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SPECTRAL GAP FOR OBSTACLE SCATTERING IN DIMENSION 2

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We study the problem of scattering by several strictly convex obstacles, with smooth boundary and satisfying a noneclipse condition. We show, in dimension 2 only, the existence of a spectral gap for the meromorphic continuation of the Laplace operator outside the obstacles. The proof of this result relies on a reduction to an *open hyperbolic quantum map*, achieved by Nonnenmacher et al. (*Ann. of Math.* (2) **179**:1 (2014), 179–251). In fact, we obtain a spectral gap for this type of object, which also has applications in potential scattering. The second main ingredient of this article is a fractal uncertainty principle. We adapt the techniques of Dyatlov et al. (*J. Amer. Math. Soc.* **35**:2 (2022), 361–465) to apply this fractal uncertainty principle in our context.

1. Introduction

Scattering by convex obstacles and spectral gap. We are interested by the problem of scattering by strictly convex obstacles in the plane; see Figure 1. Assume

$$\mathcal{O} = \bigcup_{j=1}^{J} \mathcal{O}_j,$$

where $\mathcal{O}_j$ are open, strictly convex connected obstacles in $\mathbb{R}^2$ having smooth boundary and satisfying the *Ikawa condition*: for $i \neq j \neq k$, $\overline{\mathcal{O}}_i$ does not intersect the convex hull of $\overline{\mathcal{O}}_j \cup \overline{\mathcal{O}}_k$. Let

$$\Omega = \mathbb{R}^2 \setminus \overline{\mathcal{O}}.$$

It is known that the resolvent of the Dirichlet Laplacian in Ω continues meromorphically to the logarithmic cover of $\mathbb{C}$; see for instance [Dyatlov and Zworski 2019]. More precisely, suppose that $\chi \in C_c^\infty(\mathbb{R}^2)$ is equal to 1 in a neighborhood of $\overline{\mathcal{O}}$. The map

$$\chi(-\Delta - \lambda^2)^{-1}\chi : L^2(\Omega) \to L^2(\Omega)$$

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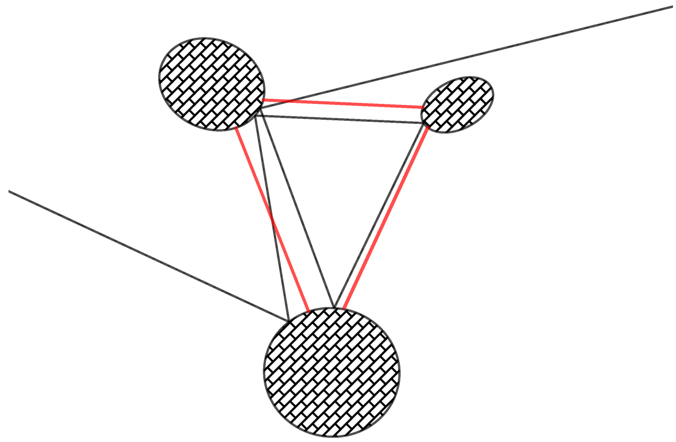

Figure 1. Scattering by three obstacles in the plane.

is holomorphic in the region $\{\operatorname{Im}\lambda > 0\}$ and it continues meromorphically to the logarithmic cover of $\mathbb{C}$. Its poles are the *scattering resonances*. We are interested in the problem of the existence of a spectral gap in the first sheet of the logarithmic cover (i.e., $\mathbb{C} \setminus i\mathbb{R}^-$). We prove the following theorem:

Theorem A. *There exist $\gamma > 0$ and $\lambda_0 > 0$ such that there is no resonance in the region*

$$[\lambda_0, +\infty[+ i[-\gamma, 0],$$

This problem has a long history in the physics and mathematics literature. The spectral gap was for instance studied by [Ikawa 1988] in dimension 3. It was experimentally investigated in [Barkhofen et al. 2013] for three- and five-disk systems. In this study, the author brings experimental evidence of the presence of a spectral gap, no matter how thin the trapped set is. For related problems concerning the distribution of scattering resonances for such systems, here is a nonexhaustive list of papers in which the reader can find pointers to a larger literature: [Gaspard and Rice 1989] for the three-disk problem, [Gérard 1988; Ikawa 1982] for the two-obstacle problem, [Petkov and Stoyanov 2010] for a link with dynamical zeta functions, [Bardos et al. 1987; Hargé and Lebeau 1994] for the diffraction by one convex obstacle, [Sjöstrand and Zworski 1999] among others papers of the two authors concerning the distribution of the scattering resonances. We will also widely use the presentation and the arguments of [Nonnenmacher et al. 2014].

The spectral gap problem is a high-frequency problem and justifies the introduction of a small parameter h, where $1/h$ corresponds to a large frequency scale. Under this rescaling, we are interested in the semiclassical operator

$$P(h) = -h^2\Delta - 1, \quad h \le h_0,$$

and spectral parameter $z \in D(0, Ch)$ for some $C > 0$.

In the semiclassical limit, the classical dynamics associated to this quantum problem is the billiard flow in $\Omega \times \mathbb{S}^1$, that is to say, the free motion outside the obstacles with normal reflection on their boundaries. A relevant dynamical object is the trapped set corresponding to the points $(x, \xi) \in \Omega \times \mathbb{S}^1$ that do not escape to infinity in the backward and forward direction of the flow. In the case of two obstacles, it is a single closed geodesic. As soon as more obstacles are involved, the structure of the trapped set becomes complex and exhibits a fractal structure. This is a consequence of the hyperbolicity of the billiard flow. It is known that the structure of the trapped set plays a crucial role in the spectral gap problem.

A good dynamical object to study this structure is the topological pressure associated to the unstable Jacobian ϕ_u. This dynamical quantity is a strictly decreasing function $s \mapsto P(s)$ which measures the instability of the flow (see Section 2 for definitions and references given there). In dimension 2, Bowen's formula shows that the Hausdorff and upper-box dimensions of the trapped set are $2s_0$, where s_0 is the unique root of the equation $P(s) = 0$. In [Nonnenmacher and Zworski 2009], the existence of a spectral gap for such systems has been proved under the pressure condition

$$P\left(\tfrac{1}{2}\right) < 0.$$

Their result holds in any dimension, with a quantitative spectral gap. Our result doesn't need this assumption anymore. In fact, it relies on the weaker pressure condition

$$P(1) < 0.$$

It is known that this condition is always satisfied in the scattering problem we consider since the trapped set is not an attractor [Bowen and Ruelle 1975]. Due to Bowen's formula, this condition can be interpreted as a fractal condition. This is this fractal property that will be crucial in the analysis.

Open hyperbolic systems and spectral gaps. The problem of scattering by obstacles falls into the wider class of spectral problems for open hyperbolic systems; see [Nonnenmacher 2011]. In these open systems, the spectral problems concern the resonances; these are generalized eigenvalues which exhibit some resonant states. Among the problems which widely interest mathematicians and physicists, resonance counting and spectral gaps are on the top of the list. Spectral gaps are known to be important to give resonance expansion (see for instance [Dyatlov and Zworski 2019]) and local energy decay (see for instance [Ikawa 1982; 1988] concerning local energy decay in the exterior of two or more obstacles in $\mathbb{R}^3$). It was conjectured in [Zworski 2017, Conjecture 3] that such systems might exhibit a spectral gap as soon as the trapped set has a fractal structure.

Potential scattering. Scattering by a compactly supported potential falls in the class of open systems. It consists of studying the semiclassical operator $P(h) = -h^2\Delta + V(x)$, where $V \in C_c^\infty(\mathbb{R}^2)$; see Figure 2. In this framework, the spectral gap problem consists of exhibiting bands in the complex plane of the form

$$[a, b] - i \times [0, h\gamma],$$

where $P(h)$ has no resonance for h small enough. In the semiclassical limit, the behavior of $P(h)$ is linked to the classical flow of the system, that is, the Hamiltonian flow generated by $p(x, \xi) = |\xi|^2 + V(x)$. Note that in potential scattering, one has to focus on some energy shell $\{p = E\}$, where $E \in \mathbb{R}$ is independent of h, with $\operatorname{Re} z$ sufficiently close to E. This specification is not necessary in obstacle scattering (implicitly, we have already decided to work with $E = 1$). The properties of the resonant states u_h, which are

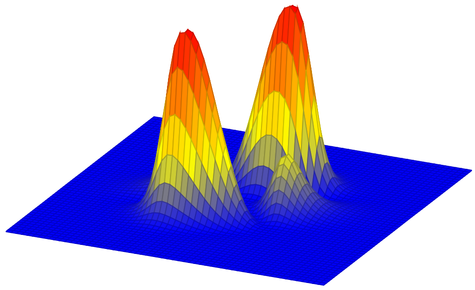

Figure 2. Scattering by a smooth compactly supported potential V.

generalized solutions of the equation $(P(h) - z)u_h = 0$, are linked to the trapped set of the flow at energy E. This trapped set K_E corresponds to all the trajectories which stay bounded for the backward and forward evolution of the flow on the energy shell $\{p = E\}$. When the flow is hyperbolic on the trapped set, this trapped set is known to exhibit a fractal structure.

In fact, a by-product of our method is that we can obtain a spectral gap in potential scattering, under the dynamical assumptions of [Nonnenmacher et al. 2011], recalled in Section 2B:

Theorem B. *Assume that the Hamiltonian flow is hyperbolic on K_E and that K_E is topologically one-dimensional. Then, there exists $\delta > 0$ such that for any $C > 0$, there exists $h_0 > 0$ such that, for $0 < h \leq h_0$, $P(h) = -h^2\Delta + V - E$ has no resonance in*

$$D(0, Ch) \cap \{\operatorname{Im} z \in [-\delta h, 0]\}.$$

It is possible to obtain a spectral gap for the more general quantum Hamiltonian presented in [Nonnenmacher et al. 2011, Section 2.1] for manifolds with Euclidean ends.

Convex cocompact hyperbolic surfaces. Another class of open hyperbolic systems exhibiting a fractal trapped set consists of the convex cocompact hyperbolic surfaces, which can be obtained as the quotient of the hyperbolic plane $\mathbb{H}^2$ by Schottky groups Γ. The spectral problem concerns the Laplacian on these surfaces and its classical counterpart is the geodesic flow on the cosphere bundle, which is known to be hyperbolic due to the negative curvature of these surfaces. In this context, it is common to write the energy variable $\lambda^2 = s(1-s)$ and study

$$(-\Delta - s(1-s))^{-1}.$$

The trapped set is linked to the limit set of Γ and the dimension δ of this limit set influences the spectrum. The Patterson–Sullivan theory (see for instance [Borthwick 2007]) tells that there is a resonance at $s = \delta$ and that the other resonances are located in $\{\operatorname{Re}(s) < \delta\}$. In particular, it gives an essential spectral gap of size $\max\big(0, \frac{1}{2} - \delta\big)$. This is consistent with the pressure condition $P(s) < \frac{1}{2}$ since in that situation, $P(s)$ is simply given by $P(s) = \delta - s$. Results where obtained by Naud [2005], where he improves the gap given by Patterson–Sullivan theory in the case $\delta \leq \frac{1}{2}$. Recent results, initiated by [Dyatlov and Zahl 2016], have improved this gap. In [Bourgain and Dyatlov 2018], the authors show that there exists an essential spectral gap for any convex cocompact hyperbolic surface. In particular, the pressure condition $\delta < \frac{1}{2}$

is no longer a necessary assumption. The new idea in these papers is the use of a fractal uncertainty principle. It will be a crucial tool of our analysis.

Reduction to open hyperbolic quantum maps. An important aspect of our analysis to prove Theorem A relies on previous results of [Nonnenmacher et al. 2014]. Their Theorem 5 (found in Section 6 of that work) reduces the study of the scattering poles to the study of the cancellation of

$$z \mapsto \det(\mathrm{I} - M(z)),$$

where

$$M(z) : L^2(\partial\mathcal{O}) \to L^2(\partial\mathcal{O}) \tag{1-1}$$

is a family of *hyperbolic open quantum maps* (see below Section 2A). The family $z \mapsto M(z)$ depends holomorphically on $z \in D(0, Ch)$ for some $C > 0$ and is sometimes called a *hyperbolic quantum monodromy operator*. The construction of this operator relies on the study of the operators $M_0(z)$ defined as follows: For $1 \le j \le J$, let $H_j(z) : C^\infty(\partial\mathcal{O}_i) \to C^\infty(\mathbb{R}^2 \setminus \mathcal{O}_j)$ be the resolvent of the problem

$$\begin{cases} (-h^2\Delta - 1 - z)(H_j(z)v) = 0, \\ H_j(z)v \text{ is outgoing}, \\ H_j(z)v = v \text{ on } \partial\mathcal{O}_j. \end{cases}$$

Let γ_j be the restriction of a smooth function $u \in C^\infty(\mathbb{R}^2)$ to $C^\infty(\partial\mathcal{O}_j)$ and define $M_0(z)$ by

$$M_0(z) = \begin{cases} 0 & \text{if } i = j, \\ -\gamma_i H_j(z) & \text{otherwise}. \end{cases}$$

Due to results of [Gérard 1988, Appendix II], this matrix is a Fourier integral operator associated with a Lagrangian relation related to the billiard flow. A priori, it excludes neither the glancing rays nor the shadow region. Ikawa's condition ensures that they do not play a role when considering the trapped set and allows the author to neglect the effects of these regions; see Section 6 in [Nonnenmacher et al. 2014]. A consequence of their analysis is that $M(z)$ is associated with a simpler Lagrangian relation $\mathcal{B}$, which is the restriction of the billiard map to a domain excluding the glancing rays. To be more precise, let us introduce

$$\begin{gathered} S^*_{\partial\mathcal{O}_j} = \{(x, \xi) \in T^*\mathbb{R}^2 : x \in \partial\mathcal{O}_j,\ |\xi| = 1\}, \\ B^*\partial\mathcal{O}_j = \{(y, \eta) \in T^*\partial\mathcal{O}_j : |\eta| \le 1\}, \\ \pi_j : S^*_{\partial\mathcal{O}_j} \to B^*\partial\mathcal{O}_j \text{ the orthogonal projection on each fiber.} \end{gathered}$$

$\mathcal{B}$ is then the union of the relations $\mathcal{B}_{ij}$ corresponding to the reflection on two obstacles: for $(\rho_i, \rho_j) \in B^*\partial\mathcal{O}_i \times B^*\partial\mathcal{O}_j$,

$$\begin{gathered} (\rho_i, \rho_j) \in \mathcal{B}_{ij} \quad \Longleftrightarrow \quad \text{there exists } t > 0 \text{ such that } \xi \in \mathbb{S}^1,\ x \in \partial\mathcal{O}_j, \\ \pi_j(x, \xi) = \rho_j, \quad \pi_i(x + t\xi, \xi) = \rho_i, \quad \nu_j(x) \cdot \xi > 0,\ \nu_i(x + t\xi) \cdot \xi < 0. \end{gathered}$$

See Figure 3. It is a standard fact in the study of chaotic billiards (see for instance [Chernov and Markarian 2006]) that the billiard map is hyperbolic due to the strict convexity assumption. Ikawa's condition ensures that the restriction of the dynamical system to the trapped set has a symbolic representation [Morita 1991].

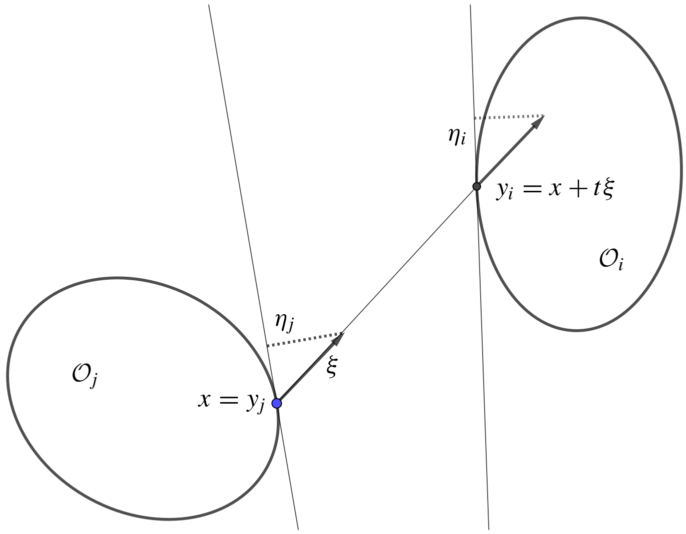

Figure 3. Description of the Lagrangian relation $\mathcal{B}_{ij}$.

Spectral gap for hyperbolic open quantum maps. Using this reduction, Theorem A will be proved once we are able to show that the spectral radius of $M(z)$ is strictly smaller than 1 for $z \in D(0, Ch) \cap \{\operatorname{Im} z \in [-\delta h, 0]\}$ for some $\delta > 0$. This will be a consequence of the following statement, which will be demonstrated in this paper (see Section 2 below for a more precise version).

Theorem C. *Let $(M(z))_z$ be the family introduced in* (1-1), *that is*, *a hyperbolic quantum monodromy operator associated with the open Lagrangian relation $\mathcal{B}$. Then*, *there exist $h_0 > 0$, $\gamma > 0$ and $\tau_{\max} > 0$ such that the spectral radius of $M(z)$*, *$\rho_{\text{spec}}(z)$*, *satisfies*, *for all $h \leq h_0$ and all $z \in D(0, Ch)$*,

$$\rho_{\text{spec}}(z) \leq e^{-\gamma - \tau_{\max} \operatorname{Im} z}.$$

When $z \in \mathbb{R}$, the operator $M(z)$ is microlocally unitary near the trapped set and its L^2 norm is essentially 1. Then, we have the trivial bound

$$\rho_{\text{spec}}(z) \leq 1.$$

The bound given by the theorem is a spectral gap since we obtain

$$\rho_{\text{spec}}(z) \leq e^{-\gamma} < 1.$$

The dependence of the bound with the parameter z is related to the symbol of the open quantum map $M(z)$.

The link between open quantum maps and the resonances of open quantum systems has also been established in [Nonnenmacher et al. 2011] for the case of potential scattering and this is why we will also obtain a spectral gap in this context. We review this reduction both in obstacle and potential scattering in Section 2 and show how it implies the spectral gap. This correspondence between open quantum maps and open quantum systems leads to a heuristic: to a resonance z for the open quantum systems, it corresponds an eigenvalue $e^{-i\tau z/h}$ of an open quantum map. Here, τ is a return time associated with the

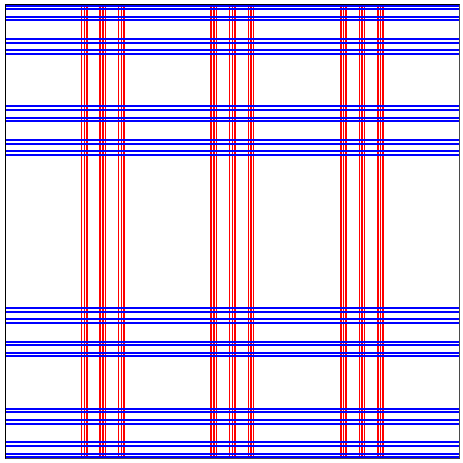

Figure 4. The fractal uncertainty principle asserts that no state can be microlocalized both in frequencies (in blue) and positions (in red) near fractal sets.

classical dynamics of the open system. In particular, the spectral gap for open quantum maps given by the theorem heuristically implies that the resonances of the open systems might satisfy $\operatorname{Im} z < -h\gamma/\tau$.

Resolvent estimates. In this paper, we use the results of [Nonnenmacher et al. 2011; 2014] as a black box. In particular, we apply directly their main theorem establishing a correspondence between scattering resonances and eigenvalues of open quantum maps. This allows us to get information on the locations of the resonances, but cannot transfer resolvent estimates from open quantum maps to the scattering resolvent directly. The main estimate of this paper (see Proposition 4.2) can be used to obtain resolvent estimates for open quantum maps. In an ongoing work, we analyze precisely the proofs in [Nonnenmacher et al. 2011; 2014] so as to explain how to deduce polynomial estimates for the cut-off resolvent both in obstacle and potential scattering. It seems to us that it should be possible to use the gluing method of [Datchev and Vasy 2012] to obtain the same kind of results (spectral gap and polynomial resolvent estimates) with other types of infinite ends, when the trapped set is hyperbolic for the flow and topologically one-dimensional.

On the fractal uncertainty principle. The fractal uncertainty principle is a recent tool in harmonic analysis in one dimension developed by Dyatlov and several collaborators. For a large survey on this topic, we refer the reader to [Dyatlov 2018]. We do not enter into the details in this introduction and give the precise definitions and statements in Section 6. We rather explain here the general idea of this principle in the spirit of our use; see Figure 4. Roughly speaking, it says that no function can be concentrated both in frequencies and positions near a fractal set. Suppose that $X, Y \subset \mathbb{R}$ are fractal sets. To fix the ideas, let's say that X and Y have upper-box dimensions δ_X and δ_Y strictly smaller than 1. For $c > 0$, we write $X(c) = X + [-c, +c]$ and the same for Y. Also denote by $\mathcal{F}_h$ the h-Fourier transform

$$\mathcal{F}_h u(\xi) = \frac{1}{(2\pi h)^{1/2}} \int_{\mathbb{R}} e^{-ix\xi/h} u(x)\, dx.$$

The fractal uncertainty principle then states that there exists $\beta > 0$ depending on X and Y (see Proposition 6.5 for the precise dependence) such that, for h small enough,

$$\|\mathbb{1}_{X(h)}\mathcal{F}_h\mathbb{1}_{Y(h)}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \le h^{\beta}.$$

Actually, one can change the scales and look for the sets $X(h^{\alpha_X})$ and $Y(h^{\alpha_Y})$, where α_X and α_Y are positive exponents. The result will stay true when these exponents satisfy the saturation condition

$$\alpha_X + \alpha_Y > 1.$$

It will be a key ingredient in the proof of the main theorem of this paper. It has been successfully used to show spectral gaps for convex cocompact hyperbolic surfaces [Dyatlov and Zahl 2016; Bourgain and Dyatlov 2017; Dyatlov and Jin 2018; Dyatlov and Zworski 2020]. A discrete version of the fractal uncertainty principle is also the main ingredient of [Dyatlov and Jin 2017], where the author proved a spectral gap for open quantum maps in a toy model case. Their results concerning the open baker's map on the torus $\mathbb{T}^2$ partly motivates our theorem on open quantum maps.

The fractal uncertainty principle has also given new results in quantum chaos on negatively curved compact surfaces. It was first successfully used for compact hyperbolic surfaces in [Dyatlov and Jin 2017], where the authors proved that semiclassical measures have full support. The hyperbolic case was treated using quantization procedures developed in [Dyatlov and Zahl 2016], which allow one to have a good semiclassical calculus for symbols very irregular in the stable direction, but smooth in the unstable one (or conversely). In [Schwartz 2021], the same ideas lead to a full delocalization of eigenstates for quantum cat maps. The quantization procedures used in these papers rely on the smoothness of the unstable and stable distributions. This smoothness is not possible for general negatively curved surfaces. However, in [Dyatlov et al. 2022], the authors bypassed this obstacle and succeeded in extending these results to the case of negatively curved surfaces. It is mainly from this paper that we borrow techniques and we adapt them in our setting.

A model example. To explain the main ideas of the proof of Theorem C, let us show how it works in an example where the trapped set is the smallest possible, a single point. In this context, we only need a simpler uncertainty principle. We focus on the case $z = 0$ in Theorem C and focus on a single open quantum map.

We consider the hyperbolic map

$$F : (x, \xi) \in \mathbb{R}^2 \mapsto (2^{-1}x, 2\xi) \in \mathbb{R}^2.$$

It has a unique hyperbolic fixed point $\rho_0 = 0$ and the stable (resp. unstable) manifold at 0 is given by $\{\xi = 0\}$ (resp. $\{x = 0\}$). The scaling operator

$$U : v \in L^2(\mathbb{R}) \mapsto \sqrt{2}v(2x)$$

is a quantum map quantizing F. To open it, consider a cut-off function $\chi \in C_c^{\infty}(\mathbb{R}^2)$ such that $\chi \equiv 1$ in $B\big(0, \frac{1}{2}\big)$ and $\operatorname{supp} \chi \Subset B(0, 1)$ and we consider the open quantum map

$$M = M(h) = \mathrm{Op}_h(\chi)U,$$

where Op_h is in this example (and only in this example) the left quantization

$$\mathrm{Op}_h(\chi)u(x) = \frac{1}{2\pi h}\int_{\mathbb{R}^2} \chi(x,\xi)e^{i(x-y)\xi/h}u(y)\,dy\,d\xi.$$

One easily checks that Egorov's property for U is true without remainder term:

$$U^*\,\mathrm{Op}_h(\chi)U = \mathrm{Op}_h(\chi\circ F), \quad U\,\mathrm{Op}_h(\chi)U^* = \mathrm{Op}_h(\chi\circ F^{-1}).$$

To show a spectral gap for M, we study M^n with

$$n = n(h) \sim -\frac{3}{4}\frac{\log h}{\log 2}.$$

This time is longer than the Ehrenfest time $-\log h/\log 2$. We write

$$M^n = U^n\,\mathrm{Op}_h(\chi\circ F^n)\cdots\mathrm{Op}_h(\chi\circ F^1).$$

The formula $[\mathrm{Op}_h(a),\mathrm{Op}_h(b)] = O(h^{1-2\delta})$ is valid for a, b symbols in S_δ (we recall the definitions of symbol classes in Section 3) and $\delta < \frac{1}{2}$. The problem here is that, for $1\le k\le n$, $\chi\circ F^k$ are uniformly in $S_{3/4}$; this is not a good symbol class. To bypass this difficulty, we observe that the symbols $\chi\circ F^k$ are uniformly in $S_{3/8}$ for $k\in\{-n/2,\dots,n/2\}$. As a consequence, for $j\in\{1,\dots,n\}$, we write

$$\begin{aligned}
[\mathrm{Op}_h(\chi\circ F^n),\mathrm{Op}_h(\chi\circ F^j)] &= U^{-n/2}[\mathrm{Op}_h(\chi\circ F^{n/2}),\mathrm{Op}_h(\chi\circ F^{j-n/2})]U^{n/2}\\
&= U^{-n/2}O(h^{1/4})U^{n/2}\\
&= O(h^{1/4}),
\end{aligned}$$

where the constants in O are uniform in j and depend only on χ. Applying this formula recursively to move the term $\mathrm{Op}_h(\chi\circ F^n)$ to the right, we get

$$M^n = U^n\,\mathrm{Op}_h(\chi\circ F^{n-1})\cdots\mathrm{Op}_h(\chi\circ F^1)\,\mathrm{Op}_h(\chi\circ F^n) + O(h^{1/4}\log h).$$

Similarly, we can write

$$M^{n+1} = \mathrm{Op}_h(\chi\circ F^{-n})\,\mathrm{Op}_h(\chi)\cdots\mathrm{Op}_h(\chi\circ F^{-n+1})U^{n+1} + O(h^{1/4}\log h).$$

Hence, we have

$$M^{2n+1} = A\,\mathrm{Op}_h(\chi\circ F^n)\,\mathrm{Op}_h(\chi\circ F^{-n})B + O(h^{1/4}\log h),$$

with

$$\begin{aligned}
A = A(h) &= U^n\,\mathrm{Op}_h(\chi\circ F^{n-1})\cdots\mathrm{Op}_h(\chi\circ F^1) = O(1),\\
B = B(h) &= \mathrm{Op}_h(\chi)\cdots\mathrm{Op}_h(\chi\circ F^{-n+1})U^{n+1} = O(1).
\end{aligned}$$

We have the following properties on the supports:

$$\operatorname{supp}\chi\circ F^n \subset \{|\xi|\le 2^{-n}\}, \quad \operatorname{supp}\chi\circ F^n \subset \{|x|\le 2^{-n}\}.$$

Assuming $n(h) \geq -\frac{3}{4}(\log h/\log 2)$, we observe that

$$\mathrm{Op}_h(\chi \circ F^n) = \mathrm{Op}_h(\chi \circ F^n)\mathbb{1}_{[-h^{3/4},h^{3/4}]}(hD_x),$$
$$\mathrm{Op}_h(\chi \circ F^{-n}) = \mathbb{1}_{[h^{-3/4},h^{3/4}]}(x)\,\mathrm{Op}_h(\chi \circ F^{-n}).$$

Finally, we have

$$M^{2n+1} = A\,\mathrm{Op}_h(\chi \circ F^n)\mathbb{1}_{[-h^{3/4},h^{3/4}]}(hD_x)\mathbb{1}_{[h^{-3/4},h^{3/4}]}(x)\,\mathrm{Op}_h(\chi \circ F^{-n})B + O(h^{1/4}\log h).$$

This is where we need an uncertainty principle:

$$\begin{aligned}\|\mathbb{1}_{[-h^{3/4},h^{3/4}]}(hD_x)\mathbb{1}_{[h^{-3/4},h^{3/4}]}(x)\|_{L^2\to L^2} &= \|\mathbb{1}_{[-h^{3/4},h^{3/4}]}\mathcal{F}_h\mathbb{1}_{[-h^{3/4},h^{3/4}]}\|_{L^2\to L^2}\\ &\leq \|\mathbb{1}_{[-h^{3/4},h^{3/4}]}\|_{L^\infty\to L^2} \times \|\mathcal{F}_h\|_{L^1\to L^\infty} \times \|\mathbb{1}_{[-h^{3/4},h^{3/4}]}\|_{L^2\to L^1}\\ &\leq Ch^{3/8} \times h^{-1/2} \times h^{3/8} = Ch^{1/4}.\end{aligned}$$

Here, the bound can be understood as a volume estimate; the box in phase space of size $h^{3/4}$ is smaller than a "quantum box". Gathering all the computations together, we see that

$$\|M^{2n+1}\|_{L^2\to L^2} = O(h^{1/4}\log h).$$

Elevating this to the power $1/(2n+1)$, we see that, for every $\varepsilon > 0$, we can find h_ε such that, for $h \leq h_\varepsilon$,

$$\rho(M) \leq (1+\varepsilon)2^{-1/6}.$$

Remark. What matters in this example is the strategy we use, and not particularly the bound, which is in fact not optimal.

Sketch of proof. The strategy presented in this simple model case is the guideline, but its direct application will encounter major pitfalls that we'll have to bypass.

- Since the trapped set is a more complex fractal set, we'll need the general fractal uncertainty principle developed by Dyatlov and his collaborators.
- Even in small coordinate charts, the trapped set cannot be written has a product of fractal sets in the unstable and stable directions. To tackle this difficulty, we build adapted coordinate charts (see Section 3E) in which we straighten the unstable manifolds. The existence of such coordinate charts is made possible by Theorem 5, in which we prove that the unstable (and stable) distribution can be extended in a neighborhood of the trapped set to a $C^{1+\beta}$ vector field.
- In the model case, there is only one point and hence one unstable Jacobian to consider which gives the Lyapouvov exponent of the map $\log J_u^1(0) = \log 2$. Generally, the growth rate of the unstable Jacobian differs from one point to another (see Section 4C) and the choice of the integer $n(h)$ is not as simple. In fact, we prefer to break the symmetry $2n(h) = n(h) + n(h)$ and split $2n(h)$ into a small logarithmic time $N_0(h)$ and a long logarithmic time $N_1(h)$ (see Section 4A). The first one is supposed to be smaller than the Ehrenfest time and allows us to use semiclassical calculus to handle M^{N_0}. As a matter of fact, the major technical difficulties concern the study of M^{N_1}.

• The study of M^{N_1} requires fine microlocal techniques. The trick used in the model case to have the commutator estimate is not possible and we have to use propagation results up to twice the Ehrenfest time. This is what we do in Section 4D but this study has to be made locally and we need to split M^{N_1} into a sum of many terms $U_{\boldsymbol{q}}$.

• We could use the fractal uncertainty principle to get the decay for single terms $M^{N_0}U_{\boldsymbol{q}}$. However, a simple triangle inequality to handle their sum will not give a decay for $M^{N_0+N_1}$ since the number of terms in the sum grows like a negative power of h. To bypass this problem, we need a more careful analysis and we gather them into clouds (see Section 4G). These clouds are supposed to interact with a few other ones, so that a Cotlar–Stein-type estimate reduces the study of the norm of the sum to the norm of each cloud. The elements of a single cloud are supposed to be close to each other, so that the fractal uncertainty principle can be applied to all of them in the same time and gives the required decay for a single cloud.

Our strategy follows the main lines of the proof of [Dyatlov et al. 2022]. In particular, their strategy allows us to apply the fractal uncertainty principle of [Bourgain and Dyatlov 2018] in a case where the unstable foliation is not smooth (and in fact, a priori defined only in a fractal set). Their strategy relies on the existence of adapted charts based on C^{2^-} regularity of the unstable foliations in negatively curved surfaces. It is based on results of [Katok and Hasselblatt 1995] for Anosov flows. We needed to prove the existence of such adapted charts in this different context. To do so, we prove that the unstable lamination can be extended into a $C^{1+\beta}$ foliation (see Section 3E). Another aspect which changes from [Dyatlov et al. 2022] is the proof of porosity. In their study, the porous sets arise as iterations of artifical “holes”, and they had to control the evolution of such holes. In our context, this study is easier since we already know that the trapped set has a fractal structure, characterized by its Hausdorff dimension. In this paper, we will rather use the upper-box dimension (but these two dimensions are equal in this context).

Restrictions. The main restriction of our theorem is that it only applies to quantum maps with two-dimensional phase space. In terms of open systems, it only concerns problems with physical space of dimension 2. Several points explain this restriction:

• The fractal uncertainty principle works in dimension 1. In higher dimensions, the result is currently not well understood and the only known cases require strong assumptions on the fractal sets; see [Dyatlov 2018, Section 6].

• Our proof strongly relies on the regularity of the stable and unstable laminations.

• The growth of the unstable Jacobian controls the contraction (resp. expansion) rate in the unique stable (resp. unstable) direction.

Plan of the paper. The paper is organized as follows:

• In Section 2, we present the main theorem of this paper and show how it gives a spectral gap in some open quantum systems.

• In Section 3, we give some background material in semiclassical analysis (pseudodifferential operators and Fourier integral operators). We also recall some standard facts about hyperbolic dynamical systems and give further results. In particular, in Theorem 5, we show that the unstable and stable distribution have $C^{1+\beta}$ regularity.

- The proof of Theorem 1 starts in Section 4, where we introduce the main ingredients needed for the proof and give several technical results.

- In Section 5, we use fine microlocal methods to microlocalize the operators we work with in small regions where the dynamic is well understood and we reduce the proof of Theorem 1 to a fractal uncertainty principle with the techniques of [Dyatlov et al. 2022].

- In Section 6, we conclude the proof of this theorem by applying the fractal uncertainty principle of [Bourgain and Dyatlov 2018], and more precisely, the version stated in [Dyatlov et al. 2022].

2. Main theorem and applications

2A. *Hyperbolic open quantum maps.* We introduce the main tools needed to state the main theorem of this paper. The following long definition is based on the definitions in the works of Nonnenmacher, Sjöstrand and Zworski [Nonnenmacher et al. 2011; 2014] specialized to the two-dimensional phase space. Consider open intervals $Y_1, \dots, Y_J$ of $\mathbb{R}$ and set

$$Y = \bigsqcup_{j=1}^{J} Y_j \subset \bigsqcup_{j=1}^{J} \mathbb{R}$$

and consider

$$U = \bigsqcup_{j=1}^{J} U_j \subset \bigsqcup_{j=1}^{J} T^*\mathbb{R}^d, \quad U_j \Subset T^*Y_j.$$

The Hilbert space $L^2(Y)$ is the orthogonal sum $\bigoplus_{i=1}^{J} L^2(Y_i)$.

Then, we introduce a smooth Lagrangian relation $F \subset U \times U$. It is a disjoint union of symplectomorphisms. For $j = 1, \dots, J$, consider open disjoint subsets $\widetilde{D}_{ij} \Subset U_j$, $1 \le i \le J$, and similarly, for $i = 1, \dots, J$, consider open disjoint subsets $\tilde{A}_{ij} \Subset U_i$, $1 \le j \le J$. We consider a family of smooth symplectomorphisms

$$F_{ij} : \widetilde{D}_{ij} \to F_{ij}(\widetilde{D}_{ij}) = \tilde{A}_{ij} \tag{2-1}$$

and define the relation F as the disjoint union of the relation F_{ij}, namely,

$$(\rho', \rho) \in F \quad \Longleftrightarrow \quad \text{there exist } 1 \le i, j \le J \text{ such that } \rho' = F_{ij}(\rho).$$

In particular, F and F^{-1} are single-valued. We will identify F with a smooth map and write by abuse of notation $\rho' = F(\rho)$ or $\rho = F^{-1}(\rho')$ instead of $(\rho', \rho) \in F$.

We let

$$\pi_L(F) = \tilde{A} = \bigsqcup_{i=1}^{J} \bigcup_{j=1}^{J} \tilde{A}_{ij}, \quad \pi_R(F) = \widetilde{D} = \bigsqcup_{j=1}^{J} \bigcup_{i=1}^{J} \widetilde{D}_{ij}.$$

We define the outgoing (resp. incoming) tail by $\mathcal{T}_+ := \{\rho \in U : F^{-n}(\rho) \in U \text{ for all } n \in \mathbb{N}\}$ (resp. $\mathcal{T}_- := \{\rho \in U : F^n(\rho) \in U \text{ for all } n \in \mathbb{N}\}$). We assume that they are closed subsets of U and that the *trapped set*

$$\mathcal{T} = \mathcal{T}_+ \cap \mathcal{T}_- \tag{2-2}$$

is compact. We denote by $f : \mathcal{T} \to \mathcal{T}$ the restriction of F to $\mathcal{T}$. For $i, j \in \{1, \dots, J\}$, we write $\mathcal{T}_i = \mathcal{T} \cap U_i$,

$$D_{ij} = \{\rho \in \mathcal{T}_j : f(\rho) \in \mathcal{T}_i\} \subset \widetilde{D}_{ij},$$
$$A_{ij} = \{\rho \in \mathcal{T}_i : f^{-1}(\rho) \in \mathcal{T}_j\} \subset \tilde{A}_{ij}.$$

Remark. F is an open canonical transformation since F (resp. F^{-1}) is defined only in $\widetilde{D}$ (resp. $\tilde{A}$). The sets $U \setminus \widetilde{D}$ (resp. $U \setminus \tilde{A}$) can be seen as holes in which a point ρ can fall in the future (resp. in the past).

We then make the following hyperbolic assumption:

$$\mathcal{T} \text{ is a hyperbolic set for } F. \tag{Hyp}$$

Namely, for every $\rho \in \mathcal{T}$, we assume that there exist stable and unstable tangent spaces $E^s(\rho)$ and $E^u(\rho)$ such that:

- $\dim E^s(\rho) = \dim E^u(\rho) = 1$.
- $T_\rho U = E^s(\rho) \oplus E^u(\rho)$.
- There exist $\lambda > 0$, $C > 0$ such that, for every $v \in E^\star(\rho)$ ($\star$ stands for u or s) and any $n \in \mathbb{N}$,

$$v \in E^s(\rho) \quad \implies \quad \|d_\rho F^n(v)\| \le Ce^{-n\lambda}\|v\|, \tag{2-3}$$
$$v \in E^u(\rho) \quad \implies \quad \|d_\rho F^{-n}(v_\star)\| \le Ce^{-n\lambda}\|v\|, \tag{2-4}$$

  where $\|\cdot\|$ is a fixed Riemannian metric on U.

The decomposition of $T_\rho U$ into stable and unstable spaces is assumed to be continuous.

Remark. • The definition is valid for any Riemannian metric and we can of course suppose that is it the standard Euclidean metric on $\mathbb{R}^2$.

• It is a standard fact (see [Mather 1968]) that there exists a smooth Riemannian metric on U, which is said to be adapted to the dynamics, such that (2-3) and (2-4) hold with $C = 1$.

• It is known that the map $\rho \mapsto E_{u/s}(\rho)$ is in fact β-Hölder for some $\beta > 0$ [Katok and Hasselblatt 1995]. We will show further an improved regularity. This will be an essential property for the proof of the main theorem.

The last assumption we'll make on $\mathcal{T}$ is a fractal assumption. To state it, we introduce the map $\phi_u : \rho \in \mathcal{T} \mapsto -\log \|d_\rho F|_{E_u(\rho)}\|$ associated with the bijection f. We suppose that

$$-\gamma_{\mathrm{cl}} := -P(-\log \|d_\rho F|_{E_u(\rho)}\|, f) > 0. \tag{Fractal}$$

Here, in terms of thermodynamics formalism, P denotes the topological pressure of the map ϕ_u. The norm $\|\cdot\|$ is associated with any Riemannian metric on U. For instance, a possible formula for the definition of the pressure is

$$P(\phi) = \lim_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log \sup_E \sum_{\rho\in E} \exp^{\sum_{k=0}^{n-1} \phi(f^k\rho)},$$

where the supremum ranges over all the (n, ε)-separated subsets $E \subset \mathcal{T}$ (E is said to be (n, ε)-separated if, for every $\rho, \rho' \in E$, there exists $k \in \{0, \dots, n-1\}$ such that $d(f^k(\rho), f^k(\rho')) > \varepsilon$).

Remark. • γ_{cl} is the classical decay rate of the dynamical system. It has the following physical interpretation: Fix a point $\rho_0 \in \mathcal{T}$ and consider the set $B_m(\rho_0, \varepsilon)$ of points $\rho \in U$ such that $|F^k(\rho) - F^k(\rho_0)| < \varepsilon$ for $0 \le k \le m-1$. Then, its Lebesgue measure if of order $e^{-m\gamma_{\mathrm{cl}}}$.

• In Section A4, we recall arguments showing that $\mathcal{T}$ is indeed "fractal". More precisely, the trace of $\mathcal{T}$ along the unstable and stable manifolds (see Lemma 3.11 for the definitions of these manifolds) have upper-box dimension strictly smaller than 1. In fact, Bowen's formula (see for instance [Barreira 2008]) gives that this upper-box dimension corresponds to the Hausdorff dimension d_H and it is the unique solution of the equation

$$P(s\phi_u, f) = 0, \quad s \in \mathbb{R}.$$

The Hausdorff dimension of the trapped set is then $2d_H$.

• This condition has to be compared with the pressure condition $P\big(\frac{1}{2}\phi_u\big) < 0$ in [Nonnenmacher and Zworski 2009], which ensured a spectral gap for chaotic systems. This condition required that $\mathcal{T}$ was sufficiently "thin", i.e., with Hausdorff dimension strictly smaller than 1. Our condition allows to go up to the limit $\dim_H \mathcal{T} = 2^-$.

We then associate to F *hyperbolic open quantum maps*, which are its quantum counterpart.

Definition 2.1. Fix $\delta \in \big[0, \frac{1}{2}\big[$. We say that $T = T(h)$ is a semiclassical Fourier integral operator associated with F, and we let $T = T(h) \in I_\delta(Y \times Y, F')$ if, for each couple $(i, j) \in \{1, \dots, J\}^2$, there exists a semiclassical Fourier integral operator $T_{ij} = T_{ij}(h) \in I_\delta(Y_j \times Y_i, F'_{ij})$ associated with F_{ij} in the sense of Definition 3.9, such that

$$T = (T_{ij})_{1\le i,j\le J} : \bigoplus_{i=1}^{J} L^2(Y_i) \to \bigoplus_{i=1}^{J} L^2(Y_i).$$

In particular $\mathrm{WF}_h(T) \subset \tilde{A} \times \widetilde{D}$. We define $I_{0^+}(Y \times Y, F') = \bigcap_{\delta>0} I_\delta(Y \times Y, F')$.

We will say that T is *microlocally unitary near* $\mathcal{T}$ if the two following conditions hold:

- $\|TT^*\| \le 1 + O(h^\varepsilon)$ for some $\varepsilon > 0$.
- There exists a neighborhood $\Omega \subset U$ of $\mathcal{T}$ such that, for every $u = (u_1, \dots, u_J) \in \bigoplus_{j=1}^{J} L^2(Y_j)$,

$$\text{for all } j \in \{1, \dots, J\},\ \ \mathrm{WF}_h(u_j) \subset \Omega \cap U_j \implies TT^*u = u + O(h^\infty)\|u\|_{L^2},\ \ T^*Tu = u + O(h^\infty)\|u\|_{L^2}.$$

Let us now briefly see what the second condition implies for the components of T^*T. First focus on the off-diagonal entries

$$(T^*T)_{ij} = \sum_{k=1}^{J} (T^*)_{ik} T_{kj} = \sum_{k=1}^{J} (T_{ki})^* T_{kj}.$$

If $k \in \{1, \dots, J\}$ and $i \neq j$, $(T_{ki})^* T_{kj} = O(h^\infty)$ since

$$\mathrm{WF}_h(T_{ki}^*) \subset \widetilde{D}_{ki} \times \tilde{A}_{ki}, \quad \mathrm{WF}_h(T_{kj}) \subset \tilde{A}_{kj} \times \widetilde{D}_{kj} \quad \text{and} \quad \tilde{A}_{kj} \cap \tilde{A}_{ki} = \varnothing.$$

As a consequence, the off-diagonal terms are always $O(h^\infty)$. For the diagonal entries,

$$(T^*T)_{ii} = \sum_{k=1}^{J} (T_{ki})^* T_{ki}.$$

Each term of this sum is a pseudodifferential operator with wavefront set

$$\mathrm{WF}_h(T_{ki}^* T_{ki}) \subset \widetilde{D}_{ki}.$$

Since the $\widetilde{D}_{ki}$ are pairwise disjoint, $T^*T = \mathrm{Id}_{L^2(Y)} + O(h^\infty)$ microlocally near $\mathcal{T}$ if and only if, for all k, i, $T_{ki}^* T_{ki} = \mathrm{Id}_{L^2(Y_i)} + O(h^\infty)$ microlocally near D_{ki}. The same computations apply to TT^*. As a consequence, T is microlocally unitary near $\mathcal{T}$ if and only if, for all (k, i), T_{ki} is a Fourier integral operator associated with F_{ki}, microlocally unitary near $D_{ki} \times A_{ki}$ (see the paragraph below Definition 3.9).

Notation. An element of $S_\delta^{\mathrm{comp}}(U)$ is a J-tuple $\alpha = (\alpha_1, \dots, \alpha_J)$, where each α_j is an element of $S^\delta_{\mathrm{comp}}(\mathbb{R}^2)$ such that $\operatorname{ess\,supp} \alpha_j \subset U_j$ (this notation is recalled in the next section).

We fix a smooth function $\Psi_Y = (\Psi_1, \dots, \Psi_J)$ such that, for $1 \le j \le J$, $\Psi_j \in C_c^\infty(Y_j, [0, 1])$ satisfies $\Psi_j = 1$ on $\pi(U_j)$ (recall that $U_j \Subset T^*Y_j$).

For $\alpha \in S_\delta^{\mathrm{comp}}(U)$, we also denote by $\mathrm{Op}_h(\alpha)$ the diagonal operator-valued matrix

$$\mathrm{Op}_h(\alpha) = \mathrm{Diag}(\Psi_1 \mathrm{Op}_h(\alpha_1)\Psi_1, \dots, \Psi_J \mathrm{Op}_h(\alpha_J)\Psi_J) : \bigoplus_{j=1}^{J} L^2(Y_j) \to \bigoplus_{j=1}^{J} L^2(Y_j).$$

Note that as operators on $L^2(\mathbb{R})$, $\mathrm{Op}_h(\alpha_j)$ and $\Psi_j \mathrm{Op}_h(\alpha_j)\Psi_j$ are equal modulo $O(h^\infty)$.

We can now state the main theorem of this paper, namely a spectral gap for hyperbolic open quantum maps. We denote by $\rho_{\mathrm{spec}}(A)$ the spectral radius of a bounded operator $A : L^2(Y) \to L^2(Y)$.

Theorem 1. *Suppose that the above assumptions on F,* (Hyp) *and* (Fractal) *are satisfied. Then, there exists $\gamma > 0$ such that the following holds*:

Let $T = T(h) \in I_{0^+}(Y \times Y, F')$ be a semiclassical Fourier integral operator associated with F in the sense of Definition 2.1 and $\alpha \in S_\delta^{\mathrm{comp}}(U)$. Assume that T is microlocally unitary in a neighborhood of $\mathcal{T}$. Then, there exists $h_0 > 0$ such that,

$$\text{for all } 0 < h \le h_0, \quad \rho_{\mathrm{spec}}(T(h)\,\mathrm{Op}_h(\alpha)) \le e^{-\gamma} \|\alpha\|_\infty,$$

where h_0 depends on (U, F), T and seminorms of α in S_δ.

For applications, we will need the following corollary (it is in fact rather a corollary of the method used to prove Theorem 1):

Corollary 1. *With the same notations and assumptions as in Theorem 1, if $R(h)$ is a family of bounded operators on $L^2(Y)$ satisfying $\|R(h)\| = O(h^\eta)$ for some $\eta > 0$, then the there exists γ' depending only on γ and η such that, for $0 < h \le h_0$,*

$$\rho_{\mathrm{spec}}(T(h)\operatorname{Op}_h(\alpha) + R(h)) \le e^{-\gamma'} \|\alpha\|_\infty.$$

Remark. • If the value h_0 depends on T and α, this is not the case of γ which depends on (U, F).

• This is a spectral gap; it has to be compared with the easy bound we could have

$$\rho_{\mathrm{spec}}(T \operatorname{Op}_h(\alpha)) \le \|\alpha\|_\infty + o(1).$$

In particular, if $\alpha \equiv 1$ in a neighborhood of $\mathcal{T}$ and $|\alpha| \le 1$ everywhere, $\rho_{\mathrm{spec}}(T(h)) \le e^{-\gamma} < 1$.

• $T \operatorname{Op}_h(\alpha)$ is the way we've chosen to write our Fourier integral operator with "gain" (or absorption depending on the modulus of α) factor α. $T \operatorname{Op}_h(\alpha)$ transforms a wave packet u_0 microlocalized near ρ_0 lying in a small neighborhood of $\mathcal{T}$ into a wave packet microlocalized near $F(\rho_0)$, with norm essentially changed by a factor $|\alpha(\rho_0)|$.

• The proof will actually show that if η is strictly bigger than some threshold, then $\gamma' = \gamma$.

Notation. Throughout the paper, the meaning of the constants C can change from line to line but these constants will only depend on our dynamical system (U, F). If there is another dependence, it will be specified.

2B. ***Applications of the theorem.*** This theorem has applications in the study of open quantum systems. We refer the reader to [Nonnenmacher 2011] for a survey on this topic. The spectral gap given by Theorem 1 will actually give a spectral gap for the resonances of semiclassical operators $P(h)$ in $\mathbb{R}^2$, or for the resonances of the Dirichlet Laplacian in the exterior of strictly convex obstacles satisfying the Ikawa noneclipse condition. We refer the reader to the review [Zworski 2017] for more background on scattering resonances or to the book [Dyatlov and Zworski 2019]. The results we will obtain from Theorem 1 give a positive answer (in dimension 2) to Conjecture 3 in [Zworski 2017], under a fractal assumption.

Scattering by strictly convex obstacles in the plane. As already explained in the Introduction the main problem motivating Theorem 1 is the problem of scattering by obstacles in the plane $\mathbb{R}^2$. It leads to:

Theorem 2. *Assume that $\mathcal{O} = \bigcup_{i=1}^J \mathcal{O}_j$, where $\mathcal{O}_j$ are open, strictly convex connected obstacles in $\mathbb{R}^2$ having smooth boundary and satisfying the **Ikawa condition**: for $i \ne j \ne k$, $\overline{\mathcal{O}}_i$ does not intersect the convex hull of $\overline{\mathcal{O}}_j \cup \overline{\mathcal{O}}_k$. Let*

$$\Omega = \mathbb{R}^2 \setminus \overline{\mathcal{O}}.$$

There exist $\gamma > 0$ and $\lambda_0 > 1$ such that the Dirichlet Laplacian $-\Delta$ on $L^2(\Omega)$ has no scattering resonance in the region

$$[\lambda_0, +\infty[+ i[-\gamma, 0].$$

Let us give the arguments to see why Theorem 1 implies this theorem. After a semiclassical reparametrization, it is enough to show that there exist $\delta > 0$ and $h_0 > 0$ such that $P(h) := -h^2\Delta - 1$ has no resonance in $D(0, Ch) \cap \{\operatorname{Im} z \in [-\delta h, 0]\}$ for any $h \le h_0$. As already explained, the implication relies on [Nonnenmacher et al. 2014, Theorem 5, Section 6]. There they prove the existence of a family of

$$(\mathcal{M}(z))_{z \in D(0,Ch)} = (\mathcal{M}(z, h)) \tag{2-5}$$

such that:

- $\mathcal{M}(z) = \Pi_h M(z) \Pi_h + O(h^L)$, where Π_h is a finite-rank projector, of rank comparable to h^{-1}, $L > 0$ is a fixed constant (which can in fact be chosen as big as we want) and $M(z)$ is described below and satisfies $\Pi_h M(z) \Pi_h = M(z) + O(h^L)$.
- $M(0)$ is an open quantum map associated with a Lagrangian relation $\mathcal{B}$ presented in the Introduction, which is microlocally unitary near $\mathcal{T}$. $\mathcal{B}$ and $M(0)$ play the roles of F and T in Theorem 1 and satisfy its assumptions.
- $M(z) = M(0) \operatorname{Op}_h(e^{iz\tau/h}) + O(h^{1-\varepsilon})$ uniformly in $D(0, Ch)$, where $\varepsilon > 0$ can be chosen arbitrarily close to zero and $\tau \in C_c^\infty(U)$ is a smooth function (which has to be seen as a return time).
- The resonances of $P(h)$ in $D(0, Ch)$ are the roots, with multiplicities, of the equation

$$\det(I - \mathcal{M}(z)) = 0.$$

Hence, to prove the theorem, it is enough to show that the spectral radius of $\mathcal{M}(z)$ is strictly smaller than 1 for $z \in D(0, Ch) \cap \{\operatorname{Im} z \in [-\delta h, 0]\}$ for some $\delta > 0$ and for h small enough. To see that, we write

$$\mathcal{M}(z) = M(0) \operatorname{Op}_h(e^{iz\tau/h}) + R(h),$$

with $R(h) = O(h^\eta)$ for any $\eta < \min(1, L)$. We apply Theorem 1 and find some γ' such that

$$\rho_{\text{spec}}(\mathcal{M}(z)) \le e^{-\gamma'} \| e^{iz\tau/h} \|_\infty \le e^{-\gamma'} e^{\delta \tau_{\max}}, \quad z \in D(0, Ch) \cap \{\operatorname{Im} z \in [-\delta h, 0]\},$$

where $\tau_{\max} = \|\tau\|_\infty$. This ensures a spectral gap of size

$$\delta < \frac{\gamma'}{\tau_{\max}}.$$

Schrödinger operators. Actually, the obstacles, seen as infinite potential barriers, can be smoothened with a potential $V \in C_c^\infty(\mathbb{R}^2)$ and we can consider the Schrödinger operators $P_0(h) = -h^2\Delta + V(x)$.

Unlike the obstacle problem, a simple rescaling does not allow to pass from energy 1 to any energy E and the behavior of the classical flow can drastically change from an energy shell to another. To study the problem at energy $E > 0$, independent of h, we rather consider

$$P(h) = P_0(h) - E.$$

The resolvent $(P(h) - z)^{-1}$ continues meromorphically from $\operatorname{Im} z > 0$ to $D(0, Ch)$ (as previously in the sense that $\chi(P(h) - z)^1 \chi$ extends meromorphically with $\chi \in C_c^\infty(\mathbb{R}^2)$) and we are interested in the existence of a spectral gap.

The classical Hamiltonian flow associated with $P(h)$ is the Hamiltonian flow Φ^t generated by $p_0(x,\xi) = |\xi|^2 + V(x)$ on the energy shell $p_0^{-1}(E)$. The trapped set is defined as above by

$$K_E := \{(x,\xi) \in T^*\mathbb{R}^2 : p_0(x,\xi) = E,\ \Phi^t(x,\xi) \text{ stays bounded as } t \to \pm\infty\}.$$

We assume that the flow is hyperbolic on K_E and that the trapped set is topologically one-dimensional. Equivalently, we assume that transversely to the flow, K_E is zero-dimensional. Under these assumptions, the authors proved (see Theorem 1 in [Nonnenmacher et al. 2011]) the existence of a family of monodromy operators associated with a Lagrangian relation F_E which is a Poincaré map of the flow on different Poincaré sections $\Sigma_1, \dots, \Sigma_J \subset p_0^{-1}(E)$. The assumption on the dimension of K_E implies that the assumption (Fractal) is satisfied since K_E cannot be an attractor [Bowen and Ruelle 1975]. Hence, Theorem 1 applies and we can prove as done in the case of obstacles:

Theorem 3. *Under the above assumptions, there exists $\delta > 0$ such that $P(h)$ has no resonances in*

$$D(0, Ch) \cap \{\operatorname{Im} z \in [-\delta h, 0]\}.$$

3. Preliminaries

3A. ***Pseudodifferential operators and Weyl quantization.*** We recall some basic notions and properties of the Weyl quantization on $\mathbb{R}^n$. We refer the reader to [Zworski 2012] for the proofs of the statements and further considerations on semiclassical analysis and quantizations. We start by defining classes of h-dependent symbols.

Definition 3.1. Let $0 \le \delta \le \frac{1}{2}$. We say that an h-dependent family $a := (a(\cdot\,;h))_{0<h\leqslant 1}$ is in the class $S_\delta(T^*\mathbb{R}^n)$ (or simply S_δ if there is no ambiguity) if, for every $\alpha \in \mathbb{N}^{2n}$, there exists $C_\alpha > 0$ such that,

$$\text{for all } 0 < h \le 1, \qquad \sup_{(x,\xi)\in\mathbb{R}^n} |\partial^\alpha a(x,\xi;h)| \le C_\alpha h^{-\delta|\alpha|}.$$

In this paper, we will mostly be concerned with $\delta < \frac{1}{2}$. We will also use the notation $S_{0^+} = \bigcap_{\delta>0} S_\delta$. We write $a = O(h^N)_{S_\delta}$ to mean that, for every $\alpha \in \mathbb{N}^{2n}$, there exists $C_{\alpha,N}$ such that,

$$\text{for all } 0 < h \le 1, \qquad \sup_{(x,\xi)\in\mathbb{R}^n} |\partial^\alpha a(x,\xi;h)| \le C_{\alpha,N} h^{-\delta|\alpha|} h^N.$$

If $a = O(h^N)_{S_\delta}$ for all $N \in N$, we'll write $a = O(h^\infty)_{S_\delta}$. A priori, the constants $C_{\alpha,N}$ depend on the symbol a. However, in this paper, we will often make them depend on different parameters but not directly on a. This will be specified when needed.

For a given symbol $a \in S_\delta(T^*\mathbb{R}^n)$, we say that a has a compact essential support if there exists a compact set K such that,

$$\text{for all } \chi \in C_c^\infty(\Omega), \quad \operatorname{supp}\chi \cap K = \varnothing \quad \Longrightarrow \quad \chi a = O(h^\infty)_{\mathcal{S}(T^*\mathbb{R}^n)}$$

(here $\mathcal{S}$ stands for the Schwartz space). We let $\operatorname{ess\,supp} a \subset K$ and say that a belongs to the class $S_\delta^{\text{comp}}(T^*\mathbb{R}^n)$. The essential support of a is then the intersection of all such compact K's. In particular, the class S_δ^{comp} contains all the symbols supported in an h-independent compact set and these symbols correspond, modulo $O(h^\infty)_{\mathcal{S}(T^*\mathbb{R})}$, to all symbols of S_δ^{comp}. For this reason, we will adopt the notation $a \in S_\delta^{\mathrm{comp}}(\Omega) \iff \operatorname{ess\,supp} a \Subset \Omega$.

For a symbol $a \in S_\delta(T^*\mathbb{R}^n)$, we'll quantize it using Weyl's quantization procedure. It is informally written as

$$(\mathrm{Op}_h(a)u)(x) = (a^W(x, hD_x)u)(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} a\left(\frac{x+y}{2}, \xi\right) u(y) e^{i((x-y)\cdot\xi)/h}\, dy\, d\xi.$$

We will denote by $\Psi_\delta(\mathbb{R}^n)$ the corresponding classes of pseudodifferential operators. By definition, the wavefront set of $A = \mathrm{Op}_h(a)$ is $\mathrm{WF}_h(A) = \operatorname{ess\,supp} a$.

We say that a family $u = u(h) \in \mathcal{D}'(\mathbb{R}^n)$ is h-tempered if, for every $\chi \in C_c^\infty(\mathbb{R}^n)$, there exist $C > 0$ and $N \in \mathbb{N}$ such that $\|\chi u\|_{H_h^{-N}} \le C h^{-N}$. For a h-tempered family u, we say that a point $\rho \in T^*\mathbb{R}^n$ does *not* belong to the wavefront set of u if there exists $a \in S^{\mathrm{comp}}(T^*\mathbb{R}^n)$ such that $a(\rho) \neq 0$ and $\mathrm{Op}_h(a)u = O(h^\infty)_{\mathcal{S}}$. We denote by $\mathrm{WF}_h(u)$ the wavefront set of u.

We say that a family of operators $B = B(h) : C_c^\infty(\mathbb{R}^{n_2}) \to \mathcal{D}'(\mathbb{R}^{n_1})$ is h-tempered if its Schwartz kernel $\mathcal{K}_B \in \mathcal{D}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is h-tempered. We define

$$\mathrm{WF}_h'(B) = \{(x, \xi, y, -\eta) \in T^*\mathbb{R}^{n_1} \times T^*\mathbb{R}^{n_2} : (x, \xi, y, \eta) \in \mathrm{WF}_h(\mathcal{K}_B)\}.$$

Let us now recall standard results in semiclassical analysis concerning the L^2-boundedness of pseudodifferential operators and their composition. We'll use the following version of the Calderón–Vaillancourt theorem [Zworski 2012, Theorem 4.23].

Theorem 4. *There exists $C_n > 0$ such that the following holds. For every $0 \le \delta < \frac{1}{2}$ and $a \in S_\delta(T^*\mathbb{R}^n)$, $\mathrm{Op}_h(a)$ is a bounded operator on L^2 and*

$$\|\mathrm{Op}_h(a)\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \le C_n \sum_{|\alpha| \le 8n} h^{|\alpha|/2} \|\partial^\alpha a\|_{L^\infty}.$$

As a consequence of the sharp Gårding inequality (see [Zworski 2012, Theorem 4.32]), we also have the precise estimate of L^2 norms of pseudodifferential operator,

Proposition 3.2. *Assume that $a \in S_\delta(\mathbb{R}^{2n})$. Then, there exists C_a depending on a finite number of seminorms of a such that*

$$\|\mathrm{Op}_h(a)\|_{L^2 \to L^2} \le \|a\|_\infty + C_a h^{1/2-\delta}.$$

We recall that the Weyl quantizations of real symbols are self-adjoint in L^2. The composition of two pseudodifferential operators in Ψ_δ is still a pseudodifferential operator. More precisely (see [Zworski 2012, Theorems 4.11 and 4.18]), if $a, b \in S_\delta$, then $\mathrm{Op}_h(a) \circ \mathrm{Op}_h(b)$ is given by $\mathrm{Op}_h(a \# b)$, where $a \# b$ is the Moyal product of a and b. It is given by

$$a \# b(\rho) = e^{ihA(D)}(a \otimes b)|_{\rho = \rho_1 = \rho_2},$$

where $a \otimes b(\rho_1, \rho_2) = a(\rho_1) b(\rho_2)$, $e^{ihA(D)}$ is a Fourier multiplier acting on functions on $\mathbb{R}^{4n}$ and, writing $\rho_i = (x_i, \xi_i)$,

$$A(D) = \tfrac{1}{2}(D_{\xi_1} \circ D_{x_2} - D_{x_1} \circ D_{\xi_2}).$$

We can estimate the Moyal product by a quadratic stationary phase and get the following expansion: for all $N \in \mathbb{N}$,

$$a \# b(\rho) = \sum_{k=0}^{N-1} \frac{i^k h^k}{k!} A(D)^k (a \otimes b)|_{\rho=\rho_1=\rho_2} + r_N,$$

where, for all $\alpha \in \mathbb{N}^{2n}$, there exists C_α, independent of a and b, such that

$$\|\partial^\alpha r_N\|_\infty \leq C_\alpha h^N \|a \otimes b\|_{C^{2N+4n+1+|\alpha|}}.$$

As a consequence of this asymptotic expansion, we have the precise product formula:

Lemma 3.3. *For every $N \in \mathbb{N}$, there exists $C_N > 0$ such that, for every $a, b \in S_\delta(\mathbb{R}^n)$,*

$$\mathrm{Op}_h(a) \circ \mathrm{Op}_h(b) = \mathrm{Op}_h\left(\sum_{k=0}^{N-1} \frac{i^k h^k}{k!} A(D)^k (a \otimes b)|_{\rho=\rho_1=\rho_2}\right) + R_N, \tag{3-1}$$

where

$$\|R_N\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq C_N h^N \|a \otimes b\|_{C^{2N+12n+1}}. \tag{3-2}$$

Remark. It will be important in the sequel to understand the derivatives of a and b involved in the k-th term of the previous expansion. A quick recurrence using the precise form of the operator $A(D)$ shows that $A(D)^k(a \otimes b)(\rho_1, \rho_2)$ is of the form

$$\sum_{|\alpha|=k, |\beta|=k} \lambda_{\alpha,\beta} \partial^\alpha a(\rho_1) \partial^\beta b(\rho_2).$$

This can be rewritten $l_k(d^k a(\rho_1), d^k b(\rho_2))$, where l_k is a bilinear form on the spaces of k-symmetric forms on $\mathbb{R}^{2n}$. Of course, we make use of the identifications $T_{\rho_1} T^*\mathbb{R}^n \simeq T_{\rho_2} T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$.

As a simple corollary, we get an expression for the commutator of pseudodifferential operators.

Corollary 3.4. *For every $N \in \mathbb{N}$, there exists $C_N > 0$ such that, for every $a, b \in S_\delta(\mathbb{R}^n)$,*

$$[\mathrm{Op}_h(a), \mathrm{Op}_h(b)] = \mathrm{Op}_h\left(\frac{h}{i}\{a, b\} + \sum_{k=2}^{N-1} h^k L_k(d^k a, d^k b)\right) + R_N,$$

where

$$\|R_N\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq C_N h^N \|a \otimes b\|_{C^{2N+12n+1}},$$

where the L_k are bilinear forms on the spaces of k-symmetric forms on $\mathbb{R}^{2n}$.

3B. *Fourier Integral operators.* We now review some aspects of the theory of Fourier integral operators. We follow [Zworski 2012, Chapter 11] and [Nonnenmacher et al. 2014]. We refer the reader to [Guillemin and Sternberg 2013] for further details. Finally, we will give the precise definition needed to understand Definition 2.1.

3B1. *Local symplectomorphisms and their quantization.* We momentarily work in dimension n. Let us denote by $\mathcal{K}$ the set of symplectomorphisms $\kappa : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ such that the following holds: there exist continuous and piecewise smooth families of smooth functions $(\kappa_t)_{t\in[0,1]}$, $(q_t)_{t\in[0,1]}$ such that:

- For all $t \in [0, 1]$, $\kappa_t : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ is a symplectomorphism.
- $\kappa_0 = \mathrm{Id}_{T^*\mathbb{R}^n}$, $\kappa_1 = \kappa$.
- For all $t \in [0, 1]$, $\kappa_t(0) = 0$.
- There exists $K \Subset T^*\mathbb{R}^n$ compact such that, for all $t \in [0, 1]$, $q_t : T^*\mathbb{R}^n \to \mathbb{R}$ and $\operatorname{supp} q_t \subset K$.
- $(d/dt)\kappa_t = (\kappa_t)^* H_{q_t}$.

If $\kappa \in \mathcal{K}$, we denote by $C = \mathrm{Gr}'(\kappa) = \{(x, \xi, y, -\eta) : (x, \xi) = \kappa(y, \eta)\}$ the twisted graph of κ. We recall [Zworski 2012, Lemma 11.4], which asserts that local symplectomorphisms can be seen as elements of $\mathcal{K}$, as soon as we have some geometric freedom.

Lemma 3.5. *Let U_0, U_1 be open and precompact subsets of $T^*\mathbb{R}^n$. Assume that $\kappa : U_0 \to U_1$ is a local symplectomorphism fixing* 0 *and which extends to $V_0 \Supset U_0$ an open star-shaped neighborhood of* 0. *Then, there exists $\tilde{\kappa} \in \mathcal{K}$ such that $\tilde{\kappa}|_{U_0} = \kappa$.*

If $\kappa \in \mathcal{K}$ and if (q_t) denotes the family of smooth functions associated with κ in its definition, we let $Q(t) = \mathrm{Op}_h(q_t)$. It is a continuous and piecewise smooth family of operators. Then the Cauchy problem

$$\begin{cases} hD_t U(t) + U(t)Q(t) = 0, \\ U(0) = \mathrm{Id} \end{cases} \tag{3-3}$$

is globally well-posed.

Following [Nonnenmacher et al. 2014, Definition 3.9], we adopt the definition:

Definition 3.6. Fix $\delta \in \left[0, \tfrac{1}{2}\right[$. We say that $U \in I_\delta(\mathbb{R}^n \times \mathbb{R}^n; C)$ if there exist $a \in S_\delta(T^*\mathbb{R}^n)$ and a path (κ_t) from Id to κ satisfying the above assumptions such that $U = \mathrm{Op}_h(a)U(1)$, where $t \mapsto U(t)$ is the solution of the Cauchy problem (3-3).

The class $I_{0^+}(\mathbb{R} \times \mathbb{R}, C)$ is by definition $\bigcap_{\delta>0} I_\delta(\mathbb{R} \times \mathbb{R}, C)$.

It is a standard result, known as Egorov's theorem (see [Zworski 2012, Theorem 11.1]) that if $U(t)$ solves the Cauchy problem (3-3) and if $a \in S_\delta$, then $U^{-1}\,\mathrm{Op}_h(a)U$ is a pseudodifferential operator in Ψ_δ and if $b = a \circ \kappa$, then $U^{-1}\,\mathrm{Op}_h(a)U - \mathrm{Op}_h(b) \in h^{1-2\delta}\Psi_\delta$.

Remark. Applying Egorov's theorem and Beal's theorem, it is possible to show that if (κ_t) is a closed path from Id to Id, and $U(t)$ solves (3-3), then $U(1) \in \Psi_0(\mathbb{R}^n)$. In other words, $I_\delta(\mathbb{R} \times \mathbb{R}, \mathrm{Gr}'(\mathrm{Id})) \subset \Psi_\delta(\mathbb{R}^n)$. But the other inclusion is trivial. Hence, this in an equality:

$$I_\delta(\mathbb{R}^n \times \mathbb{R}^n, \mathrm{Gr}'(\mathrm{Id})) = \Psi_\delta(\mathbb{R}^n).$$

The notation $I(\mathbb{R}^n \times \mathbb{R}^n, C)$ comes from the fact that the Schwartz kernel of such operators are Lagrangian distributions associated with C, and in particular have wavefront set included in C. As a consequence, if $T \in I_\delta(\mathbb{R}^n \times \mathbb{R}^n, C)$, then $\mathrm{WF}_h'(T) \subset \mathrm{Gr}(T)$.

Let us state a simple proposition concerning the composition of Fourier integral operators:

Proposition 3.7. *Let $\kappa_1, \kappa_2 \in \mathcal{K}$ and $U_1 \in I_\delta(\mathbb{R} \times \mathbb{R}, \mathrm{Gr}'(\kappa_1))$, $U_2 \in I_\delta(\mathbb{R} \times \mathbb{R}, \mathrm{Gr}'(\kappa_1))$. Then,*

$$U_1 \circ U_2 \in I_\delta(\mathbb{R} \times \mathbb{R}, \mathrm{Gr}'(\kappa_1 \circ \kappa_2)).$$

Proof. Let's write $U_1 = \mathrm{Op}_h(a_1)U_1(1)$, $U_2 = \mathrm{Op}_h(a_2)U_2(1)$ with the obvious notation associated with the Cauchy problems (3-3) for κ_1 and κ_2. Egorov's theorem asserts that $U_1(1)\,\mathrm{Op}_h(a_2)U_1(1)^{-1} = \mathrm{Op}_h(b_2)$ for some $b_2 \in S_\delta$ and $\mathrm{Op}_h(a_1)\,\mathrm{Op}_h(b_2) = \mathrm{Op}_h(a_1 \# b_2)$. It is then enough to focus on the case $a_1 = a_2 = 1$. We set

$$U_3(t) := \begin{cases} U_1(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ U_1(1) \circ U_2(2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

It solves the Cauchy problem

$$\begin{cases} hD_tU_3(t) + U_3(t)Q_3(t) = 0, \\ U(0) = \mathrm{Id}, \end{cases}$$

with

$$Q_3(t) := \begin{cases} 2Q_1(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ 2Q_2(2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

To conclude the proof, it is enough to notice that this Cauchy problem is associated with the path $\kappa_3(t)$ between $\kappa(0) = \mathrm{Id}$ and $\kappa_3(1) = \kappa_1 \circ \kappa_2$, where

$$\kappa_3(t) := \begin{cases} \kappa_1(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ \kappa_1 \circ \kappa_2(2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$ □

3B2. *Precise version of Egorov's theorem.* We will need a more quantitative version of Egorov's theorem, similar to the one in [Dyatlov et al. 2022, Lemma A.7]. The result does not show that $U(1)^{-1}\,\mathrm{Op}_h(a)U(1)$ is a pseudodifferential operator (one would need Beal's theorem to say that) but it gives a precise estimate on the remainder, depending on the seminorms of a. We now specialize to the case of dimension $n = 1$ but the following result holds in any dimension but changing the constant 15 to something of the form Mn.

Proposition 3.8. *Consider $\kappa \in \mathcal{K}$ and denote by $U(t)$ the solution of* (3-3). *There exists a family of differential operators $(D_j)_{j\in\mathbb{N}}$ of order j such that, for all $a \in S_\delta$ and all $N \in \mathbb{N}$,*

$$U(1)^{-1}\,\mathrm{Op}_h(a)U(1) = \mathrm{Op}_h\left(a \circ \kappa + \sum_{j=1}^{N-1} h^j (D_{j+1}a) \circ \kappa\right) + O_\kappa(h^N \|a\|_{C^{2N+15}}). \tag{3-4}$$

Proof. We keep the notation introduced previously. Let us first define

$$A_0(t) = U(t)\,\mathrm{Op}_h(a \circ \kappa_t)U(t)^{-1}$$

and compute

$$\begin{aligned} &U(t)^{-1}\partial_t A_0(t)U(t) \\ &\quad = -\frac{i}{h}[Q(t), \mathrm{Op}_h(a\circ\kappa_t)] + \mathrm{Op}_h(\{q_t, a\circ\kappa_t\}) \\ &\quad = \mathrm{Op}_h(\{q_t, a\circ\kappa_t\}) - \frac{i}{h}\left(\mathrm{Op}_h\left(\frac{h}{i}\{q_t, a\circ\kappa_t\} + \sum_{j=2}^{N} h^j L_j(d^j q_t, d^j(a\circ\kappa_t))\right)\right) + O(h^N \|q_t \otimes (a\circ\kappa_t)\|_{C^{2(N+1)+13}}) \\ &\quad = \mathrm{Op}_h\left(\sum_{j=1}^{N-1} -ih^j L_{j+1}(d^{j+1}q_t, d^{j+1}(a\circ\kappa_t))\right) + O_{\kappa_t}(h^N \|a\|_{C^{2N+15}}). \end{aligned}$$

We now define by induction a family of functions $a_j(t),\ j=0,\dots,N-1$, by

$$a_0(t)=a,\quad a_k(t)=\sum_{m=0}^{k-1}\int_0^t iL_{k+1-m}(d^{k+1-m}q_s,d^{k+1-m}(a_m(s)\circ\kappa_s))\circ\kappa_s^{-1}\,ds,$$

and set $A_k(t)=U(t)\operatorname{Op}_h\big(\sum_{j=0}^k h^j a_j(t)\circ\kappa_t\big)U(t)^{-1}$. We first remark by an easy induction on k, that $a_k(t)$ is of the form $D_{k+1}(t)a$, where $D_{k+1}(t)$ is a differential operator of order at most $k+1$, with coefficients depending continuously on t and on $(\kappa_t)_t$. We now check the following by induction:

$$U(t)^{-1}\partial_t A_k(t)U(t)=-i\operatorname{Op}_h\bigg(\sum_{j=k+1}^{N-1}\sum_{m=0}^{k}h^jL_{j+1-m}(d^{j+1-m}q_t,d^{j+1-m}(a_m(t)\circ\kappa_t))\bigg)+O_\kappa(h^N\|a\|_{C^{2N+15}}).$$

We've already done it for $k=0$. Let's assume that the equality holds for $k-1$ and let's prove it for $k\geq 1$:

$$U(t)^{-1}\partial_t A_k(t)U(t)=U(t)^{-1}\partial_t A_{k-1}(t)U(t)+h^kU(t)^{-1}\partial_t\operatorname{Op}_h(a_k(t)\circ\kappa_t)U(t).$$

Let's compute the second part of the right-hand side:

$$\begin{aligned}
&U(t)^{-1}\partial_t\operatorname{Op}_h(a_k(t)\circ\kappa_t)U(t)\\
&\quad=-\frac{i}{h}[Q(t),\operatorname{Op}_h(a_k(t)\circ\kappa_t)]+\operatorname{Op}_h(\{q_t,a_k(t)\circ\kappa_t\})+\operatorname{Op}_h(\partial_t a_k(t)\circ\kappa_t)\\
&\quad=-i\operatorname{Op}_h\bigg(\sum_{l=1}^{N-1-k}h^jL_{l+1}(d^{l+1}q_t,d^{l+1}(a_k(t)\circ\kappa_t))\bigg)+O_\kappa(h^{N-k}\|a_k(t)\|_{C^{2(N+1-k)+13}})+\operatorname{Op}_h(\partial_t a_k(t)\circ\kappa_t).
\end{aligned}$$

We can estimate the remainder by

$$O_\kappa(h^{N-k}\|a_k(t)\|_{C^{2(N+1-k)+13}})=O_\kappa(h^{N-k}\|a\|_{C^{2(N+1-k)+13+k+1}})=O_\kappa(h^{N-k}\|a\|_{C^{2N+15}}).$$

We now combine this with the value of

$$U(t)^{-1}\partial_t A_{k-1}(t)U(t)=-i\operatorname{Op}_h\bigg(\sum_{j=k}^{N-1}\sum_{m=0}^{k-1}h^jL_{j+1-m}(d^{j+1-m}q_t,d^{j+1-m}(a_m(t)\circ\kappa_t))\bigg)+O_\kappa(h^N\|a\|_{C^{2N+15}}).$$

By the definition of $a_k(t)$, the term $h^k\operatorname{Op}_h(\partial_t a_k(t)\circ\kappa_t)$ cancels the term corresponding to $j=k$ in the sum. Moreover, for every $j>k$, writing $j=k+l,\ l\in\{1,\dots,N-1-k\}$, the term $h^{k+l}L_{l+1}(d^{l+1}q_t,d^{l+1}(a_k(t)\circ\kappa_t))$ gives the missing term $h^jL_{j+1-k}(d^{j+1-k}q_t,d^{j+1-k}(a_k(t)\circ\kappa_t))$. This gives the required equality for $A_k(t)$.

In particular, $\partial_t A_{N-1}(t)=O_\kappa(h^N\|a\|_{C^{2N+15}})$. We now use the fact that at $t=0$, $a_0(0)=a$, $a_k(0)=0$, $k=1,\dots,N-1$, $U(0)=\mathrm{Id}$, $\kappa_0=\mathrm{Id}$, and hence $A_{N-1}(0)=\operatorname{Op}_h(a)$. Integrating between 0 and 1, we have

$$A_{N-1}(t)-\operatorname{Op}_h(a)=O_\kappa(h^N\|a\|_{C^{2N+15}}).$$

Conjugating by $U(1)$, we finally have

$$U(1)^{-1}\operatorname{Op}_h(a)U(1)=\operatorname{Op}_h\bigg(a\circ\kappa+\sum_{k=1}^{N-1}h^ka_k(1)\circ\kappa\bigg)+O_\kappa(h^N\|a\|_{C^{2N+15}}),$$

which is what we wanted, since $a_k(1)=D_{k+1}(1)a$. □

3B3. *An important example.* Let us focus on a particular case of canonical transformations. Suppose that $\kappa : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ is a canonical transformation such that

$$(x, \xi, y, \eta) \in \mathrm{Gr}(\kappa) \mapsto (x, \eta)$$

is a local diffeomorphism near $(x_0, \xi_0, y_0, \eta_0)$. Then, there exists a phase function $\psi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, Ω_x, Ω_η open sets of $\mathbb{R}^n$ and Ω a neighborhood of $(x_0, \xi_0, y_0, \eta_0)$ such that

$$\mathrm{Gr}'(\kappa) \cap \Omega = \{(x, \partial_x \psi(x, \eta), \partial_\eta \psi(x, \eta), -\eta) : x \in \Omega_x, \eta \in \Omega_\eta\}.$$

One says that ψ generates $\mathrm{Gr}'(\kappa)$. Suppose that $\alpha \in S_\delta^{\mathrm{comp}}(\Omega_x \times \Omega_\eta)$. Then, modulo a smoothing operator $O(h^\infty)$, the following operator T is an element of $I_\delta^{\mathrm{comp}}(\mathbb{R}^n \times \mathbb{R}^n, \mathrm{Gr}'(\kappa))$:

$$Tu(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{(i/h)(\psi(x,\eta) - y\cdot\eta)} \alpha(x, \eta)\, u(y)\, dy\, d\eta,$$

and if $T^*T = \mathrm{Id}$ microlocally near (y_0, η_0) then $|\alpha(x, \eta)|^2 = |\det D^2_{x\eta}\psi(x, \eta)| + O(h^{1-2\delta})_{S_\delta}$ near $(x_0, \xi_0, y_0, \eta_0)$. The converse statement holds: microlocally near $(x_0, \xi_0, y_0, \eta_0)$ and modulo $O(h^\infty)$, the elements of $I_\delta(\mathbb{R}^n \times \mathbb{R}^n, \mathrm{Gr}'(\kappa))$ can be written under this form.

3B4. *Lagrangian relations.* Recall that the Lagrangian relation F we consider is the union of local Lagrangian relations $F_{ij} \subset U_i \times U_j$. We fix a compact set $W \subset \pi_L(F)$ containing some neighborhood of $\mathcal{T}$. Our definition will depend on W. Following [Nonnenmacher et al. 2014, Section 3.4.2], we now focus on the definition of the elements of $I_\delta(Y \times Y; F')$. An element $T \in I_\delta(Y \times Y; F')$ is a matrix of operators

$$T = (T_{ij})_{1 \le i, j \le J} : \bigoplus_{j=1}^{J} L^2(Y_j) \to \bigoplus_{i=1}^{J} L^2(Y_i).$$

Each T_{ij} is an element of $I_\delta(Y_i \times Y_j, F'_{ij})$. Let's now describe the recipe to construct elements of $I_\delta(Y_i \times Y_j, F'_{ij})$. We fix $i, j \in \{1, \dots, J\}$.

- Fix some small $\varepsilon > 0$ and two open covers of U_j, $U_j \subset \bigcup_{l=1}^{L} \Omega_l$, $\Omega_l \Subset \widetilde{\Omega}_l$, with $\widetilde{\Omega}_l$ star-shaped and having diameter smaller than ε. We denote by $\mathcal{L}$ the sets of indices l such that $\Omega_l \subset \pi_R(F_{ij}) = \widetilde{D}_{ij} \subset U_j$ and we require (this is possible if ε is small enough)

$$F^{-1}(W) \cap U_j \subset \bigcup_{l \in \mathcal{L}} \Omega_l.$$

- Introduce a smooth partition of unity associated with the cover (Ω_l), $(\chi_l)_{1 \le l \le L} \in C_c^\infty(\Omega_l, [0, 1])$, $\operatorname{supp} \chi_l \subset \Omega_l$, $\sum_l \chi_l = 1$ in a neighborhood of $\overline{U}_j$.
- For each $l \in \mathcal{L}$, we denote by F_l the restriction to $\widetilde{\Omega}_l$ of F_{ij}, seen as a symplectomorphism $F_{ij} : \widetilde{D}_{ij} \subset U \to V$. By Lemma 3.5, there exists $\kappa_l \in \mathcal{K}$ which coincides with F_l on Ω_l.
- We consider $T_l = \mathrm{Op}_h(\alpha_i) U_l(1)$, where $U_l(t)$ is the solution of the Cauchy problem (3-3) associated with κ_l and $\alpha_i \in S_\delta^{\mathrm{comp}}(T^*\mathbb{R})$.

- We set
$$T^{\mathbb{R}} = \sum_{l \in \mathcal{L}} T_l \operatorname{Op}_h(\chi_l) : L^2(\mathbb{R}) \to L^2(\mathbb{R}). \tag{3-5}$$

$T^{\mathbb{R}}$ is a globally defined Fourier integral operator. We will write $T^{\mathbb{R}} \in I_\delta(\mathbb{R} \times \mathbb{R}, F'_{ij})$. Its wavefront set is included in $\tilde{A}_{ij} \times \widetilde{D}_{ij}$.

- Finally, we fix cut-off functions $(\Psi_i, \Psi_j) \in C_c^\infty(Y_i, [0,1]) \times C_c^\infty(Y_j, [0,1])$ such that $\Psi_i \equiv 1$ on $\pi(U_i)$ and $\Psi_j \equiv 1$ on $\pi(U_j)$ (here, $\pi : (x, \xi) \in T^*Y_\cdot \mapsto x \in Y_\cdot$ is the natural projection) and we adopt the following definitions:

Definition 3.9. We say that $T : \mathcal{D}'(Y_j) \to C^\infty(\overline{Y}_i)$ is a Fourier integral operator in the class $I_\delta(Y_i \times Y_j, F'_{ij})$ if there exists $T^{\mathbb{R}} \in I_\delta(\mathbb{R} \times \mathbb{R}, F')$ as constructed above such that

- $T - \Psi_i T \Psi_j = O(h^\infty)_{\mathcal{D}'(Y) \to C^\infty(\bar{Z})}$,
- $\Psi_i T \Psi_j = \Psi_i T^{\mathbb{R}} \Psi_j$.

For $U'_j \subset U_j$ and $U'_i = F(U'_j) \subset U_i$, we say that T (or $T^{\mathbb{R}}$) is microlocally unitary in $U'_i \times U'_j$ if $TT^* = \mathrm{Id}$ microlocally in U'_i and $T^*T = \mathrm{Id}$ microlocally in U'_j.

Remark. The definition of this class is not canonical since it depends in fact on the compact set W through the partition of unity.

Another version of Egorov's theorem. The precise version of Egorov's theorem in Proposition 3.8 is only stated for globally unitary Fourier integral operator defined using the Cauchy equation (3-3). We extend it here to microlocally unitary and globally defined Fourier integral operators. We fix $i, j \in \{1, \dots, J\}$.

Lemma 3.10. *Let $T \in I_\delta(\mathbb{R} \times \mathbb{R}, F'_{ij})$. Suppose that $B(\rho, 4\varepsilon) \subset U_j$ and that T is microlocally unitary in $F_{ij}(B(\rho, 4\varepsilon)) \times B(\rho, 4\varepsilon)$. Then, there exists a family $(D_k)_{k \in \mathbb{N}}$ of differential operators of order k, compactly supported in $B(\rho, 3\varepsilon)$ such that the following holds: For every $N \in \mathbb{N}$ and for all $b \in C_c^\infty(B(\rho, 2\varepsilon))$,*
$$T \operatorname{Op}_h(b) = \operatorname{Op}_h\left(b \circ \kappa^{-1} + \sum_{k=1}^{N-1} h^k (D_{k+1} b) \circ \kappa^{-1}\right) T + O(h^N \|b\|_{C^{2N+15}})_{L^2(\mathbb{R}) \to L^2(\mathbb{R})}.$$
The constants in O depend on T and F.

Proof. First, introduce some cut-off function χ such that $\chi \equiv 1$ in a neighborhood of $B(\rho, 2\varepsilon)$ and $\operatorname{supp} \chi \subset B(\rho, 3\varepsilon)$. Due to these properties and Lemma 3.3, we have
$$\operatorname{Op}_h(b) = \operatorname{Op}_h(\chi) \operatorname{Op}_h(b) \operatorname{Op}_h(\chi) \operatorname{Op}_h(\chi) + O(h^N \|b\|_{C^{2N+13}})_{L^2(\mathbb{R}) \to L^2(\mathbb{R})}.$$
Moreover, $\operatorname{Op}_h(\chi) T^* T = \operatorname{Op}_h(\chi) + O(h^\infty)$, and hence
$$T \operatorname{Op}_h(b) = T \operatorname{Op}_h(\chi) \operatorname{Op}_h(b) \operatorname{Op}_h(\chi) \operatorname{Op}_h(\chi) T^* T + O(h^N \|b\|_{C^{2N+13}})_{L^2 \to L^2} + O(h^\infty) \|\operatorname{Op}_h(b)\|_{L^2 \to L^2}.$$
The term $O(h^\infty)\|\operatorname{Op}_h(b)\|_{L^2 \to L^2}$ can be absorbed in $O(h^N \|b\|_{C^{2N+13}})_{L^2 \to L^2}$. Consider $\tilde{\kappa} \in \mathcal{K}$ extending $\kappa|_{B(\rho, 3\varepsilon)}$ and construct $U = U(1)$ by solving the Cauchy problem (3-3) associated with $\tilde{\kappa}$. Due to the

properties on composition of Fourier integral operators (Proposition 3.7), $T\,\mathrm{Op}_h(\chi)U^{-1}$ and $U\,\mathrm{Op}_h(\chi)T^*$ are pseudodifferential operators, and we denote them by $\mathrm{Op}_h(a_1)$, $\mathrm{Op}_h(a_2)$. Now write

$$\begin{aligned} T\,\mathrm{Op}_h(b) &= [T\,\mathrm{Op}_h(\chi)U^{-1}]U\,\mathrm{Op}_h(b)\,\mathrm{Op}_h(\chi)U^{-1}[U\,\mathrm{Op}_h(\chi)T^*]T + O(h^N\|b\|_{C^{2N+13}})_{L^2\to L^2} \\ &= \mathrm{Op}_h(a_1)[U\,\mathrm{Op}_h(b)\,\mathrm{Op}_h(\chi)U^{-1}]\,\mathrm{Op}_h(a_2)T + O(h^N\|b\|_{C^{2N+13}})_{L^2\to L^2}. \end{aligned}$$

By using the precise version in Proposition 3.8, one can write

$$U\,\mathrm{Op}_h(b)\,\mathrm{Op}_h(\chi)U^{-1} = \mathrm{Op}_h\biggl(b\circ\kappa^{-1} + \sum_{k=1}^{N-1}(L_{k+1}b)\circ\kappa^{-1}\biggr) + O(h^N\|b\|_{C^{2N+15}})_{L^2\to L^2}.$$

Applying Lemma 3.3, we see that we can write

$$T\,\mathrm{Op}_h(b) = \mathrm{Op}_h\biggl(b_0\circ\kappa^{-1} + \sum_{k=1}^{N-1}(D_{k+1}b)\circ\kappa^{-1}\biggr)T + O(h^N\|b\|_{C^{2N+15}})_{L^2\to L^2},$$

where $b_0 = a_1 \times b\circ\kappa^{-1} \times a_2$. Since T is microlocally unitary in $B(\rho, 4\varepsilon)$, the product a_1a_2 is equal to 1 in $B(\rho, 2\varepsilon)$, and hence, the lemma is proved. □

3C. *Hyperbolic dynamics.* We assumed that F is hyperbolic on the trapped set $\mathcal{T}$. As already mentioned, we can fix an adapted Riemannian metric on U such that the following stronger version of the hyperbolic estimates are satisfied for some $\lambda_0 > 0$: for every $\rho \in \mathcal{T}$, $n \in \mathbb{N}$,

$$v \in E_u(\rho) \quad\Longrightarrow\quad \|d_\rho F^{-n}(v)\| \le e^{-\lambda_0 n}\|v\|, \tag{3-6}$$

$$v \in E_s(\rho) \quad\Longrightarrow\quad \|d_\rho F^{n}(v)\| \le e^{-\lambda_0 n}\|v\|. \tag{3-7}$$

Notation. We now use the induced Riemannian distance on U and denote it by d.

We also use the same notation $\|\cdot\|$ to denote the subordinate norm on the space of linear maps between tangent spaces of U; namely, if $F(\rho_1) = \rho_2$,

$$\|d_{\rho_1}F\| = \sup_{v\in T_{\rho_1}U,\ \|v\|_{\rho_1}=1} \|d_{\rho_1}F(v)\|_{\rho_2}.$$

If $\rho \in \mathcal{T}$, $n \in \mathbb{Z}$, we use this Riemannian metric to define the unstable Jacobian $J_n^u(\rho)$ and stable Jacobian $J_n^s(\rho)$ at ρ by

$$v \in E_u(\rho) \quad\Longrightarrow\quad \|d_\rho F^{n}(v)\| = J_n^u(\rho)\|v\|, \tag{3-8}$$

$$v \in E_s(\rho) \quad\Longrightarrow\quad \|d_\rho F^{n}(v)\| = J_n^s(\rho)\|v\|. \tag{3-9}$$

These Jacobians quantify the local hyperbolicity of the map.

Notation. Suppose that f and g are some real-valued functions depending on the same family of parameters $\mathcal{P}$. For instance, for $J_n^u(\rho)$, $\mathcal{P} = \{n, \rho\}$. We will write $f \sim g$ to mean that there exists a constant $C \ge 1$ depending only on (U, F), but not on $\mathcal{P}$, such that $C^{-1}g \le f \le Cg$.

For instance, if we define unstable and stable Jacobians $\tilde{J}_n^u$ and $\tilde{J}_n^s$ using another Riemannian metric, then, for every $n \in \mathbb{Z}$ and $\rho \in \mathcal{T}$,

$$\tilde{J}_n^u(\rho) \sim J_n^u(\rho), \qquad \tilde{J}_n^s(\rho) \sim J_n^s(\rho).$$

From the compactness of $\mathcal{T}$, there exist $\lambda_1 \geq \lambda_0$ which satisfy

$$e^{n\lambda_0} \leq J_n^u(\rho) \leq e^{n\lambda_1} \quad \text{and} \quad e^{-n\lambda_1} \leq J_n^s(\rho) \leq e^{-n\lambda_0}, \quad n \in \mathbb{N},\ \rho \in \mathcal{T}, \tag{3-10}$$

$$e^{n\lambda_0} \leq J_{-n}^s(\rho) \leq e^{n\lambda_1} \quad \text{and} \quad e^{-n\lambda_1} \leq J_{-n}^u(\rho) \leq e^{-n\lambda_0}, \quad n \in \mathbb{N},\ \rho \in \mathcal{T}. \tag{3-11}$$

We cite here standard facts about the stable and unstable manifolds; see for instance [Katok and Hasselblatt 1995, Chapter 6].

Lemma 3.11. *For any $\rho \in \mathcal{T}$, there exist local stable and unstable manifolds $W_s(\rho)$, $W_u(\rho) \subset U$ satisfying, for some $\varepsilon_1 > 0$ (only depending on F) ($\star$ will denote a letter in $\{u, s\}$ and the use of $\pm$ with $\star$ has to be read with the convention $u \to -,\ s \to +$):*

(1) *$W_s(\rho)$, $W_u(\rho)$ are C^∞-embedded curves, with the C^∞ norms of the embedding uniformly bounded in ρ.*

(2) *The boundary of $W_\star(\rho)$ do not intersect $\overline{B(\rho, \varepsilon_1)}$.*[1]

(3) *$W_s(\rho) \cap W_u(\rho) = \{\rho\}$ and $T_\rho W_\star(\rho) = E_\star(\rho)$.*

(4) *$F^{\pm}(W_\star(\rho)) \subset W_\star(F(\rho))$.*

(5) *For each $\rho' \in W_\star(\rho)$, we have $d(F^{\pm n}(\rho), F^{\pm n}(\rho')) \to 0$.*

(6) *Let $\theta > 0$ satisfying $e^{-\lambda_0} < \theta < 1$. If $\rho' \in U$ satisfies $d(F^{\pm i}(\rho), F^{\pm i}(\rho')) \leq \varepsilon_1$ for all $i = 0, \dots, n$ then $d(\rho', W_\star(\rho)) \leq C\theta^n \varepsilon_1$ for some $C > 0$.*

(7) *If $\rho, \rho' \in \mathcal{T}$ satisfy $d(\rho, \rho') \leq \varepsilon_1$, then $W_u(\rho) \cap W_s(\rho')$ consists of exactly one point in $\mathcal{T}$.*

Since we work with the local unstable and stable manifolds, we may assume that $W_\star(\rho) \subset B(\rho, 2\varepsilon_1)$.

For our purpose, we will need a more precise version of these results. The following lemmas are an adaptation of Lemma 2.1 in [Dyatlov et al. 2022] to our setting.

Lemma 3.12. *There exists a constant $C > 0$ depending only on (U, F), such that, for all $\rho, \rho' \in U$:*

(1) *If $\rho \in \mathcal{T}$ and $\rho' \in W_s(\rho)$ then*

$$d(F^n(\rho), F^n(\rho')) \leq C J_n^s(\rho) d(\rho, \rho') \quad \textit{for all } n \in \mathbb{N}. \tag{3-12}$$

(2) *If $\rho \in \mathcal{T}$ and $\rho' \in W_u(\rho)$ then*

$$d(F^{-n}(\rho), F^{-n}(\rho')) \leq C J_{-n}^u(\rho) d(\rho, \rho') \quad \textit{for all } n \in \mathbb{N}. \tag{3-13}$$

Proof. We prove (1). Part (2) is proved in a similar way by inverting the time direction. Let $\rho \in \mathcal{T}$, $\rho' \in W_s(\rho)$. Since $T_\rho(W_s(\rho)) = E_s(\rho)$ and $d_\rho F(E_s(\rho)) = E_s(F(\rho))$, the Taylor development of F along $W_s(\rho)$ gives

$$d(F(\rho), F(\rho')) \leq J_1^s(\rho) d(\rho, \rho') + C d(\rho, \rho')^2 \leq J_1^s(\rho) d(\rho, \rho')(1 + C d(\rho, \rho')) \tag{3-14}$$

[1] In other words, there exists a smooth curve $\gamma : [-\delta, \delta] \to U$ such that $\overline{B(\rho, \varepsilon_1)} \cap W_\star(\rho) = \operatorname{Im} \gamma$, with $\gamma(0) = \rho$; it means that the size of the (un-)stable manifolds is bounded from below uniformly.

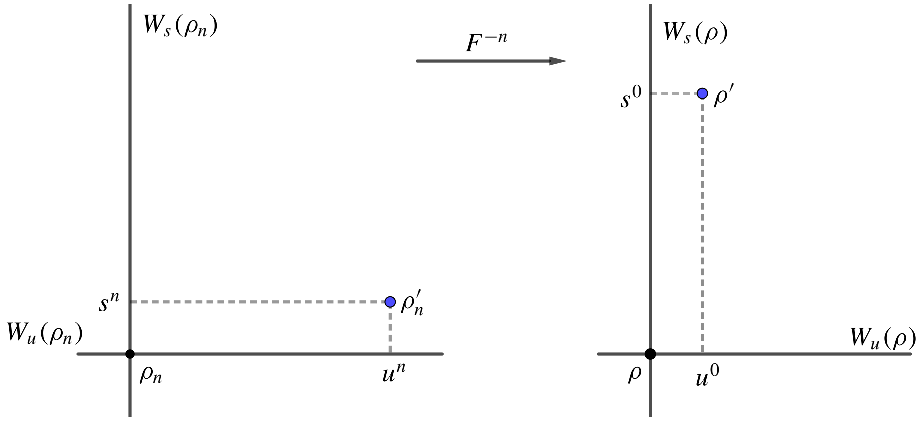

Figure 5. Framework for the proof of Lemma 3.13.

since $J_1^s \geq C^{-1}$. Applying this inequality with $F^k(\rho)$ and $F^k(\rho')$ instead of ρ and ρ', and recalling that, by Lemma 3.11, $d(F^k(\rho), F^k(\rho')) \leq C\theta^k d(\rho, \rho')$, we can write

$$d(F^{k+1}(\rho), F^{k+1}(\rho')) \leq J_1^s(F^k(\rho))\, d(F^k(\rho), F^k(\rho'))(1 + C\theta^k). \tag{3-15}$$

By this last inequality and the chain rule, we have

$$d(F^n(\rho), F^n(\rho')) \leq J_n^s(\rho) d(\rho, \rho') \prod_{k=0}^{n-1} (1 + C\theta^k) \leq C J_n^s(\rho) d(\rho, \rho'), \tag{3-16}$$

completing the proof. $\square$

The following lemma gives a stronger version of (6) in Lemma 3.11.

Lemma 3.13. *There exist $C > 0$ and $\varepsilon_1 > 0$, depending only on (U, F), such that, for all $\rho, \rho' \in U$ and $n \in \mathbb{N}$:*

(1) *If $\rho \in \mathcal{T}$ and $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_1$ for all $i \in \{0, \dots, n\}$ then*

$$d(\rho', W_s(\rho)) \leq \frac{C}{J_n^u(\rho)}, \tag{3-17}$$

$$\|d_{\rho'} F^n\| \leq C J_n^u(\rho). \tag{3-18}$$

(2) *If $\rho \in \mathcal{T}$ and $d(F^{-i}(\rho), F^{-i}(\rho')) \leq \varepsilon_1$ for all $i \in \{0, \dots, n\}$ then*

$$d(\rho', W_u(\rho)) \leq \frac{C}{J_{-n}^s(\rho)}, \tag{3-19}$$

$$\|d_{\rho'} F^{-n}\| \leq C J_{-n}^s(\rho). \tag{3-20}$$

Proof. We prove (1). Part (2) is proved in a similar way by inverting the time direction. Let $\rho \in \mathcal{T}$ and $\rho' \in U$ be such that $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_1$ for $0 \leq i \leq n$ with ε_1 to be determined. Define $\rho_k = F^k(\rho)$. The first condition on ε_1 is that it is smaller than the one of Lemma 3.11 so that we ensure the following estimates: for $k \in \{0, \dots, n\}$,

$$d(F^k(\rho'), W_s(F^k(\rho))) \le C\theta^{n-k}\varepsilon_1, \tag{3-21}$$

$$d(F^k(\rho'), W_s(F^k(\rho))) \le C\theta^{k}\varepsilon_1. \tag{3-22}$$

We will use coordinates charts $\kappa_k : \hat{\rho} \in U_k \mapsto (u^k, s^k) \in V_k$ adapted to the dynamical system; see, in [Katok and Hasselblatt 1995], Theorem 6.2.3, the explanations below and Theorem 6.2.8 for the existence of this chart. More precisely, we want these charts to satisfy:

- $\kappa_k(\rho_k) = (0, 0)$.
- $\kappa_k(W_s(\rho_k) \cap U_k) = \{(0, s) : s \in \mathbb{R}\} \cap V_k$.
- $\kappa_k(W_u(\rho_k) \cap U_k) = \{(u, 0) : u \in \mathbb{R}\} \cap V_k$.
- For $\hat{\rho} \in U_k$, we have $|u^k| \sim d(\hat{\rho}, W_s(\rho_k))$, $|s^k| \sim d(\hat{\rho}, W_u(\rho_k))$ and $|s^k|^2 + |u^k|^2 \sim d(\rho_k, \hat{\rho})^2$.
- $(\kappa_k)_{0\le k\le n}$ are uniformly bounded in the C^N topology for all N, with constant independent of ρ_0 and n. In particular, we may assume that ε_1 is chosen small enough so that $B(\rho_k, \varepsilon_1) \subset U_k$ for all $0 \le k \le n$.
- Up to changing the metric we work with (which is not problematic), we may assume that the restrictions of $d\kappa_k(\rho)$ to $E_s(\rho)$ and $E_u(\rho)$ are isometries for the metrics $|\cdot|_s$ and $|\cdot|_u$.

If we write $\widetilde{F}_k = \kappa_k \circ F \circ \kappa_{k-1}^{-1}$, we can check that in this pair of coordinates charts, the action of F^{-1} is given by

$$\widetilde{F}_k^{-1}(u^k, s^k) = (\pm J_{-1}^u(\rho_k)u^k + \alpha_k(u^k, s^k), \pm J_{-1}^s(\rho_k)s^k + \beta_k(u^k, s^k)), \tag{3-23}$$

where α_k, β_k are smooth functions, uniformly bounded in k for the C^2 topology and such that $\alpha_k(0, s^k) = 0$, $\beta_k(u^k, 0) = 0$, $d\alpha_k(0, 0) = 0$, $d\beta_k(0, 0) = 0$.

With these properties, one can check that

$$\alpha_k(u^k, s^k) \le C|u^k|\,\|(u^k, s^k)\|. \tag{3-24}$$

Let's now define $\rho'_k = F^k(\rho')$ and $(u^k, s^k) = \kappa_k(\rho'_k)$. By (3-21), (3-22), (3-23), (3-24), we can write

$$\begin{aligned} |u^{k-1}| &\le J_{-1}^u(\rho_k)|u^k| + C|u^k|\,\|(u^k, s^k)\| \\ &\le J_{-1}^u(F^k(\rho))|u^k|(1 + C\varepsilon_1(\theta_1^k + \theta_1^{n-k})) \\ &\le J_{-1}^u(F^k(\rho))|u^k|(1 + C\varepsilon_1\theta^{\min(k,n-k)}). \end{aligned}$$

Then, using the chain rule, one has

$$d(\rho', W_s(\rho)) \le C|u^0| \le CJ_{-n}^u(F^n(\rho))\prod_{k=0}^{n-1}(1 + C\varepsilon_1\theta^{\min(k,n-k)}). \tag{3-25}$$

Finally, we can estimate

$$\prod_{k=0}^{n}(1 + C\varepsilon_1\theta^{\min(k,n-k)}) \le \prod_{k=0}^{\lceil n/2\rceil}(1 + C\varepsilon_1\theta^k)^2 \le C,$$

which gives

$$d(\rho', W_s(\rho)) \leq C J^u_{-n}(F^n(\rho)) = \frac{C}{J^u_n(\rho)}. \tag{3-26}$$

This proves (3-17).

To prove (3-18), we first construct a metric which simplifies the computations. If $\rho \in \mathcal{T}$, we pick $v_\star(\rho) \in E_\star(\rho)$[2] such that $\|v_\star(\rho)\| = 1$. There exists a Riemannian metric $|\cdot|$ on $\mathcal{T}$ such that, for every $\rho \in \mathcal{T}$, $(v_u(\rho), v_s(\rho))$ is an orthonormal basis of $T_\rho U$. This metric is γ-Hölder in $\rho \in \mathcal{T}$ since stable and unstable distributions are γ-Hölder for some $\gamma \in (0,1)$.

If $\rho \in \mathcal{T}$ and $n \in \mathbb{Z}$, we denote by $\tilde{J}^{u/s}_n(\rho) \in \mathbb{R}$ the numbers such that

$$d_\rho(F^n)(v_u(\rho)) = \tilde{J}^u_n(\rho) v_u(F^n(\rho)), \quad d_\rho(F^n)(v_s(\rho)) = \tilde{J}^s_n(\rho) v_s(F^n(\rho)).$$

As already observed, $|\tilde{J}^u_n(\rho)| \sim J^u_n(\rho)$ for all n (with constants independent of n). We can also assume that $|\tilde{J}^u_1(\rho)| > |\tilde{J}^s_1(\rho)|$ for all ρ. In the orthonormal basis $(v_u(\rho), v_s(\rho))$ and $(v_u(F^n(\rho), v_s(F^n(\rho)))$, $d_\rho F^n$ has the form

$$\begin{pmatrix} \tilde{J}^u_n(\rho) & 0 \\ 0 & \tilde{J}^s_n(\rho) \end{pmatrix}.$$

Due to the orthonormality of these basis, we have that for the subordinate norms, $\|d_\rho F^n\| = |\tilde{J}^u_n(\rho)|$. Hence, the chain rule implies the following equality for this particular Riemannian metric defined on $\mathcal{T}$:

$$\text{for all } \rho \in \mathcal{T}, \quad \|d_\rho(F^n)\| = |\tilde{J}^u_n(\rho)| = \prod_{i=0}^{n-1} |\tilde{J}^u_1(F^i(\rho)| = \prod_{i=0}^{n-1} \|d_{F^i(\rho)} F\|. \tag{3-27}$$

We now claim that we can extend $|\cdot|$ to a relatively compact neighborhood V of $\mathcal{T}$ such that $\rho \in V \mapsto |\cdot|_\rho$ is still γ-Hölder. To do so, it is enough to extend the coefficients of the metric in a coordinate chart in a γ-Hölder way, which is possible (for instance, by virtue of Corollary 1 in [McShane 1934]), which still defines a nondegenerate 2-form in a sufficiently small neighborhood of $\mathcal{T}$.

We now aim at proving (3-18) for this particular metric. (3-18) will hold in the general case since two continuous metric are always uniformly equivalent in a compact neighborhood of $\mathcal{T}$.

In the following, we assume that ε_1 is small enough so that ρ belongs to the neighborhood of $\mathcal{T}$ in which $|\cdot|$ is defined. Since $\rho \mapsto \|d_\rho F\|_{T_\rho U \to T_{F(\rho)} U}$ is γ-Hölder (in the following, we will drop the subscript in the norm) we have, for all $i \in \{0, \dots, n\}$,

$$\big| \|d_{F^i(\rho')} F\| - \|d_{F^i(\rho)} F\| \big| \leq C d(F^i(\rho'), F^i(\rho))^\gamma \leq C \varepsilon_1 \theta^{\gamma \min(i, n-i)}. \tag{3-28}$$

Using the chain rule and the submultiplicativity of $\|\cdot\|$, we have

$$\|d_{\rho'} F^n\| \leq \prod_{i=0}^{n} \|d_{F^i(\rho')} F\| \leq \prod_{i=0}^{n} \|d_{F^i(\rho)} F\| (1 + C\varepsilon_1 \theta^{\gamma \min(i,n-i)}). \tag{3-29}$$

Eventually, by (3-27) and the fact that $\prod_{i=0}^{n}(1 + C\varepsilon_1\theta^{\gamma \min(i,n-i)})$ is convergent, (3-18) holds. □

As an immediate consequence of this lemma, we get:

[2]Here, we are not concerned by the orientation. It is simply a matter of direction.

Corollary 3.14. *There exist $C > 0$ and $\varepsilon_1 > 0$ (depending only on (U, F)) such that, for all $\rho, \rho' \in \mathcal{T}$ and $n \in \mathbb{N}$:*

(1) *If $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_1$ for all $i \in \{0, \dots, n\}$ then*

$$C^{-1} J_n^u(\rho) \leq J_n^u(\rho') \leq C J_n^u(\rho). \tag{3-30}$$

(2) *If $d(F^{-i}(\rho), F^{-i}(\rho')) \leq \varepsilon_1$ for all $i \in \{0, \dots, n\}$ then*

$$C^{-1} J_{-n}^s(\rho) \leq J_{-n}^s(\rho') \leq C J_{-n}^s(\rho). \tag{3-31}$$

Proof. This is a consequence of the previous lemma and of the fact that, uniformly in ρ and $n \in \mathbb{N}$,

$$\|d_\rho F^n\| \sim J_n^u(\rho),$$
$$\|d_\rho F^{-n}\| \sim J_{-n}^s(\rho). \qquad \square$$

3D. *Regularity of the invariant splitting.* It is known for Anosov diffeomorphisms that stable and unstable distributions are in fact $C^{2-\varepsilon}$ in dimension 2; see [Hurder and Katok 1990]. For our purpose, we need to extend this result to our setting, where the hyperbolic invariant set $\mathcal{T}$ is not the full phase space, but a fractal subset of it. In fact, we will show that one can extend the stable and unstable distributions to an open neighborhood of $\mathcal{T}$ and that these extensions are $C^{1,\beta}$ for some $\beta > 0$. Actually, since what happens outside a fixed neighborhood of $\mathcal{T}$ is irrelevant (one can always use cut-offs), we will prove the following theorem which might be of independent interest.

Theorem 5. *Let us denote by $\mathcal{G}_1(U)$ the Grassmannian bundle of 1-plane in TU. There exists $\beta > 0$ and sections $E_u, E_s : U \to \mathcal{G}_1(U)$ such that:*

- *For every $\rho \in \mathcal{T}$, $E_u(\rho)$ (resp. $E_s(\rho)$) is the unstable (resp. stable) distribution at ρ.*
- *E_u and E_s have regularity $C^{1,\beta}$.*

Remark. Our proof relies on the techniques of [Hirsch and Pugh 1969]. In fact, in [Katok and Hasselblatt 1995, Chapter 19, Section 1.d] the authors show how one can obtain C^1 regularity of the map $\rho \in \mathcal{T} \mapsto E_u(\rho)$ and explain how to prove $C^{1,\beta}$ regularity. Their notion of differentiability on the set $\mathcal{T}$ (which is clearly not open in our case) relies on the existence of linear approximations. Here, we choose to show a slightly different version of this regularity by proving that $\rho \in \mathcal{T} \mapsto E_u(\rho)$ can be obtained as the restriction of a $C^{1,\beta}$ map defined in an open neighborhood of $\mathcal{T}$.

3D1. *Proof of the $C^{1,\beta}$ regularity.*

Preliminaries. We recall that $\mathcal{T}$ is an invariant hyperbolic set for F. Hence, there exists a continuous splitting of $T_\mathcal{T} U$ into stable and unstable spaces $\rho \in \mathcal{T} \mapsto E_s(\rho)$, $\rho \in \mathcal{T} \mapsto E_u(\rho)$. We use a continuous Riemannian metric on $T_\mathcal{T} U$ such that $d_\rho F$ is a contraction from $E_s(\rho) \to E_s(F(\rho))$ and expanding from $E_u(\rho) \to E_u(F(\rho))$, and making $E_u(\rho)$ and $E_s(\rho)$ orthogonal.

Let $\rho \in \mathcal{T} \mapsto e_u(\rho) \in TU$ and $\rho \in \mathcal{T} \mapsto e_s(\rho) \in TU$ be two continuous sections[3] such that, for every $\rho \in \mathcal{T}$,

- $e_u(\rho)$ spans $E_u(\rho)$,
- $e_s(\rho)$ spans $E_s(\rho)$,
- $\|e_u(\rho)\| = 1$, $\|e_s(\rho)\| = 1$.

The matrix representation of $d_\rho F$[4] in these basis is

$$d_\rho F = \begin{pmatrix} \tilde{J}^u(\rho) & 0 \\ 0 & \tilde{J}^s(\rho) \end{pmatrix},$$

with $\nu := \sup_{\rho \in \mathcal{T}} \max[(|\tilde{J}^u(\rho)|)^{-1}, |\tilde{J}^s(\rho)|] < 1$.

We can extend e_u and e_s to U to continuous functions, still denoted by e_u and e_s. Let us consider smooth vector fields v_u and v_s on U approximating e_u and e_s and a smooth Riemannian metric approximating the one considered above. By slightly modifying this vector field, we can assume that, for this new metric, $(v_u(\rho), v_s(\rho))$ is an orthonormal basis for all $\rho \in U$. In these new basis, we now write

$$d_\rho F = \begin{pmatrix} a(\rho) & b(\rho) \\ c(\rho) & d(\rho) \end{pmatrix}.$$

We assume that v_u and v_s are sufficiently close to e_u and e_s to ensure that, for some $\eta > 0$ small enough,

$$\sup_{\rho \in \mathcal{T}} \max(|b(\rho)|, |c(\rho)|) \leq \eta,$$
$$\sup_{\rho \in \mathcal{T}} |d(\rho)| \leq \nu + \eta \leq 1 - 4\eta,$$
$$\inf_{\rho \in \mathcal{T}} |a(\rho)| \geq \nu^{-1} - \eta \geq 1 + 4\eta.$$

We consider an open neighborhood Ω of $\mathcal{T}$ such that the following hold:

$$\sup_{\rho \in \Omega} \max(|b(\rho)|, |c(\rho)|) \leq 2\eta,$$
$$\sup_{\rho \in \Omega} |d(\rho)| \leq \nu + 2\eta \leq 1 - 3\eta,$$
$$\inf_{\rho \in \Omega} |a(\rho)| \geq \nu^{-1} - 2\eta \geq 1 + 3\eta.$$

Our method relies on different uses of the contraction map theorem. We state the fiber contraction theorem of [Hirsch and Pugh 1969, Section 1], which will be used further. We recall that a fixed point x_0 of a continuous map $f : X \to X$ is said to be *attractive* if, for every $x \in X$, $f^n(x) \to x_0$.

[3]Note that there is no problem of orientation in constructing such global sections. Indeed, $\mathcal{T}$ is totally disconnected and hence, one can cover $\mathcal{T}$ by a disjoint union of open sets small enough so that it is possible to construct local sections in each such sets. Since these open sets are disjoint, these local sections allow us to build a global continuous section.

[4]The definition of $\tilde{J}^{u/s}$ may differ from the one of $J_1^{u/s}$ above since we don't work a priori with the same metric.

Theorem 6 (fiber contraction theorem). *Let (X, d) be a metric space and $h : X \to X$ be a map having an attractive fixed point x_0. Let us consider Y another metric space and a family of maps $(g_x : Y \to Y)_{x \in X}$ and denote by H the map*

$$H : (x, y) \in X \times Y \mapsto (h(x), g_x(y)) \in X \times Y.$$

Assume that:

- *H is continuous.*
- *For all $x \in X$, $\limsup_{n \to +\infty} L(g_{h^n(x)}) < 1$, where $L(g_{h^n(x)})$ denotes the best Lipschitz constant for $g_{h^n(x)}$.*
- *y_0 is an attractive fixed point for g_{x_0}.*

Then (x_0, y_0) is an attractive fixed point for H.

In the following, we study the regularity of the unstable distribution. The same holds for the stable distribution by changing the roles of F^{-1} and F.

E_u* *is a fixed point of a contraction. By our assumption on v_u and v_s, there exists a continuous function $\lambda : U \to \mathbb{R}$ such that

$$\mathbb{R}e_u(\rho) = \mathbb{R}(v_u(\rho) + \lambda(\rho) v_s(\rho)).$$

Hence, we will represent the extension of the unstable distribution by a continuous map $\lambda : \Omega \to \mathbb{R}$. Our aim is to show that we can find λ regular enough such that, for $\rho \in \mathcal{T}$,

$$E_u(\rho) = \mathbb{R}(v_u(\rho) + \lambda(\rho) v_s(\rho)).$$

To do so, we will start by constructing λ as a fixed point of a contraction in a nice space. This contraction will be related to invariance properties of the unstable distribution.

First of all, if $\rho' = F(\rho) \in \Omega \cap F(\Omega)$, and if $v = v_u(\rho) + \lambda v_s(\rho)$, then $d_\rho F$ maps v to

$$w = (a(\rho) + \lambda b(\rho)) v_u(\rho') + (c(\rho) + \lambda d(\rho)) v_s(\rho').$$

Hence, the line of $T_\rho U$ represented by λ is sent to the line represented by $t(\rho, \lambda)$ in $T_{\rho'} U$, where

$$t(\rho, \lambda) = \frac{\lambda d(\rho) + c(\rho)}{a(\rho) + \lambda b(\rho)}. \tag{3-32}$$

Set $\Omega_1 = \Omega \cap F(\Omega)$ and let us consider a cut-off function $\chi \in C_c^\infty(\Omega_1)$ such that $0 \le \chi \le 1$ and $\chi \equiv 1$ in a neighborhood of $\mathcal{T}$. Let us introduce the complete metric space

$$X = \{\lambda \in C(\Omega : \mathbb{R}) : \|\lambda\|_\infty \le 1\}$$

and consider the map $T : X \to X$ defined, for $\lambda \in X$ and $\rho' \in \Omega$,

$$(T\lambda)(\rho') = \chi(\rho') t\big(F^{-1}(\rho'), \lambda(F^{-1}(\rho'))\big). \tag{3-33}$$

To see that this is well-defined, first note that F^{-1} is well-defined on $\operatorname{supp} \chi$ and $F^{-1}(\operatorname{supp} \chi) \subset \Omega$. It is clear that if $\lambda \in X$, then $T\lambda$ is continuous. To see that $\|T\lambda\|_\infty \le 1$, it is enough to note that if $\rho \in \Omega$

and $|\lambda| \leq 1$,

$$|t(\rho, \lambda)| \leq \frac{|d(\rho)| + |c(\rho)|}{|a(\rho)| - |b(\rho)|} \leq \frac{1 - 3\eta + 2\eta}{1 + 3\eta - 2\eta} \leq \frac{1-\eta}{1+\eta} < 1.$$

Let us now prove the following.

Proposition 3.15. • *T is a contraction.*

- *If λ_u denotes its unique fixed point, then, for every $\rho \in \mathcal{T}$, we have $E_u(\rho) = \mathbb{R}(v_u(\rho) + \lambda_u(\rho) v_s(\rho))$.*

Proof. Let $\lambda, \mu \in X$. If $\rho' \in \Omega \setminus \operatorname{supp} \chi$, we have $T\mu(\rho') = T\lambda(\rho') = 0$. Now assume that $\rho' \in \operatorname{supp} \chi$ and write $\rho' = F(\rho)$ with $\rho \in \Omega$. Then

$$|T\lambda(\rho') - T\mu(\rho')| = |\chi(\rho')|\,|t(\rho, \lambda(\rho)) - t(\rho, \mu(\rho))| \leq |t(\rho, \lambda(\rho)) - t(\rho, \mu(\rho))|.$$

The map $\lambda \in [-1, 1] \mapsto t(\rho, \lambda)$ is smooth, so we can write

$$\|T\lambda - T\mu\|_\infty \leq \sup_{\rho' \in \operatorname{supp} \chi} |T\lambda(\rho') - T\mu(\rho')| \leq \sup_{\Omega \times [-1,1]} |\partial_\lambda t| \times \|\lambda - \mu\|_\infty.$$

It is then enough to show that $\sup_{\Omega \times [-1,1]} |\partial_\lambda t| < 1$. For $(\rho, \lambda) \in \Omega \times [-1, 1]$, we have

$$\partial_\lambda t(\rho, \lambda) = \frac{d(\rho)}{a(\rho) + \lambda b(\rho)} - b(\rho) \frac{\lambda d(\rho) + c(\rho)}{(a(\rho) + \lambda b(\rho))^2}. \tag{3-34}$$

Hence, we can control

$$|\partial_\lambda t(\rho, \lambda)| \leq \frac{1 - 3\eta}{1 + \eta} + \eta \frac{1 - \eta}{(1 + \eta)^2} = \kappa_\eta < 1$$

if η is small enough. This demonstrates that T is a contraction.

As a consequence, T has a unique fixed point, λ_u. We let $v(\rho) = v_u(\rho) + \lambda_u(\rho) v_s(\rho)$. We want to show that $v(\rho) \in \mathbb{R} e_u(\rho)$ for $\rho \in \mathcal{T}$ (recall that $e_u : U \to TU$ is continuous and that $e_u(\rho)$ spans $E_u(\rho)$ if $\rho \in \mathcal{T}$). Since $\chi = 1$ on $\mathcal{T}$, we see by the definition of T that, for every $\rho \in \mathcal{T}$,

$$d_\rho F(v(\rho)) \in \mathbb{R} v(F(\rho)). \tag{3-35}$$

If v_u is sufficiently close to e_u, we can find a continuous and bounded function μ such that

$$\mathbb{R} v(x) = \mathbb{R}(e_u(x) + \mu(x) e_s(x)).$$

From (3-35), if $\rho' = F(\rho) \in \mathcal{T}$,

$$d_\rho F(e_u(\rho) + \mu(\rho) e_s(\rho)) = \tilde{J}_1^u(\rho) \left(e_u(\rho') + \mu(\rho) \frac{\tilde{J}_1^s(\rho)}{\tilde{J}_1^u(\rho)} e_s(\rho') \right) \in \mathbb{R}(e_u(\rho') + \mu(\rho') e_s(\rho')).$$

This implies the equality

$$\mu(\rho') = \mu(\rho) \frac{\tilde{J}_1^s(\rho)}{\tilde{J}_1^u(\rho)}. \tag{3-36}$$

This equality implies that $\mu = 0$ on $\mathcal{T}$, and hence $v = e_u$ on $\mathcal{T}$, as expected. ☐

Remark. As long as $\rho' \in \{\chi = 1\}$, the vector field $v(\rho') = v_u(\rho') + \lambda(\rho')v_s(\rho')$ is invariant by dF. When $\rho' \in W_u(\rho) \cap \{\chi = 1\}$ for some $\rho \in \mathcal{T}$, we will see below that the direction given by $v(\rho')$ coincides with the tangent space to $W_u(\rho)$, namely $T_{\rho'}W_u(\rho) = \mathbb{R}v(\rho')$. When $\rho' \notin \bigcup_{\rho\in\mathcal{T}} W_u(\rho)$, there exists $n \in \mathbb{N}$ such that $F^{-n}(\rho') \notin \operatorname{supp}\chi$. Hence, $\lambda_u(\rho')$ is given by an explicit expression obtained by iterating the fixed-point formula.

Differentiability of λ_u. We go on by showing that λ is C^1 by adapting the method of [Hirsch and Pugh 1969]. We now introduce the Banach space Y of bounded continuous sections $\alpha : \Omega \to T^*\Omega$. We will use the norm on $T^*\Omega$ adapted to the metric on $T\Omega$; namely, if $\alpha \in Y$,

$$\|\alpha\|_Y = \sup_{\rho\in\Omega} \sup_{v\in T_\rho\Omega, v\neq 0} \frac{|\alpha(\rho)(v)|}{\|v\|_{T_\rho\Omega}}.$$

For $\lambda \in X$, let us introduce the map $G_\lambda : Y \to Y$, defined as follows. For $\alpha \in Y$ and $\rho' \in \Omega$,

$$(G_\lambda\alpha)(\rho') = \chi(\rho')[d_\rho t(\rho, \lambda(\rho)) + \partial_\lambda t(\rho, \lambda(\rho))\alpha(\rho)] \circ (d_\rho F)^{-1} + t(\rho, \lambda(\rho))\, d_{\rho'}\chi, \tag{3-37}$$

with $\rho = F^{-1}(\rho')$, which is well-defined since $\rho \in \Omega$ if $\rho' \in \operatorname{supp}(\chi)$. G_λ is constructed to satisfy, for $\lambda \in X$, if λ is C^1, then the following relation holds:

$$G_\lambda(d\lambda) = d(T\lambda). \tag{3-38}$$

Let us first state the key tool to show the differentiability of λ_u.

Proposition 3.16. *For every $\lambda \in X$, G_λ is a contraction with Lipschitz constant L_λ satisfying*

$$\sup_{\lambda\in X} L_\lambda < 1.$$

Before proving it, let us show how it leads us to:

Proposition 3.17. *We know λ_u is C^1.*

Proof. We use the contraction fiber theorem. Let α_u be the unique fixed point of G_{λ_u}. The map

$$H : (\lambda, \alpha) \in X \times Y \mapsto (T\lambda, G_\lambda\alpha) \in X \times Y$$

is continuous and the previous proposition shows that, for every $\lambda \in X$, $\sup_n L(G_{T^n\lambda}) < 1$. The contraction fiber theorem implies that (λ_u, α_u) is an attractive fixed point for H.

Let $\lambda \in X$ be C^1. Hence, $H^n(\lambda, d\lambda) \to (\lambda_u, \alpha_u)$. But $H^n(\lambda, d\lambda) = (T^n\lambda, \alpha_n)$, with

$$\alpha_n = G_{T^{n-1}\lambda} \circ \cdots \circ G_\lambda d\lambda.$$

It is clear that if $\lambda \in C^1$, so is $T\lambda$ and an iterative use of (3-38) implies that $\alpha_n = d(T^n\lambda)$. This shows that λ_u is C^1 and $d\lambda_u = \alpha_u$. $\square$

Let us now prove Proposition 3.16.

Proof. Let $\lambda \in X$ and fix $\alpha, \beta \in Y$. It is of course enough to control $\|G_\lambda\alpha(\rho') - G_\lambda\beta(\rho')\|$ for $\rho' \in \operatorname{supp}(\chi)$ since both $G_\lambda\alpha$ and $G_\lambda\beta$ vanish outside. Let us fix $\rho' = F(\rho) \in \operatorname{supp}(\chi)$.

$G_\lambda\alpha(\rho') - G_\lambda\beta(\rho')$ is given by

$$\chi(\rho')\partial_\lambda t(\rho, \lambda(\rho))[\alpha(\rho) - \beta(\rho)] \circ (d_\rho F)^{-1},$$

so it is enough to control $\partial_\lambda t(\rho, \lambda(\rho))\gamma(\rho) \circ (d_\rho F)^{-1}$ for $\gamma = \alpha - \beta$. With the precise expression of $\partial_\lambda t(\rho, \lambda(\rho))$ given by (3-34), we can estimate

$$|\partial_\lambda t(\rho, \lambda(\rho))| = \frac{|d(\rho)|}{|a(\rho) + \lambda(\rho)b(\rho)|} + O_\nu(\eta) = \frac{|d(\rho)|}{|a(\rho)|} + O_\nu(\eta).$$

(By the notation $O_\nu(\eta)$, we mean that this term is bounded by $C\eta$ where C is a constant depending only on ν and (F, U)).

Moreover, we have

$$\|(d_\rho F)^{-1}\| = \max\left(\frac{1}{a(\rho)}, \frac{1}{d(\rho)}\right) + O_\nu(\eta) = \frac{1}{d(\rho)} + O_\nu(\eta).$$

Hence,

$$\|\partial_\lambda t(\rho, \lambda(\rho))\gamma(\rho) \circ (d_\rho F)^{-1}\| \le \left(\frac{1}{a(\rho)} + O_\nu(\eta)\right)\|\gamma(\rho)\| \le (\nu + O_\nu(\eta))\|\gamma\|_Y.$$

Hence, if η is small enough, the proposition is proved. □

Hölder regularity of α_u. In fact, as explained at the end of [Katok and Hasselblatt 1995, Chapter 19, Section 1.d], we can improve the C^1 regularity.

To deal with Hölder regularity of sections $\alpha : \Omega \to T^*\Omega$, we will simply evaluate the distance between $\alpha(\rho_1)$ and $\alpha(\rho_2)$ for $\rho_1, \rho_2 \in \Omega$ using the natural identification $T^*\Omega = \Omega \times (\mathbb{R}^2)^*$, where we see $\alpha(\rho_1)$ as an element of $(\mathbb{R}^2)^*$. This allows us to write $\alpha(\rho_1) - \alpha(\rho_2)$ and compute $\|\alpha(\rho_1) - \alpha(\rho_2)\|$, where $\|\cdot\|$ is a norm on $(\mathbb{R}^2)^*$. There exists $C > 0$ such that, for every $\alpha \in Y$, $\sup_{\rho\in\Omega} \|\alpha(\rho)\| \le C\|\alpha\|_Y$.

Let us introduce μ a Lipschitz constant for F^{-1} on Ω and an exponent $\beta > 0$ such that

$$\nu\mu^\beta < 1. \tag{3-39}$$

This condition is called a *bunching condition* in [Katok and Hasselblatt 1995, Chapter 19, Section1.d]. Such a β exists. We will then show the following, which finally concludes the proof of Theorem 5.

Proposition 3.18. *α_u is β-Hölder, that is to say, λ_u is $C^{1,\beta}$.*

Proof. Let us introduce

$$Y^\beta := \{\alpha \in Y : \alpha \text{ is } \beta\text{-Hölder}\}.$$

Let us consider some $\varepsilon > 0$ to be determined later and we equip Y^β with the norm

$$\|\alpha\|_{Y^\beta} = \|\alpha\|_Y + \varepsilon\|\alpha\|_\beta, \quad \|\alpha\|_\beta = \sup_{\rho_1\neq\rho_2} \frac{\|\alpha(\rho_1) - \alpha(\rho_2)\|}{d(\rho_1, \rho_2)^\beta}.$$

The map $T : X \to X$ defined by (3-33) actually maps $X \cap C^1(\Omega, \mathbb{R})$ to $X \cap C^1(\Omega, \mathbb{R})$. Moreover, our previous results have proved that λ_u is an attractive fixed point for T in $X \cap C^1(\Omega, \mathbb{R})$, where $X \cap C^1(\Omega, \mathbb{R})$ is now equipped with the C^1 norm. For $\lambda \in X$ and $\alpha \in Y$, we can write

$$G_\lambda \alpha = \gamma_\lambda + \widetilde{G}_\lambda \alpha,$$

where, for $\rho' = F(\rho) \in \operatorname{supp} \chi$,

$$\gamma_\lambda(\rho') = \chi(\rho') d_\rho t(\rho, \lambda(\rho)) + t(\rho, \lambda(\rho)) d_{\rho'} \chi,$$
$$\widetilde{G}_\lambda \alpha(\rho') = \chi(\rho') \partial_\lambda t(\rho, \lambda(\rho)) \alpha(\rho) \circ (d_\rho F)^{-1}.$$

We state here some obvious facts on γ_λ and $\widetilde{G}_\lambda$:

- $C_1 := \sup_{\lambda \in X} \|\gamma_\lambda\|_\infty < +\infty$.
- If $\lambda \in X \cap C^1(\Omega, \mathbb{R})$, γ_λ is also C^1.
- According to Proposition 3.16; $\widetilde{G}_\lambda : Y \to Y$ is a contraction with Lipschitz constant L_λ and $\nu_1 := \sup_{\lambda \in X} L_\lambda < 1$.
- If $\lambda \in X \cap C^1(\Omega, \mathbb{R})$ and α is β-Hölder, then $\widetilde{G}_\lambda \alpha$ is β-Hölder.

If $M > C_1/(1-\nu_1)$ and $\lambda \in X \cap C^1(\Omega, \mathbb{R})$, then $\|d\lambda\|_Y \le M$ implies $\|d(T\lambda)\|_Y \le M$. Indeed, we have

$$\|d(T\lambda)\|_Y = \|G_\lambda(d\lambda)\|_Y = \|\gamma_\lambda + \widetilde{G}_\lambda d\lambda\|_Y \le C_1 + \nu_1 M \le M.$$

Hence, we introduce the complete metric space

$$X' = \{\lambda \in X \cap C^1(\Omega, \mathbb{R}) : \|d\lambda\|_Y \le M\}, \tag{3-40}$$

$T(X') \subset X'$ and λ_u is an attractive fixed point for (X', T).

We now wish to apply the fiber contraction theorem to

$$H_\beta : (\lambda, \alpha) \in X' \times Y^\beta \mapsto (T\lambda, G_\lambda \alpha) \in X' \times Y^\beta.$$

To do so, we need to show that, for every $\lambda \in X'$, $G_\lambda : Y^\beta \to Y^\beta$ is a contraction and find a uniform estimate for the Lipschitz constants.

Let's consider $\alpha_1, \alpha_2 \in Y^\beta$ and set $\gamma = \alpha_1 - \alpha_2$. We want to estimate the Y^β norm of $\widetilde{G}_\lambda \gamma$. We already know that $\|\widetilde{G}_\lambda \gamma\|_Y \le \nu_1 \|\gamma\|_Y$. Take $\rho_1', \rho_2' \in \Omega$ and let's estimate $\|\widetilde{G}_\lambda \gamma(\rho_1') - \widetilde{G}_\lambda \gamma(\rho_2')\|$. We distinguish three cases:

- $\rho_1', \rho_2' \notin \operatorname{supp} \chi$. There is nothing to write.
- $\rho_1' \in \operatorname{supp} \chi$, $\rho_2' \notin \Omega \cap F(\Omega)$. In this case, $d(\rho_1', \rho_2') \ge \delta > 0$, where δ is the distance between $\operatorname{supp} \chi$ and $(\Omega \cap F(\Omega))^c$. Hence,

$$\frac{\|\widetilde{G}_\lambda \gamma(\rho_1') - \widetilde{G}_\lambda(\rho_2')\|}{d(\rho_1', \rho_2')^\beta} \le \delta^{-\beta} \|\widetilde{G}_\lambda \gamma(\rho_1')\| \le \delta^{-\beta} C \|\widetilde{G}_\lambda \gamma\|_Y \le \nu_1 \delta^{-\beta} C \|\gamma\|_Y.$$

- $\rho_1', \rho_2' \in \Omega \cap F(\Omega)$. Let's write $\rho_1' = F(\rho_1)$, $\rho_2' = F(\rho_2)$ and note that $d(\rho_1, \rho_2) \leq \mu d(\rho_1', \rho_2')$. Then

$$\begin{aligned}\widetilde{G}_\lambda \gamma(\rho_1') - \widetilde{G}_\lambda \gamma(\rho_2') = \chi(\rho_1')\partial_\lambda t(\rho_1, \lambda(\rho_1))[\gamma(\rho_1) - \gamma(\rho_2)] \circ (d_{\rho_1} F)^{-1} && (*)\\ + [\chi(\rho_1')\partial_\lambda t(\rho_1, \lambda(\rho_1)) - \chi(\rho_2')\partial_\lambda t(\rho_2, \lambda(\rho_2))]\gamma(\rho_2) \circ (d_{\rho_1} F)^{-1} && (**)\\ + \chi(\rho_2')\partial_\lambda t(\rho_2, \lambda(\rho_2))\gamma(\rho_2) \circ [(d_{\rho_1} F)^{-1} - (d_{\rho_2} F)^{-1}]. && (***)\end{aligned}$$

To handle the last two terms $(**)$ and $(***)$, we notice that $\rho' \in \Omega \cap F(\Omega) \mapsto \chi(\rho')\partial_\lambda t(\rho, \lambda(\rho))$ is Lipschitz since λ is C^1, with Lipschitz constant which can be chosen uniform for $\lambda \in X'$. The same is true for $\rho \mapsto d_\rho F^{-1}$. Hence, there exists a uniform constant $C > 0$ such that

$$\|(**) + (***)\| \leq C d(\rho_1', \rho_2')^\beta \|\gamma\|_Y.$$

To deal with the first term $(*)$, we recall that by previous computations,

$$|\chi(\rho')\partial_\lambda t(\rho, \lambda(\rho))| \cdot \|(d_\rho F)^{-1}\| \leq \nu + O_\nu(\eta).$$

As consequence, we have

$$\|(*)\| \leq (\nu + O_\nu(\eta))\|\gamma\|_\beta \, d(\rho_1, \rho_2)^\beta \leq (\nu + O_\nu(\eta))\mu^\beta \|\gamma\|_\beta \, d(\rho_1', \rho_2')^\beta.$$

Henceforth, if η is small enough, so that $\nu_2 := (\nu + O_\nu(\eta))\mu^\beta < 1$,

$$\|H_\lambda \gamma\|_\beta \leq \nu_2 \|\gamma\|_\beta + C\|\gamma\|_Y.$$

Eventually,

$$\begin{aligned}\|\widetilde{G}_\lambda \gamma\|_{Y^\beta} &\leq \nu_1 \|\gamma\|_Y + \varepsilon(\nu_2 \|\gamma\|_\beta + C\|\gamma\|_Y)\\ &\leq (\nu_1 + \varepsilon C)\|\gamma\|_Y + \nu_2 \varepsilon \|\gamma\|_\beta \leq \nu_3 \|\gamma\|_{Y^\beta},\end{aligned}$$

where $\nu_3 = \max(\nu_1 + \varepsilon C, \nu_2) < 1$ if ε is small enough.

The fiber contraction theorem applies and says that (λ_u, α_u) is an attractive fixed point for H_β. We conclude as previously: Consider $\lambda \in C^{1,\beta}(\Omega, \mathbb{R}) \cap X'$ so that $(\lambda, d\lambda) \in X' \times Y^\beta$. Then $H_\beta^n(\lambda, d\lambda) = (T^n \lambda, dT^n \lambda) \to (\lambda_u, \alpha_u)$ in $X' \times Y^\beta$, which ensures that α_u is β-Hölder. $\square$

3D2. *Regularity of the stable and unstable leaves.* Once we've extended the unstable distribution to an open neighborhood of $\mathcal{T}$, we take advantage of the fact that these distributions are one-dimensional to integrate the vector field defined by their unit vector.

We set $E_u(\rho) = \mathbb{R}(v_u(\rho) + \lambda_u(\rho) v_s(\rho))$. Recall that in a compact neighborhood of $\mathcal{T}$, the relation $d_\rho F(E_u(\rho)) = E_u(F(\rho))$ is valid due to the definition of λ_u as the fixed point of T defined in (3-33). T^*U is equipped with a smooth Riemannian metric such that dF^{-1} is a contraction on $E_u(\rho)$ for $\rho \in \mathcal{T}$ and hence, in a compact neighborhood of $\mathcal{T}$, this is also true. We can consider the vector field

$$\rho \in U \mapsto e_u(\rho),$$

where $e_u(\rho)$ is a unit vector spanning $E_u(\rho)$. By our previous result, this vector field is $C^{1,\beta}$ and if ρ lies in a sufficiently small neighborhood of $\mathcal{T}$, then $d_\rho(F^{-1})(e_u(\rho)) = \tilde{J}^u(\rho) e_u(F^{-1}(\rho))$, where $|\tilde{J}^u(\rho)| \leq \nu < 1$.

We denote by $\varphi_u^t(\rho)$ the flow generated by $e_u(\rho)$ and we will show that one can identify the unstable manifolds and the flow lines of e_u in a small neighborhood of $\mathcal{T}$.

Proposition 3.19. *There exists t_0 such that, for every $\rho \in \mathcal{T}$, we have $\{\varphi_u^t(\rho) : |t| \le t_0\} \subset W_u(\rho)$.*

Proof. Consider t_0 sufficiently small that $|\tilde{J}^u(\varphi_u^t(\rho))| \le \nu < 1$ for $\rho \in \mathcal{T},\ t \in [-t_0, t_0]$. For $(t, \rho) \in \mathbb{R} \times U$, set $\mu(t, \rho) = \int_0^t \tilde{J}^u(\varphi_u^s(\rho))\, ds$ and we claim that for t_0 small enough, if $|t| \le t_0$,

$$F^{-1}(\varphi_u^t(\rho)) = \varphi_u^{\mu(t,\rho)}(F^{-1}(\rho)).$$

Indeed, in $t = 0$, both are equal to $F^{-1}(\rho)$ and a quick computation shows that both satisfy the ODE

$$\frac{d}{dt} Y(t) = J^u(\varphi_u^t(\rho)) e_u(Y(t)).$$

As a consequence, by induction, we see that one can write, for $n \in \mathbb{N}$,

$$F^{-n}(\varphi_u^t(\rho)) = \varphi_u^{\mu_n(t,\rho)}(F^{-n}(\rho)),$$

where μ_n is defined by induction by $\mu_{n+1}(t, \rho) = \mu(\mu_n(t, \rho), F^{-n}(\rho))$. Hence, if $|t| \le t_0$ and $\rho \in \mathcal{T}$, we see that $\mu_n(t, \rho)$ stays in $[-t_0, t_0]$ and moreover $|\mu_n(t, \rho)| \le \nu^n |t|$. We then see that if $|t| \le t_0$ and $\rho \in \mathcal{T}$,

$$d\big(F^{-n}(\varphi_u^t(\rho)), F^{-n}(\rho)\big) = d\big(\varphi_u^{\mu_n(t,\rho)}(F^{-n}(\rho)), F^{-n}(\rho)\big) \le C|\mu_n(t, \rho)| \le C\nu^n.$$

This shows that $\varphi_u^t(\rho)$ belongs to the global unstable manifold at ρ, and hence, if t_0 is small enough, $\varphi_u^t(\rho)$ belongs to the local manifold $W_u(\rho)$ and t_0 can be chosen uniformly with respect to $\rho \in \mathcal{T}$. □

Since the regularity of the unstable distributions implies the same regularity for the flow φ_u^t (see Lemma A.1 in the Appendix), we deduce that, up to reducing the size of the local unstable manifolds, these local unstable manifolds $W_u(\rho)$ depend $C^{1,\beta}$ on the base point $\rho \in \mathcal{T}$. We'll also use this proposition to show the same regularity for holonomy maps. Suppose that ε_0 is small enough. We know that if $\rho_1, \rho_2 \in \mathcal{T}$ satisfy $d(\rho_1, \rho_2) \le \varepsilon_0$, then $W_u(\rho_2) \cap W_s(\rho_1)$ consists of exactly one point. Let's denote it by $H_{\rho_1}^u(\rho_2)$.

Finally, we define the holonomy map

$$H_{\rho_1,\rho_2}^u : \rho_3 \in W_s(\rho_2) \cap \mathcal{T} \mapsto H_{\rho_1}^u(\rho_3) \in W_s(\rho_1) \cap \mathcal{T}.$$

Lemma 3.20. *If ε_0 is small enough, for every $\rho_1 \in \mathcal{T}$, the map*

$$H_{\rho_1}^u : \mathcal{T} \cap B(\rho_1, \varepsilon_0) \to W_s(\rho_1) \cap \mathcal{T}$$

is the restriction of a map $\widetilde{H}_{\rho_1}^u : B(\rho_1, \varepsilon_0) \to W_s(\rho_1)$ which is $C^{1,\beta}$.

Proof. Let $\rho_1 \in \mathcal{T}$. As in the proof of Lemma 3.13, consider a smooth chart $\kappa : U_1 \to V_1 \subset \mathbb{R}^2$, $\rho_1 \in U_1$, $0 \in V_1$ such that:

- $\kappa(\rho_1) = (0, 0)$.
- $\kappa(W_s(\rho_1) \cap U_1) = \{(0, s) : s \in \mathbb{R}\} \cap V_1$.
- $\kappa(W_u(\rho_1) \cap U_1) = \{(u, 0) : u \in \mathbb{R}\} \cap V_1$.
- $d_{\rho_1}\kappa(e_u(\rho_1)) = (1, 0)$.

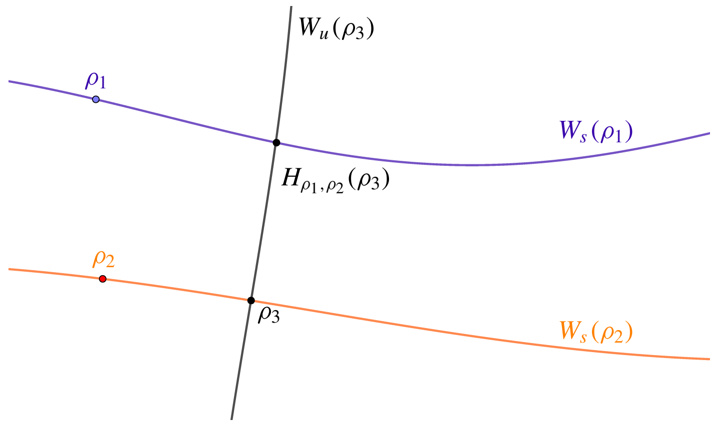

Figure 6. The holonomy map.

We now work in this chart V_1 and denote by $\Phi^t = \kappa \circ \varphi_u^t \circ \kappa^{-1}$ the flow in this chart, well-defined for t small enough. Consider the map

$$\psi(u, s) = \Phi^u(0, s);$$

ψ is $C^{1,\beta}$ and $d_0\psi = \mathrm{I}_2$. By the inverse function theorem, ψ is a local diffeomorphism between neighborhoods of 0:

$$\psi : V_2 \to V_2'.$$

Since $d_{(u,s)}(\psi^{-1}) = (d_{\psi^{-1}(u,s)}\psi)^{-1}$, we know ψ^{-1} is $C^{1,\beta}$. We now consider

$$\kappa_0 = \psi^{-1} \circ \kappa : \kappa^{-1}(V_2) := U_2 \to V_2'$$

and observe that:

- $\kappa_0(W_s(\rho_1) \cap U_2) = \{(0, s), s \in \mathbb{R}\} \cap V_2'$.
- $\kappa_0 \circ \varphi_u^t \circ \kappa_0^{-1}(u, s)) = (u + t, s)$. In other words κ_0 rectifies the unstable manifolds.

Armed with these facts, we define

$$\widetilde{H}^u_{\rho_1} : U_2 \to W_s(\rho_1), \qquad \widetilde{H}^u_{\rho_1} = \kappa_0^{-1} \circ \pi_s \circ \kappa_0,$$

where $\pi_s(u, s) = (0, s)$. $\widetilde{H}^u_{\rho_1}$ is $C^{1,\beta}$. We assume that $B(0, \varepsilon_0) \subset U_1$. Let us check that $\widetilde{H}^u_{\rho_1}$ extends the holonomy map in $B(\rho_1, \varepsilon_0)$ (if ε_0 is small enough). Let $\rho_2 \in \mathcal{T} \cap B(\rho_1, \varepsilon_0)$ and let $\rho_2' = \widetilde{H}^u_{\rho_1}(\rho_2)$. By the definition of $\widetilde{H}^u_{\rho_1}$, ρ_2' can be written $\rho_2' = \varphi_u^t(\rho_1)$ and hence, if ε_0 is small enough, $\rho_2' \in W_u(\rho_1)$. Since, $\rho_2' \in W_s(\rho_2)$, we see that $\rho_2' = H^u_{\rho_1}(\rho_2)$. $\square$

Note that by compactness, ε_0 can be chosen uniformly in $\rho_1 \in \mathcal{T}$ and the $C^{1,\beta}$ norms of $\widetilde{H}^u_{\rho_1}$ are uniform. As a corollary, we get the following:

Corollary 3.21. *Suppose that ε_0 is small enough. Then, the holonomy maps, defined for $\rho_1, \rho_2 \in \mathcal{T}$ with $d(\rho_1, \rho_2) \le \varepsilon_0$,*

$$H^u_{\rho_1,\rho_2} : W_s(\rho_2) \cap \mathcal{T} \to W_s(\rho_1) \cap \mathcal{T}$$

are the restrictions of $C^{1,\beta}$: $\widetilde{H}^u_{\rho_1,\rho_2} : W_s(\rho_1) \to W_s(\rho_2)$, with $C^{1,\beta}$ norms uniform in ρ_1, ρ_2. See Figure 6.

3E. ***Adapted charts.*** We construct charts in which the unstable manifolds are close to horizontal lines. These charts will be used at different places and their existence relies on the $C^{1+\beta}$ regularity of the unstable distribution.

Weak version. We start with a weak version of these charts.

Lemma 3.22. *Suppose that $C > 0$ is a fixed global constant and ε_0 is chosen small enough. For every $\rho_0 \in \mathcal{T}$, there exists a canonical transformation*

$$\kappa_0 : U'_{\rho_0} \to V'_{\rho_0} \subset \mathbb{R}^2$$

satisfying (we denote by (y, η) the variable in $\mathbb{R}^2$):

(1) $B(\rho_0, C\varepsilon_0) \subset U'_{\rho_0}$.

(2) $\kappa_0(\rho_0) = 0,\ d_{\rho_0}\kappa_0(E_u(\rho_0)) = \mathbb{R} \times \{0\},\ d_{\rho_0}\kappa_0(E_s(x)) = \{0\} \times \mathbb{R}$.

(3) *The image of the unstable manifold $W_u(\rho_0) \cap U'_{\rho_0}$ is exactly $\{(y, 0) : y \in \mathbb{R}\} \cap V'_{\rho_0}$.*

Moreover, for every N, the C^N norms of κ_0 are bounded uniformly with respect to $\rho_0 \in \mathcal{T}$.

Remark. The difference with the charts used in the proof of Lemma 3.13 is that we require κ_0 to be a smooth canonical transformation.

Proof. $W_u(\rho_0)$ is a C^∞ manifold; hence there exists a C^∞ defining function η defined in a neighborhood ρ_0; namely, $d_{\rho_0}\eta \neq 0$ and $W_u(\rho_0) = \{\eta = 0\}$ locally near ρ_0. Darboux's theorem gives a function y defined in a neighborhood of ρ_0 such that (y, η) forms a system of symplectic coordinates. We can assume that $y(\rho_0) = 0$. If $\kappa(\rho) = (y, \eta)$, the third point is satisfied by assumption on η and we need to ensure that $d_{\rho_0}\kappa(E_s(\rho_0)) = \{0\} \times \mathbb{R}$ by modifying η in a symplectic way.

Assume that $d_{\rho_0}\kappa(E_s(\rho_0)) = \mathbb{R}\,{}^t(a, 1)$. The symplectic matrix

$$A = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$$

maps the basis $({}^t(1, 0), {}^t(a, 1))$ to the canonical basis of $\mathbb{R}^2$ and we can set $\kappa_0 := A \circ \kappa$, which is the required canonical transformation, defined in a small neighborhood U'_{ρ_0} of ρ_0.

We can ensure that $B(\rho_0, C\varepsilon_0) \subset U'_{\rho_0}$ for ε_0 small enough and the uniformity of the C^N norms of κ thanks to the compactness of $\mathcal{T}$ and the fact that the unstable distribution depends continuously on $\rho_0 \in \mathcal{T}$. □

Straightened version. We now straighten the unstable manifolds in a stronger version of the previous charts. The construction and the use of these charts is similar to [Dyatlov et al. 2022, Lemma 2.3].

Lemma 3.23. *Suppose that ε_0 is chosen small enough. For every $\rho_0 \in \mathcal{T}$ there exists a canonical transformation*

$$\kappa = \kappa_{\rho_0} : U_{\rho_0} \subset U \to V_{\rho_0} \subset \mathbb{R}^2$$

satisfying (we denote by (y, η) the variable in $\mathbb{R}^2$):

(1) $B(\rho_0, 2\varepsilon_0) \subset U_{\rho_0}$.

(2) $\kappa(\rho_0) = 0$, $d_{\rho_0}\kappa(E_u(\rho_0)) = \mathbb{R} \times \{0\}$, $d_{\rho_0}\kappa(E_s(\rho_0)) = \{0\} \times \mathbb{R}$.

(3) *The images of the unstable manifolds $W_u(\rho)$, $\rho \in U_{\rho_0} \cap \mathcal{T}$, are described by*

$$\kappa(W_u(\rho) \cap U_{\rho_0}) = \{(y, g(y, \zeta(\rho))) : y \in \Omega\}, \tag{3-41}$$

where $\Omega \subset \mathbb{R}$ is an open set, $\zeta : U_{\rho_0} \to \mathbb{R}$ is $C^{1+\beta}$, $g : \Omega \times I \to \mathbb{R}$ is $C^{1+\beta}$ (where I is a neighborhood of $\zeta(U_{\rho_0})$) and they satisfy:

(a) *ζ is constant on the unstable manifolds.*

(b) $\zeta(\rho_0) = 0$, $g(y, 0) = 0$.

(c) $g(0, \zeta) = \zeta$.

(d) $\partial_\zeta g(y, 0) = 1$.

The derivatives of κ_{ρ_0} and the $C^{1+\beta}$ norms of g, ζ are bounded uniformly in ρ_0.

Remark. The most important condition, which will be used later on, is the last one: it makes the unstable manifolds very close to horizontal lines. The model situation we expect is when the unstable distribution is constant and horizontal.

Proof. Around a point $\rho_0 \in \mathcal{T}$, we work in the charts given by Lemma 3.22: $\kappa_0 : U'_{\rho_0} \to V'_{\rho_0}$. We recall that the unstable distribution is given by the restriction of a $C^{1+\beta}$ vector field e_u. If U'_{ρ_0} is a sufficiently small neighborhood of ρ_0, we can write, for $\rho \in U'_{\rho_0}$,

$$d_\rho \kappa_0(e_u(\rho)) \in \mathbb{R}\tilde{e}_u(\rho), \quad \text{with } \tilde{e}_u(\rho) = {}^t(1, f_0(\rho)), \tag{3-42}$$

where $f_0 : U'_{\rho_0} \to \mathbb{R}$ is a $C^{1+\beta}$ function which is nothing but the slope of the unstable direction in the chart κ_0. In the (y, η)-variable, we still write $f_0(\rho) = f_0(y, \eta)$ and we observe that due to the assumption on κ_0, we have

$$f_0(y, 0) = 0, \quad (y, 0) \in V'_{\rho_0}.$$

We consider $\Phi^t(y, \eta)$, the flow generated by the vector field $\tilde{e}_u$. Due to the form of $\tilde{e}_u$, we can write

$$\Phi^t(y, \eta) = (y + t, Z^t(y, \eta)).$$

The reparametrization made in (3-42) does not change the flow lines of the vector field $(\kappa_0)_* e_u$. In particular, by virtue of Proposition 3.19, they coincide locally with the unstable manifolds. More precisely, if we set

$$g_0(y, \eta) := Z^y(0, \eta)$$

(see Figure 7) then, for $(0, \eta) = \kappa_0(\rho) \in \kappa_0(\mathcal{T} \cap W_s(\rho_0))$,

$$\kappa_0(W_u(\rho)) \cap \{|y| < y_0\} = \{(y, g_0(y, \eta)) : |y| < y_0\}$$

for some y_0 small enough (which can be chosen uniformly in ρ_0). To define ζ, we go back up the flow: Suppose that $\rho \in U'_{\rho_0}$ and write $\kappa_0(\rho) = (y, \eta)$ and assume $|y| < y_0$. We set

$$\zeta(\rho) := Z^{-y}(y, \eta).$$

To say it differently, $\kappa_0(W_u(\rho)$ intersects the axis $\{y = 0\}$ at $(0, \zeta(\rho))$.

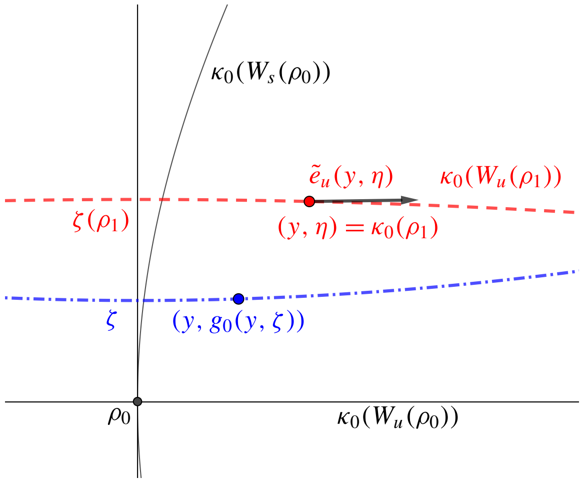

Figure 7. The definitions of g_0 and ζ use the flow generated by $\tilde{e}_u$.

We know ζ and g_0 are $C^{1+\beta}$, their $C^{1+\beta}$ norms depend uniformly on ρ_0 and they satisfy:

- By definition, ζ is constant on the flow lines, and hence, on the unstable manifolds $W_u(\rho)$ if $\rho \in \mathcal{T} \cap U'_{\rho_0} \cap \{|y| < y_0\}$.
- $\zeta(\rho_0) = 0$.
- Since $f_0(y, 0) = 0$, we have $Z^y(0, 0) = 0$ and hence $g_0(y, 0) = 0$.
- Since $Z^0(0, \eta) = \eta$, we have $g_0(0, \eta) = \eta$.

However, at this stage, the last condition ($\partial_\zeta g_0(y, 0) = 1$) is not satisfied by g_0 and we need to modify the chart. To do so, we'll make use of the following lemma, which is proved in Section A2 in the Appendix.

Lemma 3.24. *The map $y \in \{|y| < y_0\} \mapsto \partial_\eta f_0(y, 0)$ is smooth, with C^N norms bounded uniformly in ρ_0.*

We first show that this lemma implies that $y \in \{|y| < y_0\} \mapsto \partial_\eta g_0(y, 0)$ is smooth. Indeed, due to the $C^{1+\beta}$ regularity of E_u, $(t, y, \eta) \mapsto Z^t(y, \eta)$ is C^1 and satisfies

$$\frac{d}{dt}\partial_\eta Z^t(y, \eta) = \partial_\eta f_0(y + t, Z^t(y, \eta)).$$

Setting $(y, \eta) = (0, 0)$, we have

$$\frac{d}{dt}\partial_\eta Z^t(0, 0) = \partial_\eta f_0(t, 0).$$

This exactly says that $y \mapsto \partial_\eta g_0(y, 0)$ is C^1 and has $\partial_\eta f_0(y, 0)$ as derivative with respect to y and hence $y \mapsto \partial_\eta g_0(y, 0)$ is smooth, as required.

Due to the relation $g_0(0, \eta) = \eta$, we have $\partial_\eta g_0(0, 0) = 1$. As a consequence, if y_0 is small enough, we can assume that $\partial_\eta g_0(y, 0) > 0$ for $|y| < y_0$ and consider the smooth diffeomorphism defined in $\{|y| < y_0\}$

$$\psi : y \mapsto \int_0^y \partial_\eta g_0(s, 0)\, ds.$$

We then use the canonical transformation

$$\Psi : (y, \eta) \mapsto \left(\psi(y), \frac{\eta}{\psi'(y)} \right).$$

We finally consider the chart $\kappa_{\rho_0} = \Psi \circ \kappa_0$ defined in $U_{\rho_0} = U'_{\rho_0} \cap \{|y| < y_0\}$ and if ε_0 is small enough, we can ensure that $B(\rho_0, 2\varepsilon_0) \subset U_{\rho_0}$. In this chart, the graph of $g_0(\,\cdot\,, \zeta)$ is sent to the graph of the function

$$g : y \in \Omega := \psi((-y_0, y_0)) \mapsto \frac{g_0(\psi^{-1}(y), \zeta)}{\psi'(\psi^{-1}(y))}.$$

We eventually check that:

- $g(y, 0) = 0$ since $g_0(y, 0) = 0$.
- $g(0, \zeta) = \zeta$ since $\psi(0) = 0$, $\psi'(0) = 1$ and $g(0, \zeta) = \zeta$.
- $\partial_\eta g(y, 0) = 1$.
- The $C^{1+\beta}$ norm of g is bounded uniformly in ρ_0.
- The C^N norms of κ_{ρ_0} are bounded uniformly in ρ_0. □

4. Construction of a refined quantum partition

We start the proof of Theorem 1. We consider $T = T(h) \in I_{0^+}(Y \times Y, F')$ a semiclassical Fourier integral operator associated with F, microlocally unitary in a neighborhood of $\mathcal{T}$, and a symbol $\alpha \in S_{0^+}(U)$. We want to show a bound for the spectral radius of $M(h) = T(h)\operatorname{Op}_h(\alpha)$, independent of h.

4A. *Numerology.* We'll use the standard fact

$$\|M^n\|_{L^2 \to L^2} \le \rho \quad \Longrightarrow \quad \rho_{\text{spec}}(M) \le \rho^{1/n}.$$

The trivial lemma which follows reduces the theorem to the study of $\|M^n\|$ with $n = n(h) \sim \delta|\log h|$.

Lemma 4.1. *Let $\delta > 0$ and $N(h) \in \mathbb{N}$ satisfy $N(h) \sim \delta|\log h|$. Suppose that there exists $h_0 > 0$ and $\gamma > 0$ such that,*

$$\textit{for all } 0 < h < h_0, \quad \|M(h)^{N(h)}\| \le h^\gamma \|\alpha\|_\infty^{N(h)}. \tag{4-1}$$

Then, for every $\varepsilon > 0$, there exists h_ε such that, for $h \le h_\varepsilon$,

$$\rho_{\text{spec}}(M(h)) \le e^{-\gamma/\delta + \varepsilon} \|\alpha\|_\infty.$$

Proof. It suffices to observe that under the assumption (4-1), we have $\rho_{\text{spec}}(M(h)) \le e^{\gamma \log h / N(h)} \|\alpha\|_\infty$ and use the equivalence for $N(h)$. □

Remark. If we use the bound $\|M\| \le \|\alpha\|_\infty + O(h^{1/2-\varepsilon})$, one get the obvious bound $\|M^N\| \le \|\alpha\|_\infty^N (1 + o(1))$. Hence, (4-1) is a decay bound.

The proof of Theorem 1 is then reduced to the proof of the following proposition.

Proposition 4.2. *There exists $\delta > 0$, a family of integer $N(h) \sim \delta|\log(h)|$ and $\gamma > 0$ such that, for h small enough, (4-1) holds.*

Actually, this proposition is enough to show Corollary 1 concerning perturbed operators, by virtue of:

Corollary 4.3. *Suppose that $R(h) : L^2(Y) \to L^2(Y)$ is a family of bounded operators such that $R(h) = O(h^\eta)$ for some $\eta > 0$. Then, there exists $\gamma' = \gamma'(\gamma, \eta)$ such that, for h small enough,*

$$\|(M(h)+R(h))^{N(h)}\| \le h^{\gamma'} \|\alpha\|_\infty^{N(h)}.$$

Proof. We write

$$(M+R)^N = M^N + \sum_{\substack{\varepsilon\in\{0,1\}^N \\ \varepsilon\neq(1,\dots,1)}} (\varepsilon_1 M + (1-\varepsilon_1)R)\cdots(\varepsilon_N M + (1-\varepsilon_N)R).$$

Using this, we can estimate

$$\begin{aligned}\|(M+R)^N\| &\le h^\gamma \|\alpha\|_\infty^N + ((\|M\|+\|R\|)^N - \|M\|^N) \\ &\le h^\gamma \|\alpha\|_\infty^N + N\|R\|(\|M\|+\|R\|)^{N-1} \\ &\le h^\gamma \|\alpha\|_\infty^N + C|\log h| h^\eta \|\alpha\|_\infty^{N-1}(1+O(h^\eta)) \\ &= O((h^\gamma + h^{\eta-})\|\alpha\|_\infty^N).\end{aligned}$$

This gives the desired bound for any $\gamma' < \min(\gamma, \eta)$. □

Actually, the precise value of $N(h)$ we'll use is rather explicit and we now describe it. We set

$$\mathfrak{b} = \frac{1}{1+\beta}, \tag{4-2}$$

where β is the one appearing in Theorem 5 concerning the regularity of the unstable distribution. We now choose $\delta_0 \in \left(0, \frac{1}{2}\right)$ such that

$$\mathfrak{b} + \delta_0 < 1. \tag{4-3}$$

For instance, let us set

$$\delta_0 = \frac{1-\mathfrak{b}}{2} = \frac{\beta}{2(1+\beta)}.$$

Recalling the definitions of the exponent $\lambda_0 \le \lambda_1$ in (3-10) and (3-11), we introduce the notation

$$N(h) = N_0(h) + N_1(h), \quad N_0(h) = \left\lceil \frac{\delta_0}{\lambda_1}|\log(h)| \right\rceil, \quad N_1(h) = \left\lceil \frac{1}{\lambda_0}|\log(h)| \right\rceil, \tag{4-4}$$

where $N_0(h)$ (resp. $N_1(h)$) corresponds to a short (resp. long) logarithmic time. We will omit the dependence on h in the following.

To be complete with the numerology, we introduce another number $\tau < 1$ such that

$$\mathfrak{b} < \tau < 1 \quad \text{and} \quad \delta_0 \frac{\lambda_0}{\lambda_1} + \tau > 1. \tag{4-5}$$

The meaning of these conditions will be clear in the core of the proof and we will indicate where they are used. For instance, we set

$$\tau = 1 - \frac{\lambda_0}{\lambda_1}\frac{1-\mathfrak{b}}{4}. \tag{4-6}$$

An important remark. If two operators $M_1(h)$ and $M_2(h)$ are equal modulo $O(h^\infty)$, this is also the case for $M_1(h)^{N(h)}$ and $M_2(h)^{N(h)}$ as long as

- $N(h) = O(\log h)$.
- $M_1(h), M_2(h) = O(h^{-K})$ for some K.

This will be widely used in the following. In particular, recall that we work with operators acting on $L^2(Y)$ but these operators take the form $M_1(h) = \Psi_Y M_2(h) \Psi_Y$, where $\Psi_Y \in C_c^\infty(Y, [0,1])$ and $M_2(h)$ is a bounded operator on $\bigoplus_{j=1}^J L^2(\mathbb{R})$ such that $M_2(h) = \Psi_Y M_2(h) \Psi_Y + O(h^\infty)_{L^2}$. As a consequence, modulo $O(h^\infty)$, it is enough to focus on $M_2(h)^{N(h)}$. For this reason, from now on and even if we keep the same notation, we work with

$$M(h) = T(h)\operatorname{Op}_h(\alpha) : \bigoplus_{j=1}^J L^2(\mathbb{R}) \to \bigoplus_{j=1}^J L^2(\mathbb{R}),$$

where $T(h) = (T_{ij}(h))$, with $T_{ij} \in I_{0+}(\mathbb{R}\times\mathbb{R}, F'_{ij})$ and

$$\operatorname{Op}_h(\alpha) = \operatorname{Diag}(\operatorname{Op}_h(\alpha_1), \dots, \operatorname{Op}_h(\alpha_J)).$$

4B. ***Microlocal partition of unity and notations.*** We consider some $\varepsilon_0 > 0$, which is supposed small enough to satisfy all the assumptions which will appear in the following.

We consider a cover of $\mathcal{T}$ by a finite number of balls of radius ε_0,

$$\mathcal{T} \subset \bigcup_{q=1}^Q B(\rho_q, \varepsilon_0), \quad \rho_q \in \mathcal{T},$$

and we assume that for all $q \in \{1, \dots, Q\}$, there exist $j_q, l_q, m_q \in \{1, \dots, J\}$ such that

$$B(\rho_q, 2\varepsilon_0) \subset \tilde{A}_{j_q l_q} \cap \widetilde{D}_{m_q j_q} \subset U_{j_q}.$$

We also assume that T is microlocally unitary in $B(\rho_q, 4\varepsilon_0)$. We then let

$$\mathcal{V}_q = B(\rho_q, 2\varepsilon_0). \tag{4-7}$$

See Figure 8.

Remark. In the case of obstacle scattering, with obstacles satisfying the noneclipse condition, it is possible to choose a simple partition of unity, related to the coding of the trapped set according to the sequence of obstacles hit by a trajectory. Indeed, due to a result of [Morita 1991], there is a homeomorphism between the trapped set and the admissible — that is, two consecutive obstacles are different — sequences of obstacles. As a consequence, if the obstacles are numbered from 1 to J, we can partition the trapped set by open subsets $U_{\vec{\alpha}}$ indexed by

$$\{\vec{\alpha} = (\alpha_{-N}, \dots, \alpha_N) \in \{1, \dots, J\}^{2N+1} : \alpha_{i+1} \neq \alpha_i\}.$$

The diameter of such partition goes to 0 as N goes to $+\infty$ and we could get the required partition $(\mathcal{V}_q)_q$, with the additional property of being disjoint open subsets of U. This would simplify the study in this particular setting.

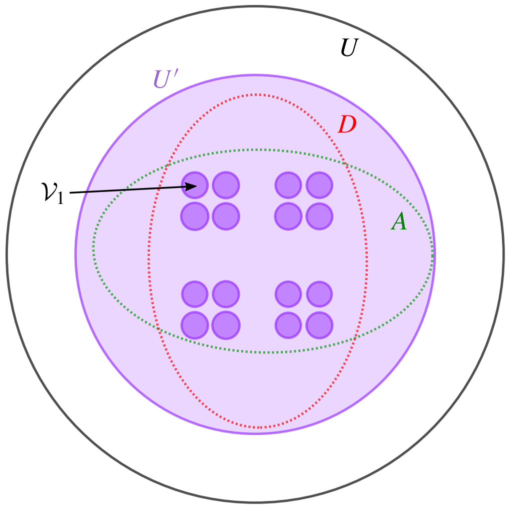

Figure 8. The partition $(\mathcal{V}_q)_{q\in\mathcal{A}_\infty}$ is made by small neighborhoods of $\mathcal{T}$ (small purple disks) and a big open set included in U'.

We complete this cover with

$$\mathcal{V}_\infty = U' \setminus \bigcup_{q=1}^{Q} \overline{B(\rho_q, \varepsilon_0)}. \tag{4-8}$$

$U' \Subset U$ is an open set such that $\mathrm{WF}_h(M) \Subset U' \times U'$. We denote by U'_j the component of U' inside U_j.

We let $\mathcal{A} = \{1, \dots, Q\}$ and $\mathcal{A}_\infty = \mathcal{A} \cup \{\infty\}$.

We then consider a partition of unity associated with the cover $\mathcal{V}_1, \dots, \mathcal{V}_Q, \mathcal{V}_\infty$, namely a family of smooth functions $\chi_q \in C_c^\infty(U)$ for $q \in \mathcal{A}_\infty$ such that:

- $\operatorname{supp} \chi_q \subset \mathcal{V}_q$.
- $0 \le \chi_q \le 1$.
- $1 = \sum_{q\in\mathcal{A}_\infty} \chi_q$ in $\bigcup_{q\in\mathcal{A}_\infty} \mathcal{V}_q$.

More precisely, if $q \in \mathcal{A}$, $\chi_q \in C^\infty(U_{j_q})$ and, for every $j \in \{1, \dots, J\}$, there exists $b_j \in C_c^\infty(U_j)$ such that on U'_j, then $1 = b_j + \sum_{q\in\mathcal{A}, j_q = j} \chi_q$. Thus, $\chi_\infty = \sum_{j=1}^{J} b_j$.

We can then quantize these symbols so as to get a pseudodifferential partition of unity. More precisely, to respect the matrix structure, we may write this quantization in a diagonal operator-valued matrix, still denoted by Op_h:

- For $q \in \mathcal{A}$, $A_q = \mathrm{Op}_h(\chi_q)$ is the diagonal matrix $\mathrm{Diag}(0, \dots, \mathrm{Op}_h(\chi_q), 0, \dots, 0)$, where the block $\mathrm{Op}_h(\chi_q)$ is in the j_q-th position.
- $\mathrm{Op}_h(\chi_\infty) = \mathrm{Diag}(\mathrm{Op}_h(b_1), \dots, \mathrm{Op}_h(b_J))$.

The family $(A_q)_{q\in\mathcal{A}_\infty}$ satisfies the properties

$$\sum_{q\in\mathcal{A}_\infty} A_q = \mathrm{Id} \text{ microlocally in } U' \quad \text{for all } q \in \mathcal{A}_\infty, \quad \|A_q\| \le 1 + O(h^{1/2}). \tag{4-9}$$

Since $M = \sum_{q \in \mathcal{A}_\infty} MA_q + O(h^\infty)$, we may write

$$M^n = \sum_{\boldsymbol{q} \in \mathcal{A}_\infty^n} U_{\boldsymbol{q}} + O(h^\infty),$$

where, for $\boldsymbol{q} = q_0 \cdots q_{n-1} \in \mathcal{A}_\infty^n$,

$$U_{\boldsymbol{q}} := MA_{q_{n-1}} \cdots MA_{q_0}. \tag{4-10}$$

For $\boldsymbol{q} = q_0 \cdots q_{n-1} \in \mathcal{A}_\infty^n$, we also define a family of refined neighborhoods, forming a refined cover of $\mathcal{T}$,

$$\mathcal{V}_{\boldsymbol{q}}^- = \bigcap_{i=0}^{n-1} F^{-i}(\mathcal{V}_{q_i}), \quad \mathcal{V}_{\boldsymbol{q}}^+ = F^n(\mathcal{V}_{\boldsymbol{q}}^-) = \bigcap_{i=0}^{n-1} F^{n-i}(\mathcal{V}_{q_i}). \tag{4-11}$$

This definition implies that a point $\rho \in \mathcal{V}_{\boldsymbol{q}}^-$ lies in $\mathcal{V}_{q_i}$ at time i (i.e., $F^i(\rho) \in \mathcal{V}_{q_i}$) for $0 \le i \le n-1$ and a point $\rho \in \mathcal{V}_{\boldsymbol{q}}^+$ lies in $\mathcal{V}_{q_{n-i}}$ at time $-i$ for $1 \le i \le n$. Roughly speaking, we expect that each operator $U_{\boldsymbol{q}}$ acts from $\mathcal{V}_{\boldsymbol{q}}^-$ to $\mathcal{V}_{\boldsymbol{q}}^+$ and is negligible (in some sense to be specified later on) elsewhere. Combining (4-9) and the bound on M, the following bound is valid (for any $\varepsilon > 0$):

$$\|U_{\boldsymbol{q}}\|_{L^2 \to L^2} \le (\|\alpha\|_\infty + O(h^{1/2-\varepsilon}))^n. \tag{4-12}$$

As soon as $|n| \le C_0 |\log h|$, we have $\|U_{\boldsymbol{q}}\|_{L^2 \to L^2} \le C \|\alpha\|_\infty^n$ for some C depending on C_0 and a finite number of seminorms of α.

Reduction to words in $\mathcal{A}$. We can find a uniform $T_0 \in \mathbb{N}$ such that if $\rho \in \mathcal{V}_\infty$, there exists $k \in \{-T_0, \dots, T_0\}$ such that $F^k(\rho)$ “falls” in the hole. By standard properties of the Fourier integral operators, each component $(M^{T_0})_{ij}$ of M^{T_0} is a Fourier integral operator associated with the component $(F^{T_0})_{ij}$ of F^{T_0}. In particular, $\mathrm{WF}_h'(M^{T_0}) \subset \mathrm{Gr}'(F^{T_0})$.

Let us study $M^{2T_0+N(h)}$. If $\boldsymbol{q} = q_0 \cdots q_{N-1} \in \mathcal{A}_\infty^N$ and if there exists an index $i \in \{0, \dots, N-1\}$ such that $q_i = \infty$, one can isolate this index i and trap A_{q_i} between two Fourier integral operators M_1, M_2, belonging to a finite family of FIO associated with F^{T_0}, so that we can write

$$M^{T_0} U_{\boldsymbol{q}} M^{T_0} = B_1 M_1 A_\infty M_2 B_2,$$

where B_1, B_2 satisfy the L^2-bound

$$\|B_1\| \times \|B_2\| \le (\|\alpha\|_\infty + O(h^{1/4}))^{N-1} = O(h^{-K})$$

for some integer K. Since

$$\mathrm{WF}_h'(M_1 A_\infty M_2) \subset \{(F^{T_0}(\rho), F^{-T_0}(\rho)) : \rho \in \mathrm{WF}_h(A_\infty)\} = \varnothing,$$

we have $M_1 A_\infty M_2 = O(h^\infty)$, with constants that can be chosen independent of $\boldsymbol{q}$. Hence, the same is true for $M^{T_0} U_{\boldsymbol{q}} M^{T_0}$. $|\mathcal{A}^N|$ is bounded by a negative power of h. So, we can write

$$M^{N+2T_0} = \sum_{\boldsymbol{q} \in \mathcal{A}_\infty^N} M^{T_0} U_{\boldsymbol{q}} M^{T_0} = \sum_{\boldsymbol{q} \in \mathcal{A}^N} M^{T_0} U_{\boldsymbol{q}} M^{T_0} + O(h^\infty) = M^{T_0} \Big(\sum_{\boldsymbol{q} \in \mathcal{A}^N} U_{\boldsymbol{q}} \Big) M^{T_0} + O(h^\infty).$$

We can then replace M by

$$\mathfrak{M} = \sum_{q \in \mathcal{A}} M A_q = M(\mathrm{Id} - A_\infty) + O(h^\infty)_{L^2 \to L^2}. \tag{4-13}$$

The decay bound

$$\|\mathfrak{M}(h)^{N(h)}\| \le h^\gamma \|\alpha\|_\infty^{N(h)} \tag{4-14}$$

will imply the required decay bound (4-1) for M with $N(h)$ replaced by $N(h) + 2T_0$. We are hence reduced to proving the decay bound (4-14).

4C. *Local Jacobian.*

A first definition. Following [Dyatlov et al. 2022], we introduce local unstable and stable Jacobians and we then state several properties. For $n \in \mathbb{N}^*$ and $\boldsymbol{q} \in \mathcal{A}^n$, let us define its local stable and unstable Jacobian:

$$J_{\boldsymbol{q}}^- := \inf_{\rho \in \mathcal{T} \cap \mathcal{V}_{\boldsymbol{q}}^-} J_n^u(\rho), \qquad J_{\boldsymbol{q}}^+ := \inf_{\rho \in \mathcal{T} \cap \mathcal{V}_{\boldsymbol{q}}^+} J_{-n}^s(\rho). \tag{4-15}$$

By the chain rule, we have, for $\rho \in \mathcal{T} \cap \mathcal{V}_{\boldsymbol{q}}^-$,

$$J_n^u(\rho) = \prod_{i=0}^{n-1} J_1^u(F^i(\rho)).$$

A similar formula is true for $\rho \in \mathcal{T} \cap \mathcal{V}_{\boldsymbol{q}}^+$:

$$J_{-n}^s(\rho) = \prod_{i=0}^{n-1} (J_1^s(F^{i-n}(\rho)))^{-1} = \prod_{i=0}^{n-1} J_{-1}^s(F^{-i}(\rho)).$$

Hence, we've got the basic estimates

$$\mathcal{T} \cap \mathcal{V}_{\boldsymbol{q}}^- \neq \varnothing \quad \Longrightarrow \quad e^{\lambda_0 n} \le J_{\boldsymbol{q}}^- \le e^{\lambda_1 n}, \tag{4-16}$$

$$\mathcal{T} \cap \mathcal{V}_{\boldsymbol{q}}^+ \neq \varnothing \quad \Longrightarrow \quad e^{\lambda_0 n} \le J_{\boldsymbol{q}}^+ \le e^{\lambda_1 n}. \tag{4-17}$$

If $\boldsymbol{q} = q_0 \cdots q_{n-1}$ and $\boldsymbol{q}_- = q_0 \cdots q_{n-2}$, then $\mathcal{V}_{\boldsymbol{q}}^- \subset \mathcal{V}_{\boldsymbol{q}_-}^-$ and thus

$$J_{\boldsymbol{q}}^- \ge e^{\lambda_0} J_{\boldsymbol{q}_-}^-. \tag{4-18}$$

Similarly, if $\boldsymbol{q}_+ = q_1 \cdots q_{n-1}$, then $\mathcal{V}_{\boldsymbol{q}}^+ \subset \mathcal{V}_{\boldsymbol{q}_+}^+$ and

$$J_{\boldsymbol{q}}^+ \ge e^{\lambda_0} J_{\boldsymbol{q}_+}^+. \tag{4-19}$$

As a consequence of Corollary 3.14, if ε_0 is small enough, the local stable and unstable Jacobians give the expansion rate of the flow at every point of $\mathcal{T} \cap \mathcal{V}_{\boldsymbol{q}}^\pm$. If $\mathcal{T} \cap \mathcal{V}_{\boldsymbol{q}}^\pm \neq \varnothing$,

$$\text{for all } \rho \in \mathcal{T} \cap \mathcal{V}_{\boldsymbol{q}}^-, \quad J_n^u(\rho) \sim J_{\boldsymbol{q}}^-, \tag{4-20}$$

$$\text{for all } \rho \in \mathcal{T} \cap \mathcal{V}_{\boldsymbol{q}}^+, \quad J_{-n}^s(\rho) \sim J_{\boldsymbol{q}}^+. \tag{4-21}$$

This definition is slightly unsatisfactory since $J_{\boldsymbol{q}}^\pm = +\infty$ as soon as $\mathcal{V}_{\boldsymbol{q}}^\pm \cap \mathcal{T} = \varnothing$. However, when $\mathcal{V}_{\boldsymbol{q}}^\pm \neq \varnothing$, this set can still stay relevant. For this purpose, we will give a definition of local stable and unstable Jacobian for such words with help of the shadowing lemma [Katok and Hasselblatt 1995, Section 18.1].

Enlarged definition. Let $n \in \mathbb{N}$ and $\boldsymbol{q} = q_0 \cdots q_{n-1} \in \mathcal{A}^n$. We focus on $\mathcal{V}_{\boldsymbol{q}}^-$, with the case of $\mathcal{V}_{\boldsymbol{q}}^+$ handled similarly by considering F^{-1} instead of F.

If $\mathcal{V}_{\boldsymbol{q}}^- \cap \mathcal{T} \neq \varnothing$, we keep the definition given in (4-15). Assume now that $\mathcal{V}_{\boldsymbol{q}}^- \neq \varnothing$ but $\mathcal{V}_{\boldsymbol{q}}^- \cap \mathcal{T} = \varnothing$. Fix $\rho \in \mathcal{V}_{\boldsymbol{q}}^-$. By definition of $\mathcal{V}_{q_i}$, for $i \in \{0, \dots, n-1\}$, we have $d(\rho_i, F^i(\rho)) \leq 2\varepsilon_0$. Hence,

$$d(F(\rho_i), \rho_{i+1}) \leq d(F(\rho_i), F^{i+1}(\rho)) + d(F^{i+1}(\rho), \rho_{i+1}) \leq C\varepsilon_0$$

for a constant C only depending on F. That is to say, $(\rho_0, \dots, \rho_{n-1})$ is a $C\varepsilon_0$ pseudo-orbit. Assume that $\delta_0 > 0$ is a small fixed parameter. By virtue of the shadowing lemma, if ε_0 is sufficiently small and $(\rho_0, \dots, \rho_{n-1})$ is δ_0 shadowed by an orbit of F, then there exists $\rho' \in \mathcal{T}$ such that, for $i \in \{0, \dots, n-1\}$, $d(\rho_i, F(\rho')) \leq \delta_0$. Consequently, $d(F^i(\rho), F^i(\rho')) \leq \delta_0 + C\varepsilon_0$. If ρ_2 is another point in $\mathcal{V}_{\boldsymbol{q}}^-$, for $i = 0, \dots, n-1$, $d(F^i(\rho_2), F^i(\rho')) \leq 2\varepsilon_0 + C\varepsilon_0 + \delta_0$. For convenience, set $\varepsilon_2 = 2\varepsilon_0 + \delta_0 + C\varepsilon_0$ and note that ε_2 can be arbitrarily small depending on ε_0. As a consequence, we have proven the following:

Lemma 4.4. *If $\mathcal{V}_{\boldsymbol{q}}^- \neq \varnothing$, then there exists $\rho' \in \mathcal{T}$ such that, for all $i \in \{0, \dots, n-1\}$ and $\rho \in \mathcal{V}_{\boldsymbol{q}}^-$, $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_2$.*

Fix any ρ' satisfying the conclusions of this lemma and we arbitrarily set

$$J_{\boldsymbol{q}}^- = J_n^u(\rho'). \tag{4-22}$$

If ρ_1' is another point satisfying this conclusion, we have $d(F^i(\rho'), F^i(\rho_1')) \leq 2\varepsilon_2$ for $i \in \{0, \dots, n-1\}$ and by virtue of Corollary 3.14,

$$J_n^u(\rho') \sim J_n^u(\rho_1').$$

Hence, up to global multiplicative constants, the definition of this unstable Jacobian is independent of the choice of ρ'. Notice that if $\mathcal{V}_{\boldsymbol{q}}^- \cap \mathcal{T} \neq \varnothing$, any $\rho' \in \mathcal{T} \cap \mathcal{V}_{\boldsymbol{q}}^-$ satisfies the conclusions of Lemma 4.4 and $J_{\boldsymbol{q}}^- \sim J_n^u(\rho')$.

To define $J_{\boldsymbol{q}}^+$, we can argue similarly and show that there exists ρ' satisfying $d(F^i(\rho'), F^i(\rho)) \leq \varepsilon_2$ for $i \in \{-n, \dots, -1\}$ and $\rho \in \mathcal{V}_{\boldsymbol{q}}^+$. We can assume that this is the same ε_2 as before and we set $J_{\boldsymbol{q}}^+ = J_{-n}^s(\rho')$ for any ρ'.

Behavior of the local Jacobian. See Figure 9. The following three lemmas are crucial to understand the behavior of the evolution of points in the sets $\mathcal{V}_{\boldsymbol{q}}^\pm$. The first one gives estimates to handle these quantities.

Lemma 4.5. *Let $n \in \mathbb{N}$ and $\boldsymbol{q}$, $\boldsymbol{p}$ in $\mathcal{A}^n$. If ε_0 is chosen small enough, then the following hold:*

(1) $\mathcal{V}_{\boldsymbol{q}}^+ \neq \varnothing \iff \mathcal{V}_{\boldsymbol{q}}^- \neq \varnothing$ *and in that case* $J_{\boldsymbol{q}}^- \sim J_{\boldsymbol{q}}^+$.

(2) *If two propagated neighborhoods intersect, the local Jacobians are comparable:*

$$\mathcal{V}_{\boldsymbol{q}}^\pm \cap \mathcal{V}_{\boldsymbol{p}}^\pm \neq \varnothing \quad \Longrightarrow \quad J_{\boldsymbol{q}}^\pm \sim J_{\boldsymbol{p}}^\pm. \tag{4-23}$$

(3) *If $\boldsymbol{q}$ can be written as the concatenation of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ of lengths n_1 and n_2 such that $n_1 + n_2 = n$ and if $\mathcal{V}_{\boldsymbol{q}}^\pm \neq \varnothing$, then*

$$J_{\boldsymbol{q}}^\pm \sim J_{\boldsymbol{q}_1}^\pm J_{\boldsymbol{q}_2}^\pm. \tag{4-24}$$

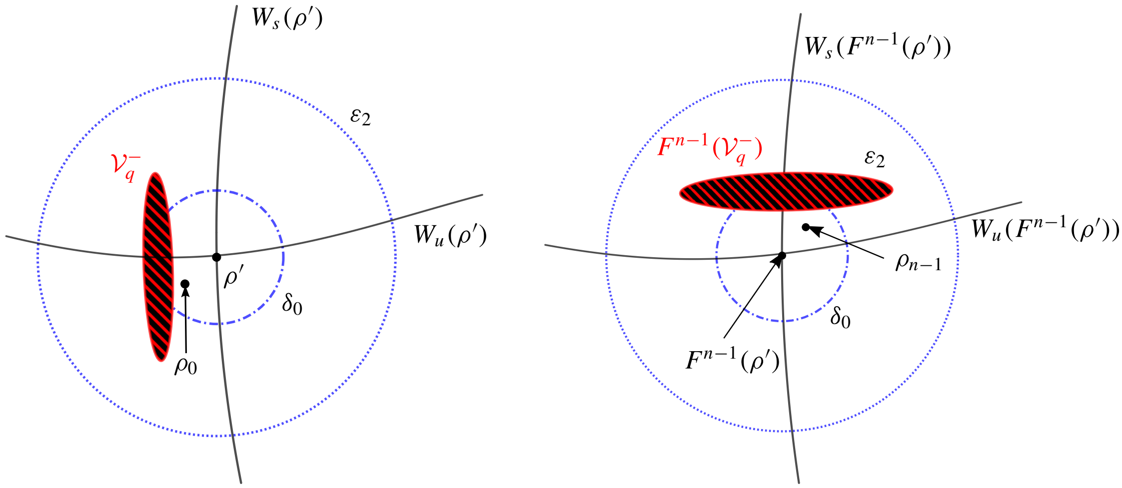

Figure 9. Evolution of the set $\mathcal{V}_{\boldsymbol{q}}^-$ (the red hatched set) at time 0 and $n-1$. The points ρ_i, $F^i(\rho')$ are represented at these times, so as the balls $B(F^i(\rho'), \varepsilon_2)$ and $B(F^i(\rho'), \delta_0)$ (their boundaries are the blue dotted lines). We've also represented the stable (resp. unstable) manifold at $F^i(\rho')$ to show the directions in which F contracts (resp. expands).

Notation. The constants in $\sim$ are independent of ρ and n. They depend on F but also on the partition $(\mathcal{V}_q)_q$. In the following, we'll be lead to use constants with the same kind of dependence. These constants will be allowed to depend also on the partition of unity $(\chi_q)_q$ and on M. Constants with such dependence will be called *global* constants.

Proof. (1) The equivalence is obvious. From the fact that F is a volume-preserving canonical transformation, we have, for some $C > 0$,

$$\text{for all } \rho \in \mathcal{T}, \text{ for all } n \in \mathbb{N}, \quad C^{-1} \leq J_n^u(\rho) J_n^s(\rho) \leq C,$$

and we write $J_n^u(\rho) \sim J_n^s(\rho)^{-1}$. From $F^{-n} \circ F^n(\rho) = \rho$, we also get $J_n^s(\rho)^{-1} = J_{-n}^s(F^n(\rho))$. Eventually, if $\rho' \in \mathcal{T}$ satisfies $d(F^i(\rho), F^i(\rho') \leq \varepsilon_2$ for $i \in \{0, \dots, n-1\}$ and $\rho \in \mathcal{V}_{\boldsymbol{q}}^-$, $F^n(\rho') = \rho^+$ satisfies $d(F^i(\rho), F^i(\rho^+)) \leq \varepsilon_2$ for $i \in \{-n, \dots, -1\}$ and $\rho \in \mathcal{V}_{\boldsymbol{q}}^+$. Hence

$$J_{\boldsymbol{q}}^+ \sim J_{-n}^s(\rho^+) \sim J_n^u(\rho') \sim J_{\boldsymbol{q}}^-.$$

Thanks to this first point, it is enough to show the remaining point only for $-$.

(2) Pick $\rho_{\boldsymbol{q}} \in \mathcal{T}$ (resp. $\rho_{\boldsymbol{p}}$) satisfying the conclusions of Lemma 4.4 for $\mathcal{V}_{\boldsymbol{q}}^-$ (resp. $\mathcal{V}_{\boldsymbol{p}}^-$). We have $d(F^i(\rho_{\boldsymbol{q}}), F^i(\rho_{\boldsymbol{p}})) \leq 2\varepsilon_2$ and hence, by virtue of Corollary 3.14, $J_n^u(\rho_{\boldsymbol{q}}) \sim J_n^u(\rho_{\boldsymbol{p}})$. This gives (2).

(3) Pick $\rho \in \mathcal{T}$ satisfying the conclusions of Lemma 4.4 for $\mathcal{V}_{\boldsymbol{q}}^-$. By the chain rule, we have $J_n^u(\rho) = J_{n_2}^u(F^{n_1}(\rho)) J_{n_1}^u(\rho)$. Note that

$$\mathcal{V}_{\boldsymbol{q}}^- = \mathcal{V}_{\boldsymbol{q}_1}^- \cap F^{-n_1}(\mathcal{V}_{\boldsymbol{q}_2}^-).$$

Hence, ρ satisfies the conclusions of Lemma 4.4 for $\boldsymbol{q}_1$ with ε_2 and the same is true for $F^{n_1}(\rho)$ and $\boldsymbol{q}_2$. It follows that $J_{\boldsymbol{q}_1}^- \sim J_{n_1}^u(\rho)$ and $J_{\boldsymbol{q}_2}^- \sim J_{n_2}^u(F^{n_1}(\rho))$. This gives (3). ☐

Remark. The first point of the previous lemma shows that we could consider only one of the two quantities. Nevertheless, we prefer keeping track of it. The reason is that a priori J^+ and J^- support two different kind of information: $J_{\boldsymbol{q}}^+$ controls the growth of F^n, whereas $J_{\boldsymbol{q}}^-$ controls the growth of F^{-n}. The fact that the two dynamics (in the past and in the future) have similar behaviors is a consequence of the fact that F is volume-preserving.

The next lemmas relate the local Jacobian to the expansion rates of the flow in the $\mathcal{V}_{\boldsymbol{q}}^{\pm}$. It will be important in our semiclassical study of operators microlocally supported in $\mathcal{V}_{\boldsymbol{q}}^{\pm}$.

Lemma 4.6 (control of expansion rate by unstable Jacobian). *If ε_0 is small enough, there exists a global constant $C > 0$ satisfying the following inequalities:*

For every $n \in \mathbb{N}^*$ and $\boldsymbol{q} \in \mathcal{A}^n$ such that $\mathcal{V}_{\boldsymbol{q}}^- \neq \varnothing$ we have

$$\sup_{\rho \in \mathcal{V}_{\boldsymbol{q}}^-} \|d_\rho F^n\| \leq C J_{\boldsymbol{q}}^-, \tag{4-25}$$

$$\sup_{\rho \in \mathcal{V}_{\boldsymbol{q}}^+} \|d_\rho F^{-n}\| \leq C J_{\boldsymbol{q}}^+. \tag{4-26}$$

Proof. This is a consequence of (3-18). Indeed, if $\mathcal{V}_{\boldsymbol{q}}^- \neq \varnothing$ and if $\rho' \in \mathcal{T}$ satisfies the conclusions of Lemma 4.4, then for every $\rho \in \mathcal{V}_{\boldsymbol{q}}^-$, $\|d_\rho F^n\| \leq C J_n^u(\rho)$ with C a global constant depending only on ε_2. $\square$

This third lemma emphasizes that $\mathcal{V}_{\boldsymbol{q}}^-$ lies in a small neighborhood of a stable manifold and $\mathcal{V}_{\boldsymbol{q}}^+$ lies in a small neighborhood of an unstable manifold, with the size of this neighborhood controlled by the local Jacobian. It is a direct consequence of Lemma 3.13.

Lemma 4.7 (localization of the $\mathcal{V}_{\boldsymbol{q}}^{\pm}$). *There exists a global constant $C > 0$ such that for all $n \in \mathbb{N}$ and $\boldsymbol{q} \in \mathcal{A}^n$:*

(1) *If $\mathcal{V}_{\boldsymbol{q}}^- \neq \varnothing$ and $\rho' \in \mathcal{T}$ satisfies the conclusion of Lemma 4.4, then, for all $\rho \in \mathcal{V}_{\boldsymbol{q}}^-$,*

$$d(\rho, W_s(\rho')) \leq \frac{C}{J_{\boldsymbol{q}}^-}. \tag{4-27}$$

(2) *If $\mathcal{V}_{\boldsymbol{q}}^+ \neq \varnothing$ and $\rho' \in \mathcal{T}$ satisfies the conclusion of Lemma 4.4 in the future (namely, $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_2$ for all $\rho \in \mathcal{V}_{\boldsymbol{q}}^+$ and $i \in \{-n, \dots, -1\}$), then, for all $\rho \in \mathcal{V}_{\boldsymbol{q}}^+$,*

$$d(\rho, W_u(\rho')) \leq \frac{C}{J_{\boldsymbol{q}}^+}. \tag{4-28}$$

4D. ***Propagation up to local Ehrenfest time.*** In this section, we show that under some control of the local Jacobian defined above, one can handle the operators $U_{\boldsymbol{q}}$ and prove the existence of symbols $a_{\boldsymbol{q}}^{\pm}$ (in exotic classes S_δ) such that

$$U_{\boldsymbol{q}} = \operatorname{Op}_h(a_{\boldsymbol{q}}^+) T^{|\boldsymbol{q}|} + O(h^\infty), \tag{4-29}$$

$$U_{\boldsymbol{q}} = T^{|\boldsymbol{q}|} \operatorname{Op}_h(a_{\boldsymbol{q}}^-) + O(h^\infty), \tag{4-30}$$

with symbols $a_{\boldsymbol{q}}^{\pm}$ supported in $\mathcal{V}_{\boldsymbol{q}}^{\pm}$. We recall that $U_{\boldsymbol{q}} = M A_{q_{n-1}} \cdots M A_{q_0}$, with $M = T \operatorname{Op}_h(\alpha)$. Let us state the precise statement we will prove.

Proposition 4.8. *Fix $0 < \delta < \delta_1 < \frac{1}{2}$ and $C_0 > 0$.*

(1) *For every $n \in \mathbb{N}$ and for all $\boldsymbol{q} \in \mathcal{A}^n$ satisfying*

$$J_{\boldsymbol{q}}^{+} \leq C_0 h^{-\delta}, \tag{4-31}$$

there exists $a_{\boldsymbol{q}}^{+} \in \|\alpha\|_{\infty}^{n} S_{\delta_1}^{\mathrm{comp}}$ such that

$$U_{\boldsymbol{q}} = \mathrm{Op}_h(a_{\boldsymbol{q}}^{+}) T^n + O(h^{\infty})_{L^2 \to L^2}, \tag{4-32}$$

$$\operatorname{supp} a_{\boldsymbol{q}}^{+} \subset \mathcal{V}_{\boldsymbol{q}}^{+}. \tag{4-33}$$

(2) *For every $n \in \mathbb{N}$ and for all $\boldsymbol{q} \in \mathcal{A}^n$ satisfying*

$$J_{\boldsymbol{q}}^{-} \leq C_0 h^{-\delta}, \tag{4-34}$$

there exists $a_{\boldsymbol{q}}^{-} \in \|\alpha\|_{\infty}^{n} S_{\delta_1}^{\mathrm{comp}}$ such that

$$U_{\boldsymbol{q}} = T^n \, \mathrm{Op}_h(a_{\boldsymbol{q}}^{-}) + O(h^{\infty})_{L^2 \to L^2}, \tag{4-35}$$

$$\operatorname{supp} a_{\boldsymbol{q}}^{-} \subset \mathcal{V}_{\boldsymbol{q}}^{-}. \tag{4-36}$$

Remark. • The implied constants appearing in the $O(h^{\infty})$ are quasiglobal; they have the same dependence as global constants but depend also on C_0, δ, δ_1. What is important is that they are independent of n and $\boldsymbol{q}$ as soon as the assumption (4-31) is satisfied.

• (4-31) implies that $\mathcal{V}_{\boldsymbol{q}}^{+} \neq \varnothing$. In particular, if $\boldsymbol{q}$ satisfies this assumption, there exists a sequence $(i_0, \dots, i_n)$ such that, for all $p \in \{0, \dots, n-1\}$, $\mathcal{V}_{q_p} \subset \widetilde{D}_{i_{p+1}, i_p} \subset U_{i_p}$.

• In fact, $\operatorname{supp} a_{\boldsymbol{q}}^{+} \subset F(\mathcal{V}_{q_{n-1}}) \subset U_{i_n}$. Hence, the operator $\mathrm{Op}_h(a_{\boldsymbol{q}}^{+})$ acting on $\bigoplus_{i=1}^{J} L^2(\mathbb{R})$ is the diagonal matrix $\mathrm{Diag}(0, \dots, \mathrm{Op}_h(a_{\boldsymbol{q}}^{+}), \dots, 0)$.

• The symbol $a_{\boldsymbol{q}}^{+}$ has an asymptotic expansion in power of h. The principal symbol is given by

$$(a_{\boldsymbol{q}}^{+})_0 = \prod_{p=1}^{n} a_{q_{n-p}} \circ F^{-p}, \tag{4-37}$$

where $a_q = \chi_q \times \alpha$. Note that if the functions $a_{q_{n-p}} \circ F^{-p}$ are not necessarily well-defined, the product is well-defined thanks to the assumptions on the supports of χ_q, namely $\operatorname{supp} \chi_q \Subset \mathcal{V}_q$. Indeed, such a symbol can be constructed inductively as the n-th term b_n of the sequence of functions $b_1 = a_{q_0} \circ F^{-1}$ and b_{i+1} is obtained from a_i by

$$b_{i+1} = (a_{q_i} \times a_i) \circ F^{-1}.$$

If we assume that $\operatorname{supp} b_i \Subset \mathcal{V}_{q_0 \cdots q_{i-1}}^{+}$, then $\operatorname{supp}(a_{q_i} \times b_i) \Subset F^{-1}(\mathcal{V}_{q_0 \cdots q_i}^{+})$. This property allows us to define b_{i+1} and $\operatorname{supp} b_{i+1} \Subset \mathcal{V}_{q_0 \cdots q_i}^{+}$.

• The same holds for $a_{\boldsymbol{q}}^{-}$ with principal symbol

$$(a_{\boldsymbol{q}}^{-})_0 = \prod_{p=0}^{n-1} a_{q_p} \circ F^{p}. \tag{4-38}$$

• Our proof follows the sketch of proof of [Dyatlov et al. 2022, Section 5] and [Rivière 2010, Section 7].

In the end of this section, we focus on proving this proposition. We only prove the first point. The second point can be proved similarly by using the same techniques.

4D1. *Iterative construction of the symbols.* Let us start by a lemma combining the precise versions of the expansion of the Moyal product (Lemma 3.3) and of Egorov theorem (Proposition 3.8). This lemma is the key ingredient for the iterative formulas below.

Lemma 4.9. *Let $q \in \mathcal{A}$ and let $a \in S^{\mathrm{comp}}_{\delta_1}$ such that* $\operatorname{supp} a \Subset U_j$ *for some* $j \in \{1, \dots, J\}$. *Then, there exists a family of differential operators $L_{k,q}$ of order $2k$, with smooth coefficients compactly supported in $\mathcal{V}_q$, such that, for every $N \in \mathbb{N}$, we have the expansion*

$$M A_q \operatorname{Op}_h(a) = \operatorname{Op}_h\left(\sum_{k=0}^{N-1} h^k (L_{k,q} a) \circ F^{-1}\right) T + O(\|a\|_{C^{2N+15}} h^N)_{L^2 \to L^2}. \tag{4-39}$$

Moreover, one has $L_{0,q} = \chi_q \times \alpha := a_q$.

Remark. • Again, since $\operatorname{supp} a \subset U_j$, $\operatorname{Op}_h(a)$ is a diagonal matrix with only one nonzero block equal to $\operatorname{Op}_h(a)$.

• Recall that we've supposed that $\mathcal{V}_q \subset \widetilde{D}_{m_q j_q}$. As a consequence, the symbols

$$a_1^{(k)} := L_{k,q} a \circ F^{-1}$$

are equal to $L_{k,q} a \circ (F_{m_q j_q})^{-1}$ and are supported in U_{m_q}; $\operatorname{Op}_h(a_1^{(k)})$ is still a diagonal matrix.

Proof. Let us first work at the order of operators $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and let us study

$$M_{m_q j_q} \operatorname{Op}_h(\chi_q) \operatorname{Op}_h(a) = T_{m_q j_q} \operatorname{Op}_h(\alpha_{j_q}) \operatorname{Op}_h(\chi_q) \operatorname{Op}_h(a).$$

Using Lemma 3.3, we write

$$\operatorname{Op}_h(\chi_q) \operatorname{Op}_h(a) = \operatorname{Op}_h\left(\sum_{k=0}^{N-1} \frac{i^k h^k}{k!} A(D)^k (\chi_q \otimes a)|_{\rho=\rho_1=\rho_2}\right) + O(h^N \|\chi_q \otimes a\|_{C^{2N+13}}),$$

the principal term of the expansion being $\chi_q a$. Set $a_{q,k}(\rho) = A(D)^k (\chi_q \otimes a)|_{\rho=\rho_1=\rho_2}$ and use Lemma 3.3 to write

$$\operatorname{Op}_h(\alpha_{j_q}) \operatorname{Op}_h(\chi_q) \operatorname{Op}_h(a) = \sum_{k_1+k_2<N} \frac{i^{k_1+k_2} h^{k_1+k_2}}{k_1!\, k_2!} \operatorname{Op}_h(A(D)^{k_2}(\alpha_{j_q} \otimes a_{q,k_1})|_{\rho=\rho_1=\rho_2}) + O(h^N \|a\|_{C^{2N+13}}).$$

The principal term in the expansion is $\alpha_{j_q} \chi_q a$. We note that

$$a \mapsto \sum_{k_1+k_2=k} A(D)^{k_2}(\alpha_{j_q} \otimes a_{q,k_1})|_{\rho=\rho_1=\rho_2}$$

is a differential operator of order $2k$. Using the precise version of Egorov theorem in Lemma 3.10, we see that, for any b with $\operatorname{supp}(b) \subset \mathcal{V}_q$,

$$T_{m_q j_q} \operatorname{Op}_h(b) = \operatorname{Op}_h\left(b \circ (F_{m_q j_q})^{-1} + \sum_{k=1}^{N-1} h^k (D_k b) \circ (F_{m_q j_q})^{-1}\right) + O(h^N \|b\|_{C^{2N+15}}),$$

where D_k are differential of order $2k$ compactly supported in $\mathcal{V}_q$. Applying this to the previous expansion, we see that we can write

$$T_{m_q j_q} \operatorname{Op}_h(\alpha_{j_q}) \operatorname{Op}_h(\chi_q) \operatorname{Op}_h(a) = \operatorname{Op}_h\left((\alpha_{j_q}\chi_q a)\circ F^{-1} + \sum_{k=1}^{N-1} k^k (L_{k,q}a)\circ F^{-1}\right) + O(h^N \|a\|_{C^{2N+15}}).$$

We now come to the entire matrix operator. Note that the matrix $M \operatorname{Op}_h(\chi_q)\operatorname{Op}_h(a)$ is of the form

$$\begin{pmatrix} 0 & \cdots & M_{1j_q}\operatorname{Op}_h(\chi_q) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & M_{Jj_q}\operatorname{Op}_h(\chi_q) & \cdots & 0 \end{pmatrix} \operatorname{Op}_h(a).$$

Recall that $\mathrm{WF}_h(\operatorname{Op}_h(\chi_q)) \subset \widetilde{D}_{m_q j_q}$ and $\mathrm{WF}'_h(M_{m_q j_q}) \subset \mathrm{Gr}'(F_{m_q j_q})$. Hence, for $m \neq m_q$, $M_{mj_q}\operatorname{Op}_h(\chi_q) = O(h^\infty)$ and the previous matrix can be written

$$\begin{pmatrix} 0 & \cdots & O(h^\infty) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & M_{m_q j_q}\operatorname{Op}_h(\chi_q) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & O(h^\infty) & \cdots & 0 \end{pmatrix} \operatorname{Op}_h(a) = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & M_{m_q j_q}\operatorname{Op}_h(\chi_q)\operatorname{Op}_h(a) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} + O(h^\infty)\|\operatorname{Op}_h(a)\|_{L^2}.$$

With constant in $O(h^\infty)$ depending on χ_q, M and $\|\operatorname{Op}_h(a)\|_{L^2\to L^2} = O(\|a\|_{C^8})$. Let's write

$$a_1^{(k)} = L_{k,q}a \circ F^{-1}$$

and observe that $\mathrm{supp}(a_1^{(k)}) \subset F(\mathrm{supp}\,\chi_q) \Subset \tilde{A}_{m_q j_q}$. Consider a cut-off function $\tilde{\chi}_q$ such that $\tilde{\chi}_q \equiv 1$ in a neighborhood of $F(\mathrm{supp}\,\chi_q)$ and $\mathrm{supp}\,\tilde{\chi}_q \subset \tilde{A}_{m_q j_q}$. Using Lemma 3.3 and the support properties of $\tilde{\chi}_q$, one has

$$\operatorname{Op}_h(a_1^{(k)}) = \operatorname{Op}_h(a_1^{(k)})\operatorname{Op}_h(\tilde{\chi}_q) + O(h^{N-k}\|a_1^{(k)}\|_{C^{2(N-k)+13}}) = \operatorname{Op}_h(a_1^{(k)})\operatorname{Op}_h(\tilde{\chi}_q) + O(h^{N-k}\|a\|_{C^{2N+13}}).$$

Then, one can write $\operatorname{Op}_h(a_1^{(k)})T$ on the form

$$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ \operatorname{Op}_h(a_1^{(k)})\operatorname{Op}_h(\tilde{\chi}_q)T_{m_q 1} & \cdots & \operatorname{Op}_h(a_1^{(k)})\operatorname{Op}_h(\tilde{\chi}_q)T_{m_q J} \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} + O(h^{N-k}\|a\|_{C^{2N+13}})$$

and, for $j \neq j_q$, $\operatorname{Op}_h(\tilde{\chi}_q)T_{m_q j} = O(h^\infty)$. We can conclude that

$$\operatorname{Op}_h(a_1^{(k)})T = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \operatorname{Op}_h(a_1^{(k)})\operatorname{Op}_h(\tilde{\chi}_q)T_{m_q j_q} & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} + O(h^\infty)\|\operatorname{Op}_h(a_1^{(k)})\|_{L^2\to L^2} + O(h^{N-k}\|a\|_{C^{2N+13}})$$

$$= \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \mathrm{Op}_h(a_1^{(k)})T_{m_q j_q} & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} + O(h^{N-k}\|a\|_{C^{2N+13}}).$$

Combining this with the version obtained with $M_{m_q j_q}$, we get (4-39). □

Let us now start the iterative construction of the symbols. Fix $N \in \mathbb{N}$ which can be taken arbitrarily large. Recall that we want to write

$$U_{\boldsymbol{q}} = \mathrm{Op}_h(a_{\boldsymbol{q}}^+)T^{|\boldsymbol{q}|} + O(h^\infty)_{L^2\to L^2}. \tag{4-40}$$

Note $U_r = U_{q_0\cdots q_{r-1}}$. We want to write

$$U_r = \mathrm{Op}_h\left(\sum_{k=0}^{N-1} h^k a_r^{(k)}\right)T^r + R_r^{(N)}. \tag{4-41}$$

We start by writing

$$U_1 = \mathrm{Op}_h\left(\sum_{k=0}^{N-1} h^k a_1^{(k)}\right)T + R_1^{(N)}, \tag{4-42}$$

which is possible by virtue of (4-39). To pass from U_r to U_{r+1}, we have the relation

$$U_{r+1} = MA_{q_r}U_r = \sum_{k=0}^{N-1} h^k MA_{q_r}\,\mathrm{Op}_h(a_r^{(k)})T^r + MA_{q_r}R_r^{(N)}.$$

So, we will construct inductively our symbols by setting

$$a_{r+1}^{(k)} = \sum_{p=0}^{k}(L_{p,q_r}a_r^{(k-p)}) \circ (F_{i_{r+1},i_r})^{-1}, \tag{4-43}$$

$$R_{r+1}^{(N)} = MA_{q_r}R_r^{(N)} + \sum_{k=0}^{N-1} O(\|a_r^{(k)}\|_{C^{2(N-k)+15}}). \tag{4-44}$$

The O encompasses the remainder terms in (4-39). The constants in the O only depend on M and the χ_q, $q \in \mathcal{A}$, but not on $\boldsymbol{q}$.

To make this construction work, we will have to prove that the symbols $a_r^{(k)}$ lie in a good symbol class $S_{\delta_1}^{\mathrm{comp}}$.

Before reaching this step, let us just note that by induction one sees that:

- $$\|R_r^{(N)}\| \le C_N h^N\left(1 + \sum_{k=0}^{N-1}\sum_{l=0}^{r-1}\|a_l^{(k)}\|_{C^{2(N-k)+15}}\right), \tag{4-45}$$

with C_N depending on N, M and the a_q, but neither on r nor $\boldsymbol{q}$.

- Since L_{p,q_r} has coefficient supported in $\mathcal{V}_{q_r}$, we see by induction that $\operatorname{supp} a_{r+1}^{(k)} \subset \mathcal{V}_{q_0\cdots q_r}^+$ as announced.
- $a_{r+1}^{(0)} = \prod_{p=1}^{r+1} a_{q_{r+1-p}} \circ F^{-p}$.

4D2. *Control of the symbols.* We aim at estimating the seminorms $\|a_r^{(k)}\|_{C^m}$ for $k < N,\ 1 \le r \le n$ and $m \in \mathbb{N}$. We will show the following:

Proposition 4.10. *For every $r \in \{1, \dots, n\},\ k \in \{0, \dots, N-1\}$ and $m \in \mathbb{N}$, there exists $C(k, m)$, such that, with $\Gamma_{k,m} = (k+1)(m+k+1)$,*

$$\|a_r^{(k)}\|_{C^m} \le C(k, m) r^{\Gamma_{k,m}} (J^+_{q_0 \cdots q_{r-1}})^{2k+m} \|\alpha\|_\infty^r. \tag{4-46}$$

Remark. • What is important in this result is the way in which the bound depends on r and $\boldsymbol{q}$. Up to the term $r^{\Gamma_{k,m}}$, which is supposed to behave like $O(|\log h|^{\Gamma_{k,m}})$, the significant part of the estimate is that we can control the symbols by the local Jacobian.

• Since $\operatorname{supp} a_r^{(k)} \subset \mathcal{V}^+_{q_0 \cdots q_{r-1}}$, we need to focus on points $\rho \in \mathcal{V}^+_{q_0 \cdots q_{r-1}}$.

• Our method is very close to the ones developed in [Rivière 2010; Dyatlov et al. 2022]. However, we've changed a few things at the cost of being less precise on the exponent $\Gamma_{k,m}$. Our aim was to treat our problem as if we wanted to control the product of r triangular matrices.

Let us pick $\rho \in \mathcal{V}^+_{q_0 \cdots q_{r-1}}$. With (4-43), one sees that if $k, m \in \mathbb{N}$, then $d^m a_{r+1}^{(k)}$ depends on $d^{m'} a_r^{(k')}(F^{-1}(\rho))$ for several m', k'. Before going deeper in the analysis of this dependence, let us note two obvious facts:

- This dependence is linear, with coefficients smoothly depending on ρ.
- If $d^m a_{r+1}^{(k)}$ depends effectively on $d^{m'} a_r^{(k')}(F^{-1}(\rho))$, then $k' \le k$ and $2k' + m' \le 2k + m$.

Precise analysis of the dependence. That being said, let us pick $m_0, k_0 \in \mathbb{N}$. Set $N_0 = 2k_0 + m_0$ and consider the (column) vector

$$A_r(\rho) := (d^m a_r^{(k)}(\rho))_{k \le k_0, 2k+m \le N_0} \in \bigoplus_{k \le k_0, 2k+m \le N_0} S^m T^*_\rho U. \tag{4-47}$$

Here $S^m T^*_\rho U$ is the space of m-linear symmetric forms on $T_\rho U$. To define a norm on the fibers $S^m T^*_\rho U$, we can use, for $f \in S^m T^*_\rho U$,

$$\|f\|_{m,\rho} = \sup_{v_1, \dots, v_m \in T_\rho U} \frac{f(v_1, \dots, v_m)}{\|v_1\|_\rho \cdots \|v_m\|_\rho}, \tag{4-48}$$

where $\|v\|_\rho$ for $v \in T_\rho U$ is the norm induced by the Riemannian metric used to define J_1^u in (3-8). Note that, for any fixed neighborhood of $\mathcal{T}$, there exists a global constant $C > 0$ such that, for each $a \in C_c^\infty(U)$ supported in this neighborhood, one has

$$C^{-1} \|a\|_{C^m} \le \sup_{m' \le m} \sup_{\rho \in U} \|d^{m'} a\|_{m',\rho} \le C \|a\|_{C^m}.$$

We will denote by γ_1, γ_2, etc. elements of $\mathcal{I} := \mathcal{I}(k_0, m_0) = \{(k, m) \in \mathbb{N}^2 : k \le k_0,\ 2k+m \le N_0\}$. We equip $\mathcal{I}$ with the lexicographic order $\prec$ and write $\#\mathcal{I} := \Gamma_{k_0,m_0}$ (see Figure 10). We order the indices of $A_r(\rho)$ with $\prec$. $A_r(\rho)$ depends linearly on $A_{r-1}(F^{-1}(\rho))$ and this dependence can be made explicit by a matrix

$$P^{(r)}(\rho) = (P^{(r)}_{\gamma_1 \gamma_2}(\rho))_{\gamma_1, \gamma_2 \in \mathcal{I}}, \quad \text{where } P^{(r)}_{\gamma_1 \gamma_2}(\rho) \in L(S^{m'} T^*_{F^{-1}(\rho)} U, S^m T^*_\rho U) \quad \text{if } \gamma_1 = (k, m),\ \gamma_2 = (k', m'),$$

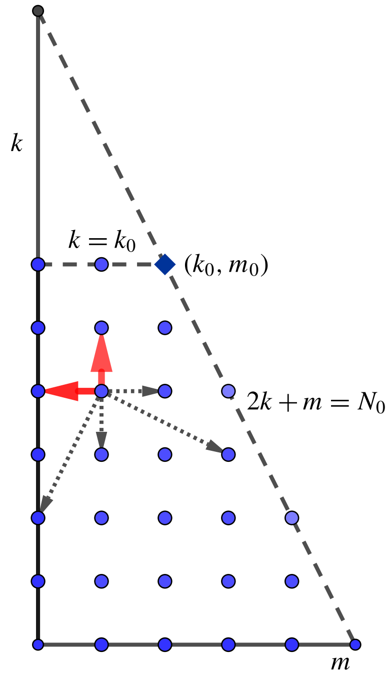

Figure 10. The starting point (k_0, m_0) is represented by a diamond. The set $\mathcal{I}$ corresponds to the couple $(k, m) \in \mathbb{N}^2$ in the region under the dotted lines $k = k_0$ and $2k + m = N_0$. We've represented a family of arrows starting from a point $\gamma_1 \in \mathcal{I}$. The dotted arrows points toward β such that $\gamma_2 \prec \gamma_1$. The big red arrows points toward points γ_2 such that $P^{(r)}_{\gamma_1\gamma_2} = 0$.

so that

$$A_r(\rho) = P^{(r)}(\rho)\, A_{r-1}(F^{-1}(\rho)). \tag{4-49}$$

Notation. If $\gamma_1 = (k, m),\ \gamma_2 = (m', k'),\ \rho, \rho' \in U$ and if $A : S^{m'} T^*_{\rho'} U \to S^m T^*_\rho U$ is a linear operator, we will denote by

$$\| \cdot \|_{\gamma_1, \rho, \gamma_2, \rho'}$$

its subordinate norm for the norms defined by (4-48).

Analyzing (4-43), it turns out that if $\gamma_1 = (k, m),\ \gamma_2 = (k', m') \in \mathcal{I}$, then:

- If $k' > k$, then $P^{(r)}_{\gamma_1\gamma_2}(\rho) = 0$.
- If $k = k'$, the contribution to $d^m a^{(k)}_r(\rho)$ of $a^{(k)}_{r-1}$ comes from

$$\begin{aligned}
&d^m((a_{q_{r-1}} a^{(k)}_{r-1}) \circ F^{-1})(\rho)\\
&= a_{q_{r-1}}(F^{-1}(\rho)) \times d^m(a^{(k)}_{r-1} \circ F^{-1})(\rho) + (\text{derivatives of order strictly less than m for } a^{(k)}_{r-1})\\
&= a_{q_{r-1}}(F^{-1}(\rho)) \times ({}^t dF^{-1}(\rho))^{\otimes m} d^m a^{(k)}_{r-1}(F^{-1}(\rho)) + (\text{derivatives of order strictly less than m for } a^{(k)}_{r-1}).
\end{aligned}$$

In particular, if $\gamma_1 = (k, m) \prec \gamma_2 = (k, m')$ doesn't hold, we see that $P^{(r)}_{\gamma_1\gamma_2}(\rho) = 0$.

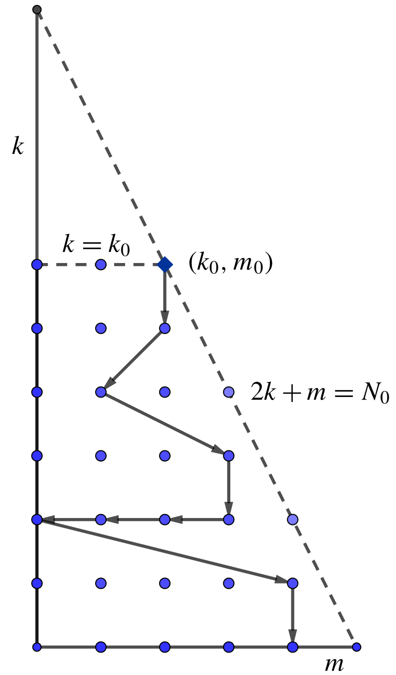

Figure 11. We've represented the reduction of an element $\vec{\gamma} \in \mathcal{E}_r(k_0, m_0)$, i.e., the arrows between γ_i and γ_{i+1} when $\gamma_i \neq \gamma_{i+1}$. During the descent, the value of m can only increase when k decreases strictly.

- If $k' < k$, we can have $P^{(r)}_{\gamma_1\gamma_2}(\rho) \neq 0$ with $m' > m$. But, the use of the lexicographic order ensures that $\gamma_1 \prec \gamma_2$ in that case.

Hence, $P^{(r)}(\rho)$ is a lower triangular matrix and the diagonal coefficients for the index $\gamma_1 = (k, m)$ are given by

$$P^{(r)}_{\gamma_1\gamma_1}(\rho) : f \in S^m T^*_{F^{-1}(\rho)} U \mapsto a_{q_{r-1}}(F^{-1}(\rho)) \times ({}^t dF^{-1}(\rho))^{\otimes m} f \in S^m T^*_\rho U. \tag{4-50}$$

Iterating (4-49), we have

$$A_r(\rho) = P^{(r)}(\rho) P^{(r-1)}(F^{-1}(\rho)) \cdots P^{(2)}(F^{-(r-2)}(\rho)) A_1(F^{1-r}(\rho)).$$

For $\gamma \in \mathcal{I}$, we define, see Figure 11,

$$\mathcal{E}_r(\gamma) = \{\vec{\gamma} = (\gamma_1, \dots, \gamma_r) \in \mathcal{I}^r : \gamma_r = \gamma,\ \gamma_i \prec \gamma_{i+1}\}.$$

The triangular property of P allows us to write

$$(A_r(\rho))_\gamma = \sum_{\vec{\gamma} \in \mathcal{E}_r(\gamma)} P^{(r)}_{\gamma_r\gamma_{r-1}}(\rho) \cdots P^{(2)}_{\gamma_2\gamma_1}(F^{-(r-2)}(\rho))(A_1(F^{1-r}(\rho)))_{\gamma_1}.$$

Control of individual terms. Let us fix $\gamma = (k, m)$ and pick $\vec{\gamma} \in \mathcal{E}_r(\gamma)$. We wish to analyze the operator

$$P_{\vec{\gamma}}(\rho) := P^{(r)}_{\gamma_r\gamma_{r-1}}(\rho) \cdots P^{(2)}_{\gamma_2\gamma_1}(F^{-(r-2)}(\rho)).$$

First of all, $\#\{i \in \{1, \dots, r-1\} : \gamma_{i+1} \neq \gamma_i\} \leq \Gamma_{k_0,m_0}$. So let us write

$$\{i \in \{1, \dots, r-1\} : \gamma_{i+1} \neq \gamma_i\} = \{t_1 < \cdots < t_d\},$$

with $d \leq \Gamma_{k_0,m_0}$. We can set $t_{d+1} = r,\ t_0 = 0$ and we can rewrite

$$\vec{\gamma} = (\underbrace{\beta_1, \dots, \beta_1}_{t_1}, \underbrace{\beta_2, \dots, \beta_2}_{t_2-t_1}, \dots, \underbrace{\beta_d, \dots, \beta_d}_{t_d-t_{d-1}}, \underbrace{\beta_{d+1}, \dots, \beta_{d+1}}_{t_{d+1}-t_d}).$$

For $p \in \{1, \dots, d+1\}$, we introduce the operator

$$D_p(\rho) = P^{(t_p)}_{\beta_p\beta_p}(F^{-(r-t_p)}(\rho)) \cdots P^{(t_{p-1}+2)}_{\beta_p\beta_p}(F^{-(r-t_{p-1}-2)}(\rho)),$$

and for $p \in \{1, \dots, d\}$

$$T_p(\rho) = P^{t_p+1}_{\beta_{p+1}\beta_p}(F^{-(r-t_p-1)}(\rho))$$

so that we can write

$$P_{\vec{\gamma}}(\rho) = D_{d+1}(\rho)T_d(\rho)D_d(\rho) \cdots T_1(\rho)D_1(\rho).$$

For $p \in \{1, \dots, d+1\}$, if $\beta_p = (k, m)$, we can see that

$$\begin{aligned} D_p(\rho) &= \Bigg[\prod_{j=t_{p-1}+1}^{t_p-1} a_{q_j} \circ F^{-(r-j)}(\rho)\Bigg][({}^t dF^{-1}(F^{-(r-t_p)}(\rho)))^{\otimes m} \circ \cdots \circ ({}^t dF^{-1}(F^{-(r-t_{p-1}-2)}(\rho)))^{\otimes m}] \\ &= \Bigg[\prod_{j=t_{p-1}+1}^{t_p-1} a_{q_j} \circ F^{-(r-j)}(\rho)\Bigg]({}^t dF^{-(t_p-t_{p-1}-1)}(F^{-(r-t_p)}(\rho)))^{\otimes m}. \end{aligned}$$

We introduce the word

$$\boldsymbol{q}_p = q_{t_{p-1}} \cdots q_{t_p-1},$$

and set $\rho_p = F^{-(r-t_p)}(\rho),\ \rho'_p = F^{-(t_p-t_{p-1}-1)}(\rho_p)$. To estimate the subordinate norm of $D_p(\rho)$, we use Lemma 4.6. Since $\rho \in \mathcal{V}_{\boldsymbol{q}^+}$, $\rho_p \in \mathcal{V}^+_{\boldsymbol{q}_p}$ and we have

$$\begin{aligned} \|D_p(\rho)\|_{\beta_p,\rho_p,\beta_p,\rho'_p} &\leq \Bigg|\prod_{j=t_{p-1}+1}^{t_p-1} a_{q_j} \circ F^{-(r-j)}(\rho)\Bigg| \sup_{\rho_p \in \mathcal{V}^+_{q_p}} \|dF^{-(t_p-t_{p-1}-1)}(\rho_p)\|^m \\ &\leq (CJ^+_{\boldsymbol{q}_p})^m \Bigg|\prod_{j=t_{p-1}+1}^{t_p-1} a_{q_j} \circ F^{-(r-j)}(\rho)\Bigg| \leq C_{k_0,m_0}(J^+_{\boldsymbol{q}_p})^{N_0} \Bigg|\prod_{j=t_{p-1}+1}^{t_p-1} a_{q_j} \circ F^{-(r-j)}(\rho)\Bigg|. \end{aligned}$$

To estimate the norms of $T_p(\rho)$, we simply note that they depend smoothly on ρ_p, which lies in a compact set, so we can bound them by a uniform constant C_1. This is not a problem since they appear d times in $P_{\vec{\gamma}}$ with $d \leq \Gamma_{k_0,m_0}$. Consequently, we can estimate $\|P_{\vec{\gamma}}(\rho)\|_{\gamma,\rho,\gamma_1,F^{-(r-1)}(\rho)}$,

$$\|P_{\vec{\gamma}}(\rho)\|_{\gamma,\rho,\gamma_1,F^{-(r-1)}(\rho)} \leq C_{k_0,m_0}(J^+_{\boldsymbol{q}_1} \cdots J^+_{\boldsymbol{q}_{d+1}})^{N_0}|a_{\boldsymbol{q},\vec{\gamma}}(\rho)| \leq C_{k_0,m_0}(J^+_{\boldsymbol{q}})^{N_0}|a_{\boldsymbol{q},\vec{\gamma}}(\rho)|, \tag{4-51}$$

where

$$a_{\boldsymbol{q},\vec{\gamma}} = \prod_{p=1}^{d+1} \prod_{j=t_{p-1}+1}^{t_p-1} a_{q_j} \circ F^{-(r-j)}. \tag{4-52}$$

Here, the last inequality holds by applying d times (4-24), with $d \leq \Gamma_{k_0,m_0}$, once we've noted that

$$\boldsymbol{q} = \boldsymbol{q}_1 \cdots \boldsymbol{q}_{d+1}.$$

Finally, if $\gamma_1 = (k_1, m_1)$, to estimate $\|(A_1(F^{1-r}(\rho)))_{\gamma_1}\|_{m_1, F^{1-r}(\rho)}$, we simply note that it depends smoothly on $F^{1-r}(\rho)$, so that we can bound it by a uniform constant. Hence, we have

$$\|P_{\vec{\gamma}}(\rho) A_1(F^{1-r}(\rho))\|_{m,\rho} \leq C_{k_0,m_0} (J_{\boldsymbol{q}}^+)^{N_0} |a_{\boldsymbol{q},\vec{\gamma}}(\rho)|. \tag{4-53}$$

Cardinality of $\mathcal{E}_r(\gamma)$. The bound we will provide is far from optimal but it will turn out to be enough for our purpose. To count the number of elements in $\mathcal{E}_r(\gamma)$, we remark that it is similar to counting the number of decreasing sequences of length r starting from γ. This number is smaller than the number of increasing sequences of length r in $\{1, \dots, \Gamma_{k_0,m_0}\}$. Recalling that the number of sequences $u_1 \leq u_2 \leq \cdots \leq u_r$ satisfying $u_1 = 1$ and $u_r = b$ is equal to $\binom{b+r-2}{r-2}$, one can estimate

$$\#\mathcal{E}_r(\gamma) \leq \sum_{b=1}^{\Gamma_{k_0,m_0}} \binom{b+r-2}{r-2} \leq \Gamma_{k_0,m_0} (r-1)^{\Gamma_{k_0,m_0}}. \tag{4-54}$$

Finally, we can compute explicitly Γ_{k_0,m_0} and we find $\Gamma_{k_0,m_0} = (k_0+1)(m_0+1+k_0)$.

Conclusion. We finally combine (4-54) and (4-53) to prove Proposition 4.10 (recall $|a_q| = |\alpha|\chi_q \leq \|\alpha\|_\infty$):

$$\begin{aligned}
\sup_{\rho \in \mathcal{V}_{q_0 \cdots q_{r-1}}} \|d^{m_0} a_r^{(k_0)}\|_{m_0,\rho} &= \sup_{\rho \in \mathcal{V}_{q_0 \cdots q_{r-1}}} \|(A_r(\rho))_{(k_0,m_0)}\|_{m_0,\rho} \\
&\leq \sum_{\vec{\gamma} \in \mathcal{E}_r(k_0,m_0)} \|P_{\vec{\gamma}}(\rho) A_1(F^{1-r}(\rho))\|_{m_0,\rho} \\
&\leq \Gamma_{k_0,m_0} r^{\Gamma_{k_0,m_0}} C_{k_0,m_0} (J_{\boldsymbol{q}}^+)^{N_0} |a_{\boldsymbol{q},\vec{\gamma}}(\rho)| \\
&\leq C_{k_0,m_0} r^{\Gamma_{k_0,m_0}} (J_{\boldsymbol{q}}^+)^{N_0} \|\alpha\|_\infty^r.
\end{aligned}$$

Finally, we get as expected

$$\|a_r^{(k_0)}\|_{C^{m_0}} \leq C_{k_0,m_0} r^{\Gamma_{k_0,m_0}} (J_{\boldsymbol{q}}^+)^{N_0} \|\alpha\|_\infty^r.$$

4D3. *End of proof of Proposition 4.8.* Armed with these estimates, we are now able to conclude the proof of Proposition 4.8 under the assumptions (4-31). Assume that this assumption is satisfied and construct inductively the symbols $a_r^{(k)}$ with the formula (4-43). Since $J_{\boldsymbol{q}}^+ \leq C h^{-\delta}$, it implies that $n = O(\log h)$. Hence, we have, for $r \leq n$,

$$\|a_r^{(k)}\|_{C^m} \leq C_{k,m} h^{-\delta m} h^{-2k\delta} |\log h|^{\Gamma_{k,m}} \|\alpha\|_\infty^r \leq C_{k,m} h^{-\delta_1 m} h^{-2k\delta_1} \|\alpha\|_\infty^r.$$

The symbol $h^{2\delta_1 k} a_r^{(k)}$ lies in $\|\alpha\|_\infty^r S_{\delta_1}^{\mathrm{comp}}(T^*\mathbb{R})$. Using Borel's theorem with the parameter $h' = h^{1-2\delta_1}$, we can construct a symbol

$$a_{q_0 \cdots q_{r-1}}^+ \sim \sum_{k=0}^{\infty} (h')^k h^{2\delta_1 k} a_r^{(k)} = \sum_{k=0}^{\infty} h^k a_r^{(k)} \in \|\alpha\|_\infty^r S_{\delta_1}^{\mathrm{comp}},$$

that is, for every $N \in \mathbb{N}$,

$$a^+_{q_0\cdots q_{r-1}} - \sum_{k=0}^{N-1} h^k a_r^{(k)} = O(h^{(1-2\delta_1)N}\|\alpha\|_\infty^r).$$

By construction of the $a_r^{(k)}$, for every $N \in \mathbb{N}$, we have

$$U_{\boldsymbol{q}}^+ - \mathrm{Op}_h(a_{\boldsymbol{q}}^+)T^{|\boldsymbol{q}|} = R_n^{(N)} + O(h^{(1-2\delta_1)}\|\alpha\|_\infty^r).$$

Fix some $K \geq 0$ such that $\min(1, \|\alpha\|_\infty^n) = O(h^{-K})$, so that $\|\alpha\|_\infty^r = O(k^{-K})$. With (4-45) and our estimates, we can control

$$\|R_n^{(N)}\| \leq C_N h^N (1 + |\log h|^{\Gamma_{k,m}+1} h^{-\delta(2N+15)} h^{-K}) \leq C_N h^{-15\delta_1 + N(1-2\delta_1)-K}.$$

Since we can choose N as large as we want, we have finally proved that

$$U_{\boldsymbol{q}}^+ - \mathrm{Op}_h(a_{\boldsymbol{q}}^+)T^{|\boldsymbol{q}|} = O(h^\infty). \qquad \square$$

4D4. *Norm of sums over many words.* We'll make use of the tools and notation developed in this subsection to prove the following proposition. To state it, we introduce the notation

$$\mathcal{Q}(n, \tau, C_0) := \{\boldsymbol{q} \in \mathcal{A}^n : J_{\boldsymbol{q}}^+ \leq C_0 h^{-\tau}\}. \tag{4-55}$$

Proposition 4.11. *There exists $C = C(C_0, \tau)$ such that, for every $Q \subset \mathcal{Q}(n, \tau, C_0)$, the following bound holds*:

$$\left| \sum_{\boldsymbol{q} \in Q} U_{\boldsymbol{q}} \right|_{L^2 \to L^2} \leq C \|\alpha\|^n |\log h|. \tag{4-56}$$

Proof. Throughout the proof, we'll denote by C quasiglobal constants, i.e., constants depending on C_0, τ and the same other parameters as global constants. We will also be led to use a constant C_1: it has the same dependence.

<u>Step 1</u>: First note that, since $J_{\boldsymbol{q}}^+ \leq C_0 h^{-\tau}$, n satisfies the bound $n = O(\log h)$.

<u>Step 2</u>: If $\boldsymbol{q} \in \mathcal{Q}(n, \tau, C_0)$, denote by $l(\boldsymbol{q}) = l$ the largest integer such that

$$J^+_{q_0\cdots q_{l-1}} \leq h^{-\tau/2}.$$

Since $J_{q_0\cdots q_l} > h^{-\tau/2}$, $J^+_{q_0\cdots q_{l-1}} > Ch^{-\tau/2}$ and hence

$$J^+_{q_l\cdots q_{n-1}} \leq C \frac{h^{-\tau}}{J^+_{q_0\cdots q_{l-1}}} \leq C_1 h^{-\tau/2}.$$

We can then write $\boldsymbol{q} = \boldsymbol{s}\boldsymbol{r}$ with $\boldsymbol{s} \in \mathcal{Q}(l, \tau/2, 1)$, $\boldsymbol{r} \in \mathcal{Q}(n-l, \tau/2, C_1)$. It follows that we can write

$$\sum_{\boldsymbol{q} \in Q} U_{\boldsymbol{q}} = \sum_{l=1}^{n} \sum_{\substack{\boldsymbol{s} \in \mathcal{Q}(l,\tau/2,1) \\ \boldsymbol{r} \in \mathcal{Q}(n-l,\tau/2,C_1)}} F_l(\boldsymbol{s}, \boldsymbol{r}) U_{\boldsymbol{r}} U_{\boldsymbol{s}},$$

with $F_l(\boldsymbol{s}, \boldsymbol{r}) = \mathbb{1}_{\boldsymbol{sr}\in\mathcal{Q}}$. It is then enough to show the bound

$$\max_{1\le l\le n}\Bigg|\sum_{\substack{\boldsymbol{s}\in\mathcal{Q}(l,\tau/2,1)\\ \boldsymbol{r}\in\mathcal{Q}(n-l,\tau/2,C_1)}} F_l(s,r)U_{\boldsymbol{r}}U_{\boldsymbol{s}}\Bigg| \le C\|\alpha\|_\infty^n. \tag{4-57}$$

In the following, we fix some $1\le l\le n$ and we'll simply write $\sum_{\boldsymbol{s},\boldsymbol{r}}$ to alleviate the notation. Note that the number of terms in the sum is bounded by

$$|\mathcal{Q}(l,\tau/2,1)\times\mathcal{Q}(n-l,\tau/2,C_1)| \le |\mathcal{A}|^l\times|\mathcal{A}|^{n-l}\le|\mathcal{A}|^n\le h^{-Q},$$

where $Q = C\log|\mathcal{A}|$.

<u>Step 3</u>: We fix some large $N\in\mathbb{N}$ and $\delta_1\in(\tau/2,1/2)$. Recall that we can write

$$U_{\boldsymbol{s}} = \Bigg(\mathrm{Op}_h\Bigg(\sum_{k=0}^{N-1}h^k a_{\boldsymbol{s}}^{(k)}\Bigg) + O_{L^2\to L^2}(h^{(1-2\delta_1)N-15\delta_1}\|\alpha\|_\infty^l)\Bigg)T^l,$$

$$U_{\boldsymbol{r}} = T^{n-l}\Bigg(\mathrm{Op}_h\Bigg(\sum_{k=0}^{N-1}h^k a_{\boldsymbol{r}}^{(k)}\Bigg) + O_{L^2\to L^2}(h^{(1-2\delta_1)N-15\delta_1}\|\alpha\|_\infty^{n-l})\Bigg),$$

with bounds on $a_{\boldsymbol{s}}^{(k)}$ and $a_{\boldsymbol{r}}^{(k)}$ given by Proposition 4.8.

We then use the formula for the composition of operators in $\Psi_{\delta_1}^{\mathrm{comp}}(T^*\mathbb{R})$ (Lemma 3.3) and for simplicity, we write $\mathcal{L}_k(a,b)(\rho) = (i^k/k!)(A(D))^k(a\otimes b)(\rho,\rho)$. For $0\le k\le N-1$, we set

$$a_{\boldsymbol{s},\boldsymbol{r},k} = \sum_{j+k_-+k_+=k}\mathcal{L}_j(a_{\boldsymbol{r}}^{(k_-)}, a_{\boldsymbol{s}}^{(k_+)}).$$

Note that if $j+k_-+k_+\ge N$,

$$\begin{aligned}\|a_{\boldsymbol{r}}^{(k_-)}\otimes a_{\boldsymbol{s}}^{(k_+)}\|_{C^{2j+13}} &\le C_j\sup_{m_++m_-=2j+13}\|a_{\boldsymbol{r}}^{(k_-)}\|_{C^{m_-}}\|a_{\boldsymbol{s}}^{(k_+)}\|_{C^{m_+}}\\ &\le C_{j,k_-,k_+}h^{-(2k_-+m_-)\delta_1}h^{-(2k_-+m_+)\delta_1}\|\alpha\|_\infty^n\\ &\le C_{j,k_-,k_+}h^{-2\delta_1(j+k_-+k_+)-13\delta_1}\|\alpha\|_\infty^n\\ &\le C_{j,k_-,k_+}h^{-2\delta_1N-13\delta_1}\|\alpha\|_\infty^n\end{aligned}$$

and henceforth,

$$O(h^{j+k_-+k_+}\|a_{\boldsymbol{r}}^{(k_-)}\otimes a_{\boldsymbol{s}}^{(k_+)}\|_{C^{2j+13}}) = O(h^{(1-2\delta_1)N-15\delta_1}\|\alpha\|_\infty^n).$$

As a consequence, we can write

$$U_{\boldsymbol{r}}U_{\boldsymbol{s}} = T^{n-l}\Bigg(\mathrm{Op}_h\Bigg(\sum_{k=0}^{N-1}h^k a_{\boldsymbol{s},\boldsymbol{r},k}\Bigg)\Bigg)T^l + O_{L^2\to L^2}(h^{(1-2\delta_1)N-15\delta_1}\|\alpha\|_\infty^n).$$

It follows that

$$\sum_{\boldsymbol{s},\boldsymbol{r}}U_{\boldsymbol{r}}U_{\boldsymbol{s}} = T^{n-l}\Bigg(\mathrm{Op}_h\Bigg(\sum_{k=0}^{N-1}h^k a^{(k)}\Bigg)\Bigg)T^l + O_{L^2\to L^2}(h^{(1-2\delta_1)N-15\delta_1-Q}\|\alpha\|_\infty^n),$$

where

$$a^{(k)} = \sum_{\boldsymbol{s},\boldsymbol{r}} F(\boldsymbol{s},\boldsymbol{r}) a_{\boldsymbol{s},\boldsymbol{r},k}. \tag{4-58}$$

Suppose that N has been chosen such that

$$(1-2\delta_1)N > 15\delta_1 + Q.$$

The remainder term is thus controlled by the desired bound since it is of order $O(\|\alpha\|_\infty^n)$.

<u>Step 4</u>: C^0 norm of $a^{(0)}$. We have

$$a^{(0)} = \sum_{\boldsymbol{s},\boldsymbol{r}} F(\boldsymbol{s},\boldsymbol{r}) a_{\boldsymbol{s}}^{(0)} a_{\boldsymbol{r}}^{(0)},$$

where, by virtue of (4-37) and (4-38),

$$a_{\boldsymbol{s}}^{(0)} = \prod_{p=1}^{l} a_{s_{l-p}} \circ F^{-p}, \quad a_{\boldsymbol{r}}^{(0)} = \prod_{p=0}^{n-l-1} a_{r_p} \circ F^{p}.$$

As a consequence, we can estimate

$$|a^{(0)}| \le \sum_{\boldsymbol{s},\boldsymbol{r}} |a_{\boldsymbol{s}}^{(0)}|\,|a_{\boldsymbol{r}}^{(0)}| \le \prod_{p=1}^{l}\biggl(\sum_{q\in\mathcal{A}} |a_q|\biggr)\circ F^{-p} \times \prod_{p=0}^{n-l-1}\biggl(\sum_{q\in\mathcal{A}} |a_q|\biggr)\circ F^{p} \le \|\alpha\|_\infty^n.$$

<u>Step 5</u>: C^m norms of $a^{(k)}$. We will show there exist constants $C_{k,m}$ (depending only on C_0, δ_1, τ and m, k) such that, for all $0 \le k \le N-1$ and $m \in \mathbb{N}$,

$$\|a^{(k)}\|_{C^m} \le C_{k,m} h^{-(2k+m)\delta_1} \|\alpha\|_\infty^n. \tag{4-59}$$

Let's compute

$$\begin{aligned}
\|a^{(k)}\|_{C^m} &\le \sum_{\boldsymbol{s},\boldsymbol{r}} \|a_{\boldsymbol{s},\boldsymbol{r},k}\|_{C^m} \le \sum_{\boldsymbol{s},\boldsymbol{r}} \sum_{j+k_+ + k_- = k} \|\mathcal{L}_j(a_{\boldsymbol{r}}^{(k_-)}, a_{\boldsymbol{s}}^{(k_+)})\|_{C^m} \\
&\le \sum_{\boldsymbol{s},\boldsymbol{r}} \sum_{j+k_+ + k_- = k} \|a_{\boldsymbol{r}}^{(k_-)} \otimes a_{\boldsymbol{s}}^{(k_+)}\|_{C^{2j+m}} \\
&\le \sum_{\boldsymbol{s},\boldsymbol{r}} \sum_{\substack{j+k_+ + k_- = k \\ m_+ + m_- \le m+2j}} \|a_{\boldsymbol{r}}^{(k_-)}\|_{C^{m_-}} \|a_{\boldsymbol{s}}^{(k_+)}\|_{C^{m_+}},
\end{aligned}$$

and hence

$$\|a^{(k)}\|_{C^m} \le C_{k,m} \sup_{\substack{j+k_+ + k_- = k \\ m_+ + m_- \le m+2j}} \sum_{\boldsymbol{s},\boldsymbol{r}} \|a_{\boldsymbol{r}}^{(k_-)}\|_{C^{m_-}} \|a_{\boldsymbol{s}}^{(k_+)}\|_{C^{m_+}}. \tag{4-60}$$

Let us fix j, k_+, k_-, m_+, m_- satisfying $j + k_+ + k_- = k$, $m_- + m_+ \le m + 2j$ and let us estimate

$$\sum_{\boldsymbol{s}} \|a_{\boldsymbol{s}}^{(k_+)}\|_{C^{m_+}} \times \sum_{\boldsymbol{r}} \|a_{\boldsymbol{r}}^{(k_-)}\|_{C^{m_-}}.$$

We estimate the sum over $\boldsymbol{s}$. The same kind of estimates will hold for $\boldsymbol{r}$ with the same methods. We reuse the tools developed in the last subsections. Namely, we set $N_+ = 2k_+ + m_+$, $\gamma_+ = (k_+, m_+)$, $\mathcal{I} = \mathcal{I}(\gamma_+)$ and

$$(A_{\boldsymbol{s}}(\rho)) = (d^m a_{\boldsymbol{s}}^{(k)})_{k \le k_+, 2k+m \le N_+}.$$

We have shown that there exists a global constant $C > 0$ such that

$$\|a_{\boldsymbol{s}}^{(k_+)}\|_{C^{m_+}} \le \sup_\rho \|A_{\boldsymbol{s}}(\rho)\| \le C \sum_{\vec{\gamma}\in\mathcal{E}_l(\gamma_+)} \|P_{\vec{\gamma}}(\rho)\| \le \sum_{\vec{\gamma}\in\mathcal{E}_l(\gamma_+)} C_{N_+,k_+}(J_{\boldsymbol{s}}^+)^{N_+}|a_{\boldsymbol{s},\vec{\gamma}}(\rho)|$$
$$\le C_{N_+,k_+} h^{-\tau N_+/2} \sum_{\vec{\gamma}\in\mathcal{E}_l(\gamma_+)} |a_{\boldsymbol{s},\vec{\gamma}}(\rho)|,$$

where C_{N_+,k_+} depends on C_0, τ, N_+, k_+ and global parameters. We hence have to estimate

$$\sum_{\boldsymbol{s}} \sum_{\vec{\gamma}\in\mathcal{E}_l(\gamma_+)} |a_{\boldsymbol{s},\vec{\gamma}}(\rho)|.$$

Fix $\vec{\gamma} \in \mathcal{E}_l(\alpha_+)$ and write it

$$\vec{\gamma} = (\underbrace{\beta_1,\dots,\beta_1}_{t_1}, \underbrace{\beta_2,\dots,\beta_2}_{t_2-t_1}, \dots, \underbrace{\beta_d,\dots,\beta_d}_{t_d-t_{d-1}}, \underbrace{\beta_{d+1},\dots,\beta_{d+1}}_{t_{d+1}-t_d}), \quad \text{where } d \le \Gamma_{k_+,m_+},$$

and recall that

$$a_{\boldsymbol{s},\vec{\gamma}} = \prod_{p=1}^{d+1} \prod_{j=t_{p-1}+1}^{t_p-1} a_{s_j} \circ F^{-(l-j)}.$$

When one sums over $\boldsymbol{s} \in \mathcal{A}^l$, the values of $\boldsymbol{s}$ at the indices $t_i,\ 1 \le i \le d$, do not play a role and we write

$$\sum_{\boldsymbol{s}} |a_{\boldsymbol{s},\vec{\gamma}}| = \sum_{s_{t_1}\in\mathcal{A}} \cdots \sum_{s_{t_d}\in\mathcal{A}} \prod_{p=1}^{d+1} \prod_{j=t_{p-1}+1}^{t_p-1} \Bigl(\sum_{s\in\mathcal{A}} |a_s|\Bigr) \circ F^{-(l-j)}$$
$$\le |\mathcal{A}|^d \sup_\rho \Bigl(\sum_{s\in\mathcal{A}} |a_s|\Bigr)^l \le K^{\Gamma_{k_+,m_+}} \|\alpha\|_\infty^l \le C_{k_+,m_+} \|\alpha\|_\infty^l.$$

As a consequence,

$$\sum_{\boldsymbol{s}} \sum_{\vec{\gamma}\in\mathcal{E}_l(\gamma_+)} |a_{\boldsymbol{s},\vec{\gamma}}| \le \#\mathcal{E}_l(\gamma_+) C_{k_+,m_+} \|\alpha\|_\infty^l \le C_{k_+,m_+} (l-1)^{\Gamma_{k_+,m_+}} \|\alpha\|_\infty^l,$$

which gives

$$\sum_{\boldsymbol{s}} \|a_{\boldsymbol{s}}^{(k_+)}\|_{C^{m_+}} \le C_{k_+,m_+} h^{-\tau N_+/2} (l-1)^{\Gamma_{k_+,m_+}} \|\alpha\|_\infty^l \le C_{k_+,m_+} h^{-\delta_1 N_+} \|\alpha\|_\infty^l,$$

where the last inequality (with a different value of C_{k_+,m_+}) follows from the fact that $l = O(\log h)$ and $\delta_1 > \tau/2$. The same kind of estimates holds for the sum over $\boldsymbol{r}$:

$$\sum_{\boldsymbol{r}} \|a_{\boldsymbol{r}}^{(k_-)}\|_{C^{m_-}} \le C_{k_-,m_-} h^{-\delta_1 N_-} \|\alpha\|_\infty^{n-l}.$$

Eventually, using (4-60), we get (4-59) since

$$N_+ + N_- = 2k_+ + m_+ + 2k_- + m_- \le 2(k_+ + k_- + j) + m = 2k + m.$$

<u>Step 6</u>: Conclusion. We can conclude the proof of the Proposition 4.11. The bound (4-59) shows that, for $0 \le k \le N-1,\ a^{(k)} \in h^{-2k\delta_1} \|\alpha\|_\infty^n S_{\delta_1}^{\text{comp}}$ and thus $\sum_{k=0}^{N-1} h^k a^{(k)} \in S_{\delta_1}^{\text{comp}} \|\alpha\|_\infty^n$. From the L^2-boundedness

of pseudodifferential operators with symbol in S_{δ_1},

$$\left\|\operatorname{Op}_h\left(\sum_{k=0}^{N-1} h^k a^{(k)}\right)\right\| \le \sum_{k=0}^{N-1}\sum_{m\le M} h^{k+m/2}\|a^{(k)}\|_{C^m} \le \sum_{k=0}^{N-1}\sum_{m\le M} C_{k,m} h^{(k+2m)(1/2-\delta_1)}\|\alpha\|_\infty^n \le C\|\alpha\|_\infty^n,$$

where C depends only on C_0, τ, δ_1. Since $\|T\| \le 1$, we get

$$\left\|\sum_{\boldsymbol{s},\boldsymbol{r}} F(\boldsymbol{s},\boldsymbol{r}) U_{\boldsymbol{r}} U_{\boldsymbol{s}}\right\| \le C\|\alpha\|_\infty^n,$$

which concludes the proof of Proposition 4.11. □

4E. *Manipulations of the $U_{\boldsymbol{q}}$.*

4E1. *First consequences.* We now make use of Proposition 4.8 to deduce several important facts. We go on following [Dyatlov et al. 2022]. In the whole subsection, we fix $0 \le \delta < \delta_1 < \frac{1}{2}$ and $C_0 > 0$. We define $\mathcal{A}^{\to} = \bigcup_{n\in\mathbb{N}} \mathcal{A}^n$.

Remark. The constants in $O(h^\infty)$ depend on $\boldsymbol{p}$ and $\boldsymbol{q}$ only through C_0, δ, δ_1, not on the precise values of $\boldsymbol{p}$ and $\boldsymbol{q}$. It will always be the case in the following and we won't mention it anymore. As already done, all the quasiglobal constants (i.e., depending on global parameters and $C_0, \delta, \tau, \delta_1$) will be noted by the letter C.

Lemma 4.12. *Let $\boldsymbol{q}, \boldsymbol{p} \in \mathcal{A}^{\to}$ satisfying $\mathcal{V}_{\boldsymbol{q}}^+ \cap \mathcal{V}_{\boldsymbol{p}}^- = \varnothing$ and $\max(J_{\boldsymbol{q}}^+, J_{\boldsymbol{p}}^-) \le C_0 h^{-\delta}$. Then*

$$U_{\boldsymbol{p}} U_{\boldsymbol{q}} = O(h^\infty)_{L^2\to L^2}.$$

Proof. By virtue of Proposition 4.8, we can write

$$U_{\boldsymbol{p}} = T^{|\boldsymbol{p}|}\operatorname{Op}_h(a_{\boldsymbol{p}}^-) + O(h^\infty),$$
$$U_{\boldsymbol{q}} = \operatorname{Op}_h(a_{\boldsymbol{q}}^+) T^{|\boldsymbol{q}|} + O(h^\infty).$$

With $a_{\boldsymbol{q}}^+ \in \|\alpha\|_\infty^{|\boldsymbol{q}|} S_{\delta_1}^{\mathrm{comp}}$, $a_{\boldsymbol{p}}^- \in \|\alpha\|_\infty^{|\boldsymbol{p}|} S_{\delta_1}^{\mathrm{comp}}$ and $\operatorname{supp} a_{\boldsymbol{p}}^- \subset \mathcal{V}_{\boldsymbol{p}}^-$, $\operatorname{supp} a_{\boldsymbol{q}}^+ \subset \mathcal{V}_{\boldsymbol{q}}^+$. Since $\mathcal{V}_{\boldsymbol{q}}^+ \cap \mathcal{V}_{\boldsymbol{p}}^- = \varnothing$, $\operatorname{Op}_h(a_{\boldsymbol{p}}^-)\operatorname{Op}_h(a_{\boldsymbol{q}}^+) = O(h^\infty)$ as a consequence of the composition of two symbols of S_{δ_1}. The constants in $O(h^\infty)$ depend on seminorms of these symbols, themselves depending on C_0, τ, δ_1. Since $T^n = O(1)$, the result is proved. □

Lemma 4.12 will have interesting consequences, starting with the following lemma which enables us to get rid (that is to say to control by $O(h^\infty)$) of words $\boldsymbol{q}$ where $\mathcal{V}_{\boldsymbol{q}}^\pm = \varnothing$, under some assumptions. In particular, it can be applied without trouble to words of "small" lengths $N \le |\log h|/(2\lambda_1)$, which could also be deduced from applying Egorov's theorem up to the global Ehrenfest time $|\log h|/(2\lambda_1)$.

Lemma 4.13. *Let $\boldsymbol{q} \in \mathcal{A}^{\to}$ such that $n = |\boldsymbol{q}| \le C_0|\log h|$ and assume that $\mathcal{V}_{\boldsymbol{q}}^- = \varnothing$. We suppose that one of the following assumptions is satisfied*:

(i) *If $m = \max\{k \in \{1,\dots,n\} : \mathcal{V}_{q_0\cdots q_{k-1}}^- \ne \varnothing\}$, then $J_{q_0\cdots q_{m-1}}^- \le C_0 h^{-2\delta}$.*

(ii) *If $m = \min\{k \in \{0,\dots,n-1\} : \mathcal{V}_{q_m\cdots q_{n-1}}^- \ne \varnothing\}$, then $J_{q_m\cdots q_{n-1}}^- \le C_0 h^{-2\delta}$.*

Then, $U_{\boldsymbol{q}} = O(h^\infty)$.

Proof. We prove this lemma under assumption (i). This is similar under (ii). We let $m=\max\{k\in\{1,\dots,n\}:\mathcal{V}^-_{q_0\cdots q_{m-1}}\neq\varnothing\}$ and assume $J^-_{q_0\cdots q_{m-1}}\le C_0h^{-2\delta}$. Due to (4-12), it is enough to show that $U_{q_0\cdots q_m}=O(h^\infty)$. Let us define $l=\max\{k\in\{1,\dots,m\}:J^-_{q_0\cdots q_{l-1}}\le h^{-\delta}\}$ and notice that $l<m$ (if h is small enough). By maximality of l, it is clear that $J^-_{q_0\cdots q_l}\ge h^{-\delta}$. According to the third point of Lemma 4.5,

$$J^-_{q_{l+1}\cdots q_{m-1}}\sim\frac{J^-_{q_0\cdots q_{m-1}}}{J^-_{q_0\cdots q_l}}\le Ch^{-\delta}.$$

Set $\boldsymbol{p}=q_l\cdots q_m$. We distinguish now between two cases:

- $\mathcal{V}^-_{\boldsymbol{p}}\neq\varnothing$: We set $\boldsymbol{r}=q_0\cdots q_{l-1}$. It follows that

$$\max(J^-_{\boldsymbol{p}},J^-_{\boldsymbol{r}})\le Ch^{-\delta}.$$

Moreover,

$$\mathcal{V}^-_{\boldsymbol{p}}\cap\mathcal{V}^+_{\boldsymbol{r}}=F^l(\mathcal{V}^-_{q_0\cdots q_m})=\varnothing.$$

By Lemma 4.12, $U_{\boldsymbol{p}}U_{\boldsymbol{r}}=U_{q_0\cdots q_m}=O(h^\infty)$.

- $\mathcal{V}^-_{\boldsymbol{p}}=\varnothing$: This time, we have $\max(J^-_{q_l\cdots q_{m-1}},J^-_{q_m})\le Ch^{-\delta}$ and $\mathcal{V}^-_{q_m}\cap\mathcal{V}^+_{q_l\cdots q_{m-1}}=\varnothing$. According to Lemma 4.12, $U_{q_l\cdots q_m}=U_{q_m}U_{q_l\cdots q_{m-1}}=O(h^\infty)$. It follows that $U_{q_0\cdots q_m}=O(h^\infty)$. □

4E2. *Orthogonality of the $U_{\boldsymbol{q}}$.* We now focus on terms $U_{\boldsymbol{q}}U^*_{\boldsymbol{p}}$ and $U^*_{\boldsymbol{q}}U_{\boldsymbol{p}}$ when $\mathcal{V}^+_{\boldsymbol{q}}$ and $\mathcal{V}^+_{\boldsymbol{p}}$ are disjoint, under growth conditions of the Jacobian. The following result shows that the operators $U_{\boldsymbol{q}}$ and $U_{\boldsymbol{p}}$ are (up to $O(h^\infty)$) orthogonal. These estimates will turn out to be important to apply Cotlar–Stein-type estimates.

Proposition 4.14. *Assume that $\boldsymbol{q},\boldsymbol{p}\in\mathcal{A}^\to$ are two words of same length $|\boldsymbol{q}|=|\boldsymbol{p}|=n$ satisfying $\mathcal{V}^+_{\boldsymbol{q}}\cap\mathcal{V}^+_{\boldsymbol{p}}=\varnothing$ and $\max(J^+_{\boldsymbol{q}},J^+_{\boldsymbol{p}})\le C_0h^{-2\delta}$. Then,*

$$U_{\boldsymbol{q}}U^*_{\boldsymbol{p}}=O(h^\infty),$$
$$U^*_{\boldsymbol{q}}U_{\boldsymbol{p}}=O(h^\infty).$$

Before proving it, we need the following lemma, whose proof relies on the iterative construction of the symbols $a^\pm_{\boldsymbol{q}}$.

Lemma 4.15. *Assume $\boldsymbol{q},\boldsymbol{p}\in\mathcal{A}^\to$ are two words of same length $|\boldsymbol{q}|=|\boldsymbol{p}|=n$ satisfying $\max(J^+_{\boldsymbol{q}},J^+_{\boldsymbol{p}})\le C_0h^{-\delta}$. Then,*

$$U_{\boldsymbol{q}}U^*_{\boldsymbol{p}}=\mathrm{Op}_h(a^+_{\boldsymbol{q}})\,\mathrm{Op}_h(a^+_{\boldsymbol{p}})^*+O(h^\infty),$$
$$U^*_{\boldsymbol{q}}U_{\boldsymbol{p}}=\mathrm{Op}_h(a^-_{\boldsymbol{q}})^*\,\mathrm{Op}_h(a^-_{\boldsymbol{p}})+O(h^\infty).$$

Proof of Lemma 4.15. We prove the first equality. The second one could be treated similarly. Recall the construction procedure of Section 4D. We adopt the same notation. We will show by induction on $r\in\{0,\dots,n-1\}$ that

$$V_r:=U_{q_0\cdots q_{r-1}}U^*_{p_0\cdots p_{r-1}}=\mathrm{Op}_h(a^+_{q_0\cdots q_{r-1}})\,\mathrm{Op}_h(a^+_{p_0\cdots p_{r-1}})^*+O(h^\infty).$$

The case $r=1$ follows from

$$MA_{q_0}A^*_{p_0}M^*=\mathrm{Op}_h(a^+_{q_0})TT^*\,\mathrm{Op}_h(a^+_{p_0})^*+O(h^\infty)=\mathrm{Op}_h(a^+_{q_0})\,\mathrm{Op}_h(a^+_{p_0})^*+O(h^\infty),$$

where we use the fact that $TT^* = I + O(h^\infty)$ microlocally in $\mathcal{V}^+_{p_0}$, Assume that the assumption is satisfied for r, namely

$$V_r = \operatorname{Op}_h(a^+_{q_0\cdots q_{r-1}}) \operatorname{Op}_h(a^+_{p_0\cdots p_{r-1}}) + O(h^\infty),$$

and let's prove it for $r+1$:

$$\begin{aligned} V_{r+1} &= M A_{q_r} V_r A^*_{p_r} M^* \\ &= M A_{q_r} \operatorname{Op}_h(a^+_{q_0\cdots q_{r-1}}) \operatorname{Op}_h(a^+_{p_0\cdots p_{r-1}})^* A^*_{p_r} M^* r + O(h^\infty) \\ &= \operatorname{Op}_h(a^+_{q_0\cdots q_r}) T T^* \operatorname{Op}_h(a^+_{p_0\cdots p_r})^* + O(h^\infty) \\ &= \operatorname{Op}_h(a^+_{q_0\cdots q_r}) \operatorname{Op}_h(a^+_{p_0\cdots p_r})^* + O(h^\infty). \end{aligned}$$

The last equality follows from $TT^* = I + O(h^\infty)$ microlocally in $\mathcal{V}^+_{p_r}$ and the one before is due to the recursive construction of the symbols $a^+_{q_0\cdots q_r}$ in the Section 4D. □

Proof of Proposition 4.14. Let us begin with the first equality. Consider the largest integer l such that

$$\max(J^+_{q_0\cdots q_{l-1}}, J^+_{p_0\cdots p_{l-1}}) \le h^{-\delta}.$$

We set $\boldsymbol{q}_\leftarrow = q_0 \cdots q_{l-1}$ and $\boldsymbol{q}_\rightarrow = q_l \cdots q_{n-1}$, and define similar notation for $\boldsymbol{p}$. We obviously have

$$U_{\boldsymbol{q}} U^*_{\boldsymbol{p}} = U_{\boldsymbol{q}_\rightarrow} U_{\boldsymbol{q}_\leftarrow} U^*_{\boldsymbol{p}_\leftarrow} U^*_{\boldsymbol{p}_\rightarrow}.$$

We then consider two cases:

- $\mathcal{V}^+_{\boldsymbol{q}_\leftarrow} \cap \mathcal{V}^+_{\boldsymbol{p}_\leftarrow} = \varnothing$: we may write

$$U_{\boldsymbol{q}_\leftarrow} U^*_{\boldsymbol{p}_\leftarrow} = T^l \operatorname{Op}_h(a^-_{\boldsymbol{q}_\leftarrow}) \operatorname{Op}_h(a^-_{\boldsymbol{q}_\leftarrow})^* T^l + O(h^\infty).$$

Since, $\mathcal{V}^-_{\boldsymbol{q}_\leftarrow} \cap \mathcal{V}^-_{\boldsymbol{p}_\leftarrow} = \varnothing$, we can use the composition formula in $S^{\mathrm{comp}}_{\delta_1}$ to conclude $\operatorname{Op}_h(a^-_{\boldsymbol{q}_\leftarrow}) \operatorname{Op}_h(a^-_{\boldsymbol{q}_\leftarrow})^* = O(h^\infty)$, which gives the desired result, recalling that $U_{\boldsymbol{q}} = O(1)$.

- $\mathcal{V}^+_{\boldsymbol{q}_\leftarrow} \cap \mathcal{V}^+_{\boldsymbol{p}_\leftarrow} \neq \varnothing$: In this case, we use the previous lemma and we can write

$$U_{\boldsymbol{q}_\leftarrow} U^*_{\boldsymbol{p}_\leftarrow} = \operatorname{Op}_h(a^+_{\boldsymbol{q}_\leftarrow}) \operatorname{Op}_h(a^+_{\boldsymbol{p}_\leftarrow})^* + O(h^\infty).$$

By virtue of the second point of Lemma 4.5, $J^+_{\boldsymbol{q}_\leftarrow} \sim J^+_{\boldsymbol{p}_\leftarrow}$. Moreover, by maximality of l, either $J^+_{\boldsymbol{q}_\leftarrow q_l} > h^{-\delta}$ or $J^+_{\boldsymbol{p}_\leftarrow p_l} > h^{-\delta}$. But

$$J^+_{\boldsymbol{q}_\leftarrow q_l} \sim J^+_{\boldsymbol{q}_\leftarrow}.$$

Hence, $J^+_{\boldsymbol{q}_\leftarrow} \sim h^{-\delta}$. Using now the third point of Lemma 4.5, we conclude that

$$J^+_{\boldsymbol{q}_\rightarrow} \sim J^+_{\boldsymbol{p}_\rightarrow} \sim h^{-\delta}.$$

This estimate allows us to write

$$U_{\boldsymbol{q}} U^*_{\boldsymbol{p}} = T^{n-l} \operatorname{Op}_h(a^-_{\boldsymbol{q}_\rightarrow}) \operatorname{Op}_h(a^+_{\boldsymbol{q}_\leftarrow}) \operatorname{Op}_h(a^+_{\boldsymbol{p}_\leftarrow})^* \operatorname{Op}_h(a^-_{\boldsymbol{p}_\rightarrow})^* (T^*)^{n-l} + O(h^\infty),$$

with all the symbols in $h^{-M} S^{\mathrm{comp}}_{\delta_1}$ for some $M > 0$. To conclude, we use the composition formula in this symbol class, noting that

$$\mathcal{V}^+_{\boldsymbol{q}_\leftarrow} \cap \mathcal{V}^-_{\boldsymbol{q}_\rightarrow} \cap \mathcal{V}^+_{\boldsymbol{p}_\leftarrow} \cap \mathcal{V}^-_{\boldsymbol{p}_\rightarrow} = F^l(\mathcal{V}^-_{\boldsymbol{q}} \cap \mathcal{V}^-_{\boldsymbol{p}}) = \varnothing.$$

To deal with the second equality, we consider the smallest integer l such that

$$\max(J^+_{q_l\cdots q_{n-1}}, J^+_{p_l\cdots p_{n-1}}) \le h^{-\delta}.$$

As before, we write $\boldsymbol{q}_\leftarrow = q_0\cdots q_{l-1}$ and $\boldsymbol{q}_\rightarrow = q_l\cdots q_{n-1}$, and define similar notation for $\boldsymbol{p}$. We obviously have

$$U^*_{\boldsymbol{q}}U_{\boldsymbol{p}} = U^*_{\boldsymbol{q}_\leftarrow}U^*_{\boldsymbol{q}_\rightarrow}U_{\boldsymbol{p}_\rightarrow}U_{\boldsymbol{p}_\leftarrow}.$$

We distinguish the cases $\mathcal{V}^+_{\boldsymbol{q}_\rightarrow}\cap\mathcal{V}^+_{\boldsymbol{p}_\rightarrow}=\varnothing$ and $\mathcal{V}^+_{\boldsymbol{q}_\rightarrow}\cap\mathcal{V}^+_{\boldsymbol{p}_\rightarrow}\neq\varnothing$ and argue similarly. □

4F. ***Reduction to subwords with precise growth of their Jacobian.*** Recall that we are interested in a decay bound for $\|\mathfrak{M}^{N_0+N_1}\|$, where $\mathfrak{M}=M(\mathrm{Id}-A_\infty)=\sum_{q\in\mathcal{A}}MA_q$. For this purpose, we take the decomposition $\mathfrak{M}^{N_1}=\sum_{\boldsymbol{q}\in\mathcal{A}^{N_1}}U_{\boldsymbol{q}}$.

If $\boldsymbol{q}\in\mathcal{A}^{N_1}$, either $\mathcal{V}^+_{\boldsymbol{q}}=\varnothing$, and in this case $J^+_{\boldsymbol{q}}=+\infty$, or $\mathcal{V}^+_{\boldsymbol{q}}\neq\varnothing$, which implies that $J^+_{\boldsymbol{q}}\ge e^{\lambda_1 N_1}\ge h^{-1}\gg h^{-\tau}$. In both cases, the following integer is well-defined:

$$n(\boldsymbol{q})=\max\{k\in\{1,N_1\}: J^+_{q_{N_1-k}\cdots q_{N_1-1}}\le h^{-\tau}\}. \tag{4-61}$$

We then set $\boldsymbol{q}_\tau=q_{N_1-n(q)-1}\cdots q_{N_1-1}$. The case $\mathcal{V}_{\boldsymbol{q}_\tau}=\varnothing$ is irrelevant. Indeed, if $\boldsymbol{q}\in\mathcal{A}^{N_1}$ and if $\mathcal{V}_{\boldsymbol{q}_\tau}=\varnothing$, then $U_{\boldsymbol{q}}=O(h^\infty)$, as an obvious consequence of Lemma 4.13. Then, we set

$$Q=\{\boldsymbol{q}\in\mathcal{A}^{N_1}:\mathcal{V}_{\boldsymbol{q}_\tau}\neq\varnothing\} \tag{4-62}$$

so that, due to the fact that $|\mathcal{A}^{N_1}|=O(h^{-M})$, for some $M>0$, we have

$$\mathfrak{M}^{N_1}=\sum_{\boldsymbol{q}\in Q}U_{\boldsymbol{q}}+O(h^\infty).$$

We partition Q in function of the length of $\boldsymbol{q}_\tau$ and the value of q_{N_1-1}. Namely, we set

$$Q_0(n,a)=\{\boldsymbol{q}\in Q:|\boldsymbol{q}_\tau|=n,\ q_{N_1-1}=a\}.$$

We finally set $Q(n,a)=\{\boldsymbol{q}_\tau:\boldsymbol{q}\in Q_0(n,a)\}$, which is simply the set of words $\boldsymbol{q}\in\mathcal{A}^n$ such that $q_{n-1}=a$ and $J^+_{q_1\cdots q_{n-1}}\le h^{-\tau}<J^+_{\boldsymbol{q}}$. Note that every word $\boldsymbol{q}\in Q_0(n,a)$ can be written in the form $\boldsymbol{q}=\boldsymbol{r}\boldsymbol{p}$, with $\boldsymbol{p}\in Q(n,a)$ and $\boldsymbol{r}\in\mathcal{A}^{N_1-n}$. We deduce that, *modulo* $O(h^\infty)$,

$$\mathfrak{M}^{N_1}=\sum_{n=1}^{N_1}\sum_{a\in\mathcal{A}}\sum_{\boldsymbol{q}\in Q_0(n,a)}U_{\boldsymbol{q}}=\sum_{n=1}^{N_1}\sum_{a\in\mathcal{A}}\sum_{\substack{\boldsymbol{p}\in Q(n,a)\\ \boldsymbol{r}\in\mathcal{A}^{N_1-n}}}U_{\boldsymbol{p}}U_{\boldsymbol{r}}=\sum_{n=1}^{N_1}\sum_{a\in\mathcal{A}}\Bigl(\sum_{\boldsymbol{q}\in Q(n,a)}U_{\boldsymbol{q}}\Bigr)\mathfrak{M}^{N_1-n}.$$

As a consequence, we get

$$\|\mathfrak{M}^{N_0+N_1}\|\le CN_1|\mathcal{A}|\sup_{\substack{1\le n\le N_1\\ a\in\mathcal{A}}}\|\mathfrak{M}^{N_0}U_{Q(n,a)}\|(\|\alpha\|_\infty)^{N_1-n}, \tag{4-63}$$

where

$$U_{Q(n,a)}=\sum_{\boldsymbol{q}\in Q(n,a)}U_{\boldsymbol{q}}. \tag{4-64}$$

Since $N_1=O(\log h)$, the proof of (4-14) is reduced to proving:

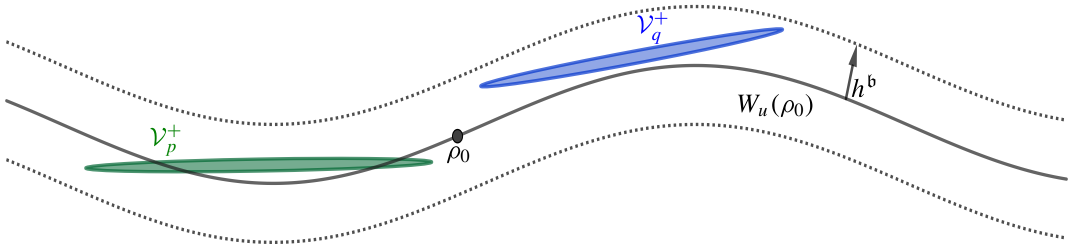

Figure 12. Two words $\boldsymbol{q}, \boldsymbol{p} \in Q(n,a)$ are close to each other if $\mathcal{V}_{\boldsymbol{q}}^+$ and $\mathcal{V}_{\boldsymbol{p}}^+$ lie in the $h^{\mathfrak{b}}$-neighborhood of the same unstable leaves, as stated in Definition 4.17.

Proposition 4.16. *There exists $\gamma > 0$ such that, for h small enough, we have*

$$\sup_{\substack{1 \leq n \leq N_1 \\ a \in \mathcal{A}}} \frac{\|\mathfrak{M}^{N_0} U_{Q(n,a)}\|}{\|\alpha\|_\infty^{n+N_0}} \leq h^\gamma. \tag{4-65}$$

4G. *Partition into clouds.* We fix $1 \leq n \leq N_1$ and $a \in \mathcal{A}$. We aim at gathering pieces of $\mathfrak{M}^{N_0} U_{Q(n,a)}$ into clouds and we want these clouds to interact (with a meaning we will define further) with only a finite and uniform number of other clouds, so that the global norm of $\|\mathfrak{M}^{N_0} U_{Q(n,a)}\|$ can be deduced from a uniform bound for each cloud.

Recall that δ_0 and τ (see (4-2), (4-3) and (4-5)) have be chosen such that

$$\mathfrak{b} + \delta_0 < 1, \quad \mathfrak{b} < \tau.$$

We start by defining a notion of closeness between two words $\boldsymbol{q}, \boldsymbol{p} \in Q(n,a)$. We choose ε_2 as in Lemma 4.4.

Definition 4.17. Let $\boldsymbol{q}, \boldsymbol{p} \in Q(n,a)$. We say that these two words are *close to each other* if there exists $\rho_0 \in \mathcal{T} \cap F(\mathcal{V}_a(\varepsilon_2))$ such that,

$$\text{for all } \rho \in \mathcal{V}_{\boldsymbol{q}}^+ \cup \mathcal{V}_{\boldsymbol{p}}^+, \quad d(\rho, W_u(\rho_0)) \leq h^{\mathfrak{b}}.$$

Otherwise, we say that $\boldsymbol{q}$ and $\boldsymbol{p}$ are *far from each other*. See Figure 12.

Remark. By the definition of $\mathcal{V}_{\boldsymbol{q}}^+$, if $\boldsymbol{q} \in \mathcal{Q}(n,a)$ and if $\rho \in \mathcal{V}_{\boldsymbol{q}}^+$, then ρ does not lie in $\mathcal{V}_a$, but $F^{-1}(\rho)$ does. Hence, we work with $F(\mathcal{V}_a)$ instead of $\mathcal{V}_a$. Moreover, the set $F(\mathcal{V}_a(\varepsilon_2))$ is chosen to fit well in the computations below and in particular in the proof of Lemma 4.19. We could replace it by $\mathcal{V}_a^+(C\varepsilon_2)$, where C is any Lipschitz constant for F.

The important fact on words $\boldsymbol{p}, \boldsymbol{q}$ far from each other is that the associated operators $\mathfrak{M}^{N_0} U_{\boldsymbol{p}}$, $\mathfrak{M}^{N_0} U_{\boldsymbol{q}}$ are almost orthogonal:

Proposition 4.18. *Assume that $\boldsymbol{q}, \boldsymbol{p} \in Q(n,a)$ are far from each other. Then,*

$$(\mathfrak{M}^{N_0} U_{\boldsymbol{q}})^* (\mathfrak{M}^{N_0} U_{\boldsymbol{p}}) = O(h^\infty), \tag{4-66}$$

$$(\mathfrak{M}^{N_0} U_{\boldsymbol{q}})(\mathfrak{M}^{N_0} U_{\boldsymbol{q}})^* = O(h^\infty). \tag{4-67}$$

We will need the following lemma.

Lemma 4.19. *If $\boldsymbol{q}, \boldsymbol{p} \in Q(n, a)$ are far from each other, there exist words $\boldsymbol{p}_1, \boldsymbol{q}_1, \boldsymbol{p}_2, \boldsymbol{q}_2$ such that*

- $|\boldsymbol{p}_1| = |\boldsymbol{q}_1|$, $|\boldsymbol{p}_2| = |\boldsymbol{q}_2|$.
- $\boldsymbol{q} = \boldsymbol{q}_1\boldsymbol{q}_2$, $\boldsymbol{p} = \boldsymbol{p}_1\boldsymbol{p}_2$.
- $\mathcal{V}^+_{\boldsymbol{q}_2} \cap \mathcal{V}^+_{\boldsymbol{p}_2} = \varnothing$.
- $\max(J^+_{\boldsymbol{q}_2}, J^+_{\boldsymbol{p}_2}) \leq Ch^{-\mathfrak{b}}$ (*for some global constant $C > 0$*).

In particular, $\mathcal{V}^+_{\boldsymbol{q}} \cap \mathcal{V}^+_{\boldsymbol{p}} = \varnothing$.

Let's momentarily admit it and prove the proposition.

Proof of Proposition 4.18. Fix $\boldsymbol{q}, \boldsymbol{p} \in Q(n, a)$ far from each other. Since $\mathcal{V}^+_{\boldsymbol{q}} \cap \mathcal{V}^+_{\boldsymbol{p}} = \varnothing$, we have $U_{\boldsymbol{q}} U^*_{\boldsymbol{p}} = O(h^\infty)$ by virtue of Proposition 4.14. Hence, using the polynomial bounds $\|\mathfrak{M}^{N_0}\| = O(h^{-M})$ (for some $M > 0$), we have

$$(\mathfrak{M}^{N_0} U_{\boldsymbol{q}})(\mathfrak{M}^{N_0} U_{\boldsymbol{p}})^* = O(h^\infty).$$

To prove the first point, we write

$$(\mathfrak{M}^{N_0} U_{\boldsymbol{q}})^*(\mathfrak{M}^{N_0} U_{\boldsymbol{p}}) = \sum_{\boldsymbol{s},\boldsymbol{t} \in \mathcal{A}^{N_0}} U^\star_{\boldsymbol{q}_1} U^*_{\boldsymbol{q}_2} U^*_{\boldsymbol{s}} U_{\boldsymbol{t}} U_{\boldsymbol{p}_2} U_{\boldsymbol{p}_1}.$$

Hence, it is enough to show that $U^*_{\boldsymbol{q}_2} U^*_{\boldsymbol{s}} U_{\boldsymbol{t}} U_{\boldsymbol{p}_2} = O(h^\infty)$ uniformly in $\boldsymbol{s}, \boldsymbol{t}$. To do so, we note that

$$\begin{gathered} \mathcal{V}^+_{\boldsymbol{q}_2\boldsymbol{s}} \cap \mathcal{V}^+_{\boldsymbol{p}_2\boldsymbol{t}} \subset F^{N_0}(\mathcal{V}^+_{\boldsymbol{q}_2} \cap \mathcal{V}^+_{\boldsymbol{p}_2}) = \varnothing, \\ J^+_{\boldsymbol{q}_2\boldsymbol{s}} \leq C J^+_{\boldsymbol{s}} J^+_{\boldsymbol{q}_2} \leq C e^{\lambda_1 N_0} h^{-\mathfrak{b}} \leq C h^{-(\delta_0 + \mathfrak{b})}, \\ J^+_{\boldsymbol{p}_2\boldsymbol{t}} \leq C h^{-(\delta_0 + \mathfrak{b})} \end{gathered}$$

and apply Proposition 4.14, with $\delta = (\delta_0 + \mathfrak{b})/2 < \frac{1}{2}$ (here we use condition (4-3)). □

We now prove the lemma.

Proof of Lemma 4.19. Consider $\boldsymbol{q}, \boldsymbol{p} \in Q(n, a)$ far from each other. Consider the smallest integer m such that $\mathcal{V}^+_{q_m \cdots q_{n-1}} \cap \mathcal{V}^+_{p_m \cdots p_{n-1}} \neq \varnothing$. We will show that $m > 0$ and set $\boldsymbol{p}_2 = p_{m-1} \cdots p_{n-1}$, $\boldsymbol{q}_2 = q_{m-1} \cdots q_{n-1}$. Pick $\rho \in \mathcal{V}^+_{q_m \cdots q_{n-1}} \cap \mathcal{V}^+_{p_m \cdots p_{n-1}}$. By choice of ε_2 after Lemma 4.4, there exists $\rho_0 \in \mathcal{T}$ such that $d(F^{-i}(\rho), F^{-i}(\rho_0)) \leq \varepsilon_2$ for $i \in \{1, \dots, n-m\}$. In particular, $d(F^{-1}(\rho), F^{-1}(\rho_0)) \leq \varepsilon_2$ and $F^{-1}(\rho) \in \mathcal{V}_a$, so that $\rho_0 \in F(\mathcal{V}_a(\varepsilon_2))$. Since, $\boldsymbol{q}, \boldsymbol{p}$ are far from each other, there exists $\rho_1 \in \mathcal{V}^+_{\boldsymbol{q}} \cup \mathcal{V}^+_{\boldsymbol{p}}$ such that $d(\rho_1, W_u(\rho_0)) > h^{\mathfrak{b}}$ (otherwise, it would contradict Definition 4.17).

Suppose for instance that $\rho_1 \in \mathcal{V}^+_{\boldsymbol{q}} \subset \mathcal{V}^+_{q_m \cdots q_{n-1}}$. Hence, $d(F^{-i}(\rho_0), F^{-i}(\rho_1)) \leq 2\varepsilon_0 + \varepsilon_2$ for $i \in \{1, \dots, n-m\}$. From (3-17), $d(\rho_1, W_u(\rho_0)) \leq C(J^{n-m}_s(\rho_0))^{-1}$ and hence, $J^{n-m}_s(\rho_0) \leq Ch^{-\mathfrak{b}}$.

But, $J^{n-m}_s(\rho_0) \sim J^+_{p_m \cdots p_{n-1}} \sim J^+_{q_m \cdots q_{n-1}}$, which gives

$$\max(J^+_{p_m \cdots p_{n-1}}, J^+_{q_m \cdots q_{n-1}}) \leq Ch^{-\mathfrak{b}}.$$

Since $\min(J^+_{\boldsymbol{q}}, J^+_{\boldsymbol{p}}) > h^{-\tau} \gg h^{-\mathfrak{b}}$ (here we use (4-5)), we cannot have $m = 0$ (if h small enough). Thus, we can set $\boldsymbol{p}_2 = p_{m-1} \cdots p_{n-1}$, $\boldsymbol{q}_2 = q_{m-1} \cdots q_{n-1}$, which satisfy the required properties by minimality of m. □

We now decompose $U_{Q(n,a)}$ into a sum of operators, each of them corresponding to a *cloud* of words. In the following, we'll use the term *cloud* to mean a subset $\mathcal{Q} \subset Q(n,a)$ and we'll adopt the notation

$$\mathcal{V}_{\mathcal{Q}}^{+} = \bigcup_{\boldsymbol{q} \in \mathcal{Q}} \mathcal{V}_{\boldsymbol{q}}^{+}$$

and the definition:

Definition 4.20. We say that two clouds $\mathcal{Q}_1, \mathcal{Q}_2$ *do not interact* if, for all pairs $(\boldsymbol{q}_1, \boldsymbol{q}_2) \in \mathcal{Q}_1 \times \mathcal{Q}_2$, $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are far from each other.

The existence of such a decomposition follows from the key proposition (see Figure 13):

Proposition 4.21. *Suppose ε_0 is small enough. There exists a partition of $Q(n,a)$ into clouds $\mathcal{Q}_1, \dots, \mathcal{Q}_r$ and a global constant $C > 0$ such that, for $i = 1, \dots, r$:*

(i) *There exists $\rho_i \in \mathcal{T}$ such that, for all $\rho \in \mathcal{V}_{\mathcal{Q}_i}^{+}$, $d(\rho, W_u(\rho_i)) \le C h^{\mathfrak{b}}$.*

(ii) *If $\mathcal{Q}_i$ interacts with exactly c_i clouds, then $c_i \le C$.*

Remark. Actually, r and the clouds $\mathcal{Q}_i$ depend on n and a. We do not write this dependence explicitly here to make the notation lighter. The second point is relevant since a priori, the only obvious bound on $r = r(n,a)$ is $|r| \le |\mathcal{A}|^n$, where $n = O(\log h)$.

Proof. Keeping in mind that, for all $\boldsymbol{q} \in Q(n,a)$, we have $\mathcal{V}_{\boldsymbol{q}}^{+} \subset \mathcal{V}_a^{+}$, we fix $\rho_a \in \mathcal{V}_a^{+}$. If ε_0 is small enough, $\mathcal{V}_a^{+}$ does not intersect the boundaries of $W_s(\rho_a)$ and $W_u(\rho_a)$.

For $\boldsymbol{q} \in Q(n,a)$, there exists $\rho_{\boldsymbol{q}} \in \mathcal{T}$ such that $d(F^{-i}(\rho), F^{-i}(\rho_{\boldsymbol{q}})) \le \varepsilon_2$ for all $\rho \in \mathcal{V}_{\boldsymbol{q}}^{+}$ and for $i = 1, \dots, n$, according to Lemma 4.4 and since $J_{\boldsymbol{q}}^{+} \sim h^{\tau}$,

$$d(\rho, W_u(\rho_{\boldsymbol{q}})) \le C h^{-\tau},$$

$d(\rho_a, \rho_{\boldsymbol{q}}) \le C(\varepsilon_2 + \varepsilon_0)$ and hence, if ε_0 is small enough, $z_{\boldsymbol{q}} := H_{\rho_a}^{u}(\rho_{\boldsymbol{q}})$ (here, $H_{\rho_a}^{u} : B(\rho_a, \varepsilon_0') \to W_s(\rho_a)$) is the unstable holonomy map defined before Lemma 3.20) is well-defined, and depends Lipschitz-continuously on $\rho_{\boldsymbol{q}}$ (with global Lipschitz constant).

Next, consider a maximal subset $\{z_1, \dots, z_r\} \subset \{z_{\boldsymbol{q}}, \boldsymbol{q} \in Q(n,a)\}$ which is $h^{\mathfrak{b}}$ separated. By maximality, for every $\boldsymbol{q} \in Q(n,a)$, there exists $i \in \{1, \dots, r\}$ such that $|z_i - z_{\boldsymbol{q}}| \le h^{\mathfrak{b}}$ and we use these z_i to partition $Q(n,a)$ into clouds $\mathcal{Q}_i$, where for $i \in \{1, \dots, r\}$, $|z_i - z_{\boldsymbol{q}}| \le h^{\mathfrak{b}}$ for all $\boldsymbol{q} \in \mathcal{Q}_i$. We now show that this partition satisfies the required properties.

Let $i \in \{1, \dots, r\}$, $\boldsymbol{q} \in \mathcal{Q}_i$ and $\rho \in \mathcal{V}_{\boldsymbol{q}}^{+}$. By local uniqueness of the unstable leaves, we may assume that ε_0 is small enough so that $W_u(\rho_{\boldsymbol{q}}) \cap \mathcal{V}_a^{+} = W_u(z_{\boldsymbol{q}}) \cap \mathcal{V}_a^{+}$. Hence,

$$d(\rho, W_u(z_{\boldsymbol{q}})) \le C h^{-\tau}.$$

Since the unstable leaves depend Lipschitz-continuously on $\rho \in \mathcal{T}$, we have

$$d(\rho, W_u(z_i)) \le C|z_i - z_{\boldsymbol{q}}| + C d(\rho, W_u(z_{\boldsymbol{q}})) \le C h^{\mathfrak{b}} + C h^{\tau} \le C h^{\mathfrak{b}}.$$

This gives (i).

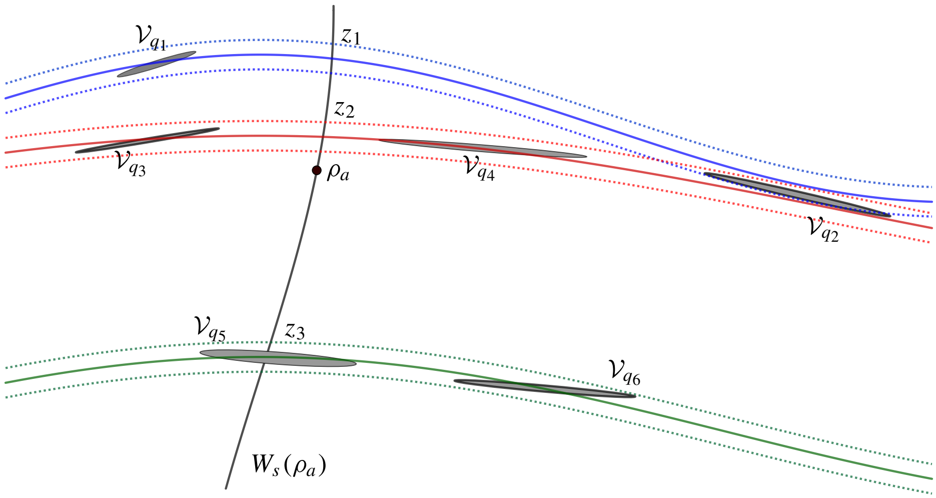

Figure 13. We gather the six small sets $\mathcal{V}_{\boldsymbol{q}}$ into three clouds corresponding to z_1, z_2 and z_3. Here, $\mathcal{Q}_1 = \{\boldsymbol{q}_1\}$, $\mathcal{Q}_2 = \{\boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4\}$ and $\mathcal{Q}_3 = \{\boldsymbol{q}_5, \boldsymbol{q}_6\}$. The clouds $\mathcal{Q}_1$ and $\mathcal{Q}_2$ interact. The dotted lines draw tubes of width $Ch^{\mathfrak{b}}$ around the unstable leaves $W_u(z_i)$. The sets $\mathcal{V}_{\boldsymbol{q}}$ have width of order h^τ.

To show (ii), suppose that $\mathcal{Q}_i$ and $\mathcal{Q}_j$ interact. Then, there exist $(\boldsymbol{q}, \boldsymbol{p}) \in \mathcal{Q}_i \times \mathcal{Q}_j$ and $\rho_0 \in \mathcal{T}$ such that, for all $\rho \in \mathcal{V}^+_{\boldsymbol{q}} \cup \mathcal{V}^+_{\boldsymbol{p}}$, $d(\rho, W_u(\rho_0)) \leq h^{\mathfrak{b}}$. It follows that $d(z_{\boldsymbol{q}}, W_u(\rho_0)) \leq Ch^\tau + h^{\mathfrak{b}} \leq Ch^{\mathfrak{b}}$ and if we denote by $z_0 = H^u_{\rho_a}(\rho_0)$ the unique point in $W_u(\rho_0) \cap W_s(\rho_a)$ then $|z_0 - z_{\boldsymbol{q}}| \leq Ch^{\mathfrak{b}}$. The same is true for $\boldsymbol{p}$ and we have $|z_{\boldsymbol{q}} - z_{\boldsymbol{p}}| \leq Ch^{\mathfrak{b}}$ and eventually, $|z_i - z_j| \leq Ch^{\mathfrak{b}}$. Since $z_1, \dots, z_r$ are $h^{\mathfrak{b}}$-separated, we see after rescaling that the number of j such that $\mathcal{Q}_i$ and $\mathcal{Q}_j$ interact is smaller than the maximal number of points in $B(0, C)$ which are 1-separated (one can for instance bound it by $(2C+1)^2$, but what matters is that it is a global constant). □

This partition into clouds allows us to decompose $\mathfrak{M}^{N_0} U_{Q(n,a)}$ into a sum of operators

$$B_i = \mathfrak{M}^{N_0} U_{\mathcal{Q}_i} = \sum_{\boldsymbol{q} \in \mathcal{Q}_i} \mathfrak{M}^{N_0} U_{\boldsymbol{q}}, \quad \mathfrak{M}^{N_0} U_{Q(n,a)} = \sum_{i=1}^r B_i. \tag{4-68}$$

The use of Cotlar–Stein theorem [Zworski 2012, Theorem C.5] reduces the control of the sum by the control of individual clouds:

Lemma 4.22. *With the above notation, there exists a global constant $C > 0$ such that*

$$\|\mathfrak{M}^{N_0} U_{Q(n,a)}\| \leq C \sup_{1 \leq i \leq r} \|B_i\| + O(h^\infty). \tag{4-69}$$

Proof. Cotlar–Stein theorem reduces to control

$$\max_i \sum_j \|B_i^* B_j\|^{1/2}, \quad \max_i \sum_j \|B_j B_i^*\|^{1/2}.$$

Fix $i \in \{1, \dots, r\}$.

If $\mathcal{Q}_i$ and $\mathcal{Q}_j$ do not interact, then $\|B_i^* B_j\|^{1/2}$ (resp. $\|B_j B_i^*\|^{1/2}$) is a sum of terms of the form $(\mathfrak{M}^{N_0} U_{\boldsymbol{q}})^*(\mathfrak{M}^{N_0} U_{\boldsymbol{p}})$ (resp. $(\mathfrak{M}^{N_0} U_{\boldsymbol{q}})(\mathfrak{M}^{N_0} U_{\boldsymbol{p}})^*$), where $\boldsymbol{p}$ and $\boldsymbol{q}$ are far from each other. By virtue of Proposition 4.14, these terms are uniformly $O(h^\infty)$ and since the number of terms in the sum grows at most polynomially with h, we can gather all these terms in a single uniform $O(h^\infty)$. As a consequence, we have

$$\begin{aligned}\sum_j \|B_i^* B_j\|^{1/2} &\le \sum_{\mathcal{Q}_i \text{ and } \mathcal{Q}_j \text{ interact}} \|B_i^* B_j\|^{1/2} + O(h^\infty)\\ &\le \sum_{\mathcal{Q}_i \text{ and } \mathcal{Q}_j \text{ interact}} \max_k \|B_k\| + O(h^\infty) \le C \max_k \|B_k\| + O(h^\infty),\end{aligned}$$

and the same holds for the second sum. This gives the desired inequalities. □

The proof of (4-14) and, as a consequence, of Proposition 4.2 is then reduced to the proof of:

Proposition 4.23. *There exists $\gamma > 0$ such that the following holds for h small enough. Assume that $\mathcal{Q} \subset \mathcal{Q}(n, a)$ satisfies, for some global constant $C > 0$,*

$$\text{there exists } \rho_0 \in \mathcal{T} \text{ such that for all } \rho \in \mathcal{V}_{\mathcal{Q}}^+, \quad d(\rho, W_u(\rho_0)) \le C h^{\mathfrak{b}},$$

where $\mathfrak{b} = 1/(1+\beta)$ is defined in (4-2). *Then,*

$$\frac{\|\mathfrak{M}^{N_0} U_{\mathcal{Q}}\|}{\|\alpha\|_\infty^{N_0+n}} \le h^\gamma.$$

5. Reduction to a fractal uncertainty principle via microlocalization properties

In this section, we reduce the proof of Proposition 4.23 to a fractal uncertainty principle. To do so, we aim at showing microlocalization properties of the operators involved. The dissymmetry between N_0 and N_1 in the decomposition $N = N_0 + N_1$ will appear clearly in this section. Since N_0 is below the Ehrenfest time, we can actually use semiclassical tools. By contrast, things are more complicated for operators $U_{\boldsymbol{q}}$, with $\boldsymbol{q} \in \mathcal{Q}(n, a)$, and we'll use methods of propagation of Lagrangian leaves. These methods are inspired by [Anantharaman and Nonnenmacher 2007a; 2007b; Nonnenmacher and Zworski 2009] and are also used in [Dyatlov et al. 2022].

5A. ***Microlocalization of $\mathfrak{M}^{N_0}$.*** We first state a microlocalization result for $\mathfrak{M}^{N_0}$. This is the easiest one to obtain since N_0 is below the Ehrenfest time. We recall the definition of $\mathcal{T}_-$, the set of the future trapped points

$$\mathcal{T}_- = \bigcap_{n \in \mathbb{N}} F^{-n}(U)$$

and focus on $\mathcal{T}_-^{\text{loc}} := \mathcal{T}_- \cap \mathcal{T}(4\varepsilon_0)$. The set $\mathcal{T}_-$ is laminated by the weak global stable leaves. Hence, if ε_0 is small enough, ensuring that the boundaries of the local stable leaves $W_s(\rho)$, $\rho \in \mathcal{T}$, do not intersect $\mathcal{T}(4\varepsilon_0)$, we have

$$\mathcal{T}_-^{\text{loc}} \subset \bigcup_{\rho \in \mathcal{T}} W_s(\rho).$$

When $\boldsymbol{q} \in \mathcal{A}^{N_0}$ and $\mathcal{V}_{\boldsymbol{q}}^- \neq \varnothing$, $\mathcal{V}_{\boldsymbol{q}}^-$ lies in an $O(h^{\delta_0\lambda_0/\lambda_1})$ neighborhood of a stable leaves, as stated in the following lemma. In the following, we write

$$\delta_2 = \delta_0 \frac{\lambda_0}{\lambda_1}. \tag{5-1}$$

We recall that we have defined $\mathfrak{b}$ in (4-2) and τ in (4-6) such that $\alpha < \tau < 1$ and $\delta_2 + \tau > 1$ (see (4-5)). Moreover, $N_0 = \lceil (\delta_0/\lambda_1)|\log h| \rceil$.

Lemma 5.1. *There exists a global constant $C_2 > 0$ such that, for all $\boldsymbol{q} \in \mathcal{A}^{N_0}$ satisfying $\mathcal{V}_{\boldsymbol{q}}^- \neq \varnothing$,*

$$d(\mathcal{V}_{\boldsymbol{q}}^-, \mathcal{T}_-^{\mathrm{loc}}) \le C_2 h^{\delta_2}.$$

Remark. In the end of this section, the use of C_2 will always refer to the constant appearing in this lemma. On other places, we keep our convention on global constants, denoting them always by C.

Proof. We already know by Lemma 4.7 that there exists $C > 0$ such that if $\mathcal{V}_{\boldsymbol{q}}^- \neq \varnothing$, there exists $\rho_0 \in \mathcal{T}$ such that

$$d(\mathcal{V}_{\boldsymbol{q}}^-, W_s(\rho_0)) \le \frac{C}{J_{\boldsymbol{q}}^-}.$$

But $J_{\boldsymbol{q}}^- \ge e^{\lambda_0 N_0} \ge C^{-1} h^{-\delta_0\lambda_0/\lambda_1}$. Finally, $d(\mathcal{V}_{\boldsymbol{q}}^-, \mathcal{T}_-^{\mathrm{loc}}) \le C h^{\delta_2}$, as required. □

The following lemma allows us to construct symbols in nice symbol classes with supports in h^δ neighborhood. Its proof can be found in [Dyatlov and Zahl 2016, Lemma 3.3].

Lemma 5.2. *Let $\varepsilon > 0$ and $\delta \in \left[0, \frac{1}{2}\right[$. Let $V_0(h) \subset V_1(h) \subset \mathbb{R}^d$ be sets depending on h and assume that, for $0 \le h \le 1$, $d(V_0(h), V_1(h)^c) > \varepsilon h^\delta$. Then, there exists a family $\chi_h \in C_c^\infty(\mathbb{R}^d)$ such that, for all $h \le 1$:*

- *$\chi_h = 1$ on $V_0(h)$.*
- supp $\chi \subset V_1(h)$.
- *For every $\alpha \in \mathbb{N}^d$, there exists C_α depending only on ε such that, for all $x \in \mathbb{R}^d$ and for all $0 < h \le 1$,*

$$|\partial^\alpha \chi_h(x)| \le C_\alpha h^{-\delta|\alpha|}.$$

Applying this lemma with $V_0(h) = \mathcal{T}_-^{\mathrm{loc}}(2C_2 h^{\delta_2})$, $V_1(h) = \mathcal{T}_-^{\mathrm{loc}}(4C_2 h^{\delta_2})$ with $\varepsilon = 2C_2$, we consider a family of smooth cut-offs $\chi_h \in S_{\delta_2}^{\mathrm{comp}}$ and we can consider it as an element of $S_{\delta_2}^{\mathrm{comp}}(U)$ since (at least for h small enough) the support of χ_h is included in U. We are now ready to state the microlocalization property of $\mathfrak{M}^{N_0}$.

Proposition 5.3.
$$\mathfrak{M}^{N_0} = \mathfrak{M}^{N_0} \operatorname{Op}_h(\chi_h) + O(h^\infty)_{L^2(Y) \to L^2(Y)}. \tag{5-2}$$

Proof. We need to show that $\mathfrak{M}^{N_0}(\operatorname{Op}_h(1-\chi_h)) = O(h^\infty)$. To do so, we take the decomposition $\mathfrak{M}^{N_0} = \sum_{\boldsymbol{q} \in \mathcal{A}^{N_0}} U_{\boldsymbol{q}}$. Since the number of terms in this sum grows polynomially with h, it is enough to show that,

$$\text{for all } \boldsymbol{q} \in \mathcal{A}^{N_0}, \quad U_{\boldsymbol{q}}(\operatorname{Op}_h(1-\chi_h)) = O(h^\infty),$$

with bounds uniform in $\boldsymbol{q}$. We then consider two cases:

• $\mathcal{V}_{\boldsymbol{q}}^{-}=\varnothing$: Lemma 4.13 applies. Indeed, if $m \leq N_0$ and $\mathcal{V}_{q_0\cdots q_{m-1}}^{-} \neq \varnothing$, we have

$$J_{q_0\cdots q_{m-1}}^{-} \leq e^{m\lambda_1} \leq e^{N_0\lambda_1} \leq Ch^{-\delta_0}.$$

Hence, $U_{\boldsymbol{q}}=O(h^{\infty})$, with global constants in the $O(h^{\infty})$.

• $\mathcal{V}_{\boldsymbol{q}}^{-}\neq\varnothing$: We apply Proposition 4.8. Since $J_{\boldsymbol{q}}^{-} \leq Ce^{\lambda_1 N_0} \leq Ch^{-\delta_0}$, we take some $\delta_1 \in \left]\delta_0, \frac{1}{2}\right[$ (in particular, $\delta_2<\delta_1$) and we can write $U_{\boldsymbol{q}}=T^{N_0}\operatorname{Op}_h(a_{\boldsymbol{q}}^{-})+O(h^{\infty})$, with $a_{\boldsymbol{q}}^{-}\in S_{\delta_1}^{\mathrm{comp}}(U)$ and $\operatorname{supp} a_{\boldsymbol{q}}^{-}\subset\mathcal{V}_{\boldsymbol{q}}^{-}$. Noticing that $\chi_h=1$ on $\mathcal{V}_{\boldsymbol{q}}^{-}\subset\mathcal{T}_{-}^{\mathrm{loc}}(2C_2h^{\delta_2})$, the composition formula in $S_{\delta_1}^{\mathrm{comp}}$ implies that $\operatorname{Op}_h(a_{\boldsymbol{q}}^{-})\operatorname{Op}_h(1-\chi_h)=O(h^{\infty})$. Since the seminorms of $a_{\boldsymbol{q}}^{-}$ are uniformly bounded in $\boldsymbol{q}$, the constants appearing in $O(h^{\infty})$ are uniform in $\boldsymbol{q}$. □

5B. ***Propagation of Lagrangian leaves and Lagrangian states.*** To study the microlocalization of $U_{\boldsymbol{q}}$ we'll use the same strategy as in [Dyatlov et al. 2022], the authors themselves inspired by [Anantharaman and Nonnenmacher 2007a; 2007b; Nonnenmacher and Zworski 2009]. We cannot show that $U_{\boldsymbol{q}}$ is a Fourier integral operator since the propagation goes behind the Ehrenfest time. Instead, we show a weaker result which will be enough for our purpose. The idea is to decompose a state u in a sum of Lagrangian states associated with Lagrangian leaves almost parallel to unstable leaves, what we will call horizontal leaves (because we will consider them in charts where the unstable leaves are close to be horizontal). Studying the precise behavior of these states, we can get fine information on the microlocalization of $U_{\boldsymbol{q}}u$. Roughly speaking, we'll show that if u is a Lagrangian state associated with an original horizontal Lagrangian $\mathcal{L}_{q_0,\theta}\subset\mathcal{V}_{q_0}$, then $U_{\boldsymbol{q}}u$ is a Lagrangian state associated with the piece of the evolved Lagrangian $F^n(\mathcal{L}_{q_0,\theta})$ inside $\mathcal{V}_{\boldsymbol{q}}^{+}$.

To define "horizontal" Lagrangian leaves, we need to work in adapted coordinate charts in which the notion of horizontality (thinking $W_u(\rho)$ as the reference) makes sense. For this purpose, for $q\in\mathcal{A}$, we consider charts centered around the points ρ_q, associated with the fixed macroscopic partition of $\mathcal{T}$ by the $\mathcal{V}_q=B(\rho_q,2\varepsilon_0)$. First, we consider symplectic maps

$$\kappa_q : W_q \subset U_{k_q} \to V_q \subset \mathbb{R}^2$$

satisfying (we denote by (x,ξ) the variable in U and (y,η) in $\mathbb{R}^2$):

(1) $B(\rho_q, C\varepsilon_0)\subset W_q$ for some global constant $C\gg 2$.

(2) $\kappa(\rho_q)=0,\ d\kappa(\rho_q)(E_u(\rho_q))=\mathbb{R}\times\{0\} : d\kappa(\rho_q)(E_s(\rho_q))=\{0\}\times\mathbb{R}$.

(3) The image of the unstable leave $W_u(\rho_q)$ is exactly $\{(y,0): y\in\mathbb{R}\}\cap\widetilde{V}_q$.

Theses charts are for instance given by Lemma 3.22 (at this stage, the strong straightening property is not necessary). In these adapted charts where $W_u(\rho_q)$ coincides with $\mathbb{R}\times\{0\}$, the horizontal Lagrangian leaves will be the of the form

$$\mathcal{C}_\theta := \{(y,\theta): y\in\mathbb{R}\}. \tag{5-3}$$

Finally, we fix unit vectors on $E_u(\rho_q)$ and $E_s(\rho_q)$, $e_u(\rho_q)$ and $e_s(\rho_q)$, used to defined the unstable and stable Jacobians in Section 3C. Let's write

$$d\kappa_q(e_u(\rho_q))=(\lambda_{q,u},0),\quad d\kappa_q(e_s(\rho_q))=(0,\lambda_{q,s}).$$

Note

$$D_q = \begin{pmatrix} \lambda_{q,u} & 0 \\ 0 & \lambda_{q,s} \end{pmatrix}.$$

We dilate the chart $\tilde{\kappa}_q$ and define

$$\tilde{\kappa}_q : \rho \in W_q \mapsto D_q \kappa_q(\rho) \in \widetilde{V}_q := D_q(V_q).$$

5B1. *Horizontal Lagrangian and their evolution.* Let us fix a word $\boldsymbol{q} \in \mathcal{A}^n$ and let us define

$$\mathcal{L}_{q_0,\theta} = \kappa_{q_0}^{-1}(\mathcal{C}_\theta \cap V_{q_0}) \cap \mathcal{V}_{q_0}. \tag{5-4}$$

Then, let's define inductively

$$\mathcal{L}_{q_0 \cdots q_j,\theta} = F(\mathcal{L}_{q_0 \cdots q_{j-1},\theta}) \cap \mathcal{V}_{q_j}, \tag{5-5}$$

which allows us to define $\mathcal{L}_{\boldsymbol{q},\theta}$. One can check that

$$\mathcal{L}_{\boldsymbol{q},\theta} = F^{-1}(\mathcal{V}_{\boldsymbol{q}}^+) \cap F^{n-1}(\mathcal{L}_{q_0,\theta}). \tag{5-6}$$

The term F^{-1} comes from the definition of $\mathcal{V}_{\boldsymbol{q}}^+$:

$$\rho \in \mathcal{V}_{\boldsymbol{q}}^+ \quad \Longleftrightarrow \quad \text{for all } 1 \le i \le n, \quad F^{-i}(\rho) \in \mathcal{V}_{q_{n-i}}.$$

Finally, let's define

$$\mathcal{C}_{\boldsymbol{q},\theta} = \kappa_{q_{n-1}}(\mathcal{L}_{\boldsymbol{q},\theta}). \tag{5-7}$$

We first focus on one step of the iterative process.

In $\widetilde{V}_q \subset \mathbb{R}^2$, we use the notation $\widetilde{B}_q(0,r)$ for the cube$]-r,r[\times]-r,r[$. We keep the subscript q to keep track of the chart in which this cube is supposed to live. Finally, we set

$$B_q(0,r) = D_q^{-1}(\widetilde{B}_q(0,r)) \subset V_q.$$

$B_q(0,r)$ is simply a rectangle centered at zero with size only depending on q (this is also a ball for some norm in $\mathbb{R}^2$). The advantage of $\widetilde{B}_q$ and $\tilde{\kappa}_q$ compared with B_q and κ_q will appear below. However, $\tilde{\kappa}_q$ is not symplectic, and for further use, it is not possible to use $\tilde{\kappa}_q$ as a symplectic change of coordinates.

Let $q, p \in \mathcal{A}$ and suppose that $\mathcal{V}_q \cap F^{-1}(\mathcal{V}_p) \neq \varnothing$. As a consequence there exists a global constant $C' > 0$ such that $d(F(\rho_q), \rho_p) \le C'\varepsilon_0$ and if C in (1) of Lemma 3.22 is large enough, we can assume that, for some global constant $C_1 > 0$,

$$\kappa_q(\mathcal{V}_q) \subset B_q(0, C_1\varepsilon_0) \subset V_q, \quad \kappa_p \circ F \circ \kappa_q^{-1}(B_q(0, C_1\varepsilon_0)) \subset V_p. \tag{5-8}$$

The following map is hence well-defined:

$$\tau_{p,q} := \kappa_p \circ F \circ \kappa_q^{-1} : B_q(0, C_1\varepsilon_0) \to \tau_{p,q}(B_q(0, C_1\varepsilon_0)) \subset V_p;$$

$\tau_{p,q}$ is nothing but the writing of F between the charts V_q and V_p. Note that since the number of possible transitions is finite, we can assume that C_1 is uniform for all $q, p \in \mathcal{A}$ such that $\mathcal{V}_q \cap F^{-1}(\mathcal{V}_p) \neq \varnothing$.

We also adopt the following definitions and notation:

Definition 5.4. Let $G_q :]-C_1\varepsilon_0, C_1\varepsilon_0[\to]-C_1\varepsilon_0, C_1\varepsilon_0[$ be a smooth map. It represents the horizontal Lagrangian

$$\mathcal{L}_{G_q} := D_q^{-1}\big(\{(y, G_q(y)) : y \in]-C_1\varepsilon_0, C_1\varepsilon_0[\}\big) \subset B_q(0, C_1\varepsilon_0) \subset V_q.$$

We say that such a Lagrangian lies in the γ-unstable cone if

$$\|G_q'\|_\infty \leq \gamma,$$

and we write $G_q \in \mathcal{C}_q^u(C_1\varepsilon_0, \gamma)$.

Remark. This is where the use of $\tilde{\kappa}_q$ and $\widetilde{B}_q$ turns out to be useful; to represent horizontal Lagrangian in V_q, we use the cube $\widetilde{B}_q(0, C_1\varepsilon_0) \subset \widetilde{V}_q$ of fixed size.

With this definition, we show in the following lemma an invariance property of the γ-unstable cones:

Lemma 5.5. *There exist global constants $C > 0$, $C_1 > 0$ such that if ε_0 is sufficiently small, then the following holds*:

For every $G_q \in \mathcal{C}_q^u(C_1\varepsilon_0, C\varepsilon_0)$, there exists $G_p \in \mathcal{C}_p^u(C_1\varepsilon_0, C\varepsilon_0)$ such that:

(i) $\tau_{p,q}(\mathcal{L}_{G_q}) \cap B_p(0, C_1\varepsilon_0) = \mathcal{L}_{G_p}$.

(ii) *For some global constants C_l, $l \geq 2$, we have $\|G_q\|_{C^l} \leq C_l \Longrightarrow \|G_p\|_{C^l} \leq C_l$.*

Moreover, let's define $\phi_{qp} :]-C_1\varepsilon_0, C_1\varepsilon_0[\to \mathbb{R}$ by

$$y_q = \phi_{qp}(y_p) \iff (y_p, G_p(y_p)) = D_p \circ \tau_{pq} \circ D_q^{-1}(\phi_{qp}(y_p), G_q \circ \phi_{qp}(y_p)).$$

Then, ϕ_{pq} is smooth contracting diffeomorphism onto its image. In particular, there exists a global constant $\nu < 1$ such that $\|\phi_{pq}'\|_\infty \leq \nu$.

Proof. Take C_1 large but fixed (with conditions further imposed) and assume that ε_0 is small enough so that (5-8) holds. Let us define $\lambda_q = J_1^u(\rho_q) > 1$ and $\mu_q = J_1^s(\rho_q) < 1$ and let us fix some global ν satisfying,

$$\text{for all } q \in \mathcal{A}, \quad \max(\lambda_q^{-1}, \mu_q) < \nu < 1.$$

Recall that e_u and e_s are $C^{1,\varepsilon}$ in ρ. We write ∂_y and ∂_η to denote the unit vector of $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$ respectively. We fix a constant $C > 0$ with conditions imposed further and we assume that $\|G_p'\|_\infty \leq C\varepsilon_0$. We let $\tilde{\tau} = D_p \circ \tau_{p,q} \circ D_q^{-1}$ (we drop the subscript for $\tilde{\tau}$ to alleviate the notation). In the computations below, the implied constants in the O are global constants (depending also on the choices on κ_q):

- $\tilde{\tau}(0) = \tilde{\kappa}_p \circ F(\rho_q) = O(\varepsilon_0)$.
- $d\tilde{\tau}(0) = d\tilde{\kappa}_p(F(\rho_q)) \circ dF(\rho_q) \circ [d\tilde{\kappa}_q(\rho_q)]^{-1}$.
- $d\tilde{\tau}(0)(\partial_y) = d\tilde{\kappa}_p(F(\rho_q))(\lambda_q e_u(F(\rho_q))) = \lambda_q(d\tilde{\kappa}_p(\rho_p) + O(\varepsilon_0))(e_u(\rho_p) + O(\varepsilon_0)) = \lambda_q\partial_y + O(\varepsilon_0)$, where we use the Lipschitz regularity of $\rho \mapsto e_u(\rho)$ in the second equality.
- Similarly, $d\tilde{\tau}(0)(\partial_\eta) = \mu_q\partial_\eta + O(\varepsilon_0)$.

(It is here that we use the renormalization of κ_q into $\tilde{\kappa}_q$). Eventually, we use the fact that $\tilde{\tau}-\tilde{\tau}(0)-d\tilde{\tau}(0)=O(C_1\varepsilon_0)_{C^1(B(0,C_1\varepsilon_0))}$ and we get

$$\tilde{\tau}(y,\eta)=(\lambda_q y+y_r(y,\eta),\,\mu_q\eta+\eta_r(y,\eta)),\quad (y,\eta)\in\widetilde{B}_q(0,C_1\varepsilon_0), \tag{5-9}$$

where $y_r(y,\eta)$ and $\eta_r(y,\eta)$ are $O(C_1\varepsilon_0)_{C^1}$. Before going further, let us show that we can fix C_1 such that

$$(y,\eta)\in\widetilde{B}_q(0,C_1\varepsilon_0)\quad\Longrightarrow\quad |\mu_q\eta+\eta_r(y,\eta))|\le C_1\varepsilon_0. \tag{5-10}$$

To do so, let us note that in fact $\tilde{\tau}-\tilde{\tau}(0)-d\tilde{\tau}(0)=O((C_1\varepsilon_0)^2)_{C^0(B(0,C_1\varepsilon_0))}$ and hence if $(y,\eta)\in\widetilde{B}_q(0,C_1\varepsilon_0)$, we have

$$|\eta_r(y,\eta)|=O(\varepsilon_0)+O((C_1\varepsilon_0)^2)_{C^0(B(0,C_1\varepsilon_0))}\le C'\varepsilon_0(1+C_1^2\varepsilon_0).$$

Assume that C_1 is large enough such that $\nu C_1+C'<C_1(\nu+1)/2$. If $(y,\eta)\in\widetilde{B}_q(0,C_1\varepsilon_0)$, we have

$$|\mu_q\eta+\eta_r(y,\eta)|\le\nu C_1\varepsilon_0+C'\varepsilon_0(1+C_1^2\varepsilon_0)\le\Big(C_1\frac{\nu+1}{2}+C_1^2\varepsilon_0\Big)\varepsilon_0.$$

This fixes C_1. Since C_1 is now a global fixed parameter, we can remove it from the O in the estimates. If ε_0 is small enough, depending on our choice of C_1, (5-10) holds.

To write the image of the leaf as a graph, we observe that, if ε_0 is small enough (depending only on global parameters) the map

$$\psi: y\in\,]-C_1\varepsilon_0,C_1\varepsilon_0[\,\mapsto\lambda_q y+y_r(y,G_q(y))$$

is expanding and we can impose $|\psi'|\ge\nu^{-1}$. In particular, $\operatorname{Im}\psi$ contains an interval of size $2\nu^{-1}C_1\varepsilon_0$. Moreover, $\psi(0)=y_r(0,G_q(0))\le\|y_r\|_{C^1}|G_q(y)|=O(\varepsilon_0^2)$. We claim that if ε_0 is small enough, $\operatorname{Im}\psi$ contains $]-C_1\varepsilon_0,C_1\varepsilon_0[$. Indeed, it suffices to have

$$\nu^{-1}C_1\varepsilon_0-|\psi(0)|\ge C_1\varepsilon_0.$$

But we have

$$C_1\varepsilon_0+|\psi(0)|\le C_1\varepsilon_0(1+O(\varepsilon_0))\le C_1\varepsilon_0\nu^{-1}$$

if $1+O(\varepsilon_0)\le\nu^{-1}$, a condition that can be satisfied if ε_0 is small enough. Hence, $\phi:=\phi_{pq}=\psi^{-1}_{|]-C_1\varepsilon_0,C_1\varepsilon_0[}$ is well-defined and we set

$$G_p(y)=\mu_q G_q(\phi(y))+\eta_r(\phi(y),G_q(\phi(y))),\quad y\in\,]-C_1\varepsilon_0,C_1\varepsilon_0[. \tag{5-11}$$

By definition, it is clear that $\tau_{p,q}(\mathcal{L}_{G_q})\cap B_p(0,C_1\varepsilon_0)=\mathcal{L}_{G_p}$ and $(y,G_p(y))=\tilde{\tau}(\phi(y),G_q(\phi(y)))$. The map ϕ is obviously a smooth contracting diffeomorphism and $\|\phi'\|\le 1/\inf|\psi'(y)|\le\nu$. Moreover, due to (5-10), $|G_p(y)|\le C_1\varepsilon_0$. To prove that $G_p\in\mathcal{C}_p^u(C_1\varepsilon_0,C\varepsilon_0)$, we compute

$$G_p'(y)=\mu_q G_q'(\phi(y))\times\phi'(y)+(\partial_y\eta_r+\partial_\eta\eta_r\times G_q'(\phi(y)))\phi'(y),$$
$$|G_p'(y)|\le\nu^2C\varepsilon_0+O(\varepsilon_0(1+C\varepsilon_0))\nu\le[\nu^2C+\nu C'(1+C\varepsilon_0)]\varepsilon_0$$

for some global $C' > 0$. If we assume $\nu^2 + \varepsilon_0 C'\nu < 1$, which is possible if ε_0 is small enough, then we can choose C large enough satisfying

$$C \times (\nu^2 + \nu C'\varepsilon_0) + \nu C' \le C.$$

This ensures that $\|G'_p\|_\infty \le C\varepsilon_0$.

Finally, we prove (ii) by induction on l: The case $l = 1$ is done. Assume that there exists a constant C_l such that $\|G_q\|_{C^l} \le C_l \implies \|G_p\|_{C^l} \le C_l$. We want to find a constant C_{l+1} fitting for the C^{l+1} norm. Using (5-11), we see by induction that the $(l+1)$-th derivative of G_p has the form

$$G_p^{(l+1)}(y) = \phi'(y)^{l+1} \times G_q^{(l+1)}(y) \times (1 + \partial_\eta \eta_r(y, \phi(y))) + P_y(G_q(y), \dots, G_q^{(l)}(y)),$$

where $P_y(\tau_0, \dots, \tau_l)$ is a polynomial with smooth coefficients in y. Hence, there exists a constant $M(C_l)$ such that for $y \in]-C_1\varepsilon_0, C_1\varepsilon_0[$, $|P_y(G_q(y), \dots, G_q^{(l)}(y))| \le M(C_l)$. Since

$$|\phi'(y)^{l+1}(1 + \partial_\eta \eta_r(y, \phi(y)))| \le \nu(1 + \varepsilon_0 C') := \nu_1$$

if ε_0 is small enough ensuring that $\nu_1 < 1$, we can take

$$C_{l+1} = \max\left(C_l, \frac{M(C_l)}{1 - \nu_1}\right).$$

Indeed, with such a constant, assuming that $\|G_q\|_{C^{l+1}} \le C_{l+1}$, we have

$$|G_p^{(l+1)}(y)| \le C_{l+1}\nu_1 + M(C_l) \le C_{l+1}. \qquad \square$$

Armed with this lemma, we can now iterate the process and get the following proposition describing the evolution of the Lagrangian $\mathcal{C}_{\boldsymbol{q},\theta}$.

Proposition 5.6. *Assume that ε_0 is small enough. Then, for every $n \in \mathbb{N}^*$, $\boldsymbol{q} \in \mathcal{A}^n$, and $\theta \in \mathbb{R}$, there exists an open subset $I_{\boldsymbol{q},\theta} \subset \mathbb{R}$ and a smooth map $G_{\boldsymbol{q},\theta}$ such that*:

- $\mathcal{C}_{\boldsymbol{q},\theta} = \{(y, G_{\boldsymbol{q},\theta}(y)) : y \in I_{\boldsymbol{q},\theta}\}$.
- $\|G'_{\boldsymbol{q},\theta}\|_\infty \le C\varepsilon_0$ *for some global constant C.*
- *For every $l \ge 2$, $\|G_{\boldsymbol{q},\theta}\|_{C^l} \le C_l$ for some global C_l.*
- *If $\phi_{\boldsymbol{q},\theta} : I_{\boldsymbol{q},\theta} \to \mathbb{R}$ is defined by*

$$\kappa_{q_{n-1}} \circ F^{n-1} \circ \kappa_{q_0}^{-1}(\phi_{\boldsymbol{q},\theta}(y), \theta) = (y, G_{\boldsymbol{q},\theta}(y)).$$

Then, for some global constants $C > 0$ and $0 < \nu < 1$, $\|\phi'_{\boldsymbol{q},\theta}\| \le C\nu^{n-1}$.

Proof. Assume that $\mathcal{L}_{\boldsymbol{q},\theta} \neq \varnothing$; otherwise, there is nothing to prove. In particular, we can restrict our attention to small θ, $|\theta| \le C_1\varepsilon_0$. As a consequence, for every $i \in \{1, \dots, n\}$, $F(\mathcal{V}_{q_{i-1}}) \cap \mathcal{V}_{q_i} \neq \varnothing$. Hence, we can consider the maps $\tau_i := \tau_{q_i,q_{i-1}}$ and since we assume that $\kappa_{q_i}(\mathcal{V}_{q_i}) \subset B_{q_i}(0, C_1\varepsilon_0)$,

$$C_{q_0 \cdots q_i,\theta} = \tau_i(C_{q_0 \cdots q_{i-1},\theta}) \cap \kappa_{q_i}(\mathcal{V}_{q_i}).$$

We start with a constant function $G_0 \in \mathcal{C}_0^u(C_1\varepsilon_0, 0)$ such that $\mathcal{L}_{G_0} = \mathcal{C}_\theta$ (it suffices to take $G_0 = \lambda_{q_0,s}\theta$) and we inductively apply the previous lemma to show the existence of a family $G_j \in \mathcal{C}_{q_j}^u(C_1\varepsilon_0, C\varepsilon_0)$, $0 \leq j \leq n-1$, such that:

(i) $\tau_i(\mathcal{L}_{G_i}) \cap B_{q_i}(0, C_1\varepsilon_0) = \mathcal{L}_{G_{i+1}}$.

(ii) $\|G_i\|_{C^l} \leq C_l$.

(iii) If we define $\phi_i :]-C_1\varepsilon_0, C_1\varepsilon_0[\to]-C_1\varepsilon_0, C_1\varepsilon_0[$ by

$$(y, G_i(y)) = D_{q_i} \circ \tau_i \circ D_{q_{i-1}}^{-1}(\phi_i(y), G_{i-1} \circ \phi_i(y))$$

then there exists $\nu < 1$ such that $\|\phi_i'\|_\infty \leq \nu$.

(iv) $\mathcal{C}_{q_0 \cdots q_i, \theta}$ is an open subset of $\mathcal{L}_{G_i}$.

We have

$$\mathcal{L}_{G_{n-1}} = D_{q_{n-1}}^{-1}(\{(y, G_{n-1}(y)) : y \in]-C_1\varepsilon_0, C_1\varepsilon_0[\}).$$

This can be also written

$$\mathcal{L}_{G_{n-1}} = \{(y, \lambda_{q_{n-1},s}^{-1} G_{n-1}(\lambda_{q_{n-1},u} y)) : |y| < \lambda_{q_{n-1},u}^{-1} C_1\varepsilon_0\}.$$

It suffices to consider

$$\begin{aligned} G_{\boldsymbol{q},\theta}(y) &= \lambda_{q_{n-1},s}^{-1} G_{n-1}(\lambda_{q_{n-1},u} y), \\ I_{\boldsymbol{q},\theta} &= \{y \in]-\lambda_{q_{n-1},u}^{-1} C_1\varepsilon_0, \lambda_{q_{n-1},u}^{-1} C_1\varepsilon_0[: (y, G_{\boldsymbol{q},\theta}(y)) \in \mathcal{C}_{\boldsymbol{q},\theta}\}, \\ \phi_{\boldsymbol{q},\theta}(y) &= \lambda_{q_1,u}^{-1} \phi_1 \circ \cdots \circ \phi_{n-1}(\lambda_{q_{n-1},u} y). \end{aligned}$$

□

5B2. *Evolution of Lagrangian states.* Once we've studied the evolution of the Lagrangian leaves starting from $\mathcal{C}_\theta$, we can study the evolution of the corresponding Lagrangian states. In our case, since the leaves stay rather horizontal, the form of the Lagrangian states we'll consider is the simplest:

$$a(x)e^{i\psi(x)/h},$$

where a is an amplitude and ψ a generating phase function. It is associated with the Lagrangian,

$$\mathcal{L} = \{(y, \psi'(y)) : y \in \operatorname{supp} a\}.$$

For $q \in \mathcal{A}$, we quantize κ_q. Remind that we denote by k_q the integer such that $\mathcal{V}_q \Subset U_{k_q}$. There exist Fourier integral operators $B_q, B_q' \in I_0^{\text{comp}}(\kappa_q) \times I_0^{\text{comp}}(\kappa_q^{-1})$,

$$B_q : L^2(Y_{k_q}) \to L^2(\mathbb{R}), \quad B_q' : L^2(\mathbb{R}) \to L^2(Y_{k_q})$$

such that they quantize κ_q in a neighborhood of $\kappa_q(\overline{\mathcal{V}}_q) \times \overline{\mathcal{V}}_q$. Moreover, we impose that $\mathrm{WF}_h(B_q B_q')$ is a compact subset of $\mathbb{R}^2$. We will still denote by B_q and B_q' the operators

$$B_q = (0, \dots, \underbrace{B_q}_{k_q}, \dots, 0) : L^2(Y) \to L^2(\mathbb{R}), \quad B_q' = {}^t(0, \dots, \underbrace{B_q'}_{k_q}, \dots, 0) : L^2(\mathbb{R}) \to L^2(Y).$$

If supp$(c_q) \subset \mathcal{V}_q$ and if C denotes the operator-valued matrix with only one nonzero entry $\mathrm{Op}_h(c_q)$ in position (k_q, k_q), then as operators $L^2(Y) \to L^2(Y)$,

$$B_q' B_q C = C + O(h^\infty), \quad C B_q' B_q = C + O(h^\infty).$$

The proposition we aim at proving is the following:

Proposition 5.7. *Fix $C_0 > 0$. For every $n \in \mathbb{N}$, $\boldsymbol{q} \in \mathcal{A}^n$ and $\theta \in \mathbb{R}$ satisfying*

$$n \le C_0 |\log h|, \quad |\theta| \le C_0 \tag{5-12}$$

and, for every $N \in \mathbb{N}$, there exists a symbol $a_{\boldsymbol{q},\theta,N} \in C_c^\infty(I_{\boldsymbol{q},\theta})$ such that

(i) $U_{\boldsymbol{q}}(B'_{q_0} e^{i(\theta \cdot)/h}) = M A_{q_{n-1}} B'_{q_{n-1}} (e^{i\psi_{\boldsymbol{q}}/h} a_{\boldsymbol{q},\theta,N}) + O(h^N)_{L^2}$,

(ii) $\|a_{\boldsymbol{q},\theta,N}\|_{C_l} \le C_{l,N} h^{-C_0 \log B}$,

(iii) *there exists $\delta > 0$ such that $d(\operatorname{supp}(a_{\boldsymbol{q},\theta,N}), \mathbb{R} \setminus I_{\boldsymbol{q},N,\theta}) \ge \delta$,*

where $\psi_{\boldsymbol{q},\theta}$ is a primitive of $G_{\boldsymbol{q},\theta}$ and $B > 0$ is a global constant.

Remark. • As usual, δ, $C_{l,N}$ and C_N depend only on F, A_q, B_q, B'_q, κ_q and the indices indicated in their notation.

• In other words, the Lagrangian state $e^{i(\theta \cdot)/h}$ is changed to a Lagrangian state associated with $\mathcal{C}_{\boldsymbol{q},\theta}$.

The end of this subsection is devoted to the proof of Proposition 5.7. In the rest of this section, we fix a constant $C_0 > 0$ and we work with a fixed word $\boldsymbol{q} \in \mathcal{A}^n$ with length $n \le C_0 |\log h|$ and a fixed momentum $|\theta| \le C_0$. From now on and until the end of the proof, the constants below will always be uniform in $\boldsymbol{q}, \theta$ satisfying the previous assumption. They will depend on global parameters and on C_0. If they depend on other parameters, we will specify it with subscripts. This is also the case for implicit constants in O (such as in $O(h^\infty)$).

Preparatory work. We first note the following fact: if $\mathcal{V}_q \cap F^{-1}(\mathcal{V}_p) = \varnothing$, $A_p M A_q = O(h^\infty)$. As a consequence, if $\mathcal{V}_{q_{i-1}} \cap F^{-1}(\mathcal{V}_{q_i}) = \varnothing$ for some i, then $U_{\boldsymbol{q}} = O(h^\infty)$. In the sequel, it is enough to consider words $\boldsymbol{q}$ for which $\mathcal{V}_{q_{i-1}} \cap F^{-1}(\mathcal{V}_{q_i}) \neq \varnothing$ for $1 \le i \le n-1$.

We consider symbols $\tilde{a}_q$ such that $\operatorname{supp}(\tilde{a}_q) \subset \mathcal{V}_q$ and $\tilde{a}_q \equiv 1$ on $\operatorname{supp}(\chi_q)$. We denote by $\tilde{A}_q = \operatorname{Op}_h(\tilde{a}_q)$ (as usual thought of as a diagonal operator-valued matrix). The following computations holds since $n = O(\log h)$ and $\|M A_q\| \le \|\alpha\|_\infty + o(1)$ uniformly in q:

$$\begin{aligned} U_{\boldsymbol{q}} B'_{q_0} &= M A_{q_{n-1}} \tilde{A}_{q_{n-1}} M A_{q_{n-2}} \tilde{A}_{q_{n-2}} \cdots M A_{q_1} \tilde{A}_{q_1} M A_{q_0} B'_{q_0} + O(h^\infty) \\ &= M A_{q_{n-1}} B'_{q_{n-1}} B_{q_{n-1}} \tilde{A}_{q_{n-1}} M \cdots M A_{q_1} B'_{q_1} B_{q_1} \tilde{A}_{q_1} M A_{q_0} B'_{q_0} + O(h^\infty). \end{aligned}$$

We set $T_{p,q} = B_p \tilde{A}_p M A_q B'_q$ and $M_q = M A_q B'_q$, which allows us to write

$$U_{\boldsymbol{q}} B'_{q_0} = M_{q_{n-1}} T_{q_{n-1},q_{n-2}} \cdots T_{q_1,q_0} + O(h^\infty).$$

For $p, q \in \mathcal{A}$ with $\mathcal{V}_q \cap F^{-1}(\mathcal{V}_p) \neq \varnothing$, we have $T_{q,p} \in I_0^{\mathrm{comp}}(\tau_{p,q})$. Moreover, the previous computations have shown that $\tau_{p,q}$ has the form

$$\tau_{p,q}(y,\eta) = (\lambda_{p,q} y + y_r(y,\eta), \mu_{p,q}\eta + \eta_r(y,\eta)), \quad (y,\eta) \in B_q(0, C_1\varepsilon_0),$$

where $y_r(y,\eta)$ and $\eta_r(y,\eta)$ are $O(\varepsilon_0)_{C^1}$. This time, $\lambda_{p,q}$, $\mu_{p,q}$ are simply constants uniformly bounded from below and from above for $p, q \in \mathcal{A}$ (recall that $B_q(0, C_1\varepsilon_0)$ is a rectangle in $\mathbb{R}^2$, built from the cube $\widetilde{B}_q(0, C_1\varepsilon_0)$ adapted to the definition of the unstable Jacobian). If ε_0 small enough, the projection $\pi : (y, \eta, x, \xi) \in \mathcal{L}_{q,p} \mapsto (y, \xi) \in \mathbb{R}^2$ is a diffeomorphism onto its image, where

$$\mathcal{L}_{q,p} = \{(\tau_{q,p}(x, \xi), x, -\xi) : (x, \xi) \in B_q(0, C_1\varepsilon_0)\}$$

is the twisted graph of $\tau_{p,q}$. As a consequence, there exists a smooth phase function $S_{p,q}$ defined in an open set $\Omega_{p,q}$ of $\mathbb{R}^2$, generating $\mathcal{L}_{p,q}$ locally, i.e.,

$$\mathcal{L}_{p,q} \cap \tau_{p,q}(B_q(0, C_1\varepsilon_0)) \times B_q(0, C_1\varepsilon_0) = \{(y, \partial_y S_{p,q}(y, \xi), \partial_\xi S_{p,q}(y, \xi), -\xi) : (y, \xi) \in \Omega_{q,p}\}.$$

Hence, $T_{p,q}$ can be written in the following form, up to a $O(h^\infty)$ remainder and for some symbol $\alpha_{p,q}(\,\cdot\,; h) \in C_c^\infty(\Omega_{p,q})$:

$$T_{p,q}u(y) = \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{(i/h)(S_{p,q}(y,\xi) - x\xi)} \alpha_{p,q}(y, \xi; h) u(x)\, dx\, d\xi. \tag{5-13}$$

Moreover, due to the operators $\tilde{A}_p$ and A_q in the definition of $T_{p,q}$, we can assume that

$$(y, \xi) \in \operatorname{supp}(\alpha_{p,q}) \quad \Longrightarrow \quad (\partial_\xi S_{p,q}(y, \xi), \xi) \in \kappa_q(\operatorname{supp} a_q), \quad (y, \partial_y S_{p,q}(y, \xi)) \in \kappa_p(\operatorname{supp} \tilde{a}_p).$$

In the sequel, we write

$$\mathcal{C}_i = \mathcal{C}_{q_0 \cdots q_i, \theta}$$

and we change the subscripts (q_{i-1}, q_i) to i in all the objects T, α, S, τ. Due to the previous results, we can write $\mathcal{C}_i = \{(y, G_i(y)) : y \in I_i\}$, with $I_i := I_{q_0 \cdots q_i, \theta}$ and $G_i := G_{q_0 \cdots q_i, \theta}$. We also have projection maps $\Phi_i : I_i \to \mathbb{R}$ defined by

$$\tau_i \circ \cdots \circ \tau_1(\Phi_i(y), \theta) = (y, G_i(y))$$

satisfying $\|\Phi_i'\|_\infty \le C\nu^i < 1$. Moreover, if we define the intermediate corresponding projection $\phi_i := \Phi_i \circ \Phi_{i-1}^{-1} : I_i \to I_{i-1}$, we observe that ϕ_i is constructed using the properties of F and G_{i-1} (see the proof of Proposition 5.6) and hence, for every l, $\|\phi_i\|_{C^l} \le C_l$ for some C_l not depending on $\boldsymbol{q}, \theta$ nor i.

For $0 \le i \le n-1$, we consider a primitive ψ_i of G_i so that $\mathcal{C}_i$ is generated by ψ_i, i.e.,

$$\mathcal{C}_i = \{(y, \psi_i'(y) : y \in I_i\}.$$

The following lemma can be found in [Nonnenmacher and Zworski 2009, Lemma 4.1]. We state it without proof, since it is the reference but it is a direct application of the stationary phase theorem.

Lemma 5.8. *Pick $i \in \{1, \dots, n-1\}$. For any $a \in C_c^\infty(I_{i-1})$, the application of T_i to the Lagrangian state $ae^{i\psi_{i-1}/h}$ associated with $\mathcal{C}_{i-1}$ gives a Lagrangian state associated with $\mathcal{C}_i$ and satisfies*

$$T_i(ae^{i\psi_{i-1}/h})(y) = e^{i\beta_i/h} e^{i\psi_i(y)/h} \left(\sum_{j=0}^{N-1} b_j(y) h^j + h^N r_N(y; h) \right), \tag{5-14}$$

where, if we let $x = \phi_i(y)$, then $b_j(y) = (L_{j,i}(x, D_x)a)(x)$ for some differential operator $L_{j,i}$ of order $2j$ with smooth coefficients supported in I_{i-1} and $\beta_i \in \mathbb{R}$. Moreover, one has:

- $b_0(y) = |\phi_i'(y)|^{1/2} a(x) \alpha_i(y, \xi) / |\det D^2_{y,\xi} S_i(y, \xi)|^{1/2}$, *with* $\xi = \psi_{i-1}'(x)$.

- $\|b_j\|_{C^l(I_i)} \le C_{l,j}\|a\|_{C^{l+2j}(I_{i-1})}$, $l \in \mathbb{N}$, $0 \le j \le N-1$.
- $\|r_N\|_{C^l(I_i)} \le C_N\|a\|_{C^{l+1+2N}(I_{i-1})}$.

The constants C_N and $C_{l,j}$ depend on τ_i, α_i, $\|\psi_i^{(m)}\|_{\infty,I_i}$.

Remark. • In particular, by virtue of Proposition 5.6, the constants $C_{l,j}$ and C_N can be chosen uniform in $\boldsymbol{q}, \theta$ as soon as they satisfy the required assumptions: $|q| \le C_0|\log h|$, $\theta \le C_0$.

• Without loss of generality, we can replace ψ_i by $\beta_i + \psi_i$ (this actually corresponds to fixing an antiderivative on $\mathcal{C}_{i+1}$) and hence we can assume that $\beta_i = 0$.

• The properties on the support of α_i imply the following ones on the support of the differential operators $L_{j,i}$:

$$y \in \operatorname{supp} L_{j,i} \quad \Longrightarrow \quad (y, \psi_i'(y)) \in \kappa_{q_i}(\operatorname{supp}\tilde{a}_{q_i}) \cap \tau_{i-1} \circ \kappa_{q_{i-1}}(\operatorname{supp} a_{q_{i-1}}). \tag{5-15}$$

Iteration formulas and analysis of the symbols. Then, we iterate this lemma starting from $\psi_0(x) = x \cdot \theta$, in the spirit of Proposition 4.1 in [Nonnenmacher and Zworski 2009]. In the sequel, we adopt the following convention: we denote by x_k the variable in I_k and we naturally define $(x_k, x_{k-1}, \dots, x_1, x_0)$, the sequence defined by $x_{i-1} = \phi_i(x_i)$. We also let

$$\beta_i(x_i) = \frac{\alpha_i(x_i,\xi)}{|\det D^2_{x_i,\xi} S_i(x_i,\xi)|^{1/2}}, \quad \xi = \psi'_{i-1}(x_{i-1}),$$
$$f_i(x_i) = \beta(x_i)|\phi_i'(x_i)|^{1/2}.$$

We fix a constant $B > 0$ (depending only on F, A_q, B_q, B_q', C_0) satisfying, for all $1 \le i \le n-1$,

$$\sup_{x_i \in I_i} |\beta_i(x_i)| \le B, \quad \|T_i\| \le B.$$

Roughly speaking, B is of order $\|\alpha\|_\infty$, but in this part, the precise value of B is not relevant. Finally, note that there exists $\nu < 1$ (again depending only on F, A_q, B_q, B_q') such that $|\phi_i'(x_i)| \le \nu$ for $x_i \in I_i$. Fix $N \in \mathbb{N}$ and define

$$\widetilde{N} = 1 + \lceil N + C_0 \log B \rceil. \tag{5-16}$$

We iteratively define a sequence of symbols $a_{i,j}$, $0 \le i \le n-1$, $0 \le j \le \widetilde{N}-1$ by $a_{0,0} = 1$, $a_{0,j} = 0$ and for $0 \le j \le \widetilde{N}-1$

$$a_{i,j}(x_i) = \sum_{p=0}^{j} L_{j-p,i}(a_{i-1,p})(x_{i-1}). \tag{5-17}$$

The following lemma controls the growth of the symbols. The proof is a precise analysis of the iteration formula (5-17) and is rather technical. We write the detailed proof in the Appendix (see Section A3) and refer the reader to [Nonnenmacher and Zworski 2009, Proposition 4.1], where the author carried out the same analysis (but in the case $B = 1$).

Lemma 5.9. *For all $j \in \{0, \dots, \widetilde{N}-1\}$, $l \in \mathbb{N}$, there exists $C_{j,l} > 0$ such that, for all $i \in \{0, \dots, n-1\}$, one has*

$$\|a_{i,j}\|_{C^l(I_i)} \le C_{j,l}(B\nu^{1/2})^i(i+1)^{l+3j}. \tag{5-18}$$

Remark. Again, what is important is the fact that $C_{j,l}$ does *not* depend on $\boldsymbol{q}$, n, θ nor i: it depends on C_0 and global parameters.

Control of the remainder. Let us call $r_{i,N}(a)$ the remainder appearing in Lemma 5.8. Define inductively $(R_{i,\widetilde{N}})$ by $R_{0,\widetilde{N}}=0$ and

$$R_{i+1,\widetilde{N}} = e^{-(i\psi_{i+1})/h} T_{i+1}(e^{i\psi_i/h} R_{i,\widetilde{N}}) + \sum_{j=0}^{\widetilde{N}-1} r_{i+1,\widetilde{N}-j}(a_{i,j}). \tag{5-19}$$

This definition ensures that, for all $1 \le i \le n$,

$$T_i \cdots T_1(e^{i\psi_0/h}) = e^{i\psi_i(y)/h}\Bigg(\sum_{j=0}^{\widetilde{N}-1} h^j a_{i,j} + h^{\widetilde{N}} R_{i,\widetilde{N}}\Bigg). \tag{5-20}$$

Lemma 5.10. *There exists $C_{\widetilde{N}}$ depending only on $\widetilde{N}$, C_0 and global parameters such that, for all $1 \le i \le n-1$,*

$$\|R_{i,\widetilde{N}}\|_{L^2(\mathbb{R})} \le C_{\widetilde{N}} B^i.$$

Proof. Recalling that $\|T_i\|_{L^2\to L^2} \le B$ and the bound on the remainder in Lemma 5.8, the recursive definition of $R_{i,\widetilde{N}}$ gives the bound

$$\|R_{i,\widetilde{N}}\|_{L^2} \le B\|R_{i-1,\widetilde{N}}\|_{L^2} + \sum_{j=0}^{\widetilde{N}-1} C_{\widetilde{N}-j} \|a_{i-1,j}\|_{C^{1+2(\widetilde{N}-j)}}.$$

By induction and using the previous bounds on $\|a_{i,j}\|_{C^l}$, we get

$$\begin{aligned}\|R_{\widetilde{N},i}\|_{L^2} &\le \sum_{p=0}^{i-1} B^{i-1-p} \sum_{j=0}^{\widetilde{N}-1} C_{\widetilde{N}-j}\|a_{p,j}\|_{C^{1+2(\widetilde{N}-j)}} \\ &\le \sum_{p=0}^{i-1} B^{i-1-p} \sum_{j=0}^{N_1-1} C_{\widetilde{N}-j} C_{\widetilde{N}-j,0} (B\nu^{1/2})^p (p+1)^{1+2\widetilde{N}+j} \\ &\le C_{\widetilde{N}} B^i \sum_{p=0}^{i-1} \nu^{p/2}(p+1)^{1+3N_1} \le C_{\widetilde{N}} B^i,\end{aligned}$$

using that the sum is absolutely convergent. □

End of proof of Proposition 5.7. We've got now all the elements to conclude the proof. We set

$$a_{\boldsymbol{q},\theta,N} := \sum_{j=0}^{\widetilde{N}-1} h^j a_{n-1,j}.$$

We know that

$$U_{\boldsymbol{q}} B'_{q_0}(e^{i\theta/h}) = M_{q_{n-1}}(e^{i\psi_{\boldsymbol{q}}\cdot/h} a_{\boldsymbol{q},\theta,N}) + M_{q_{n-1}}(h^{\widetilde{N}} R_{n-1,\widetilde{N}}).$$

Since M_q are uniformly bounded in q and $R_{n-1,\widetilde{N}} \le C_{\widetilde{N}} B^{n-1} \le C_{N_1} h^{-C_0 \log B}$, we have

$$\|M_{q_{n-1}}(h^{\widetilde{N}} R_{n-1,\widetilde{N}})\|_{L^2} \le C_N h^{\widetilde{N}-C_0\log B} \le C_N h^N.$$

Concerning the bounds on $a_{\boldsymbol{q},\theta,N}$, we have

$$\begin{aligned}\|a_{\boldsymbol{q},\theta,N}\|_{C^l} &\le \sum_{j=0}^{\widetilde{N}-1} h^j \|a_{n-1,j}\|_{C^l} \le \sum_{j=0}^{\widetilde{N}-1} C_{j,l}(B\nu^{1/2})^{n-1} n^{l+3j} h^j \\ &\le C_{l,N} n^{l+3\widetilde{N}} (B\nu^{1/2})^{n-1} \le C_{l,N} h^{-C_0 \log B} n^{l+3\widetilde{N}} \nu^{(n-1)/2} \le C_{l,N} h^{-C_0 \log B},\end{aligned}$$

where we use the fact that $n \le C_0 |\log h|$ and bound $n^{l+3\widetilde{N}} \nu^{(n-1)/2}$ by some $C_{l,\widetilde{N}}$ since $\nu < 1$.

Finally, we need to prove the property on the support of $a_{\boldsymbol{q},\theta,N}$. To do so, let us introduce, for $q \in \mathcal{A}$, an open set $\mathcal{W}_q$ satisfying

$$\operatorname{supp} \tilde{a}_q \Subset \mathcal{W}_q \subset \mathcal{V}_q .$$

This allows us to define new objects replacing $\mathcal{V}_q$ by $\mathcal{W}_q$ in the definitions

$$\mathcal{W}_{\boldsymbol{q}}^+ = \bigcap_{i=0}^{n-1} F^{n-i}(\mathcal{W}_{q_i}) \Subset \mathcal{V}_{\boldsymbol{q}}^+,$$

$$\mathcal{D}_{\boldsymbol{q},\theta} = \kappa_{q_{n-1}}(F^{-1}(\mathcal{W}_{\boldsymbol{q}}^+) \cap F^{n-1}(\mathcal{L}_{q_0,\theta})) \Subset \mathcal{C}_{\boldsymbol{q},\theta},$$

and the associated subinterval $J_{\boldsymbol{q},\theta} \Subset I_{\boldsymbol{q},\theta}$ built thanks to Proposition 5.6 such that

$$\mathcal{D}_{\boldsymbol{q},\theta} = \{(y, G_{\boldsymbol{q},\theta}(y)) : y \in J_{\boldsymbol{q},\theta}\}.$$

Let us fix $\delta > 0$ small (with further conditions imposed). We will show the stronger statement

$$d(\operatorname{supp}(a_{\boldsymbol{q},\theta,N}), \mathbb{R} \setminus J_{\boldsymbol{q},\theta}) \ge \delta.$$

Suppose this is not the case. We can find $x_{n-1} \in \operatorname{supp} a_{\boldsymbol{q},\theta,N},\ y_{n-1} \in I_{\boldsymbol{q},\theta} \setminus J_{\boldsymbol{q},\theta}$ such that $|x_{n-1} - y_{n-1}| \le \delta$. As already done, we denote by x_i (resp. y_i) the points defined by $x_{i-1} = \phi_i(x_i)$ (resp. $y_{i-1} = \phi_i(y_i)$). Since ϕ_i are contractions, we have $|x_i - y_i| \le \delta$ for $1 \le i \le n-1$. If we define

$$\rho_i = \kappa_{q_i}^{-1}(x_i, \psi_i'(x_i)), \quad \zeta_i = \kappa_{q_i}^{-1}(y_i, \psi_i'(y_i)),$$

we have, for some $C > 0$, $d(\rho_i, \zeta_i) \le C\delta$. By definition, one also has

$$F^{-i}(\rho_{n-1}) = \rho_{n-1-i}, \quad F^{-i}(\zeta_{n-1}) = \zeta_{n-1-i}.$$

By the support property (5-15) of the operators $L_{j,i},\ \rho_i \in \operatorname{supp} \tilde{a}_{q_i}$ for $0 \le i \le n-1$. Let's assume that δ is small enough so that, for all $q \in \mathcal{A}$,

$$d(\operatorname{supp} \tilde{a}_q, (\mathcal{W}_q)^c) \ge 2C\delta.$$

Hence,

$$\rho_i \in \operatorname{supp} \tilde{a}_{q_i} \text{ and } d(\rho_i, \zeta_i) \le C\delta \quad \Longrightarrow \quad \zeta_i \in \mathcal{W}_{q_i}.$$

As a consequence, for all $0 \le i \le n-1,\ F^{i+1-n}(\zeta_{n-1}) \in \mathcal{W}_{q_i}$, or equivalently $\zeta_{n-1} \in F^{-1}(\mathcal{W}_{\boldsymbol{q}}^+)$. Hence,

$$(y_{n-1}, \psi_{n-1}'(y_{n-1})) \in \mathcal{C}_{\boldsymbol{q},\theta} \cap \kappa_{q_{n-1}}(F^{-1}(\mathcal{W}_{\boldsymbol{q}}^+)) \subset \mathcal{D}_{\boldsymbol{q},\theta}$$

showing that $y_{n-1} \in J_{\boldsymbol{q},\theta}$, and giving a contradiction with $y_{n-1} \in I_{\boldsymbol{q},\theta} \setminus J_{\boldsymbol{q},\theta}$.

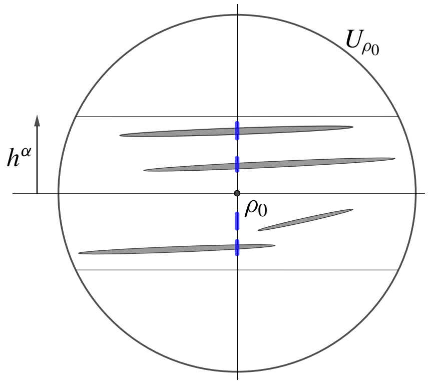

Figure 14. The definition of the sets $\Gamma^+_{\boldsymbol{q}}$. They are represented by the blue segments on the η-axis and are the projections on the η variable of the sets $\mathcal{V}^+_{\boldsymbol{q}}$ (the shaded sets). They are of width of order h^τ.

5C. ***Microlocalization of $U_{\mathcal{Q}}$.*** We now fix a cloud $\mathcal{Q} \subset \mathcal{Q}(n, a)$, centered at a point $\rho_0 \in \mathcal{T}$, namely satisfying the condition of Proposition 4.23:

$$\text{for all } \rho \in \bigcup_{\boldsymbol{q} \in \mathcal{Q}} \mathcal{V}^+_{\boldsymbol{q}}, \quad d(\rho, W_u(\rho_0)) \leq C h^{\mathfrak{b}}.$$

Let us define

$$U_{\mathcal{Q}} = \sum_{\boldsymbol{q} \in \mathcal{Q}} U_{\boldsymbol{q}} \tag{5-21}$$

and

$$\mathcal{V}^+_{\mathcal{Q}} = \bigcup_{\boldsymbol{q} \in \mathcal{Q}} \mathcal{V}^+_{\boldsymbol{q}}. \tag{5-22}$$

We fix an adapted chart $\kappa := \kappa_{\rho_0} : U_0 \to V_0$ around ρ_0 as permitted by the Lemma 3.23. We can assume that $\mathcal{V}^+_a \Subset U_0$ (if ε_0 is small enough and since the local unstable leaf $W_u(\rho_0)$ is close to points in $\mathcal{V}^+_a$)). We consider a cut-off function $\tilde{\chi}_a \in C^\infty_c(U_0)$ such that $\tilde{\chi}_a \equiv 1$ on $F(\operatorname{supp} \chi_a)$ and $\operatorname{supp} \tilde{\chi}_a \subset \mathcal{V}^+_a$. Let us write $\Xi_a = \mathrm{Op}_h(\tilde{\chi}_a)$. Since $\Xi_a M A_a = M A_a + O(h^\infty)$, $|\mathcal{Q}| = O(h^{-K})$ and $\|U_{\boldsymbol{q}}\| = O(h^{-K})$ for some $K > 0$, we have

$$\mathfrak{M}^{N_0} U_{\mathcal{Q}} = \mathfrak{M}^{N_0} \Xi_a U_{\mathcal{Q}} + O(h^\infty).$$

Let us introduce Fourier integral operators B, B' quantizing κ in $\operatorname{supp}(\chi_a)$:

$$B' B = I + O(h^\infty) \text{ microlocally in } \operatorname{supp}(\chi_a).$$

Hence

$$\mathfrak{M}^{N_0} U_{\mathcal{Q}} = \mathfrak{M}^{N_0} \Xi_a B' B U_{\mathcal{Q}} + O(h^\infty).$$

We introduce the sets

$$\Gamma^+ = \eta(\kappa(\mathcal{V}^+_{\mathcal{Q}})), \quad \Omega^+ = \Gamma^+(h^\tau), \tag{5-23}$$

and, for $\boldsymbol{q} \in \mathcal{Q}$,

$$\Gamma^+_{\boldsymbol{q}} = \eta(\kappa(\mathcal{V}^+_{\boldsymbol{q}})). \tag{5-24}$$

We will prove in the following lemma that the pieces $U_{\boldsymbol{q}}$ are microlocalized in thin horizontal rectangles (see Figure 14).

Lemma 5.11. *For every $\boldsymbol{q} \in \mathcal{Q}$,*

$$\mathbb{1}_{\Gamma_{\boldsymbol{q}}^+(h^\tau)}(hD_y)BU_{\boldsymbol{q}} = BU_{\boldsymbol{q}} + O(h^\infty)_{L^2 \to L^2}, \tag{5-25}$$

with uniform bounds in the $O(h^\infty)$.

Using the polynomial bounds $|\mathcal{Q}| = O(h^{-C})$ and $\|U_{\boldsymbol{q}}\| = O(h^{-C})$, we immediately deduce:

Proposition 5.12.
$$\mathbb{1}_{\Omega^+}(hD_y)BU_{\mathcal{Q}} = BU_{\mathcal{Q}} + O(h^\infty)_{L^2\to L^2}. \tag{5-26}$$

5C1. *Proof of Lemma 5.11.* We fix a word $\boldsymbol{q} = q_0 \cdots q_{n-2}a \in \mathcal{Q}$. Since $\mathrm{WF}_h(A_{q_0})$ is compact, we can find $\chi \in C_c^\infty(\mathbb{R})$ such that

$$A_{q_0} = A_{q_0} B'_{q_0} \chi(hD_y) B_{q_0} + O(h^\infty).$$

Since there is a finite number of symbols in $\mathcal{A}$, we can choose one single χ for all the possible symbols q_0. We are hence reduced to proving that

$$\underbrace{\mathbb{1}_{\mathbb{R}\setminus\Gamma_{\boldsymbol{q}}^+(h^\tau)}(hD_y)BU_{\boldsymbol{q}}B'_{q_0}}_{T}\chi(hD_y) = O(h^\infty)_{L^2\to L^2}. \tag{5-27}$$

If $u \in L^2(\mathbb{R})$, writing

$$(\chi(hD_y)u)(y) = \frac{1}{(2\pi h)^{1/2}} \int_{\mathbb{R}} \chi(\theta)\mathcal{F}_h u(\theta) e^{i(\theta y)/h}\, d\theta,$$

we have

$$T(\chi(hD_y)u) = \frac{1}{(2\pi h)^{1/2}} \int_{\mathbb{R}} \chi(\theta)\mathcal{F}_h u(\theta)(Te^{i(\theta\cdot)/h})\, d\theta.$$

Hence,

$$\begin{aligned}
\|T(\chi(hD_y)u)\|_{L^2} &\le \frac{1}{(2\pi h)^{1/2}} \int_{\mathbb{R}} |\chi(\theta)\mathcal{F}_h u(\theta)| \|Te^{i(\theta\cdot)/h}\|_{L^2}\, d\theta \\
&\le \frac{1}{(2\pi h)^{1/2}} \int_{\mathbb{R}} |\chi(\theta)\mathcal{F}_h u(\theta)| \sup_{\theta\in\operatorname{supp}\chi} \|Te^{i(\theta\cdot)/h}\|_{L^2} \\
&\le \frac{C_\chi}{h^{1/2}} \|\mathcal{F}_h u\|_{L^2} \sup_{\theta\in\operatorname{supp}\chi} \|Te^{i(\theta\cdot)/h}\|_{L^2} \\
&\le \frac{C_\chi}{h^{1/2}} \|u\|_{L^2} \sup_{\theta\in\operatorname{supp}\chi} \|Te^{i(\theta\cdot)/h}\|_{L^2}.
\end{aligned}$$

As a consequence, we are lead to estimate $\sup_{\theta\in\operatorname{supp}\chi}\|Te^{i(\theta\cdot)/h}\|_{L^2}$. We fix $\theta \in \operatorname{supp}\chi$. Writing that $\operatorname{supp}\chi \subset [-C_0, C_0]$ and recalling $|\boldsymbol{q}| = n \le C_0|\log h|$ for some global C_0, we are in the framework of Proposition 5.7.

We fix $N \in \mathbb{N}$ and we aim at proving that $Te^{i\theta\cdot/(h)} = O(h^N)$. By Proposition 5.7, there exists $a_{\boldsymbol{q},N,\theta} \in C_c^\infty(I_{\boldsymbol{q},\theta})$ such that

$$U_{\boldsymbol{q}} B'_{q_0}(e^{i(\theta\cdot)/h}) = MA_a B'_a(a_{\boldsymbol{q},N,\theta} e^{i\Phi_{\boldsymbol{q},\theta}/h}) + O(h^N).$$

Set $S := BMA_aB'_a$. Then S is a Fourier integral operator associated with $s := \kappa \circ F \circ \kappa_a^{-1}$. Recall that the definitions and the description of the Lagrangian

$$C_{\boldsymbol{q},\theta} = \kappa_a(F^{-1}(\mathcal{V}_{\boldsymbol{q}}^+) \cap F^{n-1}(\mathcal{L}_{q_0,\theta})) = \{(y, \Phi'_{\boldsymbol{q},\theta}(y)) : y \in I_{\boldsymbol{q},\theta}\},$$

with $\Phi_{\boldsymbol{q},\theta} \in C^\infty(I_{\boldsymbol{q},\theta})$, $\|\Phi_{\boldsymbol{q},\theta}\|_{C^1} \le C\varepsilon_0$, $\|\Phi_{\boldsymbol{q},\theta}\|_{C^l} \le C_l$.

If ε_0 is small enough, we can assume that:

• s is well-defined on $B_a(0, C_1\varepsilon_0)$ and satisfies the conclusion of Lemma 5.5. As a consequence, the Lagrangian line

$$s(\mathcal{C}_{\boldsymbol{q},\theta}) = \kappa(\mathcal{V}_{\boldsymbol{q}}^+) \cap \kappa \circ F^n(\mathcal{L}_{q_0,\theta})$$

can be written $\{(y, \Psi'(y)) : y \in I\}$ for some open $I \subset \mathbb{R}$ and some function $\Psi \in C^\infty(I)$ satisfying

$$\|\Psi\|_{C^1} \le C\varepsilon_0, \quad \|\Psi\|_{C^l} \le C_l,$$

with global constants C and C_l.

• S has the form (5-13) with a phase function and a symbol having C^l norms bounded by global constants (depending on l).

Hence, we can apply Lemma 5.8 to see that there exists $b \in C_c^\infty(I)$ such that

$$S(a_{\boldsymbol{q},N,\theta} e^{i\Phi_{q,\theta}/h}) = be^{i\Psi/h} + O(h^N)_{L^2},$$

and b satisfies the same type of bounds as $a_{\boldsymbol{q},N,\theta}$; namely,

$$\|b\|_{C^l} \le C_{l,N} h^{-C_0 \log B}.$$

Moreover, since $d(\operatorname{supp} a_{\boldsymbol{q},N,\theta}, \mathbb{R} \setminus I_{\boldsymbol{q},\theta}) \ge \delta$, there exists $\delta' > 0$ such that $d(\operatorname{supp} b, \mathbb{R} \setminus I) \ge \delta'$. The constants $C_{l,N}$ and δ' are global constants. Since N is arbitrary, to conclude the proof of Lemma 5.11, it remains to show that

$$\mathbb{1}_{\mathbb{R}\setminus\Gamma_{\boldsymbol{q}}^+(h^\tau)}(hD_y)(be^{i\Psi/h}) = O(h^N). \tag{5-28}$$

To do so, we make use of the fine Fourier localization statement from Proposition 2.7 in [Dyatlov et al. 2022]. We state it for convenience but refer the reader to the quoted paper for the proof.

Proposition 5.13. *Let $U \subset \mathbb{R}^n$ open, $K \subset U$ compact, $\Phi \in C^\infty(U)$ and $a \in C_c^\infty(U)$ with $\operatorname{supp} a \subset K$. Assume that there is a constant C_0 and constants C_N, $N \in \mathbb{N}^*$, such that*

$$\operatorname{vol}(K) \le C_0, \tag{5-29}$$

$$d(K, \mathbb{R}^n \setminus U) \ge C_0^{-1}, \tag{5-30}$$

$$\max_{0<|\alpha|\le N} \sup_U |\partial^\alpha \Phi| \le C_N, \quad N \ge 1, \tag{5-31}$$

$$\max_{0\le|\alpha|\le N} \sup_U |\partial^\alpha a| \le C_N, \quad N \ge 1. \tag{5-32}$$

Finally, assume that the projection of the Lagrangian $\{(x, \Phi'(x)) : x \in U\}$ on the momentum variable has a diameter of order h^τ; namely,

$$\operatorname{diam}(\Omega_\Phi) \le C_0 h^\tau, \quad \text{where } \Omega_\Phi = \{\Phi'(x) : x \in U\}. \tag{5-33}$$

Define the Lagrangian state

$$u(x) = a(x)e^{i\Phi(x)/h} \in C_c^\infty(U) \subset C_c^\infty(\mathbb{R}^n).$$

Then, for every $N \geq 1$, there exists C'_N such that

$$\|\mathbb{1}_{\mathbb{R}^n \setminus \Omega_\Phi(h^\tau)} u\| \leq C'_N h^N, \tag{5-34}$$

where C'_N depends on $\tau, n, N, C_0, C_{N'}$ for some $N'(n, N, \tau)$.

When $U = I,\ K = \operatorname{supp} b,\ a = h^{C_0 \log B} b,\ \Phi = \Psi$, the assumptions (5-29) to (5-32) are satisfied for some global constants C_0, C_N. In this case,

$$\Omega_\Psi = \{\Psi'(y) : y \in I\} = \eta\big(\kappa(\mathcal{V}_{\boldsymbol{q}}^+) \cap \kappa \circ F^n(\mathcal{L}_{q_0,\theta})\big).$$

Since $\Omega_\Psi \subset \Gamma_{\boldsymbol{q}}^+$, to prove (5-28), it is enough to prove it with $\Gamma_{\boldsymbol{q}}^+$ replaced by Ω_Ψ and to apply the last proposition, it remains to check that the last point (5-33) is satisfied. Since who can do more, can do less, we will show that

$$\operatorname{diam}(\Gamma_{\boldsymbol{q}}^+) \leq C_0 h^\tau.$$

This is where the strong assumption on the adapted charts will play a role. To insist on this role, we state the following lemma:

Lemma 5.14. *Let $C_0 > 0$. Assume that $\rho_1 \in \mathcal{T} \cap U_{\rho_0}$ satisfies $d(\rho_1, W_u(\rho_0)) \leq C_0 h^{\mathfrak{b}}$. If $\rho_2 \in W_u(\rho_1)$, then, for some global constant $C > 0$,*

$$|\eta(\kappa(\rho_1)) - \eta(\kappa(\rho_2))| \leq C C_0^{1+\beta} h. \tag{5-35}$$

Proof. Recall that the chart (κ, U_{ρ_0}) is the one centered at ρ_0, given by Lemma 3.23. In this chart, $\kappa(W_u(\rho_1))$ is almost horizontal; we have

$$\kappa(W_u(\rho_1)) = \{y : g(y, \zeta(\rho_1)),\ y \in \Omega\},$$

where Ω is some open bounded set of $\mathbb{R}$, with g and ζ satisfying the properties of Lemma 3.23. Hence, to prove the lemma, it is enough to estimate $|g(y, \zeta(\rho_1)) - g(0, \zeta(\rho_1))|,\ y \in \Omega$. Since $\zeta(\rho_0) = 0$ and ζ is Lipschitz, $|\zeta(\rho_1)| \leq C_0 h^{\mathfrak{b}}$. Indeed, if $\rho'_0 \in W_u(\rho_0)$ satisfies $d(\rho'_0, \rho_1) \leq 2C_0 h^{\mathfrak{b}}$,

$$|\zeta(\rho_1)| = |\zeta(\rho_1) - \zeta(\rho'_0)| \leq C d(\rho_1, \rho'_0) \leq C C_0 h^{\mathfrak{b}}.$$

Then, we have

$$\begin{aligned} |g(y, \zeta(\rho_1)) - g(0, \zeta(\rho_1))| &= |g(y, \zeta(\rho_1)) - g(y, 0) - \partial_\zeta g(y, 0)\zeta(\rho_1)| \\ &= \left| \int_0^{\zeta(\rho_1)} (\partial_\zeta g(y, \zeta) - \partial_\zeta g(y, 0))\, d\zeta \right| \\ &\leq \left| \int_0^{\zeta(\rho_1)} C \zeta^\beta\, d\zeta \right| \leq C \zeta(\rho_1)^{1+\beta} \leq C C_0^{1+\beta} h^{\mathfrak{b}(1+\beta)}. \end{aligned}$$

In the first equality, we've used the facts that $g(0, \zeta) = \zeta,\ \partial_\zeta g(y, 0) = 1$ and $g(y, 0) = 0$. This concludes the proof since, by definition (see (4-2)), $\mathfrak{b}(1 + \beta) = 1$. $\square$

Remark. This lemma explains our definition of $\mathfrak{b}$.

From this lemma, we can deduce (5-33). Indeed, recall that there exists $\rho_{\boldsymbol{q}} \in \mathcal{T}$ such that $\mathcal{V}_{\boldsymbol{q}}^+ \subset W_u(\rho_{\boldsymbol{q}})(Ch^\tau)$. If $\rho_1, \rho_2 \in \mathcal{V}_{\boldsymbol{q}}^+$, there exists $\rho_1', \rho_2' \in W_u(\rho_{\boldsymbol{q}})$ such that

$$d(\rho_i, \rho_i') \leq Ch^\tau, \quad i = 1, 2.$$

Hence, one can estimate

$$|\eta(\kappa(\rho_1)) - \eta(\kappa(\rho_2))| \leq \underbrace{|\eta(\kappa(\rho_1)) - \eta(\kappa(\rho_1'))|}_{\leq Ch^\tau} + \underbrace{|\eta(\kappa(\rho_1')) - \eta(\kappa(\rho_2'))|}_{\leq Ch} + \underbrace{|\eta(\kappa(\rho_2)) - \eta(\kappa(\rho_2'))|}_{\leq Ch^\tau}.$$

The inequality in the middle is a consequence of the previous lemma. Indeed, $\rho_1', \rho_2' \in W_u(\rho_1')$, where (recall that $\tau > \mathfrak{b}$)

$$d(\rho_1', W_u(\rho_0)) \leq d(\rho_1, \rho_1') + d(\rho_1, W_u(\rho_0)) \leq Ch^\tau + Ch^{\mathfrak{b}} \leq 2Ch^{\mathfrak{b}}.$$

5D. *Reduction to a fractal uncertainty principle.* We go on the work started in the last subsection and we keep the same notation. By Propositions 5.3 and 5.12, we can write

$$\mathfrak{M}^{N_0} U_{\mathcal{Q}} = \mathfrak{M}^{N_0} B' B \operatorname{Op}_h(\chi_h) \Xi_a B' \mathbb{1}_{\Omega^+}(hD_y) B U_{\mathcal{Q}} + O(h^\infty)_{L^2 \to L^2}, \tag{5-36}$$

where

- $\chi_h \in S_{\delta_2}^{\text{comp}}$, $\chi_h \equiv 1$ on $\mathcal{T}_-^{\text{loc}}(2C_2 h^{\delta_2})$ and $\operatorname{supp} \chi_h \in \mathcal{T}_-^{\text{loc}}(4C_2 h^{\delta_2})$ (see Proposition 5.3 and before).
- $\Xi_a = \operatorname{Op}_h(\tilde{\chi}_a)$, where $\tilde{\chi}_a \in C_c^\infty(U_0)$ is a cut-off function such that $\tilde{\chi}_a \equiv 1$ on $F(\operatorname{supp} \chi_a)$ and $\operatorname{supp} \tilde{\chi}_a \subset \mathcal{V}_a^+$ (see the beginning of Section 5C).
- $\Omega^+ = \eta(\kappa(\mathcal{V}_{\mathcal{Q}}^+))(h^\tau)$ (see (5-23) and Proposition 5.12).

In V_{ρ_0}, $U_{\mathcal{Q}}$ is microlocalized in a region $\{|\eta| \leq Ch^{\mathfrak{b}}\}$. To work with symbols in usual symbol classes, we will rather consider a bigger region $\{|\eta| \leq h^{\delta_0}\}$. For this purpose, let us define

$$\Gamma^- = y\big(\kappa(\mathcal{V}_a^+ \cap \mathcal{T}_-^{\text{loc}}(4C_2 h^{\delta_2})) \cap \{|\eta| \leq h^{\delta_0}\}\big), \quad \Omega^- = \Gamma^-(h^{\delta_0}). \tag{5-37}$$

Since $\mathcal{V}_{\mathcal{Q}}^+ \subset W_u(\rho_0)(Ch^{\mathfrak{b}})$, we have $\Omega_+ \subset [-C_0 h^{\mathfrak{b}}, C_0 h^{\mathfrak{b}}] \subset [-h^{\delta_0}, h^{\delta_0}]$ for h small enough. By Lemma 5.2, there exists $\chi_+(\eta) := \chi_+(\eta; h) \in C_c^\infty(\mathbb{R})$ such that

- $\chi_+ \equiv 1$ on Ω^+,
- $\operatorname{supp} \chi_+ \subset [-h^{\delta_0}, h^{\delta_0}]$,
- for all $k \in \mathbb{N}$ and $\eta \in \mathbb{R}$, $|\chi_+^{(k)}(\eta)| \leq C_k h^{-\delta_0 k}$ for some global constants C_k,

and χ_+ satisfies

$$\mathbb{1}_{\Omega^+}(hD_y) = \chi_+(hD_y) \mathbb{1}_{\Omega^+}(hD_y).$$

Let's now consider the following subset of Γ^-:

$$\widetilde{\Gamma}^- = y\big(\kappa(\mathcal{V}_a^+ \cap \mathcal{T}_-^{\text{loc}}(4C_2 h^{\delta_2})) \cap \{\eta \in \operatorname{supp} \chi_+\}\big).$$

The inclusion $\widetilde{\Gamma}^- \subset \Gamma^-$ comes from the support property of χ_+.

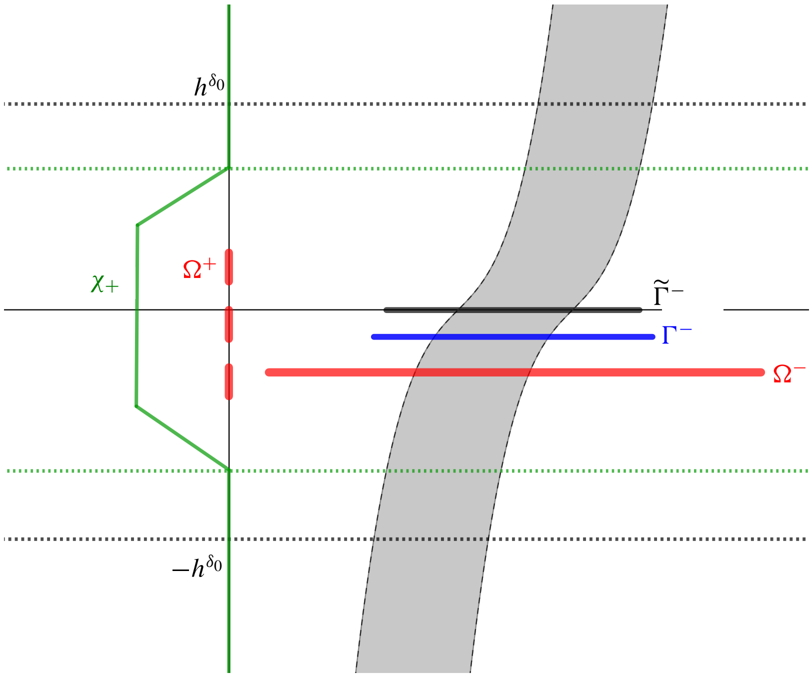

Figure 15. The set Ω^+ is represented on the η-axis, with the support of the function χ_+. On the y-axis, we project the gray set $\kappa(\mathcal{V}_a^+ \cap \mathcal{T}_-^{\mathrm{loc}}(4C_2h^{\delta_2}))$ to obtain both Γ^- and $\widetilde{\Gamma}^-$ depending on the size of the η-window. The larger set Ω^- is also represented in red.

Using again Lemma 5.2, we construct a family $\chi_-(y) := \chi_-(y; h) \in C_c^\infty(\mathbb{R})$ such that

- $\chi_- \equiv 1$ on $\widetilde{\Gamma}^-$,
- $\operatorname{supp} \chi_- \subset \Omega^- = \Gamma^-(h^{\delta_0})$,
- for all $k \in \mathbb{N}$ and $y \in \mathbb{R}$, $|\chi_-^{(k)}(y)| \le C_k h^{-\delta_0 k}$,

and χ_- allows us to write

$$\chi_-(y)\mathbb{1}_{\Omega^-}(y) = \chi_-(y).$$

We encourage the reader to use Figure 15 to fix the ideas. We now claim that

$$\mathfrak{M}^{N_0} U_{\mathcal{Q}} = \mathfrak{M}^{N_0} \operatorname{Op}_h(\chi_h)\Xi_a B' \chi_-(y)\mathbb{1}_{\Omega^-}(y)\mathbb{1}_{\Omega^+}(hD_y) B U_{\mathcal{Q}} + O(h^\infty)_{L^2 \to L^2}. \tag{5-38}$$

Due to the polynomial bounds on $\|\mathfrak{M}^{N_0}\|$ and $\|U_{\mathcal{Q}}\|$, it is then enough to show that

$$\operatorname{Op}_h(\chi_h)\Xi_a B'(1-\chi_-(y))\chi_+(hDy) = O(h^\infty).$$

Using Egorov's theorem in $\Psi_{\delta_2}(\mathbb{R})$, we see that $\Xi_0 := B \operatorname{Op}_h(\chi_h)\Xi_a B'$ is in $\Psi_{\delta_2}(\mathbb{R})$ and $\mathrm{WF}_h(\Xi_0) \subset \kappa(\operatorname{supp}\chi_a \cap \operatorname{supp}\chi_h)$. We now observe that

$$\begin{aligned}(y,\eta) \in \mathrm{WF}_h(\Xi_0) \cap \mathrm{WF}_h(1-\chi_-(y)) \cap \mathrm{WF}_h(\chi_+(hD_y))& \\ \implies \quad (y,\eta) \in \kappa(\operatorname{supp}\chi_a \cap \operatorname{supp}\chi_h), \quad \eta \in \operatorname{supp}\chi_+,\ y \notin \widetilde{\Gamma}^-,&\end{aligned}$$

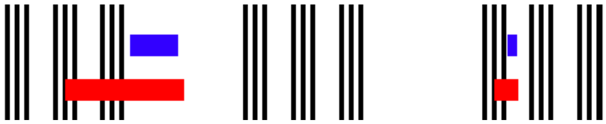

Figure 16. Example of a porous set. Its construction is based on a Cantor-like set. Red intervals correspond to choices of I, blue ones correspond to J.

But the first two conditions imply that $y \in \widetilde{\Gamma}^-$. Hence,

$$\mathrm{WF}_h(\Xi_0) \cap \mathrm{WF}_h(1-\chi_-(y)) \cap \mathrm{WF}_h(\chi_+(hD_y)) = \varnothing.$$

By the composition formulas in $\Psi_{\delta_0}(\mathbb{R})$, we have $\Xi_0(1-\chi_-(y))\chi_+(hD_y) = O(h^\infty)$. Note that the constants in $O(h^\infty)$ depend on the seminorms of $\chi_\pm$, χ_h and χ_a. Due to their construction, the seminorms of $\chi_\pm$ and χ_h are bounded by global constants. As a consequence, the constants $O(h^\infty)$ are global constants.

This proves (5-38). Recalling the bound

$$\|\mathfrak{M}^{N_0}\|_{L^2\to L^2} \le \|\alpha\|^{N_0}(1+o(1)), \quad \|U_{\mathcal{Q}}\|_{L^2\to L^2} \le C|\log h|\,\|\alpha\|_\infty^{N_1},$$

we see that the proof of Proposition 4.23 and hence of Proposition 4.2, has been reduced to proving the following proposition.

Proposition 5.15. *With the above notation, There exist $\gamma > 0$ and $h_0 > 0$ such that,*

$$\text{for all } h \le h_0, \quad \|\mathbb{1}_{\Omega^-}(y)\mathbb{1}_{\Omega^+}(hD_y)\|_{L^2\to L^2} \le h^\gamma. \tag{5-39}$$

Remark. Note γ and h_0 are global; they do not depend on the particular $\mathcal{Q} \subset \mathcal{Q}(n,a)$ satisfying the conditions of Proposition 4.23, nor on n.

The proof of this proposition is the aim of the next section and relies on a fractal uncertainty principle.

6. Application of the fractal uncertainty principle

The fractal uncertainty principle, first introduced in [Dyatlov and Zahl 2016] and further proved in full generality in [Bourgain and Dyatlov 2018], is the key tool for our decay estimate. We'll use the slightly more general version proved and used in [Dyatlov et al. 2022].

6A. *Porous sets.*

See for instance Figure 16 for an example. We start by recalling the definition of porous sets and then we state the version of the fractal uncertainty principle we'll use.

Definition 6.1. Let $\nu \in (0,1)$ and $0 \le \alpha_0 \le \alpha_1$. We say that a subset $\Omega \subset \mathbb{R}$ is ν-porous on a scale from α_0 to α_1 if, for every interval $I \subset \mathbb{R}$ of size $|I| \in [\alpha_0, \alpha_1]$, there exists a subinterval $J \subset I$ of size $|J| = \nu|I|$ such that $J \cap \Omega = \varnothing$.

The following simple lemma shows that when one fattens a porous set, one gets another porous set. For its (very elementary) proof, see [Dyatlov et al. 2022, Lemma 2.12].

Lemma 6.2. *Let $\nu \in (0, 1)$ and $0 \leq \alpha_0 < \alpha_1$. Assume that $\alpha_2 \in]0, \nu\alpha_1/3]$ and $\Omega \subset \mathbb{R}$ is ν-porous on a scale from α_0 to α_1. Then, the neighborhood $\Omega(\alpha_2) = \Omega + [-\alpha_2, \alpha_2]$ is $(\nu/3)$-porous on a scale from $\max(\alpha_0, 3\alpha_2/\nu)$ to α_1.*

The notion of porosity can be related to the different notions of fractal dimensions. Let us recall the definition of the upper-box dimension of a metric space (X, d). We denote by $N_X(\varepsilon)$ the minimal number of open balls of radius ε needed to cover X. Then, the upper-box dimension of X is defined by

$$\overline{\dim}\, X := \limsup_{\varepsilon \to 0} \frac{\log N_X(\varepsilon)}{-\log \varepsilon}. \tag{6-1}$$

In particular, if $\delta > \overline{\dim}_X$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \leq \varepsilon_0$, $N_X(\varepsilon) \leq \varepsilon^{-\delta}$. This observation motivates the following lemma:

Lemma 6.3. *Let $\Omega \subset \mathbb{R}$. Suppose that there exist $0 < \delta < 1$, $C > 0$ and $\varepsilon_0 > 0$ such that,*

$$\textit{for all } \varepsilon \leq \varepsilon_0, \quad N_\Omega(\varepsilon) \leq C\varepsilon^{-\delta}.$$

Then, there exists $\nu = \nu(\delta, \varepsilon_0, C)$ such that Ω is ν-porous on a scale from 0 to 1.

Remark. The proof will give an explicit value of ν. This quantitative statement will be important in the sequel to ensure the same porosity for all the sets $W_{u/s}(\rho_0) \cap \mathcal{T}$.

Proof. Let us set $T = \lfloor \max((6\varepsilon_0)^{-1}, (6^\delta C)^{1/(1-\delta)}) \rfloor + 1$ and $\nu = (3T)^{-1}$. We will show that Ω is ν-porous on a scale from 0 to 1. Let $I \subset \mathbb{R}$ be an interval of size $|I| \in]0, 1]$. Cut I into $3T$ consecutive closed intervals of size ν: $J_0, \dots, J_{3T-1}$. We argue by contradiction and assume that each of these intervals does intersect Ω. Let us show that

$$N_\Omega(\nu/2) \geq T. \tag{6-2}$$

Assume that $U_1, \dots, U_k$ is a family of open intervals of size ν covering Ω. For $i = 0, \dots, T-1$, there exists $x_i \in J_{3i+1}$ and $j_i \in \{1, \dots, k\}$ such that $x_i \in U_{j_i}$. It follows that $U_{j_i} \subset J_{3i} \cup J_{3i+1} \cup J_{3i+2}$ and hence $i \neq l \Longrightarrow U_{j_i} \cap U_{j_l} = \varnothing$. The map $i \in \{0, \dots, T-1\} \mapsto j_i \in \{1, \dots, k\}$ is one-to-one, and it gives (6-2). Since $T \geq 1/(6\varepsilon_0)$, we have $\nu/2 \leq \varepsilon_0$. As a consequence,

$$T \leq N(\nu/2) \leq C(6T)^\delta,$$

which implies $T^{1-\delta} \leq C6^\delta$. This contradicts the definition of T. □

In Section A5 of the Appendix, we give a result in the other way, namely, porous sets down to scale 0 have an upper-box dimension strictly smaller than 1.

For further use, we also record the easy lemma:

Lemma 6.4. *Assume (X, d), (Y, d') are metric spaces and $f : X \to Y$ is C-Lipschitz. Then, for every $\varepsilon > 0$,*

$$N_{f(X)}(\varepsilon) \leq N_X(\varepsilon/C).$$

In particular, if $N_X(\varepsilon) \leq C_1^\delta \varepsilon^\delta$ for $\varepsilon \leq \varepsilon_0$, then, for $\varepsilon \leq C\varepsilon_0$, we have $N_{f(X)}(\varepsilon) \leq (C_1 C)^\delta \varepsilon^{-\delta}$.

6B. ***Fractal uncertainty principle.*** We state here the version of the fractal uncertainty principle we'll use. This version is stated in Proposition 2.11 in [Dyatlov et al. 2022]. The difference with the original version in [Bourgain and Dyatlov 2018] is that it relaxes the assumption regarding the scales on which the sets are porous. We refer the reader to [Dyatlov 2019] to an overview on the fractal uncertainty principle with other references and applications.

Proposition 6.5 (fractal uncertainty principle). *Fix numbers $\gamma_0^\pm$, $\gamma_1^\pm$ such that*

$$0 \le \gamma_1^\pm < \gamma_0^\pm \le 1, \quad \gamma_1^+ + \gamma_1^- < 1 < \gamma_0^+ + \gamma_0^-$$

and define

$$\gamma := \min(\gamma_0^+, 1-\gamma_1^-) - \max(\gamma_1^+, 1-\gamma_0^-).$$

Then, for each $\nu > 0$, there exists $\beta = \beta(\nu) > 0$ and $C = C(\nu)$ such that the estimate

$$\| \mathbb{1}_{\Omega_-} \mathcal{F}_h \mathbb{1}_{\Omega_+} \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \le C h^{\gamma\beta} \tag{6-3}$$

holds for all $0 < h \le 1$ and all h-dependent sets $\Omega_\pm \subset \mathbb{R}$ which are ν-porous on a scale from $h^{\gamma_0^\pm}$ to $h^{\gamma_1^\pm}$.

Remark. In the sequel, we will use this result with $\gamma_1^\pm = 0$. In this case, the condition on $\gamma_0^\pm$ becomes $\gamma_0^- + \gamma_0^+ > 1$ and the exponent γ is $\gamma_0^- + \gamma_0^+ - 1$. This condition can be interpreted as a condition of saturation of the standard uncertainty principle: a rectangle of size $h^{\gamma_0^+} \times h^{\gamma_0^-}$ will be subplanckian.

6C. ***Porosity of Ω^+ and Ω^-.*** Since we want to apply Proposition 6.5 to prove Proposition 5.15, we need to show the porosity of the sets $\Omega^\pm$ defined in (5-23) and (5-37). The main tool is the following proposition.

Proposition 6.6. *There exist $\delta \in [0, 1[$, $C > 0$ and $\varepsilon_0 > 0$ such that, for every $\rho_0 \in \mathcal{T}$, if $X = W_{u/s}(\rho_0) \cap \mathcal{T} \cap U_{\rho_0}$,*

$$N_X(\varepsilon) \le C\varepsilon^{-\delta} \quad \textit{for all } \varepsilon \le \varepsilon_0.$$

Remark. Recall that $W_{u/s}(\rho_0)$ is a local unstable (resp. stable) manifold at ρ_0, and in particular a single smooth curve. U_{ρ_0} is the domain of the chart adapted κ_{ρ_0} (see Lemma 3.23).

Roughly speaking, this proposition says that the upper-box dimension of the sets $W_{u/s}(\rho) \cap \mathcal{T}$, the trace of $\mathcal{T}$ along the stable and unstable manifolds, is strictly smaller than 1. This condition on the upper-box dimension is a fractal condition. In our case, we need uniform estimates on the numbers $N_X(\varepsilon)$ for $X = W_{u/s}(\rho) \cap \mathcal{T}$. This uniformity is a consequence of the fact that the holonomy maps are C^1 with uniform C^1 bounds (and thus Lipschitz, which is enough to conclude). This result is clearly linked with Bowen's formula, which has been proved in different contexts and links the dimension of X with the topological pressure of the map $\phi_u = -\log |J_u^1|$. This is where the assumption (Fractal) is used. This proposition is proved in Section A4 of the Appendix where we borrow the arguments of [Barreira 2008, Section 4.3] to get the required bounds.

From the Proposition 6.6, we get:

Corollary 6.7. *There exists $\nu > 0$ such that, for every $\rho_0 \in \mathcal{T}$, the sets $y \circ \kappa(W_u(\rho_0) \cap \mathcal{T} \cap U_{\rho_0})$ and $\zeta(W_s(\rho_0) \cap \mathcal{T} \cap U_{\rho_0})$ are ν-porous on a scale from 0 to 1.*

Proof. The maps $y \circ \kappa$ and ζ are C-Lipschitz for a global constant C. As a consequence, the previous lemma and Lemma 6.4 give,

$$\text{for all } \varepsilon \leq \varepsilon_0/C, \quad N_\Omega(\varepsilon) \leq C^\delta \varepsilon^{-\delta}, \quad \text{where } \Omega = y \circ \kappa(W_u(\rho_0) \cap \mathcal{T} \cap U_{\rho_0}) \text{ or } \zeta(W_s(\rho_0) \cap \mathcal{T} \cap U_{\rho_0}).$$

Applying Lemma 6.3, the ν-porosity is proved for some $\nu = \nu(\delta, C, \varepsilon_0)$. ☐

To conclude, we use this corollary to show the porosity of $\Omega^\pm$. We start by studying Ω^+.

Lemma 6.8. *There exists a global constant $C > 0$ such that*

$$\Omega^+ \subset \zeta(W_s(\rho_0) \cap \mathcal{T} \cap U_{\rho_0})(Ch^\tau).$$

Proof. Since $\Omega^+ = \Gamma^+(h^\tau)$, it is enough to show the same statement for $\Gamma^+ = \eta \circ \kappa_{\rho_0}(\mathcal{V}_\mathcal{Q}^+)$.

Let $\rho \in \mathcal{V}_\mathcal{Q}^+$. By assumption on $\mathcal{Q}$ and ρ_0, $d(\rho, W_u(\rho_0)) \leq Ch^\mathfrak{b}$. Since $\rho \in \mathcal{V}_{\boldsymbol{q}}$ for some $\boldsymbol{q} \in \mathcal{Q}$, there exists $\rho_1 \in \mathcal{T}$ such that $d(\rho, W_u(\rho_1)) \leq C/J_{\boldsymbol{q}}^+(\rho_1) \leq Ch^\tau$. Fix $\rho_2 \in W_u(\rho_1)$ such that $d(\rho, \rho_2) \leq Ch^\tau$. Then

$$|\eta \circ \kappa(\rho) - \zeta(\rho_1)| = |\eta \circ \kappa(\rho) - \zeta(\rho_2)| \leq |\eta \circ \kappa(\rho) - \eta \circ \kappa(\rho_2)| + |\eta \circ \kappa(\rho_2) - \zeta(\rho_2)|.$$

Since $\eta \circ \kappa$ is Lipschitz, we can control the first term by

$$|\eta \circ \kappa(\rho) - \eta \circ \kappa(\rho_2)| \leq Cd(\rho, \rho_2) \leq Ch^\tau.$$

To estimate the second term, the same arguments used after Lemma 5.14 show that

$$|\eta \circ \kappa(\rho_2) - \zeta(\rho_2)| \leq \operatorname{diam}[\eta \circ \kappa(W_u(\rho_2) \cap U_{\rho_0})] \leq Ch.$$

It gives $|\eta \circ \kappa(\rho) - \zeta(\rho_1)| \leq Ch^\tau$. To conclude, note that there exists a unique point $\rho_1' \in W_s(\rho_0) \cap W_u(\rho_1)$ and $\zeta(\rho_1) = \zeta(\rho_1')$. ☐

As a simple corollary of this lemma and of Lemma 6.2, we get:

Corollary 6.9. *Ω^+ is $\nu/3$-porous on a scale from $(3/\nu)Ch^\tau$ to* 1.

We now turn to the study of Ω^-. We can state and prove similar results with different scales of porosity. Recall that $\delta_2 = (\lambda_0/\lambda_1)\delta_0$.

Lemma 6.10. *There exists a global constant $C > 0$ such that*

$$\Omega^- \subset y \circ \kappa(W_u(\rho_0) \cap \mathcal{T} \cap U_{\rho_0})(Ch^{\delta_2}).$$

Proof. Since $\Omega^- = \Gamma^-(h^{\delta_0})$ with $\delta_0 > \delta_2$, it is enough to prove it for

$$\Gamma^- = y \circ \kappa\big(\mathcal{V}_a^+ \cap \mathcal{T}_-^{\mathrm{loc}}(4C_2h^{\delta_2}) \cap \{|\eta| \leq h^{\delta_0}\}\big).$$

Recall that $\mathcal{T}_-^{\mathrm{loc}} \subset \bigcup_{\rho \in \mathcal{T}} W_s(\rho)$. Since in $\mathcal{V}_a^+$ all the local stable leaves intersect $W_u(\rho_0)$, we have

$$\mathcal{V}_a^+ \cap \mathcal{T}_-^{\mathrm{loc}}(4C_2h^{\delta_2}) \subset \bigcup_{\rho \in W_u(\rho_0) \cap \mathcal{T}} W_s(\rho)(4C_2h^{\delta_2}).$$

Fix $\rho \in W_u(\rho_0) \cap \mathcal{T}$. Since $d\kappa(E_s(\rho_0)) = \mathbb{R}\partial_\eta$, if ε_0 is small enough, we can write $\kappa(W_s(\rho)) = \{(G_\rho(\eta), \eta) : \eta \in O\}$, where O is some open subset of $\mathbb{R}$ and $G_\rho : O \to \mathbb{R}$ is C^∞. In particular, it is Lipschitz with a global

Lipschitz constant C. If $|\eta| \le h^{\delta_0}$, then $|G_\rho(\eta) - G_\rho(0)| \le Ch^{\delta_0}$. Recall that $\kappa(W_u(\rho_0) \cap U_{\rho_0}) \subset \mathbb{R} \times \{0\}$ and hence, $G_\rho(0) = y \circ \kappa(\rho)$. As a consequence, if $\rho_1 \in W_s(\rho) \cap \{|\eta| \le h^{\delta_0}\}$, writing $\kappa(\rho_1) = (G_\rho(\eta), \eta)$, we have

$$|y \circ \kappa(\rho_1) - y \circ \kappa(\rho)| = |G_\rho(\eta) - G_\rho(0)| \le Ch^{\delta_0}.$$

Then, if $\rho_2 \in W_s(\rho)(4C_2 h^{\delta_2})$, since κ is Lipschitz with global Lipschitz constant,

$$|y \circ \kappa(\rho_2) - y \circ \kappa(\rho)| \le Ch^{\delta_2} + Ch^{\delta_0} \le Ch^{\delta_2}.$$

This shows that $y \circ \kappa(\rho_2) \in y \circ \kappa(W_u(\rho_0) \cap \mathcal{T})(Ch^{\delta_2})$ and concludes the proof. □

As a corollary, using Lemma 6.2, we get:

Corollary 6.11. *Ω^- is $\nu/3$-porous on a scale from $(3/\nu)Ch^{\delta_2}$ to 1.*

We can now prove the last Proposition 5.15 needed to end the proof of Proposition 4.2. This is a consequence of the porosity of $\Omega^\pm$ and the fractal uncertainty principle. To apply Proposition 6.5, we need to ensure that the scale condition is satisfied, that is to say

$$\delta_2 + \tau > 1,$$

which has been supposed when defining τ in (4-5) and (4-6). Proposition 4.2 then comes with any $0 < \gamma < (\delta_2 + \tau - 1)\beta(\nu/3)$.

Appendix

A1. *Holder regularity for flows.*

Lemma A.1. *Let $U \subset \mathbb{R}^n$ be open and $Y : U \to \mathbb{R}^n$ be a complete $C^{1+\beta}$ vector field. We denote by $\phi^t(x)$ the flow generated by Y. Then, for any $T \in \mathbb{R}$ and $K \subset U$ compact, the map*

$$(t, x) \in [-T, T] \times K \mapsto \phi^t(x)$$

is $C^{1+\beta}$.

Proof. We fix T, K as in the statement. We'll use the same constants C, C' at different places, with different meaning. In addition to Y, they will depend on T, K.

Since Y is C^1, Cauchy–Lipschitz theorem gives the local existence and uniqueness of the flow. It is standard that the flow is also C^1 and satisfies

$$\partial_t d\phi^t(x) = dY(\phi^t(x)) \circ d\phi^t(x). \tag{A-1}$$

Let's define $A^t(x) = d\phi^t(x)$ and $\Xi(t, x) = dY(\phi^t(x))$. The assumption on Y implies that Ξ is β-Hölder.

Fix $(t_0, x_0), (t_1, x_1) \in [-T, T] \times K$ and let's estimate $\|A^{t_1}(x_1) - A^{t_0}(x_0)\|$. We split it into two pieces and control it with the triangle inequality:

$$\|A^{t_1}(x_1) - A^{t_0}(x_0)\| \le \|A^{t_1}(x_1) - A^{t_0}(x_1)\| + \|A^{t_0}(x_1) - A^{t_0}(x_0)\|.$$

It is not hard to control the first term of the right-hand side using (A-1) since

$$\|A^{t_1}(x_1) - A^{t_0}(x_1)\| = \left| \int_{t_0}^{t_1} \Xi(s, x_1) \circ A^s(x_1)\, ds \right| \leq C|t_1 - t_0|.$$

To estimate the second term, we estimate

$$\begin{aligned}\|\partial_t (A^t(x_1) - A^t(x_0))\| &\leq \|(\Xi(t, x_1) - \Xi(t, x_0)) \circ A^t(x_1) + \Xi(t, x_0) \circ (A^t(x_1) - A^t(x_0))\| \\ &\leq C d(x_0, x_1)^\beta + C'\|A^t(x_1) - A^t(x_0)\|.\end{aligned}$$

By Gronwall's lemma,

$$\|A^{t_0}(x_1) - A^{t_0}(x_0)\| \leq C d(x_0, x_1)^\beta e^{C' t_0} \leq C d(x_0, x_1)^\beta. \qquad \square$$

A2. ***Proof of Lemma 3.24.*** We give the missing proof of Lemma 3.24 and widely use the notation of the Section 3E. Its proof uses the construction of e_u in the proof of Theorem 5. It is inspired by techniques usually used to show the unstable manifold theorem; see for instance [Dyatlov 2018]. In fact, the smoothness of $y \mapsto f_0(y, 0)$ is a direct consequence of the smoothness of the unstable manifold $W_u(\rho_0)$. It was not clear for us if it was possible to easily deduce from this the required smoothness of $y \mapsto \partial_\eta f_0(y, 0)$. This is why we decided to give a proof of this proposition. It uses the fact that e_u has been constructed to satisfy $\mathbb{R} d_\rho F(e_u(\rho)) = \mathbb{R} e_u(F(\rho))$ for ρ in a small neighborhood of $\mathcal{T}$. To show the lemma, we need information along all the orbit of ρ_0. For this purpose, we introduce the following, for $m \in \mathbb{Z}$:

- $\rho_m = F^m(\rho_0)$.
- $\kappa_m : U_m \to V_m \subset \mathbb{R}^2$ the chart given by Lemma 3.22 centered at ρ_m and we assume that the relation $\mathbb{R} d_\rho F(e_u(\rho)) = \mathbb{R} e_u(F(\rho))$ holds for $\rho \in U_m$. We will denote by (y_m, η_m) the variable in V_m.
- $G_m = \kappa_{m+1} \circ F \circ \kappa_m^{-1} : V_m \to V_{m+1}$.
- A reparametrization of the vector field $(\kappa_m)_* e_u$: $\mathbb{R}(\kappa_m)_* e_u = \mathbb{R} e_m$, where $e_m(y_m, \eta_m) = {}^t(1, s_m(y_m, \eta_m))$, where s_m is a slope function which is known to be $C^{1+\beta}$.

Note that $s_m(y_m, 0) = 0$ due to the fact that $\kappa_m(W_u(\rho_m)) \subset \mathbb{R} \times \{0\}$. The hyperbolicity assumption on F and the properties of κ_m allow us to write

$$G_m(y_m, \eta_m) = (\lambda_m y_m + \alpha_m(y_m, \eta_m), \mu_m \eta_m + \beta_m(y_m, \eta_m)),$$

where

- For some $\nu < 1$, $0 \leq |\mu_m| \leq \nu$, $|\lambda_m| \geq \nu^{-1}$ for all $m \in \mathbb{N}$.
- $\alpha_m(0, 0) = \beta_m(0, 0) = 0$.
- $\beta_m(y_m, 0) = 0$ for $(y_m, 0) \in V_m$.
- $d\alpha_m(0, 0) = d\beta_m(0, 0) = 0$.
- We can assume that U_m are sufficiently small neighborhoods of ρ_m so that $\beta_m, \alpha_m = O(\delta_0)_{C^1(U_m)}$ for some small $\delta_0 > 0$.

The property $d_\rho F(e_u(\rho)) \in \mathbb{R}e_u(F(\rho))$ implies that $d_{(y_m,\eta_m)} G_m(e_m(y_m,\eta_m)) \in \mathbb{R}e_{m+1}(G_m(y_m,\eta_m))$. As a consequence, the transformation of the slopes gives an equation satisfied by the family of slopes $(s_m)_{m\in\mathbb{Z}}$:

$$s_{m+1}(G_m(y_m,\eta_m)) = Q_m(y_m,\eta_m,s_m(y_m,\eta_m)), \tag{A-2}$$

where Q_m is the smooth function

$$Q_m(y_m,\eta_m,s) = \frac{s\times(\mu_m+\partial_{\eta_m}\beta_m(y_m,\eta_m))+\partial_{y_m}\beta_m(y_m,\eta_m)}{\lambda_m+\partial_{y_m}\alpha_m(y_m,\eta_m)+s\times\partial_{\eta_m}\alpha_m(y_m,\eta_m)}.$$

Writing $G_m(y_m,\eta_m)=(y_{m+1},\eta_{m+1})$, we deduce by differentiation of (A-2) with respect to η_{m+1} (we omit the point of evaluation of the maps involved in the right-hand side to alleviate the line)

$$\begin{aligned}\partial_{\eta_{m+1}} s_{m+1}(y_{m+1},\eta_{m+1}) = \partial_{y_m} Q_m \times \partial_{\eta_{m+1}} y_m + \partial_{\eta_m} Q_m \times \partial_{\eta_{m+1}}\eta_m \\ + \partial_s Q_m \times (\partial_{y_m} s_m \times \partial_{\eta_{m+1}} y_m + \partial_{\eta_m} s_m \times \partial_{\eta_{m+1}}\eta_m).\end{aligned} \tag{A-3}$$

This last equation gives the transformation of vertical derivative of the slope. We now evaluate this identity at the point $(y_{m+1},0)$. In the following lines, when the variables y_m and y_{m+1} appear in the same equation, we implicitly assume that they are related by $(y_{m+1},0)=G_m(y_m,0)$, namely $y_{m+1}=\lambda_m y_m+\alpha_m(y_m,0)$. We remark that due to the fact that $\beta_m(y_m,0)=0$, we have $Q_m(y_m,0,0)=0$ and the first term of the right-hand side vanishes. The term $\partial_{y_m} s_m$ also vanishes at $(y_m,0)$. We will write

$$\begin{aligned}\sigma_m(y_m) &= \partial_{\eta_m} s_m(y_m,0),\\ h_m(y_m) &= \partial_{\eta_m} Q_m(y_m,0,0)\times\partial_{\eta_{m+1}}\eta_m(y_{m+1},0),\\ c_m(y_m) &= \partial_s Q_m(y_m,0,0)\times\partial_{\eta_{m+1}}\eta_m(y_{m+1},0).\end{aligned}$$

This notation allows us to rewrite (A-3) at $(y_{m+1},0)$:

$$\sigma_{m+1}(y_{m+1}) = h_m(y_m)+c_m(y_m)\times\sigma_m(y_m). \tag{A-4}$$

We observe that $|\partial_{\eta_{m+1}}\eta_m(y_m,0)| = |\mu_m^{-1}+O(\delta_0)_{C^0}|$ and after some computations, we see that

$$\partial_s Q_m(y_m,0,0) = \frac{\mu_m}{\lambda_m}+O(\delta_0)_{C^0}.$$

As a consequence,

$$|c_m(y_m)| = |\lambda_m^{-1}|+O(\delta_0)_{C^0}\le\nu_1, \tag{A-5}$$

where, if δ_0 is small enough, we can fix $\nu_1<1$. Moreover, c_m and h_m are smooth functions and their C^N norms are bounded uniformly in m, and actually by global constants depending only on F. Furthermore, $y_m\mapsto y_{m+1}$ is given by $y_m\mapsto\lambda_m y+\alpha_m(y_m,0)$ and is an expanding diffeomorphism provided δ_0 is small enough.

We fix some small ε such that $(-\varepsilon,\varepsilon)\times\{0\}\subset U_m$ for all m. Let's define $I=(-\varepsilon,\varepsilon)$. We will make use of the fiber contraction theorem to show that $y_m\in I\mapsto\sigma_m(y_m)$ is smooth for every m, with uniform C^N norms. For this purpose, let us introduce the following notation:

- $C_0\le C_1\le\cdots\le C_N\le\cdots$ a family of constants which will be specified in the sequel.

- The complete metric space $X_N = \{\gamma \in C^N(I) : \|\gamma\|_{C^k} \le C_k,\ 0 \le k \le N\}$ equipped with the C^N norm.
- The auxiliary metric space $X_N^{\mathrm{aux}} = \{\gamma \in C^0(I) : \|\gamma\|_\infty \le C_N\}$ equipped with the C^0 norm.
- The complete metric space $E_N = (X_N)^{\mathbb{Z}}$ equipped with the metric
$$d(\gamma_1, \gamma_2) = \sup_{m\in\mathbb{Z}} \|(\gamma_1)_m - (\gamma_2)_m\|_{C^N}.$$
- Its auxiliary counterpart $E_N^{\mathrm{aux}} = (X_N^{\mathrm{aux}})^{\mathbb{Z}}$ equipped with the metric
$$d(\gamma_1, \gamma_2) = \sup_{m\in\mathbb{Z}} \|(\gamma_1)_m - (\gamma_2)_m\|_{C^0}.$$

For $\gamma \in E_N$, let's define $T\gamma$ with the formula (A-4):
$$(T\gamma)_{m+1}(y_{m+1}) = (h_m + c_m\gamma_m)(y_m).$$
Since $y_m \mapsto y_{m+1}$ is expanding, we see that $y_{m+1} \in I \implies y_m \in I$. Hence, $(T\gamma)_{m+1}$ is well-defined on I. Our aim is to show by induction on N that for every $N \in \mathbb{N}$, $\sigma := (\sigma_m)_{m\in\mathbb{Z}}$ is in E_N and is an attractive fixed point of $T : E_N \to E_N$.

We start with the case $N = 0$. We need to check that $T(E_0) \subset E_0$. It will be the case as soon as
$$C_0\nu_1 + \sup_m \|h_m\|_\infty \le C_0.$$
For instance, take $C_0 = 2\sup_m \|h_m\|_\infty/(1-\nu_1)$. Due to the fact that $\|c_m\|_{C^0(I)} \le \nu_1$, T is a contraction with contraction rate ν_1 and hence $T : E_0 \to E_0$ has a unique attractive fixed point. This fixed point is necessarily σ since σ satisfies (A-4).

Arguing by induction, we assume that $\sigma \in E_N$, $T(E_N) \subset E_N$ and σ is an attractive fixed point for T and we want to show that the same is true for $N+1$. For this purpose, suppose that $\gamma \in E_N$ is of class C^{N+1}. Analyzing the formula defining T, we see that can write, for $m \in \mathbb{Z}$,
$$(T\gamma)_m^{(N+1)}(y_{m+1}) = h_m^{(N+1)}(y_m) + c_m(y_m) \times \left(\frac{\partial y_{m+1}}{\partial y_m}(y_m)\right)^{-N-1} \times \gamma_m^{(N+1)}(y_m) + R_{N,m}(y_m, \gamma_m(y_m), \dots, \gamma_m^{(N)}(y_m)), \quad \text{(A-6)}$$
where $R_{N,m} : I \times [-C_0, C_0] \times \cdots \times [-C_N, C_N] \to \mathbb{R}$ is a polynomial in the last $N+1$ variables with smooth coefficients in y_m, uniformly bounded in m. As a consequence, there exists a global constant C'_{N+1} such that
$$\sup_m \sup_{I\times[-C_0,C_0]\times\cdots\times[-C_N,C_N]} |R_{N,m}(y_m, \tau_0, \dots, \tau_N)| \le C'_{N+1}.$$
We can then choose $C_{N+1} \ge C_N$ such that
$$\sup_m \|h_m\|_{C^{N+1}} + C'_{N+1} + \nu_1 C_{N+1} \le C_{N+1},$$
which ensures that $T : E_{N+1} \to E_{N+1}$. We now wish to use the fiber contraction theorem (Theorem 6). If $\gamma \in E_N$, we define the map $S_\gamma : E_{N+1}^{\mathrm{aux}} \to E_{N+1}^{\mathrm{aux}}$ by
$$(S_\gamma\theta)_{m+1}(y_{m+1}) = h_m^{(N+1)}(y_m) + c_m(y_m) \times \left(\frac{\partial y_{m+1}}{\partial y_m}(y_m)\right)^{-N-1} \times \theta_m(y_m) + R_{N,m}(y_m, \gamma_m(y_m), \dots, \gamma_m^N(y_m)).$$

Due to the choice of C_{N+1}, we see that S_γ is well-defined and since we have

$$\left|\frac{\partial y_{m+1}}{\partial y_m}(y_m)\right| \geq 1$$

and $\|c_m\|_{C^0(I)} \leq \nu_1$, we know S_γ is a contraction with contraction rate ν_1 for every $\gamma \in E_N$. In particular, the map S_σ has a unique fixed point $\sigma_{N+1} \in E^{\text{aux}}_{N+1}$.

The fiber contraction theorem (Theorem 6) applies to the continuous map

$$T_N : (\gamma, \theta) \in E_N \times E^{\text{aux}}_{N+1} \mapsto (T\gamma, S_\gamma\theta) \in E_N \times E^{\text{aux}}_{N+1}$$

and (σ, σ_{N+1}) is an attractive fixed point of T_N in $E_N \times E^{\text{aux}}_{N+1}$.

In particular, if $\gamma \in E_{N+1}$, then $\tilde{\gamma} := (\gamma, \gamma^{(N+1)}) \in E_N \times E^{\text{aux}}_{N+1}$ and

$$\lim_{p\to+\infty} T_N^p\tilde{\gamma} = (\sigma, \sigma_{N+1}) \quad \text{in } E_N \times E^{\text{aux}}_{N+1}.$$

However, by the definition of S_γ,

$$T_N^p\tilde{\gamma} = (T^p\gamma, (T^p\gamma)^{(N+1)}).$$

Hence, for every fixed m, we know $(T^p\gamma)_m$ converges to σ_m in X_N and $(T^p\gamma)^{(N+1)}_m$ converges uniformly on I to σ_{N+1}. This proves that σ is C^{N+1} and $\sigma^{(N+1)} = \sigma_{N+1}$. We conclude that $\sigma \in E_{N+1}$ is then an attractive fixed point of $T : E_{N+1} \to E_{N+1}$, which proves the induction and concludes the proof of Lemma 3.24.

A3. ***Proof of Lemma 5.9.*** We give the missing proof of Lemma 5.9. The proof is a precise analysis of the iteration formula (5-17). We adopt the notation introduced for Lemma 5.9. We argue by induction on J to show the property $\mathcal{P}_J$: the bound (5-18) is valid for all $j \leq J$ and, for all $1 \leq i \leq n-1$, $l \in \mathbb{N}$, with some constants $C_{j,l}$.

1. ***Base case.*** Let us start with $\mathcal{P}_0$. The iteration formula (5-17) implies

$$a_{i,0}(x_i) = \prod_{l=1}^{i} f_l(x_l).$$

Hence, the bound $\|a_{i,0}\|_{C^0} \leq (B\nu^{1/2})^i$ is obvious and we can set $C_{0,0} = 1$. We now argue by induction on i and prove the property $\mathcal{P}_{0,i}$: the bound (5-18) is valid for $j = 0$, i and for all $l \in \mathbb{N}$, for some constants $C_{j,l}$. These bounds are trivially true for $i = 0$ and are direct consequences of Lemma 5.8 for $i = 1$. Suppose that the property holds for $i - 1$ for some $i \geq 1$ and let's show it for i.

1.1. ***Case $l = 1$.*** Let us first deal with $l = 1$ and compute the derivative of $a_{i,0}$, using the formula $a_{i,0}(x_i) = f_i(x_i)a_{i-1,0}(x_{i-1})$:

$$a'_{i,0}(x_i) = f'(x_i)a_{i-1,0}(x_{i-1}) + f_i(x_i)a'_{i-1,0}(x_{i-1})\left(\frac{\partial x_{i-1}}{\partial x_i}\right).$$

We use the (weak) bound $|\partial x_{i-1}/\partial x_i| \leq 1$ and the property $\mathcal{P}_{0,i-1}$ to show that

$$\|a_{i,0}\|_{C^1} \leq C(B\nu^{1/2})^{i-1} + C_{0,1}(B\nu^{1/2}) \times (B\nu^{1/2})^{i-1}i \leq C_{0,1}(B\nu^{1/2})^i(i+1),$$

assuming that $C_{0,1} > C(B\nu^{1/2})^{-1}$.

1.2. ***General case for $l > 0$.*** We now come back to the general case $l > 0$. By using the formula $a_{i,0}(x_i) = f_i(x_i)a_{i-1,0}(x_{i-1})$, one sees that we can write $a_{i,0}^{(l)}$ in the form

$$a_{i,0}^{(l)}(x_i) = f_i(x_i)a_{i-1,0}^{(l)}(x_{i-1})\left(\frac{\partial x_{i-1}}{\partial x_i}\right)^l + O(\|a_{i-1,0}\|_{C^{l-1}}).$$

The constants appearing in the O depend on C^l norms of f_i and ϕ_i, which, by assumption are controlled by some uniform C_l'. Hence, using the assumption $\mathcal{P}_{0,i-1}$,

$$\begin{aligned}|a_{i,0}^{(l)}(x_i)| &\leq (B\nu^{1/2})\|a_{i-1,0}\|_{C^l}\left(\frac{\partial x_{i-1}}{\partial x_i}\right)^l + C_l'\|a_{i-1,0}\|_{C^{l-1}}\\ &\leq C_{0,l}(B\nu^{1/2})(B\nu^{1/2})^{i-1}i^l + C_l'C_{0,l-1}(B\nu^{1/2})^{i-1}i^{l-1}\\ &\leq C_{0,l}(B\nu^{1/2})^i(i+1)^l,\end{aligned}$$

assuming that $C_{0,l}$ is chosen bigger than $(1/l)C_l'C_{0,l-1}(B\nu^{1/2})^{-1}$. As a consequence, we can build constants satisfying these conditions by defining inductively

$$C_{0,l} = \max\Big(C_{0,l-1}, \frac{1}{l}C_l'C_{0,l-1}(B\nu^{1/2})^{-1}\Big).$$

This ends the proof of $\mathcal{P}_{0,i}$ and hence of $\mathcal{P}_0$.

2. ***Induction step.*** We now assume that $\mathcal{P}_{j-1}$ is true for some $j \geq 1$ and aim at proving $\mathcal{P}_j$. Again, we do it by induction on i by proving the properties $\mathcal{P}_{j,i}$: the bound (5-18) is true for j, i and all $l \in \mathbb{N}$. These bounds are trivially true for $i = 0$ and are direct consequences of Lemma 5.8 for $i = 1$. Suppose that the property holds for $i - 1$ for some $i \geq 2$ and let's show it for i.

2.1 ***Case $l = 0$.*** Let's start with $l = 0$. The iteration formula shows that

$$a_{i,j}(x_i) = f_i(x_i)a_{i-1,j}(x_{i-1}) + \sum_{p=0}^{j-1} L_{j-p,i}(a_{i-1,p})(x_{i-1}).$$

By Lemma 5.8, there exist constants $C_{p,m}' > 0$ such that

$$\|L_{p,i}a\|_{C^m(I_i)} \leq C_{p,m}'\|a\|_{C^{2p+m}(I_{i-1})}.$$

Hence, assuming that (5-18) holds for $a_{i-1,j}$ with $l = 0$,

$$\begin{aligned}\|a_{i,j}\|_\infty &\leq C_{j,0}(B\nu^{1/2})(B\nu^{1/2})^{i-1}i^{3j} + \sum_{p=0}^{j-1} C_{j-p,0}'\|a_{i-1,p}\|_{C^{2(j-p)}}\\ &\leq C_{j,0}(B\nu^{1/2})^i i^{3j} + \sum_{p=0}^{j-1} C_{j-p,0}'C_{p,2(j-p)}(B\nu^{1/2})^{i-1}i^{2(j-p)+3p}\\ &\leq C_{j,0}(B\nu^{1/2})^i i^{3j} + i^{2j}(B\nu^{1/2})^{i-1}\sum_{p=0}^{j-1} C_{j-p,0}'C_{p,2(j-p)}i^p\\ &\leq C_{j,0}(B\nu^{1/2})^i i^{3j} + i^{2j}(B\nu^{1/2})^{i-1}\Big[\sup_{0\leq p\leq j-1} C_{j-p,0}'C_{p,2(j-p)}\Big]\frac{i^j-1}{i-1}\end{aligned}$$

$$\le C_{j,0}(B\nu^{1/2})^i i^{3j} + i^{3j-1}(B\nu^{1/2})^{i-1}\big[\sup_{0\le p\le j-1} C'_{j-p,0}C_{p,2(j-p)}\big]\widetilde{C}_j, \quad \text{where } \frac{i^j-1}{i-1} \le \widetilde{C}_j i^{j-1},$$
$$\le C_{j,0}(B\nu^{1/2})^i (i+1)^{3j},$$

assuming that $C_{j,0}$ is chosen bigger than

$$K_j := \frac{1}{3j}(B\nu^{1/2})^{-1}\big[\sup_{0\le p\le j-1} C'_{j-p,0}C_{p,2(j-p)}\big]\widetilde{C}_j.$$

As a consequence, the bounds hold for $l = 0$ and i, j if we set $C_{j,0} = \max(1, K_j)$.

2.2. ***Case l > 0.*** Consider now $l > 0$. As already done, one can write

$$a^{(l)}_{i,j}(x_i) = f_i(x_i)a^{(l)}_{i-1,j}(x_{i-1})\left(\frac{\partial x_{i-1}}{\partial x_i}\right)^l + O(\|a_{i-1,j}\|_{C^{l-1}}) + \sum_{p=0}^{j-1}(L_{j-p,i}(a_{i-1,p}))^{(l)}(x_{i-1}).$$

As usual, the constants in O depend on l, j but not on i and we denote by $C''_{l,j}$ the constant in this O. Hence, we can control

$$\begin{aligned}
\|a^{(l)}_{i,j}\|_\infty &\le C_{j,l}(B\nu^{1/2})(B\nu^{1/2})^{i-1}i^{l+3j} + C''_{l,j}C_{j,l-1}(B\nu^{1/2})^{i-1}i^{l+3j-1} + \sum_{p=0}^{j-1}\|L_{j-p,i}(a_{i-1,p})\|_{C^l}\\
&\le C_{j,l}(B\nu^{1/2})^i i^{l+3j} + C''_{l,j}C_{j,l-1}(B\nu^{1/2})^{i-1}i^{l+3j-1} + \sum_{p=0}^{j-1}C'_{j-p,l}\|a_{i-1,p}\|_{C^{l+2(j-p)}}\\
&\le C_{j,l}(B\nu^{1/2})^i i^{l+3j} + C''_{l,j}C_{j,l-1}(B\nu^{1/2})^{i-1}i^{l+3j-1} + \sum_{p=0}^{j-1}C'_{j-p,l}C_{p,l+2(j-p)}(B\nu^{1/2})^{i-1}i^{l+2(j-p)+3p}\\
&\le C_{j,l}(B\nu^{1/2})^i \Big(i^{l+3j} + i^{l+3j-1}\frac{1}{C_{j,l}}\underbrace{(B\nu^{1/2})^{-1}\big(C''_{l,j}C_{j,l-1} + \sup_{0\le p\le j-1}C'_{j-p,l}C_{p,l+2(j-p)}\widetilde{C}_j\big)}_{\widetilde{C}_{j,l}}\\
&\le C_{j,l}(B\nu^{1/2})^i (i+1)^{l+3j}
\end{aligned}$$

if $C_{j,l} \ge \widetilde{C}_{j,l}$. Eventually, we define by induction on l the constants $C_{j,l}$ by setting $C_{j,l} = \max(C_{j,l-1}, \widetilde{C}_{j,l})$, achieving the proof of $\mathcal{P}_j$. This concludes the proof of the lemma.

A4. ***Upper box dimension for hyperbolic set.*** This subsection is devoted to the proof of Proposition 6.6. We will simply recall some arguments which lead to give an upper bound to the upper-box dimension. We borrow these arguments from [Barreira 2008, Section 4.3] and refer the reader to this book for the definitions and properties of topological pressure (Definition 2.3.1), Markov partition (Definition 4.2.6) and other references on this theory.

We'll show that the pressure condition (Fractal) implies Proposition 6.6. We prove it for the unstable manifolds. The proof is similar in the case of stable manifolds by changing F into F^{-1}. We first begin by fixing a Markov partition for $\mathcal{T}$ with diameter at most η_0. This is possible by virtue of Theorem 18.7.3 in [Katok and Hasselblatt 1995]. We denote by $R_1, \dots, R_p \subset \mathcal{T}$ this Markov partition. Here, η_0 is smaller than the diameter of the local stable and unstable manifolds and the holonomy maps $H^{u/s}_{\rho,\rho'}$ are well-defined for $d(\rho, \rho') \le \eta_0$:

$$H^{u/s}_{\rho,\rho'} : W_{s/u}(\rho) \to W_{s/u}(\rho'), \quad \zeta \mapsto \text{the unique point in } W_u(\zeta) \cap W_s(\rho').$$

Due to our results on the regularity of the stable and unstable distributions, these maps are Lipschitz with global Lipschitz constants. In particular, if an inequality of the kind

$$N_{W_u(\rho)\cap\mathcal{T}}(\varepsilon) \le C\varepsilon^{-\delta}$$

holds for some ρ, it holds for ρ' if $d(\rho, \rho') \le \eta_0$ with C replaced by $K^\delta C$ where K is a Lipschitz constant for the holonomy maps. We fix $(\rho_1, \dots, \rho_p)$ in $(R_1, \dots, R_p)$ and we set $V = \bigcup_{i=1}^p W_u(\rho_i) \cap R_i$. It is then enough to show that

$$\overline{\dim}\, V < 1.$$

Indeed, if $\overline{\dim}\, V < 1$ for $\delta \in (\overline{\dim}\, V, 1)$, there exists $\varepsilon_0 > 0$ such that,

$$\text{for all } \varepsilon \le \varepsilon_0, \quad N_V(\varepsilon) \le \varepsilon^{-\delta},$$

and we conclude the proof of Section A4 with the above considerations on the holonomy maps.

Note $\delta := \overline{\dim}\, V$ satisfies the equation $P(\delta\phi_u) = 0$. We will actually show that $P(\delta\phi_u) \ge 0$. Since $s \mapsto P(s\phi_u)$ is strictly decreasing and has a unique root, the assumption $P(\phi_u) < 0$ will give $\delta < 1$. We will denote by

$$R_{i_0,\dots,i_n} = \bigcap_{k=0}^{n} F^{-i}(R_{i_k}), \quad V_{i_0,\dots,i_n} = R_{i_0,\dots,i_n} \cap V$$

the elements of the refined partition at time n. Similarly to the definitions of $J^+_{\boldsymbol{q}}$, we will write

$$J_{i_0,\dots,i_n} = \inf\{J^n_u(\rho),\, \rho \in R_{i_0,\dots,i_n}\}$$

and write

$$c_n(s) = \sum_{i_0,\dots,i_n} J^{-s}_{i_0,\dots,i_n} = \sum_{i_0,\dots,i_n} \exp \max_{R_{i_0,\dots,i_n}} \left(s \sum_{k=0}^{n-1} \phi_u \circ F^k \right)$$

(the last equality follows from the chain rule). Properties of Markov partitions ensure that

$$P(s\phi_u) = \lim_{n\to\infty} \frac{1}{n} \log c_n(s).$$

Fix $s > \delta$. Hence, there exists ε_1 such that, for all $\varepsilon \le \varepsilon_1$, $N_V(\varepsilon) \le \varepsilon^{-s}$.

Fix $n \in \mathbb{N}^*$. By writing $V = \bigcup_{i_0,\dots,i_n} V_{i_0,\dots,i_n}$ we have

$$N_V(\varepsilon) \le \sum_{i_0,\dots,i_n} N_{V_{i_0,\dots,i_n}}(\varepsilon).$$

Note that

$$F^n(V_{i_0,\dots,i_n}) \subset W_u(F^n(\rho_{i_0})) \cap R_{i_n}$$

and

$$H^s_{F^n(\rho_{i_0}),\rho_{i_n}}(F^n(V_{i_0,\dots,i_n})) \subset V_{i_n}.$$

Hence, if we cover V_{i_n} by N sets of diameter at most ε, $U_1, \dots, U_N$, the sets $F^{-n} \circ H^s_{\rho_{i_n}, F^n(\rho_{i_0})}(U_i)$, $1 \le i \le N$, cover $V_{i_0,\dots,i_n}$ and have diameters at most $K\varepsilon J^{-1}_{i_0,\dots,i_n}$. Hence,

$$N_{V_{i_n}}(\varepsilon) \ge N_{V_{i_0,\dots,i_n}}(K\varepsilon J^{-1}_{i_0,\dots,i_n}),$$

which gives

$$N_V(\varepsilon) \le \sum_{i_0,\dots,i_n} N_{V_{i_n}}(\varepsilon K^{-1} J_{i_0,\dots,i_n}).$$

As a consequence, if $\varepsilon < \varepsilon_1 K J_n^{-1}$, where $J_n = \sup_{i_0,\dots,i_n} J_{i_0,\dots,i_n}$, we have

$$N_V(\varepsilon) \le \sum_{i_0,\dots,i_n} K^s J^{-s}_{i_0,\dots,i_n} \varepsilon^{-s} = K^s \varepsilon^{-s} c_n(s).$$

By iterating this process, we see that, for all $m \in \mathbb{N}$, if $\varepsilon < \varepsilon_1 (K J_n^{-1})^m$,

$$N_V(\varepsilon) \le \varepsilon^{-s} K^{ms} c_n(s)^m.$$

Hence,

$$\frac{\log N_V(\varepsilon)}{-\log \varepsilon} \le s + m\frac{\log(K^s c_n(s))}{-\log \varepsilon} \le s + m\frac{\log(K^s c_n(s))}{-\log(\varepsilon_1 (K J_n^{-1})^m)}.$$

We then take the lim sup as $\varepsilon \to 0$ first and then pass to the limit as $m \to +\infty$ and find that

$$\overline{\dim}\, V \le s + \frac{\log K^s c_n(s)}{-\log K J_n^{-1}}.$$

Then, we pass to the limit $s \to \delta$ and find that $\log(K^\delta c_n(\delta)) \ge 0$. Hence,

$$P(\delta\phi_u) = \lim_{n\to\infty} \frac{1}{n} \log c_n(\delta) \ge \lim_{n\to\infty} \frac{-\delta \log K}{n} = 0.$$

This ends the proof of the required inequality and gives that $\overline{\dim}\, V < 1$.

A5. ***From porosity to upper-box dimension.*** We have shown that sets with upper-box dimension strictly smaller than 1 are porous. In this appendix, we show a result in the other way, namely, porous sets down to scale 0 have an upper-box dimension strictly smaller than 1. The following lemma gives a quantitative version of this statement. This is not useful for our use (we only needed the first implication) but we found that it could be of independent interest. Our proof is based on the proof of Lemma 5.4 in [Dyatlov and Jin 2018]. We adopt the same notation as in Section 6A.

Lemma A.2. *Let $M \in \mathbb{N}$, $\nu > 0$, $\alpha_1 > 0$. Let $X \subset [-M, M]$ be a closed set and assume that X is ν-porous on a scale from 0 to α_1. Then, there exists $C = C(\nu, \alpha_1, M) > 0$, $\varepsilon_0 = \varepsilon_0(\nu, \alpha_1, M)$ and $\delta = \delta(\nu) \in [0, 1[$ such that,*

$$\textit{for all } \varepsilon \le \varepsilon_0, \quad N_X(\varepsilon) \le C\varepsilon^{-\delta}.$$

In particular,

$$\overline{\dim}\, X \le \delta.$$

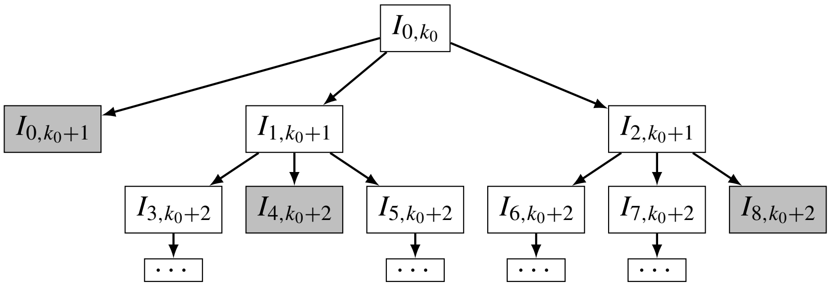

Figure 17. It illustrates the tree structure of the family of intervals $I_{k,m}$ with $L = 3$. The porosity allows us to withdraw at least one child to any parent. The missing children are shaded in gray.

Proof. We define $L = \lceil 2/\nu \rceil$ and denote by k_0 the unique integer such that

$$L^{-k_0} \le \alpha_1 < L^{-k_0+1}.$$

We will let $I_{m,k} = [mL^{-k}, (m+1)L^{-k}]$ for $k \in \mathbb{N},\ m \in \mathbb{Z}$.

We now show by induction on $k \ge k_0$ that there exists $Y_k \subset \mathbb{Z}$ such that

$$\#Y_k \le 2ML^{k_0}(L-1)^{k-k_0}, \quad \Omega \subset \bigcup_{m \in Y_k} I_{m,k}, \tag{A-7}$$

namely, at each level $k \ge k_0$, one new interval $I_{m,k}$ does not intersect Ω. See Figure 17.

The case $k = k_0$ is trivial since we simply cover Ω by the intervals I_{m,k_0} for $ML^{k_0} \le m < ML^{k_0}$.

We now assume that the result is proved for $k \ge k_0$ and we prove it for $k+1$. Fix $m \in Y_k$. We write $I = \bigcup_{j=0}^{L-1} I_{mL+j,k+1}$. We claim that among the intervals $I_{mL+j,k+1}$, at least one does not intersect Ω. Indeed, since $|I| \le L^{-k_0} \le \alpha_1$, the porosity of Ω implies the existence of an interval $J \subset I$ of size $\nu|I| = \nu L^{-k} \ge 2L^{-k-1}$ such that $J \cap \Omega = \varnothing$. Since $|J| \ge 2L^{-k-1}$, J contains at least one of the intervals $I_{mL+j,k+1}$. We denote this index by j_m. We now set

$$Y_{k+1} = \bigcup_{m \in Y_k} \{mL + j : j \in \{0, \dots, L_1\} \setminus j_m\}.$$

By the property of j_m, we have $\Omega \subset \bigcup_{m \in Y_{k+1}} I_{m,k+1}$ and $\#Y_{k+1} \le (L-1)\,\#\,Y_k \le (L-1)^{k+1-k_0} 2ML^{k_0}$.

We now consider $\varepsilon \le \frac{1}{2} L^{-k_0}$ and write k the unique integer such that

$$L^{-k} \le 2\varepsilon < L^{-k+1} \quad \text{i.e.,} \quad k = \left\lceil \frac{-\log(2\varepsilon)}{\log L} \right\rceil.$$

Since we can cover Ω by $2ML^{k_0}(L-1)^{k-k_0}$ closed intervals of size L^{-k}, we can also cover Ω by $4ML^{k_0}(L-1)^{k-k_0}$ open intervals of size 2ε. Hence,

$$N_\Omega(\varepsilon) \le 4ML^{k_0}(L-1)^{k-k_0} \le 4M\left(\frac{L}{L-1}\right)^{k_0} (L-1)^{-\log(2\varepsilon)/\log L + 1} \le C\varepsilon^{-\delta},$$

with $\delta = \log(L-1)/\log L \in [0, 1[$ and $C = 4M(L/(L-1))^{k_0}(L-1)^{1-\log 2/\log L}$. $\square$

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References

[Anantharaman and Nonnenmacher 2007a] N. Anantharaman and S. Nonnenmacher, "Entropy of semiclassical measures of the Walsh-quantized baker's map", *Ann. Henri Poincaré* **8**:1 (2007), 37–74. MR Zbl

[Anantharaman and Nonnenmacher 2007b] N. Anantharaman and S. Nonnenmacher, "Half-delocalization of eigenfunctions for the Laplacian on an Anosov manifold", *Ann. Inst. Fourier* (*Grenoble*) **57**:7 (2007), 2465–2523. MR Zbl

[Bardos et al. 1987] C. Bardos, G. Lebeau, and J. Rauch, "Scattering frequencies and Gevrey 3 singularities", *Invent. Math.* **90**:1 (1987), 77–114. MR Zbl

[Barkhofen et al. 2013] S. Barkhofen, T. Weich, A. Potzuweit, H.-J. Stöckmann, U. Kuhl, and M. Zworski, "Experimental observation of the spectral gap in microwave n-disk systems", *Phys. Rev. Lett.* **110**:16 (2013), art. id. 164102.

[Barreira 2008] L. Barreira, *Dimension and recurrence in hyperbolic dynamics*, Progr. Math. **272**, Birkhäuser, Basel, 2008. MR Zbl

[Borthwick 2007] D. Borthwick, *Spectral theory of infinite-area hyperbolic surfaces*, Progr. Math. **256**, Birkhäuser, Boston, 2007. MR Zbl

[Bourgain and Dyatlov 2017] J. Bourgain and S. Dyatlov, "Fourier dimension and spectral gaps for hyperbolic surfaces", *Geom. Funct. Anal.* **27**:4 (2017), 744–771. MR Zbl

[Bourgain and Dyatlov 2018] J. Bourgain and S. Dyatlov, "Spectral gaps without the pressure condition", *Ann. of Math.* (2) **187**:3 (2018), 825–867. MR Zbl

[Bowen and Ruelle 1975] R. Bowen and D. Ruelle, "The ergodic theory of Axiom A flows", *Invent. Math.* **29**:3 (1975), 181–202. MR Zbl

[Chernov and Markarian 2006] N. Chernov and R. Markarian, *Chaotic billiards*, Math. Surv. Monogr. **127**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[Datchev and Vasy 2012] K. Datchev and A. Vasy, "Gluing semiclassical resolvent estimates via propagation of singularities", *Int. Math. Res. Not.* **2012**:23 (2012), 5409–5443. MR Zbl

[Dyatlov 2018] S. Dyatlov, "Notes on hyperbolic dynamics", preprint, 2018. arXiv 1805.11660

[Dyatlov 2019] S. Dyatlov, "An introduction to fractal uncertainty principle", *J. Math. Phys.* **60**:8 (2019), art. id. 081505. MR Zbl

[Dyatlov and Jin 2017] S. Dyatlov and L. Jin, "Resonances for open quantum maps and a fractal uncertainty principle", *Comm. Math. Phys.* **354**:1 (2017), 269–316. MR Zbl

[Dyatlov and Jin 2018] S. Dyatlov and L. Jin, "Semiclassical measures on hyperbolic surfaces have full support", *Acta Math.* **220**:2 (2018), 297–339. MR Zbl

[Dyatlov and Zahl 2016] S. Dyatlov and J. Zahl, "Spectral gaps, additive energy, and a fractal uncertainty principle", *Geom. Funct. Anal.* **26**:4 (2016), 1011–1094. MR Zbl

[Dyatlov and Zworski 2019] S. Dyatlov and M. Zworski, *Mathematical theory of scattering resonances*, Grad. Stud. Math. **200**, Amer. Math. Soc., Providence, RI, 2019. MR Zbl

[Dyatlov and Zworski 2020] S. Dyatlov and M. Zworski, "Fractal uncertainty for transfer operators", *Int. Math. Res. Not.* **2020**:3 (2020), 781–812. MR Zbl

[Dyatlov et al. 2022] S. Dyatlov, L. Jin, and S. Nonnenmacher, "Control of eigenfunctions on surfaces of variable curvature", *J. Amer. Math. Soc.* **35**:2 (2022), 361–465. MR Zbl

[Gaspard and Rice 1989] P. Gaspard and S. A. Rice, "Scattering from a classically chaotic repellor", *J. Chem. Phys.* **90**:4 (1989), 2225–2241. MR

[Gérard 1988] C. Gérard, *Asymptotique des pôles de la matrice de scattering pour deux obstacles strictement convexes*, Mém. Soc. Math. France (N.S.) **31**, Soc. Math. France, Paris, 1988. MR Zbl

[Guillemin and Sternberg 2013] V. Guillemin and S. Sternberg, *Semi-classical analysis*, International Press, Boston, 2013. MR Zbl

[Hargé and Lebeau 1994] T. Hargé and G. Lebeau, "Diffraction par un convexe", *Invent. Math.* **118**:1 (1994), 161–196. MR Zbl

[Hirsch and Pugh 1969] M. W. Hirsch and C. C. Pugh, "Stable manifolds for hyperbolic sets", *Bull. Amer. Math. Soc.* **75** (1969), 149–152. MR Zbl

[Hurder and Katok 1990] S. Hurder and A. Katok, "Differentiability, rigidity and Godbillon–Vey classes for Anosov flows", *Inst. Hautes Études Sci. Publ. Math.* **72** (1990), 5–61. MR Zbl

[Ikawa 1982] M. Ikawa, "Decay of solutions of the wave equation in the exterior of two convex obstacles", *Osaka Math. J.* **19**:3 (1982), 459–509. MR Zbl

[Ikawa 1988] M. Ikawa, "Decay of solutions of the wave equation in the exterior of several convex bodies", *Ann. Inst. Fourier* (*Grenoble*) **38**:2 (1988), 113–146. MR Zbl

[Katok and Hasselblatt 1995] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encycl. Math. Appl. **54**, Cambridge Univ. Press, 1995. MR Zbl

[Mather 1968] J. N. Mather, "Characterization of Anosov diffeomorphisms", *Indag. Math.* **30** (1968), 479–483. MR Zbl

[McShane 1934] E. J. McShane, "Extension of range of functions", *Bull. Amer. Math. Soc.* **40**:12 (1934), 837–842. MR Zbl

[Morita 1991] T. Morita, "The symbolic representation of billiards without boundary condition", *Trans. Amer. Math. Soc.* **325**:2 (1991), 819–828. MR Zbl

[Naud 2005] F. Naud, "Expanding maps on Cantor sets and analytic continuation of zeta functions", *Ann. Sci. École Norm. Sup.* (4) **38**:1 (2005), 116–153. MR Zbl

[Nonnenmacher 2011] S. Nonnenmacher, "Spectral problems in open quantum chaos", *Nonlinearity* **24**:12 (2011), R123–R167. Zbl

[Nonnenmacher and Zworski 2009] S. Nonnenmacher and M. Zworski, "Quantum decay rates in chaotic scattering", *Acta Math.* **203**:2 (2009), 149–233. MR Zbl

[Nonnenmacher et al. 2011] S. Nonnenmacher, J. Sjöstrand, and M. Zworski, "From open quantum systems to open quantum maps", *Comm. Math. Phys.* **304**:1 (2011), 1–48. MR Zbl

[Nonnenmacher et al. 2014] S. Nonnenmacher, J. Sjöstrand, and M. Zworski, "Fractal Weyl law for open quantum chaotic maps", *Ann. of Math.* (2) **179**:1 (2014), 179–251. MR Zbl

[Petkov and Stoyanov 2010] V. Petkov and L. Stoyanov, "Analytic continuation of the resolvent of the Laplacian and the dynamical zeta function", *Anal. PDE* **3**:4 (2010), 427–489. MR Zbl

[Rivière 2010] G. Rivière, "Entropy of semiclassical measures in dimension 2", *Duke Math. J.* **155**:2 (2010), 271–336. MR Zbl

[Schwartz 2021] N. Schwartz, "The full delocalization of eigenstates for the quantized cat map", preprint, 2021. arXiv 2103.06633

[Sjöstrand and Zworski 1999] J. Sjöstrand and M. Zworski, "Asymptotic distribution of resonances for convex obstacles", *Acta Math.* **183**:2 (1999), 191–253. MR Zbl

[Zworski 2012] M. Zworski, *Semiclassical analysis*, Grad. Stud. Math. **138**, Amer. Math. Soc., Providence, RI, 2012. MR Zbl

[Zworski 2017] M. Zworski, "Mathematical study of scattering resonances", *Bull. Math. Sci.* **7**:1 (2017), 1–85. MR Zbl

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