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EXPLICIT FORMULA OF RADIATION FIELDS OF FREE WAVES WITH APPLICATIONS ON CHANNEL OF ENERGY

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We give a few explicit formulas regarding the radiation fields of linear free waves. We then apply these formulas on the channel-of-energy theory. We characterize all the radial weakly nonradiative solutions in all dimensions and give a few new exterior energy estimates.

1. Introduction

1A. *Background and topics.* The semilinear wave equation

$$\partial_t^2 u - \Delta u = \pm |u|^{p-1} u, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R},$$

especially the energy critical case $p = 1 + 4/(d-2)$, has been extensively studied by many mathematicians in the past few decades. Please see, for example, [Kapitanski 1994; Lindblad and Sogge 1995] for local existence and well-posedness, and [Ginibre, Soffer and Velo 1992; Grillakis 1990; 1992; Kenig and Merle 2008; Nakanishi 1999a; 1999b; Shatah and Struwe 1993; 1994] for global existence, regularity, scattering and blow-up. Since the semilinear wave equation can be viewed as a small perturbation of the homogenous linear wave equation in many situations, especially when we consider the asymptotic behaviors of solutions as spatial variables or time tends to infinity, it is important to first understand the asymptotic behaviors of solutions to the homogenous linear wave equation, i.e., free waves. This work is concerned with two important tools to understand the asymptotic behaviors of free waves: radiation fields and the channel of energy. We first introduce some necessary notation. Throughout this work we consider the homogenous linear wave equation with initial data in the energy space

$$\begin{cases} \partial_t^2 u - \Delta u = 0, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ u|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^d), \\ u_t|_{t=0} = u_1 \in L^2(\mathbb{R}^d). \end{cases} \tag{1}$$

In this work we also use the notation $\mathbf{S}_L(u_0, u_1)$ to represent the free wave u defined above. If it is necessary to mention the velocity u_t, we use the notation

$$\mathbf{S}_L(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u(\cdot, t) \\ u_t(\cdot, t) \end{pmatrix} \in \dot{H}^1 \times L^2.$$

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It is well known that the linear wave propagation preserves the $\dot{H}^1 \times L^2$ norm, i.e., the energy conservation law holds ($\nabla_{x,t} u = (\nabla u, u_t)$):

$$\int_{\mathbb{R}^d} |\nabla_{x,t} u(x,t)|^2 \, dx = \int_{\mathbb{R}^d} (|\nabla u_0|^2 + |u_1|^2) \, dx.$$

Now we make a brief review of radiation fields and the channel-of-energy method.

Radiation fields. The asymptotic behavior of free waves at the energy level can be characterized by the following theorem.

Theorem 1.1 (radiation field). *Assume that $d \geq 3$ and let u be a solution to the free wave equation $\partial_t^2 u - \Delta u = 0$, with initial data $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$. Then*

$$\lim_{t \to \pm\infty} \int_{\mathbb{R}^d} \left(|\nabla u(x,t)|^2 - |u_r(x,t)|^2 + \frac{|u(x,t)|^2}{|x|^2} \right) dx = 0$$

and there exist two functions $G_\pm \in L^2(\mathbb{R} \times \mathbb{S}^{d-1})$ so that

$$\lim_{t \to \pm\infty} \int_0^\infty \int_{\mathbb{S}^{d-1}} |r^{(d-1)/2} \partial_t u(r\theta, t) - G_\pm(r \mp t, \theta)|^2 \, d\theta \, dr = 0,$$

$$\lim_{t \to \pm\infty} \int_0^\infty \int_{\mathbb{S}^{d-1}} |r^{(d-1)/2} \partial_r u(r\theta, t) \pm G_\pm(r \mp t, \theta)|^2 \, d\theta \, dr = 0.$$

In addition, the maps $(u_0, u_1) \to \sqrt{2} G_\pm$ are a bijective isometries from $\dot{H}^1 \times L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R} \times \mathbb{S}^{d-1})$.

This has been known for more than 50 years, at least in the 3-dimensional case. Please see [Friedlander 1962; 1980], for example. The version of the radiation field theorem given above and a proof for all dimensions $d \geq 3$ can be found in [Duyckaerts, Kenig and Merle 2019]. In addition, there is also a 2-dimensional version of the radiation field theorem. The statement in dimension 2 can be given in almost the same way as in the higher-dimensional case, except that the limit

$$\lim_{t \to \pm\infty} \int_{\mathbb{R}^2} \frac{|u(x,t)|^2}{|x|^2} \, dx = 0$$

no longer holds. A proof by Radon transform for all dimensions $d \geq 2$ can be found in [Katayama 2013], where the statement of the theorem is slightly different. Throughout this work we call the function $G_\pm$ radiation profiles and use the notation $\boldsymbol{T}_\pm$ for the linear map $(u_0, u_1) \to G_\pm$.

Channel of energy. The second tool is the channel-of-energy method, which plays an important role in the study of wave equations in the past decade. This method is first introduced in the 3-dimensional case in [Duyckaerts, Kenig and Merle 2011] and then in the 5-dimensional case in [Kenig, Lawrie and Schlag 2014]. This method was used in the proof of the soliton resolution conjecture of the energy critical wave equation with radial data in all odd dimensions $d \geq 3$ in [Duyckaerts, Kenig and Merle 2013; 2023]. It can also be used to show the nonexistence of minimal blow-up solutions in a compactness-rigidity argument in the energy super- or subcritical case. Please see, for example, [Duyckaerts, Kenig and Merle

2014; Shen 2013]. Roughly speaking, the channel-of-energy method discusses the amount of energy located in an exterior region as time tends to infinity:

$$\lim_{t\to\pm\infty}\int_{|x|>|t|+R}|\nabla_{x,t}u(x,t)|^2\,dx.$$

Here the constant R is nonnegative. Since the energy travels at a finite speed, the energy in the exterior region $\{x:|x|>|t|+R\}$ decays as $|t|$ increases. Thus the limits above are always well-defined. We can also give the exact value of the limit in terms of the radiation field:

$$\lim_{t\to\pm\infty}\int_{|x|>|t|+R}|\nabla_{x,t}u(x,t)|^2\,dx = 2\int_R^\infty\int_{\mathbb{S}^{d-1}}|G_\pm(s,\theta)|^2\,d\theta\,ds. \tag{2}$$

We first introduce a few already known results. We start with the odd dimensions.

Proposition 1.2 [Duyckaerts, Kenig and Merle 2012]. *Assume that $d\geq 3$ is an odd integer. All solutions to $\partial_t^2 u-\Delta u=0$ satisfy*

$$\sum_{\pm}\lim_{t\to\pm\infty}\int_{|x|>|t|}|\nabla_{x,t}u(x,t)|^2\,dx=\int_{\mathbb{R}^d}|\nabla_{x,t}u(x,0)|^2\,dx. \tag{3}$$

As a result, we have:

Corollary 1.3. *Assume that $d\geq 3$ is odd. Then $u\equiv 0$ is the only free wave u satisfying*

$$\lim_{t\to\pm\infty}\int_{|x|>|t|}|\nabla_{x,t}u(x,t)|^2\,dx=0.$$

In contrast, if $R>0$, the subspace of $\dot H^1\times L^2(\mathbb{R}^d)$ defined by

$$P(R)=\left\{(u_0,u_1)\in \dot H^1\times L^2(\mathbb{R}^d):\lim_{t\to\pm\infty}\int_{|x|>R+|t|}|\nabla_{t,x}\boldsymbol{S}_L(u_0,u_1)(x,t)|^2\,dx=0\right\} \tag{4}$$

does contain initial data (u_0,u_1) whose support is essentially bigger than $\{x:|x|\leq R\}$. The free waves u satisfying

$$\lim_{t\to\pm\infty}\int_{|x|>R+|t|}|\nabla_{t,x}u(x,t)|^2\,dx=0$$

are usually called (R-weakly) nonradiative solutions. If the dimension is odd, these solutions are well-understood in the radial case:

Theorem 1.4 [Kenig, Lawrie, Liu and Schlag 2015]. *In any odd dimension $d\geq 1$, every radial solution u to (1) satisfies*

$$\max_{\pm}\lim_{t\to\pm\infty}\int_{r>|t|+R}|\nabla_{x,t}u(r,t)|^2r^{d-1}\,dr\geq \tfrac12\|\Pi^\perp_{P_{\mathrm{rad}}(R)}(u_0,u_1)\|_{\dot H^1\times L^2(r\geq R:r^{d-1}dr)}. \tag{5}$$

Here

$$P_{\mathrm{rad}}(R)\doteq \mathrm{Span}\left\{(r^{2k_1-d},0),(0,r^{2k_2-d}):k_1,k_2\in\mathbb{N};\ 1\leq k_1\leq\frac{d+2}{4},\ 1\leq k_2\leq\frac{d}{4}\right\}.$$

$\Pi^\perp_{P_{\mathrm{rad}}(R)}$ is the orthogonal projection from $\dot H^1\times L^2(r\geq R:r^{d-1}dr)$ onto the complement of the finite-dimensional subspace $P_{\mathrm{rad}}(R)$.

Note the proof of Theorem 1.4 in [Kenig, Lawrie, Liu and Schlag 2015] uses the radial Fourier transform.

The case of even dimensions is much more complicated and subtle. Côte, Kenig and Schlag [2014] showed that in general the inequality

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|>|t|} |\nabla_{x,t} u(x,t)|^2\, dx \geq C \int_{\mathbb{R}^d} |\nabla_{x,t} u(x,0)|^2\, dx$$

does not hold for any positive constant C in even dimensions. But a similar inequality holds in the radial case for either initial data $(u_0, 0)$ if $d = 0 \bmod 4$, or $(0, u_1)$ if $d = 2 \bmod 4$. More precisely we have

$$\lim_{t\to\pm\infty} \int_{|x|>|t|} |\nabla_{x,t} \boldsymbol{S}_L(u_0, 0)(x,t)|^2\, dx \geq \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u_0(x)|^2\, dx, \quad d = 4k,\ k \in \mathbb{N}, \tag{6}$$

$$\lim_{t\to\pm\infty} \int_{|x|>|t|} |\nabla_{x,t} \boldsymbol{S}_L(0, u_1)(x,t)|^2\, dx \geq \frac{1}{2} \int_{\mathbb{R}^d} |u_1(x)|^2\, dx, \quad d = 4k+2,\ k \in \mathbb{N}. \tag{7}$$

In addition, Duyckaerts, Kenig and Merle [2021] showed that the only nonradiative solution is still the zero solution in even dimensions $d \geq 4$; i.e., Corollary 1.3 still holds for even dimensions $d \geq 4$, even in the nonradial case. Much less is known about the exterior energy estimate in the region $\{x : |x| > R + |t|\}$ with $R > 0$. Duyckaerts, Kenig, Martel and Merle [2022] proves the exterior energy estimate in dimensions 4 and 6 if the initial data are radial:

$$\lim_{t\to\pm\infty} \int_{|x|>|t|+R} |\nabla_{x,t} \boldsymbol{S}_L(u_0, 0)(x,t)|^2\, dx \geq \frac{3}{10} \|\Pi_R^{\perp} u_0\|^2_{\dot{H}^1(\{x\in\mathbb{R}^4:|x|>R\})},$$

$$\lim_{t\to\pm\infty} \int_{|x|>|t|+R} |\nabla_{x,t} \boldsymbol{S}_L(0, u_1)(x,t)|^2\, dx \geq \frac{3}{10} \|\pi_R^{\perp} u_1\|^2_{L^2(\{x\in\mathbb{R}^6:|x|>R\})}.$$

Here $\Pi_R^{\perp}$ is the orthogonal projection from $\dot{H}^1(\{x \in \mathbb{R}^4 : |x| > R\})$ onto the complement space of $\operatorname{Span}\{|x|^{-2}\}$. While $\pi_R^{\perp}$ is the orthogonal projection from $L^2(\{x \in \mathbb{R}^6 : |x| > R\})$ onto the complement space of $\operatorname{Span}\{|x|^{-4}\}$.

1B. ***Main idea.*** According to (2) we may obtain exterior energy estimates conveniently from the radiation profiles $G_{\pm}$. Please note that G_- and G_+ are not independent of each other. In fact the map $\boldsymbol{T}_+ \circ \boldsymbol{T}_-^{-1} : G_- \to G_+$ is a bijective isometry. If we could find explicit expressions of the maps

$$\boldsymbol{T}_+ \circ \boldsymbol{T}_-^{-1} : G_- \to G_+, \quad \boldsymbol{T}_-^{-1} : G_- \to (u_0, u_1), \quad \boldsymbol{S}_L \circ \boldsymbol{T}_-^{-1} : G_- \to u,$$

then we would be able to:

(a) Understand how the asymptotic behavior in one time direction determines the behavior in the other time direction. This is known in the odd-dimensional case, as shown (although not stated explicitly) in the proof of Proposition 1.2 in [Duyckaerts, Kenig and Merle 2012]. In this work we will try to figure out the even-dimensional case.

(b) Characterize (weakly) nonradiative solutions, especially in the radial case. We first determine all the radiation profiles G_- so that

$$G_-(s,\theta) = G_+(s,\theta) = 0, \quad s > R \quad \Longleftrightarrow \quad \lim_{t\to\pm\infty} \int_{|x|>|t|+R} |\nabla_{x,t} u(x,t)|^2\, dx = 0;$$

then we may obtain all the nonradiative solutions (as well as their initial data) by applying the formula of $\boldsymbol{T}_-^{-1}$. In particular we prove that radial nonradiative solutions in even dimensions can be characterized in the same way as in odd dimensions.

(c) Prove exterior energy estimates. We generalize the radial exterior energy estimates in 4 and 6 dimensions to all even dimensions; we also prove a nonradial exterior energy estimate in odd dimensions. In both applications (b) and (c) we follow the same roadmap:

$$\text{exterior energy} \quad \leftrightarrow \quad \text{radiation profile} \quad \leftrightarrow \quad \text{initial data.}$$

1C. ***Main results.*** Now we give the statement of our results. The details and proofs can be found in subsequent sections.

Theorem 1.5. *Let u be a finite-energy free wave with an even spatial dimension $d \geq 2$ and G_+, G_- be the radiation profiles associated with u. Then we may give an explicit formula of the operator $\boldsymbol{T}_+ \circ \boldsymbol{T}_-^{-1} : G_- \to G_+$*

$$G_+(s, \theta) = (-1)^{d/2}(\mathcal{H}G_-)(-s, -\theta).$$

Here $\mathcal{H}$ is the Hilbert transform in the first variable, i.e.,

$$(\mathcal{H}G_-)(-s, -\theta) = \text{p.v.}\ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{G_-(\tau, -\theta)}{(-s) - \tau}\, d\tau.$$

Remark 1.6. A similar but simpler argument shows that if d is odd, then $\boldsymbol{T}_+ \circ \boldsymbol{T}_-^{-1} : G_- \to G_+$ can be explicitly given by

$$G_+(s, \theta) = (-1)^{(d-1)/2} G_-(-s, -\theta).$$

This can also be verified by assuming that the initial data is smooth and compactly supported, and considering the expression of G_- and G_+ in terms of (u_0, u_1) if d is odd. Please refer to [Duyckaerts, Kenig and Merle 2012]. Since we have $\mathcal{H}^2 = -1$. We may write the odd and even dimensions in a universal formula

$$G_+(s, \theta) = ((-\mathcal{H})^{d-1} G_-)(-s, -\theta).$$

Remark 1.7. It has been proved in Section 3.2 of [Duyckaerts, Kenig and Merle 2021] (in the language of Hankel and Laplace transforms) that the zero function is the only function $f \in L^2(\mathbb{R})$ satisfying

$$f(s) = 0, \quad s > 0, \qquad (\mathcal{H}f)(s) = 0, \quad s < 0.$$

It immediately follows that:

Corollary 1.8. *Assume $d \geq 2$. Let Ω be a region in $\mathbb{S}^{d-1}$. If a finite-energy solution u to the homogenous linear wave equation satisfies*

$$\lim_{t \to \pm\infty} \int_{|t|}^{\infty} \int_{\pm\Omega} |\nabla_{t,x} u(r\theta, t)|^2 r^{d-1}\, d\theta\, dr = 0,$$

then we have

$$\lim_{t \to \pm\infty} \int_{0}^{\infty} \int_{\pm\Omega} |\nabla_{t,x} u(r\theta, t)|^2 r^{d-1}\, d\theta\, dr = 0.$$

This is an angle-localized version of Corollary 1.3.

Applications on channel of energy. By following the idea described above, we obtain the following results about the channel of energy.

Proposition 1.9 (radial weakly nonradiative solutions). *Let $d \geq 2$ be an integer and $R > 0$ be a constant. If the initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$ are radial, then the corresponding solution to the homogeneous linear wave equation u is R-weakly nonradiative, i.e.,*

$$\lim_{t\to\pm\infty} \int_{|x|>|t|+R} |\nabla_{t,x} u(x,t)|^2 \, dx = 0,$$

if and only if the restriction of (u_0, u_1) in the region $\{x \in \mathbb{R}^d : |x| > R\}$ is contained in

$$P_{\text{rad}}(R) = \text{Span}\left\{(r^{2k_1-d}, 0), (0, r^{2k_2-d}) : 1 \leq k_1 \leq \left\lfloor \frac{d+1}{4} \right\rfloor, \ 1 \leq k_2 \leq \left\lfloor \frac{d-1}{4} \right\rfloor\right\}.$$

Here the notation $\lfloor q \rfloor$ is the integer part of q. In particular, all radial R-weakly nonradiative solutions in dimension 2 are supported in $\{(x,t) : |x| \leq |t| + R\}$.

Remark 1.10. If d is odd, we have $\lfloor (d+1)/4 \rfloor = \lfloor (d+2)/4 \rfloor$ and $\lfloor (d-1)/4 \rfloor = \lfloor d/4 \rfloor$; thus our result here is the same as the already known result in odd dimension, as given in Theorem 1.4.

Proposition 1.11 (radial exterior estimates in even dimensions). *Let $d = 4k$ with $k \in \mathbb{N}$ and $R > 0$. If initial data $u_0 \in \dot{H}^1(\mathbb{R}^d)$ are radial, then the corresponding solution u to the homogenous linear wave equation with initial data $(u_0, 0)$ satisfies*

$$\lim_{t\to\infty} \int_{|x|>R+|t|} |\nabla u(x,t)|^2 \, dx = \lim_{t\to\infty} \int_{|x|>R+|t|} |u_t(x,t)|^2 \, dx \geq \tfrac{1}{4} \|\Pi^{\perp}_{Q_k(R)} u_0\|^2_{\dot{H}^1(\{x:|x|>R\})}.$$

Here $\Pi^{\perp}_{Q_k(R)}$ is the orthogonal projection from $\dot{H}^1(\{x \in \mathbb{R}^d : |x| > R\})$ onto the complement of the k-dimensional linear space

$$Q_k(R) = \text{Span}\left\{\frac{1}{|x|^{4k-2k_1}} : 1 \leq k_1 \leq k\right\}.$$

Similarly if the dimension d satisfies $d = 4k+2 \geq 2$, with $k \in \{0\} \cup \mathbb{N}$ and the initial data $u_1 \in L^2(\mathbb{R}^d)$ are radial, then the corresponding solution u to the homogenous linear wave equation with initial data $(0, u_1)$ satisfies

$$\lim_{t\to\infty} \int_{|x|>R+|t|} |\nabla u(x,t)|^2 \, dx = \lim_{t\to\infty} \int_{|x|>R+|t|} |u_t(x,t)|^2 \, dx \geq \tfrac{1}{4} \|\Pi^{\perp}_{Q'_k(R)} u_1\|^2_{L^2(\{x:|x|>R\})}.$$

Here $\Pi^{\perp}_{Q'_k(R)}$ is the orthogonal projection from $L^2(\{x \in \mathbb{R}^d : |x| > R\})$ onto the complement of the k-dimensional linear space

$$Q'_k(R) = \text{Span}\left\{\frac{1}{|x|^{4k+2-2k_1}} : 1 \leq k_1 \leq k\right\}.$$

Remark 1.12. Given $u_0 \in \dot{H}^1(\mathbb{R}^{4k})$ or $u_1 \in L^2(\mathbb{R}^{4k+2})$, the orthogonal projection of u_0 or u_1 onto the finite-dimensional space $Q_k(R)$ or $Q'_k(R)$ gradually vanishes as $R \to 0^+$. Therefore if we make $R \to 0^+$ in Proposition 1.11, we immediately obtain (6) and (7).

Remark 1.13. At the same time as this work was done, Kenig et al. proved radial exterior estimates similar to Proposition 1.11 for even dimensions $d \geq 8$ by using the already-known result in dimension 4 and an induction argument.

Proposition 1.14 (nonradial exterior energy estimates). *Let $d \geq 3$ be an odd integer and $R > 0$ be a constant. Then the following identity holds for all $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$:*

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|>R+|t|} |\nabla_{t,x} \boldsymbol{S}_L(t)(u_0, u_1)(x,t)|^2\, dx = \|\Pi^{\perp}_{P(R)}(u_0, u_1)\|^2_{\dot{H}^1\times L^2(\mathbb{R}^d)}.$$

Here $\Pi^{\perp}_{P(R)}$ is the orthogonal projection from $\dot{H}^1 \times L^2(\mathbb{R}^d)$ onto the complement of the closed linear space

$$P(R) = \left\{(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d) : \lim_{t\to\pm\infty} \int_{|x|>R+|t|} |\nabla_{t,x} \boldsymbol{S}_L(u_0, u_1)(x,t)|^2\, dx = 0\right\}.$$

Structure of this work. This work is organized as follows. In Section 2 we deduce an explicit formula of $\boldsymbol{T}_-^{-1}$ in all dimensions. Then in Section 3 we prove the explicit formula of $\boldsymbol{T}_+ \circ \boldsymbol{T}_-^{-1}$ given in Theorem 1.5. The rest of the paper is devoted to the applications in the channel of energy. We characterize radial weakly nonradiative solutions in Section 4, prove radial exterior energy estimate for all even dimensions in Section 5 and finally give a short proof of nonradial exterior energy estimate in odd-dimensional space in Section 6. The Appendix is concerned with Hilbert transform of a family of special functions, since the Hilbert transform is involved in the even dimensions.

Notation. In this work we use the notation $C(d)$ for a nonzero constant determined solely by the dimension d. It may represent different constants in different places. This avoids the trouble of keeping track of the constants when unnecessary.

2. From radiation profile to solution

Now we assume that $G_-(r, \theta)$ is smooth and compactly supported and give an explicit formula of the operator $\boldsymbol{T}_-^{-1}$. We consider the odd dimensions first.

2A. *Odd dimensions.*

Lemma 2.1. *Assume that $d \geq 3$ is odd. Let G_- be a smooth function with* supp $G_- \subset [-R, R] \times \mathbb{S}^{d-1}$. *Then $(u_0, u_1) = \boldsymbol{T}_-^{-1} G_-$ satisfies*

$$u_0(x) = \frac{1}{(2\pi)^{(d-1)/2}} \int_{\mathbb{S}^{d-1}} G_-^{(\mu-1)}(x\cdot\omega, \omega)\, d\omega, \tag{8}$$

$$u_1(x) = \frac{1}{(2\pi)^{(d-1)/2}} \int_{\mathbb{S}^{d-1}} G_-^{(\mu)}(x\cdot\omega, \omega)\, d\omega. \tag{9}$$

Here the notation $G_-^{(k)}$ represents the partial derivative

$$G_-^{(k)}(s, \theta) = \frac{\partial^k G_-(s, \theta)}{\partial s^k}.$$

Remark 2.2. This formula in 3-dimensional case was previously known. Please refer to [Friedlander 1973], for example.

Proof. Let $(u_0, u_1) = \boldsymbol{T}_-^{-1} G_-$ and $u = \boldsymbol{S}_L(u_0, u_1)$. Given a large time $t > 0$, we choose approximated data $(v_{0,t}, v_{1,t}) \approx (u(\cdot, -t), u_t(\cdot, -t))$ as

$$v_{1,t}(r\theta) = r^{-\mu} G_-(r-t, \theta), \qquad r > 0,\ \theta \in \mathbb{S}^{d-1}, \tag{10}$$

$$v_{0,t}(r\theta) = -\chi\left(\frac{r}{t}\right) \int_r^{+\infty} r'^{-\mu} G_-(r'-t, \theta)\, dr', \quad r > 0,\ \theta \in \mathbb{S}^{d-1}. \tag{11}$$

Here $\mu = (d-1)/2$ and $\chi : \mathbb{R} \to [0, 1]$ is a smooth center cut-off function satisfying

$$\chi(s) = \begin{cases} 1, & s > \frac{1}{2}, \\ 0, & s < \frac{1}{4}. \end{cases}$$

It is clear that the data $(v_{0,t}, v_{1,t})$ are smooth and compactly supported in $\{x : |x| < R+t\}$. A straightforward calculation shows that

$$\int_0^\infty \int_{\mathbb{S}^{d-1}} |r^\mu v_{1,t}(r\theta) - G_-(r-t, \theta)|^2\, d\theta\, dr = 0,$$

$$\int_0^\infty \int_{\mathbb{S}^{d-1}} |r^\mu \partial_r v_{0,t}(r\theta) - G_-(r-t, \theta)|^2\, d\theta\, dr \lesssim \frac{1}{t},$$

$$\int_{\mathbb{R}^d} (|\nabla v_{0,t}(x)|^2 - |\partial_r v_{0,t}(x)|^2)\, dx \lesssim \frac{1}{t}.$$

Thus by the radiation field we have

$$\lim_{t \to +\infty} \|(v_{0,t}, v_{1,t}) - (u(\cdot, -t), u_t(\cdot, -t))\|_{\dot{H}^1 \times L^2(\mathbb{R}^d)} = 0.$$

Since the linear propagation operator $\boldsymbol{S}_L(t)$ preserves the $\dot{H}^1 \times L^2$ norm, we have

$$\lim_{t \to +\infty} \left\| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \boldsymbol{S}_L(t) \begin{pmatrix} v_{0,t} \\ v_{1,t} \end{pmatrix} \right\|_{\dot{H}^1 \times L^2(\mathbb{R}^d)} = 0. \tag{12}$$

Next we use the explicit expression of the linear propagation operator (see, for instance, [Evans 1998]) and write $v = \boldsymbol{S}_L(v_0, v_1)$ in terms of (v_0, v_1) when the initial data are sufficiently smooth:

$$\begin{aligned} v(x,t) &= c_d \cdot \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\mu-1}\left(t^{d-2}\int_{\mathbb{S}^{d-1}} v_0(x+t\omega)\, d\omega\right) + c_d \cdot \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\mu-1}\left(t^{d-2}\int_{\mathbb{S}^{d-1}} v_1(x+t\omega)\, d\omega\right) \\ &= c_d t^\mu \int_{\mathbb{S}^{d-1}} [((w\cdot\nabla)^\mu v_0)(x+t\omega) + ((w\cdot\nabla)^{\mu-1} v_1)(x+t\omega)]\, d\omega \\ &\quad + \sum_{0 \le k < \mu} A_{d,k} t^k \int_{\mathbb{S}^{d-1}} ((w\cdot\nabla)^k v_0)(x+t\omega)\, d\omega + \sum_{0 \le k < \mu-1} B_{d,k} t^{k+1} \int_{\mathbb{S}^{d-1}} ((w\cdot\nabla)^k v_1)(x+t\omega)\, d\omega. \end{aligned}$$

Here $c_d = 1/(2(2\pi)^{(d-1)/2})$, $A_{d,k}$, $B_{d,k}$ (and $A'_{d,k}$, $B'_{d,k}$ below) are all constants. We may differentiate and obtain

$$\begin{aligned} v_t(x,t) &= c_d t^\mu \int_{\mathbb{S}^{d-1}} [((w\cdot\nabla)^{\mu+1} v_0)(x+t\omega) + ((w\cdot\nabla)^\mu v_1)(x+t\omega)]\, d\omega \\ &\quad + \sum_{1 \le k \le \mu} A'_{d,k} t^{k-1} \int_{\mathbb{S}^{d-1}} ((w\cdot\nabla)^k v_0)(x+t\omega)\, d\omega + \sum_{0 \le k \le \mu-1} B'_{d,k} t^k \int_{\mathbb{S}^{d-1}} ((w\cdot\nabla)^k v_1)(x+t\omega)\, d\omega. \end{aligned}$$

Now we plug in $(v_0, v_1) = (v_{0,t}, v_{1,t})$ with large time t. We observe that

$$|(\omega \cdot \nabla)^k v_{j,t}(x + tw)| \lesssim t^{-\mu}, \quad j = 0, 1,\ k \geq 0, \tag{13}$$

and $(r = |x + t\omega|, \theta = (x + t\omega)/|x + t\omega|,\ k = \mu - 1, \mu)$

$$((\omega \cdot \nabla)^{k+1} v_{0,t})(x + t\omega) = (\omega \cdot \theta)^{k+1} r^{-\mu} G_-^{(k)}(r - t, \theta) + O(t^{-\mu-1}),$$
$$((\omega \cdot \nabla)^{k} v_{1,t})(x + t\omega) = (\omega \cdot \theta)^{k} r^{-\mu} G_-^{(k)}(r - t, \theta) + O(t^{-\mu-1}).$$

Thus

$$\begin{pmatrix} w_{0,t} \\ w_{1,t} \end{pmatrix} = \boldsymbol{S}_L(t) \begin{pmatrix} v_{0,t} \\ v_{1,t} \end{pmatrix}$$

satisfies

$$w_{0,t} = c_d \int_{\mathbb{S}^{d-1}} (\omega \cdot \theta)^{\mu-1}(1 + \omega \cdot \theta) G_-^{(\mu-1)}(r - t, \theta)\, d\omega + O\Big(\frac{1}{t}\Big),$$
$$w_{1,t} = c_d \int_{\mathbb{S}^{d-1}} (\omega \cdot \theta)^{\mu}(1 + \omega \cdot \theta) G_-^{(\mu)}(r - t, \theta)\, d\omega + O\Big(\frac{1}{t}\Big).$$

Please note that the implicit constants in (13), $O(t^{-\mu-1})$ and $O(1/t)$ above, may depend on x but remain uniformly bounded if x is contained in a compact subset of $\mathbb{R}^d$. Next we observe the facts

$$\theta(\omega) = \omega + O\Big(\frac{1}{t}\Big), \quad r(\omega) - t = x \cdot \omega + O\Big(\frac{1}{t}\Big),$$

and further simplify the formulas

$$w_{0,t} = 2c_d \int_{\mathbb{S}^{d-1}} G_-^{(\mu-1)}(x \cdot \omega, \omega)\, d\omega + O\Big(\frac{1}{t}\Big),$$
$$w_{1,t} = 2c_d \int_{\mathbb{S}^{d-1}} G_-^{(\mu)}(x \cdot \omega, \omega)\, d\omega + O\Big(\frac{1}{t}\Big).$$

Finally we make $t \to +\infty$, utilize (12) and obtain

$$u_0 = 2c_d \int_{\mathbb{S}^{d-1}} G_-^{(\mu-1)}(x \cdot \omega, \omega)\, d\omega,$$
$$u_1 = 2c_d \int_{\mathbb{S}^{d-1}} G_-^{(\mu)}(x \cdot \omega, \omega)\, d\omega.$$

We plug in the value of c_d and finish the proof. □

Remark 2.3. An explicit formula of the free wave $u = \boldsymbol{S}_L \boldsymbol{T}_-^{-1} G_-$ can be given by

$$u(x, t) = \frac{1}{(2\pi)^{(d-1)/2}} \int_{\mathbb{S}^{d-1}} G_-^{(\mu-1)}(x \cdot \omega + t, \omega)\, d\omega.$$

This can be verified by a straightforward calculation. One may check

- The function u above is a smooth solution to the homogenous linear wave equation.
- The initial data of u are exactly those given in Lemma 2.1.

We may differentiate and obtain

$$u_t(x,t) = \frac{1}{(2\pi)^{(d-1)/2}} \int_{\mathbb{S}^{d-1}} G_-^{(\mu)}(x\cdot\omega + t, \omega)\, d\omega,$$
$$\nabla u(x,t) = \frac{1}{(2\pi)^{(d-1)/2}} \int_{\mathbb{S}^{d-1}} G_-^{(\mu)}(x\cdot\omega + t, \omega)\, \omega\, d\omega.$$

2B. *Even dimensions.* The formula of $\boldsymbol{T}_-^{-1}$ in even dimensions is a little more complicated.

Lemma 2.4. *Assume that $d \geq 2$ is even and $G_- \in C_0^\infty(\mathbb{R}\times\mathbb{S}^{d-1})$. Then the operator $\boldsymbol{T}_-^{-1}$ is given explicitly by*

$$u_0(x) = \frac{\sqrt{2}}{(2\pi)^{d/2}} \cdot \int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{G_-^{(d/2-1)}(x\cdot\omega - \rho, \omega)}{\sqrt{\rho}}\, d\omega\, d\rho,$$
$$u_1(x) = \frac{\sqrt{2}}{(2\pi)^{d/2}} \cdot \int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{G_-^{(d/2)}(x\cdot\omega - \rho, \omega)}{\sqrt{\rho}}\, d\omega\, d\rho.$$

Proof. Without loss of generality let us assume $\operatorname{supp} G_- \subset [-R_1, R_1]\times\mathbb{S}^{d-1}$. It is sufficient to show that given any $R_2 > 0$, the formula above holds for almost every $x \in B(0, R_2)$. Let us use the notation $(u_0, u_1) = \boldsymbol{T}_-^{-1}(G_-)$ and $u = \boldsymbol{S}_L(u_0, u_1)$. We consider the approximated data

$$v_{1,t}(r\theta) = r^{-\mu} G_-(r-t, \theta),$$
$$v_{0,t}(r\theta) = -\chi\Big(\frac{r}{t}\Big) \int_r^{+\infty} r'^{-\mu} G_-(r'-t, \theta)\, dr', \quad r > 0,\ \theta \in \mathbb{S}^{d-1}.$$

and

$$\begin{pmatrix} w_{0,t} \\ w_{1,t} \end{pmatrix} = \boldsymbol{S}_L(t) \begin{pmatrix} v_{0,t} \\ v_{1,t} \end{pmatrix}.$$

Here χ is the center cut-off function as given in the previous subsection. A basic calculation shows

$$\lim_{t\to+\infty} \|(v_{0,t}, v_{1,t}) - (u(\cdot, -t), u_t(\cdot, -t))\|_{\dot{H}^1\times L^2(\mathbb{R}^d)} = 0.$$

Thus

$$\lim_{t\to+\infty} \|(w_{0,t}, w_{1,t}) - (u_0, u_1)\|_{\dot{H}^1\times L^2(\mathbb{R}^d)} = 0. \tag{14}$$

Let us first recall the explicit formula of $v = \boldsymbol{S}_L(v_0, v_1)$ in the even-dimensional case:

$$\begin{aligned} v(x,t) &= c_d \cdot \frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{(d-2)/2}\Big(t^{d-1}\int_{\mathbb{B}^d} \frac{v_0(x+ty)}{\sqrt{1-|y|^2}}\, dy\Big) + c_d \cdot \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{(d-2)/2}\Big(t^{d-1}\int_{\mathbb{B}^d} \frac{v_1(x+ty)}{\sqrt{1-|y|^2}}\, dy\Big) \\ &= c_d \cdot t^{d/2} \int_{\mathbb{B}^d} \frac{((y\cdot\nabla)^{d/2} v_0)(x+ty) + ((y\cdot\nabla)^{d/2-1} v_1)(x+ty)}{\sqrt{1-|y|^2}}\, dy \\ &\qquad + \sum_{0\leq k<d/2} A_{d,k} t^k \int_{\mathbb{B}^d} \frac{(y\cdot\nabla)^k v_0(x+ty)}{\sqrt{1-|y|^2}}\, dy + \sum_{0\leq k<d/2-1} B_{d,k} t^{k+1} \int_{\mathbb{B}^d} \frac{(y\cdot\nabla)^k v_1(x+ty)}{\sqrt{1-|y|^2}}\, dy. \end{aligned}$$

Here $\mathbb{B}_d$ is the unit ball in $\mathbb{R}^d$ and $c_d = (2\pi)^{-d/2}$ is a constant. The notations $A_{d,k}$, $B_{d,k}$ (and $A'_{d,k}$, $B'_{d,k}$ below) represent constants. We differentiate and obtain

$$v_t(x,t) = c_d \cdot t^{d/2} \int_{\mathbb{B}^d} \frac{((y\cdot\nabla)^{d/2+1} v_0)(x+ty) + ((y\cdot\nabla)^{d/2} v_1)(x+ty)}{\sqrt{1-|y|^2}}\,dy$$
$$+ \sum_{1\le k\le d/2} A'_{d,k} t^{k-1} \int_{\mathbb{B}^d} \frac{(y\cdot\nabla)^k v_0(x+ty)}{\sqrt{1-|y|^2}}\,dy + \sum_{0\le k\le d/2-1} B'_{d,k} t^k \int_{\mathbb{B}^d} \frac{(y\cdot\nabla)^k v_1(x+ty)}{\sqrt{1-|y|^2}}\,dy.$$

We plug in $(v_0, v_1) = (v_{0,t}, v_{1,t})$ and observe

$$|(y\cdot\nabla)^k v_{0,t}| \le t^{-(d-1)/2}, \quad |(y\cdot\nabla)^k v_{1,t}| \le t^{-(d-1)/2}.$$

This gives the approximation

$$w_{0,t}(x) = c_d \cdot t^{d/2} \int_{\mathbb{B}^d} \frac{((y\cdot\nabla)^{d/2} v_{0,t})(r\theta) + ((y\cdot\nabla)^{d/2-1} v_{1,t})(r\theta)}{\sqrt{1-|y|^2}}\,dy + O(t^{-1/2}),$$
$$w_{1,t}(x) = c_d \cdot t^{d/2} \int_{\mathbb{B}^d} \frac{((y\cdot\nabla)^{d/2+1} v_{0,t})(r\theta) + ((y\cdot\nabla)^{d/2} v_{1,t})(r\theta)}{\sqrt{1-|y|^2}}\,dy + O(t^{-1/2}).$$

Here $r = |x+ty|$ and $\theta = (x+ty)/|x+ty|$. Furthermore, we observe ($k = d/2, d/2-1$)

$$((y\cdot\nabla)^{k+1} v_{0,t})(r\theta) = (y\cdot\theta)^{k+1} r^{-(d-1)/2} G_-^{(k)}(r-t,\theta) + O(t^{-(d+1)/2}),$$
$$((y\cdot\nabla)^{k} v_{1,t})(r\theta) = (y\cdot\theta)^{k} r^{-(d-1)/2} G_-^{(k)}(r-t,\theta) + O(t^{-(d+1)/2}),$$

and write

$$w_{0,t}(x) = c_d \cdot t^{d/2} \int_{\mathbb{B}^d} \frac{(y\cdot\theta)^{d/2-1}(y\cdot\theta+1) r^{-(d-1)/2} G_-^{(d/2-1)}(r-t,\theta)}{\sqrt{1-|y|^2}}\,dy + O(t^{-1/2}),$$
$$w_{1,t}(x) = c_d \cdot t^{d/2} \int_{\mathbb{B}^d} \frac{(y\cdot\theta)^{d/2}(y\cdot\theta+1) r^{-(d-1)/2} G_-^{(d/2)}(r-t,\theta)}{\sqrt{1-|y|^2}}\,dy + O(t^{-1/2}).$$

Next we observe that if $|y| < 1-(R_1+R_2)/t$, then we have $r \le t|y|+|x| < t-R_1$; thus $G_-^{(k)}(r-t,\theta) = 0$. As a result, we may restrict the domain of the integral to

$$\mathbb{B}_t = \left\{ y \in \mathbb{B}^d : |y| \ge 1 - \frac{R_1+R_2}{t} \right\}.$$

Because in the region we have

$$\theta = \frac{y}{|y|} + O\Big(\frac{1}{t}\Big), \quad y\cdot\theta = 1 + O\Big(\frac{1}{t}\Big), \quad r = t + O(1),$$

we can simplify the formulas

$$w_{0,t}(x) = 2c_d \cdot t^{1/2} \int_{\mathbb{B}_t} \frac{G_-^{(d/2-1)}(r-t, y/|y|)}{\sqrt{1-|y|^2}}\,dy + O(t^{-1/2}),$$
$$w_{1,t}(x) = 2c_d \cdot t^{1/2} \int_{\mathbb{B}_t} \frac{G_-^{(d/2)}(r-t, y/|y|)}{\sqrt{1-|y|^2}}\,dy + O(t^{-1/2}).$$

Next we utilize the change of variables

$$y = \Big(1 - \frac{\rho}{t}\Big)\omega, \quad (\rho, \omega) \in (0, R_1 + R_2) \times \mathbb{S}^{d-1},$$

and the approximations

$$r - t = x \cdot \omega - \rho + O\Big(\frac{1}{t}\Big), \quad \sqrt{1 - |y|^2} = \Big(1 + O\Big(\frac{1}{t}\Big)\Big)\sqrt{\frac{2\rho}{t}}, \quad dy = \Big(1 + O\Big(\frac{1}{t}\Big)\Big)t^{-1}\, d\rho\, d\omega$$

to obtain

$$w_{0,t}(x) = \sqrt{2}c_d \cdot \int_0^{R_1+R_2}\int_{\mathbb{S}^{d-1}} \frac{G_-^{(d/2-1)}(x \cdot \omega - \rho, \omega)}{\sqrt{\rho}}\, d\omega\, d\rho + O(t^{-1/2}),$$

$$w_{1,t}(x) = \sqrt{2}c_d \cdot \int_0^{R_1+R_2}\int_{\mathbb{S}^{d-1}} \frac{G_-^{(d/2)}(x \cdot \omega - \rho, \omega)}{\sqrt{\rho}}\, d\omega\, d\rho + O(t^{-1/2}).$$

Finally we recall (14), let $t \to +\infty$ and conclude

$$u_0(x) = \sqrt{2}c_d \cdot \int_0^{R_1+R_2}\int_{\mathbb{S}^{d-1}} \frac{G_-^{(d/2-1)}(x \cdot \omega - \rho, \omega)}{\sqrt{\rho}}\, d\omega\, d\rho,$$

$$u_1(x) = \sqrt{2}c_d \cdot \int_0^{R_1+R_2}\int_{\mathbb{S}^{d-1}} \frac{G_-^{(d/2)}(x \cdot \omega - \rho, \omega)}{\sqrt{\rho}}\, d\omega\, d\rho. \qquad \square$$

Remark 2.5. If $d \geq 4$, the convergence (14) implies that $(w_{0,t}, w_{1,t})$ converges to (u_0, u_1) in $L^{2d/(d-2)} \times L^2$ by Sobolev embedding. We may combine this convergence with the local uniform convergence given above to verify the identities above. This argument breaks down in dimension 2. We given another argument below in dimension 2. Given any test function $\varphi \in C_0^\infty(\mathbb{R}^2)$, integration by parts gives an identity

$$\int w_{0,t}(x)\nabla\varphi(x)\, dx = -\int \nabla w_{0,t}(x)\varphi(x)\, dx.$$

We recall the local uniform convergence of $w_{0,t}$ given above and the L^2 convergence of $\nabla w_{0,t} \to \nabla u_0$ and then obtain

$$\int \Big(\sqrt{2}c_2 \cdot \int_0^\infty\int_{\mathbb{S}^1} \frac{G_-(x \cdot \omega - \rho, \omega)}{\sqrt{\rho}}\, d\omega\, d\rho\Big)\nabla\varphi(x)\, dx = -\int \nabla u_0(x)\varphi(x)\, dx.$$

This finishes the proof. Finally we would like to mention that we have

$$\lim_{|x| \to +\infty} \sqrt{2}c_2 \cdot \int_0^\infty\int_{\mathbb{S}^1} \frac{G_-(x \cdot \omega - \rho, \omega)}{\sqrt{\rho}}\, d\omega\, d\rho = 0.$$

Corollary 2.6. *If $G_- \in C_0^\infty(\mathbb{R} \times \mathbb{S}^{d-1})$, then $u = \boldsymbol{S}_L \boldsymbol{T}_-^{-1}(G_-)$ is given by*

$$u(x, t) = \frac{\sqrt{2}}{(2\pi)^{d/2}} \int_0^\infty\int_{\mathbb{S}^{d-1}} \frac{G_-^{(d/2-1)}(x \cdot \omega - \rho + t, \omega)}{\sqrt{\rho}}\, d\omega\, d\rho.$$

Thus

$$u_t(x, t) = \frac{\sqrt{2}}{(2\pi)^{d/2}} \int_0^\infty\int_{\mathbb{S}^{d-1}} \frac{G_-^{(d/2)}(x \cdot \omega - \rho + t, \omega)}{\sqrt{\rho}}\, d\omega\, d\rho.$$

Proof. A basic calculation shows that $u(x, t)$ solves the free wave equation with initial data given in Lemma 2.4. $\square$

2C. *Universal formula.* Now let us give a universal formula of $\boldsymbol{T}_-^{-1}$ for all dimensions. We first define two convolution operators ($1/\sqrt{\pi x}$ is understood as zero if $x < 0$)

$$\mathcal{Q}f = \frac{1}{\sqrt{\pi x}} * f, \quad \mathcal{Q}' f = \frac{1}{\sqrt{-\pi x}} * f.$$

Their Fourier symbols are

$$\frac{1 - i(\xi/|\xi|)}{2\sqrt{\pi|\xi|}} \quad \text{and} \quad \frac{1 + i(\xi/|\xi|)}{2\sqrt{\pi|\xi|}},$$

respectively. Let us also use the notation $\mathcal{D} = d/dx$ and recall that its Fourier symbol is $2\pi i\xi$. A simple calculation of symbols shows

$$\mathcal{Q}^2\mathcal{D} = 1, \quad \mathcal{Q}'^2\mathcal{D} = -1, \quad \mathcal{Q}\mathcal{Q}'\mathcal{D} = \mathcal{H}. \tag{15}$$

As a result, we may understand $\mathcal{Q}$ as $\mathcal{D}^{-1/2}$ and rewrite $u = \boldsymbol{S}_L\boldsymbol{T}_-^{-1}G_-$ in the form

$$\begin{aligned} u(x,t) &= \frac{1}{(2\pi)^{(d-1)/2}} \int_{\mathbb{S}^{d-1}} (\mathcal{Q}G_-^{(d/2-1)})(x\cdot\omega + t, \omega)\, d\omega \\ &= \frac{1}{(2\pi)^{\mu}} \int_{\mathbb{S}^{d-1}} \mathcal{D}^{\mu-1}G_-(x\cdot\omega + t, \omega)\, d\omega. \end{aligned} \tag{16}$$

Here $\mu = (d-1)/2$. This formula holds for both odd and even dimensions.

3. Between radiation profiles

In this section we give an explicit expression of the operator $\boldsymbol{T}_+ \circ \boldsymbol{T}_-^{-1}$ in the even-dimensional case, without the radial assumption.

Theorem 3.1. *Assume that $d \geq 2$ is an even integer. The operator $\boldsymbol{T}_+ \circ \boldsymbol{T}_-^{-1}$ can be explicitly given by the formula*

$$G_+(s,\theta) = (\boldsymbol{T}_+\boldsymbol{T}_-^{-1}G_-)(s,\theta) = (-1)^{d/2}(\mathcal{H}G_-)(-s,-\theta).$$

Here $\mathcal{H}$ is the Hilbert transform in the first variable, i.e.,

$$(\mathcal{H}G_-)(-s,-\theta) = \text{p.v.}\, \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{G_-(\tau,-\theta)}{-\tau - s}\, d\tau.$$

Proof. Since $\boldsymbol{T}_+ \circ \boldsymbol{T}_-^{-1}$ is a bijective isometry from $L^2(\mathbb{R}\times\mathbb{S}^{d-1})$ to itself. We only need to prove this formula for smooth and compactly supported data G_-. Without loss of generality let us assume $\operatorname{supp} G_- \subset [-R_1, R_1]\times\mathbb{S}^{d-1}$. Let us also fix a positive constant $R_2 > 0$. If $(s,\theta) \in (-R_2, R_2)\times\mathbb{S}^{d-1}$, then we may apply Corollary 2.6 and obtain

$$(t+s)^{(d-1)/2}\partial_t u((t+s)\theta, t) = \sqrt{2}c_d(t+s)^{(d-1)/2}\int_0^\infty\int_{\mathbb{S}^{d-1}} \frac{G_-^{(d/2)}((t+s)\theta\cdot\omega - \rho + t, \omega)}{\sqrt{\rho}}\, d\omega\, d\rho.$$

Let $M \gg R_1 + R_2 + 1$ be a large constant. We may split the integral above into two parts:

$$J_1 = \sqrt{2} c_d (t+s)^{(d-1)/2} \int_0^\infty \int_{\theta\cdot\omega < -1+M/t} \frac{G_-^{(d/2)}((t+s)\theta\cdot\omega - \rho + t, \omega)}{\sqrt{\rho}} \, d\omega \, d\rho,$$

$$J_2 = \sqrt{2} c_d (t+s)^{(d-1)/2} \int_0^\infty \int_{\theta\cdot\omega \geq -1+M/t} \frac{G_-^{(d/2)}((t+s)\theta\cdot\omega - \rho + t, \omega)}{\sqrt{\rho}} \, d\omega \, d\rho.$$

We may find an upper bound of J_2. In this region we have

$$(t+s)\theta\cdot\omega + t \geq M - R_2 \quad \Longrightarrow \quad G_-((t+s)\theta\cdot\omega - \rho + t) = 0 \quad \text{if } \rho < \frac{M}{2}.$$

Thus we may integrate by parts and obtain

$$J_2 = C(d)(t+s)^{(d-1)/2} \int_0^\infty \int_{\theta\cdot\omega \geq -1+M/t} \frac{G_-((t+s)\theta\cdot\omega - \rho + t, \omega)}{\rho^{(d+1)/2}} \, d\omega \, d\rho.$$

Therefore when t is sufficiently large

$$\begin{aligned} |J_2| &\lesssim t^{(d-1)/2} \int_{\theta\cdot\omega \geq -1+M/t} \int_{(t+s)\theta\cdot\omega+t-R_1}^{(t+s)\theta\cdot\omega+t+R_1} \frac{|G_-((t+s)\theta\cdot\omega - \rho + t, \omega)|}{\rho^{(d+1)/2}} \, d\rho \, d\omega \\ &\lesssim t^{(d-1)/2} \int_{\theta\cdot\omega \geq -1+M/t} \int_{(t+s)\theta\cdot\omega+t-R_1}^{(t+s)\theta\cdot\omega+t+R_1} \frac{|G_-((t+s)\theta\cdot\omega - \rho + t, \omega)|}{|(t+s)\theta\cdot\omega + t|^{(d+1)/2}} \, d\rho \, d\omega \\ &\lesssim t^{(d-1)/2} \int_{\theta\cdot\omega \geq -1+M/t} \frac{1}{|t\theta\cdot\omega + t|^{(d+1)/2}} \, d\omega \lesssim \frac{1}{M}. \end{aligned}$$

In the integral region of J_1, we have the approximation $\omega = -\theta + O(t^{-1/2})$. Thus we have

$$J_1 = \sqrt{2} c_d t^{(d-1)/2} \int_0^\infty \int_{\theta\cdot\omega < -1+M/t} \frac{G_-^{(d/2)}((t+s)\theta\cdot\omega - \rho + t, -\theta)}{\sqrt{\rho}} \, d\omega \, d\rho + O(t^{-1/2}).$$

Next we utilize the change of variables (please refer to Figure 1 for a geometrical meaning)

$$\omega = \Big(-1 + \frac{\rho'}{t}\Big)\theta + \sqrt{\Big(\frac{\rho'}{t}\Big)\Big(2 - \frac{\rho'}{t}\Big)}\varphi, \quad \rho' \in [0, M], \quad \varphi \in \mathbb{S}^{d-2} = \{\varphi \in \mathbb{S}^{d-1} : \varphi \perp \theta\},$$

$$d\omega = \Big[1 + O\Big(\frac{1}{t}\Big)\Big]\Big(\frac{2\rho'}{t}\Big)^{d/2-1} d\mathbb{S}^{d-2}(\varphi) \cdot \frac{d\rho'}{\sqrt{2\rho' t}} = \Big[1 + O\Big(\frac{1}{t}\Big)\Big](2\rho')^{(d-3)/2} t^{-(d-1)/2} \, d\mathbb{S}^{d-2}(\varphi) \, d\rho'$$

and obtain

$$J_1 = \frac{1}{2\pi^{d/2}} \int_0^\infty \int_0^M \int_{\mathbb{S}^{d-2}} G_-^{(d/2)}(\rho' - \rho - s, -\theta) \rho'^{(d-3)/2} \rho^{-1/2} \, d\varphi \, d\rho' \, d\rho + O(t^{-1/2}).$$

We observe that the integrand is independent of φ and integrate by parts

$$J_1 = \frac{(-1)^{d/2-1}}{\pi} \int_0^\infty \int_0^M \frac{G'_-(\rho' - \rho - s, -\theta)}{\sqrt{\rho\rho'}} \, d\rho' \, d\rho + O(t^{-1/2}).$$

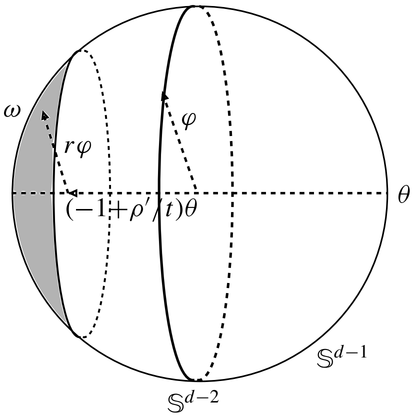

Figure 1. Change of variables, where $r = \sqrt{(\rho'/t)(2-\rho'/t)}$.

We next change the variables $\tau = \rho' - \rho$, $\eta = \rho' + \rho$, and write

$$\begin{aligned} J_1 &= \frac{(-1)^{d/2-1}}{\pi} \int_{-\infty}^{M} \int_{|\tau|}^{2M-\tau} \frac{G'_-(\tau - s, -\theta)}{\sqrt{\eta^2 - \tau^2}}\, d\eta\, d\tau + O(t^{-1/2}) \\ &= \frac{(-1)^{d/2-1}}{\pi} \int_{-\infty}^{M} G'_-(\tau - s, -\theta)\big[\ln(2M - \tau + \sqrt{4M^2 - 4M\tau}) - \ln|\tau|\big]\, d\tau + O(t^{-1/2}) \\ &= \frac{(-1)^{d/2-1}}{\pi} \int_{-R_1-R_2}^{R_1+R_2} G'_-(\tau - s, -\theta)\big[\ln(2M - \tau + \sqrt{4M^2 - 4M\tau}) - \ln|\tau|\big]\, d\tau + O(t^{-1/2}). \end{aligned}$$

The integrals above can be split into two parts:

$$\begin{aligned} I_1 &= \int_{-R_1-R_2}^{R_1+R_2} G'_-(\tau - s, -\theta)\big[\ln(2M - \tau + \sqrt{4M^2 - 4M\tau})\big]\, d\tau \\ &= \int_{-R_1-R_2}^{R_1+R_2} G'_-(\tau - s, -\theta)\big[\ln(2M - \tau + \sqrt{4M^2 - 4M\tau}) - \ln(4M)\big]\, d\tau \\ &= \int_{-R_1-R_2}^{R_1+R_2} G'_-(\tau - s, -\theta)\, O\Big(\frac{1}{M}\Big)\, d\tau = O\Big(\frac{1}{M}\Big) \end{aligned}$$

and

$$\begin{aligned} I_2 &= -\lim_{\varepsilon \to 0^+} \int_{\varepsilon < |\tau| < R_1+R_2} G'_-(\tau - s, -\theta) \ln|\tau|\, d\tau \\ &= \lim_{\varepsilon \to 0^+} \int_{\varepsilon < |\tau| < R_1+R_2} \frac{G_-(\tau - s, -\theta)}{\tau}\, d\tau = -\pi(\mathcal{H}G_-)(-s, -\theta). \end{aligned}$$

In summary we have

$$J_1 = (-1)^{d/2}(\mathcal{H}G_-)(-s, -\theta) + O\Big(\frac{1}{M}\Big) + O(t^{-1/2}).$$

Now we may combine J_1 and J_2

$$(t+s)^{(d-1)/2}\partial_t u((t+s)\theta, t) = (-1)^{d/2}(\mathcal{H}G_-)(-s, -\theta) + O\Big(\frac{1}{M}\Big) + O(t^{-1/2}).$$

Because the implicit constants in O's do not depend on $s \in [-R_2, R_2]$ or $\theta \in \mathbb{S}^{d-1}$, we may let $t \to +\infty$ then $M \to +\infty$ to conclude

$$\lim_{t\to+\infty} \int_{-R_2}^{R_2} \int_{\mathbb{S}^{d-1}} |(t+s)^{(d-1)/2} \partial_t u((t+s)\theta, t) - (-1)^{d/2} (\mathcal{H}G_-)(-s, -\theta)|^2 \, d\theta \, ds = 0. \qquad \square$$

4. Radial weakly nonradiative solutions

In this section we prove Proposition 1.9. First of all, we briefly show that any initial data in $P_{\text{rad}}(R)$ leads to an R-weakly nonradiative solution. By linearity we only need to consider the case $(u_0, u_1) = (r^{2k_1-d}, 0)$ or $(u_0, u_1) = (0, r^{2k_2-d})$. If $(u_0, u_1) = (r^{2k_1-d}, 0)$, then a basic calculation shows that if we choose $C_1, C_2, \ldots, C_{k_1-1}$ inductively, the solution

$$u_{k_1}(x,t) = \frac{1}{|x|^{d-2k_1}} + \frac{C_1 t^2}{|x|^{d-2k_1+2}} + \cdots + \frac{C_{k_1-1} t^{2k_1-2}}{|x|^{d-2}}$$

solves the linear wave equation with initial data $(|x|^{2k_1-d}, 0)$ in the region $\mathbb{R}^d \setminus \{0\}$. By finite speed of propagation, we have

$$\boldsymbol{S}_L(u_0, u_1)(x,t) = u_{k_1}(x,t), \qquad |x| > R + |t|.$$

A simple calculation shows that this is indeed a nonradiative solution. The case $(u_0, u_1) = (0, r^{2k_2-d})$ can be dealt with in the same manner by considering the solution

$$u_{k_2}(x,t) = \frac{t}{|x|^{d-2k_1}} + \frac{C_1 t^3}{|x|^{d-2k_1+2}} + \cdots + \frac{C_{k_2-1} t^{2k_1-1}}{|x|^{d-2}}.$$

Thus it is sufficient to show initial data of any nonradiative solution are contained in the space $P_{\text{rad}}(R)$. We first consider the odd dimensions.

4A. ***Odd dimensions.*** Assume that $u = \boldsymbol{S}_L(u_0, u_1)$ is a radial R-weakly nonradiative solution. Let $G_- = \boldsymbol{T}_-(u_0, u_1)$. By radial assumption G_- is independent of the angle $\omega \in \mathbb{S}^{d-1}$. Let us first consider smooth functions G_-. We may calculate ($r > R$, $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$)

$$u_0(re_1) = (2\pi)^{-\mu} \int_{\mathbb{S}^{d-1}} G_-^{(\mu-1)}(r\omega_1) \, d\omega = \frac{\sigma_{d-2}}{(2\pi)^\mu} \int_{-1}^{1} G_-^{(\mu-1)}(r\omega_1)(1-\omega_1^2)^{\mu-1} \, d\omega_1.$$

Here ω_1 is the first variable of $\mathbb{R}^d \supset \mathbb{S}^{d-1}$ and σ_{d-2} is the area of the sphere $\mathbb{S}^{d-2}$. We may integrate by parts and rescale:

$$\begin{aligned} u_0(re_1) &= \frac{(-1)^{\mu-1}\sigma_{d-2}}{(2\pi)^\mu r^{\mu-1}} \int_{-1}^{1} G_-(r\omega_1)[\partial_{\omega_1}^{\mu-1}(1-\omega_1^2)^{\mu-1}] \, d\omega_1 \\ &= \sum_{k=0}^{\lfloor(\mu-1)/2\rfloor} \frac{A_{d,k}}{r^{\mu-1}} \int_{-1}^{1} G_-(r\omega_1)\omega_1^{\mu-1-2k} \, d\omega_1 = \sum_{k=0}^{\lfloor(d-3)/4\rfloor} \frac{A_{d,k}}{r^{d-2-2k}} \int_{-R}^{R} G_-(s) s^{(d-3)/2-2k} \, ds \\ &= \sum_{k=1}^{\lfloor(d+1)/4\rfloor} \frac{A_{d,k}}{r^{d-2k}} \int_{-R}^{R} G_-(s) s^{(d+1)/2-2k} \, ds. \end{aligned}$$

Here the $A_{d,k}$ are nonzero constants. Similarly we have

$$\begin{aligned}u_1(re_1) &= (2\pi)^{-\mu}\int_{\mathbb{S}^{d-1}} G_-^{(\mu)}(r\omega_1)\,d\omega = \frac{\sigma_{d-2}}{(2\pi)^\mu}\int_{-1}^{1} G_-^{(\mu)}(r\omega_1)(1-\omega_1^2)^{\mu-1}\,d\omega_1\\ &= \frac{(-1)^\mu\sigma_{d-2}}{(2\pi)^\mu r^\mu}\int_{-1}^{1} G_-(r\omega_1)[\partial_{\omega_1}^\mu(1-\omega_1^2)^{\mu-1}]\,d\omega_1 = \sum_{k=0}^{\lfloor(\mu-2)/2\rfloor}\frac{B_{d,k}}{r^\mu}\int_{-1}^{1} G_-(r\omega_1)\omega_1^{\mu-2-2k}\,d\omega_1\\ &= \sum_{k=1}^{\lfloor(d-1)/4\rfloor}\frac{B_{d,k}}{r^{d-2k}}\int_{-R}^{R} G_-(s)s^{(d-1)/2-2k}\,ds.\end{aligned}$$

Here the $B_{d,k}$ are nonzero constants. Since smooth functions are dense in $L^2([-R,R])$, we have:

Proposition 4.1. *There exist constants $\{A_{d,k}\}_{1\le k\le\lfloor(d+1)/4\rfloor}$, $\{B_{d,k}\}_{1\le k\le\lfloor(d-1)/4\rfloor}$, so that for any $G_-\in L^2(\mathbb{R})$ supported in $[-R,R]$, the initial data $(u_0,u_1)=\boldsymbol{T}_-^{-1}G_-$ satisfy $(r>R)$*

$$u_0(r) = \sum_{k=1}^{\lfloor(d+1)/4\rfloor}\left(A_{d,k}\int_{-R}^{R} G_-(s)s^{(d+1)/2-2k}\,ds\right)r^{-d+2k},$$
$$u_1(r) = \sum_{k=1}^{\lfloor(d-1)/4\rfloor}\left(B_{d,k}\int_{-R}^{R} G_-(s)s^{(d-1)/2-2k}\,ds\right)r^{-d+2k}.$$

This clearly shows that if $u=\boldsymbol{S}_L(u_0,u_1)$ is a radial R-weakly nonradiative solution, then $(u_0,u_1)\in P_{\text{rad}}(R)$.

4B. *Even dimensions.* The even dimensions involve the Hilbert transform, and thus are much more difficult to handle. The general idea is the same. If the initial data (u_0,u_1) are radial, then $G_\pm(s)=\boldsymbol{T}_\pm(u_0,u_1)$ is independent to the angle. We also have $G_+(s)=(-1)^{d/2}\mathcal{H}G_-(-s)$. Thus $\boldsymbol{S}_L(u_0,u_1)$ is R-weakly nonradiative if and only if G_- is contained in the space

$$\mathcal{P}_{\text{rad}} = \{G_-\in L^2(\mathbb{R}) : G_-(s)=0,\ s>R,\ (\mathcal{H}G_-)(s)=0,\ s<-R\}.$$

Now recall the operators $\mathcal{Q}$, $\mathcal{Q}'$ and $\mathcal{D}$ defined in Section 2C. We claim:

Lemma 4.2. *$\mathcal{Q}'\mathcal{P}_{\text{rad}}=\dot H_0^{1/2}(-R,R)$. Here $\dot H_0^{1/2}(-R,R)$ is the completion of $C_0^\infty(-R,R)$ equipped with the $\dot H^{1/2}(\mathbb{R})$ norm.*

Proof. In order to avoid technical difficulties, we use an approximation technique. Given any $G_-\in\mathcal{P}_{\text{rad}}$, we may utilize a local smoothing kernel to generate a sequence G_k, so that:

(a) $G_k\in\mathcal{P}_{\text{rad}}(R+1/k)$.

(b) $G_k\in H^n(\mathbb{R})$ for all $n\ge 0$ and thus $G_k\in C^\infty(\mathbb{R})$.

(c) G_k converges to G_- in $L^2(\mathbb{R})$.

Let us consider the properties of the function $g_k=\mathcal{Q}'G_k\in C^\infty(\mathbb{R})$. According to part (a), $G_k(s)=0$ if $s>R+1/k$. We may use the convolution expression of $\mathcal{Q}'$ to obtain that g_k vanishes in the interval $(R+1/k,+\infty)$. Similarly $g_k=\mathcal{Q}\mathcal{H}G_k$ vanishes in the interval $(-\infty,-R-1/k)$. We recall that

$\mathcal{Q}' : L^2(\mathbb{R}) \to \dot{H}^{1/2}(\mathbb{R})$ is an isometry up to a constant. Thus $g_k \to g = \mathcal{Q}' G_-$ in $\dot{H}^{1/2}(\mathbb{R})$. This verifies $g \in \dot{H}_0^{1/2}(-R, R)$. We also need to show that given any $g \in \dot{H}_0^{1/2}(-R, R)$, then $\mathcal{Q}'^{-1} g \in \mathcal{P}_{\text{rad}}$. It is sufficient to consider $g \in C_0^\infty(-R, R)$ by smooth approximation. A simple calculation of Fourier symbols shows that $\mathcal{Q}'^{-1} = -\mathcal{Q}'\mathcal{D}$ and $\mathcal{H}\mathcal{Q}'^{-1} = \mathcal{Q}\mathcal{D}$. A combination of these identities with the convolution expressions of $\mathcal{Q}$ and $\mathcal{Q}'$ immediately verifies $\mathcal{Q}'^{-1} g \in \mathcal{P}_{\text{rad}}$. □

We also need to use the following explicit formula of $\boldsymbol{T}_-$ for radial data.

Lemma 4.3. *Assume $G \in C^\infty(\mathbb{R})$ so that $|G(s)| \lesssim |s|^{-3/2}$ for $|s| \gg 1$. Then the corresponding radial free wave $u = \boldsymbol{S}_L \boldsymbol{T}_-^{-1} G$ satisfies*

$$u(r, t) = C(d) \cdot r^{1-d/2} \int_{-1}^{1} \mathcal{Q}G(r\omega_1 + t) P_d(w_1)(1 - w_1^2)^{-1/2} \, d\omega_1. \tag{17}$$

Here P_d is an even or odd polynomial of degree $d/2 - 1$ defined by

$$\left(\frac{\partial}{\partial w_1}\right)^{d/2-1} (1 - w_1^2)^{(d-3)/2} = P_d(w_1)(1 - w_1^2)^{-1/2}.$$

Proof. If $G \in C_0^\infty(\mathbb{R})$, we use polar coordinates and integrate by parts:

$$\begin{aligned}
u(r, t) &= C(d) \int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{G^{(d/2-1)}(r\omega_1 - \rho + t)}{\sqrt{\rho}} \, d\omega \, d\rho \\
&= C(d) \int_0^\infty \int_{-1}^{1} \frac{G^{(d/2-1)}(r\omega_1 - \rho + t)}{\sqrt{\rho}} (1 - w_1^2)^{(d-3)/2} \, d\omega_1 \, d\rho \\
&= C(d) \cdot r^{1-d/2} \int_0^\infty \int_{-1}^{1} \frac{G(r\omega_1 - \rho + t)}{\sqrt{\rho}} P_d(w_1)(1 - w_1^2)^{-1/2} \, d\omega_1 \, d\rho \\
&= C(d) \cdot r^{1-d/2} \int_{-1}^{1} \mathcal{Q}G(r\omega_1 + t) P_d(w_1)(1 - w_1^2)^{-1/2} \, d\omega_1.
\end{aligned}$$

This verifies the formula if $G \in C_0^\infty(\mathbb{R})$. In order to deal with profile G without compact support, we use standard smooth cut-off techniques. More precisely, we may choose $G_k \in C_0^\infty(\mathbb{R})$ so that $G_k \to G$ in $L^2(\mathbb{R})$ and

$$|G_k(s) - G(s)| = 0, \quad s < k, \qquad |G_k(s) - G(s)| \lesssim |s|^{-3/2}, \quad s \geq k.$$

Thus we have $\|\mathcal{Q}G - \mathcal{Q}G_k\|_{L^\infty} \lesssim 1/k$. This means we have the uniform convergence for all (r, t) in any compact subset of $\mathbb{R}^+ \times \mathbb{R}$:

$$\begin{aligned}
u_k(r, t) &= \frac{C(d)}{r^{d/2-1}} \int_{-1}^{1} \mathcal{Q}G_k(r\omega_1 + t) P_d(w_1)(1 - w_1^2)^{-1/2} \, d\omega_1 \\
&\rightrightarrows \frac{C(d)}{r^{d/2-1}} \int_{-1}^{1} \mathcal{Q}G(r\omega_1 + t) P_d(w_1)(1 - w_1^2)^{-1/2} \, d\omega_1.
\end{aligned}$$

Combining this with the convergence $u_k \to u$ in $\dot{H}^1$ we finish the proof. □

Remark 4.4. If $d \geq 4$ and $G \in L^2(\mathbb{R})$, then formula (17) still holds. This follows standard smooth approximation and/or cut-off techniques. Let $G_k \in C_0^\infty(\mathbb{R})$ so that $G_k \to G$ in $L^2(\mathbb{R})$. Thus $\mathcal{Q}G_k \to \mathcal{Q}G$ in $\dot{H}^{1/2}(\mathbb{R})$. Finally we observe the fact $P_d(w_1)(1-w_1^2)^{-1/2} \in \dot{H}^{-1/2}(\mathbb{R})$, obtain a locally uniform convergence $u_k(r,t) \to u(r,t)$ and conclude the proof.

Now we are ready to give an expression of $u = \boldsymbol{S}_L \boldsymbol{T}_-^{-1} G_-$ when $G_- \in \mathcal{P}_{\text{rad}}(R)$.

Lemma 4.5. *Assume $G_- \in \mathcal{P}_{\text{rad}}(R)$. Then the following identity holds*:

$$u(r,t) = \frac{C(d)}{r^{d/2}} \int_{-R}^{R} \mathcal{Q}' G_-(s) W_d\left(\frac{s-t}{r}\right) ds.$$

Here $W_d(\sigma)$ is the Hilbert transform (the function below is understood as zero if $|w_1| > 1$)

$$W_d(\sigma) \doteq \mathcal{H}\left(\frac{P_d(w_1)}{\sqrt{1-w_1^2}}\right) = \mathcal{H}\left[\left(\frac{d}{dw_1}\right)^{d/2-1} (1-w_1^2)^{(d-3)/2}\right].$$

Proof. By Lemma 4.2, we have $\mathcal{Q}'G_- \in \dot{H}_0^{1/2}(-R,R)$. We claim that it is sufficient to consider the case $\mathcal{Q}'G_- \in C_0^\infty(-R,R)$. In fact, we may choose $G_k \in \mathcal{P}_{\text{rad}}(R)$ so that $\mathcal{Q}'G_k \in C_0^\infty(-R,R)$ so that

$$\mathcal{Q}'G_k \to \mathcal{Q}'G_- \quad \text{in } \dot{H}^{1/2}(-R,R) \quad \Longrightarrow \quad G_k \to G_- \quad \text{in } L^2(\mathbb{R}).$$

Now we observe a few important facts: we have the embedding $\dot{H}_0^{1/2}(-R,R) \hookrightarrow L^p(-R,R)$ for all $1 \leq p < +\infty$ and

$$\frac{P_d(w_1)}{\sqrt{1-w_1^2}} \in L^p(\mathbb{R}) \quad \Longrightarrow \quad W_d(\sigma) \in L^p(\mathbb{R}), \quad p \in (1,2).$$

As a result, if the identity

$$u_k(r,t) = \frac{C(d)}{r^{d/2}} \int_{-R}^{R} \mathcal{Q}'G_k(s) W_d\left(\frac{s-t}{r}\right) ds, \quad k \geq 1,$$

holds, then we may make $k \to +\infty$ in the identity above and verify that a similar identity holds for u and G_-. In fact the left-hand side converges in the space $\dot{H}^1(\mathbb{R}^d)$ for any given time t, while the right-hand side converges uniformly for (r,t) in any compact subset of $\mathbb{R}^+ \times \mathbb{R}$. Now we assume $g = \mathcal{Q}'G_- \in C_0^\infty(-R,R)$. Then $G_- = \mathcal{Q}'^{-1} g = -\mathcal{Q}'\mathcal{D}g$ satisfies the assumption of Lemma 4.3. As a result we have

$$\begin{aligned} u(r,t) &= C(d) \cdot r^{1-d/2} \int_{-1}^{1} \mathcal{Q}\mathcal{Q}'\mathcal{D}g(r\omega_1 + t) P_d(w_1)(1-w_1^2)^{-1/2}\, d\omega_1 \\ &= C(d) \cdot r^{1-d/2} \int_{-1}^{1} \mathcal{H}g(r\omega_1 + t) P_d(w_1)(1-w_1^2)^{-1/2}\, d\omega_1 \\ &= \frac{C(d)}{r^{d/2-1}} \int_{-\infty}^{\infty} g(r\sigma + t) W_d(\sigma)\, d\sigma. \end{aligned}$$

Here we use the facts $\mathcal{Q}\mathcal{Q}'\mathcal{D} = \mathcal{H}$ and

$$\int \mathcal{H}f \cdot \overline{\mathcal{H}g}\, dx = \int f \cdot \bar{g}\, dx, \quad \mathcal{H}(\mathcal{H}g(r\omega_1 + t))(\sigma) = (\mathcal{H}^2 g)(r\sigma + t) = -g(r\sigma + t).$$

Finally we apply change of variables $s = r\sigma + t$, recall the support of g and finish the proof. □

Now let us consider the Hilbert transform W_d. The key observation is the following technical lemma. This result has been known for many years; see [Solmon 1987], for instance. But we still give a brief proof in the Appendix for the purpose of completeness.

Lemma 4.6. *Assume that $P(x)$ is a polynomial of degree κ. Let W be the Hilbert transform*

$$W = \mathcal{H}\left(\frac{P(x)}{\sqrt{1-x^2}}\right).$$

Then $W(\sigma)$ is equal to a polynomial of degree $\kappa - 1$ if $\sigma \in (-1, 1)$. In particular, $W_2(\sigma) = 0$ for $\sigma \in (-1, 1)$; if $d \geq 4$, then the function $W_d(\sigma)$ is equal to an even or odd polynomial of degree $d/2-2$ in the interval $(-1, 1)$.

Proof of Proposition 1.11. According to Lemma 4.5, we have already obtained

$$u(r,t) = \frac{C(d)}{r^{d/2}} \int_{-R}^{R} \mathcal{Q}' G_-(s) W_d\left(\frac{s-t}{r}\right) ds.$$

Here $\mathcal{Q}'G_- \in \dot{H}_0^{1/2}(-R, R) \hookrightarrow L^p(-R, R)$ for all $1 < p < +\infty$. If we also have $r > |t| + R$, then

$$\left|\frac{s-t}{r}\right| < 1 \quad \text{for all } s \in (-R, R).$$

If $d = 2$, Lemma 4.6 immediately gives $u(r,t) \equiv 0$ if $r > |R| + t$ since we always have $W_2((s-t)/r) = 0$. In the higher-dimensional case $d \geq 4$, Lemma 4.6 guarantees that

$$W_d(s) = \sum_{l=1}^{\lfloor d/4 \rfloor} A_l s^{d/2-2l}, \quad -1 < s < 1,$$

is a polynomial. We plug this in the expression of u and obtain

$$u(r,t) = C(d) \sum_{l=1}^{\lfloor d/4 \rfloor} \frac{A_l}{r^{d-2l}} \int_{-R}^{R} \mathcal{Q}' G_-(s)(s-t)^{d/2-2l}\, ds, \quad r > R + |t|. \tag{18}$$

This immediately gives $(u_0, u_1) \in P_{\mathrm{rad}}(R)$.

5. Exterior energy estimates of even dimensions

In this section we prove Proposition 1.11. It suffices to consider the case $d = 4k$. The proof of $d = 4k+2$ is almost the same. Again we switch to the space of radiation profiles $G_- \in L^2(\mathbb{R} \times \mathbb{S}^{d-1})$. We start with:

Lemma 5.1. *The image of radial data in the form of $(u_0, 0)$ can be characterized by*

$$\begin{aligned}\{\boldsymbol{T}_-(u_0, 0) : u_0 \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^d)\} &= \{G_- \in L^2(\mathbb{R}) : \mathcal{H}G_-(-s) = -G_-(s)\} \\ &= \left\{\frac{G(s) - \mathcal{H}G(-s)}{2} : G \in L^2(\mathbb{R})\right\}.\end{aligned}$$

Proof. First of all, if $u_0 \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^d)$, then the free wave $u = \boldsymbol{S}_L(u_0, u_1)$ is radial and satisfies

$$u(x,t) = u(x,-t), \quad u_t(x,t) = -u_t(x,-t).$$

Therefore G_-, G_+ are radial, i.e., independent of ω and satisfy $G_+(s) = -G_-(s)$. We may apply Theorem 1.5 and obtain $G_+(s) = \mathcal{H}G_-(-s)$. As a result, G_- satisfies the identity $\mathcal{H}G_-(-s) = -G_-(s)$. Next, let us assume G_- satisfies this identity. Then we have

$$G_-(s) = \frac{G_-(s) - \mathcal{H}G_-(-s)}{2} \in \left\{ \frac{G(s) - \mathcal{H}G(-s)}{2} : G \in L^2(\mathbb{R}) \right\}.$$

Finally, if $G_-(s) = (G(s) - \mathcal{H}G(-s))/2$, we show there exists $u_0 \in \dot{H}^1_{\text{rad}}(\mathbb{R}^d)$, so that $G_- = \boldsymbol{T}_-(u_0, 0)$. In fact, we consider radial initial data $(u_0, u_1) = \boldsymbol{T}_-^{-1} G$ and free wave $u = \boldsymbol{S}_L(u_0, u_1)$. We may reverse the time and obtain $u(x, -t) = \boldsymbol{S}_L(u_0, -u_1)(x, t)$. Thus

$$\boldsymbol{T}_-(u_0, -u_1)(s) = -\boldsymbol{T}_+(u_0, u_1)(s) = -\mathcal{H}G(-s).$$

Therefore we have

$$\boldsymbol{T}_-(2u_0, 0)(s) = \boldsymbol{T}_-(u_0, u_1) + \boldsymbol{T}_-(u_0, -u_1) = G(s) - \mathcal{H}G(-s) = 2G_-(s),$$

which completes the proof. □

The key observation is the following:

Lemma 5.2. *Given $g \in L^2(\mathbb{R}^+)$, there exists a function G with $\|G\|_{L^2(\mathbb{R})} \leq 2\|g\|_{L^2(\mathbb{R}^+)}$ so that*

$$G(s) - \mathcal{H}G(-s) = 2g(s), \quad s > 0, \qquad \left\| \frac{G(s) - \mathcal{H}G(-s)}{2} \right\|_{L^2(\mathbb{R})} \leq \sqrt{2}\|g\|_{L^2(\mathbb{R}^+)}.$$

Proof. Let us first find a function G with $\|G\|_{L^2(\mathbb{R})} \leq 2\|g\|_{L^2}$ so that

$$G(s) - \frac{G(s) + \mathcal{H}G(-s)}{2} = g(s), \quad s > 0.$$

We define a linear bounded operator $\boldsymbol{T}$ from $L^2(\mathbb{R}^+)$ to itself. In the formula below we extend the domain of G to $\mathbb{R}$ by assuming $G(s) = 0$ if $s < 0$ before we apply the Hilbert transform:

$$(\boldsymbol{T}G)(s) = \frac{G(s) + \mathcal{H}G(-s)}{2} = \frac{G(s)}{2} - \frac{1}{2\pi} \int_0^\infty \frac{G(\tau)}{s + \tau} \, d\tau, \quad s > 0.$$

We may further rewrite it as

$$\boldsymbol{T}G = \frac{G}{2} - \frac{1}{2\pi} \boldsymbol{L}^2 G.$$

Here $\boldsymbol{L}$ is the Laplace transform

$$\boldsymbol{L}G(s) = \int_0^\infty G(\tau) e^{-s\tau} \, d\tau,$$

which is self-adjoint operator in $L^2(\mathbb{R}^+)$ with an operator norm $\sqrt{\pi}$. More details about the Laplace transform can be found in [Lax 2002]. As a result, we have

$$\begin{aligned}
\|\boldsymbol{T}G\|^2_{L^2(\mathbb{R}^+)} &= \tfrac{1}{4}\big\langle G - \tfrac{1}{\pi}\boldsymbol{L}^2 G, G - \tfrac{1}{\pi}\boldsymbol{L}^2 G \big\rangle \\
&= \tfrac{1}{4}\|G\|^2_{L^2} + \tfrac{1}{4\pi^2}\|\boldsymbol{L}^2 G\|^2_{L^2} - \tfrac{1}{4\pi}\langle G, \boldsymbol{L}^2 G\rangle - \tfrac{1}{4\pi}\langle \boldsymbol{L}^2 G, G\rangle \\
&\leq \tfrac{1}{4}\|G\|^2_{L^2} + \tfrac{1}{4\pi}\|\boldsymbol{L}G\|^2_{L^2} - \tfrac{1}{2\pi}\langle \boldsymbol{L}G, \boldsymbol{L}G\rangle = \tfrac{1}{4}\|G\|^2_{L^2} - \tfrac{1}{4\pi}\|\boldsymbol{L}G\|^2_{L^2}.
\end{aligned}$$

Thus the operator norm of $\boldsymbol{T}$ is less than or equal to $\frac{1}{2}$. This means that the function

$$G = \sum_{j=0}^{\infty} \boldsymbol{T}^j g \in L^2(\mathbb{R}^+)$$

satisfies the equation $G - \boldsymbol{T}G = g$ and $\|G\|_{L^2(\mathbb{R}^+)} \le 2\|g\|_{L^2(\mathbb{R}^+)}$. Finally we naturally extend the domain of G to $\mathbb{R}$ by defining $G(s) = 0$ if $s < 0$. We have

$$\frac{G(s) - \mathcal{H}G(-s)}{2} = \begin{cases} g(s), & s > 0, \\ -\frac{1}{2}\mathcal{H}G(-s), & s < 0. \end{cases}$$

Therefore we may find an upper bound of the L^2 norm

$$\left\| \frac{G(s) - \mathcal{H}G(-s)}{2} \right\|_{L^2(\mathbb{R})}^2 \le \|g\|_{L^2(\mathbb{R}^+)}^2 + \tfrac{1}{4}\|\mathcal{H}G\|_{L^2(\mathbb{R})}^2 \le 2\|g\|_{L^2(\mathbb{R}^+)}^2. \qquad \square$$

Proof of Proposition 1.11. Let $G_- = \boldsymbol{T}_-(u_0, 0)$ and $g(s)$ be its cut-off version:

$$g(s) = \begin{cases} G_-(s), & s > R, \\ 0, & s < R. \end{cases}$$

Then radiation field implies that the free wave $u = \boldsymbol{S}_L(u_0, 0)$ satisfies

$$\lim_{t \to -\infty} \int_{|x| > R + |t|} |\nabla u(x,t)|^2 \, dx = \lim_{t \to -\infty} \int_{|x| > R + |t|} |u_t(x,t)|^2 \, dx = \sigma_{4k-1} \|g\|_{L^2(\mathbb{R}^+)}^2. \tag{19}$$

Here again σ_{4k-1} is the area of the sphere $\mathbb{S}^{4k-1}$. According to Lemmas 5.1 and 5.2, there exists a function $\tilde{u}_0 \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^{4k})$, so that

$$\boldsymbol{T}_-(\tilde{u}_0, 0)(s) = g(s), \quad s > 0, \qquad \|\tilde{u}_0\|_{\dot{H}^1(\mathbb{R}^{4k})}^2 \le 4\sigma_{4k-1}\|g\|_{L^2(\mathbb{R}^+)}^2.$$

Therefore $\boldsymbol{T}_-(u_0 - \tilde{u}_0, 0)$ vanishes if $s > R$. A combination of this fact with the time symmetry gives

$$\lim_{t \to \pm\infty} \int_{|x| > |t| + R} |\nabla_{t,x} \boldsymbol{S}_L(u_0 - \tilde{u}_0, 0)(x,t)|^2 \, dx = 0.$$

As a result, we may apply Proposition 1.9 and conclude $u_0 - \tilde{u}_0 \in Q_k(R)$. This means

$$\|\Pi^{\perp}_{Q_k(R)} u_0\|_{\dot{H}^1(\{x:|x|>R\})}^2 \le \|\tilde{u}_0\|_{\dot{H}^1(\{x:|x|>R\})}^2 \le 4\sigma_{4k-1}\|g\|_{L^2(\mathbb{R}^+)}^2.$$

A combination of this inequality and identity (19) immediately verifies the conclusion of Proposition 1.11 in the negative time direction. The positive time direction follows the time symmetry.

6. Nonradial exterior energy estimates

In this section we give a short proof of Proposition 1.14. We start with:

Lemma 6.1. *Let $d \ge 3$ be an odd integer. Then*

$$\sum_{\pm} \lim_{t \to \pm\infty} \int_{|x| > R + |t|} |\nabla_{t,x} \boldsymbol{S}_L(u_0, u_1)(x,t)|^2 \, dx = 2 \int_{|s| > R} \int_{\mathbb{S}^{d-1}} |\boldsymbol{T}_-(u_0, u_1)(s, \theta)|^2 \, d\theta \, ds. \tag{20}$$

In particular, we have (see (4) for the definition of $P(R)$)

$$\boldsymbol{T}_-(P(R)) = \mathcal{P}(R) \doteq \{G_- \in L^2(\mathbb{R}\times\mathbb{S}^{d-1}) : \operatorname{supp} G_- \subset [-R, R]\times\mathbb{S}^{d-1}\}.$$

Proof. Let u be the solution of linear wave equation with initial data (u_0, u_1). Then by radiation field (Theorem 1.1) we have

$$\lim_{t\to-\infty}\int_{|x|>|t|+R} |\nabla_{t,x}u|^2\,dx = 2\int_R^\infty\int_{\mathbb{S}^{d-1}} |G_-(s,\theta)|^2\,d\theta\,ds,$$
$$\lim_{t\to-\infty}\int_{|x|<|t|-R} |\nabla_{t,x}u|^2\,dx = 2\int_{-\infty}^{-R}\int_{\mathbb{S}^{d-1}} |G_-(s,\theta)|^2\,d\theta\,ds.$$

In addition, we may apply the energy conservation law, Proposition 1.2 and obtain

$$\begin{aligned}\lim_{t\to-\infty}\int_{|x|<|t|-R} |\nabla_{t,x}u|^2\,dx &= \int_{\mathbb{R}^d}(|\nabla u_0|^2+|u_1|^2)\,dx - \lim_{t\to-\infty}\int_{|x|>|t|-R}|\nabla_{t,x}u|^2\,dx\\ &= \lim_{t\to+\infty}\int_{|x|>t+R}|\nabla_{t,x}u|^2\,dx.\end{aligned}$$

Combining these identities we have

$$\sum_{\pm}\lim_{t\to\pm\infty}\int_{|x|>R+|t|}|\nabla_{t,x}u(x,t)|^2\,dx = 2\int_{|s|>R}\int_{\mathbb{S}^{d-1}}|G_-(s,\theta)|^2\,d\theta\,ds.$$

Finally $(u_0, u_1)\in P(R)$ is equivalent to saying

$$\int_{|s|>R}\int_{\mathbb{S}^{d-1}}|G_-(s,\theta)|^2\,d\theta\,ds = 0,$$

namely $\operatorname{supp} G_- \subset [-R, R]\times\mathbb{S}^{d-1}$. □

Now we are ready to prove Proposition 1.14. Since $\sqrt{2}\boldsymbol{T}_-$ is a bijective isometry from $\dot{H}^1\times L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}\times\mathbb{S}^{d-1})$, we have

$$\boldsymbol{\Pi}^\perp_{P(R)}(u_0,u_1) = \boldsymbol{T}_-^{-1}\boldsymbol{\Pi}^\perp_{\boldsymbol{T}_-(P(R))}\boldsymbol{T}_-(u_0,u_1).$$

We next use the expression of $\mathcal{P}(R) = \boldsymbol{T}_-(P(R))$:

$$\begin{aligned}\|\boldsymbol{\Pi}^\perp_{P(R)}(u_0,u_1)\|^2_{\dot{H}^1\times L^2} &= 2\|\boldsymbol{\Pi}^\perp_{\mathcal{P}(R)}\boldsymbol{T}_-(u_0,u_1)\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}\\ &= 2\int_{|s|>R}\int_{\mathbb{S}^{d-1}}|\boldsymbol{T}_-(u_0,u_1)(s,\theta)|^2\,d\theta\,ds.\end{aligned}$$

Combining this with (20) we finish the proof.

Appendix

In this section we give a brief proof of Lemma 4.6 for completeness. We first prove this lemma for two special cases, $P(x) = 1$ and $P(x) = 1 - x^2$. We start with $P(x) = 1$. A straightforward calculation gives

$$
\begin{aligned}
\pi W(s) &= \text{p.v.} \int_{-1}^{1} \frac{(1-x^2)^{-1/2}}{s-x}\,dx \\
&= \text{p.v.} \int_{-1}^{1} \frac{(1-s^2)^{-1/2}}{s-x}\,dx + \int_{-1}^{1} \frac{(1-x^2)^{-1/2}-(1-s^2)^{-1/2}}{s-x}\,dx \\
&= (1-s^2)^{-1/2} \ln\left|\frac{1+s}{1-s}\right| + \int_{-1}^{1} \frac{(1-s^2)-(1-x^2)}{(s-x)\sqrt{1-x^2}\sqrt{1-s^2}(\sqrt{1-x^2}+\sqrt{1-s^2})}\,dx \\
&= (1-s^2)^{-1/2} \ln\left|\frac{1+s}{1-s}\right| + \frac{-s}{\sqrt{1-s^2}} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}(\sqrt{1-x^2}+\sqrt{1-s^2})}\,dx.
\end{aligned}
$$

Next we apply the change of variables $x = 2z/(1+z^2)$. We have

$$
\sqrt{1-x^2} = \frac{1-z^2}{1+z^2}\, dx = \frac{2(1-z^2)}{(1+z^2)^2}\,dz.
$$

Thus

$$
\begin{aligned}
\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}(\sqrt{1-x^2}+\sqrt{1-s^2})}\,dx &= \int_{-1}^{1} \frac{2\,dz}{1-z^2+\sqrt{1-s^2}(1+z^2)} \\
&= \frac{2}{s}\int_{0}^{1}\left(\frac{1}{(1+\sqrt{1-s^2})/s-z} + \frac{1}{(1+\sqrt{1-s^2})/s+z}\right)dz \\
&= \frac{2}{s}\ln\left|\left(\frac{1+\sqrt{1-s^2}}{s}+1\Big/\frac{1+\sqrt{1-s^2}}{s}-1\right)\right| = \frac{1}{s}\ln\left|\frac{1+s}{1-s}\right|. \qquad (21)
\end{aligned}
$$

This immediately gives $W(s) = 0$ for $s \in (-1, 1)$. Next we consider the case $P(x) = 1 - x^2$. In this case we calculate the Hilbert transform of $\sqrt{1-x^2}$:

$$
\begin{aligned}
\pi W(s) &= \text{p.v.} \int_{-1}^{1} \frac{\sqrt{1-x^2}}{s-x}\,dx \\
&= \text{p.v.} \int_{-1}^{1} \frac{\sqrt{1-s^2}}{s-x}\,dx + \int_{-1}^{1} \frac{\sqrt{1-x^2}-\sqrt{1-s^2}}{s-x}\,dx \\
&= \sqrt{1-s^2}\ln\left|\frac{1+s}{1-s}\right| + \int_{-1}^{1} \frac{(1-x^2)-(1-s^2)}{(s-x)(\sqrt{1-x^2}+\sqrt{1-s^2})}\,dx \\
&= \sqrt{1-s^2}\ln\left|\frac{1+s}{1-s}\right| + s\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}+\sqrt{1-s^2}}\,dx \\
&= \sqrt{1-s^2}\ln\left|\frac{1+s}{1-s}\right| + \pi s + s\int_{-1}^{1}\left(\frac{1}{\sqrt{1-x^2}+\sqrt{1-s^2}} - \frac{1}{\sqrt{1-x^2}}\right)dx \\
&= \sqrt{1-s^2}\ln\left|\frac{1+s}{1-s}\right| + \pi s - s\sqrt{1-s^2}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}(\sqrt{1-x^2}+\sqrt{1-s^2})}\,dx = \pi s.
\end{aligned}
$$

Here we use the integral (21) again.

Induction. Now we are ready to prove Lemma 4.6 by induction. It is clear that we only need to show the Hilbert transform of $f_\kappa(x) = x^\kappa(1-x^2)^{-1/2}$ is a polynomial of degree $\kappa - 1$ in the interval $(-1, 1)$. The cases of $\kappa = 0, 2$ have been done. Now let us consider the case of $f_1(x) = x(1-x^2)^{-1/2}$. We observe that $(s \in (-1, 1))$

$$\mathcal{H} f_1 = \mathcal{H}\frac{d}{dx}(-\sqrt{1-x^2}) = -\frac{d}{ds}\mathcal{H}(\sqrt{1-x^2}) = -1.$$

This proves the case $\kappa = 1$. Now let us assume that the cases $\kappa = 0, 1, 2, \ldots, n$ are done and consider the case $\kappa = n+1$. Here $n \geq 2$. We have

$$x^{n+1}(1-x^2)^{-1/2} = -x^{n-1}(1-x^2)^{1/2} + x^{n-1}(1-x^2)^{-1/2}.$$

The Hilbert transform of the second term in the right-hand side is known to be a polynomial of degree $n-2$. Thus we only need to consider the first term. We have

$$\begin{aligned}\frac{d}{ds}\mathcal{H}(x^{n-1}(1-x^2)^{1/2}) &= \mathcal{H}\frac{d}{dx}(x^{n-1}(1-x^2)^{1/2})\\ &= \mathcal{H}\{[-nx^n + (n-1)x^{n-2}](1-x^2)^{-1/2}\}.\end{aligned}$$

This is a polynomial of degree $n-1$ by induction hypothesis. A simple integration then finishes the proof of the case $\kappa = n+1$. Generally speaking, the derivative with respect to s as given above is in the weak sense. But since the derivative is known to be a polynomial in $(-1, 1)$, we can integrate as usual.

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