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Fibonacci sequences and the space of compact sets

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The Fibonacci numbers appear in many surprising situations. We show that Fibonacci-type sequences arise naturally in the geometry of $\mathscr{H}(\mathbb{R}^2)$, the space of all nonempty compact subsets of $\mathbb{R}^2$ under the Hausdorff metric, as the number of elements at each location between finite sets. The results provide an interesting interplay between number theory, geometry, and topology.

1. Introduction

The famous Fibonacci sequence, named after Leonardo of Pisa "son of Bonaccio", is defined recursively by $F_0 = 0$, $F_1 = 1$, and

$$F_n = F_{n-1} + F_{n-2} \tag{1}$$

for $n \geq 2$ [Sloane 2006, A000045]. The Fibonacci numbers appear in an amazing variety of interesting situations. For example, Fibonacci sequences have been noted to appear in biological settings including the patterns of petals on various flowers such as the cosmo, iris, buttercup, daisy, and the sunflower; the arrangement of pines on a pine cone; the appendages and chambers on many fruits and vegetables such as the lemon, apple, chili, and the artichoke; and spiral patterns in horns and shells [Thompson 1942; Stevens 1979; Douady and Couder 1996; Stewart 1998]. Other Fibonacci-type sequences (also called Gibonacci sequences [Benjamin and Quinn 2003]) can be obtained using the same recurrence relation (1) but with different starting values. For example, the Lucas sequence $\{L_n\}$ can be defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. This sequence is due to Édouard Lucas (1842-1891) (who also named the numbers 1, 1, 2, 3, 5, ... the Fibonacci numbers). There are some useful relations between the Fibonacci and Lucas numbers. For example, a simple induction argument can be used to show $L_n = F_{n-1} + F_{n+1}$ for $n \geq 1$. Consequently, $L_{2n} = F_{2n-1} + F_{2n+1} = F_{2n} + 2F_{2n-1}$.

MSC2000: 00A05.

Keywords: Hausdorff metric, Fibonacci, metric geometry, compact plane sets.

Mathematical applications of Fibonacci-type numbers abound. In the RSA cryptosystem, for example, if an RSA modulus is a Fibonacci number, then the cryptosystem is vulnerable [Dénes and Dénes 2001]. As another example, there are no terms in the Fibonacci or Lucas sequences whose values are equal to the cardinality of a finite nonabelian simple group [Luca 2004]. Fibonacci numbers also have interesting geometric interpretations. For example, the Fibonacci numbers describe the number of ways to tile a $2 \times (n-1)$ checkerboard with 2×1 dominoes [Graham et al. 1994]. If we let Z_n be the point (F_{n-1}, F_n) in the coordinate plane, $X_n = (F_{n-1}, 0)$, $Y_n = (0, F_n)$, and P_n the broken line from the origin O to Z_n consisting of the straight line segments $OZ_1, Z_1Z_2, \cdots, Z_{n-1}Z_n$, then P_n separates the rectangle $OX_nZ_nY_n$ into two regions of equal area when n is odd [Hilton and Pedersen 1994; Page and Sastry 1992]. In this paper, we describe how Fibonacci-type sequences arise in the geometry of $\mathcal{H}(\mathbb{R}^2)$ as the number of elements at each location between finite sets A and B.

2. The Hausdorff metric

The Hausdorff metric h was introduced by Felix Hausdorff in the early twentieth century as a way to measure the distance between compact sets. We will work in $\mathbb{R}^N$ and denote the space of all nonempty compact subsets of $\mathbb{R}^N$ as $\mathcal{H}(\mathbb{R}^N)$. (Note that $\mathcal{H}(\mathbb{R}^N)$ is also called a *hyperspace* — a topological space whose elements are subsets of another topological space.)

A metric is a function that measures distance on a space. We will denote the standard Euclidean distance between x and y in $\mathbb{R}^N$ as $d_E(x, y)$. The Hausdorff metric, defined below, imposes a geometry on the space $\mathcal{H}(\mathbb{R}^N)$ which will be the subject of our study. To distinguish between $\mathbb{R}^N$ and $\mathcal{H}(\mathbb{R}^N)$, we will refer to *points* in $\mathbb{R}^N$ and *elements* in $\mathcal{H}(\mathbb{R}^N)$.

Definition 2.1. Let A and B be elements in $\mathcal{H}(\mathbb{R}^N)$. The Hausdorff distance, $h(A, B)$, between A and B is

$$h(A, B) = \max\{d(A, B), d(B, A)\},$$

where

$$d(A, B) = \max_{x \in A}\{\min_{b \in B}\{d_E(x, b)\}\}.$$

This metric is not very intuitive, so we present three examples to illustrate.

Example 2.1. Let A be the set $\{0, 2\}$ in $\mathbb{R}$ and B the interval $[0, 2]$ in $\mathbb{R}$. Since A is a subset of B, we have $d(A, B) = 0$. However, B is not a subset of A and $d(B, A) = d_E(1, 0) = d_E(1, 2) = 1$. Thus, even though A is a subset of B, we have $h(A, B) = 1$.

Example 2.2. Let A be the unit disk and B the circle of radius 3, both centered at the origin in $\mathbb{R}^2$. Then $d(A, B) = d_E((0,0),(3,0)) = 3$, but $d(B, A) = d_E((3,0),(1,0)) = 2$. So $h(A, B) = d(A, B) = 3$.

Example 2.3. Let A be the segment from $(0,0)$ to $(1,0)$ and B the segment from $(2,-1)$ to $(2,1)$ in $\mathbb{R}^2$. Then $d(A, B) = d_E((0,0),(2,0)) = 2$ and $d(B, A) = d_E((2,1),(1,0)) = \sqrt{2}$. So $h(A, B) = 2$.

Note that these examples show $d(A, B)$ is not symmetric, so we need to use the maximum of $d(A, B)$ and $d(B, A)$ to obtain a metric in Definition 2.1. See [Barnsley 1988] for a proof that h is a metric on $\mathcal{H}(\mathbb{R}^N)$. The corresponding metric space, $(\mathcal{H}(\mathbb{R}^N), h)$, is then itself a complete metric space [Barnsley 1988]. The definition of the metric h makes it rather cumbersome to work with, but there are few good properties that h and d satisfy that help with computations. For example,

- $h(A, B) = d_E(a, b)$ for some $a \in A$ and $b \in B$,
- if $B \subseteq C$, then $d(A, B) \geq d(A, C)$ and $d(C, A) \geq d(B, A)$,
- $h(A \cup B, C \cup D)$ is less than or equal to the maximum of $h(A, C)$ and $h(B, D)$.

These properties are not difficult to verify and are left to the reader.

The geometry the metric h imposes on $\mathcal{H}(\mathbb{R}^N)$ has many interesting properties. For example, in [Bay et al. 2005] the authors show there can be infinitely many different points at a given location on a line in this geometry and that, under certain conditions, lines in this geometry can actually have end elements. In this paper, we will focus our attention on the notion of betweenness in $\mathcal{H}(\mathbb{R}^N)$.

3. Betweenness in $\mathcal{H}(\mathbb{R}^N)$

In this section we define betweenness in $\mathcal{H}(\mathbb{R}^N)$, mimicking the idea of betweenness in $\mathbb{R}^N$ under the Euclidean metric. It is in this context that we will later encounter Fibonacci-type sequences. First we need to understand the dilation of a set.

Definition 3.1. Let $A \in \mathcal{H}(\mathbb{R}^N)$ and let $s > 0$ be a real number. The dilation of A by a ball of radius s (or the s-dilation of A) is the set

$$(A)_s = \{x \in \mathbb{R}^N : d_E(x, a) \leq s \text{ for some } a \in A\}.$$

As an example, let A be the triangle with vertices $(-100, 0)$, $(100, 0)$, and $(0, 150)$. The 30-dilation of A is shown in Figure 1. In essence, the dilation of A by a ball of radius s is just the union of all closed Euclidean s-balls with centers in A. So, for example, the dilation of a single point set $A = \{a\}$ by a ball of radius s is the ball centered at a of radius s. Using dilations, we can alternatively define $h(A, B)$ as the minimum value of s so that the s-dilation of A encloses B and the

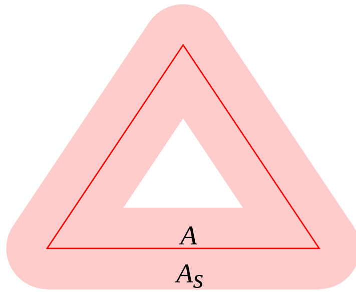


Figure 1. The dilation of a triangle.

s-dilation of B encloses A. An important and useful result about dilations is the following (Theorem 4 from [Braun et al. 2005]).

Theorem 3.1. *Let $A \in \mathcal{H}(\mathbb{R}^N)$ and let $s > 0$ be a real number. Then $(A)_s$ is a compact set that is at distance s from A. Moreover, if $C \in \mathcal{H}(\mathbb{R}^N)$ and $h(A, C) \leq s$, then $C \subseteq (A)_s$.*

Theorem 3.1 tells us that $(A)_s$ is the largest element in $\mathcal{H}(\mathbb{R}^N)$ (in terms of containment) that is a distance s from A. Now we discuss betweenness. In the standard Euclidean geometry, a point x lies between the points a and b if and only if $d_E(a, b) = d_E(a, x) + d_E(x, b)$. We extend this idea to define betweenness in $\mathcal{H}(\mathbb{R}^N)$.

Definition 3.2. Let $A, B \in \mathcal{H}(\mathbb{R}^N)$ with $A \neq B$. The element $C \in \mathcal{H}(\mathbb{R}^N)$ lies between A and B if $h(A, B) = h(A, C) + h(C, B)$.

As an example, let A be the disk centered at $(-100, 0)$ with radius 50 and B the circle centered at (100,0) with radius 25 in $\mathcal{H}(\mathbb{R}^2)$. Then $h(A, B) = d(A, B) = d_E((-150, 0), (75, 0)) = 225$ as shown in Figure 2. The element

$$C_{125} = (A)_{125} \cap (B)_{100}$$

is the grey shaded region in Figure 2. Note that $h(A, C_{125}) = d(C_{125}, A) = d_E((75, 0), (-50, 0)) = 125$ and $h(B, C_{125}) = d(C_{125}, B) = d_E(-25, 0), (75, 0) = 100$ as indicated in Figure 2. So $h(A, B) = h(A, C_{125}) + h(C_{125}, B)$ and C_{125} lies between A and B at the location 125 units from A. Moreover, any element that is s units from A and t units from B must be a subset of $C_s = (A)_s \cap (B)_t$ by Theorem 3.1. So C_{125} is the largest element $C \in \mathcal{H}(\mathbb{R}^N)$ (in the sense of containment) between A and B with $h(A, C) = 125$.

We will use the notation ACB as in [Blumenthal 1953] to indicate that C is between A and B. In Euclidean geometry, the set of points c satisfying $d_E(a, b) = d_E(a, c) + d_E(c, b)$ is the line segment $\overline{ab}$. For this reason, we will denote the set of elements $C \in \mathcal{H}(\mathbb{R}^N)$ that lie between A and B as $S(A, B)$ and call this set the

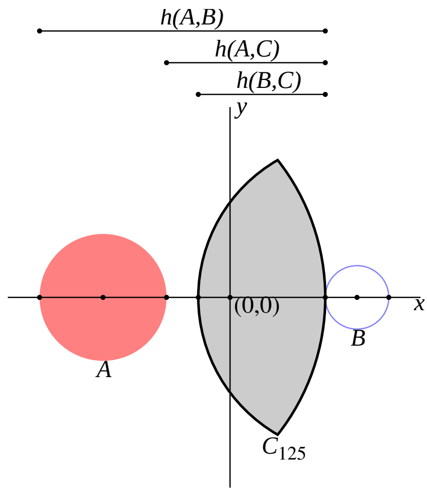


Figure 2. Distinct elements C_s and ∂C_s at the same location between A and B.

Hausdorff segment with end elements A and B. As we will see, there can be many different elements that lie at the same location between elements A and B, so there are many different collections of sets we could call a Hausdorff segment with end elements A and B. In light of Theorem 3.1, we might call $S(A, B)$ the maximal Hausdorff segment with end elements A and B, but we won't need to make that distinction in this paper.

An interesting property of Hausdorff segments is the possibility for the presence of more than one distinct element at a specific location between the end elements. For example, consider the sets A and B in Example 2.1. If we let $C = \left\{\frac{1}{2}, \frac{3}{2}\right\}$ and $C' = \left\{\frac{1}{2}, \frac{5}{2}\right\} \cup \left[\frac{3}{2}, 2\right]$, then a simple computation (left to the reader) shows C and C' satisfy ACB and $AC'B$ with $h(A, C) = h(A, C') = \frac{1}{2}$. So both C and C' lie between A and B at the same location $\frac{1}{2}$ units from A. The following definition formalizes the idea of two elements at the *same location* on a Hausdorff segment.

Definition 3.3. Let $A, B \in \mathcal{H}(\mathbb{R}^N)$ with $A \neq B$. The elements $C, C' \in S(A, B)$ are said to be at the same location between A and B if $h(A, C) = h(A, C') = s$ for some $0 < s < h(A, B)$.

As another example, if A and B are the elements in Figure 2, consider the elements $C_{125} = (A)_{125} \cap (B)_{100}$ and ∂C_{125}, the boundary of C_{125} (outlined in the figure). As Theorem 4.1 will show, these two elements, C_{125} and ∂C_{125}, both lie between A and B with $h(A, C_{125}) = 125 = h(A, \partial C_{125})$. So C_{125} and ∂C_{125} lie at the same location between A and B. In fact, Theorem 4.1 shows that any compact subset C of C_{125} that contains ∂C_{125} also satisfies ACB with $h(A, C) = s$.

4. Finding points between A and B

Let $A \neq B \in \mathcal{H}(\mathbb{R}^N)$. Hausdorff segments fall into two categories: those containing infinitely many elements at each location (except at the locations of either A or B), and those containing a finite number of elements at each location.

Lemma 4.1 [Bogdewicz 2000]. *Let $A, B \in \mathcal{H}(\mathbb{R}^N)$, $r = h(A, B)$, and let $C_s = (A)_s \cap (B)_{r-s}$ for every $s \in [0, r]$. Then $h(A, C_s) = s$ and $h(C_s, B) = r - s$.*

Bay, Lembcke, and Schlicker [Bay et al. 2005] extended Lemma 4.1 to find more elements on Hausdorff segments.

Theorem 4.1. *Let $A, B \in \mathcal{H}(\mathbb{R}^N)$ with $A \neq B$ and let $r = h(A, B)$. Let $s \in \mathbb{R}$ with $0 < s < r$, and let $t = r - s$. If C is a compact subset of $(A)_s \cap (B)_t$ containing $\partial((A)_s \cap (B)_t)$, then C satisfies ACB with $h(A, C) = s$ and $h(B, C) = t$.*

Recall that Theorem 3.1 shows us that an element $C \in \mathcal{H}(\mathbb{R}^N)$ with $h(A, C) = s$ and $h(B, C) = t$ must be a subset of both $(A)_s$ and $(B)_t$ (and so $C_s = (A)_s \cap (B)_t$ is the largest set, in the sense of containment, that is between A and B at a distance s from A). Theorem 4.1 tells us that if $(A)_s \cap (B)_t$ has an infinite interior, then there will be infinitely many elements in $\mathcal{H}(\mathbb{R}^N)$ at each location between A and B. An example of this situation occurs in Figure 2. Alternatively, if $(A)_s \cap (B)_t$ is finite, it has only finitely many subsets and therefore we can have at most a finite number of elements at each location between A and B. In [Blackburn et al. 2008], the authors show if there are finitely many elements at each location between A and B, then every point in A is the same distance from B and every point in B is that same distance from A. We label the distance from a point a to a set B as $d(a, B)$ and define it as follows.

Definition 4.1. Let $a \in \mathbb{R}^N$ and $B \in \mathcal{H}(\mathbb{R}^N)$. The distance from a to B is

$$d(a, B) = \min_{b \in B}\{d_E(a, b)\}.$$

When $d(a, B) = d(b, A)$ for all $a \in A$ and $b \in B$, it is possible for a pair of elements (A, B) to have only a finite number of elements at each location between them. Finite sets satisfying this condition are important enough that we give the following definition.

Definition 4.2. A finite configuration is a pair $[A, B]$ of elements $A, B \in \mathcal{H}(\mathbb{R}^N)$ where A and B are finite sets and $d(a, B) = d(b, A) = h(A, B)$ for all $a \in A$ and $b \in B$.

An easy example of this occurs when A and B are both single point sets. In this case, $(A)_s \cap (B)_t$ will always be a single point set for $0 < s < h(A, B)$; see [Braun et al. 2005]. In [Blackburn et al. 2008], the authors prove the following lemma that tells us about the number of elements at each location between elements A, B in a finite configuration $[A, B]$.

Lemma 4.2. *Let A, B be finite sets in $\mathcal{H}(\mathbb{R}^N)$. If all points $b \in B$ are equidistant from A and $h(A, B) = d(B, A) \geq d(A, B)$, then there is the same finite number of elements at every location between A and B.*

Lemma 4.2 shows that for a finite configuration $[A, B]$, the number of elements at each location between A and B is always the same (except at the end elements - there is only one element a distance 0 from A and only one a distance 0 from B). We denote this number by $\#([A, B])$.

A more interesting example of a configuration $[A, B]$ and the corresponding segment with a finite number of elements at each location is the following. Let $A = \{a_1, a_2, a_3, a_4\}$, where $a_1 = (2, 2)$, $a_2 = (-2, 2)$, $a_3 = (-2, -2)$, and $a_4 = (2, -2)$ are the vertices of a square and $B = \{b_1, b_2, b_3, b_4\}$, where $b_1 = (8, 0)$, $b_2 = (0, 8)$, $b_3 = (-8, 0)$, and $b_4 = (0, -8)$ are the vertices of a square eight times the size of A and rotated 45 degrees in $\mathcal{H}(\mathbb{R}^2)$. If $s, t \in \mathbb{R}^+$ with $r = h(A, B) = s + t$, then each t disk centered at a point in B is tangent to the two s-disks around the points in A closest to it as shown at left in Figure 3. Therefore, $C_s = (A)_s \cap (B)_t = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is the eight-point set that is illustrated at left in Figure 3. In fact, C_s is only one of 47 elements at each location on $S(A, B)$. Interestingly, 47 is the eighth Lucas number, L_8.

Now we find all 47 elements in $\mathcal{H}(\mathbb{R}^N)$ that lie at this location between A and B. To begin, we recall that the largest element between A and B is $C_s =$

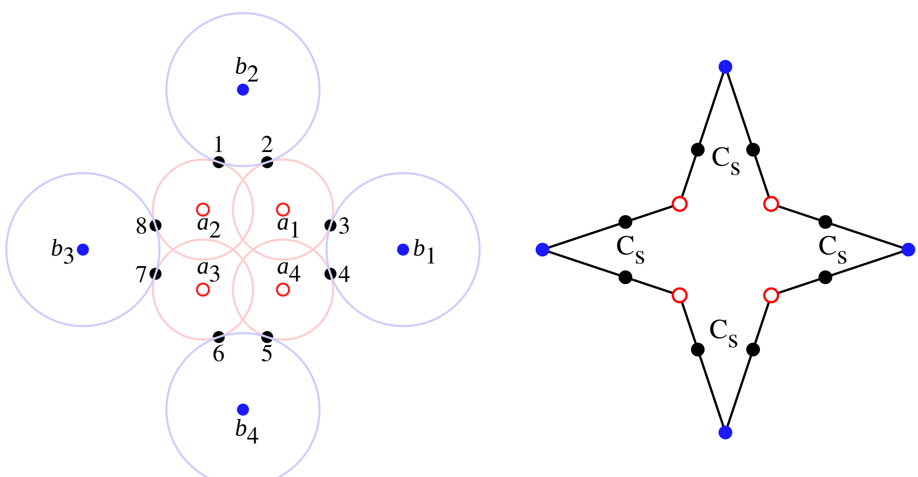


Figure 3. Left: $\#([A, B]) = 47$. Right: The trace diagram.

$\{1,2,3,4,5,6,7,8\}$. The other 46 elements C that lie between A and B at this location are certain subsets of C_s:

(1) $C = C_s - \{c\}$ where $c \in C_s$ (8 elements).

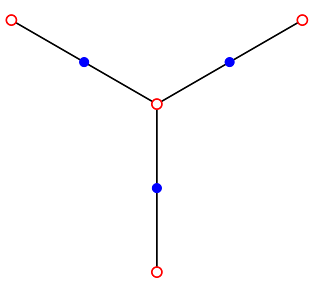
(2) $C = C_s - \{c_1, c_2\}$ where $c_1 \neq c_2 \in C_s$ and c_1 and c_2 are not labeled consecutively (mod 8) (20 elements). (To have $d(A, C) = s$, the boundary of the dilation around each point in A must contain at least one point in C. So, for example, if $2 \notin C$ and $3 \notin C$, then the boundary of $(\{a_1\})_s$ does not contain a point in C. Thus, $d(A, C) > s$ and therefore $h(A, C) > s$.)

(3) $C = C_s - \{c_1, c_2, c_3\}$ where $c_1 \neq c_2 \neq c_3 \in C_s$ and c_1 is not labeled consecutively (mod 8) with c_2 or c_3, and c_2 is not labeled consecutively (mod 8) with c_3 (16 elements).

(4) $C = C_s - \{1, 3, 5, 7\}$ and $C = C_s - \{2, 4, 6, 8\}$.

We leave it to the reader to verify that each of these elements lies at the same location as C_s on $S(A, B)$. Thus we have found all 47 elements between A and B at this location.

We can create a graphical representation of the Hausdorff segment $S(A, B)$ for this configuration by tracing out the locus of points in $C_s = (A)_s \cap (B)_t$ as s varies from 0 to $h(A, B)$ as shown at right in Figure 3 (one specific C_s is shown as the set of eight black points). We call the resulting figure a *trace diagram*. Two other trace diagrams are also shown in Figure 4, the diagram at left presents a trace of a configuration with 7 elements at each location and at right we have the trace of a configuration with 13 elements at each location.



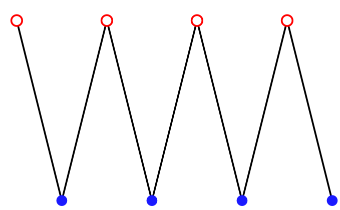


Figure 4. Trace diagrams: 7 elements (left), 13 elements (right).

5. Equivalent configurations

As we have seen, when determining $\#(X)$ for a finite configuration $X = [A, B]$ we only need to know which collection of points in $C_s = (A)_s \cap (B)_t$ we can exclude and still have a set C that satisfies ACB. The actual distance $h(A, B)$ is irrelevant; the only property of the configuration that determines the points in C_s are the points in $a \in A$ and $b \in B$ with $d_E(a, b) = h(A, B)$.

Definition 5.1. Let $[A, B]$ be a finite configuration. Two points $a \in A$ and $b \in B$ are adjacent if $d_E(a, b) = h(A, B)$.

The trace diagrams we have seen provide an obvious connection between finite configurations and graphs - where the points in a configuration $[A, B]$ provide the vertices of a graph and adjacent points in A and B correspond to adjacent points in the graph. Thus the use of the term "adjacent" in Definition 5.1. If two finite configurations X and X' have the same adjacencies, we should expect $\#(X) = \#(X')$. The next definition formalizes this notion of same adjacencies.

Definition 5.2. The finite configuration $[A', B']$ is equivalent to the finite configuration $[A, B]$ if there are bijections $f : A \to A'$ and $g : B \to B'$ such that

(1) if $d_E(a, b) = h(A, B)$ for $a \in A$ and $b \in B$, then $d_E(f(a), g(b)) = h(A', B')$ and

(2) if $d_E(a, b) > h(A, B)$ for $a \in A$ and $b \in B$, then $d_E(f(a), g(b)) > h(A', B')$.

When $[A', B']$ is equivalent to $[A, B]$ we write $[A', B'] \sim [A, B]$.

Informally, two finite configurations X and X' are equivalent if there is a bijection $\phi : X \to X'$ that preserves adjacencies and nonadjacencies. For example, the configuration shown in Figure 5 is equivalent to the configuration in Figure 3. It is easy to show that the relation $\sim$ is an equivalence relation on the set of finite configurations. One important result involving equivalent configurations is that if X and X' are equivalent configurations, then $\#(X) = \#(X')$ [Blackburn et al. 2008].

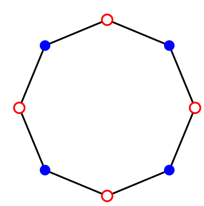


Figure 5. A configuration equivalent to the one in Figure 3.

6. Fibonacci-type sequences in $\mathcal{H}(\mathbb{R}^N)$

It may not be obvious that Fibonacci-type numbers have any connection to the idea of betweenness in the Hausdorff metric geometry. The connection lies in string and polygonal configurations.

6.1. ***String Configurations.*** Perhaps the simplest type of configuration in $\mathcal{H}(\mathbb{R}^N)$ occurs when we uniformly space points on a line segment. Let $x_1, x_2, \ldots, x_n$ be n uniformly spaced points in order on a line, $A = \{x_k : k \text{ odd}\}$, and $B = \{x_k : k \text{ even}\}$. In this case we will call any configuration equivalent to the configuration $S_n =$

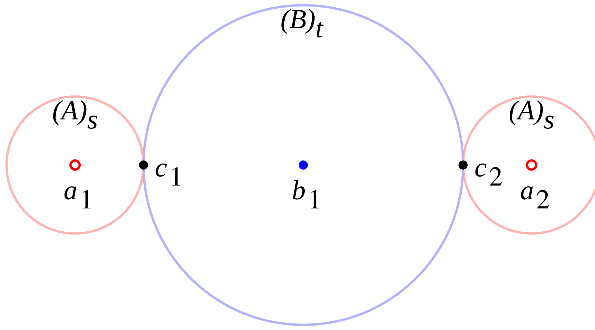


Figure 6. Configuration for S_3.

$[A, B]$ a *string configuration* and $S(A, B)$ a *string segment*. As we will see, the Fibonacci numbers are related to string configurations. We begin by finding $\#(S_n)$ for the first few values of n.

I. The simplest case occurs when $n = 2$ and $|A| = |B| = 1$ (i.e., when A and B are singleton sets), which was considered in [Braun et al. 2005]. In this case we have $\#(S_2) = 1 = F_1$.

II. Now suppose $A = \{a_1, a_2\}$ and $B = \{b_1\}$. Note that $(A)_s \cap (B)_t$ is a two point set $C_s = \{c_1, c_2\}$, with $c_1 < c_2$ as shown in Figure 6. Each element C in question will be a subset of C_s. If $c_1 \notin C$, then $h(A, C) = d_E(a_1, c_2) > s$. A similar argument shows that C contains c_2. Thus, $\#(S_3) = 1 = F_2$.

III. Consider $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Note that $(A)_s \cap (B)_t$ is a three point set $C_s = \{c_1, c_2, c_3\}$, with $d_E(a_1, c_1) = d_E(a_2, c_2) = d_E(a_2, c_3) = s$ as shown in Figure 7. Again, each element C in question will be a subset of C_s. As above, if $c_1 \notin C$, then $h(A, C) \geq d_E(a_1, c_2) > s$. A similar argument shows that C contains c_3. Notice that both $C = C_s$ and $C = \{c_1, c_3\}$ satisfy ACB with $h(A, C) = s$. Therefore, $\#(S_4) = 2 = F_3$.

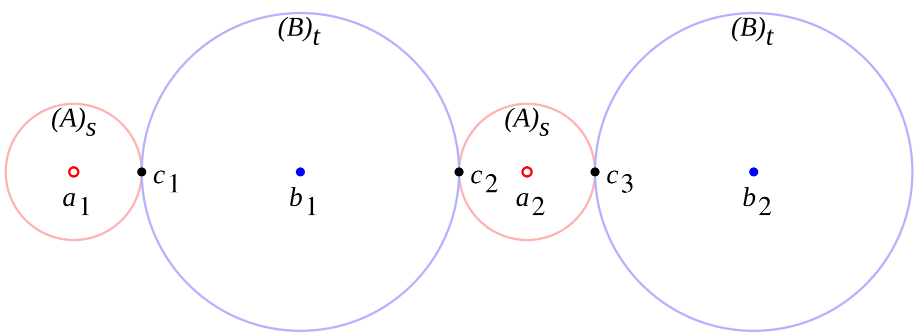


Figure 7. Configuration for S_4.

The next theorem provides the general case.

Theorem 6.1. *For each integer $n \geq 2$, $\#(S_n) = F_{n-1}$.*

Proof. Let $S_n = [A, B]$ and label the points in A in order as $a_1, a_2, \ldots, a_k$ so that $d_E(a_1, a_i) < d_E(a_1, a_j)$ when $i < j$ and the points in B as $b_1, b_2, \ldots, b_m$ so that $d_E(a_1, b_1) = h(A, B)$ and $d_E(b_1, b_i) < d_E(b_1, b_j)$ when $i < j$. Note that $k = m$ or $k = m+1$ and $n = k+m$. Let $r = h(A, B)$, $0 < s < r$ and $t = r-s$. We will determine the number of elements C in $\mathcal{H}(\mathbb{R}^N)$ satisfying ACB with $h(A, C) = s$. Theorem 3.1 tells us that C will be a subset of $C_s = (A)_s \cap (B)_t$. We have already considered the cases with $n \leq 4$. Now we argue the general case with $n \geq 5$. Then $k \geq 3$ and $m \geq 2$. The proof is by induction on n. Assume $n \geq 5$ and that $\#(S_l) = F_{l-1}$ for all $l \leq n-1$. Let $C_s = (A)_s \cap (B)_t$. We will show that there are F_{n-2} subsets C of C_s with $c_2 \in C$ satisfying ACB with $h(A, C) = s$ and F_{n-3} subsets C of C_s with $c_2 \notin C$ satisfying ACB with $h(A, C) = s$. Then $\#(S_n) = F_{n-2} + F_{n-3} = F_{n-1}$ as desired.

Now $C_s = \{c_1, c_2, c_3, \ldots, c_p\}$, with $s = d_E(a_1, c_1) < d_E(a_1, c_2) < \cdots < d_E(a_1, c_p)$ (where $p = n-1$). Note that

$$\{c_{2i-1}\} = (\{a_i\})_s \cap (\{b_i\})_t \qquad \text{and} \qquad \{c_{2i}\} = (\{a_{i+1}\})_s \cap (\{b_i\})_t.$$

Each element C satisfying ACB with $h(A, C) = s$ and $h(B, C) = t$ will be a subset of C_s. If $c_1 \notin C$, then $h(A, C) \geq d(a_1, C) = d_E(a_1, c_2) > s$. So $c_1 \in C$. Similarly, we can show $c_p \in C$. Now we consider the cases $c_2 \notin C$ and $c_2 \in C$.

Case I: $c_2 \notin C$

In order to have C satisfy ACB, we must have $d(a_2, C) = s$. We know $(\{a_2\})_s \cap (B)_t = \{c_2, c_3\}$. Since $c_2 \notin C$, it must be the case that $c_3 \in C$. We now notice that the configuration $[A', B']$ with $A' = \{a_2, a_3, \ldots, a_k\}$ and $B' = \{b_2, b_3, \ldots, b_m\}$ is a string configuration equivalent to S_{n-2} and $C = \{c_1\} \cup C'$ where C' is a set satisfying $A'C'B'$ with $h(A', C') = s$. So there is a one-to-one correspondence (given by $\phi(C) = C - \{c_1, c_2\}$) between sets C satisfying ACB and $h(A, C) = s$ and sets C' satisfying $A'C'B'$ with $h(A', C') = s$. By the induction hypothesis, the number of such sets C is $\#([A', B']) = \#(S_{n-2}) = F_{n-3}$.

Case II: $c_2 \in C$

In this case, let $A^* = \{a_2, a_3, \ldots, a_k\}$ and $C^* = C - \{c_1\}$. Since $c_2 \in C^*$ and C satisfies ABC, it is clear that C^* satisfies A^*C^*B with $h(A^*, C^*) = s$ and $h(C^*, B) = t$. Again, this provides a one-to-one correspondence ϕ between the elements C on the segment joining A and B and the elements C^* on the segment joining A^* and B, where $\phi(C) = C - \{c_1\}$. Now $[A^*, B]$ is equivalent to S_{n-1} and so there are exactly $\#(S_{n-1}) = F_{n-2}$ such elements C^* by our inductive hypothesis. Consequently, there are F_{n-2} elements C.

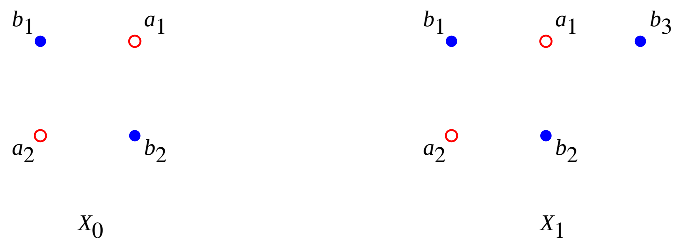


Figure 8. Left: A finite configuration X_0. Right: Adjoining a point to X_0.

Cases I and II show us that there are exactly $F_{n-3} + F_{n-2} = F_{n-1}$ elements at each location on the segment between A and B and $\#(S_n) = F_{n-1}$. □

6.2. ***Adjoining Strings to Configurations.*** We can see how other Fibonacci-type numbers arise in the Hausdorff metric geometry by successively adjoining points to finite configurations. We will illustrate the idea by adjoining a point to the finite configuration $X_0 = [A, B]$, where $A = \{(1, 1), (-1, -1)\}$ and $B = \{(-1, 1), (1, -1)\}$ in $\mathcal{H}(\mathbb{R}^2)$ as shown at left in Figure 8. Note that $d(a, B) = d(b, A) = 2 = h(A, B)$ for all $a \in A$ and $b \in B$. To adjoin a point to X_0 at a_1, we simply add a new point b_3 to B so that b_3 is adjacent to a_1 and $d_E(b_3, a_2) > 2$ as seen at right in Figure 8. This gives us a new finite configuration X_1.

The general construction is described in the next definition.

Definition 6.1. Let $[A, B]$ be a finite configuration. A finite configuration

$$[A, B](a, y)$$

obtained by adjoining a point y to $[A, B]$ at the point $a \in A$ is any configuration equivalent to the configuration $[A, B']$, where $B' = B \cup \{y\}$ and $d_E(y, a) = h(A, B)$ and $d_E(y, a') > h(A, B)$ for all other $a' \in A$.

If we adjoin points successively to a configuration X from a fixed point a, the net result is to adjoin a string configuration of some length to X at the point a. We continue our example from above by adjoining a point to X_1 to obtain finite configurations $X_2 = X_1(b_3, a_3)$, $X_3 = X_2(a_3, b_4)$, and so on as shown in Figure 9. We will show later that $\#(X_0) = 7$. Theorem 6.2 will show $\#(X_1) = 8$, $\#(X_2) = 15 = \#(X_0) + \#(X_1)$, and $\#(X_3) = 23 = \#(X_1) + \#(X_2)$. If we continue extending the configuration by adjoining more and more points, we construct a Fibonacci-type sequence $\{X_n\}$ with $\#(X_n) = \#(X_{n-1}) + \#(X_{n-2})$ for $n \geq 2$. Note that this sequence is also, among other things, the sequence A041100 in [Sloane 2006].

A general argument can be made to determine $\#([A, B](a, y))$, as shown in the next theorem.

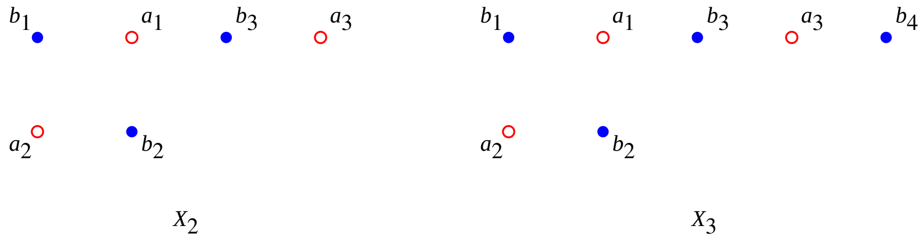


Figure 9. Adjoining points to a finite configuration.

Theorem 6.2. *Let $X = [A, B]$ be a finite configuration. Define X' to be the configuration $[A, B](a, y)$ by adjoining a point y to X at the point $a \in A$, where a is adjacent to k points $b_1, b_2, \ldots, b_k$ in B, each of which is adjacent to at least one point in A other than a. Then $\#(X') = \#(X) + \#(X - \{a\})$.*

Proof. Let X be a finite configuration defined by elements A and B. Let $X' = [A, B](a, y)$ and assume a is adjacent to k points $b_1, b_2, \ldots, b_k$ in X, each of which is adjacent to at least one point in A other than a as shown in Figure 10. Let $s, t > 0$ so that $r = h(A, B) = s + t$ and let $B^* = B \cup \{y\}$. Since $d_E(a, y) = r$, we know $(\{a\})_s \cap (\{y\})_t$ is a single point set. Let $\{c_0\} = (\{a\})_s \cap (\{y\})_t$, and let $c_i = (\{a\})_s \cap (\{b_i\})_t$ for i from 1 to k. Let C^* be an element in $\mathcal{H}(\mathbb{R}^N)$ satisfying AC^*B^* so that C^* is s units from A. Note that C^* must be a subset of $(A)_s \cap (B^*)_t$ and must also contain c_0. Now C^* either contains c_i for some $i \geq 1$ or C^* contains no c_i for $i \geq 1$. We will show that there are $\#(X)$ elements C^* satisfying AC^*B^* and $h(A, C^*) = s$ that contain c_i for some i and $\#(X - \{a\})$ elements C^* satisfying AC^*B^* and $h(A, C^*) = s$ that contain none of the c_i.

Case I: C^* contains c_i for some $i \geq 1$. Let

$$C = C^* - \{c_0\}. \tag{2}$$

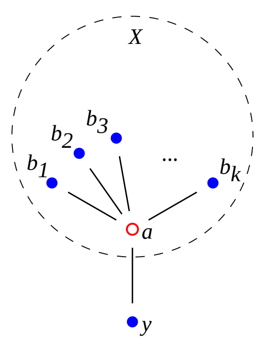


Figure 10. Adjoining a point to a configuration X.

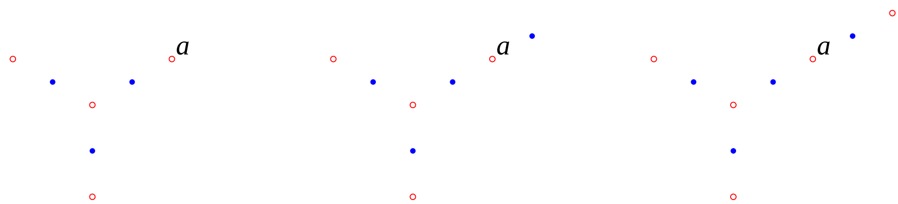


Figure 11. Configurations $\{X_n\}$ with Lucas numbers as $\#(X_n)$.

Now every point in A or B is adjacent to some point in C with a adjacent to c_i. Thus, C satisfies ACB with $h(A, C) = s$. So (2) provides a one-to-one correspondence between sets C^* and sets C. The number of such sets C is $\#(X)$.

Case II: C^* contains no c_i for $i \geq 1$. In this case, $C = C^* - \{c_0\}$ must satisfy $(A - \{a\})CB$ with $h(A - \{a\}, C) = s$. Again, (2) provides a one-to-one correspondence between sets C^* and sets C. The number of such sets C in this case is $\#([A - \{a\}, B]) = \#(X - \{a\})$.

Cases I and II show that

$$\#(X') = \#(X) + \#(X - \{a\}). \qquad \square$$

Theorem 6.2 shows how we can construct Fibonacci-type sequences by adjoining string configurations to finite configurations. Let X_0 be a finite configuration and let a be a point in X_0. If X_n is the configuration obtained by adjoining S_n to X_0 at a, then Theorem 6.2 shows

$$\#(X_n) = \#(X_{n-1}) + \#(X_{n-2})$$

for $n \geq 2$. Thus, we obtain a Fibonacci-type sequence. As another example, let X_0 be the configuration shown at left in Figure 11 and let a be the indicated point. Simple calculations show that $\#(X_0 - \{a\}) = 4$ and $\#(X_0) = 7$. In this case, $\#(X_n) = L_{n-2}$, the $(n-2)^{\text{nd}}$ Lucas number. Lucas numbers also appear in other configurations as we will see in the next section.

For one final example in this section, consider the configurations in Figure 12. In this example, $X_0 = S_6$ as shown at left. So $\#(X_0) = \#(S_6) = F_5 = 5$ and it is easy to see that $\#(X_1) = 5$ where X_1 is the configuration shown in the middle

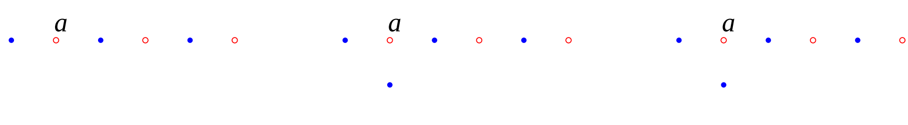


Figure 12. Configurations $\{X_n\}$ creating the sequence $5, 5, 10, 15, 25, \ldots$

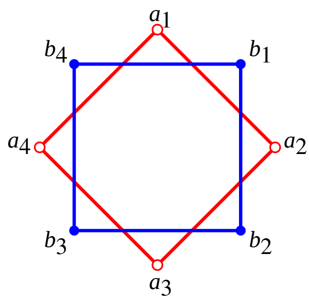


Figure 13. Example of two 4-gons with segment shown.

diagram. Therefore, the sequence generated by adjoining strings to S_6 at the point a is $5, 5, 10, 15, 25, \ldots$. This sequence is listed as A022088 in [Sloane 2006] and is described only as the Fibonacci sequence beginning with 0, 5. Now we have provided a geometric context for this sequence.

A natural question to ask is, given a positive integer k, is it possible to construct a Fibonacci-type sequence of finite configurations $\{X_n\}$ so that $\#(X_m) = k$ for some m. It turns out that this is not possible. Blackburn et al. [2008] proved the surprising result that there is no configuration X (either finite or infinite) with $\#(X) = 19$.

6.3. ***Polygonal Configurations.*** String configurations provide a simple type of finite configuration in $\mathcal{H}(\mathbb{R}^N)$. Another basic family of finite configurations is the collection of polygonal configurations. As an example, let $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3, b_4\}$ each be the set of vertices of a square, as seen in Figure 13. We see that $d(a, B) = d(b, A)$ for all a in A and all b in B. This configuration is equivalent to the one shown in Figure 3. So there are 47 elements that lie at each location on the Hausdorff segment between A and B and all such elements were exhaustively listed earlier.

The general construction of a polygonal configuration is as follows. Let A and B be vertices of regular n-gons with $n \in \mathbb{N}$ in which the n-gons share the same center point and initially are stacked such that the vertices correspond. Then B is rotated $\frac{\pi}{n}$ radians with respect to A about the center point. We call the configuration $P_n = [A, B]$ (or any configuration equivalent to it) a *polygonal configuration* and $S(A, B)$ a *polygonal segment*. As we will see, $\#(P_n) = L_{2n}$ where L_n is the n-th Lucas number.

As examples, we consider the two smallest cases: P_2 and P_3.

I. Figure 14 at left shows the configuration $P_2 = [A, B]$ with $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Let $r = h(A, B)$, $0 < s < r$, and $t = r - s$. Then $C_s = \{c_1, c_2, c_3, c_4\} = (A)_s \cap (B)_t$. To compute $\#(P_2)$ we simply count. Each element C satisfying ACB is a subset of C_s. Only those subsets that do not isolate any points in A or B from points in C are relevant. These sets are
   - $C = C_s$ (1 element),

- $C = C_s - \{c_i\}$ for any i from 1 to 4 (4 elements),
- $C = C_s - \{c_1, c_3\}$ (1 element), and
- $C = C_s - \{c_2, c_4\}$ (1 element),

for a total of 7 elements. Therefore, $\#(P_2) = 7 = F_4 + 2F_3 = L_4$.

II. Figure 14, right, shows the configuration $P_3 = [A, B]$ with $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. Let $r = h(A, B)$, $0 < s < r$, and $t = r - s$. Then $C_s = \{c_1, c_2, c_3, c_4, c_5, c_6\} = (A)_s \cap (B)_t$. To compute $\#(P_3)$ we again count. Each element C satisfying ACB is a subset of C_s. Only those subsets that do not isolate any points in A or B from points in C are relevant. These sets are

- $C = C_s$ (1 element),
- $C = C_s - \{c_i\}$ for any i from 1 to 6 (6 elements),
- $C = C_s - \{c_i, c_j\}$ for $i < j$ as long as $j \neq i + 1$ or $i = 1$ and $j = 6$ (9 elements),
- $C = C_s - \{c_1, c_3, c_5\}$ (1 element), and
- $C = C_s - \{c_2, c_4, c_6\}$ (1 element),

for a total of 18 elements. Therefore, $\#(P_3) = 18 = p_3 = 18 = F_6 + 2F_5 = L_6$.

Recall that earlier we saw $\#(P_4) = 47 = L_8$. The general case is given in the following theorem.

Theorem 6.3. *For* $n \geq 2$,

$$\#(P_n) = F_{2n} + 2 \cdot F_{2n-1} = L_{2n} \tag{3}$$

Proof. Let $n \in \mathbb{N}$ and let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, where a_1 is selected from the vertices of one n-gon and the point a_i is the i-th vertex from a_1 moving counterclockwise on the same n-gon. On the second n-gon, which was rotated $\frac{\pi}{n}$ about the center, b_1 is the first vertex that lies $\frac{\pi}{n}$ degrees counterclockwise from a_1 and the point b_j is the j-th vertex moving counterclockwise from b_1 on the same n-gon. Now let $d(a_i, B) = r = d(b_j, A)$ for each i and let $0 < s < r$ and $t = r - s$. We determine the number of elements C in $\mathcal{H}(\mathbb{R}^N)$ satisfying ACB with $h(A, C) = s$.

We have already verified this theorem for $n = 2$, 3, and 4. Now assume $n > 4$. The element $C_s = (A)_s \cap (B)_t$ is a $2n$ point set $C_s = \{c_1, c_2, c_3, \ldots, c_{2n}\}$, where

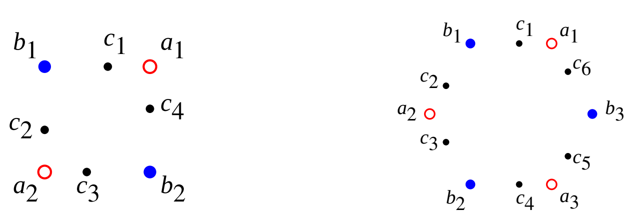


Figure 14. Left: P_2. Right: P_3.

c_{2i-1} is the point of intersection of the s-dilation about a_i and the t-dilation about b_i and c_{2i} is the point of intersection of the t-dilation about b_i and the s-dilation about a_{i+1} for $i = \{1, 2, \ldots, n\}$. Recall that every element C satisfying ACB with $h(A, C) = s$ is a subset of C_s. To find all of the elements C, we argue cases: $c_1 \notin C$, $c_2 \notin C$ and $c_1, c_2 \in C$.

Case I: $c_1 \notin C$

In order to have C satisfy ACB we must have $d(a_1, C) = s$ and $d(b_1, C) = t$. This implies $c_2, c_{2n} \in C$. We now notice the subconfiguration of alternating points from A and B, starting with b_1 and ending with a_1, is equivalent to a string configuration of $2n$ points, which we have shown to have F_{2n-1} elements satisfying ACB by Theorem 6.1.

Case II: $c_2 \notin C$

This case can be argued in a similar manner as the previous case, thus we know that there will be an additional F_{2n-1} elements which satisfy ACB.

Case III: $c_1, c_2 \in C$

We claim this case is similar to having a $2n+1$ string of alternating points from A and B, which by Theorem 6.1 will have F_{2n} elements that satisfy ACB. By assumption we have $C = \{c_1, c_2\} \cup C'$, where C' is a subset of $\{c_3, c_4, \ldots, c_{2n}\}$ such that if $c_i \notin C'$ then c_{i-1} or $c_{i+1} \in C'$ for $i = \{3, 4, 5, \ldots, 2n\}$. We can think of this as a string of alternating points starting with b_1, working in the counterclockwise direction, and ending with a new point b_*, where $b_* = b_1$, such that c_1 lies between a_1 and b_*. Then we see this is exactly the case when there is a string configuration of $2n + 1$ alternating points as desired. Therefore, by Theorem 6.1, we have F_{2n} elements which satisfy ACB.

Cases I, II and III show us that there are exactly $L_{2n} = 2F_{2n-1} + F_{2n}$ elements at each location on $S(A, B)$. □

In hindsight, the fact that string and polygonal configurations produce Fibonacci-type numbers should not be too surprising. Configurations look somewhat like graphs (with string and polygonal configurations related to paths and cycles), and in [Prodinger and Tichy 1982; Staton and Wingard 1995] the authors show that the Fibonacci and Lucas numbers occur as the number of independent vertex sets in paths and cycles.

7. Extensions to $\mathcal{H}(\mathbb{R}^N)$

All of the examples we have presented so far have been in $\mathbb{R}^2$, so it is reasonable to wonder what this paper has to do with $\mathbb{R}^N$. It should be clear that all of the examples and results we have seen extend to $\mathbb{R}^N$, but there is a more interesting connection than that. Dan Schultheis (2006, personal communication) has shown that 57 is the smallest integer for which there is a configuration X that can be constructed in $\mathbb{R}^3$ with $\#(X) = 57$, but no such configuration can be constructed in $\mathbb{R}^2$. The proof is not particularly enlightening, as it is an exhaustive analysis by cases. He identified all configurations in $\mathbb{R}$ and $\mathbb{R}^2$ such that $\#(X) \leq 58$, and showed that there were none for which $\#(X) = 57$. He did identify a finite configuration X_{57} which exists in $\mathbb{R}^3$, however. Due to the difficulty of drawing 3-dimensional configurations, we will describe this configuration $X_{57} = [A, B]$ in terms of its adjacency matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The rows of this matrix correspond to the points in A and the columns to the points in B (so $|A| = 4$ and $|B| = 3$). The entry m_{ij} of this adjacency matrix is 1 if the i-th point in A and the j-th point in B are adjacent and 0 otherwise.

This shows that there are Fibonacci-type sequences in the geometry of $\mathcal{H}(\mathbb{R}^N)$ that do not appear in $\mathcal{H}(\mathbb{R}^2)$. We expect that there are other numbers with this same property as 57, but that is an open question. It is also an open question if there are integers that appear as $\#(X)$ for finite configurations $X \in \mathbb{R}^{n+1}$ that cannot be constructed in $\mathbb{R}^n$ for $n \geq 3$. As a final note, in [Blackburn et al. 2008] the authors show that configurations X exist such that $\#(X) = k$ for all k from 1 to 18, so 19 is the smallest number that cannot be realized as $\#(X)$ for any configuration X. It is unknown for exactly which integers k there exist configurations X so that $\#(X) = k$.

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