involve

a journal of mathematics

The genus level of a group

Matthew Arbo, Krystin Benkowski, Ben Coate,
Hans Nordstrom, Chris Peterson and Aaron Wootton

mathematical sciences publishers

2009 Vol. 2, No. 3

The genus level of a group

Matthew Arbo, Krystin Benkowski, Ben Coate,
Hans Nordstrom, Chris Peterson and Aaron Wootton

(Communicated by Nigel Boston)

We introduce the notion of the genus level of a group as a tool to help classify finite conformal group actions on compact Riemann surfaces. We classify all groups of genus level 1 and use our results to outline an algorithm to classify actions of p-groups on compact Riemann surfaces. To illustrate our results, we provide a number of detailed examples.

1. Introduction

The group of conformal automorphisms of a compact Riemann surface of genus $\sigma \geq 2$ is always finite. An interesting problem is to determine all groups that can act on a surface X of genus $\sigma \geq 2$ for a fixed σ. For low genus, due to great advances in computer algebra systems, this problem has been completed. Specifically, in [Broughton 1991], all groups that can act on surfaces up to genus 3 were classified. More recently, with the aid of the computer algebra system GAP, the list of all groups that can act on surfaces of genus σ for genera $2 \leq \sigma \leq 48$ were classified in [Breuer 2000] and stored in a database in GAP. Though in principle the algorithms developed to solve this problem could be used to determine automorphism groups of surfaces of higher genus, the computations become much more complicated, and complete classification does not seem reasonable.

A different approach to classifying automorphism groups is to fix some other property than the genus of the surface X. One can, for instance, restrict how a group may act on the surface, or one can restrict the types of groups under consideration that act on X. In many cases, complete classification results are possible. Such results abound in the literature; see for example [Benim 2008; Bujalance et al. 2003; Harvey 1966; Kallel and Sjerve 2001; Wootton 2007b]. One motivation for classifying automorphism groups by imposing further restrictions on the way an

MSC2000: 14H37, 30F20.
Keywords: automorphism groups of compact Riemann surfaces, hyperelliptic surfaces, genus zero actions.
The authors were partially supported by NSF grant DMS 0649068.

automorphism group can act on a surface is that it may aid in answering other questions about the surface. For example, in certain cases, the classical problem of determining a defining model for a compact Riemann surface X is possible when assumptions are made about the existence of automorphisms on the surface acting in a particular way, as was done in [Fuertes and González-Diez 2007] or [Wootton 2007a]. Another motivational reason for classifying families of surfaces is that it may provide new techniques to the more general problem of classifying all groups that can act on surfaces of genus $\sigma \geq 2$.

Suppose G is a finite group of automorphisms of a compact Riemann surface X of genus $\sigma \geq 0$. We define the *genus level* of G to be the number of distinct genera of the quotient spaces X/H, where H runs over the nontrivial subgroups of G. We will classify all groups of genus level 1. This is an extension of [Kallel and Sjerve 2001], where full results are given for each group G that can act on a surface of genus $\sigma \geq 0$ with the property that every quotient space X/H has genus 0, where H runs over all the nontrivial subgroups of G. We use our results to provide an inductive method to determine all p-groups that act as automorphism groups on surfaces of genus $\sigma \geq 2$ (though it should be noted that with minor modifications, this algorithm would extend to a classification of nonsimple groups).

2. Preliminaries

Suppose that G is a group of conformal automorphisms of a compact Riemann surface X of genus $\sigma \geq 2$ upon which we assume G acts faithfully. We need the following definitions:

Definition 2.1. The *signature* of G is defined to be the tuple of integers

$$(g; m_1, m_2, \ldots, m_r),$$

where the orbit space X/G has genus g and the quotient map $\pi_G : X \to X/G$ is branched over r points $\alpha_1, \ldots, \alpha_r$, where the multiplicity of a point in the fiber $\pi_G^{-1}(\alpha_i)$ is equal to m_i for $1 \leq i \leq r$. We say that the *genus of* G is g and we call the numbers $m_1, \ldots, m_r$ the *periods* of G.

Remark 2.2. We note that for a general holomorphic map $F \colon X \to Y$, the multiplicities of two different points in the fiber $F^{-1}(\alpha)$ for $\alpha \in Y$ may be different. However, for normal covers, all points in a fiber will have the same multiplicity, and so our definition for the m_i is well defined. For more detail on holomorphic maps, we refer the reader to [Miranda 1995, Chapter 2].

Definition 2.3. We define the *genus level* of G to be the number of distinct genera among those of the nontrivial subgroups of G. If G has genus level n, we define the *genus level vector* to be the n-tuple of increasing integers $(t_1, \ldots, t_n)$, where t_i is the genus of some nontrivial subgroup of G.

Definition 2.4. A vector of elements $(\alpha_1, \dots, \alpha_t, \beta_1, \dots, \beta_t, \gamma_1, \dots, \gamma_r)$ is called a $(t, m_1, \dots, m_r)$-*generating vector* of G if

(i) $G = \langle \alpha_1, \dots, \alpha_t, \beta_1, \dots, \beta_t, \gamma_1, \dots, \gamma_r \rangle$,

(ii) $O(\gamma_i) = m_i$ for $1 \le i \le r$, and

(iii) $\prod_{i=1}^{t} [\alpha_i, \beta_i] \cdot \prod_{j=1}^{r} \gamma_j = e$.

We refer to the elements α_i and β_i as *hyperbolic generators* and the elements γ_i as *elliptical generators*.

The $(t; m_1, m_2, \dots, m_r)$-generating vectors of a group G provide us with a way to determine when G acts on a surface of genus $\sigma \ge 2$. Specifically, we have the following.

Theorem 2.5. *A group G acts with signature $(t; m_1, \dots, m_r)$ on a surface X with genus $\sigma \ge 2$ if G has a $(t; m_1, \dots, m_r)$-generating vector and the Riemann–Hurwitz formula holds*:

$$2\sigma - 2 = |G| \left(2t - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right).$$

Proof. See [Broughton 1991]. □

If a group G acts on a surface X, so does each of its subgroups. The signatures of the subgroups can be calculated from that of G as follows:

Theorem 2.6 [Singerman 1970]. *Suppose G has signature $(t; m_1, \dots, m_r)$ and generating vector $(\alpha_1, \beta_1, \dots, \alpha_t, \beta_t, \gamma_1, \dots, \gamma_r)$. Suppose $H \le G$, set $d = |G|/|H|$ and let $\theta : G \to S_d$ be the map induced by permutation on the left cosets of H. Then the signature of H is*

$$(s; n_{11}, n_{12}, \dots, n_{1\rho_1}, n_{21}, \dots, n_{2\rho_2}, \dots, n_{r1} \dots, n_{r\rho_r}), \tag{1}$$

where

(i) *$\theta(\gamma_j)$ has precisely ρ_j cycles of length $m_j/n_{j_1}, \dots, m_j/n_{j\rho_j}$, respectively*;

(ii) *s satisfies*

$$2s - 2 + \sum_{i=1}^{r} \sum_{j=1}^{\rho_i} \left(1 - \frac{1}{n_{ij}} \right) = d \left(2t - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right). \tag{2}$$

Conversely, if d divides $|G|$ and there exist a map $\theta : G \to S_d$ and an integer s such that conditions (i) *and* (ii) *hold, there exists $H \le G$ of index d and signature* (1).

In the special case that H is a normal subgroup of G, calculation of the signature is much easier.

Theorem 2.7. *Suppose G has signature $(t; m_1, m_2, \ldots, m_k)$ and $H \trianglelefteq G$. Then the signature of H is*

$$\Big(s; \underbrace{\frac{m_1}{n_1}, \ldots, \frac{m_1}{n_1}}_{a_1 \text{ times}}, \underbrace{\frac{m_2}{n_2}, \ldots, \frac{m_2}{n_2}}_{a_2 \text{ times}}, \ldots, \underbrace{\frac{m_r}{n_r}, \ldots, \frac{m_r}{n_r}}_{a_r \text{ times}}\Big),$$

where $O(\rho(\gamma_i)) = n_i$ under $\rho : G \to G/H$, $a_i = |G/H|/n_i$ and s satisfies

$$2s - 2 = \frac{|G|}{|H|}\Big(2t - 2 + \sum_{i=1}^{r}\Big(1 - \frac{1}{n_i}\Big)\Big).$$

Remark 2.8. We can use Theorems 2.6 and 2.7 to reconstruct the signature of G given the signature of H and the map $\theta : G \to G/H$; see Theorem 4.3 for example.

The inconsistency of the notation for the order of the elliptic generators of H in Theorems 2.6 and 2.7 is to preserve the form of the summand in the Riemann–Hurwitz formula. To avoid confusion, we shall clearly distinguish between the normal and nonnormal cases.

Remark 2.9. It is an immediate consequence of Theorem 2.6 that $t \leq s$. Note that this also implies, if G has genus vector $(t_1, \ldots, t_r)$, then G must have genus t_1.

All finite groups that can act on surfaces of genus 0 and 1 are well known; see for example [Tyszkowska and Weaver 2008] for genus 1 and [Wootton 2005] for genus 0. Most of the results and definitions we have introduced for groups acting on surfaces of genus $\sigma \geq 2$ hold for genus 0 and 1 too and will be relevant in our work. We summarize below.

Theorem 2.10. (i) *The possible finite groups and corresponding signatures for groups that can act on a surface of genus $\sigma = 1$ are shown in the table on the left. We call these signatures* toroidal signatures.

(ii) *The possible finite groups and corresponding signatures for groups that can act on a surface of genus $\sigma = 0$ are shown in the table on the right. We call these signatures* spherical signatures.

Group	Signature
$C_n \times C_m$	$(1; -)$
$(C_n \times C_m) \ltimes C_2$	$(0; 2, 2, 2, 2)$
$(C_n \times C_m) \ltimes C_3$	$(0; 3, 3, 3))$
$(C_n \times C_m) \ltimes C_4$	$(0; 2, 4, 4)$
$(C_n \times C_m) \ltimes C_6$	$(0; 2, 3, 6)$

Group	Signature
C_n	$(0; n, n)$
D_n	$(0; 2, 2, n)$
A_4	$(0; 2, 3, 3)$
S_4	$(0; 2, 3, 4)$
A_5	$(0; 2, 3, 5)$

Table 1. Groups of automorphisms of genus 1 surfaces (left) and those of genus 0 surfaces (right).

We finish this section with an explicit example to illustrate the use of these results.

Example 2.11. Consider the group

$$Q_8 = \langle x, y \mid x^4, x^2y^2, yxy^{-1} = x^{-1} \rangle$$

The vector (x^2, x^2, x, x, y, y) is a (0; 2, 2, 4, 4, 4, 4)-generating vector for Q_8. Using the Riemann–Hurwitz formula from Theorem 2.5, it follows that this is a group of automorphisms of a surface of genus

$$\begin{aligned}\sigma &= 1 + |G|(-1) + |G|/2 \sum_{i=1}^{6} (1 - 1/m_i) \\ &= 1 - 8 + 4\left(\frac{1}{2} + \frac{1}{2} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4}\right) = 9.\end{aligned}$$

Consider the normal subgroup $N = \langle x \rangle$ and the canonical homomorphism $\rho : G \to G/N$. We have $O(\rho(x^2)) = 1$, $O(\rho(x)) = 1$, and $O(\rho(y)) = 2$. By Theorem 2.6, it follows that H has signature $(s; 2, 2, 2, 2, 2, 2, 4, 4, 4, 4)$, where

$$s = 1 + |G|/|N|(g-1) + |G|/2|N| \sum_{i=1}^{6} (1 - 1/n_i) = 1 - 2 + \left(\frac{1}{2} + \frac{1}{2}\right) = 0.$$

3. Groups of genus level 1

We want to classify all genus level 1 group actions on all compact Riemann surfaces. In [Kallel and Sjerve 2001] all genus level 1 groups with genus level vector (0) are classified for groups acting on surfaces of genus $\sigma \geq 0$. Using Theorem 2.10 we get the following classification for groups acting on genus 1 surfaces with genus level 1.

Theorem 3.1. *Suppose G, a finite group, acts on a surface of genus 1 and has genus vector (1). Then $G \cong C_n \times C_m$, where C_n and C_m are finite cyclic groups.*

It remains to classify all genus level 1 groups acting on a surface of genus $\sigma \geq 2$ with genus vector (t) for $t \geq 1$. We start with the following important result.

Proposition 3.2. *Suppose G has signature $(t; m_1, m_2, \dots, m_r)$, where $t \geq 1$. Let H be a nontrivial proper subgroup of G that also has genus t. Then $t = 1$ and the elliptical generators $\gamma_i \in H$ for all i.*

Proof. We set $t = s$ in Equation (2) of Theorem 2.6 and rearrange, obtaining

$$2t - 2 + \sum_{i=1}^{r} \sum_{j=1}^{\rho_i} \left(1 - \frac{1}{n_{ij}}\right) = \frac{|G|}{|H|}\left(2t - 2\right) + \frac{|G|}{|H|} \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right),$$

$$2t-2-\frac{|G|}{|H|}\Big(2t-2\Big)=\frac{|G|}{|H|}\sum_{i=1}^{r}\Big(1-\frac{1}{m_i}\Big)-\sum_{i=1}^{r}\sum_{j=1}^{\rho_i}\Big(1-\frac{1}{n_{ij}}\Big),$$

$$2(1-t)\Big(\frac{|G|}{|H|}-1\Big)=\frac{|G|}{|H|}\sum_{i=1}^{r}\Big(1-\frac{1}{m_i}\Big)-\sum_{i=1}^{r}\sum_{j=1}^{\rho_i}\Big(1-\frac{1}{n_{ij}}\Big).$$

Now, we have $n_{ij} \le m_i$, since $n_{ij} \mid m_i$, and we know that $\rho_i \le |G|\,/\,|H|$. Furthermore, equality will hold in each case exactly when $\theta\,(\gamma_i)$ is the identity permutation, consisting of $|G|\,/\,|H|$ 1-cycles. Then

$$\begin{aligned}2(1-t)\Big(\frac{|G|}{|H|}-1\Big)&=\frac{|G|}{|H|}\sum_{i=1}^{r}\Big(1-\frac{1}{m_i}\Big)-\sum_{i=1}^{r}\sum_{j=1}^{\rho_i}\Big(1-\frac{1}{n_{ij}}\Big)\\&\ge\frac{|G|}{|H|}\sum_{i=1}^{r}\Big(1-\frac{1}{m_i}\Big)-\sum_{i=1}^{r}\frac{|G|}{|H|}\Big(1-\frac{1}{m_i}\Big)=0.\end{aligned}$$

Since $t \ge 1$, the inequality cannot be satisfied unless $t = 1$. Thus each γ_i acts as the identity permutation on the cosets of H, that is, $\gamma_i \in H$ for all i. □

Remark 3.3. A consequence of this result is that if G has genus $t > 1$, then all proper subgroups have genus strictly greater than t.

This allows us to classify all genus level 1 groups with genus level vector (t) for $t > 1$. Specifically, we have the following.

Corollary 3.4. *Suppose G acts on a surface of genus $\sigma \ge 2$ and has genus level vector (t) for some $t \ge 2$. Then $G \cong C_p$, where C_p is cyclic of prime order p; the possible signatures of G are*

$$(t;\underbrace{p,p,\dots,p}_{r\text{ times}}),$$

where r is at least 2 for all p and is even when $p = 2$; and finally, the genus σ equals $1+\frac{1}{4}p(t-1)(p-1)$.

An obvious but very useful consequence of this result is the following.

Corollary 3.5. *If G acts with signature $(1; m_1, m_2, \dots, m_r)$ and all $\gamma_i \in H$ for some subgroup H, then H has genus 1.*

The last case we need to consider is when a group with genus level vector (1) acts on a surface of genus $\sigma \ge 2$.

Theorem 3.6. *Suppose G acts on a surface of genus $\sigma \ge 2$ and has genus level vector (1). Then $G \cong C_{p^n}$ or $G \cong Q_8$.*

When $G \cong C_{p^n}$, the possible signatures are $(1;\overbrace{p,p,\dots,p}^{r\text{ times}})$, where r is at least 2 for all p and is even when $p = 2$; moreover $\sigma = 1+\frac{1}{4}p^{2n-1}(t-1)(p-1)r$.

In the opposite case the possible signatures are $(1; \underbrace{2, 2, \ldots, 2}_{r\text{ times}})$, *where r is odd*; *moreover* $\sigma = 8t + 2r - 7$.

Proof. Suppose the generating vector of G is $(\alpha, \beta, \gamma_1, \ldots, \gamma_r)$. Since every nontrivial subgroup has genus 1, Corollary 3.5 implies that all γ_i lie in each nontrivial subgroup of G. By assumption, $m_i \geq 2$, and $r \geq 1$ since $(1; -)$ does not satisfy the Riemann–Hurwitz formula for a group action on a surface of genus $\sigma \geq 2$. It follows that the intersection of all nontrivial subgroups of G contains each γ_i and hence is nontrivial, and thus G has a unique minimal nontrivial subgroup. The only groups that satisfy this condition are C_{p^n} and Q_{2^n}, by [Robinson 1995, Theorem 5.3.6]. We now proceed by cases.

First consider

$$G = Q_{2^n} = \langle x, y \mid x^{2^{n-1}}, x^{2^{n-2}} y^{-2}, yxy^{-1}x\rangle.$$

Let $H = \langle y^2 \rangle \trianglelefteq G$ and let $\theta : G \to G/H$ denote the canonical quotient map. Observe that $G/H \cong D_{2^{n-2}}$, the dihedral group of order 2^{n-1}. By our remarks above, all γ_i lie in H. Now $\alpha\beta\alpha^{-1}\beta^{-1}\gamma_1\gamma_2 \cdots \gamma_r = e$, so

$$\theta\,(e_G) = \theta(\alpha\beta\alpha^{-1}\beta^{-1}\gamma_1\gamma_2 \cdots \gamma_r) = [\theta\,(\alpha)\,, \theta\,(\beta)] = e_{G/H}.$$

It follows that $\theta\,(G)$ must be abelian, since it is generated by $\theta(\alpha)$ and $\theta(\beta)$. The only abelian dihedral group is the Klein 4 group. Hence $G \cong Q_8$.

Now consider

$$G \cong Q_8 = \langle x, y \mid x^4, x^2y^2, yxy^{-1}x\rangle.$$

Then G has a $(1; \underbrace{2, 2, \ldots, 2}_{r\text{ times}})$-generating vector $(y, x, y^2, y^2, \ldots, y^2)$ for any odd r. For even r, the generating vector must be of the form

$$(\alpha, \beta, y^2, y^2, \ldots, y^2)$$

for some α, $\beta \in G$. However, when r is even, the product of r copies of y^2 is the identity. It follows that α and β must commute and generate G, and no such generators of Q_8 exist, and hence r cannot be even. To finish, we need to show that every nontrivial subgroup of G has genus 1. First observe that the center $Z(G)$ is the intersection of all nontrivial subgroups. By Remark 2.9, since we are assuming G has genus 1, it suffices to prove that $Z(G)$ has genus 1. But this follows by Corollary 3.5. To determine σ, we simply apply Theorem 2.5.

Finally, consider $G = C_{p^n} = \langle x \mid x^{p^n}\rangle$. Since C_{p^n} is abelian and all γ_i are in the minimal subgroup $\langle x^{p^{n-1}}\rangle = \langle a\rangle$, the order of each γ_i is p. Moreover, the commutator $[\alpha, \beta]$ will be equal to the identity, and hence $\gamma_1 \ldots \gamma_r = e$. It follows that $r \geq 2$. Furthermore, if $p = 2$ then $\gamma_i = a$ for all i. In this case the product $\gamma_1 \ldots \gamma_r = e$ if and only if r is even.

To finish, we need to construct generating vectors. For r even, the vector $(x, x, a, a^{-1}, a, a^{-1}, \ldots, a, a^{-1})$ is a generating vector for G. For r odd, we know $p \neq 2$ and consequently $(x, x, a, a^{-1}, a, a^{-1}, \ldots, a, a^{-1}, a, a, a^{-2})$ is a generating vector for G. To determine σ, we simply apply Theorem 2.5. □

To summarize our results, we need the following definitions.

Definition 3.7. We define a Zassenhaus metacyclic group to be a group with presentation

$$\langle x, y \mid x^m, y^n, yxy^{-1}x^{-r}\rangle$$

, where $r^n \equiv 1 \mod (m)$ and $\gcd((r-1)n, m) = 1$. We denote such a group by $G_{m,n}(r)$.

Definition 3.8. We define a polyhedral group to be a finite subgroup of PSL $(2, \mathbb{C})$. Equivalently, polyhedral groups are the cyclic groups of order n for any n, the dihedral groups of order $2n$ for any n, the alternating groups A_4 and A_5, and the symmetric group S_4.

Combining the results of [Kallel and Sjerve 2001] with Theorems 3.1, 3.6 and Corollary 3.4, we get a complete classification of genus level 1 group actions. We summarize.

Theorem 3.9. *Suppose G is a finite group acting on a compact Riemann surface of genus $\sigma \geq 0$ and G has genus level* 1. *Then we have the following possibilities*:

(i) *If G has genus level vector* (0) *then G is either cyclic, generalized quaternion* (Q_{2^n}), *polyhedral or Zassenhaus metacyclic of the form $G_{p,4}(-1)$ for p an odd prime.*

(ii) *If G has genus level vector* (1) *and $\sigma = 1$, then $G = C_n \times C_m$, a direct product of two finite cyclic groups.*

(iii) *If G has genus level vector* (1) *and $\sigma \geq 2$, then $G \cong Q_8$, the quaternion group, or $G \cong C_{p^n}$, a cyclic group of prime power order p^n.*

(iv) *If G has genus level vector* (t) *for $t \geq 2$, then $G \cong C_p$, a cyclic group of prime order p^n.*

4. Determining p-groups

Suppose G is a nonsimple group of automorphisms acting on a surface X of genus $\sigma \geq 0$ with genus level vector $(t_1, \ldots, t_n)$. If N is a normal subgroup G with genus t_i, then G/N acts on the surface X/N of genus t_i. The genus level vector of the action of G/N on X/N is clearly related to the genus level vector of the action of G of N. The precise relationship is this:

Lemma 4.1. *Suppose G acting on X has genus level vector $(t_1, \dots, t_k)$ and $N \le G$ is a nontrivial normal subgroup of G of genus t_i. Then G/N has genus level vector $(m_1, \dots, m_l)$, where the m_j run over the genera of each subgroup $K \le G$ with $N \le K$.*

Proof. Suppose G and N are as given. If $\bar{K} \le G/N$, let $K > N$ denote its preimage in G given by the correspondence theorem. Then the quotient space $(X/N)/(\bar{K})$ is naturally homeomorphic to the quotient space X/K. In particular, the genus of the quotient space $(X/N)/(\bar{K})$ will be equal to the genus of the quotient space X/K. Conversely, using a similar argument, if $N \le K$, then the image of K in the quotient group G/N will have the same genus as K. The result follows. □

Here is a simple consequence of the relationship described in Lemma 4.1.

Corollary 4.2. *Suppose G has genus level vector $(t_1, \dots, t_k)$ for $k \ge 2$ and $N \le G$ is a nontrivial normal subgroup of G of genus t_i. Then unless G has genus level vector $(0, 1)$ and $t_i = 1$, the genus level vector of G/N is strictly shorter in length than the genus level vector of G.*

These observations suggest we can construct an inductive algorithm to help determine nonsimple groups of automorphisms of compact Riemann surfaces dependent upon the length of the genus level vector. Specifically, if the genus level of a group G is at least two, then except under the exception provided above in Corollary 4.2, the genus level of a nontrivial quotient group will be strictly smaller than the genus level of a given group. This, coupled with the classification of genus level 1 actions in Section 3, provides an inductive process for classification. Though in principle such an algorithm is possible, the calculations become difficult very quickly, especially the calculation of signatures. Therefore, rather than a general algorithm for all nonsimple groups, we shall construct an algorithm for p-groups. Henceforth assume that G is a p-group for some prime p. Before we construct the algorithm, we shall develop several results specific to p-groups.

Theorem 4.3. *Suppose that G is a p-group of order p^n acting on a surface X of genus σ with genus level vector $V = (t_1, \dots, t_k)$ and C is a normal cyclic prime subgroup of G with genus t_i, so C has signature*

$$(t_i; \underbrace{p, \dots, p}_{f \text{ times}}), \quad \text{where } f = \frac{2\sigma - 2 - 2p(t_i - 1)}{p - 1}.$$

Then the quotient group G/C acts on a surface of genus t_i with genus level vector $(t_1, t_{\beta_1}, \dots, t_{\beta_s})$, where $\beta_1, \dots, \beta_s$ run over the genera of each subgroup K with $C \le K \le G$. If the signature of G/C is $(t_1; m_1, \dots, m_r)$, the signature of G is

$$(t_1; a_1 m_1, a_2 m_2, \dots a_r m_r, \underbrace{p, \dots, p}_{l \text{ times}}), \quad \text{where } a_1, \dots, a_r \in \{1, p\},$$

and where f and l satisfy

$$f = lp^{n-1} + \sum_{i=1}^{r} \frac{(a_i - 1)p^{n-1}}{(p-1)m_i}.$$

Moreover, if $\mathcal{V} = (\alpha_1, \ldots, \alpha_t, \beta_1, \ldots, \beta_t, \gamma_1, \ldots, \gamma_r)$ is a generating vector for G, then $a_i = p$ if and only if $C \leq \langle \gamma_i \rangle$ and $\gamma_{r+i} \in C$ for all $i \geq 1$.

Proof. The specified action of G/C, the genus level vector, and generating vector are consequences of Lemma 4.1 and Theorems 2.5 and 2.7. To determine the signatures, we use a geometric argument similar to that presented in Proposition 3 of [Wootton 2005]. For brevity, let $K = G/C$. Let

$$\pi_C \colon X \to X/C, \quad \pi_K \colon X/C \to (X/C)/K \cong X/G, \quad \pi_G \colon X \to X/G,$$

denote the quotient maps, and if $F \colon Y \to Y/L$ is a holomorphic quotient map between compact Riemann surfaces and $\alpha \in Y/L$, let $M_F(\alpha)$ denote the multiplicity of a point in the fiber of $\pi_F^{-1}(\alpha)$. By the geometric definition of signature, see Definition 2.1, to determine the signature of G, we need to determine the genus of X/G and for each branch point α_i of π_G, the multiplicity of a point in the fiber $\pi_G^{-1}(\alpha_i)$. By assumption, the genus of G, and also K, is t_1. Therefore, we just need to determine the periods of G.

To determine the periods, we first make some simple observations. We have $\pi_G = \pi_K \circ \pi_C$, and for any $\alpha \in X/G$, it is easy to show that $M_{\pi_G}(\alpha) = M_{\pi_K}(\alpha) \cdot M_{\pi_C}(\beta)$, where $\beta \in \pi_K^{-1}(\alpha)$, see [Miranda 1995, page 53]. Also, since π_C is a cyclic prime cover of X/C of order p, all the periods of C will be equal to p. It follows that if $\alpha \in X/G$ is a branch point of π_G, then it will either have order m_i, order p or order $m_i p$ depending upon whether α is a branch point of π_K, the image of a branch point of π_C, or both. Since every branch point of π_K is also a branch point of π_G, it follows that G has signature

$$(t_1; a_1 m_1, a_2 m_2, \ldots a_r m_r, \underbrace{p, \ldots, p}_{l \text{ times}}),$$

where l denotes the number of branch points of π_G that are not branch points of π_K and $a_i \in \{1, p\}$.

To determine the relationship between the periods of G and C, we examine the orbits of branch points of π_C under the action of K. Suppose that β is a branch point of π_C and let $\alpha = \pi_K(\beta)$. The fiber $\pi_K^{-1}(\alpha)$ coincides with the orbit of β under K and hence has size $|K|/|\operatorname{Stab}_K(\beta)|$, where $\operatorname{Stab}_K(\beta)$ denotes the stabilizer of β under the action of K. However, $|\operatorname{Stab}_K(\beta)| = M_{\pi_K}(\alpha)$; see [Miranda 1995, Theorem 3.4]. It follows that if $\operatorname{Stab}_K(\beta)$ is trivial, there are $|K|/|\operatorname{Stab}_K(\beta)| = p^{n-1}$ points in the fiber $\pi_K^{-1}(\alpha)$. If $\operatorname{Stab}_K(\beta)$ is nontrivial, then $\operatorname{Stab}_K(\beta) = m_i$

for some i and we must have $a_i = p$ in the signature of G. In this case we get $|K|/|\operatorname{Stab}_K(\beta)| = p^{n-1}/m_i$. Since $a_j = 1$ for branch points of π_G that are not the image of a branch point of π_C, and the the number of branch points of π_G that are not branch points of π_K is equal to l, we obtain

$$f = lp^{n-1} + \sum_{i=1}^{r} \frac{(a_i - 1)p^{n-1}}{(p-1)m_i}. \qquad \square$$

As indicated by Corollary 4.2, the only barrier preventing us from constructing such an algorithm is the case where G has genus level vector (0, 1). The groups acting on a surface of genus $\sigma = 1$ with genus level vector (0, 1) are given in Theorem 2.10, so we only need to consider groups acting on surfaces of genus $\sigma \geq 2$.

Theorem 4.4. *Suppose G is a group acting on a surface X of genus $\sigma \geq 2$ with genus level vector* (0, 1) *with the property that if C is any normal cyclic prime subgroup, then C has genus* 1 *and G/C has genus vector* (0, 1). *Then one of the following completely describes G*:

(i) $G = Q_{2^n}$ *acts with signature*

$$(0; \underbrace{2, 2, \ldots, 2}_{r \text{ times}}, 4, 4, 4, 4)$$

and X has genus $\sigma = 2^{n-2}(r+2) + 1$;

(ii) *X has genus* 3 *and we have one of the following cases*:

- (a) $G = V_4$ *acts with signature* (0; 2, 2, 2, 2, 2, 2);
- (b) $G = C_4 \times C_2$ *acts with signature* (0; 2, 2, 4, 4);
- (c) $G = D_4$ *acts with signature* (0; 2, 2, 4, 4);
- (d) $G = D_4$ *acts with signature* (0; 2, 2, 2, 2, 2);
- (e) $G = C_4 \times C_4$ *acts with signature* (0; 4, 4, 4);
- (f) *G of order* 16 *with presentation* $\langle x, y \mid x^2, y^8, xyx^{-}1y^{-5}\rangle$ *acts with signature* (0; 2, 8, 8);
- (g) *G of order* 16 *with presentation*

  $$\langle x, y, z \mid x^2, y^2, z^4, [y, z], [x, z], xyx^{-}1z^{-2}y^{-}1\rangle$$

  acts with signature (0; 2, 2, 2, 4);
- (h) *G of order* 32 *with presentation*

  $$\langle x, y, z \mid x^2, y^2, z^8, yzy^{-1}z^{-5}, xyx^{-1}z^{-4}y^{-1}, xzx^{-1}z^{-3}y^{-1}\rangle$$

  acts with signature (0; 2, 4, 8).

Proof. Fix a normal cyclic prime subgroup $C \trianglelefteq G$ of genus 1. Since C has genus 1, the quotient space will have genus 1 and so by assumption, the group G/C will have genus vector (0, 1) and act on a surface of genus 1. It follows that we can only have $p = 2$ or $p = 3$. Before we determine G explicitly, we make some simple observations about the structure of G to determine all such groups that can act on surfaces of genus $\sigma \leq 3$.

First, we are assuming that both G and the quotient group G/C have genus vector (0, 1). It follows that G has genus 0, there is a subgroup K with genus 1 of order at least p^2 with $C \leq K$, and consequently G must have order at least p^3. We shall examine K. In [Broughton 1991], all groups and signatures that can act on surfaces of genus 2 and 3 are classified. By simply checking these lists, we see that for genus 2 no such K exists, and for genus 3, no such K exists for $p = 3$. To determine all 2-groups acting on genus 3 surfaces with these properties, we simply proceed through the list in [Broughton 1991] to extract all the groups and signatures that satisfy the requirements.

Henceforth, we shall assume that the genus of X is at least 4. We shall show that G has a unique cyclic prime subgroup. Suppose not and let H denote a cyclic prime subgroup different from C. Then the group $L \trianglelefteq \langle C, H \rangle$ is elementary abelian of order p^2. By Theorem 3.6, the genus of L must be 0, so L will have signature

$$(0; \underbrace{p, p, \ldots, p}_{r \text{ times}}) \quad \text{for some } r.$$

Let $\mathscr{V}$ denote a corresponding generating vector for L. Now if $K \leq L$ is any nontrivial proper subgroup, by assumption, the genus of K must be 0 or 1. Observe that the genus of $K \leq L$ is completely determined by the number of elements of $\mathscr{V}$ with nontrivial image under the quotient map $\rho \colon L \to L/K$. Specifically, if the genus of K is 0, precisely two generators will have nontrivial image. If the genus of K is 1 and $p = 2$, four will have nontrivial image; if $p = 3$, three will have nontrivial image. We consider the cases $p = 2$ and $p = 3$ separately.

If $p = 2$, then 4 elements of $\mathscr{V}$ will have nontrivial image under $\rho \colon L \to L/C$. It follows that all remaining elements of $\mathscr{V}$ lie in C, and hence will have nontrivial image under $\rho \colon L \to L/K$ for any other nontrivial subgroup $K \leq L$. This can only happen if $r = 6$ and each nontrivial subgroup $K \leq L$ contains two elements of $\mathscr{V}$, or $r = 5$ and two subgroups each contain 2 and the other contains 3. In each of these cases, we apply the Riemann–Hurwitz formula and find that either $\sigma = 2$ or $\sigma = 3$, counter to our assumption that $\sigma \geq 4$.

If $p = 3$, we use a similar argument. Specifically, the only possible signature for L is (0, 3, 3, 3, 3), where a (0, 3, 3, 3, 3)-generating vector for L has one element in each nontrivial subgroup K of L. However, it is easy to show that no such generating vector for L exists and hence no such group exists. We shall henceforth assume that G has a unique cyclic prime subgroup.

Since G has a unique cyclic prime subgroup, we can use [Robinson 1995, Theorem 5.3.6] and conclude that either $G = Q_{2^n}$ or $G = C_{p^n}$. We consider each of these cases. Suppose first that $G \cong C_{p^n}$ and let K be the subgroup of maximal order of G that has genus 1. Then the signature of K will be $(1; p, \dots, p)$. Since G/K will have genus level vector (0) and act on a surface of genus 1, it must either be C_4, or cyclic of prime order 2 or 3. Using the toroidal signatures and Theorem 4.3, we can reconstruct the signature of G for each of these cases. If G/K is cyclic of order 3, then G will have signature $(0; 3, \dots, 3, 3a, 3b, 3c)$, where a, b and c are either 1 or 3. Since G must be generated by elliptic generators, it follows that G can have order at most 9 contradicting the fact that $|G| \geq p^3$. Likewise, if G/C is cyclic of order 2, then G will have signature $(0; 2, \dots, 2, 2a, 2b, 2c, 2d)$, where a, b, c and d are either 1 or 2. It follows that G has order at most 4, again a contradiction. Finally, if $G/K = C_4$, then there would exist L with $L > K$ and $G/L = C_2$, which we have already shown cannot happen.

Now suppose that $G \cong Q_{2^n}$ and fix the presentation

$$\langle x, y \mid x^{2^{n-1}}, x^{2^{n-2}} y^{-2}, yxy^{-1}x \rangle$$

for G. The group $L = \langle x \rangle$ is a cyclic subgroup of order 2^{n-1}, where $n \geq 3$. By our previous observations, L must have genus 1. Similar to the last case, using the fact that $G/L = C_2$ and has genus level vector (0), we can reconstruct the signature for G. Specifically, the signature for G will be $(0; 2, 2, \dots, 2, 2a, 2b, 2c, 2d)$, where a, b, c, and d are either 1 or 2. Any generating vector for G will be of the form $(\gamma_1, \dots, \gamma_r, \gamma_{r+1}, \gamma_{r+2}, \gamma_{r+3}, \gamma_{r+4})$, where $\gamma_{r+i} \notin L$ for $1 \leq i \leq 4$. However, if any γ_{r+i} has order 2, then by uniqueness of C, it will lie in C and hence L. It follows that each γ_{r+i} has order 4 and so the only possible signature for G is $(0; 2, 2, \dots, 2, 4, 4, 4, 4)$. We finish by showing that this signature is admissible for every possible value of r.

We know that the product $\gamma_1 \gamma_2 \cdots \gamma_r$ is either e or y^2. We set

$$\gamma_{r+1} = yx, \; \gamma_{r+2} = y^3 x, \; \gamma_{r+3} = y.$$

If the product is e, we set $\gamma_{r+4} = y^3$; if it is y^2, we set $\gamma_{r+4} = y$. In either case, the generators multiply to e. Moreover, $x = \gamma_{r+3}\gamma_{r+2}$ and $y = \gamma_{r+3}$ generate Q_{2^n} and thus, for each possible r, we have constructed a generating vector. Application of Theorem 2.7 shows, for this generating vector, all subgroups of L have genus 1 and the Riemann–Hurwitz formula gives the signature for X. □

Putting our work together, we can now outline the inductive method to determine all p-groups that can act on a compact Riemann surface dependent upon the genus level of the group. We note that all groups are known for genus 0 and 1 surfaces, so we restrict to surfaces of genus $\sigma \geq 2$. Fix an n and assume that we know all

p-group actions for all p-groups with genus level $k \leq n-1$. To determine the possible p-groups actions, with genus level n, we do the following:

(i) If $n = 2$, determine each possible p group G with genus level vector $(0, 1)$ with the property that if C is any normal cyclic prime subgroup, then the group G/C has genus level vector $(0, 1)$. This can be done using Theorem 4.4.

(ii) Else, any such group G of order p^f admits a cyclic subgroup of order p such that G/C is a group of automorphisms of order p^{f-1} acting on a surface with a strictly shorter genus level vector than the genus level vector of G. In particular, by assumption, we know the structure of every possible $K = G/C$. Therefore, to determine G, and its signature, we can do the following:

  (a) Determine the solutions of the short exact sequence
  $$1 \to C \to G \to K \to 1$$
  for each possible K.

  (b) For a given G and K from (a), each possible signature $\mathcal{T}$of G can be determined from K using Theorem 4.3. Specifically, we run over all the possibilities by choosing $a_i = 1$ or p for each i.

  (c) For a G and signature $\mathcal{T}$ from (b), we can determine all possible generating vectors and check, for each vector, that G has genus level n. If no valid generating vector exists, then there does not exist an action by G with this signature and with genus level n.

5. Examples

We finish with some explicit examples to illustrate our results.

Example 5.1. It is easy to determine all genus level 1 group actions of a cyclic prime group G of order p on a surface of genus $\sigma \geq 0$. Specifically, the signature for such a group will be

$$(t; \underbrace{p, \ldots, p}_{r \text{ times}}), \quad \text{where } \sigma = 1 + p(t-1) + \frac{r(p-1)}{2}.$$

For $\sigma = 0$ or 1, Theorem 2.10 gives all possible signatures. For $\sigma \geq 2$, it is easy to show that a

$$(t; \underbrace{p, \ldots, p}_{r \text{ times}})\text{-generating vector for } G$$

exists for all primes p, provided $r \neq 1$ and when $p = 2$, r is even. The genus on which such a G acts is calculated using the Riemann–Hurwitz formula.

Example 5.2. We can use our results and Example 5.1 to classify all group actions by cyclic groups of order p^2 for any prime p with genus level 2 on surfaces of genus

$\sigma \geq 2$. Let G be such a group. If C is the cyclic prime subgroup of G, Example 5.1 shows the signature of G/C is

$$(t; \underbrace{p, \dots, p}_{r \text{ times}}) \quad \text{with } r \neq 1$$

and if $p = 2$, then r is even. Using Theorem 4.3, since $C \leq \langle \gamma \rangle$ for any nontrivial $\gamma \in G$, it follows that there are r elements of order p^2 in the signature of G. Therefore, the possible signatures of G are

$$(t, \underbrace{p, \dots, p}_{r_1 \text{ times}}, \underbrace{p^2, \dots, p^2}_{r \text{ times}})$$

for certain values of r_1. Using Theorem 2.7, the genus level vector of such an action will be (t, t_1), where $t_1 = 1 + p(t-1) + \frac{1}{2}r(p-1)$. We shall show that such a G exists for every possible r_1 and t. To show this, we need to show that there exists a

$$(t, \underbrace{p, \dots, p}_{r_1 \text{ times}}, \underbrace{p^2, \dots, p^2}_{r \text{ times}})\text{-generating vector for } G.$$

This can be done in a similar way to the final argument in Theorem 3.6. The genus on which such a G acts is calculated using the Riemann–Hurwitz formula.

Example 5.3. We can use our results and the Example 5.1 to classify all elementary abelian group actions of order p^2 for any prime p with genus level 2 on surfaces of genus $\sigma \geq 2$. Let G be such a group and let C be a cyclic prime subgroup of G. Using Example 5.1, the signature of G/C is

$$(t; \underbrace{p, \dots, p}_{r \text{ times}}).$$

According to Theorem 4.3, the fact that all elements of G have order p implies the signature of G is

$$(t, \underbrace{p, \dots, p}_{r_1 \text{ times}}, \underbrace{p, \dots, p}_{r \text{ times}})$$

for certain values of r_1. Using Theorem 2.7, the genus level vector of such an action is (t, t_1), where

$$t_1 = 1 + p(t-1) + \frac{r(p-1)}{2}.$$

To determine for which values of t and r_1 such a G exists, we need to construct generating vectors, or show that no such generators exists. We do this on a case-by-case basis. For the remainder of the proof, assume that $G = \langle x, y \rangle$ and let $n = r + r_1$ denote the number of elliptic generators.

First, suppose that $n=0$. Then $t \geq 2$ since we are assuming $\sigma \geq 2$. A $(t;-)$-generating vector for G is the vector $(x, y, 1, \ldots, 1)$. Using Theorem 2.7, the genus of every possible nontrivial subgroup is t_1. It follows that such a G exists for every possible choice of $t \geq 2$ with $n=0$.

Now suppose that the number of elliptic generators is $n>0$ and $t \neq 0, 1$. By assumption, G has genus level 2 and so by Proposition 3.2, every nontrivial subgroup of G must have genus t_1. Consider a generating vector

$$(\alpha_1, \ldots, \alpha_t, \beta_1, \ldots, \beta_t, \gamma_1, \ldots, \gamma_n)$$

for G. By Theorem 2.7, the genus of each nontrivial $C \leq G$ is completely determined by the number of $\gamma_i \in C$. In particular, every nontrivial subgroup $C \leq G$ has the same genus provided each contains the same number of elliptical generators, γ_i. Since there are $p+1$ nontrivial subgroups of G, it follows that the only possible signature for G is

$$(t, \underbrace{p, \ldots, p}_{n \text{ times}}),$$

where $n = r(p+1)/p$. For any such r, and $t \geq 0$, it is easy to construct a generating vector for G satisfying these properties by generalizing the final argument in Theorem 3.6. For example, if r is even and $x_1, \ldots, x_{p+1}$ is a set of nontrivial elements from distinct subgroups of G, then

$$(\underbrace{1, \ldots, 1}_{2t \text{ times}}, \underbrace{x_1, x_1^{-1}, \ldots, x_1, x_1^{-1}}_{(r/2) \text{ times}}, \ldots, \underbrace{x_{p+1}, x_{p+1}^{-1}, \ldots, x_{p+1}, x_{p+1}^{-1}}_{(r/2) \text{ times}})$$

is a generating vector for G. It follows that such a G exists for every possible choice of $t \geq 2$ provided $n = r(p+1)/p$, and no such group exists otherwise.

The next case we consider is when $t=1$. In this case, G could have subgroups with genus 1 as well as with genus t_1. If all the subgroups of G of order p have genus t_1, then it is easy to imitate the proof of the previous case to show that a generating vector for G exists only if $n = r(p+1)/p$. Therefore, we only need consider the case when at least one of the subgroups of G of order p has genus 1. Let $C \leq G$ be a subgroup with genus 1. By Proposition 3.2, then $\gamma_i \in C$ for all i. It follows that for any other subgroup $K \neq C$, $\gamma_i \notin K$ for all i. Therefore, using Theorem 2.7, the genus of every subgroup $K \neq C$ of order p will be the same. Without loss of generality, suppose that $C = \langle x \rangle$. It is easy to construct a generating vector for G satisfying these properties by generalizing the final argument in Theorem 3.6 and a generating vector always exists unless $p=2$ and n is odd.

The last case we need to consider is $t=0$. As above, if every subgroup of order p has genus t_1, then a generating vector for G exists only if $n=r(p+1)/p$. Therefore, we only need consider the case when at least one of the subgroups of G of order

p has genus 0. In [Wootton 2007b], all G containing more than one cyclic prime subgroup with genus 0 were classified and there are only two possibilities for G. In the first case, G has signature $(0; p, p, p)$ and generating vector $(x, y, (xy)^{-1})$ and three genus 0 subgroups - the ones generated by x, y, and xy. Using Theorem 2.7, all other cyclic subgroups of G will have the same genus, and hence G has genus level 2. In the second case, G has signature $(0; p, p, p, p)$ and generating vector (x, y, x^{-1}, y^{-1}) and two genus 0 subgroups - the ones generated by x and y. Using Theorem 2.7, all other cyclic subgroups of G will have the same genus, and hence G has genus level 2. Therefore, the last case we need to consider is when $t = 0$ and G contains a unique cyclic subgroup of genus 0. In this case, if C is the cyclic subgroup with genus 0 and $(\gamma_1, \ldots, \gamma_n)$ is a generating vector for G, then by Theorem 4.3, $\gamma_i \in C$ for all but two values of i, say $n-1$ and n. As remarked previously, the only way two other subgroups C_1 and C_2 can have the same genus is if they contain the same number of γ_i. This occurs exactly when G contains just two nontrivial, proper subgroups other than C, and each subgroup contains one of γ_{n-1} or γ_n. Such restrictions force $p = 2$. Through our observations, the only possible generating vectors of G are of the form

$$(\underbrace{x, \ldots, x}_{n-2\,\text{times}}, (xy), y).$$

Clearly this defines a generating vector when n is odd and does not for n even, and thus such a G exists only under these circumstances.

Acknowledgement

The authors are grateful to Michael Sawdy of Rainier Middle School for his contributions to this work.

References

[Benim 2008] R. Benim, "Classification of quasplatonic abelian groups and signatures", *Rose-Hulman Undergraduate Math. J.* **9**:1 (2008).

[Breuer 2000] T. Breuer, *Characters and automorphism groups of compact Riemann surfaces*, London Math. Soc. Lecture Note Ser. **280**, Cambridge University Press, Cambridge, 2000. MR 2002i:14034 Zbl 0952.30001

[Broughton 1991] S. A. Broughton, "Classifying finite group actions on surfaces of low genus", *J. Pure Appl. Algebra* **69**:3 (1991), 233–270. MR 92b:57021 Zbl 0722.57005

[Bujalance et al. 2003] E. Bujalance, F. J. Cirre, J. M. Gamboa, and G. Gromadzki, "On compact Riemann surfaces with dihedral groups of automorphisms", *Math. Proc. Cambridge Philos. Soc.* **134**:3 (2003), 465–477. MR 2004c:20086 Zbl 1059.30030

[Fuertes and González-Diez 2007] Y. Fuertes and G. González-Diez, "On unramified normal coverings of hyperelliptic curves", *J. Pure Appl. Algebra* **208**:3 (2007), 1063–1070. MR 2007i:14026 Zbl 1123.14019

[Harvey 1966] W. J. Harvey, "Cyclic groups of automorphisms of a compact Riemann surface", *Quart. J. Math. Oxford Ser.* (2) **17** (1966), 86–97. MR 34 #1511 Zbl 0156.08901

[Kallel and Sjerve 2001] S. Kallel and D. Sjerve, "Genus zero actions on Riemann surfaces", *Kyushu J. Math.* **55**:1 (2001), 141–164. MR 2002m:14022 Zbl 0997.30035

[Miranda 1995] R. Miranda, *Algebraic curves and Riemann surfaces*, Graduate Studies in Mathematics **5**, American Mathematical Society, Providence, RI, 1995. MR 96f:14029 Zbl 0820.14022

[Robinson 1995] D. Robinson, *A course in the theory of groups*, Grad. Texts in Math. **80**, Springer, 1995. Zbl 0836.20001

[Singerman 1970] D. Singerman, "Subgroups of Fuschian groups and finite permutation groups", *Bull. London Math. Soc.* **2** (1970), 319–323. MR 43 #7519

[Tyszkowska and Weaver 2008] E. Tyszkowska and A. Weaver, "Exceptional points in the elliptic-hyperelliptic locus", *J. Pure Appl. Algebra* **212**:6 (2008), 1415–1426. MR 2009d:30093 Zbl 1137.14020

[Wootton 2005] A. Wootton, "Non-normal Belyĭ p-gonal surfaces", pp. 95–108 in *Computational aspects of algebraic curves* (Moscow, ID, 2005), edited by T. Shaska, Lecture Notes Ser. Comput. **13**, World Sci. Publ., Hackensack, NJ, 2005. MR 2006g:14054

[Wootton 2007a] A. Wootton, "Defining equations for cyclic prime covers of the Riemann sphere", *Israel J. Math.* **157** (2007), 103–122. MR 2009b:30084 Zbl 1109.30036

[Wootton 2007b] A. Wootton, "The full automorphism group of a cyclic p-gonal surface", *J. Algebra* **312**:1 (2007), 377–396. MR 2008c:14043 Zbl 1117.30034

Received: 2008-11-05 Revised: 2009-07-02 Accepted: 2009-07-06

quicksilver.falcon@gmail.com — *University of Oregon, Department of Mathematics, Fenton Hall, 1222, Eugene, OR 97403, United States*
http://uoregon.edu/~arbo

kbenkowski@hotmail.com — *Western Michigan University, Department of Mathematics, 3319 Everett Tower, Kalamazoo, MI 49008, United States*

coateb2@rpi.edu — *Rensselaer Polytechnic Institute, School of Architecture, Greene Building, 110 8th street, Troy, NY 12180, United States*

nordstro@up.edu — *University of Portland, Department of Mathematics, 5000 North Willamette Blvd., Portland, OR 97203, United States*
http://faculty.up.edu/nordstro/

cmp02006@mymail.pomona.edu — *Pomona College, Department of Mathematics, 610 North College Avenue, Claremont, CA 91711, United States*

wootton@up.edu — *University of Portland, Department of Mathematics, 5000 North Willamette Blvd., Portland, OR 97203, United States*
http://faculty.up.edu/wootton/