item in Proposition 7.1

$$(\mu_2(\zeta_a) - \beta_a)\int_{\mathbb{R}^2} |r_\perp(\sigma,\tau)|^2\, d\sigma\, d\tau \le \widetilde{\mathcal{O}}(h^{3/8})\int_{\mathbb{R}^2} |r(\sigma,\tau)|^2\, d\sigma\, d\tau, \tag{7-16}$$

$$\int_{\mathbb{R}^2} |r(\sigma,\tau)|^2\, d\sigma\, d\tau = 1 + \widetilde{\mathcal{O}}(h^{3/8}), \tag{7-17}$$

$$\int_{\mathbb{R}^2} \big(|\partial_\tau r_\perp(\sigma,\tau)|^2 + |(b_a(\tau)\tau + \zeta_a)r_\perp(\sigma,\tau)|^2\big)\, d\sigma\, d\tau \le \widetilde{\mathcal{O}}(h^{3/8})\int_{\mathbb{R}^2} |r(\sigma,\tau)|^2\, d\sigma\, d\tau. \tag{7-18}$$

Step 3: Coming back to the definition of $r_\perp$ in (7-13), we still have to improve the error term in (7-16) to get the estimate of the third item in Proposition 7.1.

To that end, we will estimate the terms involving $\partial_\sigma v_{h,n}$ in (7-11). By (7-4) and dominated convergence, it is clear that Π_0 commutes with ∂_σ when acting on compactly supported functions of $H^1(\mathbb{R}^2)$:

$$\Pi_0\partial_\sigma = \partial_\sigma\Pi_0. \tag{7-19}$$

By (2-11), ϕ_a is orthogonal to $(b_a(\tau)\tau+\zeta_a)\phi_a$ in $L^2(\mathbb{R})$, so

$$\pi_a(b_a(\tau)\tau+\zeta_a)\pi_a = 0,$$

which implies, by taking the tensor product,

$$\Pi_0(b_a(\tau)\tau+\zeta_a)\Pi_0 = 0. \tag{7-20}$$

By (7-13), (7-19) and (7-20), we get

$$\langle r(\sigma,\tau), i(b_a(\tau)\tau+\zeta_a)\partial_\sigma r(\sigma,\tau)\rangle_{L^2(\mathbb{R}^2)} = 0.$$

Now, we inspect the term

$$\begin{aligned}
&\langle \partial_\sigma v_{h,n}, i(b_a(\tau)\tau+\zeta_a)r\rangle_{L^2(\mathbb{R}^2)}\\
&\qquad= -\langle v_{h,n}, i(b_a(\tau)\tau+\zeta_a)\partial_\sigma r\rangle_{L^2(\mathbb{R}^2)}\\
&\qquad= -\underbrace{\langle r, i(b_a(\tau)\tau+\zeta_a)\partial_\sigma r\rangle_{L^2(\mathbb{R}^2)}}_{=0} - \langle r_\perp, i(b_a(\tau)\tau+\zeta_a)\partial_\sigma r\rangle_{L^2(\mathbb{R}^2)}\\
&\qquad= -\langle r_\perp, i(b_a(\tau)\tau+\zeta_a)\partial_\sigma r\rangle_{L^2(\mathbb{R}^2)} = -\langle (b_a(\tau)\tau+\zeta_a)r_\perp, i\partial_\sigma r\rangle_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{7-21}$$

Since

$$\begin{aligned}
\|h^{3/8}\partial_\sigma r\|_{L^2(\mathbb{R}^2)} &= h^{3/8}\|\Pi_0\partial_\sigma v_{h,n}\|_{L^2(\mathbb{R}^2)} && \text{(by (7-19))}\\
&\le h^{3/8}\|\partial_\sigma v_{h,n}\|_{L^2(\mathbb{R}^2)} && \text{(by (7-3))}\\
&= \widetilde{\mathcal{O}}(h^{3/8}) && \text{(by (7-10))},
\end{aligned}$$

we get by the Cauchy–Schwarz inequality, (7-21) and (7-18)

$$h^{3/8}|\langle \partial_\sigma v_{h,n}, i(b_a(\tau)\tau+\zeta_a)r\rangle_{L^2(\mathbb{R}^2)}| \le \|(b_a(\tau)\tau+\zeta_a)r_\perp\|_{L^2(\mathbb{R}^2)}\|h^{3/8}\partial_\sigma r\|_{L^2(\mathbb{R}^2)} = \widetilde{\mathcal{O}}(h^{9/16}). \tag{7-22}$$

Now, we can estimate the following inner product term by using (7-13) and (7-22):

$$\begin{aligned}
&\langle h^{3/8}\partial_\sigma v_{h,n}, i(b_a(\tau)\tau+\zeta_a)v_{h,n}\rangle_{L^2(\mathbb{R}^2)}\\
&\qquad= \langle h^{3/8}\partial_\sigma v_{h,n}, i(b_a(\tau)\tau+\zeta_a)r_\perp\rangle_{L^2(\mathbb{R}^2)} + \langle h^{3/8}\partial_\sigma v_{h,n}, i(b_a(\tau)\tau+\zeta_a)r\rangle_{L^2(\mathbb{R}^2)}\\
&\qquad= \langle h^{3/8}\partial_\sigma v_{h,n}, i(b_a(\tau)\tau+\zeta_a)r_\perp\rangle_{L^2(\mathbb{R}^2)} + \widetilde{\mathcal{O}}(h^{9/16}).
\end{aligned} \tag{7-23}$$

By the Cauchy–Schwarz inequality, (7-10), (7-18) and (7-23), we get

$$\begin{aligned}
|\langle h^{3/8}\partial_\sigma v_{h,n}, i(b_a(\tau)\tau+\zeta_a)v_{h,n}\rangle_{L^2(\mathbb{R}^2)}| &\le \|h^{3/8}\partial_\sigma v_{h,n}\|\,\|(b_a(\tau)\tau+\zeta_a)r_\perp\| + \widetilde{\mathcal{O}}(h^{9/16})\\
&= \widetilde{\mathcal{O}}(h^{9/16}) = o(h^{1/2}).
\end{aligned} \tag{7-24}$$

Consequently,

$$\begin{aligned}
&\|h^{3/8}\partial_\sigma v_{h,n}+i(b_a(\tau)\tau+\zeta_a)v_{h,n}\|^2_{L^2(\mathbb{R}^2)}\\
&\qquad=\|h^{3/8}\partial_\sigma v_{h,n}\|^2_{L^2(\mathbb{R}^2)}+\|(b_a(\tau)\tau+\zeta_a)v_{h,n}\|^2_{L^2(\mathbb{R}^2)}+2\,\mathrm{Re}\langle h^{3/8}\partial_\sigma v_{h,n}, i(b_a(\tau)\tau+\zeta_a)v_{h,n}\rangle_{L^2(\mathbb{R}^2)}\\
&\qquad\geq \|(b_a(\tau)\tau+\zeta_a)v_{h,n}\|^2_{L^2(\mathbb{R}^2)}+o(h^{1/2}).
\end{aligned}$$

Inserting the previous inequality into (7-11) we get the following improvement of (7-12):

$$\int_{\mathbb{R}^2}(|\partial_\tau v_{h,n}|^2+|(b_a(\tau)\tau+\zeta_a)v_{h,n}|^2)\,d\tau\,d\sigma\leq\beta_a+\mathcal{O}(h^{1/2}). \tag{7-25}$$

Step 4: Now we are ready to finish the proof of items (3) and (4). By (7-15) and (7-14), we infer from (7-25) and (7-13),

$$(\mu_2(\zeta_a)-\beta_a)\int_{\mathbb{R}^2}|r_\perp(\sigma,\tau)|^2\,d\sigma\,d\tau\leq\mathcal{O}(h^{1/2})\int_{\mathbb{R}^2}|r(\sigma,\tau)|^2\,d\sigma\,d\tau,$$

$$\int_{\mathbb{R}^2}\big(|\partial_\tau r_\perp(\sigma,\tau)|^2+|(b_a(\tau)\tau+\zeta_a)r_\perp(\sigma,\tau)|^2\big)\,d\sigma\,d\tau\leq\mathcal{O}(h^{1/2})\int_{\mathbb{R}^2}|r(\sigma,\tau)|^2\,d\sigma\,d\tau.$$

With (7-17) in hand, we get the estimates of items (3) and (4) of Proposition 7.1. □

7C. ***Projection on a refined quasimode.*** We wish to improve the approximation $v_{h,n}\sim\chi(h^\eta\sigma)\chi(h^\delta\tau)\phi_a(\tau)$ obtained in Proposition 7.1 by two ways which eventually are correlated: displaying the curvature effects in $v_{h,n}$ and getting better estimates of the errors. Along the proof of Proposition 7.1, curvature effects were neglected and absorbed in the error terms. Not neglecting the curvature, we get the approximation $v_{h,n}\sim\chi(h^\eta\sigma)\chi(h^\delta\tau)\phi_{a,h}(\tau)$, where $\phi_{a,h}(\tau)$ corrects $\phi_a(\tau)$ via curvature-dependent terms (see (7-31)). This is precisely stated in Proposition 7.3 after introducing the necessary preliminaries.

7C1. *Preliminaries.* In this subsection, we write $\kappa=k(0)=k_{\max}$ and $k_2=k''(0)$. We consider the weighted L^2 space

$$X_{h,\delta}=L^2\big((-h^{-\delta},h^{-\delta});(1-h^{1/2}\kappa\tau)\,d\tau\big) \tag{7-26}$$

endowed with the Hilbertian norm

$$\|f\|_{X_{h,\delta}}=\bigg(\int_{-h^{-\delta}}^{h^{-\delta}}|f(\tau)|^2(1-h^{1/2}\kappa\tau)\,d\tau\bigg)^{1/2}.$$

This norm is equivalent to the usual norm of $L^2(-h^{-\delta},h^{-\delta})$ provided h is sufficiently small.

With domain $H^2(-h^{-\delta},h^{-\delta})\cap H^1_0(-h^{-\delta},h^{-\delta})$, consider the operator in (3-1) for $\xi=\zeta_a$:

$$\begin{aligned}
\mathcal{H}_{a,\kappa,h}=-\frac{d^2}{d\tau^2}+(b_a(\tau)\tau+\zeta_a)^2+\kappa h^{1/2}(1-\kappa h^{1/2}\tau)^{-1}\partial_\tau+2\kappa h^{1/2}\tau\Big(b_a(\tau)\tau+\zeta_a-\kappa h^{1/2}b_a(\tau)\frac{\tau^2}{2}\Big)^2\\
-\kappa h^{1/2}b_a(\tau)\tau^2(b_a(\tau)\tau+\zeta_a)+\kappa^2hb_a(\tau)^2\frac{\tau^4}{4},
\end{aligned} \tag{7-27}$$

which is self-adjoint on the space $X_{h,\delta}$. This operator can be decomposed as follows:

$$\mathcal{H}_{a,\kappa,h}=\mathfrak{h}[\zeta_a]+\kappa h^{1/2}\mathfrak{h}^{(1)}[\zeta_a]+hL_h, \tag{7-28}$$

where $\mathfrak{h}[\zeta_a]$ is introduced in (2-1) and

$$\mathfrak{h}^{(1)}[\zeta_a] = \partial_\tau + 2\tau(b_a(\tau)\tau + \zeta_a)^2 - b_a(\tau)\tau^2(b_a(\tau)\tau + \zeta_a) \tag{7-29}$$

and

$$L_h = q_{1,h}(\tau)\partial_\tau + q_{2,h}(\tau), \quad \text{with } |q_{1,h}(\tau)| \le C_1|\tau|, \ \ |q_{2,h}(\tau)| \le C_2(1+|\tau|^5), \tag{7-30}$$

where C_1, C_2 are positive constants independent of h, τ.

We introduce the following quasimode in the space $X_{h,\delta}$:

$$\phi_{a,h}(\tau) = \chi(h^\delta\tau)\big(\phi_a(\tau) + h^{1/2}\kappa\,\phi_a^{\mathrm{cor}}(\tau)\big), \tag{7-31}$$

where $\chi \in C_c^\infty(\mathbb{R}; [0, 1])$, $\operatorname{supp}\chi \subset [-1, 1]$, $\chi = 1$ on $\big[-\frac{1}{2}, \frac{1}{2}\big]$. The function ϕ_a is the positive ground state of $\mathfrak{h}[\zeta_a]$ with corresponding ground state energy β_a:

$$(\mathfrak{h}[\zeta_a] - \beta_a)\phi_a = 0.$$

We now explain the construction of ϕ_a^{cor}. By (7-28), starting from some ϕ_a^{cor} to be determined,

$$\begin{aligned}(\mathcal{H}_{a,\kappa,h} - \beta_a - h^{1/2}\kappa M_3(a))(\phi_a + h^{1/2}\kappa\,\phi_a^{\mathrm{cor}})&\\ = \kappa h^{1/2}\big((\mathfrak{h}[\zeta_a]-)\phi_a^{\mathrm{cor}} + (\mathfrak{h}^{(1)}[\zeta_a] - M_3(a))\phi_a\big) + h\mathcal{R}_{a,h},&\end{aligned} \tag{7-32}$$

where

$$\mathcal{R}_{a,h} = L_h(\phi_a + h^{1/2}\kappa\,\phi_a^{\mathrm{cor}}) + \kappa^2(\mathfrak{h}^{(1)}[\zeta_a] - M_3(a))\phi_a^{\mathrm{cor}}.$$

Note that, by Remark 2.3, $\mathfrak{h}^{(1)}[\zeta_a]\phi_a - M_3(a)\phi_a$ is orthogonal to ϕ_a in $L^2(\mathbb{R})$. Hence we can choose

$$\phi_a^{\mathrm{cor}} = -\mathfrak{R}_a(\mathfrak{h}^{(1)}[\zeta_a]\phi_a - M_3(a)\phi_a), \tag{7-33}$$

so that the coefficient of $h^{1/2}$ in (7-32) vanishes. In this way, we infer from (7-32),

$$\big(\mathcal{H}_{a,\kappa,h} - \beta_a - h^{1/2}\kappa M_3(a)\big)(\phi_a + h^{1/2}\kappa\phi_a^{\mathrm{cor}}) = h\mathcal{R}_{a,h}.$$

Notice that $\phi_{a,h}$ is constructed so that it has compact support in $(-h^{-\delta}, h^{-\delta})$ and hence satisfies the Dirichlet conditions at $\tau = \pm h^{-\delta}$. Since, ϕ_a and ϕ_a^{cor} decay exponentially at infinity by Lemma 2.4, we deduce

$$\|\mathcal{H}_{a,\kappa,h}\phi_{a,h} - (\beta_a + h^{1/2}\kappa M_3(a))\phi_{a,h}\|_{X_{h,\delta}} = \mathcal{O}(h). \tag{7-34}$$

We denote by $\phi_{a,h}^{\mathrm{gs}}$ the normalized ground state of the Dirichlet realization of $\mathcal{H}_{a,\kappa,h}$ in the weighted space $X_{h,\delta}$ (i.e., in $L^2((-h^{-\delta}, h^{-\delta}); (1-h^{1/2}\kappa\tau)d\tau)$). By (3-8), the min-max principle and Proposition 3.2, we have

$$\lambda_1(\mathcal{H}_{a,\kappa,h}) = \beta_a + h^{1/2}\kappa M_3(a) + \mathcal{O}(h) \quad \text{and} \quad \lambda_2(\mathcal{H}_{a,\kappa,h}) \ge \mu_2(\zeta_a) + o(1), \tag{7-35}$$

so we infer from (7-34) and the Hölder inequality

$$\big\langle(\mathcal{H}_{a,\kappa,h}\phi_{a,h} - \lambda_1(\mathcal{H}_{a,\kappa,h}))(\phi_{a,h}^{\mathrm{gs}} - \phi_{a,h}), \phi_{a,h}^{\mathrm{gs}} - \phi_{a,h}\big\rangle_{X_{h,\delta}} = \mathcal{O}(h)\|\phi_{a,h}^{\mathrm{gs}} - \phi_{a,h}\|_{X_{h,\delta}}.$$

Thus, by the spectral theorem,

$$\|\phi_{a,h}^{\mathrm{gs}} - \phi_{a,h}\|_{X_{h,\delta}} + \|\tau(\phi_{a,h}^{\mathrm{gs}} - \phi_{a,h})\|_{X_{h,\delta}} + \|\partial_\tau(\phi_{a,h}^{\mathrm{gs}} - \phi_{a,h})\|_{X_{h,\delta}} = \mathcal{O}(h). \tag{7-36}$$

7C2. *New projections.* We fix $h_0 > 0$ so that $1 - h_0^{1/2-\delta}\kappa > \frac{1}{2}$. In the sequel, the parameter h varies in the interval $(0, h_0)$. Consider the space

$$X^2_{h,\delta} = L^2\big(\mathbb{R} \times (-h^{-\delta}, h^{\delta}); (1 - h^{1/2}\kappa\tau)\, d\sigma\, d\tau\big) \tag{7-37}$$

endowed with the weighted norm

$$\|v\|_{X^2_{h,\delta}} = \left(\int_{\mathbb{R}} \int_{-h^{-\delta}}^{h^{-\delta}} |v(\sigma,\tau)|^2 (1 - h^{1/2}\kappa\tau)\, d\sigma\, d\tau\right)^{1/2},$$

which is equivalent to the usual norm of $L^2(\mathbb{R} \times (-h^{-\delta}, h^{\delta}))$.

We introduce the two operators

$$R_h^- : v \in X^2_{h,\delta} \mapsto \int_{\mathbb{R}} \phi_{a,h}(\tau) v(\,\cdot\,,\tau)(1 - h^{1/2}\kappa\tau)\, d\tau \in L^2(\mathbb{R}), \tag{7-38}$$

$$R_h^+ : f \in L^2(\mathbb{R}) \mapsto f \otimes \phi_{a,h} \in X^2_{h,\delta}, \quad \text{where } f \otimes \phi_{a,h}(\sigma,\tau) = f(\sigma)\phi_{a,h}(\tau). \tag{7-39}$$

The image of $R_h^+ R_h^-$ is $L^2(\mathbb{R}) \otimes \operatorname{span}(\phi_{a,h})$. Furthermore, for all $v \in X^2_{h,\delta}$, the functions $R_h^+ R_h^- v$ and $v - R_h^+ R_h^- v$ are orthogonal in $X^2_{h,\delta}$, since the operator $R_h^+ R_h^-$ can be expressed as

$$\Pi_h := R_h^+ R_h^- = I \otimes \pi_{a,h}, \tag{7-40}$$

where $\pi_{a,h}$ is the orthogonal projection, in the weighted Hilbert space $X_{h,\delta}$, on the space $\operatorname{span}\phi_{a,h}$. With this projection in hand, we can approximate the truncated bound state $v_{h,n}$, introduced in (7-6), with better error terms, thereby improving Proposition 7.1.

Proposition 7.3. *The following holds*:

$$\|v_{h,n} - \Pi_h v_{h,n}\|_{X^2_{h,\delta}} + \|\partial_\tau(v_{h,n} - \Pi_h v_{h,n})\|_{X^2_{h,\delta}} + \|\tau(v_{h,n} - \Pi_h v_{h,n})\|_{X^2_{h,\delta}} = \widetilde{\mathcal{O}}(h^{5/16}),$$

where Π_h *is the projection in* (7-40).

Remark 7.4. By (7-31) and (7-32), we observe that

$$\|(\Pi_h - \Pi_0)v_{h,n}\|_{L^2(\mathbb{R}^2)} + \|(\partial_\tau \Pi_h - \partial_\tau \Pi_0)v_{h,n}\|_{L^2(\mathbb{R}^2)} + \|\tau(\Pi_h - \Pi_0)v_{h,n}\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(h^{1/2}),$$

where Π_0 is the projection introduced in (7-4). Since the norm of $X^2_{h,\delta}$ is equivalent to the usual norm of L^2, Proposition 7.3 yields the following improvement of Proposition 7.1:

$$\|v_{h,n} - \Pi_0 v_{h,n}\|_{L^2(\mathbb{R}^2)} + \|\partial_\tau(v_{h,n} - \Pi_0 v_{h,n})\|_{L^2(\mathbb{R}^2)} + \|\tau(v_{h,n} - \Pi_0 v_{h,n})\|_{L^2(\mathbb{R}^2)} = \widetilde{\mathcal{O}}(h^{5/16}), \tag{7-41}$$

where Π_0 is the projection in (7-4). This remark will be useful in the next subsection.

Proof of Proposition 7.3. Step 1: We give here preliminary estimates that we will use in Step 3 below. Firstly, by Remark 6.1,

$$\int_{\mathbb{R}^2} \tau^4 |v_{h,n}(\sigma,\tau)|^2\, d\sigma\, d\tau = \mathcal{O}(1). \tag{7-42}$$

Secondly, we will prove that

$$\langle h^{3/8}\partial_\sigma v_{h,n}, (b_a(\tau)\tau+\zeta_a)v_{h,n}\rangle_{L^2(\mathbb{R}^2)} = \widetilde{\mathcal{O}}(h^{5/8}). \tag{7-43}$$

By (7-10) and Proposition 7.1,

$$\begin{aligned}
&|\langle h^{3/8}\partial_\sigma v_{h,n}, (b_a(\tau)\tau+\zeta_a)(v_{h,n}-\Pi_0 v_{h,n})\rangle_{L^2(\mathbb{R}^2)}| \\
&\qquad\le \|h^{3/8}\partial_\sigma v_{h,n}\|_{L^2(\mathbb{R}^2)}\|(b_a(\tau)\tau+\zeta_a)(v_{h,n}-\Pi_0 v_{h,n})\|_{L^2(\mathbb{R}^2)} = \widetilde{\mathcal{O}}(h^{5/8}).
\end{aligned}$$

Similarly, using (7-19) and Hölder's inequality, we write

$$\begin{aligned}
&|\langle (b_a(\tau)\tau+\zeta_a)h^{3/8}\partial_\sigma \Pi_0 v_{h,n}, v_{h,n}-\Pi_0 v_{h,n}\rangle_{L^2(\mathbb{R}^2)}| \\
&\qquad\le \|h^{3/8}\Pi_0\partial_\sigma v_{h,n}\|_{L^2(\mathbb{R}^2)}\|(b_a(\tau)\tau+\zeta_a)(v_{h,n}-\Pi_0 v_{h,n})\|_{L^2(\mathbb{R}^2)} = \widetilde{\mathcal{O}}(h^{5/8}).
\end{aligned}$$

Now, writing $v_{h,n}=\Pi_0 v_{h,n}+(v_{h,n}-\Pi_0 v_{h,n})$ and collecting the foregoing estimates, we get

$$\begin{aligned}
&\langle h^{3/8}\partial_\sigma v_{h,n}, (b_a(\tau)\tau+\zeta_a)v_{h,n}\rangle_{L^2(\mathbb{R}^2)} \\
&\qquad= \langle h^{3/8}\partial_\sigma v_{h,n}, (b_a(\tau)\tau+\zeta_a)\Pi_0 v_{h,n}\rangle_{L^2(\mathbb{R}^2)} + \widetilde{\mathcal{O}}(h^{5/8}) \\
&\qquad= -\langle (b_a(\tau)\tau+\zeta_a)v_{h,n}, h^{3/8}\partial_\sigma \Pi_0 v_{h,n}\rangle_{L^2(\mathbb{R}^2)} + \widetilde{\mathcal{O}}(h^{5/8}) \quad \text{(by integration by parts).}
\end{aligned}$$

Again, decomposing $v_{h,n}$ by the projection Π_0 and observing that (7-20) yields

$$\langle (b_a(\tau)\tau+\zeta_a)\Pi_0 v_{h,n}, h^{3/8}\partial_\sigma \Pi_0 v_{h,n}\rangle_{L^2(\mathbb{R}^2)} = 0,$$

we get

$$\begin{aligned}
&\langle h^{3/8}\partial_\sigma v_{h,n}, (b_a(\tau)\tau+\zeta_a)v_{h,n}\rangle_{L^2(\mathbb{R}^2)} \\
&\qquad= -\langle (b_a(\tau)\tau+\zeta_a)h^{3/8}\partial_\sigma \Pi_0 v_{h,n}, v_{h,n}-\Pi_0 v_{h,n}\rangle_{L^2(\mathbb{R}^2)} + \widetilde{\mathcal{O}}(h^{5/8}) = \widetilde{\mathcal{O}}(h^{5/8}),
\end{aligned}$$

thereby obtaining (7-43).

<u>Step 2</u>: We introduce operators involving the ground state $\phi^{\mathrm{gs}}_{a,h}$ as follows. First we introduce the operators

$$\widetilde{R}_h^- : v \in X^2_{h,\delta} \mapsto \int_{\mathbb{R}} \phi^{\mathrm{gs}}_{a,h}(\tau)v(\,\cdot\,,\tau)(1-h^{1/2}\kappa\tau)\,d\tau \in L^2(\mathbb{R}), \tag{7-44}$$

$$\widetilde{R}_h^+ : f \in L^2(\mathbb{R}) \mapsto f\otimes\phi^{\mathrm{gs}}_{a,h} \in X^2_{h,\delta}, \quad \text{where } (f\otimes\phi^{\mathrm{gs}}_{a,h})(\sigma,\tau) = f(\sigma)\phi^{\mathrm{gs}}_{a,h}(\tau). \tag{7-45}$$

Denoting by $\tilde{\pi}_{a,h}$ the orthogonal projection, in $X_{h,\delta}$, on the space span $\phi^{\mathrm{gs}}_{a,h}$, we introduce

$$\widetilde{\Pi}_h := \widetilde{R}_h^+\widetilde{R}_h^- = I\otimes\tilde{\pi}_{a,h}. \tag{7-46}$$

By (7-36) and (7-40), we observe that, for all $g\in X_{h,\delta}$ and $f\in X^2_{h,,\delta}$, we have

$$\|(\widetilde{R}_h^- - R_h^-)g\|_{X_{h,\delta}} = \mathcal{O}(h)\|g\|_{X_{h,\delta}}, \quad \|(\widetilde{\Pi}_h-\Pi_h)f\|_{X^2_{h,\delta}} = \mathcal{O}(h)\|f\|_{X^2_{h,\delta}}.$$

So if we prove that

$$\|v_{h,n}-\widetilde{\Pi}_h v_{h,n}\|_{X^2_{h,\delta}} + \|\partial_\tau(v_{h,n}-\widetilde{\Pi}_h v_{h,n})\|_{X^2_{h,\delta}} + \|\tau(v_{h,n}-\widetilde{\Pi}_h v_{h,n})\|_{X^2_{h,\delta}} = \widetilde{\mathcal{O}}(h^{5/16}), \tag{7-47}$$

then we deduce the estimate in Proposition 7.3.

Step 3: Adapting the proof of Proposition 7.1, we prove now (7-47). By Remark 6.1, we can use (5-14) with $w_h = u_{h,n},\ r_h = 1,\ m_h = \|u_{h,n}\|^2_{X^2_{h,\delta}} = 1 + \mathcal{O}(h^{1/2})$ and $\theta = \frac{1}{4}$. Thus

$$\int_{\mathbb{R}} \int_{-h^{-\delta}}^{h^{-\delta}} \Big(|\partial_\tau u_{h,n}|^2 + (1+2\kappa h^{1/2}\tau) \Big| \Big(h^{3/8}\partial_\sigma + i\Big(b_a\tau - \kappa h^{1/2} b_a \frac{\tau^2}{2}\Big)\Big) u_{h,n} \Big|^2 \Big) (1-\kappa h^{1/2}\tau)\, d\sigma\, d\tau \\ \leq (\beta_a + h^{1/2} M_3(a)\kappa + \mathcal{O}(h^{3/4})) \|u_{h,n}\|^2_{X^2_{h,\delta}}. \tag{7-48}$$

Since $u_{h,n} = e^{i\zeta_a\sigma/h^{3/8}} v_{h,n}$ (by (7-6)), we get

$$\int_{\mathbb{R}} \int_{-h^{-\delta}}^{h^{-\delta}} |\partial_\tau v_{h,n}|^2 (1-\kappa h^{1/2}\tau)\, d\sigma\, d\tau \\ + \int_{\mathbb{R}} \int_{-h^{-\delta}}^{h^{-\delta}} (1+2\kappa h^{1/2}\tau) \Big| \Big(h^{3/8}\partial_\sigma + i\Big(b_a\tau + \zeta_a - \kappa h^{1/2} b_a \frac{\tau^2}{2}\Big)\Big) v_{h,n} \Big|^2 (1-\kappa h^{1/2}\tau)\, d\sigma\, d\tau \\ \leq (\beta_a + h^{1/2} M_3(a)\kappa + \mathcal{O}(h^{3/4})) \|v_{h,n}\|^2_{X^2_{h,\delta}}. \tag{7-49}$$

Using (7-10), (7-43) and (7-42), we deduce the following estimate from (7-49):

$$\int_{\mathbb{R}} \int_{-h^{-\delta}}^{h^{-\delta}} \Big(|\partial_\tau v_{h,n}|^2 + (1+2\kappa h^{1/2}\tau) \Big| \Big(b_a\tau + \zeta_a - \kappa h^{1/2} b_a \frac{\tau^2}{2}\Big) v_{h,n} \Big|^2 \Big) (1-\kappa h^{1/2}\tau)\, d\sigma\, d\tau \\ \leq (\beta_a + h^{1/2} M_3(a)\kappa + \widetilde{\mathcal{O}}(h^{5/8})) \|v_{h,n}\|^2_{X^2_{h,\delta}}, \tag{7-50}$$

where we used also that $\|v_{h,n}\|^2_{X^2_{h,\delta}} = 1 + \mathcal{O}(h^{1/2})$, by (6-7) and (7-6).

Now we get (7-47) by decomposing $v_{h,n}$ in $X^2_{h,\delta}$ in the form

$$v_{h,n} = \tilde{r}_h + \tilde{r}_{h,\perp}, \quad \tilde{r}_h := \widetilde{\Pi}_h v_{h,n}, \quad \tilde{r}_{h,\perp} = (I - \widetilde{\Pi}_h) v_{h,n},$$

and by using the spectral asymptotics for the operator $\mathcal{H}_{h,a,\kappa}$, recalled in (7-35). □

7D. ***Quasimodes for the effective operator.*** Let us start with some heuristic considerations. The derivation of the eigenvalue upper bound of Theorem 4.1 suggested in the tangent variable the following one-dimensional effective operator (see (4-25)):

$$H_a^{\mathrm{harm}} = -c_2(a)\partial_\sigma^2 - \frac{M_3(a)k''(0)}{2}\sigma^2, \tag{7-51}$$

where $c_2(a) > 0$ is introduced in (1-12).

Moreover, by Remark 4.3, it is natural to consider the quasimode

$$v_{h,n}^{\mathrm{app}} = \big(\phi_a(\tau) + 2\mathfrak{R}_a((\zeta_a + b_a(\tau)\tau)\phi_a) i h^{3/8}\partial_\sigma + k_{\max} h^{1/2}\phi_a^{\mathrm{cor}}(\tau)\big) f_n(\sigma),$$

where $\mathfrak{R}_a$ is the regularized resolvent introduced in (2-18), ϕ_a^{cor} is the function in (7-33), and f_n is the normalized n-th eigenfunction of the operator H_a^{harm}. Denoting by $\Pi_{h,n}^{\mathrm{app}}$ the orthogonal projection, in $L^2(\mathbb{R}^2)$, on the space generated by $v_{h,n}^{\mathrm{app}}$, we observe formally, by neglecting the terms with coefficients having order lower than $h^{3/4}$,

$$c_2(a)\Pi_{h,n}^{\mathrm{app}}\mathcal{P}_h^{\mathrm{new}} \approx h^{1/2}(M_3(a)k_{\max} + h^{1/4} H_a^{\mathrm{harm}})\Pi_n^{\mathrm{new}},$$

where Π_n^{new} is the projection, in $L^2(\mathbb{R}^2)$, on the space generated by the function $\varphi_a(\tau) f_n(\sigma)$, and

$$\varphi_a(\tau) := \phi_a(\tau) - 4(b_a(\tau)\tau + \zeta_a)\mathfrak{R}_a((b_a(\tau)\tau + \zeta_a)\phi_a(\tau)). \tag{7-52}$$

Guided by these heuristic observations, we will use the truncated bound state $v_{h,n}$ in (7-6) to construct quasimodes of the operator H_a^{harm} by projecting $v_{h,n}$ on the vector space generated by the function φ_a introduced in (7-52). To that end, we introduce the operator

$$R_0^{\text{new}} : v \in L^2(\mathbb{R}^2) \mapsto \int_{\mathbb{R}} \varphi_a(\tau) v(\,\cdot\,,\tau)\, d\tau \in L^2(\mathbb{R}). \tag{7-53}$$

We will prove the following proposition.

Proposition 7.5. *Let $n \in \mathbb{N}$ be fixed. The following hold*:

(1) $\|R_0^{\text{new}} v_{h,n} - (1-4I_2(a))R_0^- v_{h,n}\|_{L^2(\mathbb{R})} = \mathcal{O}(h^{1/4})$, *where R_0^- is the operator in* (7-1) *and $I_2(a)$ is introduced in* (2-17).

(2) $\|R_0^{\text{new}} v_{h,n}\|_{L^2(\mathbb{R})} = 1 - 4I_2(a) + \mathcal{O}(h^{1/4})$.

(3) *For every $n \in \mathbb{N}$, there exists $h_n > 0$ such that*, *for all $h \in (0, h_n)$*,

$$\langle R_0^{\text{new}} v_{h,k}, R_0^{\text{new}} v_{h,k'}\rangle_{L^2(\mathbb{R})} = (1-4I_2(a))^2 \delta_{k,k'} + o(1) \quad (1 \le k, k' \le n), \tag{7-54}$$

and

$$M_n = \operatorname{span}(R_0^{\text{new}} v_{h,k},\ 1 \le k \le n) \quad \textit{satisfies} \quad \dim(M_n) = n. \tag{7-55}$$

(4) *We have as $h \to 0_+$*

$$\big\langle (H_a^{\text{harm}} - h^{-3/4}\Lambda_n(h)) R_0^{\text{new}} v_{h,n}, R_0^{\text{new}} v_{h,n}\big\rangle_{L^2(\mathbb{R})} = o(1)\|R_0^{\text{new}} v_{h,n}\|^2_{L^2(\mathbb{R})},$$

where

$$\Lambda_n(h) = h^{-1}\lambda_n(h) - \beta_a - M_3(a) k_{\max} h^{1/2},$$

and H_a^{harm} is the operator introduced in (7-51).

Proof. Proof of item (1). Consider $\Pi_0 = R_0^+ R_0^-$ the projection introduced in (7-4). By (2-17), $R_0^{\text{new}} R_0^+ = (1-4I_2(a))\mathrm{Id}$; hence, composing by R_0^- on the right gives

$$R_0^{\text{new}} \Pi_0 = (1-4I_2(a)) R_0^-.$$

Writing $v_{h,n} = \Pi_0 v_{h,n} + (v_{h,n} - \Pi_0 v_{h,n})$, we get

$$\begin{aligned} R_0^{\text{new}} v_{h,n} &= R_0^{\text{new}} \Pi_0 v_{h,n} + R_0^{\text{new}}(v_{h,n} - \Pi_0 v_{h,n}) \\ &= (1-4I_2(a)) R_0^- v_{h,n} + R_0^{\text{new}}(v_{h,n} - \Pi_0 v_{h,n}). \end{aligned}$$

Then we observe that

$$\|R_0^{\text{new}}(v_{h,n} - \Pi_0 v_{h,n})\|_{L^2(\mathbb{R})} \le \|\varphi_a\|_{L^2(\mathbb{R})} \|v_{h,n} - \Pi_0 v_{h,n}\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(h^{1/4})$$

by Hölder's inequality and Proposition 7.1. This yields the conclusion of item (1).

Proof of item (2). By (2-20), $1-4I_2(a)>0$. By (7-1) and Proposition 7.1, we have

$$\|R_0^- v_{h,n}\|_{L^2(\mathbb{R})}=\|\Pi_0 v_{h,n}\|_{L^2(\mathbb{R}^2)}=1+\mathcal{O}(h^{1/4}).$$

Now item (2) follows from item (1).

Proof of item (3). If $1\le k,k'\le n$ and $k\neq k'$, we have as $h\to 0_+$,

$$\langle v_{h,k}, v_{h,k'}\rangle_{L^2(\mathbb{R}^2)}=o(1)+\delta_{k,k'}.$$

By Proposition 7.1, we get further

$$\langle R_0^- v_{h,k}, R_0^- v_{h,k'}\rangle_{L^2(\mathbb{R})}=\langle \Pi_0 v_{h,k}, \Pi_0 v_{h,k'}\rangle_{L^2(\mathbb{R}^2)}=o(1)+\delta_{k,k'}.$$

Thus, by item (1),

$$\langle R_0^{\mathrm{new}} v_{h,k}, R_0^{\mathrm{new}} v_{h,k'}\rangle_{L^2(\mathbb{R})}=o(1)+\delta_{k,k'}.$$

With item (2) in hand, we get the conclusion of item (3).

Proof of item (4).

Step 1: We introduce the operator

$$\widetilde{R}_h^{\mathrm{new}}: v\in H^1(\mathbb{R}^2)\mapsto \int_{\mathbb{R}}\phi_{a,h}^{\mathrm{new}}(\tau, i\partial_\sigma)v(\,\cdot\,,\tau)\,d\tau\in L^2(\mathbb{R}), \tag{7-56}$$

where $\phi_{a,h}^{\mathrm{new}}(\tau,i\partial_\sigma)$ is the first-order differential operator

$$\phi_{a,h}^{\mathrm{new}}(\tau,i\partial_\sigma):=\phi_a(\tau)+2h^{3/8}\mathfrak{R}_a((b_a(\tau)\tau+\zeta_a)\phi_a(\tau))i\partial_\sigma+\kappa h^{1/2}\phi_a^{\mathrm{cor}}(\tau), \tag{7-57}$$

$\kappa=k_{\max}$ and ϕ_a^{cor} is the function introduced in (7-33).

By Hölder's inequality, there exists a constant C_1 such that, for all $v\in H^1(\mathbb{R}^2)$,

$$\|\widetilde{R}_h^{\mathrm{new}} v\|_{L^2(\mathbb{R})}\le C_1(\|v\|_{L^2(\mathbb{R}^2)}+\|\partial_\sigma v\|_{L^2(\mathbb{R}^2)}). \tag{7-58}$$

Thus, by Proposition 7.1 and Remark 7.2,

$$\|\widetilde{R}_h^{\mathrm{new}}\mathcal{P}_h^{\mathrm{new}} v_{h,n}-(h^{-1}\lambda_n(h)-\beta_a)\widetilde{R}_h^{\mathrm{new}} v_{h,n}\|_{L^2(\mathbb{R})}=\mathcal{O}(h^\infty), \tag{7-59}$$

where $\mathcal{P}_h^{\mathrm{new}}$ is the operator in (4-6).

Step 2: We prove the estimate

$$\big\langle\big(c_2(a)\widetilde{R}_h^{\mathrm{new}}\mathcal{P}_h^{\mathrm{new}}-M_3(a)k_{\max}h^{1/2}R_0^{\mathrm{new}}-h^{3/4}H_a^{\mathrm{harm}}R_0^{\mathrm{new}}\big)v_{h,n}, R_0^{\mathrm{new}}v_{h,n}\big\rangle_{L^2(\mathbb{R})}=o(h^{3/4}). \tag{7-60}$$

We first observe that it results from (7-1), (7-10), (7-56), and (7-57),

$$\|\widetilde{R}_h^{\mathrm{new}} v_{h,n}-R_0^- v_{h,n}\|_{L^2(\mathbb{R})}=\mathcal{O}(h^{1/2}). \tag{7-61}$$

For the sake of simplicity, we write $\kappa=k(0)=k_{\max}$. We introduce the functions in $L^2(\mathbb{R})$:

$$f_1=2\mathfrak{R}_a((b_a(\tau)\tau+\zeta_a)\phi_a) \tag{7-62}$$

and (see (7-29) and (7-33))

$$f_2=\phi_a^{\mathrm{cor}}=\mathfrak{R}_a\big(M_3(a)\phi_a-\phi_a'-2\tau(b_a(\tau)\tau+\zeta_a)^2\phi_a+b_a(\tau)\tau^2(b_a(\tau)\tau+\zeta_a)\phi_a\big). \tag{7-63}$$

Recall the operators P_0, P_1, P_2, P_3, Q_h introduced in (4-10) and (4-11). Noticing the decomposition in (4-9), we write, for any function v with compact support in $\mathbb{R}^2$,

$$\begin{aligned}\widetilde{R}_h^{\mathrm{new}}\mathcal{P}_h^{\mathrm{new}}v&=\int_{\mathbb{R}}\phi_a(\tau)P_0v(\sigma,\tau)\,d\tau+h^{3/8}\int_{\mathbb{R}}(if_1(\tau)\partial_\sigma P_0+\phi_a(\tau)P_1)v(\sigma,\tau)\,d\tau\\&\quad+h^{1/2}\int_{\mathbb{R}}(\phi_a(\tau)P_2+\kappa f_2(\tau)P_0)v(\sigma,\tau)\,d\tau\\&\quad+h^{3/4}\int_{\mathbb{R}}(\phi_a(\tau)P_3+if_1(\tau)\partial_\sigma P_1)v(\sigma,\tau)\,d\tau+\mathrm{R}_{h,n}v,\end{aligned}\tag{7-64}$$

where

$$\begin{aligned}\mathrm{R}_{h,n}v&=h^{7/8}\widetilde{R}_h^{\mathrm{new}}Q_hv+h^{7/8}\int_{\mathbb{R}}(if_1(\tau)\partial_\sigma P_2+\kappa f_2(\tau)P_0)v(\sigma,\tau)\,d\tau+h\kappa\int_{\mathbb{R}}f_2(\tau)P_2v(\sigma,\tau)\,d\tau\\&\quad+h^{5/4}\kappa\int_{\mathbb{R}}f_2(\tau)P_3v(\sigma,\tau)\,d\tau+h^{9/8}\kappa\int_{\mathbb{R}}if_1(\tau)\partial_\sigma P_3v(\sigma,\tau)\,d\tau.\end{aligned}\tag{7-65}$$

We now compute the first three terms on the right side of (7-64):

For the first term, since P_0 is self-adjoint in $L^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}}\phi_a(\tau)P_0v(\sigma,\tau)\,d\tau=\int_{\mathbb{R}}P_0\phi_a(\tau)v(\sigma,\tau)\,d\tau=0.$$

For the second term, we have

$$\begin{aligned}\int_{\mathbb{R}}if_1(\tau)\partial_\sigma P_0v(\sigma,\tau)\,d\tau&=\int_{\mathbb{R}}iP_0f_1(\tau)\partial_\sigma v(\sigma,\tau)\,d\tau\\&=\int_{\mathbb{R}}2i\phi_a(\tau)(b_a(\tau)\tau+\zeta_a)\partial_\sigma v(\sigma,\tau)\,d\tau.\end{aligned}$$

Hence we find, by (4-10),

$$\int_{\mathbb{R}}\big(if_1(\tau)\partial_\sigma P_0+\phi_a(\tau)P_1\big)v(\sigma,\tau)\,d\tau=0.$$

For the third term, noticing that

$$P_0f_2=M_3(a)\phi_a-\phi_a'-2\tau(b_a(\tau)\tau+\zeta_a)^2\phi_a+b_a(\tau)\tau^2(b_a(\tau)\tau+\zeta_a)\phi_a$$

and

$$\int_{\mathbb{R}}\phi_a(\tau)P_2v(\sigma,\tau)\,d\tau=\kappa\int_{\mathbb{R}}\big(-\phi_a'(\tau)+2\tau(b_a(\tau)\tau+\zeta_a)^2\phi_a(\tau)-b_a(\tau)\tau^2(b_a(\tau)\tau+\zeta_a)\phi_a(\tau)\big)v\,d\tau,$$

we get

$$\begin{aligned}(W_2v)(\sigma):=&\int_{\mathbb{R}}\big(\phi_a(\tau)P_2+\kappa f_2(\tau)P_0\big)v(\sigma,\tau)\,d\tau\\=&\int_{\mathbb{R}}\big(\phi_a(\tau)P_2+\kappa(P_0f_2(\tau))\big)v(\sigma,\tau)\,d\tau\\=&\,\kappa\int_{\mathbb{R}}\big(M_3(a)\phi_a(\tau)-2\phi_a'(\tau)\big)v(\sigma,\tau)\,d\tau.\end{aligned}$$

By the forgoing computations, (7-64) becomes

$$\widetilde{R}_h^{\text{new}}\mathcal{P}_h^{\text{new}}v = h^{1/2}W_2v + h^{3/4}W_3v + \mathrm{R}_{h,n}v, \tag{7-66}$$

with

$$(W_3v)(\sigma) := \int_{\mathbb{R}} \big(\phi_a(\tau)P_3 + if_1(\tau)\partial_\sigma P_1\big)v(\sigma,\tau)\,d\tau. \tag{7-67}$$

We estimate $W_2v_{h,n}$ by writing $v_{h,n} = \Pi_0 v_{h,n} + (v_{h,n} - \Pi_0 v_{h,n})$, with Π_0 the projection introduced in (7-4), and by using (7-41). Eventually, since $P_0\Pi_0 = 0$ and $\langle \phi_a, \phi_a'\rangle_{L^2(\mathbb{R})} = 0$, we get by Remark 2.3,

$$\|W_2v_{h,n} - M_3(a)\kappa R_0^- v_{h,n}\|_{L^2(\mathbb{R})} = o(h^{1/4}). \tag{7-68}$$

We still have to estimate the terms involving W_3 and $\mathrm{R}_{h,n}$ in (7-66) when $v = v_{h,n}$. By choosing η small enough, the error term

$$\mathrm{r}_n(\sigma,h) := \mathrm{R}_{h,n}v_{h,n}, \tag{7-69}$$

with $\mathrm{R}_{h,n}$ introduced in (7-65), satisfies

$$\langle \mathrm{r}_n(\,\cdot\,,h),\, R_0^{\text{new}}v_{h,n}\rangle_{L^2(\mathbb{R})} = o(h^{3/4}). \tag{7-70}$$

The technical proof of (7-70) is given in Appendix B. So we are left (see (7-67)) with estimating

$$W_3v_{h,n} = w_1 + w_2, \tag{7-71}$$

where

$$w_1(\sigma) := \int_{\mathbb{R}} \phi_a(\tau)P_3v_{h,n}(\sigma,\tau)\,d\tau,$$
$$w_2(\sigma) := \int_{\mathbb{R}} if_1(\tau)\partial_\sigma P_1v_{h,n}(\sigma,\tau)\,d\tau.$$

In light of the definition of P_3 in (4-10) and R_0^- in (7-1), we write

$$w_1(\sigma) = -\partial_\sigma^2 R_0^- v_{h,n}(\sigma) + \frac{k''(0)\sigma^2}{2}w(\sigma),$$

where

$$w(\sigma) = \int_{\mathbb{R}} \big(\partial_\tau + 2\tau(b_a(\tau)\tau + \zeta_a)^2 - b_a(\tau)\tau(b_a(\tau)\tau + \zeta_a)\big)\phi_a(\tau)\,v_{h,n}(\sigma,\tau)\,d\tau.$$

Using Proposition 7.1 and that $v_{h,n}$ is supported in $\{|\sigma| \le h^{-\eta}\}$, we get

$$\|\sigma^2(w - M_3(a)R_0^- v_{h,n})\|_{L^2(\mathbb{R})} = \mathcal{O}(h^{1/4-2\eta}).$$

Hence

$$\left\| w_1 - \left(-\partial_\sigma^2 + \frac{k''(0)M_3(a)}{2}\sigma^2\right)R_0^- v_{h,n}\right\|_{L^2(\mathbb{R})} = \mathcal{O}(h^{1/4-2\eta}). \tag{7-72}$$

Furthermore, by (4-10) and (7-62), the term w_2 can be expressed as

$$\begin{aligned} w_2(\sigma) &= 2\partial_\sigma^2 \int_{\mathbb{R}} f_1(\tau)(\zeta_a + b_a(\tau)\tau)v_{h,n}(\sigma,\tau)\,d\tau \\ &= 4\partial_\sigma^2 \int_{\mathbb{R}} (b_a(\tau)\tau + \zeta_a)\mathfrak{R}_a\big((b_a(\tau)\tau + \zeta_a)\phi_a(\tau)\big)v_{h,n}(\sigma,\tau)\,d\tau. \end{aligned} \tag{7-73}$$

Collecting (7-72) and (7-73), along with the definition of R_0^{new} in (7-53), we infer from (7-71)

$$\left\| W_3 v_{h,n} - \left(-\partial_\sigma^2 R_0^{\text{new}} + \frac{k''(0)M_3(a)}{2}\sigma^2 R_0^- \right) v_{h,n} \right\|_{L^2(\mathbb{R})} = \mathcal{O}(h^{1/4-2\eta}). \tag{7-74}$$

By Hölder's inequality, we infer from (7-68) and (7-74)

$$\begin{aligned} &h^{1/2}\langle (W_2 - M_3(a)\kappa R_0^-) v_{h,n},\, R_0^{\text{new}} v_{h,n} \rangle_{L^2(\mathbb{R})} \\ &\quad + h^{3/4}\left\langle W_3 v_{h,n} - \left(-\partial_\sigma^2 R_0^{\text{new}} + \frac{k''(0)M_3(a)}{2}\sigma^2 R_0^- \right) v_{h,n},\, R_0^{\text{new}} v_{h,n} \right\rangle_{L^2(\mathbb{R})} = o(h^{3/4}) \| R_0^{\text{new}} v_{h,n} \|_{L^2(\mathbb{R})}. \end{aligned}$$

By (7-66) and (7-70), we get from the above estimate

$$\big\langle \big(\widetilde{R}_h^{\text{new}} \mathcal{P}_h^{\text{new}} - h^{1/2} M_3(a)\kappa R_0^- - h^{3/4} \widetilde{H} \big) v_{h,n},\, R_0^{\text{new}} v_{h,n} \big\rangle_{L^2(\mathbb{R})} = o(h^{3/4}) \| R_0^{\text{new}} v_{h,n} \|_{L^2(\mathbb{R})},$$

where

$$\widetilde{H} := -\partial_\sigma^2 R_0^{\text{new}} + \frac{k''(0)M_3(a)}{2}\sigma^2 R_0^-.$$

Finally, by item (1) and Proposition 2.5, we get (7-60).

Step 3: Using Steps 1 and 2, we are now able to finish the proof of item (4). By (1-12) and (2-20), $c_2(a) = 1 - 4I_2(a)$; hence (7-61) and item (1) yield that

$$\| c_2(a) \widetilde{R}_h^{\text{new}} v_{h,n} - R_0^{\text{new}} v_{h,n} \|_{L^2(\mathbb{R})} = \mathcal{O}(h^{1/4}). \tag{7-75}$$

Collecting (7-59), (7-60) and (7-75), we get

$$\big\langle h^{3/4} H_a^{\text{harm}} R_0^{\text{new}} v_{h,n} - \Lambda_n(h) R_0^{\text{new}} v_{h,n},\, R_0^{\text{new}} v_{h,n} \big\rangle_{L^2(\mathbb{R})} = \mathcal{O}(|\Lambda_n(h)| h^{1/4}) + o(h^{3/4}),$$

where, by (6-4) and Theorem 4.1,

$$|\Lambda_n(h)| = |h^{-1}\lambda_n(h) - \beta_a - M_3(a) k_{\max} h^{1/2}| = o(h^{1/2}).$$

Thus, we obtain

$$\big\langle h^{3/4} H_a^{\text{harm}} R_0^{\text{new}} v_{h,n} - \Lambda_n(h) R_0^{\text{new}} v_{h,n},\, R_0^{\text{new}} v_{h,n} \big\rangle_{L^2(\mathbb{R})} = o(h^{3/4}).$$

Dividing by $h^{3/4}$ and using item (2), we get item (4). $\square$

With Proposition 7.5 in hand, we can now finish the proof of Theorem 1.2.

Proof of Theorem 1.2. The upper bound of $\lambda_n(h)$ follows from Theorem 4.1. For the lower bound of $\lambda_n(h)$, consider $u = \sum_{k=1}^n a_k R_0^{\text{new}} v_{h,k}$ such that $\|u\|_{L^2(\mathbb{R})} = 1$, where R_0^{new} is introduced in (7-53). From Proposition 7.5 we have

$$((1 - 4I_2(a))^2 + o(1)) \sum_{k=1}^n |a_k|^2 = 1$$

and

$$\langle (H_a^{\text{harm}} - h^{-3/4}\Lambda_n(h)) u,\, u \rangle_{L^2(\mathbb{R})} \le o(1) \sum_{k=1}^n |a_k|^2.$$

Consequently,

$$\max_{u\in M_n,\ \|u\|=1}\langle (H_a^{\mathrm{harm}} - h^{-3/4}\Lambda_n(h))u, u\rangle_{L^2(\mathbb{R})} = o(1),$$

where M_n is the space defined in (7-55). By the min-max principle

$$\sqrt{\frac{M_3(a)k''(0)c_2(a)}{2}}(2n-1) \le h^{-3/4}\Lambda_n(h) + o(1),$$

thereby leading to

$$\lambda_n(h) \ge \beta_a h + M_3(a)k_{\max}h^{3/2} + \sqrt{\frac{M_3(a)k''(0)c_2(a)}{2}}(2n-1)h^{7/4} + o(h^{7/4}). \qquad \square$$

Appendix A: Frenet coordinates near the magnetic edge

We introduce the Frenet coordinates near Γ. We refer the reader to [Fournais and Helffer 2010, Appendix F] and [Assaad et al. 2019] for a similar setup.

Let $s \mapsto M(s) \in \Gamma$ be the arc length parametrization of Γ such that

- $\nu(s)$ is the unit normal of Γ at the point $M(s)$ pointing towards P_1,
- $T(s)$ is the unit tangent vector of Γ at the point $M(s)$, such that $(T(s), \nu(s))$ is a direct frame, i.e., $\det(T(s), \nu(s)) = 1$.

We define the curvature k of Γ as $T'(s) = k(s)\nu(s)$. *Working under Assumption 1.1*, we assume without loss of generality that $s_0 = 0$, where s_0 is the unique maximum of the curvature at Γ ($k(0) = k_{\max}$).

For $t_0 > 0$, we define the transformation $\Phi = \Phi_{t_0}$ as

$$\Phi : \mathbb{R}\times(-t_0, t_0) \to \Gamma_{t_0} := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \Gamma) < t_0\}, \quad (s, t) \mapsto M(s) + t\nu(s). \tag{A-1}$$

We pick t_0 sufficiently small so that Φ is a diffeomorphism, whose Jacobian is

$$\mathfrak{a}(s, t) := J_\Phi(s, t) = 1 - t\,k(s). \tag{A-2}$$

We consider the following correspondence between functions u in $H^1_{\mathrm{loc}}(\Gamma_{t_0})$ and those $\tilde{u}$ in $H^1_{\mathrm{loc}}(\mathbb{R}\times(-t_0, t_0))$:

$$\tilde{u}(s, t) = u(\Phi(s, t)), \tag{A-3}$$

and vice versa.

Moreover, we assign to the potential $\boldsymbol{F}$ in (1-1) a vector field $\widetilde{\boldsymbol{F}} \in H^1_{\mathrm{loc}}(\mathbb{R}\times(-t_0, t_0))$ as

$$\boldsymbol{F}(x) = (F_1(x), F_2(x)) \mapsto \widetilde{\boldsymbol{F}}(s, t) = (\widetilde{F}_1(s, t), \widetilde{F}_2(s, t)),$$

where

$$\widetilde{F}_1(s, t) = \mathfrak{a}(s, t)\boldsymbol{F}(\Phi(s, t))\cdot T(s) \quad \text{and} \quad \widetilde{F}_2(s, t) = \boldsymbol{F}(\Phi(s, t))\cdot\nu(s). \tag{A-4}$$

Consequently,

$$(h\nabla - i\boldsymbol{F})^2 = \mathfrak{a}^{-1}(h\partial_s - i\widetilde{F}_1)\mathfrak{a}^{-1}(h\partial_s - i\widetilde{F}_1) + \mathfrak{a}^{-1}(h\partial_t - i\widetilde{F}_2)\mathfrak{a}(h\partial_t - i\widetilde{F}_2). \tag{A-5}$$

Note that

$$\operatorname{curl}\widetilde{\boldsymbol{F}}(s, t) = (1 - tk(s))\operatorname{curl}\boldsymbol{F}(\Phi(s, t)) = (1 - tk(s))(\mathbb{1}_{\{t>0\}} + a\mathbb{1}_{\{t<0\}}), \tag{A-6}$$

where $\operatorname{curl}\widetilde{\boldsymbol{F}} = \partial_s\widetilde{F}_2 - \partial_t\widetilde{F}_1$ and $\operatorname{curl}\boldsymbol{F} = \partial_{x_1}F_2 - \partial_{x_2}F_1$ is as in (1-2).

Furthermore, we present the change of variable formulas (for functions compactly supported in Γ_{t_0}):

$$\begin{aligned}\int_{\Gamma_{t_0}} |u|^2\,dx &= \int_{\mathbb{R}}\int_{-t_0}^{t_0} |\tilde{u}|^2\,\mathfrak{a}\,dt\,ds,\\ \int_{\Gamma_{t_0}} |(h\nabla - i\boldsymbol{F})u|^2\,dx &= \int_{\mathbb{R}}\int_{-t_0}^{t_0} \big(\mathfrak{a}^{-2}|(h\partial_s - i\widetilde{F}_1)\tilde{u}|^2 + |(h\partial_t - i\widetilde{F}_2)\tilde{u}|^2\big)\mathfrak{a}\,dt\,ds.\end{aligned} \tag{A-7}$$

Now, we make a global change of gauge ω as follows:

Lemma A.1. *There exists a function $\omega \in H^2(\Phi^{-1}(\Gamma_{t_0}\cap\Omega))$ such that*

$$\widetilde{\boldsymbol{F}} - \nabla_{s,t}\omega = \begin{pmatrix} -b_a(t)\big(t - \frac{1}{2}t^2k(s)\big) \\ 0 \end{pmatrix} \quad \textit{in } \Phi^{-1}(\Gamma_{t_0}\cap\Omega),$$

where $t \mapsto b_a(t)$ is defined by $b_a(t) = \mathbb{1}_{\{t>0\}} + a\,\mathbb{1}_{\{t<0\}}$.

Proof. For $(s,t) \in \Phi^{-1}(\Gamma_{t_0}\cap\Omega)$, let $\omega(s,t) = \int_0^t \widetilde{F}_2(s,t')\,dt' + \int_0^s \widetilde{F}_1(s',0)\,ds'$. This choice of ω and (A-6) establish the lemma. □

The gauge of Lemma A.1 is adequate when working with functions localized near the edge Γ. With this choice of gauge, we have the following identity which is useful to analyze the decay of functions localized near Γ.

Lemma A.2. *Assume that $\varphi \in H^2(\Omega)$ with compact support in $\Omega \cap \Gamma_{t_0}$. Let g and G be the functions defined* (*by means of* (A-3)) *as*

$$\tilde{g}(s,t) = (h^{1/2}\partial_s - i\zeta_a)\tilde{\varphi}(s,t) \quad \textit{and} \quad \widetilde{G}(s,t) = -(h^{1/2}\partial_s - i\zeta_a)(e^{2\tilde{\phi}}\tilde{g}),$$

where ζ_a is the constant in Section 2B and ϕ is a Lipschitz real-valued function on Ω. If $g \in H^2(\Omega)$, then

$$\operatorname{Re}\langle \mathcal{P}_h\varphi, G\rangle_{L^2(\Omega)} = \mathcal{Q}_h(e^{\phi}g) - h^2\||\nabla g|e^{\phi}\varphi\|^2_{L^2(\Omega)} - h^{1/2}\operatorname{Re}(\mathsf{T}_h).$$

Here $\mathcal{Q}_h$ is the quadratic form introduced in (4-11) *and*

$$\mathsf{T}_h = \big\langle (h\partial_s - i\widetilde{F}_1)\big((\partial_s\mathfrak{a}^{-1} - i\mathfrak{a}^{-1}\partial_s\widetilde{F}_1)(h\partial_s - i\widetilde{F}_1)\tilde{\varphi} - i\mathfrak{a}^{-1}(\partial_s\widetilde{F}_1)\tilde{\varphi}\big) + h^2\partial_t(\partial_s\mathfrak{a})\partial_t\tilde{\varphi}, e^{2\tilde{\phi}}\tilde{g}\big\rangle_{L^2(\mathbb{R})}.$$

Proof. We assume that $\widetilde{F}_2 = 0$ and get from (A-5) and (A-2)

$$\langle \mathcal{P}_h\varphi, G\rangle_{L^2(\Omega)} = \langle (h\partial_s - i\widetilde{F}_1)\mathfrak{a}^{-1}(h\partial_s - i\widetilde{F}_1)\varphi + h^2\partial_t\mathfrak{a}\partial_t\varphi, (h^{1/2}\partial_s - i\zeta_a)(e^{2\phi}g)\rangle_{L^2(\mathbb{R}^2)}, \tag{A-8}$$

where we dropped the tildes from the notation for the sake of simplicity. Notice that

$$\begin{aligned}(h^{1/2}\partial_s - i\zeta_a)\partial_t\mathfrak{a}\partial_t\varphi &= \partial_t((h^{1/2}\partial_s - i\zeta_a)\mathfrak{a}\partial_t\varphi)\\ &= \partial_t(\mathfrak{a}\partial_t(h^{1/2}\partial_s - i\zeta_a)\varphi) + h^{1/2}\partial_t(\partial_s\mathfrak{a})\partial_t\varphi\\ &= \partial_t\mathfrak{a}\partial_t g + h^{1/2}\partial_t(\partial_s\mathfrak{a})\partial_t\varphi,\end{aligned}$$

and

$$
\begin{aligned}
&(h^{1/2}\partial_s - i\zeta_a)(h\partial_s - i\widetilde{F}_1)\mathfrak{a}^{-1}(h\partial_s - i\widetilde{F}_1)\varphi \\
&\quad = (h\partial_s - i\widetilde{F}_1)\big((h^{1/2}\partial_s - i\zeta_a) - ih^{1/2}(\partial_s\widetilde{F}_1)\big)\mathfrak{a}^{-1}(h\partial_s - i\widetilde{F}_1)\varphi \\
&\quad = (h\partial_s - i\widetilde{F}_1)\big(\mathfrak{a}^{-1}(h\partial_s - i\widetilde{F}_1)(h^{1/2}\partial_s - i\zeta_a)\varphi - ih^{1/2}(\partial_s\widetilde{F}_1)\mathfrak{a}^{-1}(h\partial_s - i\widetilde{F}_1)\varphi\big) \\
&\qquad\qquad + h^{1/2}(h\partial_s - i\widetilde{F}_1)\big((\partial_s\mathfrak{a}^{-1})(h\partial_s - i\widetilde{F}_1)\varphi - i\mathfrak{a}^{-1}(\partial_s\widetilde{F}_1)\varphi\big) \\
&\quad = (h\partial_s - i\widetilde{F}_1)\mathfrak{a}^{-1}(h\partial_s - i\widetilde{F}_1)g + h^{1/2}(h\partial_s - i\widetilde{F}_1)\big((\partial_s\mathfrak{a}^{-1} - i\mathfrak{a}^{-1}\partial_s\widetilde{F}_1)(h\partial_s - i\widetilde{F}_1)\varphi - i\mathfrak{a}^{-1}(\partial_s\widetilde{F}_1)\varphi\big).
\end{aligned}
$$

By integration by parts, we infer from (A-8)

$$
\langle \mathcal{P}_h\varphi, G\rangle_{L^2(\Omega)} = \langle \mathcal{P}_h g, e^{2\phi}g\rangle_{L^2(\Omega)} - h^{1/2}\mathsf{T}_h. \tag{A-9}
$$

Finally, by integration by parts, we get

$$
\operatorname{Re}\langle \mathcal{P}_h g, e^{2\phi}g\rangle_{L^2(\Omega)} = \mathcal{Q}_h(e^{\phi}g) - h^2\||\nabla\phi|e^{\phi}g\|^2_{L^2(\Omega)}. \qquad \square
$$

Appendix B: Control of a remainder term

The aim of this appendix is to prove the estimate in (7-70). We fix a positive integer $n \geq 1$ and two positive constants $\eta \in \big(0, \frac{1}{8}\big)$ and $\delta \in \big(0, \frac{1}{12}\big)$.

For all $h > 0$, let $v_{h,n}$ be the function introduced in (7-6) which is supported in $\{|\sigma| < h^{-\eta},\ |\tau| < h^{-\delta}\}$. Moreover, by (7-6) and Propositions 6.2 and 6.3, we observe that,

$$
\text{for all } \theta \in \big(0, \tfrac{3}{8}\big), \text{ there exists } C_\theta > 0 \text{ such that } \|\partial_\sigma^j v_{h,n}\|_{L^2(\mathbb{R}^2)} \leq C_\theta h^{-j\theta} \quad (0 \leq j \leq 2). \tag{B-1}
$$

Consider two functions $f \in L^2(\mathbb{R})$ and $p \in L^1_{\text{loc}}(\mathbb{R}^2)$ so that,

$$
\text{for all } \alpha \geq 1, \quad \tau^\alpha f(\tau) \in L^2(\mathbb{R}),
$$

and there exist $k \geq 1$ and C such that

$$
|p(\sigma,\tau)| \leq C(|\sigma|^k + |\tau|^k + 1) \quad (\sigma, \tau \in \mathbb{R}).
$$

For $j \in \{0, 1, 2\}$, we introduce the function

$$
w_j(\sigma) = \int_{\mathbb{R}} f(\tau)p(\sigma,\tau)\partial_\sigma^j v_{h,n}(\sigma,\tau)\,d\tau, \tag{B-2}
$$

whose support is included in $\{|\sigma| < h^{-\eta}\}$, by the considerations on the support of $v_{h,n}$.

Lemma B.1. *Given $\eta \in \big(0, \frac{1}{8}\big)$, there exist two positive constants $h_0, C > 0$ such that*

$$
\|w_j\|_{L^2(\mathbb{R})} \leq C\, h^{-(k+j/2)\eta}
$$

for all $h \in (0, h_0)$ and $j \in \{0, 1, 2\}$.

Proof. By Hölder's inequality

$$
|w_j(\sigma)|^2 \leq \bigg(\int_{\mathbb{R}} |f(\tau)|^2|p(\sigma,\tau)|^2\,d\tau\bigg)\bigg(\int_{\mathbb{R}} |\partial_\sigma^j v_{h,n}(\sigma,\tau)|^2\,d\tau\bigg). \tag{B-3}
$$

For σ in the support of w_j, we have

$$\int_{\mathbb{R}} |f(\tau)|^2 |p(\sigma,\tau)|^2 \, d\tau \le C \int_{\mathbb{R}} |f(\tau)|^2 (1+|\tau|^k+|\sigma|^k)^2 \, d\tau \le \widetilde{C}_k (1+h^{-2k\eta}).$$

Inserting this into (B-3) then integrating with respect to σ, we get

$$\int_{\mathbb{R}} |w_j(\sigma)|^2 \, d\sigma \le \widetilde{C}_k (1+h^{-2k\eta}) \int_{\mathbb{R}^2} |\partial_\sigma^j v_{h,n}(\sigma,\tau)|^2 \, d\sigma \, d\tau.$$

Finally, we use (B-1) with $\theta = \eta$. □

We will encounter functions of the form

$$\mathfrak{w}_j(\sigma) = \int_{\mathbb{R}} g(\tau) q(\sigma) \partial_\tau^j v_{h,n}(\sigma,\tau) \, d\tau \quad (j \in \{1,2\},\ \sigma \in \mathbb{R}), \tag{B-4}$$

where $g \in H^j(\mathbb{R})$ and $q \in H^1_{\mathrm{loc}}(\mathbb{R})$ satisfy,

$$\text{for all } \alpha \ge 1, \quad \tau^\alpha g^{(i)}(\tau) \in L^2(\mathbb{R}) \ \ (1 \le i \le j),$$

and

$$\text{there exists } k \ge 1 \text{ such that there exists } C_k > 0 \text{ such that } \ |q(\sigma)| \le C_k (1+|\sigma|^k) \ \ (\sigma \in \mathbb{R}).$$

Lemma B.2. *Given $\eta \in \big(0, \frac{1}{8}\big)$, there exist two positive constants h_0 and C such that*

$$\|\mathfrak{w}_j\|_{L^2(\mathbb{R})} \le C h^{-(k+1)\eta}$$

for all $h \in (0, h_0]$ and $j \in \{1,2\}$.

Proof. Using integration by parts and that $v_{h,n}$ is with compact support, we get

$$\mathfrak{w}_j(\sigma) = (-1)^j \int_{\mathbb{R}} g^{(j)}(\tau) q(\sigma) v_{h,n}(\sigma,\tau) \, d\tau.$$

This function has the form of functions in Lemma B.1, with $f(\tau) = g^{(j)}(\tau)$ and $p(\sigma,\tau) = q(\sigma)$. □

The inner product of the remainder, $\mathrm{r}_n(\sigma,h)$ in (7-69), and the function, $R_0^{\mathrm{new}} v_{h,n}$ in (7-53), can be expressed as the inner product of a linear combination of functions having the forms in Lemmas B.1 and B.2. The polynomials we encounter are of degree 6 at most. More precisely,

$$\langle \mathrm{r}_n(\,\cdot\,,h), R_0^{\mathrm{new}} v_{h,n} \rangle_{L^2(\mathbb{R})} = h^{7/8} A_1 + h^{7/8} A_2 + h A_3 + h^{9/8} A_4 + h^{5/4} A_5,$$

where

$$A_1 = \langle a_{1,1}, b_1 \rangle_{L^2(\mathbb{R})} + h^{3/8} \langle a_{1,2}, b_2 \rangle_{L^2(\mathbb{R})} + h^{1/2} \langle a_{1,3}, b_1 \rangle_{L^2(\mathbb{R})},$$
$$A_2 = \langle a_{2,1}, b_2 \rangle_{L^2(\mathbb{R})} + \langle a_{2,2}, b_1 \rangle_{L^2(\mathbb{R})},$$
$$A_3 = \langle a_3, b_1 \rangle_{L^2(\mathbb{R})}, \quad A_4 = \langle a_4, b_2 \rangle_{L^2(\mathbb{R})}, \quad A_5 = \langle a_5, b_1 \rangle_{L^2(\mathbb{R})},$$

and

$$a_{1,1} = \int_{\mathbb{R}} g_1(\tau) Q_h v_{h,n} \, d\tau, \quad a_{1,2} = \int_{\mathbb{R}} g_2(\tau) Q_h v_{h,n} \, d\tau, \quad a_{1,3} = \int_{\mathbb{R}} g_3(\tau) Q_h v_{h,n} \, d\tau,$$
$$a_{2,1} = \int_{\mathbb{R}} f_1(\tau) P_2 v_{h,n} d\tau, \quad a_{2,2} = \kappa \int_{\mathbb{R}} f_2(\tau) P_0 v_{h,n} \, d\tau,$$

$$a_3 = \kappa \int_{\mathbb{R}} f_2(\tau) P_2 v_{h,n}\, d\tau, \quad a_4 = \kappa \int_{\mathbb{R}} f_1(\tau) P_3 v_{h,n}\, d\tau, \quad a_5 = \kappa \int_{\mathbb{R}} f_2(\tau) P_3 v_{h,n}\, d\tau,$$
$$b_1 = \int_{\mathbb{R}} g(\tau) v_{h,n}\, d\tau, \quad b_2 = i \int_{\mathbb{R}} g(\tau) \partial_\sigma v_{h,n}\, d\tau.$$

Here, Q_h is the operator introduced in (4-11), P_0, P_1, P_2, P_3 are the operators introduced in (4-10), f_1, f_2 are the functions introduced in (7-62)-(7-63), the functions g_1, g_2, g_3 and g are defined as follows (see (7-57) and (7-53))

$$g_1 = \phi_a, \quad g_2 = f_1 = 2\mathfrak{R}_a((b_a(\tau)\tau + \zeta_a)\phi_a),$$
$$g_3 = \kappa f_2 = \kappa \mathfrak{R}_a\big(M_3(a)\phi_a - \phi_a' - 2\tau(b_a(\tau)\tau + \zeta_a)^2\phi_a + b_a(\tau)\tau^2(b_a(\tau)\tau + \zeta_a)\phi_a\big),$$
$$g = \phi_a - 4(b_a(\tau)\tau + \zeta_a)\mathfrak{R}_a((b_a(\tau)\tau + \zeta_a)\phi_a).$$

So, we get

$$\langle \mathrm{r}_n(\,\cdot\,, h), R_0^{\mathrm{new}} v_{h,n}\rangle_{L^2(\mathbb{R})} = \mathcal{O}(h^{7/8-8\eta}).$$

By choosing $\eta < \frac{1}{64}$, we get (7-70).

Acknowledgments

Kachmar is partially supported by the Center for Advanced Mathematics Sciences (CAMS, American University of Beirut).

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Received 6 Sep 2021. Revised 25 Mar 2022. Accepted 11 Jul 2022.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

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Analysis & PDE

Volume 17 No. 2 2024