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Generating and zeta functions, structure, spectral and analytic properties of the moments of the Minkowski question mark function

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In this paper we are interested in moments of the Minkowski question mark function $?(x)$. It appears that, to some extent, the results are analogous to results obtained for objects associated with Maass wave forms: period functions, L-series, distributions. These objects can be naturally defined for $?(x)$ as well. Various previous investigations of $?(x)$ are mainly motivated from the perspective of metric number theory, Hausdorff dimension, singularity and generalizations. In this work it is shown that analytic and spectral properties of various integral transforms of $?(x)$ do reveal significant information about the question mark function. We prove asymptotic and structural results about the moments, calculate certain integrals which involve $?(x)$, define an associated zeta function, generating functions, Fourier series, and establish intrinsic relations among these objects.

1. Introduction

The aim of this paper is to continue investigations on the moments of the Minkowski question mark function, begun in [Alkauskas $\geq$ 2009]. The function $F(x)$, the *question mark function*, was introduced by Minkowski in 1904 as an example of a monotone and continuous function $F : [0, \infty) \cup \{\infty\} \to [0, 1]$, which maps rationals to dyadic rationals, and quadratic irrationals to nondyadic rationals. For nonnegative real x it is defined by the expression

$$F([a_0, a_1, a_2, a_3, \ldots]) = 1 - 2^{-a_0} + 2^{-(a_0+a_1)} - 2^{-(a_0+a_1+a_2)} + \cdots , \qquad (1)$$

where $x = [a_0, a_1, a_2, a_3, \ldots]$ stands for the representation of x by a (regular) continued fraction [Khinchin 1964]. Figure 1 shows the image of $F(x)$ for $x \in [0, 2]$. More often this function is investigated in the interval $[0, 1]$; in this case we use a standard notation $?(x) = 2F(x)$ for $x \in [0, 1]$. For rational x, the series terminates

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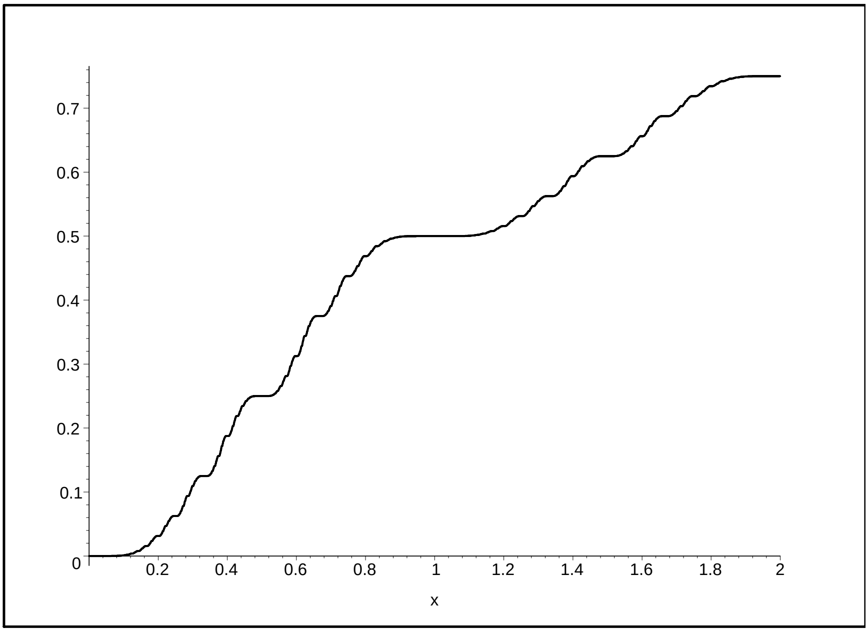


Figure 1. The Minkowski question mark function $F(x)$, $x \in [0, 2]$.

at the last nonzero partial quotient a_n of the continued fraction. This function was investigated by many authors. In particular, Denjoy [1938] showed that $?(x)$ is singular, and that the derivative vanishes almost everywhere. In fact, singularity of $?(x)$ follows from Khinchin's average value theorem on continued fractions [Khinchin 1964, chapter III]. The nature of singularity of $?(x)$ was clarified by Paradís et al. [2001]. In particular, the existence of the derivative $?'(x)$ in $\mathbb{R}$ for fixed x forces it to vanish. Salem [1943] proved (see also [Kinney 1960]) that $?(x)$ satisfies the Lipschitz condition of order $(\log 2)/(2\log\gamma)$, where $\gamma = (1+\sqrt{5})/2$, and this is in fact the best possible exponent for the Lipschitz condition. The Fourier–Stieltjes coefficients of $?(x)$, defined as $\int_0^1 e^{2\pi i n x}\,\mathrm{d}?(x)$, were also investigated in [Salem 1943]. It is worth noting that in Section 8 we will encounter analogous coefficients (see Proposition 3). Meanwhile, [Grabner et al. 2002], out of all papers in the bibliography list, is the closest in spirit to the current article. In order to derive precise error bounds for the so-called Garcia entropy of a certain measure, the authors consider the moments of the monotone, continuous singular function

$$F_2([a_1, a_2, \dots]) = \sum_{n=1}^{\infty}(-1)^{n-1}3^{-(a_1+\cdots+a_n-1)}(q_n+q_{n-1}),$$

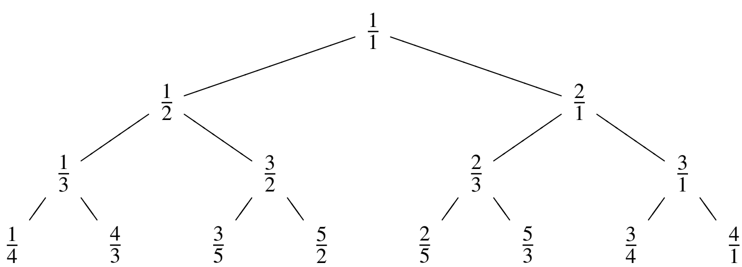
where $q_\star$ stand for a corresponding denominator of the convergent to $[a_1, a_2, \ldots]$. The moments of $F(x)$ itself were never considered before. Lamberger [2006] has shown that $F(x)$ and $F_2(x)$ are the first two members of a family (indexed by natural numbers) of mutually singular measures, derived from the subtractive Euclidean algorithm. From a number-theoretic point of view this generalization is extremely interesting and natural, and it deserves much wider attention.

We confine ourselves to a cursory overview of the properties of $?(x)$, and refer the reader to [Alkauskas $\geq$ 2009] for a short survey on available literature. These works include [Beaver and Garrity 2004; Bonanno et al. 2008; Calkin and Wilf 2000; Denjoy 1938; 1956a; 1956b; 1956c; Dushistova and Moshchevitin $\geq$ 2009; Esposti et al. $\geq$ 2009; Finch 2003; Girgensohn 1996; Grabner et al. 2002; Isola 2002; Kesseböhmer and Stratmann 2007; 2008; Kinney 1960; Lagarias 1991; Lagarias and Tresser 1995; Lamberger 2006; Moshchevitin and Vielhaber $\geq$ 2009; Okamoto and Wunsch 2007; Panti 2008; Paradís et al. 2001;1998; Ramharter 1987; Reese 1989; Reznick $\geq$ 2009; Ryde 1922; 1983; Salem 1943; Tichy and Uitz 1995; Vepštas 2004; Wirsing 2006.]

Recently, Calkin and Wilf [2000] (re)defined a binary tree which is generated by the iteration

$$\frac{a}{b} \quad \mapsto \quad \frac{a}{a+b}, \quad \frac{a+b}{b},$$

starting from the root $1/1$. Elementary considerations show that this tree contains every positive rational number once and only once, each being represented in lowest terms. The first four iterations lead to



This tree is in fact a permutation (inside each generation) of the Stern–Brocot tree. Its limitation to [0, 1] is a permutation of the Farey tree. Thus, the n-th generation consists of 2^{n-1} positive rationals. It is surprising that the iteration discovered by Newman [2003],

$$x_1 = 1, \quad x_{n+1} = 1/(2[x_n] + 1 - x_n),$$

produces exactly rationals of this tree, reading them line-by-line, and thus gives an example of a simple recurrence which produces all positive rationals (here, as usual, $[\star]$ stands for the integer part function). Recently, Dilcher and Stolarsky [2007] produced a natural analogue of this tree, replacing integers r with polynomials

$r \in (\mathbb{Z}/2\mathbb{Z})[x]$. One of the results is that these polynomials also satisfy analogous recurrence (following the proper definition of an integral part of a rational function, which comes from the Euclidean algorithm). It is important to note that the n-th generation of the Calkin–Wilf binary tree consists of exactly those rational numbers, whose elements of the continued fraction sum up to n. This fact can be easily inherited directly from the definition. First, if a rational number a/b is represented as a continued fraction $[a_0, a_1, \ldots, a_r]$, then the map $a/b \to (a+b)/b$ maps a/b to $[a_0+1, a_1 \ldots, a_r]$. Second, the map $a/b \to a/(a+b)$ maps a/b to $[0, a_1+1, \ldots, a_r]$ if $a/b < 1$, and to $[1, a_0, a_1, \ldots, a_r]$ if $a/b > 1$. This is an important fact which makes the investigations of rational numbers according to their position in the Calkin–Wilf tree highly motivated from the perspective of metric number theory and dynamics of continued fractions. The sequence of numerators

$$0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, \ldots$$

is called the Stern diatomic sequence and was introduced in [Stern 1858]. It satisfies the recurrence relations

$$s(0) = 0, \quad s(1) = 1, \quad s(2n) = s(n), \quad s(2n+1) = s(n) + s(n+1).$$

This sequence and the pairs $(s(n), s(n+1))$ have also been investigated by Reznick [$\geq$ 2009]. It is not surprising (bearing in mind the relation to the Farey tree) that the *distribution* of numerators, which are defined via the moments

$$Q_N^{(\tau)} = \sum_{n=2^N+1}^{2^{N+1}} s^{2\tau}(n), \qquad \text{for } \tau > 0,$$

has an interesting application in thermodynamics and spin physics [Contucci and Knauf 1997; Cvitanović et al. 1998].

In [Alkauskas $\geq$ 2009] it was shown that each generation of the Calkin–Wilf tree possesses a distribution function $F_n(x)$, and that $F_n(x)$ converges uniformly to $F(x)$. This is, of course, a well known fact about the Farey tree. The function $F(x)$ as a distribution function is uniquely determined by the functional equation [Alkauskas $\geq$ 2009]

$$2F(x) = \begin{cases} F(x-1)+1 & \text{if} \quad x \geq 1, \\ F(\frac{x}{1-x}) & \text{if} \quad 0 \leq x < 1. \end{cases} \tag{2}$$

This implies $F(x) + F(1/x) = 1$. The mean value of $F(x)$ has been investigated by several authors, and was proved to be $3/2$ [Alkauskas $\geq$ 2009; Reznick $\geq$ 2009; Steuding 2006; Wirsing 2006].

On the other hand, almost all the results mentioned reveal the properties of the Minkowski question mark function as a function itself. Nevertheless, the final goal and motivation of [Alkauskas $\geq$ 2009] and this work is to show that in fact there exist several unique and very interesting analytic objects associated with $F(x)$ which encode a great deal of essential information about it. These objects will be introduced in Section 2.

Lastly, and most importantly, let us point out that, surprisingly, there are striking similarities between the results proved here and in [Alkauskas $\geq$ 2009] with the results on period functions for Maass wave forms in [Lewis and Zagier 2001] . That work is an expanded and clarified exposition of an earlier paper by Lewis [1997]. The concise exposition of these objects, their properties and relations to the Selberg zeta function can be found in [Zagier 2001]. The reader who is not indifferent to the beauty of the Minkowski question mark function is strongly urged to compare results in this work with those in [Lewis and Zagier 2001]. Thus, instead of making quite numerous references to [Lewis and Zagier 2001] at various stages of the work (mainly in Sections 2, 3, 8 and 9), it is more useful to give a table of most important functions encountered there, juxtaposed with analogous objects in this work. Here is the summary (the notations on the right will be explained in Sections 2 and 9).

Maass wave form	$u(z)$	$\Psi(x)$	Periodic function on the real line
Period function	$\psi(z)$	$G(z)$	Dyadic period function
Distribution	$U(x)\,\mathrm{d}x$	$\mathrm{d}F(x)$	Minkowski's "question mark"
L-functions	$L_0(\rho)$, $L_1(\rho)$	$\zeta_{\mathcal{M}}(s)$	Dyadic zeta function
Entire function	$g(w)$	$\mathfrak{m}(t)$	Generating function of moments
Entire function	$\phi(w)$	$M(t)$	Generating function of moments
Spectral parameter	s	1/2; 1	Analogue of a spectral parameter

As a matter of fact, the first entry is the only one where the analogy is not precise. Indeed, the distribution $U(x)$ is the limit value of the Maass wave form $u(x+iy)$ on the real line (as $y \to +0$), in the sense that $u(x+iy) \sim y^{1-s}U(x) + y^{s}U(x)$, whereas $\Psi(x)$ is the same $F(x)$ made periodic. As far as the last entry of the table in concerned, the *analogue* of a spectral parameter, sometimes this role is played by 1, sometimes by $1/2$. This occurs, obviously, because the relation between the Maass forms and $F(x)$ is just an analogy which is not strictly defined.

This work is organized as follows. In Section 2 we give a summary of the previous results obtained in [Alkauskas $\geq$ 2009]. In Section 3 we give a short proof of the three-term functional (13), and prove the existence of certain distributions, which can be thought of as close relatives of $F(x)$. In Section 4 we demonstrate that there are linear relations among moments M_L, and they are presented in an explicit manner. Moreover, we formulate a conjecture, based on the analogy with periods, that these are the only possible relations. In Section 5, the estimate for the moments m_L is proved. As a consequence, $\lim_{L\to\infty}(\log m_L)/(\sqrt{L}) = -2\sqrt{\log 2}$. In Section 6 we prove the exactness of a certain sequence of functional vector spaces and linear maps related to $F(x)$ in an essential way. Section 7 is devoted to the calculation of a number of integrals, giving a rare example of a Stieltjes integral, involving the question mark function, that *can* be calculated. In Section 8 we compute the Fourier expansion of $F(x)$. It is shown that this establishes yet another relation among $\mathfrak{m}(t)$, $G(z)$ and $F(x)$ via Taylor coefficients and special values. In Section 9, the associated Dirichlet series $\zeta_{\mathcal{M}}(s)$ is introduced. In Section 10, some concluding remarks are presented, regarding future research; relations between $F(x)$ and the Calkin–Wilf tree (and the Farey tree as well) to the known objects are established. Note also that we use the word *distribution* to describe a monotone function on $[0,\infty)$ with variation 1, and also for a continuous linear functional on some space of analytic functions. In each case the meaning should be clear from the context.

2. Summary of previous results

This section provides a summary of previous results. For $L \in \mathbb{N}_0$, let

$$\begin{aligned} M_L &= \int_0^\infty x^L \, \mathrm{d}F(x), \\ m_L &= \int_0^\infty \Big(\frac{x}{x+1}\Big)^L \, \mathrm{d}F(x) = 2\int_0^1 x^L \, \mathrm{d}F(x) = \int_0^1 x^L \, \mathrm{d}?(x). \end{aligned} \tag{3}$$

Both sequences are of definite number-theoretic significance because

$$\begin{aligned} M_L &= \lim_{n\to\infty} 2^{1-n} \sum_{a_0+a_1+\cdots+a_s=n} [a_0, a_1, .., a_s]^L, \\ m_L &= \lim_{n\to\infty} 2^{2-n} \sum_{a_1+\cdots+a_s=n} [0, a_1, .., a_s]^L, \end{aligned} \tag{4}$$

(the summation takes place over rational numbers presented as continued fractions; thus, $a_0 \geq 0$, $a_i \geq 1$ for $i \geq 1$, and $a_s \geq 2$. In fact, clarification of their nature was the initial main motivation for our work. We define the exponential generating functions

$$M(t) = \sum_{L=0}^\infty \frac{M_L}{L!} t^L, \qquad \mathfrak{m}(t) = \sum_{L=0}^\infty \frac{m_L}{L!} t^L.$$

Thus,

$$M(t) = \int_0^\infty e^{xt} \, \mathrm{d}F(x), \quad \mathfrak{m}(t) = \int_0^\infty \exp\Big(\frac{xt}{x+1}\Big) \, \mathrm{d}F(x) = 2\int_0^1 e^{xt} \, \mathrm{d}F(x).$$

One easily verifies that $\mathfrak{m}(t)$ is an entire function and that the Taylor series at the origin for $M(t)$ has a radius of convergence $\log 2$. There are natural relations among values M_L and m_L, independent of a specific distribution, like $F(x)$. They encode the relations among functions x^L, $L \in \mathbb{N}_0$, and functions $\bigl(x/(x+1)\bigr)^L$, $L \in \mathbb{N}_0$, given by

$$x^L = \sum_{s \geq L} \binom{s-1}{L-1} \Bigl(\frac{x}{x+1}\Bigr)^s.$$

Therefore,

$$M_L = \sum_{s \geq L} \binom{s-1}{L-1} m_s. \tag{5}$$

On the other hand, the intrinsic information about $F(x)$ is encoded in the relations

$$m_L = M_L - \sum_{s=0}^{L-1} M_s \binom{L}{s}, \quad L \geq 0. \tag{6}$$

Further, we have

$$M(t) = \frac{1}{2-e^t}\mathfrak{m}(t), \quad \mathfrak{m}(t) = e^t \mathfrak{m}(-t). \tag{7}$$

The first relation is equivalent to the system (6), and it encodes all the information about $F(x)$ (provided we take into account the natural relations just mentioned). The second one represents only the symmetry property, given by

$$F(x) + F(1/x) = 1.$$

One of the main results about $\mathfrak{m}(t)$ is that it is uniquely determined by the regularity condition $\mathfrak{m}(-t) \ll e^{-\sqrt{t \log 2}}$, as $t \to \infty$, the boundary condition $\mathfrak{m}(0) = 1$, and the integral equation

$$\mathfrak{m}(-s) = (2e^s - 1) \int_0^\infty \mathfrak{m}'(-t) J_0(2\sqrt{st})\, \mathrm{d}t, \quad s \in \mathbb{R}_+. \tag{8}$$

(Here $J_0(*)$ stands for the Bessel function $J_0(z) = 1/\pi \int_0^\pi \cos(z \sin x)\, \mathrm{d}x$). This equation can be rewritten as a second type Fredholm integral equation [Kolmogorov and Fomin 1989, chapter 9]. In fact, if we denote

$$\psi(s) = \sqrt{2e^s - 1}, \quad \frac{J_1(2\sqrt{st})}{\psi(s)\psi(t)} = K(s,t), \quad \frac{\mathfrak{m}(-s) - 1}{\sqrt{s}\,\psi(s)} = \Upsilon(s),$$

then one has

$$\Upsilon(s) = \ell(s) - \int_0^\infty \Upsilon(t) K(s,t)\, \mathrm{d}t, \tag{9}$$

where

$$\ell(s)=-\frac{1}{\psi(s)}\int_0^\infty \frac{J_1(2\sqrt{st})}{\sqrt{t}(2e^t-1)}\,\mathrm{d}t=\frac{1}{\sqrt{s}\,\psi(s)}\Big(\sum_{n=1}^{\infty}e^{-s/n}2^{-n}-1\Big).$$

Even more importantly, all the results about the exponential generating function can be restated in terms of a generating function of moments. Let

$$G(z)=\sum_{L=1}^{\infty}m_L z^{L-1}\qquad \text{for } |z|\le 1 \tag{10}$$

(the series converge absolutely on the boundary of a unit disc as well, as is clear from Equation (5), or Theorem 3.) Then the integral

$$G(z)=\int_0^\infty \frac{\frac{x}{x+1}}{1-\frac{x}{x+1}z}\,\mathrm{d}F(x)=2\int_0^1 \frac{x}{1-xz}\,\mathrm{d}F(x) \tag{11}$$

extends $G(z)$ to the cut plane $\mathbb{C}\setminus(1,\infty)$. The generating function of moments M_L does not exists due to the factorial growth of M_L, but the generating function can still be defined in the cut plane $\mathbb{C}'=\mathbb{C}\setminus(0,\infty)$ by $\int_0^\infty\big(x/(1-xz)\big)\,\mathrm{d}F(x)$. In fact, this integral equals $G(z+1)$, which is the consequence of an algebraic identity

$$\frac{x}{1-xz}=\frac{\frac{x}{x+1}}{1-\frac{x}{x+1}(z+1)}.$$

The following result was proved in [Alkauskas ≥ 2009].

Theorem 1. *The function $G(z)$, defined initially as a power series, has an analytic continuation to the cut plane $\mathbb{C}\setminus(1,\infty)$ via Equation (11). It satisfies the functional equation*

$$-\frac{1}{1-z}-\frac{1}{(1-z)^2}\,G\Big(\frac{1}{1-z}\Big)+2G(z+1)=G(z), \tag{12}$$

and also the symmetry property

$$G(z+1)=-\frac{1}{z^2}\,G\Big(\frac{1}{z}+1\Big)-\frac{1}{z}.$$

Moreover, $G(z)\to 0$, if $z\to\infty$ and the distance from z to a half line $[0,\infty)$ tends to infinity.

Conversely, the function having these properties is unique.

Note that two functional equations for $G(z)$ can be merged into a single one. It is easy to check that the equation

$$\frac{1}{z}+\frac{1}{z^2}\,G\Big(\frac{1}{z}\Big)+2G(z+1)=G(z) \tag{13}$$

is equivalent to both of them together. In fact, the change $z \mapsto 1/z$ in the last equation gives the symmetry property, and application of it to the term $G(1/z)$ in Equation (13) gives the functional equation in Theorem 1. Nevertheless, it is sometimes convenient to separate Equation (13) into two equations. The reason for this is that in (12) all arguments belong simultaneously to $\mathbb{H}$ (the upper half plane $\Re z > 0$), $\mathbb{R}$, or $\mathbb{H}^-$ (the lower half plane), whereas in (13) they are mixed. This will become crucial later (see the Section 10).

The transition $\mathfrak{m}(t) \to G(z)$ is given by the Laplace transform:

$$1 + zG(z) = \int_0^\infty \mathfrak{m}(zt)e^{-t}\,\mathrm{d}t.$$

The same transform applied to the eigenfunctions of the Fredholm operator (9) yields the following result [Alkauskas $\geq$ 2009].

Theorem 2. *For every eigenvalue λ of the integral operator associated with the kernel $K(s,t)$, there exists at least one holomorphic function $G_\lambda(z)$ (defined for $z \in \mathbb{C} \setminus (1,\infty)$), such that*

$$2G_\lambda(z+1) = G_\lambda(z) + \frac{1}{\lambda z^2}\,G_\lambda\Big(\frac{1}{z}\Big). \tag{14}$$

Moreover, $G_\lambda(z)$ for $\Re z < 0$ satisfies all regularity conditions, imposed by it being an image under the Laplace transform [Lavrentjev and Shabat 1987, page 468*].*

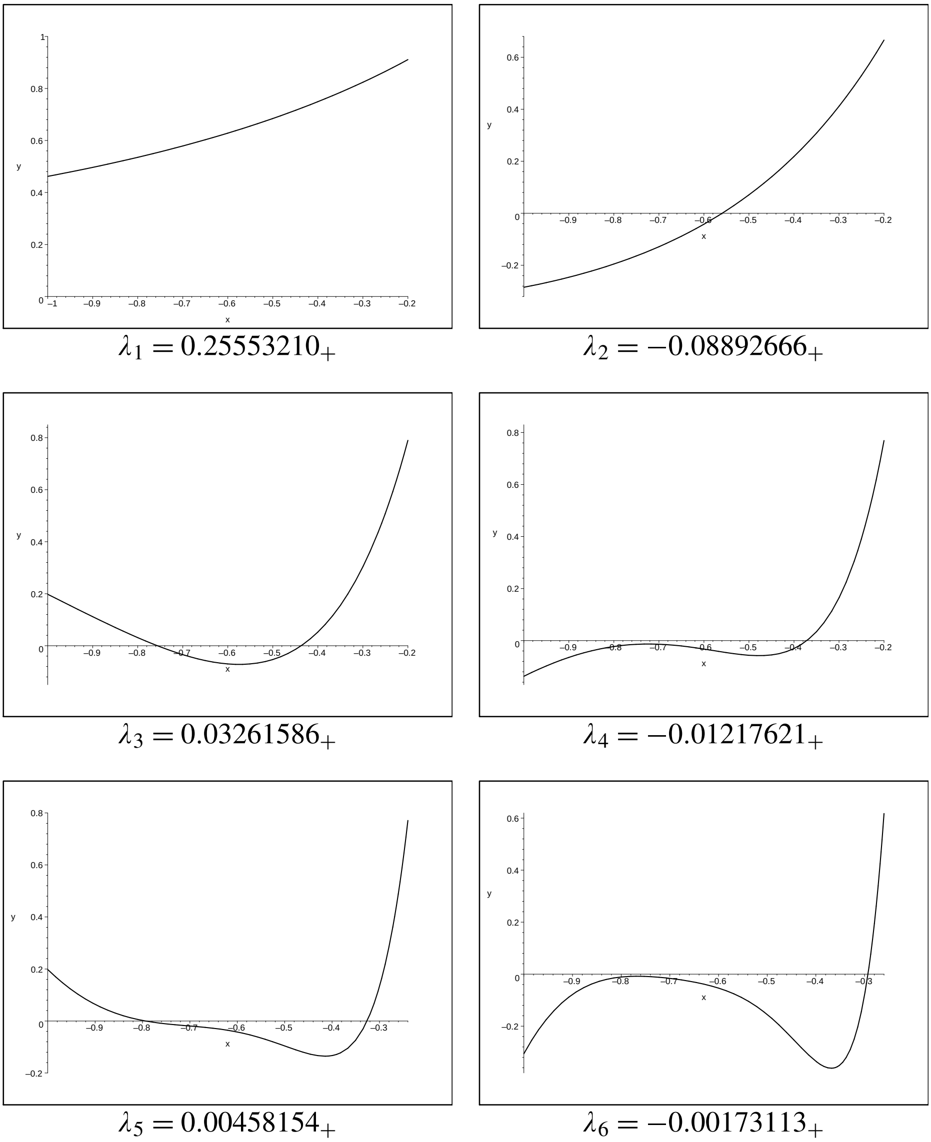
Conversely, for every λ such that there exists a function which satisfies (14) *and these conditions, λ is the eigenvalue of this operator. The set of all possible λ is countable, and $\lambda_n \to 0$, as $n \to \infty$.*

Figure 2 shows the functions $G_\lambda(z)$ (for the first six eigenvalues) for real z in the interval $[-1, -0.2]$. The choice of this interval is motivated by Theorem 2. Note also that the functional equation implies $G_\lambda(0) = \big(1/2 + 1/(2\lambda)\big)G_\lambda(-1)$. Thus, one has $G_\lambda(0)/G_\lambda(-1) \to \infty$, as $\lambda \to 0$. This can also be seen empirically from Figure 2.

Summarizing, there are three objects associated with the Minkowski question mark function.

- The distribution $F(x)$ = functional equations (2) + continuity.
- The dyadic period function $G(z)$ = three-term functional Equation (13) + mild growth condition (as in Theorem 1).
- The exponential generating function $\mathfrak{m}(t)$ = the integral Equation (8) + the boundary value and diminishing condition on the negative real line.

Each of these objects is characterized by the functional equation, and subject to some regularity conditions, is unique, and thus arises exactly from $F(x)$. The objects are described via the “equality" *Function* = *Equation* + *Condition*. This



$\lambda_1 = 0.25553210_+$ $\lambda_2 = -0.08892666_+$

$\lambda_3 = 0.03261586_+$ $\lambda_4 = -0.01217621_+$

$\lambda_5 = 0.00458154_+$ $\lambda_6 = -0.00173113_+$

Figure 2. Eigenfunctions $G_\lambda(z)$ for $z \in [-1, -0.2]$.

means that the object on the left possesses both features; conversely, any object with these properties is necessarily the object on the left.

As expected, here we encounter the phenomenon of *bootstrapping*: in all cases, regularity conditions can be significantly relaxed, and they are sufficient for the uniqueness, which automatically implies stronger regularity conditions. Here we show the rough picture of this phenomenon. In each case, we suppose that the object satisfies the corresponding functional equation. For the details, see [Alkauskas $\geq$ 2009].

(i) $F(x)$ is continuous at one point $\Rightarrow F(x)$ is continuous.

(ii) For every z with $\Re z < 0$, $G(z-x) = O(2^{x/2})$ as

$$x \to \infty \Rightarrow \quad G(z) = O(|z|^{-1}) \text{ as } \mathtt{dist}(z, \mathbb{R}_{+}) \to \infty.$$

(iii) $\mathfrak{m}'(-t) = O(t^{-1})$ as $t \to \infty \Rightarrow |\mathfrak{m}(-t)| \ll e^{-\sqrt{t \log 2}}$ as $t \to \infty$.

Corresponding converse results were proved in [Alkauskas $\geq$ 2009]. As far as $F(x)$ is concerned, this was in fact the starting point of these investigations, since the distribution of rationals in the Calkin–Wilf tree is a certain continuous function satisfying Equation (2); thus, it is exactly $F(x)$. The converse result for $\mathfrak{m}(t)$ follows from Fredholm alternative, since all eigenvalues of the operator (9) are strictly less than 1 in an absolute value. Finally, the converse theorem for $G(z)$ follows from a technical detail in the proof, which is the numerical estimate $0 < (\pi^2/12) - (\log^2 2/2) < 1$; as a matter of fact, it appears that this is essentially the same argument as in the case of $\mathfrak{m}(t)$, since this constant gives the upper bound for the moduli of eigenvalues.

One of the aims of this paper is to clarify the connections among these three objects, and to add the final fourth satellite, associated with $F(x)$. Henceforth, we have the complete list:

- The dyadic zeta function $\zeta_{\mathcal{M}}(s)$ (see Definition 1 below) = the functional equation with symmetry $s \to -s$ (27) + the regularity behavior in vertical strips.

In this case, we do not present a proof of a converse result. Indeed, the converse result for $G(z)$ is strongly motivated by its relation to the Eisenstein series $G_1(z)$ (see [Alkauskas $\geq$ 2009] and Section 10). In the case of $\zeta_{\mathcal{M}}(s)$, this question is of small importance, and we rather concentrate on the direct result and its consequences.

3. Three term functional equation, distributions $F_\lambda(x)$

In this section, we give a proof of (13) different from the one presented in [Alkauskas $\geq$ 2009], since it is considerably shorter. For our purposes, it is convenient to work in slightly greater generality. Suppose that $\lambda \in \mathbb{R}$ has the property that there exists a function $F_\lambda(x)$, $x \in [0, \infty)$, such that

$$\mathrm{d}F_\lambda(x+1) = \frac{1}{2}\,\mathrm{d}F_\lambda(x), \; \mathrm{d}F_\lambda\Big(\frac{1}{x}\Big) = \frac{1}{\lambda}\,\mathrm{d}F_\lambda(x). \tag{15}$$

We omitted the word *continuous* in the description of the function intentionally. For a moment, consider $F_\lambda(x) = F(x)$ with $\lambda = -1$. Then $F_{-1}(x)$ is certainly continuous. The reason for introducing λ will be apparent later. Let

$$G_\lambda(z) = \int_0^\infty \frac{1}{x+1-z}\,\mathrm{d}F_\lambda(x).$$

Since $F(x) + F(1/x) = 1$, we see that for $\lambda = -1$ the above definition of $G_\lambda(z)$ agrees with that of (11). This integral converges to an analytic function in the cut plane $\mathbb{C} \setminus (1, \infty)$. We have

$$\begin{aligned}
2G_\lambda(z+1) &= 2\int_0^1 \frac{1}{x-z}\,\mathrm{d}F_\lambda(x) + 2\int_1^\infty \frac{1}{x-z}\,\mathrm{d}F(x)\\
&= 2\int_0^\infty \frac{1}{\frac{x}{x+1}-z}\,\mathrm{d}F_\lambda\Big(\frac{x}{x+1}\Big) + 2\int_0^\infty \frac{1}{x+1-z}\,\mathrm{d}F_\lambda(x+1)\\
&= \frac{2}{z}\int_0^\infty \Big(\frac{x+1}{x+1-\frac{1}{z}} - 1 + 1\Big)\,\mathrm{d}F_\lambda\Big(\frac{1}{x+1}\Big) + G_\lambda(z)\\
&= \frac{\alpha}{\lambda z} + \frac{1}{\lambda z^2}G_\lambda\Big(\frac{1}{z}\Big) + G_\lambda(z), \text{ where } \alpha = \int_0^\infty \mathrm{d}F_\lambda(x).
\end{aligned}$$

For $\lambda = -1$ and $F_{-1}(x) = F(x)$, this gives Theorem 1. Further, suppose $\lambda \neq -1$. Then

$$\alpha = \int_0^\infty \mathrm{d}F_\lambda(x) = \int_1^\infty \mathrm{d}F_\lambda(x) + \int_0^1 \mathrm{d}F_\lambda(x) = \frac{\alpha}{2} - \frac{\alpha}{2\lambda} \Rightarrow \alpha = 0.$$

Therefore, the last functional equation reads as

$$2G_\lambda(z+1) = \frac{1}{\lambda z^2}\,G_\lambda\Big(\frac{1}{z}\Big) + G_\lambda(z).$$

As a matter of fact, there cannot be any reasonable function $F_\lambda(x)$ which satisfies (15). Nevertheless, the last functional equation is identical to (14). Thus, Theorem 2 gives a description of all such possible λ. This suggests that we can still find certain distributions $F_\lambda(x)$. Further, as it was mentioned, -1 is not an eigenvalue of the operator (9). Due to the minus sign in front of the operator, this is exactly the exceptional eigenvalue, which is essential in the Fredholm alternative. The above proof (rigorous at least in case $\lambda = -1$), surprisingly, proves that the next tautological sentence has a certain point: “-1 is not an eigenvalue because it is -1". Indeed, we obtain a nonhomogeneous part of the three-term functional equation only because $\lambda = -1$, since otherwise $\alpha = 0$ and the equation is homogenic.

Distributions $F_\lambda(x)$ can indeed be strictly defined, at least in the space of functions, which are analytic in the disk $\mathbf{D} = \{z : |z - (1/2)| \leq (1/2)\}$, including its boundary. This space is equipped with a topology of uniform convergence, and a distribution on this space is any continuous linear functional. Denote this space by C^{ω}. Now, since

$$\int_0^1 \frac{x}{1-xz}\,\mathrm{d}F_\lambda(x) = -\frac{\lambda}{2}\,G_\lambda(z) := \sum_{L=1}^{\infty} m_L^{(\lambda)} z^{L-1},$$

define a distribution F_λ on the space C^{ω} by $\langle z^L, F_\lambda\rangle = m_L^{(\lambda)}$, $L \geq 1$, $\langle 1, F_\lambda\rangle = 0$, and for any analytic function $B(z) \in \mathsf{C}^{\omega}$, $B(z) = \sum_{L=0}^{\infty} b_L z^L$, by

$$\langle B, F_\lambda\rangle = \sum_{L=0}^{\infty} b_L \langle z^L, F_\lambda\rangle.$$

First, $\langle *, F_\lambda\rangle$ is certainly a linear functional and is properly defined, since the functional Equation (14) implies that $G_\lambda(z)$ possesses all left derivatives at $z = 1$; as a consequence, the series $\sum_{L=1}^{\infty} L^p |m_L^{(\lambda)}|$ converges for any $p \in \mathbb{N}$ (see Theorem 3 for the estimates on moments m_L). Second, let

$$B_n(z) = \sum_{L=0}^{\infty} b_L^{(n)} z^L, \qquad n \geq 1,$$

converge uniformly to $B(z)$ in the circle $|z| \leq 1$. Thus,

$$\sup_{|z|\leq 1} |B_n(z) - B(z)| = r_n \to 0.$$

Then by Cauchy formula,

$$b_L^{(n)} = \frac{1}{2\pi i}\oint_{|z|=1} \frac{B_n(z)}{z^{L+1}}\,\mathrm{d}z.$$

This obviously implies that $|b_L^{(n)} - b_L| \leq r_n$, $L \geq 0$, and therefore $\langle *, F_\lambda\rangle$ is continuous, and hence it is a distribution. Using the condition $\mathrm{d}F_\lambda(x+1) = (1/2)\,\mathrm{d}F_\lambda(x)$, these distributions can be extended to other spaces. Summarizing, we have shown that the Minkowski question mark function has an infinite sequence of "peers" $F_\lambda(x)$ which are also related to continued fraction expansion, in somewhat similar manner. $F(x)$ is the only "nonhomogeneous" one among them.

4. Linear relations among moments M_L

In this section we clarify the nature of linear relations among the moments M_L. This was mentioned in [Alkauskas ≥ 2009], but not done in explicit form. Note that the second identity of Equation (7) gives linear relations among moments m_L:

$$m_L = \sum_{s=0}^{L} \binom{L}{s} (-1)^s m_s, \qquad L \geq 0.$$

These linear relations can be written in terms of M_L. Despite the fact that these relations form a general phenomena for symmetric distributions, in conjunction with the first identity in (7) they give an essential information about $F(x)$. Let us denote

$$\mathsf{q}(x,t)=(2-e^t)e^{xt}-(2e^t-1)e^{-xt}=\sum_{n=1}^{\infty}\mathsf{Q}_n(x)\frac{t^n}{n!}.$$

We see that $\mathsf{Q}_n(x)$ are polynomials with integer coefficients and they are given by

$$\mathsf{Q}_n(x)=2x^n-(x+1)^n-2(1-x)^n+(-x)^n. \tag{16}$$

The following table gives the first few polynomials.

n	$\mathsf{Q}_n(x)$	n	$\mathsf{Q}_n(x)$
1	$2x-3$	5	$2x^5-15x^4+10x^3-30x^2+5x-3$
2	$2x-3$	6	$6x^5-45x^4+20x^3-45x^2+6x-3$
3	$2x^3-9x^2+3x-3$	7	$2x^7-21x^6+21x^5-105x^4+35x^3-63x^2+7x-3$
4	$4x^3-18x^2+4x-3$	8	$8x^7-84x^6+56x^5-210x^4+56x^3-84x^2+8x-3$

Moreover, the following statement holds.

Proposition 1. *Polynomials $\mathsf{Q}_n(x)$ have the following properties*:

(i) $\mathsf{Q}_{2n}(x)\in L_{\mathbb{Q}}\big(\mathsf{Q}_1(x),\mathsf{Q}_3(x),\dots,\mathsf{Q}_{2n-1}(x)\big),\quad n\geq 1$;

(ii) $\deg\mathsf{Q}_{2n}=2n-1,\quad \deg\mathsf{Q}_{2n-1}=2n-1,\quad n\geq 1$;

(iii) $\widehat{\mathsf{Q}}_{2n}(x):=(\mathsf{Q}_{2n}(x)+3)/x$ *is reciprocal*: $\widehat{\mathsf{Q}}_{2n}(x)=x^{2n-2}\widehat{\mathsf{Q}}_{2n}\big(1/x\big)$;

(iv) $\int_0^\infty\mathsf{Q}_n(x)\,\mathrm{d}F(x)=0$.

Naturally, it is property (iv) which makes these polynomials very important in the study of the Minkowski question mark function. Here $L_{\mathbb{Q}}(*)$ denotes the $\mathbb{Q}$-linear space spanned by the specified polynomials.

Proof. (i) Let $\mathsf{q}_e(x,t)=(1/2)\big(\mathsf{q}(x,t)+\mathsf{q}(x,-t)\big)$, and $\mathsf{q}_o(x,t)=(1/2)\big(\mathsf{q}(x,t)-\mathsf{q}(x,-t)\big)$. Direct calculation shows that, if $e^t=T$, then

$$2\mathsf{q}_e=e^{xt}(3-T-\frac{2}{T})+e^{-xt}(3-\frac{1}{T}-2T),$$
$$2\mathsf{q}_o=e^{xt}(1-T+\frac{2}{T})-e^{-xt}(1-\frac{1}{T}+2T).$$

This yields

$$\sum_{n=1}^{\infty}\mathsf{Q}_{2n}(x)\frac{t^{2n}}{(2n)!}=\mathsf{q}_e(x,t)=\frac{T-1}{T+1}\mathsf{q}_o(x,t)=\frac{e^t-1}{e^t+1}\sum_{n=0}^{\infty}\mathsf{Q}_{2n+1}(x)\frac{t^{2n+1}}{(2n+1)!}.$$

The multiplier on the right, $(e^t-1)/(e^t+1)=\tanh(t/2)$, is independent of x, and this obviously proves the part (i). Also, part (ii) follows easily from Equation (16).

(iii) Since $\widehat{\mathrm{Q}}_{2n}(x)=(1/x)(3x^{2n}-(x+1)^{2n}-2(x-1)^{2n}+3)$, the proof is immediate.

(iv) In fact, Equation (7) gives $(2-e^t)M(t)=(2e^t-1)M(-t)$. For real $|t|<\log 2$, we have $M(t)=\int_0^\infty e^{xt}\,\mathrm{d}F(x)$. This implies

$$\int_0^\infty \mathrm{q}(x,t)\,\mathrm{d}F(x)=\sum_{n=0}^\infty \frac{t^n}{n!}\int_0^\infty \mathrm{Q}_n(x)\,\mathrm{d}F(x)\equiv 0,\quad \text{for } |t|<\log 2,$$

and this completes the proof. □

Consequently, there exist linear relations among the moments M_L. Thus, for example, part (iv) (in case $n=1$ and $n=3$) implies $2M_1-3=0$ and $2M_3-9M_2+3M_1=3$ respectively. The exact values of M_L belong to the class of constants, which can be thought as emerging from arithmetic-geometric chaos. This resembles the situation concerning polynomial relations among various periods. We will not present the definition of a period (it can be found in [Kontsevich and Zagier 2001]). In particular, the authors conjecture (and there is no support for possibility that it can be proved wrong) that *"if a period has two integral representations, then one can pass from one formula to another using only additivity, change of variables, and Newton–Leibniz formula, in which all functions and domains of integration are algebraic with coefficients in $\overline{\mathbb{Q}}$"*. Thus, for example, the conjecture predicts the possibility to prove directly that

$$\iint_{\frac{x^2}{4}+3y^2\leq 1}\mathrm{d}x\,\mathrm{d}y=\int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt[3]{(1-x)(1+x)^2}},$$

without knowing that they both are equal to $\frac{2\pi}{\sqrt{3}}$, and this indeed can be done. Similarly, returning to the topic of this paper, we believe that any finite $\mathbb{Q}$-linear relation among the constants M_L can be proved simply by applying the functional equation of $F(x)$, by means of integration by parts and change of variables. The last proposition supports this claim. In other words, we believe that there cannot be any other miraculous coincidences regarding the values of M_L. More precisely, we formulate

Conjecture 1. Suppose, $r_k\in\mathbb{Q}$, $0\leq k\leq L$, are rational numbers such that

$$\sum_{k=0}^{L} r_k M_k=0.$$

Let $\ell = \left[\frac{L-1}{2}\right]$. Then

$$\sum_{k=0}^{L} r_k x^k \in L_{\mathbb{Q}}\big(\mathsf{Q}_1(x), \mathsf{Q}_3(x), \dots, \mathsf{Q}_{2\ell+1}(x)\big).$$

This conjecture, if true, should be difficult to prove. It would imply, for example, that M_L for $L \geq 2$ are irrational. On the other hand, this conjecture seems to be much more natural and approachable, compared to similar conjectures regarding arithmetic nature of constants emerging from geometric chaos, e.g. spectral values s for Maass wave forms (say, for $\mathsf{PSL}_2(\mathbb{Z})$), or those coming from arithmetic chaos, like nontrivial zeros of Riemann's $\zeta(s)$. We cannot give any other evidence, save the last proposition, to support this conjecture.

5. Estimate for the moments m_L

This section deals with an asymptotic estimate for the moments m_L. This result was not obtained before, and in view of the expression in Equation (4), it is of certain number-theoretic interest. This result should be compared with the asymptotic formula for M_L, obtained in [Alkauskas $\geq$ 2009]:

$$M_L \sim \frac{\mathfrak{m}(\log 2)}{2\log 2}\Big(\frac{1}{\log 2}\Big)^L L!, \text{ for } L \in \mathbb{N}. \tag{17}$$

A priori, as it is implied by the fact that the radius of convergence of $G(z)$ at $z = 0$ is 1, and by Equation (5), for every $\varepsilon > 0$ and $p > 1$, one has

$$\frac{1}{L^p} \gg m_L \gg (1-\varepsilon)^L,$$

as $L \to \infty$. More precisely, we have

Theorem 3. *Let* $\mathsf{C} = e^{-2\sqrt{\log 2}} = 0.18917\dots$. *Then the following estimate holds, as* $L \to \infty$:

$$\mathsf{C}^{\sqrt{L}} \ll m_L \ll L^{1/4}\mathsf{C}^{\sqrt{L}}.$$

Both implied constants are absolute.

Proof. Fix $J \in \mathbb{N}$, and choose an increasing sequence of positive real numbers $\mu_j < 1$, $1 \leq j \leq J$. We will soon specify μ_j in such a way that $\mu_j \to 0$ uniformly as $L \to \infty$. An estimate for m_L is obtained via the defining integral (recall that

$F(x)+F(1/x)=1$):

$$m_L = \Big(\int_0^{\mu_1} + \sum_{j=1}^{J-1}\int_{\mu_j}^{\mu_{j+1}} + \int_{\mu_J}^{\infty}\Big)\Big(\frac{1}{x+1}\Big)^L \,\mathrm{d}F(x)$$
$$< F(\mu_1) + \sum_{j=1}^{J-1}\Big(\frac{1}{\mu_j+1}\Big)^L F(\mu_{j+1}) + \Big(\frac{1}{\mu_J+1}\Big)^L.$$

Indeed, in the first integral, the integrand is bounded by 1. In the middle integrals, we choose the largest value of integrand, and change bounds of integration to $[0, \mu_{j+1}]$. The same is done with the last integral, with bounds changed to $[0,\infty)$. Now choose $\mu_j = 1/(c_j\sqrt{L})$ for some decreasing sequence of constants c_j. The functional equation for $F(x)$ implies

$$F(x+n) = 1 - 2^{-n} + 2^{-n}F(x), \quad x \geq 0.$$

Thus, $1-F(x) \asymp 2^{-x}$, as $x \to \infty$ (the implied constants being min and max of the function $\Psi(x)$; see Figure 3 and Section 8). Using the identity $F(x)+F(1/x)=1$,

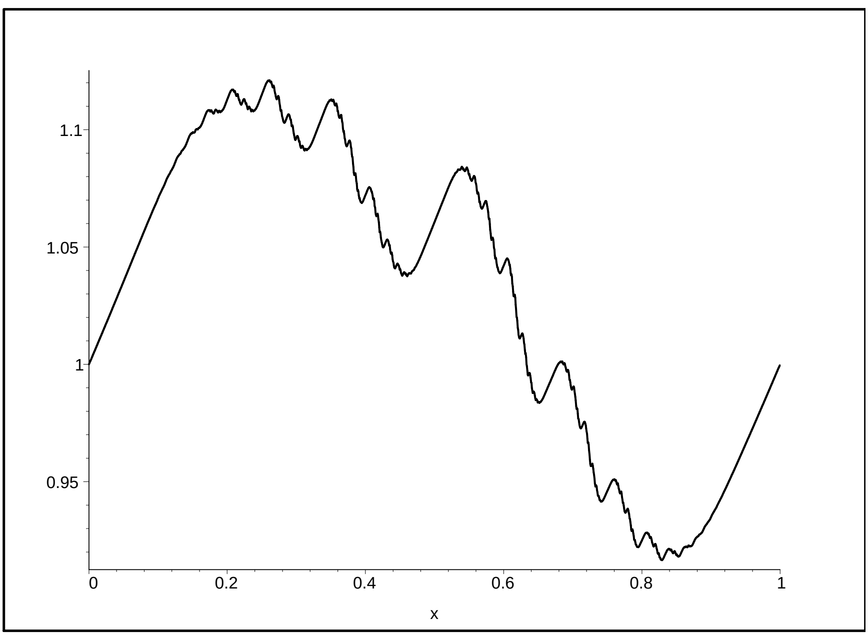


Figure 3. Periodic function $\Psi(x)$.

we therefore obtain

$$m_L \ll 2^{-c_1\sqrt{L}} + \sum_{j=1}^{J-1} \Big(\frac{1}{\frac{1}{c_j\sqrt{L}}+1} \Big)^L 2^{-c_{j+1}\sqrt{L}} + \Big(\frac{1}{\frac{1}{c_J\sqrt{L}}+1} \Big)^L$$
$$\ll e^{-\sqrt{L}c_1 \log 2} + \sum_{j=1}^{J-1} e^{-\sqrt{L}(\frac{1}{c_j}+c_{j+1}\log 2)} + e^{-\sqrt{L}\frac{1}{c_J}}. \tag{18}$$

Here we need an elementary lemma.

Lemma 1. *For given* $J \in \mathbb{N}$, *there exists a unique sequence of positive real numbers* $c_1^*, \dots, c_J^*$, *such that*

$$c_1^* = \frac{1}{c_1^*} + c_2^* = \frac{1}{c_2^*} + c_3^* = \dots = \frac{1}{c_{J-1}^*} + c_J^* = \frac{1}{c_J^*}.$$

Moreover, *this sequence* $\{c_j^*, 1 \le j \le J\}$ *is decreasing*, *and it is given by*

$$c_j^* = \frac{\sin\frac{(j+1)\pi}{J+2}}{\sin\frac{j\pi}{J+2}}, \quad j = 1, 2, \dots, J \Rightarrow c_1^* = 2\cos\frac{\pi}{J+2}.$$

Proof. Indeed, we see that $c_1^* = x$ determines the sequence c_j^* uniquely. First, $c_2 = x - 1/x = (x^2-1)/x$. Let $F_1(x) = x$, $F_2(x) = x^2 - 1$. Suppose we have shown that $c_j = F_j(x)/F_{j-1}(x)$ for certain sequences of polynomials. Then from the above equations one obtains

$$c_{j+1} = c_1 - \frac{F_{j-1}(x)}{F_j(x)} = \frac{xF_j(x) - F_{j-1}(x)}{F_j(x)}.$$

Thus, using induction we see that $c_j = F_j(x)/F_{j-1}(x)$, where polynomials $F_j(x)$ are given by the initial values $F_0(x) = 1$, $F_1(x) = x$ and then for $j \ge 1$ recurrently by $F_{j+1}(x) = xF_j(x) - F_{j-1}(x)$. This shows that $F_j(2x) = U_j(x)$, where $U(x)$ stand for the classical Chebyshev U-polynomials, given by

$$U_j(\cos\theta) = \frac{\sin(j+1)\theta}{\sin\theta}.$$

The last equation $c_1^* = 1/c_J^*$ implies $F_{J+1}(x) = 0$. Thus, $U_{J+1}(x/2) = 0$, and all possible values of c_1^* are given by $c_1^* = x = 2\cos\big((k\pi)/(J+2)\big)$, $k = 1, 2, \dots, J+1$. Thus,

$$c_j^* = \frac{F_j(x)}{F_{j-1}(x)} = \frac{U_j(x/2)}{U_{j-1}(x/2)} = \frac{\sin\frac{k(j+1)\pi}{J+2}}{\sin\frac{kj\pi}{J+2}}.$$

Since our concern is only positive solutions, this gives the last statement of lemma. Finally, monotonicity is easily verifiable. Indeed, system of equations imply $c_2^* < c_1^*$, and then we act by induction. □

Thus, $c_1^* > 2 - b/J^2$ for some constant $b > 0$. Returning to the proof of the Theorem 3, for given J, let c_j^* be the sequence in the lemma, and let $c_i^* = c_i\sqrt{\log 2}$. Thus,

$$c_1 \log 2 = \frac{1}{c_1} + c_2 \log 2 = \frac{1}{c_2} + c_3 \log 2 = \cdots = \frac{1}{c_{J-1}} + c_J \log 2 = \frac{1}{c_J}.$$

Choosing exactly this sequence for the estimate (18), and using the bound for c_1^*, we get:

$$m_L \ll (J+1)e^{-\sqrt{L}c_1 \log 2} < (J+1)\mathsf{C}^{\sqrt{L}} e^{\frac{b\sqrt{\log 2}}{J^2}\sqrt{L}}.$$

Finally, the choice $J = [L^{1/4}]$ establishes the upper bound.

The lower estimate is immediate. In fact, let $\mu = 1/(c\sqrt{L})$. Then

$$m_L > \int_0^{\mu} \Big(\frac{1}{x+1}\Big)^L \, \mathrm{d}F(x) > \Big(\frac{1}{\mu+1}\Big)^L F(\mu) \gg 2^{-c\sqrt{L}} \cdot e^{-\sqrt{L}\frac{1}{c}}$$

The choice $c = \log^{-1/2} 2$ gives the desired bound. □

The constants in Theorem 3 can also be calculated without great effort, but this is astray of the main topic of the paper.

It should be noted that, if we start directly from the last definition (3) of m_L, then in the course of the proof of Theorem 3, we use both equalities $F(x)+F(1/x)=1$ and $2F(x/(x+1)) = F(x)$. Since these two determine $F(x)$ uniquely, generally speaking, our estimate for m_L is characteristic only to $F(x)$. A direct inspection of the proof also reveals that the true asymptotic "action" in the second definition (3) of m_L takes place in the neighborhood of 1. This, obviously, is a general fact for probabilistic distributions with proper support on the interval [0, 1]. Additionally, calculations show that the sequence $m_L/(L^{1/4}\mathsf{C}^{\sqrt{L}})$ is monotonically decreasing. This is indeed the case, and there exists $\lim_{L\to\infty}(m_L/L^{1/4}\mathsf{C}^{\sqrt{L}})$ [Alkauskas 2008].

As a final remark, we note that the result of Theorem 3 must be considered in conjunction with the linear relations $m_L = \sum_{s=0}^{L}\binom{L}{s}(-1)^s m_s$, $L \geq 0$, and the natural inequalities, imposed by the fact that m_L is a sequence of moments of probabilistic distribution with support on the interval [0, 1]. We thus have Hausdorff conditions, which state that for all nonnegative integers m and n, one has

$$2\int_0^1 x^n(1-x)^m \, \mathrm{d}F(x) = \sum_{i=0}^{m}\binom{m}{i}(-1)^i m_{i+n} > 0.$$

This is, of course, the consequence of monotonicity of $F(x)$.

6. Exact sequence

In this section we prove the exactness of a sequence of continuous linear maps, intricately related to the Minkowski question mark function $F(x)$. Let $\mathsf{C}[0,1]$ denote the space of continuous, complex-valued functions on the interval $[0,1]$ with supremum norm. For $f\in\mathsf{C}[0,1]$, one has the identity

$$\int_0^1 f(x)\,\mathrm{d}F(x)=\sum_{n=1}^{\infty}\int_0^1 f\Big(\frac{1}{x+n}\Big)2^{-n}\ \mathrm{d}F(x), \tag{19}$$

Indeed, using the functional Equation (2), we have

$$\int_0^1 f(x)\,\mathrm{d}F(x)=\int_1^{\infty} f\Big(\frac{1}{x}\Big)\ \mathrm{d}F(x)=\sum_{n=1}^{\infty}\int_0^1 f\Big(\frac{1}{x+n}\Big)\ \mathrm{d}F(x+n),$$

which is exactly (19). Let C^{ω} denote, as before, the space of analytic functions in the disk $\mathbf{D}=|z-1/2|\leq 1/2$, including its boundary. We equip this space with the topology of uniform convergence (as a matter of fact, we have a wider choice of spaces; this one is chosen as an important example). Now, consider a continuous functional on C^{ω} given by $T(f)=\int_0^1 f(x)\,\mathrm{d}F(x)$, and a continuous noncompact linear operator $[\mathscr{L}f](x)=f(x)-\sum_{n=1}^{\infty}f\big(\frac{1}{x+n}\big)2^{-n}$. Finally, let i stand for the natural inclusion $i:\mathbb{C}\to\mathsf{C}^{\omega}$.

Theorem 4. *The following sequence of maps is exact*:

$$0\to\mathbb{C}\xrightarrow{i}\underset{(*)}{\mathsf{C}^{\omega}}\xrightarrow{\mathscr{L}}\underset{(**)}{\mathsf{C}^{\omega}}\xrightarrow{T}\mathbb{C}\to 0. \tag{20}$$

Proof. First, i is obviously a monomorphism. Let $f\in\mathrm{Ker}(\mathscr{L})$. This means that

$$f(x)=\sum_{n=1}^{\infty}f\big(1/(x+n)\big)2^{-n}.$$

Let $x_0\in[0,1]$ be such that $|f(x_0)|=\sup_{x\in[0,1]}|f(x)|$. Since $\sum_{n=1}^{\infty}2^{-n}=1$, this yields

$$f(1/(x_0+n))=f(x_0)\quad\text{for}\quad n\in\mathbb{N}.$$

By induction,

$$f([0,n_1,n_2,\ldots,n_I+x_0])=f(x_0)$$

for all $I\in\mathbb{N}$, and all $n_i\in\mathbb{N}$, $1\leq i\leq I$; here $[\star]$ stands for the (regular) continued fraction. Since this set is everywhere dense in $[0,1]$ and f is continuous, this forces $f(x)\equiv\mathrm{const}$ for $x\in[0,1]$. Due to the analytic continuation, this is valid for $x\in\mathbf{D}$ as well. Hence, we have the exactness at the term $(*)$.

Next, T is obviously an epimorphism. Further, the identity in Equation (19) implies that $\mathrm{Im}(\mathscr{L}) \subset \mathrm{Ker}(T)$. The task is to show that indeed we have an equality. At this stage, we need the following lemma. Denote

$$[\mathscr{S} f](x) = \sum_{n=1}^{\infty} f\big(1/(x+n)\big)2^{-n}.$$

Lemma 2. *Let $f \in \mathsf{C}^{\omega}$. Then $[\mathscr{S}^n f](x) = 2T(f) + O(\gamma^{-2n})$ for $x \in \mathbf{D}$; here $T(f)$ stands for the constant function, $\gamma = \big((1+\sqrt{5})/2\big)$ is the golden section, and the bound implied by O is uniform for $x \in \mathbf{D}$.*

Proof. In fact, the lemma is true for any function with continuous derivative. Let $x \in \mathbf{D}$. We have

$$[\mathscr{S}^r f](x) = \sum_{n_1,n_2,\ldots,n_r=1}^{\infty} 2^{-(n_1+n_2+\cdots+n_r)} f([0, n_1, n_2, \ldots, n_r + x]).$$

The direct inspection of this expression and Equation (1) shows that this is exactly twice the Riemann sum for the integral $\int_0^1 f(x)\,\mathrm{d}F(x)$, corresponding to the division of unit interval into intervals with endpoints being $[0, n_1, n_2, \ldots, n_r]$, $n_i \in \mathbb{N}$. From the basic properties of Möbius transformations we inherit that the set $[0, n_1, n_2, \ldots, n_r + x]$ for $x \in \mathbf{D}$ is a circle $\mathbf{D}_r$ whose diagonal is one of these intervals, say I_r. For fixed r, the largest of these intervals has endpoints F_{r-1}/F_r and F_r/F_{r+1}, where F_r stands for the usual Fibonacci sequence. Thus, its length is $1/(F_r F_{r+1}) \sim c\gamma^{-2r}$. Let $x_0, x_1 \in \mathbf{D}_r$, and $\sup_{x \in \mathbf{D}} |f'(x)| = A$. We have

$$\sup_{x_0,x_1 \in \mathbf{D}_r} |f(x_0) - f(x_1)| \le Ac\gamma^{-2r}.$$

Thus, the Riemann sum deviates from the Riemann integral no more than

$$|[\mathscr{S}^r f](x) - 2T(f)| \le Ac\gamma^{-2r} \sum_{n_1,n_2,\ldots,n_r=1}^{\infty} 2^{-(n_1+n_2+\cdots+n_r)} = Ac\gamma^{-2r}.$$

This proves the Lemma. □

Thus, let $f \in \mathrm{Ker}(T)$. All we need is to show that the equation $f = g - \mathscr{S}g$ has a solution $g \in \mathsf{C}^{\omega}$. Indeed, let $g = f + \sum_{n=1}^{\infty} \mathscr{S}^n f$. By the above lemma, $\|\mathscr{S}^n f\| = O(\gamma^{-2n})$. Thus, the series defining g converges uniformly and hence g is an analytic function. Finally, $g - \mathscr{S}g = f$; this shows that $\mathrm{Ker}(T) \subset \mathrm{Im}(\mathscr{L})$ and the exactness at the term $(**)$ is proved. □

These results imply that, for example, $\mathsf{Q} := \mathrm{Im}(\mathscr{L})$ is a linear subspace of C^{ω} of codimension 1. Further research proves that $\mathscr{L}|_{\mathsf{Q}}$ is an isomorphism.

The eigenfunctions of $\mathcal{S}$ acting on the space C^{ω} are given by

$$G^{\star}(-x)=\int_{0}^{-x} G_{\lambda}(z)\,\mathrm{d}z+\int_{-1}^{0} G_{\lambda}(z)\,\mathrm{d}z$$

(see Equations (22) and (23) in Section 7). Thus, the problem of convergence of $\mathcal{S}^n f$ is completely analogous to the problem of convergence for the iterates of the Gauss–Kuzmin–Wirsing operator. Let us remind that if $f\in\mathsf{C}[0,1]$, it is given by

$$[\mathbf{W}f](x)=\sum_{n=1}^{\infty}\frac{1}{(x+n)^2}f\Big(\frac{1}{x+n}\Big).$$

Dominant eigenvalue 1 correspond to an eigenfunction $1/(1+x)$. As it was proved by Kuzmin, provided that $f(x)$ has a continuous derivative, there exists $c>0$, such that

$$[\mathbf{W}^n f](x)=\frac{A}{1+x}+O(e^{-c\sqrt{n}}),\ \text{as } n\to\infty;\qquad A=\frac{1}{\log 2}\int_{0}^{1} f(x)\,\mathrm{d}x.$$

The proof can be found in [Khinchin 1964]. Note that this was already conjectured by Gauss, but he did not give the proof nor for the main neither for the error term. For the most important case, when $f(x)=1$, Lévy established the error term of the form $O(C^n)$ for $C=0.7$. Finally, Wirsing [1973/74] gave the exact result in terms of eigenfunctions of $\mathbf{W}$, establishing the error term of the form $c^n\Psi(x)+O(x(1-x)\mu^n)$, where $c=-0.303663\ldots$ is the subdominant eigenvalue (the Gauss–Kuzmin–Wirsing constant), $\Psi(x)$ is a corresponding eigenfunction, and $\mu<|c|$. Returning to our case, we have completely analogous situation: operator $\mathbf{W}$ is replaced by $\mathcal{S}$, and the measure $\mathrm{d}x$ is replaced by $\mathrm{d}F(x)$. The leading eigenvalue 1 corresponds to the constant function. However strange, Wirsing did not notice that eigenvalues of $\mathbf{W}$ are in fact eigenvalues of certain Hilbert–Schmidt operator. This was later clarified by Babenko [1978]. Recently, the Gauss–Kuzmin–Lévy theorem was generalized by Manin and Marcolli [2002]. The paper is very rich in ideas and results; in particular, it sheds a new light on the theorem just mentioned.

Concerning spaces for which Theorem 4 holds, we can investigate the space $\mathsf{C}[0,1]$ as well. However, if $f\in\mathsf{C}[0,1]$ and $f\in\mathrm{Ker}(T)$, the significant difficulty arises in proving uniform convergence of the series $\sum_{n=0}^{\infty}\mathcal{S}^n f$. Moreover, operator $\mathcal{S}$, acting on the space $\mathsf{C}[0,1]$, has additional point spectra apart from λ. Indeed, let

$$P_n(y)=y^n+\sum_{i=0}^{n-1}a_i y^i$$

be a polynomial of degree n which satisfies yet another variation of three-term functional equation

$$2P_n(1-2y) - P_n(1-y) = \frac{1}{\delta_n} P_n(y)$$

for certain δ_n. The comparison of leading terms shows that

$$\delta_n = \frac{(-1)^n}{2^{n+1}-1},$$

and that indeed for this δ_n there exists a unique polynomial, since each coefficient a_j can be uniquely determined with the knowledge of coefficients a_i for $i > j$. Thus,

$$P_1(y) = y - \frac{1}{4}, \qquad P_2(y) = y^2 - \frac{3}{5}y + \frac{1}{15},$$
$$P_3(y) = y^3 - \frac{21}{22}y^2 + \frac{3}{11}y - \frac{7}{352}, \quad P_4(y) = y^4 - \frac{30}{23}y^3 + \frac{14}{23}y^2 - \frac{45}{391}y + \frac{37}{5865}.$$

The equation for $P_n(y)$ implies that (after a substitution $y \mapsto 2^{-\ell} y$ and division by 2^{ℓ})

$$2^{1-\ell} P_n(1-2^{1-\ell} y) - 2^{-\ell} P_n(1-2^{-\ell} y) = \delta_n^{-1} 2^{-\ell} P_n(2^{-\ell} y).$$

Now, sum this over $\ell \in \mathbb{N}$, and finally substitute $y \mapsto 1-y$. This gives

$$\delta_n P_n(y) = \sum_{\ell=1}^{\infty} \frac{1}{2^{\ell}} P_n\Big(\frac{1-y}{2^{\ell}}\Big). \tag{21}$$

Then we have:

Proposition 2. *The function $P_n(F(x))$ is the eigenfunction of $\mathcal{G}$, acting on the space* $\mathsf{C}[0,1]$, *and eigenvalue* $(-1)^n/(2^{n+1}-1)$ *belongs to the point spectra of* $\mathcal{G}$.

Proof. Indeed,

$$[\mathcal{G}(P_n \circ F)](x) = \sum_{\ell=1}^{\infty} \frac{1}{2^{\ell}} P_n \circ F\Big(\frac{1}{x+\ell}\Big) \overset{(2)}{=} \sum_{\ell=1}^{\infty} \frac{1}{2^{\ell}} P_n\Big(1 - F(x+\ell)\Big) \overset{(2)}{=}$$
$$= \sum_{\ell=1}^{\infty} \frac{1}{2^{\ell}} P_n\Big(2^{-\ell} - 2^{-\ell} F(x)\Big) \overset{(21)}{=} \delta_n P_n(F(x)). \qquad \square$$

Thus, the operator $\mathcal{G}$ behaves differently in spaces $\mathsf{C}[0,1]$ and C^{ω}. We postpone the analysis of this operator in various spaces for the future.

7. Integrals involving $F(x)$

In this section we calculate certain integrals. Only rarely it is possible to express an integral involving $F(x)$ in closed form. In fact, all results we possess come from the identity $M_1 = 3/2$, and any iteration of identities similar to (19). The following theorem adds identities of quite a different sort.

Theorem 5. *Let $G_\lambda(z)$ be any function that satisfies the hypotheses of Theorem 2. Then*

(i) $\frac{\lambda}{\lambda+1}\int_0^1 G_\lambda(-x)\,\mathrm{d}x = \int_0^1 G_\lambda(-x)F(x)\,\mathrm{d}x$;

(ii) $-\int_0^1 \log x\,\mathrm{d}F(x) = 2\int_0^1 \log(1+x)\,\mathrm{d}F(x) = \int_0^1 G(-x)\,\mathrm{d}x$;

(iii) $\int_0^1 G(-x)(1+x^2)\,\mathrm{d}F(x) = \frac{1}{4}$;

(iv) $\int_0^1 G_\lambda(-x)\Big(1-\frac{x^2}{\lambda}\Big)\,\mathrm{d}F(x) = 0.$

Proof. We first prove identity (i). By (14), for every integer $n \geq 1$, we have

$$2G_\lambda(-z-n+1) - G_\lambda(-z-n) = \frac{1}{\lambda(z+n)^2}\,G_\lambda\Big(-\frac{1}{z+n}\Big).$$

Divide this by 2^n and sum over $n \geq 1$. By Theorem 1, the sum on the left is absolutely convergent. Thus,

$$G_\lambda(-z) = \sum_{n=1}^{\infty}\frac{1}{\lambda 2^n(z+n)^2}\,G_\lambda\Big(-\frac{1}{z+n}\Big).$$

Let $G^\star_\lambda(x) = \int_0^x G_\lambda(z)\,\mathrm{d}z$. In terms of $G^\star_\lambda(x)$, the last identity reads as

$$-G^\star_\lambda(-x) = \sum_{n=1}^{\infty}\frac{1}{\lambda 2^n}G^\star_\lambda\Big(-\frac{1}{x+n}\Big) - \sum_{n=1}^{\infty}\frac{1}{\lambda 2^n}G^\star_\lambda\Big(-\frac{1}{n}\Big). \tag{22}$$

In particular, setting $x=1$, one obtains

$$\sum_{n=1}^{\infty}\frac{1}{\lambda 2^n}G^\star_\lambda\Big(-\frac{1}{n}\Big) = (\frac{1}{\lambda}-1)G^\star_\lambda(-1). \tag{23}$$

Now we are able to calculate the following integral (we use integration by parts in Stieltjes integral twice).

$$\begin{aligned}
&\int_0^1 G_\lambda(-x)F(x)\,\mathrm{d}x \\
&\quad= -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}G^\star_\lambda(-x)F(x)\,\mathrm{d}x = -\frac{1}{2}G^\star_\lambda(-1) + \int_0^1 G^\star_\lambda(-x)\,\mathrm{d}F(x) \\
&\quad\overset{(22)}{=} -\frac{1}{2}G^\star_\lambda(-1) + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\lambda 2^n}G^\star_\lambda\Big(-\frac{1}{n}\Big) - \frac{1}{\lambda}\sum_{n=1}^{\infty}\int_0^1 G^\star_\lambda\Big(-\frac{1}{x+n}\Big)2^{-n}\,\mathrm{d}F(x) \\
&\quad\overset{(19),(23)}{=} -\frac{1}{2}G^\star(-1) + \frac{1}{2}(\frac{1}{\lambda}-1)G^\star_\lambda(-1) - \frac{1}{\lambda}\int_0^1 G^\star_\lambda(-x)\,\mathrm{d}F(x) \\
&\quad= -G^\star(-1) - \frac{1}{\lambda}\int_0^1 G_\lambda(-x)F(x)\,\mathrm{d}x.
\end{aligned}$$

Thus, the same integral is on the both sides, and this gives

$$\int_0^1 G_\lambda(-x)F(x)\,\mathrm{d}x = -\frac{\lambda}{\lambda+1}\,G^\star_\lambda(-1).$$

This establishes the statement (i).

Now we proceed with the second identity. Integral (11) and the Fubini theorem imply

$$\int_0^1 G(-z)\,\mathrm{d}z = 2\int_0^1\int_0^1 \frac{x}{1+xz}\,\mathrm{d}z\,\mathrm{d}F(x) = 2\int_0^1 \log(1+x)\,\mathrm{d}F(x).$$

Lastly, we apply (19) twice to obtain the needed equality. Indeed,

$$\begin{aligned}
I &= \int_0^1 \log(1+x)\,\mathrm{d}F(x) \overset{(19)}{=} \sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1 \log\Big(1+\frac{1}{x+n}\Big)\,\mathrm{d}F(x) \\
&= \sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1 \log(x+n)\,\mathrm{d}F(x) - I \overset{(19)}{=} -\int_0^1 \log x\,\mathrm{d}F(x) - I.
\end{aligned}$$

This finishes the proof of (ii).

In proving (iii), we can be more concise, since the pattern of the proof goes along the same line. One has

$$G(-z) = -\sum_{n=1}^{\infty}\frac{1}{2^n(z+n)^2}\,G\Big(-\frac{1}{z+n}\Big) + \sum_{n=1}^{\infty}\frac{1}{2^n(z+n)}.$$

Thus,

$$\int_0^1 G(-x)\,\mathrm{d}F(x) = -\sum_{n=1}^{\infty}\int_0^1 \frac{1}{2^n(x+n)^2}\,G\Big(-\frac{1}{x+n}\Big)\,\mathrm{d}F(x) + \sum_{n=1}^{\infty}\int_0^1 \frac{1}{2^n(x+n)}\,\mathrm{d}F(x) \overset{(19)}{=} -\int_0^1 x^2\,G(-x)\,\mathrm{d}F(x) + \int_0^1 x\,\mathrm{d}F(x).$$

Since $\int_0^1 x\,\mathrm{d}F(x) = \frac{m_1}{2} = \frac{1}{4}$, this finishes the proof of (iii). Part (iv) is completely analogous. □

Part (iii), unfortunately, gives no new information about the sequence m_L. Indeed, the identity can be rewritten as

$$\sum_{L=1}^{\infty} m_L(-1)^{L-1}(m_{L-1}+m_{L+1}) = 1/2,$$

which, after regrouping, turns into the identity $m_0 m_1 = 1/2$.

Concerning part (iv), and taking into account Theorem 4, one could expect that in fact $\mathrm{Ker}(T)$ is equal to the closure of vector space spanned by functions $G_\lambda(-x)(1-x^2/\lambda)$. If this is the case, then these functions, along with $G(z)(1+x^2)$, produce a Schauder basis for C^ω. Thus, if

$$x^L = \sum_{\lambda} a_L^{(\lambda)}\,G_\lambda(-x)(1-x^2/\lambda),$$

then $a_L^{(-1)} = 2m_L$. We hope to return to this point in the future.

Concerning (i), note that the values of both integrals depend on the normalization of G_λ, since it is an eigenfunction. Replacing $G_\lambda(z)$ by $cG_\lambda(z)$ for some $c \in \mathbb{R}$, we deduce that the left integral is equal to 1 or 0. Then (i) states that $\int_0^1 F(x)G_\lambda(-x)\,\mathrm{d}x = \lambda/(\lambda+1)$ or 0 (apparently, it is never equal to 0). The presence of $\lambda+1$ in the denominator should come as no surprise, minding that λ is the eigenvalue of the Hilbert–Schimdt operator. The Fredholm alternative gives us a way of solving the integral equation in terms of eigenfunctions. Since $|\lambda| \leq \lambda_1 = 0.25553210\ldots < 1$, the integral equation is *a posteriori* solvable, and $\lambda+1$ appears in the denominators. Curiously, it is possible to approach this identity numerically. One of the motivations is to check its validity, since the result heavily depends on the validity of almost all the preceding results in [Alkauskas $\geq$ 2009]. The left integral causes no problems, since Taylor coefficients of $G_\lambda(z)$ can be obtained at high precision as an eigenvector of a finite matrix, which is the truncation of an infinite one. On the other hand, the right integral can be evaluated with less precision, since it involves $F(x)$, and thus requires more time and space consuming continued fractions algorithm. Nevertheless, the author of this paper has checked it with a completely satisfactory outcome, confirming the validity.

Just as interestingly, results (i) and (iv) can be considered a reflection of a “pair-correlation" between eigenvalues λ and eigenvalue -1 (see Section 3 for some remarks on this topic). Moreover, minding properties of the distributions $F_{\mu}(x)$ (here μ simply means another eigenvalue), the following formal result can be obtained. Given the conditions enforced on F_{μ} by (15), the identity (19) is replaced by (for $f \in \mathrm{C}^{\omega}$)

$$\int_0^1 f(x)\,\mathrm{d}F_{\mu}(x) = -\frac{1}{\mu}\sum_{n=1}^{\infty}\int_0^1 f\Big(\frac{1}{x+n}\Big)2^{-n}\;\mathrm{d}F_{\mu}(x).$$

Then our trick works smoothly again, and this yields an identity

$$\int_0^1 G_{\lambda}(-x)(\lambda+\mu x^2)\,\mathrm{d}F_{\mu}(x) = 0.$$

This fact is an interesting example of pair-correlation between eigenvalues of the Hilbert–Schmidt operator in (9). Using a definition of the distribution F_{μ}, the last identity is equivalent to

$$\sum_{L=1}^{\infty}(-1)^{L}\big(m_L^{(\mu)}m_{L+1}^{(\lambda)}\lambda - m_L^{(\lambda)}m_{L+1}^{(\mu)}\mu\big) = 0,$$

and thus is a property of “orthogonality" of $G_{\lambda}(z)$. This expression is symmetric regarding μ and λ. As could be expected, it is void in case $\mu=\lambda$. As a matter of fact, the proof of the above identity is fallacious, since the definition of distributions F_{λ} does not imply properties (15) (these simply have no meaning). Nevertheless, numerical calculations suggest that the last identity truly holds. We also hope to return to this topic in the future.

8. Fourier series

The Minkowski question mark function $F(x)$, originally defined for $x \geq 0$ by Equation (1), can be extended naturally to $\mathbb{R}$ simply by the functional equation

$$F(x+1) = 1/2 + 1/2F(x).$$

Such an extension is still given by the expression (1), with the difference that a_0 can be negative integer. Naturally, the second functional equation is not preserved for negative x. Thus, we have

$$2^{x+1}(F(x+1)-1) = 2^{x}(F(x)-1) \text{ for } x \in \mathbb{R}.$$

So, $2^x(F(x)-1)$ is a periodic function, which we will denote by $-\Psi(x)$. Figure 3 gives the graph of $\Psi(x)$ for $x\in[0,1]$. Thus,

$$F(x)=-2^{-x}\Psi(x)+1.$$

Since $F(x)$ is singular, the same is true for $\Psi(x)$: it is differentiable almost everywhere, and for these regular points one has $\Psi'(x)=\log 2\cdot\Psi(x)$. As a periodic function, it has an associated Fourier series expansion

$$\Psi(x)\sim\sum_{n=-\infty}^{\infty}c_n e^{2\pi i n x}.$$

Since $F(x)$ is real function, this gives $c_{-n}=\overline{c_n}$, $n\in\mathbb{Z}$. Let for $n\geq 1$, $c_n=a_n+ib_n$, and $a_0=c_0/2$, $b_0=0$. Here we list initial numerical values for

$$c_n^{\star}=c_n(2\log 2-4\pi i n)$$

(see Proposition 3 for the reason of this normalization).

$c_0^{\star}=1.428159$, $c_3^{\star}=+0.128533-0.026840i$, $c_6^{\star}=-0.262601+0.004128i$,
$c_1^{\star}=-0.521907+0.148754i$, $c_4^{\star}=-0.140524-0.021886i$, $c_7^{\star}=+0.198742-0.013703i$,
$c_2^{\star}=-0.334910-0.017869i$, $c_5^{\star}=+0.285790+0.003744i$, $c_8^{\star}=-0.008479+0.024012i$.

It is important to note that we do not pose the question about the convergence of this Fourier series. For instance, Salem [1943] and Reese [1989] give examples of singular monotone increasing functions $f(x)$, whose Fourier–Stieltjes coefficients $\int_0^1 e^{2\pi i n x}\,\mathrm{d}f(x)$ do not vanish, as $n\to\infty$. Salem [1943] even investigated $f(x)=?(x)$. In our case, the convergence problem is far from clear. Nevertheless, in all cases we substitute $-2^{-x}\Psi(x)$ instead of $(F(x)-1)$ under an integral. Let, for example, $W(x)$ be a continuous function of at most polynomial growth, as $x\to\infty$, and let $\Psi_N(x)=\sum_{n=-N}^{N}c_n e^{2\pi i n x}$. Then

$$\Bigl|\int_0^{\infty}W(x)\Bigl((F(x)-1)+2^{-x}\Psi_N(x)\Bigr)\,\mathrm{d}x\Bigr|\ll\sum_{r=0}^{\infty}|W(r)|2^{-r}\cdot\int_0^1|2^x(F(x)-1)+\Psi_N(x)|\,\mathrm{d}x.$$

Since $2^x(F(x)-1)\in\mathscr{L}_2[0,1]$, the last integral tends to 0, as $N\to\infty$. As it was said, this makes the change of $(F(x)-1)$ into $-2^{-x}\Psi(x)$ under integral legitimate, and this also justifies term-by-term integration. Henceforth, we will omit a step of changing $\Psi(x)$ into $\Psi_N(x)$, and taking a limit $N\to\infty$.

A general formula for the Fourier coefficients is given by

Proposition 3. *Fourier coefficients* c_n *are related to special values of exponential generating function* $\mathfrak{m}(t)$ *through the equality*

$$c_n = \frac{\mathfrak{m}(\log 2 - 2\pi i n)}{2\log 2 - 4\pi i n}, \text{ and } c_n = O(n^{-1}).$$

Proof. We have (note that $F(1) = 1/2$):

$$c_n = -\int_0^1 2^x (F(x)-1)e^{-2\pi i n x}\,\mathrm{d}x = -\frac{1}{\log 2 - 2\pi i n}\int_0^1 (F(x)-1)\,\mathrm{d}e^{x(\log 2-2\pi i n)}$$
$$= \frac{1}{\log 2 - 2\pi i n}\int_0^1 e^{x(\log 2-2\pi i n)}\,\mathrm{d}F(x) = \frac{\mathfrak{m}(\log 2 - 2\pi i n)}{2\log 2 - 4\pi i n}.$$

The last assertion of the proposition is obvious. □

This proposition is a good example of intrinsic relations among the three functions $F(x)$, $G(z)$ and $\mathfrak{m}(t)$. Indeed, the moments m_L of $F(x)$ give Taylor coefficients of $G(z)$, which are proportional (up to the factorial multiplier) to Taylor coefficients of $\mathfrak{m}(t)$. Finally, special values of $\mathfrak{m}(t)$ on a discrete set of vertical line produce Fourier coefficients of $F(x)$.

Proposition 4 describes explicit relations among Fourier coefficients and the moments. Additionally, in the course of the proof we obtain the expansion of $G(z)$ for negative real z in terms of incomplete gamma integrals.

Proposition 4. *For* $L \geq 1$, *one has*

$$M_L = L!\sum_{n\in\mathbb{Z}} \frac{c_n}{(\log 2 - 2\pi i n)^L}. \tag{24}$$

Proof. Let $z < 0$ be fixed negative real. Then integration by parts gives

$$G(z+1) = \int_0^\infty \frac{x}{1-xz}\,\mathrm{d}(F(x)-1) = \int_0^\infty \frac{1}{(1-xz)^2} 2^{-x}\Psi(x)\,\mathrm{d}x$$
$$= \sum_{n=-\infty}^{\infty} c_n \int_0^\infty \frac{1}{(1-xz)^2} 2^{-x} e^{2\pi i n x}\,\mathrm{d}x = \sum_{n=-\infty}^{\infty} c_n V_n(z),$$

where

$$V_n(z) = \int_0^\infty \frac{1}{(1-xz)^2} e^{-x(\log 2 - 2\pi i n)}\,\mathrm{d}x$$
$$= \frac{1}{\log 2 - 2\pi i n}\int_0^{\infty(\log 2 - 2\pi i n)} \frac{1}{(1-\frac{yz}{\log 2 - 2\pi i n})^2} e^{-y}\,\mathrm{d}y.$$

Since by our convention $z < 0$, the function under integral does not have poles for $\Re y > 0$, and Jordan's Lemma gives

$$V_n(z) = \frac{1}{\log 2 - 2\pi i n}\int_0^{\infty}\frac{1}{(1-\frac{yz}{\log 2-2\pi i n})}e^{-y}\,\mathrm{d}y$$
$$= \frac{1}{\log 2 - 2\pi i n}\cdot V\Big(\frac{z}{\log 2 - 2\pi i n}\Big),\ \text{where } V(z) = \int_0^{\infty}\frac{1}{(1-yz)^2}e^{-y}\,\mathrm{d}y.$$

The function $V(z)$ is defined for the same values of z as $G(z+1)$ and therefore is defined in the cut plane $\mathbb{C}\setminus(0,\infty)$. Consequently, this implies

$$G(z+1) = \sum_{n\in\mathbb{Z}}\frac{c_n}{\log 2 - 2\pi i n}\cdot V\Big(\frac{z}{\log 2 - 2\pi i n}\Big). \tag{25}$$

The formula is only valid for real $z < 0$. The obtained series converges uniformly, since $|1 - y\frac{z}{\log 2 - 2\pi i n}| \geq 1$ for $n \in \mathbb{Z}$ and $z < 0$. Since

$$V\Big(\frac{1}{z}\Big) = -ze^{-z}\int_1^{\infty}\frac{1}{y^2}e^{yz}\,\mathrm{d}y,$$

this gives us the expansion of $G(z+1)$ on a negative real line in terms of incomplete gamma integrals. As noted before, and this can be seen from Equation (5), the function $G(z)$ has all left derivatives at $z = 1$. Further, the $(L-1)$-fold differentiation of $V(z)$ gives

$$V^{(L-1)}(z) = L!\int_0^{\infty}\frac{y^{L-1}}{(1-yz)^{L+1}}e^{-y}\,\mathrm{d}y \Rightarrow V^{(L-1)}(0) = L!(L-1)!.$$

Comparing (25) with (5) and (10), this gives the desired relation among moments M_L and Fourier coefficients, as stated in the proposition. □

It is important to compare this expression with the first equality of (7). Indeed, since $\mathfrak{m}(t)$ is entire, that equality via the Cauchy residue formula implies (17). It is exactly the leading term in (24), corresponding to $n = 0$.

9. Associated zeta function

Recall that for complex c and s, c^s is a multivalued complex function, defined as $e^{s\log c} = e^{s(\log|c| + i\arg(c))}$. Henceforth, we fix the branch of the logarithm by requiring that the value of $\arg c$ for c in the right half plane $\Re c > 0$ be in the range $(-\pi/2, \pi/2)$. Thus, if $s = \sigma + it$, and if we denote $r_n = \log 2 + 2\pi i n$, then $|r_n^{-s}| = |r_n|^{-\sigma}e^{t\arg r_n} \sim |r_n|^{-\sigma}e^{\pm\pi t/2}$ as $n \to \pm\infty$. Minding this convention and the identity (24), we introduce the zeta function, associated with the Minkowski question mark function.

Definition 1. The dyadic zeta function $\zeta_{\mathcal{M}}(s)$ is defined in the half plane $\Re s > 0$ by the series

$$\zeta_{\mathcal{M}}(s) = \sum_{n\in\mathbb{Z}} \frac{c_n}{(\log 2 - 2\pi i n)^s}, \tag{26}$$

where c_n are Fourier coefficients of $\Psi(x)$, and for each n, $(\log 2 - 2\pi i n)^s$ is understood in the meaning just described.

Then we have

Theorem 6. *$\zeta_{\mathcal{M}}(s)$ has an analytic continuation as an entire function to the whole plane $\mathbb{C}$, and satisfies the functional equation*

$$\zeta_{\mathcal{M}}(s)\Gamma(s) = -\zeta_{\mathcal{M}}(-s)\Gamma(-s). \tag{27}$$

Further, $\zeta_{\mathcal{M}}(L) = M_L/L!$ for $L \geq 0$. $\zeta_{\mathcal{M}}(s)$ has trivial zer s for negative integers: $\zeta_{\mathcal{M}}(-L) = 0$ for $L \geq 1$ and $\zeta_{\mathcal{M}}'(-L) = (L-1)!(-1)^L M_L$. Additionally, $\zeta_{\mathcal{M}}(s)$ is real on the real line, and thus $\zeta_{\mathcal{M}}(\overline{s}) = \overline{\zeta_{\mathcal{M}}(s)}$. The behavior of $\zeta_{\mathcal{M}}(s)$ in vertical strips is given by estimate

$$|\,\zeta_{\mathcal{M}}(\sigma + it)| \ll t^{-\sigma-1/2} \cdot e^{\pi|t|/2}$$

uniformly for $a \leq \sigma \leq b$, $|t| \to \infty$.

As we will see, these properties are immediate (subject to certain regularity conditions) for any distribution $f(x)$ with a symmetry property $f(x)+f(1/x)=1$. Nevertheless, it is a unique characteristic of $F(x)$ that the corresponding zeta function can be given a Dirichlet series expansion, like Equation (26). We do not give the proof of the converse result, since there is no motivation for this. But empirically, we see that this functional equation is equivalent exactly to the symmetry property. Additionally, the presence of a Dirichlet series expansion yields a functional equation of the kind $f(x+1) = 1/2 f(x) + 1/2$. Generally speaking, these two together are unique for $F(x)$. Note also that the functional equation implies that $\zeta_{\mathcal{M}}(it)\Gamma(1+it) = \int_0^\infty x^{it}\,\mathrm{d}F(x)$ is real for real t. Figures 4 and 5 shows its graph for $1.5 \leq t \leq 180$. Further calculations support the claim that this function has infinitely many zeros on the critical line $\Re s = 0$. On the other hand, numerical calculations of contour integrals reveal that there exist many more zeros apart from these. We need one classical integral.

Lemma 3. *Let A be real number,* $\arctan(A) = \phi \in (-\pi/2, \pi/2)$, *and* $\Re s > 0$. *Then*

$$\int_0^\infty x^{s-1}e^{-x}\cos(Ax)\,\mathrm{d}x = \frac{1}{(1+A^2)^{s/2}}\cos(\phi s)\Gamma(s).$$

The same is valid with cos *replaced by* sin *on both sides.*

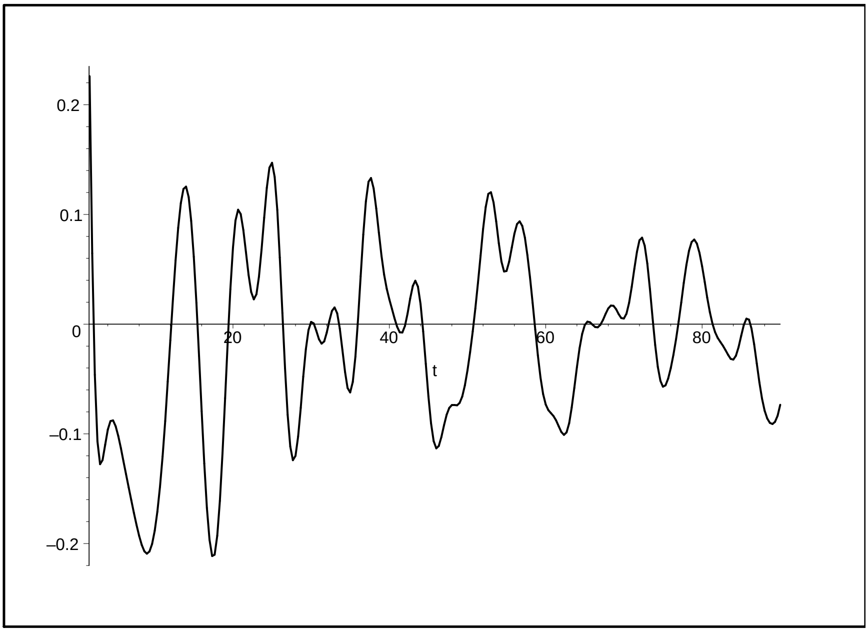


Figure 4. $\zeta_{\mathcal{M}}(it)\Gamma(1+it)$, $1.5 \le t \le 90$.

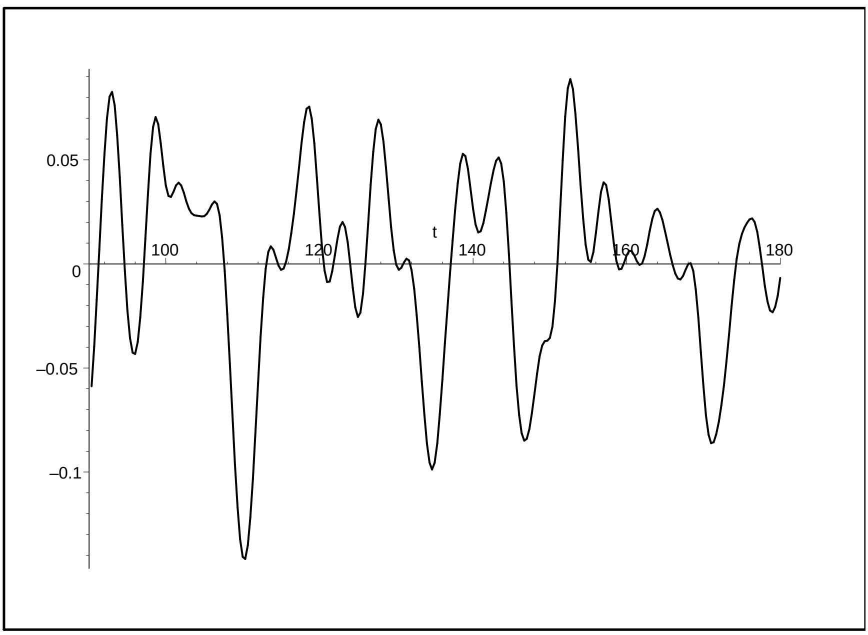


Figure 5. $\zeta_{\mathcal{M}}(it)\Gamma(1+it)$, $90 \le t \le 180$.

This can be found in any extensive table of gamma integrals or tables of Mellin transforms.

Proof of Theorem 6. Let for $n \geq 0$, $\arctan\,(2\pi n/\log 2) = \phi_n$. We will calculate the following integral. Let $\Re s > 0$. Then integrating by parts and using Lemma 3, one obtains

$$\begin{aligned}\int_0^\infty x^s \,\mathrm{d}(F(x)-1) &= s\int_0^\infty x^{s-1}2^{-x}\Psi(x)\,\mathrm{d}x = s\sum_{n\in\mathbb{Z}} c_n \int_0^\infty x^{s-1}2^{-x}e^{2\pi i n x}\,\mathrm{d}x\\ &= s\sum_{n=0}^{\infty}\int_0^\infty x^{s-1}\Big(2a_n\cos(2\pi n x) - 2b_n\sin(2\pi n x)\Big)2^{-x}\,\mathrm{d}x\\ &= 2s\Gamma(s)\sum_{n=0}^{\infty}|\log 2 + 2\pi n i|^{-s}\Big(a_n\cos(\phi_n s) - b_n\sin(\phi_n s)\Big)\\ &= s\Gamma(s)\sum_{n\in\mathbb{Z}}\frac{c_n}{(\log 2 - 2\pi i n)^s}.\end{aligned}$$

Note that the function $\int_0^\infty x^s\,\mathrm{d}F(x)$ is clearly analytic and entire. Thus, $s\Gamma(s)\,\zeta_{\mathcal{M}}(s)$ is an entire function, and this proves the first statement of the theorem. Since $F(x) + F(1/x) = 1$, this gives $\int_0^\infty x^s\,\mathrm{d}F(x) = \int_0^\infty x^{-s}\,\mathrm{d}F(x)$, and this, in turn, implies the functional equation. All other statements follow easily from this, our previous results, and known properties of the Γ function. In particular, if $s = \sigma + it$,

$$|\,\zeta_{\mathcal{M}}(s)\Gamma(s+1)| \leq \int_0^\infty |x^s|\,\mathrm{d}F(x) = \zeta_{\mathcal{M}}(\sigma)\Gamma(\sigma+1),$$

and the last statement of the theorem follows from the Stirling's formula for the Γ function: $|\Gamma(\sigma+it)| \sim \sqrt{2\pi}\,t^{\sigma-1/2}e^{-\pi|t|/2}$ uniformly for $a \leq \sigma \leq b$, as $|t| \to \infty$. □

At this stage, we remark on the similarity and differences with classical results known for the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty}(1/n^s)$. Let $\theta(x)$ denote the usual theta function $\theta(x) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 x}$, $\Im x > 0$. The following table summarizes all the ingredients which eventually produce the functional equation both for $\zeta(s)$ and $\zeta_{\mathcal{M}}(s)$.

Function	$\zeta(s)$	$\zeta_{\mathcal{M}}(s)$
Dirichlet series exp. Functional equation	Periodicity: $\theta(x+2) = \theta$ $\theta(ix) = (1/\sqrt{x})\theta(i/x)$	Periodicity: $F'(x+1) = (1/2)F'(x)$ $F'(x) = -F'(1/x)$

Since $F(x)$ is a singular function, its derivative should be considered as a distribution on the real line. For this purpose, it is sufficient to consider a distribution $U(x)$ as a derivative of a continuous function $V(x)$, for which the scalar product $\langle U, f\rangle$, defined for functions $f \in C^{\infty}(\mathbb{R})$ with compact support, equals $-\langle V, f'\rangle = -\int_{\mathbb{R}} f'(x)V(x)\,\mathrm{d}x$. Thus, both $\theta(x)$ and $2^x F'(x)$ are periodic distributions. This guarantees that the appropriate Mellin transform can be factored into the product of Dirichlet series and gamma factors. Finally, the functional equation for the distribution produces the functional equation for the Mellin transform. The difference arises from the fact that for $\theta(x)$ the functional equation is symmetry property on the imaginary line, whereas for $F'(x)$ we have the symmetry on the real line instead. This explains the unusual fact that in Equation (26) we have the summation over the discrete set of the vertical line, instead of the summation over integers.

We will finish by proving another result, which links $\zeta_{\mathcal{M}}(s)$ to the Mellin transform of $G(-z+1)$. This can be done using expansion (25), but we rather chose a direct way. Let $\int_0^{\infty} G(-z+1)z^{s-1}\,\mathrm{d}z = G^*(s)$. The symmetry property for Theorem 1 implies that $G(-z+1)$ has a simple zero, as $z \to \infty$ along the positive real line. Thus, basic properties of Mellin transform imply that $G^*(s)$ is defined for $0 < \Re s < 1$. For these values of s, we have the following classical integral:

$$\int_0^{\infty} \frac{z^{s-1}}{1+z}\,\mathrm{d}z \overset{\frac{z}{1+z}\to x}{=} \int_0^1 x^{s-1}(1-x)^{-s}\,\mathrm{d}x = \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Thus, using Equation (11), we get

$$\begin{aligned} G^*(s) &= \int_0^{\infty}\int_0^{\infty} \frac{xz^{s-1}}{1+xz}\,\mathrm{d}F(x)\,\mathrm{d}z \\ &= \int_0^{\infty}\int_0^{\infty} \frac{z^{s-1}}{1+z}x^{1-s}\,\mathrm{d}z\,\mathrm{d}F(x) = \frac{\pi}{\sin \pi s}\int_0^{\infty} x^{1-s}\,\mathrm{d}F(x). \end{aligned}$$

This holds for $0 < \Re s < 1$. Due to the analytic continuation, this gives

Proposition 5. *For $s \in \mathbb{C} \setminus \mathbb{Z}$, we have an identity $G^*(s) = \zeta_{\mathcal{M}}(s-1)\Gamma(s) \cdot \pi/(\sin \pi s)$.*

Therefore, $G^*(s)$ is a meromorphic function, $G^*(s+1) = -G^*(-s+1)$, and $\mathrm{res}_{s=L}\, G^*(s) = (-1)^L M_{L-1}$. This is the general property of the Mellin transform, since formally $G(z+1) = \sum_{L=0}^{\infty} M_L z^{L-1}$. Thus, $G(z+1) \sim \sum_{L=0}^{M} M_L z^{L-1}$ in the left neighborhood of $z = 0$.

10. Concluding remarks

Dyadic period functions in $\mathbb{H}$. As noted in [Alkauskas ≥ 2009], one encounters the surprising fact that in the upper half plane $\mathbb{H}$, Equation (12) is also satisfied by $(i/2\pi)\,G_1(z)$, where $G_1(z)$ stands for the Eisenstein series [Serre 1973]. Let $f_0(z) = G(z) - (i/2\pi)\,G_1(z)$, where $G(z)$ is the function in Theorem 1. Then for

$z \in \mathbb{H}$, $f_0(z)$ satisfies the homogeneous form of the three-term functional Equation (12); moreover, $f_0(z)$ is bounded, when $\Im z \to \infty$. Thus, if $f(z) = f_0(z)$,

$$-\frac{1}{(1-z)^2} f\Big(\frac{1}{1-z}\Big) + 2f(z+1) = f(z).$$

Therefore, denote by $\mathtt{DPF}^0$ the $\mathbb{C}$-linear vector space of solutions of this three-term functional equation, which are holomorphic in $\mathbb{H}$ and are bounded at infinity, and call it *the space of dyadic period functions in the upper half-plane*. Consequently, this space is at least one-dimensional. If we abandon the growth condition, then the corresponding space $\mathtt{DPF}$ is infinite-dimensional. This is already true for periodic solutions. Indeed, if $f(z)$ is a periodic solution, then

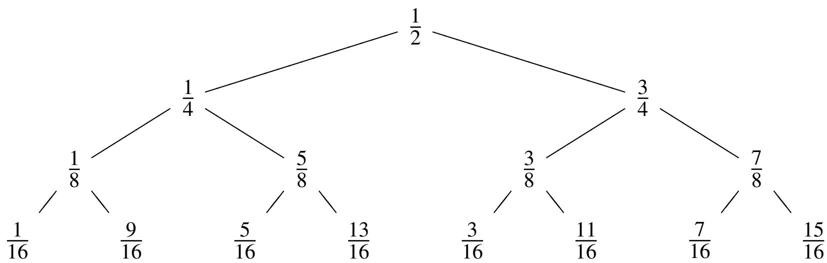
$$f(z) = (1/z^2) f(-1/z).$$

Let $P(z) \in \mathbb{C}[z]$, and suppose that $j(z)$ stands, as usually, for the j-invariant. Then any modular function of the form $j'(z)P(j(z))$ satisfies this equation. Additionally, there are nonperiodic solutions, given by $f_0(z)P(j(z))$. Therefore, $G(z)$ surprisingly enters the profound domain of classical modular forms and functions for $\mathsf{PSL}_2(\mathbb{Z})$. Hence, it is greatly desirable to give the full description and structure of spaces $\mathtt{DPF}^0$ and $\mathtt{DPF}$.

Where should the true arithmetic zeta function come from? Here we present some remarks, concerning the zeta function $\zeta_{\mathcal{M}}(s)$. This object is natural for the question mark function — its Dirichlet coefficients are the Fourier coefficients of $F(x)$, and its special values at integers are proportional to the moments M_L. Moreover, its relation to $G(z)$, $\mathfrak{m}(t)$ and $F(x)$ is the same as that of the L-series of Maass wave forms to analogous objects [Zagier 2001]. Nevertheless, one expects a richer arithmetic object associated with the Calkin–Wilf tree, since the latter consists or rational numbers, and therefore can be canonically embedded into the group of idèles $\mathbb{A}_{\mathbb{Q}}$. The p-adic distribution of rationals in the n-th generation of Calkin–Wilf tree was investigated in [Alkauskas $\geq$ 2009]. Surprisingly, the Eisenstein series $G_1(z)$ yet again manifests itself, as in case of $\mathbb{R}$ (see previous subsection). Nevertheless, there is no direct way of normalizing moments of the n-th generation in order for them to converge in the p-adic norm. There is an exception. As can easily be seen,

$$\sum_{a_0+a_1+\dots+a_s=n} [a_0, a_1, .., a_s] = 3 \cdot 2^{n-2} - 1/2,$$

and thus we have a convergence only in the 2-adic topology, namely to the value $-1/2$. The investigation of p-adic values of moments is relevant for the following reason. Let us apply $F(x)$ to each rational number in the Calkin–Wilf tree. What we obtain is the following:



Using Equation (2), we deduce that this tree starts from the root $1/2$, and then inductively each rational r produces two offsprings: $r/2$ and $r/2 + 1/2$. One is therefore led to the following.

Task. Produce a natural algorithm, which takes into account p-adic and real properties of the above tree, and generates Riemann zeta function $\zeta(s)$.

We emphasize that the choice of $\zeta(s)$ is not accidental. In fact, the $\mathbb{R}$-distribution of the above tree is a uniform one with support $[0, 1]$. Further, there is a natural algorithm to produce a *characteristic function of ring of integers of* $\mathbb{R}$ (that is, $e^{-\pi x^2}$) from the uniform distribution via the central limit theorem through the expression

$$\int_{\mathbb{R}} f(x) e^{-\pi x^2}\, \mathrm{d}x = \lim_{N\to\infty} \frac{1}{2^N} \int_{-1}^{1} \mathrm{d}x_1 \ldots \int_{-1}^{1} \mathrm{d}x_N\, f\left(\frac{x_1 + \cdots + x_N}{\sqrt{\frac{2}{3}\pi N}}\right).$$

(For clarity, here we take the uniform distribution in the interval $[-1, 1]$). This formula and this explanation and treatment of $e^{-\pi x^2}$ as a *characteristic function of the ring of integer of* $\mathbb{R}$ is borrowed from [Haran 2001, page 7]. Further, the operator which is invariant under uniform measure has the form

$$[\mathcal{U} f](x) = \frac{1}{2} f\left(\frac{x}{2}\right) + \frac{1}{2} f\left(\frac{x}{2} + \frac{1}{2}\right).$$

Indeed, for every $f \in \mathsf{C}[0, 1]$, one has $\int_0^1 [\mathcal{U} f](x)\, \mathrm{d}x = \int_0^1 f(x)\, \mathrm{d}x$. The spectral analysis of $\mathcal{U}$ shows that its eigenvalues are 2^{-n}, $n \geq 0$, with corresponding eigenfunctions being Bernoulli polynomials $B_n(x)$ [Flajolet and Vallée 1998]. These, as is well known from the time of Euler, are intricately related with $\zeta(s)$. Moreover, the partial moments of the above tree can be defined as $\sum_{i=1}^{2^N} \left((2i - 1)/2^N\right)^L$. These values are also expressed in term of Bernoulli polynomials. As we know, there are famous Kummer congruences among Bernoulli numbers, which later led to the introduction of the p-adic zeta function $\zeta_p(s)$. Thus, the real distribution of the above tree and its spectral decomposition is deeply related to the p-adic properties. This justifies the choice in the task of $\zeta(s)$. Therefore, returning to the

Calkin–Wilf tree, one expects that moments can be p-adically interpolated, and some natural arithmetic zeta function can be introduced, as a *preimage* of $\zeta(s)$ under the map F.

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