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Patrick Bahls, Voula Collins and Elizabeth Heron

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Ordering BS(1, 3) using the Magnus transformation

Patrick Bahls, Voula Collins and Elizabeth Heron

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Following a similar treatment of the Baumslag–Solitar group BS(1, 2) by Bahls, we modify a transformation developed by Magnus to linearly order the group BS(1, 3) given by the presentation $\langle a, b \mid ab = ba^3\rangle$. We demonstrate how this same method will fail to admit such a treatment of the groups BS(1, n), $n \geq 4$.

1. Introduction

This paper is heavily based on the work [Bahls 2007] on ordering the Baumslag–Solitar group BS(1, 2). The purpose of this paper is to modify the method of [Bahls 2007] to linearly order the Baumslag–Solitar group BS(1, 3) with presentation

$$P = \langle a, b \mid ab = ba^3\rangle.$$

Theorem 1.1. *The positive monoid* $\mathrm{BS}^+(1,3)$ *can be linearly ordered by an order* $\leq$ *compatible with multiplication on the left*:

$$u \leq v \Rightarrow w \cdot u \leq w \cdot v,$$

for all $u, v, w \in \mathrm{BS}^+(1,3)$. *This order passes to an order on the corresponding group* BS(1, 3) *which is also compatible with multiplication on the left.*

We will also indicate why our method does not apply to any BS(1, n), $n \geq 4$.

We would like to emphasize that it *is* possible to construct an order on BS(1, n) by other means; the significance of this current work lies in its exploration of the methods developed first in [Duchamp and Krob 1990; 1993], and [Duchamp and Thibon 1992], and later modified by Bahls [2007]. Our results here highlight both the potential and the limitations of these methods.

Before proceeding further we briefly motivate our study of orderability.

As in our theorem above, the group G is said to be *left orderable* if it admits a linear ordering $\leq$ satisfying $g_1 \leq g_2 \Rightarrow g \cdot g_1 \leq g \cdot g_2$ for all $g, g_1, g_2 \in G$. *Right*

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orderable and *biorderable* groups are defined in a similar fashion; biorderability clearly implies both left and right orderability.

Orderability works well with other algebraic conditions. For instance, it is known that if G is left orderable then it satisfies the *Zero Divisor Conjecture*: the integral group ring $\mathbb{Z}G$ has no nontrivial divisors of zero. Local indicability is another closely related property: G is said to be *locally indicable* if every nontrivial finitely generated subgroup surjects onto $\mathbb{Z}$. Such groups are known to be orderable on one side, but conversely there are examples of groups which are right orderable and not locally indicable. (See [Bergman 1991] for examples; more details can be found in [Rhemtulla 2002].)

The *braid groups* B_n are one such class: they are orderable on one side but not on both, and they are not locally indicable. Dehornoy et al. [2002] give an in-depth treatment of B_n. The braid groups are one of many topologically and geometrically significant classes of groups whose orderability has recently drawn attention. Other examples include various mapping class groups of punctured surfaces with boundary [Short and Wiest 2000] and fundamental groups of 3-manifolds [Boyer et al. 2005].

A sketch of our argument is as follows. As in [Bahls 2007], we will first define the *Magnus transformation*, μ, which maps a generator x_i of a monoid M to $1+x_i$, an element of the algebra $A_k(M)$ of formal power series freely generated by M with coefficients in the integral domain k. A more extensive discussion of the Magnus transformation in various settings can be found in the second section of [Bahls 2007], in Magnus's own classical work with Karrass and Solitar [Magnus et al. 1976], or in [Duchamp and Krob 1990; 1993; Duchamp and Thibon 1992].

Due to the simplicity of the relations governing right-angled Artin groups, Duchamp and his collaborators were able to work with μ without passing to a quotient algebra. In our present case, as in [Bahls 2007], μ is not inherently a homomorphism, so we must force it to be one by introducing a relation on the algebra, thereby passing to a quotient. After defining a normal form for the elements in $\mathrm{BS}(1,3)$ we will apply the new relation to determine a normal form for elements in the algebra. This will allow us to prove that the modified mapping μ is injective and to define a linear order on the elements of $\mathrm{BS}(1,3)$ by linearly ordering their images under μ.

2. The mapping μ and normal forms in BS(1, 3)

Let M be the noncancellative positive monoid $\mathrm{BS}^+(1,3)$ given by the presentation

$$\langle a, b \mid ab = ba^3 \rangle.$$

It is known that an element of $\mathrm{BS}^+(1,3)$ has normal form $b^m a^l$ where m and l are nonnegative integers. Similarly, an element in $\mathrm{BS}(1,3)$ has normal form $b^m a^l b^{-k}$ where k and m are nonnegative integers and l is any integer not divisible by 3.

Let $A_{\mathbb{Q}}(M)$ be the associative algebra of formal series freely generated by the elements of M with coefficients in $\mathbb{Q} \cup \{\pm\infty\}$.

Note. For reasons that will become clear in the next section we will require infinite coefficients. For any $q \in \mathbb{Q}$ we define $\infty + q = \infty$ and $\infty \cdot q = \pm\infty$, depending on the sign of q, and similarly for $-\infty$. Though strictly speaking addition is not defined on all pairs of elements in our algebra, the computation $-\infty + \infty$ will not arise.

We define $\mu : M \to A_{\mathbb{Q}}(M)$ by $x \mapsto 1 + x$ for $x \in \{a, b\}$ and extend it in the natural fashion:

$$x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_k^{\epsilon_k} \overset{\mu}{\mapsto} (1+x_1)^{\epsilon_1}(1+x_2)^{\epsilon_2} \cdots (1+x_k)^{\epsilon_k}.$$

We then make use of the existence of formal inverses in $A_{\mathbb{Q}}(M)$: given any $x \in M$,

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots ,$$

and thus we may extend μ to a mapping on $\mathrm{BS}(1,3)$ by extending the natural mapping

$$\mu(a^{-1}) = 1 - a + a^2 - a^3 + \cdots \qquad \text{and} \qquad \mu(b^{-1}) = 1 - b + b^2 - b^3 + \cdots .$$

As defined, the map μ is not a homomorphism on $\mathrm{BS}(1,3)$. In order to ensure that μ preserves the structure of the group we pass to the quotient of $A_{\mathbb{Q}}(M)$ by the image of the relation $ab = ba^3$ under μ. That is, we define

$$\mathcal{A} = A_{\mathbb{Q}}(M)/I,$$

where

$$I = \langle (1+a)(1+b) = (1+b)(1+a)^3 \rangle = \langle ba^2 = -(1/3)a^3 - a^2 - (2/3)a - ba \rangle.$$

Abusing notation, we let μ refer to the composition of the original mapping with this quotient map.

Every element in $\mathcal{A}$ can be placed in *normal form* by successive applications of the two relations

$$ab = ba^3 \qquad \text{and} \qquad ba^2 = -(1/3)a^3 - a^2 - (2/3)a - ba.$$

Such a normal form will admit additive terms in one of three forms: b^h, a^i, or $b^j a$. (The group relation allows us to move b to the left past a, and the quotient relation allows us to reduce powers of a that follow at least one b.) Images of group elements under μ take the form

$$(1+b)^m(1+a)^l(1+b)^{-k},$$

where m and k are nonnegative integers and l is an arbitrary integer. Upon expanding these binomials and applying the two above relations, we obtain a normal form

$$1+\sum_{h=1}^{\infty}\beta_h b^h+\sum_{i=1}^{\infty}\alpha_i a^i+\sum_{j=1}^{\infty}\gamma_j b^j a,$$

for rational numbers β_h, α_i, and γ_j.

3. Injectivity of μ

By passing to $\mathscr{A}$ we have ensured that μ is a homomorphism. However, before we will be able to linearly order the group by ordering its image under μ, we must prove that μ is an embedding of $\mathrm{BS}(1,3)$ into $\mathscr{A}$. To prove that μ is injective we must show that if $g \in \mathrm{BS}(1,3)$ satisfies $g \neq 1$, then $\mu(g) \neq 1$ in $\mathscr{A}$.

We will need the following:

Lemma 3.1. *For $k \in \mathbb{N}$, and $x \in (1,\infty)$, then*

$$\sum_{i=0}^{\infty}\binom{i+k-1}{k-1}\Big(1-\frac{1}{x}\Big)^i = x^k.$$

To prove Lemma 3.1 we use the following obvious fact:

Lemma 3.2. *Let $g_k(x) = \big(1/(1-x)\big)^k$. Then*

$$\frac{d^n g_k}{dx^n} = \frac{(n+k-1)!}{(k+1)!}\Big(\frac{1}{1-x}\Big)^{n+k},$$

so

$$g_k^{(n)}(0) = \frac{(n+k-1)!}{(k+1)!}.$$

Proof of Lemma 3.1. Using the binomial theorem, we see

$$\Big(\tfrac{1}{1-x}\Big)^k = \sum_{i=0}^{\infty}\binom{i+k-1}{k-1}x^i .$$

But Taylor expansion gives

$$\begin{aligned}\sum_{i=0}^{\infty}\binom{i+k-1}{k-1}\Bigl(1-\frac{1}{x}\Bigr)^i &= g_k\Bigl(1-\frac{1}{x}\Bigr)\\ &= \Bigl(\frac{1}{1-(1-(1/x))}\Bigr)^k\\ &= \Bigl(\frac{1}{(1/x)}\Bigr)^k\\ &= x^k,\end{aligned}$$

and we are done. □

Let $c(z, m)$ be the coefficient of the monoid element m as an additive term in $z \in \mathscr{A}$ written in normal form, and define H to be the image of BS(1, 3) under μ. To prove injectivity we will derive formulas for $c(z, a)$ for any arbitrary $z \in H$.

Proposition 3.3. *The mapping μ is injective.*

Proof. Let $g = b^m a^l b^{-k} \neq 1$. Clearly $c(\mu(g), b) \neq 0$ if $l = 0$ and $m \neq k$. Thus we may assume $l \neq 0$.

Suppose at least one of m or k is equal to 0 and $l \neq 0$. If $m = k = 0$, then $c(\mu(g), a) = l$, so $\mu(g) \neq 1$. If $m = 0$ and $k > 0$, then $c(\mu(g), b) = -k$; if $k = 0$ and $m > 0$, then $c(\mu(g), b) = m$. Therefore no such group elements can be mapped by μ to the identity.

Now let $g = b^m a^l b^{-k}$ where $m, k > 0$ and $l > 0$. We show that $c(\mu(g), a) \neq 0$. Expanding $\mu(b^m a^l b^{-k})$ gives a formal series with additive terms $b^h a^j b^i$ ($h \leq m$, $j \leq l$, and i arbitrarily large), before reducing to normal form. It is not difficult to compute inductively the coefficient on a in such a term once it is reduced to normal form:

Lemma 3.4. *For any $z = b^h a^j b^i$ as above,*

$$c(z, a) = (-1)^{h+i+j}\,(2/3)^{h+i}\,.$$

Proof. First apply the group relation $ab = ba^3$ to move all powers of b to the left, resulting in $b^{h+i} a^{j3^i}$.

We now show by induction on s and t that

$$c(b^s a^t, a) = (-1)^{s+t}\bigl(2/3\bigr)^s.$$

First consider $s = 1$. In the base case $t = 2$,

$$c(ba^2, a) = -2/3,$$

as desired. Suppose inductively we have shown

$$c(ba^t, a) = (-1)^{s+t}(2/3).$$

Then

$$ba^{t+1} = (ba^2)a^{t-1} = -(1/3)a^{t+2} - a^{t+1} - (2/3)a^t - ba^t.$$

Our inductive hypothesis thus gives us coefficient

$$(-1)(-1)^{t+1}(2/3) = (-1)^{[(t+1)+1]}(2/3)$$

on a, as needed.

Now suppose inductively we have shown

$$c(b^s a^t, a) = (-1)^{s+t}\big((2/3)\big)^s,$$

for any $t \geq 2$ and for some fixed s. In the base case (for $s+1$) $t = 2$, we have

$$b^{s+1}a^2 = b^s(ba^2) = -(1/3)b^s a^3 - b^s a^2 - b^s a - b^{s+1}a.$$

The last two terms contribute no a's, while inductively the first two contribute $(-1/3)\,(-1)^{s+3}\,(2/3)^s$ and $(-1)(-1)^{s+2}\,(2/3)^s$ a's, respectively. Adding these and simplifying yields the desired sum:

$$\begin{aligned}(-1/3)\,(-1)^{s+3}\,(2/3)^s + (-1)(-1)^{s+2}\,(2/3)^s &= (-1)^{s+2}\,(1/3-1)\,(2/3)^s \\ &= (-1)^{s+3}\,(2/3)\,(2/3)^s \\ &= (-1)^{(s+1)+2}\,(2/3)^{s+1},\end{aligned}$$

as needed.

Thus

$$c(z,a) = c(b^{h+i}a^{j3^i}, a) = (-1)^{h+i+j3^i}\,(2/3)^{h+i} = (-1)^{h+i+j}\,(2/3)^{h+i},$$

where the last equality holds since j and $j3^i$ have the same parity. □

We now claim that

$$c(\mu(b^m a^l b^{-k}), a) = 1 + l + \sum_{h=0}^{m}\sum_{j=1}^{l}\sum_{i=0}^{\infty}(-1)^{h+j}\left(\frac{2}{3}\right)^{h+i}\binom{m}{h}\binom{l}{j}\binom{i+k-1}{k-1}.$$

Indeed, the innermost sum, involving i, considers the contribution made by the terms in $\mu(b^{-k})$ (the formal inverse makes this sum infinite). The next sum, involving j, considers the contribution made by each term from $\mu(a^l)$. The outermost sum, involving h, considers the contribution made by each term from $\mu(b^m)$.

The binomial coefficients represent the coefficients appearing on the terms of the expanded binomials. We obtain $(-1)^{h+j}$ from $(-1)^{h+i+j}\cdot(-1)^i$, the first term arising from Lemma 3.4 and the second from the sign on the term b^i in the infinite formal inverse $(1+b)^{-k}$. Finally, the term $(2/3)^{h+i}$ appears courtesy of

Lemma 3.4. We now compute, first rearranging and then applying Lemma 3.1 to the innermost sum:

$$
\begin{aligned}
&c(\mu(b^m a^l b^{-k}), a)\\
&\quad = 1+l+\sum_{h=0}^{m}\sum_{j=1}^{l}\sum_{i=0}^{\infty}(-1)^{h+j}\Big(\frac{2}{3}\Big)^{h+i}\binom{m}{h}\binom{l}{j}\binom{i+k-1}{k-1}\\
&\quad = 1+l+\sum_{h=0}^{m}(-1)^{h}\Big(\frac{2}{3}\Big)^{h}\binom{m}{h}\sum_{j=1}^{l}(-1)^{j}\binom{l}{j}\sum_{i=0}^{\infty}\Big(\frac{2}{3}\Big)^{i}\binom{i+k-1}{k-1}\\
&\quad = 1+l+\sum_{h=0}^{m}\Big(-\frac{2}{3}\Big)^{h}\binom{m}{h}\sum_{j=1}^{l}(-1)^{j}\binom{l}{j}3^{k}\\
&\quad = 1+l+\sum_{h=0}^{m}\Big(-\frac{2}{3}\Big)^{h}\binom{m}{h}(-3^{k})\\
&\quad = 1+l+\Big(-\frac{2}{3}+1\Big)^{m}(-3^{k})\\
&\quad = 1+l+\Big(\frac{1}{3}\Big)^{m}(-3^{k}) = \qquad 1+l-3^{k-m}.
\end{aligned}
$$

If either $m > k$ or $m = k$, this yields a nonzero quantity. If $m < k$, there are two situations to consider. If m, l, k do not satisfy $3^{k-m} = l + 1$, then $c(\mu(g), a) \neq 0$. If m, l, k satisfy $3^{k-m} = l + 1$, then $c(\mu(g), a) = 0$, but since $k > m$ we know that $c(\mu(g), b) = -k + m \neq 0$, and thus $\mu(g)$ is still not the identity.

Finally, we compute $c(\mu(b^m a^l b^{-k}), a)$ when $m, k > 0$ and $l < 0$. Arguing as in the case $l > 0$, we obtain a similar formula for this coefficient, which reduces nicely by applying Lemma 3.1 once more:

$$
\begin{aligned}
&c(\mu(b^m a^l b^{-k}), a)\\
&\quad = -l+\sum_{h=0}^{m}\sum_{j=1}^{\infty}\sum_{i=0}^{\infty}(-1)^{h+i+j}(-1)^{i}(-1)^{j}\Big(\frac{2}{3}\Big)^{h+i}\binom{m}{h}\binom{j+l-1}{l-1}\binom{i+k-1}{k-1}\\
&\quad = -l+\sum_{h=0}^{m}\sum_{j=1}^{\infty}\sum_{i=0}^{\infty}(-1)^{h+2i+2j}\Big(\frac{2}{3}\Big)^{h+i}\binom{m}{h}\binom{j+l-1}{l-1}\binom{i+k-1}{k-1}\\
&\quad = -l+\sum_{h=0}^{m}(-1)^{h}\Big(\frac{2}{3}\Big)^{h}\binom{m}{h}\sum_{j=1}^{\infty}\binom{j+l-1}{l-1}\sum_{i=0}^{\infty}\Big(\frac{2}{3}\Big)^{i}\binom{i+k-1}{k-1}-\sum_{j=1}^{\infty}\binom{j+l-1}{l-1}\\
&\quad = -l+\sum_{j=1}^{\infty}\binom{j+l-1}{l-1}\Big(\sum_{h=0}^{m}\Big(-\frac{2}{3}\Big)^{h}\binom{m}{h}\sum_{i=0}^{\infty}\Big(\frac{2}{3}\Big)^{i}\binom{i+k-1}{k-1}-1\Big)
\end{aligned}
$$

$$= -l + \sum_{j=1}^{\infty} \binom{j+l-1}{l-1} \Big(\Big(-\frac{2}{3}+1\Big)^m (-(3^k)) - 1 \Big)$$
$$= -l + (3^{k-m} - 1) \sum_{j=1}^{\infty} \binom{j+l-1}{l-1}.$$

Since $\sum_{j=1}^{\infty} \binom{j+l-1}{l-1} = \infty$, the coefficient on a is either ∞ or $-\infty$ if $m \neq k$. In this case $c(\mu(g), a) \neq 0$, so $\mu(g)$ is not the identity in $\mathscr{A}$. Finally, if $m = k$, then $c(\mu(g), a) = l \neq 0$ so that in this case too $\mu(g)$ is not the identity.

As we have now shown that $\mu(g) \neq 1$ for all $1 \neq g \in \mathrm{BS}(1, 3)$, our mapping μ is injective. □

4. Ordering BS(1, 3)

Using a method like that in [Bahls 2007], we will define an order on our group H by defining a *strict positive cone* C of the algebra which satisfies the following four properties:

(C1) $C \cdot C \subseteq C$,

(C2) $hCh^{-1} \subseteq C$ for all $h \subseteq H$,

(C3) $C \cap C^{-1} = \varnothing$, and

(C4) $C \cup C^{-1} \cup \{1\} = H$.

Once we know that a set C in H satisfies the above properties, then we may define an order on H that is compatible with multiplication on the left, by demanding $h_1 < h_2$ in $H \Leftrightarrow h_1^{-1} h_2 \in C$ (as in [Bahls 2007] or [Duchamp and Thibon 1992], for example).

Let

$$x = \sum_{i=1}^{\infty} \beta_i b^i + \sum_{j=1}^{\infty} \alpha_j a^j + \sum_{h=1}^{\infty} \gamma_h b^h a \in \mathscr{A}.$$

If $c(x, b) \neq 0$, then we will define $\tau(x) = b$, otherwise $\tau(x) = a$. (We may think of τ as indicating the "dominant" term of x.) Let $\lambda(x) = c(x, \tau(x))$ and define the positive cone C by

$$C = \{1 + x \in H \mid \lambda(x) > 0\}.$$

We require a few simple technical results.

Lemma 4.1. *For positive integers i, j, i',*

$$c(b^i a^j, b^{i'} a) = \begin{cases} (-1)^{i+1} & \text{if } i = i', \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Clearly if $i < i'$ then $c(b^i a^j, b^{i'} a) = 0$ since reducing to normal form never increases the exponent on the b. We must then consider the cases where $i = i'$ and $i > i'$. However, from this point on the proof consists of a pair of nested inductions (one for $i = i'$ and one for $i > i'$), each nearly identical to those in the proof of Lemma 3.4. The details are left to the reader. □

Lemma 4.2. *Let $y \in H$. If $c(y, b) = 0$, then $c(y, b^x) = 0$ for any positive integer x.*

Proof. Indeed, $c(y, b) = 0$ implies that $y = \mu(b^m a^l b^{-m})$ for some $m \geq 0$. The only way to obtain terms of the form b^x from the product

$$(1+b)^m(1+a)^l(1-b+b^2-b^3+\cdots)^m$$

is to avoid terms with as in them, *i.e.* extracting terms b^x from

$$(1+b)^m \cdot 1 \cdot (1-b+b^2-b^3+\cdots)^m = 1,$$

which clearly cannot be done. □

Lemma 4.3. *Let $y \in H$. If $c(y, b) = 0$, then $c(y, b^x a) = 0$ for any positive integer x.*

Proof. As before, $c(y, b) = 0 \Rightarrow m = k$. Moreover, we have just shown that the only nonzero contribution to $c(y, b^x a)$ will come from reduction of terms $b^i a^j$ satisfying $i = x$. We therefore consider terms $b^h a^j b^i$ obtained from expanding

$$y = (1+b)^m(1+a)^l(1-b+b^2-b^3+\cdots)^m$$

that satisfy $i + h = x$. (Note that moving the bs to the left past as does not change the exponent on the bs.)

The contribution to $c(y, b^x a)$ coming from such unreduced terms $b^h a^j b^i$ takes the form

$$\sum_{i=0}^{x} \binom{i+m-1}{m-1}\binom{m}{x-i}(-1)^i,$$

in which $\binom{m}{x-i}$ accounts for the contribution from $(1+b)^m$ for a fixed i and $\binom{i+m-1}{m-1}(-1)^i$ accounts for the contribution from $(1-b+b^2-b^3+\cdots)^m$ for the same i. It is not hard to show that the contribution from $(1+a)^j$ is 1 as a consequence of basic combinatorics of binomial coefficients.

Thus

$$c(y, b^x a) = \sum_{i=0}^{x} \binom{i+m-1}{m-1}\binom{m}{x-i}(-1)^i = \frac{\Gamma(1)}{\Gamma(1-x)\Gamma(1+x)} = \frac{\sin(\pi x)}{\pi x}$$

by basic properties of the Gamma function. Since x is assumed to be a nonzero integer, this last quantity is 0, and we are done. □

Thus if $c(y, b) = 0$ then the normal form of y consists only of powers of a.

Proposition 4.4. *The set C defined as above satisfies* $(C1)$–$(C4)$.

Proof. Property (C4) is obvious.

For (C1), let $1+x, 1+y \in C$ be in normal form. If b does not appear as a term in these normals forms (and thus by the preceding lemmas neither do b^i or $b^i a$, $i \geq 1$) then no term of the form b^i or $b^i a$ will appear in the normal form of $(1+x)(1+y)$. Since $\tau(1+x) = a$ and $\tau(1+y) = a$, $c(1+x, a) > 0$ and $c(1+y, a) > 0$. As a result, $c((1+x)(1+y), a) > 0$ also.

If one of $1+x, 1+y \in C$ contains b as a term, by definition it will have a positive coefficient, and thus $c((1+x)(1+y), b) > 0$ as well.

For (C3), let $1+x \in C$. Assume that $1+x$ contains no terms b^i, and therefore contains only powers of a. Then

$$c((1+x)^{-1}, a) = c(1 - x + x^2 - x^3 + \cdots, a) = -c(1+x, a)$$

and thus $(1+x)^{-1} \notin C$. A similar argument may be used if terms b^i do appear in $1+x$.

For (C2), let $1+x = 1+\beta b + x'$ where $\beta > 0$ is rational and all of the terms in x' have a form in

$$B = \{b^i \mid i \geq 2\} \cup \{a^i \mid i \geq 1\} \cup \{b^i a \mid i \geq 1\}.$$

Then

$$\begin{aligned}(1+y)&(1+x)(1+y)^{-1}\\ &= (1+y)(1+\beta b + x')(1 - y + y^2 - y^3 + \cdots)\\ &= (1+\beta b + x' + y + \beta y b + y x')(1 - y + y^2 - y^3 + \cdots)\\ &= (1+\beta b + y + z)(1 - y + y^2 - y^3 + \cdots),\end{aligned}$$

where $z = x' + \beta y b + y x'$ consists of terms in B. Since terms that are not in the form γb (where $\gamma \neq 0$ is rational) do not contribute to $c(1+x, b)$ in the reduced form, none of these terms will contribute to $c(1+x, b)$ when reduced to normal form. Continuing, this becomes

$$\begin{aligned}(1+\beta b&+y+z)(1 - y + y^2 - y^3 + \cdots)\\ &= 1 + \beta b + y + z - y - \beta b y - y^2 - z y + y^2 + \beta b y^2 + \cdots\\ &= 1 + \beta b + \beta b(-y + y^2 - \cdots) + z(-y + y^2 - \cdots).\end{aligned}$$

The only term that will contribute to $c(1+x, b)$ in this equation is βb. Thus $c(1+x, b) = \beta$, and $(1+y)(1+x)(1+y)^{-1} \in C$.

Next, assume that $1+x \in C$ contains no bs. Then $1+x = (1+a)^l$ for some positive integer l. Consider $1+y \in C$. As $1+y$ is a mapping of a group element into the algebra, it will be of the form $(1+b)^i(1+a)^j(1+b)^{-k}$ for some integers i, j, k, where $i, k \geq 0$. Therefore we can rewrite $(1+y)(1+x)(1+y)^{-1}$ as

$$(1+b)^i(1+a)^j(1+b)^{-k}(1+a)^l(1+b)^k(1+a)^{-j}(1+b)^{-i}.$$

This is $\mu(b^i a^j b^{-k} a^l b^k a^{-j} b^{-i}) = \mu(b^i a^{3^k l} b^{-i})$. However, it is easily shown that $c(\mu(b^i a^{3^k l} b^{-i}), a) = 3^k l$, which is positive because $k > 0$ is nonnegative and l is positive. We can also see that $\mu(b^i a^{3^k l} b^{-i})$ will contain no bs. Hence,

$$\lambda((1+y)(1+x)(1+y)^{-1}) > 0,$$

and $(1+y)(1+x)(1+y)^{-1} \in C$. □

As discussed above, we have the following consequence:

Corollary 4.5. *The group* BS(1, 3) *is linearly orderable by an order that is compatible with multiplication on the left.*

5. BS(1, n) for $n \geq 4$

Applying the method of this article to other groups was considered briefly in the final section of [Bahls 2007]. Although analysis of other classes of groups has not been performed, we conclude this article by indicating why the method we have pursued above will fail to admit a workable mapping μ when applied analogously to $\mathrm{BS}(1, n) = \langle a, b \mid ab = ba^n \rangle$, $n \geq 4$.

As before, we may define the positive monoid M and the algebra $A_{\mathbb{Q}}(M)$ freely generated by M with coefficients in $\mathbb{Q} \cup \{\pm\infty\}$. The initial map μ taking a to $1+a$ and b to $1+b$ is still defined, and in fact we may even define $\mathcal{A}$ as before by forming the quotient of $A_{\mathbb{Q}}(M)$ by the ideal $I = \langle (1+a)(1+b) = (1+b)(1+a)^n \rangle$. This leads to a modified μ, as before.

The difficulty comes when we attempt to define a normal form for elements in $\mathcal{A}$. Expanding the relation $(1+a)(1+b) = (1+b)(1+a)^n$ yields

$$1+a+b+ab = \sum_{i=0}^{n} \binom{n}{i} a^i + \sum_{i=0}^{n} \binom{n}{i} ba^i \Rightarrow 0 = (n-1)a + \sum_{i=2}^{n} \binom{n}{i} a^i + \sum_{i=1}^{n-1} \binom{n}{i} ba^i$$

after canceling and applying the single group relation $ab = ba^n$.

What rule of reduction should we derive from this? In order that our replacement rule remain somewhat "context free" we ought to replace a single term by a sum containing $2n-2$ terms. In order that our replacement rule give a terminating sequence of reductions, the single term must be one of either a^n or ba^{n-1}, since any other choice will give rise to an infinite sequence of rewritings in which "longer" strings continually replace "shorter" ones.

Choosing the reduction rule

$$ba^{n-1} \longrightarrow \frac{1}{n}\left[(n-1)a + \sum_{i=2}^{n}\binom{n}{i}a^i + \sum_{i=1}^{n-2}\binom{n}{i}ba^i\right],$$

as before, we obtain divergent alternating series as coefficients in certain reductions. For instance, if $n = 4$, the equation

$$ba^3 = -(3/4)a - (3/2)a^2 - a^3 - (1/4)a^4 - ba - (3/2)ba^2,$$

when applied to $\mu(ba^{-1})$ gives

$$\begin{aligned}\mu(ba^{-1})&\\ &= (1+b)(1-a+a^2-a^3+a^4-\cdots)\\ &= 1-a+a^2-a^3+a^4-\cdots+b-ba+ba^2-ba^3+ba^4-\cdots\\ &= 1-a+a^2-a^3+\cdots+b-ba+ba^2+((3/4)a\\ &\qquad +(3/2)a^2+a^3+(1/4)a^4+ba+(3/2)ba^2)+ba^4-\cdots\\ &= \cdots\end{aligned}$$

Already we see a trend that will continue: the coefficient $c(\mu(ba^{-1}), a)$ will receive contributions from each term of the form ba^i, and these contributions will continually alternate in sign and grow without bound, giving a divergent alternating sum. So long as $n \geq 4$, we will have this problem.

A similar difficulty arises if we attempt the only other feasible reduction rule, replacing a^n by the remaining $2n-2$ terms. Thus our method runs aground before we even have a chance to test μ's injectivity.

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pbahls@unca.edu *University of North Carolina, Asheville, Asheville, NC 28806, United States*

vcollins@wellesley.edu *Department of Mathematics, Wellesley College, Wellesley, MA 02481, United States*

heroel8@wfu.edu *Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109, United States*