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Paths and circuits in $\mathbb{G}$-graphs

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For a group G with generating set $S = \{s_1, s_2, \ldots, s_k\}$, the $\mathbb{G}$-graph of G, denoted $\Gamma(G, S)$, is the graph whose vertices are distinct cosets of $\langle s_i \rangle$ in G. Two distinct vertices are joined by an edge when the set intersection of the cosets is nonempty. In this paper, we study the existence of Hamiltonian and Eulerian paths and circuits in $\Gamma(G, S)$.

1. Introduction

Let G be a group with a generating set $S = \{s_1, \ldots, s_k\}$. For the subgroup $\langle s_i \rangle$ of G, define the subset $T_{\langle s_i \rangle}$ of G to be a *left transversal* for $\langle s_i \rangle$ if $\{x\langle s_i \rangle \mid x \in T_{\langle s_i \rangle}\}$ is precisely the set of all left cosets of $\langle s_i \rangle$ in G. Associate a simple graph $\Gamma(G, S)$ to (G, S) with vertex set $V(\Gamma(G, S)) = \{x_j\langle s_i \rangle \mid x_j \in T_{\langle s_i \rangle}\}$. Two distinct vertices $x_j\langle s_i \rangle$ and $x_l\langle s_k \rangle$ in $V(\Gamma(G, S))$ are joined by an edge if $x_j\langle s_i \rangle \cap x_l\langle s_k \rangle$ is nonempty. The edge set, $E(\Gamma(G, S))$, consists of pairs $(x_j\langle s_i \rangle, x_l\langle s_k \rangle)$. $\Gamma(G, S)$ defined this way has no multiedge or loop. Bretto and Gillibert [2004] introduced $\Gamma(G, S)$ and a similar graph, $\overline{\Gamma}(G, S)$. $\overline{\Gamma}(G, S)$ differs from $\Gamma(G, S)$ in that it is a multigraph with a n-edge between two vertices $x_j\langle s_i \rangle$ and $x_l\langle s_k \rangle$ when $|x_j\langle s_i \rangle \cap x_l\langle s_k \rangle| = n$. The $\mathbb{G}$-graph, $\Gamma(G, S)$, is necessarily a subgraph of $\overline{\Gamma}(G, S)$.

In this paper we concentrate on results for $\Gamma(G, S)$. Many of the results from [Bretto and Gillibert 2004; 2005; Bretto et al. 2005; 2007] about $\overline{\Gamma}(G, S)$ translate easily to the simple graph $\Gamma(G, S)$.

Let $V_i = \{x_j\langle s_i \rangle \mid x_j \in T_{\langle s_i \rangle}\}$. Then $V(\Gamma(G, S)) = \cup_{i=1}^{k} V_i$. The main object of this paper is to study the existence of Hamiltonian and Eulerian paths and circuits in $\Gamma(G, S)$. To this end we recall a few results from Euler. Notice that Eulerian circuits are not considered Eulerian paths in this paper.

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Theorem 1.1 (Euler). *Let Γ be a nontrivial connected graph. Then Γ has an Eulerian circuit if and only if every vertex is of even degree.*

Theorem 1.2 (Euler). *Let Γ be a nontrivial connected graph. Then Γ has an Eulerian path if and only if Γ has exactly two vertices of odd degree. Furthermore, the path begins at one of the vertices of odd degree and terminates at the other.*

2. Preliminaries

In this section, results are proved that pertain to the degrees of vertices in $\Gamma(G, S)$. Recall that if $S = \{s_1, s_2, \dots, s_k\}$, then $\Gamma(G, S)$ is necessarily k-partite.

Lemma 2.1. *If $g \in \langle s_i \rangle \cap \langle s_j \rangle$, then $g^{-1} \in \langle s_i \rangle \cap \langle s_j \rangle$.*

Proof. Let $g \in \langle s_i \rangle \cap \langle s_j \rangle$, then there exists $m, n \in \mathbb{N}$ such that $g = s_i^m = s_j^n$. Taking the inverse, we have $g^{-1} = s_i^{-m} = s_j^{-n}$. Therefore $g^{-1} \in \langle s_i \rangle \cap \langle s_j \rangle$. $\square$

Theorem 2.1. *Let G be a group with generating set S. Let $\langle s_i \rangle \cup x_2\langle s_i \rangle \cup \cdots \cup x_{k_i}\langle s_i \rangle$ be a partition of G into cosets of $\langle s_i \rangle$ and $\langle s_j \rangle \cup y_2\langle s_j \rangle \cup \cdots \cup y_{k_j}\langle s_j \rangle$ be a partition of G into cosets of $\langle s_j \rangle$. Let*

$$V_i = \{\langle s_i \rangle, x_2\langle s_i \rangle, \dots, x_{k_i}\langle s_i \rangle\} \quad \text{and} \quad V_j = \{\langle s_j \rangle, y_2\langle s_j \rangle, \dots, y_{k_j}\langle s_j \rangle\}$$

be the appropriate subsets of the vertex set of $\Gamma(G, S)$. If

$$|\langle s_i \rangle \cap \langle s_j \rangle| = S_{i,j} \quad \text{and} \quad (x\langle s_i \rangle, y\langle s_j \rangle) \in E(\Gamma(G, S)),$$

then $|x\langle s_i \rangle \cap y\langle s_j \rangle| = S_{i,j}$.

Proof. Let $\langle s_i \rangle \cap \langle s_j \rangle = \{e, g_1, \dots, g_{S_{i,j}-1}\}$. Since $g_1 \in \langle s_i \rangle$ and $g_1 \in \langle s_j \rangle$, there exists $m, n \in \mathbb{N}$ such that $g_1 = s_i^m = s_j^n$. Let $x\langle s_i \rangle \in V_i$ and $y\langle s_j \rangle \in V_j$ such that $(x\langle s_i \rangle, y\langle s_j \rangle) \in E(\Gamma(G, S))$. Then $x\langle s_i \rangle \cap y\langle s_j \rangle \neq \varnothing$ and there exists h such that $h = xs_i^{m'} = ys_j^{n'}$. So

$$h = xs_i^{m'} = xs_i^{m'-m}s_i^m = xs_i^{m'-m}g_1.$$

Therefore $hg_1^{-1} \in x\langle s_i \rangle$. Likewise, $hg_1^{-1} \in y\langle s_j \rangle$ and $hg_1^{-1} \in x\langle s_i \rangle \cap y\langle s_j \rangle$. By similar arguments, $\{h, hg_1^{-1}, hg_2^{-1}, \dots, hg_{S_{i,j}-1}^{-1}\} \subseteq x\langle s_i \rangle \cap y\langle s_j \rangle$.

Assume there exists $g \in x\langle s_i \rangle \cap y\langle s_j \rangle$ such that $g \notin \{h, hg_1^{-1}, hg_2^{-1}, \dots, hg_{S_{i,j}-1}^{-1}\}$. Since $g \in x\langle s_i \rangle \cap y\langle s_j \rangle$ there exists $m'', n'' \in \mathbb{N}$ such that $g = xs_i^{m''} = ys_j^{n''}$. So

$$g = xs_i^{m''} = xs_i^{m'}s_i^{m''-m'} = hs_i^{m''-m'}.$$

Therefore $h^{-1}g \in \langle s_i \rangle$. Likewise $h^{-1}g \in \langle s_j \rangle$ and $h^{-1}g \in \langle s_i \rangle \cap \langle s_j \rangle$. There exists $k \in \{0, \dots, S_{i,j} - 1\}$ such that $h^{-1}g = g_k$. Since $g_k \in \langle s_i \rangle \cap \langle s_j \rangle$, $g_k^{-1} \in$

$\langle s_i\rangle \cap \langle s_j\rangle$ by Lemma 2.1. Denote g_k^{-1} by g_k'. Then $g = hg_k = h(g_k')^{-1}$ and $g \in \{h, hg_1^{-1}, hg_2^{-1}, \dots, hg_{S_{i,j}-1}^{-1}\}$. Therefore

$$\{h, hg_1^{-1}, hg_2^{-1}, \dots, hg_{S_{i,j}-1}^{-1}\} = x\langle s_i\rangle \cap y\langle s_j\rangle,$$
$$|x\langle s_i\rangle \cap y\langle s_j\rangle| = S_{i,j}.$$
□

Corollary 2.1. *The number of edges between $\langle s_i\rangle$ and V_j is given by $|s_i|/S_{i,j}$.*

Proof. Let

$$V_j = \{\langle s_j\rangle, y_2\langle s_j\rangle, \dots, y_k\langle s_j\rangle\} \quad \text{and} \quad V_j' = \{\langle s_j\rangle, y_2'\langle s_j\rangle, \dots, y_l'\langle s_j\rangle\}$$

be the set that contains all vertices in V_j that are adjacent to $\langle s_i\rangle$. Since

$$(\langle s_i\rangle, y'\langle s_j\rangle) \in E(\Gamma(G,S)) \quad \text{for all } y'\langle s_j\rangle \in V_j',\ |\langle s_i\rangle \cap y'\langle s_j\rangle| = S_{i,j}$$

by Theorem 2.1. So the number of elements in $\langle s_i\rangle$ is given by $|s_i| = S_{i,j}\cdot l$ or the number of edges between $\langle s_i\rangle$ and V_j is $|s_i|/S_{i,j}$. □

Lemma 2.2. *If G is a group with generating set $S = \{s_1, \dots, s_n\}$ and $S_{i,j} = |\langle s_i\rangle \cap \langle s_j\rangle|$, then the degree of the vertex $\langle s_i\rangle$, denoted* $\deg\langle s_i\rangle$, *is*

$$\deg\langle s_i\rangle = \Bigl(\sum_{j=1}^{n} |s_i|/S_{i,j}\Bigr) - |s_i|/S_{i,i}.$$

Proof. We proceed with induction on n. Partition the vertex set of $\Gamma(G,S)$ into n subsets $V_1, V_2, \dots, V_n$ such that $V_i = \{\langle s_i\rangle, x_2\langle s_i\rangle, \dots, x_{k_i}\langle s_i\rangle\}$. Consider the subgraph, $\Gamma_{1,2}$, of $\Gamma(G,S)$ induced by the vertex set $V_1 \cup V_2$. Let $\deg_{\Gamma_{1,2}}(\langle s_i\rangle)$ denote the degree of the vertex $\langle s_i\rangle$ in $\Gamma_{1,2}$. Then, by Corollary 2.1,

$$\deg_{\Gamma_{1,2}}(\langle s_2\rangle) = |s_2|/S_{2,1} = \Bigl(\sum_{j=1}^{2} |s_2|/S_{2,j}\Bigr) - |s_2|/S_{2,2}.$$

Likewise

$$\deg_{\Gamma_{1,2}}(\langle s_1\rangle) = |s_1|/S_{1,2} = \Bigl(\sum_{j=1}^{2} |s_1|/S_{1,j}\Bigr) - |s_1|/S_{1,1},$$

and the formula holds for $n = 2$.

Consider the subgraph, $\Gamma_{1,2,\dots,n-1}$, of $\Gamma(G,S)$ induced by the vertex set $V_1 \cup V_2 \cup \dots \cup V_{n-1}$. Let $\deg_{\Gamma_{1,2,\dots,n-1}}(\langle s_i\rangle)$ denote the degree of the vertex $\langle s_i\rangle$ in $\Gamma_{1,2,\dots,n-1}$. Assume that the theorem holds for $n-1$, that is,

$$\deg_{\Gamma_{1,2,\dots,n-1}}(\langle s_i\rangle) = \Bigl(\sum_{j=1}^{n-1} |s_i|/S_{i,j}\Bigr) - |s_i|/S_{i,i}.$$

Now consider the entire graph, $\Gamma(G, S)$. The number of edges between $\langle s_i \rangle$ and V_n is $|s_i|/S_{i,n}$. So

$$\deg\langle s_i \rangle = |s_i|/S_{i,n} + \Big(\sum_{j=1}^{n-1} |s_i|/S_{i,j}\Big) - |s_i|/S_{i,i} = \Big(\sum_{j=1}^{n} |s_i|/S_{i,j}\Big) - |s_i|/S_{i,i}. \quad \square$$

Remark 1. Notice that $|s_i|/S_{i,i} = 1$, since $S_{i,i} = |\langle s_i \rangle \cap \langle s_i \rangle| = |s_i|$.

Corollary 2.2. *If G is a group with generating set $S = \{s_1, s_2, \ldots, s_n\}$, then* $\deg\langle s_i \rangle$ *equals* $\deg g\langle s_i \rangle$ *for all $g\langle s_i \rangle$ in V_i, that is, every vertex in the same vertex set has the same degree.*

Proof. Let G be a group with generating set $S = \{s_1, s_2, \ldots, s_n\}$ and $S_{i,j} = |\langle s_i \rangle \cap \langle s_j \rangle|$. From Theorem 2.1, if $g, h \in G$ such that $(g\langle s_i \rangle, h\langle s_j \rangle) \in E(\Gamma(G, S))$, then $|g\langle s_i \rangle \cap h\langle s_j \rangle| = S_{i,j}$. From Lemma 2.2,

$$\deg g\langle s_i \rangle = \Big(\sum_{j=1}^{n} \frac{|g\langle s_i \rangle|}{S_{i,j}}\Big) - 1 = \Big(\sum_{j=1}^{n} \frac{|\langle s_i \rangle|}{S_{i,j}}\Big) - 1 = \deg\langle s_i \rangle. \qquad \square$$

Theorem 2.2. *If G is a group with generating set $S = \{s_1, s_2, \ldots, s_n\}$ and $S_{i,j} = |\langle s_i \rangle \cap \langle s_j \rangle|$, then $\Gamma(G, S)$ is complete n-partite if and only if*

$$\Big(\sum_{j=1}^{n} \frac{|\langle s_i \rangle|}{S_{i,j}}\Big) - 1 = \Big(\sum_{k=1}^{n} |V_k|\Big) - |V_i|.$$

3. Abelian groups of rank ≤ 2

In this section, we let G be an abelian group of rank ≤ 2 and let $|S| = 2$. G is isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_m$ for some m and n. Notice that if G is infinite then it is isomorphic to $\mathbb{Z} \approx \mathbb{Z} \times \mathbb{Z}_1$ and the theorems of this section apply.

Theorem 3.1. *Let $G = \mathbb{Z}_n \times \mathbb{Z}_m$ and $S = \{(1, 0), (0, 1)\}$, then $\Gamma(G, S)$ has a Hamiltonian path if and only if $|m - n| \leq 1$.*

Proof. $(\Rightarrow)$ Let $\Gamma(G, S)$ contain a Hamiltonian path. $\Gamma(G, S)$ is $K_{m,n}$ [Daniel $\geq$ 2008]. Assume that $n \geq m$. $|(1, 0)| = n$ and $|(0, 1)| = m$ and $V = V_1 \cup V_2$ where

$$\begin{aligned} V_1 =& \{a_1 + \langle(1, 0)\rangle, a_2 + \langle(1, 0)\rangle, \ldots, a_m + \langle(1, 0)\rangle\} \text{ and} \\ V_2 =& \{b_1 + \langle(0, 1)\rangle, b_2 + \langle(1, 0)\rangle, \ldots, b_n + \langle(1, 0)\rangle\}. \end{aligned}$$

Let $H_1 = \langle(1, 0)\rangle$ and $H_2 = \langle(0, 1)\rangle$. Since $n \geq m$, any Hamiltonian path must start with a vertex in V_2, that is, $b_{i_1} + H_2$.

$$(b_{i_1} + H_2, a_{j_1} + H_1), (a_{j_1} + H_1, b_{i_2} + H_2), (b_{i_2} + H_2, a_{j_2} + H_1), \ldots,$$
$$(a_{j_{m-1}} + H_1, b_{i_m} + H_2), (b_{i_m} + H_2, a_{j_m} + H_1), \ldots$$

Notice that all the vertices in V_1 have been exhausted. So either the path ends here and $n = m$ or it ends with the edge $(a_{j_m} + H_1, b_{i_{m+1}} + H_2)$ and $n = m + 1$. Therefore $|m - n| \leq 1$. The proof for $m \geq n$ is similar.

($\Leftarrow$) Let $|m - n| \leq 1$, $|(1, 0)| = n$, and $|(0, 1)| = m$. Let

$$a_1 + H_1 \cup a_2 + H_1 \cup \cdots \cup a_m + H_1$$

be a partition of G into cosets of $\langle(1, 0)\rangle$ and let

$$b_1 + H_2 \cup b_2 + H_2 \cup \cdots \cup b_n + H_2$$

be a partition of G into cosets of $\langle(0, 1)\rangle$. Since $\Gamma(G, S)$ is $K_{m,n}$, there exists an edge between $a_i + H_1$ and $b_j + H_2$ for all i, j.

(i) $m = n + 1$ and $(a_1 + H_1, b_1 + H_2)$, $(b_1 + H_2, a_2 + H_1)$, $\ldots$, $(a_n + H_1, b_n + H_2)$, $(b_n + H_2, a_m + H_1)$ is a Hamiltonian path.

(ii) $n = m + 1$ and $(b_1 + H_2, a_1 + H_1)$, $(a_1 + H_1, b_2 + H_2)$,$\ldots$, $(b_m + H_2, a_m + H_1)$, $(a_m + H_1, b_n + H_2)$ is a Hamiltonian path.

(iii) $m = n$ and $(a_1 + H_1, b_1 + H_2)$, $(b_1 + H_2, a_2 + H_1)$, $\ldots$, $(b_{n-1} + H_2, a_n + H_1)$, $(a_n + H_1, b_n + H_2)$ is a Hamiltonian path. □

Theorem 3.2. *Let $G = \mathbb{Z}_n \times \mathbb{Z}_m$ and $S = \{(1, 0), (0, 1)\}$, then $\Gamma(G, S)$ has a Hamiltonian circuit if and only if $m = n$.*

Proof. ($\Rightarrow$) Let $\Gamma(G,S)$ contain a Hamiltonian circuit. $\Gamma(G,S)$ is $K_{m,n}$ [Daniel $\geq$ 2008]. $|(1,0)| = n$ and $|(0,1)| = m$ and $V = V_1 \cup V_2$ where

$$\begin{aligned} V_1 =&\{a_1 + \langle(1, 0)\rangle, a_2 + \langle(1, 0)\rangle, \ldots, a_m + \langle(1, 0)\rangle\},\\ V_2 =&\{b_1 + \langle(0, 1)\rangle, b_2 + \langle(1, 0)\rangle, \ldots, b_n + \langle(1, 0)\rangle\}. \end{aligned}$$

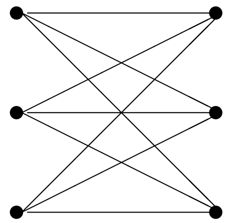
Let $H_1 = \langle(1, 0)\rangle$ and $H_2 = \langle(0, 1)\rangle$. Start with a vertex in V_2, that is, $b_{i_1} + H_2$ and trace the Hamiltonian circuit

$$(b_{i_1} + H_2, a_{j_1} + H_1), (a_{j_1} + H_1, b_{i_2} + H_2), (b_{i_2} + H_2, a_{j_2} + H_1), \ldots,\\ (a_{j_{m-1}} + H_1, b_{i_m} + H_2), (b_{i_m} + H_2, a_{j_m} + H_1), \ldots$$

Notice that all the vertices in V_1 have been exhausted. So the path ends here and to complete the circuit we need the edge $(a_{j_m} + H_1, b_{i_1} + H_2)$. Therefore $n = m$. The proof starting with a vertex in V_1 is similar.

($\Leftarrow$) Let $m = n$ and $a_1 + H_1 \cup a_2 + H_1 \cup \cdots \cup a_m + H_1$ be partition of G into cosets of $\langle(1, 0)\rangle$ Since $\Gamma(G, S)$ is $K_{m,m}$, there exist an edge between $a_i + H_1$ and $b_j + H_2$ for all i, j. Then $(a_1 + H_1, b_1 + H_2)$, $(b_1 + H_2, a_2 + H_1)$, $\ldots$, $(a_m + H_1, b_m + H_2)$, $(b_m + H_2, a_1 + H_1)$ is a Hamiltonian circuit. □

Example 1. Let $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $S = \{(1, 0), (0, 1)\}$, then $\Gamma(G, S) = K_{3,3}$ (see figure) and $\Gamma(G, S)$ contains both a Hamiltonian path and circuit.



Theorem 3.3. *Let $G = \mathbb{Z}_m \times \mathbb{Z}_n$ and $S = \{(1, 0), (0, 1)\}$, then $\Gamma(G, S)$ has an Eulerian circuit if and only if m and n are both even.*

Proof. ($\Rightarrow$) Let $\Gamma(G, S)$ have an Eulerian circuit. From [Daniel $\geq$ 2008], $S_{1,2} = S_{2,1} = 1$ so $\deg\langle(1, 0)\rangle = n$ and $\deg\langle(0, 1)\rangle = m$. Since every vertex is even, m and n are even.

($\Leftarrow$) Let m and n be even. From [Daniel $\geq$ 2008], $\Gamma(G, S)$ is $K_{n,m}$. Therefore $\deg\langle(1, 0)\rangle = m$ and $\deg\langle(0, 1)\rangle = n$. Since m and n are both even, $\Gamma(G, S)$ contains an Eulerian circuit. □

Theorem 3.4. *Let $G = \mathbb{Z}_m \times \mathbb{Z}_n$ and $S = \{(1, 0), (0, 1)\}$, then $\Gamma(G, S)$ has an Eulerian path if and only if m is odd and $n = 2$ or n is odd and $m = 2$.*

Proof. ($\Rightarrow$) Let $\Gamma(G, S)$ contain an Eulerian path. Then $\Gamma(G, S)$ contains exactly 2 vertices of odd degree. Since $\Gamma(G, S)$ is bipartite, there exists i such that V_i contains the two vertices of odd degree.

Let V_1 contains the two vertices of odd degree. $S_{1,2} = S_{2,1} = 1$ so $\deg\langle(1, 0)\rangle = n$, for n odd, and $\deg\langle(0, 1)\rangle = 2$. Likewise if V_2 contains the two vertices of odd degree, $\deg\langle(1, 0)\rangle = 2$ and $\deg\langle(0, 1)\rangle = m$, for m odd.

($\Leftarrow$) First, assume $m = 2$ and n is odd. $\Gamma(G, S)$ is $K_{2,n}$ and $\deg\langle(1, 0)\rangle = n$ and $\deg\langle(0, 1)\rangle = 2$. Since $|(1, 0)| = 2$, then there are exactly 2 vertices of odd degree.

Now, assume instead that m is odd and $n = 2$. Then $\Gamma(G, S)$ is $K_{m,2}$ and $\deg\langle(1, 0)\rangle = 2$ and $\deg\langle(0, 1)\rangle = m$. Since $|(0, 1)| = 2$, then there are exactly 2 vertices of odd degree. Therefore $\Gamma(G, S)$ contains an Eulerian path. □

4. Dihedral groups

For the dihedral group, D_n, let r be a rotation of $360°/n$ and let f and rf be two different reflections. In [Daniel $\geq$ 2008], it was shown that $\Gamma(G, S) = K_{2,n}$ for $G = D_n$ and $S = \{r, f\}$ and that $\Gamma(G, S)$ is the cycle of length $2n$, C_{2n}, for $G = D_n$ and $S = \{f, rf\}$.

Theorem 4.1. *Let $G = D_n$ and $S = \{f, rf\}$, then $\Gamma(G, S)$ contains an Eulerian circuit.*

Proof. Let $V = V_1 \cup V_2$ such that $V_1 = \{\langle f\rangle, r\langle f\rangle, r^2\langle f\rangle, \ldots, r^{n-1}\langle f\rangle\}$ and

$$V_2 = \{\langle rf\rangle, r\langle rf\rangle, r^2\langle rf\rangle, \ldots, r^{n-1}\langle rf\rangle\}.$$

We have $\langle rf\rangle = \{rf, e\}$ so $\langle rf\rangle$ shares an edge with $\langle f\rangle$ and $r\langle f\rangle$ and $\deg\langle rf\rangle = 2$. By Corollary 2.2, every vertex in V_2 has degree 2. Likewise $\langle f\rangle = \{f, e\}$, $\langle f\rangle$ shares an edge with $\langle rf\rangle$ and $r^{n-1}\langle rf\rangle$ and every vertex in V_1 has degree 2. Since every vertex has degree 2, Theorem 1.1 says that $\Gamma(G, S)$ contains an Eulerian circuit. □

Corollary 4.1. *Let $G = D_n$ and $S = \{f, rf\}$, then $\Gamma(G, S)$ does not contains an Eulerian path.*

Proof. Because the degree of every vertex is 2, $\Gamma(G, S)$ does not contain two vertices of odd degree. □

Theorem 4.2. *Let $G = D_n$ and $S = \{f, rf\}$, then $\Gamma(G, S)$ contains a Hamiltonian circuit.*

Proof. Let $V = V_1 \cup V_2$ such that $V_1 = \{\langle f\rangle, r\langle f\rangle, r^2\langle f\rangle, \ldots, r^{n-1}\langle f\rangle\}$ and

$$V_2 = \{\langle rf\rangle, r\langle rf\rangle, r^2\langle rf\rangle, \ldots, r^{n-1}\langle rf\rangle\}.$$

A Hamiltonian circuit is then given by $\big(\langle f\rangle, \langle rf\rangle\big)$, $\big(\langle rf\rangle, r\langle f\rangle\big)$, $\big(r\langle f\rangle, r\langle rf\rangle\big)$, $\big(r\langle rf\rangle, r^2\langle f\rangle\big), \ldots, \big(r^{n-1}\langle f\rangle, r^{n-1}\langle rf\rangle\big)$, $\big(r^{n-1}\langle rf\rangle, \langle f\rangle\big)$. □

Corollary 4.2. *Let $G = D_n$ and $S = \{f, rf\}$, then $\Gamma(G, S)$ contains a Hamiltonian path.*

Theorem 4.3. *Let $G = D_n$ and $S = \{r, f\}$, then $\Gamma(G, S)$ contains an Eulerian circuit if and only if n is even.*

Proof. ($\Rightarrow$) Let $\Gamma(G, S)$ contain an Eulerian circuit. Then every vertex must be of even degree. Let $V = V_1 \cup V_2$ such that

$$V_1 = \{\langle r\rangle, f\langle r\rangle\} \text{ and } V_2 = \{\langle f\rangle, r\langle f\rangle, r^2\langle f\rangle, \ldots, r^{n-1}\langle f\rangle\}.$$

We have

$$\langle r\rangle \cap r^m\langle f\rangle = \{r^m\} \qquad \text{for all } m = 0, \ldots, n-1,$$

so the edge $(\langle r\rangle, r^m\langle f\rangle)$ is in $\Gamma(G, S)$ for $m = 0, \ldots, n-1$ and $\deg\langle r\rangle = n$. Likewise

$$f\langle r\rangle \cap r^m\langle f\rangle = \{r^m f\} \qquad \text{for all } m = 0, \ldots, n-1,$$

so the edge $(f\langle r\rangle, r^m\langle f\rangle)$ is in $\Gamma(G, S)$ for $m = 0, \ldots, n-1$ and $\deg f\langle r\rangle = n$. Therefore, n must be even.

($\Leftarrow$) Assume that n is even. Then the vertices in V_1 are of even degree from above. Choose a vertex in V_2, $r^m\langle f\rangle$. $r^m\langle f\rangle$ shares an edge with $\langle r\rangle$ and $f\langle r\rangle$. Therefore $\deg r^m\langle f\rangle = 2$ and every vertex in V_2 is of degree 2. Since all the vertices of $\Gamma(G, S)$ are of even degree, $\Gamma(G, S)$ contains an Eulerian circuit. □

Theorem 4.4. *Let $G = D_n$ and $S = \{r, f\}$, then $\Gamma(G, S)$ contains an Eulerian path if and only if n is odd.*

Proof. ($\Rightarrow$) Let $\Gamma(G, S)$ contain an Eulerian path. Then $\Gamma(G, S)$ contains exactly two vertices of odd degree. Let $V = V_1 \cup V_2$. There are n vertices in V_2 and they are of degree 2. There are two vertices in V_1 and they are of degree n. Therefore, n must be odd.

($\Leftarrow$) Assume that n is odd. Then the two vertices in V_1 are of odd degree and the n vertices in V_2 are of degree 2. Therefore $\Gamma(G, S)$ contains an Eulerian path. $\square$

Theorem 4.5. *Let $G = D_n$ and $S = \{r, f\}$, then $\Gamma(G, S)$ contains a Hamiltonian path if and only if $n = 2$ or 3.*

Proof. ($\Rightarrow$) Let $\Gamma(G, S)$ contain a Hamiltonian path. $\Gamma(G, S)$ is $K_{2,n}$ [Daniel $\geq$ 2008]. Then $V = V_1 \cup V_2$ where

$$V_1 = \{\langle r\rangle, f\langle r\rangle\} \text{ and } V_2 = \{\langle f\rangle, r\langle f\rangle, r^2\langle f\rangle, \dots, r^{n-1}\langle f\rangle\}.$$

Since $n \geq 2$, any Hamiltonian path must start with a vertex in V_2.

$$\big(r^{i_1}\langle f\rangle, f^{j_1}\langle r\rangle\big), \big(f^{j_1}\langle r\rangle, r^{i_2}\langle f\rangle\big), \big(r^{i_2}\langle f\rangle, f^{j_2}\langle r\rangle\big), \dots$$

Notice that all the vertices in V_1 have been exhausted. So either the path ends here and $n = 2$ or it ends with the edge $(f^{j_2}\langle r\rangle, r^{i_3}\langle f\rangle)$ and $n = 3$. Therefore $n = 2$ or 3.

($\Leftarrow$) Assume that n is 2 or 3. If $n = 2$ then $V_2 = \{\langle f\rangle, r\langle f\rangle\}$ and

$$\big(\langle r\rangle, \langle f\rangle\big), \big(\langle f\rangle, f\langle r\rangle\big), \big(f\langle r\rangle, r\langle f\rangle\big)$$

is a Hamiltonian path. If $n = 3$ then $V_2 = \{\langle f\rangle, r\langle f\rangle, r^2\langle f\rangle\}$ and

$$\big(\langle f\rangle, \langle r\rangle\big), \big(\langle r\rangle, r\langle f\rangle\big), \big(r\langle f\rangle, f\langle r\rangle\big), \big(f\langle r\rangle, r^2\langle f\rangle\big)$$

is a Hamiltonian path. $\square$

Theorem 4.6. *Let $G = D_n$ and $S = \{r, f\}$, then $\Gamma(G, S)$ contains a Hamiltonian circuit if and only if $n = 2$.*

Proof. ($\Rightarrow$) Let $\Gamma(G, S)$ contain a Hamiltonian circuit. Start with a vertex in V_2 and trace the Hamiltonian circuit

$$(r^{i_1}\langle f\rangle, f^{j_1}\langle r\rangle), (f^{j_1}\langle r\rangle, r^{i_2}\langle f\rangle), (r^{i_2}\langle f\rangle, f^{j_2}\langle r\rangle), \dots$$

Notice that all the vertices in V_1 have been exhausted so the circuit must end with the edge $(f^{j_2}\langle r\rangle, r^{i_1}\langle f\rangle)$ and n must be 2. The proof starting with a vertex in V_1 is similar.

($\Leftarrow$) Assume that n is 2. Then $V_2 = \{\langle f\rangle, r\langle f\rangle\}$ and $\big(\langle r\rangle, \langle f\rangle\big)$, $\big(\langle f\rangle, f\langle r\rangle\big)$, $\big(f\langle r\rangle, r\langle f\rangle\big)$, $\big(r\langle f\rangle, \langle r\rangle\big)$ is a Hamiltonian circuit. $\square$

5. Eulerian circuits and paths

Now we investigate the existence of Eulerian circuits and paths in $\Gamma(G, S)$ for a generic group G.

Theorem 5.1. *Let G be a group with generating set $S = \{s_1, s_2, \ldots, s_n\}$ such that $|\langle s_i\rangle \cap \langle s_j\rangle| = 1$ for all $i \neq j$; then $\Gamma(G, S)$ contains an Eulerian circuit if and only if $|s_i|$ is even for all i, or n is odd.*

Proof. From Lemma 2.2,

$$\deg\langle s_i\rangle = \Big(\sum_{j=1}^{n} |s_i|/S_{i,j}\Big) - |s_i|/S_{i,i}.$$

Also $\deg\langle s_i\rangle = (n-1)|s_i|$, since $S_{i,j} = 1$ for $i \neq j$. Then $\Gamma(G, S)$ contains an Eulerian circuit if and only if $|s_i|$ is even for all i or the number of generators, n, is odd. □

Theorem 5.2. *Let G be a group with generating set $S = \{s_1, s_2, \ldots, s_n\}$ such that $|\langle s_i\rangle \cap \langle s_j\rangle| = m$ for all $i \neq j$, then $\Gamma(G, S)$ contains an Eulerian circuit if and only if $2m|(n-1)(|s_i|)$ for all i.*

Proof. From Lemma 2.2,

$$\deg\langle s_i\rangle = \Big(\sum_{j=1}^{n} \frac{|s_i|}{S_{i,j}}\Big) - |s_i|/S_{i,i}.$$

Also, $\deg\langle s_i\rangle = (n-1)|s_i|/m$, since $S_{i,j} = m$ for $i \neq j$. Since $\Gamma(G, S)$ contains an Eulerian circuit if and only if $\deg\langle s_i\rangle$ is even for all i, then $\Gamma(G, S)$ contains an Eulerian circuit if and only if $2m|(n-1)(|s_i|)$ for all i. □

Theorem 5.3. *Let G be a group with generating set $S = \{s_1, s_2, \ldots, s_n\}$, then $\Gamma(G, S)$ contains an Eulerian circuit if and only if*

$$2|(n-1)(|s_i|)\Big(\sum_{j=1}^{n} \frac{1}{S_{i,j}}\Big), \quad \textit{for all } i.$$

Proof. From Lemma 2.2,

$$\deg\langle s_i\rangle = \Big(\sum_{j=1}^{n} \frac{|s_i|}{S_{i,j}}\Big) - |s_i|/S_{i,i}. \quad S_{i,i} = |s_i|, \quad \deg\langle s_i\rangle = (n-1)(|s_i|)\Big(\sum_{j=1}^{n} \frac{1}{S_{i,j}}\Big).$$

Also, $\Gamma(G, S)$ contains an Eulerian circuit if and only if

$$2|(n-1)(|s_i|)\Big(\sum_{j=1}^{n} \frac{1}{S_{i,j}}\Big), \quad \text{for all } i. \qquad \square$$

Theorem 5.4. *Let G be a group with generating set $S = \{s_1, s_2, \ldots, s_n\}$, if $\Gamma(G, S)$ contains an Eulerian path then one of these cases apply*

(i) *there exists i such that $|V_i| = 2$ with $\deg\langle s_i\rangle$ odd and $\deg\langle s_j\rangle$ even for all $j \neq i$, or*

(ii) *there exists i, j such that $|V_i| = |V_j| = 1$ with $\deg\langle s_i\rangle$ and $\deg\langle s_j\rangle$ odd and $\deg\langle s_k\rangle$ even for all $k \neq i, j$.*

Corollary 5.1. *Let G be a group with generating set $S = \{s_1, s_2, \ldots, s_n\}$, if $\Gamma(G, S)$ contains an Eulerian path then G is of even order or G is cyclic.*

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