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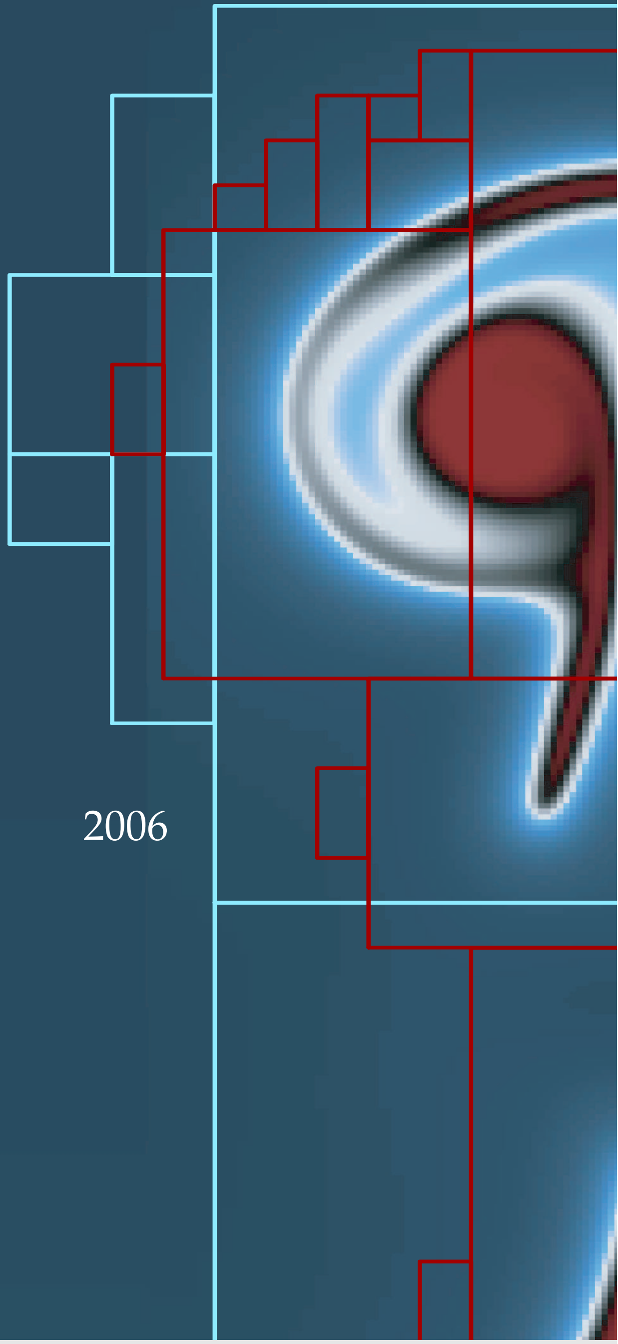

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ON THE ACCURACY OF FINITE DIFFERENCE METHODS FOR ELLIPTIC PROBLEMS WITH INTERFACES

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In problems with interfaces, the unknown or its derivatives may have jump discontinuities. Finite difference methods, including the method of A. Mayo and the immersed interface method of R. LeVeque and Z. Li, maintain accuracy by adding corrections, found from the jumps, to the difference operator at grid points near the interface and by modifying the operator if necessary. It has long been observed that the solution can be computed with uniform $O(h^2)$ accuracy even if the truncation error is $O(h)$ at the interface, while $O(h^2)$ in the interior. We prove this fact for a class of static interface problems of elliptic type using discrete analogues of estimates for elliptic equations. Moreover, we show that the gradient is uniformly accurate to $O(h^2 \log(1/h))$. Various implications are discussed, including the accuracy of these methods for steady fluid flow governed by the Stokes equations. Two-fluid problems can be handled by first solving an integral equation for an unknown jump. Numerical examples are presented which confirm the analytical conclusions, although the observed error in the gradient is $O(h^2)$.

1. Introduction

Often in problems of fluid flow or wave propagation an interface between different regions exerts a force on the material, or an interface separates regions of different material properties. The static problem is formulated as an elliptic partial differential equation with possible discontinuities in the coefficients and nonhomogeneous terms, and with possible jump conditions for the unknown and its derivative across the interface. For the numerical solution a finite difference method is straightforward away from the interface, but accuracy will be lost near the interface unless special care is taken. A class of practical methods has been developed, including the method of A. Mayo [32; 34; 31] and the immersed interface method of R. LeVeque and Z. Li [24; 27; 26], in which the specified jumps at the

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interface are used to derive corrections to the difference operator when the stencil crosses the interface, and, if needed, modification of the difference operator as well [24; 27; 26]. Using Taylor expansions and incorporating jumps, the truncation error is corrected to a desired order. It has long been observed that, with grid size h and $O(h^2)$ truncation error in the interior, but only $O(h)$ truncation error near the interface, the solution is still uniformly accurate to $O(h^2)$. In this paper we provide a rigorous explanation for this fact in certain cases. Although we treat steady problems here, this class of methods is naturally suited for time-dependent problems with moving boundaries such as Stokes flow of a viscous fluid; see [25].

We consider a problem in a rectangular region Ω in R^d, $d = 2$ or 3, of the form

$$\beta_-\Delta u_- = f_- \quad \text{in } \Omega_-, \qquad \beta_+\Delta u_+ = f_+ \quad \text{in } \Omega_+, \tag{1–1}$$

$$[u] = g_0 \quad \text{on } S, \qquad [\beta\partial_n u] = g_1 \quad \text{on } S \tag{1–2}$$

in which a closed curve S ($d = 2$), or a closed surface S ($d = 3$), separates an inner region Ω_- from an outer region Ω_+, with $\Omega = \Omega_- \cup S \cup \Omega_+$. Here $[u] = u_+ - u_-$ on S and similarly for $\beta\partial_n u = \beta\partial u/\partial n$, where n is the normal to S, outward from Ω_-. We assume here that $\beta_\pm$ are positive constants, although operators in divergence form with variable coefficients are dealt with by the immersed interface method. We suppose u is given on $\partial\Omega$. If the problem is given in free space, the solution might first be computed on $\partial\Omega$ from an integral representation (see Section 4). Our results hold for other boundary conditions as well; the simplest would be periodicity on $\partial\Omega$.

We first treat the case $\beta_- = \beta_+$; in that case $[\partial_n u]$ is known on S. In Sections 2 and 3 we prove that, with the truncation error as above, the computed solution is uniformly $O(h^2)$ accurate, and moreover the gradient can be found uniformly to $O(h^2 \log(1/h))$. We verify that this result holds for the methods of Mayo and of LeVeque and Li. The gain in accuracy is shown to be a consequence of two facts. First, since the $O(h)$ truncation error is on a set of relative size $O(h)$, it can be written as the discrete divergence of a function which is only $O(h^2)$ in magnitude. Second, the gain in regularity in solving the discrete elliptic problem means that this part of the truncation error contributes an error to the solution which is $O(h^2)$ in a higher norm. To make this plausible, we consider an analogous estimate with continuous variable: If v is a localized function of $x \in R^d$ and $\Delta v = \sum_{k=1}^d \partial_k F_k$, then $v = \sum \partial_k G \star F_k$, where G is the fundamental solution, $\partial_k = \partial/\partial x_k$, and $\star$ denotes convolution. The kernel $\partial_k G$ is locally integrable, and if F_k is bounded, then v is bounded. Moreover, estimates for $\partial_\ell\partial_k G$ show that $\partial_\ell v \in L^p$ for any $p < \infty$. We follow a related line of argument for the discrete problem, using a discrete Green's function.

Various extensions and applications are discussed in Section 4. For the case where β_-, β_+ are unequal positive constants, the problem (1–1), (1–2) in free space can be treated by first solving an integral equation on S for $[\partial_n u]$ and then proceeding as before. The theory of Sections 2 and 3 shows that the immersed interface method for steady fluid flow governed by the Stokes equations as in [25] is second-order accurate. The two-fluid case can again be treated by first solving an integral equation. The analysis can be applied to higher-order methods; use of the nine-point Laplacian in two dimensions, rather than the usual five-point Laplacian, leads to uniform $O(h^4)$ accuracy. Mayo [32] noted that a boundary value problem could be treated as an interface problem by writing the solution as a layer potential on S and first solving a classical integral equation for the strength of the potential. A different but related method introduced by Mayo [33] and expanded on in [2] for solving interface problems or boundary value problems can also be viewed with the present analysis. This approach is to compute the solution near S as a nearly singular integral, form the discrete Laplacian, and then invert. Computational examples of the several types of problems are given in Section 5. We observe $O(h^2)$ accuracy in the gradient, indicating that the $O(h^2 \log(1/h))$ estimate proved here may not be sharp.

The gain in accuracy which is established here has been noted and analyzed since these methods were introduced [32; 33; 24]. The ideas in the Appendix of [33] are related to those used here. In [32] it was shown that, with $O(h)$ truncation error at the irregular points, the error in L^2 norm is at most $O(h^{3/2})$. Proofs of $O(h^2)$ accuracy for general equations in one dimension have been given in [3; 37; 17]. Theorems with a conclusion similar to the present one were proved in [27], Theorems 5.1 and 5.2, for a more general class of equations with Dirichlet boundary condition, using the maximum principle and comparison functions. However, this result required a hypothesis, related to the position of the interface with respect to the grid points, which does not hold in general. In particular, the hypothesis implies that, where the slope of the curve is close to horizontal, the curve cannot cross a vertical grid line closer than $C_0 h$ to a grid point, or the curve must be within $C_1 h^{1+\sigma}$ of the grid point for some $\sigma > 0$ independent of h. This hypothesis is violated for any parabola $x_2 = a x_1^2 + b$ for arbitrarily small h; this is shown in the Appendix.

Related but different methods for solving Dirichlet problems in general regions by embedding in a larger domain and using a regular grid have been used since the 1930's. At internal grid points the standard discrete Laplacian can be used, but a modified stencil must be used at the boundary of the region. A line of analysis beginning with Gerschgorin, and presented in [15], Section 23, shows that the order of accuracy can exceed that of the truncation error at the boundary by 2 under certain circumstances; for example, the accuracy of the solution can be $O(h^2)$

when the truncation error is $O(h^2)$ in the interior, but only $O(1)$ near the boundary. The method of proof is based on the maximum principle, and the gain in accuracy depends on the modification of the difference operator at the boundary. A general approach for such results using discrete Green's functions was developed in [5; 6], and a convergence proof for a class of methods with interpolation at the boundary was given by Böhmer [4]. For a recent review, see Jomaa and Macaskill [21]. Analysis and examples in [21] indicate that an $O(h)$ truncation error at the interface is preferable despite the theoretical results. As noted in [32],[37], the interface methods studied here can be used to solve boundary value problems, extending past the boundary to a computational box. This approach has the important difference from the one just described that the stencil of the differential operator is not modified at the boundary.

Elliptic problems with interfaces can be solved by finite element methods. Convergence results include [9; 11; 29]. In [28] a Cartesian grid method using a finite element formulation is introduced, and the various numerical approaches to interface problems are discussed and compared. Discrete elliptic estimates like Lemma 2.3 below are well known for finite element approximations to elliptic problems; see [10], Chapter 8 and [12], Section 21.

The main result is presented in Section 2 as Theorem 2.1. It gives the error estimate for the solution of Equations (1–1), (1–2) with $\beta_\pm = 1$, assuming estimates for the truncation error. The theorem follows from two facts: Lemma 2.2 shows that a grid function localized near the interface can be written as the divergence of a function smaller in norm, and Lemma 2.3 gives a maximum norm estimate for a discrete elliptic problem with a nonhomogeneous term of divergence form. The applicability to the methods of Mayo, LeVeque, and Li is explained, including a discussion of smoothness properties needed to justify the truncation error. The lemmas are proved in Section 3. Extensions and applications are given in Section 4, and computational examples are presented in Section 5.

2. Main results

We consider the interface problem with $\beta_+ = \beta_-$, a positive constant. For simplicity, we assume $\beta_\pm = 1$. We write the problem (1–1), (1–2) as

$$\Delta u_\pm = f_\pm \quad \text{in } \Omega_\pm, \qquad [u] = g_0 \quad \text{on } S, \quad [\partial_n u] = g_1 \quad \text{on } S \tag{2–1}$$

where $S \subseteq \Omega$, $\overline{\Omega}_- = \Omega_- \cup S$, $\overline{\Omega}_+ = \Omega_+ \cup S$; $f_\pm, u_\pm$ are defined on $\overline{\Omega}_\pm$; and g_0, g_1 are on S. To complete the problem we assume u is specified on $\partial\Omega$,

$$u = u_0 \quad \text{on } \partial\Omega, \tag{2–2}$$

although other boundary conditions are considered below. We assume that $f_\pm$, g_0, g_1, and S are fairly smooth, with a possible jump in $f_\pm$ at S. For appropriate u_0 on $\partial\Omega$, or for other boundary conditions, it follows that $u_\pm$ is smooth on $\overline{\Omega}_\pm$, as discussed below. In order to estimate truncation errors in difference schemes, we suppose for now that $u_\pm$ is C^4 on $\overline{\Omega}_\pm$ and also that S is C^4. It follows that each of $u_\pm$ has a C^4 extension to an open set containing S; this fact will be used to justify the corrections at the interface. Sufficient conditions for the regularity of u are given in Lemma 2.4.

To discuss discretization, we write the region Ω as

$$\Omega = \{x \in R^d : 0 < x_k < A_k,\ 1 \le k \le d\}. \tag{2–3}$$

For simplicity, we assume ratios of the lengths A_k are rational, so that the domain can be partitioned by grid cubes of size h for arbitrarily small h. We assume h is chosen so that $A_k = N_k h$ with integer N_k for each k. The computational domain is

$$\Omega_h = \{jh \in hZ^d : 1 \le j_k \le N_k - 1,\ 1 \le k \le d\}, \tag{2–4}$$

with boundary

$$\partial\Omega_h = \{jh : 0 \le j_k \le N_k,\ 1 \le k \le d\,;\, j_k = 0 \text{ or } N_k \text{ for some } k\}. \tag{2–5}$$

The closure is $\overline{\Omega}_h = \Omega_h \cup \partial\Omega_h$. We also need the partial boundary

$$\partial^0\Omega_h = \{jh : 0 \le j_k \le N_k - 1,\ 1 \le k \le d\,;\, j_k = 0 \text{ for some } k\}. \tag{2–6}$$

We use the usual second-order discrete Laplacian, defined for a function u^h on $\overline{\Omega}_h$ as

$$\Delta_h u^h = \sum_{k=1}^{d} D_k^- D_k^+ u, \tag{2–7}$$

where $D_k^\pm$ is the usual forward or backward difference operator in the k-th direction; for example, with $d = 2$,

$$D_1^+ u(j_1 h, j_2 h) = \big(u((j_1+1)h, j_2 h) - u(j_1 h, j_2 h)\big)/h.$$

We write $\nabla_h^\pm u$ for the discrete gradient whose components are $D_k^\pm$. We will use the discrete L^p norm and maximum norm,

$$\|u^h\|_{p,\Omega_h} = \Big(\sum_{jh\in\Omega_h} |u^h(jh)|^p h^d\Big)^{1/p}, \quad \|u^h\|_{\max,\Omega_h} = \max_{jh\in\Omega_h} |u^h(jh)|. \tag{2–8}$$

Now suppose the grid size h is chosen and each grid point $jh \in \overline{\Omega}_h$ is labeled as a point in $\overline{\Omega}_+$ or $\overline{\Omega}_-$; points lying on S can be assigned arbitrarily. We say a grid point is *regular* with respect to S if all grid points in the stencil of the discrete

Laplacian at that point are in the same closed region. Otherwise it is *irregular*. Let u^e be the exact solution of (2–1), (2–2). At each regular point we have the usual truncation error

$$\Delta_h u^e(jh) = f_\pm(jh) + \tau^h(jh), \qquad |\tau^h(jh)| \le Ch^2, \tag{2–9}$$

with $f_\pm$ chosen according to whether $jh \in \overline{\Omega}_+$ or $\overline{\Omega}_-$. This holds even if there are boundary points within h of jh, since the $u^e_\pm$ have smooth extensions independent of h; the usual Taylor expansion applies to the extended $u^e_\pm$, once h is small enough. Next we consider the error at the irregular points. Suppose we identify the leading terms in $\Delta_h u^e(jh)$, as is done in the methods under discussion, and explained further below see (2–23)–(2–26), with a first order error remaining. That is, we find $T^h(jh)$, determined by the jumps, so that

$$\Delta_h u^e(jh) = f_\pm(jh) + T^h(jh) + \tau^h(jh), \qquad |\tau^h(jh)| \le Ch. \tag{2–10}$$

(If $jh \in S$, $f_\pm$ is chosen to be consistent with the labeling of jh.) Now define f^h on Ω_h by

$$f^h(jh) = \begin{cases} f_\pm(jh) + T^h(jh), & jh \text{ irregular}, \\ f_\pm(jh), & jh \text{ regular}. \end{cases} \tag{2–11}$$

Finally, as in [32; 34; 24], we take u^h to be the solution of

$$\Delta_h u^h = f^h \quad \text{in } \Omega_h, \qquad u^h = u_0 \quad \text{on } \partial\Omega_h. \tag{2–12}$$

Then the error $u^h - u^e$ satisfies

$$\Delta_h(u^h - u^e) = -\tau^h \quad \text{in } \Omega_h, \qquad u^h - u^e = 0 \quad \text{on } \partial\Omega_h. \tag{2–13}$$

We can now state our main result. We assume that (2–9) and (2–10) hold, rather than making assumptions about the smoothness of the problem. After the theorem and related lemmas, we describe the assumptions which guarantee the needed smoothness and then review the derivation of (2–10). The theorem implies that the error in (2–13) is uniformly $O(h^2)$, with a similar estimate for the discrete gradient.

Theorem 2.1. *Let u^e be the exact solution of the problem* (2–1), (2–2) *with S at least C^1. Suppose $\Delta^h u^e$ has the form given by* (2–9), (2–10), *with $|\tau_h(jh)| \le Ch$ at irregular grid points and $|\tau_h(jh)| \le Ch^2$ at regular grid points. Let u^h be the solution of* (2–11), (2–12). *Then*

$$|u^h(jh) - u^e(jh)| \le C_0 h^2, \qquad jh \in \Omega_h \tag{2–14}$$

and for $1 \le \ell \le d$,

$$|D^+_\ell u^h(jh) - D^+_\ell u^e(jh)| \le C_1 h^2 \log(1/h), \qquad jh \in \Omega_h \cup \partial^0\Omega_h \tag{2–15}$$

with C_0, C_1 dependent on u^e but independent of h.

The discrete gradient estimate (2–15) can be interpreted as an estimate for

$$D_\ell^-(u^h - u^e)$$

at a slightly different set of points, and thus a similar estimate also holds for centered differences on Ω_h. An accurate approximation to ∇u^e can thus be found; see Corollary 2.5 and Equation (2–27) below.

Theorem 2.1 will follow directly from the next two lemmas, which are proved in Section 3.

Lemma 2.2. *Suppose f^{irr} is a function on Ω_h which is nonzero only on the set of irregular points. Assume S is C^1. Then there exist functions F_k on $\Omega_h \cup \partial^0\Omega_h$, $1 \le k \le d$, such that $F_k = 0$ on $\partial^0\Omega_h$,*

$$f^{\text{irr}} = \sum_{k=1}^{d} D_k^- F_k \quad \text{in } \Omega_h \tag{2–16}$$

and

$$\|F_k\|_{\max,\Omega_h\cup\partial^0\Omega_h} \le Ch\|f^{\text{irr}}\|_{\max,\Omega_h}, \quad 1 \le k \le d, \tag{2–17}$$

where C depends on S but is independent of h.

Lemma 2.3. *Suppose*

$$\Delta_h v = f^{\text{reg}} + \sum_{k=1}^{d} D_k^- F_k \quad \text{in } \Omega_h, \qquad v = 0 \quad \text{on } \partial\Omega_h, \tag{2–18}$$

where

$$v : \overline{\Omega}_h \to R, \quad f^{\text{reg}} : \Omega_h \to R, \tag{2–19}$$

$$F_k : \Omega_h \cup \partial^0\Omega_h \to R, \quad 1 \le k \le d \tag{2–20}$$

and $F_k(jh) = 0$ for each $jh \in \partial^0\Omega_h$ with $j_\ell = 0$ for some $\ell \ne k$. Then

$$\|v\|_{\max,\Omega_h} \le C_0\Bigl(\|f^{\text{reg}}\|_{2,\Omega_h} + \sum_{k=1}^{d} \|F_k\|_{\max,\Omega_h\cup\partial^0\Omega_h}\Bigr), \tag{2–21}$$

$$\|D_\ell^+ v\|_{\max,\Omega_h\cup\partial^0\Omega_h} \le C_1 \log(1/h)\Bigl(\|f^{\text{reg}}\|_{\max,\Omega_h} + \sum_{k=1}^{d} \|F_k\|_{\max,\Omega_h\cup\partial^0\Omega_h}\Bigr) \tag{2–22}$$

for $1 \le \ell \le d$, where C_0, C_1 depend only on the lengths A_k.

To derive the theorem, we set f^{irr} equal to the restriction of τ_h to the irregular points and use Lemma 2.2, concluding that $F_k = O(h \cdot h) = O(h^2)$. Then we apply Lemma 2.3 to $v = u^h - u^e$, using (2–13) with f^{reg} equal to the regular part of τ_h. The entire right side of (2–21) is $O(h^2)$, and similarly for (2–22). Theorem 2.1 and the lemmas also hold with periodic or Neumann boundary conditions, rather than Dirichlet, as discussed below. For the discrete Dirichlet problem (2–18), it is well known that the maximum of v can be estimated by the maximum of the right side, using the discrete maximum principle, but (2–21) is sharper in dependence on F_k.

In order to verify Equations (2–9), (2–10) we need general conditions on the problem (2–1), (2–2) to ensure the smoothness of $u_\pm$. An existence and regularity theorem for a general class of interface problems is given in [22], Section 16. The statement of higher regularity given below for the present case is based on potential theory and the classical Schauder estimates for elliptic equations. A brief justification is given in Section 3. This statement can be extended to the case with a discontinuous coefficient in the jump in normal derivative; see Section 4. We say that $f \in C^{m+\alpha}(\overline{\Omega})$, for integer m and $0 < \alpha < 1$, if $f \in C^m(\overline{\Omega})$ and $D^m f$ is uniformly Hölder continuous with exponent α on $\overline{\Omega}$.

Lemma 2.4. *Suppose $u_\pm$ in Equation* (2–1) *is the restriction to Ω of a solution to the extended problem in R^d. Suppose S is $C^{4+\alpha}$,*

$$f_- \in C^{2+\alpha}(\overline{\Omega}_-), \quad f_+ \in C^{2+\alpha}(R^d - \Omega_-), \quad g_0 \in C^{4+\alpha}(S), \quad \textit{and} \quad g_1 \in C^{3+\alpha}(S),$$

for some $0 < \alpha < 1$. *Then* $u_\pm \in C^{4+\alpha}(\overline{\Omega}_\pm)$.

We now describe the derivation of (2–10) as in Mayo's method [32; 34; 31], the related work of Wiegmann and Bube [37], or the immersed interface method of LeVeque and Li [24; 27; 26]. All these methods start with the observation that jumps in higher derivatives of $u_\pm$ in (2–1) can be found by differentiating the jumps in $u_\pm, \partial_n u_\pm$ along S and using $\Delta u_\pm = f_\pm$. To be specific, we emphasize Mayo's point of view. For dimension 2, writing $(x, y) \in R^2$, the jumps in first and second derivatives are

$$[u_x] = x' g_0' + y' g_1, \qquad [u_y] = y' g_0' - x' g_1, \tag{2–23}$$

$$[u_{xx}] = g_2 + y'^2 [f], \qquad [u_{yy}] = -g_2 + x'^2 [f], \tag{2–24}$$

where

$$g_2 \equiv 2\kappa x' y' g_0' + (x'^2 - y'^2)(g_0'' - \kappa g_1) + 2x' y' g_1', \tag{2–25}$$

and where primes denote arclength derivative d/ds along S and κ is the curvature $\kappa = x'' y' - x' y''$. These jump formulas, or equivalent ones, are used to find the corrections T_h at the irregular grid points. Suppose, for example, that

$$(j_1 h, j_2 h) \in \overline{\Omega}_- \quad \text{but} \quad ((j_1 + 1)h, j_2 h) \in \overline{\Omega}_+.$$

To correct $\Delta_h u(j_1 h, j_2 h)$, we find a point $((j_1+\theta)h, j_2 h) \in S,\ 0 \le \theta \le 1$. A Taylor expansion gives

$$u_+((j_1+1)h, j_2 h) - u_-(j_1 h, j_2 h) = hu_{-,x} + \tfrac{1}{2}h^2 u_{-,xx} \\ + [u] + (1-\theta)h[u_x] + \tfrac{1}{2}(1-\theta)^2 h^2 [u_{xx}] + O(h^3), \quad (2\text{–}26)$$

where $u_{-,x}$,$u_{-,xx}$ are evaluated at $(j_1 h, j_2 h)$ and the jumps are located at

$$((j_1+\theta)h, j_2 h).$$

This expression is valid even if S intersects the segment at more than one point; the Taylor expansion for $u_\pm$ applies to the extended functions under the smoothness assumptions of Lemma 2.4. To approximate $\Delta_h u(j_1 h, j_2 h)$ we consider four such segments, finding jump terms if needed, add expressions similar to (2–26), and divide by h^2, to obtain an equation in the form (2–10), thus identifying $T^h(j_1 h, j_2 h)$. The procedure for the immersed interface method [24] is very similar, but for each irregular point $(j_1 h, j_2 h)$, one nearby boundary point is chosen, and a Taylor expansion in (x, y) about this point is used for each of the points in the stencil. In either case the derivation of (2–9), (2–10) is justified, and Theorem 2.1 applies:

Corollary 2.5. *For the problem* (2–1), (2–2), *with the smoothness assumptions of Lemma 2.4*, *either Mayo's method* [32; 34] *or the immersed interface method of LeVeque and Li* [24], *with corrections of the form* (2–10), *gives a computed solution* u_h *with* $|u_h - u_e| \le Ch^2$ *uniformly. Moreover*, ∇u_e *can be found on* Ω_h *from* u_h *with error uniformly* $O(h^2 \log(1/h))$.

It remains to verify the last statement of the corollary. For regular points the centered difference of u_h gives a value of ∇u_e accurate to $O(h^2 \log(1/h))$, according to (2–15). At irregular points we can correct the centered difference to the same order using formulas such as (2–26). For example, suppose $(j_1 h, j_2 h)$ and $((j_1-1)h, j_2 h)$ are in $\overline{\Omega}_-$ but $((j_1+1)h, j_2 h) \in \overline{\Omega}_+$. We find, for the exact solution,

$$u_+\big((j_1+1)h, j_2 h\big) - u_-\big((j_1-1)h, j_2 h\big) \\ = 2hu_{-,x}(j_1 h, j_2 h) + [u] + (1-\theta)h[u_x] + \tfrac{1}{2}(1-\theta)^2 h^2 [u_{xx}] + O(h^3). \quad (2\text{–}27)$$

From this we obtain a computed value of ∇u which is again accurate to

$$O(h^2 \log(1/h)).$$

Similar results hold if we impose a boundary condition on $\partial\Omega$ other than (2–2). No change is needed if we use the homogeneous Dirichlet condition $u = 0$ on $\partial\Omega$, provided f is the restriction to Ω of an odd, periodic function, with period $2A_k$ in direction k, which is smooth except for the jump at S and its reflections.

(For example, this would be true if $f_+ = 0$ near $\partial\Omega$.) The solution is then smooth, since the problem extends to R^d with u odd and periodic. Alternatively, we could use periodic boundary conditions for u on $\overline{\Omega}$, if f extends smoothly to a periodic function with periods A_k. In this case we have the necessary condition

$$\int_{\Omega_-} f_- + \int_{\Omega_+} f_+ + \int_S [g_1]\, dS = 0 \tag{2–28}$$

and u has an arbitrary constant. Finally, we could impose the Neumann, or no-flux, condition

$$\partial_n u = 0 \quad \text{on } \partial\Omega, \tag{2–29}$$

again with condition (2–28), if f has a smooth, even, periodic extension. In this case we solve for u^h on $\overline{\Omega}_h$, with u^h extended past $\partial\Omega_h$ so that

$$u(-h, j_2 h) = u(h, j_2 h),$$

etc., consistent with (2–29). The exact and discrete Neumann problems both extend to even, periodic problems, and the analysis for the periodic case applies to this case as well.

We discuss the modifications of the analysis for the periodic boundary condition. We cannot solve $\Delta_h u^h = f^h$ exactly with u^h periodic; instead we solve

$$\Delta_h u^h = f^h - f_0^h, \tag{2–30}$$

where f_0^h is the mean value of f^h. Since $\Delta_h u^h$ has mean value zero, and the number of irregular points is $O(h^{-d+1})$, it follows from (2–9), (2–10) that

$$f_0^h = O(h^2),$$

so that this term does not affect the error estimate. Lemma 2.2 must be replaced by the version below. The proof of Lemma 2.3 is similar to the earlier case but simpler. The new term F_0 is treated in the theorem like the term f^{reg}.

Lemma 2.6. *Suppose f^{irr} is a function on Ω_h which is nonzero only on the set of irregular points. Assume S is C^1. Then there exist periodic functions F_k on $\Omega_h \cup \partial^0\Omega_h$, $0 \le k \le d$, so that $F_k = 0$ on $\partial^0\Omega_h$ for $1 \le k \le d$,*

$$f^{\text{irr}} = F_0 + \sum_{k=1}^{3} D_k^- F_k \quad \text{in } \Omega_h \tag{2–31}$$

and

$$\|F_k\|_{\max,\Omega_h \cup \partial^0\Omega_h} \le Ch \|f^{\text{irr}}\|_{\max,\Omega_h}, \quad 0 \le k \le d \tag{2–32}$$

where C depends on S but is independent of h.

3. Proofs of the lemmas

Proof of Lemma 2.2. For simplicity, we assume dimension $d = 3$. We wish to work with pieces of S for which one spatial coordinate can be written as a function of the others. We can localize using a partition of unity (see, for example, [14], p. 13): Since S is C^1 and compact, there are finitely many open sets $U_i, V_i \subseteq \Omega$ and C^1 functions $\zeta_i \geq 0$ on Ω so that $\overline{V_i} \subseteq U_i$; the V_i cover S; each ζ_i is supported in V_i; $\sum_i \zeta_i(x) = 1$ for each x in an open neighborhood $\mathcal{N}$ of S; and for each i we can choose one coordinate, say x_3, so that the part of S in U_i consists of

$$S \cap U_i \;=\; \{(x_1, x_2, Z_i(x_1, x_2)) : (x_1, x_2) \in U_i'\}, \tag{3–1}$$

where U_i' is an open subset of R^2 and $Z : U_i' \to R$ is a C^1 function. Since the irregular points are within distance h of S, they are contained in $\mathcal{N}$ once h is small enough. For f^{irr} as specified, we can then write $f^{\text{irr}} = \sum_i \zeta_i f^{\text{irr}}$. It will suffice to prove the lemma for each $f^{(i)} = \zeta_i f^{\text{irr}}$.

Having localized the problem to considering $f^{(i)}$ on V_i, we first estimate the number of irregular points in V_i with given projection on U_i'. Let V_i' be the projection of V_i on U_i'. Suppose $x' = jh = (j_1 h, j_2 h) \in V_i'$. If $p = (x', z) \in V_i$ is an irregular point, then there is some $q \in S$ with $|q - p| \leq h$, say $q = (x'', z'')$ with $x'' \in U_i'$. Then $|x'' - x'| \leq h$, $|z'' - z| \leq h$, and $z'' = Z_i(x'')$. If M is a bound for $|\nabla Z_i|$, then $|Z_i(x'') - Z_i(x')| \leq Mh$, and

$$|z - Z_i(x')| \leq |z - z''| + |Z_i(x'') - Z_i(x')| \leq (1 + M)h. \tag{3–2}$$

Thus z is restricted to an interval of length $2(M + 1)h$, and the number of irregular points in V_i projecting onto x' is at most $C_1 \equiv 2M + 3$, a number bounded independent of $x' = jh$.

We will write $f^{(i)}$ as $D_3^- F^{(i)}$ for some $F^{(i)}$. We set $F^{(i)} = 0$ on $\partial^0 \Omega_h$, and for $(jh, kh) = (j_1 h, j_2 h, kh) \in \Omega_h$ we define

$$F^{(i)}(jh, kh) \;=\; \sum_{\ell=1}^{k} f^{(i)}(jh, \ell h)\, h. \tag{3–3}$$

Then, since kh is the third coordinate,

$$D_3^- F^{(i)} \;=\; f^{(i)} \quad \text{in } \Omega_h. \tag{3–4}$$

The function $f^{(i)}$ can only be nonzero at irregular points, and as noted above, the number of such points contributing to the sum (3–3) has a uniform upper bound. The estimate (2–17) for $F^{(i)}$ follows, and the proof is completed by summing over i. □

In proving Lemma 2.3 we will use a discrete Green's function G_h on hZ^d, satisfying

$$\Delta_h G_h(x) = \delta_h(x), \quad x \in hZ^d, \tag{3–5}$$

where $\delta_h(x) = h^{-d}$ for $x = 0$ and $\delta_h(x) = 0$ for $x \neq 0$. For $d = 2$ or 3 such G_h exists, with pointwise estimates analogous to those for the fundamental solution of the exact Laplacian,

$$|G_h(x)| \leq C_{00} + C_0 |\log(|x| + h)|, \quad d = 2, \tag{3–6}$$

$$|G_h(x)| \leq C_0(|x| + h)^{-1}, \qquad d = 3, \tag{3–7}$$

and for the first and second differences in directions k or ℓ, $1 \leq k, \ell \leq d$,

$$|D_k^+ G_h(x)| \leq C_1(|x| + h)^{1-d}, \qquad d = 2, 3, \tag{3–8}$$

$$|D_\ell^+ D_k^+ G_h(x)| \leq C_2(|x| + h)^{-d}, \qquad d = 2, 3. \tag{3–9}$$

For example, for $h = 1$, G_1 is introduced in [23] in terms of the expected number of visits to x by a random walk on Z^d starting at 0. The estimates (3–6)–(3–9) follow from those in [23], (pp. 32, 40), after rescaling G_1 to G_h. (For $d = 2$, G_h must also be adjusted by a constant. For second differences, [23] gives an estimate for a repeated difference in any direction, but $D_k^+ D_\ell^+$ can be reduced to this case by writing, with $h = 1$,

$$(S_k - I)(S_\ell - I) = \tfrac{1}{2}\big((S_k - I)^2 + (S_\ell - I)^2 - S_\ell^2(S_k S_\ell^{-1} - I)^2\big) \tag{3–10}$$

where S_k is the forward shift in direction k.) If w is a function on hZ^d supported in a bounded set, then

$$w(x) = \sum_{y \in hZ^d} G_h(x - y)(\Delta_h w)(y)\, h^d. \tag{3–11}$$

This follows from (3–5) and the uniqueness of solution of the discrete Poisson problem.

We will need estimates for norms of G_h, $D_k^+ G_h$, and $D_\ell^- D_k^+ G_h$ which follow directly from the pointwise estimates (3–6)–(3–9). With $B_h(R) = \{x \in hZ^d : |x| < R\}$, we have

$$\|G_h\|_{2,B_h(R)} \leq C_0(R), \quad \|D_k^+ G_h\|_{1,B_h(R)} \leq C_1(R), \tag{3–12}$$

$$\|D_\ell^+ D_k^+ G_h\|_{1,B_h(R)} \leq C_2(R) \log(1/h), \tag{3–13}$$

with constants depending on R. Discrete Green's functions for more general elliptic operators and domains were constructed by Bramble et al. [7] and pointwise estimates for G_h were found using the maximum principle [8].

Proof of Lemma 2.3. First we check that

$$\|\nabla_h^+ v\|_{2,\Omega_h\cup\partial^0\Omega_h} \le C\big(\|f^{\text{reg}}\|_{2,\Omega_h} + \sum_k \|F_k\|_{2,\Omega_h}\big). \tag{3–14}$$

To show this, we multiply by v in (2–18), sum over Ω_h, and then sum by parts on the left and in the F_k terms, using the boundary conditions for v and F_k, to obtain

$$\langle\nabla_h^+ v, \nabla_h^+ v\rangle_{\Omega_h\cup\partial^0\Omega_h} = -\langle f^{\text{reg}}, v\rangle_{\Omega_h} + \sum_k \langle F_k, D_k^+ v\rangle_{\Omega_h}, \tag{3–15}$$

where brackets denote the usual discrete inner product, for example,

$$\langle v, w\rangle_{\Omega_h} = \sum_{jh\in\Omega_h} v(jh)w(jh)h^d, \qquad \|v\|_{2,\Omega_h} = \langle v, v\rangle_{\Omega_h}^{1/2}. \tag{3–16}$$

We can then derive (3–15) from the Cauchy–Schwarz inequality and the discrete Poincaré inequality, valid since $v = 0$ on $\partial\Omega$,

$$\|v\|_{2,\Omega_h} \le \|\nabla_h^+ v\|_{2,\Omega_h\cup\partial^0\Omega_h}. \tag{3–17}$$

Next we extend the Poisson equation from Ω_h to hZ^d. Let $\tilde f$ be the odd, periodic extension of f^{reg}, with period $2N_k h$ in direction k, with $\tilde f = 0$ on the faces

$$j_k h = 0, N_k h$$

and their images. Let $\tilde\phi_k$ be the similar odd periodic extension of $D_k^- F_k$, and $\tilde v$ the odd periodic extension of v. Then

$$\Delta_h \tilde v = \tilde f + \sum_k \tilde\phi_k \quad \text{in } hZ^d. \tag{3–18}$$

We want to write $\tilde\phi_k$ as D_k^- of some extension $\tilde F_k$ of F_k. For example, if $k = 1$ and $d = 3$, for $1 \le j_1 \le N_1$ and $0 \le j_k \le N_k - 1$, $k = 2, 3$, we define

$$\tilde F_1(-j_1 h, j_2 h, j_3 h) = F_1((j_1 - 1)h, j_2 h, j_3)h.$$

We then extend $\tilde F_1$ to all $j_1 h$, with period $2N_1 h$. Finally we extend $\tilde F_1$ to be odd and periodic in $j_2 h$, $j_3 h$, with $\tilde F_1(j_1 h, j_2 h, j_3 h) = 0$ if $j_k h$ is a multiple of $N_k h$ for $k = 2$ or 3. With this definition, and a similar one for each $\tilde F_k$, we have

$$\tilde\phi_k = D_k^- \tilde F_k \quad \text{in } hZ^d. \tag{3–19}$$

We can now derive the maximum estimate for v. Choose a smooth function $\zeta : R^d \to [0, 1]$ with $\zeta(x) = 1$ for an open set containing $\overline{\Omega}$ and $\zeta = 0$ outside a bounded set B. Then

$$\Delta_h(\zeta\tilde v) = \zeta\tilde f + \zeta\nabla_h^- \cdot \tilde F - \nabla_h^\pm \zeta \cdot \nabla_h^\pm \tilde v - (\Delta_h\zeta)\tilde v \quad \text{in } hZ^d, \tag{3–20}$$

where $\tilde{F}$ is the vector with components $\tilde{F}_k$ and $\pm$ indicates two terms. We use the discrete Green's function G_h to write, for $x \in \Omega_h$,

$$v(x) = T_1 + T_2 + T_3 + T_4, \tag{3–21}$$

where

$$T_1 = \sum_{y \in hZ^d} G_h(x-y)\zeta(y)\tilde{f}(y)\, h^d, \quad T_2 = \sum_{y \in hZ^d} G_h(x-y)\zeta(y)\nabla_h^- \cdot \tilde{F}(y)\, h^d$$

or, after summation by parts,

$$T_2 = \sum_{y \in hZ^d} (\nabla_h^+ G_h)(x-y)\zeta(y)\tilde{F}(y)\, h^d \\ - \sum_{y \in hZ^d} G_h(x-y)(\nabla_h^+ \zeta)(y)\tilde{F}(y)\, h^d \tag{3–22}$$

and similarly T_3, T_4 are discrete convolutions of G_h with $\nabla_h^\pm \zeta \cdot \nabla_h^\pm \tilde{v}$ and $(\Delta_h \zeta)\tilde{v}$.

To estimate these terms, let $\tilde{B} \subseteq R^d$ be a bounded set which contains all points $x - y$ with $x \in \overline{\Omega}$ and $y \in B$, and let $B_h = B \cap hZ^d$, $\tilde{B}_h = \tilde{B} \cap hZ^d$. Then for each $x \in \Omega_h$,

$$\begin{aligned} |T_1| &\le \|G_h\|_{2,\tilde{B}_h} \|\tilde{f}\|_{2,B_h}, \\ |T_2| &\le \left(\|\nabla_h^+ G_h\|_{1,\tilde{B}_h} + C_2 \|G_h\|_{2,\tilde{B}_h}\right) \end{aligned} \tag{3–23}$$

and

$$\begin{aligned} |T_3| &\le C_3 \|G_h\|_{2,\tilde{B}_h} \left(\|\nabla_h^+ \tilde{v}\|_{2,B_h} + \|\nabla_h^- \tilde{v}\|_{2,B_h}\right), \\ |T_4| &\le C_4 \|G_h\|_{2,\tilde{B}_h} \|\tilde{v}\|_{2,B_h}. \end{aligned} \tag{3–24}$$

The extension of f and F was such that

$$\|\tilde{f}\|_{2,B_h} \le C \|f^{\mathrm{reg}}\|_{2,\Omega_h}, \quad \|\tilde{F}\|_{\max,B_h} \le C \|F\|_{\max,\Omega_h \cup \partial^0 \Omega_h} \tag{3–25}$$

and using (3–12) we get

$$|T_1| + |T_2| \le C \left(\|f^{\mathrm{reg}}\|_{2,\Omega_h} + \|F\|_{\max,\Omega_h \cup \partial^0 \Omega_h}\right). \tag{3–26}$$

Also v was extended so that

$$\|\tilde{v}\|_{2,B_h} \le C \|v\|_{2,\Omega_h}, \qquad \|\nabla_h^\pm \tilde{v}\|_{2,B_h} \le C \|\nabla_h^+ v\|_{2,\Omega_h \cup \partial^0 \Omega_h}. \tag{3–27}$$

Combining this with (3–24), (3–14), (3–17), and (3–12), we see that T_3, T_4 have the same estimate as in (3–26), and (2–21) is now established.

The proof of (2–22) is very similar. We apply D_ℓ^+ to (3–21) with T_2 in the form (3–22); in each term D_ℓ^- acts on the x-variable in G_h. In T_3 and T_4, $D_\ell^+ G_h$ is uniformly bounded for $x \in \Omega_h$ since the support of $\nabla_h^\pm \zeta$ is away from Ω_h. $\square$

Proof of Lemma 2.4. We first reduce to the case $f_\pm = 0$, as in [18]. From the Schauder regularity theory, the presumed solution u_+ is $C^{4+\alpha}$ away from S. Using this fact and the Schauder theory, we see that there exists v_+ in $C^{4+\alpha}(R^d - \Omega_-)$ such that $\Delta v_+ = f_+$ and $v_+ = 0$ on S, and there exists v_- in $C^{4+\alpha}(\overline{\Omega}_-)$ such that $\Delta v_- = f_-$ and $v_- = 0$ on S. Subtracting $v_\pm$, we now consider the reduced problem with $f_\pm = 0$. We can write a solution as the sum of a double layer potential and a single layer, with strengths g_0 and g_1 respectively. The double layer potential has boundary values on each side of S in $C^{4+\alpha}$, the same as for g_0, and it follows from the Schauder theory that it has the desired regularity in $\overline{\Omega}_\pm$. A similar remark applies to the Neumann boundary condition for the single layer potential. This solution may not be the same as $u_\pm$, since we have not imposed a condition at infinity, but the difference is harmonic throughout and therefore is smooth. □

Proof of Lemma 2.6. We proceed as in the proof of Lemma 2.2, but in place of (3–3), we set $F^{(i)}(jh, 0) = 0$ and

$$F^{(i)}(jh,kh) = \sum_{\ell=1}^{k} \big(f^{(i)}(jh,\ell h) - F_0^{(i)}(jh)\big)\, h, \quad 1 \le k \le N_3, \tag{3–28}$$

where $F_0^{(i)}(jh)$ is the average of $f_0^{(i)}(jh,\ell h)$ over ℓ, that is,

$$F_0^{(i)}(jh) = A_3^{-1} \sum_{\ell=0}^{N_3-1} f^{(i)}(jh,\ell h)\, h \tag{3–29}$$

for $jh \in V_i'$ and $F_0^{(i)}(jh) = 0$ otherwise. Then $F^{(i)}(jh, N_3 h) = 0$, so that $F^{(i)}$ extends periodically, and

$$f^{(i)} = D_3^- F^{(i)} + F_0^{(i)}. \tag{3–30}$$

For each j, there are at most $C_1 = O(1)$ terms in the sum (3–28), and thus

$$\|F_0^{(i)}\|_{\max} \le A_3^{-1} C_1 h \|f^{(i)}\|_{\max} \le A_3^{-1} C_1 h \|f^{\mathrm{irr}}\|_{\max}. \tag{3–31}$$

Then (2–17) holds for $F^{(i)}$, as defined in (3–28). Finally, we sum over i. □

4. Applications and extensions

Piecewise constant coefficients. In Section 2 we treated the problem (1–1), (1–2) in the special case $\beta_+ = \beta_-$. We now return to the problem where β_+, β_- are unequal, positive constants, perhaps representing different material properties. The important change is that $[\partial_n u]$ is not known, although $[\beta \partial_n u]$ is known. One possible approach is to enlarge the system of equations for the discretized elliptic system ([37; 27]). Here we use a different strategy, assuming the problem is in free space: We first solve an integral equation on S for the unknown $[\partial_n u]$, based on an integral representation for the solution, thus reducing the problem to the earlier case. A similar strategy is used below for Stokes flow with two fluids, using such a representation, as described, for example, in [36].

Suppose the problem (1–1), (1–2) is the restriction to Ω of a problem in R^2 in which $f_+ = 0$ outside Ω and $u \to 0$ at infinity. We will assume u is continuous across S, that is, $g_0 = 0$ in (1–2), but $[\beta \partial_n u] = g_1$ may be nonzero. The extra step of solving for $[\partial_n u]$ is needed even if $g_1 = 0$. The unknown u can be thought of as a weak solution of

$$\nabla \cdot (\beta \nabla u) \;=\; f \;+\; g_1 \delta_S, \tag{4–1}$$

where δ_S is the measure that restricts to S. A recent analytical treatment of such problems can be found in [18]. The solution has the form

$$u(x) \;=\; \int_\Omega G(x-y)\frac{f(y)}{\beta(y)}\,dy \;+\; \int_S G(x-y)q(y)\,ds(y) \tag{4–2}$$

for some q defined on S, where $G(x) = (2\pi)^{-1}\log|x|$, $\beta(y) = \beta_\pm$ for $y \in \Omega_\pm$ and $f = f_\pm$. The last term is a single layer potential with strength q, to be determined. From potential theory we have an expression for the normal derivative of $u_\pm$ at S:

$$\partial_n u_\pm(x) = \int_\Omega \partial_{n(x)} G(x-y)\frac{f(y)}{\beta(y)}\,dy + \int_S \partial_{n(x)} G(x-y)q(y)\,ds(y) \pm \tfrac{1}{2}\,q(x). \tag{4–3}$$

Here

$$\partial_n(x) G(x-y) \;=\; n(x)\cdot \nabla G(x-y), \quad \nabla G(x-y) \;=\; \frac{x-y}{2\pi|x-y|^2}. \tag{4–4}$$

Subtracting, we see that

$$[\partial_n u(x)] \;=\; q(x) \tag{4–5}$$

so that, once q is known, we have reduced the problem to one of the earlier type for the unknown u. To find q we multiply (4–3) by $\beta_\pm$, subtract, and use the second condition in (1–2), obtaining the integral equation

$$\tfrac{1}{2}(\beta_+ + \beta_-)q \;+\; (\beta_+ - \beta_-)\int_S (\partial_n G) q\,ds \;=\; g_1 \;-\; (\beta_+ - \beta_-)\int_\Omega (\partial_n G)\,(f/\beta)\,dy \tag{4–6}$$

([36], Section 5.3). The equation has a unique solution since

$$|(\beta_+ + \beta_-)/(\beta_+ - \beta_-)| > 1$$

(see, for example, [36], Section 5.4). In this two-dimensional case, $\partial_n(x)G(x-y)$ is smooth for $x, y \in S$, whereas the second integrand in (4–2) has an integrable singularity. If $f_+ = 0$ near $\partial\Omega$, u can be found on $\partial\Omega$ from (4–2) in a routine way, and the solution u can be found as in Section 2 using (4–5). We see from (2–23)–(2–26) that we need to solve for q with accuracy $O(h^2)$ in order to obtain $O(h)$ truncation error near S, and thereby $O(h^2)$ accuracy for the solution u, according to the theory of Section 2. The solution of (4–6) is discussed further in Section 5; see (5–10)–(5–11).

Higher order accuracy. In principle, the theory of Sections 2 and 3 can be applied to higher order methods. In dimension $d = 2$, the nine-point Laplacian (see, for example, [19], Section 7.3) has truncation error $O(h^2)$ proportional to the Laplacian, with remaining error $O(h^4)$. The right-hand side of the discrete Poisson equation can be modified so that the truncation error is $O(h^4)$. In this way the methods outlined in Section 2 can then be improved so that the error in the solution is uniformly $O(h^4)$. The jump conditions of (2–23)-(2–26) can be carried to the fourth derivatives so that the truncation error τ^h in (2–10) remaining at the irregular points after correction is $O(h^3)$, while at the regular points the truncation error is $O(h^4)$. The analogue of Lemma 2.3 holds for the nine-point Laplacian; the estimates (3–6), (3–8), (3–9) apply to the discrete Green's function for this operator, as can be seen from Theorem 2 in [16], and thus (3–12), (3–13) hold as well. It then follows from Lemma 2.2 and the modified version of Lemma 2.3 that the conclusion of Theorem 2.1 holds, with h^4 in place of h^2 in the estimates (2–14), (2–15). Fourth order methods of this type have been given in [30] and [20].

Nearly singular integrals. Mayo [33] suggested a procedure to solve a problem for a harmonic function with prescribed jumps, such as (2.1) with $f_\pm = 0$, distinct from the approach of [32]. The first step is to write the solution as a layer potential and calculate it at grid points near the interface, directly as a nearly singular integral. The discrete Laplacian is then formed at the irregular points from these values and extended to be zero at regular points. Finally, a fast Poisson solver is used to find the solution at all grid points. Mayo was able to solve a boundary value problem in this manner by regarding the boundary as an interface and solving an integral equation for the strength of the layer potential. Beale and Lai [2] developed a method for computing nearly singular integrals and used the approach of [33] to solve Dirichlet problems ([2], Section 4). An error estimate for the solution resulting from this procedure is justified by the theory of Section 2: Suppose we find the solution at the irregular points, accurate to $O(h^3)$, as was essentially done in [2], Section 4. The discrete Laplacian formed from these values at the irregular grid points near the interface is accurate at least to $O(h)$. The discrete Laplacian is set to zero elsewhere, with truncation error $O(h^2)$. Thus it follows from Theorem

2.1 that the computed solution is uniformly accurate to $O(h^2)$. The estimate for the gradient applies as well. Mayo gave an argument for a similar conclusion in the Appendix of [33].

Stokes flow. The methods studied here have been used to solve the Stokes equations, describing creeping flow of a very viscous fluid, with an interface separating regions. The immersed interface method of LeVeque and Li was applied to problems with moving interfaces in one fluid in a periodic region [25]. Here we discuss a very similar method for the steady problem in free space. We emphasize the implications of the present results for the error estimates. We see that choices for corrections as in [25] lead to uniform $O(h^2)$ accuracy for both pressure and velocity. We first discuss the case of 2D flow with one fluid, and then explain how the procedure can be extended to the two-fluid case by first solving an integral equation on the interface.

We write the problem as

$$-\mu\Delta v + \nabla p = f\delta_S, \quad \nabla\cdot v = 0, \tag{4–7}$$

where $v = (v_1, v_2)$ is the fluid velocity, μ is the viscosity, p is the pressure, and $f = (f_1, f_2)$ is a specified force on the interface S. The associated stress tensor is

$$\sigma_{ij} = -p\delta_{ij} + \mu\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big), \tag{4–8}$$

with $i, j = 1, 2$. We assume for now that μ has the same value on both sides, but later we consider the case of different viscosities. We always assume the velocity is continuous across S. The delta function terms in the Stokes equations amount to a jump condition on σ (see [35]),

$$[\sigma_{ij}]n_j = -f_i, \qquad i = 1, 2 \tag{4–9}$$

with sum over j understood. From the jump in stress we obtain jump conditions for p and $\partial v/\partial n$ [35; 25]

$$[p] = f\cdot n, \qquad \mu\left[\frac{\partial v}{\partial n}\right] = -(f\cdot\tau)\tau. \tag{4–10}$$

We also need the jump condition for $\partial p/\partial n$, derived in [25],

$$\left[\frac{\partial p}{\partial n}\right] = \frac{\partial(f\cdot\tau)}{\partial s}, \tag{4–11}$$

where s is the arclength parameter on S.

To solve (4–7) we proceed in steps, as in [25], solving first for p and then for v. We choose a computational rectangle Ω containing S and use a square grid as before. We solve the free space problem, assuming decay at infinity. On the computational boundary we prescribe the exact solution, which is known in integral form (for example, see [36; 13]). The pressure is

$$p(x) = \int_S \nabla G(x-y)\cdot f(y)\,ds(y), \tag{4–12}$$

with ∇G as in (4–4). The velocity is

$$v_i(x) = \frac{1}{\mu}\int_S V_{ij}(x-y) f_j(y)\,ds(y), \tag{4–13}$$

$$V_{ij}(x) = -\frac{\delta_{ij}}{4\pi}\log|x| + \frac{x_i x_j}{4\pi|x|^2}. \tag{4–14}$$

The pressure p is determined by a problem of the form (2–1), with $\Delta p = 0$ in $\Omega_\pm$ and jump conditions for $p, \partial p/\partial n$ given in (4–10), (4–11). We solve for p using the procedure of Section 2, adding corrections to the discrete Laplacian. In this way we obtain a solution p^h with error uniformly $O(h^2)$. Next we solve for the velocity components v_1^h, v_2^h. For the exact v we have $\mu\Delta v = \nabla p$ in $\Omega_\pm$. We find a computed velocity v^h as the solution of

$$\mu\Delta_h v^h = \nabla^h p^h + T^h, \tag{4–15}$$

where ∇^h is the centered difference operator for ∇. In T^h we include correction terms to account for the jumps in $\partial v/\partial n$ and in ∇p, given in (4–10), (4–11), as well as corrections for the difference approximation to ∇p, as in (2–27). According to Theorem 2.1, the resulting v^h would be uniformly second-order accurate if p^h were exact. However, since $p^h - p^e = O(h^2)$ uniformly, the error on the right-hand side of the form $\nabla^h(p^h - p^e)$ contributes an error to the solution which is uniformly $O(h^2)$, according to Lemma 2.3. (As noted in [25], we only need correct the difference $\nabla^h p$ to $O(h)$ near S to obtain $O(h^2)$ accuracy for the velocity.)

Stokes flow with two fluids. Next we consider the case of two different viscosities, $\mu_\pm$. With the Stokes equations (4–7) otherwise the same, we have the same jump condition (4–9) for the normal stress. The solution can be written in integral form, derived in [36], Section 5.3:

$$p(x) = \mu_\pm \int_S \nabla G(x-y)\cdot q(y)\,ds(y), \tag{4–16}$$

$$v_i(x) = \int_S V_{ij}(x-y) q_j(y)\,ds(y). \tag{4–17}$$

Here $q = (q_1, q_2)$ is a function on S which solves the integral equation

$$\tfrac{1}{2} q_i(x) = \alpha n_k(x)\int_S T_{ijk}(x-y) q_j(y)\,ds(y) + \beta f_i(x), \tag{4–18}$$

with

$$T_{ijk} = -\frac{x_i x_j x_k}{\pi |x|^4}, \qquad \alpha = \frac{\mu_+ - \mu_-}{\mu_+ + \mu_-}, \qquad \beta = \frac{1}{\mu_+ + \mu_-}. \tag{4–19}$$

(See equation (5.3.9) in [36], noting the factors of 4π should be replaced by 2π in the two-dimensional case.) To solve the flow problem, we begin by solving this integral equation for q. The solvability is discussed in [36], Section 5.4. The kernel $n_k T_{ijk}$ is smooth on S, and the integrals can be computed in a standard way. After solving for q, we can think of v, $p/\mu_\pm$ on $\Omega_\pm$ as the solution of the Stokes equations with $\mu_\pm$ replaced by 1 and with

$$[\sigma_{ij}^{(1)}] n_j = -q_i, \tag{4–20}$$

where

$$\sigma_{ij}^{(1)} = -\frac{p}{\mu_\pm} \delta_{ij} + \Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \Big). \tag{4–21}$$

It follows that v, $p/\mu_\pm$ have jumps as in (4–10), (4–11) but with f replaced by q. Once these jumps are known, we can solve for $p/\mu_\pm$ and then v as in the earlier one-fluid case. In view of (2–23)–(2–26), we need to find q to accuracy $O(h^3)$ to obtain an $O(h)$ truncation error near S in the problem for p of the form (4–10), in order to solve for p, and then v, with $O(h^2)$ accuracy. It is not difficult to solve the integral equation (4–18) to this accuracy provided S and f are smooth enough.

5. Numerical examples

Interface problem with $\beta_+ = \beta_-$. In the first set of examples, we consider the interface problem with $\beta_\pm = 1$:

$$\Delta u_\pm = f_\pm \text{ in } \Omega_\pm, \tag{5–1}$$

where the interface S is given by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{5–2}$$

and $\Omega = [-1.1, 1.1] \times [-1.1, 1.1]$.

The first example we consider has a solution given by

$$u_- = \sin x \cos y, \;\; u_+ = 0. \tag{5–3}$$

With $u_\pm$ specified, $f_\pm$ in (5–1) and the jump conditions g_0 and g_1 can be determined. Two choices for the semi-axes of the ellipse were used, first, $(a, b) = (0.7, 0.9)$, and then $(a, b) = (0.9, 0.1)$. In the latter, the curvature $\kappa = -90$ at $(\pm a, 0)$, leading to a more severe test. The solution and its gradient were computed using the technique of Mayo [32] and using the immersed interface method of LeVeque and Li [24]. Results for the two ellipses are reported in Table 1. Solutions were obtained for $N = 40$, 80, 160, 320, and 640, where N denotes half of the number of subintervals in each dimension. Normalized errors in the L^r-norm, defined as $\|u^h - u^e\|_r / \|u^h\|_r$, are shown for $r = 2$ and ∞. These results show $O(h^2)$

N	u		u_x		u_y	
	L^2	L^∞	L^2	L^∞	L^2	L^∞
Mayo's technique						
40	5.106E-5	3.451E-5	1.448E-4	1.370E-4	2.120E-4	1.436E-4
80	1.415E-5	1.045E-5	3.565E-5	3.275E-5	5.475E-5	3.594E-5
160	3.416E-6	2.458E-6	8.980E-6	8.269E-6	1.336E-5	8.965E-6
320	8.217E-7	5.728E-7	2.262E-6	2.089E-6	3.269E-6	2.352E-6
640	1.180E-7	8.756E-8	4.189E-7	3.945E-7	5.532E-7	4.186E-7
Immersed interface method						
40	2.345E-5	1.773E-5	1.107E-4	1.035E-4	1.362E-4	1.302E-4
80	2.748E-5	2.632E-5	2.748E-5	2.632E-5	3.613E-5	3.510E-5
160	1.510E-6	1.139E-6	6.912E-6	6.656E-6	8.856E-6	8.989E-6
320	3.722E-7	2.805E-7	1.731E-6	1.664E-6	2.199E-6	2.323E-6
640	9.732E-8	7.645E-8	4.305E-7	4.166E-7	5.184E-7	5.967E-7

N	u		u_x		u_y	
	L^2	L^∞	L^2	L^∞	L^2	L^∞
Mayo's technique						
20	2.377E-5	1.209E-5	4.926E-4	4.900E-4	1.506E-3	9.577E-4
40	6.020E-6	2.730E-6	1.231E-4	1.219E-4	3.637E-4	2.269E-4
80	2.261E-6	1.003E-6	3.057E-5	3.001E-5	1.424E-4	1.376E-4
160	5.730E-7	2.340E-7	7.637E-6	7.532E-6	3.649E-5	3.331E-5
320	1.297E-7	5.360E-8	1.915E-6	1.951E-6	8.189E-6	8.309E-6
Immersed interface method						
20	7.134E-4	6.933E-4	1.601E-3	3.051E-3	6.267E-2	7.441E-2
40	1.311E-5	7.441E-6	1.221E-4	1.611E-4	9.386E-4	1.559E-3
80	4.750E-6	2.043E-6	2.993E-5	3.538E-5	3.019E-4	2.047E-4
160	1.210E-6	5.156E-7	7.468E-6	7.949E-6	7.696E-5	5.145E-5
320	2.972E-7	1.249E-7	1.869E-6	2.036E-6	1.881E-5	1.277E-5

Table 1. Results for interface problem with $\beta = 1$, example (5–3). Normalized errors in computed solution and first derivatives. Top: $a = 0.7$, $b = 0.9$; bottom: $a = 0.9$, $b = 0.1$.

convergence in the solution u, consistent with Theorem 2.1. In Section 2, we proved that ∇u can be approximated from u^h with error uniformly $O(h^2 \log(1/h))$. However, results in both tables show $O(h^2)$ accuracy in ∇u^h.

We then consider a second example where the solution is given by

$$u_- = x^9 y^8, \quad u_+ = 0. \tag{5–4}$$

This example is constructed such that the solution has large high-order derivatives. In particular, $|\partial^3 u/\partial x^3|$ and $|\partial^3 u/\partial y^3|$, which occur in the lowest-order uncorrected terms in both Mayo's method and the immersed interface method, are large. Tables 2 and 3 show normalized errors in the solution and its gradient. Results in Table 2, computed for the ellipse $(a, b) = (0.7, 0.9)$, show $O(h^2)$ convergence, although the magnitude of the errors is larger in this example. In particular, as in the previous example, $O(h^2)$ accuracy was obtained for ∇u^h.

The ellipse $(a, b) = (0.9, 0.1)$ used in the next example has large curvature $|\kappa| \le 90$, compared to $|\kappa| < 1.84$ in the previous example. As shown in Table 3, the computed solution has large errors, compared to all previous examples. In particular, solution errors are $> 100\%$ for $N = 40$. These large errors can be attributed to the $O(h^3)$ error terms neglected in (2–26) by Mayo's technique and in the analogous expression by the immersed interface method. The magnitude of these $O(h^3)$ error terms is proportional to κ and to $\nabla^3 u$ — both of which are large in this example by

N	u L^2	u L^∞	u_x L^2	u_x L^∞	u_y L^2	u_y L^∞
Mayo's technique						
40	2.370E-2	6.401E-3	2.489E-2	1.994E-2	1.340E-2	1.046E-2
80	7.469E-3	2.764E-3	5.895E-3	5.520E-3	3.249E-3	2.385E-3
160	1.362E-3	4.802E-4	1.508E-3	1.411E-3	7.825E-4	6.834E-4
320	3.232E-4	9.426E-5	3.858E-4	3.532E-4	1.985E-4	1.768E-4
640	8.363E-5	2.952E-5	9.389E-5	9.304E-5	4.839E-5	4.211E-5
Immersed interface method						
40	2.140E-2	5.699E-3	2.557E-2	2.096E-2	1.390E-2	1.144E-2
80	6.963E-3	2.478E-3	6.139E-3	5.840E-3	3.546E-3	2.695E-3
160	1.236E-3	3.719E-4	1.565E-3	1.489E-3	8.571E-4	7.456E-4
320	2.815E-4	7.053E-5	3.992E-4	3.749E-4	2.170E-4	1.931E-4
640	7.858E-5	2.502E-5	9.742E-5	9.795E-5	5.365E-5	4.618E-5

Table 2. Results for interface problem with $\beta = 1$, example (5–4). Normalized errors in computed solution and first derivatives, obtained for $a = 0.7$, $b = 0.9$.

construction. Thus, for a sufficiently coarse grid, these uncorrected $O(h^3)$ error terms result in large solution errors. Nonetheless, for sufficiently large N, the approximations show $O(h^2)$ convergence, although the error magnitude remains large.

Interface problem with piecewise-constant β. Next we consider the problem of an interface with piecewise-constant coefficients $\beta_\pm$

$$\beta_\pm \Delta u_\pm = f_\pm \quad \text{in } \Omega_\pm, \tag{5–5}$$

$$[u] = 0, \qquad [\beta \partial_n u] = g_1, \tag{5–6}$$

where the interface S is given by an ellipse (5–2) with $a = 0.9$ and $b = 0.7$, and $\Omega = [-1.3, 1.3] \times [-1.3, 1.3]$. The solution is given in elliptic coordinates to be

$$\begin{aligned} u_- &= a_0^3 \left(\cosh^2 \rho \sinh \rho \cos^2 \theta \sin \theta + \sinh^3 \rho \sin^3 \theta \right), \\ u_+ &= c e^{-3\rho} \sin 3\theta + d e^{-\rho} \sin \theta, \end{aligned} \tag{5–7}$$

where $\rho \in [0, \infty)$ and $\theta \in [0, 2\pi]$; ρ and θ are defined by the conformal mapping

$$x + \imath y = a \cosh(\rho + \imath \theta) \tag{5–8}$$

such that

$$x = a_0 \cosh \rho \cos \theta, \;\; y = a_0 \sinh \rho \sin \theta. \tag{5–9}$$

N	u L^2	u L^∞	u_x L^2	u_x L^∞	u_y L^2	u_y L^∞
Mayo's technique						
40	5.068E0	2.356E0	4.146E0	3.575E0	8.513E-1	6.492E-1
80	6.787E-1	1.946E-1	6.443E-1	4.322E-1	2.004E-1	1.523E-1
160	1.388E-1	3.359E-2	1.236E-1	1.107E-1	4.859E-2	4.095E-2
320	3.777E-2	9.627E-3	2.523E-2	2.869E-2	1.120E-2	1.128E-2
640	8.629E-3	2.126E-2	5.548E-3	7.501E-3	3.003E-3	3.019E-3
Immersed interface method						
40	4.683E0	2.291E0	4.183E0	3.663E0	8.542E-1	6.511E-1
80	6.262E-1	1.791E-1	6.452E-1	3.264E-1	2.009E-1	1.527E-1
160	1.391E-1	3.015E-2	1.251E-1	1.133E-1	4.871E-2	4.101E-2
320	3.493E-2	8.695E-3	2.549E-2	2.944E-2	1.202E-2	1.130E-2
640	8.665E-3	1.921E-3	5.731E-3	7.663E-3	3.009E-3	3.024E-3

Table 3. Results for interface problem with $\beta = 1$, example (5–4). Normalized errors in computed solution and first derivatives, obtained for $a = 0.9$, $b = 0.1$.

Note that u_+ is harmonic, that is, $f_+ = 0$. The coefficients c and d in (5–7) are set to 1.26713535 and 1.12854242, respectively, so that $[u] = 0$.

To compute the solution for (5–5) and (5–6), $[\partial_n u] \equiv q$ is first computed by solving (4–6) iteratively:

$$q^{[k+1]} = \frac{2}{\beta_+ + \beta_-}\big(g_1 - (\beta_+ - \beta_-)\big(\int_S (\partial_n G) q^{[k]}\, ds + \int_{\Omega_-} \big(\partial_n G\big)(f_-/\beta)\, dy\big)\big) \tag{5–10}$$

Because $\partial_{n(x)} G(x-y)$ is nearly singular for y near (though not on) S, a naïve integration of the second integral containing $\partial_n G$ yields only $O(h)$ accuracy. To attain $O(h^2)$ accuracy, we follow a standard procedure and subtract

$$\partial_{n(x)} G(x-y) f_-(x)/\beta$$

from the integrand, where $x \in S$ is the point at which $q(x)$ in (5–10) is being evaluated; then we add an $O(h^2)$ approximation to $f_-(x)/\beta$ times

$$\int_\Omega \partial_{n(x)} G(x-y)\, dy = -\int_S n(x)\cdot n(y) G(x-y)\, ds(y). \tag{5–11}$$

The resulting interface condition $[\partial_n u] = q$, together with $[u] = 0$, is then used to solve (5–5): $f_\pm$ is divided by $\beta_\pm$, and then u is computed as in the previous example with constant coefficient $\beta = 1$.

Normalized errors in u are shown in Table 4 for two pairs of coefficients. In the first case, $\beta_+ = 2$ and $\beta_- = 0.5$; in the second case the difference between β's is increased substantially: $\beta_+ = 100$ and $\beta_- = 0.2$. Mayo's technique was used to compute correction terms for the finite-difference stencil. The results in Table 4 suggest that, for this problem, not only is $O(h^2)$ accuracy obtained as predicted by Theorem 2.1, but the accuracy of the method is insensitive to the difference between the β's. We did observe a small increase ($\sim$20%) in the number

N	L^2	L^∞
$\beta_+ = 2,\ \beta_- = 0.5$		
40	1.979E-2	7.252E-2
80	5.637E-4	1.058E-3
160	1.381E-4	2.565E-4
320	3.485E-5	6.507E-5

N	L^2	L^∞
$\beta_+ = 100,\ \beta_- = 0.2$		
40	2.002E-2	7.298E-2
80	4.618E-4	9.524E-4
160	1.123E-4	2.302E-4
320	2.847E-5	5.857E-5

Table 4. Normalized errors in computed solution for the interface problem with piecewise-constant $\beta_\pm$. N denotes half of the number of subintervals in each dimension and along the interface S.

of iterations required for (5–10) to converge, when the ratio β_+/β_- was increased from 4 to 50. Similar results were also obtained for the immersed interface method, and for the cases where $\beta_- > \beta_+$.

Stokes equations. In the third example we solved the Stokes equations (4–7) for two fluids. In [13], Cortez derived analytic solutions for the one-fluid case where the enclosed boundary is a unit circle; see examples 4a and 4b in [13]. In each of those examples, the boundary force has either a normal or a tangential component. To obtain a more general example with nontrivial jumps $[p]$, $[\partial p/\partial n]$, $[\partial v/\partial n]$, we combined those two examples by adding the two solutions, and extended the resulting example to the two-fluid case. To that end, we assumed the same v as in [13], scaled p by the appropriate viscosity $\mu_\pm$, and computed boundary forces using (4–8) and (4–9). The resulting pressure and velocities are given by

$$p(r,\theta) = \begin{cases} \mu_+ r^{-3}(\sin 3\theta - \cos 3\theta), & r \geq 1, \\ -\mu_- r^3(\sin 3\theta + \cos 3\theta), & r < 1, \end{cases} \tag{5–12}$$

$$v_1(r,\theta) = \begin{cases} \tfrac{1}{8}r^{-2}(\sin 2\theta - \cos 2\theta) + \tfrac{1}{16}r^{-4}(-3\sin 4\theta + 5\cos 4\theta) \\ \qquad\qquad + \tfrac{1}{4}r^{-2}(\sin 4\theta - \cos 4\theta), & r \geq 1, \\ \tfrac{1}{8}r^{2}(3\sin 2\theta + \cos 2\theta) + \tfrac{1}{16}r^{4}(\sin 4\theta + \cos 4\theta) \\ \qquad\qquad + \tfrac{1}{4}r^{4}(-\sin 2\theta - \cos 2\theta), & r < 1, \end{cases} \tag{5–13}$$

$$v_2(r,\theta) = \begin{cases} \tfrac{1}{8}r^{-2}(\sin 2\theta + \cos 2\theta) + \tfrac{1}{16}r^{-4}(5\sin 4\theta + 3\cos 4\theta) \\ \qquad\qquad + \tfrac{1}{4}r^{-2}(-\sin 4\theta - \cos 4\theta), & r \geq 1, \\ \tfrac{1}{8}r^{2}(3\sin 2\theta - \sin 2\theta) + \tfrac{1}{16}r^{4}(\sin 4\theta - \cos 4\theta) \\ \qquad\qquad + \tfrac{1}{4}r^{4}(\sin 2\theta - \cos 2\theta), & r < 1. \end{cases} \tag{5–14}$$

With p and v chosen, the boundary force f is determined by (4–8), (4–9). The viscosities μ^+ and μ^- were set to 0.5 and 2, respectively. The computational domain Ω was chosen to be $[-3.0, 3.0] \times [-3.0, 3.0]$.

The solution was computed following the procedure described in Section 4. As noted there, the integral equation (4–18) must be solved to $O(h^3)$ accuracy so that the corrections at the interface lead to an $O(h^2)$ solution of the problem. The integral in (4–18) was approximated using the trapezoid rule, providing the necessary accuracy when S is a unit circle. Table 5 shows normalized errors in the solution obtained using Mayo's technique [32]. These results show evidence of the expected $O(h^2)$ convergence. The immersed interface method yielded similar accuracy.

N	p		u		v	
	L^2	L^∞	L^2	L^∞	L^2	L^∞
40	4.769E-2	7.646E-2	5.358E-2	4.558E-2	4.004E-2	3.726E-2
80	1.264E-2	2.192E-2	1.687E-2	1.377E-2	9.338E-3	8.745E-3
160	3.233E-3	5.485E-3	2.597E-3	2.268E-3	2.712E-3	2.564E-3
320	7.811E-4	1.352E-3	6.641E-4	5.825E-4	6.003E-4	5.458E-4
640	1.973E-4	3.385E-4	1.507E-4	1.326E-4	1.605E-4	1.480E-4

Table 5. Normalized errors in computed solution for the Stokes equations. N denotes half of the number of subintervals in each dimension and along the interface S. Results show second-order convergence.

Appendix

The following lemma shows that a parabola $y = ax^2 + b$ can cross a vertical grid line, near the vertex, such that the vertical distance from a grid point is of any specified order in h for small h. Thus a hypothesis such as in Theorem 5.2 of [27] is often violated.

Lemma A.1. *Given $a, b, \sigma \in R$, with $a \neq 0$ and $0 < \sigma < 1$, there are infinitely many integers $N > 0$ such that, with $h = 1/N$, there is a point $(x, y) \in R^2$ on the curve $y = ax^2 + b$ of the form*

$$x = jh, \quad y = kh + ch^{1+\sigma} \tag{A–1}$$

for some j, k, c depending on N, where j and k are integers, $\frac{1}{2} < c < 2$, and $x = jh = O(h^{(1+\sigma)/2})$.

Proof. According to a theorem of Dirichlet (see [1], Section 6.1, for instance), if b is irrational, there are infinitely many fractions m/N such that $b = m/N + \theta/N^2$ with $|\theta| < 1$. If b is irrational, we choose these N; if b is rational, we can choose infinitely many N so that $b = m/N$ for some m, and we take $\theta = 0$ in the argument to follow.

Substituting for x, y and b in $ax^2 + b = y$, we seek j, k, c so that

$$aj^2h^2 + mh + \theta h^2 = kh + ch^{1+\sigma} \tag{A–2}$$

or, multiplying by $N^2 = h^{-2}$,

$$aj^2 + mN + \theta = kN + cN^{1-\sigma}. \tag{A–3}$$

We choose $k = m$ and divide by a, so that the equation is now

$$j^2 + \theta/a = (c/a)N^{1-\sigma}. \tag{A–4}$$

We will first choose j as an approximate solution, ignoring θ and c, and then choose c. Let $r = \sqrt{N^{1-\sigma}/a}$, and let j be the greatest integer $\leq r$. Finally, define c so that (A.4) holds, that is, $c = (j^2 + \theta/a)/r^2$. It is easy to check that $c \to 1$ as $r \to \infty$, that is, as $N \to \infty$. Finally, $j = O(N^{1/2-\sigma/2})$, and $x = jh = O(N^{-1/2-\sigma/2})$. $\square$

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