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# NECESSARY DENSITY CONDITIONS FOR SAMPLING AND INTERPOLATION IN SPECTRAL SUBSPACES OF ELLIPTIC DIFFERENTIAL OPERATORS

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# NECESSARY DENSITY CONDITIONS FOR SAMPLING AND INTERPOLATION IN SPECTRAL SUBSPACES OF ELLIPTIC DIFFERENTIAL OPERATORS

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We prove necessary density conditions for sampling in spectral subspaces of a second-order uniformly elliptic differential operator on $\mathbb{R}^d$ with slowly oscillating symbol. For constant-coefficient operators, these are precisely Landau's necessary density conditions for bandlimited functions, but for more general elliptic differential operators it has been unknown whether such a critical density even exists. Our results prove the existence of a suitable critical sampling density and compute it in terms of the geometry defined by the elliptic operator. In dimension $d = 1$, functions in a spectral subspace can be interpreted as functions with variable bandwidth, and we obtain a new critical density for variable bandwidth. The methods are a combination of the spectral theory and the regularity theory of elliptic partial differential operators, some elements of limit operators, certain compactifications of $\mathbb{R}^d$, and the theory of reproducing kernel Hilbert spaces.

## 1. Introduction

The classical Paley–Wiener space is the subspace $\mathrm{PW}_\Omega = \{f \in L^2(\mathbb{R}) : \mathrm{supp}\,\hat{f} \subseteq [-\Omega, \Omega]\}$ of $L^2(\mathbb{R})$. Using Fourier inversion, one sees that the point evaluation $f \mapsto f(x)$ is bounded on $\mathrm{PW}_\Omega$. The fundamental questions about $\mathrm{PW}_\Omega$ are originally motivated by problems in signal processing and information theory: when is $f \in \mathrm{PW}_\Omega$ completely and stably determined by its samples $\{f(s) : s \in S\}$ on a set $S \subseteq \mathbb{R}$? On which sets $S \subseteq \mathbb{R}$ can every sequence $(a_s)_{s \in S} \in \ell^2(S)$ be interpolated by a function $f$ in $\mathrm{PW}_\Omega$, so that $f(s) = a_s$ for all $s \in S$? These questions were answered by Beurling [1989] and Landau [1967].

**Theorem A.** (i) *Assume that $S$ is uniformly separated and*

$$A\|f\|_2^2 \leq \sum_{s \in S} |f(s)|^2 \leq B\|f\|_2^2 \quad \text{for all } f \in \mathrm{PW}_\Omega. \tag{1-1}$$

*Then*

$$D^-(S) = \liminf_{r \to \infty} \inf_{x \in \mathbb{R}} \frac{\#(S \cap [x-r, x+r])}{2r} \geq \frac{\Omega}{\pi}. \tag{1-2}$$

(ii) *If for all $a \in \ell^2(S)$ there exists $f \in \mathrm{PW}_\Omega$ such that $f(s) = a_s$, $s \in S$, then*

$$D^+(S) = \limsup_{r \to \infty} \sup_{x \in \mathbb{R}} \frac{\#(S \cap [x-r, x+r])}{2r} \leq \frac{\Omega}{\pi}. \tag{1-3}$$

In the established terminology, a set that satisfies a sampling inequality of the form (1-1) is called a sampling set for the underlying space $PW_\Omega$, or a set of stable sampling. A set on which arbitrary $\ell^2$-data can be interpolated is called a set of interpolation. The expressions $D^-(S)$ and $D^+(S)$ are called the lower and the upper Beurling density.

The number $\Omega/\pi$ in (1-2) and (1-3) is an important invariant of the space $PW_\Omega$ and has an interpretation in information theory. Since, roughly speaking, the densities $D^\pm(S)$ measure the average number of samples in $S$ per unit length, the *necessary density conditions* of Theorem A say that at least $\Omega/\pi$ samples per unit length are required to recover a function in $PW_\Omega$ from $f|_S$, whereas at most $\Omega/\pi$ values per unit length are permitted to solve the interpolation problem in $PW_\Omega$. Thus the density $\Omega/\pi$ represents a critical value below which (stable) sampling is impossible, and above which interpolation is impossible. Indeed, these questions about sampling and interpolation were at the origin of Shannon's information theory [1948], and the uniform sampling theorem with $S = \alpha\mathbb{Z}$ is still considered the basis of analog-digital conversion in modern signal processing. The ratio $D^\pm(S)/\Omega$ is a measure for the redundancy, thus for the performance quality, of the sampling set $S$. The theory of Beurling, Kahane, and Landau provides a rigorous mathematical formulation for the existence of a critical density for arbitrary sets $S$ (in place of $\alpha\mathbb{Z}$). Although we will not touch this question here, we mention that the conditions of Theorem A yield almost a characterization of sets of sampling and of interpolation: *in dimension $d = 1$, if $S$ is uniformly separated and $D^-(S) > 1$, then $S$ is a sampling set for* $PW_\Omega$*, and if $D^+(S) < 1$, then $S$ is a set of interpolation for* $PW_\Omega$. See [Kahane 1962; Beurling 1989; Seip 2004] for an exposition of the sampling theory in the classical Paley–Wiener space.

The connection with partial differential operators comes from the observation that $PW_\Omega$ is a spectral subspace of the differential operator $H = -d^2/dx^2$. Using the Fourier transform $\mathscr{F}$, this differential operator is unitarily equivalent to the multiplication operator $\mathscr{F}(-d^2f/dx^2)(\xi) = \xi^2 \hat{f}(\xi)$. In this representation of $-d^2/dx^2$ the spectral projection on the interval $[0, \Omega]$ is given by $\chi_{[0,\Omega]}(H)f = \mathscr{F}^{-1}(\chi_{[0,\Omega]}(\xi^2)\hat{f})$. This implies

$$PW_\Omega = \chi_{[0,\Omega^2]}(H)L^2(\mathbb{R}).$$

This observation is the starting point for many generalizations of Paley–Wiener spaces and sampling theorems. In this work we study the question of necessary density conditions for sampling and interpolation in the spectral subspaces of a self-adjoint uniformly elliptic differential operator

$$H_a = -\sum_{j,k=1}^d \partial_j a_{jk}(x) \partial_k$$

acting on $L^2(\mathbb{R}^d)$ with a smooth positive definite (matrix) symbol $a = (a_{jk}(x))_{j,k=1,\dots,d}$. The Paley–Wiener space associated to $H_a$ is the spectral subspace

$$PW_\Omega(H_a) = \chi_{[0,\Omega]}(H_a)L^2(\mathbb{R}^d),$$

where, as usual, $\chi_{[0,\Omega]}(H_a)$ is the orthogonal projection corresponding to the spectrum $[0, \Omega]$.

If the symbol $a(x) = a$ is constant, then $H_a$ is similar to the Laplace operator, and the corresponding spectral subspace can be described with Fourier techniques. For this case necessary density conditions for sampling and interpolation are already contained in [Landau 1967]. Optimal sufficient conditions

for sampling in $\mathbb{R}^d$ in terms of a covering density were obtained in [Beurling 1966]. However, if $H_a$ is a uniformly elliptic differential operator with variable coefficients, then the standard techniques break down, and it was an open question whether a critical density exists for sampling and interpolation in the spectral subspaces of $H_a$, and how to compute this critical density.

We will answer this question for a class uniformly elliptic operators. We say a smooth symbol with all derivatives bounded, $a \in C_b^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$, is slowly oscillating if $\lim_{|x| \to \infty} |\partial_k a(x)| = 0$ for $k = 1, \ldots, d$.

**Theorem B.** *If $a$ is slowly oscillating, then there exists a critical density for sampling and interpolation for $\mathrm{PW}_\Omega(H_a)$.*

Adapting the measure to the geometry associated to the differential operator $H_a$, the critical density can be determined explicitly. This is our main result.

**Theorem C.** *Assume $H_a = -\sum_{j,k=1}^d \partial_j a_{jk} \partial_k$ is a self-adjoint uniformly elliptic operator with slowly oscillating symbol $a \in C_b^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$. Let $d\nu(x) = (\det a(x))^{-1/2} \, dx$ be the associated measure.*

(i) *If $S \subseteq \mathbb{R}^d$ is a set of stable sampling for $\mathrm{PW}_\Omega(H_a)$ then*

$$D_\nu^-(S) = \liminf_{r \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\#(S \cap B_r(x))}{\nu(B_r(x))} \geq \frac{|B_1|}{(2\pi)^d} \Omega^{d/2}.$$

(ii) *If $S \subseteq \mathbb{R}^d$ is a set of interpolation for $\mathrm{PW}_\Omega(H_a)$, then*

$$D_\nu^+(S) = \limsup_{r \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\#(S \cap B_r(x))}{\nu(B_r(x))} \leq \frac{|B_1|}{(2\pi)^d} \Omega^{d/2}.$$

Except for the modified definition of the density, the formulation of the theorem is identical to Landau's theorem [1967]. By contrast, the method of proof is vastly different as Fourier methods are not available for the proof of Theorem C. In addition we draw the new insight that the appropriate notion of density must be linked to the geometry defined by $a$. For compact manifolds the link between density and geometry was already observed in [Ortega-Cerdà and Pridhnani 2012].

For the special case of a symbol that is asymptotically constant at infinity we can use the standard Beurling densities in $\mathbb{R}^d$ from (1-2) and (1-3) (with intervals replaced by Euclidean balls $B_r(x)$) and obtain the following consequence.

**Corollary D.** *Assume that $a \in C_b^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$ is asymptotically constant, i.e., $\lim_{x \to \infty} a(x) = b$. Let $\Sigma_\Omega^b = \{\xi \in \mathbb{R}^d : b\xi \cdot \xi \leq \Omega\}$.*

(i) *If $S \subseteq \mathbb{R}^d$ is a set of sampling for the Paley–Wiener space $\mathrm{PW}_\Omega(H_a)$, then*

$$D^-(S) \geq \frac{|\Sigma_\Omega^b|}{(2\pi)^d} = (\det b)^{-1/2} \frac{|B_1|}{(2\pi)^d} \Omega^{d/2}.$$

(ii) *If $S \subseteq \mathbb{R}^d$ is a set of interpolation for the Paley–Wiener space $\mathrm{PW}_\Omega(H_a)$, then*

$$D^+(S) \leq \frac{|\Sigma_\Omega^b|}{(2\pi)^d}.$$

We note that the same critical density holds for the Paley–Wiener space of the constant-coefficient differential operator $H_b$. Since $H_a$ may be considered a perturbation of $H_b$ and since the Beurling density $D^{\pm}(S)$ is an asymptotic quantity, it is to be expected that the necessary density for $\mathrm{PW}_\Omega(H_a)$ coincides with the necessary density for $\mathrm{PW}_\Omega(H_b)$.

Let us put these statements into context.

*Sampling in spectral subspaces.* Several researchers have created an extensive qualitative theory of sampling in spectral subspaces of a general unbounded, positive, self-adjoint operator $H$ on a Hilbert space $\mathcal{H}$. In this case the abstract Paley–Wiener space is defined as $\mathrm{PW}_{[0,\Omega]}(H) = \chi_{[0,\Omega]}(H)\mathcal{H}$. Usually $\mathcal{H} = L^2(X, \mu)$ and $\mathrm{PW}_{[0,\Omega]}(H)$ is a reproducing kernel Hilbert space. In this situation many authors have proved the existence of sampling sets [Coulhon et al. 2012; Feichtinger et al. 2016; Filbir and Mhaskar 2011; Feichtinger and Pesenson 2004; Pesenson 2000; 2001; Pesenson and Zayed 2009]. In particular the set-up of [Coulhon et al. 2012; Pesenson 1999; Pesenson and Zayed 2009] covers the case of $H$ being a self-adjoint uniformly elliptic differential operator on $L^2(\mathbb{R}^d)$. The construction of sampling sets in these abstract Paley–Wiener spaces requires some smoothness properties of functions in $\mathrm{PW}_\Omega(H)$ and a Bernstein-type inequality (see (2-1) below). The result then is that a "sufficiently dense" subset in $X$ is a sampling set and a "sufficiently sparse" subset of $X$ is a set of interpolation. What remained unknown is the existence of a critical density against which one could compare the quality of the construction. Theorems B, C, and Corollary D address this gap for uniformly elliptic differential operators. Once a critical sampling density is established, one may aim for sampling sets near the critical density. The question of optimal sampling sets in spectral subspaces is wide open; in fact, it has become meaningful only after the critical density is known explicitly. This problem is already difficult for multivariate bandlimited functions $\mathrm{PW}_K = \{f \in L^2(\mathbb{R}^d) : \mathrm{supp}\, \hat{f} \subseteq K\}$ for compact spectrum $K \subseteq \mathbb{R}^d$ and was solved only recently in [Matei and Meyer 2010; Olevskiĭ and Ulanovskii 2008]. A possible general approach is via the construction of Fekete sets and weak limits, as was carried out in [Gröchenig et al. 2019] for Fock spaces with a general weight.

*Insight for partial differential operators.* Although the spectral subspaces of a partial differential operator are natural objects, they seem to have received little attention. To the best of our knowledge, nothing is known about the nature of the corresponding reproducing kernel and the behavior of functions in the spectral subspaces $\mathrm{PW}_\Omega$. Our investigation reveals several properties of the reproducing kernel, such as the behavior of its diagonal and some form of off-diagonal decay. These are key properties for the proofs of Theorems B and C, and we hope that these also hold some interest for partial differential operators.

*Variable bandwidth.* Our original motivation comes from a new concept of variable bandwidth. In [Gröchenig and Klotz 2017] we argued that the spectral subspaces of the Sturm–Liouville operator $-(d/dx)a(d/dx)$ on $L^2(\mathbb{R})$ for some function $a > 0$ can be taken as spaces with variable bandwidth. We proved that $a(x)^{-1/2}$ is a measure for the bandwidth near $x$ (the largest active frequency at position $x$). The function $a$ thus parametrizes the local bandwidth. For $a = \mathrm{const.}$, the spectral subspace is just the classical Paley–Wiener space $\mathrm{PW}_\Omega$. For the special case of an eventually constant parametrizing function $a$, i.e., $a$ is constant outside an interval $[-R, R]$, we computed the critical density for sampling in

$PW_\Omega(-(d/dx)a(d/dx))$. The proof required intricate details of the scattering theory of one-dimensional Schrödinger operators. Theorem C, formulated for dimension $d = 1$, yields a significant extension of the density theorem for the sampling of functions of variable bandwidth.

**Corollary E.** *Assume that $a \in C_b^\infty(\mathbb{R})$ is bounded, $a > 0$, and $\lim_{x \to \pm\infty} a'(x) = 0$. Let $PW_\Omega(H_a)$ be the Paley–Wiener space associated to $H_a$.*

*If $S$ is a sampling set for $PW_\Omega(H_a)$, then*

$$D_v^-(S) \geq \frac{\Omega^{1/2}}{\pi}.$$

*Similarly, if $S$ is a set of interpolation for $PW_\Omega(H_a)$, then*

$$D_v^+(S) \leq \frac{\Omega^{1/2}}{\pi}.$$

*Methods.* The proofs of Theorems B and C combine ideas and techniques from several areas of analysis.

*Critical density in reproducing kernel Hilbert spaces.* Originally, density theorems in the style of Landau — and there are dozens in analysis — were proved from scratch. In our approach we apply the results on sampling and interpolation in general reproducing kernel Hilbert spaces from [Führ et al. 2017]. The main insight was that it suffices to verify some geometric conditions on the measure space, such as a doubling condition of the underlying measure, and of the reproducing kernel, such as some form of off-diagonal decay. Once these conditions are satisfied, one obtains the existence of a critical density and can calculate it in terms of the averaged trace of the reproducing kernel. Since the geometric conditions are trivially satisfied for $\mathbb{R}^d$, our main technical difficulty is to understand the reproducing kernel of the spectral subspaces of a self-adjoint uniformly elliptic differential operator.

*Regularity theory and heat kernel estimates.* To study this reproducing kernel, we use the fundamental results of the regularity theory of elliptic differential operators. With these tools we investigate the smoothness of the reproducing kernel and compare various Sobolev norms on $PW_\Omega(H_a)$. See Lemma 2.1 and Proposition 2.2. For an important technical detail (Proposition 2.2) we will need Gaussian estimates for the heat kernel, which we expect to play a key role in extensions of our theory.

*Limit operators and slowly varying symbols.* To connect asymptotic properties of the symbol $a$ of a partial differential operator $H_a$ to the spectral theory of $H_a$, we use the notion of limit operators. Although we do not use any elaborate results from this theory (see [Georgescu 2011; Rabinovich et al. 2004a; Špakula and Willett 2017]), limit operators are central to our arguments.

*Higson compactification of $\mathbb{R}^d$.* An important structure underlying the proof of Theorem C is a compactification of $\mathbb{R}^d$, the so-called Higson compactification. This is the compactification arising as the maximal ideal space of the $C^*$-algebra of slowly oscillating functions on $\mathbb{R}^d$. By Gelfand theory every slowly oscillating function can be identified with a continuous function on the Higson compactification $h\mathbb{R}^d$; see, e.g., [Rabinovich et al. 2004a; Roe 2003; Shteinberg 2000]. On a technical level we will show that for slowly oscillating symbols the mapping $x \to T_{-x}k_x$ of centered reproducing kernels can be extended continuously to the compactification $h\mathbb{R}^d$ (Proposition 6.3).

The underlying philosophy is summarized in the following diagram; we write $T_x f(z) = f(z - x)$ for the translation operator and $k_x$ for the reproducing kernel of $\mathrm{PW}_\Omega(H_a)$:

$$\{T_{-x}a : x \in \mathbb{R}^d\} \text{ compact} \quad \Longrightarrow \quad \{T_{-x}H_aT_x : x \in \mathbb{R}^d\} \text{ compact} \quad \Longrightarrow \quad \{T_{-x}k_x : x \in \mathbb{R}^d\} \text{ compact}.$$

Thus, if $T_{x_n}a \to b$ in a suitable topology, then $T_{-x_n}H_aT_{x_n} \to H_b$ and the sequence of centered reproducing kernels $T_{-x_n}k_{x_n}$ converges to the reproducing kernel of $\mathrm{PW}_\Omega(H_b)$. In the considered examples the limit operator $H_b$ is simpler than the original operator $H_a$, and this facilitates information about the reproducing kernel of $\mathrm{PW}_\Omega(H_a)$.

The paper is organized as follows. Section 2 prepares the background material on regularity theory, symbol classes for partial differential operators, and reproducing kernel Hilbert spaces. We prove the basic properties of the Paley–Wiener space $\mathrm{PW}_\Omega(H_a)$. Section 3 gives the precise formulation of the general density theorem for $\mathrm{PW}_\Omega(H_a)$. Its proof is given in Sections 4 and 5. In Section 6 we calculate the critical density for sampling in $\mathrm{PW}_\Omega(H_a)$ for the class of slowly varying symbols (Theorem C and Corollary D). We conclude with an outlook and collect additional material in Appendices A and B.

## 2. Preliminaries

**2A. Notation.** For a function $f$ on $\mathbb{R}^d$ and $x, z \in \mathbb{R}^d$ we define the translation operator $T_x f(z) = f(z - x)$. The open Euclidean ball of radius $r$ at $x$ is $B_r(x)$, and $B_r = B_r(0)$.

We use standard multi-index notation; thus the differential operator $D^\alpha$ is $\partial^{|\alpha|}/(\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d})$ and the multivariate binomial symbol is $\binom{\alpha}{\gamma} = \prod_{j=1}^d \binom{\alpha_j}{\gamma_j}$ for multi-indices $\alpha, \gamma \in \mathbb{N}_0^d$.

We will denote the space of uniformly continuous and bounded functions on $\mathbb{R}^d$ with values in a Banach space $X$ by $C_b^u(\mathbb{R}^d, X)$. The indices $c$, $\infty$, and $0$ refer to the subspaces of compactly supported, smooth, and vanishing-at-infinity functions in $C(\mathbb{R}^d)$. Thus $C_b^\infty(\mathbb{R}^d, X)$ consists of all smooth $X$-valued functions with bounded derivatives of all orders. The space $C^\infty(\mathbb{R}^d, X)$ has the Fréchet space topology induced by the seminorms $|f|_{R,\alpha} = \sup_{x \in B_R(0)} \|D^\alpha f(x)\|_X$. If $X = \mathbb{C}$, we write $C_b^\infty(\mathbb{R}^d)$, etc.

The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is

$$\mathscr{F}f(\omega) = \hat{f}(\omega) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\omega}\, dx,$$

and $\mathscr{F}$ extends to a unitary operator on $L^2(\mathbb{R}^d)$ as usual. For every $s \geq 0$ the Sobolev space $W_2^s$ is defined by

$$W_2^s = \left\{ f \in L^2(\mathbb{R}^d) : \|f\|_{W_2^s} = \left[ (2\pi)^{-d/2} \int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 (1 + |\omega|^2)^s \, d\omega \right]^{1/2} < \infty \right\}.$$

If $s \in \mathbb{N}$, then $\|f\|_{W_2^s} \asymp \sum_{|\alpha| \leq s} \|D^\alpha f\|_2$. By the Sobolev embedding theorem, $W_2^s \hookrightarrow C_0(\mathbb{R}^d)$ for $s > d/2$.

Recall that a reproducing kernel Hilbert space $\mathscr{H}$ is a Hilbert space of functions defined on a set $X$ such that $f(x) = \langle f, k_x \rangle_\mathscr{H}$ for all $f \in \mathscr{H}$ and $x \in X$. We write $k(x, y) = \overline{k_x(y)}$ for the reproducing kernel of $\mathscr{H}$. See, e.g., [Aronszajn 1950]. In particular, the Sobolev space $W_2^s$ is a reproducing kernel Hilbert space with reproducing kernel $T_x\kappa$, $x \in \mathbb{R}^d$, where $\hat{\kappa}(\omega) = \hat{\kappa}_s(\omega) = (1 + |\omega|^2)^{-s}$, by direct computation or by [Wendland 2005].

**2B.** *The generalized Paley–Wiener space and its basic properties.* Pesenson's idea [1998; 2001] (see also [Pesenson and Zayed 2009]) was to define an abstract Paley–Wiener space as a spectral subspace associated to an arbitrary positive, self-adjoint operator $H \geq 0$ with domain $\mathscr{D}(H)$ on a Hilbert space $\mathscr{H}$ and a spectral interval $[0, \Omega]$. Let $\chi_{[0,\Omega]}(H)$ be the spectral projection of $H$. Then the generalized Paley–Wiener space is defined as

$$\mathrm{PW}_\Omega(H) = \chi_{[0,\Omega]}(H)\mathscr{H}.$$

Equivalently, for a positive, self-adjoint operator, one can define the Paley–Wiener space $\mathrm{PW}_\Omega(H)$ by a Bernstein-type inequality: $f \in \mathrm{PW}_\Omega(H)$, if and only if $f \in \mathscr{D}(H^k)$ for all $k \in \mathbb{N}$, and

$$\|H^k f\|_2 \leq \Omega^k \|f\|_2 \quad \text{for all } k \in \mathbb{N}. \tag{2-1}$$

This is an easy consequence of the spectral theorem; see [Gröchenig and Klotz 2010; Pesenson 2001; Pesenson and Zayed 2009].

If $H = -d^2/dx^2$ on $L^2(\mathbb{R})$, then

$$\mathrm{PW}_\Omega(H) = \{f \in L^2(\mathbb{R}) : \mathrm{supp}\, \hat{f} \subseteq [-\sqrt{\Omega}, \sqrt{\Omega}]\}$$

is precisely the classical Paley–Wiener space, or in engineering language the space of band-limited functions with bandwidth $2\sqrt{\Omega}$.

*Convention.* In this work we consider positive, formally self-adjoint differential expressions $H = H_a$ of the form

$$H_a f = -\sum_{j,k=1}^d \partial_j a_{jk} \partial_k f, \quad f \in W_2^2. \tag{2-2}$$

Here the *matrix symbol* $a \in C_b^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$ is positive definite; i.e., $a_{jk} = \bar{a}_{kj} \in C_b^\infty(\mathbb{R}^d)$ and there exists $\theta > 0$ such that $a(x)\xi \cdot \xi \geq \theta|\xi|^2$ for all $\xi, x \in \mathbb{R}^d$. Then $H_a$ is a positive, uniformly elliptic self-adjoint operator on $\mathbb{R}^d$ with domain $\mathscr{D}(H_a) = W_2^2$. In particular $C_c^\infty(\mathbb{R}^d)$ is a core for $H_a$; i.e., $H_a$ is the operator closure of $H_a|_{C_c^\infty(\mathbb{R}^d)}$. The regularity theory of elliptic differential operators asserts that for every $k \in \mathbb{N}_0$ there is a $c_k \in \mathbb{R}$ such that

$$H_a^k + c_k : W_2^{2k} \to L^2(\mathbb{R}^d)$$

is a Hilbert space isomorphism. See [Zimmer 1990, Theorem 6.3.12] or the standard references [Agmon 1965; Shubin 1992]. For further use we record the fact that a uniformly elliptic operator is one-to-one on its domain and thus

$$0 \text{ is not an eigenvalue of } H_a. \tag{2-3}$$

To see this, we use the ellipticity and $f \in W_2^2$. Then the identity

$$\langle H_a f, f \rangle = \int \sum_{j,k} a_{jk} \partial_k f(x) \overline{\partial_j f(x)} \, dx = 0$$

implies that $\partial_j f \equiv 0$; thus $f = 0$.

**Remark.** We regard the mapping $a \mapsto H_a$ as a mapping from functions to operators (a symbolic calculus) and refer to $a$ as the (matrix) symbol of the operator. This terminology differs slightly from

the usage in PDE, where the (principal) symbol of the differential operator $\sum_{|\alpha| \leq m} a_\alpha D^\alpha$ is the function $p(x, \xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ on $\mathbb{R}^{2d}$. For the second-order differential operator $H_a$ in (2-2) the principal symbol is $p(x, \xi) = a(x)\xi \cdot \xi$. Since $H_a$ is self-adjoint, the coefficients $a_\alpha$ are all real for $|\alpha| = 2$.

First we verify that $\mathrm{PW}_\Omega(H_a)$ embeds in every Sobolev space.

**Lemma 2.1.** *The Paley–Wiener space* $\mathrm{PW}_\Omega(H_a)$ *is continuously embedded in all Sobolev spaces* $W_2^s$, $s \geq 0$, *and in* $C_0^\infty(\mathbb{R}^d)$. *As a consequence, on* $\mathrm{PW}_\Omega(H_a)$, *the* $L^2$*-norm and the Sobolev norms are equivalent.*

*Proof.* Let $f \in \mathrm{PW}_\Omega(H_a)$ and $k \in \mathbb{N}$. By elliptic regularity and Bernstein's inequality (2-1), $\|f\|_{W_2^{2k}} \asymp \|(H^k + c_k)f\|_2 \leq (\Omega^k + |c_k|)\|f\|_2$. Consequently, $f \in \bigcap_{k \in \mathbb{N}} W_2^{2k} = \bigcap_{s \geq 0} W_2^s \subseteq C_0^\infty(\mathbb{R}^d)$ via the Sobolev embedding. $\qquad\square$

Embeddings of Paley–Wiener spaces different from Lemma 2.1 can be found in [Feichtinger and Pesenson 2004].

Next we show that $\mathrm{PW}_\Omega(H_a)$ is a reproducing kernel Hilbert space in $L^2(\mathbb{R}^d)$.

**Proposition 2.2.** *There exists a reproducing kernel* $k_x \in \mathrm{PW}_\Omega(H_a)$ *such that* $\chi_{[0,\Omega]}(H_a)f(x) = \langle f, k_x \rangle$ *for all* $f \in L^2(\mathbb{R}^d)$ *and all* $x \in \mathbb{R}^d$. *In addition, there are positive constants* $c$, $C$ *such that*

$$0 < c \leq \|k_x\|_2 \leq C \quad \text{for all } x \in \mathbb{R}^d. \tag{2-4}$$

*Proof.* Let $f \in \mathrm{PW}_\Omega(H_a)$ and $s > d/2$. By Lemma 2.1, $f \in W_2^s$ and $\|f\|_2 \asymp \|f\|_{W_2^s}$. Since $W_2^s$ is a reproducing kernel Hilbert space, we obtain

$$|f(x)| = |\langle f, T_x \kappa \rangle_{W_2^s}| \leq \|T_x \kappa\|_{W_2^s} \|f\|_{W_2^s} \leq C\|f\|_2.$$

Thus $\mathrm{PW}_\Omega(H_a)$ is a reproducing kernel Hilbert space with kernel $k_x \in \mathrm{PW}_\Omega(H_a)$.

For the lower bound in (2-4) we do not have a proof based exclusively on regularity theory. Instead we refer to [Coulhon et al. 2012, Lemma 3.19], where the lower bound for the reproducing kernel was derived by means of heat kernel estimates. As some details and notation differ, we reproduce the proof in Appendix B. $\qquad\square$

**Proposition 2.3.** *The mapping* $x \mapsto k_x$ *is continuous from* $\mathbb{R}^d$ *to* $W_2^s$, $s \geq 0$.

*Proof.* Since $k_x \in PW_\Omega(H_a)$ and $\|k_x\|_2$ is bounded by (2-4), Lemma 2.1 and the Sobolev embedding theorem imply that $C_1 = \sup_{x,y \in \mathbb{R}} |\nabla k_x(y)|$ is finite; therefore

$$\|k_x - k_y\|_{W_2^s}^2 \leq C\|k_x - k_y\|_2^2 = C\big(k_x(x) - k_x(y) - k_y(x) + k_y(y)\big)$$
$$\leq 2C \sup_{z,w} |\nabla k_z(w)||x - y| \leq C'|x - y|.$$

Consequently $x \to k_x$ is continuous. $\qquad\square$

## 2C. *Sampling and interpolation in* $\mathrm{PW}_\Omega(H_a)$ *and the Beurling densities.*

Let $\mu$ be a Borel measure on $\mathbb{R}^d$ that is equivalent to Lebesgue measure in the sense that $d\mu = h\, dx$ for a measurable function $h$ with $0 < c \leq h(x) \leq C$ for all $x \in \mathbb{R}^d$.

The lower Beurling density of $S$ with respect to $\mu$ is defined as

$$D_\mu^-(S) = \liminf_{r \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\#(S \cap B_r(x))}{\mu(B_r(x))}, \tag{2-5}$$

and the upper Beurling density of $S$ is

$$D_\mu^+(S) = \limsup_{r \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\#(S \cap B_r(x))}{\mu(B_r(x))}.$$

If $d\mu = dx$ we omit the subscript and write $D^\pm(S)$.

For sampling in reproducing kernel Hilbert spaces the relevant measure is $d\mu(x) = k(x, x)\, dx$. We call the Beurling density with respect to this measure the dimension-free density and write $D_0^\pm(S)$ for $D_\mu^\pm(S)$.

We say that the reproducing kernel $k$ of a reproducing kernel Hilbert space $\mathcal{H} \subseteq L^2(\mathbb{R}^d, dx)$ satisfies the *weak localization* property (WL) if for every $\varepsilon > 0$ there is a constant $r = r(\varepsilon)$ such that

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus B_r(x)} |k(x, y)|^2\, dy < \varepsilon^2. \tag{WL}$$

The discrete analog of the weak localization is the so-called *homogeneous approximation property* (HAP) of the reproducing kernel: Assume that $S$ is such that $\{k_s : s \in S\}$ is a *Bessel sequence* for $\mathcal{H}$; i.e., $S$ satisfies the upper sampling inequality $\sum_{s \in S}|f(s)|^2 \le C\|f\|_2^2$ for all $f \in \mathcal{H}$. Then for every $\varepsilon > 0$ there is a constant $r = r(\varepsilon)$ such that

$$\sup_{x \in \mathbb{R}^d} \sum_{s \in S \setminus B_r(x)} |k(x, s)|^2 < \varepsilon^2. \tag{HAP}$$

Under the assumptions of weak localization (WL) and (2-4), an upper sampling inequality implies that for some (and hence all) $\rho > 0$

$$\max_{x \in \mathbb{R}^d} \#(S \cap B_\rho(x)) < \infty.$$

We call such a set $S$ *relatively separated*. See also [Führ et al. 2017, Lemma 3.7].

The two localization properties (WL) and (HAP) are the key properties of the reproducing kernel required for an abstract density theorem to hold. For reproducing kernel Hilbert spaces embedded in $L^2(\mathbb{R}^d)$ this can be stated as follows [Führ et al. 2017, Corollary 4.1].

**Theorem 2.4.** *Let $\mathcal{H} \subseteq L^2(\mathbb{R}^d, dx)$ be a reproducing kernel Hilbert space with kernel $k$. Assume that $k$ satisfies the boundedness property (2-4) on the diagonal, the weak localization (WL) and the homogeneous approximation property (HAP).*

(i) *If $S$ is a sampling set for $\mathcal{H}$, then $D_0^-(S) \ge 1$.*

(ii) *If $S$ is an interpolating set for $\mathcal{H}$, then $D_0^+(S) \le 1$.*

This result holds under a set of natural assumptions on metric measure spaces and conditions on the reproducing kernel. We will not dwell on the geometric conditions, e.g., doubling measure, as these are clearly satisfied for $\mathbb{R}^d$ with $\mu$ equivalent to Lebesgue measure. We want to verify Theorem 2.4 for $\mathcal{H} = \mathrm{PW}_\Omega(H_a)$ for a suitable class of symbols $a$. The boundedness of the diagonal of the kernel was

already established in Proposition 2.2, (2-4). To prove Theorems B and C we therefore need to verify the properties (WL) and (HAP) for the reproducing kernel Hilbert space $\mathrm{PW}_\Omega(H_a)$.

Observe that (WL) is equivalent to the condition

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus B_r(0)} |T_{-x}k_x(y)|^2\,dy < \varepsilon^2 \tag{2-6}$$

for the *centered* reproducing kernels. We will show the stronger statement that the set $\{T_{-x}k_x : x\in\mathbb{R}^d\}$ is relatively compact in $L^2(\mathbb{R}^d)$. The Riesz–Kolmogorov compactness theorem then implies (2-6) and thus (WL).

The proof of (HAP) requires some additional local regularity of $k_x$. We will use prominently elliptic regularity theory to show that $\{T_{-x}k_x : x\in\mathbb{R}^d\}$ is relatively compact in all Sobolev spaces $W_2^s$. For the proof of (HAP) it is fundamental that the point evaluation on $\mathrm{PW}_\Omega(H_a)$ can be expressed two-fold as

$$f(x) = \langle f, k_x\rangle_{L^2} = \langle f, T_x\kappa\rangle_{W_2^s} \quad \text{for all } f\in \mathrm{PW}_\Omega(H_a). \tag{2-7}$$

**2D. *Classes of symbols, limit operators.*** First we define the relevant symbol classes. Let

$$\tau_x(H_a) = T_{-x}H_aT_x = H_{T_{-x}a} \tag{2-8}$$

be the conjugation of $H_a$ by the translation $T_x$. If $a\in C_b^\infty(\mathbb{R}^d,\mathbb{C}^{d\times d})$, observe that $\tau_x(H_a)$ is again a self-adjoint, uniformly elliptic operator with domain $W_2^2$ and core $C_c^\infty(\mathbb{R}^d)$. In this section we describe symbol classes that ensure that $\{\tau_x(H_a)f : x\in\mathbb{R}^d\}$ is relatively compact in $L^2(\mathbb{R}^d)$ for all $f\in C_c^\infty(\mathbb{R}^d)$. Equivalently, every sequence $\tau_{x_k}(H_a)f$ has a norm-convergent subsequence. If $(x_k)$ is bounded, this follows from the continuity of $x\mapsto T_xf$. To treat unbounded sequences we need some terminology.

Since in Section 6 we will deal with a nonmetrizable compactification of $\mathbb{R}^d$, we formulate most results for nets $(x_\lambda)_{\lambda\in\Lambda}$ instead of sequences. (Here $\Lambda$ is a directed set with a partial order $\succeq$ and we write $\lim_\lambda x_\lambda$ for the limit of a net when it exists.)

**Definition 2.5.** Assume $a\in C_b^\infty(\mathbb{R}^d,\mathbb{C}^{d\times d})$. If the net $(x_\lambda)_{\lambda\in\Lambda}\subset\mathbb{R}^d$ diverges to infinity and there is an operator $H\in\mathscr{B}(W_2^2, L^2(\mathbb{R}^d))$ such that $\lim_\lambda \tau_{x_\lambda}(H_a)f = Hf$ for all $f\in C_c^\infty(\mathbb{R}^d)$, then $H$ is called a *limit operator* of $H_a$.

**Remark 2.6.** (i) Existence and uniqueness of the limit operator follow from the Banach–Steinhaus theorem.

(ii) We do not even scratch the surface of the method of limit operators: see, amongst many others, [Rabinovich et al. 2004a; 2004b; Špakula and Willett 2017], and in the $C^*$-algebra setting [Davies and Georgescu 2013; Georgescu 2011; 2018].

(iii) Limit operators are related to compactifications of $\mathbb{R}^d$. An example can be found in Section 6B.

**2D1. *Compact orbits.*** Identity (2-8) suggests that compactness properties of $\{\tau_x(H_a) : x\in\mathbb{R}^d\}$ are related to compactness properties of $\{T_{-x}a : x\in\mathbb{R}^d\}$, so we investigate these first.

**Lemma 2.7.** (i) *If $f\in C_b^\infty(\mathbb{R}^d)$, then $\{T_xf : x\in\mathbb{R}^d\}$ is relatively compact in the Fréchet space $C^\infty(\mathbb{R}^d)$ with respect to its topology of uniform convergence of all derivatives on compact sets.*

(ii) *In particular, if* $\lim_\lambda T_{x_\lambda} f = g$ *pointwise, then* $\lim_\lambda T_{x_\lambda} f = g$ *in* $C^\infty(\mathbb{R}^d)$. *The limit function* $g$ *is again in* $C_b^\infty(\mathbb{R}^d)$.

*Proof.* (i) The space $C^\infty(\mathbb{R}^d)$ has the Heine–Borel property [Rudin 1973, 1.46], so it suffices to verify that $\{T_x f : x \in \mathbb{R}^d\}$ is bounded in $C^\infty(\mathbb{R}^d)$, which means that

$$\|D^\alpha T_x f\|_{L^\infty(B_r(0))} < C_{\alpha,r} \quad \text{for all } x \in \mathbb{R}^d \text{ and all } r > 0, \ \alpha \in \mathbb{N}_0^d.$$

But this is trivial for $f \in C_b^\infty(\mathbb{R}^d)$, since all derivatives are globally bounded.

(ii) We apply the following observation: A net converges to a limit $g$ if and only if every subnet has a subnet that converges to $g$. By (i) every subnet of $(T_{x_\lambda} f)_{k \in \mathbb{N}}$ has a subnet $(T_{z_\lambda} f)_{k \in \mathbb{N}}$ that converges in $C^\infty(\mathbb{R}^d)$ (to the limit function $g$). We conclude that $(T_{x_\lambda} f)_{k \in \mathbb{N}}$ converges to $g$ in $C^\infty(\mathbb{R}^d)$. As all functions and their derivatives of all orders are bounded and continuous, this is true for the limit as well. $\square$

**Proposition 2.8.** *Let* $a \in C_b^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$, $k, m \in \mathbb{N}_0$, *and assume* $\lim_\lambda T_{-x_\lambda} a = b$ *pointwise. Then, for every* $f \in W_2^{2m+2k}$

$$\lim_\lambda \|(\tau_{x_\lambda}(H_a^k) - H_b^k)f\|_{W_2^{2m}} = 0.$$

*Proof.* We treat the case $k = 1$ first and assume for the moment that $f \in C_c^\infty(\mathbb{R}^d)$. Set $a^{(\lambda)} = T_{-x_\lambda} a$. We can express $H_a$ in the form $H_a = \sum_{|\beta| \leq 2} a_\beta D^\beta$, with coefficients $a_\beta \in C_b^\infty(\mathbb{R}^d)$, and estimate, for every multiindex $\alpha$ with $|\alpha| \leq 2m$,

$$|D^\alpha(H_{a^{(\lambda)}} - H_b)f| = \left| D^\alpha \sum_{|\beta| \leq 2} (a_\beta^{(\lambda)} - b_\beta) D^\beta f \right| = \left| \sum_{|\beta| \leq 2} \sum_{|\gamma| \leq |\alpha|} \binom{\alpha}{\gamma} D^\gamma(a_\beta^{(\lambda)} - b_\beta) D^{\alpha - \gamma + \beta} f \right|.$$

By Lemma 2.7 we have $\lim_\lambda D^\gamma a^{(\lambda)} = D^\gamma b$ uniformly on compact sets, so the convergence is actually uniform on supp $f$, and thus

$$\lim_\lambda \|D^\alpha(H_{a^{(\lambda)}} - H_b)f\|_\infty = 0.$$

Consequently

$$\|(H_{a^{(\lambda)}} - H_b)f\|_{W_2^{2m}} \leq C \max_{|\alpha| \leq 2m} \|D^\alpha(H_{a^{(\lambda)}} - H_b)f\|_2$$
$$\leq C|\text{supp } f|^{1/2} \max_{|\alpha| \leq 2m} \|D^\alpha(H_{a^{(\lambda)}} - H_b)f\|_\infty \to 0.$$

As $C_c^\infty(\mathbb{R}^d)$ is dense in $W_2^{2m+2}$, and the operators $H_{a^{(\lambda)}}$ are uniformly bounded from $W_2^{2m+2}$ to $W_2^{2m}$, a standard density argument (see, e.g., [Teschl 2009, Lemma 1.14]) implies $\|(H_{a^{(\lambda)}} - H_b)f\|_{W_2^{2m}} \to 0$ for all $f \in W_2^{2m+2}$.

For $k > 1$ observe that

$$H_a^k f - H_b^k f = H_a^{k-1}(H_a f - H_b f) + (H_a^{k-1} - H_b^{k-1})H_b f.$$

As $\lim_\lambda \|(H_{a^{(\lambda)}} - H_b)f\|_{W_2^{2m}} = 0$ for $f \in W_2^{2m+2}$, the result follows by induction on $k$. $\square$

**Remark.** The statement of the proposition and its proof are valid under the following more general conditions: $a_\lambda, b \in C_b^\infty(\mathbb{R}^d)$, $a_\lambda \xrightarrow{C^\infty} b$, and $(H_{a_\lambda})$ is uniformly bounded from $W_2^2$ to $L^2(\mathbb{R}^d)$.

Though not needed in the sequel, we state an interesting corollary that shows how compactness properties of the orbit $\{T_x a : x \in \mathbb{R}^d\}$ are transferred to compactness properties of $\{\tau_x(H_a) : x \in \mathbb{R}^d\}$.

**Corollary 2.9.** *If $a \in C_b^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$ and $f \in C_c^\infty(\mathbb{R}^d)$ the set $\{\tau_x(H_a)f : x \in \mathbb{R}^d\}$ is relatively compact in every Sobolev space $W_2^s$, $s > 0$.*

*Proof.* The set $\{T_x a : x \in \mathbb{R}^d\}$ is relatively compact in $C^\infty(\mathbb{R}^d)$ by Lemma 2.7, and Proposition 2.8 says that the mapping $a \mapsto H_a f$ is continuous from $\overline{\{T_x a : x \in \mathbb{R}^d\}}^{C^\infty(\mathbb{R}^d)}$ to $W_2^s$.                    □

**2D2.** *Slowly oscillating symbols.* In the next step we single out a subclass of operators for which the spectral theory is sufficiently simple. In our approach it is essential that the limit operators do not have the endpoint 0 and $\Omega$ of the spectrum as eigenvalues. The limits of translates of *slowly oscillating* symbols are constant, if they exist (Lemma 2.13 below), so the limit operators are similar to the Laplacian. This will be used in Section 6 to compute the critical density.

**Definition 2.10.** An $X$-valued function $f \in C_b^u(\mathbb{R}^d, X)$ is slowly oscillating[1] if for all compact subsets $M \subset \mathbb{R}^d$

$$\lim_{|x| \to \infty} \sup_{m \in M} \|f(x) - f(x+m)\|_X = 0.$$

In fact, it suffices to use the closed unit ball $\overline{B}_1$ instead of an arbitrary compact set $M$.

We denote the space of all slowly oscillating functions on $\mathbb{R}^d$ by $C_h(\mathbb{R}^d, X)$ and define $C_h^\infty(\mathbb{R}^d, X) = C_h(\mathbb{R}^d, X) \cap C_b^\infty(\mathbb{R}^d, X)$.

The space $C_h(\mathbb{R}^d)$ with the $\|\cdot\|_\infty$-norm and pointwise multiplication is a commutative $C^*$-subalgebra of $C_b^u(\mathbb{R}^d)$.

We will need the following characterization of $C_h^\infty(\mathbb{R}^d, X)$. The statement is folklore, but we do not know a formal reference. For completeness we sketch the simple proof.

**Lemma 2.11.** *A function $f$ is in $C_h^\infty(\mathbb{R}^d, X)$ if and only if $f \in C_b^\infty(\mathbb{R}^d, X)$ and $\lim_{|x| \to \infty} \partial_k f(x) = 0$ for all $1 \le k \le d$.*

*Proof.* Assume that $f \in C_b^\infty(\mathbb{R}^d, X)$ and $\lim_{|x| \to \infty} \partial_k f(x) = 0$ for all $1 \le k \le d$ and choose $M = [-h, h]^d$. Writing $m \in M$ as $m = \sum_{k=1}^d h_k e_k$ with $|h_k| \le h$, the difference in Definition 2.10 is

$$f\left(x + \sum_{k=1}^d h_k e_k\right) - f(x) = \sum_{k=0}^{d-1} \int_{x + \sum_{l \le k} h_l e_l}^{x + \sum_{l \le k+1} h_l e_l} \partial_{k+1} f.$$

This implies that $\sup_{m \in M} \|f(x+m) - f(x)\|_X \to 0$ for $|x| \to \infty$.

Conversely, assume that $f \in C_h^\infty(\mathbb{R}^d, X)$. Fix $\varepsilon > 0$ and let $\eta \in C_c^\infty(\mathbb{R}^d)$ with $\text{supp}\,\eta \subset B_1$, $\eta \ge 0$, $\int_{\mathbb{R}^d} \eta(x)\,dx = 1$. Then $\eta_\tau(x) = \tau^{-d}\eta(\tau^{-1}x)$, $\tau > 0$, is an approximate unit. This implies that for $f \in C_u^b(\mathbb{R}^d, X)$ bounded and uniformly continuous

$$\lim_{\tau \to 0+} \sup_{x \in \mathbb{R}^d} \|f(x) - f * \eta_\tau(x)\|_X = 0. \tag{2-9}$$

To estimate the partial derivative of $f \in C_h^\infty(\mathbb{R}^d, X)$ we introduce the approximate unit:

$$\|\partial_k f(x)\|_X \le \|\partial_k f(x) - \eta_\tau * \partial_k f(x)\|_X + \|\eta_\tau * \partial_k f(x)\|_X = \mathrm{I}_\tau + \mathrm{II}_\tau.$$

---

[1]In the literature $f$ is also called "of vanishing oscillation at infinity" or a Higson function.

Since all derivatives of $f \in C_h^\infty(\mathbb{R}^d, X)$ are bounded and uniformly continuous, by (2-9) there exists $\tau_0$ such that $I_\tau < \varepsilon/2$ for $t < \tau_0$. Fix $\tau < \tau_0$ and observe that $\eta_\tau * \partial_k f = \partial_k \eta_\tau * f$ and that $\int_{\mathbb{R}^d} \partial_k \eta_\tau(y)\, dy = 0$. So

$$
\begin{aligned}
\mathrm{II}_\tau = \|\eta_\tau * \partial_k f(x)\|_X &= \left\| \int_{\mathbb{R}^d} (f(x) - f(y)) \partial_k \eta_\tau(x - y)\, dy \right\|_X \\
&\leq \sup_{y \in B_\tau(x)} \|f(x) - f(y)\|_X \int_{\mathbb{R}^d} |\partial_k \eta_\tau(x - y)|\, dy \\
&\leq C_\tau \sup_{y \in B_\tau(0)} \|f(x) - f(x + y)\|_X.
\end{aligned}
$$

As $f \in C_h^\infty(\mathbb{R}^d, X)$, there is $R > 0$ such that $\mathrm{II}_\tau \leq \varepsilon/(2C_\tau)$ for $|x| > R$, and thus $\|\eta_\tau * \partial_k f(x)\|_X < \varepsilon$ for $|x| > R$. $\qquad\square$

**Example 2.12.** A typical example of a genuinely slowly oscillating function is $a(x) = \sin|x|^{1/2}(1 - \varphi(x))$ for some $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi(x) = 1$ near 0. (The cut-off of the singularity at 0 serves to make all derivatives of $a$ bounded, but, of course, it is immaterial for the asymptotic behavior.)

Our interest in $C_h(\mathbb{R}^d, X)$ stems from the following fact (see [Rabinovich et al. 2004a, Proposition 2.4.1]):

**Lemma 2.13.** *Assume that $f \in C_h(\mathbb{R}^d, X)$ and $(x_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R}^d$ diverges to infinity, $|x_\lambda| \to \infty$. If $\lim_\lambda T_{-x_\lambda} f(x) = g(x)$ exists for all $x \in \mathbb{R}^d$, then $g$ is constant.*

*Proof.* Let $x, x' \in \mathbb{R}^d$. Definition 2.10 with $M = \{x, x'\}$ shows that for all $\varepsilon > 0$ there exists an index $\lambda_\varepsilon = \lambda_\varepsilon(x, x')$ such that $\|f(x + x_\lambda) - f(x_\lambda)\|_X < \varepsilon/2$ and $\|f(x' + x_\lambda) - f(x_\lambda)\|_X < \varepsilon/2$ for all $\lambda \geq \lambda_\varepsilon$. So $\|f(x + x_\lambda) - f(x' + x_\lambda)\|_X < \varepsilon$ for all $\lambda \geq \lambda_\varepsilon$. If $g = \lim_\lambda T_{-x_\lambda} f$ exists, it follows that $\|g(x) - g(x')\|_X \leq \varepsilon$. As $\varepsilon > 0$ was arbitrary, $g$ must be constant. $\qquad\square$

## 3. Statement of the density theorem

We state our main theorems. A first version describes a general setup for symbols in the class $C_b^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$ under additional assumptions on the spectra of the limit operators. We then formulate a corollary for slowly oscillating symbols, where the assumptions on the limit operators are automatically satisfied. We discuss possible applications of the general version in Section 7.

**Theorem 3.1.** *Assume that $H_a = -\sum_{j,k=1}^d \partial_j a_{jk} \partial_k$ is uniformly elliptic with symbol $a \in C_b^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$. Let $\mathrm{PW}_\Omega(H_a)$ be the Paley–Wiener space as defined in Section 2B. Assume that $\Omega$ is not an eigenvalue of any limit operator $H_b$.*

- *If $S$ is a set of stable sampling for $\mathrm{PW}_\Omega(H_a)$, then*

$$D_0^-(S) \geq 1.$$

- *If $S$ is a set of interpolation for $\mathrm{PW}_\Omega(H_a)$, then*

$$D_0^+(S) \leq 1.$$

The following consequence is a more explicit version of Theorem B of the Introduction, where we have used the equivalence of Lemma 2.11 to avoid the formal definition of $C_h^\infty(\mathbb{R}^d)$.

**Corollary 3.2.** *Assume that $H_a = -\sum_{j,k=1}^d \partial_j a_{jk} \partial_k$ is uniformly elliptic with symbol $a \in C_h^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$.*

- *If $S$ is a set of stable sampling for $\mathrm{PW}_\Omega(H_a)$, then $D_0^-(S) \geq 1$.*
- *If $S$ is a set of interpolation for $\mathrm{PW}_\Omega(H_a)$, then $D_0^+(S) \leq 1$.*

*Proof of Corollary 3.2.* If $a \in C_h^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$, then by Lemma 2.7 every net $(x_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R}^d$ that diverges to infinity has a subnet $(x_\mu)_{\mu \in M}$ such that $\lim_\mu T_{-x_\mu} a = b$ in the topology of $C^\infty(\mathbb{R}^d)$ for a symbol $b$. This symbol $b$ is constant by Lemma 2.13 and positive definite; so $H_b$ is similar to the Laplacian and has no point spectrum. $\qquad\square$

## 4. Proof of weak localization of the kernel

To prove Theorem 3.1 we invoke Theorem 2.4 and verify its main hypotheses (WL) and (HAP) on the reproducing kernel.

Let $Q_h = [-h/2, h/2]^d$ be the cube of side-length $h$, and let $\varphi_x^h(y) = h^{-d} \chi_{Q_h}((y-x)/h) = T_x \varphi_0^h(y)$ be the usual approximate unit.

**Lemma 4.1.** *We have $\lim_{h \to 0} \|\chi_{[0,\Omega]}(H_a)\varphi_x^h - k_x\|_2 = 0$ uniformly in $x \in \mathbb{R}^d$.*

*Proof.* Let $f \in \mathrm{PW}_\Omega(H_a)$. Then

$$
\begin{aligned}
|\langle f, \chi_{[0,\Omega]}(H_a)\varphi_x^h - k_x \rangle| = |\langle f, \varphi_x^h \rangle - f(x)| &= h^{-d} \left| \int_{Q_h(x)} (f(y) - f(x))\, dy \right| \\
&\leq h^{-d} \int_{Q_h(x)} |f(y) - f(x)|\, dy \leq \sup_{z \in \mathbb{R}^d} |\nabla f(z)| h^{-d} \int_{Q_h(x)} |y - x|\, dy \\
&\leq C \|\nabla f\|_\infty h.
\end{aligned}
$$

Since $f \in W_2^s(\mathbb{R}^d)$ for all $s \geq 0$, we apply first the Sobolev embedding (with $s > d/2 + 1$) and then Lemma 2.1 and obtain

$$
\|\nabla f\|_\infty \leq C_1 \|f\|_{W_2^s} \leq C \|f\|_2,
$$

since $f \in \mathrm{PW}_\Omega(H_a)$. Consequently,

$$
|\langle f, \chi_{[0,\Omega]}(H_a)\varphi_x^h - k_x \rangle| \leq Ch \|f\|_2.
$$

Taking the supremum over $f \in \mathrm{PW}_\Omega(H_a)$, we obtain

$$
\|\chi_{[0,\Omega]}(H_a)\varphi_x^h - k_x\|_2 = \sup_{f \in \mathrm{PW}_\Omega(H_a), \|f\|_2 = 1} \langle f, \chi_{[0,\Omega]}(H_a)\varphi_x^h - k_x \rangle \leq Ch.
$$

As this estimate is independent of $x$, we have shown that $\chi_{[0,\Omega]}(H_a)\varphi_x^h \to k_x$ in $L^2(\mathbb{R}^d)$ uniformly in $x$. $\quad\square$

The following result relates the reproducing kernel of a limit operator of $H_a$ to the original kernel. It expresses a form of continuous dependence of the reproducing kernel of the matrix symbol of $H_a$. We will denote the point spectrum of an operator $H$ by $\sigma_p(H)$.

**Theorem 4.2.** *Let $H_a$ with symbol $a \in C_b^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$, and let $(x_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R}^d$ be an unbounded net such that* $\lim_\lambda T_{-x_\lambda} a = b$ *pointwise. Assume that* $\Omega \notin \sigma_p(H_b)$. *Let* $\tilde{k}$ *be the reproducing kernel of* $\mathrm{PW}_\Omega(H_b)$. *Then*

$$\lim_\lambda T_{-x_\lambda} k_{x_\lambda} = \tilde{k}_0,$$

*with convergence in $W_2^s$ for every $s \geq 0$.*

Before the proof we remind the reader of the following standard facts of spectral theory; see, e.g., [Teschl 2009, Chapter 6.6]. Although in the literature these results are formulated for sequences of operators, the statements and proofs are equally valid for nets.[2]

Let $H_\lambda$, $\lambda \in \Lambda$, and $H_b$ be self-adjoint operators with a common core $\mathscr{D}$. If $H_\lambda f \to H_b f$ for all $f \in \mathscr{D}$, then, for every $F \in C_b(\mathbb{R})$,

$$F(H_\lambda)f \to F(H_b)f \quad \text{for all } f \in L^2(\mathbb{R}^d). \tag{4-1}$$

Furthermore, if $\chi_{\{\alpha\}}(H_b) = \chi_{\{\beta\}}(H_b) = 0$, i.e., $\alpha, \beta \notin \sigma_p(H_b)$, then

$$\chi_{[\alpha,\beta]}(H_\lambda)f \to \chi_{[\alpha,\beta]}(H_b)f \quad \text{for all } f \in L^2(\mathbb{R}^d). \tag{4-2}$$

*Proof of Theorem 4.2.* We split the difference $T_{-x_\lambda} k_{x_\lambda} - \tilde{k}_0$ into three terms and then estimate their $W_2^s$-norms separately:

$$\|T_{-x_\lambda} k_{x_\lambda} - \tilde{k}_0\|_{W_2^s}$$
$$\leq \|T_{-x_\lambda} k_{x_\lambda} - T_{-x_\lambda} \chi_{[0,\Omega]}(H_a)\varphi_{x_\lambda}^h\|_{W_2^s} + \|T_{-x_\lambda} \chi_{[0,\Omega]}(H_a)\varphi_{x_\lambda}^h - \chi_{[0,\Omega]}(H_b)\varphi_0^h\|_{W_2^s} + \|\chi_{[0,\Omega]}(H_b)\varphi_0^h - \tilde{k}_0\|_{W_2^s}$$
$$= (I) + (II) + (III).$$

Choose $\varepsilon > 0$.

Step 1: Expression $(I)$ can be estimated by

$$\|T_{-x_\lambda} k_{x_\lambda} - T_{-x_\lambda} \chi_{[0,\Omega]}(H_a)\varphi_{x_\lambda}^h\|_{W_2^s} = \|k_{x_\lambda} - \chi_{[0,\Omega]}(H_a)\varphi_{x_\lambda}^h\|_{W_2^s} \leq C_s \|k_{x_\lambda} - \chi_{[0,\Omega]}(H_a)\varphi_{x_\lambda}^h\|_2.$$

The first equality holds by the translation invariance of the Sobolev norm; the second inequality is a consequence of Lemma 2.1. By Lemma 4.1 there exists $h_\varepsilon > 0$ such that, for every $0 < h < h_\varepsilon$,

$$\|k_x - \chi_{[0,\Omega]}(H_a)\varphi_x^h\|_2 < \frac{\varepsilon}{3C_s}$$

for *all* $x \in \mathbb{R}^d$. So for $h < h_\varepsilon$, we obtain $(I) < \varepsilon/3$. Similarly, we achieve $(III) < \varepsilon/3$ for every $h < h_\varepsilon'$.

Step 2: To bound the decisive term $(II)$, we bring in limit operators and elliptic regularity theory. Set $a_\lambda = T_{-x_\lambda} a$. First note that

$$T_{-x_\lambda} \chi_{[0,\Omega]}(H_a)\varphi_{x_\lambda}^h = T_{-x_\lambda} \chi_{[0,\Omega]}(H_a)T_{x_\lambda}\varphi_0^h = \chi_{[0,\Omega]}(\tau_{x_\lambda} H_a)\varphi_0^h = \chi_{[0,\Omega]}(H_{a_\lambda})\varphi_0^h.$$

We have to verify that

$$\lim_\lambda \|\chi_{[0,\Omega]}(H_{a_\lambda})\varphi_0^h - \chi_{[0,\Omega]}(H_b)\varphi_0^h\|_{W_2^s} = 0. \tag{4-3}$$

---

[2]The cited results use the strong operator topology. As this topology is metrizable on bounded sets, the convergence of nets is equivalent to the convergence of sequences.

For $L^2$-convergence ($s = 0$) we argue as follows. By Lemma 2.7 the translates $T_{-x_\lambda}a$ converge to the matrix $b$ uniformly on compact sets. Proposition 2.8 implies that $H_{T_{-x_\lambda}a}f \to H_b f$ for $f \in W_2^s$, $s \geq 0$. To apply (4-2), we note that $C_c^\infty(\mathbb{R}^d)$ is a common core for all $H_{a_\lambda}$ and for $H_b$ and that $0 \notin \sigma_p(H_b)$ by (2-3) and $\Omega \notin \sigma_p(H_b)$ by assumption. Therefore (4-3) follows from (4-2).

For the convergence of (4-3) in general Sobolev spaces $W_2^s$ it suffices to treat the case $s = 2k$ for every integer $k$. Recall that by the results on elliptic regularity in Section 2B the operator $(H_a^k + c_k)$ defines an isomorphism from $W_2^{2k}(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$, and since $\tau_{x_\lambda}(H_a^k + c_k) = H_{a_\lambda}^k + c_k$ we obtain

$$\|H_{a_\lambda}^k + c_k\|_{W_2^{2k} \to L^2} = \|H_a^k + c_k\|_{W_2^{2k} \to L^2} < \infty.$$

The Sobolev norm can be estimated by the $L^2$-norm

$$\|f\|_{W_2^{2k}} = \|T_x f\|_{W_2^{2k}} \leq C_s \|(H_a^k + c_k)T_x f\|_2 = C_s \|T_{-x}(H_a^k + c_k)T_x f\|_2$$

independently of $x \in \mathbb{R}^d$. Thus $(II)$ can be estimated by the $L^2$-norm, namely

$$\begin{aligned}\|\chi_{[0,\Omega]}(H_{a_\lambda})\varphi_0^h - \chi_{[0,\Omega]}(H_b)\varphi_0^h\|_{W_2^s} &\leq C_s \|(H_{a_\lambda}^k + c_k)\chi_{[0,\Omega]}(H_{a_\lambda})\varphi_0^h - (H_{a_\lambda}^k + c_k)\chi_{[0,\Omega]}(H_b)\varphi_0^h\|_2 \\ &\leq C_s \|(H_{a_\lambda}^k + c_k)\chi_{[0,\Omega]}(H_{a_\lambda})\varphi_0^h - (H_b^k + c_k)\chi_{[0,\Omega]}(H_b)\varphi_0^h\|_2 \\ &\quad + C_s \|(H_b^k + c_k)\chi_{[0,\Omega]}(H_b)\varphi_0^h - (H_{a_\lambda}^k + c_k)\chi_{[0,\Omega]}(H_b)\varphi_0^h\|_2 \\ &= A_\lambda + B_\lambda.\end{aligned}$$

By Proposition 2.8 we have $(H_{a_\lambda}^k + c_k)f \to (H_b^k + c_k)f$ in $L^2$-norm for all $f \in W_2^{2k}$. In particular, this holds for $f = \chi_{[0,\Omega]}(H_b)\varphi_0^h$; thus $\lim_\lambda B_\lambda = 0$.

For the first term we use spectral theory again. Define $F \in C_c(\mathbb{R})$ such that its restriction to $[0, \Omega]$ satisfies

$$F(t) = t^k + c_k \quad \text{for } t \in [0, \Omega].$$

Then $F(t)\chi_{[0,\Omega]}(t) = (t^k + c_k)\chi_{[0,\Omega]}(t)$, and $\lim_\lambda F(\tau_{x_\lambda}(H_a))f = F(H_b)f$ for all $f \in L^2(\mathbb{R}^d)$ by (4-1). Since the product of bounded operators is continuous in the strong operator topology, it follows that

$$\begin{aligned}\lim_\lambda (H_{a_\lambda}^k + c_k)\chi_{[0,\Omega]}(H_{a_\lambda})\varphi_0^h &= \lim_\lambda F(H_{a_\lambda})\left(\lim_\lambda \chi_{[0,\Omega]}(H_{a_\lambda})\varphi_0^h\right) \\ &= F(H_b)\chi_{[0,\Omega]}(H_b)\varphi_0^h = (H_b^k + c_k)\chi_{[0,\Omega]}(H_b)\varphi_0^h,\end{aligned}$$

and so $\lim_\lambda A_\lambda = 0$.

We can finish the proof as follows. We have already chosen $h < \min\{h_\varepsilon, h_\varepsilon'\}$ so that the terms $(I)$ and $(III)$ are $< \varepsilon/3$ for all $\lambda \in \Lambda$. For this fixed $h > 0$ we can find an index $\lambda_0$ such that

$$(II) \leq C_s \|(H_{a_\lambda}^k + c_k)\chi_{[0,\Omega]}(H_{a_\lambda})\varphi_0^h - (H_{a_\lambda}^k + c_k)\chi_{[0,\Omega]}(H_b)\varphi_0^h\|_2 < \frac{\varepsilon}{3}$$

for all $\lambda \succeq \lambda_0$. Altogether we obtain $\|T_{-x_\lambda}k_{x_\lambda} - \tilde{k}_0\|_2 \leq (I) + (II) + (III) < \varepsilon$.  □

**Theorem 4.3.** *Assume that $H_a$ is uniformly elliptic with symbol $a \in C_b^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$ and that no limit operator has the eigenvalue $\Omega$. Then the set $\{T_{-x}k_x : x \in \mathbb{R}^d\}$ is relatively compact in $W_2^s$ for every $s \geq 0$.*

*Proof.* This follows directly from Theorem 4.2. Let $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}^d$ be an arbitrary sequence. By Lemma 2.7 the sequence $T_{-x_n} a$ has a $C^\infty$-convergent subsequence $T_{-x_{n_l}} a$. If $(x_{n_l})_{l \in \mathbb{N}}$ is bounded, we can assume without loss of generality that $x_{n_l} \to x \in \mathbb{R}^d$, and $T_{-x_{n_l}} k_{x_{n_l}} \to T_{-x} k_x$ in $W_2^s$ by the continuity of the translations and Proposition 2.3. If $(x_{n_l})_{l \in \mathbb{N}}$ is unbounded, we can assume $|x_{n_l}| \to \infty$. This case is settled by Theorem 4.2 and yields the convergence of $T_{-x_{n_l}} k_{x_{n_l}}$. $\qquad\square$

A combination of the above arguments yields the weak localization (WL) .

**Theorem 4.4.** *Assume that $H_a$ is uniformly elliptic with symbol $a \in C_b^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$ and that no limit operator has the eigenvalue $\Omega$. Let $k$ be the reproducing kernel of $\mathrm{PW}_\Omega(H_a)$. Then $k$ satisfies the weak localization property* (WL), *i.e.,*

$$\lim_{R \to \infty} \int_{|y-x|>R} |k(x,y)|^2 \, dy = 0.$$

*Proof.* By Theorem 4.3 (for $s = 0$) the set $\{T_{-x} k_x : x \in \mathbb{R}^d\}$ is relatively compact in $L^2(\mathbb{R}^d)$. The Riesz–Kolmogorov theorem implies that for all $\varepsilon > 0$ there is $R > 0$ such that for all $x \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d \setminus B_R(0)} |T_{-x} k_x(y)|^2 \, dy < \varepsilon^2.$$

By a change of variable this expression reads as

$$\int_{|y-x|>R} |k(x,y)|^2 \, dy < \varepsilon^2,$$

and this is (WL). $\qquad\square$

## 5. Proof of the homogeneous approximation property (HAP)

Next we prove the homogeneous approximation property. Recall that $T_x \kappa$ is the reproducing kernel for $W_2^s$ with $\hat{\kappa}(\omega) = (1 + |\omega|^2)^{-s}$.

**Lemma 5.1.** *If $S$ is a relatively separated set in $\mathbb{R}^d$, then $\{T_x \kappa : x \in S\}$ is a Bessel sequence for $W_2^s$, $s > d/2$.*

*Proof.* By standard facts of frame theory (see, e.g., [Heil 2011, Theorem 7.6]) the Bessel property is equivalent to the boundedness of the Gramian $G = (\langle T_x \kappa, T_y \kappa \rangle_{W_2^s})_{x,y \in S}$ on $\ell^2(S)$. To deduce the boundedness of $G$ we first show that $G$ possesses exponential off-diagonal decay and then apply Schur's test. The off-diagonal decay follows from a (well-known) calculation. Let $\mathscr{J}_r$ denote the Bessel function of the first kind and $\mathscr{K}_r$ the modified Bessel function of the second kind. Then by [Wendland 2005, Theorem 6.13] or [Grafakos 2004, Appendix B]

$$\begin{aligned}
\langle T_x \kappa, T_y \kappa \rangle_{W_2^s} &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \widehat{T_x \kappa}(\omega) \overline{\widehat{T_y \kappa}}(\omega) (1 + |\omega|^2)^s \, d\omega \\
&= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i(x-y)\omega} (1 + |\omega|^2)^{-s} \, d\omega \\
&= C |x-y|^{-(d-2)/2} \int_0^\infty (1 + r^2)^{-s} \mathscr{J}_{(d-2)/2}(r|x-y|) r^{d/2} \, dr \\
&= C' |x-y|^{s-d/2} \mathscr{K}_{s-d/2}(|x-y|).
\end{aligned}$$

Using the asymptotic decay $\mathscr{K}_r(x) \sim \sqrt{\pi/(2x)}e^{-x}$ for $x \to \infty$, see, e.g., [DLMF 2020, equation 10.25.3], the off-diagonal decay of $G$ is

$$|\langle T_x\kappa, T_y\kappa\rangle_{W_2^s}| \leq C''|x-y|^{s-d/2-1/2}e^{-|x-y|} \quad (|x-y| \to \infty). \tag{5-1}$$

The off-diagonal decay of the Gramian implies the boundedness of the Gramian as follows. By (5-1) there exists $N_0 \in \mathbb{N}$ such that $|G_{xy}| \leq Ce^{-c|x-y|}$ if $|x-y| > N_0$. Obviously, $|G_{xy}| \leq \|\kappa\|_{W_2^s}^2$ is bounded for all $x, y$.

For $x \in S$ and $k \in \mathbb{N}_0$ set $A_k(x) = \{y \in S : k < |y-x| \leq k+1\}$. Since $S \subset \mathbb{R}^d$ is relatively separated, there exists $r > 0$ such that

$$\max \#(S \cap B_r(x)) < \infty.$$

A covering argument (of a large ball $B_R(z)$ by balls $B_r(x)$) implies that $\#(S \cap B_R(z)) = \mathcal{O}(R^d)$ for arbitrary $R > 0$. Consequently we also obtain $\#A_k(x) \leq Ck^d$ independent of $x$. Then

$$\sum_{y \in S}|G_{xy}| = \sum_{k=0}^{\infty}\sum_{y \in A_k(x)}|G_{xy}| = \sum_{k=0}^{N_0}\sum_{y \in A_k(x)}|G_{xy}| + \sum_{k > N_0}\sum_{y \in A_k(x)}|G_{xy}|$$

$$\leq C_0 \#(B_{N_0+1}(x) \cap S) + C\sum_{k > N_0}e^{-ck}\#A_k(x)$$

$$\leq C_1(N_0+1)^d + C_2\sum_{k > N_0}e^{-ck}k^d.$$

This expression is bounded independently of $x$. Now Schur's test implies that the Gramian is bounded on $\ell^2(S)$. $\qquad\square$

**Theorem 5.2** (HAP). *Assume that $H_a$ is uniformly elliptic with symbol $a \in C_b^{\infty}(\mathbb{R}^d, \mathbb{C}^{d \times d})$ and that $\Omega \notin \sigma_p(H_b)$ for every limit operator $H_b$. Let $\{k_x : x \in S\}$ be a Bessel sequence in $\mathrm{PW}_\Omega(H_a)$. Then for every $\varepsilon > 0$ there exists an $R > 0$ such that for all $y \in \mathbb{R}^d$*

$$\sum_{x \in S \setminus B_R(y)}|k(y,x)|^2 < \varepsilon^2.$$

*Proof.* If $\{k_x : x \in S\}$ is a Bessel sequence of reproducing kernels, then $S$ is relatively separated in $\mathbb{R}^d$ (see [Führ et al. 2017, Lemma 3.7]). Lemma 5.1 implies that $\{T_x\kappa : x \in S\}$ is also a Bessel sequence in $W_2^s$ for $s > d/2$.

Choose $\varepsilon > 0$. Since $\{T_{-x}k_x : x \in \mathbb{R}^d\}$ is relatively compact in $W_2^s$ for $s \geq 0$ by Theorem 4.3, the Riesz–Kolmogorov theorem for translation-invariant Banach spaces [Feichtinger 1984] asserts that there exists a $R = R_\varepsilon > 0$ and a function $\psi \in C_c^{\infty}(\mathbb{R}^d)$ satisfying $\psi|_{B_{R/2}(0)} = 1$, supp $\psi \subseteq B_R(0)$ such that

$$\|T_{-x}k_x(1-\psi)\|_{W_2^s} \leq \varepsilon \quad \text{for all } x \in \mathbb{R}^d.$$

We now use the fundamental observation (2-7) that the point evaluation in $\mathrm{PW}_\Omega(H_a)$ can be expressed in two ways. For $f = k_x$ we have

$$k(y,x) = \langle k_y, k_x\rangle_{L^2} = \langle k_y, T_x\kappa\rangle_{W_2^s}. \tag{5-2}$$

Since $\{T_x\kappa : x \in S\}$ is a Bessel sequence in $W_2^s$ with bound $B$, the set $\{T_{x-y}\kappa : x \in S\}$ is a Bessel sequence with the same bound. Observe that for $|u| > R$ we obtain

$$\langle \psi T_{-y}k_y, T_u\kappa \rangle_{W_2^s} = T_{-y}k_y(u)\psi(u) = 0.$$

This implies

$$\sum_{x \in S \setminus B_R(y)} |\langle k_y, T_x\kappa \rangle_{W_2^s}|^2 = \sum_{x \in S \setminus B_R(y)} |\langle T_{-y}k_y, T_{x-y}\kappa \rangle_{W_2^s}|^2$$

$$= \sum_{x \in S \setminus B_R(y)} |\langle (1 - \psi)T_{-y}k_y, T_{x-y}\kappa \rangle_{W_2^s}|^2$$

$$\leq B \|(1 - \psi)T_{-y}k_y\|_{W_2^s}^2 \leq B\varepsilon^2,$$

and this is the homogeneous approximation property, (HAP). $\qquad\square$

*Proof of Theorem 3.1.* After the verification of the properties (WL) and (HAP) of the reproducing kernel, the version for the dimension-free density of Theorem 2.4 is applicable and yields Theorem 3.1. The statement asserts the existence of a critical density for sampling and interpolation with the dimension-free Beurling density $D_0^{\pm}(S)$. $\qquad\square$

## 6. Geometric Beurling densities

In this section we derive results for the geometric densities (2-5). According to Theorem 2.4 this step requires the computation of the averaged trace $|\mu(B_r(x))|^{-1} \int_{B_r(x)} k(x, x) \, d\mu(x)$ of the reproducing kernel. This version of the density theorems is of interest because it relates the critical density in $\mathrm{PW}_\Omega(H_a)$ to the geometry defined by the differential operator $H_a$. The explicit computation of the averaged trace becomes possible by introducing a suitable compactification of $\mathbb{R}^d$ and then extending the centered kernels $T_{-x}k_x$ to this compactification.

**6A. *The basic computation*: *constant coefficients*.** For reference we mention the case when $H_b = -\sum_{j,k} \partial_j b_{jk} \partial_k = -\sum_{j,k} b_{jk}\partial_j\partial_k$ is a differential operator with constant coefficients $b_{jk}$. Define

$$\Sigma_\Omega^b = \{\xi \in \mathbb{R}^d : b\xi \cdot \xi \leq \Omega\} = b^{-1/2}\overline{B_{\Omega^{1/2}}(0)},$$

with volume

$$|\Sigma_\Omega^b| = \det(b^{-1/2})|B_{\Omega^{1/2}}(0)| = \det(b^{-1/2})\Omega^{d/2}|B_1|. \tag{6-1}$$

Since $\widehat{H_b f}(\xi) = \sum_{j,k} b_{jk}\xi_j\xi_k \hat{f}(\xi) = (b\xi \cdot \xi)\hat{f}(\xi)$, the spectral subspace is

$$\mathrm{PW}_\Omega(H_b) = \chi_{[0,\Omega]}(H_b)L^2(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \mathrm{supp}\,\hat{f} \subseteq \Sigma_\Omega^b\}.$$

The kernel of $\mathrm{PW}_\Omega(H_b)$ is

$$\tilde{k}(x, y) = (2\pi)^{-d/2}(\mathscr{F}^{-1}\chi_{\Sigma_\Omega^b})(x - y), \tag{6-2}$$

whence

$$\tilde{k}(x, x) = (2\pi)^{-d/2}(\mathscr{F}^{-1}\chi_{\Sigma_\Omega^b})(0) = \frac{|\Sigma_\Omega^b|}{(2\pi)^d} = \frac{|B_1|}{(2\pi)^d}\det(b^{-1/2})\Omega^{d/2}.$$

By Landau's theorem [1967] a sampling set $S$ for $\mathrm{PW}_\Omega(H_b)$ has Beurling density at least $D^-(S) \geq |\Sigma_\Omega^b|/(2\pi)^d$.

**6B.** *The Higson compactification.* We recall how a compactification arises in Gelfand theory. Let $C_\gamma(\mathbb{R}^d)$ be a unital $C^*$ algebra of functions on $\mathbb{R}^d$ satisfying the embeddings $C_0(\mathbb{R}^d) \subset C_\gamma(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$. The *maximal ideal space $M_\gamma$* of $C_\gamma(\mathbb{R}^d)$ is the space of all multiplicative homomorphisms $\varphi : C_\gamma(\mathbb{R}^d) \to \mathbb{C}$. Equipped with the weak-star topology $M_\gamma$ is a compact Hausdorff space. The point evaluations $\delta_x(f) = f(x)$ constitute an embedding $\gamma$ of $\mathbb{R}^d$ into $M_\gamma$ via $\gamma(x) = \delta_x$, and $\gamma(\mathbb{R}^d)$ is dense in $M_\gamma$ and homeomorphic to $\mathbb{R}^d$. Thus, the pair $(\gamma, M_\gamma)$ is a compactification of $\mathbb{R}^d$, which we will call $\gamma\mathbb{R}^d$. The corona of $\gamma\mathbb{R}^d$ is $\partial_\gamma\mathbb{R}^d = \gamma\mathbb{R}^d \setminus \gamma(\mathbb{R}^d)$. By abuse of notation we will identify a point $x \in \mathbb{R}^d$ with its point evaluation $\delta_x$. Then $C_\gamma(\mathbb{R}^d)$ is isometrically isomorphic to $C(\gamma\mathbb{R}^d)$. We denote the image of $f \in C_\gamma(\mathbb{R}^d)$ in $C(\gamma\mathbb{R}^d)$ by $\bar{f}$. See, e.g., [Engelking 1977] for the basics of compactifications, and [Gamelin 1969] for compactifications of function algebras.

As noted in Section 2D2 the space $C_h(\mathbb{R}^d)$ of slowly oscillating functions with supremum norm is a commutative unital $C^*$-algebra. Thus there is a compactification $h\mathbb{R}^d$ of $\mathbb{R}^d$, the *Higson compactification*, such that $C_h(\mathbb{R}^d)$ is isometrically isomorphic to $C(h\mathbb{R}^d)$. It is known that $h\mathbb{R}^d$ is nonmetrizable, and even more, points of the corona $h\mathbb{R}^d \setminus \mathbb{R}^d$ can only be reached by nets; see, e.g., [Rabinovich et al. 2004a, 2.4.10]. Therefore we need to work with nets instead of sequences.

The relevance of the Higson compactification and the algebra of slowly oscillating functions in our context is given by the fact that translations act trivially on the corona $\partial_h\mathbb{R}^d$.

**Lemma 6.1.** *If $x_\lambda \to \eta \in \partial_h\mathbb{R}^d$, then $x + x_\lambda \to \eta$ for all $x \in \mathbb{R}^d$.*

*Proof.* By definition, $x_\lambda \to \eta \in \partial_h\mathbb{R}^d$ if $f(x_\lambda) \to \bar{f}(\eta) = \eta(f)$ for every $f \in C_h(\mathbb{R}^d)$. From the definition of $C_h(\mathbb{R}^d)$ we obtain

$$\lim_\lambda |f(x_\lambda) - f(x + x_\lambda)| = 0$$

for every $x \in \mathbb{R}^d$, so $f(x_\lambda + x) \to \bar{f}(\eta)$ for every $f \in C_h(\mathbb{R}^d)$ as well. $\square$

One can show that $h\mathbb{R}^d$ is the *maximal* compactification of $\mathbb{R}^d$ with this property: every $C_\gamma(\mathbb{R}^d)$ as above with translations acting trivially on $\partial_\gamma\mathbb{R}^d$ is a subalgebra of $C_h(\mathbb{R}^d)$.

We need the following fact [Roe 2003].

**Proposition 6.2.** *Let $C_\gamma(\mathbb{R}^d)$ be a $C^*$-algebra of functions on $\mathbb{R}^d$ as above with corresponding compactification $\gamma\mathbb{R}^d$ of $\mathbb{R}^d$. If $f \in C_\gamma(\mathbb{R}^d)$ satisfies*

$$\bar{f}|_{\partial_\gamma\mathbb{R}^d} \equiv 0$$

*then $f \in C_0(\mathbb{R}^d)$.*

*Proof.* Let $(x_\lambda)_{\lambda\in\Lambda} \subset \mathbb{R}^d$ be an unbounded net, $\lim_\lambda |x_\lambda| = \infty$. As $\gamma\mathbb{R}^d$ is compact, every subnet of $(x_\lambda)$ has a convergent subnet $(x_\mu)_{\mu\in M}$, and $\lim_\mu x_\mu = \eta \in \partial_\gamma\mathbb{R}^d$ by the assumption of unboundedness. So $\lim_\mu f(x_\mu) = \bar{f}(\eta) = 0$ for a subnet of a given subnet, and therefore $\lim_\lambda f(x_\lambda) = 0$. This means $f \in C_0(\mathbb{R}^d)$. $\square$

We next study uniformly elliptic operators $H_a$ with a symbol in $a \in C_h^\infty(\mathbb{R}^d, \mathbb{C}^{d\times d})$. By definition $a$ has a continuous extension to $h\mathbb{R}^d$. Thus for $\eta \in \partial_h\mathbb{R}^d$ the symbol $\bar{a}(\eta) = \lim_{x\to\eta} a(x)$ is well-defined, and by Lemma 2.13 $H_{\bar{a}(\eta)}$ is a differential operator with constant coefficients. Let $k^\eta$ denote the reproducing kernel of $\mathrm{PW}_\Omega(H_{\bar{a}(\eta)})$. We show that the mapping $x \mapsto T_{-x}k_x$ has a continuous extension to $h\mathbb{R}^d$.

**Proposition 6.3.** *Let the symbol $a \in C_h^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$ be the symbol of the operator $H_a$ and let $k_x$ be the reproducing kernel of $\mathrm{PW}_\Omega(H_a)$. Set $K(x) = T_{-x} k_x \in L^2(\mathbb{R}^d)$. Then $K$ extends to a continuous function from $h\mathbb{R}^d$ to $L^2(\mathbb{R}^d)$ by setting $K(\eta) = k_0^\eta$ for $\eta \in h\mathbb{R}^d$. In particular, the diagonal $k(x,x) = \|k_x\|_2^2$ is a slowly oscillating function.*

*Proof.* By Proposition 2.3 the centered reproducing kernel $K$ is continuous from $\mathbb{R}^d$ to $L^2(\mathbb{R}^d)$. To show that $K \in C_h(\mathbb{R}^d, L^2)$, we need to extend $K$ to the Higson corona $\partial_h \mathbb{R}^d$.

This is accomplished by means of Theorem 4.2. Let $(x_\lambda) \subseteq \mathbb{R}^d$ be an unbounded net such that $x_\lambda \to \eta \in \partial_h \mathbb{R}^d$. Since $a \in C_h(\mathbb{R}^d, \mathbb{C}^{d \times d})$, there is a continuous function $\bar{a} \in C(h\mathbb{R}^d, \mathbb{C}^{d \times d})$ such that $\lim_\lambda a(x_\lambda) = \bar{a}(\eta)$. Furthermore, for $x \in \mathbb{R}^d$ arbitrary, $x + x_\lambda \to \eta$ by Lemma 6.1, and this fact implies the pointwise convergence

$$\lim_\lambda T_{-x_\lambda} a(x) = \bar{a}(\eta).$$

Clearly, the spectrum of the (constant-coefficient) operator $H_{\bar{a}(\eta)}$ is continuous and does not contain any eigenvalues. The assumptions of Theorem 4.2 are thus satisfied.

To formulate its conclusion, denote the reproducing kernel of $\mathrm{PW}_\Omega(H_{\bar{a}(\eta)})$ by $k^\eta$. Then by Theorem 4.2

$$\lim_\lambda T_{-x_\lambda} k_{x_\lambda} = k_0^\eta$$

in the $L^2$-norm, and this holds for every net $(x_\lambda)$ with $x_\lambda \to \eta$. Thus we must take the limiting function to be

$$K(\eta) = k_0^\eta = (2\pi)^{-d/2} \mathscr{F}^{-1}(\chi_{\Sigma_\Omega^{\bar{a}(\eta)}}),$$

with the explicit formula for the kernel given by (6-2).

It remains to be shown that the limiting kernel $K$ is continuous on $\partial_h \mathbb{R}^d$. Let $\eta_\lambda \to \eta \in \partial_h \mathbb{R}^d$. Then with the definition of $\Sigma_\Omega^b$ and (6-1) we obtain

$$\|k_0^{\eta_\lambda} - k_0^\eta\|_2^2 = (2\pi)^{-d} \|\chi_{\Sigma_\Omega^{\bar{a}(\eta_\lambda)}} - \chi_{\Sigma_\Omega^{\bar{a}(\eta)}}\|_2^2$$
$$= (2\pi)^{-d}(|\Sigma_\Omega^{\bar{a}(\eta_\lambda)}| + |\Sigma_\Omega^{\bar{a}(\eta)}| - 2|\Sigma_\Omega^{\bar{a}(\eta_\lambda)} \cap \Sigma_\Omega^{\bar{a}(\eta)}|).$$

As $a \in C_h(\mathbb{R}^d, \mathbb{C}^{d \times d})$, $\bar{a}$ is continuous on $\partial_h \mathbb{R}^d$, and this expression tends to $0$, whence $K$ is also continuous on the corona $\partial_h \mathbb{R}^d$. $\qquad\square$

**6C. *Geometric densities for slowly oscillating symbols.*** In order to obtain values for the critical densities $D_\mu^\pm(S)$ we need the averaged traces

$$\mathrm{tr}_\mu^-(k) = \liminf_{r \to \infty} \inf_{x \in \mathbb{R}^d} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} k(z, z) \, dz$$

and $\mathrm{tr}_\mu^+(k)$. For the comparison of averaged traces we will need the following well-known fact, whose proof is supplied in Appendix A for completeness.

For $f \in C_b(\mathbb{R}^d)$ set

$$\mathrm{tr}^-(f) = \liminf_{r \to \infty} \inf_{y \in \mathbb{R}^d} \frac{1}{|B_r(y)|} \int_{B_r(y)} f(x) \, dx,$$

and define $\mathrm{tr}^+(f)$ similarly with sup instead of inf.

**Lemma 6.4.** *Assume that* $f, g \in C_b(\mathbb{R}^d)$ *and* $\lim_{x \to \infty} |f(x) - g(x)| = 0$. *Then*

$$\mathrm{tr}^-(f) = \mathrm{tr}^-(g) \quad \text{and} \quad \mathrm{tr}^+(f) = \mathrm{tr}^+(g).$$

**Proposition 6.5.** *If the symbol* $a$ *is in* $C_h^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$, *then the trace of the reproducing kernel satisfies*

$$\lim_{r \to \infty} \sup_{y \in \mathbb{R}^d} \left| \frac{1}{|B_r(y)|} \int_{B_r(y)} \left( k(x, x) - \frac{|B_1|}{(2\pi)^d} \frac{\Omega^{d/2}}{\det a(x)^{1/2}} \right) dx \right| = 0. \tag{6-3}$$

*Equivalently, using the Borel measure* $\nu(B) = \int_B \det a(x)^{-1/2} \, dx$,

$$\lim_{r \to \infty} \sup_{y \in \mathbb{R}^d} \left| \frac{1}{\nu(B_r(y))} \int_{B_r(y)} k(x, x) \, dx - \frac{|B_1|}{(2\pi)^d} \Omega^{d/2} \right| = 0. \tag{6-4}$$

*Consequently, the averaged trace is*

$$\mathrm{tr}_\nu^+(k) = \mathrm{tr}_\nu^-(k) = \frac{|B_1|}{(2\pi)^d} \Omega^{d/2}. \tag{6-5}$$

*Proof.* We apply Lemma 6.4 to the functions $f(x) = k(x, x)$ and $g(x) = (|B_1| \Omega^{d/2}/(2\pi)^d) \det a(x)^{-1/2}$. Then $k(x, x)$ is bounded by Proposition 2.2 and continuous by Proposition 2.3. Likewise $\det a(x)^{-1/2}$ is bounded and continuous by elliptic regularity. By assumption on $a$ and Proposition 6.3 both functions are in $C_h(\mathbb{R}^d)$ and thus possess the limits $\bar{a}(\eta)$ and $\|K(\eta)\|_2^2$; in particular for $a$ this means that

$$\lim_{x_\lambda \to \eta} \det a(x_\lambda)^{-1/2} = \det \bar{a}(\eta)^{-1/2}.$$

Using the notation of Section 6A and Proposition 6.3, we obtain

$$\lim_{x_\lambda \to \eta} \|k_{x_\lambda}\|_2^2 = \lim_{x_\lambda \to \eta} \|T_{-x_\lambda} k_{x_\lambda}\|_2^2 = \|k_0^\eta\|_2^2 = |\Sigma_\Omega^{\bar{a}(\eta)}| = \frac{|B_1| \Omega^{d/2}}{(2\pi)^d} \det \bar{a}(\eta)^{-1/2}.$$

We conclude that both $f$ and $g$ have the same limit function, and therefore $f - g \in C_0(\mathbb{R}^d)$ by means of Proposition 6.2. Lemma 6.4 now yields (6-3). Equation (6-4) follows after multiplying with $|B_r(y)|/\nu(B_r(y))$ and taking limits. Finally, (6-5) is a direct consequence of (6-4). □

Equation (6-4) allows us to state our main result on geometric Beurling densities for operators with slowly oscillating symbols in a simple form. In order to do so we need an elementary result on the relation between density and a change of measure.

**Lemma 6.6.** *Let* $d\mu = h \, dx$ *for a positive, continuous function* $h$ *on* $\mathbb{R}^d$, *bounded above and below,* $0 < c \le h(z) \le C$ *for all* $z \in \mathbb{R}^d$. *Then the dimension-free density condition*

$$D_0^-(S) = \liminf_{r \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\#(S \cap B_r(x))}{\int_{B_r(x)} k(y, y) \, dy} \ge 1$$

*holds, if and only if*

$$D_\mu^-(S) \ge \mathrm{tr}_\mu^-(k).$$

*Similarly*

$$D_0^+(S) \le 1 \quad \text{if and only if} \quad D_\mu^+(S) \le \mathrm{tr}_\mu^-(k).$$

*Proof.* The inequality $D_0^-(S) \geq 1$ means that for all $\varepsilon > 0$ there is an $r_\varepsilon > 0$ such that for all $r > r_\varepsilon$

$$\#(S \cap B_r(x)) \geq (1 - \varepsilon) \int_{B_r(x)} k(y, y) \, dy,$$

or equivalently,

$$\frac{\#(S \cap B_r(x))}{\mu(B_r(x))} \geq (1 - \varepsilon) \frac{1}{\mu(B_r(x))} \int_{B_r(x)} k(y, y) \, dy.$$

Written in terms of the Beurling density, this is

$$D_\mu^-(S) \geq \liminf_{r \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\int_{B_r(x)} k(y, y) \, dy}{\mu(B_r(x))} = \mathrm{tr}_\mu^-(k).$$

The converse is obtained by reading the argument backwards. $\qquad\square$

As a direct consequence we obtain the main result on geometric Beurling densities for uniformly elliptic operators with slowly oscillating symbols. This is Theorem C of the Introduction.

**Theorem 6.7.** *Assume that $H_a = -\sum_{j,k=1}^d \partial_j a_{jk} \partial_k$ is uniformly elliptic with symbol $a \in C_h^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$. Let $\mathrm{PW}_\Omega(H_a) = \chi_{[0,\Omega]}(H_a) L^2(\mathbb{R}^d)$ be the corresponding Paley–Wiener space and set $d\nu(x) = \det(a(x))^{-1/2} \, dx$.*

- *If $S \subseteq \mathbb{R}^d$ is a set of stable sampling for $\mathrm{PW}_\Omega(H_a)$ then*

$$D_\nu^-(S) \geq \frac{|B_1|}{(2\pi)^d} \Omega^{d/2}.$$

- *If $S \subseteq \mathbb{R}^d$ is a set of interpolation for $\mathrm{PW}_\Omega(H_a)$, then*

$$D_\nu^+(S) \leq \frac{|B_1|}{(2\pi)^d} \Omega^{d/2}.$$

*Proof.* We only verify the first assertion. By Corollary 3.2, if $S$ is a set of stable sampling, then $D_0^-(S) \geq 1$. By Lemma 6.6 this is equivalent to

$$D_\nu^-(S) \geq \mathrm{tr}_\nu^-(k).$$

The averaged trace $\mathrm{tr}_\nu^-(k)$ was computed in (6-5) to be $(|B_1|/(2\pi)^d)\Omega^{d/2}$. $\qquad\square$

**Example 6.8.** We consider some special cases of Theorem 6.7.

(i) *Asymptotically constant symbols.* Assume that $a \in C_b^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$ and $\lim_{x \to \infty} a(x) = b$. Then it is straightforward to verify that $a \in C_h^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$ and $D_\nu^\pm(S) = (\det b)^{1/2} D^\pm(S)$. Thus we may use the original Beurling density, and Theorem 6.7 implies that a sampling set $S \subseteq \mathbb{R}^d$ for $\mathrm{PW}_\Omega(H_a)$ must have density

$$D^-(S) = (\det b)^{-1/2} D_\nu^-(S) \geq (\det b)^{-1/2} \frac{|B_1|}{(2\pi)^d} \Omega^{d/2} = \frac{|\Sigma_\Omega^b|}{(2\pi)^d},$$

and a set of interpolation $S$ in $\mathrm{PW}_\Omega(H_a)$ must satisfy $D^+(S) \leq |\Sigma_\Omega^b|/(2\pi)^d$. This is Corollary D of the Introduction. As was to be expected, this coincides with the critical density for the classical Paley–Wiener space $\mathrm{PW}_\Omega(H_b)$; see [Landau 1967].

(ii) *Symbols with radial limits.* Let us consider the class of symbols that possess radial limits at $\infty$. We say that $a \in C_b^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$ is spherically continuous, if it possesses radial limits in the following sense. There exists a continuous matrix function $b \in C(S^{d-1}, \mathbb{C}^{d \times d})$ such that

$$\lim_{r \to \infty} \sup_{\eta \in S^{d-1}} \|a(r\eta) - b(\eta)\|_X = 0.$$

A $3\varepsilon$-argument shows that these symbols are slowly oscillating. Consequently Theorem 6.7 holds for spherically continuous symbols in $C_b^\infty(\mathbb{R}^d, \mathbb{C}^{d \times d})$. Spherically continuous symbols are related to another compactification, the spherical compactification with corona $S^{d-1}$. In contrast to the Higson compactification, it is metrizable, but it is much smaller. See [Cordes 1979] for its use in partial differential equations.

**6D. *Variable bandwidth in dimension $d = 1$.*** Let $H_a$ be the differential operator

$$H_a f = -\frac{d}{dx}\left(a\frac{d}{dx}f\right)$$

on $L^2(\mathbb{R})$. This is a Sturm–Liouville operator on $\mathbb{R}$, and the ellipticity assumption amounts to the conditions $\inf_{x \in \mathbb{R}} a(x) > 0$ and $a \in C_b^\infty(\mathbb{R}^d)$. In [Gröchenig and Klotz 2017] we argued that the spectral subspaces of $H_a$ can be interpreted as spaces of locally variable bandwidth. Intuitively, the quantity $a(x)^{-1/2}$ is a measure for the bandwidth in a neighborhood of $x$. We apply Theorem 6.7 to $H_a$. The relevant measure is $d\nu(x) = a^{-1/2}(x)\,dx$, and $\nu(I) = \int_I a(x)^{-1/2}\,dx$ for $I \subseteq \mathbb{R}$. Then we have the following necessary density condition for functions of variable bandwidth (Corollary E of the Introduction).

**Corollary 6.9.** *Assume that $a \in C_b^\infty(\mathbb{R})$ and $\lim_{x \to \pm\infty} a'(x) = 0$. Let $\mathrm{PW}_\Omega(H_a)$ be the Paley–Wiener space associated to $H_a$.*

(i) *If $S$ is a sampling set for $\mathrm{PW}_\Omega(H_a)$, then*

$$D_\nu^-(S) = \liminf_{r \to \infty} \inf_{x \in \mathbb{R}} \frac{\#(S \cap [x-r, x+r])}{\nu([x-r, x+r])} \geq \frac{\Omega^{1/2}}{\pi}. \tag{6-6}$$

(ii) *If $S$ is a set of interpolation for $\mathrm{PW}_\Omega(H_a)$, then $D_\nu^+(S) \leq \Omega^{1/2}/\pi$.*

Arguing as in Lemma 6.6, equation (6-6) says that for $\varepsilon > 0$ and $r$ large enough we have

$$\#(S \cap [x-r, x+r]) \geq \left(\frac{\Omega^{1/2}}{\pi} - \varepsilon\right) \int_{x-r}^{x+r} a(y)^{-1/2}\,dy.$$

Thus the number of samples in an interval $[x-r, x+r]$ is determined by $a(x)^{-1/2}$, which is in line with our interpretation of $a^{-1/2}$ as the local bandwidth.

Corollary 6.9 is precisely the formulation of the necessary density conditions in [Gröchenig and Klotz 2017]. However, the main result of that work was proved under the restrictive assumption that $a$ is constant outside an interval $[-R, R]$. The proof there dwelt heavily on the scattering theory of one-dimensional Schrödinger operators. The method of this paper yields a significantly more general result with a completely different method of proof. Corollary 6.9 was our dream that motivated this work.

Finally we remark that the density conditions of Theorem 6.7 suggest that the Paley–Wiener spaces $\mathrm{PW}_\Omega(H_a)$ associated to a uniformly elliptic differential operator may be taken as an appropriate generalization of variable bandwidth to higher dimensions.

## 7. Outlook

We have proved necessary density conditions for sampling and interpolation in spectral subspaces of uniformly elliptic partial differential operators with slowly oscillating coefficients. These spectral subspaces may be taken as a suitable generalization of the notion of variable bandwidth to higher dimensions. The emphasis has been on a new method that combines elements from limit operators, regularity theory and heat kernel estimates, and the use of compactifications.

Clearly one can envision manifold extensions of our results and methods. Theorem 3.1 is stated for a significantly larger class of operators and symbols. For instance, it could be applied to higher-order partial differential operators or to Schrödinger operators and to symbols with less smoothness or to almost periodic symbols. However, the spectral theory of such operators is more involved and one needs to find conditions that prevent their limit operators from having a point spectrum at the ends of the spectral interval. As these questions belong to spectral theory rather than sampling theory, we plan to pursue them in a separate publication.

In a different direction one may consider the graph Laplacian on an infinite graph or even a metric measure space endowed with a kernel that satisfies Gaussian estimates [Coulhon et al. 2012]. While many steps of our proofs remain in place, this set-up opens numerous new questions.

Finally several hidden connections beg to be explored. The identity (6-3) resembles the famous Weyl formula for the asymptotic density of eigenvalues in a spectral interval [Hörmander 1968]. This observation invites the comparison of the Beurling density with the density of states in spectral theory. We plan to investigate some of these issues in future work.

## Appendix A: Averaged traces

For completeness we provide the proof of Lemma 6.4. Recall that

$$\mathrm{tr}^-(f) = \liminf_{r \to \infty} \inf_{y \in \mathbb{R}^d} \frac{1}{|B_r(y)|} \int_{B_r(y)} f(x)\, dx.$$

*If $f, g \in C_b(\mathbb{R}^d)$ and $\lim_{|x| \to \infty} |f(x) - g(x)| = 0$, then*

$$\mathrm{tr}^-(f) = \mathrm{tr}^-(g) \quad \text{and} \quad \mathrm{tr}^+(f) = \mathrm{tr}^+(g).$$

*Proof.* Set $h = f - g$, then $\lim_{x \to \infty} h(x) = 0$. We split the relevant averages as

$$\frac{1}{|B_r(y)|} \int_{B_r(y)} |h(x)|\, dx = \frac{1}{|B_r(y)|} \left[ \int_{B_r(y) \cap B_R^c(0)} + \int_{B_r(y) \cap B_R(0)} \right] |h(x)|\, dx = (I) + (II).$$

Given $\varepsilon > 0$, there exists an $R_\varepsilon > 0$ such that $\sup_{|x| \geq R_\varepsilon} |h(x)| < \varepsilon/2$. So,

$$(I) < \frac{|B_r(y) \cap B_{R_\varepsilon}^c(0)|}{|B_r(y)|} \frac{\varepsilon}{2} < \frac{\varepsilon}{2},$$

independent of $y$. For the second term observe that

$$(II) < \|h\|_\infty \frac{|B_r(y) \cap B_{R_\varepsilon}(0)|}{|B_r(y)|} \leq \|h\|_\infty \frac{|B_{R_\varepsilon}|}{|B_r(y)|} < \frac{\varepsilon}{2}$$

for $r > \left(\frac{2}{\varepsilon} \|h\|_\infty\right)^{1/d} R_\varepsilon$. Consequently,

$$\lim_{r \to \infty} \sup_{y \in \mathbb{R}^d} \frac{1}{|B_r(y)|} \int_{B_r(y)} |f - g| = 0.$$

It follows that

$$\mathrm{tr}^-(f) \leq \liminf_{r \to \infty} \inf_{y \in \mathbb{R}^d} \frac{1}{|B_r(y)|} \int_{B_r(y)} g + \lim_{r \to \infty} \sup_{y \in \mathbb{R}^d} \frac{1}{|B_r(y)|} \int_{B_r(y)} |f - g| = \mathrm{tr}^-(g).$$

Interchanging $f$ and $g$ yields equality. The equality $\mathrm{tr}^+(f) = \mathrm{tr}^+(g)$ is proved in the same way. □

## Appendix B: The lower bound for the reproducing kernel

We verify the lower bound for $\|k_x\|_2$ in the proof of Proposition 2.2. This fact is proved in [Coulhon et al. 2012, Lemma 3.19(a)]. For completeness we reproduce that proof with some necessary modifications and adjustments. The idea is to relate the reproducing kernel to the heat kernel of $e^{-tH_a}$ via functional calculus.

We write $k^\Omega$ for the reproducing kernel of $\mathrm{PW}_\Omega(H_a)$. For a bounded, nonnegative Borel function $F \geq 0$ with support in $[0, \Omega]$ we define $k_x^F = F(H_a)k_x^\Omega$ and the corresponding integral kernel $k^F(x, y) := F(H_a)(x, y) := k_x^F(y)$. The last expression is well-defined, as $k_x^F \in \mathrm{PW}_\Omega(H_a)$. The kernel $k^F(x, y)$ is symmetric, because $F(H_a)$ is self-adjoint:

$$k_x^F(y) = \langle F(H_a)k_x^\Omega, k_y^\Omega \rangle = \langle k_x^\Omega, F(H_a)k_y^\Omega \rangle = \overline{\langle F(H_a)k_y^\Omega, k_x^\Omega \rangle} = \overline{k_y^F(x)}.$$

Consequently, $F(H_a)$ is an integral operator. For $f \in L^2(\mathbb{R}^d)$

$$F(H_a)f(x) = \langle F(H_a)f, k_x^\Omega \rangle = \langle f, F(H_a)k_x^\Omega \rangle = \langle f, k_x^F \rangle = \int_{\mathbb{R}^d} k^F(y, x) f(y) \, dy.$$

If $0 \leq G \leq F$, then

$$0 \leq k^G(x, x) \leq k^F(x, x) \tag{B-1}$$

for all $x \in \mathbb{R}^d$. For the proof observe that $F - G \geq 0$ implies that

$$k^F(x, x) = \langle F(H_a)k_x^\Omega, k_x^\Omega \rangle \geq \langle G(H_a)k_x^\Omega, k_x^\Omega \rangle = k^G(x, x).$$

The heat operator $e^{-tH_a}$ is bounded and has a kernel $p_t(x, y)$ that satisfies *on diagonal estimates*. There are positive constants $c, C$ such that for all $x \in \mathbb{R}^d$ and $t > 0$

$$ct^{-d/2} \le p_t(x, x) \le Ct^{-d/2}.$$

This is well known; see, e.g., [Ouhabaz 2006].

**Claim.** *We have $0 < c < k^\Omega(x, x) < C$ for all $x \in \mathbb{R}^d$.*

*Proof.* As $\chi_\Omega(u) \le e \cdot e^{-u/\Omega} \chi_{[0,\infty)}(u)$ and $\chi_{[0,\infty)}(H_a) = \mathrm{Id}$, we obtain

$$k^\Omega(x, x) = \chi_{[0,\Omega]}(H_a)(x, x) \le ee^{-\Omega^{-1}H_a}(x, x) \le C\Omega^{d/2}, \tag{B-2}$$

which gives an explicit upper bound for $\|k_x\|_2^2$ in Proposition 2.2.

For the proof of the lower bound, we use a dyadic decomposition:

$$\chi_{[0,T)}(u)e^{-tu} \le \chi_{[0,\infty)}(u)e^{-tu} = \chi_{[0,\Omega]}(u)e^{-tu} + \sum_{k \ge 0} \chi_{(2^k\Omega, 2^{k+1}\Omega]}(u)e^{-tu}$$

$$\le \chi_{[0,\Omega]}(u) + \sum_{k \ge 0} \chi_{[0,2^{k+1}\Omega]}(u)e^{-t2^k\Omega}, \quad t > 0.$$

One can verify that this inequality remains true as an operator inequality

$$\chi_{[0,T]}(H_a)e^{-tH_a} \le \chi_{[0,\Omega]}(H_a) + \sum_{k \ge 0} \chi_{[0,2^{k+1}\Omega]}(H_a)e^{-t2^k\Omega},$$

with strong convergence of the sum, and every term is an integral operator. By (B-1) and (B-2) the operator inequality can be transferred to a corresponding inequality of the diagonals of the integral kernel as follows:

$$(\chi_{[0,T]}(H_a)e^{-tH_a})(x, x) \le \chi_{[0,\Omega]}(H_a)(x, x) + \sum_{k \ge 0} \chi_{[0,2^{k+1}\Omega]}(H_a)(x, x)e^{-t2^k\Omega}$$

$$\le \chi_{[0,\Omega]}(H_a)(x, x) + C\Omega^{d/2} \sum_{k \ge 0} 2^{(k+1)d/2} e^{-t2^k\Omega}.$$

In [Coulhon et al. 2012, equation (3.46)] it is shown that $p_t(x, y) = \lim_{T \to \infty}(\chi_{[0,T]}(H_a)e^{-tH_a})(x, y)$, consequently

$$ct^{-d/2} \le p_t(x, x) \le \chi_{[0,\Omega]}(H_a)(x, x) + C\Omega^{d/2} \sum_{k \ge 0} 2^{(k+1)d/2} e^{-t2^k\Omega}.$$

We choose $t = 2^r/\Omega$ for $r \in \mathbb{N}$ to be specified later. Then

$$c\Omega^{d/2}2^{-rd/2} \le \chi_{[0,\Omega]}(H_a)(x, x) + C\Omega^{d/2} \sum_{k \ge 0} e^{-2^k2^r} 2^{(k+1)d/2}$$

$$= \chi_{[0,\Omega]}(H_a)(x, x) + C2^{d/2}\Omega^{d/2}2^{-rd/2} \sum_{k \ge 0} e^{-2^{k+r}} 2^{(k+r)d/2}$$

$$\le \chi_{[0,\Omega]}(H_a)(x, x) + C2^{d/2}\Omega^{d/2}2^{-rd/2} \sum_{k \ge r} e^{-2^k} 2^{kd/2}.$$

Hence,

$$\Omega^{d/2} 2^{-rd/2} \left( c - C' 2^{d/2} \sum_{k \geq r} e^{-2^k} 2^{kd/2} \right) \leq \chi_{[0,\Omega]}(H_a)(x,x) = k^{\Omega}(x,x).$$

For $r \in \mathbb{N}$ sufficiently large, this implies the lower bound for $k^{\Omega}(x,x)$. $\qquad\square$

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