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ON L^∞ ESTIMATES FOR MONGE–AMPÈRE AND HESSIAN EQUATIONS ON NEF CLASSES

BIN GUO, DUONG H. PHONG, FREID TONG AND CHUWEN WANG

The PDE approach developed earlier by the first three authors for L^∞ estimates for fully nonlinear equations on Kähler manifolds is shown to apply as well to Monge–Ampère and Hessian equations on nef classes. In particular, one obtains a new proof of the estimates of Boucksom, Eyssidieux, Guedj and Zeriahi (2010) and Fu, Guo and Song (2020) for the Monge–Ampère equation, together with their generalization to Hessian equations.

1. Introduction

The goal of this short note is to show that the PDE approach introduced in [Guo et al. 2023a; 2023b] for L^∞ and Trudinger-type estimates for general classes of fully nonlinear equations on a compact Kähler manifold applies as well to Monge–Ampère and Hessian equations on nef classes.

The key to the approach in [Guo et al. 2023a; 2023b] is an estimate of Trudinger-type, obtained by comparing the solution φ of the given equation to the solution of an auxiliary Monge–Ampère equation with the energy of the sublevel set function $-\varphi+s$ on the right-hand side. We shall see that, in the present case of nef classes, the argument can still be made to work by replacing φ by $\varphi-V$, where V is the envelope of the nef class. Applied to the Monge–Ampère equation, this gives a PDE proof of the estimates obtained earlier for nef classes by Boucksom, Eyssidieux, Guedj and Zeriahi [Boucksom et al. 2010] and Fu, Guo and Song [Fu et al. 2020]. The estimates which we obtain with this method applied to Hessian equations seem new.

We note that the use of an auxiliary Monge–Ampère equation was instrumental in the recent progress of Chen and Cheng [2021] on the constant scalar curvature Kähler metrics problem. There the auxiliary equation involved the entropy, and not the energy, of sublevel set functions as in our case. More generally, auxiliary equations have often been used in the theory of partial differential equations, notably by De Giorgi [1961] and more recently by Dinew and Kołodziej [Demailly et al. 2014; Dinew and Kołodziej 2014] in their approach to Hölder estimates for the complex Monge–Ampère equation.

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2. The Monge–Ampère equation

We begin with the Monge–Ampère equation. Let (X, ω) be a compact Kähler manifold, and, without loss of generality, let us assume that $\int_X \omega^n = 1$. Let χ be a closed $(1, 1)$-form on X. We assume the cohomology class $[\chi]$ is nef and let $\nu \in \{0, 1, \dots, n\}$ be the numerical dimension of $[\chi]$, i.e.,

$$\nu = \max\{k \mid [\chi]^k \neq 0 \text{ in } H^{k,k}(X, \mathbb{C})\}.$$

When $\nu = n$ we say the class $[\chi]$ is *big*.

Let $\hat{\omega}_t = \chi + t\omega$ for $t \in (0, 1]$. The form $\hat{\omega}_t$ may not be positive but its class is Kähler. We consider the family of complex Monge–Ampère equations

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n = c_t e^F \omega^n, \quad \sup_X \varphi_t = 0, \tag{2-1}$$

where $c_t = [\hat{\omega}_t^n] = O(t^{n-\nu})$ is a normalizing constant and $F \in C^\infty(X)$ satisfies $\int_X e^F \omega^n = \int_X \omega^n$. This equation admits a unique smooth solution φ_t by Yau's theorem [1978].

The form χ is not assumed to be semipositive, so the usual L^∞ estimate of φ_t may not hold [Kołodziej 1998]. As in [Boucksom et al. 2010; Fu et al. 2020], we need to modify the solution φ_t by an envelope V_t of the class $[\hat{\omega}_t]$, defined as

$$V_t = \sup\{v \mid v \in \mathrm{PSH}(X, \hat{\omega}_t),\ v \le 0\}.$$

Then we have:

Theorem 1. *Consider (2-1), and assume that the cohomology class of χ is nef. For any $s > 0$, let $\Omega_s = \{\varphi_t - V_t \le -s\}$ be the sublevel set of $\varphi_t - V_t$.*

(a) *There are constants $C = C(n, \omega, \chi) > 0$ and $\alpha_0 = \alpha_0(n, \omega, \chi) > 0$ such that*

$$\int_{\Omega_s} \exp\left\{\alpha_0 \left(\frac{-(\varphi_t - V_t + s)}{A_s^{1/(1+n)}}\right)^{(n+1)/n}\right\} \omega^n \le C \exp(C E_t), \tag{2-2}$$

where $A_s = \int_{\Omega_s} (-\varphi_t + V_t - s) e^F \omega^n$ and $E_t = \int_X (-\varphi_t + V_t) e^F \omega^n$.

(b) *Fix $p > n$. There is a constant $C(n, p, \omega, \chi, \|e^F\|_{L^1(\log L)^p})$ such that, for all $t \in (0, 1]$, we have*

$$0 \le -\varphi_t + V_t \le C(n, p, \omega, \chi, \|e^F\|_{L^1(\log L)^p}). \tag{2-3}$$

We remark that the estimates in Theorem 1 continue to hold for a family of Kähler metrics (maybe with distinct complex structures) which satisfy a uniform α-invariant-type estimate.

Proof. We would like to find an auxiliary equation with smooth coefficients, so that its solvability can be guaranteed by Yau's theorem. For this, we need a lemma due to Berman [2019] on a smooth approximation for V_t; see also Lemma 4 below. Fix a time $t \in (0, 1]$.

Lemma 2. *Let u_β be the smooth solution to the complex Monge–Ampère equation*

$$(\hat{\omega}_t + i\partial\bar{\partial}u_\beta)^n = e^{\beta u_\beta} \omega^n.$$

Then u_β converges uniformly to V_t as $\beta \to \infty$.

We remark that by [Chu et al. 2018], V_t is a $C^{1,1}$ function on X, although this fact is not used in this note. We now return to the proof of Theorem 1(a).

We choose a sequence of smooth positive functions $\tau_k : \mathbb{R} \to \mathbb{R}_+$ such that $\tau_k(x)$ decreases to $x \cdot \chi_{\mathbb{R}_+}(x)$ as $k \to \infty$. Fix a smooth function u_β as in Lemma 2. The function u_β depends on t, but for simplicity we omit the subscript t. We solve the following auxiliary Monge–Ampère equation on X,

$$(\hat{\omega}_t + i\partial\bar{\partial}\psi_{t,k})^n = c_t \frac{\tau_k(-\varphi_t + u_\beta - s)}{A_{s,k,\beta}} e^F \omega^n, \quad \sup_X \psi_{t,k} = 0, \tag{2-4}$$

where

$$A_{s,k,\beta} = \int_X \tau_k(-\varphi_t + u_\beta - s) e^F \omega^n.$$

Since $\psi_{t,k} \le V_t$ and u_β converges uniformly to V_t, by taking β large enough, we may assume $\psi_{t,k} < u_\beta + 1$.

Define a function

$$\Phi = -\varepsilon(-\psi_{t,k} + u_\beta + 1 + \Lambda)^{n/(n+1)} - (\varphi_t - u_\beta + s),$$

with the constants

$$\varepsilon^{n+1} = A_{s,k,\beta} n^{-n}(n+1)^n, \quad \Lambda = n^{n+1}(n+1)^{-n-1}\varepsilon^{n+1}. \tag{2-5}$$

As a smooth function on the compact manifold X, we know Φ must achieve its maximum at some $x_0 \in X$. If $x_0 \in X \backslash \Omega_s^\circ$, then

$$\Phi(x_0) \le -(\varphi_t - u_\beta + s) \le -V_t + u_\beta \le \epsilon_\beta,$$

where $\epsilon_\beta \to 0$ as $\beta \to \infty$. On the other hand, if $x_0 \in \Omega_s^\circ$, we calculate (Δ_t denotes the Laplacian with respect to the metric $\omega_t = \hat{\omega}_t + i\partial\bar{\partial}\varphi_t$)

$$\begin{aligned}
0 &\ge \Delta_t \Phi(x_0) \\
&= -\varepsilon \frac{n}{n+1}(-\psi_{t,k} + u_\beta + \Lambda + 1)^{-1/(n+1)} \operatorname{tr}_{\omega_t}(-i\partial\bar{\partial}\psi_{t,k} + i\partial\bar{\partial}u_\beta) - \operatorname{tr}_{\omega_t}(i\partial\bar{\partial}\varphi_t - i\partial\bar{\partial}u_\beta) \\
&\qquad + \frac{n\varepsilon}{(n+1)^2}(-\psi_{t,k} + u_\beta + 1 + \Lambda)^{-(n+2)/(n+1)} \operatorname{tr}_{\omega_t} i\partial(\psi_{t,k} - u_\beta) \wedge \bar{\partial}(\psi_{t,k} - u_\beta) \\
&\ge \frac{n\varepsilon}{n+1}(-\psi_{t,k} + u_\beta + \Lambda + 1)^{-1/(n+1)} \operatorname{tr}_{\omega_t}(\hat{\omega}_{t,\psi_{t,k}} - \hat{\omega}_{t,u_\beta}) - n + \operatorname{tr}_{\omega_t} \hat{\omega}_{t,u_\beta} \\
&\ge \frac{n\varepsilon}{n+1}(-\psi_{t,k} + u_\beta + \Lambda + 1)^{-1/(n+1)} n \left(\frac{\hat{\omega}^n_{t,\psi_{t,k}}}{\omega_t^n} \right)^{1/n} \\
&\qquad - n + \left(1 - \frac{n\varepsilon}{n+1}(-\psi_{t,k} + u_\beta + \Lambda + 1)^{-1/(n+1)}\right) \operatorname{tr}_{\omega_t} \hat{\omega}_{t,u_\beta} \\
&\ge \frac{n^2\varepsilon}{n+1}(-\psi_{t,k} + u_\beta + \Lambda + 1)^{-1/(n+1)} (\tau_k(-\varphi_t + u_\beta - s) A^{-1}_{s,k,\beta})^{1/n} \\
&\qquad - n + \left(1 - \frac{n\varepsilon}{n+1}\Lambda^{-1/(n+1)}\right) \operatorname{tr}_{\omega_t} \hat{\omega}_{t,u_\beta} \\
&\ge \frac{n^2\varepsilon}{n+1}(-\psi_{t,k} + u_\beta + \Lambda + 1)^{-1/(n+1)} (-\varphi_t + u_\beta - s)^{1/n} A^{-1/n}_{s,k,\beta} - n.
\end{aligned}$$

Therefore, at $x_0 \in \Omega_s^\circ$,

$$-(\varphi_t - u_\beta + s) \le \left(\frac{n+1}{n\varepsilon}\right)^n A_{s,k,\beta}(-\psi_{t,k} + u_\beta + \Lambda + 1)^{n/(n+1)} = \varepsilon(-\psi_{t,k} + u_\beta + \Lambda + 1)^{n/(n+1)},$$

i.e., $\Phi(x_0) \le 0$. Combining the two cases, we conclude that $\sup_X \Phi \le \epsilon_\beta \to 0$ as $\beta \to \infty$. It then follows that, on Ω_s,

$$(-\varphi_t + u_\beta - s)^{(n+1)/n} \le C_n A_{s,k,\beta}^{1/n}(-\psi_{t,k} + u_\beta + 1 + A_{s,k,\beta}) + \epsilon_\beta^{(n+1)/n}.$$

Letting $\beta \to \infty$, we have

$$(-\varphi_t + V_t - s)^{(n+1)/n} \le C_n A_{s,k}^{1/n}(-\psi_{t,k} + V_t + 1 + A_{s,k}),$$

where $A_{s,k} = \int_X \tau_k(-\varphi_t + V_t + s)e^F \omega^n$. Observe that $V_t \le 0$ by definition and, by the α-invariant estimate [Hörmander 1966; Tian 1987], there exists an $\alpha_0(n, \omega, \chi)$ such that

$$\int_{\Omega_s} \exp\left(\alpha_0 \frac{(-\varphi_t + V_t - s)^{(n+1)/n}}{A_{s,k}^{1/n}}\right)\omega^n \le \int_{\Omega_s} \exp(\alpha_0 C_n(-\psi_{t,k} + 1 + A_{s,k}))\omega^n \le Ce^{CA_{s,k}}. \tag{2-6}$$

Letting $k \to \infty$, we obtain

$$\int_{\Omega_s} \exp\left(\alpha_0 \frac{(-\varphi_t + V_t - s)^{(n+1)/n}}{A_s^{1/n}}\right)\omega^n \le Ce^{CA_s}.$$

Theorem 1(a) is proved by noting that $A_s \le E_t$ for any $s > 0$.

Once Theorem 1(a) has been proved, part (b) can be proved by following closely the arguments in [Guo et al. 2023a].

Fix $p > n$, and define $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ by $\eta(x) = (\log(1+x))^p$. Note that η is a strictly increasing function with $\eta(0) = 0$, and let η^{-1} be its inverse function. Write

$$v := \frac{\alpha_0}{2}\left(\frac{-\varphi_t + V_t - s}{A_s^{1/(n+1)}}\right)^{(n+1)/n}. \tag{2-7}$$

Then by the generalized Young's inequality with respect to η, for any $z \in \Omega_s$,

$$\begin{aligned} v(z)^p e^{F(z)} &\le \int_0^{\exp(F(z))} \eta(x)\,dx + \int_0^{v(z)^p} \eta^{-1}(y)\,dy \le \exp(F(z))(1+|F(z)|)^p + \int_0^{v(z)^p} (e^{y^{1/p}} - 1)\,dy \\ &\le \exp(F(z))(1+|F(z)|)^p + p\int_0^{v(z)} e^y y^{p-1}\,dy \le \exp(F(z))(1+|F(z)|)^p + C(p)\exp(2v(z)). \end{aligned}$$

We integrate both sides in the inequality above over $z \in \Omega_s$ and get by Theorem 1(a) that

$$\int_{\Omega_s} v(z)^p e^{F(z)}\omega^n \le \int_{\Omega_s} e^F(1+|F(z)|)^p\omega^n + \int_{\Omega_s} e^{2v(z)}\omega^n \le \|e^F\|_{L^1(\log L)^p} + C + Ce^{CE_t},$$

where the constant $C > 0$ depends only on n, ω_X and χ. In view of the definition of v, this implies

$$\int_{\Omega_s} (-\varphi_t + V_t - s)^{(n+1)p/n} e^{F(z)}\omega^n \le 2^p \alpha_0^{-p} A_s^{p/n}(\|e^F\|_{L^1(\log L)^p} + C + Ce^{CE_t}). \tag{2-8}$$

From the definition of A_s, it follows from Hölder's inequality that

$$\begin{aligned} A_s = \int_{\Omega_s} (-\varphi_t + V_t - s) e^F \omega^n &\le \left(\int_{\Omega_s} (-\varphi_t + V_t - s)^{(n+1)p/n} e^F \omega^n \right)^{n/((n+1)p)} \cdot \left(\int_{\Omega_s} e^F \omega^n \right)^{1/q} \\ &\le A_s^{1/(n+1)} (2^p \alpha_0^{-p} (\|e^F\|_{L^1(\log L)^p} + C + Ce^{CE_t}))^{n/((n+1)p)} \cdot \left(\int_{\Omega_s} e^F \omega^n \right)^{1/q}, \end{aligned}$$

where $q > 1$ satisfies $n/(p(n+1)) + 1/q = 1$, i.e., $q = p(n+1)/(p(n+1) - n)$. The inequality above yields

$$A_s \le (2^p \alpha_0^{-p} (\|e^F\|_{L^1(\log L)^p} + C + Ce^{CE_t}))^{1/p} \cdot \left(\int_{\Omega_s} e^F \omega^n \right)^{(1+n)/(qn)}. \tag{2-9}$$

Observe that the exponent of the integral on the right-hand of (2-9) satisfies

$$\frac{1+n}{qn} = \frac{pn + p - n}{pn} = 1 + \delta_0 > 1$$

for $\delta_0 := (p-n)/(pn) > 0$. For convenience of notation, set

$$B_0 := (2^p \alpha_0^{-p} (\|e^F\|_{L^1(\log L)^p} + C + Ce^{CE_t}))^{1/p}. \tag{2-10}$$

From (2-9) we then get

$$A_s \le B_0 \left(\int_{\Omega_s} e^F \omega^n \right)^{1+\delta_0}. \tag{2-11}$$

If we define $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\phi(s) := \int_{\Omega_s} e^F \omega^n$, then (2-11) and the definition of A_s implies

$$r\phi(s+r) \le B_0 \phi(s)^{1+\delta_0} \quad \text{for all } r \in [0, 1] \text{ and } s \ge 0. \tag{2-12}$$

Since ϕ is clearly nonincreasing and continuous, a De Giorgi-type iteration argument shows that there is some S_∞ such that $\phi(s) = 0$ for any $s \ge S_\infty$. This finishes the proof of the L^∞ estimate of $\varphi_t - V_t$, combining with a bound on E_t by $\|e^F\|_{L^1(\log L)^1}$ which follows from Jensen's inequality; see Lemma 6 in [Guo et al. 2023a]. $\square$

Finally, we note the recent advances in the theory of envelopes in [Guedj and Lu 2021; 2023], which can provide an approach to L^∞ estimates for Monge–Ampère equations on Hermitian manifolds.

3. Complex Hessian equations

We explain in this section how the proof of Theorem 1 can be modified to give a similar result for a degenerate family of complex Hessian equations. With the same notations as above, we consider the σ_k-equations

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^k \wedge \omega^{n-k} = c_t e^F \omega^n, \quad \sup_X \varphi_t = 0. \tag{3-1}$$

Define the envelope corresponding to the Γ_k-cone

$$\widetilde{V}_{t,k} = \sup\{ v \mid v \in \mathrm{SH}_k(X, \omega, \hat{\omega}_t) \cap C^2,\ v \le 0 \},$$

where $v \in \mathrm{SH}_k(X, \omega, \hat{\omega}_t) \cap C^2$ indicates that the vector of eigenvalues of the linear transformation $\omega^{-1} \cdot (\hat{\omega}_t + i\partial\bar{\partial} v)$ lies in the Γ_k-cone, which is the convex cone in $\mathbb{R}^n$ given by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_1(\lambda) > 0, \dots, \sigma_k(\lambda) > 0\},$$

where $\sigma_j(\lambda)$ denotes the j-th elementary symmetric polynomial of $\lambda \in \mathbb{R}^n$.

Let

$$E_t(\varphi_t) = \int_X (-\varphi_t + \widetilde{V}_{t,k}) e^{nF/k} \omega^n$$

be the entropy associated to (3-1) as in [Guo et al. 2023a], and let $\bar{E}_t$ be an upper bound of $E_t(\varphi_t)$. Then the following L^∞ estimate holds for the solution φ_t to (3-1).

Theorem 3. *Let φ_t be the solution to (3-1). There exists a constant depending on*

$$\bar{E}_t, \quad \|e^{(n/k)F}\|_{L^1(\log L)^p}, \quad \frac{c_t}{[\hat{\omega}_t^k][\omega^{n-k}]} \quad \text{and} \quad p > n$$

such that

$$0 \le -\varphi_t + \widetilde{V}_{t,k} \le C.$$

This theorem can be derived using a similar argument as in Section 2 with suitable modifications for σ_k equations — see [Guo et al. 2023a] — so we omit the details. The only novel ingredient is the smooth approximation of $\widetilde{V}_{t,k}$ as in Lemma 2. One can adapt the method in [Berman 2019] to derive this required approximation. For the convenience of the reader, we present a sketch of the proof.

Lemma 4. *Fix $t \in (0, 1]$. There exists a sequence of smooth functions $u_\beta \in \mathrm{SH}_k(X, \omega, \hat{\omega}_t)$ converging uniformly to $\widetilde{V}_{t,k}$ as $\beta \to \infty$.*

Proof. Let $u_\beta \in \mathrm{SH}_k(X, \omega, \hat{\omega}_t)$ be the solution to the σ_k-equations

$$(\hat{\omega}_t + i\partial\bar{\partial} u_\beta)^k \wedge \omega^{n-k} = c_t e^{\beta u_\beta} \omega^n, \tag{3-2}$$

which admits a unique smooth solution by [Dinew and Kołodziej 2017]. We claim that there is a constant $C_t > 0$ such that

$$\sup_X |u_\beta - \widetilde{V}_{t,k}| \le \frac{C_t \log \beta}{\beta},$$

from which the lemma follows.

By the maximum principle, at a maximum point of u_β we have $i\partial\bar{\partial} u_\beta \le 0$, so

$$\beta u_\beta \le \log \frac{\hat{\omega}_t^k \wedge \omega^{n-k}}{c_t \omega^n} \le C_t,$$

that is, $u_\beta - C_t/\beta \le 0$. By the definition of $\widetilde{V}_{t,k}$, it follows that

$$u_\beta - \frac{C_t}{\beta} \le \widetilde{V}_{t,k}. \tag{3-3}$$

On the other hand, we fix a smooth $u \le 0$ such that $\hat{\omega}_t + i\partial\bar{\partial}u > 0$. Such a u exists because $[\hat{\omega}_t]$ is a Kähler class by assumption. For any $v \in \mathrm{SH}_k(X, \omega, \hat{\omega}_t) \cap C^2$ with $v \le 0$, we consider the barrier function

$$\tilde{u} = \frac{1}{\beta}u + \left(1 - \frac{1}{\beta}\right)v - \frac{C_t' \log\beta}{\beta},$$

where $C_t' > 0$ is a large constant to be determined. By direct calculation, we have

$$(\hat{\omega}_t + i\partial\bar{\partial}\tilde{u})^k \wedge \omega^{n-k} \ge \frac{1}{\beta^k}(\hat{\omega}_t + i\partial\bar{\partial}u)^k \wedge \omega^{n-k} \ge e^{\beta\tilde{u}}\omega^n,$$

where the last inequality holds if we choose C_t' large enough such that

$$e^{-C_t'\log\beta} \le \frac{1}{\beta^k} \min_X \frac{(\hat{\omega}_t + i\partial\bar{\partial}u)^k \wedge \omega^{n-k}}{\omega^n}.$$

Therefore, we get

$$(\hat{\omega}_t + i\partial\bar{\partial}\tilde{u})^k \wedge \omega^{n-k} \ge e^{\beta(\tilde{u}-u_\beta)}(\hat{\omega}_t + i\partial\bar{\partial}u_\beta)^k \wedge \omega^{n-k}.$$

At the maximum point of $\tilde{u} - u_\beta$, we have $(\hat{\omega}_t + i\partial\bar{\partial}\tilde{u})^k \wedge \omega^{n-k} \le (\hat{\omega}_t + i\partial\bar{\partial}u_\beta)^k \wedge \omega^{n-k}$. This shows that $\tilde{u} - u_\beta \le 0$ on X. Taking the supremum over all such v in $\tilde{u}$, it follows that

$$\left(1 - \frac{1}{\beta}\right)\widetilde{V}_{t,k} \le u_\beta + \frac{C_t \log\beta}{\beta}.$$

The lemma follows from this and (3-3). $\square$

References

[Berman 2019] R. J. Berman, "From Monge–Ampère equations to envelopes and geodesic rays in the zero temperature limit", *Math. Z.* **291**:1-2 (2019), 365–394. MR Zbl

[Boucksom et al. 2010] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi, "Monge–Ampère equations in big cohomology classes", *Acta Math.* **205**:2 (2010), 199–262. MR Zbl

[Chen and Cheng 2021] X. Chen and J. Cheng, "On the constant scalar curvature Kähler metrics, I: A priori estimates", *J. Amer. Math. Soc.* **34**:4 (2021), 909–936. MR Zbl

[Chu et al. 2018] J. Chu, V. Tosatti, and B. Weinkove, "$C^{1,1}$ regularity for degenerate complex Monge–Ampère equations and geodesic rays", *Comm. Partial Differential Equations* **43**:2 (2018), 292–312. MR Zbl

[De Giorgi 1961] E. De Giorgi, *Frontiere orientate di misura minima*, Editrice Tecnico Sci., Pisa, 1961. MR Zbl

[Demailly et al. 2014] J.-P. Demailly, S. Dinew, V. Guedj, H. H. Pham, S. Kołodziej, and A. Zeriahi, "Hölder continuous solutions to Monge–Ampère equations", *J. Eur. Math. Soc.* **16**:4 (2014), 619–647. MR Zbl

[Dinew and Kołodziej 2014] S. Dinew and S. Kołodziej, "A priori estimates for complex Hessian equations", *Anal. PDE* **7**:1 (2014), 227–244. MR Zbl

[Dinew and Kołodziej 2017] S. Dinew and S. Kołodziej, "Liouville and Calabi–Yau type theorems for complex Hessian equations", *Amer. J. Math.* **139**:2 (2017), 403–415. MR Zbl

[Fu et al. 2020] X. Fu, B. Guo, and J. Song, "Geometric estimates for complex Monge–Ampère equations", *J. Reine Angew. Math.* **765** (2020), 69–99. MR Zbl

[Guedj and Lu 2021] V. Guedj and C. H. Lu, "Quasi-plurisubharmonic envelopes, I: Uniform estimates on Kähler manifolds", preprint, 2021. arXiv 2106.04273

[Guedj and Lu 2023] V. Guedj and C. H. Lu, "Quasi-plurisubharmonic envelopes, III: Solving Monge–Ampère equations on Hermitian manifolds", *J. Reine Angew. Math.* **800** (2023), 259–298. MR Zbl

[Guo et al. 2023a] B. Guo, D. H. Phong, and F. Tong, "On L^∞ estimates for complex Monge–Ampère equations", *Ann. of Math.* (2) **198**:1 (2023), 393–418. MR Zbl

[Guo et al. 2023b] B. Guo, D. H. Phong, and F. Tong, "Stability estimates for the complex Monge–Ampère and Hessian equations", *Calc. Var. Partial Differential Equations* **62**:1 (2023), art. id. 7. MR Zbl

[Hörmander 1966] L. Hörmander, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, NJ, 1966. MR Zbl

[Kołodziej 1998] S. Kołodziej, "The complex Monge–Ampère equation", *Acta Math.* **180**:1 (1998), 69–117. MR Zbl

[Tian 1987] G. Tian, "On Kähler–Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$", *Invent. Math.* **89**:2 (1987), 225–246. MR Zbl

[Yau 1978] S. T. Yau, "On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I", *Comm. Pure Appl. Math.* **31**:3 (1978), 339–411. MR Zbl

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