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SHEAR FLOW LAMINARIZATION AND ACCELERATION BY SUSPENDED HEAVY PARTICLES: A MATHEMATICAL MODEL AND GEOPHYSICAL APPLICATIONS

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A modified model of turbulent shear flow of a suspension of small heavy particles in a fluid is presented. The modification is based on the assumption that in the flow there are two sorts of particles. For the particles of the first sort the velocity of free fall a_1 is larger than the characteristic velocity fluctuation, for the particles of the second sort the velocity of free fall a_2 is less than the characteristic velocity of fluctuation.

Introduction

The energy of turbulent vortices (energy of turbulence) in a horizontal or slightly inclined shear flow is reduced by suspended heavy particles, and this reduction leads to flow acceleration. The basic model of this seemingly paradoxical phenomenon was suggested by A.N. Kolmogorov (see [13]), and developed quantitatively by the present author ([1; 2], see also the book of Monin and Yaglom [15], pp. 412–416). Later this model, properly modified, was applied to several natural flow phenomena, in particular to dust storms, both terrestrial and Martian [10], and lower quasi-homogeneous layers of the ocean [8; 9]. It is important to mention that in the basic model and its applications it was always assumed that the particles are identical.

Meanwhile, Sir James Lighthill (see his published paper [14]) proposed the "sandwich model" of tropical hurricanes. A detailed analysis of the observations (especially of the expedition on the Russian vessel "Priliv") led Lighthill to the fundamental assumption that a specific feature of hurricanes is the availability of an intermediate layer between the sea and air; Lighthill called it "ocean spray". In this layer, air is filled by suspended water droplets, formed during the process of the breaking of surface water waves. Lighthill specially emphasized "the need to fill the gaps in knowledge about ocean spray at extreme wind speeds".

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Following the direct suggestion of M.J. Lighthill, the original basic model was applied by A.J. Chorin, V.M. Prostokishin and the present author [7] to the flow in ocean spray. The main result obtained in this paper is that indeed the droplets accelerate the wind, and, if they are large, "ocean spray" plays the role of a *lubrication layer* for the wind: that is the reason for the wind acceleration. However, the calculated increase of wind velocity happened to be less than was expected.

In the present paper a modified model is proposed for ocean spray. The key point of the modification is that it is assumed that in ocean spray there are droplets of different sizes: small and large ones. The most important result, obtained using this assumption, is that ocean spray is acting not only as a lubrication layer for the wind, but also as a source of smaller droplets which are suspended by the wind and which suppress the turbulence in the core of the air flow. Suppression of turbulence by small droplets in the core of the wind is, according to the modified model, the basic cause of extreme wind speeds.

The same modified model can be suggested for dust storms and for large fires, in particular, forest and grass fires. In particular it allows us to understand the nature of the firestorms observed in great fires (e.g., Chicago 1871, Dresden, 1945, and Hiroshima, 1945). These topics are considered in the present paper.

1. Kolmogorov's example

A.N. Kolmogorov, whose ideas shaped the modern theory of turbulence, posed at the beginning of his course on turbulence at Moscow State University in 1954 the following question: what would the velocity be at the surface of the river Volga (in Russia, its parameters are close to those of the Mississippi River), if, by a miracle, the river, having preserved its geometry, became laminar. It was clear for the listeners that the velocity at the surface will increase, but to what extent?

Naturally, Kolmogorov modelled the river by a weakly inclined (the slope i is small, $i \ll 1$), spatially homogeneous open channel (Figure 1a). In this simple case of a laminar flow in a channel the basic Navier-Stokes equations are reduced to a single equation, and the easily obtainable solution to this equation has the form

$$u = \frac{\rho g i H^2}{2\eta}\left(\frac{2z}{H} - \frac{z^2}{H^2}\right). \tag{1-1}$$

Here u is the velocity component along the bottom, η the dynamic viscosity of water, ρ its density, z is the coordinate perpendicular to the bottom, and reckoned from it, and g is the acceleration of gravity, so the the velocity u_{surf} at the surface

$z = H$ is equal to

$$u_{\text{surf}} = \frac{\rho g i H^2}{2\eta}. \tag{1-2}$$

Now, substitute into (1.2) realistic values of the parameters: $\eta/\rho = 10^{-2}\, cm^2/s$, $H = 20\, m = 2 \cdot 10^3\, cm$, $i = 10^{-4}$, $g = 10^3\, cm/s^2$. We obtain a value $u_{\text{surf}} = 2 \cdot 10^7 \text{cm}/s = 200 \text{km}/s \cong 400,000$ miles per hour! The reason for this obviously absurd result is that the flow in the river is not laminar, it is "stuffed" with vortices (Figure 1b). These vortices make the flow field random; they transfer the momentum across the flow immensely faster than the thermal oscillations of the water molecules in the laminar flow (which is the mechanism of the molecular fluid viscosity). This basic idea was introduced by the French applied mathematician J. Boussinesq, who in fact was the first to study turbulent flows mathematically. Boussinesq introduced the basic concept of the "eddy viscosity" (viscosité tourbillonaire) η_{turb}: the effective viscosity of the turbulent flow, created by vortices which remains one of the basic concepts in turbulence studies. We emphasize that contrary to the molecular viscosity η, the eddy viscosity η_{turb} is no longer a *fluid property*, it is a *flow property*, different at different places. In the present case the value of η_{turb} needed in (1.2) to obtain a realistic value of the velocity at the river surface is $\sim 200,000\eta$!

However, the Kolmogorov example is especially significant, even fundamental due to the following reason. It demonstrates clearly the huge reserves of energy available in natural fluid flows.

These reserves can be revealed if somehow even a partial flow laminarization is achieved. And this happens in reality: such partial laminarization is achieved in dust storms (the laminarizing factor is the suspended dust particles), tropical hurricanes (the laminarizing factor is the water droplets formed on the oceanic surface when

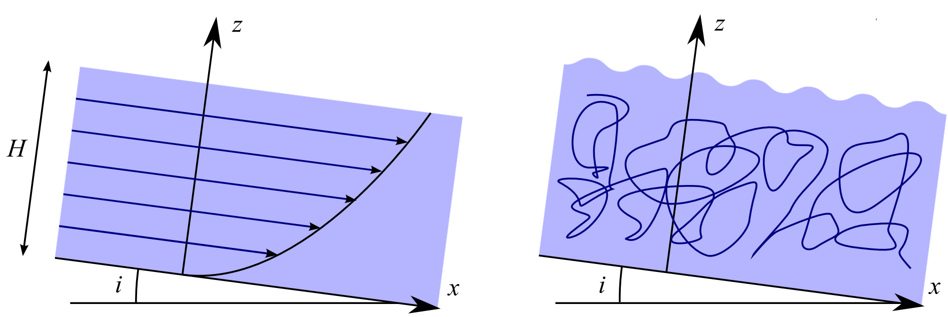

Figure 1. Kolmogorov's example (a) Laminar flow in a channel, (b) Turbulent flow in a river

water waves are breaking), firestorms (the laminarizing factor is unburnt debris and soot particles), and other natural flows.

2. Turbulent shear flows. The Kolmogorov-Prandtl model

Turbulent flows are random, and turbulence studies operate with the averages of flow field properties. In theoretical studies the "ensemble", or "probability" averages are used (the averages over the whole ensemble of possible turbulent flow realizations under given external conditions (e.g., pressure drop at the ends of a pipe).

Shear flow is a steady flow, homogeneous in the direction of average velocity. All properties of the shear flow field vary only in the direction z perpendicular to the direction of the mean velocity.

Studies of turbulent shear flows are of special importance for theoretical and experimental investigations. They allow us, due to substantial simplifications, to advance deeper without accepting doubtful assumptions. Indeed, in general turbulent flows are non-local, both in time and space: The properties of a flow field at a certain point and at a certain moment depend on the flow properties in a certain neighborhood around the point, and at a certain time interval. This is not the case for shear flows: The average flow field at a point can be assumed to be a local property, depending only upon the flow characteristics at this point. Also, an important advantage of shear flows from a practical viewpoint is that ensemble (probability) averages can be replaced (the "ergodicity" property) by averaging over time intervals (due to steadiness) or longitudinal space intervals (due to the spacial homogeneity along the flow direction).

By averaging the Navier-Stokes equations and integrating we obtain only one equation for shear flows due to steadiness and the homogeneity of the mean flow:

$$\frac{d}{dz}(-\rho\overline{u'w'}) = 0, \qquad -\rho\overline{u'w'} = \text{Const} = \tau. \tag{2-1}$$

Here the velocity components u, v, w correspond to the axis x, y, z; bars denote the mean values and primes denotethe fluctuations. In Equation (2.1) the contribution of the molecular viscosity was neglected in comparison with the contribution of the eddy viscosity: we consider the turbulence as a "developed" one. The term neglected is important in the close vicinity of the wall which we exclude from consideration. The term $-\rho\overline{u'w'}$ represents the turbulent flux of momentum, the "Reynolds stress".

Of fundamental importance for future consideration is the equation of turbulent energy balance. For shear flow the equation obtained in this way assumes the form

$$(-\rho\overline{u'w'})\frac{du}{dz} - \rho\epsilon = 0. \tag{2-2}$$

The first term is the rate of inflow of turbulent energy per unit volume from the energy of mean motion, and

$$\epsilon = \frac{\nu}{2}\overline{(\partial_\alpha u'_\beta + \partial_\beta u'_\alpha)(\partial_\alpha u'_\beta + \partial_\beta u'_\alpha)} \tag{2-3}$$

(summation by Greek indexes from 1 to 3 is assumed) is the rate of turbulent energy dissipation into heat per unit mass.

In Equation (2.2) the term neglected is responsible for the contribution of turbulent diffusion of turbulent energy. This assumption is plausible in the main core of shear flow, but not close to the boundaries (see e.g. Monin and Yaglom, 1971).

In the Kolmogorov[12]-Prandtl[16] semi-empirical theory for shear flow, the coefficient of turbulent momentum exchange, $k = (-\rho\overline{u'w'})/\rho(du/dz)$, is the kinematic eddy viscosity. This introduction for shear flow is not a new hypothesis. Equations (2.1) and (2.2) take the form

$$k\frac{du}{dz} = {u_*}^2, \qquad k\left(\frac{du}{dz}\right)^2 - \epsilon = 0. \tag{2-4}$$

Here the quantity $u_* = (\tau/\rho)^{1/2}$ is an important governing parameter of shear flow: "dynamic" or "friction" velocity.

The basic hypothesis underlying the Kolmogorov-Prandtl theory can be presented in the following way: at large Reynolds numbers the local structure of the set of vortices around any point is statistically identical for all shear flows at a given Reynolds number; only the time and space scales are different. Therefore, leaving aside the Reynolds number dependence, all dimensionless flow properties should be identical. This means that all kinematic flow properties at a certain point including the momentum exchange coefficient k and the dissipation rate ϵ could be determined by the local values of two kinematic properties having different dimensions. Properties such as the turbulent energy of the unit mass

$$b = \frac{\overline{u'^2 + v'^2 + w'^2}}{2} \tag{2-5}$$

and the external length scale (mean length scale of vortices) ℓ, can be selected (b, ℓ version). Also in wide use is the (b, ϵ) version, where as basic quantities b and ϵ—the dissipation rate—are selected. We will use the (b, ℓ) version, in fact both versions are logically equivalent.

Dimensional analysis leads to the following relations:

$$k = \ell\sqrt{b}, \qquad \epsilon = \gamma^4 b^{3/2}/\ell. \tag{2-6}$$

The coefficient in the first Equation (2.6) can be selected equal to one because the length scale is determined with accuracy up to a constant factor. The constant γ

is a Reynolds number-dependent quantity; at large Reynolds numbers this quantity is close to 0.5 (see the book of Monin and Yaglom [15]).

Thus, Equations (2.4) assume the form

$$\ell\sqrt{b}\frac{du}{dz} = {u_*}^2, \qquad \ell\sqrt{b}\left(\frac{du}{dz}\right)^2 - \gamma^4\frac{b^{3/2}}{\ell} = 0. \tag{2-7}$$

It is important that from Equations (2.7) without any assumptions concerning the length scale ℓ, the relation for turbulent energy can be obtained:

$$b = \frac{u_*^2}{\gamma^2}. \tag{2-8}$$

Relation (2.8) shows that the dynamic, or friction velocity u_*, determines the order of magnitude of the velocity fluctuations.

Thus, if the length scale is known, the mean velocity u can be easily obtained from the first equation of the system. The situation of determining the length scale is, however, non-trivial. Using dimensional analysis a relation is obtained

$$\ell = z\Phi\left(\mathrm{Re}, \frac{u_* z}{\nu}\right). \tag{2-9}$$

We remind the reader that shear flow at large Reynolds numbers is considered, and also that the value $u_* z/\nu$ is large outside the close vicinity of the boundary $z = 0$, which, as was mentioned before, is excluded from consideration. Therefore, traditionally "complete" similarity (see, e.g., [4]) in both parameters Re and $u_* z/\nu$ is assumed. This means that function Φ can be replaced by its limit $\Phi(\infty, \infty) = \kappa\gamma$, which is assumed to be finite. The new constant κ is known as the von Kármán constant. The relation $\ell = \kappa\gamma z$ and relation (2.8) are substituted into the first Equation of (2.7), and the resulting relation is integrated, so the equation traditionally obtained is

$$\frac{u}{u_*} = \frac{1}{\kappa}\ln\left(\frac{u_* z}{\nu}\right) + B, \tag{2-10}$$

known as the universal (Reynolds number-independent) von Kármán-Prandtl logarithmic law. It is also tacitly assumed that the constant B is finite and Reynolds number-independent. The values $\kappa = 0.4$, $B = 5.1$ are usually accepted, although large deviations from these values have been reported in processing the experimental data.

However, as it was shown in a cycle of works by A.J. Chorin, V.M. Prostokishin and the present author (see [5; 6] and monograph [4] as well as the references presented there) this is not the case. There is "incomplete similarity" (see e.g. [4]) in parameter $u_* z/\nu$ and no similarity in parameter Re. In fact, at large Reynolds numbers and large $u_* z/\nu$ the mean velocity is represented by a family of Reynolds number-dependent power laws:

$$\frac{u}{u_*} = \left(\frac{1}{\sqrt{3}}\ln \mathrm{Re} + \frac{5}{2}\right)\left(\frac{u_* z}{\nu}\right)^{3/2\ln \mathrm{Re}}. \tag{2-11}$$

Furthermore, in the basic working interval of $u_* z/\nu$, the family of velocity distribution curves (2.11) can be represented in the form of a *Reynolds number-dependent* logarithmic law:

$$\frac{u}{u_*} = \frac{1}{\kappa(\mathrm{Re})}\ln\left(\frac{u_* z}{\nu}\right) + B(\mathrm{Re}), \tag{2-12}$$

where

$$\kappa(\mathrm{Re}) = \frac{e^{-3/2}}{\sqrt{3}/2 + 15/(4\ln \mathrm{Re})}, \qquad B(\mathrm{Re}) = -\frac{e^{3/2}\ln \mathrm{Re}}{2\sqrt{3}} - \frac{5}{4}e^{3/2}. \tag{2-13}$$

We mention several important properties of (2.12), (2.13). Firstly, at $\mathrm{Re} \to \infty$ the quantity $\kappa(\mathrm{Re})$ tends to a limit $\kappa_\infty = 2\sqrt{3}e^{3/2} \simeq 0.2776$. However, this tendency to the limit is very slow, so approximating the limiting value of κ_∞ with accuracy, for example, 10% κ requires huge values of Re, $\mathrm{Re} \sim 10^{20}$. For realistic lower values of Re, $\kappa(\mathrm{Re})$ are significantly less than κ_∞, so the slope of the straight line u/u_* vs $\ln(u_* z/\nu)$ is steeper than the slope of the straight line representing the usually accepted universal logarithmic law. *At the same time the additive constant $B(\mathrm{Re})$ at $\mathrm{Re} \to \infty$ tends not to a finite limit but to minus infinity.* All that means is that at large but realistic Re the universal (Reynolds number-independent) law for velocity distribution is not valid, although the velocity distributions in the $\ln(u_* z/\nu)$, u/u_* plane are represented by a family of Reynolds number-dependent straight lines (logarithmic laws) in the significant interval of the values of $u_* z/\nu$. These properties of velocity distributions obtained an instructive confirmation in the experiments by Zagarola [18], performed in pipe flows. Summing up we obtain an expression for the length scale ℓ, using formulae (2.7), (2.8) and (2.12):

$$\ell = \kappa(\mathrm{Re})\gamma z. \tag{2-14}$$

From (2.14) and the first Equation of (2.7) it follows

$$\frac{du}{dz} = \frac{u_*}{\kappa(\mathrm{Re})z}. \tag{2-15}$$

By integration we return to the relation (2.12), the constant of integration cannot be assumed to be a universal one.

3. Shear flow laminarization by suspended heavy particles. Mono-disperse particles size distribution

Consider a horizontal or slightly inclined shear flow in a gravity field loaded by small suspended heavy particles. We assume that both volume and mass concentrations of particles are small. Nevertheless as we will see, the dynamic effect of particles can be large: dust storms, firestorms, and tropical hurricanes give instructive examples. The reason for such large influence of heavy particles is that due to large gravitational force the vortices in turbulent flow have to spend a substantial part of their energy on suspending the particles, and this energy is not returned to the flow when the particles fall down but is dissipated into heat via viscosity. Namely, that is the main cause of a substantial laminarization and acceleration of natural flows. An instructive example: The Martian atmosphere is very subtle; the thickness of the sand layer in absence of a wind is a certain fraction of a millimeter only, but this tiny amount of sand was enough in the year 1972 to create a dust storm that quickly destroyed American and Soviet landing vehicles.

The suspended particles are assumed to be smaller than the internal turbulence length scale (the Kolmogorov scale) below which turbulent vortices begin to be affected by viscosity. Therefore the time of viscous relaxation of the velocity of particles can be considered as negligibly small. This means that it can be assumed that the horizontal components of the *instantaneous* velocity of particles coincide with those of fluid whereas the vertical component of the instantaneous velocity of particles is equal to that of fluid minus a constant quantity: the velocity of the free fall of a particle in an infinite fluid a (the concentration of particles, we remind you, is assumed to be small).

The density of the fluid-particles mixture is equal to $\rho_f(1-s)+\rho_p s=\rho_f(1+\sigma s)$, $\sigma=(\rho_p-\rho_f)/\rho_f$, where ρ_p is the density of particles, ρ_f-the density of fluid, and s is the volume concentration of the particles. In agreement with natural observation it can be assumed also that $\sigma s \ll 1$, so the density of the mixture can be taken equal to the density of the fluid everywhere that the difference of fluid density and density of mixture is not multiplied by the gravity acceleration. Therefore the transverse component of the momentum balance equation

$$-\overline{u'w'}={u_*}^2 \tag{3-1}$$

can be taken identical to the corresponding equation for pure fluid.

The balance of particles leads to a simple equation: the turbulent flux of particles is equal to the amount of falling particles per unit time and unit area:

$$-\overline{s'w'}-as=0 \tag{3-2}$$

(we denote by s the average concentration of particles and by s' its fluctuation).

The difference of the energy balance equation for pure fluid and fluid-particles mixture is the key point. Indeed, the inflow rate of turbulent energy from the mean flow is balanced for the mixture not only by the rate of viscous dissipation into heat, but, in addition by the work of suspension of particles by turbulent vortices which, we repeat, is not returned to the mean flow when the particles fall down. This work (per unit time, unit area and unit mass) is equal to the mean turbulent flux of particles $\overline{s'w'}$ times extra-weight (weight minus to Archimedean force $(\rho_p - \rho_f)g$ per unit volume of particles), divided by the fluid density ρ_f. We obtain for this specific work the expression $\sigma g \overline{s'w'}$, so that the equation of balance of turbulent energy for the fluid-particles mixture takes the form:

$$\overline{u'w'}\frac{du}{dz} + \epsilon + \sigma g \overline{s'w'} = 0. \tag{3-3}$$

We emphasize that the last term of (3.3) is the only term where the concentration enters, and it is significant because it contains a large factor — gravity acceleration g.[1] Equation (3.3) can be rewritten in the form, emphasizing its difference from the corresponding equation for pure fluid (2.2):

$$\overline{u'w'}(1 - \mathrm{Ko})\frac{du}{dz} + \epsilon = 0, \tag{3-4}$$

where the dimensionless parameter

$$\mathrm{Ko} = -(\sigma g \overline{s'w'})/(\overline{u'w'}(du/dz)), \tag{3-5}$$

named *the Kolmogorov parameter (number)* represents the relative part of the turbulent energy influx from the mean flow, spent for the work of suspension of particles by turbulent vortices.

Our further consideration follows the lines of the Kolmogorov-Prandtl analysis of turbulent shear flow.

By analogy with the coefficient of the turbulent momentum exchange k we introduce the coefficient of the particle exchange k_s:

$$k_s = -\overline{s'w'}/(ds/dz). \tag{3-6}$$

As is the case of eddy viscosity k the introduction of k_s, for shear flow, is not a new hypothesis. Assuming, following the Kolmogorov-Prandtl shear flow model, the similarity hypothesis we obtain:

$$k_s = \alpha_s \ell \sqrt{b}. \tag{3-7}$$

[1] As far as is known to the author, the expression for the work of suspension of particles was first obtained by M.A. Velikanov[18]. However, Velikanov deliberately included this work in the equation of the energy balance of the mean flow, not the turbulent energy balance, which cannot be considered as quite correct.

The quantity α_s, which can be called the turbulent Prandtl number for the fluid-particles mixture is a Reynolds number-dependent quantity. Generalizing the considerations of the length scale in the previous section we assume

$$\ell = \kappa(\mathrm{Re})\gamma z \Phi_\ell(\mathrm{Re}, \mathrm{Ko}), \tag{3-8}$$

where the function $\Phi_\ell(\mathrm{Re}, \mathrm{Ko})$ is equal to one for $\mathrm{Ko} = 0$ (pure fluid) and is less than one for $\mathrm{Ko} > 0$,

Using similarity relations (2.6): $k = \ell\sqrt{b}$, $k_s = \alpha_s \ell \sqrt{b}$, $\epsilon = \gamma^4 b^{3/2}/\ell$, we come to a closed system of equations of our model

$$\begin{aligned}
&\ell\sqrt{b}\,\frac{du}{dz} = {u_*}^2, \\
&\alpha_s \ell \sqrt{b}\,\frac{ds}{dz} + as = 0, \\
&b^2 = \frac{{u_*}^4}{\gamma^4}(1 - \mathrm{Ko}), \\
&\ell = \kappa(\mathrm{Re})\gamma z \Phi_\ell(\mathrm{Re}, \mathrm{Ko}).
\end{aligned} \tag{3-9}$$

The Kolmogorov number can be presented in the following form:

$$\begin{aligned}
\mathrm{Ko} &= -\frac{\sigma g \overline{s'w'}}{\overline{u'w'}(du/dz)} = -\frac{\sigma g \alpha_s (ds/dz)}{(du/dz)^2} = -\frac{\sigma g a s}{{u_*}^2 (du/dz)} \\
&= \frac{\sigma g a s \cdot \ell\sqrt{b}}{{u_*}^4} = \frac{\sigma g a^2 s^2}{{u_*}^4 \alpha_s (ds/dz)} = \frac{\omega^2}{dR/dZ},
\end{aligned} \tag{3-10}$$

where

$$R = \frac{1}{s}, \qquad Z = \frac{\alpha_s \sigma g \kappa^2}{{u_*}^2} z, \qquad \omega = \frac{a}{\kappa \alpha_s u_*}. \tag{3-11}$$

The system (3.9) can be reduced to a single equation of first order

$$\frac{1}{\omega^2}\frac{dR}{dZ}\Phi_\ell\Big(\frac{\omega^2}{(dR/dZ)}\Big)\Big(1 - \frac{\omega^2}{(dR/dZ)}\Big)^{1/4} = \frac{R}{\omega Z}. \tag{3-12}$$

We recognize that both κ and Φ_ℓ depend on Reynolds number Re, however we omit this argument in the following formulae.

We emphasize that parameter $\omega = a/\kappa\alpha_s u_*$ plays a basic role in our model. Its physical meaning is transparent: with accuracy up to a constant of the order one it is the ratio of the particle free fall velocity to the characteristic value of the velocity fluctuation.

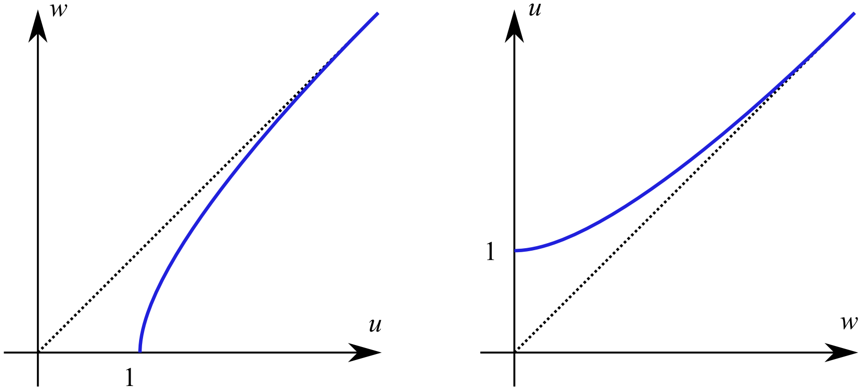

Figure 2. Functions $u(w)$ and $w(u)$ (see the text).

We introduce the function $w(u)$

$$w = u\Phi_\ell\left(\frac{1}{u}\right)\left(1-\frac{1}{u}\right)^{1/4}, \tag{3-13}$$

and the function $u(w)$, the inverse to it. They are presented in Figure 2. Both functions at $w, u \to \infty$ have an asymptote $u = w$, represented by the bisectrix of the first quadratures in the u, w plane.

Equation (3.12) can be rewritten in the form

$$\frac{dR}{dZ} = \omega^2 u\left(\frac{R}{\omega Z}\right). \tag{3-14}$$

Equation (3.14) is a homogeneous one, and it can be integrated by quadratures. Introducing a new variable $P = R/\omega Z$, so that $dR/dZ = \omega P + \omega Z(dP/dZ)$, and we obtain

$$Z\frac{dP}{dZ} = \omega u(P) - P, \tag{3-15}$$

or, after integration,

$$\ln Z + \text{Const} = \ln\frac{z}{z_0} = \int\limits_{P_0}^{P}\frac{dP}{\omega u(P) - P}. \tag{3-16}$$

Here z_0, P_0 are constants.

The structure of the integral curves of Equation (3.15) in the $P \ln Z$ plane is substantially different in $\omega < 1$ and $\omega > 1$. In the case $\omega < 1$ there exits one and only one root $P = P_*$ of equation $\omega u(P) - P = 0$; this is clear from elementary geometric considerations. There are two classes of integral curves separated by the straight line $P = P_*$. All integral curves approach the asymptotics $P = P_*$ at $\ln(z/z_0) \to \infty$. Returning to the plane s, z, we obtain a picture of integral curves, represented in Figure 3. At $z \to \infty$ all distributions of the concentration of particles tend asymptotically to the curve $s = 1/P_*\omega Z$. The curves of class 1, lying under the separatrix $s = 1/P_*\omega Z$, approach the bottom $z = 0$ asymptotically. The curves of class 2 go to $s = \infty$ at a certain finite value of z. The integral curves of each class can be obtained one from another by shifting along the $\ln Z$ axis. Therefore, the initial height $z = z_0$, where the concentration $s = s_0$ can be prescribed, can be crossed by the integral curves of both classes.

It follows from the previous investigation that at large z the distributions of the concentration of particles, if ω is less than one independently of the boundary condition at a certain level $z = z_0$, are described by the curve $s = 1/P_*\omega Z$. Physically this means that if the velocity fluctuations are sufficiently large, and exceed the free fall velocity a, turbulent flow "takes" as much of the particles as it can, i.e., as much as is allowed by the prescribed shear stress $\tau = \rho {u_*}^2$. Therefore this asymptotic regime is called "the regime of limiting saturation". The regime of limiting saturation corresponds to a constant value of the Kolmogorov number $\mathrm{Ko} = \mathrm{Ko}^*$, which can be obtained from the following equation:

$$\Phi_\ell(\mathrm{Ko}^*)(1 - \mathrm{Ko}^*)^{1/4} = \omega. \tag{3-17}$$

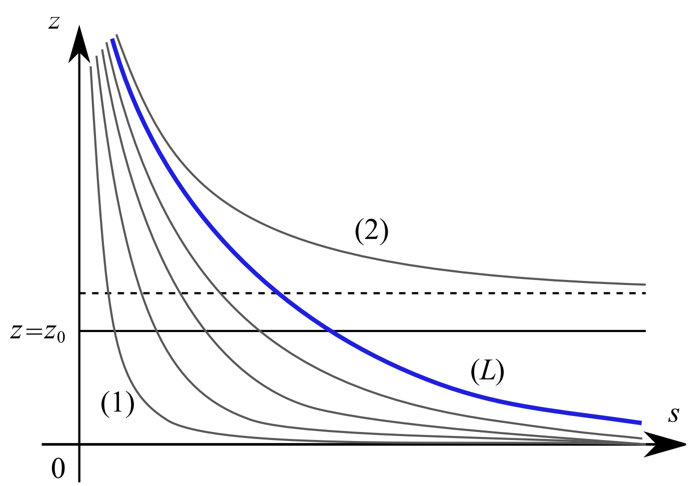

Figure 3. The field of concentration distributions for the case $\omega < 1$. The regime of limiting saturation (Curve L) for which $s \sim \mathrm{Const}/z$ attracts all curves, corresponding to the regimes with various boundary conditions at $z = z_0$. It should be emphasized that these curves have physical meaning only at $s \ll 1$.

Furthermore, using Equation (3.17) we obtain from system (3.9)

$$\frac{du}{dz} = \frac{u_*}{\kappa(\mathrm{Re})\omega z}. \tag{3-18}$$

This means that the velocity gradient at the core of the flow, where the regime of limiting saturation is achieved is $(1/\omega)$ times larger than the velocity gradient in pure fluid flow, given by Equation (2.16). The case when ω is much less than one (very small particles) is of special interest. In this case the Kolmogorov number (in the regime of limiting saturation) is close to one, so nearly the whole inflow of turbulent energy from the mean motion is spent for the suspension of particles. Turbulent energy is strongly reduced. The distribution of concentration in this case takes the form

$$s = \frac{1}{\omega^2 Z} = \frac{\alpha_s {u_*}^4}{a^2 \sigma g z}. \tag{3-19}$$

For the case $\omega > 1$, when the velocity fluctuations are smaller than free fall velocity of particles the situation is different. The denominator of the integrand in (3.16) is positive everywhere. The concentration distributions go to infinity at a certain z, like the curves of the second class in the case $\omega < 1$. Clearly, when s is no longer sufficiently small, these curves make no physical sense. It is important that there is a strong difference between the cases $\omega > 1$ and $\omega < 1$ in the behavior of integral curves, i.e., concentration distributions at large z. It is easy to show that as $z \to \infty$ the distributions behave as $s \sim \mathrm{Const}/z^{\omega}$, and the Kolmogorov number goes to zero as $\mathrm{Const}/z^{\omega-1}$. This means that at large heights the velocity gradient du/dz becomes undisturbed by particles, and is given by relation (2.15). The work of suspension of particles is negligible, and there is no flow acceleration in the core of the flow. The particles create a "lubrication layer", so that the velocities increase at any height but only due to "lubrication".

4. Geophysical applications. The modified model

An instructive application of the theoretical construction presented in the previous section is the mathematical modelling of the tropical hurricane. The basis for further consideration will be the "sandwich" model of the hurricane proposed by Sir James Lighthill (see his posthumously published paper [14]). According to this model between the air and the sea there exists an intermediate layer (see Figure 4). Lighthill called it "ocean spray"—which consists of suspended water droplets formed in the process of the breaking of water waves and air. Lighthill proposed to consider ocean spray as a "third fluid", and strongly emphasized "the need to fill the gap in knowledge about ocean spray at extreme wind speeds". Lighthill himself concentrated on the thermodynamic side of the modelling. In paper [7] a model, complementary to the Lighthill one was proposed. We emphasize that the possibility of constructing such a model was anticipated by Sir James Lighthill, who discussed it with the authors. However, further analysis showed that a substantially modified model was needed, and it is presented below.

We concentrate here, as in paper [7] mentioned above, on a single effect: flow acceleration in ocean spray by water droplets. We leave aside the effect of the Coriolis force, as well as the cooling effect due to evaporation of droplets and other thermal effects. These effects can be incorporated into the model as was done previously when modelling other atmospheric and oceanic phenomena, see for example [8; 9].

The essence of the modification of the model is as follows. In paper [7] it was naturally assumed as the first step that all the water droplets in ocean spray are identical, and the construction described in the previous section was applied. It was assumed that the water droplets are large, so that the basic parameter ω is larger than one.

The effect of flow acceleration was obtained, but it was less than expected, in spite of the large values of the parameter ω and large concentrations that were assumed in the numerical calculations.

We will demonstrate below that taking into account the availability in ocean spray both of large and small droplets changes the situation. It is difficult to take into consideration the whole spectrum of droplet sizes, in particular, because it is unknown, and it is changing due to various factors, basically unknown. However, it happens to be enough to assume that in ocean spray there are droplets of two sizes, corresponding to the values of parameter $\omega = \omega_1 > 1$ and $\omega = \omega_2 < 1$. Under such a simplified assumption much larger wind accelerations are obtained. The general

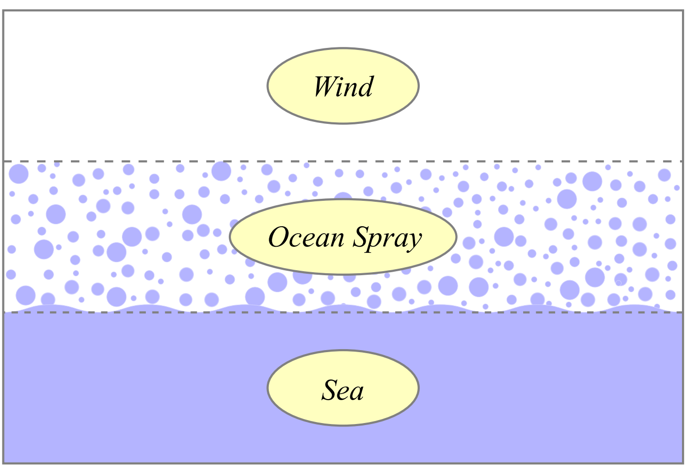

Figure 4. The Lighthill "sandwich model" of a tropical hurricane

consideration of a more realistic case of the continuous spectrum of particle sizes will also be presented below.

As considered previously, we assume that ocean spray occupies the region $z \geq z_0$, where z_0 is the thickness of the layer where the droplets are produced, and the vertical coordinate z is reckoned from the average sea surface. As before we neglect the coalescence of the droplets and the variation of their size due to evaporation. Thus, we assume that two sorts of particles are available in the flow in ocean spray; due to smallness of the concentrations s_1 and s_2 of both kinds of droplets the interference of droplets can be neglected.

The basic system of equations of the modified model is taken in the form suggested by previous analysis, presented in Section 3:

$$\ell\sqrt{b}\,\frac{du}{dz} = {u_*}^2, \tag{4-1}$$

- the momentum balance equation

$$\alpha_s \ell\sqrt{b}\frac{ds_1}{dz} + a_1 s_1 = 0, \tag{4-2}$$

$$\alpha_s \ell\sqrt{b}\frac{ds_2}{dz} + a_2 s_2 = 0, \tag{4-3}$$

- the equations of conservation of both sorts of droplets,

$$b^2 = \frac{u_*^4}{\gamma^4}(1 - \mathrm{Ko}); \tag{4-4}$$

- the equation of turbulent energy balance.

Here the Kolmogorov number $\mathrm{Ko} = \mathrm{Ko}_1 + \mathrm{Ko}_2$, Ko_1 is the Kolmogorov number, corresponding to larger droplets:

$$\mathrm{Ko}_1 = -\frac{\alpha_s \sigma g (ds_1/dz)}{(du/dz)^2}, \tag{4-5}$$

whereas Ko_2 is the Kolmogorov number, corresponding to smaller droplets:

$$\mathrm{Ko}_2 = -\frac{\alpha_s \sigma g (ds_2/dz)}{(du/dz)^2}. \tag{4-6}$$

Thus, a separate balance of droplets of both sizes and their contributions to the work of suspension are considered.

For the length scale the following relation is proposed

$$\ell = \kappa\gamma\, z \Phi_\ell(\mathrm{Ko}), \tag{4-7}$$

naturally generalizing relation (3.8) for the monodisperse mixture; the Reynolds number dependence of the parameters α_s, κ, γ and the function $\Phi_\ell(\mathrm{Ko})$ are also assumed.

Although system (4.1)–(4.7) seems to be more complicated than the system for the monodisperse mixture, it can also be reduced to a Cauchy problem for an ordinary differential equation of first order due to existence of a first integral. This reduction allows us to perform an asymptotic analysis.

Indeed, we obtain from Equations (4.2), (4.3)

$$\begin{aligned} \frac{ds_1}{ds_2} &= \frac{\omega_1}{\omega_2}\frac{s_1}{s_2}, \\ \omega_1 &= \frac{a_1}{\kappa \alpha_s u_*}, \\ \omega_2 &= \frac{a_2}{\kappa \alpha_s u_*}, \end{aligned} \tag{4-8}$$

and, by integration

$$\frac{s_2}{s_{20}} = \left(\frac{s_1}{s_{10}}\right)^{\omega_2/\omega_1}, \tag{4-9}$$

where s_{10} and s_{20} are the concentrations of both kinds of droplets at $z = z_0$. Also, we obtain, similarly to what was done previously,

$$\begin{aligned} \mathrm{Ko}_1 &= \frac{\sigma g a_1 s_1}{{u_*}^2 (du/dz)}, \\ \mathrm{Ko}_2 &= \frac{\sigma g a_2 s_2}{{u_*}^2 (du/dz)}. \end{aligned} \tag{4-10}$$

As previously, it is convenient to pass to dimensionless variables

$$U = \frac{\kappa u}{u_*}, \quad Z = \frac{\alpha_s \kappa^2 \sigma g}{{u_*}^2} z, \quad S_1 = \frac{s_1}{s_0}, \quad S_2 = \frac{s_2}{s_0}. \tag{4-11}$$

We assumed here for simplicity $s_{10} = s_{20} = s_0$, and we reduced the system to the form

$$\begin{aligned} \Phi_\ell(\mathrm{Ko})(1-\mathrm{Ko})^{1/4} Z \frac{dU}{dZ} &= 1 \\ \Phi_\ell(\mathrm{Ko})(1-\mathrm{Ko})^{1/4} Z \frac{dS_1}{dZ} + \omega_1 S_1 &= 0 \\ \Phi_\ell(\mathrm{Ko})(1-\mathrm{Ko})^{1/4} Z \frac{dS_2}{dZ} + \omega_2 S_2 &= 0. \end{aligned} \tag{4-12}$$

The boundary conditions we take are of the form

$$S_1 = S_2 = 1, \qquad U = 0 \; at \; Z = Z_0 = \frac{\alpha_s \kappa^2 \sigma g}{u_*^2} z_0. \tag{4-13}$$

Let's estimate the orders of magnitude of all quantities that enter the problem: $z_0 = 10^2 - 10^3$ cm– is the range of the amplitudes of the waves; $\alpha_s \kappa^2$ is of the order of one, $\sigma g \sim 10^6$ cm/s^2, u_* is of the order of 10^2 cm/s, therefore Z_0 is in the range $10^4 - 10^6$. Furthermore, s_0 should be in the range of $10^{-6} - 10^{-4}$, so the values of the parameter $A = s_0 Z_0$ can be assumed to be in the range $10^{-1} - 10$.

Introducing the variable $\zeta = \ln(Z/Z_0) = \ln(z/z_0)$ we come to the ultimate system of equations and initial conditions

$$(1-\mathrm{Ko})^{1/4}\Phi_\ell(\mathrm{Ko})\frac{dU}{d\zeta} = 1 \tag{4-14}$$

$$(1-\mathrm{Ko})^{1/4}\Phi_\ell(\mathrm{Ko})\frac{dS_1}{d\zeta} + \omega_1 S_1 = 0 \tag{4-15}$$

$$(1-\mathrm{Ko})^{1/4}\Phi_\ell(\mathrm{Ko})\frac{dS_2}{d\zeta} + \omega_2 S_2 = 0 \tag{4-16}$$

$$\mathrm{Ko} = \frac{Ae^\zeta(\omega_1 S_1 + \omega_2 S_2)}{dU/d\zeta} \tag{4-17}$$

with the boundary conditions $S_1 = S_2 = 1$, $U = 0$ at $\zeta = 0$. The first integral (4.9) takes the form

$$S_2 = S_1^{\omega_2/\omega_1}. \tag{4-18}$$

From system (4.13), (4.14), (4.16), (4.17) a relation for Ko can be obtained:

$$\mathrm{Ko} = Ae^\zeta {\omega_1}^2(1+\theta R_1^{1-\theta})/(dR_1/d\zeta), \tag{4-19}$$

where

$$\begin{aligned} R_1 &= 1/S_1, \\ \theta &= \omega_2/\omega_1. \end{aligned} \tag{4-20}$$

After division by S_1^2 Equation (4.14) can be reduced to the form:

$$(1-\mathrm{Ko})^{1/4}\Phi_\ell(\mathrm{Ko})\frac{dR_1}{d\zeta} - \omega_1 R_1 = 0. \tag{4-21}$$

Finally, dividing by $Ae^\zeta {\omega_1}^2(1+\theta R_1^{1-\theta})$ we obtain

$$\begin{aligned} &\frac{dR_1}{d\zeta}\frac{1}{A\omega_1^2 e^\zeta(1+\theta R_1^{1-\theta})}\left(1-\frac{Ae^\zeta\omega_1^2(1+\theta R_1^{1-\theta})}{dR_1/d\zeta}\right)^{1/4}\Phi_\ell\left(\frac{Ae^\zeta\omega_1^2(1+\theta R_1^{1-\theta})}{dR_1/d\zeta}\right) \\ &\qquad = \frac{R_1}{Ae^\zeta\omega_1(1+\theta R_1^{1-\theta})}. \end{aligned} \tag{4-22}$$

Using the function $u(w)$ introduced by the relation (3.13), we present Equation (4.21) in the form:

$$\frac{dR_1}{d\zeta} = A\omega_1^2 e^{\zeta}(1+\theta R_1^{1-\theta})u\left(\frac{R_1}{Ae^{\zeta}\omega_1(1+\theta R_1^{1-\theta})}\right). \tag{4-23}$$

This is an ordinary differential equation of first order, which is to be solved under the initial condition

$$R_1 = 1 \; at \; \zeta = 0. \tag{4-24}$$

Under the assumption that $\theta = \omega_2/\omega_1$ is small, the solution to Equation (4.22) can be investigated asymptotically. Indeed, θR_1 is much less than one, i.e., $s_1 \gg \theta s_0$ in a certain interval $0 \leq \zeta \leq \zeta_*$. In this interval the term $\theta R_1^{1-\theta}$ in (4.22) can be neglected in comparison with 1, and Equation (4.22) takes the form

$$\frac{dR_1}{d\zeta} = A\omega_1^2 e^{\zeta} u\left(\frac{R_1}{Ae^{\zeta}\omega_1}\right). \tag{4-25}$$

This equation coincides with Equation (3.12) for $\omega = \omega_1$ (monodisperse flow of large particles). Furthermore, the function $u(w)$ is larger than one, i.e., R_1 is growing faster than $A\omega_1^2 e^{\zeta}$ at all ζ. Therefore there exist a number ζ_{**} where θR_1 becomes much larger than one, and Equation (4.22) takes the form

$$\frac{dR_1}{d\zeta} = A\omega_1^2 e^{\zeta}\theta R_1^{1-\theta} u\left(\frac{R_1^{\theta}}{Ae^{\zeta}\omega_1\theta}\right), \tag{4-26}$$

which can be transformed easily using the integral (4.17) to the form

$$\frac{dR_2}{d\zeta} = A\omega_2^2 e^{\zeta} u\left(\frac{R_2}{Ae^{\zeta}\omega_2}\right), \tag{4-27}$$

i.e., to Equation (3.12) for $\omega = \omega_2$ (monodisperse flow of small particles).

In the interval $0 \leq \zeta < \zeta_*$, according to the investigation of the monodisperse flows, the Kolmogorov number is decreasing; in the interval $\zeta_{**} < \zeta$ it is increasing, reaching the value Ko* satisfying the equation

$$\Phi_\ell(\mathrm{Ko}^*)(1-\mathrm{Ko}^*)^{1/4} = \omega_2. \tag{4-28}$$

Somewhere in between ζ_* and ζ_{**} a minimum of the Kolmogorov number is reached. Therefore the flow is separated in two regions: the lower region, where the Kolmogorov number is decreasing and reaching a minimum, and the upper region where the Kolmogorov number is growing from the minimum to the final value Ko*. It is natural to consider the lower region as a “lubrication layer” and the upper region as a “suspension layer”. The graphs presented in Figures 5,6, constructed by Dr. C.H. Rycroft on the basis of numerical computations, illustrate a typical structure of the flow in ocean spray if the availability of droplets of two

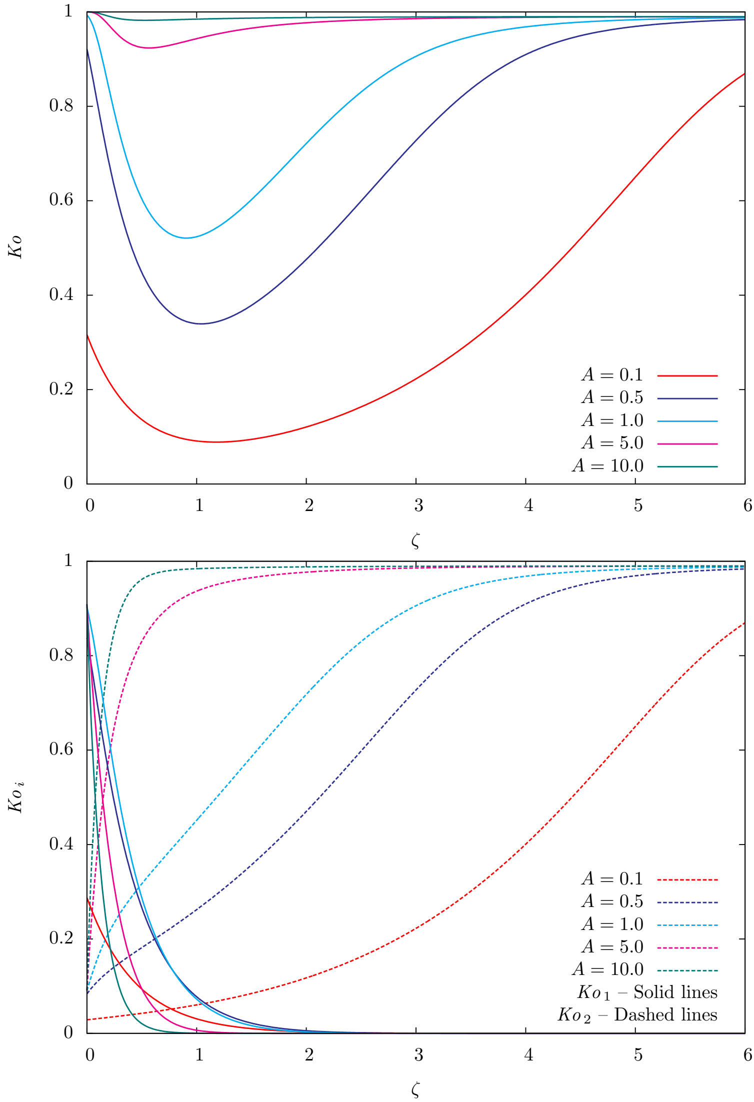

Figure 5. The distribution of the total Kolmogorov number Ko and the Kolmogorov number corresponding to large and small particles Ko_1, Ko_2 for various values of parameter $A = Z_0 s_0$

sizes, large ones ($\omega_1 > 1$) and small ones ($\omega_2 < 1$), is taken into account. In the numerical computations, function $\Phi_\ell(\mathrm{Ko})$ was taken equal to one, and the values of ω of order one were taken in both cases: $\omega_1 = \sqrt{10}$, $\omega_2 = 1/\sqrt{10}$. However, the ratio $\theta = \omega_2/\omega_1 = 1/10$ is a small parameter, allowing an asymptotic analysis. Computations support the results of the asymptotic analysis.

Figure 6 is especially instructive: it demonstrates the strong increase of wind speed in ocean spray in comparison with pure air flow ($S_1 = S_2 = 0$) and also with the flow of fluid-particle mixtures where only large particles are available, $S_2 = 0$.

The analysis presented above can be extended to the case of a continuous spectrum of particle sizes: $\Omega_1 \geq \omega \geq \Omega_2$, where $\Omega_1 > 1$, $\Omega_2 < 1$. Equations (4.13), (4.14) remain valid if ω_1 is a certain reference parameter of value $1 < \omega_1 < \Omega_1$), whereas Equation (4.15) is replaced by the equation of conservation of particles for arbitrary ω in the interval $\Omega_1 \geq \omega \geq \Omega_2$

$$(1-\mathrm{Ko})^{1/4}\Phi_\ell(\mathrm{Ko})\frac{dS}{d\zeta} + \omega S = 0. \tag{4-29}$$

The first integral takes the form $S = S_1^{\omega/\omega_1}$. Here it is assumed that the concentration at the boundary $z = z_0$ of particles in the range between ω and $\omega + d\omega$ is $s_0(\omega)d\omega$. The expression for the Kolmogorov number is given by the following relation:

$$\mathrm{Ko} = \frac{e^\zeta \int_{\omega_2}^{\omega_1} A(\omega) S_1^{\omega/\omega_1}\, d\omega}{dU/d\zeta}, \tag{4-30}$$

where $A(\omega) = s_0(\omega)Z_0$. The previous case of the two-point spectrum corresponds to

$$s_0(\omega) = s_{01}\delta(\omega-\omega_1) + s_{02}\delta(\omega-\omega_2), \tag{4-31}$$

where $\delta(\omega)$ is the Dirac delta function. There is no principal distinction in the results obtained for the case of the continuous spectrum or the two-point spectrum.

5. Conclusion and discussion

The modified model of turbulent shear flow of a suspension of small heavy particles in a fluid is presented. The modification is based on the assumption that in the flow there are two sorts of particles. For the particles of the first sort the velocity of free fall a_1 is larger than the characteristic velocity fluctuation, for the particles of the second sort the velocity of free fall a_2 is less than the characteristic velocity of fluctuation. Considering a_2/a_1 as a small parameter allowed an effective asymptotic analysis of the model equations that were obtained. The investigation was simplified by the existence of a first integral found for the system. The numerical computations are in agreement with the asymptotic analysis.

The main result is that a two-layered flow structure is obtained. In the lower layer, which we called the *lubrication layer*, the Kolmogorov number—the ratio of the work spent on the suspension of particles to the turbulent energy influx from the mean flow—is decreasing. In the upper layer, which we called the *suspension layer*, the Kolmogorov number is increasing after reaching a minimum, until it reaches

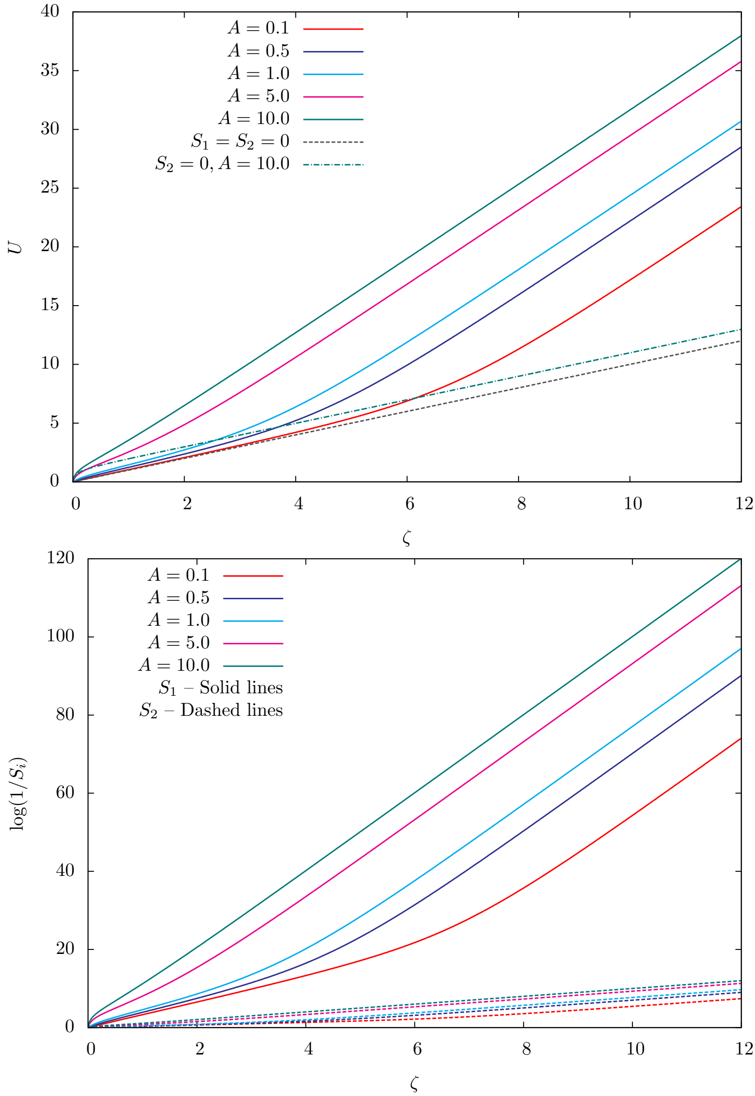

Figure 6. The distributions of dimensionless velocity U and inverse concentrations $1/S_i$ for various values of parameter $A = Z_0 s_0$

the ultimate value at large heights. The basic flow acceleration occurs in the upper layer, where the velocity gradient is small.

Numerical investigation showed that significant laminarization of the flow can be obtained by the addition of heavy particles. What is specifically significant is that the large particles could be of moderate size for reaching high flow speed.

The modified model is applied to the flow in the oceanic spray of a tropical hurricane. It seems that it gives a more realistic structure of the flow than the previously used mono-disperse model.

The modified model can also be applied to dust storms and to big forest and grass fires as well as to other fires when the debris (larger particles) and particles of soot (small particles) are caught by the wind. If the process of combustion is an intensive one so that a sufficiently large amount of small (e.g., soot) particles is produced in the combustion zone, a suspension layer can be formed, and the transition to firestorms—large wind accelerations by intensive fire—can happen, as apparently was the case in the large Chicago fire, 1871. Such firestorms due to intense fires created by large scale bombing (Dresden, February 1945; Hiroshima, August 1945) were also observed.

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