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AN IMPROVED REGULARITY CRITERION AND ABSENCE OF SPLASH-LIKE SINGULARITIES FOR G-SQG PATCHES

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We prove that splash-like singularities cannot occur for sufficiently regular patch solutions to the generalized surface quasi-geostrophic equation on the plane or half-plane with parameter $\alpha \le \frac{1}{4}$. This includes potential touches of more than two patch boundary segments in the same location, an eventuality that has not been excluded previously and presents nontrivial complications (in fact, if we do a priori exclude it, then our results extend to all $\alpha \in (0, 1)$). As a corollary, we obtain an improved global regularity criterion for H^3 patch solutions when $\alpha \le \frac{1}{4}$, namely that finite time singularities cannot occur while the H^3 norms of patch boundaries remain bounded.

1. Introduction

The *g-SQG* (*generalized surface quasi-geostrophic*) *equation* is the active scalar PDE

$$\partial_t \omega + u \cdot \nabla \omega = 0, \tag{1-1}$$

where the scalar $\omega : \mathbb{R}^2 \times (0, \infty) \to \mathbb{R}$ is advected by the velocity field

$$u := \nabla^\perp (-\Delta)^{-1+\alpha} \omega. \tag{1-2}$$

Here $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$ and $\alpha \in (0, 1)$ is a given parameter. Note that (1-1) is the vorticity form of the (incompressible) two-dimensional Euler equation when $\alpha = 0$, which models the motion of ideal fluids, with u the fluid velocity and $\omega := \nabla^\perp \cdot u$ its vorticity. When $\alpha = \frac{1}{2}$, it is the SQG equation, which is used in atmospheric science models [Pedlosky 1979] and was first analyzed rigorously by Constantin, Majda, and Tabak [Constantin et al. 1994]. The g-SQG equation with $\alpha \in (0, 1)$ is its generalization and has also been studied in both geophysical and mathematical literature, including in [Chae et al. 2012; Constantin et al. 2008; Córdoba et al. 2005; Gancedo 2008; Kiselev and Luo 2023; Kiselev et al. 2016; 2017; Pierrehumbert et al. 1994; Smith et al. 2002].

Global regularity for (smooth or bounded) solutions has been known in the Euler case $\alpha = 0$ since the works of Hölder [1933], Wolibner [1933], and Yudovich [1963], but it is still an open problem in the g-SQG case with any $\alpha \in (0, 1)$. In this work we consider so-called *patch solutions* to (1-1), that is, weak solutions that are linear combinations of characteristic functions of some time-dependent sets $\Omega_n(t) \subseteq \mathbb{R}^2$ (often only a single such set/patch is considered but the extension to multiple sets is typically

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straightforward). The main question now is that of global well-posedness for these solutions: if the boundary of each initial patch $\Omega_n(0)$ is a simple closed curve of some prescribed regularity (H^k or $C^{k,\gamma}$) and these curves are pairwise disjoint, does this setup persist forever or may it cease existing in finite time? This of course involves not only the required regularity of each $\partial\Omega_n(t)$, but also that they all remain pairwise disjoint simple closed curves.

Chemin [1993] showed the answer to be in the affirmative when $\alpha = 0$, but the question remains open for any $\alpha \in (0, 1)$. Local existence for these models was proved for $\alpha \in \left(0, \frac{1}{2}\right]$ and H^3 patches by Gancedo [2008], who also obtained uniqueness for $\alpha \in \left(0, \frac{1}{2}\right)$ and those solutions that satisfy a related *contour equation* (well-posedness for $\alpha = \frac{1}{2}$ in a special class of patches was earlier proved by Rodrigo [2005]). Local existence was also proved for $\alpha \in \left[\frac{1}{2}, 1\right)$ and H^4 patches by Chae, Constantin, Córdoba, Gancedo, and Wu [Chae et al. 2012]. Kiselev, Yao, and Zlatoš [Kiselev et al. 2017] later proved full local well-posedness for $\alpha \in \left(0, \frac{1}{2}\right)$ and H^3 patches (they also considered the related half-plane case, in which global well-posedness was proved to fail by Kiselev, Yao, Ryzhik, and Zlatoš [Kiselev et al. 2016]). In addition, Córdoba, Córdoba, and Gancedo achieved this for $\alpha = \frac{1}{2}$ and H^3 patches [Córdoba et al. 2018], Gancedo and Patel for $\alpha \in \left(0, \frac{1}{2}\right)$ and H^2 patches as well as for $\alpha \in \left(\frac{1}{2}, 1\right)$ and H^3 patches [Gancedo and Patel 2021], and Gancedo, Nguyen, and Patel for $\alpha = \frac{1}{2}$ and $H^{2+\gamma}$ patches [Gancedo et al. 2022].

The singularity-formation mechanism on the half-plane from [Kiselev et al. 2016], which was motivated by numerical simulations for the three-dimensional Euler equation due to Luo and Hou [2014a; 2014b] and by the related proof of double exponential growth of gradients for smooth solutions to the two-dimensional Euler equation on a bounded domain by Kiselev and Šverák [2014], does not seem to extend to the whole plane case. It is therefore still unknown whether global well-posedness holds on $\mathbb{R}^2$ for any $\alpha \in (0, 1)$. Nevertheless, the local well-posedness result in [Kiselev et al. 2017] does show that, at least for $\alpha \in \left(0, \frac{1}{2}\right)$ and H^3 patches, finite time singularity can only occur if either a patch boundary loses H^3 regularity or a touch happens. The latter might involve two or more patch boundary segments, which might belong to different patches or to a single patch.

The main result of this paper is that, for $\alpha \in \left(0, \frac{1}{4}\right]$, a touch cannot occur without the loss of $C^{1,2\alpha/(1-2\alpha)}$ (and hence also H^3) regularity of a patch boundary at the same time (this is also suggested by numerical simulations of Córdoba, Fontelos, Mancho, and Rodrigo [Córdoba et al. 2005]). If it did occur and the $C^{1,\gamma}$ norm of the patch boundary would stay uniformly bounded for some $\gamma > 0$, the resulting singularity would be called a *splash*. One might think that its existence for the free boundary Euler equation, demonstrated by Castro, Córdoba, Fefferman, Gancedo, and Gómez-Serrano [Castro et al. 2013] and Coutand and Shkoller [2014], would suggest its possibility for g-SQG patches as well. But these two cases are very different: the converging boundary segments are separated by vacuum in the free boundary case, while for (1-1) they are separated by the (incompressible) fluid medium, which must be "squeezed out" of the region between them before a touch can occur.

One might also think that impossibility of general splash singularities was already proved by Gancedo and Strain [2014] for the SQG case $\alpha = \frac{1}{2}$ and smooth patches, who showed that a touch of two patch boundary segments (which we call a *simple splash*) is indeed impossible at any specific location without a loss of boundary smoothness; their argument extends to all $\alpha \in \left(0, \frac{1}{2}\right)$. However, they proved this assuming that no singularity occurs elsewhere, and the result also does not exclude simultaneous touches of three or more boundary segments. Crucially, their proof does not extend to this case either. In it, they place the two segments in a coordinate system in which both are close to horizontal, and use the fact that normal vectors at two points that minimize the vertical distance of the two boundary segments (at any given time) are automatically parallel. This causes important cancellations in the integral evaluating the approach velocity of the two points, which bound this velocity by a multiple of the product of the distance of the two points and the log of this distance. Grönwall's inequality then yields at most double exponential in time approach rate of the two segments.

We can even obtain a simple exponential bound for $C^{2,\gamma}$ patches with $\gamma > 0$ by instead minimizing the distance (rather than vertical distance) of the two boundary segments, in which case the normals at the closest points both lie on the line connecting these points. The resulting computation then bounds the approach velocity by only a multiple of the distance, and it even extends to all $\alpha \in (0, 1)$ with appropriate γ (see Section 2E).

However, when a third boundary segment is present nearby, its normal vector at the point where it intersects the above line need not lie on that line, which significantly compromises the cancellations involved. One then needs to obtain very precise bounds on the resulting errors in this case, which we will achieve by using the uniform $C^{1,2\alpha/(1-2\alpha)}$ bound on the patch boundary to estimate the angle between this normal vector and the line, in terms of the distance of the third segment from the two closest points on the first two segments. When this distance is small, the error will be controlled because the angle must be small; this control worsens when the distance is larger, but then the effect of the third segment on the two points decreases as well. This will yield the needed bound on the approach velocity of the closest points, and this estimate will even extend to the case of arbitrarily many boundary segments folded on top of each other and attempting to create a *complex splash* singularity.

As a result, we will obtain an improved regularity criterion for H^3 patch solutions to (1-1), requiring only a uniform bound on the $C^{1,2\alpha/(1-2\alpha)}$ norm of the patch boundaries. Nevertheless, this approach only works when $\alpha \in \left(0, \frac{1}{4}\right]$, and the obtained estimates are insufficient for larger α (specifically, Lemma 2.5). The reason for this is not just technical, and simply assuming higher boundary regularity will not suffice to overcome the new complications involved. We believe that a different (dynamical) approach will be needed for $\alpha > \frac{1}{4}$ (if the result extends to this range at all), which likely makes it a very difficult problem.

Let us now state rigorously the definition of patch solutions to (1-1) from [Kiselev et al. 2017] (which even allows patches to be nested) and our main result. Below we let $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$.

Definition 1.1. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set whose boundary $\partial\Omega$ is a simple closed C^1 curve with arc-length $|\partial\Omega|$. We call a *constant-speed parametrization* of $\partial\Omega$ any counterclockwise parametrization $z\colon \mathbb{T} \to \mathbb{R}^2$ of $\partial\Omega$ with $|z'| \equiv |\partial\Omega|/(2\pi)$ on $\mathbb{T}$ (these are all translations of each other), and we define $\|\Omega\|_{C^{k,\gamma}} := \|z\|_{C^{k,\gamma}(\mathbb{T})}$ and $\|\Omega\|_{H^k} := \|z\|_{H^k}$ for $(k, \gamma) \in \mathbb{N}_0 \times [0, 1]$.

Next we note that, when $\alpha \in \left(0, \frac{1}{2}\right)$, the velocity u from (1-2) satisfies the explicit formula

$$u(x,t) := c_\alpha \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \omega(y,t)\, dy \tag{1-3}$$

for bounded ω, with $v^\perp := (-v_2, v_1)$ and $c_\alpha > 0$ an appropriate constant; see Section 2E for the necessary adjustments when $\alpha \in \left[\frac{1}{2}, 1\right)$. For any $\Gamma \subseteq \mathbb{R}^2$, vector field $v\colon \Gamma \to \mathbb{R}^2$, and $h \in \mathbb{R}$, we let the set to which Γ is advected by v in time h be

$$X_v^h[\Gamma] := \{x + hv(x) \mid x \in \Gamma\}.$$

Definition 1.2. Let $\theta_1, \dots, \theta_N \in \mathbb{R} \setminus \{0\}$, and for each $t \in [0, T)$, let $\Omega_1(t), \dots, \Omega_N(t) \subseteq \mathbb{R}^2$ be bounded open sets whose boundaries are pairwise disjoint simple closed curves such that each $\partial\Omega_n(t)$ is also continuous in $t \in [0, T)$ with respect to Hausdorff distance d_H of sets. Define $\partial\Omega(t) := \bigcup_{n=1}^N \partial\Omega_n(t)$ and $\|\Omega(t)\|_Y := \sum_{n=1}^N \|\Omega_n(t)\|_Y$ for $Y \in \{C^{k,\gamma}, H^k\}$, and let

$$\omega(\,\cdot\,, t) := \sum_{n=1}^N \theta_n \chi_{\Omega_n(t)}. \tag{1-4}$$

If for each $t \in (0, T)$ we have

$$\lim_{h\to 0} \frac{d_H(\partial\Omega(t+h), X^h_{u(\,\cdot\,,t)}[\partial\Omega(t)])}{h} = 0, \tag{1-5}$$

with u from (1-3), then ω is a *a patch solution* to (1-1)–(1-2) on the time interval $[0, T)$. If we also have $\sup_{t\in[0,T']} \|\Omega(t)\|_Y < \infty$ for some $Y \in \{C^{k,\gamma}, H^k\}$ and each $T' \in (0, T)$, then ω is a *Y patch solution* to (1-1)–(1-2) on $[0, T)$.

While (1-5) is stated for each single time t (akin to the definition of strong or classical solutions to a PDE), it agrees with the usual flow-map based definition of solutions to the two-dimensional Euler equation; see the remarks after Definition 1.2 in [Kiselev et al. 2017]. Since u is only Hölder continuous at the patch boundaries when $\alpha > 0$ (and hence the flow map may not be unique), this definition is more appropriate in the g-SQG case.

Theorem 1.5 in [Kiselev et al. 2017] shows that for any $\theta_1, \dots, \theta_N \in \mathbb{R} \setminus \{0\}$ and any bounded open sets $\Omega_1(0), \dots, \Omega_N(0) \subseteq \mathbb{R}^2$ whose boundaries are pairwise disjoint simple closed H^3 curves, there is a time $T \in (0, \infty]$ such that a unique H^3 patch solution $\omega = \sum_{n=1}^N \theta_n \chi_{\Omega_n(\,\cdot\,)}$ to (1-1)–(1-2) exists on $[0, T)$. And if the maximal such T is finite, then either $\sup_{t\in[0,T)} \|\Omega(t)\|_{H^3} = \infty$ or

$$\sup_{t\in[0,T)} \sup_{\substack{(n,\xi),(j,\eta)\in\{1,\dots,N\}\times\mathbb{T} \\ (n,\xi)\neq(j,\eta)}} \frac{|n-j| + |\xi-\eta|}{|z_n(\xi, t) - z_j(\eta, t)|} = \infty, \tag{1-6}$$

where $z_n(\,\cdot\,, t)$ is a constant-speed parametrization of $\partial\Omega_n(t)$ and $|\xi-\eta|$ is distance on $\mathbb{T}$. Note that if (1-6) holds with $n = j$ — which means that the *arc-chord ratio* for some $\Omega_n(\,\cdot\,)$ becomes unbounded as $t \to T$ — then this must be realized by a touch of "distinct" segments (which we call *folds*) of $\partial\Omega_n(\,\cdot\,)$ whenever $\sup_{t\in[0,T)} \|\Omega(t)\|_{C^{1,\gamma}} < \infty$ for some $\gamma > 0$. Indeed, since $|\partial_\xi z_n(\xi, t)|$ is then bounded below by a positive constant uniformly in (n, ξ, t) (see (2-4)), it follows that the fraction in (1-6) is uniformly bounded above when $n = j$ and $|\xi - \eta|$ is small enough. Therefore (1-6) is the correct definition of a *splash-like singularity* at time T (i.e., a touch of either boundaries of distinct patches or folds of the same patch boundary, including both simple and complex splashes) when $\sup_{t\in[0,T)} \|\Omega(t)\|_{C^{1,\gamma}} < \infty$ for some $\gamma > 0$.

The following theorem is now our main result.

Theorem 1.3. *If* $\alpha \in \left(0, \frac{1}{4}\right]$ *and a* $C^{1,2\alpha/(1-2\alpha)}$ *patch solution to* (1-1)–(1-2) *on the time interval* $[0, T)$ *with* $T < \infty$ *satisfies* $\sup_{t\in[0,T)} \|\Omega(t)\|_{C^{1,2\alpha/(1-2\alpha)}} < \infty$, *then* (1-6) *fails* (*so no splash-like singularity can occur*). *In particular*, *if the maximal time* T *of existence of an* H^3 *patch solution from* [Kiselev et al. 2017, Theorem 1.5] *is finite and* $\alpha \in \left(0, \frac{1}{4}\right]$, *then* $\sup_{t\in[0,T)} \|\Omega(t)\|_{H^3} = \infty$.

Remark. (1) Our proof shows that the left-hand side of (1-6) with $\sup_{t\in[0,T)}$ removed can grow at most exponentially in time (up to time T) if $\sup_{t\in[0,T)} \|\Omega(t)\|_{C^{1,2\alpha/(1-2\alpha)}} < \infty$. Hence boundaries of distinct patches, as well as folds of the same patch boundary, can only approach each other exponentially quickly in this case.

(2) While we do not know whether this result extends to some $\alpha > \frac{1}{4}$, in Section 2E we provide an extension to all $\alpha \in (0, 1)$ when one a priori requires that only simple splashes can occur (i.e., no more than two segments of $\partial\Omega$ are allowed to touch in the same location) and $\sup_{t\in[0,T)} \|\Omega(t)\|_{C^{k,\gamma}} < \infty$ holds for $k = 1$ and some $\gamma \geq 2\alpha$, when $\alpha \in \left(0, \frac{1}{2}\right]$, or for $k = 2$ and some $\gamma \geq 2\alpha - 1$, when $\alpha \in \left[\frac{1}{2}, 1\right)$. The obtained bound on the approach rate of two patches/folds is now double exponential when γ is equal to the minimal value above (2α or $2\alpha - 1$), and exponential otherwise. We note that when the potential simple splash is assumed to have a predetermined location and development of singularities elsewhere is a priori excluded, then this was also proved for $\alpha = \frac{1}{2}$ in [Gancedo and Strain 2014] (for smooth patches and with a double exponential bound on the approach rate), and for all $\alpha \in (0, 1)$ in [Kiselev and Luo 2023] (this work was done contemporaneously with and independently of ours).

Finally, here is an extension to the half-plane; see Section 3 for the relevant adjustments.

Theorem 1.4. *Theorem 1.3 extends to patch solutions on the half-plane*, *with the second claim involving* H^3 *patch solutions from* [Kiselev et al. 2017, Theorem 1.4] *and* $\alpha \in \left(0, \frac{1}{24}\right)$, *or* H^2 *patch solutions from* [Gancedo and Patel 2021, Theorem 1.1] *and* $\alpha \in \left(0, \frac{1}{6}\right)$.

It was proved in [Kiselev et al. 2016] that, for any $\alpha \in \left(0, \frac{1}{24}\right)$, there are H^3 patch solutions on the half-plane that become singular in finite time. For $\alpha \in \left(0, \frac{1}{6}\right)$ and H^2 patch solutions this was proved in [Gancedo and Patel 2021]. Theorem 1.4 shows that this cannot happen only via a splash-like singularity and always involves blow-up of their H^3 and H^2 norms, respectively.

2. Proof of Theorem 1.3

2A. ***The single patch case.*** For the sake of simplicity of notation, let us first consider the case of a single patch on which $\omega \equiv 1$; that is, $\omega(\cdot, t) = \chi_{\Omega(t)}$. Then (1-3) becomes

$$u(x, t) := \int_{\Omega(t)} \frac{(x - y)^{\perp}}{|x - y|^{2+2\alpha}}\, dy \tag{2-1}$$

after rescaling (1-1) in time by c_α (which we do in order to remove the constant).

We will not assume $\alpha \leq \frac{1}{4}$ until it is needed, so that it is clear where this hypothesis enters into our argument. We will therefore consider a $C^{1,\gamma}$ patch solution with any $\gamma \in (0, 1]$ below. If now $z(\cdot, t)$ is any constant-speed parametrization of $\partial\Omega(t)$ for $t \in [0, T)$, we assume that

$$M := \sup_{t\in[0,T)} \|z(\cdot\,, t)\|_{C^{1,\gamma}} < \infty. \tag{2-2}$$

We now want to show that this implies

$$\sup_{t\in[0,T)} \sup_{\substack{\xi,\eta\in\mathbb{T}\\ \xi\neq\eta}} \frac{|\xi-\eta|}{|z(\xi,t)-z(\eta,t)|} < \infty. \tag{2-3}$$

Since a C^1 patch solution is also a weak solution to (1-1)–(1-2) with $|\Omega(t)|$ being conserved (see Remark 3 after Definition 1.2 in [Kiselev et al. 2017]), the isoperimetric inequality shows that

$$M' := \inf_{(\xi,t)\in\mathbb{T}\times[0,T)} |\partial_\xi z(\xi,t)| > 0. \tag{2-4}$$

Now for any $t \in [0, T)$ and $\xi, \eta \in \mathbb{T}$, there are $\xi_1, \xi_2 \in \mathbb{T}$ between ξ and η such that

$$|z(\xi,t)-z(\eta,t)| = |\xi-\eta|\,|(\partial_\xi z_1(\xi_1,t), \partial_\xi z_2(\xi_2,t))| \geq |\xi-\eta|(|\partial_\xi z(\xi,t)| - 2M|\xi-\eta|^\gamma).$$

Hence if we let $\delta := (M'/4M)^{1/\gamma}$, then

$$|z(\xi,t)-z(\eta,t)| \geq \tfrac{1}{2}M'|\xi-\eta| \tag{2-5}$$

whenever $|\xi-\eta| \leq \delta$. To conclude (2-3), it now suffices to show

$$\inf_{t\in[0,T)} \min_{\substack{\xi,\eta\in\mathbb{T}\\ |\xi-\eta|\geq\delta}} |z(\xi,t)-z(\eta,t)| > 0. \tag{2-6}$$

We therefore let

$$m(t) := \min_{\substack{\xi,\eta\in\mathbb{T}\\ |\xi-\eta|\geq\delta}} |z(\xi,t)-z(\eta,t)| \geq 0, \tag{2-7}$$

and let $\xi_t, \eta_t \in \mathbb{T}$ be such that $|z(\xi_t,t)-z(\eta_t,t)| = m(t)$. If (2-6) fails, then clearly for all $t < T$ close enough to T we have $m(t) < \frac{1}{2}M'\delta$, which shows that $|\xi_t - \eta_t| > \delta$ for these t because (2-5) holds. It suffices to consider only such t. Then, following an argument in [Constantin and Escher 1998], one can easily see that $m(t)$ is locally Lipschitz (and so differentiable at almost all such t) and we have

$$m'(t) = \frac{z(\xi_t,t)-z(\eta_t,t)}{m(t)} \cdot (u(z(\xi_t,t),t) - u(z(\eta_t,t),t)) \tag{2-8}$$

for almost every such t. Hence Grönwall's inequality shows that it suffices to prove

$$-(u(z(\xi_t,t),t) - u(z(\eta_t,t),t)) \cdot n_t \leq Cm(t) \tag{2-9}$$

for some t-independent $C < \infty$ and all t such that $m(t) \in \big(0, \frac{1}{2}M'\delta\big)$, where $n_t := (z(\xi_t,t)-z(\eta_t,t))/m(t)$ is the unit vector in the direction $z(\xi_t,t)-z(\eta_t,t)$. Of course, the definition of ξ_t, η_t shows that n_t is also normal to $\partial\Omega(t)$ at both $z(\xi_t,t)$ and $z(\eta_t,t)$.

Since (2-9) only involves quantities at a single time, we will now assume that t is close to T and drop the dependence of Ω, z, u, and m on t from our notation. The above also shows that after a translation and rotation we can assume:

(1) $z(\eta_t) = (0, 0)$ and $z(\xi_t) = (0, m)$, with $m \in \left(0, \frac{1}{2}M'\delta\right)$.

(2) $\partial_\xi z(\xi_t), \partial_\xi z(\eta_t) \perp (0, 1)$.

We will do so, and then (2-9) becomes just

$$u_2(0, 0) - u_2(0, m) \leq Cm. \tag{2-10}$$

We will prove this in the next three subsections. We note that all constants below may depend on α, γ, M, M' (recall that δ also depends on these), but will be independent of m and t.

2B. ***Some geometric lemmas.*** We first state some geometric lemmas that will be used throughout. The first of these is a trivial consequence of $C^{1,\gamma}$-regularity of $\partial\Omega$, which says that near any $z(\xi)$, the curve z is the graph of some function $f : \mathbb{R} \to \mathbb{R}$ defined with respect to the coordinate system centered at $z(\xi)$ and with the horizontal axis not too far from $\partial_\xi z(\xi)$.

Lemma 2.1. *There are $A \geq 1$ and $R_0 > 0$ such that, for any $\xi \in \mathbb{T}$ and any $v \in \mathbb{S}^1$ with $|\partial_\xi z(\xi) \cdot v| \geq \frac{1}{2}|\partial_\xi z(\xi)|$, there is $f : [-R_0, R_0] \to \mathbb{R}$ with $\|f\|_{C^{1,\gamma}} \leq A$ such that*

$$\{z(\xi) + hv + f(h)v^\perp \mid h \in [-R_1, R_1]\} = z([\xi - \xi_1, \xi + \xi_2])$$

for each $R_1 \in [0, R_0]$ and some $\xi_1, \xi_2 \in [R_1/M, 3R_1/M']$. Then

$$f(0) = 0 \quad \textit{and} \quad f'(0) = \frac{\partial_\xi z(\xi) \cdot v^\perp}{\partial_\xi z(\xi) \cdot v}.$$

The next lemma shows that when two folds of $\partial\Omega$ are close to each other, the angles between their tangent lines are controlled by their distance.

Lemma 2.2. *There are $B, R > 0$ such that, for any $\xi, \eta \in \mathbb{T}$ with $|z(\xi) - z(\eta)| \leq R$ we have*

$$|\tan\theta| \leq B|z(\xi) - z(\eta)|^{\gamma/(1+\gamma)}, \tag{2-11}$$

where θ is the angle between $\partial_\xi z(\xi)$ and $\partial_\xi z(\eta)$.

Proof. Let A, R_0 be from Lemma 2.1. First note that it suffices to prove

$$|\sin\theta| \leq B|z(\xi) - z(\eta)|^{\gamma/(1+\gamma)} \tag{2-12}$$

instead of (2-11). Indeed, we then only need to replace R by $\min\{R, (2B)^{-(1+\gamma)/\gamma}\}$, which yields $|\cos\theta| \geq \frac{1}{2}$, and then double B.

Take $C := 9A$, and let $R := \min\left\{\frac{1}{2}C^{-(2+2\gamma)/\gamma}, (R_0/(3C))^2\right\}$ and $B := 3C^2$. Without loss assume that $z(\eta) = 0$ and $\partial_\xi z(\eta)/|\partial_\xi z(\eta)| = (1, 0)$, and then let $r := |z(\xi)| \leq R$ and $r' := Cr^{1/(1+\gamma)} \leq CR^{1/2} \leq \frac{1}{3}R_0$. Then Lemma 2.1 with $v := (1, 0)$ shows that z near the origin is a curve connecting the vertical sides of the rectangle $Q := [-3r', 3r'] \times [-C^3 r, C^3 r]$ because

$$A(3r')^\gamma(3r') \leq 9AC^2 r \leq C^3 r$$

(note that the definition of R shows that $C^3 r < r'$, so the vertical sides are the shorter ones).

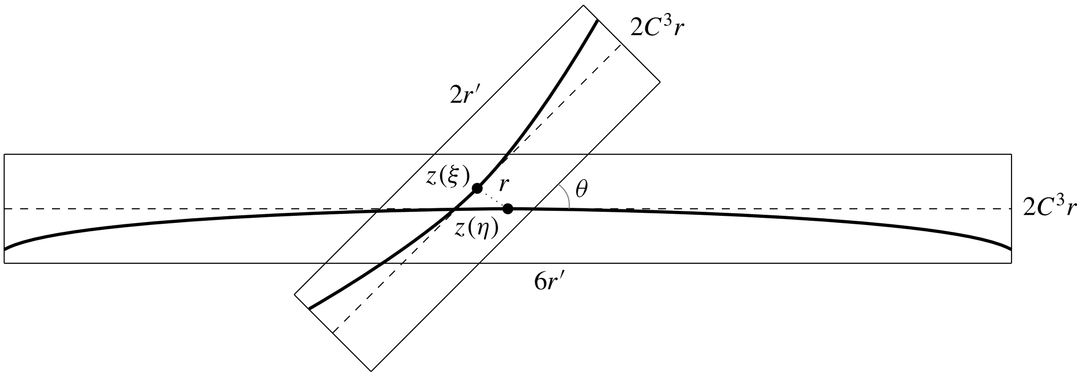

Figure 1. The curve z inside the rectangles Q and Q'.

Apply the same argument with $v := \partial_\xi z(\xi)/|\partial_\xi z(\xi)|$ and the rectangle Q' centered at $z(\xi)$ whose longer axis connects the points $z(\xi) \pm vr'$ and whose shorter sides have again length $2C^3 r$. It shows that z near $z(\xi)$ is a curve connecting the shorter sides of Q'. If (2-12) is violated, then one of these sides lies fully in $(-3r', 3r') \times (C^3 r, \infty)$ and the other in $(-3r', 3r') \times (-\infty, -C^3 r)$ (see Figure 1) because $r + r' + C^3 r < 3r'$ and

$$r' \sin\theta > BCr \geq 3C^3 r > 2C^3 r + r.$$

But this means that the two curves must intersect, a contradiction with our assumption that no touch has occurred before time T. □

We can now combine Lemmas 2.1 and 2.2 to obtain the following constraint on the geometry of $\partial\Omega$ near the origin.

Lemma 2.3. *In the setting of* (1) *and* (2), *there are* $A, B, R > 0$ *with*

$$B(3R)^{\gamma/(1+\gamma)} \leq \tfrac{1}{2} \quad \textit{and} \quad M(4R)^\gamma \leq (M')^{1+\gamma}$$

such that, for any $\xi \in \mathbb{T}$ *with* $z(\xi) \in [-R, R] \times [-2R, 2R]$, *there is* $f : [-R, R] \to \mathbb{R}$ *with*

$$\|f\|_{C^{1,\gamma}} \leq A \quad \textit{and} \quad |f'(z_1(\xi))| \leq B|z(\xi)|^{\gamma/(1+\gamma)}$$

such that the graph of f *is a segment of the curve* z *around* $z(\xi)$. *In particular*,

$$|f(h) - z_2(\xi) - f'(z_1(\xi))(h - z_1(\xi))| \leq A|h - z_1(\xi)|^{1+\gamma}$$

for all $h \in [-R, R]$. *And if* $|f(h')| > 2R$ *for some* $h' \in [-R, R]$, *then* $|f(h)| > R$ *for all* $h \in [-R, R]$.

Proof. The first statement is an immediate consequence of Lemmas 2.1 and 2.2, with A from Lemma 2.1, B from Lemma 2.2, and R being the minimum of one third of R from Lemma 2.2 and

$$\min\left\{\tfrac{1}{3}(2B)^{-(1+\gamma)/\gamma}, \tfrac{1}{4}(M')^{(1+\gamma)/\gamma} M^{-1/\gamma}\right\}$$

(Lemma 2.1 is applied with $v := (1, 0)$ and $z_2(\xi)$ is added to the obtained f). The second statement is its immediate consequence, while the third holds by $B(3R)^{\gamma/(1+\gamma)} \leq \frac{1}{2}$. □

The third claim shows that any connected component of $\partial\Omega \cap ([-R, R] \times [-2R, 2R])$ that intersects $[-R, R]^2$ is a graph of a function $f : [-R, R] \to [-2R, 2R]$ that satisfies the lemma. Note also that since the arc-length of any such component is at least $2R$ and the arc-length of $\partial\Omega$ is uniformly bounded above because so is $\|\partial\Omega\|_{C^1}$, it follows that the number of such components is bounded above by some constant K. That is, if we assume (2-2), only a finite number of folds of $\partial\Omega$ might potentially create a single touch (splash) at time T; we will show below that this is in fact not possible when $\alpha \le \frac{1}{4}$.

2C. ***Reduction to regions near individual boundary segments.*** Take A, B, R from Lemma 2.3, and K from the above discussion. From (2-1) we see that the left-hand side of (2-10) is the sum of the terms

$$I := \int_{\Omega\cap[-R,R]^2} \left(\frac{y_1}{|y|^{2+2\alpha}} - \frac{y_1}{|y-(0,m)|^{2+2\alpha}} \right) dy$$

and

$$I' := \int_{\Omega\setminus[-R,R]^2} \left(\frac{y_1}{|y|^{2+2\alpha}} - \frac{y_1}{|y-(0,m)|^{2+2\alpha}} \right) dy.$$

To prove (2-10), it clearly suffices to assume that $m \le \frac{1}{2}R$, in which case clearly $|I'| \le Cm$ for some constant C. Hence we only need to show that $I \le Cm$.

Assume that $f_1, \dots, f_k \colon [-R, R] \to [-2R, 2R]$ are distinct functions whose graphs are all the connected components of $\partial\Omega$ from the paragraph after Lemma 2.3 (so $k \le K$), and order them so that $f_1(0) < \cdots < f_k(0)$. Let

$$g_i := \operatorname{sgn}(f_i) \min\{|f_i|, R\},$$

so that

$$I \le \sum_{i=1}^{k+1} \left| \int_{-R}^{R} \int_{g_{i-1}(h)}^{g_i(h)} \left(\frac{h}{(h^2+v^2)^{1+\alpha}} - \frac{h}{(h^2+(v-m)^2)^{1+\alpha}} \right) dv\, dh \right|, \tag{2-13}$$

where $g_0 \equiv -R$ and $g_{k+1} \equiv R$. Since the integrand is odd in h, its integral on any region symmetric with respect to the vertical axis is zero. Since $[-R, R] \times [g_{i-1}(0), g_i(0)]$ is such a region we can replace the integral $\int_{g_{i-1}(h)}^{g_i(h)}$ in (2-13) by the sum of integrals $\int_{g_{i-1}(h)}^{g_{i-1}(0)}$ and $\int_{g_i(0)}^{g_i(h)}$ (with the same integrand). We therefore obtain $|I| \le 2\sum_{i=1}^{k} |I_i|$, where

$$I_i := \int_{-R}^{R} \int_{g_i(0)}^{g_i(h)} \left(\frac{h}{(h^2+v^2)^{1+\alpha}} - \frac{h}{(h^2+(v-m)^2)^{1+\alpha}}\, dv \right) dh.$$

Hence, we are left with showing $|I_i| \le Cm$ for each i. When doing this, we can just assume that $g_i = f_i$ because the error we incur by this involves only integration over $\Omega \setminus [-R, R]^2$ and therefore is no more than Cm (similarly to I').

2D. ***Estimating the individual integrals.*** We thus consider any $f \colon [-R, R] \to [-2R, 2R]$ whose graph is a segment of the curve z passing through a point in $[-R, R]^2$, let

$$J := \int_{-R}^{R} \int_{f(0)}^{f(h)} \left(\frac{h}{(h^2+v^2)^{1+\alpha}} - \frac{h}{(h^2+(v-m)^2)^{1+\alpha}} \right) dv\, dh, \tag{2-14}$$

and need to show that $|J| \le Cm$. We further divide this integral into two pieces:

$$J_1 := \int_{-R}^{R} \int_{f(0)}^{f(0)+hf'(0)} \left(\frac{h}{(h^2+v^2)^{1+\alpha}} - \frac{h}{(h^2+(v-m)^2)^{1+\alpha}} \right) dv\, dh,$$

$$J_2 := \int_{-R}^{R} \int_{f(0)+hf'(0)}^{f(h)} \left(\frac{h}{(h^2+v^2)^{1+\alpha}} - \frac{h}{(h^2+(v-m)^2)^{1+\alpha}} \right) dv\, dh,$$

and estimate J_2 first.

Lemma 2.4. *We have $|J_2| \le Cm$ when $\gamma > 2\alpha$, and $|J_2| \le Cm(1+\ln_- m)$ when $\gamma = 2\alpha$, for some constant C.*

Proof. By Lemma 2.3, we have

$$|J_2| \le \int_{-R}^{R} \int_{f(0)+hf'(0)-A|h|^{1+\gamma}}^{f(0)+hf'(0)+A|h|^{1+\gamma}} \left| \frac{h}{(h^2+v^2)^{1+\alpha}} - \frac{h}{(h^2+(v-m)^2)^{1+\alpha}} \right| dv\, dh.$$

The mean value theorem yields

$$\left| \frac{h}{(h^2+v^2)^{1+\alpha}} - \frac{h}{(h^2+(v-m)^2)^{1+\alpha}} \right| = \frac{(2+2\alpha)|h|(h^2+\bar{v}^2)^{\alpha}|\bar{v}|}{(h^2+v^2)^{1+\alpha}(h^2+(v-m)^2)^{1+\alpha}} m$$

for some $\bar{v} \in [v-m, v]$. Hence

$$\left| \frac{h}{(h^2+v^2)^{1+\alpha}} - \frac{h}{(h^2+(v-m)^2)^{1+\alpha}} \right| \le \frac{3m}{|h|^{2+2\alpha}},$$

and if $V := \max\{|v|, |v-m|)\} \ge |h|$, then we also have

$$\left| \frac{h}{(h^2+v^2)^{1+\alpha}} - \frac{h}{(h^2+(v-m)^2)^{1+\alpha}} \right| \le \frac{3m|h|2^{\alpha}V^{1+2\alpha}}{|h|^{2+2\alpha}V^{2+2\alpha}} \le \frac{6m}{|h|^{1+2\alpha}V}.$$

Since $V \ge \frac{1}{2}m$ and we assume that $m \le \frac{1}{2}R$ (see the start of Section 2C), we obtain

$$|J_2| \le 2\int_0^{m/2} \frac{12}{|h|^{1+2\alpha}} 2A|h|^{1+\gamma}\, dh + 2\int_{m/2}^{R} \frac{6m}{|h|^{2+2\alpha}} 2A|h|^{1+\gamma}\, dh.$$

This is less than Cm if $\gamma > 2\alpha$ and less than $Cm(1+\ln_- m)$ if $\gamma = 2\alpha$ (for some C). □

To estimate J_1, it suffices to assume that $f(0) \notin [0, m]$. Indeed, if $f(0) \in \{0, m\}$, then the graph of f contains either $(0, 0)$ or $(0, m)$, so (1) and (2) above (2-10) imply $f'(0) = 0$ and therefore $J_1 = 0$. And if $f(0) \in (0, m)$, then the definition of m shows that there must be $\eta \in \mathbb{T}$ with $|\eta - \eta_t| < \delta$ such that $z(\eta) = (0, f(0))$. Here (2-5) yields $|\eta - \eta_t| \le 4R/M'$, and hence for all ξ between η and η_t we have

$$|\partial_\xi z(\xi) - \partial_\xi z(\eta_t)| \le M\left(\frac{4R}{M'}\right)^{\gamma} \le M' \le |\partial_\xi z(\eta_t)|$$

by Lemma 2.3. This shows that $\partial_\xi z(\xi) \cdot \partial_\xi z(\eta_t) \ge 0$ for all these ξ, which clearly contradicts

$$(z(\eta) - z(\eta_t)) \cdot \partial_\xi z(\eta_t) = 0.$$

So $f(0) \notin [0, m]$, and we define $a := -f(0) > 0$ when $f(0) < 0$, and $a := f(0) - m > 0$ when $f(0) > m$. In both cases Lemma 2.3 yields

$$|f'(0)| \le Ba^{\gamma/(1+\gamma)} \le BR^{\gamma/(1+\gamma)} \le \tfrac{1}{2}. \tag{2-15}$$

Lemma 2.5. *We have* $|J_1| \le Ca^{\gamma/(1+\gamma)}(a+m)^{-2\alpha}m$ *for some constant* C.

Proof. Note that the definition of m shows that $a \ge m$, so we could replace $a+m$ by a. We will not use this so that this result also applies in Section 3. We will assume $f(0) < 0$ since the proof for the other case is virtually identical. We can then rewrite J_1 as

$$J_1 = \int_{-R}^{R} \int_0^{hf'(0)} \left(\frac{h}{(h^2+(v-a)^2)^{1+\alpha}} - \frac{h}{(h^2+(v-a-m)^2)^{1+\alpha}} \right) dv\, dh.$$

We split the integral into two parts:

$$J_3 := \int_{|h|<a+m} \int_0^{hf'(0)} \left(\frac{h}{(h^2+(v-a)^2)^{1+\alpha}} - \frac{h}{(h^2+(v-a-m)^2)^{1+\alpha}} \right) dv\, dh,$$
$$J_4 := \int_{a+m\le|h|\le R} \int_0^{hf'(0)} \left(\frac{h}{(h^2+(v-a)^2)^{1+\alpha}} - \frac{h}{(h^2+(v-a-m)^2)^{1+\alpha}} \right) dv\, dh.$$

For J_3, note that for any v in the domain of integration, we have

$$v-a-m \le |(a+m)f'(0)| - a - m \le -\tfrac{1}{2}(a+m).$$

This also shows that $v-a \le \frac{1}{2}(m-a)$, so $|v-a| \le |v-a-m|$. The mean value theorem then gives, for some $\bar{v} \in [v-a-m, v-a]$,

$$\left| \frac{h}{(h^2+(v-a)^2)^{1+\alpha}} - \frac{h}{(h^2+(v-a-m)^2)^{1+\alpha}} \right| = \frac{(2+2\alpha)|h|(h^2+\bar{v}^2)^{\alpha}|\bar{v}|}{(h^2+(v-a)^2)^{1+\alpha}(h^2+(v-a-m)^2)^{1+\alpha}} m$$
$$\le \frac{6m}{|h|^{1+2\alpha}(a+m)}$$

because $\max\bigl\{|\bar{v}|, \frac{1}{2}(a+m)\bigr\} \le |v-a-m|$. This and (2-15) yield

$$|J_3| \le \int_{-a-m}^{a+m} \frac{6m}{|h|^{1+2\alpha}(a+m)} |hf'(0)|\, dh \le \frac{12m}{1-2\alpha} \frac{|f'(0)|}{(a+m)^{2\alpha}} \le \frac{12B}{1-2\alpha} a^{\gamma/(1+\gamma)} \frac{m}{(a+m)^{2\alpha}}.$$

As for J_4, the mean value theorem yields

$$\left| \frac{h}{(h^2+(v-a)^2)^{1+\alpha}} - \frac{h}{(h^2+(v-a-m)^2)^{1+\alpha}} \right| \le \frac{3m}{|h|^{2+2\alpha}}.$$

From this and (2-15) we obtain

$$|J_4| \le 2\int_{a+m}^{R} \frac{3m}{|h|^{2+2\alpha}} |hf'(0)|\, dh \le \frac{3m}{\alpha} \frac{|f'(0)|}{(a+m)^{2\alpha}} \le \frac{3B}{\alpha} a^{\gamma/(1+\gamma)} \frac{m}{(a+m)^{2\alpha}}. \qquad \square$$

The last two lemmas, together with the estimate $|I'| \le Cm$ above and $k \le K$, show that (2-10) holds when $\gamma > 2\alpha$ and $\gamma/(1+\gamma) \ge 2\alpha$. Since $\gamma = 2\alpha/(1-2\alpha)$ satisfies this (and (2-2) holds for it by the hypothesis), the proof of the single-patch case of Theorem 1.3 (and of Remark (1) after it) is finished.

2E. ***Absence of simple splashes for all $\alpha \in (0, 1)$.*** Let us now assume that only simple splashes can happen for a $C^{1,\gamma}$ patch and $\alpha \in \left(0, \frac{1}{2}\right)$. That is, there is $R > 0$ such that, for all t close enough to T and any $\xi_t, \eta_t \in \mathbb{T}$ satisfying $|z(\xi_t, t) - z(\eta_t, t)| = m(t)$, there is no $\xi \in \mathbb{T}$ such that $\min\{|\xi - \xi_t|, |\xi - \eta_t|\} \ge \delta$ and also $|z(\xi, t) - z(\eta_t, t)| \le R$. This essentially means that any potential splash only involves two folds of $\partial\Omega$, although this requirement is in fact weaker than that: multiple folds are allowed but not near minimizers of (2-7). Then in Lemma 2.5 we have $f'(0) = 0$ and so $J_1 = 0$. Hence Lemma 2.4 shows that a simple splash cannot occur by time T if $\sup_{t\in[0,T)} \|\Omega(t)\|_{C^{1,\gamma}} < \infty$ for some $\gamma \in [2\alpha, 1]$, and $m(t)$ can decrease at most exponentially when $\gamma > 2\alpha$ and at most double exponentially when $\gamma = 2\alpha$.

In fact, one can extend this result to all $\alpha \in \left[\frac{1}{2}, 1\right)$. In this case one must replace u in (1-5) (which becomes infinite on $\partial\Omega(t)$) by its normal "component"

$$u_n(x, t) := \text{p.v.}\int_{\Omega(t)} c_\alpha \frac{(x-y)^\perp \cdot n_{x,t}}{|x-y|^{2+2\alpha}}\, dy\, n_{x,t}$$

(which is finite), with $n_{x,t}$ the unit outer normal vector to $\Omega(t)$ at $x \in \partial\Omega(t)$; see also [Kiselev et al. 2017, Remark 2 after Definition 1.2] or [Kiselev and Luo 2023]. If we now assume $\sup_{t\in[0,T)} \|\Omega(t)\|_{C^{2,\gamma}} < \infty$, one can use (2) above (2-10) to show that $\left|f(h) - f(0) - \frac{1}{2} f''(0)h^2\right| \le A|h|^{2+\gamma}$ in (2-14). Then the oddness of the integrand in h will yield the estimate

$$|J_2| \le 2\int_0^{m/2} \frac{16}{|h|^{1+2\alpha}} 2A|h|^{2+\gamma}\, dh + 2\int_{m/2}^{R} \frac{8m}{|h|^{2+2\alpha}} 2A|h|^{2+\gamma}\, dh$$

in the proof of Lemma 2.4 whenever $\gamma \in [2\alpha - 1, 1]$. Hence no finite time simple splash can happen by time T in this case either, and we again obtain an exponential (resp. double exponential) lower bound on $m(t)$ when $\gamma > 2\alpha - 1$ (resp. $\gamma = 2\alpha - 1$). Note also that for $\alpha = \frac{1}{2}$ it even suffices to assume $\sup_{t\in[0,T)} \|\Omega(t)\|_{C^{1,1}} < \infty$, with $2+\gamma$ replaced by 2 and with a double exponential lower bound on $m(t)$.

2F. ***The multiple patches case.*** In the general multiple patches case, (2-3) becomes

$$\sup_{t\in[0,T)} \sup_{\substack{(n,\xi),(j,\eta)\in Z_N\times\mathbb{T} \\ (n,\xi)\neq(j,\eta)}} \frac{|n-j| + |\xi - \eta|}{|z_n(\xi, t) - z_j(\eta, t)|} < \infty,$$

where $Z_N := \{1, \dots, N\}$ and $z_n(\cdot, t)$ is a constant-speed parametrization of $\partial\Omega_n(t)$. We choose the same δ (with all z_n included in the definitions of M and M'), and then

$$m(t) := \min_{\substack{(n,\xi),(j,\eta)\in Z_N\times\mathbb{T} \\ |(n,\xi)-(j,\eta)|\ge\delta}} |z_n(\xi, t) - z_j(\eta, t)| \ge 0. \tag{2-16}$$

The points ξ_t and η_t may now be on the boundaries of distinct patches, but that does not change our analysis, which only deals with the individual patch segments in a small rectangle centered at η_t. The geometric lemmas are unchanged; the estimates on integrals I' and I_i in Section 2C only change by the factor $|\theta_1|+\cdots+|\theta_N|$ and hence so does the rest of the argument. This finishes the proof of Theorem 1.3 (and of Remark (1) after it) as stated. The claim in Remark (2) after Theorem 1.3 also extends to this case.

3. Proof of Theorem 1.4

Let us now turn to the half-plane case $D := \mathbb{R}\times\mathbb{R}^+$, when the proof is essentially identical to Theorem 1.3 (and the H^3 and H^2 local well-posedness results from [Gancedo and Patel 2021; Kiselev et al. 2017] require $\alpha \in \left(0, \frac{1}{24}\right)$ and $\alpha \in \left(0, \frac{1}{6}\right)$, respectively).

Let us first recall the definition of patch solutions in this setting from [Kiselev et al. 2017]. Equation (1-1) is unchanged, and Δ in (1-2) is the Dirichlet Laplacian on D. If we assume that $\alpha \in \left(0, \frac{1}{2}\right)$, this means that for an appropriate constant $c_\alpha > 0$ we have

$$u(x,t) = c_\alpha \int_D \left(\frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} - \frac{(x-\bar{y})^\perp}{|x-\bar{y}|^{2+2\alpha}} \right) \omega(y,t)\, dy \tag{3-1}$$

for each $x \in \overline{D}$, where $\bar{y} := (y_1, -y_2)$. Definition 1.2 is as before, but with the patches $\Omega_1(t), \dots, \Omega_N(t)$ now contained in D instead of $\mathbb{R}^2$, and with u from (3-1) instead of (1-3).

We define M, M', and δ as before and $m(t)$ via (2-16). We also consider the reflected patches $\overline{\Omega}_n(t) := \{y \in \mathbb{R}^2 \setminus D \mid \bar{y} \in \Omega_n(t)\}$, which allows us to write (after dropping c_α via rescaling)

$$u(x,t) = \sum_{n=1}^N \theta_n \int_{\Omega_n(t)} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}}\, dy - \sum_{n=1}^N \theta_n \int_{\overline{\Omega}_n(t)} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}}\, dy.$$

Theorem 1.4 will now follow once we show (2-9) with this u. This is proved in the same way as on $\mathbb{R}^2$, but now the boundary segments defining functions f_i in Section 2C can belong to both the original and the reflected patches. Note that the distance of $\partial\Omega_n(t)$ and $\partial\overline{\Omega}_n(t)$ can be less than $m(t)$ — even 0 because they can touch at ∂D, in which case their normal vectors coincide at any point of touch. But they obviously cannot cross — this is why we did not assume $a \geq m$ in Lemma 2.5 — which allows us to use the same estimates as in Section 2, modulo a factor of 2 due to the number of patches now being doubled.

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