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Multiplicity results for semipositone two-point boundary value problems

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In this paper we address two-point boundary value problems of the form

$$u'' + f(u) = 0, \qquad \text{in } (0, 1),\ u(0) = u(1) = 0,$$

where the function f resembles $f(u) = \lambda(\exp(au/(a+u)) - c)$ for some constants $c \geq 0,\ \lambda > 0,\ a > 4$. We prove the existence of positive solutions for the semipositone case where $f(0) < 0$, and further prove multiplicity under certain conditions. In particular we extend theorems from Henderson and Thompson to the semipositone case.

1. Introduction

In this paper we address two-point boundary value problems of the form

$$u'' + f(u) = 0, \quad \text{in } (0, 1), \qquad u(0) = u(1) = 0, \tag{1}$$

where the function f resembles $f(u) = \lambda(\exp(au/(a+u)) - c)$ for some constants $c \geq 0,\ \lambda > 0,$ and $a > 4$. Boundary value problems of this sort are motivated by a variety of applications, such as nonlinear heat generation and combustion [Brown et al. 1981], and have been studied extensively since the early work of authors such as Keller and Cohen [1967]. These references deal exclusively with the *positone* case, the case where f is *posi*tive and mono*tone*.

In this paper we are interested in finding multiple positive solutions for the semipositone case where $f(0) < 0$. In particular we extend theorems from [Henderson and Thompson 2000] to the semipositone case. Our results complement those in [Brown et al. 1981; Castro and Shivaji 1998], and many related papers that discuss S-shaped bifurcation curves for positone and semipositone problems. Related PDE results can be found in [Drábek and Robinson 2006; Robinson and Rudd 2006]. Drábek and Robinson [2006] generalizes the main theorem in [Henderson and Thompson 2000] to the PDE case over arbitrary smooth bounded domains.

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[Robinson and Rudd 2006] generalizes our ODE results to the analogous PDE problem on the unit ball.

Our proofs characterize solutions as critical points of the functional

$$J(u)=\frac{1}{2}\int_0^1 \left(u'\right)^2-\int_0^1 F(u), \quad u\in H_0^1(0,1),$$

where $F(u):=\int_0^u f$. Using step functions as a simple model for f we produce lower solutions, $\{\underline{u}_1,\underline{u}_2\}$, and upper solutions, $\{\overline{u}_1,\overline{u}_2\}$, with the ordering

$$\underline{u}_1\le\overline{u}_1\le\underline{u}_2\le\overline{u}_2.$$

Standard arguments show that J has a local minimum in each of the generalized intervals $\left[\underline{u}_1,\overline{u}_1\right]$ and $\left[\underline{u}_2,\overline{u}_2\right]$. The third solution is characterized as a saddle point lying between the two minima. Our theorems show that one of the minima is positive and the other is negative, and, under certain conditions, the saddle point solution is also positive. We provide two separate criteria that guarantee a second positive solution.

The theorems in [Brown et al. 1981] and [Henderson and Thompson 2000] are representative of two different approaches to very similar problems, so it is of some interest to provide an explicit comparison of these theorems. In Section 6 we provide such a comparison for the positone PDE case. In particular, we show that the conditions in [Drábek and Robinson 2006], where the main theorem of [Henderson and Thompson 2000] is generalized to the PDE case, are more general than those in [Brown et al. 1981].

2. Preliminaries

The expression $\underline{u}\in C^2(0,1)\bigcap C\,[0,1]$ is called a lower solution of Equation (1) if

$$\underline{u}''+f(\underline{u})\ge 0,\quad \underline{u}(0)\le 0,\quad \underline{u}(1)\le 0.$$

Upper solutions are defined similarly with reversed inequalities.

Since f is a bounded continuous function it is straightforward to show the J is a C^1 functional that satisfies the Palais–Smale condition, and that the following minimization and mountain pass lemmas are true [Struwe 1990].

Lemma 2.1. *Suppose that $\underline{u}$ and $\overline{u}$ are lower and upper solutions of Equation* (1), *respectively*, *and suppose that $\underline{u}\le\overline{u}$ on* [0, 1]. *Then J achieves a local minimum at some critical point $u\in\left[\underline{u},\overline{u}\right]:=\{u\in H_0^1(0,1):\underline{u}\le u\le\overline{u}\}$.*

See [Struwe 1990, Theorem 2.4] for an elegant proof.

Lemma 2.2. *Suppose that $\underline{u}$ and $\overline{u}$ are lower and upper solutions of* (1), *respectively*, *and suppose that u_1, u_2 are distinct local minima of J in $\left[\underline{u},\overline{u}\right]$. Then there* *is a third critical point of* J, $u_3 \in [\underline{u}, \overline{u}]$, *which satisfies*

$$J(u_3) = c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], [\underline{u}, \overline{u}]) : \gamma(0) = u_1, \gamma(1) = u_2\}$.

Note that our solutions must be symmetric about $x = 1/2$, and it will often be convenient to look at the problem over the interval $[0, 1/2]$ with the condition $u'(1/2) = 0$.

With the given construction of lower and upper solutions it is also possible to construct proofs using degree theory. The connection between upper and lower solutions and degree theory is developed nicely in [Amann 1976; Shivaji 1987], and is used in [Brown et al. 1981; Castro and Shivaji 1998; Drábek and Robinson 2006; Robinson and Rudd 2006], and many related papers.

3. The ideal case

In this section we perform a detailed analysis of an important special case. We study Equation (1) assuming that f is a step function with description

$$f(u) := \begin{cases} k, & u < 1, \\ K, & u \geq 1. \end{cases} \tag{2}$$

We will identify a region of the (k, K) plane where this ideal problem has three solutions. For points in the region where $k > 0$ all three solutions are positive. For points in the region where $k < 0$ it will always be the case that one solution is positive, one solution is negative, and the third solution is either sign-changing or positive. We will characterize the subregion where two of the solutions are positive and one is negative.

The solution of the ideal problem can be broken into two pieces corresponding to the subintervals where $u < 1$ and $u \geq 1$. Let $u = u_1$ on $\{x : u(x) < 1\}$, so $u_1 = -kx^2/2 + ax + b$, where we choose $b = 0$ in order to satisfy $u(0) = 0$. Let $u = u_2$ on $\{x : u(x) \geq 1\}$, so $u_2 = -Kx^2/2 + cx + d$, where $c = K/2$ in order to satisfy $u'(1/2) = 0$.

A solution whose maximum does not exceed 1 will satisfy

$$u \equiv u_1 = -\frac{k}{2}x^2 + \frac{k}{2}x,$$

where we have chosen $a = k/2$ in order to guarantee $u_1'(1/2) = 0$. If $k > 0$, then u is positive with $1 \geq \max u = k/8$. Of course, if $k \leq 0$, then u is nonpositive. Hence, a solution with $\max u \leq 1$ exists if and only if $k \leq 8$.

It remains to discover solutions whose maximum exceeds 1. This necessitates $K > 0$, else the solution would never have an interior maximum above 1. In order to explicitly construct these solutions we must satisfy continuity conditions by finding an $x_0 \in (0, 1/2)$ such that $u_1(x_0) = 1 = u_2(x_0)$. We must also satisfy a smoothness condition $u_1'(x_0) = u_2'(x_0)$. The smoothness condition can be used to solve for a, which can then be substituted into the first continuity condition to get

$$\left(\frac{k}{2} - K\right)x_0^2 + \frac{K}{2}x_0 - 1 = 0.$$

Basic curve sketching techniques from calculus show that this equation has exactly one root $x_0 \in [0, 1/2]$ when (k, K) is on the upper branch of the parabola

$$K^2 - 16K + 8k = 0,$$

the graph of $K = 8 + 2\sqrt{16 - 2k}$. We will refer to this curve as Γ_1. When (k, K) lies above Γ_1 then we get two roots. Once (k, K) has been chosen we can easily use the second continuity condition to solve for d. The two solutions thus obtained are either both positive or one is positive and one is sign-changing. Distinguishing between the latter two possibilities reduces to determining when the initial slope of the solution is nonnegative. This can be done for a particular (k, K) by using the conditions above to solve for $a = u'(0)$. To discover the condition that separates the sign-changing case from the positive case, we set $a = 0$ and solve. This curve, call it Γ_2, is described by

$$K = \frac{(8 + 2\sqrt{-2k})k}{k + 8}, \quad -\infty < k < -8.$$

It is straightforward to show that Γ_2 lies above Γ_1, and that the two curves are asymptotic as $k \to -\infty$. If a pair (k, K) lies on Γ_1, then the ideal problem has exactly one positive solution. If the pair lies above Γ_1 and below Γ_2, then the problem has two positive solutions. If the pair lies above Γ_2, then the problem has one positive solution and one sign-changing solution.

4. A three solutions theorem

In this section we see that the ideal case generalizes in a straightforward way.

Theorem 4.1. *Let (k, K) be a point on the curve Γ_1, let $0 < b$, and suppose that $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function such that*

(a) $f(0) < 0$,

(b) $kb \le f(u)$, *for* $u < b$,

(c) $f(u) \ge Kb$, *for* $b \le u \le Mb$,

where M is the maximum of the solution to Equation (1) *assuming the ideal conditions* (2). *Then the boundary value problem* (1) *has at least three symmetric solutions, one of which is positive and one of which is negative.*

Proof. We use symmetry to reduce the argument to the half interval [0, 1/2], with the boundary conditions $u(0) = 0$ and $u'(1/2) = 0$. As a further simplification we rescale the problem so that, without loss of generality, $b = 1$. Simply let $v = u/b$ and note that $v'' + f(bv)/b = 0$, and that $\overline{f}(\cdot) := f(b\,\cdot)/b$ satisfies $\overline{f}(v) \geq k$ for $v \geq 1$, etc.

It is easy to check that $\overline{u}_1 \equiv 0$ and $\overline{u}_2 = -Cx(1-x)/2$ are upper solutions, where C is chosen so that $f(u) < C$ for all u. It is also easy to check that $\underline{u}_1 = -kx(1-x)/2$ is a lower solution. Now consider the positive function $\underline{u}_2 = \psi$, where ψ is the solution of

$$\psi'' = \begin{cases} -k, & \psi < 1, \\ -K, & \psi \geq 1, \end{cases} \qquad \psi(0) = 0, \quad \psi'(1/2) = 0,$$

as described in Section 3. Let $M := \max_{[0,1/2]} \psi = \psi(1/2)$. It follows that $f(\underline{u}_2) \geq K$ at points where $\psi \geq 1$, where $1 \leq \psi \leq M$, and that $f(\psi) \geq k$ where $\psi < 1$, where $0 \leq \psi < 1$. Hence $\underline{u}_2$ is a positive lower solution.

Theorem 4.1c implies $C > K$, so we have $\overline{u}_2'' < \underline{u}_2''$, $\overline{u}_2(0) = \underline{u}_2(0) = 0$, and $\overline{u}_2'(1/2) = \underline{u}_2'(1/2) = 0$. A simple comparison implies that $\underline{u}_2 \leq \overline{u}_2$. Other comparisons are easy, and lead to $\underline{u}_1 \leq \overline{u}_1 \leq \underline{u}_2 \leq \overline{u}_2$ in [0, 1/2].

Applying the variational methods described in Section 2 we infer the existence of three solutions. The solution lying in the generalized interval $\left[\underline{u}_1, \overline{u}_1\right]$ is clearly negative, and the solution lying in the generalized interval $\left[\underline{u}_2, \overline{u}_2\right]$ is clearly positive. The third solution, the saddle point solution, cannot be easily described without further conditions on f. □

5. Criteria for two positive solutions

In this section we state criteria that guarantee two positive solutions. Since our interest is in positive solutions we assume throughout this section, without loss of generality, that $f(u) = f(0)$ for $u \leq 0$. This introduces the convenience that Equation (1) has a unique nonpositive solution satisfying $u'' + f(0) = 0$, so every other solution must be either sign-changing or positive.

In the ideal problem we get two positive solutions when $k \geq -8$. The analogous result follows for the more general problem.

Theorem 5.1. *Let (k, K) be a point on the curve Γ_1, let $0 < a < b$, and suppose that $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function such that*

(a) $f(0) < 0$,

(b) $kb \leq f(u)$, *for* $u \leq b$,

(c) $f(u) \geq Kb$, *for* $b \leq u \leq Mb$,

(d) $-8a \leq f(u) \leq 0$ *for* $0 \leq u \leq a$, *where M is the maximum of the solution to* (1) *assuming the ideal conditions* (2). *Then* (1) *has two positive solutions.*

Proof. Without loss of generality, rescale the problem so that $a = 1$. Apply Theorem 4.1 to get three solutions, one of which is negative, one of which is positive, and one of which is not yet well described. Let u be this third solution and observe, as above, that u must have positive values somewhere on its domain. By Theorem 5.1d, u cannot achieve a positive maximum below height $a = 1$. In fact, if x_1 is the first point where $u(x) = 1$, then $u'' \geq 0$ on $[0, x_1]$ implies that $u'(x_1) > 0$, so u must achieve a maximum strictly above 1.

Suppose that $u'(0) \leq 0$, and compare u to $v = 4x^2$. We know that $u(0) = 0 = v(0)$, $u'(0) \leq 0 = v'(0)$, and $u'' \leq 8 = v''$ on $\{x : u(x) < 1\}$. It follows that $u \leq v$ at least until the first point where $u = 1$, which cannot happen until after v reaches 1. But $v \leq 1$ on $[0, 1/2]$, so u cannot achieve a maximum greater than 1. This is a contradiction, so it must be that $u'(0) > 0$. A similar comparison leads to a contradiction if $u(x) = 0$ and $u'(x) \leq 0$ at any other point $x \in (0, 1/2)$. Thus u must be positive. □

It is interesting to note that for the analogous PDE problem on the unit ball, there is no theorem similar to Theorem 5.1. In fact, for any $k < 0$, it is possible to construct a sign-changing third solution for the ideal case [Robinson and Rudd 2006].

For $k < -8$ we have seen that the ideal problem has two positive solutions for (k, K) in the region above Γ_1 and below Γ_2. One might conjecture, and at one time these authors did, that the general problem will have two positive solutions if the K in Theorem 4.1 satisfies $K \leq (8 + 2\sqrt{-2k})k/(k + 8)$. It turns out that an explicit counterexample can be constructed, as we shall soon demonstrate. However, the next theorem shows that an alternative upper bound on K does guarantee the existence of two positive solutions.

Theorem 5.2. *Let (k, K) be a point on the curve Γ_1 with $k < -8$, let $0 < a < b$, and suppose that $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function such that*

(a) $ka \leq f(u) < 0$ *for* $0 \leq u \leq a$,

(b) $kb \leq f(u)$, *for* $u \leq b$,

(c) $f(u) \geq Kb$, *for* $b \leq u \leq Mb$,

(d) $f(u) < 16ka/(k + 8)$ *for all* u,

where M is the maximum of the solution to Equation (1) *assuming the ideal conditions of* (2). *Then* (1) *has two positive solutions.*

Proof. Rescale the problem so that, without loss of generality, $a = 1$. Let u represent the third solution as in the previous proof. Once again we use a comparison argument to show that assuming $u'(0) \leq 0$ leads to a contradiction. The case where we assume $u(x) = 0$ and $u'(x) \leq 0$ leads to a similar contradiction.

Let $k < -8$ be fixed, and consider a family of comparison functions indexed by $t \in \left[t_0, \sqrt{-2/k}\right]$,

$$v_t := \begin{cases} -\dfrac{k}{2}x^2, & x \in [0,t]\,, \\ -kt(x-t) - \dfrac{k}{2}t^2, & x \in [t, x_t]\,, \\ -\dfrac{K_t}{2}x^2, & x \in \left[x_t, \dfrac{1}{2}\right], \end{cases}$$

where

$$t_0 := \frac{1}{2} + \frac{1}{2k}\sqrt{k^2+8k}, \qquad x_t := \frac{kt^2-2}{2kt}, \qquad K_t = \frac{2k^2t^2}{kt^2-kt-2}.$$

This function can be visualized in three pieces: the first is a parabola emerging from the origin with 0 slope, the second is a tangent line to the parabola at the point $(t, -kt^2/2)$, and the third is a parabolic cap that meets the tangent line smoothly at $(x_t, 1)$ and then reaches a critical point at $x = 1/2$. t_0 describes the t value such that $x_t = 1/2$, and thus describes the infimum of the t values such that this comparison function makes sense, and, not coincidentally, identifies a vertical asymptote for K_t. Computing two derivatives with respect to x, except at $x = t$ and $x = x_t$, we see that

$$v_t'' := \begin{cases} -k, & x \in [0,t)\ , \\ 0, & x \in (t, x_t), \\ -K_t, & x \in (x_t, \frac{1}{2}]. \end{cases}$$

Recall, as in the previous proof, that u must reach a positive maximum above 1 at some point. How do u and v_t compare? If $u'(0) \leq 0$, then $u \leq -kx^2/2$ on $\left[0, \sqrt{-2/k}\right]$, because $u'' \leq -k$ while $u < 1$, and $-kx^2/2$ reaches height 1 at $\sqrt{-2/k}$. It is clearly possible to adjust the choice of t so that x_t represents the first point where $u(x) = 1$. Since $u'' \geq 0$ on $[0, x_t]$, $u(t) \leq v_t(t)$, and $u(x_t) = v_t(x_t)$ it follows that $u \leq v_t$ on $[0, x_t]$ and that $u'(x_t) \geq v_t'(x_t) > 0$. By the mean value theorem, there is an $x \in (x_t, 1/2)$ such that

$$-u''(x) = -\frac{u'\left(\frac{1}{2}\right) - u'(x_t)}{\frac{1}{2} - x_t} = \frac{u'(x_t)}{\frac{1}{2} - x_t} \geq \frac{v'(x_t)}{\frac{1}{2} - x_t} = K_t.$$

Elementary calculus reveals that K_t achieves a minimum of

$$K_t = \frac{16k}{k+8}, \qquad \text{at } t = -\frac{4}{k}.$$

Thus

$$-u''(x) \geq 16k/(k+8) > f(u(x)),$$

and so a contradiction has been reached. □

It is important to note that $16k/(k+8) < (8+2\sqrt{-2k})k/(k+8)$ for $k < -8$, and so the comparison functions, v_t, satisfy the conditions of Theorem 4.1 and the additional restriction $-v_t'' < (8+2\sqrt{-2k})k/(k+8)$. Since the inequality is strict we can slightly modify v_t so that it has negative slope at 0, and is thus sign-changing, but still satisfies conditions Theorem 5.2a–c, as well as the given estimate on its second derivative. This provides the counterexample to the conjecture, expressed above, that Γ_2 provides a boundary guaranteeing two positive solutions for the general case.

Finally, if f is to satisfy the conditions of Theorem 5.2, and if C represents the upper bound for f, then we must have $8+2\sqrt{16-2k} \leq C < 16k/(k+8)$. A careful comparison of expressions on the left and right of this inequality shows that their graphs cross in the (k, K) plane at the point $(-24, 24)$. Thus Theorem 5.2 is only applicable for $-24 < k < -8$. It seems reasonable to conjecture that finer estimates and comparison arguments will discover criteria for two positive solutions when $k < -24$.

6. A comparison of solvability conditions

The methods and results in [Brown et al. 1981] and [Henderson and Thompson 2000] represent two different, and complimentary, approaches to similar problems. The more obvious differences are that [Henderson and Thompson 2000] does not impose the same monotonicity and smoothness conditions used in [Brown et al. 1981], and is, in that sense, more general. On the other hand the results in [Brown et al. 1981] deal with both the ODE and PDE cases, and take good advantage of the more restrictive conditions to prove more precise results, especially in the ODE case.

The relationship between the solvability conditions in the two papers is not as obvious. In this section we explore that relationship. In particular we prove that if the conditions in [Brown et al. 1981] are satisfied, then so are the conditions in [Henderson and Thompson 2000]. In order to demonstrate this in some generality we consider the PDE case,

$$\Delta u + \lambda f(u) = 0 \text{ in } D, \qquad u|_{\partial D} = 0, \tag{3}$$

where $D \subset \mathbb{R}^n$ is a smooth bounded domain and $f : [0, \infty) \to [0, \infty)$ is continuous. [Henderson and Thompson 2000] presented purely ODE results, but their work is generalized in [Drábek and Robinson 2006], so we will actually compare the conditions in [Brown et al. 1981] and [Drábek and Robinson 2006].

It is helpful to begin by defining several constants. First, let $m := \max_D \phi$, where ϕ is the unique positive solution of

$$\Delta\phi + 1 = 0 \text{ in } D, \qquad \phi|_{\partial D} = 0.$$

Second, consider the problem

$$\Delta\psi + Kh(\psi) = 0 \text{ in } D, \qquad \phi|_{\partial D} = 0, \tag{4}$$

where $h(u) \equiv 0$ when $u < 1$ and $h(u) \equiv 1$ when $u \geq 1$. It is proved in [Drábek and Robinson 2006] that there is a minimal positive K such that Equation (4) has a positive solution, and we assume throughout the arguments below that K is this minimal constant. Let $M := \max_D \psi$.

Drábek and Robinson [2006] proved that (3) has three nonnegative solutions if

(a) $\lambda f(u) < ka$ on $[0, a]$,

(b) $\lambda f(u) \geq Kb$ on $[b, Mb]$,

(c) $\lambda f(u) \leq kc$ on $[0, c]$,

where $0 < a < b < Mb < c$ and $k := 1/m$.

Before stating the solvability conditions in [Brown et al. 1981] we describe yet another constant. Consider a subdomain $\Omega \subset\subset D$ and consider

$$\Delta\eta + \chi_\Omega = 0 \text{ in } D, \qquad \eta|_{\partial D} = 0.$$

Define $M_2 := [\inf_\Omega \eta]^{-1}$. Observe that $\upsilon = M_2\eta$ satisfies

$$\Delta\upsilon + M_2\chi_\Omega = 0 \text{ in } D, \qquad \eta|_{\partial D} = 0,$$

with $\upsilon \geq 1$ on Ω. In particular we have $\Delta\upsilon + M_2 h(\upsilon) \geq 0$, so υ is a positive lower solution for (4). Combining this with a simple constant upper solution we can show that (4) has a positive solution when M_2 is substituted for K. Since K is the minimal constant with this property we see that $K \leq M_2$. For a more detailed discussion of K and its properties see [Drábek and Robinson 2006].

Brown et al. [1981] proved that (3) has three nonnegative solutions if f is a smooth and bounded function, which is increasing on $\left[0, c'\right]$, and which satisfies

$$M_2\left(\frac{l_2}{f(l_2)}\right) \leq \lambda \leq \min\left\{M_1\left(\frac{l_1}{f(l_1)}\right), M_3\left(\frac{c'}{f(l_1)}\right)\right\}, \tag{5}$$

where $0 < l_1 < l_2 < c'$.

In order to compare solvability conditions it remains to do a careful reading of the proof in [Brown et al. 1981] to see how the constants are chosen and how they compare to those in [Drábek and Robinson 2006]. First, it turns out that

$M_1 = M_3 = 1/m$. Hence, the inequality

$$\lambda \leq M_1 \left(\frac{l_1}{f(l_1)} \right),$$

implies that $\lambda f(a) \leq ka$ if we substitute $l_1 = a$ and $k = 1/m$. Moreover, the monotonicity assumption on f leads to $\lambda f(u) \leq ka$ for u in $[0, a]$. The inequality

$$M_2 \left(\frac{l_2}{f(l_2)} \right) \leq \lambda,$$

leads us to $Kb \leq \lambda f(b)$, where we have substituted $l_2 = b$ and $M_2 \geq K$. Once again monotonicity implies that $Kb \leq \lambda f(u)$, for $u \in \left[b, c'\right]$. Substituting $M_3 = 1/m$, $b = l_2$, and $c = c'$ into the inequality

$$M_2 \left(\frac{l_2}{f(l_2)} \right) \leq M_3 \left(\frac{c'}{f(l_2)} \right),$$

gives $mM_2 b \leq c$, and thus $c \geq mKb$. By the definition of m we know that $K\phi(x) \leq mK$ for all $x \in D$. Also, $\Delta(K\phi) = -K \leq -Kh(\psi) = \Delta\psi$ in D, with strict inequality over the set $D \setminus \Omega$, so the maximum principle implies that $K\phi(x) > \psi(x)$ in D. Hence $Km > M$, and so $c > Mb$.

So far we have used Equation (5) to verify conditions (a) and (b) on page 131 for Equation (3) having nonnegative solutions, with the modest exception of obtaining a strict inequality for condition (a). The purpose of the strict inequality in [Drábek and Robinson 2006] is to guarantee that the intermediate lower solution is strict, which helps in distinguishing the three different solutions. This hair can easily be split by allowing equality and then using the monotonicity condition on f to recover. Finally, condition (c) follows easily from the fact that f is bounded. Hence the solvability conditions in [Brown et al. 1981] imply those in [Drábek and Robinson 2006].

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Paths and circuits in $\mathbb{G}$-graphs

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For a group G with generating set $S = \{s_1, s_2, \ldots, s_k\}$, the $\mathbb{G}$-graph of G, denoted $\Gamma(G, S)$, is the graph whose vertices are distinct cosets of $\langle s_i \rangle$ in G. Two distinct vertices are joined by an edge when the set intersection of the cosets is nonempty. In this paper, we study the existence of Hamiltonian and Eulerian paths and circuits in $\Gamma(G, S)$.

1. Introduction

Let G be a group with a generating set $S = \{s_1, \ldots, s_k\}$. For the subgroup $\langle s_i \rangle$ of G, define the subset $T_{\langle s_i \rangle}$ of G to be a *left transversal* for $\langle s_i \rangle$ if $\{x\langle s_i \rangle \mid x \in T_{\langle s_i \rangle}\}$ is precisely the set of all left cosets of $\langle s_i \rangle$ in G. Associate a simple graph $\Gamma(G, S)$ to (G, S) with vertex set $V(\Gamma(G, S)) = \{x_j\langle s_i \rangle \mid x_j \in T_{\langle s_i \rangle}\}$. Two distinct vertices $x_j\langle s_i \rangle$ and $x_l\langle s_k \rangle$ in $V(\Gamma(G, S))$ are joined by an edge if $x_j\langle s_i \rangle \cap x_l\langle s_k \rangle$ is nonempty. The edge set, $E(\Gamma(G, S))$, consists of pairs $(x_j\langle s_i \rangle, x_l\langle s_k \rangle)$. $\Gamma(G, S)$ defined this way has no multiedge or loop. Bretto and Gillibert [2004] introduced $\Gamma(G, S)$ and a similar graph, $\overline{\Gamma}(G, S)$. $\overline{\Gamma}(G, S)$ differs from $\Gamma(G, S)$ in that it is a multigraph with a n-edge between two vertices $x_j\langle s_i \rangle$ and $x_l\langle s_k \rangle$ when $|x_j\langle s_i \rangle \cap x_l\langle s_k \rangle| = n$. The $\mathbb{G}$-graph, $\Gamma(G, S)$, is necessarily a subgraph of $\overline{\Gamma}(G, S)$.

In this paper we concentrate on results for $\Gamma(G, S)$. Many of the results from [Bretto and Gillibert 2004; 2005; Bretto et al. 2005; 2007] about $\overline{\Gamma}(G, S)$ translate easily to the simple graph $\Gamma(G, S)$.

Let $V_i = \{x_j\langle s_i \rangle \mid x_j \in T_{\langle s_i \rangle}\}$. Then $V(\Gamma(G, S)) = \cup_{i=1}^{k} V_i$. The main object of this paper is to study the existence of Hamiltonian and Eulerian paths and circuits in $\Gamma(G, S)$. To this end we recall a few results from Euler. Notice that Eulerian circuits are not considered Eulerian paths in this paper.

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Theorem 1.1 (Euler). *Let Γ be a nontrivial connected graph. Then Γ has an Eulerian circuit if and only if every vertex is of even degree.*

Theorem 1.2 (Euler). *Let Γ be a nontrivial connected graph. Then Γ has an Eulerian path if and only if Γ has exactly two vertices of odd degree. Furthermore, the path begins at one of the vertices of odd degree and terminates at the other.*

2. Preliminaries

In this section, results are proved that pertain to the degrees of vertices in $\Gamma(G, S)$. Recall that if $S = \{s_1, s_2, \ldots, s_k\}$, then $\Gamma(G, S)$ is necessarily k-partite.

Lemma 2.1. *If $g \in \langle s_i \rangle \cap \langle s_j \rangle$, then $g^{-1} \in \langle s_i \rangle \cap \langle s_j \rangle$.*

Proof. Let $g \in \langle s_i \rangle \cap \langle s_j \rangle$, then there exists $m, n \in \mathbb{N}$ such that $g = s_i^m = s_j^n$. Taking the inverse, we have $g^{-1} = s_i^{-m} = s_j^{-n}$. Therefore $g^{-1} \in \langle s_i \rangle \cap \langle s_j \rangle$. □

Theorem 2.1. *Let G be a group with generating set S. Let $\langle s_i \rangle \cup x_2\langle s_i \rangle \cup \cdots \cup x_{k_i}\langle s_i \rangle$ be a partition of G into cosets of $\langle s_i \rangle$ and $\langle s_j \rangle \cup y_2\langle s_j \rangle \cup \cdots \cup y_{k_j}\langle s_j \rangle$ be a partition of G into cosets of $\langle s_j \rangle$. Let*

$$V_i = \{\langle s_i \rangle, x_2\langle s_i \rangle, \ldots, x_{k_i}\langle s_i \rangle\} \quad \text{and} \quad V_j = \{\langle s_j \rangle, y_2\langle s_j \rangle, \ldots, y_{k_j}\langle s_j \rangle\}$$

be the appropriate subsets of the vertex set of $\Gamma(G, S)$. If

$$|\langle s_i \rangle \cap \langle s_j \rangle| = S_{i,j} \quad \text{and} \quad (x\langle s_i \rangle, y\langle s_j \rangle) \in E(\Gamma(G, S)),$$

then $|x\langle s_i \rangle \cap y\langle s_j \rangle| = S_{i,j}$.

Proof. Let $\langle s_i \rangle \cap \langle s_j \rangle = \{e, g_1, \ldots, g_{S_{i,j}-1}\}$. Since $g_1 \in \langle s_i \rangle$ and $g_1 \in \langle s_j \rangle$, there exists $m, n \in \mathbb{N}$ such that $g_1 = s_i^m = s_j^n$. Let $x\langle s_i \rangle \in V_i$ and $y\langle s_j \rangle \in V_j$ such that $(x\langle s_i \rangle, y\langle s_j \rangle) \in E(\Gamma(G, S))$. Then $x\langle s_i \rangle \cap y\langle s_j \rangle \neq \varnothing$ and there exists h such that $h = xs_i^{m'} = ys_j^{n'}$. So

$$h = xs_i^{m'} = xs_i^{m'-m}s_i^m = xs_i^{m'-m}g_1.$$

Therefore $hg_1^{-1} \in x\langle s_i \rangle$. Likewise, $hg_1^{-1} \in y\langle s_j \rangle$ and $hg_1^{-1} \in x\langle s_i \rangle \cap y\langle s_j \rangle$. By similar arguments, $\{h, hg_1^{-1}, hg_2^{-1}, \ldots, hg_{S_{i,j}-1}^{-1}\} \subseteq x\langle s_i \rangle \cap y\langle s_j \rangle$.

Assume there exists $g \in x\langle s_i \rangle \cap y\langle s_j \rangle$ such that $g \notin \{h, hg_1^{-1}, hg_2^{-1}, \ldots, hg_{S_{i,j}-1}^{-1}\}$. Since $g \in x\langle s_i \rangle \cap y\langle s_j \rangle$ there exists $m'', n'' \in \mathbb{N}$ such that $g = xs_i^{m''} = ys_j^{n''}$. So

$$g = xs_i^{m''} = xs_i^{m'}s_i^{m''-m'} = hs_i^{m''-m'}.$$

Therefore $h^{-1}g \in \langle s_i \rangle$. Likewise $h^{-1}g \in \langle s_j \rangle$ and $h^{-1}g \in \langle s_i \rangle \cap \langle s_j \rangle$. There exists $k \in \{0, \ldots, S_{i,j} - 1\}$ such that $h^{-1}g = g_k$. Since $g_k \in \langle s_i \rangle \cap \langle s_j \rangle$, $g_k^{-1} \in$

$\langle s_i \rangle \cap \langle s_j \rangle$ by Lemma 2.1. Denote g_k^{-1} by g_k'. Then $g = hg_k = h(g_k')^{-1}$ and $g \in \{h, hg_1^{-1}, hg_2^{-1}, \ldots, hg_{S_{i,j}-1}^{-1}\}$. Therefore

$$\{h, hg_1^{-1}, hg_2^{-1}, \ldots, hg_{S_{i,j}-1}^{-1}\} = x\langle s_i \rangle \cap y\langle s_j \rangle,$$
$$|x\langle s_i \rangle \cap y\langle s_j \rangle| = S_{i,j}.$$
□

Corollary 2.1. *The number of edges between* $\langle s_i \rangle$ *and* V_j *is given by* $|s_i|/S_{i,j}$.

Proof. Let

$$V_j = \{\langle s_j \rangle, y_2\langle s_j \rangle, \ldots, y_k\langle s_j \rangle\} \quad \text{and} \quad V_j' = \{\langle s_j \rangle, y_2'\langle s_j \rangle, \ldots, y_l'\langle s_j \rangle\}$$

be the set that contains all vertices in V_j that are adjacent to $\langle s_i \rangle$. Since

$$(\langle s_i \rangle, y'\langle s_j \rangle) \in E(\Gamma(G, S)) \quad \text{for all } y'\langle s_j \rangle \in V_j',\ |\langle s_i \rangle \cap y'\langle s_j \rangle| = S_{i,j}$$

by Theorem 2.1. So the number of elements in $\langle s_i \rangle$ is given by $|s_i| = S_{i,j} \cdot l$ or the number of edges between $\langle s_i \rangle$ and V_j is $|s_i|/S_{i,j}$. □

Lemma 2.2. *If* G *is a group with generating set* $S = \{s_1, \ldots, s_n\}$ *and* $S_{i,j} = |\langle s_i \rangle \cap \langle s_j \rangle|$, *then the degree of the vertex* $\langle s_i \rangle$, *denoted* $\deg\langle s_i \rangle$, *is*

$$\deg\langle s_i \rangle = \Big(\sum_{j=1}^{n} |s_i|/S_{i,j}\Big) - |s_i|/S_{i,i}.$$

Proof. We proceed with induction on n. Partition the vertex set of $\Gamma(G, S)$ into n subsets $V_1, V_2, \ldots, V_n$ such that $V_i = \{\langle s_i \rangle, x_2\langle s_i \rangle, \ldots, x_{k_i}\langle s_i \rangle\}$. Consider the subgraph, $\Gamma_{1,2}$, of $\Gamma(G, S)$ induced by the vertex set $V_1 \cup V_2$. Let $\deg_{\Gamma_{1,2}}(\langle s_i \rangle)$ denote the degree of the vertex $\langle s_i \rangle$ in $\Gamma_{1,2}$. Then, by Corollary 2.1,

$$\deg_{\Gamma_{1,2}}(\langle s_2 \rangle) = |s_2|/S_{2,1} = \Big(\sum_{j=1}^{2} |s_2|/S_{2,j}\Big) - |s_2|/S_{2,2}.$$

Likewise

$$\deg_{\Gamma_{1,2}}(\langle s_1 \rangle) = |s_1|/S_{1,2} = \Big(\sum_{j=1}^{2} |s_1|/S_{1,j}\Big) - |s_1|/S_{1,1},$$

and the formula holds for $n = 2$.

Consider the subgraph, $\Gamma_{1,2,\ldots,n-1}$, of $\Gamma(G, S)$ induced by the vertex set $V_1 \cup V_2 \cup \cdots \cup V_{n-1}$. Let $\deg_{\Gamma_{1,2,\ldots,n-1}}(\langle s_i \rangle)$ denote the degree of the vertex $\langle s_i \rangle$ in $\Gamma_{1,2,\ldots,n-1}$. Assume that the theorem holds for $n - 1$, that is,

$$\deg_{\Gamma_{1,2,\ldots,n-1}}(\langle s_i \rangle) = \Big(\sum_{j=1}^{n-1} |s_i|/S_{i,j}\Big) - |s_i|/S_{i,i}.$$

Now consider the entire graph, $\Gamma(G, S)$. The number of edges between $\langle s_i \rangle$ and V_n is $|s_i|/S_{i,n}$. So

$$\deg\langle s_i \rangle = |s_i|/S_{i,n} + \Big(\sum_{j=1}^{n-1} |s_i|/S_{i,j}\Big) - |s_i|/S_{i,i} = \Big(\sum_{j=1}^{n} |s_i|/S_{i,j}\Big) - |s_i|/S_{i,i}. \quad \square$$

Remark 1. Notice that $|s_i|/S_{i,i} = 1$, since $S_{i,i} = |\langle s_i \rangle \cap \langle s_i \rangle| = |s_i|$.

Corollary 2.2. *If G is a group with generating set $S = \{s_1, s_2, \ldots, s_n\}$, then* $\deg\langle s_i \rangle$ *equals* $\deg g\langle s_i \rangle$ *for all $g\langle s_i \rangle$ in V_i, that is, every vertex in the same vertex set has the same degree.*

Proof. Let G be a group with generating set $S = \{s_1, s_2, \ldots, s_n\}$ and $S_{i,j} = |\langle s_i \rangle \cap \langle s_j \rangle|$. From Theorem 2.1, if $g, h \in G$ such that $(g\langle s_i \rangle, h\langle s_j \rangle) \in E(\Gamma(G, S))$, then $|g\langle s_i \rangle \cap h\langle s_j \rangle| = S_{i,j}$. From Lemma 2.2,

$$\deg g\langle s_i \rangle = \Big(\sum_{j=1}^{n} \frac{|g\langle s_i \rangle|}{S_{i,j}}\Big) - 1 = \Big(\sum_{j=1}^{n} \frac{|\langle s_i \rangle|}{S_{i,j}}\Big) - 1 = \deg\langle s_i \rangle. \quad \square$$

Theorem 2.2. *If G is a group with generating set $S = \{s_1, s_2, \ldots, s_n\}$ and $S_{i,j} = |\langle s_i \rangle \cap \langle s_j \rangle|$, then $\Gamma(G, S)$ is complete n-partite if and only if*

$$\Big(\sum_{j=1}^{n} \frac{|\langle s_i \rangle|}{S_{i,j}}\Big) - 1 = \Big(\sum_{k=1}^{n} |V_k|\Big) - |V_i|.$$

3. Abelian groups of rank ≤ 2

In this section, we let G be an abelian group of rank ≤ 2 and let $|S| = 2$. G is isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_m$ for some m and n. Notice that if G is infinite then it is isomorphic to $\mathbb{Z} \approx \mathbb{Z} \times \mathbb{Z}_1$ and the theorems of this section apply.

Theorem 3.1. *Let $G = \mathbb{Z}_n \times \mathbb{Z}_m$ and $S = \{(1, 0), (0, 1)\}$, then $\Gamma(G, S)$ has a Hamiltonian path if and only if $|m - n| \leq 1$.*

Proof. ($\Rightarrow$) Let $\Gamma(G, S)$ contain a Hamiltonian path. $\Gamma(G, S)$ is $K_{m,n}$ [Daniel $\geq$ 2008]. Assume that $n \geq m$. $|(1, 0)| = n$ and $|(0, 1)| = m$ and $V = V_1 \cup V_2$ where

$$\begin{aligned} V_1 &= \{a_1 + \langle (1, 0) \rangle, a_2 + \langle (1, 0) \rangle, \ldots, a_m + \langle (1, 0) \rangle\} \text{ and} \\ V_2 &= \{b_1 + \langle (0, 1) \rangle, b_2 + \langle (1, 0) \rangle, \ldots, b_n + \langle (1, 0) \rangle\}. \end{aligned}$$

Let $H_1 = \langle (1, 0) \rangle$ and $H_2 = \langle (0, 1) \rangle$. Since $n \geq m$, any Hamiltonian path must start with a vertex in V_2, that is, $b_{i_1} + H_2$.

$$(b_{i_1} + H_2, a_{j_1} + H_1), (a_{j_1} + H_1, b_{i_2} + H_2), (b_{i_2} + H_2, a_{j_2} + H_1), \ldots, \\ (a_{j_{m-1}} + H_1, b_{i_m} + H_2), (b_{i_m} + H_2, a_{j_m} + H_1), \ldots$$

Notice that all the vertices in V_1 have been exhausted. So either the path ends here and $n = m$ or it ends with the edge $(a_{j_m} + H_1, b_{i_{m+1}} + H_2)$ and $n = m + 1$. Therefore $|m - n| \leq 1$. The proof for $m \geq n$ is similar.

($\Leftarrow$) Let $|m - n| \leq 1$, $|(1, 0)| = n$, and $|(0, 1)| = m$. Let

$$a_1 + H_1 \cup a_2 + H_1 \cup \cdots \cup a_m + H_1$$

be a partition of G into cosets of $\langle(1, 0)\rangle$ and let

$$b_1 + H_2 \cup b_2 + H_2 \cup \cdots \cup b_n + H_2$$

be a partition of G into cosets of $\langle(0, 1)\rangle$. Since $\Gamma(G, S)$ is $K_{m,n}$, there exists an edge between $a_i + H_1$ and $b_j + H_2$ for all i, j.

(i) $m = n + 1$ and $(a_1 + H_1, b_1 + H_2)$, $(b_1 + H_2, a_2 + H_1)$, …, $(a_n + H_1, b_n + H_2)$, $(b_n + H_2, a_m + H_1)$ is a Hamiltonian path.

(ii) $n = m + 1$ and $(b_1 + H_2, a_1 + H_1)$, $(a_1 + H_1, b_2 + H_2)$,…, $(b_m + H_2, a_m + H_1)$, $(a_m + H_1, b_n + H_2)$ is a Hamiltonian path.

(iii) $m = n$ and $(a_1 + H_1, b_1 + H_2)$, $(b_1 + H_2, a_2 + H_1)$, …, $(b_{n-1} + H_2, a_n + H_1)$, $(a_n + H_1, b_n + H_2)$ is a Hamiltonian path. □

Theorem 3.2. *Let $G = \mathbb{Z}_n \times \mathbb{Z}_m$ and $S = \{(1, 0), (0, 1)\}$, then $\Gamma(G, S)$ has a Hamiltonian circuit if and only if $m = n$.*

Proof. ($\Rightarrow$) Let $\Gamma(G,S)$ contain a Hamiltonian circuit. $\Gamma(G,S)$ is $K_{m,n}$ [Daniel ≥ 2008]. $|(1,0)| = n$ and $|(0,1)| = m$ and $V = V_1 \cup V_2$ where

$$V_1 = \{a_1 + \langle(1, 0)\rangle, a_2 + \langle(1, 0)\rangle, \ldots, a_m + \langle(1, 0)\rangle\},$$
$$V_2 = \{b_1 + \langle(0, 1)\rangle, b_2 + \langle(1, 0)\rangle, \ldots, b_n + \langle(1, 0)\rangle\}.$$

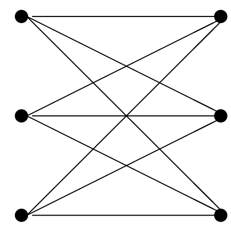
Let $H_1 = \langle(1, 0)\rangle$ and $H_2 = \langle(0, 1)\rangle$. Start with a vertex in V_2, that is, $b_{i_1} + H_2$ and trace the Hamiltonian circuit

$$(b_{i_1} + H_2, a_{j_1} + H_1), (a_{j_1} + H_1, b_{i_2} + H_2), (b_{i_2} + H_2, a_{j_2} + H_1), \ldots,$$
$$(a_{j_{m-1}} + H_1, b_{i_m} + H_2), (b_{i_m} + H_2, a_{j_m} + H_1), \ldots$$

Notice that all the vertices in V_1 have been exhausted. So the path ends here and to complete the circuit we need the edge $(a_{j_m} + H_1, b_{i_1} + H_2)$. Therefore $n = m$. The proof starting with a vertex in V_1 is similar.

($\Leftarrow$) Let $m = n$ and $a_1 + H_1 \cup a_2 + H_1 \cup \cdots \cup a_m + H_1$ be partition of G into cosets of $\langle(1, 0)\rangle$ Since $\Gamma(G, S)$ is $K_{m,m}$, there exist an edge between $a_i + H_1$ and $b_j + H_2$ for all i, j. Then $(a_1 + H_1, b_1 + H_2)$, $(b_1 + H_2, a_2 + H_1)$, …, $(a_m + H_1, b_m + H_2)$, $(b_m + H_2, a_1 + H_1)$ is a Hamiltonian circuit. □

Example 1. Let $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $S = \{(1, 0), (0, 1)\}$, then $\Gamma(G, S) = K_{3,3}$ (see figure) and $\Gamma(G, S)$ contains both a Hamiltonian path and circuit.



Theorem 3.3. *Let $G = \mathbb{Z}_m \times \mathbb{Z}_n$ and $S = \{(1, 0), (0, 1)\}$, then $\Gamma(G, S)$ has an Eulerian circuit if and only if m and n are both even.*

Proof. ($\Rightarrow$) Let $\Gamma(G, S)$ have an Eulerian circuit. From [Daniel $\geq$ 2008], $S_{1,2} = S_{2,1} = 1$ so $\deg\langle(1, 0)\rangle = n$ and $\deg\langle(0, 1)\rangle = m$. Since every vertex is even, m and n are even.

($\Leftarrow$) Let m and n be even. From [Daniel $\geq$ 2008], $\Gamma(G, S)$ is $K_{n,m}$. Therefore $\deg\langle(1, 0)\rangle = m$ and $\deg\langle(0, 1)\rangle = n$. Since m and n are both even, $\Gamma(G, S)$ contains an Eulerian circuit. □

Theorem 3.4. *Let $G = \mathbb{Z}_m \times \mathbb{Z}_n$ and $S = \{(1, 0), (0, 1)\}$, then $\Gamma(G, S)$ has an Eulerian path if and only if m is odd and $n = 2$ or n is odd and $m = 2$.*

Proof. ($\Rightarrow$) Let $\Gamma(G, S)$ contain an Eulerian path. Then $\Gamma(G, S)$ contains exactly 2 vertices of odd degree. Since $\Gamma(G, S)$ is bipartite, there exists i such that V_i contains the two vertices of odd degree.

Let V_1 contains the two vertices of odd degree. $S_{1,2} = S_{2,1} = 1$ so $\deg\langle(1, 0)\rangle = n$, for n odd, and $\deg\langle(0, 1)\rangle = 2$. Likewise if V_2 contains the two vertices of odd degree, $\deg\langle(1, 0)\rangle = 2$ and $\deg\langle(0, 1)\rangle = m$, for m odd.

($\Leftarrow$) First, assume $m = 2$ and n is odd. $\Gamma(G, S)$ is $K_{2,n}$ and $\deg\langle(1, 0)\rangle = n$ and $\deg\langle(0, 1)\rangle = 2$. Since $|(1, 0)| = 2$, then there are exactly 2 vertices of odd degree.

Now, assume instead that m is odd and $n = 2$. Then $\Gamma(G, S)$ is $K_{m,2}$ and $\deg\langle(1, 0)\rangle = 2$ and $\deg\langle(0, 1)\rangle = m$. Since $|(0, 1)| = 2$, then there are exactly 2 vertices of odd degree. Therefore $\Gamma(G, S)$ contains an Eulerian path. □

4. Dihedral groups

For the dihedral group, D_n, let r be a rotation of $360°/n$ and let f and rf be two different reflections. In [Daniel $\geq$ 2008], it was shown that $\Gamma(G, S) = K_{2,n}$ for $G = D_n$ and $S = \{r, f\}$ and that $\Gamma(G, S)$ is the cycle of length $2n$, C_{2n}, for $G = D_n$ and $S = \{f, rf\}$.

Theorem 4.1. *Let $G = D_n$ and $S = \{f, rf\}$, then $\Gamma(G, S)$ contains an Eulerian circuit.*

Proof. Let $V = V_1 \cup V_2$ such that $V_1 = \{\langle f\rangle, r\langle f\rangle, r^2\langle f\rangle, \dots, r^{n-1}\langle f\rangle\}$ and

$$V_2 = \{\langle rf\rangle, r\langle rf\rangle, r^2\langle rf\rangle, \dots, r^{n-1}\langle rf\rangle\}.$$

We have $\langle rf\rangle = \{rf, e\}$ so $\langle rf\rangle$ shares an edge with $\langle f\rangle$ and $r\langle f\rangle$ and $\deg\langle rf\rangle = 2$. By Corollary 2.2, every vertex in V_2 has degree 2. Likewise $\langle f\rangle = \{f, e\}$, $\langle f\rangle$ shares an edge with $\langle rf\rangle$ and $r^{n-1}\langle rf\rangle$ and every vertex in V_1 has degree 2. Since every vertex has degree 2, Theorem 1.1 says that $\Gamma(G, S)$ contains an Eulerian circuit. □

Corollary 4.1. *Let $G = D_n$ and $S = \{f, rf\}$, then $\Gamma(G, S)$ does not contains an Eulerian path.*

Proof. Because the degree of every vertex is 2, $\Gamma(G, S)$ does not contain two vertices of odd degree. □

Theorem 4.2. *Let $G = D_n$ and $S = \{f, rf\}$, then $\Gamma(G, S)$ contains a Hamiltonian circuit.*

Proof. Let $V = V_1 \cup V_2$ such that $V_1 = \{\langle f\rangle, r\langle f\rangle, r^2\langle f\rangle, \dots, r^{n-1}\langle f\rangle\}$ and

$$V_2 = \{\langle rf\rangle, r\langle rf\rangle, r^2\langle rf\rangle, \dots, r^{n-1}\langle rf\rangle\}.$$

A Hamiltonian circuit is then given by $(\langle f\rangle, \langle rf\rangle)$, $(\langle rf\rangle, r\langle f\rangle)$, $(r\langle f\rangle, r\langle rf\rangle)$, $(r\langle rf\rangle, r^2\langle f\rangle)$, ..., $(r^{n-1}\langle f\rangle, r^{n-1}\langle rf\rangle)$, $(r^{n-1}\langle rf\rangle, \langle f\rangle)$. □

Corollary 4.2. *Let $G = D_n$ and $S = \{f, rf\}$, then $\Gamma(G, S)$ contains a Hamiltonian path.*

Theorem 4.3. *Let $G = D_n$ and $S = \{r, f\}$, then $\Gamma(G, S)$ contains an Eulerian circuit if and only if n is even.*

Proof. ($\Rightarrow$) Let $\Gamma(G, S)$ contain an Eulerian circuit. Then every vertex must be of even degree. Let $V = V_1 \cup V_2$ such that

$$V_1 = \{\langle r\rangle, f\langle r\rangle\} \text{ and } V_2 = \{\langle f\rangle, r\langle f\rangle, r^2\langle f\rangle, \dots, r^{n-1}\langle f\rangle\}.$$

We have

$$\langle r\rangle \cap r^m\langle f\rangle = \{r^m\} \qquad \text{for all } m = 0, \dots, n-1,$$

so the edge $(\langle r\rangle, r^m\langle f\rangle)$ is in $\Gamma(G, S)$ for $m = 0, \dots, n-1$ and $\deg\langle r\rangle = n$. Likewise

$$f\langle r\rangle \cap r^m\langle f\rangle = \{r^m f\} \qquad \text{for all } m = 0, \dots, n-1,$$

so the edge $(f\langle r\rangle, r^m\langle f\rangle)$ is in $\Gamma(G, S)$ for $m = 0, \dots, n-1$ and $\deg f\langle r\rangle = n$. Therefore, n must be even.

($\Leftarrow$) Assume that n is even. Then the vertices in V_1 are of even degree from above. Choose a vertex in V_2, $r^m\langle f\rangle$. $r^m\langle f\rangle$ shares an edge with $\langle r\rangle$ and $f\langle r\rangle$. Therefore $\deg r^m\langle f\rangle = 2$ and every vertex in V_2 is of degree 2. Since all the vertices of $\Gamma(G, S)$ are of even degree, $\Gamma(G, S)$ contains an Eulerian circuit. □

Theorem 4.4. *Let $G = D_n$ and $S = \{r, f\}$, then $\Gamma(G, S)$ contains an Eulerian path if and only if n is odd.*

Proof. ($\Rightarrow$) Let $\Gamma(G, S)$ contain an Eulerian path. Then $\Gamma(G, S)$ contains exactly two vertices of odd degree. Let $V = V_1 \cup V_2$. There are n vertices in V_2 and they are of degree 2. There are two vertices in V_1 and they are of degree n. Therefore, n must be odd.

($\Leftarrow$) Assume that n is odd. Then the two vertices in V_1 are of odd degree and the n vertices in V_2 are of degree 2. Therefore $\Gamma(G, S)$ contains an Eulerian path. □

Theorem 4.5. *Let $G = D_n$ and $S = \{r, f\}$, then $\Gamma(G, S)$ contains a Hamiltonian path if and only if $n = 2$ or 3.*

Proof. ($\Rightarrow$) Let $\Gamma(G, S)$ contain a Hamiltonian path. $\Gamma(G, S)$ is $K_{2,n}$ [Daniel $\geq$ 2008]. Then $V = V_1 \cup V_2$ where

$$V_1 = \{\langle r\rangle, f\langle r\rangle\} \text{ and } V_2 = \{\langle f\rangle, r\langle f\rangle, r^2\langle f\rangle, \ldots, r^{n-1}\langle f\rangle\}.$$

Since $n \geq 2$, any Hamiltonian path must start with a vertex in V_2.

$$\big(r^{i_1}\langle f\rangle, f^{j_1}\langle r\rangle\big), \big(f^{j_1}\langle r\rangle, r^{i_2}\langle f\rangle\big), \big(r^{i_2}\langle f\rangle, f^{j_2}\langle r\rangle\big), \ldots$$

Notice that all the vertices in V_1 have been exhausted. So either the path ends here and $n = 2$ or it ends with the edge $(f^{j_2}\langle r\rangle, r^{i_3}\langle f\rangle)$ and $n = 3$. Therefore $n = 2$ or 3.

($\Leftarrow$) Assume that n is 2 or 3. If $n = 2$ then $V_2 = \{\langle f\rangle, r\langle f\rangle\}$ and

$$\big(\langle r\rangle, \langle f\rangle\big), \big(\langle f\rangle, f\langle r\rangle\big), \big(f\langle r\rangle, r\langle f\rangle\big)$$

is a Hamiltonian path. If $n = 3$ then $V_2 = \{\langle f\rangle, r\langle f\rangle, r^2\langle f\rangle\}$ and

$$\big(\langle f\rangle, \langle r\rangle\big), \big(\langle r\rangle, r\langle f\rangle\big), \big(r\langle f\rangle, f\langle r\rangle\big), \big(f\langle r\rangle, r^2\langle f\rangle\big)$$

is a Hamiltonian path. □

Theorem 4.6. *Let $G = D_n$ and $S = \{r, f\}$, then $\Gamma(G, S)$ contains a Hamiltonian circuit if and only if $n = 2$.*

Proof. ($\Rightarrow$) Let $\Gamma(G, S)$ contain a Hamiltonian circuit. Start with a vertex in V_2 and trace the Hamiltonian circuit

$$(r^{i_1}\langle f\rangle, f^{j_1}\langle r\rangle), (f^{j_1}\langle r\rangle, r^{i_2}\langle f\rangle), (r^{i_2}\langle f\rangle, f^{j_2}\langle r\rangle), \ldots$$

Notice that all the vertices in V_1 have been exhausted so the circuit must end with the edge $(f^{j_2}\langle r\rangle, r^{i_1}\langle f\rangle)$ and n must be 2. The proof starting with a vertex in V_1 is similar.

($\Leftarrow$) Assume that n is 2. Then $V_2 = \{\langle f\rangle, r\langle f\rangle\}$ and $\big(\langle r\rangle, \langle f\rangle\big)$, $\big(\langle f\rangle, f\langle r\rangle\big)$, $\big(f\langle r\rangle, r\langle f\rangle\big)$, $\big(r\langle f\rangle, \langle r\rangle\big)$ is a Hamiltonian circuit. □

5. Eulerian circuits and paths

Now we investigate the existence of Eulerian circuits and paths in $\Gamma(G, S)$ for a generic group G.

Theorem 5.1. *Let G be a group with generating set $S = \{s_1, s_2, \dots, s_n\}$ such that $|\langle s_i\rangle \cap \langle s_j\rangle| = 1$ for all $i \neq j$; then $\Gamma(G, S)$ contains an Eulerian circuit if and only if $|s_i|$ is even for all i, or n is odd.*

Proof. From Lemma 2.2,

$$\deg\langle s_i\rangle = \Big(\sum_{j=1}^{n} |s_i|/S_{i,j}\Big) - |s_i|/S_{i,i}.$$

Also $\deg\langle s_i\rangle = (n-1)|s_i|$, since $S_{i,j} = 1$ for $i \neq j$. Then $\Gamma(G, S)$ contains an Eulerian circuit if and only if $|s_i|$ is even for all i or the number of generators, n, is odd. □

Theorem 5.2. *Let G be a group with generating set $S = \{s_1, s_2, \dots, s_n\}$ such that $|\langle s_i\rangle \cap \langle s_j\rangle| = m$ for all $i \neq j$, then $\Gamma(G, S)$ contains an Eulerian circuit if and only if $2m|(n-1)(|s_i|)$ for all i.*

Proof. From Lemma 2.2,

$$\deg\langle s_i\rangle = \Big(\sum_{j=1}^{n} \frac{|s_i|}{S_{i,j}}\Big) - |s_i|/S_{i,i}.$$

Also, $\deg\langle s_i\rangle = (n-1)|s_i|/m$, since $S_{i,j} = m$ for $i \neq j$. Since $\Gamma(G, S)$ contains an Eulerian circuit if and only if $\deg\langle s_i\rangle$ is even for all i, then $\Gamma(G, S)$ contains an Eulerian circuit if and only if $2m|(n-1)(|s_i|)$ for all i. □

Theorem 5.3. *Let G be a group with generating set $S = \{s_1, s_2, \dots, s_n\}$, then $\Gamma(G, S)$ contains an Eulerian circuit if and only if*

$$2|(n-1)(|s_i|)\Big(\sum_{j=1}^{n} \frac{1}{S_{i,j}}\Big), \qquad \textit{for all } i.$$

Proof. From Lemma 2.2,

$$\deg\langle s_i\rangle = \Big(\sum_{j=1}^{n} \frac{|s_i|}{S_{i,j}}\Big) - |s_i|/S_{i,i}. \quad S_{i,i} = |s_i|, \quad \deg\langle s_i\rangle = (n-1)(|s_i|)\Big(\sum_{j=1}^{n} \frac{1}{S_{i,j}}\Big).$$

Also, $\Gamma(G, S)$ contains an Eulerian circuit if and only if

$$2|(n-1)(|s_i|)\Big(\sum_{j=1}^{n} \frac{1}{S_{i,j}}\Big), \qquad \text{for all } i.$$ □

Theorem 5.4. *Let G be a group with generating set $S = \{s_1, s_2, \ldots, s_n\}$, if $\Gamma(G, S)$ contains an Eulerian path then one of these cases apply*

(i) *there exists i such that $|V_i| = 2$ with $\deg\langle s_i\rangle$ odd and $\deg\langle s_j\rangle$ even for all $j \neq i$, or*

(ii) *there exists i, j such that $|V_i| = |V_j| = 1$ with $\deg\langle s_i\rangle$ and $\deg\langle s_j\rangle$ odd and $\deg\langle s_k\rangle$ even for all $k \neq i, j$.*

Corollary 5.1. *Let G be a group with generating set $S = \{s_1, s_2, \ldots, s_n\}$, if $\Gamma(G, S)$ contains an Eulerian path then G is of even order or G is cyclic.*

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On graphs for which every planar immersion lifts to a knotted spatial embedding

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We call a graph G intrinsically linkable if there is a way to assign over/under information to any planar immersion of G such that the associated spatial embedding contains a pair of nonsplittably linked cycles. We define intrinsically knottable graphs analogously. We show there exist intrinsically linkable graphs that are not intrinsically linked. (Recall a graph is intrinsically linked if it contains a pair of nonsplittably linked cycles in every spatial embedding.) We also show there are intrinsically knottable graphs that are not intrinsically knotted. In addition, we demonstrate that the property of being intrinsically linkable (knottable) is not preserved by vertex expansion.

1. Introduction

We start with a quick review of some definitions. A *graph* G consists of a finite nonempty set $V(G)$ of *vertices* together with a set $E(G)$ of unordered pairs of (usually distinct) vertices, called *edges*. If $x = (u, v) \in E(G)$, for $u, v \in V(G)$, we say that u and v are *adjacent* vertices, and that vertex u and edge x are incident with each other, as are v and x.

A *walk* in a graph G is an alternating sequence of vertices and edges

$$v_0, x_1, v_1, \ldots, v_{n-1}, x_n, v_n$$

beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. A *cycle* is a walk with $n \geq 2$ vertices and with all vertices distinct except $v_0 = v_n$. We say such a cycle has *length* n.

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Let G be a graph with

$$V(G) = \{v_1, v_2, \ldots, v_n\} \qquad \text{and} \qquad E(G) = \{x_1, x_2, \ldots, x_m\}.$$

A *spatial embedding* of G is a map f of G to a subspace $G(M)$ of $\mathbb{R}^3$ such that

$$G(M) = \left(\bigcup_{i=1}^{n} v_i(M) \right) \cup \left(\bigcup_{j=1}^{m} x_j(M) \right),$$

where

(i) $v_1(M), v_2(M), \ldots, v_n(M)$ are n distinct points of $\mathbb{R}^3$ with $f(v_i) = v_i(M)$;

(ii) $x_1(M), \ldots, x_m(M)$ are m mutually disjoint open arcs in $\mathbb{R}^3$ with

$$f(x_i) = x_i(M);$$

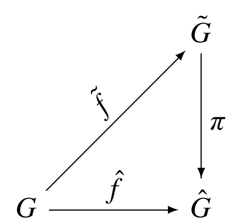
(iii) $x_j(M) \cap v_i(M) = \varnothing$, $i = 1, \ldots, n$, $j = 1, \ldots, m$;

(iv) if $x_j = (v_{j_1}, v_{j_2})$, then the open arc $x_j(M)$ has $v_{j_1}(M)$ and $v_{j_2}(M)$ as end points for $j = 1, \ldots, m$.

In the above definition, an *arc* in $\mathbb{R}^3$ is a homeomorphic image of $[0, 1]$; an *open arc* is an arc less its two end points, the images of 0 and 1. More informally, a spatial embedding is a way to place a given graph in space.

We define a *planar immersion* of a graph G similar to a spatial embedding of G, except the codomain is $\mathbb{R}^2$ instead of $\mathbb{R}^3$, and we allow the image of edges of G to intersect, though we require that no three edges can intersect at the same point and we require the image of our edges to intersect transversely (they intersect locally in only one point, and they are not tangent to each other). We will assume that all embeddings and immersions are *tame*, that is, can be approximated by a finite collection of line segments. We will often simply use the term *immersion* instead of planar immersion. We use $\hat{G}$ to denote the image of an immersion of G under the map $\hat{f}$. If H is a subgraph of G, we similarly denote by $\hat{H}$ the image of H under $\hat{f}$.

Given an immersion $\hat{f}$ of a graph G with image $\hat{G}$, one can, by assigning over/under information to its double points, lift the immersion into 3-space, thereby creating a well-defined spatial embedding $\tilde{f}$ with image $\tilde{G}$. If π is the standard projection $\pi(x, y, z) = (x, y)$, and $\hat{f} = \pi \circ \tilde{f}$, we have the commutative diagram



If there exists a lift of the immersion $\hat{f}$ whose image contains a pair of nonsplittably linked cycles (in other words, cannot be deformed to have a planar projection with no crossings between strands from two different components), then we say the immersion is *linkable*. We define the graph G to be *intrinsically linkable* if every immersion of G is linkable. We define *knottable* and *intrinsically knottable* analogously.

The study of intrinsically linkable graphs was inspired by two different ideas: intrinsically linked graphs, and graphs with a knot inevitable projection. The property of having a knot inevitable projection was introduced by Taniyama [1995] and studied by others (for example, Sugiura and Suzuki [2000], and Tamura [2004]). A (planar) graph has a *knot inevitable projection* if there exists a regular projection (that is, a planar immersion) of the graph such that every choice of over/under-crossings induces a spatial embedding that is knotted (in other words, cannot be deformed to a spatial embedding that has a planar projection without crossings).

The first results concerning intrinsically linked graphs were written up by Conway and Gordon [1983], and by Sachs [1983], who independently showed that every spatial embedding of K_6 (the graph on 6 vertices that contains all 15 possible edges between vertices) contains a pair of disjoint cycles that form a nonsplittable link, that is, K_6 is *intrinsically linked*. (See [Adams 2004] for a good background on knot theory in general, and on intrinsically linked and knotted graphs in particular.)

Conway and Gordon [1983] also showed that every spatial embedding of K_7 contains a cycle that forms a nontrivial knot, that is, K_7 is *intrinsically knotted*. Robertson et al. [1995] later showed that the collection of minor-minimal intrinsically linked graphs is exactly the *Petersen family*, that is, the seven graphs obtainable from the classic Petersen graph by repeated Δ-Y and Y-Δ exchanges. No one has yet classified the minor-minimal intrinsically knotted graphs, though they are known to be finite in number [Robertson and Seymour 2004].

Recall that a graph H is a *minor* of a graph G if H can be obtained from G by a sequence of deletions and/or contractions of edges and/or deletions of vertices. A graph G is *minor minimal* with respect to a given property if it has the property, but no minor of G has the property. Let a, b, and c be vertices of a graph G such that edges (a, b), (a, c), and (b, c) exist. Then a *Δ-Y exchange* on a triangle (a, b, c) of graph G is as follows. Vertex v is added to G, edges (a, b), (a, c), and (b, c) are deleted, and edges (a, v), (b, v), and (c, v) are added. A Y-Δ exchange is the reverse operation.

Clearly, an intrinsically linked (knotted) graph is also intrinsically linkable (knottable), but the converse is not true. In this paper, we present several intrinsically linkable graphs, each of which is a proper minor of some graph in the Petersen family (and hence not intrinsically linked), and several intrinsically knottable graphs, which are all in the Petersen family (and not intrinsically knotted).

Recall that a *vertex expansion* of a vertex v in a graph G is achieved by replacing v with two vertices v' and v'', adding the edge (v', v'') and connecting a subset of the edges that were incident to v to v' and the rest of the edges that were incident to v to v''. A graph G is considered to be an *expansion* of a graph H if G can be obtained by vertex expansions of H. It is well known that vertex expansions preserve intrinsic linking and intrinsic knotting; see [Nešetřil and Thomas 1985; Fellows and Langston 1988]. We demonstrate several intrinsically linkable (knottable) graphs for which vertex expansion destroys intrinsic linkability (knottability). We thus conjecture that vertex expansion preserves intrinsic linkability (knottability) only for those graphs that are intrinsically linked (knotted).

2. Intrinsically linkable graphs

We start this section with a quick introduction to the linking number. Recall that given a link of two components, L_1 and L_2 (two disjoint circles embedded in space), one computes the linking number of the link by examining a projection (with over and under-crossing information) of the link. Choose an orientation for each component of the link. At each crossing between two components, one of the pictures in Figure 1 will hold. We count $+1$ for each crossing of the first type (where you can rotate the over-strand counterclockwise to line up with the under-strand) and -1 for each crossing of the second type. To get the linking number, $lk(L_1, L_2)$, take the sum of $+1$'s and -1's and divide by 2. One can show that the absolute value of the linking number is independent of projection, and of chosen orientations (see [Adams 2004] for further explanation). Note that if $lk(L_1, L_2) \neq 0$, then the associated link is nonsplit. The converse does not hold. That is, there are nonsplit links with linking number 0 (the Whitehead link is a famous example, see again [Adams 2004]).

Lemma 2.1. *Let a graph G consist of two disjoint cycles A and B. A planar immersion $\hat{f}$ of G is linkable if and only if $\hat{A}$ and $\hat{B}$ intersect.*

Proof. Suppose there is a planar immersion $\hat{f}$ with disjoint cycles $\hat{A}$ and $\hat{B}$ that intersect. We will construct from $\hat{f}$ a spatial embedding $\tilde{f}$ in which the linking number $lk(\tilde{A}, \tilde{B})$ is nonzero. Arbitrarily choose orientations for $\hat{A}$ and $\hat{B}$, and then choose each crossing in $\hat{G}$ to be positive. It is assumed that $\hat{A}$ and $\hat{B}$ intersect, so there exists at least one crossing between them. We now have an induced spatial embedding $\tilde{f}$ in which $lk(\tilde{A}, \tilde{B}) > 0$.

The other implication is trivial to prove. □

Here, we provide a sufficient condition for a graph to be intrinsically linkable:

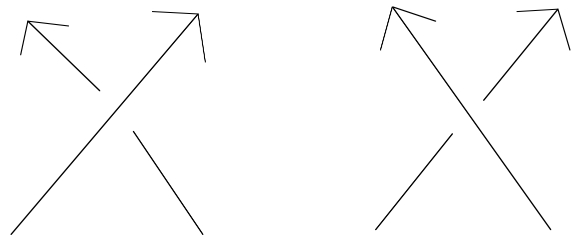


Figure 1. Computing the linking number.

Lemma 2.2. *A graph G is intrinsically linkable if it contains a nonplanar subgraph H such that for any pair $\{e_1, e_2\}$ of nonadjacent edges in H, e_1 and e_2 belong to disjoint cycles in G.*

Proof. Let G be any graph that satisfies the above condition and let $\hat{f}$ be any immersion of G. Since H is nonplanar, there exists in $\hat{H}$ at least one pair $\{e_1, e_2\}$ of nonadjacent edges that intersect. By hypothesis there are disjoint cycles, C_1 and C_2, that contain e_1 and e_2 respectively. Since $\hat{C}_1$ and $\hat{C}_2$ intersect, by Lemma 2.1 $\hat{f}$ is linkable. □

Remark (A remark on notation). We use the notation $G - e_{m,n}$ to denote the subgraph of G obtained by removing an edge connecting a vertex of degree m to a vertex of degree n. This notation is used only when the edge classes of G are uniquely determined by the degree of the incident vertices. If no subscript is present on e, then all edges of G belong to the same class. (Recall that the *degree* of a vertex is the number of edges incident to that vertex.)

We denote the graph in the Petersen family obtained from K_6 by a single Δ-Y exchange by P_7, and we denote the graph in the Petersen family obtained from $K_{3,3,1}$ by a single Δ-Y exchange by P_8. Finally, we denote the graph in the Petersen family obtained from P_8 by a single Δ-Y exchange by P_9. (Recall that $K_{3,3,1}$ is the graph of 7 vertices with vertices in three classes: $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$ and $\{v_7\}$ and edges between two vertices if and only if they lie in different classes. The graph $K_{4,4}$ is defined similarly on 8 vertices with two vertex classes of size 4.)

Theorem 2.3. *The following graphs are intrinsically linkable*: $K_6 - e$, $K_{3,3,1} - e_{4,6}$, $P_7 - e_{4,5}$, $P_7 - e_{5,5}$, $(K_{4,4} - e) - e_{4,4}$, *and* $P_8 - e_{4,5}$.

Proof. We will show that $G = K_{3,3,1} - e_{4,6}$ is intrinsically linkable. Proofs for the remaining graphs are similar.

Label the vertices as in Figure 2. Notice that in this labeling scheme the vertex classes are $S = \{s_1\}$, $U = \{u_1\}$, $V = \{v_1, v_2, v_3\}$, and $W = \{w_1, w_2\}$. We say that

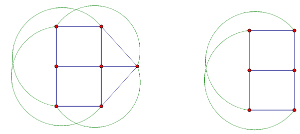


Figure 2. Vertex classes of $K_{3,3,1} - e_{4,6}$ and the subgraph H.

an edge is in the class SV if it connects a vertex in S with a vertex in V. Naming the other edge classes similarly, we have four edge classes in total: SV, SW, UV, and VW.

Take any immersion $\hat{f}$ of G. Let H be the subgraph induced by

$$\{u_1, v_1, v_2, v_3, w_1, w_2\}.$$

Since H is isomorphic to $K_{3,3}$, H is nonplanar and thus $\hat{H}$ has a pair of nonadjacent intersecting edges. There are two cases.

Case 1: Suppose one edge belongs to UV and the other to VW. We may assume the two edges to be (u_1, v_2) and (v_1, w_1). Then the disjoint cycles

$$(s_1, v_1, w_1) \quad \text{and} \quad (u_1, v_2, w_2, v_3)$$

intersect in $\hat{G}$.

Case 2: Suppose both edges belong to VW. We may assume the two edges to be (v_1, w_1) and (v_2, w_2). Then the disjoint cycles

$$(s_1, v_1, w_1) \quad \text{and} \quad (u_1, v_2, w_2, v_3)$$

intersect in $\hat{G}$.

Thus in either case we have a pair of disjoint cycles that intersect in $\hat{G}$. By Lemma 2.2, G is intrinsically linkable. □

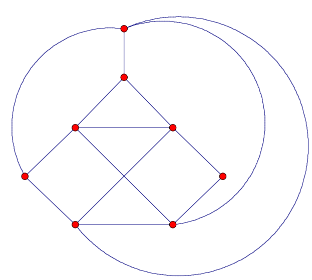


Figure 3. An immersion of $P_8 - e_{3,3}$ with only one crossing.

Since vertex expansion, Δ-Y exchange, and Y-Δ exchange preserve intrinsic linking [Nešetřil and Thomas 1985; Fellows and Langston 1988; Motwani et al. 1988; Robertson et al. 1995], it is natural to ask if these same graph operations preserve intrinsic linkability. In general, this is not the case. For example, $P_8 - e_{4,4}$ can be obtained from $P_7 - e_{4,5}$ by Δ-Y exchange, but $P_8 - e_{4,4}$ is not intrinsically linkable (See Figure 3).

In addition, certain expansions of $K_6 - e$ and $K_{3,3,1} - e_{4,6}$, which are exhibited in Figure 4 (notice that the expanded immersions contain only one crossing), are not intrinsically linkable. Any intrinsically linkable graph for which vertex expansion does preserve linkability, we call *strongly linkable*. Having found many examples in which expansion kills intrinsic linkability, we conjecture the following:

Conjecture 2.4. *A graph is strongly linkable if and only if it is intrinsically linked.*

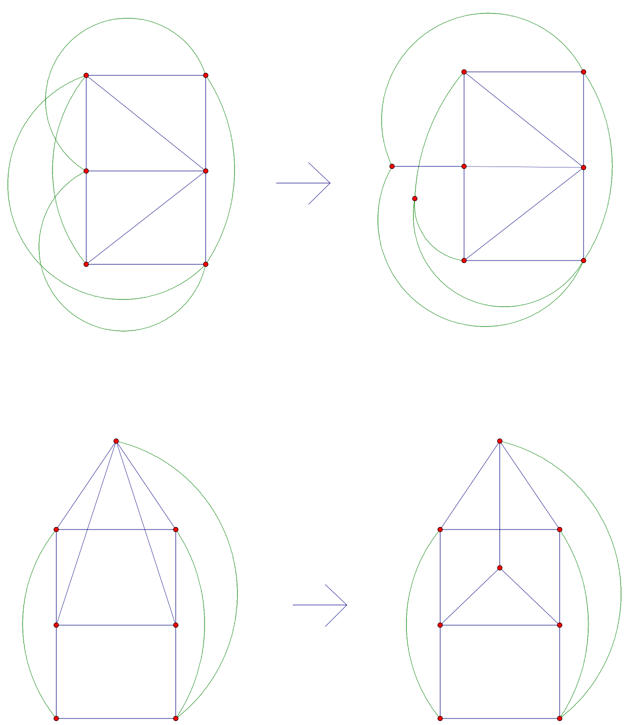


Figure 4. Two graphs for which vertex expansion destroys intrinsic linkability.

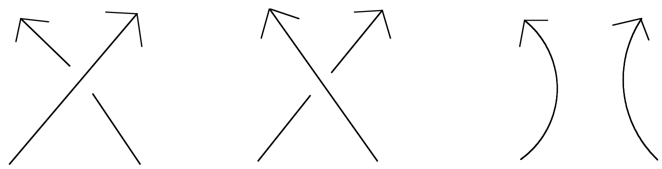


Figure 5. The neighborhoods involved in Lemma 3.1.

3. Intrinsically knottable graphs

3A. ***Introduction.*** The following lemma about knots is from [Kauffman 1983]. Note that we use $lk_2(L_1, L_2)$ to denote the mod 2 linking number for link components L_1 and L_2. Recall that a *knot* is a tame embedding of S^1 into $\mathbb{R}^3$.

Lemma 3.1. *For a knot K, the Arf invariant $\alpha(K)$ is the second coefficient of the Conway polynomial* (*mod* 2)*. It satisfies the following Skein relation* (*see Figure 5*):

$$\alpha(K_+) = \alpha(K_-) + lk_2(L_1, L_2).$$

Note that if $\alpha(K) \neq 0$, then K is nontrivial. (There are, however, many nontrivial knots with vanishing Arf invariant).

We use the following lemma from [Taniyama and Yasuhara 2001] (see also [Foisy 2002]). This lemma uses the second coefficient of the Conway polynomial of a knot, which is denoted by $a_2(K)$, for a knot K (again, if $a_2(K) \neq 0$, then K is nontrivial). Recall that a Hamiltonian cycle in a graph is a cycle that uses every vertex of the graph.

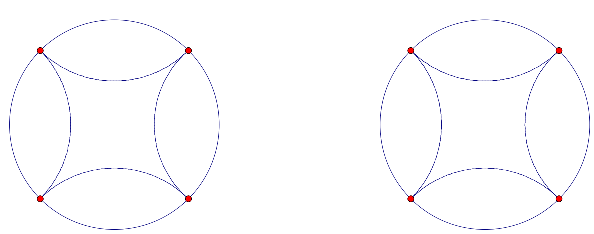


Figure 6. A planar embedding of D_4.

Lemma 3.2. *Consider the graph D_4, labeled as in Figure 6. Let f be a function embedding D_4 in space. Let S_0 and S_1 be sets of Hamiltonian cycles where*

$$S_0 = \{\, (a_i b_j c_k d_l) \mid i+j+k+l \text{ is even} \,\},$$
$$S_1 = \{\, (a_i b_j c_k d_l) \mid i+j+k+l \text{ is odd} \,\}.$$

Let

$$\lambda(f) = \sum_{C \in S_0} a_2(f(C)) - \sum_{C \in S_1} a_2(f(C)).$$

Then

$$\lambda(f) = \big| lk(C_1, C_3) \cdot lk(C_2, C_4) \big|.$$

In particular, if $\lambda(f)$ is nonzero, one of the Hamiltonian cycles must be knotted.

The following corollary is an immediate consequence; see [Taniyama and Yasuhara 2001; Foisy 2002].

Corollary 3.3. *If for a given embedding of G, there is an expansion of D_4 contained as an embedded subgraph with*

$$lk(C_1, C_3) \cdot lk(C_2, C_4) > 0,$$

then the embedded G contains a knotted cycle.

3B. ***Nontrivial examples of intrinsically knottable graphs.*** We explore the connection between intrinsic linking and intrinsic knottability by looking at the Petersen graphs. We originally conjectured that an intrinsically linked graph would necessarily be intrinsically knottable, but we quickly found counterexamples. It is easy to see that an immersion must have at least three crossings in order to be knottable. There are immersions of P_9, PG, and P_8 that have only two crossings (see, for example Figure 7), so clearly these graphs are not intrinsically knottable.

Theorem 3.4. *The graph K_6 is intrinsically knottable.*

Our proof of this theorem relies heavily on the following lemma which is similar to Lemma 3.2.

Lemma 3.5. *Let D_4' be a graph with four vertices, two nonadjacent 2-cycles C_1 and C_2, and two nonadjacent edges A_1 and A_2 that connect C_1 and C_2 (see Figure 8). Given any immersion of D_4', if C_1 and C_2 cross and A_1 and A_2 cross, then the immersion is knottable.*

Proof. Take any immersion of D_4' such that C_1 and C_2 cross and A_1 and A_2 are crossed. Assign over/under information to the crossings of C_1 and C_2 such that $lk_2(C_1, C_2) = 1$. We will show that there is a way to assign over/under information to the crossings on A_1 and A_2 such that the resulting embedding contains a knot.

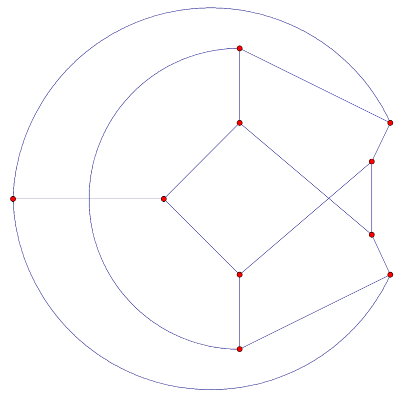


Figure 7. An immersion of the classic Petersen graph with only two crossings.

Let S be the set of all Hamiltonian cycles of D'_4. Given any embedding of D'_4, we can define σ as follows:

$$\sigma = \sum_{C \in S} \alpha(C).$$

For disjoint arcs a_1 and a_2 in an embedding of D'_4, define $\omega(a_1, a_2) \in \mathbb{Z}_2$ to be the number of times mod 2 that a_1 crosses over a_2. Note that by definition, for any embedding of D'_4,

$$\omega(e_1, e_3) + \omega(e_1, e_4) + \omega(e_2, e_3) + \omega(e_2, e_4) = lk_2(C_1, C_2).$$

Assign arbitrarily all crossings of A_1 with A_2 but one. Consider the crossing that has not been assigned. Let D_+ denote the embedding of D'_4 in which A_1 crosses over A_2 at that crossing and D_- denote the embedding of D'_4 in which A_2 crosses over A_1. Consider the change $\Delta\sigma$ in σ that will result from changing the crossing on A_1 and A_2.

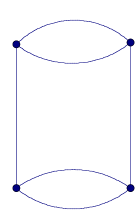


Figure 8. The graph D'_4.

Let C be a Hamiltonian cycle containing A_1 and A_2 and $\epsilon(C)$ be the change in $\alpha(C)$ induced by the crossing change. Now by Lemma 3.1 above,

$$\epsilon(C) = \alpha(C_+) + \alpha(C_-) = lk_2(L_1, L_2) = \sum_{E_1 \in L_1, E_2 \in L_2} \omega(E_1, E_2).$$

Now, summing up $\epsilon(C)$ over all Hamiltonian cycles C gives the change in σ. Fortunately most of the terms cancel out and we are left with

$$\begin{aligned} \Delta\sigma &= \sum_{C \in S} \epsilon(C) \\ &= \omega(e_1, e_3) + \omega(e_1, e_4) + \omega(e_2, e_3) + \omega(e_2, e_4) \\ &= lk_2(C_1, C_2) = 1. \end{aligned}$$

This means that either D_+ or D_- contains a knot. □

Proof of Theorem 3.4. Take any lift of any immersion of K_6. Since K_6 is intrinsically linked, there is a pair of linked triangles, C_1 and C_2, in the resulting embedding.

Suppose that we temporarily ignore the edges of C_1 and C_2. We are left with $K_{3,3}$, which has a crossing in nonadjacent edges, say A_1 and A_2. Notice that A_1 and A_2 connect the cycles C_1 and C_2. The cycles C_1 and C_2, along with the edges A_1 and A_2, make up a subgraph of K_6 that is D_4' (with some extra degree 2 vertices). Since C_1 and C_2 are linkable and A_1 and A_2 cross, this subgraph immersion is knottable, by Lemma 3.5. Thus K_6 is knottable. □

Now we show that $K_{4,4} - e$ is intrinsically knottable. First we need the following lemma.

Lemma 3.6. *Suppose G is a graph that contains in every immersion two pairs of linkable cycles*, C_1 *and* C_2, C_3 *and* C_4. *Suppose the union of the cycles is an expansion of D_4 with C_1 and C_2 opposite each other and C_3 and C_4 opposite each other* (*so C_1 and C_2 are disjoint*, *C_3 and C_4 are disjoint*, *and all other pairs of C_i and C_j*, *for $i \neq j$*, *intersect in either a vertex*, *an edge*, *or a simple path*). *If there is a way to orient the cycles consistently*, *then G is intrinsically knottable.*

Proof. Orient the cycles in a consistent way, and assign all crossings to be positive. Then $lk(C_1, C_2)$ and $lk(C_3, C_4)$ are both positive. Since the cycles C_1, C_2, C_3, and C_4 form a subgraph of G that is an expansion of D_4 with the desired linking properties, we can apply Corollary 3.3 and conclude that the resulting embedding contains a knot. □

Theorem 3.7. *The graph $K_{4,4} - e$ is intrinsically knottable.*

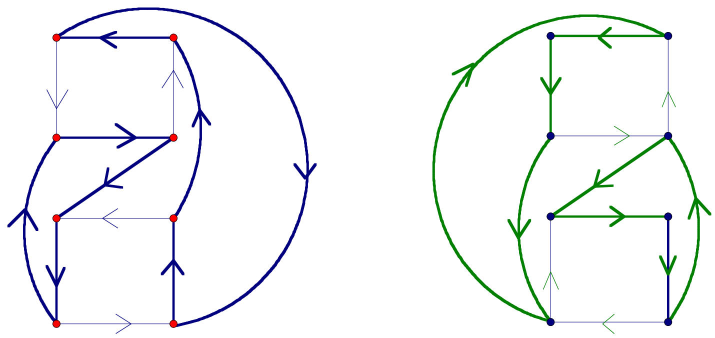


Figure 9. Case 1 (left): C_3 shares exactly one edge with C_1 and one edge with C_2. Case 2 (right): C_3 shares exactly one edge with C_1 and one edge with C_2.

Proof. We first label the vertices of $K_{4,4}-e$ as $v_1, \dots, v_4, w_1, \dots, w_4$, where every v_i belongs to one partition and every w_i belongs to the other partition. Let (v_1, w_3) be the missing edge.

Take any lift of any immersion of $K_{4,4}-e$. Since $K_{4,4}-e$ is intrinsically linked, there is a pair of nonsplittably linked (thus linkable) 4-cycles in the lift embedding. We again denote these 4-cycles as C_1 and C_2 where C_1 is (v_1, w_1, v_2, w_2) and C_2 is (v_3, w_3, v_4, w_4). (Up to symmetry this is the only way to get disjoint 4-cycles.)

Now the subgraph of $K_{4,4}-e$ resulting from the removal of (v_1, w_1) is intrinsically linkable by Theorem 2.3 above. So there is a pair of linkable cycles, C_3 and C_4 in the subgraph. There are two ways in which C_3 and C_4 can be related to C_1 and C_2: C_3 shares exactly one edge with C_1 and one edge with C_2, or C_3 shares exactly one edge with C_1 and one edge with C_2.

In each case, there is a way to orient the cycles C_1, C_2, C_3, and C_4 consistently. (See Figure 9.) Since the cycles C_1, C_2, C_3, and C_4 form a subgraph of $K_{4,4}-e$ that is an expansion of D_4 with the desired linkability properties, we can apply Lemma 3.6 and conclude that $K_{4,4}-e$ is intrinsically knottable. □

The techniques of this proof can also be applied to prove that K_6, P_7 and $K_{3,3,1}$ are intrinsically knottable.

3C. *Strongly knottable graphs.* We say that a graph G is *strongly knottable* if every expansion of G is intrinsically knottable.

Proposition 3.8. *The graphs K_6, $K_{3,3,1}$, $K_{4,4}$, and P_7 are not strongly knottable.*

Proof. In Figures 10 and 11, we exhibit immersions of expansions of K_6 and $K_{4,4}$, such that each immersion has only two crossings, and thus certainly is not knottable. Similar immersions for P_7 and $K_{3,3,1}$ exist. □

This leads us to the following conjecture.

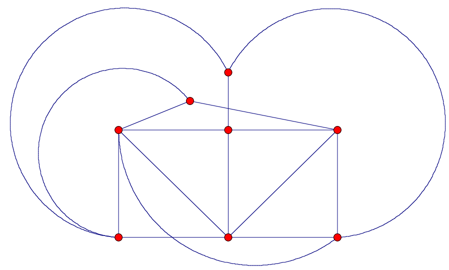


Figure 10. An immersion of an expansion of K_6 with only two crossings.

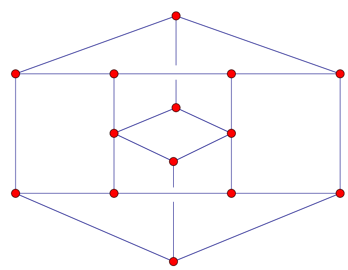


Figure 11. An immersion of an expansion of $K_{4,4}$ with only two crossings.

Conjecture 3.9. *A graph is strongly knottable if and only if it is intrinsically knotted.*

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Invariant polynomials and minimal zero sequences

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A connection is developed between polynomials invariant under abelian permutation of their variables and minimal zero sequences in a finite abelian group. This connection is exploited to count the number of minimal invariant polynomials for various abelian groups.

1. Introduction

Invariant theory has a long and beautiful history, with early work by Hilbert [1893] and Noether [1915]. Classically, it is concerned with polynomials over $\mathbb{R}$ or $\mathbb{C}$ that are invariant over certain permutations of their variables. For an introduction to this subject, see any of [Dolgachev 2003; Neusel and Smith 2002; Olver 1999].

Minimal zero sequences (also called minimal zero-sum sequences) have also been the subject of considerable study (for example, see [Chapman et al. 2001; Gao and Geroldinger 1999; Geroldinger and Schneider 1992; Mazur 1992; van Emde Boas and Kruyswijk 1967]). They are multisets of elements from a fixed finite abelian group G subject to the restriction that the sum (according to multiplicity) must be zero in G. This forms a semigroup under the multiset sum operation. For an introduction, see one of [Caro 1996; Gao and Geroldinger 2006; Geroldinger and Halter-Koch 2006; Halter-Koch 1997].

Our main result, Theorem 1, connects these two areas of mathematics. Let G be a finite abelian group, and let $\mathfrak{I}$ be the subalgebra of the polynomial ring on the $|G|$ variables that is invariant under the variable permutation induced by G. We provide a canonical representation for $\mathfrak{I}$ under which the natural set of generators are bijective with minimal zero sequences of G. Since the 1948 paper of Strom [1948], which settled the case where G has rank one, only partial progress [Kraft and Procesi 1996; Schmid 1991] has been made in this area.

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Theorem 1. *There exists a canonical set of generators of $\mathfrak{I}$ in bijective correspondence with the set of minimal zero sequences of G, where generators of degree k correspond to sequences of cardinality k.*

2. Applications

Our result permits us to count canonical generators of $\mathfrak{I}$ more efficiently, both by degree and in total. These results, found in Table 1[1], use minimal zero sequence counting algorithms such as that found in [Finklea et al. $\geq$ 2008] which recursively finds zero-free sequences. We are thus able to extend the table found in [Strom 1948] substantially. The total number of canonical generators for cyclic G (the rightmost column of Table 1) is extended in Table 2.[2] We can similarly report the total number of canonical generators for some groups of the form $\mathbb{Z}_m \oplus \mathbb{Z}_n$ in Table 3. Some of these are of rank one and also appear in Table 1; they are included for completeness.

The relation between these two areas has great potential for mutual benefit. For example, two conjectures of Elashvili, as stated in [Harris and Wehlau 2006], have already been partially proved in [Ponomarenko 2004] and fully proved in [Yuan 2007], by considering Theorem 1.

3. Proof of main theorem

Fix the finite abelian group $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$. We consider the polynomial ring in the variables x_g, for each $g \in G$. We let $h \in G$ act on the variables via $h : x_g \to x_{h+g}$. Let $\mathfrak{I}$ denote the subring that is invariant under all $|G|$ such actions, and equivalently invariant under the k actions

$$e_1 = (-1, 0, \ldots, 0),$$
$$e_2 = (0, -1, \ldots, 0), \ldots,$$
$$e_k = (0, 0, \ldots, -1).$$

(The actions are chosen to be the negatives of the standard basis for technical reasons, to be evident later. These elements generate G.)

We will describe a degree-preserving change of variables that will preserve $\mathfrak{I}$. After this change, the group action on the original variables will act on the new canonical variables as scalar multiplication.

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[1] Space considerations limit the size of these tables; larger versions are available (together with the software used to generate them) up to $\mathbb{Z}_{64}$ at `http://www-rohan.sdsu.edu/~vadim/research.html`

[2] These results, through other methods, were also found by A. Elashvili and V. Tsiskaridze [Elashvili and Tsiskaridze $\geq$ 2008]. Their unpublished data matches ours, and equally continues to $\mathbb{Z}_{64}$.

G	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Total
$\mathbb{Z}_1$	1															1
$\mathbb{Z}_2$	1	1														2
$\mathbb{Z}_3$	1	1	2													4
$\mathbb{Z}_4$	1	2	2	2												7
$\mathbb{Z}_5$	1	2	4	4	4											15
$\mathbb{Z}_6$	1	3	6	6	2	2										20
$\mathbb{Z}_7$	1	3	8	12	12	6	6									48
$\mathbb{Z}_8$	1	4	10	18	16	8	4	4								65
$\mathbb{Z}_9$	1	4	14	26	32	18	12	6	6							119
$\mathbb{Z}_{10}$	1	5	16	36	48	32	12	8	4	4						166
$\mathbb{Z}_{11}$	1	5	20	50	82	70	50	30	20	10	10					348
$\mathbb{Z}_{12}$	1	6	24	64	104	84	36	20	12	8	4	4				367
$\mathbb{Z}_{13}$	1	6	28	84	168	180	132	84	60	36	24	12	12			827
$\mathbb{Z}_{14}$	1	7	32	104	216	242	162	96	42	30	18	12	6	6		974
$\mathbb{Z}_{15}$	1	7	38	130	306	388	264	120	88	56	40	24	16	8	8	1494

Table 1. Number of canonical generators of $\mathfrak{J}$, by degree.

For all $m \in \mathbb{N}$, we set $\varepsilon_m = e^{\frac{2\pi\sqrt{-1}}{m}}$, where e is the usual transcendental 2.718 . . . . We will need two well-known properties (for example, see [Ahlfors 1978] or [Davenport 2000]).

$\mathbb{Z}_1$	1	$\mathbb{Z}_{16}$	2135	$\mathbb{Z}_{31}$	280352	$\mathbb{Z}_{46}$	7581158
$\mathbb{Z}_2$	2	$\mathbb{Z}_{17}$	3913	$\mathbb{Z}_{32}$	295291	$\mathbb{Z}_{47}$	10761816
$\mathbb{Z}_3$	4	$\mathbb{Z}_{18}$	4038	$\mathbb{Z}_{33}$	405919	$\mathbb{Z}_{48}$	9772607
$\mathbb{Z}_4$	7	$\mathbb{Z}_{19}$	7936	$\mathbb{Z}_{34}$	508162	$\mathbb{Z}_{49}$	15214301
$\mathbb{Z}_5$	15	$\mathbb{Z}_{20}$	8247	$\mathbb{Z}_{35}$	674630	$\mathbb{Z}_{50}$	15826998
$\mathbb{Z}_6$	20	$\mathbb{Z}_{21}$	12967	$\mathbb{Z}_{36}$	708819	$\mathbb{Z}_{51}$	20930012
$\mathbb{Z}_7$	48	$\mathbb{Z}_{22}$	17476	$\mathbb{Z}_{37}$	1230259	$\mathbb{Z}_{52}$	23378075
$\mathbb{Z}_8$	65	$\mathbb{Z}_{23}$	29162	$\mathbb{Z}_{38}$	1325732	$\mathbb{Z}_{53}$	34502651
$\mathbb{Z}_9$	119	$\mathbb{Z}_{24}$	28065	$\mathbb{Z}_{39}$	1709230	$\mathbb{Z}_{54}$	32192586
$\mathbb{Z}_{10}$	166	$\mathbb{Z}_{25}$	49609	$\mathbb{Z}_{40}$	1868565	$\mathbb{Z}_{55}$	44961550
$\mathbb{Z}_{11}$	348	$\mathbb{Z}_{26}$	59358	$\mathbb{Z}_{41}$	3045109	$\mathbb{Z}_{56}$	47162627
$\mathbb{Z}_{12}$	367	$\mathbb{Z}_{27}$	83420	$\mathbb{Z}_{42}$	2804474	$\mathbb{Z}_{57}$	63662925
$\mathbb{Z}_{13}$	827	$\mathbb{Z}_{28}$	97243	$\mathbb{Z}_{43}$	4694718	$\mathbb{Z}_{58}$	74515122
$\mathbb{Z}_{14}$	974	$\mathbb{Z}_{29}$	164967	$\mathbb{Z}_{44}$	4695997	$\mathbb{Z}_{59}$	102060484
$\mathbb{Z}_{15}$	1494	$\mathbb{Z}_{30}$	152548	$\mathbb{Z}_{45}$	5902561	$\mathbb{Z}_{60}$	85954379

Table 2. Total number of canonical generators for $G = \mathbb{Z}_n$.

	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_5$	$\mathbb{Z}_6$	$\mathbb{Z}_7$
$\mathbb{Z}_2$	5	20	39	166	253	974
$\mathbb{Z}_3$	20	69	367	1494	2642	12967
$\mathbb{Z}_4$	39	367	1107	8247	19463	97243
$\mathbb{Z}_5$	166	1494	8247	31029	152548	674630
$\mathbb{Z}_6$	253	2642	19463	152548	390861	2804474
$\mathbb{Z}_7$	974	12967	97243	674630	2804474	9540473

Table 3. Total number of canonical generators for $G = \mathbb{Z}_m \oplus \mathbb{Z}_n$.

Proposition 1. *Let ε_m be as above. Then*

(1) $(\varepsilon_m)^k = 1$ *if and only if m divides k.*

(2) *Let $j \in \mathbb{Z}$. Then* $\sum_{k=0}^{m-1} (\varepsilon_m)^{jk} = \begin{cases} m, & \text{if } m \text{ divides } j; \\ 0, & \text{otherwise.} \end{cases}$

For $g \in G$, we use $(g)_i \in \mathbb{Z}$ to denote the projection of g onto the i-th coordinate (for $1 \leq i \leq k$). For each $h \in G$, we define new variables y_h via:

$$y_h = \sum_{g \in G} \Big(\prod_{i=1}^{k} (\varepsilon_{n_i})^{(g)_i (h)_i} \Big) x_g.$$

The inverse change of basis is given explicitly below; hence this basis change is degree-preserving.

Lemma 1. *For all $g \in G$ we have*

$$x_g = \frac{1}{|G|} \sum_{h \in G} \Big(\prod_{j=1}^{k} (\varepsilon_{n_j})^{(h)_j (-(g)_j)} \Big) y_h.$$

Proof. We substitute for y_h into the right hand side to get:

$$\frac{1}{|G|} \sum_{h \in G} \Big(\prod_{j=1}^{k} (\varepsilon_{n_j})^{(h)_j (-(g)_j)} \Big) \sum_{g' \in G} \Big(\prod_{i=1}^{k} (\varepsilon_{n_i})^{(g')_i (h)_i} \Big) x_{g'} =$$

$$\frac{1}{|G|} \sum_{g' \in G} x_{g'} \sum_{h \in G} \Big(\prod_{i=1}^{k} (\varepsilon_{n_i})^{(h)_i ((g')_i - (g)_i)} \Big) = \frac{1}{|G|} \sum_{g' \in G} x_{g'} \left\{ \begin{array}{ll} |G|, & \text{if } g = g'; \\ 0, & \text{otherwise.} \end{array} \right\}$$

In the last step, if $g = g'$, then each term in the innermost product is 1. Otherwise, for some w, we have $(g')_w - (g)_w \neq 0$. We now collect the summands n_w at a time, where the w-th coordinate assumes all possible values and the other coordinates are fixed. We pull out the common factors and apply Proposition 1 to get 0. □

Under the canonical basis $\{y_h\}$, the k actions permuting the variables act as scalar multiplication.

Lemma 2. $e_j : y_h \to (\varepsilon_{n_j})^{(h)_j} y_h$.

Proof. We have

$$e_j(y_h) = \sum_{g \in G} \Big(\prod_{i=1}^{k} (\varepsilon_{n_i})^{(g)_i (h)_i} \Big) x_{g+e_j}$$

$$= \sum_{(g+e_j) \in G} \Big(\prod_{i=1}^{k} (\varepsilon_{n_i})^{(g+e_j-e_j)_i (h)_i} \Big) x_{g+e_j} = \sum_{g \in G} \Big(\prod_{i=1}^{k} (\varepsilon_{n_i})^{(g-e_j)_i (h)_i} \Big) x_g$$

$$= y_h (\varepsilon_{n_j})^{-(e_j)_j (h)_j} = y_h (\varepsilon_{n_j})^{(h)_j}. \qquad \square$$

An immediate consequence of the above is that $e_j : y_h^a \to (\varepsilon_{n_j})^{a(h)_j} y_h^a$. More generally, we can calculate the effect of e_j on an arbitrary monomial.

Lemma 3. *For constant* α, $e_j : \alpha \prod_{h \in G} y_h^{a_h} \to \Big((\varepsilon_{n_j})^{\sum_{h \in G} a_h (h)_j} \Big) \alpha \prod_{h \in G} y_h^{a_h}$.

Observe that under the canonical basis all invariant polynomials may be written as the sum of invariant monomials. Further, each invariant monomial may be written as the product of invariant monomials. Hence, there is a canonical set of generators of $\mathfrak{I}$ under the canonical basis, namely the set of irreducible invariant monomials.

Consider an irreducible monomial $\prod_{h \in G} y_h^{a_h}$. We must have

$$\sum_{h \in G} a_h (h)_j \equiv 0 \pmod{n_j}$$

for each j. Combining these j requirements, we get

$$\sum_{h \in G} a_h h = 0,$$

where 0 is the zero element in G. Therefore, we can consider the a_h as multiplicities for each element $h \in G$, and since the sum is zero we have a zero sequence. Further, this must be a minimal zero sequence by the irreducibility of the generator. Conversely, every minimal zero sequence yields an irreducible monomial.

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Boundary data smoothness for solutions of nonlocal boundary value problems for n-th order differential equations

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Under certain conditions, solutions of the boundary value problem

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}),$$

$y^{(i-1)}(x_1) = y_i$ for $1 \le i \le n-1$, and $y(x_2) - \sum_{i=1}^{m} r_i y(\eta_i) = y_n$, are differentiated with respect to boundary conditions, where $a < x_1 < \eta_1 < \dots < \eta_m < x_2 < b$, and $r_1, \dots, r_m, y_1, \dots, y_n \in \mathbb{R}$.

1. Introduction

In this paper, we will be concerned with differentiating solutions of certain nonlocal boundary value problems with respect to boundary data for the n-th order ordinary differential equation

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \quad a < x < b, \tag{1}$$

satisfying

$$y^{(i-1)}(x_1) = y_i, \ 1 \le i \le n-1, \qquad y(x_2) - \sum_{k=1}^{m} r_k y(\eta_k) = y_n, \tag{2}$$

where $a < x_1 < \eta_1 < \dots < \eta_m < x_2 < b$, and $y_1, \dots, y_n, r_1, \dots, r_m \in \mathbb{R}$, and where we assume

(i) $f(x, u_1, \dots, u_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous,

(ii) $\partial f / \partial u_i (x, u_1, \dots, u_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ are continuous, $i = 1, 2, \dots, n$, and

(iii) solutions of initial value problems for (1) extend to (a, b).

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We remark that condition (iii) is not necessary for the spirit of this work's results, however, by assuming (iii), we avoid continually making statements in terms of solutions' maximal intervals of existence.

Under uniqueness assumptions on solutions of (1) and (2), we will establish analogues of a result that Hartman [1964] attributes to Peano concerning differentiation of solutions of (1) with respect to initial conditions. For our differentiation with respect to the boundary conditions results, given a solution $y(x)$ of (1), we will give much attention to the *variational equation for* (1) *along* $y(x)$, which is defined by

$$z^{(n)} = \sum_{k=1}^{n} \frac{\partial f}{\partial u_k}(x, y(x),\ y'(x), \ldots, y^{(n-1)}(x)) z^{(k-1)}. \tag{3}$$

There has long been interest in multipoint nonlocal boundary value problems for ordinary differential equations, with much attention given to positive solutions. To see only a few of these papers, we refer the reader to [Bai and Fang 2003; Gupta and Trofimchuk 1998; Ma 1997; 2002; Yang 2002].

Likewise, many papers have been devoted to smoothness of solutions of boundary value problems with respect to boundary data. For a view of how this work has evolved, involving not only boundary value problems for ordinary differential equations, but also discrete versions, functional differential equations versions and dynamic equations on time scales versions, we suggest results from among the many papers [Datta 1998; Ehme 1993; Ehme et al. 1993; Ehme and Henderson 1996; Ehme and Lawrence 2000; Hartman 1964; Henderson 1984; 1987; Henderson et al. 2005; Henderson and Lawrence 1996; Lawrence 2002; Peterson 1976; 1978; 1981; 1987; Spencer 1975]. In fact, smoothness results have been given some consideration for (1) and (2) when $n = 2$ and for specific and general values of m [Ehrke et al. 2007; Henderson and Tisdell 2004].

The theorem for which we seek an analogue and attributed to Peano by Hartman can be stated in the context of (1) as follows

Theorem 1.1. [Peano] *Assume that, with respect to* (1), *conditions* (i)–(iii) *are satisfied. Let* $x_0 \in (a, b)$ *and* $y(x) \equiv y(x, x_0, c_1, c_2, \ldots, c_n)$ *denote the solution of* (1) *satisfying the initial conditions* $y^{(i-1)}(x_0) = c_i,\ 1 \le i \le n$. *Then,*

(i) *For each* $1 \le i \le n$, $\partial y/\partial c_i$ *exists on* (a, b) *and* $\alpha_i \equiv \partial y/\partial c_i$ *is a solution of the variational equation* (3) *along* $y(x)$ *and satisfies the initial condition,*

$$\alpha_i^{(j-1)}(x_0) = \delta_{ij}, \quad 1 \le i, j \le n.$$

(ii) $\partial y/\partial x_0$ *exists on* (a, b), *and* $\beta \equiv \partial y/\partial x_0$ *is the solution of the variational equation* (3) *along* $y(x)$ *satisfying the initial conditions,*

$$\beta^{(i-1)}(x_0) = -y^{(i)}(x_0), \quad 1 \le i \le n.$$

(iii) $\partial y/\partial x_0(x) = -\sum_{k=1}^{n} y^{(k)}(x_0)\partial y/\partial c_k(x)$.

In addition, our analogue of Theorem 1.1 depends on uniqueness of solutions of (1) and (2), a condition we list as an assumption.

(iv) Given $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$, if $y^{(i-1)}(x_1) = z^{(i-1)}(x_1)$ for each $1 \le i \le n-1$, and $y(x_2) - \sum_{k=1}^{m} r_k y(\eta_k) = z(x_2) - \sum_{k=1}^{m} r_k z(\eta_k)$, where $y(x)$ and $z(x)$ are solutions of (1), then $y(x) \equiv z(x)$.

We will also make extensive use of a similar uniqueness condition on (3) along solutions $y(x)$ of (1).

(v) Given $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$, and a solution $y(x)$ of (1), if $u^{(i-1)}(x_1) = 0$, $1 \le i \le n-1$, and $u(x_2) - \sum_{k=1}^{m} r_k u(\eta_k) = 0$, where $u(x)$ is a solution of (3) along $y(x)$, then $u(x) \equiv 0$.

2. An analogue of Peano's Theorem for Equations (1) and (2)

In this section, we derive our analogue of Theorem 1.1 for boundary value problem (1), (2). For such a differentiation result, we need continuous dependence of solutions on boundary conditions. The arguments for this continuous dependence follow much along the lines of those in [Henderson and Tisdell 2004], when (1) is of second order. For that reason, we omit the details of the proof.

Theorem 2.1. *Assume* (i)–(iv) *are satisfied with respect to* (1). *Let* $u(x)$ *be a solution of* (1) *on* (a, b), *and let* $a < c < x_1 < \eta_1 < \cdots < \eta_m < x_2 < d < b$ *be given. Then, there exists a* $\delta > 0$ *such that, for*

$$|x_i - t_i| < \delta, \qquad i = 1, 2,$$

$$|\eta_i - \tau_i| < \delta \quad \text{and} \quad |r_i - \rho_i| < \delta, \qquad 1 \le i \le m,$$

$$|u^{(i-1)}(x_1) - y_i| < \delta, \qquad 1 \le i \le n-1$$

$$\Big|u(x_2) - \sum_{k=1}^{m} r_k u(\eta_k) - y_n\Big| < \delta,$$

there exists a unique solution $u_\delta(x)$ *of* (1) *such that*

$$u_\delta^{(i-1)}(t_1) = y_i, \quad 1 \le i \le n-1,$$

$$u_\delta(t_2) - \sum_{k=1}^{m} \rho_k u_\delta(\tau_k) = y_n$$

and $\{u_\delta^{(j-1)}(x)\}$ *converges uniformly to* $u^{(j-1)}(x)$, *as* $\delta \to 0$, *on* $[c, d]$, *for* $1 \le j \le n$.

We now present the result of the paper.

Theorem 2.2. *Assume conditions* (i)–(v) *are satisfied. Let* $u(x)$ *be a solution* (1) *on* (a,b). *Let* $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$ *be given, so that*

$$u(x) = u(x, x_1, x_2, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m),$$

where $u^{(i-1)}(x_1) = u_i$, $1 \le i \le n-1$, *and* $u(x_2) - \sum_{k=1}^{m} r_k u(\eta_k) = u_n$. *Then,*

(i) *For each* $1 \le i \le n$, $\partial u/\partial u_i$ *exists on* (a,b). *Moreover, for each* $1 \le j \le n-1$, $y_j \equiv \partial u/\partial u_j$ *solves Equation* (3) *along* $u(x)$ *and satisfies the boundary conditions,*

$$y_j^{(i-1)}(x_1) = \delta_{ij},\ 1 \le i \le n-1, \quad y_j(x_2) - \sum_{k=1}^{m} r_k y_j(\eta_k) = 0,$$

and $y_n \equiv \partial u/\partial u_n$ *solves* (3) *along* $u(x)$ *and satisfies the boundary conditions,*

$$y_n^{(i-1)}(x_1) = 0,\ 1 \le i \le n-1, \quad y_n(x_2) - \sum_{k=1}^{m} r_k y_n(\eta_k) = 1.$$

(ii) $\partial u/\partial x_1$ *and* $\partial u/\partial x_2$ *exist on* (a,b), *and* $z_i \equiv \partial u/\partial x_i$, $i = 1, 2$, *are solutions of* (3) *along* $u(x)$ *and satisfy the respective boundary conditions,*

$$z_1^{(i-1)}(x_1) = -u^{(i)}(x_1),\ 1 \le i \le n-1, \qquad z_1(x_2) - \sum_{k=1}^{m} r_k z_1(\eta_k) = 0,$$

$$z_2^{(i-1)}(x_1) = 0,\ 1 \le i \le n-1, \qquad z_2(x_2) - \sum_{k=1}^{m} r_k z_2(\eta_k) = -u'(x_2).$$

(iii) *For* $1 \le j \le m$, $\partial u/\partial \eta_j$ *exists on* (a,b), *and* $w_j \equiv \partial u/\partial \eta_j$, $j = 1, \ldots, m$, *is a solution of* (3) *along* $u(x)$ *and satisfies*

$$w_j^{(i-1)}(x_1) = 0,\ 1 \le i \le n-1, \quad w_j(x_2) - \sum_{k=1}^{m} r_k w_j(\eta_k) = r_j u'(\eta_j).$$

(iv) *For* $1 \le j \le m$, $\partial u/\partial r_j$ *exists on* (a,b), *and* $v_j \equiv \partial u/\partial r_j$, $j = 1, \ldots, m$, *is a solution of* (3) *along* $u(x)$ *and satisfies,*

$$v_j^{(i-1)}(x_1) = 0,\ 1 \le i \le n-1, \quad v_j(x_2) - \sum_{k=1}^{m} r_k v_j(\eta_k) = u(\eta_j).$$

Proof. For part (i), let $1 \le j \le n-1$, and consider $\partial u/\partial u_j$, since the argument for $\partial u/\partial u_n$ is similar. In this case we designate, for brevity, $u(x, x_1, x_2, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$ by $u(x, u_j)$.

Let $\delta > 0$ be as in Theorem 2.1. Let $0 < |h| < \delta$ be given and define

$$y_{jh}(x) = \frac{1}{h}\big[u(x, u_j + h) - u(x, u_j)\big].$$

Note that $u^{(j-1)}(x_1, u_j + h) = u_j + h$, and $u^{(j-1)}(x_1, u_j) = u_j$, so that, for every $h \neq 0$,

$$y_{jh}^{(j-1)}(x_1) = \frac{1}{h}[u_j + h - u_j] = 1.$$

Also, for every $h \neq 0,\ 1 \le i \le n-1,\ i \neq j$,

$$y_{jh}^{(i-1)}(x_1) = \frac{1}{h}\big[u^{(i-1)}(x_1, u_j + h) - u^{(i-1)}(x_1, u_j)\big] = \frac{1}{h}[u_i - u_i] = 0,$$

and

$$y_{jh}(x_2) - \sum_{k=1}^{m} r_k y_{jh}(\eta_k) = \frac{1}{h}\big[u(x_2, u_j + h) - u(x_2, u_j)\big] - \sum_{k=1}^{m} \frac{r_k}{h}\big[u(\eta_k, u_j + h) - u(\eta_k, u_j)\big] = \frac{1}{h}[u_n - u_n] = 0.$$

Let $\beta = u^{(n-1)}(x_1, u_j)$, and $\epsilon = \epsilon(h) = u^{(n-1)}(x_1, u_j + h) - \beta$. By Theorem 2.1, $\epsilon = \epsilon(h) \to 0$, as $h \to 0$. Using the notation of Theorem 1.1 for solutions of initial value problems for Equation (1) and viewing the solutions u as solutions of initial value problems and denoting $y(x, x_1, u_1, \ldots, u_j, \ldots, u_{n-1}, \beta)$ by $y(x, x_1, u_j, \beta)$, we have

$$y_{jh}(x) = \frac{1}{h}\big[y(x, x_1, u_j + h, \beta + \epsilon) - y(x, x_1, u_j, \beta)\big].$$

Then, by utilizing a telescoping sum, we have

$$y_{jh}(x) = \frac{1}{h}\big[\{y(x, x_1, u_j + h, \beta + \epsilon) - y(x, x_1, u_j, \beta + \epsilon)\} + \{y(x, x_1, u_j, \beta + \epsilon) - y(x, x_1, u_j, \beta)\}\big].$$

By Theorem 1.1 and the Mean Value Theorem, we obtain

$$y_{jh}(x) = \frac{1}{h}\alpha_j\big(x, y(x, x_1, u_j + \bar{h},\ \beta + \epsilon)\big)(u_j + h - u_j) + \frac{1}{h}\alpha_n\big(x, y(x, x_1, u_j, \beta + \bar{\epsilon})\big)(\beta + \epsilon - \beta),$$

where $\alpha_k(x, y(\cdot)),\ k \in \{j, n\}$, is the solution of the variational Equation (3) along $y(\cdot)$ and satisfies, in each case,

$$\alpha_j^{(i-1)}(x_1) = \delta_{ij} \quad \alpha_n^{(i-1)}(x_1) = \delta_{in}, \qquad 1 \le i \le n,$$

respectively. Furthermore, $u_j + \bar{h}$ is between u_j and $u_j + h$, and $\beta + \bar{\epsilon}$ is between β and $\beta + \epsilon$. Now simplifying,

$$y_{jh}(x) = \alpha_j\big(x, y(x, x_1, u_j + \bar{h}, \beta + \epsilon)\big) + \frac{\epsilon}{h}\alpha_n\big(x, y(x, x_1, u_j, \beta + \bar{\epsilon})\big).$$

Thus, to show $\lim_{h\to 0} y_{jh}(x)$ exists, it suffices to show $\lim_{h\to 0} \epsilon/h$ exists.

Now $\alpha_n(x, y(\cdot))$ is a nontrivial solution of Equation (3) along $y(\cdot)$, and

$$\alpha_n^{(i-1)}(x_1, y(\cdot)) = 0, \qquad 1 \le i \le n-1.$$

So, by assumption (v), $\alpha_n(x_2, y(\cdot)) - \sum_{k=1}^m r_k \alpha_n(\eta_k, y(\cdot)) \ne 0$. However, we observed that $y_{jh}(x_2) - \sum_{k=1}^m r_k y_{jh}(\eta_k) = 0$, from which we obtain

$$\frac{\epsilon}{h} = \frac{\sum_{k=1}^m r_k \alpha_j\big(\eta_k, y(x, x_1, u_j + \bar{h}, \beta + \epsilon)\big) - \alpha_j\big(x_2, y(x, x_1, u_j + \bar{h}, \beta + \epsilon)\big)}{\alpha_n\big(x_2, y(x, x_1, u_j, \beta + \bar{\epsilon})\big) - \sum_{k=1}^m r_k \alpha_n\big(\eta_k, y(x, x_1, u_j, \beta + \bar{\epsilon})\big)}.$$

As a consequence of continuous dependence, we can let $h \to 0$, so that

$$\begin{aligned}\lim_{h\to 0} \frac{\epsilon}{h} &= -\frac{\alpha_j\big(x_2, y(x, x_1, u_j, \beta_2)\big) - \sum_{k=1}^m r_k \alpha_j\big(\eta_k, y(x, x_1, u_j, \beta)\big)}{\alpha_n\big(x_2, y(x, x_1, u_j, \beta)\big) - \sum_{k=1}^m r_k \alpha_n\big(\eta_k, y(x, x_1, u_j, \beta)\big)} \\ &= -\frac{\alpha_j(x_2, u(x)) - \sum_{k=1}^m r_k \alpha_j(\eta_k, u(x))}{\alpha_n(x_2, u(x)) - \sum_{k=1}^m r_k \alpha_n(\eta_k, u(x))} =: D.\end{aligned}$$

Let $y_j(x) = \lim_{h\to 0} y_{jh}(x)$, and note by construction of $y_{jh}(x)$ that

$$y_j(x) = \frac{\partial u}{\partial u_j}(x).$$

Furthermore,

$$y_j(x) = \lim_{h\to 0} y_{jh}(x) = \alpha_j(x, y(x, x_1, u_j, \beta)) + D\alpha_n(x, (u(x)),$$

which is a solution of the variational Equation (3) along $u(x)$. In addition because of the boundary conditions satisfied by $y_{jh}(x)$, we also have

$$y_j^{(i-1)}(x_1) = \delta_{ij}, \ 1 \le i \le n-1, \quad y_j(x_2) - \sum_{k=1}^m r_k y_j(\eta_k) = 0.$$

This completes the argument for $\partial u/\partial u_j$.

In part (ii) of the theorem, we will produce the details for $\partial u/\partial x_1$, with the arguments for $\partial u/\partial x_2$ being similar. This time, we designate

$$u(x, x_1, x_2, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$$

by $u(x, x_1)$.

So, let $\delta > 0$ be as in Theorem 2.1, let $0 < |h| < \delta$ be given, and define

$$z_{1h}(x) = \frac{1}{h}[u(x, x_1 + h) - u(x, x_1)].$$

Note that, for $1 \le i \le n-1$,

$$\begin{aligned} z_{1h}^{(i-1)}(x_1) &= \frac{1}{h}[u^{(i-1)}(x_1, x_1+h) - u^{(i-1)}(x_1, x_1)] \\ &= \frac{1}{h}[u^{(i-1)}(x_1, x_1+h) - u^{(i-1)}(x_1+h, x_1+h)] \\ &= -\frac{1}{h}[u^{(i)}(c_{x_1,h}, x_1+h)\cdot h] \\ &= -u^{(i)}(c_{x_1,h}, x_1+h), \end{aligned}$$

where $c_{x_1,h}$ lies between x_1 and x_1+h. In addition, we note that, for every $h \neq 0$,

$$\begin{aligned} z_{1h}(x_2) - \sum_{k=1}^{m} r_k z_{1h}(\eta_k) = \frac{1}{h}[u(x_2, x_1+h) - \sum_{k=1}^{m} r_k u(\eta_k, x_1+h) \\ - \{u(x_2, x_1) - \sum_{k=1}^{m} r_k u(\eta_k, x_1)\}] = \frac{1}{h}[u_n - u_n] = 0. \end{aligned}$$

Next, let

$$\begin{aligned} \beta &= u^{(n-1)}(x_1, x_1), \\ \epsilon_j &= \epsilon_j(h) = u^{(j-1)}(x_1, x_1+h) - u_j, \\ \epsilon_\beta &= \epsilon_\beta(h) = u^{(n-1)}(x_1, x_1+h) - \beta. \end{aligned}$$

Let us note at this point that

$$\frac{\epsilon_j}{h} = z_{1h}^{(j-1)}(x_1) = -u^{(j)}(c_{x_1,h}, x_1+h).$$

By Theorem 2.1, both $\epsilon_j \to 0$ and $\epsilon_\beta \to 0$, as $h \to 0$. As in part (i), we employ the notation of Theorem 1.1 for solutions of initial value problems for (1). Viewing the solutions u as solutions of initial value problems, and denoting

$$y(x, x_1, u_1, \ldots, u_j, \ldots, u_n - 1, \beta)$$

by $y(x, x_1, u_j, \beta)$, we have

$$\begin{aligned} z_{1h}(x) &= \frac{1}{h}[y(x, x_1, u_j+\epsilon_j, \beta+\epsilon_\beta) - y(x, x_1, u_j, \beta)] \\ &= \frac{1}{h}[y(x, x_1, u_j+\epsilon_j, \beta+\epsilon_\beta) - y(x, x_1, u_j, \beta+\epsilon_\beta) \\ &\qquad + y(x, x_1, u_j, \beta+\epsilon_\beta) - y(x, x_1, u_j, \beta)]. \end{aligned}$$

By the Mean Value Theorem,

$$z_{1h}(x) = \frac{1}{h}\big[\epsilon_j \alpha_j\big(x, y(x, x_1, u_j+\bar{\epsilon}_j, \beta+\epsilon_\beta)\big) + \epsilon_\beta \alpha_n\big(x, y(x, x_1, u_j, \beta+\bar{\epsilon}_\beta)\big)\big],$$

where $u_j+\bar{\epsilon}_j$ lies between u_j and $u_j+\epsilon_j$, $\beta+\bar{\epsilon}_\beta$ lies between β and $\beta+\epsilon_\beta$, and $\alpha_j(x, y(\cdot))$ and $\alpha_n(x, y(\cdot))$ are the solutions of Equation (3) along $y(\cdot)$ and satisfy, respectively,

$$\alpha_j^{(i-1)}(x_1)=\delta_{ij}, \quad 1\le i\le n,$$
$$\alpha_n^{(i-1)}(x_1)=\delta_{in}, \quad 1\le i\le n.$$

As before, to show that $\lim_{h\to 0} z_{1h}(x)$ exists, it suffices to show that

$$\lim_{h\to 0}\frac{\epsilon_j}{h} \quad \text{and} \quad \lim_{h\to 0}\frac{\epsilon_\beta}{h}$$

exist. Now, from above,

$$\lim_{h\to 0}\frac{\epsilon_j}{h}=\lim_{h\to 0} z_{1h}^{(j-1)}(x_1)=\lim_{h\to 0} u^{(j)}(c_{x_1,h}, x_1+h)=-u^{(j)}(x_1).$$

Since $\alpha_n(x,y(\cdot))$ is a nontrivial solution of (3) along $y(\cdot)$ and

$$\alpha_n^{(i-1)}(x_1,y(\cdot))=0, \qquad 1\le i\le n-1,$$

it follows from assumption (v) that

$$\alpha_n(x_2,y(\cdot))-\sum_{k=1}^{m} r_k\alpha_n(\eta_k,y(\cdot))\neq 0.$$

Since

$$z_{1h}(x_2)-\sum_{k=1}^{m} r_k z_{1h}(\eta_k)=0,$$

we have

$$\frac{\epsilon_\beta}{h}=\Big(\frac{-\epsilon_j}{h}\Big)\frac{A}{\alpha_n\big(x_2, y(x,x_1,u_j,\beta+\bar{\epsilon}_\beta)\big)-\sum_{k=1}^{m} r_k\alpha_n\big(\eta_k, y(x,x_1,u_j,\beta+\bar{\epsilon}_\beta)\big)},$$

where

$$A=\alpha_j\big(x_2, y(x,x_1,u_j+\bar{\epsilon}_j,\beta+\epsilon_\beta)\big)-\sum_{k=1}^{m} r_k\alpha_j\big(\eta_k, y(x,x_1,u_j+\bar{\epsilon}_j,\beta+\epsilon_\beta)\big).$$

And so,

$$\lim_{h\to 0}\frac{\epsilon_\beta}{h}=\frac{u^{(j)}(x_1)\big[\alpha_j\big(x_2, y(x,x_1,u_j,\beta)\big)-\sum_{i=1}^{m} r_i\alpha_j\big(\eta_i, y(x,x_1,u_j,\beta)\big)\big]}{\alpha_n\big(x_2, y(x,x_1,u_j,\beta)\big)-\sum_{i=1}^{m} r_i\alpha_n\big(\eta_i, y(x,x_1,u_j,\beta)\big)}$$
$$=\frac{u^{(j)}(x_1)\big[\alpha_j\big(x_2,u(x)\big)-\sum_{i=1}^{m} r_i\alpha_j\big(\eta_i,u(x)\big)\big]}{\alpha_n\big(x_2,u(x)\big)-\sum_{i=1}^{m} r_i\alpha_n\big(\eta_i,u(x)\big)} =: E.$$

From the above expression,

$$z_{1h}(x) = \frac{\epsilon_j}{h}\alpha_j\big(x, y(x, x_1, u_j + \bar{\epsilon}_j, \beta + \epsilon_\beta)\big) + \frac{\epsilon_\beta}{h}\alpha_n\big(x, y(x, x_1, u_j, \beta + \bar{\epsilon}_\beta)\big),$$

and we can evaluate the limit as $h \to 0$. If we let $z_1(x) = \lim_{h\to 0} z_{1h}(x)$, then $z_1(x) = \partial u/\partial x_1$, and

$$\begin{aligned} z_1(x) &= \lim_{h\to 0} z_{1h}(x) \\ &= -u^{(j)}(x_1)\alpha_j\big(x, y(x, x_1, u_j, \beta)\big) + E\alpha_n\big(x, y(x, x_1, u_j, \beta)\big) \\ &= -u^{(j)}(x_1)\alpha_j\big(x, u(x)\big) + E\alpha_n\big(x, u(x)\big), \end{aligned}$$

which is a solution of Equation (3) along $u(x)$. In addition, from above observations, $z_1(x)$ satisfies the boundary conditions,

$$z_1^{(i-1)}(x_1) = \lim_{h\to 0} z_{1h}^{(i-1)}(x_1) = -u^{(i)}(x_1),\ 1 \le i \le n-1,$$

and

$$z_1(x_2) - \sum_{k=1}^{m} r_k z_1(\eta_k)) = \lim_{h\to 0}(z_{1h}(x_2) - \sum_{k=1}^{m} r_k z_{1h}(\eta_k)) = 0.$$

This completes the proof for $\partial u/\partial x_1$.

The proofs of (iii) and (iv) are in very much the same spirit. For (iii), we fix $1 \le j \le m$, and this time we designate

$$u(x, x_1, x_2, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$$

by $u(x, \eta_j)$. Let $\delta > 0$ be as in Theorem 2.1 and $0 < |h| < \delta$ be given. Define

$$w_{jh}(x) = \frac{1}{h}[u(x, \eta_j + h) - u(x, \eta_j)].$$

Note that for every $h \ne 0$,

$$w_{jh}^{(i-1)}(x_1) = 0,\ 1 \le i \le n-1.$$

Next, let $\beta = u^{(n-1)}(x_1, \eta_j)$, and

$$\epsilon = \epsilon(h) = u^{(n-1)}(x_1, \eta_j + h) - \beta.$$

By Theorem 2.1, $\epsilon \to 0$, as $h \to 0$. Again, we use the notation of Theorem 1.1 for solutions of initial value problems for (1); viewing the solutions u as solutions of initial value problems and denoting

$$y(x, x_1, u_1, \ldots, u_{n-1}, \beta)$$

by $y(x, x_1, \beta)$, we have

$$w_{jh}(x) = \frac{1}{h}[y(x, x_1, \beta+\epsilon) - y(x, x_1, \beta)].$$

By the Mean Value Theorem,

$$w_{jh}(x) = \frac{\epsilon}{h}\alpha_n(x, y(x, x_1, \beta+\bar{\epsilon})),$$

where $\alpha_n(x, y(\cdot))$ is the solution of Equation (3) along $y(\cdot)$ and satisfies

$$\alpha_n^{(i-1)}(x_1) = \delta_{in}, \quad 1 \le i \le n-1,$$

and $\beta+\bar{\epsilon}$ lies between β and $\beta+\epsilon$. Once again, to show $\lim_{h\to 0} w_{jh}(x)$ exists, it suffices to show $\lim_{h\to 0} \epsilon/h$ exists.

Since $\alpha_n(x,y(\cdot))$ is a nontrivial solution of (3) along $y(\cdot)$ and

$$\alpha_n^{(i-1)}(x_1,y(\cdot)) = 0, \quad 1 \le i \le n-1,$$

it follows from assumption (v) that

$$\alpha_n(x_2,y(\cdot)) - \sum_{k=1}^{m} r_k\alpha_n(\eta_k,y(\cdot)) \neq 0.$$

Hence,

$$\frac{\epsilon}{h} = \frac{w_{jh}(x_2) - \sum_{k=1}^{m} r_k w_{jh}(\eta_i)}{\alpha_n\big(x_2, y(x, x_1, \beta_2+\bar{\epsilon})\big) - \sum_{k=1}^{m} r_k\alpha_n\big(\eta_k, y(x, x_1, \beta+\bar{\epsilon})\big)}.$$

We look in more detail at the numerator of this quotient. Consider

$$\begin{aligned}
&w_{jh}(x_2) - \sum_{k=1}^{m} r_k w_{jh}(\eta_k)\\
&\quad = \frac{1}{h}\Big[u(x_2, \eta_j+h) - \sum_{k=1}^{m} r_k u(\eta_k, \eta_j+h) - \Big[u(x_2, \eta_j) - \sum_{k=1}^{m} r_k u(\eta_k, \eta_j)\Big]\Big]\\
&\quad = \frac{1}{h}\Big[u(x_2, \eta_j+h) - \sum_{k\in\{1,\dots,m\}\setminus\{j\}} r_k u(\eta_k, \eta_j+h)\\
&\qquad\quad - r_j u(\eta_j+h, \eta_j+h) + r_j u(\eta_j+h, \eta_j+h)\\
&\qquad\qquad - r_j u(\eta_j, \eta_j+h)\Big] - \frac{u_n}{h}\\
&\quad = \frac{u_n}{h} - \frac{u_n}{h} + \frac{r_j u(\eta_j+h, \eta_j+h) - r_j u(\eta_j, \eta_j+h)}{h}\\
&\quad = \frac{r_j}{h}\big[u(\eta_j+h, \eta_j+h) - u(\eta_j, \eta_j+h)\big]
\end{aligned}$$

$$= \frac{r_j}{h} \int_{\eta_j}^{\eta_j+h} u'(s, \eta_j + h)\, ds$$
$$= \frac{r_j}{h} u'(c_{j,h}, \eta_j + h)(\eta_j + h - \eta_j) = r_j u'(c_{j,h}, \eta_j + h),$$

where $c_{j,h}$ is between η_j and $\eta_j + h$. So, as $h \to 0$ we obtain

$$r_j u'(c_h, \eta_j + h) \to r_j u'(\eta_j, \eta_j) = r_j u'(\eta_j).$$

When we return to the quotient defining ϵ/h, we compute the limit,

$$\lim_{h\to 0} \frac{\epsilon}{h} = \frac{r_j u'(\eta_j)}{\alpha_n\big(x_2, y(x, x_1, u_1, \beta)\big) - \sum_{k=1}^m r_k \alpha_n\big(\eta_k, y(x, x_1, u_1, \beta)\big)}$$
$$= \frac{r_j u'(\eta_j)}{\alpha_n\big(x_2, u(x)\big) - \sum_{k=1}^m r_k \alpha_n\big(\eta_k, u(x)\big)} =: E_j.$$

From

$$w_{jh}(x) = \frac{\epsilon}{h} \alpha_n\big(x, y(x, x_1, u_1, \beta + \bar{\epsilon})\big),$$

if we let $w_j(x) = \lim_{h\to 0} w_{jh}(x)$, then $w_j(x) = \partial u/\partial \eta_j$, and

$$w_j(x) = \lim_{h\to 0} w_{jh}(x) = E_j \alpha_n(x, y(x, x_1, u_1, \beta)) = E_j \alpha_n(x, u(x)),$$

which is a solution of Equation (3) along $u(x)$. In addition, from above observations, $w_j(x)$ satisfies the boundary conditions,

$$w_j^{(i-1)}(x_1) = \lim_{h\to 0} w_{jh}^{(i-1)}(x_1) = 0, \quad 1 \le i \le n-1,$$
$$w_j(x_2) - \sum_{k=1}^m r_k w_j(\eta_k) = r_j u'(\eta_j).$$

This concludes the proof of (iii). It remains to verify part (iv).

Fix $1 \le j \le m$ as before and consider $\partial u/\partial r_j$. Again, let $\delta > 0$ be as in Theorem 2.1 and $0 < |h| < \delta$. Define

$$v_{jh}(x) = \frac{1}{h}[u(x, r_j + h) - u(x, r_j)],$$

where, for brevity, we designate

$$u(x, x_1, x_2, u_1, \dots, u_n, \eta_1, \dots, \eta_m, r_1, \dots, r_m)$$

by $u(x, r_j)$. Note that

$$v_{jh}^{(i-1)}(x_1) = \frac{1}{h}(u_i - u_i) = 0,$$

for every $h \neq 0$ and $1 \leq i \leq n-1$. Also, we see that

$$\begin{aligned}
&v_{jh}(x_2) - \sum_{k=1}^{m} r_k v_{jh}(\eta_k) \\
&\quad = \frac{1}{h}\Big[u(x_2, r_j + h) - u(x_2, r_j) - \sum_{k=1}^{m} r_k \big(u(\eta_k, r_j + h) - u(\eta_k, r_j)\big)\Big] \\
&\quad = \frac{1}{h}\Big[u(x_2, r_j + h) - u(x_2, r_j) - \sum_{k=1}^{m} r_k u(\eta_k, r_j + h) + \sum_{k=1}^{m} r_k u(\eta_k, r_j)\Big] \\
&\quad = \frac{1}{h} u(x_2, r_j + h) - \frac{1}{h}\sum_{k=1}^{m} r_k u(\eta_k, r_j + h) - \frac{u_n}{h} \\
&\quad = \frac{1}{h}\Big[u(x_2, r_j + h) - \sum_{k\in\{1,\dots,m\}\setminus\{j\}} r_k u(\eta_k, r_j + h) \\
&\qquad\qquad - r_j u(\eta_j, r_j + h) - h u(\eta_j, r_j + h) + h u(\eta_j, r_j + h)\Big] - \frac{u_n}{h} \\
&\quad = \frac{1}{h}\Big[u(x_2, r_j + h) - \sum_{k\in\{1,\dots,m\}\setminus\{j\}} r_k u(\eta_k, r_j + h) \\
&\qquad\qquad - (r_j + h) u(\eta_j, r_j + h)\Big] + u(\eta_j, r_j + h) - \frac{u_n}{h} \\
&\quad = \frac{u_n}{h} + u(\eta_j, r_j + h) - \frac{u_n}{h} \\
&\quad = u(\eta_j, r_j + h).
\end{aligned}$$

And so by Theorem 2.1,

$$\lim_{h\to 0} v_{jh}(x_2) - \sum_{k=1}^{m} r_k v_{jh}(\eta_k) = u(\eta_j, r_j).$$

Now recall that $u^{(n-2)}(x_1, r_j) = u_{n-1}$, and define

$$\beta = u^{(n-1)}(x_1, r_j), \quad \text{and} \quad \epsilon = \epsilon(h) = u^{(n-1)}(x_1, r_j + h) - \beta.$$

As usual, $\epsilon \to 0$ as $h \to 0$. Once again, using the notation for solutions of initial value problems for (1) and denoting $y(x, x_1, u_1, \dots, u_{n-1}, \beta)$ by $y(x, x_1, \beta)$, we have

$$v_{jh}(x) = \frac{1}{h}\big[y(x, x_1, \beta + \epsilon) - y(x, x_1, \beta)\big].$$

By the Mean Value Theorem,

$$\begin{aligned}
v_{jh}(x) &= \frac{1}{h}\alpha_n\big(x, y(x, x_1, \beta + \bar{\epsilon})\big)(\beta + \epsilon - \beta) \\
&= \frac{\epsilon}{h}\alpha_n\big(x, y(x, x_1, \beta + \bar{\epsilon})\big),
\end{aligned}$$

where $\alpha_n(x, y(\cdot))$ is the solution of Equation (3) along $y(\cdot)$ and satisfies

$$\alpha_n^{(i-1)}(x_1, y(\cdot)) = 0, \qquad 1 \le i \le n-1,$$
$$\alpha_n^{(n-1)}(x_1, y(\cdot)) = 1,$$

and $\beta+\bar{\epsilon}$ lies between β and $\beta+\epsilon$. As in previous cases, it follows from assumption (v) that

$$\alpha_n(x_2, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot)) \neq 0.$$

Hence,

$$\frac{\epsilon}{h} = \frac{v_{jh}(x_2) - \sum_{k=1}^{m} r_k v_{jh}(\eta_k)}{\alpha_n\big(x_2, y(x, x_1, \beta+\bar{\epsilon})\big) - \sum_{k=1}^{m} r_k \alpha_n\big(\eta_k, y(x, x_1, \beta+\bar{\epsilon})\big)},$$

and so from above,

$$\begin{aligned}\lim_{h\to 0}\frac{\epsilon}{h} &= \frac{r_j u(\eta_j)}{\alpha_n\big(x_2, y(x, x_1, \beta)\big) - \sum_{k=1}^{m} r_k \alpha_n\big(\eta_k, y(x, x_1, \beta)\big)} \\ &= \frac{r_j u(\eta_j)}{\alpha_n\big(x_2, u(x)\big) - \sum_{k=1}^{m} r_k \alpha_n\big(\eta_k, u(x)\big)} =: E_j.\end{aligned}$$

From

$$v_{jh}(x) = \frac{\epsilon}{h}\alpha_n\big(x, y(x, x_1, \beta+\bar{\epsilon})\big),$$

if we set $v_j(x) = \lim_{h\to 0} v_{jh}(x)$, we obtain $v_j(x) = \partial u/\partial r_j$. In particular,

$$v_j(x) = \lim_{h\to 0} v_{jh}(x) = E_j \alpha_n(x, y(x, x_1, \beta)) = E_j \alpha_n(x, u(x)),$$

which is a solution of (3) along $u(x)$. In addition, $v_j(x)$ satisfies the boundary conditions,

$$v_j(x_1) = \lim_{h\to 0} v_{jh}^{(i-1)}(x_1) = 0, \quad 1 \le i \le n-1,$$
$$v_j(x_2) - \sum_{k=1}^{m} r_k v_j(\eta_k) = u(\eta_j).$$

This completes case (iv), which in turn completes the proof of the theorem. □

We conclude the paper with a corollary to Theorem 2.2, whose verification is a consequence of the n-dimensionality of the solution space for the variational Equation (3). In addition, this corollary establishes an analogue of part (iii) of Theorem 1.1.

Corollary 2.2.1. *Assume the conditions of Theorem 2.2. Then,*

$$\frac{\partial u}{\partial x_1} = -\sum_{k=1}^{n-1} u^{(k)}(x_1)\frac{\partial u}{\partial u_i} \qquad \text{and} \qquad \frac{\partial u}{\partial x_2} = -u'(x_2)\frac{\partial u}{\partial u_n},$$

and for $1 \le j \le m$,

$$\frac{\partial u}{\partial \eta_j} = r_j \frac{u'(\eta_j)}{u(\eta_j)} \frac{\partial u}{\partial r_j}.$$

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Gap functions and existence of solutions for generalized vector quasivariational inequalities

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The gap functions for generalized vector quasivariational inequalities in Hausdorff topological vector spaces are introduced, then using Fan–Knaster–Kuratowski–Mazurkiewicz (FKKM) theorem, some existence theorems for a class of generalized vector quasivariational inequalities under suitable assumptions are established. The obtained results extend and unify corresponding results in the literature.

1. Introduction

The vector variational inequality in a finite-dimensional Euclidean space was first introduced by Giannessi [1980]. It is the vector-valued version of the variational inequality of Hartman and Stampacchia [1966]. Later on, many authors have extensively studied various types of vector variational inequalities in abstract space (see, for example, [Ansari 1995; Chen 1992; Chen et al. 1997; Chen et al. 2005; Ding and Tarafdar 2000; Giannessi 2000; Göpfert et al. 2003; Huang and Fang 2005; Huang and Gao 2003; Huang and Li 2006; Khanh and Luu 2004; Konnov and Yao 1997; Lee and Lee 2000; Lee et al. 1996; Li and He 2005; Siddiqi et al. 1997; Yang 2003; Yang and Yao 2002; Yu and Yao 1996] and the references therein).

The gap function approach is an important research method in the study of variational inequalities. One advantage of the gap function for the variational inequality is that the variational inequality can be transformed into the optimization problem. Thus, powerful optimization solution methods and algorithms can be applied to find solutions of variational inequalities. Recently, many authors have investigated the gap functions for vector variational inequalities. Chen et al. [1997] introduced

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two set-valued functions as the gap functions for two classes of vector variational inequalities. Yang and Yao [2002] introduced the gap function for the multivalued vector variational inequality. Li and He [2005] generalized the results of Yang and Yao [2002] to the generalized vector variational inequality. They introduced a gap function for a class of generalized vector variational inequalities and proved the existence of some solutions for such problems. For some related works, we refer to [Li and Mastroeni 2008] and [Yang 2003].

Inspired and motivated by the reseach mentioned above, we introduce in this paper some new gap functions for generalized vector quasivariational inequalities in Hausdorff topological vector spaces. By using FKKM theorem, we prove a number of existence theorems for a class of generalized vector quasivariational inequalities under certain assumptions. The results presented in this paper extend, improve and unify some corresponding results in the literature.

2. Gap functions for generalized vector quasivariational inequalities

Let X and Y be two real Hausdorff topological vector spaces and E a nonempty subset of X. Let $L(X, Y)$ be the space of all the continuous linear operators from X into Y and σ is the family of bounded subsets of X whose union is total in X, that is, the linear hull of $\cup\{S : S \in \sigma\}$ is dense in X. Let $\mathbf{B}$ be a neighborhood base of 0 in Y. When S runs through σ, V through $\mathbf{B}$, the family

$$M(S, V) = \{t \in L(X, Y) : \cup_{x\in S}\langle t, x\rangle \subset V\}$$

is a neighborhood base of 0 in $L(X, Y)$ for a unique translation-invariant topology, called the topology of uniform convergence on the sets $S \in \sigma$, or, briefly the σ-topology where $\langle t, x\rangle$ denotes the valuation of the linear operator $t \in L(X, Y)$ at $x \in X$ (see, [Schaefer 1971]). By the corollary of Schaefer [1971], $L(X, Y)$ becomes a locally convex topological vector space under the σ-topology, where Y is assumed a locally convex topological vector space.

Lemma 2.1 ([Ding and Tarafdar 2000]). *Let X and Y be two real Hausdorff topological vector spaces and $L(X, Y)$ be the topological vector space under the σ-topology. Then the bilinear mapping*

$$\langle\cdot, \cdot\rangle : L(X, Y) \times X \to Y$$

is continuous in $L(X, Y) \times X$.

Let E be a nonempty compact subset of X, and $C \subseteq Y$ be a closed, convex, pointed cone in Y with apex at the origin and $\operatorname{int} C \neq \varnothing$. Assume that $K : E \to 2^E$ is a lower semicontinuous with compact-valued mapping, and $T : E \times E \to 2^{L(X,Y)}$ is set-valued mapping such that $T(x, x)$ is compact for any $x \in E$. Assume that $\eta : E \times E \to E$ and $h : E \times E \to Y$ are two continuous functions with respect to the first argument. Let $\eta(x, x) = 0$ and $h(x, x) = 0$ for any $x \in E$. In this section, we consider the following three generalized vector quasivariational inequalities (for short, GVQVIs):

(I) find $x^* \in E$ and $t^* \in T(x^*, x^*)$ such that

$$x^* \in K(x^*) \quad and \quad \langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} C, \qquad \text{for all } y \in K(x^*) \tag{1}$$

(II) find $x^* \in E$ and $t^* \in T(x^*, x^*)$ such that

$$x^* \in K(x^*) \quad and \quad \langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -C\backslash\{0\}, \qquad \text{for all } y \in K(x^*) \tag{2}$$

Remark 2.1. It is clear that any solution of GVQVI (2) is a solution of GVQVI (1). But the converse is not true in general.

Remark 2.2. If $T(x, x) = T(x)$ and $K = I$ (where I is the identity mapping) for any $x \in E$, then GVQVI (1) and (2) reduce to the following generalized vector variational inequalities (for short, GVVI), respectively:

(I) find $x^* \in E$ and $t^* \in T(x^*)$ such that

$$\langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} C, \qquad \text{for all } y \in E; \tag{3}$$

(II) find $x^* \in E$ and $t^* \in T(x^*)$ such that

$$\langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -C\backslash\{0\}, \qquad \text{for all } y \in E. \tag{4}$$

GVVI (3) and (4) were studied by Li and He [2005].

Remark 2.3. If $\eta(x, y) = x - y$ and $h(x, y) = 0$ for any $x, y \in E$, then GVVI (3) and (4) reduce to the following multivalued vector variational inequalities (for short, MVVI), respectively:

(I) find $x^* \in E$ and $t^* \in T(x^*)$ such that

$$\langle t^*, y - x^*\rangle \notin -\operatorname{int} C, \qquad \text{for all } y \in E; \tag{5}$$

(II) find $x^* \in E$ and $t^* \in T(x^*)$ such that

$$\langle t^*, y - x^*\rangle \notin -C\backslash\{0\}, \qquad \text{for all } y \in E. \tag{6}$$

MVVI (5) and (6) were studied by Yang and Yao [2002].

In the rest of this section, let R^l be an l-dimensional vector space, and let

$$R^l_+ = \{(r_1, \ldots, r_l) \in R^l \mid r_i \geq 0, i = 1, 2, \ldots, l\}$$

be the nonnegative orthant of R^l. Let $Y = R^l$ and $C = R^l_+$. Now we introduce some gap functions for GVQVI (1) and (2). Set

$$S = \{x \in E \mid x \in K(x)\}.$$

Definition 2.1. $\phi : S \to R$ is said to be a gap function for GVQVI (1) (resp. (2)) if it satisfies the following properties:

(i) $\phi(x) \le 0$ for all $x \in S$;

(ii) $\phi(x^*) = 0$ if and only if x^* solves GVQVI (1) (resp. (2)).

Let $x \in S$, $y \in K(x)$ and $t \in T(x, x)$. Denote

$$\langle t, \eta(y, x)\rangle + h(y, x) = ([\langle t, \eta(y, x)\rangle + h(y, x)]_1, \ldots, [\langle t, \eta(y, x)\rangle + h(y, x)]_l).$$

Now, we introduce the mappings $\varphi_1 : S \times L(X, R^l) \to R$ and $\varphi : S \to R$ as follows:

$$\varphi_1(x, t) = \min_{y \in K(x)} \max_{1 \le i \le l} (\langle t, \eta(y, x)\rangle + h(y, x))_i$$

and

$$\varphi(x) = \max\{\varphi_1(x, t) \mid t \in T(x, x)\}. \tag{7}$$

Since $K(x)$ is compact, η is continuous and h is continuous with respect to the first argument respectively, $\varphi_1(x, t)$ is well-defined. By Lemma 2.1, $\varphi(x)$ is well-defined. For any $x \in S$ and $t \in T(x, x)$, it is easy to see that

$$\varphi_1(x, t) = \min_{y \in K(x)} \max_{1 \le i \le l} (\langle t, \eta(y, x)\rangle + h(y, x))_i \le 0.$$

Theorem 2.1. *The function $\varphi(x)$ defined by Equation* (7) *is a gap function for GVQVI* (1).

Proof. Since

$$\varphi_1(x, t) \le 0, \qquad \text{for all } x \in S, \quad t \in T(x, x), \tag{8}$$

it follows that

$$\varphi(x) = \max\{\varphi_1(x, t) \mid t \in T(x, x)\} \le 0, \qquad \text{for all } x \in S.$$

If $\varphi(x^*) = 0$, then there exists a $t^* \in T(x^*, x^*)$ such that $\varphi_1(x^*, t^*) = 0$. Thus,

$$\min_{y \in K(x^*)} \max_{1 \le i \le l} (\langle t^*, \eta(y, x^*)\rangle + h(y, x^*))_i = 0.$$

From which it follows that, for any $y \in K(x^*)$,

$$\max_{1 \le i \le l} (\langle t^*, \eta(y, x^*)\rangle + h(y, x^*))_i \ge 0,$$

which implies that for any $y \in K(x^*)$,

$$\langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} R^l_+,$$

that is, x^* is a solution of GVQVI (1).

Conversely, if x^* is a solution of GVQVI (1), then there exists a $t^* \in T(x^*, x^*)$ such that

$$x^* \in K(x^*) \quad \textit{and} \quad \langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} R^l_+, \qquad \text{for all } y \in K(x^*).$$

It follows that for any $y \in K(x^*)$,

$$\max_{1\le i\le l} (\langle t^*, \eta(y, x^*)\rangle + h(y, x^*))_i \ge 0.$$

Hence, we have

$$\varphi_1(x^*, t^*) = \min_{y\in K(x^*)} \max_{1\le i\le l} (\langle t^*, \eta(y, x^*)\rangle + h(y, x^*))_i \ge 0. \tag{9}$$

It follows from (8) and (9) that $\varphi_1(x^*, t^*) = 0$. Again, from (8), we obtain

$$\varphi_1(x^*, t) \le 0, \quad t \in T(x^*, x^*).$$

Therefore, $\varphi(x^*) = 0$. This completes the proof. □

From Remark 2.1 and Theorem 2.1, it is easy to see that the following result holds.

Corollary 2.1. *If x^* is a solution of GVQVI* (2), *then $\varphi(x^*) = 0$.*

3. Existence theorems for generalized vector quasivariational inequalities

Let X and Y be two Hausdorff topological vector spaces and E be a nonempty subset of X. Let $L(X, Y)$ be a set of all the continuous linear operators from X into Y. Let $C : E \to 2^Y$ be a set-valued mapping such that for any $x \in E$, $C(x)$ is a point, closed and convex cone in Y with $\operatorname{int} C(x) \neq \varnothing$. Assume that $K : E \to 2^E$ and $T : E\times E \to 2^{L(X,Y)}$ are two set-valued mappings, $\eta : E\times E \to E$ and $h : E\times E \to Y$ are two vector-valued functions. In this section, we consider GVQVI with moving cone $C(x)$: find $x^* \in E$ and $t^* \in T(x^*, x^*)$ such that $x^* \in K(x^*)$ and

$$\langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} C(x^*), \qquad \text{for all } y \in K(x^*). \tag{10}$$

The following problems are special cases of GVQVI (10).

(1) If $T(x, x) = T(x)$ and $K = I$ (where I is the identity mapping) for any $x \in E$, then problem (10) reduces to the following problem: find $x^* \in E$ and $t^* \in T(x^*)$ such that

$$\langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} C(x^*), \qquad \text{for all } y \in E, \tag{11}$$

which was considered by Lee and Lee [2000] and Li and He [2005].

(2) If $\eta(y, x) = y - x$ and $h(y, x) = 0$ for any $x, y \in E$, then problem (11) reduces to the following problem: find $x^* \in E$ and $t^* \in T(x^*)$ such that

$$\langle t^*, y - x^*)\rangle \notin -\operatorname{int} C(x^*), \qquad \text{for all } y \in E, \tag{12}$$

which was considered by Konnov and Yao [1997].

(3) If T is a single-valued mapping, then problem (12) reduces the to following problem: find $x^* \in E$ such that

$$\langle T(x), y - x^*)\rangle \notin -\operatorname{int} C(x^*), \quad \text{for all } y \in E, \tag{13}$$

which was considered by Chen [1992] and Yu and Yao [1996].

(4) If T is a single-valued mapping, $\eta(y, x) = y - g(x)$ and $h(y, x) = 0$ for any $x, y \in E$, where $g : E \to E$, then problem (11) reduces to the following problem: find $x^* \in E$ such that

$$\langle T(x), y - g(x)\rangle \notin -\operatorname{int} C(x^*), \quad \text{for all } y \in E, \tag{14}$$

which was considered by Siddiqi et al. [1997].

In order to prove our main results, we need the following definitions and lemma.

Definition 3.1 ([Fan 1960/1961]). A multivalued mapping $G : X \to 2^X$ is called a KKM-mapping if for any finite subset $\{x_1, x_2, \dots, x_n\}$ of X, $co\{x_1, x_2, \dots, x_n\}$ is contained in $\bigcup_{i=1}^n G(x_i)$, where coA denotes the convex hull of the set A.

Lemma 3.1 ([Fan 1960/1961]). *Let M be a nonempty subset of a Hausdorff topological vector space X. Let $G : M \to 2^X$ be a KKM-mapping such that $G(x)$ is closed for any $x \in M$ and is compact for at least one $x \in M$. Then $\bigcap_{y \in M} G(y) \neq \varnothing$.*

Definition 3.2. Let $h : E \times E \to Y$ be a vector-valued mapping. Then $h(\cdot, x)$ is said to be $C(x)$-convex on E for a fixed $x \in E$ if, for any $y_1, y_2 \in E$ and $\lambda \in [0, 1]$,

$$h(\lambda y_1 + (1 - \lambda) y_2, x) \in \lambda h(y_1, x) + (1 - \lambda) h(y_2, x) - C(x).$$

Remark 3.1. It is easy to say that $h(\cdot, x)$ is $C(x)$-convex if and only if for any given $x \in E$,

$$h(\sum_{i=1}^n \lambda_i y_i, x) \in \sum_{i=1}^n \lambda_i h(y_i, x) - C(x),$$

for any $y_i \in E$ and $\lambda_i \in [0, 1]$ $(i = 1, 2, \dots, n)$ with $\sum_{i=1}^n \lambda_i = 1$.

Theorem 3.1. *Assume that the following conditions hold*:

(i) *E is a compact subset of X and $E \cap K(x)$ is nonempty and convex for any $x \in E$*;

(ii) *K is a closed mapping and $K^{-1}(y)$ is open in E for any $y \in E$*;

(iii) *for any $x \in E$, $\eta(x, x) = h(x, x) = 0$*;

(iv) *for any $x \in E$, the mapping $y \to h(y, x)$ is $C(x)$-convex*;

(v) *for any fixed $x, y \in E$ and each $t \in T(x, x)$, the mapping $y \to \langle t, \eta(y, x)\rangle$ is $C(x)$-convex*;

(vi) *for any* $y \in E$, $\{x \in E : \exists t \in T(x, x), \langle t, \eta(y, x)\rangle + h(y, x) \notin -\operatorname{int} C(x)\}$ *is closed.*

Then GVQVI (10) *has a solution.*

Proof. For any $x, y \in E$, set

$$S = \{x \in E : x \in K(x)\},$$
$$P(x) = \{z \in E : \langle T(x, x), \eta(z, x)\rangle + h(z, x) \subset -\operatorname{int} C(x)\},$$
$$\varphi(x) = \begin{cases} K(x) \cap P(x), & \text{if} \quad x \in S, \\ E \cap K(x), & \text{if} \quad x \in E\backslash S \end{cases}$$

and

$$Q(y) = E\backslash\varphi^{-1}(y).$$

First, we show that Q is a KKM-mapping. Indeed, suppose that there exists a finite subset $N = \{y_1, y_2, \ldots, y_n\} \subseteq E$ and that $\alpha_i \geq 0$, $i = 1, 2, \ldots, n$ with $\sum_{i=1}^n \alpha_i = 1$ such that $x = \sum_{i=1}^n \alpha_i y_i \notin \bigcup_{i=1}^n Q(y_i)$. Then, $x \notin Q(y_i)$; that is, $y_i \in \varphi(x)$ for $i = 1, 2, \ldots, n$. If $x \in S$, then

$$\varphi(x) = K(x) \cap P(x).$$

Thus, $y_i \in P(x), \quad i = 1, 2, \ldots, n$, which implies that

$$\langle T(x, x), \eta(y_i, x)\rangle + h(y_i, x) \subset -\operatorname{int} C(x).$$

It follows that

$$\sum_{i=1}^n \alpha_i \langle T(x, x), \eta(y_i, x)\rangle + \sum_{i=1}^n \alpha_i h(y_i, x) \subset -\operatorname{int} C(x). \tag{15}$$

By conditions (iii)–(v) of Theorem 3.1 and (15), we have for any $x \in E$ and $t \in T(x, x)$

$$\begin{aligned} 0 &= \langle t, \eta(x, x)\rangle + h(x, x) \\ &\in \sum_{i=1}^n \alpha_i \langle t, \eta(y_i, x)\rangle - C(x) + \sum_{i=1}^n \alpha_i h(y_i, x) - C(x) \\ &\subseteq -\operatorname{int} C(x) - C(x) - C(x) \\ &\subseteq -\operatorname{int} C(x). \end{aligned}$$

Therefore, $0 \in -\operatorname{int} C(x)$, which is a contradiction. So, the only possibility is $x \in E\backslash S$. By the definition of S, $x \notin K(x)$. On the other hand, for $i = 1, 2, \ldots, n$

$$y_i \in \varphi(x) = E \cap K(x).$$

Hence,

$$x = \sum_{i=1}^{n} \alpha_i y_i \in K(x),$$

represents another contradiction. Thus, Q is a KKM-mapping.

Next, we show that $Q(y)$ is a closed set for any $y \in E$. In fact, we have

$$\begin{aligned}\varphi^{-1}(y) &= \{x \in S : y \in K(x) \cap P(x)\} \cup \{x \in E \backslash S : y \in K(x)\}\\ &= \{x \in S : x \in K^{-1}(y) \cap P^{-1}(y)\} \cup \{x \in E \backslash S : x \in K^{-1}(y)\}\\ &= [S \cap K^{-1}(y) \cap P^{-1}(y)] \cup [(E \backslash S) \cap K^{-1}(y)]\\ &= [(E \backslash S) \cup P^{-1}(y)] \cap K^{-1}(y).\end{aligned}$$

Therefore,

$$\begin{aligned}Q(y) &= E \backslash \{[(E \backslash S) \cup P^{-1}(y)] \cap K^{-1}(y)\}\\ &= \{E \backslash [(E \backslash S) \cup P^{-1}(y)]\} \cup [E \backslash K^{-1}(y)]\\ &= [S \cap E \backslash P^{-1}(y)] \cup [E \backslash K^{-1}(y)]. \qquad (16)\end{aligned}$$

Since K is closed mapping, S is closed set. From the definition of $P(x)$, we have

$$\begin{aligned}E \backslash P^{-1}(y) &= \{x \in E : y \notin P(x)\}\\ &= \{x \in E : \exists t \in T(x, x), \langle t, \eta(y, x)\rangle + h(y, x) \notin -\operatorname{int} C(x)\},\end{aligned}$$

which is closed by condition (vi). It follows from condition (ii) and (16) that $Q(y)$ is closed for any $y \in E$. Since E is compact, so is $Q(y)$. Therefore, by Lemma 3.1, we have that there exists $x^* \in E$ such that

$$x^* \in \bigcap_{y \in E} Q(y) = E \backslash \bigcup_{y \in E} \varphi^{-1}(y).$$

Thus, for any $y \in E$, $x^* \notin \varphi^{-1}(y)$; that is, $\varphi(x^*) = \varnothing$. If $x^* \in E \backslash S$, then we have

$$\varphi(x^*) = E \cap K(x^*) = \varnothing,$$

which contradicts condition (i).

If $x^* \in S$, that is, $x^* \in K(x^*)$, then

$$\varnothing = \varphi(x^*) = K(x^*) \cap P(x^*).$$

Thus, for any $y \in K(x^*)$, $y \notin P(x^*)$. It follows that there exists $t \in T(x^*, x^*)$ such that

$$\langle t, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} C(x^*), \qquad \text{for all } y \in K(x^*).$$

This completes the proof. □

Example 3.1. Let $X = Y = R$, $E = [0, 1]$, $C(x) = R_+$,

$$K(x) = [0, \frac{1}{2}(x+1)], \quad \text{for all } x \in [0, 1],$$

$$T(x, x) = \begin{cases} [0, 2], & \text{if} \quad x = 0.5, \\ [4x, 4], & \text{if} \quad x \neq 0.5, \end{cases}$$

$$\eta(y, x) = \begin{cases} \frac{x-y}{2}, & \text{if} \quad x \geq y, \\ \frac{y-x}{2}, & \text{if} \quad x < y, \end{cases}$$

$$h(y, x) = y^2 - x^2.$$

It is easy to verify that assumptions (i)–(iii) of Theorem 3.1 are fulfilled and for any $y \in E$, $K^{-1}(y)$ is an open set which was shown in [Khanh and Luu 2004]. Since

$$\begin{aligned} &\lambda h(y_1, x) + (1-\lambda)h(y_2, x) - h(\lambda y_1 + (1-\lambda)y_2, x) = \\ &\quad = \lambda(y_1^2 - x^2) + (1-\lambda)(y_2^2 - x^2) - [(\lambda y_1 + (1-\lambda)y_2)^2 - x^2] \\ &\quad = \lambda y_1^2 + (1-\lambda)y_2^2 - x^2 - [(\lambda y_1 + (1-\lambda)y_2)^2 - x^2] \\ &\quad = \lambda(1-\lambda)(y_1 - y_2)^2 \\ &\quad \geq 0, \end{aligned}$$

then condition (iv) of Theorem 3.1 is satisfied.

Let $\lambda y_1 + (1-\lambda)y_2 > x$. If $y_1 > x$ and $y_2 > x$, then

$$\lambda\langle t, \eta(y_1, x)\rangle + (1-\lambda)\langle t, \eta(y_2, x)\rangle - \langle t, \eta(\lambda y_1 + (1-\lambda)y_2, x)\rangle = 0.$$

If $y_1 > x$ and $y_2 \leq x$, then we have

$$\begin{aligned} &\lambda\langle t, \eta(y_1, x)\rangle + (1-\lambda)\langle t, \eta(y_2, x)\rangle - \langle t, \eta(\lambda y_1 + (1-\lambda)y_2, x)\rangle \\ &\qquad = \langle t, \frac{2(1-\lambda)(x - y_2)}{2}\rangle \geq 0. \end{aligned}$$

If $y_1 \leq x$ and $y_2 > x$, then

$$\begin{aligned} &\lambda\langle t, \eta(y_1, x)\rangle + (1-\lambda)\langle t, \eta(y_2, x)\rangle - \langle t, \eta(\lambda y_1 + (1-\lambda)y_2, x)\rangle \\ &\qquad = \langle t, \frac{2\lambda(x - y_1)}{2}\rangle \geq 0. \end{aligned}$$

Let $\lambda y_1 + (1-\lambda)y_2 \leq x$. If $y_1 \leq x$ and $y_2 \leq x$, then we have

$$\lambda\langle t, \eta(y_1, x)\rangle + (1-\lambda)\langle t, \eta(y_2, x)\rangle - \langle t, \eta(\lambda y_1 + (1-\lambda)y_2, x)\rangle = 0.$$

If $y_1 > x$ and $y_2 \leq x$, then

$$\lambda\langle t, \eta(y_1, x)\rangle + (1-\lambda)\langle t, \eta(y_2, x)\rangle - \langle t, \eta(\lambda y_1 + (1-\lambda)y_2, x)\rangle = \\ = \langle t, \frac{2\lambda(y_1 - x)}{2}\rangle > 0.$$

If $y_1 \leq x$ and $y_2 > x$, then we have

$$\lambda\langle t, \eta(y_1, x)\rangle + (1-\lambda)\langle t, \eta(y_2, x)\rangle - \langle t, \eta(\lambda y_1 + (1-\lambda)y_2, x)\rangle = \\ = \langle t, \frac{2(1-\lambda)(y_2 - x)}{2}\rangle > 0.$$

Therefore, condition (v) of Theorem 3.1 is satisfied.

If $x = 0.5$, $x \geq y$ and let $t = 2$, then

$$\langle t, \eta(y, x)\rangle + h(y, x) = \langle 2, \frac{x-y}{2}\rangle + y^2 - \frac{1}{4} = (y - \frac{1}{2})^2 \geq 0.$$

If $x = 0.5$, $x < y$ and let $t = 0$, then we have

$$\langle t, \eta(y, x)\rangle + h(y, x) = \langle 0, \frac{y-x}{2}\rangle + y^2 - \frac{1}{4} = y^2 - \frac{1}{4} > 0.$$

If $x \neq 0.5$, $x \geq y$ and let $t = 4x$, then

$$\langle t, \eta(y, x)\rangle + h(y, x) = \langle 4x, \frac{y-x}{2}\rangle + y^2 - x^2 = (x-y)^2 \geq 0.$$

If $x \neq 0.5$, $x < y$ and let $t = 4$, then we have

$$\langle t, \eta(y, x)\rangle + h(y, x) = \langle 4, \frac{y-x}{2}\rangle + y^2 - x^2 = (y+1)^2 - (x+1)^2 > 0.$$

Thus, for any $y \in E$,

$$\{x \in E : \exists t \in T(x, x), \langle t, \eta(y, x)\rangle + h(y, x) \notin -\operatorname{int} C(x)\} = [0, 1]$$

is a closed set. Therefore, all the assumptions of Theorem 3.1 are satisfied. It is easy to see that $x = 1$ and $t = 4$ is a solution of GVQVI (10).

Remark 3.2. Theorem 3.1 extends and unifies corresponding results of [Chen 1992; Konnov and Yao 1997; Lee and Lee 2000; Lee et al. 1996; Li and He 2005; Siddiqi et al. 1997; Yang 2003; Yang and Yao 2002; Yu and Yao 1996]. Furthermore, our proof is different from the methods used in these papers.

Corollary 3.1. *Assume that conditions* (*i*)–(*v*) *of Theorem 3.1 hold and the following assumptions are satisfied*:

(a) *if* $x_\alpha \to x$, $y_\alpha \to y$ *in* E *and if* $t_\alpha \in T(x_\alpha, x_\alpha)$, *then there exists* $t \in T(x, x)$ *and subnets* x_β, y_β *and* $t_\beta \in T(x_\beta, x_\beta)$ *such that* $(t_\beta, y_\beta) \to (t, y)$;

(b) *for any* $y \in E$, *the mappings* $x \to \eta(y, x)$ *and* $x \to h(y, x)$ *are continuous*;

(c) *the mapping* $x \to Y\backslash(-\operatorname{int} C(x))$ *is closed.*

Then GVQVI (10) *has a solution.*

Proof. By Theorem 3.1, it is sufficient to show that for any $y \in E$, the set

$$M = \{x \in E : \exists t \in T(x, x), \langle t, \eta(y, x)\rangle + h(y, x) \notin -\operatorname{int} C(x)\}$$

is closed. Let $\{x_\alpha\} \subset M$ and $x_\alpha \to x^*$. Then, there exists $t_\alpha \in T(x_\alpha, x_\alpha)$ such that

$$\langle t_\alpha, \eta(y, x_\alpha)\rangle + h(y, x_\alpha) \notin -\operatorname{int} C(x_\alpha).$$

By assumptions (a) and (b) of Corollary 3.2, there exists $t^* \in T(x^*, x^*)$ and subnets x_β and $t_\beta \in T(x_\beta, x_\beta)$ such that

$$\langle t_\beta, \eta(y, x_\beta)\rangle + h(y, x_\beta) \to \langle t^*, \eta(y, x^*)\rangle + h(y, x^*).$$

It follows from condition (c) that

$$\langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} C(x^*).$$

Therefore, $x^* \in M$. This means M is a closed set. This completes the proof. □

Remark 3.3. Example 2.1 in [Khanh and Luu 2004] illustrates that assumption (a) is satisfied.

Remark 3.4. If $T(x, x) = T(x)$, $\eta(y, x) = y - g(x)$ and $\eta(y, x) = 0$, where $g : E \to E$ is continuous mapping, then Corollary 3.1 reduces Theorem 2.1 in [Khanh and Luu 2004].

Corollary 3.2. *Assume that all conditions in Theorem 3.1 hold, except the assumed compactness of E which is replaced by one of the following conditions*:

(a) *there exists* $y^* \in E$ *such that* $E\backslash K^{-1}(y^*)$ *is compact and there exists a compact subset* $B \subset E$ *such that*

$$\langle T(x, x), \eta(y^*, x)\rangle + h(y^*, x) \subset -\operatorname{int} C(x), \quad \textit{for all } x \in E\backslash B;$$

(b) *there exists* $y^* \in E$ *such that* $E\backslash K^{-1}(y^*)$ *is compact and* S *is compact.*

Then GVQVI (10) *has a solution.*

Proof. From (16), it is sufficient to verify the compactness of $S \cap E\backslash P^{-1}(y^*)$ so that the FKKM theorem can be applied.

In case (a), we can obtain $E\backslash B \subset P^{-1}(y^*)$. Thus, $E\backslash P^{-1}(y^*) \subset B$. Since $E\backslash P^{-1}(y^*)$ is closed and B is compact, then both $E\backslash P^{-1}(y^*)$ and $S \cap E\backslash P^{-1}(y^*)$ are compact. For case (b), since S is compact and $E\backslash P^{-1}(y^*)$ is closed, then $S \cap E\backslash P^{-1}(y^*)$ is compact. Therefore, the FKKM theorem can be applied in cases (a) and (b). By Theorem 3.1, GVQVI (10) has a solution. This completes the proof. □

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Fibonacci sequences and the space of compact sets

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The Fibonacci numbers appear in many surprising situations. We show that Fibonacci-type sequences arise naturally in the geometry of $\mathcal{H}(\mathbb{R}^2)$, the space of all nonempty compact subsets of $\mathbb{R}^2$ under the Hausdorff metric, as the number of elements at each location between finite sets. The results provide an interesting interplay between number theory, geometry, and topology.

1. Introduction

The famous Fibonacci sequence, named after Leonardo of Pisa "son of Bonaccio", is defined recursively by $F_0 = 0$, $F_1 = 1$, and

$$F_n = F_{n-1} + F_{n-2} \tag{1}$$

for $n \geq 2$ [Sloane 2006, A000045]. The Fibonacci numbers appear in an amazing variety of interesting situations. For example, Fibonacci sequences have been noted to appear in biological settings including the patterns of petals on various flowers such as the cosmo, iris, buttercup, daisy, and the sunflower; the arrangement of pines on a pine cone; the appendages and chambers on many fruits and vegetables such as the lemon, apple, chili, and the artichoke; and spiral patterns in horns and shells [Thompson 1942; Stevens 1979; Douady and Couder 1996; Stewart 1998]. Other Fibonacci-type sequences (also called Gibonacci sequences [Benjamin and Quinn 2003]) can be obtained using the same recurrence relation (1) but with different starting values. For example, the Lucas sequence $\{L_n\}$ can be defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. This sequence is due to Édouard Lucas (1842-1891) (who also named the numbers 1, 1, 2, 3, 5, ... the Fibonacci numbers). There are some useful relations between the Fibonacci and Lucas numbers. For example, a simple induction argument can be used to show $L_n = F_{n-1} + F_{n+1}$ for $n \geq 1$. Consequently, $L_{2n} = F_{2n-1} + F_{2n+1} = F_{2n} + 2F_{2n-1}$.

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Mathematical applications of Fibonacci-type numbers abound. In the RSA cryptosystem, for example, if an RSA modulus is a Fibonacci number, then the cryptosystem is vulnerable [Dénes and Dénes 2001]. As another example, there are no terms in the Fibonacci or Lucas sequences whose values are equal to the cardinality of a finite nonabelian simple group [Luca 2004]. Fibonacci numbers also have interesting geometric interpretations. For example, the Fibonacci numbers describe the number of ways to tile a $2 \times (n-1)$ checkerboard with 2×1 dominoes [Graham et al. 1994]. If we let Z_n be the point (F_{n-1}, F_n) in the coordinate plane, $X_n = (F_{n-1}, 0)$, $Y_n = (0, F_n)$, and P_n the broken line from the origin O to Z_n consisting of the straight line segments $OZ_1, Z_1Z_2, \cdots, Z_{n-1}Z_n$, then P_n separates the rectangle $OX_nZ_nY_n$ into two regions of equal area when n is odd [Hilton and Pedersen 1994; Page and Sastry 1992]. In this paper, we describe how Fibonacci-type sequences arise in the geometry of $\mathcal{H}(\mathbb{R}^2)$ as the number of elements at each location between finite sets A and B.

2. The Hausdorff metric

The Hausdorff metric h was introduced by Felix Hausdorff in the early twentieth century as a way to measure the distance between compact sets. We will work in $\mathbb{R}^N$ and denote the space of all nonempty compact subsets of $\mathbb{R}^N$ as $\mathcal{H}(\mathbb{R}^N)$. (Note that $\mathcal{H}(\mathbb{R}^N)$ is also called a *hyperspace* — a topological space whose elements are subsets of another topological space.)

A metric is a function that measures distance on a space. We will denote the standard Euclidean distance between x and y in $\mathbb{R}^N$ as $d_E(x, y)$. The Hausdorff metric, defined below, imposes a geometry on the space $\mathcal{H}(\mathbb{R}^N)$ which will be the subject of our study. To distinguish between $\mathbb{R}^N$ and $\mathcal{H}(\mathbb{R}^N)$, we will refer to *points* in $\mathbb{R}^N$ and *elements* in $\mathcal{H}(\mathbb{R}^N)$.

Definition 2.1. Let A and B be elements in $\mathcal{H}(\mathbb{R}^N)$. The Hausdorff distance, $h(A, B)$, between A and B is

$$h(A, B) = \max\{d(A, B), d(B, A)\},$$

where

$$d(A, B) = \max_{x \in A}\{\min_{b \in B}\{d_E(x, b)\}\}.$$

This metric is not very intuitive, so we present three examples to illustrate.

Example 2.1. Let A be the set $\{0, 2\}$ in $\mathbb{R}$ and B the interval $[0, 2]$ in $\mathbb{R}$. Since A is a subset of B, we have $d(A, B) = 0$. However, B is not a subset of A and $d(B, A) = d_E(1, 0) = d_E(1, 2) = 1$. Thus, even though A is a subset of B, we have $h(A, B) = 1$.

Example 2.2. Let A be the unit disk and B the circle of radius 3, both centered at the origin in $\mathbb{R}^2$. Then $d(A, B) = d_E((0,0),(3,0)) = 3$, but $d(B, A) = d_E((3,0),(1,0)) = 2$. So $h(A, B) = d(A, B) = 3$.

Example 2.3. Let A be the segment from $(0,0)$ to $(1,0)$ and B the segment from $(2,-1)$ to $(2,1)$ in $\mathbb{R}^2$. Then $d(A, B) = d_E((0,0),(2,0)) = 2$ and $d(B, A) = d_E((2,1),(1,0)) = \sqrt{2}$. So $h(A, B) = 2$.

Note that these examples show $d(A, B)$ is not symmetric, so we need to use the maximum of $d(A, B)$ and $d(B, A)$ to obtain a metric in Definition 2.1. See [Barnsley 1988] for a proof that h is a metric on $\mathcal{H}(\mathbb{R}^N)$. The corresponding metric space, $(\mathcal{H}(\mathbb{R}^N), h)$, is then itself a complete metric space [Barnsley 1988]. The definition of the metric h makes it rather cumbersome to work with, but there are few good properties that h and d satisfy that help with computations. For example,

- $h(A, B) = d_E(a, b)$ for some $a \in A$ and $b \in B$,
- if $B \subseteq C$, then $d(A, B) \geq d(A, C)$ and $d(C, A) \geq d(B, A)$,
- $h(A \cup B, C \cup D)$ is less than or equal to the maximum of $h(A, C)$ and $h(B, D)$.

These properties are not difficult to verify and are left to the reader.

The geometry the metric h imposes on $\mathcal{H}(\mathbb{R}^N)$ has many interesting properties. For example, in [Bay et al. 2005] the authors show there can be infinitely many different points at a given location on a line in this geometry and that, under certain conditions, lines in this geometry can actually have end elements. In this paper, we will focus our attention on the notion of betweenness in $\mathcal{H}(\mathbb{R}^N)$.

3. Betweenness in $\mathcal{H}(\mathbb{R}^N)$

In this section we define betweenness in $\mathcal{H}(\mathbb{R}^N)$, mimicking the idea of betweenness in $\mathbb{R}^N$ under the Euclidean metric. It is in this context that we will later encounter Fibonacci-type sequences. First we need to understand the dilation of a set.

Definition 3.1. Let $A \in \mathcal{H}(\mathbb{R}^N)$ and let $s > 0$ be a real number. The dilation of A by a ball of radius s (or the s-dilation of A) is the set

$$(A)_s = \{x \in \mathbb{R}^N : d_E(x, a) \leq s \text{ for some } a \in A\}.$$

As an example, let A be the triangle with vertices $(-100, 0)$, $(100, 0)$, and $(0, 150)$. The 30-dilation of A is shown in Figure 1. In essence, the dilation of A by a ball of radius s is just the union of all closed Euclidean s-balls with centers in A. So, for example, the dilation of a single point set $A = \{a\}$ by a ball of radius s is the ball centered at a of radius s. Using dilations, we can alternatively define $h(A, B)$ as the minimum value of s so that the s-dilation of A encloses B and the

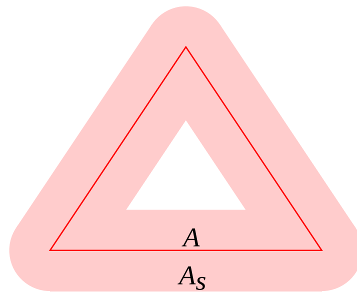

Figure 1. The dilation of a triangle.

s-dilation of B encloses A. An important and useful result about dilations is the following (Theorem 4 from [Braun et al. 2005]).

Theorem 3.1. *Let $A \in \mathscr{H}(\mathbb{R}^N)$ and let $s > 0$ be a real number. Then $(A)_s$ is a compact set that is at distance s from A. Moreover, if $C \in \mathscr{H}(\mathbb{R}^N)$ and $h(A, C) \leq s$, then $C \subseteq (A)_s$.*

Theorem 3.1 tells us that $(A)_s$ is the largest element in $\mathscr{H}(\mathbb{R}^N)$ (in terms of containment) that is a distance s from A. Now we discuss betweenness. In the standard Euclidean geometry, a point x lies between the points a and b if and only if $d_E(a, b) = d_E(a, x) + d_E(x, b)$. We extend this idea to define betweenness in $\mathscr{H}(\mathbb{R}^N)$.

Definition 3.2. Let $A, B \in \mathscr{H}(\mathbb{R}^N)$ with $A \neq B$. The element $C \in \mathscr{H}(\mathbb{R}^N)$ lies between A and B if $h(A, B) = h(A, C) + h(C, B)$.

As an example, let A be the disk centered at $(-100, 0)$ with radius 50 and B the circle centered at (100,0) with radius 25 in $\mathscr{H}(\mathbb{R}^2)$. Then $h(A, B) = d(A, B) = d_E((-150, 0), (75, 0)) = 225$ as shown in Figure 2. The element

$$C_{125} = (A)_{125} \cap (B)_{100}$$

is the grey shaded region in Figure 2. Note that $h(A, C_{125}) = d(C_{125}, A) = d_E((75, 0), (-50, 0)) = 125$ and $h(B, C_{125}) = d(C_{125}, B) = d_E(-25, 0), (75, 0) = 100$ as indicated in Figure 2. So $h(A, B) = h(A, C_{125}) + h(C_{125}, B)$ and C_{125} lies between A and B at the location 125 units from A. Moreover, any element that is s units from A and t units from B must be a subset of $C_s = (A)_s \cap (B)_t$ by Theorem 3.1. So C_{125} is the largest element $C \in \mathscr{H}(\mathbb{R}^N)$ (in the sense of containment) between A and B with $h(A, C) = 125$.

We will use the notation ACB as in [Blumenthal 1953] to indicate that C is between A and B. In Euclidean geometry, the set of points c satisfying $d_E(a, b) = d_E(a, c) + d_E(c, b)$ is the line segment $\overline{ab}$. For this reason, we will denote the set of elements $C \in \mathscr{H}(\mathbb{R}^N)$ that lie between A and B as $S(A, B)$ and call this set the

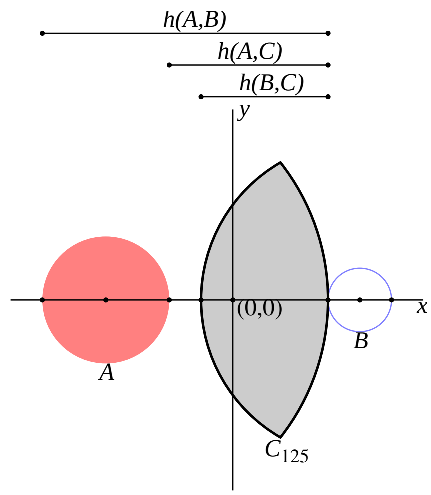

Figure 2. Distinct elements C_s and ∂C_s at the same location between A and B.

Hausdorff segment with end elements A and B. As we will see, there can be many different elements that lie at the same location between elements A and B, so there are many different collections of sets we could call a Hausdorff segment with end elements A and B. In light of Theorem 3.1, we might call $S(A, B)$ the maximal Hausdorff segment with end elements A and B, but we won't need to make that distinction in this paper.

An interesting property of Hausdorff segments is the possibility for the presence of more than one distinct element at a specific location between the end elements. For example, consider the sets A and B in Example 2.1. If we let $C = \left\{\frac{1}{2}, \frac{3}{2}\right\}$ and $C' = \left\{\frac{1}{2}, \frac{5}{2}\right\} \cup \left[\frac{3}{2}, 2\right]$, then a simple computation (left to the reader) shows C and C' satisfy ACB and $AC'B$ with $h(A, C) = h(A, C') = \frac{1}{2}$. So both C and C' lie between A and B at the same location $\frac{1}{2}$ units from A. The following definition formalizes the idea of two elements at the *same location* on a Hausdorff segment.

Definition 3.3. Let $A, B \in \mathcal{H}(\mathbb{R}^N)$ with $A \neq B$. The elements $C, C' \in S(A, B)$ are said to be at the same location between A and B if $h(A, C) = h(A, C') = s$ for some $0 < s < h(A, B)$.

As another example, if A and B are the elements in Figure 2, consider the elements $C_{125} = (A)_{125} \cap (B)_{100}$ and ∂C_{125}, the boundary of C_{125} (outlined in the figure). As Theorem 4.1 will show, these two elements, C_{125} and ∂C_{125}, both lie between A and B with $h(A, C_{125}) = 125 = h(A, \partial C_{125})$. So C_{125} and ∂C_{125} lie at the same location between A and B. In fact, Theorem 4.1 shows that any compact subset C of C_{125} that contains ∂C_{125} also satisfies ACB with $h(A, C) = s$.

4. Finding points between A and B

Let $A \neq B \in \mathcal{H}(\mathbb{R}^N)$. Hausdorff segments fall into two categories: those containing infinitely many elements at each location (except at the locations of either A or B), and those containing a finite number of elements at each location.

Lemma 4.1 [Bogdewicz 2000]. *Let $A, B \in \mathcal{H}(\mathbb{R}^N)$, $r = h(A, B)$, and let $C_s = (A)_s \cap (B)_{r-s}$ for every $s \in [0, r]$. Then $h(A, C_s) = s$ and $h(C_s, B) = r - s$.*

Bay, Lembcke, and Schlicker [Bay et al. 2005] extended Lemma 4.1 to find more elements on Hausdorff segments.

Theorem 4.1. *Let $A, B \in \mathcal{H}(\mathbb{R}^N)$ with $A \neq B$ and let $r = h(A, B)$. Let $s \in \mathbb{R}$ with $0 < s < r$, and let $t = r - s$. If C is a compact subset of $(A)_s \cap (B)_t$ containing $\partial((A)_s \cap (B)_t)$, then C satisfies ACB with $h(A, C) = s$ and $h(B, C) = t$.*

Recall that Theorem 3.1 shows us that an element $C \in \mathcal{H}(\mathbb{R}^N)$ with $h(A, C) = s$ and $h(B, C) = t$ must be a subset of both $(A)_s$ and $(B)_t$ (and so $C_s = (A)_s \cap (B)_t$ is the largest set, in the sense of containment, that is between A and B at a distance s from A). Theorem 4.1 tells us that if $(A)_s \cap (B)_t$ has an infinite interior, then there will be infinitely many elements in $\mathcal{H}(\mathbb{R}^N)$ at each location between A and B. An example of this situation occurs in Figure 2. Alternatively, if $(A)_s \cap (B)_t$ is finite, it has only finitely many subsets and therefore we can have at most a finite number of elements at each location between A and B. In [Blackburn et al. 2008], the authors show if there are finitely many elements at each location between A and B, then every point in A is the same distance from B and every point in B is that same distance from A. We label the distance from a point a to a set B as $d(a, B)$ and define it as follows.

Definition 4.1. Let $a \in \mathbb{R}^N$ and $B \in \mathcal{H}(\mathbb{R}^N)$. The distance from a to B is

$$d(a, B) = \min_{b \in B}\{d_E(a, b)\}.$$

When $d(a, B) = d(b, A)$ for all $a \in A$ and $b \in B$, it is possible for a pair of elements (A, B) to have only a finite number of elements at each location between them. Finite sets satisfying this condition are important enough that we give the following definition.

Definition 4.2. A finite configuration is a pair $[A, B]$ of elements $A, B \in \mathcal{H}(\mathbb{R}^N)$ where A and B are finite sets and $d(a, B) = d(b, A) = h(A, B)$ for all $a \in A$ and $b \in B$.

An easy example of this occurs when A and B are both single point sets. In this case, $(A)_s \cap (B)_t$ will always be a single point set for $0 < s < h(A, B)$; see [Braun et al. 2005]. In [Blackburn et al. 2008], the authors prove the following lemma that tells us about the number of elements at each location between elements A, B in a finite configuration $[A, B]$.

Lemma 4.2. *Let A, B be finite sets in $\mathcal{H}(\mathbb{R}^N)$. If all points $b \in B$ are equidistant from A and $h(A, B) = d(B, A) \geq d(A, B)$, then there is the same finite number of elements at every location between A and B.*

Lemma 4.2 shows that for a finite configuration $[A, B]$, the number of elements at each location between A and B is always the same (except at the end elements - there is only one element a distance 0 from A and only one a distance 0 from B). We denote this number by $\#([A, B])$.

A more interesting example of a configuration $[A, B]$ and the corresponding segment with a finite number of elements at each location is the following. Let $A = \{a_1, a_2, a_3, a_4\}$, where $a_1 = (2, 2)$, $a_2 = (-2, 2)$, $a_3 = (-2, -2)$, and $a_4 = (2, -2)$ are the vertices of a square and $B = \{b_1, b_2, b_3, b_4\}$, where $b_1 = (8, 0)$, $b_2 = (0, 8)$, $b_3 = (-8, 0)$, and $b_4 = (0, -8)$ are the vertices of a square eight times the size of A and rotated 45 degrees in $\mathcal{H}(\mathbb{R}^2)$. If $s, t \in \mathbb{R}^+$ with $r = h(A, B) = s + t$, then each t disk centered at a point in B is tangent to the two s-disks around the points in A closest to it as shown at left in Figure 3. Therefore, $C_s = (A)_s \cap (B)_t = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is the eight-point set that is illustrated at left in Figure 3. In fact, C_s is only one of 47 elements at each location on $S(A, B)$. Interestingly, 47 is the eighth Lucas number, L_8.

Now we find all 47 elements in $\mathcal{H}(\mathbb{R}^N)$ that lie at this location between A and B. To begin, we recall that the largest element between A and B is $C_s =$

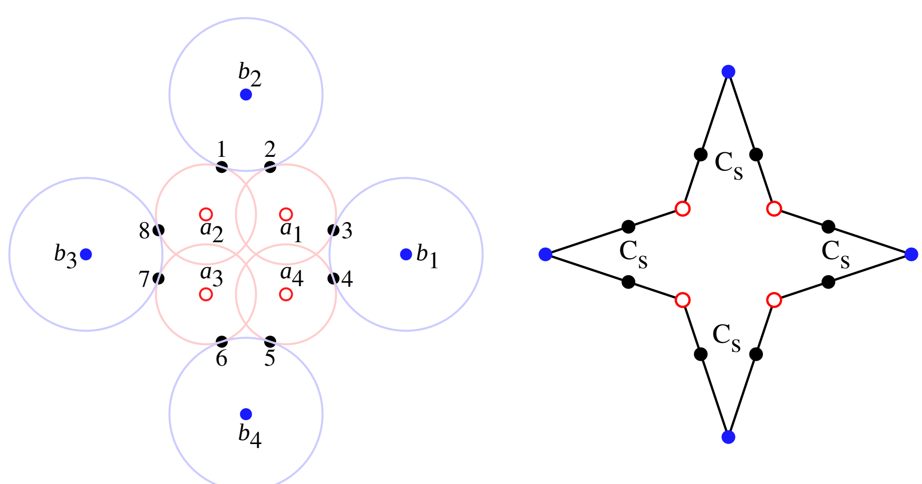

Figure 3. Left: $\#([A, B]) = 47$. Right: The trace diagram.

$\{1, 2, 3, 4, 5, 6, 7, 8\}$. The other 46 elements C that lie between A and B at this location are certain subsets of C_s:

(1) $C = C_s - \{c\}$ where $c \in C_s$ (8 elements).

(2) $C = C_s - \{c_1, c_2\}$ where $c_1 \neq c_2 \in C_s$ and c_1 and c_2 are not labeled consecutively (mod 8) (20 elements). (To have $d(A, C) = s$, the boundary of the dilation around each point in A must contain at least one point in C. So, for example, if $2 \notin C$ and $3 \notin C$, then the boundary of $(\{a_1\})_s$ does not contain a point in C. Thus, $d(A, C) > s$ and therefore $h(A, C) > s$.)

(3) $C = C_s - \{c_1, c_2, c_3\}$ where $c_1 \neq c_2 \neq c_3 \in C_s$ and c_1 is not labeled consecutively (mod 8) with c_2 or c_3, and c_2 is not labeled consecutively (mod 8) with c_3 (16 elements).

(4) $C = C_s - \{1, 3, 5, 7\}$ and $C = C_s - \{2, 4, 6, 8\}$.

We leave it to the reader to verify that each of these elements lies at the same location as C_s on $S(A, B)$. Thus we have found all 47 elements between A and B at this location.

We can create a graphical representation of the Hausdorff segment $S(A, B)$ for this configuration by tracing out the locus of points in $C_s = (A)_s \cap (B)_t$ as s varies from 0 to $h(A, B)$ as shown at right in Figure 3 (one specific C_s is shown as the set of eight black points). We call the resulting figure a *trace diagram*. Two other trace diagrams are also shown in Figure 4, the diagram at left presents a trace of a configuration with 7 elements at each location and at right we have the trace of a configuration with 13 elements at each location.

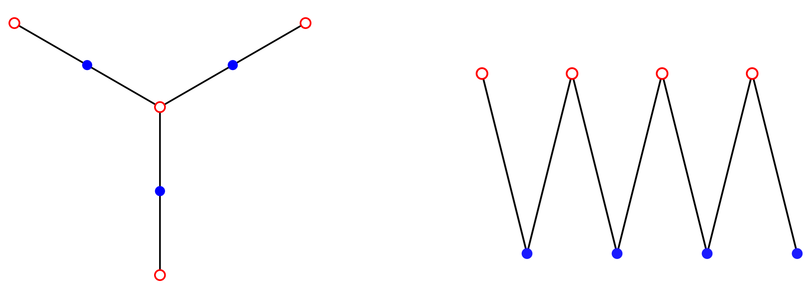

Figure 4. Trace diagrams: 7 elements (left), 13 elements (right).

5. Equivalent configurations

As we have seen, when determining $\#(X)$ for a finite configuration $X = [A, B]$ we only need to know which collection of points in $C_s = (A)_s \cap (B)_t$ we can exclude and still have a set C that satisfies ACB. The actual distance $h(A, B)$ is irrelevant; the only property of the configuration that determines the points in C_s are the points in $a \in A$ and $b \in B$ with $d_E(a, b) = h(A, B)$.

Definition 5.1. Let $[A, B]$ be a finite configuration. Two points $a \in A$ and $b \in B$ are adjacent if $d_E(a, b) = h(A, B)$.

The trace diagrams we have seen provide an obvious connection between finite configurations and graphs - where the points in a configuration $[A, B]$ provide the vertices of a graph and adjacent points in A and B correspond to adjacent points in the graph. Thus the use of the term "adjacent" in Definition 5.1. If two finite configurations X and X' have the same adjacencies, we should expect $\#(X) = \#(X')$. The next definition formalizes this notion of same adjacencies.

Definition 5.2. The finite configuration $[A', B']$ is equivalent to the finite configuration $[A, B]$ if there are bijections $f : A \to A'$ and $g : B \to B'$ such that

(1) if $d_E(a, b) = h(A, B)$ for $a \in A$ and $b \in B$, then $d_E(f(a), g(b)) = h(A', B')$ and

(2) if $d_E(a, b) > h(A, B)$ for $a \in A$ and $b \in B$, then $d_E(f(a), g(b)) > h(A', B')$.

When $[A', B']$ is equivalent to $[A, B]$ we write $[A', B'] \sim [A, B]$.

Informally, two finite configurations X and X' are equivalent if there is a bijection $\phi : X \to X'$ that preserves adjacencies and nonadjacencies. For example, the configuration shown in Figure 5 is equivalent to the configuration in Figure 3. It is easy to show that the relation $\sim$ is an equivalence relation on the set of finite configurations. One important result involving equivalent configurations is that if X and X' are equivalent configurations, then $\#(X) = \#(X')$ [Blackburn et al. 2008].

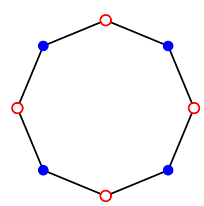

Figure 5. A configuration equivalent to the one in Figure 3.

6. Fibonacci-type sequences in $\mathscr{H}(\mathbb{R}^N)$

It may not be obvious that Fibonacci-type numbers have any connection to the idea of betweenness in the Hausdorff metric geometry. The connection lies in string and polygonal configurations.

6.1. ***String Configurations.*** Perhaps the simplest type of configuration in $\mathscr{H}(\mathbb{R}^N)$ occurs when we uniformly space points on a line segment. Let $x_1, x_2, \ldots, x_n$ be n uniformly spaced points in order on a line, $A = \{x_k : k \text{ odd}\}$, and $B = \{x_k : k \text{ even}\}$. In this case we will call any configuration equivalent to the configuration $S_n =$

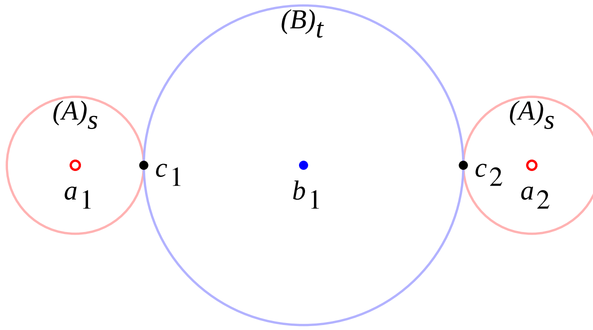

Figure 6. Configuration for S_3.

$[A, B]$ a *string configuration* and $S(A, B)$ a *string segment*. As we will see, the Fibonacci numbers are related to string configurations. We begin by finding $\#(S_n)$ for the first few values of n.

I. The simplest case occurs when $n = 2$ and $|A| = |B| = 1$ (i.e., when A and B are singleton sets), which was considered in [Braun et al. 2005]. In this case we have $\#(S_2) = 1 = F_1$.

II. Now suppose $A = \{a_1, a_2\}$ and $B = \{b_1\}$. Note that $(A)_s \cap (B)_t$ is a two point set $C_s = \{c_1, c_2\}$, with $c_1 < c_2$ as shown in Figure 6. Each element C in question will be a subset of C_s. If $c_1 \notin C$, then $h(A, C) = d_E(a_1, c_2) > s$. A similar argument shows that C contains c_2. Thus, $\#(S_3) = 1 = F_2$.

III. Consider $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Note that $(A)_s \cap (B)_t$ is a three point set $C_s = \{c_1, c_2, c_3\}$, with $d_E(a_1, c_1) = d_E(a_2, c_2) = d_E(a_2, c_3) = s$ as shown in Figure 7. Again, each element C in question will be a subset of C_s. As above, if $c_1 \notin C$, then $h(A, C) \geq d_E(a_1, c_2) > s$. A similar argument shows that C contains c_3. Notice that both $C = C_s$ and $C = \{c_1, c_3\}$ satisfy ACB with $h(A, C) = s$. Therefore, $\#(S_4) = 2 = F_3$.

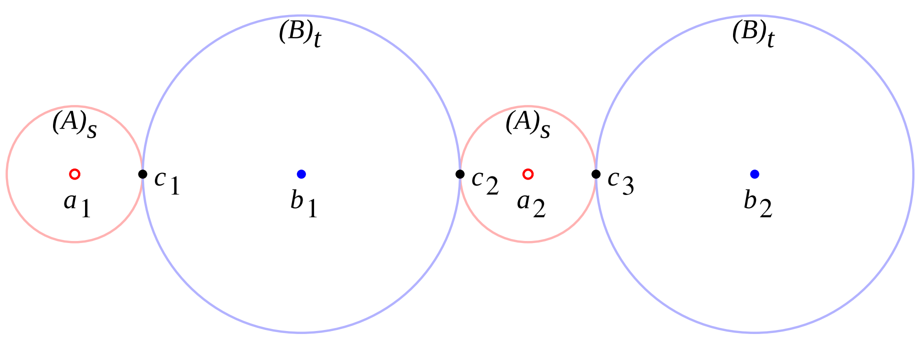

Figure 7. Configuration for S_4.

The next theorem provides the general case.

Theorem 6.1. *For each integer $n \geq 2$, $\#(S_n) = F_{n-1}$.*

Proof. Let $S_n = [A, B]$ and label the points in A in order as $a_1, a_2, \dots, a_k$ so that $d_E(a_1, a_i) < d_E(a_1, a_j)$ when $i < j$ and the points in B as $b_1, b_2, \dots, b_m$ so that $d_E(a_1, b_1) = h(A, B)$ and $d_E(b_1, b_i) < d_E(b_1, b_j)$ when $i < j$. Note that $k = m$ or $k = m+1$ and $n = k+m$. Let $r = h(A, B)$, $0 < s < r$ and $t = r-s$. We will determine the number of elements C in $\mathcal{H}(\mathbb{R}^N)$ satisfying ACB with $h(A, C) = s$. Theorem 3.1 tells us that C will be a subset of $C_s = (A)_s \cap (B)_t$. We have already considered the cases with $n \leq 4$. Now we argue the general case with $n \geq 5$. Then $k \geq 3$ and $m \geq 2$. The proof is by induction on n. Assume $n \geq 5$ and that $\#(S_l) = F_{l-1}$ for all $l \leq n-1$. Let $C_s = (A)_s \cap (B)_t$. We will show that there are F_{n-2} subsets C of C_s with $c_2 \in C$ satisfying ACB with $h(A, C) = s$ and F_{n-3} subsets C of C_s with $c_2 \notin C$ satisfying ACB with $h(A, C) = s$. Then $\#(S_n) = F_{n-2} + F_{n-3} = F_{n-1}$ as desired.

Now $C_s = \{c_1, c_2, c_3, \dots, c_p\}$, with $s = d_E(a_1, c_1) < d_E(a_1, c_2) < \cdots < d_E(a_1, c_p)$ (where $p = n-1$). Note that

$$\{c_{2i-1}\} = (\{a_i\})_s \cap (\{b_i\})_t \qquad \text{and} \qquad \{c_{2i}\} = (\{a_{i+1}\})_s \cap (\{b_i\})_t.$$

Each element C satisfying ACB with $h(A, C) = s$ and $h(B, C) = t$ will be a subset of C_s. If $c_1 \notin C$, then $h(A, C) \geq d(a_1, C) = d_E(a_1, c_2) > s$. So $c_1 \in C$. Similarly, we can show $c_p \in C$. Now we consider the cases $c_2 \notin C$ and $c_2 \in C$.

Case I: $c_2 \notin C$

In order to have C satisfy ACB, we must have $d(a_2, C) = s$. We know $(\{a_2\})_s \cap (B)_t = \{c_2, c_3\}$. Since $c_2 \notin C$, it must be the case that $c_3 \in C$. We now notice that the configuration $[A', B']$ with $A' = \{a_2, a_3, \dots, a_k\}$ and $B' = \{b_2, b_3, \dots, b_m\}$ is a string configuration equivalent to S_{n-2} and $C = \{c_1\} \cup C'$ where C' is a set satisfying $A'C'B'$ with $h(A', C') = s$. So there is a one-to-one correspondence (given by $\phi(C) = C - \{c_1, c_2\}$) between sets C satisfying ACB and $h(A, C) = s$ and sets C' satisfying $A'C'B'$ with $h(A', C') = s$. By the induction hypothesis, the number of such sets C is $\#([A', B']) = \#(S_{n-2}) = F_{n-3}$.

Case II: $c_2 \in C$

In this case, let $A^* = \{a_2, a_3, \dots, a_k\}$ and $C^* = C - \{c_1\}$. Since $c_2 \in C^*$ and C satisfies ABC, it is clear that C^* satisfies A^*C^*B with $h(A^*, C^*) = s$ and $h(C^*, B) = t$. Again, this provides a one-to-one correspondence ϕ between the elements C on the segment joining A and B and the elements C^* on the segment joining A^* and B, where $\phi(C) = C - \{c_1\}$. Now $[A^*, B]$ is equivalent to S_{n-1} and so there are exactly $\#(S_{n-1}) = F_{n-2}$ such elements C^* by our inductive hypothesis. Consequently, there are F_{n-2} elements C.

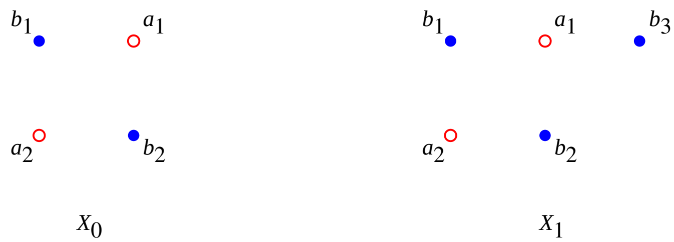

Figure 8. Left: A finite configuration X_0. Right: Adjoining a point to X_0.

Cases I and II show us that there are exactly $F_{n-3} + F_{n-2} = F_{n-1}$ elements at each location on the segment between A and B and $\#(S_n) = F_{n-1}$. □

6.2. ***Adjoining Strings to Configurations.*** We can see how other Fibonacci-type numbers arise in the Hausdorff metric geometry by successively adjoining points to finite configurations. We will illustrate the idea by adjoining a point to the finite configuration $X_0 = [A, B]$, where $A = \{(1, 1), (-1, -1)\}$ and $B = \{(-1, 1), (1, -1)\}$ in $\mathcal{H}(\mathbb{R}^2)$ as shown at left in Figure 8. Note that $d(a, B) = d(b, A) = 2 = h(A, B)$ for all $a \in A$ and $b \in B$. To adjoin a point to X_0 at a_1, we simply add a new point b_3 to B so that b_3 is adjacent to a_1 and $d_E(b_3, a_2) > 2$ as seen at right in Figure 8. This gives us a new finite configuration X_1.

The general construction is described in the next definition.

Definition 6.1. Let $[A, B]$ be a finite configuration. A finite configuration

$$[A, B](a, y)$$

obtained by adjoining a point y to $[A, B]$ at the point $a \in A$ is any configuration equivalent to the configuration $[A, B']$, where $B' = B \cup \{y\}$ and $d_E(y, a) = h(A, B)$ and $d_E(y, a') > h(A, B)$ for all other $a' \in A$.

If we adjoin points successively to a configuration X from a fixed point a, the net result is to adjoin a string configuration of some length to X at the point a. We continue our example from above by adjoining a point to X_1 to obtain finite configurations $X_2 = X_1(b_3, a_3)$, $X_3 = X_2(a_3, b_4)$, and so on as shown in Figure 9. We will show later that $\#(X_0) = 7$. Theorem 6.2 will show $\#(X_1) = 8$, $\#(X_2) = 15 = \#(X_0) + \#(X_1)$, and $\#(X_3) = 23 = \#(X_1) + \#(X_2)$. If we continue extending the configuration by adjoining more and more points, we construct a Fibonacci-type sequence $\{X_n\}$ with $\#(X_n) = \#(X_{n-1}) + \#(X_{n-2})$ for $n \geq 2$. Note that this sequence is also, among other things, the sequence A041100 in [Sloane 2006].

A general argument can be made to determine $\#([A, B](a, y))$, as shown in the next theorem.

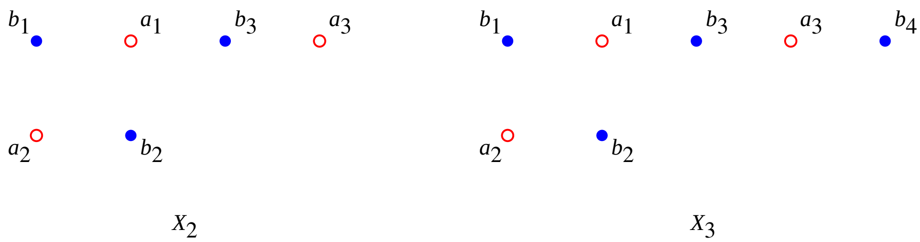

Figure 9. Adjoining points to a finite configuration.

Theorem 6.2. *Let* $X = [A, B]$ *be a finite configuration. Define* X' *to be the configuration* $[A, B](a, y)$ *by adjoining a point* y *to* X *at the point* $a \in A$, *where* a *is adjacent to* k *points* $b_1, b_2, \ldots, b_k$ *in* B, *each of which is adjacent to at least one point in* A *other than* a. *Then* $\#(X') = \#(X) + \#(X - \{a\})$.

Proof. Let X be a finite configuration defined by elements A and B. Let $X' = [A, B](a, y)$ and assume a is adjacent to k points $b_1, b_2, \ldots, b_k$ in X, each of which is adjacent to at least one point in A other than a as shown in Figure 10. Let $s, t > 0$ so that $r = h(A, B) = s + t$ and let $B^* = B \cup \{y\}$. Since $d_E(a, y) = r$, we know $(\{a\})_s \cap (\{y\})_t$ is a single point set. Let $\{c_0\} = (\{a\})_s \cap (\{y\})_t$, and let $c_i = (\{a\})_s \cap (\{b_i\})_t$ for i from 1 to k. Let C^* be an element in $\mathcal{H}(\mathbb{R}^N)$ satisfying AC^*B^* so that C^* is s units from A. Note that C^* must be a subset of $(A)_s \cap (B^*)_t$ and must also contain c_0. Now C^* either contains c_i for some $i \geq 1$ or C^* contains no c_i for $i \geq 1$. We will show that there are $\#(X)$ elements C^* satisfying AC^*B^* and $h(A, C^*) = s$ that contain c_i for some i and $\#(X - \{a\})$ elements C^* satisfying AC^*B^* and $h(A, C^*) = s$ that contain none of the c_i.

Case I: C^* contains c_i for some $i \geq 1$. Let

$$C = C^* - \{c_0\}. \tag{2}$$

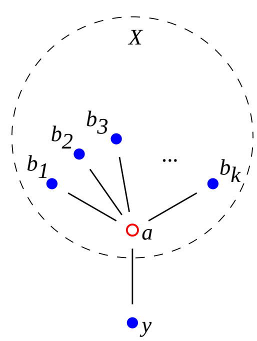

Figure 10. Adjoining a point to a configuration X.

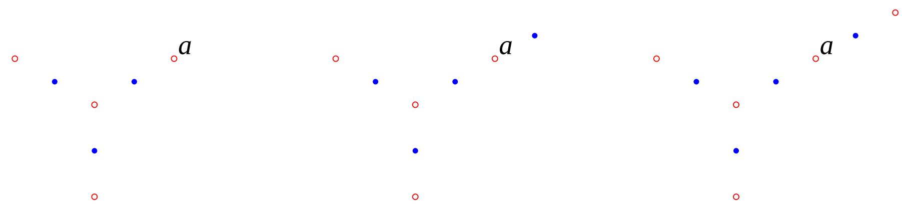

Figure 11. Configurations $\{X_n\}$ with Lucas numbers as $\#(X_n)$.

Now every point in A or B is adjacent to some point in C with a adjacent to c_i. Thus, C satisfies ACB with $h(A, C) = s$. So (2) provides a one-to-one correspondence between sets C^* and sets C. The number of such sets C is $\#(X)$.

Case II: C^* contains no c_i for $i \geq 1$. In this case, $C = C^* - \{c_0\}$ must satisfy $(A - \{a\})CB$ with $h(A - \{a\}, C) = s$. Again, (2) provides a one-to-one correspondence between sets C^* and sets C. The number of such sets C in this case is $\#([A - \{a\}, B]) = \#(X - \{a\})$.

Cases I and II show that

$$\#(X') = \#(X) + \#(X - \{a\}). \qquad \square$$

Theorem 6.2 shows how we can construct Fibonacci-type sequences by adjoining string configurations to finite configurations. Let X_0 be a finite configuration and let a be a point in X_0. If X_n is the configuration obtained by adjoining S_n to X_0 at a, then Theorem 6.2 shows

$$\#(X_n) = \#(X_{n-1}) + \#(X_{n-2})$$

for $n \geq 2$. Thus, we obtain a Fibonacci-type sequence. As another example, let X_0 be the configuration shown at left in Figure 11 and let a be the indicated point. Simple calculations show that $\#(X_0 - \{a\}) = 4$ and $\#(X_0) = 7$. In this case, $\#(X_n) = L_{n-2}$, the $(n-2)^{\text{nd}}$ Lucas number. Lucas numbers also appear in other configurations as we will see in the next section.

For one final example in this section, consider the configurations in Figure 12. In this example, $X_0 = S_6$ as shown at left. So $\#(X_0) = \#(S_6) = F_5 = 5$ and it is easy to see that $\#(X_1) = 5$ where X_1 is the configuration shown in the middle

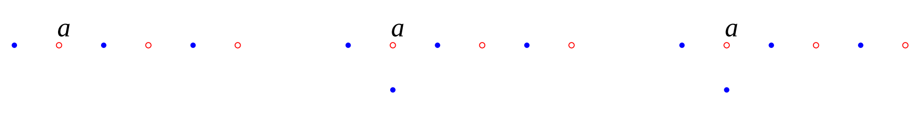

Figure 12. Configurations $\{X_n\}$ creating the sequence 5, 5, 10, 15, 25, ...

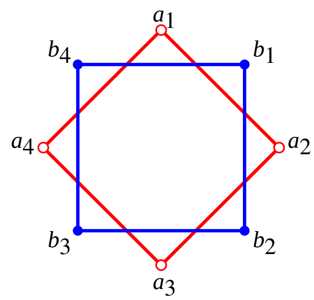

Figure 13. Example of two 4-gons with segment shown.

diagram. Therefore, the sequence generated by adjoining strings to S_6 at the point a is $5, 5, 10, 15, 25, \ldots$. This sequence is listed as A022088 in [Sloane 2006] and is described only as the Fibonacci sequence beginning with 0, 5. Now we have provided a geometric context for this sequence.

A natural question to ask is, given a positive integer k, is it possible to construct a Fibonacci-type sequence of finite configurations $\{X_n\}$ so that $\#(X_m) = k$ for some m. It turns out that this is not possible. Blackburn et al. [2008] proved the surprising result that there is no configuration X (either finite or infinite) with $\#(X) = 19$.

6.3. ***Polygonal Configurations.*** String configurations provide a simple type of finite configuration in $\mathcal{H}(\mathbb{R}^N)$. Another basic family of finite configurations is the collection of polygonal configurations. As an example, let $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3, b_4\}$ each be the set of vertices of a square, as seen in Figure 13. We see that $d(a, B) = d(b, A)$ for all a in A and all b in B. This configuration is equivalent to the one shown in Figure 3. So there are 47 elements that lie at each location on the Hausdorff segment between A and B and all such elements were exhaustively listed earlier.

The general construction of a polygonal configuration is as follows. Let A and B be vertices of regular n-gons with $n \in \mathbb{N}$ in which the n-gons share the same center point and initially are stacked such that the vertices correspond. Then B is rotated $\frac{\pi}{n}$ radians with respect to A about the center point. We call the configuration $P_n = [A, B]$ (or any configuration equivalent to it) a *polygonal configuration* and $S(A, B)$ a *polygonal segment*. As we will see, $\#(P_n) = L_{2n}$ where L_n is the n-th Lucas number.

As examples, we consider the two smallest cases: P_2 and P_3.

I. Figure 14 at left shows the configuration $P_2 = [A, B]$ with $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Let $r = h(A, B)$, $0 < s < r$, and $t = r - s$. Then $C_s = \{c_1, c_2, c_3, c_4\} = (A)_s \cap (B)_t$. To compute $\#(P_2)$ we simply count. Each element C satisfying ACB is a subset of C_s. Only those subsets that do not isolate any points in A or B from points in C are relevant. These sets are

- $C = C_s$ (1 element),

- $C = C_s - \{c_i\}$ for any i from 1 to 4 (4 elements),
- $C = C_s - \{c_1, c_3\}$ (1 element), and
- $C = C_s - \{c_2, c_4\}$ (1 element),

for a total of 7 elements. Therefore, $\#(P_2) = 7 = F_4 + 2F_3 = L_4$.

II. Figure 14, right, shows the configuration $P_3 = [A, B]$ with $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. Let $r = h(A, B)$, $0 < s < r$, and $t = r - s$. Then $C_s = \{c_1, c_2, c_3, c_4, c_5, c_6\} = (A)_s \cap (B)_t$. To compute $\#(P_3)$ we again count. Each element C satisfying ACB is a subset of C_s. Only those subsets that do not isolate any points in A or B from points in C are relevant. These sets are

- $C = C_s$ (1 element),
- $C = C_s - \{c_i\}$ for any i from 1 to 6 (6 elements),
- $C = C_s - \{c_i, c_j\}$ for $i < j$ as long as $j \neq i + 1$ or $i = 1$ and $j = 6$ (9 elements),
- $C = C_s - \{c_1, c_3, c_5\}$ (1 element), and
- $C = C_s - \{c_2, c_4, c_6\}$ (1 element),

for a total of 18 elements. Therefore, $\#(P_3) = 18 = p_3 = 18 = F_6 + 2F_5 = L_6$.

Recall that earlier we saw $\#(P_4) = 47 = L_8$. The general case is given in the following theorem.

Theorem 6.3. *For* $n \geq 2$,

$$\#(P_n) = F_{2n} + 2 \cdot F_{2n-1} = L_{2n} \tag{3}$$

Proof. Let $n \in \mathbb{N}$ and let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, where a_1 is selected from the vertices of one n-gon and the point a_i is the i-th vertex from a_1 moving counterclockwise on the same n-gon. On the second n-gon, which was rotated $\frac{\pi}{n}$ about the center, b_1 is the first vertex that lies $\frac{\pi}{n}$ degrees counterclockwise from a_1 and the point b_j is the j-th vertex moving counterclockwise from b_1 on the same n-gon. Now let $d(a_i, B) = r = d(b_j, A)$ for each i and let $0 < s < r$ and $t = r - s$. We determine the number of elements C in $\mathcal{H}(\mathbb{R}^N)$ satisfying ACB with $h(A, C) = s$.

We have already verified this theorem for $n = 2, 3$, and 4. Now assume $n > 4$. The element $C_s = (A)_s \cap (B)_t$ is a $2n$ point set $C_s = \{c_1, c_2, c_3, \ldots, c_{2n}\}$, where

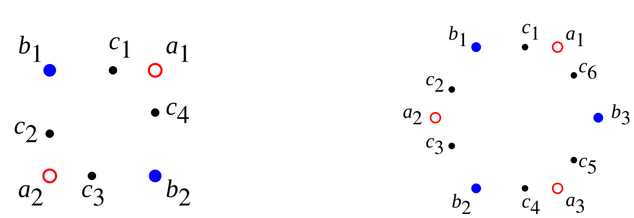

Figure 14. Left: P_2. Right: P_3.

c_{2i-1} is the point of intersection of the s-dilation about a_i and the t-dilation about b_i and c_{2i} is the point of intersection of the t-dilation about b_i and the s-dilation about a_{i+1} for $i = \{1, 2, \ldots, n\}$. Recall that every element C satisfying ACB with $h(A, C) = s$ is a subset of C_s. To find all of the elements C, we argue cases: $c_1 \notin C$, $c_2 \notin C$ and $c_1, c_2 \in C$.

Case I: $c_1 \notin C$

In order to have C satisfy ACB we must have $d(a_1, C) = s$ and $d(b_1, C) = t$. This implies $c_2, c_{2n} \in C$. We now notice the subconfiguration of alternating points from A and B, starting with b_1 and ending with a_1, is equivalent to a string configuration of $2n$ points, which we have shown to have F_{2n-1} elements satisfying ACB by Theorem 6.1.

Case II: $c_2 \notin C$

This case can be argued in a similar manner as the previous case, thus we know that there will be an additional F_{2n-1} elements which satisfy ACB.

Case III: $c_1, c_2 \in C$

We claim this case is similar to having a $2n+1$ string of alternating points from A and B, which by Theorem 6.1 will have F_{2n} elements that satisfy ACB. By assumption we have $C = \{c_1, c_2\} \cup C'$, where C' is a subset of $\{c_3, c_4, \ldots, c_{2n}\}$ such that if $c_i \notin C'$ then c_{i-1} or $c_{i+1} \in C'$ for $i = \{3, 4, 5, \ldots, 2n\}$. We can think of this as a string of alternating points starting with b_1, working in the counterclockwise direction, and ending with a new point b_*, where $b_* = b_1$, such that c_1 lies between a_1 and b_*. Then we see this is exactly the case when there is a string configuration of $2n + 1$ alternating points as desired. Therefore, by Theorem 6.1, we have F_{2n} elements which satisfy ACB.

Cases I, II and III show us that there are exactly $L_{2n} = 2F_{2n-1} + F_{2n}$ elements at each location on $S(A, B)$. □

In hindsight, the fact that string and polygonal configurations produce Fibonacci-type numbers should not be too surprising. Configurations look somewhat like graphs (with string and polygonal configurations related to paths and cycles), and in [Prodinger and Tichy 1982; Staton and Wingard 1995] the authors show that the Fibonacci and Lucas numbers occur as the number of independent vertex sets in paths and cycles.

7. Extensions to $\mathcal{H}(\mathbb{R}^N)$

All of the examples we have presented so far have been in $\mathbb{R}^2$, so it is reasonable to wonder what this paper has to do with $\mathbb{R}^N$. It should be clear that all of the examples and results we have seen extend to $\mathbb{R}^N$, but there is a more interesting connection than that. Dan Schultheis (2006, personal communication) has shown that 57 is the smallest integer for which there is a configuration X that can be constructed in $\mathbb{R}^3$ with $\#(X) = 57$, but no such configuration can be constructed in $\mathbb{R}^2$. The proof is not particularly enlightening, as it is an exhaustive analysis by cases. He identified all configurations in $\mathbb{R}$ and $\mathbb{R}^2$ such that $\#(X) \leq 58$, and showed that there were none for which $\#(X) = 57$. He did identify a finite configuration X_{57} which exists in $\mathbb{R}^3$, however. Due to the difficulty of drawing 3-dimensional configurations, we will describe this configuration $X_{57} = [A, B]$ in terms of its adjacency matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The rows of this matrix correspond to the points in A and the columns to the points in B (so $|A| = 4$ and $|B| = 3$). The entry m_{ij} of this adjacency matrix is 1 if the i-th point in A and the j-th point in B are adjacent and 0 otherwise.

This shows that there are Fibonacci-type sequences in the geometry of $\mathcal{H}(\mathbb{R}^N)$ that do not appear in $\mathcal{H}(\mathbb{R}^2)$. We expect that there are other numbers with this same property as 57, but that is an open question. It is also an open question if there are integers that appear as $\#(X)$ for finite configurations $X \in \mathbb{R}^{n+1}$ that cannot be constructed in $\mathbb{R}^n$ for $n \geq 3$. As a final note, in [Blackburn et al. 2008] the authors show that configurations X exist such that $\#(X) = k$ for all k from 1 to 18, so 19 is the smallest number that cannot be realized as $\#(X)$ for any configuration X. It is unknown for exactly which integers k there exist configurations X so that $\#(X) = k$.

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The coefficients of the Ihara zeta function

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(Communicated by Andrew Granville)

In her Ph.D. Thesis, Czarneski began a preliminary study of the coefficients of the reciprocal of the Ihara zeta function of a finite graph. We give a survey of the results in this area and then give a complete characterization of the coefficients. As an application, we give a (very poor) bound on the number of Eulerian circuits in a graph. We also use these ideas to compute the zeta function of graphs which are cycles with a single chord. We conclude by posing several questions for future work.

1. Introduction

Ihara wrote two papers [1966a; 1966b] in which he set forth the framework to define the Ihara zeta function of a finite k-regular graph. Then Bass [1992] gave an expression for the zeta function that applied to all graphs, regardless of the regularity. Since then a great deal of work has been done on this function. We refer the reader to the series [Stark and Terras 1996; 2000; Terras and Stark 2007] for a very comprehensive overview. In general, the zeta function of a graph is the reciprocal of a polynomial and can be computed in polynomial time. The aim of this paper is to study the coefficients of this polynomial with an eye towards relating each coefficient to a specific structure in the graph.

Answering this question opens the door to some very interesting questions for future study. By understanding the polynomial, we have a solid ground to investigate families of graphs which are uniquely determined by their zeta functions. This type of question is addressed in a survey by Noy [2003] for several other important polynomial invariants. In addition, the roots of this polynomial connect to the Ramanujan condition on a graph [Bass 1992; Stark and Terras 1996; Kotani and Sunada 2000], and it would be very interesting to be able to construct polynomials

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which are reciprocals of Ihara zeta functions and then to find which graph gives rise to it. We pose some of these questions at the end of the paper.

For the rest of this section, we give a definition of the Ihara zeta function and then survey the work that has been done on the coefficients. We also present our main result at the end of this section. In Section 2, we give Kotani and Sunada's "oriented line graph" construction [2000], which will allow us to write the zeta function as

$$\det(I - uT)^{-1},$$

where T is the adjacency operator on the oriented line graph. Our results come from analyzing this determinant expression, much as Biggs [1994] analyzed the coefficients of the characteristic polynomial. In Section 2, we explicitly compute the zeta function of graphs which are cycles with a single chord. In addition, we give a rough bound on the number of Eulerian circuits in a graph in Section 3. Finally, we conclude by posing several questions for future work.

We begin by defining graphs, digraphs, and the symmetric digraph associated to a graph. All structures treated here are finite. We refer the reader to the books [Harary 1969b; Chartrand and Lesniak 1986] for a good overview of these structures.

A *graph* $X = (V, E)$ is a finite nonempty set V of *vertices* and a finite multiset E of unordered pairs of vertices, called *edges*. If $\{u, v\} \in E$, we say that u is *adjacent* to v and write

$$u \sim v.$$

A graph X is *simple* if there is no edge of the form $\{v, v\}$ and if there is no repeated edge.

A *directed graph* or *digraph* $D = (V, E)$ is a finite nonempty set V of *vertices* and a finite multiset E of ordered pairs of vertices, called *arcs*. For an arc $e = (u, w)$, we define the *origin* of e to be $o(e) = u$ and the *terminus* of e to be $t(e) = w$. The *inverse arc* of e, written as $\overline{e}$, is the arc formed by switching the origin and terminus of e: $\overline{e} = (w, u)$. In general, the inverse arc of an arc need not be present in the arc set of a digraph.

A digraph D is called *symmetric* if whenever (u, w) is an arc of D, its inverse arc (w, u) is as well. There is a natural one-to-one correspondence between the set of symmetric digraphs and the set of graphs, given by identifying an edge of the graph to an arc and its inverse arc on the digraph on the same vertices. We denote by $D(X)$ the symmetric digraph associated with the graph X. We give an example in Figure 1.

To define the Ihara zeta function, we need several cycle definitions. Let X be a graph and $D(X)$ its symmetric digraph. A *cycle c of length n* in X is a sequence $c = (e_1, \ldots, e_n)$ of n arcs in $D(X)$ such that $t(e_i) = o(e_{i+1})$ for $1 \le i \le n-1$ and $t(e_n) = o(e_1)$. We say that c has *backtracking* if $\overline{e_{i+1}} = e_i$ for some i satisfying

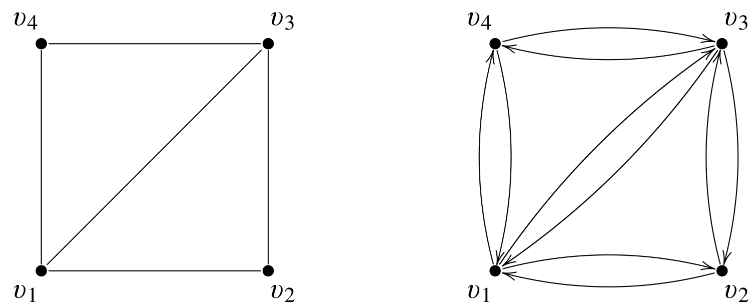

Figure 1. The complete graph minus an edge and its symmetric digraph.

$1 \le i \le n-1$. Also, c has a *tail* if $e_1 = \overline{e_n}$. We will primarily be interested in cycles with no backtracking or tail.

The *r-multiple* of the cycle c is the cycle c^r formed by going r times around c. We say a cycle is *primitive* if it is not the r-multiple of some other cycle b for $r \ge 2$. We impose an equivalence relation on cycles via cyclic permutation; that is, two cycles $b = (e_1, \ldots, e_n)$ and $c = (f_1, \ldots, f_n)$ are *equivalent* if there is a fixed $\alpha \in \mathbb{Z}/n\mathbb{Z}$ such that $e_i = f_{i+\alpha}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$ (all indices are considered modulo n). Note that the direction of travel does matter so traversing a cycle in the opposite direction does not give a cycle equivalent to the original one. A *prime cycle* is the equivalence class of primitive cycles which have no backtracking or tail, written as $[c]$.

The *Ihara zeta function* of a graph X is defined as a function of $u \in \mathbb{C}$ for $|u|$ sufficiently small by

$$Z_X(u) = \prod_{[c]} (1 - u^{l(c)})^{-1},$$

where the product is over the prime cycles in X and $l(c)$ is the *length* of the cycle c. Typically, this is an infinite product; however, the function is always rational. In fact, $Z_X(u)$ is always the reciprocal of a polynomial of maximum degree $2|E|$.

For a graph X, we let $n = |V|$ and $m = |E|$. We write

$$\frac{1}{Z_X(u)} = Z_X(u)^{-1} = c_0 + c_1 u + c_2 u^2 + c_3 u^3 + \cdots + c_{2m} u^{2m}.$$

We are concerned with determining the coefficients c_i in terms of structure in the graph X. We cite the known results and then give our main result.

From the definition of $Z_X(u)$, it is immediate that $c_0 = 1$. The first result in this area was given by Kotani and Sunada [2000], which is an expression for c_{2m}.

Theorem 1. *Let X be a graph and $Z_X(u)$ its Ihara zeta function as written above. We take $n = |V|$ and $m = |E|$. We denote by $d(v)$ the* degree *of vertex v which is* *the number of edges to which v is incident. Then*,

$$c_{2m} = (-1)^{m-n} \prod_{v_i \in V} (d(v_i) - 1).$$

Czarneski computed c_1 in her dissertation [2005]:

Theorem 2. *Let X be a graph and $Z_X(u)$ its Ihara zeta function as written above. Then the coefficient c_1 is the negative of twice the number of loops in X.*

In his dissertation, Storm [2007] computed c_3 from the number of triangles in X. The method used in the next section for an arbitrary coefficient is an extension of the one used for this theorem. We will look at it in more detail in the next section.

Theorem 3. *Let X be a simple graph and $Z_X(u)$ its Ihara zeta function as written above. Then the coefficient c_3 is the negative of twice the number of triangles in X.*

The final result in this area comes from Horton's dissertation [2006]. It encompasses Theorem 3; however, it is harder to generalize to realize the other coefficients. He shows that the girth of X can be recovered from the zeta function and relates a coefficient of the zeta function to this. We give two definitions and then his theorem.

Definition 4. Let X be a graph. The *girth* of X is the length of the shortest cycle in X. A *k-gon* in X is a subgraph of X which is isomorphic to the cycle graph C_k–C_k is the connected graph on k vertices such that the degree of every vertex is 2.

Theorem 5. *Let g be the girth of a simple connected graph X with zeta function $Z_X(u)$ written as above. Then, $c_k = 0$ for $1 \leq k < g$. Moreover, c_g is the negative of twice the number of g-gons in X.*

To state our more general result, we need a few more digraph definitions. The *indegree* of a vertex v, $in(v)$, in a digraph D is the number of arcs with terminus v. Similarly, the *outdegree* of v, $out(v)$, is the number of arcs with origin v. A *subgraph* of a digraph D is a digraph having all of its vertices and arcs in D. A *spanning subgraph* is a subgraph containing all of the vertices of D. Finally, a *linear subgraph of a digraph D* is a spanning subgraph in which each vertex has indegree one and outdegree one. A linear subgraph is thus a disjoint spanning collection of directed cycles.

Definition 6. Let D be a digraph. We denote by $\mathcal{S}_k(D)$ the set of subgraphs of D which have exactly k vertices. For an element $\tilde{D}$ of $\mathcal{S}_k(D)$, we denote by $\mathcal{E}_k(\tilde{D})$ the number of linear subgraphs of $\tilde{D}$ which consists of an even number of cycles of even length. Similarly we denote by $\mathcal{O}_k(\tilde{D})$ the number of linear subgraphs of $\tilde{D}$ with an odd number of cycles of even length.

We now state our main theorem:

Theorem 7. *Let X be a connected graph with oriented line graph L^oX (defined in the next section) and $Z_X(u)$ its Ihara zeta function as before. We also take the notation of Definition 6 as applied to the digraph L^oX. Then for $1 \le k \le 2m$, the coefficient c_k can be realized as*

$$c_k = \sum_{D \in \mathcal{S}_k(L^oX)} (-1)^k (\mathscr{E}_k(D) - \mathbb{O}_k(D)).$$

We prove this theorem in the next section and explore some of its consequences. In particular, we can realize Theorems 3 and 5 as corollaries to this. We will also give a practical list of things the Ihara zeta function must determine about a graph as a consequence of this theorem. In particular, Corollary 14 points out that the Ihara zeta function of a simple graph determines the number of triangles, squares, and pentagons in the graph.

2. Explicit representation of the coefficients

The first step to analyzing the coefficients of the zeta function is to realize the zeta function as a determinant expression. To do this, we construct an oriented line graph, a technique which was first proposed by Kotani and Sunada [2000].

We begin with a graph X and form its symmetric digraph $D(X)$. Hence $D(X)$ has $2|E(X)|$ arcs. Now we construct the oriented line graph $L^oX = (V_L, E_L^o)$ by

$$V_L = E(D(X)),$$
$$E_L^o = \{(e_i, e_j) \in E(D(X)) \times E(D(X));\ \overline{e_i} \neq e_j,\ t(e_i) = o(e_j)\}.$$

We give an example of this construction in Figure 2. The intuitive idea is that we are building a digraph which models all of the "legal" moves we could take to get prime cycles in X. It is for this reason that we disallow going from an arc to its inverse arc. We are particularly concerned with the adjacency matrix of this digraph.

Definition 8. Let D be a digraph with n vertices, written as $\{v_1, \dots, v_n\}$. The *adjacency matrix* T of D is the $n \times n$ matrix given by setting the (i, j)-entry $T_{i,j}$ to be 1 if there is an arc with origin v_i and terminus v_j, and zero otherwise.

Thus for the oriented line graph, the matrix T is a $2|E(X)| \times 2|E(X)|$ matrix which catalogues whether it is legal for an arc in $D(X)$ to feed into another arc. This matrix is given several different names in the zeta function literature. Stark and Terras [1996] refer to it as an "edge routing matrix". Kotani and Sunada [2000] call it the Perron–Frobenius matrix.

The following proposition, found in [Kotani and Sunada 2000], makes it clear why we are concerned with the oriented line graph and its adjacency matrix.

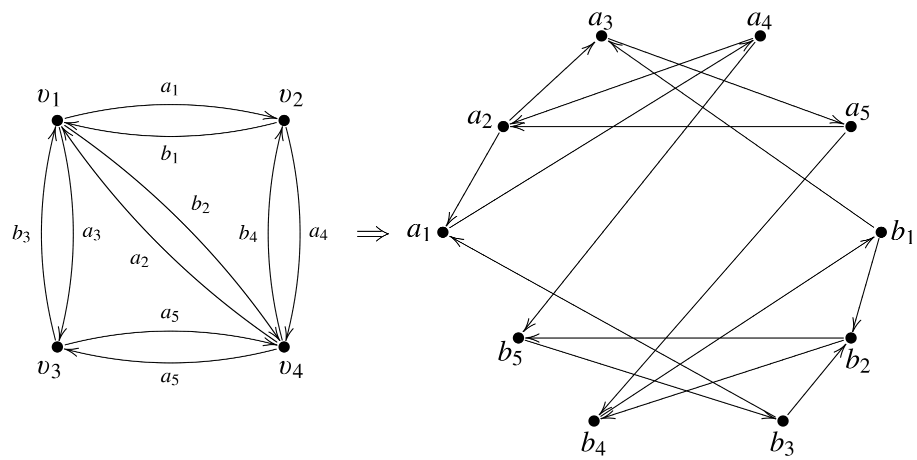

Figure 2. Construction of an oriented line graph of K_4 minus an edge.

Proposition 9. *There is a one-to-one correspondence between primitive cycles with no backtracking or tail in X and primitive cycles in L^oX. Moreover, if X is a connected graph, the zeta function $Z_X(u)$ can be written as*

$$Z_X(u) = \det(I - uT)^{-1},$$

where T is the adjacency matrix of the oriented line graph L^oX.

Studying this determinant expression will give us insight into the coefficients. For the rest of this section we let $m = |E(X)|$. We first note that the coefficients of the characteristic polynomial of T and those of the reciprocal of the zeta function are intimately related.

Lemma 10. *Let T be the adjacency matrix of the oriented line graph associated with the connected graph X. We write the characteristic polynomial of T as*

$$\chi_T(u) = \det(T - uI) = u^{2m} + c_1u^{2m-1} + \cdots + c_{2m}.$$

Then the reciprocal of the Ihara zeta function of X can be written as

$$\frac{1}{Z_X(u)} = Z_X(u)^{-1} = 1 + c_1u + c_2u^2 + c_3u^3 + \cdots + c_{2m}u^{2m}.$$

Proof. We begin by considering $\chi_T(u) = \det(T - uI)$. We rewrite this as

$$\det(T - uI) = (-u)^{2m} \det\Big(I - \frac{1}{u}T\Big).$$

We now replace u by $1/u$ and the result follows. □

This is very helpful since the coefficients of characteristic polynomials are very well understood as the sum of the principal minors of the matrix involved.

Definition 11. A *principal minor* of a square matrix M is the determinant of a submatrix of M formed by selecting a subset of the matrix's rows and the columns indexed by the same subset.

We make use of a useful linear algebra fact:

Lemma 12. *Let M be an $n \times n$ square matrix with characteristic polynomial*

$$\chi_M(u) = u^n + c_1 u^{n-1} + \cdots + c_n.$$

Then the coefficient c_i is $(-1)^i$ times the sum of all $i \times i$ principal minors of M.

We wish to apply Lemma 12 to the characteristic polynomial of the adjacency matrix T of the oriented line graph of X. This will then give us the information we need about coefficients of the reciprocal of the Ihara zeta function.

How can we interpret a principal minor of the matrix T? We let I be the index set which determines which rows we are keeping when we pass to the principal minor. Each row and the corresponding column represent a vertex in the oriented line graph. These vertices in turn represent arcs in the symmetric digraph $D(X)$. Then by reducing the matrix T to only keeping the rows and columns indexed by I, we are in fact looking at the matrix $\tilde{T}$ we would get by taking the subgraph induced on $D(X)$ by the arcs indexed by I and then forming the submatrix's oriented line graph and adjacency matrix. Thus an $i \times i$ principal minor can be computed by taking the appropriate subgraph of $D(X)$ induced by i edges, forming its $\tilde{T}$ matrix, and then taking the determinant.

This leaves only the question: how can we compute the determinant of the adjacency matrix of a digraph? Fortunately Harary [1962] answers this by

Lemma 13. *Let D be a digraph whose linear subgraphs are D_i, for $i = 1, \ldots, n$, and suppose each D_i has e_i even cycles. Then*

$$\det A = \sum_{i=1}^{n} (-1)^{e_i},$$

where A is the adjacency matrix of D.

Proof of Theorem 7. We consider the coefficient c_k for $2 \le k < 2m$. By Lemma 12, we must consider all of the $k \times k$ principal minors of T. Each such principal minor corresponds to picking k vertices of $L^o X$ and then taking the subdigraph induced by those vertices. Such a subgraph is then a member of $\mathcal{S}_k(L^o X)$. We call this subgraph $\tilde{D}$.

Then the principal minor corresponds to the determinant of the adjacency operator $\tilde{T}$ of $\tilde{D}$. To take this determinant, we use Lemma 13. We let $\tilde{D}_i$ for $i = 1, \ldots, j$

be the linear subgraphs of $\tilde{D}$. Then

$$\det \tilde{T} = \sum_{i=1}^{j} (-1)^{e_i},$$

where e_i is the number of even cycles in $\tilde{D}_i$. Using the notation of Definition 6, we have

$$\det \tilde{T} = \mathscr{E}_k(\tilde{D}) - \mathbb{O}_k(\tilde{D}).$$

We combine this statement with Lemma 12 to get the result

$$c_k = \sum_{\tilde{D} \in \mathscr{S}_k(L^o X)} (-1)^k \big(\mathscr{E}_k(\tilde{D}) - \mathbb{O}_k(\tilde{D})\big). \qquad \square$$

With this theorem it is fairly easy to compute the coefficients of smaller powers of u. We use Proposition 9 to take information about cycles in the oriented line graph back to information about cycles in the graph. In particular, notice that a linear subgraph of $L^o X$ corresponds to an edge-disjoint collection of backtrack-free, tailless cycles in the symmetric digraph of the original graph. Therefore, each subgraph of the symmetric digraph that has k edges and consists only of edge-disjoint backtrackfree, tailless cycles contributes to the coefficient c_k. This approach, of course, is not a practical way to compute higher powers; fortunately, we can get a great deal of information from the lower powers. We first give a very explicit statement; then, we give a second corollary which is more general.

Corollary 14. *Let X be a connected graph with Ihara zeta function as above.*

(1) *If X has loops, the coefficient c_1 can be computed by Theorem 2.*

(2) *If X does not have loops, then the coefficient c_2 is the negative of twice the number of primitive cycles of length 2 in X. Also, the coefficient c_3 is the negative of twice the number of triangles in X. In addition, c_4 is the number of primitive cycles of length 2 plus twice the number of pairs of primitive cycles of length 2 that share an edge plus four times the number of edge disjoint pairs of primitive cycles of length 2 minus twice the number of squares in X.*

(3) *If X is a simple graph, the coefficients c_3, c_4, and c_5 are the negative of twice the number of triangles, squares, and pentagons in X respectively. Also, c_6 is the negative of twice the number of hexagons in X plus four times the number of pairs of edge disjoint triangles plus twice the number of pairs of triangles with a common edge, while c_7 is the negative of twice the number of heptagons in X plus four times the number of edge disjoint pairs of one triangle and one square plus twice the number of pairs of one triangle and one square that share a common edge.*

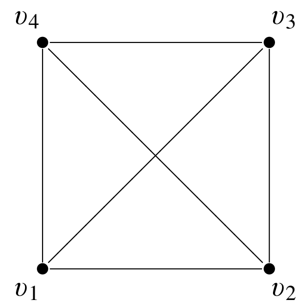

Figure 3. The complete graph K_4.

Proof. We leave the proof as an exercise to the reader. Particular care should be taken to get the coefficient c_4 as detailed in the second statement. The possible ways to orient the smaller cycles show up in the number of subgraphs on 4 vertices of the oriented line graph. □

Corollary 14 provides a very concrete way to compute the coefficients of smaller powers of u. We give a definition and then a more general statement.

Definition 15. Let X be a graph with two cyclic subgraphs C_n and C_m. We call C_n and C_m *compatible* if it is possible to orient the edges of X so that C_n and C_m both become oriented cycles.

Example 16. We consider the complete graph K_4 shown in Figure 3. For our first cycle, we choose the cycle which goes from v_1 to v_2 to v_3 to v_4 and back to v_1. This is a copy of C_4. Now consider the copy of C_3 given by going from v_1 to v_2 to v_3 and back to v_1. These two cycles are compatible. Any orientation which makes our copy of C_4 into an oriented cycle will work so long as we orient the edge $\{v_1, v_3\}$ correctly.

Let's look at an example of some cycles which are not compatible. We keep the same graph and the same initial copy of C_4. Now we choose a second copy of C_4 given by going from v_1 to v_2 to v_4 to v_3 and back to v_1. These two cycles are not compatible. Orient the first cycle so that you get an oriented cycle. Now either the edge $\{v_1, v_2\}$ or $\{v_3, v_4\}$ will not be oriented correctly to make the second cycle into an oriented cycle, irrespective of how the edges $\{v_1, v_3\}$ and $\{v_2, v_4\}$ are oriented.

Compatible cycles play an important role in this analysis since, whenever two cycles are compatible, they give rise to edge-disjoint cycles in the symmetric digraph — simply take one cycle as oriented then reverse the edge orientations for the other cycles so that neither of them ever use an edge in the same direction. These edge-disjoint cycles then show up in the oriented line graph as disjoint unions of cycles, exactly the structures that contribute to the coefficients of the zeta function. Now that we have this connection in general, we can state a more general corollary.

Corollary 17. *Let X be a connected graph with girth g and Ihara zeta function as above.*

(1) *Whenever $0 < i < g$, the coefficient c_i equals 0.*

(2) *Whenever $g \le i < 2g$, the coefficient c_i is the negative of twice the number of i-gons in X.*

(3) *Whenever $2g \le i < 3g$, the coefficient c_i is the sum of the following terms:*

- *the negative of twice the number of i-gons in X,*
- *four times the number of edge disjoint pairs of a k-gon and a $(c_i - k)$-gon for $g \le k < 2g$,*
- *twice the number of pairs of a k-gon and a $(c_i - k)$-gon that share at least one edge and are compatible for $g \le k < 2g$, and*
- *twice the number of edge disjoint pairs of a k_1-gon and a k_2-gon that have a path of length $\frac{1}{2}(c_i - k_1 - k_2)$ between them and are compatible for $k_1 + k_2 < 3g$.*

Corollary 17 thus encompasses Theorems 3 and 5. It also makes it possible to write down the zeta function of certain graphs, particularly graphs which have very few cycles. The fewer the number of cycles, the easier it is to identify where the linear subgraphs are showing up in the calculations of Theorem 7. We look at the graphs C_n and the graphs which are a cycle with a single chord. The zeta function of C_n is easy to compute directly from the definition, but it is instructive to apply the corollary to these graphs.

Example 18. Consider the graph C_n which is the graph that is a cycle on n vertices. Then its zeta function is given by

$$Z_{C_n}(u)^{-1} = 1 - 2u^n + u^{2n}.$$

The graph C_n has girth n, so all of the coefficients up to c_n are zero. In addition, there is a single n-gon, so the coefficient c_n is given by -2. There are no other k-gons, so all of the rest of the coefficients up to c_{2n} must be zero. Finally, c_{2n} can be computed by Czarneski's result or as a consequence of there being only one n-gon and no primitive cycle of length $2n$.

Cycles which have exactly one chord are a bit more delicate since there are more cycles to consider. With due care, we can still work out the zeta function. We define the graph $CH_{n,k}$ by starting with the cycle graph C_n and adding an additional edge so that the smallest cycle in $CH_{n,k}$ has length $k+1$. We illustrate $CH_{10,4}$ in Figure 4.

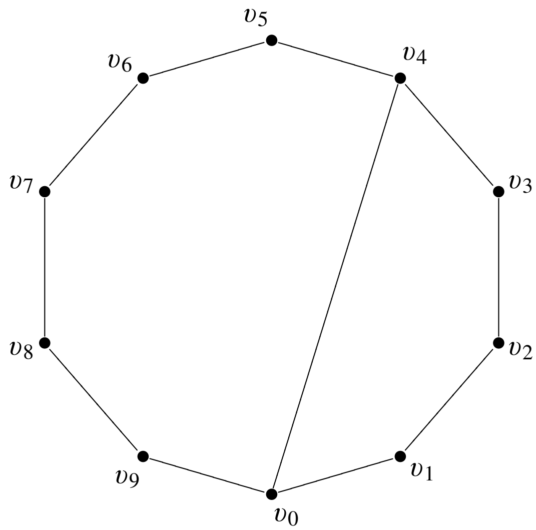

Figure 4. $CH_{10,4}$.

Corollary 19. *The zeta function of $CH_{n,k}$ is given by*

$$Z_{CH_{n,k}}(u)^{-1} = 1 - 2u^{k+1} - 2u^{n-k+1} - 2u^{n} + 2u^{2n-k+1} + 2u^{n+2} + 2u^{n+k+1} + u^{2k+2} + u^{2n-2k+2} + u^{2n} - 4u^{2n+2}.$$

Proof. By inspection, it is easy to see that there are exactly six backtrackfree, tailless cycles in the symmetric digraph $D(CH_{n,k})$. Specifically, $D(CH_{n,k})$ contains clockwise and counterclockwise copies of C_n, C_{n-k+1}, and C_{k+1}. Taken individually, these cycles contribute the second, third and fourth term of $Z_{C_{n,k}}(u)^{-1}$ above. There are nine different subgraphs of the symmetric digraph that consist of exactly two of these cycles; these subgraphs contribute the next six terms. Finally there are four linear subgraphs of the oriented line graph of $CH_{n,k}$, giving us the final u^{2n+2} term. We leave it to the reader to find these four linear subgraphs and verify that they break up into an odd number of cycles. There is no further backtrackfree, tailless cycle in $D(CH_{n,k})$. □

Example 20. We return to the example of $CH_{10,4}$. By direct calculation, using the formula $\det(I - uT)$, we see that

$$Z_{CH_{10,4}}(u)^{-1} = 1 - 2u^5 - 2u^7 - u^{10} + 2u^{12} + u^{14} + 2u^{15} + 2u^{17} + u^{20} - 4u^{22}.$$

Corollary 19 would have us write the function as

$$Z_{CH_{10,4}}(u)^{-1} = 1 - 2u^5 - 2u^7 - 2u^{10} + 2u^{17} + 2u^{12} + 2u^{15} + u^{10} + u^{14} + u^{20} - 4u^{22}.$$

By collecting common powers in the second expression, we see that we do have the same polynomial.

Thus the reciprocal of the Ihara zeta function of a graph encodes a great deal of structural information about the graph, particularly about the graph's primitive cycles. We have high hopes that, in general, it encodes enough information to allow us to conclude that certain families of graphs are determined by their zeta functions. We will pose this question and a few others in the next section.

3. Conclusion

In this section, we first explore the question of bounding the number of Eulerian circuits in an Eulerian graph. Recently, the problem of counting the number of Eulerian circuits has been shown to be $\#P$-complete in the class of undirected graphs by Brightwell and Winkler [2004]. There has been very little success at even bounding this number. We give a fairly rough bound, which is very inaccurate.

We let X be an (undirected) Eulerian graph and denote by $\operatorname{eul} X$ the number of Eulerian circuits on X. An *Eulerian circuit* is a cycle which uses every edge of X exactly once. As such, it is a primitive cycle of length m where m is the number of edges in X. When counting Eulerian circuits, we do distinguish direction of travel, so given one circuit, we can get another by traversing the same edge sequence by in reverse order. Thus the cycle graphs C_n satisfy $\operatorname{eul} C_n = 2$.

To state our bound, we need to define the *permanent* of a matrix M.

Definition 21. Let $M = (m_{i,j})$ be an $n \times n$ square matrix. The *permanent* of M the "signless determinant", that is,

$$\operatorname{perm} M = \sum_{\sigma \in S_n} \prod_{i=1}^{n} m_{i,\sigma(i)},$$

where S_n is the symmetric group over the set $\{1, \ldots, n\}$ (the group of permutations of this set).

The permanent shows up in several interesting ways in graph theory. For instance, it gives the number of perfect matchings of a bipartite graph [Harary 1969a]. For a general (0, 1)-square matrix, the same reference gives us a useful expression connecting the permanent to linear subgraphs of a digraph:

Lemma 22. *We use the notation from Definition 6. Let D be a digraph with n vertices and adjacency matrix A. Then the permanent of A is given by*

$$\operatorname{perm} A = \mathscr{E}_n(D) + \mathscr{O}_n(D).$$

In other words, the permanent of A counts the number of linear subgraphs of D.

With this interpretation, it is easier to be persuaded of the validity of the next result:

Theorem 23. *Let X be an Eulerian graph with oriented line graph $L^o X$. We let T be the adjacency matrix of $L^o X$. Then*

$$\operatorname{eul} X \leq \det T + \operatorname{perm} T.$$

We will use the notation $\vartheta(X) = \det T + \operatorname{perm} T$.

Proof. For a particular Eulerian circuit c, we denote by $\overline{c}$ the Eulerian circuit formed by traversing the edges in the opposite direction. Thus the pair of Eulerian circuits c and $\overline{c}$ induce a linear subgraph of the oriented line graph $L^o X$. This linear subgraph is composed of exactly 2 cycles. They are either both even or both odd cycles. In either case, this linear subgraph contributes positive 1 to the computation of $\det T$.

We can think of the determinant of T as the sum of the positive contribution minus the negative contribution. The permanent, however, is the sum of the positive contribution plus the negative contribution. Thus if we take $\det T + \operatorname{perm} T$, we get twice the positive contribution. Since two Eulerian circuits add exactly 1 to the positive contribution, we get the desired result. □

There is a fairly serious flaw with this bound. In general, computing the permanent of a (0, 1)-matrix is a $\#P$-complete problem [Valiant 1979], so we do not seem to have really improved matters. Fortunately, there are polynomial probabilistic algorithms that can compute the permanent within a specific amount of error [Jerrum et al. 2004]. As there is no known polynomial algorithm to even estimate the number of Eulerian circuits in a graph, we have actually managed to say something.

We present in Table 1 the results of computing the number of Eulerian circuits as well as the sum of the determinant and the permanent of the adjacency matrix of the oriented line graph for all connected Eulerian graphs on 6 vertices. We denote by n the number of vertices, by m the number of edges, by $-\chi$ the negative of the *Euler number* (which is $n - m$), by eul the exact number of Eulerian circuits, and by ϑ the given bound. The graphs here are given in graph6 format. We use the program Nauty [McKay 2007] to generate the graphs. All calculations were done in SAGE [Stein 2008]. The exact number of Eulerian circuits was computed using an algorithm that is currently being worked on by Klyve and Storm. The graphs are small enough and the algorithm is developed well enough so we are certain of the calculations presented.

From the data presented, we see that this bound fluctuates wildly in terms of error and would be completely ineffective for a graph with decent size.

This work suggests a great many problems for further research. We present a few of them here, in no particular order of perceived difficulty.

Graph	n	m	$-\chi$	eul	ϑ	error
EqGW	6	6	0	2	2	0
E@ro	6	7	1	4	6	2
E_lw	6	8	2	12	90	78
E?~o	6	8	2	12	90	78
EElw	6	9	3	32	702	670
ET\w	6	10	4	88	6642	6554
Er\w	6	11	5	264	58806	58542
E}lw	6	12	6	744	532170	531426

Table 1. Computations for all connected Eulerian graphs on 6 vertices.

Problem 24 (Graphs determined by their zeta functions). Recently, there has been some good work showing that several infinite families of graphs are determined by their Tutte polynomials [de Mier and Noy 2004]. One of the keys to these proofs is that the Tutte polynomial determines the number of triangles and squares in a graph. We saw in the previous section that the zeta function determines the number of triangles, squares, and pentagons in a simple graph. This gives us some hope that some large families of graphs are determined by their zeta functions. This would be particularly interesting since the Ihara zeta function can be computed in polynomial time.

A reader interested in this problem may want to start with the survey by Stark and Terras [1996] to become familiar with the edge zeta function as this function is necessary to determine if every vertex of a graph has degree greater than or equal to 2 or not. Cooper [2006] also has some preliminary work towards identifying other graph invariants determined by the zeta function that could prove useful.

We conjecture that the wheel graphs W_n defined by taking the cycle C_n and adding a vertex which is adjacency to every other vertex are uniquely determined by the Ihara zeta function among the connected graphs for which every vertex has degree at least 2. Through a computer search, we have verified

Theorem 25. *Within the family of connected graphs such that the degree of every vertex in a graph is at least* 2, *the graphs* W_3, W_4, W_5, W_6, W_7, W_8 *and* W_9 *are determined by their Ihara zeta functions. If*, *instead*, *we consider the edge zeta function defined by Stark and Terras [1996]*, *we can remove the condition on the degrees of the vertices.*

In the left half of Table 2, we count the number of connected graphs on n vertices for $n = 4, \ldots, 8$ as well as how many distinct zeta functions, characteristic polynomials, and pairs of zeta function and characteristic polynomial. In the right half, we only count graphs which are “md2” — every vertex has degree at least

Vertices	Graphs	Distinct Zetas	Distinct Spectra	Distinct Pairs	md2 Graphs	Distinct Zetas	Distinct Spectra	Distinct Pairs
4	6	5	6	6	3	3	3	3
5	21	16	21	21	11	11	11	11
6	112	77	111	112	61	61	61	61
7	853	584	821	850	507	507	494	507
8	11117	10423	8025	11106	7442	7441	7064	7442

Table 2. Graph and zeta function counting.

2 — as these are the more natural classes to consider zeta functions. The column referring to “Spectra” is counting the number of unique adjacency matrix spectra which appear. We see that the zeta function does remarkably well at distinguishing graphs, suggesting that there could be a lot of opportunities to show that families are uniquely determined.

Problem 26 (The inverse problem). Though we have given a characterization of the coefficients of the reciprocal of the Ihara zeta function, we have not answered some important questions.

(1) Given a polynomial $p(u)$, determine if it is the reciprocal of the Ihara zeta function of some graph.

(2) Given a polynomial $p(u)$ which is the reciprocal of the Ihara zeta function of a graph, construct an oriented line graph which gives rise to it. This is equivalent to constructing a graph which gives rise to it, as Cooper’s algorithm [2006] recovers the graph from its oriented line graph.

(3) Construct a polynomial which satisfies the graph “Riemann” hypothesis (see [Kotani and Sunada 2000; Stark and Terras 1996] for details) and which is also the reciprocal of the Ihara zeta function of some graph.

Solving the last two questions would provide a new construction of Ramanujan graphs. We also suggest [Horton et al. 2006; Murty 2003] for more information on Ramanujan graphs and their connection to the Ihara zeta function.

Problem 27 (A better Eulerian circuit count bound). It should be possible to give a better bound than the one found in Theorem 23. In our examination of Eulerian circuits, we really only scratched the surface of the structure that the zeta function tells us about. A deeper study may prove fruitful.

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