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Multiplicity results for semipositone two-point boundary value problems

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In this paper we address two-point boundary value problems of the form

$$u'' + f(u) = 0, \qquad \text{in } (0, 1),\ u(0) = u(1) = 0,$$

where the function f resembles $f(u) = \lambda(\exp(au/(a+u)) - c)$ for some constants $c \geq 0,\ \lambda > 0,\ a > 4$. We prove the existence of positive solutions for the semipositone case where $f(0) < 0$, and further prove multiplicity under certain conditions. In particular we extend theorems from Henderson and Thompson to the semipositone case.

1. Introduction

In this paper we address two-point boundary value problems of the form

$$u'' + f(u) = 0, \quad \text{in } (0, 1), \qquad u(0) = u(1) = 0, \tag{1}$$

where the function f resembles $f(u) = \lambda(\exp(au/(a+u)) - c)$ for some constants $c \geq 0,\ \lambda > 0,$ and $a > 4$. Boundary value problems of this sort are motivated by a variety of applications, such as nonlinear heat generation and combustion [Brown et al. 1981], and have been studied extensively since the early work of authors such as Keller and Cohen [1967]. These references deal exclusively with the *positone* case, the case where f is *posi*tive and mono*tone*.

In this paper we are interested in finding multiple positive solutions for the semipositone case where $f(0) < 0$. In particular we extend theorems from [Henderson and Thompson 2000] to the semipositone case. Our results complement those in [Brown et al. 1981; Castro and Shivaji 1998], and many related papers that discuss S-shaped bifurcation curves for positone and semipositone problems. Related PDE results can be found in [Drábek and Robinson 2006; Robinson and Rudd 2006]. Drábek and Robinson [2006] generalizes the main theorem in [Henderson and Thompson 2000] to the PDE case over arbitrary smooth bounded domains.

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[Robinson and Rudd 2006] generalizes our ODE results to the analogous PDE problem on the unit ball.

Our proofs characterize solutions as critical points of the functional

$$J(u)=\frac{1}{2}\int_0^1 \left(u'\right)^2-\int_0^1 F(u), \quad u\in H_0^1(0,1),$$

where $F(u):=\int_0^u f$. Using step functions as a simple model for f we produce lower solutions, $\{\underline{u}_1,\underline{u}_2\}$, and upper solutions, $\{\overline{u}_1,\overline{u}_2\}$, with the ordering

$$\underline{u}_1\le\overline{u}_1\le\underline{u}_2\le\overline{u}_2.$$

Standard arguments show that J has a local minimum in each of the generalized intervals $\left[\underline{u}_1,\overline{u}_1\right]$ and $\left[\underline{u}_2,\overline{u}_2\right]$. The third solution is characterized as a saddle point lying between the two minima. Our theorems show that one of the minima is positive and the other is negative, and, under certain conditions, the saddle point solution is also positive. We provide two separate criteria that guarantee a second positive solution.

The theorems in [Brown et al. 1981] and [Henderson and Thompson 2000] are representative of two different approaches to very similar problems, so it is of some interest to provide an explicit comparison of these theorems. In Section 6 we provide such a comparison for the positone PDE case. In particular, we show that the conditions in [Drábek and Robinson 2006], where the main theorem of [Henderson and Thompson 2000] is generalized to the PDE case, are more general than those in [Brown et al. 1981].

2. Preliminaries

The expression $\underline{u}\in C^2(0,1)\bigcap C[0,1]$ is called a lower solution of Equation (1) if

$$\underline{u}''+f(\underline{u})\ge 0,\quad \underline{u}(0)\le 0,\quad \underline{u}(1)\le 0.$$

Upper solutions are defined similarly with reversed inequalities.

Since f is a bounded continuous function it is straightforward to show the J is a C^1 functional that satisfies the Palais–Smale condition, and that the following minimization and mountain pass lemmas are true [Struwe 1990].

Lemma 2.1. *Suppose that $\underline{u}$ and $\overline{u}$ are lower and upper solutions of Equation (1), respectively, and suppose that $\underline{u}\le\overline{u}$ on $[0,1]$. Then J achieves a local minimum at some critical point $u\in\left[\underline{u},\overline{u}\right]:=\{u\in H_0^1(0,1):\underline{u}\le u\le\overline{u}\}$.*

See [Struwe 1990, Theorem 2.4] for an elegant proof.

Lemma 2.2. *Suppose that $\underline{u}$ and $\overline{u}$ are lower and upper solutions of (1), respectively, and suppose that u_1,u_2 are distinct local minima of J in $\left[\underline{u},\overline{u}\right]$. Then there* *is a third critical point of* J, $u_3 \in [\underline{u}, \overline{u}]$, *which satisfies*

$$J(u_3) = c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], [\underline{u}, \overline{u}]) : \gamma(0) = u_1, \gamma(1) = u_2\}$.

Note that our solutions must be symmetric about $x = 1/2$, and it will often be convenient to look at the problem over the interval $[0, 1/2]$ with the condition $u'(1/2) = 0$.

With the given construction of lower and upper solutions it is also possible to construct proofs using degree theory. The connection between upper and lower solutions and degree theory is developed nicely in [Amann 1976; Shivaji 1987], and is used in [Brown et al. 1981; Castro and Shivaji 1998; Drábek and Robinson 2006; Robinson and Rudd 2006], and many related papers.

3. The ideal case

In this section we perform a detailed analysis of an important special case. We study Equation (1) assuming that f is a step function with description

$$f(u) := \begin{cases} k, & u < 1, \\ K, & u \geq 1. \end{cases} \tag{2}$$

We will identify a region of the (k, K) plane where this ideal problem has three solutions. For points in the region where $k > 0$ all three solutions are positive. For points in the region where $k < 0$ it will always be the case that one solution is positive, one solution is negative, and the third solution is either sign-changing or positive. We will characterize the subregion where two of the solutions are positive and one is negative.

The solution of the ideal problem can be broken into two pieces corresponding to the subintervals where $u < 1$ and $u \geq 1$. Let $u = u_1$ on $\{x : u(x) < 1\}$, so $u_1 = -kx^2/2 + ax + b$, where we choose $b = 0$ in order to satisfy $u(0) = 0$. Let $u = u_2$ on $\{x : u(x) \geq 1\}$, so $u_2 = -Kx^2/2 + cx + d$, where $c = K/2$ in order to satisfy $u'(1/2) = 0$.

A solution whose maximum does not exceed 1 will satisfy

$$u \equiv u_1 = -\frac{k}{2}x^2 + \frac{k}{2}x,$$

where we have chosen $a = k/2$ in order to guarantee $u_1'(1/2) = 0$. If $k > 0$, then u is positive with $1 \geq \max u = k/8$. Of course, if $k \leq 0$, then u is nonpositive. Hence, a solution with $\max u \leq 1$ exists if and only if $k \leq 8$.

It remains to discover solutions whose maximum exceeds 1. This necessitates $K > 0$, else the solution would never have an interior maximum above 1. In order to explicitly construct these solutions we must satisfy continuity conditions by finding an $x_0 \in (0, 1/2)$ such that $u_1(x_0) = 1 = u_2(x_0)$. We must also satisfy a smoothness condition $u_1'(x_0) = u_2'(x_0)$. The smoothness condition can be used to solve for a, which can then be substituted into the first continuity condition to get

$$\left(\frac{k}{2} - K\right)x_0^2 + \frac{K}{2}x_0 - 1 = 0.$$

Basic curve sketching techniques from calculus show that this equation has exactly one root $x_0 \in [0, 1/2]$ when (k, K) is on the upper branch of the parabola

$$K^2 - 16K + 8k = 0,$$

the graph of $K = 8 + 2\sqrt{16 - 2k}$. We will refer to this curve as Γ_1. When (k, K) lies above Γ_1 then we get two roots. Once (k, K) has been chosen we can easily use the second continuity condition to solve for d. The two solutions thus obtained are either both positive or one is positive and one is sign-changing. Distinguishing between the latter two possibilities reduces to determining when the initial slope of the solution is nonnegative. This can be done for a particular (k, K) by using the conditions above to solve for $a = u'(0)$. To discover the condition that separates the sign-changing case from the positive case, we set $a = 0$ and solve. This curve, call it Γ_2, is described by

$$K = \frac{(8 + 2\sqrt{-2k})k}{k + 8}, \quad -\infty < k < -8.$$

It is straightforward to show that Γ_2 lies above Γ_1, and that the two curves are asymptotic as $k \to -\infty$. If a pair (k, K) lies on Γ_1, then the ideal problem has exactly one positive solution. If the pair lies above Γ_1 and below Γ_2, then the problem has two positive solutions. If the pair lies above Γ_2, then the problem has one positive solution and one sign-changing solution.

4. A three solutions theorem

In this section we see that the ideal case generalizes in a straightforward way.

Theorem 4.1. *Let (k, K) be a point on the curve Γ_1, let $0 < b$, and suppose that $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function such that*

(a) $f(0) < 0$,

(b) $kb \le f(u)$, *for* $u < b$,

(c) $f(u) \ge Kb$, *for* $b \le u \le Mb$,

where M is the maximum of the solution to Equation (1) assuming the ideal conditions (2). Then the boundary value problem (1) has at least three symmetric solutions, one of which is positive and one of which is negative.

Proof. We use symmetry to reduce the argument to the half interval $[0, 1/2]$, with the boundary conditions $u(0) = 0$ and $u'(1/2) = 0$. As a further simplification we rescale the problem so that, without loss of generality, $b = 1$. Simply let $v = u/b$ and note that $v'' + f(bv)/b = 0$, and that $\overline{f}(\cdot) := f(b\,\cdot)/b$ satisfies $\overline{f}(v) \geq k$ for $v \geq 1$, etc.

It is easy to check that $\overline{u}_1 \equiv 0$ and $\overline{u}_2 = -Cx(1-x)/2$ are upper solutions, where C is chosen so that $f(u) < C$ for all u. It is also easy to check that $\underline{u}_1 = -kx(1-x)/2$ is a lower solution. Now consider the positive function $\underline{u}_2 = \psi$, where ψ is the solution of

$$\psi'' = \begin{cases} -k, & \psi < 1, \\ -K, & \psi \geq 1, \end{cases} \qquad \psi(0) = 0, \quad \psi'(1/2) = 0,$$

as described in Section 3. Let $M := \max_{[0,1/2]} \psi = \psi(1/2)$. It follows that $f(\underline{u}_2) \geq K$ at points where $\psi \geq 1$, where $1 \leq \psi \leq M$, and that $f(\psi) \geq k$ where $\psi < 1$, where $0 \leq \psi < 1$. Hence $\underline{u}_2$ is a positive lower solution.

Theorem 4.1c implies $C > K$, so we have $\overline{u}_2'' < \underline{u}_2''$, $\overline{u}_2(0) = \underline{u}_2(0) = 0$, and $\overline{u}_2'(1/2) = \underline{u}_2'(1/2) = 0$. A simple comparison implies that $\underline{u}_2 \leq \overline{u}_2$. Other comparisons are easy, and lead to $\underline{u}_1 \leq \overline{u}_1 \leq \underline{u}_2 \leq \overline{u}_2$ in $[0, 1/2]$.

Applying the variational methods described in Section 2 we infer the existence of three solutions. The solution lying in the generalized interval $\left[\underline{u}_1, \overline{u}_1\right]$ is clearly negative, and the solution lying in the generalized interval $\left[\underline{u}_2, \overline{u}_2\right]$ is clearly positive. The third solution, the saddle point solution, cannot be easily described without further conditions on f. □

5. Criteria for two positive solutions

In this section we state criteria that guarantee two positive solutions. Since our interest is in positive solutions we assume throughout this section, without loss of generality, that $f(u) = f(0)$ for $u \leq 0$. This introduces the convenience that Equation (1) has a unique nonpositive solution satisfying $u'' + f(0) = 0$, so every other solution must be either sign-changing or positive.

In the ideal problem we get two positive solutions when $k \geq -8$. The analogous result follows for the more general problem.

Theorem 5.1. *Let* (k, K) *be a point on the curve* Γ_1, *let* $0 < a < b$, *and suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is a bounded continuous function such that*

(a) $f(0) < 0$,

(b) $kb \leq f(u)$, *for* $u \leq b$,

(c) $f(u) \geq Kb$, *for* $b \leq u \leq Mb$,

(d) $-8a \leq f(u) \leq 0$ *for* $0 \leq u \leq a$, *where M is the maximum of the solution to (1) assuming the ideal conditions (2). Then (1) has two positive solutions.*

Proof. Without loss of generality, rescale the problem so that $a = 1$. Apply Theorem 4.1 to get three solutions, one of which is negative, one of which is positive, and one of which is not yet well described. Let u be this third solution and observe, as above, that u must have positive values somewhere on its domain. By Theorem 5.1d, u cannot achieve a positive maximum below height $a = 1$. In fact, if x_1 is the first point where $u(x) = 1$, then $u'' \geq 0$ on $[0, x_1]$ implies that $u'(x_1) > 0$, so u must achieve a maximum strictly above 1.

Suppose that $u'(0) \leq 0$, and compare u to $v = 4x^2$. We know that $u(0) = 0 = v(0)$, $u'(0) \leq 0 = v'(0)$, and $u'' \leq 8 = v''$ on $\{x : u(x) < 1\}$. It follows that $u \leq v$ at least until the first point where $u = 1$, which cannot happen until after v reaches 1. But $v \leq 1$ on $[0, 1/2]$, so u cannot achieve a maximum greater than 1. This is a contradiction, so it must be that $u'(0) > 0$. A similar comparison leads to a contradiction if $u(x) = 0$ and $u'(x) \leq 0$ at any other point $x \in (0, 1/2)$. Thus u must be positive. □

It is interesting to note that for the analogous PDE problem on the unit ball, there is no theorem similar to Theorem 5.1. In fact, for any $k < 0$, it is possible to construct a sign-changing third solution for the ideal case [Robinson and Rudd 2006].

For $k < -8$ we have seen that the ideal problem has two positive solutions for (k, K) in the region above Γ_1 and below Γ_2. One might conjecture, and at one time these authors did, that the general problem will have two positive solutions if the K in Theorem 4.1 satisfies $K \leq (8 + 2\sqrt{-2k})k/(k+8)$. It turns out that an explicit counterexample can be constructed, as we shall soon demonstrate. However, the next theorem shows that an alternative upper bound on K does guarantee the existence of two positive solutions.

Theorem 5.2. *Let (k, K) be a point on the curve Γ_1 with $k < -8$, let $0 < a < b$, and suppose that $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function such that*

(a) $ka \leq f(u) < 0$ *for* $0 \leq u \leq a$,

(b) $kb \leq f(u)$, *for* $u \leq b$,

(c) $f(u) \geq Kb$, *for* $b \leq u \leq Mb$,

(d) $f(u) < 16ka/(k+8)$ *for all* u,

where M is the maximum of the solution to Equation (1) assuming the ideal conditions of (2). Then (1) has two positive solutions.

Proof. Rescale the problem so that, without loss of generality, $a = 1$. Let u represent the third solution as in the previous proof. Once again we use a comparison argument to show that assuming $u'(0) \leq 0$ leads to a contradiction. The case where we assume $u(x) = 0$ and $u'(x) \leq 0$ leads to a similar contradiction.

Let $k < -8$ be fixed, and consider a family of comparison functions indexed by $t \in \left[t_0, \sqrt{-2/k}\right]$,

$$v_t := \begin{cases} -\dfrac{k}{2}x^2, & x \in [0,t], \\ -kt(x-t) - \dfrac{k}{2}t^2, & x \in [t, x_t], \\ -\dfrac{K_t}{2}x^2, & x \in \left[x_t, \dfrac{1}{2}\right], \end{cases}$$

where

$$t_0 := \frac{1}{2} + \frac{1}{2k}\sqrt{k^2+8k}, \qquad x_t := \frac{kt^2-2}{2kt}, \qquad K_t = \frac{2k^2t^2}{kt^2-kt-2}.$$

This function can be visualized in three pieces: the first is a parabola emerging from the origin with 0 slope, the second is a tangent line to the parabola at the point $(t, -kt^2/2)$, and the third is a parabolic cap that meets the tangent line smoothly at $(x_t, 1)$ and then reaches a critical point at $x = 1/2$. t_0 describes the t value such that $x_t = 1/2$, and thus describes the infimum of the t values such that this comparison function makes sense, and, not coincidentally, identifies a vertical asymptote for K_t. Computing two derivatives with respect to x, except at $x = t$ and $x = x_t$, we see that

$$v_t'' := \begin{cases} -k, & x \in [0,t), \\ 0, & x \in (t, x_t), \\ -K_t, & x \in (x_t, \frac{1}{2}]. \end{cases}$$

Recall, as in the previous proof, that u must reach a positive maximum above 1 at some point. How do u and v_t compare? If $u'(0) \leq 0$, then $u \leq -kx^2/2$ on $\left[0, \sqrt{-2/k}\right]$, because $u'' \leq -k$ while $u < 1$, and $-kx^2/2$ reaches height 1 at $\sqrt{-2/k}$. It is clearly possible to adjust the choice of t so that x_t represents the first point where $u(x) = 1$. Since $u'' \geq 0$ on $[0, x_t]$, $u(t) \leq v_t(t)$, and $u(x_t) = v_t(x_t)$ it follows that $u \leq v_t$ on $[0, x_t]$ and that $u'(x_t) \geq v_t'(x_t) > 0$. By the mean value theorem, there is an $x \in (x_t, 1/2)$ such that

$$-u''(x) = -\frac{u'\left(\frac{1}{2}\right) - u'(x_t)}{\frac{1}{2} - x_t} = \frac{u'(x_t)}{\frac{1}{2} - x_t} \geq \frac{v'(x_t)}{\frac{1}{2} - x_t} = K_t.$$

Elementary calculus reveals that K_t achieves a minimum of

$$K_t = \frac{16k}{k+8}, \qquad \text{at } t = -\frac{4}{k}.$$

Thus

$$-u''(x) \geq 16k/(k+8) > f(u(x)),$$

and so a contradiction has been reached. □

It is important to note that $16k/(k+8) < (8+2\sqrt{-2k})k/(k+8)$ for $k < -8$, and so the comparison functions, v_t, satisfy the conditions of Theorem 4.1 and the additional restriction $-v_t'' < (8+2\sqrt{-2k})k/(k+8)$. Since the inequality is strict we can slightly modify v_t so that it has negative slope at 0, and is thus sign-changing, but still satisfies conditions Theorem 5.2a–c, as well as the given estimate on its second derivative. This provides the counterexample to the conjecture, expressed above, that Γ_2 provides a boundary guaranteeing two positive solutions for the general case.

Finally, if f is to satisfy the conditions of Theorem 5.2, and if C represents the upper bound for f, then we must have $8+2\sqrt{16-2k} \leq C < 16k/(k+8)$. A careful comparison of expressions on the left and right of this inequality shows that their graphs cross in the (k, K) plane at the point $(-24, 24)$. Thus Theorem 5.2 is only applicable for $-24 < k < -8$. It seems reasonable to conjecture that finer estimates and comparison arguments will discover criteria for two positive solutions when $k < -24$.

6. A comparison of solvability conditions

The methods and results in [Brown et al. 1981] and [Henderson and Thompson 2000] represent two different, and complimentary, approaches to similar problems. The more obvious differences are that [Henderson and Thompson 2000] does not impose the same monotonicity and smoothness conditions used in [Brown et al. 1981], and is, in that sense, more general. On the other hand the results in [Brown et al. 1981] deal with both the ODE and PDE cases, and take good advantage of the more restrictive conditions to prove more precise results, especially in the ODE case.

The relationship between the solvability conditions in the two papers is not as obvious. In this section we explore that relationship. In particular we prove that if the conditions in [Brown et al. 1981] are satisfied, then so are the conditions in [Henderson and Thompson 2000]. In order to demonstrate this in some generality we consider the PDE case,

$$\Delta u + \lambda f(u) = 0 \text{ in } D, \qquad u|_{\partial D} = 0, \tag{3}$$

where $D \subset \mathbb{R}^n$ is a smooth bounded domain and $f : [0, \infty) \to [0, \infty)$ is continuous. [Henderson and Thompson 2000] presented purely ODE results, but their work is generalized in [Drábek and Robinson 2006], so we will actually compare the conditions in [Brown et al. 1981] and [Drábek and Robinson 2006].

It is helpful to begin by defining several constants. First, let $m := \max_D \phi$, where ϕ is the unique positive solution of

$$\Delta\phi + 1 = 0 \text{ in } D, \qquad \phi|_{\partial D} = 0.$$

Second, consider the problem

$$\Delta\psi + Kh(\psi) = 0 \text{ in } D, \qquad \phi|_{\partial D} = 0, \tag{4}$$

where $h(u) \equiv 0$ when $u < 1$ and $h(u) \equiv 1$ when $u \geq 1$. It is proved in [Drábek and Robinson 2006] that there is a minimal positive K such that Equation (4) has a positive solution, and we assume throughout the arguments below that K is this minimal constant. Let $M := \max_D \psi$.

Drábek and Robinson [2006] proved that (3) has three nonnegative solutions if

(a) $\lambda f(u) < ka$ on $[0, a]$,

(b) $\lambda f(u) \geq Kb$ on $[b, Mb]$,

(c) $\lambda f(u) \leq kc$ on $[0, c]$,

where $0 < a < b < Mb < c$ and $k := 1/m$.

Before stating the solvability conditions in [Brown et al. 1981] we describe yet another constant. Consider a subdomain $\Omega \subset\subset D$ and consider

$$\Delta\eta + \chi_\Omega = 0 \text{ in } D, \qquad \eta|_{\partial D} = 0.$$

Define $M_2 := [\inf_\Omega \eta]^{-1}$. Observe that $v = M_2\eta$ satisfies

$$\Delta v + M_2\chi_\Omega = 0 \text{ in } D, \qquad \eta|_{\partial D} = 0,$$

with $v \geq 1$ on Ω. In particular we have $\Delta v + M_2 h(v) \geq 0$, so v is a positive lower solution for (4). Combining this with a simple constant upper solution we can show that (4) has a positive solution when M_2 is substituted for K. Since K is the minimal constant with this property we see that $K \leq M_2$. For a more detailed discussion of K and its properties see [Drábek and Robinson 2006].

Brown et al. [1981] proved that (3) has three nonnegative solutions if f is a smooth and bounded function, which is increasing on $[0, c']$, and which satisfies

$$M_2\left(\frac{l_2}{f(l_2)}\right) \leq \lambda \leq \min\left\{M_1\left(\frac{l_1}{f(l_1)}\right), M_3\left(\frac{c'}{f(l_1)}\right)\right\}, \tag{5}$$

where $0 < l_1 < l_2 < c'$.

In order to compare solvability conditions it remains to do a careful reading of the proof in [Brown et al. 1981] to see how the constants are chosen and how they compare to those in [Drábek and Robinson 2006]. First, it turns out that

$M_1 = M_3 = 1/m$. Hence, the inequality

$$\lambda \leq M_1 \left(\frac{l_1}{f(l_1)} \right),$$

implies that $\lambda f(a) \leq ka$ if we substitute $l_1 = a$ and $k = 1/m$. Moreover, the monotonicity assumption on f leads to $\lambda f(u) \leq ka$ for u in $[0, a]$. The inequality

$$M_2 \left(\frac{l_2}{f(l_2)} \right) \leq \lambda,$$

leads us to $Kb \leq \lambda f(b)$, where we have substituted $l_2 = b$ and $M_2 \geq K$. Once again monotonicity implies that $Kb \leq \lambda f(u)$, for $u \in \left[b, c'\right]$. Substituting $M_3 = 1/m$, $b = l_2$, and $c = c'$ into the inequality

$$M_2 \left(\frac{l_2}{f(l_2)} \right) \leq M_3 \left(\frac{c'}{f(l_2)} \right),$$

gives $mM_2 b \leq c$, and thus $c \geq mKb$. By the definition of m we know that $K\phi(x) \leq mK$ for all $x \in D$. Also, $\Delta(K\phi) = -K \leq -Kh(\psi) = \Delta\psi$ in D , with strict inequality over the set $D \setminus \Omega$, so the maximum principle implies that $K\phi(x) > \psi(x)$ in D. Hence $Km > M$, and so $c > Mb$.

So far we have used Equation (5) to verify conditions (a) and (b) on page 131 for Equation (3) having nonnegative solutions, with the modest exception of obtaining a strict inequality for condition (a). The purpose of the strict inequality in [Drábek and Robinson 2006] is to guarantee that the intermediate lower solution is strict, which helps in distinguishing the three different solutions. This hair can easily be split by allowing equality and then using the monotonicity condition on f to recover. Finally, condition (c) follows easily from the fact that f is bounded. Hence the solvability conditions in [Brown et al. 1981] imply those in [Drábek and Robinson 2006].

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