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GUILLAUME COQUERET

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APPROXIMATION OF PROBABILISTIC LAPLACE TRANSFORMS AND THEIR INVERSES

GUILLAUME COQUERET

We present a method to approximate the law of positive random variables defined by their Laplace transforms. It is based on the study of the error in the Laplace domain and allows for many behaviors of the law, both at 0 and infinity. In most cases, both the Kantorovich/Wasserstein error and the Kolmogorov–Smirnov error can be accurately computed. Two detailed examples illustrate our results.

1. Introduction

The topic of Laplace transform inversion is an old problem which relates to physics, probability theory, analysis and numerical methods. The number of publications dedicated to it is so large that it is possible to write surveys of surveys on the subject; see [5, Chapter 9]. If f is a positive integrable function on $\mathbb{R}_+$, we define the Laplace transform operator as follows,

$$\mathbb{L}[f(x)](t) = L(t) = \int_0^\infty e^{-tx} f(x)\, ds,$$

When L is given, the inverse Laplace transform operator $\mathbb{L}^{-1}$ applied to L yields the original function f. Two of the most important results related to $\mathbb{L}^{-1}$ are the *Bromwich integral* (see section 2.2 in [5]):

$$\mathbb{L}^{-1}[L(t)](x) = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{xt} L(t)\, dt, \tag{1-1}$$

for some c chosen so that the path of integration makes sense for L, and the *Post–Widder formula*: (see [8], section VII.6):

$$f(x) = \lim_{n\to+\infty} \frac{(-1)^n}{n!} \left(\frac{n}{x}\right)^{n+1} L^{(n)}(n/x). \tag{1-2}$$

However, there are many other ingenious ways to obtain f from L. Most techniques of Laplace transform inversion belong to one of the five families of methods listed in Table 1.

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	Method based on	References
i)	the Bromwich integral	[4], [26], [5, Chapters 4 and 6]
ii)	the Post–Widder formula	[2], [25], [24], [5, Chapter 7]
iii)	Fredholm equations of the first kind	[7], [22, §12.5-3], [14], [5, Chapter 8]
iv)	rational approximation	[12], [15], [16], [5, Chapter 5]
v)	series expansion	[5, Chapter 3]

Table 1

The references related to these methods are of course far from exhaustive.

All of these approaches lead to numerical approximations. A recent survey on the efficiency of some of these procedures was recently carried out by Masol and Teugels in [18]. The families of techniques can further be categorized into two broader sets:

- **i)** + **ii)** + **iii)**: methods for which the initial function, L, is exact, but the inversion approximate (discretization of the integral in (1-1) or choice of a large, but finite, n in (1-2))
- **iii)** + **iv)** + **v)**: methods for which the inversion is exact, but the target function is an approximation of the initial Laplace transform

(Some methods from the third family can belong to both sets.)

The approximations stemming from the second set of techniques take the form

$$L(t) \approx \sum_{k=1}^{N} c_k L_k(t) \quad \Longleftrightarrow \quad f(x) \approx \sum_{k=1}^{N} c_k f_k(x) \tag{1-3}$$

The core idea of this paper is to take advantage of the properties of Laplace transforms in probability to choose the L_k (and thus f_k) wisely, depending on some properties of f. We present an iterative procedure which progressively reduces the Kantorovich error induced by the approximation. The main contribution of our approach is that when f is bounded from above, this method provides a uniform maximum for the error made on the cumulative distribution function. Such results are quite rare in the literature, but one reference in a slightly different setting is [23].

This method can be used, for instance, to approximate the law of positive infinitely divisible distributions, which are usually characterized by their Laplace transform.

The paper is organized as follows: in Section 2, we detail some of the properties of f which can be inferred from L and which will be used further on. In Section 3, we detail our method and some error related results, and, lastly, we provide numerical examples in Section 4.

2. Some properties of the density

The aim of this section is to recall a few classical results which show that many properties of f can be derived from a thorough study of L.

We begin with some notations. Throughout the paper, we will consider two positive random variables X and Y with densities f and g, cumulative distribution functions (CDFs) F and G and Laplace transforms L and M respectively. We also denote by $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$ their survival functions. The function L (resp. F) will be the original Laplace transform (resp. CDF) and M (resp. G) its approximation.

For a function $f = f^{(0)}$, $f^{(n)}$ will denote its n-th derivative and in some asymptotic settings, we will write $f(x) \sim g(x)$ for $f(x)/g(x) \to 1$.

Support. The first basic piece of information required to characterize a distribution is its support.

Theorem 2.1. *Let A denote the left point of the support of the positive random variable X. Then if B is the set of real numbers b such that $e^{bt}L(t) = O(1)$ as $t \to \infty$, we have*

$$A = \sup_{b \in B} b$$

Proof. If $A = 0$, then, for any $x < 0$, $e^{xt}L(t) \to 0$ $(t \to +\infty)$. For $x > 0$,

$$e^{xt}L(t) \geq \int_0^x e^{st} f(x-s)\, ds \geq \eta e^{\delta t} \mathrm{Leb}\{s \in [\delta, x] : f(x-s) \geq \eta\} \to \infty, \quad t \to +\infty,$$

where Leb is the Lebesgue measure and $\delta, \eta > 0$ were chosen such that

$$\mathrm{Leb}\{s \in [\delta, x] : f(x-s) \geq \eta\} > 0,$$

which is possible, since $A = 0$. The case $A > 0$ follows by direct translation. □

Another way to obtain the lower bound of the support of X is in fact to compute the limit of $-L'(t)/L(t)$ when $t \to \infty$. Indeed, by Hölder's inequality, $\operatorname{Log} L$ is convex, hence L'/L is increasing. Since it is bounded above by zero, it converges to some negative limit. A simple analysis shows that this limit at infinity is in fact $-A$.

In order to find the upper bound of the support of X, we propose a test, based on the following proposition. Note that it is easy to compute $\mathbb{E}[X]$ with the sole knowledge of L, since $\mathbb{E}[X] = -L'(0)$.

Proposition 2.2. *If the positive random variable X is almost surely bounded above by C, then for any $A > 0$ and $\gamma \geq 1$,*

$$L(t) \leq 1 - \mathbb{E}[X^\gamma]\frac{1-e^{-A}}{A^\gamma}t^\gamma \quad \textit{for all } t \in [0, A/C].$$

Proof. The proof relies on the inequality

$$y^\gamma - x^\gamma \geq e^{-x}y^\gamma - e^{-y}x^\gamma, \quad \gamma \geq 1, \quad 0 < x < y.$$

Setting $x = tX$, $y = A$ and applying the expectation operator yields the result. □

Hence, if $L(t) > 1 - t\,\mathbb{E}[X](1-e^{-A})/A$ in the vicinity of 0, then X is unbounded. The test usually performs better for $A \ll 1$.

In the same spirit, note that Theorem 7(b) in [10] makes it possible to build another test based on $\mathbb{E}[X^\gamma]$ for $\gamma < 1$. Since they depend on the interval $[0, A/C]$, these results make it even possible to derive bounds for C.

By Theorem 2.1 and Proposition 2.2, we will henceforth, without much loss of generality, restrict ourselves to distributions with *supports on the whole positive real line*.

Tail behaviors. This subsection recalls classical Tauberian theorems in probability (see for instance [8, XIII.5]). These results show the strong link that exists between the behavior of f near zero and that of L near infinity and vice-versa. The general form of the de Bruijn exponential Tauberian theorem can be found in [3], Theorem 4.12.9, but we recall below a more peculiar form, derived from Corollary 4.12.6 of the same monograph.

Theorem 2.3. *Let $0 < \gamma < 1$, $\delta \in \mathbb{R}$, $C > 0$ and X a positive random variable. Then,*

$$\log \mathbb{E}[e^{-tX}] \sim -Ct^\gamma(\log(t))^\delta, \quad t \to \infty$$

if and only if

$$\log P[X \leq x] \sim -[C\gamma^\gamma(1-\gamma)^{1-\gamma-\delta}x^{-\gamma}(-\log x)^\delta]^{1/(1-\gamma)}, \quad x \downarrow 0.$$

In a series of papers, Nakagawa provides conditions on L to determine whether a distribution has a heavy or a light tail. We state one of his results below (see [19] and the references therein). For a complex number $z = a + ib$, if $L(z)$ converges for $a > a_0$ and diverges for $a < a_0$, then a_0 is said to be the abscissa of convergence of $L(z)$.

Theorem 2.4. *If a_0 is the abscissa of L such that $-\infty < a_0 < 0$ and a_0 is a pole of L, then*

$$\lim_{x\to\infty} \frac{1}{x} \log P[X > x] = a_0.$$

When the asymptotic behaviors are not exponential but of power form, we can also resort to [3, Corollary 8.1.7] and [8, XIII.5, Theorem 2], which we recall below (with $l(x) = C\log(x)^{\beta}$).

Proposition 2.5. *For* $0 \le \alpha < 1$, $\beta \ge 0$ *and* $C > 0$, *the following are equivalent*:

$$1 - L(t) \sim -Ct^{\alpha}\log(t)^{\beta}, \quad t \downarrow 0.$$
$$1 - F(x) \sim C\frac{\log(x)^{\beta}}{x^{\alpha}\Gamma(1-\alpha)}, \quad x \to \infty.$$

Proposition 2.6. *For* $\alpha, \beta \ge 0$ *and* $C > 0$, *the following are equivalent*:

$$L(t) \sim C\,\frac{\log(t)^{\beta}}{t^{\alpha}}, \qquad t \to \infty.$$
$$F(x) \sim -C\,\frac{\log(x)x^{\alpha}}{\Gamma(1+\alpha)}, \quad x \downarrow 0.$$

The first result allows one to accurately determine the tail of a distribution when it is very heavy. For other power tail behaviors (when $\alpha > 1$), we refer to Theorem 8.1.6 in [3].

Lastly, we recall the initial value theorem:

$$f(0+) = \lim_{t\to\infty} tL(t).$$

Boundedness. It can be very convenient to know whether a distribution has a bounded density. In order to do so, it is possible to build a test based on the following corollary of the Post–Widder formula.

Lemma 2.7. *A function* L, *defined on* $\mathbb{R}_+$ *is the Laplace transform of a probability density bounded above by* c *if and only if* $L(0) = 1$ *and*

$$0 \le (-1)^n L^{(n)}(t) \le \frac{cn!}{t^{n+1}}$$

for all $n = 0, 1, \ldots$ *and* $t > 0$.

3. The approximation method

3.1. *Introductory remarks.* We shall henceforth consider a given positive function L defined on $\mathbb{R}_+$, satisfying $L(0) = 1$,

$$(-1)^n L^{(n)}(t) \ge 0 \quad \text{for all } t > 0 \text{ and all } n \ge 0 \tag{3-1}$$

and

$$\lim_{t\to\infty} tL(t) < \infty. \tag{3-2}$$

Any function L for which (3-1) holds is called a complete monotone function. Such functions have the following well-known property (see Theorem 7.11 in [27] for instance).

Theorem 3.1. *A function h is completely monotone on $\mathbb{R}_+$ if and only if it is the Laplace transform of a nonnegative finite Borel measure ν, i.e., if*

$$h(x) = \int_0^\infty e^{-xt}\nu(dt).$$

Therefore, (3-1) and $L(0) = 1$ are necessary and sufficient conditions for probabilistic Laplace transforms.

Our approach is essentially error driven: many inversion techniques do not allow to compute the error made on $f(x)$ or $F(x)$. Some techniques come with error bounds, but these bounds usually increase with x (see for instance (4.61) and (6.19) in [5]). Our method will focus on $\int_0^\infty |F(x) - G(x)|\,dx$, where F is the original CDF and G the approximate one. As we will see, this is an appropriate choice, because when f is bounded, it yields a uniform bound on $|F - G|$, which is a strong result.

The main problem with the L^1 error on F is that it cannot be retrieved from $\left|\int_0^\infty (F(x) - G(x))\,dx\right|$, unless the sign of $F - G$ does not change. Notice that focusing on cumulative distribution functions is critical since it can occur that G is dominated by F on $\mathbb{R}_+$ while this is impossible for two probability densities. The aim of our method is thus to find G as close to F as possible, satisfying $G(x) \geq F(x)$ (or $G(x) \leq F(x)$) for all $x \geq 0$.

This property is connected to a notion called stochastic ordering. We will say that the positive random variable X is less than Y *in the usual stochastic order*, abbreviated i.u.s.o., if

$$1 - F(x) = P[X \geq x] \leq P[Y \geq x] = 1 - G(x) \quad \text{for all } x \geq 0.$$

If X is less than Y i.u.s.o., then an integration by parts yields $L(t) \geq M(t)$ for all $t \geq 0$. Sadly, the converse is not always true. A counter-example is given by the densities $f(x) = \frac{1}{2}(\mathbf{1}_{(0,1)}(x) + \mathbf{1}_{(2,3)}(x))$ and $g(x) = \mathbf{1}_{(1,2)}(x)$. In this case, the CDFs are not ordered, while

$$L(t) = \frac{1 - e^{-t} + e^{-2t} - e^{-3t}}{2t} \geq \frac{e^{-t} - e^{-2t}}{t} = M(t), \quad t \geq 0.$$

In order to make sure that $G(x) \geq F(x)$ for all $x \geq 0$, we will quite logically resort to completely monotone functions. Given L, our aim is to find (or build) another probabilistic Laplace transform M, as close as possible (in some sense) to L and such that

$$t \mapsto \mathbb{L}[G(x) - F(x)](t) = \frac{M(t) - L(t)}{t}$$

is a completely monotone function. Under these conditions, the error made on the cumulative distribution functions will have a constant sign. In fact, this idea can be applied any finite number of times in order to get G as close to F as desired.

Practical implementation. We proceed in two steps.

Step 1. The first step is to find M, a *rough* proxy of L. Inspired by the results on tail behaviors (pages 234 and 235), we propose families of approximants depending on tail behaviors.

If X is light-tailed, then a relevant tool to work with is the gamma distribution. Indeed, its tail is light and it allows for any power behavior near the origin, including $f(0+) > 0$. M and G then have the form

$$M(t) = \frac{a^b}{(a+t)^b}, \quad G(x) = \gamma(b, ax)/\Gamma(b), \quad g(x) = \frac{a^b x^{b-1} e^{-ax}}{\Gamma(b)}, \quad a, b > 0,$$

where $\gamma(\cdot, \cdot)$ is the lower gamma function.

If X has heavy tails, then the choice of the Pareto distribution seems quite straightforward when $f(0+) > 0$. That is,

$$M(t) = ba^b e^{at} t^b \Gamma(-b, at), \quad G(x) = 1 - \frac{a^b}{(a+x)^b}, \quad g(x) = \frac{ba^b}{(a+x)^{b+1}}, \quad a, b > 0,$$

where $\Gamma(\cdot, \cdot)$ is the upper gamma function. In this case, $M(t) \sim ba^{-1}/t$, $t \to \infty$.

If $f(0+) = 0$ (and X is heavy-tailed), we propose the following two choices:

- If f goes slowly to 0 ($x \downarrow 0$), set

  $$M(t) = \frac{b(b-1)}{at}(1 - e^{at}(at)^b(at+b)\Gamma(-b, at)), \quad G(x) = 1 - a^{b-1}\frac{a+bx}{(a+x)^b},$$
  $$g(x) = \frac{b(b-1)a^{b-1}x}{(a+x)^{b+1}}, \quad a > 0,\ b > 1.$$

- If f goes rapidly to 0, set

  $$M(t) = \frac{2}{\Gamma(b)}(at)^{b/2} K_b(2\sqrt{at}), \quad G(x) = \frac{\Gamma(b, a/x)}{\Gamma(b)}, \quad g(x) = \frac{a^b e^{-a/x}}{\Gamma(b) x^{b+1}},$$

  with $a, b > 0$, where $K_\nu(x)$ is the modified Bessel function of the second kind with index ν (see 3.471-9 in [9] for the computation of the Laplace transform). This is a generalization of both the Lévy and the inverse chi-square laws, often referred to as the inverse gamma distribution.

The purpose of the rough proxy is to mimic as well as possible the behavior of L at 0 and/or infinity while satisfying the condition that $\mp(M - L)$ is a completely monotonic function.

Step 2. Without loss of generality, we consider $M - L > 0$. The error made with the rough proxy $N(t) = M(t) - L(t)$ is usually not satisfactory and requires improvement. The trick is thus to find an easily Laplace-inverted minorant μ of N (i.e., $\mu(x) < N(x)$ for all $x > 0$) such that $(N(t) - \mu(t))/t$ is a completely monotone function. Consequently,

$$G(x) \geq G(x) - \mathbb{L}^{-1}[\mu(t)](x) \geq F(x), \quad x \geq 0,$$

and the new approximation is better than the preceding one at any point in $\mathbb{R}_+$.

The aim of step 2 is to reduce the error of a prior approximation, hence it can be carried out several times. However, in our examples, we will show that only one iteration of step 2 may be sufficient to obtain a reasonably small error.

Because μ must satisfy $\mu(0) = \lim_{t\to 0} \mu(t) = 0$, good candidates for μ are differences of Laplace transforms of stochastically ordered distributions. Taking, for instance, gamma or $\frac{1}{2}$-stable laws yields the forms

$$\mu(t) = c\left(\frac{a^\nu}{(a+t)^\nu} - \frac{b^\nu}{(b+t)^\nu}\right), \quad c, \nu > 0, \quad a > b > 0 \tag{3-3}$$

and

$$\mu(t) = c\big(e^{-\sqrt{at}} - e^{-\sqrt{bt}}\big), \quad c > 0, \quad b > a > 0. \tag{3-4}$$

We underline that the choice of a proper μ is crucial as it will enhance the approximation in a very peculiar way. For instance, choosing (3-3) will have a considerable impact on the tail of the Laplace approximation and thus on the behavior of the new CDF near 0; however, on the contrary, μ defined in (3-4) is negligible near infinity, but not near zero, thereby having the opposite effect on the target CDF: little impact near zero, but a significant modification of the tail of the approximating distribution.

Once μ is chosen (this task usually requires a fitting tool from a quantitative software), the critical point is to check that $h(t)/t := (N(t) - \mu(t))/t$ is indeed a completely monotone function. We recall that, for any C^n function h,

$$\begin{aligned}\left(\frac{h(\cdot)}{\cdot}\right)^{(n)}(t) &= \frac{1}{t^{n+1}} \sum_{i=0}^{n} \frac{(-1)^i n!}{(n-i)!} t^{n-i} h^{(n-i)}(t) \\ &= \frac{h^{(n)}(t) - n\,(h(\cdot)/\cdot)^{(n-1)}\,(t)}{t},\end{aligned} \tag{3-5}$$

which can be proven iteratively.

The function $h(t) - th'(t)$ requires a particular focus since it is associated with the first derivative. If the functions $t^n h^{(n)}(t)$ are smooth then some patterns can be identified for n small enough. If $h(\cdot)/\cdot$ is indeed completely monotone, then, in (3-5), the relative weight of $h^{(n)}$ compared to that of $n(h(\cdot)/\cdot)^{(n-1)}$ will decrease as n increases. The idea, based on empirical results, is to test to what extent

$$d_n(t) := -t\frac{(h(\cdot)/\cdot)^{(n)}(t)}{(h(\cdot)/\cdot)^{(n-1)}(t)} = n - \frac{h^{(n)}(t)}{(h(\cdot)/\cdot)^{(n-1)}(t)} \approx n \quad \text{for all } t \geq 0.$$

We provide examples below to illustrate this matter (Figures 1 and 2).

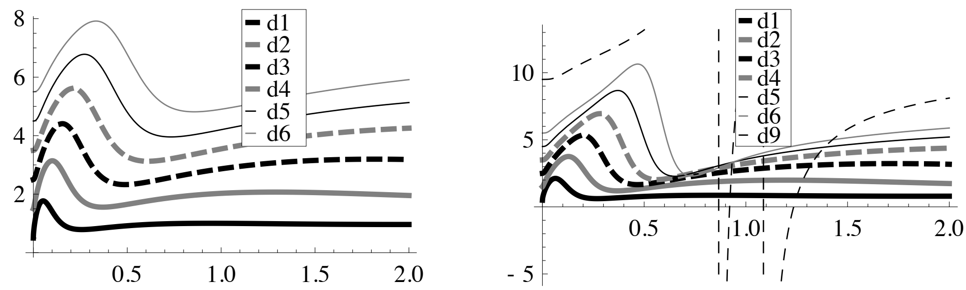

Figure 1. Graph of d_n for various n in two cases, for Example 4.1.

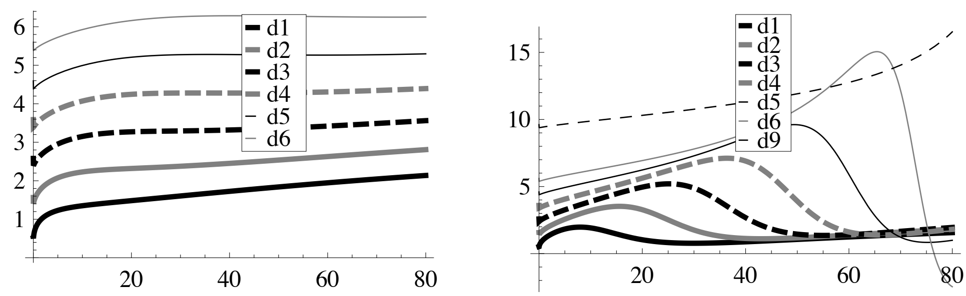

Figure 2. Graph of d_n for various n in two cases, for Example 4.2.

Among these four graphs, the two on the right fail the test: not only does d_n drift away from n, but there is a sign change at some point. The wave shape in three of the graphs is due to the fact that the function $h(t) - th'(t)$ has a local minimum away from zero. In this case, the successive derivatives may progressively (as n increases) hit zero in the vicinity of this local minimum. When there is no local minimum away from zero, our empirical tests have shown that the d_n are close to a constant or a slightly increasing affine function (as in the left graph of Figure 2).

Error results. We define $N = M - L$, $H = G - F$ and recall that $\mathbb{L}[H(x)](t) = N(t)/t$ is the Laplace transform of the error on the CDFs. The following proposition provides the Mellin transform of H (given N) and the Kantorovich distance between X and Y, which we define by

$$K(X, Y) = \sup\left\{\int_0^\infty f(x)(F(dx) - G(dx));\ f \in \mathrm{Lip}\right\}, \tag{3-6}$$

where Lip is the set of 1-Lipschitz functions. Dall'Aglio proved in [6] that, in fact,

$$K(X, Y) = \int_0^1 |F^{-1}(x) - G^{-1}(x)|\, dx = \int_0^\infty |F(x) - G(x)|\, dx$$

because the support of X and Y is $\mathbb{R}_+$.

We recall that in our setting, the functions N and H are either nonnegative or nonpositive. For simplicity, and without any loss of generality, we henceforth assume that they are nonnegative.

Proposition 3.2. *For* $0 < b < 1$, *whenever these integrals make sense*,

$$\int_0^\infty \frac{N(t)}{t^{1+b}} dt = \Gamma(1-b) \int_0^\infty x^{b-1} H(x)\, dx = \frac{\Gamma(1-b)}{b} \int_0^\infty H(x^{1/b})\, dx$$

Moreover,

$$\lim_{t \downarrow 0} N(t)/t = \int_0^\infty H(x)\, dx = K(X, Y) \tag{3-7}$$

Proof. The first equality is simply Fubini's theorem combined with the identity $\int_0^\infty e^{-xt} t^{-b} dt = \Gamma(1-b) x^{b-1}$ and a standard change of variable; the second equality is obvious. □

In some cases, it is possible to obtain an upper bound for the L^p quasi-norm of H for $p \in (0, 1)$, using Jensen's (reversed) inequality.

Lastly, we would like to recall the link between the Kantorovich distance and the Kolmogorov–Smirnov (uniform) distance $\sup_{x \geq 0} |F(x) - G(x)|$. Intercalating the Lévy and Prohorov metrics (using the results from [11, pp. 35–36] and [21, p. 43]), we get

$$\sup_{x \geq 0} |F(x) - G(x)| \leq \left((1+c) \int_0^\infty |F(x) - G(x)|\, dx \right)^{1/2}$$

where c is the maximum value (over $\mathbb{R}_+$) of $f = F'$, the density of X.

4. Examples

We test our method on two heavy-tailed distributions for which a rather simple closed form for f or F is available. The driving criterion for our approximations will be to get a finite Kantorovich distance.

Example 4.1. ***A generalized Mittag-Leffler distribution.*** We follow the notations of [13]. Generalized Mittag-Leffler distributions are a two-parameter family of laws with Laplace transforms

$$L(t) = (1 + t^\alpha)^{-\beta}, \quad \beta > 0,\ 0 < \alpha \leq 1$$

and cumulative distribution function

$$F(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\beta+k) x^{\alpha(\beta+k)}}{k!\,\Gamma(\beta)\Gamma(1+\alpha(\beta+k))}$$

We will focus on the simple case $\alpha = 1/2$, and $\beta = 2$. First notice that since $\frac{\Gamma(2k+2)}{\Gamma(k+2)(2k)!} = \frac{1}{k!}(2-1/(k+1))$,

$$\sum_{k=0}^{\infty} \frac{\Gamma(2+2k) x^{(2+2k)/2}}{(2k)!\,\Gamma(1+(2+2k)/2)} = e^x(2x-1)+1$$

Next, the odd integers are dealt with using the infinite series representation of the error function (8.253-1 in [9])

$$e^x \operatorname{erf}(\sqrt{x}) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{2^k x^{k+1/2}}{(2k+1)!!}$$

where $(2k+1)!! = 1 \cdot 3 \cdot 5 \ldots (2k{+}1)$, and the identity

$$\frac{\Gamma(2k+3)}{(2k+1)!\,\Gamma(k+5/2)} = (2k+2)\frac{2^{k+2}}{\sqrt{\pi}(2k+3)!!}$$

which yields in the end

$$F(x) = e^x(2x-1)\operatorname{erfc}(\sqrt{x}) - 2\sqrt{x/\pi} + 1$$

From $L(t) = (1+\sqrt{t})^{-2}$, we know that $f(0+) = 1$ and that f has a heavy tail. We thus choose the Pareto family with $a = n$ in order to have the proper asymptotic behavior for L (when $t \to \infty$). In fact, the domination condition imposes $a, n \geq 1/2$ and a few tests show that $a = n = 1/2$ is a relevant choice, yielding

$$M(t) = \frac{\sqrt{t}e^{t/2}\Gamma(-1/2, t/2)}{2\sqrt{2}}, \quad t \geq 0$$

As expected, the approximation is not satisfactory and we must resort to an appropriate μ.

We wish to stress the importance of the choice of μ and we will test the performance of two functions, namely μ_1 and μ_2. The first naive choice was to take μ of the form

$$\mu_1(t) = c\left(\frac{a^{3/2}}{(a+t)^{3/2}} - \frac{b^{3/2}}{(b+t)^{3/2}}\right)$$

and an admissible set of parameters was $a = 3$, $b = 0.05$ and $c = 0.135$. This triple was the result of a fitting algorithm from a quantitative software.

Unfortunately, this approximation does not allow to compute the Kantorovich error because it is not good enough near 0. In order to be able to compute (3-7), we recall the expansion of L at zero (derived from that of $(1+t)^{-2}$):

$$(1+\sqrt{t})^{-2} = 1 - 2\sqrt{t} + 3t - 4t^{3/2} + O(t^2), \quad t \downarrow 0$$

Therefore, a strong improvement of the approximation should satisfy

$$M(t) - \mu_2(t) \sim 1 - 2\sqrt{t} + O(t), \quad t \downarrow 0.$$

By [20, 45:5:2] combined with [1, 6.5.17 and 7.1.5], we have

$$\frac{\sqrt{t}e^{t/2}\Gamma(-1/2, t/2)}{2\sqrt{2}} = 1 - \sqrt{\frac{\pi t e^t}{2}}\big(1 - \operatorname{erf}(\sqrt{t/2})\big) = 1 - \sqrt{\pi t/2} + t + O(t^{3/2})$$

as $t \downarrow 0$. Moreover,

$$e^{-\sqrt{at}} = 1 - \sqrt{at} + \tfrac{1}{2}at + O(t^{3/2}), \quad t \downarrow 0;$$

hence we propose $\mu_2(t) = c(e^{-\sqrt{at}} - e^{-\sqrt{bt}})$ with a, b, c satisfying $c(\sqrt{a} - \sqrt{b}) = -2 + \sqrt{\pi/2}$. The triple $a = 0.777$, $b = 20$ and $c = 0.206$ yields promising results with a Kantorovich distance of approximately 0.02 (computed via (3-7)).

Of course, in both cases, we have checked, using the d_n for $n \in \{1, \ldots, 9\}$, that the error h was such that $h(t)/t$ was a completely monotone function.

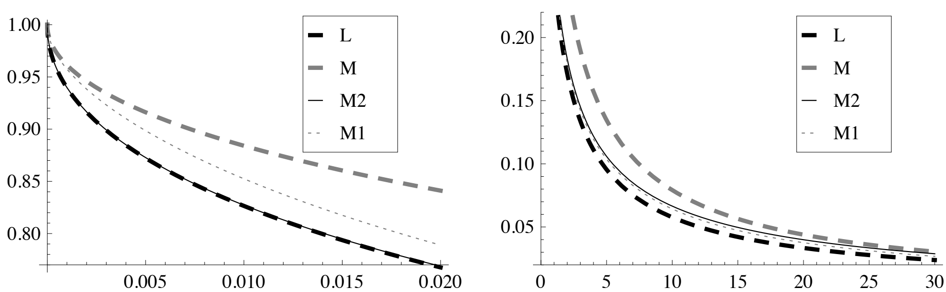

Figure 3. Graph of L and its proxies for $t \in (0, 0.02)$ and $t \in [0.02, 30]$.

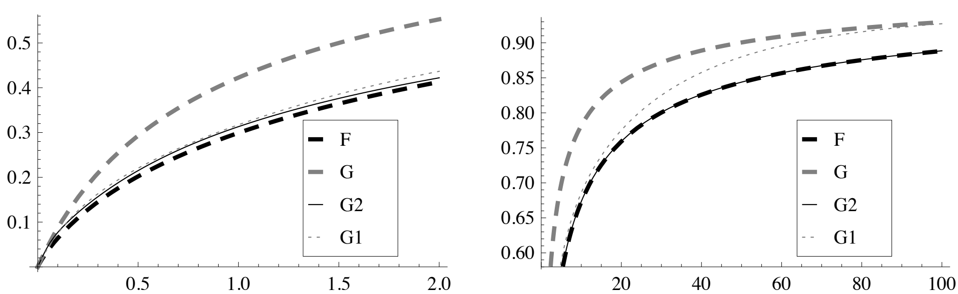

Figure 4. Graph of F and its proxies for $x \in (0, 2)$ and $x \in (2, 100)$.

We provide the graphical results below (Figures 3 and 4). M is the Laplace transform of the Pareto distribution with $a = n = 1/2$, $M_1(t) = M(t) - \mu_1(t)$ and $M_2(t) = M(t) - \mu_2(t)$. Their CDF counterparts are G, G_1 and G_2. It is plain on the graphs that M_1 and M_2 are quite close, except near zero; this explains why only G_2 is a good fit for F for x large (as expected).

Example 4.2. ***A positive stable distribution.*** Our next example is the one-parameter one-sided stable laws with Laplace transform

$$L(t) = e^{-t^{\alpha}}, \quad \alpha \in (0, 1)$$

The case $\alpha = \frac{1}{2}$ is sometimes referred to as the Lévy distribution, which is connected with the first passage time of the Brownian motion over fixed levels. The case $\alpha = \frac{1}{3}$ also has a closed-form density (see B.25 in [17] for instance):

$$f(x) = \frac{K_{1/3}\left(\sqrt{4/(27x)}\right)}{3\pi x^{3/2}}, \quad x \geq 0$$

We will thus aim at approximating $L(t) = e^{-t^{1/3}}$. In this case, $f(0+) = 0$ and f has a fat tail. Moreover,

$$L(t) = 1 - t^{1/3} + \tfrac{1}{2}t^{2/3} - \tfrac{1}{6}t + O(t^{4/3}), \quad t \downarrow 0 \tag{4-1}$$

The choice of the inverse gamma family with $n = 1/3$ seems relevant, as it satisfies (see 51:6:1 in [20])

$$M(t) = \frac{2}{\Gamma\left(\frac{1}{3}\right)}(at)^{1/6}K_{1/3}(2\sqrt{at}) = 1 + \frac{a^{1/3}\Gamma\left(-\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}t^{1/3} + 3at/2 + O(t^{4/3}), \quad t \downarrow 0.$$

Hence, for $a = -\Gamma\left(\frac{1}{3}\right)^3/\Gamma\left(-\frac{1}{3}\right)^3$, the $t^{1/3}$ term in the error will vanish, by (4-1), but the $t^{2/3}$ term will remain. This leads to the following choice of μ:

$$\begin{aligned}\mu(t) &= 2c\left(\frac{(bt)^{1/3}K_{2/3}(2\sqrt{bt})}{\Gamma\left(\frac{2}{3}\right)} - \frac{(dt)^{1/3}K_{2/3}(2\sqrt{dt})}{\Gamma\left(\frac{2}{3}\right)}\right) \\ &= c\frac{\Gamma\left(-\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}(b^{2/3} - d^{2/3})t^{2/3} + 3c(b-d)t + O(t^{5/3}), \quad t \downarrow 0.\end{aligned}$$

Notice that this time, the ordering is in the opposite way: $L(t) \geq M(t)$ for all $t \geq 0$. In this setting, an admissible set of parameters is $c = 6$, $b = 0.4$ and $d = 0.43$, which yields a Kantorovich distance of less than 0.06 (see Figures 5 and 6 on the next page, where $M_1(t) = M(t) + \mu(t)$).

Of course, in both examples, it is possible to further reduce the error by repeating step 2 on page 237 at least one time (using a minorant μ^* of $M - \mu - L$ for instance).

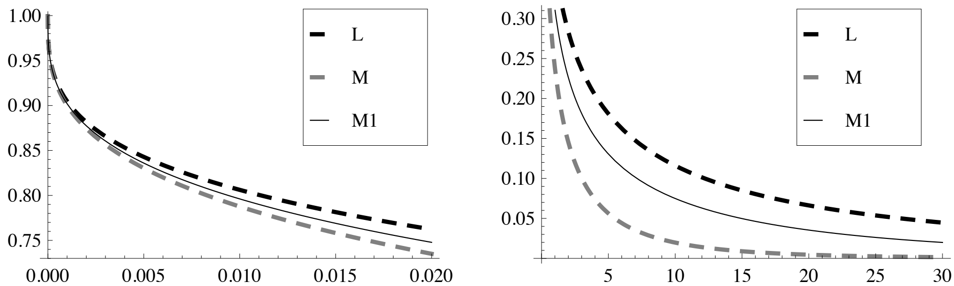

Figure 5. Graph of L and its proxies for $t \in (0, 0.02)$ and $t \in [0.02, 30]$.

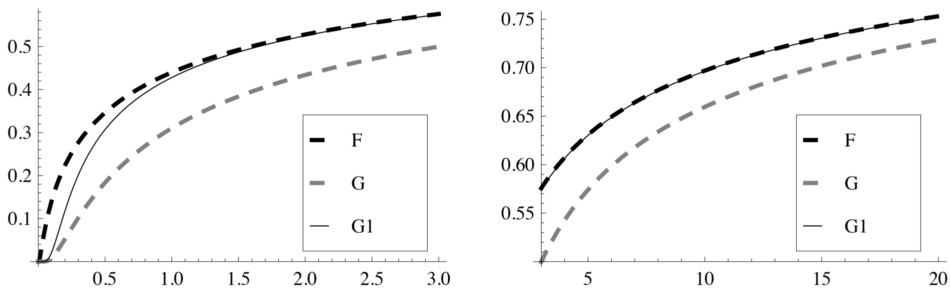

Figure 6. Graph of F and its proxies for $x \in (0, 2)$ and $x \in (2, 100)$.

Remarks. We did not study distributions with lighter tails in the examples because when $1 - F(x) \leq cx^{-\alpha}$, with $c > 0$ and $\alpha > 1$ for any large x, then it is much easier to obtain a finite Kantorovich error, as the original survival function is already integrable.

Using the exact same procedure as in the second example, it would thus take $n - 2$ iterations of step 2 to obtain an approximation with finite Kantorovich error for the stable law with Laplace transform equal to $e^{-t^{1/n}}$. The same holds for generalized Mittag-Leffler distributions defined by $L(t) = (1 + t^{1/n})^{-p}$, for any real $p \geq n$. These assertions are a consequence of the Taylor expansion of L at 0. In the stable case, when $\alpha \in \left(\frac{1}{2}, 1\right)$, it is possible to obtain a finite Kantorovich measure by taking $M(t) = e^{-\sqrt{t}}$ and μ such that $\mu(t) \sim t^{\alpha} - \sqrt{t}$ when $t \downarrow 0$.

Finally, we stress that even though we have assumed (for simplicity) that the law of X was absolutely continuous, our method remains valid for most positive laws. It is indeed possible to make do without densities throughout the whole process, especially if the law of X has a finite number of atoms and an absolutely continuous part. However, it is not clear whether this method can perform well for some rather unusual distributions, such as those which possess an infinite number of atoms.

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GUILLAUME COQUERET: guillaume.coqueret@essec.edu
ESSEC Business School / Université de Lille-1, Avenue Bernard Hirsch, 95000 Cergy-Pontoise, France

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